Secure Equilibria in Weighted Games*

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Abstract. We consider two-player non zero-sum infinite duration games played on weighted graphs. We extend the notion of secure equilibrium introduced by Chatterjee et al., from the Boolean setting to this quantitative setting. As for the Boolean setting, our notion of secure equilibrium refines the classical notion of Nash equilibrium. We prove that secure equilibria always exist in a large class of weighted games which includes common measures like sup, inf, lim sup, lim inf, mean-payoff, and discounted sum. Moreover we show that it is possible to synthesize such strategy profiles that are finite-memory and use few memory. Finally, we prove that the constrained existence problem for secure equilibria is decidable for sup, inf, lim sup, lim inf and mean-payoff measures. Our solutions rely on new results for zero-sum quantitative games with lexicographic objectives that are interesting on their own right.

1 Introduction

Two-player zero-sum infinite duration games played on graphs is a useful framework to formalize many important problems in computer science. System synthesis, and especially synthesis of reactive systems, is one of those important problems, see for example [27,26]. In this application, the vertices and the edges of the graph represent the states and the transitions of the system; one player represents the system to synthesize, whereas the other represents the environment with which the system interacts. In the classical setting, the environment is considered as *antagonist* and the objectives of the two players are *complementary*, leading to a so-called *zero-sum* game. There are many known results about those zero-sum games [23,15,17,19].

Modeling the environment as completely adversarial is usually a bold abstraction of reality. However it is often used as it is simple and sound: a winning strategy against a completely adversarial model of the environment, is winning against any environment which pursues its own objective. But this approach may fail to find a winning strategy whereas there exists a solution when the objective of the environment is taken into account. In this case, we need to consider the more general framework of *non zero-sum* games. The classical notion of rational

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behavior in this context is commonly formalized as Nash equilibria [24]. Nash equilibria capture rational behaviors when the players only care about their own payoff (internal criteria), and they are indifferent to the payoff of the other player (external criteria). In the setting of synthesis, the more appropriate notion is the *adversarial external criteria*, where the players are as harmful as possible to the other players without sabotaging with their own objectives. This has inspired the study of refinements of Nash equilibria, like the notion of *secure equilibria* that captures the adversarial external criteria and is at the basis of *compositional synthesis algorithms* [13]. In secure equilibria, lexicographic objectives are considered: each player first tries to maximize his own payoff, and then to minimize the opponent's payoff. It is shown in [13] that secure equilibria are those Nash equilibria which represent enforceable contracts between the two players.

In this paper, we extend the notion of secure equilibria from the Boolean setting with ω-regular objectives of [13] to a quantitative setting where objectives are non necessarily ω-regular. More precisely, we consider two-player non zero-sum turn-based games played on weighted graphs, called *weighted games* with objectives defined by the classical measures considered in the literature for infinite plays: *sup*, *inf*, *lim sup*, *lim inf*, *mean-payoff*, and *discounted sum* [9]. In our setting, the edges of the weighted graph are labelled with pairs of rational values that are used to assign two values to each infinite play: one value models the reward of player 1, and the other the reward of player 2.

Contributions. Our contributions are threefold. (1) We show that all weighted games with the classical measures have *secure equilibria*. We also establish that there exist simple profiles of strategies that witness such equilibria: finite-memory strategies are suffisant with a linear memory size for most of the measures (polynomial size for inf, sup measures). (2) We provide polynomial or pseudo-polynomial (depending on the measures) algorithms for the automatic synthesis of such strategy profiles. So we provide the necessary algorithms to extend the compositional synthesis framework of [12] to a quantitative setting. (3) We prove that one can decide the existence of a secure equilibrium whose outcome satisfies some constraints, for all measures except the discounted sum. In the latter case, we show that this problem is connected to a challenging open problem [3,10]. Our solutions rely on the analysis of two-player zero-sum games with lexicographic objectives (for all the measures) for which we provide worst-case optimal algorithms. These so-called *lexicographic games* are interesting on their own right and, to the best of our knowledge, were not studied in the literature.

Related work. Secure equilibria were first introduced in [13] in a Boolean setting, and for ω-regular objectives. We extend this work to a quantitative setting in which objectives are not necessarily ω-regular. Our work (1) and (2) is very much inspired by a recent work by Brihaye et al. [7] that provides general results for the existence of Nash equilibria in a large class of multi-player weighted graphs. We show that their main theorem is extendable to secure equilibria in the case of two-player games. However, to adapt their theorem we need nontrivial new results on lexicographic games, while for Nash equilibria they can rely on well-known results for (non lexicographic) zero-sum two-player games. Previously to [7], existence results about different kinds of equilibria (Nash, secure, perfect, and subgame perfect) have been established for *quantitative reachability* objectives in multi-player weighted games [5,6,22]. In [29], among other results, the authors study the decision problem of both the existence and the constraint existence of Nash equilibria in multi-player weighted games for the mean-payoff measure. A part of our work (3) is inspired by some of their techniques. In [18] is studied the *rational synthesis* problem, when a system interacts with agents that all have their own objectives. This problem asks to construct a strategy profile that enforces the objective of the system, and which is an equilibrium for the agents. The objectives are ω-regular or defined by deterministic latticed Büchi word automata (they do not include mean-payoff and discounted sum objectives), and secure equilibria are not considered explicitly. Lexicographic games were first considered in [2], for a mean-payoff measure that differs from the one studied in this paper. The proof technique that we develop is similar to their approach but requires non trivial adaptations. To the best of our knowledge, lexicographic games for all the other objectives (sup, inf, lim sup, lim inf, and discounted sum) have not been studied previously.

Structure of the paper. In Section 2, we first recall classical notions on games and equilibria, we then introduce the three problems studied in this paper and our solutions for the sup, inf, lim sup, lim inf, mean-payoff, and discounted sum measures (Theorems 1-3). In Section 3, for each of the three problems, we provide a general framework of weighted games in which the problem can be solved (Theorems 4-6). Those frameworks impose in particular the determinacy of two (one for each player) lexicographic games associated with the initial game. Lexicographic games are introduced in Section 3 and proved to be determined in Section 4, where their complexity is also established. With these results, in Section 3, we are able to prove Theorems 1-2, because games with the considered measures all fall in the frameworks proposed for solving Problems 1-2. The framework proposed for the last problem requires an additional hypothesis that we study in Section 5. Theorem 3 can be then derived for all the measures, except the discounted sum. We also show that Problem 3 for this measure is linked to a challenging open problem [3,10]. In Section 6, we give a conclusion.

2 Weighted Games and Studied Problems

In this section, we recall the notions of weighted game, Nash equilibrium, secure equilibrium, and we state the problems that we want to solve.

2.1 Weighted Games

We here consider two-player turn-based non zero-sum weighted games such that the weights are seen as rewards, and the two players want to maximize their payoff (this payoff is computed from the weights, for example as in Definition 2).

Definition 1. *A two-player non zero-sum* weighted game *is a tuple* $\mathcal{G} = (V, V_1, V_2, E, r, \mathsf{Payoff})$ *where*

- (V, E) *is a finite directed graph, the arena of the game, with vertices* V *and edges* $E \subseteq V \times V$*, such that for each* $v \in V$*, there exists* $e = (v, v') \in E$ *for some* $v' \in V$ *(no deadlock),*
- V_1, V_2 *forms a partition of* V *such that* V_i *is the set of vertices controlled by player* $i \in \{1, 2\}$*,*
- $r = (r_1, r_2)$ *is the weight function such that* $r_i : E \to \mathbb{Q}$ *associates a rational reward to each edge for player* $i \in \{1, 2\}$*,*
- $\mathsf{Payoff} = (\mathsf{Payoff}_1, \mathsf{Payoff}_2)$ *is the payoff function such that* $\mathsf{Payoff}_i : V^\omega \to \mathbb{R}$ *is the payoff function of player* $i \in \{1, 2\}$.

When an initial vertex $v_0 \in V$ is fixed, we call $(\mathcal{G}, v_0)$ an *initialized weighted game.* A *play* of $(\mathcal{G}, v_0)$ is an infinite sequence $\rho = \rho_0\rho_1 \ldots \in V^\omega$ such that $\rho_0 = v_0$ and $(\rho_i, \rho_{i+1}) \in E$ for all $i \in \mathbb{N}$. *Histories* of $(\mathcal{G}, v_0)$ are finite sequences $h = h_0 \ldots h_n \in V^+$ defined in the same way. A *prefix* (resp. *suffix*) of a play ρ is a finite sequence $\rho_0 \ldots \rho_n$ (resp. infinite sequence $\rho_n\rho_{n+1} \ldots$) denoted by $\rho_{\leq n}$ (resp. $\rho_{\geq n}$). The *length* of $\rho_{\leq n}$ is the number n of its edges. For a play $\rho \in V^\omega$ and a player $i \in \{1, 2\}$, we note $r_i(\rho) = r_i(\rho_0, \rho_1)r_i(\rho_1, \rho_2) \ldots$ the sequence of player i weights along ρ. The *payoff* of ρ for player i is given by $\mathsf{Payoff}_i(\rho)$, and $\mathsf{Payoff}(\rho) = (\mathsf{Payoff}_1(\rho), \mathsf{Payoff}_2(\rho))$ is the payoff of ρ in $\mathcal{G}$.

A *strategy* σ for player $i \in \{1, 2\}$ is a function $\sigma : V^*V_i \to V$ assigning to each history $hv \in V^*V_i$ a vertex $v' = \sigma(hv)$ such that $(v, v') \in E$. We denote by Σ_i the set of strategies of player i. A strategy σ for player i is *positional* if $\sigma(h) = \sigma(h')$ for all histories h, h' ending with the same vertex (σ only depends on the last vertex of the history). In particular, a positional strategy is a function $\sigma : V_i \to V$ (instead of $\sigma : V^*V_i \to V$). A strategy σ is a *finite-memory* strategy if it only needs finite memory of the history (recorded by a finite strategy automaton). Formally a *finite strategy automaton* for player $i \in \{1, 2\}$ over a weighted game $\mathcal{G} = (V, V_1, V_2, E, r, \mathsf{Payoff})$ is a Mealy automaton $\mathcal{M} = (M, m_0, V, \delta, \nu)$ where:

- M is a non-empty, finite set of memory states,
- $m_0 \in M$ is the initial memory state,
- $\delta : M \times V \to M$ is the memory update function,
- $\nu : M \times V_i \to V$ is the transition choice function for player i, such that $(v, \nu(m, v)) \in E$ for all $m \in M$ and $v \in V_i$.

This automaton defines the finite-memory strategy $\sigma_{\mathcal{M}}$ such that $\sigma_{\mathcal{M}}(hv) = \nu(\hat{\delta}(m_0, h), v)$ for all $hv \in V^*V_i$, where $\hat{\delta}$ extends δ to histories starting from m_0 as expected. The memory size of $\sigma_{\mathcal{M}}$ is defined as the size of M.

Given a strategy $\sigma \in \Sigma_i$ with $i \in \{1, 2\}$, we say that a play ρ of $\mathcal{G}$ is *consistent* with σ if $\rho_{k+1} = \sigma(\rho_{\leq k})$ for all $k \in \mathbb{N}$ such that $\rho_k \in V_i$. A *strategy profile* of $\mathcal{G}$ is a pair (σ_1, σ_2) of strategies, with $\sigma_i \in \Sigma_i$ for each $i \in \{1, 2\}$. Given an initial vertex v_0, such a strategy profile determines a unique play of $(\mathcal{G}, v_0)$ that is consistent with both strategies. This play is called the *outcome* of (σ_1, σ_2) and is denoted by $\langle \sigma_1, \sigma_2 \rangle_{v_0}$. For a history $hv \in V^*V$, and a strategy profile (σ_1, σ_2), we note

$\langle\sigma_1|_h, \sigma_2\rangle_v$ the outcome of $(\sigma_1|_h, \sigma_2)$ in the initialized game $(\mathcal{G}, v)$, where $\sigma_1|_h$ is the strategy defined by $\sigma_1|_h(h'v') = \sigma_1(hh'v')$ for all histories $h'v' \in V^*V_1$ that begins with v. The outcome $\langle\sigma_1, \sigma_2|_h\rangle_v$ is defined similarly. We say that a player *deviates* from a strategy (resp. from a play) if he does not carefully follow this strategy (resp. play). We say that a strategy profile (σ_1, σ_2) is a positional (resp. finite-memory) strategy profile if σ_1 and σ_2 are positional (resp. finite-memory) strategies. The memory size of a finite-memory strategy profile is equal to the maximum memory size of its strategies.

In this paper, we focus on several well-known payoff functions (see for instance [9]).

Definition 2. *Given a weighted game $\mathcal{G} = (V, V_1, V_2, E, r, \mathsf{Payoff})$, we define the payoff function* Payoff *as one of the measures in* $\{\mathsf{Inf}, \mathsf{Sup}, \mathsf{LimInf}, \mathsf{LimSup}, \mathsf{InfMP}, \mathsf{SupMP}, \mathsf{Disc}^\lambda$ *for* $\lambda \in]0,1[\}$*, where for all* $i \in \{1,2\}$ *and* $\rho \in V^\omega$*:*

- $\mathsf{Inf}_i(\rho) = \inf_{n\in\mathbb{N}} r_i(\rho_n, \rho_{n+1})$,
- $\mathsf{Sup}_i(\rho) = \sup_{n\in\mathbb{N}} r_i(\rho_n, \rho_{n+1})$,
- $\mathsf{LimInf}_i(\rho) = \liminf\limits_{n\to\infty} r_i(\rho_n, \rho_{n+1})$,
- $\mathsf{LimSup}_i(\rho) = \limsup\limits_{n\to\infty} r_i(\rho_n, \rho_{n+1})$,
- $\mathsf{InfMP}_i(\rho) = \liminf\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=0}^{n-1} r_i(\rho_k, \rho_{k+1})$,
- $\mathsf{SupMP}_i(\rho) = \limsup\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=0}^{n-1} r_i(\rho_k, \rho_{k+1})$,
- $\mathsf{Disc}_i^\lambda(\rho) = (1-\lambda) \cdot \sum_{n=0}^{\infty} \lambda^n r_i(\rho_n, \rho_{n+1})$.

We also call these games Payoff weighted games.

2.2 Equilibria

We now recall the concept of Nash equilibrium and secure equilibrium. In this aim we need to fix two lexicographic orders on $\mathbb{R}^2$: a lexicographic order $\preccurlyeq_1$ w.r.t. the first component and a lexicographic order $\preccurlyeq_2$ w.r.t. the second component such that for all $(x_1, x_2), (x'_1, x'_2) \in \mathbb{R}^2$,

$$(x_1, x_2) \preccurlyeq_1 (x'_1, x'_2) \text{ iff } (x_1 < x'_1) \vee (x_1 = x'_1 \wedge x_2 \geq x'_2),$$
$$(x_1, x_2) \preccurlyeq_2 (x'_1, x'_2) \text{ iff } (x_2 < x'_2) \vee (x_2 = x'_2 \wedge x_1 \geq x'_1).$$

Notice that $(\mathbb{R}^2, \preccurlyeq_1)$ and $(\mathbb{R}^2, \preccurlyeq_2)$ are totally ordered sets.

A strategy profile (σ_1, σ_2) in an initialized weighted game $(\mathcal{G}, v_0)$ is a Nash equilibrium if player 1 (resp. player 2) has no incentive to deviate unilaterally from σ_1 (resp. σ_2), since he cannot strictly increase his payoff when using σ'_1 (resp. σ'_2) instead of σ_1 (resp. σ_2). The notion of secure equilibrium is stronger in the sense that player i has no incentive to deviate from σ_i with respect to the order $\preccurlyeq_i$ (instead of the usual order $\leq$ on his payoffs).

Definition 3. *Let $(\mathcal{G}, v_0)$ be an initialized weighted game. A strategy profile (σ_1, σ_2) with $\sigma_i \in \Sigma_i$, $i \in \{1, 2\}$, is a* Nash equilibrium *in $(\mathcal{G}, v_0)$ if, for each strategy $\sigma'_i \in \Sigma_i$, $i \in \{1, 2\}$,*

$$\mathsf{Payoff}_1(\langle \sigma'_1, \sigma_2 \rangle_{v_0}) \leq \mathsf{Payoff}_1(\langle \sigma_1, \sigma_2 \rangle_{v_0}),$$
$$\mathsf{Payoff}_2(\langle \sigma_1, \sigma'_2 \rangle_{v_0}) \leq \mathsf{Payoff}_2(\langle \sigma_1, \sigma_2 \rangle_{v_0}).$$

It is a secure equilibrium *in $(\mathcal{G}, v_0)$ if, for each strategy $\sigma'_i \in \Sigma_i$, $i \in \{1, 2\}$,*

$$\mathsf{Payoff}(\langle \sigma'_1, \sigma_2 \rangle_{v_0}) \preccurlyeq_1 \mathsf{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}),$$
$$\mathsf{Payoff}(\langle \sigma_1, \sigma'_2 \rangle_{v_0}) \preccurlyeq_2 \mathsf{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}).$$

Let us remark that the notion of secure equilibrium is a refinement of that of Nash equilibrium. In a Nash equilibrium, each player only cares about his own payoff (he maximizes it), whereas in a secure equilibrium, he cares about the payoff of both players (he maximizes his payoff and then minimizes that of the other player).

With the notation of Definition 3, we say that σ'_1 is a *profitable deviation* for player 1 w.r.t. (σ_1, σ_2) if $\mathsf{Payoff}_1(\langle \sigma_1, \sigma_2 \rangle_{v_0}) < \mathsf{Payoff}_1(\langle \sigma'_1, \sigma_2 \rangle_{v_0})$ (resp. $\mathsf{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}) \prec_1 \mathsf{Payoff}(\langle \sigma'_1, \sigma_2 \rangle_{v_0})$). Profitable deviations for player 2 are defined similarly. With these terms, we can say that (σ_1, σ_2) is a Nash equilibrium (resp. secure equilibrium) if no player has a profitable deviation w.r.t. (σ_1, σ_2) for the relation $<$ (resp. $\prec_i$).

Example 1. Consider the initialized InfMP weighted game $(\mathcal{G}, v_0)$ depicted in Figure 1. Circle (resp. square) vertices are player 1 (resp. player 2) vertices, and the weights $(0, 0)$ are not specified. In this simple game, both players have two positional strategies that are respectively denoted by σ_i and σ'_i for each player $i \in \{1, 2\}$. These strategies are depicted in Figure 1.

