

A characterization of functions over the integers computable in polynomial time using discrete differential equations *

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September 27, 2022

Abstract

This paper studies the expressive and computational power of discrete Ordinary Differential Equations (ODEs), a.k.a. (Ordinary) Difference Equations. It presents a new framework using these equations as a central tool for computation and algorithm design. We present the general theory of discrete ODEs for computation theory, we illustrate this with various examples of algorithms, and we provide several implicit characterizations of complexity and computability classes.

The proposed framework presents an original point of view on complexity and computation classes. It unifies several constructions that have been proposed for characterizing these classes including classical approaches in implicit complexity using restricted recursion schemes, as well as recent characterizations of computability and complexity by classes of continuous ordinary differential equations. It also helps understanding the relationships between analog computations and classical discrete models of computation theory.

At a more technical point of view, this paper points out the fundamental role of linear (discrete) ODEs and classical ODE tools such as changes of variables to capture computability and complexity measures, or as a tool for programming many algorithms.

Note The current article is a journal version of [4].

1 Introduction

Since the beginning of the foundations of computer science, the classification of the difficulty of problems, with various models of computation, either by their complexity or by their computability properties, is a thriving field. Nowadays, classical computer science problems also deal with continuous data coming from different areas and modeling involves the use of tools like numerical analysis, probability theory or differential equations. Thus new characterizations related to these fields have been proposed. On a dual way, the quest for new types of computers recently led to revisit the power of some models for analog machines based on differential equations, and to compare them to modern digital models. In both contexts, when discussing the related computability or complexity issues, one has to overcome the fact that today's (digital) computers are in essence discrete machines while the objects under study are continuous and naturally correspond to Ordinary Differential Equations (ODEs).

We consider here an original approach in between the two worlds: discrete based computation with difference equations.

ODEs appear to be a natural way of expressing properties and are intensively used, in particular in applied science. The theory of classical (continuous) ODEs has an abundant literature (see

*This work was funded by the ANR - project Difference

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e.g. [1, 3, 12]) and is rather well understood under many aspects. We are interested here in a discrete counterpart of classical continuous ODEs: discrete ODEs, also known as difference equations. Its associated derivative notion, called *finite differences*, has been widely studied in numerical optimization for function approximation [15] and in *discrete calculus* [17, 16, 18, 23] for combinatorial analysis (remark that similarities between discrete and continuous statements have also been historically observed, under the terminology of *umbral* or *symbolic calculus* as early as in the 19th century). However, even if the underlying computational content of finite differences theory is clear and has been pointed out many times, no fundamental connections with algorithms and complexity have been exhibited so far.

In this article, our goal is to demonstrate that discrete ODEs are a very natural tool for algorithm design and to prove that complexity and computability notions can be elegantly and simply captured using discrete ODEs. We illustrate this by providing a characterization of **FPTIME**, the class of polynomial time computable functions. To this aim, we will also demonstrate how some notions from the analog world such as linearity of differential equations or derivatives along some particular functions (i.e. changes of variables) are representative of a certain computational hardness and can be used to solve efficiently some (classical, digital) problems.

As far as we know, this is the first time that computations with discrete ODEs and their related complexity aspects are considered. By contrast, complexity results have been recently obtained about classical (continuous) ODEs for various classes of functions, mostly in the framework of computable analysis. The hardness of solving continuous ODEs has been intensively discussed: for example [22] establishes some bases of the complexity aspects of ODEs and more recent works like [21] or [13] establish links between complexity or effective aspects of such differential equations. We believe that investigating the expressive power of discrete ODE, can help to better understand complexity of computation for both the discrete and continuous settings. Indeed, on the one hand, our results offers a new machine independent perspective on classical discrete computations, i.e. computations that deal with bits, words, or integers. And, on the other hand, it relates classical (discrete) complexity classes to analog computations, i.e. computations over the reals, as analog computation have been related in various ways to continuous ordinary differential equations, and as discrete ODEs provide clear hints about their continuous counterparts. A mid-term goal of this line of research is also to bring insights from complexity theory to the problem of solving ODE (discrete and, hopefully also, numerical). A descriptive approach such as the one initiated in the paper could help classifying large classes of ODEs by their computational hardness and bring some uniformity to methods of this field.

From restricted recursion scheme to discrete differential equations Recursion schemes constitutes a major approach of computability theory and to some extent of complexity theory. A foundational result in that spirit is due to Cobham, who gave in [11] a characterization of functions computable in polynomial time through the notion of *bounded recursion on notations* (BRN, for short) (see Section 2.2 for a review). Later, notions such as safe recursion [2] or ramification ([25, 24] have allowed syntactical characterizations of polynomial time or other classes [26] that do not require the use of an explicit bound but at the expense of rather sophisticated schemes. These works have therefore been at the origin of the very vivid field of *implicit complexity* at the interplay of logic and theory of programming.

A discrete function can also be described by its (right discrete) derivative i.e. the value $\mathbf{f}(x + 1, \mathbf{y}) - \mathbf{f}(x, \mathbf{y})$. It is straightforward to rewrite primitive recursion in this setting, though one may have to cope with possibly negative values. To capture more fine grained time and space measures we come up in the proposed context of discrete ODEs with at least two original concepts which are very natural in the continuous setting:

- deriving along a function: when such a function is suitably chosen this allows to control the number of steps in the computation;
- linearity that permits to control object sizes.

By combining these two approaches, we provide a characterization of **FPTIME** that does not require to specify an explicit bound in the recursion, in contrast to Cobham's work, nor to assign a

specific role or type to variables, in contrast to safe recursion or ramification. The characterization happens to be very simple from a syntactical point of view using only natural notions from the world of ODE.

This characterization is also a first step to convince the reader that deep connections between complexity and ODE solving do exist and that these connections are worth to be further studied.

Related works on analog computations As many historical or even possibly futuristic analog machines are naturally described by (continuous) ODEs, the quest of understanding how the computational power of analog models compares to classical digital ones have led to several results relating classical complexity to various classes of continuous ODEs. In particular, a series of papers has been devoted to study various classes of the so-called \mathbb{R} -recursive functions, after their introduction in [28] as a theoretical model for computations over the reals. At the complexity level, characterizations of complexity classes such as **PTIME** and **NP** using \mathbb{R} -recursive algebra have been obtained [30], motivated in particular by the idea of transferring classical questions from complexity theory to the context of real and complex analysis [27, 30, 29]. But this has been done with the addition of limit schemata and with a rather different settings.

More recently, revisiting the model of General Purpose Analog Computer of Claude Shannon, it has been proved that polynomial differential equations can be considered as a very simple and elegant model in which computable functions over the reals and polynomial time computable functions over the reals can be defined without any reference to concepts from discrete computation theory [5, 32]. We believe the current work is a substantial step to understand the underlying power and theory of such analog models, by providing concepts, definitions and results relating the two worlds.

Refer to [6] for an up to date survey about various works on analog computations, in particular in a computation theory perspective.

How to read the paper As this article is mixing considerations coming from various fields, we tried to write it as much as possible in a self-contained manner. In particular,

- for a reader familiar with discrete differences, a.k.a. discrete ODEs, Section 3 can be mostly skipped, as it is mainly reformulating some known results. The only point with respect to classical literature is that we often write f' instead of Δf to help readers not familiar with that field to grasp the intuition of some of the statements with respect to their usual continuous counterpart. Our concept of falling exponential (Definition 3.5) is just a notation, and seems possibly non-standard, even if clearly inspired from the very standard falling power.
- A reader familiar with computability theory may mostly skip Section 2, as it is mostly only recalling some well-known notions and definition from computability theory. The only point is that we present all the various classes in some algebraic fashion, but all the presented statements are well-known in computability theory. We later observe in Section 4 that all these schemas can be seen as particular natural discrete ODEs schemas. Maybe the most original observation is about the role played by linear ODEs in that framework, and its relations with the Grzegorzczuk hierarchy.
- For a reader familiar with complexity theory, Subsection 2.2 is pointing out that not only computability classes but also complexity classes can be characterized algebraically. This might be less-known, but we are only reviewing here some basic results from so-called implicity complexity theory.

The truly original parts of this article are about the characterization of complexity classes, starting from Section 5. Another clear originality is related to our discussion on various ways to program with discrete ODEs, with several examples: In particular discussions in Sections 4.1 and 5.1.

Structure of the paper In Section 2, we review some basics of the literature of computability and complexity theory. In Section 3 we provide a short introduction to discrete difference equations and some general basics definitions and results. In Section 4 we provide by an illustration, through examples, of the programming ability of discrete ODE. We then provide formal definitions of discrete ODE schemas together with characterizations of classical computability classes. From Section 5, the focus is put on complexity theory. We introduce the notion of length-ODE which is central, together with the notion of (essentially) linear differential equation, for the characterization of **FPTIME** (Section 6). A conclusion is given in Section 7 where we also discuss some possible extensions of the results.

Note A preliminary version coauthored with Sabrina Ouazzani is available on <https://arxiv.org/abs/1810.02241>.

2 Computability and complexity: a quick introduction

2.1 Computability theory and bounded schemes

Computable functions, that is functions computable by Turing machines or equivalent models have originally been seen through the prism of classical recursion theory. In this approach, instead of words, one deals with functions over integers, that is to say with functions $\mathbf{f} : \mathbb{N}^p \rightarrow \mathbb{N}^d$ for some positive integers p, d .

It is well known that all main classes of classical recursion theory can then be characterized as closures of a set of basic functions by a finite number of basic rules (operations) to build new functions: See e.g. [33, 31, 10].

In that context, the smallest set of functions containing functions f_1, f_2, \dots, f_k that is closed under operations $op_1, op_2, \dots, op_\ell$ is often denoted as

$$[f_1, f_2, \dots, f_k; op_1, op_2, \dots, op_\ell].$$

For example, we have.

Theorem 2.1 (Total computable functions). *The set \mathcal{C} of total computable functions correspond to*

$$\mathcal{C} = [0, \pi_i^p, \mathbf{s}; \text{composition}, \text{primitive recursion}, \text{safe minimization}] :$$

That is to say, a total function over the integers is computable if and only if it belongs to the smallest set of functions that contains constant function 0 , the projection functions π_i^p , the functions successor \mathbf{s} , that is closed under composition, primitive recursion and safe minimization.

In this statement, 0 , π_i^p and \mathbf{s} are respectively the functions from $\mathbb{N} \rightarrow \mathbb{N}$, $\mathbb{N}^p \rightarrow \mathbb{N}$ and $\mathbb{N} \rightarrow \mathbb{N}$, defined as $n \mapsto 0$, $(n_1, \dots, n_p) \mapsto n_i$, and $n \mapsto n + 1$. The above statement relies on two important notions, primitive recursion and safe minimization whose definitions are recalled below.

Definition 2.2 (Primitive recursion). *Given functions $g : \mathbb{N}^p \rightarrow \mathbb{N}$ and $h : \mathbb{N}^{p+2} \rightarrow \mathbb{N}$, function $f = \text{REC}(g, h)$ defined by primitive recursion from g and h is the function $\mathbb{N}^{p+1} \rightarrow \mathbb{N}$ satisfying*

$$\begin{aligned} f(0, \mathbf{y}) &= g(\mathbf{y}) \\ f(x + 1, \mathbf{y}) &= h(f(x, \mathbf{y}), x, \mathbf{y}). \end{aligned}$$

Definition 2.3 ((Safe) Minimization). *Given function $g : \mathbb{N}^{p+1} \rightarrow \mathbb{N}$, such that for all x there exists \mathbf{y} with $g(x, \mathbf{y}) = 0$, function $f = \text{SMIN}(g)$ defined by (safe) minimization from g is the (total) function $\mathbb{N}^p \rightarrow \mathbb{N}$ satisfying $\text{SMIN}(g) : \mathbf{y} \mapsto \min\{x; g(x, \mathbf{y}) = 0\}$.*

This latter definition of safe minimization is considered here instead of classical minimization as we focus in this article only on total functions. Forgetting minimization, one obtains the following well-known class.

Definition 2.4 (Primitive recursive functions). *We denote by*

$$\mathcal{PR} = [\mathbf{0}, \pi_i^p, \mathbf{s}; \text{composition}, \text{primitive recursion}]$$

the class of primitive recursive functions: A function over the integers is primitive recursive if and only if it belongs to the smallest set of functions that contains constant function $\mathbf{0}$, the projection functions π_i^p , the functions successor \mathbf{s} , that is closed under composition and primitive recursion.

Primitive recursive functions have been stratified into various subclasses. We recall here the Grzegorzczuk hierarchy in the rest of this subsection.

