SOME TREELIKE OBJECTS

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'I count a lot of things that there's no need to count,' Cameron said. 'Just because that's the way I am. But I count all the things that need to be counted,'

Richard Brautigan, 'The Hawkline Monster: a Gothic Western' (Picador, London, 1976, p. 104).

1. Introduction

THE Catalan numbers 1, 2, 5, 14, 42, ... $\left(\text{the general term is } \binom{2n}{n}\Big/(n+1)\right)$ form one of the most celebrated integer sequences in combinatorial mathematics. One of their striking features is the number of superficially different situations in which they occur; these include binary trees, dissections of polygons, ballot problems, characters of symmetric groups, and bracketings of nonassociative products.

There are, however, many closely related sequences, some of which have received attention too. In Neil Sloane's "Handbook of Integer Sequences" [**21**] we find sequences 123 (boron trees), 298 (Wedderburn–Etherington numbers), 339 (dissections of a polygon), 466 (series-parallel networks), 558 (series-reduced planted trees), 577 (Catalan numbers), 808 (coefficients of Hermite polynomials), 942 and 1163 (dissections of a polygon), 1217 (double factorials), and 1465 (Schröder's fourth problem). (Sequence 1170, Schröder's second problem, agrees with 1163 in all terms except the fourth, which appears to be a misprint.)

The purpose of this paper is to explore, and attempt to clarify, some of the connections between these and related sequences. Each of these sequences enumerates a class of trees sometimes with extra structure, by number of leaves (vertices of valency 1). Moreover, each object can be described by one or more relations on the set of leaves. The classes of relational structures which arise in this way all have the amalgamation property (see Section 2). This implies that, for each sequence, there is a group G of permutations on a countably infinite set X, for which the number of orbits of G on n-element subsets of X is the nth term in the sequence.

These groups are also of relevance to permutation group theorists, and some of them have appeared in other contexts, such as work of Cameron [**6** IV] on the relationship between transitivity on ordered and unordered sets, and more recently a beautiful theorem of Adeleke and Neumann [**2**] on infinite Jordan groups. In addition, they are particularly relevant to a theorem of Macpherson [**18**]. This theorem asserts that, if the group G acts primitively on the infinite set X, then the sequence enumerating orbits of G on n-element subsets of X either is constant with value 1, or grows at least exponentially. Many of the sequences described here do have exponential growth rate (more precisely, the logarithm of the nth term is asymptotically a linear function of n), and they comprise almost all known examples of orbit-counting sequences whose growth is no faster than exponential.

It is also my purpose to propose a systematic notation for these objects. The symbols can be regarded as standing for the classes of finite structures, for the infinite homogeneous structures, for the automorphism groups of the latter, or even (at a pinch) for the sequences of natural numbers.

I gratefully acknowledge the help I have received from discussions with S. Adeleke, J. Covington, H. D. Macpherson, P. M. Neumann, and R. E. Woodrow; this paper is greatly improved as a result.

2. Homogeneous structures

We consider relational structures over a language with finitely many relation symbols. (Each relation symbol is equipped with an arity, and a structure consists of a set carrying relations corresponding to the relation symbols of the language and having the appropriate arities.) A substructure consists of a subset together with the restrictions of all the relations to it. For any structure X, the *age of X* is the class of all structures isomorphic to finite substructures of X; we denote it by $\mathcal{F}(x)$.

The structure X is called *homogeneous* if any isomorphism between finite substructures of X extends to an automorphism of X. Thus, if X is homogeneous, then the number of orbits of the automorphism group Aut (X) on n-element subsets of X is equal to the number of isomorphism types of n-element structures in $\mathcal{F}(X)$. (Note that we count unlabelled structures.)

