Arch. Math. Logic 43, 1–18 (2004)
Digital Object Identifier (DOI):
10.1007/s00153-003-0190-2

Mathematical Logic

Ingo Lepper

Simply terminating rewrite systems with long derivations

Received: 30 September 1999 / Revised version: 20 March 2003 /
Published online: 12 December 2003 –

Abstract. A term rewrite system is called simply terminating if its termination can be shown by means of a simplification ordering. According to a result of Weiermann, the derivation length function of any simply terminating finite rewrite system is eventually dominated by a Hardy function of ordinal less than the small Veblen ordinal. This bound had appeared to be of rather theoretical nature, because all known examples had had multiple recursive complexities, until recently Touzet constructed simply (and even totally) terminating examples with complexities beyond multiple recursion. This was established by simulating the Hydra battle for all ordinal segments below the proof-theoretic ordinal of Peano arithmetic. By extending this result to the small Veblen ordinal we prove the huge bound of Weiermann to be sharp. As a spin-off we can show that total termination allows for complexities as high as those of simple termination.

1. Introduction

One of the main topics in term rewriting theory is termination, and a vast variety of techniques for proving termination has evolved. Adapting a question of Kreisel to this field, one may ask: What else do we know about a rewrite system, once its termination is proved? Interesting answers concerning the (derivation) complexities of simply terminating rewrite systems have been given. A rewrite system is called *simply terminating* if its termination can be shown via a simplification ordering. The *complexity* of a rewrite system is the function mapping each natural number to the length of a longest possible rewrite sequence starting with a term of depth bounded by this number, provided that such a sequence exists. This is a natural measure for the worst case behaviour of a rewrite system.

Simplification orderings constitute an important tool for proving termination. Amongst the multitude of such orderings are the multiset path ordering MPO, the Knuth-Bendix ordering KBO, and the lexicographic path ordering LPO. If termination of a finite rewrite system is provable by MPO this implies, according

I. Lepper: Institut für Mathematische Logik und Grundlagenforschung der Westfälischen Wilhelms-Universität Münster, Einsteinstraße 62, 48149 Münster, Germany.
e-mail: `lepper@uni-muenster.de`

This paper is part of the author's doctoral dissertation project (under the supervision of Professor A. Weiermann at the University of Münster).
The work on this paper was supported by DFG grant WE 2178/2–1
Mathematics Subject Classification (2000): 03D20, 68Q15, 68Q42
Key words or phrases: Simple termination – Hardy hierarchy – Fundamental sequence – Hydra

to Hofbauer [9], primitive recursive complexity. For KBO Hofbauer [8] established a (low) multiple recursive complexity bound, and Weiermann [20] showed that termination via LPO yields a multiple recursive complexity bound, too. For the general case of a simply terminating finite rewrite system (over a finite signature with fixed arities), Weiermann [19] showed that its complexity is eventually dominated by the Hardy function whose index is the *small Veblen ordinal*, which is $\bar{\theta}_{\Omega^\omega}(0)$ in Schütte [14] style. This was established by calculating the proof-theoretic strength of Kruskal's Tree Theorem, which is the backbone of well-foundedness proofs for simplification orderings.

Ferreira and Zantema [7] call rewrite systems allowing for a total termination ordering *totally terminating*. Each totally terminating finite rewrite systems over a finite signature is simply terminating. As the converse does not hold, one may ask whether total termination is blessed with a lower complexity bound.

Cichon [3] conjectured that the complexity of a rewrite system for which termination is provable using a termination ordering of order type α is eventually dominated by a function from the slow growing hierarchy of ordinal determined by α. For quite a while, the most complex known simply terminating finite rewrite systems had had multiple recursive complexities. Multiple recursion is identical with the concept of $<\omega_3$-recursion (cf., for example, [12]), where $\omega_0 := 1$ and $\omega_{n+1} := \omega^{\omega_n}$. Since the Hardy hierarchy below ω_3 and the slow growing hierarchy below the small Veblen ordinal coincide, this had led to the conjecture that all simply terminating finite rewrite systems have multiple recursive complexities. However, Touzet [15, 16] constructed totally terminating finite rewrite systems with proper ω_n-recursive complexities for all $n \geq 3$. Termination of these rewrite systems is still provable in Peano arithmetic. Thus there remains a huge gap between all known examples of simply terminating rewrite systems and the bound of Weiermann.

We bridge the gap by showing that the bound of Weiermann is sharp. This closes the case for problem 81 in the RTA list of open problems [5], whose revision in [6] asks what complexity can be achieved by simply terminating finite rewrite systems over finite signatures.

According to Weiermann [19], the bound cannot be reached by a single finite rewrite system. Thus we have to define a hierarchy of rewrite systems whose increasing complexities approach the bound. All these rewrite systems are totally terminating, hence, concerning largest possible complexities, total termination is as powerful as simple termination.

The basic idea, taken from Touzet [16], consists in simulating the *Battle of Hercules and the Hydra* from Kirby and Paris [10] with a totally terminating finite rewrite system. Since the Hydra battle and the Hardy hierarchy are closely connected, we have to do so for all ordinals below $\bar{\theta}_{\Omega^\omega}(0)$. The $\bar{\theta}$ function is very powerful and, due to its definition using closure processes, not easy to handle by finite rewrite systems. For our purposes it seems natural to use k-ary fixed-point free Veblen φ functions ψ instead of the $\bar{\theta}$ function, since the absence of fixed-points significantly simplifies calculations and since

$$\bar{\theta}_{\Omega^\omega}(0) = \sup_{k\geq 1} \psi(1, \underbrace{0, \ldots, 0}_{k \text{ times}})$$

holds. In view of this, a canonical approach consists in defining rewrite systems $\mathcal{R}_k$ which simulate Hydra battles for all ordinals below $\psi(1, 0, \ldots, 0)$ where ψ is $k+1$-ary. This is done in Section 4, where we use a special encoding for the ordinals below $\psi(1, 0, \ldots, 0)$ introduced in Section 3. Total termination of $\mathcal{R}_k$ is then established in Section 5 using a technically smooth characterisation of total termination which stems from Touzet [16]. The $\mathcal{R}_k$ are given in a uniform manner, and for $k > l$ the rewrite system $\mathcal{R}_k$ can be regarded as a proper extension of $\mathcal{R}_l$. Thus the $(\mathcal{R}_k)_{k\in\mathbb{N}}$ form a hierarchy of totally terminating finite rewrite systems, and the complexity of any simply terminating finite rewrite system is eventually dominated by the complexity of some $\mathcal{R}_k$.

2. Preliminaries

Let us first settle a few conventions. Natural numbers are denoted by letters between i and n; ordinals are usually denoted by $\alpha, \beta, \gamma, \delta, \mu$ and λ; and t, u, v, d range over closed terms. We abbreviate finite sequences of terms like $t_1, \ldots, t_n$ by $\bar{t}$ and do the same with ordinals and variables. Empty sequences are allowed and will show up later. The length of a sequence will (hopefully) always be clear from the context.

2.1. Term rewriting

We assume some basic knowledge of term rewriting theory, which can be found, for example, in the text of Dershowitz and Jouannaud [4]. The following definitions and results are variations on [7, 16].

Let $\mathcal{F}$ be a *finite* signature whose function symbols have fixed arity and let $\mathcal{X}$ be a set of variables. The *term algebra* generated by $\mathcal{F}$ and $\mathcal{X}$ is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{X})$, and $\mathcal{T}(\mathcal{F})$ is the set of closed (or *ground*) terms of this algebra. Let $\mathcal{R}$ be a rewrite system over $\mathcal{F}$. We write $\to_{\mathcal{R}}$ for the associated rewrite relation, $\stackrel{+}{\to}_{\mathcal{R}}$ for the transitive hull of $\to_{\mathcal{R}}$, and $\stackrel{*}{\to}_{\mathcal{R}}$ for the reflexive closure of $\stackrel{+}{\to}_{\mathcal{R}}$. If $\to_{\mathcal{R}}$ is Noetherian then we say that $\mathcal{R}$ *terminates*. A rewrite system is called *finite* if it contains only finitely many rewrite rules. The *complexity* of a terminating finite rewrite system is measured by the *derivation length* function $\mathrm{Dl}_{\mathcal{R}} \colon \mathbb{N} \to \mathbb{N}$. First one defines $\mathrm{dl}_{\mathcal{R}} \colon \mathcal{T}(\mathcal{F}) \to \mathbb{N}$ by $t \mapsto \max\{\mathrm{dl}_{\mathcal{R}}(u) + 1 \colon t \to_{\mathcal{R}} u\}$, and then

$$\mathrm{Dl}_{\mathcal{R}}(n) := \max\{\mathrm{dl}_{\mathcal{R}}(t) \colon t \in \mathcal{T}(\mathcal{F}) \wedge \mathrm{dp}(t) \leq n\}$$

where $\mathrm{dp}(t)$ is the *depth* of t.

