Existence of equilibria in countable games: an algebraic approach

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Highlights of Logic, Games and Automata
Paris, September 2013

Literature

- Nash(1950, 1951) used fixed point theorems to prove that any finite game admits an equilibrium in mixed strategies.
- The result fails in general if the strategy sets are not finite.
- Several authors have provided conditions under which even infinite games admit an equilibrium. Among them Debreu (1952), Glicksberg (1952), Fan (1952) and, more recently, Dasgupta and Maskin (1986), Simon (1987), Simon and Zame (1990), Reny (1999), Carmona (2005, 2010), Barelli and Soza (2010), Barelli, Govindan, and Wilson (2012), Bich and Laraki (2012) and several others.
- Most of this literature uses topological conditions on the strategy sets and the payoff functions.

Countable games

- Wald (1945) considered two-person zero-sum games and showed that the mixed extension of a game has a value if one of the strategy sets is finite, but in general it doesn't if they are both countable.
- One way to overcome the lack of equilibria in some games is to enlarge the set of mixed strategy by including also finitely additive probability measures.

Finitely additive strategies

- The main issue along this road is that in general a mixed extension is not well defined.
- Several approaches were proposed in the framework of zero-sum two-person games, for instance by Yanovskaya (1970), Kindler (1983), Heath and Sudderth (1972), Schervish and Seidenfeld (1996), and others.
- Finitely additive mixed strategies have been used in more general contexts by Maitra and Sudderth (1993, 1998), Cotter (1991), Marinacci (1997), Harris, Stinchcombe, and Zame (2005), Stinchcombe (2005, 2011), Capraro and Morrison (2011).

This talk

- We prove an existence result for Nash equilibria of countable games by imposing some algebraic conditions on the payoff functions.
- The strategy set of each player is assumed to be a countable group and the payoff functions depend on their arguments only through the group operation.
- No topological condition is required.
- We allow finitely additive mixed strategies defined on the power set of the group.
- We show the expression of the equilibrium strategies.
- The equilibrium payoffs have a very simple form.
- We extend the result to uncountable games.

Matching pennies

	A	B
A	$-1, \quad 1$	$1, \quad -1$
B	$1, \quad -1$	$-1, \quad 1$

The unique equilibrium of this game is the profile of mixed strategies $((1/2, 1/2), (1/2, 1/2))$.

Groups

- A set G with a binary operation $*$ is called a group if the operation is associative, it has a unit element, and every element has an inverse.
- If the operation is commutative, then the group is called abelian.
- Make the set $\{A, B\}$ a finite group by endowing it with the binary operation $*$ defined as

$$A * A = B * B = B, \quad A * B = B * A = A.$$

- Define $\phi : \{A, B\} \to \mathbb{R}$ as follows:

$$\phi(x) = \begin{cases} 1 & \text{for } x = A, \\ -1 & \text{for } x = B. \end{cases}$$

Matching pennies, re-written

- Consider a game played by players 1 and 2, where each player's pure strategy set is $\{A, B\}$ and the payoffs are

$$u_1(x, y) = -u_2(x, y) = \phi(x * y), \quad \text{for } x, y \in \{A, B\}.$$

- The game that we just described is nothing else than the matching pennies game defined before.

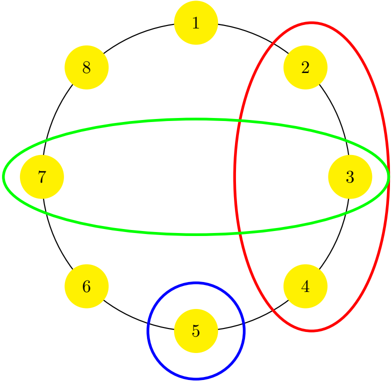
Generalization

- Consider a finite group $(G, *)$ and N functions $\phi_1, \ldots, \phi_N : G \to \mathbb{R}$.
- Given a set of players $P = \{1, \ldots, N\}$, for $i \in P$ let $u_i : G^N \to \mathbb{R}$ be defined as

$$u_i(x_1, \ldots, x_N) = \phi_i(x_1 * \cdots * x_N). \tag{1}$$

- For $\phi := (\phi_1, \ldots, \phi_N)$, call $\mathscr{G}(P, G, \phi)$ the game where the set of players is P, each player's set of pure strategies is G, and player i's payoff function is given by (1).

Example



Equilibrium

- Call $\mathscr{P}(G)$ the set of all probability measures on 2^G.
- A probability measure $\lambda \in \mathscr{P}(G)$ is invariant if for all $x, y \in G$ we have $\lambda(x) = \lambda(x * y)$.
- Finite groups have a unique invariant measure, that is, the uniform measure.

Proposition

The game $\mathscr{G}(P, G, \phi)$ admits an equilibrium in mixed strategies $(\lambda, \ldots, \lambda)$, with λ invariant on G.