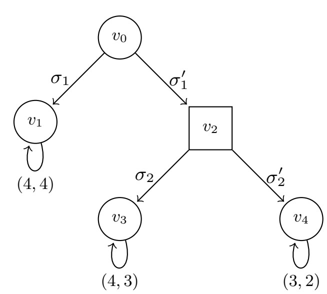

Fig. 1. A simple weighted game

The strategy profiles (σ_1, σ'_2) and (σ'_1, σ_2) are secure equilibria in $(\mathcal{G}, v_0)$. Indeed, for the first one, only player 1 has control on the play $\langle \sigma_1, \sigma'_2 \rangle_{v_0}$ with payoff $(4,4)$, and he decreases his payoff if he plays strategy σ_1' instead of σ_1 $((3,2) \prec_1 (4,4))$. For the second one, both players have control on the play $\langle\sigma_1', \sigma_2\rangle_{v_0}$ with payoff $(4,3)$. If player 1 plays σ_1 instead of σ_1', he keeps the same payoff but increases the payoff of player 2 $((4,4) \prec_1 (4,3))$. If player 2 plays σ_2' instead of σ_2, he decreases his payoff $((3,2) \prec_2 (4,3))$. Hence in both examples no player has a profitable deviation, and these strategy profiles are secure equilibria (and then Nash equilibria) in $(\mathcal{G}, v_0)$.

However the strategy profile (σ_1, σ_2) is a Nash equilibrium which is not a secure equilibrium. Indeed, σ_1' is a profitable deviation for player 1 since by playing σ_1' instead of σ_1, he keeps his own payoff but can decrease player 2 payoff.

Finally, the strategy profile (σ_1', σ_2') is neither a secure equilibrium nor a Nash equilibrium.

2.3 Problems and Main Theorems

To conclude this section, we state the three problems studied in the paper and our results. Let $(\mathcal{G}, v_0)$ be an initialized two-player non zero-sum weighted game.

Problem 1. Does there exist a secure equilibrium in $(\mathcal{G}, v_0)$? If it is the case, does there exist a finite-memory secure equilibrium?

Problem 2. What is the complexity of constructing a (finite-memory) secure equilibrium in $(\mathcal{G}, v_0)$ if one exists?

Problem 3. Given two thresholds $\mu, \nu \in (\mathbb{Q} \cup \{\pm\infty\})^2$, is it decidable whether there exists a secure equilibrium (σ_1, σ_2) in $(\mathcal{G}, v_0)$ such that

$$\mu \leq \mathsf{Payoff}(\langle\sigma_1, \sigma_2\rangle_{v_0}) \leq \nu,$$

i.e. $\mu_i \leq \mathsf{Payoff}_i(\langle\sigma_1, \sigma_2\rangle_{v_0}) \leq \nu_i$ for $i \in \{1,2\}$?

As expected, if no restriction is given on the payoff function used in weighted games, the answer to Problem 1 is negative. An example of a weighted game with no Nash equilibrium (and thus with no secure equilibrium) is given in [7]. In this paper, we solve the stated problems for weighted games with the payoff functions given in Definition 2. Our solutions are as follows.

Theorem 1. *Let $(\mathcal{G}, v_0)$ be an initialized weighted game. Then $(\mathcal{G}, v_0)$ has a secure equilibrium with memory at most $|V| + 2$ (resp. $|V| \cdot |E|^2 + 3$) for payoff functions* InfMP*,* SupMP*,* LimInf*,* LimSup *and* Disc^λ *(resp.* Inf *and* Sup*).*

Theorem 2. *Let $(\mathcal{G}, v_0)$ be an initialized weighted game. Then one can compute a finite-memory secure equilibrium in $(\mathcal{G}, v_0)$ in pseudo-polynomial time (resp. polynomial time) for payoff functions* InfMP*,* SupMP *and* Disc^λ *(resp.* LimInf*,* LimSup*,* Inf *and* Sup*).*

Theorem 3. *Let $(\mathcal{G}, v_0)$ be an initialized weighted game and $\mu, \nu \in (\mathbb{Q} \cup \{\pm\infty\})^2$ be two thresholds. Then one can decide in* $\mathsf{NP} \cap \mathsf{co\text{-}NP}$ *(resp. in* P*) whether there exists a secure equilibrium (σ_1, σ_2) in $(\mathcal{G}, v_0)$ such that $\mu \leq \mathsf{Payoff}(\langle\sigma_1, \sigma_2\rangle_{v_0}) \leq \nu$ for payoff functions* InfMP *and* SupMP *(resp.* LimInf*,* LimSup*,* Inf *and* Sup*).*

3 Lexicographic Payoff Games and Equilibria

To solve Problems 1-3, we follow the approach proposed in [7] to solve the first problem for Nash equilibria (instead of secure equilibria). The general idea is the following one. Given an initialized (non zero-sum) weighted game $(\mathcal{G}, v_0)$, we derive two well-chosen two-player (zero-sum) games $\mathcal{G}^{\preccurlyeq_1}$ and $\mathcal{G}^{\preccurlyeq_2}$, and under adequate hypotheses, we obtain properties about secure equilibria in $(\mathcal{G}, v_0)$ through determinacy results and characterization of the optimal strategies of $\mathcal{G}^{\preccurlyeq_1}$ and $\mathcal{G}^{\preccurlyeq_2}$.

In this section, we first introduce the games $\mathcal{G}^{\preccurlyeq_1}$ and $\mathcal{G}^{\preccurlyeq_2}$. Then for each of the three studied problems, we propose a general framework (i.e. adequate general hypotheses on $(\mathcal{G}, v_0)$, $\mathcal{G}^{\preccurlyeq_1}$ and $\mathcal{G}^{\preccurlyeq_2}$) under which we are able to solve the considered problem (Theorems 4-6).

Later in Sections 4 and 5, we will prove that these hypotheses are satisfied for most of the weighted games with the payoff functions of Definition 2. We will thus be able to prove Theorems 1-3 as a consequence of the general Theorems 4-6.

3.1 Lexicographic Payoff Games

Definition 4. *From a weighted game $\mathcal{G} = (V, V_1, V_2, E, r, \mathsf{Payoff})$ as in Definition 1, we derive a zero-sum* lexicographic payoff game $\mathcal{G}^{\preccurlyeq_1}$ *of the form $(V, V_1, V_2, E, r, \mathsf{Payoff}, \preccurlyeq_1)$ such that the lexicographic order $\preccurlyeq_1$ is used to compare payoffs of the game.*

In the zero-sum game $\mathcal{G}^{\preccurlyeq_1}$, the two players have antagonistic goals. For each play $\rho \in V^\omega$, player 1 receives the *payoff* $\mathsf{Payoff}(\rho)$ that he wants to maximize w.r.t. the lexicographic order $\preccurlyeq_1$, while player 2 pays the *cost* $\mathsf{Payoff}(\rho)$ that he wants to minimize w.r.t. $\preccurlyeq_1$. When Payoff is one among the payoff functions of Definition 2, we say that $\mathcal{G}^{\preccurlyeq_1}$ is a Payoff *lexicographic payoff game*.

In the sequel, we need to consider a dual lexicographic payoff game $\mathcal{G}^{\preccurlyeq_2} = (V, V_2, V_1, E, r, \mathsf{Payoff}, \preccurlyeq_2)$ where the roles of the two players are exchanged and the used lexicographic order is $\preccurlyeq_2$. In this game player 2 wants to maximize $\mathsf{Payoff}(\rho)$ w.r.t. $\preccurlyeq_2$, while player 1 wants to minimize it.

Definition 5. *Given a lexicographic payoff game $\mathcal{G}^{\preccurlyeq_1}$, we define for every vertex $v \in V$ the* upper value $\overline{\mathrm{Val}}(v)$ *and the* lower value $\underline{\mathrm{Val}}(v)$ *respectively as:*

$$\overline{\mathrm{Val}}(v) = \inf_{\sigma_2 \in \Sigma_2} \sup_{\sigma_1 \in \Sigma_1} \mathsf{Payoff}(\langle \sigma_1, \sigma_2 \rangle_v),$$
$$\underline{\mathrm{Val}}(v) = \sup_{\sigma_1 \in \Sigma_1} \inf_{\sigma_2 \in \Sigma_2} \mathsf{Payoff}(\langle \sigma_1, \sigma_2 \rangle_v).$$

The game $\mathcal{G}^{\preccurlyeq_1}$ is determined *if, for every $v \in V$, we have $\overline{\mathrm{Val}}(v) = \underline{\mathrm{Val}}(v)$. We also say that $\mathcal{G}^{\preccurlyeq_1}$ has a* value *from v, and we write $\mathrm{Val}(v) = \overline{\mathrm{Val}}(v) = \underline{\mathrm{Val}}(v)$.*

In the previous definition, let us remind that the infimum and supremum functions are applied on a set of payoffs lexicographically ordered with $\preccurlyeq_1$.

Definition 6. *Given a lexicographic payoff game* $\mathcal{G}^{\preccurlyeq_1}$ *and a vertex* $v \in V$*, we say that* $\sigma_1^\star \in \Sigma_1$ *is an* optimal strategy *for player 1 if, for each strategy* $\sigma_2 \in \Sigma_2$ *and each vertex* $v \in V$*, we have*

$$\underline{\mathrm{Val}}(v) \preccurlyeq_1 \mathsf{Payoff}(\langle \sigma_1^\star, \sigma_2 \rangle_v).$$

Similarly, $\sigma_2^\star \in \Sigma_2$ *is an optimal strategy for player 2 if, for each strategy* $\sigma_1 \in \Sigma_1$ *and each vertex* $v \in V$*, we have*

$$\mathsf{Payoff}(\langle \sigma_1, \sigma_2^\star \rangle_v) \preccurlyeq_1 \overline{\mathrm{Val}}(v).$$

We say that $\mathcal{G}^{\preccurlyeq_1}$ *is* positionally-determined *if it is determined and has positional optimal strategies for both players and all vertices. Additionally, we say that* $\mathcal{G}^{\preccurlyeq_1}$ *is* uniformly-determined *if the positional optimal strategies* $\sigma_1^\star, \sigma_2^\star$ *can be chosen globally independently of the initial vertex* v*. We call these strategies* uniform.

Example 2. We come back to the weighted game $\mathcal{G}$ of Figure 1, and consider the InfMP lexicographic payoff game $\mathcal{G}^{\preccurlyeq_1}$. Let us show that this game is uniformly-determined. Indeed each vertex has a value. In case of vertices v_1, v_3 and v_4, the value is trivially the weight of the loop on these vertices. Vertex v_2 has value $(3,2)$ since the worse that player 2 can do is to play strategy σ_2' ($(3,2) \prec_1 (4,3)$). Vertex v_0 has value $(4,4)$ (realised by strategy σ_1 of player 1). The positional optimal strategies are σ_1 and σ_2' for player 1 and player 2 respectively, they are uniform.

Additionally to the notion of lexicographic payoff game, we need to define the next properties in a way to solve Problems 1-3.

Definition 7 ([7]). *Let* $\mathcal{G} = (V, V_1, V_2, E, r, \mathsf{Payoff})$ *be a weighted game. The payoff function* Payoff_i*,* $i \in \{1,2\}$*, is* prefix-linear *in* $\mathcal{G}$ *if, for every vertex* $v \in V$ *and history* $hv \in V^+$*, there exists* $a \in \mathbb{R}$ *and* $b \in \mathbb{R}^+$ *such that, for every play* $\rho \in V^\omega$ *whose first vertex is* v*, we have :*

$$\mathsf{Payoff}_i(h\rho) = a + b \cdot \mathsf{Payoff}_i(\rho).$$

Moreover, Payoff_i *is* prefix-independent *if for all* $v \in V$*,* $hv \in V^+$ *and* $\rho \in V^\omega$ *whose first vertex is* v*, we have* $\mathsf{Payoff}_i(h\rho) = \mathsf{Payoff}_i(\rho)$*.*

Clearly the notion of prefix-independent payoff function is a particular case of that of prefix-linear.

Remark 1. The two components of the payoff functions LimInf, LimSup, InfMP and SupMP are clearly prefix-independent.

For payoff function Disc^λ, the components are not prefix-independent but rather prefix-linear (see [7]), since for $h = \rho_0 \dots \rho_{n-1}$ and $\rho = \rho_n \rho_{n+1} \dots$, we have for $i \in \{1,2\}$ that $\mathsf{Disc}_i^\lambda(h\rho) = (1-\lambda) \cdot \sum_{k=0}^{\infty} \lambda^k \cdot r_i(\rho_k, \rho_{k+1}) = (1-\lambda) \cdot \sum_{k=0}^{n-1} \lambda^k \cdot r_i(\rho_k, \rho_{k+1}) + \lambda^n \cdot \mathsf{Disc}_i^\lambda(\rho)$.

Finally neither Inf nor Sup functions have prefix-linear components. Indeed, it is easy to find a history h and two plays ρ, ρ' such that $\mathsf{Inf}(h\rho) < \mathsf{Inf}(\rho), \mathsf{Inf}(\rho')$ with $\mathsf{Inf}(\rho) \neq \mathsf{Inf}(\rho')$, which implies that $a = \mathsf{Inf}(h\rho)$ and $b = 0$ in Definition 7. We then get a contradiction by taking ρ'' such that $\mathsf{Inf}(\rho'') < a$.

3.2 Existence of Secure Equilibria

We can now state the general framework under which Problem 1 is positively solved. This framework is composed of all initialized weighted games $(\mathcal{G}, v_0)$ such that each lexicographic payoff game $\mathcal{G}^{\preccurlyeq_i}$ is uniformly-determined, and each payoff function Payoff_i is prefix-linear, with $i \in \{1, 2\}$.

Theorem 4. *Let $\mathcal{G} = (V, V_1, V_2, E, r, \mathsf{Payoff})$ be a weighted game and v_0 be an initial vertex. If for each $i \in \{1, 2\}$, Payoff_i is prefix-linear, and $\mathcal{G}^{\preccurlyeq_i}$ is uniformly-determined, then there exists a finite-memory secure equilibrium in $(\mathcal{G}, v_0)$ with memory size at most $|V| + 2$.*

This theorem is an adaptation to secure equilibria in two-player games of a theorem[3] given in [7,14] for the existence of Nash equilibria in multi-player games. The main difference is that we here need to work with (two-dimensional) lexicographic payoff games instead of classical (one-dimensional, non lexicographic) zero-sum quantitative games. The proof of this theorem is similar to the one given in [7,14]. Nevertheless, we give this proof in a way to have a self-contained paper. In the next sections, we show that the hypotheses of Theorem 4 are satisfied for most of payoff functions in Definition 2.

Proof. Let $\mathcal{G} = (V, V_1, V_2, E, r, \mathsf{Payoff})$ be a weighted game and let $v_0 \in V$ be an initial vertex. We know that the lexicographic payoff game $\mathcal{G}^{\preccurlyeq_i}$ is uniformly-determined for each $i \in \{1, 2\}$. Let us fix some notation. In $\mathcal{G}^{\preccurlyeq_i}$, player i wants to maximize his payoff against the other player who wants to minimize it. To emphasize this situation, we denote by player $-i$ (instead of player $3 - i$) the player that is opposed to player i in $\mathcal{G}^{\preccurlyeq_i}$. Moreover we denote by $\sigma_i^\star$ and $\sigma_{-i}^\star$ the uniform optimal strategies for player i and $-i$ respectively in $\mathcal{G}^{\preccurlyeq_i}$. In other words, $\sigma_1^\star$ denotes a uniform optimal strategy of player 1 in $\mathcal{G}^{\preccurlyeq_1}$, and $\sigma_{-1}^\star$ denotes a uniform optimal strategy of player 2 in $\mathcal{G}^{\preccurlyeq_1}$. Similarly, $\sigma_2^\star$ (resp. $\sigma_{-2}^\star$) denotes a uniform optimal strategy of player 2 (resp. player 1) in $\mathcal{G}^{\preccurlyeq_2}$.

We first show that there exists a secure equilibrium (τ_1, τ_2) in $(\mathcal{G}, v_0)$, as follows: each player i plays according to his strategy $\sigma_i^\star$ in $\mathcal{G}^{\preccurlyeq_i}$, and punishes the other player j if he is the first player to deviate from his strategy $\sigma_j^\star$ in $\mathcal{G}^{\preccurlyeq_j}$, by playing according to $\sigma_{-j}^\star$ in $\mathcal{G}^{\preccurlyeq_j}$.

Formally, let $\rho = \langle \sigma_1^\star, \sigma_2^\star \rangle_{v_0}$ be the outcome of the optimal strategies $(\sigma_1^\star, \sigma_2^\star)$ from v_0. We need to specify a punishment function $P : V^+ \to \{\perp, 1, 2\}$ that detects who is the first player to deviate from the play ρ, i.e. who has to be punished. For the initial vertex v_0, we define $P(v_0) = \perp$ and for every history $hv \in V^+V$ starting in v_0, we let:

$$P(hv) = \begin{cases} \perp & \text{if } P(h) = \perp \text{ and } hv \text{ is a prefix of } \rho, \\ i & \text{if } P(h) = \perp, hv \text{ is not a prefix of } \rho, \text{ and } h \in V^*V_i, \\ P(h) & \text{otherwise } (P(h) \neq \perp). \end{cases}$$

[3] In Theorem 10 of [7], one hypothesis is missing: determinacy must be uniform determinacy.

For each player $i \in \{1,2\}$ we define the strategy τ_i such that for all $hv \in V^* V_i$,

$$\tau_i(hv) = \begin{cases} \sigma_i^\star(v) & \text{if } P(hv) = \bot \text{ or } i, \\ \sigma_{-j}^\star(v) & \text{otherwise where } j = 3 - i \text{ is the other player.} \end{cases} \tag{1}$$

Clearly, the outcome of (τ_1, τ_2) is the play $\rho = \langle \sigma_1^\star, \sigma_2^\star \rangle_{v_0}$. Let us show that (τ_1, τ_2) is a secure equilibrium in $(\mathcal{G}, v_0)$. We first prove that player 1 has no profitable deviation. As a contradiction, let us assume that τ_1' is a profitable deviation for player 1 w.r.t. (τ_1, τ_2). We thus have that:

$$\mathsf{Payoff}(\rho) \prec_1 \mathsf{Payoff}(\rho') \tag{2}$$

with $\rho' = \langle \tau_1', \tau_2 \rangle_{v_0}$.

Let $hv \in V^*V$ be the longest prefix common to ρ and ρ'. This prefix exists and is finite as both plays ρ and ρ' start from vertex v_0 and $\rho \neq \rho'$. As the optimal strategies $\sigma_1^\star, \sigma_2^\star$ are uniform, we can write that $\rho = h \langle \sigma_1^\star, \sigma_2^\star \rangle_v$. In the case of ρ', player 1 does not follow his strategy $\sigma_1^\star$ any more from vertex v, and so, player 2 punishes him by playing according to his optimal strategy $\sigma_{-1}^\star$ after history hv. Therefore we have $\rho' = h \left\langle \tau_1'|_h, \sigma_{-1}^\star \right\rangle_v$.

