# A Coinductive Treatment of Infinitary Rewriting*

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## Abstract

We introduce a coinductive definition of infinitary term rewriting. The setup is surprisingly simple, and has in contrast to the usual definitions of infinitary rewriting, neither need for ordinals nor for metric convergence. While the idea of a coinductive treatment of infinitary rewriting is not new, all previous approaches were limited to reductions of length $\leq \omega$. The approach presented in this paper is the first to capture the full infinitary term rewriting with reductions of arbitrary ordinal length.

We find that using least ($\mu x$) and greatest fixed points ($\nu x$), infinitary rewriting has the following succinct definition:

$$\twoheadrightarrow\!\!\!\!\!\!\!\twoheadrightarrow \quad = \quad \mu\, x.\, \nu\, y.\, (\to_\varepsilon \cup\, \overline{x})^* \circ \overline{y}$$

where $\overline{R} = \{\, \langle f(s_1,\ldots,s_n),\, f(t_1,\ldots,t_n)\rangle \mid s_1\ R\ t_1,\ldots,s_n\ R\ t_n \,\} \cup \mathsf{id}$.

Additionally, our approach gives rise — in a very natural way — to novel notions of infinitary equational reasoning $=^\infty$ and bi-infinite rewriting ${}^\infty\!\!\to^\infty$:

$$\begin{aligned} =^\infty \quad &= \quad \nu\, x.\, (\leftarrow_\varepsilon \cup \to_\varepsilon \cup\, \overline{x})^* \\ {}^\infty\!\!\to^\infty \quad &= \quad \nu\, x.\, (\to_\varepsilon \cup\, \overline{x})^* \end{aligned}$$

Especially, the theory of infinitary equational reasoning is largely underdeveloped.


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## 1 Coinductive Rewriting

Infinitary rewriting is a generalization of the ordinary finitary rewriting to infinite terms and infinite reductions (including reductions of ordinal lengths larger than $\omega$). We present a coinductive treatment of infinitary rewriting free of ordinals, metric convergence and partial orders which have been essential in earlier definitions of the concept [4, 8, 10, 5, 13, 11, 9, 12, 7, 2, 1, 3, 6]. In a slogan one could say: *Infinitary rewriting has never been easier!*

Let us describe the idea. Let $R$ be a term rewriting system (TRS). We write $\to_\varepsilon$ for root steps with respect to $R$, that is, we define $\to_\varepsilon\, = \{\, (\ell\sigma, r\sigma) \mid \ell \to r \in R,\ \sigma \text{ a substitution} \,\}$. The crucial ingredient of our definition of infinitary rewriting $\twoheadrightarrow\!\!\!\!\!\!\!\twoheadrightarrow$ are the *coinductive* rules

$$\frac{s\ (\to_\varepsilon \cup \overline{\twoheadrightarrow\!\!\!\!\!\!\!\twoheadrightarrow})^*\ t}{s \twoheadrightarrow\!\!\!\!\!\!\!\twoheadrightarrow t} \qquad\qquad \frac{s_1 \twoheadrightarrow\!\!\!\!\!\!\!\twoheadrightarrow t_1 \quad \ldots \quad s_n \twoheadrightarrow\!\!\!\!\!\!\!\twoheadrightarrow t_n}{f(s_1,s_2,\ldots,s_n)\ \overline{\twoheadrightarrow\!\!\!\!\!\!\!\twoheadrightarrow}\ f(t_1,t_2,\ldots,t_n)} \tag{1}$$

Here $\twoheadrightarrow\!\!\!\!\!\!\!\twoheadrightarrow$ and $\overline{\twoheadrightarrow\!\!\!\!\!\!\!\twoheadrightarrow}$ stand for finite and infinite reductions where $\overline{\twoheadrightarrow\!\!\!\!\!\!\!\twoheadrightarrow}$ contains only steps below the root. Note that these relations are defined mutually. The coinductive nature of the rules means that the derivation trees, the proof trees, need not be well-founded. (As we shall see, for the standard notion of infinitary rewriting, we need to restrict the derivation trees.)

To illustrate the use of the rules, let us immediately consider an example.

