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Recursive sequences attached to modular representations of finite groups

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The core of a finite-dimensional modular representation M of a finite group G is its largest non-projective summand. We prove that the dimensions of the cores of $M^{\otimes n}$ have algebraic Hilbert series when M is Omega-algebraic, in the sense that the non-projective summands of $M^{\otimes n}$ fall into finitely many orbits under the action of the syzygy operator Ω. Similarly, we prove that these dimension sequences are eventually linearly recursive when M is what we term Ω^+-algebraic. This partially answers a conjecture by Benson and Symonds. Along the way, we also prove a number of auxiliary permanence results for linear recurrence under operations on multi-variable sequences.

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Introduction

Let G be a finite group, k a field whose characteristic p divides $|G|$, and mod kG the category of G-modules, finite-dimensional over k. The paper [3] studies the asymptotic behavior as $n \to \infty$ of the *cores* of the tensor powers $M^{\otimes n}$ for $M \in \text{mod } kG$, where by definition

$$core(M) = core_G(M) := \text{ the largest non-projective summand of } M.$$

The initial motivation for the present paper was [3, Conjecture 13.3], stating that the dimensions

$$c_n^G(M) := \dim core\left(M^{\otimes n}\right) \tag{0.1}$$

form an eventually linearly recursive sequence. A likely more tractable version is [3, Conjecture 14.2], which restricts the class of G-modules under consideration. To make sense of that statement, recall (e.g. [1, §1.5] or [3, discussion following Lemma 2.7]) that for a finite-dimensional G-module M one writes

- ΩM for the kernel of a *projective cover* $P \to M$;
- $\Omega^{-1}M$ for the cokernel of an *injective hull* $M \to I$.

These are not quite endofunctors on the category of modules, because projective/injective covers are not functorial, but they do descend to endofunctors of the *stable module category*

$$\text{stmod } kG := \text{mod } kG/\text{proj},$$

defined as having the same objects as the category mod kG of finite-dimensional G-modules and whose morphisms are obtained by annihilating those module morphisms that factor through projective (or equivalently, injective) objects; see e.g. [14, Chapter I].

In stmod kG Ω and Ω^{-1} are indeed (as the notation suggests) mutually inverse functors:

$$\Omega\left(\Omega^{-1}M\right) \cong core(M) \cong \Omega^{-1}\left(\Omega M\right)$$

already holds in mod kG, and stabilization has the effect of identifying M and its core. Given that

- we often ignore projective summands, as the problems under consideration require;
- and $\Omega^{\pm 1}$ are endofunctors of stmod kG,

we will often treat them as functors, referring to them as such, composing them, etc. With this in place, recall [3, Definition 14.1]:

Definition 0.1. A G-module is *Omega-algebraic* (or Ω-algebraic) if the non-projective indecomposable summands of the various tensor powers $M^{\otimes n}$ fall into finitely many orbits under the action of $\mathbb{Z}$ via Ω. ♦

This means that the functor $M \otimes -$ can be recast as a matrix T with entries in the Laurent polynomial ring $\mathbb{Z}[\Omega^{\pm 1}]$. We can restrict this further (see Section 4) for a fuller discussion:

Definition 0.2. $M \in \operatorname{mod} kG$ is *Omega*$^+$ *(or* Ω^+*)-algebraic* if

- it is Ω-algebraic in the sense of Definition 0.1, and
- the representatives

$$N_1 = k,\ N_2,\ \cdots$$

  for the Ω-orbits of the non-projective indecomposable summands of $M^{\otimes n}$, $n \in \mathbb{N}$ can be chosen so that the entries of the matrix T given by $M \otimes -$ are polynomials in $\mathbb{N}[\Omega]$ (rather than *Laurent* polynomials).

We define *Omega*$^-$*-algebraic* modules similarly, substituting $\mathbb{N}[\Omega^{-1}]$ for $\mathbb{N}[\Omega]$ above. ♦

Our main results pertaining to these classes of modules are as follows. First, regarding [3, Conjecture 14.2], we have (Theorem 4.4 and Corollary 4.5)

Theorem. *Let $M \in \operatorname{mod} kG$. The sequence (0.1) is eventually linearly recursive if M is either Ω^+ or Ω^--algebraic.*

Consequently, the same holds if M is of the form $\Omega^d N$ for Ω-algebraic N and sufficiently large (or sufficiently small) $d \in \mathbb{Z}$. ■

A sequence $\mathbf{a} = (a_n)$ is eventually linearly recursive precisely when its *Hilbert series*

$$H_{\mathbf{a}}(t) = \sum_n a_n t^n$$

is rational (see Section 1 below for a lengthier discussion of linear recursion). This condition can be weakened in various ways, e.g. by requiring that $H_{\mathbf{a}}$ be only *algebraic* (i.e. that it satisfy a polynomial equation with coefficients in the field of rational functions in t). To return to G-modules, for Ω- (rather than $\Omega^{\pm}$-)algebraic modules we have Theorem 4.6:

Theorem. *For an Ω-algebraic $M \in \operatorname{mod} kG$ the sequence (0.1) has algebraic Hilbert series.* ■

This will require a bit of a detour, as we need various results to the effect that recursion and related properties (e.g. having an algebraic Hilbert series) are invariant under various constructions involving sequences or, more generally, *multi-sequences* (§1.1). Such results are presumably of some independent interest, and they appear throughout Sections 1 and 2. A small sampling (Definition 1.13 and Proposition 1.15):

Proposition. *Consider*

- *an eventually-linearly-recursive sequence $(P_n)_n$ of polynomials in x over a field $\mathbb{K}$;*
- *an eventually-linearly-recursive sequence $\mathbf{a} = (a_n)_n$ in $\mathbb{K}$,*

and denote by

$$P \triangleright \mathbf{a} = \sum_k c_k a_k$$

*the **convolution** of a polynomial $P(x) = \sum c_k x^k$ with $\mathbf{a}$.*

Then, the sequence $(P_n \triangleright \mathbf{a})_n$ is eventually linearly recursive. ■

Such convolution operations feature prominently in the proofs of the above-mentioned theorems, and they form the focus of Section 2 and part of Section 1.

In Section 3 we prove that various generalizations of $c_n^G(M)$ are eventually linearly recursive or algebraic, broadening the scope of the discussion. Specifically, an aggregate of Theorem 3.8 and Theorem 3.9 reads

Theorem. *Let $M \in \operatorname{mod} kG$ and F a functor from $\operatorname{mod} kG$ to finite-dimensional vector spaces that is either exact or of the form $\operatorname{Hom}_G(S, -)$ for a simple G-module S.*

(a) *If $(P_n)_n$ is an eventually linearly recursive sequence of polynomials in $\mathbb{N}[x]$ then the sequences*

$$n \mapsto \dim F\left(P_n \Omega M\right) \quad \text{or} \quad \dim F\left(P_n \Omega^{-1} M\right)$$

are eventually linearly recursive.

(b) *On the other hand, if P_n are **Laurent** polynomials, the same sequences have algebraic Hilbert series.* ■

Finally, Section 5 contains examples of sequences $c_n^G(M)$ and analogues for specific modules/groups, illustrating the main results outlined above.

Some notation

We write $\mathbb{N}$ for $\mathbb{Z}_{\geq 0}$. Throughout,

- G is a finite group;
- k is a field of positive characteristic p (typically dividing $|G|$; otherwise most of the discussion below will be trivial);
- Vect (respectively Vect^f) means (finite-dimensional) k-vector spaces,
- and as in the Introduction, mod kG denotes the category of k-finite-dimensional G-modules.

We write $\ell(M)$ for the length of a module M, so $\ell = \dim$ for plain vector spaces.

Recall the quantities $c_n^G(M)$ from the Introduction (0.1). Prompted by [3, Remark 2.5(i)] on the resilience of the invariant $\gamma_G(M)$ to replacing $c_n^G(M)$ with the length or the length of the socle of *core* $(M^{\otimes n})$ we write

- $d_n^G(M)$ for the length of the socle of *core* $(M^{\otimes n})$;
- $l_n^G(M)$ for the length of *core* $(M^{\otimes n})$;
- $s_n^G(M)$ for the number of indecomposable summands of *core* $(M^{\otimes n})$.

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1. Generalities on recursion

1.1. Multi-sequences

[20, Chapter 4] is a good reference for the material on linear recursive sequences needed below. Since we are interested in sequences (and sometimes multi-sequences, i.e. $a_{m,n,\cdots}$) of either complex numbers or polynomials, it will be convenient to keep in mind that most of the discussion below makes sense over commutative rings.

Definition 1.1. Let R be a commutative ring and r a positive integer.

The elements of the product space $X = R^{\mathbb{N}^r}$ are *R-valued r-sequences*, or *r-dimensional (multi-) sequences*. When we do not specify r we use the phrase *multi-sequence*.

For $1 \leq i \leq r$, the i^{th} *shift* S_i on X is the operator that shifts the i^{th} index of a multi-sequence up (and hence shifts the multi-sequence "leftward" along the i^{th} direction).

When $r = 1$ (i.e. we work with plain sequences) we will often write S for the only shift operator S_1.

For a tuple $\mathbf{n} = (n_1, \cdots, n_r)$ of non-negative integers, we write $S^{\mathbf{n}}$ for the product

$$S_1^{n_1} \cdots S_r^{n^r}. \quad \blacklozenge$$

To illustrate:

Example 1.2. For $r = 1$, given a sequence $\mathbf{a} = (a_n)_n$, its shift $S\mathbf{a}$ is $(a_{n+1})_n$.

On the other hand, for $r = 2$ and $\mathbf{a} = (a_{m,n})_{m,n}$ we have

$$S_2\mathbf{a} = (a_{m,n+1})_{m,n}. \quad \blacklozenge$$

The fundamental result, to be used extensively below, is the characterization of (eventually) linear recursive sequences given in [20, Theorem 4.1.1]. Paraphrasing that result slightly, extending it to algebraically-closed fields more general than $\mathbb{C}$ (e.g. [15, Theorems 4.1 and 4.3]), and supplementing it with a shift criterion, we have

Theorem 1.3. *Let $\mathbf{a} = (a_n)_n$ be a sequence valued in a field $\mathbb{K}$. The following conditions are equivalent.*

(a) *The Hilbert series*

$$H_{\mathbf{a}}(t) := \sum_n a_n t^n$$

attached to the sequence is a rational function in t.

(b) *The sequence is eventually linear recursive, in the sense that*

$$a_{n+T} + c_1 a_{n+T-1} + \cdots + c_T a_n = 0$$

for some complex numbers c_i and sufficiently large n.

(c) *There are polynomials p_s and elements γ_s of the algebraic closure $\overline{\mathbb{K}} \supseteq \mathbb{K}$ such that*

$$a_n = \sum_s p_s(n)\gamma_s^n$$

for sufficiently large n.

