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FINITELY CONSTRAINED CLASSES OF HOMOGENEOUS DIRECTED GRAPHS

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Abstract. Given a finite relational language L is there an algorithm that, given two finite sets $\mathscr{A}$ and $\mathscr{B}$ of structures in the language, determines how many homogeneous L structures there are omitting every structure in $\mathscr{B}$ and embedding every structure in $\mathscr{A}$?

For directed graphs this question reduces to: Is there an algorithm that, given a finite set of tournaments Γ, determines whether $\mathscr{C}_\Gamma$, the class of finite tournaments omitting every tournament in Γ, is well-quasi-order?

First, we give a nonconstructive proof of the existence of an algorithm for the case in which Γ consists of one tournament. Then we determine explicitly the set of tournaments each of which does not have an antichain omitting it. Two antichains are exhibited and a summary is given of two structure theorems which allow the application of Kruskal's Tree Theorem. Detailed proofs of these structure theorems will be given elsewhere.

The case in which Γ consists of two tournaments is also discussed.

§1. Introduction.

1.1. The classification problem. A relational language is a language with only relation symbols. We say a structure in a relational language is *homogeneous* if any isomorphism between finite substructures can be extended to an automorphism of the entire structure. It is not known in general if the homogeneous structures for a given relational language can be classified (throughout this paper all structures are assumed to be countable). The classification is known in the cases of directed graphs and tournaments.

DEFINITION 1.1. A *directed graph D* is a structure consisting of a set of vertices together with an irreflexive asymmetric binary relation E, the arc relation.

DEFINITION 1.2. A *tournament T* is a directed graph whose arc relation satisfies: for every $x \neq y$ in T, either $E(x,y)$ or $E(y,x)$ holds.

Lachlan classified the countable homogeneous tournaments; there are exactly five [12], [6]. On the other hand, Henson [9] showed that there are $2^{\aleph_0}$ countable homogeneous directed graphs, the maximum conceivable. Cherlin has classified the countable homogeneous directed graphs [3], [4], [5]. There are countably many "exceptional" ones that are distributed among finitely many families. There are $2^{\aleph_0}$ "generic" ones that are generated using a method developed by Henson. Lachlan has shown that, given any finite relational language, there are only countably many stable homogenous structures [13]. The structures are contained in finitely many families.

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Cherlin has summarized in [7] what is known if the stability condition is dropped. He has also formulated the classification problem in general terms. Let L be a finite relational language. Given a finite set $\mathscr{A}$ of finite structures in the language, called the positive constraints, and a finite set $\mathscr{B}$ of finite structures, called the negative constraints, a solution to $(\mathscr{A}, \mathscr{B})$ is a homogeneous L-structure that embeds every structure in the set of positive constraints and omits every structure in the set of negative constraints. We consider the general question: given a finite relational language, is there an algorithm that given two finite sets $\mathscr{A}$ and $\mathscr{B}$ of structures in the language determines how many solutions there are to $(\mathscr{A}, \mathscr{B})$?

1.2. The classification problem for directed graphs.

DEFINITION 1.3. A quasi-ordering is a set with a binary relation that is reflexive and transitive. A quasi-ordering is said to be a *well-quasi-ordering*, *wqo*, if it contains no infinite strictly descending sequence and no infinite antichain.

DEFINITION 1.4. Let T_1 and T_2 be tournaments. We say that T_1 *embeds* in T_2, $T_1 \leq T_2$, if there is an injective function $f: T_1 \to T_2$ such that if $E(x, y)$ holds in T_1 then $E(f(x), f(y))$ holds in T_2.

The embeddability relation defines a quasi-ordering on the class of all tournaments, $\mathscr{T}$.

DEFINITION 1.5. We say that the tournament T_2 *omits* the tournament T_1 if $T_1 \not\leq T_2$.

DEFINITION 1.6. Let Γ be a finite set of tournaments. Define $\mathscr{Q}_\Gamma$ to be the class of finite tournaments T such that T omits every element of Γ. We say that a class of tournaments $\mathscr{T}$ *omits* Γ if $\mathscr{T} \subseteq \mathscr{Q}_\Gamma$.

Using Cherlin's classification of the countable homogeneous directed graphs, we show, following [7], that for directed graphs the general question posed above reduces to: Is there an algorithm that, given a finite set of finite tournaments Γ, determines whether $\mathscr{Q}_T$ is wqo?

DEFINITION 1.7. Let $\{T_n\}_{n \in I}$ be a set of tournaments. $\{T_n\}_{n \in I}$ is an *antichain* if $T_i \not\leq T_j$ for all $i \neq j$ in the index set I.

Since $\mathscr{Q}_\Gamma$ has no infinite strictly descending sequence, $\mathscr{Q}_\Gamma$ is wqo if and only if it has no infinite antichain.

If Γ is a set with a single element T, then we abbreviate $\mathscr{Q}_\Gamma$ to $\mathscr{Q}_T$. We say a finite tournament T is *tight* if $\mathscr{Q}_T$ wqo. we say T is *loose* if $\mathscr{Q}_T$ contains an infinite antichain. We let $\mathscr{Q}_1$ be the class of tight tournaments. This terminology reflects the fact that the tournament T is playing the role of a constraint here, and the corresponding class is to be considered tightly constrained if it is a wqo under embeddability. The generalization $\mathscr{Q}_k$ will be introduced in §4.

We give a nonconstructive proof of the existence of an algorithm for determining membership in $\mathscr{Q}_1$. We then determine the elements of $\mathscr{Q}_1$ explicitly. This is accomplished by giving the set Q of isomorphism types of the maximal nontransitive elements in $\mathscr{Q}_1$ and the set M of isomorphism types of the minimal elements not in $\mathscr{Q}_1$. To show that these sets are what we say they are, three tasks must be accomplished.

(1) We must construct antichains which together omit every element of M.

(2) We must verify that $Q \subseteq \mathscr{Q}_1$.

(3) We must show that if a nontransitive tournament T does not embed an element of M, then T embeds in an element of Q.

We accomplish the first and last tasks in §5. The sketch of the proofs required for the second task are given in §6 with the detailed proofs to be given elsewhere. In §7 the case in which Γ consists of two tournaments is discussed.

§2. The reduction to one set of constraints.

2.1. Cherlin's classification theorem. In Cherlin's classification of the homogeneous directed graphs there are countably many exceptional examples and uncountably many that are generated by a Henson style construction.

Our analysis of the classification problem for finitely constrained classes of directed graphs depends on a closer examination of the homogeneous directed graphs that are generated using a Henson style construction. As this class is best described in terms of Fraissé's theory of amalgamation classes [8], we first review this theory in the context of directed graphs.

An amalgamation class $\mathbb{A}$ is a collection of finite directed graphs which is closed under isomorphism, substructure, and has the amalgamation property:

> whenever $D_0, D_1, D_2 \in \mathbb{A}$ and $D_0 \leq D_1$, $D_0 \leq D_2$, then there is a $D_3 \in \mathbb{A}$ and embeddings $D_1 \leq D_3$, $D_2 \leq D_3$ such that the image of D_0 in D_3 under the D_1 embedding is the same as the image of D_0 under the D_2 embedding.

