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Ranks of fuzzy matrices. Applications in state reduction of fuzzy automata ☆

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Received 10 June 2015; received in revised form 29 May 2017; accepted 30 May 2017

Abstract

In this paper we consider different types of ranks of fuzzy matrices over residuated lattices. We investigate relations between ranks and prove that row rank, column rank and Schein rank of idempotent fuzzy matrices are equal. In particular, ranks and corresponding decompositions of fuzzy matrices representing fuzzy quasi-orders are studied in detail. We show that fuzzy matrix decomposition by ranks can be used in the state reduction of fuzzy automata. Moreover, we prove that using rank decomposition of fuzzy matrices improves results of any state reduction method based on merging indistinguishable states of fuzzy automata.

Keywords: Fuzzy matrices; Fuzzy matrix decomposition; Matrix subdecomposition; Fuzzy quasi-order; State reduction; Residuated lattice

1. Introduction

The development of fuzzy matrix theory started in 1971, when Zadeh introduced fuzzy equivalence relations [32]. Fuzzy matrices emerged as a mean of representing fuzzy equivalences and fuzzy relations between finite sets in general. Since then, together with fuzzy relations, fuzzy matrices appeared to be useful in many different contexts such as: fuzzy control, approximate reasoning, fuzzy cluster analysis, fuzzy neural networks, fuzzy decision making, fuzzy cognitive and fuzzy relational mapping, etc. However, independently of fuzzy relations, development of a fuzzy matrix theory began in 1980, when K. H. Kim and F. W. Roush introduced a general framework of the theory of fuzzy matrices as a generalization of the Boolean matrix theory and the theories of matrices over nonnegative real numbers, nonnegative integers, and over other types of semirings. They established basic concepts of linear algebra over fuzzy matrices: spaces of fuzzy vectors, row and column basis, Schein rank, row and column rank, regular fuzzy matrices, eigenvectors and eigenvalues, etc. Ever since, fuzzy matrices appeared to be the topic of many studies in different areas of mathematics: computation of generalized inverses of regular fuzzy matrices, as well as solvability

☆ Research supported by Ministry of Education and Science, Republic of Serbia, Grant No. 174013.
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http://dx.doi.org/10.1016/j.fss.2017.05.028
0165-0114/

Please cite this article in press as: A. Stamenković et al., Ranks of fuzzy matrices. Applications in state reduction of fuzzy automata, Fuzzy Sets Syst. (2017), http://dx.doi.org/10.1016/j.fss.2017.05.028

and solution computation of linear fuzzy matrix equations and other classical problems in linear algebra [1–3,15,16,24–26,29].

In most of the aforementioned papers, fuzzy matrices over the fuzzy algebra (also known as the Gödel structure) and over chain semirings were studied, whereas in the last ten years the researchers focused on fuzzy relations, and consequently on fuzzy matrices over more general structures – residuated lattices. In 1999, Bělohlávek started his work on formal concept analysis by introducing fuzzy Galois connections as a generalization of Galois connections from the point of view of fuzzy logic with membership values in residuated lattices [4]. Later, in [5] fixed points of fuzzy Galois connections were proven to be special cases of *maximal subdecompositions* of fuzzy matrices. Through a series of papers, while studying formal concept analysis, Bělohlávek and his coworkers provided significant results regarding properties of row and column spaces associated to fuzzy matrices [11,12], optimal decompositions of fuzzy matrices [8,9], and efficient computation of suboptimal decompositions of fuzzy matrices [14]. It is worth noting that Ćirić et al. [17–21,23,30,31] made a great contribution to the theory of fuzzy matrices over complete residuated lattices, and offered a comprehensive survey on the theory of fuzzy automata, particularly relevant to problems of state reduction, determinization and of bisimulation of fuzzy automata.

Basic notions and notation concerning subdecompositions and decompositions, row and column spaces, Schein ranks, and row and column ranks of fuzzy matrices over residuated lattices are introduced in Section 2. In Section 3, we investigate and enclose several properties of fuzzy matrices having equal column (row) spaces. We also offer some characterizations of ranks of fuzzy matrices and prove that for a given fuzzy matrix A and any of its ranks, there exists a corresponding decomposition. By using these results, we prove that all ranks of idempotent fuzzy matrices are equal, which lead us to the same result regarding fuzzy-quasi order matrix. It is well known (cf. [22,31]) that the cardinality of a set of columns of a fuzzy quasi-order matrix Q is equal to the cardinality of the set of rows of Q, and that number we denote $d(Q)$. We first show that the rank of fuzzy quasi-order matrices can be strictly smaller than $d(Q)$. Therefore, $d(Q)$ is a "good" candidate for estimating the rank of fuzzy quasi-order matrix Q, in the sense that it is efficiently computable and the corresponding $d(Q)$-decomposition of Q is also efficiently computable.

In Section 4, we continue to investigate ranks and various types of subdecompositions of fuzzy quasi-order matrices over residuated lattices. We first introduce maximal subdecompositions and maximal decompositions of matrices as the main tool for our further research. After examining basic properties of maximal subdecompositions and maximal decompositions, we prove that the cardinality of any set of columns (resp. rows) of a fuzzy quasi-order matrix Q, which is the spanning set of the column space $\mathscr{C}(Q)$ (resp. row space $\mathscr{R}(Q)$), is also good for estimation of its rank. We provide a procedure for testing whether a set of rows (resp. columns) of a fuzzy quasi-order matrix is the spanning set for the column space (resp. row space) of that matrix. That lead us to the conclusion that the best candidate of that kind is the cardinality of the spanning set of columns (rows), which is minimal w.r.t. set inclusion. However, we give an example of a fuzzy quasi-order matrix Q with many different minimal sets of columns (resp. rows) which are spanning sets of the column space (resp. row space) and have different sizes. Ultimately, we give the main result of this section – characterization of residuated lattices over which any fuzzy quasi-order matrix Q has rank $\rho(Q)$ equal to $d(Q)$. It is worth noting that all of the results in this section are proven using maximal subdecompositions and maximal decompositions of fuzzy matrices, and since they are both finite suprema of certain formal concepts, tools and results of Bělohlávek and his coworkers are intensely used.

In Section 5, we explain how decomposition of fuzzy matrices can improve results of state reduction methods based on merging undistinguishable states of fuzzy automata. The basic idea of reducing the number of states of non-deterministic automata is to compute and merge indistinguishable states. It resembles the minimization algorithm for deterministic automata, but is more complicated. In the fuzzy case, indistinguishability is modeled by crisp equivalences [28,30], fuzzy equivalences [17,18], and fuzzy quasi-orders [30,31]. In all the aforementioned papers matrices of fuzzy relations used in state reductions are fuzzy quasi-order matrices and are solutions to the *general system*. Moreover, the state reduction of a fuzzy automaton using a fuzzy quasi-order matrix Q produces a fuzzy automaton that is equivalent to the starting automaton and its number of states is equal to $d(Q)$. Our improvement is not based on computing solutions to the general system, but on using already computed solutions in a better way. Namely, if Q is a solution to the general system, we compute a k-decomposition (L, R) of Q and make a new fuzzy automaton called (L, R)-*transformation* of the starting fuzzy automaton. An (L, R)-transformation of a fuzzy automaton is equivalent to starting one and has k states. In the case when Q is a fuzzy quasi-order and $k < d(Q)$, provided (L, R)-transformation is efficiently computable, our method yields an improvement.

Please cite this article in press as: A. Stamenković et al., Ranks of fuzzy matrices. Applications in state reduction of fuzzy automata, Fuzzy Sets Syst. (2017), http://dx.doi.org/10.1016/j.fss.2017.05.028

2. Preliminaries

Throughout this paper residuated lattices will be used as structures of membership values.

2.1. Residuated lattices

A *residuated lattice* is an algebra $\mathscr{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ such that

(L1) $(L, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,
(L2) $(L, \otimes, 1)$ is a commutative monoid with the unit 1,
(L3) $\otimes$ and $\to$ form an *adjoint pair*, i.e., they satisfy the *adjunction property*: for all $x, y, z \in L$,

$$x \otimes y \leqslant z \;\Leftrightarrow\; x \leqslant y \to z. \tag{1}$$

If, in addition, $(L, \wedge, \vee, 0, 1)$ is a complete lattice, then $\mathscr{L}$ is called a *complete residuated lattice*.

