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THE BIVARIATE GENERATING FUNCTION AND TWO PROBLEMS IN DISCRETE STOCHASTIC PROCESSES[1]

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1. INTRODUCTION

THE DESIGN OF A COMMUNICATIONS SYSTEM can give rise to many problems in probability theory. There come to mind such examples as the transmission and detection of signal in noise, and the evaluation of system reliability. It is our purpose in this paper to consider two questions which have recently been generated in this manner, and to expound a uniform method for their solution. The problems, both novel as far as we know, and potentially of wider interest and applicability than their original context suggests, may be stated in general terms as follows:

I. Given a Markoff chain having two states 0 and 1, what is the probability of getting m ones in n consecutive trials.

II. Given a sequence of Bernoulli trials consisting of successes and failures, what is the expected value of the number of failures in n trials if the number of consecutive successes cannot exceed s.

Problem I appears as part of an investigation by Masonson [2], on the frame synchronization of a digital data stream at a transmission terminal, while Problem II occurs in a study by Zabronsky [3], on an inspection procedure for testing reels of cable in a large-scale communication system.

It was our observation that both problems may be most simply and expeditiously treated by the same means, namely, the method of bivariate generating functions. Now, whereas the generating function of one variable (or Z-transform) has been extensively used in the theory of probability [1, p. 248 et seq.] and elsewhere, the bivariate generating function, although defined in the literature [1, p. 261], is relatively unknown to the applied analyst as a working tool. This is regrettable as it is a powerful technique, particularly applicable to many questions in probability which involve two discrete parameters.

Accordingly, in an attempt to fill this lacuna, we shall devote the next section, (§2), to a brief exposition of the bivariate generating function, in particular, setting up a short table of "transform pairs" to facilitate the use of the method in our application. In the subsequent sections, we apply the material of §2 to solve the stated problems.

2. THE BIVARIATE GENERATING FUNCTION*

In the case of a single-variable, a sequence $f(n)$, $(n = 0, 1, 2, \cdots)$ is associated with its corresponding generating function $F(x) = \sum_{n=0}^{\infty} f(n)x^n$. The

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* As is well-known, an equivalent alternative approach is the two dimensional Laplace transform. The method discussed in this section seems more appropriate to us since we are concerned with sequences rather than with functions of continuous variables.

extension to two variables is immediate. Given a function $f(m, n)$, of two discrete variable m and n, where $m = 0, 1, 2, \cdots ; n = 0, 1, 2, \cdots$, we associate with it the function of two variables defined by the double power series

$$F(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m, n) x^m y^n. \tag{2.1}$$

$F(x, y)$ is said to be the bivariate generating function corresponding to $f(m, n)$ if the series in (2.1) converges absolutely in some neighborhood of the origin, $x = 0, y = 0$. We shall also refer to $f(m, n)$ as a double sequence and to $F(x, y)$ as its corresponding transform.

In what follows, we shall not be concerned with the convergence of the series defining the generating functions, but we shall proceed to develop a purely formal theory. It may easily be verified that for all the applications in this paper our series satisfy the convergence requirement.

Equation (2.1) defines a unique generating function $F(x, y)$ for a given $f(m, n)$. The converse is also true. By the identity theorem for power series, each generating function $F(x, y)$ yields a unique double sequence $f(m, n)$, $(m = 0, 1, \cdots, n = 0, 1, 2, \cdots)$, namely the coefficients of its unique expansion into a double power series about the origin.

It is immediately evident that the transformation with which we are dealing, i.e. from double sequence to bivariate generating function, is a linear operation. By this we mean that the double sequence: $k_1 f_1(m, n) + k_2 f_2(m, n)$, k_1 and k_2 constants, has $k_1 F_1(x, y) + k_2 F_2(x, y)$ as associated generating function, where F_1 and F_2 are the transforms of f_1 and f_2 respectively.