Fraissé's theory of amalgamation classes tells us that every countably infinite homogeneous directed graph D corresponds to an amalgamation class $\mathbb{A}(D) = \{F : F \text{ finite}, F \leq D\}$ of finite directed graphs, and that, conversely, for any amalgamation class $\mathbb{A}$ of finite directed graphs, there is a unique countably infinite homogeneous directed graph associated with it, embedding every element of $\mathbb{A}$.

Let χ be a set of finite tournaments and let $\mathbb{A}_\chi$ be the class of finite directed graphs F such that all subtournaments of F are isomorphic to subtournaments of elements of χ. Then $\mathbb{A}_\chi$ is an amalgamation class, and the tournaments in $\mathbb{A}_\chi$ are exactly those which embed in some element of χ. By Fraissé's theory, the corresponding homogeneous directed graph embeds those tournaments, and only those tournaments, which embed in some element of χ. In particular, if $\mathscr{I}$ is an infinite antichain of finite tournaments, and χ, χ' are distinct subsets of $\mathscr{I}$, then the homogeneous directed graphs associated with $\mathbb{A}_\chi$ and $\mathbb{A}_{\chi'}$ are nonisomorphic. This is Henson's construction which products the $2^{\aleph_0}$ homogeneous directed graphs in Cherlin's classification.

If the antichain $\mathscr{I}$ omits a set of tournaments, then the above construction yields $2^{\aleph_0}$ homogeneous directed graphs omitting that set of tournaments.

2.2. Decision problems for finitely constrained classes of directed graphs. Given a finite relational language L, let $\mathscr{F}$ be the class of pairs $(\mathscr{A}, \mathscr{B})$ of positive and negative constraints such that there are at most finitely many solutions to $(\mathscr{A}, \mathscr{B})$. Let $\mathscr{C}$ be the class of pairs $(\mathscr{A}, \mathscr{B})$ such that there are countably many solutions to $(\mathscr{A}, \mathscr{B})$. Let $\mathscr{U}$ be the class of pairs $(\mathscr{A}, \mathscr{B})$ such that there are uncountably many solutions to $(\mathscr{A}, \mathscr{B})$.

Fix L to be the language of directed graphs. We will show that $\mathscr{F}$ is recursive. Then the question of the existence of an algorithm determining the number of solutions to a given pair of constraints reduces to showing that $\mathscr{U}$ is recursive. We show that $\mathscr{U}$ is recursive if there is an algorithm that given a finite set of tournaments Γ, determines whether there is an infinite antichain of tournaments which omits Γ. Since it is possible to determine how many solutions to a given pair of constraints occur among the exceptional homogeneous directed graphs of Cherlin's classification, it is enough to show that $\mathscr{F}$ and $\mathscr{U}$ are recursive when restricted to the class of homogeneous directed graphs that are generated using a Henson style construction.

Let $\mathscr{F}^*$ be the class of pairs $(\mathscr{A}, \mathscr{B})$ of positive and negative constraints such that there are at most finitely many solutions to $(\mathscr{A}, \mathscr{B})$ of the form $\mathbb{A}_\chi$ with χ a set of finite tournaments. We define $\mathscr{C}^*$ and $\mathscr{U}^*$ similarly, but we note that by the classification theorem, $\mathscr{C}^* = \mathscr{C}$ and $\mathscr{U}^* = \mathscr{U}$.

In the proofs below we identify a solution D with its amalgamation class $\mathbb{A}(D)$ whenever it is convenient. The constraints are the pair $(\mathscr{A}, \mathscr{B})$.

We define two conditions and show that each of these conditions provides a test for membership in a particular class.

Condition 1. For some $B \in \mathscr{B}$ every nontransitive subtournament of B embeds in some $A \in \mathscr{A}$.

Condition 2. There is a set of finite tournaments Γ such that $\mathscr{Q}_\Gamma$ contains an infinite antichain and

(1) every $T \in \Gamma$ embeds in some $B \in \mathscr{B}$ and is omitted by every $A \in \mathscr{A}$;

(2) for every $B \in \mathscr{B}$, some $T \in \Gamma$ embeds in B.

LEMMA 2.1. *$\mathscr{F}^*$ is recursive.*

PROOF. Suppose Condition 1 holds for some $B \in \mathscr{B}$ with n vertices. Suppose $\mathbb{A}_\chi$ is a solution to the pair of constraints $(\mathscr{A}, \mathscr{B})$. If χ does not omit the transitive tournament on n vertices, then $\mathbb{A}_\chi$ contains B, so there is no solution. If χ omits the transitive tournament on n vertices, the elements of χ are of uniformly bounded size. In either case, $(\mathscr{A}, \mathscr{B}) \in \mathscr{F}^*$.

If Condition 1 fails and L_n is transitive and larger than any $A \in \mathscr{A}$, then infinitely many solutions are determined by the following sets of tournaments:

χ_0 is the set of all subtournaments of elements of $\mathscr{A}$.

$\chi_i = \chi_0 \cup \{L_{n+i}\}$. $\square$

LEMMA 2.2. *The following are equivalent for a finite set of finite tournaments* Γ.

(1) *there are $2^{\aleph_0}$ countable homogeneous directed graphs omitting every tournament in* Γ.

(2) *there is an infinite antichain of tournaments omitting* Γ

PROOF. $1 \Rightarrow 2$. Suppose there is no such infinite antichain and χ is a downward closed set of finite tournaments which omits Γ. Let $\mathscr{Y}$ be the set of minimal tournaments not in χ. Then $\mathscr{Y}$ is an antichain and $\mathscr{Y} \backslash \Gamma$ omits Γ. Thus $\mathscr{Y} \backslash \Gamma$ is finite and so $\mathscr{Y}$ is finite. As $\mathscr{Y}$ determines χ, there are only countably many $\mathbb{A}_\chi$, a contradiction.

$2 \Rightarrow 1$. Let $\mathscr{I}$ be an infinite antichain in $\mathscr{Q}_\Gamma$. If χ, χ' are distinct subsets of $\mathscr{I}$, then the homogeneous directed graphs associated with $\mathbb{A}_\chi$ and $\mathbb{A}'_\chi$ are nonisomorphic. $\square$

We now show that the question of $\mathscr{U}$ being recursive can be reduced to a question of the existence of an infinite antichain of tournaments.

LEMMA 2.3. *If there is an algorithm which given a finite set of tournaments Γ, determines whether $\mathscr{Q}_\Gamma$ contains an infinite antichain, then there is an algorithm which, given a pair $(\mathscr{A}, \mathscr{B})$ of finite sets of directed graphs, determines whether there are $2^{\aleph_0}$ solutions to $(\mathscr{A}, \mathscr{B})$.*

PROOF. Since $\mathscr{B}$ is finite, there are only finitely many such Γ which satisfy clauses (1) and (2) of Condition 2, so Condition 2 is easily checked recursively. We show that $(\mathscr{A}, \mathscr{B}) \in \mathscr{U}^*$ if and only if Condition 2 holds.

