SET SYSTEMS: ORDER TYPES, CONTINUOUS NONDETERMINISTIC DEFORMATIONS, AND QUASI-ORDERS

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Abstract. By reformulating a learning process of a set system L as a game between Teacher and Learner, we define the order type of L to be the order type of the game tree, if the tree is well-founded. The features of the order type of L ($\dim L$ in symbol) are (1) We can represent any well-quasi-order (WQO for short) by the set system L of the upper-closed sets of the WQO such that the *maximal order type* of the WQO is equal to $\dim L$. (2) $\dim L$ is an upper bound of the mind-change complexity of L. $\dim L$ is defined iff L has a finite elasticity (FE for short), where, according to computational learning theory, if an indexed family of recursive languages has FE then it is learnable by an algorithm from positive data. Regarding set systems as subspaces of Cantor spaces, we prove that FE of set systems is preserved by any continuous function which is monotone with respect to the set-inclusion. By it, we prove that finite elasticity is preserved by various (nondeterministic) language operators (Kleene-closure, shuffle-closure, union, product, intersection,. . ..) The monotone continuous functions represent nondeterministic computations. If a monotone continuous function has a computation tree with each node followed by at most n immediate successors and the order type of a set system L is α, then the direct image of L is a set system of order type at most n-adic diagonal Ramsey number of α. Furthermore, we provide an order-type-preserving contravariant embedding from the category of quasi-orders and finitely branching simulations between them, into the complete category of subspaces of Cantor spaces and monotone continuous functions having Girard's linearity between them. Keyword: finite elasticity, shuffle-closure, Ramsey's theorem, finitely branching simulation, game, order type

1. Introduction

A set system $\mathcal{L}$ over a set T, a subfamily of the power set $P(T)$, is a topic of (extremal) combinatorics [1, 2], as well as a target of an algorithm to learn in computational learning theory of languages [3].

By reformulating a learning process of a set system $\mathcal{L}$ as a game between Teacher and $\mathcal{L}$earner, we define the order type of $\mathcal{L} \subseteq P(T)$ to be the order type of the game tree. The features of the order type of $\mathcal{L}$ ($\dim \mathcal{L}$ in symbol) are followings:

- We can represent any well-quasi-order (WQO for short) by the set system of the upper-closed sets of the WQO such that the *maximal order type* [4] of the WQO is equal to the $\dim \mathcal{L}$.
- $\dim \mathcal{L}$ is an upper bound of the mind-change complexity [5] of $\mathcal{L}$ which is recently studied in relation to Noetherian property of algebras, set-theoretical topology and reverse mathematics [6, 7, 8, 9]. $\dim L$ is defined if and only if L has a finite elasticity (FE for short), where, according to computational

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learning theory [10, 3], if an indexed family of recursive languages has FE then it is learnable by an algorithm from positive data.

In computational learning of languages, a set system algorithmically learnable from positive data is often a combination of set systems (e.g. extended pattern languages [3].) To discuss which combinatorial operations for set systems preserve FE, quantitatively with the order type of the set systems, let us consider a motivating example. Suppose $\mathcal{L}$ is the class of arithmetical progressions over $\mathbb{N}$. Observe the class of binary unions of arithmetical progressions over $\mathbb{N}$, that is, $\mathcal{L} \mathbin{\widetilde{\cup}} \mathcal{L} := \{L \cup M \ ; \ L, M \in \mathcal{L}\}$ is more difficult to learn than $\mathcal{L} \mathbin{\widetilde{\uplus}} \mathcal{L} := \{\mathcal{L} \uplus M \ ; \ L, M \in \mathcal{L}\}$, where $L \uplus M$ is the disjoint union of L and M, i.e., the union of the progression L colored red and the progression M colored black. The difficulty of $\mathcal{L} \mathbin{\widetilde{\cup}} \mathcal{L}$ is because the discoloration brings *nondeterminism* to Teacher and/or Learner. By the discoloration of $L \uplus M$, we mean $L \cup M$, and by that of $\mathcal{L} \mathbin{\widetilde{\uplus}} \mathcal{L}$, we mean $\mathcal{L} \mathbin{\widetilde{\cup}} \mathcal{L}$. We can notice that the discolorization of the direct product $L \times M$ of languages L, M is the concatenation $L \cdot M$, and observe that $\mathcal{L} \mathbin{\widetilde{\times}} \mathcal{L} = \{L \times M \ ; \ L, M \in \mathcal{L}\}$ is easier to learn than the discolorization $\mathcal{L} \mathbin{\widetilde{\cdot}} \mathcal{L} = \{L \cdot M \ ; \ L \in \mathcal{L}\}$.

Following questions are central in this paper:

Question 1. *Does discoloration preserve finite elasticity?*

Question 2 ([11, 12, 13, 14])**.** *Which operations for set systems preserve finite elasticity?*

Question 3. *What is the nondeterminism brought by operations that preserve finite elasticity?*

Question 4. *How much do such operations increase the order type of set systems?*

Question 1 is yes, because Ramsey's theorem [15] implies any dichromatic coloring of any infinite game sequence of $\mathcal{L} \mathbin{\widetilde{\cup}} \mathcal{L}$ has an infinite, monochromatic game subsequence of $\mathcal{L}$. This is another saying of Motoki-Shinohara-Wright's theorem [16, 10]. This argument leads to a solution of Question 4 with Ramsey number [15].

For Question 2, first observe that the discoloration $L \cup M$ of $L \uplus M$ is the *inverse image* $R^{-1}[L \uplus M] = \{s \ ; \ \exists u \in L \uplus M. R(s, u)\}$ by a following finitely branching relation: $R(s, u) :\iff u = \langle\, s, \mathit{red}\,\rangle$ or $u = \langle\, s, \mathit{black}\,\rangle$. For a relation $R \subseteq X \times Y$, the inverse images of a set M and a set system $\mathcal{M}$ are, by definition, respectively

$$(1) \quad R^{-1}[M] := \{x \in X \ ; \ \exists y \in M. \ R(x, y)\}, \quad \widetilde{R^{-1}}[\mathcal{M}] := \{R^{-1}[M] \ ; \ M \in \mathcal{M}\}.$$

Let us abbreviate "a set system with finite elasticity" by an FESS. In [13, 14], Kanazawa derived "the inverse image of an FESS by a finitely branching relation again an FESS" from König's lemma, and established not only the union but also the permutation closure and so on preserves FESSs. We generalize his lemma further as: "the direct image $\mathcal{L}$ of an FESS $\mathcal{M}$ by a *continuous* function which is monotone with respect to the set-inclusion is again an FESS." Here we regard $\mathcal{L}$ and $\mathcal{M}$ as subspaces of Cantor spaces, which are the product topological spaces $\{0, 1\}^{\bigcup \mathcal{L}}$, $\{0, 1\}^{\bigcup \mathcal{M}}$ of copies of finite discrete topological space $\{0, 1\}$.

Interestingly, a monotone, continuous function is a stable function [17] plus a modest nondeterministic computation, so to say. To explain the relation among monotone, continuous functions, (linear) stable functions and nondeterminism, let us consider a following characterization by Tychonoff's theorem: a monotone, continuous function is a function $\mathfrak{D} : \mathcal{M} \to \mathcal{L}$ such that there is a finitely branching relation $R \subseteq (\bigcup \mathcal{L}) \times [\bigcup \mathcal{M}]^{<\omega}$ satisfying that for all $x \in \bigcup \mathcal{L}$ and all $M \in \mathcal{M}$,

$$\mathfrak{D}(1_M)(x) = \begin{cases} 1, & (\exists v \subseteq M.\ R(x, v))\,; \\ 0, & \text{(otherwise,)} \end{cases}$$

where $[\bigcup \mathcal{M}]^{<\omega}$ is the class of finite subsets of $\bigcup \mathcal{M}$ and 1_M is the indicator function of the set M. From *linear logic* [17] point of view, when $\mathcal{L}$ and $\mathcal{M}$ are coherence spaces and $\#\{v\ ;\ R(x, v)\} \leq 1$ for all x, then $\mathfrak{D}$ becomes a stable function from $\mathcal{L}$ to $\mathcal{M}$, and if further $\forall x \forall v. (R(x, v) \Rightarrow \#v \leq 1)$ holds, then $\mathfrak{D}$ becomes a *linear* stable function [17]. Kanazawa's lemma is nothing but "the direct image of an FESS by a *linear*, monotone, continuous function is again an FESS" where the relation R in the lemma is the *trace* [17] of the linear function.

For Question 3, the nondeterminism brought by the (linear) monotone, continuous functions $\mathfrak{D}$ are the "finite OR-parallelism" caused by finite sets v's. The degree of the nondeterminism is $\#\{v\ ;\ R(x, v)\}$. In other words, the trace R of the monotone, continuous function is finitely branching, while that of stable function has at most one branching. So we can easily prove that there are monotone, continuous functions $1_L \mapsto 1_{L^*}$ and $1_L \mapsto 1_{L^{\circledast}}$ where $L^{\circledast}$ is the shuffle-closure [18] of L. Here are a non-example and an example of nondeterminism.

- Because a Π-continuous function [6] can represent an unbounded search unlike monotone, continuous functions, the direct image of an FESS by a Π-continuous function is not necessarily an FESS (see Theorem 6.)
- We define the category $\mathbb{QO}_{FinSim}$ of *quasi-orders* and finitely branching simulations between them. Here a usual order-homomorphism is an instance of a finitely branching *simulation* which appears in concurrency theory. Let $\mathbb{SS}$ be the complete category of set systems and monotone, continuous functions between them. We provide an *order-type-preserving* contravariant embedding from $\mathbb{QO}_{FinSim}$ to $\mathbb{SS}$. By this embedding, each quasi-order is sent to the family of upper-closed sets. When the branching of the relation is at most 1, it is sent to a stable (*sequential*) function [17] in $\mathbb{SS}$. In fact, the category of coherence spaces and stable functions between them, introduced in [17] embeds in $\mathbb{SS}$.

As for Question 4, the Ramsey number argument for Question 1 establishes : If a monotone, continuous function $\mathfrak{D}$ with the trace R has n such $\#\{v\ ;\ R(s, v)\} \leq n$ for each s, then the direct image of $\mathcal{L}$ by $\mathfrak{D}$ has order type at most the n-adic diagonal Ramsey number of $\dim \mathcal{L} + 2$.

This paper is organized as follows. In the next section, we review parts of order theory, various (closure) operations of languages from algebraic theory [18, 19] of languages and automata, and finite elasticity of computational learning theory. In Section 3, we introduce the order type of a set system, and then represent every quasi-order by a set system having the same order type as the quasi-order. We prove that if the set system $\mathcal{L}$ is an indexed family of recursive languages, as in the case of computational learning theory, and if moreover the indexing is without repetition, then $\dim \mathcal{L}$ is exactly a recursive ordinal. In Section 4, we prove "the direct image of an FESS by a monotone, continuous function is again an FESS." In Section 5, we employ Ramsey numbers to answer Question 4. In Section 6, we embed the category $\mathbb{QO}_{FinSim}$ and a categorical model of linear logic in the category $\mathbb{SS}$. In A, we record the proof of Theorem 9 on the categorical structure of $\mathbb{SS}$, $\mathbb{SS}_{lin}$ and $\mathbb{SS}_{seq}$, where $\mathbb{SS}_{lin}$ is the subcategory induced by linear functions and $\mathbb{SS}_{seq}$ by sequential functions. We prove the category $\mathbb{SS}_{seq}$ does not have a binary coproduct because the sequential function does not represent a nondeterministic computation. And then we discuss whether $\mathbb{SS}$ has the duality operator and the bang operator as the category of coherence spaces.

2. Preliminaries

Let $R \subseteq \mathbf{S} \times \mathbf{U}$ be a relation. If the cardinality $B_R(s)$ of $\{u \in \mathbf{U}\ ;\ R(s,u)\}$ is finite for all $s \in \mathbf{S}$, then we say R is *finitely branching*. If $B_R(s) \leq 1$ for all $s \in \mathbf{S}$, then we say R is a *partial function*. For a set $\mathbf{U}$, let $[\mathbf{U}]^{<\alpha}$ be the class of subsets A of $\mathbf{U}$ such that $\#A < \alpha$.

2.1. Order theory.

A quasi-order (QO for short) over a set X is a pair $\mathcal{X} = (X, \preceq)$ where $\preceq$ is a reflexive, transitive relation. A *bad* sequence is a possibly infinite sequence $\langle a_0, a_1, \ldots, a_n(, \ldots)\rangle$ such that $a_i \not\preceq a_j$ whenever $i < j$. A *well-quasi-order* (WQO for short) is a quasi-order that has no infinite bad sequences. For $A \subseteq X$, let $A \uparrow \mathcal{X} := \{x \in \mathcal{X}\ ;\ \exists a \in A.\, a \preceq x\ \}$.

Definition 1. *For a quasi-order $\mathcal{X} = (X, \preceq)$, let a set system* $\mathrm{ss}(\mathcal{X})$ *be the complete lattice of upper-closed subset of X with respect to $\mathcal{X}$.*

Proposition 1 ([20, Theorem 2.1])**.** *For every quasi-ordered set $\mathcal{X}$, the following are equivalent:*

1. *$\mathcal{X}$ is a* WQO.
2. Finite basis property*: Every $A \uparrow \mathcal{X}$ is $B \uparrow \mathcal{X}$ for some $B \in [X]^{<\omega}$.*
3. Ascending chain condition*:* $\mathrm{ss}(\mathcal{X})$ *is a complete lattice with ascending chain condition. That is, there is no infinite, strictly ascending sequence of members.*

The length of a sequence $\sigma = \langle b_1, \ldots, b_m\rangle$ is, by definition, $ln(\sigma) = m$, and the length of an infinite sequence σ is, by definition, $ln(\sigma) = \infty$.

By a *tree*, we mean a set T of finite sequences such that any initial segment of a sequence in T is in T. A tree T is said to be *well-founded* if there is no infinite sequence $\langle a_1, a_2, \ldots\rangle$ such that $\langle a_1, \ldots, a_n\rangle$ is in T for each n.

Let T be a well-founded tree. For each node σ of T, let the ordinal number $|\sigma|$ be the supremum of $|\sigma'| + 1$ such that $\sigma' \in T$ is an immediate extension of σ. Then the *order type* $|T|$ of the well-founded tree T is defined by the ordinal number $|\langle\,\rangle|$ assigned to the root $\langle\,\rangle$ of T. For a tree T which is not well-founded, let $|T|$ be ∞. For the sake of convenience, we set $\alpha < \infty$ for all ordinal numbers α. As in [21], we define the *order type* $\mathrm{otp}(\mathcal{X})$ of a WQO $\mathcal{X}$ to be the order type of the well-founded tree of bad sequences in $\mathcal{X}$. According to [21, Sect. 2], $\mathrm{otp}(\mathcal{X})$ is equal to the *maximal order type* of de Jongh-Parikh [4].

By an *embedding* from a tree T to a tree T', we mean an injection $f : T \to T'$ such that $f(v \sqcup u) = f(v) \sqcup f(u)$ for all vertices u, v in T, where $v \sqcup u$ is the greatest common ancestor of a pair of vertices u, v.

Fact 1. *If there is an embedding from a tree T to a tree T', then $|T| \leq |T'|$.*

2.2. **Computational learning theory for languages.** A *set system* is a subfamily of a power set. We use $\mathcal{L}, \mathcal{M}, \mathcal{N}, \dots$ to represent set systems.

