PROGRESS MEASURES, IMMEDIATE DETERMINACY, AND A SUBSET CONSTRUCTION FOR TREE AUTOMATA

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Abstract

Using the concept of progress measure, we give a simplified proof of Rabin's fundamental result that the languages defined by tree automata are closed under complementation.

To do this we show that for infinite games based on tree automata, the *forgetful determinacy* property of Gurevich and Harrington can be strengthened to an *immediate determinacy* property for the player

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who is trying to win according to a Rabin acceptance condition. Moreover, we show a graph theoretic duality theorem for such acceptance conditions. We also present a strengthened version of Safra's determinization construction. Together these results and the determinacy of Borel games yield a straightforward method for complementing tree automata.

Our construction is almost optimal, i.e. the state space blow-up is essentially exponential—thus roughly the same as for automata on finite or infinite words. To our knowledge, no prior constructions have been better than double exponential.

1 Introduction

The complementation problem has long been a central topic in the theory of automata on infinite objects. Its solution is crucial to decidability results for restricted second order logics [24]. More recently, this problem—and especially the need to solve it as efficiently as possible—has also been a focus of research in temporal logics [3, 7, 21, 22], where decision problems can often be reduced to automata-theoretic problems.

Using the classical subset construction [17], one can easily complement nondeterministic automata on finite words. The complexity of this procedure, i.e. the increase in the size of the automaton, is exponential and this can be shown to be a lower bound. For automata defining languages of infinite words, the complementation problem is substantially more challenging. Although Büchi solved the problem in 1962 [1], it was only recently that methods of essentially exponential complexity were given [19, 22].[1] For automata on infinite trees, the complementation problem was first solved in a long and very difficult proof by Rabin in 1969 [16]. Because of its significance, the complementation problem for tree automata has since been addressed in several articles [2, 4, 6, 14, 15, 23, 26.]. The alternative proofs suggested are, however, still so complex or indirect that, to the author's knowledge, an exponential upper bound has not been achieved before. Analyses of earlier published work appear at best to give double-exponential methods [20.].

In this article we show that complementation of nondeterministic au-

[1]By an essentially exponential method we mean that for some $p > 0$, the size of the complemented automaton is $2^{O(p^n)}$ where n is the size of the original automaton.

tomata on infinite trees can be carried out with only an essentially exponential increase in size. This method is based on the technique of *progress measures* [7, 9, 10], which allow to reason locally about global properties of infinite paths in a graph. Our method relies on the following new results.

First, we generalize the result by Emerson and Jutla [4] that for certain infinite automata games the *forgetful determinacy* result—that winning strategies need only finite memory about the past—of Gurevich and Harrington [6] can be strengthened to *immediate determinacy*—that winning strategies can be made memoryless. Our result applies to strategies for a player who tries to win according to a *Rabin condition* (or *pairs condition*) [16], which is a special disjunctive normal form of conditions expressing that designated subsets of states are encountered infinitely often.

Second, we show a graph theoretic duality theorem for Rabin and Streett conditions. This result implies that to establish the global property of a Rabin condition, it suffices to guess a finitary approximation to a progress measure and then verify a Streett condition.

Third, we present a strengthened version of Safra's determinization construction [19].

Together these results yield a straightforward method for complementing tree automata equipped with a Streett condition. Our approach is motivated by the idea of directly using the subset construction. But as opposed to most of the previous approaches, we do not here use mathematical induction (except for what is built into Rabin progress measures) or translate automata to and from second-order or fixpoint logics.

According to results in [21] the lower bound for complementation of Streett automata on infinite words is essentially exponential. This bound carries over to Streett tree automata, making our method optimal to within an at most polynomial gap in the exponent.

Previous approaches

For Büchi automata on infinite words Sistla, Vardi, and Wolper [22] gave an essentially exponential complementation construction based on ideas in Büchi's original proof. Safra [19] showed that the subset construction could serve to determinize and complement. He obtained an optimal method of complexity $2^{O(n \cdot \log n)}$, which in addition greatly simplifies McNaughton's determinization proof from 1966 [12]. Another optimal method for complementing, but not determinizing, Büchi automata is based on progress measures [7].

Rabin's original proof [16] of the complementation property for tree automata relies on complicated ordinal induction. The proof by Muchnik [14] is based on an elegant induction on the number of states.

Using infinite games, Büchi [2], and later Gurevich and Harrington [6], made significant advances in understanding the problem. The idea here is to view a run of the automaton as a strategy for an *automaton player*, who tries to satisfy the acceptance condition; the other player, the *pathfinder*, tries to find a path in the input tree dissatisfying the acceptance condition. A *forgetful* strategy for the pathfinder is such that his choices need depend only on a fixed amount of information about the history of previous automaton states played.

Gurevich and Harrington proved that the automaton player has a winning forgetful strategy iff the automaton player does not have a winning strategy. Such strategies, also called *restricted memory* strategies, allow a subset construction to be carried through. Although simplifying previous work, the article contains some complicated inductive arguments.

Yakhnis and Yakhnis [26.] generalized the Gurevich-Harrington method to incorporate certain restraints on the players. The complexity of their method, however, has not been analyzed.

Muller and Schupp [15] simplified the complementation proof by using alternating automata. In addition to having branching transitions, these automata may proceed along each branch to Boolean combinations of new states. The problem with this approach is that closure under projection—a fundamental property used for modeling existential quantifiers in second order logics—is about as hard as complementing usual nondeterministic tree automata.

The approach by Emerson and Jutla [4] uses a simple acceptance condition, called the parity condition. For the parity condition Emerson and Jutla show that more than forgetful determinacy holds: a winning strategy can be made memoryless—a property that we here call *immediate determinacy.* Their tree automaton complementation method is based on translations between alternating tree automata and the logic of μ-calculus. The parity condition appears to be exponentially less succinct than the usual Rabin or Streett conditions.

The immediate determinacy property for games with the parity condition was discovered independently by A.W. Mostowski [13].

Safra's construction has been generalized previously in [4], where it is shown how to express the property that a Streett condition holds along all paths calculated by a subset automaton by means of a Rabin condition on subsets. We show how to express this property by means of a Streett condition on subsets.

A new construction by Safra [20] allows one to express by means of a Streett condition that a Rabin condition is satisfied on all paths. Together with our immediate determinacy result (or the result in [4] for the parity condition), this construction yields an alternative method for complementing tree automata with an essentially exponential blow-up.

2 Rabin and Streett conditions

A *graph* $G = (V, E)$ consists of a countable set of vertices (or states) V and a set of directed edges $E \subseteq V \times V$. A *basic pair* (R, I) on V consists of a set $R \subseteq V$ of *reconfirming* states and a set $I \subseteq V$ of *invalidating* states. A *pairs set* C is a set $\{(R_\chi, I_\chi) \mid \chi \in X\}$ of basic pairs; here X is a finite set of *colors*, and basic pair (R_χ, I_χ) is said to have color χ. We assume that no pair in C is repeated and say that $|C| = |X|$ is the *size* of C. For technical reasons, we always assume without loss of generality that $0 \in X$ and that $I_0 = \emptyset$ (one can always add the pair $(\emptyset, \emptyset)$ without changing the semantics of satisfaction defined next). We say that v and v' are *equivalent with respect to* C, and we write $v \equiv_C v'$, if for all $\chi \in X, v \in R_\chi$ iff $v' \in R_\chi$ (note that R_χ sets alone determine equivalence).

A *Rabin condition* $\mathbf{R}C$ is defined by a pairs set C. We say that an infinite sequence $v_0 v_1 \cdots$ *satisfies* $\mathbf{R}C$, and we write $v_0 v_1 \cdots \models \mathbf{R}C$, if for some $\chi, v_k \in R_\chi$ infinitely often and $v_k \in I_\chi$ only finitely often. We say that a graph $G = (V, E)$ satisfies a Rabin condition $\mathbf{R}C$ on V and write $G \models \mathbf{R}C$ if every infinite path $v_0 v_1 \cdots$ in G satisfies $\mathbf{R}C$. A *Streett condition* $\mathbf{S}C$ is the dual of Rabin condition and also given by a pairs set C. We let $v_0 v_1 \cdots \models$

SC denote that $v_0 v_1 \cdots \not\models \mathbf{R}C$, and let $G \models \mathbf{S}C$ denote that for all paths $v_0 v_1 \cdots$ in G, $v_0 v_1 \cdots \models \mathbf{S}G$.

3 Measures, surgery, and duality

Rabin progress measures are defined in terms of pointer trees, also known as direction trees. A *pointer tree* T is a countable, prefix-closed subset of ω_1^*, the set of finite sequences of countable ordinals.[2] Each sequence $t = t^1 \cdots t^\ell$ in T represents a *node*, which has *children* $td \in T$, where td denotes t concatenated with the single element sequence d. Here $d \in \omega_1$ is the *pointer* to td from t. The root of T is the empty sequence denoted ϵ. We visualize pointer trees as growing upwards; see Figure 1, where children are depicted from left to right in descending order. If t' is a prefix of $t \in T$, denoted $t' \overset{\leq}{\cdot} t$, then t' is called an *ancestor* of t. The *highest common ancestor* $t \uparrow t'$ of nodes $t = t^1 \cdots t^\ell$ and $t' = t'^1 \cdots t'^{\ell'}$ is the node $t^1 \cdots t^\lambda$, where λ is maximal such that $t^1 \cdots t^\lambda = t'^1 \cdots t'^\lambda$. The *level* $|t|$ of a node $t = t^1 \cdots t^\ell$ is the number ℓ; the level of ϵ is 0. The *prefix up to level* λ of $t = t^1 \cdots t^n$ is $t^1 \cdots t^{\min\{n,\lambda\}}$ denoted $t \downarrow \lambda$. The *height* of T is the maximum node level (if it exists).

