Focused Natural Deduction*

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Abstract. Natural deduction for intuitionistic linear logic is known to be full of non-deterministic choices. In order to control these choices, we combine ideas from intercalation and focusing to arrive at the calculus of *focused natural deduction*. The calculus is shown to be sound and complete with respect to first-order intuitionistic linear natural deduction and the backward linear focusing calculus.

1 Introduction

The idea of focusing goes back to Andreoli [1] and gives an answer to the question on how to control non-determinism in proof search for the classical sequent calculus for fragments of linear logic. It removes the "bureaucracy" that arises due to the permutability of inference rules. Historically, the idea of focusing has influenced a great deal of research [2,3,5], all centered around sequent style systems. There is one area, however, which has been suspiciously neglected in all of this: natural deduction. In our view, there are two explanations.

First, for various reasons we don't want to get into here, most theorem proving systems are based on the sequent calculus. Therefore it is not surprising that most of the effort has gone into the study of sequent calculi as evidenced by results in uniform derivations [8] and of course focusing itself.

Second, it is possible to characterize the set of more "normalized" proofs for natural deduction in intuitionistic logic. Many such characterizations have been given in the past, for example, the intercalation calculus [9,10]. Backward chaining from the conclusion will only use introduction rules until only atomic formulas are leftover, and similarly, forward chaining will only use elimination rules.

Theorem provers for natural deduction, implementations of logical frameworks and bi-directional type checkers are all inspired by this idea of intercalation. One of its advantages is that the aforementioned "bureaucracy" never arises in part because the book-keeping regarding hypotheses is done externally and not within the formal system. In this paper we refine intercalation by ideas from focusing, resulting in a calculus with an even stricter notion of proof. This is useful when searching for and working with natural deduction derivations.

The hallmark characteristic of focusing is its two phases. First, invertible rules are applied eagerly, until both context and goal are non-invertible. This phase

* This work was in part supported by NABITT grant 2106-07-0019 of the Danish Strategic Research Council.

C. Fermüller and A. Voronkov (Eds.): LPAR-17, LNCS 6397, pp. 157–171, 2010.

is called the inversion or *asynchronous* phase. Second, a single formula taken from the context or the goal is selected and *focused* upon, applying a maximal sequence of non-invertible rules to this formula and its subformulas. This is known as the focusing or *synchronous* phase. The central idea of focusing is to polarize connectives into two groups. A connective is flagged *negative* when the respective introduction rule is invertible but the elimination rule(s) are not, and it is considered *positive* in the opposite case. Each connective of linear logic is either positive or negative, the polarity of atoms is uncommitted.

The main motivation for our work is to explore how proof theory, natural deduction in particular, can be used to explain concurrent systems. In that our goals are similar to those of the concurrent logical framework research project CLF [11]. Pragmatically speaking, in this paper, we remove all bureaucratic non-determinism from logical natural deduction derivations (as described by focusing), so that we can, in future work, use the remaining non-determinism to characterize concurrent systems. That the resulting focused natural deduction calculus is complete is perhaps the most important contribution of the paper.

Consider for example the judgment $a, a \multimap \mathbf{1} \Vdash \mathbf{1} \oplus b \Uparrow$, for which we would like to find a derivation in the intercalation calculus. The judgment should be read as find a *canonical* proof of $\mathbf{1} \oplus b$ from the linear assumptions a and $a \multimap \mathbf{1}$. As nothing is known about b, one might hope (at least intuitively) for one unique derivation. However, there are two, which we will next inspect in turn.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{a \multimap \mathbf{1} \Vdash a \multimap \mathbf{1} \Downarrow}\,\mathrm{hyp} \qquad \dfrac{\dfrac{}{a \Vdash a \Downarrow}\,\mathrm{hyp}}{a \Vdash a \Uparrow}\,{\Downarrow\Uparrow}}{a, a \multimap \mathbf{1} \Vdash \mathbf{1} \Downarrow}\,{\multimap}\mathrm{E} \qquad \dfrac{}{\cdot \Vdash \mathbf{1} \Uparrow}\,\mathbf{1}\mathrm{I}}{a, a \multimap \mathbf{1} \Vdash \mathbf{1} \Uparrow}\,\mathbf{1}\mathrm{E}}{a, a \multimap \mathbf{1} \Vdash \mathbf{1} \oplus b \Uparrow}\,\oplus\mathrm{I}_1
$$

The experienced eye might have already spotted, that this derivation is *negatively focused*, because the two upper rules in the left-most chain of inference rules hyp and $\multimap$E form a focus. However, it is not *positively focused*. The rule $\oplus \mathrm{I}_1$ and the rule $\mathbf{1}$I ought to form a focus (both are positively polarized), but the $\mathbf{1}$E rule breaks it. This rule is the only inversion rule in this derivation, and hence it is *maximally inverted*.

$$
\dfrac{\dfrac{\dfrac{}{a \multimap \mathbf{1} \Vdash a \multimap \mathbf{1} \Downarrow}\,\mathrm{hyp} \qquad \dfrac{\dfrac{}{a \Vdash a \Downarrow}\,\mathrm{hyp}}{a \Vdash a \Uparrow}\,{\Downarrow\Uparrow}}{a, a \multimap \mathbf{1} \Vdash \mathbf{1} \Downarrow}\,{\multimap}\mathrm{E} \qquad \dfrac{\dfrac{}{\cdot \Vdash \mathbf{1} \Uparrow}\,\mathbf{1}\mathrm{I}}{\cdot \Vdash \mathbf{1} \oplus b \Uparrow}\,\oplus\mathrm{I}_1}{a, a \multimap \mathbf{1} \Vdash \mathbf{1} \oplus b \Uparrow}\,\mathbf{1}\mathrm{E}
$$

By permuting $\mathbf{1}$E and $\oplus \mathrm{I}_1$ we can restore the focus, and again, the inversion phase of this derivation ($\mathbf{1}$E) is maximal. In summary, for intuitionistic linear logic, the intercalation calculus does not rule out as many derivations as it should.

In the remainder of this paper, we define and discuss the calculus of *focused natural deductions*. It is a simple, yet deep, generalization of the intercalation formulation. Among the two derivations above only the second one is derivable in focused natural deduction. The main idea behind this calculus is to distinguish negative from positive connectives and to make the related coercions explicit as connectives ${}^{\uparrow}P$ and ${}^{\downarrow}N$. We call these connectives *delay* connectives, because they effectively delay positive and negative focusing phases. To ensure maximal asynchronous phases, we use a generalization of the *patterns* introduced by Zeilberger in [12].

The paper is organized as follows. In Section 2 we introduce the focused natural deduction calculus. Next, we show soundness and completeness with respect to first-order intuitionistic linear natural deduction in Section 3 and with respect to the backward linear focusing calculus of Pfenning, Chaudhuri and Price [4] in Section 4. We conclude and assess results in Section 5.