We proceed to develop certain transform pairs which will be required in our applications. Consider first the forward-shifted double sequence $f(m, n + 1)$, $(m = 0, 1, 2, \cdots ; n = 0, 1, 2, \cdots)$. Let $G(x, y)$ be the transform of $f(m, n + 1)$ and $F(x, y)$ the transform of $f(m, n)$. By definition,

$$G(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m, n + 1) x^m y^n.$$

Then

$$yG(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m, n + 1) x^m y^{n+1} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m, n) x^m y^n - \sum_{m=0}^{\infty} f(m, 0) x^m.$$

Thus

$$G(x, y) = \frac{1}{y} \left[F(x, y) - \sum_{m=0}^{\infty} f(m, 0) x^m \right].$$

In a similar manner, we get formulas for the transforms corresponding to $f(m + 1, n), f(m + 1, n + 1)$, and $f(m, n + 2)$, all for m and n both varying from 0 to ∞. These are listed in Table 1.

The backward-shifted double sequences are still simpler to handle. Consider $f(m - 1, n)$, $(m = 0, 1, 2, \cdots ; n = 0, 1, 2)$ and let us adopt the convention that $f(m, n) = 0$ if either m or n is negative. (This is analogous to the situation

TABLE 1
BIVARIATE TRANSFORM PAIRS

Double Sequence $m, n \geqq 0$	Bivariate Generating Function
1) $f(m, n)$	$F(x, y)$
2) $k_1 f_1(m, n) + k_2 f_2(m, n)$, k_1, k_2 constant	$k_1 F_1(x, y) + k_2 F_2(x, y)$
3) $f(m, n + 1)$	$y^{-1}[F(x, y) - \sum_{m=0}^{\infty} f(m, 0)x^m]$
4) $f(m + 1, n)$	$x^{-1}[Fx, y) - \sum_{n=0}^{\infty} f(0, n)y^n]$
5) $f(m + 1, n + 1)$	$(xy)^{-1}[F(x, y) - f(0, 0) - \sum_{m=1}^{\infty} f(m, 0)x^m - \sum_{n=1}^{\infty} f(0, n)y^n]$
6) $f(m, n + 2)$	$y^{-2}[F(x, y) - \sum_{m=0}^{\infty} f(m, 0)x^m - y \sum_{m=0}^{\infty} f(m, 1)x^m]$
7) $f(m - 1, n)$, $(f(-1, n) = 0)$	$xF(x, y)$
8) $f(m, n - 1)$, $(f(m, -1) = 0)$	$yF(x, y)$
9) $f(m - 1, n - 1)$, $(f(-1, n) = f(m, -1) = 0)$	$xyF(x, y)$
10) $f(m - 1, n + 1)$, $(f(-1, n + 1) = 0)$	$xy^{-1}[F(x, y) = \sum_{m=0}^{\infty} f(m, 0)x^m]$
11) $mf(m, n)$	$x \dfrac{\partial F(x, y)}{\partial x}$
12) $nf(m, n)$	$y \dfrac{\partial F(x, y)}{\partial y}$
13) $mnf(m, n)$	$xy \dfrac{\partial F(x, y)}{\partial x \partial y}$
14) $\alpha^m f(m, n)$	$F(\alpha x, y)$
15) $\beta^n f(m, n)$	$F(x, \beta y)$
16) $\alpha^m \beta^n f(m, n)$	$F(\alpha x, \beta y)$

in Laplace transforms). Then, with $G(x, y)$ the transform of $f(m - 1, n)$ we have

$$\begin{aligned} G(x, y) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m - 1, n)x^m y^n = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} f(m - 1, n)x^m y^n \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m, n)x^{m+1} y^n = xF(x, y), \end{aligned}$$

where $F(x, y)$ is the transform of $f(m, n)$. The double sequences $f(m, n - 1)$ and $f(m - 1, n - 1)$ are treated similarly and yield $yF(x, y)$ and $xyF(x, y)$ respectively as transforms. A mixed case such as $f(m - 1, n + 1)$ results in $xG(x, y)$ where $G(x, y)$ is the transform of $f(m, n + 1)$, previously calculated.