Operations $\otimes$ (called *multiplication*) and $\to$ (called *residuum*) are intended for modeling conjunction and implication of the corresponding logical calculus, while supremum ($\bigvee$) and infimum ($\bigwedge$) are intended for modeling the existential and general quantifier, respectively. It is easy to verify that if $\mathscr{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ is a residuated lattice then $(L, \vee, \otimes, 0, 1)$ is a commutative, additively idempotent semiring. For other properties of residuated lattices one can refer to [6,13].

The most studied and applied examples of residuated lattices, defined on the real unit interval $[0, 1]$ with $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$, are the *Łukasiewicz structure* ($x \otimes y = \max(x + y - 1, 0)$, $x \to y = \min(1 - x + y, 1)$), the *Goguen* (*product*) *structure* ($x \otimes y = x \cdot y$, $x \to y = 1$ if $x \leqslant y$ and $= y/x$ otherwise) and the *Gödel structure* ($x \otimes y = \min(x, y)$, $x \to y = 1$ if $x \leqslant y$ and $= y$ otherwise). All of the aforementioned residuated lattices are complete. More generally, an algebra $([0, 1], \wedge, \vee, \otimes, \to, 0, 1)$ is a complete residuated lattice if and only if $\otimes$ is a left-continuous t-norm and the residuum is defined by $x \to y = \bigvee\{u \in [0, 1] \mid u \otimes x \leqslant y\}$. A special case of the latter algebras is the two-element Boolean algebra of classical logic with the support $\{0, 1\}$. The only adjoint pair on the two-element Boolean algebra consists of the classical conjunction and implication operations. This structure of truth values we call the *Boolean structure*.

2.2. Fuzzy matrices

Let A be a matrix. The entry (or the element) in the ith row and jth column of A will be denoted by a_{ij}. A *fuzzy matrix* is a matrix whose all entries belong to some residuated lattice. For the sake of simplicity, in this paper fuzzy matrices will be called matrices. However, if necessary, a matrix whose entries belong to a (complete) residuated lattice $\mathscr{L}$ will be called a *matrix over a (complete) residuated lattice* $\mathscr{L}$.

Let $\mathscr{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated lattice and let A be a matrix over $\mathscr{L}$. If matrix A has all elements equal to 0 it will be called a *zero* matrix, and will be denoted by $A = 0$. The *equality* of matrices A and B of the same size $m \times n$ is defined as usual, i.e. $A = B$ if and only if $a_{ij} = b_{ij}$, for every $i \in \{1, 2, \dots, m\}$, $j \in \{1, 2, \dots, n\}$. The *transpose* of an $m \times n$ matrix A is a matrix $A^\top$ of type $n \times m$ defined by $a^\top_{ij} = a_{ji}$, for every $i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, m\}$. For matrices A and B of size $m \times n$ and an arbitrary $\lambda \in L$, matrices $C = A \vee B$ and $D = \lambda \otimes A$ are $m \times n$ matrices defined by

$$c_{ij} = a_{ij} \vee b_{ij}, \quad d_{ij} = \lambda \otimes a_{ij}, \tag{2}$$

for every $i \in \{1, 2, \dots, m\}$, $j \in \{1, 2, \dots, n\}$.

For a matrix A of type $m \times n$ and a matrix B, of type $n \times k$, their *product* is an $m \times k$ matrix denoted by $A \circ B$ and if $C = A \circ B$, then c_{ij} is defined by

$$c_{ij} = \bigvee_{l=1}^{n} a_{ik} \otimes b_{kj}, \tag{3}$$

for every $i \in \{1, 2, \dots, m\}$, $j \in \{1, 2, \dots, k\}$. If $C = A \circ B$, matrices A and B will be called *factors* of the matrix C, and the matrix pair (A, B) will be called a *decomposition* or a *factorization* of C. A decomposition (A, B) of C, where A is an $m \times k$ matrix and B is a $k \times n$ matrix, is called a *k-decomposition* of a matrix C. If $k \in \mathbb{N}$ is given, then any k-decomposition of C, if one exists, is called a *decomposition corresponding to k*. For a given $n \in \mathbb{N}$, the *identity matrix I_n* is an $n \times n$ matrix whose elements d_{ij} are defined as

$$d_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}, \tag{4}$$

for every $i, j \in \{1, 2, \dots, n\}$. For every matrix A of type $m \times n$, and identity matrices I_m and I_n, the following stands $I_m \circ A = A \circ I_n = A$.

Let us denote by $V_{m\times n}$ the set of all $m \times n$ matrices over a residuated lattice $\mathscr{L}$. Matrices of type of $1 \times n$ will be called *row vectors* and matrices of type $n \times 1$ will be called *column vectors*. Matrix A will be called a *vector* if A is either a row or a column vector. For a matrix A of type $m \times n$, its ith row iA is a row vector and its jth column Aj is a column vector defined by

$$(iA)_j = a_{ij} \quad \text{and} \quad (Aj)_i = a_{ij},$$

for every $i \in \{1, 2, \dots, m\}$ and every $j \in \{1, 2, \dots, n\}$. Let $Row(A)$ denote the set of all rows of a matrix A, and let $Col(A)$ denotes the set of all columns of A. For an arbitrary $R \subseteq Row(A)$, if $R = \{r_1, r_2, \dots, r_k\}$, by A_R will be denoted an $k \times n$ matrix whose ith row is a row vector r_i, for every $i \in \{1, 2, \dots, k\}$. Dually, for any $C \subseteq Col(A)$, if $C = \{c_1, c_2, \dots, c_k\}$, by A^C will be denoted an $m \times k$ matrix whose jth column is a column vector c_j, for every $j \in \{1, 2, \dots, k\}$. In the sequel, we assume that there is an order by indices on the set C defined in the following way: If $c_i = Ai'$ and $c_j = Aj'$ for some i', $j' \in \{1, 2, \dots, m\}$, then

$$i \leqslant j \quad \Leftrightarrow \quad i' \leqslant j',$$

for every $i, j \in \{1, 2, \dots, k\}$. Otherwise, definition of A^C would not be correct, since there would be many different matrices A^C, depending on the order of elements in the set C. The order on indices of the set B will be imposed in the same way.

Linear combination of row vectors a_i of type $1 \times n$, where $i \in \{1, 2, \dots, m\}$, is a row vector $b = \bigvee_{i=1}^{m} \lambda_i \otimes a_i$, where $\lambda_i \in L$, for every $i \in \{1, 2, \dots, m\}$, i.e.

$$b = \lambda \circ A, \tag{5}$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, and A is a matrix of type $m \times n$ whose rows are row vectors a_i, $i \in \{1, 2, \dots, m\}$. Dually, a linear combination of column vectors a_i, $i \in \{1, 2, \dots, m\}$ is a column vector $b = \bigvee_{i=1}^{m} \lambda_i \otimes a_i$, where $\lambda_i \in S$, for every $i \in \{1, 2, \dots, m\}$, or equivalently

$$b = A \circ \lambda, \tag{6}$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)^{\mathrm{T}}$, and A is an $n \times m$ matrix, whose columns are columns vectors a_i, $i \in \{1, 2, \dots, m\}$. For a set R of row (column) vectors, the set of all linear combinations of vectors from R is called the *span* of R, denoted $\langle R\rangle$. If $\langle R\rangle = W$, then R is called a *spanning set* for W. The *row space* $\mathscr{R}(A)$ of an $m \times n$ matrix A is the span of the set of all rows of that matrix. The *row rank* $\rho_{\mathrm{r}}(A)$ is the smallest possible size (or cardinality) of a spanning set for the row space. The *column space* $\mathscr{C}(A)$ and the *column rank* $\rho_{\mathrm{c}}(A)$ are defined in a dual fashion. Let us also write

$$S_{\mathrm{c}}(A) = \{C \subseteq Col(A) \mid \langle C\rangle = \mathscr{C}(A)\} \quad \text{and} \quad S_{\mathrm{r}}(A) = \{R \subseteq Row(A) \mid \langle R\rangle = \mathscr{R}(A)\}, \tag{7}$$

or equivalently,

$$S_{\mathrm{c}}(A) = \{C \subseteq Col(A) \mid \mathscr{C}(A^C) = \mathscr{C}(A)\} \quad \text{and} \quad S_{\mathrm{r}}(A) = \{R \subseteq Row(A) \mid \mathscr{R}(A_R) = \mathscr{R}(A)\}. \tag{8}$$

In addition, by $S_{\mathrm{c}}^{\min}(A)$ (resp. $S_{\mathrm{r}}^{\min}(A)$) we denote the set of all minimal elements (w.r.t $\subseteq$ relation) of the set $S_{\mathrm{c}}(A)$ (resp. $S_{\mathrm{r}}(A)$).

