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Characterizations of rational ω-languages by means of right congruences

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Abstract

By means of right congruences, we characterize in a unified way different classical classes of rational ω-languages. A new congruence associated with an ω-automaton is introduced named cycle congruence. The family of rational ω-languages which have, by morphism, a unique minimal recognizing ω-automaton is characterized. It appears that for such a minimal ω-automaton, the cycle congruence coincides with the syntactic congruence of the recognized ω-language. We prove that the other rational ω-languages have an infinite number of minimal automaton.

Résumé

Différentes classes de ω-langages sont caractérisées de manière uniforme en utilisant des demi-congruences. Nous introduisons une nouvelle congruence associée à un ω-automate appelée congruence des cycles. Nous caractérisons la famille des ω-langages rationnels qui admettent, par homomorphisme, un unique automate minimal. La congruence des cycles de ces automates est la congruence syntaxique du ω-langage reconnu. Les autres ω-langages rationnels ont une infinité d'automates minimaux.

1. Introduction

Büchi [2] first introduced two notions of acceptance of ω-languages (languages of infinite words) by ω-automata and by congruences (or equivalently, by semigroups) He proved that an ω-language is accepted by a Büchi automaton iff it is saturated by a finite congruence or equivalently it is a rational ω-language. Unlike in [4, 11, 12],

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where a classification of rational ω-languages is based on semigroup properties, we characterize here, in an unified way, the families of deterministic (*DRAT*), co-deterministic (*Co-DRAT*) ω-languages and also the family $DRAT \cap Co\text{-}DRAT$ by means of right congruences. These three classes have been studied in [5, 15, 18, 16]. This kind of characterization has been used in [6] to establish a canonical bijection between the family of deterministic table transition automata and the family of right congruences of finite index $\sim$ which satisfy the following properties:

- $\forall u, v \in \Sigma^*, \forall w \in \Sigma^\omega$: $u \sim v$ implies $\{uw \in L \Leftrightarrow vw \in L\}$.
- $\forall x, u, v \in \Sigma^*$, $x \sim xu \sim xv$ implies $\{x(u^+v^+)^\omega \cap L \neq \emptyset \Rightarrow x(u^+v^+)^\omega \subseteq L\}$.

Concretely, let L be a rational ω-language in Σ^ω and let $RC(L)$ be the family of the right congruences of finite index $\sim$ that satisfies $\forall u, v \in \Sigma^*, \forall w \in \Sigma^\omega$: $u \sim v$ implies $\{uw \in L \Leftrightarrow vw \in L\}$. We have the following characterizations:

(i) $L \in DRAT \Leftrightarrow$ there exists $\sim \in RC(L)$ such that $\forall x, u, v \in \Sigma^*$: $x \sim xu \sim xv$ implies $\{x(u^*v^*)^\omega \cap L \neq \emptyset \Rightarrow x(u^+v^+)^\omega \subseteq L\}$.

(ii) $L \in Co\text{-}DRAT \Leftrightarrow$ there exists $\sim \in RC(L)$ such that $\forall x, u, v \in \Sigma^*$: $x \sim xu \sim xv$ implies $\{x(u^+v^+)^\omega \cap L \neq \emptyset \Rightarrow x(u^*v^*)^\omega \subseteq L\}$.

(iii) $L \in DRAT \cap Co\text{-}DRAT \Leftrightarrow$ there exists $\sim \in RC(L)$ such that $\forall x, u, v \in \Sigma^*$: $x \sim xu \sim xv$ implies $\{x(u^*v^*)^\omega \cap L \neq \emptyset \Rightarrow x(u^*v^*)^\omega \subseteq L\}$.

The associated deterministic transition table ω-automata are described. These characterizations are similar to those proposed in [5, 15, 18, 16].

Next, we introduce a new congruence associated with an ω-automaton: *the cycle congruence*. The definition of this congruence uses the notion of accepting cycle. A cycle is an automaton path whose origin coincides with its end. An accepting cycle is a cycle which is "useful" to recognize ω-words of the considered rational ω-language. For a Büchi automaton, it means that such a cycle uses one of the elected states of the automaton, for a Müller automaton that the set of all the used states of the cycle is an element of the acceptance table, and for table transition automata, that the set of all the used transitions of the cycle is an element of the acceptance table. So, we defined two finite words to be equivalent for the cycle congruence if they have firstly, the same computation in the automaton and secondly, if they can be substituted in the label of any cycle without changing the fact that it is an accepting cycle or not. The definition of the cycle congruence is identical for Büchi, Müller or table transitions automata. Moreover, it can be interpreted as an analogous, for automata, of the syntatic congruence defined in [1]: the syntatic congruence uses the notion of period of ultimately periodic words of the considered ω-language, this notion naturally corresponds to those of cycle in automata.

It is well known that the minimization problem for ω-automata is not as simple as the one for automata. There are two possible approaches of the minimization problem for ω-automata. In the first one, an ω-automaton is minimal if there is no other ω-automatan with strictly less states which accepts the same ω-language (such an automaton is called s-minimal). The second approach is more algebraic: an ω-automatan is minimal if it is irreducible using surjective automaton morphisms (such an automaton is called m-minimal). For the languages, the two definitions are equivalent: a language is accepted by a unique minimal deterministic automaton which is the homomorphic image of any other accepting deterministic automaton. For rational ω-languages, the two notions of minimizations are different. A rational ω-language has always a finite number of s-minimal deterministic automata which are, obviously, m-minimal, but if a deterministic automaton is m-minimal it is not necessarily s-minimal.

Here, we essentially treat the minimization using deterministic automaton morphism. Obviously, if an ω-language is accepted by a *unique* m-minimal deterministic table transition automaton (up to isomorphism) then this automaton is s-minimal and we say that it is m-minimum. We characterize the rational ω-languages which are accepted by a m-minimum deterministic transition table automaton. In this case the cycle congruence associated with such an automaton is the syntatic congruence of the corresponding ω-language. This family is composed by the rational ω-languages L which are recognized by their right congruence of prefixes (i.e. the largest right congruence of the family $RC(L)$). Moreover, we prove that all the other rational ω-languages admit an infinite number of m-minimal automata.

Section 2 contains preliminary notions. In Section 3, we characterize various classes of rational ω-languages by means of right congruences. In Section 4, we study the cycle congruence associated with an ω-automatan. Section 5 is devoted to the minimization problem for ω-automata.

2. Preliminaries

Let Σ be a finite alphabet. We denote by Σ^* and Σ^ω the sets of all finite and infinite words over Σ, respectively. Let $x \in \Sigma^*$, we denote by $|x|$ the length of x, for any $1 \leqslant i \leqslant |x|$, x_i is the ith letter of x and $x[i]$ the word $x_1 \cdots x_i$. The notation $x' \leqslant x$ or $x' < x$ means x' is a prefix, or a proper prefix of x, respectively. For any subset $X \neq \emptyset$ of Σ^*, we denote $X^+ = \{x_1 x_2 \cdots x_n,\ n \geqslant 1,\ x_i \in X\}$, $X^* = X^+ \cup \{\varepsilon\}$, where ε is the empty word, and $X^\omega = \{x_1 x_2 \cdots \forall i \geqslant 1,\ x_i \in X \backslash \{\varepsilon\}\}$.

A language is a subset of Σ^* and a language of infinite words (ω-language) is a subset of Σ^ω. We denote by $\bar{L}$ the complement of a language L and $UP(L)$ the set of its ultimately periodic words: $UP(L) = \{xy^\omega \in L,\ x, y \in \Sigma^+\}$. An ω-language L is rational if it is a finite union of ω-languages of the form CB^ω where C, B are rational languages in Σ^*.

A transition system TS is a triple (Q, I, δ), where Q is a finite set of states, $I \subset Q$ a set of initial states and δ a transition function, i.e., a partial mapping of $Q \times \Sigma$ into 2^Q. As usual, we extend δ to $Q \times \Sigma^*$ by setting, for all $q \in Q$, $a \in \Sigma$ and $w \in \Sigma^*$, $\delta(q, \varepsilon) = \{q\}$ and $\delta(q, aw) = \bigcup_{p \in \delta(q,a)} \delta(p, w)$. A transition system TS is complete if for any pair $(q, a) \in Q \times \Sigma$, $\delta(q, a)$ is never empty. A transition system TS is deterministic if $Card(I) = 1$ and for any pair $(q, a) \in Q \times \Sigma$, there is at most one state q' in $\delta(q, a)$. By abuse, we will write $\delta(q, a) = q'$, when TS is deterministic.

Let $TS = (Q, I, \delta)$ be a transition system. A transition is an element (q, a, q') of $Q \times \Sigma \times Q$ such that $q' \in \delta(q, a)$. We denote by Δ the set of transitions of TS.

A computation c in TS is a finite or infinite sequence $\delta_0\delta_1\delta_2\cdots$ of transitions, where for each $i, \delta_i = (q_i, a_i, q_{i+1}) \in \Delta$. The word $w = a_0a_1\cdots$ is called the label of c and the state q_0 the origin of c. We use the notation $c(q_0, w, q_{n+1})$ to denote a finite computation c with the origin q_0, the end q_{n+1} and the label $w = a_0a_1\cdots a_n$.