As the payoff functions are prefix-linear, there exist $a = (a_1, a_2) \in \mathbb{R}^2$ and $b = (b_1, b_2) \in (\mathbb{R}^+)^2$ such that

$$\mathsf{Payoff}(\rho') = a + b \cdot \mathsf{Payoff}\left(\left\langle \tau_1'|_h, \sigma_{-1}^\star \right\rangle_v \right), \; and$$

$$\mathsf{Payoff}(\rho) = a + b \cdot \mathsf{Payoff}\left(\langle \sigma_1^\star, \sigma_2^\star \rangle_v \right).$$

Since $\sigma_{-1}^\star$ is an optimal strategy for player 2 in the lexicographic payoff game $\mathcal{G}^{\preccurlyeq_1}$, we have:

$$\mathsf{Payoff}(\rho') \preccurlyeq_1 a + b \cdot \mathrm{Val}^1(v), \tag{3}$$

where $\mathrm{Val}^1(v)$ is the value of v in $\mathcal{G}^{\preccurlyeq_1}$. Furthermore, as $\sigma_1^\star$ is an optimal strategy for player 1 in $\mathcal{G}^{\preccurlyeq_1}$, it follows that:

$$a + b \cdot \mathrm{Val}^1(v) \preccurlyeq_1 \mathsf{Payoff}(\rho). \tag{4}$$

From Equations (3) and (4) we have that $\mathsf{Payoff}(\rho') \preccurlyeq_1 \mathsf{Payoff}(\rho)$ in contradiction with Equation (2). This proves that player 1 has no profitable deviation. We can show that player 2 has not profitable deviation in the same way.

We now prove that (τ_1, τ_2) is a finite-memory strategy profile such that the memory size of both τ_1 and τ_2 is bounded by $|V| + 2$. For this purpose, we define a finite strategy automaton for both players that remembers the play ρ and who has to be punished. As the play ρ is the outcome of the uniform strategy profile $(\sigma_1^\star, \sigma_2^\star)$, we can write $\rho = v_0 \dots v_{k-1}(v_k \dots v_n)^\omega$ where $0 \leq k \leq n \leq |V|$, $v_l \in V$ for all $0 \leq l \leq n$ and these vertices are all different. For any $i \in \{1,2\}$, let $\mathcal{M}_i = (M, m_0, V, \delta, \nu)$ be the strategy automaton of player i, where[4]:

[4] In this definition, vertex v_{n+1} means vertex v_k.

- $M = \{v_0v_0, v_0v_1, \ldots, v_{n-1}v_n, v_nv_k\} \cup \{1,2\}$. With M we remember the next edge that should be chosen to be sure that both players follow ρ, or the player $j \in \{1,2\}$ that has first deviated.
- $m_0 = v_0v_0$. This memory element means that the play has not begun yet.
- $\delta : M \times V \to M$ is defined for all $m \in M$ and $v \in V$ as follows:

$$\delta(m,v) = \begin{cases} v_lv_{l+1} & \text{if } m = uv_l \text{ and } v = v_l, \text{ with } u \in V, l \in \{0,\ldots,n\} \\ \\ j & \text{if } m = j \in \{1,2\} \text{ or } (m = uv_l, v \neq v_l, \\ & \text{ with } u \in V_j, l \in \{0,\ldots,n\}) \end{cases}$$

  Function δ udpates the memory either to the next edge of ρ in case of no deviation, or to j if player j was the first to deviate.
- $\nu : M \times V_i \to V$ is defined for all $m \in M$ and $v \in V_i$ as follows:

$$\nu(m,v) = \begin{cases} \sigma_i^\star(v) & \text{if } \delta(m,v) = v_lv_{l+1}, \text{ with } l \in \{0,\ldots,n\} \\ \sigma_{-j}^\star(v) & \text{if } \delta(m,v) = j \in \{1,2\} \end{cases}$$

  Function ν proposes to play according to Equation (1).

Obviously, the strategy $\sigma_{\mathcal{M}_i}$ computed by the strategy automaton $\mathcal{M}_i$ exactly corresponds to the strategy τ_i of the secure equilibrium. Notice that in the definition of M, we can forget the memory element i. Indeed if player i follows the strategy computed by $\mathcal{M}_i$, then he never deviates from the play ρ. Therefore, M has size at most $|V| + 2$, and strategy τ_i requires a memory of size at most $|V| + 2$. □

Example 3. We come back again to the weighted game $\mathcal{G}$ of Figure 1 with the initial vertex v_0. Uniform optimal strategies in game $\mathcal{G}^{\preceq_1}$ are $\sigma_1^\star = \sigma_1$ and $\sigma_{-1}^\star = \sigma_2'$ (we use the notations of the previous proof). In $\mathcal{G}^{\preceq_2}$, they are $\sigma_{-2}^\star = \sigma_1'$ and $\sigma_2^\star = \sigma_2$. We thus obtain the secure equilibrium (τ_1, τ_2) leading to the outcome $\rho = \langle \sigma_1, \sigma_2 \rangle_{v_0}$. If player 1 deviates from ρ (in order to decrease player 2 payoff), then player 2 punishes him by playing σ_2' which is the worst for player 1 (since it decreases his payoff).

Notice that the proof of Theorem 4 also holds for payoff functions that mix different measures for the two players, like for example InfMP_1 for player 1 and Sup_2 for player 2.

We can also prove a theorem similar to Theorem 4 such that the hypotheses are replaced by the following ones: $(\mathcal{G}, v_0)$ is an initialized weighted game such that for each $i \in \{1,2\}$, $\mathcal{G}^{\preceq_i}$ is determined (instead of uniformly-determined) and Payoff_i is prefix-independent (instead of prefix-linear). The same kind of result is proved in [14] in the case of Nash equilibria. The proof can be easily adapted to the case of secure equilibria as done for the proof of Theorem 4.

3.3 Construction of Secure Equilibria

In the previous section, we have proposed a rather general framework of weighted games for which Problem 1 has a positive answer. In this section we show that we can solve Problem 2 in the same general framework, with the additional hypothesis that one can compute uniform optimal strategies in $\mathcal{G}^{\preceq_i}$, for $i \in \{1,2\}$.

Theorem 5. *Let $(\mathcal{G}, v_0)$ be an initialized weighted game such that for each $i \in \{1,2\}$, Payoff_i is prefix-linear, and $\mathcal{G}^{\preceq_i}$ is uniformly-determined with computable uniform optimal strategies. If such strategies can be computed in C time for both players, then a finite-memory secure equilibrium in $(\mathcal{G}, v_0)$ can also be constructed in C time.*

Proof. By Theorem 4, we know that there exists a finite-memory secure equilibrium (τ_1, τ_2) in $(\mathcal{G}, v_0)$. Moreover the proof of this theorem indicates how to construct (τ_1, τ_2) from uniform optimal strategies in $\mathcal{G}^{\preceq_1}$ and $\mathcal{G}^{\preceq_2}$ (see (1) and the finite strategy automaton proposed for both players at the end of this proof). It follows that a finite-memory secure equilibrium can be constructed in C time. □

3.4 Constrained Existence of Secure Equilibria

Let us turn to Problem 3. First notice that contrarily to Problem 2, strategy profiles for solving this problem require infinite memory, as shown by the next example.

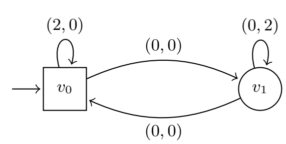

Fig. 2. An weighted game that shows the need of infinite memory for Problem 3

Example 4. Consider the initialized InfMP weighted game $(\mathcal{G}, v_0)$ of Figure 2. Take thresholds $\mu = (1,1)$ and $\nu = (+\infty, +\infty)$.

As explained in [31][Proof of Lemma 7], any strategy profile that is finite-memory produces an ultimately periodic outcome ρ such that $\mathsf{Payoff}_1(\rho) + \mathsf{Payoff}_2(\rho) < 1$, thus not satisfying threshold $(1,1)$. However there exists an infinite path ρ^* in $(\mathcal{G}, v_0)$ with payoff equal to $(1,1)$: ρ^* visits n times vertex v_0 and then n times vertex v_1, and repeats this forever with increasing value of n.

Let us define a strategy profile (σ_1, σ_2) such that its outcome is equal to ρ^*, and as soon as one player deviates from ρ^*, the other player punishes him by always chosing an edge with weights $(0,0)$. This strategy profile is a secure equilibrium since one verifies that the player who first deviates receives a payoff of 0 instead of 1.

Therefore there exists a secure equilibrium (σ_1, σ_2) in $(\mathcal{G}, v_0)$ that satisfies $\mu \leq \mathsf{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}) \leq \nu$, but it cannot be finite-memory.

We show that Problem 3 is decidable in the same framework as for Problem 1, with the additional hypotheses that one can compute values in $\mathcal{G}^{\preccurlyeq_i}$, for $i \in \{1,2\}$, and the next problem is decidable.

Problem 4. Let $G = (V, E, v_0, r, \mathrm{Val})$ be a finite directed graph with an initial vertex v_0, a weight function $r = (r_1, r_2)$ with $r_i : E \to \mathbb{Q}$, and a value function $\mathrm{Val} = (\mathrm{Val}^1, \mathrm{Val}^2)$, with $\mathrm{Val}^i : V \to \mathbb{Q}$. Let $\mu, \nu \in (\mathbb{Q} \cup \{\pm\infty\})^2$ be two thresholds. Is it decidable whether there exists an infinite path ρ in G starting in v_0 such that

- $\forall k \geq 0, \forall i \in \{1,2\}, \mathrm{Val}^i(\rho_k) \preccurlyeq_i \mathsf{Payoff}(\rho_{\geq k})$, and
- $\mu \leq \mathsf{Payoff}(\rho) \leq \nu$?

Theorem 6. *Let $(\mathcal{G}, v_0)$ be an initialized weighted game and $\mu, \nu \in (\mathbb{Q} \cup \{\pm\infty\})^2$ be two thresholds. Suppose that*

- *for each $i \in \{1,2\}$, Payoff_i is prefix-linear, and $\mathcal{G}^{\preccurlyeq_i}$ is uniformly-determined with computable values,*
- *Problem 4 is decidable for the graph G constructed from $\mathcal{G}$ such that both functions Val^i, $i \in \{1,2\}$, are constructed from the values $\mathrm{Val}^i(v)$, $v \in V$, in $\mathcal{G}^{\preccurlyeq_i}$.*

Then one can decide whether there exists a secure equilibrium (σ_1, σ_2) in $(\mathcal{G}, v_0)$ such that $\mu \leq \mathsf{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}) \leq \nu$.

The proof of this theorem is based on the next lemma that characterizes the outcomes of secure equilibria in $(\mathcal{G}, v_0)$.

Lemma 1. *Let ρ be a play in $\mathcal{G}$ starting in v_0. Then ρ is the outcome of a secure equilibrium in $(\mathcal{G}, v_0)$ if and only if*

$$\forall k \geq 0, \forall i \in \{1,2\}, \quad \mathrm{Val}^i(\rho_k) \preccurlyeq_i \mathsf{Payoff}(\rho_{\geq k}). \tag{5}$$

Proof. As in the proof of Theorem 4, we denote by $\sigma_i^\star$ and $\sigma_{-i}^\star$ the uniform optimal strategies for player i and $-i$ respectively in $\mathcal{G}^{\preccurlyeq_i}$.

First, let (τ_1, τ_2) be a secure equilibrium in $(\mathcal{G}, v_0)$, and $\rho = \langle \tau_1, \tau_2 \rangle_{v_0}$ be its outcome. By contradiction, assume that there exists $k \geq 0$ and $i \in \{1,2\}$ such that $\mathsf{Payoff}(\rho_{\geq k}) \prec_i \mathrm{Val}^i(\rho_k)$. Take such a k as small as possible and let $h = \rho_{\leq k-1}$. Suppose that $i = 1$ (the case $i = 2$ is similar). We can then construct a profitable deviation τ_1' for player 1: he follows the strategy τ_1 until vertex ρ_k from which he uses his optimal strategy $\sigma_1^\star$ in $\mathcal{G}^{\preccurlyeq_1}$. As Payoff is prefix-linear and $\sigma_1^\star$ is uniform, it follows that for some $a \in \mathbb{R}^2$ and $b \in (\mathbb{R}^+)^2$, we have

$$\begin{aligned}
\mathsf{Payoff}(\langle \tau_1, \tau_2 \rangle_{v_0}) &= a + b \cdot \mathsf{Payoff}(\rho_{\geq k}) \\
&\prec_1 a + b \cdot \mathrm{Val}^1(\rho_k) \\
&\preccurlyeq_1 a + b \cdot \mathsf{Payoff}(\langle \sigma_1^\star, \tau_2|_h \rangle_{\rho_k}) \\
&= \mathsf{Payoff}(\langle \tau_1', \tau_2 \rangle_{v_0}).
\end{aligned}$$

This is impossible since (τ_1, τ_2) is a secure equilibrium.

Next, let ρ be a play that starts with v_0 and satisfies (5). We define a strategy profile (τ_1, τ_2) such that $\langle \tau_1, \tau_2 \rangle_{v_0} = \rho$, and as soon as player i deviates, the other player uses his strategy $\sigma^\star_{-i}$ in $\mathcal{G}^{\preccurlyeq_i}$ to punish him. Let us prove that (τ_1, τ_2) is a secure equilibrium in $(\mathcal{G}, v_0)$. Let τ'_1 be a strategy for player 1 such that $\rho' = \langle \tau'_1, \tau_2 \rangle_{v_0} \neq \rho$, and let $h\rho_k$ be the longest common prefix of ρ and ρ'. We have for some $a \in \mathbb{R}^2$ and $b \in (\mathbb{R}^+)^2$:

$$\begin{aligned} \mathsf{Payoff}(\langle \tau'_1, \tau_2 \rangle_{v_0}) &= a + b \cdot \mathsf{Payoff}(\langle \tau'_1|_h, \sigma^\star_{-1} \rangle_{\rho_k}) \\ &\preccurlyeq_1 a + b \cdot \mathrm{Val}^1(\rho_k) \\ &\preccurlyeq_1 a + b \cdot \mathsf{Payoff}(\rho_{\geq k}) \\ &= \mathsf{Payoff}(\rho). \end{aligned}$$

This shows that player 1 has no profitable deviation. The same holds for player 2. Hence (τ_1, τ_2) is a secure equilibrium. □

Proof (of Theorem 6). We propose the next algorithm. First compute $\mathrm{Val}^i(v)$ for each vertex v of $\mathcal{G}^{\preccurlyeq_i}$, $i \in \{1, 2\}$. Then construct from $(\mathcal{G}, v_0)$ the graph $G = (V, E, v_0, r, \mathrm{Val})$. Finally test whether there exists a path ρ in G starting in v_0 such that $\forall k \geq 0, \forall i \in \{1, 2\}, \mathrm{Val}^i(\rho_k) \preccurlyeq_i \mathsf{Payoff}(\rho_{\geq k})$, and $\mu \leq \mathsf{Payoff}(\rho) \leq \nu$. Notice that it is indeed an algorithm since each $\mathcal{G}^{\preccurlyeq_i}$ is computationally uniformly-determined, and Problem 4 is decidable. This algorithm is correct by Lemma 1. □

Again, as for Theorems 4-5, Theorem 6 also holds for distinct payoff functions that mix different measures for the two players

4 Determinacy of Lexicographic Payoff Games

In Section 3, we have established strong links between secure equilibria in initialized weighted games $(\mathcal{G}, v_0)$ and determinacy of the two lexicographic payoff games $\mathcal{G}^{\preccurlyeq_i}$, $i \in \{1, 2\}$. We have shown how to solve Problems 1-3 under some adequate general hypotheses. In this section, we study the determinacy of lexicographic payoff games for the payoff functions proposed in Definition 2. We show that for all payoffs, these games are uniformly-determined, except for the Inf and Sup payoffs for which they are only positionally-determined. We also study the complexity of computing values and optimal strategies for these games. Besides being important ingredients to solve Problems 1-3, these results are also very interesting on their own right. Our results are the following ones (we state them only for $\mathcal{G}^{\preccurlyeq_1}$):

Theorem 7. *Let $\mathcal{G}^{\preccurlyeq_1}$ be a lexicographic payoff game. Then $\mathcal{G}^{\preccurlyeq_1}$ is uniformly-determined (resp. positionally-determined) for payoff functions* InfMP*,* SupMP*,* LimInf*,* LimSup *and* Disc^λ *(resp.* Inf *and* Sup*).*

Theorem 8. *Let $\mathcal{G}^{\preccurlyeq_1}$ be a lexicographic payoff game.*

1. *Let v be a vertex of $\mathcal{G}^{\preccurlyeq_1}$ and $(\alpha, \beta) \in \mathbb{Q}^2$ be a pair of rationals. Deciding whether $(\alpha, \beta) \preccurlyeq_1 \mathrm{Val}(v)$ is in* $\mathsf{NP} \cap \mathsf{co\text{-}NP}$ *for payoff functions* InfMP, SupMP *and* Disc^λ, *and* P*-complete for payoff functions* LimInf, LimSup, Inf *and* Sup.
2. *The (rational) value of each vertex of $\mathcal{G}^{\preccurlyeq_1}$ can be computed in pseudo-polynomial time for payoff functions* InfMP, SupMP *and* Disc^λ, *and in polynomial time for payoff functions* LimInf, LimSup, Inf *and* Sup.
3. *Uniform (resp. positional) optimal strategies for both players can be computed in pseudo-polynomial time for payoff functions* InfMP, SupMP *and* Disc^λ, *and in polynomial time for payoff functions* LimInf *and* LimSup *(resp.* Inf *and* Sup*).*

Theorem 7 states that every Inf lexicographic payoff game is positionally-determined. The next example shows that there is no hope to have uniformly-determined such games.

Example 5. Consider the Inf lexicographic payoff game depicted in Figure 3. Clearly, this game is positionally-determined, such that each vertex has value $(2, 0)$ except vertex v_4 that has value $(3, 1)$. Let us show that it is not uniformly-determined. To guarantee value $(2, 0)$ for v_0, player 1 has to use the positional strategy $\sigma_1^\star$ such that $\sigma_1^\star(v_4) = v_3$. Indeed, this is the only way to have $(2, 0) \preccurlyeq_1 \mathsf{Inf}(\langle \sigma_1^\star, \sigma_2 \rangle_{v_0})$ for all strategy σ_2 of player 2. However, with this strategy, player 1 cannot guarantee value $(3, 1)$ for v_4.

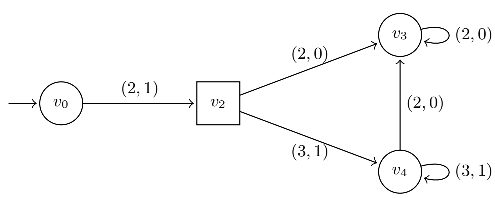

Fig. 3. An Inf lexicographic payoff game that is not uniformly-determined.