2 Natural Deduction for Intuitionistic Linear Logic

First we give the definition of the intercalation calculus. The syntax of linear logic is standard.

$$A, B, C ::= a \mid A \otimes B \mid \mathbf{1} \mid A \oplus B \mid \mathbf{0} \mid \exists x.A \mid A \,\&\, B \mid \top \mid A \multimap B \mid !A \mid \forall x.A$$

As for the judgment, we use a two zone formulation with a turnstile with double bars, $\Gamma; \Delta \Vdash A \Uparrow$, which reads as there is a canonical proof of A from intuitionistic assumptions Γ and linear assumptions Δ. Conversely, we define $\Gamma; \Delta \Vdash A \Downarrow$ for atomic derivation of A. The inference rules are standard and depicted in Figure 1.

2.1 Focused Natural Deduction

We split the connectives of linear logic into two disjoint groups based on their inversion properties. Connectives that are invertible on the right of the turnstile and non-invertible on the left become negative connectives. Conversely, connectives that are invertible on the left and non-invertible on the right become positive connectives. This gives us the following syntax of polarized formulas:

$$P, Q ::= a^+ \mid P \otimes Q \mid \mathbf{1} \mid P \oplus Q \mid \mathbf{0} \mid \exists x.P \mid !N \mid {}^{\downarrow}N$$
$$N, M ::= a^- \mid N \,\&\, M \mid \top \mid P \multimap N \mid \forall x.N \mid {}^{\uparrow}P$$

We use P, Q for positive propositions, and N, M for negative propositions. We use A, B for propositions where the polarity does not matter. The syntax is similar to the ones presented in [2,7]. Additionally we use the following shorthand:

$$\gamma^+ ::= {}^{\uparrow}P \mid a^-, \qquad \gamma^- ::= {}^{\downarrow}N \mid a^+$$

$$
\frac{\Gamma; \Delta \Vdash a \Downarrow}{\Gamma; \Delta \Vdash a \Uparrow}\ {\Downarrow\Uparrow}
\qquad
\frac{}{\Gamma; A \Vdash A \Downarrow}\ \mathrm{hyp}
\qquad
\frac{}{\Gamma, A; \cdot \Vdash A \Downarrow}\ \mathrm{uhyp}
$$

$$
\frac{}{\Gamma; \cdot \Vdash \mathbf{1} \Uparrow}\ \mathbf{1}\mathrm{I}
\qquad
\frac{\Gamma; \Delta_1 \Vdash \mathbf{1} \Downarrow \quad \Gamma; \Delta_2 \Vdash C \Uparrow}{\Gamma; \Delta_1, \Delta_2 \Vdash C \Uparrow}\ \mathbf{1}\mathrm{E}
\qquad
\frac{}{\Gamma; \Delta \Vdash \top \Uparrow}\ \top\mathrm{I}
\qquad
\frac{\Gamma; \Delta_1 \Vdash \mathbf{0} \Downarrow}{\Gamma; \Delta_1, \Delta_2 \Vdash C \Uparrow}\ \mathbf{0}\mathrm{E}
$$

$$
\frac{\Gamma; \Delta_1 \Vdash A \Uparrow \quad \Gamma; \Delta_2 \Vdash B \Uparrow}{\Gamma; \Delta_1, \Delta_2 \Vdash A \otimes B \Uparrow}\ \otimes\mathrm{I}
\qquad
\frac{\Gamma; \Delta_1 \Vdash A \otimes B \Downarrow \quad \Gamma; \Delta_2, A, B \Vdash C \Uparrow}{\Gamma; \Delta_1, \Delta_2 \Vdash C \Uparrow}\ \otimes\mathrm{E}
$$

$$
\frac{\Gamma; \Delta \Vdash A_i \Uparrow}{\Gamma; \Delta \Vdash A_1 \oplus A_2 \Uparrow}\ \oplus\mathrm{I}_i
\qquad
\frac{\Gamma; \Delta_1 \Vdash A \oplus B \Downarrow \quad \Gamma; \Delta_2, A \Vdash C \Uparrow \quad \Gamma; \Delta_2, B \Vdash C \Uparrow}{\Gamma; \Delta_1, \Delta_2 \Vdash C \Uparrow}\ \oplus\mathrm{E}
$$

$$
\frac{\Gamma; \Delta \Vdash A \Uparrow \quad \Gamma; \Delta \Vdash B \Uparrow}{\Gamma; \Delta \Vdash A \,\&\, B \Uparrow}\ \&\mathrm{I}
\qquad
\frac{\Gamma; \Delta \Vdash A_1 \,\&\, A_2 \Downarrow}{\Gamma; \Delta \Vdash A_i \Downarrow}\ \&\mathrm{E}_i
$$

$$
\frac{\Gamma; \Delta \Vdash [a/x]A \Uparrow}{\Gamma; \Delta \Vdash \forall x.A \Uparrow}\ \forall\mathrm{I}^a
\qquad
\frac{\Gamma; \Delta \Vdash \forall x.A \Downarrow}{\Gamma; \Delta \Vdash [t/x]A \Downarrow}\ \forall\mathrm{E}
$$

$$
\frac{\Gamma; \Delta, A \Vdash B \Uparrow}{\Gamma; \Delta \Vdash A \multimap B \Uparrow}\ {\multimap}\mathrm{I}
\qquad
\frac{\Gamma; \Delta_1 \Vdash A \multimap B \Downarrow \quad \Gamma; \Delta_2 \Vdash A \Uparrow}{\Gamma; \Delta_1, \Delta_2 \Vdash B \Downarrow}\ {\multimap}\mathrm{E}
$$

$$
\frac{\Gamma; \cdot \Vdash A \Uparrow}{\Gamma; \cdot \Vdash !A \Uparrow}\ !\mathrm{I}
\qquad
\frac{\Gamma; \Delta_1 \Vdash !A \Downarrow \quad \Gamma, A; \Delta_2 \Vdash C \Uparrow}{\Gamma; \Delta_1, \Delta_2 \Vdash C \Uparrow}\ !\mathrm{E}
$$

$$
\frac{\Gamma; \Delta \Vdash [t/x]A \Uparrow}{\Gamma; \Delta \Vdash \exists x.A \Uparrow}\ \exists\mathrm{I}
\qquad
\frac{\Gamma; \Delta_1 \Vdash \exists x.A \Downarrow \quad \Gamma; \Delta_2, [a/x]A \Vdash C \Uparrow}{\Gamma; \Delta_1, \Delta_2 \Vdash C \Uparrow}\ \exists\mathrm{E}^a
$$

Fig. 1. Linear natural deduction (in intercalation notation)

Patterns. We use the concept of patterns, as seen in [6,12], to capture the decomposition of formulas that takes place during the asynchronous phase of focusing. The previous notion of patterns is extended to work with unrestricted hypotheses and quantifiers.

Pattern judgments have the form $\Gamma; \Delta \vDash P$ and $\Gamma; \Delta \vDash N > \gamma^+$, the latter of which corresponds to decomposing N into γ^+ using only negative elimination rules. These judgments are derived using the inference rules given in Figure 2.

We require that the variable a in the judgments concerning the quantifiers satisfies the eigenvariable condition, and does not appear below the rule that mentions said variable.