Definition 1. *For an ordering $(A, \succ)$ and $[\cdot] \colon \mathcal{T}(\mathcal{F}) \to A$ we say that*

i. $[\cdot]$ *is* monotone *if for all $u, v, t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F})$, for all $f \in \mathcal{F}$*

$$[u] \succeq [v] \Rightarrow [f(t_1, \ldots, u, \ldots, t_n)] \succeq [f(t_1, \ldots, v, \ldots, t_n)], \tag{1}$$

ii. $[\cdot]$ *is* strictly monotone *if* (1) *holds where both $\succeq$ are replaced by $\succ$,*

iii. $[\cdot]$ *has the* subterm property *if for all* $u_1, \ldots, u_n \in \mathcal{T}(\mathcal{F})$, *for all* $f \in \mathcal{F}$

$$\forall i \in \{1, \ldots, n\} \quad [f(u_1, \ldots, u_n)] \succ [u_i], \tag{2}$$

iv. $[\cdot]$ normalizes *a rewrite system* $\mathcal{R}$ *if*

$$\forall (l, r) \in \mathcal{R} \quad \forall \sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F}) \quad [l\sigma] \succ [r\sigma].$$

Inspired by (2), *some* $G\colon A^n \to A$ *has the* subterm property *if*

$$\forall a_1, \ldots, a_n \in A \;\; \forall i \in \{1, \ldots, n\} \quad G(a_1, \ldots, a_n) \succ a_i$$

holds.

Given a well-founded ordering $(A, \succ)$, an *interpretation* for $\mathcal{R}$ on A is a mapping $[\cdot]\colon \mathcal{T}(\mathcal{F}) \to A$ that fulfils

$$\forall u, v \in \mathcal{T}(\mathcal{F}) \quad u \to_{\mathcal{R}} v \Rightarrow [u] \succ [v]. \tag{3}$$

Since $(A, \succ)$ is well-founded, this establishes termination. Usually, interpretations are defined in a *homomorphic* manner: to each symbol f of the signature is assigned an *interpreting function* $[f]$ on A of appropriate arity, and then the morphism $[\cdot]$ is defined by recursion on $\mathcal{T}(\mathcal{F})$ via

$$[f(t_1, \ldots, t_n)] := [f]([t_1], \ldots, [t_n]).$$

Lemma 2. *A homomorphic morphism is (strictly) monotone if each of the interpreting functions is (strictly) monotone, and it has the subterm property if each of the interpreting functions has the subterm property.*

The concept of *total termination* was introduced by Ferreira and Zantema [7].

Definition 3. *A rewrite system* $\mathcal{R}$ *is* totally terminating *if there exist a well-ordering* $(A, \succ)$ *and a strictly monotone morphism* $[\cdot]\colon \mathcal{T}(\mathcal{F}) \to A$ *which normalizes* $\mathcal{R}$.

Total termination of $\mathcal{R}$ is equivalent to the existence of a strictly monotone interpretation for $\mathcal{R}$ on some well-ordering $(A, \succ)$. According to [7], any totally terminating finite rewrite system is simply terminating. Thus the bound of Weiermann eventually dominates the complexity of any totally terminating finite rewrite system.

One main obstacle to proving total termination for a given rewrite system is to exhibit a *strictly* monotone morphism, whereas many natural candidates are monotone. The following result of Touzet, which is of central importance for [16] and our paper, shows that strictness can be dropped under a certain assumption.

Theorem 4 (Touzet). *A rewrite system* $\mathcal{R}$ *is totally terminating iff there exist a well-ordering* $(A, \succ)$ *and a monotone morphism* $[\cdot]\colon \mathcal{T}(\mathcal{F}) \to A$ *which has the subterm property and normalizes* $\mathcal{R}$.

A proof is given in [16]. The basic idea is to construct a strictly monotone interpretation $\mathcal{I}$ for $\mathcal{R}$ on the well-ordering $(\mathsf{mul}(A), \succ_{\mathsf{mul}})$, where $\mathsf{mul}(A)$ denotes the set of finite multisets on A and $\succ_{\mathsf{mul}}$ the multiset extension of $\succ$ on $\mathsf{mul}(A)$, by putting

$$\mathcal{I}(f(t_1, \ldots, t_n)) := \{[f(t_1, \ldots, t_n)]\} \cup \mathcal{I}(t_1) \cup \cdots \cup \mathcal{I}(t_n),$$

where $\cup$ stands for the union of multisets.

2.2. The fixed-point free Veblen function

We will now introduce the Veblen [17] φ function and its fixed-point free variant ψ. The connection between ψ and $\bar{\theta}$ was illuminated by Schmidt [13]. Some familiarity with ordinal theory, as to be found, for example, in the books of Pohlers [11] and Schütte [14], is assumed.

The class of ordinals is denoted by On, and the class of limit ordinals by Lim. An ordinal $\alpha > 0$ is *principal* if for all $\beta, \gamma < \alpha$ we have $\beta + \gamma < \alpha$. The principal ordinals constitute the class H.

For $\alpha_1, \ldots, \alpha_k \in \mathsf{On}$ with $k > 0$ we intend to recursively define the branch $\varphi_{\bar{\alpha}} \colon \mathsf{On} \to \mathsf{On}$ of the *Veblen function*. It is helpful to interchangeably use $\varphi_{\bar{\alpha}}(\beta)$ and $\varphi(\bar{\alpha}, \beta)$, thus regarding φ as a function from the ordinal sequences of lengths larger than 1 into the ordinals. The definition is as follows: $\varphi_{\bar{0}}$ enumerates H; if $\alpha_k > 0$ then $\varphi_{\bar{\alpha}}$ is the enumerating function of the proper class

$$\{\beta \colon \forall \gamma < \alpha_k \ \varphi(\alpha_1, \ldots, \alpha_{k-1}, \gamma, \beta) = \beta\},$$

and otherwise we have $(\alpha_1, \ldots, \alpha_k) = (\alpha_1, \ldots, \alpha_i, \bar{0}, 0)$ with $\alpha_i > 0$. Here we let $\varphi_{\bar{\alpha}}$ be the enumerating function of

$$\{\beta \colon \forall \gamma < \alpha_i \ \varphi(\alpha_1, \ldots, \alpha_{i-1}, \gamma, \beta, \bar{0}, 0) = \beta\}.$$

Obviously $\varphi_{\bar{0}, \bar{\alpha}} = \varphi_{\bar{\alpha}}$ holds. The φ function lacks the subterm property since it admits fixed-points. Therefore we concentrate on ψ, the fixed-point free version of φ. We let $\psi(\alpha_1, \ldots, \alpha_k, \beta)$ be $\varphi(\bar{\alpha}, \beta + 1)$ if $\beta = \beta_0 + n$ for some $\beta_0 \in \mathsf{Lim} \cup \{0\}$ with $\varphi(\bar{\alpha}, \beta_0) \in \{\alpha_1, \ldots, \alpha_k, \beta_0\}$, and otherwise $\psi(\bar{\alpha}, \beta)$ is $\varphi(\bar{\alpha}, \beta)$.

For the remainder of the text we keep some $k > 0$ fixed and focus on the $k + 1$-ary ψ.

We denote the first infinite ordinal closed under $+$ and the $k + 1$-ary ψ function by Δ_k, thus $\Delta_k = \varphi(1, \bar{0})$ where $\bar{0}$ has length $k + 1$. According to [13]

$$\bar{\theta}_{\Omega^\omega}(0) = \sup_{k \in \mathbb{N}} \Delta_k$$

holds. For every ordinal $\alpha > 0$ there are uniquely determined principal ordinals $\alpha_0 \geq \cdots \geq \alpha_n$ such that $\alpha = \alpha_0 + \cdots + \alpha_n$ holds. In addition, for every principal $\alpha < \Delta_k$ there are uniquely determined $\alpha_1, \ldots, \alpha_{k+1}$ below α satisfying $\alpha = \psi(\bar{\alpha})$. So every $\alpha < \Delta_k$ can be associated with a unique representation solely built up from 0, $+$ and the $k + 1$-ary ψ. We call this representation the (k-) *normal form* of α.

The next Lemma lists some of the basic properties of ψ, but before we can state it we need some additional notations. The lexicographic ordering of ordinal sequences is denoted by $>_{\mathsf{lex}}$. If $\gamma_1, \ldots, \gamma_n$ is a nonempty sequence and $M = \{1, \ldots, n\}$, then $\alpha > \bar{\gamma}$ abbreviates $\forall i \in M \ (\alpha > \gamma_i)$, and $\alpha \leq \bar{\gamma}$ stands for $\exists i \in M (\alpha \leq \gamma_i)$.

Lemma 5. *Let $\alpha_1, \ldots, \alpha_{k+1}$ and $\gamma_1, \ldots, \gamma_{k+1}$ be given.*

i. *Each $\psi(\bar{\alpha})$ is a principal ordinal, and, except for $\psi(\bar{0}) = 1$, a limit ordinal.*
ii. *The function ψ has the subterm property and is strictly monotone in all arguments.*
iii. *$\psi(\bar{\alpha}) > \psi(\bar{\gamma})$ is equivalent to $[\bar{\alpha} >_{\mathsf{lex}} \bar{\gamma} \wedge \psi(\bar{\alpha}) > \bar{\gamma}] \vee \psi(\bar{\gamma}) \leq \bar{\alpha}$.*

Since we are interested in iterations of the ψ function we need some additional notations. For a unary ordinal function f we define the nth iteration f^n by $f^0(y) := y$ and $f^{n+1}(y) := f(f^n(y))$. We will make heavy use of this notation for functions of higher arity where all but one arguments are fixed. In these cases a $\cdot$ indicates the free position.