We have now obtained all the transform pairs which we will require in the subsequent sections. However, we have added a few additional pairs to the table to make it more complete.

A particular type of double sequence occurs frequently in the applications to probability. Let X_n be a random variable depending upon a parameter n, n a non-negative integer, and suppose X_n assumes integral values, $0, 1, 2, \cdots, n$. Write $\Pr[X_n = m] = p(m, n)$, $n \geqq 0$. Of course, in this case $p(m, n) = 0$ if $m > n$, but we do not make this replacement explicitly in the following formulas since they are valid also when X_n may assume any non-negative integral value and in this case $p(m, n)$ need not be zero for $m > n$.

The bivariate generating function corresponding to $p(m, n)$,

$$P(x, y) = \sum_{m,n=0}^{\infty} p(m, n)x^m y^n, \tag{2.2}$$

may be rewritten in the form

$$P(x, y) = \sum_{n=0}^{\infty} P_n(x)y^n, \tag{2.3}$$

where

$$P_n(x) = \sum_{m=0}^{\infty} p(m, n)x^m \tag{2.4}$$

is the one-variable generating function corresponding to the probability distribution of X_n. Thus a knowledge of the bivariate generating function (2.2) implies in the re-arrangement (2.3) a knowledge of the one-variable generating function (2.4). Oft-times, it is possible to express $P(x, y)$ in (2.2), simply, in closed form, whereas $P_n(x)$ in (2.4) has no such simple representation. Then, it is quite advantageous to determine first the bivariate generating function $P(x, y)$.

Similarly, we may obtain directly from equation (2.2) a one-variable generating function corresponding to the sequence of mathematical expectations $E(X_n)$, $(n = 0, 1, 2, \cdots)$. As is well known, $E(X_n) = P_n'(1)$. From (2.3), it therefore follows that

$$\left[\frac{\partial P(x, y)}{\partial x}\right]_{x=1} = \sum_{n=0}^{\infty} E(X_n)y^n, \tag{2.5}$$

which is the required one-variable generating function.

Equation (2.5) is again very useful when the expectations $E(X_n)$ are required but the generating function (2.4) is too complicated to find explicitly, while the bivariate generating function (2.2) can be found. For a case in point see the solution of Problem II in §4. Single variable generating functions for sequences of higher moments can be found in the same way.

Example

A simple example will illustrate some of these ideas and indicate how one operates with the bivariate generating function. Consider a sequence of Bernoulli trials, i.e. a sequence of independent trials, each with just two possible outcomes, say 1 and 0, with p and q respectively the probabilities of getting 1 or 0 at each trial. Suppose we wish to find, by our method, the well-known probability, $p(m, n)$, of getting m ones in n trials, $n > 0$, $m \geqq 0$. We first set up a difference equation for the probabilities $p(m, n)$. Since we can get m ones in $n + 1$ trials, only by adding a one or a zero respectively to a sequence of n trials having $m - 1$ or m ones respectively, it follows that

$$p(m, n + 1) = p \cdot p(m - 1, n) + q \cdot p(m, n). \tag{2.6}$$

The relation (2.6) holds for $m = 0, 1, 2, \cdots, n = 0, 1, 2, \cdots$, if we define $p(0, 0) = 1$, $p(m, 0) = 0$, $m \geqq 1$. Equation (2.6) is a linear partial difference equation (or recursion relation) with constant coefficients, and since it has just two discrete variables m and n, it is amenable to solution by means of the bivariate generating function.

Let $P(x, y)$ be the transform of $p(m, n)$. Then in Table 1 we use the transform pairs (1), (3), (7), the operational rule (2) and the uniqueness of the transform, to convert equation (2.6) into the equation

$$(2.7) \qquad \frac{1}{y}\left[P(x,y) - \sum_{m=0}^{\infty} p(m,0)x^m\right] = px\,P(x,y) + q\,P(x,y)$$

in the transform domain.