Note that pattern derivations $\Gamma; \Delta \vDash P$ and $\Gamma; \Delta \vDash N > \gamma^+$ are entirely driven by the structure of P and N. In particular, this means that when we quantify over all patterns $\Gamma; \Delta \vDash P \multimap N > \gamma^+$ for a given formula $P \multimap N$, this is equivalent to quantifying over all patterns $\Gamma_1; \Delta_1 \vDash P$ and $\Gamma_2; \Delta_2 \vDash N > \gamma^+$.

A crucial part of the definition of this system, is that there are only finitely many patterns for any given polarized formula. For this reason, we treat the unrestricted context in the patterns as a multiset of formulas. This also means that we need to treat two patterns as equal if they only differ in the choice of eigenvariables in the pattern rules for the quantifiers. This is a reasonable decision, as the eigenvariable condition ensures that the variables are fresh, hence the actual names of the variables are irrelevant. With these conditions in place,

$$
\frac{}{\cdot;\gamma^- \vDash \gamma^-} \qquad \frac{}{\cdot;\cdot \vDash \mathbf{1}} \qquad \frac{\Gamma_1;\Delta_1 \vDash P \quad \Gamma_2;\Delta_2 \vDash Q}{\Gamma_1,\Gamma_2;\Delta_1,\Delta_2 \vDash P \otimes Q}
$$

$$
\frac{\Gamma;\Delta \vDash P_i}{\Gamma;\Delta \vDash P_1 \oplus P_2} \qquad \frac{}{N;\cdot \vDash\ !N} \qquad \frac{\Gamma;\Delta \vDash [a/x]P}{\Gamma;\Delta \vDash \exists x.P}
$$

$$
\frac{}{\cdot;\cdot \vDash \gamma^+ > \gamma^+} \qquad \frac{\Gamma;\Delta \vDash N_i > \gamma^+}{\Gamma;\Delta \vDash N_1 \,\&\, N_2 > \gamma^+}
$$

$$
\frac{\Gamma_1;\Delta_1 \vDash P \quad \Gamma_2;\Delta_2 \vDash N > \gamma^+}{\Gamma_1,\Gamma_2;\Delta_1,\Delta_2 \vDash P \multimap N > \gamma^+} \qquad \frac{\Gamma;\Delta \vDash [a/x]N > \gamma^+}{\Gamma;\Delta \vDash \forall x.N > \gamma^+}
$$

Fig. 2. Positive and negative pattern judgments

it is straightforward to prove that for any given formula, there are only finitely many patterns for said formula.

Inference rules. For the judgments in the polarized system, we use a turnstile with a single vertical bar. The inference rules can be seen in Figure 3.

The quantification used in the ${}^{\uparrow}$E and ${}^{\downarrow}$I rules is merely a notational convenience to ease the formulation of the rules. It is a shorthand for the finitely many premises corresponding to the patterns of the principal formula. Thus, the number of premises for these rules may vary depending on the principal formula. This does not, however, pose a problem when checking that a rule has been applied properly, as we can reconstruct the necessary subgoals from the formulas ${}^{\uparrow}P$ and ${}^{\downarrow}N$. As usual, we require that the eigenvariables introduced by the patterns must be fresh, i.e. may not occur further down in the derivation.

As an example, we show the only possible proof of the statement given in the introduction. We have chosen to polarize both atoms positively, and elided the empty intuitionistic context.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{{}^{\downarrow}(a \multimap {}^{\uparrow}\mathbf{1}) \vdash {}^{\downarrow}(a \multimap {}^{\uparrow}\mathbf{1}) \Downarrow}\,\text{hyp}}{{}^{\downarrow}(a \multimap {}^{\uparrow}\mathbf{1}) \vdash a \multimap {}^{\uparrow}\mathbf{1} \Downarrow}\,{}^{\downarrow}\text{E} \qquad \dfrac{\dfrac{}{a \vdash a \Downarrow}\,\text{hyp}}{a \vdash a \Uparrow}\,{\Downarrow\Uparrow}}{a, {}^{\downarrow}(a \multimap {}^{\uparrow}\mathbf{1}) \vdash {}^{\uparrow}\mathbf{1} \Downarrow}\,{\multimap}\text{E} \qquad \dfrac{\dfrac{\dfrac{}{\cdot \vdash \mathbf{1} \Uparrow}\,\mathbf{1}\text{I}}{\cdot \vdash \mathbf{1} \oplus b \Uparrow}\,\oplus\text{I}_1}{\cdot \vdash {}^{\uparrow}(\mathbf{1} \oplus b) \Uparrow}\,{}^{\uparrow}\text{I}}{a, {}^{\downarrow}(a \multimap {}^{\uparrow}\mathbf{1}) \vdash {}^{\uparrow}(\mathbf{1} \oplus b) \Uparrow}\,{}^{\uparrow}\text{E}
$$

Note that the pattern judgment for **1** does not appear in this derivation. Indeed, the pattern judgments are only used to specify which premises should be present in a valid application of the ${}^{\uparrow}$E and ${}^{\downarrow}$I rules.

The use of patterns in the ${}^{\uparrow}$E and ${}^{\downarrow}$I rules collapses the positive and negative asynchronous phases into a single rule application. Thus we equate proofs that only differ in the order in which the negative introduction and positive elimination rules are applied. For instance, the assumption $(A \otimes B) \otimes (C \otimes D)$ can be eliminated two ways:

$$
\dfrac{\Gamma; \Delta \vdash a \Downarrow}{\Gamma; \Delta \vdash a \Uparrow} \Downarrow\Uparrow
\qquad
\dfrac{}{\Gamma; P \vdash P \Downarrow} \text{hyp}
\qquad
\dfrac{}{\Gamma, N; \cdot \vdash N \Downarrow} \text{uhyp}
\qquad
\dfrac{}{\Gamma; \cdot \vdash \mathbf{1} \Uparrow} \mathbf{1}\text{I}
$$

$$
\dfrac{\Gamma; \Delta_1 \vdash P \Uparrow \qquad \Gamma; \Delta_2 \vdash Q \Uparrow}{\Gamma; \Delta_1, \Delta_2 \vdash P \otimes Q \Uparrow} \otimes\text{I}
\qquad
\dfrac{\Gamma; \Delta \vdash P_i \Uparrow}{\Gamma; \Delta \vdash P_1 \oplus P_2 \Uparrow} \oplus\text{I}_i
\qquad
\dfrac{\Gamma; \Delta \vdash [t/x]P \Uparrow}{\Gamma; \Delta \vdash \exists x.P \Uparrow} \exists\text{I}
$$

$$
\dfrac{\Gamma; \cdot \vdash {\downarrow}N \Uparrow}{\Gamma; \cdot \vdash !N \Uparrow} !\text{I}
\qquad
\dfrac{\Gamma; \Delta \vdash N_1 \,\&\, N_2 \Downarrow}{\Gamma; \Delta \vdash N_i \Downarrow} \&\text{E}_i
\qquad
\dfrac{\Gamma; \Delta \vdash \forall x.N \Downarrow}{\Gamma; \Delta \vdash [t/x]N \Downarrow} \forall\text{E}
$$

