Timed Negotiations

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Abstract. Negotiations were introduced in [6] as a model for concurrent systems with multiparty decisions. What is very appealing with negotiations is that it is one of the very few non-trivial concurrent models where several interesting problems, such as soundness, i.e. absence of deadlocks, can be solved in PTIME [2]. In this paper, we introduce the model of timed negotiations and consider the problem of computing the minimum and the maximum execution time of a negotiation. The latter can be solved using the algorithm of [10] computing costs in negotiations, but surprisingly minimum execution time cannot.
In this paper, we propose new algorithms to compute both minimum and maximum execution time, that work in much more general classes of negotiations than [10], that only considered sound and deterministic negotiations. Further, we uncover the precise complexities of these questions, ranging from PTIME to Δ_2^P-complete. In particular, we show that computing the minimum execution time is more complex than computing the maximum execution time in most classes of negotiations we consider.

1 Introduction

Distributed systems are notoriously difficult to analyze, mainly due to the explosion of the number of configurations that have to be considered to answer even simple questions. A challenging task is then to propose models on which analysis can be performed with tractable complexities, preferably within polynomial time. Free choice Petri nets are a classical model of distributed systems that allow for efficient verification, in particular when the nets are 1-safe [5, 4].

Recently, [6] introduced a new model called *negotiations* for workflows and business processes. A negotiation describes how processes interact in a distributed system: a subset of processes in a node of the system take a synchronous decisions among several *outcomes*. The effect of this outcome sends contributing processes to a new set of nodes. The execution of a negotiation ends when processes reach a *final configuration*. Negotiations can be deterministic (once an outcome is fixed, each process knows its unique successor node) or not.

Negotiations are an interesting model since several properties can be decided with a reasonable complexity. The question of *soundness*, i.e., deadlock-freedom: whether from every reachable configuration one can reach a final configuration, is PSPACE-complete. However, for deterministic negotiations, it can be decided in PTIME [7]. The decision procedure uses reduction rules. Reduction techniques were originally proposed for Petri nets [1, 8, 12, 17]. The main idea is to define transformations rules that produce a model of smaller size w.r.t. the original model, while preserving the property under analysis. In the context of negotiations, [7, 2] proposed a sound and complete set of soundness-preserving reduction rules and algorithms to apply these rules efficiently. The question of soundness for deterministic negotiations was revisited in [9] and showed NLOGSPACE-complete using anti patterns instead of reduction rules. Further, they show that the PTIME result holds even when relaxing determinism [9]. Negotiation games have also been considered to decide whether one particular process can force termination of a negotiation. While this question is EXPTIME complete in general, for sound and deterministic negotiations, it becomes PTIME [13].

While it is natural to consider cost or time in negotiations (e.g. think of the Brexit negotiation where time is of the essence, and which we model as running example in this paper), the original model of negotiations proposed by [6] is only qualitative. Recently, [10] has proposed a framework to associate costs to the executions of negotiations, and adapt a static analysis technique based on reduction rules to compute end-to end cost functions that are not sensitive to scheduling of concurrent nodes. For sound *and* deterministic negotiations, the end-to end cost can be computed in $O(n.(C + n))$, where n is the size of the negotiation and C the time needed to compute the cost of an execution. Requiring soundness or determinism seem perfectly reasonable, but asking sound *and* deterministic negotiations is too restrictive: it prevents a process from waiting for decisions of other processes to know how to proceed.

In this paper, we revisit time in negotiations. We attach time intervals to outcomes of nodes. We want to compute maximal and minimal executions times, for negotiations that are not necessarily sound and deterministic. Since we are interested in minimal and maximal execution time, cycles in negotiations can be either bypassed or lead to infinite maximal time. Hence, we restrict this study to acyclic negotiations. Notice that time can be modeled as a cost, following [10], and the maximal execution time of a sound and deterministic negotiation can be computed in PTIME using the algorithm from [10]. Surprisingly however, we give an example (Example 3) for which the minimal execution time cannot be computed in PTIME by this algorithm.

The first contribution of the paper shows that reachability (whether at least one run of a negotiation terminates) is NP-complete, already for (untimed) deterministic acyclic negotiations. This implies that computing minimal or maximal execution time for deterministic (but unsound) acyclic negotiations cannot be done in PTIME (unless NP=PTIME). We characterize precisely the complexities of different decision variants (threshold, equality, etc.), with complexities ranging from (co-)NP-complete to Δ_2^P.

We thus turn to negotiations that are sound but not necessarily deterministic. Our second contribution is a new algorithm, not based on reduction rules, to compute the maximal execution time in PTIME for sound negotiations. It is based on computing the maximal execution time of critical paths in the negotiations. However, we show that *minimal* execution time cannot be computed in PTIME for sound negotiations (unless NP=PTIME): deciding whether the minimal execution time is lower than T is NP-complete, even for T given in unary, using a reduction from a Bin packing problem. This shows that minimal execution time is harder to compute than maximal execution time.

Our third contribution consists in defining a class in which the minimal execution time can be computed in (pseudo) PTIME. To do so, we define the class of k-layered negotiations, for k fixed, that is negotiations where nodes can be organized into layers of at most k nodes at the same depth. These negotiations can be executed without remembering more than k nodes at a time. In this case, we show that computing the maximal execution time is PTIME, even if the negotiation is neither deterministic nor sound. The algorithm, not based on reduction rules, uses the k-layer restriction in order to navigate in the negotiation while considering only a polynomial number of configurations. For minimal execution time, we provide a pseudo PTIME algorithm, that is PTIME if constants are given in unary. Finally, we show that the size of constants do matter: deciding whether the minimal execution time of a k-layered negotiation is less than T is NP-complete, when T is given in binary. We show this by reducing from a Knapsack problem, yet again emphasizing that the minimal execution time of a negotiation is harder to compute than its maximal execution time.

This paper is organized as follows. Section 2 introduces the key ingredients of negotiations, determinism and soundness, known results in the untimed setting, and provides our running example modeling the Brexit negotiation. Section 3 introduces time in negotiations, gives a semantics to this new model, and formalizes several decision problems on maximal and minimal durations of runs in timed negotiations. We recall the main results of the paper in Section 4. Then, Section 5 considers timed execution problems for deterministic negotiations, Section 6 for sound negotiations, and section 7 for layered negotiations. Proof details for the last three technical sections are given in the Appendices A, B and C.

2 Negotiations: Definitions and Brexit example

In this section, we recall the definition of negotiations, of some subclasses (acyclic and deterministic), as well as important problems (soundness and reachability).

Definition 1 (Negotiation [6, 10]). *A* negotiation *over a finite set of processes P is a tuple $\mathcal{N} = (N, n_0, n_f, \mathcal{X})$, where:*

- *N is a finite set of nodes. Each node is a pair $n = (P_n, R_n)$ where $P_n \subseteq P$ is a non empty set of processes participating in node n, and R_n is a finite set of outcomes of node n (also called results), with $R_{n_f} = \{r_f\}$. We denote by R the union of all outcomes of nodes in N.*
- *n_0 is the first node of the negotiation and n_f is the final node. Every process in P participates in both n_0 and n_f.*
- *For all $n \in N$, $\mathcal{X}_n : P_n \times R_n \to 2^N$ is a map defining the transition relation from node n, with $\mathcal{X}_n(p, r) = \emptyset$ iff $n = n_f, r = r_f$. We denote $\mathcal{X} : N \times P \times$*

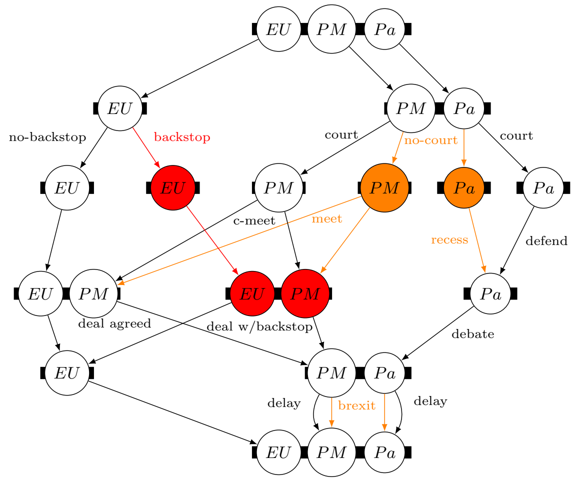

Fig. 1. A (sound but non-deterministic) negotiation modeling Brexit.

$R \to 2^N$ the partial map defined on $\bigcup_{n \in N}(\{n\} \times P_n \times R_n)$, with $\mathcal{X}(n, p, a) = \mathcal{X}_n(p, a)$ for all p, a.

Intuitively, at a node $n = (P_n, R_n)$ in a negotiation, all processes of P_n have to agree on a common outcome r chosen from R_n. Once this outcome r is chosen, every process $p \in P_n$ is ready to move to any node prescribed by $\mathcal{X}(n, p, r)$. A new node m can only start when all processes of P_m are ready to move to m.

Example 1. We illustrate negotiations by considering a simplified model of the Brexit negotiation, see Figure 1. There are 3 processes, $P = \{EU, PM, Pa\}$. At first EU decides whether or not to enforce a backstop in any deal (outcome backstop) or not (outcome no-backstop). In the meantime, PM decides to proroge Pa, and Pa can choose or not to appeal to court (outcome court/no court). If it goes to court, then PM and Pa will take some time in court (c-meet, defend), before PM can meet EU to agree on a deal. Otherwirse, Pa goes to recess, and PM can meet EU directly. Once EU and PM agreed on a deal, PM tries to convince Pa to vote the deal. The final outcome is whether the deal is voted, or whether Brexit is delayed.

Definition 2 (Deterministic negotiations). *A process $p \in P$ is* deterministic *iff, for every $n \in N$ and every outcome r of n, $\mathcal{X}(n, p, r)$ is a singleton. A negotiation is* deterministic *iff all its processes are deterministic. It is* weakly non-deterministic *[9] (called weakly deterministic in [2]) iff, for every node n, one of the processes in P_n is deterministic. Last, it is* very weakly non-deterministic [9] *(called weakly deterministic in [6]) iff, for every n, every $p \in P_n$ and every outcome r of n, there exists a deterministic process q such that $q \in P_{n'}$ for every $n' \in \mathcal{X}(n, p, r)$.*

In deterministic negotiations, once an outcome is chosen, each process knows the next node it will be involved in. In (very)-weakly non-deterministic negotiations, the next node might depend upon the outcome chosen in other nodes by other processes. However, once the outcomes have been chosen for all current nodes, there is only one next node possible for each process. Observe that the class of deterministic negotiations is isomorphic to the class of free choice workflow nets [10]. Coming back to example 1, the Brexit negotiation is non-deterministic, because process PM is non-deterministic. Indeed, consider outcomes *c-meet*: it allows two nodes, according to whether the backstop is enforced or not, which is a decision taken by process EU. However, the Brexit negotiation is very weakly non-deterministic, as the other processes are deterministic.

Semantics: A *configuration* [2] of a negotiation is a mapping $M : P \to 2^N$. Intuitively, it tells for each process p the set $M(p)$ of nodes p is ready to engage in. The semantics of a negotiation is defined in terms of moves from a configuration to the next one. The *initial* M_0 and *final* M_f configurations, are given by $M_0(p) = \{n_0\}$ and $M_f(p) = \emptyset$ respectively for every process $p \in P$. A configuration M *enables* node n if $n \in M(p)$ for every $p \in P_n$. When n is enabled, a decision at node n can occur, and the participants at this node choose an outcome $r \in R_n$. The occurrence of (n, r) produces the configuration M' given by $M'(p) = \mathcal{X}(n, p, r)$ for every $p \in P_n$ and $M'(p) = M(p)$ for remaining processes in $P \setminus P_n$. Moving from M to M' after choosing (n, r) is called a *step*, denoted $M \xrightarrow{n,r} M'$. A *run* of $\mathcal{N}$ is a sequence $(n_1, r_1), (n_2, r_2)...(n_k, r_k)$ such that there is a sequence of configurations $M_0, M_1, \dots, M_k$ and every (n_i, r_i) is a step between M_{i-1} and M_i. A run starting from the initial configuration and ending in the final configuration is called a *final run*. By definition, its last step is (n_f, r_f).

An important class of negotiations in the context of timed negotiations are acyclic negotiations, where infinite sequence of steps are impossible:

Definition 3 (Acyclic negotiations). *The* graph *of a negotiation $\mathcal{N}$ is the labeled graph $G_{\mathcal{N}} = (V, E)$ where $V = N$, and $E = \{((n, (p, r), n') \mid n' \in \mathcal{X}(n, p, r)\}$, with pairs of the form (p, r) being the labels. A negotiation is* acyclic *iff its graph is acyclic. We denote by $Paths(G_{\mathcal{N}})$ the set of paths in the graph of a negotiation. These paths are of the form $\pi = (n_0, (p_0, r_0), n_1) \dots (n_{k-1}, (p_k, r_k), n_k)$.*

The Brexit negotiation of Fig.1 is an example of acyclic negotiation. Despite their apparent simplicity, negotiations may express involved behaviors as shown with the Brexit example. Indeed two important questions in this setting are whether there is some way to reach a final node in the negotiation from (i) the initial node and (ii) any reachable node in the negotiation.

Definition 4 (Soundness and Reachability).

1. *A negotiation is* sound *iff every run from the initial configuration can be extended to a final run. The problem of soundness is to check if a given negotiation is sound.*
2. *The problem of reachability asks if a given negotiation has a final run.*

Notice that the Brexit negotiation of Fig.1 is sound (but not deterministic). It seems hard to preserve the important features of this negotiation while being both sound *and* deterministic. The problem of soundness has received considerable attention. We summarize the results about soudness in the next theorem:

Theorem 1. *Determining whether a negotiation is sound is PSPACE-Complete. For (very-)weakly non-deterministic negotiations, it is co-NP-complete [9]. For acyclic negotiations, it is in DP and co-NP-Hard [6]. Determining whether an acyclic weakly non-deterministic negotiation is sound is in PTIME [2, 9]. Finally, deciding soundness for deterministic negotiation is NLOGSPACE-complete [9].*

Checking reachability is NP-complete, even for deterministic acyclic negotiations (surprisingly, we did not find this result stated before in the literature):

Proposition 1. *Reachability is NP-complete for acyclic negotiations, even if the negotiation is deterministic.*

Proof (sketch). One can easily guess a run of size $\leq |\mathcal{N}|$ in polynomial time, and verify if it reaches n_f, which gives the inclusion in NP. The hardness part comes from a reduction from 3-CNF-SAT that can be found in the proof of Theorem 3. □

k-Layered Acyclic Negotiations

We introduce a new class of negotiations which has good algorithmic properties, namely k-layered acyclic negotiations, for k fixed. Roughly speaking, nodes of a k-layered acyclic negotiations can be arranged in layers, and these layers contain at most k nodes. Before giving a formal definition, we need to define the depth of nodes in $\mathcal{N}$.

First, a *path* in a negotiation is a sequence of nodes $n_0 \dots n_\ell$ such that for all $i \in \{1, \dots, \ell-1\}$, there exists p_i, r_i with $n_{i+1} \in \mathcal{X}(n_i, p_i, r_i)$. The *length* of a path $n_0, \dots, n_\ell$ is ℓ. The *depth* $\text{depth}(n)$ of a node n is the maximal length of a path from n_0 to n (recall that $\mathcal{N}$ is acyclic, so this number is always finite).

Definition 5. *An acyclic negotiation is* layered *if for all node n, every path reaching n has length $depth(n)$. An acyclic negotiation is k*-layered *if it is layered, and for all $\ell \in \mathbb{N}$, there are at most k nodes at depth ℓ.*

The Brexit example of Fig.1 is 6-layered. Notice that a layered negotiation is necessarily k-layered for some $k \leq |\mathcal{N}| - 2$. Note also that we can always transform an acyclic negotiation $\mathcal{N}$ into a layered acyclic negotiation $\mathcal{N}'$, by adding dummy nodes: for every node $m \in \mathcal{X}(n, p, r)$ with $\text{depth}(m) > \text{depth}(n) + 1$, we can add several nodes $n_1, \dots n_\ell$ with $\ell = \text{depth}(m) - (\text{depth}(n) + 1)$, and processes $P_{n_i} = \{p\}$. We compute a new relation $\mathcal{X}'$ such that $\mathcal{X}'(n, p, r) = \{n_1\}$, $\mathcal{X}(n_\ell, p, r) = \{m\}$ and for every $i \in 1..\ell-1$, $\mathcal{X}(n_i, p, r) = n_{i+1}$. This transformation is polynomial: the resulting negotiation is of size up to $|\mathcal{N}| \times |\mathcal{X}| \times |P|$. The proof of the following Theorem can be found in appendix C.

