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Symmetric Functions and P-Recursiveness

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1. INTRODUCTION

Many enumeration problems, such as that of counting nonnegative integer matrices with given row and column sums, have solutions which can be expressed as coefficients of symmetric functions. We show here how useful formulas can be obtained from these symmetric function generating functions. In some cases, the symmetric functions yield reasonably simple explicit formulas or generating functions for the coefficients. In other cases, such as in counting several types of regular graphs, explicit formulas that are too unwieldy to be useful in computation can still be used to show that a sequence of coefficients is P-recursive; that is, it satisfies a linear homogeneous recurrence with polynomial coefficients.

In Sections 2 and 3 we review the basic facts about symmetric functions and introduce the method we use (due to Read [35]) for coefficient extraction: The coefficient of $x_{i_1}^{\lambda_1}x_{i_2}^{\lambda_2}\cdots x_{i_k}^{\lambda_k}$ in the symmetric function f is the scalar product $\langle f, h_{\lambda_1}\cdots h_{\lambda_k}\rangle$, where h_n is the complete symmetric function. To evaluate this scalar product, we expand f and the h_{λ_i} in power sum symmetric functions and use the orthogonality of the power sum symmetric functions. We give examples of explicit formulas and generating functions obtained by this method for problems of counting permutations, trees, and partitions.

In Sections 4 and 5 we introduce the basic facts about P-recursive functions and D-finite power series. We define D-finite symmetric functions and show that their coefficients give rise to P-recursive sequences, proving a conjecture of Gouden and Jackson [18] that the counting sequence for k-regular graphs is P-recursive for all k. In Section 6, we apply the theory to Schur functions and use some formulas of Gordon and Houten [15],

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Gordon [16], and Bender and Knuth [5] to show how the exponential generating function for the number of standard tableaux with at most k rows can be expressed in terms of a determinant of Bessel functions. We obtain the explicit formulas of Regev [39] and Gouyou-Beauchamps [20] for the case $k \leqslant 5$. In Section 7 we consider symmetric functions in two sets of variables, and show that the counting sequence for $n \times n$ nonnegative integer matrices with every row and column sum equal to k (with k fixed, as a function of n) is P-recursive, and similarly for 0–1 matrices. We also derive a doubly exponential generating function for permutations with longest increasing subsequence of length at most k, as a determinant of Bessel functions, and find an explicit formula for the coefficients in the case $k = 3$. In Section 8 we show how our method can sometimes be applied to generating functions that are almost, but not quite, symmetric, such as in counting $n \times n$ nonnegative integer matrices with every row and column sum k, and zeroes on the diagonal. Finally, in Section 9, we give a brief indication of how the unwieldy explicit formulas obtained from symmetric functions can yield asymptotic approximations.

A preliminary version of some of this material appeared in [14].

2. Symmetric Functions

We recall some basic facts about symmetric functions. Proofs and details can be found in Macdonald's book [29].

We work with symmetric functions in the infinitely many variables $x_1, x_2, \ldots$ with coefficients in a field of characteristic zero. We shall be concerned with the following particular symmetric functions:

The *elementary* symmetric function e_n is defined by

$$e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition, i.e., a nonincreasing sequence of nonnegative integers, we define $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$.

The *complete* symmetric function h_n is defined by

$$h_n = \sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

It is convenient to define h_λ to be $h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}$ for any sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of nonnegative integers, not necessarily a partition. We set $h = \sum_{n=0}^{\infty} h_n$ and $e = \sum_{n=0}^{\infty} e_n$, where $h_0 = e_0 = 1$. We also define h_n to be 0 for $n < 0$.

The *monomial* symmetric function m_λ is the sum of all distinct monomials of the form $x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$, where $i_1, ..., i_k$ are distinct.

The *power sum* symmetric function p_n is defined by

$$p_n = \sum_i x_i^n.$$

More generally, if $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ is a partition, we define $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$.

The *Schur function* s_λ is the determinant $\det(h_{\lambda_i + j - i})_{1 \leqslant i, j \leqslant k}$, and more generally, the *skew Schur function* $s_{\lambda/\mu}$ is the determinant $\det(h_{\lambda_i - \mu_j + j - i})_{1 \leqslant i, j \leqslant k}$. We refer the reader to Stanley [46] or Macdonald [29] for their combinatorial interpretation.

It is known that each of the sets $\{e_\lambda\}$, $\{h_\lambda\}$, $\{p_\lambda\}$, $\{m_\lambda\}$, and $\{s_\lambda\}$, where λ ranges over all partitions of n, is a basis for the vector space of symmetric functions homogeneous of degree n.

It is convenient to use the notation $(1^{r_1} 2^{r_2} \cdots k^{r_k})$ for the partition with r_i parts equal to i. If $\lambda = (1^{r_1} 2^{r_2} \cdots k^{r_k})$ then we define z_λ to be $1^{r_1} 2^{r_2} \cdots k^{r_k} r_1! \, r_2! \cdots r_k!$. It is also convenient to identify partitions which differ only in the number of zero parts. The empty partition is denoted by 0.

There is a symmetric scalar product $\langle \, , \rangle$ defined on symmetric functions that has the properties

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu} \tag{1}$$

and

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \, \delta_{\lambda\mu}, \tag{2}$$

where $\delta_{\lambda\mu}$ is 1 if $\lambda = \mu$ and 0 otherwise. This scalar product was introduced by J. H. Redfield [38] in 1927 in his then-ignored but now-famous paper on what later became known as Pólya theory. Redfield called it the "cap product." The scalar product was rediscovered by Hall [21] in 1957 and is sometimes attributed to him. It is equivalent to the usual scalar product on characters of symmetric groups.

Note that Eq. (1) implies that if f is a symmetric function, then the coefficient of $x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$ in f is $\langle f, h_\lambda \rangle$. Equation (2) allows us to evaluate scalar products of symmetric functions that can be expressed explicitly in terms of power sum symmetric functions. To use this technique for coefficient extraction, we need to express the complete symmetric functions in terms of power sum symmetric functions. This is accomplished by the formula

$$\sum_{n=0}^{\infty} h_n = \exp\left(\sum_{k=1}^{\infty} \frac{p_k}{k} \right), \tag{3}$$

which implies that

$$h_n = \sum_{\lambda} \frac{p_\lambda}{z_\lambda},$$

where the sum is over all partitions λ of n. In particular, we have

$$\begin{aligned}
h_1 &= p_1\\
h_2 &= p_1^2/2 + p_2/2\\
h_3 &= p_1^3/6 + p_1 p_2/2 + p_3/3\\
h_4 &= p_1^4/24 + p_1^2 p_2/4 + p_1 p_3/3 + p_2^2/8 + p_4/4\\
h_5 &= p_1^5/120 + p_1^3 p_2/12 + p_1^2 p_3/12 + p_1 p_2^2/8 + p_1 p_4/4 + p_2 p_3/6 + p_5/5.
\end{aligned}$$

Next we recall the operation of *internal* (also called *inner*) product on symmetric functions which is defined by

$$p_\lambda * p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda \tag{4}$$

and extended by linearity to all symmetric functions. The internal product was also discovered by Redfield [38] in 1927, who called it the "cup product," and it was rediscovered by D. E. Littlewood [27] in 1956. It is equivalent to pointwise multiplication of characters of symmetric groups, which corresponds to the tensor (or Kronecker) product of representations of groups.

Finally, we need to consider the operation of *composition* (also called *plethysm*) for symmetric functions. To motivate the general definition, first suppose that g is a symmetric function which can be expressed in the form $t_1 + t_2 + \cdots$, where each t_j is of the form $x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$. (The terms t_j need not be distinct.) Then for any symmetric function $f = f(x_1, x_2, \ldots)$ we define the composition $f(g)$ to be $f(t_1, t_2, \ldots)$.

In the general case, composition may be defined as follows: If f_1 and f_2 are symmetric functions then $(f_1 + f_2)(g) = f_1(g) + f_2(g)$ and $(f_1 f_2)(g) = f_1(g) f_2(g)$ so it is sufficient to define $p_n(g)$. This is accomplished by the formula $p_n(g) = g(p_n)$, where $g(p_n)$ may be determined by the special case given in the previous paragraph, or by the formula $p_m(p_n) = p_{mn}$.