$$
\dfrac{\Gamma; \Delta_1 \vdash P \multimap N \Downarrow \qquad \Gamma; \Delta_2 \vdash P \Uparrow}{\Gamma; \Delta_1, \Delta_2 \vdash N \Downarrow} \multimap\text{E}
$$

$$
\dfrac{\Gamma; \Delta \vdash P \Uparrow}{\Gamma; \Delta \vdash {\uparrow}P \Uparrow} {\uparrow}\text{I}
\qquad
\dfrac{\Gamma; \Delta_1 \vdash {\uparrow}P \Downarrow \qquad \begin{array}{c}\text{for all } \Gamma_P; \Delta_P \vDash P \\ \Gamma, \Gamma_P; \Delta_2, \Delta_P \vdash \gamma^+ \Uparrow\end{array}}{\Gamma_1; \Delta_1, \Delta_2 \vdash \gamma^+ \Downarrow} {\uparrow}\text{E}
$$

$$
\dfrac{\Gamma; \Delta \vdash {\downarrow}N \Downarrow}{\Gamma; \Delta \vdash N \Downarrow} {\downarrow}\text{E}
\qquad
\dfrac{\begin{array}{c}\text{for all } \Gamma_N; \Delta_N \vDash N > \gamma^+ \\ \Gamma, \Gamma_N; \Delta, \Delta_N \vdash \gamma^+ \Uparrow\end{array}}{\Gamma; \Delta \vdash {\downarrow}N \Uparrow} {\downarrow}\text{I}
$$

Fig. 3. Focused linear natural deduction

$$
(A \otimes B) \otimes (C \otimes D) \rightsquigarrow A \otimes B, C \otimes D \rightsquigarrow A, B, C \otimes D \rightsquigarrow A, B, C, D
$$
$$
(A \otimes B) \otimes (C \otimes D) \rightsquigarrow A \otimes B, C \otimes D \rightsquigarrow A \otimes B, C, D \rightsquigarrow A, B, C, D
$$

corresponding to which assumptions act as the principal formula for the $\otimes$E rule. By using patterns, this unnecessary distinction is avoided, as there is only one pattern for $(A \otimes B) \otimes (C \otimes D)$, specifically the pattern $\cdot; A, B, C, D$.

Note that the rules $\top$I and $\mathbf{0}$E are captured by the above rules, as there are no patterns of the form $\Gamma; \Delta \vDash \mathbf{0}$ and $\Gamma; \Delta \vDash \top > \gamma^+$.

The polarity restrictions on the hyp and uhyp rules are justified by noting that $\multimap$ is the internalized version of the linear hypothetical judgments. In particular, this means that the linear context can only contain positive formulas; any negative formulas must be in delayed form ${\downarrow}N$. Unrestricted formulas, on the other hand, are not delayed, as choosing whether or not to use an unrestricted resource is always a non-deterministic (hence synchronous) choice.

By inspecting the polarized rules, we may observe the following:

1. In a derivation of the judgment $\Gamma; \Delta \vdash A \Uparrow$ where A is positive (e.g. $A = P \otimes Q$), the final rule must be the positive introduction rule corresponding to A, or $\Downarrow\Uparrow$ if A is an atom. This follows from the fact that positive eliminations are only applicable when the succedent is negative.
2. The only rule applicable to the judgment $\Gamma; \Delta \vdash A \Downarrow$ where A is negative (e.g. $A = P \multimap N$), is the appropriate elimination rule for A, or $\Downarrow\Uparrow$ if A is an atom.

Based on the above observations, we define the following synchronous phases based on the polarity of the succedent and whether it is atomic or canonical.

$\Gamma; \Delta \vdash P \Uparrow$	Positive focusing, initiated by ${}^{\uparrow}\mathrm{I}$
$\Gamma; \Delta \vdash N \Downarrow$	Negative focusing, initiated by ${}^{\downarrow}\mathrm{E}$

By our use of patterns, we collapse the asynchronous phases, which would otherwise have corresponded to the judgments $\Gamma; \Delta \vdash P \Downarrow$ and $\Gamma; \Delta \vdash N \Uparrow$.

The positive focusing phase proceeds in a bottom-up fashion, and is initiated by using the ${}^{\uparrow}\mathrm{I}$ rule to remove the positive shift in front of the formula to be focused. The focused formula is then decomposed by a sequence of positive introduction rules. This phase ends when the succedent becomes negative or atomic.

The negative focusing phase proceeds in a top-down fashion, and is initiated by choosing a negative or delayed negative formula in either the linear or unrestricted context, and applying a sequence of negative elimination rules to the judgment $\Gamma; {}^{\downarrow}N \vdash {}^{\downarrow}N \Downarrow$ or $\Gamma, N; \cdot \vdash N \Downarrow$, given by either the hyp rule or the uhyp rule. The phase terminates when the succedent is either positive or atomic. In the former case, the subderivation must end in the ${}^{\uparrow}\mathrm{E}$ rule, and in the latter case in the $\Downarrow\Uparrow$ rule.

Note that because the positive elimination rules restrict the succedent of the conclusion to be of the form γ^{+}, it is not possible to apply the ${}^{\uparrow}\mathrm{E}$ rule inside a positive focusing phase. As negative focusing ends in positive elimination or coercion, it is not possible to perform negative focusing inside a positive focusing phase. Likewise, the negative focusing phase cannot be interrupted.

It is in this sense the above system is *maximally focused* — once a formula is chosen for focusing, it must be decomposed fully (i.e. keep the focus) before other formulas can be focused or otherwise eliminated.

3 Soundness and Completeness

A note on notation. To formulate the soundness and completeness theorems, we need to be able to talk about when the entire linear context is covered by some pattern. This is done using the judgment $\Gamma'; \Delta' \vDash \Delta$. The following inference rules define this judgment:

$$\frac{}{\cdot; \cdot \vDash \cdot} \qquad \frac{\Gamma'; \Delta' \vDash \Delta \quad \Gamma_P; \Delta_P \vDash P}{\Gamma', \Gamma_P; \Delta', \Delta_P \vDash \Delta, P}$$

We will tacitly use the fact that this judgment is well-behaved with regard to splitting the context Δ. In other words, that $\Gamma'; \Delta' \vDash \Delta_1, \Delta_2$ if and only if $\Gamma' = \Gamma'_1, \Gamma'_2$ and $\Delta' = \Delta'_1, \Delta'_2$ where $\Gamma'_i; \Delta'_i \vDash \Delta_i$ for $i = 1, 2$.

Soundness. To prove soundness, we define a function from polarized formulas to regular formulas. This is simply the function $(-)^e$ that erases all occurrences of the positive and negative shifts, i.e. $({}^{\uparrow}P)^e = P^e$ and $({}^{\downarrow}N)^e = N^e$, whilst $(P \multimap N)^e = P^e \multimap N^e$ and $(\forall x.N)^e = \forall x.N^e$ and so on.