Theorem 2. *Let $k \in \mathbb{N}^+$. Checking reachability or soundness for a k-layered acyclic negotiation $\mathcal{N}$ can be done in PTIME.*

3 Timed Negotiations

In many negotiations, time is an important feature to take into account. For instance, in the Brexit example, with an initial node starting at the begining of September 2019, there are 9 weeks to pass a deal till the 31^{st} October deadline.

We extend negotiations by introducing timing constraints on outcomes of nodes, inspired by time Petri nets [15] and by the notion of negotiations with costs [10]. We use time intervals to specify lower and upper bounds for the duration of negotiations. More precisely, we attach time intervals to pairs (n, r) where n is a node and r an outcome. In the rest of the paper, we denote by $\mathcal{I}$ the set of intervals with endpoints that are non-negative integers or ∞. For convenience we only use closed intervals in this paper (except for ∞), but the results we show can also be extended to open intervals with some notational overhead. Intuitively, outcome r can be taken at a node n with associated time interval $[a, b]$ only after a time units have elapsed from the time all processes contributing to n are ready to engage in n, and at most b time units later.

Definition 6. *A* timed negotiation *is a pair $(\mathcal{N}, \gamma)$ where $\mathcal{N}$ is a negotiation, and $\gamma : N \times R \to \mathcal{I}$ associates an interval to each pair (n, r) of node and outcome such that $r \in R_n$. For a given node n and outcome r, we denote by $\gamma^-(n, r)$ (resp. $\gamma^+(n, r)$) the lower bound (resp. the upper bound) of $\gamma(n, r)$.*

Example 2. In the Brexit example, we define the following timed constraints γ. We only specify the outcome names, as the timing only depends upon them. Backstop and no-backstop both take between 1 and 2 weeks: $\gamma(\text{backstop}) = \gamma(\text{no-backstop}) = [1, 2]$. In case of no-court, recess takes 5 weeks $\gamma(\text{recess}) = [5, 5]$, and PM can meet EU immediatly $\gamma(\text{meet}) = [0, 0]$. In case of court action, PM needs to spend 2 weeks in court $\gamma(\text{c-meet}) = [2, 2]$, and depending on the court delay and decision, Pa needs between 3 (court overules recess) to 5 (court confirms recess) weeks, $\gamma(\text{defend}) = [3, 5]$. Agreeing on a deal can take anywhere from 2 weeks to 2 years (104 weeks): $\gamma(\text{deal agreed}) = [2, 104]$ - some would say infinite time is even possible! It needs more time with the backstop, $\gamma(\text{deal w/backstop}) = [5, 104]$. All others outcomes are assumed to be immediate, i.e., associated with $[0, 0]$.

Semantics: A *timed valuation* is a map $\mu : P \to \mathbb{R}^{\geq 0}$ that associates a non-negative real value to every process. A *timed configuration* is a pair (M, μ) where M is a configuration and μ a timed valuation. There is a *timed step* from (M, μ) to (M', μ'), denoted $(M, \mu) \xrightarrow{(n,r)} (M', \mu')$, if (i) $M \xrightarrow{(n,r)} M'$, (ii) $p \notin P_n$ implies $\mu'(p) = \mu(p)$ (iii) $p \in P_n$ implies $(\mu'(p) - \max_{p' \in P_n} \mu(p')) \in \gamma(n, r)$

Intuitively a timed step $(M, \mu) \xrightarrow{(n,r)} (M', \mu')$ depicts a decision taken at node n, and how long each process of P_n waited in that node before taking decision (n, r). The last process engaged in n must wait for a duration contained in $\gamma(n, r)$. However, other processes may spend a time greater than $\gamma^+(n, r)$.

A *timed run* is a sequence of steps $\rho = (M_1, \mu_1) \xrightarrow{e_1} (M_2, \mu_2) \ldots (M_k, \mu_k)$ where each $(M_i, \mu_i) \xrightarrow{e_i} (M_{i+1}, \mu_{i+1})$ is a timed step. It is *final* if $M_k = M_f$. Its *execution time* $\delta(\rho)$ is defined as $\delta(\rho) = \max_{p \in P} \mu_k(p)$.

Notice that we only attached timing to processes, not to individual steps. With our definition of runs, timing on steps may not be monotonous (i.e., non-decreasing) along the run, while timing on processes is. Viewed by the lens of concurrent systems, the timing is monotonous on the partial orders of the system rather than the linearization. It is not hard to restrict paths, if necessary, to have a monotonous timing on steps as well. In this paper, we are only interested in execution time, which does not depend on the linearization considered.

Given a timed negotiation $\mathcal{N}$, we can now define the minimum and maximum execution time, which correspond to optimistic or pessimistic views:

Definition 7. *Let $\mathcal{N}$ be a timed negotiation. Its* minimum execution time*, denoted $mintime(\mathcal{N})$ is the minimal $\delta(\rho)$ over all final timed run ρ of $\mathcal{N}$. We define the* maximal execution time *$maxtime(\mathcal{N})$ of $\mathcal{N}$ similarly.*

Given $T \in \mathbb{N}$, the main problems we consider in this paper are the following:

- The mintime problem, i.e., do we have $mintime(\mathcal{N}) \leq T$**?**.
  In other words, does there exist a final timed run ρ with $\delta(\rho) \leq T$?
- The maxtime problem, i.e., do we have $maxtime(\mathcal{N}) \leq T$**?**.
  In other words, does $\delta(\rho) \leq T$ for every final timed run ρ?

These questions have a practical interest : in the Brexit example, the question "is there a way to have a vote on a deal within 9 weeks ?" is indeed a minimum execution time problem. We also address the equality variant of these decision problems, i.e., $mintime(\mathcal{N}) = T$ **:** is there a final run of $\mathcal{N}$ that terminates in exactly T time units and no other final run takes less than T time units? Similarly for $maxtime(\mathcal{N}) = T$.

Example 3. We use Fig. 1 to show that it is not easy to compute the minimal execution time, and in particular one cannot use the algorithm from [10] to compute it. Consider the node n with $P_n = \{PM, Pa\}$ and $R_n = \{\text{court}, \text{no\_court}\}$. If the outcome is court, then PM needs 2 weeks before he can talk to EU and Pa needs at least 3 weeks before he can debate. However, if the outcome is no_court, then PM need not wait before he can talk to EU, but Pa wastes 5 weeks in recess. This means that one needs to remember different alternatives which could be faster in the end, depending on the future. On the other hand, the algorithm from [10] attaches one minimal time to process Pa, and one minimal time to process PM. No matter the choices (0 or 2 for PM and 3 or 5 for Pa), there will be futures in which the chosen number will over or underapproximate the real minimal execution time (this choice is not explicit in [10])[4]. For maximum execution time, it is not an issue to attach to each node a unique maximal execution time. The reason for the asymmetry between minimal execution time and maximal execution time of a negotiation is that the execution time of a path is $\max_{p \in P} \mu_k(p)$, for μ_k the last timed valuation, hence breaking the symmetry between min and max.

[4] the authors of [10] acknowledged the issue with their algorithm for mintime.

4 High Level view of the main results

In this section, we give a high-level description of our main results. Formal statements can be found in the sections where they are proved. We gather in Fig. 2 the precise complexities for the minimal and the maximal execution time problems for 3 classes of negotiations that we describe in the following. Since we are interested in minimum and maximum execution time, cycles in negotiations can be either bypassed or lead to infinite maximal time. Hence, while we define timed negotiations in general, we always restrict to acyclic negotiations (such as Brexit) while stating and proving results.

In [10], a PTIME algorithm is given to compute different costs for negotiations that are both sound *and* deterministic. One limitation of this result is that it cannot compute the minimum execution time, as explained in Example 3. A second limitation is that the class of sound and deterministic negotiations is quite restrictive: it cannot model situations where the next node a process participates in depends on the outcome from another process, as in the Brexit example. We thus consider classes where one of these restrictions is dropped.

We first consider (Section 5) negotiations that are deterministic, but without the soundness restriction. We show that for this class, no timed problem we consider can be solved in PTIME (unless NP=PTIME). Further, we show that the equality problems ($maxtime/mintime(\mathcal{N}) = T$), are complete for the complexity class DP, i.e., at the second level of the Boolean Hierarchy [16].

We then consider (Section 6) the class of negotiations that are sound, but not necessarily deterministic. We show that maximum execution time can be solved in PTIME, and propose a new algorithm. However, the minimum execution time cannot be computed in PTIME (unless NP=PTIME). Again for the mintime equality problem we have a matching DP-completeness result.

Finally, in order to obtain a polytime algorithm to compute the minimum execution time, we consider the class of k-layered negotiations (see Section 7): Given $k \in \mathbb{N}$, we can show that $maxtime(\mathcal{N})$ can be computed in PTIME for k-layered negotiations. We also show that while the $mintime(\mathcal{N}) \leq T$? problem is weakly NP-complete for k-layered negotiations, we can compute $mintime(\mathcal{N})$ in pseudo-PTIME, i.e. in PTIME if constants are given in unary.

<table>
<tr><th></th><th>Deterministic</th><th>Sound</th><th>k-layered</th></tr>
<tr><td>Max $\leq T$</td><td>co-NP-complete (Thm. 3)</td><td rowspan="2">PTIME (Prop. 3)</td><td rowspan="2">PTIME (Thm. 6)</td></tr>
<tr><td>Max $= T$</td><td>DP-complete (Prop. 2)</td></tr>
<tr><td>Min $\leq T$</td><td>NP-complete (Thm. 3)</td><td>NP-complete* (Thm. 5)</td><td>pseudo-PTIME (Thm. 8)
NP-complete** (Thm. 7)</td></tr>
<tr><td>Min $= T$</td><td>DP-complete (Prop. 2)</td><td>DP-complete* (Prop. 4)</td><td>pseudo-PTIME (Thm. 8)</td></tr>
</table>

Fig. 2. Results for acyclic timed negotiations. *DP* refers to the complexity class, Difference Polynomial time [16], the second level of the Boolean Hierarchy.
* hardness holds even for very weakly non-deterministic negotiations, and T in unary.
** hardness holds even for sound and very weakly non-deterministic negotiations.

5 Deterministic Negotiations

We start by considering the class of deterministic acyclic negotiations. We show that both maximal and minimal execution time cannot be computed in PTIME (unless NP=PTIME), as the threshold problems are (co-)NP-complete.

Theorem 3. *The $mintime(\mathcal{N}) \leq T$ decision problem is NP complete, and the $maxtime(\mathcal{N}) \leq T$ decision problem is co-NP complete for acyclic deterministic timed negotiations.*

Proof. For $mintime(\mathcal{N}) \leq T$, containment in NP is easy: we just need to guess a run ρ (of polynomial size as $\mathcal{N}$ is acyclic), consider the associated timed run ρ^- where all decisions are taken at their earliest possible dates, and check whether $\delta(\rho^-) \leq T$, which can be done in time $O(|\mathcal{N}| + \log T)$.

For the hardness, we give the proof in two steps. First, we start with a proof of Proposition 1 that reachability problem is NP-hard using reduction of 3-CNF SAT, i.e., given a formula ϕ, we build a deterministic negotiation $\mathcal{N}_\phi$ s.t. ϕ is satisfiable iff $\mathcal{N}_\phi$ has a final run. In a second step, we introduce timings on this negotiation and show that $mintime(\mathcal{N}_\phi) \leq T$ iff ϕ is satisfiable.

Step 1: Reducing 3-CNF-SAT to Reachability problem.

Given a boolean formula ϕ with variables $v_i, 1 \leq i \leq n$ and clauses $c_j, 1 \leq j \leq m$, for each variable v_i we define the sets of clauses $S_{i,\mathtt{t}} = \{c_j | v_i \text{ is present in } c_j\}$ and $S_{i,\mathtt{f}} = \{c_j | \neg v_i \text{ is present in } c_j\}$. Clauses in $S_{i,\mathtt{t}}$ and $S_{i,\mathtt{f}}$ are naturally ordered: $c_i < c_j$ iff $i < j$. We denote these elements $S_{i,\mathtt{t}}(1) < S_{i,\mathtt{t}}(2) < \ldots$. Similarly for set $S_{i,\mathtt{f}}$.

Now, we construct a negotiation $\mathcal{N}_\phi$ (as depicted in Figure 3) with a process V_i for each variable v_i and a process C_j for each clause c_j:

- Initial node n_0 has a single outcome r taking each process C_j to node $Lone_{c_j}$, and each process V_i to node $Lone_{v_i}$.
- $Lone_{c_j}$ has three outcomes: if literal $v_i \in c_j$, then t_i is an outcome, taking V_i to $Pair_{c_j,v_i}$, and if literal $\neg v_i \in c_j$, then f_i is an outcome, taking V_i to $Pair_{c_j,\neg v_i}$.
- The outcomes of $Lone_{v_i}$are `true` and `false`. Outcome `true` brings v_i to node $Tlone_{v_i,1}$ and outcome `false` brings v_i to node $Flone_{v_i,1}$.
- We have a node $Tlone_{v_i,j}$ for each $j \leq |S_{i,\mathtt{t}}|$ and $Flone_{v_i,j}$ for each $j \leq |S_{i,\mathtt{f}}|$, with V_i as only process. Let $c_r = S_{i,\mathtt{t}}(j)$. Node $Tlone_{v_i,j}$ has two outcomes $vton$ bringing V_i to $Tlone_{v_i,j+1}$ (or n_f if $j = |S_{i,\mathtt{t}}|$), and $vtoc_{i,r}$ bringing V_i to $Pair_{c_r,v_i}$. The two outcomes from $Flone_{v_i,j}$ are similar.
- Node $Pair_{c_r,v_i}$ has V_i and C_r as its processes and one outcome $ctof$ which takes process C_j to final node n_f and process V_i to $Tlone_{v_i,j+1}$ (with $c_r = S_{i,\mathtt{t}}(j)$), or to n_f if $j = |S_{i,\mathtt{t}}|$. Node $Pair_{c_r,\neg v_i}$ is defined in the same way from $Flone_{v_i,j}$.

With this we claim that $\mathcal{N}_\phi$ has a final run iff ϕ is satisfiable which completes the first step of the proof. We give a formal proof of this claim in Appendix A. Observe that the negotiation $\mathcal{N}_\phi$ constructed is deterministic and acyclic (but it is not sound).