Before giving the proofs of Theorems 7 and 8, notice that with these results, we are now able to provide a proof of Theorems 1 and 2, except for Inf and Sup payoffs for which we give a separate proof.

Proof (of Theorems 1 and 2, except for Inf *and* Sup *payoffs).* Let $(\mathcal{G}, v_0)$ be an initialized weighted game with a payoff function equal to one among InfMP, SupMP, LimInf, LimSup and Disc^λ. By Remark 1, this function is prefix-linear. Moreover, by Theorem 7, each game $\mathcal{G}^{\preccurlyeq_i}$, $i \in \{1, 2\}$, is uniformly-determined. Therefore, since the hypotheses of Theorem 4 are satisfied, there exists a finite-memory secure equilibrium in $(\mathcal{G}, v_0)$ with memory size at most $|V|+2$, leading to Theorem 1. The complexities stated in Theorem 2 are obtained as a consequence of Theorems 5 and 8. $\square$

The proof of Theorem 1 for Inf and Sup payoffs cannot be based on Theorem 4 since none of its hypotheses is satisfied. Indeed, the Inf payoff function is not prefix-linear (see Remark 1), and there exist Inf lexicographic payoff games that are not uniformly-determined (see Example 5). However the case of Inf and Sup payoffs can be solved thanks to a reduction to LimInf and LimSup payoffs as follows.

Proof (of Theorems 1 and 2 for Inf *and* Sup *payoffs).* Let $\mathcal{G} = (V, V_1, V_2, E, r, \mathsf{Inf})$ be an Inf weighted game and v_0 be an initial vertex (the proof is similar for Sup payoff). Let $R = \{r_1(e) \mid e \in E\} \cup \{r_2(e) \mid e \in E\} \cup \{+\infty\}$ be the set of weights labeling the edges of E with, in addition, a highest new weight $+\infty$. We construct an initialized LimInf weighted game $(\mathcal{G}', v_0')$ with $\mathcal{G}' = (V', V_1', V_2', E', r', \mathsf{LimInf})$ as follows:

- $V' = V \times R \times R$ is partitioned into $V_1' = V_1 \times R \times R$ and $V_2' = V_2 \times R \times R$,
- $E' \subseteq V' \times V'$ is the set of edges of E augmented with the current smallest weights, that is, $e' = ((v, w_1, w_2), (v', w_1', w_2')) \in E'$ iff $e = (v, v') \in E$ and $w_i' = \min\{w_i, r_i(e)\}$, with $i \in \{1, 2\}$,
- $r' = (r_1', r_2')$ is the weight function that also remembers the current smallest weights, that is, for each $i \in \{1, 2\}$, $r_i'(e') = w_i'$ for each edge $e' = ((v, w_1, w_2), (v', w_1', w_2')) \in E'$,
- $v_0' = (v_0, +\infty, +\infty)$ is the initial vertex augmented with the highest weight $+\infty$ for both components.

Clearly, to any play ρ in $(\mathcal{G}, v_0)$ corresponds a unique play ρ' in $(\mathcal{G}', v_0')$, and conversely. Moreover, by construction of $\mathcal{G}'$, for all $n > 0$, we have $\rho_n' = (\rho_n, w_n^1, w_n^2)$ with

$$w_n^i = \min_{0 \leq k < n} r_i((\rho_k, \rho_{k+1}))$$

for $i \in \{1, 2\}$. Since there is a finite number of possible weight values, the decreasing weight components of ρ' eventually stabilise along the play. It follows that $\mathsf{Inf}(\rho) = \mathsf{LimInf}(\rho')$.

Clearly any finite-memory secure equilibrium in $(\mathcal{G}, v_0)$ is a finite-memory secure equilibrium in $(\mathcal{G}', v_0')$, and conversely. By Theorems 1 and 2 for LimInf payoff, $(\mathcal{G}', v_0')$ has a secure equilibrium with memory at most $|V'| + 2$ that can be computed in polynomial time. Since $|V'| \leq |V| \cdot |E|^2 + 1$, we can conclude that $(\mathcal{G}, v_0)$ has a secure equilibrium with memory at most $|V| \cdot |E|^2 + 3$ that can also be computed in polynomial time. $\square$

In the next sections 4.2-4.5, we are going to prove Theorems 7 and 8 for the various payoff functions. In this aim, let us first briefly recall some useful notions and results about several classes of one-dimensional (instead of two-dimensional) games.

4.1 Useful Background

Mean-payoff games (resp. *Liminf games*, *discounted games*) are games as in Definition 4 except that functions r and $\mathsf{Payoff} = \mathsf{InfMP}$ (resp. LimInf, Disc^λ) are one-dimensional instead of two-dimensional. The comparison of two payoffs is thus done with usual order $\leq$ instead of $\preccurlyeq_1$. As for lexicographic payoff games, the notions of value and optimal strategies can be defined for such games. It is a classical result that mean-payoff games are uniformly-determined [15,32]. Moreover deciding whether the value of a vertex is greater than or equal to a given threshold is in in $\mathsf{NP} \cap \mathsf{co\text{-}NP}$, there exists a pseudo-polynomial time algorithm to compute the values of each vertex and optimal strategies for both players [32]. The same results hold for discounted games [32,19]. LimInf games are known to be uniformly-determined [8].

In [15,1], the authors study mean-payoff games by considering the finite duration version of these games, such that a play is stopped as soon as some vertex is repeated. More precisely, the arena of a *finite cycle-forming game* $\mathcal{F}$ is a finite directed graph (V, E) such that the vertices are partitioned into V_1 and V_2, and the edges are labeled by weights in $\mathbb{Q}$. The game starts in some initial vertex v_0, and players play in a turn-based manner until some vertex v is visited for the second time. At this point, the game stops, a history hc called *$\mathcal{F}$-play*, has been constructed by both players with c a cycle from v to v, and the cycle c of this $\mathcal{F}$-play is analyzed. A payoff is then associated with hc depending on the edge weights of c. The next theorem is proved in [1][Section 7.2].

Theorem 9. *A finite cycle-forming game $\mathcal{F}$ such that the payoff of an $\mathcal{F}$-play depends only on the vertices that appear in the formed cycle (modulo cyclic permutations) is uniformly-determined.*

We also need the next proposition about the cycle decomposition of an infinite path in an arena (V, E). Given a path ρ, we consider its *cycle decomposition* into a multiset of simple cycles as follows.[5] We push successively vertices $\rho_0, \rho_1, \ldots$ onto a stack. Whenever we push a vertex ρ_m equal to a vertex ρ_n in the stack (i.e. a simple cycle $\rho_n \ldots \rho_m$ is formed), we remove it from the stack (i.e. we remove all the vertices above ρ_n, but not ρ_n) and add it to the cycle decomposition multiset of ρ. Notice that at any moment, the stack contains the vertices of a simple path, thus at most $|V|$ vertices.

Proposition 1. *Let (V, E) be an arena. For each infinite path ρ in (V, E) and its cycle decomposition, the next properties hold. Let $\mathcal{F}$ be a finite cycle-forming game with arena (V, E).*

1. *If c is a cycle in the cycle decomposition of ρ, then c appears in an $\mathcal{F}$-play hc in the game $\mathcal{F}$, for some history h starting in ρ_0 and composed of edges of ρ.*
2. *There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, (ρ_n, ρ_{n+1}) is an edge of a cycle that appears in an $\mathcal{F}$-play hc in the game $\mathcal{F}$, for some history h starting in ρ_0 and composed of edges of ρ.*

Proof. Let us begin with the first statement. During the construction of the cycle decomposition multiset of ρ, consider the time when cycle c is formed, and let h

[5] We can similarly consider the cycle decomposition of finite paths ρ.

be the path composed by the vertices currently stored in the stack. Then hc is the required $\mathcal{F}$-play in $\mathcal{F}$.

For the second statement, as ρ is infinite, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, each vertex ρ_n is infinitely often repeated in $\rho_{\geq n_0}$. Consider the edge (ρ_n, ρ_{n+1}), with $n \geq n_0$. When $q = \rho_n$ is put on the stack, either a cycle is formed and removed from the stack, or a cycle is not yet been formed. In both cases, q is at the top of the stack. Then $q' = \rho_{n+1}$ is pushed on the stack above q, as well as $\rho_{n+2}, \rho_{n+3}, \ldots$ and formed simple cycles are successively removed. At a certain point, a simple cycle c containing the edge (q, q') will be formed since q appears infinitely often along ρ. By the first statement of Proposition 1, the proof is completed. $\square$

Let us now recall some useful properties about *reachability games* (resp. *safety games*, *co-Büchi* games, *Rabin games*)[19,16,26]. They are two-player zero-sum games played on an arena which is a finite directed graph (V, E) with no weights. A set $A \subseteq V$ of vertices is given in the case of reachability, safety and co-Büchi games, whereas a finite set of pairs (A_k, B_k), with $A_k, B_k \subseteq V$ is given in the case of Rabin games. The two players play in a turn-based manner from a given initial vertex v_0. In the case of reachability (resp. safety) game, the produced play ρ is won by player 1 if ρ visits some vertex of A (resp. no vertex of A), and by player 2 otherwise. The set of initial vertices v_0 from which player 1 has a winning strategy can be computed in time $O(|V| + |E|)$, and a winning strategy can be chosen positional and computed in time $O(|V| + |E|)$ [19]. Moreover deciding whether player 1 has a winning strategy from an initial vertex v_0 is known to be P-complete [20]. For co-Büchi games, play ρ is won by player 1 if $\mathsf{Inf}(\rho) \cap A = \emptyset$ (with $\mathsf{Inf}(\rho)$ being the set of vertices infinitely visited along ρ), and by player 2 otherwise. For Rabin games, this condition is replaced by $\exists k : \mathsf{Inf}(\rho) \cap A_k = \emptyset \wedge \mathsf{Inf}(\rho) \cap B_k \neq \emptyset$. One can decide if player 1 has a winning strategy in time $O(n^2)$ [11] for co-Büchi games and in time $O(n^{p+1} \cdot p!)$ [25] for Rabin games, with n the number of vertices and p the number of pairs. Moreover for co-Büchi games, this problem is known to be P-complete, as this is already the case for the AND-OR graph reachability problem [20].

We conclude this section with a remark that will be useful when studying the determinacy of lexicographic payoff games.

Remark 2. Without lost of generality we can assume that the game $\mathcal{G}^{\preccurlyeq_1} = (V, V_1, V_2, E, r, \mathsf{Payoff}, \preccurlyeq_1)$ uses a weight function $r = (r_1, r_2)$ such that $r_i : E \to \mathbb{N}$ (instead of $r_i : E \to \mathbb{Q}$) for each $i \in \{1, 2\}$. Indeed all the weights appearing in $\mathcal{G}^{\preccurlyeq_1}$ have the form $\frac{a}{b}$ with $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Let a^* be the smallest weight numerator (0 if weights are positive in $\mathcal{G}^{\preccurlyeq_1}$) and b^* be the least common multiple of the weight denominators. Then we define a new lexicographic payoff game $\mathcal{G}'^{\preccurlyeq_1} = (V, V_1, V_2, E, r', \mathsf{Payoff}, \preccurlyeq_1)$ such that $r'_i(e) = r_i(e) \cdot b^* - a^* \cdot b^*$ for all $e \in E$ and $i \in \{1, 2\}$. For the payoff functions Payoff of Definition 2, given a play ρ in $\mathcal{G}^{\preccurlyeq_1}$, its payoff is multiplied by b^* and shifted by the value $(-a^* \cdot b^*, -a^* \cdot b^*)$ when it is seen as a play in $\mathcal{G}'^{\preccurlyeq_1}$. The same holds for values (if they exist), and positional optimal strategies correspond in both games.

4.2 InfMP and SupMP Lexicographic Payoff Games

In this section we prove that every InfMP lexicographic payoff game is uniformly-determined (see Theorem 7). We also give the proof of Theorem 8 for InfMP payoff. Notice that we do not provide the proofs for SupMP payoff since they are similar.

Theorems 7 and 8 for InfMP payoff are nontrivial extensions to dimension 2 of well-known results about (one-dimensional) mean-payoff games [15,32]. In [2], similar results have been proved for a class of zero-sum games called lexicographic mean-payoff games. It should be noted that this class is different from our class of InfMP lexicographic payoff games. Indeed, the authors use a lexicographic order $\preccurlyeq'_1$ on $\mathbb{R}^2$ different from ours: $(x_1, x_2) \preccurlyeq'_1 (x'_1, x'_2)$ iff $(x_1 < x'_1) \vee (x_1 = x'_1 \wedge x_2 \leq x'_2)$, and most importantly the payoff of a play ρ is computed as $\liminf\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=0}^{n-1} (r_1(\rho_k, \rho_{k+1}), r_2(\rho_k, \rho_{k+1}))$ at the level of two-dimensional vectors and with the $\liminf$ using order $\preccurlyeq'_1$ (whereas we proceed componentwise). An example showing that the two definitions are not equivalent is given in [2][6].

Let us first proceed to the proof of Theorem 7 with InfMP payoff. It is inspired from the proof given in [2], however with several nontrivial adaptations. At the end of this section, we will establish the related complexity stated in Theorem 8.

Uniform-Determinacy. Let $\mathcal{G}^{\preccurlyeq_1} = (V, V_1, V_2, E, r, \mathsf{InfMP}, \preccurlyeq_1)$ be a lexicographic payoff game with a weight function with natural weights (see Remark 2). For each $i \in \{1, 2\}$, we denote by $|r_i| = \max\{r_i(e) \mid e \in E\}$ the maximal weight for component i, and by $|R| = \max\{|r_1|, |r_2|\}$ the maximal weight of the game.

We derive from $\mathcal{G}^{\preccurlyeq_1}$ the mean-payoff game $\mathcal{M} = (V, V_1, V_2, E, r_1, \mathsf{InfMP}_1)$ such that the weights are limited to the first component r_1 of r. Recall that such a game is uniformly-determined [15,1]. Therefore each vertex v has a value $\mathrm{Val}(v)$ in $\mathcal{M}$. Notice that it is not possible for player 1 to move to a vertex with a higher value, and player 1 will never choose to move to a vertex with a lower value. Similarly, player 2 cannot move to a vertex with a lower value and will never decide to move to a vertex with a higher value.

Thus, for each value l, we can consider the subarena[7] of (V, E) restricted to V^l being the set of all vertices $v \in V$ such that $\mathrm{Val}(v) = l$. We denote by $\mathcal{G}^{\preccurlyeq_1}_{[l]}$ the lexicographic subgame of $\mathcal{G}^{\preccurlyeq_1}$ restricted to V^l. We also consider a finite cycle-forming game denoted by $\mathcal{F}^l$ restricted to V^l and with the following payoff function Cost. From a given initial vertex v_0 in $\mathcal{F}^l$, the two players play until a cycle c is formed. Let $\rho = \rho_0 \dots \rho_m \dots \rho_n$ be the constructed $\mathcal{F}^l$-play such that $v_0 = \rho_0$ and $c = \rho_m \dots \rho_n$ with $\rho_n = \rho_m$. We denote for each $i \in \{1, 2\}$

$$\mathsf{MP}_i(c) = \frac{1}{n-m} \sum_{k=m}^{n-1} r_i(\rho_k, \rho_{k+1})$$

[6] This example is given in the ArXiv version of [2], following discussions with one of the authors.

[7] The notion of subarena is classical. Recall that it requires that there is no deadlock.

the mean-payoff of the cycle c according to component i. We define:

$$\mathsf{Cost}(\rho) = \begin{cases} -1 & \text{if } \mathsf{MP}_1(c) > l \\ |r_2| + 1 & \text{if } \mathsf{MP}_1(c) < l \\ \mathsf{MP}_2(c) & \text{if } \mathsf{MP}_1(c) = l. \end{cases}$$

In the finite cycle-forming game $\mathcal{F}^l$, player 1 wants to *minimize* the payoff while player 2 wants to *maximize* it.[8] Recall that $0 \leq \mathsf{MP}_2(c) \leq |r_2|$ since the weights are supposed positive (see Remark 2) and $|r_2|$ denotes the maximal weight for the second component.

By Theorem 9, the finite cycle-forming game $\mathcal{F}^l$ is uniformly-determined. We have the next lemma.

Lemma 2. *Let v be a vertex in $\mathcal{F}^l$ with value*[9] *$\beta^l(v)$, then $0 \leq \beta^l(v) \leq |r_2|$.*

Proof. Let us first prove that $\beta^l(v) \neq |r_2| + 1$. As $\mathcal{M}$ is uniformly-determined, player 1 has a uniform optimal strategy $\sigma_1^\star$ in $\mathcal{M}$, such that for each strategy σ_2 of player 2, $\mathsf{InfMP}_1(\langle\sigma_1^\star, \sigma_2\rangle_v) \geq l$.

Let us show that $\sigma_1^\star$, restricted to V^l and seen as a strategy in $\mathcal{F}^l$, ensures that each $\mathcal{F}^l$-play ρ in $\mathcal{F}^l$ that starts from v and is consistent with $\sigma_1^\star$ satisfies $\mathsf{Cost}(\rho) \neq |r_2| + 1$. It will follow that $\beta^l(v) \neq |r_2| + 1$. Assume the contrary and suppose that player 2 has a strategy σ_2 in $\mathcal{F}^l$ such that for the $\mathcal{F}^l$-play $\rho = \langle\sigma_1^\star, \sigma_2\rangle_v = hc$ with c a cycle, we have $\mathsf{Cost}(\rho) = |r_2| + 1$, or equivalently $\mathsf{MP}_1(c) < l$. With σ_2, extended arbitrarily to vertices of $V \setminus V^l$ and seen as a strategy in $\mathcal{M}$, we have that the infinite play $\rho' = \langle\sigma_1^\star, \sigma_2\rangle_v$ cycles infinitely in c (remind that $\sigma_1^\star$ is uniform). As $\mathsf{InfMP}_1(\rho') = \mathsf{MP}_1(c)$, we get in contradiction with the optimality of $\sigma_1^\star$ in $\mathcal{M}$.

We can show that $\beta^l(v) \neq -1$ with dual arguments with player 2. It follows that $0 \leq \beta^l(v) \leq |r_2|$ by definition of Cost. $\square$

The next lemma (see for instance [9]) will also be useful in Lemma 4.