Suppose Condition 2 holds. Let $\mathscr{I}$ be an infinite antichain in $\mathscr{Q}_\Gamma$. Let $\mathscr{A}^*$ be the set of subtournaments of elements of $\mathscr{A}$. Since $\mathscr{A}^*$ is finite, we many assume that $\mathscr{A}^* \cap \mathscr{I} = \varnothing$. Let χ be a subset of $\mathscr{I}$. Consider the amalgamation class, $\mathbb{A} = \mathbb{A}_{\chi \cup \mathscr{A}^*}$. $\mathscr{A} \subset \mathbb{A}$, and by clause (2) of Condition 2, no element of $\mathscr{B}$ is in $\mathbb{A}$. So $\mathbb{A}$ is a solution to $(\mathscr{A}, \mathscr{B})$. Distinct subsets of $\mathscr{I}$ give us nonisomorphic solutions, so there are $2^{\aleph_0}$ solutions to $(\mathscr{A}, \mathscr{B})$.

Suppose Condition 2 fails. Let $\mathbb{A}$ be a solution to $(\mathscr{A}, \mathscr{B})$. $\mathbb{A}$ is determined by the upward closed class $\mathscr{Y}$ of tournaments not in $\mathbb{A}$. Let Γ be the set of subtournaments of elements $\mathscr{B}$ that are not in $\mathbb{A}$. $\Gamma \subset \mathscr{Y}$ and Γ is finite. As Γ satisfies clauses (1) and (2) of Condition 2, $\mathscr{Q}_\Gamma$ contains no infinite antichain. Let $\mathscr{Y}^M =$ (the minimal elements of $\mathscr{Y}$)$\backslash\Gamma$. $\mathscr{Y}^M$ is an antichain in $\mathscr{Q}_\Gamma$, and hence is finite. Since $\mathbb{A}$ is determined by the finite set $\mathscr{Y}^M \cup \Gamma$ there can be only countably many such $\mathbb{A}$, for each class Γ. As there are only finitely many classes Γ satisfying clauses (1) and (2) of Condition 2, $(\mathscr{A}, \mathscr{B}) \in \mathscr{C}$. $\square$

Before proceeding with the proof that the algorithm of Lemma 2.3 exists for the case in which Γ consists of one tournament, we need some terminology and results relating to tournaments.

§3. Some basic definitions and results relating to tournaments.

Let x, y be vertices in a tournament T. We will use the notation $x \to y$ to mean $E(x, y)$ holds in T. If we say that xy is an arc in T, we mean $x \to y$.

If T_1, T_2 are subtournaments of T, we use the notation $T_1 \Rightarrow T_2$ if for every vertex x in T_1 and every vertex y in $T_2, x \to y$. In particular, if T_1 is a single vertex, x, then $x \Rightarrow T_2$ means $x \to y$ for every vertex y in T_2. Similarly, for $T_1 \Rightarrow x$.

By $|T|$ we mean the cardinality of the vertex set of T.

DEFINITION 3.1. If V is a subset of the vertex set of the tournament T, then the *induced subtournament* on V is the tournament with the vertex set V whose arc relation is the restriction of the arc relation of T to $V \times V$.

DEFINITION 3.2. The *dual* of a tournament T is the tournament obtained by reversing the orientation of every arc in T.

DEFINITION 3.3. Given a tournament T and a vertex a in T, let $'a$ and a' be the induced subtournaments on the vertex set $\{b \in T : b \to a\}$ and $\{b \in T : a \to b\}$, respectively.

3.1. Special tournaments and p-arcs.

DEFINITION 3.4. A tournament T is *transitive* if the arc relation is transitive.

There are several tournaments which are discussed frequently enough to merit standard names. The most common ones are as follows:

I is the tournament on a unique vertex and no arcs.

L_n is a transitive tournament on the vertex set $\{1, \dots, n\}$ with the arc relation $i \to j$ if $i < j$.
N_n is the tournament obtained from L_n by reversing the orientation of the arc between i and $i + 1$.
C_3 is the tournament on the vertex set $\{a, b, c\}$ with arcs ab, bc, and ca.
$C_3(A_1, A_2, A_3)$ is the tournament obtained by replacing the vertices of C_3 with the tournaments A_1, A_2, and A_3.
The tournament $T_1 T_2$ has subtournaments T_1, T_2, and $T_1 \Rightarrow T_2$.
The Paley tournament [1] on p vertices, p a prime congruent to 3 mod 4, has as its vertex set $\mathbb{Z}/p\mathbb{Z}$ with the arc relation defined by

$$i \to j \text{ if and only if } j - i \text{ is a square mod } p.$$

PROPOSITION 3.5. *The automorphism group of the Paley tournament acts transitively on the arcs (and, in particular, on the vertices).*

PROOF. Let ij be an arc. It is enough to show that there is an automorphism sending the arc 01 to ij. $\theta(x) = (j - i)x$ and $\phi(x) = x + i$ are automorphisms of the Paley tournament on p vertices. $\phi \circ \theta$ is the desired automorphism. $\square$

B_5, B_6 are the tournaments on 5 or 6 vertices, derived from the Paley tournament on 7 vertices by omitting 2 vertices or 1 vertex, respectively. By the preceding proposition, these tournaments are unique up to isomorphism. (We usually take the vertex sets for B_5 and B_6 to be $\mathbb{Z}/7\mathbb{Z}$ with $\{5, 6\}$ or $\{6\}$ removed, respectively.)

We notice that a tournament T is transitive if and only if its omits the tournament C_3. Another useful observation is:

LEMMA 3.6. *There are four tournaments on four vertices, up to isomorphism, namely, L_4, IC_3, C_3I, and one additional tournament denoted by S_4.*

PROOF. By inspection, considering the possible sequences of outdegrees. $\square$

DEFINITION 3.7. An arc xy is said to be a *p-arc* if xy occurs in at least p distinct copies of C_3. We call the vertex z a *witness* for a p-arc xy if $xyz \simeq C_3$.

3.2. Types.

DEFINITION 3.8. Given two subtournaments T_1, T_2 of a tournament T, the *type* of T_1 over T_2 is the function with domain $T_1 \times T_2$ which assigns to the pair (x, y) the orientation of the corresponding arc. If T_1 has a unique vertex, then the type of T_1 over T_2 is called a *1-type*. When necessary, the type of T_1 over T_2 can be viewed as the sequence of 1-types realized by the elements of T_1. We say that A is the *locus* of a 1-type over T_2 if A is a maximal set of vertices in T, each of which has the same type over T_2. We say that two 1-types over T_2 are *dual* if they assign the opposite orientation to the arc adjacent to the vertex y for every $y \in T_2$. Two loci are dual if their types are dual.

DEFINITION 3.9. Let $S \subseteq T$, and let x be a vertex of $T \backslash S$. We say x has *mixed type* over S if there are vertices $y, z \in S$ such that $x \to y$ and $x \leftarrow z$. Otherwise, we say that x has *unmixed type* over S.