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▶ **Example 1.** Let $R$ be the TRS consisting solely of the following rewrite rule $\mathsf{a} \to \mathsf{C}(\mathsf{a})$. We write $\mathsf{C}^\omega$ to denote the infinite term $\mathsf{C}(\mathsf{C}(\mathsf{C}(\dots)))$, the solution of the equation $\mathsf{C}^\omega = \mathsf{C}(\mathsf{C}^\omega)$. We then have $\mathsf{a} \twoheadrightarrow \mathsf{C}^\omega$, that is, an infinite reduction from $\mathsf{a}$ to $\mathsf{C}^\omega$ in the limit:

$$\mathsf{a} \to \mathsf{C}(\mathsf{a}) \to \mathsf{C}(\mathsf{C}(\mathsf{a})) \to \mathsf{C}(\mathsf{C}(\mathsf{C}(\mathsf{a}))) \to \dots \to^\omega \mathsf{C}^\omega$$

Using the rules above, we can derive $\mathsf{a} \twoheadrightarrow \mathsf{C}^\omega$ as shown in Figure 1. This is an infinite proof

$$\dfrac{\mathsf{a} \to_\varepsilon \mathsf{C}(\mathsf{a}) \qquad \dfrac{\dfrac{}{\mathsf{a} \twoheadrightarrow \mathsf{C}^\omega}}{\mathsf{C}(\mathsf{a}) \mathrel{\rightarrow\!\!\!\!\!\!/\!\!/\!\!/} \mathsf{C}^\omega}}{\mathsf{a} \twoheadrightarrow \mathsf{C}^\omega}$$

**Figure 1** A reduction $\mathsf{a} \twoheadrightarrow \mathsf{C}^\omega$ of length $\omega$.

tree as indicated by the loop $\cdots\!\!>$ in which the rewrite sequence $\mathsf{a} \to_\varepsilon \mathsf{C}(\mathsf{a}) \mathrel{\rightarrow\!\!\!\!\!\!/\!\!/\!\!/} \mathsf{C}^\omega$ is written in the form $\mathsf{a} \to_\varepsilon \mathsf{C}(\mathsf{a}) \quad \mathsf{C}(\mathsf{a}) \mathrel{\rightarrow\!\!\!\!\!\!/\!\!/\!\!/} \mathsf{C}^\omega$, that is, two separate steps such that the target of the first equals the source of the second step; this is made precise in Notation 1, below.

Put in words, the proof tree in Figure 1 can be described as follows. We have an infinitary rewrite sequence $\twoheadrightarrow$ from $\mathsf{a}$ to $\mathsf{C}^\omega$ since we have a root step from $\mathsf{a}$ to $\mathsf{C}(\mathsf{a})$, and an infinitary reduction below the root $\mathrel{\rightarrow\!\!\!\!\!\!/\!\!/\!\!/}$ from $\mathsf{C}(\mathsf{a})$ to $\mathsf{C}^\omega$. The latter reduction $\mathsf{C}(\mathsf{a}) \mathrel{\rightarrow\!\!\!\!\!\!/\!\!/\!\!/} \mathsf{C}^\omega$ is in turn witnessed by the infinitary rewrite sequence $\mathsf{a} \twoheadrightarrow \mathsf{C}^\omega$ on the direct subterms. ◀

▶ Notation 1. Instead of introducing derivation rules for transitivity, in particular $(\to_\varepsilon \cup \mathrel{\rightarrow\!\!\!\!\!\!/\!\!/\!\!/})^*$, we will write rewrite sequences $s_0 \leadsto_0 s_1 \leadsto_1 \dots \leadsto_{n-1} s_n$ where $\leadsto_i \in \{\to_\varepsilon, \mathrel{\rightarrow\!\!\!\!\!\!/\!\!/\!\!/}\}$ as sequence of single steps $s_0 \leadsto_0 s_1 \quad s_1 \leadsto_1 s_2 \quad \dots \quad s_{n-1} \leadsto_{n-1} s_n$. That is:

$$\dfrac{s_0 \leadsto_0 s_1 \quad s_1 \leadsto_1 s_2 \quad \dots \quad s_{n-1} \leadsto_{n-1} s_n}{s_0 \twoheadrightarrow s_n}$$

This notation is more convenient since it avoids the need for explicitly introducing rules for transitivity, and thereby keeps the proof trees small.

