

# NIP $\omega$ -categorical structures: the rank 1 case

Pierre Simon\*

## Abstract

We classify primitive, rank 1,  $\omega$ -categorical structures having polynomially many types over finite sets. For a fixed number of 4-types, we show that there are only finitely many such structures and that all are built out of finitely many linear or circular orders interacting in a restricted number of ways. As an example of application, we deduce the classification of primitive structures homogeneous in a language consisting of  $n$  linear orders as well as all reducts of such structures.

## 1 Introduction

Since the work of Lachlan on finite homogeneous structures, interactions between homogeneous structures and model theory have been very fruitful in both directions. Lachlan [Lac84] realized that the property of stability and the toolbox that comes with it were relevant in the finite case. Geometric stability theory had its birth in Zilber's work on totally categorical structures [Zil] and this in turn led to a fairly detailed understanding of the  $\omega$ -stable  $\omega$ -categorical structures ([CHL85], [Hru89]). Following a suggestion of Lachlan, this analysis was then generalized to smoothly approximable structures, first by Kantor, Liebeck, Macpherson [KLM89] in the primitive case using classification of finite simple groups and by Cherlin and Hrushovski [CH03] in the general case by model-theoretic methods. In that latter work, many features of simple theories first appeared. The present paper fits in this line of research and begins the study of yet another class of  $\omega$ -categorical structures defined by a model theoretic condition.

To define this class, let us restrict first to the case of structures homogeneous in a finite relational language (which we also call *finitely homogeneous*). If  $M$  is such a structure, then given any finite  $A \subseteq M$ , the number of 1-types over  $A$  (that is, the number of orbits under the stabilizer of  $A$ ) is finite. For a given  $n$ , we let  $f_M(n)$  be the maximal number of 1-types over a set  $A \subseteq M$  of size  $n$ . For instance, if  $M = (\mathbb{Q}, \leq)$ , then  $f_M(n) = 2n + 1$ . If  $M = (G, R)$  is a model of the random graph, then  $f_M(n) = 2^n + n$ . A well-known theorem of Sauer and Shelah implies that this function has either polynomial or exponential growth.

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\*Partially supported by NSF (grants no. 1665491 and 1848562) and a Sloan fellowship.

Following the unfortunate model-theoretic terminology, we call a finitely homogeneous structure  $M$  *NIP* if the function  $f_M$  has polynomial growth. (**NIP stands for the negation of the independence property**. We like to think of those structures as being *geometric* in some sense.) For instance, dense linear orders are NIP, whereas the random graph is not. Intuitively, NIP structures have no random-like behavior. Another important example of NIP structure is the Fraïssé limit of **finite trees (where a tree  $(T, \leq, \wedge)$  is a partial order such that the predecessors of a point form a chain, and  $a \wedge b$  is the infimum of  $\{a, b\}$ )**.

Within structures homogeneous in a finite relational language, there is another characterization of NIP obtained by counting orbits on unordered  $k$ -tuples, or equivalently finite substructures of size  $k$  up to isomorphism. If  $M$  is homogeneous in a finite relational language (or more generally an  $\omega$ -categorical relational structure), define  $\pi_M(k)$  as being the number of substructures of  $M$  of size  $k$ . Cameron showed in [Cam81] that this function is always non-decreasing and in [Cam76] he classified the case where  $\pi_M$  is constant equal to 1. Macpherson [Mac85] showed that if  $M$  is primitive, then  $\pi_M$  is either constant equal to 1 or grows at least exponentially. A number of structures for which the growth is no faster than exponential are given by Cameron in [Cam87]: they are all order-like or tree-like structures. Cameron also remarks there that those seem to be essentially the only examples of such structures known at the time. In [Mac87], Macpherson shows that for structures homogeneous in a finite relational language, there is a gap in the possible growth rates of the function  $\pi_M$ . Using the aforementioned Sauer–Shelah theorem, we can state a stronger version of his result: if  $M$  is NIP, then  $\pi_M(k) = O(2^{ck \ln k})$  for some  $c > 0$  (see the remark after Fact 2.8). If  $M$  has IP (is not NIP), then  $\pi_M(k) \geq 2^{p(k)}$  for some polynomial  $p(X)$  of degree at least 2. Hence homogeneous structures with  $\pi_M$  of exponential growth are a subclass of NIP homogeneous structures. See e.g. [Mac11, Section 6.3] for many more results on this function.

We conjecture that NIP finitely homogeneous structures can be reasonably well classified, and in particular that there are only countably many up to bi-interpretability. We will give some precise conjectures at the end of this paper. What we have in mind is that those structures are all built out of linear orders, possibly branching into trees. However, we are for now not capable of saying much in the general case, and introduce another condition, which should be thought of as forbidding trees in the structure: we ask that there is a rank function on definable sets satisfying certain axioms. This limits the size of a nested sequence of definable equivalence relations. In model theory, this condition is called *rosiness*. It is always satisfied by binary structures, so one may want to think of this work as studying binary NIP homogeneous structures, though our actual hypothesis are *a priori* more general. We will actually relax the homogeneity assumption to  $\omega$ -categoricity. Similarly, NIP, which we defined by counting types, becomes a condition on formulas. Under those hypotheses, we conjecture that the results on  $\omega$ -categorical  $\omega$ -stable and quasi-finite structures essentially go through *mutatis mutandis*. In particular, we should have coordinatization by rank 1 sets and quasi-finite axiomatization. We deal here only

with the rank 1 primitive case, for which we give a complete classification, up to inter-definability. The general finite rank case will be studied in subsequent work with Alf Onshuus.

As a rather straightforward application, we classify primitive homogenous multi-orders (also called finite-dimensional permutation structures): that is primitive structures homogeneous in a language consisting of  $n$  linear orders. For  $n = 2$ , this was solved by Cameron [Cam02] and for  $n = 3$  by Braunfeld [Bra18], where the general case is conjectured. We show that for any  $n$ , there is only one primitive homogeneous multi-order, where no two orders are equal or reverse of each other: the Fraïssé limit of finite sets with  $n$  orders. We also classify all reducts of such structures, generalizing the work of Linman and Pinsker [LP15] on the case  $n = 2$ .

Looking at it from the point of view of model theory, one can see this work as a development of the study of (rosy) NIP structures along the lines of stable theories. We hope that it will eventually lead to new insights into general NIP structures. At any rate, the results demonstrate that there is a richer theory of NIP than one suspected only a few years ago and that this world is much more structured and closer to stability than was expected. It does not seem completely unreasonable to hope for classification results for some subclasses of NIP in the spirit of Shelah’s classification for superstable theories, where cardinal dimensions will be complemented by isomorphism types of linear orders (which are shown to exist in [Sim18]). But we are not quite there yet.

## 1.1 Summary of results

We are concerned with structures  $M$  such that:

( $\star$ )  $M$  is an  $\omega$ -categorical, rank 1, primitive, unstable NIP structure, where “rank 1” means that there is no definable set  $D$  and uniformly definable family  $(X_t)_{t \in D}$  of infinite subsets of  $M$  which is  $k$ -inconsistent for some  $k$ : that is for any  $k$  values  $t_1, \dots, t_k \in D$ , we have  $X_{t_1} \cap \dots \cap X_{t_k} = \emptyset$ . Those hypotheses will be fully enforced only in Section 6. In sections before that, we study  $\omega$ -categorical linear and circular orders under a weakening of the rank 1 assumption, but make no use of NIP. Results there might be of some use in the classification of other classes of ordered homogeneous structures. We then give a fairly explicit description of structures satisfying ( $\star$ ) up to inter-definability. They all admit an interpretable finite cover composed of a disjoint union of linear and circular order, independent of each other.

Here are some examples of structures that satisfy the hypotheses.

EXAMPLE 1.1. • *A dense linear order or any of its 3 non-trivial reducts: a betweenness relation, circular order or separation relation.*

- *The Fraïssé limit of finite sets equipped with  $n$  orders.*
- *The class of structures equipped with two linear orders  $\leq_1, \leq_2$  and a binary relation  $R$  that satisfies  $a' \leq_1 a \ R \ b \leq_2 b' \Rightarrow a' R b'$  and  $\neg a R a$  is a*

*Fraïssé class. Its Fraïssé limit satisfies  $(\star)$ . This kind of structure will be studied in Section 3.1.*

- *The class of finite sets equipped with a circular order  $C$  and an equivalence relation  $E$  all of whose classes have exactly two elements is a Fraïssé class. The quotient by  $E$  of the Fraïssé limit of this class satisfies  $(\star)$ . It does not admit a circular order definable over  $\text{acl}^{eq}(\emptyset)$  but does have one definable over any one parameter.*

As a consequence of the classification we obtain the following theorems (the terminology will be explained later).

**Theorem 1.2.** *Given an integer  $n$ , there are, up to inter-definability, finitely many  $\omega$ -categorical primitive NIP structures  $M$  of rank 1 having at most  $n$  4-types.*

**Theorem 1.3.** *If  $M$  is an  $\omega$ -categorical, primitive, rank 1, NIP, unstable structure, then:*

1. *over  $\emptyset$ , there is an interpretable set  $W$ , which is a finite union of circular orders and admits a finite-to-one map to  $M$ ;*
2. *up to inter-definability,  $M$  is homogeneous in a finite relational language and finitely axiomatizable;*
3. *after naming a finite set of points,  $M$  admits elimination of quantifiers in a binary language and has a definable linear order;*
4.  *$M$  is distal of finite op-dimension;*
5.  *$M$  has trivial geometry:  $\text{acl}(A) = A$  for every  $A \subseteq M$ , equivalently the stabilizer of any finite  $A \subseteq M$  in the automorphism group of  $M$  has no finite orbit on  $M \setminus A$ .*

Statement 1 follows from the construction in Section 6. Statements 2 and 3 are proved in Section 6.6, along with distality. Statement 5 also follows from the discussion there. Finiteness of op-dimension is Proposition 6.6.

As regards homogeneous multi-orders, we prove the following.

**Theorem 1.4.** *Let  $(M; \leq_1, \dots, \leq_n)$  be homogeneous, primitive and such that each  $\leq_i$  defines a linear order on  $M$ . Assume that no two of those orders are equal or reverse of each other. Then  $M$  is the Fraïssé limit of finite sets with  $n$  orders.*

The proof of this last theorem requires only a small part of the paper, namely Sections 2, 3 and 7. The imprimitive case is classified in [BS], joint with Samuel Braunfeld.

## 1.2 Overview of the proof

Let  $M$  be an  $L$ -structure that satisfies  $(\star)$ . The starting point for this work is the result proved in [Sim18] that any NIP  $\omega$ -categorical unstable structure interprets a linear order. In fact more is true: Guingona and Hill introduce in [GH15] the notion of op-dimension, which tells us the maximal number of independent orders that a structure (or type) can have. The main theorem of [Sim18] says—in the  $\omega$ -categorical case—that if  $M$  is NIP of op-dimension at least  $n$ , then we can find some infinite definable set  $X$  on which we can interpret  $n$  linear orders. By transitivity of  $M$ , the family of conjugates of  $X$  covers  $M$ .

In Section 3, we show that any extra structure on a rank 1 linear order must be dense with respect to the order and that different definable orders can interact only in a few prescribed ways. This is extended to circular orders in Section 4. (Those sections make no use of NIP.) This allows us to glue the orders coming from various conjugates of  $X$  together. Each order might then wrap around itself, yielding a circular order. We construct in this way a 0-definable finite family  $W$  of linear and circular orders. We also show that this  $W$  is a finite cover of  $M$ , that is admits a finite-to-one map onto  $M$ .

We then have to analyze the additional structure on  $W$ . Using op-dimension, we show that any additional structure must come from stable formulas. By rank 1, those formulas cannot fork. Using finiteness of the number of non-forking extensions, those formulas can be defined from *local* equivalence relations with finitely many classes. Here *local* means that the equivalence relation is only defined locally, on bounded intervals of the orders, and may not glue as an equivalence relation on the whole structure. Such relations are studied in Section 5, in which a purely topological discussion shows that they must come from connected finite covers of circular orders.

## Acknowledgments

Thanks to Alf Onshuus, Dugald Macpherson, Udi Hrushovski, Gregory Cherlin and Sam Braunfeld for helpful discussions and for looking over parts of those results. Thanks also to David Bradley-Williams for pointing out the application to multi-orders. Finally, thanks to the referee for a very detailed report that helped a lot in improving the paper.

## 2 Preliminaries

### 2.1 Model theoretic terminology

We will use standard model-theoretic notation and terminology. Lowercase letters such as  $a, b, c$  will usually denote finite tuples of variables:  $a \in M$ , means  $a \in M^{|a|}$ . Similarly, variables  $x, y, z$  denote in general finite tuples of variables. We will sometimes write say  $\bar{a}, \bar{x}$  if we want to emphasize this.

For the sake of completeness, we recall some basic definitions. More details can be found in any introductory book on model theory, for instance [Mar02] or [TZ12]. We first give the general definitions that make sense in arbitrary structures, and then give equivalent formulations in terms of automorphism groups, that are only valid in the  $\omega$ -categorical case, over finite set of parameters.

We work in a structure  $M$ , in a countable language  $L$ . Let  $B \subseteq M$  be any set. A subset  $X \subseteq M^k$  is *definable over  $B$* , or  *$B$ -definable* if it is the solution set of a first-order formula  $\phi(x; b)$ , where  $b$  is a tuple of parameters from  $B$ . A set is *definable* if it is definable over some  $B$ . We write 0-definable to mean  $\emptyset$ -definable. The notation  $M \models \phi(a; b)$  and  $a \models \phi(x; b)$  both mean that  $\phi(a; b)$  is true in  $M$ . Since we will work throughout in a fixed structure  $M$ , we will usually not indicate it and simply write  $\models \phi(a; b)$  instead of  $M \models \phi(a; b)$ . If  $\pi(x)$  is a set of formulas all with variable  $x$ , we write  $a \models \pi(x)$  to mean  $a \models \phi(x)$  for all  $\phi(x) \in \pi(x)$ .

The *type* (or *complete type*) of a tuple  $a \in M^k$  over  $B$ , denoted  $\text{tp}(a/B)$ , is the set of formulas  $\phi(x; b)$ ,  $|x| = |a|$ , with parameters  $b$  in  $B$  that hold of  $a$ . If  $B = \emptyset$ , we may omit it. If  $p = \text{tp}(a/B)$ , we usually write  $p \vdash \phi(x; b)$  to mean  $\phi(x; b) \in p$ . The set of types in  $k$  variables over  $B$  is denoted  $S_k(B)$ . We sometimes omit  $k$  if it is clear from the context, or irrelevant. We write  $a \equiv_B a'$  to mean that  $a$  and  $a'$  have the same type over  $B$ .

The *definable closure* of a tuple  $a$ , denoted  $\text{dcl}(a)$ , is the set of elements  $c \in M$  for which there exists a formula  $\phi(x; a)$  of which  $c$  is the only solution in  $M$ . Similarly the *algebraic closure* of a tuple  $a$ , denoted  $\text{acl}(a)$ , is the set of elements  $c \in M$  for which there exists a formula  $\phi(x; a)$  satisfied by  $c$  and which has only finitely many solutions in  $M$ .

It is often important to consider not only definable subsets of  $M$  (or  $M^k$ ), but also quotients of definable subsets by definable equivalence relations. A convenient way to do this is to introduce a multisorted structure  $M^{eq}$  in which such quotients are represented by definable sets. More precisely,  $M^{eq}$  has a sort  $M_E$  for every 0-definable equivalence relation  $E$  on some  $M^n$ . The sort  $M_E$  is interpreted as the quotient of  $M$  by  $E$ . The sort  $M_=$  is identified with  $M$  and equipped with the same structure as  $M$ . Furthermore, for each  $E$  as above, we equip  $M^{eq}$  with the canonical projection map  $\pi_E$  from  $M^n$  to  $M_E$ . One can then show that, for any  $A \subseteq M$ , a subset of  $M_E$  is  $A$ -definable in  $M^{eq}$  if and only if its pre-image under  $\pi_E$  is  $A$ -definable in the original  $M$ . In particular, the original  $M$  and the copy of  $M$  inside  $M^{eq}$  have the same definable sets.

We recall that a countable structure  $M$  is  $\omega$ -categorical if any of the following equivalent conditions is satisfied:

- For any  $n < \omega$ , there are finitely many types of the form  $\text{tp}(a/\emptyset)$ , with  $a \in M^n$ .
- For any finite  $A \subseteq M$ , there are finitely many 1-types  $\text{tp}(a/A)$ , with  $a \in M$  a singleton.
- For any  $n < \omega$ , the action of  $\text{Aut}(M)$  on  $M^n$  has finitely many orbits.
- Any countable  $N$  elementarily equivalent to  $M$  is isomorphic to it.

Assume from now on that  $M$  is  $\omega$ -categorical. Then one can define most

model-theoretic notions using the automorphism group alone (at least over finite parameter sets). Let  $A \subseteq M$  be finite. A subset  $X \subseteq M^n$  is  $A$ -definable if and only if it is (setwise) invariant under the group  $\text{Aut}(M)_A$  of automorphisms fixing  $A$  pointwise. In particular,  $X$  is 0-definable if and only if it is  $\text{Aut}(M)$ -invariant.

Still assuming that  $A$  is finite, two tuples  $a$  and  $a'$  have the same type over  $A$ , denoted  $a \equiv_A a'$ , if and only if there is an automorphism of  $M$  fixing  $A$  pointwise and sending  $a$  to  $a'$ . Thus types over  $A$  are in natural bijection with orbits of  $\text{Aut}(M)_A$ .

An element  $c$  of  $M$  is in the definable closure of  $A$  if and only if it is fixed by  $\text{Aut}(M)_A$ . Similarly,  $c$  is in the algebraic closure of  $A$  if and only if its orbit under  $\text{Aut}(M)_A$  is finite.

We will often consider the algebraic closure evaluated in  $M^{eq}$ :  $\text{acl}^{eq}(a)$ . This can be thought of as containing a name for each equivalence class of  $a$  under a  $\emptyset$ -definable equivalence relation with finitely many classes. In particular, a subset  $X \subseteq M^n$  is definable over  $\text{acl}^{eq}(\emptyset)$  if and only if it has finitely many conjugates under the automorphism group of  $M$ . The strong type of  $a$  over  $A$  is the type of  $a$  over  $\text{acl}^{eq}(A)$ : two elements have the same strong type over  $A$  if they are equivalent for every  $A$ -definable equivalence relation with finitely many classes.

Let  $A \subseteq M$  be any set of parameters and let  $X \subseteq M^k$  be an  $A$ -definable set. We say that  $X$  is *transitive* over  $A$  if any two elements of  $X$  have the same type over  $A$ . Note that since  $X$  is  $A$ -definable, any element of  $M^k$  having the same type as a member of  $X$  is itself in  $X$ . Thus an  $A$ -definable set  $X$  is transitive over  $A$  if and only if  $\text{Aut}(M)_A$  acts transitively on it. Similarly, we say that the  $A$ -definable set  $X$  is *primitive* over  $A$  if the action of  $\text{Aut}(M)_A$  on  $X$  is primitive, or equivalently  $X$  does not admit any non-trivial  $A$ -definable equivalence relation. If  $A = \emptyset$ , then we will usually omit “over  $A$ ”.

Finally, we say that two structures  $M$  and  $N$  are *inter-definable* if they have the same universe and the same 0-definable sets (in all cartesian powers). Hence  $M$  and  $N$  are essentially the same structure, but in possibly different languages.

**Assumption:** Throughout this paper, we work in an  $\omega$ -categorical structure  $M$  in a language  $L$ . That assumption will in general not be recalled, and is implicitly assumed in all statements.

## 2.2 Homogeneous structures

We will call a countable structure  $M$  in a relational language  $L$  *homogeneous* if for any finite  $A \subseteq M$  and  $\sigma: A \rightarrow M$  a partial isomorphism (that is,  $\sigma: A \rightarrow \sigma(A)$  is an isomorphism, where  $A$  and  $\sigma(A)$  are equipped with the induced structure from  $M$ ), there is an automorphism  $\tilde{\sigma}: M \rightarrow M$  that extends  $\sigma$ . This is also sometimes called *ultrahomogeneous*.

We call a structure  $M$  *finitely homogeneous* if it is homogeneous and its language is finite and relational. A structure  $M$  is *finitely homogenizable* if it is interdefinable with a finitely homogeneous structure. Note that any finitely

homogenizable structure is  $\omega$ -categorical: since the language is finite relational, the number of isomorphism types of substructures of a fixed size  $n < \omega$  is finite, hence by homogeneity, the action of  $\text{Aut}(M)$  on  $M^n$  has finitely many orbits.

It is easy to see that a structure  $M$  is finitely homogenizable if and only if there is  $k < \omega$  such that the following two conditions hold:

- There are finitely many types of  $k$ -tuples of elements of  $M$ .
- For any  $n < \omega$ , any two  $n$ -types  $p(\bar{x})$  and  $q(\bar{x})$  of tuples of elements of  $M$  are equal if and only if they have the same restriction to any set of  $k$ -variables.

### 2.3 Linear orders and their reducts

There is only one countable homogeneous linear order:  $(\mathbb{Q}, \leq)$ . It is also the only  $\omega$ -categorical linear order with transitive automorphism group. Its reducts follow from Cameron's result on highly homogeneous permutations groups [Cam76]: there are five of them. Apart from the trivial reduct to pure equality, there are three unstable proper reducts:

- the generic betweenness relation  $(\mathbb{Q}; B(x, y, z))$ , where

$$B(x, y, z) \leftrightarrow (x \leq y \leq z) \vee (z \leq y \leq x);$$

- the generic circular order  $(\mathbb{Q}; C(x, y, z))$ , where

$$C(x, y, z) \leftrightarrow (x \leq y \leq z) \vee (z \leq x \leq y) \vee (y \leq z \leq x);$$

- the generic separation relation  $(\mathbb{Q}; S(x, y, z, t))$ , where

$$S(x, y, z, t) \leftrightarrow (C(x, y, z) \wedge C(y, z, t) \wedge C(z, t, x) \wedge C(t, x, y)) \vee \\ (C(t, z, y) \wedge C(z, y, x) \wedge C(y, x, t) \wedge C(x, t, z)).$$

The automorphism group of the betweenness relation is generated by the automorphism of the linear order along with a bijection that reverses the order, for instance  $x \mapsto -x$ . Similarly, the automorphism group of the separation relation is generated from that of the circular order along with an order-reversing bijection.

Depending on the context, order will mean either linear order or circular order; by default linear. Linear and circular orders will play an essential role in this paper, but the betweenness and separation relations will not explicitly appear. They will be accounted for in the analysis by having every order come with a dual in order-reversing bijection with it. Thus the betweenness relation for example will be present in our classification as the quotient of two linear orders in order-reversing bijection.



## 2.4 Rank

We define rank as in [CH03], Section 2.2.1, restricting to the  $\omega$ -categorical context. This notion of rank also coincides with what is now called thorn-rank, which is defined for any structure: see [Ons06, Definition 4.1.1, Remark 4.1.9].

**Definition 2.1.** Given a definable set  $D \subseteq M^k$  and ordinal  $\alpha$ , we define inductively  $\text{rk}(D) \geq \alpha$ :

- $\text{rk}(D) \geq 1$  if  $D$  is infinite;
- $\text{rk}(D) \geq \alpha + 1$  if there is, in  $M^{eq}$ , an infinite uniformly definable family  $(X_t : t \in E)$  of subsets of  $D$  which is  $k$ -inconsistent for some  $k$  and such that  $\text{rk}(X_t) \geq \alpha$  for each  $t \in E$ ;
- for limit  $\lambda$ ,  $\text{rk}(D) \geq \lambda$  if  $\text{rk}(D) \geq \alpha$  for all  $\alpha < \lambda$ .

The rank of a definable set  $D$  is either an ordinal or  $\infty$  in the case where  $\text{rk}(D) \geq \alpha$  for all  $\alpha$ . We say that a structure  $M$  is ranked if  $\text{rk}(M) < \infty$ . The rank of a type  $\text{tp}(a/b)$ , denoted  $\text{rk}(a/b)$ , is the minimal rank of a  $b$ -definable set containing  $a$ .

This definition does not coincide with the one in [CH03], but is equivalent to it: to  $D, D_1, D_2, f, \pi$  as in [CH03, Definition 2.2.1], associate the family  $X_t := \pi(f^{-1}(t))$ ,  $t \in D_2$ . Conversely, to a family  $(X_t : t \in E)$  as in Definition 2.1, associate the sets  $D_1 = \{(a, t) \in D \times E : a \in X_t\}$  and  $D_2 = E$  with the canonical projection maps.

We state some basic properties of the rank, which will be used in the text without mention. See [CH03], Section 2.2.1 for proofs.

**Proposition 2.2.** 1.  $\text{rk}(a/b) = 0$  if and only if  $a \in \text{acl}(b)$ .

$$2. \text{rk}(D_1 \cup D_2) = \max(\text{rk}(D_1), \text{rk}(D_2)).$$

$$3. \text{ If } B_1 \subseteq B_2, \text{ then } \text{rk}(a/B_1) \geq \text{rk}(a/B_2)$$

$$4. \text{ If } D \text{ is definable over } B, \text{ then there is } a \in D \text{ such that } \text{rk}(a/B) = \text{rk}(D).$$

$$5. \text{ We have } \text{rk}(a/b) \geq n + 1 \text{ if and only if there are } a', c \in M^{eq} \text{ with } a' \in \text{acl}^{eq}(abc) \setminus \text{acl}^{eq}(ac) \text{ and } \text{rk}(a/a'bc) \geq n.$$

$$6. \text{ If } \text{rk}(a/bc) \text{ and } \text{rk}(b/c) \text{ are finite, then so is } \text{rk}(ab/c) \text{ and we have}$$

$$\text{rk}(ab/c) = \text{rk}(a/bc) + \text{rk}(b/c).$$

In particular, if  $a' \in \text{acl}^{eq}(ab)$ , then by point 1 above,  $\text{rk}(a'/ab) = 0$ , hence  $\text{rk}(a'/b) \leq \text{rk}(aa'/b) = \text{rk}(a/b)$ .

From point 6, we deduce that if  $M$  has finite rank, then any finite tuple of elements of  $M$ , or indeed  $M^{eq}$ , has finite rank.

Another important consequence of point 6 is that if  $\text{rk}(M) = 1$ , then the algebraic closure relation defines a *pregeometry* on  $M$ : the operator  $\text{acl}$  always

defines a closure relation in the sense that  $\text{acl}(\text{acl}(A)) = \text{acl}(A)$  for all  $A$  and  $\text{acl}(A) \subseteq \text{acl}(B)$  whenever  $A \subseteq B$ . Assuming that  $\text{rk}(M) = 1$ , then it furthermore satisfies the exchange property: for all  $A \subseteq M$  and two singletons  $a, b \in M$ , we have:

$$b \in \text{acl}(Aa) \setminus \text{acl}(A) \iff a \in \text{acl}(Ab) \setminus \text{acl}(A).$$

We can then define independent sets and bases as one does for vector spaces, with  $\text{rk}$  playing the role of dimension. We will only make mild use of this fact.

Still following [CH03], we define rank independence.

**Definition 2.3.** ( $M$  has finite rank.) Say that two tuples  $a$  and  $b$  are independent over  $E$  and write  $a \perp_E b$  if

$$\text{rk}(ab/E) = \text{rk}(a/E) + \text{rk}(b/E).$$

This is a symmetric notion in  $a$  and  $b$  and it satisfies transitivity:  $a$  and  $bc$  are independent over  $E$  if and only if  $a$  and  $b$  are independent over  $Ec$  and  $a$  and  $b$  are independent over  $E$ .

## 2.5 Stability

Recall that a formula  $\phi(x; y)$  is stable (in some structure  $M$ ) if for some integer  $k$ , we cannot find tuples  $(a_i : i < k)$  and  $(b_j : j < k)$  such that

$$\phi(a_i; b_j) \iff i \leq j.$$

We say that the structure  $M$  is stable if all formulas are stable. Stability is preserved under elementary equivalence and we say that a theory  $T$  is stable if some/any model of  $T$  is stable.

We are concerned in this paper with unstable structures, but stable formulas will appear briefly at the end of the analysis in Section 6.5. There, we will need the following fact, which the reader not well acquainted with stability theory can take as a black box.

**Fact 2.4.** Let  $M$  be a ranked  $\omega$ -categorical structure and let  $\phi(x; y)$  be a stable formula. Let  $p \in S(A)$  be a type over some set  $A \subseteq M$  and let  $B \subseteq M$  be any set, then the set

$$\left\{ \text{tp}_\phi(a/B) : a \models p, a \perp_A B \right\}$$

is finite.

*Proof.* (Assuming knowledge of stability theory: see for instance [Pil96], Chapter 1.) First, note that by [Ons06, Theorem 5.1.1], forking and thorn-forking are the same for stable formulas. Hence if  $a \models p, a \perp_A B$ , then the partial type  $p \cup \text{tp}_\phi(a/B)$  does not fork over  $A$ . Since  $\phi$  is stable, there are only finitely many non-forking extensions of  $p$  to a  $\phi$ -type over  $B$ .  $\square$

### 2.5.1 Strongly minimal sets

We check that Theorem 1.2 holds in the stable case and for that assume familiarity with stability theory. None of this will be used later.

