On the Computational Content of the Brouwer Fixed Point Theorem

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Abstract. We study the computational content of the Brouwer Fixed Point Theorem in the Weihrauch lattice. One of our main results is that for any fixed dimension the Brouwer Fixed Point Theorem of that dimension is computably equivalent to connected choice of the Euclidean unit cube of the same dimension. Connected choice is the operation that finds a point in a non-empty connected closed set given by negative information. Another main result is that connected choice is complete for dimension greater or equal to three in the sense that it is computably equivalent to Weak Kőnig's Lemma. In contrast to this, the connected choice operations in dimensions zero, one and two form a strictly increasing sequence of Weihrauch degrees, where connected choice of dimension one is known to be equivalent to the Intermediate Value Theorem. Whether connected choice of dimension two is strictly below connected choice of dimension three or equivalent to it is unknown, but we conjecture that the reduction is strict. As a side result we also prove that finding a connectedness component in a closed subset of the Euclidean unit cube of any dimension greater than or equal to one is equivalent to Weak Kőnig's Lemma.

1 Introduction

In this paper we continue with the programme to classify the computational content of mathematical theorems in the Weihrauch lattice (see [6,4,3,14,13,5,8]). This lattice is induced by Weihrauch reducibility, which is a reducibility for partial multi-valued functions $f :\subseteq X \rightrightarrows Y$ on represented spaces X, Y. Intuitively, $f \leq_{\mathrm{W}} g$ reflects the fact that the function f can be realized with a single application of the function g as an oracle. Hence, if two functions are equivalent in the sense that they are mutually reducible to each other, then they are equivalent as computational resources, as far as computability is concerned.

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Many theorems in mathematics are actually of the logical form

$$(\forall x \in X)(\exists y \in Y)\ P(x, y)$$

and such theorems can straightforwardly be represented by a multi-valued function $f : X \rightrightarrows Y$ with $f(x) := \{y \in Y : P(x, y)\}$ (sometimes partial f are needed, where the domain captures additional requirements that this input x has to satisfy). In some sense the multi-valued function f directly reflects the computational task of the theorem to find some suitable y for any x. Hence, in a very natural way the classification of a theorem can be achieved via a classification of the corresponding multi-valued function that represents the theorem. In this paper we attempt to classify the Brouwer Fixed Point Theorem.

Theorem 1 (Brouwer Fixed Point Theorem 1911). *Every continuous function $f : [0, 1]^n \to [0, 1]^n$ has a fixed point $x \in [0, 1]^n$.*

The fact that Brouwer's Fixed Point Theorem cannot be proved constructively has been confirmed in many different ways, most relevant for us is the counterexample in Russian constructive analysis by Orevkov [12], which was transferred into computable analysis by Baigger [1].

Constructions similar to those used for the above counterexamples have been utilized in order to prove that the Brouwer Fixed Point Theorem is equivalent to Weak Kőnig's Lemma in reverse mathematics [17,16] and to analyze computability properties of fixable sets [11], but a careful analysis of these reductions reveals that none of them can be straightforwardly transferred into a *uniform* reduction in the sense that we are seeking here. The results cited above essentially characterize the complexity of fixed points themselves, whereas we want to characterize the complexity of finding the fixed point, given the function. This requires full uniformity.

In the Weihrauch lattice the Brouwer Fixed Point Theorem of dimension n is represented by the multi-valued function $\mathsf{BFT}_n : \mathcal{C}([0, 1]^n, [0, 1]^n) \rightrightarrows [0, 1]^n$ that maps any continuous function $f : [0, 1]^n \to [0, 1]^n$ to the set of its fixed points $\mathsf{BFT}_n(f) \subseteq [0, 1]^n$. The question now is where BFT_n is located in the Weihrauch lattice? It easily follows from a meta theorem presented in [3] that the Brouwer Fixed Point Theorem BFT_n is reducible to Weak Kőnig's Lemma WKL for any dimension n, i.e., $\mathsf{BFT}_n \leq_{\mathrm{W}} \mathsf{WKL}$. However, for which dimensions n do we also obtain the inverse reduction? Clearly not for $n = 0$, since BFT_0 is computable, and clearly not for $n = 1$, since BFT_1 is equivalent to the Intermediate Value Theorem IVT and hence not equivalent to WKL, as proved in [3].[1]

In order to approach this question for a general dimension n, we introduce a choice principle CC_n that we call *connected choice* and which is just the closed choice operation restricted to connected subsets. That is, in the sense discussed above, CC_n is the multi-valued function that represents the following mathematical statement: every non-empty connected closed set $A \subseteq [0, 1]^n$ has a point

[1] In constructive reverse mathematics the Intermediate Value Theorem is equivalent to Weak Kőnig's Lemma [9], since parallelization is freely available in this framework.