As a second example, let us consider a rewrite sequence of length beyond $\omega$.

▶ **Example 2.** We consider the term rewriting system consisting of the following rules:

$$\mathsf{f}(x, x) \to \mathsf{D} \qquad \mathsf{a} \to \mathsf{C}(\mathsf{a}) \qquad \mathsf{b} \to \mathsf{C}(\mathsf{b})$$

Then we have the following reduction of length $\omega + 1$:

$$\mathsf{f}(\mathsf{a}, \mathsf{b}) \to \mathsf{f}(\mathsf{C}(\mathsf{a}), \mathsf{b}) \to \mathsf{f}(\mathsf{C}(\mathsf{a}), \mathsf{C}(\mathsf{b})) \to \dots \to^\omega \mathsf{f}(\mathsf{C}^\omega, \mathsf{C}^\omega) \to \mathsf{D}$$

That is, after an infinite rewrite sequence of length $\omega$, we reach the limit term $\mathsf{f}(\mathsf{C}^\omega, \mathsf{C}^\omega)$, and we then continue with a rewrite step from $\mathsf{f}(\mathsf{C}^\omega, \mathsf{C}^\omega)$ to $\mathsf{D}$. Figure 2 shows how this rewrite sequence $\mathsf{f}(\mathsf{a}, \mathsf{b}) \twoheadrightarrow \mathsf{D}$ can be derived in our setup. The precise meaning of the symbol $\overset{\leq}{\mathrel{\rightarrow\!\!\!\!\!\!/\!\!/\!\!/}}$ in the figure will be explained later; for the moment, we may think of $\overset{\leq}{\mathrel{\rightarrow\!\!\!\!\!\!/\!\!/\!\!/}}$ to be $\mathrel{\rightarrow\!\!\!\!\!\!/\!\!/\!\!/}$.
We note that the rewrite sequence $\mathsf{f}(\mathsf{a}, \mathsf{b}) \twoheadrightarrow \mathsf{D}$ cannot be 'compressed' to length $\omega$. That is, there exists no reduction $\mathsf{f}(\mathsf{a}, \mathsf{b}) \to^{\leq\omega} \mathsf{D}$. ◀

For the definition of rewrite sequences of ordinal length, there is a design choice concerning the connectedness at limit ordinals: (a) Cauchy convergence, or (b) strong convergence. The purpose of the connectedness condition is to exclude jumps at limit ordinals, as illustrated in

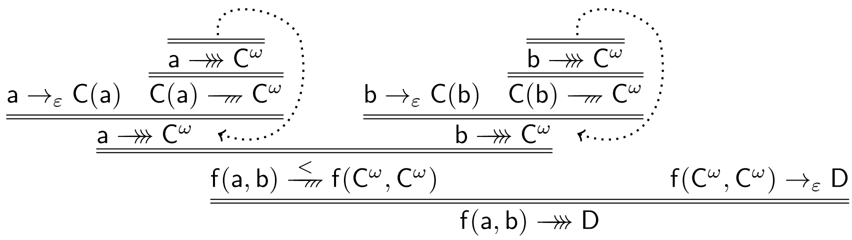

**Figure 2** A reduction $\mathsf{f}(\mathsf{a},\mathsf{b}) \twoheadrightarrow \mathsf{D}$ of length $\omega + 1$.

the following non-connected rewrite sequence (where $R = \{\, \mathsf{a} \to \mathsf{a},\ \mathsf{b} \to \mathsf{b} \,\}$):

$$\underbrace{\mathsf{a} \to \mathsf{a} \to \mathsf{a} \to \dots}_{\omega\text{-many steps}}\ \mathsf{b} \to \mathsf{b}$$

The rewrite sequence stays $\omega$ steps at $\mathsf{a}$ and in the limit step 'jumps' to $\mathsf{b}$.

