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Stability of Spatially Homogeneous Periodic Solutions of Reaction-Diffusion Equations

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When a certain condition is satisfied, a reaction-diffusion equation has a spatially homogeneous periodic solution, i.e. a temporally periodic solution that does not depend on spatial variables. We analyse the orbital stability of this periodic solution. A sufficient condition is given for the homogeneity breaking instability, which is stated in terms of the manner of dependency of its temporal period on a certain parameter of the system.

1. INTRODUCTION

In this paper we consider the following partial differential equation with a temporal variable t and n spatial variables $\mathbf{x} = (x_1, ..., x_n)$.

$$\frac{\partial}{\partial t}\mathbf{u} = D\Delta\mathbf{u} + \mathbf{f}(\mathbf{u}), \tag{1}$$

$$-\infty < x_i < \infty, \qquad i = 1, ..., n.$$

Here

$$D = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_m \end{pmatrix}, \qquad \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

$$\mathbf{u} = (u_1, ..., u_m)',$$

$$\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), ..., f_m(\mathbf{u}))'.$$

The $'$ denotes the transposition of a vector. It is assumed that $f_i(\mathbf{u})$ are smooth functions of $\mathbf{u}$ and σ_i are nonnegative constants such that $\sigma_1 + \cdots + \sigma_m > 0$. Equations of this type, called the reaction-diffusion equations, appear in various fields in chemistry and biology ([3], [6]). For example a spatially distributed chemical system is described by an equation of the form (1), in which $\mathbf{u}(\mathbf{x}, t)$ corresponds to the spatial distribution of chemical components and $f_i(\mathbf{u})$ and σ_i correspond to the reaction rate and diffusibility of each component respectively.

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One can observe a spatially synchronized oscillation of chemical components in a chemical reaction-diffusion system called the Zhabotinski system [7]. Namely this reaction-diffusion system has a spatially homogeneous periodic solution $\mathbf{u} = \boldsymbol{\phi}(t)$, i.e. a temporally periodic solution which does not depend on $\mathbf{x}$. Clearly $\boldsymbol{\phi}(t)$ is given as a periodic solution of the ordinary differential equation

$$\frac{d}{dt}\,\bar{\mathbf{u}} = \mathbf{f}(\bar{\mathbf{u}}). \tag{2}$$

The solution $\mathbf{u} = \boldsymbol{\phi}(t)$ of Eq. (1) is not always stable even if $\bar{\mathbf{u}} = \boldsymbol{\phi}(t)$ is a stable periodic solution of Eq. (2). It is shown by a numerical analysis of the Zhabotinski system that $\mathbf{u} = \boldsymbol{\phi}(t)$ becomes unstable when the values of σ_i are chosen suitably. If an appropriate small disturbance which is spatially inhomogeneous is added to the spatially homogeneous periodic solution $\mathbf{u} = \boldsymbol{\phi}(t)$, this inhomogeneity grows and the solution of Eq. (1) begins to deviate from $\mathbf{u} = \boldsymbol{\phi}(t)$. The instability of this type is called the homogeneity breaking instability. In this paper we aim to obtain a sufficient condition for the occurrence of this instability.

In the following, we assume that the ordinary differential equation (2) has a periodic solution $\bar{\mathbf{u}} = \boldsymbol{\phi}(t)$, and we denote by T its minimum period, i.e.

$$\frac{d}{dt}\,\boldsymbol{\phi}(t) = \mathbf{f}(\boldsymbol{\phi}(t)), \qquad \boldsymbol{\phi}(t) = \boldsymbol{\phi}(t+T). \tag{3}$$

We further assume that $\bar{\mathbf{u}} = \boldsymbol{\phi}(t)$ is a stable periodic solution of Eq. (2). It is easily proved that the spatially homogeneous periodic solution $\mathbf{u} = \boldsymbol{\phi}(t)$ of Eq. (1) is always unstable if $\bar{\mathbf{u}} = \boldsymbol{\phi}(t)$ is an unstable periodic solution of Eq. (2).