The following Lemma contains all properties of ψ we will need.

Lemma 6. *Let $\alpha_1, \ldots, \alpha_{k+1}$, $\gamma_1, \ldots, \gamma_{k+1}$ and $i \in \{1, \ldots, k\}$ be given. We abbreviate $\psi(\alpha_1, \ldots, \alpha_i, \cdot, \alpha_{i+1}, \ldots, \alpha_k)^n(x)$ with $\alpha^n(x)$.*

i. *If $n > m$ and $\alpha_j \neq 0$ for some j then $\psi(\bar{\alpha}) \cdot n > \psi(\bar{\alpha}) \cdot m$.*
ii. *If $\psi(\bar{\alpha}) > \psi(\bar{\gamma})$ then $\psi(\bar{\alpha}) > \psi(\bar{\gamma}) \cdot n + m$.*
iii. *If $n \geq m$, $\delta \geq \delta'$ and at least one of the inequalities is proper then $\alpha^n(\delta) > \alpha^m(\delta')$.*
iv. *If $(\gamma_1, \ldots, \gamma_i) >_{\mathsf{lex}} (\alpha_1, \ldots, \alpha_i)$ and $\psi(\bar{\gamma}) > \alpha_1, \ldots, \alpha_k, \delta$ then $\psi(\bar{\gamma}) > \alpha^n(\delta)$ for all n.*

Proof. The first point follows from $\psi(\bar{\alpha}) > 1$. Under the conditions of (ii), $\psi(\bar{\alpha})$ is a principal limit ordinal. For (iii) we utilize the subterm property and the strict monotonicity of ψ, respectively. Finally, (iv) is established by induction on n using Lemma 5.iii. □

2.3. *Fundamental sequences and Hydrae*

Fundamental sequences are an important tool in proof theory: To each ordinal α below some fixed ordinal Λ is uniformly assigned an ordinal sequence $(\alpha[n])_{n\in\mathbb{N}}$. If α is a limit ordinal then this sequence is strictly increasing and its supremum is α. The assignment has the *Bachmann property* if, for all α, β below Λ,

$$\alpha[n] < \beta < \alpha[n+1] \Rightarrow \alpha[n] \leq \beta[0]$$

holds. Before we can introduce such an assignment for the ordinals below Δ_k, we have to wade through some technical definitions.

Definition 7. *Let $\alpha_1, \ldots, \alpha_k < \Delta_k$ be given. The collection of fixed-points of $\varphi_{\bar{\alpha}}$ is*

$$\mathsf{Fix}(\bar{\alpha}) := \{\psi(\bar{\gamma}, \delta) \colon \bar{\gamma} >_{\mathsf{lex}} \bar{\alpha} \wedge \psi(\bar{\gamma}, \delta) > \bar{\alpha}\}.$$

For $\beta < \Delta_k$ we need the notation

$$\psi(\bar{\alpha}, \beta)^* := \begin{cases} \psi(\bar{\alpha}, \beta_0) & \text{if } \beta = \beta_0 + 1 \\ \beta & \text{otherwise,} \end{cases}$$

and for $\gamma < \Delta_k$ in normal form we are going to define the set $\mathrm{IS}_{\bar{\alpha}}(\gamma)$ of (relative to $\bar{\alpha}$) interesting *subterms of γ by induction: We begin with $\mathrm{IS}_{\bar{\alpha}}(0) := \{0\}$ and $\mathrm{IS}_{\bar{\alpha}}(\gamma_1 + \cdots + \gamma_m) := \bigcup_{1\leq i\leq m} \mathrm{IS}_{\bar{\alpha}}(\gamma_i)$ for $m > 1$. The nontrivial case is given by*

$$\mathrm{IS}_{\bar{\alpha}}(\psi(\bar{\gamma})) := \begin{cases} \{\psi(\bar{\gamma})\} & \text{if } (\gamma_1, \ldots, \gamma_k) \geq_{\mathsf{lex}} (\alpha_1, \ldots, \alpha_k) \\ \bigcup_{1\leq i\leq k+1} \mathrm{IS}_{\bar{\alpha}}(\gamma_i) & \text{otherwise.} \end{cases}$$

The (relative to $\bar{\alpha}$*)* maximal *interesting subterm* $\mathrm{MS}_{\bar{\alpha}}(\gamma_1, \ldots, \gamma_n)$ *of a nonempty sequence* $\bar{\gamma}$ *can then by defined to be the maximum of the ordinals occurring in the* $\mathrm{IS}_{\bar{\alpha}}(\gamma_i)$.

The following definition corresponds to the assignment of fundamental sequences for ordinals below Δ_k from Weiermann [21], which is based on work of Buchholz [1].

Definition 8. *We define* $\alpha[n]$ *by recursion on* $\alpha < \Delta_k$ *in normal form:*

$$\begin{aligned}
0[n] &:= 0 \\
(\alpha_1 + \cdots + \alpha_m)[n] &:= \alpha_1 + \cdots + \alpha_{m-1} + \alpha_m[n] \quad \text{if } m > 1 \\
\psi(\bar{0})[n] &:= 0 \\
\psi(\bar{\alpha}, \lambda)[n] &:= \psi(\bar{\alpha}, \lambda[n]) \text{ if } \lambda \in \mathsf{Lim} \setminus \mathsf{Fix}(\bar{\alpha}) \\
\psi(\bar{0}, \beta)[n] &:= \alpha^* \cdot (n+1) \\
\psi(\alpha_1, \ldots, \alpha_i + 1, \bar{0}, \beta)[n] &:= \psi(\alpha_1, \ldots, \alpha_i, \cdot, \bar{0})^{n+1}(\alpha^*) \\
\psi(\alpha_1, \ldots, \alpha_i, \bar{0}, 0)[n] &:= \psi(\alpha_1, \ldots, \alpha_i[n], \bar{0}, \mathrm{MS}_{\bar{\alpha},\bar{0}}(\bar{\alpha})) \\
\psi(\alpha_1, \ldots, \alpha_i, \bar{0}, \beta)[n] &:= \psi(\alpha_1, \ldots, \alpha_i[n], \bar{0}, \alpha^*).
\end{aligned}$$

It should be clear that $\alpha_i \in \mathsf{Lim}$ *holds for the last two lines.*

Lemma 9. *Let* $\alpha < \Delta_k$ *and* n *be given. If* $\alpha > 0$ *then we have* $\alpha > \alpha[n]$*, for* $\alpha > 1$ *we get* $\alpha[n] > 0$*, and if* α *is a limit ordinal then* $\alpha[n+1] > \alpha[n]$ *and* $\alpha = \sup_{n \in \mathbb{N}} \alpha[n]$ *hold. The assignment of fundamental sequences has the Bachmann property.*

A proof can be found in [21].

The *Battle of Hercules and the Hydra* from Kirby and Paris [10] is closely connected to the Hardy hierarchy, which is emphasized by the fact that *Hardy* is an anagram for *Hydra*. A *Hydra* is an ordinal below Δ_k. For each Hydra α the ordered pair $c := (\alpha, n)$ is called a *configuration*. The *next* configuration c^+ for c is $(\alpha[n], n+1)$, and the (standard) *Hydra battle* for the configuration c is the sequence of configurations $(c_m)_{m \in \mathbb{N}}$ with $c_0 = c$ and $c_{m+1} = c_m^+$. As an immediate consequence of Lemma 9 we see that for each configuration there is a minimal m such that the Hydra in c_m is 0, called the *length* of the battle.

Theorem 10. *If* $\alpha = \omega^\alpha$ *then the function which maps* n *to the length of the battle for the configuration* (α, n) *is not* $<\alpha$*-recursive since it eventually dominates all* $<\alpha$*-recursive functions.*

Proof. The length of the battle for (α, n) is larger than $H_\alpha(n)$, where H_α is the αth element of the Hardy hierarchy, cf. [18]. Since the assignment of fundamental sequences has the Bachmann property, the claim is established by [2]. □

3. Encoding all Hydrae below Δ_k

The signature $\mathcal{F}_0$ consists of the constant $\mathsf{0}$, the unary (successor) S, the binary $\boldsymbol{+}$ and the $k+1$-ary P, which stands for ψ. Each $t \in \mathcal{T}(\mathcal{F}_0)$ has a canonical *value* $\mathrm{val}(t) < \Delta_k$, which is calculated by interpreting $\mathsf{0}$, S, $\boldsymbol{+}$, P with 0, the successor function, the ordinal sum, and ψ, respectively.

The symbol S is not needed for coding Hydrae since the terms $\boldsymbol{+}(t, \mathsf{P}(\bar{\mathsf{0}}))$ and $\mathsf{S}t$ have the same value. We use S to separate limit and successor ordinals, additionally its presence will simplify various calculations.