Countable games

- Given a set of players $P = \{1, \ldots, N\}$, a countable set S and bounded functions $u_i : S^N \to [0, 1]$, $i \in P$, consider a game $\mathscr{G} = \langle P, S, (u_i)_{i \in P} \rangle$, where S is the strategy set of all players, and u_i is the payoff function of player i.
- Existence of mixed equilibria may fail if only countably additive mixed strategies are allowed.
- Therefore we consider a mixed extension of the game $\mathscr{G}$ where the space of mixed strategies is $\mathscr{P}(S)$, the space of all finitely additive probability measures on S.

Selection

- A selection problem immediately arises.
- Given $\mu_1, \dots, \mu_N \in \mathscr{P}(S)$, a product measure $\otimes_{i=1}^N \mu_i$ is uniquely defined only on the algebra generated by the cylinders $S \times \cdots \times S \times A \times S \cdots \times S$, for all $A \subset S$.
- This product measure can be (non-uniquely) extended to the power set $2^{S \times \cdots \times S}$. Different extensions correspond to different values of the expected payoff $\int_{S \times \cdots \times S} u \, \mathrm{d} \otimes_{i=1}^N \mu_i$.
- As a consequence, Fubini's theorem cannot be applied to this situation and in general the order of integration of a multiple integral matters.
- We consider a parametric class of possible extensions that has the advantage of being easily computable.

Selection, continued

- The extension is obtained by averaging over all possible orders of integration. Formally:
- Call $\Sigma(P)$ the space of permutations of P.
- Let $\nu \in \mathscr{P}(\Sigma(P))$ and $\mu_1, \ldots, \mu_N \in \mathscr{P}(S)$.
- For $i \in P$, let $u_i : S^N \to [0, 1]$. Define

$$
u_i^\nu(\mu_1, \ldots, \mu_N) := \sum_{\pi \in \Sigma(P)} \nu(\pi) \int_S \cdots \int_S u_i(x_1, \ldots, x_N) \, \mathrm{d}\mu_{\pi(1)}(x_{\pi(1)}) \ldots \mathrm{d}\mu_{\pi(N)}(x_{\pi(N)}). \quad (2
$$

- This defines an extension of $\otimes_{i=1}^N \mu_i$ to $2^{S \times \cdots \times S}$.

Theorem

- Let $(G, *)$ be a countable group and given $\phi_1, \ldots, \phi_N : G \to [0, 1]$, define

$$u_i(s_1, \ldots, s_N) = \phi_i(s_1 * \cdots * s_N). \tag{3}$$

- For $\phi = (\phi_1, \ldots, \phi_N)$ call $\mathscr{G}(P, S, \phi, \nu)$ the mixed extension of the game $\mathscr{G}$ when u_i is defined as in (3) and the product measure of the finitely additive mixed strategies is selected as in (2).

Theorem

If $(G, *)$ is a countable abelian group, then the game $\mathscr{G}(P, G, \phi, \nu)$ admits a Nash equilibrium that does not depend on ν.

Invariant measures

Let G be a countable group, $A \subset G$ and $g \in G$. Call

$$g * A = \{g * a : a \in A\} \qquad A * g = \{a * g : a \in A\}.$$

Definition

A finitely additive probability measure μ on 2^G is called invariant mean, if

$$\mu(A) = \mu(g * A) = \mu(A * g) \quad \text{for all } g \in G \text{ and } A \subset G,$$

For a given countable group G, we call $\mathscr{I}(G)$ the set of invariant means on G.

Abelian groups admit invariant means.

How many invariant measures?

How many invariant measures does a countable Abelian group have?

(a) Zero.
(b) One.
(c) A finite number.
(d) A countable number.
(e) A continuum.
(f) None of the above.

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Examples of invariant means on $\mathbb{Z}$

- Let α_n be the uniform measure on $\{-n, \ldots, n\}$. As $n \to \infty$ we get an invariant mean α such that $\alpha(\mathbb{N}) = 1/2$.
- Let β_n be the uniform measure on $\{-n, \ldots, 2n\}$. As $n \to \infty$ we get an invariant mean β such that $\beta(\mathbb{N}) = 2/3$.
- Let γ_n be the uniform measure on $\{-n, \ldots, n^2\}$. As $n \to \infty$ we get an invariant mean γ such that $\gamma(\mathbb{N}) = 1$.

Variational principle

Theorem

If G is a countable abelian group and $f : G \to [0,1]$, then for all $\pi \in \Sigma(P)$ and all $\lambda_2, \ldots, \lambda_N \in \mathscr{I}(G)$ the functional $\Psi : \mathscr{P}(G) \to \mathbb{R}$ defined as

$$\Psi(\mu) = \int \ldots \int \int f(x_1 * \cdots * x_N)\, \mathrm{d}\mu(x_{\pi(1)})\, \mathrm{d}\lambda_2(x_{\pi(2)}) \ldots \mathrm{d}\lambda_N(x_{\pi(N)})$$

attains its maximum at some $\lambda \in \mathscr{I}(G)$.