A structure  $M$  is strongly minimal if for any  $N$  elementarily equivalent to  $M$ , any definable (over parameters) subset of  $N$  is either finite or cofinite. If  $M$  is  $\omega$ -categorical, then it is enough to check the condition for  $N = M$ . The classification of strongly minimal primitive  $\omega$ -categorical structures was established by Zilber [Zil] using model-theoretic methods. The paper [CHL85] gives an exposition of this result, as well as a shorter proof attributed to Cherlin and Mills, using the classification of finite simple groups. The results are expressed in terms of the geometry coming from algebraic closure. We have explained in Section 2.4 how any rank 1 structure is equipped with a pregeometry whose closure operation is given by algebraic closure. If  $M$  is primitive, this pregeometry is in fact a *geometry*, meaning that  $\text{acl}(a) = \{a\}$  for any element  $a \in M$ . By the acl-geometry of  $M$ , we mean the set  $M$  equipped with the closure operator  $\text{acl}$ .

**Fact 2.5.** *If  $M$  is strongly minimal, primitive and  $\omega$ -categorical, then either:*

1.  *$M$  is a pure set;*
2. *the acl-geometry on  $M$  is that of an infinite-dimensional affine space over a finite field;*
3. *the acl-geometry on  $M$  is that of an infinite-dimensional projective space over a finite field.*

Cases 2 and 3 do not completely determine  $M$  up to inter-definability, but they do determine it up to finitely many possibilities corresponding to automorphism groups  $G$  with  $\text{AGL}_\omega(F_q) \subseteq G \subseteq \text{A}\Gamma\text{L}_\omega(F_q)$  in the affine case and  $\text{PGL}_\omega(F_q) \subseteq G \subseteq \text{P}\Gamma\text{L}_\omega(F_q)$  in the projective case.

**Proposition 2.6.** *For a given  $n < \omega$ , there are, up to inter-definability, finitely many rank 1, primitive, stable,  $\omega$ -categorical structures  $M$  having at most  $n$  4-types.*

*Proof.* If  $M$  is stable of finite rank, then rank-independence is the same thing as forking-independence: see [Ons06, Theorem 5.1.1]. Thus if  $M$  is stable of rank 1, it is superstable of  $U$ -rank 1. If  $M$  is furthermore primitive, then  $x = x$  is a complete strong type over  $\emptyset$  and therefore for any definable set  $D \subseteq M$ , either  $D$  or its complement forks over  $\emptyset$ . Hence by  $U$ -rank 1, either  $D$  or its complement is finite. Therefore a stable, rank 1, primitive,  $\omega$ -categorical structure is strongly minimal.

Fact 2.5 describes the possibilities. We can assume that  $M$  is not a pure set. Assume first that  $M$  is affine over a field  $F_q$ ,  $q = p^n$ . Then if we fix a point  $a$  as the origin, making  $M$  linear, and take  $b, c$  colinear, we have  $c = \lambda \cdot b$  for some  $\lambda \in F_q$ , defined in the worst case up to an element of  $\text{Gal}(F_q/F_p)$ . That Galois group has size  $n$  and therefore the number of orbits goes to infinity with  $q$ . Hence so does the number of 3-types. The projective case is similar, except that

we need to name two points to serve as 0 and  $\infty$  and obtain that the number of 4-types goes to infinity with  $q$ .  $\square$

## 2.6 NIP and op-dimension

We recall some basic facts about NIP theories and refer the reader to [Sim15] for more details.

**Definition 2.7.** A formula  $\phi(x; y)$  is NIP in  $M$  if for some integer  $k$ , we cannot find tuples  $(a_i : i < k)$  and  $(b_J : J \in \mathfrak{P}(k))$  in  $M$  with:

$$M \models \phi(a_i; b_J) \iff i \in J.$$

If a formula  $\phi(x; y)$  is NIP, then it stays so in any structure  $N$  elementarily equivalent to  $M$ . We say that the theory  $T$  is NIP if for some/any model of  $T$ , all formulas are NIP.

By a result of Shelah, if all formulas  $\phi(x; y)$  with  $|x| = 1$  are NIP, then the theory is NIP. Stable theories are NIP and so is for example the theory of dense linear orders.

The NIP condition can be characterized by counting  $\phi$ -types over finite sets. See [Sim15, Chapter 6]. In the finitely homogeneous case, this becomes a particularly natural condition.

**Fact 2.8.** A structure  $M$  homogeneous in a finite relational language is NIP if and only if there is a polynomial  $P(X)$  such that the number of types over any finite set  $A$  is bounded by  $P(|A|)$ .

Note in particular, that if  $M$  is homogeneous in a finite relational language, then the size of  $S_n(\emptyset)$  is bounded by  $P(1) \cdot P(2) \cdots P(n-1)$ , where  $P(X)$  is the polynomial given by the previous fact. Hence  $|S_n(\emptyset)| = O(2^{cn \ln(n)})$  for some  $c > 0$ .

We now give a short account of [Sim18] which establishes that NIP unstable theories interpret linear orders. First, we define op-dimension as in [GH15], which will allow us to determine how many independent orders we can hope to find.

**Definition 2.9.** An ird-pattern of length  $\kappa$  for the partial type  $\pi(x)$  is given by:

- a family  $(\phi_\alpha(x; y_\alpha) : \alpha < \kappa)$  of formulas;
- an array  $(b_{\alpha,k} : \alpha < \kappa, k < \omega)$  of tuples, with  $|b_{\alpha,k}| = |y_\alpha|$ ;

such that for any  $\eta : \kappa \rightarrow \omega$ , there is  $a_\eta \models \pi(x)$  such that for any  $\alpha < \kappa$  and  $k < \omega$ , we have

$$\models \phi_\alpha(a_\eta; b_{\alpha,k}) \iff \eta(\alpha) < k.$$

*Remark 2.10.* This definition is from [She90], III.7.1. The letters *ird* stand for *independent orders*.

**Definition 2.11.** We say that  $T$  has op-dimension less than  $\kappa$ , and write  $\text{opD}(T) < \kappa$  if, in a saturated model of  $T$ , there is no ird-pattern of length  $\kappa$  for the partial type  $x = x$ .

If a structure is NIP, then it has op-dimension less than  $|T|^+$  (otherwise, we can assume  $\phi_\alpha = \phi$  is constant and then  $\phi$  has IP: we can take  $\{b_{\alpha,0} : \alpha < \omega\}$  as the  $a_i$ 's in Definition 2.7). Conversely, if for some cardinal  $\kappa$ , we have  $\text{opD}(T) < \kappa$ , then  $T$  is NIP. (If  $\phi(x; y)$  has IP, we can find by compactness an ird-pattern of any length with  $\phi_\alpha = \phi$ .)

By a linear quasi-order, we mean a reflexive, transitive relation  $\leq$  for which any two elements are related. If  $\leq$  is a linear quasi-order, then the associated strict order  $<$  is defined by

$$a < b \iff a \leq b \wedge \neg(b \leq a).$$

Furthermore, the relation  $aEb \iff (a \leq b) \wedge (b \leq a)$  is an equivalence relation and  $\leq$  induces a linear order on the quotient.

The main result of [Sim18] in the  $\omega$ -categorical case is the following.

**Fact 2.12** ([Sim18], Theorem 6.14). *If the theory  $T$  is  $\omega$ -categorical, NIP,  $\text{opD}(x = x) \geq n > 0$ , then there is a finite set  $A$ , a set  $D$  definable and transitive over  $A$  and  $n$   $A$ -definable linear quasi-orders  $\leq_1, \dots, \leq_n$  on  $D$ , such that the structure  $(D; \leq_1, \dots, \leq_n)$  contains an isomorphic copy of every finite structure  $(X_0; \leq_1, \dots, \leq_n)$  equipped with  $n$  linear orders.*

Note that by transitivity, for each  $i$ , the quotient of  $D$  by the equivalence relation associated with  $\leq_i$  is infinite and, using  $\omega$ -categoricity,  $\leq_i$  induces on it a dense linear order without endpoints.

### 2.6.1 Distality

Distality was introduced in [Sim13]. It is meant to capture the notion of a purely unstable NIP structure. We give here the equivalent definition from [CS15].

**Definition 2.13.** A structure  $M$  is called *distal* if for every formula  $\phi(x; y)$ , there is a formula  $\psi(x; z)$  such that for any finite set  $A \subseteq M$  and tuple  $a \in M^{|x|}$ , there is  $d \in A^{|z|}$  such that  $\psi(a; d)$  holds and for any instance  $\phi(x; b) \in \text{tp}(a/A)$ , we have the implication

$$M \models (\forall x) \psi(x; d) \rightarrow \phi(x; b).$$

Assume that  $M$  is homogeneous in a finite relational language. Then if  $M$  is distal, there is an integer  $k$  such that for any finite set  $A$  and singleton  $a \in M$ , there is  $A_0 \subseteq A$  of size  $\leq k$  such that  $\text{tp}(a/A_0) \vdash \text{tp}(a/A)$ . (That is, if  $\text{tp}(a'/A_0) = \text{tp}(a/A_0)$ , then  $\text{tp}(a'/A) = \text{tp}(a/A)$ .) In fact, the converse is also true, as can be seen by induction on  $|x|$  in the definition above, but we will not need this.

For instance, DLO is distal, and we can take  $k = 2$ . We will see in Theorem 8.3 that a distal finitely homogeneous structure is always finitely axiomatizable.

### 3 Linear orders

We will consider definable linear orders  $(V, \leq)$ , meaning that the underlying set  $V$  is parameter-definable (possibly over parameters), and so is the order relation  $\leq$ . We will often abuse notation by denoting the pair  $(V, \leq)$  by  $V$ , sometimes by  $\leq$ . If we have two definable orders  $(V_0, \leq_0)$ ,  $(V_1, \leq_1)$ , it may happen that the underlying sets  $V_0, V_1$  are equal. This will, however, be irrelevant for most of what we say and it might be more convenient to think of  $V_0$  and  $V_1$  as two disjoint copies of the same set. In any case,  $V_0$  will mean the set equipped with the order  $\leq_0$  and  $V_1$  the set equipped with the order  $\leq_1$ . The reverse of the order  $(V, \leq)$  is  $(V, \geq)$ .

Orders are always equipped with the order topology, and product of orders with the product topology. Hence, in the situation above,  $V_0 \times V_1$  is equipped with the product topology coming from  $\leq_0$  on the first coordinate and  $\leq_1$  on the second, regardless of whether the underlying sets  $V_0$  and  $V_1$  are equal or not.

**Lemma 3.1.** *Let  $(V, \leq)$  be a 0-definable infinite linear order, which is a complete type over  $\emptyset$ . Then the order  $\leq$  is dense and for any  $a \in V$ ,  $\text{acl}(a) \cap V = \{a\}$ .*

*Proof.* If  $\leq$  is not dense, then some point  $a \in V$  has an immediate successor. Since  $V$  is a complete type over  $\emptyset$ , all points have a successor and hence the order is discrete. By  $\omega$ -categoricity,  $V$  is finite.

If  $b \in \text{acl}(a) \cap V$ , say  $b > a$ , then again as  $V$  is a complete type, there is  $b_1 > b$ ,  $b_1 \in \text{acl}(b)$  and iteratively  $b_{k+1} > b_k$ ,  $b_{k+1} \in \text{acl}(b_k)$ . This gives infinitely many elements in  $\text{acl}(a)$ , contradicting  $\omega$ -categoricity.  $\square$

A convex equivalence relation on an order  $(V, \leq)$  is an equivalence relation with convex classes. Such a relation is non-trivial if it has more than one class and is not equality.

**Definition 3.2.** Let  $(V, \leq)$  be an  $A$ -definable linear order.

- We say that  $(V, \leq)$  has *topological rank 1* if it does not admit any definable (over parameters) convex equivalence relation  $E$  with infinitely many infinite classes.
- We say that  $(V, \leq)$  is *weakly transitive* over  $A$  if it is a dense order without first or last element and any  $A$ -definable subset of  $V$  is either empty or dense in  $V$ .
- We say that  $(V, \leq)$  is *minimal* over  $A$  if it is weakly transitive over  $A$ , and has topological rank 1. If  $A = \emptyset$ , then we omit it.

The name *topological rank 1* comes from the fact that a rank 1 structure, in the sense of Section 2.4, cannot have a definable equivalence relation with infinitely many infinite classes. Here, we forbid such equivalence relations that have convex classes. We will not define topological rank in general.

Note that if  $(V, \leq)$  is transitive over  $A$ , in the sense that it is a complete type over  $A$ , then it is weakly transitive over  $A$ . As an example, consider the structure  $(\mathbb{Q}; \leq, P)$ , where  $\leq$  is the usual order on  $\mathbb{Q}$  and  $P(x)$  is a unary predicate that is dense co-dense in  $\mathbb{Q}$ . Then the order  $(\mathbb{Q}; \leq)$  is weakly transitive (over  $\emptyset$ ), but is not a complete type.

**Lemma 3.3.** 1. *A definable subset of a topological rank 1 linear order has itself topological rank 1.*

2. *If  $(V, \leq)$  is an  $A$ -definable dense order without first or last element, and  $W \subseteq V$  is a dense  $A$ -definable subset of  $V$ , then  $V$  has topological rank 1 (resp. is weakly transitive over  $A$ , resp. is minimal over  $A$ ) if and only if  $W$  has the same property*

*Proof.* 1. Let  $(V, \leq)$  have topological rank 1 and  $W \subseteq V$  be definable (over some parameters). Let  $E$  be a definable convex equivalence relation on  $W$  with infinitely many infinite classes. Define a relation  $\bar{E}$  on  $V$  by:  $\bar{E}(a, b)$  holds if all the points of  $W$  in the interval  $a \leq x \leq b$  are in one  $E$ -equivalence class. Then  $\bar{E}$  is a definable convex equivalence relation on  $V$  with infinitely many infinite classes. This contradicts  $V$  having topological rank 1.

2. If  $E$  is a definable convex equivalence relation on  $V$ , then its restriction  $E|_W$  to  $W$  is also a definable convex equivalence relation. Furthermore if  $E$  has infinitely many infinite classes on  $V$ , each of those classes has infinite intersection with  $W$  by density, hence  $E|_W$  shows that  $W$  does not have topological rank 1. Along with the first point, this shows that  $V$  has topological rank 1 if and only if  $W$  has topological rank 1.

If  $X \subseteq V$  is  $A$ -definable and neither empty nor dense, then  $W \cap \bar{X}$  has the same property, where  $\bar{X}$  denotes the closure of  $X$  in  $V$ . This shows that if  $W$  is weakly transitive over  $A$ , then so is  $V$ . The reverse implication is obvious.  $\square$

**Lemma 3.4.** *Let  $(V, \leq)$  be a definable dense order of topological rank 1. Then any definable closed (or open) subset of  $V$  is a finite union of convex sets.*

*Proof.* Let  $X \subseteq V$  be a definable closed subset. Consider the equivalence relation  $E_X$  which holds of a pair  $(a, b)$  in  $V^2$  if either  $a = b$  or there is no element of  $X$  in the interval  $a \leq x \leq b$ . This is a convex equivalence relation. Moreover, any  $E_X$ -class is either of the form  $\{a\}$ ,  $a \in X$ , or of the form  $a < x < b$ , with  $a, b \in X \cup \{\pm\infty\}$ . Since  $(V, \leq)$  is dense, classes of the second type are infinite. By topological rank 1, there can be only finitely many such classes. This implies that the complement of  $X$  is a finite union of convex sets. Then so is  $X$ .  $\square$

### 3.1 Intertwinings

Let  $(V, \leq)$  be an  $A$ -definable dense order with no first or last element. By a *cut* in  $V$  we mean an initial segment of it which is neither empty nor the whole of  $V$  and has no last element. We let  $\bar{V}$  be the set of definable (over any parameters) cuts of  $V$ . Let  $\phi(x; y)$  be a formula without parameters. The set  $C_\phi := \{b : \phi(V; b) \text{ is a cut of } V\}$  is definable over  $A$ . The set of cuts of  $V$  definable by a formula of the form  $\phi(x; b)$  can be identified with the quotient of  $C_\phi$  by the

equivalence relation  $b \sim b' \iff (\forall x \in V)(\phi(x; b) \leftrightarrow \phi(x; b'))$ . Hence the set of cuts in  $V$  that can be defined by an instance of  $\phi(x; y)$  is naturally an  $A$ -definable set in  $M^{eq}$ . If  $\Phi$  is a finite set of formulas as above, write  $C_\Phi = \bigcup_{\phi \in \Phi} C_\phi$ . This is also an  $A$ -definable set in  $M^{eq}$ . Now  $\overline{V} = \bigcup_{\Phi} C_\Phi$ , where  $\Phi$  runs over all finite set of formulas of the form  $\phi(x; y)$ , is naturally a directed union of  $A$ -definable sets. (It would more rigorous to describe it as a direct limit of  $A$ -definable sets, but we will do without introducing such formalism.) In all arguments using  $\overline{V}$ , one can replace  $\overline{V}$  with a big enough definable subset of it of the form  $C_\Phi$ .

A function  $f: X \rightarrow \overline{V}$  is said to be definable over some  $B \supseteq A$  if there is a  $B$ -definable binary relation  $F \subseteq X \times V$  such that for all  $a \in X$ , the fiber  $F_a := \{x \in V : (a, x) \in F\} \subseteq V$  is equal to  $f(a)$ . This is consistent with the view of  $\overline{V}$  as a union of definable sets: a function  $f: X \rightarrow \overline{V}$  is  $B$ -definable if and only if it takes values inside a fixed definable subset  $C_\Phi$  of  $\overline{V}$  and is  $B$ -definable in the usual sense.

We identify  $V$  with a (definable) subset of  $\overline{V}$  by  $a \mapsto \{x \in V : x < a\}$ . The order  $\leq$  naturally extends to  $\overline{V}$ , where it coincides with inclusion. Note that  $V$  is dense in  $\overline{V}$ . Note also that if  $a \in V$  and  $c \in \overline{V}$ , then  $a \in c$  is equivalent to  $a < c$ , where  $<$  is meant in  $\overline{V}$  with the identification just discussed. We will use both notations.

**Lemma 3.5.** *Let  $(V, \leq)$  be definable and minimal over some  $A$ . Any  $A$ -definable non-empty subset of  $\overline{V}$  is dense in  $\overline{V}$ .*

*Proof.* Let  $X \subseteq \overline{V}$  be  $A$ -definable. We define a relation  $E_X$  on  $V$  by:

$$a E_X b \iff (a = b) \vee (\forall x \in \overline{V})(a \leq x \leq b \rightarrow x \notin X).$$

Then  $E_X$  is a convex  $A$ -definable equivalence relation on  $V$  and by topological rank 1, it has only finitely many infinite classes. Assume it has an infinite class, then that class is  $A$ -definable and by weak transitivity, it is the whole of  $V$ . This implies that  $X$  is empty. If there is no infinite class, then by density of  $V$ , all classes have one element, which implies that  $X$  is dense in  $\overline{V}$ .  $\square$

**Lemma 3.6.** *Let  $(V, \leq)$  be definable and minimal over some  $A$  and let  $W \subseteq \overline{V}$  be an  $A$ -definable subset of  $\overline{V}$  containing  $V$ . Then  $W$  is minimal over  $A$ .*

*Proof.* We know that  $V$  is dense in  $\overline{V}$ , hence also in  $W$ . The result then follows from Lemma 3.3.  $\square$

**Lemma 3.7.** *Given a finite tuple  $\bar{a}$  and an  $\bar{a}$ -definable dense order  $V$ ,  $\text{dcl}^{eq}(\bar{a}) \cap \overline{V}$  is finite.*

*Proof.* Formally, the conclusion says that there is some number  $k < \omega$  such that  $\text{dcl}^{eq}(\bar{a}) \cap V_0$  has size at most  $k$  for all  $\bar{a}$ -definable  $V_0 \subseteq \overline{V}$ . Let  $V_0$  be such a set and let  $m_1 < \dots < m_n$  be in  $\text{dcl}^{eq}(\bar{a}) \cap V_0$ . By density of  $V$  in  $\overline{V}$ , we can find  $b_0, b_1, \dots, b_{n-1} \in V$  with  $m_1 < b_1 < m_2 < \dots < b_{n-1} < m_n$ . Each  $b_i$  has a different type over  $\bar{a}$ , and hence there are at least  $n - 1$  different types of elements of  $V$  over  $\bar{a}$ . Hence  $\text{dcl}^{eq}(\bar{a}) \cap V_0$  has size bounded by the number of 1-types over  $\bar{a}$  of elements of  $V$ , which is finite by  $\omega$ -categoricity.  $\square$



**Definition 3.8.** Let  $(V, \leq_V)$  and  $(W, \leq_W)$  be orders, definable and weakly transitive over  $A$ . We say that they are *intertwined* over  $A$  if there is an  $A$ -definable non-decreasing map  $f: V \rightarrow \overline{W}$ . If  $A$  is clear from the context, we omit it.

Note that this is the same thing as saying that there is an  $A$ -definable binary relation  $R \subseteq V \times W$  such that

$$(a R b) \wedge (a' \leq_V a) \wedge (b \leq_W b') \implies a' R b'.$$

Indeed, the relation  $R$  is defined from  $f$  by

$$x R y \iff f(x) \leq_{\overline{W}} y \iff \neg F(x, y),$$

where  $F$  is associated to  $f$  as above. Observe also that by weak transitivity, no element of  $\overline{W}$  is definable over  $A$ , hence the image of  $f$  has to be cofinal and coinital in  $\overline{W}$ .

**Lemma 3.9.** *For any fixed  $A$ , intertwining is an equivalence relation on orders that are definable and weakly transitive over  $A$ .*

*Proof.* Any order is intertwined with itself via the identity function. If  $R$  as above is an intertwining relation from  $V$  to  $W$ , then  $R'$  defined by  $x R' y \iff \neg y R x$  is an intertwining relation from  $W$  to  $V$ . Finally if  $R$  is an intertwining relation from  $V$  to  $W$  and  $S$  an intertwining relation from  $W$  to  $Z$ , then  $T$  defined by  $x T y \iff (\exists z)(x R z \wedge z S y)$  intertwines  $V$  and  $Z$ .  $\square$

Working over some base  $A$ , let  $V$  and  $W$  be two weakly transitive orders and  $f: V \rightarrow \overline{W}$  an intertwining map. If  $W$  has topological rank 1, then the image of  $f$  must be dense in  $\overline{W}$  (otherwise we can define an equivalence relation as in the proof of Lemma 3.4; it cannot have finitely many classes as  $W$  is weakly transitive). If  $V$  has topological rank 1, then  $f$  is injective:  $f(x) = f(y)$  is a convex equivalence relation on  $V$ ; it cannot have finitely many infinite classes by weak transitivity and cannot have infinitely many by topological rank 1. Hence all classes are singletons and  $f$  is injective. We conclude that if both  $V$  and  $W$  have topological rank 1, an intertwining gives an increasing injection of  $V$  into a dense subset of  $\overline{W}$ . Furthermore, the map  $f$  extends to an increasing bijection  $\tilde{f}: \overline{V} \rightarrow \overline{W}$  defined as follows: if  $c \in \overline{V}$  is a cut in  $V$ , seen as a subset of  $V$ , we let  $\tilde{f}(c) = \{y \in W : y < f(x) \text{ for some } x \in c\}$ . Since  $f$  is increasing and  $c$  has no last element,  $\tilde{f}(c)$  also has no last element and is a definable cut in  $W$ . One sees at once that  $\tilde{f}$  extends  $f$  and is increasing. Also if  $d \in \overline{W}$  is a definable cut in  $W$ , then  $c := \{x \in V : f(x) < d\}$  is a definable cut in  $V$  and  $\tilde{f}(c) = d$ . Hence  $\tilde{f}: \overline{V} \rightarrow \overline{W}$  is a bijection. It follows that we can—and will—think of  $V$  and  $W$  as having a common definable completion, or equivalently as being dense in each other's completion.

**Lemma 3.10.** *Working over  $A$ , if  $V$  and  $W$  are minimal linear orders which are intertwined, then there is a unique  $A$ -definable intertwining map  $f: V \rightarrow \overline{W}$ .*

*Proof.* Assume that we are given two increasing maps  $f, g: V \rightarrow \overline{W}$ , both definable over  $A$ . Keeping only the parameters needed to define  $V, W, f$  and  $g$ , we may assume that  $A$  is finite. The two maps  $f$  and  $g$  extend uniquely to increasing bijections from  $\overline{V}$  to  $\overline{W}$ , still denoted by  $f$  and  $g$ . If for some  $a \in V$ ,  $f(a) < g(a)$ , then we have  $a < f^{-1}(g(a))$  and hence  $g(a) < g(f^{-1}(g(a)))$ . Continuing in this way we find

$$a < f^{-1}(g(a)) < f^{-1}(g(f^{-1}(g(a)))) < \dots,$$

which gives infinitely many elements in  $\text{dcl}(Aa) \cap \overline{V}$ , contradicting Lemma 3.7.  $\square$

It will follow from Lemma 3.17 that even over a larger set of parameters, there cannot be another intertwining map from an interval of  $V$  to an interval of  $W$ .

We now study definable subsets of cartesian powers of a minimal order.

**Proposition 3.11.** *Working over some  $A$ , let  $(V, \leq)$  be a minimal definable linear order. Let  $p(x_0, \dots, x_{n-1}) \in S(A)$  be a type in  $V^n$  such that  $p \vdash x_0 < x_1 < \dots < x_{n-1}$ . Then given open intervals  $I_0 < \dots < I_{n-1}$  of  $V$ , we can find  $a_i \in I_i$  such that  $(a_0, \dots, a_{n-1}) \models p$ .*

*Proof.* For simplicity of notation, assume  $A = \emptyset$ . The strategy of the proof is as follows: we first ignore the type  $p$  and produce by induction on  $l < \omega$ , types  $r_l \in S_l(\emptyset)$  which satisfy the conclusion of the proposition. We then show how the existence of  $r_{2n}$  implies that  $p$  itself has the required density property by sandwiching elements of a realization of  $p$  between elements of a realization of  $r_{2n}$ .

For any finite tuple  $\vec{d}$ , let  $m(\vec{d})$  denote the maximal element of  $\text{dcl}^{eq}(\vec{d}) \cap \overline{V}$ . Note that for a fixed tuple of variables  $\vec{y}$ , the relation  $\phi(x; \vec{y}) := x > m(\vec{y})$  is invariant under  $\text{Aut}(M)$ , and therefore definable.

We construct an increasing sequence of types  $r_l(x_0, \dots, x_{l-1}) \in S(\emptyset)$ ,  $l > 0$ , of elements of  $V^l$ . For  $l = 1$ , let  $a_0 \in V$  be any element and set  $r_1 = \text{tp}(a_0)$  and  $m_0 = m(a_0) \in \overline{V}$ . Pick any point  $a_1 > m_0$  and let  $r_2 = \text{tp}(a_0, a_1)$ . We continue in this way: having constructed  $r_l = \text{tp}(a_0, \dots, a_{l-1})$ , let  $m_{l-1} = m(a_{\leq l-1})^1$ . Pick any  $a_l > m_{l-1}$  and set  $r_{l+1} = \text{tp}(a_0, \dots, a_l)$ . We note that

$$r_{l+1}(x_0, \dots, x_l) \vdash x_l > m(x_0, \dots, x_{l-1}).$$

This being done, let  $I_0 < \dots < I_{l-1}$  be open intervals of  $V$ . We claim that we can find  $(b_0, \dots, b_{l-1}) \models r_l$  such that  $m(b_{\leq k})$  lies in  $I_k$  for each  $k$ . We do this by induction. Assume that  $b_{< k}$  have been selected and set  $m = m(b_{< k})$  (if  $k = 0$ , take  $m = -\infty$ ). Define the relation  $E_k$  on  $V_{> m}$  by  $v E_k w$  if either  $v = w$ , or for no  $s$  with  $\text{tp}(b_{< k}, s) = r_{k+1}$  do we have  $v \leq m(b_{< k}s) \leq w$ . This is an equivalence relation with convex classes. By the topological rank 1 assumption, it must have finitely many infinite classes. The infima and suprema of those classes

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<sup>1</sup>Where  $a_{\leq l} := a_0, \dots, a_l$

are elements of  $\bar{V}$  definable over  $b_{<k}$ . However, by definition, no cut above  $m(b_{<k})$  is definable over  $b_{<k}$ . Hence all classes of  $E_k$  are finite and by density of the order, all classes have one element. It follows that we can find  $b_k$  with  $\text{tp}(b_{\leq k}) = r_{k+1}$  and  $m(b_{\leq k})$  lying in  $I_k$ .