We denote by $Q\_fin(c)$ (resp. $Q\_inf(c)$) the set of states which have finitely (resp. infinitely) many occurrences in c. In the same way, we denote by $T\_fin(c)$ (resp. $T\_inf(c)$) the set of transitions which have a finitely (resp. infinitely) many occurrences in c. A subset E of Q is *coherent* if there exists a computation c with label in $UP(\Sigma^\omega)$ such that $Q\_inf(c) = E$. A subset T of Δ is *coherent* if there exists a computation c in TS with label in $UP(\Sigma^\omega)$ such that $T\_inf(c) = T$.

Definition 2.1. A *Büchi automaton* [2] is a quadruple $\mathscr{A} = (Q, I, \delta, E)$, where (Q, I, δ) is a transition system and E a subset of Q. An infinite word w is accepted by $\mathscr{A}$ if there exists a computation c in $\mathscr{A}$ with the origin in I and the label w such that $Q\_inf(c) \cap E \neq \emptyset$.

Definition 2.2. A *Müller automaton* [10] is a quadruple $\mathscr{A} = (Q, I, \delta, \mathscr{E})$, where (Q, I, δ) is a transition system and $\mathscr{E}$ a family of coherent subsets of Q. An infinite word w is accepted by $\mathscr{A}$ if there exists a computation c in $\mathscr{A}$ with the origin in I and the label w such that $Q\_inf(c) \in \mathscr{E}$.

Definition 2.3. A *transition table automaton* [6] is a quadruple $\mathscr{A} = (Q, I, \delta, \mathscr{T})$, where (Q, I, δ) is a transition system and $\mathscr{T}$ a set of coherent subsets of transitions. An infinite word $w \in \Sigma^\omega$ is accepted by $\mathscr{A}$ if there exists a computation c in $\mathscr{A}$ with the origin in I and the label w such that $T\_inf(c) \in \mathscr{T}$.

The Büchi automata and the Müller automata accept exactly the rational ω-languages [2, 10, 9]. The recognition power of Müller automata does not decrease if one restricts to the deterministic ones [19, 13, 7]. But it is not the case for Büchi automata: the family of rational ω-languages is the boolean closure of the family of ω-languages accepted by a deterministic Büchi automaton [2, 5]. The transition table automata have the same recognition power as the Müller automata, that is, they also accept the rational ω-languages and their recognition power does not decrease if we only consider the deterministic ones [6]. In the sequel, we denote $\mathscr{L}(\mathscr{A})$ the ω-language accepted by an automaton $\mathscr{A}$.

Let $TS = (Q, I, \delta)$ be a transition system. A cycle is a finite computation whose origin coincide with its end. We denote by $\mathscr{C}$ the set of cycles of TS.

Definition 2.4. Let $\mathscr{A}_B = (Q, I, \delta, E)$, $\mathscr{A}_M = (Q, I, \delta, \mathscr{E})$ and $\mathscr{A}_T = (Q, I, \delta, \mathscr{T})$ be a Büchi, a Müller and a transition table automaton, respectively. Their sets of *accepting cycles* are defined as follows:

$$\mathscr{C}(\mathscr{A}_B) = \{c(q, w, q) \in \mathscr{C} \mid Q\_fin(c(q, w, q)) \cap E \neq \emptyset\},$$

$$\mathscr{C}(\mathscr{A}_M) = \{c\{q, w, q\} \in \mathscr{C} \mid Q\_fin(c(q, w, q)) \in \mathscr{E}\}$$

and

$$\mathscr{C}(\mathscr{A}_T) = \{c(q,w,q) \in \mathscr{C} \mid T\_fin(c(q,w,q)) \in \mathscr{T}\}.$$

Now, let E be a subset of Q, and $\mathscr{E}$ be the set of all the coherent subsets $E' \subset Q$ such that $E' \cap E \neq \emptyset$. Let $\mathscr{T}$ be the family of the coherent subsets $T \subset \Delta$ such that there exists $E' \in \mathscr{E}$ satisfying $E' = \{q \in Q \mid (q,a,q') \in T\}$. We have the following fact.

Fact 2.5. *The Büchi automaton* $\mathscr{A}_B = (Q,I,\delta,E)$, *the Müller automaton* $\mathscr{A}_M = (Q,I,\delta,\mathscr{E})$ *and the transition table automaton* $\mathscr{A}_T = (Q,I,\delta,\mathscr{T})$ *accept the same* ω*-language and* $\mathscr{C}(\mathscr{A}_B) = \mathscr{C}(\mathscr{A}_M) = \mathscr{C}(\mathscr{A}_T)$.

Fact 2.6. *Let* $\mathscr{A}_T = (Q,I,\delta,\mathscr{T})$ *be a table transition automaton. Then, the Müller automaton* $\mathscr{A}_M = ((Q\times\Sigma\times Q)\cup I, I, \delta', \mathscr{E})$, *where* $\delta'((q'',b,q),a) = \{(q,a,q') \mid q' \in \delta(q,a)\}$ *and* $\delta'(q,a) = \{(q,a,q') \mid q' \in \delta(q,a),\ q \in I\}$ *accepts the same* ω*-langage as* $\mathscr{A}_T$.

Remark 2.7. The two previous facts explain why we use table transition automata instead of Müller automata: For every Müller automaton, there is a table transition automaton accepting the same ω-language and having a same transition relation (but a different table), whereas in the converse direction the structure must be changed. For instance, consider the following example: Let $L = \{a,b\}^*a^\omega$. It is a rational ω-language and it is accepted by a deterministic table transition automaton with one state and two transitions on this state, respectively, labelled a and b. But, the corresponding Müller automaton does not accept L and it is easy to see that we always need more than two states to build a Müller automaton which recognizes this ω-language.

Büchi proved that the rational ω-languages are characterized by their ultimately periodic words [2].

Lemma 2.8. *Let* L, L' *be rational* ω*-languages. Then*

$$L \neq \emptyset \Leftrightarrow UP(L) \neq \emptyset \quad \text{and} \quad L \subseteq L' \Leftrightarrow UP(L) \subseteq UP(L').$$

Let $\equiv$ be a equivalence relation on Σ^*. We denote by $x_\equiv$ the class of the word x and we say that $\equiv$ is of finite index if it has a finite number of classes. An equivalence relation $\equiv$ is larger than another equivalence relation $\approx$ if $\forall u \in \Sigma^*$, we have $u_\approx \subseteq u_\equiv$. A right congruence $\sim$ on Σ^* is an equivalence relation satisfying: $\forall u,v \in \Sigma^*, \forall w \in \Sigma^*, u \sim v$ implies $uw \sim vw$. A congruence $\approx$ on Σ^* is an equivalence relation satisfying: $\forall u,v \in \Sigma^*$, $\forall w,w' \in \Sigma^*$, $u \approx v$ implies $wvw' \approx wvw'$. As usual, with any congruence $\approx$ on Σ^*, we associate a monoid $M_\approx$ and a morphism $\varphi_\approx : \Sigma^* \to M_\approx$ defined by $M_\approx = \{u_\approx, u \in \Sigma^*\}$ and for any $u \in \Sigma^*$, $\varphi(u) = u_\approx$.

Definition 2.9 (*Büchi* [2]). A congruence $\approx$ on Σ^* *saturates* an ω-language L iff for any pair m,f in $M_\approx$, we have $\varphi_\approx^{-1}(m)[\varphi_\approx^{-1}(f)]^\omega \cap L \neq \emptyset$ implies $\varphi_\approx^{-1}(m)[\varphi_\approx^{-1}(f)]^\omega \subset L$. We also say that $M_\approx$ and $\varphi_\approx$ saturate L.

In a finite semigroup, every element has a power which is an idempotent. So, using a congruence of finite index, we can rewrite the previous definition in the following way.

Definiton 2.10. A congruence of finite index $\approx$ on Σ^* *saturates* an ω-language L if for any pair $m,f \in M_\approx$ such that $mf = m$ and $f^2 = f$, we have $\varphi_\approx^{-1}(m)[\varphi_\approx^{-1}(f)]^\omega \cap L \neq \emptyset$ implies $\varphi_\approx^{-1}(m)[\varphi_\approx^{-1}(f)]^\omega \subset L$.

Theorem 2.11 (Büchi [2]). *An ω-language L is rational iff there exists a congruence of finite index $\approx$ on Σ^* which saturates L.*

Remark 2.12. It is easy to see that if $\equiv$ and $\approx$ are congruences on Σ^* such that $\equiv$ saturates L and is larger than $\approx$ then $\approx$ also saturates L.