Obviously, each ordinal below Δ_k can be denoted by terms of $\mathcal{T}(\mathcal{F}_0)$. To mimic the fundamental sequences on terms we introduce a set of *standard terms*. Because our $\boldsymbol{+}$ has fixed arity and we do not bother about distinguishing between $\boldsymbol{+}(\boldsymbol{+}(t,u),v)$ and $\boldsymbol{+}(t,\boldsymbol{+}(u,v))$, there will usually be different standard terms denoting the same ordinal. Furthermore, for γ not a successor and $n > 0$ we want to denote $\gamma + n$ by $\mathsf{S}^n t$ (where t is a previously defined standard term for γ), thus using $\boldsymbol{+}$ only for certain additions of (standard terms for) limit ordinals. This will be helpful later when we treat reductions of standard terms.

The *natural sum* on ordinals is denoted by $\oplus$. A pair (λ, μ) of *limit* ordinals is called *compatible* if $\lambda + \mu = \lambda \oplus \mu$ holds.

Definition 11. *The set $\mathcal{S} \subset \mathcal{T}(\mathcal{F}_0)$ of* standard terms *is the smallest superset of* $\{\mathsf{0}\}$ *which is closed under* S *and these rules:*

i. $\bar{t} \in \mathcal{S}$ *and* $\bar{t} \neq \bar{\mathsf{0}} \Rightarrow \mathsf{P}(\bar{t}) \in \mathcal{S}$
ii. $t, u \in \mathcal{S}$ *and* $(\mathrm{val}(t), \mathrm{val}(u))$ *compatible* $\Rightarrow \boldsymbol{+}(t,u) \in \mathcal{S}$.

With $\mathcal{S}(\alpha)$ we denote the set of standard terms with value α.

If u is a proper subterm of $t \in \mathcal{S}(\alpha)$ then there exists $\beta < \alpha$ with $u \in \mathcal{S}(\beta)$. An induction on $\alpha < \Delta_k$ shows $\mathcal{S}(\alpha) \neq \emptyset$, hence $\mathcal{S}$ is the union of the $\mathcal{S}(\alpha)$ with $\alpha < \Delta_k$.

For a unary function symbol f we adapt the nth iteration f^n given just above Lemma 6. A formal multiplication for $t \in \mathcal{T}(\mathcal{F}_0, \mathcal{X})$ and $n > 0$ is defined by $t \times n := \boldsymbol{+}(\cdot, t)^{n-1}(t)$. It is important for our purposes that the recursion occurs at the first position.

Our aim is now to mimic the definition of $\alpha[n]$ for $\mathcal{S}$. Here, we encounter some difficulties: Though in Definition 8 the ordinals on the left are supposed to be in normal form, the ordinal representations on the right are not always in normal form. In the same way, a formal equivalent to fundamental sequences for standard terms will not always produce standard terms. For example, if we defined $\mathsf{P}(\bar{\mathsf{0}}, \mathsf{S}t)[n]$ to be $\mathsf{P}(\bar{\mathsf{0}}, t) \times (n+1)$, this would result in occurrences of the nonstandard term $\mathsf{P}(\bar{\mathsf{0}})$ for $t = \mathsf{0}$. We overcome this obstacle for $d \in \mathcal{S}(\alpha)$ by simultaneously defining $d\langle n\rangle$ and $d[n]$, where $d\langle n\rangle$ is a formal equivalent to $\alpha[n]$ which need not be a standard term but has a uniform definition, while $d[n]$ is a refinement of $d\langle n\rangle$ and an element of $\mathcal{S}(\alpha[n])$. Later we will work in a rewrite system in which $d\langle n\rangle$ reduces to $d[n]$. For the sake of transparency we split up the cases in Definition 8 where α^* is involved. With $\mathcal{S}(\mathsf{Lim})$ we denote the set of standard terms whose values are limit ordinals, while the analogies to $\mathsf{Fix}(\bar{\alpha})$ and $\mathrm{MS}_{\bar{\alpha}}(\bar{\beta})$ on $\mathcal{S}$ are called $\mathsf{Fix}(\bar{t})$ and $\mathrm{MS}_{\bar{t}}(\bar{u})$.

Definition 12. *For $d \in \mathcal{S}$ and $n \in \mathbb{N}$ we simultaneously define $d\langle n\rangle$ and $d[n]$ by recursion on $\mathcal{S}$:*

$$\mathsf{0}\langle n\rangle := \mathsf{0} \tag{4a}$$

$$\mathsf{S}t\langle n\rangle := t \tag{4b}$$

$$\mathsf{+}(t,u)\langle n\rangle := \mathsf{+}(t,u[n]) \tag{4c}$$

$$\mathsf{P}(\bar{t},u)\langle n\rangle := \mathsf{P}(\bar{t},u[n]) \text{ if } u \in \mathcal{S}(\mathsf{Lim})\setminus\mathsf{Fix}(\bar{t}) \tag{4d}$$

$$\mathsf{P}(\bar{\mathsf{0}},\mathsf{S}u)\langle n\rangle := \mathsf{P}(\bar{\mathsf{0}},u)\times(n+1) \tag{4e}$$

$$\mathsf{P}(\bar{\mathsf{0}},u)\langle n\rangle := u\times(n+1) \tag{4f}$$

$$\mathsf{P}(t_1,\ldots,\mathsf{S}t_i,\bar{\mathsf{0}},\mathsf{S}u)\langle n\rangle := \mathsf{P}(\bar{t},\cdot,\bar{\mathsf{0}})^{n+1}(\mathsf{P}(t_1,\ldots,\mathsf{S}t_i,\bar{\mathsf{0}},u)) \tag{4g}$$

$$\mathsf{P}(t_1,\ldots,\mathsf{S}t_i,\bar{\mathsf{0}},u)\langle n\rangle := \mathsf{P}(\bar{t},\cdot,\bar{\mathsf{0}})^{n+1}(u) \tag{4h}$$

$$\mathsf{P}(t_1,\ldots,t_i,\bar{\mathsf{0}},\mathsf{0})\langle n\rangle := \mathsf{P}(t_1,\ldots,t_i[n],\bar{\mathsf{0}},\mathrm{MS}_{\bar{t},\bar{\mathsf{0}}}(\bar{t})) \tag{4i}$$

$$\mathsf{P}(t_1,\ldots,t_i,\bar{\mathsf{0}},\mathsf{S}u)\langle n\rangle := \mathsf{P}(t_1,\ldots,t_i[n],\bar{\mathsf{0}},\mathsf{P}(t_1,\ldots,t_i,\bar{\mathsf{0}},u)) \tag{4j}$$

$$\mathsf{P}(t_1,\ldots,t_i,\bar{\mathsf{0}},u)\langle n\rangle := \mathsf{P}(t_1,\ldots,t_i[n],\bar{\mathsf{0}},u) \tag{4k}$$

Similar to Definition 8, *$t_i \neq \mathsf{0}$ is required for* (4i)–(4k).

If $d = \mathsf{+}(t,u)$ and $u[n] = \mathsf{S}^i u'$ where i is as large as possible, we put

$$\mathsf{+}(t,u)[n] := \begin{cases} \mathsf{S}^i t & \text{if } u' = \mathsf{0} \\ \mathsf{S}^i\mathsf{+}(t,u') & \text{otherwise.} \end{cases}$$

Moreover, we demand $\mathsf{P}(\bar{\mathsf{0}},\mathsf{S0})[n] := \mathsf{S}^{n+1}\mathsf{0}$ as well as

$$\mathsf{P}(\mathsf{0},\ldots,\mathsf{S0},\bar{\mathsf{0}},\mathsf{0})[n] := \mathsf{P}(\mathsf{0},\ldots,\mathsf{0},\cdot,\bar{\mathsf{0}})^n(\mathsf{S0}),$$

and in all remaining cases we put $d[n] := d\langle n\rangle$.

We now show that this definition is correct and meets our requirements.

Lemma 13. *For $d \in \mathcal{S}(\alpha)$ we have $\mathrm{val}(d\langle n\rangle) = \alpha[n]$ and $d[n] \in \mathcal{S}(\alpha[n])$.*

Proof by induction on $\mathcal{S}$. As the definition of $d\langle n\rangle$ just copies Definition 8, the first statement is obvious by the induction hypothesis. Because in the above definition recursion is only used for standard terms t and u denoting limit ordinals, we have $t[n] \neq \mathsf{0}$ and $u[n] \neq \mathsf{0}$ according to the induction hypothesis and Lemma 9. Since $\mathrm{MS}_{\bar{t},\bar{\mathsf{0}}}(\bar{t})$, being a subterm of one $t_j \in \mathcal{S}$, cannot be $\mathsf{P}(\bar{\mathsf{0}})$, the only possible occurrences of $\mathsf{P}(\bar{\mathsf{0}})$ in $d\langle n\rangle$ are the ones we gave special treatment in the definition of $d[n]$. In both cases it is obvious that $d\langle n\rangle$ and $d[n]$ have the same value and that $d[n]$ is standard.

Now let $d = \mathsf{+}(t,u)$ and let i, u' be as in the definition of $d[n]$. Since d is standard, we know the pair (τ,μ) with $\tau := \mathrm{val}(t)$ and $\mu := \mathrm{val}(u)$ is compatible. The statement obviously holds if $u' = \mathsf{0}$. So let u' denote a limit ordinal, say μ'. By Definition 11 we have to show (τ,μ') is compatible, which can be established by proving $\mu' \leq \mu$. The induction hypothesis yields that $u[n]$ is a standard term with $\mathrm{val}(u[n]) = \mu[n]$. Because of $\mu \in \mathsf{Lim}$ and Lemma 9 we get $\mu' \leq \mu[n] < \mu$. Thus $d[n]$ is standard and has the correct value. In the remaining cases the statement easily follows from the induction hypothesis. □

4. Simulating all Hydra battles below Δ_k

We are now prepared to gradually define the rewrite system $\mathcal{R}$, which is intended to simulate all Hydra battles below Δ_k. As $\mathcal{F}_0$ has too little expressive power for our aims, we have to add new symbols to it. The intended meaning of these symbols will be elucidated in the following definitions.