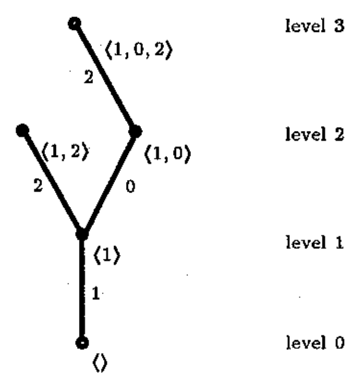


Figure 1: A pointer tree.

Definition 1. (Kleene-Brouwer Ordering) The ordering $\succ$ on T is defined

[2] X^* denotes the set of sequences of elements of X. X^+ denotes the set of non-empty sequences of elements of X.

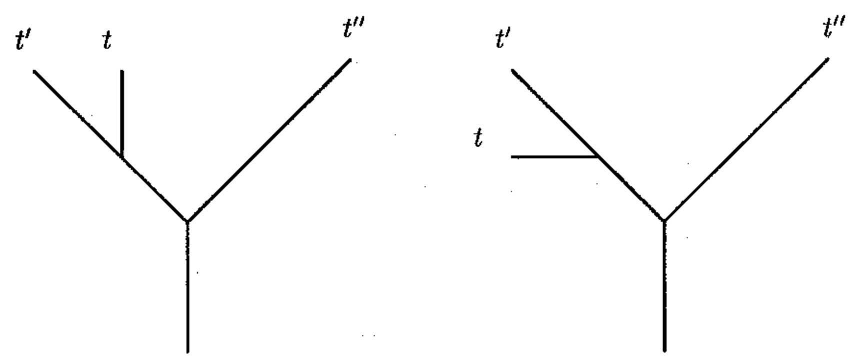
by: $t \succ t'$ if there is λ such that $t \downarrow \lambda = t' \downarrow \lambda$ and either $\lambda = |t| < |t'|$ or $\lambda < |t|, |t'|$ and $t^{\lambda+1} > t'^{\lambda+1}$. The ordering $\succeq$ is defined as $t \succeq t'$ if $t \succ t'$ or $t = t'$.

In other words $t \succeq t'$ if t is an ancestor of t' or if t' branches off to the right of t (assuming T is depicted as in Figure 1).

Lemma 1. (Kleene-Brouwer Ordering) If T is finite-path, then $\succ$ is well-ordered.

Proof. See [18]. $\square$

The Kleene-Brouwer Ordering has a property that may be explained informally as follows: if t' is to the left of t'' and if t is such that t'' branches off from t below where t' branches off from t, then t is to the left of t''—as illustrated by two examples:



Formally, we have

Lemma 2. $t' \succeq t''$ and $t \uparrow t'' \overset{<}{\cdot} t \uparrow t'$ implies $t \succ t''$.

Proof. Assume $t' \succeq t''$ and $t \uparrow t'' \overset{<}{\cdot} t \uparrow t'$. Let $\lambda = |t' \uparrow t''|$. From $t \uparrow t'' \overset{<}{\cdot} t \uparrow t'$, it follows that $|t|, |t'| > \lambda$ and that $t \downarrow \lambda + 1 = t' \downarrow \lambda + 1$. Moreover, since $t' \succeq t''$, it holds that $|t''| > \lambda$. Since also $|t'| > \lambda$, it follows that $t' \downarrow \lambda + 1 \succ t'' \downarrow \lambda + 1$. Thus $t \downarrow \lambda + 1 = t' \downarrow \lambda + 1 \succ t'' \downarrow \lambda + 1$. It follows that $t \succ t''$. $\square$

A *colored pointer tree* (T, ξ) is a pointer tree T with a partial mapping

$\xi : T \hookrightarrow X$, where X is a set of colors, assigning a color $\xi(t) \in X$ to each node t in domξ. For $\chi \in X$, we write "$\chi \in t$" as a shorthand for "$\exists \ell : \chi = \xi(t \downarrow \ell)$".

From [9] we slightly modify the notion of a Rabin progress measure:

Definition 2. A *Rabin progress measure* (or just *Rabin measure*) (μ, T, ξ) for (G,$\mathbf{R}C$) is a mapping $\mu : V \to T$, where (T, ξ) is finite-path colored pointer tree, such that

(I) for all $v \in V$ and all $\chi \in \mu(v), v \notin I_\chi$, and
(R) for all $(u, v) \in E, u \rhd_\mu v$,

where

$(\rhd)$

$$u \rhd_\mu v \text{ if } \mu(u) \succ (v), \text{ or} \\ \text{if there exists } \chi \in \mu(u) \uparrow \mu(v) \\ \text{such that } v \in \mathbf{R}_\chi.$$

Thus the value $\mu(v)$ of a Rabin progress measure μ denotes a list of colors such that (I) none of them are invalidating and such that (R) progress according to $\rhd$ takes place across any edge. The main result of [9] is:

Theorem 1. $G \models \mathbf{R}C$ iff there is a Rabin progress measure for (G, $\mathbf{R}C$)

Proof. "$\Leftarrow$" Let $v_0 v_1 \cdots$ be an infinite path in G. Then by (R), $v_0 \rhd_\mu v_1 \rhd_\mu \cdots$. It is not hard to see that there is a unique node $t \in T$ such that almost always $t \overset{\leq}{\cdot} \mu(v_k)$ and infinitely often $t = \mu(v_k) \uparrow \mu(v_{k+1})$. Let $\ell = |t|$. Suppose for a contradiction that there is no $\hat{t} \overset{\leq}{\cdot} t$ such that $v_k \in R_{\xi(\hat{t})}$ infinitely often. Then by (R) and the definition of t, it can be seen that almost always, it holds that $\mu(v_k) \downarrow (\ell+1) \succeq \mu(v_{k+1}) \downarrow (\ell+1)$. Moreover, since $t = \mu(v_k) \uparrow \mu(v_{k+1})$ holds infinitely often, also $\mu(v_k) \downarrow (\ell+1) \succ \mu(v_{k+1}) \downarrow (\ell+1)$ holds infinitely often. Thus there exists a K such that $\mu(v_K) \downarrow (\ell+1) \succeq \mu(v_{K+1}) \downarrow (\ell+1) \succeq \cdots$, where infinitely many of the inequalities are strict. This contradicts the Kleene-Brouwer Ordering Lemma. Thus there is a $\hat{t} \overset{\leq}{\cdot} t$ such that $v_k \in R_{\xi(\hat{t})}$

infinitely often. By (I) and definition of t, it holds that $v_k \in I_{\xi(\hat{t})}$ only finitely often. We conclude that $v_0 v_1 \cdots \models \mathbf{R}(R_{\xi(\hat{t})}, I_{\xi(\hat{t})})$, whence $v_0 v_1 \cdots \models \mathbf{R}C$.

"$\Rightarrow$" See [9] (the assumption that for color $\chi = 0, I_\chi = \emptyset$, is used here, and $\xi(\epsilon)$ is defined to be 0. $\square$

From the proof of Theorem 1 follows:

Lemma 3. If $(G, \mathbf{R}C)$ has Rabin measure (μ, T, ξ) and $v_0 v_1 \cdots$ is a path in G, then there is a $t \in T$ such that almost always $\mu(v_k) \overset{\leq}{\cdot} t$ and $v_0 v_1 \cdots \mathbf{R} \models (R_{\xi(t)}, I_{\xi(t)})$.

The next lemma is instrumental for the proof of immediate determinacy:

Lemma 4. (Surgery)

(1) If $v \rhd_\mu v',\ v' \equiv_C v''$, and $\mu(v') \succeq \mu(v'')$, then $v \rhd_\mu v''$.

(2) Thus if in addition $G \models \mathbf{R}C$, then all edges (v, v'') for v, v', and v'' satisfying (1) can be added to G and $G \models \mathbf{R}C$ will still hold.

Proof. Assume $v \rhd_\mu v',\ v' \equiv_C v''$, and $\mu(v') \succeq \mu(v'')$.

(1) Case 1: $\mu(v) \succ \mu(v')$. Since $\succeq$ is an order, $\mu(v) \succeq \mu(v'')$, and it follows that $(\rhd)$ is satisfied for v and v''. Case 2: There is a $\hat{t} \overset{\leq}{\cdot} \mu(v) \uparrow \mu(v')$ such that $v' \in R_{\xi(t)}$. If $\hat{t} \overset{\leq}{\cdot} \mu(v) \uparrow \mu(v'')$, then $(\rhd)$ is satisfied for v and v'', because $v' \equiv_C v''$ and thus $v'' \in R_{\xi(t)}$; otherwise, $\mu(v) \uparrow \mu(v') \overset{<}{\cdot} \hat{t} \overset{\leq}{\cdot} \mu(v) \uparrow \mu(v')$ and since $\mu(v') \succeq \mu(v'')$, application of Lemma 2 gives $\mu(v) \succ \mu(v'')$, whence $(\rhd)$ also holds for v and v''.