2. Main Result

Let us consider a periodic solution of the following ordinary differential equation.

$$(E + \nu D)\,\frac{d}{dt}\,\mathbf{u} = \mathbf{f}(\mathbf{u}). \tag{4}$$

Here E is an $m \times m$ unit matrix and ν is a real parameter. It follows from (3) that Eq. (4) in the case of $\nu = 0$ has a periodic solution $\mathbf{u} = \boldsymbol{\phi}(t)$. Since this periodic solution is orbitally stable, $\nu = 0$ is not a bifurcation point of the parameter ν (see [2]). Namely, if $|\nu|$ is sufficiently small, Eq. (4) has a periodic solution which smoothly depends on ν and approaches $\mathbf{u} = \boldsymbol{\phi}(t)$ as $\nu \to 0$. Let $\boldsymbol{\psi}(t, \nu)$ and $L(\nu)$ denote this periodic solution and its minimum period respectively. In order to fix the phase of the periodic solution, we assume the additional condition

$$\frac{\partial}{\partial t}\psi_1(0, \nu) = 0,$$

where $\psi_1(t, \nu)$ is the first component of $\boldsymbol{\psi}(t, \nu)$. The vector function $\boldsymbol{\psi}(t, \nu)$ satisfies the following equalities.

$$(E + \nu D)\frac{\partial}{\partial t}\boldsymbol{\psi}(t, \nu) = \mathbf{f}(\boldsymbol{\psi}(t, \nu)),$$
$$\boldsymbol{\psi}(t, \nu) = \boldsymbol{\psi}(t + L(\nu), \nu), \tag{5}$$

$$\boldsymbol{\psi}(t, 0) = \boldsymbol{\phi}(t), \qquad L(0) = T. \tag{6}$$

The main result of this paper is given as follows.

THEOREM. *Assume that* $\bar{\mathbf{u}} = \boldsymbol{\phi}(t)$ *is a stable periodic solution of Eq.* (2). *The spatially homogeneous periodic solution* $\mathbf{u} = \boldsymbol{\phi}(t)$ *of Eq.* (1) *is unstable and the homogeneity breaking instability takes place if* $L'(0) < 0$.

This theorem is obtained as a result of two lemmas that are formulated in the next section.

3. Stability Analysis

Let $\bar{\mathbf{u}}(t)$ be a solution of Eq. (2) slightly deviating from the periodic solution $\boldsymbol{\phi}(t)$. The deviation $\bar{\mathbf{v}}(t) \equiv \bar{\mathbf{u}}(t) - \boldsymbol{\phi}(t)$ satisfies the following linearized perturbation equation if higher order terms are neglected.

$$\frac{d}{dt}\bar{\mathbf{v}}(t) = \frac{\partial \mathbf{f}(\boldsymbol{\phi}(t))}{\partial \mathbf{u}}\bar{\mathbf{v}}(t), \tag{7}$$

where $\partial \mathbf{f}(\boldsymbol{\phi})/\partial \mathbf{u}$ is an $m \times m$ matrix given by $[\partial \mathbf{f}(\boldsymbol{\phi})/\partial \mathbf{u}]_{ij} = \partial f_i(\boldsymbol{\phi})/\partial u_j$. Let $\Psi(t)$ denote the fundamental solution of Eq. (7), i.e. an $m \times m$ matrix which satisfies $(d/dt)\Psi(t) = [\partial \mathbf{f}(\boldsymbol{\phi}(t))/\partial \mathbf{u}]\Psi(t)$ and $\Psi(0) = E$. By the use of this matrix, a solution $\bar{\mathbf{v}}(t)$ with initial data $\bar{\mathbf{v}}(0)$ is given by $\bar{\mathbf{v}}(t) = \Psi(t)\bar{\mathbf{v}}(0)$.