The connectedness condition with respect to *Cauchy convergence* requires that for every limit ordinal $\gamma$, the terms $t_\alpha$ converge with limit $t_\gamma$ as $\alpha$ approaches $\gamma$ from below. The *strong convergence* requires additionally that the depth of the rewrite steps $t_\alpha \to t_{\alpha+1}$ tends to infinity as $\alpha$ approaches $\gamma$ from below. The standard notion of infinitary rewriting [14, 6] is based on strong convergence as it gives rise to a more elegant rewriting theory; for example, allowing to trace symbols and redexes over limit ordinals. This is the notion that we are concerned with in this paper.

The rules (1) give rise to infinitary rewrite sequences in a very natural way, without the need for ordinals, metric convergence, or depth requirements. The depth requirement in the definition of strong convergence arises naturally in the rules (1) by employing coinduction over the term structure. Indeed, it is not difficult to see that the coinductive rules (1) capture all infinitary strongly convergent reductions $s \twoheadrightarrow t$. This is a consequence of a result due to Klop and de Vrijer [13] which states that every strongly convergent rewrite sequence contains only a finite number of steps at any depth $d \in \mathbb{N}$. Thus, in particular, only a finite number of root steps $\to_\varepsilon$, and before, in-between and after these root steps, there are strongly convergent rewrite sequences on the arguments. As a consequence, every strongly convergent rewrite sequence is of the shape $(\,\overline{\twoheadrightarrow} \circ \to_\varepsilon)^* \circ \overline{\twoheadrightarrow}$. Since strongly convergent rewrite sequences are closed under transitivity, we allow the slightly more general $(\to_\varepsilon \cup \overline{\twoheadrightarrow})^*$ in (1).

While this argument shows that every strongly convergent reduction $s \twoheadrightarrow t$ can be derived using the rules (1), it does not guarantee that we can derive precisely the strongly convergent reductions. Actually, the rules do allow to derive more, as the following example shows.

▶ **Example 3.** Let $R$ consist of the rewrite rule $\mathsf{C}(\mathsf{a}) \to \mathsf{a}$. Using the rules (1), we can derive $\mathsf{C}^\omega \twoheadrightarrow \mathsf{a}$ as shown in Figure 3.

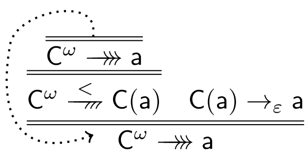

**Figure 3** A derivation of $\mathsf{C}^\omega \twoheadrightarrow \mathsf{a}$.

We emphasize that with respect to the standard notion of infinitary rewriting $\twoheadrightarrow$ in the literature we do not have $\mathsf{C}^\omega \twoheadrightarrow \mathsf{a}$ since $\mathsf{C}^\omega$ is a normal form (does not contain an occurrence of the left-hand side $\mathsf{C}(\mathsf{a})$ of the rule). Note that the rule $\mathsf{C}(\mathsf{x}) \to \mathsf{x}$ also gives rise to $\mathsf{C}^\omega \mathrel{\to\!\!\!\!\!\twoheadrightarrow} \mathsf{a}$ by the same derivation as in Figure 3. ◀

This example illustrates that, without further restrictions, the rules (1) give rise to a notion of infinitary rewriting that allows rewrite sequences to extend infinitely forwards, but also infinitely backwards. Here backwards does *not* refer to reversing the arrow $\leftarrow_\varepsilon$. While this is a non-standard notion of infinitary rewriting, it is nevertheless interesting, especially for a theory of infinitary equational reasoning, a field that has remained largely underdeveloped.

From the rules (1), a theory of *infinitary equational reasoning* arises naturally by replacing $\to_\varepsilon$ with $\leftarrow_\varepsilon \cup \to_\varepsilon$ in the first rule. This notion of infinitary equational reasoning has the property of strong convergence built in, and thereby allows to trace redex occurrences forwards as well as backwards over rewriting sequences of arbitrary length. As a consequence, this concept can profit from the well-developed theory of term rewriting and infinitary term rewriting.