Theorem 1 (Soundness of polarized derivations). *The following properties hold:*

1. *If* $\Gamma;\Delta \vdash A \Uparrow$ *then* $\Gamma^e;\Delta^e \Vdash A^e \Uparrow$.
2. *If* $\Gamma;\Delta \vdash A \Downarrow$ *then* $\Gamma^e;\Delta^e \Vdash A^e \Downarrow$.
3. *If for all* $\Gamma';\Delta' \vDash \Omega$ *and* $\Gamma_A;\Delta_A \vDash A > \gamma^+$ *we have* $\Gamma,\Gamma',\Gamma_A;\Delta,\Delta',\Delta_A \vdash \gamma^+ \Uparrow$, *then* $\Gamma^e;\Delta^e,\Omega^e \Vdash A^e \Uparrow$.

Proof. The first two claims are proved by induction on the structure of the given derivations. In the case of the ${}^{\uparrow}$E and ${}^{\downarrow}$I rules, we reconstruct the asynchronous phases, hence the third hypothesis is needed.

We prove the third claim by an induction on the number of connectives in Ω and A. This is needed when reconstructing a tensor, as Ω will then grow in size. The base case for this induction is when all formulas in Ω are of the form γ^- and A is of the form γ^+. In this case, we appeal to the first induction hypothesis. We show a few representative cases here:

Case $A = P \multimap N$**:** By inversion on $\Gamma_A;\Delta_A \vDash A > \gamma^+$, we get $\Gamma_A = \Gamma_P, \Gamma_N$ and $\Delta_A = \Delta_P, \Delta_N$ such that $\Gamma_P;\Delta_P \vDash P$ and $\Gamma_N;\Delta_N \vDash N > \gamma^+$, hence $\Gamma',\Gamma_P;\Delta',\Delta_P \vDash \Omega, P$ and by the induction hypothesis $\Gamma^e;\Delta^e,\Omega^e,P^e \Vdash N^e \Uparrow$, hence by applying $\multimap$I we get the desired derivation of $\Gamma^e;\Delta^e,\Omega^e \Vdash (P \multimap N)^e \Uparrow$.

Case $\Omega = \Omega_1, P \otimes Q, \Omega_2$**:** By assumption, $\Gamma'_1,\Gamma_{PQ},\Gamma'_2;\Delta'_1,\Delta_{PQ},\Delta'_2 \vDash \Omega_1, P \otimes Q, \Omega_2$, hence by inversion we have $\Gamma_{PQ} = \Gamma_P, \Gamma_Q$ and $\Delta_{PQ} = \Delta_P, \Delta_Q$ such that $\Gamma_P;\Delta_P \vDash P$ and $\Gamma_Q;\Delta_Q \vDash Q$. Then $\Gamma'_1,\Gamma_P,\Gamma_Q,\Gamma'_2;\Delta'_1,\Delta_P,\Delta_Q,\Delta'_2 \vDash \Omega_1, P, Q, \Omega_2$, hence by the induction hypothesis $\Gamma^e;\Delta^e,\Omega_1^e,P^e,Q^e,\Omega_2^e \Vdash C^e \Uparrow$. Applying $\otimes$E to this judgment and the hypothesis judgment $\Gamma^e;(P \otimes Q)^e \Vdash P^e \otimes Q^e \Downarrow$, we get the desired derivation of $\Gamma^e;\Delta^e,\Omega_1^e,(P \otimes Q)^e,\Omega_2^e \Vdash C^e \Uparrow$. $\square$

Polarizing formulae. To prove that the polarized system is complete with regard to the natural deduction calculus depicted in Figure 1, we first need to find a way of converting regular propositions into polarized propositions. To do this, we define the following two mutually recursive functions, $(-)^p$ and $(-)^n$, by structural induction on the syntax of unpolarized formulas.

$$
\begin{array}{ccc}
\mathbf{1}^p = \mathbf{1} & \mathbf{0}^p = \mathbf{0} & \top^n = \top \\
(A \otimes B)^p = A^p \otimes B^p & (A \oplus B)^p = A^p \oplus B^p & (!A)^p = !A^n \\
(A \,\&\, B)^n = A^n \,\&\, B^n & (A \multimap B)^n = A^p \multimap B^n & \\
(\exists x.A)^p = \exists x.A^p & (\forall x.A)^n = \forall x.A^n &
\end{array}
$$

The above definitions handle the cases where the polarity of the formula inside the parentheses matches the polarizing function that is applied, i.e. cases of the form N^n and P^p. All that remains is to handle the cases where the polarity does not match, which we do with the following definition:

$$
\begin{array}{ll}
A^n = {}^{\uparrow}A^p & \text{when } A \text{ is positive} \\
A^p = {}^{\downarrow}A^n & \text{when } A \text{ is negative}
\end{array}
$$

In the case of atoms we assume that there is some fixed assignment of polarity to atoms, and define $a^p = a$ for positive atoms and $a^n = a$ for negative atoms. Atoms with the wrong polarity, e.g. a^n for a positive atom a are handled by the P^n case above, i.e. $a^n = {\uparrow}a^p = {\uparrow}a$ for positive atoms, and conversely for the negative atoms.

In short, whenever a subformula of positive polarity appears in a place where negative polarity is expected, we add a positive shift to account for this fact, and vice versa. Thus, a formula such as $(a \,\&\, b) \multimap (c \otimes d)$ is mapped by $(-)^n$ to ${\downarrow}({\uparrow}a \,\&\, b) \multimap {\uparrow}(c \otimes {\downarrow}d)$, when a, c are positive, and b, d are negative. These functions are extended in the obvious way to contexts, in the sense that Δ^p means applying the function $(-)^p$ to each formula in Δ.

Of course, this is not the only way of polarizing a given formula, as we may always add redundant shifts ${\uparrow\downarrow}N$ and ${\downarrow\uparrow}P$ inside our formulas. The above procedure gives a *minimal polarization* in that there are no redundant negative or positive shifts. Note that $(A^p)^e = (A^n)^e = A$ for all unpolarized formulae A, and $(P^e)^p = P$ and $(N^e)^n = N$ for minimally polarized formulae P and N.

Completeness. Before we can prove completeness, we must prove a series of lemmas. First of all, we need the usual substitution properties.

Lemma 1 (Substituting properties for linear resources). *The following substitution properties hold:*

1. *If* $\Gamma; \Delta_1 \vdash N \Downarrow$ *and* $\Gamma; \Delta_2, {\downarrow}N \vdash C \Uparrow$ *then* $\Gamma; \Delta_1, \Delta_2 \vdash C \Uparrow$
2. *If* $\Gamma; \Delta_1 \vdash N \Downarrow$ *and* $\Gamma; \Delta_2, {\downarrow}N \vdash C \Downarrow$ *and the latter derivation is not an instance of the* hyp *rule then* $\Gamma; \Delta_1, \Delta_2 \vdash C \Downarrow$.