Definition 2.13 (*Arnold* [1]). Let $L \in \Sigma^\omega$ and let $\approx_s$ be the congruence on Σ^* defined by

$$\forall u,v \in \Sigma^*,\ u \approx_s v \Leftrightarrow \begin{cases} \forall (x,w) \in \Sigma^* \times \Sigma^\omega,\ xuw \in L \Leftrightarrow xvw \in L, \\ \forall x,w,w' \in \Sigma^* \text{ with } ww' \neq \varepsilon,\ x(wuw')^\omega \in L \Leftrightarrow x(wvw')^\omega \in L. \end{cases}$$

We denote by M_s the monoid associated with $\approx_s$ and $\varphi_s : \Sigma^* \to M_s$ the morphism defined by $\forall u \in \Sigma^*$, $\varphi_s(u) = u_{\approx_s}$. The congruence $\approx_s$ (resp. the monoid M_s, the morphism φ_s) is called the syntactic congruence (resp. monoid, morphism) of L. The congruence $\approx_s$ is the largest congruence of finite index which saturates a rational ω-language [1].

3. Right congruences

In this section, we characterize classical classes of rational ω-languages by means of right congruences. We propose different recognition criteria by right congruences and study the corresponding table transition automata.

Fact 3.1. *There is a canonical bijection between the family of right congruences of finite index and the family of complete deterministic transition systems.*

This canonical bijection is defined in the following way: With any right congruence of finite index $\sim$, we associate a deterministic transition system $TS_\sim = (Q_\sim, I_\sim, \delta_\sim)$, where $Q_\sim = \{x_\sim, x \in \Sigma^*\}$, $I_\sim = \{\varepsilon_\sim\}$, $\delta_\sim$ is the function from $Q_\sim \times \Sigma$ to $Q_\sim$ defined by $\forall (x_\sim, a) \in Q_\sim \times \Sigma$, $\delta_\sim(x_\sim, a) = (xa)_\sim$. With any deterministic transition system

$TS = (Q, \{q_0\}, \delta)$, we associate a right congruence $\sim_Q$ defined by $\forall u, v \in \Sigma^*, u \sim_Q v$ iff $\delta(q_0, u) = \delta(q_0, v)$. The right congruence $\sim_Q$ is of finite index.

It is easy to see that the right congruence associated with $TS_\sim$ is exactly $\sim$ and, conversely, that the transition system associated with $\sim_Q$ is TS.

Definition 3.2 [*Le Saëc* [7]). We say that a right congruence $\sim$ on Σ^* saturates an ω-language L if the following properties hold:
(1) $\forall (u,v) \in \Sigma^* \times \Sigma^*, \forall w \in \Sigma^\omega$: $u \sim v$ implies $\{uw \in L \Leftrightarrow vw \in L\}$.
(2) $\forall (x,u,v) \in \Sigma^* \times \Sigma^* \times \Sigma^*$, $x \sim xu \sim xv$ implies $\{x(u^+v^+)^\omega \cap L \neq \emptyset \Rightarrow x(u^+v^+)^\omega \subseteq L\}$.

Definition 3.3. Let L be a rational ω-language. We denote by $\mathscr{A}_\sim = (Q_\sim, I_\sim, \delta_\sim, \mathscr{T}_\sim)$ the deterministic transition table automaton associated with $\sim$, where $(Q_\sim, I_\sim, \delta_\sim)$ is the deterministic transition system $TS_\sim$ and $\mathscr{T}_\sim = \{t\text{-}inf(c)$, where c is a computation with the origin $\varepsilon_\sim$ and the label in $UP(L)\}$.

If $\sim$ is of finite index and saturates L, then this transition table automaton $\mathscr{A}_\sim$ accepts L [7].

The following result has been established in [7].

Theorem 3.4. *Let L be a rational ω-language and $\sim$ be a right congruence of finite index saturating L.*
(1) *There exists a unique (up to isomorphism) complete deterministic table transition automaton $\mathscr{A}_\sim$ accepting L such that $\sim_Q = \sim$.*
(2) *The right congruence $\sim_Q$ associated with any deterministic Büchi, Müller or table transition automaton accepting L, saturates L.*

Remark 3.5. The previous result has an immediate corollary. If $L \in RAT$, then there exists a right congruence of finite index saturating L. The converse does not hold. Consider, for instance, the following ω-language proposed by Staiger, $L = \bigcup_{w,u,v \in \Sigma^*} w\{u,v\}^\omega$. This ω-language is not regular (it is different from Σ^ω and its ultimately periodic words are those of Σ^ω), but it is saturated by the trivial right congruence with one class Σ^*.

Remark 3.6. This result establishes a canonical bijection between the family of the complete deterministic table transition automata which accept a given rational ω-language and the family of right congruences of finite index which saturate this ω-language.

The first part of Theorem 3.4 does not hold for deterministic Müller automata: the right congruence $\sim$ having only one class $\{a,b\}^*$ saturates the ω-language $\{a,b\}^*a^\omega$, but it does not enable us to build a deterministic Müller automaton $\mathscr{A}_M$ such that $\sim = \sim_Q$ (see Remark 2.7).

Definition 3.7. A right congruence $\sim$ on Σ^* is a *Landweber right congruence* saturating L if the following conditions hold:
(1) $\forall (u,v) \in \Sigma^* \times \Sigma^*, \forall w \in \Sigma^\omega$: $u \sim v$ implies $\{uw \in L \Leftrightarrow vw \in L\}$.
(2) $\forall (x,u,v) \in \Sigma^* \times \Sigma^* \times \Sigma^*$, $x \sim xu \sim xv$ implies $\{x(u^*v^*)^\omega \cap L \neq \emptyset \Rightarrow x(u^+v^+)^\omega \subseteq L\}$.

Definition 3.8. We say that a transition table automaton $\mathscr{A} = (Q, I, \delta, \mathscr{T})$ is *cyclically open* if $\mathscr{T}$ contains together with every element T all its coherent supersets $T' \supseteq T$. This property has been introduced in [5] for Müller automata.

Theorem 3.9. *Let L be a rational ω-language. The following three conditions are equivalent:*
(1) $L \in DRAT$.
(2) *Every right congruence of finite index that saturates L is a Landweber right congruence.*
(3) *Every deterministic transition table automaton that accepts L is cyclically open.*

Proof. The equivalence (1) $\Leftrightarrow$ (3) has been proved in [6]. Now, we prove that (2) $\Leftrightarrow$ (3).

Assume that (2) holds and let $\mathscr{A} = (Q, \{q_0\}, \delta, \mathscr{T})$ be a complete deterministic transition table automaton accepting L and let $\sim_Q$ be the right congruence associated with $\mathscr{A}$. From Theorem 3.4, it is of finite index that saturates L. So, by hypothesis, it is also a Landweber right congruence which saturates L.

Let us prove that $\mathscr{A}$ is cyclically open. Let $(T, T') \in \mathscr{T} \times 2^{\Delta}$ such that $T \cup T'$ is coherent. Then T and T' have a common state $x_{\sim Q}$.

By definition, there exists a computation c_v with the origin $\varepsilon_{\sim Q}$ and the label xu^ω such that $u \in \Sigma^+$ and $x \sim_Q xu$ and $T\_inf(c_v) = T$. There also exists a computation c_v with origin $\varepsilon_{\sim Q}$ and label xv^ω such that $v \in \Sigma^+$ and $x \sim_Q xv$ and $T\_inf(c_v) = T'$. Since $x \sim xu \sim xv$ and $\sim_Q$ is a Landweber right congruence, we have $\{xu^\omega \in x(u^*v^*)^\omega \cap L \neq \emptyset \Rightarrow x(u^+v^+)^\omega \subseteq L\}$. Since $\mathscr{A}$ is deterministic, $x(uv)^\omega \in L$ implies that $T \cup T' \in \mathscr{T}$ and $\mathscr{A}$ is cyclically open.

Conversely, assume that property (3) holds and let $\sim$ be a right congruence of finite index that saturates a rational ω-language L and let $\mathscr{A}_\sim = (Q_\sim, \{q_{0\sim}\}, \delta_\sim, \mathscr{T}_\sim)$ be the complete deterministic transition table automaton associated with $\sim$ defined in Definition 3.3. From Theorem 3.4, this automaton accepts L and by hypothesis, it is cyclically open. The right congruence $\sim_Q$ associated with $\mathscr{A}$ is equal to $\sim$.

Now, let $x, u, v \in \Sigma^*$ such that $x \sim_Q xu \sim_Q xv$. Assume that $x(u^*v^*)^\omega \cap L \neq \emptyset$ and let c be an infinite computation in $\mathscr{A}$ with the origin q_0 and the label $w \in x(u^*v^*)^\omega \cap L$.
(1) If $w \in x(u^+v^+)^\omega$ then, since $\sim$ saturates L, we have $x(u^+v^+)^\omega \subseteq L$.
(2) If $w \in x\{u,v\}^*u^\omega$. Let c_u and c_v be the computations with the origin q_0 and the label xu^ω and xv^ω, respectively. Since $\mathscr{A}_\sim$ is deterministic, $T\_inf(c) = Tinf(c_u) \in \mathscr{T}_\sim$.