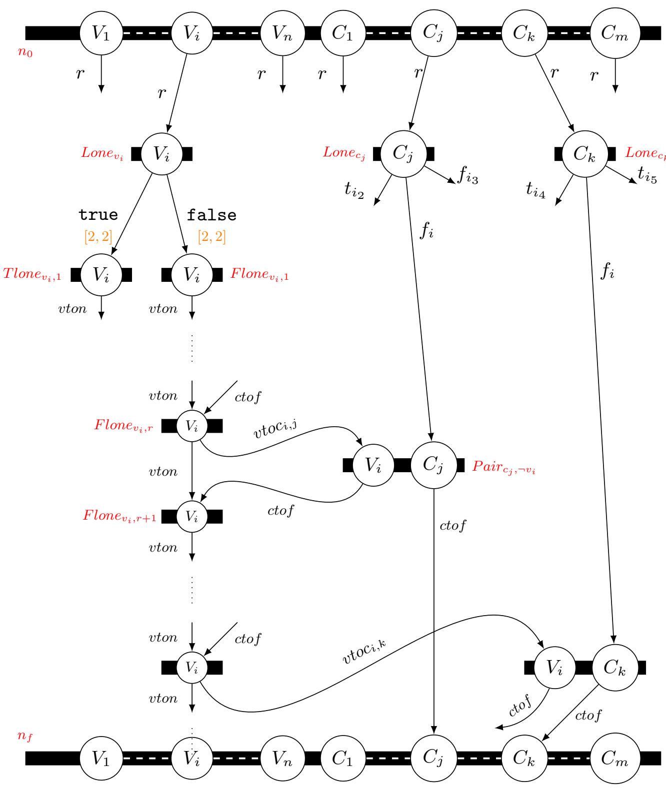

Fig. 3. A part of $\mathcal{N}_\phi$ where clause c_j is $(i_2 \vee \neg i \vee \neg i_3)$ and clause c_k is $(i_4 \vee \neg i \vee i_5)$. Timing is $[0, 0]$ whereever not mentioned

Step 2: Before we introduce timing on $\mathcal{N}_\phi$, we introduce a new outcome r' at n_0 which takes all processes to n_f. Now, the timing function γ associated with the $\mathcal{N}_\phi$ is: $\gamma(n_0, r) = [2, 2]$ and $\gamma(n_0, r') = [3, 3]$ and $\gamma(n, r) = [0, 0]$, for all node $n \neq n_0$ and all $r \in R_n$. Then, $mintime(\mathcal{N}_\phi) \leq 2$ iff ϕ has a satisfiable assignment: if $mintime(\mathcal{N}_\phi) \leq 2$, there is a run with decision r taken at n_0 which is final. But existence of any such final run implies satisfiability of ϕ. For reverse implication, if ϕ is satisfiable, then the corresponding run for satisfying assignment takes 2 units time, which means that $mintime(\mathcal{N}_\phi) \leq 2$.

Similarly, we can prove that the MaxTime problem is co-NP complete by changing $\gamma(n_0, r^{'}) = [1, 1]$ and asking if $maxtime(\mathcal{N}_\phi) > 1$ for the new $\mathcal{N}_\phi$. The answer will be yes iff ϕ is satisfiable. $\square$

We now consider the related problem of checking if $mintime(\mathcal{N}) = T$ (or if $maxtime(\mathcal{N}) = T$). These problems are harder than their threshold variant under usual complexity assumptions: they are DP-complete (Difference Polynomial

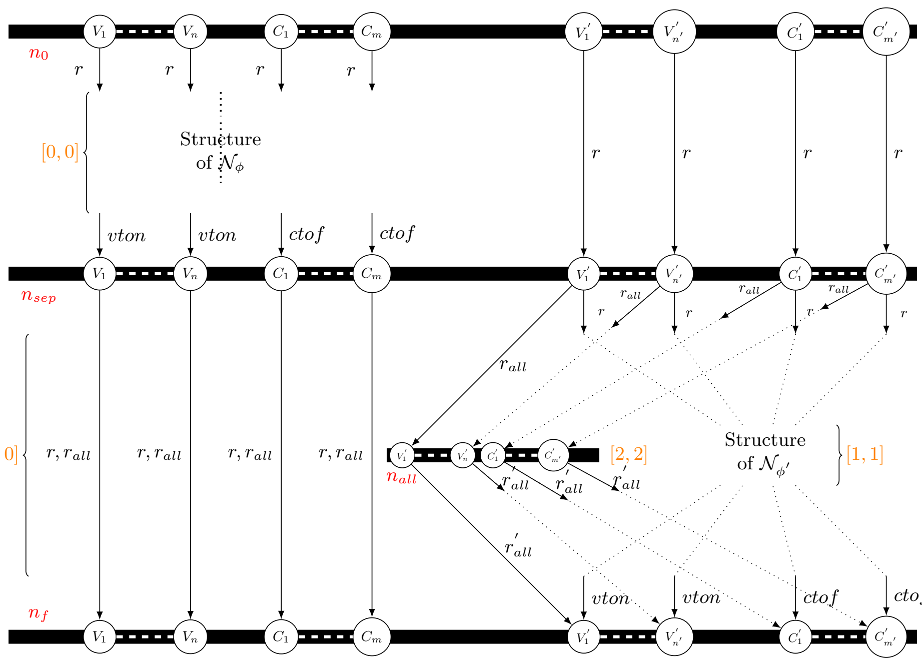

Fig. 4. Structure of $\mathcal{N}_{\phi,\phi'}$

time class, i.e., second level of the Boolean Hierarchy, defined as intersection of a problem in NP and one in co-NP [16]).

Proposition 2. *The $mintime(\mathcal{N}) = T$ and $maxtime(\mathcal{N}) = T$ decision problems are DP-complete for acyclic deterministic negotiations.*

Proof. We only give the proof for *mintime* (the proof for *maxtime* is given in Appendix A). Indeed, it is easy to see that this problem is in DP, as it can be written as $mintime(\mathcal{N}) \leq T$ which is in NP and $\neg(mintime(\mathcal{N}) \leq T - 1))$, which is in co-NP. To show hardness, we use the negotiation constructed in the above proof as a gadget, and show a reduction from the SAT-UNSAT problem (a standard DP-complete problem).

The SAT-UNSAT Problem asks given two Boolean expressions ϕ and ϕ', both in CNF forms with three literals per clause, is it true that ϕ is satisfiable and ϕ' is unsatisfiable? SAT-UNSAT is known to be DP-complete [16]. We reduce this problem to $mintime(\mathcal{N}) = T$.

Given ϕ, ϕ', we first make the corresponding negotiations $\mathcal{N}_\phi$ and $\mathcal{N}_{\phi'}$ as in the previous proof. Let n_0 and n_f be the initial and final nodes of $\mathcal{N}_\phi$ and n_0' and n_f' be the initial and final nodes of $\mathcal{N}_{\phi'}$.(Similarly, for other nodes we write $'$ above the nodes to signify they belong to $\mathcal{N}_{\phi'}$).

In the negotiation $\mathcal{N}_{\phi'}$, we introduce a new node n_{all}, in which all the processes participate (see Figure 4). The node n_{all} has a single outcome r'_{all} which sends all the processes to n_f. Also, for node $n_0^{'}$, apart from the outcome r which sends all processes to different nodes, there is another outcome r_{all} which sends all the processes to n_{all}. Now we merge the nodes n_f and $n_0^{'}$ and call the merged node n_{sep}. Also nodes n_0 and n_f' now have all the processes of $\mathcal{N}_\phi$ and $\mathcal{N}_{\phi'}$ participating in them. This merged process gives us a new negotiation $\mathcal{N}_{\phi,\phi'}$ in which the structure above n_{sep} is same as $\mathcal{N}_\phi$ while below it is same as $\mathcal{N}_{\phi'}$. Node n_{sep} now has all the processes of $\mathcal{N}_\phi$ and $\mathcal{N}_{\phi'}$ participating in it. The outcomes of n_{sep} will be same as that of n_0' (r_{all}, r). For both the outcomes of n_{sep} the processes corresponding to $\mathcal{N}_\phi$ directly go to n_f of the $\mathcal{N}_{\phi,\phi'}$. Similarly n_0 of $\mathcal{N}_{\phi,\phi'}$ which is same n_0 of $\mathcal{N}_{\phi,}$ sends processes corresponding to $\mathcal{N}_{\phi'}$ directly to n_{sep} for all its outcomes. We now define timing function γ for $\mathcal{N}_{\phi,\phi'}$ which is as follows: $\gamma(Lone_{v_i}^{'}, r) = [1,1]$ for all $v_i \in \phi^{'}$ and $r \in \{\texttt{true, false}\}$, $\gamma(n_{all}, r'_{all}) = [2,2]$ and $\gamma(n,r) = [0,0]$ for all other outcomes of nodes. With this construction, one can conclude that $mintime(\mathcal{N}_{\phi,\phi'}) = 2$ iff ϕ is satisfiable and $\phi^{'}$ is unsatisfiable (see Appendix for details). This completes the reduction and hence proves DP-hardness. ☐

Finally, we consider a related problem of computing the min and max time. To consider the decision variant, we rephrase this problem as checking whether an arbitrary bit of the minimum execution time is 1. Perhaps surprisingly, we obtain that this problem goes even beyond DP, the second level of the Boolean Hierarchy and is in fact hard for Δ_2^P (second level of the *polynomial* hierarchy), which contains the entire Boolean Hierarchy. Formally,

Theorem 4. *Given an acyclic deterministic timed negotiation and a positive integer k,computing the k^{th} bit of the maximum/minimum execution time is Δ_2^P-complete.*

Finally, we remark that if we were interested in the optimization variant and not the decision variant of the problem, the above proof can be adapted to show that these variants are OptP-complete (as defined in [14]). But as optimization is not the focus of this paper, we avoid formal details of this proof.

6 Sound Negotiations

Sound negotiations are negotiations in which every run can be extended to a final run, as in Fig. 1. In this section, we show that $maxtime(\mathcal{N})$ can be computed in PTIME for sound negotiations, hence giving PTIME complexities for the $maxtime(\mathcal{N}) \leq T$? and $maxtime(\mathcal{N}) = T$? questions. However, we show that $mintime(\mathcal{N}) \leq T$ is NP-complete for sound negotiations, and that $mintime(\mathcal{N}) = T$ is DP-complete, even if T is given in unary.

Consider the graph $G_\mathcal{N}$ of a negotiation $\mathcal{N}$. Let $\pi = (n_0, (p_0, r_0), n_1) \cdots (n_k, (p_k, r_k), n_{k+1})$ be a path of $G_\mathcal{N}$. We define the *maximal execution time* of a path π as the value $\delta^+(\pi) = \sum_{i \in 0..k} \gamma^+(n_i, r_i)$. We say that a path $\pi = (n_0, (p_0, r_0), n_1) \cdots (n_\ell, (p_\ell, r_\ell), n_{\ell+1})$ is a path of some run $\rho = (M_1, \mu_1) \xrightarrow{(n_1, r_1')} \cdots (M_k, \mu_k)$ if $r_0, \ldots, r_\ell$ is a subword of $r_1', \ldots, r_k'$.

Lemma 1. *Let $\mathcal{N}$ be an acyclic and sound timed negotiation. Then $maxtime(\mathcal{N}) = \max_{\pi \in Paths(G_\mathcal{N})} \delta^+(\pi) + \gamma^+(n_f, r_f)$.*

Proof. Let us first prove that $maxtime(\mathcal{N}) \geq \max_{\pi \in Paths(G_\mathcal{N})} \delta^+(\pi) + \gamma^+(n_f, r_f)$. Consider any path π of $G_\mathcal{N}$, ending in some node n. First, as $\mathcal{N}$ is sound, we can compute a run ρ_π such that π is a path of ρ_π, and ρ_π ends in a configuration in which n is enabled. We associate with ρ_π the timed run ρ^+_π which associates to every node the latest possible execution date. We have easily $\delta(\rho^+_\pi) \geq \delta(\pi)$, and then we obtain $\max_{\pi \in Paths(G_\mathcal{N})} \delta(\rho^+_\pi) \geq \max_{\pi \in Paths(G_\mathcal{N})} \delta(\pi)$. As $maxtime(\mathcal{N})$ is the maximal duration over all runs, it is hence necessarily greater than $\max_{\pi \in Paths(G_\mathcal{N})} \delta(\rho^+_\pi) + \gamma^+(n_f, r_f)$.

We now prove that $maxtime(\mathcal{N}) \leq \max_{\pi \in Paths(G_\mathcal{N})} \delta^+(\pi) + \gamma^+(n_f, r_f)$. Take any timed run $\rho = (M_1, \mu_1) \xrightarrow{(n_1, r_1)} \cdots (M_k, \mu_k)$ of $\mathcal{N}$ with a unique maximal node n_k. We show that there exists a path π of ρ such that $\delta(\rho) \leq \delta^+(\pi)$ by induction on the length k of ρ. The initialization is trivial for $k = 1$. Let $k \in \mathbb{N}$. Because n_k is the unique maximal node of ρ, we have $\delta(\rho) = \max_{p \in P_{n_k}} \mu_{k-1}(p) + \gamma^+(n_k, r_k)$. We choose one p_{k-1} maximizing $\mu_{k-1}(p)$. Let $\ell < k$ be the maximal index of a decision involving process p_{k-1} (i.e. $p_{k-1} \in P_{n_\ell}$). Now, consider the timed run ρ' subword of ρ, but with n_ℓ as unique maximal node (that is, it is ρ where nodes $n_i, i > \ell$ has been removed, but also where some nodes $n_i, i < \ell$ have been removed if they are not causally before n_ℓ (in particular, $P_{n_i} \cap P_{n_\ell} = \emptyset$).

By definition, we have that $\delta(\rho) = \delta(\rho') + \gamma^+(n_\ell, r_\ell) + \gamma^+(n_k, r_k)$. We apply the induction hypothesis on ρ', and obtain a path π' of ρ' ending in n_ℓ such that $\delta(\rho') + \gamma^+(n_\ell, r_\ell) \leq \delta^+(\pi')$. It suffices to consider the path $\pi = \pi'.(n_\ell, (p_{k-1}, r_\ell), n_k)$ to prove the inductive step $\delta(\rho) \leq \delta^+(\pi) + \gamma^+(n_k, r_k)$.

Thus $maxtime(\mathcal{N}) = \max \delta(\rho) \leq \max_{\pi \in Paths(G_\mathcal{N})} \delta^+(\pi) + \gamma^+(n_f, r_f)$. ☐

Lemma 1 gives a way to evaluate the maximal execution time. This amounts to finding a path of maximal weight in an acyclic graph, which is a standard PTIME problem that can be solved using standard max-cost calculation.

Proposition 3. *Computing the maximal execution time for an acyclic sound negotiation $\mathcal{N} = (N, n_0, n_f, \mathcal{X})$ can be done in time $O(|N| + |\mathcal{X}|)$.*

A direct consequence is that $maxtime(\mathcal{N}) \leq T$ and $maxtime(\mathcal{N}) = T$ problems can be solved in polynomial time when $\mathcal{N}$ is sound. Notice that if $\mathcal{N}$ is deterministic but not sound, then Lemma 1 does not hold: we only have an inequality.

We now turn to $mintime(\mathcal{N})$. We show that it is strictly harder to compute for sound negotiations than $maxtime(\mathcal{N})$.

Theorem 5. *$mintime(\mathcal{N}) \leq T$ is NP-complete in the strong sense for sound acyclic negotiations, even if $\mathcal{N}$ is very weakly non-deterministic.*

Proof (sketch). First, we can decide $mintime(\mathcal{N}) \leq T$ in NP. Indeed, one can guess a final (untimed) run ρ of size $\leq |N|$, consider ρ^- the timed run corresponding to ρ where all outcomes are taken at the earliest possible dates, and compute in linear time $\delta(\rho^-)$, and check that $\delta(\rho^-) \leq T$.

The hardness part is obtained by reduction from the **Bin Packing** problem. The reduction is similar to Knapsack, that we will present in Thm. 7. The difference is that we use ℓ bins in parallel, rather than 2 process, one for the weight and one for the value. The hardness is thus strong, but the negotiation is not k-layered for a bounded k (It is $2\ell + 1$ bounded, with ℓ depending on the input). A detailed proof is given in Appendix B. $\square$

We show that $mintime(\mathcal{N}) = T$ is harder to decide than $mintime(\mathcal{N}) \leq T$, with a proof similar to Prop. 2.

Proposition 4. *The $mintime(\mathcal{N}) = T$? decision problem is DP-complete for sound acyclic negotiations, even if it is very weakly non-deterministic.*

An open question is whether the minimal execution time can be computed in PTIME if the negotiation is both sound and deterministic. The reduction from Bin Packing does not work with deterministic (and sound) negotiations.

7 k-Layered Negotiations

In the previous sections, we have considered sound negotiations, and deterministic negotiations. For both classes, computing the minimal execution time cannot be done in PTIME (unless NP=PTIME), even if constants are given in unary. In this section, we consider k-layeredness (see Section 2), a syntactic property that can be efficiently verified (it suffices to compute the depth of each node, which can be done in polynomial time).