3. Coefficient Extraction for Symmetric Functions

There are several useful methods for extracting coefficients from symmetric functions. Goulden and Jackson [17, Section 3.5; 18] and Goulden, Jackson, and Reilly [19] used a method based on expanding a symmetric function in power sums to find recurrences for the coefficients. This method is useful in many of the problems considered in this paper, such as counting regular graphs. Read [36] used the method of evaluating $\langle f, h_\lambda \rangle$ by expanding f and h_λ in Schur functions, and using the orthogonality of Schur functions. This method seems most useful in computing numbers for small cases, rather than in deriving general formulas. A related technique, which is occasionally useful, is to expand f in Schur functions and use what is known about the coefficients of Schur functions (the Kostka numbers). This technique can be used, for example, in studying degree sequences of graphs. Sometimes it is possible to evaluate coefficients of symmetric functions by applying linear functionals to products of polynomials. (See, for example, Even and Gillis [9], Askey and Ismail [1], and Zeng [52].)

In this paper we shall develop the method sketched in the previous section for extracting coefficients from symmetric functions: The coefficient of $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k}$ in the symmetric function f is $\langle f, h_\lambda \rangle$. To evaluate this scalar product, we expand f and h_λ in power sum symmetric functions. This method was described clearly by Read [35], but does not seem to be well known. Applying it usually gives us a simple, and often well-known, formula for the case $\lambda = (1^n)$, or at least a simple exponential generating function. We often get a fairly simple formula when all parts of λ are at most 2, and a formula which is reasonable to write down when all parts are at most 3.

The coefficient of $x_1 x_2 \cdots x_n$ in a symmetric function is easily extracted by the following theorem, of which the underlying idea, if not the explicit formulation, is well known (cf. Gessel [11, Theorem 3.5] and Goulden and Jackson [17, Lemma 4.2.5, p. 233]):

THEOREM 1. *Let θ be the homomorphism from the ring of symmetric functions to the ring of formal power series in X defined by $\theta(p_1) = X$, $\theta(p_n) = 0$ for $n > 1$. Then if f is a symmetric function,*

$$\theta(f) = \sum_{n=0}^{\infty} a_n \frac{X^n}{n!},$$

where a_n is the coefficient of $x_1 x_2 \cdots x_n$ in f. In particular, $\theta(h_n) = X^n/n!$.

Proof. Let $f(p_1, p_2, ...)$ be a symmetric function and let

$$f(p_1, 0, 0, ...) = \sum_{n=0}^{\infty} a_n \frac{p_1^n}{n!}.$$

Then the coefficient of $x_1 x_2 \cdots x_n$ in f is

$$\langle f, h_1^n \rangle = \langle f(p_1, 0, 0, ...), p_1^n \rangle = \left\langle \sum_{m=0}^{\infty} a_m \frac{p_1^m}{m!}, p_1^n \right\rangle = a_n.$$

It is interesting to note that this principle was not known to MacMahon, who worked extensively with symmetric functions, but failed to find the simple exponential generating functions, such as those for Eulerian numbers, that his symmetric generating functions implied.

There is a q-analog of the homomorphism θ which is useful in studying partitions (especially plane partitions) and in counting permutations by greater index, although we will not discuss it further in this paper. Define the homomorphism Θ from symmetric functions to formal power series by $\Theta(f(x_1, x_2, ...)) = f(X, qX, q^2X, ...)$. Then $\Theta(h_n) = X^n/(q)_n$ and $\Theta(e_n) = q^{\binom{n}{2}}X^n/(q)_n$, where $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$. Moreover, $\theta(f) = \lim_{q\to 1} \Theta(f)((1-q)X)$.

We now give an unusual example of the use of Theorem 1, suggested by Goulden and Jackson [17, pp. 73–74]. A sequence of integers from 1 to n is of *increasing support* if $12\cdots n$ occurs as a (not necessarily consecutive) subsequence, or equivalently, if the sequence has a strictly increasing subsequence of length n. Goulden and Jackson showed that the number $I_{(\lambda_1, \lambda_2, ..., \lambda_n)}(n)$ of permutations of increasing support of the multiset $\{1^{\lambda_1+1}, 2^{\lambda_2+1}, ..., n^{\lambda_n+1}\}$ is the coefficient of $x_1^{\lambda_1}\cdots x_n^{\lambda_n}$ in

$$A = (1-x)^{-1} \prod_{j=1}^{n} (1-x+x_j)^{-1},$$

where $x = x_1 + \cdots + x_n$. In this form, A is a symmetric function of the n variables $x_1, ..., x_n$. Now we may write A as

$$\begin{aligned}
&(1-x)^{-(n+1)} \prod_{j=1}^{n} \left(1+\frac{x_j}{1-x}\right)^{-1} \\
&\quad = (1-x)^{-(n+1)} \exp \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (-1)^k \frac{x_j^k}{k(1-x)^k} \\
&\quad = (1-p_1)^{-(n+1)} \exp \sum_{k=1}^{\infty} (-1)^k \frac{p_k}{k(1-p_1)^k}, \qquad (5)
\end{aligned}$$

where the p_i are in the variables $x_1, ..., x_n$. It is clear that as long as λ has at most n parts, the coefficient of $x_1^{\lambda_1}\cdots x_n^{\lambda_n}$ will be unchanged if we take infinitely many variables in (5). Thus Theorem 1 implies that if $m \leqslant n$ then the coefficient of $x_1 \cdots x_m$ in A, which corresponds to the multiset $\{1^2, 2^2, ..., m^2, m+1, ..., n\}$, is the coefficient of $X^m/m!$ in

$$(1-X)^{-(n+1)} \exp\left(-\frac{X}{1-X}\right),$$

and thus

$$I_{(1^m)}(n) = \sum_{r=0}^{m} (-1)^r \frac{m!}{r!} \binom{m+n}{m-r}.$$

In terms of the Laguerre polynomials, which may be defined by the generating function

$$(1-X)^{-(n+1)} \exp\left(-\frac{uX}{1-X}\right) = \sum_{m=0}^{\infty} L_m^{(n)}(u)\, X^m,$$

we have $I_{(1^m)}(n) = m!\, L_m^{(n)}(1)$. Note that $I_{(1^m)}(n)$ makes sense only for $m \leqslant n$, and in fact $m!\, L_m^{(n)}(1)$ need not be nonnegative when $m > n$. The first few values of $I_{(1^m)}(n)$ are as follows:

$$\begin{aligned}
I_0(n) &= 1\\
I_{(1)}(n) &= n\\
I_{(1^2)}(n) &= n^2 + n - 1\\
I_{(1^3)}(n) &= n^3 + 3n^2 - n - 4\\
I_{(1^4)}(n) &= n^4 + 6n^3 + 5n^2 - 16n - 15\\
I_{(1^5)}(n) &= n^5 + 10n^4 + 25n^3 - 20n^2 - 111n - 56.
\end{aligned}$$

An analog of Theorem 1 for the coefficient of $x_1 \cdots x_m x_{m+1}^2 \cdots x_{m+n}^2$ will be given in Theorem 4. First we consider some explicit formulas for coefficients.

THEOREM 2. *Let*

$$f(p_1, p_2) = \sum_{r,s} f_{r,s} \frac{p_1^r}{r!} \frac{(p_2/2)^s}{s!}.$$

Then the coefficient of $x_1 \cdots x_m x_{m+1}^2 \cdots x_{m+n}^2$ *in* $f(p_1, p_2)$ *is*

$$2^{-n} \sum_{j=0}^{n} \binom{n}{j} f_{m+2j,\, n-j}.$$

Proof. We want the coefficient of $(\alpha^m/m!)(\beta^n/n!)$ in

$$\begin{aligned}
\langle f, e^{\alpha h_1} e^{\beta h_2} \rangle &= \langle f, \exp(\alpha p_1 + \beta p_1^2/2 + \beta p_2/2) \rangle\\
&= \left\langle \sum_{r,s} f_{r,s} \frac{p_1^r}{r!} \frac{(p_2/2)^s}{s!}, \sum_{m,j,k} \alpha^m \beta^{j+k} \frac{p_1^{m+2j} p_2^k}{m!\, 2^{j+k} j!\, k!} \right\rangle
\end{aligned}$$

$$= \sum_{m,j,k} \alpha^m \beta^{j+k} \frac{f_{m+2j,k}}{m!\, 2^{j+k} j!\, k!}$$

$$= \sum_{m,j,k} \frac{\alpha^m}{m!} \frac{\beta^{j+k}}{(j+k)!} 2^{-(j+k)} \binom{j+k}{k} f_{m+2j,k}$$

and the theorem follows.