We say a set system $\mathcal{L}$ over X has an *infinite elasticity*, if there are infinite sequences $t_0, t_1, \dots \in X$ and $L_1, L_2, \dots \in \mathcal{L}$ such that $\{t_0, \dots, t_{i-1}\} \subseteq L_i \not\ni t_i$ for every positive integer i. Otherwise, we say $\mathcal{L}$ has a *finite elasticity* (FE.) A set system with an FE is abbreviated as an FESS.

Let $\mathbb{N}$ be the set of nonnegative integers.

Example 1. (1) The class of integer lattices contained in $\mathbb{Z}^d$ and the class of ideals over $\mathbb{Z}[x, y, z]$ are FESSs, because $\mathbb{Z}^d$ and $\mathbb{Z}[x, y, z]$ are is a Noetherian module and a Noetherian ring respectively [22, p. 112].

(2) The class of finitely generated free sub-semigroups of $(\mathbb{N}^2, +)$ is not an FESS ([23].)

(3) The class of (extended) pattern languages with bounded number of variables is an indexed family of recursive languages, and is an FESS ([10]. For an elementary proof, see [23].)

(4) $\mathcal{S}ingl := \{\{x\} \ ; \ x \in \mathbb{N}\}$ of singletons is an FESS.

(5) The class $\mathcal{D}cl := \{\{y \ ; y \le x\} \ ; \ x \in \mathbb{N}\} \subseteq P(\mathbb{N})$ is not an FESS.

Definition 2. *By an* indexed family of recursive languages (IFRL for short)*, we mean a pair $\mathcal{L} = (\nu : J \to X, \ \gamma : I \times J \to \{0, 1\})$ such that $I, J \subseteq \mathbb{N}$, ν is a bijection, and γ is recursive. Put $\mathcal{L}^i := \{\nu(j) \in X \ ; \ \gamma(i, j) = 1\}$ for $i \in I$. An* IFRL *without repetition is just an* IFRL *such that $\mathcal{L}^i \ne L^j$ for distinct i, j.*

Proposition 2 ([16, 10])**.** *Every* IFRL *with an* FE *is learnable from positive data by an algorithm.*

By an alphabet we mean a finite nonempty set. Let Σ be an alphabet. Denote the empty word by ε. For words $u, v \in \Sigma^*$, the *shuffle product* $u \diamond v$ of u and v is, by definition, the set of all the words $u_1v_1u_2v_2...u_nv_n$ such that $\exists n \ge 1 \exists u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n \in \Sigma^*$ we have $u = u_1u_2 \cdots u_n$ and $v = v_1v_2 \cdots v_n$. For $L, M \subseteq \Sigma^*$, let $L \diamond M := \bigcup\{u \diamond v \ ; \ u \in L, v \in M\}$. Put $L^\diamond := L \cup (L \diamond L) \cup (L \diamond L \diamond L) \cup \cdots$. Let us call $L^{\circledast} := L^\diamond \cup \{\varepsilon\}$ *the shuffle-closure of* L. The shuffle-product and shuffle-closure are studied in algebraic theory of automata and languages [18, 19], for example.

Let a disjoint union of languages B_i $(i \in I)$ be

$$\biguplus_{i \in I} B_i := \{\langle\, b, \, i \,\rangle \ ; \ b \in B_i, i \in I \ \}.$$

For a language M, let M^m be $\overbrace{M \cdot M \cdot \cdots \cdot M}^{m}$ $(m \ge 1)$, let us call $M^+ := \bigcup_{m \ge 1} M^m$ *the positive Kleene-closure*, and let M^* be the Kleene-closure. Let $M^{\circledast}$ be the shuffle-closure of M, $\frac{1}{2}(M)$ be the half initial segment.

For all $\mathcal{L}_i \subseteq P(X_i)$ $(1 \le i \le n)$ and an operation $\odot$ on languages of arity n, put

$$\widetilde{\odot}\, (\mathcal{L}_1, \dots, \mathcal{L}_n) := \{\odot(L_1, \dots, L_n) \ ; \ L_i \in \mathcal{L}_i, (1 \le i \le n) \ \}.$$

Here is an application of Ramsey's theorem:

Proposition 3 (Motoki-Shinohara-Wright [16, 10])**.** *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are* FESS*s, so is $\mathcal{L}_1 \mathbin{\widetilde{\cup}} \mathcal{L}_2$.*

In fact, it is derived from a weak principle: König's lemma.

Proposition 4 (Moriyama-Sato [12]). *For a fixed finite alphabet, the family of language classes with* FE *is closed under* $\cup$, $\widetilde{\cup}$, $\widetilde{\cdot}$, $\widetilde{\cap}$, $\widetilde{\ ^*}$, $\widetilde{\ ^+}$, *and* $\widetilde{\ ^m}$ *for every positive integer* m*, but not under elementwise complement.*

Independently, Kanazawa [13, 14] proved a following nice result by using König's lemma:

Proposition 5 (Kanazawa [13, 14]). *If* $\mathcal{M} \subseteq P(Y)$ *is an* FESS *and* $R \subseteq X \times Y$ *is finitely branching, then* $\widetilde{R^{-1}}[\mathcal{M}] \subseteq P(X)$ *is so.*

In fact, without invoking König's lemma, he showed

Lemma 1 ([13, 14]). *If* $\dim \mathcal{L}, \dim \mathcal{M} < \infty$*, then* $\dim \mathcal{L} \,\widetilde{\uplus}\, \mathcal{M}$.

Then he proved various language operations preserves FESSs, by applying Proposition 5.

Corollary 1 (Kanazawa [13, 14]). *For a fixed alphabet, the family of language classes with an* FE *is closed under* $\widetilde{\cup}$*, elementwise permutation closures,* $\widetilde{\diamond}$*, and* $\widetilde{\frac{1}{2}}(\cdot)$*. For each nonerasing homomorphism* $h : \Sigma_1^* \to \Sigma_2^*$*, if a language class* $\mathcal{L} \subseteq P(\Sigma_1^*)$ *has an* FE*, so does* $\widetilde{h}[\mathcal{L}] \subseteq P(\Sigma_2^*)$*. If* $\mathcal{L}$ *is a class of* ε*-free languages with an* FE*, then so is* $\{L_1 \cdot L_2 \cdot \cdots \cdot L_n \,;\, n \geq 1, L_1, \ldots, L_n \in \mathcal{L}\}$.

3. Order types of set systems and WQOs

We introduce order types of set systems, study the set system of upper-closed subsets of a QO from viewpoint of order types and algebraic theory of lattices [24].

We regard a learning process of a set system $\mathcal{L} \subseteq P(T)$, as a game between Teacher T and Learner $\mathcal{L}$ where in each inning $i \geq 1$ Teacher presents a "fresh" example $e_{i-1} \in T$ and Learner submits a hypothesis $H_i \in \mathcal{L}$ that explains examples presented so far, that is, $\{e_0, \ldots, e_{i-1}\} \subseteq H_i$. By a "fresh"example e_{i-1}, we mean $e_i \notin H_i$. The well-foundedness of the game tree coincides with the *finite elasticity* [16, 10] of the set system $\mathcal{L}$, which was introduced in computational learning theory of languages [3]. If $\mathcal{L}$ is further an IFRL, then some algorithm can learn $\mathcal{L}$ from positive data [16, 10]. First, we introduce *the order type* $\dim \mathcal{L}$ *of the set system* $\mathcal{L}$ by the order type of the game tree.

Definition 3 (Production sequence). *A* production sequence *of a set system* $\mathcal{L}$ *is a sequence* $\langle\langle t_0, L_1\rangle, \langle t_1, L_2\rangle, \ldots, \langle t_{m-1}, L_m\rangle\rangle$ $(m \geq 0)$ *or an infinite sequence* $\langle\langle t_0, L_1\rangle, \langle t_1, L_2\rangle, \ldots\rangle$ *such that*

$$\{t_0, \ldots, t_{i-1}\} \subseteq L_i \in \mathcal{L} \ (i = 1, 2, \ldots (, m)) \quad \textit{and} \quad L_j \not\ni t_j \ (j = 1, 2, \ldots (, m-1).)$$

Let $\mathrm{Prod}(\mathcal{L})$ *be the set of all production sequences of* $\mathcal{L}$.

Clearly a sequence $\langle L_1, L_2, \ldots\rangle$ is a bad sequence in a poset $(\mathcal{L}, \supseteq)$, because $i < j$ implies $L_j \setminus L_i \ni t_i$.

Definition 4 (Dimension). *The* dimension *of* $\mathcal{L}$*, denoted by* $\dim \mathcal{L}$*, is defined to be* $|\mathrm{Prod}(\mathcal{L})|$.

By Fact 1, $\mathcal{L} \subseteq \mathcal{L}'$ implies $\dim \mathcal{L} \leq \dim \mathcal{L}'$.

Let us see examples of order types of set systems.

We note that any ordinal α is the dimension of the set system of upper-closed subsets of α.

As in [25, p. 384], we understand that a *recursive ordinal* is an ordinal number α such that $\alpha = |T|$ for some recursive well-founded tree T.

Theorem 1. *If an* IFRL $\mathcal{L}$ *without repetition is an* FESS*, then* $\dim \mathcal{L}$ *is a recursive ordinal. Conversely, for every recursive ordinal* α *there is an* IFRL $\mathcal{L}$ *without repetition such that* $\alpha = \dim \mathcal{L}$*.*

Proof. Let $\mathcal{L}$ be an IFRL without repetition by a pair of functions ν, γ. Define a set $T_{\mathcal{L}} \subset \mathbb{N}$ inductively as follows. We also use symbols '$\langle$' and '$\rangle$' for sequence numbers, and Odifreddi's notation [25, p. 88] of operations on sequence numbers. (1) $\langle\,\rangle \in T_{\mathcal{L}}$. (2) If $\gamma(e, j_0) = 1$, then $\langle\langle j_0, e\rangle\rangle \in T_{\mathcal{L}}$. (3) If $\sigma \in T_{\mathcal{L}}$, $\gamma\left(\left((\sigma)_{ln(\sigma)-1}\right)_1, j\right) = 0$, and $\gamma(e, j) = 1 = \gamma\left(e,\ ((\sigma)_k)_0\right)$ $\forall k < \ln(\sigma)$, then $\sigma * \langle\langle j, e\rangle\rangle \in T_{\mathcal{L}}$. Clearly $T_{\mathcal{L}}$ is a recursive tree.

Let $\varphi : \mathrm{Prod}(\mathcal{L}) \to T_{\mathcal{L}}$ take any $\langle\langle t_0, L_1\rangle, \langle t_1, L_2\rangle, \dots, \langle t_{l-1}, L_l\rangle\rangle \in \mathrm{Prod}(\mathcal{L})$ to a node $\langle\langle\, j_0,\, e_1\,\rangle, \langle\, j_1,\, e_2\,\rangle, \dots, \langle\, j_{l-1},\, e_l\,\rangle\rangle$ of $T_{\mathcal{L}}$ where e_i is the unique number such that $L_i = \mathcal{L}^{e_i}$ and j_i is the unique number such that $\nu(j_i) = t_i$. Then j_i is well-defined because ν is bijective, and e_i is too because $\mathcal{L}$ is an IFRL *without repetition*. The function φ is obviously an surjective order-homomorphism that preserves glb's, and in fact an injection because $\mathcal{L}$ is an IFRL without repetition. Therefore $\dim \mathcal{L} = |T_{\mathcal{L}}|$. Since $\mathcal{L}$ is an FESS, $T_{\mathcal{L}}$ is a recursive well-founded tree, so $\dim \mathcal{L}$ is a recursive ordinal number.

Next we prove the second assertion. The ordinal number α is constructive by [26, Theorem XX, Ch.11]. So there is a recursively related, univalent system assigning a notation to α by [26, Theorem XIX, Ch.11]. Therefore, there is an injective function ν from some set $J \subseteq \mathbb{N}$ onto an initial segment $\{\beta\ ;\ 0 \le \beta \le \alpha\} = \alpha + 1$ of the ordinal numbers such that

$$\{\langle x, y\rangle\ ;\ x, y \in J,\ \nu(x) \le \nu(y)\} \subseteq \mathbb{N} \text{ is recursive.} \tag{2}$$

In particular, J is recursive. When J is infinite, there is a recursive strictly monotone function d with the range being J. Put $\mathcal{L} := \{L^i\ ;\ i \in \mathbb{N}\}$, and $L^i := \{\beta\ ;\ \exists j \in J.\ \beta = \nu(j) \ge \nu(d(i))\ \}$. Because ν is a bijection to $\alpha + 1$, L^i is an upper-closed subset of $\alpha + 1$. Define a function $\gamma : \mathbb{N} \times J \to \{0, 1\}$ by $\gamma(i, j) = 1$ if $\nu(j) \ge \nu(d(i))$, 0 otherwise. From (2), γ is recursive. In fact, $\mathcal{L}$ is $\mathrm{ss}\,((\alpha + 1, \le))$ because the range of d is exactly J and $\nu(J) = \alpha + 1$. Then $\mathcal{L}$ is an IFRL. Moreover $\mathcal{L}$ is an IFRL without repetition because d and ν are injective. By Theorem 2 (1), $\dim \mathcal{L} = \mathrm{otp}\,((\alpha + 1, \le)) = \alpha$. When J is finite, we can prove the assertion similarly.

Next we introduce a left-inverse of $\mathrm{ss}\,(\bullet)$.

Definition 5. *For a set system* $\mathcal{L} \subseteq P(X)$*, define a quasi-order*

$$x \preceq_{\mathcal{L}} y :\iff \forall L \in \mathcal{L}\,(x \in L \Rightarrow y \in L.) \quad \mathrm{qo}\,(\mathcal{L}) := (X, \preceq_{\mathcal{L}}.)$$

Below, we prove that $\mathrm{ss}\,(\bullet)$ is an order-type preserving representation of QOs by set systems. In other words, the order type of a WQO turns out to be the difficulty in learning the class of upper-closed subsets of the WQO. Then we prove that $\mathrm{ss}\,(\bullet)$ indeed has $\mathrm{qo}\,(\bullet)$ as the left-inverse.

Theorem 2 (Representation of QO)**.** *Let* $\mathcal{X} = (X, \preceq)$ *be a quasi-order.*

(1) $\mathrm{otp}(\mathcal{X}) = \dim \mathrm{ss}\,(\mathcal{X})$.
(2) $\mathcal{X} = \mathrm{qo}\,(\mathrm{ss}\,(\mathcal{X}))$.

Proof. We prove the assertion (1), by a transfinite induction using

$$\exists L_1, \ldots, L_l.\ \langle\langle t_0, L_1\rangle, \langle t_1, L_2\rangle, \ldots, \langle t_{l-1}, L_l\rangle\rangle \in \mathrm{Prod}(\mathrm{ss}\,(\mathcal{X})) \\ \iff \langle t_0, \ldots, t_{l-1}\rangle \text{ is a bad sequence of } \mathcal{X}.$$

The $\Rightarrow$-part is demonstrated as follows: We have $\{t_0, \ldots, t_{i-1}\} \subseteq L_i \not\ni t_i$ $(1 \le i \le l-1)$. Because each $L_i \in \mathrm{ss}\,(\mathcal{X})$ is upper-closed, for any nonnegative integers $j < i \le l-1$, $t_j \not\preceq t_i$. The $\Leftarrow$-part is witnessed by $L_i := \{t_0, \ldots, t_{i-1}\}\uparrow\mathcal{X}$.

(2) Assume $x \preceq_{\mathrm{ss}(\mathcal{X})} y$. Then $\forall L \in \mathrm{ss}\,(\mathcal{X})\,.\,(x \in L \Rightarrow y \in L)$. Take $L := \{y \in X\ ;\ x \preceq y\} \in \mathrm{ss}\,(\mathcal{X})$. Hence $x \preceq y$. Conversely, assume $x \preceq y$. Then because every $L \in \mathrm{ss}\,(\mathcal{X})$ is upper-closed with respect to $\preceq$, $x \in L$ implies $y \in L$. Therefore $x \preceq_{\mathrm{ss}(\mathcal{X})} y$.