(2) After adding all such edges, μ is still a Rabin measure. Thus, by Theorem 1, $G \models \mathbf{R}C$ still holds.

$\square$

(A variant of this lemma is used in [8] for a proof of a version of Theorem 1 formulated for reasoning about fairness of programs.)

The variation on Rabin measures introduced next is a further simplification of the *quasi Rabin measures* in [7]. The idea is to get rid of the infinite range of a progress measure so that the values of the measure can be guessed by a finite-state machine.

Definition 3. The *edge graph* $\mathcal{E}G$ of $G = (V, E)$ is the graph whose vertices are E and whose edges are on the form $((u, v), (u', v'))$ with $v = u'$.

Definition 4. A *reduced quasi Rabin measure* $\mu : V \to X^*$ for $(G,\mathbf{R}C)$ assigns to each $v \in V$ a list $\mu(v) = \chi^0 \cdots \chi^\ell, \ell \leq |X|$, of colors such that for every $\chi \in \mu(v), v \notin I_\chi$. For a list $\chi_0 \cdots \chi^\ell$, define $|\chi^0 \cdots \chi^\ell| = \ell$. Given a reduced quasi Rabin measure μ, we define a pairs set $C^\mu = \{(\overline{R}_\ell, \overline{I}_\ell) \mid \ell \leq |X|\}$ on $\mathcal{E}G$ by:

$$
\begin{array}{ll}
(u,v) \in \overline{R}_\ell \text{ iff} & |\mu(u) \uparrow \mu(v)| \leq \ell. \\
(u,v) \in \overline{I}_\ell \text{ iff} & |\mu(u) \uparrow \mu(v)| < \ell, \text{ or} \\
& |\mu(u) \uparrow \mu(v)| \geq \ell \text{ and} \\
& v \in R_{\mu(v)^{\hat{\ell}}} \text{ for some } \hat{\ell} \leq \ell \text{ .}
\end{array}
$$

C^μ is called the *dual pairs set of C with respect to μ.*

Intuitively, a reduced quasi Rabin measure assigns to each vertex v list $\mu(v) = \chi^0 \cdots \chi^\ell$ that is prioritized with respect to the invalidating states: if $\ell < \ell'$, then along a path from v it is less likely that an I_{χ^ℓ}-invalidating state than an $I_{\chi^{\ell'}}$- invalidating state is encountered. Thus colors at lower positions represent pairs that are more likely to be satisfied than those at higher positions. The set $\overline{R}_\ell$ is constructed so that for an infinite sequence $v_0 v_1 \cdots, (v_i, v_{i+1}) \in \overline{R}_\ell$ holds infinitely often iff $\ell \geq \liminf_i |\mu(v_i) \uparrow \mu(v_{i+1})|$, i.e. if ℓ is, or is above, the highest level with an eventually constant color. The set $\overline{I}_\ell$ is constructed so that for an infinite sequence $v_0 v_1 \cdots, (v_i, v_{i+1}) \in \overline{I}_\ell$ holds infinitely often iff either $\ell > \liminf |\mu(v_i) \uparrow \mu(v_{i+1})|$, i.e. if ℓ is above the highest level with an eventually constant color, or if $\ell \leq \liminf |\mu(v_i) \uparrow \mu(v_{i+1})|$ and infinitely often the state v_i is recurrent for a basic pair denoted by some color at or below ℓ.

A Streett condition $\mathbf{S}C^\mu$ on all paths in the edge graph can be used to check that *no* path in the original graph satisfies a given Streett condition $\mathbf{S}C$ (i.e. that all paths satisfy $\mathbf{R}C$):

Lemma 5. (Rabin Duality) $G \models \mathbf{R}C$ iff there is a reduced quasi Rabin measure μ such that $\mathcal{E}G \models \mathbf{S}C^\mu$.

Proof. "$\Leftarrow$" Assume that $\mathcal{E}G \models \mathbf{S}C^\mu$. Let $v_0 v_1 \cdots$ be an infinite path in G. Note that $(v_0, v_1)(v_1, v_2)\cdots$ is an infinite path in $\mathcal{E}G$ and let $\ell_k = |\mu(v_k) \uparrow \mu(v_{k+1})|, \ell = \lim\inf_k \ell_k$. Also for $\hat{\ell} \le \ell$, let $\chi^{\hat{\ell}}$ be the eventually constant value of $\mu(v_k)^{\hat{\ell}}$.

From the definition of ℓ and $\overline{R}_\ell$, it follows that $(v_k, v_{k+1}) \in \overline{R}_\ell$ holds infinitely often. Thus by assumption that $\mathcal{E}G \models \mathbf{S}C^\mu$, $(v_k, v_{k+1}) \in \overline{I}_\ell$ holds infinitely often. By definition of $\overline{I}_\ell$ and since $\ell_k \ge \ell$ holds almost always, $v_k \in R_{\chi^{\hat{\ell}}}$ holds infinitely often for some $\hat{\ell} \le \ell$. Also, since $|\mu(v_k)| \ge \ell$ holds almost always, it follows that $v_k \notin I_{\chi^{\hat{\ell}}}$ holds almost always. Thus $v_0 v_1 \cdots \models \mathbf{R}(R_{\chi^{\hat{\ell}}}, I_{\chi^{\hat{\ell}}})$, whence $v_0 v_1 \cdots \models \mathbf{R}C$.

"$\Rightarrow$" Assume $G \models \mathbf{R}C$. By Theorem 1, there is a Rabin measure $(\tilde{\mu}, T, \xi)$ for $(G, \mathbf{R}C)$. The mapping $\tilde{\mu} : V \to T$ naturally induces a mapping $\mu : V \to X^*$ defined by $\mu(v) = \xi(\epsilon)\cdots\xi(\tilde{\mu}(v))$ and μ is easily seen to be a reduced quasi Rabin measure. To see that $\mathcal{E}G \models \mathbf{S}C^\mu$, consider a path $(v_0, v_1)(v_1, v_2)\cdots$ in $\mathcal{E}G$. Let $\ell_k = |\mu(v_k) \uparrow \mu(v_{k+1})| = |\tilde{\mu}(v_k) \uparrow \tilde{\mu}(v_{k+1})|$ and $\ell = \lim\inf_k \ell_k$. For $\hat{\ell} \le \ell$, let $\chi^{\hat{\ell}}$ be the eventually constant value of $\mu(v_k)^{\hat{\ell}} = \xi(\tilde{\mu}(v_k) \downarrow \hat{\ell})$. Fix $\lambda \le |X|$. Assume that $(v_k, v_{k+1}) \in \overline{R}_\lambda$ holds infinitely often. We must prove that $(v_k, v_{k+1}) \in \overline{I}_\lambda$ holds infinitely often.

Case 1: $\lambda > \ell$. By definitions of λ, ℓ, and $\overline{I}_\lambda$, it holds infinitely often that $(v_k, v_{k+1}) \in \overline{I}_\lambda$

Case 2: $\lambda \le \ell$. It most hold that $\ell = \lambda$; for if $\lambda < \ell$ then it holds almost always that $\ell_k \ge \ell > \lambda$, whence v_k would be in $\overline{R}_\lambda$ only finitely often. Since $\hat{\mu}$ is a Rabin measure, we have by Lemma 3 that $v_0 v_1 \cdots \models \mathbf{R}(R_{\xi(t)}, I_{\xi(t)})$ for some $t \in T$ such that almost always $t \overset{\le}{\cdot} \hat{\mu}(v_k)$. Let $\hat{\ell} = |t|$. By definition of $\mu, \hat{\ell} \le \ell$ and $\chi^{\hat{\ell}} = \xi(t)$. Thus, $v_0 v_1 \cdots \models \mathbf{R}(R_{\chi^{\hat{\ell}}}, I_{\chi^{\hat{\ell}}})$; in particular, $v_k \in R_{\chi^\ell}$ infinitely often. It follows that $(v_k, v_{k+1}) \in \overline{I}_\lambda$ holds infinitely often. $\square$

4 Tree automata and games

An infinite word is an infinite sequence of letters. Labeled trees are a generalization that allow a branching structure to the infinite objects.

Definition 5. A Σ-*labeled tree* $\tau : \mathbb{B}^* \to \Sigma$ assigns a letter $\tau(\delta)$ to each $\delta \in \mathbb{B}^*$.

We will sometimes call 1 the *left direction* and 0 the *right direction*. Tee automata are simple machine models designed to give a finite presentation of sets of infinite labeled trees.

Definition 6. A *tree automaton* $\mathbb{U} = (\Sigma, V, \to, V^0)$ consists of an *alphabet* Σ, *state set* V, *transition relation* $\to \subseteq V \times \mathbb{B} \times \Sigma \times V$, and a set of *initial states* $V^0 \subseteq V$. The *size* of $\mathbb{U}$ is denoted $|\mathbb{U}|$ and is the number of states, i.e. $|V|$. An *acceptance condition* is a condition on infinite state sequences. Automaton $\mathbb{U}$ equipped with acceptance condition $\mathbf{S}C$ is denoted $\mathbb{U}\mathbf{S}C$ and is called a *Streett tree automaton*.