Proof. We proceed by mutual induction on the derivations of $\Gamma; \Delta_2, {\uparrow}N \vdash C \Downarrow$ and $\Gamma; \Delta_2, {\uparrow}N \vdash C \Uparrow$. In all cases, we proceed by applying the induction hypothesis to the premise which contains the formula ${\uparrow}N$, and then reapply the appropriate rule. This is possible if the premise is not a derivation of $\Gamma; {\downarrow}N \vdash {\downarrow}N \Downarrow$ using the hyp rule. This can only happen in the case of the ${\downarrow}$E rule, in which case, $C = N$ and Δ_2 is empty. Hence we may then apply the following reduction:

$$\dfrac{\dfrac{}{\Gamma; {\downarrow}N \vdash {\downarrow}N \Downarrow}\,\text{hyp}}{\Gamma; {\downarrow}N \vdash N \Downarrow}\,{\downarrow}\text{E} \quad \rightsquigarrow \quad \begin{array}{c}\vdots \\ \Gamma; \Delta_1 \vdash N \Downarrow\end{array}$$

□

Corollary 1. *If* $\Gamma; \Delta_1 \vdash A^n \Downarrow$ *and for all patterns* $\Gamma_A; \Delta_A \vDash A^p$ *we have* $\Gamma, \Gamma_A; \Delta_2, \Delta_A \vdash \gamma^+ \Uparrow$ *then* $\Gamma; \Delta_1, \Delta_2 \vdash \gamma^+ \Uparrow$.

Proof. If A is positive, then $A^n = {\uparrow}A^p$, and the result follows by applying the ${\uparrow}$E rule. If A is negative, then $A^p = {\downarrow}A^n$, hence there is only one pattern for A^p, specifically $\cdot; {\downarrow}A^n \vDash {\downarrow}A^n$ and the result follows from Lemma 1. □

For the synchronous rules, we need to capture the fact that we may push positive elimination rules below positive introduction rules. To formulate this, we introduce a new function $(-)^d$ that always produces a delayed formula. Thus, we define $A^d = {}^{\uparrow}A^p$ when A is positive, and $A^d = {}^{\downarrow}A^n$ when A is negative. We first note the following

Lemma 2. *If for all $\Gamma_A; \Delta_A \vDash A^n > \gamma^+$ we have $\Gamma, \Gamma_A; \Delta, \Delta_A \vdash \gamma^+ \Uparrow$, then $\Gamma; \Delta \vdash A^d \Uparrow$.*

Proof. By case analysis on the polarity of A. If A is positive then $A^n = {}^{\uparrow}A^p$, hence $\cdot; \cdot \vDash {}^{\uparrow}A^p > {}^{\uparrow}A^p$ is a pattern for A^n. Thus, $\Gamma; \Delta \vdash {}^{\uparrow}A^p \Uparrow$ as desired.

If A is negative, then $A^d = {}^{\downarrow}A^n$, hence by the rule ${}^{\downarrow}$I, we get the desired result. □

We are now able to prove that under certain circumstances we can change a positively polarized canonical premise to the corresponding delayed canonical premise.

Lemma 3. *For any admissible rule of the form*

$$\frac{\Gamma; \Delta \vdash A^p \Uparrow}{\Gamma; \Delta, \Delta' \vdash \gamma^+ \Uparrow}\, \Sigma$$

the following rule is admissible

$$\frac{\Gamma; \Delta \vdash A^d \Uparrow}{\Gamma; \Delta, \Delta' \vdash \gamma^+ \Uparrow}\, \Sigma^d$$

Proof. By case analysis on the polarity of A and induction on the derivation of $\Gamma; \Delta \vdash A^d \Uparrow$. If A is negative, $A^d = {}^{\downarrow}A^n = A^p$, hence the result follows immediately by applying Σ. If A is positive, $A^d = {}^{\uparrow}A^p$, hence the final rule of the derivation is either ${}^{\uparrow}$I or ${}^{\uparrow}$E. If the derivation ends in ${}^{\uparrow}$I, the result again by applying Σ to the premise of this rule. If the derivation ends in ${}^{\uparrow}$E, we apply the induction hypothesis to the second premise, and reapply the ${}^{\uparrow}$E rule. □

The above argument easily extends to admissible rules with multiple premises, hence we get the following

Corollary 2. *The following rules are admissible:*

$$\frac{\Gamma; \Delta_1 \vdash A^d \Uparrow \qquad \Gamma; \Delta_2 \vdash B^d \Uparrow}{\Gamma; \Delta_1, \Delta_2 \vdash (A \otimes B)^n \Uparrow}\, \otimes^d\mathrm{I} \qquad \frac{\Gamma; \Delta \vdash A_i^d \Uparrow}{\Gamma; \Delta \vdash (A_1 \oplus A_2)^n \Uparrow}\, \oplus_i^d\mathrm{I}$$

$$\frac{\Gamma; \Delta \vdash ([t/x]A)^d \Uparrow}{\Gamma; \Delta \vdash (\exists x.A)^n \Uparrow}\, \exists^d\mathrm{I}$$

Also, if $\Gamma; \Delta_1 \vdash A^d \Downarrow$ and for all $\Gamma_B; \Delta_B \vDash B^p$ we have $\Gamma, \Gamma_B; \Delta_2, \Delta_B \vdash \gamma^+ \Uparrow$, then $\Gamma; \Delta_1, \Delta_2, (A \multimap B)^p \vdash \gamma^+ \Uparrow$.

Proof. The first few rules are easy applications of the previous lemma to the positive introduction rules. The property holds by using the lemma on the following argument: If $\Gamma;\Delta_1 \vdash A^p \Uparrow$, then $\Gamma;\Delta_1,(A \multimap B)^p \vdash B^n \Uparrow$ by $\multimap$E, ${}^{\downarrow}$E and hyp on $(A \multimap B)^p$. With the additional assumption that $\Gamma,\Gamma_B;\Delta_2,\Delta_B \vdash \gamma^+$ for all patterns $\Gamma_B;\Delta_B \vDash B^p$, by applying Lemma 1, we get the desired result $\Gamma;\Delta_1,\Delta_2,(A \multimap B)^p \vdash \gamma^+ \Uparrow$. □

Before we formulate the completeness theorem, we have to decide which polarizing function to apply to the hypotheses and the conclusion. We would like translate a derivation of $\Gamma;\Delta \Vdash A \Uparrow$ into a derivation $\Gamma^x;\Delta^y \vdash A^z \Uparrow$, where each of x, y and z is one of the above functions. In the case of x and y, the nature of the uhyp and hyp rules force us to choose $x = n$ and $y = p$. If $z = p$, we should be able to derive $\cdot;(a \otimes b)^p \vdash a^p \otimes b^p \Uparrow$, as this is clearly derivable in the unpolarized system. In the polarized system, however, we are forced to enter a focusing phase prematurely requiring us to split a single element context into two parts. None of the resulting subgoals are provable. Therefore $z = n$. Thus, as far as the canonical judgments are concerned, the completeness theorem will take derivations of $\Gamma;\Delta \Vdash A \Uparrow$ to derivations of $\Gamma^n;\Delta^p \vdash A^n \Uparrow$.