$x \in [0,1]^n$. Since closed sets are represented by negative information (i.e., by an enumeration of open balls that exhaust the complement), the computational task of CC_n consists in finding a point in a closed set $A \subseteq [0,1]^n$ that is promised to be non-empty and connected and that is given by negative information.

One of our main results, presented in Section 4, is that the Brouwer Fixed Point Theorem is equivalent to connected choice for each fixed dimension n, i.e., $\mathsf{BFT}_n \equiv_{\mathrm{W}} \mathsf{CC}_n$. This result allows us to study the Brouwer Fixed Point Theorem in terms of the function CC_n, which is easier to handle since it involves neither function spaces nor fixed points. This is also another instance of the observation that several important theorems are equivalent to certain choice principles (see [3]) and many important classes of computable functions can be calibrated in terms of choice (see [2]). For instance, closed choice on Cantor space $\mathsf{C}_{\{0,1\}^{\mathbb{N}}}$ and on the unit cube $\mathsf{C}_{[0,1]^n}$ are both easily seen to be equivalent to Weak Kőnig's Lemma WKL, i.e., $\mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{C}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{C}_{[0,1]^n}$ for any $n \geq 1$. Studying the Brouwer Fixed Point Theorem in the form of CC_n now amounts to comparing $\mathsf{C}_{[0,1]^n}$ with its restriction CC_n.

Our second main result, given in Section 5, is that from dimension three onwards connected choice is equivalent to Weak Kőnig's Lemma, i.e., $\mathsf{CC}_n \equiv_{\mathrm{W}} \mathsf{C}_{[0,1]}$ for $n \geq 3$. The backwards reduction is based on a geometrical construction that seems to require at least dimension three in a crucial sense. It is easy to see that connected choice operations for dimensions $0, 1$ and 2 form a strictly increasing sequence of Weihrauch degrees, i.e., $\mathsf{CC}_0 <_{\mathrm{W}} \mathsf{CC}_1 <_{\mathrm{W}} \mathsf{CC}_2 \leq_{\mathrm{W}} \mathsf{CC}_3 \equiv_{\mathrm{W}} \mathsf{C}_{[0,1]}$. The status of connected choice CC_2 of dimension two remains unresolved and we conjecture that it is strictly weaker than choice of dimension three, i.e., $\mathsf{CC}_2 <_{\mathrm{W}} \mathsf{CC}_3$.

In order to prove our results, we use a representation of closed sets by trees of so-called rational complexes, which we introduce in Section 3. It can be seen as a generalization of the well-known representation of co-c.e. closed subsets of Cantor space $\{0,1\}^{\mathbb{N}}$ by trees. As a side result we mention that finding a connectedness component in a closed set for any fixed dimension from one upwards is equivalent to Weak Kőnig's Lemma. This yields conclusions along the line of earlier studies of connected components in [10]. In the following Section 2 we start with a short summary of relevant definitions and results regarding the Weihrauch lattice.

This extended abstract does not contain any proofs.

2 The Weihrauch Lattice

In this section we briefly recall some basic results and definitions regarding the Weihrauch lattice. The original definition of Weihrauch reducibility is due to Weihrauch and has been studied for many years (see [18,19,20,7]). Only recently it has been noticed that a certain variant of this reducibility yields a lattice that is very suitable for the classification of mathematical theorems (see [6,4,3,14,2,13,5]). The basic reference for all notions from computable analysis is [21]. The Weihrauch lattice is a lattice of multi-valued functions on represented spaces. A *representation* δ of a set X is just a surjective partial map

$\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. In this situation we call (X, δ) a *represented space*. In general we use the symbol "$\subseteq$" in order to indicate that a function is potentially partial. Using represented spaces we can define the concept of a realizer. We denote the composition of two (multi-valued) functions f and g either by $f \circ g$ or by fg.