Moreover, $\mathscr{A}$ is cyclically open and $T\_inf(c_u) \cup T\_inf(c_v)$ is coherent, so for any computation c' with the origin q_0 and the label w' in $x(u^+v^+)^\omega$, we have $T\_inf(c') = T\_inf(c_u) \cup T\_inf(c_v) \in \mathscr{T}_\sim$ and $x(u^+v^+)^\omega \subseteq L$. The case $w \in x\{u,v\}^*v^\omega$ is similar.

So the finite right congruence $\sim_Q = \sim$ is a Landweber right congruence of finite index. $\square$

Definition 3.10. A right congruence $\sim$ on Σ^* is a *co-Landweber right congruence* saturating L if the following properties hold:
(1) $\forall u,v \in \Sigma^* \times \Sigma^*, \forall w \in \Sigma^\omega$: $u \sim v$ implies $\{uw \in L \Leftrightarrow vw \in L\}$.
(2) $\forall x,u,v \in \Sigma^*$, $x \sim xu \sim xv$ implies $\{x(u^+v^+)^\omega \cap L \neq \emptyset \Rightarrow x(u^*v^*)^\omega \subseteq L\}$.

We denote by *co-DRAT* the family of the rational ω-languages whose complement is in *DRAT*. We say that a transition table automaton $\mathscr{A} = (Q, I, \delta, \mathscr{T})$ is *cyclically closed* if $\mathscr{T}$ contains together with every element T all its coherent subsets $T' \subseteq T$. The family *co-DRAT* has been studied in [15, 18, 16].

Lemma 3.11. *If $\sim$ is a Lanbweber right congruence saturating on ω-language L then $\sim$ is a co-Landweber right congruence saturating the complement $\bar{L}$ of L.*

Proof. Let $\sim$ be a Landweber right congruence that saturates L. By definition, we have:
(1) $\forall u,v \in \Sigma^*, \forall w \in \Sigma^\omega$: $u \sim v$ implies $\{uw \in L \Leftrightarrow vw \in L\}$.
(2) $\forall x,u,v \in \Sigma^*$, $x \sim xu \sim xv$ implies $\{x(u^*v^*)^\omega \cap L \neq \emptyset \Rightarrow x(u^+v^+)^\omega \subseteq L\}$.

Thus, $\forall u,v \in \Sigma^*, \forall w \in \Sigma^\omega$: $u \sim v$ implies $\{uw \in \bar{L} \Leftrightarrow vw \in \bar{L}\}$. Now, let $x,u,v \in \Sigma^*$ such that $x \sim xu \sim xv$ and $x(u^+v^+)^\omega \cap \bar{L} \neq \emptyset$. Since $\sim$ is a Landweber right congruence, $x(u^*v^*)^\omega \cap L = \emptyset$. Thus, $x(u^*v^*)^\omega \subseteq \bar{L}$. $\square$

Corollary 3.12. *Let L be a rational ω-language. The following three conditions are equivalent:*
(1) *$L \in$ co-DRAT.*
(2) *Every right congruence of finite index that saturates L is co-Landweber.*
(3) *Every deterministic transition table automaton accepting L is cyclically closed.*

Proof. The fact that (1) $\Leftrightarrow$ (2) is a direct consequence of Theorem 3.9 and Lemma 3.11. On what concerning (1) $\Leftrightarrow$ (3), again by Theorem 3.9, it suffices to remark that, if the deterministic transition table automaton $\mathscr{A} = (Q, I, \delta, \mathscr{T})$ is complete and accepts L then $\bar{\mathscr{A}} = (Q, I, \delta, 2^\Delta \setminus \mathscr{T})$ accepts $\bar{L}$ and $\mathscr{A}$ is cyclically open iff $\bar{\mathscr{A}}$ is cyclically closed. $\square$

Definition 3.13. A right congruence $\sim$ is *an open-closed right congruence* saturating L if it satisfies the following properties:
(1) $\forall u,v \in \Sigma^*, \forall w \in \Sigma^\omega$: $u \sim v$ implies $\{uw \in L \Leftrightarrow vw \in L\}$.
(2) $\forall x,u,v \in \Sigma^*$, $x \sim xu \sim xv$ implies $\{x(u^*v^*)^\omega \cap L \neq \emptyset \Rightarrow x(u^*v^*)^\omega \subseteq L\}$.

A transition table automaton $\mathscr{A} = (Q, I, \delta, \mathscr{T})$ is *open-closed* if it is both cyclically open and cyclically closed. Obviously, any open-closed right congruence is both Landweber and co-Landweber one. So, combining Theorem 3.9 and Corollary 3.12, we obtain the following corollary.

Corollary 3.14. *Let L be a rational ω-language. The following three conditions are equivalent*:
(1) *$L \in DRAT \cap co\text{-}DRAT$.*
(2) *Every finite right congruence that saturates L is open-closed.*
(3) *Every deterministic transition table automaton that accepts L is open-closed.*

Remark 3.15. In virtue of Theorem 3.9, we have: With any $\sim$, a Landweber right congruence of finite index saturating $L \in DRAT$ (resp. $co\text{-}DRAT$, $DRAT \cap co\text{-}DRAT$), is associated a unique complete cyclically open (resp. closed, open-closed) deterministic table transition automaton $\mathscr{A}_\sim$ accepting L such that $\sim_Q = \sim$. Moreover, the right congruence $\sim_Q$ associated with a cyclically closed deterministic table transition automaton accepting L, is a Landweber (resp. co-Landweber, open-closed) right congruence.

4. The cycle congruence of an automaton

Definition 4.1. Let $\mathscr{A} = (Q, I, \delta, C)$ be a automaton and $\mathscr{C}(\mathscr{A})$ the set of accepting cycles of $\mathscr{A}$. We define the equivalence relation $\approx_c$ in the following way:

$$\forall u, v \in \Sigma^*, \quad u \approx_c v \iff \begin{cases} \forall q \in Q, \ \delta(q,u) = \delta(q,v), \\ \forall q \in Q, \ w, w' \in \Sigma^* \ \textit{such that} \ ww' \neq \varepsilon, \\ \exists c(q, wuw', q) \in \mathscr{C}(\mathscr{A}) \iff \exists c'(q, wvw', q) \in \mathscr{C}(\mathscr{A}). \end{cases}$$

Proposition 4.2. *If $\mathscr{A}$ is an automaton accepting L then $\approx_c$ is a congruence of finite index that saturates L.*

This congruence $\approx_c$ is called *the cycle congruence* of $\mathscr{A}$.

Proof. Let $\mathscr{A}(Q, I, \delta, \mathscr{C})$ be a Büchi, Müller or transition table automaton. The equivalence relation $\approx_c$ is a congruence, since for any $w, w' \in \Sigma^*$, we have $\forall q \in Q$, $\delta(q,u) = \delta(q,v) \Rightarrow \forall q \in Q$, $\delta(q, wuw') = \delta(q, wvw')$ and $\forall q \in Q$, $\forall w, w' \in \Sigma^*$ such that $ww' \neq \varepsilon$, $\exists c(q, wuw', q) \in \mathscr{C}(\mathscr{A}) \iff \exists c'(q, wvw', q) \in \mathscr{C}(\mathscr{A})$ implies $\forall q \in Q$, $\forall w, w' \in \Sigma^*$, $\exists c(q, waubw', q) \in \mathscr{C}(\mathscr{A}) \iff \exists c'(q, wavbw', q) \in \mathscr{C}(\mathscr{A})$.

If two words $u, v \in \Sigma^*$ satisfy $\forall q \in Q$, $\delta(q,u) = \delta(q,v)$ and $\forall q, q' \in Q$, $T_{fin}(c(q,u,q')) = T_{fin}(c'(q,v,q'))$ then they are equivalent for the congruence $\approx_c$. So, it is clear that the congruence $\approx_c$ is of finite index.

It remains to prove that $\approx_c$ saturates L. So let $m, f \in M_{\approx_c}$ such that $\varphi_{\approx_c}^{-1}(m)[\varphi_{\approx_c}^{-1}(f)]^\omega \cap L \neq \emptyset$. Since $\approx_c$ is of finite index, we can assume that $mf = m$ and $f^2 = f$. Moreover, $\varphi_{\approx_c}^{-1}(m)$ and $\varphi_{\approx_c}^{-1}(f)$ are rational languages, by Lemma 2.8, we just need to prove that $UP(\varphi_{\approx_c}^{-1}(m)[\varphi_{\approx_c}^{-1}(f)]^\omega) \subset L$. It is easy to see that for any $w \in UP(\varphi_{\approx_c}^{-1}(m)[\varphi_{\approx_c}^{-1}(f)]^\omega)$, there exist $x, y \in \Sigma^*$ such that $w = xy^\omega$, $x \in \varphi_{\approx_c}^{-1}(m)$ and $y \in \varphi_{\approx_c}^{-1}(f)$.

