# Efficient Analysis of Probabilistic Programs with an Unbounded Counter

Tomáš Brázdil[1], Stefan Kiefer[2], and Antonín Kučera[1]

[1] Faculty of Informatics, Masaryk University, Czech Republic.
`{brazdil,kucera}@fi.muni.cz`
[2] Oxford University Computing Laboratory, United Kingdom.
`stefan.kiefer@comlab.ox.ac.uk`

**Abstract.** We show that a subclass of infinite-state probabilistic programs that can be modeled by probabilistic one-counter automata (pOC) admits an efficient quantitative analysis. In particular, we show that the expected termination time can be approximated up to an arbitrarily small relative error with polynomially many arithmetic operations, and the same holds for the probability of all runs that satisfy a given $\omega$-regular property. Further, our results establish a powerful link between pOC and martingale theory, which leads to fundamental observations about quantitative properties of runs in pOC. In particular, we provide a "divergence gap theorem", which bounds a positive non-termination probability in pOC away from zero.

## 1 Introduction

In this paper we aim at designing *efficient* algorithms for analyzing basic properties of probabilistic programs operating on unbounded data domains that can be abstracted into a non-negative integer counter. Consider, e.g., the recursive program of Fig. 1 which evaluates a given AND-OR tree, i.e., a tree whose root is an AND node, all descendants of AND nodes are either leaves or OR nodes, and all descendants of OR nodes are either leaves or AND nodes. Note that the program evaluates a subtree only when necessary. In general, the program may not terminate and we cannot say anything about its expected termination time. Now assume that we *do* have some knowledge about the actual input domain of the program, which might have been gathered empirically:

- an AND node has about $a$ descendants on average;
- an OR node has about $o$ descendants on average;
- the length of a branch is $b$ on average;
- the probability that a leaf evaluates to 1 is $z$.

Further, let us assume that the actual number of descendants and the actual length of a branch are *geometrically* distributed (which is a reasonably good approximation in many cases). Hence, the probability that an AND node has *exactly n* descendants is $(1-x_a)^{n-1}x_a$ with $x_a = \frac{1}{a}$. Under these assumption, the behaviour of the program is well-defined in the probabilistic sense, and we may ask the following questions:

1) Does the program terminate with probability one? If not, what is the termination probability?
2) If we restrict ourselves to terminating runs, what is the expected termination time? (Note that this conditional expected value is defined even if our program does not terminate with probability one.)

```
procedure AND(node)                    procedure OR(node)
if node is a leaf                      if node is a leaf
    then return node.value                 then return node.value
else                                   else
    for each successor s of node do        for each successor s of node do
        if OR(s) = 0 then return 0             if AND(s) = 1 then return 1
    end for                                end for
    return 1                               return 0
end if                                 end if
```

**Fig. 1.** A recursive program for evaluating AND-OR trees.

These questions are not trivial, and at first glance it is not clear how to approach them. Apart of the expected termination time, which is a fundamental characteristic of terminating runs, we are also interested in the properties on *non-terminating* runs, specified by linear-time logics or automata on infinite words. Here, we ask for the probability of all runs satisfying a given linear-time property. Using the results of this paper, answers to such questions can be computed *efficiently* for a large class of programs, including the one of Fig. 1. More precisely, the first question about the probability of termination can be answered using the existing results [14]; the original contributions of this paper are efficient algorithms for computing answers to the remaining questions.

The abstract class of probabilistic programs considered in this paper corresponds to *probabilistic one-counter automata (pOC)*. Informally, a pOC has finitely many control states $p, q, \dots$ that can store global data, and a single non-negative counter that can be incremented, decremented, and tested for zero. The dynamics of a given pOC is described by finite sets of *positive* and *zero* rules of the form $p \xrightarrow{x,c}_{>0} q$ and $p \xrightarrow{x,c}_{=0} q$, respectively, where $p, q$ are control states, $x$ is the *probability* of the rule, and $c \in \{-1, 0, 1\}$ is the *counter change* which must be non-negative in zero rules. A *configuration* $p(i)$ is given by the current control state $p$ and the current counter value $i$. If $i$ is positive/zero, then positive/zero rules can be applied to $p(i)$ in the natural way. Thus, every pOC determines an infinite-state Markov chain where states are the configurations and transitions are determined by the rules. As an example, consider a pOC model of the program of Fig. 1. We use the counter to abstract the stack of activation records. Since the procedures AND and OR alternate regularly in the stack, we keep just the current stack height in the counter, and maintain the "type" of the current procedure in the finite control (when we increase or decrease the counter, the "type" is swapped). The return values of the two procedures are also stored in the finite control. Thus, we obtain the pOC model of Fig. 2 with 6 control states and 12 positive rules (zero rules are irrelevant and hence not shown in Fig. 2). The initial configuration is $(and,init)(1)$, and the pOC terminates either in $(or,return,0)(0)$ or $(or,return,1)(0)$, which corresponds to evaluating the input tree to 0 and 1, respectively. We set $x_a := 1/a$, $x_o := 1/o$ and $y := 1/b$ in order to obtain the average numbers $a, o, b$ from the beginning.

As we already indicated, pOC can model recursive programs operating on unbounded data structures such as trees, queues, or lists, assuming that the structure can be faithfully abstracted into a counter. Let us note that modeling general recursive programs requires more powerful formalisms such as *probabilistic pushdown automata (pPDA)* or *recursive Markov chains (RMC)*. However, as it is mentioned below, pPDA and RMC do not admit *efficient* quantitative analysis for fundamental reasons. Hence, we must inevitably sacri-
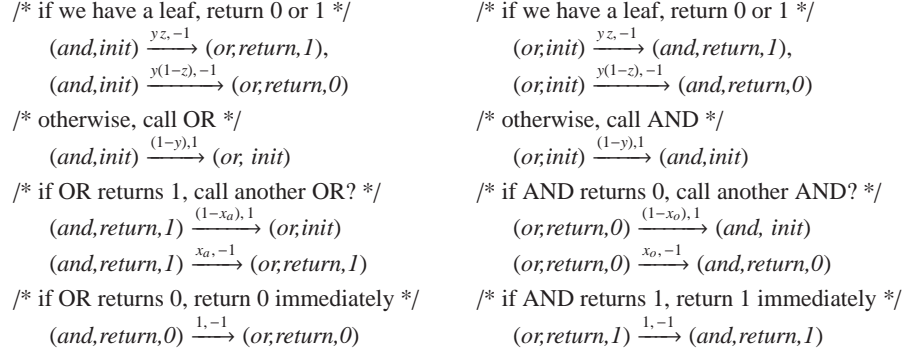
/* if we have a leaf, return 0 or 1 */
$\quad$ $(and, init) \xrightarrow{yz,-1} (or, return, 1)$,
$\quad$ $(and, init) \xrightarrow{y(1-z),-1} (or, return, 0)$
/* otherwise, call OR */
$\quad$ $(and, init) \xrightarrow{(1-y),1} (or,\ init)$
/* if OR returns 1, call another OR? */
$\quad$ $(and, return, 1) \xrightarrow{(1-x_a),1} (or, init)$
$\quad$ $(and, return, 1) \xrightarrow{x_a,-1} (or, return, 1)$
/* if OR returns 0, return 0 immediately */
$\quad$ $(and, return, 0) \xrightarrow{1,-1} (or, return, 0)$

/* if we have a leaf, return 0 or 1 */
$\quad$ $(or, init) \xrightarrow{yz,-1} (and, return, 1)$,
$\quad$ $(or, init) \xrightarrow{y(1-z),-1} (and, return, 0)$
/* otherwise, call AND */
$\quad$ $(or, init) \xrightarrow{(1-y),1} (and, init)$
/* if AND returns 0, call another AND? */
$\quad$ $(or, return, 0) \xrightarrow{(1-x_o),1} (and,\ init)$
$\quad$ $(or, return, 0) \xrightarrow{x_o,-1} (and, return, 0)$
/* if AND returns 1, return 1 immediately */
$\quad$ $(or, return, 1) \xrightarrow{1,-1} (and, return, 1)$

**Fig. 2.** A pOC model for the program of Fig. 1.

fice a part of pPDA modeling power to gain efficiency in algorithmic analysis, and pOC seem to be a convenient compromise for achieving this goal.

The relevance of pOC is not limited just to recursive programs. As observed in [14], pOC are equivalent, in a well-defined sense, to discrete-time *Quasi-Birth-Death processes (QBDs)*, a well-established stochastic model that has been deeply studied since late 60s. Thus, the applicability of pOC extends to queuing theory, performance evaluation, etc., where QBDs are considered as a fundamental formalism. Very recently, games over (probabilistic) one-counter automata, also called "energy games", were considered in several independent works [9, 10, 4, 3]. The study is motivated by optimizing the use of resources (such as energy) in modern computational devices.

**Previous work.** In [12, 17], it has been shown that the vector of termination probabilities in pPDA and RMC is the least solution of an effectively constructible system of quadratic equations. The termination probabilities may take irrational values, but can be effectively approximated up to an arbitrarily small absolute error $\varepsilon > 0$ in polynomial space by employing the decision procedure for the existential fragment of Tarski algebra (i.e., first order theory of the reals) [8]. Due to the results of [17], it is possible to approximate termination probabilities in pPDA and RMC "iteratively" by using the decomposed Newton's method. However, this approach may need exponentially many iterations of the method before it starts to produce one bit of precision per iteration [19]. Further, any non-trivial approximation of the non-termination probabilities is at least as hard as the SQUAREROOTSUM problem [17], whose exact complexity is a long-standing open question in exact numerical computations (the best known upper bound for SQUAREROOTSUM is PSPACE). Computing termination probabilities in pPDA and RMC up to a given *relative* error $\varepsilon > 0$, which is more relevant from the point of view of this paper, is *provably* infeasible because the termination probabilities can be doubly-exponentially small in the size of a given pPDA or RMC [17].

The expected termination time and the expected reward per transition in pPDA and RMC has been studied in [13]. In particular, it has been shown that the tuple of expected termination times is the least solution of an effectively constructible system of linear equations, where the (products of) termination probabilities are used as coefficients. Hence, the equational system can be represented only symbolically, and the corresponding approximation algorithm again employs the decision procedure for Tarski algebra. There also other results for pPDA and RMC, which concern model-checking problems for linear-

3

time [15, 16] and branching-time [7] logics, long-run average properties [5], discounted properties of runs [2], etc.

**Our contribution.** In this paper, we build on the previously established results for pPDA and RMC, and on the recent results of [14] where is shown that the decomposed Newton method of [20] can be used to compute termination probabilities in pOC up to a given *relative* error $\varepsilon > 0$ in time which is *polynomial* in the size of pOC and $\log(1/\varepsilon)$, assuming the unit-cost rational arithmetic RAM (i.e., Blum-Shub-Smale) model of computation. Adopting the same model, we show the following:

1. The expected termination time in a pOC $\mathscr{A}$ is computable up to an arbitrarily small relative error $\varepsilon > 0$ in time polynomial in $|\mathscr{A}|$ and $\log(1/\varepsilon)$. Actually, we can even compute the expected termination time up to an arbitrarily small *absolute* error, which is a better estimate because the expected termination time is always at least 1.
2. The probability of all runs in a pOC $\mathscr{A}$ satisfying an $\omega$-regular property encoded by a deterministic Rabin automaton $\mathcal{R}$ is computable up to an arbitrarily small relative error $\varepsilon > 0$ in time polynomial in $|\mathscr{A}|$, $|\mathcal{R}|$, and $\log(1/\varepsilon)$.

The crucial step towards obtaining these results is the construction of a suitable *martingale* for a given pOC, which allows to apply powerful results of martingale theory (such as the optional stopping theorem or Azuma's inequality, see, e.g., [21, 22]) to the quantitative analysis of pOC. In particular, we use this martingale to establish the crucial *divergence gap theorem* in Section 4, which bounds a positive divergence probability in pOC away from 0. The divergence gap theorem is indispensable in analysing properties of non-terminating runs, and together with the constructed martingale provide generic tools for designing efficient approximation algorithms for other interesting quantitative properties of pOC.

Although our algorithms have polynomial worst-case complexity, the obtained bounds look complicated and it is not immediately clear whether the algorithms are practically usable. Therefore, we created a simple experimental implementation which computes the expected termination time for pOC, and used this tool to analyse the pOC model of Fig. 2. The details are given in Section 5.

## 2   Definitions

We use $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Q}$, and $\mathbb{R}$ to denote the set of all integers, positive integers, non-negative integers, rational numbers, and real numbers, respectively. Let $\delta > 0$, $x \in \mathbb{Q}$, and $y \in \mathbb{R}$. We say that $x$ approximates $y$ up to a relative error $\delta$, if either $y \neq 0$ and $|x - y|/|y| \leq \delta$, or $x = y = 0$. Further, we say that $x$ approximates $y$ up to an absolute error $\delta$ if $|x - y| \leq \delta$. We use standard notation for intervals, e.g., $(0, 1]$ denotes $\{x \in \mathbb{R} \mid 0 < x \leq 1\}$.

Given a finite set $Q$, we regard elements of $\mathbb{R}^Q$ as vectors over $Q$. We use boldface symbols like $\boldsymbol{u}, \boldsymbol{v}$ for vectors. In particular we write $\boldsymbol{1}$ for the vector whose entries are all 1. Similarly, matrices are elements of $\mathbb{R}^{Q \times Q}$.

Let $\mathcal{V} = (V, \rightarrow)$, where $V$ is a non-empty set of vertices and $\rightarrow \subseteq V \times V$ a *total* relation (i.e., for every $v \in V$ there is some $u \in V$ such that $v \rightarrow u$). The reflexive and transitive closure of $\rightarrow$ is denoted by $\rightarrow^*$. A *finite path* in $\mathcal{V}$ of *length* $k \geq 0$ is a finite sequence of vertices $v_0, \ldots, v_k$, where $v_i \rightarrow v_{i+1}$ for all $0 \leq i < k$. The length of a finite path $w$ is denoted by $length(w)$. A *run* in $\mathcal{V}$ is an infinite sequence $w$ of vertices such that every finite prefix of $w$ is a finite path in $\mathcal{V}$. The individual vertices of $w$ are denoted by

4

$w(0), w(1), \ldots$ The sets of all finite paths and all runs in $\mathcal{V}$ are denoted by $FPath_{\mathcal{V}}$ and $Run_{\mathcal{V}}$, respectively. The sets of all finite paths and all runs in $\mathcal{V}$ that start with a given finite path $w$ are denoted by $FPath_{\mathcal{V}}(w)$ and $Run_{\mathcal{V}}(w)$, respectively. A *bottom strongly connected component (BSCC)* of $\mathcal{V}$ is a subset $B \subseteq V$ such that for all $v, u \in B$ we have that $v \rightarrow^* u$, and whenever $v \rightarrow u'$ for some $u' \in V$, then $u' \in B$.

We assume familiarity with basic notions of probability theory, e.g., *probability space*, *random variable*, or the *expected value*. As usual, a *probability distribution* over a finite or countably infinite set $X$ is a function $f : X \rightarrow [0, 1]$ such that $\sum_{x \in X} f(x) = 1$. We call $f$ *positive* if $f(x) > 0$ for every $x \in X$, and *rational* if $f(x) \in \mathbb{Q}$ for every $x \in X$.

**Definition 1.** *A* Markov chain *is a triple* $\mathcal{M} = (S, \rightarrow, Prob)$ *where $S$ is a finite or countably infinite set of* states*, $\rightarrow \subseteq S \times S$ is a total transition relation, and Prob is a function that assigns to each state $s \in S$ a positive probability distribution over the outgoing transitions of $s$. As usual, we write $s \xrightarrow{x} t$ when $s \rightarrow t$ and $x$ is the probability of $s \rightarrow t$.*

A Markov chain $\mathcal{M}$ can be also represented by its *transition matrix* $M \in [0, 1]^{S \times S}$, where $M_{s,t} = 0$ if $s \nrightarrow t$, and $M_{s,t} = x$ if $s \xrightarrow{x} t$.

To every $s \in S$ we associate the probability space $(Run_{\mathcal{M}}(s), \mathcal{F}, \mathcal{P})$ of runs starting at $s$, where $\mathcal{F}$ is the $\sigma$-field generated by all *basic cylinders*, $Run_{\mathcal{M}}(w)$, where $w$ is a finite path starting at $s$, and $\mathcal{P} : \mathcal{F} \rightarrow [0, 1]$ is the unique probability measure such that $\mathcal{P}(Run_{\mathcal{M}}(w)) = \prod_{i=1}^{length(w)} x_i$ where $w(i-1) \xrightarrow{x_i} w(i)$ for every $1 \leq i \leq length(w)$. If $length(w) = 0$, we put $\mathcal{P}(Run_{\mathcal{M}}(w)) = 1$.