Lemma 3. *For all sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ of reals numbers, we have:*

$$\begin{aligned} \limsup_{n\to\infty}(a_n + b_n) &\leq \limsup_{n\to\infty}(a_n) + \limsup_{n\to\infty}(b_n) \\ \limsup_{n\to\infty}(a_n - b_n) &\geq \limsup_{n\to\infty}(a_n) - \limsup_{n\to\infty}(b_n) \\ \liminf_{n\to\infty}(a_n + b_n) &\geq \liminf_{n\to\infty}(a_n) + \liminf_{n\to\infty}(b_n) \\ \liminf_{n\to\infty}(a_n - b_n) &\leq \liminf_{n\to\infty}(a_n) - \liminf_{n\to\infty}(b_n). \end{aligned}$$

Lemma 4 relates values in game $\mathcal{F}^l$ to values in game $\mathcal{G}^{\preccurlyeq_1}_{[l]}$ (notice that it is not the whole game $\mathcal{G}^{\preccurlyeq_1}$), as well as for optimal strategies. It constitutes the last step to get Theorem 7.

[8] Here the payoff is rather a cost as Player 1 wants to minimize it.

[9] It is its value in $\mathcal{F}^l$, not to be confused with its value l in $\mathcal{M}$. We use superscript l in $\beta^l(v)$ to recall the underlying game $\mathcal{F}^l$.

Lemma 4. *1. Let v be a vertex such that its value in $\mathcal{M}$ is* $\mathrm{Val}(v) = l$ *and its value in $\mathcal{F}^l$ is $\beta^l(v)$. Then its value in $\mathcal{G}_{[l]}^{\preccurlyeq_1}$ is* $(\mathrm{Val}(v), \beta^l(v))$.

2. A uniform optimal strategy in $\mathcal{F}^l$ is optimal in $\mathcal{G}_{[l]}^{\preccurlyeq_1}$.

Proof. By Theorem 9, the finite cycle-forming game $\mathcal{F}^l$ is uniformly-determined. A uniform optimal strategy $\tau_1^\star$ (resp. $\tau_2^\star$) thus exists for player 1 (resp. player 2) in $\mathcal{F}^l$. To prove the lemma, we show that for each vertex $v \in V^l$, strategy $\tau_1^\star$ ensures at least[10] the value $(\mathrm{Val}(v), \beta^l(v))$ in $\mathcal{G}_{[l]}^{\preccurlyeq_1}$, and that strategy $\tau_2^\star$ ensures at most $(\mathrm{Val}(v), \beta^l(v))$ in $\mathcal{G}_{[l]}^{\preccurlyeq_1}$. We first prove this claim for player 1 and then for player 2.

Let us fix strategy $\tau_1^\star$ and consider any strategy σ_2 of player 2 in $\mathcal{G}_{[l]}^{\preccurlyeq_1}$. Let $v \in V^l$ and $\rho = \langle \tau_1^\star, \sigma_2 \rangle_v$. Let $\rho_{\leq n}$ be any prefix of ρ. By Proposition 1, each simple cycle c of the cycle decomposition multiset of $\rho_{\leq n}$ appears in an $\mathcal{F}$-play hc from v consistent with $\tau_1^\star$ in the finite cycle-forming game $\mathcal{F}^l$. By Lemma 2, we have $\beta^l(v) \leq |r_2|$. Since $\tau_1^\star$ is optimal in $\mathcal{F}^l$, it follows that either $\mathrm{Cost}(hc) = -1$ or $\mathrm{Cost}(hc) = \mathsf{MP}_2(c) \leq \beta^l(v)$. Equivalently, with $\beta = \beta^l(v)$ we have

$$\text{either} \quad \mathsf{MP}_1(c) = l \text{ and } \mathsf{MP}_2(c) \leq \beta \tag{6}$$

$$\text{or} \quad \mathsf{MP}_1(c) \geq \bar{l} > l. \tag{7}$$

For each prefix $\rho_{\leq n}$ of ρ of length n, let us denote by $J_1(n)$ (resp. $J_2(n)$) the sum of the lengths of all simple cycles in its cycle decomposition that satisfy Property (6) (resp. Property (7)). Recall that $n - J_1(n) - J_2(n) \leq |V|$ by definition of the cycle decomposition. First observe that for $i \in \{1, 2\}$,

$$0 \leq \liminf_{\mathrm{n}\to\infty} \left(\frac{J_i(n)}{n} \right) \leq \limsup_{\mathrm{n}\to\infty} \left(\frac{J_i(n)}{n} \right) \leq 1.$$

and that

$$\lim_{n\to\infty} \frac{J_1(n) + J_2(n)}{n} = 1,$$

since

$$1 = \lim_{n\to\infty} \frac{n}{n} \geq \lim_{n\to\infty} \frac{J_1(n) + J_2(n)}{n} \geq \lim_{n\to\infty} \frac{n - |V|}{n} = 1.$$

For all $n \in \mathbb{N}$ we have

$$\sum_{k=0}^{n-1} r_1(\rho_k, \rho_{k+1}) \geq J_1(n) \cdot l + J_2(n) \cdot \bar{l}, \text{ and}$$

$$\sum_{k=0}^{n-1} r_2(\rho_k, \rho_{k+1}) \leq J_1(n) \cdot \beta + J_2(n) \cdot |R| + |V| \cdot |R|.$$

[10] according to $\preccurlyeq_1$.

Let $\kappa = \liminf_{n\to\infty} \frac{J_2(n)}{n}$. We have either $\kappa > 0$ or $\kappa = 0$. We have (notice the usage of Lemma 3):

$$\begin{aligned}
\mathsf{InfMP}_1(\rho) &\geq \liminf_{n\to\infty} \left(\frac{J_1(n)}{n}\cdot l + \frac{J_2(n)}{n}\cdot \bar{l}\right) \\
&\geq \liminf_{n\to\infty} \left(\frac{J_1(n)+J_2(n)}{n}\cdot l\right) + \liminf_{n\to\infty}\left(\frac{J_2(n)}{n}\cdot(\bar{l}-l)\right) \\
&= l + \kappa\cdot(\bar{l}-l).
\end{aligned}$$

We can now prove that $(l,\beta) \preccurlyeq_1 \mathsf{InfMP}(\rho)$. This will show that strategy $\tau_1^\star$ ensures at least value (l,β) from vertex v in $\mathcal{G}_{[l]}^{\preccurlyeq_1}$ against any strategy of player 2.

(a) If $\kappa > 0$, since $\bar{l} > l$, we have

$$\mathsf{InfMP}_1(\rho) \geq l + \kappa\cdot(\bar{l}-l) > l$$

and so $(l,\beta) \preccurlyeq_1 \mathsf{InfMP}(\rho)$.

(b) If $\kappa = 0$, then

$$\mathsf{InfMP}_1(\rho) \geq l,$$

and (using again Lemma 3 and recalling that $|R|, \beta \geq 0$)

$$\begin{aligned}
\mathsf{InfMP}_2(\rho) &\leq \liminf_{n\to\infty}\left(\frac{J_1(n)}{n}\cdot\beta + \frac{J_2(n)}{n}\cdot|R| + \frac{|V|}{n}\cdot|R|\right) \\
&\leq \liminf_{n\to\infty}\left(\frac{J_2(n)}{n}|R|\right) - \liminf_{n\to\infty}\left(\frac{-J_1(n)}{n}\cdot\beta + \frac{-|V|}{n}\cdot|R|\right) \\
&\leq \liminf_{n\to\infty}\left(\frac{J_2(n)}{n}\right)\cdot|R| + \limsup_{n\to\infty}\left(\frac{J_1(n)}{n}\right)\cdot\beta + \limsup_{n\to\infty}\left(\frac{|V|}{n}\cdot|R|\right) \\
&\leq 0\cdot|R| + 1\cdot\beta + 0 = \beta.
\end{aligned}$$

In this case we thus also have $(l,\beta) \preccurlyeq_1 \mathsf{InfMP}(\rho)$.

Let us now fix strategy $\tau_2^\star$ and consider any strategy σ_1 of player 1 in $\mathcal{G}_{[l]}^{\preccurlyeq_1}$. Let $v \in V$ and $\rho = \langle\sigma_1, \tau_2^\star\rangle_v$. We adapt the previous arguments to show that $\mathsf{InfMP}(\rho) \preccurlyeq_1 (l,\beta)$ (we use the same notations as above).

By lemma 2, we have $0 \leq \beta$ and Properties (6) and (7) are now replaced by the next ones:

$$\text{either} \quad \mathsf{MP}_1(c) = l \text{ and } \mathsf{MP}_2(c) \geq \beta \tag{8}$$
$$\text{or} \quad \mathsf{MP}_1(c) \leq \bar{l} < l. \tag{9}$$

For each prefix $\rho_{\leq n}$ of ρ of length n, let us denote by $J_1(n)$ (resp. $J_2(n)$) the sum of the lengths of all simple cycles in its cycle decomposition that satisfy Property (8) (resp. Property (9)).

For any $n \in \mathbb{N}$ we have

$$\sum_{k=0}^{n-1} r_1(\rho_k, \rho_{k+1}) \leq J_1(n) \cdot l + J_2(n) \cdot \bar{l} + |V| \cdot |R|, \text{ and}$$

$$\sum_{k=0}^{n-1} r_2(\rho_k, \rho_{k+1}) \geq J_1(n) \cdot \beta.$$

It follows that

$$\begin{aligned}
\mathsf{InfMP}_1(\rho) &\leq \liminf_{\mathrm{n}\to\infty} \left(\frac{J_1(n)}{n}(l - \bar{l} + \bar{l}) + \frac{J_2(n)}{n} \cdot \bar{l} + \frac{|V|}{n} \cdot |R| \right) \\
&\leq \liminf_{\mathrm{n}\to\infty} \left(\frac{J_1(n)}{n} \cdot (l - \bar{l}) \right) - \liminf_{\mathrm{n}\to\infty} \left(-\frac{J_1(n) + J_2(n)}{n} \cdot \bar{l} - \frac{|V|}{n} \cdot |R| \right) \\
&\leq \liminf_{\mathrm{n}\to\infty} \left(\frac{J_1(n)}{n} \cdot (l - \bar{l}) \right) + \limsup_{\mathrm{n}\to\infty} \left(\frac{J_1(n) + J_2(n)}{n} \cdot \bar{l} \right) \\
&= \liminf_{\mathrm{n}\to\infty} \left(\frac{J_1(n)}{n} \right) \cdot (l - \bar{l}) + \bar{l}.
\end{aligned}$$

Recall that $\mu = \liminf_{\mathrm{n}\to\infty} \left(\frac{J_1(n)}{n} \right) \leq 1$.

(a) If $\mu < 1$, then $\mathsf{InfMP}_1(\rho) < l$.
(b) If $\mu = 1$, then $\mathsf{InfMP}_1(\rho) \leq l$, and

$$\mathsf{InfMP}_2(\rho) \geq \liminf_{\mathrm{n}\to\infty} \left(\frac{J_1(n)}{n} .\beta \right) = \beta.$$

We can conclude that $\mathsf{InfMP}(\rho) \preccurlyeq_1 (l, \beta)$. $\square$

We can now complete the proof of Theorem 7.

Proof (of Theorem 7 with InfMP *payoff).* Let $\mathcal{G}^{\preccurlyeq_1}$ be an InfMP lexicographic payoff game. By Lemma 4, we know that the lexicographic subgame $\mathcal{G}^{\preccurlyeq_1}_{[l]}$ is uniformly-determined, for each value l. Let $\sigma^\star_1[l]$ (resp. $\sigma^\star_2[l]$) be a uniform optimal strategy of player 1 (resp. player 2) in $\mathcal{G}^{\preccurlyeq_1}_{[l]}$. We construct a uniform strategy $\sigma^\star_1$ (resp. $\sigma^\star_2$) of player 1 (resp. player 2) such that its restriction to V^l is equal to $\sigma^\star_1[l]$ (resp. $\sigma^\star_2[l]$). These strategies are well-defined since the different sets V^l form a partition of V. Let us prove that the values in game $\mathcal{G}^{\preccurlyeq_1}$ are the same as in the subgames $\mathcal{G}^{\preccurlyeq_1}_{[l]}$ and that strategies $\sigma^\star_1, \sigma^\star_2$ are optimal. We only give the proof for player 1 since it is similar for player 2.

Let σ_2 be any strategy of player 2. Let $v \in V$ be such that its value in $\mathcal{M}$ is $\mathrm{Val}(v) = l$ and its value in $\mathcal{F}^l$ is $\beta^l(v) = \beta$. By Lemma 4, its value in $\mathcal{G}^{\preccurlyeq_1}_{[l]}$ is equal to (l, β). We are going to show that $(l, \beta) \preccurlyeq_1 \mathsf{InfMP}(\rho)$ with $\rho = \langle \sigma^\star_1, \sigma_2 \rangle_v$.

Clearly, play ρ eventually stays inside some set V^L since there is a finite number of such sets. Moreover, we have $L \geq l$. Indeed, by definition of $\sigma_1^\star$, only player 2 can force the play to leave V^{l_1} to go to another V^{l_2}, and only for values $l_2 > l_1$.

Suppose first that $L = l$, i.e., the whole play ρ stays in V^l. Then by Lemma 4, we get $(l, \beta) \preccurlyeq_1 \mathsf{InfMP}(\rho)$. Suppose now that $L > l$, and let $n > 0$ such that $\rho' = \rho_{\geq n} \in (V^L)^\omega$. As InfMP function is prefix-independent, $\mathsf{InfMP}(\rho') = \mathsf{InfMP}(\rho)$, and we can limit our study to ρ'. By definition of $\sigma_1^\star$ and Lemma 4, we have $(\mathrm{Val}(\rho_n), \beta^L(\rho_n)) \preccurlyeq_1 \mathsf{InfMP}(\rho')$. It follows by definition of $\preccurlyeq_1$ that $\mathsf{InfMP}_1(\rho') \geq \mathrm{Val}(\rho_n) = L > l$. Consequently, $(l, \beta) \preccurlyeq_1 \mathsf{InfMP}(\rho')$. $\square$

Complexity Results. We have proved that every InfMP lexicographic payoff game is uniformly-determined. Let us now established the related complexities as given in Theorem 8.

Proof (of Theorem 8 for InfMP *payoff).* The proof is inspired by a proof given in [2]: we are going to sketch a reduction of InfMP lexicographic payoff games to mean-payoff games for optimal strategies. This reduction keeps the same arena (V, E) but replaces the weight function (r_1, r_2) by a single weight function r^*. We again suppose that the weights are positive integers by Remark 2. We know by Theorem 7 that $\mathcal{G}^{\preccurlyeq_1}$ is uniformly-determined. We can thus restrict our attention to simple cycles. The mean-payoff of a simple cycle C in $\mathcal{G}^{\preccurlyeq_1}$ is of the form $(\frac{a}{n}, \frac{b}{n})$ such that each component is a bounded rational (between 0 and $|r_1|$ (resp. $|r_2|$)), and the denominator is at most equal to $|V|$. We define $m = |V|^2 \cdot |r_2| + 1$ and $r^* = r_1 \cdot m - r_2$. It follows that in the resulting mean-payoff game, the cycle C has a mean-payoff equal to $\frac{a \cdot m - b}{n}$. Notice that given two simple cycles C_1 and C_2 in $\mathcal{G}^{\preccurlyeq_1}$, if the first components of their mean-payoff are distinct, that is, $|\frac{a_1}{n_1} - \frac{a_2}{n_2}| > 0$, then $|\frac{a_1}{n_1} - \frac{a_2}{n_2}| \geq \frac{1}{|V|^2}$ because the weights are integers. By the choice of m, let us show that

$$\frac{a_1}{n_1} < \frac{a_2}{n_2} \quad \Rightarrow \quad \frac{a_1 \cdot m - b_1}{n_1} < \frac{a_2 \cdot m - b_2}{n_2}. \tag{10}$$

Indeed, we have

$$\begin{aligned} \frac{a_1}{n_1} + \frac{1}{|V|^2} &\leq \frac{a_2}{n_2} \\ \frac{a_1}{n_1} \cdot m + |r_2| + \frac{1}{|V|^2} &\leq \frac{a_2}{n_2} \cdot m \\ \frac{a_1 \cdot m - b_1}{n_1} \leq \frac{a_1}{n_1} \cdot m &< \frac{a_2}{n_2} \cdot m - |r_2| \leq \frac{a_2 \cdot m - b_2}{n_2}. \end{aligned}$$

To prove the correctness of the proposed reduction, it is enough to prove that $(\frac{a_1}{n_1}, \frac{b_1}{n_1}) \preccurlyeq_1 (\frac{a_2}{n_2}, \frac{b_2}{n_2})$ in $\mathcal{G}^{\preccurlyeq_1}$ if and only if $\frac{a_1 \cdot m - b_1}{n_1} \leq \frac{a_2 \cdot m - b_2}{n_2}$ in the resulting mean-payoff game (the comparaison of two cycle payoffs coincide in both games). Suppose first that $(\frac{a_1}{n_1}, \frac{b_1}{n_1}) \preccurlyeq_1 (\frac{a_2}{n_2}, \frac{b_2}{n_2})$, i.e., either $\frac{a_1}{n_1} < \frac{a_2}{n_2}$, or $(\frac{a_1}{n_1} = \frac{a_2}{n_2}$ and

$\frac{b_1}{n_1} \geq \frac{b_2}{n_2}$). In both cases we get $\frac{a_1 \cdot m - b_1}{n_1} \leq \frac{a_2 \cdot m - b_2}{n_2}$ (we use (10) in the first case). Suppose now that $\frac{a_1 \cdot m - b_1}{n_1} \leq \frac{a_2 \cdot m - b_2}{n_2}$. It follows that $\frac{a_1}{n_1} \leq \frac{a_2}{n_2}$ by (10). Moreover if $\frac{a_1}{n_1} = \frac{a_2}{n_2}$, then $\frac{b_1}{n_1} \geq \frac{b_2}{n_2}$. Therefore $(\frac{a_1}{n_1}, \frac{b_1}{n_1}) \preccurlyeq_1 (\frac{a_2}{n_2}, \frac{b_2}{n_2})$.

By this reduction and classical results on mean-payoff games [15,32], we get the three statements of Theorem 8. □

4.3 LimInf and LimSup Lexicographic Payoff Games

In this section, we consider LimInf lexicographic payoff games and we provide proofs of Theorems 7 and 8 for these games. The proofs are not given for LimSup payoff because they are simple adaptations of the ones given for LimInf payoff.

Uniform-Determinacy. The proof of Theorem 7 for LimInf payoff has the same structure as for InfMP payoff. Let $\mathcal{G}^{\preccurlyeq_1}$ be a LimInf lexicographic payoff game. Without lost of generality, we can suppose the weights used in $\mathcal{G}^{\preccurlyeq_1}$ are natural numbers (see Remark 2). As for lexicographic payoff games with InfMP payoff function, we derive two games with one-dimensional weight function.