Definition 2.5 (Bounded sum). *Given functions $g(\mathbf{y}) : \mathbb{N}^p \rightarrow \mathbb{N}$,*

- *function $f = \text{BSUM}(g) : \mathbb{N}^{p+1} \rightarrow \mathbb{N}$ is defined by $f : (x, \mathbf{y}) \mapsto \sum_{z \leq x} g(z, \mathbf{y})$.*
- *function $f = \text{BSUM}_{<}(g) : \mathbb{N}^{p+1} \rightarrow \mathbb{N}$ is defined by $f : (x, \mathbf{y}) \mapsto \sum_{z < x} g(z, \mathbf{y})$ for $x \neq 0$, and 0 for $x = 0$.*

Definition 2.6 (Bounded product). *Given functions $g : \mathbb{N}^p \rightarrow \mathbb{N}$,*

- *function $f = \text{BPROD}(g) : \mathbb{N}^{p+1} \rightarrow \mathbb{N}$ is defined by $f : (x, \mathbf{y}) \mapsto \prod_{z \leq x} g(z, \mathbf{y})$.*
- *function $f = \text{BPROD}_{<}(g)$ is defined by $f : (x, \mathbf{y}) \mapsto \prod_{z < x} g(z, \mathbf{y})$ for $x \neq 0$, and 1 for $x = 0$.*

We have

$$\begin{aligned} \text{BSUM}(g)(x, \mathbf{y}) &= \text{BSUM}_{<}(g)(x, \mathbf{y}) + g(x, \mathbf{y}) \\ \text{BPROD}(g)(x, \mathbf{y}) &= \text{BPROD}_{<}(g)(x, \mathbf{y}) \cdot g(x, \mathbf{y}). \end{aligned}$$

Definition 2.7 (Elementary functions). *We denote by*

$$\mathcal{E} = [\mathbf{0}, \pi_i^p, \mathbf{s}, +, \ominus; \text{composition}, \text{BSUM}, \text{BPROD}]$$

the class of elementary functions: A function over the integers is elementary if and only if it belongs to the smallest set of functions that contains constant function $\mathbf{0}$, the projection functions π_i^p , the functions successor \mathbf{s} , addition $+$, limited subtraction $\ominus : (n_1, n_2) \mapsto \max(0, n_1 - n_2)$, and that is closed under composition, bounded sum and bounded product.

The class \mathcal{E} contains many classical functions. In particular:

Lemma 2.8 ([33, Lemma 2.5, page 6]). $(x, y) \mapsto \lfloor x/y \rfloor$ is in \mathcal{E} .

Lemma 2.9 ([33]). $(x, y) \mapsto x \cdot y$ is in \mathcal{E} .

The following normal form is also well-known.

Theorem 2.10 (Normal form for computable functions [20, 33]). *Any total recursive function f can be written as $f = g(\text{SMIN}(h))$ for some elementary functions g and h .*

Consider the family of functions \mathbf{E}_n defined by induction as follows. When f is a function, $f^{[d]}$ denotes its d -th iterate: $f^{[0]}(\mathbf{x}) = x$, $f^{[d+1]}(\mathbf{x}) = f(f^{[d]}(\mathbf{x}))$:

$$\begin{aligned} \mathbf{E}_0(x) &= s(x) = x + 1, \\ \mathbf{E}_1(x, y) &= x + y, \\ \mathbf{E}_2(x, y) &= (x + 1) \cdot (y + 1), \\ \mathbf{E}_3(x) &= 2^x, \\ \mathbf{E}_{n+1}(x) &= \mathbf{E}_n^{[x]}(1) \text{ for } n \geq 3. \end{aligned}$$

Definition 2.11 (Bounded recursion). *Given functions $g(\mathbf{y}) : \mathbb{N}^p \rightarrow \mathbb{N}$, $h(f, x, \mathbf{y}) : \mathbb{N}^{p+2} \rightarrow \mathbb{N}$ and $i(x, \mathbf{y}) : \mathbb{N}^{p+1} \rightarrow \mathbb{N}$, the function $f = \text{BR}(g, h, i)$ defined by bounded recursion from g , h and i is defined as the function $\mathbb{N}^{p+1} \rightarrow \mathbb{N}$ verifying*

$$\begin{aligned} f(0, \mathbf{y}) &= g(\mathbf{y}) \\ f(x+1, \mathbf{y}) &= h(f(x, \mathbf{y}), x, \mathbf{y}) \end{aligned}$$

under the condition that:

$$f(x, \mathbf{y}) \leq i(x, \mathbf{y}).$$

Definition 2.12 (Grzegorzczak hierarchy (see [33])). *Class*

$$\mathcal{E}_0 = [\mathbf{0}, \pi_i^p, \mathbf{s}; \text{composition}, \text{BR}]$$

denotes the class that contains the constant function $\mathbf{0}$, the projection functions π_i^p , the successor function \mathbf{s} , and that is closed under composition and bounded recursion.

For every $n \geq 1$, the class

$$\mathcal{E}_n = [\mathbf{0}, \pi_i^p, \mathbf{s}, \mathbf{E}_n; \text{composition}, \text{BR}]$$

is defined similarly except that functions \max and \mathbf{E}_n are added to the list of initial functions.

Proposition 2.13 (Odi92, Campagnolo, CamMooCos02). *Let $n \geq 3$.*

$$\mathcal{E}_n = [\mathbf{0}, \pi_i^p, \mathbf{s}, +, \ominus, \mathbf{E}_n; \text{composition}, \text{BSUM}, \text{BPROD}].$$

The above proposition means that closure under bounded recursion is equivalent to using both closure under bounded sum and closure under bounded product. Indeed, as explained in chapter 1 of [33] (see Theorem 3.1 for details), bounded recursion can be expressed as a minimization of bounded sums and bounded products, itself being expressed as a bounded sum of bounded products. The following facts are known:

Proposition 2.14 ([33, 31, 10]).

$$\begin{aligned} \mathcal{E}_3 &= \mathcal{E} \subsetneq \mathcal{PR} \\ \mathcal{E}_n &\subsetneq \mathcal{E}_{n+1} \text{ for } n \geq 3 \\ \mathcal{PR} &= \bigcup_i \mathcal{E}_i \end{aligned}$$

2.2 Complexity theory and bounded schemes

We suppose the reader familiar with the basics of complexity theory (see e.g. [34]): Complexity theory is a finer theory whose aim is to discuss the resources such as time or space that are needed to compute a given function. In the context, of functions over the integers similar to the framework of previous discussion, the complexity of a function is measured in terms of the length (written in binary) of its arguments.

As usual, we denote by **PTIME** (also denoted **P** in the literature), resp. **NP**, resp. **PSPACE**, the classes of decision problems decidable in deterministic polynomial time, resp. non deterministic polynomial time, resp polynomial space, on Turing machines. We denote by **FPTIME** (or, shorter, **FP**), resp. **FPSPACE** the classes of functions, $f : \mathbb{N}^k \rightarrow \mathbb{N}$ with $k \in \mathbb{N}$, computable in polynomial time, resp. polynomial space, on deterministic Turing machines. Note that while **FPTIME** is closed by composition, it is not the case of **FPSPACE** since the size of the output can be exponentially larger than the size of the input.

It turns out that the main complexity classes have also been characterized algebraically, by restricted form of recursion scheme. A foundational result in that spirit is due to Cobham, who gave in [11] a characterization of function computable in polynomial time. The idea is to consider schemes similar to primitive recursion, but with restricting the number of recursion steps.

Let $\mathbf{0}(\cdot)$ and $\mathbf{1}(\cdot)$ be the successor functions defined by $\mathbf{0}(x) = 2.x$ and $\mathbf{1}(x) = 2.x + 1$.

Definition 2.15 (Bounded recursion on notations). *A function f is defined by bounded recursion scheme on notations from g, h_0, h_1, k , denoted by $f = \text{BRN}(g, h_0, h_1, k)$, if*

$$\begin{aligned} f(0, \mathbf{y}) &= g(\mathbf{y}) \\ f(\mathbf{0}(x), \mathbf{y}) &= h_0(f(x, \mathbf{y}), x, \mathbf{y}) \text{ for } x \neq 0 \\ f(\mathbf{1}(x), \mathbf{y}) &= h_1(f(x, \mathbf{y}), x, \mathbf{y}) \end{aligned}$$

under the condition that:

$$f(x, \mathbf{y}) \leq k(x, \mathbf{y})$$

for all x, \mathbf{y} .

Based on this scheme, Cobham proposed the following class of functions: We write $\ell(x)$ for the length (written in binary) of x .

Definition 2.16 (\mathcal{F}_p). *Define*

$$\mathcal{F}_p = [\mathbf{0}, \pi_i^p, \mathbf{0}(x), \mathbf{1}(x), \#; \text{composition}, \text{BRN}] :$$

this is the smallest class of primitive recursive functions containing $\mathbf{0}$, the projections π_i^p , the successor functions $\mathbf{0}(x) = 2.x$ and $\mathbf{1}(x) = 2.x + 1$, the function $\#$ defined by $x \# y = 2^{\ell(x) \times \ell(y)}$ and closed by composition and by bounded recursion scheme on notations.

This class turns out to be a characterization of polynomial time:

Theorem 2.17 ([11], see [9] for a proof).

$$\mathcal{F}_p = \mathbf{P} \mathbf{T} \mathbf{I} \mathbf{M} \mathbf{E}.$$

Cobham's result opened the way to various characterizations of complexity classes, or various ways to control recursion schemes. This includes the famous characterization of **P**TIME from Bellantoni and Cook in [2] and by Leivant in [24]. Refer to [9, 10] for monographies presenting a whole series of results in that spirit.

The task to capture **F**SPACE is less easy since the principle of such characterizations is to use classes of functions closed by composition. However, for functions with a reasonable output size some characterizations have been obtained. Let us denote by $\mathcal{F}_{\mathbf{P} \mathbf{S} \mathbf{P} \mathbf{A} \mathbf{C} \mathbf{E}}$, the subclass of **F**SPACE of functions of polynomial growth i.e. of functions $f : \mathbb{N}^k \rightarrow \mathbb{N}$, such that, for all $\mathbf{x} \in \mathbb{N}^k$, $\ell(f(\mathbf{x})) = O(\max_{1 \leq i \leq k} \ell(x_i))$. The following then holds:

Theorem 2.18 ([35], [10, Theorem 6.3.16]).

$$\mathcal{F}_{\mathbf{P} \mathbf{S} \mathbf{P} \mathbf{A} \mathbf{C} \mathbf{E}} = [\mathbf{0}, \pi_i^p, \mathbf{s}, \#; \text{composition}, \text{BR}].$$

That is to say, a function over the integers is in $\mathcal{F}_{\mathbf{P} \mathbf{S} \mathbf{P} \mathbf{A} \mathbf{C} \mathbf{E}}$ if and only if it belongs to the smallest set of functions that contains the constant function $\mathbf{0}$, the projection functions π_i^p , the functions successor \mathbf{s} , function $\#$ that is closed under composition and bounded recursion.

3 Discrete difference equations and discrete ODEs

From now on, on this section, we review some basic notions of discrete calculus to help intuition in the rest of the paper (refer to [19, 15] for a more complete review). Discrete derivatives, also known as discrete differences, are usually intended to concern functions over the integers of type $\mathbf{f} : \mathbb{N}^p \rightarrow \mathbb{Z}^d$, for some integers p, d , but the statements and concepts considered in our discussions are also valid more generally for functions of type $\mathbf{f} : \mathbb{Z}^p \rightarrow \mathbb{Z}^d$ or even functions $\mathbf{f} : \mathbb{R}^p \rightarrow \mathbb{R}^d$. The basic idea is to consider the following concept of derivative:

Remark 3.1. *We first discuss the case where $p = 1$, i.e. functions $\mathbf{f} : \mathbb{N} \rightarrow \mathbb{Z}^d$. We will later on consider more general functions, with partial derivatives instead of derivatives.*

Definition 3.2 (Discrete Derivative). *The discrete derivative of $\mathbf{f}(x)$ is defined as $\Delta\mathbf{f}(x) = \mathbf{f}(x+1) - \mathbf{f}(x)$. We will also write \mathbf{f}' for $\Delta\mathbf{f}(x)$ to help readers not familiar with discrete differences to understand statements with respect to their classical continuous counterparts.*

Several results from classical derivatives generalize to the settings of discrete differences: this includes linearity of derivation $(a \cdot f(x) + b \cdot g(x))' = a \cdot f'(x) + b \cdot g'(x)$, formulas for products and division such as $(f(x) \cdot g(x))' = f'(x) \cdot g(x+1) + f(x) \cdot g'(x) = f(x+1)g'(x) + f'(x)g(x)$. Notice that, however, there is no simple equivalent of the chain rule.

A fundamental concept is the following:

Definition 3.3 (Discrete Integral). *Given some function $\mathbf{f}(x)$, we write $\int_a^b \mathbf{f}(x)\delta x$ as a synonym for $\int_a^b \mathbf{f}(x)\delta x = \sum_{x=a}^{x=b-1} \mathbf{f}(x)$ with the convention that it takes value 0 when $a = b$ and $\int_a^b \mathbf{f}(x)\delta x = -\int_b^a \mathbf{f}(x)\delta x$ when $a > b$.*

The telescope formula yields the so-called Fundamental Theorem of Finite Calculus:

Theorem 3.4 (Fundamental Theorem of Finite Calculus). *Let $\mathbf{F}(x)$ be some function. Then,*

$$\int_a^b \mathbf{F}'(x)\delta x = \mathbf{F}(b) - \mathbf{F}(a).$$

As for classical functions, a given function has several primitives. These primitives are defined up to some additive constant. Several techniques from the classical settings generalize to the discrete settings: this includes the technique of integration by parts.

