Equivalence Testing of Weighted Automata over Partially Commutative Monoids

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Abstract -

We study the *equivalence* testing of automata over partially commutative monoids (pc monoids) and show efficient algorithms in special cases, exploiting the structure of the underlying non-commutation graph of the monoid.

Specifically, if the clique edge cover number of the non-commutation graph of the pc monoid is a constant, we obtain a deterministic quasi-polynomial time algorithm. As a consequence, we also obtain the first deterministic quasi-polynomial time algorithms for equivalence testing of k-tape weighted automata and for equivalence testing of deterministic k-tape automata for constant k. Prior to this, a randomized polynomial-time algorithm for the above problems was shown by Worrell [24].

We also consider pc monoids for which the non-commutation graphs have cover consisting of at most k cliques and star graphs for any constant k. We obtain randomized polynomial-time algorithm for equivalence testing of weighted automata over such monoids.

Our results are obtained by designing efficient zero testing algorithms for weighted automata over such pc monoids.

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1 Introduction

Testing equivalence of multi-tape finite automata is a fundamental problem in automata theory. For a k-tape automaton, we usually denote by $\Sigma_1, \ldots, \Sigma_k$ the mutually disjoint alphabets for the k tapes, and the automaton accepts a subset of the product monoid $\Sigma_1^* \times \cdots \times \Sigma_k^*$. Two multi-tape automata are equivalent if they accept the same subset. It is well-known that equivalence testing of multi-tape non-deterministic automata is undecidable [13].

For 2-tape deterministic automata equivalence testing was shown to be decidable in the 1970's [4, 23]. In [2] an exponential upper bound was shown. Subsequently, a polynomial-time algorithm was obtained [12] and the authors conjectured that equivalence testing of deterministic k-tape automata for any constant k is in polynomial time.

A closely related problem is testing the *multiplicity equivalence* of multi-tape automata. Intuitively, the multiplicity equivalence testing problem is to decide whether for each tuple in the product monoid $\Sigma_1^* \times \cdots \times \Sigma_k^*$, the number of accepting paths in the two input automata are the same. Since a deterministic automaton has at most one accepting path for each word, equivalence testing of two deterministic k-tape automata coincides with multiplicity equivalence testing. More generally, for weighted automata, equivalence testing is to decide if the coefficient of each word (over a field or ring) is the same in the given automata. For the weighted case, the equivalence testing is in deterministic polynomial time for one-tape automata [20, 22]. Such an algorithm for the k-tape case remained elusive for a long time. Eventually the equivalence testing of k-tape non-deterministic weighted automata was shown decidable by Harju and Karhumäki [14] using the theory of free groups ¹. No nice complexity-theoretic upper bound was known, until recently Worrell [24] obtained a randomized polynomial-time algorithm for testing the equivalence of k-tape weighted non-deterministic automata (and equivalence testing of deterministic k-tape automata) for any constant k. Worrell takes a different approach via Polynomial Identity Testing (PIT). In [24], Worrell explicitly raised the problem of finding an efficient deterministic algorithm for equivalence problem for k-tape weighted automata for any fixed k.

In this paper, we show that the equivalence testing for k-tape weighted automata can be solved in deterministic quasi-polynomial time. This immediately yields the first deterministic quasi-polynomial time algorithm to check the equivalence of deterministic k-tape automata, making progress on a question asked earlier [12, 14]. In fact, our proof technique shows a stronger result that we explain now. The product monoid $M = \Sigma_1^* \times \cdots \times \Sigma_k^*$ associated with k-tape automata is a partially commutative monoid (henceforth pc monoid), in the sense that any two variables $x \in \Sigma_i, y \in \Sigma_j, i \neq j$ commute with each other whereas the variables in the same tape alphabet Σ_i are mutually non-commuting. We associate a non-commutation graph G_M with M to describe the non-commutation relations: (x,y) is an edge if and only if x and y do not commute. If there is no edge (x,y) in G_M , the words xy and yx are considered to be equivalent as x and y commute. The notion of words over a pc monoid and equivalence of two words are discussed in details in Section 3. For the k-tape case, the vertex set of G_M is $\Sigma_1 \cup \ldots \cup \Sigma_k$ and G_M is clearly the union of k disjoint cliques, induced by each Σ_i , forming a clique edge cover of size k. For convenience, each isolated vertex is a clique of size one.

In this paper, we obtain an equivalence testing algorithm for weighted automata over any pc monoid whose non-commutation graph has a constant size clique edge cover (*not necessarily* disjoint, and all the isolated vertices are part of the cover). In short, we call such

They were also the first to settle the decidability of equivalence problem for deterministic multi-tape automata.

monoids as k-clique monoids where the clique edge cover size is bounded by k. Since two weighted automata A and B are equivalent if and only if the difference automaton C = A - B is a zero weighted automaton, we prefer to describe the results in terms of zero testing of weighted automaton. Here the difference of two weighted automata has an obvious meaning: the weight of each word w in C is the difference between the weights of w in A and B. The words over any pc monoid are defined with respect to the equivalence relation induced by the non-commutation graph of the pc monoid. This is explained in Section 3. Let $\mathbb F$ be an infinite field from where the weights are taken.

▶ Theorem 1. Let A be a given \mathbb{F} -weighted automaton of size s over a pc monoid M for which the non-commutation graph G_M has a clique edge cover of size k. Then, the zero testing of A can be decided in deterministic $(nks)^{O(k^2 \log ns)}$ time. Here n is the size of the alphabet of M, and the clique edge cover is given as part of the input.

As an immediate corollary, the above theorem yields a deterministic quasi-polynomial time algorithm for equivalence testing of k-tape weighted automata (also for equivalence testing of deterministic k-tape automata). Notice that, for the k-tape case, the clique edge cover of size k is also part of the input since for each $1 \le i \le k$, the i^{th} tape alphabet Σ_i is explicitly given.

Next we address equivalence testing over more general pc monoids M. M is a k-monoid if its non-commutation graph G_M is a union $G_1 \cup G_2$ of two graphs, where G_1 has a clique edge cover of size at most k' and G_2 has a vertex cover of size at most k - k' (hence the edges of G_2 can be covered by k - k' many star graphs). We show that equivalence testing over k-monoids has a randomized polynomial-time algorithm. One can also see this result as a generalization of Worrell's result [24].