Now, since $\varphi_{\approx_c}^{-1}(m)[\varphi_{\approx_c}^{-1}(f)]^\omega \cap L \neq \emptyset$, there exist $u, v \in \Sigma^*$ such that $uv^\omega \in L$, $u \in \varphi_{\approx_c}^{-1}(m)$ and $v \in \varphi_{\approx_c}^{-1}(f)$. So, there exist $q_0 \in I$, $q \in Q$ and a computation $c = c_1(q_0, uv, q)c_2(q, v, q)^\omega$ of the word uv^ω such that $c_2(q, v, q) \in \mathscr{C}(\mathscr{A})$. Any $w \in UP(\varphi_{\approx_c}^{-1}(m)[\varphi_{\approx_c}^{-1}(f)]^\omega)$ has a computation $c' = c'_1(q_0, x, q)c'_2(q, y, q)^\omega$ such that $w = xy^\omega$, $uv \approx_c x$ and $v \approx_c y$. So, $c'_2(q, y, q) \in \mathscr{C}(\mathscr{A})$ and $\varphi_{\approx_c}^{-1}(m)[\varphi_{\approx_c}^{-1}(f)]^\omega \subset L$. $\square$

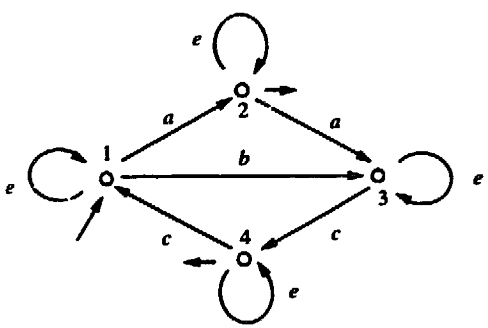
Remark 4.3. Using Fact 2.6, it is clear that the congruence $\approx_c$ associated with a Büchi or Müller automaton coincides with the relation associated with the corresponding transition table automaton.

Remark 4.4. In order to prove Theorem 2.11, Büchi associated with any Büchi automaton $\mathscr{A} = (Q, I, \delta, E)$ the following congruence on Σ^* denoted by $\approx_B$:

$$\forall u, v \in \Sigma^*, \quad u \approx_B v \Leftrightarrow \begin{cases} \forall q \in Q, \ \delta(q, u) = \delta(q, v), \\ \forall q \in Q, \ \delta(q, u)_E = \delta(q, v)_E, \end{cases}$$

where $\delta(q, w)_E$ is the set of states q' that satisfy the following condition: there exist $w_1 w_2 = w$ and $e \in E$ such that $e \in \delta(q, w_1)$ and $q' \in \delta(e, w_2)$. The congruence $\approx_B$ is of finite index and saturates the ω-language that $\mathscr{A}$ accepts.

For Büchi automata, it is easy to see that the cycle congruence $\approx_c$ is larger than the Büchi's congruence $\approx_B$. Moreover, the congruence $\approx_c$ may have strictly less classes than $\approx_B$: Consider the deterministic Büchi automaton $\mathscr{A} = (Q, \{q_0\}, \delta, E)$, where $Q = \{1, 2, 3, 4\}$, $q_0 = 1$, $E = \{2, 4\}$ and δ is defined by the following graph:



The words aa and b have the same computation in $\mathscr{A}$. Moreover, any infinite computation that use infinitely often the word aa visits infinitely often the state 4 in E and any infinite computation that uses infinitely often the word b also visits infinitely often the state 4 in E. So it is clear that aa is equivalent to b for $\approx_c$. But it is not the case for the congruence $\approx_B$ since the computation of aa uses the state 2 in E, whereas the computation of b uses only the states 1 and 3 which are not in E.

Definition 4.5. Given a right congruence of finite index $\sim$ on Σ^* and a ω-language L. We define the cycle congruence $\simeq_c$ of $\sim$ by $\forall u, v \in \Sigma^+$

$$u \simeq_c v \Leftrightarrow \begin{cases} \forall w \in \Sigma^*,\ wu \sim wv, \\ \forall x, w, w' \in \Sigma^* \text{ with } ww' \in \Sigma^+,\ x \sim xwuw' \sim xwvw' \text{ imply} \\ \quad \{x(wuw')^\omega \in L \Leftrightarrow x(wvw')^\omega \in L\}. \end{cases}$$

The congruence $\simeq_c$ is called cycle congruence associated with the right congruence $\sim$.

Proposition 4.6. *Let $\sim$ a right congruence of finite index that saturates a rational ω-language L. The cycle congruence $\simeq_c$ is of finite index and saturates L.*

Proof. Let $\sim$ be a finite right congruence saturating L and $\mathscr{A}_\sim = (Q_\sim, I_\sim, \delta_\sim, \mathscr{T}_\sim)$ the deterministic transition table automaton associated with $\sim$. This transition table automaton $\mathscr{A}_\sim$ accepts L. It is easy to see that the cycle congruence $\simeq_c$ associated with the right congruence $\sim_c$ of $A_\sim$ coincides with the cycle congruence $\sim$. So from Theorem 3.4, it is of finite index and saturates L. □

5. On the minimization problem

5.1. Introduction and definitions

Definition 5.1. Let $\mathscr{A}(Q, I, \Delta, \mathscr{T})$ and $\mathscr{A}'(Q', I', \Delta', \mathscr{T}')$ be two deterministic table transition automaton. A deterministic automaton morphism which associates $\mathscr{A}'$ with $\mathscr{A}$ is an application φ from Q into Q' such that $\varphi(Q) = Q'$, $\varphi(I) = I'$, $\Delta' = \{(\varphi(q), a, \varphi(q')) \text{ such that } (q, a, q') \in \Delta\}$ and $\mathscr{T}' = \{\varphi(T): T \in \mathscr{T}'\}$.

Fact 5.2. *If φ is a deterministic automaton morphism then we have $\mathscr{L}(\mathscr{A}) \subseteq \mathscr{L}(\varphi(\mathscr{A}))$.*

Definition 5.3. (a) A complete deterministic table transition automaton $\mathscr{A}$ accepting a rational ω-language L is s-*minimal* if there does not exist another complete deterministic table transition automaton accepting L with strictly less states than $\mathscr{A}$. If the ω-language L admits a unique s-minimal deterministic table transition automaton (up to an isomorphism) this automaton is s-minimum.

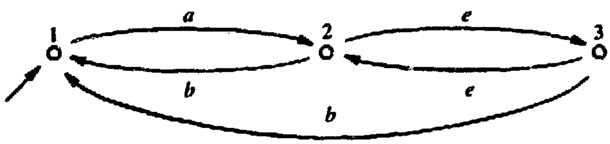
(b) A complete deterministic table transition automaton $\mathscr{A}$ accepting L is m-*minimal* if any automaton morphism φ such that $\varphi(\mathscr{A})$ is complete and $\mathscr{L}(\varphi(\mathscr{A})) = L$, is an isomorphism. If the ω-language L admits a unique m-minimal deterministic table transition automaton (up to an isomorphism) this automaton is m-minimum.

Fact 5.4. (1) *A m-minimum deterministic transition table automaton accepting a rational ω-language L is s-minimum.*

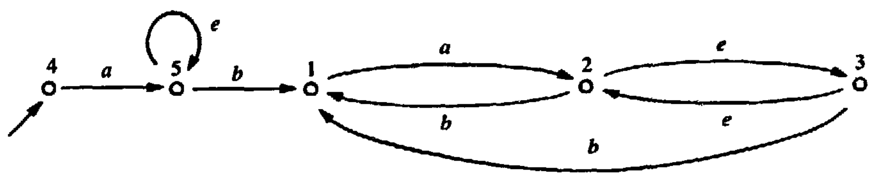
(2) *A s-minimal deterministic transition table automaton accepting a rational ω-language L is m-minimal.*

The converse of the two properties of Fact 5.4 does not hold.

Example 5.5. Consider the ω-language $L = (ae^*b)^*(a(ee)^+b(ab)^*)^\omega$. It is accepted by the deterministic table transition automaton $\mathscr{A}_1(Q_1, q_{01}, \Delta_1, \mathscr{T}_1)$, where $Q_1 = \{1,2,3\}$, $q_{01} = \{1\}$, $\mathscr{T}_1 = \{\{(1,a,2); (2,e,3); (3,e,2); (2,b,1)\}\}$ and δ_1 is given by the following graph:



This ω-language is also accepted by the following deterministic table transition automaton: $\mathscr{A}_2(Q_2, q_{02}, \Delta_2, \mathscr{T}_2)$, where $Q_2 = \{4,5,1,2,3\}$, $q_{02} = \{4\}$, $\mathscr{T}_2 = \{\{(1,a,2); (2,e,3); (3,e,2); (2,b,1)\}\}$ and δ_2 is given by the following graph:

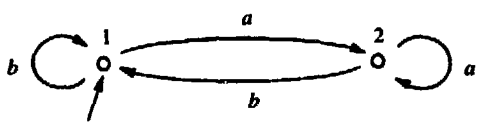


The automaton $\mathscr{A}_1$ is s-minimum and $\mathscr{A}_2$ is m-minimal so $\mathscr{A}_1$ is not m-minimum and $\mathscr{A}_2$ is not s-minimal.