(d) *The vector subspace of $\mathbb{K}^{\mathbb{N}}$ spanned by the shifts $S^d\mathbf{a}$, $d \in \mathbb{N}$ is finite-dimensional.*
$\blacksquare$

We will soon see that for multi-sequences things are more complicated. For instance, the rationality of the Hilbert series ((a) of Theorem 1.3) and the finite-dimensionality of the space of shifts (condition (d)) part ways.

For a start, we identify a particularly well-behaved class of multi-sequences: the *multi-C-finite* ones of [22, §2.2.2].

Theorem 1.4. *Let* $\mathbf{a} = (a_{\mathbf{n}})_{\mathbf{n}=(n_1,\cdots,n_r)}$ *be a multi-sequence valued in a field* $\mathbb{K}$. *The following conditions are equivalent.*

(a) *The Hilbert series*

$$H_{\mathbf{a}}(t_1,\cdots,t_r) := \sum_{\mathbf{n}} a_{\mathbf{n}} t_1^{n_1}\cdots t_r^{n_r}$$

is a function of the form

$$\frac{P(t_1,\cdots,t_r)}{Q_1(t_1)\cdots Q_r(t_r)}$$

for an r*-variable polynomial* P *and single-variable polynomials* Q_i.

(b) *There are* r*-variable polynomials* p_s *and tuples*

$$\gamma_s = (\gamma_{s,1},\cdots,\gamma_{s,r}) \in \overline{\mathbb{K}}^r$$

such that

$$\begin{aligned} a_{\mathbf{n}} = a_{n_1,\cdots,n_r} &= \sum_s p_s(n_1,\cdots,n_r)\gamma_s^{\mathbf{n}} \\ &= \sum_s p_s(n_1,\cdots,n_r)\gamma_{s,1}^{n_1}\cdots\gamma_{s,r}^{n_r} \end{aligned} \tag{1.1}$$

for all but finitely many tuples $\mathbf{n} = (n_1,\cdots,n_r)$.

(c) *The vector subspace of* $\mathbb{K}^{\mathbb{N}^r}$ *spanned by the shifts* $S^{\mathbf{n}}\mathbf{a}$, $\mathbf{n} \in \mathbb{N}^r$ *is finite-dimensional.*

(d) *For each* $1 \le i \le r$, *the vector subspace of* $\mathbb{K}^{\mathbb{N}^r}$ *spanned by the shifts* $S_i^n\mathbf{a}$, $n \in \mathbb{N}$ *is finite-dimensional.*

Proof. **(a) ⇒ (b).** By [13, Theorem 1] we may as well assume that the polynomials Q_i have non-vanishing free terms, and hence we can factor them as

$$Q_i(t_i) = (1-\mu_{1,i}t_i)^{m_{1,i}}\cdots(1-\mu_{k_i,i}t_i)^{m_{k_i,i}}$$

for distinct (possibly vanishing) elements $\mu_{\bullet,i}$ in the algebraic closure $\overline{\mathbb{K}}$. A simple computation now shows that we can take the tuples γ_s in (b) to be

$$(\gamma_{s,1},\cdots,\gamma_{s,r}) = (\mu_{\bullet,1},\cdots,\mu_{\bullet,r})$$

for various choices of '$\bullet$'.

(b) ⇒ (d). The multi-sequences satisfying either of the two conditions form a linear space, so it is enough to consider a "monomial" multi-sequence, of the form

$$a_{n_1,\cdots,n_r} = n_1^{k_1}\cdots n_r^{k_r}\gamma_1^{n_1}\cdots\gamma_r^{n_r}.$$

Without loss of generality, it is enough to show that $S_1^n \mathbf{a}$, $n \in \mathbb{N}$ span a finite-dimensional space. Rescaling S_1 by γ_1, the exponential part $\gamma_1^{n_1} \cdots \gamma_r^{n_r}$ can be dropped entirely, as can all factors independent of n_1. In short, we can consider

$$a_{n_1,\cdots,n_r} = n_1^{k_1}$$

instead. We are now back in the plain-sequence case, where we can fall back on Theorem 1.3.

(c) ⇔ (d). The rightward implication is obvious, whereas its converse follows from the fact that the shifts S_i (for $1 \le i \le r$) commute: by (d), for each i there is some N_i such that every $S_i^d \mathbf{a}$ is a linear combination of the $S_i^n \mathbf{a}$ for $n < N_i$. But then, by the noted commutation,

$$S_1^{d_1} \cdots S_r^{d_r} \mathbf{a} = \text{a linear combination of } S_1^{n_1} \cdots S_r^{n_r} \mathbf{a}, \quad n_i < N_i.$$

(d) ⇒ (a). Condition (d) says that for each $1 \le i \le r$ there is some polynomial $Q_i(t_i)$ such that the exponents of t_i in $Q_i(t_i) H_{\mathbf{a}}(t_1, \cdots, t_r)$ are uniformly bounded. Applying this to all i, the product

$$Q_1(t_1) \cdots Q_r(t_r) H_{\mathbf{a}}(t_1, \cdots, t_r)$$

has only finitely many monomials; in other words, it is a polynomial. □

Definition 1.5. A multi-sequence over a field $\mathbb{K}$ is

- *C-finite* if it satisfies the equivalent conditions of Theorem 1.4.
- *rational* if its Hilbert series is rational.
- *algebraic* if its Hilbert series $H(t)$ is algebraic, in the sense that it satisfies an equation

$$P_d(t) H(t)^d + \cdots + P_1(t) H(t) + P_0(t) = 0$$

in $\mathbb{K}[[t]]$, where the P_i are polynomials (not all vanishing);

see [19, Definition 6.1.1].

C-finiteness and rationality are equivalent in the 1-dimensional case, where we also refer to such (plain, 1-dimensional) sequences as *eventually linear(ly) recursive.* ⧫

Remark 1.6. Our linear (or linearly) recursive sequences are those studied in [20, §4.1], as well as the *recurrence sequences* of [12, §1.1.1]: it is assumed, in particular, that they satisfy recurrence relations of the form

$$a_{n+T} = c_{T-1} a_{n+T-1} + \cdots + c_0 a_n, \quad c_0 \text{ not a zero divisor} \tag{1.2}$$

(in whatever commutative ring the coefficients c_i belong to) for *all* n. This convention rules out, for instance, sequences that are eventually zero. This is the reason for requiring the modifier 'eventually' in Definition 1.5 and for introducing the pithier term 'rational' (justified by Theorem 1.4). ♦

Example 1.7. The rationality of Definition 1.5 is weaker than C-finiteness. This is clear for instance from condition (a) of Theorem 1.4, which requires that the denominator be separable as a product of univariate polynomials, but can also be seen by exhibiting a 2-dimensional sequence with rational Hilbert series whose shifts span an infinite-dimensional space.

Take, say,

$$\mathbf{a} = (a_{m,n})_{m,n}, \quad a_{m,n} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

The shifts $S_2^d \mathbf{a}$ are linearly independent, but the Hilbert series is the rational 2-variable function $\frac{1}{1-xy}$. ♦

Theorem 1.8. *Rationality for multi-sequences in the sense of Definition 1.5 enjoys the following permanence properties.*

(a) *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two* r*-sequences over a field* $\mathbb{K}$*, with*
- $\mathbf{a}$ *C-finite, and*
- $\mathbf{b}$ *rational or C-finite.*

then, their product

$$\mathbf{ab} := (a_{\mathbf{n}} b_{\mathbf{n}})_{\mathbf{n}=(n_1,\cdots,n_r)}$$

is rational or C-finite respectively.

(b) *If* $\mathbf{a}$ *is a rational* r*-sequence over* $\mathbb{K}$ *and for each* $(r-1)$*-tuple* $(n_1, \cdots, n_{r-1})$ *only finitely many* $a_{n_1,\cdots,n_r}$ *are non-zero, the truncation*

$$\mathbf{a}' \in \mathbb{K}^{\mathbb{N}^{r-1}}, \quad a'_{n_1,\cdots,n_{r-1}} = \sum_{n_r} a_{n_1,\cdots,n_r}$$

is again rational.

(c) *If* $\mathbf{a}$ *is a rational 2-sequence such that for each* m *the number of non-zero* $a_{m,n}$ *is finite and* $\mathbf{b}$ *is a rational sequence, then the "matrix product" sequence*

$$(\mathbf{a} \bullet \mathbf{b})_m := \left(\sum_n a_{m,n} b_n \right)_m$$

is rational.

(d) *If* $\mathbf{a}$ *is a rational (C-finite) r-sequence then so is*

$$(\mathbf{b_n})_{\mathbf{n}=(n_1,\cdots,n_r)} := \left(\sum_{i=0}^{n_1} a_{i,n_2\cdots n_r}\right)_{n_1,\cdots,n_r}.$$

(e) *If* $\mathbf{a}$ *is a rational (C-finite) r-sequence then so is*

$$(\mathbf{b_n})_{\mathbf{n}=(n_1,\cdots,n_r)} := (a_{dn_1,n_2\cdots n_r})_{n_1,\cdots,n_r},$$

for any positive integer d.

Proof. (a) Since the proofs of the two claims are substantively different, we treat them separately.

(Case 1: a C-finite, b rational) The argument resembles that in the proof of [19, Proposition 6.1.11], except it is simpler because we are handling only rational (rather than algebraic) power series.

The rationality of $\mathbf{a} \bullet \mathbf{b}$ will not be affected by altering finitely many terms of $\mathbf{a}$, so we may as well assume we have an expression (1.1) for $\mathbf{a}$. Moreover, by linearity, we can simplify this to

$$a_{n_1,\cdots,n_r} = n_1^{k_1} \cdots n_r^{k_r} \gamma_1^{n_1} \cdots \gamma_r^{n_r}.$$

Since the goal is to show that

$$\sum_{n_i} b_{n_1,\cdots,n_r} a_{n_1,\cdots,n_r} t_1^{n_1} \cdots t_r^{n_r}$$

is rational, the change of variables $t_i \mapsto \gamma_i t_i$ further reduces this to

$$a_{n_1,\cdots,n_r} = n_1^{k_1} \cdots n_r^{k_r},$$

and inducting separately on the k_i finally boils down the goal to proving that if

$$H_{\mathbf{b}}(t_i) = \sum_{\mathbf{n}=(n_1,\cdots,n_r)} b_{\mathbf{n}} t_1^{n_1} \cdots t_r^{n_r}$$

is rational then so is

$$\widetilde{H}_{\mathbf{b}}(t_i) := \sum_{\mathbf{n}=(n_1,\cdots,n_r)} n_1 b_{\mathbf{n}} t_1^{n_1} \cdots t_r^{n_r}.$$

This is immediate though, because we have

$$\widetilde{H}_{\mathbf{b}}(t_i) = t_i \frac{\partial H_{\mathbf{b}}(t_i)}{\partial t_i}.$$

(Case 2: a and b C-finite) This time around it is the proof of [15, Theorem 4.2, point 2.] that we adapt.