If $S \simeq C_3$ and x has mixed type over S, then we notice that the induced subtournament on $\{x\} \cup S$ is isomorphic to S_4.

3.3. Congruences.

DEFINITION 3.10. A *block* B in a tournament T is a nonempty subset of vertices each of which has the same type over $T \backslash B$.

LEMMA 3.11. *If B_1, B_2 are blocks in a tournament T and $B_1 \cap B_2 \neq \varnothing$, then $B_1 \cup B_2$ is a block.*

PROOF. Let x, y be vertices in $B_1 \cup B_2$, and let z be a vertex in $B_1 \cap B_2$. If w is a vertex in $T \backslash (B_1 \cup B_2)$, then x and y have the same type over w as z. □

DEFINITION 3.12. A *congruence relation* $\sim$ on a tournament T is an equivalence relation such that each of its equivalence classes is a block. $T/\sim$ is the quotient tournament, whose vertices are the $\sim$-classes, and whose arc relation is induced by the arc relation on T.

DEFINITION 3.13. Let T and S be finite tournaments. The tournament $T[S]$ is constructed by replacing each vertex in T with a copy of S.

In particular, there is a congruence on $T[S]$ such that every congruence class is a copy of S and the quotient is a copy of T.

DEFINITION 3.14. We say that a tournament T is *indecomposable* if there is no nontrivial congruence relation on T.

3.4. Local orders. S_{2n+1} is the tournament on $\mathbb{Z}/(2n+1)\mathbb{Z}$ with the arc relation defined by

$$a \to b \quad \text{if and only if} \quad b - a \in \{1, \dots, n\}.$$

($C_3 \simeq S_3$ but we will always use the notation C_3.)

LEMMA 3.15. *The automorphism group of S_{2n+1} is transitive.*

PROOF. $\phi(a) = a + 1$ is an automorphism of S_{2n+1}. □

We will show that S_{2n+1} can be characterized as the indecomposable union of two transitive tournaments and that $\{S_{2n+1}\}_{1 \leq n < \omega}$ are exactly the indecomposable nontransitive tournaments in $\mathscr{Q}_{\{IC_3, C_3I\}}$. The first step in this direction is a theorem of Cameron's [2].

THEOREM 3.16. *For a tournament T the following are equivalent:*

(1) *T omits IC_3 and C_3I.*

(2) *$T = L_1 \cup L_2$, where L_1, L_2 are transitive, and if L is the tournament obtained from T by reversing the orientation of the arcs between L_1 and L_2, then L is transitive.*

PROOF. $1 \Rightarrow 2$. Fix a vertex a in T. Let $L_1 = \{a\} \cup a'$, and let $L_2 = {}'a$. Since T omits IC_3 and C_3I, L_1 and L_2 are transitive.

Consider the tournament L derived from T by taking $L = L_1 \cup L_2$ but with the arcs between L_1 and L_2 given the reverse of their orientation in T. We will show that L is transitive. If $A \leq L$ and $A \simeq C_3$, then A meets both L_1 and L_2, and hence, in T we have $A \simeq L_3$. Label the vertices of A x, y, z with $x \to y \to z$ in L. Here x is in L_1 and y, z are in L_2 or vice versa.

It is easy to see that a is not in A for example, if $a = x$, then it must realize two distinct types over y and z in L, which yields a contradiction. Accordingly, if y, z are in L_2, then $axz \simeq C_3$ and $y \Rightarrow axz$, while if y, z are in L_1, then $ayx \simeq C_3$ and $ayx \Rightarrow z$.

$2 \Rightarrow 1$. We show that T omits IC_3. The other claim follows by duality. Suppose on the contrary that $A \leq T$, $A \simeq C_3$, and there is a vertex a in T with $a \Rightarrow A$.

Label L_1 and L_2 so that a is a vertex in L_2. Since A meets both L_1 and L_2, A contains an arc xy from L_1 to L_2. Then $axy \simeq C_3$ in L, a contradiction. □

DEFINITION 3.17. A tournament which satisfies the two equivalent conditions in Theorem 3.16 is called a *local order.*

In particular S_4 is the unique nontransitive local order on four vertices. Notice also that $S_4 \simeq C_3(I, I, L_2)$ is decomposable.

Knowing the structure of tournaments omitting both IC_3 and C_3I leads us easily to the structure of tournaments omitting at least one of the two.

COROLLARY 3.18. *If the tournament T omits C_3I (IC_3), then $T = LS$ (SL), where L is transitive and S is a local order.*

PROOF. Define S to be the set of vertices in T which participate in copies of C_3. Then $L = T \backslash S$ is transitive. We claim that $L \Rightarrow S$. Suppose not. Then there are vertices a in L and x in S such that $x \to a$. By the definition of S, there are vertices y, z in S such that $xyz \simeq C_3$. Then a cannot have mixed type over xyz, so $xyz \Rightarrow a$, a contradiction.

It remains to show that S is a local order. Suppose not. Then by Theorem 3.16, S contains a copy of IC_3. Let a, b, c, x be vertices in S such that $abc \simeq C_3$ and $x \Rightarrow abc$. Then there are vertices y, z in S such that $xyz \simeq C_3$. We may assume that z is distinct from a, b, c, x. Each of a, b, c has mixed type over xyz, so we may assume that $xaz \simeq C_3$. Since $xa \Rightarrow b$, we have that $b \to z$ and $bzx \simeq C_3$. Now $ab \Rightarrow z$ so $z \to c$, hence $bzx \Rightarrow c$, a contradiction. □

We now show that $\{S_{2n+1}\}_{1 \leq n < \omega}$ are exactly the indecomposable nontransitive local orders.

LEMMA 3.19. *S_{2n+1} is an indecomposable local order.*

PROOF. S_{2n+1} has a transitive automorphism group; therefore, to show that S_{2n+1} omits IC_3 and C_3I it is enough to show that $0'$ and $'0$ omit C_3. This is clear since $0' \simeq L_n$ and $'0 \simeq L_n$.

Let C be a nontrivial congruence class in S_{2n+1}. We may suppose that $0, a \in C$ for some $a \neq 0$ with $0 \to a$. Since $a \to n+1 \to 0$, $n+1 \in C$. As $0, n+1 \in C$, easily $C = S_{2n+1}$. □

LEMMA 3.20. *If S is an indecomposable local order on $k > 2$ vertices, then $k = 2n+1$, for some n, and $S \simeq S_{2n+1}$.*

PROOF. Since $k > 2$, we have that S is nontransitive. By Theorem 3.16, $S = L_1 \cup L_2$, where L_i is transitive and if $L = L_1 \cup L_2$ with the arcs reversed between L_1 and L_2, then L is transitive, and hence, totally ordered by the arc relation. If u and v are vertices in L_i which are consecutive in L, then $\{u, v\}$ is a block in S, a contradiction. So the vertices of L_1 and L_2 alternate in L. Number the vertices in S according to their order in L, so that v_1 is the initial vertex in L and v_k is the terminal vertex. We may assume $v_1 \in L_1$. If k is even, then $v_k \in L_2$. In S we have that v_1 dominates every vertex in $L_1 \backslash \{v_1\}$ and is dominated by every vertex in $L_2 \backslash \{v_k\}$. The same holds for v_k, hence $v_1 \sim v_k$ in S, a contradiction.