7.1 Algorithmic properties

Let k be a fixed integer. We first show that the maximum execution time can be computed in PTIME for k-layered negotiations. Let N_i be the set of nodes at layer i. We define for every layer i the set S_i of subsets of nodes $X \subseteq N_i$ which can be jointly enabled and such that for every process p, there is exactly one node $n(X, p)$ in X with $p \in n(X, p)$. Formally, we define S_i inductively. We start with $S_0 = \{n_0\}$. We then define S_{i+1} from the contents of layer S_i: we have $Y \in S_{i+1}$ iff $\bigcup_{n \in Y} P_n = P$ and there exist $X \in S_i$ and an outcome $r_m \in R_m$ for every $m \in X$, such that $n \in \mathcal{X}(n(X, p), p, r_m)$ for each $n \in Y$ and $p \in P_n$.

Theorem 6. *Let $k \in \mathbb{N}^+$. Computing the maximum execution time for a k-layered acyclic negotiation $\mathcal{N}$ can be done in PTIME. More precisely, the worst-case time complexity is $O(|P| \cdot |\mathcal{N}|^{k+1})$.*

Proof (Sketch). The first step is to compute S_i layer by layer, by following its inductive definition. The set S_i is of size at most 2^k, as $|N_i| < k$ by definition of k-layeredness. Knowing S_i, it is easy to build S_{i+1} by induction. This takes time in $O(|P||\mathcal{N}|^{k+1})$: We need to consider all k-uple of outcomes for each layer. There can be $|\mathcal{N}|^k$ such tuples. We need to do that for all processes ($|P|$), and for all layers (at most $|\mathcal{N}|$).

We then keep for each subset $X \in S_i$ and each node $n \in X$, the maximal time $f_i(n, X) \in \mathbb{N}$ associated with n and X. From S_{i+1} and f_i, we inductively compute f_{i+1} in the following way: for all $X \in S_i$ with successor $Y \in S_{i+1}$ for outcomes $(r_p)_{p\in P}$, we denote $f_{i+1}(Y, n, X) = \max_{p\in P(n)} f_i(X, n(X, p)) + \gamma^+(n(X, p), r_p)$. If there are several choices of $(r_p)_{p\in P}$ leading to the same Y, we take r_p with the maximal $f_i(X, n(X, p)) + \gamma^+(n(X, p), r_p)$. We then define $f_{i+1}(Y, n) = \max_{X\in S_i} f_{i+1}(Y, n, X)$. Again, the initialization is trivial, with $f_0(\{n_0\}, n_0) = 0$. The maximal execution time of $\mathcal{N}$ is $f(\{n_f\}, n_f)$. □

We can bound the complexity precisely by $O(d(\mathcal{N}) \cdot C(\mathcal{N}) \cdot ||R||^{k^*})$, with:

- $d(\mathcal{N}) \leq |\mathcal{N}|$ the depth of n_f, that is the number of layers of $\mathcal{N}$, and $||R||$ is the maximum number of outcomes of a node,
- $C(\mathcal{N}) = \max_i |S_i| \leq 2^k$, which we will call the *number of contexts of* $\mathcal{N}$, and which is often much smaller than 2^k.
- $k^* = \max_{X\in\bigcup_i S_i} |X| \leq k$. We say that $\mathcal{N}$ is k^**-thread bounded*, meaning that there cannot be more that k^* nodes in the same context X of any layer. Usually, k^* is strictly smaller than $k = \max_i |N_i|$, as $N_i = \bigcup_{X\in S_i} X$.

Consider again the Brexit example Figure 1. We have $(k + 1) = 7$, while we have the depth $d(\mathcal{N}) = 6$, the negotiation is $k^* = 3$-thread bounded (k^* is bounded by the number of processes), $||R|| = 2$, and the number of contexts is at most $C(\mathcal{N}) = 4$ (EU chooses to enforce backstop or not, and Pa chooses to go to court or not).

7.2 Minimal Execution Time

As with sound negotiations, computing minimal time is much harder than computing the maximal time for k-layered negotiations:

Theorem 7. *Let $k \geq 6$. The $Min \leq T$ problem is NP-Complete for k-layered acyclic negotiations, even if the negotiation is sound and very weakly non-deterministic.*

Proof. One can guess in polynomial time a final run of size $\leq |\mathcal{N}|$. If the execution time of this final run is smaller than T then we have found a final run witnessing $mintime(\mathcal{N}) \leq T$. Hence the problem is in NP.

Let us now show that the problem is NP-hard. We proceed by reduction from the **Knapsack** decision problem. Let us consider a set of items $U = \{u_1, \ldots u_n\}$ of respective values $v_1, \ldots v_n$ and weight $w_1, \ldots, w_n$ and a knapsack of maximal capacity W. The knapsack problem asks, given a value V whether there exists a subset of items $U' \subseteq U$ such that $\sum_{u_i\in U'} v_i \geq V$ and such that $\sum_{u_i\in U'} w_i \leq W$.

We build a negotiation with $2n$ processes $P = \{p_1, \ldots p_{2n}\}$, as shown in Fig. 5. Intuitively, $p_i, i \leq n$ will serve to encode the value of selected items as timing, while $p_i, i > n$ will serve to encode the weight of selected items as timing.

Concerning timing constraints for outcomes we do the following: Outcomes 0, yes and no are associated with $[0, 0]$. Outcome c_i is associated with $[w_i, w_i]$, the weight of u_i. Last, outcome b_i is associated with a more complex function,

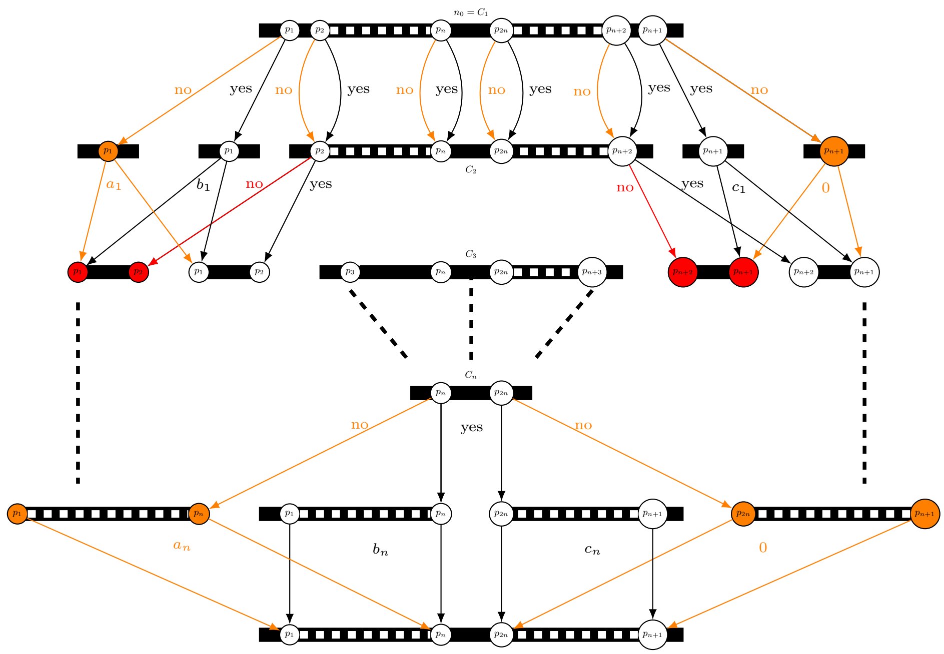

Fig. 5. The negotiation encoding Knapsack

such that $\sum_i b_i \leq W$ iff $\sum_i v_i \geq V$. For that, we set $[\frac{(v_{max}-v_i)W}{n \cdot v_{max}-V}, \frac{v_{max}W}{n \cdot v_{max}-v_i}]$ for outcome b_i, where v_{max} is the largest value of an item, and V is the total value we want to reach at least. Also, we set $[\frac{(v_{max})W}{n \cdot v_{max}-V}, \frac{v_{max}W}{n \cdot v_{max}-v_i}]$ for outcome a_i. We set $T = W$, the maximal weight of the knapsack.

Now, consider a final run ρ in $\mathcal{N}$. The only choices in ρ are outcomes *yes* or *no* from $C_1, \ldots, C_n$. Let I be the set of indices such that *yes* is the outcome from all C_i in this path. We obtain $\delta(\rho) = \max(\sum_{i \notin I} a_i + \sum_{i \in I} b_i, \sum_{i \in I} c_i)$. We have $\delta(\rho) \leq T = W$ iff $\sum_{i \in I} w_i \leq W$, that is the sum of the weights is lower than W, and $\sum_{i \notin I} \frac{(v_{max})W}{n \cdot v_{max}-V} + \sum_{i \in I} \frac{(v_{max}-v_i)W}{n \cdot v_{max}-V} \leq W$. That is, $n \cdot v_{max} - \sum_{i \in I} v_i \leq n \cdot v_{max} - V$, i.e. $\sum_{i \in I} v_i \geq V$. Hence, there exists a path ρ with $\delta(\rho) \leq T = W$ iff there exists a set of items of weight less than W and of value more than V. $\square$

It is well known that Knapsack is weakly NP-hard, that is, it is NP-hard only when weights/values are given in binary. This means that Thm. 7 shows that minimum execution time $\leq T$ is NP-hard only when T is given in binary. We can actually show that for k-layered negotiations, the $mintime(\mathcal{N}) \leq T$ problem can be decided in PTIME if T is given in unary (i.e. if T is not too large):

Theorem 8. *Let $k \in \mathbb{N}$. Given a k-layered negotiation $\mathcal{N}$ and T written in unary, one can decide in PTIME whether the minimum execution time of $\mathcal{N}$ is $\leq T$. The worst-case time complexity is $O(|\mathcal{N}| \cdot |P| \cdot (T \cdot |\mathcal{N}|)^k)$.*

Proof. We will remember for each layer i a set $\mathcal{T}_i$ of functions τ from nodes N_i of layer i to a value in $\{1, \dots, T, \bot\}$. Basically, we have $\tau \in \mathcal{T}_i$ if there exists a path ρ reaching $X = \{n \in N_i \mid f(n) \neq \bot\}$, and this path reaches node $n \in X$ after $\tau(n)$ time units. As for S_i, for all p, we should have a unique node $n(\tau, p)$ such that $p \in n(f, p)$ and $\tau(n(\tau, p)) \neq \bot$. Again, it is easy to initialize $\mathcal{T}_0 = \{\tau_0\}$, with $\tau_0(n_0) = 0$, and $\tau_0(n) = \bot$ for all $n \neq n_0$.

Inducively, we build $\mathcal{T}_{i+1}$ in the following way: $\tau_{i+1} \in \mathcal{T}_{i+1}$ iff there exists a $\tau_i \in \mathcal{T}_i$ and $r_p \in R_{n(\tau_i,p)}$ for all $p \in P$ such that for all n with $\tau_{i+1}(n) \neq \bot$, we have $\tau_{i+1}(n) = \max_p \tau_i(n(\tau_i, p)) + \gamma(n(\tau_i, p), r_p)$.

We have that the minimum execution time for $\mathcal{N}$ is $\min_{\tau \in \mathcal{T}_n} \tau(n_\tau)$, for n the depth of n_f. There are at most T^k functions τ in any $\mathcal{T}_i$, and there are at most $|\mathcal{N}|$ layers to consider, giving the complexity. $\square$

As with Thm. 6, we can more accurately state the complexity as $O(d(\mathcal{N}) \cdot C(\mathcal{N}) \cdot ||R||^{k^*} \cdot T^{k^*-1})$. The $k^* - 1$ is because we only need to remember minimal functions $\tau \in \mathcal{T}_i$: if $\tau'(n) \geq \tau(n)$ for all n, then we do not need to keep τ' in $\mathcal{T}_i$. In particular, for the knapsack encoding in the proof of Thm. 7, we have $k^* = 3$, $||R|| = 2$ and $C(\mathcal{N}) = 4$.

Notice that if k is part of the input, then the problem is strongly NP-hard, even if T is given in unary, as e.g. encoding bin packing with ℓ bins result to a $2\ell + 1$-layered negotiations.

8 Conclusion

In this paper, we considered timed negotiations. We believe that time is of the essence in negotiations, as examplified by the Brexit negotiation. It is thus important to be able to compute in a tractable way the minimal and maximal execution time of negotiations.

We showed that we can compute in PTIME the maximal execution time for acyclic negotiations that are either sound or k-layered, for k fixed. We showed that we cannot compute in PTIME the maximal execution time for negotiations that are not sound nor k-layered, even if they are deterministic and acyclic (unless NP=PTIME). We also showed that surprisingly, computing the minimal execution time is much harder, with strong NP-hardness results in most of the classes of negotiations, contradicting a claim in [10]. We came up with a new reasonable class of negotiations, namely k-layered negotiations, which enjoys a pseudo PTIME algorithm to compute the minimal execution time. That is, the algorithm is PTIME when the timing constants are given in unary. We showed that this restriction is necessary, as the problem becomes NP-hard for constants given in binary, even when the negotiation is sound and very weakly non-deterministic. The problem to know whether the minimal execution time can be computed in PTIME for deterministic and sound negotiation remains open.

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Appendix A: Deterministic Negotiations

We start by considering the class of deterministic acyclic negotiations. We show that both maximal and minimal execution time cannot be computed in PTIME (unless NP=PTIME), as the threshold problems are (co-)NP-complete.

Theorem 3. *The $mintime(\mathcal{N}) \leq T$ decision problem is NP complete, and the $maxtime(\mathcal{N}) \leq T$ decision problem is co-NP complete for acyclic deterministic timed negotiations.*

Proof. For $mintime(\mathcal{N}) \leq T$, containment in NP is easy: we just need to guess a run ρ (of polynomial size as $\mathcal{N}$ is acyclic), consider the associate timed run ρ^- where all decisions are taken at their earliest possible dates, and check whether $\delta(\rho^-) \leq T$, which can be done in time $O(|\mathcal{N}|+\log T)$.

For the hardness, we give the proof in two steps. First, we start with a proof of Proposition 1 that reachability problem is NP-hard using reduction of 3-CNF SAT, i.e., given a formula ϕ, we build a deterministic negotiation $\mathcal{N}_\phi$ s.t. ϕ is satisfiable iff $\mathcal{N}_\phi$ has a final run. In a second step, we introduce timings on this negotiation and show that $mintime(\mathcal{N}_\phi) \leq T$ iff ϕ is satisfiable.

Step 1: Reducing 3-CNF-SAT to Reachability problem.

Given a boolean formula ϕ with variables $v_i, 1 \leq i \leq n$ and clauses $c_j, 1 \leq j \leq m$, for each variable v_i we define the sets of clauses $S_{i,\mathtt{t}} = \{c_j | v_i \text{ is present in } c_j\}$ and $S_{i,\mathtt{f}} = \{c_j | \neg v_i \text{ is present in } c_j\}$. Clauses in $S_{i,\mathtt{t}}$ and $S_{i,\mathtt{f}}$ are naturally ordered: $c_i < c_j$ iff $i < j$. We denote these elements $S_{i,\mathtt{t}}(1) < S_{i,\mathtt{t}}(2) < \ldots$. Similarly for set $S_{i,\mathtt{f}}$.

Now, we construct a negotiation $\mathcal{N}_\phi$ with a process V_i for each variable v_i and a process C_j for each clause c_j:

- Initial node n_0 has a single outcome r taking each process C_j to node $Lone_{c_j}$, and each process V_i to node $Lone_{v_i}$.
- $Lone_{c_j}$ has three outcomes: if literal $v_i \in c_j$, then t_i is an outcome, taking V_i to $Pair_{c_j,v_i}$, and if literal $\neg v_i \in c_j$, then f_i is an outcome, taking V_i to $Pair_{c_j,\neg v_i}$.
- The outcomes of $Lone_{v_i}$are `true` and `false`. Outcome `true` brings v_i to node $Tlone_{v_i,1}$ and outcome `false` brings v_i to node $Flone_{v_i,1}$.
- We have a node $Tlone_{v_i,j}$ for each $j \leq |S_{i,\mathtt{t}}|$ and $Flone_{v_i,j}$ for each $j \leq |S_{i,\mathtt{f}}|$, with V_i as only process. Let $c_r = S_{i,\mathtt{t}}(j)$. Node $Tlone_{v_i,j}$ has two outcomes $vton$ bringing V_i to $Tlone_{v_i,j+1}$ (or n_f if $j = |S_{i,\mathtt{t}}|$), and $vtoc_{i,r}$ bringing V_i to $Pair_{c_r,v_i}$. The two outcomes from $Flone_{v_i,j}$ are similar.
- Node $Pair_{c_r,v_i}$ has V_i and C_r as its processes and one outcome $ctof$ which takes process C_j to final node n_f and process V_i to $Tlone_{v_i,j+1}$ (with $c_r = S_{i,\mathtt{t}}(j)$), or to n_f if $j = |S_{i,\mathtt{t}}|$. Node $Pair_{c_r,\neg v_i}$ is defined in the same way from $Flone_{v_i,j}$.