Multiplication

$$(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} \bullet \mathbf{b}$$

is bilinear, inducing a linear map $m : A \otimes A \to A$ for $A := \mathbb{K}^{\mathbb{N}^r}$. That map is compatible with shifting (i.e. the shifts act as algebra endomorphisms), so the shifts $S^{\mathbf{n}}(\mathbf{a} \bullet \mathbf{b})$ are contained in the image of

$$(\text{span of shifts of } \mathbf{a}) \otimes (\text{span of shifts of } \mathbf{b}) \le A \otimes A$$

through the multiplication map m. Since both tensorands are finite-dimensional by assumption, so is

$$\operatorname{span}\{S^{\mathbf{n}}(\mathbf{a} \bullet \mathbf{b}) \mid \mathbf{n} \in \mathbb{N}^r\}.$$

(b) The Hilbert series

$$H_{\mathbf{a}'}(t_1, \cdots, t_{r-1})$$

of $\mathbf{a}'$ is obtained from that of $\mathbf{a}$ by substituting 1 for the r^{th} variable t_r (we need the vanishing hypothesis for this to make sense). Since we are assuming rationality, we have

$$H_{\mathbf{a}}(t_1, \cdots, t_r) = \frac{A(t_1, \cdots, t_r)}{B(t_1, \cdots, t_r)}, \quad A, B \in \mathbb{K}[t_1, \cdots, t_r];$$

and hence will obtain a rational function upon making the substitution $t_r = 1$.

(c) The matrix product sequence $\mathbf{a} \bullet \mathbf{b}$ is obtained by

- constructing the 2-sequence $\overline{\mathbf{b}}$ defined by

$$\overline{b}_{m,n} = b_n,$$

  which is C-finite (for instance because it satisfies condition (d) of Theorem 1.4);
- then forming the product $\mathbf{a}\overline{\mathbf{b}}$, which is rational by part (a);
- and then summing out the second component of the resulting 2-sequence:

$$(\mathbf{a} \bullet \mathbf{b})_m = \sum_n \left(\mathbf{a}\overline{\mathbf{b}}\right)_{m,n};$$

  this again produces a rational sequence by part (b).

(d) The monomial $b_{n_1\cdots n_r}t_1^{n_1}\cdots t_r^{n_r}$ of the Hilbert series $H_{\mathbf{b}}(t_i)$ is, by definition,

$$a_{0,n_2\cdots n_r}t_1^{n_1}\cdots t_r^{n_r}+\cdots+a_{n_1,n_2\cdots n_r}t_1^{n_1}\cdots t_r^{n_r}.$$

These are

- the $(0,n_2,\cdots,n_r)$ term of $H_{\mathbf{a}}(t_i)$ multiplied by $t_1^{n_1}$;
- the $(1,n_2,\cdots,n_r)$ term of $H_{\mathbf{a}}(t_i)$ multiplied by $t_1^{n_1-1}$;
- $\cdots$
- the $(n_1,n_2,\cdots,n_r)$ term of $H_{\mathbf{a}}(t_i)$ (multiplied by $1=t_1^0$).

Summing over all tuples $\mathbf{n}$, this means that $H_{\mathbf{b}}$ is obtained from $H_{\mathbf{a}}$ by multiplying each term of the latter by $1+t_1+t_1^2+\cdots$. In short:

$$H_{\mathbf{b}}(t_1,\cdots,t_r)=\frac{1}{1-t_1}H_{\mathbf{a}}(t_1,\cdots,t_r).$$

Both versions (rational and C-finite) of the claim follow from this (using part (a) of Theorem 1.4 for the C-finite arm of the argument).

(e) The Hilbert series $H_{\mathbf{b}}$ is obtained from the original one $H_{\mathbf{a}}$ by

- dropping all monomials $t_1^{n_1}\cdots t_r^{n_r}$ where n_1 is *not* divisible by d,
- and then substituting t_1 for t_1^d throughout.

The first step (dropping monomials) can be achieved by taking the Hadamard product with the C-finite Hilbert series

$$H(t_1,\cdots,t_r)=\frac{1}{1-t_1^d},$$

so it preserves both rationality and C-finiteness by part (a) of the present result. As for the second step, write

$$H(t_1,\cdots,t_r)=\frac{P(t_1,\cdots,t_r)}{Q(t_1,\cdots,t_r)}$$

for coprime polynomials P and Q (this makes sense because polynomial rings are unique factorization domains [9, §9.3, Theorem 7]), with Q either a plain polynomial or a special one, separable as a product $Q_1(t_1)\cdots Q_r(t_r)$ as in part (a) of Theorem 1.4. By Lemma 1.9 P and Q are both polynomials in t_1^d and $t_2,\cdots,t_r$, so replacing t_1^d by t_1 throughout again keeps us rational/C-finite. $\square$

Lemma 1.9. *Let $H(t_1,\cdots,t_r)\in\mathbb{K}[[t_1,\cdots,t_r]]$ be a formal power series which*

(a) *is rational, in the sense that it is expressible as*

$$H(t_1, \cdots, t_r) = \frac{P(t_1, \cdots, t_r)}{Q(t_1, \cdots, t_r)} \tag{1.3}$$

for polynomials $P, Q \in \mathbb{K}[t_1, \cdots, t_r]$*, and*

(b) *is expressible as a formal power series of* t_1^d *and* $t_2, \cdots, t_r$*, i.e. contains only monomials in which the exponent of* t_1 *is divisible by* d*, where* d *is some fixed positive integer.*

Then, we can write (1.3) for polynomials P *and* Q *in* t_1^d *and* $t_2, \cdots, t_r$*.*

Proof. We will assume that we have an expression (1.3) for *coprime* P and Q, and seek to show that they are polynomials in t_1^d and $t_2, \cdots, t_r$.

Since we can induct on the number of prime divisors of d, we may as well assume that the latter is prime to begin with. There are now two cases to treat:

Case 1: The prime d is not char $\mathbb{K}$. Let $\zeta \in \overline{\mathbb{K}}$ be a primitive d^{th} root of unity (one exists, precisely because $d \neq \text{char } \mathbb{K}$). Condition (b) says that

$$H(\zeta t_1, t_2, \cdots, t_r) = H(t_1, t_2, \cdots, t_r),$$

and hence the same holds for $\frac{P}{Q}$. Now, the coprimality of P and Q and the fact that Q has non-vanishing free term [13, Theorem 1] imply that

$$P(\zeta t_1, t_2 \cdots, t_r) = P(t_1, t_2 \cdots, t_r) \quad \text{and} \quad Q(\zeta t_1, t_2 \cdots, t_r) = Q(t_1, t_2 \cdots, t_r).$$

In turn, this is precisely the desired conclusion.

Case 2: $d = \text{char } \mathbb{K}$. This time around condition (b) is expressible as

$$\frac{\partial H}{\partial t_1} = 0 \Rightarrow \frac{\partial}{\partial t_1}\left(\frac{P}{Q}\right) = 0.$$

We can henceforth ignore H and work only with polynomials, which we will evaluate at tuples of elements in the algebraic closure $\overline{\mathbb{K}}$.

We can evaluate P and Q at some tuple $(t_2, \cdots, t_r) \in \overline{\mathbb{K}}^{r-1}$ so as to ensure that

$$p(t) := P(t, t_2, \cdots, t_r) \quad \text{and} \quad q(t) := Q(t, t_2, \cdots, t_r)$$

are coprime. Our goal is now to show that if $\frac{p}{q}$ has vanishing formal derivative (with respect to t), then p and q are both polynomials in t^d. Write

$$p(t) = (t - \lambda_1)^{m_1} \cdots (t - \lambda_l)^{m_l}$$

and

$$q(t) = (t - \mu_1)^{n_1} \cdots (t - \mu_k)^{n_k}$$

for distinct λ_i and μ_j in $\overline{\mathbb{K}}$. Any m_i and n_j that are divisible by d can be eliminated, since for any polynomial u we have

$$\frac{d}{dt}\left(u^d \frac{p}{q}\right) = 0 \iff \frac{d}{dt}\left(\frac{p}{q}\right) = 0;$$

in other words, we may as well assume that *all* m_i and n_j are coprime to d. If there is at least one numerator factor $t - \lambda_1$, the derivative $\left(\frac{p}{q}\right)'$ will have vanishing order $m_1 - 1$ at λ_1, and hence not vanish. One argues similarly for the denominator factors, concluding that the original p and q must have been d^{th} powers (and hence polynomials in d, since $d = \mathrm{char}\,\mathbb{K}$) to begin with. □

Remark 1.10. Parts (b) and (c) of Theorem 1.8 are very much in the spirit of [16, Theorem 3.8 (vi) and (viii)] respectively, which are the analogous results for *D-finite* (rather than rational) power series.

Parts (d) and (e) (which, although stated only for the index n_1 for brevity, have obvious variants valid for the other n_i) are multi-variable analogues of [15, Theorem 4.2, parts 3. and 4.] respectively. ♦

The product $\mathbf{a} \bullet \mathbf{b}$ in Theorem 1.8, (a) is what is usually referred to as the *Hadamard product*: see e.g. [15, §2.1] or [19, discussion preceding Proposition 6.1.11] (we apply the term freely to both multi-sequences and their corresponding power series). Given

- Theorem 1.8, (a), which ensures the rationality of the Hadamard product if one of the factors is C-finite;
- which specializes to the well-known fact that for plain, 1-dimensional sequences rationality is closed under Hadamard products [15, Theorem 4.2 2.];
- while at the same time the Hadamard product of *algebraic* power series need not be algebraic ([15, discussion immediately preceding §6.5] and [19, paragraph preceding Proposition 6.1.11]),

one might naturally ask whether the Hadamard product of two rational multi-variable power series is again rational. This is not true in general, in more than one variable. To place the example in context, recall ([19, Definition 6.3.1]):

Definition 1.11. The *diagonal* of a multi-variable power series

$$H(t_1, \cdots, t_r) = \sum_{n_i} a_{n_1 \cdots n_r} t_1^{n_1} \cdots t_r^{n_r}$$

is the series

$$\operatorname{diag} H(t) := \sum_n a_{n\cdots n} t^n.$$

The same terminology applies to (multi-)sequences: the diagonal of the multi-sequence

$$\mathbf{a} = (a_{\mathbf{n}})_{\mathbf{n}=(n_1,\cdots,n_r)}$$

is the sequence

$$\operatorname{diag} \mathbf{a} := (a_{n,\cdots,n})_n. \quad \blacklozenge$$

Picking out the constituent terms $a_{n\cdots n}$ of the diagonal sequence attached to $\mathbf{a}$ can be achieved by forming the Hadamard product

$$H_{\mathbf{a}}(t_1,\cdots,t_r) \bullet \frac{1}{1-t_1\cdots t_r}.$$

Since diagonals of rational power series need not be rational ([19, Example 6.3.2]), this allows us to construct rational series with non-rational Hadamard product (see also [17, p.403]).