A classical concept in discrete calculus is the one of falling power defined as

$$x^{\underline{m}} = x \cdot (x-1) \cdot (x-2) \cdots (x-(m-1)).$$

This notion is motivated by the fact that it satisfies a derivative formula $(x^{\underline{m}})' = m \cdot x^{\underline{m-1}}$ similar to the classical one for powers in the continuous setting. In a similar spirit, we introduce the concept of falling exponential.

Definition 3.5 (Falling exponential). *Given some function $\mathbf{U}(x)$, the expression \mathbf{U} to the falling exponential x , denoted by $\bar{2}^{\mathbf{U}(x)}$, stands for*

$$\begin{aligned} \bar{2}^{\mathbf{U}(x)} &= (1 + \mathbf{U}'(x-1)) \cdots (1 + \mathbf{U}'(1)) \cdot (1 + \mathbf{U}'(0)) \\ &= \prod_{t=0}^{t=x-1} (1 + \mathbf{U}'(t)), \end{aligned}$$

with the convention that $\prod_0^0 = \prod_0^{-1} = \mathbf{id}$, where \mathbf{id} is the identity (sometimes denoted 1 hereafter)

This is motivated by the remarks that $2^x = \bar{2}^x$, and that the discrete derivative of a falling exponential is given by

$$\left(\bar{2}^{\mathbf{U}(x)}\right)' = \mathbf{U}'(x) \cdot \bar{2}^{\mathbf{U}(x)}$$

for all $x \in \mathbb{N}$.

Lemma 3.6 (Derivation of an integral with parameters). *Consider*

$$\mathbf{F}(x) = \int_{a(x)}^{b(x)} \mathbf{f}(x, t)\delta t.$$

Then

$$\mathbf{F}'(x) = \int_{a(x)}^{b(x)} \frac{\partial \mathbf{f}}{\partial x}(x, t)\delta t + \int_0^{-a'(x)} \mathbf{f}(x+1, a(x+1)+t)\delta t + \int_0^{b'(x)} \mathbf{f}(x+1, b(x)+t)\delta t.$$

In particular, when $a(x) = a$ and $b(x) = b$ are constant functions, $\mathbf{F}'(x) = \int_a^b \frac{\partial \mathbf{f}}{\partial x}(x, t)\delta t$, and when $a(x) = a$ and $b(x) = x$, $\mathbf{F}'(x) = \int_a^x \frac{\partial \mathbf{f}}{\partial x}(x, t)\delta t + \mathbf{f}(x+1, x)$.

Proof.

$$\begin{aligned}
\mathbf{F}'(x) &= \mathbf{F}(x+1) - \mathbf{F}(x) \\
&= \sum_{t=a(x+1)}^{b(x+1)-1} \mathbf{f}(x+1, t) - \sum_{t=a(x)}^{b(x)-1} \mathbf{f}(x, t) \\
&= \sum_{t=a(x)}^{b(x)-1} (\mathbf{f}(x+1, t) - \mathbf{f}(x, t)) + \sum_{t=a(x+1)}^{t=a(x)-1} \mathbf{f}(x+1, t) + \sum_{t=b(x)}^{b(x+1)-1} \mathbf{f}(x+1, t) \\
&= \sum_{t=a(x)}^{b(x)-1} \frac{\partial \mathbf{f}}{\partial x}(x, t) + \sum_{t=a(x+1)}^{t=a(x)-1} \mathbf{f}(x+1, t) + \sum_{t=b(x)}^{b(x+1)-1} \mathbf{f}(x+1, t) \\
&= \sum_{t=a(x)}^{b(x)-1} \frac{\partial \mathbf{f}}{\partial x}(x, t) + \sum_{t=0}^{t=-a(x+1)+a(x)-1} \mathbf{f}(x+1, a(x+1) + t) + \sum_{t=0}^{b(x+1)-b(x)-1} \mathbf{f}(x+1, b(x) + t).
\end{aligned}$$

□

We will focus in this article on discrete Ordinary Difference Equations (ODE) on functions with several variables, that is to say for example on equations of the (possibly vectorial) form:

$$\frac{\partial \mathbf{f}(x, \mathbf{y})}{\partial x} = \mathbf{h}(\mathbf{f}(x, \mathbf{y}), x, \mathbf{y}), \quad (1)$$

where $\frac{\partial \mathbf{f}(x, \mathbf{y})}{\partial x}$ stands as expected for the derivative of functions $\mathbf{f}(x, \mathbf{y})$ considered as a function of x , when \mathbf{y} is fixed i.e.

$$\frac{\partial \mathbf{f}(x, \mathbf{y})}{\partial x} = \mathbf{f}(x+1, \mathbf{y}) - \mathbf{f}(x, \mathbf{y}).$$

When some initial value $\mathbf{f}(0, \mathbf{y}) = \mathbf{g}(\mathbf{y})$ is added, this is called an Initial Value Problem (IVP) or a Cauchy Problem. An IVP can always be put in integral form

$$\mathbf{f}(x_0, \mathbf{y}) = \mathbf{f}(0, \mathbf{y}) + \int_0^{x_0} \mathbf{h}(\mathbf{f}(u, \mathbf{y}), u, \mathbf{y}) \delta u.$$

Our aim here is to discuss total functions whose domain and range is either of the form $\mathcal{D} = \mathbb{N}$, \mathbb{Z} , or possibly a finite product $\mathcal{D} = \mathcal{D}_1 \times \dots \times \mathcal{D}_k$ where each $\mathcal{D}_i = \mathbb{N}, \mathbb{Z}$. By considering that $\mathbb{N} \subset \mathbb{Z}$, we assume that the codomain is always \mathbb{Z}^d for some d . The concept of solution for such ODEs is as expected: given $h : \mathbb{Z}^d \times \mathbb{N} \times \mathbb{Z}^p \rightarrow \mathbb{Z}$ (or $h : \mathbb{Z}^d \times \mathbb{Z} \times \mathbb{Z}^p \rightarrow \mathbb{Z}$), a solution over \mathcal{D} is a function $f : \mathcal{D} \times \mathbb{Z}^p \rightarrow \mathbb{Z}^d$ that satisfies the equations for all x, \mathbf{y} .

We will only consider well-defined ODEs such as above in this article (but variants with partially defined functions could be considered as well). Observe that an IVP of the form (1) always admits a (necessarily unique) solution over \mathbb{N} since f can be defined inductively with

$$\left\{ \begin{array}{l} \mathbf{f}(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \\ \mathbf{f}(x+1, \mathbf{y}) = \mathbf{f}(x, \mathbf{y}) + \mathbf{h}(\mathbf{f}(x, \mathbf{y}), x, \mathbf{y}). \end{array} \right. \quad (2)$$

Remark 3.7. Notice that this is not necessarily true over \mathbb{Z} : As an example, consider $f'(x) = -f(x) + 1$, $f(0) = 0$. By definition of $f'(x)$, we must have $f(x+1) = 1$ for all x , but if $x = -1$, $f(0) = 1 \neq 0$.

Remark 3.8 (Sign function). It is very instructive to realize that the solution of the above IVP over \mathbb{N} is the sign $\text{sg}_{\mathbb{N}}(x)$ function defined by

$$\text{sg}_{\mathbb{N}}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

3.1 Discrete linear ODE

Discrete *linear* (also called *Affine*) ODEs i.e. discrete ordinary differential equations of the form $\mathbf{f}'(x) = \mathbf{A}(x) \cdot \mathbf{f}(x) + \mathbf{B}(x)$ will play an important role in what follows,

Remark 3.9. Recall that the solution of $f'(x) = a(x)f(x) + b(x)$ for classical continuous derivatives turns out to be given by (usually obtained using the method of variation of parameters):

$$f(x) = f(0)e^{\int_0^x a(t)dt} + \int_0^x b(u)e^{\int_u^x a(t)dt} du.$$

This generalizes with our definitions to discrete ODEs, and this works even vectorially as shown by the following lemma.

Lemma 3.10 (Solution of linear ODE). *For matrices \mathbf{A} and vectors \mathbf{B} and \mathbf{G} with coefficients in x and \mathbf{y} , the solution of equation $\mathbf{f}'(x, \mathbf{y}) = \mathbf{A}(x, \mathbf{y}) \cdot \mathbf{f}(x, \mathbf{y}) + \mathbf{B}(x, \mathbf{y})$ with initial conditions $\mathbf{f}(0, \mathbf{y}) = \mathbf{G}(\mathbf{y})$ is*

$$\mathbf{f}(x, \mathbf{y}) = \left(\bar{2}^{\int_0^x \mathbf{A}(t, \mathbf{y}) \delta t} \right) \cdot \mathbf{G}(\mathbf{y}) + \int_0^x \left(\bar{2}^{\int_{u+1}^x \mathbf{A}(t, \mathbf{y}) \delta t} \right) \cdot \mathbf{B}(u, \mathbf{y}) \delta u. \quad (3)$$

Remark 3.11. Notice that this can be rewritten in the familiar sum and product notation as

$$\mathbf{f}(x, \mathbf{y}) = \sum_{u=-1}^{x-1} \left(\prod_{t=u+1}^{x-1} (1 + \mathbf{A}(t, \mathbf{y})) \right) \cdot \mathbf{B}(u, \mathbf{y})$$

with the (not so usual) conventions that for any function $\kappa(\cdot)$, $\prod_x^{x-1} \kappa(x) = 1$ and $\mathbf{B}(-1, \mathbf{y}) = \mathbf{G}(\mathbf{y})$. Such equivalent expressions both have a clear computational content. They can be interpreted as an algorithm unrolling the computation of $\mathbf{f}(x+1, \mathbf{y})$ from the computation of $\mathbf{f}(x, \mathbf{y}), \mathbf{f}(x-1, \mathbf{y}), \dots, \mathbf{f}(0, \mathbf{y})$.

Actually, we can even prove a generalization of the previous statement. Matrices \mathbf{A} and \mathbf{B} may indeed be some arbitrary function of x . In particular, we observe that this work even if we assume \mathbf{A}, \mathbf{B} to be some function of \mathbf{f} and of some external function \mathbf{h} . The statement of the previous lemma is clearly a particular case of what follows.

Lemma 3.12 (Solution of linear ODE). *For matrices \mathbf{A} and vectors \mathbf{B} and \mathbf{G} , the solution of equation $\mathbf{f}'(x, \mathbf{y}) = \mathbf{A}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}) \cdot \mathbf{f}(x, \mathbf{y}) + \mathbf{B}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y})$ with initial conditions $\mathbf{f}(0, \mathbf{y}) = \mathbf{G}(\mathbf{y})$ is*

$$\begin{aligned} \mathbf{f}(x, \mathbf{y}) &= \left(\bar{2}^{\int_0^x \mathbf{A}(\mathbf{f}(t, \mathbf{y}), \mathbf{h}(t, \mathbf{y}), t, \mathbf{y}) \delta t} \right) \cdot \mathbf{G}(\mathbf{y}) \\ &+ \int_0^x \left(\bar{2}^{\int_{u+1}^x \mathbf{A}(\mathbf{f}(t, \mathbf{y}), \mathbf{h}(t, \mathbf{y}), t, \mathbf{y}) \delta t} \right) \cdot \mathbf{B}(\mathbf{f}(u, \mathbf{y}), \mathbf{h}(u, \mathbf{y}), u, \mathbf{y}) \delta u. \end{aligned}$$

Proof. Denoting the right-hand side by $\mathbf{rhs}(x, \mathbf{y})$, we have

$$\begin{aligned} \mathbf{rhs}'(x, \mathbf{y}) &= \mathbf{A}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}) \cdot \left(\bar{2}^{\int_0^x \mathbf{A}(\mathbf{f}(t, \mathbf{y}), \mathbf{h}(t, \mathbf{y}), t, \mathbf{y}) \delta t} \right) \cdot \mathbf{G}(\mathbf{y}) \\ &+ \int_0^x \left(\bar{2}^{\int_{u+1}^x \mathbf{A}(\mathbf{f}(t, \mathbf{y}), \mathbf{h}(t, \mathbf{y}), t, \mathbf{y}) \delta t} \right)' \cdot \mathbf{B}(\mathbf{f}(u, \mathbf{y}), \mathbf{h}(u, \mathbf{y}), u, \mathbf{y}) \delta u \\ &+ \left(\bar{2}^{\int_{x+1}^x \mathbf{A}(\mathbf{f}(t, \mathbf{y}), \mathbf{h}(t, \mathbf{y}), t, \mathbf{y}) \delta t} \right) \cdot \mathbf{B}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}) \\ &= \mathbf{A}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}) \cdot \left(\bar{2}^{\int_0^x \mathbf{A}(\mathbf{f}(t, \mathbf{y}), \mathbf{h}(t, \mathbf{y}), t, \mathbf{y}) \delta t} \right) \cdot \mathbf{G}(\mathbf{y}) \\ &+ \mathbf{A}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}) \cdot \int_0^x \left(\bar{2}^{\int_{u+1}^x \mathbf{A}(\mathbf{f}(t, \mathbf{y}), \mathbf{h}(t, \mathbf{y}), t, \mathbf{y}) \delta t} \right) \mathbf{B}(\mathbf{f}(u, \mathbf{y}), \mathbf{h}(u, \mathbf{y}), u, \mathbf{y}) \delta u \\ &+ \mathbf{B}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}) \\ &= \mathbf{A}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}) \cdot \mathbf{rhs}(x, \mathbf{y}) + \mathbf{B}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}) \end{aligned}$$

where we have used linearity of derivation and definition of falling exponential for the first term, and derivation of an integral (Lemma 3.6) providing the other terms to get the first equality, and then the definition of falling exponential. This proves the property by unicity of solutions of a discrete ODE, observing that $\mathbf{rhs}(0, \mathbf{y}) = \mathbf{G}(\mathbf{y})$. \square