**Definition 2.** *A* probabilistic one-counter automaton (pOC) *is a tuple,* $\mathscr{A} = (Q, \delta^{=0}, \delta^{>0}, P^{=0}, P^{>0})$, *where*

- *$Q$ is a finite set of* states*,*
- *$\delta^{>0} \subseteq Q \times \{-1, 0, 1\} \times Q$ and $\delta^{=0} \subseteq Q \times \{0, 1\} \times Q$ are the sets of* positive *and* zero rules *such that each $p \in Q$ has an outgoing positive rule and an outgoing zero rule;*
- *$P^{>0}$ and $P^{=0}$ are* probability assignments*: both assign to each $p \in Q$, a positive rational probability distribution over the outgoing rules in $\delta^{>0}$ and $\delta^{=0}$, respectively, of $p$.*

In the following, we often write $p \xrightarrow{x,c}_{=0} q$ to denote that $(p, c, q) \in \delta^{=0}$ and $P^{=0}(p, c, q) = x$, and similarly $p \xrightarrow{x,c}_{>0} q$ to denote that $(p, c, q) \in \delta^{>0}$ and $P^{>0}(p, c, q) = x$. The size of $\mathscr{A}$, denoted by $|\mathscr{A}|$, is the length of the string which represents $\mathscr{A}$, where the probabilities of rules are written in binary. A *configuration* of $\mathscr{A}$ is an element of $Q \times \mathbb{N}_0$, written as $p(i)$. To $\mathscr{A}$ we associate an infinite-state Markov chain $\mathcal{M}_{\mathscr{A}}$ whose states are the configurations of $\mathscr{A}$, and for all $p, q \in Q$, $i \in \mathbb{N}$, and $c \in \mathbb{N}_0$ we have that $p(0) \xrightarrow{x} q(c)$ iff $p \xrightarrow{x,c}_{=0} q$, and $p(i) \xrightarrow{x} q(c)$ iff $p \xrightarrow{x,c-i}_{>0} q$. For all $p, q \in Q$, let

- $Run_{\mathscr{A}}(p{\downarrow}q)$ be the set of all runs in $\mathcal{M}_{\mathscr{A}}$ initiated in $p(1)$ that visit $q(0)$ and the counter stays positive in all configurations preceding this visit;
- $Run_{\mathscr{A}}(p{\uparrow})$ be the set of all runs in $\mathcal{M}_{\mathscr{A}}$ initiated in $p(1)$ where the counter never reaches zero.

We omit the "$\mathscr{A}$" in $Run_{\mathscr{A}}(p{\downarrow}q)$ and $Run_{\mathscr{A}}(p{\uparrow})$ when it is clear from the context, and we use $[p{\downarrow}q]$ and $[p{\uparrow}]$ to denote the probability of $Run(p{\downarrow}q)$ and $Run(p{\uparrow})$, respectively. Observe that $[p{\uparrow}] = 1 - \sum_{q \in Q}[p{\downarrow}q]$ for every $p \in Q$.

At various places in this paper we rely on the following proposition proven in [14] (recall that we adopt the unit-cost rational arithmetic RAM model of computation):

**Proposition 3.** *Let* $\mathscr{A} = (Q, \delta^{=0}, \delta^{>0}, P^{=0}, P^{>0})$ *be a pOC, and* $p, q \in Q$.

- *The problem whether* $[p{\downarrow}q] > 0$ *is decidable in polynomial time.*
- *If* $[p{\downarrow}q] > 0$, *then* $[p{\downarrow}q] \geq x_{\min}^{|Q|^3}$, *where* $x_{\min}$ *is the least (positive) probability used in the rules of* $\mathscr{A}$.
- *The probability* $[p{\downarrow}q]$ *can be approximated up to an arbitrarily small relative error* $\varepsilon > 0$ *in a time polynomial in* $|\mathscr{A}|$ *and* $\log(1/\varepsilon)$.

Due to Proposition 3, the set $T^{>0}$ of all pairs $(p, q) \in Q \times Q$ satisfying $[p{\downarrow}q] > 0$ is computable in polynomial time.

## 3 Expected Termination Time

In this section we give an efficient algorithm which approximates the expected termination time in pOC up to an arbitrarily small relative (or even absolute) error $\varepsilon > 0$.

For the rest of this section, we fix a pOC $\mathscr{A} = (Q, \delta^{=0}, \delta^{>0}, P^{=0}, P^{>0})$. For all $p, q \in Q$, let $R_{p{\downarrow}q} : Run(p(1)) \to \mathbb{N}_0$ be a random variable defined as follows:

$$R_{p{\downarrow}q}(w) = \begin{cases} k & \text{if } w \in Run(p{\downarrow}q) \text{ and } k \text{ is the least index such that } w(k) = q(0); \\ 0 & \text{otherwise.} \end{cases}$$

If $(p, q) \in T^{>0}$, we use $E(p{\downarrow}q)$ to denote the conditional expectation $\mathbb{E}[R_{p{\downarrow}q} \mid Run(p{\downarrow}q)]$. Note that $E(p{\downarrow}q)$ can be finite even if $[p{\downarrow}q] < 1$.

The first problem we have to deal with is that the expectation $E(p{\downarrow}q)$ can be infinite, as illustrated by the following example.

*Example 4.* Consider a simple pOC with only one control state $p$ and two positive rules $(p, -1, p)$ and $(p, 1, p)$ that are both assigned the probability $1/2$. Then $[p{\downarrow}p] = 1$, and due to results of [13], $E(p{\downarrow}p)$ is the least solution (in $\mathbb{R}^+ \cup \{\infty\}$) of the equation $x = 1/2 + 1/2(1 + 2x)$, which is $\infty$.

We proceed as follows. First, we show that the problem whether $E(p{\downarrow}q) = \infty$ is decidable in polynomial time (Section 3.1). Then, we eliminate all infinite expectations, and show how to approximate the finite values of the remaining $E(p{\downarrow}q)$ up to a given absolute (and hence also relative) error $\varepsilon > 0$ efficiently (Section 3.2).

### 3.1 Finiteness of the expected termination time

Our aim is to prove the following:

**Theorem 5.** *Let* $(p, q) \in T^{>0}$. *The problem whether* $E(p{\downarrow}q)$ *is finite is decidable in polynomial time.*

Theorem 5 is proven by analysing the underlying finite-state Markov chain $X$ of the considered pOC $\mathscr{A}$. The transition matrix $A \in [0, 1]^{Q \times Q}$ of $X$ is given by

$$A_{p,q} = \sum_{(p,c,q) \in \delta^{>0}} P^{>0}(p, c, q).$$

We start by assuming that $X$ is strongly connected (i.e. that for all $p, q \in Q$ there is a path from $p$ to $q$ in $X$). Later we show how to generalize our results to an arbitrary $X$.

**Strongly connected $X$:** Let $\alpha \in (0, 1]^Q$ be the *invariant distribution* of $X$, i.e., the unique (row) vector satisfying $\alpha A = \alpha$ and $\alpha \mathbf{1} = 1$ (see, e.g., [18, Theorem 5.1.2]). Further, we define the (column) vector $s \in \mathbb{R}^Q$ of *expected counter changes* by

$$s_p = \sum_{(p,c,q) \in \delta^{>0}} P^{>0}(p, c, q) \cdot c$$

and the *trend* $t \in \mathbb{R}$ of $X$ by $t = \alpha s$. Note that $t$ is easily computable in polynomial time. Now consider some $E(p{\downarrow}q)$, where $(p, q) \in T^{>0}$. We show the following:

(A) If $t \neq 0$, then $E(p{\downarrow}q)$ is finite.
(B) If $t = 0$, then $E(p{\downarrow}q) = \infty$ iff the set $Pre^*(q(0)) \cap Post^*(p(1))$ is infinite, where
  - $Pre^*(q(0))$ consists of all $r(k)$ that can reach $q(0)$ along a run $w$ in $\mathcal{M}_{\mathscr{A}}$ such that the counter stays positive in all configurations preceding the visit to $q(0)$;
  - $Post^*(p(1))$ consists of all $r(k)$ that can be reached from $p(1)$ along a run $w$ in $\mathcal{M}_{\mathscr{A}}$ where the counter stays positive in all configurations preceding the visit to $r(k)$.

Note that the conditions of Claims (A) and (B) are easy to verify in polynomial time. (Due to [11], there are finite-state automata constructible in polynomial time recognizing the sets $Pre^*(q(0))$ and $Post^*(p(1))$. Hence, we can efficiently compute a finite-state automaton $\mathcal{F}$ recognizing the set $Pre^*(q(0)) \cap Post^*(p(1))$ and check whether the language accepted by $\mathcal{F}$ is infinite.) Thus, if $X$ is strongly connected and $(p, q) \in T^{>0}$, we can decide in polynomial time whether $E(p{\downarrow}q)$ is finite.

It remains to prove Claims (A) and (B). This is achieved by employing a generic observation which connects the study of pOC to martingale theory. Recall that a stochastic process $m^{(0)}, m^{(1)}, \ldots$ is a martingale if, for all $i \in \mathbb{N}$, $\mathbb{E}(|m^{(i)}|) < \infty$, and $\mathbb{E}(m^{(i+1)} \mid m^{(1)}, \ldots, m^{(i)}) = m^{(i)}$ almost surely. Let us fix some initial configuration $r(c) \in Q \times \mathbb{N}$. Our aim is to construct a suitable martingale over $Run(r(c))$. Let $p^{(i)}$ and $c^{(i)}$ be random variables which to every run $w \in Run(r(c))$ assign the control state and the counter value of the configuration $w(i)$, respectively. Note that if the vector $s$ of expected counter changes is constant, i.e., $s = \mathbf{1} \cdot t$ where $t$ is the trend of $X$, then we can define a martingale $m^{(0)}, m^{(1)}, \ldots$ simply by

$$m^{(i)} = \begin{cases} c^{(i)} - i \cdot t & \text{if } c^{(j)} \geq 1 \text{ for all } 0 \leq j < i; \\ m^{(i-1)} & \text{otherwise.} \end{cases}$$

Since $s$ is generally not constant, we might try to "compensate" the difference among the individual control states by a suitable vector $v \in \mathbb{R}^Q$. The next proposition shows that this is indeed possible.

**Proposition 6.** *There is a vector $v \in \mathbb{R}^Q$ such that the stochastic process $m^{(0)}, m^{(1)}, \ldots$ defined by*

$$m^{(i)} = \begin{cases} c^{(i)} + v_{p^{(i)}} - i \cdot t & \text{if } c^{(j)} \geq 1 \text{ for all } 0 \leq j < i; \\ m^{(i-1)} & \text{otherwise} \end{cases}$$

*is a martingale, where $t$ is the trend of $X$.*

*Moreover, the vector $\boldsymbol{v}$ satisfies $\boldsymbol{v}_{\max} - \boldsymbol{v}_{\min} \le 2|Q|/x_{\min}^{|Q|}$, where $x_{\min}$ is the smallest positive transition probability in $X$, and $\boldsymbol{v}_{\max}$ and $\boldsymbol{v}_{\min}$ are the maximal and the minimal components of $\boldsymbol{v}$, respectively.*

Due to Proposition 6, powerful results of martingale theory such as optional stopping theorem or Azuma's inequality (see, e.g., [21, 22]) become applicable to pOC. In this paper, we use the constructed martingale to complete the proof of Claims (A) and (B), and to establish the crucial *divergence gap theorem* in Section 4 (due to space constraints, we only include brief sketches of Propositions 7 and 9 which demonstrate the use of Azuma's inequality and optional stopping theorem). The range of possible applications of Proposition 6 is of course wider.

**A proof of Claim A.** For every $i \in \mathbb{N}$, let $Run(p{\downarrow}q, i)$ be the set of all $w \in Run(p{\downarrow}q)$ that visit $q(0)$ in *exactly $i$* transitions, and let $[p{\downarrow}q, i]$ be the probability of $Run(p{\downarrow}q, i)$. Claim (A) is proven by demonstrating that if $t \ne 0$, then the probabilities $[p{\downarrow}q, i]$ decay exponentially in $i$. Hence, $E(p{\downarrow}q) = \sum_{i=1}^{\infty} i \cdot [p{\downarrow}q, i]/[p{\downarrow}q]$ is finite.

**Proposition 7.** *There are $0 < a < 1$ and $h \in \mathbb{N}$ such that for all $i \ge h$ we have that $[p{\downarrow}q, i] \le a^i$.*

*Proof (Sketch).* Consider the martingale $m^{(0)}, m^{(1)}, \dots$ over $Run(p(1))$ as defined in Proposition 6. A relatively straightforward computation reveals that for sufficiently large $h \in \mathbb{N}$ and all $i \ge h$ we have the following: If $t < 0$, then $[p{\downarrow}q, i] \le \mathcal{P}\left(m^{(i)} - m^{(0)} \ge (i/2) \cdot (-t)\right)$, and if $t > 0$, then $[p{\downarrow}q, i] \le \mathcal{P}\left(m^{(0)} - m^{(i)} \ge (i/2) \cdot t\right)$. In each step, the martingale value changes by at most $\boldsymbol{v}_{\max} - \boldsymbol{v}_{\min} + t + 1$, where $\boldsymbol{v}$ is from Proposition 6. Hence Azuma's inequality (see [22]) asserts for $t \ne 0$ and $i \ge h$:

$$
\begin{aligned}
[p{\downarrow}q, i] &\le \exp\left(-\frac{(i/2)^2 t^2}{2i(\boldsymbol{v}_{\max} - \boldsymbol{v}_{\min} + t + 1)^2}\right) \qquad \text{(Azuma's inequality)} \\
&= a^i .
\end{aligned}
$$

Here $a = \exp\left(-t^2 \,/\, 8(\boldsymbol{v}_{\max} - \boldsymbol{v}_{\min} + t + 1)^2\right)$. $\qquad\square$

It follows directly from Proposition 7 that

$$
E(p{\downarrow}q) \;=\; \sum_{i=1}^{\infty} i \cdot \frac{[p{\downarrow}q, i]}{[p{\downarrow}q]} \;\le\; \frac{1}{[p{\downarrow}q]}\left(\sum_{i=1}^{h-1} i \cdot [p{\downarrow}q, i] + \sum_{i=h}^{\infty} i \cdot a^i\right) \;<\; \infty
$$

**A proof of Claim B.** We start with the "$\Rightarrow$" direction of Claim (B), which is easy to prove by contradiction. Intuitively, if $Pre^*(q(0)) \cap Post^*(p(1))$ is finite, then we can transform the states of $Pre^*(q(0)) \cap Post^*(p(1))$ into a finite-state Markov chain and show that $E(p{\downarrow}q)$ is finite.

**Proposition 8.** *If $Pre^*(q(0)) \cap Post^*(p(1))$ is finite, then $E(p{\downarrow}q)$ is also finite.*

The other direction of Claim (B) is more complicated. Let us first introduce some notation. For every $k \in \mathbb{N}_0$, let $Q(k)$ be the set of all configurations where the counter value equals $k$. Let $p, q \in Q$ and $\ell, k \in \mathbb{N}_0$, where $\ell > k$. An *honest path from $p(\ell)$ to $q(k)$* is a finite path $w$ from $p(\ell)$ to $q(k)$ such that the counter stays above $k$ in all configurations of $w$ except for the last one. We use $hpath(p(\ell), Q(k))$ to denote the set of all honest paths from $p(\ell)$ to some $q(k) \in Q(k)$. For a given $P \subseteq hpath(p(\ell), Q(k))$, the *expected lenght of an honest path in $P$* is defined as $\sum_{w \in P} \mathcal{P}(Run(w)) \cdot length(w)$. Using the above constructed martingale, we show the following:

**Proposition 9.** *If $Pre^*(q(0))$ is infinite, then almost all runs initiated in an arbitrary configuration reach $Q(0)$. Moreover, there is $k_1 \in \mathbb{N}$ such that, for all $\ell \geq k_1$, the expected length of an honest path from $r(\ell)$ to $Q(0)$ is infinite.*

*Proof (Sketch).* Assume that $Pre^*(q(0))$ is infinite. The fact that almost all runs initiated in an arbitrary configuration reach $Q(0)$ follows from results of [4].

Consider an initial configuration $r(\ell)$ with $\ell + v_r > v_{\max}$. We will show that the expected length of an honest path from $r(\ell)$ to $Q(0)$ is infinite; i.e., we can take $k_1 := \lceil v_{\max} - v_{\min} + 1 \rceil$. Consider the martingale $m^{(0)}, m^{(1)}, \ldots$ defined in Proposition 6 over $Run(r(\ell))$. Note that as $t = 0$, the term $i \cdot t$ vanishes from the definition of the martingale.