If k is odd the same analysis shows that S is unique, and we have already checked that S_k is one possibility. □

We now show that every local order is embeddable in one of the form $S_{2n+1}[L_m]$.

LEMMA 3.21. *Let S be a local order, and let $\sim$ be a congruence relation on S. If B is a $\sim$-class, then B is transitive or $B = S$.*

PROOF. Suppose $B \neq S$ is a $\sim$-class containing a copy C of C_3. Let $u \in S \backslash B$. Then $\{u\} \cup C$ is a copy of IC_3 or $C_3 I$, a contradiction. □

LEMMA 3.22. *Let S be a local order. Then $S \leq S_{2n+1}[L_m]$ for some $n \geq 0$ and $m \geq 1$.*

PROOF. If $|S| \leq 2$, then $S \simeq L_m$, where $m = |S|$. We consider the case $|S| > 2$. If S is indecomposable, then $S \simeq S_{2n+1}$ for some n, and we take $m = 1$. So we may suppose that S is decomposable. Let $\sim$ be a nontrivial congruence relation on S such that the quotient is indecomposable. Then $S/\sim$ is isomorphic to S_{2n+1} for some n. Let m be the maximum of the sizes of the $\sim$-classes. Then every $\sim$-class embeds in L_m, and $S \leq S_{2n+1}[L_m]$. □

§**4. A nonconstructive proof of decidability of $\mathscr{Q}_1$.** Recall that $\mathscr{Q}_1$ is the class of tournaments T such that $\mathscr{Q}_T$ is wqo. In the next section it will be shown that

$$(*) \qquad C_3L_2, L_2C_3, \text{ and } C_3(I, I, L_4) \text{ are not in } \mathscr{Q}_1.$$

This gives us a quick proof that $\mathscr{Q}_1$ is recursive. In fact, it gives us an even stronger result.

THEOREM 4.1. *If T is a finite, nontransitive tournament with $|T| \geq 24$, then $T \notin \mathscr{Q}_1$.*

Note. Since the acceptance of this paper, my students J. Komsa, E. Murray, and E. Weiss modified the proof to reduce 24 to 18.

This theorem certainly gives the decidability of $\mathscr{Q}_1$, in polynomial time, but does not actually yield a decision procedure.

We will use the following fact repeatedly: if we have a tournament $C \simeq C_3$ and a vertex x outside C realizing a mixed type over C, then x belongs to a copy of C_3 contained in $\{x\} \cup C$. So x serves as a witness to an arc in C. This may be seen at once by inspection.

PROOF OF THEOREM 4.1. Let T be a finite, nontransitive tournament. It is enough to show that if $|T| \geq 24$ then T embeds C_3L_2, L_2C_3, or $C_3(I, I, L_4)$, since, by $(*)$, none of these tournaments is tight.

Since T is nontransitive, there is a $C \leq T$ such that $C \simeq C_3$. Every vertex in $T \backslash C$ has one of 2^3 possible types over C. Either T embeds L_2C_3, embeds C_3L_2, or there are two types over C with at most one vertex. If the third case holds, then there are 19 or more vertices having one of the 6 remaining types. By the pigeonhole principle, we have a subtournament T_1 of T such that every vertex of T_1 has the same type over C and $|T_1| \geq 4$. So T_1 contains a subtournament T_2 isomorphic to L_3.

If $T_2 \Rightarrow C$, then $L_2C_3 \leq T_2C_3 \leq T$. If $C \Rightarrow T_2$, then $C_3L_2 \leq C_3T_2 \leq T$. In the remaining case, each vertex x of T_2 is a witness to the same arc ab in C. Let c be the third vertex of C. Then c is also a witness to ab and $T_2 \cup \{c\} \simeq L_4$. Then $C_3(I, I, L_4) \simeq T_2 \cup C \leq T$. □

We define $\mathscr{Q}_k$ to be the class of k-sets of tournaments $\{T_1, \ldots, T_k\}$ such that $\mathscr{Q}_{\{T_1, \ldots; T_k\}}$ is wqo. We would like to be able to give a similar argument to the one above to show that $\mathscr{Q}_2$ is decidable. The obstacles to this program are discussed at the end of this paper.

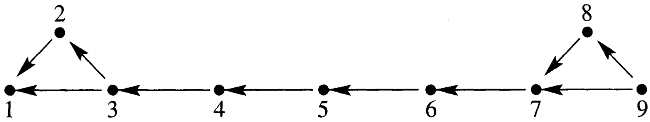

FIGURE 1. Some arcs of T_9.

§5. **The constructive proof of decidability.** In this section we will determine the elements of $\mathscr{Q}_1$ explicitly. This will be accomplished by showing that

$$Q = \{N_5, IC_3(I, I, L_3), C_3(I, I, L_3)I\}$$

is the set of isomorphism types of the maximal nontransitive elements in $\mathscr{Q}_1$ and that

$$M = \{S_5, C_3(I, L_2, L_2), C_3(I, I, L_4), L_2C_3, C_3L_2, IC_3I, C_3(I, I, C_3), B_5\}$$

is the set of isomorphism types of the minimal elements not in $\mathscr{Q}_1$. To show that these sets are what we say they are, three tasks must be accomplished.

(1) We must construct antichains which together omit every element of M.
(2) We must verify that $Q \subseteq \mathscr{Q}_1$.
(3) We must show that if a nontransitive tournament T does not embed an element of M, then T embeds in an element of Q.

We accomplish the first and last tasks in this section. The sketch of the proofs required for the second task are given in §6 with the detailed proofs to be given elsewhere.

Let $\mathscr{L}$ be the class of all finite transitive tournaments. By a straightforward application of Ramsey's Theorem we have that $\mathscr{L} \subset \mathscr{Q}_1$. Let $S(Q)$ be the class of all tournaments embedding in tournaments of Q. We claim that $\mathscr{Q}_1 = \mathscr{L} \cup S(Q)$.

Let $\mathscr{T}$ be the class of finite tournaments. Define $\mathscr{M}$ to be the class of minimal tournaments in $\mathscr{T} \backslash (\mathscr{L} \cup S(Q))$. Since $\mathscr{T} \backslash \mathscr{Q}_1$ is upward closed, it is enough to prove that every tournament T in Q is tight and every tournament in $\mathscr{M}$ is loose. All of this is contained in the following three propositions.

PROPOSITION 5.1. *$\mathscr{M}$ is the closure of M under isomorphism.*

PROPOSITION 5.2. *Every tournament in M is loose.*

PROPOSITION 5.3. *Every tournament in Q is tight.*

We construct two antichains which can be used to demonstrate that every element of M is loose. We then prove that M is the set of isomorphism classes of $\mathscr{M}$.