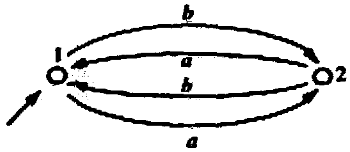
Obviously, any rational ω-language admits a finite number of s-minimal automata, but there is not always a unique s-minimal deterministic table transition automaton for a given rational ω-language.

Example 5.6. Consider the rational ω-language $\{a,b\}^*(ab)^\omega$. This ω-language has seven different s-minimal deterministic table transition automata.

Two of them are: $\mathscr{A}_1(Q_1, q_{01}, \Delta_1, \mathscr{T}_1)$, where $Q_1 = \{1,2\}$, $q_{01} = \{1\}$, $\mathscr{T}_1 = \{\{(1,a,2); (2,b,1)\}\}$ and δ_1 is given by the graph



and $\mathscr{A}_2(Q_2, q_{02}, \Delta_2, \mathscr{T}_2)$, where $Q_2 = \{1,2\}$, $q_{02} = \{1\}$, $\mathscr{T}_2 = \{\{(1,a,2); (2,b,1)\}; \{(1,b,2);(2,a,1)\}\}$ and δ_2 is given by the graph



Remark 5.7. The deterministic automaton morphism reductions are not confluent. Consider, for instance, the product of the two table transition automata given in Example 5.6. This new deterministic automaton also accepts the ω-language $\{a,b\}^*(ab)^\omega$ and it can be easily reduced, using deterministic automaton morphisms, in $\mathscr{A}_1$ or $\mathscr{A}_2$ which are both m-minimal.

In the next section, we characaterize the family of rational ω-languages which have a m-minimum deterministic transition table automaton and we prove that the other rational ω-languages admit an infinite number of m-minimal automata.

5.2. *Prefix recognizable ω-languages*

Definition 5.8. The *right congruence of prefixes* $\sim_p$ of an ω-language L is defined by

$$\forall u,v \in \Sigma^*, \quad u \sim_p v \text{ if and only if } \forall w \in \Sigma^\omega, uw \in L \Leftrightarrow vw \in L.$$

The right congruence of prefixes $\sim_p$ a rational ω-language is of finite index [17].

We denote by $\mathscr{A}_L = (Q_{\sim_p}, I_{\sim_p}, \delta_{\sim_p}, \mathscr{T}_{\sim_p})$ the complete deterministic transition table automaton associated with $\sim_p$ defined by $(Q_{\sim_p}, I_{\sim_p}, \delta_{\sim_p})$ is the deterministic transition system $TS_{\sim_p}$ and $\mathscr{T}_{\sim_p} = \{t\text{-}inf(c)$, where c is a computation with origin $\varepsilon_{\sim_p}$ and label in $L\}$. The right congruence $\sim_Q$ associated with $\mathscr{A}_L$ is equal to $\sim_p$.

Definition 5.9. A rational ω-language saturated by its right congruence of prefixes is called a *prefix recognizable ω-language* (P-ω-language for short).

Let L be a P-ω-language. By definition, the right congruence $\sim_p$ saturates L. So the deterministic complete transition table automaton $\mathscr{A}_L$ accepts L. From Theorem 3.4, the right congruence $\sim_p$ is larger than any right congruence which saturates L, so the automaton $\mathscr{A}_L$ is the homomorphic image of any deterministic transition table automaton accepting L, so it is m-minimum and s-minimum.

Fact 5.10. *Let L be a P-ω-language. Then the automaton $\mathscr{A}_L$ accepts L, it is the homomorphic image of any deterministic table transition automaton accepting L and so $\mathscr{A}_L$ is m-minimum.*

Remark 5.11. Clearly, any ω-language accepted by a deterministic table transition automaton $\mathscr{A}_L$ such that $\sim_Q = \sim_p$ is a P-ω-language.

Let $\mathscr{A} = (Q, \{q_0\}, \delta, \mathscr{T})$ be a deterministic table transition automaton accepting an ω-language L. For all $x \in \Sigma^*$, we denote by $Q_{x_{\sim_p}}$ the set of states $q \in Q$ such that there exists $w \in x_{\sim_p}$ satisfying $\delta(q_0, w) = q$. Since $\mathscr{A}$ is deterministic, we have the following property: $\forall q \in Q_{x_{\sim_p}}, \forall w \in \Sigma^*, \delta(q_0, w) = q \Rightarrow w \in x_{\sim_p}$.

Fact 5.12. *If φ is a deterministic automaton morphism such that $\mathscr{L}(\mathscr{A}) = \mathscr{L}(\varphi(\mathscr{A}))$ then $\forall x \in \Sigma^*, \varphi(Q_{x_{\sim_p}}) = Q'_{x_{\sim_p}}$.*

Lemma 5.13. *If L is not P-ω-language then there exist $x \in \Sigma^*$ and $u, v \in \Sigma^+$ with $x \sim_p xu \sim_p xv$ and $u[1] \neq v[1]$ satisfying, for any deterministic table transition automaton $\mathscr{A} = (Q, \{q_0\}, \delta, \mathscr{T})$ accepting L, $\forall q \in Q_{x_{\sim_p}}, \delta(q, u) = q \Rightarrow \delta(q, v) \neq q$.*

Proof. Let L be a rational ω-language which is not a P-ω-language. Then, the right congruence of prefixes of L does not saturate L. So, by definition, there exist $x \in \Sigma^*$ and $u, v \in \Sigma^+$ satisfying $x \sim_p xu \sim_p xv$ such that $x(u^+v^+)^\omega \cap L \neq \emptyset$ and $x(u^+v^+)^\omega \cap \bar{L} \neq \emptyset$. Let w be the longest common prefix of u and v. Let $u = wu'$ and $v = wv'$. If $u' \neq \varepsilon$ and $v' \neq \varepsilon$ then we have $x((wu')^+(wv')^+)^\omega = xw((u'w)^+(v'w)^+)^\omega$ and it is clear, replacing respectively, x, u and v by $xw', u'w'$ and wv', that we can choose u and v such that $u[1] \neq v[1]$. It is not possible to have both $u' = \varepsilon$ and $v' = \varepsilon$ since, in this case, $x((wu')^+(wv')^+)^\omega = \{xw^\omega\}$. Now, if u is a prefix of v (or v is a prefix of u) one can set $v = uu^pu'w$ with $u'u'' = u$, $p \geqslant 0$ and $u''[1] \neq w[1]$. So we have $x(u^+v^+)^\omega = xu^{p+1}u'((u''u')^+(wu^{p+1}u'))^\omega$ and, replacing respectively, x, u and v by $xu^{p+1}u', u''u'$ and $wu^{p+1}u'$ it is clear that we can choose u and v such that $u[1] \neq v[1]$.

Now, let $\mathscr{A} = (Q, \{q_0\}, \delta, \mathscr{T})$ be a deterministic table transition automaton such that $\mathscr{L}(\mathscr{A}) = L$. Assume there exists $q \in Q_{x_{\sim_p}}$ such that $\delta(q, u) = q$ and $\delta(q, v) = q$. Since, $x(u^+v^+)^\omega \cap L \neq \emptyset$ and $q \in Q_{x_{\sim_p}}$, we have $\forall w \in (u^+v^+)^\omega$, $t\_inf(q, w) = t\_inf(q, (uv)^\omega) \in \mathscr{T}$ so $\forall w \in (u^+v^+)^\omega$, $xw \in L$. A contradiction. $\square$

Theorem 5.14. *Let L be rational ω-language. If L is not a P-ω-language then L admits an infinite number of m-minimal deterministic table transition automata.*

Proof. Let L be a rational ω-language which is not a P-ω-language. Let $x, u, v \in \Sigma^+$ such that $x \sim_p xu \sim_p xv$ and $u[1] \neq v[1]$ satisfying the condition of Lemma 5.13.

Let $x = x_1 \cdots x_k$ and $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_m$. Let $\mathscr{A} = (Q, \{q_0\}, \delta, \mathscr{T})$ be a deterministic table transition automaton such that $\mathscr{L}(\mathscr{A}) = L$.

Assume that L has a finite number of m-minimal deterministic table transition automata. Let $l - 1$ be the number of states of the biggest m-minimal deterministic table transition automaton of L.

We will construct a deterministic table transition automaton accepting L with a number of states greater than l. Such an automaton is not m-minimal, but we will

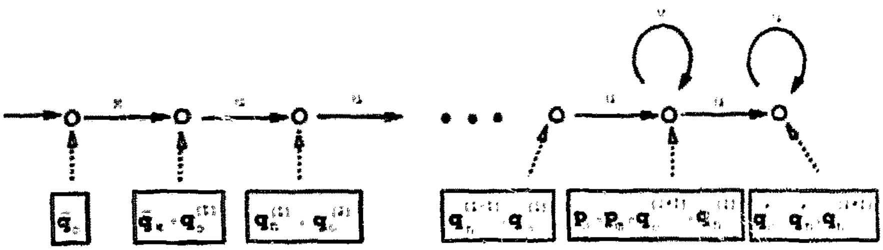


Fig. 1.

prove that it cannot be reduced using automaton morphism, so we will obtain a contraction with our hypothesis. The construction of this automaton is partially illustrated by Fig. 1.