- ▶ Theorem 2. Let A be a given \mathbb{F} -weighted automaton of size s over a k-monoid M. Then the zero testing of A can be decided in randomized $(ns)^{O(k)}$ time. Here n is the size of the alphabet of M.
- ▶ Remark 3. What is the complexity of equivalence testing for weighted automata over general pc monoids? The non-commutation graph G_M of any pc monoid $M = (X^*, I)$ has a clique edge covering of size bounded by $\binom{|X|}{2}$. Hence, the above results give an exponential-time algorithm. Note that if G_M has an induced matching of size more than k then M is not a k-monoid. Call M a matching monoid if G_M is a perfect matching. It follows from Lemma 8 shown in Section 3, that equivalence testing over arbitrary pc monoids is deterministic polynomial-time reducible to equivalence testing over matching monoids (if G_M has isolated vertices, one can add a new vertex (variable) for each isolated vertex and introduce a matching edge between them).

Various automata-theoretic problems have been studied in the setting of pc monoids. For example, pc monoids have found applications in modelling the behaviour of concurrent systems [16]. Droste and Gastin [10] have studied the relation between recognizability and rationality over pc monoids. Broadly, it is interesting to understand and identify the results in algebraic automata theory that can be generalized to the setting of pc monoids.

Proof Overview: Now we briefly discuss the main ideas behind our results. Worrell's key insight [24] is to reduce k-tape automata equivalence problem to a suitable instance of polynomial identity testing over non-commuting variables, which can be solved in randomized polynomial time [1, 5, 17]. Our strategy too is to carry out reductions to polynomial identity testing problem. Since we are considering automata over general pc monoids and we aim to design efficient deterministic algorithms, we require additional ideas. First, we suitably

apply a classical algebraic framework to transfer the zero testing problem over general po monoids to pc monoids whose non-commutation graphs are disjoint union of cliques [6, 8]. This allows us to prove a Schützenberger [20] type theorem over *general pc monoids* which says that any nonzero weighted automata of size s over any pc monoid, must have a nonzero word within length poly(s, n) where n is the size of the alphabet. Furthermore, this also allows us to reduce the zero testing of weighted automata to polynomial identity testing for algebraic branching programs over pc monoids. It turns out that the latter problem can be solved by suitably adapting a black-box polynomial identity test for noncommutative algebraic branching programs based on hitting sets due to Forbes and Shpilka [11]. Our algorithm recursively builds on this result, ensuring that the resulting hitting set remains of quasi-polynomial size, like the Forbes-Shpilka hitting set [11]. This requires coupling a result of Schützenberger related to the Hadamard product of weighted automata ([19], Theorem 3.2) with our algebraic framework. The proof of Theorem 2 also follows a similar line of argument. First we give a randomized polynomial-time identity testing algorithm over pc monoids whose non-commutation graph is a star graph. Then a composition lemma yields an identity testing algorithm over k-monoids.

The paper is organized as follows. In Section 2, we give some background. We prove a Schützenberger type theorem for automata over pc monoids in Section 3. Theorem 1 is presented in Section 4, and Theorem 2 in Section 5. Some proof details are in the appendix.

2 Preliminaries

We recall some basic definitions and results, mainly from automata theory and arithmetic circuit complexity, and define some notation used in the subsequent sections.

Notation: Let \mathbb{F} be an infinite field. Let $\mathcal{M}_t(\mathbb{F})$ denote the ring of $t \times t$ matrices over \mathbb{F} . For matrices A and B of sizes $m \times n$ and $p \times q$ respectively, their Tensor (Kronecker) product $A \otimes B$ is defined as $(a_{ij}B)_{1 \leq i \leq m, 1 \leq j \leq n}$. The dimension of $A \otimes B$ is $pm \times qn$. Given bases $\{v_i\}$ and $\{w_j\}$ for the vector spaces V and W, the vector space $V \otimes W$ is the tensor product space with a basis $\{v_i \otimes w_j\}$.

For a series (resp. polynomial) S and a word (resp. monomial) w, let [w]S denote the coefficient of w in the series S (resp. polynomial). In this paper, we consider weighted automata over a field \mathbb{F} and alphabet (or variables) $X = \{x_1, \ldots, x_n\}$.

We also consider coverings of graphs: a graph G=(X,E) is said to have a graph covering $\{G_i=(X_i,E_i)\}_{i=1}^k$ of size k if $X=\cup_{i=1}^k X_i$ and $E=\cup_{i=1}^k E_i$.

Automata Theory: We recall some basic definitions from automata theory. More details can be found in the Berstel-Reutenauer book [3].

Let K be a semiring and X be an alphabet. A K-weighted automaton over X is a 4-tuple, A = (Q, I, E, T), where Q is a finite set of states, and the mappings $I, T: Q \to K$ are weight functions for entering and leaving a state respectively, and $E: Q \times X \times Q \to K$ is the weight of each transition. We define |Q|, the number of states, to be the size of the automaton. A path is a sequence of edges: $(q_0, a_1, q_1)(q_1, a_2, q_2) \cdots (q_{t-1}, a_t, q_t)$. The weight of the path is the product of the weights of the edges. The formal series $S \in K\langle\!\langle X \rangle\!\rangle$ which is the (possibly infinite) sum of the weights over all the paths is recognized by A. Then, for each word $w = a_1 a_2 \cdots a_t \in X^*$, the contribution of all the paths for the word w is given by $[w]S = \sum_{q_0, \dots, q_t \in Q} I(q_0) \cdot E(q_0, a_1, q_1) \cdots E(q_{t-1}, a_t, q_t) \cdot T(q_t)$.

A K-weighted automaton A with ϵ -transitions over X is defined with E modified, such that $E: Q \times \{X \cup \epsilon\} \times Q \to K$. Let $A_0 \in \mathbb{M}_{|Q|}(K)$ be the transition matrix for the ϵ -transitions. An automaton computes a valid formal series in $K(\!\langle X \rangle\!\rangle$, if and only if $\sum_k A_0^k$

converges. In that case, another automaton A' without ϵ -transitions computing the same series can be constructed efficiently [15]. Henceforth, we consider all automata are valid and free of ϵ -transitions.

The following basic result by Schützenberger [20] is important for the algorithmic results presented in this paper.

▶ Theorem 4 (Schützenberger). Let K be a subring 2 of a division ring and A be a K-weighted automaton with s states representing a series S in $K\langle\!\langle X \rangle\!\rangle$. Then S is a nonzero series if and only if there is a word $w \in X^*$ of length at most s-1, such that [w]S is nonzero.

