Unique Normal Forms in Infinitary Weakly Orthogonal Term Rewriting*

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Dedicated to Roel de Vrijer on the occasion of his 60th birthday.

Abstract

The theory of finite and infinitary term rewriting is extensively developed for orthogonal rewrite systems, but to a lesser degree for weakly orthogonal rewrite systems. In this note we present some contributions to the latter case of weak orthogonality, where critial pairs are admitted provided they are trivial.

We start with a refinement of the by now classical Compression Lemma, as a tool for establishing infinitary confluence CR^{∞}, and hence the infinitary normal form property UN^{∞}, for the case of weakly orthogonal TRSs that do not contain collapsing rewrite rules.

That this restriction of collapse-freeness is crucial, is shown in an elaboration of a simple TRS which is weakly orthogonal, but has two collapsing rules. It turns out that all the usual theory breaks down dramatically.

We conclude with establishing a positive fact: the diamond property for infinitary developments for weakly orthogonal TRSs, by means of a detailed analysis initiated by van Oostrom for the finite case.

1 Preliminaries

An *infinitary rewrite rule* is a pair $\langle s, t\rangle$ with $s \in \mathit{Ter}(\Sigma)$ and $t \in \mathit{Ter}^{\infty}(\Sigma)$ such that s is not a variable and every variable in t occurs in s. A rewrite rule $\langle s, t\rangle$ is *left-linear* if no variable has more than one occurrence in s.

An *infinitary term rewriting system (iTRS)* is a pair $\langle \Sigma, R\rangle$ consisting of a signature Σ and a set R of infinitary rewrite rules. An iTRS is called *weakly orthogonal* if all rules are left-linear and all critical pairs $\langle s_1, s_2\rangle$ are trivial ($s_1 \equiv s_2$).

As a preparation for Section 3 we will prove the following lemma, which is a refined version of the Compression Lemma in left-linear TRSs. In its original

*An earlier version of this paper appeared in [3].

formulation (e.g. see Theorem 12.7.1 on page 689 in [5]), it states that strongly convergent rewrite sequences in left-linear TRSs can be compressed to length less or equal to ω.

Lemma 1.1 (Refined Compression Lemma)**.** *Let* R *be a left-linear iTRS. Let* $\kappa : s \to_R^\alpha t$ *be a rewrite sequence,* d *the minimal depth of a step in* κ*, and* n *the number of steps at depth* d *in* κ*. Then there exists a rewrite sequence* $\kappa' : s \to_R^{\leq\omega} t$ *in which all steps take place at depth* $\geq d$*, and where precisely* n *steps contract redexes at depth* d*.*

Proof. We proceed by transfinite induction on the ordinal length α of rewrite sequences $\kappa : s \to_R^\alpha t$ with d the minimal depth of a step in κ, and n the number of steps at depth d in κ.

In case that $\alpha = 0$ nothing needs to be shown.

Suppose α is a successor ordinal. Then $\alpha = \beta + 1$ for some ordinal β, and κ is of the form $s \to^\beta s' \to t$. Applying the induction hypothesis to $s \to^\beta s'$ yields a rewrite sequence $s \to^\gamma s'$ of length $\gamma \leq \omega$ that contains the same number of steps at depth d, and no steps at depth less than d.

If $\gamma < \omega$, then $s \to^\gamma s' \to t$ is a rewrite sequence of length $\gamma + 1 < \omega$, in which all steps take place at depth $\geq d$ and precisely n steps at depth d.

If $\gamma = \omega$, we obtain a rewrite sequence of the form $s \equiv s_0 \to s_1 \to \ldots \to^\omega s_\omega \to t$. Let $\ell \to r \in R$ be the rule applied in the final step $s_\omega \to t$, that is, $s_\omega \equiv C[\ell\sigma] \to C[r\sigma] \equiv t$ for some context C and substitution σ. Moreover, let d_h be the depth of the hole in C, and d_p the depth of the pattern of ℓ. Since the reduction $s_0 \to^\omega s_\omega$ is strongly convergent, there exists $n \in \mathbb{N}$ such that all rewrite steps in $s_n \to^\omega s_\omega$ have depth $> d_h + d_p$, and hence are below the pattern of the redex contracted in the last step $s_\omega \to t$. As a consequence, there exists a substitution τ such that $s_n \equiv C[\ell\tau]$, and since $s_n \equiv C[\ell\tau] \to^\omega C[\ell\sigma] \equiv s_\omega$, it follows that $\forall x \in \mathit{Var}(\ell).\ \tau(x) \to^{\leq\omega} \sigma(x)$. We now prepend the final step $s_\omega \to t$ to s_n, that is: $s_n \equiv C[\ell\tau] \to C[r\tau]$. Even if r is an infinite term, this creates at most ω-many copies of subterms $\tau(x)$ with reduction sequences $\tau(x) \to^{\leq\omega} \sigma(x)$ of length $\leq \omega$. Since these reductions are independent of each other there exists an interleaving $C[r\tau] \to^{\leq\omega} C[r\sigma]$ of length at most ω (the idea is similar to establishing countability of ω^2 by dovetailing). Hence we obtain a rewrite sequence $\kappa' : s \to^{\leq\omega} t$, since $s \to^n s_n \equiv C[\ell\tau] \to C[r\tau] \to^{\leq\omega} C[r\sigma] \equiv t$. It remains to be shown that κ' contains only steps at depth $\geq d$, and that it has the same number of steps as the original sequence κ at depth d. This follows from the induction hypothesis and the fact that all steps in $s_n \to^\omega s_\omega$ have depth $> d_h + d_p$ and thus also all steps of the interleaving $C[r\tau] \to^{\leq\omega} C[r\sigma]$ have depth $> d_h + d_p - d_p = d_h \geq d$ (the application of $\ell \to r$ can lift steps at most by the pattern depth d_p of ℓ).