The focus of this paper is the standard notion of infinitary rewriting. *How to obtain the strongly convergent rewrite sequences* $s \mathrel{\to\!\!\!\!\!\twoheadrightarrow} t$*?* For this purpose it suffices to impose a syntactic restriction on the shape of the proof trees obtained from the rules (1). The idea is that all rewrite sequences $\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}$ in $(\to_\varepsilon \cup \mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/})^*$, that are before a root step $\to_\varepsilon$, should be shorter than the rewrite sequence that we are defining. To this end, we change $(\to_\varepsilon \cup \mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/})^*$ to $(\to_\varepsilon \cup \overset{\le}{\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}})^* \circ \mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}$ where $\overset{\le}{\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}}$ is a marked equivalent of $\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}$, and we employ the marker to exclude infinite nesting of $\overset{\le}{\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}}$. Then we have an infinitary strongly convergent rewrite sequence from $s$ to $t$ if and only if $s \mathrel{\to\!\!\!\!\!\twoheadrightarrow} t$ can be derived by the rules

$$\frac{s\ (\to_\varepsilon \cup \overset{\le}{\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}})^* \circ \mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}\ t}{s \mathrel{\to\!\!\!\!\!\twoheadrightarrow} t} \qquad \frac{s_1 \mathrel{\to\!\!\!\!\!\twoheadrightarrow} t_1 \quad \dots \quad s_n \mathrel{\to\!\!\!\!\!\twoheadrightarrow} t_n}{f(s_1, s_2, \dots, s_n) \overset{(<)}{\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}} f(t_1, t_2, \dots, t_n)} \qquad \frac{}{s \overset{(<)}{\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}} s} \tag{2}$$

in a (not necessarily well-founded) proof tree without infinite nesting of $\overset{\le}{\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}}$. In other words, we only allow those proof trees in which all paths (ascending through the proof tree) contain only finitely many occurrences of $\overset{\le}{\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}}$.

We note that the second and third rule are abbreviations for two rules each: the symbol $\overset{(<)}{\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}}$ stands for $\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}$ and for $\overset{\le}{\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}}$. Intuitively, $\overset{\le}{\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}}$ can be thought of as infinitary rewrite sequence below the root that is 'smaller' than the sequence we are defining. Here 'smaller' refers to the nesting depth of $\overset{\le}{\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}}$, but can equivalently be thought of the length of the reduction (in some well-founded order).

▶ **Example 4.** Let us revisit Examples 1, 2 and 3. Example 1 contains no occurrences of $\overset{\le}{\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}}$. The proof tree in Example 2 has a single occurrence of $\overset{\le}{\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}}$, but this occurrence is not contained in the indicated loops, and thus not infinitely nested. Only Example 3 contains a symbol $\overset{\le}{\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}}$ on a loop, and hence a path with infinitely many occurrences of $\overset{\le}{\mathrel{\rightarrow\!\!\!\!\!\!\!\!/\!/\!/}}$, and thus the proof tree is excluded by the syntactic restriction. ◀

## 2 Conclusion

We have proposed a coinductive framework of infinitary rewriting. From the framework arise three natural variants of infinitary rewriting:

(a) infinitary equational reasoning,
(b) bi-infinite rewriting, and
(c) infinitary rewriting,

of which (c) is the standard definition of infinitary rewriting with respect to strong convergence. The variants (a) and (b) are novel and have to the best knowledge of the authors not yet been studied. For example, we are interested in a comparison of the Church-Rosser properties $=^{\infty} \subseteq \twoheadrightarrow \circ \twoheadleftarrow$ and $(\twoheadleftarrow \circ \twoheadrightarrow)^{*} \subseteq \twoheadrightarrow \circ \twoheadleftarrow$ . As a consequence of the coinduction over the term structure, all of the infinitary rewriting notions (a), (b) and (c) have the strong convergence built in, and thus can profit from the well-developed techniques (such as tracing) in infinitary rewriting.

We emphasize that our framework captures the full infinitary rewriting with rewrite sequences of arbitrary ordinal length. Previously, coinductive definitions of infinitary rewriting have been limited to rewrite sequences of length at most $\omega$.

Last but not least, our work contributes towards a formalization of infinitary rewriting in theorem provers.

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