Now, we recall the definition of weighted **multi-tape automata** following Worrell's work [24]. Let M be the pc monoid over variables $X = X_1 \cup \cdots \cup X_k$ defined as follows: the variables in each X_i are non-commuting, but for all $i \neq j$ and any $x \in X_i, y \in X_j$ we have xy = yx. As defined already, the transition function E is a mapping $Q \times X \times Q \to K$. A path is a sequence of edges: $(q_0, x_1, q_1)(q_1, x_2, q_2) \cdots (q_{t-1}, x_t, q_t)$ where each $x_i \in X_j$ for some j. The label of the run is $m = x_1x_2 \cdots x_t$ in the pc monoid M, and $[m]\mathcal{A}$ is the total contribution of all the runs having the label equivalent to m.

An automaton is *deterministic* if the set of states can be partitioned as $Q = Q^{(1)} \cup ... \cup Q^{(k)}$, where states in $Q^{(i)}$ read input only from the set X_i which is the alphabet of i^{th} tape, and each state has a single transition for every input variable. Thus, a deterministic automaton has at most one accepting path for each input $m \in M$.

Arithmetic Circuit Complexity: An algebraic branching program (ABP) is a directed acyclic graph with one in-degree-0 vertex called source, and one out-degree-0 vertex called sink. The vertex set of the graph is partitioned into layers $0, 1, \ldots, \ell$, with directed edges only between adjacent layers (i to i+1). The source and the sink are at layers zero and ℓ respectively. Each edge is labeled by an affine linear form over \mathbb{F} . The polynomial computed by the ABP is the sum over all source-to-sink directed paths of the product of linear forms that label the edges of the path. The maximum number of nodes in any layer is called the width of the algebraic branching program. The size of the branching program is taken to be the total number of nodes.

Equivalently, the computation of an algebraic branching program can be defined via the iterated matrix product $\mathbf{u}^T M_1 M_2 \cdots M_\ell \mathbf{v}$, where \mathbf{u}, \mathbf{v} are vectors in \mathbb{F}^w and each M_i is a $w \times w$ matrix whose entries are affine linear forms over X. Here w corresponds to the ABP width and ℓ corresponds to the number of layers in the ABP. If X is a set of non-commuting variables then the ABP is a noncommutative algebraic branching program (e.g., see [18]).

Now we recall some results from noncommutative polynomial identity testing. Let $S \subset \mathbb{F}\langle X \rangle$ be a subset of polynomials in the noncommutative polynomial ring $\mathbb{F}\langle X \rangle$ where $X = \{x_1, \ldots, x_n\}$. Given a mapping $v: X \to \mathcal{M}_t(\mathbb{F})$ from variables to $t \times t$ matrices, it defines an evaluation map defined for any polynomial $f \in \mathbb{F}\langle X \rangle$ as $v(f) = f(v(x_1), \ldots, v(x_n))$. A collection H of such evaluation maps is a hitting set for S, if for every nonzero f in S, there is an evaluation $v \in H$ such that $v(f) \neq 0$.

Let $S_{n,d,s}$ denote the subset of polynomials f in $\mathbb{F}\langle X\rangle$ such that f has an algebraic branching program of size s and d layers. Forbes and Shpilka [11] have shown that a hitting set $H_{n,d,s}$ of quasi-polynomial size for $S_{n,d,s}$ can be constructed in quasi-polynomial time.

▶ Theorem 5 (Forbes-Shpilka). For all $s,d,n \in \mathbb{N}$ if $|\mathbb{F}| \geq \operatorname{poly}(d,n,s)$, then there is a set $H_{n,d,s}$ which is a hitting set for $S_{n,d,s}$. Further $|H_{n,d,s}| \leq (sdn)^{O(\log d)}$ and there is a

 $^{^2}$ For some applications, this could also be subsemirings as originally proved [20].

deterministic algorithm to output the set $H_{n,d,s}$ in time $(sdn)^{O(\log d)}$.

3 A Schützenberger Type Theorem for Partially Commutative Monoids

In this section, we prove a theorem in the spirit of Theorem 4 over general pc monoids.

Pc monoids and associated partitioned pc monoids: Let X be a finite alphabet (equivalently, variable set). A pc monoid M over X is usually denoted as $M = (X^*, I)$ where $I \subseteq X \times X$ is a symmetric and reflexive binary relation such that $(x_1, x_2) \in I$ if and only if $x_1x_2 = x_2x_1$ in M. Let \tilde{I} be the congruence generated by I using the transitive closure. The monoid elements are defined as the congruence classes \tilde{m} for $m \in X^*$. In other words, M is a factor monoid of X^* generated by \tilde{I} . The non-commutation graph $G_M = (X, E)$ of M is a simple undirected graph such that $(x_1, x_2) \in E$ if and only if $(x_1, x_2) \notin I$.

A k-partitioned pc monoid is a pc monoid for which the non-commutation graph can be partitioned into k vertex-disjoint subgraphs. Given any pc monoid M, we can associate a partitioned pc monoid M' with it, such that M is isomorphic to a submonoid of M', as follows. Let $\{G_i\}_{i=1}^k$ be the k-cover for G_M where $G_i = (X_i, E_i)$. Consider a set of variables $\widehat{X} = \{x_{ti} : 1 \leq t \leq n, 1 \leq i \leq k\}$. Do a new labelling of the graph G_i by changing the variable $x_t \in X_i$ by x_{ti} . In this process obtain the graphs G'_1, \ldots, G'_k which are vertex disjoint. The edges in G'_i are naturally induced by G_i . For each $1 \leq i \leq k$, the new pc monoid M'_i has G'_i as its non-commutation graph. Finally, M' be the pc monoid generated by M'_1, \ldots, M'_k and the alphabet $X' = \bigcup_{i=1}^k V(G'_i)$. By construction, the non-commutation graph $G_{M'}$ is the disjoint union of G'_1, \ldots, G'_k . As \mathbb{F} -algebra $\mathbb{F}\langle M' \rangle$ is isomorphic to the tensor product of the \mathbb{F} -algebras $\mathbb{F}\langle M'_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle M'_k \rangle$.

It is a classical result that M is isomorphic to a submonoid of M' [6, 8, 9] via the map ψ , which we define next.