Let now  $p(x_0, \dots, x_{n-1})$  be as in the statement of the lemma and  $\bar{a} \models p$ . Let  $r = r_{2n}$ . Then by the previous paragraph, we can find  $\bar{b} \models r$  such that for each  $k$ ,  $m(b_{\leq 2k}) < a_k < m(b_{\leq 2k+1})$ . Pick open intervals  $I_0 < \dots < I_{n-1}$  of  $V$ . For each  $k$ , let  $J_{2k} < J_{2k+1}$  be two subintervals of  $I_k$ . Applying the previous paragraph again, we can find  $\bar{b}' \models r$  such that for each  $i$ ,  $m(b'_{\leq i}) \in J_i$ . Since  $\bar{b}$  and  $\bar{b}'$  have the same type, there is  $\sigma \in \text{Aut}(M)$  with  $\sigma(\bar{b}) = \bar{b}'$ . Let  $\bar{a}' = \sigma(\bar{a})$ . We then have  $m(b'_{\leq 2k}) < a'_k < m(b'_{\leq 2k+1})$  for each  $k$ . By the choice of  $\bar{b}'$ , this implies  $a'_k \in I_k$  as required.  $\square$

*Remark 3.12.* Let  $(V, \leq)$  be definable and minimal over  $A$ . Let  $p(x_0, \dots, x_{n-1}) \in S(A)$  be a type in  $\bar{V}^n$  such that  $p \vdash x_0 < \dots < x_{n-1}$ . Then there is some  $A$ -definable  $W \subseteq \bar{V}$  containing  $V$  such that  $p$  lies in  $W^n$ . By Lemma 3.6,  $W$  is also minimal over  $A$  and we can apply the previous proposition with  $W$  instead of  $V$ . This shows that Proposition 3.11 can be applied to types in  $\bar{V}^n$  instead of  $V^n$ .

**Corollary 3.13.** *Let  $(V, \leq)$  be a minimal definable linear order over some  $A$ . Let  $X \subseteq V^n$  be an  $A$ -definable subset, then the topological closure of  $X$  is a boolean combination of sets of the form  $x_i \leq x_j$ .*

*Proof.* We can write  $X = \bigcup_{i < n} Y_i$ , where the  $Y_i$ 's are pairwise disjoint and each  $Y_i$  is  $A$ -definable and defines a complete type over  $A$ . Since the closure of  $X$  is the union of the closures of the  $Y_i$ 's, it is enough to prove the statement for each  $Y_i$ . We may therefore assume that  $X$  defines a complete type over  $A$ . Let  $(a_0, \dots, a_{n-1}) \in X$ . For some permutation  $\sigma$  of  $\{0, \dots, n-1\}$ , we have  $a_{\sigma(0)} \leq \dots \leq a_{\sigma(n-1)}$ . If the coordinates of  $\bar{a}$  are pairwise distinct, then the previous proposition implies that  $X$  is dense in the set defined by  $x_{\sigma(0)} \leq \dots \leq x_{\sigma(n-1)}$ . In general,  $X$  is dense in the intersection of that set with the set defined by the conjunction of the equations  $x_{\sigma(i)} = x_{\sigma(i+1)}$  that hold in  $\bar{a}$ .  $\square$

In the end of this section, we give a more concrete description of intertwined orders and show that there is only one transitive structure composed of  $n$  intertwined orders, up to isomorphism and permutation of the orders. (See Proposition 3.15 for a precise statement.)

**Proposition 3.14.** *Consider the language  $L_n = \{\leq, P_0, \dots, P_{n-1}, f_1, \dots, f_{n-1}\}$ , where the  $P_i$ 's are unary predicates and the  $f_i$ 's unary functions. Let the theory  $T_n$  say that:*

- $\leq$  defines a dense linear order without endpoints;
- the  $P_i$ 's partition the universe and are dense (with respect to  $\leq$ );
- the function  $f_i$  is the identity outside of  $P_0$ ; its restriction to  $P_0$  is a bijection between  $P_0$  and  $P_i$ ;

- for all  $x \in P_0$ , we have  $x < f_1(x) < f_2(x) < \dots < f_{n-1}(x)$ ;
- given any open intervals  $I_0 < I_1 < \dots < I_n$ , there is  $x \in I_0$  such that  $f_i(x) \in I_i$  for each  $1 \leq i < n$ .

Then the theory  $T_n$  is complete,  $\omega$ -categorical and has elimination of quantifiers.

*Proof.* This can be shown by a straightforward back-and-forth argument. Alternatively, one can see that  $T_n$  is the Fraïssé limit of the class of finite  $L_n$  structures satisfying:

- $\leq$  defines a linear order;
- the  $P_i$ 's partition the universe;
- the function  $f_i$  is the identity outside of  $P_0$  and for  $x \in P_0$ , we have  $P_i(f_i(x))$  and  $x < f_1(x) < f_2(x) < \dots < f_{n-1}(x)$ .

It follows that  $T_n$  has elimination of quantifiers. Hence it is complete and  $\omega$ -categorical (because the structure generated by a set of size  $m$  has size at most  $nm$ ).  $\square$

Let now  $(V; \leq_0, \dots, \leq_{n-1})$  be a structure equipped with  $n$  distinct linear orders. Assume that each order  $V_i := (V, \leq_i)$  has topological rank 1 and that any two  $V_i, V_j$  are intertwined. Further assume that the structure  $V$  is transitive (that is, there is a unique 1-type over  $\emptyset$ ). For each  $i < n$ , there is by Lemma 3.10 a unique increasing 0-definable map  $f_i: V_i \rightarrow \overline{V_0}$ . Inside  $V$ , we interpret an  $L_n$ -structure  $V_*$  as follows: the universe of  $V_*$  is the union of  $n$  disjoint copies of  $V$ , which we think of as representing the orders  $V_0$  to  $V_{n-1}$ . The unary predicate  $P_i$  names the  $i$ -th copy of  $V$ , which we identify with the image  $f_i(V_i)$  inside  $\overline{V_0}$ . The order  $\leq$  on  $V_*$  is then given by the order on  $\overline{V_0}$  using those identifications. Finally, the function  $f_i$  sends a point  $x \in P_0(V_*)$  to the corresponding point in  $P_i(V_*)$ : remember, that both are just copies of  $V$ , so  $f_i$  is just the canonical identification of one copy of  $V$  with the other. Define also  $f_0$  as being the identity function on  $V_*$ .

Since we assumed that  $V$  has a unique 1-type over  $\emptyset$ , then for some permutation  $\sigma$  of  $\{0, \dots, n-1\}$ , we have that for all  $x \in P_0(V_*)$ ,

$$f_{\sigma(0)}(x) < f_{\sigma(1)}(x) < \dots < f_{\sigma(n)}(x).$$

If  $\sigma$  is the identity, then  $V_*$  is a model of  $T_n$  as defined above. Otherwise, we obtain a model of  $T_n$  by applying the same construction to the structure  $(V; \leq_{\sigma^{-1}(0)}, \dots, \leq_{\sigma^{-1}(n-1)})$ . Note that there is a unique  $\sigma$  with this property.

Conversely, given a (countable) model  $(V_*; \leq, P_0, \dots, P_{n-1})$  of  $T_n$ , we can construct a structure  $(V^{(n)}; \leq_0, \dots, \leq_{n-1})$  by taking as universe  $V^{(n)} = P_0(V_*)$ , interpreting  $\leq_0$  as  $\leq$  and  $\leq_i, i > 0$  by:

$$x \leq_i y \iff f_i(x) \leq f_i(y).$$

Note that by  $\omega$ -categoricity of  $T_n$ , the structure  $V^{(n)}$  is uniquely defined up to isomorphism. For each  $i \leq n$ , let  $V_i^{(n)}$  be the definable linear order  $(V^{(n)}; \leq_i)$ . It might seem that by going to  $V^{(n)}$ , we have lost the intertwining between the orders, but in fact this is not the case. Indeed, the orders  $V_0^{(n)}$  and  $V_i^{(n)}$ ,  $i < n$  are intertwined in  $V^{(n)}$ : let  $x \in V^{(n)}$  and consider the set

$$g_i(x) := \{y \in V^{(n)} : (\forall z <_i y) z <_0 x\}.$$

Then  $g_i(x)$  is a cut of  $V_i^{(n)}$  and we leave it to the reader to check that  $g_i$  does define an intertwining from  $V_0^{(n)}$  to  $V_i^{(n)}$ .

If we apply the first construction above to  $V^{(n)}$ , then we recover the  $V_*$  we started with. The following proposition now follows from this discussion.

**Proposition 3.15.** *Let  $(V; \leq_0, \dots, \leq_{n-1})$  be a transitive (countable) structure equipped with  $n$  distinct linear orders. Assume that each order  $V_i := (V, \leq_i)$  has topological rank 1, any two  $V_i, V_j$  are intertwined. Then for some unique permutation  $\sigma$  of  $\{0, \dots, n-1\}$ ,  $(V; \leq_{\sigma(0)}, \dots, \leq_{\sigma(n-1)})$  is isomorphic to the structure  $(V^{(n)}; \leq_0, \dots, \leq_{n-1})$  defined above. In particular, there are exactly  $n!$  such structures up to isomorphism.*

## 3.2 Independent orders

**Definition 3.16.** Let  $V$  and  $W$  be two orders, definable over some  $A$ . We say that  $V$  and  $W$  are *independent* if there does not exist:

- a set of parameters  $B \supseteq A$ ,
- $B$ -definable infinite subsets  $X \subseteq V$  and  $Y \subseteq W$ , both weakly transitive over  $B$ , which we equip with the induced orders from  $V$  and  $W$  respectively,
- a  $B$ -definable intertwining from  $X$  to either  $Y$  or the reverse of  $Y$ .

Note that independence is a symmetric relation.

**Lemma 3.17.** *Let  $(V, \leq)$  be definable and minimal over some  $A$ . Let  $B \supseteq A$  and  $I, J \subseteq V$  be two infinite  $B$ -definable disjoint convex subsets, weakly transitive over  $B$ , then  $(I, \leq)$  and  $(J, \leq)$  are independent.*

*Proof.* Without loss of generality,  $B$  is finite. Assume first that there is an intertwining map  $f$  from  $I$  to  $J$ , definable over  $B$ . Then  $f$  extends to an increasing bijection from  $\bar{I} \rightarrow \bar{J}$ , which we still denote by  $f$ . Assume for definiteness that  $I < J$ . Let  $c_1, c_2$  be the infimum and supremum of  $I$  respectively, seen as elements of  $\bar{V}$ . Define similarly  $d_1, d_2$  for  $J$ . Hence we have  $c_1 < c_2 < d_1 < d_2$ . By Proposition 3.11 (and Remark 3.12), we can find  $(c'_1, c'_2, d'_1, d'_2) \equiv_A (c_1, c_2, d_1, d_2)$  such that

$$c_1 < c'_1 < c'_2 < c_2 < d'_1 < d_1 < d_2 < d'_2.$$

Let  $\sigma \in \text{Aut}(M)_A$  send  $(c_1, c_2, d_1, d_2)$  to  $(c'_1, c'_2, d'_1, d'_2)$ . Let  $I', J'$  be the images of  $I, J$  respectively under  $\sigma$  and set  $g = \sigma \circ f \circ \sigma^{-1}$ , so that  $g$  is an increasing map from  $\bar{I}'$  to  $\bar{J}'$ .

Let  $a \in I \setminus I'$ , say  $c_1 < a < c'_1$ . Then  $f$  sends  $a$  to a point in  $J \subset J'$ . So  $g^{-1}$  is defined on  $f(a)$  and sends it to a point in  $\overline{I'}$ , hence  $a < g^{-1}(f(a))$ . Applying  $f$ , we obtain  $f(a) < f(g^{-1}(f(a)))$ , thus  $g^{-1}(f(a)) < g^{-1}(f(g^{-1}(f(a))))$ . Iterating, we find infinitely many elements in  $\text{dcl}^{eq}(aB\sigma(B)) \cap \overline{V}$ :

$$a < g^{-1} \circ f(a) < g^{-1} \circ f \circ g^{-1} \circ f(a) < g^{-1} \circ f \circ g^{-1} \circ f \circ g^{-1} \circ f(a) < \dots$$

This contradicts Lemma 3.7. The same argument shows that  $I$  is not intertwined over  $B$  with the reverse of  $J$ .

Now take  $B' \supseteq B$  and  $X \subseteq I$ ,  $Y \subseteq J$  two infinite subsets, definable and weakly transitive over  $B'$ . Since  $V$  is minimal, the closure of  $X$  is a finite union of convex subsets. By weak transitivity, it is just one convex subset  $I'$ . Similarly the closure of  $Y$  is a convex subset  $J'$ . An intertwining between  $X$  and  $Y$  induces naturally an intertwining between  $I'$  and  $J'$ . Using the previous paragraph we see that there is no such intertwining. We conclude that  $I$  and  $J$  are independent.  $\square$

**Corollary 3.18.** *Let  $(V, \leq)$  be definable and minimal over some  $A$ . Then  $(V, \leq)$  is not intertwined with its reverse  $(V, \geq)$ .*

*Proof.* If there is an  $A$ -definable decreasing map  $f: V \rightarrow \overline{V}$ , then we can find an open interval  $I \subseteq V$  such that the convex hull of the image  $f(I)$  is disjoint from  $I$ . Let  $J$  be the intersection of the convex hull of  $f(I)$  with  $V$ . Then  $I$  and  $J$  contradict the previous lemma.  $\square$

**Lemma 3.19.** *Let  $V_0, V_1$  be two linear orders definable and minimal over some  $A$ . Assume that they are not independent. Then there is either an  $A$ -definable intertwining from  $V_0$  to  $V_1$  or an  $A$ -definable intertwining from  $V_0$  to the reverse of  $V_1$ .*

*Proof.* Let  $B \supseteq A$ . Assume that we have some  $B$ -definable  $X_0 \subseteq V_0$  and  $X_1 \subseteq V_1$  both weakly transitive over  $B$  and a  $B$ -definable increasing map  $f: X_0 \rightarrow \overline{X_1}$  (if there is a decreasing map from  $X_0$  to  $\overline{X_1}$ , replace  $V_1$  by its reverse). Restricting  $X_0$ , we may assume that it is transitive over  $B$ . Let  $a \in X_0$ . By topological rank 1, both  $X_0$  and  $X_1$  are dense in their convex hulls and  $f$  extends to an increasing map  $\overline{X_0} \rightarrow \overline{X_1}$ . Assume that  $f(a) \notin \text{dcl}(Aa)$ . Then, for some interval  $I$  of  $V_0$  containing  $a$ , we can find another increasing map  $f': I \rightarrow \overline{V_1}$ , a conjugate of  $f$  defined over some  $B'$  with  $f'(a) \neq f(a)$ . Reducing  $I$  further, we can assume that  $f(I)$  and  $f'(I)$  are disjoint. But then  $f' \circ f^{-1}$  gives an intertwining map from  $f(I)$  to  $f'(I)$ , which contradicts Lemma 3.17.

It follows that  $f(a) \in \text{dcl}(Aa)$ . Let  $g$  be the  $A$ -definable map sending  $a$  to  $f(a)$  and for simplicity assume  $V_0$  is transitive over  $A$  (otherwise, replace it by the locus of  $\text{tp}(a/A)$ ). Then by transitivity of  $X_0$ ,  $g$  coincides with  $f$  on  $X_0$  and therefore is increasing on it. Let  $X'_0$  be a conjugate of  $X_0$  over  $A$ . Then  $g$  is also increasing on  $X'_0$ . Assume that the convex hulls of  $X_0$  and  $X'_0$  have an open interval  $Z$  in their intersection. We can construct two increasing maps from  $Z$  to  $\overline{V_1}$ : one induced by  $g|_{X_0}$  and one induced by  $g|_{X'_0}$ . By Lemma 3.10, those two maps coincide. By transitivity, the conjugates of  $X_0$  cover the convex hull

of  $X_0$ . It follows that  $g$  is increasing on the convex hull of  $X_0$ . Therefore  $g$  is locally increasing on  $V_0$ : for each  $a \in V_0$ , there is an open convex subset of  $V_0$  containing  $a$  on which  $g$  is increasing. Let  $C_a$  denote the maximal such set. The sets  $C_a$  form an  $A$ -definable partition of  $V_0$  into infinite convex sets. As  $V_0$  has topological rank 1,  $C_a = V_0$  for all  $a$  and  $g$  is increasing on  $\text{tp}(a/A)$ . It follows that  $g$  intertwines  $V_0$  and  $V_1$ .  $\square$

**Lemma 3.20.** *Working over some  $A$ , let  $(V_0, \leq_0)$ ,  $(V_1, \leq_1)$  be two minimal independent definable orders. Let  $f_0: V_0 \rightarrow \overline{V_0}$  and  $f_1: V_0 \rightarrow \overline{V_1}$  be two  $A$ -definable functions. Then the set*

$$\{f_0(x), f_1(x) : x \in V_0\}$$

*is dense in  $\overline{V_0} \times \overline{V_1}$ .*

*Proof.* First, the images of  $f_0$  and  $f_1$  are definable subsets of  $\overline{V_0}$  and  $\overline{V_1}$  respectively. By Lemma 3.6, we can replace  $V_0$  by  $V_0 \cup f_0(V_0)$  and  $V_1$  by  $V_1 \cup f_1(V_0)$  and assume that  $f_0$  and  $f_1$  take values in  $V_0$  and  $V_1$  respectively.

Let  $V \subseteq V_0$  be  $A$ -definable and transitive over  $A$ . Then by minimality,  $V$  is dense in  $V_0$  and it is enough to prove that  $\{f_0(x), f_1(x) : x \in V\}$  is dense in  $\overline{V_0} \times \overline{V_1}$ . Next, notice that since  $V_0$  and  $V_1$  are minimal over  $A$  and  $f_0, f_1, V$  are  $A$ -definable,  $f_0(V)$  is dense in  $\overline{V_0}$  and  $f_1(V)$  is dense in  $\overline{V_1}$ . Fix  $a \in V$  and consider the set

$$X_a = \{f_0(x) : x \in V, f_1(x) <_1 f_1(a)\}.$$

This set is non-empty by the previous sentence. Let also  $Y_a$  be the closure of  $X_a$ . Then by Lemma 3.4,  $Y_a$  is a finite union of convex sets.

The infima and suprema of those convex sets are either  $\pm\infty$  or elements of  $\overline{V_0}$ . Let  $W \subseteq \overline{V_0}$  be an  $A$ -definable subset containing  $V_0$  along with all those elements. By Lemma 3.6,  $W$  is minimal over  $A$ . Assume that  $Y_a$  contains a bounded interval

$$c \leq_0 x \leq_0 d, \quad c, d \in W,$$

and this interval is maximal in  $Y_a$ . By Proposition 3.11 applied to  $W$ , there is an automorphism  $\sigma$  such that  $c <_0 \sigma(c) <_0 d <_0 \sigma(d)$ . But then, we have neither  $Y_{\sigma(a)} \subseteq Y_a$ , nor  $Y_a \subseteq Y_{\sigma(a)}$  and this is impossible by the definition of  $X_a$ . We can do the same thing if  $Y_a$  contains two disjoint unbounded intervals. We conclude that  $Y_a$  is either an initial segment, an end segment, or the whole of  $\overline{V_0}$ .

Assume that  $Y_a$  is an initial segment and define  $g(a) \in W$  to be its supremum. Then as  $V$  is a complete type,  $Y_{a'}$  is an initial segment for each  $a' \in V$ . Let  $h: f_1(V) \rightarrow V_0$  send a point  $b = f_1(a')$  to  $g(a')$ . This is well defined as  $X_{a'}$  and hence  $g(a')$  depends only on  $f_1(a')$ . Note that  $h$  is non-decreasing and therefore intertwines  $f_1(V)$  and  $V_0$ . This contradicts independence. Similarly, if  $Y_a$  is an end segment, we obtain an intertwining from  $f_1(V)$  to the reverse of  $V_0$ . We therefore conclude that  $Y_a$  is equal to  $\overline{V_0}$ . We also have symmetrically that  $\{f_0(x) : x \in V, f_1(x) >_1 f_1(a)\}$  is dense in  $\overline{V_0}$  for all  $a \in V$ .

Assume now towards a contradiction that for some bounded open interval  $I \subset V_0$ , the set

$$H(I) := \{f_1(x) : x \in V_0, f_0(x) \in I\}$$

is not dense in  $\overline{V_1}$  (where we have identified  $I$  with its convex closure in  $\overline{V_0}$ ). Let  $J \subset V_0$  be any bounded interval. By Proposition 3.11, there is an automorphism  $\sigma$  over  $A$  such that  $\sigma(J) \subseteq I$ . Then  $H(\sigma(J)) \subseteq H(I)$  is not dense in  $\overline{V_1}$ . Therefore, also  $H(J)$  is not dense in  $\overline{V_1}$ .

By what we know so far,  $H(I)$  is cofinal and coinital in  $\overline{V_1}$  (since for any  $d \in f_1(V)$ , the sets  $\{f_0(x) : x \in V, f_1(x) <_1 d\}$  and  $\{f_0(x) : x \in V, f_1(x) >_1 d\}$  are dense in  $\overline{V_0}$  and  $f_1(V)$  is dense in  $V_1$ ). Let  $C(I) = V_1 \setminus \overline{H(I)}$ . Then  $C(I)$  is a non-empty finite union of bounded open intervals. Let  $\tilde{C}(I)$  be its convex hull. If  $I \subseteq J$ , then  $H(I) \subseteq H(J)$ , so  $C(I) \supseteq C(J)$  and  $\tilde{C}(I) \supseteq \tilde{C}(J)$ . As any two intervals are contained in a third one, any two intervals of the form  $\tilde{C}(J)$  intersect, where  $J$  is any open interval of  $V_0$ . Let  $a \in V_1$  to the left of  $\tilde{C}(I)$  and  $b \in V_1$  to the right of it of same type as  $a$ . Then there is an automorphism  $\sigma$  over  $A$  sending  $a$  to  $b$ . Then  $\sigma(\tilde{C}(I)) = \tilde{C}(\sigma(I))$  is disjoint from  $\tilde{C}(I)$ . This is a contradiction.  $\square$

Having described the closed definable subsets of weakly transitive orders, and hence of products of intertwined orders, we now complete the picture with the case of pairwise independent orders.

**Proposition 3.21.** *Working over some set  $A$ , let  $V_0, \dots, V_{n-1}$  be pairwise independent minimal definable orders. Then any  $A$ -definable closed set  $X \subseteq V_0^{k_0} \times \dots \times V_{n-1}^{k_{n-1}}$  is a finite union of products of the form  $D_0 \times \dots \times D_{n-1}$ , where each  $D_i$  is an  $A$ -definable closed subset of  $V_i^{k_i}$ .*

*Proof.* We assume for simplicity that  $A = \emptyset$ . Say that a type  $p \in S(\emptyset)$  on  $V_0^{k_0} \times \dots \times V_{n-1}^{k_{n-1}}$  has property  $\boxtimes$  if the closure of its set of realizations is a product of closed 0-definable sets  $D_i \subseteq V_i^{k_i}$ . We prove the following two statements by induction on  $n$ :

( $A_n$ ) Let  $V_0, \dots, V_{n-1}$  be pairwise independent minimal orders. Let  $f_i: V_0 \rightarrow \overline{V_i}$ ,  $i < n$ , be 0-definable functions, then  $\{(f_0(x), f_1(x), \dots, f_{n-1}(x)) : x \in V_0\}$  is dense in  $\overline{V_0} \times \dots \times \overline{V_{n-1}}$ .

( $B_n$ ) Let  $V_0, \dots, V_{n-1}$  be pairwise independent minimal orders. Then any type  $p \in S(\emptyset)$  on  $V_0^{k_0} \times \dots \times V_{n-1}^{k_{n-1}}$  has property  $\boxtimes$ .

The statement of the theorem then follows from ( $B_n$ ) since by  $\omega$ -categoricity, any definable set is a finite union of types.

Property ( $A_1$ ) follows from minimality and ( $B_1$ ) holds trivially. We will show that ( $A_n$ ) and ( $B_n$ ) together imply ( $A_{n+1}$ ) and then that ( $A_{n+1}$ ) implies ( $B_{n+1}$ ).

( $A_n$ ) + ( $B_n$ )  $\Rightarrow$  ( $A_{n+1}$ ): The property ( $A_2$ ) is Lemma 3.20, so we can assume  $n > 1$ . We follow closely the proof of Lemma 3.20. Fix  $a \in V_0$  and define

$$X_a = \{(f_0(x), f_1(x), \dots, f_{n-1}(x)) : x \in V_0, f_n(x) < f_n(a)\} \subseteq V_0 \times \dots \times V_{n-1}.$$



For each  $i < n$ , let  $Y_i \subseteq V_i$  be a complete type over  $a$ . Note that  $\overline{Y_i}$  is convex in  $\overline{V_i}$  (it is a finite union of convex sets by minimality and then is convex since it defines a complete type over  $a$ ). Set

$$\hat{Y} = \prod_{i < n} \overline{Y_i} \subseteq \prod_{i < n} \overline{V_i}.$$

Working over the parameter  $a$ , the  $Y_i$ 's are pairwise independent minimal orders. The property  $(B_n)$  then implies that  $X_a \cap \hat{Y}$  is either dense in  $\hat{Y}$  or empty. It now follows that the closure  $\overline{X_a}$  of  $X_a$  in  $\prod_{i < n} \overline{V_i}$  is a union of finitely many rectangles of the form  $\prod_{i < n} I_i$ , where each  $I_i \subseteq \overline{V_i}$  is a convex set. The tuple of endpoints (or endcuts) of those convex sets is an element of some  $\overline{V_1}^{k_1} \times \cdots \times \overline{V_{n-1}}^{k_{n-1}}$ . Let  $p$  be the type of that tuple over  $\emptyset$ . Replacing each  $V_i$  with a large enough definable subset of  $\overline{V_i}$  and applying  $(B_n)$ , we see that  $p$  has property  $\boxtimes$ . In addition, it follows from the definition of  $X_a$  that for any  $a'$  having the same type as  $a$ , one of  $\overline{X_a}$  and  $\overline{X_{a'}}$  is included in the other. Since property  $\boxtimes$  allows us to move the endpoints of the convex sets defining  $X_a$  freely, this is only possible if  $\overline{X_a}$  is either the full product  $\prod_{i < n} \overline{V_i}$ , or is a rectangle, unbounded on all but at most one coordinate. However, by  $(B_2)$ , we know that  $\overline{X_a}$  must have full projection on each coordinate. Hence the only possibility is that  $\overline{X_a} = \prod_{i < n} \overline{V_i}$ .

We end as in Lemma 3.20. Density of  $X_a$  in the product implies that for any product  $\hat{I} = \prod_{i < n} I_i$  of open intervals, the set

$$s(\hat{I}) := \{f_n(x) : (f_0(x), f_1(x), \dots, f_{n-1}(x)) \in \hat{I}\}$$

is coinital in  $V_n$ . By applying the same argument to the reverse order, we get that it is also cofinal. Furthermore, by minimality, the closure of  $s(\hat{I})$  is a finite union of convex sets. Hence, given any  $\hat{I}$ , there is a unique minimal bounded convex set  $c(\hat{I}) \subseteq V_n$  such that  $s(\hat{I})$  is dense in  $V_n \setminus c(\hat{I})$ . If  $\hat{I} \subseteq \hat{I}'$ , then  $c(\hat{I}) \supseteq c(\hat{I}')$ . As  $V_n$  is weakly transitive, the intersection of all  $c(\hat{I})$  is empty. Since the family of  $\hat{I}$ 's is upward-directed under inclusion,  $c(\hat{I}_*)$  must be empty for some  $\hat{I}_*$ . But then by  $(B_n)$ , for any  $\hat{I}'$ , one can find  $\hat{I}'_* \subseteq \hat{I}'$  which is a conjugate of  $\hat{I}_*$ . Hence  $c(\hat{I}')$  is also empty and  $s(\hat{I}')$  is dense in  $V_n$ . Since this holds for any  $\hat{I}'$ ,  $(A_{n+1})$  follows.

$(A_{n+1}) \Rightarrow (B_{n+1})$ : As in the proof of Proposition 3.11, to show that all types have property  $\boxtimes$ , it is enough to find, for all  $k < \omega$ , one type in  $\overline{V_0}^k \times \cdots \times \overline{V_n}^k$  having property  $\boxtimes$  and for which no two coordinates are equal. To this end, take  $b_0 \in V_0$ . For each  $i \leq n$ , let  $m_i(b_0)$  denote the largest element of  $\overline{V_i}$  definable from  $b_0$ . Set  $a_{0,i} = m_i(b_0)$ . Then by  $(A_{n+1})$  applied to the functions  $m_i$ , we see that  $p_1 := \text{tp}(a_{0,i} : i \leq n)$  has property  $\boxtimes$ : its set of realizations is dense in  $\overline{V_0} \times \cdots \times \overline{V_n}$ .

Assume that  $b_l, a_{l,i}$  have been constructed for  $l < k, i \leq n$ , with  $a_{l,i} = m_i(b_{\leq l})$ . For  $i \leq n$ , let  $X_i$  be a complete type over  $b_{< k}$  of elements in  $V_i$ , greater than  $a_{k-1,i}$ . So  $X_i$  is dense in  $\{x \in V_i : x > a_{k-1,i}\}$ . Work over  $b_{< k}$  and consider the sets  $X_0, \dots, X_n$  equipped with the induced orders. They are pairwise independent. Pick any  $b_k \in X_0$  and define  $a_{k,i} = m_i(b_{\leq k}), i \leq n$ . Then again

by  $(A_{n+1})$ , the set of realizations of  $\text{tp}(a_{k,i} : i \leq n)$  is dense in  $\overline{X_0} \times \cdots \times \overline{X_n}$ . It follows inductively that the resulting type  $p_k := \text{tp}(a_{l,i} : l \leq k, i \leq n)$  satisfies  $\boxtimes$ .  $\square$

We say that a betweenness relation has topological rank 1 if one (or equivalently both) of its associated linear orders has topological rank 1.