Definition 14. *The signature $\mathcal{F}$ consists of $\mathcal{F}_0$ enriched by the unary $\bullet$, $\circ$, $[]$, the $k+1$-ary M, the $i+1$-ary J_i, for $1 \leq i \leq k$, the $i+1$-ary Q_{ij}, for $1 \leq j \leq i \leq k$, and the $i+2$-ary R_i, for $1 \leq i \leq k$.*

The notions f^n and $t \times n$ canonically extend to $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Because we focus on the rewrite system $\mathcal{R}$ from now on, we will often drop a subscript $\mathcal{R}$, for example we write $\to$ for $\to_{\mathcal{R}}$. If (AB) is a rule from $\mathcal{R}$ then $\to_{AB}$ indicates a reduction due to this rule.

It will sometimes be necessary to reduce a term to one of its subterms. Since $\mathcal{R}$ is intended to be simplifying, we can introduce, for all symbols $f \in \mathcal{F}$ with arity $n > 0$ and for $1 \leq i \leq n$, the new rules

$$(\mathrm{S}_i f) \quad f(x_1, \ldots, x_n) \to x_i .$$

Lemma 15. *$\mathcal{R}$ is confluent, and each $t \in \mathcal{T}(\mathcal{F})$ reduces in at most $\mathrm{dp}(t)$ steps to its unique normal form $\mathsf{0}$. If u is a subterm of t then $t \xrightarrow{*} u$.*

In order to reduce $d\langle n\rangle$ to $d[n]$ we need the rules

$$(\mathrm{F1}) \quad \mathsf{P}(\bar{\mathsf{0}}) \to \mathsf{S0} \qquad (\mathrm{F2}) \quad +(x, \mathsf{S}y) \to \mathsf{S}+(x, y).$$

Lemma 16. *For $d \in \mathcal{S}$ we have $d\langle n\rangle \xrightarrow{*} d[n]$.*

Proof. The difference between $d\langle n\rangle$ and $d[n]$ for $d = \mathsf{P}(\bar{\mathsf{0}}, \mathsf{S0}, \bar{\mathsf{0}}, \mathsf{0})$ consists of one single $\mathsf{P}(\bar{\mathsf{0}})$ which is replaced by $\mathsf{S0}$. This can be handled by (F1). For $d = \mathsf{P}(\bar{\mathsf{0}}, \mathsf{S0})$ and $n > 0$ we have to show $\mathsf{P}(\bar{\mathsf{0}}) \times n \xrightarrow{+} \mathsf{S}^n\mathsf{0}$, which is done by induction on n. The case $n = 1$ is again established by (F1), while the induction hypothesis yields $\mathsf{P}(\bar{\mathsf{0}}) \times (n+1) \xrightarrow{+} +(\mathsf{S}^n\mathsf{0}, \mathsf{P}(\bar{\mathsf{0}}))$. Now we get

$$+(\mathsf{S}^n\mathsf{0}, \mathsf{P}(\bar{\mathsf{0}})) \to_{\mathrm{F1}} +(\mathsf{S}^n\mathsf{0}, \mathsf{S0}) \to_{\mathrm{F2}} \mathsf{S}+(\mathsf{S}^n\mathsf{0}, \mathsf{0}) \to_{\mathrm{S_1}+} \mathsf{S}^{n+1}\mathsf{0}.$$

It is sufficient for the remaining case $d = +(t, u)$ that

$$+(t, \mathsf{S}^i u') \xrightarrow{*}_{\mathrm{F2}} \mathsf{S}^i+(t, u') \to_{\mathrm{S_1}+} \mathsf{S}^i t$$

is possible in $\mathcal{R}$ for arbitrary i and u'. □

Following an idea of Touzet [16], our rewrite system $\mathcal{R}$ is to regard $\bullet[]^{n+1}d$ with $d \in \mathcal{S}$ as a term which encodes the battle configuration $(\mathrm{val}(d), n)$. Since we want to simulate Hydra battles at full length we intend, for $d \neq \mathsf{0}$, to establish reductions of the kind

$$\bullet[]^{n+1}d \xrightarrow{+} \bullet[]^{n+2}d[n],$$

which can then be carried further, until $\bullet[]^{n+m}\mathsf{0}$ is reached.

For some calculations it will be necessary to facilitate $\bullet[]^{n+1}d \xrightarrow{+} []^{n+1}\bullet^{n+2}d$, so that $\bullet^{n+2}$ can then be used to create $\bullet[]^{n+1}$ in front of subterms of d in order to start recursions. When we reach a point where d can safely be modified into something close to $d\langle n\rangle$, we reduce to this modification and put $\circ$ in front of the term generated from d. This $\circ$ will enable us to create $\bullet[]$ in front of $[]^{n+1}d[n]$, furthermore, recursions like the ones needed for (4c) can be simulated. The required rules are these:

$$\begin{array}{llll} (\mathrm{N1}) & \bullet[]x \to []\bullet\bullet x & (\mathrm{N2}) & []\circ x \to \circ[][]x \\ (\mathrm{N3}) & \circ x \to []x & (\mathrm{N4}) & []x \to \bullet x. \end{array}$$

Lemma 17. *For $n > 0$ and $t \in \mathcal{T}(\mathcal{F})$ we have*

i. $[]^n t \xrightarrow{+} \bullet^n t \xrightarrow{+} t$
ii. $\bullet[]^n t \xrightarrow{+} []^n \bullet^{2^n} t$
iii. $[]^n \circ t \xrightarrow{+} \circ t$ *as well as* $[]^n \circ t \xrightarrow{+} \bullet[]^{n+1} t$.

Proof. For (i) we rely on (N4) and ($\mathrm{S}_1\bullet$), while (ii) is shown by induction on n using

$$\bullet^m []t \xrightarrow{*}_{\mathrm{N1}} []\bullet^{2m} t, \tag{5}$$

which in turn is shown by induction on $m \geq 0$ using (N1). The first part of (iii) is an application of (i), and the second part follows from

$$[]^n \circ t \xrightarrow{+}_{\mathrm{N2}} \circ[]^{2n} t \xrightarrow{*} \circ[]^{n+1} t \to_{\mathrm{N3}} [][]^{n+1} t \to_{\mathrm{N4}} \bullet[]^{n+1} t,$$

where the first step is proved like (5), and the second step relies on $2n \geq n + 1$ and (i). □

To simulate cases like (4c) we have to import $\bullet[]^n$ into standard terms. For $f \in \{\mathsf{S}, +, \mathsf{P}\}$ with arity m and for $1 \leq i \leq m$ we introduce the rules

$$(\mathrm{D}_i f) \quad \bullet f(\bar{x}) \to f(x_1, \ldots, []x_i, \ldots, x_m).$$

Lemma 18. *For t, u, $\bar{t} \in \mathcal{T}(\mathcal{F})$, $n > 0$ and $1 \leq i \leq k + 1$ we have*

i. $\bullet^{n+1}+(t, u) \xrightarrow{+} +(t, \bullet[]^n u)$
ii. $\bullet^{n+1}\mathsf{P}(\bar{t}) \xrightarrow{+} \bullet\mathsf{P}(t_1, \ldots, []^n t_i, \ldots, t_{k+1})$
iii. $\bullet^{n+1}\mathsf{P}(\bar{t}) \xrightarrow{+} \mathsf{P}(t_1, \ldots, \bullet[]^n t_i, \ldots, t_{k+1})$
iv. $\bullet^{n+1}\mathsf{P}(t_1, \ldots, \mathsf{S}t_i, \ldots, t_{k+1}) \xrightarrow{+} \mathsf{P}(t_1, \ldots, \mathsf{S}[]^n t_i, \ldots, t_{k+1})$.