As for atomic derivations, these cannot in general be transferred to atomic derivations in the focused system. The reader may verify this by checking that $\cdot;(a \otimes b),(a \otimes b) \multimap c \Vdash c \Downarrow$ is derivable in the unpolarized system, whereas $\cdot;(a \otimes b),{}^{\downarrow}((a \otimes b) \multimap c) \vdash c \Downarrow$ is not derivable in the polarized system. Thus, we need to generalize the induction hypothesis in the completeness theorem.

Theorem 2 (Completeness). *The following properties hold:*

1. *Given $\Gamma;\Delta \Vdash A \Uparrow$ and patterns $\Gamma';\Delta' \vDash \Delta^p$ and $\Gamma_A;\Delta_A \vDash A^n > \gamma^+$, then $\Gamma^n,\Gamma',\Gamma_A;\Delta',\Delta_A \vdash \gamma^+ \Uparrow$.*
2. *If $\Gamma_1;\Delta_1 \Vdash A \Downarrow$ and $\Gamma';\Delta' \vDash \Delta_1^p$ and for all $\Gamma_A;\Delta_A \vDash A^p$, then we have $\Gamma_2,\Gamma_A;\Delta_2,\Delta_A \vdash \gamma^+ \Uparrow$ then $\Gamma_1^n,\Gamma_2,\Gamma';\Delta_1,\Delta_2,\Delta' \vdash \gamma^+ \Uparrow$.*

Proof. By induction on the derivations of $\Gamma;\Delta \Vdash A \Uparrow$ and $\Gamma;\Delta \Vdash A \Downarrow$. We give a few representative cases.

Case $\otimes$**E:**

$$\dfrac{\begin{matrix}\mathcal{D} \\ \Gamma;\Delta_1 \Vdash A \otimes B \Downarrow\end{matrix} \qquad \begin{matrix}\mathcal{E} \\ \Gamma;\Delta_2,A,B \Vdash C \Uparrow\end{matrix}}{\Gamma;\Delta_1,\Delta_2 \Vdash C \Uparrow}\otimes\mathrm{E}$$

(1)	$\Gamma_1',\Gamma_2';\Delta_1',\Delta_2' \vDash \Delta_1^p,\Delta_2^p$	Assumption.
(2)	$\Gamma_C;\Delta_C \vDash C^n > \gamma^+$	Assumption.
(3)	$\Gamma_A,\Gamma_B;\Delta_A,\Delta_B \vDash A^p,B^p$	Assumption.
(4)	$\Gamma_2',\Gamma_A,\Gamma_B;\Delta_1',\Delta_2',\Delta_A,\Delta_B \vDash \Delta_2^p,A^p,B^p$	By (1) and (3).
(5)	$\Gamma^n,\Gamma_2',\Gamma_A,\Gamma_B,\Gamma_C;\Delta_2',\Delta_A,\Delta_B,\Delta_C \vdash \gamma^+ \Uparrow$	By i.h.1 on $\mathcal{E}$, (2), (4).
	$\Gamma_A,\Gamma_B;\Delta_A,\Delta_B \vDash A^p \otimes B^p$	By the pattern rule for $\otimes$.

$\Gamma_A, \Gamma_B; \Delta_A, \Delta_B \vDash (A \otimes B)^p$ By the defn. of $(-)^p$.

$\Gamma^n, \Gamma_1', \Gamma_2', \Gamma_C; \Delta_1', \Delta_2', \Delta_C \vdash \gamma^+ \Uparrow$ By i.h. 2 on $\mathcal{D}$ and (5).

Case $\multimap$I:

$$\dfrac{\overset{\mathcal{D}}{\Gamma; \Delta, A \Vdash B \Uparrow}}{\Gamma; \Delta \Vdash A \multimap B \Uparrow} \multimap \mathrm{I}$$

(1) $\Gamma'; \Delta' \vDash \Delta^p$ Assumption.

$\Gamma_{AB}; \Delta_{AB} \vDash (A \multimap B)^n$ Assumption.

$\Gamma_{AB}; \Delta_{AB} \vDash A^p \multimap B^n$ By defn. of $(-)^n$.

(2) $\Gamma_{AB} = \Gamma_A, \Gamma_B$ and $\Delta_{AB} = \Delta_A, \Delta_B$ such that

(3) $\Gamma_A; \Delta_A \vDash A^p$,

$\Gamma_B; \Delta_B \vDash B^n > \gamma^+$ By inversion.

$\Gamma^n, \Gamma', \Gamma_A, \Gamma_B; \Delta', \Delta_A, \Delta_B \vdash \gamma^+ \Uparrow$ By i.h. 1 on $\mathcal{D}$, (1) and (3).

$\Gamma^n, \Gamma', \Gamma_{AB}; \Delta', \Delta_{AB} \vdash \gamma^+ \Uparrow$ By (2).

Case !I:

$$\dfrac{\overset{\mathcal{D}}{\Gamma; \cdot \Vdash A \Uparrow}}{\Gamma; \cdot \Vdash !A \Uparrow} !\mathrm{I}$$

$\Gamma'; \Delta' \vDash \cdot$ Assumption.

$\Gamma' = \Delta' = \cdot$ By inversion.

$\Gamma_A; \Delta_A \vDash (!A)^n > \gamma^+$ Assumption.

$\Gamma_A = \Delta_A = \cdot,\ \gamma^+ = {}^{\uparrow}(!A^n)$ By inversion.

(1) $\Gamma_A'; \Delta_A' \vDash A^n > \gamma^+$ Assumption

$\Gamma^n, \Gamma_A'; \Delta_A' \vdash \gamma^+ \Uparrow$ By i.h. 1 on $\mathcal{D}$ and (1).

$\forall(\Gamma_A'; \Delta_A' \vDash A^n > \gamma^+) : \Gamma^n, \Gamma_A'; \Delta_A' \vdash \gamma^+ \Uparrow$ By discharging (1).

$\Gamma^n; \cdot \vdash {}^{\downarrow}A^n \Uparrow$ By ${}^{\downarrow}$I.

$\Gamma^n; \cdot \vdash !A^n$ By !I.

$\Gamma^n; \cdot \vdash {}^{\uparrow}(!A^n)$ By ${}^{\uparrow}$I.

Case $\otimes$I:

$$\dfrac{\overset{\mathcal{D}}{\Gamma; \Delta_1 \Vdash A \Uparrow} \quad \overset{\mathcal{E}}{\Gamma; \Delta_2 \Vdash B \Uparrow}}{\Gamma; \Delta_1, \Delta_2 \Vdash A \otimes B \Uparrow} \otimes \mathrm{I}$$

$\Gamma_1', \Gamma_2'; \Delta_1', \Delta_2' \vDash \Delta_1^p, \Delta_2^p$ Assumption.

$\Gamma_{AB}; \Delta_{AB} \vDash (A \otimes B)^n > \gamma^+$ Assumption.
$\Gamma_{AB} = \Delta_{AB} = \cdot, \gamma^+ = (A \otimes B)^n$ By inversion.
$\Gamma^n, \Gamma_1'; \Delta_1' \vdash A^n \Uparrow$ By i.h. 1 on $\mathcal{D}$.
$\Gamma^n, \Gamma_2'; \Delta_2' \vdash B^n \Uparrow$ By i.h. 1 on $\mathcal{E}$.
$\Gamma^n, \Gamma_1', \Gamma_2'; \Delta_1', \Delta_2' \vdash (A \otimes B)^n \Uparrow$ By $\otimes^d$I. □

As the mapping of derivations that is implicit in the proof of the soundness theorem preserves (maximal) synchronous phases and reconstructs (maximal) asynchronous phases, one may prove the following corollary.