5.1. An antichain of modified transitive orders. We construct an antichain that omits the tournaments $S_5, B_5, C_3(I, L_2, L_2)$, $C_3(I, I, C_3)$, and $C_3(I, I, L_4)$.

We define a set of tournaments $\mathscr{I} = \{T_n\}_{7 \leq n < \omega}$. Let $\{1, \ldots, n\}$ be the vertices of N_n. We define T_n by reversing the arcs $(1, 3)$ and $(n-2, n)$ in N_n. The right to left arcs of T_9 are shown in Figure 1.

LEMMA 5.4. *$\mathscr{I}$ is an antichain.*

PROOF. First we identify the 2-arcs in T_n. The arcs $(i+1, i)$ are 2-arcs in N_n for $i = 3, \ldots, n-3$ and remain 2-arcs in T_n. The copies of C_3 destroyed by reversing the arcs $(1, 3)$ and $(n-2, n)$ are $(1, 3, 2)$ and $(n-2, n, n-1)$. The copies of C_3 created are $(3, 1, 4)$ and $(n, n-2, n-3)$. So $(4, 3)$ and $(n-2, n-3)$ are 3-arcs.

$(3,2)$ and $(n-1, n-2)$ are 1-arcs but not 2-arcs. So there are two 3-arcs in T_n and no new 2-arcs have been created.

Suppose $T_m \leq T_n$ for $m \leq n$. The 3-arcs of T_m must map to the 3-arcs of T_n, and the remaining 2-arcs of T_m must map to the remaining 2-arcs of T_n, so clearly, from the pattern of 3-arcs and 2-arcs, $m = n$. □

COROLLARY 5.5. *S_5 is loose.*

PROOF. S_5 contains the cycle of 2-arcs (02413). If T_n is a tournament in $\mathscr{I}$, then T_n has no cycle of 2-arcs. □

LEMMA 5.6. *$C_3(I, L_2, L_2)$ is loose.*

PROOF. $C_3(I, L_2, L_2)$ has a vertex that is common to four distinct 2-arcs. If T_n is a tournament in $\mathscr{I}$, every vertex of T is common to at most two 2-arcs. □

LEMMA 5.7. *$C_3(I, I, L_4)$ is loose.*

PROOF. $C_3(I, I, L_4)$ has a 4-arc. If T_n is a tournament in $\mathscr{I}$, then 3 is the largest p for which T_n has a p-arc. □

LEMMA 5.8. *$C_3(I, I, C_3)$ is loose.*

PROOF. $C_3(I, I, C_3)$ contains a 3-arc, where the witnesses form a C_3. If T_n is a tournament in $\mathscr{I}$, then the witnesses for every 3-arc in T_n form an L_3. □

LEMMA 5.9. *B_5 is loose.*

PROOF. We notice that if T_n is a tournament in $\mathscr{I}$, then any two distinct 2-arcs in T_n which share a common witness also share a common vertex. In B_5, the 2-arcs 30 and 41 both have the vertex 2 as a witness. □

5.2. An antichain of modified local orders.

LEMMA 5.10. *There is an antichain $\{T_n\}_{4 \leq n < \omega}$ which omits IC_3I, L_2C_3, and C_3L_2.*

PROOF. Let T_n be realized by $\mathbb{Z}/(2n+1)\mathbb{Z}$, $n \geq 2$, with the arc relation

$$a \to b \quad \text{if and only if} \quad b - a \in \{1, 2, \ldots, n-1, n+1\}.$$

T_n is isomorphic to its dual under the isomorphism $\phi(x) = -x$, so if T_n omits L_2C_3, then it also omits C_3L_2. So it is enough to check that x' omits IC_3 and C_3I for every vertex x. As the automorphism group of T_n is transitive, it is enough to check $0'$. $0' = C_3(n+1, 1, \{2, \ldots, n-1\})$, and $\{2, \ldots, n-1\} \simeq L_{n-2}$; hence, $0'$ does omit IC_3 and C_3I.

We claim that $\{T_n\}_{4 \leq n < \omega}$ is an antichain. Suppose for some $m \leq n$ we have $T_m \leq T_n$. We study the mapping of the vertices of T_m to the vertices of T_n. We denote the vertices of T_m and T_n by x and y respectively.

We know that $x' = C_3(x+m+1, x+1, \{x+2, \ldots, x+m-1\})$. So the arc $x+(m+1) \to x+1$ is the unique $(m-2)$-arc in x', for $(m-2) \geq 2$. If $x \mapsto y$, then the $(m-2)$-arc in x' maps to the $(n-2)$-arc in y', so necessarily $(x+1) \mapsto (y+1)$. Hence, $x = x + (2m+1) \mapsto y + (2m+1)$, so $2m+1 \equiv 0 \pmod{2n+1}$. Since $m \leq n$, we have that $m = n$. □

Proposition 5.2 has now been proven.

5.3. The minimal elements of $\mathscr{T} \backslash \mathscr{Q}_1$. We now proceed with the proof of Proposition 5.1. Recall that

$$M = \{S_5, C_3(I, L_2, L_2), C_3(I, I, L_4), L_2C_3, C_3L_2, IC_3I, C_3(I, I, C_3), B_5\}.$$

LEMMA 5.11. *If $T = \{x\} \cup N_5$, then T embeds a tournament of M.*

PROOF. Let $\{1, \ldots, 5\}$ be the vertices of N_5. Call the vertices i, j in N_5 consecutive if $j = i + 1$. Let $\{\hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{4}\}$ be the vertices of B_5.

Claim. If x dominates or is dominated by three consecutive vertices in N_5, then T embeds a tournament of M.

PROOF OF CLAIM. We may assume that x has mixed type over 321 and 543 since otherwise T embeds IC_3I, L_2C_3, C_3L_2, or $C_3(I, I, C_3)$. Suppose $x \Rightarrow 432$. Then $15 \Rightarrow x$ and $\{1,2,3,5,x\} \simeq B_5$ via $(1,2,3,5,x) \mapsto (\hat{0},\hat{3},\hat{2},\hat{4},\hat{1})$. Similarly, if $432 \Rightarrow x$, then $x \Rightarrow 15$ and $\{1,3,4,5,x\} \simeq B_5$ via the mapping $(1,3,4,5,x) \mapsto (\hat{0},\hat{2},\hat{1},\hat{4},\hat{3})$. The claim is proven. □

Case 1. x dominates or is dominated by two consecutive vertices in 432.

Suppose, for example, that $x \Rightarrow 32$. Then we may assume, by the claim, that $14 \Rightarrow x$, and so $C_3(2, 14, x3) \leq T$.

Case 2. There are four consecutive vertices in N_5 such that x dominates or is dominated by exactly the first and third vertices.

Suppose, for example, that $x \Rightarrow 13$ and $24 \Rightarrow x$. Then $C_3(3, 2, x14) \leq T$ and $x14 \simeq C_3$. On the other hand, if $13 \Rightarrow x$ and $x \Rightarrow 24$, then $\{1,2,3,4,x\} \simeq B_5$ via $(1,2,3,4,x) \mapsto (\hat{0},\hat{3},\hat{1},\hat{4},\hat{2})$.