Proof. To settle point (i), (D_2+) is applied $n+1$ times followed by one application of (N4). With little changes, using ($\mathrm{D}_i\mathsf{P}$) and ($\mathrm{D}_1\mathsf{S}$) instead of (D_2+), the remaining points follow. □

As mentioned earlier, importing $\bullet[]^n$ or $[]^n$ shall enable us to locally reduce until a $\circ$ appears. Sometimes this $\circ$ has to be exported, which is done by these rules:

$$(\mathrm{E}_2\mathsf{+}) \qquad \mathsf{+}(x, \circ y) \to \circ\mathsf{+}(x, y)$$

$$(\mathrm{E}_i\mathsf{P}) \quad \mathsf{P}(x_1, \ldots, \circ x_i, \ldots, x_{k+1}) \to \circ\mathsf{P}(\bar{x}) \qquad \text{for } 1 \leq i \leq k+1.$$

In order to simulate (4e)–(4h) we need rewrite rules for a special kind of multiplication and for iterations of P:

$$(\mathrm{R}\mathsf{M}) \qquad \mathsf{M}(\bar{x}, []y) \to \mathsf{+}(\mathsf{M}(\bar{x}, y), \mathsf{P}(\bar{x}, y))$$

$$(\mathrm{R}\mathsf{J}_i) \quad \mathsf{J}_i(x_1, \ldots, []x_i, y) \to \mathsf{P}(\bar{x}, \mathsf{J}_i(\bar{x}, y), \bar{\mathsf{0}}) \qquad \text{for } 1 \leq i \leq k.$$

Lemma 19. *For $\bar{t}$, $u \in \mathcal{T}(\mathcal{F})$, $n > 0$, and $1 \leq i \leq k$ we have*

i. $\mathsf{M}(\bar{t}, []^n u) \xrightarrow{+} \mathsf{P}(\bar{t}, u) \times n$
ii. $\mathsf{J}_i(t_1, \ldots, []^n t_i, u) \xrightarrow{+} \mathsf{P}(t_1, \ldots, t_i, \cdot, \bar{\mathsf{0}})^n(u)$.

Proof. As both statements are treated similarly by induction on n, we only prove (i) in detail. For the start we have

$$\mathsf{M}(\bar{t}, []u) \to_{\mathrm{R}\mathsf{M}} \mathsf{+}(\mathsf{M}(\bar{t}, u), \mathsf{P}(\bar{t}, u)) \to_{\mathrm{S}_2\mathsf{+}} \mathsf{P}(\bar{t}, u),$$

and the induction step is given by

$$\begin{aligned} \mathsf{M}(\bar{t}, []^{n+1}u) \to_{\mathrm{R}\mathsf{M}} & \mathsf{+}(\mathsf{M}(\bar{t}, []^n u), \mathsf{P}(\bar{t}, []^n u)) \\ \xrightarrow{+} \quad & \mathsf{+}(\mathsf{M}(\bar{t}, []^n u), \mathsf{P}(\bar{t}, u)) \quad \xrightarrow{+} \mathsf{+}(\mathsf{P}(\bar{t}, u) \times n, \mathsf{P}(\bar{t}, u)), \end{aligned}$$

where we used Lemma 17.i and the induction hypothesis for the last two reductions. Statement (ii) relies on $(\mathrm{R}\mathsf{J}_i)$ and $(\mathrm{S}_{i+1}\mathsf{J}_i)$ instead of $(\mathrm{R}\mathsf{M})$ and $(\mathrm{S}_2\mathsf{+})$. □

We now present the rules intended to carry out the transformations prescribed by Definition 12. Because it is not easy to distinguish between cases like (4d) and (4k), Lemma 18 and the rules $(\mathrm{E}_i\mathsf{P})$ facilitate both possible transformations. Hence $\mathcal{R}$ is able to simulate wrong battles. In the wrong cases the ordinals denoted are smaller, thus this shall pose no problem to our intended normalising morphism. In the same spirit, transformations like the one needed in (4h) are made possible for arbitrary terms u. Similarly, the u in (4f) is supposed to be an element of $\mathsf{Fix}(\bar{\mathsf{0}})$, whereas the associated rule (H3) treats arbitrary terms beginning with P. The final rules are:

$$\begin{aligned}
&(\mathrm{H1}) & \bullet\mathsf{S}x &\to \circ x \\
&(\mathrm{H2}) & \mathsf{P}(\bar{\mathsf{0}}, \mathsf{S}y) &\to \circ\mathsf{M}(\bar{\mathsf{0}}, y) \\
&(\mathrm{H3}) & \mathsf{P}(\bar{\mathsf{0}}, \mathsf{P}(\bar{x}, y)) &\to \circ\mathsf{M}(\bar{x}, y) \\
&(\mathrm{H}_i4) & \mathsf{P}(x_1, \ldots, \mathsf{S}x_i, \bar{\mathsf{0}}, y) &\to \circ\mathsf{J}_i(\bar{x}, y) \\
&(\mathrm{H}_i5) & \mathsf{P}(x_1, \ldots, \mathsf{S}x_i, \bar{\mathsf{0}}, \mathsf{S}y) &\to \circ\mathsf{J}_i(\bar{x}, \mathsf{P}(x_1, \ldots, \mathsf{S}x_i, \bar{\mathsf{0}}, y)) \\
&(\mathrm{H}_{ij}6) & \bullet\mathsf{P}(x_1, \ldots, x_i, \bar{\mathsf{0}}, \mathsf{0}) &\to \mathsf{Q}_{ij}(x_1, \ldots, \bullet x_i, x_j) \\
&(\mathrm{R}\mathsf{Q}_{ij}) & \mathsf{Q}_{ij}(x_1, \ldots, \circ x_i, y) &\to \circ\mathsf{P}(\bar{x}, \bar{\mathsf{0}}, y) \\
&(\mathrm{H}_i7) & \bullet\mathsf{P}(x_1, \ldots, x_i, \bar{\mathsf{0}}, \mathsf{S}y) &\to \mathsf{R}_i(x_1, \ldots, \bullet x_i, x_i, y) \\
&(\mathrm{R}\mathsf{R}_i) & \mathsf{R}_i(x_1, \ldots, \circ x_i, y, z) &\to \circ\mathsf{P}(\bar{x}, \bar{\mathsf{0}}, \mathsf{P}(x_1, \ldots, x_{i-1}, y, \bar{\mathsf{0}}, z))
\end{aligned}$$

for i and j with $1 \leq j \leq i \leq k$.

Proposition 20. *For* $d \in \mathcal{S}$ *with* $d \neq \mathsf{0}$ *and* $n = m + 1$ *we have*

$$\bullet[]^n d \xrightarrow{+} []^n \bullet^{n+1} d \xrightarrow{+} []^n \circ d[m] \xrightarrow{+} \begin{cases} \circ d[m] \\ \bullet[]^{n+1} d[m]. \end{cases}$$

Thus, by Lemma 13, $\mathcal{R}$ is able to simulate Hydra battles for all configurations (α, n) with $0 < \alpha < \Delta_k$ at full length:

$$\bullet[]^{m+1} d \xrightarrow{+} \bullet[]^{m+2} d[m] \xrightarrow{+} \bullet[]^{m+3} d[m][m+1] \xrightarrow{+} \ldots \xrightarrow{+} \bullet[]^{m+l} \mathsf{0} \xrightarrow{+} \mathsf{0}.$$

It is obviously not wise to strive after a similar result for $d = \mathsf{0}$.

Proof by induction on $\mathcal{S}$. As mentioned earlier, a close look at Definition 12 shows that whenever $v[m]$ (with v being a subterm of d) is used to define $d[m]$ we have $v \neq \mathsf{0}$. This observation enables us to rely on the induction hypothesis when required.

A quick glance at Lemmata 16 and 17 assures us that it suffices to show

$$\bullet^{n+1} d \xrightarrow{+} \circ d\langle m \rangle.$$

For (4b) we employ Lemma 17.i to get

$$\bullet^{n+1} \mathsf{S} t \xrightarrow{+} \bullet \mathsf{S} t \rightarrow_{\mathrm{H1}} \circ t = \circ d\langle m \rangle,$$

while (4c) is handled by Lemma 18 and the induction hypothesis:

$$\bullet^{n+1} \mathsf{+}(t, u) \xrightarrow{+} \mathsf{+}(t, \bullet[]^n u) \xrightarrow{+} \mathsf{+}(t, \circ u[m]) \rightarrow_{\mathrm{E}_2\mathsf{+}} \circ\mathsf{+}(t, u[m]).$$

The treatment of (4d) and (4k) is very close to this, relying on ($\mathrm{E}_i\mathsf{P}$) instead of ($\mathrm{E}_2\mathsf{+}$). For (4f) we note that $d = \mathsf{P}(\bar{\mathsf{0}}, v)$ with $v \in \mathsf{Fix}(\bar{\mathsf{0}})$ holds. According to Definition 7, v is some $\mathsf{P}(\bar{t}, u)$. Applying Lemmata 18.iii and 17.i (both twice) and Lemma 19 we see

$$\begin{aligned} \bullet^{n+1} \mathsf{P}(\bar{\mathsf{0}}, \mathsf{P}(\bar{t}, u)) \xrightarrow{+} \quad & \mathsf{P}(\bar{\mathsf{0}}, \mathsf{P}(\bar{t}, \bullet[]^n u)) \xrightarrow{+} \mathsf{P}(\bar{\mathsf{0}}, \mathsf{P}(\bar{t}, []^n u)) \\ \rightarrow_{\mathrm{H3}} & \circ\mathsf{M}(\bar{t}, []^n u) \qquad \xrightarrow{+} \circ(\mathsf{P}(\bar{t}, u) \times (m+1)). \end{aligned}$$