**Corollary 3.22.** *Let  $V$  be a definable transitive set and let  $B_1, \dots, B_n$  be distinct  $\mathcal{O}$ -definable betweenness relations on  $V$  of topological rank 1. Then for any subset  $I \subseteq n$ , we can find  $a_I, b_I, c_I \in V$  such that  $B_i(a_I, b_I, c_I)$  holds if and only if  $i \in I$ .*

We now extend Proposition 3.15 to a structure equipped with any  $n$  minimal linear orders.

**Proposition 3.23.** *Let  $(M; \leq_1, \dots, \leq_n)$  be countable,  $\omega$ -categorical, transitive over  $\mathcal{O}$ . Assume that each  $M_i := (M, \leq_i)$  is a linear order of topological rank 1 and that no two of them are equal or reverse of each other. Then for each  $i \neq j \leq n$ , exactly one of the following holds:*

- $\leq_i$  and  $\leq_j$  are independent;
- $\leq_i$  is intertwined with  $\leq_j$  and if  $f_{ij}: M_i \rightarrow \overline{M_j}$  is the unique  $\mathcal{O}$ -definable increasing map, we have  $f_{ij}(x) <_j x$  for all  $x$ ;
- $\leq_i$  is intertwined with  $\leq_j$  and we have  $f_{ij}(x) >_j x$  for all  $x$ ;
- $\leq_i$  is intertwined with the reverse of  $\leq_j$  and if  $f_{ij}: M_i \rightarrow \overline{M_j}$  is the unique  $\mathcal{O}$ -definable decreasing map, we have  $f_{ij}(x) <_j x$  for all  $x$ ;
- $\leq_i$  is intertwined with the reverse of  $\leq_j$  and we have  $f_{ij}(x) >_j x$  for all  $x$ .

Furthermore, the data of which of those cases holds for each pair  $i \neq j$  completely determines the isomorphism type of  $M$ .

*Proof.* The argument is similar to that of Proposition 3.15, which we present a little bit differently. First note that by Corollary 3.18, by replacing some orders  $\leq_i$  with their reverses, we can assume that the last two cases never occur. Let then  $E$  be the equivalence relation on  $\{1, \dots, n\}$  which holds for  $i, j$  if  $M_i$  and  $M_j$  are intertwined. Let  $s_1, \dots, s_k$  be representatives of the  $E$ -classes and for each  $i \leq n$ , let  $t(i)$  be such that  $i E s_{t(i)}$ . Define also  $\iota_i: M_i \rightarrow \overline{M_{s_{t(i)}}}$  be the unique increasing  $\mathcal{O}$ -definable map intertwining  $M_i$  and  $M_{s_{t(i)}}$ .

For  $t \leq k$ , define  $V_t \subseteq \overline{M_{s_t}}$  as the union

$$V_t = \bigcup_{i E s_t} \iota_i(M_i),$$

and let  $\preceq_t$  be its canonical linear order. Then  $(V_t, \preceq_t)$  is a minimal,  $\mathcal{O}$ -definable order. Define

$$\Gamma = \{(\iota_1(x), \dots, \iota_n(x)) : x \in M\} \subseteq \prod_{i \leq n} V_{t(i)}.$$

Now by the previous proposition,  $\Gamma$  is dense in a product  $D_1 \times \cdots \times D_k$  of closed subsets of the  $V_i$ 's. By Corollary 3.13,  $D_k$  is dense in a set defined by a boolean combination of inequalities on variables  $x_i \leq x_j$ . Those inequalities are determined by inequalities  $\iota_i(x) \leq_{s_{t(i)}} x$  that are true in  $M$  and are part of the data that we are given. We conclude by a direct back-and-forth argument as in Proposition 3.14.  $\square$

## 4 Circular orders

Most of the results above generalize to circular orders, though some extra arguments are required.

Let  $(V, C)$  be a circular order. We will abuse notation by writing say  $a < b < c < d$  to mean that  $a, b, c, d$  are pairwise distinct and  $(a, b, c, d)$  lie in this order on  $V$ : that is  $C(a, b, c) \wedge C(b, c, d) \wedge C(c, d, a) \wedge C(d, a, b)$ . So  $a < b$  only means that  $a \neq b$  and  $a < b < c$  is equivalent to  $a \neq b \neq c \wedge C(a, b, c)$ . Hopefully, this will not lead to confusion. For any  $a < b$  on  $V$ , the set defined by  $a < x < b$  is called an *open interval* of  $V$ . Any interval has a canonical linear order on it coming from the circular order on  $V$ . The notations are consistent in the sense that if  $I \subseteq V$  is an open interval, and  $c, d, e \in I$ , then we have  $c < d < e$  in the sense of the circular order if and only if we have  $c < d < e$  in the sense of the induced linear order on  $I$ .

For  $a \in V$ , we let  $V_{a \rightarrow} = V \setminus \{a\}$ , equipped with the linear order inherited from  $C$ . We say that  $V$  has topological rank 1 if it does not admit a parameter-definable convex equivalence relation with infinitely many infinite classes. Then  $V$  has topological rank 1 if and only if some/any  $V_{a \rightarrow}$  has topological rank 1.

Let  $V$  be circularly ordered. A subset  $I \subseteq V$  is *convex* if for any  $a \neq b \in I$ , one of the two intervals  $a < x < b$  and  $b < x < a$  is included in  $I$ . A convex set  $I$  is *bounded* if its complement is infinite. Note that if  $V$  is dense, then any open interval is bounded. A bounded convex set  $I \subseteq V$  has a well defined linear order induced by the circular order on  $V$ . If  $I$  and  $J$  are two bounded convex subsets of  $V$  with no last element (in their induced linear orders), we say that  $I$  and  $J$  define the same cut in  $V$  if one is an end segment of the other.

We define the completion  $\overline{V}$  of  $V$  as the set of definable convex subsets of  $V$  quotiented by the equivalence relation of defining the same cut. As for linear orders, this is naturally a countable union of interpretable sets (or rather a direct limit). In fact, given  $a \in V$ ,  $\overline{V}$  can be canonically identified with  $\overline{V_{a \rightarrow}} \cup \{a\}$  for any  $a \in V$ : the element  $a \in V$  is identified with the class of an open interval  $b < x < a$  and any cut of  $V_{a \rightarrow}$  is a bounded convex subset of  $V$  and is identified with its class in  $\overline{V}$ . As in the case of linear orders,  $\overline{V}$  is naturally equipped with a circular order, and there is a canonical embedding of  $V$  in  $\overline{V}$  which sends  $V$  to a dense subset of  $\overline{V}$ .

We say that  $V$  is *weakly transitive* if it is densely ordered and no element in  $\overline{V}$  is algebraic over  $\emptyset$ .

**Lemma 4.1.** *If  $(V, C)$  is weakly transitive of topological rank 1, then any  $\emptyset$ -definable subset of  $V$  is dense in  $V$ .*

*Proof.* By topological rank 1, any closed  $\emptyset$ -definable subset of  $V$  is a finite union of convex sets. The cuts defining these convex sets are algebraic over  $\emptyset$ , but there can be no such cut by weak transitivity.  $\square$

If  $V$  and  $W$  are two weakly transitive circular orders, we say that they are intertwined over  $A$  if there is an  $A$ -definable order-preserving injective map  $f: V \rightarrow \overline{W}$ . As for linear orders, this is an equivalence relation. It is no longer true that such a map has to be unique, however, we will see that there can be at most finitely many.

**Definition 4.2.** A self-intertwining of a circular order  $(V, C)$  is an intertwining map  $f: V \rightarrow \overline{V}$  which is not the identity.

Let  $(V, C)$  be a 0-definable circular order of topological rank 1 and fix some  $a \in V$ . Then we can write  $V = F \cup V_1 \cup \dots \cup V_n$ , where  $F = \text{dcl}(a) \cap V$  and the  $V_i$ 's are convex subsets of  $V$ , definable and weakly transitive over  $a$ , with  $V_1 < V_2 < \dots < V_n$ . By Lemma 3.10, for any  $i, j \leq n$ , there is at most one intertwining map  $f_{ij}: V_i \rightarrow \overline{V_j}$ . If it exists,  $f_{ij}$  has dense image.

Let now  $f: V \rightarrow \overline{V}$  be a self-intertwining map (defined over any set of parameters). For each  $i \leq n$ , there is a partition of  $V_i$  such that  $f$  coincides with some  $f_{ij}$  on each set in the partition (otherwise, by composing with some  $f_{ij}^{-1}$ , we would get a intertwining from a subset of  $V_i$  to itself, which contradicts Lemma 3.17). By continuity of  $f$ , there must be some  $j$  such that  $f$  coincides with  $f_{ij}$  on the whole of  $V_i$ . So  $f$  sends  $V_i$  to some  $\overline{V_j}$  via  $f_{ij}$ . Assume that for some  $i$ ,  $f$  sends  $V_i$  to  $\overline{V_{i+k}}$ . Then as  $f$  preserves the order, it must send  $V_{i+1}$  to  $\overline{V_{i+k+1}}$  (addition modulo  $n$ ) and iteratively send any  $V_j$  to  $\overline{V_{j+k}}$ . The number  $k$  completely determines  $f$ , as does therefore the image of  $a$ . The possibilities for  $k$  form a subgroup in  $\mathbb{Z}/n\mathbb{Z}$ . Hence the set of self-intertwinings along with the identity map, equipped with composition, is isomorphic to  $\mathbb{Z}/\delta\mathbb{Z}$  for some integer  $\delta$ .

**Definition 4.3.** A circular order  $V$  is *minimal* if it is weakly transitive, of topological rank 1 and admits no self-intertwining.

**Lemma 4.4.** *Let  $V$  be a circular order and let  $X_a, a \in D$ , be a uniformly definable family of non-empty subsets of  $V$  which is directed: for any  $a, a' \in D$  there is  $a'' \in D$  such that  $X_{a''} \subseteq X_a \cap X_{a'}$ . Then there is some  $c \in \overline{V}$  such that for any  $a \in D$  and any neighborhood  $I$  of  $c$  in  $V$ ,  $I \cap X_a \neq \emptyset$ .*

*Proof.* We fix a point  $d \in V$  and work in the linear order  $V_{d \rightarrow}$ . Let  $c \in \overline{V}$  be equal to  $\inf_{a \in D}(\sup X_a)$ . (If  $c = \pm\infty$ , then set  $c = d$ .) Then  $c$  has the required property.  $\square$

**Proposition 4.5.** *Working over some  $A$ , let  $V$  be a minimal definable circular order. Then for any type  $p(x_1, \dots, x_n) \vdash x_1 < \dots < x_n$  over  $A$ , and any open intervals  $I_1 < \dots < I_n$  of  $V$ , we can find  $a_i \in I_i$  with  $(a_1, \dots, a_n) \models p$ .*

*Proof.* For simplicity, assume  $A = \emptyset$ . Fix  $a < b$  in  $V$  and let  $q(x, y) = \text{tp}(a, b)$ . Call a convex subset  $I$  of  $V$  *small* if there are no  $a' < b'$  in  $I$  with  $\text{tp}(a', b') = q$  (where the order  $<$  is the canonical one on  $I$ ). Assume that there is some small interval. Then by weak transitivity, for any point  $c$  of  $V$ , there is a small open interval containing  $c$ . For any  $c \in V$ , let  $s(c)$  be the maximal cut in  $V_{c \rightarrow}$  so that  $(c, s(c))$  is small. We have

$$c < d < s(c) \implies c < d < s(c) \leq s(d).$$

Note that if  $c < d < s(c) = s(d)$ , then  $s(c) = s(e)$  for any  $e, c < e < d$ . Hence the preimage of a cut by  $s$  is a convex set. If the preimage of some cut is infinite, then this is true for infinitely many cuts in  $\overline{V}$  by weak transitivity. But then the relation  $s(x) = s(y)$  is a convex equivalence relation with infinitely many infinite classes, contradicting topological rank 1. It follows that  $s$  is injective. Hence  $s: V \rightarrow \overline{V}$  is a self-intertwining, which contradicts minimality. We have established that no interval is small.

Given  $a \in V$ , let  $m(a)$  denote the maximal cut in  $V_{a \rightarrow}$  definable over  $a$  (and  $m(a) = a$  if there is no such cut). Choose  $a_* \leq m(a_*) < b_*$  in  $V$  and set  $q = \text{tp}(a_*, b_*)$ . Now pick  $I_0 < I_1 < \dots < I_n$  open intervals of  $V$ . By the previous paragraph, we can find some pair  $(a, b) \models q$  such that  $a, b \in I_0$ . The interval  $x > m(a)$  in  $V_{a \rightarrow}$  is a linear order which is weakly transitive over  $a$ . Let  $p_{1n}(x_1, x_n)$  be the restriction of  $p$  to the variables  $(x_1, x_n)$ . Applying the previous paragraph to  $p_{1n}$ , we see that there is a realization of  $p$  in  $\{x \in V_{a \rightarrow} : x > m(a)\}$  composed of elements in increasing order. By Lemma 3.11, we can find  $a_1 \in I_1, \dots, a_n \in I_n$  with  $\text{tp}(a_1, \dots, a_n) = p$ .  $\square$

**Lemma 4.6.** *Let  $V$  be a minimal circular order and  $I, J \subseteq V$  two disjoint open intervals, then  $I$  and  $J$  are independent (as linear orders).*

*Proof.* Assume that some two disjoint intervals  $I, J$  of  $V$  are intertwined (over some set of parameters). Then by the previous proposition, we can find  $I' \subset I$  and  $J' \supset J$  disjoint such that the pair  $(I', J')$  is a conjugate of  $(I, J)$ . In particular  $I'$  and  $J'$  are intertwined and we conclude as in Lemma 3.17.  $\square$

Say that two circular orders  $V$  and  $W$  are *independent* if any open interval of  $V$  is independent (as a linear order) from any open interval of  $W$ .

**Lemma 4.7.** *Working over some  $A$ , let  $V$  be an weakly transitive definable circular order of topological rank 1 and  $W$  a weakly transitive definable linear order of topological rank 1. Then an open interval of  $V$  is independent with any open interval of  $W$ .*

*Proof.* Assume that  $I \subseteq V$  and  $J \subseteq W$  are two open intervals definable and weakly transitive over some  $B$ , which are intertwined. Let  $a \in I$ . If  $I_0$  is an open interval containing  $a$  intertwined with some interval  $J_0$  of  $W$ , then by uniqueness of intertwinings (and Lemma 3.17), the intertwining maps  $I_0 \rightarrow \overline{W}$  and  $I \rightarrow \overline{W}$  must coincide on  $I_0 \cap I$ . It follows that the element of  $\overline{W}$  to which  $a$  is mapped lies in  $\text{dcl}(Aa)$ : say it is equal to  $g(a)$  for some  $A$ -definable function  $g$ . Then  $g: V \rightarrow \overline{W}$  is locally increasing, which is impossible.  $\square$

**Proposition 4.8.** *Working over some  $A$ , let  $V_0, \dots, V_{n-1}$  be minimal definable circular orders, pairwise independent. Then any  $A$ -definable closed set  $X \subseteq V_0^{k_0} \times \dots \times V_{n-1}^{k_{n-1}}$  is a finite union of products of the form  $D_0 \times \dots \times D_{n-1}$ , where each  $D_i$  is an  $A$ -definable closed subset of  $V_i^{k_i}$ .*

*Proof.* Assume  $A = \emptyset$ . We show the following two statements by induction on  $n$ . Note that  $(B_n)$  implies what we want since by  $\omega$ -categoricity, any definable set is a finite union of types.

( $A_n$ ) Let  $p(\bar{x}_i : i < n)$  be a type in some product  $V_0^{l_0} \times \dots \times V_{n-1}^{l_{n-1}}$ , then given any intervals  $I_i \subseteq V_i$ , we can find  $(\bar{a}_i : i < n) \models p$  with  $\bar{a}_i \in I_i$  for each  $i < n$ .

( $B_n$ ) For any type  $p$  over  $\emptyset$  on  $V_0^{k_0} \times \dots \times V_{n-1}^{k_{n-1}}$ , the closure  $X$  of the set of realizations of  $p$  is equal to the product of its projections to each factor  $V_i^{k_i}$ .

( $B_n$ ): Assume we know ( $A_n$ ) and we show that ( $B_n$ ) follows.

Let  $X$  be given as in ( $B_n$ ) and for  $i < n$ , let  $D_i$  be the projection of  $X$  to  $V_i^{k_i}$ . For each  $i < n$ , let  $T_i \subseteq V_i$  be an open interval and set  $T = T_0^{k_0} \times \dots \times T_{n-1}^{k_{n-1}}$ . Since we can choose  $T$  to contain any given finite set, it is enough to show the result for  $X \cap T$  instead of  $X$ .

Let  $\bar{e}$  be any tuple of parameters containing at least two points from each  $V_i$ ,  $i < n$ . For each  $i < n$ , let  $a_i, b_i \in \text{dcl}(\bar{e}) \cap \bar{V}_i$  be such that the complement of the convex set  $a_i \leq x \leq b_i$  in  $V_i$  is infinite and weakly transitive over  $\bar{e}$ . By ( $A_n$ ), we may choose  $\bar{e}$  so that each convex set  $a_i \leq x \leq b_i$  is disjoint from  $T_i$ . Then over  $\bar{e}$ , the  $T_i$ 's are intervals in some weakly transitive  $\bar{e}$ -definable linear orders, which are pairwise independent. Therefore by Proposition 3.21, the restriction of  $X$  to  $T$  is the product of its projections to each factor, as required.

( $A_n$ ): Assume that we know ( $B_{n-1}$ ) and we prove ( $A_n$ ).

Let  $V = V_0$  and  $W = \prod_{0 < i < n} V_i$ . Given a point  $d \in \prod_{0 < i < n} V_i$ , a neighborhood of  $d$  will mean a product  $\prod_{0 < i < n} J_i$ , where each  $J_i$  is an open interval containing  $d_i$ .

Let  $c \in V$ . Say that a subset  $J = \prod_{0 < i < n} J_i \subseteq W$  is *good for  $c$*  if for any open interval  $I \subset V$  containing  $c$ , there is  $(\bar{b}_i)_{i < n} \models p$ , with  $\bar{b}_0 \in I$  and  $\bar{b}_i \in J_i$ ,  $i > 0$ . We claim that there are bounded convex sets  $J_i \subset V_i$ ,  $i < n$  such that  $\prod_{0 < i < n} J_i$  is good for  $c$ . To see this, take for each  $i < n$ ,  $K_{i,1}, \dots, K_{i,t}$  disjoint open intervals of  $V_i$ , with  $t > |\bar{b}_i|$  and set  $J_{i,s} = V_i \setminus K_{i,s}$ : a bounded convex subset of  $V_i$ . By Proposition 4.5, for any neighborhood  $I$  of  $c$ , there is  $(\bar{b}_i)_{i < n} \models p$  with  $\bar{b}_0 \in I$  and  $\bar{b}_i \in V_i$ ,  $0 < i < n$ . For each  $0 < i < n$ , there must be some  $s(i)$  such that no coordinate of  $\bar{b}_i$  lies in  $K_{i,s(i)}$ . As the family of possible  $I$  is directed downwards, there is a choice of  $s(i)$  which works for all  $I$ . Let  $J_i = J_{i,s(i)}$ ,  $i < n$ . Then the set  $\prod_{0 < i < n} J_i$  is good for  $c$ .

For any  $J \subseteq W$  a product of bounded convex sets, let  $X(J) \subseteq V$  be the set of elements  $c \in V$  for which  $J$  is good. Note that  $X(J)$  is closed in the

order topology and hence is a finite union of closed intervals. For  $d \in W$ , the family  $\{X(J) : J \text{ neighborhood of } d\}$  is directed. By Lemma 4.4, there is some  $c \in \bar{V}$  which lies in the closures of each such  $X(J)$ . We then have the following property: for any neighborhoods  $I$  of  $c$  and  $J$  of  $d$ , there is  $(\bar{b}_i)_{i < n} \models p$ , with  $\bar{b}_0 \in I$  and  $\bar{b}_i \in J_i$ ,  $i > 0$ . Take a set of parameters  $\bar{e}$  containing two points from each  $V_i$  and such that neither  $c$  nor  $d$  lies in  $\text{acl}(\bar{e})$ . Then, over  $\bar{e}$ , there are intervals  $J_i \subseteq V_i$  that are weakly transitive and with  $c \in J_0$  and  $d_i \in J_i$ . By assumption, the  $J_i$ 's are pairwise independent. Therefore by Lemma 3.21, given any subintervals  $J'_i \subseteq J_i$ , we can find a realization of  $p$  in  $\prod_{i < n} J'_i$ .

Given  $d \in W$ , let  $Z(d)$  be the set of points  $c \in V$  such that any neighborhood  $d$  is good for  $c$ . By the previous paragraph, there is  $d$  such that  $Z(d)$  has non-empty interior. Then by Proposition 4.5, for any open interval  $I_*$  of  $V$ , there is  $d_* \in W$  such that  $Z(d_*) \supseteq I_*$ . Let  $Z_*(I_*)$  denote the set of such points  $d_*$ . For  $i < n$  let  $\pi_i: \prod_{0 < j < n} V_j \rightarrow V_i$  be the canonical projection. Fix  $c \in V$  and for each  $0 < i < n$  consider the family  $\{\pi_i(Z_*(I)) : I \text{ open interval disjoint from } c\}$ . Since the map  $Z_*$  is decreasing, that family is directed and by Lemma 4.4 there is  $e_i \in \bar{V}_i$  in the closure of all of its elements. Now  $(B_{n-1})$  implies that the closure of  $Z_*(I)$  can be written as a union of definable subsets of the form  $D_1 \times \dots \times D_{n-1}$ , where each  $D_i$  is a closed definable subset of  $V_i$ . Restricting it, we can assume that it is equal to one such set. We then have that  $e = (e_1, \dots, e_{n-1})$  is in the closure of each  $Z_*(I)$ ,  $I$  open interval disjoint from  $c$ .

We have thus obtained the following property: for any neighborhood  $J$  of  $e$  in  $W$  and any open interval  $I$  of  $V$  not containing  $c$ , there are  $\bar{a} \in I$  and  $\bar{b} \in J$  such that  $(\bar{a}, \bar{b}) \models p$ . Since any open interval contains a subinterval not containing  $c$ , we can remove the requirement that  $I$  does not contain  $c$ . By  $(A_{n-1})$ , the locus of  $e$  is dense in  $\bar{W}$ : any product of open intervals in  $W$  contains a conjugate of  $e$ . This shows that for any open  $I \subseteq V$  and  $J_i \subseteq V_i$ , we can find  $(\bar{b}_i)_{i < n} \models p$ , with  $\bar{b}_0 \in I$  and  $\bar{b}_i \in J_i$ ,  $0 < i$ , as required.  $\square$

**Lemma 4.9.** *Working over some  $A$ , let  $V_0, \dots, V_{n-1}$  be pairwise independent, minimal definable circular orders. Let  $V_n, \dots, V_{m-1}$  be pairwise independent minimal definable linear orders. Let  $p(\bar{x}_i : i < m)$  be a type over  $A$  in some product  $V_0^{I_0} \times \dots \times V_{m-1}^{I_{m-1}}$ , then given any open intervals  $I_i \subseteq V_i$ ,  $i < n$  and initial segments  $I_i \subseteq V_i$ ,  $n \leq i < m$ , we can find  $(\bar{a}_i : i < m) \models p$  with  $\bar{a}_i \in I_i$  for each  $i < m$ .*

*Proof.* Fix some intervals  $I_i \subseteq V_i$ ,  $i < n$ . Then by Proposition 4.8, we can find  $(\bar{a}_i : i < m) \models p$  with  $\bar{a}_i \in I_i$  for each  $i < n$ . For each  $n \leq t < m$ , let  $c_t \in \bar{V}_t$  be the maximal cut such that there does not exist  $(\bar{a}_i : i < m) \models p$  with  $\bar{a}_i \in I_i$  for  $i < n$  and  $\bar{a}_t < c_t$ , if such a  $c_t$  exists and  $c_t = -\infty$  otherwise. If  $c_t = -\infty$  for all  $t$ , then by Proposition 3.21, for any initial segments  $I_i \subseteq V_i$ ,  $n \leq i < m$ , we can find  $(\bar{a}_i : i < m) \models p$ , with  $\bar{a}_i \in I_i$ ,  $i < m$ .

Assume now that say  $c_n \neq -\infty$ . Let  $\tilde{I} = \prod_{i < n} I_i^{I_i}$ . Observe that by Proposition 4.8, there are finitely many automorphisms  $\sigma_1, \dots, \sigma_k$  such that  $\bigcup_{i \leq k} \sigma_i(\tilde{I})$  covers  $\prod_{i < n} V_i^{I_i}$ . But then, if  $c_* = \inf_{i \leq k} \sigma_i(c_n)$ , we see that there is no realization of  $p$  with its  $V_n$ -part below  $c_*$ . This contradicts weak transitivity of  $V_n$ .  $\square$

**Theorem 4.10.** *Working over some  $A$ , let  $V_0, \dots, V_{n-1}$  be pairwise independent, minimal definable circular orders. Let  $V_n, \dots, V_{m-1}$  be pairwise independent minimal definable linear orders. Then any  $A$ -definable closed subset  $D \subseteq V_0^{k_0} \times \dots \times V_{m-1}^{k_{m-1}}$  is a finite union of products of the form  $D_0 \times \dots \times D_{m-1}$ , where each  $D_i$  is an  $A$ -definable closed subset of  $V_i^{k_i}$ .*

*Proof.* The proof is very similar to that of  $(B_n)$  in Proposition 4.8, using Lemma 4.9. Assume  $A = \emptyset$ .

Let  $D$  be given as in the statement and assume that it is the closure of a complete type. For  $i < m$ , let  $D_i$  be the projection of  $D$  to  $V_i^{k_i}$ . For each  $i < m$ , let  $T_i \subseteq V_i$  be a bounded interval and set  $T = T_0^{k_0} \times \dots \times T_{m-1}^{k_{m-1}}$ . It is enough to show the result for  $D \cap T$  instead of  $D$ .

Let  $\bar{e}$  be any tuple of parameters containing at least two points from each  $V_i$ ,  $i < m$ . For each  $i < n$ , let  $a_i, b_i \in \text{dcl}(\bar{e}) \cap \bar{V}_i$  be such that the complement of the interval  $a_i \leq x \leq b_i$  in  $V_i$  is infinite and weakly transitive over  $\bar{e}$ . For each  $n \leq i < m$ , let  $d_i \in \text{dcl}(\bar{e}) \cap \bar{V}_i$  such that the end-segment  $x > d_i$  is weakly transitive over  $\bar{e}$ .

By Lemma 4.9, we may choose  $\bar{e}$  so that each interval  $a_i \leq x \leq b_i$  is disjoint from  $T_i$  for  $i < n$ , and for  $n \leq i < m$ , we have  $d_i < T_i$ . Then over  $\bar{e}$ , the  $T_i$ 's are intervals in some weakly transitive  $\bar{e}$ -definable linear orders, which are pairwise independent. Therefore by Proposition 3.21, the restriction of  $X$  to  $T$  is the product of its projections to each factor, as required.  $\square$

**Corollary 4.11.** *Let  $V_0, \dots, V_{n-1}$  be pairwise independent, minimal 0-definable circular orders. Let  $V_n, \dots, V_{m-1}$  be pairwise independent minimal 0-definable linear orders. Let  $D \subseteq V_0^{k_0} \times \dots \times V_{m-1}^{k_{m-1}}$  be a closed subset, definable over some parameters  $A$ . Then  $D$  is a finite union of products of the form  $D_0 \times \dots \times D_{m-1}$ , where each  $D_i$  is an  $A$ -definable closed subset of  $V_i^{k_i}$ .*

*Proof.* Each  $V_0$  breaks over  $A$  into finitely many  $A$ -definable points and  $A$ -definable convex subsets, each weakly transitive over  $A$ . Any two such convex subsets are independent by Lemmas 3.17 and 4.6. We then conclude by Theorem 4.10.  $\square$

With the same argument as for Proposition 3.23, we can show the following classification result.

**Corollary 4.12.** *Let  $(M; C_1, \dots, C_m, \leq_1, \dots, \leq_n)$  be countable,  $\omega$ -categorical, transitive, equipped with  $m$  circular orders and  $n$  linear orders, each minimal. Write  $M_i = (M, \leq_i)$ . Then the isomorphism type of  $M$  is completely determined up to automorphism by the following information:*

- For any  $i, j \leq m$ , whether  $C_i$  and  $C_j$  are equal, equal up to reversal, intertwined, intertwined up to reversal, or independent.
- For any  $i, j \leq n$ , whether  $\leq_i$  and  $\leq_j$  are equal, equal up to reversal, intertwined, intertwined up to reversal, or independent.