Remark 3.13. Notice that this can still be rewritten as

$$\mathbf{f}(x, \mathbf{y}) = \sum_{u=-1}^{x-1} \left(\prod_{t=u+1}^{x-1} (1 + \mathbf{A}(\mathbf{f}(t, \mathbf{y}), \mathbf{h}(t, \mathbf{y}), t, \mathbf{y})) \right) \cdot \mathbf{B}(\mathbf{f}(u, \mathbf{y}), \mathbf{h}(u, \mathbf{y}), u, \mathbf{y}).$$

Again, this can be interpreted as an algorithm unrolling the computation of $\mathbf{f}(x+1, \mathbf{y})$ from the computation of $\mathbf{f}(x, \mathbf{y}), \mathbf{f}(x-1, \mathbf{y}), \dots, \mathbf{f}(0, \mathbf{y})$ in a dynamic programming way. The next section will build on that remark through some examples.

Note that it could be that $\mathbf{A}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}) = \mathbf{f}(x, \mathbf{y})$, i.e. we would be considering $\mathbf{f}'(x, \mathbf{y}) = \mathbf{f}(x, \mathbf{y})^2$ that would not be expected to be called a linear ODE (but rather a quadratic or polynomial one). Later, when talking about computability and complexity issues, we will forbid such possibilities, in order to really focus on linear ODEs. But at that point, as the previous statements hold in full generality, we stated them as such.

4 Computability and Discrete ODEs

We now go to the heart of the subject of the current paper, and we discuss discrete ODEs as a way to program. We will illustrate this with examples in the simpler context of computability, and we will then later go to complexity theory.

Indeed, the computational dimension of calculus of finite differences has been widely stressed in mathematical analysis. However, no fundamental connection has been established with algorithmic and complexity. In this section, we show that several algorithms can actually be naturally expressed as discrete ODEs.

4.1 Programming with discrete ODEs: First examples

Notice first that the theory of discrete ODEs also provides very natural alternative ways to compute various quantities. First observe that discrete ODEs allow to express easily search functions:

Searching with discrete ODEs As an illustration, suppose that we want to compute the minimum of a function

$$\min f : x \mapsto \min\{f(y) : 0 \leq y \leq x\}.$$

This is given by $F(x, x)$ where F is solution of the discrete ODE

$$\begin{aligned} F(0, x) &= f(0); \\ \frac{\partial F(t, x)}{\partial t} &= H(F(t, x), f(x), t, x), \end{aligned}$$

where

$$H(F, f, t, x) = \begin{cases} 0 & \text{if } F < f, \\ f - F & \text{if } F \geq f. \end{cases}$$

In integral form, we have:

$$F(x, y) = f(0) + \int_0^x H(F(t, y), t, y) \delta t.$$

Conversely such an integral above (equivalently discrete ODE) can always be considered as a (recursive) algorithm: compute the integral from its definition as a sum. On this example, this corresponds basically to compute $F(x, x)$ recursively by

$$F(t+1, x) = \text{if}(F(t, x) < f(x), F(t, x), f(x)).$$

where $\text{if}(a, b, c)$ is b when a is true, c otherwise.

Remark 4.1. Note that this algorithm is not polynomial in the length of its argument x , as it takes time $\mathcal{O}(x)$ to compute $\min f$. Getting to polynomial algorithms will be at the heart of coming discussions starting from Section 5.

The fact that discrete ODEs provide very natural alternative ways (possibly not efficient) to compute various quantities is very clear when considering numeric functions such as \tan , \sin , etc.

Computing \tan with discrete ODEs, iterative algorithms As an illustration, suppose one wants to compute $\tan(x_0)$ for say $x_0 = 72$. One way to do it is to observe that

$$\tan(x)' = \tan(1) \cdot (1 + \tan(x) \tan(x + 1)). \quad (4)$$

From fundamental theorem of finite calculus we can hence write:

$$\tan(x_0) = 0 + \int_0^{x_0} \tan'(x) \delta x \quad (5)$$

$$= 0 + \tan(1) \cdot \int_0^{x_0} (1 + \tan(x) \tan(x + 1)) \delta x. \quad (6)$$

Inspired from previous remarks, the point is that Equation (5) can be interpreted as an algorithm: it provides a way to compute $\tan(x_0)$ as an integral (or if you prefer as a sum).

Thinking about what this integral means, discrete ODE (4), also encoded by (6), can also be interpreted as

$$\tan(x + 1) - \tan(x) = \tan(1) \cdot [1 + \tan(x) \tan(x + 1)]$$

that is to say

$$\tan(x + 1) = f(\tan(x)) \quad \text{Ratnec}$$

where $f(X) = \frac{X + \tan(1)}{1 - \tan(1)X}$. Hence, this is suggesting a way to compute $\tan(72)$ by a method close to express that $\tan(x_0) = f^{[x_0]}(0)$. That is to say Equations (5) and (6) can be interpreted as providing a way to compute $\tan(72)$ using an iterative algorithm: they basically encode some recursive way of computing \tan .

Of course, a similar principle would hold for \sin , or \cos using discrete ODEs for these functions, and for many other functions starting from expression of their derivative.

Remark 4.2. Given x_0 , (even if we put aside how to deal with involved real quantities) a point is that computing $\tan(x_0)$ using this method can not be considered as polynomial time, as the (usual) convention is that time complexity is measured in term of the length of x_0 , and not on x_0 .

Again, this example is illustrative and is not claimed to be an efficient algorithm. Getting to discrete ODEs that would yield to a better complexity is at the heart of the remaining of the article starting from Section 5. In particular, by playing with change of variable so that the integral becomes computable in polynomial time.

Before getting to this efficiency issues, we first consider functions defined by discrete ODEs under the prism of computability. It serves as an introductory illustration of the close relationship between discrete ODEs and recursive schemata before moving to efficient algorithms and complexity classes characterizations with a finer approach.

4.2 Computability theory and discrete ODEs

Through this section, we will see that the class of functions that can be programmed using discrete ODEs is actually (and precisely) the whole class of computable functions. We state actually finer results, using various classes of computability theory reviewed in Section 2. This part is clearly inspired by ideas from [7, 8], but adapted here for our framework of discrete ODEs.

Definition 4.3 (Discrete ODE schemata). *Given $g : \mathbb{N}^p \rightarrow \mathbb{N}$ and $h : \mathbb{Z} \times \mathbb{N}^{p+1} \rightarrow \mathbb{Z}$, we say that f is defined by discrete ODE solving from g and h , denoted by $f = \text{ODE}(g, h)$, if $f : \mathbb{N}^{p+1} \rightarrow \mathbb{Z}$ corresponds to the (necessarily unique) solution of Initial Value Problem*

$$\begin{aligned} f(0, \mathbf{y}) &= g(\mathbf{y}), \\ \frac{\partial f(x, \mathbf{y})}{\partial x} &= h(f(x, \mathbf{y}), x, \mathbf{y}). \end{aligned} \tag{7}$$

Remark 4.4. *To be more general, we could take $g : \mathbb{N}^p \rightarrow \mathbb{Z}$. However, this would be of no use in the context of this paper. We could also consider vectorial equations as in what follows which are very natural in the context of ODEs, and we would get similar statements.*

It is clear that primitive recursion schemes can be reformulated as discrete ODE schemata.

Theorem 4.5.

$$\mathcal{PR} = \mathbb{N}^{\mathbb{N}} \cap [\mathbf{0}, \pi_i^p, \mathbf{s}; \text{composition}, \text{ODE}].$$

The restriction to Linear ODEs is very natural, in particular as this class of ODEs has a highly developed theory for the continuous setting. It is very instructive to realize that the class of functions definable by Linear ODEs is exactly the well-known class of elementary functions [20] as we will see now.

Definition 4.6 (Linear ODE schemata). *Given a vector $\mathbf{G} = (G_i)_{1 \leq i \leq k}$, matrix $\mathbf{A} = (A_{i,j})_{1 \leq i,j \leq k}$, $\mathbf{B} = (B_i)_{1 \leq i \leq k}$ whose coefficients corresponds to functions $g_i : \mathbb{N}^p \rightarrow \mathbb{N}^k$, and $a_{i,j} : \mathbb{N}^{p+1} \rightarrow \mathbb{Z}$ and $b_{i,j} : \mathbb{N}^{p+1} \rightarrow \mathbb{Z}$ respectively, we say that \mathbf{f} is obtained by linear ODE solving from G, A and B , denoted by $\mathbf{f} = \text{LI}(\mathbf{G}, \mathbf{A}, \mathbf{B})$, if $f : \mathbb{N}^{p+1} \rightarrow \mathbb{Z}^k$ corresponds to the (necessarily unique) solution of Initial Value Problem*

$$\begin{aligned} \mathbf{f}(0, \mathbf{y}) &= \mathbf{G}(\mathbf{y}) \\ \frac{\partial \mathbf{f}(x, \mathbf{y})}{\partial x} &= \mathbf{A}(x, \mathbf{y}) \cdot \mathbf{f}(x, \mathbf{y}) + \mathbf{B}(x, \mathbf{y}). \end{aligned} \tag{8}$$

Bounded sum and product, two of the very natural operations at the core of the definition of elementary computable functions, easily come as solutions of linear ODE.

Lemma 4.7 (Bounded sum and product). *Let $g : \mathbb{N}^{p+1} \rightarrow \mathbb{N}$.*

- *Function $f = \text{BSUM}_{<}(g)$ is the unique solution of initial value problem*

$$\begin{aligned} f(0, \mathbf{y}) &= 0, \\ \frac{\partial f(x, \mathbf{y})}{\partial x} &= g(x, \mathbf{y}); \end{aligned}$$

- *Function $f = \text{BPROD}_{<}(g)$ is the unique solution of initial value problem*

$$\begin{aligned} f(0, \mathbf{y}) &= 1, \\ \frac{\partial f(x, \mathbf{y})}{\partial x} &= f(x, \mathbf{y}) \cdot (g(x, \mathbf{y}) - 1). \end{aligned}$$

One key observation behind the coming characterizations is the following:

Lemma 4.8 (Elementary vs Linear ODEs). *Consider \mathbf{G}, \mathbf{A} and \mathbf{B} as in Definition 4.6. Then Let $\mathbf{f} = \text{LI}(\mathbf{G}, \mathbf{A}, \mathbf{B})$. Then \mathbf{f} is elementary when \mathbf{G}, \mathbf{A} and \mathbf{B} are.*

Proof. We do the proof in the scalar case, writing a, b, g for $\mathbf{A}, \mathbf{B}, \mathbf{G}$. The general (vectorial) case follows from similar arguments. By Lemma 3.12, it follows that:

$$\begin{aligned}
f(x, \mathbf{y}) &= \left(\prod_{t=0}^{t=x-1} (1 + a(t, \mathbf{y})) \right) \cdot g(\mathbf{y}) \\
&\quad + b(x-1, \mathbf{y}) \\
&\quad + \sum_{u=0}^{x-2} \left(\prod_{t=u+1}^{x-1} (1 + a(t, \mathbf{y})) \right) \cdot b(u, \mathbf{y}).
\end{aligned}$$

Clearly, $\prod_{t=0}^{t=x-1} (1 + a(t, \mathbf{y})) = \text{BPROD}_{<}(1 + a(t, \mathbf{y}))(x, \mathbf{y})$. Similarly, let

$$\begin{aligned}
p(u, x, \mathbf{y}) &=_{\text{def}} \prod_{t=u+1}^{x-1} (1 + a(t, \mathbf{y})) \\
&= \frac{\text{BPROD}_{<}(1 + a(t, \mathbf{y}))(x, \mathbf{y})}{\text{BPROD}_{<}(1 + a(t, \mathbf{y}))(u+1, \mathbf{y})}.
\end{aligned}$$

As the function $(x, y) \mapsto \lfloor x/y \rfloor$ is elementary from Lemma 2.8, we get that $p(u, x, \mathbf{y})$ is elementary. As multiplication is elementary, it follows that

$$\sum_{u=0}^{x-2} p(u, x, \mathbf{y}) b(u, \mathbf{y}) = \text{BSUM}_{<}(p(u, x, \mathbf{y}) b(u, \mathbf{y}))(x-2, \mathbf{y})$$

is also elementary, and f is elementary using closure by composition and multiplication. \square

The proof of this theorem easily follows.