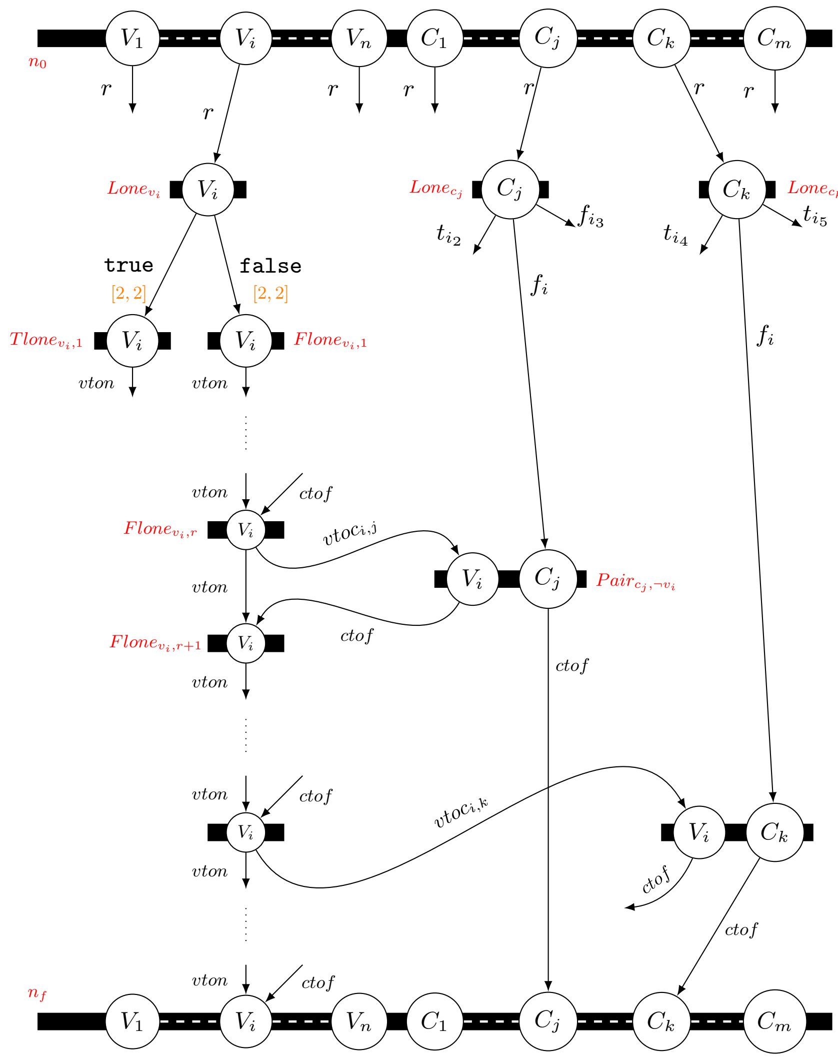

Fig. 3. A part of $\mathcal{N}_\phi$ where clause c_j is $(i_2 \vee \neg i \vee \neg i_3)$ and clause c_k is $(i_4 \vee \neg i \vee i_5)$. Timing is $[0,0]$ whereever not mentioned

Claim. $\mathcal{N}_\phi$ has a final run iff ϕ is satisfiable.

Proof. First we show that if there is a run ρ from n_0 to n_f then ϕ is satisfiable. In ρ, all processes reached n_f. So each process V_i takes either outcome `true` or `false` in ρ. Let val the valuation associated each variable v_i with the choice `true` or `false` by V_i. We now show that all clause c_r have at least one literal true in val. In ρ, process C_r reaches the final node n_f: it must have gone via one node either $Pair_{c_r,v_i}$ or $Pair_{c_r,\neg v_i}$, for some i. Wlog, let us assume that C_r went to $Pair_{c_r\neg v_i}$. The only way it is possible is for process V_i to have been in $Flone_{v_i,j}$, with $c_r = S_{i,\mathtt{f}}(j)$. This is possible only if V_i decided outcome `false` at $Lone_{v_i}$. So this implies that literal $\neg v_i$ of c_j is true in val. Hence ϕ is satisfiable.

Conversely, we show that if ϕ is satisfiable then $\mathcal{N}_\phi$ has final run. Let val a satisfiable assignment $val : V \to \{\mathtt{true,false}\}$ for ϕ. We build a run ρ which is final. After reaching $Lone_{v_i}$, V_i will decide the outcome according to the value of $val(v_i)$ and reach $Flone_{v_i,1}$ or $Tlone_{v_i,1}$ accordingly. Let $G_i(val)$ be the set of clause c_j such that i is the minimal literal of c_j true under val. When there is a choice between two outcomes $vton$ and $vtoc_{i,k}$ for process V_i, the run chooses $vtoc_{i,k}$ iff $k \in G_i(val)$. Concerning C_j, it appears in exactly one $G_i(val)$, because val satisfies ϕ. If $val(v_i) = \mathtt{true}$, run ρ chooses outcome t_i for V_i in node $Lone_{c_j}$, and outcome f_i if $val(v_i) = \mathtt{false}$. Observe that the same variable v_i can be associated with several clauses c_j, but then all these clauses go to the same type of nodes i.e. $Pair_{c_j,v_i}$ if $val(v_i) = \mathtt{true}$ and $Pair_{c_j,\neg v_i}$ if $val(v_i) = \mathtt{false}$.

This run ρ is final: Every process C_j reaches n_f after participating in exactly one node $Pair_{c_j,v_i}$ or $Pair_{c_j,\neg v_i}$. Every process V_i reaches n_f after participating in zero or more node $Pair_{c_j,v_i}$ or $Pair_{c_j,\neg v_i}$ (it participates in exactly $|G_i|$ such nodes). $\square$

With this we claim that $\mathcal{N}_\phi$ has a final run iff ϕ is satisfiable which completes the first step of the proof. Observe that the negotiation $\mathcal{N}_\phi$ constructed is deterministic and acyclic (but it is not sound).

Step 2: Before we introduce timing on $\mathcal{N}_\phi$, we introduce a new outcome r' at n_0 which takes all processes to n_f. Now, the timing function γ associated with the $\mathcal{N}_\phi$ is: $\gamma(n_0, r) = [2, 2]$ and $\gamma(n_0, r') = [3, 3]$ and $\gamma(n, r) = [0, 0]$, for all node $n \neq n_0$ and all $r \in R_n$. Then, $mintime(\mathcal{N}_\phi) \leq 2$ iff ϕ has a satisfiable assignment: if $mintime(\mathcal{N}_\phi) \leq 2$, there is a run with decision r taken at n_0 which is final. But existence of any such final run implies satisfiability of ϕ. For reverse implication, if ϕ is satisfiable, then the corresponding run for satisfying assignment takes 2 units time, which means that $mintime(\mathcal{N}_\phi) \leq 2$.

Similarly, we can prove that the MaxTime problem is Co-NP complete by changing $\gamma(n_0, r^{'}) = [1, 1]$ and asking if $maxtime(\mathcal{N}_\phi) > 1$ for the new $\mathcal{N}_\phi$. The answer will be yes iff ϕ is satisfiable.

As a side note, we observe that the NP-hardness for mintime could also have been proved without introducing the new result r' but then it would have been possible that $\mathcal{N}_\phi$ had no final run making $mintime(\mathcal{N}_\phi) \leq 2$ vacuous. $\square$

We now consider the related problem of checking if $mintime(\mathcal{N}) = T$ (or if $maxtime(\mathcal{N}) = T$). These problems are harder than their threshold variant un-

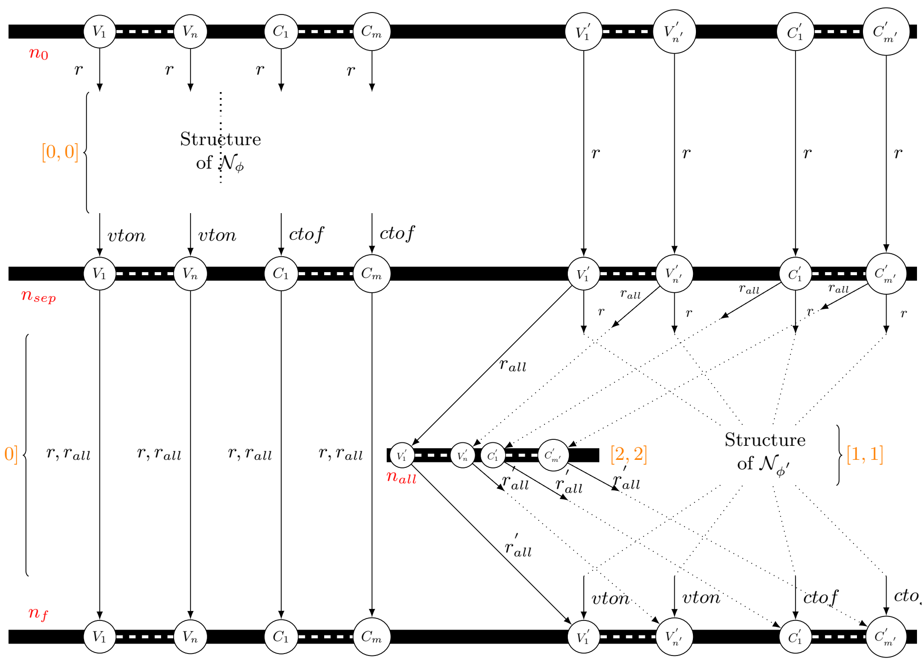

Fig. 4. Structure of $\mathcal{N}_{\phi,\phi'}$

der usual complexity assumptions: they are DP-complete (Difference Polynomial time class, i.e., second level of the Boolean Hierarchy, defined as intersection of a problem in NP and co-NP [16]).

Proposition 2. *The* $mintime(\mathcal{N}) = T$ *and* $maxtime(\mathcal{N}) = T$ *decision problems are DP-complete for acyclic deterministic negotiations.*

Proof. Indeed, it is easy to see that this problem is in DP, as it can be written as $mintime(\mathcal{N}) \leq T$ which is in NP and $\neg(mintime(\mathcal{N}) \leq T-1))$, which is in co-NP. To show hardness, we use the negotiation constructed in the above proof as a gadget, and show a reduction from the SAT-UNSAT problem (a standard DP-complete problem).

SAT-UNSAT Problem : Given two Boolean expressions ϕ and $\phi^{'}$, both in CNF forms with three literals per clause, is it true that ϕ is satisfiable and $\phi^{'}$ is unsatisfiable? SAT-UNSAT is known to be DP-Complete [16]. We reduce this problem to $mintime(\mathcal{N}) = T$.

Given ϕ, $\phi^{'}$, we first make the corresponding negotiations $\mathcal{N}_\phi$ and $\mathcal{N}_{\phi'}$ as in the previous proof. Let n_0 and n_f be the initial and final nodes of $\mathcal{N}_\phi$ and $n_0^{'}$ and $n_f^{'}$ be the inital and final nodes of $\mathcal{N}_{\phi'}$.(Similarly, for other nodes we write $'$ above the nodes to signify they belong to $\mathcal{N}_{\phi'}$). In the negotiation $\mathcal{N}_{\phi'}$, we introduce a new node n_{all} (see Figure 4), in which all the processes participate. The node n_{all} has a single outcome r'_{all} which sends all the processes to n_f. Also, for node

$n_0^{'}$, apart from the outcome r which sends all processes to different nodes, there is another outcome r_{all} which sends all the processes to n_{all}.
Now we merge the nodes n_f and $n_0^{'}$ and call the merged node n_{sep}. Also nodes n_0 and n_f' now have all the processes of $\mathcal{N}_\phi$ and $\mathcal{N}_{\phi'}$ participating in them.

This merged process gives us a new negotiation $\mathcal{N}_{\phi,\phi'}$ in which the structure above n_{sep} is same as $\mathcal{N}_\phi$ while below it is same as $\mathcal{N}_{\phi'}$. Node n_{sep} now has all the processes of $\mathcal{N}_\phi$ and $\mathcal{N}_{\phi'}$ participating in it. The outcomes of n_{sep} will be same as that of n_0' (r_{all}, r). For both the outcomes of n_{sep} the processes corresponding to $\mathcal{N}_\phi$ directly go to n_f of the $\mathcal{N}_{\phi,\phi'}$. Similarly n_0 of $\mathcal{N}_{\phi,\phi'}$ which is same n_0 of $\mathcal{N}_{\phi,}$ sends processes corresponding to $\mathcal{N}_{\phi'}$ directly to n_{sep} for all its outcomes. We now define timing function γ for $\mathcal{N}_{\phi,\phi'}$ which is as follows:

- $\gamma(Lone^{'}_{v_i,r}) = [1,1]$ for all $v_i \in \phi^{'}$ and $r \in \{\texttt{true, false}\}$,
- $\gamma(n_{all}, r'_{all}) = [2,2]$ and
- $\gamma(n,r) = [0,0]$ for all other outcomes of nodes.

The claim is that

Claim. $mintime(\mathcal{N}_{\phi,\phi'}) = 2$ iff ϕ is satisfiable and $\phi^{'}$ is unsatisfiable.

Proof. If $mintime(\mathcal{N}_{\phi,\phi'}) = 2$, this implies that ϕ is satisfiable, for if it was not satifiable then for no run, all the processes corresponding to ϕ could reach n_{sep} and therefore the negotiation could not complete and hence MinTime would be infinite. Also $\phi^{'}$ is unsatisfiable because if it would have been satisfiable then there would have been a final run in which the processes after reaching n_{sep} choose the outcome r from n_{sep} and complete the negotiation. The time for that run would be 1 unit and therefore the $mintime(\mathcal{N}_{\phi,\phi'}) \neq 2$.

For the other side of the implication, we can argue similarly that if ϕ is satisfiable then the processes of $\mathcal{N}_\phi$ would complete the structure above n_{sep} and reach n_{sep} in 0 units of time. From there the processes would have to choose the outcome r_{all} to reach n_f because otherwise, the run would not be final. The time taken for the path would be 2 units. So total time associated will this run will be 2 units which will also be the $mintime(\mathcal{N}_{\phi,\phi'})$. $\square$

For equality decision problem of MaxTime, the proof is similar; only the $\gamma(Lone^{'}_{v_i}, r) = [2,2]$ for all $v_i \in \phi^{'}$, $\gamma(n_{all}, r'_{all}) = [1,1]$ and $\gamma(n,r) = [0,0]$ for all other nodes. The question asked is $maxtime(\mathcal{N}_{\phi,\phi'}) = 2$ which is true if only if ϕ is satisfiable and $\phi^{'}$ is unsatisfiable. $\square$

Finally, we consider a related problem of deciding if a bit of $mintime(\mathcal{N})$ is 1 (or similarly with $maxtime(\mathcal{N})$). Perhaps surprisingly, we obtain that these problems goes even beyond DP (the second level of the Boolean Hierarchy) and is in fact hard for Δ_2^P, which contains the whole Boolean Hierarchy:

Theorem 4. *Given an acyclic deterministic timed negotiation and a positive integer k, computing the k^{th} bit of the maximum/minimum execution time is Δ_2^P complete.*

Proof. Containment is again relatively easy. Given an acyclic deterministic timed negotiation, we can compute the largest possible time attainable as a function of the number of nodes and maximal constant in each node. Now guess the min/max time (in binary) and then check it using NP-oracle or equivalently Co-NP oracle calls.

