A short note on Simulation and Abstraction

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This short note is written in celebration of David Schmidt's sixtieth birthday. He has now been active in the program analysis research community for over thirty years and we have enjoyed many interactions with him. His work on characterising simulations between Kripke structures using Galois connections was particularly influential in our own work on using probabilistic abstract interpretation to study Larsen and Skou's notion of probabilistic bisimulation. We briefly review this work and discuss some recent applications of these ideas in a variety of different application areas.

1 Introduction

Since his earliest contributions on state transition machines for lambda calculus expressions [13], David Schmidt has been at the forefront of research in programming language theory, particularly program analysis.

His work on program analysis started through his collaboration with Neil Jones. No doubt partly inspired by Patrick and Radhia Cousot's work on abstract interpretation [2], he has made a study of various aspects of Galois connections. An early contribution was [7]. A later example, which was influential in our own work, was [15] where he shows how to characterise simulation relations using Galois Connections.

The early work of our group was also inspired by the Cousots [1]. Over the last fourteen years, we have been working on the analysis of probabilistic and quantitative programming languages and systems [12]. This has led to a framework called *probabilistic abstract interpretation* (PAI). Rather than using lattices and Galois Connections, PAI uses Hilbert Spaces and Moore-Penrose Pseudo Inverses. Inspired by [15], we have used PAI to characterise probabilistic bisimulation [8, 10]; we demonstrated the probabilistc analogue of Dave's earlier results which amounted to characterising Larsen and Skou's probabilistic bisimulation [5] using Moore-Penrose Pseudo Inverses. A major feature of our approach is that it becomes natural to introduce a notion of approximate bisimulation – this has proved to be very useful in studies of language-based security [9].

The author first met David Schmidt nearly thirty years ago. He visited Imperial College whilst developing the work in [7], his visit coincided with our own work on the ideas in [1]. We found a lot to talk about and it was the start of many further interactions. In addition to his scientific contributions, he has always given time to developing text books; his [14] was used to educate several generations of Imperial students in the principles of programming language design. The author has also worked with Dave on many programme committees; perhaps the high point was when they were co-General Chairs of POPL in 2001. Dave has a had a long and successful career and we wish him many more years. The author would like to add his congratulations to Dave on this important milestone.

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2 Simulation and Galois Connections

In [15], Dave addresses, *inter alia*, the question of characterising simulation relations between Kripke structures using Galois connections.

Recall that we define (L,α,γ,M) to be a *Galois connection* between the complete lattices $(L,\sqsubseteq)$ and $(M,\sqsubseteq)$ if and only if

$$\alpha : L \to M \text{ and } \gamma : M \to L \text{ are monotone functions}$$

that satisfy:

$$\begin{aligned} \gamma \circ \alpha &\sqsupseteq id_L \\ \alpha \circ \gamma &\sqsubseteq id_M \end{aligned}$$

For Kripke structures $C = \langle \Sigma_C, \to_C, \mathscr{I}_C \rangle$ and $A = \langle \Sigma_A, \to_A, \mathscr{I}_A \rangle$, a binary relation $\mathscr{R} \subseteq \Sigma_C \times \Sigma_A$ is a *simulation* of C by A ($C \lhd_{\mathscr{R}} A$), if for every $c \in \Sigma_C$, $a \in \Sigma_A$:

$$\text{if } c\,\mathscr{R}\,a \text{ and } c \to_C c' \text{ then } \exists a' \in \Sigma_A [a \to_A a' \text{ and } c'\,\mathscr{R}\,a']$$

In the framework studied in [15], the concrete Kripke structures are often infinite state structures representing programs and the abstract Kripke structures are some program analysis. In this setting, we should abstract sets of states of the concrete structure to a single state in the abstract structure. Given a Galois connection, $(\alpha : \mathscr{P}(\Sigma_C) \to \Sigma_A, \gamma : \Sigma_A \to \mathscr{P}(\Sigma_C))$, we can construct the relation, $\mathscr{R}_{(\alpha,\gamma)} \subseteq \mathscr{P}(\Sigma_C) \times \Sigma_A$:

$$S\,\mathscr{R}_{(\alpha,\gamma)} a \text{ if and only if } \alpha(S) \sqsubseteq_A a$$

This can be shown to be the suitable basis for a simulation relation.

Whilst [15] achieves much more than described here, it was these ideas that inspired our own work on probabilistic bisimulation (originally introduced by Larsen and Skou [5]) for reactive systems (also called fully probabilistic systems):

Definition 1 *A* probabilistic bisimulation *is an equivalence relation $\sim_b$ on states of a probabilistic transition system satisfying for all actions $a \in A$:*

$$p \sim_b q \text{ and } p \to_a \pi \Leftrightarrow q \to_a \rho \text{ and } \pi \sim_b \rho.$$

where π and ρ are distributions of states.