- For any  $i < j \leq n$  such that  $\leq_i$  and  $\leq_j$  are intertwined (possibly up to reversal) but not equal, if  $f_{ij}: M_i \rightarrow \overline{M}_j$  is the intertwining map, whether we have  $f_{ij}(x) <_j x$  or  $x <_j f_{ij}(x)$  for some/any  $x \in M$ .

## 5 Local relations

We now aim at describing a certain kind of relations on products of minimal orders, which we call local. This will only be used at the very end of the analysis to show the finiteness result of Theorem 1.2. Very little of it is needed to prove the statements in Theorem 1.3, which already imply that there are only countably many structures satisfying  $(\star)$ . We advise the reader to skip this section at first and come back to it when it is called for.

We start by giving examples of local relations.

EXAMPLE 5.1. *All structures are assumed to be countable.*

1. Let  $(V, \leq)$  be a dense linear order without endpoints and let  $E$  be an equivalence relation on  $V$  with finitely many classes, each of which is dense co-dense. In the structure  $(V; \leq, E)$ , the order  $(V, \leq)$  is weakly transitive and rank 1. The isomorphism type of this structure is determined by the number of classes. One could further expand this structure by adding any structure on the finite quotient  $V/E$ . We will see that those are the only weakly transitive, rank 1 and op-dimension 1 expansions of a linear order.
2. Let  $(V, C)$  be a dense circular order. We may similarly expand it by adding an equivalence relation  $E$  with finitely many classes, each of which is dense co-dense. Again, the isomorphism type of the expansion is determined by the number of classes and one can expand the resulting structure by putting any structure on the quotient  $V/E$ .
3. Take  $(V, C)$  a dense (countable) circular order. Let  $\pi: W \rightarrow V$  be a connected  $k$ -fold cover of  $V$ : that is  $W$  is itself a circular order, the map  $\pi$  is locally an isomorphism and is  $k$ -to-one. Up to isomorphism, there is a unique such structure. Now let  $s: V \rightarrow W$  be a section of  $\pi$  which is generic in the sense that on any small interval of  $V$ ,  $s$  takes values in the  $k$  sheets of the cover above that interval. Again, those conditions determine the isomorphism type of  $(W, V; \pi, s)$ .

The induced structure on  $V$  can be described in various ways. If  $k > 1$ , let  $R(x, y)$  be the binary relation which holds for two points  $a, b$  if  $\pi$  is injective on the interval  $s(a) < x < s(b)$ . Note that the circular order on  $V$  is definable from  $R$  and in fact the whole structure is bi-interpretable with  $(V; R)$ . Those structures  $(V; R)$  are sometimes named  $S(k)$  in the literature. We will call them finite covers of  $V$  (in general a finite cover of a structure  $M$  is a structure  $N$  equipped with a finite-to-one projection map onto  $M$ ).

Another way to encode the structure on  $V$  which will be more natural to us is as a local equivalence relation. Define a 4-ary predicate

$$E(s, t; x, y) \equiv (s < x = y < t) \vee (s < x < y < t \wedge R(x, y)) \vee \\ \vee (s < y < x < t \wedge R(y, x)).$$

Then for any  $a \neq b$ , the relation  $E(a, b; x, y)$  is an equivalence relation on the interval  $a < x < b$ . It is in this form that those structures will appear in our analysis.

4. We can combine examples (2) and (3). Fix some integers  $(k_1, \dots, k_m)$ . Let  $(V, C)$  be a dense circular order, equipped with an equivalence relation  $E$  with  $m$  dense co-dense classes. On the  $i$ -th class, we have a  $k_i$ -fold cover coded by a local equivalence relation  $E_i$  as in (3). The isomorphism type of the structure  $(V; C, E_1, \dots, E_m)$  is determined by the tuple  $(k_1, \dots, k_m)$ .

As we will see eventually, those are, up to inter-definability, the only minimal, rank 1, op-dimension 1, expansions of circular orders.

5. Let  $(V, C)$  be a dense circular order equipped with two equivalence relations  $E$  and  $F$  such that  $F$  has two dense classes, each  $E$ -class consists of exactly one element from each, and the structure is generic such. Let  $M$  be the quotient of  $V$  by  $E$ . Then  $M$  satisfies  $(\star)$  and is a proper expansion of the last structure in Example 1.1 (obtained from  $M$  by forgetting about  $F$ ). We then have an equivalence relation with two classes on the set  $W_*$  of pairs  $(a, b) \in V^2$ , with  $a E b$ . This is another example of a local equivalence relation. In this case it is a bona fide equivalence relation, although not on the structure  $M$  itself, but on the finite cover  $W_*$ .

Let  $(V_k^* : k < m_*)$  be a finite family of 0-definable minimal linear and circular orders so that any two are independent. Let  $\bar{c} = (c_i)_{i < n_*}$  enumerate a relatively algebraically closed subset of  $\bigcup V_k^*$  such that  $\bar{c} \in \text{acl}(c_i)$  for each  $i$ . For  $i < n_*$ , let  $k(i) < m_*$  be such that  $c_i \in V_{k(i)}^*$  and set  $V_i = V_{k(i)}^*$ . Reordering  $\bar{c}$  if necessary, assume that for some  $n_c < n_*$ ,  $V_i$  is circular for  $i < n_c$  and linear otherwise. Let  $p_0 = \text{tp}(\bar{c})$  and  $W_* \subseteq \prod_{i < n_*} V_i$  the locus of  $p_0$ .

By the  $L_0$ -structure, we mean the structure having one sort for each  $V_k^*$  equipped with its linear or circular order and a unary predicate for  $W_*$  as a subset of  $\prod_{i < n_*} V_{k(i)}^*$ . We start by describing the  $L_0$ -structure and will then study additional *local* structure. In the next section, we will show that under a hypothesis on the op-dimension, any additional structure on  $W_*$  has to be local.

For each  $i$ , the projection  $W_i$  of  $W_*$  on  $V_i$  is dense in  $V_i$  and is a transitive set (in the original structure, and therefore also in the  $L_0$ -structure). If  $i \neq j$  and  $V_i = V_j$  are linear, then  $W_i \neq W_j$  since algebraic closure must be trivial on  $W_i$ . However, if  $V_i = V_j$  is circular, then we could have either  $W_i \neq W_j$  or  $W_i = W_j$ . By construction of  $\bar{c}$ , if  $\bar{d}, \bar{e} \in W_*$  are such that  $d_i = e_j$  for some  $i, j$ , then  $\bar{d}$  is a permutation of  $\bar{e}$ . Let  $G \leq \mathfrak{S}(n_*)$  be the group of permutations  $\sigma$  such that

$(c_{\sigma(1)}, \dots, c_{\sigma(n_*-1)}) \models p_0$ . Note that  $G$  is non-trivial if only if for some  $i \neq j$ , we have  $W_i = W_j$ . (If  $W_i = W_j$ , then given  $\bar{c} \in W_*$ , there is an automorphism sending  $c_i$  to  $c_j$  which must induce a permutation of the tuple  $\bar{c}$ .)

Theorem 4.10 and a back-and-forth argument shows that the isomorphism type of the  $L_0$ -structure is entirely determined by:

- the number  $m_*$  of orders, the type (linear or circular) of each;
- the integer  $n_*$  and the map  $k: n_* \rightarrow m_*$ ;
- for each  $i, j < n_*$  such that  $V_i = V_j$  is linear, whether or not  $p_0(\bar{x}) \vdash x_i < x_j$ ;
- for each  $i, j, k < n_*$  such that  $V_i = V_j = V_k$  is circular, whether or not  $p_0(\bar{x}) \vdash x_i < x_j < x_k$ ;
- the group  $G$  as a subgroup of  $\mathfrak{S}(n_*)$ .

### 5.0.1 Small cells, paths and simple connectedness

A bounded interval of a linear or circular order is an interval of the form  $a < x < b$ , with  $a < b$ .

A *small cell* of  $W_*$  is the intersection with  $W_*$  of a product  $\prod_{i < n_*} I_i$  such that:

- each  $I_i \subseteq V_i$  is a bounded interval and any two  $I_i, I_j, i \neq j$ , are disjoint;
- if  $i, j$  are such that  $V_i = V_j$  are linear and  $p_0(\bar{x}) \vdash x_i < x_j$ , then  $I_i < I_j$ ;
- if  $i, j, k$  are such that  $V_i = V_j = V_k$  and  $p_0(\bar{x}) \vdash x_i < x_j < x_k$ , then  $I_i < I_j < I_k$ .

Note that by Theorem 4.10,  $W_*$  is dense in such a product. Also each projection  $\pi_i$  is injective on a small cell.

**Lemma 5.2.** *Let  $X \subseteq W_*$  be a non-empty definable open set and let  $C \subseteq m_*$  be a small cell, then there is  $C' \subseteq X$ , such that  $C'$  is a conjugate of  $C$ .*

*Proof.* This follows at once from Theorem 4.10: we can choose the end points of the intervals defining  $C'$  arbitrarily.  $\square$

**Definition 5.3.** Let  $C_{\bar{a}}$  be a small cell defined over some tuple of parameters  $\bar{a}$  and let  $E_{\bar{a}}$  be a definable equivalence relation on  $C_{\bar{a}}$ . We say that  $E_{\bar{a}}$  is a *local equivalence relation* if for any  $\bar{a}' \equiv \bar{a}$  such that  $C_{\bar{a}'} \subseteq C_{\bar{a}}$ ,  $E_{\bar{a}'}$  and  $E_{\bar{a}}$  coincide on  $C_{\bar{a}'}$ .

**Lemma 5.4.** *Let  $E_{\bar{a}}$  be a local equivalence relation defined on  $C_{\bar{a}}$  and take  $\bar{a}' \equiv \bar{a}$ . Let  $C_0 \subseteq C_{\bar{a}} \cap C_{\bar{a}'}$  be a small cell, then  $E_{\bar{a}}$  and  $E_{\bar{a}'}$  coincide on  $C_0$ .*

*Proof.* For any finite  $F \subseteq C_0$ , there is by Theorem 4.10,  $\bar{a}'' \equiv \bar{a}$  such that  $F \subseteq C_{\bar{a}''} \subseteq C_{\bar{a}} \cap C_{\bar{a}'}$ . The result therefore follows by the definition of a local equivalence relation.  $\square$

Observe that a non-empty intersection of two small cells need not be a small cell: the intersection of two intervals in a circular order may be two disjoint intervals.

Fix a local equivalence relation  $E_{\vec{a}}$  and let  $\mathcal{E}$  be the family  $\{E_{\vec{a}'} : \vec{a}' \equiv \vec{a}\}$ . We will also refer to  $\mathcal{E}$  as a local equivalence relation. For any small cell  $C$ , we can find  $E \in \mathcal{E}$  whose domain contains  $C$ . Then by the previous lemma,  $E|_C$  does not depend on the choice of  $E \in \mathcal{E}$ . We will denote that equivalence relation by  $\mathcal{E}(C)$  and its set of classes by  $C/\mathcal{E}$ .

**Lemma 5.5.** *For any small cell  $C$ , any  $\mathcal{E}(C)$ -class is dense in  $C$ .*

*Proof.* By Corollary 4.11, closures of  $\mathcal{E}(C)$ -classes are boolean combinations of small cells. Assume that some  $\mathcal{E}(C)$ -class was not dense in  $C$ . Then there would be some cut in some order definable from any parameters defining  $C$ . Furthermore, if  $C' \supseteq C$  is a conjugate of  $C$ , then the same cut would be definable from parameters defining  $C'$ . By Theorem 4.10, this is impossible.  $\square$

If  $C_0, C_1$  are small cells such that  $C_0 \cap C_1$  is also a small cell, then we have a natural bijection  $f: C_0/\mathcal{E} \rightarrow C_1/\mathcal{E}$  given by identifying both  $C_0/\mathcal{E}$  and  $C_1/\mathcal{E}$  with  $C_0 \cap C_1/\mathcal{E}$ .

**Definition 5.6.** A *path* is a family  $\mathbf{p} = (C_i)_{i < n}$  such that each  $C_i$  is a small cell and each  $C_i \cap C_{i+1}$  is a small cell.

Given a path  $\mathbf{p} = (C_i)_{i < n}$ , we can define a map  $f_{\mathbf{p}}: C_0/\mathcal{E} \rightarrow C_{n-1}/\mathcal{E}$  given by composing the natural bijections  $f_i: C_i/\mathcal{E} \rightarrow C_{i+1}/\mathcal{E}$  defined above.

**Definition 5.7.** Say that a path  $\mathbf{p}' = (C'_i)_{i < n'}$  refines a path  $\mathbf{p} = (C_i)_{i < n}$  if there exists indices

$$0 = i_0 < \dots < i_{n-1} < i_n = n'$$

such that  $i_k \leq i < i_{k+1}$  implies  $C'_i \subseteq C_k$ .

**Proposition 5.8.** 1. *If a path  $\mathbf{p} = (C_i)_{i < n}$  satisfies that all the  $C_i$ 's lie in some given small cell  $C$ , then  $f_{\mathbf{p}}: C_0/\mathcal{E} \rightarrow C_{n-1}/\mathcal{E}$  is given by the identification of  $C_0/\mathcal{E}$  and  $C_{n-1}/\mathcal{E}$  to  $C/\mathcal{E}$ .*

2. *If a path  $\mathbf{p}'$  refines  $\mathbf{p}$ , then  $f_{\mathbf{p}'}$  is equal to  $f_{\mathbf{p}}$ , modulo the canonical identifications of the domain and range given by inclusion maps.*

*Proof.* The proof of (1) is immediate by induction on  $n$ .

To prove (2), let  $0 = i_0 < \dots < i_{n-1} < i_n = n'$  be as in Definition 5.7. The map from  $C'_0/\mathcal{E}$  to  $C'_{i_1-1}/\mathcal{E}$  obtained following  $\mathbf{p}'$  is given by the identification of both to  $C_0/\mathcal{E}$ . Then since  $C'_{i_1-1} \cap C'_{i_1} \subseteq C_0 \cap C_1$ , the map  $C'_{i_1-1}/\mathcal{E} \rightarrow C'_{i_1}/\mathcal{E}$  is the same—up to canonical identification of domain and range—as the one  $C_0/\mathcal{E} \rightarrow C_1/\mathcal{E}$ . Going on in this way proves the result.  $\square$

**Definition 5.9.** 1. An open definable set  $X \subseteq W_*$  is *path-connected* if for any two points  $a, b \in X$ , there is a path  $\mathbf{p} = (C_i : i < n)$  with  $a \in C_0$  and  $b \in C_{n-1}$ .

2. An open set  $X \subseteq W_*$  is *simply connected* if it is path-connected and for any two paths  $\mathfrak{p} = (C_i : i < n)$  and  $\mathfrak{p}' = (C'_i : i < n')$  with  $C_0 = C'_0$ ,  $C_{n-1} = C'_{n'-1}$ , the maps  $f_{\mathfrak{p}}$  and  $f_{\mathfrak{p}'}$  are equal.

Let  $X \subseteq W_*$  be a simply connected open set. Let  $a, b \in X$  and take a path  $\mathfrak{p}$  from some small cell  $C_a$  containing  $a$  to a small cell  $C_b$  containing  $b$ . This induces a map  $f_{\mathfrak{p}}: C_a/\mathcal{E} \rightarrow C_b/\mathcal{E}$ . Say that  $a$  and  $b$  are  $\mathcal{E}(X)$ -related if  $f_{\mathfrak{p}}$  maps the  $\mathcal{E}(C_a)$  class of  $a$  to the  $\mathcal{E}(C_b)$ -class of  $b$ . This notion does not depend on the choice of  $\mathfrak{p}$  by definition. It also does not depend on the choice of  $C_a$  and  $C_b$ , since if we make a different choice, say  $C'_a$  and  $C'_b$ , related by a path  $\mathfrak{p}'$ , then we can find  $C''_a \subseteq C_a \cap C'_a$  and  $C''_b \subseteq C_b \cap C'_b$  and any map  $f_{\mathfrak{p}''}: C''_a/\mathcal{E} \rightarrow C''_b/\mathcal{E}$  coming from a path must coincide (modulo canonical identifications) with  $f_{\mathfrak{p}}$  and  $f_{\mathfrak{p}'}$ .

We therefore see that  $\mathcal{E}(X)$  is an equivalence relation on  $X$ . Furthermore, it follows by construction that if  $Y \subseteq X$  are both simply connected, then  $\mathcal{E}(X)$  and  $\mathcal{E}(Y)$  coincide on  $Y$ . Also if  $C$  is a small cell, then by Proposition 5.8 (1), this definition of  $\mathcal{E}(C)$  coincides with the previous one.

**Lemma 5.10.** 1. If  $X$  is simply connected, then any  $\mathcal{E}(X)$ -class is dense in  $X$ .

2. If  $X$  and  $Y$  are simply connected, then the equivalence relations  $\mathcal{E}(X)$  and  $\mathcal{E}(Y)$  have the same number of classes.

*Proof.* 1. Let  $X$  be simply connected and let  $C_0, C_1 \subseteq X$  be small cells. Then there is a path  $\mathfrak{p}$  from some  $C'_0 \subseteq C_0$  to some  $C'_1 \subseteq C_1$ . This path induces a bijection  $f_{\mathfrak{p}}: C'_0/\mathcal{E} \rightarrow C'_1/\mathcal{E}$  which in turns induces a bijection  $C_0/\mathcal{E} \rightarrow C_1/\mathcal{E}$  via the canonical identifications induced by the inclusion maps.

2. Each of  $\mathcal{E}(X)$  and  $\mathcal{E}(Y)$  has the same number of classes as  $\mathcal{E}(C)$  for some/any small cell  $C$ .  $\square$

**Lemma 5.11.** Let  $X$  be an open-subset of  $W_*$ . Assume that we have a family  $\mathcal{F}$  of definable (over parameters) open subsets of  $X$  such that:

1. for any finite collection  $\{C_1, \dots, C_k\}$  of small cells, there is a finite set  $F \subseteq \mathcal{F}$  whose union contains all the  $C_i$ 's;
2. for any non-empty finite set  $F \subseteq \mathcal{F}$  the intersection of all the sets in  $F$  is non-empty and simply connected.

Then  $X$  is simply connected.

*Proof.* To see that  $X$  is connected, let  $a, b \in X$ . We can find two sets  $X_a, X_b \in \mathcal{F}$  that contain  $a$  and  $b$  respectively. By assumption  $X_a \cap X_b$  is non-empty; pick a point  $c$  in it. Then since both  $X_a$  and  $X_b$  are connected, there are paths from  $a$  to  $c$  and from  $c$  to  $b$ , which we can compose to obtain a path from  $a$  to  $b$ .

Let  $\mathfrak{p} = (C_i : i < n)$  and  $\mathfrak{p}' = (C'_i : i < n')$  be two paths with  $C_0 = C'_0$ ,  $C_{n-1} = C'_{n'-1}$ . Let  $F$  be the finite set promised by condition 1 for the family  $\{C_0, \dots, C_{n-1}, C'_0, \dots, C'_{n'-1}\}$ . Refining the two paths, we may assume that each  $C_i$  and  $C'_i$  lies in a unique member of the family. Let  $F_\infty$  be the intersection of all the sets in  $F$ . By hypothesis  $F_\infty$  is simply connected, so  $\mathcal{E}(F_\infty)$  is well defined. Then we see that the transition maps from  $C_i/\mathcal{E} \rightarrow C_{i+1}/\mathcal{E}$  coincide

with the identification of both domain and range with  $F_\infty/\mathcal{E}$ , and same for the primed family. Hence the two maps  $f_p$  and  $f_{p'}$  are also defined in this way and therefore coincide.  $\square$

**Lemma 5.12.** *For each  $i < n_*$ , let  $I_i \subseteq V_i$  be either an open interval of  $V_i$  or the whole of  $V_i$ . Assume that for each  $k < m_*$  such that  $V_k^*$  is circular, there is exactly one value of  $i$  for which  $V_i = V_k^*$  and  $I_i \neq V_i$ . Then  $X := W_* \cap \prod_{i < n_*} I_i$  is empty or simply connected.*

*Proof.* We first explain what this corresponds to in a standard topological framework. Let  $\tilde{V}_k^*, k < m_*$ , be 1-dimensional manifolds, which are thus homeomorphic to either  $\mathbb{R}$  or the circle  $S_1$ . Let  $\tilde{V}_i, i < n_*$  be each equal to one of the  $\tilde{V}_k^*$  and let  $U \subseteq \prod_{i < n_*} \tilde{V}_i$  be the set of tuples with distinct coordinates. Let  $\tilde{W}_*$  be a connected component of  $\tilde{U}$ . Choose open intervals  $\tilde{I}_i \subseteq \tilde{V}_i$  satisfying the same condition as in the statement of the lemma. Then the set  $\tilde{X} = \tilde{W}_* \cap \prod_{i < n_*} \tilde{I}_i$  is simply connected. In fact this space is contractible. This is not hard to see: First, we can assume that  $m_* = 1$ , since the space decomposes as a product of spaces each involving one  $\tilde{V}_k^*$  and a product of contractible spaces is contractible. Let us assume for example that  $\tilde{V}_0^*$  is circular. At least one coordinate, say  $i = 0$  is constrained inside a proper interval  $\tilde{I}_0$ . Fix any element  $\bar{a} \in \tilde{X}$ . Then we can send any other element  $\bar{a}'$  to  $\bar{a}$ , by sending  $a'_0$  to  $a_0$  via a shortest path (and moving the other coordinates with it so that no two cross). We then move only the other coordinates in the circle minus  $\{a_0\}$ , and this reduces to the linear case which is clear.

Now, we just have to translate this topological intuition into an argument in our context. The reader who is already convinced will not lose anything by skipping the rest of this proof. Assume that  $X$  is not empty. As above, we can assume that  $m_* = 1$ : all points live in the same order  $V_0^*$ , since coordinates in different  $V_k^*$  are completely independent of each other. If  $n_* = 1$ , then this follows from Proposition 5.8 (1): any finite set of bounded intervals are included in one bounded interval, so any two paths are included in one common bounded interval and thus define the same functions  $f_p$ .

Assume that  $V_0^*$  is linear, and we prove the result by induction on  $n_*$ . Without loss  $p_0(\bar{x}) \vdash x_0 < \dots < x_{n_*-1}$ . Consider the family  $\mathcal{F}$  of non-empty sets of the form  $X \cap J_0 \times \prod_{i < n_*} J_i$ , where  $J_0$  is an initial segment of  $V_0^*$  and  $J_1$  the complementary end segment. Any finite intersection of those sets is a non-empty set of the form  $X \cap L_0 \times \prod_{i < n_*} L_i$ , where  $L_0$  is an initial segment and  $L_1$  some end segment of  $V_0^*$ . In such a set, the first coordinate lives in the linear order  $L_0$  and the others in  $L_1$  which is independent from it. By induction, that set is simply connected and we conclude by Lemma 5.11.

Assume next that  $V_0^*$  is circular. Without loss,  $I_0$  is a proper interval and  $I_i = V_i$  for  $i > 0$ . We may also assume that  $p_0(\bar{x}) \vdash x_0 < x_1 < \dots < x_{n_*-1}$ . Fix some  $I_* \subset I_0$  a proper subinterval that has no endpoint in common with  $I_0$  and let  $J_*$  be the complement of  $I_*$ . Define  $F$  to be  $W_* \cap I_* \times \prod_{0 < i < n_*} J_* \subseteq X$ . By the linear case,  $F$  is simply connected.

Identify  $\{0, \dots, n_* - 1\}$  with  $\mathbb{Z}/n_*\mathbb{Z}$ . Let  $\mathcal{S}$  be the set of pairs  $(t, k) \in \mathbb{Z}/n_*\mathbb{Z}^2$  such that the sequence  $(t, t+1, \dots, t+k)$  contains 0. For  $(t, k) \in \mathcal{S}$ , let  $G_{t,k} \subseteq X$  be the set of tuples  $\bar{a} \in X$  for which  $a_t, \dots, a_{t+k}$  lie in  $I_0$  in that order and no other  $a_i$  is in  $I_0$ . Again using the linear case, any such set is simply connected. Note also that two distinct  $G_{t,k}$  are disjoint. For  $(t, k) \in \mathcal{S}$ ,  $G_{t,k} \cap F$  has the form  $\prod_{i < n_*} I_i$ , where the  $I_i$ 's are intervals, any two of which are either equal or disjoint. From the linear case, it follows that  $G_{t,k} \cap F$  is simply connected. Enumerate the elements of  $\mathcal{S}$  arbitrarily as  $s_1, \dots, s_v$ . For  $r \leq v$ , let  $F_r = F \cup \bigcup_{i < r} G_{s_i}$ . By induction using the remarks above and Lemma 5.11 with the two element family  $\{F_{r-1}, G_{s_r}\}$ , we see that each  $F_r$  is simply connected. Since  $F_v = X$ , we are done.  $\square$

## 5.1 Classification of local equivalence relations

Let  $\mathcal{E}$  be a local equivalence relation as above. Fix arbitrarily a small cell  $C_{\bar{a}}$  and define the relation  $E(\bar{t}; \bar{x}, \bar{y})$  which holds for  $\bar{x}, \bar{y} \in W_*$  and  $\bar{t} \equiv \bar{a}$  if  $\bar{x}, \bar{y}$  are in  $C_{\bar{t}}$  and are  $\mathcal{E}(C_{\bar{t}})$ -equivalent. Let  $L_{\mathcal{E}}$  be the language  $L_0 \cup \{E\}$  and our goal now is to describe the possibilities for the isomorphism type of the expansion of the  $L_0$  structure to  $L_{\mathcal{E}}$ . For a fixed choice of  $C_{\bar{a}}$  (which is irrelevant for us), we will see that the isomorphism types are classified by an action of some  $\mathbb{Z}^n$  on a finite set  $X$  obtained by monodromy.

Let  $\mathcal{C}$  be the set of indices  $k < m_*$  for which  $V_k^*$  is circular.

For each  $k \in \mathcal{C}$ , let three distinct points  $\alpha_k < \beta_k < \gamma_k \in V_k^*$  be given. Define three intervals  $C_{k,0} := \alpha_k < x < \beta_k$ ,  $C_{k,1} := \beta_k < x < \gamma_k$  and  $C_{k,2} := \gamma_k < x < \alpha_k$  of  $V_k^*$ . The indices  $0, 1, 2$  in  $C_{k,0}, \dots$  are considered as elements of the cyclic group  $\mathbb{Z}_3$ . Let also  $A = \{\alpha_k, \beta_k, \gamma_k : k \in \mathcal{C}\}$ .

Given a tuple  $\bar{t} = (t_k : k \in \mathcal{C})$  of elements of  $\mathbb{Z}_3$ , let  $C_{\bar{t}} = W_* \cap \prod_{i < n_*} C_{\bar{t},i}$ , where

$$C_{\bar{t},i} = \begin{cases} C_{k,t_k} & \text{if } V_i = V_k^* \text{ is circular and } V_i \neq V_j \text{ for } j < i, \\ V_i & \text{otherwise.} \end{cases}$$

A *big cell* of  $W_*$  is a set of the form  $C_{\bar{t}}$ , with  $\bar{t}$  as above. Note that any big cell of  $W_*$  is definable over  $A$ . We say that two big cells  $C_{\bar{t}}$  and  $C_{\bar{s}}$  are adjacent if  $\bar{t} - \bar{s}$  has exactly one non-zero coordinate. By Lemma 5.12, each big cell is simply connected. Given any two adjacent big cells  $C_{\bar{t}}$  and  $C_{\bar{s}}$  of  $W_*$ , their union is included in an open simply connected set  $D(\bar{t}, \bar{s})$  which is equal to  $C_{\bar{t}} \cup C_{\bar{s}}$ , plus possibly finitely many points having at least one of  $\alpha_k, \beta_k, \gamma_k$  as coordinate which lie in the convex closure of that union.

Let  $\mathcal{E}$  be a local equivalence relation on  $W_*$ . Then  $\mathcal{E}(C)$  is a well defined equivalence relation on each big cell  $C$  of  $W_*$ . Also  $\mathcal{E}(D(\bar{t}, \bar{s}))$  is a well defined equivalence relation on each  $D(\bar{t}, \bar{s})$ . The latter induces a bijection between  $C_{\bar{t}}/\mathcal{E}$  and  $C_{\bar{s}}/\mathcal{E}$ , which we will denote by  $f_{\bar{t}, \bar{s}}$ .