Now let us fix $k \in \mathbb{N}$ such that $\ell + v_r < v_{\max} + k$ and define a *stopping time* $\tau$ (see e.g. [22]) which returns the first point in time in which either $m^{(\tau)} \geq v_{\max} + k$, or $m^{(\tau)} \leq v_{\max}$. A routine application of optional stopping theorem gives us the following

$$\mathcal{P}(m^{(\tau)} \geq v_{\max} + k) \quad \geq \quad \frac{\ell + v_r - v_{\max}}{k + M}. \tag{1}$$

Denote by $T$ the number of steps to hit $Q(0)$. Note that $m^{(\tau)} \geq v_{\max} + k$ implies $c^{(\tau)} = m^{(\tau)} - v_{p^{(\tau)}} \geq v_{\max} + k - v_{p^{(\tau)}} \geq k$, and thus also $T \geq k$, as at least $k$ steps are required to decrease the counter value from $k$ to $0$. It follows that $\mathcal{P}(m^{(\tau)} \geq v_{\max} + k) \leq \mathcal{P}(T \geq k)$. By putting this inequality together with the inequality (1) we obtain

$$\mathbb{E}T \;=\; \sum_{k \in \mathbb{N}} \mathcal{P}(T \geq k) \;\geq\; \sum_{k=\ell+1}^{\infty} \mathcal{P}(T \geq k) \;\geq\; \sum_{k=\ell+1}^{\infty} \frac{\ell + v_r - v_{\max}}{k + M} \;=\; \infty.$$

$\square$

Further, we need the following observation about the structure of $\mathcal{M}_{\mathscr{A}}$, which holds also for non-probabilistic one-counter automata:

**Proposition 10.** *There is $k_2 \in \mathbb{N}$ such that for every configuration $r(\ell) \in Pre^*(q(0))$, where $\ell \geq k_2$, we have that if $r(\ell) \to r'(\ell')$, then $r'(\ell') \in Pre^*(q(0))$.*

To show that $E(p{\downarrow}q) = \infty$, it suffices to identify a subset $W \subseteq R(p{\downarrow}q)$ such that $\mathcal{P}(W) > 0$ and $\mathbb{E}[R_{p{\downarrow}q} \mid W] = \infty$. Now observe that if $Pre^*(q(0)) \cap Post^*(p(1))$ is infinite, there is a configuration $r(\ell) \in Pre^*(q(0))$ reachable from $p(1)$ along a finite path $u$ such that $\ell \geq k_1 + k_2$, where $k_1$ and $k_2$ are the constants of Propositions 9 and 10.

Due to Proposition 9, the expected length of an honest path from $r(\ell - k_2)$ to $Q(0)$ is infinite. Howeover, then also the expected length of an honest path from $r(\ell)$ to $Q(k_2)$ is infinite. This means that there is a state $s \in Q$ such that the expected length of an honest path from $r(\ell)$ to $s(k_2)$ in infinite. Further, it follows directly from Proposition 10 that $s(k_2) \in Pre^*(q(0))$ because there is an honest path from $r(\ell)$ to $s(k_2)$.

Now consider the set $W$ of all runs $w$ initiated in $p(1)$ that start with the finite path $u$, then follow an honest path from $r(\ell)$ to $s(k_2)$, and then follow an honest path from $s(k_2)$ to $q(0)$. Obviously, $\mathcal{P}(W) > 0$, and $\mathbb{E}[R_{p{\downarrow}q} \mid W] = \infty$ because the expected length of the middle subpath is infinite. Hence, $E(p{\downarrow}q) = \infty$ as needed.

**Non-strongly connected $\mathcal{X}$:** The general case still requires some extra care. First, realize that each BSCC $\mathscr{B}$ of $\mathcal{X}$ can be seen as a strongly connected finite-state Markov chain, and hence all notions and arguments of the previous subsection can be applied to $\mathscr{B}$ immediately (in particular, we can compute the trend of $\mathscr{B}$ in polynomial time). We prove the following claims:

(C) If $q$ does not belong to a BSCC of $X$, then $E(p{\downarrow}q)$ is finite.

(D) If $q$ belongs to a BSCC $\mathscr{B}$ of $X$ such that the trend of $\mathscr{B}$ is different from 0, then $E(p{\downarrow}q)$ is finite.

(E) If $q$ belongs to a BSCC $\mathscr{B}$ of $X$ such that the trend of $\mathscr{B}$ is 0, then $E(p{\downarrow}q) = \infty$ iff the set $Pre^*(q(0)) \cap Post^*(p(1))$ is infinite.

Note that the conditions of Claims (C)-(E) are verifiable in polynomial time.

Intuitively, Claim (C) is proven by observing that if $q$ does not belong to a BSCC of $X$, then for all $s(\ell) \in Post^*(p(1))$, where $\ell \geq |Q|$, we have that $s(\ell)$ can reach a configuration outside $Pre^*(q(0))$ in at most $|Q|$ transitions. It follows that the probability of performing an honest path from $p(1)$ to $q(0)$ of length $i$ decays exponentially in $i$, and hence $\mathbb{E}(p{\downarrow}q)$ is finite.

Claim (D) is obtained by combining the arguments of Claim (A) together with the fact that the conditional expected number of transitions needed to reach $\mathscr{B}$ from $p(0)$, under the condition that $\mathscr{B}$ is indeed reached from $p(0)$, is finite (this is a standard result for finite-state Markov chains).

Finally, Claim (E) follows by re-using the arguments of Claim (B).

## 3.2 Efficient approximation of finite expected termination time

Let us denote by $T^{\geq 0}_{<\infty}$ the set of all pairs $(p, q) \in T^{>0}$ satisfying $E(p{\downarrow}q) < \infty$. Our aim is to prove the following:

**Theorem 11.** *For all $(p, q) \in T^{\geq 0}_{<\infty}$, the value of $E(p{\downarrow}q)$ can be approximated up to an arbitrarily small absolute error $\varepsilon > 0$ in time polynomial in $|\mathscr{A}|$ and $\log(1/\varepsilon)$.*

Note that if $y$ approximates $E(p{\downarrow}q)$ up to an absolute error $1 > \varepsilon > 0$, then $y$ approximates $E(p{\downarrow}q)$ also up to the relative error $\varepsilon$ because $E(p{\downarrow}q) \geq 1$.

The proof of Theorem 11 is based on the fact that the vector of all $E(p{\downarrow}q)$, where $(p, q) \in T^{\geq 0}_{<\infty}$, is the unique solution of a system of linear equations whose coefficients can be efficiently approximated (see below). Hence, it suffices to approximate the coefficients, solve the approximated equations, and then bound the error of the approximation using standard arguments from numerical analysis.

Let us start by setting up the system of linear equations for $E(p{\downarrow}q)$. For all $p, q \in T^{>0}$, we fix a fresh variable $V(p{\downarrow}q)$, and construct the following system of linear equations, $\mathcal{L}$, where the termination probabilities are treated as constants:

$$
V(p{\downarrow}q) = \sum_{(p,-1,q)\in\delta^{>0}} \frac{P^{>0}(p,-1,q)}{[p{\downarrow}q]} + \sum_{(p,0,t)\in\delta^{>0}} \frac{P^{>0}(p,0,t) \cdot [t{\downarrow}q]}{[p{\downarrow}q]} \cdot \left(1 + V(t{\downarrow}q)\right)
$$

$$
+ \sum_{(p,1,t)\in\delta^{>0}} \sum_{r\in Q} \frac{P^{>0}(p,1,t) \cdot [t{\downarrow}r] \cdot [r{\downarrow}q]}{[p{\downarrow}q]} \cdot \left(1 + V(t{\downarrow}r) + V(r{\downarrow}q)\right)
$$

It has been shown in [13] that the tuple of all $E(p{\downarrow}q)$, where $(p, q) \in T^{>0}$, is the least solution of $\mathcal{L}$ in $\mathbb{R}^+ \cup \{\infty\}$ with respect to component-wise ordering (where $\infty$ is treated according to the standard conventions). Due to Theorem 5, we can further simplify the system $\mathcal{L}$ by erasing the defining equations for all $V(p{\downarrow}q)$ such that $E(p{\downarrow}q) = \infty$ (note that if $E(p{\downarrow}q) < \infty$, then the defining equation for $V(p{\downarrow}q)$ in $\mathcal{L}$ cannot contain any variable $V(r{\downarrow}t)$ such that $E(r{\downarrow}t) = \infty$).

Thus, we obtain the system $\mathcal{L}'$. It is straightforward to show that the vector of all finite $E(p{\downarrow}q)$ is the *unique* solution of the system $\mathcal{L}'$ (see, e.g., Lemma 6.2.3 and Lemma 6.2.4 in [1]). If we rewrite $\mathcal{L}'$ into a standard matrix form, we obtain a system $V = H \cdot V + b$, where $H$ is a nonsingular nonnegative matrix, $V$ is the vector of variables in $\mathcal{L}'$, and $b$ is a vector. Further, we have that $b = 1$, i.e., the constant coefficients are all 1. This follows from the following equality (see [12, 17]):

$$[p{\downarrow}q] = \sum_{(p,-1,q)\in\delta^{>0}} P^{>0}(p,-1,q) \; + \sum_{(p,0,t)\in\delta^{>0}} P^{>0}(p,0,t) \cdot [t{\downarrow}q]$$
$$+ \sum_{(p,1,t)\in\delta^{>0}} \sum_{r\in Q} P^{>0}(p,1,t) \cdot [t{\downarrow}r] \cdot [r{\downarrow}q] \tag{2}$$

Hence, $\mathcal{L}'$ takes the form $V = H \cdot V + 1$. Unfortunately, the entries of $H$ can take irrational values and cannot be computed precisely in general. However, they can be approximated up to an arbitrarily small relative error using Proposition 3. Denote by $G$ an approximated version of $H$. We aim at bounding the error of the solution of the "perturbed" system $V = G \cdot V + 1$ in terms of the error of $G$. To measure these errors, we use the $l_\infty$ norm of vectors and matrices, defined as follows: For a vector $V$ we have that $\|V\| = \max_i |V_i|$, and for a matrix $M$ we have $\|M\| = \max_i \sum_j |M_{ij}|$. Hence, $\|M\| = \|M \cdot 1\|$ if $M$ is nonnegative. We show the following:

**Proposition 12.** *Let $b \geq \max\left\{E(p{\downarrow}q) \mid (p,q) \in T_{<\infty}^{>0}\right\}$. Then for each $\varepsilon$, where $0 < \varepsilon < 1$, let $\delta = \varepsilon/(12 \cdot b^2)$. If $\|G - H\| \leq \delta$, then the perturbed system $V = G \cdot V + 1$ has a unique solution $F$, and in addition, we have that*

$$|E(p{\downarrow}q) - F_{pq}| \quad \leq \quad \varepsilon \qquad \text{for all } (p,q) \in T_{<\infty}^{>0}.$$

*Here $F_{pq}$ is the component of $F$ corresponding to the variable $V(p{\downarrow}q)$.*

The proof of Proposition 12 is based on estimating the size of the condition number $\kappa = \|1 - H\| \cdot \|(1 - H)^{-1}\|$ and applying standard results of numerical analysis. The $b$ in Proposition 12 can be estimated as follows:

**Proposition 13.** *Let $x_{\min}$ denote the smallest nonzero probability in A. Then we have:*

$$E(p{\downarrow}q) \quad \leq \quad 85000 \cdot |Q|^6 / \left( x_{\min}^{6|Q|^3} \cdot t_{\min}^4 \right) \qquad \text{for all } (p,q) \in T_{<\infty}^{>0},$$

*where $t_{\min} = \{|t| \neq 0 \mid t \text{ is the trend in a BSCC of } X\}$.*

Although $b$ appears large, it is really the value of $\log(1/b)$ which matters, and it is still reasonable. Theorem 11 now follows by combining Propositions 13, 12 and 3, because the approximated matrix $G$ can be computed using a number of arithmetical operations which is polynomial in $|\mathscr{A}|$ and $\log(1/\varepsilon)$.

## 4 Quantitative Model-Checking of $\omega$-regular Properties

In this section, we show that for every $\omega$-regular property encoded by a deterministic Rabin automaton, the probability of all runs in a given pOC that satisfy the property can be approximated up to an arbitrarily small relative error $\varepsilon > 0$ in polynomial time. This

is achieved by designing and analyzing a new quantitative model-checking algorithm for pOC and $\omega$-regular properties, which is *not* based on techniques developed for pPDA and RMC in [12, 15, 16].

Recall that a deterministic Rabin automaton (DRA) over a finite alphabet $\Sigma$ is a deterministic finite-state automaton $\mathcal{R}$ with total transition function and *Rabin acceptance condition* $(E_1, F_1), \ldots, (E_k, F_k)$, where $k \in \mathbb{N}$, and all $E_i$, $F_i$ are subsets of control states of $\mathcal{R}$. For a given infinite word $w$ over $\Sigma$, let $\inf(w)$ be the set of all control states visited infinitely often along the unique run of $\mathcal{R}$ on $w$. The word $w$ is accepted by $\mathcal{R}$ if there is $i \leq k$ such that $\inf(w) \cap E_i = \emptyset$ and $\inf(w) \cap F_i \neq \emptyset$.

Let $\Sigma$ be a finite alphabet, $\mathcal{R}$ a DRA over $\Sigma$, and $\mathscr{A} = (Q, \delta^{=0}, \delta^{>0}, P^{=0}, P^{>0})$ a pOC. A *valuation* is a function $\nu$ which to every configuration $p(i)$ of $\mathscr{A}$ assigns a unique letter of $\Sigma$. For simplicity, we assume that $\nu(p(i))$ depends only on the control state $p$ and the information whether $i \geq 1$. Intuitively, the letters of $\Sigma$ correspond to collections of predicates that are valid in a given configuration of $\mathscr{A}$. Thus, every run $w \in Run_{\mathscr{A}}(p(i))$ determines a unique infinite word $\nu(w)$ over $\Sigma$ which is either accepted by $\mathcal{R}$ or not. The main result of this section is the following theorem:

**Theorem 14.** *For every $p \in Q$, the probability of all $w \in Run_{\mathscr{A}}(p(0))$ such that $\nu(w)$ is accepted by $\mathcal{R}$ can be approximated up to an arbitrarily small relative error $\varepsilon > 0$ in time polynomial in $|\mathscr{A}|$, $|\mathcal{R}|$, and $\log(1/\varepsilon)$.*

Our proof of Theorem 14 consists of three steps:

1. We show that the problem of our interest is equivalent to the problem of computing the probability of all accepting runs in a given pOC $\mathscr{A}$ with Rabin acceptance condition.
2. We introduce a finite-state Markov chain $\mathcal{G}$ (with possibly irrational transition probabilities) such that the probability of all accepting runs in $\mathcal{M}_{\mathscr{A}}$ is equal to the probability of reaching a "good" BSCC in $\mathcal{G}$.
3. We show how to compute the probability of reaching a "good" BSCC in $\mathcal{G}$ with relative error at most $\varepsilon$ in time polynomial in $|\mathscr{A}|$ and $\log(1/\varepsilon)$.

Let us note that Steps 1 and 2 are relatively simple, but Step 3 requires several insights. In particular, we cannot solve Step 3 without bounding a positive non-termination probability in pOC (i.e., a positive probability of the form $[p\uparrow]$) away from zero. This is achieved in our "divergence gap theorem" (i.e., Theorem 20), which is based on applying Azuma's inequality to the martingale constructed in Section 3. Now we elaborate the three steps in more detail.

**Step 1.** For the rest of this section, we fix a pOC $\mathscr{A} = (Q, \delta^{=0}, \delta^{>0}, P^{=0}, P^{>0})$, and a *Rabin acceptance condition* $(\mathcal{E}_1, \mathcal{F}_1), \ldots, (\mathcal{E}_k, \mathcal{F}_k)$, where $k \in \mathbb{N}$ and $\mathcal{E}_i, \mathcal{F}_i \subseteq Q$ for all $1 \leq i \leq k$. For every run $w \in Run_{\mathscr{A}}$, let $\inf(w)$ be the set of all $p \in Q$ visited infinitely often along $w$. We use $Run_{\mathscr{A}}(p(0), acc)$ to denote the set of all *accepting runs* $w \in Run_{\mathscr{A}}(p(0))$ such that $\inf(w) \cap \mathcal{E}_i = \emptyset$ and $\inf(w) \cap \mathcal{F}_i \neq \emptyset$ for some $i \leq k$. Sometimes we also write $Run_{\mathscr{A}}(p(0), rej)$ to denote the set $Run_{\mathscr{A}}(p(0)) \setminus Run_{\mathscr{A}}(p(0), acc)$ of *rejecting* runs.

Our next proposition says that the problem of computing/approximating the probability of all runs $w$ in a given pOC that are accepted by a given DRA is efficiently reducible to the problem of computing/approximating the probability of all accepting runs in a given pOC with Rabin acceptance condition. The proof is very simple (we just "synchronize" a given pOC with a given DRA, and setup the Rabin acceptance condition accordingly).