Let λ_i and $\bar{\mathbf{v}}_i(0)$, $i = 1,..., m$, denote the eigenvalues and eigenvectors of the matrix $\Psi(T)$ respectively, i.e.

$$\lambda_i\bar{\mathbf{v}}_i(0) = \Psi(T)\bar{\mathbf{v}}_i(0) = \bar{\mathbf{v}}_i(T), \qquad i = 1,..., m, \tag{8}$$

where $\bar{\mathbf{v}}_i(t)$ are vector functions given by $\bar{\mathbf{v}}_i(t) = \Psi(t)\bar{\mathbf{v}}_i(0)$. The eigenvalues λ_i

are called the Floquet multipliers of the periodic solution $\phi(t)$. The following equalities are obtained by differentiating (3) with respect to t.

$$\frac{d}{dt}\frac{d\phi}{dt} = \frac{\partial \mathbf{f}(\phi(t))}{\partial \mathbf{u}} \frac{d\phi}{dt},$$

$$\frac{d\phi}{dt}(0) = \frac{d\phi}{dt}(T).$$

Hence, without losing any generality, we may assume that

$$\bar{\mathbf{v}}_1(t) = \frac{d\phi}{dt}(t), \qquad \lambda_1 = 1. \tag{9}$$

Since the periodic solution $\phi(t)$ is assumed to be stable, the other Floquet multipliers must satisfy

$$|\lambda_i| < 1, \qquad i = 2,..., m,$$

(see [1]).

Next let us consider a solution $\mathbf{u}(x, t)$ of Eq. (1) slightly deviating from the spatially homogeneous periodic solution $\mathbf{u} = \phi(t)$. The deviation $\mathbf{v}(x, t) \equiv \mathbf{u}(x, t) - \phi(t)$ satisfies the following linearized perturbation equation if higher order terms are neglected.

$$\frac{\partial}{\partial t}\mathbf{v} = D\Delta\mathbf{v} + \frac{\partial \mathbf{f}(\phi(t))}{\partial \mathbf{u}}\mathbf{v}, \tag{10}$$

$$-\infty < x_i < \infty, \qquad = 1,..., n.$$

Let us consider a solution of the form $\mathbf{v}(x, t) = \mathbf{a}(t)\exp(i\langle\boldsymbol{\mu}, \mathbf{x}\rangle)$, where $\boldsymbol{\mu}$ is a real n-vector which corresponds to a spatial frequency and $\langle\boldsymbol{\mu}, \mathbf{x}\rangle = \mu_1 x_1 + \cdots + \mu_n x_n$. It is easily verified that $\mathbf{a}(t)$ satisfies the ordinary differential equation

$$\frac{d}{dt}\mathbf{a}(t) = \left\{-\alpha D + \frac{\partial \mathbf{f}(\phi(t))}{\partial \mathbf{u}}\right\}\mathbf{a}(t), \tag{11}$$

where $\alpha = \|\boldsymbol{\mu}\|^2 \equiv \langle\boldsymbol{\mu}, \boldsymbol{\mu}\rangle$.

Let $\Phi(t, \alpha)$ be the fundamental solution of Eq. (11) and $\omega_i(\alpha)$ and $\mathbf{a}_i(0, \alpha)$ be the eigenvalues and the eigenvectors of the matrix $\Phi(T, \alpha)$ respectively, i.e.

$$\omega_i(\alpha)\,\mathbf{a}_i(0, \alpha) = \Phi(T, \alpha)\,\mathbf{a}_i(0, \alpha) = \mathbf{a}_i(T, \alpha), \tag{12}$$

where $\mathbf{a}_i(t, \alpha)$, $i = 1,..., m$, are vector functions given by $\mathbf{a}_i(t, \alpha) = \Phi(t, \alpha)\,\mathbf{a}_i(0, \alpha)$.