Finally, suppose that α is a limit ordinal $> \omega$. We refer to Figure 1 for a sketch of the proof. Since κ is strongly convergent, only a finite number of steps take place at depth d. Hence there exists $\beta < \alpha$ such that s_β is the target of the last step at depth d in κ. We have $s \to^\beta s_\beta \to^{\leq\alpha} t$ and all rewrite steps in $s_\beta \to^{\leq\alpha} t$ are at depth $> d$. By induction hypothesis there exists a rewrite sequence $\xi : s \to^{\leq\omega} s_\beta$ containing an equal amount of steps at depth d as

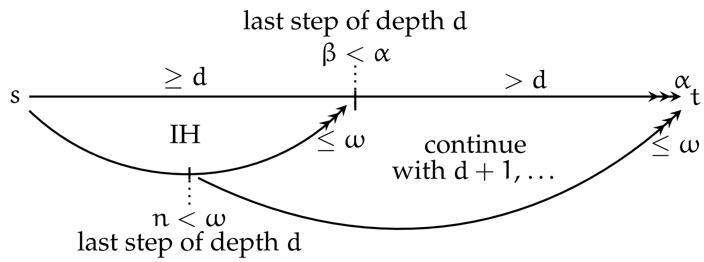

Figure 1: *Compression Lemma, in case α is a limit ordinal.*

$s \to^{\beta} s_\beta$. Consider the last step of depth d in ξ. This step has a finite index $n < \omega$. Thus we have $s \to^* s_n \to^{\leq \alpha} t$, and all steps in $s_n \to^{\leq \alpha} t$ are at depth $> d$. By successively applying this argument to $s_n \to^{\leq \alpha} t$ we construct finite initial segments $s \to^* s_n$ with strictly increasing minimal rewrite depth d. Concatenating these finite initial segments yields a reduction $s \to^{\leq \omega} t$ containing as many steps at depth d as the original sequence. □

With this refined compression lemma we now prove that also divergent rewrite sequences can be compressed to length less or equal to ω.

Theorem 1.2. *Let R be a left-linear iTRS. For every divergent rewrite sequence κ : $s \to_R^{\alpha}$ of length α there exists a divergent rewrite sequence κ' : $s \to_R^{\leq \omega}$ of length less or equal to ω.*

Proof. Let $\kappa : s \to_R^{\alpha}$ be a divergent rewrite sequence. Then there exist $k \in \mathbb{N}$ such that infinitely many steps in κ take place at depth k. Let d be the minimum of all numbers k with that property. Let β be the index of the last step above depth d in κ, $\kappa : s \to^{\beta} s_\beta \to^{\leq \alpha}$. Then by Lemma 1.1 the rewrite sequence $s \to^{\beta} s_\beta$ can be compressed to a rewrite sequence $s \to^{\leq \omega} s_\beta$ such that $s_\beta \to^{\leq \alpha}$ consists only of steps at depth $\geq d$, among which infinitely many steps are at depth d. Let n be the index of the last step of depth $\leq d$ in the rewrite sequence $s \to^{\leq \omega} s_\beta$. Then $s \to^* s_n \to^{\leq \omega} s_\beta \to^{\leq \alpha}$, and $s_n \to^{\leq \omega} s_\beta \to^{\leq \alpha}$ contains only steps at depth $\geq d$. Thus all steps with depth less than d occur in the finite prefix $s \to^* s_n$.

Now consider the rewrite sequence $\kappa_1 : s_n \to^{\leq \omega} \cdot \to^{\leq \alpha}$, say $\kappa_1 : s_n \to^{\gamma}$ for short, containing infinitely many steps at depth d. Let γ' be the first step at depth d in κ_1. Then $\kappa_1 : s_n \to^{\gamma'} u \to^{\leq \gamma}$ for some term u and $s_n \to^{\gamma'} u$ can be compressed to $s_n \to^{\leq \omega} u$ containing exactly one step at depth d. Now let m be the index of this step, then $s_n \to^{m} u' \to^{\leq \omega} u \to^{\leq \gamma}$ where $s_n \to^{m} u'$ contains one step at depth d. Repeatedly applying this construction to $u' \to^{\leq \omega} u \to^{\leq \gamma}$ we obtain a divergent rewrite sequence $\kappa' : s \to^* s_n \to^* u' \to^* u'' \to \ldots$ that contains infinitely many steps at depth d, and hence is divergent. □

Remark 1.3. A slightly weaker version of Lemma 1.1, as well as Theorem 1.2 was formulated by the second author in a private communication with Hans

Zantema. These statements have been published, in a reworked form, in [6] (see Lemma 3 and Theorem 4 there). The weaker version of Lemma 1.1, which is sufficient to obtain a proof of Theorem 1.2, states the following: Every strongly convergent rewrite sequence $\kappa : s \to_R^\alpha t$ with d the minimal depth of its steps can be compressed to a rewrite sequence $\kappa' : s \to_R^{\leq\omega} t$ of length $\leq \omega$ with the same or more steps at minimal depth d.

We note that very closely related statements have been formulated for infinitary CRSs (Combinatory Reduction Systems) by Jeroen Ketema in [1] (see Theorem 2.7 and Lemma 5.2 there).

2 Infinitary Unique Normal Forms

In [4], Klop and de Vrijer have shown that every orthogonal TRS exhibits the infinitary unique normal forms (UN^∞) property. By way of contrast, we will now give a counterexample showing that the UN^∞ property does *not* generalize to weakly orthogonal TRSs. The counterexample is very simple: its signature consists of the unary symbols P and S with the reduction rules:

$$\mathsf{P}(\mathsf{S}(x)) \to x \qquad\qquad \mathsf{S}(\mathsf{P}(x)) \to x\,.$$

It is easily seen that this TRS is indeed weakly orthogonal.

Using S and P we have infinite terms such as $\mathsf{SPSPSP}\ldots$, where we drop the brackets that are associating to the right. We write S^ω for $\mathsf{SSS}\ldots$ and P^ω for $\mathsf{PPP}\ldots$. In fact, S^ω and P^ω are the only infinite normal forms.