Theorem 3. *If* $\mathrm{qo}\,(\mathcal{L})$ *is a* WQO*,* $\mathcal{L}$ *is an* FESS *but not conversely. Actually* $\dim \mathcal{L} \le \mathrm{otp}(\mathrm{qo}\,(\mathcal{L})) \le \infty$ *and* $\dim(\mathcal{S}\mathit{ingl}) = 1 < \mathrm{otp}(\mathrm{qo}\,(\mathcal{S}\mathit{ingl}))$.

Proof. Observe that for every $\langle\langle t_0, L_1\rangle, \langle t_1, L_2\rangle, \ldots, \langle t_{l-1}, L_l\rangle\rangle \in \mathrm{Prod}(\mathcal{L})$, a sequence $\langle t_0, t_1, \ldots, t_{l-1}\rangle$ is a bad sequence of $\mathrm{qo}\,(\mathcal{L})$. So we can prove the inequality by a transfinite induction [27] on $\mathrm{Prod}(\mathcal{L})$. The equality $\dim \mathcal{L} = \mathrm{otp}(\mathrm{qo}\,(\mathcal{L}))$ is not necessarily true. For example, although $\dim \mathcal{S}\mathit{ingl} = 1$, a quasi-order $\mathrm{qo}\,(\mathcal{S}\mathit{ingl}) = (\mathbb{N}, =)$ has an infinite bad sequence $\langle 0, 1, 2, 3, \ldots\rangle$, which implies $\mathrm{otp}(\mathrm{qo}\,(\mathcal{S}\mathit{ingl})) = \infty$.

Proposition 6 ([9, p. 41])**.** *If* $\mathcal{L}$ *has a finite thickness and* $\mathcal{L}$ *has no infinite antichain with respect to* $\subseteq$*, then* $\mathrm{qo}\,(\mathcal{L})$ *is a* WQO.

We will study structure of the representation of QOs by set systems from viewpoint of algebraic theory of lattices [24]. From [24], we recall “atom,” “atomic,” and “compact” (and the dual notions.)

Let L be a complete lattice. By a *coatom* of L, we mean any nontop element C such that every nontop $c \in L$ is codisjoint from C (i.e. $c \cup C$ is top) or less than or equal to C. A *coatomic*, complete lattice is, by definition, a complete lattice such that for any nontop element C_0 there is a coatom greater than or equal to C_0. We say an element c in a complete lattice L is called *compact* if whenever $c \le \bigcup S$ there exists a finite subset $T \subseteq S$ with $c \le \bigcup T$.

Proposition 7 ([24])**.** *Every element of a complete lattice* L *is compact if and only if* L *satisfies the ascending chain condition.*

Theorem 4. *Let* $\mathcal{X} = (X, \preceq)$ *be a quasi-order.*

(1) *The following are equivalent:*
    (a) $\mathcal{X}$ *is a* WQO.
    (b) $\mathrm{ss}\,(\mathcal{X})$ *is an* FESS.
    (c) $\mathrm{ss}\,(\mathcal{X})$ *is a complete lattice such that every element is compact.*
(2) *If* $\mathcal{X}$ *is a* WQO*, then* $\mathrm{ss}\,(\mathcal{X})$ *is a coatomic, complete lattice.*

Proof. As for the assertion (1), the equivalence between the conditions (a) and (b) follows from Theorem 4 (1). The equivalence between the conditions (a) and (c) is by Proposition 1 and Proposition 7.

(2) Let $\mathcal{X} = (X, \preceq)$. $\{\emptyset, X\}$ is obviously a coatomic, complete lattice. So assume $\mathrm{ss}(\mathcal{X}) \neq \{\emptyset, X\}$. By the assertion (1), the complete lattice $\mathrm{ss}(\mathcal{X})$ is an FESS. If $\mathrm{ss}(\mathcal{X})$ is not coatomic, then there exists $C_0 \in \mathrm{ss}(\mathcal{X}) \setminus \{X\}$ such that

(3) $\quad \forall C \in \mathrm{ss}(\mathcal{X}) \setminus \{X\} \left(C_0 \subseteq C \implies \exists c \in \mathrm{ss}(\mathcal{X}) \ (c \cup C \neq X \ \& \ c \setminus C \neq \emptyset)\right).$

We can construct an infinite $\langle\langle x_0, C_1\rangle, \langle x_1, C_2\rangle, \ldots\rangle \in \mathrm{Prod}(\mathrm{ss}(\mathcal{X}))$ as follows: Because $\mathrm{ss}(\mathcal{X}) \supsetneq \{\emptyset, X\}$, we can take a pair of $C_1 \in \mathrm{ss}(\mathcal{X}) \setminus \{X\}$ and $x_0 \in C_1$ such that $C_0 \subseteq C_1$. Suppose we have a pair of $C_i \in \mathrm{ss}(\mathcal{X}) \setminus \{X\}$ and $x_{i-1} \in C_i$ such that $C_0 \subseteq C_i$. Once we can find a pair of $C_{i+1} \in \mathrm{ss}(\mathcal{X}) \setminus \{X\}$ and $x_i \in C_{i+1} \setminus C_i$ such that $C_0 \subseteq C_{i+1}$, then by iterating this process, we can construct an infinite production sequence of $\mathrm{ss}(\mathcal{X})$. Because $C_0 \subseteq C_i$ and (3), there exist $c_i \in \mathrm{ss}(\mathcal{X})$ and $x_i \in c_i \setminus C_i$ such that $c_i \cup C_i \neq X$. So, let $C_{i+1} := C_i \cup c_i$. Then it is in $\mathrm{ss}(\mathcal{X}) \setminus \{X\}$ because $\mathrm{ss}(\mathcal{X})$ is closed under the union. Moreover $x_i \in C_{i+1} \setminus C_i$ because $x_i \in c_i \setminus C_i$. Clearly $C_0 \subseteq C_i \subseteq C_i \cup c_i = C_{i+1}$.

This section suggests a close similarity between WQOs and finitely elastic set systems, so it is worth studying whether the closure properties for WQOs solve the questions of which operation on set systems preserves finite elasticity. According to [28], the study on closure properties for WQOs (Higman's theorem for WQOs on finite sequences [20], Kruskal's theorem for WQOs on finite trees [29], Nash-Williams' theorem for *better-quasi-order*s on transfinite sequences [30],...) can be advanced via set-theoretic topological methods and a Ramsey-type argument. So, to advance the study on the questions of which operation on set system preserves finite elasticity, it is natural for us to employ set-theoretic topology (see Section 4) and a Ramsey-type argument (see Section 5.)

4. Continuous deformations of set systems

For nonempty finite set U, the product topological space $\{0,1\}^U$ is called a *Cantor* space. Subspaces of Cantor spaces are represented by $\mathcal{C}, \mathcal{D}, \mathcal{E}, \ldots$.

Definition 6. *For every set system $\mathcal{L} \subseteq P(X)$, define a function*

$$\mathsf{i} \ : \ \mathcal{L} \to \mathsf{i}\mathcal{L} := \left\{1_L \in \{0,1\}^{\bigcup \mathcal{L}} \ ; \ L \in \mathcal{L}\right\} \ ; \ L \mapsto 1_L \ .$$

Then $\mathsf{i}\mathcal{L}$ is a topological space, induced from a Cantor space $\{0,1\}^{\bigcup \mathcal{L}}$. For $\mathcal{C} \subseteq \{0,1\}^X$, put

$$\mathrm{fld}(\mathcal{C}) := \bigcup \mathsf{i}^{-1}(\mathcal{C}) \subseteq X.$$

Let us identify $g \in \mathcal{D}$ with an infinite sequence $(g(y))_{y \in \mathrm{fld}(\mathcal{D})}$. For each $x \in \mathrm{fld}(\mathcal{C})$, let $\pi_x : \mathcal{C} \to \{0,1\}$ be the canonical projection to the x-th component. So $\pi_x(f) = f(x)$ for every $x \in \mathrm{fld}(\mathcal{C})$. Recall that a Cantor space $\{0,1\}^{\mathrm{fld}(\mathcal{C})}$ is generated by a class of sets $\pi_x^{-1}[\{b\}]$ such that $x \in \mathrm{fld}(\mathcal{C})$ and $b \in \{0,1\}$. Let us call each $\pi_x^{-1}[\{b\}]$ a generator of $\mathcal{C}$. Then an open set of $\mathcal{C}$ is exactly an arbitrary union of finite intersections of generators. Note that each generator of $\mathcal{C}$ is clopen.

A *Boolean formula over a set* Y is built up from the truth values 0, 1, or elements of Y, by means of negation, finite conjunction, and finite disjunction.

Lemma 2. *A function $\mathfrak{O} : \mathcal{D} \to \mathcal{C}$ is continuous, if and only if there is a sequence $(B_x)_{x \in \mathrm{fld}(\mathcal{C})}$ of Boolean formulas over $\mathrm{fld}(\mathcal{D})$ such that for every $g \in \mathcal{D}$ and every $x \in \mathrm{fld}(\mathcal{C})$, the value $\mathfrak{O}(g)(x)$ is the truth value of B_x under the truth assignment g.*

Proof. (If-part) The inverse image $\mathfrak{O}^{-1}\left[\pi_x^{-1}[\{b\}]\right]$ of a generator $\pi_x^{-1}[\{b\}]$ is the class of the truth assignments $g \in \mathcal{D}$ under which the truth value of B_x is b. Because the Boolean formula B_x is equivalent to a finite disjunction of finite conjunctions of elements of $\mathrm{fld}(\mathcal{D})$ and the negations of elements of $\mathrm{fld}(\mathcal{D})$, the inverse image $\mathfrak{O}^{-1}\left[\pi_x^{-1}[\{b\}]\right]$ is just a finite union of finite intersections of generators of $\mathcal{D}$, while $\mathfrak{O}^{-1}\left[\pi_x^{-1}[\{0\}]\right]$ is just a finite intersection of finite unions of generators of $\mathcal{D}$. Therefore, the inverse image $\mathfrak{O}^{-1}\left[\pi_x^{-1}[\{b\}]\right]$ is open.

(Only-if-part) Because $\mathfrak{O}$ is continuous and $\{b\}$ $(b = 0, 1)$ is clopen in the finite discrete topology $\{0, 1\}$, the inverse image $\mathfrak{O}^{-1}\left[\pi_x^{-1}[\{b\}]\right]$ of a generator $\pi_x^{-1}[\{1\}]$ by $\mathfrak{O}$ is clopen, which is an arbitrary union of intersections of generators.

Because $\{0, 1\}$ is compact, Tychonoff's theorem implies the compactness of $\{0, 1\}^{\mathrm{fld}(\mathcal{C})}$ and thus that of $\mathcal{C}$. Moreover, $\mathcal{C}$ is a Hausdorff space, because for all distinct $f, g \in \mathcal{C}$, there is $x \in \mathrm{fld}(\mathcal{C})$ such that $f(x) \neq g(x)$, which implies that $\pi_x^{-1}[\{f(x)\}]$ and $\pi_x^{-1}[\{g(x)\}]$ are open sets such that $f \in \pi_x^{-1}[\{f(x)\}]$ and $g \in \pi_x^{-1}[\{g(x)\}]$.

Since every closed subset of a compact Hausdorff space is compact, the clopen set $\mathfrak{O}^{-1}\left[\pi_x^{-1}\{b\}\right]$ is $\bigcup_{i=1}^{m}\bigcap_{j=1}^{n_i}\pi_{y_{ij}}^{-1}[\{b_{ij}\}]$ for some nonnegative integers m, n_i $(1 \leq i \leq m)$, some $y_{ij} \in \mathrm{fld}(\mathcal{D})$, and some $b_{ij} \in \{0, 1\}$ $(1 \leq i \leq m,\ 1 \leq j \leq n_i.)$ So, define a Boolean formula over $\mathrm{fld}(\mathcal{D})$ by $\bigvee_{i=1}^{m}\bigwedge_{j=1}^{n_i}(b_{ij} \leftrightarrow y_{ij})$, where each $b_{ij} \leftrightarrow y_{ij}$ represents a Boolean formula y_{ij} for $b_{ij} = 1$ and the negation $\overline{y_{ij}}$ for $b_{ij} = 0$. Clearly we have $\mathfrak{O}(g)(x) = 1$ iff $g \in \mathfrak{O}^{-1}\left[\pi_x^{-1}[\{1\}]\right]$ iff g satisfies B_x.

For functions $f, g \in \{0, 1\}^Z$, we write $f \leq g$ if $f(z) \leq g(z)$ for all $z \in Z$.

Definition 7 (Monotone functions)**.** *Let $\mathcal{C} \subseteq \{0, 1\}^X$ and $\mathcal{D} \subseteq \{0, 1\}^Y$. We say a function $\mathfrak{O} : \mathcal{D} \to \mathcal{C}$ is* monotone*, if $f \leq g$ implies $\mathfrak{O}(f) \leq \mathfrak{O}(g)$.*

We say a *Boolean formula* positive if it does not contain a negation.

Definition 8. *Let $\mathbf{S}$ and $\mathbf{U}$ be two (not necessarily distinct) sets of objects, and R be a $R \subseteq \mathbf{S} \times [\mathbf{U}]^{<\omega}$. For $M \subseteq \mathbf{U}$ and $\mathcal{M} \subseteq P(\mathbf{U})$, define*

$$R^{-1}[[M]] := \left\{s\ ;\ \exists v \in [M]^{<\omega}\ .\ R(s, v)\right\}, \qquad \widetilde{R^{-1}}[[\mathcal{M}]] := \left\{R^{-1}[[M]]\ ;\ M \in \mathcal{M}\right\}.$$

Define $!\mathcal{M} := \left\{[M]^{<\omega}\ ;\ M \in \mathcal{M}\right\}$*. Then* $\bigcup !\mathcal{M} \subseteq \left[\bigcup \mathcal{M}\right]^{<\omega}$.

Lemma 3. (1) *Following conditions are equivalent:*

(a) *A function $\mathfrak{O} : \mathcal{D} \to \mathcal{C}$ is monotone and continuous.*

(b) *$\mathfrak{O}$ is a function of $g \in \mathcal{D}$ and $x \in \mathrm{fld}(\mathcal{C})$ such that it first produces a positive Boolean formula B_x over $\mathrm{fld}(\mathcal{D})$, and then queries to an oracle g whether g satisfies B_x or not.*

(2) *If $R \subseteq \mathrm{fld}(\mathcal{C}) \times [\mathrm{fld}(\mathcal{D})]^{<\omega}$ is a finitely branching relation, then*

$$\mathfrak{O}_R(g)(x) := \bigvee_{R(x,v)} \bigwedge_{y \in v} (g(y) = 1), \quad (g \in \mathcal{D},\ x \in \mathrm{fld}(\mathcal{C})) \tag{4}$$

defines a monotone, continuous function from $\mathcal{D}$ to $\mathcal{C}$ such that,

$$\widetilde{R^{-1}}[[\mathcal{M}]] \ = \ \mathsf{i}^{-1}\left(\mathfrak{O}_R\left[\mathsf{i}\mathcal{M}\right]\right) \ \subseteq \ P\left(\mathrm{fld}(\mathcal{C})\right), \qquad \left(\mathcal{M} \subseteq P(\mathrm{fld}(\mathcal{D}))\right.) \tag{5}$$

In fact, every monotone, continuous function from $\mathcal{D}$ to $\mathcal{C}$ is written as (4).