Definition 1 (Realizer). *Let $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ be a multi-valued function on represented spaces. A function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is called a* realizer *of f, in symbols $F \vdash f$, if $\delta_Y F(p) \in f\delta_X(p)$ for all $p \in \mathrm{dom}(f\delta_X)$.*

Realizers allow us to transfer the notions of computability and continuity and other notions available for Baire space to any represented space; a function between represented spaces will be called *computable*, if it has a computable realizer, etc. Now we can define Weihrauch reducibility.

Definition 2 (Weihrauch reducibility). *Let f, g be multi-valued functions on represented spaces. Then f is said to be* Weihrauch reducible *to g, in symbols $f \leq_{\mathrm{W}} g$, if there are computable functions $K, H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $K\langle \mathrm{id}, GH\rangle \vdash f$ for all $G \vdash g$. Moreover, f is said to be* strongly Weihrauch reducible *to g, in symbols $f \leq_{\mathrm{sW}} g$, if there are computable functions K, H such that $KGH \vdash f$ for all $G \vdash g$.*

Here $\langle , \rangle$ denotes some standard pairing on Baire space. We note that the relations $\leq_{\mathrm{W}}$, $\leq_{\mathrm{sW}}$ and $\vdash$ implicitly refer to the underlying representations, which we mention explicitly only when necessary. It is known that these relations only depend on the underlying equivalence classes of representations, but not on the specific representatives (see Lemma 2.11 in [4]). We use $\equiv_{\mathrm{W}}$ and $\equiv_{\mathrm{sW}}$ to denote the respective equivalences regarding $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$, and by $<_{\mathrm{W}}$ and $<_{\mathrm{sW}}$ we denote strict reducibility.

A particularly useful multi-valued function in the Weihrauch lattice is closed choice (see [6,4,3,2]) and it is known that many notions of computability can be calibrated using the right version of choice. We shall focus on closed choice for computable metric spaces, which are separable metric spaces such that the distance function is computable on the given dense subset. We assume that computable metric spaces are represented via their Cauchy representation (see [21] for details).

By $\mathcal{A}_-(X)$ we denote the set of closed subsets of a metric space X, where the index "$-$" indicates that we work with negative information. This information is given by a representation $\psi_- : \mathbb{N}^{\mathbb{N}} \to \mathcal{A}_-(X)$, defined by $\psi_-(p) := X \setminus \bigcup_{i=0}^{\infty} B_{p(i)}$, where B_n is some standard enumeration of the open balls of X with center in the dense subset and rational radius. The computable points in $\mathcal{A}_-(X)$ are called *co-c.e. closed sets*. We now define closed choice for the case of computable metric spaces.

Definition 3 (Closed Choice). *Let X be a computable metric space. Then the* closed choice *operation of this space is defined by $\mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$ with $\mathrm{dom}(\mathsf{C}_X) := \{A \in \mathcal{A}_-(X) : A \neq \emptyset\}$.*

Intuitively, C_X takes as input a non-empty closed set in negative representation (i.e., given by ψ_-) and it produces an arbitrary point of this set as output. Hence,

$A \mapsto A$ means that the multi-valued map C_X maps the input $A \in \mathcal{A}_-(X)$ to the set $A \subseteq X$ as a set of possible outputs.

3 Closed Sets and Trees of Rational Complexes

In this section we want to describe a representation of closed sets $A \subseteq [0,1]^n$ that is useful for the study of connectedness. It is well-known that closed subsets of Cantor space can be characterized exactly as sets of infinite paths of trees. We describe a similar representation of closed subsets of the Euclidean unit cube $[0,1]^n$. While in the case of Cantor space clopen balls are associated to each node of the tree, we now associate finite complexes of rational balls to each node. While infinite paths lead to points of the closed set in case of Cantor space, they now lead to connectedness components.

This representation of closed subsets $A \subseteq [0,1]^n$ of the unit cube will enable us to analyze the relation between connected choice and the Brouwer Fixed Point Theorem in the next section. In this section we shall use this representation in order to show that finding a connectedness component of a closed set A is computably exactly as difficult as Weak Kőnig's Lemma.

We first fix some topological terminology. We work with the *maximum norm* $|| \ ||$ on $\mathbb{R}^n$. By $B(x,r) := \{y \in \mathbb{R}^n : ||x-y|| < r\}$ we denote the *open ball* with center x and radius r and by $B[x,r] := \{y \in \mathbb{R}^n : ||x-y|| \leq r\}$ the corresponding *closed ball*. Since we are using the maximum norm, all these balls are open or closed cubes, respectively (if the radius is positive). By ∂A we denote the topological *boundary*, by $\overline{A}$ the *closure* and by A° the *interior* of a set A. If the underlying space X is clear from the context, then $A^c := X \setminus A$ denotes the *complement* of A. We are now prepared to define rational complexes.