A class $\mathcal{C}$ of structures is called *hereditary* if it is closed under taking induced substructures; $\mathcal{C}$ has the *joint embedding property* if, whenever $F_1, F_2 \in \mathcal{C}$, there exists $F_3 \in \mathcal{C}$ and embeddings g_i: $F_i \to F_3$ $(i = 1, 2)$; and $\mathcal{C}$ has the *amalgamation property* if, whenever $F_0, F_1, F_2 \in \mathcal{C}$ and f_i: $F_0 \to F_i$ are embeddings $(i = 1, 2)$, then there exists $F_3 \in \mathcal{C}$ and embeddings g_i: $F_i \to F_3$ $(i = 1, 2)$ such that $f_1 g_1 = f_2 g_2$. It is well known that a class of finite structures (over a finite relational language) is the age of some countable structure if and only if it is closed under isomorphism, is hereditary, and has the joint embedding property. The following theorem of Fraïssé [**11**] gives conditions for such a class to be the age of a countable homogeneous structure.

THEOREM 2.1. *Let $\mathscr{C}$ be a class of finite structures over a finite relational language L, which is closed under isomorphism, is hereditary, and has the amalgamation property. Then there exists a finite or countably infinite homogeneous structure X over L with $\mathscr{F}(X)=\mathscr{C}$, unique up to isomorphism.*

As we have seen, if X is homogeneous, then the number of orbits of Aut (X) on n-sets is equal to the number of unlabelled n-element structures in $\mathscr{F}(X)$. However, it often happens that there is a related group for which the number of orbits on n-sets is equal to the number of labelled n-element structures in (X). We say that a class $\mathscr{C}$ has the *strong amalgamation property* if, in the conclusion of the amalgamation property, the embeddings g_1 and g_2 can be chosen so that, if $yg_1 = zg_2$ for some $y \in F_1$, $z \in F_2$, then there exists $x \in F_0$ with $xf_1 = y$, $xf_2 = z$. (In other words, two structures F_1 and F_2 can be amalgamated over a common substructure F_0 without making any identifications of points outside F_0.)

PROPOSITION 2.2. *Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be classes of finite structures over disjoint finite languages L_1 and L_2, which are isomorphism-closed and hereditary and have the strong amalgamation property. Let $\mathscr{C}_1 \wedge \mathscr{C}_2$ be the class of structures over $L_1 \cup L_2$ which consist of a member of $\mathscr{C}_1$ and a member of $\mathscr{C}_2$ sharing the same point set. Then $\mathscr{C}_1 \wedge \mathscr{C}_2$ is also isomorphism-closed and hereditary and has the strong amalgamation property. In particular, if $\mathscr{C}_2$ is the class of finite total orders, then the number of unlabelled n-element structures in $\mathscr{C}_1 \wedge \mathscr{C}_2$ is equal to the number of labelled n-element structures in $\mathscr{C}_1$.*

Proof. The first part is clear; the second holds because a labelling of a finite structure is equivalent to a total order on its point set. Note that the class of total orders does satisfy our hypotheses; the corresponding countable homogeneous total order is $\mathbb{Q}$, so all groups constructed by the Proposition are subgroups of the group of order-automorphisms of $\mathbb{Q}$.

In the sequel, if Σ is a symbol for a class $\mathscr{C}_1$ of finite structures satisfying the conditions of Propositions 2.2, then $L\Sigma$ (mnemonic "labelled Σ") will be the symbol for the class $\mathscr{C}_1 \wedge \mathscr{C}_2$, where $\mathscr{C}_2$ is the class of finite total orders.

3. On the leaves of a tree

In order to associate a group with a class of trees, we must first give a recipe for producing relational structures from trees.

In this section and the next, we take the underlying set X of the structure to be the set of leaves (vertices of valency 1) of a finite tree. There is a quaternary relation on X defined as follows. given any four vertices, consider the tree formed by the vertices and edges lying on paths joining those vertices, and suppress all divalent vertices. The result

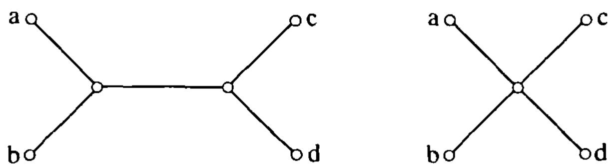

FIG. 1

must have one of the two forms shown in Figure 1. We write $ab \mid cd$ if the tree is of the first type.