Proof. (1) By Lemma 2. Positivity of a Boolean formula is equivalent to absence of negation in the formula. (2) follows from (1).

Lemma 4. $\widetilde{R^{-1}}[[\mathcal{M}]] = \widetilde{R^{-1}}[!\mathcal{M}]$.

Proof. $L \in \widetilde{R^{-1}}[[\mathcal{M}]]$ iff there exists $M \in \mathcal{M}$ such that $L = R^{-1}[[M]] = \{x\ ;\ \exists v \in [M]^{<\omega} .\ R(x,v)\} = R^{-1}\left[[M]^{<\omega}\right] \in \widetilde{R^{-1}}\,[!\mathcal{M}]$.

Theorem 5. *If $\mathcal{M}$ is an* FESS*, so is $!\mathcal{M}$.*

Proof. Otherwise there exist an infinite sequence $v_0, v_1, \ldots$ of elements of $\bigcup !\mathcal{M}$ and an infinite sequence $[M_1]^{<\omega}, [M_2]^{<\omega}, \ldots$ of elements of $!\mathcal{M}$ such that for each $n \geq 1$ we have $\{v_0, \ldots, v_{n-1}\} \subseteq [M_n]^{<\omega} \not\ni v_n$, which implies $\bigcup_{i=1}^{n-1} v_i \subseteq M_n \not\supseteq v_n$. Put $v_i' := v_i \setminus M_i$ $(i = 0, 1, \ldots.)$ Then $v_i' \cap v_j' = \emptyset$ $(0 \leq i < j)$ and each v_i' is a nonempty finite set. Therefore $\{v_i'\ ;\ i \in \mathbb{N}\}$ satisfies the Hall's condition of the marriage theorem [31, Theorem 3.41]: for each finite set $F \subset \mathbb{N}$ we have $\#\left(\bigcup_{i\in F} v_i'\right) \geq \#F$. By the marriage theorem, $\{v_i'\ ;\ i \in \mathbb{N}\}$ has a system of distinct representative $\{y_i\ ;\ i \in \mathbb{N}\}$, i.e., $y_i \neq y_j$ $(0 \leq i < j)$ and $y_i \in v_i'$ $(i = 0, 1, \ldots.)$ Then for each $n \geq 1$ $\{y_0, \ldots, y_{n-1}\} \subseteq \bigcup_{i=0}^{n-1} v_i \subseteq M_n$, while $y_n \notin M_n$ because $y_n \in v_n' = v_n \setminus M_n$. This contradicts the FE of $\mathcal{M}$.

The previous theorem generalizes Proposition 5 which is useful in Section 4.

Corollary 2. *Let $\mathcal{M} \subseteq P(\mathbf{U})$ be an* FESS *and let $R \subseteq \mathbf{S} \times [\mathbf{U}]^{<\omega}$ be a finitely branching relation. Then $\mathcal{L} = \widetilde{R^{-1}}[[\mathcal{M}]] \subseteq P(\mathbf{S})$ is also an* FESS.

Proof. By Theorem 5, Lemma 4 and Proposition 5.

Conversely, Theorem 5 follows from Corollary 2 with $\mathbf{U} := \bigcup \mathcal{M}$, $\mathbf{S} := [\bigcup \mathcal{M}]^{<\omega}$, and a following:

Definition 9.

$$R_! := \left\{(v,v)\ ;\ v \in \left[\bigcup \mathcal{M}\right]^{<\omega}\right\},\ \textit{and}\ \ !\mathcal{M} = \widetilde{R_!^{-1}}[[\mathcal{M}]]$$

In terms of topology, the previous corollary becomes a following:

Corollary 3. *Assume $\mathcal{L}$ and $\mathcal{M}$ are set systems and $\mathfrak{O} : \mathsf{i}\mathcal{M} \to \mathsf{i}\mathcal{L}$ is a monotone, continuous function. Then if $\mathcal{M}$ is an* FESS*, so is $\mathsf{i}^{-1}(\mathfrak{O}[\mathsf{i}\mathcal{M}])$.*

Proof. By Lemma 3 (2), $\mathfrak{O}(1_M)(x) = \bigvee_{R(x,v)} \bigwedge_{y\in v}(y \in M)$ where $R \subseteq \bigcup \mathcal{L} \times [\bigcup \mathcal{M}]^{<\omega}$ is a finitely branching relation. Therefore we have $L = R^{-1}[[M]] \iff L = \{x\ ;\ \exists v\,(R(x,v)\ \&\ v \subseteq M)\} \iff 1_L = \mathfrak{O}(1_M)$. Hence the family $\mathsf{i}^{-1}(\mathfrak{O}[\mathsf{i}\mathcal{M}])$ is $\widetilde{R^{-1}}[[\mathcal{M}]]$, which is an FESS by Corollary 2.

Although the mind-change complexity of language identification from positive data is characterized by using the *positive information topology* [6, 8, 9], Corollary 3 does not hold for positive information topology. Recall that the positive information topology $\mathcal{C} \subseteq \{0,1\}^X$ is induced by the product topology of the topology $\{0,1\}$ where the only nontrivial open subset of $\{0,1\}$ is $\{1\}$. So the basic open sets of the positive information topology are

(6) $\quad U_F^{\mathcal{C}} = \{f \in \mathcal{C}; f[F] = \{1\}\}$ where F is an arbitrary finite subset of X.

Let us abbreviate "continuous with respect to the positive information topology" by "Π-continuous."

Lemma 5. *A monotone, continuous function is Π-continuous.*

Proof. Let $\mathfrak{O} : \mathcal{D} \to \mathcal{C}$ be a monotone, continuous function and let $U_F^{\mathcal{C}}$ be a basic open set of the positive information topology $\mathcal{C}$ where F is a finite subset of $\mathrm{fld}(\mathcal{C})$. By Lemma 3 (1), there are positive Boolean formulas B_x over $\mathrm{fld}(D)$ ($x \in F$) such that for every $F \in [\mathrm{fld}(\mathcal{C})]^{<\omega}$ the inverse image $\mathfrak{O}^{-1}\left[U_F^{\mathcal{C}}\right]$ is $\bigcap_{x\in F} \{g \in \mathcal{D}\ ;\ g \text{ satisfies } B_x\}$. Observe that each B_x is equivalent to $\bigvee_{i=1}^{n_x} \bigwedge F_{x,i}$ for some $n_x \geq 0$ and some $F_{x,i} \in [\mathrm{fld}(\mathcal{D})]^{<\omega}$ ($1 \leq i \leq n_x$.) Therefore $\mathfrak{O}^{-1}\left[U_F^{\mathcal{C}}\right]$ is $\bigcap_{x\in F}\bigcup_{i=1}^{n_x} U_{F_{x,i}}^{\mathcal{D}}$, which is open with respect to positive information topology because F is finite.

Recall that $\mathcal{S}ingl = \{\{n\}\ ;\ n \in \mathbb{N}\}$ is an FESS. For $L \subseteq \mathbb{N}$, let $\downarrow L \subseteq \mathbb{N}$ be the downward closure $\{n\ ;\ n \leq m\ (\exists m \in L)\}$ of L. To decide whether $n \in \downarrow L$, we must carry out *unbounded* search to find some $m \in L \cap [n, \infty)$.

Theorem 6. (1) *A function* $\mathfrak{O}_\downarrow : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ *that sends* 1_L *to* $1_{\downarrow L}$ *is monotone and* Π*-continuous but* $\mathsf{i}^{-1}\mathfrak{O}_\downarrow[\,\mathsf{i}\mathcal{S}ingl\,]$ *is not an* FESS*; and*
(2) *There is a non-monotone, continuous, non-*Π*-continuous function* $\mathfrak{O}_\neg : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ *such that* $\mathsf{i}^{-1}\mathfrak{O}_\neg[\,\mathsf{i}\mathcal{S}ingl\,]$ *is not an* FESS.

Proof. A basic open set (6) with $\mathcal{C} = \{0,1\}^{\mathbb{N}}$ is simply written U_F below: (1) The monotonicity of $\mathfrak{O}_\downarrow$ is obvious. The function $\mathfrak{O}_\downarrow$ is Π-continuous, because for every basic open set U_F with finite $F \subseteq \mathbb{N}$, the inverse image by $\mathfrak{O}_\downarrow$ is an open set $\bigcup_{B\subseteq F}\left(U_{F\setminus B} \cap \bigcap_{b\in B}\bigcup_{x>b} U_{\{x\}}\right)$ where $\bigcap_{b\in B}\cdots$ is $\{0,1\}^{\mathbb{N}}$ if $B = \emptyset$. However $\mathsf{i}^{-1}\mathfrak{O}_\downarrow[\,\mathsf{i}\,\mathcal{S}ingl\,] = \mathcal{D}cl$ is not an FESS.

(2) Moriyama-Sato [12] observed that the elementwise complement does not preserve the FE of set systems. Define

$$\mathfrak{O}_\neg(g)(x) = 1 - g(x).$$

Then $\mathsf{i}^{-1}\mathfrak{O}_\neg[\,\mathsf{i}\mathcal{S}ingl\,] = \{\mathbb{N}\setminus\{y\}\ ;\ y \in \mathbb{N}\}$ has an infinite elasticity: 0, $\mathbb{N}\setminus\{1\}$, 1, $\mathbb{N}\setminus\{2\}, 2, \ \ldots$. If $\mathfrak{O}_\neg$ is Π-continuous, then $\mathfrak{O}_\neg^{-1}\left[U_{\{0\}}\right]$ should be $\bigcup_G U_G$ where G ranges over a certain class of finite subsets of $\mathbb{N}$. For such a finite set G, $g = 1_G$ belongs to the inverse image by $\mathfrak{O}_\neg$, but the support should be $\mathbb{N}\setminus\{0\}$. Contradiction.

5. The order types of nondeterministically deformed set systems

We present a typical application of Corollary 3, and answer Question 4 "How much do such operations increase the order type of set systems?" by a Ramsey number argument.

Fix an alphabet Σ. To know whether a word w belongs to the Kleene closure $L^* = \bigcup_{n\geq 0} L^n$ of a language L, we need to guess n nondeterministically. Nondeterministic operations such as the Kleene closure operator $(\cdot)^*$ and the shuffle-closure operator $(\cdot)^{\circledast}$ are representable by monotone, continuous functions. So Corollary 3 is useful in deriving the following:

$$\mathcal{M} \subseteq P(\Sigma^*) \text{ is an FESS} \ \Rightarrow\ \mathcal{M}^{\widetilde{*}} \text{ and } \mathcal{M}^{\widetilde{+}} \text{ are FESSs.} \tag{7}$$

Let us see the proof to generalize for the case of the shuffle-closure. Assume $\mathcal{M}$ is an FESS. Let ε be the empty word and let $\mathfrak{O}_1(1_M) := 1_{M\setminus\{\varepsilon\}}$ and $\mathfrak{O}_3(1_M) := 1_{M\cup\{\varepsilon\}}$. Then $\mathfrak{O}_1$ and $\mathfrak{O}_3$ are monotone and continuous. Let $\mathfrak{O}_2(1_L)$ be computed by a Turing machine with the oracle tape being 1_L as follows: if an input $s \in \Sigma^*$ is ε then the oracle Turing machine returns 0. Otherwise, it tries to find a partition $s_1, \ldots, s_m$ of s such that $s = s_1 \cdots s_m$, $ln(s_i) > 0$ ($1 \leq i \leq m$), $m \geq 1$ and

$\{s_1, \ldots, s_m\} \subseteq L$. If such a partition is found, then the oracle Turing machine returns 1, and 0 otherwise. The number of queries the oracle Turing machine makes is bounded by the number of partitions of s, which implies the continuity of $\mathfrak{D}_2$. It is easy to see $\mathfrak{D}_2$ is monotone. Observe $\mathfrak{D}_2(1_L) = 1_{L^+}$ for all $L \subseteq \Sigma^* \setminus \{\varepsilon\}$. We can prove, for every $M \subseteq \Sigma^*$,

$$(M \setminus \{\varepsilon\})^+ = M^+ \setminus \{\varepsilon\}, \qquad (M \setminus \{\varepsilon\})^+ \cup \{\varepsilon\} = M^*.$$

So we have $\mathfrak{D}_3 \circ \mathfrak{D}_2 \circ \mathfrak{D}_1(1_M) = 1_{M^*}$. By Corollary 3, $\mathcal{M}^{\widetilde{*}}$ is an FESS.

Assume $\mathcal{M}^{\widetilde{+}}$ has an infinite production sequence $\langle \langle t_0, M_1^+\rangle, \langle t_1, M_2^+\rangle, \ldots\rangle$. Note that there is at most one i such that $t_i = \varepsilon$. Removal of such $\langle t_i, M_i\rangle$ from the infinite production sequence of $\mathcal{M}^{\widetilde{+}}$ results in still an infinite production sequence of $\mathcal{M}^{\widetilde{+}}$. By adjoining the empty word ε to each language in the infinite production sequence, we have an infinite production sequence of $\mathcal{M}^{\widetilde{*}}$, because $M^+ \cup \{\varepsilon\} = M^*$. But this is a contradiction against the FE of $\mathcal{M}^{\widetilde{*}}$. So, $\mathcal{M}^{\widetilde{+}}$ is an FESS.

Remind that to find such a partition can be done by a nondeterministic computation. We can prove the counterpart of (7) for the shuffle-closures $(\cdot)^{\circledast}$, as follows:

Corollary 4. *If $\mathcal{L} \subseteq P(\Sigma^*)$ is an* FESS*, so is $\mathcal{L}^{\widetilde{\circledast}}$.*

Proof. The proof is similar to that of (7) except $\mathfrak{D}_2(1_L)$ is computed by another Turing machine with the oracle tape being 1_L as follows: if an input $s \in \Sigma^*$ is ε, then it returns 0. Otherwise, it tries to find a sequence $s_1, \ldots, s_m$ such that s is an "interleaving merge" of $s_1, \ldots, s_m$, $ln(s_i) > 0$ $(1 \le i \le m)$, $m \ge 1$ and $\{s_1, \ldots, s_m\} \subseteq L$. Then $\mathfrak{D}_2$ is clearly monotone and continuous. Moreover $\mathfrak{D}_2(1_L) = 1_{L^\diamond}$ for every $L \subseteq \Sigma^* \setminus \{\varepsilon\}$. We can prove, for every $M \subseteq \Sigma^*$,

$$(M \setminus \{\varepsilon\})^\diamond = M^\diamond \setminus \{\varepsilon\}, \qquad (M \setminus \{\varepsilon\})^\diamond \cup \{\varepsilon\} = M^{\circledast}.$$

So we have $\mathfrak{D}_3 \circ \mathfrak{D}_2 \circ \mathfrak{D}_1(1_M) = 1_{M^{\circledast}}$. By Corollary 3, we have done.

We can see that (7) also holds for tree languages [32].

Next we answer Question 4 "How much do such operations increase the order type of set systems?" by a Ramsey number argument.

The finitely branching relation

$$R_n(s,u) :\Longleftrightarrow \bigvee_{i=0}^{n-1} (u = \{\langle s, i\rangle\}\,.) \qquad (n = 2, 3, \ldots) \tag{8}$$

satisfies $\widetilde{R_n^{-1}}\left[[\mathcal{L}_1 \mathbin{\widetilde{\uplus}} \cdots \mathbin{\widetilde{\uplus}} \mathcal{L}_n]\right] = \mathcal{L}_1 \mathbin{\widetilde{\cup}} \cdots \mathbin{\widetilde{\cup}} \mathcal{L}_n$. So, if $\mathcal{L}_i$'s are all FESSs, then so is $\mathcal{L}_1 \mathbin{\widetilde{\uplus}} \cdots \mathbin{\widetilde{\uplus}} \mathcal{L}_n$ by Lemma 1. By Corollary 2, $\mathcal{L}_1 \mathbin{\widetilde{\cup}} \cdots \mathbin{\widetilde{\cup}} \mathcal{L}_n$ is an FESS, too.