Theorem 4.9 (A discrete ODE characterization of elementary functions).

$$\mathcal{E} = \mathbb{N}^{\mathbb{N}} \cap [\mathbf{0}, \pi_i^p, \mathbf{s}, +, -; \text{composition}, \text{LI}].$$

That is to say, the set of elementary functions \mathcal{E} is the intersection with $\mathbb{N}^{\mathbb{N}}$ of the smallest set of functions that contains the zero functions $\mathbf{0}$, the projection functions π_i^p , the successor function \mathbf{s} , addition $+$, subtraction $-$, and that is closed under composition and discrete linear ODE schemata LI.

By adding suitable towers of exponential as basis functions, the above result can be generalized to characterize the various levels of the Grzegorzcz hierarchy.

Proposition 4.10. *Let $n \geq 3$.*

$$\mathcal{E}_n = \mathbb{N}^{\mathbb{N}} \cap [\mathbf{0}, \pi_i^p, \mathbf{s}, \mathbf{E}_n; \text{composition}, \text{LI}].$$

We can also reexpress Kleene's minimization:

Theorem 4.11 (Discrete ODE computability and classical computability are equivalent). *A total function $f : \mathbb{N}^p \rightarrow \mathbb{N}$ is total recursive iff there exist some functions $h_1, h_2 : \mathbb{N}^{p+1} \rightarrow \mathbb{N}^2$ in the smallest set of functions that contains the zero functions $\mathbf{0}$, the projection functions π_i^p , the successor function \mathbf{s} , and that is closed under composition and discrete linear ODE schemata such that: for all \mathbf{y} ,*

- *there exists some $T = T(\mathbf{y})$ with $h_2(T, \mathbf{y}) = 0$;*
- *$f(\mathbf{y}) = h_1(T, \mathbf{y})$ where T is the smallest such T .*

5 Restricted recursion and integration schemes

As illustrated so far, discrete ODEs are convenient tools to define functions. From their very definition, such schemes come with an evaluation mechanism that makes their solution function computable. However, if one is interested in realistically computable functions, such as polynomial time ones, we are still lacking ODE schemes that structurally guarantee that their solution can be efficiently computed. We focus on this aspect in the rest of the paper.

5.1 Programming with discrete ODEs: Going to efficient algorithms

The previous examples discussed in Section 4.1 were not polynomial. We now want to go to efficiency issues. To do so, for now, we suppose that composition of functions, constant and the following basic functions can be used freely as functions from \mathbb{Z} to \mathbb{Z} :

- arithmetic operations: $+$, $-$, \times ;
- $\ell(x)$ returns the length of $|x|$ written in binary;
- $\text{sg}(x) : \mathbb{Z} \rightarrow \mathbb{Z}$ (respectively: $\text{sg}_{\mathbb{N}}(x) : \mathbb{N} \rightarrow \mathbb{Z}$) that takes value 1 for $x > 0$ and 0 in the other case;

From these basic functions, for readability, one may define useful functions as synonyms:

- $\bar{\text{sg}}(x)$ stands for $\bar{\text{sg}}(x) = (1 - \text{sg}(x)) \times (1 - \text{sg}(-x))$: it takes value in $\{0, 1\}$ and values 1 iff $x = 0$ for $x \in \mathbb{Z}$;
- $\bar{\text{sg}}_{\mathbb{N}}(x)$ stands for $\bar{\text{sg}}_{\mathbb{N}}(x) = 1 - \text{sg}_{\mathbb{N}}(x)$: it takes value in $\{0, 1\}$ and values 1 iff $x = 0$ for $x \in \mathbb{N}$.
- $\text{if}(x, y, z)$ stands for $\text{if}(x, y, z) = z + \bar{\text{sg}}(x) \cdot (y - z)$ and $\text{if}_{\mathbb{N}}(x, y, z)$ stands for $\text{if}_{\mathbb{N}}(x, y, z) = z + \bar{\text{sg}}_{\mathbb{N}}(x) \cdot (y - z)$: They value y when $x = 0$ and z otherwise. $\text{if}(x < x', y, z)$ will be a synonym for $\text{if}(\text{sg}(x - x' + 1), y, z)$. Similarly, $\text{if}(x \geq x', y, z)$ will be a synonym for $\text{if}(\text{sg}(x - x' + 1), z, y)$ and $\text{if}(x = x', y, z)$ will be a synonym for $\text{if}(1 - \bar{\text{sg}}(x - x'), y, z)$.

Doing a change of variable We illustrate our discussion through an example, before getting to the general theory.

Example 5.1 (Computing the integer part and divisions, going to Length-ODE). *Suppose that we want to compute*

$$\lfloor \sqrt{x} \rfloor = \max\{y \leq x : y \cdot y \leq x\}$$

and

$$\left\lfloor \frac{x}{y} \right\rfloor = \max\{z \leq x : z \cdot y \leq x\}.$$

It can be done by the following general method. Let f, h be some functions with h being non decreasing. We compute some_h with $\text{some}_h(x) = y$ s.t. $|f(x) - h(y)|$ is minimal. When $h(x) = x^2$ and $f(x) = x$, it holds that:

$$\lfloor \sqrt{x} \rfloor = \text{if}(\text{some}_h(x)^2 \leq x, \text{some}_h(x), \text{some}_h(x) - 1).$$

The relation some_h can be computed (in non-polynomial time) as a solution of an ODE similar to what we did to compute the minimum of a function.

However, there is a more efficient (polynomial time) way to do it based on what one usually does with classical ordinary differential equations: performing a change of variable so that the search becomes logarithmic in x through dichotomic search. Indeed, we can write $\text{some}_h(x) = G(\ell(x), x)$ for some function $G(t, x)$, that is a solution of

$$\begin{aligned} G(0, x) &= x; \\ \frac{\partial G(t, x)}{\partial t} &= E(G(t, x), t, x) \end{aligned}$$

where

$$E(G, t, x) = \begin{cases} 2^{\ell(x)-t-1} & \text{whenever } h(G) > f(x), \\ 0 & \text{whenever } h(G) = f(x), \\ -2^{\ell(x)-t-1} & \text{whenever } h(G) < f(x). \end{cases}$$

The example above is indeed a discrete ODE whose solution is converging fast (in polynomial time) to what we want.

Reformulating what we just did, we wrote $\text{some}_h(x) = G(\ell(x), x)$ using the solution of the above discrete ODE, i.e. the solution of $G(T, y) = x + \int_0^T E(G(t, y), t, y) \delta t$. This provides a polynomial time algorithm to solve our problems using a new parameter $t = \ell(x)$ logarithmic in x . Such techniques will be at the heart of the coming results.

Non-numeric examples Discrete ODEs turn out to be very natural in many other contexts, in particular non numerical ones, where they would probably not be expected.

Example 5.2 (Computing suffixes with discrete ODEs). *The suffix function, $\text{suffix}(x, y)$ takes as input two integers x and y and outputs the $\ell(y) = t$ least significant bits of the binary decomposition of x . We describe below a way to compute a suffix working over a parameter t , that is logarithmic in x . Consider the following unusual algorithm that can be interpreted as a fix-point definition of the function: $\text{suffix}(x, y) = F(\ell(x), y)$ where*

$$\begin{aligned} F(0, x) &= x; \\ F(t+1, x) &= \text{if}(\ell(F(t, x)) = 1, F(t, x), F(t, x) - 2^{\ell(F(t, x))-1}). \end{aligned}$$

This can be interpreted as a discrete ODE, whose solution is converging fast again (i.e. in polynomial time) to what we want. In other words, $\text{suffix}(x, y) = F(\ell(x), x)$ using the solution of the IVP:

$$\begin{aligned} F(0, x) &= x, \\ \frac{\partial F(t, x)}{\partial t} &= \text{if}(\ell(F(t, x)) = 1, 0, -2^{\ell(F(t, x))-1}). \end{aligned}$$

5.2 Derivation along a function: the concept of \mathcal{L} -ODE

In order to talk about complexity instead of simple computability, we need to add some restrictions on the integration scheme. We introduce the following variation on the notion of derivation and consider derivation along some function $\mathcal{L}(x, \mathbf{y})$.

Definition 5.3 (\mathcal{L} -ODE). *Let $\mathcal{L} : \mathbb{N}^{p+1} \rightarrow \mathbb{Z}$ and \mathbf{h} some function. We write*

$$\frac{\partial \mathbf{f}(x, \mathbf{y})}{\partial \mathcal{L}} = \frac{\partial \mathbf{f}(x, \mathbf{y})}{\partial \mathcal{L}(x, \mathbf{y})} = \mathbf{h}(\mathbf{f}(x, \mathbf{y}), x, \mathbf{y}), \quad (9)$$

as a formal synonym for

$$\mathbf{f}(x+1, \mathbf{y}) = \mathbf{f}(x, \mathbf{y}) + (\mathcal{L}(x+1, \mathbf{y}) - \mathcal{L}(x, \mathbf{y})) \cdot \mathbf{h}(\mathbf{f}(x, \mathbf{y}), x, \mathbf{y}).$$

When $\mathcal{L}(x, \mathbf{y}) = \ell(x)$, the length function, we will call this special case a length-ODE

Remark 5.4. This is motivated by the fact that the latter expression is similar to classical formula for classical continuous ODEs:

$$\frac{\delta f(x, \mathbf{y})}{\delta x} = \frac{\delta \mathcal{L}(x, \mathbf{y})}{\delta x} \cdot \frac{\delta f(x, \mathbf{y})}{\delta \mathcal{L}(x, \mathbf{y})}.$$

This will allow us to simulate suitable changes of variables using this analogy. We will talk about \mathcal{L} -IVP when some initial condition is added.

Example 5.5. It is easily seen that:

- function $f : x \mapsto 2^{\ell(x)}$ satisfies the equation $\frac{\partial f(x)}{\partial \ell} = \ell(x)' \cdot f(x)$

For this, we have basically observed the fact that

$$\begin{aligned} (2^{\ell(x)})' &= 2^{\ell(x+1)} - 2^{\ell(x)} = (2^{\ell(x)'} - 1) \cdot 2^{\ell(x)} \\ &= \ell(x)' \cdot 2^{\ell(x)} \end{aligned}$$

where in last line, we have used the fact that $2^e - 1 = e$ for $e \in \{0, 1\}$.

- function $f : x \mapsto 2^{\ell(x)^2}$ satisfies the equation $\frac{\partial f(x)}{\partial \mathcal{L}} = (\ell(x)^2)' \cdot f(x)$ considering $\mathcal{L}(x) = \ell(x)^2$.

Example 5.6. More generally $f(x, y) = 2^{\ell(x) \cdot \ell(y)}$ is the solution of the following Length-IVP:

$$\begin{aligned} f(0, y) &= 2^{\ell(y)} \\ \frac{\partial f(x, y)}{\partial \ell} &= f(x, y) \cdot (2^{\ell(y)} - 1), \end{aligned}$$

using $2^{e \cdot \ell(y)} - 1 = e \cdot (2^{\ell(y)} - 1)$ for $e \in \{0, 1\}$ and

$$\begin{aligned} \left(2^{\ell(x) \cdot \ell(y)}\right)' &= \left(2^{\ell(x)' \cdot \ell(y)} - 1\right) \cdot 2^{\ell(x) \cdot \ell(y)} \\ &= \ell(x)' \cdot \left(2^{\ell(y)} - 1\right) \cdot 2^{\ell(x) \cdot \ell(y)}. \end{aligned}$$

5.3 Computing values for solutions of \mathcal{L} -ODE

The main result of this part illustrates one key property of the \mathcal{L} -ODE scheme from a computational point of view: its dependence on the number of distinct values of function \mathcal{L} . So computing values $f(x)$ of a function f solution of some \mathcal{L} -ODE system depends on the number of distinct values taken by \mathcal{L} between 0 and x .

Definition 5.7 (*Jump $_{\mathcal{L}}$*). Let $\mathcal{L} : \mathbb{N}^{p+1} \rightarrow \mathbb{Z}$ be some function. Fixing $\mathbf{y} \in \mathbb{N}^p$, let

$$\text{Jump}_{\mathcal{L}}(x, \mathbf{y}) = \{0 \leq i \leq x - 1 \mid \mathcal{L}(i + 1, \mathbf{y}) \neq \mathcal{L}(i, \mathbf{y})\}$$

be the set of non-negative integers less than x after which the value of \mathcal{L} changes.