Rank of a nonzero matrix A, denoted $\rho(A)$, is the smallest integer k such that there exists a k-decomposition of A. Rank of a zero matrix is equal to 0. Note that, not only in the Boolean matrix theory, but also in the theory of fuzzy matrices [9,12,25], the above defined number $\rho(A)$ is known as the *Schein rank* of a matrix. In addition, in [15], the number $\rho(A)$ is called the *fuzzy rank* of A. Denote by $d_{\mathrm{r}}(A)$ the cardinality of the set $Row(A)$, and as $d_{\mathrm{c}}(A)$ the cardinality of the set $Col(A)$, i.e. $d_{\mathrm{r}}(A) = |Row(A)|$ and $d_{\mathrm{c}} = |Col(A)|$. It is easy to verify that for every matrix A, the following stands:

$$\rho(A) \leqslant \rho_{\mathrm{r}}(A) \leqslant d_{\mathrm{r}}(A) \quad \text{and} \quad \rho(A) \leqslant \rho_{\mathrm{c}}(A) \leqslant d_{\mathrm{c}}(A) \tag{9}$$

A matrix A is a *rank*-1 matrix, if $\rho(A) = 1$. The following well-known result was proven in [25] for matrices over a commutative semiring and in [2] for matrices over an arbitrary semiring.

Lemma 2.1. *Rank of a nonzero matrix A is the minimal number of rank-*1 *matrices whose supremum is equal to A.*

The following result is a synthesis of Proposition 2.4 and Theorem 2.5 of [25] and it will be useful in the sequel.

Lemma 2.2. *For arbitrary matrices A, B, the following statements hold:*

(i) $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ *if and only if there exists a matrix X such that $B = X \circ A$.*
(ii) $\mathscr{C}(B) \subseteq \mathscr{C}(A)$ *if and only if there exists a matrix Y such that $B = A \circ Y$.*
(iii) $\mathscr{R}(A) = \mathscr{R}(B)$ *if and only if there exist X, Y such that $B = X \circ A$ and $A = Y \circ B$.*
(iv) $\mathscr{C}(A) = \mathscr{C}(B)$ *if and only if there exist X, Y such that $B = A \circ X$ and $A = B \circ Y$.*

Let us note that the previous lemma, although proven for matrices over a commutative semiring, stands also for matrices over an arbitrary semiring.

If matrices A and B of the same type are over some residuated lattice, one can define a *partial order* $A \leqslant B$ by: $A \leqslant B$ if and only if $a_{ij} \leqslant b_{ij}$, for every $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$. In that case, the matrix product is isotone in both arguments, i.e.

$$A \leqslant B \quad \Rightarrow \quad A \circ C \leqslant B \circ C, \tag{10}$$

$$A \leqslant B \quad \Rightarrow \quad C \circ A \leqslant C \circ B, \tag{11}$$

is satisfied for arbitrary matrices A, B, C of compatible sizes. Moreover, a pair of matrices (A, B) is called a *subdecomposition* of a matrix C if $A \circ B \leqslant C$. If, moreover, A is an $m \times k$ matrix and B is a $k \times n$ matrix, then the subdecomposition (A, B) is called a k*-subdecomposition* of C.

Let $\{A^\alpha\}_{\alpha \in Y}$ be an arbitrary family of matrices over a complete residuated lattice. Supremum (join) $\bigvee_{\alpha \in Y} A^\alpha$ of the family $\{A^\alpha\}_{\alpha \in Y}$ of $m \times n$ matrices is the matrix B of the same type defined by

$$b_{ij} = \bigvee_{\alpha \in Y} a_{ij}^\alpha,$$

for every $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$. Moreover,

$$C \circ \Big(\bigvee_{\alpha \in Y} A^\alpha\Big) = \bigvee_{\alpha \in Y} C \circ A^\alpha \quad \text{and} \quad \Big(\bigvee_{\alpha \in Y} A^\alpha\Big) \circ D = \bigvee_{\alpha \in Y} A^\alpha \circ D \tag{12}$$

are satisfied for arbitrary matrices C and D of appropriate dimensions.

3. Ranks and rank decompositions of matrices

In this section, we give some basic properties of ranks and the corresponding decompositions of matrices over residuated lattice. In particular, we investigate matrices having equal column (row) spaces in order to give certain conditions under whose ranks of matrices are equal. Ultimately, we compare ranks of idempotent matrices with the focus on fuzzy quasi-order matrices.

The following result deals with the equality of column and row spaces of matrices:

Lemma 3.1. *For arbitrary matrices A, B, the following statements hold:*

(i) *If matrix A is a factor of the matrix B, and B is a factor of A, then $\rho(A) = \rho(B)$.*
(ii) $\mathscr{R}(A) = \mathscr{R}(B)$ *implies $\rho_{\mathrm{r}}(A) = \rho_{\mathrm{r}}(B)$ and $\rho(A) = \rho(B)$.*
(iii) $\mathscr{C}(A) = \mathscr{C}(B)$ *implies $\rho_{\mathrm{c}}(A) = \rho_{\mathrm{c}}(B)$ and $\rho(A) = \rho(B)$.*

Proof. (i) Let A be a factor of a matrix B, and let B be a factor of A. Then, there exist matrices C and D, of adequate dimensions, such that $A = B \circ C$ and $B = A \circ D$. If $\rho(A) = k$, then there exists a k-decomposition (A', A'') of a matrix A. Now, we have:

$$B = A \circ D = (A' \circ A'') \circ D = A' \circ (A'' \circ D).$$

Since $(A', A'' \circ D)$ is a k-decomposition of the matrix B, by definition of the rank of matrices, we conclude $\rho(B) \leqslant \rho(A)$. Similarly, we can prove that $\rho(A) \leqslant \rho(B)$, and therefore $\rho(A) = \rho(B)$. Note here that for all other cases when A is a factor of B is a factor of A, the fact that $\rho(A) = \rho(B)$ can be proven analogously.
(ii) If $\mathscr{R}(A) = \mathscr{R}(B)$ then those spaces have the same spanning set and thus $\rho_{\mathrm{r}}(A) = \rho_{\mathrm{r}}(B)$. Further, by (iii) of Lemma 2.2 we have that A is the factor of B and B is the factor of A, which by (i) yields $\rho(A) = \rho(B)$.

The proof of the statement (iii) can be deduced in a dual fashion as proof for (ii). □

The next result characterizes Shein rank, column rank, and row rank of matrices.

Theorem 3.2. *For an arbitrary matrix $A \in V_{m\times n}$, the following stands:*

(i) $\rho_{\mathrm{r}}(A) = k$ *if and only if k is the least integer such that there exist a $k \times n$ matrix B, an $m \times k$ matrix C, and a $k \times m$ matrix D for which*

$$A = C \circ B \quad \text{and} \quad B = D \circ A.$$

(ii) $\rho_{\mathrm{c}}(A) = k$ *if and only if k is the smallest integer such that there exist an $m \times k$ matrix B, a $k \times n$ matrix C, and an $n \times k$ matrix D for which is*

$$A = B \circ C \quad \text{and} \quad B = A \circ D.$$

(iii) *If $\rho(A) = k$, and (B, C) is a k-decomposition of the matrix A, then*

$$\rho(B) = \rho(C) = k \quad \text{and} \quad \rho_{\mathrm{c}}(B) = \rho_{\mathrm{r}}(C) = k.$$

Proof. (i) For an arbitrary finite spanning set R for $\mathscr{R}(A)$, let k be the number of its elements, and let B be a $k \times n$ matrix whose rows are row vectors from R. Since $\mathscr{R}(B) = \mathscr{R}(A)$, by Lemma 2.2 we have that $A = C \circ B$ and $B = D \circ A$, for some matrices C and D. Now, if $\rho_{\mathrm{r}}(A) = k$, by the previous observation we have that a spanning set R for $\mathscr{R}(A)$ with k elements has the smallest size if and only if k is the smallest integer such that for some $k \times n$ matrix B, $m \times k$ matrix C, and $k \times m$ matrix D, the following is true: $A = C \circ B$ and $B = D \circ A$.
(ii) Can be deduced dually to (i).
(iii) Let $\rho(A) = k$, and let (B, C) be a k-decomposition of a matrix A. Obviously $\rho(B) \leqslant k$. Thus, suppose $\rho(B) = k' < k$. Then, there exists (B', B''), a k'-decomposition of the matrix B. From the previous we obtain $A = B' \circ (B'' \circ C)$, which lead to the contradiction $\rho(A) \leqslant k' < k$. Thus $\rho(B) = k$.