One can give a similar formula for the coefficient of a monomial in which all powers are at most 3, but for simplicity we give only the case in which all powers are 1 or 3:

THEOREM 3. *Let*

$$f(p_1, p_2, p_3) = \sum_{r,s,t} f_{r,s,t} \frac{p_1^r}{r!} \frac{(p_2/2)^s}{s!} \frac{(p_3/3)^t}{t!}.$$

Then the coefficient of $x_1 \cdots x_m x_{m+1}^3 \cdots x_{m+n}^3$ *in* $f(p_1, p_2, p_3)$ *is*

$$\sum_{i+j+k=n} 2^{-(i+j)} 3^{-(i+k)} \frac{(i+j+k)!}{i!\, j!\, k!} f_{m+3i+j,j,k}.$$

Proof. We want the coefficient of $(\alpha^m/m!)(\beta^n/n!)$ in

$$\langle f, e^{\alpha h_1 + \beta h_3} \rangle = \langle f, \exp(\alpha p_1 + \beta p_1^3/6 + \beta p_1 p_2/2 + \beta p_3/3) \rangle$$

$$= \left\langle f, \sum_{i,j,k} \alpha^m \beta^{i+j+k} \frac{p_1^{3i+j} p_2^j p_3^k}{m!\, 6^i i!\, 2^j j!\, 3^k k!} \right\rangle$$

$$= \sum_{i,j,k} \alpha^m \beta^{i+j+k} \frac{f_{m+3i+j,j,k}}{m!\, 6^i i!\, 2^j j!\, 3^k k!}$$

$$= \sum_{i,j,k} \frac{\alpha^m}{m!} \frac{\beta^{i+j+k}}{(i+j+k)!} 2^{-(i+j)} 3^{-(i+k)} \frac{(i+j+k)!}{i!\, j!\, k!} f_{m+3i+j,j,k}.$$

We now give some examples of Theorems 2 and 3. For other explicit formulas that can be obtained by this method, see Read [34, 35].

Carlitz [6] (see also Goulden and Jackson [17, Corollary 4.2.7, p. 234]) showed that if $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ is even then the coefficient of $x_{j_1}^{\lambda_1} x_{j_2}^{\lambda_2} \cdots x_{j_k}^{\lambda_k}$ in

$$E = 1 \Big/ \sum_{n=0}^{\infty} (-1)^n h_{2n}$$

is equal to the number of permutations $a_1 a_2 \cdots a_{|\lambda|}$ of the multiset

$\{j_1^{\lambda_1}, j_2^{\lambda_2}, ..., j_k^{\lambda_k}\}$ satisfying $a_1 \leqslant a_2 > a_3 \leqslant a_4 \cdots \leqslant a_{|\lambda|}$. Now with $i = \sqrt{-1}$ we have

$$\begin{aligned} E &= 2 \Big/ \left(\sum_{n=0}^{\infty} i^n h_n + \sum_{n=0}^{\infty} (-i)^n h_n \right) \\ &= 2 \Big/ \left(\exp \sum_{n=1}^{\infty} i^n \frac{p_n}{n} + \exp \sum_{n=1}^{\infty} (-i)^n \frac{p_n}{n} \right) \\ &= \exp \left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{p_{2n}}{2n} \right) \sec \left(\sum_{n=0}^{\infty} (-1)^n \frac{p_{2n+1}}{2n+1} \right). \end{aligned}$$

The *Euler* or *secant* numbers E_n are defined by

$$\sec x = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}$$

(so $E_n = 0$ when n is odd). For any partition λ let us define E_λ to be $\langle E, h_\lambda \rangle$. Then Theorem 1 yields $E_{(1^n)} = E_n$, as is well known. On setting $p_i = 0$ for $i > 3$, E becomes

$$e^{p_2/2} \sec(p_1 - p_3/3) = \sum_{r,s,t} (-1)^t E_{r+t} \frac{p_1^r}{r!} \frac{(p_2/2)^s}{s!} \frac{(p_3/3)^t}{t!}.$$

Then from Theorems 2 and 3 we have

$$E_{(1^m 2^n)} = 2^{-n} \sum_{i=0}^{n} \binom{n}{i} E_{m+2i}, \tag{6}$$

as found by Goulden and Jackson [17, Exercise 4.2.2, p. 240, Solution, pp. 459–460], and

$$\begin{aligned} E_{(1^m 3^n)} &= \sum_{i+j+k=n} 2^{-(i+j)} 3^{-(i+k)} \frac{(i+j+k)!}{i!\,j!\,k!} (-1)^k E_{m+3i+j+k} \\ &= 6^{-n} \sum_{i=0}^{n} \binom{n}{i} E_{m+n+2i}. \end{aligned} \tag{7}$$

Note that these formulas imply the identity $E_{(1^{m+n} 2^n)} = 3^n E_{(1^m 3^n)}$. It follows from (6) that the sum

$$\sum_{i=0}^{n} \binom{n}{i} E_{m+2i}$$

is divisible by 2^n for all m and from (7) that this sum is divisible by 6^n for $m \geqslant n$. (See Gessel [12] for further divisibility properties of these numbers.)

The first few values of the numbers $E_{(2^n)}$ and $E_{(3^n)}$ are as follows:

n:	0	1	2	3	4	5	6	7	8
$E_{(2^n)}$:	1	1	2	10	104	1816	47312	1714000	82285184
$E_{(3^n)}$:	1	0	2	0	2248	0	54103952	0	9573516562048

Carlitz [6] also showed (see also Goulden and Jackson [17, Corollary 4.2.20, p. 238]) that if $|\lambda|$ is odd then the coefficient of $x_{j_1}^{\lambda_1} x_{j_2}^{\lambda_2} \cdots x_{j_k}^{\lambda_k}$ in

$$T = \sum_{n=0}^{\infty} (-1)^n h_{2n+1} \Big/ \sum_{n=0}^{\infty} (-1)^n h_{2n}$$

is equal to the number of permutations $a_1 a_2 \cdots a_{|\lambda|}$ of the multiset $\{j_1^{\lambda_1}, j_2^{\lambda_2}, ..., j_k^{\lambda_k}\}$ satisfying $a_1 \leqslant a_2 > a_3 \leqslant a_4 \cdots > a_{|\lambda|}$.

As before, we can show that

$$T = \tan\left(\sum_{n=0}^{\infty} (-1)^n \frac{p_{2n+1}}{2n+1}\right).$$

Since T involves does not involve p_m for m even, we will get simpler formulas from T than from E. The *tangent numbers* T_{2n+1} are defined by

$$\tan x = \sum_{n=0}^{\infty} T_{2n+1} \frac{x^{2n+1}}{(2n+1)!}.$$

For any partition λ, let us define T_λ to be $\langle T, h_\lambda \rangle$. Then Theorem 1 yields $T_{(1^n)} = T_n$ and Theorem 2 yields $T_{(1^m 2^n)} = 2^{-n} T_{m+2n}$. A straightforward calculation shows that

$$T_{(1^l 2^m 3^n)} = 2^{-m} 3^{-n} \sum_{i=0}^{n} (-1)^{n-i} 2^{-i} \binom{n}{i} T_{l+2m+n+2i}$$

and that

$$E_{(1^m 4^n)} = 3^{-n} \sum_{i=0}^{n} (-1)^{n-i} 8^{-i} \binom{n}{i} T_{m+2n+i}.$$

The first few values of the numbers $T_{(3^n)}$, for n odd, are as follows:

n:	1	3	5	7
$T_{(3^n)}$:	0	30	217800	16294301520

Symmetric functions arise in Pólya theory as "cycle indices." If f is a symmetric function interpretated in this way, then the coefficient of $x_1^{\lambda_1}\cdots x_n^{\lambda_n}$ in f is the number of "labeled objects" counted by f in which i occurs as a label λ_i times. We give one simple example. It follows from the work of G. Labelle [24, Corollary A2] that the part of the cycle index for rooted trees involving only p_1 and p_2 is

$$\sum_{\substack{r,s\\ r>0}} r^r(r+2s)^{s-1}\frac{p_1^r}{r!}\frac{(p_2/2)^s}{s!}.$$

Then by Theorem 2, the number of rooted trees in which m labels appear once and n labels appear twice is

$$2^{-n}\sum_{j=0}^{n}\binom{n}{j}(m+2j)^{m+2j}(m+2n)^{n-j-1},$$

where 0^0 is interpreted as 0.