Since the matrix $\Phi(t, \alpha)$ satisfies $\Phi(t, 0) = \Psi(t)$, we may assume without losing generality that

$$\mathbf{a}_1(t, 0) = \bar{\mathbf{v}}_1(t) \equiv \frac{d\boldsymbol{\phi}}{dt}(t), \qquad \omega_1(0) = \lambda_1 \equiv 1. \tag{13}$$

Moreover since $\Phi(T, \alpha)$ depends smoothly on the parameter α and $\lambda_1 = 1$ is a simple eigenvalue of $\Psi(T)$, we may assume that $\omega_1(\alpha)$ and $\mathbf{a}_1(t, \alpha)$ are smooth functions of α if $|\alpha|$ is sufficiently small.

If there exists a vector $\boldsymbol{\mu}_0 \neq 0$ such that an eigenvalue $\omega_i(\alpha_0)$ of $\Phi(T, \alpha_0)$, $\alpha_0 \equiv \|\boldsymbol{\mu}_0\|^2 > 0$, satisfies $|\omega_i(\alpha_0)| > 1$, the absolute value of $\mathbf{a}_i(kT, \alpha_0)$ becomes large as the integer k increases. Namely the deviation $\mathbf{v}(x, t)$ with the spatial frequence $\boldsymbol{\mu}_0$ becomes large in the course of time. Hence, taking into account of (13), we obtain a sufficient condition for the instability of $\mathbf{u} = \boldsymbol{\phi}(t)$ as follows.

LEMMA 1. *The spatially homogeneous periodic solution $\mathbf{u} = \boldsymbol{\phi}(t)$ is unstable and the homogeneity breaking instability takes place if $\omega_1'(0) > 0$.*

The vector function $\mathbf{a}_1(t, \alpha)$ satisfies the equation

$$\frac{\partial}{\partial t}\mathbf{a}_1(t, \alpha) = \left\{-\alpha D + \frac{\partial \mathbf{f}(\boldsymbol{\phi}(t))}{\partial \mathbf{u}}\right\} \mathbf{a}_1(t, \alpha).$$

Since $\mathbf{a}_1(t, \alpha)$ depends smoothly on α when $|\alpha|$ is sufficiently small, we can define the partial derivative $\mathbf{a}_{1\alpha}(t, \alpha) \equiv \partial \mathbf{a}_1(t, \alpha)/\partial \alpha$ at $\alpha = 0$ on the finite interval $[0, T]$ of the variable t. By differentiating the above equality with respect to α, we obtain the equality

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{a}_{1\alpha}(t, 0) &= \frac{\partial \mathbf{f}(\boldsymbol{\phi}(t))}{\partial \mathbf{u}} \mathbf{a}_{1\alpha}(t, 0) - D\mathbf{a}_1(t, 0) \\ &= \frac{\partial \mathbf{f}(\boldsymbol{\phi}(t))}{\partial \mathbf{u}} \mathbf{a}_{1\alpha}(t, 0) - D\frac{d\boldsymbol{\phi}}{dt}(t). \end{aligned} \tag{14}$$

Similarly, by differentiating the equality (12) for $i = 1$ with respect to α and setting $\alpha = 0$, we obtain

$$\begin{aligned} \mathbf{a}_{1\alpha}(T, 0) &= \omega_1(0)\, \mathbf{a}_{1\alpha}(0, 0) + \omega_1'(0)\, \mathbf{a}_1(0, 0) \\ &= \mathbf{a}_{1\alpha}(0, 0) + \omega_1'(0) \frac{d\boldsymbol{\phi}}{dt}(0). \end{aligned} \tag{15}$$

Next let us consider the function $\boldsymbol{\psi}(t, \nu)$. Since this is a smooth function of ν when $|\nu|$ is sufficiently small, we can define the partial derivative $\boldsymbol{\psi}_\nu(t, \nu) \equiv$

$\partial\psi(t, \nu)/\partial\nu$ at $\nu = 0$ on the interval $[0, T]$. By differentiating (5) with respect to ν and setting $\nu = 0$, we obtain

$$\frac{\partial}{\partial t}\psi_\nu(t, 0) + D\frac{\partial}{\partial t}\psi(t, 0) = \frac{\partial \mathbf{f}(\psi(t, 0))}{\partial \mathbf{u}}\psi_\nu(t, 0),$$

$$\psi_\nu(0, 0) = \psi_\nu(L(0), 0) + \frac{\partial}{\partial t}\psi(L(0), 0)\, L'(0).$$