Now, since $\rho_{\mathrm{c}}(B)$ is less or equal to k, which is the number of columns of B, and since from (9) we have $\rho(B) \leqslant \rho_{\mathrm{c}}(B) = k$, it stands that $\rho_{\mathrm{c}}(B) = k$. In a similar way one can prove that $\rho(C) = \rho_{\mathrm{r}}(C) = k$. □

As a consequence, we have the following result:

Theorem 3.3. *If A is an arbitrary matrix and $k \in \{\rho(A), \rho_{\mathrm{r}}(A), \rho_{\mathrm{c}}(A), d_{\mathrm{r}}(A), d_{\mathrm{c}}(A)\}$, then there exists a k-decomposition of A.*

Proof. By the definition of a rank of matrices we have that there exists a $\rho(A)$-decomposition of an arbitrary matrix A. If $k \in \{\rho_{\mathrm{r}}(A), \rho_{\mathrm{c}}(A)\}$, the existence of a k-decomposition of A is an immediate consequence of the previous theorem.

Let $m \times n$ be the size of a matrix A and $C = Col(A)$. Obviously, the matrix A^C is an $m \times d_{\mathrm{c}}(A)$ matrix. Since every column of the matrix A is a linear combination of columns of A^C, we have that

$$A^C \circ B = A,$$

for some matrix B, and thus (A^C, B) is a d_{c}-decomposition of A.

The existence of a $d_{\mathrm{r}}(A)$-decomposition of A can be proven similarly. □

A matrix $A \in V_{m\times m}$ is *idempotent* if $A^2 = A$. Idempotent matrices will play a crucial role in this section, especially in studying ranks of regular matrices. Namely, we have the following result.

Theorem 3.4. *Let A be an arbitrary idempotent matrix. Then*

$$\rho(A) = \rho_{\mathrm{c}}(A) = \rho_{\mathrm{r}}(A). \tag{13}$$

Proof. Let A be an idempotent matrix of type $m \times m$, let $\rho(A) = k$, and let $A = B \circ C$, where B is an $m \times k$ and C is a $k \times m$ matrix. Put $B' = A \circ B$ and $C' = C \circ A$. First we have the following:

$$B' \circ C' = (A \circ B) \circ (C \circ A) = A \circ (B \circ C) \circ A = A^3 = A.$$

By (iii) of Theorem 3.2 we have $\rho(B') = \rho_{\mathrm{c}}(B') = \rho(A)$. Further, since $A \circ B' = B'$, by Lemma 2.2 and Lemma 3.1, along with previous observation, we have $\rho_{\mathrm{c}}(A) = \rho_{\mathrm{c}}(B')$, and thus $\rho_{\mathrm{c}}(A) = \rho(A)$. Similarly, we prove $\rho_{\mathrm{r}}(A) = \rho(A)$. □

It is worth noting that there is a more general context in which all the previous results are true: matrices over a semiring.

A square matrix $Q \in V_{m\times m}$ is a matrix *representing a fuzzy quasi-order* relation, or simply a *fuzzy quasi-order matrix* if

$$Q^2 \leqslant Q \quad \text{and} \quad I_m \leqslant Q, \tag{14}$$

where I is the $m \times m$ identity matrix.

Theorem 3.5. *Let Q be an arbitrary fuzzy quasi-order matrix. Then*

$$\rho(Q) = \rho_{\mathrm{c}}(Q) = \rho_{\mathrm{r}}(Q). \tag{15}$$

Proof. The proof follows immediately from the previous result and the fact that Q is an idempotent matrix. □

One of the basic properties of fuzzy quasi-order matrices, proven in [31], is the following one:

$$iQ = jQ \quad \Leftrightarrow \quad Qi = Qj \quad \Leftrightarrow \quad q_{ij} = q_{ji} = 1, \tag{16}$$

for all $i, j \in \{1, 2, \dots, m\}$, so $d_{\mathrm{r}}(Q) = d_{\mathrm{c}}(Q)$. For the sake of simplicity, the number $d_{\mathrm{r}}(Q) = d_{\mathrm{c}}(Q)$ will be denoted by $d(Q)$. For other properties of fuzzy quasi-order matrices we refer to [22,31].

The next example shows that for some fuzzy quasi-order matrices, their rank is strictly smaller than $d(Q)$.

Example 3.6. Let S be an arbitrary finite set, and let $\mathscr{L}$ be an algebra $\mathscr{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$, where $L = P(S)$ is a powerset of S, $\wedge = \cap$ is the intersection and $\vee = \cup$ is the union of subsets of S, $0 = \emptyset$, $1 = S$, and the operation $\to$ is defined by: $a \to b = S \setminus (a \setminus b)$, for every $a, b \in P(S)$. It is easy to check (see [6,13]) that an algebra $\mathscr{L}$ is a complete residuated lattice. Let $S = \{p, q, r\}$ and let

$$0 = \emptyset, \quad a = \{p\}, \quad b = \{q\}, \quad c = \{r\}, \quad d = \{p, q\}, \quad e = \{p, r\} \quad f = \{q, r\}, \quad 1 = S.$$

For a fuzzy quasi-order matrix Q over $\mathscr{L}$, we have

$$Q = \begin{bmatrix} 1 & e & a & c \\ e & 1 & d & f \\ a & d & 1 & b \\ c & f & b & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ b & e \\ f & a \\ d & c \end{bmatrix} \circ \begin{bmatrix} 0 & b & f & d \\ 1 & e & a & c \end{bmatrix}.$$

Therefore, $\rho(A) \leqslant 2$ while $d(A) = 4$. Moreover, let us note that $\rho(Q) = 2$, since rank-1 fuzzy quasi-order matrices are matrices whose all elements are equal to $1 \in L$.

4. Ranks, subdecompositions and decompositions of fuzzy quasi-order matrices

In this section, we further investigate ranks, subdecompositions and decompositions of fuzzy quasi-order matrices over residuated lattices.

For that purpose, we introduce operators $R_A : V_{m\times k} \to V_{k\times n}$ and $L_A : V_{k\times n} \to V_{m\times k}$, where $k \in \mathbb{N}$ and A is an $m \times n$ matrix, in the following way:

$$\text{if } R_A(B) = C \text{ and } L_A(Q) = P \quad \text{then} \quad c_{ij} = \bigwedge_{l=1}^{m} b_{li} \to a_{lj} \quad \text{and} \quad p_{ij} = \bigwedge_{l=1}^{n} q_{jl} \to a_{il}, \tag{17}$$

for arbitrary $B \in V_{m\times k}$, and $Q \in V_{k\times n}$. By definition (17), for arbitrary matrices A, B and C of appropriate types, both $(B, R_A(B))$ and $(L_A(Q), Q)$ are subdecompositions of the matrix A. In addition, it is easy to prove using antitonicity in the first argument of the $\to$ operation, that both operators L_A and R_A are antitonic.

Operators R_A and L_A play a crucial role in solving linear and weakly linear systems of fuzzy relational equations and inequalities [19–23]. Namely, it is well-known that a matrix $R_A(B)$ (resp. $L_A(Q)$) is the greatest solution to the linear matrix inequality

$$B \circ X \leqslant A, \quad (\text{resp. } X \circ Q \leqslant A), \tag{18}$$

where X is an unknown matrix. Let us also recall that a fuzzy quasi-order matrix Q of type $m \times m$ is the greatest solution to the inequalities

$$X \circ Q \leqslant Q, \quad Q \circ X \leqslant Q, \tag{19}$$

where X is an unknown matrix, which implies

$$q_{ij} = \bigwedge_{k=1}^{m} q_{ki} \to q_{kj} = \bigwedge_{k=1}^{m} q_{jk} \to q_{ik}, \tag{20}$$

for every $i, j \in \{1, 2, \dots, m\}$. Therefore $L_Q(Q) = R_Q(Q) = Q$. The following result will be useful in the sequel:

Theorem 4.1. *Let Q be an arbitrary $m \times m$ fuzzy quasi-order matrix, $C \subseteq Col(Q)$ and $R \subseteq Row(Q)$. If $C = \{Qj \mid j \in Y\}$ and $R = \{iQ \mid i \in Y\}$, for some $Y \subseteq \{1, 2, \dots, m\}$, then the following statements are equivalent:*

(i) $C \in S_c(Q)$,
(ii) $R \in S_r(Q)$,
(iii) $Q^C \circ Q_R = Q$.