The hardness is not so simple to obtain. We first notice that it suffices to show the problem of whether $maxtime/mintime(\mathcal{N}) =$ odd ? is Δ_2^p hard. This is because odd or even is the same as the last bit. We first show that $maxtime(\mathcal{N})$ = odd is Δ_2^p complete.

Consider the following problem: Given a Boolean formula $\phi(x_1, x_2, ...x_n)$, is $x_n = 1$ in the lexicographically largest satisfying assignment of ϕ?

The above problem is known to be Δ_2^p complete [14] and we reduce it to the decision problem of $maxtime(\mathcal{N}) =$ odd? First, we convert ϕ to 3-CNF form using Tseitin transformation. Let the new variables introduced be called $t_1, t_2, ...t_k$. So $\phi(x_1, x_2, ...x_n)$ is equisatisfiable to 3-CNF $\phi^{'}(v_1, v_2, ...v_n, v_{n+1}, ...v_{n+k})$ where $v_i = x_i$ for $i \leq n$ and $v_i = t_i$ for $i > n$. We convert $\phi^{'}$ to a negotiation $\mathcal{N}_{\phi'}$. $\mathcal{N}_{\phi'}$ has the same structure as that of $\mathcal{N}_\phi$ which was constructed in Theorem 3 apart from some change in arcs and participation of processes in nodes.
Paritcipation changes are the following : The node $Lone_{v_i}$ associated with each variable v_i of $\phi^{'}$ now involve two processes namely V_i and V_{i-1}. ($Lone_{v_1}$ has only V_1 as process). Both of the outcomes, `true` and `false` associated with $Lone_{v_i}$ take V_{i-1} to n_f while `true` takes V_i to $TLone_{v_i,1}$ and `false` takes V_i to $Flone_{v_i,1}$. Change in arcs is the following: The outcome *vton* of $FLone_{v_i,r}$ where $r = |S_{i,\mathtt{f}}|$ and $TLone_{v_i,r'}$ where $r' = |S_{i,\mathtt{t}}|$ takes V_i to $Lone_{v_{i+1}}$ (Except for $i = n + k$ for which there is no change). We now define timing function γ as follows:

– $\gamma(Lone_{v_i}, \mathtt{true}) = [2^{n-i}, 2^{n-i}]$ for all $i \leq n$ and
– $\gamma(n, r) = 0$ for all other combination of nodes and outcomes.

The claim is that $maxtime(\mathcal{N}_{\phi'})$ = odd iff $x_n = 1$ in the lexicographically largest satisfying assignment of ϕ.

To prove the claim, we prove a stronger outcome that there is a run which is final and takes time t iff there is a satisfying assignment to ϕ whose lexicographic value is same as t in binary.

To prove the forward implication, consider any run σ which is final. Now, just like the proof in 3, the process V_i must have choosen either `true` or `false` at node $Lone_{v_i}$. The assignment f, corresponding to this outcome chosen by each V_i is essentially the one whose lexicographic value is same as t. The fact that this assignment is satisifable follows from the proof of theorem 3. To show that that lexicorgraphic value is same, first of all the observe that time taken t can be written as $2^{n-i_1} + 2^{n-i_2} + ... + 2^{n-i_k}$ where V_{i_j} are those processes which chose `true` at $Lone_{v_i}$. Moreover $i_j \leq n$, which implies that all these variables are also present in ϕ. Also the the contribution of a variable x_{i_j} (which is same as v_{i_j}) in lexicographic value will be 2^{n-i_j} which is same as its contribution in t. Hence the forward implication.

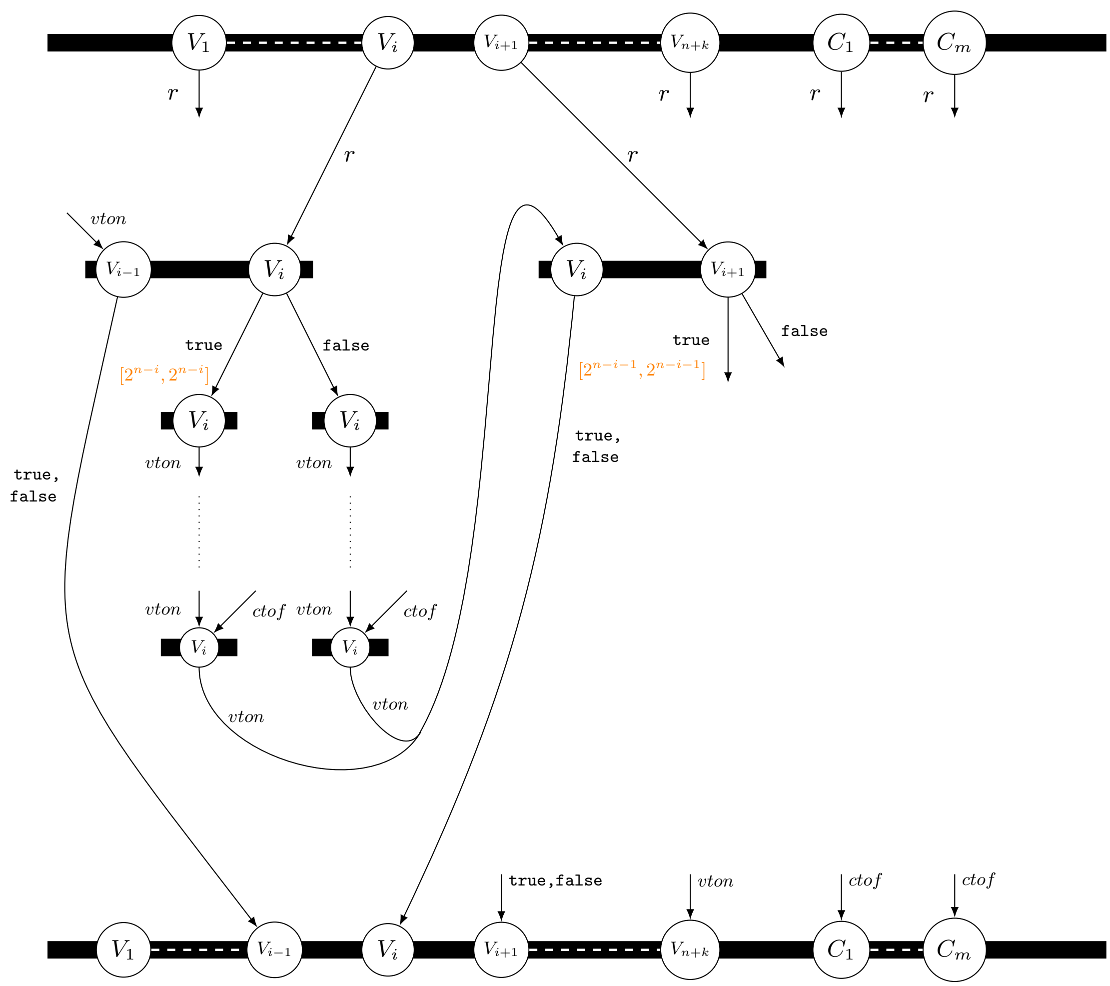

Fig. 5. A part of $\mathcal{N}_\phi$. Here if $i > n$, then timing with arcs `true` and `false` will be $[0, 0]$.

For backward implication, consider any satisfiable assignment f of ϕ. Since ϕ and $\phi^{'}$ are equisatisfiable hence there will exist an satisfiable assignment $f^{'}$ to $\phi^{'}$, such that $f'(x_i) = f(v_i)$ for $i \leq n$. Now following the proof of

Thm. 3, it is easy see that the run σ corresoponding to the assignment $f^{'}$ will be final. Moreover the time taken for the path will be $2^{n-i_1} + 2^{n-i_2} + ... + 2^{n-i_k}$ where $f'(v_{i_k}) = \texttt{true}$. Since all these $i_j \leq n$, these variables will also be present in ϕ and their contribution in lexicographic value of f would also be 2^{n-i_j}. And hence the backward implication.

This proves the claim and shows that $maxtime(\mathcal{N}_{\phi'})$ = odd iff $x_n = 1$ in the lexicographically largest satisfying assignment of ϕ. ☐

Finally, we note that if we were interested in the optimization and not the decision variant of the problem, the above proof can be adapted to show that the optimization variants are **OptP-Complete** (as defined in [14]).

Appendix B: Sound Negotiations

Sound negotiations are negotiations in which every run can be extended to a final run, as in Fig. 1. In this section, we show that $maxtime(\mathcal{N})$ can be computed in PTIME for sound negotiations, hence giving PTIME complexities for the $maxtime(\mathcal{N}) \leq T$? and $maxtime(\mathcal{N}) = T$? questions. However, we show that $mintime(\mathcal{N}) \leq T$ is NP-complete for sound negotiations, and that $mintime(\mathcal{N}) = T$ is DP-complete, even if T is given in unary.

Consider the graph $G_{\mathcal{N}}$ of a negotiation $\mathcal{N}$. Let $\pi = (n_0, (p_0, r_0), n_1) \cdots (n_k, (p_k, r_k), n_{k+1})$ be a path of $G_{\mathcal{N}}$. We define the *maximal execution time* of a path π as the value $\delta^+(\pi) = \sum_{i \in 0..k} \gamma^+(n_i, r_i)$. We say that a path $\pi = (n_0, (p_0, r_0), n_1) \cdots (n_\ell, (p_\ell, r_\ell), n_{\ell+1})$ is a path of some run $\rho = (M_1, \mu_1) \xrightarrow{(n_1, r_1')} \cdots (M_k, \mu_k)$ if $r_0, \ldots, r_\ell$ is a subword of $r_1', \ldots, r_k'$.

Lemma 1. *Let $\mathcal{N}$ be an acyclic and sound timed negotiation. Then $maxtime(\mathcal{N})$ $= \max_{\pi \in Paths(G_{\mathcal{N}})} \delta^+(\pi) + \gamma^+(n_f, r_f)$.*

Proof. Let us first prove that $maxtime(\mathcal{N}) \geq \max_{\pi \in Paths(G_{\mathcal{N}})} \delta^+(\pi) + \gamma^+(n_f, r_f)$. Consider any path π of $G_{\mathcal{N}}$, ending in some node n. First, as $\mathcal{N}$ is sound, we can compute a run ρ_π such that π is a path of ρ_π, and ρ_π ends in a configuration in which n is enabled. We associate with ρ_π the timed run ρ_π^+ which associates to every node the latest possible execution date. We have easily $\delta(\rho_\pi^+) \geq \delta(\pi)$, and then we obtain $\max_{\pi \in Paths(G_{\mathcal{N}})} \delta(\rho_\pi^+) \geq \max_{\pi \in Paths(G_{\mathcal{N}})} \delta(\pi)$. As $maxtime(\mathcal{N})$ is the maximal duration over all runs, it is hence necessarily greater than $\max_{\pi \in Paths(G_{\mathcal{N}})} \delta(\rho_\pi^+) + \gamma^+(n_f, r_f)$

We now prove that $maxtime(\mathcal{N}) \leq \max_{\pi \in Paths(G_{\mathcal{N}})} \delta^+(\pi) + \gamma^+(n_f, r_f)$.

Take any timed run $\rho = (M_1, \mu_1) \xrightarrow{(n_1, r_1)} \cdots (M_k, \mu_k)$ of $\mathcal{N}$ with a unique maximal node n_k. We show that there exists a path π of ρ such that $\delta(\rho) \leq \delta^+(\pi)$ by induction on the length k of ρ. The initialization is trivial for $k = 1$. Let $k \in \mathbb{N}$. Because n_k is the unique maximal node of ρ, we have $\delta(\rho) = \max_{p \in P_{n_k}} \mu_{k-1}(p) +$

$\gamma^+(n_k, r_k)$. We choose one p_{k-1} maximizing $\mu_{k-1}(p)$. Let $\ell < k$ be maximal index of a decision involving process p_{k-1} (i.e. $p_{k-1} \in P_{n_\ell}$). Now, consider the timed run ρ' subword of ρ, but with n_ℓ as unique maximal node (that is, it is ρ where nodes $n_i, i > \ell$ has been removed, but also where some nodes $n_i, i < \ell$ have been removed if they are not causally before n_ℓ (in particular, $P_{n_i} \cap P_{n_\ell} = \emptyset$).

By definition, we have that $\delta(\rho) = \delta(\rho') + \gamma^+(n_\ell, r_\ell) + \gamma^+(n_k, r_k)$. We apply the induction hypothesis on ρ', and obtain a path π' of ρ' ending in n_ℓ such that $\delta(\rho') + \gamma^+(n_\ell, r_\ell) \leq \delta^+(\pi')$. It suffices to consider the path $\pi = \pi'(n_\ell, (p_{k-1}, r_\ell), n_k)$ to prove the inductive step $\delta(\rho) \leq \delta^+(\pi) + \gamma^+(n_k, r_k)$.

Thus $maxtime(\mathcal{N}) = \max \delta(\rho) \leq \max_{\pi \in Paths(G_\mathcal{N})} \delta^+(\pi) + \gamma^+(n_f, r_f)$. $\square$

Lemma 1 gives a way to evaluate the maximal execution time. This amounts to finding a path of maximal weight, which is a standard PTIME graph problem that can be solved using standard max-cost calculation.

Proposition 3. *Computing the maximal execution time for an acyclic sound negotiation $\mathcal{N} = (N, n_0, n_f, \mathcal{X})$ can be done in time $O(|N| + |\mathcal{X}|)$.*

Proof. First of all, we compute a topological order $<$ on nodes of the graph $G_\mathcal{N}$, that is for all $n' \in \mathcal{X}(n, r)$, we have $n < n'$. This can be done in $O(|N| + |\mathcal{X}|)$ [3]. Then, we follow the total order $<$ on nodes of $G_\mathcal{N}$ and attach to each node n a maximal time $\delta^+(n)$ for runs ending at node n in the following way: $\delta^+(n_0) = 0$ and for each node n, we let $\delta^+(n) = \max_{n' | (n', (p,r), n) \in G_\mathcal{N}} (\gamma^+(n', r) + \delta^+(n'))$. It is easy to see that $\delta^+(n)$ is the maximal $\delta(\pi)$ over all paths π from n_0 to n. As every transition of $G_\mathcal{N}$ is considered only once, the computation of δ^+ can be done in $O(|N| + |\mathcal{X}|)$. It then suffices to return $\delta^+(n_f) + \gamma^+(n_f, r_f)$. $\square$

A direct consequence is that $maxtime(\mathcal{N}) \leq T$ and $maxtime(\mathcal{N}) = T$ problems can be solved in polynomial time when $\mathcal{N}$ is. Notice that if $\mathcal{N}$ is deterministic but not sound, then Lemma 1 does not hold: we only have an inequality.

We now turn to $mintime(\mathcal{N})$. We show that it is strictly harder to compute for sound negotiations than $maxtime(\mathcal{N})$.

Theorem 5. *$mintime(\mathcal{N}) \leq T$ is NP-complete in the strong sense for sound acyclic negotiations, even if $\mathcal{N}$ is very weakly non-deterministic.*

Proof. First, we can decide $mintime(\mathcal{N}) \leq T$ in NP. Indeed, one can guess a final (untimed) run ρ of size $\leq |N|$, consider ρ^- the timed run corresponding to ρ where all outcomes are taken at the earliest possible dates, and compute in linear time $\delta(\rho^-)$, and check that $\delta(\rho^-) \leq T$.

The hardness part is obtained by reduction from the **Bin Packing** problem. We give a set U of items, a size $s(u) \in \mathbb{N}$ for each $u \in U$, a positive integer B defining a bin capacity. The bin packing problem asks whether there exists a partition of U into k disjoint subsets $U_1, U_2 ... U_k$ such that the sum of sizes of items in each U_i is smaller or equal to B. Bin Packing is known to be NP-Complete [11] in the strong sense, that is even if the constants are given in unary. Let us now show that every instance of Bin Packing can be reduced to a min-time problem for very-weakly non-deterministic sound negotiations.