- ▶ **Lemma 6.** Let $\psi : \mathbb{F}\langle M \rangle \to \mathbb{F}\langle M' \rangle$ be the map such that $\psi(m) = m_1 \otimes m_2 \otimes \cdots \otimes m_k$ for any monomial m in M and extend by linearity. Here for $1 \leq i \leq k$, the monomial m_i is obtained from the part of m (after erasing the letters not in X_i) by labelling x_t in X_i by x_{ti} . Then, ψ is an injective homomorphism.
- ▶ Remark 7. To fit with our notation, we include a self-contained proof in the appendix.

Using Lemma 6, we can show that the zero testing for weighted automata over pc monoids reduces to zero testing of weighted automata over partitioned pc monoids in deterministic polynomial time. More formally, we show the following result.

- ▶ Lemma 8. Let A be the given \mathbb{F} -weighted automaton of size s over a pc monoid M, for which the non-commutation graph G_M has k-covering $\{G_i = (X_i, E_i)\}_{i=1}^k$. Then the zero testing of A reduces to the zero testing of another \mathbb{F} -weighted automaton B over the associated partitioned pc monoid M' in deterministic polynomial time. Moreover the size of the automaton B is poly(n, s, k).
- **Proof.** The automaton B is simply obtained by applying the map ψ on the variables in M. For a variable x_t , let $J_t \subseteq \{1, 2, \ldots, k\}$ be the set of indices such that, $i \in J_t$ if and only if $x_t \in X_i$. Then $\psi(x_t) = \eta_{i_1} \otimes \cdots \otimes \eta_{i_{|J_t|}}$ where $i_1 < i_2 < \cdots < i_{|J_t|}$ and for each $j, i_j \in J_t$. Now for each $q_0, q_k \in Q$ such that $(q_0, x_t, q_k) \in E$ and $wt(q_0, x_t, q_k) = \alpha \in \mathbb{F}$, we introduce new states $q_1, \ldots, q_{|J_t|-1}$ and for each $j \leq |J_t|-1$, add the edge $e_j = (q_{j-1}, \eta_{i_j}, q_j)$ in E and $wt(e_1) = \alpha$ and for other newly added edges the weight is 1. Since the number of edges in A

is $O(ns^2)$, it is easy to see the number of nodes in B is $O(ns^2k)$. The fact that A is zero if and only if B is zero follows from Lemma 6.

Worrell has already proved that the zero testing of weighted automata over partitioned monoids whose non-commutation graphs are the union of disjoint cliques, can be reduced to the identity testing of noncommutative ABPs [24]. We restate the following proposition from Worrell's paper in a form that fits with our framework.

▶ Proposition 9 (Adaptation of Proposition 5 of [24]). Let A be a given \mathbb{F} -weighted automaton of size s over a partitioned pc monoid M computing a series S. Moreover the non-commutation graph G_M is the disjoint union of k cliques. Let N be the transition matrix of A. Then S is a zero series if and only if the ABPs $\mathbf{u}^T N^{\ell} \mathbf{v} = 0$ for each $0 \leq \ell \leq s - 1$, where u, v are vectors in \mathbb{F}^s .

Combining Lemma 8 and Proposition 9 we obtain the following generalization of Schützenberger's theorem [20] over arbitrary pc monoids.

▶ Theorem 10 (A Schützenberger type theorem). Let A be a given \mathbb{F} -weighted automaton of size s over any pc monoid M representing a series S. Then S is a nonzero series if and only if there exists a word $w \in X^*$ such that [w]S is nonzero and the length of w is bounded by $O(n^3s^2)$.

Proof. Observe that the non-commutation graph G_M has a trivial clique edge cover of size $\leq n^2$ where n is the size of the alphabet. Then we apply Lemma 8 to conclude that S is a zero series if and only if the series S' computed by the \mathbb{F} -weighted automaton B over the associated partitioned pc monoid (whose non-commutation graph is a disjoint union of cliques) is zero. The size s' of B is bounded by $O(n^3s^2)$. Now we use Proposition 9 to see that S' is identically zero if and only if the ABPs $\mathbf{u}^T N^\ell \mathbf{v} = 0$ for each $0 \leq \ell \leq s' - 1$ are identically zero where N is the transition matrix of B. Now notice that under the image of ψ map, the length of any word can only increase. In other words, for any word $w: |\psi(w)| \geq |w|$. Using this, we conclude that $(S' = \psi(S))^{\leq s'-1}$ is a nonzero polynomial. Since ψ is injective, it must be the case that $S^{\leq s'-1}$ is also a nonzero polynomial and the proof of the theorem follows.

4 Deterministic Algorithm for Zero Testing of Weighted Automata Over *k*-Clique Monoids

Recall from Section 1, that a k-clique monoid is a pc monoid M whose non-commutation graph G_M has a clique edge cover of size k. In this section, we show that the zero testing problem for automata over k-clique monoids for constant k can be solved in deterministic quasi-polynomial time. In fact, using Lemma 8 and Proposition 9, it is straightforward to observe that the zero testing problem reduces to the polynomial identity testing of ABPs over partitioned pc monoids whose non-commutation graph is a disjoint union of k cliques. Thus the main purpose of this section is to develop identity testing algorithm for ABPs computing polynomials in $\mathbb{F}\langle X_1\rangle\otimes\cdots\otimes\mathbb{F}\langle X_k\rangle$, where each set $X_j=\{x_{ij}\}_{1\leq i\leq n}$ is of size n, and the sets are mutually disjoint. The parameter k is a constant. This will suffice to prove Theorem 1.

We first formally define the concept of evaluation and partial evaluation of polynomials over algebra.

Evaluation of a polynomial over algebras: Given a polynomial $f \in \mathbb{F}\langle X_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle X_k \rangle$ and a k-tuple of \mathbb{F} -algebras $\mathbf{A} = (A_1, \dots, A_k)$, an evaluation of f in \mathbf{A} is given by a k-tuple of maps $\mathbf{v} = (v_1, v_2, \dots, v_k)$, where $v_i : X_i \to A_i$. We can extend it to the map $\mathbf{v} : \mathbb{F}\langle X_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle X_k \rangle \to A_1 \otimes \cdots \otimes A_k$ as follows: For any monomial $m = m_1 \otimes \cdots \otimes m_k$ where $m_i \in X_i^*$, let $\mathbf{v}(m) = v_1(m_1) \otimes \cdots \otimes v_k(m_k)$. In particular, for each $x \in X_j$ let $\mathbf{v}(x) = 1_1 \otimes \cdots \otimes v_j(x) \otimes \cdots \otimes 1_k$ where 1_j is the multiplicative identity of A_j . We can now extend \mathbf{v} by linearity to all polynomials in the domain $\mathbb{F}\langle X_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle X_k \rangle$.

