

On well-quasi-ordering finite trees

By C. ST. J. A. NASH-WILLIAMS

King's College, Aberdeen

(Received 9 March 1963)

Abstract. A new and simple proof is given of the known theorem that, if T_1, T_2, \dots is an infinite sequence of finite trees, then there exist i and j such that $i < j$ and T_i is homeomorphic to a subtree of T_j .

A *quasi-ordered set* is a set Q on which a reflexive and transitive relation \leq is defined. Q and Q' will denote quasi-ordered sets. An infinite sequence q_1, q_2, \dots of elements of Q will be called *good* if there exist positive integers i, j such that $i < j$ and $q_i \leq q_j$; if not, the sequence will be called *bad*. A quasi-ordered set Q is *well-quasi-ordered* (*wqo*) if every infinite sequence of elements of Q is good.

A *graph* G consists (for our purposes) of a **finite** set $V(G)$ of elements called *vertices* of G and a subset $E(G)$ of the Cartesian product $V(G) \times V(G)$. The elements of $E(G)$ are called *edges* of G . If $(\xi, \eta) \in E(G)$, we call η a *successor* of ξ . If $\xi, \eta \in V(G)$, a $\xi\eta$ -*path* is a sequence ξ_0, \dots, ξ_n of vertices of G such that $\xi_0 = \xi$, $\xi_n = \eta$ and $(\xi_{i-1}, \xi_i) \in E(G)$ for $i = 1, \dots, n$. The sequence with sole term ξ is accepted as a $\xi\xi$ -path. If there exists a $\xi\eta$ -path, we say that η *follows* ξ . For the purposes of this paper, a *tree* is a graph T possessing a vertex $\rho(T)$ (called its *root*) such that, for every $\xi \in V(T)$, there exists a unique $\rho(T)\xi$ -path in T . The letter T (with or without dashes or suffixes) will always denote a tree. For the purposes of this paper, a *homeomorphism* of T into T' is a function $\phi: V(T) \rightarrow V(T')$ such that, for every $\xi \in V(T)$, the images under ϕ of the successors of ξ follow *distinct* successors of $\phi(\xi)$. The set of all trees will be quasi-ordered by the rule that $T \leq T'$ if and only if there exists a homeomorphism of T into T' . This paper presents a new and shorter proof of the following theorem of Kruskal (2).

THEOREM 1. *The set of all trees is wqo.*

If A, B are subsets of Q , a mapping $f: A \rightarrow B$ is *non-descending* if $a \leq f(a)$ for every $a \in A$. The class of finite subsets of Q will be denoted by SQ , and will be quasi-ordered by the rule that $A \leq B$ if and only if there exists a one-to-one non-descending mapping of A into B , where A, B denote members of SQ . The Cartesian product $Q \times Q'$ will be quasi-ordered by the rule that $(q_1, q'_1) \leq (q_2, q'_2)$ if and only if $q_1 \leq q_2$ and $q'_1 \leq q'_2$. The cardinal number of a set A will be denoted by $|A|$.

The following two lemmas are well known (see (1)), but for the reader's convenience their proofs are given here.

LEMMA 1. *If Q, Q' are wqo, then $Q \times Q'$ is wqo.*

Proof. We must prove an arbitrary infinite sequence $(q_1, q'_1), (q_2, q'_2), \dots$ of elements of $Q \times Q'$ to be good. Call q_m *terminal* if there is no $n > m$ such that $q_m \leq q_n$. The number

of q_m which are terminal must be finite, since otherwise they would form a bad subsequence of q_1, q_2, \dots . Therefore there is an N such that q_r is not terminal if $r > N$. We can therefore select a positive integer $f(1) > N$, then an $f(2) > f(1)$ such that $q_{f(1)} \leq q_{f(2)}$, then an $f(3) > f(2)$ such that $q_{f(2)} \leq q_{f(3)}$ and so on. Since Q' is wqo, there exist i, j such that $i < j$ and $q'_{f(i)} \leq q'_{f(j)}$, whence $(q_{f(i)}, q'_{f(i)}) \leq (q_{f(j)}, q'_{f(j)})$ and therefore our original sequence is good.

LEMMA 2. *If Q is wqo, then SQ is wqo.*

Proof. Assume that the lemma is false. Select an $A_1 \in SQ$ such that A_1 is the first term of a bad sequence of members of SQ and $|A_1|$ is as small as possible. Then select an A_2 such that A_1, A_2 (in that order) are the first two terms of a bad sequence of members of SQ and $|A_2|$ is as small as possible. Then select an A_3 such that A_1, A_2, A_3 (in that order) are the first three terms of a bad sequence of members of SQ and $|A_3|$ is as small as possible. Assuming the Axiom of Choice, this process yields a bad sequence A_1, A_2, A_3, \dots . Since this sequence is bad, no A_i is empty: therefore we can select an element a_i from each A_i . Let $B_i = A_i - \{a_i\}$. If there existed a bad sequence $B_{f(1)}, B_{f(2)}, \dots$ such that $f(1) \leq f(i)$ for all i , the sequence