Proof. (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iii): If $C \in S_c(Q)$, then by (8) we have $\mathscr{C}(Q^C) = \mathscr{C}(Q)$, which by Lemma 3.1 implies $Q = Q^C \circ B$, for some matrix B. Without loss of generality, assume that $B = R_C(Q^C)$. Since for every $i \in Y$, there exists $i' \in \{1, 2, \dots m\}$ such that ith column of Q^C is equal to the i'th column of Q, by (17) and (20), we obtain

$$\begin{aligned}(iB)_j = b_{ij} &= \bigwedge_{k=1}^{m} (Q^C)_{ki} \to q_{kj} = \bigwedge_{k=1}^{m} (Q^C i)_k \to (Qj)_k = \bigwedge_{k=1}^{m} (Qi')_k \to (Qj)_k = \bigwedge_{k=1}^{m} q_{ki'} \to q_{kj} \\ &= q_{i'j} = (i'Q)_j,\end{aligned}$$

for every $i \in I$ and $j \in \{1, 2, \dots, k\}$. Therefore $iB = i'Q$, i.e. the ith row of a matrix B is equal to the i'th row of Q. Consequently $B = Q_R$, and thus $Q^C \circ Q_R = Q$. Similarly one can prove (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii): Conversely, let $C = \{Qj \mid j \in Y\}$ and $R = \{iQ \mid i \in Y\}$, where $Y \subseteq \{1, 2, \dots, m\}$. If $Q^C \circ Q_R = Q$, then by Lemma 2.2., we have $\mathscr{C}(Q) \subseteq \mathscr{C}(Q^C)$ and since $C \subseteq Col(Q)$ we have $\mathscr{C}(Q^C) \subseteq \mathscr{C}(Q)$, and therefore $C \in S_c(Q)$. Similarly, we prove $R \in S_r(Q)$. □

In the case when $C = Col(Q)$ the corresponding set of rows of Q is $R = Row(Q)$, and Q^C (resp. Q_R) will be denoted by Q^{l} (resp. Q^{r}). By the previous theorem, we have that $(Q^{\mathrm{l}}, Q^{\mathrm{r}})$ is a decomposition of the matrix Q. A decomposition $(Q^{\mathrm{l}}, Q^{\mathrm{r}})$ is a $d(Q)$-decomposition of Q, since $d(Q) = |Col(Q)| = |Row(Q)|$. The next example shows how the previous result can be used as a tool to test whether a set C of columns of a fuzzy quasi-order matrix Q is a spanning set for $\mathscr{C}(Q)$.

Example 4.2. Let $\mathscr{L}$ be an algebra $\mathscr{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$, as in Example 3.6. For a fuzzy quasi-order matrix Q over $\mathscr{L}$ we have

$$Q = \begin{bmatrix} 1 & e & a & c \\ e & 1 & d & f \\ a & d & 1 & b \\ c & f & b & 1 \end{bmatrix} = \begin{bmatrix} 1 & a & c \\ e & d & f \\ a & 1 & b \\ c & b & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & e & a & c \\ a & d & 1 & b \\ c & f & b & 1 \end{bmatrix}.$$

Hence, a set $C = \{Q1, Q3, Q4\}$ of columns of a matrix Q is a spanning set for $\mathscr{C}(Q)$, i.e. $C \in S_c(Q)$. Moreover, in this way it is easy to check that beside C and $Col(Q)$, there are no other sets of columns of Q that are spanning sets for $\mathscr{C}(Q)$. Therefore, $C \in S_c^{\min}(Q)$, i.e. $C = \{Q1, Q3, Q4\}$ is the only minimal spanning set for $\mathscr{C}(Q)$ w.r.t. set inclusion. Let us recall (Example 3.6) that $\rho(A) = 2$, which means that, in general, for a fuzzy quasi-order matrix Q one cannot find the spanning set of $\mathscr{C}(Q)$ (resp. $\mathscr{R}(Q)$) among the columns (resp. rows) of Q whose cardinality is equal to $\rho(Q)$.

However, a fuzzy quasi-order matrix does not generally have a unique minimal spanning set for the column space. Namely, there are matrices for which $S_c^{\min}$ contains many different elements with different cardinalities. For a fuzzy quasi-order matrix R, we have

$$R = \begin{bmatrix} 1 & 0 & a & c & b \\ 0 & 1 & 0 & d & a \\ a & 0 & 1 & 0 & c \\ c & d & 0 & 1 & a \\ b & a & c & a & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ a & 0 & 1 \\ c & d & 0 \\ b & a & c \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & a & c & b \\ 0 & 1 & 0 & d & a \\ a & 0 & 1 & 0 & c \end{bmatrix} = \begin{bmatrix} 0 & a & c & b \\ 1 & 0 & d & a \\ 0 & 1 & 0 & c \\ d & 0 & 1 & a \\ a & c & a & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 & 0 & d & a \\ a & 0 & 1 & 0 & c \\ c & d & 0 & 1 & a \\ b & a & c & a & 1 \end{bmatrix}.$$

Using Theorem 4.1, it can be easily verified that $C_1, C_2 \in S_c^{\min}(R)$, where $C_1 = \{R1, R2, R3\}$ and $C_2 = \{R2, R3, R4, R5\}$. Moreover, three is the smallest number in the set $\{|C| \mid C \in S_c^{\min}(R)\}$, since neither $\{R1, R4\}$ nor $\{R1, R5\}$ is a spanning set for $\mathscr{C}(R)$.

Subdecomposition (B', C') of a matrix A *contains* another subdecomposition (B, C) of A if $B \leqslant B'$ and $C \leqslant C'$. Subdecomposition (B, C) of A is *maximal* if for any subdecomposition (P, Q) of A that contains (B, C) it must be that $B = P$ and $C = Q$, i.e. if the following stands:

$$B \leqslant P \ \text{ and } \ C \leqslant Q \quad \Rightarrow \quad B = P \ \text{ and } \ C = Q.$$

Decomposition (B, C) of the matrix A is a *maximal decomposition* of A if (B, C) is a maximal subdecomposition of A.

Let us recall that *formal concepts* used in a formal concept analysis (cf. [4–14]) are instances of maximal subdecompositions. Namely, a pair (B, C) is a formal concept of A iff it is a maximal 1-subdecomposition of A. In addition, in these papers maximal decompositions of a matrix A, are called *suboptimal decompositions*, while $\rho(A)$-decompositions of A are called *optimal decompositions* of A.

The next lemma summarizes some of the results proven in Theorem 5 of [9]:

Lemma 4.3. *For an $m \times n$ matrix A, the following stand:*

(i) *(B, C) is a maximal 1-subdecomposition of A if and only if $C = R_A(B)$ and $B = L_A(C)$.*
(ii) *For any k-decomposition (B, C) of A there exist a k-decomposition (B', C') of A containing (B, C), such that $(B'i, iC')$ is a maximal subdecomposition of A, for every $i \in \{1, 2, \dots, k\}$.*

The next result gives some characterizations of maximal subdecompositions of matrices:

Theorem 4.4. *Let (B, C) be a k-subdecomposition of an $m \times n$ matrix A. The following statements are equivalent:*

(i) *(B, C) is a maximal subdecomposition of A,*
(ii) *$C = R_A(B)$ and $B = L_A(C)$,*
(iii) *(Bi, iC) is a maximal subdecomposition of A, for every $i \in \{1, 2, \dots, k\}$.*

Proof. (i) ⇒ (iii): If (B, C) is a maximal k-subdecomposition of A, then (B, C) is a maximal k-decomposition of $A' = B \circ C$. By (ii) of Lemma 4.3., there exists a k-decomposition (B',C') of A', such that $(B, C) \leqslant (B', C')$, and $(B'i, iC')$ is a maximal subdecomposition of A, for every $i \in \{1, 2, \dots, k\}$. It is easy to verify that (B', C') is a maximal subdecomposition of A, by which we obtain $(B, C) = (B', C')$. Thus, (Bi, iC) is a maximal subdecomposition of A, for every $i \in \{1, 2, \dots, k\}$.

(iii) ⇒ (ii): Let (Bi, iC) be a maximal 1-subdecomposition of A, for every $i \in \{1, 2, \dots, k\}$. Then by (i) of Lemma 4.3, we have that $iC = R_A(Bi)$ and $Bi = L_A(iC)$, for every $i \in \{1, 2, \dots, k\}$, which leads to the following:

$$(iC)_j = \bigwedge_{k=1}^{m} (Bi)_k \to a_{kj} \quad \text{and} \quad (Bi)_l = \bigwedge_{k=1}^{n} a_{lk} \to (iC)_k, \quad \text{or equivalently}$$

$$c_{ij} = \bigwedge_{k=1}^{m} b_{ki} \to a_{kj} \quad \text{and} \quad b_{li} = \bigwedge_{k=1}^{n} a_{lk} \to c_{ik},$$

for every $i \in \{1, 2, \dots, k\}$, $j \in \{1, 2, \dots, n\}$, $l \in \{1, 2, \dots, m\}$. Thus, $C = R_A(B)$ and $B = L_A(C)$.