So let $\bar{\mathscr{A}} = (\bar{Q}, \{\bar{q}_0\}, \bar{\delta}, \bar{\mathscr{T}})$ be the complete deterministic table transition automaton defined by

- $\bar{Q} = Q \cup \{\bar{q}_0, \ldots, \bar{q}_k\} \cup \bigcup_{i=1}^{i=l+1} \{q_0^{(i)}, \ldots, q_n^{(i)}\} \cup \{p_0, \ldots, p_m\} \cup \{q'_0, \ldots, q'_n\}$ such that
  (1) $Q \cap (\{\bar{q}_0, \ldots, \bar{q}_k\} \cup \bigcup_{i=1}^{i=l+1} \{q_0^{(i)}, \ldots, q_n^{(i)}\} \cup \{p_0, \ldots, p_m\} \cup \{q'_0, \ldots, q'_n\}) = \emptyset$,
  (2) $\bar{q}_k = q_0^{(1)}$, $\forall 1 \leqslant i \leqslant l$, $q_0^{(i+1)} = q_n^{(i)}$, $q_0^{(l+1)} = p_0 = p_m$, $q_n^{(l+1)} = q'_0 = q'_n$.
- $\bar{\delta}$: $\bar{Q} \times \Sigma$ into $\bar{Q}$ is defined by
  (1) $\forall 1 \leqslant j \leqslant k$, $\bar{\delta}(\bar{q}_0, x[j]) = \bar{q}_j$,
  (2) $\forall 1 \leqslant i \leqslant l+1$, $\forall 1 \leqslant j \leqslant n$, $\bar{\delta}(q_0^{(i)}, u[j]) = q_j^{(i)}$,
  (3) $\forall 1 \leqslant j \leqslant m$, $\bar{\delta}(p_0, v[j]) = p_j$,
  (4) $\forall 1 \leqslant j \leqslant n$, $\bar{\delta}(q'_0, u[j]) = q'_j$,
  (5) $\forall 0 \leqslant j < k$, $\forall a \neq x_{j+1}$, $\bar{\delta}(\bar{q}_0, x[j]a) = \delta(q_0, x[j]a)$,
  (6) $\forall 0 \leqslant j < n$, $\forall a \neq u_{j+1}$, $\forall 1 \leqslant i \leqslant l$, $\bar{\delta}(q_0^{(i)}, u[j]a) = \delta(q_0, xu[j]a)$,
  (7) $\forall a \neq u_1$, $\forall a \neq v_1$, $\bar{\delta}(q_0^{(l+1)}, a) = \delta(q_0, xa)$,
  (8) $\forall 0 < j < m$, $\forall a \neq v_{j+1}$, $\bar{\delta}(q_0^{(l+1)}, v[j]a) = \delta(q_0, xv[j]a)$,
  (9) $\forall 0 < j < n$, $\forall a \neq u_{j+1}$, $\bar{\delta}(q_0^{(l+1)}, u[j]a) = \delta(q_0, xu[j]a)$,
  (10) $\forall a \neq u_1$, $\bar{\delta}(q_0^{(l+1)}, ua) = \delta(q_0, xa)$,
  (11) $\forall 0 < j < n$, $\forall a \neq u_{j+1}$, $\bar{\delta}(q'_0, u[j]a) = \delta(q_0, xu[j]a)$.
- $\bar{\mathscr{T}} = \mathscr{T} \cup U \cup V$ with $U = \{t\_inf(\bar{q}_0, xu^\omega)\}$, if $xu^\omega \in L$ and $U = \emptyset$ otherwise and $V = \{t\_inf(\bar{q}_0, xu^l v^\omega)\}$, if $xv^\omega \in L$ and $V = \emptyset$ otherwise.

The structure in $\bar{\mathscr{A}}$ associated with the items (1)–(4) of the definition of $\bar{\delta}$ is illustrated by Fig. 1. The items (5)–(11) describe a connection, with respect of $\sim_p$, of the previously described structure to the automaton $\mathscr{A}$.

By construction, the automaton $\bar{\mathscr{A}}$ is complete and, since $u[1] \neq v[1]$, $\bar{\mathscr{A}}$ is deterministic and it satisfies:

(P1) $\forall w \in \Sigma^*$, $\bar{\delta}(\bar{q}_0, w) = q \in Q$ implies $\delta(q_0, w) = q'$ and $q, q' \in Q_{w \sim_p}$.
(P2) $\forall w \in \Sigma^\omega$: $t\_inf(\bar{q}_0, w) \cap Q = \emptyset \Leftrightarrow \{w \in xu^l v^* u^\omega$ or $w = xu^l v^\omega\}$.

Let us prove that $L = \mathscr{L}(\bar{\mathscr{A}})$:

Let $w \in \mathscr{L}(\bar{\mathscr{A}})$. Two cases may be considered.

(a) $\exists v' \in \Sigma^*$, $v'' \in \Sigma^\omega$ such that $w = v'v''$ and $\bar{\delta}(\bar{q}_0, v') = q \in Q$.

In this case, we have $t\_inf(\bar{q}_0, w) \subseteq Q$. Since, $w \in \mathscr{L}(\bar{\mathscr{A}})$, $t\_inf(\bar{q}_0, w)$ belongs to $\mathscr{T}$. But, from (P1), we also have $\delta(q_0, v') = q'$ with $q, q' \in Q_{v'\sim_p}$, so $t\_inf(\bar{q}_0, w) \in \mathscr{T}$ implies $t\_inf(\bar{q}_0, w) \in \mathscr{T}$ and $w \in \mathscr{L}(\mathscr{A})$.

(b) $\forall v' \in \Sigma^*$, $v'' \in \Sigma^\omega$ such that $w = v'v''$, we have $\bar{\delta}(\bar{q}_0, v') \notin Q$.
Then, from (P2), we have $w \in xu^l v^* u^\omega$ or $w = xu^l v^\omega$. Since, for any $p \geq 0$, $xu^l v^p \sim_p x$, by construction, $w \in L$.

Conversely, let $w \in \mathscr{L}(\mathscr{A})$. If $w \in xu^l v^* u^\omega$ or $w = xu^l v^\omega$ then, by construction, $w \in \mathscr{L}(\bar{\mathscr{A}})$ else we have $t\_inf(\bar{q}_0, w) \subseteq Q$. In this case, there exist $v' \in \Sigma^*$, $v'' \in \Sigma^\omega$ such that $w = v'v''$ and $\bar{\delta}(\bar{q}_0, v') = q \in Q$. But, from (P1), we also have $\delta(q_0, v') = q'$ with q, $q' \in Q_{v'\sim_p}$, so $t\_inf(q_0, w) \in \mathscr{T}$ implies $t\_inf(\bar{q}_0, w) \in \mathscr{T}$ and $w = v'v'' \in \mathscr{L}(\bar{\mathscr{A}})$.

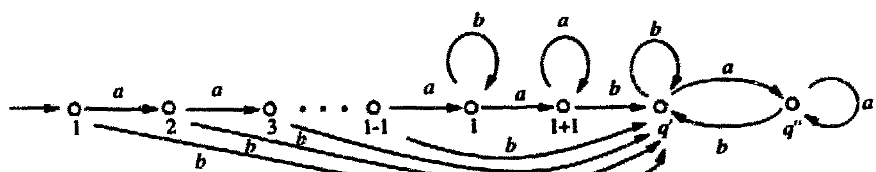
Since L admits a finite number of m-minimal deterministic table transition automata and since $l - 1$ is the cardinality of the biggest one, there exist a deterministic automaton morphism ψ and a m-minimal automaton $\mathscr{A}_m = (Q_m, \{q_{0m}\}, \delta_m, \mathscr{T}_m)$ accepting L such that $\psi(q_0^{(j)}) = \psi(q_0^{(j+h)})$ with $1 \leq j < j + h \leq l + 1$ (we choose j, h as short as possible).