Games

Given a tree automaton $\mathbb{U}$ and a Σ-labeled tree τ, called the *input tree*, we define the game $(\mathbb{U}, \tau)$, which is played between the *automaton player*, denoted A, and the *pathfinder player*, denoted PF. The players A and PF alternate. A plays states according to the transition relation, and PF plays directions indicating a path through τ. More precisely, the game goes as follows:

Round 0: $\{$
- A plays an initial state $v_0 \in V^0$.

Round 1: $\{$
- PF plays a direction $d_1 \in \mathbb{B}$, and
- A plays a state v_1 such that $v_0, d_1 \xrightarrow{\tau(\epsilon)} v_1$.

...

...

Round i: $\{$
- PF plays a direction $d_i \in \mathbb{B}$, and
- A plays a state v_i such that $v_{i-1}, d_i \xrightarrow{\tau(d_1 \cdots d_{i-1})} v_i$.

...

...

Note the asymmetry in the game $(\mathbb{U}, \tau)$: any $d \in \mathbb{B}$ is a legal move for player PF, whereas player A must obey the transition relation of $\mathbb{U}$. At any point during the course of a game, the *position* is a pair (ν,δ) describing the moves leading up that point. A *play* is the sequence of positions encountered, namely $(\epsilon, \epsilon), (v_0, \epsilon), (v_0, d_1), (v_0 v_1, d_1), \ldots$ In general a position is on the form (ν, δ) if it is A's turn and $(\nu v, \delta)$ if it is PF's turn, where in both cases $|\nu| = |\delta|$. In round i, the state *visited* is v_i, and if $i > 0$, the *input position* is $d_1 \cdots d_{i-1}$, and the letter *read* by $\mathbb{U}$ is $\tau(d_1 \cdots d_{i-1})$.

In the *acceptance game* $(\mathbb{U}\mathbf{S}C,\tau)$, A's goal is to play states satisfying $\mathbf{S}C$. Thus A *wins* the play if $v_0 v_1 \cdots \models \mathbf{S}C$; otherwise PF *wins*.

Strategies for player A

An A *strategy* α in the game $(\mathbb{U}, \tau)$ is a V-labeled tree that determines A's moves as a function of PF's moves; player A *plays according to* α if A's move in round i is $\alpha(d_1 \cdots d_i)$. Since the strategy α must denote legal moves, it must satisfy:

$$(\alpha 0) \qquad \alpha(\epsilon) \in V^0 \text{ and}$$

$$(\alpha 1) \qquad \alpha(\delta), d \xrightarrow{\tau(\delta)} (\delta d)$$

The strategy α is a *winning strategy for* A in the acceptance game $(\mathbb{U}\mathbf{S}C,\tau)$ if A wins all plays according to α. Automaton $\mathbb{U}\mathbf{S}C$ *accepts* τ if A has a winning strategy in the acceptance game. A winning strategy is also called a *run* over τ. The *language* $L(\mathbb{U}\mathbf{S}C)$ defined by $\mathbb{U}\mathbf{S}C$ is the set of all τ accepted by $\mathbb{U}\mathbf{S}C$.

Streett tree automata are as powerful as tree automata with the Rabin or Muller acceptance condition [24]. It is not hard to see that languages accepted by Streett automata are closed under union, intersection, and projection (homomorphism).

5 Complementing tree automata

The problem that we are to solve is: given $\mathbb{U}\mathbf{S}C$, find $\overline{\mathbb{U}}\mathbf{S}\overline{C}$ such that $\tau \notin L(\mathbb{U}\mathbf{S}C)$ iff $\tau \in L(\overline{\mathbb{U}}\mathbf{S}\overline{C})$. The proof follows the following recipe.

- We define strategies for PF. By the determinacy of Borel games, either A or PF has a winning strategy.
- To each PF strategy ρ we construct a graph $\mathcal{G}_\rho$, called a ρ-graph, that describes all possible games when PF plays according to ρ and A plays in any legal way. We show that a PF strategy ρ is winning iff $\mathcal{G}_\rho \models \mathbf{R}C$.
- We prove the immediate determinacy property that if PF has a winning strategy, then PF has a *memoryless winning strategy*, i.e. one that depends only on the position in the input tree and the last move by A.
- By the Rabin Duality Lemma, checking that $\mathcal{G}_\rho \models \mathbf{R}C$ is equivalent to checking that for some reduced quasi Rabin measure $\mathcal{E}\mathcal{G}_\rho \models \mathbf{S}C^\mu$. We show how the edge graph of $\mathcal{G}_\rho$ can be represented as a ρ-graph of an automaton $\mathcal{E}\mathbb{U}$, called the *edge automaton*.
- To complement, we apply a *subset construction* and obtain an automaton $\mathcal{P}\mathcal{E}\mathbb{U}$. This automaton guesses a memoryless PF strategy ρ and calculates piecemeal the graph describing all possible games that can be played when PF plays ρ.
- In order to make $\mathcal{P}\mathcal{E}\mathbb{U}$ verify that for some μ, $\mathcal{E}\mathcal{G}_\rho \models \mathbf{S}C^\mu$ holds, we modify it into an automaton $\mathcal{M}\mathcal{P}\mathcal{E}\mathbb{U}$, called the *measure automaton* that guesses the value of a reduced quasi progress measure μ for each vertex in $\mathcal{G}_\rho$. A Streett condition $\mathbf{S}C^{\mathcal{M}}$ on the graph calculated by $\mathcal{P}\mathcal{E}\mathbb{U}$ expresses $\mathbf{S}C^\mu$ according to the progress values guessed by $\mathcal{M}\mathcal{P}\mathcal{E}\mathbb{U}$.
- The final step consists of changing $\mathcal{M}\mathcal{P}\mathcal{E}\mathbb{U}$ into a Streett automaton that verifies that along all paths through the subsets, the Streett condition $\mathbf{S}C^{\mathcal{M}}$ holds. For each basic pair $(\overline{R}, \overline{I})$ in $C^{\mathcal{M}}$, we use an extended version of Safra's construction that allows a Streett condition on the states of $\mathcal{M}\mathcal{P}\mathcal{E}\mathbb{U}$ to express that for all paths, $\mathbf{S}(\overline{R}, \overline{I})$ holds. We use a cross product on the Safra constructions to express the conjunction of the basic pairs in $C^{\mathcal{M}}$.

Strategies for player PF

Strategies for PF specify PF's moves as functions of A's moves. We could define a strategy ρ for PF to be a function from V^+ to $\mathbb{B}$, but in order later to introduce memoryless strategies, we define ρ to be a partial function $V^+ \times \mathbb{B}^* \to \mathbb{B}$. Each element in dom$\rho$ is on the form $(\nu v, \delta), |\nu| = |\delta|$, and denotes the position after A has played νv and PF has played δ. The value of $\rho(\nu v, \delta)$ denotes PF's next move according to ρ. Formally, ρ must satisfy

$(\rho 0)$ $\quad (v, \epsilon) \in \mathrm{dom}\rho$ iff $v \in V^0$ and

$$(\rho 1) \quad \left.\begin{array}{l} (\nu v, \delta) \in \mathrm{dom}\rho \\ d = \rho(\nu v, \delta), \text{and} \\ v, d \xrightarrow{\tau(\delta)} v' \end{array}\right\} \text{iff } (\nu v v', \delta d) \in \mathrm{dom}\rho$$

Condition $(\rho 0)$ states that ρ defines a move for PF after any initial move by A. Condition $(\rho 1)$ states that from any position $(\nu v, \delta)$ in a game played according to ρ, PF plays $d = \rho(\nu v, \delta)$ and for any move v' that A then may play, the position $(\nu v v', \delta d)$ must be described by ρ; in the other direction, condition $(\rho 1)$ states that any position $(\nu v v', \delta d)$ described must be accessible by moves according to the strategy.

Lemma 6. (Determinacy) Either A or PF has a winning strategy in the acceptance game $(\mathbb{U}\mathbf{S}C,\ \tau)$.

Proof. The set of sequences satisfying a Streett condition is a Boolean combination of $\prod_2^0$ sets (sets at the second level of the Borel hierarchy also called $\mathbf{G}_\delta$ sets). Such combinations are contained in the third level of the Borel hierarchy. Then by Martin's Theorem [11] (or the simpler result in [5]) the acceptance game is determined, i.e. one of the players A or PF has a winning strategy. $\square$

Definition 7. The graph $\mathcal{G}_\rho(\mathbb{U}, \tau)$, called the ρ-*graph*, of a strategy ρ in the game $(\mathbb{U}, \tau)$ is the graph on domρ whose edges are $((\nu v, \delta), (\nu v v', \delta d))$, where $d = \rho(\nu v, \delta)$ and $v,\ d \xrightarrow{\tau(\delta)} v'$, corresponding to the condition $(\rho 1)$. This graph describes all possible ways that A can play given that PF plays according to ρ. $\mathcal{G}_\rho(\mathbb{U}, \tau)$inherits a pairs set C in the natural way; for example,

$R \subseteq V$ is inherited as $\{(\nu v, \delta) \mid v \in R, |\nu| = |\delta|\}$. The inherited pairs set is also denoted C.