In some cases, Theorems 2 and 3 can be used to derive reasonably simple exponential generating functions. We consider here only the case of Theorem 2.

THEOREM 4. *Let m be a fixed nonnegative integer and let c_n be the coefficient of $x_1\cdots x_m x_{m+1}^2\cdots x_{m+n}^2$ in the symmetric function $f(p_1, p_2)$. Let ψ be the linear operator on power series in y defined by*

$$\psi\left(\sum_{r=0}^{\infty} a_r\frac{y^r}{r!}\right)=\sum_{j=0}^{\infty} a_{m+2j}\frac{(X/2)^j}{j!}.$$

Then

$$\sum_{n=0}^{\infty} c_n\frac{X^n}{n!}=\psi(f(y, X)).$$

Proof. Let $f_{r,s}$ be as in Theorem 2. Then by Theorem 2,

$$\begin{aligned}\sum_{n=0}^{\infty} c_n\frac{X^n}{n!}&=\sum_{n=0}^{\infty}\frac{(X/2)^n}{n!}\sum_{j=0}^{\infty}\binom{n}{j}f_{m+2j,n-j}\\&=\sum_{j,s=0}^{\infty} f_{m+2j,s}\frac{(X/2)^j}{j!}\frac{(X/2)^s}{s!}\\&=\psi(f(y, X)).\end{aligned}$$

As our first example of Theorem 4, let w_n be the number of $n\times n$ nonnegative integer matrices with every row and column sum 2. It is not difficult to see that w_n is the coefficient of $x_1^2 \cdots x_n^2$ in h_2^n. Since the coefficient of $x_1^2 \cdots x_n^2$ in h_2^k is 0 for $k \neq n$, $w_n/n!$ is the coefficient of $x_1^2 \cdots x_n^2$ in

$$e^{h_2} = e^{(p_1^2 + p_2)/2}.$$

Then applying Theorem 4 with $m = 0$ we have

$$\begin{aligned}
\sum_{n=0}^{\infty} w_n \frac{X^n}{n!^2} &= \psi(e^{(y^2 + X)/2}) = e^{X/2}\psi(e^{y^2/2}) \\
&= e^{X/2}\psi\left(\sum_{r=0}^{\infty} \frac{(2r)!}{2^r r!} \frac{y^{2r}}{(2r)!}\right) \\
&= e^{X/2} \sum_{j=0}^{\infty} \frac{(2j)!}{2^j j!} \frac{(X/2)^j}{j!} \\
&= e^{x/2}(1-x)^{-1/2}.
\end{aligned}$$

This generating function was first found by Anand, Dumir, and Gupta [2]. See also Stanley [45, Example 6.11] and Goulden and Jackson [17, Exercise 3.4.15, p. 212, Solution, pp. 449–450, pp. 221–224].

Next we apply Theorem 4 to the problem of counting partitions of the multiset $\{1, 2, ..., m, (m+1)^2, ..., (m+n)^2\}$. It is clear that the number of partitions of the multiset $\{1^{\lambda_1}, ..., l^{\lambda_l}\}$ with k blocks is the coefficient of $t^k x_1^{\lambda_1} \cdots x_l^{\lambda_l}$ in

$$\sum_{k=0}^{\infty} t^k h_k(h-1) = \exp\left[\sum_{i=1}^{\infty} \frac{t^i}{i}\left(\exp\left(\sum_{j=1}^{\infty} \frac{p_{ij}}{j}\right) - 1\right)\right]. \tag{8}$$

The part of (8) involving only p_1 and p_2 is

$$\exp\left(t(e^{p_1 + p_2/2} - 1) + \frac{t^2}{2}(e^{p_2} - 1)\right).$$

Now let $T_m(n, k)$ be the number of partitions of the multiset $\{1, 2, ..., m, (m+1)^2, ..., (m+n)^2\}$ with k blocks. Then it follows from Theorem 4 that

$$\sum_{n,k=0}^{\infty} t^k \frac{X^n}{n!} T_m(n, k) = \exp\left(\frac{t^2}{2}(e^X - 1)\right) \psi(e^{t(e^{y + X/2} - 1)}). \tag{9}$$

Let $S(n, k)$ be the Stirling number of the second kind. Then we have

$$\begin{aligned}
\psi(e^{t(e^{y+X/2} - 1)}) &= \psi\left(\sum_{n,k} t^k \frac{(y + X/2)^n}{n!} S(n, k)\right) \\
&= \psi\left(\sum_{l,r,k} t^k \frac{y^r}{r!} \frac{(X/2)^l}{l!} S(l + r, k)\right)
\end{aligned}$$

$$= \sum_{l,j,k} t^k \frac{(X/2)^{j+l}}{(j+l)!} S(l+m+2j, k) \binom{j+l}{j}$$

$$= \sum_{n,k} t^k \frac{X^n}{n!} 2^{-n} \sum_{j=0}^{n} \binom{n}{j} S(m+n+j, k). \tag{10}$$

It follows from (9) and (10) that

$$\sum_{n,k=0}^{\infty} t^k \frac{X^n}{n!} T_m(n, k) = \exp\left(\frac{t^2}{2}(e^X - 1)\right) \times \sum_{n,k=0}^{\infty} t^k \frac{X^n}{n!} 2^{-n} \sum_{j=0}^{n} \binom{n}{j} S(m+n+j, k). \tag{11}$$

Now let $C_m(n)$ be the total number of partitions of the multiset $\{1, 2, ..., m, (m+1)^2, ..., (m+n)^2\}$, so

$$C_m(n) = \sum_{k=0}^{2n+m} T_m(n, k).$$

Then setting $t = 1$ in (11) we obtain

$$\sum_{n=0}^{\infty} \frac{X^n}{n!} C_m(n) = \exp\left(\frac{1}{2}(e^X - 1)\right) \times \sum_{n=0}^{\infty} \frac{X^n}{n!} 2^{-n} \sum_{j=0}^{n} \binom{n}{j} B_{m+n+j}, \tag{12}$$

where B_n is the Bell number. The right side of (12) may be expressed symbolically as

$$B^m \exp\left(\frac{1}{2}(e^X - 1) + \binom{B+1}{2} X\right),$$

where after expansion, B^n is replaced by B_n (cf. Baroti [3]).

We can obtain another expression for the generating function for $T_m(n, k)$ by writing (9) as

$$\sum_{n=0}^{\infty} T_m(n, k) \frac{X^n}{n!} = \exp\left(-t + \frac{t^2}{2}(e^X - 1)\right) \psi(e^{te^{X/2}e^y}). \tag{13}$$

Note that

$$\psi(e^{uy}) = \psi\left(\sum_{r=0}^{\infty} u^r \frac{y^r}{r!}\right) = \sum_{j=0}^{\infty} u^{m+2j} \frac{(X/2)^j}{j!} = u^m e^{u^2 X/2}.$$

So if v is an indeterminate,

$$\psi(e^{ve^{y}}) = \sum_{j=0}^{\infty} \frac{v^j}{j!} \psi(e^{jy}) = \sum_{j=0}^{\infty} \frac{v^j}{j!} j^m e^{j^2 X/2}.$$

Thus

$$\psi(e^{te^{X/2}e^{y}}) = \sum_{j=0}^{\infty} \frac{(te^{X/2})^j}{j!} j^m e^{j^2 X/2} = \sum_{j=0}^{\infty} \frac{t^j}{j!} j^m e^{\binom{j+1}{2} X}. \tag{14}$$

It follows from (13) and (14) that

$$\sum_{n,k=0}^{\infty} t^k \frac{X^n}{n!} T_m(n,k) = \exp\left(-t + \frac{t^2}{2}(e^X - 1)\right) \sum_{j=0}^{\infty} \frac{t^j}{j!} j^m e^{\binom{j+1}{2} X}. \tag{15}$$

Formula (15) was found (in the case $m=0$) by Bender [4]. Partitions of the multiset $\{1^2, ..., n^2\}$ were also studied in connection with a problem in genetics by Thompson [48] and Lloyd [28]. The closely related problem of counting partitions of multisets in which every block is a set was studied by Comtet [7], Baroti [3], Bender [4], Reilly [40], Devitt and Jackson [8], and Goulden and Jackson [17, pp. 201–204, Exercises 3.5.6–3.5.8, p. 229, Solutions, pp. 455–457].