**Proposition 15.** *Let $\Sigma$ be a finite alphabet, $\mathcal{A}$ a pOC, $\nu$ a valuation, $\mathcal{R}$ a DRA over $\Sigma$, and $p(0)$ a configuration of $\mathcal{A}$. Then there is a pOC $\mathcal{A}'$ with Rabin acceptance condition and a configuration $p'(0)$ of $\mathcal{A}'$ constructible in polynomial time such that the probability of all $w \in Run_{\mathcal{A}}(p(0))$ where $\nu(w)$ is accepted by $\mathcal{R}$ is equal to the probability of all accepting $w \in Run_{\mathcal{A}'}(p'(0))$.*

**Step 2.** Let $\mathcal{G}$ be a finite-state Markov chain, where $Q \times \{0, 1\} \cup \{acc, rej\}$ is the set of states (the elements of $Q \times \{0, 1\}$ are written as $p(i)$, where $i \in \{0, 1\}$), and the transitions of $\mathcal{G}$ are determined as follows:

- $p(0) \xrightarrow{x} q(j)$ is a transition of $\mathcal{G}$ iff $p(0) \xrightarrow{x} q(j)$ is a transition of $\mathcal{M}_{\mathcal{A}}$;
- $p(1) \xrightarrow{x} q(0)$ iff $x = [p{\downarrow}q] > 0$;
- $p(1) \xrightarrow{x} acc$ iff $x = \mathcal{P}(Run_{\mathcal{A}}(p(1), acc) \cap Run_{\mathcal{A}}(p{\uparrow})) > 0$;
- $p(1) \xrightarrow{x} rej$ iff $x = \mathcal{P}(Run_{\mathcal{A}}(p(1), rej) \cap Run_{\mathcal{A}}(p{\uparrow})) > 0$;
- $acc \xrightarrow{1} acc$, $rej \xrightarrow{1} rej$;
- there are no other transitions.

A BSCC $B$ of $\mathcal{G}$ is *good* if either $B = \{acc\}$, or there is some $i \leq k$ such that $\mathcal{E}_i \cap Q(B) = \emptyset$ and $\mathcal{F}_i \cap Q(B) \neq \emptyset$, where $Q(B) = \{p \in Q \mid p(j) \in B \text{ for some } j \in \{0, 1\}\}$. For every $p \in Q$, let $Run_{\mathcal{G}}(p(0), good)$ be the set of all $w \in Run_{\mathcal{G}}(p(0))$ that visit a good BSCC of $\mathcal{G}$. The next proposition is obtained by a simple case analysis of accepting runs in $\mathcal{M}_{\mathcal{A}}$.

**Proposition 16.** *For every $p \in Q$ we have $\mathcal{P}(Run_{\mathcal{A}}(p(0), acc)) = \mathcal{P}(Run_{\mathcal{G}}(p(0), good))$.*

**Step 3.** Due to Proposition 16, the problem of our interest reduces to the problem of approximating the probability of visiting a good BSCC in the finite-state Markov chain $\mathcal{G}$. Since the termination probabilities in $\mathcal{A}$ can be approximated efficiently (see Proposition 3), the main problem with $\mathcal{G}$ is approximating the probabilities $x$ and $y$ in transitions of the form $p(1) \xrightarrow{x} acc$ and $p(1) \xrightarrow{y} rej$. Recall that $x$ and $y$ are the probabilities of all $w \in Run_{\mathcal{A}}(p{\uparrow})$ that are accepting and rejecting, respectively. A crucial observation is that almost all $w \in Run_{\mathcal{A}}(p{\uparrow})$ still behave accordingly with the underlying finite-state Markov chain $X$ of $\mathcal{A}$ (see Section 3). More precisely, we have the following:

**Proposition 17.** *Let $p \in Q$. For almost all $w \in Run_{\mathcal{A}}(p{\uparrow})$ we have that $w$ visits a BSCC $B$ of $X$ after finitely many transitions, and then it visits all states of $B$ infinitely often.*

A BSCC $B$ of $X$ is *consistent* with the considered Rabin acceptance condition if there is $i \leq k$ such that $B \cap \mathcal{E}_i = \emptyset$ and $B \cap \mathcal{F}_i \neq \emptyset$. If $B$ is not consistent, it is *inconsistent*. An immediate corollary to Proposition 17 is the following:

**Corollary 18.** *Let $Run_{\mathcal{A}}(p(1), cons)$ and $Run_{\mathcal{A}}(p(1), inco)$ be the sets of all $w \in Run_{\mathcal{A}}(p(1))$ such that $w$ visit a control state of some consistent and inconsistent BSCC of $X$, respectively. Then*

- $\mathcal{P}(Run_{\mathcal{A}}(p(1), acc) \cap Run_{\mathcal{A}}(p{\uparrow})) = \mathcal{P}(Run_{\mathcal{A}}(p(1), cons) \cap Run_{\mathcal{A}}(p{\uparrow}))$
- $\mathcal{P}(Run_{\mathcal{A}}(p(1), rej) \cap Run(p{\uparrow})) = \mathcal{P}(Run_{\mathcal{A}}(p(1), inco) \cap Run_{\mathcal{A}}(p{\uparrow}))$

Due to Corollary 18, we can reduce the problem of computing the probabilities of transitions of the form $p(1) \xrightarrow{x} acc$ and $p(1) \xrightarrow{y} rej$ to the problem of computing the probability of non-termination in pOC. More precisely, we construct pOC's $\mathcal{A}_{cons}$ and $\mathcal{A}_{inco}$ which are the same as $\mathcal{A}$, except that for each control state $q$ of an inconsistent (or consistent, resp.) BSCC of $X$, all positive outgoing rules of $q$ are replaced with $q \xrightarrow{1,-1}_{>0} q$. Then $x = \mathcal{P}(Run_{\mathcal{A}_{cons}}(p{\uparrow}))$ and $y = \mathcal{P}(Run_{\mathcal{A}_{inco}}(p{\uparrow}))$.

Due to [4], the problem whether a given non-termination probability is positive (in a given pOC) is decidable in polynomial time. This means that the underlying graph of $\mathcal{G}$ is computable in polynomial time, and hence the sets $G_0$ and $G_1$ consisting of all states $s$ of $\mathcal{G}$ such that $\mathcal{P}(Run_{\mathcal{G}}(s, good))$ is equal to 0 and 1, respectively, are constructible in polynomial time. Let $G$ be the set of all states of $\mathcal{G}$ that are not contained in $G_0 \cup G_1$, and let $X_{\mathcal{G}}$ be the stochastic matrix of $\mathcal{G}$. For every $s \in G$ we fix a fresh variable $V_s$ and the equation

$$V_s \quad = \quad \sum_{s' \in G} X_{\mathcal{G}}(s, s') \cdot V_{s'} \quad + \quad \sum_{s' \in G_1} X_{\mathcal{G}}(s, s')$$

Thus, we obtain a system of linear equations $V = AV + b$ whose unique solution $V^*$ in $\mathbb{R}$ is the vector of probabilities of reaching a good BSCC from the states of $G$. This system can also be written as $(I - A)V = b$. Since the elements of $A$ and $b$ correspond to (sums of) transition probabilities in $\mathcal{G}$, it suffices to compute the transition probabilities of $\mathcal{G}$ with a sufficiently small relative error so that the approximate $A$ and $b$ produce an approximate solution where the relative error of each component is bounded by the $\varepsilon$. By combining standard results for finite-state Markov chains with techniques of numerical analysis, we show the following:

**Proposition 19.** *Let $c = 2|Q|$. For every $s \in G$, let $R_s$ be the probability of visiting a BSCC of $\mathcal{G}$ from $s$ in at most $c$ transitions, and let $R = \min\{R_s \mid s \in G\}$. Then $R > 0$ and if all transition probabilities in $\mathcal{G}$ are computed with relative error at most $\varepsilon R^3/8(c + 1)^2$, then the resulting system $(I - A')V = b'$ has a unique solution $U^*$ such that $|V_s^* - U_s^*|/V_s^* \leq \varepsilon$ for every $s \in G$.*

Note that the constant $R$ of Proposition 19 can be bounded from below by $x_t^{|Q|-1} \cdot x_n$, where

- $x_t = \min\{X_{\mathcal{G}}(s, s') \mid s, s' \in G\}$, i.e., $x_t$ is the minimal probability that is either explicitly used in $\mathscr{A}$, or equal to some positive termination probability in $\mathscr{A}$;
- $x_n = \min\{X_{\mathcal{G}}(s, s') \mid s \in G, s' \in G_1\}$, i.e., $x_n$ is the minimal probability that is either a positive termination probability in $\mathscr{A}$, or a positive non-termination probability in the pOC's $\mathscr{A}_{cons}$ and $\mathscr{A}_{inco}$ constructed above.

Now we need to employ the promised divergence gap theorem, which bounds a positive non-termination probability in pOC away from zero (for all $p, q \in Q$, we use $[p, q]$ to denote the probability of all runs $w$ initiated in $p(1)$ that visit a configuration $q(k)$, where $k \geq 1$ and the counter stays positive in all configurations preceding this visit).

**Theorem 20.** *Let $\mathscr{A} = (Q, \delta^{=0}, \delta^{>0}, P^{=0}, P^{>0})$ be a pOC and $X$ the underlying finite-state Markov chain of $\mathscr{A}$. Let $p \in Q$ such that $[p\uparrow] > 0$. Then there are two possibilities:*

1. *There is $q \in Q$ such that $[p, q] > 0$ and $[q\uparrow] = 1$. Hence, $[p\uparrow] \geq [p, q]$.*
2. *There is a BSCC $\mathscr{B}$ of $X$ and a state $q$ of $\mathscr{B}$ such that $[p, q] > 0$, $t > 0$, and $v_q = v_{\max}$ (here $t$ is the trend, $v$ is the vector of Proposition 6, and $v_{\max}$ is the maximal component of $v$; all of these are considered in $\mathscr{B}$). Further,*

$$[p\uparrow] \quad \geq \quad [p, q] \cdot \frac{t^3}{12(2(v_{\max} - v_{\min}) + 4)^3} \, .$$

Hence, denoting the relative precision $\varepsilon R^3/8(c + 1)^2$ of Proposition 19 by $\delta$, we obtain that $\log(1/\delta)$ is bounded by a polynomial in $|\mathscr{A}|$ and $\log(1/\varepsilon)$. Further, the transition probabilities of $\mathcal{G}$ can be approximated up to the relative error $\delta$ in time polynomial in $|\mathscr{A}|$ and $\log(1/\varepsilon)$ by approximating the termination probabilities of $\mathscr{A}$ (see Proposition 3). This proves Theorem 14.

| | $[a{\downarrow}]$ | $[a{\downarrow}0]$ | $[a{\downarrow}1]$ | $E[a{\downarrow}0]$ | $E[a{\downarrow}1]$ |
|---|---|---|---|---|---|
| $z = 0.5, y = 0.4, x_a = 0.2, x_o = 0.2$ | 0.800 | 0.500 | 0.300 | 11.000 | 7.667 |
| $z = 0.5, y = 0.4, x_a = 0.2, x_o = 0.4$ | 0.967 | 0.667 | 0.300 | 104.750 | 38.917 |
| $z = 0.5, y = 0.4, x_a = 0.2, x_o = 0.6$ | 1.000 | 0.720 | 0.280 | 20.368 | 5.489 |
| $z = 0.5, y = 0.4, x_a = 0.2, x_o = 0.8$ | 1.000 | 0.732 | 0.268 | 10.778 | 2.758 |
| $z = 0.5, y = 0.5, x_a = 0.1, x_o = 0.1$ | 0.861 | 0.556 | 0.306 | 11.400 | 5.509 |
| $z = 0.5, y = 0.5, x_a = 0.2, x_o = 0.1$ | 0.931 | 0.556 | 0.375 | 23.133 | 20.644 |
| $z = 0.5, y = 0.5, x_a = 0.3, x_o = 0.1$ | 1.000 | 0.546 | 0.454 | 83.199 | 111.801 |
| $z = 0.5, y = 0.5, x_a = 0.4, x_o = 0.1$ | 1.000 | 0.507 | 0.493 | 12.959 | 21.555 |
| $z = 0.2, y = 0.4, x_a = 0.2, x_o = 0.2$ | 0.810 | 0.696 | 0.115 | 7.827 | 6.266 |
| $z = 0.3, y = 0.4, x_a = 0.2, x_o = 0.2$ | 0.811 | 0.636 | 0.175 | 8.928 | 6.783 |
| $z = 0.4, y = 0.4, x_a = 0.2, x_o = 0.2$ | 0.808 | 0.571 | 0.236 | 10.005 | 7.258 |
| $z = 0.5, y = 0.4, x_a = 0.2, x_o = 0.2$ | 0.800 | 0.500 | 0.300 | 11.000 | 7.667 |

**Fig. 3.** Quantities of the pOC from Fig. 2

## 5 Experimental results, future work

We have implemented a prototype tool in the form of a Maple worksheet[3], which allows to compute the termination probabilities of pOC, as well as the conditional expected termination times. Our tool employs Newton's method to approximate the termination probabilities within a sufficient accuracy so that the expected termination time is computed with absolute error (at most) one by solving the linear equation system from Section 3.2.

We applied our tool to the pOC from Fig. 2 for various values of the parameters. Fig. 3 shows the results. We also show the associated termination probabilities, rounded to three digits. We write $[a{\downarrow}0]$ etc. to abbreviate $[(and,init){\downarrow}(or,return,0)]$ etc., and $[a{\downarrow}]$ for $[a{\downarrow}0] + [a{\downarrow}1]$).

We believe that other interesting quantities and numerical characteristics of pOC, related to both finite paths and infinite runs, can also be efficiently approximated using the methods developed in this paper. An efficient implementation of the associated algorithms would result in a verification tool capable of analyzing an interesting class of infinite-state stochastic programs, which is beyond the scope of currently available tools limited to finite-state systems only.

## References

1. T. Brázdil. *Verification of Probabilistic Recursive Sequential Programs*. PhD thesis, Masaryk University, Faculty of Informatics, 2007.
2. T. Brázdil, V. Brožek, J. Holeček, and A. Kučera. Discounted properties of probabilistic pushdown automata. In *Proceedings of LPAR 2008*, volume 5330 of *LNCS*, pages 230–242. Springer, 2008.
3. T. Brázdil, V. Brožek, and K. Etessami. One-counter stochastic games. In *Proceedings of FST&TCS 2010*, volume 8 of *LIPIcs*, pages 108–119. Schloss Dagstuhl, 2010.
4. T. Brázdil, V. Brožek, K. Etessami, A. Kučera, and D. Wojtczak. One-counter Markov decision processes. In *Proceedings of SODA 2010*, pages 863–874. SIAM, 2010.

---

[3] Available at `http://www.comlab.ox.ac.uk/people/stefan.kiefer/pOC.mws`.

5. T. Brázdil, J. Esparza, and A. Kučera. Analysis and prediction of the long-run behavior of probabilistic sequential programs with recursion. In *Proceedings of FOCS 2005*, pages 521–530. IEEE, 2005.

6. T. Brázdil, P. Jančar, and A. Kučera. One-counter Markov decision processes. *CoRR*, abs/0904.2511, 2010.

7. T. Brázdil, A. Kučera, and O. Stražovský. On the decidability of temporal properties of probabilistic pushdown automata. In *Proceedings of STACS 2005*, volume 3404 of *LNCS*, pages 145–157. Springer, 2005.

8. J. Canny. Some algebraic and geometric computations in PSPACE. In *Proceedings of STOC'88*, pages 460–467. ACM Press, 1988.

9. K. Chatterjee and L. Doyen. Energy parity games. In *Proceedings of ICALP 2010, Part II*, volume 6199 of *LNCS*, pages 599–610. Springer, 2010.

10. K. Chatterjee, L. Doyen, T. Henzinger, and J.-F. Raskin. Generalized mean-payoff and energy games. In *Proceedings of FST&TCS 2010*, volume 8 of *LIPIcs*, pages 505–516. Schloss Dagstuhl, 2010.

11. J. Esparza, D. Hansel, P. Rossmanith, and S. Schwoon. Efficient algorithms for model checking pushdown systems. In *Proceedings of CAV 2000*, volume 1855 of *LNCS*, pages 232–247. Springer, 2000.

12. J. Esparza, A. Kučera, and R. Mayr. Model-checking probabilistic pushdown automata. In *Proceedings of LICS 2004*, pages 12–21. IEEE, 2004.

13. J. Esparza, A. Kučera, and R. Mayr. Quantitative analysis of probabilistic pushdown automata: Expectations and variances. In *Proceedings of LICS 2005*, pages 117–126. IEEE, 2005.

14. K. Etessami, D. Wojtczak, and M. Yannakakis. Quasi-birth-death processes, tree-like QBDs, probabilistic 1-counter automata, and pushdown systems. In *Proceedings of 5th Int. Conf. on Quantitative Evaluation of Systems (QEST'08)*. IEEE, 2008.