By the use of (6), these are rewritten as follows.

$$\frac{\partial}{\partial t}\psi_\nu(t, 0) = \frac{\partial \mathbf{f}(\phi(t))}{\partial \mathbf{u}}\psi_\nu(t, 0) - D\frac{d\phi}{dt}(t) \tag{16}$$

$$\psi_\nu(T, 0) = \psi_\nu(0, 0) - L'(0)\frac{d\phi}{dt}(0). \tag{17}$$

It follows from (16) that $\psi_\nu(t, 0)$ is a particular solution of Eq. (14). Hence $\mathbf{a}_{1\alpha}(t, 0)$ must be written as

$$\mathbf{a}_{1\alpha}(t, 0) = \psi_\nu(t, 0) + \sum_{i=1}^{m} \beta_i \bar{\mathbf{v}}_i(t),$$

where β_1 is an appropriate constant and $\beta_i(t)$, $i = 2, \ldots, m$, and appropriate constants or polynomials of t. Substituting this in (15) we obtain

$$\psi_\nu(T, 0) + \sum_{i=1}^{m} \beta_i(T)\,\bar{\mathbf{v}}_i(T) = \psi_\nu(0, 0) + \sum_{i=1}^{m} \beta_i(0)\,\bar{\mathbf{v}}_i(0) + \omega_1'(0)\frac{d\phi}{dt}(0).$$

By the use of (8), (9) and (17), this is rewritten as

$$\{L'(0) + \omega_1(0)\}\frac{d\phi}{dt}(0) = \sum_{i=2}^{m} \{\lambda_i\beta_i(T) - \beta_i(0)\}\,\bar{\mathbf{v}}_i(0).$$

Since the vector $\bar{\mathbf{v}}_1(0) \equiv (d\phi/dt)(0)$, $\bar{\mathbf{v}}_2(0), \ldots, \bar{\mathbf{v}}_m(0)$ are linearly independent eigenvectors of the matrix $\Psi(T)$, it follows that

$$L'(0) + \omega_1'(0) = 0.$$

Hence the next lemma holds.

LEMMA 2. *The inequality* $\omega_1'(0) > 0$ *holds when* $L(\nu)$ *satisfies* $L'(0) < 0$.

The theorem in Section 2 is an immediate consequence of Lemma 1 and Lemma 2.

4. Example

Let us apply Theorem to the following second order system.

$$
\begin{aligned}
C_1 \frac{\partial}{\partial t} u &= \sigma_1 \Delta u + f(u) - w, \\
C_2 \frac{\partial}{\partial t} w &= \sigma_2 \Delta w + bu - kw,
\end{aligned}
\tag{18}
$$

$$-\infty < x_i < \infty, \qquad i = 1, \ldots, n.$$

This system was introduced in [5] as a simplified mathematical description of the spontaneous spatial pattern formation (morphogenesis) in spatially distributed biological and chemical system [6]. We assume that $f(u)$ is a smooth function of u that satisfies

$$
\begin{aligned}
f(u) &= -f(-u), \\
f(0) &= 0, \qquad m \equiv f'(0) > 0, \\
f'(u) &\leqslant 0 \qquad \text{for} \quad |u| \geqslant a > 0, \\
uf''(u) &< 0 \qquad \text{for} \quad u \neq 0,
\end{aligned}
$$

and C_i, σ_i, b and k are positive constants that satisfy

$$mk < b < 2mk.$$

This equation has a spatially homogeneous stationary solution $(u, w) = (0, 0)$ because $(\bar{u}, \bar{w}) = (0, 0)$ is a stationary solution of the ordinary differential equation

$$
\begin{aligned}
C_1 \frac{d}{dt} \bar{u} &= f(\bar{u}) - \bar{w}, \\
C_2 \frac{d}{dt} \bar{w} &= b\bar{u} - k\bar{w}.
\end{aligned}
\tag{19}
$$