Corollary 3. *For any proof of a judgment $\Gamma; \Delta \Vdash A \Uparrow$ there exists a proof of the same judgment with maximal focusing and inversion phases.*

4 Relation to the Backward Linear Focusing Calculus

In this section we consider the connection between our system of focused linear natural deduction, and the backward linear focusing calculus of Pfenning, Chaudhuri and Price [4]. The main result of this section is a very direct soundness and completeness result between these two systems. The syntax of formulas in the sequent system is the same as the unpolarized system in Section 2. The same distinction between positive and negative connectives is made, but no shifts are present to move from positive to negative and vice versa. Additionally, the shorthand P^- and N^+ is used in a similar fashion to our γ^- and γ^+.

The judgments of their system have either two or three contexts of the following form: Γ is a context of unrestricted hypotheses, Δ is a linear context of hypotheses of the form N^+. A third context Ω which is both linear and ordered may also be present. This context may contain formulas of any polarity.

There are four different kinds of sequents:

$\Gamma; \Delta \gg A$	*right-focal* sequent with A under focus.
$\Gamma; \Delta; A \ll Q^-$	*left-focal* sequent with A under focus.
$\Gamma; \Delta; \Omega \Longrightarrow \cdot; Q^-$	
$\Gamma; \Delta; \Omega \Longrightarrow C; \cdot$	Active sequents

The focal sequents correspond to the synchronous phases where a positive or negative formula is decomposed maximally. Conversely, the active sequents correspond to the asynchronous phase where non-focal formulas in Ω and the formula C may be decomposed asynchronously. The goal $\cdot; Q^-$ signifies that Q^- has been inverted maximally, and is no longer active.

Theorem 3 (Soundness w.r.t. backward linear focusing calculus). *The following properties hold:*

1. *If $\Gamma; \Delta \vdash P \Uparrow$ then $\Gamma^e; \Delta^e \gg P^e$.*
2. *If $\Gamma; \Delta \vdash N \Downarrow$ and $\Gamma^e; \Delta'; N^e \ll Q^-$ then $\Gamma^e; \Delta^e, \Delta'; \cdot \Longrightarrow \cdot; Q^-$.*

3. *If* $\Gamma'; \Gamma_C; \Delta, \Delta', \Delta_C \vdash \gamma^+ \Uparrow$ *for all* $\Gamma'; \Delta' \vDash \Omega$ *and* $\Gamma_C; \Delta_C \vDash C > \gamma^+$*, then* $\Gamma^e; \Delta^e; \Omega^e \Longrightarrow C^e; \cdot$.
4. *If* $\Gamma; \Gamma'; \Delta, \Delta' \vdash \gamma^+ \Uparrow$ *for all* $\Gamma'; \Delta' \vDash \Omega$*, then* $\Gamma^e; \Delta^e; \Omega^e \Longrightarrow \cdot; (\gamma^+)^e$.

Theorem 4 (Completeness w.r.t. backward linear focusing calculus). *The following properties hold:*

1. *If* $\Gamma; \Delta \gg A$ *then* $\Gamma^n; \Delta^p \vdash A^p \Uparrow$.
2. *If* $\Gamma; \Delta; A \ll Q^-$ *and* $\Gamma^n; \Delta' \vdash A^n \Downarrow$ *then* $\Gamma^n; \Delta^p, \Delta' \vdash (Q^-)^n \Uparrow$.
3. *If* $\Gamma; \Delta; \Omega \Longrightarrow C; \cdot$ *then* $\Gamma^n, \Gamma', \Gamma_C; \Delta^p, \Delta', \Delta_C \vdash \gamma^+ \Uparrow$ *for all* $\Gamma'; \Delta' \vDash \Omega^p$ *and* $\Gamma_C; \Delta_C \vDash C^n > \gamma^+$.
4. *If* $\Gamma; \Delta; \Omega \Longrightarrow \cdot; Q^-$ *then* $\Gamma^n, \Gamma'; \Delta^p, \Delta' \vdash (Q^-)^n \Uparrow$ *for all* $\Gamma'; \Delta' \vDash \Omega^p$.

The proofs of the above theorems may be seen as a way of transferring proofs between the two systems, and its action may be summarized as follows:

– The soundness proof maps synchronous to synchronous phases, and reconstructs inversion phases from the principal formulas in the ${}^{\uparrow}$E and ${}^{\downarrow}$I rules.
– The completeness proof takes synchronous phases to synchronous phases, and collapses asynchronous phases into ${}^{\uparrow}$E and ${}^{\downarrow}$I rules.

In particular, this leads to the following

Corollary 4. *The mapping of proofs induced by the soundness theorem is injective on proofs of judgments of the form* $\Gamma; \Delta \vdash A \Uparrow$ *where* Γ*,* Δ *and* A *are minimally polarized.*

Consequently, if we consider proofs of minimally polarized judgments, the current system has the same number or fewer proofs of that judgment than the backward focused sequent calculus has proofs of the corresponding sequent.

5 Conclusion and Related Work

Using the concepts of focusing and patterns, we have presented a natural deduction formulation of first-order intuitionistic linear logic that ensures the maximality of both synchronous and asynchronous phases. This removes a large part of the bureaucracy and unnecessary choices that were present in the previous formulation.

In [4], completeness with regard to the unfocused linear sequent calculus is established by proving focused versions of cut admissibility and identity. Because of the four different kinds of sequents and the three contexts, the proof of cut admissibility consists of more than 20 different kinds of cuts, all connected in an intricate induction order. In contrast, the proofs of the soundness and completeness results we have established are relatively straightforward. This is in part because we only have two judgments, and also because the intercalation formulation of linear natural deduction is already negatively focused.

From Corollary 4, it follows that our system, when restricted to minimally polarized formulas, has the same number or fewer proofs than the backward linear focusing calculus. This is only the case if we consider minimally polarized formulas, however. In particular this opens the possibility of using different polarization strategies to capture different classes of proofs.

In our formulation, we have chosen to use patterns only as a means of ensuring maximal asynchronous phases. It is possible to extend the use of patterns to the synchronous phases as well, but there are several reasons why we have chosen not to do this. The first and most compelling reason is that it is not necessary to ensure maximal synchronous phases. The restrictions on the succedent of the $^{\uparrow}$E rule suffices to ensure the maximality of the synchronous phases. Second, the use of patterns for the asynchronous phase extends easily to the case of quantifiers because the actual choice of eigenvariables does not matter — only freshness is important. Interpreting the pattern rules for quantifiers in a synchronous setting, we would need to substitute appropriately chosen *terms* for these variables, to capture the behavior of the $\forall$E and $\exists$I rules. This would complicate the system considerably.

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