Since, by definition, $p(m, 0) = 0$ if $m > 0$, the series in this equation reduces to the single term $p(0, 0) = 1$, and we find upon solving for $P(x, y)$ that

$$(2.8) \qquad P(x,y) = \frac{1}{1 - qy - pxy}.$$

This is the bivariate generating function for $p(m, n)$.

By using the geometric series to expand $P(x, y)$ in equation (2.8), there results

$$P(x,y) = \sum_{n=0}^{\infty} (q + px)^n\, y^n,$$

which is the form corresponding to equation (2.3). Thus we get back the familiar one-variable generating function for $p(m, n)$, n fixed, namely $(q + px)^n$, from which the value $p(m, n)$ follows at once.

Applying equation (2.5) to our present $P(x, y)$ of equation (2.8), we get

$$\left[\frac{\partial}{\partial x} P(x,y)\right]_{x=1} = \left[\frac{py}{(1 - qy - pxy)^2}\right]_{x=1} = \frac{py}{(1 - y)^2} = \sum_{n=0}^{\infty} pny^n,$$

where the last form may be obtained by using the binomial theorem to expand $(1 - y)^{-2}$. The coefficients of the series are immediately recognizable as the expected values associated with the Bernoulli sequence of n trials. We have obtained them directly from equation (2.8).

3. PROBLEM I—THE MARKOFF CHAIN

Consider a Markoff chain* having two states 1 and 0, and having a constant transition probability matrix$\begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$, where, of course $p_{00} + p_{01} = p_{10} + p_{11} = 1$. Let us start at a point in the chain where the probability of a one is p and that of a zero is q. We require the probability $p(m, n)$ of getting m ones in n consecutive trials of the chain starting from this point, $n > 0$, $m \geqq 0$. This problem is one of the simplest generalizations of the example discussed in §2 to the case of dependent trials.

In our solution, we shall actually use a simultaneous transform calculus in that we define two new double sequences $p_0(m, n)$ and $p_1(m, n)$ and work simul-

* For this section, the reader need know only the definition of a Markoff chain, and the meaning of its associated transition probability matrix; cf. [1, pp. 338–340].

taneously with their corresponding transforms, $P_0(x, y)$ and $P_1(x, y)$. However, this extension of the method is immediate and calls for no special remarks.

Let $p_i(m, n)$, $(i = 0, 1)$ be respectively the probability of getting m ones in n consecutive trials of the chain, with the nth place being respectively a zero or a one. It follows from this definition that

$$
\begin{aligned}
&p_0(0, n) = qp_{00}^{n-1}, \qquad p_1(0, n) = 0, && (n \geqq 1);\\
(3.1)\quad &p_0(m, 1) = p_1(m, 1) = 0 && (m \geqq 2),\\
&p_0(1, 1) = 0, \qquad p_0(0, 1) = q, \qquad p_1(0, 1) = 0, \qquad p_1(1, 1) = p.
\end{aligned}
$$

Obviously,

$$
(3.2) \qquad p(m, n) = p_0(m, n) + p_1(m, n).
$$

We now set up simultaneous partial difference equations for $p_0(m, n)$ and $p_1(m, n)$. A sequence of $n + 2$ trials $(n \geqq 0)$ containing m ones and ending in a zero can be obtained only by adding a zero to a sequence of $n + 1$ trials containing m ones and ending either in a zero, or a one. Similarly, a sequence of $n + 2$ trials with m ones and ending in a one can only be obtained only by adding a one to a sequence of $n + 1$ trials containing $m - 1$ ones and ending either in a zero or a one. This means, in terms of probabilities, that for all non-negative integral m and n

$$
(3.3) \qquad
\begin{aligned}
p_0(m, n + 2) &= p_{00}p_0(m, n + 1) + p_{10}p_1(m, n + 1),\\
p_1(m, n + 2) &= p_{01}p_0(m - 1, n + 1) + p_{11}p_1(m - 1, n + 1).
\end{aligned}
$$