If one tree is obtained from another by suppressing a divalent vertex, then the corresponding relational structures are identical. So we may, if required, assume that our trees are *series-reduced*, that is, have no divalent vertices. Now the relational structure determines the tree:

PROPOSITION 3.1. *Any isomorphism between the relational structures derived as above from series-reduced trees t_1 and t_2 extends uniquely to an isomorphism from t_1 to t_2.*

Proof. To establish the result by induction, it clearly suffices to show that the relational structure obtained by adding a new leaf x uniquely determines the point at which x is attached to the tree. This point of attachment is either a new divalent vertex inserted into an existing edge e, or an existing vertex v.

Call a subset of the leaves different from x *compatible* if $xa \mid bc$ holds whenever b and c belong to the subset and a to its complement. Clearly the partition of the tree into connected components after the deletion of e or v induces a partition of the leaves into compatible subsets; moreover, no compatible subset (other than the set of all leaves) can contain leaves from different components. Thus, the components are determined as the maximal proper compatible sets, and so e or v is determined.

PROPOSITION 3.2. *The class of relational structures obtained above is isomorphism-closed and hereditary and has the strong amalgamation property.*

Proof. The first two properties are clear. So let X_1 and X_2 be structures with a common substructure X_0. By Proposition 3.1, we can assume that X_1 and X_2 are the sets of leaves of trees t_1 and t_2 with a common subtree t_0 (up to divalent vertices), and that X_0 is the set of leaves of t_0. Now each of t_1 and t_2 can be obtained from t_0 by adding "branches", either at existing vertices or at new divalent vertices, these branches being just rooted subtrees attached at the root. If we add both sets of branches, we obtain the required strong amalgam.

We use the notation T for this class of finite relational structures (or for the corresponding homogeneous structure).

A number of variations on this construction are possible.

First variation. Instead of considering all finite trees, we may take only those in which the valencies of the non-leaves do not exceed k, where k is a fixed integer which is at least 3. (We must modify the proof of the strong amalgamation property slightly. In the previous argument, we might be required to add branches of t_1 and t_2 at the same point of t_0 in such a way that its valency would exceed k. If the point in question is a new divalent vertex, we can simply create two divalent vertices in the same edge, and attach the branches of t_1 to one and those of t_2 to the other. On the other hand, if it is an existing non-leaf, we grow one new edge at that vertex; at its other end, we grow two new edges, and attach the branches of t_1 to the other end of one, those of t_2 to the other.) These structures will be denoted by T_k.

The case T_3 is particularly interesting. The second type of tree in Figure 1 cannot occur; so any four leaves, in some order (in fact, in just 8 of their 24 possible orders), satisfy the quaternary relation. Furthermore, there is a unique *boron tree* (series-reduced tree with maximum valency 3) having 5 leaves (see Figure 2). So the resulting group is the one constructed rather differently in [**6** IV], which acts transitively on unordered 5-sets but is not more than triply transitive. The sequence enumerating boron trees is Sloane 123.

Second variation. As remarked above, all these classes of structures satisfy the hypotheses of Proposition 2.2. Thus we may add a binary relation symbol to the language and interpret it as a total order (in other words, consider labelled structures), for either the original or the first variation.