15. K. Etessami and M. Yannakakis. Algorithmic verification of recursive probabilistic systems. In *Proceedings of TACAS 2005*, volume 3440 of *LNCS*, pages 253–270. Springer, 2005.

16. K. Etessami and M. Yannakakis. Checking LTL properties of recursive Markov chains. In *Proceedings of 2nd Int. Conf. on Quantitative Evaluation of Systems (QEST'05)*, pages 155–165. IEEE, 2005.

17. K. Etessami and M. Yannakakis. Recursive Markov chains, stochastic grammars, and monotone systems of non-linear equations. In *Proceedings of STACS 2005*, volume 3404 of *LNCS*, pages 340–352. Springer, 2005.

18. J.G. Kemeny and J.L. Snell. *Finite Markov Chains*. D. Van Nostrand Company, 1960.

19. S. Kiefer, M. Luttenberger, and J. Esparza. On the convergence of Newton's method for monotone systems of polynomial equations. In *STOC 2007*, pages 217–226, 2007.

20. S. Kiefer, M. Luttenberger, and J. Esparza. On the convergence of Newton's method for monotone systems of polynomial equations. In *Proceedings of STOC 2007*, pages 217–226. ACM Press, 2007.

21. J.S. Rosenthal. *A first look at rigorous probability theory*. World Scientific Publishing, 2006.

22. D. Williams. *Probability with Martingales*. Cambridge University Press, 1991.

## A   Proofs

In this section we give the proofs that were omitted in the main body of the paper. The appendix is structured according to sections and subsections of the main part.

### A.1   Finiteness of the expected termination time (Section 3.1)

Recall that $\mathscr{A} = (Q, \delta^{=0}, \delta^{>0}, P^{=0}, P^{>0})$ is a fixed pOC, $\mathcal{X}$ is the underlaying Markov chain of $\mathscr{A}$, and $A$ is the transition matrix of $\mathcal{X}$.

This section has two parts. In the first part (Section A.1.1) we provide the proofs that apply specifically to the case where $\mathcal{X}$ is strongly connected. In the second part (Section A.1.2) we deal with the general case, showing Theorem 5.

### A.1.1 Strongly connected $\mathcal{X}$

Recall that

- $\boldsymbol{\alpha} \in (0, 1]^Q$ is the invariant distribution of $\mathcal{X}$,
- $\boldsymbol{s} \in \mathbb{R}^Q$ is the vector expected counter changes defined by

$$s_p = \sum_{(p,c,q)\in\delta^{>0}} P^{>0}(p, c, q) \cdot c$$

- $t$ is the trend of $\mathcal{X}$ given by $t = \boldsymbol{\alpha s}$.

A *potential* is any vector $\boldsymbol{v}$ that satisfies $\boldsymbol{s} + A\boldsymbol{v} = \boldsymbol{v} + \mathbf{1}t$. The intuitive meaning of a potential $\boldsymbol{v}$ is that, starting in any state $p \in Q$, the expected counter increase after $i$ steps for large $i$ is $it + \boldsymbol{v}_p$. Given a potential $\boldsymbol{v}$, we define $|\boldsymbol{v}| := \boldsymbol{v}_{\max} - \boldsymbol{v}_{\min}$, where $\boldsymbol{v}_{\max}$ and $\boldsymbol{v}_{\min}$ are the largest and the smallest component of $\boldsymbol{v}$, respectively. Now we prove two lemmata that together imply Proposition 6.

**Lemma 21.** *We have the following:*

*(a) Let $W := \mathbf{1}\boldsymbol{\alpha}$, i.e., each row of $W$ equals $\boldsymbol{\alpha}$. Let $Z := (I - A + W)^{-1}$. The matrix $Z$ exists and the vector $Z\boldsymbol{s}$ is a potential.*

*(b) Denote by $x_{\min}$ the smallest nonzero coefficient of $A$. There exists a potential $\boldsymbol{v}$ with $|\boldsymbol{v}| \le 2|Q|/x_{\min}^{|Q|}$.*

*Proof.*

(a) The matrix $Z := (I - A + W)^{-1}$ exists by [18, Theorem 5.1.3]. (The matrix $Z$ is sometimes called the *fundamental matrix* of the finite Markov chain induced by $A$.) Furthermore, by [18, Theorem 5.1.3(d)] the fundamental matrix $Z$ satisfies $I + AZ = Z + W$. Multiplying with $\boldsymbol{s}$ and setting $\boldsymbol{u} := Z\boldsymbol{s}$, we obtain $\boldsymbol{s} + A\boldsymbol{u} = \boldsymbol{u} + \mathbf{1}\boldsymbol{\alpha s}$; i.e., $Z\boldsymbol{s}$ is a potential.

(b) Let $\boldsymbol{u}$ be the potential from (a); i.e., we have

$$(I - A)\boldsymbol{u} = \boldsymbol{s} - \mathbf{1}t. \tag{3}$$

By the Perron-Frobenius theorem for strongly connected matrices, there exists a positive vector $\boldsymbol{d} \in (0, 1]^Q$ with $A\boldsymbol{d} = \boldsymbol{d}$; i.e., $(I - A)\boldsymbol{d} = \mathbf{0}$. Observe that $\boldsymbol{u} + r\boldsymbol{d}$ is a

potential for all $r \in \mathbb{R}$. Choose $r$ such that $\boldsymbol{v} := \boldsymbol{u} + r\boldsymbol{d}$ satisfies $\boldsymbol{v}_{\max} = 2|Q|/x_{\min}^{|Q|}$. It suffices to prove $\boldsymbol{v}_{\min} \geq 0$. Let $q \in Q$ such that $\boldsymbol{v}_q = \boldsymbol{v}_{\max}$. Define the *distance* of a state $p \in Q$ as the distance of $p$ from $q$ in the graph induced by $A$. Note that $q$ has distance 0 and all states have distance at most $n - 1$, as $A$ is strongly connected. We prove by induction that a state $p$ with distance $i$ satisfies $\boldsymbol{v}_p \geq 2(n-i)/x_{\min}^{n-i}$. The claim is obvious for the induction base ($i = 0$). For the induction step, let $p$ be a state with distance $i + 1$ and $i \geq 0$. Let $r$ be a state with distance $i$ and $A_{pr} > 0$. We have:

$$
\begin{aligned}
\boldsymbol{v}_p &= (A\boldsymbol{v})_p + s_p - t & \text{(as } \boldsymbol{v} \text{ is a potential)} \\
&\geq (A\boldsymbol{v})_p - 2 & \text{(as } s_p, t \in [-1, 1]) \\
&\geq x_{\min}\boldsymbol{v}_r - 2 & \text{(as } A_{pr} > 0 \text{ implies } A_{pr} \geq x_{\min}) \\
&\geq x_{\min} \cdot 2(n - i)/x_{\min}^{n-i} - 2 & \text{(by induction hypothesis)} \\
&= 2(n - i)/x_{\min}^{n-(i+1)} - 2 & \\
&\geq 2(n - (i + 1))/x_{\min}^{n-(i+1)} & \text{(as } x_{\min} \leq 1).
\end{aligned}
$$

This completes the induction step. Hence we have $\boldsymbol{v}_{\min} \geq 0$ as desired.

$\square$

In the following, the vector $\boldsymbol{v}$ is always a potential. Recall that $p^{(i)}$ and $c^{(i)}$ are random variables which to every run $w \in Run(r(c))$ assign the control state and the counter value of the configuration $w(i)$, respectively, and $m^{(i)}$ is a random variable defined by

$$
m^{(i)} = \begin{cases} c^{(i)} + \boldsymbol{v}_{p^{(i)}} - it & \text{if } c^{(j)} \geq 1 \text{ for all } 0 \leq j < i \\ m^{(i-1)} & \text{otherwise} \end{cases}
$$

**Lemma 22.** *The sequence $m^{(0)}, m^{(1)}, \ldots$ is a martingale.*

*Proof.* Fix a path $u \in FPath(p^{(0)}(c^{(0)}))$ of length $i \geq 1$. First assume that $c^{(j)} \geq 1$ does not hold for all $j \in \{0, \ldots, i - 1\}$. Then for every run $w \in Run(u)$ we have $m^{(i)}(w) = m^{(i-1)}(w)$. Now assume that $c^{(j)} \geq 1$ holds for all $j \in \{0, \ldots, i - 1\}$. Then we have:

$$
\begin{aligned}
\mathbb{E}\left[m^{(i)} \mid Run(u)\right] &= \mathbb{E}\left[c^{(i)} + \boldsymbol{v}_{p^{(i)}} - it \mid Run(u)\right] \\
&= c^{(i-1)} + \sum_{\substack{(p^{(i-1)},a,q) \in \delta > 0 \\ P > 0(p^{(i-1)},a,q) = x}} x \cdot a + \sum_{\substack{(p^{(i-1)},a,q) \in \delta > 0 \\ P > 0(p^{(i-1)},a,q) = x}} x \cdot \boldsymbol{v}_q - it \\
&= c^{(i-1)} + s_{p^{(i-1)}} + (A\boldsymbol{v})_{p^{(i-1)}} - it \\
&= m^{(i-1)} + s_{p^{(i-1)}} + (A\boldsymbol{v})_{p^{(i-1)}} - \boldsymbol{v}_{p^{(i-1)}} - t \\
&= m^{(i-1)},
\end{aligned}
$$

where the last equality holds because $\boldsymbol{v}$ is a potential.

$\square$

A direct corollary to Lemma 21 and Lemma 22 is the following:

**Proposition 6.** *There is a vector $\boldsymbol{v} \in \mathbb{R}^Q$ such that the stochastic process $m^{(1)}, m^{(2)}, \ldots$ defined by*

$$
m^{(i)} = \begin{cases} c^{(i)} + \boldsymbol{v}_{p^{(i)}} - i \cdot t & \text{if } c^{(j)} \geq 1 \text{ for all } 0 \leq j < i; \\ m^{(i-1)} & \text{otherwise} \end{cases}
$$

18

*is a martingale, where t is the trend of $X$.*

*Moreover, the vector $\mathbf{v}$ satisfies $\mathbf{v}_{\max} - \mathbf{v}_{\min} \leq 2|Q|/x_{\min}^{|Q|}$, where $x_{\min}$ is the smallest positive transition probability in $X$, and $\mathbf{v}_{\max}$ and $\mathbf{v}_{\min}$ are the maximal and the minimal components of $\mathbf{v}$, respectively.*

Now we prove the propositions needed to justify Claims (A) and (B) of Section 3.1.

**Proposition 7.** *Let $p(k)$ be an initial configuration, and let $H_i$ be set of all runs initiated in $p(k)$ that visit a configuration with zero counter in exactly $i$ transitions. Let*

$$a = \exp\left(-\frac{t^2}{8(|\mathbf{v}| + t + 1)^2}\right).$$

*Note that $0 < a < 1$. Further, let*

$$h = \begin{cases} 2 \cdot \frac{-|\mathbf{v}| - c^{(0)}}{t} & \text{if } t < 0 \\ 2 \cdot \frac{|\mathbf{v}| - c^{(0)}}{t} & \text{if } t > 0 . \end{cases}$$

*Then for all $i \in \mathbb{N}$ with $i \geq h$ we have that $\mathcal{P}(H_i) \leq a^i$.*

*Proof.* For all runs in $H_i$ we have $m^{(i)} = \mathbf{v}_{p^{(i)}} - it$ and so

$$m^{(0)} - m^{(i)} = c^{(0)} + \mathbf{v}_{p^{(0)}} - \mathbf{v}_{p^{(i)}} + it . \tag{4}$$

**Case $t < 0$:** By (4) we have for $i \geq h$:

$$
\begin{aligned}
\mathcal{P}(H_i) &= \mathcal{P}(H_i \wedge m^{(i)} - m^{(0)} = -c^{(0)} - \mathbf{v}_{p^{(0)}} + \mathbf{v}_{p^{(i)}} - it) \\
&\leq \mathcal{P}(m^{(i)} - m^{(0)} = -c^{(0)} - \mathbf{v}_{p^{(0)}} + \mathbf{v}_{p^{(i)}} - it) \\
&\leq \mathcal{P}(m^{(i)} - m^{(0)} \geq -c^{(0)} - |\mathbf{v}| - it) \\
&= \mathcal{P}\left(m^{(i)} - m^{(0)} \geq (i - h/2) \cdot (-t)\right) \\
&\leq \mathcal{P}\left(m^{(i)} - m^{(0)} \geq (i/2) \cdot (-t)\right) .
\end{aligned}
$$

**Case $t > 0$:** By (4) we have for $i \geq h$:

$$
\begin{aligned}
\mathcal{P}(H_i) &= \mathcal{P}(H_i \wedge m^{(0)} - m^{(i)} = c^{(0)} + \mathbf{v}_{p^{(0)}} - \mathbf{v}_{p^{(i)}} + it) \\
&\leq \mathcal{P}(m^{(0)} - m^{(i)} = c^{(0)} + \mathbf{v}_{p^{(0)}} - \mathbf{v}_{p^{(i)}} + it) \\
&\leq \mathcal{P}(m^{(0)} - m^{(i)} \geq c^{(0)} - |\mathbf{v}| + it) \\
&= \mathcal{P}\left(m^{(0)} - m^{(i)} \geq (i - h/2) \cdot t\right) \\
&\leq \mathcal{P}\left(m^{(0)} - m^{(i)} \geq (i/2) \cdot t\right) .
\end{aligned}
$$

In each step, the martingale value changes by at most $|\mathbf{v}| + t + 1$. Hence Azuma's inequality (see [22]) asserts for $t \neq 0$ and $i \geq h$:

$$
\begin{aligned}
\mathcal{P}(H_i) &\leq \exp\left(-\frac{(i/2)^2 t^2}{2i(|\mathbf{v}| + t + 1)^2}\right) \qquad\qquad \text{(Azuma's inequality)} \\
&= a^i .
\end{aligned}
$$

$\square$

**Proposition 9.** *Assume that Pre*<sup>*</sup>*(q(0)) is infinite. Then almost all runs initiated in an arbitrary configuration reach Q(0). Moreover, there is $k_1 \in \mathbb{N}$ such that, for all $\ell \geq k_1$, the expected length of an honest path from $r(\ell)$ to $Q(0)$ is infinite.*

*Proof.* As $Pre^*(q(0)) = \infty$ and $X$ is strongly connected, $Q(0)$ is reachable from every configuration with positive probability. Also, recall that $t = 0$. Using strong law of large numbers (see e.g. [22]) and results of [6] (in particular Lemma 19), one can show that $Q(0)$ is reached from any configuration with probability one.

Consider an initial configuration $r(\ell)$ with $\ell + v_r > v_{\max}$. We will show that the expected length of an honest path from $r(\ell)$ to $Q(0)$ is infinite; i.e., we can take $k_1 := \lceil |v| + 1 \rceil$. Consider the martingale $m^{(1)}, m^{(2)}, \ldots$ defined in Proposition 6 over $Run(r(\ell))$. Note that as $t = 0$, the definition of the martingale simplifies to

$$
m^{(i)} = \begin{cases} c^{(i)} + v_{p^{(i)}} & \text{if } c^{(j)} \geq 1 \text{ for all } 0 \leq j < i; \\ m^{(i-1)} & \text{otherwise} \end{cases}
$$

Observe that $m^{(0)} = \ell + v_r$ and that the martingale value changes by at most $M := \lceil |v| \rceil + 1$ in a single step. Let us fix $k \in \mathbb{N}$ such that $\ell + v_r < v_{\max} + k$. Define a *stopping time* $\tau$ (see e.g. [22]) which returns the first point in time in which either $m^{(\tau)} \geq v_{\max} + k$, or $m^{(\tau)} \leq v_{\max}$. Observe that $\tau$ is almost surely finite and that $m^{(\tau)} \in [v_{\max} - M, v_{\max}] \cup [v_{\max} + k, v_{\max} + k + M]$. Define $x := \mathcal{P}(m^{(\tau)} \geq v_{\max} + k)$. Then

$$
\mathbb{E}[m^{(\tau)}] \leq x \cdot (v_{\max} + k + M) + (1 - x) \cdot v_{\max} = v_{\max} + x \cdot (k + M) \tag{5}
$$

and by the optional stopping theorem (see e.g. [22]),

$$
\mathbb{E}[m^{(\tau)}] = \mathbb{E}[m^{(0)}] = \ell + v_r. \tag{6}
$$

By putting the equations (5) and (6) together, we obtain that

$$
\mathcal{P}(m^{(\tau)} \geq v_{\max} + k) \geq \frac{\ell + v_r - v_{\max}}{k + M}. \tag{7}
$$

Denote by $T$ the time to hit $Q(0)$. We need to show $\mathbb{E}T = \infty$. For any run $w$ with $m^{(\tau)} \geq v_{\max} + k$ we have

$$
c^{(\tau)} = m^{(\tau)} - v_{p^{(\tau)}} \geq v_{\max} + k - v_{p^{(\tau)}} \geq k,
$$

hence we have $T \geq k$ for $w$, as at least $k$ steps are required to decrease the counter value from $k$ to 0. It follows $\mathcal{P}(m^{(\tau)} \geq v_{\max} + k) \leq \mathcal{P}(T \geq k)$. Hence:

$$
\mathbb{E}T = \sum_{k \in \mathbb{N}} \mathcal{P}(T \geq k) \geq \sum_{k=\ell+1}^{\infty} \mathcal{P}(T \geq k)
$$

$$
\geq \sum_{k=\ell+1}^{\infty} \mathcal{P}(m^{(\tau)} \geq v_{\max} + k) \overset{(7)}{\geq} \sum_{k=\ell+1}^{\infty} \frac{\ell + v_r - v_{\max}}{k + M} = \infty.
$$

$\square$

**Proposition 10.** *There is $k_2 \in \mathbb{N}$ such that for every configuration $r(\ell) \in Pre^*(q(0))$, where $\ell \geq k_2$, we have that if $r(\ell) \to r'(\ell')$, then $r'(\ell') \in Pre^*(q(0))$.*

*Proof.* We start by observing that $Pre^*(q(0))$ has an "ultimately periodic" structure. For every $i \in \mathbb{N}_0$, let $Pre(i) = \{r \in Q \mid r(i) \in Pre^*(q(0))\}$. Note that if $Pre(i) = Pre(j)$ for some $i, j \in \mathbb{N}_0$, then also $Pre(i+1) = Pre(j+1)$. Let $m_1$ be the least index such that $Pre(m_1) = Pre(j)$ for some $j > m_1$, and let $m_2$ be the least $j$ with this property. Further, we put $m = m_2 - m_1$. Observe that $m_1, m_2 \leq 2^{|Q|}$, and for every $\ell \geq m_2$ we have that $Pre(\ell) = Pre(\ell+m)$.