The proof of (4e) is very similar (using (H2)) and therefore left out here. For (4h) we need Lemmata 18.iv and 19.ii:

$$\begin{aligned} \bullet^{n+1} \mathsf{P}(t_1, \ldots, \mathsf{S} t_i, \bar{\mathsf{0}}, u) \xrightarrow{+} \quad & \mathsf{P}(t_1, \ldots, \mathsf{S}[]^n t_i, \bar{\mathsf{0}}, u) \\ \rightarrow_{\mathrm{H}_i 4} & \circ\mathsf{J}_i(t_1, \ldots, []^n t_i, u) \\ \xrightarrow{+} \quad & \circ\mathsf{P}(t_1, \ldots, t_i, \cdot, \bar{\mathsf{0}})^{m+1}(u), \end{aligned}$$

while take care of (4g) in much the same way replacing ($\mathrm{H}_i 4$) by ($\mathrm{H}_i 5$). The treatment of $\mathrm{MS}_{\bar{t},\bar{\mathsf{0}}}(\bar{t})$ in (4i) requires a new idea, since $\mathcal{R}$ cannot know which one of the t_j has $\mathrm{MS}_{\bar{t},\bar{0}}(\bar{t})$ as a subterm. For each j with $1 \leq j \leq i$ and for $t_i \neq \mathsf{0}$, by virtue of Lemma 18.ii and the induction hypothesis, we can show

$$\begin{aligned} \bullet^{n+1}\mathsf{P}(t_1, \ldots, t_i, \bar{\mathsf{0}}, \mathsf{0}) &\xrightarrow{+} \bullet\mathsf{P}(t_1, \ldots, []^n t_i, \bar{\mathsf{0}}, \mathsf{0}) && (6a)\\ &\xrightarrow{+} \mathsf{Q}_{ij}(t_1, \ldots, \bullet[]^n t_i, t_j) && (6b)\\ &\xrightarrow{+} \mathsf{Q}_{ij}(t_1, \ldots, \circ t_i[m], t_j) \\ &\to_{\mathsf{RQ}_{ij}} \circ\mathsf{P}(t_1, \ldots, t_i[m], \bar{\mathsf{0}}, t_j). && (6c) \end{aligned}$$

To get from (6a) to (6b), we make use of ($\mathrm{H}_{ij}6$) and, in case of $j = i$, Lemma 17.i. Now we may incorporate Lemma 15 to reduce the second t_j in (6c) to any of its subterms. Since $\mathrm{MS}_{\bar{t},\bar{0}}(\bar{t})$ is a subterm of some t_j, we reach our goal. When we use (H_i7) and (RR_i) instead of ($\mathrm{H}_{ij}6$) and (RQ_{ij}) the result for (4j) is easily obtained. □

Corollary 21. *If $\mathcal{R}$ terminates then $\mathrm{Dl}_{\mathcal{R}}$ is not $<\Delta_k$-recursive since it eventually dominates all $<\Delta_k$-recursive functions.*

Proof. As mentioned earlier, $\Delta_k = \psi(1, \bar{0})$ holds where $\bar{0}$ has length $k + 1$. According to Theorem 10 the function which maps n to the length of the Hydra battle for $c_n = (\Delta_k[n], n + 1)$ eventually dominates all $<\Delta_k$-recursive functions. Taking a short digression from our fixed k to $k + 1$, we remember $\psi_{0,\bar{\alpha}} = \psi_{\bar{\alpha}}$ and see that

$$\Delta_k[n] = \psi(0, \cdot, \bar{0})^{n+1}(0) = \psi(\cdot, \bar{0})^{n+1}(0)$$

holds where the first ψ is $k+2$-ary and the second one is, as usual, $k+1$-ary. Since $t_n := \bullet[]^{n+2}\mathsf{P}(\cdot, \bar{\mathsf{0}})^n(\mathsf{S0})$ encodes c_n, the length of the battle for c_n is majorized by $\mathrm{dl}_{\mathcal{R}}(t_n)$. Because $\mathrm{dp}(t_{n+1}) = \mathrm{dp}(t_n) + 2$ holds, $\mathrm{Dl}_{\mathcal{R}}$ grows too fast to be $<\Delta_k$-recursive. □

Since the complexity of a rewrite system terminating via MPO, KBO or LPO is bounded by a multiple recursive function, termination of $\mathcal{R}$ cannot be established by one of these orderings. We want to demonstrate where LPO fails.

$\mathcal{R}$ owes much of its strength to the interplay between $\bullet$, $[]$, and $\circ$ on the one hand and $+$ and P on the other hand. LPO is not able to prove termination of any rewrite system containing the rules (N3), (N4), ($\mathrm{D}_1\mathsf{P}$), and ($\mathrm{E}_1\mathsf{P}$), since the first three rules require $\circ \succ [] \succ \bullet \succ \mathsf{P}$ while the fourth rule implies $\mathsf{P} \succ \circ$.

5. Proof of total termination

We order $\mathcal{P} := (\Delta_k \backslash \{0\}) \times \omega \times \omega$ by the lexicographical combination of the usual $>$ on these sets of ordinals, called $\succ$. Hence $(\mathcal{P}, \succ)$ is a well-ordering. We identify $(\alpha, 0, 0) \in \mathcal{P}$ with α to avoid lengthy notations. Thus $\alpha > \beta$ implies $\alpha \succ (\beta, m, n) \succeq \beta$.

We now define interpreting functions for all symbols of our rewrite system. The homomorphic morphism based on these functions will establish total termination for $\mathcal{R}$ via Theorem 4. Note that this homomorphic morphism may not be an interpretation for $\mathcal{R}$ in the sense of (3), it is rather used to define such an interpretation via Theorem 4. In spite of this, we call $[f]$ the interpretation of $f \in \mathcal{F}$.

Definition 22. *For arbitrary elements $p = (\alpha, m, n)$, $p_l = (\alpha_l, m_l, n_l)$ (with $1 \le l \le k$), $q = (\beta, m', n')$ and $r = (\gamma, m'', n'')$ of $\mathcal{P}$ we put*

$$\begin{aligned}
[\mathsf{0}] &:= 1 \\
[\mathsf{S}](p) &:= \alpha + 1 \\
[\mathsf{+}](p, q) &:= \alpha \oplus \beta \oplus \beta \\
[\mathsf{P}](\bar{p}, q) &:= \psi(\bar{\alpha}, \beta) \\
[\bullet](p) &:= (\alpha, m, m + n + 1) \\
[[]](p) &:= (\alpha, 2m + 2, n) \\
[\circ](p) &:= \alpha + 1 \\
[\mathsf{M}](\bar{p}, q) &:= \psi(\bar{\alpha}, \beta) \cdot (2m' + 1) \\
[\mathsf{J}_i](p_1, \dots, p_i, q) &:= \psi(\alpha_1, \dots, \alpha_i, \cdot, \bar{1})^{m_i + 1}(\beta) \\
[\mathsf{Q}_{ij}](p_1, \dots, p_i, q) &:= \psi(\alpha_1, \dots, \max\{\alpha_j, \beta\}, \dots, \alpha_i, \bar{1}) \\
[\mathsf{R}_i](p_1, \dots, p_i, q, r) &:= \psi(\alpha_1, \dots, \max\{\alpha_i, \beta\}, \bar{1}, \gamma + 1).
\end{aligned}$$

Note that some components of q, r and p_l are never used. Furthermore, the interpretations of the symbols from $\mathcal{F}_0$ intentionally forget the two trailing components of p, q and p_l. Hence, for $t, u, \bar{t} \in \mathcal{T}(\mathcal{F})$, we have

$$\begin{aligned}
&[\mathsf{S}[]t] = [\mathsf{S}t] = [\circ t] = [\circ[]t], \\
&[\mathsf{+}([]t, u)] = [\mathsf{+}(t, []u)] = [\mathsf{+}(t, u)], \qquad (7) \\
&[\mathsf{P}(t_1, \dots, []t_i, \dots, t_{k+1})] = [\mathsf{P}(\bar{t})].
\end{aligned}$$

This remains true when [] is replaced by $\bullet$.

The use of m_i in the interpretation of J_i is based on the interpretation of []. If $n > 0$ and $[t_i] = (\alpha_i, m', m'')$ then $[[]^n t_i]$ reduces to $p' := (\alpha_i, n', m'')$ with n' considerably larger than n. Thus n' can be used to give a bound for the effects n will have on the possible reductions of terms containing $[]^n t_i$. For example, one glance at Lemma 19 urges us to satisfy

$$[\mathsf{J}_i(t_1, \dots, []^n t_i, u)] \succ [\mathsf{P}(t_1, \dots, t_i, \cdot, \bar{\mathsf{0}})^n(u)]$$

for all closed terms $t_1, \dots, t_{i-1}, u$. Since $[\mathsf{P}(\bar{t}, \cdot, \bar{\mathsf{0}})^{n'}(u)] \succ [\mathsf{P}(\bar{t}, \cdot, \bar{\mathsf{0}})^n(u)]$ holds (because $[\mathsf{P}]$ has the subterm property), it seems reasonable to require $[t_l] = p_l$ and $[u] = q$ that $[\mathsf{J}_i](p_1, \dots, p_{i-1}, p', q)$ shall be close to $[\mathsf{P}(\bar{t}, \cdot, \bar{\mathsf{0}})^{n'}(u)]$. This is exactly what $[\mathsf{J}_i]$ does. In the same way $[\mathsf{M}]$ is defined.

The definitions of $[\mathsf{Q}_{ij}]$ and $[\mathsf{R}_i]$ reflect the duplication of x_j and x_i in (H_{ij}6) and (H_i7), respectively. Here we profit from Theorem 4, since taking the maximum is a violation of strict monotonicity.