(ii) ⇒ (i): Let $B = L_A(C)$ and $C = R_A(B)$. By definition of the operator L_A and (18), we have the following:

$$B \circ C = L(C) \circ C \leqslant A,$$

that is, (B, C) is a subdecomposition of A. Let (P, Q) be a subdecomposition of A such that $B \leqslant P$ and $C \leqslant Q$. Since L_A and R_A are antitonic operators, we have:

$$Q \leqslant R_A(P) \leqslant R_A(B) = C \quad \text{and} \quad P \leqslant L_A(Q) \leqslant L_A(C) = B,$$

which yields $Q = C$ and $P = B$. □

The following result will be very useful in our further research:

Theorem 4.5. *For any subdecomposition (B, C) of a matrix A, there exists a maximal subdecomposition (B', C') of A that contains (B, C).*

Proof. By (ii) of Lemma 4.3., for a subdecomposition (B, C) of a matrix A there exists a decomposition (B', C') of A containing (B, C), such that $(B'i, iC')$ is a maximal subdecomposition of A, for every $i \in \{1, 2, \dots, k\}$. By (iii) of the previous theorem, (B', C') is a maximal decomposition of A. □

Note that for a subdecomposition (B, C) of a matrix A, pairs of matrices $(L_A(R_A(B)), R_A(B))$ and $(L_A(C), R_A(L_A(C)))$ are subdecompositions of A that are maximal ones containing (B, C). Further, we have the following:

Lemma 4.6. *Let Q be a fuzzy quasi-order matrix over a complete residuated lattice $\mathscr{L}$, and let I be the identity matrix of an appropriate size. The following assertions hold:*

(i) *If (A, B) is a maximal subdecomposition of Q, then $Q \circ A = A$ and $B \circ Q = B$.*
(ii) *If $\{(A^\alpha, B^\alpha)\}_{\alpha \in Y}$ is a family of maximal subdecompositions of Q, then $\bigvee_{\alpha \in Y} A^\alpha \circ B^\alpha = Q$ if and only if $I \leqslant \bigvee_{\alpha \in Y} A^\alpha \circ B^\alpha$.*

Proof. (i) Let (A, B) be a maximal subdecomposition of Q. Then $(Q \circ A, B \circ Q)$ is a subdecomposition of Q, since

$$(Q \circ A) \circ (B \circ Q) = Q \circ (A \circ B) \circ Q \leqslant Q^3 = Q.$$

However, by (10), (11) and the definition of fuzzy quasi-order matrices, we have $A = I \circ A \leqslant Q \circ A$ and $B = B \circ I \leqslant B \circ Q$, and therefore $A = Q \circ A$ and $B = B \circ Q$.
(ii) Suppose $I \leqslant \bigvee_{\alpha \in In} A^\alpha \circ B^\alpha$. Accordingly, by (12) and (i) of this lemma, we have

$$Q = Q \circ I \leqslant Q \circ \Big(\bigvee_{\alpha \in Y} A^\alpha \circ B^\alpha\Big) = \bigvee_{\alpha \in Y} Q \circ A^\alpha \circ B^\alpha = \bigvee_{\alpha \in Y} A^\alpha \circ B^\alpha \leqslant Q.$$

The converse is obvious. □

Let us note that the previous lemma can be proven for fuzzy quasi-order matrices over an arbitrary residuated lattice, provided the index set Y is finite. Prior to give the main result of the section, let us give the following:

Lemma 4.7. *Let $A = (a_1, a_2, \dots, a_m)^\top$ be a column vector and $B = (b_1, b_2, \dots, b_m)$ be a row vector such that (A, B) is a maximal sudecomposition of an $m \times m$ fuzzy quasi-order matrix Q. Then, $A = Qi$ and $B = iQ$ if and only if $a_j = b_j = 1$, for some $j \in \{1, 2, \dots, m\}$.*

Proof. Let $a_j = b_j = 1$, for some $j \in \{1, 2, \dots, m\}$. We have the following

$$a_i = a_i \otimes 1 = a_i \otimes b_j \leqslant^* q_{ij} = (Qj)_i,$$

for every $i \in \{1, 2, \dots, m\}$, i.e. $A \leqslant Qj$. Let us note that the inequality marked with $*$ follows from the fact that $A \circ B \leqslant Q \Leftrightarrow a_i \otimes b_j \leqslant q_{ij}$, for every $i, j \in \{1, 2, \dots, m\}$. On the other hand, from Lemma 4.6, we have $Q \circ A = A$ and therefore

$$(Qj)_i = q_{ij} = q_{ij} \otimes 1 = q_{ij} \otimes a_j \leqslant (Q \circ A)_i = a_i,$$

for every $i \in \{1, 2, \dots, m\}$. From the previous, we conclude the converse inequality $Qj \leqslant A$. Thus, $A = Qj$. Similarly, we prove $B = jQ$.

If, $A = Qi$ and $B = iQ$, for some $i \in \{1, 2, \dots, m\}$, then $a_i = (Qi)_i = q_{ii} = 1$. Similarly, we obtain $b_i = 1$. □

One of the main goals in this section is to establish necessary and sufficient conditions for a residuated lattice $\mathscr{L}$ under which for every fuzzy quasi-order matrix Q over $\mathscr{L}$, the identity $d(Q) = \rho(Q)$ stands. With this purpose, observe the following condition for a bounded lattice $(L, \wedge, \vee, 0, 1)$:

$$a \vee b = 1 \quad \Rightarrow \quad a = 1 \quad \text{or} \quad b = 1, \tag{21}$$

for every $a, b \in L$. Note that all linearly ordered lattices, such as those used in the definition of residuated lattices taking truth values in the real unit interval $[0, 1]$, satisfy the condition (21). Next result gives a characterization of lattices satisfying (21).

Theorem 4.8. *Let $\mathscr{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated lattice. A bounded lattice $(L, \wedge, \vee, 0, 1)$ satisfies the condition (21) if and only if for every fuzzy quasi-order matrix Q over $\mathscr{L}$, the identity $\rho(Q) = d(Q)$ stands.*

Proof. Let Q be an $m \times m$ fuzzy quasi-order matrix over a residuated lattice $\mathscr{L}$, and let a lattice $(L, \wedge, \vee, 0, 1)$ satisfies (21). Also, let $\rho(Q) = k$ and $Y = \{1, 2, \dots, k\}$. By Lemma 2.1 and the fact that a rank-1 matrix is a product of a column and a row matrix, we have

$$Q = \bigvee_{\alpha=1}^{k} A^\alpha \circ B^\alpha,$$

where $A^\alpha = (a_1^\alpha, a_2^\alpha, \dots, a_m^\alpha)$ and $B^\alpha = (b_1^\alpha, b_2^\alpha, \dots, b_m^\alpha)$. According to Theorem 4.5, we can assume that (A_α, B_α) are maximal subdecompositions of Q, for all $\alpha \in Y$.

Suppose $a_i^\alpha < 1$ or $b_i^\alpha < 1$ for some $\alpha \in Y$ and every $i \in \{1, 2, \dots, m\}$. That assumption and the fact that $I_m \leqslant \bigvee_{\alpha=1}^{k} A^\alpha \circ B^\alpha$ (Lemma 4.6), yield the following

$$\Big(\bigvee_{\beta \in Y \setminus \{\alpha\}} a_i^\beta \otimes b_i^\beta\Big) \vee (a_i^\alpha \otimes b_i^\alpha) = q_{ii} = 1, \quad \text{where} \ a_i^\alpha \otimes b_i^\alpha < 1,$$

for every $i \in \{1, 2, \dots, m\}$, which by (21) implies $\bigvee_{\beta \in Y \setminus \{\alpha\}} a_i^\beta \otimes b_i^\beta = 1$, for every $i \in \{1, 2, \dots, m\}$. In other words, according to a Lemma 4.6 we have $Q = \bigvee_{\beta \in Y \setminus \{\alpha\}} A^\beta \circ B^\beta$, and thus $\rho(Q) \leqslant k - 1$, which contradicts our first assumption. In conclusion, for every $\alpha \in Y$, there is an $i \in \{1, 2, \dots, m\}$ such that $a_i^\alpha = b_i^\alpha = 1$, i.e. by Lemma 4.7, for every $\alpha \in Y$ there is an $i \in \{1, 2, \dots, m\}$ such that $A^\alpha = Qi$ and $B^\alpha = iQ$, that is, A^α is a column and B^α is the corresponding row of Q, for every $\alpha \in Y$.