Given an infinite SP-term t we can plot in a graph the surplus number of S's of t when traversing from the root of t downwards to infinity (or to the right if t is written horizontally). This graph is obtained by counting S for $+1$ and P for -1. We define $\mathrm{sum}(t, n)$ as the result of this counting up to depth n in the term t. For $t = \mathsf{SPSPSP}\ldots$ the graph takes values, consecutively, $1, 0, 1, 0, \ldots$, while for $t = \mathsf{S}^\omega$ the excess number is $1, 2, 3, \ldots$, for $t = \mathsf{P}^\omega$ we have $-1, -2, -3, \ldots$.

The upper and lower S-height of t are defined as the supremum and infimum obtained by this graph, i.e., $\sup_{n\in\mathbb{N}} \mathrm{sum}(t, n)\}$ and $\inf_{n\in\mathbb{N}} \mathrm{sum}(t, n)\}$, respectively. So the upper (lower) S-height of $(\mathsf{SP})^\omega$ is 1 (0), of S^ω it is ∞ (0), and of P^ω 0 it is ($-\infty$).

Now we have:

Proposition 2.1.

(i) $t \to^\omega \mathsf{S}^\omega$ *if and only if the upper* S*-height of* t *is* ∞*,*

(ii) $t \to^\omega \mathsf{P}^\omega$ *if and only if the lower* S*-height of* t *is* $-\infty$*.*

Proof. Left for Roel. ☐

Now let us take a term q with upper S-height ∞ *and* lower S-height $-\infty$! Then q reduces to both S^ω and P^ω, both normal forms. Hence UN^∞ fails. Is

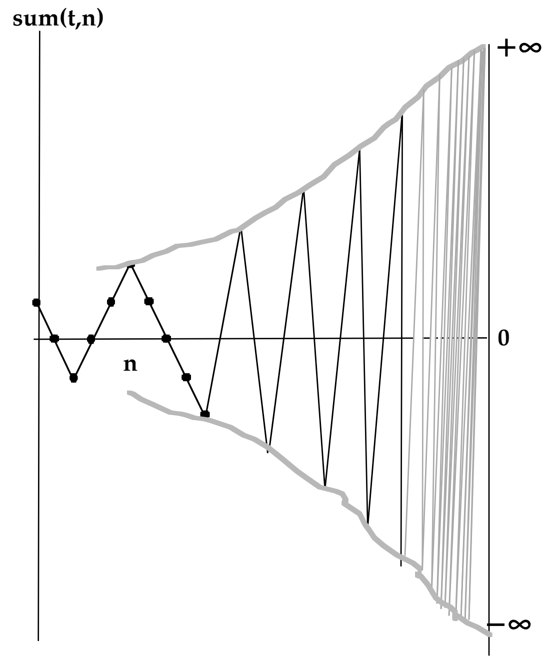

Figure 2: *Graph for the oscillating* SP-*term* $q = S^1\,P^2\,S^3\,\ldots$.

there indeed such a q? Yes there is (see also Figure 2):

$$q = \mathsf{S\,PP\,SSS\,PPPP\,SSSSS\,PPPPPP}\ \ldots$$

To see that q indeed reduces to both S^ω and P^ω: shifting the "P-blocks to the right, so that they are 'absorbed' by the S-blocks, yields ever more S's. On the other hand, shifting the S-blocks to the right so that they are absorbed by the P-blocks, leaves infinitely many P's.

The failure of UN^∞ for two collapsing rules raises the following question:

Question 2.2. What if we admit only *one* collapsing rule?

We leave this question to future work.

We will now provide a more detailed analysis of various classes of SP-terms. In Figure 3, the extension of these classes is pictured. Here A is the set of infinite SP-terms reducing to S^ω, and B that of those reducing to P^ω with a shaded non-empty intersection containing the counterexample term q mentioned above. The term $r = S\,P\,S^2\,P^2\,S^3\,P^3\,\ldots$ is an element of $(A \setminus B) \cap RA$, where RA is the set of root-active terms. The dotted part $\subseteq A \cup B$ is SN^∞. The set RA is characterized as follows:

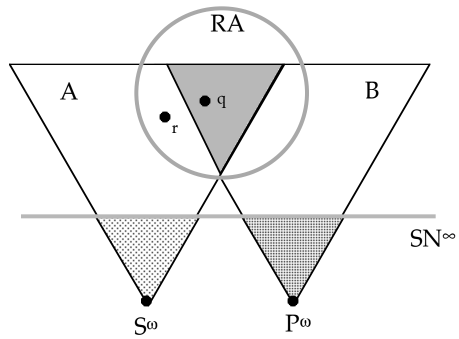

Figure 3: *Classes of infinite* SP*-terms.*

Proposition 2.3.

(i) An SP*-term* t *is root-active if and only if the partial sums* $\mathrm{sum}(t, n)$ *equal* 0 *for infinitely many* n*. Equivalently:*

(ii) An SP*-term* t *is root-active if and only if* t *is the concatenation of infinitely many 'finite zero words'* $t_0, t_1, t_2, \ldots$*. Here a zero word is a finite* SP*-term with the same number of* S*'s and* P*'s. If* w *is a zero word, then* w *clearly reduces to the empty word.*

Proof. The direction $\Leftarrow$ is obvious.