4. D-Finite Power Series and P-Recursive Functions

A formal power series $f(x)$ is said to be *D-finite* (or *differentiably finite*) if f satisfies a linear homogeneous differentiable equation with polynomial coefficients. An equivalent condition is that the derivatives of f span a finite-dimensional vector space over the field of rational functions in x. A function $a(n)$ defined on the nonnegative integers is said to be *P-recursive* (or *polynomially recursive*) if there exist polynomials $r_0(n), r_1(n), ..., r_k(n)$ such that

$$\sum_{i=0}^{k} r_i(n)\, a(n+i) = 0$$

for all nonnegative integers n.

The fundamental fact relating these two concepts is that $a(n)$ is P-recursive if and only if its generating function $\sum_{n=0}^{\infty} a(n)\, x^n$ is D-finite. We refer the reader to Stanley [47] for the proof of this and other basic facts.

In the next few sections we use symmetric functions to show that counting sequences for certain combinatorial problems are P-recursive.

To do this we need a multivariable generalization of the theory of D-finiteness and P-recursiveness. Such a generalization was first proposed by Zeilberger [50]. However, Zeilberger's definition of multivariable P-recursiveness does not imply D-finiteness. For example, if we set

$$f(x, y) = \sum_{m,n=0}^{\infty} a(m, n)\, x^m y^n = \sum_{j=0}^{\infty} x^j y^{j^2}$$

then $a(m, n)$ is P-recursive by Zeilberger's definition since $(n-m^2)\, a(m, n) = 0$ for all m and n, but $f(x, y)$ is not D-finite. (This example was suggested by Richard Stanley.) Lipshitz [26] has given a definition of multivariable P-recursiveness that does correspond to D-finiteness, and Zeilberger [51] has written a corrected version of [50]. However, we shall work only with multivariable D-finiteness. In the next section we define multivariable D-finiteness and describe some of its properties. All the properties follow easily from the definition except for the fact that Hadamard products of D-finite series are D-finite. In the one-variable case, this is straightforward using P-recursiveness; one need only show that the product of P-recursive functions is P-recursive. In the multivariable case the proof is more difficult and is due to Lipshitz [25].

5. D-Finite Power Series in Several Variables

First we discuss the theory of D-finite power series. Let I be an integral domain and let $f(x_1, x_2, ..., x_n)$ be a formal power series in $I[[x_1, x_2, ..., x_n]]$. Let F be the quotient field of $I[x_1, x_2, ..., x_n]$. We say that $f(x)$ is *D-finite* in the variables $x_1, x_2, ..., x_n$ if the set of all partial derivatives $(\partial^{i_1 + \cdots + i_n} f)/(\partial x_1^{i_1} \cdots \partial x_n^{i_n})$ spans a finite-dimensional vector space over F (as a subspace of the tensor product $F \otimes_{I[x_1, ..., x_n]} I[[x_1, x_2, ..., x_n]]$).

The following lemma contains some of the basic facts about D-finite power series in several variables that we will need. By D-finite we mean D-finite in the variables $x_1, x_2, ..., x_n$ except where otherwise stated.

LEMMA 5. (i) *The set of D-finite power series forms an I-subalgebra of* $I[[x_1, x_2, ..., x_n]]$.

(ii) *If f is D-finite in $x_1, x_2, ..., x_n$ then f is D-finite in any subset of $x_1, x_2, ..., x_n$.*

(iii) *If $f(x_1, x_2, ..., x_n)$ is D-finite in $x_1, x_2, ..., x_n$ and for each i, r_i is a polynomial in the variables $y_1, y_2, ..., y_m$ (which may include some or all of* *the* x_i), *then* $f(r_1, r_2, \ldots, r_n)$ *is D-finite in* $y_1, y_2, \ldots, y_m$, *as long as it is well defined as a formal power series.*

(iv) *If* $P(x)$ *is a polynomial in* $x_1, x_2, \ldots, x_n$ *then* $e^{P(x)}$ *is D-finite.*

The proofs of these statements are straightforward and are similar to proofs for the one-variable case given by Stanley [47]. We need one further fact about D-finite power series in several variables, due to Lipshitz [25]. If $A(x) = \sum a(i_1, \ldots, i_n)\, x_1^{i_1} \cdots x_n^{i_n}$ and $B(x) = \sum b(i_1, \ldots, i_n)\, x_1^{i_n} \cdots x_n^{i_n}$, then the Hadamard product $A(x) \odot B(x)$ with respect to the variables $x_1, x_2, \ldots, x_n$ is defined to be

$$\sum_{i_1, \ldots, i_n} a(i_1, \ldots, i_n)\, b(i_1, \ldots, i_n)\, x_1^{i_1} \cdots x_n^{i_n}.$$

Note that the a's and b's may involve other variables.

LEMMA 6 (Lipshitz [25]). *Suppose that* A *and* B *are D-finite in the variables* $x_1, x_2, \ldots, x_{m+n}$. *Then the Hadamard product* $A \odot B$ *with respect to the variables* $x_1, x_2, \ldots, x_n$ *is D-finite in* $x_1, x_2, \ldots, x_{m+n}$.

Now suppose that f is a formal power series in an infinite set X of variables. For any subsets S of X let f_S be the formal power series in the variables in S obtained by setting to zero all the variables in $X - S$. We shall say that f is D-finite in X if f_S is D-finite in S for every finite subset S of X. With this definition, all the properties of D-finite series in finitely many variables given by Lemma 5 are easily seen to remain valid for D-finite series in infinitely many variables, except that in Lemma 5(iii) we may only substitute for finitely many variables.

We shall say that a symmetric function is D-finite if it is D-finite when considered as a power series in the power sum symmetric functions p_i. This definition is useful because it allows us to show that functions obtained from coefficients of D-finite symmetric functions are P-recursive.

THEOREM 7. *Suppose that* f *and* g *are symmetric functions which are D-finite in the* p_i *and possibly in some other variables. Then* $f * g$ *is D-finite in these variables.*

Proof. Note that $f * g = f \odot g \odot u$, where $\odot$ is the Hadamard product in the p_i, and u is the symmetric function given by

$$u = \sum z_\lambda p_\lambda,$$

where the sum is over all partitions λ. Now

$$\begin{aligned} u &= \sum_{r_1, r_2, \ldots} 1^{r_1} 2^{r_2} \cdots r_1!\, r_2! \cdots p_1^{r_1} p_2^{r_2} \cdots \\ &= \left(\sum_{r_1} r_1!\, (1p_1)^{r_1}\right)\left(\sum_{r_2} r_2!\, (2p_2)^{r_2}\right) \cdots \\ &= A(1p_1)\, A(2p_2) \cdots, \end{aligned}$$

where $A(y) = \sum_{n=0}^{\infty} n!\, y^n$. Since u is D-finite, $f * g$ is a Hadamard product of three D-finite power series, and is thus D-finite by Lipshitz's theorem.

COROLLARY 8. *Let f and g be symmetric functions which are D-finite in the p_i and in another variable t, and suppose that g involves only finitely many of the p_i. Then $\langle f, g \rangle$ is D-finite in t as long as it is well defined as a formal power series.*

Proof. By the previous theorem, $f * g$ is D-finite in the p_i and in t, and involves only finitely many of the p_i. Then $\langle f, g \rangle$ is obtained from $f * g$ by setting each p_i equal to 1, and thus the conclusion follows from Lemma 5(iii).

Note that without the restriction on g Theorem 8 would not be true: according to our definitions, $\sum_{n=0}^{\infty} c_n p_n$ is D-finite for any coefficients c_n and thus any power series in t can be obtained as a scalar product of two D-finite symmetric functions.