Example 1.12. Consider the rational multi-sequences $\mathbf{a}$ and $\mathbf{b}$ with Hilbert series

$$H_{\mathbf{a}}(s,t) = \frac{1}{1-s-t} \quad \text{and} \quad H_{\mathbf{b}}(s,t) = \frac{1}{1-st}.$$

As follows from the aforementioned [19, Example 6.3.2], we have

$$H_{\mathbf{a}\bullet\mathbf{b}}(s,t) = \sum_n \binom{2n}{n} s^n t^n = \frac{1}{\sqrt{1-4st}},$$

which is not rational. $\blacklozenge$

1.2. Recursive polynomial sequences

We will need a "composition" operation between sequences of polynomials and plain complex number sequences, as detailed below.

Definition 1.13. For a polynomial

$$P(x) = \sum_k c_k x^k \tag{1.4}$$

and a sequence $\mathbf{a} = (a_n)_n$ we write

$$P \triangleright \mathbf{a} = \sum_k c_k a_k.$$

Similarly, for a sequence $\mathcal{P} = (P_n)_n$ of polynomials and a complex number sequence $\mathbf{a} = (a_n)_n$ we write $\mathcal{P} \triangleright \mathbf{a}$ for the sequence $(P_n \triangleright \mathbf{a})_n$. ♦

In other words, one simply substitutes a_k for x^k in the polynomials P_n and evaluates to obtain the n^{th} term b_n.

Definition 1.14. With $\mathcal{P} = (P_n)_n$ and $(a_n)_n$ as in Definition 1.13 we say that the two sequences $\mathcal{P}$ and $\mathbf{a}$ are *recursively compatible* if the sequence $\mathcal{P} \triangleright \mathbf{a}$ is eventually linear recursive.

The sequence $\mathcal{P}$ of polynomials is *recursively well-adjusted* (or just 'well-adjusted' for short) if it is recursively compatible with every eventually linear recursive sequence $\mathbf{a} = (a_n)_n$. ♦

Note that the operation

$$(\mathcal{P}, \mathbf{a}) \mapsto \mathcal{P} \triangleright \mathbf{a}$$

is additive in both variables, and hence so is the recursive compatibility relation.

Proposition 1.15. *Every eventually linear recursive polynomial sequence $\mathcal{P} = (P_n)_n$ is recursively well-adjusted in the sense of Definition 1.14.*

Proof. Given

- the additivity of recursive compatibility noted just before the statement,
- the characterization of eventually recursive sequences in of Theorem 1.3 above,
- and the fact that we can ignore finitely many initial sequence terms a_i, $0 \le i \le k$ by Lemma 1.16 below,

it is enough to prove that $\mathcal{P}$ is recursively compatible with the sequence $\mathbf{a} = (a_n)_n$ given by

$$a_n = n^d \gamma^n$$

for some non-negative integer d and some $\gamma \in \mathbb{C}$.

First, note that when $d = 0$ and hence $a_n = \gamma^n$ we have

$$\mathcal{P} \triangleright \mathbf{a} = (P_n(\gamma))_n$$

and hence the conclusion is immediate. In general, consider a linear recurrence of $\mathcal{P}$ (for large n):

$$P_{n+T} = c_{T-1} P_{n+T-1} + \cdots + c_0 P_n \tag{1.5}$$

for polynomials c_i. Then, the derived polynomials satisfy the relation

$$P'_{n+T} = c_{T-1}P'_{n+T-1} + \cdots + c_0 P'_n + c'_{T-1}P_{n+T-1} + \cdots + c'_0 P_n. \tag{1.6}$$

If $d = 1$ then the conclusion amounts to proving that $(P'_n(\gamma))_n$ is eventually recursive; this, in turn, follows from (1.6) and the fact that $(P_n(\gamma))_n$ is (eventually) recursive.

For $d = 2$ repeat the procedure: (1.6) once more shows that $(P'_n(\gamma))_n$ is recurrent (as, of course, is $(P_n(\gamma))_n$). Differentiating once more we obtain

$$P''_{n+T} = \sum_{i=0}^{T-1} (c_i P''_{n+i} + 2c'_i P'_{n+i} + c''_i P_{n+i}).$$

This, in turn, shows that $(P''_n(\gamma))_n$ is recurrent and hence so is $\mathcal{P} \triangleright (n^2\gamma^n)_n$.

It should be clear now how to continue this recursive process to conclude for arbitrary d. □

Proof (alternative). The fact that $(P_n(x))_n$ is eventually linearly recursive implies that its Hilbert series

$$H_{\mathcal{P}}(x, y) = \sum_{m,n} c_{m,n} x^m y^n := \sum_{n \geq 0} P_n(x) y^n,$$

which a priori is an element of $\mathbb{K}[x][[y]]$ (formal y-power series over the polynomial ring in x), is rational:

$$H_{\mathcal{P}}(x, y) = \frac{A(x, y)}{B(x, y)}, \quad A, B \in \mathbb{K}[x, y].$$

The Hilbert series $H_{\mathcal{P}\triangleright\mathbf{a}}(y)$ of $\mathcal{P} \triangleright \mathbf{a}$ is obtained from $H_{\mathcal{P}}(x, y)$ by substituting a_m for each x^m; in other words, the coefficient of y^n in $H_{\mathcal{P}\triangleright\mathbf{a}}$ is

$$\sum_m c_{m,n} a_m.$$

The fact that the resulting power series is rational (and hence $\mathcal{P} \triangleright \mathbf{a}$ is eventually linearly recursive by Theorem 1.3) follows from part (c) of Theorem 1.8. □

Lemma 1.16. *Under the hypotheses of Proposition 1.15, $\mathcal{P} \triangleright \mathbf{a}$ is eventually linearly recursive if $\mathbf{a}$ eventually vanishes.*

Proof. This is almost immediate: if $a_i = 0$ for $i > k$ then the n^{th} term of $\mathcal{P} \triangleright \mathbf{a}$ is a linear combination (with constant coefficients) of the first $k + 1$ coefficients of P_n, and a recurrence relation (1.5) induces one for each coefficient a_i, $0 \leq i \leq k$. □

1.3. Laurent polynomials

It will be useful later on, in Section 4, to have a Laurent-polynomial analogue of sorts for Proposition 1.15. To elaborate, we will have

- a linearly recursive sequence $\mathcal{P} = (P_n)_n$ of *Laurent* polynomials;
- and C-finite sequences $(\mathbf{a}_n)$ and $(\mathbf{b}_n)$;
- and the goal of showing that $\mathcal{P} \triangleright (\mathbf{a}, \mathbf{b})$ is well-behaved (C-finite, algebraic, D-finite, etc.),

the latter symbol is the object of the following expansion of Definition 1.13.

Definition 1.17. For a Laurent polynomial (1.4) and sequences $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ and $\mathbf{b} = (b_n)_{n\in\mathbb{N}}$ we write

$$P \triangleright (\mathbf{a}, \mathbf{b}) = \sum_{k\geq 0} c_k a_k + \sum_{k<0} c_k b_{-k-1}.$$

In other words, $\mathcal{P} \triangleright (\mathbf{a}, \mathbf{b})$ is defined similarly to $\mathcal{P} \triangleright \mathbf{a}$, except this time around

- a_k is substituted for each x^k appearing in P,
- while b_k is substituted for each x^{-k-1} appearing in P.

For a sequence $\mathcal{P} = (P_n)_n$ of Laurent polynomials we write

$$\mathcal{P} \triangleright (\mathbf{a}, \mathbf{b}) := (P_n \triangleright (\mathbf{a}, \mathbf{b}))_n. \quad \blacklozenge$$

We can now state

Theorem 1.18. *Let $\mathcal{P} = (P_n)_n$ be an eventually linearly recursive sequence of Laurent polynomials and $\mathbf{a}$, $\mathbf{b}$ two eventually linearly recursive sequences. Then, $\mathcal{P} \triangleright (\mathbf{a}, \mathbf{b})$ is algebraic.*

Proof. Since

$$\mathcal{P} \triangleright (\mathbf{a}, \mathbf{b}) = \mathcal{P} \triangleright (\mathbf{a}, \mathbf{0}) + \mathcal{P} \triangleright (\mathbf{0}, \mathbf{b}),$$

it is enough to work with a single sequence $\mathbf{a}$ and hence try to argue that

$$\mathcal{P} \triangleright \mathbf{a} := \mathcal{P} \triangleright (\mathbf{a}, \mathbf{0})$$

is algebraic. This means that we are substituting a_n for the non-negative-exponent x^n appearing in the P_m, and dropping the negative-exponent x^{-n-1}, $n \in \mathbb{N}$.

In this setting, we can replace Laurent with ordinary polynomials at the cost of replacing '$\triangleright$' with a more sophisticated operation. To see this, first consider a recursion (1.5)

$$P_{n+T}(x) = \sum_{i=1}^{T} c_{T-i}(x) P_{n+T-i}(x),$$

holding for all n sufficiently large (say for $n \geq \bar{N}$), where $c_\bullet(x)$ are Laurent polynomials and $T \geq 1$ is a positive integer. Choosing natural numbers A and B such that $x^{Ai} c_{T-i}(x)$ and $x^{Aj+B} P_j(x)$ are polynomials for $1 \leq i \leq T$ and $0 \leq j < \bar{N} + T$, we have

$$\begin{aligned} x^{A(n+T)+B} P_{n+T}(x) &= \sum_{i=1}^{T} x^{A(n+T)+B} c_{T-i}(x) P_{n+T-i}(x) \\ &= \sum_{i=1}^{T} x^{Ai} c_{T-i}(x) (x^{A(n+T-i)+B} P_{n+T-i}(x)) \end{aligned}$$

for $n \geq \bar{N}$. Replacing the original P with the *polynomials* $x^{An+B} P_n(x)$ satisfying a linear recurrence with respective polynomial coefficients $x^{Ai} c_{T-i}(x)$ in place of c_{T-i}, we may as well assume that everything in sight is a plain (as opposed to Laurent) polynomial.

The substitution of terms a_n for powers of x in P_m, though, now takes on a different character. We will have some polynomial $q(n) = An + B$ such that

- we substitute a_0 for each $x^{q(n)}$ in each $P_n(x)$ (note the correlation: as n grows, we start substituting as for xs in P_n starting with larger and larger exponents $q(n)$);
- similarly, we substitute a_1 for each $x^{q(n)+1}$;
- etc.

The recursion (1.5) shows that

$$\deg P_{n+T} \leq \max_{0 \leq i \leq T-1} \deg c_i P_{n+i},$$

which puts a bound of Dn (for fixed D) on the degree of P_n. We may thus assume that the substitution of as for xs takes place over a range of exponents for monomials of P_n: starting with $x^{q(n)} = x^{An+B}$ and ending with x^{Dn}.