Definition 4 (Rational complex). *We call a set $R := \{B[c_1,r_1],...,B[c_k,r_k]\}$ of finitely many closed balls $B[c_i,r_i]$ with rational center $c_i \in \mathbb{Q}^n$ and positive rational radius $r_i \in \mathbb{Q}$ an (n–dimensional) rational complex if $\bigcup R$ is connected and $B_1, B_2 \in R$ with $B_1 \neq B_2$ implies $B_1^\circ \cap B_2^\circ = \emptyset$. By $\mathrm{C}\mathbb{Q}^n$ we denote the set of n–dimensional rational complexes.*

Each rational complex R can be represented by a list of the corresponding rational numbers $c_1, r_1, ..., c_k, r_k$ and we implicitly assume in the following that this representation is used for the set of rational complexes $\mathrm{C}\mathbb{Q}^n$.

In order to organize the rational complexes that are used to approximate sets it is suitable to use trees. We recall that a *tree* is a set $T \subseteq \mathbb{N}^*$ which is closed under prefix, i.e., $u \sqsubseteq v$ and $v \in T$ implies $u \in T$. A function $b : \mathbb{N} \to \mathbb{N}$ is called a *bound* of a tree T if $w \in T$ implies $w(i) \leq b(i)$ for all $i = 0, ..., |w|-1$, where $|w|$ denotes the *length* of the word w. A tree is called *finitely branching*, if it has a bound. A tree of rational complexes is understood to be a finitely branching tree T (together with a bound) such that to each node of the tree a rational complex is associated with the property that these complexes are compactly included in each other if we proceed along paths of the tree and they are disjoint on any particular level of the tree. We write $A \Subset B$ for two sets $A, B \subseteq \mathbb{R}^n$ if the closure

$\overline{A}$ of A is included in the interior B° of B and we say that A is *compactly included* in B in this case.

Definition 5 (Tree of rational complexes). *We call (T, f) a* tree of rational complexes *if $T \subseteq \mathbb{N}^*$ is a finitely branching tree and $f : T \to \mathrm{C}\mathbb{Q}^n$ is a function such that for all $u, v \in T$ with $u \neq v$*

1. $u \sqsubseteq v \Longrightarrow \bigcup f(v) \Subset \bigcup f(u)$,
2. $|u| = |v| \Longrightarrow \bigcup f(u) \cap \bigcup f(v) = \emptyset$.

In the following we assume that finitely branching trees T are represented as a pair (χ_T, b), where $\chi_T : \mathbb{N}^* \to \{0, 1\}$ is the characteristic function of T and $b : \mathbb{N} \to \mathbb{N}$ is a bound of T. Correspondingly, trees (T, f) of rational complexes are then represented in a canonical way by (χ_T, b, f). We now define which set $A \subseteq [0, 1]^n$ is represented by such a tree (T, f) of rational complexes.

Definition 6 (Closed sets and trees of rational complexes). *We say that a closed set $A \subseteq \mathbb{R}^n$ is* represented *by a tree (T, f) of n–dimensional rational complexes if one obtains $A = \bigcap_{i=0}^{\infty} \bigcup_{w \in T \cap \mathbb{N}^i} \bigcup f(w)$.*

It is clear that in this way any tree (T, f) of rational complexes actually represents a compact set A. This is because $\bigcup f(w)$ is compact for each $w \in T$ and since T is finitely branching, the set $T \cap \mathbb{N}^i$ is finite for each i, hence $\bigcup_{w \in T \cap \mathbb{N}^i} \bigcup f(w)$ is compact and hence A is compact too. Vice versa, every compact set $A \subseteq \mathbb{R}^n$ can be represented by a tree of n–dimensional rational complexes. For $[0, 1]^n$ we mention the uniform result that even the map $(T, f) \mapsto A$ is computable and has a computable multi-valued right inverse. We assume that trees of rational complexes are represented as specified above and closed sets A are represented as points in $\mathcal{A}_-([0, 1]^n)$.