COROLLARY 9. *Let f be a D-finite symmetric function and let S be a finite set of integers. Define integers s_n as follows: s_n is the sum over all n-tuples $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in S^n$ of the coefficient of $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in f. Then $s(t) = \sum_{n=0}^{\infty} s_n t^n$ is D-finite.*

Proof. The coefficient of $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in f is $\langle f, h_\lambda \rangle$, and so $s(t) = \langle f, g \rangle$, where

$$g = \sum_{n=0}^{\infty} \left(t \sum_{i \in S} h_i\right)^n = \left(1 - t \sum_{i \in S} h_i\right)^{-1}$$

and the assertion follows from Corollary 8.

THEOREM 10. *Suppose that g is a polynomial in the p_n. Then $h(g)$ and $e(g)$ are D-finite.*

Proof. By (3) we have

$$h(g) = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(g)}{k}\right) = \exp\left(\sum_{k=1}^{\infty} \frac{g(p_k)}{k}\right). \tag{16}$$

To show that $h(g)$ is D-finite we need only show that $h(g)$ is D-finite in the variables $p_1, p_2, ..., p_n$ for each n. But by (16), we have

$$h(g) = \exp\left(\sum_{k=1}^{n} \frac{g(p_k)}{k}\right) \exp\left(\sum_{k=n+1}^{\infty} \frac{g(p_k)}{k}\right),$$

where the second factor on the right does not involve $p_1, p_2, ..., p_n$. Then by Lemma 5(vi), $h(g)$ is D-finite in $p_1, p_2, ..., p_n$. The proof for $e(g)$ is similar.

Theorem 10 implies that the following products are all D-finite:

$$h(h_2) = \prod_{i \leqslant j} \frac{1}{1 - x_i x_j} \tag{17}$$

$$h(e_2) = \prod_{i < j} \frac{1}{1 - x_i x_j} \tag{18}$$

$$e(h_2) = \prod_{i \leqslant j} (1 + x_i x_j) \tag{19}$$

$$e(e_2) = \prod_{i < j} (1 + x_i x_j). \tag{20}$$

Each may be interpreted as counting a class of graphs. Thus the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots$ in (17) is the number of graphs on the vertex set $\{1, 2, ...\}$, with multiple edges and loops allowed, such that the degree of vertex i is λ_i, where a loop contributes 2 to the degree of its vertex. Similarly, (18) counts graphs with multiple edges but no loops, (19) counts graphs with loops allowed, but not multiple edges, and (20) counts graphs without loops or multiple edges. Graphs with loops in which a loop contributes only 1 to the degree of its vertex are counted by $h(e_1 + e_2)$ (multiple edges allowed) and $e(e_1 + e_2)$ (multiple edges not allowed). Similarly, k-uniform hypergraphs are counted by $e(e_k)$, and so on.

From Corollary 9 and the preceeding observations we may conclude the following:

COROLLARY 11. *Let S be a finite set of integers and let G be one of the classes of graphs described above. Let $a(n)$ be the number of graphs in G on an n-element set in which the degree of every vertex is in S. Then $a(n)$ is P-recursive.*

Corollary 11 (for (17) and (20)) was conjectured by Goulden and Jackson [18]. Recurrences for various types of graphs and digraphs with degree restrictions have been given by Goulden and Jackson [17, Section 3.5; 18], Goulden, Jackson, and Reilly [19], Read [34, 35], Read and Wormald [37], and Wormald [49].

6. Schur Functions

We recall that the coefficient of $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in the skew Schur function $s_{\mu/\nu}$ is the number of tableaux (or column-strict skew plane partitions) of shape $\mu - \nu$ and weight λ. (See Stanley [46] or Macdonald [29] for details.)

It is well known that $\sum_\lambda s_\lambda = h(e_1 + e_2)$, where the sum is over all partitions λ [29, Exercise 4, p. 45], and thus it follows that for fixed j, the number of tableaux of weight (j^n) is a P-recursive function of n. More generally, for fixed ν, $\sum_\mu s_{\mu/\nu}$ is also D-finite, since it is equal to

$$h(e_1 + e_2) \sum_\mu s_{\nu/\mu}.$$

(See, for example, Sagan and Stanley [42].)

We now consider tableaux with at most r rows, which are counted by skew Schur functions $s_{\lambda/\mu}$ in which λ has at most r parts.

THEOREM 12. *Let μ be a fixed partition and let*

$$B_k(\mu) = \sum_\lambda s_{\lambda/\mu},$$

where the sum is over all partitions λ with at most k parts. Then $B_k(\mu)$ is D-finite.

Proof. By a result of Gordon and Houten [15, Lemma 1] (see also [16, Lemma 1]), $B_k(\mu)$ can be expressed as a polynomial in h and in the symmetric functions

$$c_i = \sum_{n=0}^{\infty} h_n h_{n+i}. \tag{21}$$

Each c_i is D-finite since it is obtained by setting $t = 1$ in the Hadamard product (with respect to t)

$$\sum_{n=0}^{\infty} h_n t^{n+i} \odot \sum_{n=0}^{\infty} h_n t^n.$$

COROLLARY 13. *For j and k fixed, let $v(n)$ be the number of tableaux with at most k rows, of weight (j^n). Then $v(n)$ is P-recursive.*

Since Gordon and Houten's results are not well known, we state them here, together with simplifications in the case $\mu = 0$ due to Gordon [16] and Bender and Knuth [5]. (I am grateful to Richard Stanley for bringing these papers to my attention.)

THEOREM 14. *For $n > 0$ set*

$$d_n = c_0 + 2(c_1 + \cdots + c_{n-1}) + c_n,$$

where c_i is given by (21), and set $d_{-n} = -d_n$, with $d_0 = 0$. Then for k even, $B_k(\mu)$ is the Pfaffian of the skew-symmetric $k \times k$ matrix,

$$D_k = (d_{\mu_i - \mu_j + j - i})_{1 \leqslant i, j \leqslant k}$$

and for k odd, $B_k(\mu)$ is the Pfaffian of the skew-symmetric $(k+1) \times (k+1)$ matrix

$$\begin{pmatrix} 0 & H \\ -H^t & D_k \end{pmatrix}$$

obtained by bordering D_k with a row H of h's, a column $-H^t$ of $-h$'s, and a zero.

In the case in which μ is empty, we have (with $c_{-i} = c_i$)

$$\begin{aligned} B_{2m}(0) &= \det(c_{i-j} + c_{i+j-1})_{1 \leqslant i, j \leqslant m} \\ B_{2m+1}(0) &= h \det(c_{i-j} - c_{i+j})_{1 \leqslant i, j \leqslant m}. \end{aligned} \tag{22}$$

Gordon [16] showed that for $\mu = 0$ the Pfaffian could be reduced to a determinant, and he further simplified the determinant for the specialization he was interested in. Bender and Knuth [5] observed that Gordon's simplification would apply in general for $\mu = 0$ to yield (22). These authors stated their results in terms of determinants of indeterminates, rather than Schur functions, but as Bender and Knuth noted, the determinants become Schur functions when the h_i are substituted for the indeterminates.

The first few cases of (22) are, with $B_k = B_k(0)$:

$$\begin{aligned} B_1 &= h \\ B_2 &= c_0 + c_1 \\ B_3 &= h(c_0 - c_2) \\ B_4 &= c_0^2 + c_0 c_1 + c_0 c_3 + c_1 c_3 - c_1^2 - 2c_1 c_2 - c_2^2 \\ B_5 &= h(c_0^2 - c_0 c_2 - c_0 c_4 + c_2 c_4 - c_1^2 + 2c_1 c_3 - c_3^2). \end{aligned} \tag{23}$$

If we apply the homomorphism θ to the formulas of (23), we obtain exponential generating functions for standard tableaux. (The images of (23) under the q-analog Θ are also interesting.)