For the direction $\Rightarrow$ we label all S's and P's in the start term, say by numbering them from left to right, so e.g. the labelled t could be:

$$\mathsf{S}_0\,\mathsf{S}_1\,\mathsf{P}_2\,\mathsf{S}_3\,\mathsf{P}_4\,\mathsf{P}_5\ \ldots$$

Then in a reduction of t any S or P can be traced back to a unique ancestor in t. Furthermore let w_i be the prefix of t of length i. Observation: if S_i or P_i gets at a root position in a reduction of t, then w_i is a 0-word. Proof: easy. □

Corollary 2.4. SN^∞ *is the set of* SP*-terms that are infinitarily strongly normalizing. Then* $t \in \mathrm{SN}^\infty$ *if and only if each value* $\mathrm{sum}(t, n)$ *for* $n = 0, 1 \ldots$ *occurs only finitely often, or equivalently* $\lim_{n\to\infty} \mathrm{sum}(t, n)$ *exists (then it is* ∞ *or* $-\infty$*).*

Proof. Note that then the normal form is unique, since there is no oscillation. □

3 Infinitary Confluence

In the previous section we have seen that the property UN^∞ fails dramatically for weakly orthogonal TRSs when collapsing rules are present, and hence also

CR^∞. Now we show that weakly orthogonal TRSs without collapsing rules are infinitary confluent (CR^∞), and as a consequence also have the property UN^∞.

We adapt the projection of parallel steps in weakly orthogonal TRSs from [5, Section 8.8.4.] to infinite terms. The basic idea is to orthogonalize the parallel steps, and then project the orthogonalized steps. The orthogonalization uses that overlapping redexes have the same effect and hence can be replaced by each other. In case of overlaps we replace the outermost redex by the innermost one. This is possible for infinitary parallel steps since there can never be infinite chains of overlapping, nested redexes (see Figure 11). For a treatment of infinitary developments where such chains can occur, we refer to Section 4. See further [5, Proposition 8.8.23] for orthogonalization in the finitary case.

Proposition 3.1. *Let $\phi : s \twoheadrightarrow t_1$, $\psi : s \twoheadrightarrow t_2$ be parallel steps in a weakly orthogonal TRS. Then there exists an* orthogonalization $\langle \phi', \psi' \rangle$ of ϕ and ψ, *that is, a pair of orthogonal parallel steps such that $\phi' : s \twoheadrightarrow t_1$, $\psi' : s \twoheadrightarrow t_2$.*

Proof. In case of overlaps between ϕ and ψ, then for every overlap we replace the outermost redex by the innermost one (if there are multiple inner redexes overlapping, then we choose the left-most among the top-most redexes). If there are two redexes at the same position but with respect to different rules, then we replace the redex in ψ with the one in ϕ. See also Figure 4. □

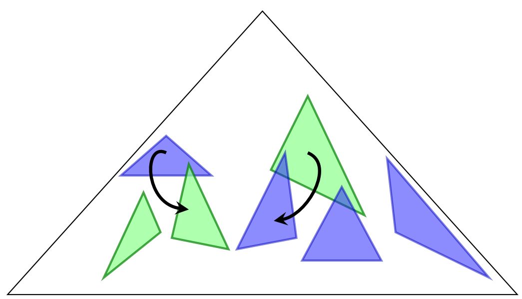

Figure 4: *Orthogonalization of parallel steps; the arrow indicates replacement.*

Definition 3.2. Let $\phi : s \twoheadrightarrow t_1$, $\psi : s \twoheadrightarrow t_2$ be parallel steps in a weakly orthogonal TRS. The *weakly orthogonal projection ϕ/ψ of ϕ over ψ* is defined as the orthogonal projection ϕ'/ψ' where $\langle \phi', \psi' \rangle$ is the orthogonalization of ϕ and ψ.

Remark 3.3. The weakly orthogonal projection does not give rise to a residual system in the sense of [5]. The projection fulfils the three identities $\phi/\phi \approx 1$, $\phi/1 \approx \phi$, and $1/\phi \approx 1$, but not the *cube identity* $(\phi/\psi)/(\chi/\psi) \approx (\phi/\chi)/(\psi/\chi)$, depicted in Figure 5.

Lemma 3.4. *Let $\phi : s \twoheadrightarrow t_1$, $\psi : s \twoheadrightarrow t_2$ be parallel steps in a weakly orthogonal TRS R. Let d_ϕ and d_ψ be the minimal depth of a step in ϕ and ψ, respectively. Then the minimal depth of the weakly orthogonal projections ϕ/ψ and ψ/ϕ is greater or*

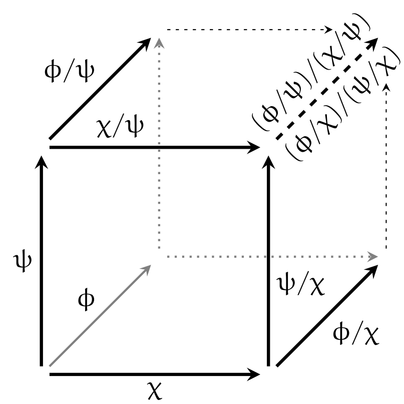

Figure 5: *Cube identity* $(\phi/\psi)/(\chi/\psi) \approx (\phi/\chi)/(\psi/\chi)$.

equal $\min(d_\phi, d_\psi)$. *If* R *contains no collapsing rules then the minimal depth of* ϕ/ψ *and* ψ/ϕ *is greater or equal* $\min(d_\phi, d_\psi + 1)$ *and* $\min(d_\psi, d_\phi + 1)$, *respectively.*

Proof. Immediate from the definition of the orthogonalization (for overlaps the innermost redex is chosen) and the fact that in the orthogonal projection a non-collapsing rule applied at depth d can lift nested redexes at most to depth $d+1$ (but not above). $\square$

Lemma 3.5 (Strip/Lift Lemma)**.** *Let* R *be a weakly orthogonal TRS,* $\kappa : s \to^{\alpha} t_1$ *a rewrite sequence, and* $\phi : s \mathrel{+\!\!\!\!\!\!\!\longrightarrow} t_2$ *a parallel rewrite step. Let* d_κ *and* d_ξ *be the minimal depth of a step in* κ *and* ϕ*, respectively. Then there exist a term* u*, a rewrite sequence* $\xi : t_2 \to^{\leq\omega} u$ *and a parallel step* $\psi : t_1 \mathrel{+\!\!\!\!\!\!\!\longrightarrow} u$ *such that the minimal depth of the rewrite steps in* ξ *and* ψ *is* $\min(d_\kappa, d_\xi)$*; see Figure 6.*

$$\begin{tikzcd}[column sep=huge]
s \arrow[r, "\geq d_\kappa", ->>>] \arrow[d, "\geq d_\xi"', "+" marking] & t_1 \arrow[d, "\geq \min(d_\kappa, d_\xi)", "+"' marking] \\
t_2 \arrow[r, dashed, ->>>, "\geq \min(d_\kappa, d_\xi)"'] & u
\end{tikzcd}$$