Proposition 1 (Closed sets and complexes). *Let $n \geq 1$. The map $(T, f) \mapsto A$ that maps every tree of n–dimensional rational complexes (T, f) to the closed set $A \subseteq [0, 1]^n$ represented by it, is computable and has a multi-valued computable right inverse.*

The representation of closed sets $A \subseteq [0, 1]^n$ by trees of rational complexes also has the advantage that connectedness components of A can easily be expressed in terms of the tree structure. This is made precise by the following lemma. By $[T] := \{p \in \mathbb{N}^{\mathbb{N}} : (\forall i)\ p|_i \in T\}$ we denote the set of *infinite paths* of T, which is also called the *body* of T. Here $p|_i = p(0)...p(i-1) \in \mathbb{N}^*$ denotes the *prefix* of p of length i for each $i \in \mathbb{N}$. We recall that a *connectedness component* of a set A is a connected subset of A that is not included in any larger connected subset of A. Any connectedness component of a subset A is closed in A. According to the following lemma there is bijection between $[T]$ and the set of connectedness components of a non-empty closed set $A \subseteq [0, 1]^n$.

Lemma 1 (Connectedness components). *Let (T, f) be a tree of rational complexes and let $A \subseteq [0, 1]^n$ be the closed set represented by (T, f). Then the sets $C_p := \bigcap_{i=0}^{\infty} \bigcup f(p|_i)$ for $p \in [T]$ are exactly all connectedness components of A (without repetitions).*

As another interesting result we can deduce from Proposition 1 a classification of the operation that determines a connectedness component. We first define this operation. For short we use the notation $\mathcal{A}_n := \{A \in \mathcal{A}_-([0,1]^n) : A \neq \emptyset\}$ for the space of non-empty closed subsets with representation ψ_-.

Definition 7 (Connectedness components). *By* $\mathsf{Con}_n : \mathcal{A}_n \rightrightarrows \mathcal{A}_n$ *we denote the map with* $\mathsf{Con}_n(A) := \{C : C$ *is a connectedness component of* $A\}$ *for every* $n \geq 1$.

The problem Con_n of finding a connectedness component of a closed set has the same strong Weihrauch degree as Weak Kőnig's Lemma for every dimension $n \geq 1$.

Theorem 2 (Connectedness components). $\mathsf{Con}_n \equiv_{\mathrm{sW}} \mathsf{WKL}$ *for* $n \geq 1$.

4 Brouwer's Fixed Point Theorem and Connected Choice

In this section we want to show that the Brouwer Fixed Point Theorem is computably equivalent to connected choice for any fixed dimension. We first define these two operations. By $\mathcal{C}(X,Y)$ we denote the *set of continuous functions* $f : X \to Y$ and for short we write $\mathcal{C}_n := \mathcal{C}([0,1]^n, [0,1]^n)$.

Definition 8 (Brouwer Fixed Point Theorem). *By* $\mathsf{BFT}_n : \mathcal{C}_n \rightrightarrows [0,1]^n$ *we denote the operation defined by* $\mathsf{BFT}_n(f) := \{x \in [0,1]^n : f(x) = x\}$ *for* $n \in \mathbb{N}$.

We note that BFT_n is well-defined, i.e., $\mathsf{BFT}_n(f)$ is non-empty for all f, since by the Brouwer Fixed Point Theorem every $f \in \mathcal{C}_n$ admits a fixed point x, i.e., with $f(x) = x$. We now define connected choice.

Definition 9 (Connected choice). *By* $\mathsf{CC}_n :\subseteq \mathcal{A}_n \rightrightarrows [0,1]^n$ *we denote the operation defined by* $\mathsf{CC}_n(A) := A$ *for all non-empty connected closed* $A \subseteq [0,1]^n$ *and* $n \in \mathbb{N}$. *We call* CC_n connected choice *(of dimension* n*).*

Hence, connected choice is just the restriction of closed choice $\mathsf{C}_{[0,1]^n}$ to connected sets. We also use the following notation for the set of fixed points of a functions $f \in \mathcal{C}_n$.

Definition 10 (Set of fixed points). *By* $\mathsf{Fix}_n : \mathcal{C}_n \to \mathcal{A}_n$ *we denote the function with* $\mathsf{Fix}_n(f) := \{x \in [0,1]^n : f(x) = x\}$.

It is easy to see that Fix_n is computable, since $\mathsf{Fix}_n(f) := (f - \mathrm{id})^{-1}\{0\}$ and it is well-known that closed sets in $\mathcal{A}_n$ can also be represented as zero sets of continuous functions (see [21]). We note that the Brouwer Fixed Point Theorem can be decomposed to $\mathsf{BFT}_n = \mathsf{CC}_n \circ \mathsf{Con}_n \circ \mathsf{Fix}_n$.