Let $b_i = \theta(c_i)$. Then

$$b_i = \sum_{n=0}^{\infty} \frac{X^{2n+i}}{n!\,(n+i)!} = \sum_{n=0}^{\infty} \binom{2n+i}{n} \frac{X^{2n+i}}{(2n+i)!},$$

which is the modified Bessel function $I_i(2X)$. Vandermonde's theorem implies the following formula for the coefficients of $b_i b_j$:

$$b_i b_j = \sum_{n=0}^{\infty} \frac{X^{2l+i+j}}{(2l+i+j)!} \cdot \binom{2l+i+j}{l}\binom{2l+i+j}{l+j}. \tag{24}$$

We also note the recurrence

$$Xb_k = (1-k)\, b_{k-1} + Xb_{k-2}, \tag{25}$$

which yields the following simplifications for $\theta(B_k)$:

$$\begin{aligned}
\theta(B_1) &= e^X \\
\theta(B_2) &= b_0 + b_1 \\
\theta(B_3) &= e^X X^{-1} b_1 \\
\theta(B_4) &= X^{-2} b_1^2 + 2X^{-2}(b_1^2 - b_0 b_2) \\
\theta(B_5) &= e^X(4X^{-2} b_1^2 - 4X^{-2} b_0 b_2 - 2X^{-3} b_1 b_2).
\end{aligned}$$

Expanding and (for $k=4$ and 5) simplifying, we obtain the following explicit formulas:

THEOREM 15. *Let $y_k(n)$ be the number of standard tableaux with n entries and at most k rows, and let C_n be the Catalan number $(1/(n+1))\binom{2n}{n}$. Then*

$$\begin{aligned}
y_2(n) &= \binom{n}{\lfloor n/2 \rfloor} \\
y_3(n) &= \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} C_i \\
y_4(n) &= C_{\lfloor (n+1)/2 \rfloor} C_{\lceil (n+1)/2 \rceil} \\
y_5(n) &= 6 \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} C_i \frac{(2i+2)!}{(i+2)!\,(i+3)!}.
\end{aligned}$$

These formulas are due to Regev [39] for $k=3$ and to Gouyou-Beauchamps [20] for $k=4$ and 5. In general, if k is even then (22) expresses $\theta(B_k)$ as a sum of at most $2^{k/2}(k/2)!$ terms, each of which is a product of $k/2$ of the b_i. Using (24) we can express such a product as a $\lceil k/4 \rceil$-fold sum. Thus $y_k(n)$ can be expressed as a sum of at most $2^{k/2}(k/2)!$ terms, each of which is a $\lceil k/4 \rceil - 1 = \lfloor (k-1)/4 \rfloor$-fold sum of products of binomial coefficients. Similarly, if k is odd, then $y_k(n)$ can be expressed as a sum of at most $2^{\lfloor k/2 \rfloor}\lfloor k/2 \rfloor!$ terms, each of which is a $\lfloor (k+1)/4 \rfloor$-fold sum of products of binomial coefficients. It remains to be seen whether these sums simplify in general as they do for $k \leqslant 5$.

7. Symmetric Functions in Several Sets of Variables

For some applications it is necessary to work with symmetric functions in two or more sets of variables. For simplicity, we consider here only the case of two sets of variables, which we use to count nonnegative integer matrices with prescribed row and column sums (or equivalently, digraphs with prescribed indegrees and outdegrees, or two-colored graphs with prescribed degrees).

Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be two disjoint sets of variables. We shall consider power series in these variables which are symmetric in the x's and symmetric in the y's. It is easy to see that such a symmetric function can be expressed in the form

$$\sum_{\lambda, \mu} a_{\lambda\mu}\, p_\lambda(x)\, p_\mu(y),$$

where $p_\lambda(x)$ means $p_\lambda(x_1, x_2, \ldots)$ and similarly for $p_\mu(y)$. We call these series D-finite if they are D-finite in the $p_i(x)$ and $p_j(y)$.

We may extend the scalar product $\langle\ ,\ \rangle$ to symmetric functions in two sets of variables by setting $\langle f_1(x)\, f_2(y), g_1(x)\, g_2(y)\rangle = \langle f_1(x), g_1(x)\rangle \langle f_2(y), g_2(y)\rangle$ and using linearity.

If f is a symmetric function, by $f(xy)$ we mean $f(x_1 y_1, x_1 y_2, \ldots, x_i y_j, \ldots)$. Thus, for example, we have

$$h(xy) = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

This product is easily seen to be D-finite using the fact that $p_n(xy) = p_n(x)\, p_n(y)$. It is then clear that the coefficient of $x_1^{\lambda_1} \cdots x_n^{\lambda_n} y_1^{\mu_1} \cdots y_n^{\mu_n}$ in $h(xy)$ is the number of digraphs on $\{1, 2, \ldots, n\}$ in which vertex i has outdegree λ_i and indegree μ_i (with multiple edges allowed), or equivalently, the number of $n \times n$ matrices of nonnegative integers in which the sum of the ith row is λ_i and the sum of the jth column is μ_j. This coefficient is equal to $\langle h(xy), h_\lambda(x)\, h_\mu(y)\rangle$. (It is not difficult to show that this coefficient is also equal to $\langle h_\lambda(x), h_\mu(x)\rangle$, so this example could be done using only symmetric functions in one set of variables.)

Now let m_n be the number of $n\times n$ nonnegative integer matrices with every row and column sum equal to k. It follows from what we have said that

$$m(t)=\sum_{n=0}^{\infty} m_n t^n = \langle h(xy), u(x, y)\rangle,$$

where $u(x, y)$ is given by $u(x, y)=(1-th_k(x)\, h_k(y))^{-1}$, and thus as before, $m(t)$ is D-finite. Similarly, $e(xy)$ counts 0–1 matrices with prescribed row and column sums, or equivalently, digraphs without multiple edges with prescribed indegrees and outdegrees.

Next we consider the sum

$$R_k(x, y)=\sum_{\lambda} s_\lambda(x)\, s_\lambda(y),$$

over all partitions λ with at most k parts. (Similar reasoning would apply to the more general sum $\sum_\lambda s_{\lambda/\alpha}(x)\, s_{\lambda/\beta}(y)$ for fixed α and β.) The following theorem implies that $R_k(x, y)$ is D-finite:

THEOREM 16. *For each integer i let*

$$A_i=\sum_{l=0}^{\infty} h_{l+i}(x)\, h_l(y).$$

Then

$$R_k(x, y)=\det(A_{j-i}).$$

Proof. Let $M(x)$ be the $\infty\times k$ matrix $(h_{i-j}(x))_{i\geqslant 1, 1\leqslant j\leqslant k}$ and for any set S of positive integers, let $M_S(x)$ be the $k\times k$ minor of $M(x)$ obtained from the rows indexed by elements of S. Now let $\lambda=(\lambda_1, ..., \lambda_k)$ be a partition and let $S=\{\lambda_{k+1-i}+i \mid 1\leqslant i\leqslant k\}$. Then $M_S(x)=\det(h_{\lambda_{k+1-i}+i-j}(x))_{1\leqslant i, j\leqslant k}$. Reversing the order of the rows and columns in this determinant yields $M_S(x)=\det(h_{\lambda_i+j-i}(x))=s_\lambda(x)$. It follows that

$$R_k(x, y)=\sum M_S(x)\, M_S(y),$$

where the sum is over all k-element sets S of positive integers. Then by the Cauchy–Binet theorem, $R_k(x, y) = \det M^t M$. Now

$$(M^t M)_{ij} = \sum_l h_{l-i}(x)\, h_{l-j}(y) = A_{j-i},$$

and the theorem follows.

Let us now write θ_x for θ and θ_y for the corresponding homomorphism in the y's, taking $h_n(y)$ to $Y^n/n!$. Then $\theta_y(R_k(x, y))$ is D-finite and it follows from Knuth's generalization [22] of Schensted's correspondence [43] that the coefficient of $Y^n/n!$ in $\theta_y(R_k(x, y))$ is the generating function in the x_i for sequences of length n with longest increasing subsequence of length at most k. It also follows that $\theta_x\theta_y(R_k(x, y))$ is the exponential generating function in X and Y for permutations with longest increasing subsequence of length at most k, and thus that for fixed k, the number of permutations with longest increasing subsequence of length at most k is P-recursive.