If $C_2/C_1 > k/m$, the solution $(\bar{u}, \bar{w}) = (0, 0)$ is unstable and Eq. (19) has a stable periodic solution $(\bar{u}, \bar{w}) = (\phi_1(t), \phi_2(t))$ (see Chap. 11 of [4]). On the other hand, if $C_2/C_1 < k/m$, $(\bar{u}, \bar{w}) = (0, 0)$ is asymptotically stable and Eq. (19) has no non-trivial periodic solutions.

First let us consider the stability of the homogeneous stationary solution $(u, w) = (0, 0)$ of Eq. (18). This solution is not always stable even if $C_2/C_1 < k/m$. It is proved in [5] that, if σ_1 and σ_2 satisfy

$$
\begin{aligned}
m\sigma_2 - k\sigma_1 &> 0, \\
(m\sigma_2 + k\sigma_1)^2 - 4b\sigma_1\sigma_2 &> 0,
\end{aligned}
\tag{20}
$$

and if an appropriate small disturbance is added to $(u, w) = (0, 0)$, there appears spatial inhomogeneity with a certain periodic spatial structure in the solution of Eq. (18), and this inhomogeneous solution begins to deviate from $(u, w) = (0, 0)$. Namely, a sort of homogeneity breaking instability takes place in this case. The condition (20) is satisfied when $\sigma_2 \gg \sigma_1$.

Next let us consider the case $C_2/C_1 > k/m$ and study the stability of the spatially homogeneous periodic solution $(u, w) = (\phi_1(t), \phi_2(t))$ of Eq. (18). It is conjectured from the above consideration that this spatially homogeneous solution is unstable if σ_1 and σ_2 are chosen so that $\sigma_2 \gg \sigma_1$ and if the value of C_2/C_1 is chosen suitably.

The minimum period $L(C_1, C_2)$ of this periodic solution is obtained as follows by the use of Poincaré's method in the theory of nonlinear oscillations (see [8]).

$$L(C_1, C_2) \doteqdot L_1(C_1, C_2) = 2\pi \Big/ \left(\frac{b}{C_1 C_2} - \left(\frac{k}{C_1} \right)^2 \right)^{1/2}.$$

Here $L_1(C_1, C_2)$ denotes the first approximation of the period. (The inequality $b/C_1C_2 - (k/C_2)^2 > 0$ follows from the assumptions $b > mk$ and $C_2/C_1 > k/m$.) It is shown that $L_1(C_1, C_2)$ is sufficiently close to $L(C_1, C_2)$ if the value of C_2/C_1 ($>k/m$) is chosen so that $C_2/C_1 \doteqdot k/m$. On the other hand, $L_1(C_1, C_2)$ satisfies

$$\frac{\partial}{\partial C_2} L_1(C_1, C_2) < 0$$

if C_2/C_1 is in the range $(k/m, 2k^2/b)$. (The inequality $2k^2/b > k/m$ follows from the assumption $2mk > b$.) Hence, if σ_2/σ_1 is sufficiently large and if C_2/C_1 is chosen suitably, the period $L(C_1, C_2)$ satisfies

$$\begin{aligned} &\frac{d}{d\nu} L(C_1 + \sigma_1 \nu, C_2 + \sigma_2 \nu) \big|_{\nu=0} \\ &\qquad = \sigma_1 \frac{\partial}{\partial C_1} L(C_1, C_2) + \sigma_2 \frac{\partial}{\partial C_2} L(C_1, C_2) < 0. \end{aligned}$$

Thus it follows from the theorem in Section 2 that the spatially homogeneous periodic solution $(u, w) = (\phi_1(t), \phi_2(t))$ is unstable and the homogeneity breaking instability takes place in this case.

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