Let  $\bar{t} \in \mathbb{Z}_3^{\mathcal{C}}$  and take  $\bar{e}_0, \bar{e}_1 \in \mathbb{Z}_3^{\mathcal{C}}$  having each exactly one non-zero coordinate, with  $\bar{e}_0 \neq \pm \bar{e}_1$ . Then the 4 sets  $D(\bar{t}, \bar{t} + \bar{e}_0)$ ,  $D(\bar{t}, \bar{t} + \bar{e}_1)$ ,  $D(\bar{t} + \bar{e}_0, \bar{t} + \bar{e}_0 +$

$\bar{e}_1$ ),  $D(\bar{t} + \bar{e}_1, \bar{t} + \bar{e}_0 + \bar{e}_1)$  are included in a common simply connected set. It follows that we have the commutation relation:

$$(\square) \quad f_{\bar{t}+\bar{e}_0, \bar{t}+\bar{e}_0+\bar{e}_1} \circ f_{\bar{t}, \bar{t}+\bar{e}_0} = f_{\bar{t}+\bar{e}_1, \bar{t}+\bar{e}_0+\bar{e}_1} \circ f_{\bar{t}, \bar{t}+\bar{e}_1}.$$

Denote by  $\bar{0} \in \mathbb{Z}_3^{\mathcal{C}}$  the tuple all of whose coordinates are 0 and let  $X = C_{\bar{0}}/\mathcal{E}$ . We may identify each  $C_{\bar{t}}/\mathcal{E}$  with  $X$  by following a path of bijections between  $C_{\bar{0}}$  and  $C_{\bar{t}}$  that never *wraps around*. More formally, order  $\mathbb{Z}_3$  by identifying it with  $\{0, 1, 2\}$ . If  $C_{\bar{t}_0}, \dots, C_{\bar{t}_n}$  and  $C_{\bar{s}_0}, \dots, C_{\bar{s}_n}$  are two sequences of cells with

$$\bar{t}_0 \leq \bar{t}_1 \leq \dots \leq \bar{t}_n, \bar{s}_0 \leq \bar{s}_1 \leq \dots \leq \bar{s}_n, \text{ and } \bar{t}_0 = \bar{s}_0, \bar{t}_n = \bar{s}_n$$

and both

$$f_{\bar{t}_{n-1}, \bar{t}_n} \circ \dots \circ f_{\bar{t}_0, \bar{t}_1} \text{ and } f_{\bar{s}_{n-1}, \bar{s}_n} \circ \dots \circ f_{\bar{s}_0, \bar{s}_1}$$

well defined, then those two compositions are equal by iterations of  $(\square)$ . We identify  $C_{\bar{t}}/\mathcal{E}$  with  $X = C_{\bar{0}}/\mathcal{E}$  by following any sequence of adjacent big cells from  $C_{\bar{0}}$  to  $C_{\bar{t}}$  as above.

For any  $i \in \mathcal{C}$ , let  $\bar{e}_i \in \mathbb{Z}_3^{\mathcal{C}}$  be the element with coordinates 0 everywhere except for 2 at the  $i$ -th place. Now to describe  $\mathcal{E}$ , it is enough to describe the maps  $f_{\bar{t}, \bar{t}+\bar{e}_i}$  when the  $i$ -th coordinate of  $\bar{t}$  is equal to 0. (All other maps  $f_{\bar{t}, \bar{s}}$  are the identity on  $X$  by our identification.) In fact, we can further simplify by noticing that such an  $f_{\bar{t}, \bar{t}+\bar{e}_i}$  is equal to  $f_{\bar{0}, \bar{e}_i}$ : let  $g$  be a composition of maps  $f_{\bar{t}, \bar{s}}$ , which do not wrap around (that is change a coordinate from 2 to 0 or vice-versa), such that  $t_i = s_i = 2$  so that the  $i$ -th coordinate is not changed and  $g \circ f_{\bar{t}, \bar{t}+\bar{e}_i}$  maps  $C_{\bar{t}}/\mathcal{E}$  to  $C_{\bar{e}_i}/\mathcal{E}$ . Let  $h$  be the same composition as  $g$ , but with all  $i$ -th coordinate being equal to 0 instead of 2. Then  $h$  sends  $C_{\bar{t}}/\mathcal{E}$  to  $C_{\bar{0}}/\mathcal{E}$  and  $f_{\bar{0}, \bar{e}_i} \circ h$  also sends  $C_{\bar{t}}/\mathcal{E}$  to  $C_{\bar{e}_i}/\mathcal{E}$ . As neither  $g$  nor  $h$  wraps around,  $g$  and  $h$  induce the identity map on  $X$ . Furthermore, by successive applications of  $(\square)$ , one sees that

$$g \circ f_{\bar{t}, \bar{t}+\bar{e}_i} = f_{\bar{0}, \bar{e}_i} \circ h.$$

Hence, seen as maps from  $X$  to  $X$ , we have  $f_{\bar{t}, \bar{t}+\bar{e}_i} = f_{\bar{0}, \bar{e}_i}$ .

For each index  $i$ , set  $h_i = f_{\bar{0}, \bar{e}_i}$ , seen as a map from  $X$  to  $X$ . Using  $(\square)$  and following the standard argument that the fundamental group of a torus is  $\mathbb{Z}^2$ , one obtains that  $h_i$  and  $h_j$  commute for all  $i, j$ . (Deform the path corresponding to  $h_i \circ h_j$  to that corresponding to  $h_j \circ h_i$  by successive applications of  $(\square)$ .)

We have thus associated to the local equivalence relation  $\mathcal{E}$  a family of pairwise commuting maps  $h_i: X \rightarrow X$ , or equivalently, an action of  $\mathbb{Z}^{\mathcal{C}}$  on  $X$ . We will call this the *monodromy action* of  $\mathcal{E}$ . Given a decomposition of  $W_*$  into big cells, this action is well defined only up to conjugation by a permutation of  $X$ . Furthermore, it follows from the analysis above that another choice of big cells would lead to the same family of maps (again up to conjugation).

Assume for now that the group  $G$  defined at the beginning of the section is trivial. Then the monodromy action determines the  $L_{\mathcal{E}}$  structure up to isomorphism, as can be seen from a simple back-and-forth argument: assume that  $M$  and  $M'$  are  $L_{\mathcal{E}}$  structures with isomorphic  $L_0$ -reducts,  $G$  trivial and the same



monodromy action. Take elementary extensions  $M \prec N$  and  $M \prec N'$  and choose points  $\alpha_k, \beta_k, \gamma_k$  in  $N \setminus M$  (resp.  $\alpha'_k, \beta'_k, \gamma'_k$  in  $N' \setminus M'$ ) to define big cells. Each big cell has the same number of equivalence classes and they are all dense. We can identify the classes on big cells in  $M$  and  $M'$  so as to respect the monodromy action and then carry out a back-and-forth construction between them following this identification. The  $L_{\mathcal{E}}$ -structure can then be recovered from the big cells, the classes in each big cell and the monodromy action, so the two  $L_{\mathcal{E}}$ -structures are isomorphic.

Although this will not be needed, we give explicit construction of those structures, at least in the case where  $W_*$  is primitive (and still assuming that  $G$  is trivial). An action of  $\mathbb{Z}^{\mathcal{C}}$  on a finite set  $X$  is entirely described up to a permutation of  $X$  by the number and size of each orbit, and for each orbit, the stabilizer  $S \leq \mathbb{Z}^{\mathcal{C}}$  of any of its elements, which has the form  $S = \prod_{k \in \mathcal{C}} l_k \mathbb{Z}$  for some  $l_0, \dots, l_{\mathcal{C}-1} \in \mathbb{N}$  (elements of  $\mathcal{C}$  are integers and the coordinates of  $\mathbb{Z}^{\mathcal{C}}$  are ordered naturally).

Consider first the case  $n_c = n_* = 1$  and let some action of  $\mathbb{Z}$  on a finite set  $X$  be given. Let there be  $m$  orbits and  $k_1, \dots, k_m$  the index of their stabilizers in  $\mathbb{Z}$ . Expand  $V_1$  by the structure described in Example 5.1 (4). This has the required monodromy.

Claim 1: If  $W_*$  is primitive, then the monodromy action is transitive.

*Proof:* Say that two points  $a, b \in W_*$  are  $\bar{\mathcal{E}}$ -related if there is a path  $\mathfrak{p} = (C_i)_{i < n}$  in  $W$  with  $a \in C_0, b \in C_{n-1}$  and  $f_{\mathfrak{p}}$  sends the  $\mathcal{E}(C_0)$ -class of  $a$  to the  $\mathcal{E}(C_{n-1})$ -class of  $b$ . Then  $\bar{\mathcal{E}}$  is an  $\emptyset$ -definable equivalence relation on  $W_*$ . By primitivity, it is trivial. This implies that the monodromy action is transitive.

Assume now that the  $\mathbb{Z}^{\mathcal{C}}$  action has a unique orbit, which by the previous claim is always true if  $W_*$  is primitive. Let  $S = \prod_{k \in \mathcal{C}} l_k \mathbb{Z}$  be the stabilizer of some/any element of  $X$ . For each  $k$ , let  $i < n_*$  minimal such that  $V_i = V_k^*$  and expand  $W_i$  by a local equivalence relation coding a finite cover of degree  $l_i$  as in Example 5.1 (3). Call  $\mathcal{E}_i$  the local equivalence relation thus constructed on  $W_i$ . For all other values of  $i$ , let  $\mathcal{E}_i$  be the trivial equivalence relation. Let  $W_*$  be placed with respect to this expansion so that:

- For any small cell  $C = W_* \cap \prod_{i < n_*} I_i$  and for any choice of  $e_i \in I_i / \mathcal{E}_i$ ,  $i < n_*$ , there is  $\bar{c} \in C$  whose  $i$ -th coordinate is in the class  $e_i$ .

There is now a definable local equivalence relation  $\mathcal{E}$  on  $W_*$  which is locally the intersection of the relations  $\mathcal{E}_i$  on each coordinate and has the required monodromy.

Claim 2: The local relations  $\mathcal{E}_i$  are definable from the relation  $\mathcal{E}$ .

*Proof:* Let  $i < n_*$  be such that  $V_i = V_k^*$  is circular and  $V_i \neq V_j$  for  $j < i$ . Let  $I \subseteq V_i$  be a bounded interval. Then two points  $a, b \in I$  are  $\mathcal{E}_i(I)$ -equivalent if and only if there is in  $\pi_i^{-1}(I) \subseteq W_*$  a path  $\mathfrak{p} = (C_i)_{i < n}$  with  $\pi_i^{-1}(a) \in C_0, \pi_i^{-1}(b) \in C_{n-1}$  and the map  $f_{\mathfrak{p}}$  maps the  $\mathcal{E}(C_0)$ -class of  $\pi_i^{-1}(a)$  to the  $\mathcal{E}(C_{n-1})$ -class of  $\pi_i^{-1}(b)$ . (Since inside  $\pi_i^{-1}(I)$ , we can move freely along the finite covers of other circular orders.)

It remains to deal with the case where  $G$  is not trivial, equivalently  $W_i = W_j$  for some  $i \neq j$ , as in Example 5.1 (5). It seems more complicated to describe all the resulting structures, and it is no longer true that any monodromy action can occur, so we will only show a finiteness result. When  $G$  is trivial, having fixed a system of big cells, the type of a tuple  $\bar{c} \in W_*$  over a finite set  $A$  (over which the big cells are defined) is entirely given by order relations—which also determine the big cell in which  $\bar{c}$  lies—and the equivalence class of  $\bar{c}$  in that big cell. If  $G$  is non-trivial, we have to give in addition the equivalence classes of each  $\sigma(\bar{c})$ ,  $\sigma \in G$ . Note that those tuple might lie in the same or different big cells. Having fixed big cells and a number of equivalence classes, there are only finitely many possibilities for the tuple  $(\sigma(\bar{c})/\mathcal{E} : \sigma \in G)$ ,  $\bar{c} \in W_*$ , where  $\sigma(\bar{c})/\mathcal{E}$  denotes the  $\mathcal{E}(C)$ -class of  $\sigma(\bar{c})$  where  $C$  is the big cell to which  $\bar{c}$  belongs. Each such tuple that occurs in the structure must occur on a dense subset of some big cell. Hence as before, knowing the group  $G$  and which of those tuples occur determines the  $L_{\mathcal{E}}$ -structure up to isomorphism. In particular, having fixed the number of classes and the size of  $\bar{c}$ , there are only finitely many possibilities.

## 5.2 Local relations

Say that two small cells  $C_0, C_1$  of  $W_*$  are *strongly disjoint* if for any  $i, j < n_*$  so that  $V_i = V_j$ , the projections  $\pi_i(C_0)$  and  $\pi_j(C_1)$  to  $V_i$  and  $V_j$  are disjoint.

**Definition 5.13.** A relation  $R(x_1, \dots, x_k) \subseteq W_*^k$  is *local* if there is a local equivalence relation  $\mathcal{E}_R$  on  $W_*$  such that given strongly disjoint small cells  $C_1, \dots, C_k$  and two tuples  $(a_1, \dots, a_k), (a'_1, \dots, a'_k) \in C_1 \times \dots \times C_k$ ,

$$\bigwedge (a_i, a'_i) \in \mathcal{E}_R(C_i) \implies (R(a_1, \dots, a_k) \leftrightarrow R(a'_1, \dots, a'_k)).$$

**Proposition 5.14.** Let  $R(x_1, \dots, x_k)$  be a local relation. Let  $\bar{a} = (a_1, \dots, a_k), \bar{b} = (b_1, \dots, b_k) \in W_*^k$  be two tuples of pairwise distinct elements. Assume that  $\bar{a}$  and  $\bar{b}$  have the same  $L_0$ -type and that for each  $i \leq k$ , there is a big cell  $C$  of  $W_*$  containing both  $a_i$  and  $b_i$  with  $(a_i, b_i) \in \mathcal{E}_R(C)$ . Then we have

$$R(a_1, \dots, a_k) \leftrightarrow R(b_1, \dots, b_k).$$

*Proof.* (Sketch) For any two  $k$ -tuples  $\bar{c}$  and  $\bar{d}$  of elements of  $W_*$ , write  $\bar{c} \rightarrow \bar{d}$  if for each  $i \leq k$ , there is a big cell  $C_i$  of  $W_*$  and a small cell  $C'_i \subseteq C_i$  that contains  $c_i$  and  $d_i$  and such that  $(c_i, d_i) \in \mathcal{E}_R(C'_i)$  and the  $C'_i$ 's are strongly disjoint. To prove the proposition, it is sufficient to find a sequence  $\bar{a} = \bar{a}^0 \rightarrow \bar{a}^1 \rightarrow \dots \rightarrow \bar{a}^m = \bar{b}$ . The fact that the  $L_0$ -types of  $\bar{a}$  and  $\bar{b}$  are the same implies that the relative order of the elements in the tuple are the same. Thus we can always find such a path from  $\bar{a}$  to  $\bar{b}$  by moving the points one by one.  $\square$

It follows that a local relation  $R$  is definable over the parameters  $A$  used to define the big cells along with parameters defining the equivalence relations  $\mathcal{E}_R$  on each big cell and a name for each  $\mathcal{E}_R$ -equivalence class inside each big cell.

## 6 Classification of rank 1 structures

### 6.1 Prolongation of orders

The following proposition will be fundamental for us: it shows that certain binary functions are actually unary.

**Proposition 6.1.** *Assume that  $M$  has finite rank and is NIP. Let  $a, b$  be two finite tuples and set  $p(x, y) = \text{tp}(a, b)$ . Assume that either  $a \perp b$  or  $\text{rk}(a) = 1$ . Let also  $V$  be a linear or circular order of topological rank 1 and let  $f: p(M) \rightarrow \overline{V}$  be a  $\mathcal{O}$ -definable function. Then  $f(a, b) \in \text{acl}(a) \cup \text{acl}(b)$ .*

*Proof.* If  $a \in \text{acl}(b)$ , there is nothing to show. If  $\text{rk}(a) = 1$  and  $a \notin \text{acl}(b)$ , then  $a \perp b$ . Hence we can assume  $a \perp b$ . Let  $a_1, \dots, a_n \in M$  be rank-independent realizations of  $p(x, b)$ . For  $i \leq n$ , set  $c_i = f(a_i, b)$ . If  $c_i$  is algebraic either over  $b$  or over  $a_i$ , we are done. Otherwise, by independence, the  $c_i$ 's are pairwise distinct. Set  $\bar{a} = (a_1, \dots, a_n)$  and we claim that  $c_i \notin \text{acl}(\bar{a})$ . We have  $\text{rk}(b, a_i, c_i) = \text{rk}(b, a_i)$  and  $\text{rk}(a_i, c_i) > \text{rk}(a_i)$ . It follows from Proposition 2.2 (6) that  $m := \text{rk}(b/a_i) > \text{rk}(b/a_i c_i)$ . Now  $\text{rk}(b/\bar{a}) = m$  by independence, hence  $\text{rk}(b/\bar{a} c_i) \leq \text{rk}(b/a_i c_i) < m = \text{rk}(b/\bar{a})$ . Hence  $c_i \notin \text{acl}(\bar{a})$ .

Let  $Z_i$  be the locus of  $c_i$  over  $\bar{a}$ . The closures of the  $Z_i$ 's in  $V$  are convex sets, which are pairwise either equal or disjoint. If they are equal, then by Proposition 3.11 (or 4.5 in the circular case), for any subset  $I \subseteq \omega$ , we can find  $b \equiv_{\bar{a}} b'$  such that  $f(a_i, b') < c_i \iff i \in I$ . If they are disjoint, then the same holds using Proposition 3.21. In either case, we contradict NIP.  $\square$

**Corollary 6.2.** *Assume that  $M$  has rank 1 and is NIP. Let  $(V, \leq)$  be a minimal  $\mathcal{O}$ -definable linear order. Let  $\overline{V}(a)$  denote  $\text{acl}(a) \cap \overline{V} = \text{dcl}(a) \cap \overline{V}$ . Then:*

1. *for any  $a_0, \dots, a_{n-1} \in M$ , we have  $\overline{V}(a_0, \dots, a_{n-1}) = \bigcup_{i < n} \overline{V}(a_i)$ ;*
2.  *$\overline{V}$  is definable, minimal and has rank 1.*

*Proof.* (1) Let  $c \in \overline{V}(a_0, \dots, a_{n-1})$  and set  $p = \text{tp}(a_0, a_1 \hat{\ } \dots \hat{\ } a_{n-1})$ . Then for some  $\mathcal{O}$ -definable function  $f$  defined on realizations of  $p$ ,  $f(a_0, a_1 \hat{\ } \dots \hat{\ } a_{n-1}) = c$ . By Proposition 6.1,  $c \in \text{dcl}(a_0) \cup \text{dcl}(a_1 \hat{\ } \dots \hat{\ } a_{n-1})$ . We conclude by induction on  $n$ .

(2) Let  $a$  be a singleton, then  $\overline{V}(a)$  is finite, by Lemma 3.7. Hence the sets  $\overline{V}(a)$  for  $a$  a singleton live in finitely many sorts. If  $M$  has rank 1, then each of these sorts has rank 1, since an element in it is in the definable closure of a singleton. By (1), those finitely many sorts are enough to encode all of  $\overline{V}$ .  $\square$

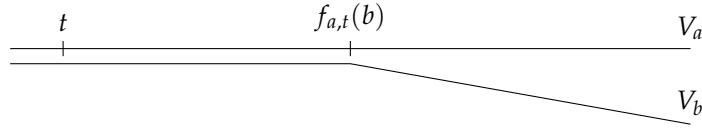
In what follows, we will consider a definable family  $(V_a, \leq_a)$ ,  $a \in D$  of linear orders, by which we mean that  $D$  is a definable set and there are formulas  $\phi(x; t)$  and  $\psi(x, y; t)$  such that for any  $a \in D$ , the formula  $\psi(x, y; a)$  defines a linear order denoted  $\leq_a$  on  $V_a := \phi(M; a)$ .

**Proposition 6.3.** Let  $D$  be a 0-definable set and let  $(V_a, \leq_a)$ ,  $a \in D$ , be a definable family of linearly ordered sets, with  $V_a$  minimal over  $a$ . Assume that  $D$  is ranked and let  $a, b \in D$ . Let  $I \subseteq V_a$  be a non-empty convex subset which admits a parameter-definable increasing map  $h: I \rightarrow \overline{V_b}$ . Assume that  $I$  is maximal such. Set  $J$  be the intersection of the convex hull of  $h(I)$  with  $V_b$ . Then one of the following holds:

1.  $I = V_a$ ;
2.  $J = V_b$ ;
3.  $I$  is a proper initial segment of  $V_a$  and  $J$  is a proper end segment of  $V_b$ ;
4.  $I$  is a proper end segment of  $V_a$  and  $J$  is a proper initial segment of  $V_b$ .

*Proof.* Assume that  $I, h, J$  are as in the statement and that neither  $I$  nor  $J$  is cofinal in  $V_a$  or  $V_b$  respectively. Fix some  $t \in I$  and  $u \in J$ .

For  $b' \in D$  define  $f_{a,t}(b')$  as the maximal element  $s \in \overline{V_a}$ ,  $s > t$ , such that the interval  $t < x < s$  is intertwined with a convex subset of  $V_{b'}$ , if such an element exists, and  $f_{a,t}(b')$  is undefined otherwise. Thus  $f_{a,t}(b) = \sup(I)$ . Similarly, we have  $f_{b,u}(a) = \sup(J)$ . Define also the equivalence relation  $E_{a,t}(x, y)$  by  $f_{a,t}(x) = f_{a,t}(y)$ .



**Figure 1:** Definition of  $f_{a,t}(b)$ . Intertwined intervals are represented as parallel lines.

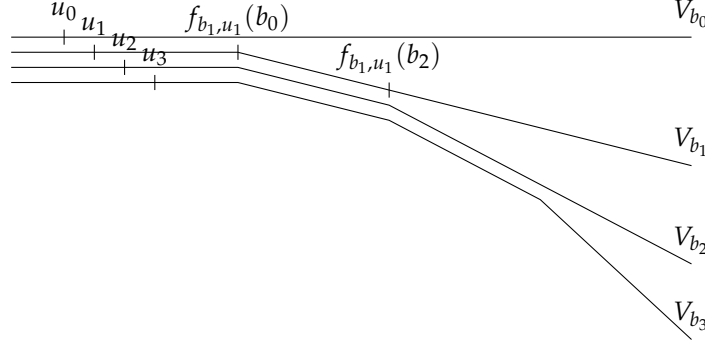
By weak transitivity, the image of  $f_{b,u}$  is dense in the end segment  $x > u$  of  $V_b$ . Let  $a_0 = a, a_1, a_2, \dots$  be such that  $f_{b,u}(a_0) < f_{b,u}(a_1) < f_{b,u}(a_2) < \dots$ . Observe that if  $f_{b,u}(a') > f_{b,u}(a)$ , then  $f_{a,t}(a') = f_{a,t}(b)$ . This shows that the  $E_{a,t}$ -class of  $b$  is infinite and contains  $a_1, a_2, \dots$ . Furthermore, the  $E_{b,u}$ -class of each  $a_i$ ,  $i > 0$  is included in the  $E_{a,t}$ -class of  $b$ . Hence the  $E_{a,t}$ -class of  $b$  is cut into infinitely many infinite  $E_{b,u}$ -classes.

Now define inductively  $(b_i : i < \omega)$  in  $D$  and points  $(u_i : i < \omega)$ ,  $u_i \in V_{b_i}$ , by:

- $(b_0, u_0) = (a, t)$ ,  $(b_1, u_1) = (b, u)$ ;
- given  $(b_k, u_k)$ , let  $b_{k+1}$  be such that  $f_{b_k, u_k}(b_{k+1}) > f_{b_k, u_k}(b_{k-1})$ ;
- set  $u_{k+1} \in V_{b_{k+1}}$  to be such that some neighborhood of  $u_{k+1}$  in  $V_{b_{k+1}}$  is intertwined with a neighborhood of  $u_k$  in  $V_{b_k}$ .

Finally define  $E_k = E_{b_k, u_k}$ . Then as above we have that for  $k < l < l'$ ,  $b_l$  and  $b_{l'}$  are  $E_k$ -equivalent and the  $E_k$ -class of  $b_l$  is split into infinitely many  $E_{l'}$ -classes. At each stage, we have infinitely many choices for the  $E_k$ -class of  $b_{k+1}$  and hence can choose one which is not algebraic over  $(b_k, u_k)$ . This contradicts  $D$

being ranked.



**Figure 2:** Construction of  $(b_k, u_k)$ , showing  $f_{b_1, u_1}(b_2) > f_{b_1, u_1}(b_0)$ .

Coming back to the initial  $I$  and  $J$ , this shows that either  $I$  is an end segment of  $V_a$  or  $J$  is an end segment of  $V_b$ . Similarly, either  $I$  is an initial segment of  $V_a$  or  $J$  is an initial segment of  $V_b$ . Thus the only possibilities are those in the statement of the proposition.  $\square$

## 6.2 Gluing definable orders

Let  $(V_a, \leq_a)$ ,  $a \in D$ , be a 0-definable family of linearly ordered sets, with  $V_a$  minimal over  $a$ . Assume that  $D$  is ranked. Our goal in this section is to glue the orders  $V_a$  together as much as possible along definable intertwinings between subintervals so as to construct a 0-definable family of pairwise independent orders.

More precisely, we will prove the following.

**Theorem 6.4.** *Let  $(V_a, \leq_a)_{a \in D}$  be a 0-definable family of linearly ordered sets, with  $V_a$  minimal over  $a$  and such that  $D$  is ranked. Then there is a 0-interpretable set  $F$  and a 0-interpretable family  $(W_e, \leq_e)_{e \in F}$  of linear or circular orders such that:*

- for  $e \in F$ ,  $W_e$  is minimal over  $e$ ;
- for any  $e \neq e' \in F$ ,  $W_e$  and  $W_{e'}$  are either independent or in definable order-reversing bijection;
- for any  $a \in D$ , there is (a necessarily unique)  $e \in F$  such that  $V_a$  admits a definable order-preserving map into  $W_e$ .

In order to carry out the construction, we begin by increasing  $D$  and assume:

- ( $\triangle$ ) the family  $(V_a)_{a \in D}$  is closed under restricting to an open sub-interval and reversing the order.

The first stage of the construction is to *thicken* the  $V_a$ 's by replacing each by a large enough definable subset of  $\overline{V_a}$ , which will be called  $W_a$ .

Define an equivalence relation  $\sim$  on pairs  $(a, t)$ ,  $a \in D, t \in V_a$  by  $(a, t) \sim (b, u)$  if some neighborhood of  $t$  in  $\overline{V_a}$  is in definable increasing bijection with a neighborhood of  $u$  in  $\overline{V_b}$ , and that bijection sends  $t$  to  $u$ . Note that this is equivalent to saying that some neighborhood of  $t$  in  $V_a$  is intertwined with a neighborhood of  $u$  in  $V_b$  and the (unique) intertwining map sends  $t$  to  $u$ . By Lemma 3.17, for  $t, u \in V_a$  distinct, we have  $(a, t) \approx (a, u)$ .

Let  $[a, t]_\sim$  denote the  $\sim$ -class of  $(a, t)$  and let  $W$  be the set of  $\sim$ -classes. For  $a \in D$ , let  $W_a$  be

$\{[a', t]_\sim : \text{a neighborhood of } t \text{ in } V_{a'} \text{ admits a parameter-definable intertwining map into } V_a\}$ .

So  $W_a$  is naturally in increasing bijection with a dense subset of  $\overline{V_a}$  and will be identified with it. In particular, it inherits the definable order  $\leq_a$ , and is minimal over  $a$ . Furthermore, if an open interval  $I \subseteq W_a$  admits an intertwining map into  $W_b$ , then  $I$  is a subset of  $W_b$ .

Let  $a, b \in D$ . If  $W_a \cap W_b$  is non-empty and say  $e \in W_a \cap W_b$ , then by construction there is a neighborhood of  $e$  in  $W_a$  that coincides with a neighborhood of  $e$  in  $W_b$ . It then follows from Proposition 6.3 and the remarks above that one of the following occurs:

1.  $W_a \cap W_b = \emptyset$ ;
2.  $W_a \subseteq W_b$ , or  $W_b \subseteq W_a$ ;
3. an end segment of  $W_a$  is equal to an initial segment of  $W_b$ : we write  $W_a \trianglelefteq W_b$ ;
4. an initial segment of  $W_a$  is equal to an end segment of  $W_b$ :  $W_b \trianglelefteq W_a$ .

Note that (3) and (4) could both be true, even if  $W_a \neq W_b$ : this happens for example if we have a definable circular order  $V$  and each  $V_a$  is obtained by removing the point  $a$  from  $V$ .

We now glue the  $W_a$ 's together.

Say that  $t \in W$  is a left end-point of  $W_a$  if there is  $b \in D$  such that  $t \in W_b$  and  $W_a \cap W_b$  is an end segment of  $W_b$  of the form  $(t, +\infty)$ . Note that if  $W_b \subset W_a$  is an interval of the form  $(t, s)$  in  $W_a$ , then  $t$  is a left end-point of  $W_b$  (using  $(\Delta)$ ).

**Claim 1:** A set  $W_a$  has at most one left end-point.

*Proof:* Assume that  $t, t' \in W$  are both left end-points of  $W_a$  as witnessed by  $W_b$  and  $W_{b'}$  respectively. Then some interval of the form  $(t, u)$  in  $W_b$  is equal to an interval of the form  $(t', u')$  in  $W_{b'}$ . But then by the discussion above (or Proposition 6.3), we must have  $t = t'$ .

We define right end-points similarly.