We first derive from $\mathcal{G}^{\preccurlyeq_1}$ the LimInf game $\mathcal{L} = (V, V_1, V_2, E, r_1, \mathsf{LimInf}_1)$. This game is uniformly-determined [8]. We denote by $\mathrm{Val}(v)$ the value of each vertex v in $\mathcal{L}$. As done in the proof of Theorem 7 for InfMP payoff, for each value l, we can consider the subarena of (V, E) restricted to V^l being the set of all vertices $v \in V$ such that $\mathrm{Val}(v) = l$. We denote by $\mathcal{G}^{\preccurlyeq_1}_{[l]}$ the lexicographic subgame of $\mathcal{G}^{\preccurlyeq_1}$ restricted to V^l. We also consider a finite cycle-forming game $\mathcal{F}^l$ restricted to V^l and with the following payoff function Cost. Given an initial vertex v in $\mathcal{F}^l$, the two players play until a cycle c is formed. Let $\rho = \rho_0 \dots \rho_m \dots \rho_n$ be the constructed $\mathcal{F}^l$-play such that $v = \rho_0$ and $c = \rho_m \dots \rho_n$ with $\rho_n = \rho_m$. We denote for each $i \in \{1, 2\}$

$$\mathsf{Min}_i(c) = \min_{m \leq k < n} r_i(\rho_k, \rho_{k+1})$$

and we define

$$\mathsf{Cost}(\rho) = \begin{cases} -1 & \text{if } \mathsf{Min}_1(c) > l \\ |r_2| + 1 & \text{if } \mathsf{Min}_1(c) < l \\ \mathsf{Min}_2(c) & \text{if } \mathsf{Min}_1(c) = l. \end{cases}$$

In the finite cycle-forming game $\mathcal{F}^l$, player 1 wants to *minimize* the payoff while player 2 wants to *maximize* it. This finite cycle-forming game is uniformly-determined by Theorem 9. Lemma 2 also holds (the proof is the same) as well as the next lemma (which is the counterpart of Lemma 4).

Lemma 5. *1. Let v be a vertex such that its value in $\mathcal{L}$ is $\mathrm{Val}(v) = l$ and its value in $\mathcal{F}^l$ is $\beta^l(v)$. Then its value in $\mathcal{G}^{\preccurlyeq_1}_{[l]}$ is $(\mathrm{Val}(v), \beta^l(v))$.*
2. A uniform optimal strategy in $\mathcal{F}^l$ is optimal in $\mathcal{G}^{\preccurlyeq_1}_{[l]}$.

Proof. Let $\tau_1^\star$ (resp. $\tau_2^\star$) be a uniform optimal strategy for player 1 (resp. player 2) in $\mathcal{F}^l$. We are going to show that for each vertex v, strategy $\tau_1^\star$ (resp. $\tau_2^\star$) ensures at least (resp. at most) the value $(\mathrm{Val}(v), \beta^l(v))$ in $\mathcal{G}_{[l]}^{\preccurlyeq_1}$. We begin with strategy $\tau_1^\star$.

Consider any strategy σ_2 of player 2 in $\mathcal{G}_{[l]}^{\preccurlyeq_1}$, and let $\rho = \langle \tau_1^\star, \sigma_2 \rangle_v$ be a play in $\mathcal{G}_{[l]}^{\preccurlyeq_1}$ starting in $v \in V^l$. By Proposition 1, each simple cycle c of the cycle decomposition multiset of ρ appears in an $\mathcal{F}^l$-play hc from v consistent with $\tau_1^\star$ in the finite cycle-forming game $\mathcal{F}^l$. By Lemma 2 and since $\tau_1^\star$ is optimal in $\mathcal{F}^l$, it follows, with $l = \mathrm{Val}(v)$ and $\beta = \beta^l(v)$, that

$$\text{either} \quad \mathsf{Min}_1(c) = l \text{ and } \mathsf{Min}_2(c) \leq \beta \tag{11}$$

$$\text{or} \quad \mathsf{Min}_1(c) \geq \bar{l} > l. \tag{12}$$

By Proposition 1, there is an index $n_0 \in \mathbb{N}$ such that each edge (ρ_n, ρ_{n+1}) with $n \geq n_0$, is in a cycle c_n that appears in an $\mathcal{F}^l$-play hc in the game $\mathcal{F}^l$, for some history h starting in v. Then, for $n \geq n_0$, we define

$$u_n = \begin{cases} 0 \text{ if cycle } c_n \text{ satisfies Property (11)} \\ 1 \text{ if cycle } c_n \text{ satisfies Property (12)} \end{cases}$$

Remark that $\liminf_{n\to\infty} u_n = 0$ or 1.

If $\liminf_{n\to\infty} u_n = 0$, there are infinitely many cycles c_n, with $n \geq n_0$ such that $\mathsf{Min}_1(c_n) = l$ and $\mathsf{Min}_2(c_n) \leq \beta$, whereas the other ones satisfy $\mathsf{Min}_1(c_n) \geq \bar{l} > l$. Therefore $\mathsf{LimInf}_1(\rho) = l$ and $\mathsf{LimInf}_2(\rho) \leq \beta$.

If $\liminf_{n\to\infty} u_n = 1$, then there exists some $n_1 \geq n_0$ such that all cycles c_n, for $n \geq n_1$, satisfy $\mathsf{Min}_1(c_n) \geq \bar{l} > l$. Thus $\mathsf{LimInf}_1(\rho) > l$.

It follows that in both cases $(l, \beta) \preccurlyeq_1 \mathsf{LimInf}(\rho)$.

Consider now any strategy σ_1 of player 1 in $\mathcal{G}_{[l]}^{\preccurlyeq_1}$, and let $\rho = \langle \sigma_1, \tau_2^\star \rangle_v$ be a play in $\mathcal{G}_{[l]}^{\preccurlyeq_1}$ starting in $v \in V^l$. The arguments are similar to the previous ones. We just sketch the proof using the same notations. Properties (11) and (12) are replaced by the next ones:

$$\text{either} \quad \mathsf{Min}_1(c) = l \text{ and } \mathsf{Min}_2(c) \geq \beta \tag{13}$$

$$\text{or} \quad \mathsf{Min}_1(c) \leq \bar{l} < l. \tag{14}$$

We define

$$u_n = \begin{cases} 1 \text{ if cycle } c_n \text{ satisfies Property (13)} \\ 0 \text{ if cycle } c_n \text{ satisfies Property (14)} \end{cases}$$

Then, either $\liminf_{n\to\infty} u_n = 0$ and $\mathsf{LimInf}_1(\rho) < l$, or $\liminf_{n\to\infty} u_n = 1$ and $\mathsf{LimInf}_1(\rho) = l$, $\mathsf{LimInf}_2(\rho) \geq \beta$. We conclude that $\mathsf{LimInf}(\rho) \preccurlyeq_1 (l, \beta)$. □

The proof of Theorem 7 for LimInf payoff follows from Lemma 5, in the same way as it was proved from Lemma 4 for InfMP payoff.

Complexity Results. Let us now turn to complexity results and give a proof of Theorem 8 for LimInf payoff. To get the announced polynomial complexities, the previous reduction to finite cycle-forming games is too expensive. We here propose another reduction to co-Büchi and (one pair) Rabin games.

We have the next lemma. The proof of Theorem 8 will follow. We suppose again that weights are natural numbers by Remark 2.

Lemma 6. *Let $\mathcal{G}^{\preccurlyeq_1}$ be a LimInf lexicographic payoff game, $v \in V$ be a vertex and $(\alpha, \beta) \in \mathbb{N}^2$ be a pair of naturals. Deciding whether player 1 has a strategy σ_1 from v such that $(\alpha, \beta) \preccurlyeq_1 \mathsf{LimInf}(\langle\sigma_1, \sigma_2\rangle_v)$ for all strategies σ_2 of player 2 is P-complete.*

Proof. We first prove that one can decide in $O((|V| + |E|)^2)$ whether player 1 has a strategy σ_1 from v such that $(\alpha, \beta) \preccurlyeq_1 \mathsf{LimInf}(\langle\sigma_1, \sigma_2\rangle_v)$ for all strategies σ_2 of player 2. Let us fix $v \in V$, $(\alpha, \beta) \in \mathbb{N}^2$, and a strategy σ_1 of player 1 in $\mathcal{G}^{\preccurlyeq_1}$. Consider a strategy σ_2 of player 2 and the outcome $\rho = \langle\sigma_1, \sigma_2\rangle_v$. We have that $(\alpha, \beta) \preccurlyeq_1 \mathsf{LimInf}(\rho)$ iff

- there exists $n \geq 0$ such that $r_1(\rho_k, \rho_{k+1}) \geq \alpha + 1$ for all $k \geq n$, or
- there exists $n \geq 0$ such that $r_1(\rho_k, \rho_{k+1}) \geq \alpha$ for all $k \geq n$ and $r_2(\rho_l, \rho_{l+1}) \leq \beta$ for infinitely many l.

Using LTL notation, we abbreviate these conditions by

$$\rho \models (\Diamond\Box\ r_1 \geq \alpha + 1) \quad \vee \quad \rho \models (\Diamond\Box\ r_1 \geq \alpha) \wedge (\Box\Diamond\ r_2 \leq \beta). \tag{15}$$

The first condition is a co-Büchi condition, while the second one is a Rabin condition.

Let us construct the following co-Büchi game $\mathcal{C}$ in relation with condition $\rho \models (\Diamond\Box\ r_1 \geq \alpha + 1)$. Firstly, from the arena (V, E) of $\mathcal{G}^{\preccurlyeq_1}$, we construct a new arena (V', E') in a way to have weights depending on vertices instead of edges. We proceed as follows: each edge $e = (v, v') \in E$ is split into two consecutive edges. The new intermediate vertex belongs to player 1[11], and it is decorated with $(r_1(e), r_2(e))$. The vertices of V are decorated with $(+\infty, +\infty)$, and V' has $n' = |V| + |E|$ vertices. Secondly, we define a set $A \subseteq V'$ composed of all vertices decorated by a pair (a, b) such that $a < \alpha + 1$. Thirdly, player 1 has a winning strategy in the constructed co-Büchi game from vertex v iff player 1 has a strategy σ_1 in $\mathcal{G}^{\preccurlyeq_1}$ such that for all strategies σ_2, we have $\rho \models (\Diamond\Box\ r_1 \geq \alpha + 1)$ with $\rho = \langle\sigma_1, \sigma_2\rangle_v$. Testing if player 1 has a winning strategy in the co-Büchi game from vertex v can be done in time $O(n'^2)$.

Let us now construct the following Rabin game in relation with condition $\rho \models (\Diamond\Box\ r_1 \geq \alpha) \wedge (\Box\Diamond\ r_2 \leq \beta)$ of (15). We first construct the same arena (V', E') as before, with the same vertex decorations. We then define one Rabin pair (A, B) such that $A \subseteq V'$ (resp. $B \subseteq V'$) is composed of all vertices decorated by a pair (a, b) such that $a < \alpha$ (resp. $b \leq \beta$). It follows that player 1 has a winning strategy in the constructed Rabin game from vertex v iff player 1 has a strategy

[11] It could belong to player 2 since there is exactly one outgoing edge.

σ_1 in $\mathcal{G}^{\preccurlyeq_1}$ such that for all strategies σ_2, we have $\rho \models (\Diamond\Box\, r_1 \geq \alpha) \wedge (\Box\Diamond\, r_2 \leq \beta)$. This property can be checked in time $O(n'^{p+1} \cdot p!)$ with $p = 1$ (there is one Rabin pair).

By this two-case analysis, one can thus decide in time $O((|V|+|E|)^2)$ whether player 1 has a strategy σ_1 from v such that $(\alpha, \beta) \preccurlyeq_1 \mathsf{LimInf}(\langle\sigma_1, \sigma_2\rangle_v)$ for all strategies σ_2 of player 2.

To conclude the proof of Lemma 6, we show that for each co-Büchi game $\mathcal{C}$, we can construct a LimInf lexicographic payoff game $\mathcal{G}_{\mathcal{C}}^{\preccurlyeq_1}$ and a pair of naturals (α, β) such that the problem of deciding if player 1 has a winning strategy from a vertex v in $\mathcal{C}$ is equivalent to the problem stated in Lemma 6. Let (V, E) be the arena of $\mathcal{C}$ and $A \subseteq V$ be the set given for the co-Büchi condition. We set $(\alpha, \beta) = (1, 1)$ and we define $\mathcal{G}_{\mathcal{C}}^{\preccurlyeq_1} = (V, V_1, V_2, E, r, \mathsf{LimInf})$ with the reward function r such that for all $(v, v') \in E$, $r(v, v') = (0, 1)$ if $v' \in A$, and $r(v, v') = (1, 1)$ otherwise. Clearly, for every play ρ starting in v, we have $\mathsf{Inf}(\rho) \cap A = \emptyset$ iff $\mathsf{LimInf}(\rho) = (1, 1)$. This completes the proof. $\square$

Proof (of Theorem 8 for LimInf *payoff).* The first part is a direct consequence of Lemma 6 by definition of the value.

The second is also obtained as a corollary of Lemma 6. Indeed, given a vertex v, it suffices to apply it to all pairs (α, β) such that $\alpha = r_1(e)$ and $\beta = r_2(e')$ for some edges $e, e' \in E$. Notice that there are $|E|^2$ pairs. The value of v is thus the maximum among those pairs (α, β) for which player 1 has a strategy σ_1 from v such that $(\alpha, \beta) \preccurlyeq_1 \mathsf{LimInf}(\langle\sigma_1, \sigma_2\rangle_v)$ for all strategies σ_2 of player 2. This argument shows that computing the value of v can be done in polynomial time.

The third statement is a consequence of the second one, using a dichotomy on the edges of E as proposed in [32]. We first start by computing the values $\mathrm{Val}(v)$ for each v in $\mathcal{G}^{\preccurlyeq_1}$. If all the vertices of V have outdegree one, then player 1 has a unique uniform optimal strategy. Otherwise, let v be a vertex with outdegree $d > 1$. Let us remove $\lceil \frac{d}{2} \rceil$ of the edges leaving v, and let us compute the value $\mathrm{Val}'(v)$ of v in the resulting graph. If $\mathrm{Val}'(v) = \mathrm{Val}(v)$ (resp. $\mathrm{Val}'(v) \neq \mathrm{Val}(v)$), then there is a uniform optimal strategy for player 1 which does not use any of the removed edges (uses one of the removed edges). In both cases, we can restrict the computation to a subgraph of $\mathcal{G}^{\preccurlyeq_1}$ with at least $\lfloor \frac{d}{2} \rfloor$ fewer edges. After a polynomial[12] number of such experiments, we get the required uniform optimal strategy for player 1 with the announced complexity. A uniform optimal strategy for player 2 can be found in the same way. $\square$

4.4 Inf and Sup Lexicographic Payoff Games

In this section, we prove Theorems 7 and 8 for Inf and Sup lexicographic payoff games. Without lost of generality, we limit the proofs to the Inf payoff.

Positional-Determinacy. We begin by proving that these games are positionally-determined. Some preliminary lemmas are necessary.

[12] $O(\Sigma_{v \in V_1} \log d(v))$ with $d(v)$ being the outdegree of vertex v in $\mathcal{G}^{\preccurlyeq_1}$.

Lemma 7. *Every* Inf *lexicographic payoff game* $\mathcal{G}^{\preccurlyeq_1}$ *is determined, and its values can be computed in polynomial time.*

Proof. Given $\mathcal{G}^{\preccurlyeq_1}$ and an initial vertex v_0, we derive a LimInf lexicographic payoff game $\mathcal{G}'^{\preccurlyeq_1}$ and an initial vertex $v'_0 = (v_0, +\infty, +\infty)$, with the same construction as in the proof of Theorem 1 for Inf payoff (we augment the vertices with the current smallest seen weights). The latter game is uniformly-determined by Theorem 7 and its values can be computed in polynomial by Theorem 8 (for LimInf payoff). In $\mathcal{G}'^{\preccurlyeq_1}$, let (α, β) be the value of v'_0 and $\sigma_1'^{\star}$, $\sigma_2'^{\star}$ be a pair of uniform optimal strategies for player 1 and player 2 respectively. We naturally derive finite-memory strategies $\sigma_1^{\star}$, $\sigma_2^{\star}$ in $\mathcal{G}^{\preccurlyeq_1}$ that mimic strategies $\sigma_1'^{\star}$, $\sigma_2'^{\star}$. Let us show that in $\mathcal{G}^{\preccurlyeq_1}$, v_0 has value (α, β) and $\sigma_1^{\star}$, $\sigma_2^{\star}$ are optimal strategies[13]. In $\mathcal{G}^{\preccurlyeq_1}$, let σ_2 be any strategy for player 2, and let us prove that $(\alpha, \beta) \preccurlyeq_1 \mathsf{Inf}(\langle \sigma_1^{\star}, \sigma_2 \rangle_{v_0})$ (see Definition 6). This inequality follows from $(\alpha, \beta) \preccurlyeq_1 \mathsf{LimInf}(\langle \sigma_1'^{\star}, \sigma'_2 \rangle_{v'_0})$ in $\mathcal{G}'^{\preccurlyeq_1}$ and $\mathsf{Inf}(\langle \sigma_1^{\star}, \sigma_2 \rangle_{v_0}) = \mathsf{LimInf}(\langle \sigma_1'^{\star}, \sigma'_2 \rangle_{v'_0})$, where σ'_2 is the strategy that mimics σ_2 in $\mathcal{G}'^{\preccurlyeq_1}$. Similarly we have $\inf(\langle \sigma_1, \sigma_2^{\star} \rangle_{v_0}) \preccurlyeq_1 (\alpha, \beta)$ for all strategy σ_1 of player 1 in $\mathcal{G}^{\preccurlyeq_1}$. Therefore from this reduction to LimInf lexicographic payoff game, $\mathcal{G}^{\preccurlyeq_1}$ is determined, and its values can be computed in polynomial time. $\square$

Lemma 8. *Let* (α, β) *be a pair of rational numbers.*

1. *There exists a partition of* V *into* W_1 *and* W_2 *such that* $v \in W_1$ *iff player 1 has a (positional) strategy* σ_1^* *from* v *in* $\mathcal{G}^{\preccurlyeq_1}$ *such that* $(\alpha, \beta) \preccurlyeq_1 \mathsf{Inf}(\langle \sigma_1^*, \sigma_2 \rangle_v)$ *for all strategies* σ_2 *of player 2.*
2. *Similarly, there exists a partition of* V *into* T_1 *and* T_2 *such that* $v \in T_2$ *iff player 2 has a (positional) strategy* σ_2^* *from* v *in* $\mathcal{G}^{\preccurlyeq_1}$ *such that* $\mathsf{Inf}(\langle \sigma_1, \sigma_2^* \rangle_v) \preccurlyeq_1 (\alpha, \beta)$ *for all strategies* σ_1 *of player 1.*

Moreover, the two partitions of V *and the two strategies* σ_1^*, σ_2^* *can be computed in polynomial time.*

Proof. We only prove the first statement, the second one is proved similarly. From $\mathcal{G}^{\preccurlyeq_1}$, we construct the same arena $\mathcal{A} = (V', E')$ as in Lemma 6 in a way to have weights depending on vertices instead of edges. Recall that $V' = V \cup E$ such that each edge e of $\mathcal{G}^{\preccurlyeq_1}$ is split into two consecutive edges where the new intermediate vertex s is decorated with $(r_1(s), r_2(s))$ and belongs to player 1. The vertices v of V are decorated with $(r_1(v), r_2(v)) = (+\infty, +\infty)$.