Lemma 23. *The morphism $[\cdot]$ is monotone and has the subterm property.*

Proof. Due to Lemma 2 it suffices to show that the functions interpreting the symbols of $\mathcal{F}$ are monotone and have the subterm property. We start with the latter one.

The proof for $[\mathsf{+}]$ uses the fact that the first components of elements of $\mathcal{P}$ are larger than 0, and all interpretations involving ψ rely on its subterm property.

The monotonicity is obvious for all interpretations from $[\mathsf{0}]$ to $[\circ]$, and a moment's reflection establishes it for $[\mathsf{Q}_{ij}]$ and $[\mathsf{R}_i]$. The result for $[\mathsf{M}]$ and $[\mathsf{J}_i]$ relies on Lemma 6. □

Note that only the interpreting functions of $\bullet$ and $[]$ are strictly monotone, as the other functions ignore the third component of their first arguments.

Proposition 24. *$\mathcal{R}$ is totally terminating.*

Proof. Because of Theorem 4 and Lemma 23 it remains to prove that $[\cdot]$ normalizes $\mathcal{R}$. Let $\sigma\colon \mathcal{X} \to \mathcal{T}(\mathcal{F})$ be given. We denote the values of σ for x, x_i (with $1 \leq i \leq k+1$), y and z by t, t_i, u and v with interpretations (α, m, n), (α_i, m_i, n_i), (β, m', n') and (γ, m'', n''). Our goal will be established by showing

$$\forall (l \to r) \in \mathcal{R} \quad p := [l\sigma] \succ [r\sigma] =: p'.$$

For all rules $(\mathrm{S}_i f)$ we can fall back upon the subterm property of the interpretations in question. The subterm properties of $[\bullet]$ and $[\circ]$, sometimes in combination with (7), settle (N2), (N3), $(\mathrm{D}_i f)$, (H1), $(\mathrm{H}_{ij}6)$ and $(\mathrm{H}_i 7)$. For example, we can treat (N3) by $p = [\circ t] = [\circ[]t] \succ [[]t] = p'$, and

$$\begin{aligned} p &= [\bullet\mathsf{P}(t_1, \ldots, t_i, \bar{\mathsf{0}}, \mathsf{S}u)] \succ [\mathsf{P}(t_1, \ldots, t_i, \bar{\mathsf{0}}, \mathsf{S}u)] \\ &= [\mathsf{P}(t_1, \ldots, \bullet t_i, \bar{\mathsf{0}}, \mathsf{S}u)] = [\mathsf{R}_i(t_1, \ldots, \bullet t_i, t_i, u)] = p' \end{aligned}$$

suffices for $(\mathrm{H}_i 7)$. Regarding (N1) we get

$$p = (\alpha, 2m+2, 2m+n+3) \succ (\alpha, 2m+2, 2m+n+2) = p',$$

and likewise (N4) uses $(\alpha, 2m+2, n) \succ (\alpha, m, m+n+1)$, while (F2) and $(\mathrm{E}_2\mathsf{+})$ both rely on $\alpha \oplus (\beta+1) \oplus (\beta+1) > \alpha \oplus \beta \oplus \beta + 1$. The strict monotonicity of ψ and Lemma 6.ii yield the result we are after for (F1), $(\mathrm{E}_i\mathsf{P})$, and (H2), while (i) and (iii) of Lemma 6 settle $(\mathrm{R}\mathsf{M})$ and $(\mathrm{R}\mathsf{J}_i)$. The subterm property and the strict monotonicity of ψ, joined with (ii) and (iv) of Lemma 6, provide us with everything we need for (H3), $(\mathrm{H}_i 4)$ and $(\mathrm{H}_i 5)$. It remains to take care of $(\mathrm{R}\mathsf{Q}_{ij})$ and $(\mathrm{R}\mathsf{R}_i)$. For the former we have $p \succeq \psi(\alpha_1, \ldots, \alpha_i + 1, \bar{1})$ by monotonicity, and $p \succ \alpha_1, \ldots, \alpha_i, 1, \beta$ by the subterm property of ψ. Under these conditions Lemma 5.iii implies $p \succ p'$. Via similar reasoning we can handle $(\mathrm{R}\mathsf{R}_i)$. □

6. Conclusion

Combining Corollary 21, Proposition 24, and the aforementioned result of Weiermann we arrive at our conclusion.

Theorem 25. *For every $\alpha < \bar{\theta}_{\Omega^\omega}(0)$ there exists a simply (and even totally) terminating finite rewrite system whose complexity is not $<\alpha$-recursive as it eventually dominates all $<\alpha$-recursive functions.*

On the other hand, for every simply terminating finite rewrite system there exists $\alpha < \bar{\theta}_{\Omega^\omega}(0)$ *such that the complexity of the rewrite system is dominated by a* $<\alpha$*-recursive function.*

As if multiple recursion isn't complex enough, we see that simple termination has a surprising computational power. We wonder whether there is a natural class of simplification orderings whose corresponding rewrite systems can have complexities beyond multiple recursion.

Acknowledgement. I wish to express my sincere thanks to Andreas Weiermann for invaluable help and lots of suggestions.

References

1. Buchholz, W.: Fundamentalfolgen für $\bar{\theta}(\nu)$, Handwritten notes in German, 1980 See also the unpublished manuscript "Ordinal notations and Fundamental Sequences," Available at `http://www.mathematiu.uni-muenchem.de/~buchholz`.
2. Buchholz, W., Cichon, E.A., Weiermann, A.: A uniform approach to fundamental sequences and hierarchies. Math. Logic Quarterly **40**, 273–286 (1994)
3. Cichon, E.A.: Termination orderings and complexity characterisations. In: P. Aczel, H. Simmons, S.S. Wainer (eds), Proof Theory, pp. 171–193. Cambridge University Press, 1992
4. Dershowitz, N., Jouannaud, J.-P.: Rewrite systems. In: J. van Leeuwen (ed.), Handbook of Theoretical Computer Science. Vol. B, pp. 243–320. Elsevier, 1990
5. Dershowitz, N., Jouannaud, J.-P., Klop, J.W.: Problems in rewriting III. In: J. Hsiang (ed.), Proceedings of the 6th RTA, vol. 914 of Lect. Notes Comput. Sci. pp. 457–471. Springer-Verlag, Berlin, 1995
6. Dershowitz, N., Treinen, R.: An on-line problem database. In T. Nipkow (ed.), Proceedings of the 9th RTA, vol. 1379 of Lect. Notes Comput. Sci. pp. 332–342. Springer-Verlag, Berlin, 1998. The list is available at `http://www.lsv.ens-cachan.fr/rtaloop`
7. Ferreira, M.D.C.F., Zantema, H.: Total termination of term rewriting. In: C. Kirchner (ed.), Proceedings of the 5th RTA, vol. 690 of Lect. Notes Comput. Sci. pp. 213–227, Springer-Verlag, Berlin, 1993
8. Hofbauer, D. Termination Proofs and Derivation Lengths in Term Rewriting Systems. Ph.D. thesis, Technische Universität Berlin, 1991
9. Hofbauer, D.: Termination proofs by multiset path orderings imply primitive recursive derivation lengths. Theoret. Comput. Sci. **105** (1), 129–140 (1992)
10. Kirby, L., Paris, J.: Accessible independence results for Peano arithmetic. Bull. London Math. Soc. **14**, 285–293 (1982)
11. Pohlers, W.: Proof Theory. No. 1407 in Lect. Notes Math. Springer-Verlag, Berlin, 1989
12. Rose, H.E.: Subrecursion: Functions and Hierarchies. Clarendon Press, Oxford, 1984
13. Schmidt, D.: Well-Partial Orderings and their Maximal Order Types. Habilitationsschrift, Fakultät für Mathematik der Ruprecht-Karl-Universität Heidelberg, 1979
14. Schütte, K.: Proof Theory. Springer-Verlag, Berlin, 1977
15. Touzet, H.: *Propriétés combinatoires pour la terminaison de systèmes de réécriture*. Ph.D. thesis, Université Henri Poincaré, Nancy, 1997 `http://www.lifl.fr/~touzet/Publications/publi.html`. In French.
16. Touzet, H.: Encoding the Hydra battle as a rewrite system. In: L. Brim, J. Gruska, J. Zlatuška (eds), Mathematical Foundations of Computer Science, vol. 1450 of Lect. Notes Comput. Sci., pp. 267–276. Springer-Verlag, Berlin, 1998
17. Veblen, O.: Continuous increasing functions of finite and transfinite ordinals. Trans. Amer. Math. Soc. **9**, 280–292 (1908)

18. Wainer, S.S.: Ordinal recursion, and a refinement of the extended Grzegorczyk hierarchy. J. Symbolic Logic **37** (2), 281–292 (1972)
19. Weiermann, A.: Complexity bounds for some finite forms of Kruskal's theorem. J. Symbolic Computation **18** (5), 463–488 (1994)
20. Weiermann, A.: Termination proofs for term rewriting systems by lexicographic path orderings imply multiply recursive derivation lengths. Theoret. Comput. Sci. **139** (1–2), 355–362 (1995)
21. Weiermann, A.: Ordinalzahlbezeichnungssysteme und subrekursive Hierarchien, 1997, Handwritten notes in German.