We also write:

- $J_{\mathcal{L}} = |\text{Jump}_{\mathcal{L}}(x, \mathbf{y})|$ for its cardinality;
- $\alpha : [0..J_{\mathcal{L}} - 1] \rightarrow \text{Jump}_{\mathcal{L}}(x, \mathbf{y})$ for an increasing function enumerating the elements of $\text{Jump}_{\mathcal{L}}(x, \mathbf{y})$: If $i_0 < i_1 < i_2 < \dots < i_{J_{\mathcal{L}}-1}$ denote all elements of $\text{Jump}_{\mathcal{L}}(x, \mathbf{y})$, then $\alpha(j) = i_j \in \text{Jump}_{\mathcal{L}}(x, \mathbf{y})$.

Technically, α should be written $\alpha_{\mathbf{y}}$, i.e. depends on \mathbf{y} . For simplicity of writing, we will write α without putting explicitly this dependency.

Lemma 5.8. Let $\mathbf{f} : \mathbb{N}^{p+1} \rightarrow \mathbb{Z}^d$ and $\mathcal{L} : \mathbb{N}^{p+1} \rightarrow \mathbb{Z}$ be some functions. Assume that (9) holds. Then $\mathbf{f}(x, \mathbf{y})$ is equal to:

$$\mathbf{f}(0, \mathbf{y}) + \int_0^{J_{\mathcal{L}}} \Delta \mathcal{L}(\alpha(u), \mathbf{y}) \cdot \mathbf{h}(\mathbf{f}(\alpha(u), \mathbf{y}), \alpha(u), \mathbf{y}) \delta u.$$

Proof. By definition, we have

$$\mathbf{f}(x+1, \mathbf{y}) = \mathbf{f}(x, \mathbf{y}) + (\mathcal{L}(x+1, \mathbf{y}) - \mathcal{L}(x, \mathbf{y})) \cdot \mathbf{h}(\mathbf{f}(x, \mathbf{y}), x, \mathbf{y}).$$

Hence,

- as soon as $i \notin \text{Jump}_{\mathcal{L}}(x, \mathbf{y})$, then $\mathbf{f}(i+1, \mathbf{y}) = \mathbf{f}(i, \mathbf{y})$, since $\mathcal{L}(i+1, \mathbf{y}) - \mathcal{L}(i, \mathbf{y}) = 0$. In other words, $\Delta \mathbf{f}(i, \mathbf{y}) = 0$.
- as soon as $i \in \text{Jump}_{\mathcal{L}}(x, \mathbf{y})$, say $i = i_j$, then

$$\Delta \mathbf{f}(i_j, \mathbf{y}) = (\mathcal{L}(i_j+1, \mathbf{y}) - \mathcal{L}(i_j, \mathbf{y})) \cdot \mathbf{h}(\mathbf{f}(i_j, \mathbf{y}), i_j, \mathbf{y})$$

$$\text{I.e. } \Delta \mathbf{f}(i_j, \mathbf{y}) = \Delta \mathcal{L}(i_j, \mathbf{y}) \cdot \mathbf{h}(\mathbf{f}(i_j, \mathbf{y}), i_j, \mathbf{y}).$$

Now

$$\begin{aligned} \mathbf{f}(x, \mathbf{y}) &= \mathbf{f}(0, \mathbf{y}) + \int_0^x \Delta \mathbf{f}(t, \mathbf{y}) \delta t \\ &= \mathbf{f}(0, \mathbf{y}) + \sum_{t=0}^{x-1} \Delta \mathbf{f}(t, \mathbf{y}) \\ &= \mathbf{f}(0, \mathbf{y}) + \sum_{i_j \in \text{Jump}_{\mathcal{L}}(x, \mathbf{y})} \Delta \mathbf{f}(i_j, \mathbf{y}) \\ &= \mathbf{f}(0, \mathbf{y}) + \sum_{i_j \in \text{Jump}_{\mathcal{L}}(x, \mathbf{y})} \Delta \mathcal{L}(i_j, \mathbf{y}) \cdot \mathbf{h}(\mathbf{f}(i_j, \mathbf{y}), i_j, \mathbf{y}) \\ &= \mathbf{f}(0, \mathbf{y}) + \sum_{j=0}^{J_{\mathcal{L}}-1} \Delta \mathcal{L}(\alpha(j), \mathbf{y}) \cdot \mathbf{h}(\mathbf{f}(\alpha(j), \mathbf{y}), \alpha(j), \mathbf{y}) \\ &= \mathbf{f}(0, \mathbf{y}) + \int_0^{J_{\mathcal{L}}} \Delta \mathcal{L}(\alpha(u), \mathbf{y}) \cdot \mathbf{h}(\mathbf{f}(\alpha(u), \mathbf{y}), \alpha(u), \mathbf{y}) \delta u \end{aligned}$$

which corresponds to the expression. \square

Remark 5.9. Note that the above result would still hold with \mathcal{L} and h taking their images in \mathbb{R} .

The above proof is based on (and illustrates) some fundamental aspect of \mathcal{L} -ODE from their definition: for fixed \mathbf{y} , the value of $\mathbf{f}(x, \mathbf{y})$ only changes when the value of $\mathcal{L}(x, \mathbf{y})$ changes. Under the previous hypotheses, there is then an alternative view to understand the integral, by using a change of variable, and by building a discrete ODE that mimics the computation of the integral.

Lemma 5.10 (Alternative view). Let $f : \mathbb{N}^{p+1} \rightarrow \mathbb{Z}^d$, $\mathcal{L} : \mathbb{N}^{p+1} \rightarrow \mathbb{Z}$ be some functions and assume that (9) holds. Then $\mathbf{f}(x, \mathbf{y})$ is given by $\mathbf{f}(x, \mathbf{y}) = \bar{\mathbf{F}}(J_{\mathcal{L}}(x, \mathbf{y}), \mathbf{y})$ where $\bar{\mathbf{F}}$ is the solution of initial value problem

$$\begin{aligned} \bar{\mathbf{F}}(0, \mathbf{y}) &= \mathbf{f}(0, \mathbf{y}), \\ \frac{\partial \bar{\mathbf{F}}(t, \mathbf{y})}{\partial t} &= \Delta \mathcal{L}(\alpha(t), \mathbf{y}) \cdot \mathbf{h}(\bar{\mathbf{F}}(t, \mathbf{y}), \alpha(t), \mathbf{y}). \end{aligned}$$

Proof. If we rewrite the previous integral as an ODE, we get that $\mathbf{f}(x, \mathbf{y}) = \bar{\mathbf{F}}(J_{\mathcal{L}}(x, \mathbf{y}), \mathbf{y})$ where $\bar{\mathbf{F}}$ is the solution of initial value problem

$$\begin{aligned}\bar{\mathbf{F}}(0, \mathbf{y}) &= \mathbf{f}(0, \mathbf{y}), \\ \frac{\partial \bar{\mathbf{F}}(t, \mathbf{y})}{\partial t} &= \Delta \mathcal{L}(\alpha(t), \mathbf{y}) \cdot \mathbf{h}(\mathbf{f}(\alpha(t), \mathbf{y}), \alpha(t), \mathbf{y}).\end{aligned}$$

But by induction, $\mathbf{f}(\alpha(t), \mathbf{y})$ can be rewritten as $\bar{\mathbf{F}}(J_{\mathcal{L}}(\alpha(t), \mathbf{y}), \mathbf{y}) = \bar{\mathbf{F}}(t, \mathbf{y})$ as $J_{\mathcal{L}}(\alpha(t), \mathbf{y}) = t$. \square

In the special case of a length ODE, that is where $\mathcal{L}(x, \mathbf{y}) = \ell(x)$, it holds that $J_{\mathcal{L}} = J_{\ell}(x, \mathbf{y}) = |\text{Jump}_{\mathcal{L}}(x, \mathbf{y})| = \ell(x) - 1$. Hence, the preceding result can also be formulated as:

Lemma 5.11 (Alternative view, case of Length ODEs). *Let $f : \mathbb{N}^{p+1} \rightarrow \mathbb{Z}^d$, $\mathcal{L} : \mathbb{N}^{p+1} \rightarrow \mathbb{Z}$ be some functions and assume that (9) holds considering $\mathcal{L}(x, \mathbf{y}) = \ell(x)$. Then $\mathbf{f}(x, \mathbf{y})$ is given by $\mathbf{f}(x, \mathbf{y}) = \mathbf{F}(\ell(x), \mathbf{y})$ where \mathbf{F} is the solution of initial value problem*

$$\begin{aligned}\mathbf{F}(1, \mathbf{y}) &= \mathbf{f}(0, \mathbf{y}), \\ \frac{\partial \mathbf{F}(t, \mathbf{y})}{\partial t} &= \mathbf{h}(\mathbf{F}(t, \mathbf{y}), 2^t - 1, \mathbf{y}).\end{aligned}$$

Proof. Consider $\mathbf{F}(t, \mathbf{y}) = \bar{\mathbf{F}}(t - 1, \mathbf{y})$, observing that $\frac{\partial \mathbf{F}(t, \mathbf{y})}{\partial t} = \mathbf{F}(t + 1, \mathbf{y}) - \mathbf{F}(t, \mathbf{y}) = \bar{\mathbf{F}}(t, \mathbf{y}) - \bar{\mathbf{F}}(t - 1, \mathbf{y}) = \frac{\partial \bar{\mathbf{F}}(t - 1, \mathbf{y})}{\partial t} = \Delta \mathcal{L}(\alpha(t - 1), \mathbf{y}) \cdot \mathbf{h}(\bar{\mathbf{F}}(t - 1, \mathbf{y}), \alpha(t - 1), \mathbf{y}) = 1 \cdot \mathbf{h}(\mathbf{F}(t, \mathbf{y}), \alpha(t - 1), \mathbf{y})$. Observing that $\alpha(t) = 2^{t+1} - 1$, the assertion follows. \square

Example 5.12 (Back to function $2^{\ell(x)}$). *To compute function $f : x \mapsto 2^{\ell(x)}$, one can also remark that $f(x) = F(\ell(x))$ where $F(t) = 2^t$ is solution of IVP: $F'(t) = F(t)$, $F(0) = 1$. We have used a change of variable $t = \ell(x)$.*

5.4 Linear \mathcal{L} -ODE

In this section we adapt the concept of linearity to \mathcal{L} -ODE and prove that the functions that are solutions of the so-called linear \mathcal{L} -ODE are intrinsically polynomial time computable when the chosen \mathcal{L} function do not change of values often, as it is the case for the length function $\ell(x)$.

Let's first consider the following length-ODE:

$$\begin{aligned}f(0) &= 2 \\ \frac{\partial f}{\partial \ell}(x) &= f(x) \cdot f(x) - f(x).\end{aligned}$$

The unique solution f of the equation is $f(x) = 2^{2^{\ell(x)}}$. This example illustrates that it is possible to derive fast growing functions by simple length-ODE. This is due to the presence of non linear terms such as $f(x)^2$ in the right-hand side of an equation. To control the growth of functions defined by ODE one possibility is to restrict the way functions that appear in equations use their argument. While doing so, one challenge is then to design a restriction that is flexible and powerful enough to permit a natural and simple description of a rich rest of functions, in particular polynomial time computable functions.

Definition 5.13. A *sg-polynomial expression* $P(x_1, \dots, x_h)$ is an expression built-on $+$, $-$, \times (often denoted \cdot) and *sg()* functions over a set of variables/terms $X = \{x_1, \dots, x_h\}$ and integer constants. The degree $\deg(x, P)$ of a term $x \in X$ in P is defined inductively as follows:

- $\deg(x, x) = 1$ and for $x' \in X \cup \mathbb{Z}$ such that $x' \neq x$, $\deg(x, x') = 0$;
- $\deg(x, P + Q) = \max\{\deg(x, P), \deg(x, Q)\}$;

- $\deg(x, P \times Q) = \deg(x, P) + \deg(x, Q)$;
- $\deg(x, \text{sg}(P)) = 0$.

A **sg-polynomial expression** P is essentially constant in x if $\deg(x, P) = 0$. It is essentially linear in x if $\deg(x, P) = 1$ i.e. if there exist **sg-polynomial expression** P_1, P_2 such that $P = Q_1 \cdot x + Q_2$ and $\deg(x, Q_1) = \deg(x, Q_2) = 0$.

A vectorial function (resp. a matrix or a vector) is said to be a **sg-polynomial expression** if all its coordinates (resp. coefficients) are. It is said to be essentially constant (resp. essentially linear) if all its coefficients are.

Compared to the classical notion of degree in a polynomial expression, here all subterms that are within the scope of a sign function contributes 0 to the degree.

Example 5.14. Let us consider the following **sg-polynomial expressions**.

- The expression $P(x, y, z) = x \cdot \text{sg}((x^2 - z) \cdot y) + y^3$ is essentially linear in x , essentially constant in z and not linear in y .
- The expression $P(x, 2^{\ell(y)}, z) = \text{sg}(x^2 - z) \cdot z^2 + 2^{\ell(y)}$ is essentially constant in x , essentially linear in $2^{\ell(y)}$ (but not essentially constant) and not essentially linear in z .
- The expression: $\text{if}(x, y, z) = z + \text{sg}(x) \cdot (y - z) = z + (1 - \text{sg}(x)) \cdot (1 - \text{sg}(-x)) \cdot (y - z)$ is essentially constant in x and linear in y and z .
- The following matrix is essentially linear in z and y and constant in x .

$$A(x, y, z) = \begin{pmatrix} \text{sg}(x - y) & \text{sg}(x) \cdot y \\ \text{sg}(z^5 - x^3) & z \end{pmatrix}$$

We are now ready to define the following main concept of ODE.