In what follows, a *path* from  $s$  to  $t$  is a triple  $\mathbf{p} = (s, t, (p_0, \dots, p_{n-1}))$ , where  $s, t \in W$  and  $(p_0, \dots, p_{n-1})$  is a finite tuple of elements of  $D$  such that, setting  $W_{\mathbf{p}, i} = W_{p_i}$ :

- $s$  is the left end-point of  $W_{p,0}$ ;
- $t$  is the right end-point of  $W_{p,n-1}$ ;
- $W_{p,i} \leq W_{p,i+1}$  for all  $i < n - 1$ .

If  $p = (s, t, \bar{p})$  and  $p' = (s', t', \bar{p}')$  are paths with  $t = s'$ , then we can form (non-uniquely) a concatenation  $p'' = (s, t', \bar{p} \cap p_* \cap \bar{p}')$ , where  $p_*$  is chosen so that  $W_{p_*}$  is a small enough open interval around  $t = s'$  in any  $W_\bullet$  containing it. This exists by  $(\Delta)$ .

A path  $p = (s, t, (p_0, \dots, p_{n-1}))$  is *simple* if:

- For each  $i \neq j < n$ , we have  $W_{p,i} \not\subseteq W_{p,j}$ ;
- For each  $i < j < n$ , we have  $W_{p,j} \not\subseteq W_{p,i}$ .

Note that if  $p$  is a simple path and  $i < j < n$ , then  $W_{p,i} \cap W_{p,j}$  is either an end segment of  $W_{p,i}$  and an initial segment of  $W_{p,j}$ , or empty.

If  $p$  is a simple path, we define

$$W_p = \bigcup_{i < n} W_{p,i}.$$

This set is equipped with a linear order  $\leq_p$  defined as follows: for  $t, u \in W_p$ , we have  $t \leq_p u$  if one of the following occurs:

1. for some  $i < n$ ,  $t, u \in W_{p,i}$  and we have  $t \leq u$  in  $W_{p,i}$ ;
2. for some  $i < j < n$ ,  $t \in W_{p,i} \setminus W_{p,j}$  and  $u \in W_{p,j}$ ;
3. for some  $i < j < n$ ,  $t \in W_{p,i}$  and  $u \in W_{p,j} \setminus W_{p,i}$ .

The simplicity assumption implies that this does define a linear order on  $W_p$ . Note that if  $p$  and  $p'$  are simple paths, then the orders  $\leq_p$  and  $\leq_{p'}$  must coincide on the intersection  $W_p \cap W_{p'}$ , since they locally agree with the orders on the  $W_a$ 's.

Say that two simple paths  $p$  and  $p'$  are equivalent if  $W_p = W_{p'}$ .

**Claim 2:** If  $p$  and  $p'$  are two simple paths with initial point  $s$ , then one of  $W_p$  and  $W_{p'}$  is an initial segment of the other.

*Proof:* Since  $W_p$  and  $W_{p'}$  have the same left end-point, they have an initial segment in common. Take a maximal  $W_0 \subseteq W$  which is an initial segment of both  $W_p$  and  $W_{p'}$ . If it is a proper initial segment of both, we can find some indices  $i, j$  such that  $W_0 \cap W_{p,i}$  is a proper initial segment of  $W_{p,i}$  and  $W_0 \cap W_{p',j}$  is a proper initial segment of  $W_{p',j}$ . Then  $W_{p,i}$  and  $W_{p',j}$  contradict Proposition 6.3.

Say that  $s, t \in W$  are connected if there is a path from  $s$  to  $t$ , or from  $t$  to  $s$ . The set of elements connected to  $s$  will be denoted by  $W(s)$ . Being connected is an equivalence relation, which we denote by  $E$ .

Say that an element  $s \in W$  is of *circular type* if there is a simple path from  $s$  to  $s$ . Otherwise, say that  $s$  is of *linear type*.

We leave the proofs of the following statements to the reader; they are routine using the previous results, but cumbersome to write down in details:

- If  $s \in W$  is of circular type as witnessed by  $W_p$ , then  $W(s) = W_p \cup \{s\}$  and every element in  $W(s)$  is of circular type. There is a definable circular order on  $W(s)$  defined by  $C(u, v, w)$  if there is a simple path  $p$  from  $u$  to  $w$  with  $v \in W_p$ .
- If  $s \in W$  is of linear type, then for any two  $W_p, W_{p'} \subseteq W(s)$ , the orders  $\leq_p$  and  $\leq_{p'}$  coincide on  $W_p \cap W_{p'}$ . There is a definable linear order on  $W(s)$  obtained by taking the union of those orders. Equivalently, for  $u, v \in W(s)$ , we have  $u \leq v$  if there is a path from  $u$  to  $v$ .

To summarize the situation: we have on  $W$  a  $\emptyset$ -definable equivalence relation  $E$  each class of which is equipped with either a linear order or a circular order, definable over a code for the class<sup>2</sup>. By  $(\Delta)$ , for every  $E$ -class  $V$ , there is an  $E$ -class  $V'$  which admits a (necessarily unique) definable order-reversing bijection with  $V$ . If  $V, V'$  are distinct  $E$ -classes which are not in order-reversing bijection, then they are independent.

**Claim 3:** Each  $E$ -class  $V$  is minimal over its code  $e \in W/E$ .

*Proof:* Assume first that  $V$  is circular. Then it is covered by finitely many sets of the form  $W_a$ . Since each  $W_a$  has topological rank 1, so does  $V$ . Furthermore, if there is an  $e$ -definable element of  $\overline{V}$ , then for some  $a$  with  $W_a \subseteq V$ , that element is in  $\overline{W}_a$ . Since  $e \in \text{dcl}(a)$ , this contradicts weak transitivity of  $W_a$ . The fact that circular classes have no self-intertwining follows at once from the construction.

If  $V$  is linear, then by  $\omega$ -categoricity, there is an integer  $n$  such that any bounded interval of  $V$  lies in the union of some  $n$  many sets of the form  $W_a$ . Hence if there was some definable convex equivalence relation with infinitely many infinite classes, this would already be true for some  $W_a$ . Similarly, an  $e$ -definable element in  $\overline{V}$  leads to an  $a$ -definable element in some  $W_a$ .

Setting  $F = W/E$ , this finishes the proof of Theorem 6.4.

### 6.3 Analysis of a rank 1 structure

In this section we assume:

- ( $\star$ )  $M$  is an  $\omega$ -categorical, rank 1, primitive, unstable NIP structure.

By rank 1 and primitivity, every singleton is acl-closed.

Assume that  $\text{opD}(M) \geq n$ , then Fact 2.12 provides us with a finite tuple of parameters  $d$  and a  $d$ -definable subset  $X_d$  transitive over  $d$ ,  $d$ -definable equivalence relations  $E_{d,1}, \dots, E_{d,n}$  on  $X_d$  with infinitely many classes and  $d$ -definable

<sup>2</sup>By a code for a class  $V$ , we mean the imaginary element corresponding to  $V$  in the definable quotient  $W/E$ .



linear quasi-orders  $\leq_{d,1}, \dots, \leq_{d,n}$  on  $X_d$  such that each  $\leq_{d,i}$  induces a linear order on the quotient  $X_d/E_{d,i}$ . Since  $M$  has rank 1, all  $E_{d,i}$ -classes are finite and the quotients  $(X_d/E_{d,i}, \leq_{d,i})$  are minimal. Let  $E_d(x, y)$  be the equivalence relation on  $X_d$  defined by  $\text{acl}(dx) = \text{acl}(dy)$ . Since by minimality  $\text{acl}$  is trivial on  $X_d/E_{d,i}$ , we see that  $E_{d,i} = E_d$  for each  $i$ . Define  $Y_d := X_d/E_d$  and  $\pi_d: X_d \rightarrow Y_d$  the canonical projection. Then  $Y_d$  is equipped with  $n$  minimal linear orders. Theorem 4.10 describes the possibilities. If some pairs of orders are intertwined, we restrict to a subset of  $Y_d$  (definable over additional parameters from  $Y_d$ ) where they are independent. We can therefore assume that the  $n$  orders are independent.

Having obtained this, we want to glue those orders together. To place ourselves in the context of the previous section, we forget for a moment that we have found a set  $Y_d$  with  $n$  independent orders on it and think of it as  $n$  different interpretable orders  $(V_{d,i}, \leq_{d,i})$ , where  $V_{d,i} = Y_d$  for each  $i$ . This forms a uniformly definable family  $V_a$  of linear orders, where  $a$  ranges on the set  $D$  of pairs  $(d, i)$  for which  $\leq_{d,i}$  is defined. We can then apply the results of the previous section to this family  $(V_a)_{a \in D}$ , providing us with an interpretable set  $W$  equipped with an equivalence relation  $E$ , so that the family  $(W_e)_{e \in W/E}$  is as in Theorem 6.4.

We now make use of the rank 1 hypothesis to obtain additional properties.

**Claim 4:** Let  $e \in W/E$ . Take  $a = (d, i)$  in  $D$  and  $t \in X_d$  so that  $[a, \pi_d(t)]_\sim$  is in the  $E$ -class coded by  $e$ . Then  $[a, \pi_d(t)]_\sim$  is algebraic over  $(e, t)$ .

*Proof:* Working over  $e$ , define the  $f_a: X_d \rightarrow V$  by  $f_a: t \mapsto [a, \pi_d(t)]_\sim$ . By Proposition 6.1,  $f_a(t) \in \text{acl}(e, t)$  (as  $t$  is not algebraic over  $a$ ).

**Claim 5:** Let  $a = (d, i)$  in  $D$  and  $t \in X_d$ , then  $t$  is algebraic over  $[a, \pi_d(t)]_\sim$ .

*Proof:* By the previous claim,  $[a, \pi_d(t)]_\sim \in \text{acl}(e, t) \setminus \text{acl}(e)$ . As  $\text{rk}(M) = 1$ , it follows that  $t \in \text{acl}(e, [a, \pi_d(t)]_\sim) \subseteq \text{acl}([a, \pi_d(t)]_\sim)$ .

**Claim 6:** There are finitely many  $E$ -classes.

*Proof:* Assume that there are infinitely many  $E$ -classes. For  $e \in W/E$ , consider the set  $M(e) := \{t \in M : \text{for some } a \in D, [a, t]_\sim \text{ lies in the class coded by } e\}$ . As  $M$  has rank 1, there is an infinite subset  $X \subseteq W/E$  such that the intersection  $\bigcap_{e \in X} M(e)$  is infinite. Fix some finite subset  $X_0 \subseteq X$  of pairwise independent classes and let  $X_1 \supseteq X_0$  be a finite set containing at least one point in each class coded in  $X_0$ . Hence all those classes are linearly ordered over  $X_1$ . Let  $Z_0 \subseteq \bigcap_{e \in X_0} M(e)$  be an infinite  $X_1$ -definable set, transitive over  $X_1$ . Let  $Z_1$  be the quotient of  $Z_0$  by the relation of inter-algebraicity over  $X_1$ . By the previous claim, for each  $e \in X_0$ ,  $Z_1$  admits an  $X_1$ -definable injection in the class coded by  $e$ : send each  $a$  to the smallest element algebraic over  $(a, e)$ . This induces a linear order on  $Z_1$ . Those orders are pairwise independent and uniformly definable. By Theorem 4.10 and NIP, their number is bounded by some integer  $N$ . This is a contradiction since  $X_0$  can be chosen as large as we want.

**Claim 7:** There is a 0-definable map  $\pi: W \rightarrow M$  with finite fibers which maps each  $E$ -class surjectively on  $M$ .

*Proof:* It follows from the claims above, that if  $t \in X_d$  and  $a = (d, i)$ , then any  $[a, \pi_d(t)]_\sim$  is inter-algebraic with  $t$ . Since in  $M$  singletons are algebraically closed, we deduce that  $(a, \pi_d(t)) \sim (b, \pi_d(u))$  implies  $t = u$ . This defines a map  $\pi: W \rightarrow M$  sending  $(a, \pi_d(t))$  to  $t$ . As  $M$  is primitive, each  $E$ -class maps surjectively onto  $M$ . Furthermore any  $x \in W$  is algebraic over  $\pi(x)$ , hence the map  $\pi$  has finite fibers.

Note that the proof of the claim also gives that  $\pi_d$  is injective, hence is the identity. Therefore  $Y_d = X_d$  for all  $d$  and the orders  $V_a, a \in D$  have as universe definable subsets of  $M$ .

Given a point  $a \in M$ , define  $W(a) = \pi^{-1}(a)$ . Note that we also have  $W(a) = \text{acl}^{eq}(a) \cap W$  as  $\text{acl}^{eq}(a) \cap M = M$ . For any  $V \subseteq W$ , define also  $V(a) = \pi^{-1}(a) \cap V = \text{acl}^{eq}(a) \cap V$ .

**Lemma 6.5.** *There are three points  $a, b, c \in M$  such that:*

- *there is a set  $W_{or} \subseteq W$ , definable over  $abc$ , which is a union of  $E$ -classes and contains exactly one class in each pair of classes in order-reversing bijection;*
- *$\text{dcl}^{eq}(abc)$  intersects each  $E$ -class in at least 3 points;*
- *for every  $V_t, t \in D$ , there is an  $\text{acl}^{eq}(a)$ -definable linear or circular order on  $M$  that extends  $\leq_t$  on  $V_t$ .*

*Proof.* Choose three points  $a, b, c \in M$  so that for every class  $V$ , we have either  $V(a) < V(b) < V(c)$  or  $V(c) < V(b) < V(a)$  (meaning that those inequalities holds for any choice of one element in each tuple). This is possible by Theorem 4.10. If  $V$  and  $V'$  are two classes with an order-reversing definable bijection, then for exactly one of  $V$  or  $V'$  do we have  $V(a) < V(b) < V(c)$ . Take  $W_{or} \subseteq W$  to be the union of classes  $V$  for which  $V(a) < V(b) < V(c)$ .

Let  $V$  be a circular class in  $W$  of code  $e \in W/E$ . Then  $V$  admits a linear order definable over  $V(a)$  so that either  $V(a) < V(b) < V(c)$  or  $V(c) < V(b) < V(a)$  holds in the linear order, for example by placing the appropriate element of  $V(a)$  as either first or last element. Then for any  $d \in M$ , every element of  $W(d)$  is definable over  $W(a)d$ . Let  $t \in D$ . Then there is a unique  $E$ -class  $V_t$  and unique definable order-preserving injection  $g_t$  of  $V_t$  into  $V$ . Then  $g_t$  is a section of  $\pi$  and we can extend that section to a section  $f$  of  $\pi$  defined over  $a$ . We can then pullback the circular or linear order from the class  $V$  to  $M$  using  $f$ .  $\square$

**Proposition 6.6.** *The structure  $M$  has finite op-dimension, bounded by the number of 4-types of elements of  $M$ .*

*Proof.* Assume that  $\text{opD}(M) \geq n$ . Then by the discussion at the beginning of this section, we can choose the family  $(V_a, \leq_a: a \in D)$  so that for each  $a \in D$ , there are  $a_1, \dots, a_n \in D$  with  $V_{a_i} = V_a$  and the orders  $\leq_{a_i}$  are pairwise independent. Pick some  $a \in D$  and  $a_i$ 's as above. Let  $a_* \in M$  be any point. Then by Lemma 6.5 (3) and transitivity of  $M$ , each order  $\leq_{a_i}$  extends to an  $\text{acl}^{eq}(a_*)$ -definable circular order on  $M$ , say  $C_i$ . The  $C_i$ 's are pairwise independent.

Let  $D_1, \dots, D_m$  be the distinct separation relations on  $M$  coming from the  $C_i$ 's and all their conjugates over  $a_*$ . Then  $m \geq n$ . Fix some  $k \leq m$ . Let  $X$  be a complete type over  $a_*$ , then the  $D_i$ 's induce  $m$  many pairwise distinct betweenness relations on  $X$ . By Corollary 3.22, we can find  $b_k, c_k, d_k \in M$  such that  $D_i(a_*, b_k, c_k, d_k)$  holds for exactly  $k$  values of  $i$ . Then the tuples  $(b_k, c_k, d_k)$ ,  $k \leq n$ , all have different types over  $a_*$ . Hence  $M$  has at least  $m + 1$  4-types.  $\square$

## 6.4 The skeletal structure

Let  $n = \text{opD}(M)$ . From now on, we fix a family  $(V_a, \leq_a : a \in D)$  satisfying that for each  $a \in D$ , there are  $a_1, \dots, a_n \in D$  with  $V_{a_i} = V_a$  and the orders  $\leq_{a_i}$  are pairwise independent. From this family, we construct  $W$  and  $E$  as in previous sections.

Consider the reduct of  $W$  to:

- the equivalence relation  $E$  and the structure induced on the quotient  $W/E$ ;
- the linear and circular orders on each  $E$ -class along with existing definable order-reversing bijections between them;
- an equivalence relation  $E_\pi$  whose classes are the fibers of  $\pi$  along with the structure on each such fiber.

We will call this structure the skeletal structure on  $W$ .

Note that each  $E_\pi$ -class is a relatively acl-closed subset of  $W$  (since every singleton in  $M$  is algebraically closed), and all of its elements are inter-algebraic. Any two  $E_\pi$ -class intersect any given  $E$ -class in the same number of elements by primitivity of  $M$ .

Let  $V$  be a linear  $E$ -class and assume that each  $E_\pi$  intersects  $V$  in  $n$  elements. For each  $k < n$ , we define  $V_k \subseteq V$  as the set of elements  $a \in V$  such that there are exactly  $k$  elements below  $a$  and inter-definable with it. Then each  $V_k$  is dense in  $V$ , definable over  $\text{acl}^{eq}(\emptyset)$ , and is in definable bijection with  $M$ . The  $V_k$ 's are thus complete types over  $\text{acl}^{eq}(\emptyset)$ .

If however  $V$  is circular, then it can be that an  $E_\pi$ -class intersects a strong type of  $V$  in more than one element.

A back-and-forth argument shows that this skeletal structure is completely described up to isomorphism by:

- the number of  $E$ -classes, the type (linear/circular) of each and the pairing of them in pairs with an order-reversing bijection between them;
- for every class  $V$ , the number of points that an  $E_\pi$  class has in  $V$ ;
- the structure on the finite quotient  $W/E$ ;
- the structure on some/any  $E_\pi$ -class.

We comment on the last point. Let  $a \in M$  and consider the  $E_\pi$ -class  $A := \pi^{-1}(a)$ . This is a finite set definable over  $a$ . It admits an  $a$ -definable canonical surjection to  $W/E$  and inherits whatever  $\mathcal{O}$ -definable structure there is on that finite quotient. The sets of elements of  $A$  lying the same class inherit the linear or circular order from that class. If all classes are linear, then  $A$  is rigid over its image in  $W/E$ , so there is no additional structure. However if there are circular classes, there may be additional structure on  $A$ .

**Lemma 6.7.** *For a given number  $n$ , there are, up to isomorphism, finitely many possible skeletal structures  $W$  associated to structures  $M$  with at most  $n$  4-types.*

*Proof.* Let  $F \subseteq W/E$  be a set containing exactly one point in every pair of classes in definable order-reversing bijection. Fix  $a \in M$  and let  $\bar{a}$  enumerate the elements in  $W(a)$  that lie in the preimage of  $F$  and write  $\bar{a} = (a_1, \dots, a_m)$ . By Proposition 3.21, given small enough intervals  $I_1, \dots, I_m$  around each  $a_i$ , the locus of  $\text{tp}(\bar{a})$  is dense in  $\prod_{i \leq m} I_i$ . This shows that  $\text{opD}(\text{tp}(\bar{a})) \geq m$  and hence  $\text{opD}(M) \geq m$  as  $\bar{a}$  is in the algebraic closure of an element of  $M$ . By Proposition 6.6,  $m$  is less than the number of 4-types. Both  $|W/E|$  and the size of an  $E_\pi$ -class being bounded, there are only finitely many possibilities for the skeletal type of  $W$ .  $\square$

## 6.5 The additional local structure

We now show that the structure on  $W$ , in addition to the skeletal structure, comes from local equivalence relations.

Let  $F \subseteq W/E$  be a set containing exactly one point in every pair of classes in definable order-reversing bijection. Hence any two different classes of  $F$  are independent. From now on, we work over  $F$ . Take some  $a \in M$  and let  $a_*$  enumerate the intersection of  $\text{acl}^{eq}(a)$  with the classes in  $F$ .

**Claim 8:** The size  $N$  of the tuple  $a_*$  is equal to the op-dimension of  $M$ .

*Proof:* By the proof of Lemma 6.7, the size of  $a_*$  is at most the op-dimension of  $M$ . By the choice of  $D$  at the beginning of Section 6.4, it is at least the op-dimension of  $M$ .

Let  $W_*$  be the locus of  $\text{tp}(a_*/F)$ . We are now in the context of Section 5 and we use the terminology from there.

Let  $\phi(\bar{x}; y) = \phi(x_1, \dots, x_k, y)$  be a formula over  $F$ , where  $y$ , as well as each  $x_i$  ranges over  $W_*$ . Fix  $\bar{a} \in W_*^k$  and  $b \in W_* \setminus \text{acl}(\bar{a})$ . Let  $U \subseteq W_*^k$  be a product of small cells containing  $\bar{a}$  and  $V \subseteq W_*$  a small cell containing  $b$ . Assume that  $U$  and  $V$  are small enough so that  $V$  is strongly disjoint from any small cell appearing in the product defining  $U$ . Then for any  $(b_0, b_1) \in V^2$  and finite subset  $\bar{u}_0$  of  $U$ , the skeletal types of  $(\bar{u}_0, b_0)$  and  $(\bar{u}_0, b_1)$  are the same.

**Claim 9:** The formula  $\phi_{UV}(\bar{x}; y) \equiv \phi(\bar{x}; y) \wedge \bar{x} \in U \wedge y \in V$  is stable.

*Proof:* Assume not, then we can find sequences  $(\bar{a}_i)_{i < \omega}$  in  $U$  and  $(b_i)_{i < \omega}$  in  $V$  such that  $\phi(\bar{a}_i; b_j)$  holds if and only if  $i \leq j$ . For every  $j, n < \omega$ , the set of realizations of  $\text{tp}(b_j/\bar{a}_{<n})$  is dense in a set definable in the skeletal structure

over  $\bar{a}_{<n}$ . Since it has a point in  $V$ , it is dense in  $V$ . Hence the set of realizations of the full type  $\text{tp}(b_j/\bar{a}_{<\omega})$  is dense in  $V$ .

For each coordinate  $i$  of  $W_*$ , let the formulas  $(\zeta_{i,k}(y) : k < \omega)$  define the preimages of disjoint intervals on the  $i$ -th coordinate. Then the family

$$(\zeta_{i,k}(y) : k < \omega, i < N)$$

forms an ird-pattern of size  $N$  inside  $V$ . By density of  $\text{tp}(b_j/\bar{a}_{<\omega})$ , we can add to it the line  $(\phi(\bar{a}_i; y) : i < \omega)$ , giving us an ird-pattern of size  $N + 1$ . This contradicts the fact that  $N = \text{opD}(M) = \text{opD}(W_*)$  and proves the claim.

Let  $\bar{c}_U$  (resp.  $\bar{c}_V$ ) be the tuple of end-points of the intervals in each  $E$ -class defining  $U$  (resp.  $V$ ) and set  $\bar{c} = \bar{c}_U \hat{\ } \bar{c}_V$ .

**Claim 11:** The formula  $\phi(x_1, \dots, x_k, y)$  is local.

*Proof:* Let  $E_{UV}$  be the equivalence relation on  $V$  defined over  $\bar{c}$  by:

$$b E_{UV} b' \iff (\forall \bar{a}' \in U)(\phi(\bar{a}', b) \leftrightarrow \phi(\bar{a}', b')).$$

Note that for the tuples that we consider  $\phi$  is the same thing as  $\phi_{UV}$ . We claim that  $E_{UV}$  has finitely many classes. To see this first note that if  $b \in V$  and  $\bar{a} \in U$ , then we have  $b \perp_{\bar{c}} \bar{a}$ . This is because  $\text{rk}(b/\bar{c}) = 1$  and by construction of  $U$  and  $V$ ,  $b$  cannot be algebraic over  $\bar{a}\bar{c}$ . Hence  $\text{rk}(b/\bar{a}\bar{c}) = 1$ .

Now, for  $b \in V$ , there are finitely many possibilities for  $\text{tp}(b/\bar{c})$ . Fix such a type  $p = \text{tp}(b/\bar{c})$ . By Fact 2.4 and the previous paragraph, the set

$$\{\text{tp}_{\phi_{UV}}(b/U) : b \models p\}$$

is finite. Hence, there are finitely many possibilities for  $\text{tp}_{\phi_{UV}}(b/U)$ ,  $b \in V$ .

We now show that  $E_{UV}$  actually only depends on  $V$  and not on  $U$ . This will follow from similar argument as in Section 5. To simplify notation, we write e.g.  $U \equiv_V U'$  to mean  $\bar{c}_U \equiv_{\bar{c}_V} \bar{c}_{U'}$ . If  $U \equiv_V U'$  and  $U' \subseteq U$ , then  $E_{UV}$  and  $E_{U'V}$  coincide, since they must have the same number of classes. Next, if  $U \equiv_V U'$ , are such that  $U \cap U' \neq \emptyset$ , then there is  $U'' \subseteq U \cap U'$  such that  $U'' \equiv_V U$  and we conclude that  $E_{UV}$  and  $E_{U'V}$  coincide. Finally, any  $U' \equiv_V U$  can be linked to  $U$  by a finite chain  $U' = U_0, \dots, U_m = U$ , with  $U_i \equiv_V U$ ,  $U_i \cap U_{i+1} \neq \emptyset$ .

It follows that the relation  $E_{UV}$  is definable over  $\bar{c}_V$  and depends only on  $V$  and  $\text{tp}(U/V)$ . If  $V' \subseteq V$ , then  $E_{UV}$  and  $E_{UV'}$  coincide on  $V'$ , hence  $E_{UV}$  is a local equivalence relation. This relation depends only on  $\text{tp}(\bar{a}, b/F)$ .

Now, do the same starting with any type of tuple  $(\bar{a}, b)$  and any permutation of the variables of  $\phi$ . Let  $\mathcal{E}_\phi$  be the intersection of all the local equivalence relations obtained. Then  $\mathcal{E}_\phi$  is a local equivalence relation definable over  $F$  which witnesses the fact that  $\phi$  is a local formula.

We can now prove our main theorem.

**Theorem 6.8.** *Given an integer  $n$ , there are, up to inter-definability, finitely many  $\omega$ -categorical primitive NIP structures  $M$  of rank 1 having at most  $n$  4-types.*

*Proof.* We have already seen that for a given number of 4 types, there are only finitely many possibilities for the skeletal structure. Let  $a, b, c \in M$  be given by Lemma 6.5. Then the set  $F$  we used to define  $W_*$  is definable over  $abc$ . Furthermore, each class  $V$  has three points  $\alpha_V, \beta_V, \gamma_V$  definable over  $abc$ .

Let  $\mathcal{E}$  be the finest  $\emptyset$ -definable (equivalently  $\text{acl}^{eq}(\emptyset)$ -definable) local equivalence relation on  $W_*$ . Define big cells  $C_{\bar{i}}$  as in Section 5 using  $\alpha_V, \beta_V, \gamma_V$ . Let  $e$  be any  $\mathcal{E}(C_{\bar{i}})$ -class. Then the  $\mathcal{E}(C_{\bar{0}})$ -class  $e_0$  canonically identified with  $e$  is definable from  $e$  (along with  $abc$ ), since we obtain one from the other by following a sequence of transition maps  $f_{\bar{i}, \bar{s}}$ , which are all definable over  $abc$ . Similarly, any class in  $\mathcal{E}(C_{\bar{0}})$  in the orbit of  $e_0$  under the monodromy action is definable from  $e$ . Furthermore, given any set  $A \subseteq W_*$ , the union of the  $\mathcal{E}(C_{\bar{i}})$ -classes that one can reach from points in  $A$  following maps  $f_{\bar{i}, \bar{s}}$  is definable from  $A$  alone (that is, without  $abc$ ), since that set does not depend on the choice of big cells and can be also defined by following arbitrary paths of small cells.

Given  $d \in M$ , there is  $\bar{d} \in W_*$  interalgebraic with  $d$  and definable over  $abc$ . Define the group  $G$  as in the beginning of Section 5. The set  $\{\sigma(\bar{d}) : \sigma \in G\}$  is interdefinable with  $d$ . By primitivity of  $M$  and the previous paragraph, all  $\mathcal{E}(C_{\bar{i}})$ -classes are definable from it along with  $abc$ . Since we can take  $d = c$ , all those classes are definable over  $abc$ . We conclude that the number of classes of  $\mathcal{E}$  is bounded above by the number of types of elements of  $M$  over  $abc$ .