For every configuration $r(\ell)$ of $\mathcal{A}$, let $C(r(\ell))$ be the set of all configurations $r(\ell + i)$ such that $0 \leq i < m$ and $r \in Pre(\ell+i)$. Note that $C(r(\ell))$ has at most $m$ elements, and we define the *index* of $r(\ell)$ as the cardinality of $C(r(\ell))$. Due the periodicity of $Pre^*(q(0))$, we immediately obtain that for every $r(\ell)$ and $j \in \mathbb{N}_0$, where $\ell \geq m_1$, the index of $r(\ell)$ is the same as the index of $r(\ell+j)$.

Let $k_2 = m_1 + |Q| + 1$, and assume that there is a transition $r(\ell) \to r'(\ell')$ such that $r \in Pre(\ell)$, $r' \notin Pre(\ell')$, and $\ell \geq k_2$. Then $r(\ell+i) \to r'(\ell'+i)$ for all $0 \leq i < m$. Obviously, if $r' \in Pre(\ell'+i)$, then also $r \in Pre(\ell+i)$, which means that the index of $r'(\ell')$ is *strictly smaller* that the index of $r(\ell)$. Since $X$ is strongly connected, there is finite path from $r'(\ell')$ to $r(n)$ of length at most $|Q|$, where $n \geq m_1$. This means that there is a finite path from $r'(\ell'+i)$ to $r(n+i)$ for every $0 \leq i < m$. Hence, the index of $r'(\ell')$ is at least as large as the index of $r(n)$. Since the indexes of $r(n)$ and $r(\ell)$ are the same, we have a contradiction. $\quad\square$

### A.1.2 General Case

**Lemma 23.** *Consider a finite Markov chain on a set $Q$ of states with $|Q| = n$. Let $x$ denote the smallest nonzero transition probability in the chain. Let $p \in Q$ be any state and $S \subseteq Q$ any subset of $Q$. Define the random variable $T$ on runs starting in $p$ by*

$$T := \begin{cases} k & \text{if the run hits a state in } S \text{ for the first time after exactly } k \text{ steps} \\ \text{undefined} & \text{if the run never hits a state in } S. \end{cases}$$

*We have $\mathcal{P}(T \geq k) \leq 2c^k$ for all $k \geq n$, where $c := \exp(-x^n/n)$.*

*Proof.* If $x = 1$ then all states that are visited are visited after at most $n - 1$ steps and hence $\mathcal{P}(T \geq n) = 0$. Assume $x < 1$ in the following. Since for each state the sum of the probabilities of the outgoing edges is 1, we must have $x \leq 1/2$. Call *crash* the event of, within the first $n - 1$ steps, either hitting $S$ or some state $r \in Q$ from which $S$ is not reachable. The probability of a crash is at least $x^{n-1} \geq x^n$, regardless of the starting state. Let $k \geq n$. For the event where $T \geq k$, a crash has to be avoided at least $\lfloor \frac{k-1}{n-1} \rfloor$ times; i.e.,

$$\mathcal{P}(T \geq k) \leq (1 - x^n)^{\lfloor \frac{k-1}{n-1} \rfloor}.$$

As $\lfloor \frac{k-1}{n-1} \rfloor \geq \frac{k-1}{n-1} - 1 \geq \frac{k}{n} - 1$, we have

$$\mathcal{P}(T \geq k) \leq \frac{1}{1 - x^n} \cdot \left( (1 - x^n)^{1/n} \right)^k \leq 2 \cdot \left( (1 - x^n)^{1/n} \right)^k$$

$$= 2 \cdot \exp\left( \frac{1}{n} \log(1 - x^n) \right)^k \leq 2 \cdot \exp\left( \frac{1}{n} \cdot (-x^n) \right)^k = 2 \cdot c^k.$$

$\square$

**Lemma 24.** *Let $p, q \in Q$ such that $[p{\downarrow}q] > 0$ and $q$ is not in a BSCC of $\mathcal{X}$. Then*

$$E(p{\downarrow}q) \le \frac{5|Q|}{x_{\min}^{|Q|+|Q|^3}} \, .$$

*Proof.* Consider the finite Markov chain $\mathcal{X}$. Define, for runs in $\mathcal{X}$ starting in $p$, the random variable $\widehat{R}$ as the time to hit $q$, and set $\widehat{R} := undefined$ for runs that do not hit $q$. There is a straightforward probability-preserving mapping that maps runs in $\mathcal{M}_{\mathscr{A}}$ with $R_{p{\downarrow}q} = k$ to runs in $\mathcal{X}$ with $\widehat{R} = k$. Hence, $\mathcal{P}(R_{p{\downarrow}q} = k) \le \mathcal{P}(\widehat{R} = k)$ for all $k \in \mathbb{N}_0$ and so

$$E(p{\downarrow}q) \cdot [p{\downarrow}q] = \sum_{k \in \mathbb{N}_0} \mathcal{P}(R_{p{\downarrow}q} = k) \cdot k \le \sum_{k \in \mathbb{N}_0} \mathcal{P}(\widehat{R} = k) \cdot k$$

$$= \sum_{k \in \mathbb{N}} \mathcal{P}(\widehat{R} \ge k) \le \sum_{k=1}^{|Q|} 1 + \sum_{k=0}^{\infty} 2c^k = |Q| + \frac{2}{1-c} \qquad \text{(Lemma 23)}\,.$$

We have $1 - c = 1 - \exp(-x_{\min}^{|Q|}/|Q|) \ge x_{\min}^{|Q|}/(2|Q|)$, hence

$$E(p{\downarrow}q) \cdot [p{\downarrow}q] \le |Q| + \frac{4|Q|}{x_{\min}^{|Q|}} \le \frac{5|Q|}{x_{\min}^{|Q|}} \, .$$

As $[p{\downarrow}q] \ge x_{\min}^{|Q|^3}$ by Proposition 3, it follows

$$E(p{\downarrow}q) \le \frac{5|Q|}{x_{\min}^{|Q|+|Q|^3}} \, .$$

$\square$

**Lemma 25.** *Let $p, q \in Q$ such that $[p{\downarrow}q] > 0$ and $q$ is in a BSCC with trend $t \ne 0$. Then*

$$E(p{\downarrow}q) \le 85000 \cdot \frac{|Q|^6}{x_{\min}^{5|Q|+|Q|^3} \cdot t^4} \, .$$

*Proof.* Let $B$ denote the BSCC of $q$. For a run $w \in Run(p{\downarrow}q)$, define $R^{(1)}(w)$ as the time to hit $B$, and $R^{(2)}(w)$ as the time to reach $q(0)$ after hitting $B$. For other runs $w$ let $R^{(1)}(w) := undefined$ and $R^{(2)}(w) := undefined$. Note that $R_{p{\downarrow}q}(w) = R^{(1)}(w) + R^{(2)}(w)$ whenever $R^{(1)}(w)$ and $R^{(2)}(w)$ are defined. We have:

$$E(p{\downarrow}q) \cdot [p{\downarrow}q] = \sum_{k \in \mathbb{N}_0} \mathcal{P}(R_{p{\downarrow}q} = k) \cdot k$$

$$= \sum_{k \in \mathbb{N}_0} \mathcal{P}(R^{(1)} + R^{(2)} = k) \cdot k$$

$$= \sum_{k_1, k_2 \in \mathbb{N}_0} \mathcal{P}(R^{(1)} = k_1 \,\wedge\, R^{(2)} = k_2) \cdot (k_1 + k_2)$$

$$= \sum_{k_1, k_2 \in \mathbb{N}_0} \mathcal{P}(R^{(1)} = k_1) \cdot \mathcal{P}(R^{(2)} = k_2 \mid R^{(1)} = k_1) \cdot (k_1 + k_2)$$

$$= E_1 + E_2 \, ,$$

where

$$E_1 := \sum_{k_1,k_2 \in \mathbb{N}_0} \mathcal{P}(R^{(1)} = k_1) \cdot \mathcal{P}(R^{(2)} = k_2 \mid R^{(1)} = k_1) \cdot k_1 \qquad \text{and}$$

$$E_2 := \sum_{k_1,k_2 \in \mathbb{N}_0} \mathcal{P}(R^{(1)} = k_1) \cdot \mathcal{P}(R^{(2)} = k_2 \mid R^{(1)} = k_1) \cdot k_2 \, .$$

For a bound on $E_1$ we have

$$E_1 = \sum_{k_1 \in \mathbb{N}_0} \mathcal{P}(R^{(1)} = k_1) \cdot k_1 \cdot \sum_{k_2 \in \mathbb{N}_0} \mathcal{P}(R^{(2)} = k_2 \mid R^{(1)} = k_1)$$

$$\leq \sum_{k_1 \in \mathbb{N}_0} \mathcal{P}(R^{(1)} = k_1) \cdot k_1$$

Consider the finite Markov chain $\mathcal{X}$. Define, for runs in $\mathcal{X}$ starting in $p$, the random variable $\widehat{R^{(1)}}$ as the time to hit $B$, and set $\widehat{R^{(1)}} :=$ *undefined* for runs that do not hit $B$. There is a straightforward probability-preserving mapping that maps runs in $\mathcal{M}_{\mathscr{A}}$ with $R^{(1)} = k_1$ to runs in $\mathcal{X}$ with $\widehat{R^{(1)}} = k_1$. Hence, $\mathcal{P}(R^{(1)} = k_1) \leq \mathcal{P}(\widehat{R^{(1)}} = k_1)$ for all $k_1 \in \mathbb{N}_0$ and so

$$E_1 \leq \sum_{k_1 \in \mathbb{N}_0} \mathcal{P}(\widehat{R^{(1)}} = k_1) \cdot k_1 \quad = \quad \sum_{k_1 \in \mathbb{N}} \mathcal{P}(\widehat{R^{(1)}} \geq k_1) \quad \leq \quad \frac{2}{1-c} \tag{8}$$

with $c$ from Lemma 23.

For a bound on $E_2$, fix any $k_1 \in \mathbb{N}_0$. We have:

$$\sum_{k_2 \in \mathbb{N}_0} \mathcal{P}(R^{(2)} = k_2 \mid R^{(1)} = k_1) \cdot k_2$$

$$= \sum_{j=0}^{k_1+1} \sum_{k_2 \in \mathbb{N}_0} \underbrace{\mathcal{P}(R^{(2)} = k_2 \mid R^{(1)} = k_1, \ c^{(0)} = j)}_{=\mathcal{P}(R^{(2)}=k_2 \mid c^{(0)}=j)} \cdot k_2 \cdot \mathcal{P}(c^{(0)} = j \mid R^{(1)} = k_1),$$

where we denote by $c^{(0)}$ the counter value when hitting $B$. In the last equality we used the fact that in each step the counter value can increase by at most 1, thus $R^{(1)} = k_1$ implies $c^{(0)} \leq k_1 + 1$. Denote by $m(k_1) \in \{0, \ldots, k_1 + 1\}$ the value of $j$ that maximizes $\sum_{k_2 \in \mathbb{N}_0} \mathcal{P}(R^{(2)} = k_2 \mid c^{(0)} = j) \cdot k_2$. Then we can continue:

$$\leq \sum_{k_2 \in \mathbb{N}_0} \mathcal{P}(R^{(2)} = k_2 \mid c^{(0)} = m(k_1)) \cdot k_2 \cdot \underbrace{\sum_{j=0}^{k_1+1} \mathcal{P}(c^{(0)} = j \mid R^{(1)} = k_1)}_{=1}$$

Denote by $h(c^{(0)})$ the $h$ from Lemma 7. We have $h(m(k_1)) \leq 2\frac{|v|+m(k_1)}{|t|} \leq 2\frac{|v|+k_1+1}{|t|} =: \hat{h}(k_1)$. So we can continue:

$$\leq \sum_{k_2=0}^{\lfloor \hat{h}(k_1) \rfloor} k_2 + \sum_{k_2=\lceil \hat{h}(k_1) \rceil}^{\infty} a^{k_2} \cdot k_2 \qquad \text{(with } a \text{ from Proposition 7)}$$

$$\leq \hat{h}(k_1)^2 + \frac{a}{(1-a)^2} = \frac{4(|v| + k_1 + 1)^2}{t^2} + \frac{a}{(1-a)^2} \, .$$

23

With this inequality and the random variable $\widehat{R^{(2)}}$ from above at hand we get a bound on $E_2$:

$$E_2 = \sum_{k_1 \in \mathbb{N}_0} \underbrace{\mathcal{P}(R^{(1)} = k_1)}_{\underbrace{\leq \mathcal{P}(\widehat{R^{(1)}} = k_1)}_{\leq \mathcal{P}(\widehat{R^{(1)}} \geq k_1)}} \cdot \underbrace{\sum_{k_2 \in \mathbb{N}_0} \mathcal{P}(R^{(2)} = k_2 \mid R^{(1)} = k_1) \cdot k_2}_{\leq \frac{4(|v|+k_1+1)^2}{t^2} + \frac{a}{(1-a)^2}}$$

$$\leq \sum_{k_1=0}^{|Q|-1} \left( \frac{4(|v| + k_1 + 1)^2}{t^2} + \frac{a}{(1-a)^2} \right) + \sum_{k_1=0}^{\infty} 2c^{k_1} \frac{a}{(1-a)^2} + \sum_{k_1=0}^{\infty} 2c^{k_1} \frac{4(|v| + k_1 + 1)^2}{t^2}$$

$$\leq \frac{4|Q|(|v| + |Q|)^2}{t^2} + \frac{2|Q|}{(1-c)(1-a)^2} + \frac{8}{t^2} \sum_{k_1=0}^{\infty} c^{k_1} (|v| + k_1 + 1)^2$$

The last series can be bounded as follows:

$$\sum_{k_1=0}^{\infty} c^{k_1}(|v| + k_1 + 1)^2 \leq \sum_{k_1=0}^{\lfloor |v|+1 \rfloor} (2(|v| + 1))^2 + \sum_{k_1=\lfloor |v|+1 \rfloor + 1}^{\infty} c^{k_1} \cdot (2k_1)^2$$

$$\leq 4(|v| + 2)^3 + 4 \sum_{k_1=0}^{\infty} c^{k_1} \cdot k_1^2 = 4(|v| + 2)^3 + 4\frac{c(c+1)}{(1-c)^3}$$

$$\leq 4(|v| + 2)^3 + \frac{8}{(1-c)^3}$$

It follows:

$$E_2 \leq \frac{4|Q|(|v| + |Q|)^2}{t^2} + \frac{2|Q|}{(1-c)(1-a)^2} + \frac{32}{t^2}\left( (|v| + 2)^3 + \frac{2}{(1-c)^3} \right) \tag{9}$$

Recall the following bounds:

$$|v| \leq 2|Q|/x_{\min}^{|Q|} \qquad \text{(Lemma 21)}$$

$$1 - c = 1 - \exp(-x_{\min}^{|Q|}/|Q|) \geq x_{\min}^{|Q|}/(2|Q|) \qquad \text{(Lemma 23)}$$

$$1 - a = 1 - \exp\left(-t^2/\left(8(|v| + 2)^2\right)\right) \geq t^2/\left(16(|v| + 2)^2\right) \qquad \text{(Proposition 7)}$$

$$[p{\downarrow}q] \geq x_{\min}^{|Q|^3} \qquad \text{(Proposition 3)}$$

After plugging those bounds into (8) and (9) we obtain using straightforward calculations:

$$E_1 \leq 4\frac{|Q|}{x_{\min}^{|Q|}} \qquad \text{and} \qquad E_2 \leq 84356\frac{|Q|^6}{x_{\min}^{5|Q|} \cdot t^4}, \qquad \text{hence}$$

$$E(p{\downarrow}q) = \frac{E_1 + E_2}{[p{\downarrow}q]} \leq 85000 \cdot \frac{|Q|^6}{x_{\min}^{5|Q|+|Q|^3} \cdot t^4}.$$

$\square$

**Lemma 26.** *Let $p, q \in Q$. If $Pre^*(q(0)) \cap Post^*(p(1))$ is finite, then*

$$|Pre^*(q(0)) \cap Post^*(p(1))| \quad \leq \quad |Q|^2 \cdot (|Q| + 2)$$

*Proof.* In this proof we use some notions and results of [11] (in particular, we use the notion of $\mathcal{P}$-automata as defined in Section 2.1 of [11]). Consider the pOC as a (non-probabilistic) pushdown system with one letter stack alphabet, say $\Gamma = \{X\}$ (the counter of height $n$ then corresponds to the stack content $X^n$).