Let Qi be an arbitrary column of Q. Since $q_{ii} = 1$, by (21) we have that $a_i^\alpha \otimes b_i^\alpha = 1$, for some $\alpha \in Y$, that is, $a_i^\alpha = b_i^\alpha = 1$, for some $\alpha \in Y$. By Lemma 4.7, we conclude that $A^\alpha = Qi$ and $B^\alpha = iQ$. Thus, the set $\{A^\alpha | \alpha \in Y\}$ is the set of all columns of Q, and since its cardinality is $k = \rho(Q)$ we have $d(Q) = \rho(Q)$.

Let us assume that $(L, \wedge, \vee, 0, 1)$ is a bounded lattice that does not satisfy (21). Then, there exist $a, b \in L$ such that $a \vee b = 1$, and $a, b < 1$. Matrix

$$Q = \begin{bmatrix} 1 & b & 1 \\ a & 1 & 1 \\ a & b & 1 \end{bmatrix},$$

is a fuzzy quasi-order matrix with $d(Q) = 3$. However, since

$$Q = \begin{bmatrix} 1 & b \\ a & 1 \\ a & b \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

we conclude that $\rho(Q) \leqslant 2$. □

An immediate consequence of the previous result is the following theorem:

Theorem 4.9. *Let $\mathscr{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated lattice, and Q a fuzzy quasi-order matrix over $\mathscr{L}$. If the lattice $(L, \wedge, \vee, 0, 1)$ satisfies the condition (21) then $S_c^{\min}(Q) = S_c(Q) = \{Col(Q)\}$ (resp. $S_r^{\min}(Q) = S_r(Q) = \{Row(Q)\}$).*

Proof. For an arbitrary $C \in S_c(Q)$ from the previous theorem and the fact that

$$\rho(Q) \leqslant |C| \leqslant d(Q),$$

we have that $|C| = \rho(Q) = d(Q)$. Consequently, $C \in S_c^{\min}(Q)$ and $C = Col(Q)$, which proves the theorem. □

5. Applications of matrix decompositions in the state reduction of fuzzy automata

Let $\mathscr{L}$ be a residuated lattice. A *fuzzy automaton over* $\mathscr{L}$ is defined as a five-tuple $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$, where A is the *set of states* and X is the *input alphabet*, $\delta^A : A \times X \times A \to L$ is called the *fuzzy transition function*, $\sigma^A \in L^A$ is the fuzzy set of *initial states*, and $\tau^A \in L^A$ is the fuzzy set of *terminal states*. The input alphabet X will be always finite, but for methodological reasons one usually allows the set of states A to be infinite. A fuzzy automaton whose set of states is finite is called a *fuzzy finite automaton*. Since in this paper we deal with fuzzy finite automata, we will call them simply fuzzy automata.

The number of states a fuzzy automaton $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ is denoted by $|\mathscr{A}|$. In the case when $\mathscr{A}$ has m states, i.e. $|\mathscr{A}| = m$, let us denote the set of states of $\mathscr{A}$ by $A = \{1, 2, \dots, m\}$. For any $x \in X$, and any $i, j \in A$, define an $m \times m$ matrix δ_x^A on A by

$$(\delta_x^A)_{ij} = \delta^A(i, x, j). \tag{22}$$

Matrix δ_x will be called the *transition matrix* determined by x. The fuzzy set of initial states σ^A and a fuzzy set of terminal states τ^A are defined as $1 \times m$ and an $m \times 1$ matrices in the following way

$$\sigma_i^A = \sigma^A(i), \quad \tau_i^A = \tau^A(i), \tag{23}$$

for every $i \in A$.

Please cite this article in press as: A. Stamenković et al., Ranks of fuzzy matrices. Applications in state reduction of fuzzy automata, Fuzzy Sets Syst. (2017), http://dx.doi.org/10.1016/j.fss.2017.05.028

A *fuzzy language* in X^* over $\mathscr{L}$, or briefly a *fuzzy language*, is any fuzzy subset of a free monoid X^*, i.e. any mapping from X^* into L. A *fuzzy language recognized by a fuzzy automaton* $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$, denoted $[\![\mathscr{A}]\!]$, is a fuzzy language defined by

$$\begin{aligned} [\![\mathscr{A}]\!](\varepsilon) &= \sigma^A \circ \tau^A, \\ [\![\mathscr{A}]\!](u) &= \sigma^A \circ \delta^A_{x_1} \circ \delta^A_{x_2} \circ \cdots \circ \delta^A_{x_n} \circ \tau^A, \end{aligned} \tag{24}$$

for any $u = x_1x_2 \dots x_n \in X^+$, where $x_1, x_2, \dots, x_n \in X$ and ε is an empty word. Fuzzy automata $\mathscr{A}$ and $\mathscr{B}$ are *equivalent* if they recognize the same fuzzy language, i.e. if:

$$[\![\mathscr{A}]\!](u) = [\![\mathscr{B}]\!](u), \tag{25}$$

for every $u \in X^*$.

Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton with m states. Let L be an $m \times k$ matrix and R be a $k \times m$ matrix, for an arbitrary $k \in \mathbb{N}$. A fuzzy automaton $\mathscr{B} = (B, X, \delta^B, \sigma^B, \tau^B)$ is an (L, R)-*transformation* of a fuzzy automaton $\mathscr{A}$ if $B = \{1, 2, \dots, k\}$ is the set of states of $\mathscr{B}$ and matrices δ^B_x, δ^B_y, σ^B and τ^B are defined by:

$$\begin{aligned} \delta^B_x &= R \circ \delta^A_x \circ L, \\ \sigma^B &= \sigma^A \circ L, \\ \tau^B &= R \circ \tau^A. \end{aligned} \tag{26}$$

Theorem 5.1. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton with m states and let $\mathscr{B} = (B, X, \delta^B, \sigma^B, \tau^B)$ be an (L, R)*-transformation *of $\mathscr{A}$ for a $k \times m$ matrix L and an $m \times k$ matrix R, where $k \in \mathbb{N}$. Automata $\mathscr{A}$ and $\mathscr{B}$ are equivalent if and only if matrix $Q = L \circ R$ is a solution to the system of matrix equations*

$$\begin{aligned} &\sigma^A \circ \tau^A = \sigma^A \circ U \circ \tau^A, \\ &\sigma^A \circ \delta^A_{x_1} \circ \delta^A_{x_2} \circ \cdots \circ \delta^A_{x_n} \circ \tau^A = \sigma^A \circ U \circ \delta^A_{x_1} \circ U \circ \delta^A_{x_2} \circ U \circ \cdots \circ U \circ \delta^A_{x_n} \circ U \circ \tau^A, \end{aligned} \tag{27}$$

for all $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n \in X$, where U is an unknown $m \times m$ matrix.

Proof. The proof is evident and follows directly from (24), (25) and (26). □

In the sequel the system (27) will be called the *general system*.

State reduction of a fuzzy automaton $\mathscr{A}$ by using arbitrary (L, R)-transformation of $\mathscr{A}$ is performed as follows: For an arbitrary matrix Q, which is a solution to the general system, one has to find as good as possible k-decomposition (L, R) of Q, in the sense that k has to be as small as possible. Afterwards, one can simply build an (L, R)-transformation of $\mathscr{A}$ by computing all its transition matrices. An (L, R)-transformation of $\mathscr{A}$ is the resulting fuzzy automaton which is, by Theorem 5.1, equivalent to $\mathscr{A}$ and has k states. Using (L, R)-transformations in the state reduction gives two big opportunities: The first one is that for the state reduction one can use an arbitrary matrix Q that is a solution to the general system. The second one is that the state reduction using (L, R)-transformations gives better results. Namely, if Q is a fuzzy quasi-order matrix used for the state reduction of a fuzzy automaton $\mathscr{A}$, in the case when $\rho(Q) < d(Q)$, the resulting fuzzy automaton obtained by using (L, R)-transformations has $\rho(Q)$ states, i.e. it is smaller than the one obtained using any of the methods described in [17,18,28,30,31]. The reasons are as follows: Matrices used in all reduction methods are fuzzy quasi-order matrices. The resulting fuzzy automaton (factor fuzzy automaton or aftereset fuzzy automaton) computed by any of the reduction methods is isomorphic to $(Q^{\mathrm{l}}, Q^{\mathrm{r}})$-transformation of the staring fuzzy automaton and therefore has $d(Q)$ states.