Next, we define a partial evaluation of $f \in \mathbb{F}\langle X_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle X_k \rangle$ in \mathbf{A} . Let k' < k and $\hat{\mathbf{A}} = (A_1, \ldots, A_{k'})$ be a k'-tuple of \mathbb{F} -algebras. A partial evaluation of $\mathbb{F}\langle X_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle X_k \rangle$ in $\hat{\mathbf{A}}$ is given by a k'-tuple of maps $\hat{\boldsymbol{v}} = (v_1, \ldots, v_{k'})$, where $v_i : X_i \to A_i$. Now, we can define $\hat{\boldsymbol{v}} : \mathbb{F}\langle X_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle X_k \rangle \to A_1 \otimes \cdots \otimes A_{k'} \otimes \mathbb{F}\langle X_{k+1} \rangle \otimes \cdots \otimes \mathbb{F}\langle X_k \rangle$ as follows. For a monomial $m = (m_1 \otimes \cdots \otimes m_k)$, $m_i \in X_i^*$, we let $\hat{\boldsymbol{v}}(m) = v_1(m_1) \otimes \cdots \otimes v_{k'}(m_{k'}) \otimes m_{k'+1} \otimes \cdots \otimes m_k$. By linearity, the partial evaluation $\hat{\boldsymbol{v}}$ is defined for any $f \in \mathbb{F}\langle X_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle X_k \rangle$ where $\hat{\boldsymbol{v}}$ takes values in $A_1 \otimes \cdots \otimes A_{k'} \otimes \mathbb{F}\langle X_{k'+1} \rangle \otimes \cdots \mathbb{F}\langle X_k \rangle$.

Although it is already implicit, we formally recall that when we consider ABPs over $\mathbb{F}\langle X_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle X_k \rangle$ the linear forms are defined over tensors of the form $1 \otimes \cdots \otimes x_{ij} \otimes \cdots \otimes 1$. These tensors play the role of a variable in the tensor product structure.

A few more useful notations: Let $S_{k,n,d,s}$ denote the set of all polynomials in $\mathbb{F}\langle X_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle X_k \rangle$ computed by ABPs of size s and d layers. Following the notation in Theorem 5, let $\mathcal{H}_{k,n,d,s}$ be a hitting set for $S_{k,n,d,s}$. That is, $\mathcal{H}_{k,n,d,s}$ is a collection of evaluations $\mathbf{v} = (v_1, \ldots, v_k)$, such that for any nonzero polynomial $f \in S_{k,n,d,s}$ there is an evaluation $\mathbf{v} = (v_1, \ldots, v_k) \in \mathcal{H}_{k,n,d,s}$ such that $\mathbf{v}(f)$ is a nonzero matrix. Forbes and Shpilka [11] have constructed a quasi-polynomial size hitting set $\mathcal{H}_{1,n,d,s}$. (see Theorem 5). The following lemma shows an efficient bootstrapped construction of a hitting set $\mathcal{H}_{k,n,d,s}$ for the set $S_{k,n,d,s}$ of polynomials, using the hitting set $\mathcal{H}_{1,n,d,s}$.

More formally, we state the following lemma.

▶ Lemma 11. There is a set of evaluation maps $\mathcal{H}_{k,n,d,s} = \{(v_1,\ldots,v_k) : v_i \in \mathcal{H}_{1,n,d,s_k}\}$ where $s_k = s(d+1)^{(k-1)}$ such that, for $i \in [k]$, we have $v_i : X_i \to \mathcal{M}_{d+1}(\mathbb{F})$, and $\mathcal{H}_{k,n,d,s}$ is a hitting set for the class of polynomials $S_{k,n,d,s}$. Moreover, the size of the set is at most $(nskd)^{O(k^2 \log d)}$, and it can be constructed in deterministic $(nskd)^{O(k^2 \log d)}$ time.

Once we prove the above lemma, we will be done with the identity test, since we need to only evaluate the input polynomial on the points in the hitting set and check whether the polynomial evaluates to nonzero on any such point.

Before presenting the proof, we discuss two important ingredients. A polynomial f in $\mathbb{F}\langle X_1\rangle\otimes\cdots\otimes\mathbb{F}\langle X_k\rangle$ can be written as $f=\sum_{m\in X_k^*}f_m\otimes m$ where each m is a monomial over variables X_k and $f_m\in\mathbb{F}\langle X_1\rangle\otimes\cdots\otimes\mathbb{F}\langle X_{k-1}\rangle$. Given that f has a small ABP, we first show that each polynomial f_m also has a small ABP.

▶ Lemma 12. For each $f \in S_{k,n,d,s}$ and $m \in X_k^*$, the polynomial $f_m \in \mathbb{F}\langle X_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle X_{k-1} \rangle$ has an ABP of size s(d+1) and d layers.

Proof. Suppose $f \in \mathbb{F}\langle X_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle X_k \rangle$ has an ABP B of size s and $m = x_{i_1k}x_{i_2k}\cdots x_{i_\ell k}$ where some of the indices could be repeated. We create a copy of f in $\mathbb{F}\langle X \rangle$ where $X = \bigcup_{i=1}^k X_i$ in an obvious way: Just substitute $1 \otimes \cdots \otimes x_{ij} \otimes \cdots \otimes 1$ terms present on the edge label of the ABP for f by x_{ij} . Call this copy g and its ABP B (with a little abuse of notation). Now, we construct an automaton A that isolates precisely those words (monomials) $w \in X^*$ from g such that $w|_{X_k} = m$. The automaton A is depicted in Figure 1.

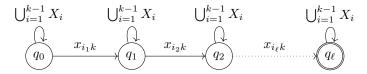


Figure 1 The transition diagram of the automaton A

The automaton simply loops around in each state q_t if the input letter is in $\bigcup_{i=1}^{k-1} X_i$. It makes a forward transition from q_t to q_{t+1} only on reading $x_{i_{t+1}k}$, for $0 \le t \le \ell - 1$.