The space $\mathscr{P}(G)$ is compact in the weak* topology, but the functional Ψ is not continuous. So the existence of its maximum is a priori not guaranteed.

Equilibrium strategies

Assume that G is a countable abelian group and, for $i \in P$, let $\phi_i : G \to [0, 1]$ and

$$\overline{\lambda}_i \in \arg\max \left\{ \int \phi_i(x) \, \mathrm{d}\lambda(x) : \lambda \in \mathscr{I}(G) \right\}.$$

Theorem

If G is a countable abelian group, then the profile of strategies $(\overline{\lambda}_1, \ldots, \overline{\lambda}_N)$ is a Nash equilibrium for the game $\mathscr{G}(P, G, \phi, \nu)$.

Transformation of group operations

Let $\eta_1, \ldots, \eta_N : G \to G$ be bijections and let

$$u_i^{\eta}(x_1, \ldots, x_N) = \phi_i(\eta_1(x_1) * \cdots * \eta_N(x_N)) \quad \text{for } i \in P. \tag{4}$$

For $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_N)$ call $\mathscr{G}(P, G, \phi, \boldsymbol{\eta}, \nu)$ the game where the payoffs are given by the mixed extensions of (4), as in (2).

Theorem

If G is a countable abelian group, then the game $\mathscr{G}(P, G, \phi, \boldsymbol{\eta}, \nu)$ admits a Nash equilibrium.

Graph games

- For every $i \in P$ let $P_i \subset P$ be such that $i \in P_i$ and $|P_i| \geqslant 2$.
- Players are nodes of a directed graph and P_i are the neighbors of player i (including i herself).
- Consider functions $\phi_i : G \to [0, 1]$ such that

$$u_i(x_1, \ldots, x_N) = \phi_i(*_{j \in P_i} x_j), \tag{5}$$

  player i's payoff depends only on her neighbors' strategies.
- Call $\boldsymbol{P} = (P_1, \ldots, P_N)$ and define the game $\mathscr{G}(P, \boldsymbol{P}, G, \phi, \nu)$ where the payoff functions are as in (5).

Theorem

If G is a countable abelian group, then the game $\mathscr{G}(P, \boldsymbol{P}, G, \phi, \nu)$ admits a Nash equilibrium.

Games on $\mathbb{Z}$

When the game is played on $\mathbb{Z}$, each equilibrium strategy has its mass either entirely adherent to $-\infty$ or to $+\infty$, or if it is split between the two, then it can be split in any possible way.

Proposition

Consider a game $\mathscr{G}(P, \mathbb{Z}, \phi, \nu)$. Let $(\lambda_1, \ldots, \lambda_N)$ be an equilibrium for this game, where $\lambda_1, \ldots, \lambda_N \in \mathscr{I}(\mathbb{Z})$. Then, for all $i \in P$, one of the following three possibilities is true

(a) $\lambda_i(\mathbb{N}) = 0$,

(b) $\lambda_i(\mathbb{N}) = 1$,

(c) *if $0 < \lambda_i(\mathbb{N}) < 1$, then for any $\delta \in [0, 1]$ there exists another equilibrium strategy $\lambda_i' \in \mathscr{I}(\mathbb{Z})$ with $\lambda_i'(\mathbb{N}) = \delta$.*

Countable matching pennies

- The strategy set of each of the two players is $\mathbb{Z}$ and the payoff functions are

$$u_1(x, y) = 1 - u_2(x, y) = \mathbb{1}_{2\mathbb{Z}}(x + y),$$

  where $k\mathbb{Z}$ is the set of multiples of k.
- This game is equivalent to the one where players choose only Odd or Even and player 1 wins if both players make the same choice.
- Any profile of strategies (μ_1, μ_2) such that $\mu_1(2\mathbb{Z}) = \mu_2(2\mathbb{Z}) = 1/2$ is an equilibrium of the game.

Generalization of matching pennies

- Consider a partition $A_1, \ldots, A_N$ of $\mathbb{Z}$ and payoff functions
$$u_i(x_1, \ldots, x_N) = \mathbb{1}_{A_i}(x_1 + \cdots + x_N).$$
- If for each $i \in P$ the measure
$$\overline{\lambda}_i \in \arg \max_{\lambda_i \in \mathscr{I}(\mathbb{Z})} \lambda_i(A_i),$$
then, by Theorem 5, the profile $(\overline{\lambda}_1, \ldots, \overline{\lambda}_N)$ is a Nash equilibrium of the game.
- If all sets $A_1, \ldots, A_N$ are periodic (not necessarily with the same period), then for all $\lambda, \lambda' \in \mathscr{I}(\mathbb{Z})$ we have
$$\lambda(A_i) = \lambda'(A_i),$$
- Therefore any profile of invariant measures is an equilibrium.