A $\mathcal{P}$-automaton $\mathscr{A}_{q(0)}$ accepting the set of configurations $\{q(0)\}$ can be defined to have the set of states $Q$, no transitions, and $q$ as the only accepting state. Let $\mathscr{A}_{pre^*}$ be the $\mathcal{P}$-automaton accepting $Pre^*(q(0))$ constructed using the procedure from Section 4 of [11]. The automaton $\mathscr{A}_{pre^*}$ has the same set of states, $Q$, as $\mathscr{A}_{q(0)}$.

A $\mathcal{P}$-automaton $\mathscr{A}_{p(1)}$ accepting the set of configurations $\{p(1)\}$ can be defined to have the set of states $Q \cup \{p_{acc}\}$, one transition $(p, X, p_{acc})$, and $q_{acc}$ as the only accepting state. Let $\mathscr{A}_{post^*}$ be the automaton accepting $Post^*(p(1))$ constructed using the procedure from Section 6 of [11]. The automaton $\mathscr{A}_{post^*}$ has at most $|Q| + 2$ states.

Using standard product construction we obtain a $\mathcal{P}$-automaton $\mathscr{A}$ accepting $Pre^*(q(0)) \cap Post^*(p(1))$, which has $|Q| \cdot (|Q| + 2)$ states. Now note that if $Pre^*(q(0)) \cap Post^*(p(1))$ is finite, then a standard pumping argument for finite automata implies that the length of every word accepted by $\mathscr{A}$ is bounded by $|Q| \cdot (|Q| + 2)$. It follows that there are only $|Q|^2 \cdot (|Q| + 2)$ configurations in $Pre^*(q(0)) \cap Post^*(p(1))$. $\qquad\square$

**Lemma 27.** *Let $p, q \in Q$ such that $Pre^*(q(0)) \cap Post^*(p(1))$ is finite. Then*

$$E(p{\downarrow}q) \le E(p{\downarrow}q) \le \frac{15|Q|^3}{x_{min}^{4|Q|^3}}$$

*Proof.* We construct a finite Markov chain $\mathcal{Y}$ as follows. The states of $\mathcal{Y}$ are the states in $Pre^*(q(0)) \cap Post^*(p(1)) \cup \{o\}$, where $o$ is a fresh symbol. In general, the transitions in $\mathcal{Y}$ are as in the infinite Markov chain $\mathcal{M}_{\mathscr{A}}$, with the following exceptions:

 – all transitions leaving the set $Pre^*(q(0)) \cap Post^*(p(1))$ are redirected to $o$;
 – all transitions leading to a configuration $r(0)$ with $r \ne q$ are redirected to $o$;
 – $o$ gets a probability 1 self-loop.

Let $T$ denote the time that a run in $\mathcal{Y}$ starting from $p(1)$ hits $q(0)$ in exactly $k$ steps. This construction of $\mathcal{Y}$ makes sure that $\mathcal{P}(T = k) = \mathcal{P}(R_{p{\downarrow}q} = k)$. Note that by Lemma 26 the chain $\mathcal{Y}$ has at most $\ell := 3|Q|^3$ states. So we have:

$$
\begin{aligned}
[p{\downarrow}q] \cdot E(p{\downarrow}q) &\le \sum_{k \in \mathbb{N}} \mathcal{P}(R_{p{\downarrow}q} \ge k) = \sum_{k \in \mathbb{N}} \mathcal{P}(T \ge k) \\
&= \sum_{k=1}^{\ell-1} \mathcal{P}(T \ge k) + \sum_{k=\ell}^{\infty} \mathcal{P}(T \ge k) \\
&\le \ell + \sum_{k=0}^{\infty} 2c^k = \ell + \frac{2}{1-c} \qquad\qquad \text{(Lemma 23)}
\end{aligned}
$$

We have $1 - c = 1 - \exp(-x_{min}^{\ell}/\ell) \ge x_{min}^{\ell}/(2\ell)$, hence

$$[p{\downarrow}q] \cdot E(p{\downarrow}q) \le 3|Q|^3 + \frac{12|Q|^3}{x_{min}^{3|Q|^3}} \le \frac{15|Q|^3}{x_{min}^{3|Q|^3}},$$

and so, by Proposition 3,

$$E(p{\downarrow}q) \le \frac{15|Q|^3}{x_{\min}^{4|Q|^3}}\ .$$

<div align="right">□</div>

By combining Lemmata 24, 25 and 27 we obtain the following proposition, which directly implies Theorem 5:

**Proposition 28.** *Let* $(p, q) \in T^{>0}$. *Let* $\mathcal{B}$ *be the SCC of* $q$ *in* $X$. *Let* $x_{\min}$ *denote the smallest nonzero probability in A. Then we have:*

- *If* $Pre^*(q(0)) \cap Post^*(p(1))$ *is a finite set, then* $E(p{\downarrow}q) \le 15|Q|^3/x_{\min}^{4|Q|^3}$;
- *otherwise, if* $\mathcal{B}$ *is not a BSCC of* $X$, *then* $E(p{\downarrow}q) \le 5|Q|/\left(x_{\min}^{|Q|+|Q|^3}\right)$;
- *otherwise, if* $\mathcal{B}$ *has trend* $t \neq 0$, *then* $E(p{\downarrow}q) \le 85000|Q|^6/\left(x_{\min}^{5|Q|+|Q|^3} \cdot t^4\right)$.
- *otherwise,* $E(p{\downarrow}q)$ *is infinite.*

### A.2 Efficient approximation of finite expected termination time (Section 3.2)

We will use the following theorem from numerical analysis (see, e.g., [14]):

**Theorem 29.** *Consider a system of linear equations,* $B \cdot V = b$, *where* $B \in \mathbb{R}^{n \times n}$ *and* $b \in \mathbb{R}^n$. *Suppose that* $B$ *is regular and* $b \neq 0$. *Let* $V^* = B^{-1} \cdot b$ *be the unique solution of this system and suppose that* $V^* \neq 0$. *Denote by* $\kappa(B) = \|B\| \cdot \|B^{-1}\|$ *the condition number of* $B$. *Consider a system of equations* $(B + \Delta) \cdot V = b + \zeta$ *where* $\Delta \in \mathbb{R}^{n \times n}$ *and* $\zeta \in \mathbb{R}^n$. *If* $\|\Delta\| < \frac{1}{\|B^{-1}\|}$, *then the system* $(B + \Delta) \cdot V = b + \zeta$ *has a unique solution* $V_p^*$. *Moreover, for every* $\delta > 0$ *satisfying* $\frac{\|\Delta\|}{\|B\|} \le \delta$ *and* $\frac{\|\zeta\|}{\|b\|} \le \delta$ *and* $4 \cdot \delta \cdot \kappa(B) < 1$ *the solution* $V_p^*$ *satisfies*

$$\frac{\|V^* - V_p^*\|}{\|V^*\|} \quad \le \quad 4 \cdot \delta \cdot \kappa(B)$$

**Proposition 30.** *Consider a system of linear equations,* $C \cdot W = c$, *where* $C \in \mathbb{R}^{n \times n}$ *and* $c \in \mathbb{R}^n$. *Suppose that* $C$ *is nonsingular and* $c \neq 0$. *Let* $W^* = C^{-1} \cdot c$ *be the unique solution of this system. Let* $\|\cdot\|$ *be the* $l_\infty$ *norm. Consider a system* $(C + \mathcal{E}) \cdot W = c$ *where* $\mathcal{E} \in \mathbb{R}^{n \times n}$. *Let* $\|C\| \le u \ge 1$ *and* $\|C^{-1}\| \le v \ge 1$. *If* $\|\mathcal{E}\| < 1/v$, *then the system* $(C + \mathcal{E}) \cdot W = c$ *has a unique solution* $W_p^*$. *Moreover, if* $\|\mathcal{E}\| \le \delta < 1/(4uv)$, *then* $W_p^*$ *satisfies*

$$\frac{\|W^* - W_p^*\|}{\|W^*\|} \quad \le \quad \delta \cdot 4uv$$

*Proof.* We apply Theorem 29 with

$$B := \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad b := \begin{pmatrix} c \\ 1 \end{pmatrix} \quad \text{and} \quad \Delta := \begin{pmatrix} \mathcal{E} & 0 \\ 0 & 0 \end{pmatrix};$$

i.e., a single equation $x = 1$, for a new variable $x$ is added to the system, without new errors. Notice that

$$B^{-1} = \begin{pmatrix} C^{-1} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad V^* := \begin{pmatrix} W^* \\ 1 \end{pmatrix}.$$

Further $\|B^{-1}\| = \max\{1, \|C^{-1}\|\}$. So we have $\|\varDelta\| = \|\mathcal{E}\| < 1/v \le 1/\max\{1, \|C^{-1}\|\} = 1/\|B^{-1}\|$. Thus, by Theorem 29 there is a unique solution of $(B + \varDelta) \cdot V = b$, hence $W_p^*$ is unique too. Moreover, we have

$$\frac{\|\varDelta\|}{\|B\|} = \frac{\|\varDelta\|}{\max\{1, \|C\|\}} \le \|\varDelta\| = \|\mathcal{E}\| \le \delta \qquad \text{and}$$

$$4 \cdot \delta \cdot \kappa(B) = 4 \cdot \delta \cdot \max\{1, \|C\|\} \cdot \max\{1, \|C^{-1}\|\} \le 4 \cdot \delta \cdot u \cdot v < 1,$$

so Theorem 29 implies

$$\frac{\|W^* - W_p^*\|}{\|W^*\|} \quad \le \quad 4 \cdot \delta \cdot \kappa(B) \quad \le \quad \delta \cdot 4uv.$$

$\square$

With this at hand we can prove Proposition 12:

**Proposition 12.** *Let $b \in \mathbb{R}^+$ satisfy $E(p{\downarrow}q) \le b$ for all $(p, q) \in T_{<\infty}^{>0}$. For each $\varepsilon$, where $0 < \varepsilon < 1$, let $\delta = \varepsilon / (12 \cdot b^2)$. If $\|G - H\| \le \delta$, then the perturbed system $V = G \cdot V + 1$ has a unique solution $F$. Moreover, we have that*

$$|E(p{\downarrow}q) - F_{pq}| \quad \le \quad \varepsilon \qquad \text{for all } (p, q) \in T_{<\infty}^{>0}.$$

*Here $F_{pq}$ is the component of $F$ corresponding to the variable $V(p{\downarrow}q)$.*

*Proof.* Denote by $E$ the vector of expected termination times, i.e., the unique solution of $\mathcal{L}'$, i.e., $E = (I - H)^{-1}\mathbf{1}$. Recall that all components of $E$ are finite.

We will apply Proposition 30 using the following assignments: $C = I - H, C + \mathcal{E} = I - G, c = \mathbf{1}, W^* = E, W_p^* = F$. To find a suitable $u$, we need to find a bound on $\|I - H\|$. By comparing $\mathcal{L}'$ with (2) it follows that $\|H\mathbf{1}\| \le 2$ and hence

$$\|I - H\| \quad \le \quad 1 + \|H\| \quad = \quad 1 + \|H\mathbf{1}\| \quad \le \quad 3 \ =: \ u. \tag{10}$$

Further, we set $v := b$, so we need to show $\|(I - H)^{-1}\| \le b$. By our assumption, $\|E\| \le b$. Recall that $E = (I - H)^{-1}\mathbf{1}$, so if $(I - H)^{-1}$ is nonnegative, then $\|(I - H)^{-1}\| = \|(I - H)^{-1}\mathbf{1}\| = \|E\| \le b$, hence it remains to show that $(I - H)^{-1}$ is nonnegative. To see this, note that $E$ is the (unique) fixed point of a linear function $\mathcal{F}$ which to every $V$ assigns $H \cdot V + \mathbf{1}$. This function is continuous and monotone, so by Kleene's theorem we get that $E = \sup_{i \in \mathbb{N}} \mathcal{F}^i(\mathbf{0}) = \sum_{i=0}^{\infty} H^i \mathbf{1}$. Recall that $E$ is finite, so the matrix series $H^* := \sum_{i=0}^{\infty} H^i$ converges and thus equals $(I - H)^{-1}$. Hence $(I - H)^{-1} = H^*$, which is nonnegative as $H$ is nonnegative.

Now we are ready to apply Theorem 30. Since $\|G - H\| \le \varepsilon / (12 \cdot b^2) < 1/v$, the perturbed system $V = G \cdot V + \mathbf{1}$ has a unique solution $F$ as desired. By applying the second part of Theorem 30 we get

$$\frac{\|E - F\|}{\|E\|} \quad \le \quad \delta \cdot 12 \cdot b \qquad \text{for } \|G - H\| \le \delta \le 1/(12 \cdot b). \tag{11}$$

Hence,

$$\begin{aligned}
|E(p{\downarrow}q) - F_{pq}| &\le \|E - F\| && \text{(by the definition of the norm)} \\
&\le b \cdot \frac{\|E - F\|}{\|E\|} && \text{by } \|E\| \le b \\
&\le b \cdot \delta \cdot 12 \cdot b && \text{(by (11))} \\
&= \varepsilon && \text{(by the definition of } \delta\text{).}
\end{aligned}$$

$\square$

**Proposition 13.** *Let $x_{\min}$ denote the smallest nonzero probability in A. Then we have:*

$$E(p{\downarrow}q) \;\;\leq\;\; 85000 \cdot |Q|^6 / \left( x_{\min}^{6|Q|^3} \cdot t_{\min}^4 \right) \qquad \text{for all } (p,q) \in T_{<\infty}^{>0},$$

*where $t_{\min} = \{|t| \neq 0 \mid t \text{ is the trend in a BSCC of } X\}$.*

*Proof.* The proof follows directly from Proposition 28. $\square$

### A.3  Quantitative Model-Checking of $\omega$-regular Properties (Section 4)

**Proposition 15.** *Let $\Sigma$ be a finite alphabet, $\mathscr{A}$ a pOC, $\nu$ a valuation, $\mathcal{R}$ a DRA over $\Sigma$, and $p(0)$ a configuration of $\mathscr{A}$. Then there is a pOC $\mathscr{A}'$ with Rabin acceptance condition and a configuration $p'(0)$ of $\mathscr{A}'$ constructible in polynomial time such that the probability of all $w \in Run_{\mathscr{A}}(p(0))$ where $\nu(w)$ is accepted by $\mathcal{R}$ is equal to the probability of all accepting $w \in Run_{\mathscr{A}'}(p'(0))$.*

*Proof.* Let $(E_1, F_1), \ldots, (E_k, F_k)$ be the Rabin acceptance condition of $\mathcal{R}$. The automaton $\mathscr{A}'$ is the synchronized product of $\mathscr{A}$ and $\mathcal{R}$ where

- $Q \times R$ is the set of control states, where $R$ is the set of states of $\mathcal{R}$;
- $(p,r) \xrightarrow{x,c}_{>0} (p',r')$ iff $p \xrightarrow{x,c}_{>0} p'$ and $r \xrightarrow{\nu(p(1))} r'$ is a transition in $\mathcal{R}$;
- $(p,r) \xrightarrow{x,c}_{=0} (p',r')$ iff $p \xrightarrow{x,c}_{=0} p'$ and $r \xrightarrow{\nu(p(0))} r'$ is a transition in $\mathcal{R}$.