Let (α, β) be a pair of rational numbers. We are going to construct a partition of V' into two sets W_1 and W_2 such that for all $v' \in V'$, we have that $v' \in W_1 \cap V$ iff player 1 has a strategy σ_1^* (that can be chosen positional) from v' in $\mathcal{G}^{\preccurlyeq_1}$ such that $(\alpha, \beta) \preccurlyeq_1 \mathsf{Inf}(\langle \sigma_1^*, \sigma_2 \rangle_{v'})$ for all strategies σ_2 of player 2. The restriction to V of this partition leads to the required partition of V of Lemma 8. Recall that $(\alpha, \beta) \preccurlyeq_1 \mathsf{Inf}(\rho)$, with $\rho = \langle \sigma_1^*, \sigma_2 \rangle_{v'}$ iff

– either $r_1(\rho_n, \rho_{n+1}) > \alpha$ for all n,

[13] In the proof of Theorem 7 for Inf payoff, we will explain how to obtain positional optimal strategies.

– or $r_1(\rho_n, \rho_{n+1}) \geq \alpha$ for all n, and $r_2(\rho_n, \rho_{n+1}) \leq \beta$ for one n.

In the sequel, let us use notation $\langle\langle i \rangle\rangle^{\mathcal{A}'} \phi$ for the set of vertices v' inside a game restricted to the subarena $\mathcal{A}'$ of $\mathcal{A}$, from which player i has a strategy that ensures an LTL formula ϕ, with ϕ describing a reachability objective. To get the partition of V' into W_1 and W_2, we proceed step by step.

1. We consider $W_2^{<\alpha} = \langle\langle 2 \rangle\rangle^{\mathcal{A}} \Diamond \{s \in V' \mid r_1(s) < \alpha\}$, that is, the set of vertices v' in $\mathcal{A}$ from which player 2 can ensure to reach a vertex s decorated with $r_1(s) < \alpha$. Clearly, from such vertices $v' \in V \cap W_2^{<\alpha}$ seen as vertices in $\mathcal{G}^{\preccurlyeq_1}$, player 1 cannot ensure $(\alpha, \beta) \preccurlyeq_1 \mathsf{Inf}(\langle \sigma_1, \sigma_2 \rangle_{v'})$ against all strategies σ_2 of player 2. In other words, $W_2^{<\alpha} \subseteq W_2$.
2. Let $\mathcal{A}_1 = \mathcal{A}|_{V' \setminus W_2^{<\alpha}}$ be the subarena $\mathcal{A}$ restricted to vertices in $V^1 = V' \setminus W_2^{<\alpha}$. Notice that in subarena $\mathcal{A}_1$, every vertex $s \in V^1$ is decorated with $r_1(s) \geq \alpha$ since otherwise $s \in W_2^{<\alpha}$. We consider the set $W_1^{\leq\beta} = \langle\langle 1 \rangle\rangle^{\mathcal{A}_1} \Diamond \left\{s \in V^1 \mid r_2(s) \leq \beta\right\}$. We get that $W_1^{\leq\beta} \subseteq W_1$. Moreover, for all $v' \in W_1^{\leq\beta} \cap V$, player 1 has a positional strategy σ_1^* from v' in $\mathcal{G}^{\preccurlyeq_1}$ such that $(\alpha, \beta) \preccurlyeq_1 \mathsf{Inf}(\langle \sigma_1^*, \sigma_2 \rangle_{v'})$. Indeed this strategy is given by the positional strategy that is used in the reachability game played on $\mathcal{A}_1$ to reach a vertex s decorated with $r_2(s) \leq \beta$.
3. We define the subarena $\mathcal{A}_2 = \mathcal{A}|_{V^1 \setminus W_1^{\leq\beta}}$ of $\mathcal{A}$ restricted to $V^2 = V^1 \setminus W_1^{\leq\beta}$. Each vertex s of $\mathcal{A}_2$ is decorated with $r_1(s) \geq \alpha$ and $r_2(s) > \beta$. With the set $W_2^{=\alpha} = \langle\langle 2 \rangle\rangle^{\mathcal{A}_2} \Diamond \left\{s \in V^2 \mid r_1(s) = \alpha\right\}$, we have that $W_2^{=\alpha} \subseteq W_2$.
4. Since each vertex v' of $V' \setminus (W_2^{<\alpha} \cup W_1^{\leq\beta} \cup W_2^{=\alpha})$ is decorated with $r_1(v') > \alpha$, it follows that v' belongs to W_1. Moreover, there exists a positional strategy σ_1^* of player 1 from v' in $\mathcal{G}^{\preccurlyeq_1}$ such that $(\alpha, \beta) \preccurlyeq_1 \mathsf{Inf}(\langle \sigma_1^*, \sigma_2 \rangle_{v'})$. It is given by a positional strategy used in the safety game played on $\mathcal{A}_2$ to avoid all vertices s decorated with $r_1(s) = \alpha$.

Therefore, we get required partition of V' into $W_2 = W_2^{<\alpha} \cup W_2^{=\alpha}$ and $W_1 = V' \setminus W_2$. Moreover, as this partition has been obtained thanks to three reachability games, the sets $W_2^{<\alpha}$, $W_1^{\leq\beta}$ and $W_2^{=\alpha}$ can be computed in $O(|V| + 3 \cdot |E|)$[14], as well as the positional strategy σ_1^* [19]. □

Proof (of Theorem 7 for Inf *payoff).* Let $\mathcal{G}^{\preccurlyeq_1}$ be an Inf lexicographic payoff game. By Lemma 7, we know that $\mathcal{G}^{\preccurlyeq_1}$ is determined. Given an initial vertex v_0 with value (α, β), let us indicate how to construct positional optimal strategies for both players. We partition V into the sets W_1 and W_2 (resp. T_1 and T_2) as indicated in Lemma 8. By Definition 6, we have that $v_0 \in W_1 \cap T_2$, and Lemma 8 provides positional strategies σ_1^* and σ_2^* that are optimal. Notice that these strategies depend on v_0 since they depend on (α, β). □

Complexity Results. Let us now turn to the proof of Theorem 8.

[14] The arena $\mathcal{A}$ has $|V| + |E|$ vertices and $2 \cdot |E|$ edges.

Proof (of Theorem 8 for Inf *payoff).* The second statement is direct consequence of Lemma 7. A polynomial time algorithm for the first and third statements follows from Lemma 8.

To establish P-completeness for the first statement, we show that for each safety game $\mathcal{S}$, we can construct an Inf lexicographic payoff game $\mathcal{G}_{\mathcal{S}}^{\preceq_1}$ and a pair of naturals (α, β) such that the problem of deciding if player 1 has a winning strategy from a vertex v_0 in $\mathcal{S}$ is equivalent to the value problem stated in the first statement of Theorem 8 for Inf payoff. Let (V, E) be the arena of $\mathcal{S}$ and $A \subseteq V$ be the set of vertices that player 1 wants to avoid. We set $(\alpha, \beta) = (1, 1)$ and we define $\mathcal{G}_{\mathcal{S}}^{\preceq_1} = (V \cup \{w\}, V_1 \cup \{w\}, V_2, E' = E \cup \{(w, v_0)\}, r, \mathsf{Inf})$ with $w \notin V$ and r such that for all $(v, v') \in E'$, $r(v, v') = (0, 1)$ if $v' \in A$, and $r(v, v') = (1, 1)$ otherwise. Clearly, for every play ρ starting in v_0 in $\mathcal{S}$, we have a corresponding path $w\rho$ in $\mathcal{G}_{\mathcal{S}}^{\preceq_1}$ such that $\rho_n \notin A$ for all $n \geq 0$ iff $\mathsf{Inf}(w\rho) = (1, 1)$. Therefore, deciding whether player 1 has a winning strategy from v_0 in $\mathcal{S}$ is equivalent to decide whether the value of w in $\mathcal{G}_{\mathcal{S}}^{\preceq_1}$ is at least equal to $(\alpha, \beta) = (1, 1)$. □

4.5 Disc^{λ} Lexicographic Payoff Games

In this section we study the Disc^{λ} payoff function for which we prove Theorems 7 and 8.

Uniform-Determinacy. The proof of Theorem 7 for Disc^{λ} lexicographic payoff games follows from the simple next idea: player 1 first tries to maximize his payoff limited to the first component and then to minimize his payoff limited to the second component. Player 1 can use a uniform strategy to achieve this goal because at each step, the game reduces to a one-dimensional discounted game.

Let us now go into the details. The proof of Theorem 7 for Disc^{λ} payoff will be obtained as a consequence of Lemmas 9 and 10 given below.

Let $\lambda \in]0, 1[$ and $\mathcal{G}^{\preceq_1} = (V, V_1, V_2, E, r, \mathsf{Disc}^{\lambda}, \preceq_1)$ be a lexicographic payoff game with weight function Disc^{λ}. We derive from $\mathcal{G}^{\preceq_1}$ the discounted game $\mathcal{D} = (V, V_1, V_2, E, r_1, \mathsf{Disc}_1^{\lambda})$ such that the weights are limited to the first component r_1 of r, and player 1 wants to maximize the payoff $\mathsf{Disc}_1^{\lambda}$ while player 2 wants to minimize it. By [32], this game is uniformly-determined, and the value of each vertex $v \in V$ and uniform strategies can be defined by the following system of equations:

$$\mathrm{Val}(v) = \begin{cases} \max_{(v,v') \in E} \{(1-\lambda) \cdot r_1(v, v') + \lambda \cdot \mathrm{Val}(v')\} & \text{if } v \in V_1, \\ \min_{(v,v') \in E} \{(1-\lambda) \cdot r_1(v, v') + \lambda \cdot \mathrm{Val}(v')\} & \text{if } v \in V_2. \end{cases} \tag{16}$$

We denote by E' the set of all optimal edges of E, i.e. edges realising the maximum (resp. minimum) for player 1 (resp. player 2) in system (16).

The next lemma states that against an optimal strategy of player 1 in $\mathcal{D}$, player 2 should rather choose edges in E'.

Lemma 9. *1. Let $\sigma_1^{\star}$ be a uniform optimal strategy of player 1 in $\mathcal{D}$, and let σ_2 be a strategy of player 2 in $\mathcal{D}$. Let v be a vertex in V with value* $\mathrm{Val}(v)$

in $\mathcal{D}$. If $\rho = \langle \sigma_1^\star, \sigma_2 \rangle_v$ contains two vertices ρ_l, ρ_{l+1} with $(\rho_l, \rho_{l+1}) \in E \setminus E'$, then $\mathsf{Disc}_1^\lambda(\langle \sigma_1^\star, \sigma_2 \rangle_v) > \mathrm{Val}(v)$.

2. *Similarly let $\sigma_2^\star$ be a uniform optimal strategy of player 2, and σ_1 be a strategy of player 1. If $\rho = \langle \sigma_1, \sigma_2^\star \rangle_v$ contains two vertices ρ_l, ρ_{l+1} with $(\rho_l, \rho_{l+1}) \in E \setminus E'$, then $\mathsf{Disc}_1^\lambda(\langle \sigma_1, \sigma_2^\star \rangle_v) < \mathrm{Val}(v)$.*

Proof. We only give the proof for the first statement, the proof being similar for the second one. Notice that player 1 only uses edges in E'.

Suppose that player 2 follows a strategy σ_2 such that $\rho = \langle \sigma_1^\star, \sigma_2 \rangle_v$ with $\rho_0 = v$, uses at least one edge in $E \setminus E'$. We consider the smallest index $l \geq 0$ such that $\rho_l \in V_2$ and $(\rho_l, \rho_{l+1}) \in E \setminus E'$.

On one hand, we have by (16)

$$\mathrm{Val}(v) = (1-\lambda) \cdot \sum_{k=0}^{l-1} \lambda^k \cdot r_1(\rho_k, \rho_{k+1}) + \lambda^l \cdot \mathrm{Val}(\rho_l). \tag{17}$$

On the other hand, by optimality of $\sigma_1^\star$, we have that

$$\begin{aligned}
\mathsf{Disc}_1^\lambda\left(\langle \sigma_1^\star, \sigma_2 \rangle_v\right) &= (1-\lambda) \cdot \sum_{k=0}^{l} \lambda^k \cdot r_1(\rho_k, \rho_{k+1}) + \lambda^{l+1} \cdot \mathsf{Disc}_1^\lambda\left(\left\langle \sigma_1^\star, \sigma_2|_{\rho_{\leq l}} \right\rangle_{\rho_{l+1}}\right) \\
&\geq (1-\lambda) \cdot \sum_{k=0}^{l} \lambda^k \cdot r_1(\rho_k, \rho_{k+1}) + \lambda^{l+1} \cdot \mathrm{Val}(\rho_{l+1}).
\end{aligned}$$

Then by definition of l, we have

$$\mathsf{Disc}_1^\lambda\left(\langle \sigma_1^\star, \sigma_2 \rangle_v\right) > (1-\lambda) \cdot \sum_{k=0}^{l-1} \lambda^k \cdot r_1(\rho_k, \rho_{k+1}) + \lambda^l \cdot \mathrm{Val}(\rho_l). \tag{18}$$

By (17) and (18), we can conclude that $\mathsf{Disc}_1^\lambda\left(\langle \sigma_1^\star, \sigma_2 \rangle_v\right) > \mathrm{Val}(v)$. $\square$

We have shown that if both players play optimally in $\mathcal{D}$, then the produced outcome uses edges of E' only. We now consider the discounted game $\mathcal{D}' = (V, V_1, V_2, E', r_2, \mathsf{Disc}_2^\lambda)$ such that the edges are limited to E' and the weight function is now r_2. Another difference is that player 1 wants to *minimize* his payoff (now seen as a cost) while player 2 wants to *maximize* it.

The following lemma describes values and optimal strategies of the lexicographic payoff game $\mathcal{G}^{\preccurlyeq_1}$ in relation with the ones of discounted games $\mathcal{D}$ and $\mathcal{D}'$. Theorem 7 for Disc^λ payoff is a consequence of this lemma.

Lemma 10. 1. *Let $v \in V$ be a vertex. If v has value $\mathrm{Val}(v)$ in $\mathcal{D}$ and $\beta(v)$ in $\mathcal{D}'$, then v has value $(\mathrm{Val}(v), \beta(v))$ in $\mathcal{G}^{\preccurlyeq_1}$.*

2. *A uniform optimal strategy in $\mathcal{D}'$ is optimal in $\mathcal{G}^{\preccurlyeq_1}$.*

Proof. Let $\tau_1^\star$ be a uniform optimal strategy for player 1 in $\mathcal{D}'$, and σ_2 be a strategy of player 2 in $\mathcal{G}^{\preccurlyeq_1}$. Suppose vertex v has value $\mathrm{Val}(v)$ in $\mathcal{D}$ and $\beta(v)$ in $\mathcal{D}'$. Let $\rho = \langle \tau_1^\star, \sigma_2 \rangle_v$. We are going to show that $(\mathrm{Val}(v), \beta(v)) \preccurlyeq_1 \mathsf{Disc}^\lambda(\rho)$.

As $\tau_1^\star$ is a strategy in $\mathcal{D}'$, it is an optimal strategy in $\mathcal{D}$. Therefore $\mathsf{Disc}_1^\lambda(\rho) \geq \mathrm{Val}(v)$. If this inequality is strict, then $(\mathrm{Val}(v), \beta(v)) \preccurlyeq_1 \mathsf{Disc}^\lambda(\rho)$. Let us thus suppose that $\mathsf{Disc}_1^\lambda(\rho) = \mathrm{Val}(v)$. By Lemma 9, the outcome ρ uses edges of E' only and is then an outcome in $\mathcal{D}'$. By optimality of $\tau_1^\star$ in $\mathcal{D}'$, it follows that $\mathsf{Disc}_2^\lambda(\rho) \leq \beta(v)$, and thus $(\mathrm{Val}(v), \beta(v)) \preccurlyeq_1 \mathsf{Disc}^\lambda(\rho)$.

With $\tau_2^\star$ a uniform optimal strategy for player 2 in $\mathcal{D}'$ and σ_1 a strategy of player 1 in $\mathcal{G}^{\preccurlyeq_1}$, we show similarly that $\mathsf{Disc}^\lambda(\langle \sigma_1, \tau_2^\star \rangle_v) \preccurlyeq_1 (\mathrm{Val}(v), \beta(v))$. □

Complexity Results. Let us now provide a proof of Theorem 8 for Disc^λ payoff.

Proof (of Theorem 8 for Disc^λ *payoff).* We begin with the second statement. By Lemma 10, we know that if v has value $\mathrm{Val}(v)$ in $\mathcal{D}$ and $\beta(v)$ in $\mathcal{D}'$, then v has value $(\mathrm{Val}(v), \beta(v))$ in $\mathcal{G}^{\preccurlyeq_1}$. The value $\mathrm{Val}(v)$ of each v in the discounted game $\mathcal{D}$ can be computed in pseudo-polynomial time [32]. Once these values are computed in $\mathcal{D}$, the subgame $\mathcal{D}'$ can be computed in polynomial time by substituting each value $\mathrm{Val}(v)$ in system (16) and removing each edge that does not realise the maximum (resp. minimum) for player 1 (resp. player 2) in the resulting system. The value $\beta(v)$ of each vertex v in the discounted game $\mathcal{D}'$ can be computed in pseudo-polynomial time.

Notice that we also have an algorithm in $\mathsf{NP} \cap \mathsf{co\text{-}NP}$ to compute the value $(\mathrm{Val}(v), \beta(v))$ of a given vertex v of $\mathcal{G}^{\preccurlyeq_1}$. Let us explain the main ideas. Recall that in discounting games, deciding whether the value of a vertex is greater than or equal to a given threshold is in $\mathsf{NP} \cap \mathsf{co\text{-}NP}$, and that the solutions to a system like (16) can be written with polynomially many bits [32,19]. Hence, there is an exponential number of possibilities for the values of a discounted game. To compute such a value, we can apply a dichotomy that uses the previous decision problem in $\mathsf{NP} \cap \mathsf{co\text{-}NP}$ a polynomial number of times. We thus have an algorithm that uses a polynomial number of calls to an oracle in $\mathsf{NP} \cap \mathsf{co\text{-}NP}$. Since $\mathsf{P}^{\mathsf{NP} \cap \mathsf{co\text{-}NP}} = \mathsf{NP} \cap \mathsf{co\text{-}NP}$ [4], we get an algorithm in $\mathsf{NP} \cap \mathsf{co\text{-}NP}$ to compute the value $(\mathrm{Val}(v), \beta(v))$ of vertex v in $\mathcal{G}^{\preccurlyeq_1}$.