Given a set U of items, a bin capacity B and number k of bins, we build a timed negotiation $\mathcal{N}_{U,k}$ with k processes $u_{i,1}, u_{i,2}, ..., u_{i,k}$ for each item $u_i \in U$, and k additional processes $v_1, v_2, ..v_k$. The timing of a process v_i will encode the total size of items put in the bin i. We then show that Bin Packing with items U, k bins, and a bound B has a solution iff $mintime(\mathcal{N}_{U,k}) \leq B$.

We describe the negotiation $\mathcal{N}_{U,k}$ layer by layer. In total we will have $|U|+1$ layers: intuitively, we will consider one item in each layer, and make one global decision to decide in which bin this item goes. The first layer has only the initial node n_0. The set of processes involved in n_0 is the set of all processes. The outcomes from the initial node are $r_{1,1}, \ldots, r_{1,k}$, which tell in which bin $1, \ldots, k$ the first item is placed. Outcome $r_{1,i}$ leads process $u_{i,1}$ and v_i to node YES_i^1. It leads processes $u_{j,1}$ and v_j to NO_j^1 for every $j \neq i$. Last, it leads all other processes in $\{u_{j,m} \mid j > 1, 1 \leq m \leq k\}$ to node n_1. Intuitively, moving to node YES_i^1 means that item u_1 is placed in bin i. The second layer has $2k+1$ nodes: $\mathrm{YES}_1^2 \ldots \mathrm{YES}_k^2$, $\mathrm{NO}_1^2 \ldots \mathrm{NO}_k^2$ and n_1. The timing of outcome $r_{1,i}$ from node n_0 is $\gamma(n_0, r_{1,i}) = [0,0]$.

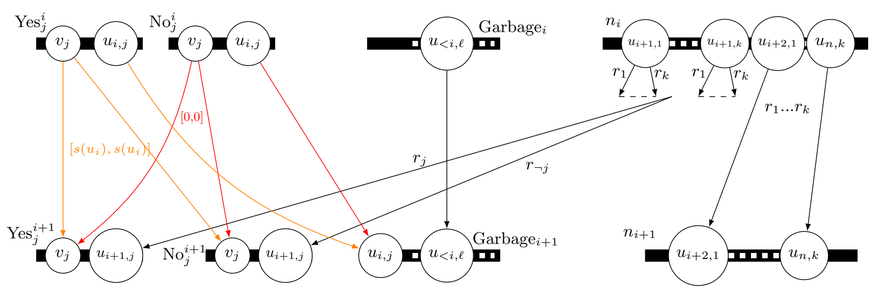

Fig. 6. Layer i of the very weakly non-deterministic $\mathcal{N}(U,k)$

Inductively, layer i is defined as in Fig 6. Node n_i contains processes $u_{j,\ell}$ for all $j > i$ and all ℓ. It is similar to n_0, with outcome $r_{i+1,1}, \ldots, r_{i+1,k}$. Outcome $r_{i+1,\ell}$ leads process $u_{i+1,\ell}$ to node Yes_ℓ^{i+1}, and process $u_{i+1,j}$ to No_j^{i+1} for all $j \neq \ell$. Other processes $u_{i',j}$ with $i' > i+1$ are sent to n_{i+1}. The associated timings are $[0,0]$.

Node $\mathrm{Garbage}_i$ collects all nodes $u_{\ell,j}$ with $\ell < i$. There is a unique outcome, with associated timing $[0,0]$, leading all processed to $\mathrm{Garbage}_{i+1}$.

Node YES_j^i has a unique outcome r, with timing $\gamma(\mathrm{YES}_j^i, r) = [s(u_i), s(u_i)]$, and with $\mathcal{X}(\mathrm{YES}_j^i, r) = \{\mathrm{YES}_j^{i+1}, \mathrm{No}_j^{i+1}\}$. That is, node YES_j^i is non deterministic, and it awaits the decision from $u_{i+1,j}$ to known whether it will go to YES_j^{i+1} or to No_j^{i+1}. Last, $u_{i,j}$ is sent to node $\mathrm{Garbage}_{i+1}$. This allows each nodes to have at least one deterministic process, as v_i only are non-deterministic.

In the same way, NO_j^i has a unique outcome r, timed with $\gamma(\mathrm{NO}_j^i, r) = [0,0]$, and with $\mathcal{X}(\mathrm{NO}_j^i, r) = \{\mathrm{YES}_j^{i+1}, \mathrm{No}_j^{i+1}\}$. It sends process $u_{j,i}$ to $\mathrm{Garbage}_{i+1}$.

The last layer has only node n_f. Nodes Yes_i^k and No_i^k both have a single outcome which take all their processes to n_f.

The timing function γ is defined as follows: $\gamma(Yes_j^i, r_i) = [s(u_i), s(u_i)]$ and $\gamma(n, r) = [0, 0]$ for all other node and outcome r.

We now prove that $MinTime(\mathcal{N}_{U,k}) \leq B$ iff the answer to Bin Packing is positive. The maximal execution time over runs ρ of $\mathcal{N}_{U,k}$ is the maximal value of all valuations $\mu(v_j)$ and $\mu(u_{i,j})$, with $i \in 1..|U|, j \in 1..k$. Take the valuation μ at the last step before (n_f, r_f). Consider $t = \max_j \mu(v_j)$. We have easily that $\mu(u_{i,j}) \leq t$ for all i, j by construction, because each $u_{i,j}$ had the same timing as v_j before reaching a garbage node. Now, we have $\mu(v_j) = \sum_{(\mathrm{Yes}_j^i, r_i) \in \rho} s(u_i)$. Hence, $\delta(\rho) = \max_{j \in 1..k} \mu(v_j)$. That is, $mintime(\mathcal{N}(U, B, k) \leq B$ iff there is a path ρ such that $\mu(v_j) = \sum_{(\mathrm{Yes}_j^i, r_i) \in \rho} s(u_i) \leq B$ for all j, ie there exists a valuation such that each item is in one bin, and no bin exceeds its bound B.

Last, we now show that $\mathcal{N}_{U,k}$ is a very weakly non-deterministic, sound and layered negotiations. First, the only processes that have non-deterministic transitions are processes $v_1, \dots v_k$, from Yes_j^i and NO_j^i nodes. However, both nodes also have the same deterministic process u_j^i. Thus $\mathcal{N}_{U,k}$ is very weakly non-deterministic. Let us now prove soundness. The only choices are made from node n_i, the rest just follow in a unique way. From any configuration M, let i such that $M(u_{i+1}, j) = \{n_i\}$ for some j. By construction, i is unique. We can then do steps $r_{i+1,1} \dots r_{n,1}$, that is chosing to place items $i+1, \dots, n$ to the first bin. The steps from other processes are uniquely derived, and all processes reach n_f. The layeredness comes from the definition. Actually, $\mathcal{N}_{U,k}$ is $2k+2$-layered, for k the number of bins. However, as k is part of the input, it does not fall in our k-layered restriction. $\square$

We show that $mintime(\mathcal{N}) = T$ is harder to decide than $mintime(\mathcal{N}) \leq T$:

Proposition 4. *The $mintime(\mathcal{N}) = T$? decision problem is DP-complete for sound acyclic negotiations, even if it is very weakly non-deterministic.*

Proof. The reduction is very similar to proof of Proposition 2. First, we define the complement of Bin-Packing Problem, **Non-Bin-Packing Problem**:
Given a set U of items, a size $s(u) \in \mathbb{N}$ for each $u \in U$, a positive integer bin capacity B, does for any partition U into k disjoint subsets $U_1, U_2 ... U_k$ there exist a subset U_i such that the sum of sizes of the items in U_i is more than B?
Since the Bin-Packing Problem is NP-Complete, so the Non-Bin-Problem is co-NP Complete. Now consider the following **Bin-Non-Bin Problem** :

Given two instances of Bin-Packing parameters, $P_1 = (U_1, s_1, B_1, k_1)$ and $P_2 = (U_2, s_2, B_2, k_2)$, does P_1 satisfy Bin-Packing Problem and P_2 satisfy Non-Bin-Packing Problem?

Bin-Non-Bin Problem is DP-Complete, so we reduce it to our equality decision problem of min time. First, we construct the negotiations $\mathcal{N}_{U_1', B_1', k_1}$ and $\mathcal{N}_{U_2', B_2', k_2}$ like in proof of Theorem 5, but only after tripling each $s(u)$ in U_1 and doubling each $s(u)$ in U_2. Likewise we triple B_1 and double B_2, so that new

$B_1^{'} = 3 * B_1$ and $B_2^{'} = 2 * B_2$.

In $\mathcal{N}_{U_1',B_1',k_1}$, we add a new node n_0 with a single outcome r which now acts as the first node. The older n_0 is now called n_0'. We also add a new process a_1, which goes to another new node n_{a_1}(has only a_1 as process) from n_0 for its single outcome r. Outcome r sends all other processs from n_0 to n_0'. Node n_{a_1} has a single outcome r_1 which takes a_1 to n_f. Also, $\gamma(n_{a_1}, r_1) = [3 * B_1 + 1, 3 * B_1 + 1]$ while $\gamma(n_0, r) = [0, 0]$.

Similarly in $\mathcal{N}_{U_2',B_2',k_2}$, we add a new node n_0 with two outcomes r and r_{new} which now acts as the first node. The older n_0 is now called n_0'. We also add a new process a_2, which goes to another new node n_{a_2}(has only a_2 as process) from n_0 for its outcome r. Outcome r sends all other processes from n_0 to n_0'. Node n_{a_2} has a single outcome r_2 which takes a_2 to n_f. Also, $\gamma(n_{a_2}, r_2) = [2*B_1, 2*B_1]$ while $\gamma(n_0, r) = [0, 0]$. For outcome r_{new} of n_0, all processes(including a_2) directly go to n_f. Also, $\gamma(n_0, r_{new}) = [2 * B_2 + 1, 2 * B_2 + 1]$.

Now we merge the two negotiations $\mathcal{N}_{U_1',B_1',k_1}$ and $\mathcal{N}_{U_2',B_2',k_2}$ in the same way as we merged in Corollary 2, merging the n_f of $\mathcal{N}_{U_1',B_1',k_1}$ with n_0 of $\mathcal{N}_{U_2',B_2',k_2}$ and making other similar changes we did in Corollary 2. We call this new negotiation $\mathcal{N}_{P_1',P_2'}$. Note the negotiation $\mathcal{N}_{P_1',P_2'}$ is sound as well as very weakly non-deterministic.

The claim is that $mintime(\mathcal{N}_{P_1',P_2'}) = 3 * B_1 + 2 * B_2 + 2$ iff (P_1, P_2) satisfy Bin-Non-Bin Problem.

We first show the reverse implication i.e if (P_1, P_2) satisfy Bin-Non-Bin Problem, then $mintime(\mathcal{N}_{P_1',P_2'}) = 3*B_1+2*B_2+2$. Since P_1 is satisfiable, so the mintime to complete the structure above n_{sep} of $\mathcal{N}_{P_1',P_2'}$ is $3*B_1+1$. This is because all the processes corresponding to $\mathcal{N}_{U_1',B_1',k_1}$ take $(\leq 3 * B)$ time to reach n_{sep} (because P_1 is satisfies Bin-Packing) while a_1 takes $3*B_1+1$ units of time. After reaching n_{sep}, processes can now take either outcome r_2 or r_{new}. If processes choose outcome r_2, then the timetaken by any final run will be $(\geq 2*(B_2+1))$ because P_2 satisfies Non-Bin-Packing. On the other hand, if processes choose r_{new} to reach n_f, then the time taken will be $2 * B_2 + 1$. So it is clear mintime for part below n_{sep} is $2 * B_2 + 1$. So, overall the $mintime(\mathcal{N}_{P_1',P_2'}) = 3 * B_1 + 2 * B_2 + 2$.

For forward implication, we consider all four scenerios of (P_1, P_2) and argue that P_1 satisfies Bin-Packing and P_2 satisfies Non-Bin-Packing is the only possibility. First let's assume that P_1 does not satisfy Bin-Packing. Then the mintime to complete the structure above n_{sep} is $(\geq 3 * (B_1 + 1))$. This is beacuse processes corresponding to $\mathcal{N}_{U_1',B_1',k_1}$ take at least $3 * (B_1 + 1)$ time to reach n_{sep} while a_1 take $3 * B_1 + 1$. Now since the mintime which can be taken to reach n_f from n_{sep} in either case whether P_2 satisfies Non-Bin-Packing or not is $(\geq 2 * B_2)$ so the min time to complete $(\mathcal{N}_{P_1',P_2'}) \geq 3 * B_1 + 2 * B_2 + 3$. Hence this shows that P_1 satisfies Bin-Packing. This also shows the final run corresponding to mintime of $\mathcal{N}_{P_1',P_2'}$ takes exactly $3 * B_1 + 1$ units of time to reach n_{sep} from n_0 (i.e. all processes have reached n_{sep}) if $mintime(\mathcal{N}_{P_1',P_2'}) = 3 * B_1 + 2 * B_2 + 2$.

Now if we assume the P_2 does not satisfy Non-Bin-Packing, then the mintime to reach n_f from n_{sep} is $2 * B_2$. And we already know that mintime to reach n_{sep}

from n_0 is $3*B_1+1$. So $mintime(\mathcal{N}_{P_1',P_2'}) = 2*B_2+3*B_1+1$. Hence this leaves us with the only case when P_1 satisfies Bin-Packing and P_2 satisfies Non-Bin-Packing for which we already know that the min time taken is $3*B_1+2*B_2+2$ from the reverse implication. $\square$

An open question is whether the minimal execution time can be computed in PTIME if the negotiation is both sound and deterministic. The reduction to bin packing does not work with deterministic (and sound) negotiations.

Appendix C: k-Layered Negotiations

In the previous sections, we have considered sound negotiations, and deterministic negotiations. For both classes, computing the minimal execution time cannot be done in PTIME (unless NP=PTIME), even if constants are given in unary. In this section, we consider k-layeredness (see Section 2), a syntactic property that can be efficiently verified (it suffices to compute the depth of each node, which can be done in polynomial time).

8.1 Algorithmic properties

Let k be a fixed integer. We first show that Reachability, Soundness and maximum execution time can be checked in PTIME for k-layered negotiations (the two first results were stated in Section 2). Let N_i be the set of nodes at layer i. We define for every layer i the set S_i of subsets of nodes $X \subseteq N_i$ which can be jointly enabled and such that for every process p, there is exactly one node $n(X,p)$ in X with $p \in n(X,p)$. Formally, we define S_i inductively. We start with $S_0 = \{n_0\}$. We then define S_{i+1} from the contents of layer S_i: we have $Y \in S_{i+1}$ iff $\bigcup_{n \in Y} P_n = P$ and there exist $X \in S_i$ and an outcome $r_m \in R_m$ for every $m \in X$, such that $n \in \mathcal{X}(n(X,p),p,r_m)$ for each $n \in Y$ and $p \in P_n$.

Theorem 6. *Let $k \in \mathbb{N}^+$. Checking reachability, soundness and computing the maximum execution time for a k-layered acyclic negotiation $\mathcal{N}$ can be done in PTIME. More precisely, the worst-case time complexity is $O(|P| \cdot |\mathcal{N}|^{k+1})$.*

Proof (Sketch of Proof). The algorithm has the same form for all problems. The basis is to compute S_i layer by layer, by following its inductive definition. The set S_i is of size at most 2^k, as $|N_i| < k$ by definition of k-layerness. Knowing S_i, it is easy to build S_{i+1} by induction. This takes time at most $O(|P||\mathcal{N}|^{k+1})$: We need to consider all k-uple of outcomes for each layer. There can be $|\mathcal{N}|^k$ such tuples. We need to do that for all processes ($|P|$), and for all layers (at most $|\mathcal{N}|$).

For reachability, we just need to check whether layer $S_d = \{n_f\}$, where d is the depth of n_f.

For soundness, let us denote by $Next(X,(r_n)_{n \in X})$ the set of nodes that are successors of nodes in X after outcomes $(r_n)_{n \in X}$. We need to check that for all layer i, for all set $X \in S_i$ and all tuple of outcomes $(r_n)_{n \in X}$, there is a $Y \subseteq Next(X,(r_n)_{n\in X})$ such that every process p is in exactly one node $n(Y,p)$ of Y. All nodes of $Next(X,(r_n)_{n\in X})$ are at depth $i+1$, and thus there are at most k nodes in $Next(X,(r_n)_{n\in X})$. There are thus at most 2^k subset $Y \subseteq Next(X,(r_n)_{n\in X})$ and we can check them one by one.