The main result of this section will be that the Brouwer Fixed Point Theorem and connected choice are (strongly) equivalent for any fixed dimension n (see Theorem 3 below). An important tool for both directions of the proof is the representation of closed sets by trees of rational complexes. The direction

$\mathsf{CC}_n \leq_{\mathrm{sW}} \mathsf{BFT}_n$ can be seen as a uniformization of an earlier construction of Baigger [1] that is in turn built on results of Orevkov [12]. For the other direction $\mathsf{BFT}_n \leq_{\mathrm{sW}} \mathsf{CC}_n$ of the reduction we uniformize ideas of Joseph S. Miller [11] and we use again the representation of closed sets by trees of rational complexes. We also exploit the fact that each rational complex can easily be converted into a simplicial complex. We recall that a *proper n–dimensional rational simplex* is the convex hull of $n+1$ geometrically independent rational points in $[0,1]^n$ and a *proper rational simplicial complex* is a set of finitely many proper simplexes such that the interiors of distinct simplexes are disjoint. By $\mathrm{S}\mathbb{Q}^n$ we denote the set of all such proper rational simplicial complexes and we assume that each such complex is represented by a specification of a list of $n+1$ geometrically independent rational points for each simplex in the complex. Hence, it is clear that there is a computable $f : \mathrm{C}\mathbb{Q}^n \to \mathrm{S}\mathbb{Q}^n$ with $\bigcup f(R) = \bigcup R$. That means that we can easily translate each tree of rational complexes into a corresponding tree of rational simplicial complexes (understood in the analogous way). We essentially use Miller's ideas to reduce the Brouwer Fixed Point Theorem uniformly to connected choice. The first observation is that the map $\mathsf{Con}_n \circ \mathsf{Fix}_n$ is computable (which might be surprising in light of Theorem 2).

Proposition 2. $\mathsf{Con}_n \circ \mathsf{Fix}_n : \mathcal{C}_n \rightrightarrows \mathcal{A}_n$ *is computable for all* $n \in \mathbb{N}$.

Since $\mathsf{BFT}_n \supseteq \mathsf{CC}_n \circ \mathsf{Con}_n \circ \mathsf{Fix}_n$ we can directly conclude $\mathsf{BFT}_n \leq_{\mathrm{sW}} \mathsf{CC}_n$ for all n. Together with $\mathsf{CC}_n \leq_{\mathrm{sW}} \mathsf{BFT}_n$ we obtain the following theorem.

Theorem 3 (Brouwer Fixed Point Theorem). $\mathsf{BFT}_n \equiv_{\mathrm{sW}} \mathsf{CC}_n$ *for all* n.

It is easy to see that in general the Brouwer Fixed Point Theorem and connected choice are not independent of the dimension. In case of $n = 0$ the space $[0,1]^n$ is the one-point space $\{0\}$ and hence $\mathsf{BFT}_0 \equiv_{\mathrm{sW}} \mathsf{CC}_0$ are both computable. In case of $n = 1$ connected choice was already studied in [3] and it was proved that it is equivalent to the Intermediate Value Theorem IVT (see Definition 6.1 and Theorem 6.2 in [3]).

Corollary 1 (Intermediate Value Theorem). $\mathsf{IVT} \equiv_{\mathrm{sW}} \mathsf{BFT}_1 \equiv_{\mathrm{sW}} \mathsf{CC}_1$.

It is also easy to see that the Brouwer Fixed Point Theorem BFT_2 in dimension two is more complicated than in dimension one. For instance, it is known that the Intermediate Value Theorem IVT always offers a computable function value for a computable input, whereas this is not the case for the Brouwer Fixed Point Theorem BFT_2 by Baigger's counterexample [1]. We continue to discuss this topic in Section 5.

Here we point out that Proposition 2 implies that the fixed point set $\mathsf{Fix}_n(f)$ of every computable function $f : [0,1]^n \to [0,1]^n$ has a co-c.e. closed connectedness component. Joseph S. Miller observed that also any co-c.e. closed superset of such a set is the fixed point set of some computable function and the following result is a uniform version of this observation. We denote by $(f, g) :\subseteq X \rightrightarrows Y \times Z$ the *juxtaposition* of two functions $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq X \rightrightarrows Z$, defined by $(f, g)(x) = (f(x), g(x))$.