Naturally, the ABP B can be thought of as a \mathbb{F} -weighted acyclic automaton B without any ϵ -transition 3 computing same g. Now we compute the Hadamard product of B with A, denoted by $B \odot A$ over the *free* monoid computing $g \odot A$. By a basic result of Schützenberger [19, Theorem 3.2, pp. 428], it is known that $B \odot A$ has an automaton of size $s(\ell+1)$. This is basically the computation of intersection of two weighted automata and it can be easily observed that the resulting automata is also an ABP.

For a polynomial f in $\mathbb{F}\langle X_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle X_k \rangle$, consider a partial evaluation $\mathbf{v} = (v_1, \dots, v_{k-1})$ such that each $v_i : X_i \to \mathcal{M}_{t_i}(\mathbb{F})$. The evaluation $\mathbf{v}(f)$ is a $T \times T$ matrix with entries from $\mathbb{F}\langle X_k \rangle$, where $T = t_1 t_2 \cdots t_{k-1}$.

▶ **Lemma 13.** For each $p, q \in [T]$, the $(p, q)^{th}$ entry of $\mathbf{v}(f)$ can be computed by an ABP of size sT and d layers.

The proof is routine and included in the appendix. Now we are ready to prove Lemma 11.

Proof of Lemma 11. The proof is by induction on k. For the base case k=1 the hitting set $\mathcal{H}_{1,n,d,s}$ from Theorem 5 suffices. Note that any nonzero $f \in S_{k,n,d,s}$ can be written as $f = \sum_{m \in X_k^*} f_m \otimes m$ where each m is a monomial over X_k and $f_m \in \mathbb{F}\langle X_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle X_{k-1} \rangle$. Since $f \not\equiv 0$ we must have $f_m \not\equiv 0$ for some $m \in X_k^*$. Moreover, by Lemma 12 we know that for each $m \in X_k^*$ the polynomial $f_m \in \mathbb{F}\langle X_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle X'_{k-1} \rangle$ can be computed by an ABP of size s(d+1). Let s' = s(d+1).

By the inductive hypothesis f_m evaluates to nonzero on some point in the set : $\mathcal{H}_{k-1,n,d,s'} = \{(v_1,v_2,\ldots,v_{k-1})|v_i\in\mathcal{H}_{1,n,d,s',-1}\}$ where $s'_{k-1} = s'(d+1)^{k-2} = s(d+1)^{k-1}$.

Hence, there is an evaluation $\mathbf{v'} \in \mathcal{H}_{k-1,n,d,s'}$ such that $\mathbf{v'}(f_m)$ is a nonzero matrix of dimension $(d+1)^{k-1}$. Interpreting $\mathbf{v'}$ as a partial evaluation for f, we observe that $\mathbf{v'}(f)$ is a $(d+1)^{k-1} \times (d+1)^{k-1}$ matrix with entries from $\mathbb{F}\langle X_k \rangle$. Since $\mathbf{v'}(f_m) \neq 0$, it follows that some $(p,q)^{th}$ entry of $\mathbf{v'}(f)$ is a nonzero polynomial $g \in \mathbb{F}\langle X_k \rangle$. By Lemma 13, each entry of $\mathbf{v'}(f)$ has an ABP of size $s(d+1)^{k-1}$. In particular, $g \in S_{1,n,d,s(d+1)^{k-1}}$ and it follows from Theorem 5 that there is a an evaluation v'' in $\mathcal{H}_{1,n,d,s(d+1)^{k-1}}$ such that v''(g) is a nonzero matrix of dimension $(d+1) \times (d+1)$.

Thus, for the combined evaluation map $\mathbf{v} = (\mathbf{v'}, v'')$, it follows that $\mathbf{v}(f)$ is a nonzero matrix of dimension $(d+1)^k \times (d+1)^k$. Define $\mathcal{H}_{k,n,d,s} = \{(v_1, \ldots, v_k) : v_i \in \mathcal{H}_{1,n,d,s_k}\}$, where $s_k = s(d+1)^{k-1}$. However, from the inductive hypothesis, we know that $\mathbf{v'} = (v_1, \ldots, v_{k-1}) \in \mathcal{H}_{k-1,n,d,s(d+1)}$ where each $v_i \in \mathcal{H}_{1,n,d,s(d+1)^{k-1}}$. Therefore, $\mathbf{v} = (\mathbf{v'}, \mathbf{v''}) \in \mathcal{H}_{k,n,d,s}$ and $\mathcal{H}_{k,n,d,s}$ is a hitting set for the class of polynomials $S_{k,n,d,s}$.

³ In fact any ABP can be also represented by an weighted acyclic automaton of similar size, such that the polynomial computed by the ABP and the finite series computed by the automaton are the same.

Finally, note that $|\mathcal{H}_{k,n,d,s}| = |\mathcal{H}_{1,n,d,s_k}|^k$. Since $|\mathcal{H}_{1,n,d,s_k}| \leq (nds_k)^{O(\log d)}$, it follows that $|\mathcal{H}_{k,n,d,s}| \leq (nskd)^{O(k^2 \log d)}$. Clearly, the set $\mathcal{H}_{k,n,d,s}$ can be constructed in the claimed running time.

5 Randomized Algorithm for Zero Testing of Weighted Automata Over *k*-Monoids

We now consider pc monoids more general than k-clique monoids, over which too we can do efficient zero testing of automata. A k-monoid is a pc monoid M whose non-commutation graph G_M is a union of subgraphs $G_M = G_1 \cup G_2$ such that G_1 has a clique edge cover of size k' and G_2 has a vertex cover of size k - k'. It follows that G_M has a k-covering of cliques and star graphs. For the application, we will assume that this k-covering of G_M is explicitly given as part of the input. In this section $\mathbb{F}\langle M \rangle$ is used to denote the \mathbb{F} -algebra generated by the monoid M.

▶ Lemma 14. Let $\{M_i\}_{i=1}^k$ be pc monoids defined over disjoint variable sets $\{X_i\}_{i=1}^k$, respectively. For each i, suppose A_i is a randomized procedure that outputs an evaluation $v_i : \mathbb{F}\langle M_i \rangle \to \mathcal{M}_{t_i(d)}(\mathbb{F})$ such that for any polynomial g_i in $\mathbb{F}\langle M_i \rangle$ of degree at most d, g_i is nonzero if and only if $v_i(g_i)$ is a nonzero matrix with probability at least $1 - \frac{1}{2k}$.