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3 Probabilistic Bisimulation and Moore-Penrose Pseudo Inverses

In [10, 8] we introduced an approximate version of bisimulation and confinement where the approximation can be used as a measure ε for the information leakage of the system under analysis. We represented probabilistic transition systems by linear operators, i.e. by their transition matrices $\mathbf{M}$. In the case of probabilistic programs and systems these matrices $\mathbf{M}$ are the usual well known stochastic matrices which are the generators of the corresponding Markov chains (for details see [10, 8]).

We then showed that two systems $\mathbf{M_1}$ and $\mathbf{M_2}$ are bisimilar if there exist simplified, or abstracted, versions of $\mathbf{M_1}$ and $\mathbf{M_2}$, represented by matrices $\mathbf{M_1^\#}$ and $\mathbf{M_2^\#}$, such that $\mathbf{M_1^\#} = \mathbf{M_2^\#}$. In the probabilistic abstract interpretation setting that we use, a bounded linear operator and its Moore-Penrose Pseudo Inverse are the analogue of the adjoined pair of monotonic functions in a Galois insertion. The abstract systems are obtained by *lumping* states, i.e. by identifying each concrete state s_i with a class C_j of states which are all behavioural equivalent to each other.

Concretely, we compute this via $n \times m$ matrices $\mathbf{K}$ (where n is the number of concrete states and m the number of abstract classes) with $\mathbf{K_{ij}} = \mathbf{1}$ iff $s_i \in C_j$ and 0 otherwise. We refer to such matrices which have exactly one entry 1 in each row while all other entries are 0 as *classification matrices*, and denote the set of all classification matrices by $\mathscr{K}$. The abstract systems are then given by $\mathbf{M_i^\#} = \mathbf{K_i^\dagger M_i K_i}$ with $\mathbf{K_i}$ some classification matrix and † constructing the so called *Moore-Penrose pseudo-inverse* – in the case of classification matrices $\mathbf{K}^\dagger$ can be constructed as the row-normalised transpose of $\mathbf{K}$.

The problem of showing that two systems $\mathbf{M_1}$ and $\mathbf{M_2}$ are behaviourally equivalent, i.e. are (probabilistically) bisimilar, is now translated into finding two classification matrices $\mathbf{K_i} \in \mathscr{K}$ such that

$$\mathbf{M_1^\#} = \mathbf{K_1^\dagger M_1 K_1} = \mathbf{K_2^\dagger M_2 K_2} = \mathbf{M_2^\#}.$$

In case that two systems are not bisimilar we can still define a quantity ε which describes how (non-)bisimilar the two systems are. This ε is formally defined in terms of the norm of a linear operator representing the partition induced by the 'minimal' bisimulation on the set of the states of a given system, i.e. the one minimising the observational difference between the system's components (see again [10] for further details, in particular regarding labeled probabilistic transition systems):

Definition 2 *Let* $\mathbf{M_1}$ *and* $\mathbf{M_2}$ *be the matrix representations of two probabilistic transition systems. We say that* $\mathbf{M_1}$ *and* $\mathbf{M_2}$ *are* ε-bisimilar, *denoted by* $\mathbf{M_1} \sim_{\mathbf{b}}^{\varepsilon} \mathbf{M_2}$, *iff*

$$\inf_{\mathbf{K_1},\mathbf{K_2} \in \mathscr{K}} \|\mathbf{K_1^\dagger M_1 K_1} - \mathbf{K_2^\dagger M_2 K_2}\| = \varepsilon$$

where $\|.\|$ *denotes an appropriate norm, e.g. the supremum norm* $\|.\|_\infty$.

In [10] we show that, when $\varepsilon = 0$ this gives the standard notion of probabilistic bisimulation.

4 Conclusion

This short note has sketched some early work by David which forms part of his deep study of the use of Galois connections and relations in program analysis. Our own work on characterising probabilistic bisimulation using probabilistic abstract interpretation has found a number of applications, including:

- the detection and removal of timing channels in probabilistic transition systems [11] – we study a concept called probabilistic time bisimilarity and use it to detect timing channels;
- the detection of sub-communities in social media [6] – we evaluate a number of algorithms including one using the notion of stability from [3] which effectively lumps nodes together if their mutual interactions are "stronger" than interactions outside the group; and
- the abstraction of stochastic and Bayesian games to provide decision support in cyber security [4] – where we hope to apply probabilistic abstract interpretation directly to the underlying probabilistic transition systems in the games, thereby developing a principled way of reducing the state spaces to achieve tractability of game solutions.

We look forward to discussing some of this work with David in the future but, in the meantime, reiterate our best wishes on this important anniversary.

5 Acknowledgements

Much of the work discussed above was done in collaboration with Alessandra Di Pierro and Herbert Wiklicky. More recently, I have enjoyed working on the application of these ideas to other areas with Erwan Le Martelot and Pasquale Malacaria.

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