Figure 6: *Strip/Lift Lemma with collapsing rules.*

If additionally R *contains no collapsing rules, then the minimal depth of a step in* ξ *and* ψ *is* $\min(d_\kappa, d_\xi + 1)$ *and* $\min(d_\xi, d_\kappa + 1)$*, respectively. See also Figure 7.*

Proof. By compression we may assume $\alpha \leq \omega$ in $\kappa : s \to^{\leq\omega} t_1$ (note that, the minimal depth d is preserved by compression). Let $\kappa : s \equiv s_0 \to s_1 \to s_2 \to \ldots$, and define $\psi_0 = \psi$. Furthermore, let $\kappa_{\leq n}$ denote the prefix of κ of length n, that is, $s_0 \to \ldots \to s_n$ and let $\kappa_{\geq n}$ denote the suffix $s_n \to s_{n+1} \to \ldots$ of κ. We employ the projection of parallel steps to close the elementary

$$
\begin{tikzcd}
s \arrow[r, twoheadrightarrow, "\geq d_\kappa"] \arrow[d, "\geq d_\xi"'] & t_1 \arrow[d, "\geq \min(d_\xi, d_\kappa + 1)"] \\
t_2 \arrow[r, dashed, twoheadrightarrow, "\geq \min(d_\kappa, d_\xi + 1)"'] & u
\end{tikzcd}
$$

Figure 7: *Strip/Lift Lemma without collapsing rules.*

diagrams with top $s_n \to s_{n+1}$ and left $\psi_n : s_n \twoheadrightarrow s'_n$, that is, we construct the projections $\psi_{i+1} = \psi_i/(s_i \to s_{i+1})$ (right) and $(s_i \to s_{i+1})/\psi_i$ (bottom). Then by induction on n using Lemma 3.4 there exists for every $1 \leq n \leq \alpha$ a term s'_n, and parallel steps $\phi_n : s_n \twoheadrightarrow s'_n$ and $s'_{n-1} \twoheadrightarrow s'_n$. See Figure 8 for an overview.

$$
\begin{tikzcd}
s \equiv s_0 \arrow[r] \arrow[dd, "\phi = \phi_0"'] & s_1 \arrow[r] \arrow[dd, "\phi_1"] & \cdots \arrow[r] & s_{n_0} \arrow[r, twoheadrightarrow, "\geq d"] \arrow[d, "\phi_{n_0,<d}"] & s_{m_0} \arrow[r, twoheadrightarrow, "\geq d + p"] \arrow[d, "\psi \subseteq \phi_{n_0,<d}"] & t_1 \arrow[d, "\psi"] \\
 & & & s''_{n_0} \arrow[d, "\phi_{n_0,\geq d}"] & s''_{m_0} \arrow[r, dashed, twoheadrightarrow, "\geq d"] \arrow[d, "\phi_{m_0,\geq d}"] & t''_1 \arrow[d, dashed, "\geq d"] \\
t_2 \equiv s'_0 \arrow[r] & s'_1 \arrow[r] & \cdots \arrow[r] & s'_{n_0} \arrow[r, dashed, twoheadrightarrow] & s'_{m_0} \arrow[r, dashed, twoheadrightarrow, "\geq d"] & u
\end{tikzcd}
$$

Figure 8: *Strip/Lift Lemma, proof overview.*

We show that the rewrite sequence constructed at the bottom $s'_0 \twoheadrightarrow s'_1 \twoheadrightarrow \ldots$ of Figures 6 and 7 is strongly convergent, and that the parallel steps ϕ_i have a limit for $i \to \infty$ (parallel steps are always strongly convergent).

Let $d \in \mathbb{N}$ be arbitrary. By strong convergence of κ there exists $n_0 \in \mathbb{N}$ such that all steps in $\kappa_{\geq n_0}$ are at depth $\geq d$. Since ϕ_{n_0} is a parallel step there are only finitely many redexes $\phi_{n_0,<d} \subseteq \phi_{n_0}$ in ϕ_{n_0} rooted above depth d. By projection of ϕ_{n_0} along $\kappa_{\geq n_0}$ no fresh redexes above depth d can be created. The steps in $\phi_{n_0,<d}$ may be cancelled out due to overlaps, nevertheless, for all $m \geq n_0$ the set of steps above depth d in ϕ_m is a subset of $\phi_{n_0,<d}$.

Let p be the maximal depth of a left-hand side of a rule applied in $\phi_{n_0,<d}$. By strong convergence of κ there exists $m_0 \geq n_0 \in \mathbb{N}$ such that all steps in $\kappa_{\geq n_0}$ are at depth $\geq d + p$. As a consequence the steps ψ in ϕ_{m_0} rooted above depth d will stay fixed throughout the remainder of the projection. Then for all $m \geq m_0$ the parallel step ϕ_m can be split into $\phi_m = s_m \twoheadrightarrow_\psi s''_m \twoheadrightarrow_{\phi_{m,\geq d}} s'_m$ where $\phi_{m,\geq d}$ consists of the steps of ϕ_m at depth $\geq d$. Since d was arbitrary, it follows that projection of ϕ over κ has a limit. Moreover the steps of the projection of $\kappa_{\geq m_0}$ over ϕ_{m_0} are at depth $\geq d + p - p = d$ since rules with pattern depth $\leq p$ can lift steps by at most by p. Again, since d was arbitrary, it follows that the projection of κ over ϕ is strongly convergent.