Example. In Figure 2 we have depicted the ρ-graph of a strategy for PF in a game $(\mathbb{U}, \tau)$. The tree τ is shown by representing the edges as "dividing screens." The only initial state v^0 is shown to the left; the black square here denotes the game position (v^0, ϵ) after A has played v^0 in Round 0. In Round 1, PF's strategy is to play "left," i.e. $\rho(v^0, \epsilon) = 1$. There are three possible states that A may play in turn, namely v', v'', and v'''; that is, these states are all v such that $v^0, 1 \overset{\tau(\epsilon)}{\to} v$. The move "left" by PF and the choices of A in Round 1 are depicted as three edges along the "dividing screen" along the left direction from the root. There are three possible game positions corresponding to the position 1 in the input tree, namely $(v^0v', 1), (v^0v'', 1)$, and $(v^0v''', 1)$. In round 2, PF plays "left" in the first position, giving A two choices (the dotted edge will be explained later); "left" in the second position, giving A three choices; and "right" in the third position, giving A two choices. Note that in the position 11 of the input tree, there are *two* ways of reaching the state $\hat{v}$: through v' and through v''. The strategy is conflicting: in the first case it specifies that PF play "left," in the second "right." In general, there is no bound on the number of different copies that may exist in a given position of the input tree. This is the fundamental problem that is overcome by means of memoryless strategies.

Definition 8. If $w = d_1 d_2 \cdots$ is an infinite path in τ, then the subgraph of $\mathcal{G}_\rho(\mathbb{U}, \tau)$ consisting of all nodes of the form $(\nu v, d_1 \cdots d_k)$, where $|\nu| = |k|$, is called the *w-bundle* of $\mathcal{G}_\rho(\mathbb{U}, \tau)$.

Note that each w-bundle is a forest; in fact, all of $\mathcal{G}_\rho(\mathbb{U}, \tau)$ is a forest.

Lemma 7. In the game $(\mathbb{U}\mathbf{S}C,\ \tau)$, ρ is a winning strategy for PF iff $\mathcal{G}_\rho(\mathbb{U}, \tau) \models \mathbf{R}C$.

Proof. Follows from the fact that the infinite paths (starting in $(v^0, \epsilon), v^0 \in V^0$) of $\mathcal{G}_\rho(\mathbb{U}, \tau)$ correspond to all possible outcomes of the game when PF

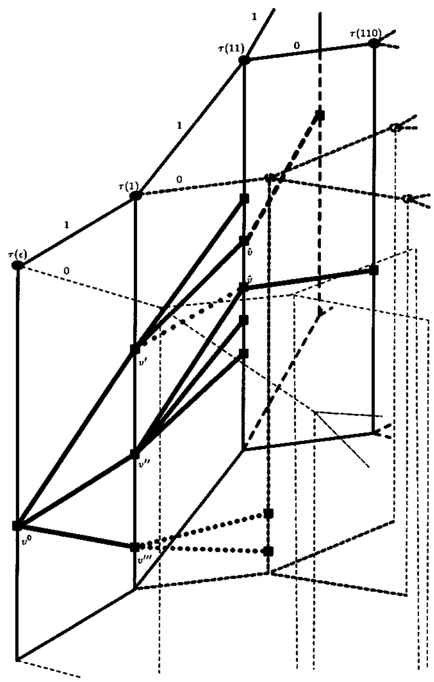


Figure 2: Immediate determinacy.

plays according to ρ. $\square$

Memoryless PF strategies

A *memoryless* PF *strategy* ρ depends on all the previous directions played by PF but only on the very last move by A. Thus it can be described by a partial function $V \times \mathbb{B}^* \hookrightarrow \mathbb{B}$. An element $(v, \delta) \in \mathrm{dom}\rho$ describes a position in a game where PF has played δ according to ρ and A has played νv for some ν, where $|\nu| = |\delta|$; PF's next move is then $\rho(v, \delta)$. A memoryless strategy ρ defines a strategy $\overline{\rho}$ by $\overline{\rho}(\nu v, \delta) = \rho(v, \delta)$. Thus ρ must satisfy:

(ρMLO) $\quad (v, \epsilon) \in \mathrm{dom}\rho$ iff $v \in V^0$ and

$$(\rho ML1) \quad \left.\begin{array}{l} \exists v : (v, \delta) \in \mathrm{dom}\rho \\ d = \rho(v, \delta)\ , \text{ and} \\ v, d \xrightarrow{\tau(\delta)} v' \end{array}\right\} \text{iff } (v', \delta d) \in \ \mathrm{dom}\rho$$

Definition 9. The graph $\mathcal{G}_\rho^{ML}(\mathbb{U}, \tau)$, called the *memoryless ρ-graph*, of a strategy ρ in the game $(\mathbb{U}, \tau)$ is the graph on $\mathrm{dom}\rho$ whose edges are $((v, \delta), (v', \delta d))$, where $d = \rho(\delta)$ and $v, d \xrightarrow{\tau(\delta)} v'$, corresponding to the condition $(\rho ML1)$. As in Definition 7, a Rabin condition C on V is inherited and also denoted C.

Definition 10. If $w = d_1 d_2 \cdots$ is an infinite path in τ, then the subgraph of $\mathcal{G}_\rho^{ML}(\mathbb{U}, \tau)$ consisting of all nodes of the form $(v, d_1 \cdots d_k)$ is called the *w-bundle* of $\mathcal{G}_\rho^{ML}(\mathbb{U}, \tau)$.

Note that a w-bundle of $\mathcal{G}_\rho^{ML}(\mathbb{U}, \tau)$ is a not a forest, but a graph whose "width" is at most $|V|$; in other words for each δ, there are at most $|V|$ vertices on the form (v, δ).

Lemma 8. (Memoryless ρ-graph) In the game $(\mathbb{U}\mathbf{S}C, \rho)$, the memoryless PF strategy ρ is a winning strategy iff $\mathcal{G}_\rho^{ML}(\mathbb{U}, \tau) \models \mathbf{R}C$.

Proof. Cf. proof of Lemma 7. □

Example (Continued). Consider again the ρ-graph in Figure 2. Assume that the strategy for PF is winning with respect to some Rabin condition on the automaton states. Then there is a Rabin measure μ for the graph. Suppose that $\mu(v^0 v' \hat{v}, 11) \geq \mu(v^0 v'' \hat{v}, 11)$. Now according to the Sewing Lemma, an edge from $(v_0, v', 1)$ to $(v^0 v' \hat{v}, 11)$ can be added with μ remaining a progress measure. The edge from $(v_0, v', 1)$ to $(v^0 v' \hat{v}, 11)$ can now be deleted (together with the unattached subtree rooted in $(v^0 v'' \hat{v}, 11)$). The graph will still represent a winning strategy since it has a progress measure. The strategy ρ at 11 has become memoryless: when $\hat{v}$ was the last state played by A, ρ specifies that PF plays "right" regardless of how $\hat{v}$ was reached.

By using the idea in the example above, we easily prove:

Lemma 9. (Immediate Determinacy) In the game $(\mathbb{U}\mathbf{S}C,\tau)$, there is a winning PF strategy iff there is a memoryless winning PF strategy.

Proof. "$\Leftarrow$" Obvious.

"$\Rightarrow$" Assume that ρ is a winning strategy for PF. We will define a memoryless winning strategy $\overline{\rho}$ by applying surgery to the ρ-graph representation $(\widehat{V},\widehat{E}) = \mathcal{G}_\rho(\mathbb{U},\tau)$. For $\hat{v} = (\nu v, \delta) \in \widehat{V}$, define $\mathbf{v}\hat{v} = v$ and $\delta\hat{v} = \delta$. The idea is for each (v,δ) to select a canonical position $\hat{v} = m(v,\delta)$ in $(\widehat{V},\widehat{E})$ with $v = \mathbf{v}\hat{v}$ and $\delta = \delta\hat{v}$. Then $\overline{\rho}(v,\delta)$ is defined to be $\rho(\hat{v})$. According to Lemma 7 and Theorem 1 there is a Rabin measure μ for $(\mathcal{G}_\rho(\mathbb{U},\tau),\mathbf{R}C)$. The canonical position $m(v,\delta)$ is now chosen according to μ: define $m(v,\delta) = \hat{v}$ such that $v = \mathbf{v}\hat{v}$, $\delta = \delta\hat{v}$, and $\mu(\hat{v})$ is minimal with respect to the Kleene-Brouwer ordering (if several $\hat{v}$ qualify, choose one).

The graph $(\widehat{V},\widehat{E})$ restricted to the canonical positions would not correspond to a strategy: from $\hat{v} = m(v,\delta)$ there may emanate an edge $(\hat{v},\hat{v}')$ to a position $\hat{v}'$ that is not a canonical position. The position $\hat{v}'$ is such that $\mathbf{v}\hat{v}' = v'$ for some v' with $v, d \xrightarrow{\tau(\delta)} v'$ and
$mathbf\delta\hat{v}' = d$, where $d = \rho(\hat{v})$. Now consider the vertex $\hat{v}'' = m(v', \delta d)$. Since $(\hat{v},\hat{v}') \in \widehat{E}$, it holds that $\hat{v} \rhd_\mu \hat{v}'$; since the pairs set C on $\widehat{V}$ is inherited from V and $\mathbf{v}\hat{v}' = \mathbf{v}\hat{v}'' = v'$, it holds that $\hat{v}' \equiv_C \hat{v}''$; and by the definition of $\hat{v}''$, it holds that $\mu(\hat{v}') \succeq \mu(\hat{v}'')$.