Theorem 4 (Fixability). $(\mathsf{Fix}_n, \mathsf{Con}_n \circ \mathsf{Fix}_n)$ *is computable and has a multi-valued computable right inverse for all* $n \in \mathbb{N}$.

Roughly speaking a closed set $A \in \mathcal{A}_n$ together with one of its connectedness components is as good as a continuous function $f \in \mathcal{C}_n$ with A as set of fixed points. As a non-uniform corollary we obtain immediately Miller's original result.

Corollary 2 (Fixable sets, Miller 2002). *A set* $A \subseteq [0,1]^n$ *is the set of fixed points of a computable function* $f : [0,1]^n \to [0,1]^n$ *if and only if it is non-empty and co-c.e. closed and contains a co-c.e. closed connectedness component.*

5 Aspects of Dimension

In this section we want to discuss aspects of dimension of connected choice and the Brouwer Fixed Point Theorem. Our main result is that connected choice is computably universal or complete from dimension three onwards in the sense that it is strongly equivalent to Weak Kőnig's Lemma, which is one of the degrees of major importance. In order to prove this result, we use the following geometric construction.

Proposition 3 (Twisted cube). *The function* $T :\subseteq \mathcal{A}_{-}[0,1] \to \mathcal{A}_3$ *with* $T(A) = (A \times [0,1] \times \{0\}) \cup (A \times A \times [0,1]) \cup ([0,1] \times A \times \{1\})$ *is computable and maps non-empty closed sets* $A \subseteq [0,1]$ *to non-empty connected closed sets* $T(A) \subseteq [0,1]^3$.

Here tuples $(x_1, x_2, x_3) \in T(A)$ have the property that at least one of the first two components provide a solution $x_i \in A$, but the third component provides the additional information which one surely does. If x_3 is close to 1, then surely $x_2 \in A$ and if x_3 is close to 0, then surely $x_1 \in A$. If x_3 is neither close to 0 nor 1, then both $x_1, x_2 \in A$. Hence, there is a computable function H such that $\mathsf{C}_{[0,1]} = H \circ \mathsf{CC}_3 \circ T$, which proves $\mathsf{C}_{[0,1]} \leq_{\mathrm{sW}} \mathsf{CC}_3$. Together with Theorem 3 we obtain the following conclusion.

Theorem 5 (Completeness of three dimensions). *For* $n \geq 3$ *we obtain* $\mathsf{CC}_n \equiv_{\mathrm{sW}} \mathsf{BFT}_n \equiv_{\mathrm{sW}} \mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]}$.

We note that the reduction $\mathsf{CC}_n \leq_{\mathrm{sW}} \mathsf{C}_{[0,1]^n}$ holds for all $n \in \mathbb{N}$, since connected choice is just a restriction of closed choice and $\mathsf{C}_{[0,1]^n} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{WKL}$ is known for all $n \geq 1$ (see [2]).

In particular, we get the Baigger counterexample for dimension $n \geq 3$ as a consequence of Theorem 5. A superficial reading of the results of Orevkov [12] and Baigger [1] can lead to the wrong conclusion that they actually provide a reduction of Weak Kőnig's Lemma to the Brouwer Fixed Point Theorem BFT_n of any dimension $n \geq 2$. However, this is only correct in a non-uniform way and the corresponding uniform result is still open and does not follow from the known constructions. The Orevkov-Baigger result is built on the following fact.

Proposition 4 (Mixed cube). *The function* $M :\subseteq \mathcal{A}_-[0,1] \to \mathcal{A}_2$ *with* $M(A) = (A\times[0,1])\cup([0,1]\times A)$ *is computable and maps non-empty closed sets* $A \subseteq [0,1]$ *to non-empty connected closed sets* $M(A) \subseteq [0,1]^2$.

It follows straightforwardly from the definition that the pairs $(x,y) \in M(A)$ are such that one out of two components x, y is actually in A. In order to express the uniform content of this fact, we introduce the concept of a fraction.

Definition 11 (Fractions). *Let* $f :\subseteq X \rightrightarrows Y$ *be a multi-valued function and* $0 < n \le m \in \mathbb{N}$. *We define the* fraction $\frac{n}{m}f :\subseteq X \rightrightarrows Y^m$ *such that* $\frac{n}{m}f(x)$ *is the set of all* $(y_1, ..., y_m) \in \mathrm{range}(f)^m$ *with* $|\{i : y_i \in f(x)\}| \ge n$ *for all* $x \in \mathrm{dom}(\frac{n}{m}f) := \mathrm{dom}(f)$.