Finally, both constructed rewrite sequences (bottom and right) converge towards the same limit u since all terms $\{s'_m, s''_m \mid m \geq m_0\}$ coincide up to depth $d-1$ (the terms $\{s_m \mid m \geq m_0\}$ coincide up to depth $d+p-1$ and the lifting effect of the steps ϕ_m is limited by p). $\square$

Theorem 3.6. *Every weakly orthogonal TRS without collapsing rules is infinitary confluent.*

$$\begin{tikzcd}[column sep=huge, row sep=huge]
s \arrow[r, ->>, "\geq d"] \arrow[d, ->>, "\geq d"'] \arrow[dr, phantom, "\text{finitary diagram}"] & s_1 \arrow[r, ->>>, "> d"] \arrow[d, dashed, ->>, "\nparallel \geq d"] \arrow[dr, phantom, "\text{strip lemma}"] & t_1 \arrow[d, dashed, ->>, "\geq d"] \\
s_2 \arrow[r, dashed, ->>, "\geq d"] \arrow[d, ->>, "> d"'] \arrow[dr, phantom, "\text{strip lemma}"] & s' \arrow[r, dashed, ->>>, "> d"] \arrow[d, dashed, ->>, "> d"] \arrow[dr, phantom, "\begin{array}{c}\text{coinduction/}\\ \text{repeat construction}\\ \text{with } d+1\end{array}"] & t'_1 \arrow[d, dashed, ->>] \\
t_2 \arrow[r, dashed, ->>>, "\geq d"] & t'_2 \arrow[r, dashed, ->>>] & u
\end{tikzcd}$$

Figure 9: *Infinitary confluence.*

Proof. An overview of the proof is given in Figure 9. Let $\kappa : s \twoheadrightarrow^{\alpha} t_1$ and $\xi : s \twoheadrightarrow^{\beta} t_2$ be two rewrite sequences. By compression we may assume $\alpha \leq \omega$ and $\beta \leq \omega$. Let d be the minimal depth of any rewrite step in κ and ψ. Then κ and ξ are of the form $\kappa : s \twoheadrightarrow^{*} s_1 \twoheadrightarrow^{\leq\omega} t_1$ and $\xi : s \twoheadrightarrow^{*} s_2 \twoheadrightarrow^{\leq\omega} t_2$ such that all steps in $s_1 \twoheadrightarrow^{\leq\omega} t_1$ and $s_2 \twoheadrightarrow^{\leq\omega} t_2$ at depth $> d$.

Then $s \twoheadrightarrow^{*} s_1$ and $s \twoheadrightarrow^{*} s_2$ can be joined by finitary diagram completion employing the diamond property for parallel steps. If follows that there exists a term s' and finite sequences of (possibly infinite) parallel steps $s_1 \mathrel{-\!\!\parallel\!\!\!\to}^{*} s'$ and $s_2 \mathrel{-\!\!\parallel\!\!\!\to}^{*} s'$ all steps of which are at depth $\geq d$ (Lemma 3.4). We project $s_1 \twoheadrightarrow^{\leq\omega} t_1$ over $s_1 \mathrel{-\!\!\parallel\!\!\!\to}^{*} s'$ $s_2 \twoheadrightarrow^{\leq\omega} t_2$ over $s_2 \mathrel{-\!\!\parallel\!\!\!\to}^{*} s'$ by repeated application of the Lemma 3.5, obtaining rewrite sequences $t_1 \twoheadrightarrow^{\infty} t'_1$, $s' \twoheadrightarrow^{\infty} t'_1$, $t_2 \twoheadrightarrow^{\infty} t'_2$, and $s' \twoheadrightarrow^{\infty} t'_2$ with depth $\geq d$, $> d$, $\geq d$, and $> d$, respectively. As a consequence we have t'_1, s' and t'_2 coincide up to (including) depth d. Recursively applying the construction to the rewrite sequences $s' \twoheadrightarrow^{\infty} t'_1$ and $s' \twoheadrightarrow^{\infty} t'_2$ yields strongly convergent rewrite sequences $t_2 \twoheadrightarrow^{\infty} t'_2 \twoheadrightarrow^{\infty} t''_2 \twoheadrightarrow^{\infty} \ldots$ and $t_1 \twoheadrightarrow^{\infty} t'_1 \twoheadrightarrow^{\infty} t''_1 \twoheadrightarrow^{\infty} \ldots$ where the terms $t_1^{(n)}$ and $t_2^{(n)}$ coincide up to depth $d+n-1$. Thus these rewrite sequences converge towards the same limit u. $\square$

We consider an example to illustrate that non-collapsingness is a necessary condition for Theorem 3.6.

Example 3.7. Let R be a TRS over the signature $\{f, a, b\}$ consisting of the rule:

$$f(x, y) \to x$$

Then, using a self-explaining recursive notation, the term $s = f(f(s, b), a)$ rewrites in ω many steps to $t_1 = f(t_1, a)$ as well as $t_2 = f(t_2, b)$ which have no common reduct. The TRS R is weakly orthogonal (even orthogonal) but not confluent.

4 The Diamond Property for Developments

We prove that infinitary developments in weakly orthogonal TRSs without collapsing rules have the diamond property. For this purpose we establish an orthogonalization algorithm for co-initial developments, that is, we make the developments orthogonal to each other by elimination of overlaps. Since overlapping steps in weakly orthogonal TRSs have the same targets, we can replace one by the other. The challenge is to reorganize the steps in such a way that no new overlaps are created.