The most widely studied methods of the state reduction of fuzzy automata use left and right invariant fuzzy equivalences and fuzzy quasi-orders [17,18,30,31]. The following example shows how results of the state reduction of a particular fuzzy automaton performed using left invariant fuzzy quasi-order matrix can be improved using (L, R)-transformation.

Example 5.2. Let $\mathscr{L}$ be a residuated lattice $\mathscr{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ defined as in Example 3.6, where $L = P(S)$ is the powerset of $S = \{p, q\}$, and

$$0 = \emptyset, \quad a = \{p\}, \quad b = \{q\} \quad \text{and} \quad 1 = S.$$

Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton over $\mathscr{L}$, where $A = \{1, 2, 3\}$, $X = \{x, y\}$, and δ_x^A, δ_y^A, σ^A and τ^A are given by matrices

$$\delta_x^A = \begin{bmatrix} a & 1 & 1 \\ 0 & 0 & 0 \\ a & b & 1 \end{bmatrix}, \quad \delta_y^A = \begin{bmatrix} a & b & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \sigma^A = \begin{bmatrix} a & b & 1 \end{bmatrix}, \quad \tau^A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

It is easy to verify that the greatest left invariant fuzzy quasi-order relation on $\mathscr{A}$ which is a solution to a fuzzy relation equation

$$\sigma^A \circ X = X,$$

is represented by a fuzzy quasi-order matrix Q given by

$$Q = \begin{bmatrix} 1 & b & 1 \\ a & 1 & 1 \\ a & b & 1 \end{bmatrix}.$$

Since $d(Q) = 3$, both resulting fuzzy automata – the aftereset fuzzy automaton $\mathscr{A}/Q$ and the forest fuzzy automaton $\mathscr{A} \setminus Q$ w.r.t. $\mathscr{A}$ have three states, as the starting fuzzy automaton $\mathscr{A}$. Therefore, the state reduction of $\mathscr{A}$ by using Q does not give satisfying results.

However, since

$$Q = L \circ R, \quad \text{where} \quad L = \begin{bmatrix} 1 & b \\ a & 1 \\ a & b \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

a fuzzy automaton $\mathscr{B} = (B, X, \delta^B, \sigma^B, \tau^B)$ which is an (L, R)-transformation of $\mathscr{A}$ has two states, and if $B = \{1, 2\}$ then δ_x^B, δ_y^B, σ^B and τ^B are matrices

$$\delta_x^B = R \circ \delta_x^A \circ L = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} a & 1 & 1 \\ 0 & 0 & 0 \\ a & b & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & b \\ a & 1 \\ a & b \end{bmatrix} = \begin{bmatrix} a & 1 \\ a & b \end{bmatrix},$$

$$\delta_y^B = R \circ \delta_x^A \circ L = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} a & b & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & b \\ a & 1 \\ a & b \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix},$$

$$\sigma^B = \sigma^A \circ L = \begin{bmatrix} a & b & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & b \\ a & 1 \\ a & b \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix},$$

$$\tau^A = R \circ \tau^A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Let $\mathscr{A}$ be a fuzzy automaton and let A be a matrix that is a solution to the system general system. According to Theorem 3.3, there exists a $\rho(A)$-decomposition of a matrix A, by which one can achieve the best possible reduction of $\mathscr{A}$ using (L, R)-transformations. However, in general, finding rank of A and computing the corresponding decomposition is a problem [14,27]. Even computing the set $C \in S_{\mathrm{c}}^{\min}$ (or $R \in S_{\mathrm{r}}^{\min}$) that has the smallest possible size is computationally expensive, since one has to check all the subsets of columns (rows) of A whether they are spanning sets of $\mathscr{C}(A)$ ($\mathscr{R}(A)$) in order to find the smallest one. Therefore, one has to efficiently compute as small as possible $k \in \mathbb{N}$ such that k-decomposition of A exists and can be also efficiently computed.

Please cite this article in press as: A. Stamenković et al., Ranks of fuzzy matrices. Applications in state reduction of fuzzy automata, Fuzzy Sets Syst. (2017), http://dx.doi.org/10.1016/j.fss.2017.05.028

Theorem 5.3. *Let A be an arbitrary matrix. Then, the following stands*

(a) *For an arbitrary $C \in S_c^{\min}(A)$, if k is the cardinality of C then $\rho_c(A) \leqslant k \leqslant d_c(A)$, and there exists a k-decomposition of A.*
(b) *There exists $C \in S_c^{\min}(A)$ such that $|C| = k$ and the corresponding k-decomposition of A can be efficiently computed.*

Proof. Let $m \times n$ be the size of the matrix A.
(a) Let k be the cardinality of the set $C \in S_c^{\min}(A)$, and let $C = \{c_1, c_2, \ldots, c_k\}$. Set C is a spanning set for $\mathscr{C}(A)$, and therefore we have $\rho_c(A) \leqslant k \leqslant d_c(A)$. Matrix A^C is of the size $m \times k$ and $\mathscr{C}(A^C) = \mathscr{C}(A)$, which by Lemma 3.1 leads to $A = A^C \circ B$, for some matrix B. Therefore, (A^C, B) is the k-decomposition of A.
(b) Let us describe a simple and efficient procedure for computing $C \in S_c^{\min}(A)$ and the corresponding k-decomposition of A, where k is the cardinality of C.

Algorithm 5.4 *(computing $C \in S_c^{\min}(A)$ and a corresponding k-decomposition of A).* Let $(C_i)_{i\in\mathbb{N}}$ be a sequence of subsets of $Col(A)$ inductively obtained as follows:

(A1) $C_1 = Col(A)$.
(A2) Assume C_i is computed. If there is a column vector $c \in C_i$ such that $\langle C_i \setminus \{c\}\rangle = \mathscr{C}(A)$, then $C_{i+1} = C_i \setminus \{c\}$. Otherwise $C = C_i$, $k = n - i + 1$, and the procedure finishes.

By construction, it is obvious that $C \in S_c^{\min}(A)$. Let us note that the procedure finishes after a finite number of basic steps (A2). When the sequence $(C_i)_{i\in\mathbb{N}}$ stabilizes, i.e. when C is computed we build an $m \times k$ matrix A^C, where $k = |C|$. Since $A^C \circ R_A(A^C) = A$, the pair $(A^C, R_A(A^C))$ is a k-decomposition of a matrix A.

In each of the basic steps of Algorithm 5.4 one has to test whether there exists $c \in C_i$ such that $C_i \setminus \{c\}$ is a spanning set for $\mathscr{C}(A)$. In other words, for each set C_i and every $c \in C_i$, one has to check whether $\langle C_i \setminus \{c\}\rangle = \mathscr{C}(A)$. That can be done using the following equivalence:

$$\langle C_i \setminus \{c\}\rangle = \mathscr{C}(A) \quad \Leftrightarrow \quad A^{D_i} \circ R_A(D_i) = A, \tag{28}$$

where $D_i = C_i \setminus \{c\}$. If $A^{D_i} \circ R_A(D_i) = A$, then $C_{i+1} = D_i$, otherwise $C = C_i$ and the sequence stabilizes. It is worth noting that our computing procedure is efficient since there are at most $n - 1$ basic steps, in each of which one has to perform at most $|C_i| - 1 \leqslant n - 1$ checks if the equality (28) stands, while computing $R_A(L)$ for arbitrary matrices A and L along with verifying equalities of matrices can be efficiently done. □

It has to be noted that the number k computed by the previous procedure depends on the column vectors chosen to be deleted from the set C_i, in each ith basic step. Therefore, its cardinality does not have to be minimum of the set $\{|C| \mid C \in S_c^{\min}(A)\}$. Let us also note that a k-decomposition $(A^C, R_A(A^C))$ of A, which is the output of our algorithm, can be used to find maximal k-decomposition of A, by computing $L_C(R_A(A^C))$. In that case, a k-decomposition $(L_C(R_A(A^C)), R_A(A^C))$ of A would be the maximal one. In addition, in the case A is a matrix over a linearly ordered lattice, a faster algorithm for computing maximal decompositions of matrices is presented in [14]. Clearly, the corresponding results regarding rows, row spaces and row ranks can be deduced dually.

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