On the other hand, in [16], Wright proved that "if $\mathcal{L}_1$ and $\mathcal{L}_2$ are FESSs, then so is $\mathcal{L}_1 \mathbin{\widetilde{\cup}} \mathcal{L}_2$," by using Ramsey theorem "for any dichromatic coloring of an infinite complete graph, there is a monochromatic infinite complete subgraph." By adapting his proof, we can provide an explicit upper bound of the dimension by using a *Ramsey number* [33]:

Proposition 8 ([33, Sect 1.1])**.** *For all positive integers $n, l_1, \ldots, l_n$, there exists a positive integer k such that any edge-coloring with colors $1, \ldots, n$ for the complete graph of size k has a complete subgraph of size l_i colored homogeneously by some color $i \in \{1, \ldots, n\}$. Such minimum integer k, denoted by* $\mathrm{Ram}(l_1, \ldots, l_n, n)$*, is* *called the* Ramsey number of $l_1, \ldots, l_n$. *When* $l_1 = \cdots = l_n$, *we call it the* n-adic diagonal Ramsey number of l_1, *and write it as* $\mathrm{Ram}(l_1; n)$. *For the sake of convenience, put* $\mathrm{Ram}(l; 1) = l$ *for every nonzero ordinal number (and hence every positive integer)* l.

By [15, Section 4.2], $\mathrm{Ram}(l, m) \leq \begin{pmatrix} m+l-2 \\ l-1 \end{pmatrix} \leq c4^{\max(l,m)}/\sqrt{\max(l,m)}$ for some constant c.

Lemma 6. *For every positive integer* n, *if* $\dim \mathcal{L}_i < \omega$ $(i = 1, \ldots, n)$, *then*

$$\dim\left(\mathcal{L}_1 \,\widetilde{\cup} \cdots \widetilde{\cup}\, \mathcal{L}_n\right) + 1 < \mathrm{Ram}(\dim(\mathcal{L}_1) + 2, \ldots, \dim(\mathcal{L}_n) + 2).$$

Proof. When $n = 1$ the assertion is trivial. Consider the case $n = 2$. Suppose that $k + 1 \geq \mathrm{Ram}(\dim(\mathcal{L}) + 2, \dim(\mathcal{M}) + 2)$, and suppose there are a sequence $t_0, \ldots, t_{k-1}$, a sequence $L_1, \ldots, L_k$ of $\mathcal{L}$ and a sequence $M_1, \ldots, M_k$ of $\mathcal{M}$ such that

$$\{t_0, \ldots, t_{i-1}\} \subseteq L_i \cup M_i \ (i = 1, 2, \ldots, k) \quad \text{and} \quad L_j \not\ni t_j \quad (j = 1, \ldots k-1.) \tag{9}$$

By the definition of Ramsey number,

$$k \geq \dim(\mathcal{L}) + 1, \dim(\mathcal{M}) + 1.$$

Consider a complete graph G with the vertices being $0, \ldots, k-1$. For any edge $\{i, j\} (i \neq j)$, color it by red if $0 \leq i < j \leq k-1$ and $t_i \in L_j$, while color it by black otherwise.

Assume $k + 1 \geq \mathrm{Ram}(\dim(\mathcal{L}) + 2,\ \dim(\mathcal{M}) + 2)$. By Ramsey's theorem, the colored complete graph G has either a red clique of size $\dim(\mathcal{L}) + 2$ or a black clique of size $\dim(\mathcal{M}) + 2$. When a red clique of size $\dim(\mathcal{L}) + 2$ exists, write it as $\{u_0 < \cdots < u_{\dim(\mathcal{L})+1}\}$. Then we have $\{t_{u_0}, \ldots, t_{u_{i-1}}\} \subseteq L_{u_i}$ $(i = 1, \ldots, \dim(\mathcal{L})+1)$ but $L_{u_j} \not\ni t_{u_j} \quad (j = 1, 2, \ldots, \dim(\mathcal{L}))$, which contradicts the definition of $\dim \mathcal{L}$.

Otherwise, a black clique of size $\dim(\mathcal{M})+2$ exists, so we write it as $\{u_0 < \cdots < u_{\dim(\mathcal{M})+1}\}$. Then we have $\{t_{u_0}, \ldots, t_{u_{i-1}}\} \cap L_{u_i} = \emptyset \quad (i = 1, \ldots, \dim(\mathcal{M})+1)$. By (9), we have $\{t_{u_0}, \ldots, t_{u_{i-1}}\} \subseteq L_{u_i} \cup M_{u_i}$ and $L_{u_j} \cup M_{u_j} \not\ni t_{u_j}$ $(j = 1, \ldots, \dim(\mathcal{M}))$, so $\{t_{u_0}, \ldots, t_{u_{i-1}}\} \subseteq M_{u_i}$ and $M_{u_j} \not\ni t_{u_j}$ $(j = 1, \ldots, \dim \mathcal{M})$, which contradicts the definition of $\dim(\mathcal{M})$.

Consider the case $n \geq 3$. Suppose that $k+1 \geq \mathrm{Ram}(\dim \mathcal{L}_1 + 2, \ldots, \dim \mathcal{L}_n + 2)$, and suppose there are a sequence $t_0, \ldots, t_{k-1}$ and a sequence $L_1^{(l)}, \ldots, L_k^{(l)}$ of $\mathcal{L}_l$ $(1 \leq l \leq n)$ such that $\{t_0, \ldots, t_{i-1}\} \subseteq \bigcup_{l=1}^n L_i^{(l)}$ $(i = 1, \ldots, k)$ and $\bigcup_{l=1}^n L_j^{(l)} \not\ni t_j$ $(j = 1, \ldots, k-1.)$ For any edge $\{i, j\}$ $(i \neq j)$, if $0 \leq i < j \leq k-1$ and $t_i \in L_j^{(1)}$, color $\{i, j\}$ by the color 1; else if $0 \leq i < j \leq k-1$ and $t_i \in L_j^{(2)}$, color it by the color 2; else if $\ldots$; else if $0 \leq i < j \leq k-1$ and $t_i \in L_j^{(n-1)}$, color it by the color $n-1$; else color $\{i, j\}$ by the color n. Then apply the same argument as above.

The lemma generalizes for any relation R with $\sup_s \#\{v\ ;\ R(s, v)\} \leq n$.

Theorem 7. *Assume* $R \subseteq \bigcup \mathcal{L} \times [\bigcup \mathcal{M}]^{<\omega}$ *has a bound* $n \geq 1$ *of* $\#\{v\ ;\ R(x, v)\}$ $(x \in \bigcup \mathcal{L})$. *If* $\mathcal{M}$ *is an* FESS, *then*

$$\dim \widetilde{R^{-1}}[[\mathcal{M}]] + 1 < \mathrm{Ram}(\dim \mathcal{M} + 2; n),$$

provided $\dim \mathcal{M}$ *is finite or* $n = 1$. *Actually, when* $n = 1$,

$$\dim\ \widetilde{R^{-1}}\,[[\mathcal{M}]] \leq \dim\ \mathcal{M} \tag{10}$$

where the equality holds if each $y \in \bigcup \mathcal{M}$ *has* $\xi(y) \in \bigcup \mathcal{L}$ *such that* $R(\xi(y), \{y\})$.

Proof. To show the inequality for $n = 1$, by Fact 1, it is sufficient to build an embedding f from a well-founded tree $\mathrm{Prod}(\widetilde{R^{-1}}[[\mathcal{M}]])$ to a well-founded tree $\mathrm{Prod}(\mathcal{M})$. Suppose

$$a = \langle\langle\, s_0,\, L_1\,\rangle, \ldots, \langle\, s_{l-1},\, L_l\,\rangle\rangle \in \mathrm{Prod}(\widetilde{R^{-1}}[[\mathcal{M}]]).$$

For each $L \in \widetilde{R^{-1}}[[\mathcal{M}]]$, choose $M(L)$ from $\{M \in \mathcal{M}\ ;\ L = R^{-1}[[M]]\} \neq \emptyset$. For each $i = 0, \ldots, l-1$, because $n = 1$, there exists exactly one v_i such that $R(s_i, v_i)$ and $v_i \in [M(L_{i+1})]^{<\omega}$. Since $s_i \notin L_i$, $v_i \not\subseteq M(L_i)$. Because the class of finite sets $v_i \setminus M(L_i)$ satisfies the Hall's condition of the marriage [31, Theorem 3.41] theorem, we have a system $\{y_i\ ;\ i \in \mathbb{N}\}$ of distinct representative. Obviously $y_i \notin M(L_i)$. Define

$$f(a) := \langle\langle\, y_0,\, M(L_1)\,\rangle\,, \ldots, \langle\, y_{l-1},\, M(L_l)\,\rangle\,\rangle\,.$$

We have indeed $f(a) \in \mathrm{Prod}(\mathcal{M})$, because for $0 \leq i < j \leq l$, since $s_i \in L_j$, $y_i \in v_i \subseteq M(L_j)$. The mapping f is indeed injective by the construction. Clearly f preserves the greatest upper bounds.

The verification of the *equality* is as follows: By $n = 1$ and the assumption of Theorem 7, we have $\exists v \in [M_i]^{<\omega}\,.\ R(\xi(y_j), v) \iff \{y_j\} = v \subseteq M_i$. So ξ is injective. Moreover

$$\xi(y_j) \in R^{-1}[[M_i]] \iff y_j \in M_i. \tag{11}$$

Define a function g as:

$$\begin{aligned} & b = \langle\langle\, y_0,\, M_1\,\rangle, \ldots, \langle\, y_{l-1},\, M_l\,\rangle\rangle \in \mathrm{Prod}(\mathcal{M}) \\ \mapsto\quad & g(b) := \big\langle\langle \xi(y_0), R^{-1}[[M_1]]\,\rangle, \ldots, \langle\, \xi(y_{l-1}),\, R^{-1}[[M_l]]\,\rangle\big\rangle\,. \end{aligned}$$

Then $g(b) \in \mathrm{Prod}(\widetilde{R^{-1}}\,[[\mathcal{M}]])$ by (11). The injectivity of g is from that of ξ. The preservation of glb's by g is easy.

Next we prove the case where $n > 1$ and $\dim \mathcal{M} < \omega$. There are relations $R_i \subseteq \bigcup \mathcal{L} \times [\bigcup \mathcal{M}]^{<\omega}$ such that for all $x \in \bigcup \mathcal{L}$ and all $v \in [\bigcup \mathcal{M}]^{<\omega}$

$$R = \bigcup_{i=1}^{n} R_i \ \text{ and } \ \#\{v\ ;\ R_i(x, v)\} \leq 1.$$

Then for all $M \in \mathcal{M}$, we have $R^{-1}[[M]] = \bigcup_{i=1}^{n} R_i^{-1}[[M]]$, because the left-hand side is $\{s \in \bigcup \mathcal{L}\ ;\ \exists v \subseteq M.\ R(s, v)\} = \bigcup_{i=1}^{n} \{s \in \bigcup \mathcal{L}\ ;\ \exists v \subseteq M.\ R_i(s, v)\}$ which is the right-hand side. So we have

$$\widetilde{R^{-1}}[[\mathcal{M}]] \subseteq \widetilde{R_1^{-1}}[[\mathcal{M}]]\ \widetilde{\cup} \cdots \widetilde{\cup}\ \widetilde{R_n^{-1}}[[\mathcal{M}]].$$

By Lemma 6, we have

$$\dim \widetilde{R^{-1}}[[\mathcal{M}]] + 1 \ \leq\ \mathrm{Ram}(\dim \widetilde{R_1^{-1}}[[\mathcal{M}]] + 2, \ldots, \dim \widetilde{R_n^{-1}}[[\mathcal{M}]] + 2).$$

Since we have already proved (10), we can use (10) to derive $\dim \widetilde{R_i^{-1}}[[\mathcal{M}]] \leq \dim \mathcal{M} < \omega$. The monotonicity of Ram concludes the desired consequence.

We will use Theorem 7 again to derive the linearization (Corollary 6) of WQOs. Let us see an example of the inequality (10) of Theorem 7.

Corollary 5. *Let $\mathcal{L}$ and $\mathcal{M}$ be* FESS*s.*

(1) *Let R be $\{\ (s, \{\langle\, s,\, s\,\rangle\})\ ;\ s \in \bigcup \mathcal{L} \cap \bigcup \mathcal{M}\}$.*
  (a) $\widetilde{R^{-1}}\,\big[[\mathcal{L}\ \widetilde{\times}\ \mathcal{M}]\big] = \mathcal{L}\ \widetilde{\cap}\ \mathcal{M}$.
  (b) $\mathcal{L}\ \widetilde{\times}\ \mathcal{M}$ *is an* FESS.

(c) *If* $\dim \mathcal{L} < \omega$ *and* $\dim \mathcal{M} < \omega$*, then*

$$\dim(\mathcal{L} \mathbin{\tilde{\times}} \mathcal{M}) \geq \dim \mathcal{L} + \dim \mathcal{M} - 1 \geq \dim(\mathcal{L} \mathbin{\tilde{\cap}} \mathcal{M}).$$

The inequalities are best possible.

(2) $\dim \mathcal{M} = \dim {!\mathcal{M}}$.

Proof. (1a) is immediate. To prove (1b), assume $\mathcal{L} \mathbin{\tilde{\times}} \mathcal{M}$ is not an FESS. Then we have an infinite production sequence

$$\Big\langle \langle (t_0, p_0), L_1 \times M_1 \rangle,\ \langle (t_1, p_1),\ L_2 \times M_2 \rangle, \dots \Big\rangle \in \mathrm{Prod}(\mathcal{L} \mathbin{\tilde{\times}} \mathcal{M}).$$

When $k := \sup\{i\ ;\ L_i \not\ni t_i\} < \infty$, then for all $i > k$, we have $M_i \not\ni p_i$, and thus an infinite production sequence $\Big\langle \langle p_k, M_{k+1} \rangle, \langle p_{k+1}, M_{k+2} \rangle, \dots \Big\rangle$ of $\mathcal{M}$, contradicting the FE of $\mathcal{M}$. Otherwise, we have an infinite sequence $i_0, i_1, \dots$ such that $\Big\langle \langle t_{i_0}, L_{i_1} \rangle,\ \langle t_{i_1},\ L_{i_2} \rangle, \dots \Big\rangle \in \mathrm{Prod}(\mathcal{L})$, contradicting the FE of $\mathcal{L}$.