Generalization of matching pennies, continued

- Let m be the lowest common multiple of the periods m_i of the A_i's.
- Consider now the m congruence classes $m\mathbb{Z} + k$.
- Any profile of probability measures $(\mu_1, \ldots, \mu_N)$ such that for $i \in P$ and $k \in \{0, \ldots, m-1\}$

$$\mu_i(m\mathbb{Z} + k) = \overline{\lambda}_j(m\mathbb{Z} + k) = 1/m$$

is an equilibrium, too.

Wald's game

- Wald (1945) introduced this game as a counterexample to the existence of minmax in zero-sum two-person games when the sets of strategies for both players are infinite.
- Let the strategy set be $\mathbb{Z}$ and

$$u_1(x,y) = 1 - u_2(x,y) = \begin{cases} 1 & \text{if } x > y, \\ 1/2 & \text{if } x = y, \\ 0 & \text{if } x < y. \end{cases}$$

Wald's game, continued

- Call $z := -y$; then the payoff function becomes

$$u_1(x, z) = \begin{cases} 1 & \text{if } x + z > 0, \\ 1/2 & \text{if } x + z = 0, \\ 0 & \text{if } x + z < 0. \end{cases}$$

- Applying Theorem 6 we obtain the equilibrium (λ, ρ) with $\lambda, \rho \in \mathscr{I}(\mathbb{Z})$ and $\lambda(\mathbb{N}) = \rho(-\mathbb{N}) = 1$.
- This shows the striking difference between countably additive and finitely additive extensions of countable games.

Games on $\mathbb{Q} \cap [0, 1]$ mod 1

- Take $G = \mathbb{Q} \cap [0, 1]$ equipped with the sum modulo 1.
- Then G is a countable abelian group.

Proposition

For every $\lambda \in \mathscr{I}(G)$ and every $a, b \in [0, 1]$, $a < b$

$$\lambda([a, b]) = b - a,$$

where $[a, b] := \{x \in G : a \leqslant x \leqslant b\}$.

- Therefore any two invariant means on G coincide on the algebra generated by intervals in G, but they can be extended in many different ways to 2^G.

Cardinality

Proposition (Chou (1976))

Call $\mathfrak{c}$ the cardinality of $\mathbb{R}$. Then $|\mathscr{I}(G)| \geqslant 2^{\mathfrak{c}}$.

- Consider now a game on G where for $i \in P$,

$$\phi_i(x) = \mathbb{1}_{A_i}(x).$$

- If $A_1, \ldots, A_N$ are in the algebra generated by intervals, then every profile $(\lambda_1, \ldots, \lambda_N)$, with $\lambda_i \in \mathscr{I}(G)$ is an equilibrium.
- Otherwise the equilibrium strategy for player i is

$$\overline{\lambda}_i \in \arg \max_{\lambda \in \mathscr{I}(G)} \lambda(A_i).$$

Uncountable games

Definition

A locally compact topological group G is called amenable if there exists a linear operator $T : L^\infty(G) \to \mathbb{R}$ verifying the following properties:

Positivity. If $f : G \to \mathbb{R}_+$, then $T(f) \geqslant 0$.

Normalization. If $f \equiv 1$, then $T(f) = 1$.

Invariance. For all $f \in L^\infty(G)$ and for all $g \in G$, one has

$$T(L_g f) = T(f) = T(R_g f). \tag{6}$$

A positive and normalized linear operator T that verifies the first (second) equality in (6) is called left-invariant (right-invariant) mean; an operator that is both left- and right-invariant is called invariant mean.

Theorem

Let G be an amenable, locally compact, metric group such that left-invariant and right-invariant means coincide. Then the game $\mathscr{G}(P, G, u, \nu)$ admits Nash equilibria which do not depend on ν.

Takeaways

- We have considered a class of games where the strategy sets are a countable group and the payoff functions depend on the strategies only through the group operation.
- We have shown that finitely additive equilibria exist for this class of games.
- We have not used any topological conditions, just the algebraic structure of the payoffs.
- The most important tool is the invariant mean.
- The only measure theoretical assumption in our paper refers to the selection of the product of finitely additive mixed strategies.
- The algebraic condition that we use includes cases that are not covered in the literature.

Open problems

- What are the properties of the class of product measures of a fixed set of finitely additive marginals?
- What is its cardinality?
- Can the extreme points of this class be identified?
- Can algebraic methods be used to find equilibria for some other class of games?
- For instance, can they be used for countable harmonic games?