For LT we obtain Sloane 1465 (Schröder's fourth problem); for LT_3 Sloane 1217 (double factorials). The proof of the first assertion is an exercise. For the second, use induction on n. Let $f(n)$ be the number of T_3-structures on $\{1, \ldots, n\}$. To add a new leaf $n + 1$, we must choose an edge of the existing tree (of which there are $2n - 3$, since a boron tree

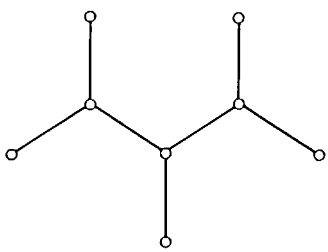

FIG. 2

with n leaves has $n-2$ non-leaves), and attach the new leaf to a divalent vertex inserted in that dege. This gives the recurrence

$$f_2 = 1, \qquad f_{n+1} = (2n-3)f_n \quad \text{for} \quad n \geq 2,$$ P-recursive

whence

$$f_n = (2n-5)!! \quad \text{for} \quad n \geq 2.$$

The recurrence for LT or LT_k ($k \geq 3$) is more complicated, since the new leaf may be joined to an existing vertex. We will say more about this in Section 10.

Third variation. We may, instead, regard our trees as being embedded in the plane. The effect of this is to impose a circular order on the leaves. However, unlike the last variation, the circular order is not independent of the quaternary relation: if a, b, c, d appear in that order around the circle, then $ac \mid bd$ is impossible (Figure 3).

For general trees, strong amalgamation holds as before: the structures will be called PT ($P =$ "plane"). However, for trees of bounded valency, the amalgamation property fails in general. If t_1 and t_2 have branches attached to the same vertex of t_0 but in different sectors of the plane, then it is not possible to identify these branches or to grow them from a common stem, and the valency of the vertex may thus be forced above k. Only in the case $k=3$ is this not a problem, since a branch can only be added at a new divalent vertex, and we can create further divalent vertices as required. Thus we obtain a class PT_3 with the amalgamation property.

As a further twist, in either of these examples, we may allow reversal of the circular order, replacing it by its derived quaternary separation relation. In group-theoretic terms, we obtain a new group which contains the old one as a subgroup of index 2. We denote these by P^*T and P^*T_3 respectively. The relevant sequences are Sloane 339 (partitions of a polygon) and Sloane 942 (dissections of a polygon).

Finally, we could label any of the four structures. If this is done for PT_3, we obtain a sequence LPT_3 whose nth term is $2^{n-2}(2n-5)!!$ for

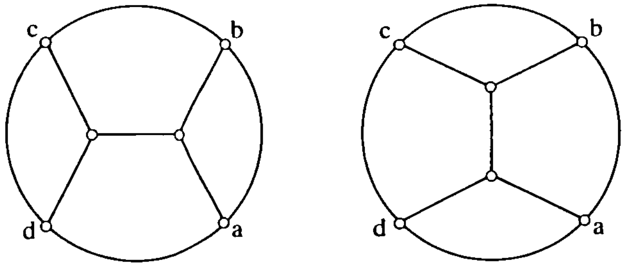

FIG. 3

$n > 2$. (This is proved by an induction similar to that for LT_3. This time, when adding a new leaf, we must choose both the edge in which to insert the divalent vertex, and on which side of this edge to grow the leaf, giving the recurrence $f(n+1) = 2(2n-3)f(n)$.) This sequence appears as Sloane 808 (coefficients of Hermite polynomials); in fact it records the constant terms of suitably normalised Hermite polynomials, a fact most easily verified by using the recurrence relation for these polynomials. The sequence has the additional feature that the $(n+1)$st term is the product of $n!$ and the nth Catalan number, presaging the appearance of Catalan numbers in the next section, where the sequences PT_3 and P^*T_3 will be expressed in terms of Catalan numbers.

4. A distinguished point

In many enumeration problems, it is easier to count structures with a distinguished point than to count arbitrary structures. (Consider the relation between rooted trees and trees, see [**13**]). We denote the operation of distinguishing a point by the prefix ∂ (suggesting differentiation, see Section 10).