To show $\dim(\mathcal{L} \mathbin{\tilde{\times}} \mathcal{M}) \geq \dim \mathcal{L} + \dim \mathcal{M} - 1$ of (1c), let

$$\Big\langle \langle t_0, L_1 \rangle,\ \langle t_1, L_2 \rangle, \dots, \langle t_{l-1}, L_l \rangle \Big\rangle \in \mathrm{Prod}(\mathcal{L}),$$

$$\Big\langle \langle p_0, M_1 \rangle, \langle p_1, M_2 \rangle, \dots, \langle p_{m-1}, M_m \rangle \Big\rangle \in \mathrm{Prod}(\mathcal{M}).$$

Then the class $\mathcal{L} \mathbin{\tilde{\times}} \mathcal{M}$ has a following production sequence consisting of $(l+m-1)$ members of $\mathcal{L} \mathbin{\tilde{\times}} \mathcal{M}$:

$$\Big\langle \langle (t_0, p_0), L_1 \times M_1 \rangle, \quad \langle (t_0, p_1), L_1 \times M_2 \rangle, \quad \dots, \quad \langle (t_0, p_{m-1}),\ L_1 \times M_m \rangle,$$
$$\langle (t_1, p_{m-1}), L_2 \times M_m \rangle,\ \langle (t_2, p_{m-1}), L_3 \times M_m \rangle, \dots, \langle (t_{l-1}, p_{m-1}), L_l \times M_m \rangle \Big\rangle.$$

Thus $l + m - 1 \leq \dim(\mathcal{L} \mathbin{\tilde{\times}} \mathcal{M})$. Since $\dim \mathcal{L} < \omega$ and $\dim \mathcal{M} < \omega$, we have $\dim \mathcal{L} + \dim \mathcal{M} - 1 \leq \dim(\mathcal{L} \mathbin{\tilde{\times}} \mathcal{M})$. The equality is attained by $\mathcal{L} = \mathcal{M} = \{\{1\}\}$.

To verify the inequality $\dim \mathcal{L} + \dim \mathcal{M} - 1 \geq \dim(\mathcal{L} \mathbin{\tilde{\cap}} \mathcal{M})$ of (1c), let

$$\Big\langle \langle t_0, L_1 \cap M_1 \rangle,\ \langle t_1, L_2 \cap M_2 \rangle, \dots, \langle t_{n-1}, L_n \cap M_n \rangle \Big\rangle \in \mathrm{Prod}(\mathcal{L} \mathbin{\tilde{\cap}} \mathcal{M}).$$

Then $t_i \in \bigcup \mathcal{L} \cap \bigcup \mathcal{M}$, and for every positive integer $i \leq n$, we have $\{t_0, \dots, t_{i-1}\} \subseteq L_i \cap M_i \not\ni t_i$. So for each positive $i \leq n-1$, $L_i \not\ni t_i$ or $M_i \not\ni t_i$. Let $i_1, \dots, i_l$ be the strictly ascending list of positive integers i such that $L_i \not\ni t_i$, and $j_1, \dots, j_m$ be the strictly ascending list of positive integers j such that $M_j \not\ni t_j$. Then $i_l \neq n$, $j_m \neq n$, and so $\Big\langle \langle t_0, L_{i_1} \rangle, \langle t_{i_1}, L_{i_2} \rangle, \dots, \langle t_{i_{l-1}}, L_{i_l} \rangle, \langle t_{i_l}, L_n \rangle \Big\rangle \in \mathrm{Prod}(\mathcal{L})$, and $\Big\langle \langle t_0, M_{j_1} \rangle, \langle t_{j_1}, M_{j_2} \rangle, \dots, \langle t_{j_{m-1}}, M_{j_m} \rangle, \langle t_{j_m}, M_n \rangle \Big\rangle \in \mathrm{Prod}(\mathcal{M})$. Therefore $l + 1 \leq \dim \mathcal{L}$ as well as $m+1 \leq \dim \mathcal{M}$. Because $n-1 \leq l+m$, we have $\dim(\mathcal{L} \mathbin{\tilde{\cap}} \mathcal{M}) - 1 \leq (\dim \mathcal{L} - 1) + (\dim \mathcal{M} - 1)$, from which the conclusion follows. The latter inequality of Corollary 5 (1c) is best possible. The equality holds for

$$\mathcal{L} = \{\emptyset, \{0\}, \{0,1,2\}\} \text{ and } \mathcal{M} = \{\emptyset, \{1\}, \{0,1,2\}\}, \tag{12}$$

because

$$\mathcal{L} \mathbin{\tilde{\cap}} \mathcal{M} = \{\emptyset, \{0\}, \{1\}, \{0,1,2\}\},\ \dim(\mathcal{L} \mathbin{\tilde{\cap}} \mathcal{M}) = 3,\ \dim \mathcal{L} = \dim \mathcal{M} = 2. \tag{13}$$

The assertion (2) holds, because of Definition 9, $n = 1$, and $\xi(y) = \{y\}$.

There are many equivalent definitions of WQOs (see [20, Theorem 2.1] and [4].) In [34], Cholak-Marcone-Solomon studied for which definition of WQO and which subsystem of second order arithmetic [35] proves

$$\mathcal{X} \text{ and } \mathcal{Y} \text{ are WQOs} \Rightarrow \mathcal{X} \cap \mathcal{Y} \text{ and } \mathcal{X} \times \mathcal{Y} \text{ are WQOs.}$$

The results are certainly related to a question "for which ordinal number do we have $\operatorname{otp}(\mathcal{X}), \operatorname{otp}(\mathcal{Y}) < \alpha \Rightarrow \operatorname{otp}(\mathcal{X} \cap \mathcal{Y}) < \alpha$?" We conjecture that we can take as α the proof-theoretic ordinal Γ_0. According to Simpson [35, Ch. V], Γ_0 is the proof-theoretic ordinal of a formal system which can formalize and develop significant parts of order (type) theory. I wonder whether we can take as α the the first nonrecursive ordinal. If $\dim(\mathcal{L})$ were almost equal to $\operatorname{otp}(\operatorname{qo}(\mathcal{L}))$ (cf. Remark 3), then we would smoothly study which ordinal numbers satisfy

$$\dim(\mathcal{L}), \dim(\mathcal{M}) < \alpha \;\Rightarrow\; \dim(\mathcal{L} \odot \mathcal{M}) < \alpha, \quad (\odot = \widetilde{\times}, \widetilde{\cup}, \widetilde{\cap}, \ldots.)$$

A Ramsey number argument used in the proof of Lemma 6 establishes an upper bound of a WQO obtained as the intersection of WQOs.

Theorem 8.

$$\operatorname{otp}(\mathcal{X}), \operatorname{otp}(\mathcal{Y}) < \omega \;\Rightarrow\; \operatorname{otp}(\mathcal{X} \cap \mathcal{Y}) < \operatorname{Ram}(\operatorname{otp}(\mathcal{X}) + 1, \operatorname{otp}(\mathcal{Y}) + 1, 2).$$

Proof. The proof is similar as that of Lemma 6. Assume $\mathcal{X} = (X, \preceq)$, $\mathcal{Y} = (Y, \sqsubseteq)$ and $\langle t_1, t_2, \ldots, t_m \rangle$ is a bad sequence of $\mathcal{X} \cap \mathcal{Y}$. Then for all i, j with $1 \le i < j \le m$, we have $t_i \not\preceq t_j$ or $t_i \not\sqsubseteq t_j$. For the complete graph consisting of $\{1, \ldots, m\}$, color all edges $\{i, j\}$ $(i \ne j)$ by red if $t_i \not\preceq t_j$, and color the other edges by black. Then there is a red complete graph consisting of size $\operatorname{otp}(\mathcal{X}) + 1$, or a black complete graph of size $\operatorname{otp}(\mathcal{Y}) + 1$. For the former case, the bad sequence $\langle t_1, t_2, \ldots, t_m \rangle$ has a bad subsequence, which consists of terms with the suffixes from the red graph's vertices. This bad sequence of $\mathcal{X}$ has the length $\operatorname{otp}(\mathcal{X})+1$, a contradiction. For the latter case, the black complete graph of size $\operatorname{otp}(\mathcal{Y}) + 1$ induces a bad subsequence of $\mathcal{Y}$ having the length $\operatorname{otp}(\mathcal{Y}) + 1$, a contradiction. Thus, we have the desired consequence.

One may conjecture

$$\operatorname{ss}(\mathcal{X} \cap \mathcal{Y}) \subset \operatorname{ss}(\mathcal{X}) \,\widetilde{\cap}\, \operatorname{ss}(\mathcal{Y}) \tag{14}$$

in order to derive a following asymptotic improvement of Theorem 8

$$\operatorname{otp}(\mathcal{X} \cap \mathcal{Y}) < \operatorname{otp}(\mathcal{X}) + \operatorname{otp}(\mathcal{Y}) \text{ for } \operatorname{otp}(\mathcal{X}), \operatorname{otp}(\mathcal{Y}) < \omega, \tag{15}$$

with an argument below: By (14) and Theorem 2 (1), we have $\operatorname{otp}(\mathcal{X} \cap \mathcal{Y}) \le \dim\big(\operatorname{ss}(\mathcal{X}) \,\widetilde{\cap}\, \operatorname{ss}(\mathcal{Y})\big)$, but Corollary 5 (1c) implies the latter is less than or equal to $\dim \operatorname{ss}(\mathcal{X}) + \dim \operatorname{ss}(\mathcal{Y}) - 1 = \operatorname{otp}(\mathcal{X}) + \operatorname{otp}(\mathcal{Y}) - 1$.

However the inclusion of (14) is actually opposite, when $\mathcal{X}, \mathcal{Y}$ are following WQOs $\le_0$ and $\le_1$. Let $\le_i$ $(i = 0, 1)$ be a WQO over $\{0, 1, 2\}$ such that the pair of $\operatorname{ss}(\le_i)$ $(i = 1, 2)$ is the pair of $\mathcal{L}$ and $\mathcal{M}$ presented in (12), which attains $\dim \mathcal{L} + \dim \mathcal{M} - 1 = \dim(\mathcal{L} \,\widetilde{\cap}\, \mathcal{M})$. Namely, $\le_i$ is such that two elements other than i are mutually related by $\le_i$ and are strictly lower than i by $\le_i$. Then $\le_0 \cap \le_1$ becomes a WQO such that the elements 0 and 1 are not comparable but are strictly greater than the element 2. Thus $\operatorname{ss}(\le_0 \cap \le_1) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}, \{0, 1, 2\}\} \supset \operatorname{ss}(\le_1) \,\widetilde{\cap}\, \operatorname{ss}(\le_2) = \{\emptyset, \{0\}, \{1\}, \{0, 1, 2\}\}$.

6. Embedding of the category of quasi-orders and finitely branching simulations

In hope that we could import idea and results on closure properties of WQOs and BQOs to study those of FEs, we show that $\mathrm{ss}(\bullet)$ studied in Section 3 becomes a neat embedding from the category of quasi-orders and *finitely branching simulations* to the category of set systems and *linear monotone, continuous functions.* Here a "*simulation*" is used widely in theoretical computer science (see [36].) "*Linear*" is used in the model theory of linear logic [17] and we will point out that it corresponds to Kanazawa's relation $R \subseteq X \times Y$ (see Proposition 5.)

By "neat embedding," we mean that $\mathrm{ss}(\bullet)$ not only preserves order types but also, in the jargon of category theory [37], becomes an injective-on-objects, full and faithful contravariant functor right adjoint to a functor that $\mathrm{qo}(\bullet)$ (see Section 3) induces.

Definition 10 (Finitely Branching Simulation)**.** *Let $\mathcal{X} = (X, \preceq)$ and $\mathcal{Y} = (Y, \sqsubseteq)$ be quasi-orders. We say a relation R is a* simulation *of $\mathcal{X}$ by $\mathcal{Y}$, provided $R \subseteq X \times Y$ and whenever $R(x, y)$ and $x \preceq x'$, there exists $y' \sqsupseteq y$ such that $R(x', y')$. We say a simulation R is* finitely branching *if $\#\{y\ ;\ R(x, y)\} < \infty$ for every x.*

Example 2 (Lineariztion)**.**
1. For an order-homomorphism $f : \mathcal{X} \to \mathcal{Y}$, a relation $R_f := \{(x, y)\ ;\ f(x) = y\}$ is a finitely branching simulation of $\mathcal{X}$ by $\mathcal{Y}$.
2. For every surjective order-homomorphism f from a quasi-order $\mathcal{X}$ to a linear order $\mathcal{Y}$, the relation R_f is a finitely branching simulation of $\mathcal{X}$ by $\mathcal{Y}$. In this case, we call $\mathcal{Y}$ a *linearization* of $\mathcal{X}$.

Lemma 7. *Let $\mathcal{X}$ and $\mathcal{Y}$ be quasi-orders. If R is a simulation of $\mathcal{X}$ by $\mathcal{Y}$, then $\widetilde{R^{-1}}[\mathrm{ss}(\mathcal{Y})] \subseteq \mathrm{ss}(\mathcal{X})$.*

Proof. Let $\mathcal{X} = (X, \preceq)$ and $\mathcal{Y} = (Y, \sqsubseteq)$. Any member of $\widetilde{R^{-1}}[\mathrm{ss}(\mathcal{Y})]$ is written as a set $L := \{x \in X\ ;\ \exists g \in M. \exists y \sqsupseteq g.\ R(x, y)\}$ for some $M \in \mathrm{ss}(\mathcal{Y})$. Suppose $x' \succeq x \in L$. Then because R is a simulation, there is y' such that $y' \sqsupseteq y$ and $R(x', y')$. By the transitivity of $\sqsubseteq$, we have $y' \sqsupseteq g$. Therefore $x' \in L$. Thus L is an upper-closed set, which implies $L \in \mathrm{ss}(\mathcal{X})$.

By Lemma 7, we have a so-called linearization lower bound [21, Sect. 2]:

Corollary 6. *For any linearization $\mathcal{Y}$ of $\mathcal{X}$, $\mathrm{otp}(\mathcal{Y}) \leq \mathrm{otp}(\mathcal{X})$.*

Proof. By the premise, there is a surjective order-homomorphism $f : \mathcal{X} \to \mathcal{Y}$. Because a relation R_f is a simulation, Lemma 7 implies $\dim \mathrm{ss}(\mathcal{X}) \geq \dim \widetilde{R_f^{-1}}[\mathrm{ss}(\mathcal{Y})]$ from which Theorem 2 (1) implies

$$\mathrm{otp}(\mathcal{X}) \geq \dim \widetilde{R_f^{-1}}[\mathrm{ss}(\mathcal{Y})]. \tag{16}$$

Put

$$Q_f := \big\{(x, \{y\})\ ;\ f(x) = y\big\} \subseteq \bigcup \mathrm{ss}(\mathcal{X}) \times \Big[\bigcup \mathrm{ss}(\mathcal{Y})\Big]^{<\omega}.$$

Then $\#\{v\ ;\ Q_f(s, v)\} = \#\{v\ ;\ \{f(s)\} = v\} \leq 1$ and

$$\dim \widetilde{R_f^{-1}}[\mathrm{ss}(\mathcal{Y})] = \dim \widetilde{Q_f^{-1}}[[\mathrm{ss}(\mathcal{Y})]], \tag{17}$$

because $\widetilde{R_f^{-1}}[\mathrm{ss}\,(\mathcal{Y})] = \{R_f^{-1}[M]\ ;\ M \in \mathrm{ss}\,(\mathcal{Y})\} = \big\{\{x \in \bigcup \mathrm{ss}\,(\mathcal{X})\ ;\ \exists y \in M.\, f(x) = y\}\ ;\ M \in \mathrm{ss}\,(\mathcal{Y})\big\} = \big\{\{x \in \bigcup \mathrm{ss}\,(\mathcal{X})\ ;\ \exists v \in [M]^{<\omega}.\ Q_f(x,v)\}\ ;\ M \in \mathrm{ss}\,(\mathcal{Y})\big\} = \{Q_f^{-1}[[M]]\ ;\ M \in \mathrm{ss}\,(\mathcal{Y})\} = \widetilde{Q_f^{-1}}[[\mathrm{ss}\,(\mathcal{Y})]]$.