All edges $(\hat{v},\hat{v}'')$, where $\hat{v},\hat{v}''$ are gotten as above, are added to $(\widehat{V},\widehat{E})$. By the Surgery Lemma (2) the resulting graph, also denoted $(\widehat{V},\widehat{E})$, has a Rabin measure; moreover, $(\widehat{V},\widehat{E})$ restricted to canonical positions now describes a memoryless strategy $\overline{\rho}$. Thus by Theorem 1, $\overline{\rho}$ is a winning strategy. □

Edge automata

Definition 11. Given $\mathbb{U}$, define the *edge automaton* $\mathcal{E}\mathbb{U}$ by

$$\mathcal{E}\mathbb{U} = (\Sigma, (V \cup \{\perp\}) \times V, \to_\sim, \{\perp\} \times V^0),$$

where $(u,v), d \xrightarrow{\sigma}_\sim (u',v')$ iff $v = u'$ and $u', d \xrightarrow{\sigma} v'$.

The edge graphs of $\mathbb{U}$'s ρ-graphs are essentially the same as $\mathcal{E}\mathbb{U}$'s ρ-graphs:

Lemma 10. (Edge Automaton)

(1) To each PF strategy ρ in $(\mathbb{U}, \tau)$ corresponds a PF strategy $\tilde{\rho}$ in $(\mathcal{E}\mathbb{U}, \tau)$ and this correspondence is bijective.

(2) $\mathcal{G}_{\tilde{\rho}}(\mathcal{E}\mathbb{U}, \tau) \setminus \tilde{V}^0 \simeq \mathcal{E}\mathcal{G}_{\rho}(\mathbb{U}, \tau)$, where $\tilde{V}^0 = \{((\perp, v^0), \epsilon) \mid v^0 \in V^0\}$.

Proof.

(1) The correspondence is given by $\begin{array}{rcl} \tilde{\rho}((\perp, v_0), \epsilon) & = & \rho(v_0, \epsilon) \\ \tilde{\rho}((\perp, v_0)(v_0, v_1) \cdots (v_{k-1}, v - k), \delta) & = & \rho(v_0 \cdots v_k, \delta) \\ k \geq 1, |\delta| & = & k. \end{array}$

(2) An isomorphism $h : \mathcal{G}_{\sim\rho}(\mathcal{E}\mathbb{U}, \tau) \backslash \sim V^0 \to \mathcal{E}\mathcal{G}_{\rho}(\mathbb{U}, \tau)$ is defined by:

$$h((\perp, v_0)(v_0, v_1) \cdots (v_{k-1}, v_k), \delta d) = \\ ((v_0 \cdots v_{k-1}, \delta), (v_0 \cdots v_k, \delta d)),$$

where $k \geq 1, |\delta| = k - 1$.

$\square$

Subset automata

We define an automaton $\mathcal{P}\mathbb{U}$ whose runs or A strategies are the PF strategies of $\mathbb{U}$:

Definition 12. Given $\mathbb{U} = (\Sigma, V, \to, V^0)$, define the *subset automaton* $\mathcal{P}\mathbb{U}$ of A by

$$\mathcal{P}\mathbb{U} = (\Sigma, (V \hookrightarrow \mathbb{B}), \to_{\mathcal{P}}, \{v' \mid \exists \mathrm{dom}\Delta = V^0\}),$$

where $\Delta, d \xrightarrow{\sigma}_{\mathcal{P}} \Delta'$ iff $\mathrm{dom}\Delta' = \{v' \mid \exists v \in \mathrm{dom}\Delta : \Delta(v) = d \text{ and } v, d \xrightarrow{\sigma} v'\}$.

Intuitively, each $\Delta : V \hookrightarrow \mathbb{B}$ prescribes the direction $\Delta(v)$ that PF chooses for $v \in \mathrm{dom}\Delta$.

Definition 13. If α is an A strategy for $(\mathcal{P}\mathbb{U}, \tau)$, define the *$\alpha$-graph* $\mathcal{G}_\alpha(\mathcal{P}\mathbb{U}, \tau)$ as the graph whose vertices are (v, δ), where $v \in \mathrm{dom}\alpha(\delta)$, and whose edges are $((v, \delta), (v', \delta d))$, where $v, d \stackrel{\tau(\delta)}{\rightarrow} v'$.

Lemma 11. (Subset Automaton)

(1) To each memoryless PF strategy ρ in the game $(\mathbb{U}, \tau)$ corresponds an A strategy α in the game $(\mathcal{P}\mathbb{U}, \tau)$ and this correspondence is bijective.

(2) Moreover, $\mathcal{G}_\rho^{ML}(\mathbb{U}, \tau) \simeq \mathcal{G}_\alpha(\mathcal{P}\mathbb{U}, \tau)$.

Proof. (sketch) The proof consists of straightforward applications of the natural isomorphism between $V \times \mathbb{B}^* \hookrightarrow \mathbb{B}$ and $\mathbb{B}^* \rightarrow (V \hookrightarrow \mathbb{B})$, since memoryless PF strategies for $(\mathbb{U}, \tau)$ are functions of the first kind and A strategies for $(\mathcal{P}\mathbb{U}, \tau)$ are functions of the second kind. $\square$

Note that each path $w = d_1 d_2 \cdots$ in τ also defines a w-bundle consisting of all paths that go through subsets along w according to the transition relation.

Automata extensions

We will need to augment the information carried by the states of an automaton $\mathbb{U}$ by adding a component W to its state space while preserving its basic behavior. Formally, we define an *extension* $\mathcal{X}\mathbb{U}$ of $\mathbb{U}$ to be an automaton on the form $(\Sigma, V \times W, \rightarrow_\mathcal{X}, V^0 \times \{w^0\})$ such that

$$v, d \stackrel{a}{\rightarrow} v' \text{ and } w \in W \text{ implies}$$

$$(v, w), d \stackrel{a}{\rightarrow}_\mathcal{X} (v', w') \text{ for some unique } w' \in W.$$

A run α of $\mathbb{U}$ corresponds to a unique run, also denoted α, of an extension $\mathcal{X}\mathbb{U}$. The *size factor* $|\mathcal{X}|$ of $\mathcal{X}\mathbb{U}$ is defined as $|W|$.

If $\mathcal{X}_i\mathbb{U}$, where i ranges over some finite set, are extensions of $\mathbb{U}$, then they can be combined in a cross product construction to form an extension denoted $(\otimes_i \mathcal{X}_i)\mathbb{U}$; this automaton has state space $V \times \Pi W_i$. In the following we consider *Streett extensions* on the form $\mathcal{X}\mathbb{U}\mathbf{S}C$, where C describes a pairs set on the automaton $\mathcal{X}\mathbb{U}$. Assume that $\mathcal{X}_i\mathbb{U}\mathbf{S}C_i$ are Streett extensions. Then $(\otimes_i \mathcal{X}_i)\mathbb{U}\mathbf{S}\bigcup_i C_i$ denotes the Streett extension that has as acceptance condition the conjunction of the Streett conditions $\mathbf{S}\overline{C}_i$'s.

Lemma 12. (And'ing Extensions) The strategy α is a winning strategy for all Streett extensions $\mathcal{X}_i\mathbb{U}\mathbf{S}C_i$ iff α is a winning strategy for the Streett extension $(\otimes_i \mathcal{X}_i)\mathbb{U}\mathbf{S}\cup_i C_i$.

Proof. By definition of $(\otimes_i \mathcal{X}_i)\mathbb{U}\mathbf{S}\cup_i C_i$ □

Extended Safra construction

Safra's subset construction for automata on infinite words shows how to obtain a deterministic Rabin automaton accepting the language of a nondeterministic Büchi automaton [19]. By duality, if we are given an automaton on infinite words and a Streett condition on the form $\mathbf{S}(R, \emptyset)$—which is satisfied iff R is encountered only finitely often—then Safra's result tells us how to construct a deterministic Streett automaton that accepts a word iff *all* runs over the word satisfies $\mathbf{S}(R, \emptyset)$. It is easy to extend this subset construction to a tree automaton $\mathbb{U}$: there exists a Streett extension $\mathcal{X}\mathcal{P}\mathbb{U}\mathbf{S}\mathcal{C}$ such that α is a winning strategy for A in $(\mathcal{X}\mathcal{P}\mathbb{U}\mathbf{S}\mathcal{C},\, \tau)$ iff all paths in $\mathcal{G}_\alpha(\mathcal{P}\mathbb{U},\, \tau)$ satisfy $\mathbf{S}(R, \emptyset)$. Thus, $\mathcal{X}\mathcal{P}\mathbb{U}\mathbf{S}\mathcal{C}$ accepts τ iff there is a winning strategy for PF in the game $(\mathbb{U}\mathbf{R}(R, \emptyset), \tau)$ (i.e. whatever A does, a play of $(\mathbb{U}, \tau)$ results satisfying $\mathbf{S}(R, \emptyset)$). Conditions on the form $\mathbf{S}(R, I)$ require a more detailed consideration.