Then, for the evaluation $\mathbf{v}: \mathbb{F}\langle M_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle M_k \rangle \to \mathcal{M}_{t_1(d)}(\mathbb{F}) \otimes \cdots \otimes \mathcal{M}_{t_k(d)}(\mathbb{F})$ such that $\mathbf{v} = (v_1, \ldots, v_k)$ and any nonzero polynomial $f \in \mathbb{F}\langle M_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle M_k \rangle$ of degree at most d, the matrix $\mathbf{v}(f)$ is nonzero with probability at least 1/2.

Proof. The proof is by induction on k. For the base case k=1, it is trivial. Let us fix an $f \in \mathbb{F}\langle M_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle M_k \rangle$ of degree at most d such that $f \not\equiv 0$. The polynomial f can be written as $f = \sum_{m \in \mathcal{M}_k} f_m \otimes m$ where m are the words over the pc monoid M_k and $f_m \in \mathbb{F}\langle M_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle M_{k-1} \rangle$. Since $f \not\equiv 0$ we must have $f_m \not\equiv 0$ for some $m \in M_k$.

Now, inductively we have the evaluation $\mathbf{v'} = (v_1, \dots, v_{k-1})$ for the class of polynomials in $\mathbb{F}\langle M_1 \rangle \otimes \cdots \otimes \mathbb{F}\langle M_{k-1} \rangle$ of degree at most d. Since $f_m \not\equiv 0$, with high probability $\mathbf{v'}(f_m)$ is a nonzero matrix of dimension $\prod_{i=1}^{k-1} t_i(d)$. By induction the failure probability is bounded by $\frac{k-1}{2^k}$.

As $\mathbf{v'}$ is a partial evaluation for f, we observe that $\mathbf{v'}(f)$ is a matrix of dimension $\prod_{i=1}^{k-1} t_i(d)$ whose entries are polynomials in $\mathbb{F}\langle M_k \rangle$. Since $\mathbf{v'}(f_m) \neq 0$ we conclude that some $(p,q)^{th}$ entry of $\mathbf{v'}(f)$ contains a nonzero polynomial $g \in \mathbb{F}\langle M_k \rangle$ of degree at most d. Choose the evaluation $v_k \in S_k$ which is the output of the randomized procedure A_k , such that $v_k(g)$ is a nonzero matrix of dimension $t_k(d)$. Hence, for the combined evaluation $\mathbf{v} = (\mathbf{v'}, v_k), \mathbf{v}(f)$ is a nonzero matrix of dimension $\prod_{i=1}^k t_i(d)$. Using an union bound the failure probability can be bounded by 1/2.

For the proof of Theorem 2, we first give a randomized polynomial-time identity testing algorithm for polynomials over pc monoids whose non-commutation graph is a star graph.

- ▶ **Lemma 15.** Let $M = ((X \cup y)^*, I)$ be a monoid whose non-commutation graph G_M is a star graph with center y. Then for any constant k, there is a randomized procedure that outputs an evaluation $v: X \cup \{y\} \to \mathcal{M}_{t(d)}(\mathbb{F})$ where t(d) is at most d, such that for any polynomial $f \in \mathbb{F}\langle M \rangle$ of degree at most d, the polynomial f is nonzero if and only if v(f) is a nonzero matrix. The success probability of the algorithm is at least $1 \frac{1}{2k}$.
- **Proof.** If f is nonzero, then there exists a monomial m in M with nonzero coefficient. The idea is to isolate all monomials in $\{X \cup y\}^*$ that are equivalent to m in M. Let the degree of y in monomial m be $\ell \leq d$. Then m can be written as $m = m_1 y m_2 \cdots m_\ell y m_{\ell+1}$ where each

 m_i is a word in X^* . As X is a commuting set of variables, any permutation of m_i produces a monomial equivalent to m in M. Now consider the automaton in Figure 2.

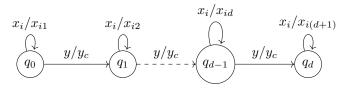


Figure 2 The transition diagram of the automaton

Let m as $m = m_1 y m_2 \cdots m_\ell y m_{\ell+1}$, where each m_i is a maximal substring of m in X^* . We refer to the m_i as blocks. The above automaton keeps count of blocks as it scans the monomial m. As it scans m, if the automaton is in the j^{th} block, it substitutes each variable $x_i \in X$ read by a corresponding commuting variable x_{ij} where the index j encodes the block number. The y variable is renamed by a commutative variable y_c . In effect, we substitute each x_i and y by the transition matrices N_{x_i} and N_y of dimension d+1. The transition matrices are explicitly given below.

$$N_{x_i} = \begin{bmatrix} x_{i1} & 0 & 0 & \dots & 0 \\ 0 & x_{i2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & x_{id} & 0 \\ 0 & 0 & \dots & 0 & x_{i(d+1)} \end{bmatrix}, \qquad N_y = \begin{bmatrix} 0 & y_c & 0 & \dots & 0 \\ 0 & 0 & y_c & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & y_c \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Now we explain this matrix substitution. Let $f = \sum_m \alpha_m m$, where $\alpha_m \in \mathbb{F}$. We write $f = \sum_{\ell=1}^d f_{\ell}$, where $f_{\ell} = \sum_{m: \deg_y(m) = \ell} \alpha_m m$. That is, f_{ℓ} is the part of f consisting of monomials m with y-degree $\deg_y(m) = \ell$.

From the description of the automaton, we can see that for each $\ell \in [d]$, the $(0,\ell)^{th}$ entry of the output matrix is the commutative polynomial $f_{\ell}^c \in \mathbb{F}[\{x_{i,j}\}_{1 \leq i \leq n, 1 \leq j \leq d+1}, y_c]$. The construction ensures the following.

▶ **Observation 1.** For each $0 \le \ell \le d$, $f_{\ell} = 0$ if and only if $f_{\ell}^{c} = 0$.

The randomized identity test is by substituting random scalar values for the commuting variables x_{ij} and y_c from a set $S \subseteq \mathbb{F}$ of size at least 2kd, such that the output matrix becomes nonzero. The bound on the success probability follows from Polynomial Identity Lemma [25, 21, 7].

Now we are ready to prove Theorem 2.