Since $\theta_x\theta_y(R_k(x, y))$ is a power series in XY, one variable is redundant. Let U_k be obtained by setting $Y = X$ in $\theta_x\theta_y(R_k(x, y))$ and let

$$U_k = \sum_{n=0}^{\infty} u_k(n) \frac{X^{2n}}{n!^2},$$

so that $u_k(n)$ is the number of permutations of $\{1, 2, ..., n\}$ with longest increasing subsequence of length at most k. Note that setting $Y = X$ in $\theta_x\theta_y(A_i)$, yields the series b_i considered in Section 5. Then

$$U_k = \det(b_{|i-j|})_{1 \leqslant i, j \leqslant k}$$

and we have

$$U_1 = b_0$$

$$U_2 = b_0^2 - b_1^2$$

$$U_3 = b_0^3 + 2b_1^2 b_2 - 2b_1^2 b_0 - b_0 b_2^2$$

To find the coefficients of U_k, we rewrite the case $i + j = 2r$ of (24) as

$$b_i b_j = \sum_{n=0}^{\infty} \frac{X^{2n}}{n!^2} \frac{n(n-1)\cdots(n-r+1)}{(n+1)(n+2)\cdots(n+r)} \binom{2n}{n-r+j}. \tag{26}$$

Then to calculate $u_2(n)$, we have

$$b_0^2 = \sum_{n=0}^{\infty} \frac{X^{2n}}{n!^2} \binom{2n}{n}$$

and

$$b_1^2 = \sum_{n=0}^{\infty} \frac{X^{2n}}{n!^2} \frac{n}{n+1} \binom{2n}{n}.$$

Thus

$$u_2(n) = \binom{2n}{n} - \frac{n}{n+1} \binom{2n}{n} = \frac{1}{n+1} \binom{2n}{n}$$

as found by MacMahon [33, Vol. 1, pp. 130–131]. See also Knuth [23, p. 64].

Using the recurrence (25), we find that

$$U_3 = 2X^{-1} b_0^2 b_1 - 2X^{-1} b_1^3 - X^{-2} b_0 b_1^2$$

and we obtain the following explicit formula for $u_3(n)$:

$$u_3(n) = 2 \sum_{k=0}^{n} \binom{2k}{k} \binom{n}{k}^2 \frac{3k^2 + 2k + 1 - n - 2kn}{(k+1)^2 (k+2)(n-k+1)}.$$

The first few values of $u_3(n)$ are as follows:

n:	0	1	2	3	4	5	6	7	8	9	10
$u_3(n)$:	1	1	2	6	23	103	513	2761	15767	94359	586590

Rogers [41] has studied some special cases of $u_k(n)$.

8. Nonsymmetric Functions

In some cases coefficients of generating functions which are close to symmetric can be evaluated using the theory of symmetric functions. Suppose that f is a symmetric function. Then the coefficient of $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in $x_1^{\mu_1} \cdots x_n^{\mu_n} f$ is equal to the coefficient of $x_1^{\lambda_1 - \mu_1} \cdots x_n^{\lambda_n - \mu_n}$ in f, and is thus $\langle h_{\lambda_1 - \mu_1} \cdots h_{\lambda_n - \mu_n}, f \rangle$. We can use this idea to find coefficients in power series which are products of symmetric functions and power series with simple coefficients.

For example, let $N_k(a, b)$ be the number of graphs (without multiple edges) on $\{1, 2, ..., a+b\}$ in which every vertex has degree k and loops are forbidden on vertices $1, 2, ..., a$, but allowed on the other vertices. Then $N_k(a, b)$ is the coefficient of $x_1^k x_2^k \cdots x_{a+b}^k$ in

$$\left(\prod_{i=1}^{a} \frac{1}{1+x_i^2} \right) \left(\prod_{i \leqslant j} (1 + x_i x_j) \right) = \left(\prod_{i=1}^{a} (1 - x_i^2 + x_i^4 - \cdots) \right) e(h_2),$$

and it follows that

$$N_k(a, b) = \langle (h_k - h_{k-2} + h_{k-4} - \cdots)^a h_k^b, e(h_2) \rangle.$$

Thus for fixed k, $N_k(n, n)$ is a P-recursive function of n.

As another example, let $t_k(n)$ be the number of $n \times n$ nonnegative integer matrices with every row and column sum k, and with zeroes on the diagonal. Then $t_k(n)$ is the coefficient of $x_1^k \cdots x_n^k y_1^k \cdots y_n^k$ in

$$\prod_{i \neq j} \frac{1}{1 - x_i y_j} = \left(\prod_i (1 - x_i y_i) \right) h(xy),$$

so

$$t_k(n) = \langle (h_k(x) h_k(y) - h_{k-1}(x) h_{k-1}(y))^n, h(xy) \rangle.$$

It follows that for fixed k, $t_k(n)$ is P-recursive.

9. Asymptotics

Although the explicit formulas obtained from symmetric functions are often unwieldy, they can be used to obtain asymptotic approximations, since in many cases all but the initial terms are asymptotically negligible.

We give here a brief indication, with details omitted, of how asymptotic approximations can be obtained from symmetric functions. The number of $n \times n$ nonnegative integer matrices with every row and column sum k is $\langle h_k^n, h_k^n \rangle$. The explicit formula obtained by expanding this expression is complicated, but it can be shown that asymptotically (for fixed k as $n \to \infty$) we may approximate h_k by its first two terms, yielding

$$\left\langle \left(\frac{p_1^k}{k!} + \frac{p_1^{k-2}}{(k-2)!} \frac{p_2}{2} \right)^n, \left(\frac{p_1^k}{k!} + \frac{p_1^{k-2}}{(k-2)!} \frac{p_2}{2} \right)^n \right\rangle$$
$$= \frac{(kn)!}{k!^{2n}} \sum_{i=0}^{n} \frac{(k(k-1))^{2i}}{i!\, 2^i} \frac{(n(n-1) \cdots (n-i+1))^2}{kn(kn-1) \cdots (kn-2i+1)}.$$

(This approximation is exact when $k = 2$.) Asymptotically each summand may be approximated by its limit as $n \to \infty$, and the sum is thus asymptotic to

$$\frac{(kn)!}{k!^{2n}} \sum_{i=0}^{\infty} \frac{(k^2(k-1)^2/2)^i}{i!} \left(\frac{1}{k} \right)^{2i} = \frac{(kn)!}{k!^{2n}} e^{(k-1)^2/2},$$

as found by Everett and Stein [10], who also used symmetric functions.

10. Further Remarks

Some counting problems have solutions which can be expressed as coefficients of generating functions that are not quite symmetric functions. For example, it is easy to see that the number of $3 \times n$ Latin rectangles is the coefficient of $x_1 \cdots x_n y_1 \cdots y_n z_1 \cdots z_n$ in

$$\left(\sum_{i,j,k} x_i y_j z_k\right)^n, \tag{27}$$

where the sum is over all triples of distinct integers i, j, k, and Goulden and Jackson [17, Exercise 4.5.6, p. 288, Solution, pp. 506–507)] derived the explicit formula for three-line Latin rectangles from this generating function. The power series (27) has the property that it is unchanged by any permutation of the subscripts which acts the same on x's, y's, and z's, but it is not a symmetric function in three sets of variables in the usual sense, and the standard theory of symmetric functions does not apply to it. MacMahon [30, 33] studied these symmetric function extensively, calling them "symmetric functions of several systems of quantities," and in [32] he applied them to the problem of counting Latin rectangles. Except for Schur functions, the usual theory of symmetric functions, including the application to P-recursiveness discussed here, can be generalized to these symmetric functions of MacMahon, as shown in Gessel [14], and they can be used to derive the explicit formula for Latin rectangles given in [13]. These symmetric functions can also be used, in the problem considered in Section 7 of counting matrices of nonnegative integers with zeros on the diagonal, where the generating function is $\prod_{i \neq j} (1 - x_i y_j)^{-1}$.

The problem of counting k-regular graphs with a specified set of forbidden subgraphs is one whose P-recursiveness remains open. Wormald [49] showed that the counting sequence for 3-regular graphs without triangles is P-recursive. It seems likely that an explicit formula for these graphs can be obtained using symmetric functions and Möbius inversion. However, for 4-regular graphs without triangles or 3-regular graphs without 4-cycles the situation is more complicated, and it seems likely that new techniques will be necessary to deal with the general case.

Another possible candidate for P-recursiveness is the problem of counting permutations (or more generally sequences) with forbidden subsequences determined by inequalities, for example permutations $a_1 a_2 \cdots a_n$ of $\{1, 2, ..., n\}$ with no subsequence $a_i a_j a_k$ satisfying $a_i < a_k < a_j$. Simion and Schmidt [44] have given a comprehensive study of restrictions involving subsequences with length at most 3. The case of forbidden increasing or decreasing subsequences is covered by the remarkable properties of Schensted's correspondence [43] and Knuth's generalization [22]. The general problem seems to be difficult.

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