Lemma 13. (Extended Safra) Let $\mathbb{U} = (\Sigma, V, \rightarrow, V^0)$ and let (R, I) be a basic pair on V. There is a Streett extension $\mathcal{X}\mathcal{P}\mathbb{U}\mathbf{S}\mathcal{C}$ of $\mathcal{P}\mathbb{U}$ such that

$\mathcal{G}_\alpha(\mathcal{P}\mathbb{U},\, \tau) \models \mathbf{S}(R, I)$ iff

α is a winning strategy for A in the game $(\mathcal{X}\mathcal{P}\mathbb{U}\mathbf{S}\mathcal{C},\, \tau)$.

Moreover, $|\mathcal{X}| = 2^{O(n \cdot \log n)}$ and $|C| = n$, where $n = |V|$.

Proof. Let $\mathbb{U} = (\Sigma, V, \rightarrow, V^0)$ be a tree automaton and let (R, I) be a basic pair on V. Let τ be an infinite Σ-labeled tree and let α be a run of $\mathcal{P}\mathbb{U}$ over τ, i.e. a PF strategy in $(\mathbb{U}, \tau)$. We extend the subset automaton $\mathcal{P}\mathbb{U}$ with a component W that traces paths through the bundles of $\mathcal{G}_\alpha(\mathcal{P}\mathbb{U},\, \tau)$. The following describes how W and the corresponding Streett extension $\mathcal{X}\mathcal{P}\mathbb{U}\mathbf{S}\mathcal{C}$ are to be defined.

The component W designates at each position a tree Z for which each node $z \in Z$ denotes a set $\ell(z)$ of states from the current subset as calculated by $\mathcal{P}\mathbb{U}$. Each such node z traces a part of $\mathcal{G}_\alpha(\mathcal{P}\mathbb{U}, \tau)$ that corresponds to paths where R might occur infinitely often but I does not occur. Note that $\mathbf{S}(R, I)$ is satisfied on $\mathcal{G}_\alpha(\mathcal{P}\mathbb{U}, \tau)$ iff no such part exists. Thus by defining $\mathcal{C} = \{(\widehat{R}_z, \widehat{I}_z) \mid z \text{ is a node}\}$, we may express by a Streett condition $\mathbf{S}\mathcal{C}$ that for each node z, if R occurs again and again in the part designated by $\ell(z)$, then eventually $\ell(z)$ becomes empty. When $\ell(z)$ becomes empty the node z can be reused to trace another part of the graph. This will allow us to use only $n + 1$ different nodes. It is not hard to see that the representation of a tree with $n + 1$ different nodes can be accomplished in $O(n{\cdot} \log n)$ bits.

Making the Streett condition $\mathbf{S}\mathcal{C}$ sufficient for $\mathbf{S}(R, I)$ to hold calls for a sophisticated management of the different parts traced. The tree Z is used to dynamically maintain a hierarchy expressing the possible parts of the graph where $\mathbf{S}(R, I)$ might not hold. Initially Z consists only of the root, designated z^0. The label of the root is always the current subset; thus initially $\ell(z^0) = \{V^0\}$. Except for the root, a node z can be removed and inserted repeatedly in Z. A node z not in Z is free. When a node is in Z, its *last insertion position* is the position in τ when it was last inserted. If the last insertion position of z is u and the last insertion position of z' is u' then z is older than z' if $u \overset{<}{\cdot} u'$.

The tree Z always satisfies the following requirements, where the tree is viewed as growing upwards:

(1) The children of a node are ordered from left to right according to decreasing last insertion position. In general we say that a node z is to *the left* of a node z' if their highest common ancestor z'' is a proper ancestor of both z and z' and if the child of z'' on the path to z is to the left of the child of z'' on the path to z'.

(2) The labeling of the tree is always such that whenever z is an ancestor of z' then $\ell(z) \supseteq \ell(z')$.

(3) All $\ell(z')$, where z' is a child of z, are disjoint.

Note that when a node z is to the left of z', then $\ell(z)$ and $\ell(z')$ are disjoint. For a state v in the current subset, the *topnode* z of v is defined as the highest node z such that $v \in \ell(z)$. By the requirements above, the top-node of v is uniquely defined. Vice versa, the top-nodes of all v in the current subset uniquely determine the labeling ℓ. Therefore, the labeling ℓ can be specified using $O(n \cdot \log n)$ bits. Since both the description of the shape of Z and the labeling of Z is $O(n \cdot \log n)$, the size factor $|\mathcal{X}| = |W|$ is $O(2^{n \cdot \log n})$.

In the algorithm in Figure 3 it is described how Z is changed along a transition from input position δ to δd. Also, it is indicated for each $z \neq z^0$ whether the resulting state of the extended tree automaton is in $\widehat{R}_z$ ("flash $\widehat{R}_z$") or $\widehat{I}_z$ ("flash $\widehat{I}_z$"). We use $\mathcal{R}(\delta d, \widehat{V})$ to denote $\{v' \mid \exists v \in \widehat{V} : \Delta(v) = d \text{ and } v, d \ \overrightarrow{\tau(\delta)} \ v'\}$, which is the set of successors of $\widehat{V}$ in direction d of $\mathcal{G}_\alpha(\mathcal{P}\mathbb{U}, \tau)$; here Δ is the state of the subset automaton at δ.

In Step (1) of the algorithm every set $\ell(z)$ is expanded to its set of successors. This step may violate requirement (c).

In Step (2) vertices that are found in different branches of the tree are deleted from all branches except the rightmost one. This reestablishes (c). In Step (3) all invalidating vertices are removed from all nodes except the root. If this makes the label of a node z empty, then $\widehat{I}_z$ is flashed in Step (4) (intuitively, since all paths in $\ell(z)$ have ended in an invalidating state or moved to another node, this part of the graph is good.) In Step (5) $\widehat{R}_z$ is flashed and all descendants are removed when $\ell(z) = \bigcup_{z' \text{ is a child of } z}$. As we shall see this means that all paths not passing through I passed through R since $\widehat{R}_z$ was flashed last time. In Step (6) a node $v \in R \setminus I$ is moved up to a new child of z if not already in a child. Intuitively this is done because a promising part of $\mathcal{G}_\alpha(\mathcal{P}\mathbb{U}, \tau)$ where $\mathbf{S}(R, I)$ is not fulfilled, may be narrowed down to only contain this vertex.

To finish the proof, we need to show:

Claim 1. $\mathcal{G}_\alpha(\mathcal{P}\mathbb{U}, \tau) \models \mathbf{S}(R, I)$ iff
α is a winning strategy for A in the game $(\mathcal{X}\mathcal{P}\mathbb{U}\mathbf{S}\mathcal{C}, \tau)$.

(1) for all $z \in Z$:

$\ell(z) := \mathcal{R}(\delta d, \ell(z))$;

(2) for all $z, z' \in Z$ and all $v \in V$:

if z is to the left of z' and $v \in \ell(z)$ and $v \in \ell(z')$, then remove v from $\ell(z)$;

(3) for all $z \in Z \backslash \{z^0\}$ and for all $v \in \ell(z)$:

if $v \in I$, then remove v from $\ell(z)$;

(4) for all $z \in Z$ such that $\ell(z) = \emptyset$:

remove z from Z and flash $\widehat{I}_z$;

(5) for all $z \in Z \backslash \{z^0\}$:

if $\ell(z) = \bigcup_{z' \text{ is a child of } z} \ell(z')$, then remove every descendant of z and flash $\widehat{R}_z$;

(6) for all $z \in Z$ and for all $v \in \ell(z)$:

if $v \in R$, $v \notin I$, and v is not in any $\ell(z')$, where z' is child of z, then add a new node z'' to Z, make z'' a left most child of z, and let $\ell(z'') = \{v\}$.

Figure 3: Extended Safra Construction

Proof. "$\Rightarrow$" Assume that for some z and for some input path w through τ, $\widehat{R}_z$ is flashed infinitely often and $\widehat{I}_z$ is flashed only finitely often. It is then sufficient to prove that for some path along w in $\mathcal{G}_\alpha(\mathcal{P}\mathbb{U}, \tau)$, R occurs infinitely often and I only finitely often. Now since $\widehat{I}_z$ can only be flashed in Step (4) and this happens a finite number of times, then from some point on, $\ell(z)$ is always non-empty and contains no I-invalidating state. Thus it suffices to find a path that is R-reconfirming infinitely often in the part of the graph described by the $\ell(z)$ subsets along w. By assumption, the condition in Step (5) is satisfied infinitely many times. Consider two positions δ and δ' where this happens and $\delta \overset{<}{\cdot} \delta'$. Then every finite path in $\mathcal{G}_\alpha(\mathcal{P}\mathbb{U}, \tau)$ from (v, δ) to (v, δ') described by the subsets $\ell(z)$ passes through an R-reconfirming state according to Step (6). We can then use König's Lemma to show that the infinite subgraph made of these finite paths contains an infinite path that passes through R infinitely often.