It follows that any local relation on  $W_*$  is definable over  $abc$ , hence the whole structure on  $W_*$  is definable over  $abc$ . From Section 5, it follows that, for a fixed number of 4 types, there are finitely many possibilities for  $\mathcal{E}$ . All together, there are only finitely possibilities for  $W$  up to inter-definability, and hence also for  $M$ .  $\square$

## 6.6 Homogeneity and finite axiomatizability

We keep the same notation  $M, W, \dots$  as in the previous section. Fix a finite set  $A \subseteq M$  so that all elements of  $W/E$  are definable over  $A$  and each  $E$ -class has at least three points definable over  $A$ . Then the fibers of the projection  $\pi: W \rightarrow M$  are rigid. We can therefore enumerate the elements of  $\pi^{-1}(a)$  as  $(a_1, \dots, a_N)$  in an  $A$ -definable way, so that for  $a, b \in M$ , we have  $(a_1, \dots, a_N) \equiv (b_1, \dots, b_N)$ .

As in the proof of Theorem 6.8, we can define some collection of big cells using parameters from  $A$  and for each such cell  $C$ , define all  $\mathcal{E}(C)$ -equivalence classes, where  $\mathcal{E}$  is the finest local equivalence relation on  $W$ . In particular, if two points in  $W$  have the same type over  $A$ , they are in some common big cell  $C$  and are  $\mathcal{E}(C)$ -equivalent.

Let  $L_A$  be the language consisting of:

- a constant for each element of  $A$ ;
- unary sets naming the complete types over  $A$ ;
- for each non-algebraic type  $p(x)$  over  $A$  and each  $i < N$ , a binary relation  $\leq_{p,i}$  interpreted as follows: the elements  $b_i$  for  $b \models p$  lie in some minimal

$A$ -definable proper interval of an  $E$ -class and for  $b, c \models p$ , we set  $b \leq_{p,i} c$  if  $b_i \leq c_i$  according to the order on that interval;

- for each  $(p, i)$  and  $(q, j)$  as in the previous point, such that  $b_i$  for  $b \models p$  and  $c_j$  for  $c \models q$  lie in the same minimal  $A$ -definable interval of an  $E$ -class, a binary relation  $R_{p,i,q,j}$  coding the unique intertwining between the order  $\leq_{p,i}$  on the locus of  $p$  and  $\leq_{q,j}$  on the locus of  $q$ .

The set  $W$  along with its full structure is interpretable in  $M$  seen as an  $L_A$ -structure. Hence so is the full structure on  $M$ . Furthermore, the  $L_A$ -structure on  $M$  is composed of finitely many unary sets, finitely many dense orders on them which are either independent or have a quantifier-free definable intertwining. By a back-and-forth argument,  $M$  admits elimination of quantifiers in  $L_A$ . This structure is binary, finitely axiomatizable and distal. Distality and non-distality are preserved by naming constants, so  $M$  is distal in its original language.

To finish the proof of Theorem 1.3, it remains to show that the original structure  $M$  admits a finite relational language for which it is homogeneous.

**Lemma 6.9.** *Let  $M$  be an  $\omega$ -categorical structure. Assume that for some integer  $r$ , for any set  $A \subseteq M$  of size  $r$ , the expansion of  $M$  naming every  $\text{acl}^{eq}(A)$ -definable set is finitely homogenizable. Then  $M$  is finitely homogenizable.*

*Proof.* We need to show that for some integer  $k$ , any  $n$ -type  $p(x_1, \dots, x_n)$  is implied by the conjunction of its restrictions to sets of  $k$  variables. Fix an  $r$ -type  $q$  and  $\bar{a} \models q$ . Let  $L_q = \{\phi_1(\bar{x}_1), \dots, \phi_l(\bar{x}_l)\}$  be a set of  $\text{acl}^{eq}(\bar{a})$ -definable formulas such that  $M$  has quantifier elimination in a language with a predicate for each of those formulas. Assume that  $L_q$  is closed under  $\text{Aut}(\text{acl}^{eq}(\bar{a})/\bar{a})$  and that the maximal arity of those formulas is  $m$ . For any finite set  $C \subseteq M$ , define an equivalence relation  $E_C^q$  on  $L_q$  by saying that two formulas  $\phi(\bar{x})$  and  $\phi'(\bar{x})$  are  $E_C^q$ -equivalent if they are conjugated over  $\bar{a}$  and for any tuple  $\bar{c}$  of elements of  $C$ , we have

$$M \models \phi(\bar{c}) \leftrightarrow \phi'(\bar{c}).$$

If a pair  $(\phi, \phi')$  is not in  $E_C^q$ , then there is a subset  $C_0 \subseteq C$  of size at most  $m$  such that  $(\phi, \phi')$  is not in  $E_{C_0}^q$ . It follows that for any  $C$ , there is  $C_* \subseteq C$  of size at most  $N(q) = l^2 m$  such that  $E_C^q = E_{C_*}^q$ .

Let  $p = \text{tp}(a_1, \dots, a_n)$  be any type in finitely many variables. Without loss, all the  $a_i$ 's are distinct. Set  $\bar{a} = (a_1, \dots, a_r)$  and  $q = \text{tp}(\bar{a})$ . Let  $C = \{a_1, \dots, a_n\}$  and take  $C_* \subseteq \{a_1, \dots, a_n\}$  of size at most  $N(q)$  so that  $E_{C_*}^q = E_C^q$ . By construction of  $E_C^q$ , for any  $\bar{d}$  subtuple of  $(a_1, \dots, a_n)$ , the type  $\text{tp}(\bar{d}/\bar{a}C_*)$  implies the quantifier-free  $L_q$ -type of  $\bar{d}$ . By assumption on  $L_q$ , it follows that  $\text{tp}(a_1, \dots, a_n)$  is implied by the conjunction of  $\text{tp}(a_{i_1}, \dots, a_{i_m}/\bar{a}C_*)$  for any choice of  $i_1, \dots, i_m$ . Therefore  $k := r + m + \max_q N(q)$  has the required property.  $\square$

**Question 6.10.** *In the previous lemma, can we replace “for any set  $A \subseteq M$ ” by “for some set  $A \subseteq M$ ”?*

**Proposition 6.11.** *The structure  $M$  is inter-definable with a structure in a finite relational language which is homogeneous and finitely axiomatizable.*

*Proof.* All  $W/E$ -classes are definable over  $\text{acl}^{eq}(\emptyset)$  and for any set  $A \subseteq M$  of size 3, there are at least 3  $\text{acl}^{eq}(A)$ -definable elements in each  $E$ -class. It follows from the previous discussion that the expansion of  $M$  obtained by naming all  $\text{acl}^{eq}(A)$ -definable sets is finitely homogeneous. By Lemma 6.9,  $M$  itself is finitely homogeneous.

Assume that  $M$  is equipped with such a finite relational language  $L$  for which it is homogeneous. We have seen that after naming some appropriate finite set of points  $A$ ,  $M$  becomes homogeneous in a binary language for which it is finitely axiomatizable. It follows that  $M$  is finitely axiomatizable in the language  $\bar{L}(A)$  equal to  $L$  augmented by a finite set of constants to name the elements of  $A$ . Then by quantifying on  $A$ , we see that  $M$  is finitely axiomatizable in  $L$ .  $\square$

## 6.7 Reducts

Using the classification, one can relatively easily describe the reducts of any given structure satisfying  $(\star)$ . First notice that by Theorem 6.8 every such structure has only finitely many reducts, confirming a famous conjecture of Thomas in this case (see e.g. [BM16]). Let  $M$  satisfy  $(\star)$  and  $W$  the finite cover associated to it. Let  $M'$  be a reduct of  $M$ . If  $M'$  is stable, then by Proposition 2.6 it is strongly minimal. Then since algebraic closure is trivial on  $M$ , it has to be pure equality. If  $M'$  is unstable, then we can construct a finite cover  $W'$  of it as above. Any linear order definable in  $M$  with parameters and with universe a subset of  $M$  is in order-preserving definable bijection with a subset of one of the  $E$ -classes of  $W$ . This follows from the construction of  $W$ . Therefore any  $E$ -class in  $W'$  is in definable order-preserving bijection with a (necessarily dense) subset of an  $E$ -class of  $W$ . For a given  $W$ , one can then by inspection determine all the possibilities for  $W'$ . Instead of attempting to write a general statement, we will examine two special cases: the case where  $M = (M; \leq_1, \dots, \leq_n)$  is equipped with  $n$  independent linear orders and the case where  $W$  has just two circular orders in order-reversing bijection, each extending to a unique strong type over  $\emptyset$ .

Assume that  $M = (M; \leq_1, \dots, \leq_n)$  is the Fraïssé limit of sets equipped with  $n$  linear orders and define  $W$  and  $E$  as usual. Then  $W$  is composed of  $2n$  linear orders pairwise in order-reversing bijection and otherwise independent, and the fibers of the projection  $\pi: W \rightarrow M$  pick out exactly one element per linear order. Let  $M'$  be a reduct of  $M$  and  $W'$  the corresponding finite cover, with equivalence relation  $E'$ . We think of  $W'$  as a set interpretable in  $M$ . As observed above, every  $E'$ -class is locally isomorphic to a subset of some  $E$ -class. Since  $E$ -classes are complete types over  $\emptyset$ , every  $E'$ -class is in definable bijection with some  $E$ -class. Furthermore, the projection map  $\pi': W' \rightarrow M'$  cannot pick out more than one element per  $E'$ -class, since algebraic closure in  $M'$  cannot be larger than in  $M$ . It follows that  $W'$  is obtained from  $W$  by



removing some classes, making some classes circular, and possibly adding automorphisms permuting the classes.

One can associate to each reduct of  $M$  a triple  $(V_l, V_c, G)$  where  $V_l, V_c$  are two disjoint subsets of  $\{1, \dots, n\}$  of cardinalities  $m_l$  and  $m_c$  respectively, and  $G$  is a subgroup of the wreath product  $\mathbb{Z}_2 \wr (\mathfrak{S}_{m_l} \times \mathfrak{S}_{m_c})$ . The subsets  $V_l, V_c$  indicate respectively which of the  $n$  orders are kept as linear orders and which are kept as circular orders. The subgroup  $G$  is the group of automorphism on the quotient  $W'/E'$ . The reducts of  $M$  are completely classified by such triples and every triple corresponds to a reduct.

For instance for  $n = 2$ , we have  $3^2 = 9$  choices for the pair  $(V_l, V_r)$ . If either of the two sets has cardinality 2, then we get 10 possibilities for  $G$  (the group  $\mathbb{Z}_2 \wr \mathfrak{S}_2$  is isomorphic to the dihedral group  $D_8$  and has 10 subgroups). If the two sets have cardinality 1, we get 5 possibilities for  $G$  corresponding to subgroups of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ , if one set has cardinality 1 and the other 0, we have two possibilities for  $G$  and finally, if both sets are empty, we have one possibility for  $G$ . Summing it all up, we obtain  $10 \cdot 2 + 5 \cdot 2 + 2 \cdot 4 + 1 = 39$  reducts. We thus recover the result of Linman and Pinsker [LP15].

Let us now turn to the second example. Assume that  $W$  has two  $E$ -classes, which are circular, in order-reversing bijection, conjugated by an automorphism, and the fibers of the projection  $\pi$  contain exactly  $n$  points per class. The associated  $M$  can be obtained by taking the Fraïssé limit of separations relations with an equivalence relation  $F$  having classes of cardinality  $n$  and quotienting by  $F$ .

Let  $M'$  be an unstable reduct of  $M$  and  $W'$  its associated finite cover, which we again think of as interpreted in  $M$ . Let  $V$  be any one of the two  $E$ -classes of  $W$ . Every  $E'$ -class is in definable bijection with  $V$ . Since the map  $\pi': W' \rightarrow M'$  is also interpretable in  $M$ , fibers of  $\pi'$  have to contain at least  $n$  points from each  $E'$ -class (otherwise there would be in  $W$  an  $\text{acl}^{eq}(\emptyset)$ -definable equivalence relation on  $V$  with classes of size  $< n$ , which is not the case). Hence as above, since algebraic closure cannot be larger in  $M'$  as it is in  $M$ ,  $W'$  has two  $E'$ -classes in order reversing bijection and  $\pi'$  is  $n$ -to-one on each of them. But then we see that  $W'$  is isomorphic to  $W$  and there can be no additional automorphisms on the set of classes. So  $M'$  is equal to  $M$ .

This shows that  $M$  has no proper non-trivial reduct. This gives a new example of an infinite family of  $\omega$ -categorical structures with no proper reduct, or equivalently of maximal closed (oligomorphic) permutation groups. (See e.g., [BM16] or [KS16] for more about maximal closed permutation groups.)

## 7 Binary structures and multi-orders

We say that a structure  $M$  is binary if it eliminates quantifier in a finite binary relational language.

**Lemma 7.1.** *Let  $M$  be a binary structure. Then  $M$  has finite rank.*

*Proof.* Assume not and fix some integer  $N$  large enough. Then as  $\text{rk}(M) > N$ , we can build:

- an increasing sequence of finite tuples  $(c(n) : n < N)$ ;
- for each  $n < N$ , a  $c(n)$ -definable set  $D_n$ , transitive over  $c(n)$ ;
- a  $c(n)$ -definable set of parameters  $E_n$ , transitive over  $c(n)$ ;
- a  $c(n)$ -definable family  $(X_t : t \in E_n)$  of infinite subsets of  $D_n$  which is  $k(n)$ -inconsistent for some  $k(n) < \omega$ , such that for some  $t \in E_n$ ,  $D_{n+1} \subseteq X_t$ .

**Claim:** For each  $n$ , there are  $x, y \in D_n$  such that for no  $t \in E_n$  do we have both  $x \in X_t$  and  $y \in X_t$ .

*Proof:* For any  $x \in D_n$  there is a finite tuple  $(t_1, \dots, t_n)$  of elements of  $E_n$  such that  $x$  is in each  $X_{t_i}$  and in no other  $X_t$ . Since  $E_n$  is transitive over  $c(n)$ , no element of  $E_n$  is algebraic over  $c(n)$  and we can find a tuple  $(t'_1, \dots, t'_n) \equiv (t_1, \dots, t_n)$  with  $t'_i \neq t_j$  for all  $i, j \leq n$ . Now take  $y$  so that  $(y, t'_1, \dots, t'_n) \equiv (x, t_1, \dots, t_n)$ .

For each  $n$ , let  $\phi_n(x; y)$  be the relation saying that for some  $t \in E_n$ ,  $x, y \in X_t$ . This relation is definable over  $c(n)$ . As the structure is binary and all elements of  $D_n$  have the same type over  $c(n)$ , there is a formula  $\psi_n(x; y)$  definable over  $\emptyset$  which coincides with  $\phi_n(x; y)$  on  $D_n$ . For every  $n$ , there are  $a, b \in D_n$  with  $\neg\phi_n(a; b)$ . However we must have  $\phi_m(a; b)$  for all  $m < n$ . Hence all formulas  $\phi_n(x; y)$  are distinct. Taking  $N$  large enough, this is a contradiction.  $\square$

**Question 7.2.** Let  $M$  be a primitive binary structure. Must  $M$  have rank 1?

We say that  $(M, \leq)$  is topologically primitive, where  $\leq$  is a linear order, if it does not admit a  $\emptyset$ -definable convex non-trivial equivalence relation.

**Lemma 7.3.** Let  $(M, \leq, \dots)$  be a ranked  $\omega$ -categorical structure, where  $\leq$  is a linear order on  $M$ . Assume that  $(M, \leq)$  is topologically primitive. Then  $(M, \leq)$  has topological rank 1.

*Proof.* Assume that over parameters  $\bar{a}$ , there is some definable convex equivalence relation  $E_{\bar{a}}$  with infinitely many classes. By  $\omega$ -categoricity, the order induced by  $\leq$  on the quotient  $M/E_{\bar{a}}$  is not discrete. Thus there are  $c < d$  in  $M$  such that there are infinitely many  $E_{\bar{a}}$ -classes between  $c$  and  $d$ . The relation  $R(x, y)$  saying that for every  $\bar{b} \equiv \bar{a}$ , there are finitely many  $E_{\bar{b}}$  classes between  $x$  and  $y$  is a 0-definable equivalence relation with convex classes. As  $M$  is topologically primitive,  $R$  is trivial: its classes are singletons. It follows that for every open interval  $I$ , we can find some  $\bar{b} \equiv \bar{a}$  such that  $E_{\bar{b}}$  has infinitely many classes in  $I$ . This easily implies that  $M$  has unbounded rank and contradicts the previous lemma.  $\square$

**Theorem 7.4.** Let  $(M, \leq_1, \dots, \leq_n)$  be a homogeneous multi-order such that no two orders  $\leq_i$  and  $\leq_j$  are equal or opposite of each other. Assume that each  $(M, \leq_i)$  is topologically primitive, then  $M$  is the Fraïssé limit of finite sets equipped with  $n$  orders.

*Proof.* The assumptions along with the previous lemmas imply that each order  $(M, \leq_i)$  has topological rank 1 and is a complete type over  $\emptyset$ . Proposition 3.23 describes the possibilities. The only homogeneous structures in the list are the ones with no intertwining (other than equalities between orders), since the intertwining relations  $R_{ij}$  are not quantifier-free definable from the orders.  $\square$

The classification of imprimitive homogeneous multi-orders is carried out in [BS], making further use of techniques from this paper.

More generally, a primitive set equipped with  $n$  orders definable in a binary structure satisfies the hypotheses of Proposition 3.23. This might help in classifying other classes of ordered homogeneous structures.

## 8 The general NIP case

We hope to be able eventually to classify all finitely homogeneous NIP structures, and possibly even all  $\omega$ -categorical structures having polynomially many types over finite sets.

**Conjecture 8.1.** *Let  $M$  be finitely homogeneous and NIP, then:*

1. *The automorphism group  $\text{Aut}(M)$  acts oligomorphically on the space of types  $S_1(M)$ .*
2. *The structure  $M$  is interpretable in a distal, finitely homogeneous structure.*
3. *There is  $M'$  bi-interpretable with  $M$  whose theory is quasi-finitely axiomatizable.*
4. *If  $M$  is not distal, then its theory is not finitely axiomatizable.*

Points (2) and (3) each imply that there are only countably many such structures (for point (2), this follows from Theorem 8.3 below). If  $M$  is stable finitely homogeneous, then it is  $\omega$ -stable and the conjecture is known to be true: (1) by [CHL85, Theorem 6.2], (2) by [Lac87], (3) by [Hru89] and (4) by [CHL85, Corollary 7.4].

Note that we cannot expect an analogue of Theorem 6.8: For  $k < \omega$ , let  $M_k$  be the Fraïssé limit of finite trees with  $\leq k$  branching at each node. Then for  $k \geq 4$ , the structures  $M_k$  all have the same 4-types.

The previous conjecture was stated for the finitely homogeneous case, but we could have stated it also for  $\omega$ -categorical structures with polynomially many types over finite sets, or finite dp-rank, which is *a priori* weaker. (For a definition of dp-rank, see e.g. [Sim15, Chapter 4].) However, even the stable case is then unknown.

**Question 8.2.** *Let  $M$  be  $\omega$ -categorical, stable of finite dp-rank. Is  $M$   $\omega$ -stable?*

One intuition we have on NIP structures is that they are somehow combinations of stable and distal ones. At the very least, we expect that reasonable statements that hold true for stable and distal structures are true for all

NIP structures. If  $M$  is finitely homogeneous and stable, then we know that it is quasi-finitely axiomatizable. Somewhat surprisingly, the distal case can be proved directly rather easily: see Theorem 8.3 below. We consider this as strong evidence towards this part of the conjecture. It is possible that the other parts could also be proved directly for distal structures, without having any kind of classification, but we have not managed to do so.

**Theorem 8.3.** *Let  $M$  be homogeneous in a finite relational language  $L$  and distal. Then the theory of  $M$  is finitely axiomatizable.*

*Proof.* Let  $r$  be the maximal arity of a relation in  $L$ . By distality, there is  $k$  such that for any finite set  $A \subseteq M$  and element  $a \in M$ , there is  $A_0 \subseteq A$  of size  $\leq k$  with  $\text{tp}(a/A_0) \vdash \text{tp}(a/A)$ . Let  $n_0 = kr + k + r + 1$ . Consider the theory  $T_*$  composed of:

1. all formulas of the form  $(\forall \bar{x})\phi(\bar{x})$ , with  $|\bar{x}| \leq n_0$  and  $\phi$  quantifier-free that are true in  $M$ ;
2. all formulas of the form  $(\forall \bar{x})(\theta(\bar{x}) \rightarrow (\exists y)\phi(\bar{x}, y))$  with  $|\bar{x}| \leq k$ ,  $|y| = 1$  and  $\theta, \phi$  quantifier-free that are true in  $M$ .

Up to logical equivalence,  $T_*$  contains finitely many formulas. Since  $M$  is a model of  $T_*$ , that theory is consistent. Let  $N$  be any countable model of it and we will show that  $N$  is isomorphic to  $M$ .

Claim 0: Let

$$Y \equiv (\forall x, \bar{y}, \bar{z})(\theta(x, \bar{y}) \wedge \psi(\bar{y}, \bar{z}) \rightarrow \phi(x, \bar{z})),$$

with  $|x| = 1$ ,  $|\bar{y}| \leq k$  and where each of  $\theta, \psi, \phi$  is quantifier-free and describes a complete type. Then if  $M$  satisfies  $Y$ , so does  $N$ .

*Proof:* Since the arity of  $L$  is bounded by  $r$ ,  $\phi(x, \bar{z})$  is a conjunction of formulas of the form  $\phi'(x, \bar{z}')$ , where  $\bar{z}' \subseteq \bar{z}$  is a subtuple of size  $\leq r$ . For each such formula, we have

$$M \models (\forall x, \bar{y}, \bar{z}')(\theta(x, \bar{y}) \wedge \psi'(\bar{y}, \bar{z}') \rightarrow \phi'(x, \bar{z}'))$$

where  $\psi'(\bar{y}, \bar{z}')$  is a complete quantifier-free formula implied by  $\psi(\bar{y}, \bar{z})$  with variables  $(\bar{y}, \bar{z}')$ . This formula is in  $T_*$ , since it is universal and has less than  $n_0$  variables, so  $N$  also satisfies it.

Claim 1:  $N$  satisfies the universal theory of  $M$ : for any finite set  $B \subseteq N$ , there is  $B' \subseteq M$  which is isomorphic to it.

*Proof:* We prove the result by induction on the cardinality of  $B$ . For  $|B| \leq n_0$ , this follows from the construction of  $T_*$ . Assume that we know the result for some  $n \geq n_0$  and are given a finite subset  $B \subseteq N$  of size  $n$  and an additional point  $d \in N$ . We want to find an isomorphic copy of  $B \cup \{d\}$  in  $M$ . Pick any  $r$  distinct elements  $b_0, \dots, b_{r-1}$  in  $B$ . For  $i < r$ , set  $B_i = B \setminus \{b_i\}$ . The set  $B_i \cup \{d\}$  has an isomorphic copy in  $M$ . It follows by distality of  $M$  that there is  $B'_i \subseteq B_i$  of size  $\leq k$  such that

$$(\Delta) \quad M \models \text{tp}(d, B'_i) \wedge \text{tp}(B'_i, B_i) \rightarrow \text{tp}(d, B_i).$$

By Claim 0,  $N$  also satisfies that formula. Let  $B_r = \bigcup_{i < r} B'_i$ . By the case  $n = kr + 1 < n_0$ , the set  $B_r \cup \{d\}$  is isomorphic to some  $A_r \cup \{c\}$  in  $M$ . By homogeneity of  $M$  and induction, we can find  $A \supseteq A_r$  such that  $\text{tp}(A_r, A) = \text{tp}(B_r, B)$ . For  $i < r$ , define  $A_i$  is the image of  $B_i$  under this isomorphism. By  $(\Delta)$ , which holds both in  $M$  and in  $N$ , we have  $\text{tp}(d, B_i) = \text{tp}(c, A_i)$  for each  $A$ . Since the arity of the language is at most  $r$  and any  $r$  elements from  $Bd$  either lie in  $B$  or in some  $B_i d$ , we conclude that  $Bd$  and  $Ac$  are isomorphic. This finishes the induction.

We now show by back-and-forth that  $N$  is isomorphic to  $M$ . Assume we have a partial isomorphism  $f$  from a finite subset  $A \subseteq M$  to  $N$ . Let  $c \in M$ . By distality, there is  $A_0 \subseteq A$  of size  $\leq k$  such that  $\text{tp}(c/A_0) \vdash \text{tp}(c/A)$ . Let  $B_0$  be the image of  $A_0$  in  $B$ . By assumption on  $T_*$ , there is  $d \in N$  such that  $\text{tp}(d, B_0) = \text{tp}(c, A_0)$ . By Claim 0, we have  $\text{tp}(d, B) = \text{tp}(c, A)$ , hence we can extend the partial isomorphism  $f$  by setting  $f(c) = d$ . The back direction follows at once from Claim 1 and homogeneity of  $M$ .  $\square$

## References

- [BM16] Manuel Bodirsky and Dugald Macpherson. Reducts of structures and maximal-closed permutation groups. *Journal of Symbolic Logic*, 81(3):1087–1114, 2016.
- [Bra18] Samuel Braunfeld. Homogeneous 3-dimensional permutation structures. *The Electronic Journal of Combinatorics*, 25(2):Paper 52, 2018.
- [BS] Samuel Braunfeld and Pierre Simon. The classification of homogeneous finite-dimensional permutation structures. *Electronic Journal of Combinatorics (to appear)*.
- [Cam76] Peter Cameron. Transitivity of permutation groups on unordered sets. *Mathematische Zeitschrift*, 148:127–139, 1976.
- [Cam81] Peter Cameron. Orbits of permutation groups on unordered sets, II. *J. Lond. Math. Soc. (2)*, 23:249–264, 1981.
- [Cam87] Peter Cameron. Some treelike objects. *Quart. J. Math. Oxford (2)*, 38:155–183, 1987.
- [Cam02] Peter Cameron. Homogeneous permutations. *The Electronic Journal of Combinatorics*, 9(2), 2002.
- [CH03] G.L. Cherlin and E. Hrushovski. *Finite Structures with Few Types*. Annals of mathematics studies. Princeton University Press, 2003.
- [CHL85] G. Cherlin, L. Harrington, and A.H. Lachlan.  $\omega$ -categorical,  $\omega$ -stable structures. *Annals of Pure and Applied Logic*, 28(2):103 – 135, 1985.

- [CS15] Artem Chernikov and Pierre Simon. Externally definable sets and dependent pairs II. *Transactions of the American Mathematical Society*, 367(7):5217–5235, 2015.
- [GH15] Vincent Guingona and Cameron Donnay Hill. On a common generalization of shelah’s 2-rank, dp-rank, and o-minimal dimension. *Annals of Pure and Applied Logic*, 166(4):502–525, 2015.
- [Hru89] Ehud Hrushovski. Totally categorical structures. *Trans. Amer. Math. Soc.*, 313:131–159, 1989.
- [KLM89] W.M. Kantor, Martin W. Liebeck, and H. D. Macpherson.  $\aleph_0$ -categorical structures smoothly approximated by finite substructures. *Proc. London Math. Soc.* (3), 59:439–463, 1989.
- [KS16] Itay Kaplan and Pierre Simon. The affine and projective groups are maximal. *Transaction of the American Mathematical Society*, 368(7):5229–5245, 2016.
- [Lac84] A. H. Lachlan. On countable stable structures which are homogeneous for a finite relational language. *Israel Journal of Mathematics*, 49(1):69–153, Sep 1984.
- [Lac87] A. H. Lachlan. Structures coordinatized by indiscernible sets. *Annals of Pure and Applied Logic*, 34:245–273, 1987.
- [LP15] Julie Linman and Michael Pinsker. Permutations of the random permutation. *The Electronic Journal of Combinatorics*, 22(2), 2015.
- [Mac85] H. D. Macpherson. Orbits of infinite permutation groups. *Proc. London Math. Soc.* (3), 51:246–284, 1985.
- [Mac87] H. D. Macpherson. Infinite permutation groups of rapid growth. *J. London Math. Soc.* (2), 35:276–286, 1987.
- [Mac11] Dugald Macpherson. A survey of homogeneous structures. *Discrete Mathematics*, 311(15):1599 – 1634, 2011. Infinite Graphs: Introductions, Connections, Surveys.
- [Mar02] David Marker. *Model Theory: An Introduction*. Springer, 2002.
- [Ons06] Alf Onshuus. Properties and consequences of thorn-independence. *J. Symbolic Logic*, 71(1):1–21, 2006.
- [Pil96] A. Pillay. *Geometric stability theory*. Oxford logic guides. Clarendon Press, 1996.
- [She90] Saharon Shelah. *Classification theory and the number of nonisomorphic models*, volume 92 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, second edition, 1990.

- [Sim13] Pierre Simon. Distal and non-distal theories. *Annals of Pure and Applied Logic*, 164(3):294–318, 2013.
- [Sim15] Pierre Simon. *A Guide to NIP theories*. Lecture Notes in Logic. Cambridge University Press, 2015.
- [Sim18] Pierre Simon. Linear orders in NIP theories. preprint, 2018.
- [TZ12] K. Tent and M. Ziegler. *A Course in Model Theory*. Lecture Notes in Logic. Cambridge University Press, 2012.
- [Zil] B. Zilber. *Uncountably Categorical Theories*. Translations of mathematical monographs. American Mathematical Society.