The idea of a fraction $\frac{n}{m}f$ is that it provides m potential answers for f, at least $n \le m$ of which have to be correct. The uniform content of the Orevkov-Baigger construction is then summarized in the following result.

Proposition 5 (Dimension two). $\frac{1}{2}\mathsf{C}_{[0,1]} \le_{\mathrm{sW}} \mathsf{CC}_2 \le_{\mathrm{sW}} \mathsf{C}_{[0,1]}$.

Proof. With Proposition 4 we obtain $\frac{1}{2}\mathsf{C}_{[0,1]} = \mathsf{CC}_2 \circ M$ and hence $\frac{1}{2}\mathsf{C}_{[0,1]} \le_{\mathrm{sW}} \mathsf{CC}_2$. The other reduction follows from $\mathsf{CC}_2 \le_{\mathrm{sW}} \mathsf{C}_{[0,1]^2} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]}$.

That is, given a closed set $A \subseteq [0,1]$ we can utilize connected choice CC_2 of dimension 2 in order to find a pair of points (x,y) one of which is in A. This result directly implies the counterexample of Baigger [1] because the fact that there are non-empty co-c.e. closed sets $A \subseteq [0,1]$ without computable point immediately implies that $\frac{1}{2}\mathsf{C}_{[0,1]}$ is not non-uniformly computable (i.e., there are computable inputs without computable outputs) and hence CC_2 is also not non-uniformly computable.

Corollary 3 (Orevkov 1963, Baigger 1985). *There exists a computable function* $f : [0,1]^2 \to [0,1]^2$ *that has no computable fixed point* $x \in [0,1]^2$. *There exists a non-empty connected co-c.e. closed subset* $A \subseteq [0,1]^2$ *without computable point.*

We mention that Proposition 5 does not directly imply $\mathsf{C}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{CC}_2$, since $\frac{1}{2}\mathsf{C}_{[0,1]} <_{\mathrm{W}} \mathsf{CC}_2$. In the following result we summarize the known relations for connected choice in dependency of the dimension.

Proposition 6. *We obtain* $\mathsf{CC}_0 <_{\mathrm{W}} \mathsf{CC}_1 <_{\mathrm{W}} \mathsf{CC}_2 \le_{\mathrm{W}} \mathsf{CC}_n \equiv_{\mathrm{W}} \mathsf{C}_{[0,1]}$ *for* $n \ge 3$.

Altogether, we are left with the major open problem whether $\mathsf{C}_{[0,1]} \le_{\mathrm{W}} \mathsf{CC}_2$ holds or not. We have a conjecture but currently no proof of it.

Conjecture 1 (Brouwer Fixed Point Theorem in dimension two). We conjecture that $\mathsf{CC}_2 <_{\mathrm{W}} \mathsf{C}_{[0,1]}$.

We mention that this conjecture is equivalent to the property that CC_2 is not parallelizable, i.e., to the property that $\widehat{\mathsf{CC}_2} \equiv_{\mathrm{W}} \mathsf{CC}_2$ does not hold. This is because $\widehat{\mathsf{CC}_2} \equiv_{\mathrm{W}} \mathsf{C}_{[0,1]}$ follows from $\mathsf{C}_{\{0,1\}} \le_{\mathrm{sW}} \mathsf{CC}_2$ and $\widehat{\mathsf{C}_{\{0,1\}}} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]}$ and the fact that parallelization is a closure operator, which are known results (see [3]).

6 Conclusions

We have systematically studied the uniform computational content of the Brouwer Fixed Point Theorem for any fixed dimension and we have obtained a systematic classification that leaves only the status of the two-dimensional case unresolved. Besides a solution of this open problem, one can proceed into several different directions.

For one, one could study generalizations of the Brouwer Fixed Point Theorem, such as the Schauder Fixed Point Theorem or the Kakutani Fixed Point Theorem. On the other hand, one could study results that are based on the Brouwer Fixed Point Theorem, such as equilibrium existence theorems in computable economics (see for instance [15]). Nash equilibria existence theorems have been studied in [13] and they can be seen to be strictly simpler than the general Brouwer Fixed Point Theorem (in fact they can be considered as linear version of it). In this context the question arises of how the Brouwer Fixed Point Theorem can be classified for other subclasses of continuous functions, such as Lipschitz continuous functions?

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