"$\Leftarrow$" Assume that $\mathcal{G}_\alpha \models \mathbf{S}(R, I)$ does not hold. Then some infinite path in $\mathcal{G}_\alpha \models \mathbf{S}(R, I)$ is R-reconfirming infinitely often and I-invalidating finitely often. Let w be the corresponding input path in τ and let $v_0 v_1 \cdots$ be the corresponding states. It suffices to prove that along w there is some z such that $\widehat{R}_z$ is flashed infinitely often and $\widehat{I}_z$ is flashed only finitely often.

Claim 2. There is a node $z \neq z^0$ such that along w, z is almost always an ancestor of the top-node of v_k and is infinitely often identical to the top-node.

Proof. From some point on $v_k \notin I$. From Step (5) and (6) it can be seen that once the top-node then becomes different from the root, the top-node is always above the root. Thus from some point on we can observe the node z_k^1 that is defined as the child of the root on the path to the top-node of v_k. This node can move only to the right and since nodes are inserted from the left, z_k^1 eventually settles down to some final value denoted z^1. If z^1 is infinitely often the top-node, the conclusion in Claim 2 follows.

Otherwise, a similar argument can be applied to obtain a child z^2 of z_1^1 that is almost always an ancestor of the top-node of z_k. If z^2 is not infinitely often a top-node, we obtain a z^3 etc. Since the number of nodes is finite, we eventually obtain some node z_k satisfying Claim 2. $\square$

Let z be the node of Claim 2. Since the label of any ancestor of the top-node of v_k contains v_k and since z is almost always an ancestor of the top-node, $\widehat{I}_z$ is flashed only finitely many times according to Step (4). Suppose now for a contradiction that $\widehat{R}_z$ is flashed only finitely many times. Following the last such time and the last time v_k is I-invalidating, v_k is eventually moved to a child of z by Step (6), since v_k is R-reconfirming infinitely often. The state v_k may subsequently be passed towards the right from one child to another but will never be removed since v_k is not I-invalidating. Thus v_k will eventually remain in a fixed child of z, contradicting that z satisfies Claim 2. $\square$

$\square$

Putting it all together

Theorem 2. (Complementation) There is an automaton $\overline{\mathbb{U}}\mathbf{S}\overline{C}$ accepting the complement of $L(\mathbb{U}\mathbf{S}C)$. Moreover, $|\overline{\mathbb{U}}| = 2^{O(n^2 \cdot m \cdot \log(n \cdot m))}$ and $|\overline{C}| = n^2 \cdot m$, where $n = |\mathbb{U}|$ and $m = |C|$. In particular, when $m = O(n)$, $|\mathbb{U}| = 2^{O(n^3 \cdot \log n)}$ and $|\overline{C}| = n^3$.

(1) $\tau \notin L(\mathfrak{A}\mathbf{S}C)$ iff	by definition of acceptance
(2) A does not have a w.s. in $(\mathfrak{A}\mathbf{S}C, \tau)$ iff	by (Borel Determinacy)
(3) PF has a w.s. in $(\mathfrak{A}\mathbf{S}C, \tau)$ iff	by (Immediate Determinacy)
(4) PF has a memoryless w.s. in $(\mathfrak{A}\mathbf{S}C, \tau)$ iff	by (Memoryless ρ-graph)
(5) $\exists \rho : \mathcal{G}_\rho(\mathfrak{A}, \tau) \models \mathbf{R}C$ iff	by (Rabin Duality)
(6) $\exists \mu : \exists \rho : \mathcal{E}\mathcal{G}_\rho(\mathfrak{A}, \tau) \models \mathbf{S}C^\mu$ iff	by (Edge Automaton)
(7) $\exists \mu : \exists \rho : \mathcal{G}_\rho(\mathcal{E}\mathfrak{A}, \tau) \models \mathbf{S}C^\mu$ iff	by (Subset Construction)
(8) $\exists \mu : \exists \alpha : \mathcal{G}_\alpha(\mathcal{P}\mathcal{E}\mathfrak{A}, \tau) \models \mathbf{S}C^\mu$ iff	see text
(9) $\exists \alpha : \mathcal{G}_\alpha(\mathcal{M}\mathcal{P}\mathcal{E}\mathfrak{A}, \tau) \models \mathbf{S}C^\mathcal{M}$ iff	by definition of Streett acceptance
(10) $\exists \alpha : \forall (R, I) \in C^\mathcal{M} : \mathcal{G}_\alpha(\mathcal{M}\mathcal{P}\mathcal{E}\mathfrak{A}, \tau) \models \mathbf{S}(R, I)$ iff	by (Extended Safra)
(11) $\exists \alpha : \forall (R, I) \in C^\mathcal{M} : \alpha$ is a w.s. in $(\underset{(R,I)}{\mathcal{X}} \mathcal{M}\mathcal{P}\mathcal{E}\mathfrak{A}\mathbf{S}\underset{(R,I)}{\mathcal{C}}, \tau)$ iff	by (And'ing Extensions)
(12) $\exists \alpha : \alpha$ is a w.s. in $((\bigotimes_{\substack{(R,I) \\ \in C^\mathcal{M}}} \underset{(R,I)}{\mathcal{X}}) \mathcal{M}\mathcal{P}\mathcal{E}\mathfrak{A}\mathbf{S}(\bigcup_{\substack{(R,I) \\ \in C^\mathcal{M}}} \underset{(R,I)}{\mathcal{C}}), \tau)$ iff	by definition of acceptance
(13) $\tau \in L((\bigotimes_{\substack{(R,I) \\ \in C^\mathcal{M}}} \underset{(R,I)}{\mathcal{X}}) \mathcal{M}\mathcal{P}\mathcal{E}\mathfrak{A}\mathbf{S}(\bigcup_{\substack{(R,I) \\ \in C^\mathcal{M}}} \underset{(R,I)}{\mathcal{C}}))$	

Figure 4: Putting it all together. (Here "w.s." abbreviates "winning strategy.")

Proof. The ingredients in the proof are put together in Figure 4, according to the recipe in Section 5. Thus we define

$$\overline{\mathfrak{A}\mathbf{S}\overline{S}} = (\bigotimes_{\substack{(R,I) \\ \in C^\mathcal{M}}} \underset{(R,I)}{\mathcal{X}}) \mathcal{M}\mathcal{P}\mathcal{E}\mathfrak{A}\mathbf{S}(\bigcup_{\substack{(R,I) \\ \in C^\mathcal{M}}} \underset{(R,I)}{\mathcal{C}}),$$

where the notation $\underset{(R,I)}{\mathcal{X}}$ and $\underset{(R,I)}{\mathcal{C}}$ indicates that the Extended Safra construction is applied for each pair (R, I) in the pairs set $C^\mathcal{M}$ defined below. A few comments are needed.

The graphs in (6), (7), and (8) are essentially isomorphic (by Lemma Edge Automaton and Lemma Subset Automaton). Thus we denote the dual pairs condition by C^μ for all three graphs.

Note that in (8) the subsets calculated by $\mathcal{P}\mathcal{E}\mathbb{U}$ contains pairs on the form $(\perp, v)$ or (v, v'), where $v, v' \in V$ and V is the set of states of $\mathbb{U}$.

In (9) the $\mathcal{M}$ operator denotes the transformation that makes the automaton guess each value of the reduced quasi progress measure. Note that on a transition, the resulting automaton must check that if (v, v') is in the old subset and (v', v'') is in the new subset, then $\mu(v')$ guessed for the old subset is the same as $\mu(v'')$ guessed for the new subset.

Since the state of $\mathcal{MPE}\mathbb{U}$ contains the guesses of $\mu(v)$ and $\mu(v')$ for each (v, v') in the current subset, the dual pairs condition C^μ can be calculated from the state of automaton according to Definition 4. Because the Streett condition $\mathrm{S}C^\mu$ now is defined solely in terms of information in the state of $\mathcal{MPE}\mathbb{U}$, the condition for $\mathcal{MPE}\mathbb{U}$ is denoted $\mathbf{S}C^{\mathcal{M}}$.

In (10) the Extended Safra construction is applied to the measure automaton $\mathcal{MPE}\mathbb{U}$ rather than directly to the subset automaton. This is inconsequential since all information available in $\mathcal{PE}\mathbb{U}$ is available in $\mathcal{MPE}\mathbb{U}$.

The size of $\mathcal{PE}\mathbb{U}$ is $2^{O(n^2)}$. Since $\mathcal{MPE}\mathbb{U}$ guesses a progress value, which is a permutation of at most m colors, for each state of $\mathcal{PE}\mathbb{U}$, the size of $\mathcal{MPE}\mathbb{U}$ is $2^{O(n^2 \cdot m \ \log\ m)}$. Since $|C^{\mathcal{M}}| = |C| = m$ and each application of the Extended Safra construction yields a size factor of $2^{O(n^2 \cdot \log\ n)}$, the total size factor is $2^{O(n^2 \cdot m \cdot \log\ n)}$ The size of $\overline{\mathbb{U}}$ is then $2^{O(n^2 \cdot m \cdot \log\ m)} \cdot 2^{O(n^2 \cdot m\ \log\ m)} = 2^{O(n^2 \cdot m \cdot \log(n \cdot m))}$. Moreover, since each application of the Extended Safra construction yields a pairs set of size n^2, the size of $\overline{C}$ is $n^2 \cdot m$. □

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