We can now proceed along the lines of the alternative proof of Proposition 1.15:

- consider the rational Hilbert series

$$H_{\mathcal{P}}(x, y) = \sum_{m,n} c_{m,n} x^m y^n := \sum_{n \geq 0} P_n(x) y^n$$

with its attached rational 2-sequence $(c_{m,n})$;

- note that it will make no difference to change finitely many members of **a**, because the difference to the original sequence would then be eventually vanishing, and the problem would reduce to arguing that the sequences

$$(b_{q(n),n})_n,\ (b_{q(n)+1,n})_n, \cdots, (b_{q(n)+\ell,n})_n$$

  are algebraic, for fixed ℓ. In turn, this follows from the fact that the diagonal of a rational 2-sequence is algebraic [19, Theorem 6.3.3].
- but then we may as well assume that **a** is of the form

$$a_n = \sum_{i=1}^{s} Q_i(n)\gamma_i^n$$

  for *all* n, and hence is extendable to negative n by the same formula, and then also extendable to the C-finite 2-sequence

$$(a_{m-q(n)})_{m,n} = \sum_{i=1}^{s} Q_i(m-q(n))\gamma_i^{m-q(n)};$$

  the C-finiteness follows because the 2-sequence has the shape described in part (b) of Theorem 1.4.
- now form the (also rational, by Theorem 1.8 (a)) 2-sequence $a'_{m,n} := c_{m,n}a_{m-q(n)}$;
- which then yields a 2-sequence

$$b_{m,n} := a'_{q(m),n} + a'_{q(m)+1,n} + \cdots + a'_{Dm,n},$$

  rational by parts (d) and (e) of Theorem 1.8;
- which in turn has an algebraic *diagonal* sequence

$$b_n := b_{n,n} = a'_{q(n),n} + a'_{q(n)+1,n} + \cdots + a'_{Dn,n}$$

  by [19, Theorem 6.3.3].

b_n is our target sequence, and the conclusion that it is algebraic is precisely what we were after. □

An example will illustrate the substitution $x^n \to a_n$ in the discussion above.

Example 1.19. Take $P_n = (\frac{1}{x} + x)^n$ and $a_n = \delta_{0,n}$ (i.e. $\mathbf{a} = (1, 0, 0, \cdots)$). Furthermore, take $q(n) = n$. Then for all n,

$$x^n P_n = (1 + x^2)^n.$$

We consider $\mathcal{P} \triangleright (\mathbf{a}, \mathbf{0})$. The substitution in question will pick out the coefficient of x^n in $x^n P_n$, i.e. will return the sequence

$$b_n = \begin{cases} \binom{n}{\frac{n}{2}} & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

This is an algebraic sequence: by [19, Example 6.3.2], its Hilbert series is $\frac{1}{\sqrt{1-4x^2}}$. ♦

2. Results concerning $(P_n \triangleright \mathbf{a})_n$

Definition 1.13 can be extended to define $P \triangleright \mathbf{a}$ where P is a polynomial of ℓ variables, and $\mathbf{a}$ is an ℓ-sequence. Let

$$P(x_1, ..., x_\ell) = \sum_{s_1,...,s_\ell} c_{s_1,...,s_\ell} x_1^{s_1} \cdots x_\ell^{s_\ell},$$

and let $\mathbf{a} = (a_{i_1,...,i_\ell})_{i_1,...,i_\ell}$ be an ℓ-sequence. Then

$$P \triangleright \mathbf{a} = \sum_{s_1,...,s_\ell} c_{s_1,...,s_\ell} a_{s_1,...,s_\ell}.$$

Further, if $\mathcal{P} = (P_n)_n$ then $\mathcal{P} \triangleright \mathbf{a} = (P_n \triangleright \mathbf{a})_n$. In this section, we consider $(P_n \triangleright \mathbf{a})_n$ where, unless otherwise stated, $(P_n)_n$ is an eventually linear recursive sequence of complex polynomials (of several variables) and $\mathbf{a}$ is a multi-sequence of complex numbers. In particular, we consider the sequence $(P_n \triangleright \mathbf{a})_n$ where $\mathbf{a}$ has a given property (rational, algebraic, P-recursive).

2.1. P-recursive multi-sequences

Recall the definition of P-recursive (multi-) sequences as given in [16, Definition 3.2] and as restated below:

Definition 2.1. A sequence $a(i_1, i_2, ..., i_\ell)$ is P-recursive if there is a natural number m such that

1. For each $j = 1, 2, ..., \ell$ and each $\mathbf{v} = (v_1, v_2, ..., v_\ell) \in \{0, 1, ..., m\}^\ell$ there is a polynomial $p_{\mathbf{v}}^{(j)}$ (with at least one $p_{\mathbf{v}}^{(j)} \neq 0$ for each j) such that

$$\sum_{\mathbf{v}} p_{\mathbf{v}}^{(j)}(i_j)\, a(i_1 - v_1, i_2 - v_2, ..., i_\ell - v_\ell) = 0$$

   for all $i_1, i_2, ..., i_\ell \geq m$, and
2. if $\ell > 1$ then all the m-sections of $a(i_1, i_2, ..., i_\ell)$ are P-recursive. ♦

We find that if $(P_n)_n$ is an eventually linear recursive sequence of polynomials and $\mathbf{a}$ is a P-recursive multi-sequence, then $(P_n \triangleright \mathbf{a})_n$ is a P-recursive sequence, as stated in Proposition 2.3. To prove this result, we will require the following theorem from [16, Theorem 3.8, (i) and (vi)].

Theorem 2.2.

(a) *The P-recursive sequences (of dimension ℓ) form an algebra over $\mathbb{C}[i_1, i_2, ..., i_\ell]$.*
(b) *If $(a_{i_1,...,i_\ell})_{i_1,...,i_\ell}$ is P-recursive and $\sum_{i_\ell} a_{i_1,...,i_\ell}$ converges for every $i_1, ..., i_{\ell-1}$ then the sequence $(b_{i_1,...,i_{\ell-1}})_{i_1,...,i_{\ell-1}}$ given by*

$$b_{i_1,...,i_{\ell-1}} = \sum_{i_\ell} a_{i_1,...,i_{\ell-1},i_\ell}$$

is P-recursive.

Proposition 2.3. *Let $(P_n)_n$ be an eventually linear recursive sequence of complex polynomials and let $\mathbf{a} : \mathbb{N}^\ell \to \mathbb{C}$ be a P-recursive multi-sequence. Then $(P_n \triangleright \mathbf{a})_n$ is a P-recursive sequence.*

Proof. Let $\mathcal{P} = (P_n(x_1, ..., x_\ell))_n$ be an eventually linear recursive sequence of polynomials. For each n, we write

$$P_n(x_1, ..., x_\ell) = \sum_{s_1,...,s_\ell} c_{n,s_1,...,s_\ell} x_1^{s_1} \cdots x_\ell^{s_\ell}.$$

Since $(P_n(x_1, ..., x_\ell))_n$ is an eventually linear recursive sequence, by Theorem 1.3(a),

$$H_{\mathcal{P}}(t) = \sum_n P_n(x_1, ..., x_\ell) t^n = \sum_{n,s_1,...,s_\ell} c_{n,s_1,...,s_\ell} x_1^{s_1} \cdots x_\ell^{s_\ell} t^n$$

is rational. It follows that the attached multi-sequence $(c_{n,s_1,...,s_\ell})_{n,s_1,...,s_\ell}$ is rational and therefore a P-recursive multi-sequence (by [16, Proposition 2.3, (ii)]). Let $\mathbf{a} = (a_{i_1,...,i_\ell})_{i_1,...,i_\ell}$ be a P-recursive multi-sequence. For each n, define

$$\bar{a}_{n,i_1,...,i_\ell} = a_{i_1,...,i_\ell}.$$

Since $(a_{i_1,...,i_\ell})_{i_1,...,i_\ell}$ is P-recursive, $(\bar{a}_{n,i_1,...,i_\ell})_{n,i_1,...,i_\ell}$ is also P-recursive. Then for each n,

$$P_n \triangleright \mathbf{a} = \sum_{i_1,...,i_\ell} c_{n,i_1,...,i_\ell} a_{i_1,...,i_\ell} = \sum_{i_1,...,i_\ell} c_{n,i_1,...,i_\ell} \bar{a}_{n,i_1,...,i_\ell}.$$

By Theorem 2.2 part (a), $(c_{n,i_1,...,i_\ell} \bar{a}_{n,i_1,...,i_\ell})_{n,i_1,...,i_\ell}$ is a P-recursive multi-sequence. Since $(c_{n,s_1,...,s_\ell})_{s_1,...,s_\ell}$ are the coefficients of the terms of $P_n(x_1, ..., x_\ell)$, for a fixed n

there are finitely many $s_1, ..., s_\ell$ such that $c_{n,s_1,...,s_\ell} \neq 0$. Thus, by applying Theorem 2.2 part (b) a number of times, we find that

$$\left(\sum_{i_1,...,i_\ell} c_{n,i_1,...,i_\ell}\bar{a}_{n,i_1,...,i_\ell}\right)_n = \left(\sum_{i_1,...,i_\ell} c_{n,i_1,...,i_\ell}a_{i_1,...,i_\ell}\right)_n$$

is a P-recursive sequence. That is, $(P_n \triangleright \mathbf{a})_n$ is a P-recursive sequence. □

2.2. Examples

When $\mathbf{a}$ is a rational or algebraic multi-sequence, $(P_n \triangleright \mathbf{a})_n$ is P-recursive by Proposition 2.3. We wonder if this result could be improved: If $\mathbf{a}$ is a rational (or algebraic) multi-sequence, is $(P_n \triangleright \mathbf{a})_n$ necessarily rational (or algebraic)? In the following examples we see that this is not always the case over $\mathbb{C}$.