Transforming these equations by means of (2), (3), (6), and (10) of Table 1, we get

$$
\begin{aligned}
&\frac{1}{y^2}\left[P_0(x, y) - \sum_{m=0}^{\infty} p_0(m, 0)x^m - y\sum_{m=0}^{\infty} p_0(m, 1)x^m\right]\\
&\quad = p_{00}\frac{1}{y}\left[P_0(x, y) - \sum_{m=0}^{\infty} p_0(m, 0)x^m\right] + p_{10}\frac{1}{y}\left[P_1(x, y) - \sum_{m=0}^{\infty} p_1(m, 0)x^m\right],\\
(3.4)\quad &\frac{1}{y^2}\left[P_1(x, y) - \sum_{m=0}^{\infty} p_1(m, 0)x^m - y\sum_{m=0}^{\infty} p_1(m, 1)x^m\right]\\
&\quad = p_{01}\frac{x}{y}\left[P_0(x, y) - \sum_{m=0}^{\infty} p_0(m, 0)x^m\right]\\
&\qquad\qquad + p_{11}\frac{x}{y}\left[P_1(x, y) - \sum_{m=0}^{\infty} p_1(m, 0)x^m\right].
\end{aligned}
$$

For convenience, define $p_0(0, 0) = 1$ and $p_1(0, 0) = 0$, $p_0(m, 0) = p_1(m, 0) = 0$, $m \geqq 1$. Then with these values and those provided by equation (3.1), the series in equations (3.4) may be written out explicitly. We find that there is considerable simplification in doing so, and equations (3.4) can be brought into the form

$$(1 - p_{00}y)P_0(x, y) - p_{10}yP_1(x, y) = 1 + (q - p_{00})y,$$
$$-p_{01}xyP_0(x, y) + (1 - p_{11}xy)P_1(x, y) = (p - p_{01})xy.$$

The solution of this set of linear equations in P_0 and P_1 is

$$(3.5)\qquad P_0 = 1 + \frac{y(q + \alpha xy)}{\Delta}, \qquad P_1 = \frac{xy(p + \beta y)}{\Delta},$$

where we have used the abbreviations

$$\alpha = pp_{10} - qp_{11},$$
$$\beta = qp_{01} - pp_{00},$$
$$\delta = p_{00}p_{11} - p_{01}p_{10},$$
$$\Delta = 1 - p_{00}y - p_{11}xy + \delta xy^2.$$

Finally, if $P(x, y)$ denotes the transform of $p(m, n)$, then equation (3.2) implies that $P(x, y) = P_0(x, y) + P_1(x, y)$. Using equation (3.5) we get, after some simplification,

$$(3.6)\qquad P(x, y) = \frac{1 + \beta y + \alpha xy}{1 - p_{00}y - p_{11}xy + \delta xy^2}.$$

This is the required bivariate generating function for $p(m, n)$.

By expanding it into a power series in x and y, explicit expressions in the form of finite sums may be obtained for the probabilities $p(m, n)$. One way of doing this is to write $P(x, y)$ in the form

$$P(x, y) = \frac{1 + \beta y + \alpha xy}{(1 - p_{00}y)(1 - p_{11}xy)} \cdot \frac{1}{1 - \dfrac{p_{01}p_{10}xy^2}{(1 - p_{00}y)(1 - p_{11}xy)}},$$

and to expand the second fraction by means of the geometric series. This gives

$$(3.7)\qquad P(x, y) = (1 + \beta y + \alpha xy) \cdot \sum_{k=0}^{\infty} \frac{p_{01}^{\ k}p_{10}^{\ k}x^k y^{2k}}{(1 - p_{00}y)^{k+1}(1 - p_{11}xy)^{k+1}}.$$