Moreover, ψ is a deterministic automaton morphism, $\bar{\delta}(q_0^{(j)}, u^h) = q_0^{(j+h)}$ and $\bar{\delta}(q_0^{(j+h)}, u^{l+1-j-h+1}) = q_n^{(l+1)}$, so there exists $0 \leq \alpha < h$ such that $\psi(q_0^{(j+\alpha)}) = \psi(q_n^{(l+1)})$. Now, $\delta_m(\psi(q_n^{(l+1)}), u) = \psi(\bar{\delta}(q_n^{(l+1)}, u)) = q_n^{(l+1)} = q_0^{(j+\alpha)}$, so from the determinism of ψ, $\forall j \leq s \leq l + 1$, $\psi(q_0^{(s)}) = \psi(q_n^{(l+1)})$. In particular, we have $\psi(q_0^{(l)}) = \psi(\bar{\delta}(q_0^{(l)}, v)) = \psi(q_n^{(l+1)}) = \psi(\bar{\delta}(q_0^{(l+1)}, u))$. So $\delta_m(\psi(q_0^{(l)}), u) = \delta_m(\psi(q_0^{(l)}), v) = \psi(q_0^{(l)})$. Since $\mathscr{A}_m$ is a deterministic table transition automaton accepting L, $\psi(q_0^{(l)}) \in Q_{mx\sim_p}$ and x, u, v satisfy the hypothesis of Lemma 5.13 so there is a contradiction. Finally, L has an infinite number of m-minimal deterministic table transition automata. □

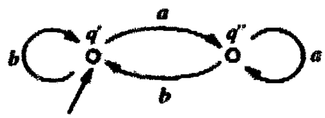
In Fact 5.10, we show that a P-ω-language has a m-minimal deterministic table transition automaton so we have the following corollary.

Corollary 5.15. *A rational ω-language has a m-minimum deterministic table transition $\mathscr{A}$ iff it is a rational P-ω-language.*

Example 5.16. Let $L = \{a, b\}^*(ab)^\omega$. This ω-language is not a P-ω-language, it has an infinite number of minimal automaton. Let $\mathscr{A}_l = \{Q_l, I_l, \delta_l, \mathscr{T}_l)$ be the deterministic table transition automaton, where $Q_l = \{0, 1, 2, \ldots, l, l + 1\} \cup \{q', q''\}$, $I_l = \{0\}$, $\mathscr{T}_l = \{\{(q', a, q''), (q'', b, q')\}\}$ and where δ_l is defined by the graph



The automaton $\mathscr{A}_l$ has been obtained using the construction of Theorem 5.14 for the following deterministic table transition automaton $\mathscr{A} = \{Q, I, \delta, \mathscr{T}\}$ with $Q = \{q', q''\}$, $I = \{q'\}$, $\mathscr{T} = \{\{(q', a, q''), (q'', b, q')\}\}$ and where δ is defined by the following graph:



It is easy to see that, for any $l \leqslant 0$, the deterministic table transition automaton $\mathscr{A}_l$ is m-minimal.

5.3. Some properties of P-ω-languages

Fact 5.17. *The family of rational P-ω-languages is closed by complement and it is not closed by intersection.*

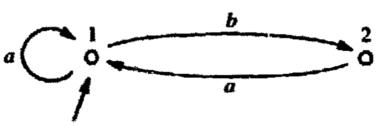
The fact that the family of rational P-ω-languages is closed by complement, is a direct consequence of Lemma 3.11. The family of rational P-ω-languages is not closed by intersection:

Example 5.18. Consider the following ω-languages $L_1 = \{a,b\}^*(ab)^\omega + (a^*b)^*c^\omega$ and $L_2 = \{a,b\}^*(ab)^\omega + (b^*a)^+c^\omega$. It is easy to see that L_1 and L_2 are P-ω-languages whereas their intersection $L_1 \cap L_2 = \{a,b\}^*(ab)^\omega$ is the ω-language considered in Example 5.6 which is not a P-ω-language.

Fact 5.19. *The family $DRAT \cap Co\text{-}DRAT$ is included in the family of P-ω-languages and this inclusion is strict.*

In [14], it is shown that the ω-languages in $DRAT \cap Co\text{-}DRAT$ are accepted by a unique (up to isomorphism) deterministic Müller automaton $\mathscr{A}_M$ such that $\sim_Q = \sim_P$ so this family is included in one of the P-ω-languages (see Fact 2.5). But there exist ω-languages which are not in $DRAT \cap Co\text{-}DRAT$ accepted by a deterministic Müller automaton such that $\sim_Q = \sim_P$.

Example 5.20. Consider, for instance, the ω-language $L = \{a, ba\}^*a^\omega$. This ω-language is not in $DRAT$, but it is a P-ω-language. It is accepting by the Müller automaton $\mathscr{A}_L = (Q, I, \delta, \mathscr{E})$ with $Q = \{1,2\}$, $I = \{1\}$, $\mathscr{E} = \{\{1\}\}$ and where δ is defined by the graph



If we consider Müller automata, we can define a family of ω-languages analogous to the P-ω-languages. This family is constituted by the ω-languages accepted by a deterministic Müller automaton such that $\sim_Q = \sim_p$. Such ω-languages, here called P_M-ω-languages, have been studied in [8]. They clearly admit a m-minimum deterministic Müller automaton which accepts them and satisfies $\sim_Q = \sim_p$ [3, 8]. Moreover, it is easy to rewrite the proof of Theorem 5.14 using Müller automata so we have the following proposition.

Proposition 5.21. *Let L be rational ω-language. If L is not a P_M-ω-language then L admits an infinite number of m-minimal deterministic Müller automata.*

In this paper we choose to consider table transition automata, the reason lies in the following fact.

Fact 5.22. *The family of rational P_M-ω-languages, is strictly included in the family of rational P-ω-languages.*

Using Fact 2.5, the inclusion is clear. Now, consider the ω-language $L = \{a,b\}^*a^\omega$. It is a rational P-ω-language. Its right congruence of prefixes has only one class $\{a,b\}^*$ and it is accepted by a deterministic table transition automaton with one state and two transitions on this state, respectively, labelled a and b. But the corresponding Müller automaton does not accept L. □

Fact 5.23. (1) *There exist ω-languages not being P-ω-language which admits a unique s-minimal deterministic transition table automaton (cf. Example 5.5)*

(2) *There exist rational ω-languages which have several s-minimal automata (cf. Example 5.6).*

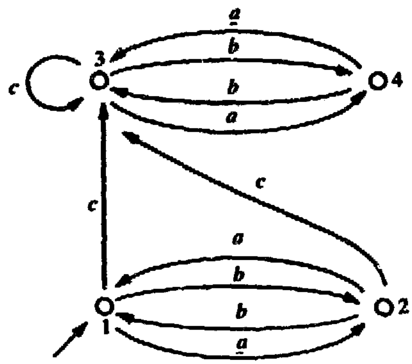
Proposition 5.24. *Let L be a P-ω-language. Then the syntactic congruence $\approx_s$ of L, the cycle congruence $\approx_c$ of the automaton $\mathscr{A}_L$ are equal.*

Proof. Since $\approx_s$ is larger than $\approx_c$, it is sufficient to prove that for any $u, v \in \Sigma^*$, $u \approx_s v$ implies $u \approx_c v$. We first have, $\forall w \in \Sigma^*$, $\forall w' \in \Sigma^\omega$, $wuw' \in L \Leftrightarrow wvw' \in L$. Hence, for any $w \in \Sigma^*$, we have $wu \sim_p wv$. So in $\mathscr{A}_L$, we have $\delta(q_0, w) = \delta(q_0, wv)$.

Next, we have $\forall x, w, w' \in \Sigma^*$ such that $ww' \neq \varepsilon$, $x(wuw')^\omega \in L \Leftrightarrow x(wvw')^\omega \in L$. Now, since $\mathscr{A}_L$ is deterministic, if there exists a cycle $c(q, wuw', q) \in \mathscr{C}(\mathscr{A}_L)$ then there exists $x \in \Sigma^*$ such that $\delta(q_0, x) = q$ and $x(wuw')^\omega \in L$. Since, for any $\bar{w} \in \Sigma^*$, we have $\delta(q_0, \bar{w}u) = \delta(q_0, \bar{w}v)$ and $\delta(q_0, xwu) = \delta(q_0, xwv)$. Thus, there exists $c'(q, wvw', q) \in \mathscr{C}$. But, from the definition of $\approx_s$, we have $x(wvw')^\omega \in L$. So, $c'(q, wvw', q)$ is an accepting cycle of $\mathscr{A}_L$. □

Fact 5.25. *There exist rational ω-languages not being P-ω-language which have a m-minimal deterministic table transition automaton such that $\approx_c = \approx_s$.*

Example 5.26. Consider the ω-language $L = \{a,b\}^*(ab)^\omega \cup \{a,b\}^*c\{c,aa,ab,ba,bb\}^*c^\omega$. This ω-language is accepted by the s-minimal deterministic table transition automaton: $\mathscr{A} = (Q,\{q_0\},\delta,\mathscr{T})$, where $Q = \{1,2,3,4\}$, $q_0 = 1$, $\mathscr{T} = \{\{(1,a,2);(2,b,1)\};\{(1,b,2);(2,a,1)\};\{(3,c,3)\}\}$ and δ is defined by the graph



The ω-language L is not a P-ω-language since the class of ε for the right congruence of prefixes is $\{a,b\}^*$. The cycle congruence of the automaton $\mathscr{A}$ is equal to the syntactic congruence of L. It is an open problem to characterize the family of ω-languages which have a deterministic accepting table transition automaton such that its cycle congruence is the syntactic congruence of the considered ω-language.

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