Example 2.4. If $\mathbf{a} : \mathbb{N}^2 \to \mathbb{C}$ is a rational multi-sequence, and $(P_n(x_1, x_2))_n$ is a linear recursive sequence of complex polynomials then $(P_n \triangleright \mathbf{a})_n$ may not be a rational sequence. Let $\mathbf{a} = (a_{i_1,i_2})_{i_1,i_2}$ be given by:

$$a_{i_1,i_2} = \begin{cases} 1 & \text{for } i_1 = i_2 \\ 0 & \text{else} \end{cases}$$

Then,

$$H_{\mathbf{a}}(x_1, x_2) := \sum_{i_1,i_2} a_{i_1,i_2} x_1^{i_1} x_2^{i_2} = \sum_i x_1^i x_2^i = \frac{1}{1 - x_1 x_2}$$

and $(a_{i_1,i_2})_{i_1,i_2}$ is a rational multi-sequence. Let $P_n(x_1, x_2) = (x_1 + x_2)^{2n}$. That is, $P_n(x_1, x_2) = (x_1 + x_2)^2 P_{n-1}(x_1, x_2)$ and $(P_n(x_1, x_2))_n$ is a linear recursive sequence. Notice that,

$$H_{\mathcal{P}\triangleright\mathbf{a}}(t) := \sum_n (P_n \triangleright \mathbf{a})\, t^n = \sum_n \binom{2n}{n} t^n = \frac{1}{\sqrt{1 - 4t}}$$

as in [19, Example 6.3.2]. Therefore, in this case, $(P_n \triangleright \mathbf{a})_n$ is not a rational sequence.♦

Example 2.5. Let $\mathbf{a} : \mathbb{N}^\ell \to \mathbb{C}$ be a rational multi-sequence, and $\mathcal{P} = (P_n(x_1, ..., x_\ell))_n$ be a linear recursive sequence of complex polynomials, where $\ell > 2$. $(P_n \triangleright \mathbf{a})_n$ may not be an algebraic sequence. Let $\ell > 2$ be a natural number. For each n, let $P_n(x_1, ..., x_\ell) = (x_1 + \cdots + x_\ell)^{\ell n}$. Then

$$P_n(x_1, ..., x_\ell) = (x_1 + \cdots + x_\ell)^\ell P_{n-1}(x_1, ..., x_\ell)$$

and $(P_n(x_1, ..., x_\ell))_n$ is a linear recursive sequence. Let $\mathbf{a} = (a_{i_1,...,i_\ell})_{i_1,...,i_\ell}$ be defined by:

$$a_{i_1,...,i_\ell} = \begin{cases} 1 & \text{for } i_1 = \cdots = i_\ell \\ 0 & \text{else} \end{cases}.$$

Then

$$H_{\mathbf{a}}(x_1, ..., x_\ell) := \sum_{i_1,...,i_\ell} a_{i_1,...,i_\ell} x_1^{i_1} \cdots x_\ell^{i_\ell} = \sum_i x_1^i \cdots x_\ell^i = \frac{1}{1 - x_1 \cdots x_\ell}$$

and $(a_{i_1,...,i_\ell})_{i_1,i_2,...,i_\ell}$ is a rational multi-sequence. Notice that

$$H_{\mathcal{P} \triangleright \mathbf{a}}(t) := \sum_n (P_n \triangleright \mathbf{a})\, t^n = \sum_n \binom{\ell n}{n, n, ..., n} t^n.$$

However, this series is transcendental over any field of characteristic zero (see, for example [21, Theorem 3.8]). Therefore, in this case, $(P_n \triangleright \mathbf{a})_n$ is not an algebraic sequence. ♦

Example 2.6. If $\mathbf{a} : \mathbb{N}^2 \to \mathbb{C}$ is an algebraic multi-sequence and $(P_n(x_1, x_2))_n$ is a linear recursive sequence of complex polynomials, $(P_n \triangleright \mathbf{a})_n$ may not be algebraic. Let $\mathbf{a} = (a_{i_1,i_2})_{i_1,i_2}$ be given by:

$$a_{i_1,i_2} = \begin{cases} \binom{2i_1}{i_1} & \text{for } i_1 = i_2 \\ 0 & \text{else} \end{cases}$$

Then,

$$H_{\mathbf{a}}(x_1, x_2) := \sum_{i_1,i_2} a_{i_1,i_2} x_1^{i_1} x_2^{i_2} = \sum_i \binom{2i}{i} x_1^i x_2^i = \frac{1}{\sqrt{1 - 4x_1x_2}}$$

as in [19, Example 6.3.2]. That is, $(a_{i_1,i_2})_{i_1,i_2}$ is an algebraic multi-sequence (but not rational). Let $P_n(x_1, x_2) = (x_1 + x_2)^{2n}$. Then $P_n(x_1, x_2) = (x_1 + x_2)^2 P_{n-1}(x_1, x_2)$ and $(P_n(x_1, x_2))_n$ is a linear recursive sequence. Notice that,

$$H_{\mathcal{P} \triangleright \mathbf{a}}(t) := \sum_n (P_n(x_1, x_2) \triangleright \mathbf{a})\, t^n = \sum_n \binom{2n}{n}^2 t^n$$

which is transcendental over $\mathbb{C}$ (see for example [18, §4, Example (g)]). Therefore, in this case, $(P_n \triangleright \mathbf{a})_n$ is not an algebraic sequence. ♦

Over fields of characteristic p, examples analogous to Example 2.5 and Example 2.6 cannot be found, as stated in Proposition 2.7.

Proposition 2.7. *Let $\mathbb{K}$ be a field of characteristic p. Let $\mathbf{a} : \mathbb{N}^\ell \to \mathbb{K}$ be an algebraic multi-sequence and let $(P_n)_n$ be an eventually recursive sequence of polynomials in $\mathbb{K}[x_1, ..., x_\ell]$. Then $(P_n \triangleright \mathbf{a})_n$ is an algebraic sequence.*

The proof of Proposition 2.7 is similar to the proof of Proposition 2.3, however instead of applying Theorem 2.2, we need the following result (stated in [17, Introduction, p.395]), as well as Proposition 2.9.

Theorem 2.8. *If $\mathbb{K}$ is a field of characteristic $p > 0$ and if f, g are algebraic series over $\mathbb{K}$, then the Hadamard product of f and g is again an algebraic series over $\mathbb{K}$.*

Proposition 2.9. *Let $\mathbb{K}$ be a field of characteristic $p > 0$, and let $\mathbf{a} : \mathbb{N}^\ell \to \mathbb{K}$ be a multi-sequence. If $\mathbf{a} = (a_{i_1,...,i_\ell})_{i_1,...,i_\ell}$ is an algebraic multi-sequence and for each $(\ell-1)$-tuple $(i_1, ..., i_{\ell-1})$, $a_{i_1,...,i_\ell} \neq 0$ for only finitely many i_ℓ, then $\mathbf{b} = (b_{i_1,...,i_{\ell-1}})_{i_1,...,i_{\ell-1}}$ given by*

$$b_{i_1,...,i_{\ell-1}} = \sum_{i_\ell} a_{i_1,...,i_\ell}$$

is an algebraic (multi-)sequence.

Proof. Since $\mathbf{a} = (a_{i_1,...,i_\ell})_{i_1,...,i_\ell}$ is an algebraic multi-sequence,

$$H_{\mathbf{a}}(x_1, ..., x_\ell) = \sum_{s_1,...,s_\ell} a_{s_1,...,s_\ell} x_1^{s_1} \cdots x_\ell^{s_\ell}$$

satisfies

$$P_d(x_1, ..., x_\ell) H_{\mathbf{a}}(x_1, ..., x_\ell)^d + \cdots + P_1(x_1, ..., x_\ell) H_{\mathbf{a}}(x_1, ..., x_\ell) + P_0(x_1, ..., x_\ell) = 0 \quad (2.1)$$

for some d, where $\{P_i(x_1, ..., x_\ell)\}_{i=0}^d$ are polynomials, not all of which vanish. Further, we can assume that these polynomials share no common factors; if there were such a common factor, it could be factored from Equation (2.1), yielding an equation in this form where the polynomials do not share a common factor. Since for each $(i_1, ..., i_{\ell-1})$, $a_{i_1,...,i_\ell} \neq 0$ for only finitely many i_ℓ, we consider $H_{\mathbf{a}}(x_1, ..., x_{\ell-1}, 1)$ as follows

$$H_{\mathbf{a}}(x_1, ..., x_{\ell-1}, 1) = \sum_{s_1,...,s_\ell} a_{s_1,...,s_\ell} x_1^{s_1} \cdots x_{\ell-1}^{s_{\ell-1}} = \sum_{s_1,...,s_{\ell-1}} \left(\sum_{s_\ell} a_{s_1,...,s_\ell} \right) x_1^{s_1} \cdots x_{\ell-1}^{s_{\ell-1}}.$$

In particular, we notice that $H_{\mathbf{b}}(x_1, ..., x_{\ell-1}) = H_{\mathbf{a}}(x_1, ..., x_{\ell-1}, 1)$. Letting $x_\ell = 1$ in Equation (2.1), we obtain

$$\begin{aligned} P_d(x_1, ..., x_{\ell-1}, 1) H_{\mathbf{b}}(x_1, ..., x_{\ell-1})^d + \cdots + P_1(x_1, ..., x_{\ell-1}, 1) H_{\mathbf{b}}(x_1, ..., x_{\ell-1}) \\ + P_0(x_1, ..., x_{\ell-1}, 1) = 0. \end{aligned}$$

It is not the case that for all $1 \leq i \leq d$, $P_i(x_1, ..., x_{\ell-1}, 1) = 0$, since the polynomials $\{P_i(x_1, ..., x_\ell)\}_{i=0}^d$ share no common factors, therefore $\mathbf{b}$ is an algebraic multi-sequence. □

3. Polynomial invariants

Since for Ω-algebraic modules M we are reduced to examining the entries of the powers of a matrix over $\mathbb{Z}[\Omega^{\pm 1}]$, the question arises of whether $\dim \Omega^n M$ is eventually polynomial in n. More generally, one can ask this of the "size" of $\Omega^n M$ in various guises: dimension, length, length of the socle, etc.

Certainly, $\dim \Omega^n M$ has polynomial *growth*: there is a smallest non-negative integer s such that

$$\dim \Omega^n M = O(n^s) \tag{3.1}$$

is standard big-O notation. That s is precisely the *complexity* $\mathrm{cx}(M) = \mathrm{cx}_G(M)$, as covered for instance in [4, §2.24].

Requiring that dim $\Omega^n M$ be eventually polynomial in n is too much to ask though: when $\mathrm{cx}(M) = 1$ the module M is *periodic*, in the sense that $\Omega^T M \cong M$ for some T and the sequence is simply periodic. This remark and [5, Theorem 3.4] are suggestive of the possibility that perhaps (3.1) is always decomposable as a disjoint union of eventually-polynomial sequences. We will see below that this is indeed the case.

Recall the following notion, e.g. from [20, §4.4].

Definition 3.1. A sequence (a_n) is *quasipolynomial of quasiperiod* T if there are polynomials P_i, $0 \leq i \leq T-1$ such that

$$a_n = P_{n \bmod T}(n),\ \forall n.$$

It is *eventually* quasipolynomial if this constraint holds for sufficiently large n. ♦

For a simple $S \in \operatorname{mod} kG$ and a finite-dimensional G-module M we write $\ell_S M$ for the multiplicity of S in M as a composition factor. We then have the following result (essentially contained in [2, §5.3]).

Proposition 3.2. *For a finite group G, a finite-dimensional G-module M and a simple G-module S the sequence*

$$n \mapsto \ell_S(\mathrm{soc}\ \Omega^n M)$$

is eventually quasipolynomial in n. The same goes for Ω^{-n} in place of Ω^n.