The general term of the series in equation (3.7) is now developed into a double power series by making use of the result

$$\frac{1}{(1 - z)^{k+1}} = \sum_{s=k}^{\infty} \binom{s}{k} z^{s-k}$$

(which follows from the binomial theorem) for $z = p_{11}xy$ and $z = p_{00}y$. We get

$$(3.8)\qquad \begin{aligned} P(x, y) &= (1 + \beta y + \alpha xy) \\ &\cdot \sum_{k=0}^{\infty} \sum_{s_1=k}^{\infty} \sum_{s_2=k}^{\infty} p_{01}^{\ k}p_{10}^{\ k}p_{11}^{\ s_1-k}p_{00}^{\ s_2-k} \binom{s_1}{k}\binom{s_2}{k} x^{s_1}y^{s_1+s_2}. \end{aligned}$$

Picking out the coefficients of $x^m y^n$ in the right member of equation (3.8) gives, finally,

$$p(m, n) = \sum_{k \geqq 0} p_{01}{}^{k} p_{10}{}^{k} p_{00}{}^{n-m-k-1} p_{11}{}^{m-k-1}$$

$$\tag{3.9} \cdot \left[p_{00} p_{11} \binom{m}{k} \binom{n-m}{k} + \beta p_{11} \binom{m}{k} \binom{n-m-1}{k} + \alpha p_{00} \binom{m-1}{k} \binom{n-m}{k} \right].$$

A purely combinatorial proof of formula (3.9) can also be given.

There seems to be no simple closed-form expression for the one-variable generating function $P_n(x) = \sum_{m=0}^{\infty} p(m, n)x^m$. However, we can obtain such an expression for the generating function

$$Q_m(y) = \sum_{n=m}^{\infty} p(m, n) y^n.$$

For, $P(x, y)$ may be rewritten as

$$\tag{3.10} P(x, y) = \sum_{m=0}^{\infty} Q_m(y) x^m.$$

Now from equation (3.6),

$$P(x, y) = \frac{1 + \beta y + \alpha x y}{1 - p_{00} y} \cdot \frac{1}{1 - \dfrac{xy(p_{11} - \delta y)}{(1 - p_{00} y)}}.$$

Again expanding the second fraction by means of the geometric series, we find

$$P(x, y) = (1 + \beta y + \alpha x y) \sum_{k=0}^{\infty} x^k y^k \cdot \frac{(p_{11} - \delta y)^k}{(1 - p_{00} y)^{k+1}}.$$

A comparison of this equation with equation (3.10) yields

$$Q_m(y) = \frac{y^m (1 + \beta y)(p_{11} - \delta y)^m}{(1 - p_{00} y)^{m+1}} + \frac{\alpha y^m (p_{11} - \delta y)^{m-1}}{(1 - p_{00} y)^m};$$

which becomes, after simplification

$$Q_m(y) = \frac{y^m (1 - \delta y)(p + \beta y)(p_{11} - \delta y)^{m-1}}{(1 - p_{00} y)^{m+1}}.$$

This is the generating function obtained by Masonson [2] using another approach.

4. PROBLEM II—SUCCESS RUNS

Consider a sequence of n Bernoulli trials in which the outcome of each trial is either a success S or a failure F, and let p and q denote respectively the probability of a success or of a failure occurring at a single trial. Define $p(m, n)$ as the probability of getting m F's $(0 \leqq m \leqq n)$ in n trials, $(n > 0,)$ subject to the restriction R: that the number of consecutive successes cannot exceed s. We require the expected number of failures, E_n, under these conditions, i.e., we must find

$$E_n = \sum_{m=0}^{n} mp(m, n), \qquad (n \geqq 1).$$

In studying this problem, we will carry our analysis to the point where we obtain a one-variable generating function for the sequence E_n, $n = 1, 2, \cdots$. In the course of our work, we also develop a bivariate generating function for $p(m, n)$ itself. As in the previous Problem I, we introduce certain auxiliary double sequences and apply the simultaneous bivariate transform technique to these.