Definition 5.15 (linear \mathcal{L} -ODE). Function \mathbf{f} is linear \mathcal{L} -ODE definable (from \mathbf{u} , \mathbf{g} and \mathbf{h}) if it corresponds to the solution of \mathcal{L} -IVP

$$\begin{aligned} \mathbf{f}(0, \mathbf{y}) &= \mathbf{g}(\mathbf{y}), \\ \frac{\partial \mathbf{f}(x, \mathbf{y})}{\partial \mathcal{L}} &= \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}) \end{aligned} \tag{10}$$

where \mathbf{u} is essentially linear in $\mathbf{f}(x, \mathbf{y})$. When $\mathcal{L}(x, \mathbf{y}) = \ell(x)$, such a system is called linear length-ODE.

In other words, function \mathbf{f} is linear \mathcal{L} -ODE definable if there exist $\mathbf{A}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y})$ and $\mathbf{B}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y})$ that are **sg-polynomial expressions** essentially constant in $\mathbf{f}(x, \mathbf{y})$ such that

$$\frac{\partial \mathbf{f}(x, \mathbf{y})}{\partial \mathcal{L}} = \mathbf{A}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}) \cdot \mathbf{f}(x, \mathbf{y}) + \mathbf{B}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}).$$

In all previous reasoning, we considered that a function over the integers is polynomial time computable if it is in the length of all its arguments, as this is the usual convention. When not explicitly stated, this is our convention. As usual, we also say that some vectorial function (respectively: matrix) is polynomial time computable if all its components are. We need sometimes to consider also polynomial dependency directly in some of the variables and not on their length. This happens in the next fundamental lemma where we consider linear ODE but derivation on a variable x (and not along a function \mathcal{L}).

We use the sup norm for the length of vectors and matrices. Hence, given some matrix $\mathbf{A} = (A_{i,j})_{1 \leq i \leq n, 1 \leq j \leq m}$, we set $\ell(\mathbf{A}) = \max_{i,j} \ell(A_{i,j})$.

Lemma 5.16 (Fundamental observation). *Consider the ODE*

$$\mathbf{f}'(x, \mathbf{y}) = \mathbf{A}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}) \cdot \mathbf{f}(x, \mathbf{y}) + \mathbf{B}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}). \quad (11)$$

Assume:

1. *The initial condition $\mathbf{G}(\mathbf{y}) =_{\text{def}} \mathbf{f}(0, \mathbf{y})$, as well as $\mathbf{h}(x, \mathbf{y})$ are polynomial time computable in x and in the length of \mathbf{y} .*
2. *$\mathbf{A}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y})$ and $\mathbf{B}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y})$ are sg-polynomial expressions essentially constant in $\mathbf{f}(x, \mathbf{y})$.*

Then, there exists a polynomial p such that $\ell(\mathbf{f}(x, \mathbf{y})) \leq p(x, \ell(\mathbf{y}))$ and $\mathbf{f}(x, \mathbf{y})$ is polynomial time computable in x and the length of \mathbf{y} .

Proof. We know by Remark 3.13 following Lemma 3.12 that we must have:

$$\mathbf{f}(x, \mathbf{y}) = \sum_{u=-1}^{x-1} \left(\prod_{t=u+1}^{x-1} (1 + \mathbf{A}(\mathbf{f}(t, \mathbf{y}), \mathbf{h}(t, \mathbf{y}), t, \mathbf{y})) \right) \cdot \mathbf{B}(\mathbf{f}(u, \mathbf{y}), \mathbf{h}(u, \mathbf{y}), u, \mathbf{y}).$$

with the conventions that $\prod_x^{x-1} \kappa(x) = 1$ and $\mathbf{B}(\cdot, -1, \mathbf{y}) = \mathbf{G}(\mathbf{y})$.

This formula permits to evaluate $\mathbf{f}(x, \mathbf{y})$ using a dynamic programming approach in a number of arithmetic steps that is polynomial in x and $\ell(\mathbf{y})$. Indeed: For any $-1 \leq u \leq x$, $\mathbf{A}(\mathbf{f}(u, \mathbf{y}), \mathbf{h}(u, \mathbf{y}), u, \mathbf{y})$ and $\mathbf{B}(\mathbf{f}(u, \mathbf{y}), \mathbf{h}(u, \mathbf{y}), u, \mathbf{y})$ are matrices whose coefficients are sg-polynomial. So assuming, by induction, that each $\mathbf{f}(u, \mathbf{y})$, for $-1 \leq u < x$, can be computed in a number of steps polynomial in u and $\ell(\mathbf{y})$, so are the coefficients of $\mathbf{A}(\mathbf{f}(u, \mathbf{y}), \mathbf{h}(u, \mathbf{y}), u, \mathbf{y})$ and $\mathbf{B}(\mathbf{f}(u, \mathbf{y}), \mathbf{h}(u, \mathbf{y}), u, \mathbf{y})$ which involve finitely many arithmetic operations or sign operations from its inputs. Once this is done, computing $\mathbf{f}(x, \mathbf{y})$ requires polynomially in x many arithmetic operations: basically, once the values for \mathbf{A} and \mathbf{B} are known we have to sum up $x + 1$ terms, each of them involving at most $x - 1$ multiplications.

We need now to prove that not only the arithmetic complexity is polynomial in x and $\ell(\mathbf{y})$, but also the bit complexity. As the bit complexity of a sum, product, etc is polynomial in the size of its arguments, it is sufficient to show that the growth rate of function $\mathbf{f}(x, \mathbf{y})$ can be polynomially dominated. For this, recall that, for any $-1 \leq u \leq x$, coefficients of $\mathbf{A}(\mathbf{f}(u, \mathbf{y}), \mathbf{h}(u, \mathbf{y}), u, \mathbf{y})$ and $\mathbf{B}(\mathbf{f}(u, \mathbf{y}), \mathbf{h}(u, \mathbf{y}), u, \mathbf{y})$ are essentially constant in $\mathbf{f}(u, \mathbf{y})$. Hence, the size of these coefficients do not depend on $\ell(\mathbf{f}(u, \mathbf{y}))$. Since, in addition, \mathbf{h} is computable in polynomial time in x and $\ell(\mathbf{y})$, there exists a polynomial p_M such that:

$$\max(\ell(\mathbf{A}(\mathbf{f}(u, \mathbf{y}), \mathbf{h}(u, \mathbf{y}), u, \mathbf{y})), \ell(\mathbf{B}(\mathbf{f}(u, \mathbf{y}), \mathbf{h}(u, \mathbf{y}), u, \mathbf{y}))) \leq p_M(u, \ell(\mathbf{y})).$$

It then holds that,

$$\ell(\mathbf{f}(x + 1, \mathbf{y})) \leq p_M(x, \mathbf{y}) + \ell(\mathbf{f}(x, \mathbf{y})) + 1$$

It follows from an easy induction that we must have $\ell(\mathbf{f}(x, \mathbf{y})) \leq \ell(G(\mathbf{y})) + (x + 1) \cdot p_M(x, \ell(\mathbf{y}))$ which gives the desired bound on the length of values for function \mathbf{f} . \square

The previous statements lead to the following:

Lemma 5.17 (Intrinsic complexity of linear \mathcal{L} -ODE). *Assume that \mathbf{f} is the solution of (10) and that functions $\mathbf{u}, \mathbf{g}, \mathbf{h}, \mathcal{L}$ and elements of $\text{Jump}_{\mathcal{L}}$ are computable in polynomial time. Then, \mathbf{f} is computable in polynomial time.*

Proof. Let \mathbf{f} be a solution of the linear \mathcal{L} -ODE (10)

where \mathbf{u} is *essentially linear* in $\mathbf{f}(x, \mathbf{y})$. From Lemma 5.10, $\mathbf{f}(x, \mathbf{y})$ can also be given by $\mathbf{f}(x, \mathbf{y}) = \bar{\mathbf{F}}(J_{\mathcal{L}}(x, \mathbf{y}), \mathbf{y})$ where $\bar{\mathbf{F}}$ is the solution of initial value problem¹

$$\begin{aligned}\bar{\mathbf{F}}(0, \mathbf{y}) &= \mathbf{g}(\mathbf{y}), \\ \frac{\partial \bar{\mathbf{F}}(t, \mathbf{y})}{\partial t} &= \Delta \mathcal{L}(\alpha(t), \mathbf{y}) \cdot \mathbf{u}(\bar{\mathbf{F}}(t, \mathbf{y}), \mathbf{h}(\alpha(t), \mathbf{y}), \alpha(t), \mathbf{y}).\end{aligned}$$

Functions \mathbf{u} are *sg-polynomial* expressions that are essentially linear in $\mathbf{f}(x, \mathbf{y})$. So there exist matrices \mathbf{A}, \mathbf{B} that are essentially constants in $\mathbf{f}(t, \mathbf{y})$ such that

$$\frac{\partial \mathbf{f}(x, \mathbf{y})}{\partial \mathcal{L}} = \mathbf{A}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}) \cdot \mathbf{f}(x, \mathbf{y}) + \mathbf{B}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}).$$

In other words, it holds

$$\mathbf{F}'(t, \mathbf{y}) = \bar{\mathbf{A}}(\mathbf{F}(t, \mathbf{y}), t, \mathbf{y}) \cdot \mathbf{F}(t, \mathbf{y}) + \bar{\mathbf{B}}(\mathbf{F}(t, \mathbf{y}), t, \mathbf{y}).$$

by setting

$$\begin{aligned}\bar{\mathbf{A}}(\mathbf{F}(t, \mathbf{y}), t, \mathbf{y}) &= \Delta \mathcal{L}(\alpha(t), \mathbf{y}) \cdot \mathbf{A}(\bar{\mathbf{F}}(t, \mathbf{y}), \mathbf{h}(\alpha(t), \mathbf{y}), \alpha(t), \mathbf{y}) \\ \bar{\mathbf{B}}(\mathbf{F}(t, \mathbf{y}), t, \mathbf{y}) &= \Delta \mathcal{L}(\alpha(t), \mathbf{y}) \cdot \mathbf{B}(\bar{\mathbf{F}}(t, \mathbf{y}), \mathbf{h}(\alpha(t), \mathbf{y}), \alpha(t), \mathbf{y})\end{aligned}$$

The corresponding matrix $\bar{\mathbf{A}}$ and vector $\bar{\mathbf{B}}$ are essentially constant in $\mathbf{F}(t, \mathbf{y})$. Also, functions \mathbf{g}, \mathbf{h} are computable in polynomial time, more precisely polynomial in $\ell(x)$, hence in t , and $\ell(y)$. Function $\text{Jump}_{\mathcal{L}}$ is polynomial time computable in $\ell(x)$ and $\ell(y)$. So given t , obtaining $\alpha(t)$ is immediate. This guarantees that all hypotheses of Lemma 5.16 are true. We can then conclude remarking, again, that $t = \mathcal{L}(x)$. \square

We are now ready to state the main result of this subsection as a corollary of the previous results.

Corollary 5.18. *Assume that \mathbf{f} is the solution of a linear length-ODE from polynomial time computable functions \mathbf{u}, \mathbf{g} and \mathbf{h} . Then, \mathbf{f} is computable in polynomial time.*

6 A characterization of polynomial time

The objective of this section is to provide a characterization by ODE schemes of the function computable in polynomial time. Before proving this result, we briefly describe the computation model that will be used in proofs.

6.1 Register machines

A register machine program (a.k.a. *goto* program) is a finite sequence of ordered labeled instructions acting on a finite set of registers of one of the following type:

- increment the j th register R_j by the value of i th register R_i and go the next instruction:
 $R_j := R_j + R_i$;
- decrement the j th register R_j by the value of i th register R_i and go the next instruction:
 $R_j := R_j - R_i$;
- set the j th register R_j to integer a , for $a \in \{0, 1\}$ and go the next instruction: $R_j := a$;
- if register j is greater or equal to 0, go to instruction p else go to next instruction: if $R_j \geq 0$ goto p ;

¹In the statement of Lemma 5.10, functions \mathbf{h} can be considered as part of \mathbf{u} . They are presented separately here since \mathbf{u} will be considered as linear and \mathbf{h} plays the role of auxiliary functions that may have been computed before

- halt the program: halt

In the following, since coping with negative numbers on classical models of computation can be done through simple encodings, we will restrict ourselves to non-negative numbers.