Proof. The two versions are interchanged by duality, so it suffices to prove the claim for the cosyzygy functors Ω^{-n}.

According to [11, Theorem 8.1] the cohomology

$$\mathrm{Ext}^n(S, M) \cong H^n(G, M \otimes S^*)$$

is a finitely generated graded module over the finitely generated skew-commutative graded ring $H^*(G)$. It follows from standard Hilbert-Samuel theory (e.g. [2, Proposition 5.3.1]) that the Hilbert series of

$$n \mapsto \dim\ H^n(G, M \otimes S^*) = \dim\ \mathrm{Ext}^n(S, M) \tag{3.2}$$

is of the form $\frac{P(n)}{Q(n)}$ for polynomials P and Q with the zeroes of Q being roots of unity. It then follows from [20, Proposition 4.4.1] that (3.2) is eventually quasipolynomial. Since for $n \geq 1$ we have

$$\dim\ \mathrm{Ext}^n(S, M) = \dim \mathrm{Hom}(S, \Omega^{-n}M) = \text{number of } S \text{ summands of soc } \Omega^{-n}M,$$

this finishes the proof. $\square$

Corollary 3.3. *Let G be a finite group and $F : \mathrm{mod}\ kG \to \mathrm{Vect}^f$ a linear functor. For a finite-dimensional G-module M the sequence*

$$n \mapsto \dim F(\mathrm{soc}\ \Omega^n M)$$

is eventually quasipolynomial in n. The same goes for Ω^{-n} in place of Ω^n.

Proof. Immediate from Proposition 3.2, given that

$$F(\mathrm{soc}\ \Omega^n M) \cong \bigoplus_{\text{simple } S} F(S)^{\ell_S(\mathrm{soc}\ \Omega^n M)}$$

and hence

$$\dim F(\mathrm{soc}\ \Omega^n M) = \sum_{\text{simple } S} \ell_s \dim F(S). \quad \square$$

We also have the following version, for $\Omega^n M$ rather than their socles.

Theorem 3.4. *Let G be a finite group and $F : \mathrm{mod}\ kG \to \mathrm{Vect}^f$ an exact functor. For a finite-dimensional G-module M the sequence*

$$n \mapsto \dim F(\Omega^n M)$$

is eventually quasipolynomial in n. The same goes for Ω^{-n} in place of Ω^n.

Proof. For variety, we focus on Ω^{-n} this time around.

Consider a minimal injective resolution

$$0 \to M \to I_0 \to I_1 \to \cdots \tag{3.3}$$

As argued in [2, §5.3], for each simple S the multiplicity $m_{S,n}$ of its injective hull I_S as a summand of I_n has a Hilbert series as in the proof of Proposition 3.2: rational, with root-of-unity poles. It once more follows from [20, Proposition 4.4.1] that $n \mapsto m_{S,n}$ is eventually quasipolynomial, and hence so is

$$n \mapsto \dim FI_n = \sum_{\text{simple } S} m_{S,n} \dim FI_S. \tag{3.4}$$

Applying the exact functor F to (3.3) produces a long exact sequence,

$$0 \to FM \to FI_0 \to FI_1 \to \cdots$$

resulting from splicing together the short exact sequences

$$0 \to F\Omega^{-n+1}M \to FI_{n-1} \to F\Omega^{-n}M \to 0,\ n \geq 1.$$

These short exact sequences in turn imply that

$$\dim F\Omega^{-n}M = \dim FI_{n-1} - \dim FI_{n-2} + \dim FI_{n-3} - \cdots + (-1)^n \dim FM$$

(note that the signs alternate).

We thus obtain

$$\dim F\Omega^{-(n+2)}M - \dim F\Omega^{-n}M = \dim FI_{n+1} - \dim FI_n,$$

and hence the conclusion follows from the quasipolynomial character of (3.4). □

As an immediate consequence, we have the announced result on dimensions:

Corollary 3.5. *For G and M as in Theorem 3.4 the sequence*

$$n \mapsto \dim \Omega^n M$$

is eventually quasipolynomial in n, and similarly for Ω^{-n}

Proof. Simply take F of Theorem 3.4 to be the forgetful functor from G-modules to vector spaces. □

The same goes for lengths rather than dimensions:

Corollary 3.6. *For G and M as in Theorem 3.4 and a simple module $S \in \operatorname{mod} kG$ the sequence*

$$n \mapsto \ell_S(\Omega^n M)$$

is eventually quasipolynomial in n, and similarly for Ω^{-n}

Proof. This is an application of Theorem 3.4 with $F := \operatorname{Hom}_G(P_S, -)$, where $P_S \to S$ is the projective cover: ℓ_S can be recovered as $\dim F(-)$. □

As far as recursion goes, we now have

Corollary 3.7. *Let G be a finite group as before, and $S, M \in \operatorname{mod} kG$ a simple and an arbitrary G-module respectively. For exact functors $F : \operatorname{mod} kG \to \operatorname{Vect}^f$ as in Theorem 3.4 or $F = \operatorname{Hom}_G(S, -)$ the sequence*

$$n \mapsto \dim F(\Omega^n M)$$

is eventually linearly recursive, and the same goes for Ω^{-n}.

Proof. This follows from Proposition 3.2 and Theorem 3.4 and the fact that eventually quasipolynomial sequences are eventually linearly recursive. □

Next, note that for every Laurent polynomial $P \in \mathbb{N}[x^{\pm 1}]$ we can talk about the functor $P(\Omega)$ (written $P\Omega$ for brevity), with addition being interpreted as direct sum. We have the following amplification of Corollary 3.7.

Theorem 3.8. *For F and M as in Corollary 3.7 and an eventually linearly recursive sequence of polynomials*

$$\mathcal{P} = (P_n)_n \subset \mathbb{N}[x]$$

the sequences

$$n \mapsto \dim F(P_n \Omega M)$$

and

$$n \mapsto \dim F(P_n \Omega^{-1} M)$$

are eventually linearly recursive.

Proof. To fix ideas, we prove the version about Ω. Denoting

$$a_n = \dim F(\Omega^n M) \text{ and } b_n = \dim F(P_n \Omega M)$$

we have

$$\mathbf{b} := (b_n)_n = \mathcal{P} \triangleright \mathbf{a} \text{ for } \mathbf{a} := (a_n)_n$$

with '$\triangleright$' as in Definition 1.13. The conclusion thus follows from Proposition 1.15. □

On the other hand, for *Laurent* (as supposed to ordinary) polynomials we have the following version.

Theorem 3.9. *For F and M as in Corollary 3.7 and an eventually linearly recursive sequence of Laurent polynomials*

$$\mathcal{P} = (P_n)_n \subset \mathbb{N}[x^{\pm 1}]$$

the sequence

$$n \mapsto \dim F(P_n \Omega M) \tag{3.5}$$

is algebraic.

Proof. We can proceed as in the proof of Theorem 3.8, this time using Theorem 1.18 and noting that the sequence (3.5) is (essentially, up to irrelevant shifts) $\mathcal{P} \triangleright (\mathbf{a}, \mathbf{b})$ for

$$\mathbf{a} = (\dim F(\Omega^n M))_n \quad \text{and} \quad \mathbf{b} = (\dim F(\Omega^{-n} M))_n. \quad \square$$

4. Invariant sequences for Omega-algebraic modules

Recall the definitions of Ω and $\Omega^{\pm}$-algebraic modules from the Introduction (Definition 0.1 and Definition 0.2). We introduce some notation:

- Let M be a finite-dimensional Ω-algebraic kG-module.
- Let $N_1, ..., N_s$ be the Ω-orbit representatives of the various non-projective indecomposable summands that appear in $M^{\otimes n}$, including $N_1 = k$ for $k = M^{\otimes 0}$.
- Let $T = (t_{ij})$ be the $k \times k$ matrix whose rows give the effect of tensoring with M. So,

$$core_G(M \otimes N_i) = \bigoplus_{j=1}^{s} t_{ij}(N_j) \tag{4.1}$$

where t_{ij}'s are Laurent polynomials in Ω.

Proposition 4.1. *If M is an Omega-algebraic non-projective indecomposable G-module, then the sequence $c_n^G(M)$ is the sum of the dimensions of the entries of the first row of the matrix T^n, i.e.*

$$c_n^G(M) = \sum_{j=1}^{s} \dim t_{1j}^{(n)}(N_j)$$

where $T^n = (t_{ij}^{(n)}) \in M_s(\mathbb{N}[\Omega^{\pm 1}])$.

Proof. Letting $N_1 = k$ as mentioned before, we have

$$\begin{aligned} core_G(M) &= \bigoplus_{j=1}^{s} t_{1j}(N_j) \\ core_G(M^{\otimes 2}) &= \bigoplus_{j=1}^{s} core_G(t_{1j}(M \otimes N_j)) \\ &= \bigoplus_{j=1}^{s} t_{1j}\Big(\bigoplus_{l=1}^{s} t_{jl}(N_l)\Big), \end{aligned}$$

etc. The proof follows by induction. □

Remark 4.2. Proposition 4.1 hinges on the fact that when regarded as functors on the *stable module category* of G (e.g. [1, §2.1]) the functors $M \otimes -$ and $\Omega^{\pm 1}$ commute. ♦

Corollary 4.3. *Let M be an Omega-algebraic G-module. Let $N_1 = k, \cdots, N_s$ be the Ω-orbit representatives of the various non-projective indecomposable summands that appear in $M^{\otimes n}$ with $N_1, ..., N_r$ being the Ω-orbit representatives of the non-projective indecomposable summands of M, with*

$$core_G(M) = \bigoplus_{i=1}^{r} q_i(N_i)$$

where q_i's are Laurent polynomials in Ω. Let T be the matrix that gives the effect of tensoring with M. Then we have,

$$c_n^G(M) = \sum_{i=1}^{r} \sum_{j=1}^{s} \dim q_i p_{ij}(N_j)$$

where $T^n = (p_{ij})$ with p_{ij}'s being Laurent polynomials in Ω.

Proof. Let

$$core_G(M \otimes N_i) = \bigoplus_{j=1}^{s} t_{ij}(N_j)$$

where t_{ij}'s are Laurent polynomials in Ω.

$$\begin{aligned} core_G(M \otimes M) &= core_G(M \otimes \bigoplus_{i=1}^{r} q_i(N_i)) \\ &= core_G(\bigoplus_{i=1}^{r} q_i(M \otimes N_i)) \\ &= \bigoplus_{i=1}^{r} q_i\Big(\bigoplus_{j=1}^{s} t_{ij}(N_j)\Big) \end{aligned}$$

The proof now follows from Proposition 4.1. □

Theorem 4.4. *[3, Conjecture 14.2] holds for Omega$^+$ and Omega$^-$-algebraic modules M.*

Proof. The two claims are analogous, so we focus on the Omega$^+$ case.