Case 3. We are not in Case 1 or Case 2.

If $x \to 3$ then, since we are not in Case 1, $24 \Rightarrow x$. Since we are not in Case 2, $15 \Rightarrow x$, and so $C_3(3, 25, 4x) \leq T$. Similarly, if $3 \to x$, then $x \Rightarrow \{1,2,4,5\}$ and $C_3(3, x2, 14) \leq T$. □

PROOF OF PROPOSITION 5.1. Let T be a nontransitive tournament which omits M. We claim that $T \leq N_5$, $T \leq IC_3(I, I, L_3)$, or $T \leq C_3(I, I, L_3)I$.

Case 1. Suppose T omits C_3I.

Then, by Corollary 3.18 and Lemma 3.22, $T \simeq L_kS$, where $S \leq S_{2n+1}[L_m]$ for some k, n, and m. Since T omits L_2C_3, $k \leq 1$. Since T omits $C_3(I, I, L_4)$, $m \leq 3$. Since T omits S_5, $n \leq 1$. Since T omits $C_3(I, L_2, L_2)$, $S \leq C_3(I, I, L_m)$. Hence, $T \leq IC_3(I, I, L_3)$.

Case 2. Suppose T omits IC_3. This case is dual to Case 1.

Case 3. T embeds both C_3I and IC_3.

If there is a vertex in T which dominates all others or is dominated by all others, then $IC_3I \leq T$. Let C be a copy of C_3 in T dominated by a vertex x. Then there is a vertex y in T such that $y \to x$. Since T omits L_2C_3 and $C_3(I, I, C_3)$, y has mixed type over C. If y dominates exactly one vertex of C, then $B_5 \leq T$, so y dominates exactly two vertices of C, and hence, $\{x, y\} \cup C \simeq N_5$.

By Lemma 5.11, there are no proper extensions of N_5 which omit M; hence $T \simeq N_5$. □

§6. A summary of the structure theorems.

To complete the constructive analysis of $\mathscr{Q}_1$, we must show that $IC_3(I, I, L_3)$, $C_3(I, I, L_3)I$, and N_5 are tight. By duality, we only need to show that N_5 and $IC_3(I, I, L_3)$ are tight. This is accomplished by analyzing the structure of the tournaments T omitting a given tournament T_0 to the point at which Higman's Lemma [10] or Kruskal's Tree Theorem [11], [14] applies. We summarize the results for N_5 and $IC_3(I, I, L_3)$ whose proofs are given elsewhere.

6.1. The structure of tournaments omitting N_5. We define $\mathscr{W}(\mathscr{A})$ to be the class of tournaments that are generated from the tournaments in the class $\mathscr{A}$ by closing under isomorphism, the formation of subtournaments, and composition. In particular, if T is constructed from an element of $\mathscr{W}(\mathscr{A})$ by replacing a vertex by a copy of some tournament in $\mathscr{W}(\mathscr{A})$, then T is in $\mathscr{W}(\mathscr{A})$.

Let $\mathscr{A}$ be the class of indecomposable tournaments which omit N_5. It follows that $\mathscr{Q}_{N_5} = \mathscr{W}(\mathscr{A})$. Elsewhere, we show that $\mathscr{A}$ is the union of the following three classes:

(1) the indecomposable subtournaments of the Paley tournament on 7 vertices;
(2) the indecomposable local orders;
(3) a set of indecomposable tournaments $\{D_{2n+1}\}_{n\geq 1}$ whose elements are defined in a manner very close to the representation of S_{2n+1} as a union of two transitive tournaments.

The tournaments in $\mathscr{A}$ can be encoded with a finite alphabet in the sense of Definition 7.2. The tournaments in $\mathscr{W}(\mathscr{A})$ can be encoded by directed labelled trees using this finite alphabet for the labels. These trees are rooted, each maximal directed path has the root as its initial vertex, and the set of labelled arcs with a common initial vertex are transitively ordered. $\mathscr{W}(\mathscr{A})$ can then be shown to be wqo by an application of Kruskal's Tree Theorem.

6.2. The structure of tournaments omitting $IC_3(I,I,L_3)$. We study $\mathscr{Q}_{IC_3(I,I,L_3)}$ by first understanding the structure of tournaments in $\mathscr{Q}_{C_3(I,I,L_3)}$. Elsewhere, we show that each tournaments in $\mathscr{Q}_{C_3(I,I,L_3)}$ has a congruence relation whose quotient embeds in $L[C_3]$, where L is a transitive tournament. Each congruence class either has fewer than 293 vertices or is isomorphic to N_n for some n. It is then an application of Kruskal's Tree Theorem to show that $\mathscr{Q}_{C_3(I,I,L_3)}$ is wqo.

Let $\mathscr{Q} = \mathscr{Q}_{IC_3(I,I,L_3)} \setminus \mathscr{Q}_{C_3(I,I,L_3)}$. Our approach to showing that $\mathscr{Q}$ is wqo is to study the relationships between the loci of the types in a tournament $T \in \mathscr{Q}$ over a copy C of $C_3(I,I,L_3)$. After removing a small set of vertices R, if two loci are nondual, then one dominates the other. This defines a directed graph E whose vertices are the loci. The union of two dual loci has a congruence relation whose quotient is a local order. Moreover, each congruence class is a subset of one of the loci. If A is a nonempty locus of a type over C, then it is not the case that $A \Rightarrow C$, so for some vertex c in C we have that $c \Rightarrow A$. Hence, the induced subtournament on the vertices of a congruence class contained in A is an element of $\mathscr{Q}_{C_3(I,I,L_3)}$. The tournament T can be encoded by R, E, the local orders, and the elements of $\mathscr{Q}_{C_3(I,I,L_3)}$. The remaining obstacle to applying Kruskal's Tree Theorem is then to show that for R fixed, the set of tournaments $(N_n \cup R : n \geq 1\} \cap \mathscr{Q}_{IC_3(I,I,L_3)}$ is wqo.

§7. A discussion of $\mathscr{Q}_2$. Recall that $\mathscr{Q}_2$ is defined to be the class of 2-sets of tournaments $\{T_1, T_2\}$ such that $\mathscr{Q}_{\{T_1,T_2\}}$ is wqo. Clearly, if $T_1 \in \mathscr{Q}_1$ or $T_2 \in \mathscr{Q}_1$ then $\{T_1, T_2\} \in \mathscr{Q}_2$. Let

$$M_1 = \{L_2C_3, C_3L_2, IC_3I\}$$

and

$$M_2 = \{S_5, C_3(I, L_2, L_2), C_3(I, I, L_4), C_3(I, I, C_3), B_5\}.$$

If T_1, T_2 each embed an element of M_i for the same $i = 1$ or 2, then $\{T_1, T_2\} \notin \mathscr{Q}_2$ since the antichain which omits every element of M_i will omit T_1 and T_2.