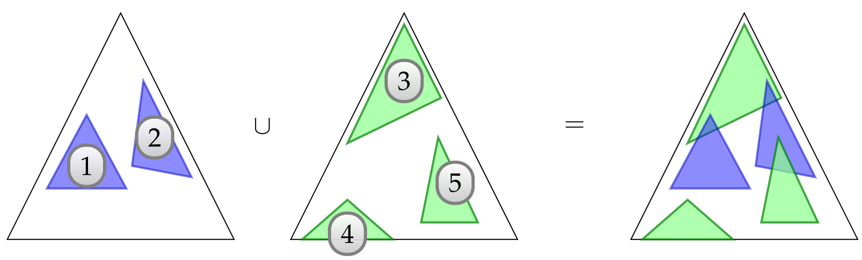

Figure 10: *Orthogonalization in a weakly orthogonal TRS.*

Consider for example Figure 10, where the redexes 2 and 3 overlap with each other. When trying to solve this overlap, we have to be careful since replacing the redex 2 by 3 as well as 3 by 2 creates new conflicts.

The case of finitary weakly orthogonal rewriting is treated by van Oostrom and de Vrijer in [5, Theorem 8.8.23]. They employ an inside-out algorithm, that is, inductively extend an orthogonalization of the subtrees to the whole tree. The basic observation is that you overcome the difficulties pointed out above by starting at the bottom of the tree and solving overlaps by choosing the deeper (innermost) redex.

Example 4.1. We consider Figure 10 and apply the orthogonalization algorithm from [5, Theorem 8.8.23]. We start at the bottom of the tree. The first overlap we find is between the redexes 2 and 5; this is removed by replacing 2 with 5. Then the overlap between 2 and 3 has also disappeared. The only remaining overlap is between the redexes 3 and 1. Hence we replace 3 by 1. As result we obtain two orthogonal developments $\{1, 5\}$ and $\{1, 4, 5\}$.

Note that the above algorithm does not carry over to the case of infinitary developments since we may have infinite chains of overlapping redexes and thus have no bottom to start at. This is illustrated in Figure 11.

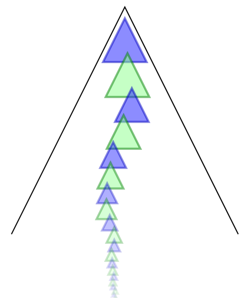

Figure 11: *Infinite chain of overlaps.*

Example 4.2. As an example where such an infinite chain of overlaps arises we consider the TRS R consisting of the rule:

$$A(A(A(x))) \to A(x)$$

together with two developments of blue and green redexes in the term A^ω:

$$A(A(A(A(A(A(A(A(A(A(A(A(A(A(A(\ldots)\ldots)$$

The blue redexes are marked by overlining, the green redexes by underlining.

Definition 4.3. Let R be a weakly orthogonal TRS, and $t \in \mathit{Ter}^\infty(\Sigma)$ a term. For redexes u and v in t we write $u \leftrightsquigarrow v$ if the pattern of u and v overlap in t.

Definition 4.4. Let R be a weakly orthogonal TRS, $t \in \mathit{Ter}^\infty(\Sigma)$ a term, and σ, δ two developments of sets of redexes U and V in t, respectively. We call σ and δ *orthogonal (to each other)* if for all redexes $u \in U$ and $v \in V$ with $u \leftrightsquigarrow v$ we have that $u = v$ (redexes are the same, that is, with respect to the same rule and position).

An *orthogonalization* $\langle \sigma', \delta' \rangle$ *of* σ *and* δ consists of orthogonal developments σ', δ' of redexes in t such that the results (targets) of σ' and δ' coincide with the results of σ and δ, respectively.

The overlap relation $\leftrightsquigarrow$ is symmetric, and hence the redexes form clusters with respect to the transitive closure $\leftrightsquigarrow^*$. If such a cluster contains parallel redexes, then the redex-cluster is called *Y-cluster* in [2]. The redexes in a Y-cluster we call Y-redexes; they can be defined as follows.

Definition 4.5. Let R be a weakly orthogonal TRS, and $t \in \mathit{Ter}^\infty(\Sigma)$ a term. A redex u in t is called *Y-redex* if there exist redexes v_1, v_2 at disjoint positions in t such that $u \leftrightsquigarrow^* v_1$ and $u \leftrightsquigarrow^* v_2$ (see also Figure 12, cases (ii) and (iv)).

At first sight one might expect that Y-redexes are due to trivial rules of the form $\ell \to r$ with $\ell \equiv r$. However, the following example illustrates that this is in general not the case (for another example see [5, p.508, middle]).

Example 4.6. Let R consist of the following rules:

$$(\rho_1)\ \ f(g(x,y)) \to f(g(y,x)) \qquad (\rho_2)\ \ g(a,a) \to g(a,a) \qquad (\rho_3)\ \ a \to a$$

We consider the term $t \equiv f(g(a,a))$ which contains a ρ_1-redex u_ϵ at the root, a ρ_2-redex u_1 at position 1, and two ρ_3-redexes u_{11} and u_{12} at position 11 and 12, respectively. We have $u_\epsilon \leftrightsquigarrow u_1$, $u_1 \leftrightsquigarrow u_{11}$ and $u_1 \leftrightsquigarrow u_{12}$. Since u_{11}, and u_{12} are at disjoint positions, it follows that u_ϵ is a Y-redex. However, the rule ρ_1 permutes its subterms, and thus in general may very well have an effect.

Despite the above example, it is always safe to drop Y-redexes from developments without changing the outcome of the development. This result is implicit in [2]. In particular in [2, Remark 4.38] it is mentioned that Y-clusters are a generalisation of Takahashi-configurations.

Lemma 4.7. *Let R be a weakly orthogonal TRS, $t \in Ter^\infty(\Sigma)$ a term, and U a set of non-overlapping redexes in t which have a development. If $u \in U$ is a Y-redex, then the development of U results in the same term as the development of $U \setminus \{u\}$.*

Proof. Since u is a Y-redex there exist redexes v_1, v_2 at disjoint positions in t such that $v_1 \leftrightsquigarrow^* u \leftrightsquigarrow^* v_2$. By weak orthogonality overlapping redexes have the same effect. Whenever $w_1 \leftrightsquigarrow w_2 \leftrightsquigarrow w_3$, it follows that w_1 has the same effect as w_3. Consequently all redexes in the Y-cluster $Y_u = \{v \mid u \leftrightsquigarrow^* v\}$ of u have the same effect. However, v_1 and v_2 are at disjoint positions, and thus it follows that $(*)$ contraction of any redex in Y_u leaves t unchanged.