Definition 6.1. Let $t : \mathbb{N} \rightarrow \mathbb{N}$. A function $f : \mathbb{N}^p \rightarrow \mathbb{Z}$ is computable in time t by a register machine M with k registers if:

- when starting in initial configuration with registers $R_1, \dots, R_{\min(p,k)}$ set to $x_1, \dots, x_{\min(p,k)}$ and all other registers to 0 and
- starting on the first instruction (of label 0);

machine M ends its computation after at most $t(\ell(\mathbf{x}))$ instructions where $\ell(\mathbf{x}) = \ell(x_1) + \dots + \ell(x_p)$ and with register R_0 containing $f(x_1, \dots, x_p)$. A function is computable in polynomial time by M if there exists $c \in \mathbb{N}$ such that $t(\ell(\mathbf{x})) \leq \ell(\mathbf{x})^c$ for all $\mathbf{x} = (x_1, \dots, x_p)$.

The definition of register machines might look rudimentary however, the following is easy (but tedious) to prove for any reasonable encoding of integers by Turing machines.

Theorem 6.2. A function f from $\mathbb{N}^p \rightarrow \mathbb{Z}$ is computable in polynomial time on Turing machines iff it is computable in polynomial time on register machines.

6.2 A characterization of polynomial time

The results of Section 5.4 show that functions defined by linear length-ODE from functions computable in polynomial time, are indeed polynomial time. We are now ready to prove a kind of reciprocal result. For this, we will introduce a recursion scheme based on solving linear differential equations.

Remark 6.3. Since the functions we define take their values in \mathbb{N} and have output in \mathbb{Z} , composition is an issue. Instead of considering restrictions of these functions with output in \mathbb{N} (which is always possible, even by syntactically expressible constraints), we simply admit that composition may not be defined in some cases. In other words, we consider that composition is a partial operator.

Definition 6.4 (Linear Derivation on Length ², $\mathbb{L}\mathbb{D}\mathbb{L}$). Let

$$\mathbb{L}\mathbb{D}\mathbb{L} = [\mathbf{0}, \mathbf{1}, \pi_i^p, \ell(x), +, -, \times, \mathbf{sg}(x) ; \text{composition, linear length ODE}].$$

That is to say, $\mathbb{L}\mathbb{D}\mathbb{L}$ is the smallest subset of functions, that contains $\mathbf{0}$, $\mathbf{1}$, projections π_i^p , the length function $\ell(x)$, the addition function $x+y$, the subtraction function $x-y$, the multiplication function $x \times y$ (often denoted $x \cdot y$), the sign function $\mathbf{sg}(x)$ and closed under composition (when defined) and linear length-ODE scheme.

Remark 6.5. As our proofs show, the definition of $\mathbb{L}\mathbb{D}\mathbb{L}$ would remain the same by considering closure under any kind of \mathcal{L} -ODE with \mathcal{L} satisfying the hypothesis of Lemma 5.17.

Example 6.6. A number of natural functions are in $\mathbb{L}\mathbb{D}\mathbb{L}$. Functions $2^{\ell(x)}$, $2^{\ell(x) \cdot \ell(y)}$, $\text{if}(x, y, z)$, $\text{suffix}(x, y)$, $\lfloor \sqrt{x} \rfloor$, $\lfloor \frac{x}{y} \rfloor$, $2^{\lfloor \sqrt{x} \rfloor}$ all belong to $\mathbb{L}\mathbb{D}\mathbb{L}$ by some linear length-ODE left as an exercise.

We can now state the main complexity result of this paper.

Theorem 6.7. $\mathbb{L}\mathbb{D}\mathbb{L} = \mathbf{FPTIME}$

²In the conference version of this paper ([4]), the class was called $\mathbb{D}\mathbb{L}$. It seems more appropriate to emphasize on the linearity also in the name of the class

Proof. The inclusion $\mathbf{LDL} \subseteq \mathbf{FPTIME}$ is a consequence of Corollary 5.18 and on the fact that arithmetic operations that are allowed can be computed in polynomial time and that \mathbf{FPTIME} is closed under composition of functions.

We now prove that $\mathbf{FPTIME} \subseteq \mathbf{LDL}$. Let $f : \mathbb{N}^p \rightarrow \mathbb{N}$ be computable in polynomial time and M a k registers machine that compute f in time $\ell(\mathbf{x})^c$ for some $c \in \mathbb{N}$. We first describe the computation of M by simultaneous recursion scheme on length for functions $R_0(t, \mathbf{x}), \dots, R_k(t, \mathbf{x})$ and $\text{inst}(t, \mathbf{x})$ that give, respectively, the values of each register and the label of the current instruction at time $\ell(t)$.

We start with an informal description of the characterization. Initializations of the functions are given by: $R_0(0, \mathbf{x}) = 0, R_1(0, \mathbf{x}) = x_1, \dots, R_p(0, \mathbf{x}) = x_p, R_{p+1}(0, \mathbf{x}) = \dots = R_k(0, \mathbf{x}) = 0$ et $\text{inst}(0, \mathbf{x}) = 0$. Let $m \in \mathbb{N}$ be the number of instructions of M and let $l \leq m$. Recall that, for a function f , $\frac{\partial f}{\partial L}(t, \mathbf{x})$ represents a manner to describe $f(t+1, \mathbf{x})$ from $f(t, \mathbf{x})$ when $L(t+1) = L(t)+1$. We denote by, $\text{next}_l^I, \text{next}_l^h, h \leq k$, the evolution of the instruction function and of register R_h after applying instruction l at any such instant t . They are defined as follows:

- If instruction of label l if of the type $R_j := R_j + R_i$, then:
 - $\text{next}_l^I = 1$ since $\text{inst}(t+1, \mathbf{x}) = \text{inst}(t, \mathbf{x}) + 1$
 - $\text{next}_l^j = R_i(t, \mathbf{x})$ since $R_j(t+1, \mathbf{x}) = R_j(t, \mathbf{x}) + R_i(t, \mathbf{x})$
 - $\text{next}_l^h = 0$ since $R_h(t, \mathbf{x})$ does not change for $h \neq j$.
- If instruction of label l if of the type $R_j := R_j - R_i$, then:
 - $\text{next}_l^I = 1$ since $\text{inst}(t+1, \mathbf{x}) = \text{inst}(t, \mathbf{x}) + 1$
 - $\text{next}_l^j = -R_i(t, \mathbf{x})$ since $R_j(t+1, \mathbf{x}) = R_j(t, \mathbf{x}) - R_i(t, \mathbf{x})$
 - $\text{next}_l^h = 0$ since $R_h(t, \mathbf{x})$ does not change for $h \neq j$.
- If instruction of label l if of the type $R_j := a$, for $a \in \{0, 1\}$ then:
 - $\text{next}_l^I = 1$ since $\text{inst}(t+1, \mathbf{x}) = \text{inst}(t, \mathbf{x}) + 1$
 - $\text{next}_l^j = a - R_j(t, \mathbf{x})$ since $R_j(t+1, \mathbf{x}) = a$
 - $\text{next}_l^h = 0$ since $R_h(t, \mathbf{x})$ does not change for $h \neq j$.
- If instruction of label l if of the type if $R_j \geq 0$ goto p , then:
 - $\text{next}_l^I = \text{if}(R_j(t, \mathbf{x}) \geq 0, p - \text{inst}(t, \mathbf{x}), 1)$ since, in case $R_j(t, \mathbf{x}) \geq 0$ instruction number goes from $\text{inst}(t, \mathbf{x})$ to p .
 - $\text{next}_l^h = 0$.
- If instruction of label l if of the type **Halt**, then:
 - $\text{next}_l^I = 0$ since the machine stays in the same instruction when halting
 - $\text{next}_l^h = 0$.

The definition of function inst by derivation on length is now given by (we use a more readable "by case" presentation):

$$\frac{\partial \text{inst}}{\partial \ell}(t, \mathbf{x}) = \text{case} \begin{cases} \text{inst}(t, \mathbf{x}) = 1 & \text{next}_1^I \\ \text{inst}(t, \mathbf{x}) = 2 & \text{next}_2^I \\ \vdots & \\ \text{inst}(t, \mathbf{x}) = m & \text{next}_m^I. \end{cases}$$

Expanded as an arithmetic expression, this give:

$$\frac{\partial \text{inst}}{\partial \ell}(t, \mathbf{x}) = \sum_{l=0}^m \left(\prod_{i=0}^{l-1} \text{sg}(\text{inst}(t, \mathbf{x}) - i) \right) \cdot \bar{\text{sg}}(\text{inst}(t, \mathbf{x}) - l) \cdot \text{next}_l^I.$$

Note that each next_l^I is an expression in terms of $\text{inst}(t, \mathbf{x})$ and, in some cases, in $\text{sg}(R_j(t, \mathbf{x}))$, too (for a conditional statement). Similarly, for each $j \leq k$:

$$\frac{\partial R_j}{\partial \ell}(t, \mathbf{x}) = \sum_{l=0}^m \left(\prod_{i=0}^{l-1} \text{sg}(\text{inst}(t, \mathbf{x}) - i) \right) \cdot \bar{\text{sg}}(\text{inst}(t, \mathbf{x}) - l) \cdot \text{next}_l^j.$$

It is easily seen that, in each of these expressions above, there is at most one occurrence of $\text{inst}(t, \mathbf{x})$ and $R_j(t, \mathbf{x})$ that is not under the scope of an essentially constant function (i.e. the sign function). Hence, the expressions are of the prescribed form i.e. linear.

We know M works in time $\ell(\mathbf{x})^c$ for some fixed $c \in \mathbb{N}$. Both functions $\ell(\mathbf{x}) = \ell(x_1) + \dots + \ell(x_p)$ and $B(\mathbf{x}) = 2^{\ell(\mathbf{x}) \cdot \ell(\mathbf{x})}$ are in \mathbb{LDL} . It is easily seen that $\ell(\mathbf{x})^c \leq B^{(c)}(\ell(\mathbf{x}))$ where $B^{(c)}$ is the c -fold composition of function B . We can conclude by setting $f(\mathbf{x}) = R_0(B^{(c)}(\max(\mathbf{x})), \mathbf{x})$. \square

Definition 6.8 (Normal linear \mathcal{L} -ODE (N \mathcal{L} -ODE)). *Function \mathbf{f} is definable by a normal linear \mathcal{L} -ODE if it corresponds to the solution of \mathcal{L} -IVP*

$$\begin{aligned} \mathbf{f}(0, \mathbf{y}) &= \mathbf{v}(\mathbf{y}) \\ \frac{\partial \mathbf{f}(x, \mathbf{y})}{\partial \mathcal{L}} &= \mathbf{u}(\mathbf{f}(x, \mathbf{y}), x, \mathbf{y}), \end{aligned}$$

where \mathbf{u} is essentially linear in $\mathbf{f}(x, \mathbf{y})$ and \mathbf{v} is either the identity, a projection or a constant function.

From the proof of Theorem 6.7 the result below can be easily obtained. It expresses that composition needs to be used only once as exemplified in the above definition.

Theorem 6.9 (Alternative characterization of **FPTIME**). *A function $\mathbf{f} : \mathbb{N}^p \rightarrow \mathbb{Z}$ is in **FPTIME** iff $\mathbf{f}(\mathbf{y}) = \mathbf{g}(\ell(\mathbf{y})^c, \mathbf{y})$ for some integer c and some $\mathbf{g} : \mathbb{N}^{p+1} \rightarrow \mathbb{Z}^k$ solution of a normal linear length-ODE*

$$\frac{\partial \mathbf{g}(x, \mathbf{y})}{\partial \ell(x)} = \mathbf{u}(\mathbf{g}(x, \mathbf{y}), x, \mathbf{y}).$$

Remark 6.10. *From similar arguments, **FPTIME** also corresponds to the class defined as \mathbb{LDL} but where linear length-ODE scheme is replaced by normal linear length-ODE scheme: this forbids a function already defined with some ODE scheme to be used into some other ODE scheme.*

7 Discussions and further works

Our aim in this article was to give the basis of a presentation of complexity theory based on discrete Ordinary Differential Equations and their basic properties. We demonstrated the particular role played by affine ordinary differential equations in complexity theory, as well as the concept of derivation along some particular function (i.e. change of variable) to guarantee a low complexity.

Previous ideas used here for **FPTIME** opens the way to provide a characterization of other complexity classes: This includes the possibility of characterizing non-deterministic polynomial time, using an existential quantification, or the class $\mathbf{P}_{[0,1]}$ of functions computable in polynomial time over the reals in the sense of computable analysis, or more general classes of classical discrete complexity theory such as **FSPACE**. For the clarity of the current exposition, as this would require to introduce other types of schemata of ODEs, we leave this characterization (and improvement of the corresponding schemata to “the most natural and powerful form”) for future work, but we believe the current article basically provides the key types of arguments to conceive that this is indeed possible.

More generally, it is also very instructive to revisit classical algorithmic under this viewpoint, and for example one may realize that even inside class **PTIME**, the Master Theorem (see e.g. [14, Theorem 4.1] for a formal statement) can be basically read as a result on (the growth of) a particular class of discrete time length ODEs. Several recursive algorithms can then be reexpressed as particular discrete ODEs of specific type.

Acknowledgements

We would like to thank Sabrina Ouazzani for many scientific discussions about the results in this article.

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