For maximal execution time, we keep for each subset $X \in S_i$ and each node $n \in X$, the maximal time $f_i(n,X) \in \mathbb{N}$ associated with n and X. From S_{i+1} and f_i, we inductively compute f_{i+1} in the following way: for all $X \in S_i$ with successor $Y \in S_{i+1}$ for outcomes $(r_p)_{p\in P}$, we denote $f_{i+1}(Y,n,X) = \max_{p\in P(n)} f_i(X,n(X,p))+\gamma^+(n(X,p),r_p)$. If there are several choices of $(r_p)_{p\in P}$ leading to the same Y, we take r_p with the maximal $f_i(X,n(X,p))+\gamma^+(n(X,p),r_p)$. We then define $f_{i+1}(Y,n) = \max_{X\in S_i} f_{i+1}(Y,n,X)$. Again, the initialization is trivial, with $f_0(\{n_0\},n_0)=0$. The maximal execution time of $\mathcal{N}$ is $f(\{n_f\},n_f)$. That is, for all nodes (at most $|\mathcal{N}|$), we have to consider every k-uple of outcomes, and there are at most $|\mathcal{N}|^k$ of them, and every process to compute the max, and the complexity is still in $O(|P|\cdot|\mathcal{N}|^{k+1})$. ☐

We can bound the complexity precisely by $O(d(\mathcal{N})\cdot C(\mathcal{N})\cdot ||R||^{k^*})$, with:

- $d(\mathcal{N}) \leq |\mathcal{N}|$ the depth of n_f, that is the number of layers of $\mathcal{N}$, and $||R||$ is the maximum number of outcomes of a node,
- $C(\mathcal{N}) = \max_i |S_i| \leq 2^k$, which we will call the *number of contexts of* $\mathcal{N}$, and which is often much smaller than 2^k.
- $k^* = \max_{X\in\bigcup_i S_i} |X| \leq k$. We say that $\mathcal{N}$ is k^**-thread bounded*, meaning that there cannot be more that k^* nodes in the same context X of any layer. Usually, k^* is strictly smaller than $k = \max_i |N_i|$, as $N_i = \bigcup_{X\in S_i} X$.

Consider again the Brexit example Figure 1. We have $(k+1)=7$, while we have the depth $d(\mathcal{N})=6$, the negotiation is $k^*=3$-thread bounded (k^* is bounded by the number of processes), and the number of contexts is at most $C(\mathcal{N})=4$ (EU chooses to enforce backstop or not, and Pa chooses to go to court or not).

8.2 Minimal Execution Time

As with sound negotiations, computing minimal time is much harder than computing the maximal time for k-layered negotiations:

Theorem 7. *Let $k \geq 6$. The $Min \leq T$ problem is NP-Complete for k-layered acyclic negotiations, even if the negotiation is sound and very weakly non-deterministic.*

Proof. One can guess in polynomial time a final run of size $\leq |\mathcal{N}|$. If the execution time of this final run is smaller than T then we have found a final run witnessing $Min(\mathcal{N}) \leq T$. Hence the problem is in NP.

Let us now show that the problem is NP-hard. We proceed by reduction from the knapsack decision problem. Let us consider a set of items $U=\{u_1,\dots u_n\}$ of respective values $v_1,\dots v_n$ and weight $w_1,\dots,w_n$ and a knapsack of maximal capacity W. The knapsack problem asks, given a value V whether there exists a subset of items $U' \subseteq U$ such that $\sum_{u_i\in U'} v_i \geq V$ and such that $\sum_{u_i\in U'} w_i \leq W$.

We build a negotiation with $2n$ processes $P = \{p_1, \dots p_{2n}\}$. Intuitively, $p_i, i \leq n$ will serve to encode the value as timing, while $p_i, i > n$ will serve to encode the weight as timing. We set the set of nodes $N = \{n_0, n_f\} \cup \{C_i \mid i \in 1..n\} \cup \{n_{L,0,i}, n_{L,1,i}, n_{R,0,i}, n_{R,1,i} \mid i \in 1..n\}$. Intuitively, node $n_{L,1,i}$ (resp $n_{R,1,i}$) will be used to remember that item i is placed in the knapsack and that its value (resp. weight) needs to be added. For all i, node $n_{L,1,i}$ (resp. $n_{R,1,i}$) has a unique possible outcome, b_i (resp. c_i). Nodes of the form $n_{L,0,i}$ remember that item i has not been placed in the knapsack, and they have outcome a_i. Nodes of the form $n_{R,0,i}$ remember that item i has not been placed in the knapsack, and they all have outcome 0. This outcome does not change the execution time, matching the fact that the current weight and value of the knapsack is not increased.

Last, nodes of the form C_i will just remember the items that have already been considered. These nodes have two outputs, yes and no, telling whether the item i should be placed in the knapsack or not, consistently for weight and value processes.

We set $P_{n_0} = P_{n_f} = P$, and for other nodes $n_{L,0,i}$, $P_{n_{L,0,i}} = P_{n_{L,1,i}} = \{p_1 \dots p_i\}$ and $P_{n_{R,0,i}} = P_{n_{R,1,i}} = \{p_{n+1} \dots p_{n+i}\}$. Last $P_{C_i} = \{p_{i+1} \dots p_n . p_{n+i} \dots p_{2n}\}$.

We define the transition relation as follows: $\mathcal{X}(n_0, \text{yes}, p_1) = \{n_{L,1,i}\}$, and $\mathcal{X}(n_0, \text{no}, p_1) = \{n_{L,0,1}\}$, such that process p_1 remembers that the item is picked/notpicked. In the same way, $\mathcal{X}(n_0, \text{no}, p_{n+1}) = \{n_{R,0,1}\}$ and $\mathcal{X}(n_0, \text{yes}, p_{n+1}) = \{n_{R,1,1}\}$ for process p_{i+1}. Hence both process p_1, p_{n+1} will have the same information about whether the first item is picked or not. Finally, for every $k \in 2..n$, we define $\mathcal{X}(n_0, \text{no}, p_k) = \mathcal{X}(n_0, \text{no}, p_{k+n}) = \mathcal{X}(n_0, \text{yes}, p_k) = \mathcal{X}(n_0, \text{no}, p_{k+n}) = \{C_1\}$.

Other layers are similar: for $i \in 1..n$, we have $\mathcal{X}(C_i, \text{no}, p_i) = \{n_{L,0,i+1}\}$ $\mathcal{X}(C_i, \text{yes}, p_i) = \{n_{L,1,i+1}\}$, Similarly, for every $i \in 1..n$, $\mathcal{X}(C_i, \text{no}, p_{i+n}) = \{n_{R,0,i+1}\}$, and $\mathcal{X}(C_i, \text{yes}, p_{i+n}) = \{n_{R,1,i+1}\}$. We set $\mathcal{X}(C_i, \text{no}, p_j) = \mathcal{X}(C_i, \text{yes}, p_j) = \{C_{i+1}\}$ for every $j \in [i+1, n-1] \cup [n+i+1, 2n]$.

The most interesting set of transitions are to interface $n_{L,0,i}, n_{L,1,i}, n_{R,0,i}, n_{R,1,i}$ with the next layer, in a non deterministic way because they dont know whether the next item will be picked or not: $\mathcal{X}(n_{L,0,i}, a_i, p_j) = \mathcal{X}(n_{L,1,i}, b_i, p_j) = \{n_{L,0,i+1}, n_{L,1,i+1}\}$ for $j \in 1..i$ and, $\mathcal{X}(n_{R,0,i}, 0, p_j) = \mathcal{X}(n_{R,1,i}, c_i, p_j) = \{n_{R,0,i+1}, n_{R,1,i+1}\}$ for $j \in n+1..n+i$.

Last, all processes synchronize on n_f by setting)$\mathcal{X}(n_{L,0,n}, 0, p_j) = \mathcal{X}(n_{L,1,n}, b_n, p_j) = \mathcal{X}(n_{R,0,n}, 0, p_j) = \mathcal{X}(n_{R,1,n}, c_n, p_j) = \{n_f\}$

We now have to set timing constraints for outcomes. Outcomes 0, yes and no are associated with $[0, 0]$. Outcome c_i is associated with $[w_i, w_i]$, the weight of u_i. Last, outcome b_i is associated with a more complex function, such that $\sum_i b_i \leq W$ iff $\sum_i v_i \geq V$. For that, we set $[\frac{(v_{max}-v_i)W}{n \cdot v_{max}-V}, \frac{v_{max}W}{n \cdot v_{max}-v_i}]$ for outcome b_i, where v_{max} is the largest value of an item, and V is the total value we want to reach at least. Also, we set $[\frac{(v_{max})W}{n \cdot v_{max}-V}, \frac{v_{max}W}{n \cdot v_{max}-v_i}]$ for outcome a_i. We set $T = W$, the maximal weight of the knapsack.

Now, consider a final run ρ in $\mathcal{N}$. The only choice is about yes, no from C_i. Let I be the set of indices such that yes is the outcome from all C_i in this path. We obtain $\delta(\rho) = \max(\sum_{i \notin I} a_i + \sum_{i \in I} b_i, \sum_{i \in I} c_i)$. We have $\delta(\rho) \leq T = W$ iff $\sum_{i \in I} w_i \leq W$, that is the sum of the weight are lower than W, and

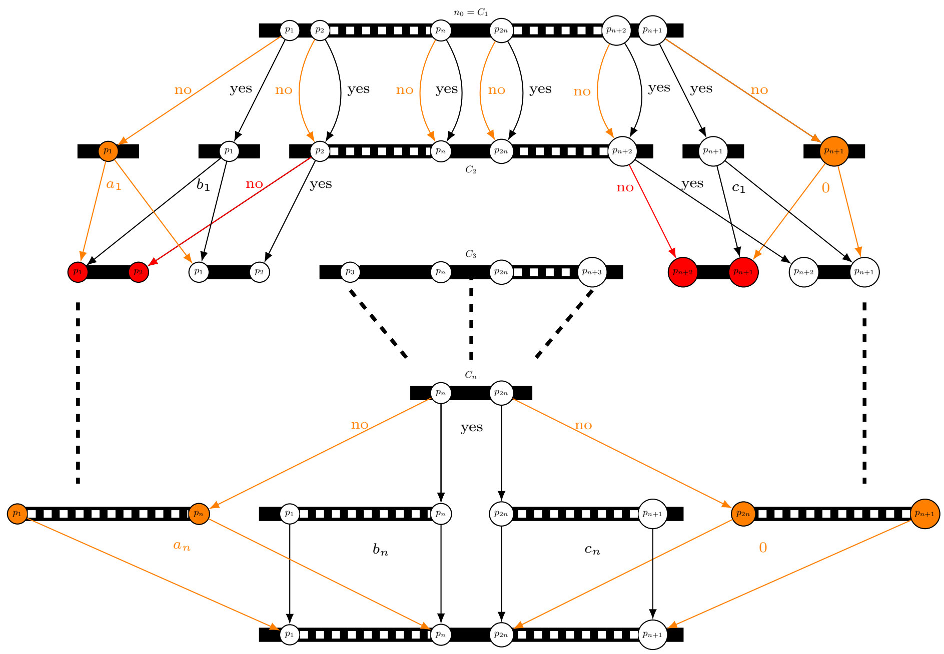

Fig. 5. The negotiation encoding Knapsack

$\sum_{i\notin I} \frac{(v_{max})W}{n\cdot v_{max}-V} + \sum_{i\in I} \frac{(v_{max}-v_i)W}{n\cdot v_{max}-V} \leq W$. That is, $n\cdot v_{max} - \sum_{i\in I} v_i \leq n\cdot v_{max} - V$, i.e. $\sum_{i\in I} v_i \geq V$. Hence, there exists a path ρ with $\delta(\rho) \leq T = W$ iff there exists a set of items of weight less than W and of value more than V.

So, given a knapsack of size n, a value V and a weight limit W one can build a negotiation $\mathcal{N}_V^{Knap}$ with $O(3n+2)$ nodes. We can encode all weights with $\log(v_{max}.W) + \log(n.v_{max})$ bits. One can notice that $\mathcal{N}_V^{Knap}$ is 5-layered and sound.

However, it is not (weakly) non-deterministic because of nodes $n_{L,0,i}, n_{L,1,i}, n_{R,0,i}, n_{R,1,i}$. It is easy to add two processes V (resp. W), present in all nodes $n_{L,0,i}, n_{L,1,i}$ (resp $n_{R,0,i}, n_{R,1,i}$), and make process P_i (resp. P_{n+i} leave these nodes, deterministically leading to a new node garbage$_{i+1}$ at layer $i+1$. Then the negotiation is very weakly deterministic, and 6-layered. □

Following the same lines as for the proofs of Propositions 2 and 4, a consequence of Theorem 7 is that the $Min = T$ problem is in DP for k-layered acyclic negotiations.

It is well known that Knapsack is weakly NP-hard, that is it NP-hard only when weights/values are given in binary. This means that Thm. 7 shows that minimum execution time $\leq T$ is NP-hard only when T is given in binary. We can actually show that for k-layered negotiations, the $mintime(\mathcal{N}) \leq T$ problem can be decided in PTIME if T is given in unary (i.e. if T is not too large):

Theorem 8. *Let $k \in \mathbb{N}$. Given a k-layered negotiation $\mathcal{N}$ and T written in unary, one can decide in PTIME whether the minimum execution time of $\mathcal{N}$ is $\leq T$. The worst-case time complexity is $O(|\mathcal{N}| \cdot |P| \cdot (T \cdot |\mathcal{N}|)^k)$.*

Proof. We will remember for each layer i a set $\mathcal{T}_i$ of functions τ from nodes N_i of layer i to a value in $\{1, \ldots, T, \perp\}$. Basically, we have $\tau \in \mathcal{T}_i$ if there exists a path ρ reaching $X = \{n \in N_i \mid f(n) \neq \perp\}$, and this path reaches node $n \in X$ after $\tau(n)$ time units. As for S_i, for all p, we should have a unique node $n(\tau, p)$ such that $p \in n(f,p)$ and $\tau(n(\tau,p)) \neq \perp$. Again, it is easy to initialize $\mathcal{T}_0 = \{\tau_0\}$, with $\tau_0(n_0) = 0$, and $\tau_0(n) = \perp$ for all $n \neq n_0$.

Inducively, we build $\mathcal{T}_{i+1}$ in the following way: $\tau_{i+1} \in \mathcal{T}_{i+1}$ iff there exists a $\tau_i \in \mathcal{T}_i$ and $r_p \in R_{n(\tau_i,p)}$ for all $p \in P$ such that for all n with $\tau_{i+1}(n) \neq \perp$, we have $\tau_{i+1}(n) = \max_p \tau_i(n(\tau_i,p)) + \gamma(n(\tau_i,p), r_p)$.

We have that the minimum execution time for $\mathcal{N}$ is $\min_{\tau \in \mathcal{T}_n} \tau(n_\tau)$, for n the depth of n_f. There are at most T^k functions τ in any $\mathcal{T}_i$, and there are at most $|\mathcal{N}|$ layers to consider, giving the complexity. ◻

As with Thm. 6, we can more accurately state the complexity as $O(d(\mathcal{N}) \cdot C(\mathcal{N}) \cdot ||R||^{k^*} \cdot T^{k^*-1})$. The k^*-1 is because we only need to remember minimal functions $\tau \in \mathcal{T}_i$: if $\tau'(n) \geq \tau(n)$ for all n, then we do not need to keep τ' in $\mathcal{T}_i$. In particular, for the knapsack encoding in the proof of Thm. 7, we have $k^* = 3$, $||R|| = 2$ and $C(\mathcal{N}) = 4$.

Notice that if k is part of the input, then the problem is strongly NP-hard, even if T is given in unary, as e.g. encoding bin packing with k bins result to a $k+1$-layered negotiations.