Let us turn to the first statement of Theorem 8. Let (α, β) be a pair of rational numbers, and let v be a vertex. To check whether $(\alpha, \beta) \preccurlyeq_1 (\mathrm{Val}(v), \beta(v))$, we compute $(\mathrm{Val}(v), \beta(v))$ in $\mathsf{NP} \cap \mathsf{co\text{-}NP}$ as just explained, and then make the comparison with (α, β).

For the third statement, by Lemma 10, we also know that a uniform optimal strategy in $\mathcal{D}'$ is optimal in $\mathcal{G}^{\preccurlyeq_1}$. Uniform optimal strategies in discounted games can be constructed in pseudo-polynomial time [32], and we have seen before that $\mathcal{D}'$ can also be constructed in pseudo-polynomial time. □

5 Study of Problems 3 and 4

In Section 4, we have studied the determinacy of lexicographic payoff games and the related complexities. From these results we were able to prove Theorems 1 and 2 about the existence and the construction of a secure equilibrium in weighted games with the payoff functions of Definition 2.

The aim of this section is to provide a proof of Theorem 3 about the constrained existence of a secure equilibrium in such games (see Problem 3). Thanks to Theorem 6 that provides a general framework for solving Problem 3, it remains to study Problem 4 about the constraint existence of a path in a graph with weights and values. Hence we first study the latter problem. We then derive a proof of Theorem 3 for $\mathsf{InfMP}, \mathsf{SupMP}, \mathsf{LimInf}, \mathsf{LimSup}, \mathsf{Inf}$ and Sup payoffs. We leave the discounted case open. However we show that a solution in this case would provide a solution to an open problem mentioned in [10] itself related to other difficult open problems in mathematics [3].

5.1 Solution for Problem 4

Let us first show that Problem 4 is decidable for InfMP, SupMP, LimInf and LimSup payoffs. The approach that we develop is inspired by proof techniques proposed in [29].

Theorem 10. *Let $G = (V, E, v_0, r, \mathrm{Val})$ be a finite directed graph with an initial vertex v_0, a weight function r, and a value function* Val. *Let $\mu, \nu \in (\mathbb{Q} \cup \{\pm\infty\})^2$ be two thresholds. Then for $\mathsf{Payoff} = \mathsf{InfMP}$, SupMP, LimInf and LimSup, one can decide in polynomial time whether there exists an infinite path ρ in G such that $\forall k \geq 0, \forall i \in \{1,2\}, \mathrm{Val}^i(\rho_k) \preccurlyeq_i \mathsf{Payoff}(\rho_{\geq k})$, and $\mu \leq \mathsf{Payoff}(\rho) \leq \nu$.*

The proof of this theorem is based on the next lemma.

Lemma 11. *One can decide in polynomial time whether there exists an infinite path ρ in G such that for each $i \in \{1,2\}$, $\mu_i \sim_i \mathsf{Payoff}_i(\rho) \sim'_i \nu_i$ with $\sim_i, \sim'_i \in \{<, \leq\}$.*

Proof. We begin with InfMP payoff (the proof can be easily adapted to SupMP payoff). Lemma 11 is stated in [29] and proved in [30], but for $\sim_i, \sim'_i$ equal to $\leq$ only. The proposed proof reduces the existence of the required path ρ to the existence of a solution of a linear program (there is no linear objective function to optimize, just a finite set of linear constraints to solve). One can check in polynomial time whether there exists a solution to a linear program [28]. In our case, the proof of [30] leads to a set of linear constraints with both strict and non strict inequalities. One can also check in polynomial time the existence of a solution of such a set of constraints (see Lemma 13 in the appendix).

In the case of LimInf payoff, let us show a reduction to the emptiness problem for Rabin automata (the proof is similar for LimSup payoff). Let ρ be a path in G starting in v_0 such that for each $i \in \{1,2\}$, $\mu_i \sim_i \mathsf{LimInf}_i(\rho) \sim'_i \nu_i$. Equivalently, for each $i \in \{1,2\}$, there exists $n_i \geq 0$ such that $\mu_i \sim_i r_i(\rho_k, \rho_{k+1})$ for all $k \geq n_i$, and $r_i(\rho_l, \rho_{l+1}) \sim'_i \nu_i$ for infinitely many l. This is the conjonction of two Rabin conditions, one for each component i. From G, we construct a Rabin automaton $\mathcal{R}_i$, $i \in \{1,2\}$, with initial vertex v_0, and sets of vertices and edges V'_i, E'_i defined as follows. Each edge $e = (v, v') \in E$ is split into two consecutive edges, and the new intermediate vertex is decorated with $r_i(e)$. The set V'_i has thus $|V| + |E|$ vertices, such that vertices of V are decorated with

Algorithm 1 $ConstrainedExistence(G, \mu, \nu)$

1: **for** each $s_1, s_2 \in V$ **do**
2: Compute the subgraph G' of G whose vertices v are such that $\mathrm{Val}^i(v) \preccurlyeq_i \mathrm{Val}^i(s_i)$, for $i \in \{1, 2\}$
3: **if** there exists a play ρ in G' such that $\mathrm{Val}^i(s_i) \preccurlyeq_i \mathsf{Payoff}(\rho)$, for $i \in \{1, 2\}$, and $\mu \leq \mathsf{Payoff}(\rho) \leq \nu$ **then**
4: **return** True
5: **end if**
6: **end for**
7: **return** False

$+\infty$. We then define one Rabin pair (A_i, B_i) such that $A_i \subseteq V_i'$ (resp. $B_i \subseteq V_i'$) is composed of all vertices decorated by a such that $\neg(\mu_i \sim_i a)$ (resp. by b such that $b \sim_i' \nu_i$). Consider the Rabin automaton $\mathcal{R}$ being the intersection of the automata $\mathcal{R}_1$ and $\mathcal{R}_2$. We have that there exists an infinite path ρ in G such that $\forall i \in \{1, 2\}, \mu_i \sim_i \mathsf{Payoff}_i(\rho) \sim_i' \nu_i$ if and only if there exists an accepting path in $\mathcal{R}$. The latter property can be checked in polynomial time [21]. $\square$

Proof (of Theorem 10). Notice that each Payoff_i is prefix-independent in Theorem 10 (see Remark 1). Hence condition $\forall k \geq 0, \forall i \in \{1, 2\}, \mathrm{Val}^i(\rho_k) \preccurlyeq_i \mathsf{Payoff}(\rho_{\geq k})$ can be replaced by

$$\forall k \geq 0, \forall i \in \{1, 2\}, \quad \mathrm{Val}^i(\rho_k) \preccurlyeq_i \mathsf{Payoff}(\rho). \tag{19}$$

We thus propose Algorithm 1 to solve Problem 4.

To prove its soundness, assume that the algorithm returns True. Hence, there exist a subgraph G' of G depending on two vertices s_1, s_2, and a play ρ in G' such that $\mathrm{Val}^i(s_i) \preccurlyeq_i \mathsf{Payoff}(\rho)$, for $i \in \{1, 2\}$, and $\mu \leq \mathsf{Payoff}(\rho) \leq \nu$. It follows that ρ is a play in G that satisfies (19) and $\mu \leq \mathsf{Payoff}(\rho) \leq \nu$.

To prove that the algorithm is complete, let ρ be a play that satisfies (19) and $\mu \leq \mathsf{Payoff}(\rho) \leq \nu$. For each i, let $s_i \in V$ be such that $\mathrm{Val}^i(s_i)$ is the maximum value among $\mathrm{Val}^i(\rho_k)$, $k \geq 0$ (for the order $\preccurlyeq_i$). Therefore $\mathrm{Val}^i(s_i) \preccurlyeq_i \mathsf{Payoff}(\rho)$, for $i \in \{1, 2\}$. It follows that ρ is a play in the subgraph G' of G whose vertices v are such that $\mathrm{Val}^i(v) \preccurlyeq_i \mathrm{Val}^i(s_i)$, for $i \in \{1, 2\}$. As $\mu \leq \mathsf{Payoff}(\rho) \leq \nu$, the algorithm will return True.

Let us now explain how to check the existence of a path as indicated in line 3 of Algorithm 1. If one recalls the definition of the orders $\preccurlyeq_1$ and $\preccurlyeq_2$, one notices that the condition that a path ρ has to satisfy in line 3 is equivalent to the disjunction of four conditions of the form $\forall i \in \{1, 2\}, x_i \sim_i \mathsf{Payoff}_i(\rho) \sim_i' y_i$ with $x, y \in (\mathbb{Q} \cup \{\pm\infty\})^2$ and $\sim_i, \sim_i' \in \{<, \leq\}$. The latter conditions can be checked in polynomial time by Lemma 11.

Hence, Algorithm 1 is correct and works in polynomial time. $\square$

5.2 Solution to Problem 3

We now have all the required material to prove Theorem 3 for all the payoffs except Disc^λ payoff. We begin with InfMP, SupMP, LimInf and LimSup payoffs, since Inf and Sup payoffs require a separate proof.

Proof (of Theorem 3 for InfMP*,* SupMP*,* LimInf *and* LimSup *payoffs).* By Remark 1, and Theorems 7, 8 and 10, we see that the hypotheses of Theorem 6 are satisfied. Therefore, given an initialized weighted game $(\mathcal{G}, v_0)$ and two thresholds $\mu, \nu \in (\mathbb{Q} \cup \{\pm\infty\})^2$, one can decide whether there exists a secure equilibrium (σ_1, σ_2) in $(\mathcal{G}, v_0)$ such that $\mu \leq \mathsf{Payoff}(\langle\sigma_1, \sigma_2\rangle_{v_0}) \leq \nu$. Let us come back to the algorithm that was proposed to solve this problem and let us study its complexity (see the proof of Theorem 6): (1) we compute $\mathrm{Val}^i(v)$ for each vertex v of $\mathcal{G}^{\preccurlyeq_i}$, $i \in \{1, 2\}$; (2) we construct from $(\mathcal{G}, v_0)$ the graph $G = (V, E, v_0, r, \mathrm{Val})$ on which Algorithm 1 is applied. For each payoff $\mathsf{InfMP}, \mathsf{SupMP}, \mathsf{LimInf}$ and LimSup, step (2) can be done in polynomial time by Theorem 10. For LimInf and LimSup payoffs, step (1) can also be done in polynomial time by Theorem 8.

To complete the proof, it remains to show that step (1) is in $\mathsf{NP} \cap \mathsf{co\text{-}NP}$ for InfMP and SupMP payoffs. By Theorem 8, we know that (a) optimal strategies are uniform for InfMP and SupMP lexicographic payoff games, and (b) deciding whether the value of a vertex is greater than or equal to a given threshold is in $\mathsf{NP} \cap \mathsf{co\text{-}NP}$. Therefore, by (a) the value of a vertex is a bounded rational (between 0 and $|R|$[15]) whose denominator is at most equal to $|V|$ (the maximal length of a simple cycle), thus leading to an exponential number of possibilities. To compute this value, we can apply a simple dichotomy that uses (b). We thus have an algorithm using a polynomial number of calls to an oracle in $\mathsf{NP} \cap \mathsf{co\text{-}NP}$. Since $\mathsf{P}^{\mathsf{NP} \cap \mathsf{co\text{-}NP}} = \mathsf{NP} \cap \mathsf{co\text{-}NP}$ [4], step (1) is in $\mathsf{NP} \cap \mathsf{co\text{-}NP}$ as announced. □

Proof (of Theorem 3 for Inf *and* Sup *payoffs).* We use the same reduction to LimInf and LimSup weighted games as done in the proof of Theorems 1 and 2 for Inf and Sup payoffs. As Theorem 3 holds for LimInf and LimSup payoffs, it also holds for Inf and Sup payoffs. □

5.3 Particular Case of Disc^λ Payoff

In this section, we consider Disc^λ weighted games. We are going to show that Problem 3 for these games is related to the next problem whose decidability is unknown [10]. Moreover the latter problem is related to other hard open problems in diverse mathematical fields according to [3].

Problem 5. Given three rational numbers a, b and t, and a rational discount factor $\lambda \in]0, 1[$, does there exist an infinite sequence $w = w_0 w_1 \ldots \in \{a, b\}^\omega$ such that $\sum_{k=0}^{\infty} w_k \lambda^k$ is equal to t?

[15] Under Remark 2, recall that $|R|$ is the maximal weight of the game.

Let us provide the next reduction of Problem 5 to Problem 3 for Disc^λ payoff. Let $a, b, t \in \mathbb{Q}$ and a rational factor $\lambda \in]0,1[$. We consider the Disc^λ weighted game $(\mathcal{G}^{a,b}, v_0)$ played on the arena depicted in Figure 4, where c is a rational number such that $c > \max\{|a|, |b|\}$.[16] We also consider the thresholds $\mu = ((1-\lambda)\cdot t, -(1-\lambda)\cdot t)$ and $\nu = (+\infty, +\infty)$. Notice that threshold ν imposes no constraint. We want to show the next proposition:

Proposition 2. *There exists a sequence $w \in \{a,b\}^\omega$ such that $\sum_{k=0}^{\infty} w_k \lambda^k = t$ if and only if there exists a secure equilibrium in $(\mathcal{G}^{a,b}, v_0)$ with outcome ρ such that $\mu \leq \mathsf{Disc}^\lambda(\rho) \leq \nu$.*

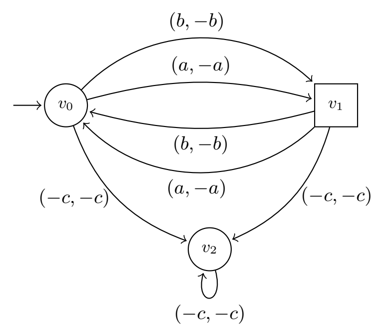

Fig. 4. The Disc^λ weighted game $\mathcal{G}^{a,b}$

In the next lemma, we denote by $\mathcal{A}$ the subarena of $\mathcal{G}^{a,b}$ restricted to the set of vertices $\{v_0, v_1\}$.

Lemma 12. *For all infinite paths ρ of $\mathcal{G}^{a,b}$ starting in v_0, ρ is a path in $\mathcal{A}$ if and only if there exists a secure equilibrium (σ_1, σ_2) in $(\mathcal{G}^{a,b}, v_0)$ with outcome ρ.*

Proof. Suppose that ρ is a path in $\mathcal{A}$. We constuct the strategy profile (σ_1, σ_2) as follows. Both strategies follow ρ, and as soon as a player deviates from this path, the other player immediately deviates to vertex v_2. It is clear that no player has an incentive to deviate. Indeed both players want to maximize their discounted sum, and the self loop on vertex v_2 is labelled with $-c$ which is strictly smaller than all the weights in the subarena $\mathcal{A}$. So no deviation from ρ is profitable, and (σ_1, σ_2) is a secure equilibrium.

Suppose now that ρ is the outcome of a secure equilibrium (σ_1, σ_2) in $(\mathcal{G}^{a,b}, v_0)$. By contradiction, assume that ρ visits vertex v_2. Consider the smallest k such that $\rho_{k+1} = v_2$. Without lost of generality suppose that $\rho_k = v_0$. Then player 1 has a strategy σ_1' that goes to v_1 and thus avoids going to v_2 (for at least one

[16] Notice that $(\mathcal{G}^{a,b}, v_0)$ have mutiple edges between two vertices. Adequate intermediate vertices should be added to respect Definition 1.

more round). This is a profitable deviation for him, because all the edges that go to v_2 have reward $(-c,-c)$, which is strictly smaller than rewards in the sub-arena $\mathcal{A}$. This is in contradiction with (σ_1, σ_2) being a secure equilibrium. □

Proof (of Proposition 2). Lemma 12 states that paths ρ in $\mathcal{A}$ that start in v_0 are exactly outcomes of secure equilibria in $(\mathcal{G}^{a,b}, v_0)$. Moreover, by definition of the weight function of $\mathcal{G}^{a,b}$, we have $\mathsf{Disc}_1^\lambda(\rho) = -\mathsf{Disc}_2^\lambda(\rho)$. Therefore there exists a sequence $w \in \{a,b\}^\omega$ such that $\sum_{k=0}^{\infty} w_k \lambda^k = t$ if and only if there exists a secure equilibrium in $(\mathcal{G}^{a,b}, v_0)$ with outcome ρ such that $\mu \leq \mathsf{Disc}^\lambda(\rho)$ $(\leq \nu)$. □

We have thus shown that Problem 5 reduces to Problem 3 for Disc^λ payoff. Therefore Problem 5 is decidable if Problem 3 is itself decidable for this payoff.

6 Conclusion

In this paper, we have studied Problems 1-3 about secure equilibria in two-player weighted games, and proposed three general frameworks in which we can solve them. We have proved that weighted games with a classical payoff function like InfMP, SupMP, LimInf, LimSup, Inf, Sup and Disc^λ, all fall in these frameworks, except the Disc^λ payoff for Problem 3. We have shown that this particular problem is linked to another challenging open problem [3,10].

Our approach was inspired by the recent work [7] that states the existence of Nash equilibria in a large class of multi-player weighted games. As in [7], we have considered two particular zero-sum games $\mathcal{G}^{\preccurlyeq_1}$, $\mathcal{G}^{\preccurlyeq_2}$ associated with the initial game $\mathcal{G}$; we have proved that they are uniformly-determined and studied their complexity for all the classical payoffs. These results are very interesting on their own right, and to the best of our knowledge, were not studied in the literature, except for a variant of the InfMP payoff studied in [2]. For InfMP payoff, our proofs were inspired by techniques developed in [2] and [29], however with far from trivial adaptations.

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Appendix

Lemma 13. *Let*

$$\begin{cases} \bar{A}_i \cdot \bar{x} \quad > b_i & 1 \leq i \leq n \\ \bar{A}_{n+j} \cdot \bar{x} \geq b_{n+j} & 1 \leq j \leq m \end{cases}$$

be a system of n strict linear constraints and m non strict linear constraints. Then one can decide in polynomial time whether this system has a solution.

Proof. The problem of deciding if the previous system S has a solution can be reduced to the following linear program P where y is a new variable:

$$\max y$$

$$\begin{cases} \bar{A}_i \cdot \bar{x} - y \geq b_i & 1 \leq i \leq n \\ \bar{A}_{n+j} \cdot \bar{x} \quad \geq b_{n+j} & 1 \leq j \leq m \end{cases}$$

If P has no solution, then S has no solution neither. If P has a solution, depending on whether the optimal value y satisfies $y > 0$ or $y \leq 0$, then S has a solution or not. $\square$