$$A_1, A_2, \dots, A_{f(1)-1}, B_{f(1)}, B_{f(2)}, \dots$$

would be bad (since $A_i \leq B_j$ entails $A_i \leq A_j$ and is therefore impossible if $i < j$). Since this would contradict the definition of $A_{f(1)}$, there can be no bad sequence $B_{f(1)}, B_{f(2)}, \dots$ such that $f(1) \leq f(i)$ for all i . It follows that the class $(\mathfrak{B}, \text{say})$ of sets B_i is wqo, since any bad sequence of sets B_i would have a (bad) infinite subsequence in which no suffix was less than the first. Therefore, by Lemma 1, $Q \times \mathfrak{B}$ is wqo. Therefore there exist i, j such that $i < j$ and $(a_i, B_i) \leq (a_j, B_j)$, which implies that $A_i \leq A_j$ and thus contradicts the badness of A_1, A_2, \dots . This contradiction proves the lemma.

The *branch* of T at a vertex ξ is the tree R such that $V(R)$ is the set of those vertices of T which follow ξ and

$$E(R) = E(T) \cap (V(R) \times V(R)).$$

Proof of Theorem 1. Assume that the theorem is false. Select a tree T_1 such that T_1 is the first term of a bad sequence of trees and $|V(T_1)|$ is as small as possible. Then select a T_2 such that T_1, T_2 (in that order) are the first two terms of a bad sequence of trees and $|V(T_2)|$ is as small as possible. Continuing this process as in the proof of Lemma 2 yields a bad sequence T_1, T_2, \dots . Let B_i be the set of branches of T_i at the successors of its root, and let $B = B_1 \cup B_2 \cup \dots$. If there existed a bad sequence R_1, R_2, \dots such that $R_i \in B_{f(i)}$ and $f(1) \leq f(i)$ for every i , the sequence

$$T_1, T_2, \dots, T_{f(1)-1}, R_1, R_2, \dots$$

would be bad (since $T_i \leq R \in B_j$ entails $T_i \leq T_j$ and is therefore impossible if $i < j$). Since this would contradict the definition of $T_{f(1)}$, there can be no bad sequence R_1, R_2, \dots such that $R_i \in B_{f(i)}$ and $f(1) \leq f(i)$ for every i . Since any bad sequence of elements of B would have a bad subsequence of this form, it follows that no sequence of elements of B is bad. Therefore B is wqo and hence, by Lemma 2, SB is wqo. Therefore

$B_i \leq B_j$ for some pair i, j such that $i < j$. Therefore there is a one-to-one non-descending mapping $\phi: B_i \rightarrow B_j$. For each $R \in B_i$, $R \leq \phi(R)$ and so there exists a homeomorphism h_R of R into $\phi(R)$. A homeomorphism h of T_i into T_j may thus be defined by writing $h(\rho(T_i)) = \rho(T_j)$ and making h coincide with h_R on the vertices of each $R \in B_i$. Therefore $T_i \leq T_j$, which contradicts the badness of T_1, T_2, \dots and thus proves Theorem 1.

The Tree Theorem of (2) is stronger than Theorem 1 of the present paper, but the above proof of Theorem 1 can easily be adapted to prove the Tree Theorem by considering $X \times F(B)$ in place of SB (where X, F have the meanings stated in (2)). Because the necessary changes are easy to make, I have sacrificed this much generality in the interests of readability.

Note added 10 August 1963. It has been brought to the author's notice that Kruskal's proof of the Tree Theorem (2) anticipated a somewhat similar proof obtained independently by S. Tarkowski (*Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* 8 (1960), 39–41).

REFERENCES

- (1) HIGMAN, G. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc.* (3), 2 (1952), 326–336.
- (2) KRUSKAL, J. B. Well-quasi-ordering, the tree theorem, and Vázsonyi's conjecture. *Trans. American Math. Soc.* 95 (1960), 210–225.

An application of harmonic coordinates in general relativity

By R. H. BOYER

Department of Applied Mathematics, University of Liverpool

Communicated by L. ROSENHEAD

(Received 23 November 1962 and, in revised form, 17 April 1963)

We suppose the first derivatives of the components of a continuous metric tensor to exhibit jumps across a non-null hypersurface. We shall show that harmonic coordinates (see Fock (1), p. 175) lead to an automatic smoothing of the metric.

Let x^m ($m = 1, 2, 3, 4$) be coordinates in a certain region of Riemannian space-time. We write $F \in (C^N, C^{N+K})$ to mean that $F(x^m)$ has continuous N th partial derivatives throughout the region with its $(N+1)$ th, ..., $(N+K)$ th derivatives discontinuous only across a hypersurface $u(x^m) = 0$. Lichnerowicz ((2), p. 5) requires the metric tensor $g_{ij}(x^m)$ in 'admissible' coordinates to satisfy $g_{ij} \in (C^1, C^3)$ with respect to hypersurfaces for which $u \in (C^2, C^4)$. This state of affairs is preserved by a (C^2, C^4) coordinate transformation. Now suppose we leave the class of admissible coordinate systems by