Since $f : \mathcal{X} \to \mathcal{Y}$ is surjective and $\bigcup \mathrm{ss}\,(\mathcal{Y})$ is the underlying set of $\mathcal{Y}$, there is a right-inverse $\xi : \bigcup \mathrm{ss}\,(\mathcal{Y}) \to \bigcup \mathrm{ss}\,(\mathcal{X})$ of f. In other words, each $y \in \bigcup \mathrm{ss}\,(\mathcal{Y})$ has $\xi(y) \in \bigcup \mathrm{ss}\,(\mathcal{X})$ such that $f(\xi(y)) = y$, i.e., $R_f\,(\xi(y), y)$. Hence $Q_f\,(\xi(y), \{y\})$. By Theorem 7, $\dim \widetilde{Q_f^{-1}}[[\mathrm{ss}\,(\mathcal{Y})]] = \dim \mathrm{ss}\,(\mathcal{Y}) = \mathrm{otp}(\mathcal{Y})$. By (17), $\dim \widetilde{R_f^{-1}}[\mathrm{ss}\,(\mathcal{Y})] = \mathrm{otp}(\mathcal{Y})$. By (16), we have the desired consequence.

We will define the category $\mathbb{QO}_{FinSim}$ of quasi-orders and finitely branching simulations between them, as well as a suitable category of set systems and monotone, continuous functions between them, and then will show that the operation $\mathrm{ss}\,(\bullet)$ becomes a contravariant, functor from the former category $\mathbb{QO}_{FinSim}$ to the latter category, and that the functor $\mathrm{ss}\,(\bullet)$ is order-type-preserving, injective-on-objects, full and faithful. For notion of category theory, see [37].

Definition 11. The category $\mathbb{QO}_{FinSim}$ of quasi-orders and finitely branching simulations between them *is defined as follows: The objects are quasi-orders* $(X, \preceq)$*. The identity morphism of object* $\mathcal{X} = (X, \preceq)$ *is* $\mathrm{id}_{\mathcal{X}} = \{(x,x)\ ;\ x \in X\}$*. The morphisms from* $\mathcal{X} = (X, \preceq)$ *to* $\mathcal{Y} = (Y, \sqsubseteq)$ *are finitely branching simulations* $R \subseteq X \times Y$*. For morphisms* $R : (X, \preceq) \to (Y, \sqsubseteq)$ *and* $S : (Y, \sqsubseteq) \to (Z, \trianglelefteq)$*, the composition is defined as the relational composition*

$$S \circ R = \{(x,z)\ ;\ R(x,y) \text{ and } S(y,z) \text{ for some } y \in Y\}.$$

Let $\mathbb{QO}$ *be the category of quasi-ordered sets and order-homomorphisms between them. Then there is a faithful, identity-on-objects functor from* $\mathbb{QO}$ *to* $\mathbb{QO}_{FinSim}$*, because of Example 2 (1).*

Definition 12 (Linear, Sequential)**.** *Let* $\mathcal{D}$ *and* $\mathcal{C}$ *be set systems and* $\mathfrak{O} : \mathcal{D} \to \mathcal{C}$ *be a monotone, continuous function.* $\mathfrak{O}$ *is said to be* linear*, if there is* $R \subseteq \mathrm{fld}(\mathcal{C}) \times [\mathrm{fld}(\mathcal{D})]^{<2}$ *such that* $\mathfrak{O} = \mathfrak{O}_R$*.* $\mathfrak{O}$ *is said to be* sequential*, if there is* $R \subseteq \mathrm{fld}(\mathcal{C}) \times [\mathrm{fld}(\mathcal{D})]^{<\omega}$ *such that* $\mathfrak{O} = \mathfrak{O}_R$ *and* $\#\{v\ ;\ R(s,v)\} \leq 1$ *for all* $s \in \mathrm{fld}(\mathcal{C})$*.*

Let $\mathbb{SS}$ be the category of set systems and monotone, continuous functions between them. *Let* $\mathbb{SS}_{lin}$ *(*$\mathbb{SS}_{seq}$*, resp.) be the category of set systems and linear (sequential, resp.) monotone, continuous functions between them.*

Thus every object $\mathcal{C}$ of $\mathbb{SS}$ is written as $\mathsf{i}\mathcal{L}$ for some set system $\mathcal{L}$.

Let $\mathbb{COH}_{stable}$ be the cartesian closed category of coherence spaces and stable functions between them, introduced by Girard [17]. Here a stable function was originally introduced by Berry in an attempt to give a semantic characterization of *sequential* algorithms. Defining coproducts in $\mathbb{COH}_{stable}$ is difficult according to [17]. However not in $\mathbb{SS}$ and $\mathbb{SS}_{lin}$. It is because the morphisms of the two categories can represent nondeterministic computations as we saw in the proof of Section 5.

Theorem 9. (1) $\mathbb{SS}$ *and* $\mathbb{SS}_{lin}$ *are indeed complete categories with all finite coproducts. In* $\mathbb{SS}$ *and* $\mathbb{SS}_{lin}$*, for objects* $\mathsf{i}\mathcal{L}_j$ $(j \in J)$*, the coproduct is*

$$\mathsf{i}\mathcal{L} := \bigoplus_{j \in J} \mathsf{i}\mathcal{L}_j \text{ where } \mathcal{L} = \{\ L \times \{j\}\ ;\ L \in \mathcal{L}_j,\ j \in J\ \}, \tag{18}$$

and the product is $\mathsf{i}\left(\widetilde{\biguplus}_{j\in J}\mathcal{L}_j\right)$.

(2) *A following* ι *is a full functor from* $\mathbb{COH}_{stable}$ *to* $\mathbb{SS}_{seq}$ *:*

$$\iota(\mathcal{A}) = \mathsf{i}\mathcal{A}, \qquad \iota(\mathcal{A} \xrightarrow{F} \mathcal{B}) = \mathsf{i}\mathcal{A} \xrightarrow{\mathsf{i}^{-1}} \mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{\mathsf{i}} \mathsf{i}\mathcal{B}.$$

Proof. See A.

In $\mathbb{SS}_{seq}$, the dimension of an object is a categorical notion.

Theorem 10. *If* $\mathcal{C}$ *and* $\mathcal{D}$ *are isomorphic objects in* $\mathbb{SS}_{seq}$*,* $\dim \mathsf{i}^{-1}\mathcal{C} = \dim \mathsf{i}^{-1}\mathcal{D}$.

Proof. Because one object is the image of the other object by a sequential function, the former dimension is less than or equal to the latter dimension by Theorem 7 (10).

Following proves a part of Proposition 4 by Moriyama-Sato [12].

Theorem 11. *If* $\mathcal{L}$ *and* $\mathcal{M}$ *are* FESS*s,* $\dim \mathsf{i}^{-1}(\mathsf{i}\mathcal{L}_1 \oplus \mathsf{i}\mathcal{L}_2) = \max(\dim \mathcal{L}_1, \dim \mathcal{L}_2)$ *and the union* $\mathcal{L}_1 \cup \mathcal{L}_2$ *is an* FESS*.*

Proof. Because $\mathsf{i}^{-1}(\mathsf{i}\mathcal{L}_1 \oplus \mathsf{i}\mathcal{L}_2) = \{L \times \{j\}\ ;\ L \in \mathcal{L}_j, j = 1, 2\}$, any production sequence of it is exactly written as $\langle\langle(t_0, j), N_1\times\{j\}\rangle, \langle(t_1, j), N_2\times\{j\}\rangle, \ldots, \langle(t_{n-1}, j), N_n\times\{j\}\rangle\rangle$ for some n, $N_i \in \mathcal{L}_j$ $(1 \le i \le n)$ and $t_i \in N_i$ $(0 \le i \le n-1)$. Therefore, $\mathrm{Prod}(\mathsf{i}^{-1}(\mathsf{i}\mathcal{L}_1 \oplus \mathsf{i}\mathcal{L}_2))$ is the disjoint sum of $\mathrm{Prod}(\mathcal{L}_1)$ and $\mathrm{Prod}(\mathcal{L}_2)$, from which the conclusion follows.

The second assertion is because $\mathsf{i}(\mathcal{L} \cup \mathcal{M})$ is the direct image by the monotone, continuous function $\mathfrak{O}_{R_2} : \mathsf{i}\mathcal{L} \oplus \mathsf{i}\mathcal{M} \to \mathsf{i}(\mathcal{L} \cup \mathcal{M})$ of $\mathsf{i}\mathcal{L} \oplus \mathsf{i}\mathcal{M}$ where R_2 is defined in (8).

Definition 13. (1) *Define a contravariant functor* Ss *from* $\mathbb{QO}_{FinSim}$ *to* $\mathbb{SS}_{lin}$ *as follows. Let* $\mathcal{X} = (X, \preceq), \mathcal{Y} = (Y, \sqsubseteq)$ *be objects of* $\mathbb{QO}_{FinSim}$*. Put* $\mathrm{Ss}(\mathcal{X}) := \mathsf{i}(\mathrm{ss}(\mathcal{X}))$*. For each morphism* R *from* $\mathcal{X}$ *to* $\mathcal{Y}$*, let* $\mathrm{Ss}(R)$ *be the monotone, linear, continuous function* $\mathfrak{O}_{\hat{R}} : \mathrm{Ss}(\mathcal{Y}) \to \mathrm{Ss}(\mathcal{X})$ *with the trace* $\hat{R} = \{(x, \{y\})\ ;\ R(x, y)\}$*.*

(2) *Define a contravariant functor* Qo *from* $\mathbb{SS}_{lin}$ *to* $\mathbb{QO}_{FinSim}$ *as follows. Let* $\mathcal{C}$ *be an object of* $\mathbb{SS}_{lin}$*. Put* $\mathrm{Qo}(\mathcal{C}) := \mathrm{qo}\left(\mathsf{i}^{-1}\mathcal{C}\right)$*. For each morphism* $\mathfrak{O}_R : \mathcal{D} \to \mathcal{C}$ *in* $\mathbb{SS}_{lin}$*, let* $\mathrm{Qo}(\mathfrak{O}_R)$ *be a finitely branching simulation* $\check{R} := \{(x, y)\ ;\ R(x, \{y\})\} \subseteq \mathrm{fld}(\mathcal{C}) \times \mathrm{fld}(\mathcal{D})$ *of* $\mathbb{QO}_{FinSim}$*.*

Lemma 8. (1) Ss *is indeed a functor* $\mathbb{QO}^{\mathrm{op}}_{FinSim}$ *from to* $\mathbb{SS}_{lin}$ *.*

(2) Qo *is indeed a functor from* $\mathbb{SS}_{lin}$ *to* $\mathbb{QO}^{\mathrm{op}}_{FinSim}$*.*

Proof. (1) For every morphism $R : \mathcal{X} \to \mathcal{Y}$ of $\mathbb{QO}_{FinSim}$, $\mathrm{Ss}(R)[\mathsf{i}(\mathrm{ss}(\mathcal{Y}))] = \mathfrak{O}_R[\mathsf{i}(\mathrm{ss}(\mathcal{Y}))]$ is $\mathsf{i}\left(\widetilde{R^{-1}}[\mathrm{ss}(\mathcal{Y})]\right)$ by (5), a subset of $\mathsf{i}(\mathrm{ss}(\mathcal{X}))$ by Lemma 7 with R being a simulation. Thus $\mathrm{Ss}(R)$ is indeed a function from $\mathsf{i}(\mathrm{ss}(\mathcal{Y}))$ to $\mathsf{i}(\mathrm{ss}(\mathcal{X}))$. The functoriality is because

$$\begin{aligned} \mathfrak{O}_{S\circ R}(g)(x) &= \bigvee\nolimits_{R(x,y)} \bigvee\nolimits_{S(y,z)} (g(z) = 1) = \bigvee\nolimits_{R(x,y)} (\mathfrak{O}_S(g)(y) = 1) \\ &= \mathfrak{O}_R(\mathfrak{O}_S(g))(x) = (\mathfrak{O}_R \circ \mathfrak{O}_S)(g)(x). \end{aligned} \tag{19}$$

(2) Firstly, we establish the well-definedness of Qo. For any finitely branching relations $R, S \subseteq X \times Y$, $\mathfrak{O}_R = \mathfrak{O}_S$ implies $R = S$. To see it, suppose $\mathfrak{O}_R = \mathfrak{O}_S$. If $R(x, y)$, then $\mathfrak{O}_R(1_{\{y\}})(x) = 1 = \mathfrak{O}_S(1_{\{y\}})(x) = \bigvee_{S(x,y')}(y' = y)$, which implies $S(x, y)$. Therefore $R = S$.

Next, Qo preserves the identity morphism, because $\mathrm{Qo}\,(\mathfrak{O}_\Delta) = \Delta$ for every set A and for every diagonal relation on $A \times A$. $\mathrm{Qo}\,(\mathfrak{O}_R \circ \mathfrak{O}_S) = \mathrm{Qo}\,(\mathfrak{O}_S) \circ \mathrm{Qo}\,(\mathfrak{O}_R)$ follows from (19).

According to [37, Theorem IV.1.1, Theorem IV.1.2], a functor $G : A \to X$ is a left adjoint functor to a functor $F : X \to A$ if and only if there are natural transformations $\eta : \mathrm{Id}_X \dot{\to} GF$ and $\varepsilon : FG \dot{\to} \mathrm{Id}_A$ such that both the following composites are the identity natural transformations (of G, resp. F.)

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G, \quad F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F. \tag{20}$$

η is called the *unit* and ε is called the *counit*. The *opposite category* of a category A is denoted by A^{op}.

Theorem 12. *The functor* $\mathrm{Qo} : \mathbb{SS}_{lin} \to \mathbb{QO}^{\mathrm{op}}_{FinSim}$ *is a left adjoint functor to the functor* $\mathrm{Ss} : \mathbb{QO}^{\mathrm{op}}_{FinSim} \to \mathbb{SS}_{lin}$ *where the counit of the adjunction is the identity natural transformation of the identity functor* $\mathrm{Id}_{\mathbb{QO}^{\mathrm{op}}_{lin}}$.

Proof. By Theorem 2 (1) and the definition, the composite $\mathrm{Qo} \circ \mathrm{Ss}$ is the identity functor of $\mathbb{QO}_{lin}$. Define the unit $\eta_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ss}(\mathrm{Qo}\,(\mathcal{C}))$ by the inclusion map. Then (20) follows immediately.

Corollary 7. *The functor* Ss *is an injective-on-objects, full and faithful functor from* $\mathbb{QO}_{FinSim}$ *to* $\mathbb{SS}^{\mathrm{op}}_{lin}$. *Moreover* $\mathrm{otp}(\mathcal{X}) = \dim \mathrm{i}^{-1}\mathrm{Ss}(\mathcal{X})$ *for every object* $\mathcal{X}$ *of* $\mathbb{QO}_{FinSim}$.

Proof. The first assertion follows from Theorem 12, because every right adjoint functor is full and faithful whenever every component of the counit is an isomorphism [37, Theorem IV.3.1]. The other assertion follows from Theorem 2 (1).

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Appendix A. The categories of set systems and linear/sequential monotone, continuous functions

First we will prove Theorem 9. $\mathsf{i}\mathcal{L}$ and $\mathsf{i}L_j$ are as in the Theorem. See Figure 1 (right).