The Rabin acceptance condition of $\mathscr{A}'$ is $(Q \times E_1, Q \times F_1), \ldots, (Q \times E_k, Q \times F_k)$. $\square$

**Proposition 19.** *Let $c = 2|Q|$. For every $s \in G$, let $R_s$ be the probability of visiting a BSCC of $\mathcal{G}$ from $s$ in at most $c$ transitions, and let $R = \min\{R_s \mid s \in G\}$. Then $R > 0$ and if all transition probabilities in $\mathcal{G}$ are computed with relative error at most $\varepsilon R^3 / 8(c+1)^2$, then the resulting system $(I - A')V = b'$ has a unique solution $U^*$ such that $|V_s^* - U_s^*|/V_s^* \leq \varepsilon$ for every $s \in G$.*

*Proof.* The first step towards applying Theorem 29 is to estimate the condition number $\kappa = \|I - A\| \cdot \|(I - A)^{-1}\|$. Obviously, $\|I - A\| \leq 2$. Further, $\|(I - A)^{-1}\|$ is bounded by the expected number of steps needed to reach a BSCC of $\mathcal{G}$ from a state of $G$ (here we use a standard result about absorbing finite-state Markov chains). Since $G$ has at most $c$ states, we have that $R_s > 0$, and hence also $R > 0$. Obviously, the probability on *non-visiting* a BSCC of $\mathcal{G}$ in at most $i$ transitions from a state of $G$ is bounded by $(1-R)^{\lfloor i/c \rfloor}$. Hence, the probability of visiting a BSCC of $\mathcal{G}$ from a state of $G$ after *exactly* $i$ transitions is bounded by $(1-R)^{\lfloor (i-1)/c \rfloor}$. Further, a simple calculation shows that

$$\begin{aligned}
\|(I - A)^{-1}\| &\leq \sum_{i=1}^{\infty} i \cdot (1-R)^{\lfloor (i-1)/c \rfloor} &=& \sum_{i=0}^{\infty} \left( \frac{c(c+1)}{2} + ic^2 \right) \cdot (1-R)^i \\
&= \frac{c(c+1)}{2R} + \frac{c^2(1-R)}{R^2} &\leq& \left( \frac{c+1}{R} \right)^2
\end{aligned}$$

28

Hence, $\kappa \leq 2(c+1)^2/R^2$. Let $\boldsymbol{V}^*$ be the unique solution of $(I-A)\boldsymbol{V} = \boldsymbol{b}$. Since $\|\boldsymbol{V}^*\| \leq 1$ and $\boldsymbol{V}_s^* \geq R$ for every $s \in G$, it suffices to compute an approximate solution $\boldsymbol{U}^*$ such that

$$\frac{\|\boldsymbol{V}^* - \boldsymbol{U}^*\|}{\|\boldsymbol{V}^*\|} \quad \leq \quad \varepsilon \cdot R$$

By Theorem 29, we have that

$$\frac{\|\boldsymbol{V}^* - \boldsymbol{U}^*\|}{\|\boldsymbol{V}^*\|} \quad \leq \quad 4\tau\kappa \quad \leq \quad \frac{8\tau(c+1)^2}{R^2}$$

where $\tau$ is the relative error of $A$ and $\boldsymbol{b}$. Hence, it suffices to choose $\tau$ so that

$$\tau \quad \leq \quad \frac{\varepsilon R^3}{8(c+1)^2}$$

and compute all transition probabilities in $G$ up to the relative error $\tau$. Note that the approximation $A'$ of the matrix $A$ which is obtained in this way is still regular, because

$$\|A - A'\| \quad \leq \quad \tau \quad \leq \quad \frac{\varepsilon R^3}{8(c+1)^2} \quad < \quad \frac{R^2}{(c+1)^2} \quad \leq \quad \frac{1}{\|(I-A)^{-1}\|}$$

$\square$

Now we prove the divergence gap theorem. Some preliminary lemmata are needed.

**Lemma 31.** *Let $A$ be strongly connected and $t \geq 0$. Assume $[p\!\downarrow] > 0$ for all $p \in Q$. Let $c^{(0)} \geq 1$ and $p^{(0)} \in Q$ such that $\boldsymbol{v}_{p^{(0)}} = \boldsymbol{v}_{\max}$. Let $b \in \mathbb{N}$. Then*

$$\mathcal{P}\left(\exists i : c^{(i)} \geq b \wedge \forall j \leq i : c^{(j)} \geq 1 \,\big|\, Run(p^{(0)}(c^{(0)}))\right) \quad \geq \quad \frac{1}{b+1+|\boldsymbol{v}|}.$$

*Proof.* If $c^{(0)} \geq b$, the lemma holds trivially. So we can assume that $c^{(0)} < b$. For a run $w \in Run(p^{(0)}(c^{(0)}))$, we define a so-called *stopping time* $\tau$ as follows:

$$\tau := \inf\{i \in \mathbb{N}_0 \mid m^{(i)} \leq \boldsymbol{v}_{\max} \ \vee \ m^{(i)} \geq b + \boldsymbol{v}_{\max}\}$$

Note that $1 + \boldsymbol{v}_{\max} \leq m^{(0)} < b + \boldsymbol{v}_{\max}$, i.e., $\tau \geq 1$. Let $E$ denote the subset of runs in $Run(p^{(0)}(c^{(0)}))$ where $\tau < \infty$ and $m^{(\tau)} \geq b + \boldsymbol{v}_{\max}$; i.e., $E$ is the event that the martingale $m^{(i)}$ reaches a value of $b + \boldsymbol{v}_{\max}$ or higher without previously reaching a value of $\boldsymbol{v}_{\max}$ or lower. Similarly, let $D$ denote the subset of runs in $Run(p^{(0)}(c^{(0)}))$ such that the counter reaches a value of $b$ ore higher without previously hitting 0. To prove the lemma we need to show $\mathcal{P}(D) \geq 1/(b+1+|\boldsymbol{v}|)$. We will do that by showing that $D \supseteq E$ and $\mathcal{P}(E) \geq 1/(b+1+|\boldsymbol{v}|)$.

First we show $D \supseteq E$. Consider any run in $E$; i.e., $m^{(\tau)} \geq b + \boldsymbol{v}_{\max}$ and $m^{(i)} > \boldsymbol{v}_{\max}$ for all $i \leq \tau$. So, for all $i \leq \tau$ we have $m^{(i)} = c^{(i)} + \boldsymbol{v}_{p^{(i)}} - it > \boldsymbol{v}_{\max}$, implying $c^{(i)} > 0$. Similarly, $m^{(\tau)} = c^{(\tau)} + \boldsymbol{v}_{p^{(\tau)}} - \tau t \geq b + \boldsymbol{v}_{\max}$, implying $c^{(\tau)} \geq b$. Hence, the run is in $D$, implying $D \supseteq E$. Hence it remains to show $\mathcal{P}(E) \geq 1/(b+1+|\boldsymbol{v}|)$.

Next we argue that $\mathbb{E}\tau$ is finite: Since $[p\!\downarrow] > 0$ for all $p \in Q$, there are constants $k \in \mathbb{N}$ and $x \in (0,1]$ such that, given any configuration $p(c)$ with $p \in Q$ and $c \geq 1$, the probability of reaching in at most $k$ steps a configuration $q(c-1)$ for some $q \in Q$ is at least $x$. Since $A$ is strongly connected, it follows that there are constants $k' \in \mathbb{N}$ and

$x' \in (0, 1]$ such that, given any configuration $p(c)$ with $p \in Q$ and $c \geq 1$, the probability of reaching in at most $k'$ steps either a configuration with zero counter or a configuration $p(c - b)$ is at least $x'$. It follows that whenever $m^{(i)} < b + v_{\max}$ the probability that there is $j \leq k'$ with $m^{(i+j)} \leq v_{\max}$ is at least $x'$. Hence we have

$$\mathbb{E}\tau = \sum_{\ell=0}^{\infty} \mathcal{P}(\tau > \ell) \leq k' \sum_{\ell=0}^{\infty} \mathcal{P}(\tau > k'\ell) \leq k' \sum_{\ell=0}^{\infty} (1 - x')^{\ell} = k'/x' \; ;$$

i.e., $\mathbb{E}\tau$ is finite. Consequently, the *Optional Stopping Theorem* [22] is applicable and asserts

$$\mathbb{E}m^{(\tau)} = \mathbb{E}m^{(0)} = m^{(0)} \geq 1 + v_{\max} . \tag{12}$$

For runs in $E$ we have $m^{(\tau-1)} < b + v_{\max}$. Since the value of $m^{(i)}$ can increase by at most $1 + |v|$ in a single step, we have $m^{(\tau)} \leq b + v_{\max} + 1 + |v|$ for runs in $E$. It follows that

$$\mathbb{E}m^{(\tau)} \leq \mathcal{P}(E) \cdot (b + v_{\max} + 1 + |v|) + (1 - \mathcal{P}(E)) \cdot v_{\max}$$
$$= v_{\max} + \mathcal{P}(E) \cdot (b + 1 + |v|) .$$

Combining this inequality with (12) yields $\mathcal{P}(E) \geq 1/(b + 1 + |v|)$. This completes the proof. $\qquad\square$

Let $[p^{(0)}(c^{(0)})\!\downarrow\!]$ denote the probability that a run initiated in $p^{(0)}(c^{(0)})$ eventually reaches counter value zero. The following lemma gives an upper bound on $[p^{(0)}(c^{(0)})\!\downarrow\!]$.

**Lemma 32.** *Let A be strongly connected and $t > 0$. Let*

$$a := \exp\left(-\frac{t^2}{2(|v| + t + 1)^2}\right) .$$

*Note that $0 < a < 1$. Let $c^{(0)} \geq |v|$. Then we have*

$$[p^{(0)}(c^{(0)})\!\downarrow\!] \quad \leq \quad \frac{a^{c^{(0)}}}{1 - a} \qquad \text{for all } p^{(0)} \in Q.$$

*Moreover, if $c^{(0)} \geq 6(|v| + t + 1)^3/t^3$, then $[p^{(0)}(c^{(0)})\!\downarrow\!] \leq 1/2$ for all $p^{(0)} \in Q$.*

*Proof.* Define $H_i$ as the event that the counter reaches zero for the first time after exactly $i$ steps; i.e., $H_i := \{w \in Run(p^{(0)}(c^{(0)})) \mid c^{(i)} = 0 \ \wedge \ \forall 0 \leq j < i : c^{(j)} \geq 1\}$. We have $[p^{(0)}(c^{(0)})\!\downarrow\!] = \mathcal{P}(H_0 \cup H_1 \cup \cdots)$. Observe that $H_i = \emptyset$ for $i < c^{(0)}$, because in each step the counter value can decrease by at most 1. For all runs in $H_i$ we have $m^{(i)} = v_{p^{(i)}} - it$ and so

$$m^{(0)} - m^{(i)} = c^{(0)} + v_{p^{(0)}} - v_{p^{(i)}} + it .$$

It follows that

$$\mathcal{P}(H_i) = \mathcal{P}(H_i \ \wedge \ m^{(0)} - m^{(i)} = c^{(0)} + v_{p^{(0)}} - v_{p^{(i)}} + it)$$
$$\leq \mathcal{P}(m^{(0)} - m^{(i)} = c^{(0)} + v_{p^{(0)}} - v_{p^{(i)}} + it)$$
$$\leq \mathcal{P}(m^{(0)} - m^{(i)} \geq c^{(0)} - |v| + it)$$
$$\leq \mathcal{P}(m^{(0)} - m^{(i)} \geq it) \qquad\qquad (\text{as } c^{(0)} \geq |v|) .$$

In each step, the martingale value changes by at most $|v| + t + 1$. Hence Azuma's inequality (see [22]) asserts

$$\mathcal{P}(H_i) \leq \exp\left(-\frac{it^2}{2(|v| + t + 1)^2}\right) \qquad \text{(Azuma's inequality)}$$

$$= a^i.$$

It follows that

$$[p^{(0)}(c^{(0)})\!\downarrow] = \sum_{i=0}^{\infty} \mathcal{P}(H_i) = \sum_{i=c^{(0)}}^{\infty} \mathcal{P}(H_i) \qquad \text{(as } H_i = \emptyset \text{ for } i < c^{(0)})$$

$$\leq \sum_{i=c^{(0)}}^{\infty} a^i \qquad \text{(by the computation above)}$$

$$= a^{c^{(0)}}/(1 - a).$$

This proves the first statement. For the second statement, we need to find a condition on $c^{(0)}$ such that $[p^{(0)}(c^{(0)})\!\downarrow] \leq 1/2$. The condition provided by the first statement is equivalent to

$$c^{(0)} \geq \frac{\ln(1 - a) - \ln 2}{\ln a}.$$

Define $d := \frac{t^2}{2(|v|+t+1)^2}$. Note that $a = \exp(-d)$ and $0 < d < 1$. It is straightforward to verify that

$$\frac{\ln(1 - \exp(-d)) - \ln 2}{-d} \leq \frac{2}{d^{3/2}} \quad \text{for all } 0 < d < 1.$$

Since

$$\frac{2}{d^{3/2}} = \frac{2 \cdot 2^{3/2} \cdot (|v| + t + 1)^3}{t^3} \leq \frac{6(|v| + t + 1)^3}{t^3},$$

the second statement follows. $\qquad\qquad\square$

**Proposition 33.** *Let A be strongly connected and $t > 0$ and $[p\!\downarrow] > 0$ for all $p \in Q$. Let $p \in Q$ with $v_p = v_{\max}$. Then*

$$[p\!\uparrow] \geq \frac{t^3}{12(2|v| + 4)^3}.$$

*Proof.* Define $b$ as the smallest integer $b \geq 6(|v| + t + 1)^3/t^3$. By Lemma 31 we have

$$\mathcal{P}\left(\exists i : c^{(i)} \geq b \wedge \forall j \leq i : c^{(j)} \geq 1 \mid Run(p(1))\right) \geq \frac{1}{b + 1 + |v|}.$$

Since $0 < t \leq 1$, we have

$$b + 1 + |v| \leq 6(|v| + t + 2)^3/t^3 + 1 + |v| \leq 6(2|v| + 4)^3/t^3$$

and so

$$\mathcal{P}\left(\exists i : c^{(i)} \geq b \wedge \forall j \leq i : c^{(j)} \geq 1 \mid Run(p(1))\right) \geq \frac{t^3}{6(2|v| + 4)^3}.$$

Using the Markov property and Lemma 32 we obtain

$$[p\!\uparrow] \geq \frac{t^3}{12(2|v| + 4)^3}.$$

$$\square$$

Now let us drop the assumption that $A$ is strongly connected. Each BSCC $\mathscr{B}$ of $A$ induces a strongly connected pOC in which we have a trend $t$ and a potential $\boldsymbol{v}$.

**Theorem 20.** *Let $\mathscr{A} = (Q, \delta^{=0}, \delta^{>0}, P^{=0}, P^{>0})$ be a pOC and $X$ the underlying finite-state Markov chain of $\mathscr{A}$. Let $p \in Q$ such that $[p\uparrow] > 0$. Then there are two possibilities:*

1. *There is $q \in Q$ such that $[p, q] > 0$ and $[q\uparrow] = 1$. Hence, $[p\uparrow] \geq [p, q]$.*
2. *There is a BSCC $\mathscr{B}$ of $X$ and a state $q$ of $\mathscr{B}$ such that $[p, q] > 0$, $t > 0$, and $\boldsymbol{v}_q = \boldsymbol{v}_{\max}$ (here $t$ is the trend, $\boldsymbol{v}$ is the vector of Proposition 6, and $\boldsymbol{v}_{\max}$ is the maximal component of $\boldsymbol{v}$; all of these are considered in $\mathscr{B}$). Further,*

$$[p\uparrow] \quad \geq \quad [p, q] \cdot \frac{t^3}{12(2|\boldsymbol{v}| + 4)^3} .$$

*Proof.* Assume that $[q\uparrow] < 1$ for all $q \in Q$. Given a BSCC $\mathscr{B}$, denote by $R_{\mathscr{B}}$ the set of runs of $Run(p\uparrow)$ that reach $\mathscr{B}$. Almost all runs of $Run(p\uparrow)$ belong to $\bigcup_{\mathscr{B}} R_{\mathscr{B}}$. Moreover, using strong law of large numbers (see e.g. [22]) and results of [6] (in particular Lemma 19), one can show that almost every run of $Run(p\uparrow)$ belongs to some $R_{\mathscr{B}}$ satisfying $t > 0$. It follows that there is a BSCC $\mathscr{B}$ such that $t > 0$ and $\mathcal{P}(R_{\mathscr{B}}) > 0$. Now almost all runs of $R_{\mathscr{B}}$ either terminate, or visit all states of $\mathscr{B}$ infinitely many times. In particular, almost all runs of $R_{\mathscr{B}}$ reach a state $q$ satisfying $\boldsymbol{v}_q = \boldsymbol{v}_{\max}$, and thus $[p, q] > 0$. □