Proof. Let M' be a pc monoid whose non-commutation graph $G_{M'}$ is a clique. Let $g \in \mathbb{F}\langle M' \rangle$ be a nonzero polynomial of degree at most d. By the Amitsur-Levitzki Theorem [1], if we substitute variables $x_i \in M'$ by generic matrix of size d over the variables $\{x_{u,v}^{(i)}\}_{1 \leq u,v \leq d}$, the output matrix is nonzero. Moreover, the entries of the output matrix are commutative polynomials of degree at most d in the variables $\{x_{u,v}^{(i)}\}_{1 \leq i \leq n, 1 \leq u,v \leq d}$. It suffices to randomly substitute for each $x_{u,v}^{(i)}$ variable from a set $S \subseteq \mathbb{F}$ of size at least 2kd. This defines the evaluation map $v : \mathbb{F}\langle M' \rangle \to \mathbb{M}_d(\mathbb{F})$. The resulting identity test succeeds with probability at least $1 - \frac{1}{2k}$. For the star graphs, the evaluation map is already defined in Lemma 15.

⁴ In fact the Amitsur-Levitzki theorem guarantees that generic matrices of size $\lceil \frac{d}{2} \rceil + 1$ suffice [1].

Given a \mathbb{F} -weighted automaton A of size s over a k-monoid $M=(X^*,I)$, by Theorem 10, the zero testing of A reduces to identity testing of a collection of ABPs of the form : $f=u^TN^dv$ over $\mathbb{F}\langle M\rangle$, where N is the transition matrix of A and d is bounded by $O(n^3s^2)$. Now, to test identity of f where M is a k-monoid, it suffices to test identity of $\psi(f)$ where ψ is the injective homomorphism from Lemma 6. Now $\psi(f)$ in $\mathbb{F}\langle M_1'\rangle\otimes\cdots\otimes\mathbb{F}\langle M_k'\rangle$, where for each $i\in[k]$ the non-commutation graph of M_i' is either a clique or a star.

By Lemma 14, we can construct the evaluation map $\mathbf{v} = v_1 \otimes v_2 \otimes \cdots \otimes v_k$ where for each $i \in [k]$, v_i is an evaluation map for either a clique or a star graph depending on M'_i . The range of \mathbf{v} is matrices of dimension at most d^k , which is bounded by $(sn)^{O(k)}$ as d is bounded by $O(n^3s^2)$. This completes the proof of Theorem 2.

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A The Proof of Lemma 6

Proof. It is straightforward to check that ψ is a ring homomorphism. To show the injectivity, it is enough to show that $\psi(m) = \psi(m')$ implies m = m' in M for any words $m, m' \in M$. We prove the claim by induction on the length of words in M. Suppose that for words $m \in M$ of length at most ℓ , if m' is not \tilde{I} -equivalent to m then $\psi(m) \neq \psi(m')$. The base case, for $\ell = 0$ clearly holds.

Now, suppose $m = x \cdot m_1 \in X^{\ell+1}$ for $x \in X$ and $\psi(m) = \psi(m')$.

 \triangleright Claim 16. For some $m_2 \in M$, $m' = x \cdot m_2$ in M.

Proof. Assume, to the contrary, that there is no $m_2 \in M$ such that $m' = x \cdot m_2$. Let $J = \{j \in [k] \mid x \in X_j\}$. If the variable x does not occur in m' then $m|_{X_j} \neq m'|_{X_j}$ for each $j \in J$. This implies that $\psi(m) \neq \psi(m')$ which is a contradiction.

On other hand, suppose x occurs in m' and it cannot be moved to the leftmost position in m' using the commutation relations in I. Then we must have m' = ayxb for some $y \in X_j$ and $j \in J$, where $a, b \in X^*$, for the leftmost occurrence of x in m'. Hence $m|_{X_j} \neq m'|_{X_j}$, because x is the first variable in $m|_{X_j}$ and x comes after y in $m'|_{X_j}$. Therefore, $\psi(m) \neq \psi(m')$ which is a contradiction.

Now, $\psi(x \cdot m_1) = \psi(x \cdot m_2)$ implies that $\psi(m_1) = \psi(m_2)$. Both m_1 and m_2 are of length ℓ . By induction hypothesis it follows that $m_1 = m_2$, and hence m = m'.

B The Proof of Lemma 13

Proof. In effect the edges of the input branching program B are now labelled by matrices of dimension T with entries are linear forms over the variables X'_k . To show that each entry of the final $T \times T$ matrix can be computed by an ABP of size sT, let us fix some (i, j) such that $1 \le i, j \le T$ and construct an ABP B'_{ij} computing the polynomial in the $(i, j)^{th}$ entry.

The construction of B'_{ij} is as follows. We make T copies of each node p (except the source and sink node) of B and label it as (p,k) for each $k \in [T]$. Let us fix two nodes p and q from B such that there is a $T \times T$ matrix M_{pq} labelling the edge (p,q) after the substitution. Then, for each $j_1, j_2 \in [T]$, add an edge between (p, j_1) and (q, j_2) in B'_{ij} and label it by the $(j_1, j_2)^{th}$ entry of M_{pq} . When p is the source node, for each $j_2 \in T$, add an edge between the source node and (q, j_2) in B'_{ij} and label it by the $(i, j_2)^{th}$ entry of M_{pq} . Similarly, when q is the sink node, for each $j_1 \in T$, add an edge between (p, j_1) and the sink node in B'_{ij} and label it by the $(j_1, j)^{th}$ entry of M_{pq} .

We just need to argue that the intermediate edge connections simulate matrix multiplications correctly. This is simple to observe, since for each path $\mathcal{P} = \{(s,p_1),(p_1,p_2),\ldots,(p_{\ell-1},t)\}$ in B (where s,t are the source and sink nodes respectively) and each $(j_1,\ldots,j_{\ell-1})$ such that $1 \leq j_1,\ldots,j_{\ell-1} \leq T$, there is a path $(s,(p_1,j_1)),((p_1,j_1),(p_2,j_2)),\ldots,((p_{\ell-1},j_{\ell-1}),t)$ in B'_{ij} that computes $M_{(s,p_1)}[i,j_1]M_{(p_1,p_2)}[j_1,j_2]\cdots M_{p_{\ell-1},t}[j_{\ell-1},j]$ where $M_{(p,q)}$ is the $T\times T$ matrix labelling the edge (p,q) in B. The size of B'_{ij} is sT, and the number of layers is d.