So it remains to consider the case where, without loss of generality, T_1 embeds an element of M_1 and T_2 embeds an element of M_2. Given the nonconstructive result for $\mathscr{Q}_1$, it is natural to conjecture that if T_1 or T_2 is sufficiently large and T_1, T_2 are each loose then $\{T_1, T_2\} \notin \mathscr{Q}_2$. We demonstrate that this conjecture is false by constructing 2-sets in $\mathscr{Q}_2$ whose elements can be arbitrarily large. This requires the application of Higman's Lemma [10] which says that the set of finite words in a finite alphabet is wqo with respect to a suitable quasi-ordering.

Given a quasi-ordering $\mathscr{E}, \mathscr{E}^*$ is the set of finite strings of elements of $\mathscr{E}$. Let $S = s_1 \cdots s_n$ and $S' = s'_1 \cdots s'_m$ be elements of $\mathscr{E}^*$. We say $S \leq S'$ if and only if $n \leq m$ and there is a strictly increasing function α from $[n]$ to $[m]$ such that $s_i \leq s'_{\alpha(i)}$. We call the elements of $\mathscr{E}^*$ words in the alphabet $\mathscr{E}$.

LEMMA 7.1 (Higman). *If $\mathscr{E}$ is finite, then $\mathscr{E}^*$ is wqo.*

Let $\mathscr{Q}$ be a class of tournaments. If there is a finite alphabet $\mathscr{E}$ and a function f from $\mathscr{E}^*$ to the set of isomorphism classes of all finite tournaments such that

the set of isomorphism classes of tournaments in $\mathscr{Q}$ is in the image of f
and f preserves embeddability,

then $\mathscr{Q}$ is wqo. In this case we say that $\mathscr{E}$ encodes the tournaments in $\mathscr{Q}$.

DEFINITION 7.2. A class of tournaments $\mathscr{Q}$ is said to have a *vertex coding* if there is a finite alphabet which encodes the tournaments in $\mathscr{Q}$ such that every letter in a word encoding a tournament encodes a vertex in that tournament, and the orientation of the arc between two vertices is determined by the letters encoding the vertices and the order in which these letters occur in the encoding word.

LEMMA 7.3. *If for all T in a class of tournaments $\mathscr{Q}$, $T = R \cup T_0$, where R is one of a finite set of tournaments and the tournaments T_0 can be encoded by a vertex coding, then there is a finite alphabet which encodes the tournaments in $\mathscr{Q}$.*

PROOF. Let $\mathscr{E}$ be a finite alphabet in a vertex coding for the tournaments T_0.

R is one of finitely many possible tournaments, $\{R_j, j = 1, \ldots, p\}$. Let the vertex set of R_j be V_j. Every vertex in $T \backslash R_j$ has one of at most $2^{|R_j|}$ possible types over R_j. We will show that any tournament T in $\mathscr{Q}$ can be encoded with the finite alphabet $\bigcup_{j=1}^{p} [p] \cup \mathscr{E} \times 2^{|R_j|}$ where $2^{|R_j|}$ is the set of binary strings of length $|R_j|$. The ordering on this alphabet is defined as follows:

If $i, j \in [p]$, then $i \leq j$ if and only if $i = j$.
If (a_1, b_1) and (a_2, b_2) are in $\mathscr{E} \times 2^{|R_j|}$, then $(a_1, b_1) \leq (a_2, b_2)$ if and only if $a_1 \leq a_2$ in $\mathscr{E}$ and $b_1 = b_2$.

Given $T = R \cup T_0$, $R \simeq R_j$, let $a_1 \cdots a_{|T_0|}$ be the word encoding T_0. Let v_i be the vertex in T_0 corresponding to the letter a_i. Then the type of v_i over R is the type of a single vertex v over R_j and can be encoded by a binary string. The kth bit in the string is 0 if $v \rightarrow r_k$ and 1 otherwise, where r_k is a vertex in R_j. Let b_i be the binary string associated with v_i. Then we can encode T by the word $j(a_1, b_1) \cdots (a_{|T_0|}, b_{|T_0|})$. The first letter of the word encodes the tournament R and the numbering on its vertices that correspond to the ordering of the binary strings. $\square$

We now construct a 2-set $\mathscr{Q}_2$ whose elements are arbitrarily large.

PROPOSITION 7.4. *Let L be a fixed transitive tournament, and let M be a fixed integer. Let $\{T_n\}_{n<\omega}$ be an antichain. Then*

(1) *There is an integer N such that for all $n \geq N$, T_n has M disjoint copies of C_3.*

(2) *If $\{T_n\}_{n<\omega}$ omits LC_3, then it does not omit $C_3(I, I, L)$.*

PROOF OF 1. Suppose not. By taking an infinite subsequence if necessary, we may assume that T_n has fewer then M disjoint copies of C_3 for all n. Then there is an integer k such that for all n, $T_n = T_{n,0} \cup R_n$, where $T_{n,0}$ is transitive and $|R_n| \leq k$. By Lemma 7.3, $\{T_n\}_{n<\omega}$ can be encoded by a finite alphabet, and hence, is wqo, a contradiction.

PROOF OF 2. Suppose $\{T_n\}_{n<\omega}$ omits LC_3. Choose n large enough that T_n contains M disjoint copies of C_3, where the integer M is chosen such that if we color pairs of disjoint copies of C_3 by the types of one copy of C_3 over another, then we have a monochromatic transitive ordered set of size $k = |L| + 1$. Let $C_1, \dots, C_k$ be this set. Since T_n omits LC_3, there is no vertex x_i in C_i such that $x_i \Rightarrow C_k$ since otherwise, $\{x_1, \dots, x_{k-1}\} \cup C_k \simeq LC_3$. If no vertex in C_i has mixed type over C_k, then we have $C_k \Rightarrow C_i$; hence, $C_j \Rightarrow C_i$ for $j > i$. Then $\{x_2, \dots, x_k\} \cup C_1 \simeq LC_3$. So some vertex x_i in C_i has mixed type over C_k. We may assume that x_i is a witness for the arc ab in C_k. But then we have $C_3(a, b, \{x_1, \dots, x_{k-1}\}) \leq T_n$ and $\{x_1, \dots, x_{k-1}\} \simeq L$.

Thus, the pair $\{LC_3, C_3(I, I, L)\}$ is tight when L is transitive, although LC_3 and $C_3(I, I, L)$ are each loose for $|L| > 4$. In particular, it requires at least two antichains to demonstrate that each of the tournaments in M is loose.

By a similar argument, one can show that if an antichain omits $L[C_3]$, then it does not omit $C_3(I, I, L)$.

$\mathscr{Q}_1$ contains only one infinite family of tournaments, namely $\mathscr{L}$. We have not yet examined whether there are other infinite families of 2-sets in $\mathscr{Q}_2$ other than $\{\{LC_3, C_3(I, I, L)\}\}, \{\{L[C_3], C_3(I, I, L)\}\}$, and their duals.

Understanding $\mathscr{Q}_1$ explicitly has enabled us to identify some of the elements of $\mathscr{Q}_2$ and also some families of elements in the complement of $\mathscr{Q}_2$. Not much light has yet been shed upon the question: For which values of k is $\mathscr{Q}_k$ decidable?

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