We now develop U using an innermost strategy (from bottom to top). Then reducing redexes below the pattern of Y_u leaves the Y-cluster unchanged, and if a redex v of Y_u (among which is u) is reduced, then by $(*)$ the term is left unchanged, and by innermost strategy there are no redexes nested in v which could be influenced (copied/ deleted). Thus contracting u has no effect. $\square$

We also give the following alternative proof using results from [2].

Proof (by Vincent van Oostrom). We cut out the Y-cluster (union of the redex patterns), and introduce distinct new variables for the cut-off subterms. Contracting a redex in the Y-cluster leaves the Y-cluster unchanged. In particular no subterms matched by variables are moved, copied or deleted. As a consequence contracting u can only affect redexes which are in the Y-cluster of u, and thus in turn their contraction does not change the term. It follows that u can be dropped from the development. $\square$

We now devise a top–down orthogonalization algorithm. Roughly, we start at the top of the term and replace overlapping redexes with the outermost one. However, care has to be taken in situations as depicted in Figure 10.

Theorem 4.8. *Let* R *be a weakly orthogonal TRS,* $t \in \mathit{Ter}^{\infty}(\Sigma)$ *a (possibly infinite) term, and* σ, δ *two developments of sets of redexes* U *and* V*, respectively. Then there exists an orthogonalization of* U *and* V*.*

Proof. We obtain an orthogonalization of U and V as the limit of the following process. If there are no overlaps between U and V, then we are finished. Here, by overlap we mean non-identical redexes whose patterns overlap. Otherwise, if there exist overlaps, let $u \in (U \cup V)$ be a topmost redex (that is, having minimal depth) among the redexes which have an overlap. Without loss of generality (by symmetry) we assume that $u \in U$ and let $v \in V$ be a topmost redex among the redexes in V overlapping u. We distinguish the following cases:

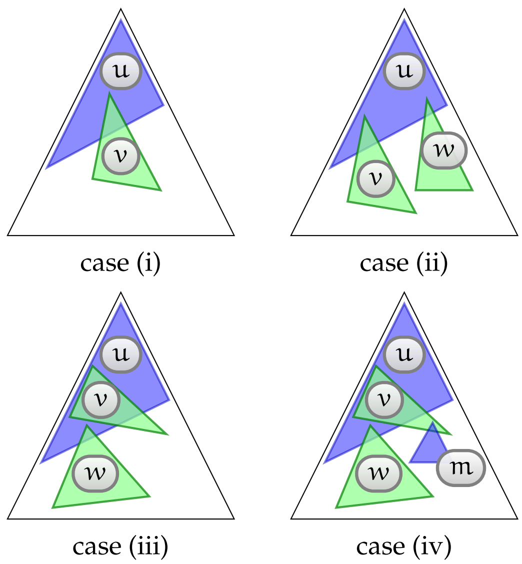

Figure 12: *Case distinction for the orthogonalization algorithm.*

(i) If v is the only redex in V that overlaps with u, case (i) of Figure 12, then we can safely replace v by u.

Otherwise there is a redex $w \in V$, $w \neq v$ and w overlaps u.

(ii) Assume that v and v' are at disjoint positions, case (ii) of Figure 12. Then u, v, w are Y-redexes and can be dropped from U and V by Lemma 4.7.

Otherwise, v and v' are not disjoint, and then w must be nested inside v.

(iii) If u is the only redex from U overlapping v, case (iii) of Figure 12, then we can replace u by v.

(iv) In the remaining case there must be a redex $m \in U$, $m \neq u$ and m overlaps with the redex v, see case (iv) of Figure 12. Since U and V are developments u cannot overlap with m, and v cannot overlap with w. We have that w is nested in v, both overlapping u, but m is below the pattern of u, overlapping v. Hence w and m must be at disjoint positions (v cannot tunnel through w to touch m); this has also been shown in [2]. Then by Lemma 4.7 all redexes u, m, v and w are Y-redexes and can be removed.

We have shown that it is always possible to solve outermost conflicts without creating fresh ones. Hence we can push the conflicts down to infinity and thereby obtain the orthogonalization of U and V. □

We obtain the diamond property as corollary.

Corollary 4.9. *For every weakly orthogonal TRS without collapsing rules, (infinite) developments have the diamond property.*

Proof. Let σ, δ be two coinitial developments $t_1 \xleftarrow{\sigma} s \xrightarrow{\delta} t_2$. Then by Theorem 4.8 there exists an orthogonalization $\langle \sigma', \delta' \rangle$ of σ, δ. The orthogonal projections σ'/δ' and δ'/σ' are developments again, which are strongly convergent since the rules are not collapsing. Hence $t_1 \xrightarrow{\delta'/\sigma'} s' \xleftarrow{\sigma'/\delta'} t_2$. □

Note that in Corollary 4.9 the non-collapsingness is a necessary condition. To see this, reconsider Example 3.7 and observe that the non-confluent derivations are developments.

In a similar vein, we can prove the triangle property for infinitary weakly orthogonal developments without collapsing rules, but we will postpone this to future work.

5 Conclusions

We have shown the failure of UN^∞ for weakly orthogonal TRSs in the presence of two collapsing rules. For weakly orthogonal TRSs without collapsing rules we establish that CR^∞ (and hence UN^∞) holds. This result is optimal in the sense that allowing only one collapsing rule may invalidate CR^∞.

However, Question 2.2 remains open:

Does UN^∞ *hold for weakly orthogonal TRSs with one collapsing rule?*

Furthermore, we have shown that infinitary developments in weakly orthogonal TRSs without collapsing rules have the diamond property. In general this property fails already in the presence of only one collapsing rule.

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