Let $p_i(m, n)$, $i = 0, 1, 2, \cdots, s$, denote respectively the probability of getting m F's in n consecutive trials subject to restriction R, but in addition subject to the further condition that exactly the last i places of the total of n be successes (i.e. the sequence of trials is of the form $\cdots F \cdot \underbrace{SS \cdots S}_{i} \cdot$). In particular, for $i = 0$, the nth place must be an F.

Evidently

$$p(m, n) = \sum_{i=0}^{s} p_i(m, n). \tag{4.1}$$

We note that the only way of getting an ending consisting of exactly i S's $(1 \leqq i \leqq s)$ in $n + 1$ trials, is to add an S to a sequence of n trials ending in exactly $(i - 1)$ S's. This, of course, does not increase the number of F's in the $n + 1$ place sequence. On the other hand, to get an $(n + 1)$-place sequence ending in F, we simply affix an F to any n-place sequence, the total number of F's thus increasing by one. Interpreting these remarks from the standpoint of probabilities, the following difference equations connecting the $p_i(m, n)$ hold

$$\begin{aligned} p_0(m + 1, n + 1) &= q\sum_{i=0}^{s} p_i(m, n), \\ p_i(m, n + 1) &= p \cdot p_{i-1}(m, n) \qquad (i = 1, 2, \cdots, s). \end{aligned} \tag{4.2}$$

If we define $p_0(0, 0) = 1$, $p_i(0, 0) = 0$, $(i \geqq 1)$, $p_i(m, n) = 0$, $(m > n \geqq 0, i \geqq 0)$, then equations (4.2) hold for all non-negative integral values of m and n.

We now transform these equations using (1), (2), (3) and (5) of Table 1. With the notation $P_i(x, y)$ to designate the transforms of $p_i(m, n)$ and the observation that, by definition, $p_i(m, 0) = 0$ for $m \geqq 1$, and $p_0(0, n) = 0$ for $n \geqq 1$, we find that equations (4.2) become in the transform domain

$$\begin{aligned} \frac{P_0(x, y) - 1}{xy} &= q[P_0(x, y) + P_1(x, y) + \cdots + P_s(x, y)] \\ \frac{P_i(x, y)}{y} &= pP_{i-1}(x, y) \qquad (1 \leqq i \leqq s) \end{aligned} \tag{4.3}$$

By iterating the relation in the second equation of (4.3) we get

$$P_i(x, y) = pyP_{i-1}(x, y) = p^2y^2P_{i-2} = \cdots = p^iy^iP_0(x, y), \qquad (1 \leqq i \leqq s). \tag{4.4}$$

Substituting the expression for each $P_i(x, y)$, $(1 \leqq i \leqq s)$ in terms of $P_0(x, y)$ into equation (4.3) and solving for $P_0(x, y)$ yields

$$P_0(x, y) = \frac{1 - py}{1 - py - qxy + qxp^{s+1}y^{s+2}}. \tag{4.5}$$

From equation (4.4), it follows that

$$P_i(x, y) = \frac{p^i y^i(1 - py)}{1 - py - qxy + qxp^{s+1}y^{s+2}}, \qquad (1 \leqq i \leqq s). \tag{4.6}$$

If now, $P(x, y)$ is the transform of $p(m, n)$, then by virtue of equation (4.1) together with equations (4.5) and (4.6)

$$P(x, y) = \sum_{i=0}^{s} P_i(x, y) = \frac{1 - p^{s+1}y^{s+1}}{1 - py - qxy + qxp^{s+1}y^{s+2}}. \tag{4.7}$$

This is the bivariate generating function for $p(m, n)$.

We now apply equation (2.5) to the function $P(x, y)$ of equation (4.7). This gives the required one-variable generating function for E_n, viz.

$$\sum_{n=1}^{\infty} E_n y^n = \left[\frac{\partial P(x, y)}{\partial x}\right]_{x=1} = qy \frac{(1 - p^{s+1}y^{s+1})^2}{(1 - y + qp^{s+1}y^{s+2})^2}.$$

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