For each $j \in J$, the injection $\iota_j : \mathsf{i}\mathcal{L}_j \to \mathsf{i}\mathcal{L}$ is $\mathfrak{O}_{T_j}$ where

$$T_j := \left\{ (\langle x,\, j\rangle, \{x\}) \ ; \ x \in \bigcup \mathcal{L}_j,\ j \in J \right\} \ \subseteq\ \mathrm{fld}(\mathsf{i}\mathcal{L}) \times [\mathrm{fld}(\mathsf{i}\mathcal{L}_j)]^{<2} .$$

For any set $\left\{\mathfrak{O}_{S_j} : \mathsf{i}\mathcal{L}_j \to \mathcal{D} \ ;\ j \in J\right\}$ of morphisms of $\mathbb{SS}$, define

$$S \ :=\ \{(y, v_j \times \{j\})\ ;\ S_j(y, v_j),\ j \in J\ \}\ \subseteq\ \mathrm{fld}(\mathcal{D}) \times P\left(\mathrm{fld}(\mathsf{i}\mathcal{L})\right),$$

and a possibly non-continuous function $\mathfrak{F} : \mathsf{i}\mathcal{L} \to \mathcal{D}$ by

$$\mathfrak{F}(h)(y) := \bigvee_{S(y,v)} \bigwedge_{x \in v} h(x) = 1. \quad (h \in \mathsf{i}\mathcal{L},\ y \in \mathrm{fld}(\mathcal{D}).) \tag{21}$$

Theorem 9. (1) The two categories are closed under composition because of (4).

The terminal object is $\{\emptyset\} = \{0,1\}^{\emptyset}$. Any monotone, continuous function $\mathfrak{O}_R : \mathcal{C} \to \{0,1\}^{\emptyset}$ has $R \subseteq \emptyset \times [\mathrm{fld}(\mathcal{C})]^{<\omega}$ and thus $R = \emptyset$. Actually, for any $g \in \mathcal{C}$ and $x \in \emptyset$, we have $\mathfrak{O}_R(g)(x) = \bigvee_{R(x,v)} \bigwedge_{y \in v} (g(y) = 1)$.

For arbitrary nonempty set Λ, the product of objects $\mathsf{i}(\mathcal{L}_\lambda)$ $(\lambda \in \Lambda)$ is just the $\mathcal{C} = \mathsf{i}\left(\widetilde{\biguplus}_{\lambda\in\Lambda} \mathcal{L}_\lambda\right)$. For each $\lambda \in \Lambda$, the projection $\Pi_\lambda : \mathcal{C} \to \mathsf{i}\mathcal{L}_\lambda$ is $\Pi_\lambda(h) := h\left(\langle \bullet, \lambda\rangle\right)$ for all $h \in \mathcal{C}$. For any $\mathfrak{O}_{R_\lambda} : \mathcal{D} \to \mathsf{i}\mathcal{L}_\lambda$ $(\lambda \in \Lambda)$, the mediating morphism $\mathfrak{O}_R$ of Figure 1 is defined by $R \subseteq \mathrm{fld}(\mathcal{C}) \times [\mathrm{fld}(\mathcal{D})]^{<\omega}$ where $R := \{(\langle s,\ \lambda\rangle,\ v)\ ;\ R_\lambda(s,v)\}$. The $\mathfrak{O}_R$ is a morphism of $\mathbb{SS}$ (and $\mathbb{SS}_{lin}$ resp.) if $\mathfrak{O}_{R_\lambda}$'s are.

The equalizer $\mathfrak{O} : \mathsf{i}\mathcal{N} \to \mathsf{i}\mathcal{M}$ of a pair of functions $\mathfrak{O}_1, \mathfrak{O}_2 : \mathsf{i}\mathcal{M} \rightrightarrows \mathsf{i}\mathcal{L}$ is defined by

$$\mathsf{i}\mathcal{N} := \{g \in \mathsf{i}\mathcal{M}\ ;\ \mathfrak{O}_1(g) = \mathfrak{O}_2(g)\}, \quad \mathfrak{O}(g)(x) := g(x) \quad (g \in \mathsf{i}\mathcal{N}, x \in \bigcup \mathcal{M}.)$$

For Figure 1 (middle), the mediating morphism $\widetilde{\mathfrak{O}} : \mathcal{D} \to \mathsf{i}\mathcal{N}$ is defined by $\widetilde{\mathfrak{O}}(g)(y) = \mathfrak{O}'(g)(y)$ for any $g \in \mathcal{D}$ and any $y \in \bigcup \mathcal{N}$.

The initial object is $\emptyset$. Any function from $\emptyset$ to $\mathcal{C}$ is the function $\emptyset$, which is monotone, continuous because for any $g \in \emptyset$ and any $x \in \mathrm{fld}(\mathcal{C})$, we have $\emptyset(g)(x) = \mathfrak{O}_\emptyset(g)(x) = \bigvee_{\emptyset(x,v)} \bigwedge_{y \in v} (g(y) = 1)$.

The existence of a binary coproduct is because the finiteness of J implies the $\mathfrak{F}$ is indeed a morphism of $\mathbb{SS}$ ($\mathbb{SS}_{lin}$ resp.) if $\mathfrak{O}_{S_j}$'s are.

$$
\begin{tikzcd}
\mathcal{D} \arrow[d, dashed, "\mathfrak{O}_R"'] \arrow[dr, "\mathfrak{O}_{R_\lambda}"] & \\
\mathsf{i}(\widetilde{\biguplus}_{\lambda\in\Lambda} \mathcal{L}_\lambda) \arrow[r, "\Pi_\lambda"'] & \mathsf{i}\mathcal{L}_\lambda
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{i}\mathcal{D} \arrow[d, dashed, "\widetilde{\mathfrak{S}}"'] \arrow[dr, "\mathfrak{O}'"] & & \\
\mathsf{i}\mathcal{N} \arrow[r, "\mathfrak{O}"'] & \mathsf{i}\mathcal{M} \arrow[r, shift left, "\mathfrak{O}_1"] \arrow[r, shift right, "\mathfrak{O}_2"'] & \mathsf{i}\mathcal{L}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D} & \\
\mathsf{i}\mathcal{L} \arrow[u, dashed, "\mathfrak{F}"] & \mathsf{i}\mathcal{L}_j \arrow[l, "\iota_j=\mathfrak{O}_{T_j}"] \arrow[ul, "\mathfrak{O}_{S_j}"']
\end{tikzcd}
$$

$$h \longmapsto h\left(\langle \bullet, \lambda\rangle\right)$$

Figure 1. Product of $\mathsf{i}\mathcal{L}_\lambda$'s is $\mathsf{i}\left(\widetilde{\biguplus}_{\lambda\in\Lambda} \mathcal{L}_\lambda\right)$ (left), the equalizer $\mathsf{i}\mathcal{N}$ is constructed in a standard manner (middle), and coproduct of $\mathsf{i}\mathcal{L}_j$'s is $\mathsf{i}\{\ \mathcal{L}_j \times \{j\}\ ;\ j \in J\ \}$ where j ranges over a finite set J (right).

(2) According to [17, Section 8.5], the stable function $F : \mathcal{A} \to \mathcal{B}$ is exactly a function from $\mathcal{A}$ to $\mathcal{B}$ having a *trace*. Here the trace of F is the set R of pairs $(x, v) \in \bigcup \mathcal{B} \times [\bigcup \mathcal{A}]^{<\omega}$ such that v is a minimal (and actually the minimum) among L's such that $x \in F(L)$. A stable function F is recovered from the trace R by $F(L) = \{x \in \bigcup \mathcal{B}\ ;\ \exists v \subseteq L.\ R(x, v)\}$ for all $L \in \mathcal{A}$. So $\iota(F) = \mathsf{i} \circ F \circ \mathsf{i}^{-1}$ is written as $\mathfrak{O}_R$. Because v is minimum, and is in particular unique, $\mathfrak{O}_R$ is sequential, i.e., $\mathfrak{O}_R \in \mathbb{SS}_{seq}$. So ι is indeed well-defined. We can easily see that ι is indeed a functor.

Next we verify that the functor ι is indeed full. Let $\mathfrak{O}_R : \iota(\mathcal{A}) \to \iota(\mathcal{B})$ be a morphism of the category $\mathbb{SS}_{seq}$. Recall $\mathfrak{O}_R(1_L)(x) = 1 \iff \exists v \subseteq L.\ R(x, v)$. Because $\mathfrak{O}_R$ is sequential, each x has at most one v such that $R(x, v)$. So R is the set of pairs (x, v) such that v is minimum among L's such that $\mathfrak{O}_R(1_L)(x) = 1$. Thus $\mathsf{i}^{-1} \circ F \circ \mathsf{i}$ is the stable function with the trace being R.

Lemma 9. *None of $\mathbb{SS}$, $\mathbb{SS}_{lin}$ and $\mathbb{SS}_{seq}$ does not have the object $\mathsf{i}\mathcal{L}$ of (18) as a coproduct if J is infinite. Even $\mathbb{SS}_{seq}$ does not for $2 \leq \#J \leq \infty$.*

Proof. We show that $\mathfrak{F} : \mathsf{i}\mathcal{L} \to \mathcal{D}$ of (21) is not a morphism of $\mathbb{SS}$, when

$$S_\mu := \{(y, \emptyset)\ ;\ y \in \mathrm{fld}(\mathcal{D})\} \subseteq \mathrm{fld}(\mathcal{D}) \times [\mathrm{fld}(\mathsf{i}\mathcal{L})]^{<2}, \quad (\mu \in J) \tag{22}$$

Let $y \in \mathrm{fld}(\mathcal{D})$. Because J is infinite but $S \subseteq \mathrm{fld}(\mathcal{D}) \times [\mathrm{fld}(\mathsf{i}\mathcal{L})]^{<\omega}$, there is $\mu \in J \setminus \{j\ ;\ \exists v \in [\mathrm{fld}(\mathsf{i}\mathcal{L})]^{<\omega}.\ S(y, v)\ \wedge \exists \xi \in v \exists a.\ \xi = \langle\, a,\ j\,\rangle\ \}$. Let $f \in \mathsf{i}\mathcal{L}_\mu$ such that $f(x) = 1$ for all $x \in \mathrm{fld}(\mathsf{i}\mathcal{L}_\mu)$. Then by Figure 1 (right), we have $(\mathfrak{O}_S \circ \iota_\mu)(f)(y) = \mathfrak{O}_{S_\mu}(f)(y)$. Therefore

$$\bigvee_{S(y,v)} \bigwedge_{\xi\in v} (\iota_\mu(f)(\xi) = 1) = \bigvee_{S_\mu(y,u)} \bigwedge_{x\in u} (f(x) = 1).$$

Here $\xi \in v$ is written as $\xi = \langle a, j\rangle$ for some $j \neq \mu$. So $\iota_\mu(f)(\langle a, j\rangle) = 0$, which implies the left-hand side is 0. But, the right-hand side is 1 by (22).

It is difficult to relate $\dim(\mathcal{L} \mathbin{\widetilde{\times}} \mathcal{M})$ with $\dim(\mathcal{L} \mathbin{\widetilde{\cup}} \mathcal{M})$. When $\mathcal{L}$ and $\mathcal{M}$ are both coherence spaces, $\mathcal{L} \mathbin{\widetilde{\times}} \mathcal{M}$ is the "tensor product" $\mathcal{L} \otimes \mathcal{M}$.

Lemma 10. *Let $\mathcal{L}$ and $\mathcal{M}$ be set systems with $\bigcup \mathcal{L}$ infinite and $\bigcup \mathcal{M} \neq \emptyset$. Then,*

1. *There is no monotone, continuous function $\mathfrak{O} : \mathsf{i}(\mathcal{L} \mathbin{\widetilde{\times}} \mathcal{M}) \to \mathsf{i}(\mathcal{L} \mathbin{\widetilde{\cup}} \mathcal{M})$ such that $\mathfrak{O}(1_{L\times M}) = 1_{L\cup M}$ for all $L \in \mathcal{L}$ and $M \in \mathcal{M}$.*

(2) *There is no monotone, continuous function* $\mathfrak{O} : \mathsf{i}(\mathcal{L} \mathbin{\widetilde{\times}} \mathcal{M}) \to \mathsf{i}\mathcal{M}$ *such that* $\mathfrak{O}(1_{L\times M}) = 1_M$ *for all* $L \in \mathcal{L}$ *and* $M \in \mathcal{M}$.

Proof. (1) Let $X := \bigcup \mathcal{L}$ and $Y := \bigcup \mathcal{M}$. Assume there is such $\mathfrak{O}$. Then for each $s \in X \cup Y$, there exists a positive Boolean formula B_s over $\{v_{(x,y)}\ ;\ x \in X,\ y \in Y\}$, such that $\mathfrak{O}(1_{L\times M})(s)$ is the truth value of B_s under the truth assignment $1_{L\times M}$ for all $L \in \mathcal{L}$ and all $M \in \mathcal{M}$. Choose some $y \in Y$. There is a variable $v_{(x,y)}$ such that it does not appear B_y, because X is infinite. Therefore the truth value of B_y under the truth assignment $1_{\{(x,y)\}}$ is 0 because B_y does not contain negations of Boolean variables. On the other hand $\mathfrak{O}(1_{\{(x,y)\}})(y) = 1_{\{x\}\cup\{y\}}(y) = 1$. Contradiction. The assertion (2) is similarly proved.

The bang operator of a coherence space have following counterparts in $\mathbb{SS}$:

$$!\mathsf{i}\mathcal{L} := \mathsf{i}!\mathcal{L}$$

where the '!' in the right-hand side is defined in Theorem 5. Then $!\mathcal{L} \mathbin{\widetilde{\times}} !\mathcal{M}$ is isomorphic to $\mathcal{L} \mathbin{\widetilde{\uplus}} \mathcal{M}$, as in the case of $\mathbb{COH}_{stable}$. The duality operator of a coherence space, however, seems to have no exact counterpart in $\mathbb{SS}$, when we take an FE seriously. Since the elementwise complement of an FESS is not necessarily an FESS, the complement operation seems useless in defining the duality operator in $\mathbb{SS}$. So let us examine the exchange of Teacher and Learner. To be precise, For a set system $\mathcal{L}$ and $x \in \mathcal{L}$, put $\mathcal{L}(x) := \{L \in \mathcal{L}\ ;\ x \in L\}$, and $\mathcal{L}^{\perp} := \{\ \mathcal{L}(x);\ x \in \bigcup \mathcal{L}\ \}$. Then $\bigcup \left(\mathcal{L}^{\perp}\right) := \mathcal{L} \setminus \{\emptyset\}$. If $\mathcal{L}$ is the class of open sets of a sober space, then $\left(\mathcal{L}^{\perp}\right)^{\perp}$ is isomorphic to $\mathcal{L}$ in $\mathbb{SS}$. Since $L \in \mathcal{L}^{\perp}(x)$ iff $x \in L$,

$$\begin{aligned} &\langle\langle\, t_0,\, L_1 \,\rangle, \langle\, t_1,\, L_2 \,\rangle, \ldots, \langle\, t_{l-2},\, L_{l-1} \,\rangle, \langle\, t_{l-1},\, L_l \,\rangle\rangle \in \mathrm{Prod}(\mathcal{L}) \\ &\Longleftrightarrow \\ &\langle\langle\, L_l,\, \mathcal{L}(t_{l-1}) \,\rangle, \langle\, L_{l-1},\, \mathcal{L}(t_{l-2}) \,\rangle, \ldots, \langle\, L_2,\, \mathcal{L}(t_1) \,\rangle, \langle\, L_1,\, \mathcal{L}(t_0) \,\rangle\rangle \in \mathrm{Prod}(\mathcal{L}^{\perp}) \end{aligned}$$

We have an embedding from $\mathrm{Prod}(\mathcal{L})$ to $\mathrm{Prod}(\left(\mathcal{L}^{\perp}\right)^{\perp})$, by $\bigcap\bigcap \left(\left(\mathcal{L}^{\perp}\right)^{\perp}(L)\right) = L$. Thus $\dim \mathcal{L} \le \dim \left(\mathcal{L}^{\perp}\right)^{\perp}$.

Further categorical structures will be studied elsewhere.

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