By Proposition 4.1 and the assumption that M is Omega$^+$-algebraic $c_n^G(M)$ is the sum of the dimensions of

$$t_{1j}^{(n)} N_j,\ 1 \le j \le s,$$

where $t_{1j}^{(n)}$ are the respective entries of the n^{th} power T^n of an $s \times s$ matrix over $\mathbb{N}[\Omega]$.

By the Cayley-Hamilton theorem (over the ring of polynomials in Ω) the sequences $(t_{1j}^{(n)})_n$ are all recursive. The conclusion now follows from Theorem 3.8 applied to said sequences (with N_j respectively in place of M). □

As an immediate consequence, we have

Corollary 4.5. *For any Omega-algebraic $M \in \operatorname{mod} kG$ the sequences*

$$(c_n^G(\Omega^d M))_n \text{ and } (c_n^G(\Omega^{-d} M))_n$$

are eventually linearly recurrent for sufficiently large d.

Proof. Indeed, for sufficiently large d $\Omega^d M$ is Omega$^+$-algebraic while $\Omega^{-d} M$ is Omega$^-$-algebraic. The conclusion follows from Theorem 4.4. □

As to the Ω-algebraic analogue of Theorem 4.4:

Theorem 4.6. *For an Ω-algebraic M the sequence $(c_n^G(M))_n$ is algebraic.*

Proof. As in the proof of Theorem 4.4, except now the polynomials are Laurent and we use Theorem 3.9 in place of Theorem 3.8. □

On the other hand, the number $s_n^G(M)$ of indecomposable summands of $core_G(M^{\otimes n})$ is better behaved:

Theorem 4.7. *For an Ω-algebraic M the sequence $(s_n^G(M))_n$ is eventually linearly recursive.*

Proof. Since Ω preserves (in)decomposability, the effect on s_n of tensoring by M is given as in (4.1), upon substituting 1 for Ω in the matrix entries t_{ij} and also 1 for each indecomposable N_j. In other words, $s_n^G(M)$ can be recovered as the sum of the entries of A^n for a scalar matrix A; clearly, this is a recursive sequence. □

Remark 4.8. It follows from Theorem 4.7 and [3, Theorem 13.2] that for an Ω-algebraic module M, the invariant $\gamma_G(M)$, as defined in [3, Definition 1.1], will always be an algebraic integer. ♦

5. Examples

Recall that $c_n^G(M)$ is the sequence of dimensions of the core of $M^{\otimes n}$ whereas $s_n^G(M)$ is the sequence of the number of indecomposable summands of the core of $M^{\otimes n}$. In this section, we will see some examples to demonstrate that this sequence is eventually polynomial or recurrent. These computations are performed using the computer algebra system Magma [6].

(1) Let G be the cyclic group of order 7, k be a field of characteristic 7 and M be the indecomposable kG-module of dimension 2. Then, the sequence

$$c_n^G(M) =< 2, 4, 8, 16, 32, 57, 114, 193, 386, 639, 1278, 2094, 6829, ... >,$$

and

$$s_n^G(M) =< 1, 2, 3, 6, 10, 19, 33, 61, 108, 197, 352, 638, 1145, 2069, ... > .$$

The sequences above satisfy the relation

$$x_n = 5x_{n-2} - 6x_{n-4} + x_{n-6}.$$

(2) Let $G = \mathcal{S}_{10}$, k be a field of characteristic 5 and M be the permutation module of the symmetric group $\mathcal{S}_{10}$ labelled by the partition $\lambda = (9, 1)$. Then, the sequence

$$s_n^G(M) =< 1, 4, 19, 94, 469, 2344, ... >,$$

which satisfies the relation

$$x_n = x_{n-1} + 25x_{n-2} - 25x_{n-3}.$$

(3) Let $G = \mathcal{S}_9$, k be a field of characteristic 3. Young modules are indecomposable summands of the permutation modules of the symmetric group and are also labelled by partitions of n as shown in [10]. Let $M = Y^\lambda$, the Young module corresponding to the partition $\lambda = (7, 2)$. Then, the sequence

$$s_n^G(M) =< 1, 4, 35, 310, 2789, 25096, ... >,$$

which satisfies the relation

$$x_n = 9x_{n-1} + x_{n-2} - 9x_{n-3}.$$

(4) Let $G = \langle g, h\rangle \cong \mathbb{Z}/3 \times \mathbb{Z}/3$ and $k = \mathbb{F}_3$. Let M be the six-dimensional module given by the following matrices:

$$g \mapsto \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad h \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

This is precisely [3, Example 15.1], on which we now elaborate. First, as noted in [3], M is Omega-algebraic: if $N = k_{\langle g\rangle} \uparrow^G$, then

$$\begin{aligned} core_G(M \otimes M) &\cong \Omega(M) \oplus \Omega^{-1}(M^*) \oplus N \\ core_G(M \otimes M^*) &\cong \Omega^{-1}(M) \oplus \Omega(M^*) \oplus N \\ core_G(M \otimes N) &\cong 3\Omega(N). \end{aligned} \tag{5.1}$$

Remark 5.1. It is also mentioned in [3, Example 15.1] that M is not algebraic. Indeed, it can be shown that in Craven's taxonomy of 6-dimensional indecomposable G-modules, it belongs to class P in the table from [8, §3.3.5]. Indeed, M

- has socle layers of dimensions 2,2,2, as can easily be seen either directly or from the diagram displayed next to the two matrices in [3, Example 15.1];
- has dual with socle layers of dimensions 2,3,1, as is again easily seen from the fact that in passing from M to M^* one can simply transpose the matrices corresponding to the generators g and h.

Jointly, these remarks eliminate all possibilities in [8, table, §3.3.5] except for classes P and I*. The only distinction noted in [8] between the two is the cardinality of the set of conjugates under the action of the automorphism group Aut G: 4 for P and 8 for I*. Now, Aut G has order 48, so it will be enough to check whether the isotropy group of (the isomorphism class of) M contains a subgroup of order 4: if it does the class must be P, and it will be I* otherwise.

To conclude, simply note that the Klein 4-group generated by the automorphisms that square one of the two generators and fix the other one fixes M: conjugation by $\mathrm{diag}(2,1,2,1,2,1)$ maps

$$g \mapsto g \quad \text{and} \quad h \mapsto h^2,$$

whereas conjugation by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

gives the other automorphism

$$g \mapsto g^2 \quad \text{and} \quad h \mapsto h.$$

To reiterate, this means that the isotropy group of M in Aut G has order divisible by 4, and hence the size of the orbit must divide $\frac{48}{4} = 12$. In particular that size cannot be 8, ruling out class I* from [8, table, §3.3.5]. ♦

Let T be the 3×3 matrix whose rows give the effect of tensoring the non-projectives with M.

	M	M^*	N
M	Ω	Ω^{-1}	1
M^*	Ω^{-1}	Ω	1
N	0	0	3Ω

Hence,

$$T = \begin{pmatrix} \Omega & \Omega^{-1} & 1 \\ \Omega^{-1} & \Omega & 1 \\ 0 & 0 & 3\Omega \end{pmatrix} \qquad \text{and} \qquad T^n = \begin{pmatrix} A_n & B_n & C_n \\ B_n & A_n & C_n \\ 0 & 0 & (3\Omega)^n \end{pmatrix}$$

where A_n, B_n and C_n are Laurent polynomials in Ω described as follows:

$$A_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} \Omega^{(n-4i)} \tag{5.2}$$

$$B_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i+1} \Omega^{(n-(4i+2))} \tag{5.3}$$

$$C_n = \sum_{k=1}^{n} \alpha_n^{(k)} \Omega^{(n-(2k-1))}$$

where

$$\alpha_n^{(k)} = \sum_{i=0}^{k} (-1)^i \left[\binom{k+1}{i+1} + 2\binom{k}{i} \right] \alpha_{n-1-i}^{(k)}$$

are linear recurrence relations with the initial conditions:

$$\alpha_t^{(k)} = 0, \quad \text{if} \quad 0 \le t < k$$
$$\alpha_k^{(k)} = 1.$$

The characteristic equation of T is

$$x^3 - 5\Omega x^2 - (\Omega^{-2} - 7\Omega^2)x - (3\Omega^3 - 3\Omega^{-1}) = 0.$$

So by the Cayley-Hamilton Theorem over the ring $\mathbb{C}[\Omega, \Omega^{-1}]$, the sequences A_n, B_n and C_n satisfy the recurrence relation

$$x_n = 5\Omega x_{n-1} + (\Omega^{-2} - 7\Omega^2)x_{n-2} + (3\Omega^3 - 3\Omega^{-1})x_{n-3}$$

for $n \ge 4$.
The number $c_n^G(M)$ can now be recovered as

$$c_n^G(M) = \dim A_n(M) + \dim B_n(M^*) + \dim C_n(N). \tag{5.4}$$

We do not know whether this ends up being eventually linearly recursive (as opposed to just algebraic Theorem 4.6), but we end with a few remarks on the matter.
First, note that the third summand $\dim C_n(N)$ is unproblematic here, as it is indeed linearly recursive. To see this, note that N is periodic because its restriction to the maximal subgroup $\langle h \rangle \subset G$ is projective [4, Corollary 2.24.7]. Furthermore, this implies that it is periodic of period 1 or 2 [7, Theorem 6.3]. But then the recursion

$$C_{n+1} = (\Omega + \Omega^{-1})C_n + (3\Omega)^n$$

implies that we can substitute Ω for Ω^{-1} in the formula above, and can hence conclude as in the Ω^+-algebraic case covered by Theorem 4.4.

The other two terms in (5.4) seem more difficult to tackle. Observe that since M is not periodic, it must have complexity 2 (because we are working over a 3-group of rank 2 [4, Theorem 2.24.4 (xv)]). This implies that $\dim \Omega^n M$ and all of its analogues ($\dim \Omega^n(M^*)$, etc.) are eventually polynomials of degree 1.

It follows from the above, for instance, that the multiplicity of M in $core_G(M^{\otimes n})$ cannot be eventually linearly recursive: the number of terms in $A_n(M)$ and $B_n(M^*)$ (per the decompositions in (5.2) and (5.3)) that can be isomorphic to M is, for dimension reasons, uniformly bounded in n and concentrated around the middle of the range in either of those two sums, so the multiplicities in question are sums of binomial coefficients of the form

$$\binom{n}{\lfloor \frac{n}{2} \rfloor + k}$$

with k ranging over a fixed interval centered at 0. Such binomial coefficients do not form linearly recursive sequences: see e.g. [19, Example 6.3.2].

On the other hand, as per Theorem 4.7, $s_n := s_n^G(M)$ is a recursive sequence: (5.1), together with the fact that $M \otimes -$ and $\Omega^{\pm 1}$ commute module projective summands, makes it clear that each iteration of tensoring with M will triple the number of indecomposable, non-projective summands. We thus have $s_n = 3^{n-1}$.

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