If we take a tree and distinguish one of its leaves ∞ as the "root", the quaternary relation induces a ternary relation on the remaining leaves, by the rule that $x \mid yz$ holds if and only if $\infty x \mid yz$ holds (the first situation in Figure 4). Moreover, the quaternary relation can be recovered from the ternary: if $x \mid yz$, then $\infty x \mid yz$; and if either $x \mid zw$, $y \mid zw$, or $z \mid xy$, $w \mid xy$, then $xy \mid zw$ (Figure 5).

Of course, the amalgamation property holds if this procedure is applied to any of the examples of Section 3. Indeed, the automorphism group of the homogeneous structure is just the stabiliser of a point in the automorphism group of the corresponding structure from Section 3. Thus, each of those examples gives us a new orbit-coupling sequence. In accordance with the observation at the start of this section, many of these structures are more familiar than their "unrooted" counterparts.

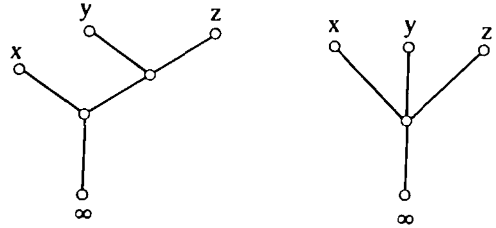

FIG. 4

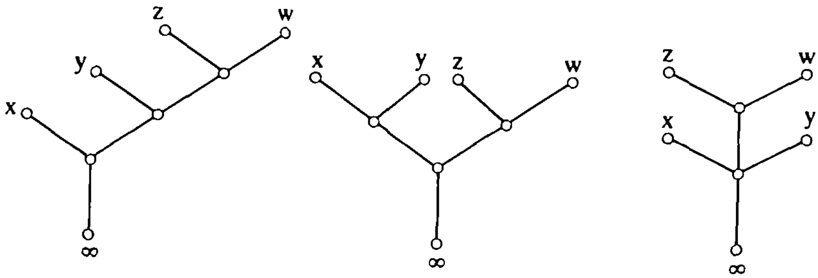

FIG. 5

First, let us observe that for labelled structures, the new sequence is obtained simply by displacing the old sequence one place to the left. For, given labelled structure on $\{1, \ldots, n+1\}$, one may just choose the root to be $n+1$ and consider the induced labelled structure on $\{1, \ldots, n\}$.

Of the remaining structures, three give sequences of particular interest, namely ∂T (Sloane 558: series-reduced planted trees), ∂T_3 (Sloane 298: Wedderburn–Etherington numbers), and ∂PT_3 (Sloane 577: Catalan numbers). Note that both the second and the third of these count boron trees rooted at a leaf (that is, binary trees), but in different ways: in the former, we do not distinguish between the left and right branches at any internal vertex, whereas in the latter (as usual in applications such as computer science) we do. In another interpretation, both sequences count the number of ways of bracketing a nonassociative product of n terms where, for the Wedderburn–Etherington numbers, we assume that the product is commutative, while for the Catalan numbers, we do not.

A circular order, "rooted" at a point ∞ as before, yields a linear (total) order: $x<y$ if ∞, x, y come in that sequence in the circular order. Thus the objects counted by ∂PT and ∂PT_3 possess a natural total order. In fact, it is just the order in which the leaves are visited by the usual tree-search algorithm. Now, labelling the leaves of ∂PT_3 (giving $L\ \partial PT_3$) involves adding a second, arbitrary, total order, which can be done in $n!$ different ways. Thus, the number of labelled ∂PT_3 structures is $n!C_n$, where C_n is the nth Catalan number. This, with our observation that rooting labelled structures just shifts the sequence one place left, justifies the earlier assertion that

$$n!C_n = 2^{n-1}(2n-3)!!.$$

Before leaving this topic, we note that the numbers of structures of various types associated with plane boron trees can be expressed in terms of Catalan numbers, as follows:

$$\begin{aligned} \partial PT_3&: \; C_n \\ \partial P^*T_3&: \; \tfrac{1}{2}(C_n + C_{\frac{1}{2}n}) \\ PT_3&: \; (1/n)C_{n-1} + \tfrac{1}{2}C_{\frac{1}{2}n} + \tfrac{2}{3}C_{\frac{1}{3}n} \\ P^*T_3&: \; \begin{cases} \tfrac{1}{2}nC_{n-1} + \tfrac{1}{3}C_{\frac{1}{3}n} + \tfrac{1}{2}C_{\frac{1}{2}(n-1)} & (n \text{ odd}), \\ \tfrac{1}{2}nC_{n-1} + \tfrac{1}{3}C_{\frac{1}{3}n} + \tfrac{3}{4}C_{\frac{1}{2}n} & (n \text{ even}), \end{cases} \end{aligned}$$

with the convention that $C_q = 0$ if q is not an integer.

The proof is an application of "Burnside's lemma". For ∂P^*T_3, we count orbits of the group consisting of the identity and left-right reversal; the former fixes all C_n objects, the latter the $C_{\frac{1}{2}n}$ for which the right-hand subtree is the reflection of the left-hand one. For PT_3, we consider the action of the cyclic groupf of order n on the set of pairs consisting of a plane boron tree and a distinguished leaf in that three, where the generator of the group fixes the three and distinguishes the next leaf in the anticlockwise sense. The number of such pairs is C_{n-1}. Only automorphisms of order 1, 2 or 3 fix anything; and involution fixes $(n/2)C_{n/2}$ pairs, while an element of order 3 fixes $(n/3)C_{n/3}$.

For unrooted trees up to reflection, we argue similarly with the dihedral group; details are omitted. For both even and odd n the result can be expressed as

$$\tfrac{1}{2}(\gamma_n + C_{\lfloor \frac{1}{2}n \rfloor}),$$

where γ_n is the number without reflection (that is, PT_3).

5. *N*-free objects

An *N-free graph* is one which contains no path of length 3 (as induced subgraph); an *N-free poset* is one which does not contain the poset of Figure 6 (as induced sub-poset). These objects have been studied under many other names: for example, N-free graphs are called cographs [9] or hereditary Dacey graphs [22], and are closely related to series-parallel networks (see below); N-free posets are called reticles [19] or *TSP*-diagraphs [15], and also appear in statistics for describing experimental designs whose analysis can be obtained by nesting and crossing [3].

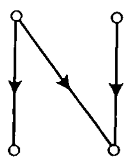

FIG. 6

Their main properties are summarised in the following (well-known) omnibus result:

PROPOSITION 5.1. (a) *The class of finite N-free graphs is the smallest class of graphs which is closed under isomorphism, disjoint unions, and complementation, and contains a* 1*-vertex graph.*

(b) *The complement of a connected finite N-free graph on more than one vertex is disconnected* (*and* vice versa).

(c) *The comparability graph of an N-free poset is an N-free graph; and conversely, the edges of an N-free graph can be directed so as to form an N-free poset.*

Outline of proof. (a) The smallest class $\mathscr{C}$ closed under the given operations is defined inductively as follows: let $\Sigma_0 = \Delta_0$ be the class of 1-vertex graphs, Σ_{n+1} the class of disjoint unions of graphs in Δ_n, and Δ_{n+1} the class of complements of graphs in Σ_{n+1}; then $\mathscr{C} = \bigcup(\Sigma_n \cup \Delta_n) = \bigcup \Sigma_n$, since $\Delta_n \subseteq \Sigma_{n+1}$. An easy induction shows that all graphs in $\mathscr{C}$ are N-free. Suppose, for a contradiction, that Γ is the smallest N-free graph not in $\mathscr{C}$. By the construction of $\mathscr{C}$, Γ and its complement $\bar{\Gamma}$ are connected, but, for any vertex x, either $\Gamma\{x\}$ or $\bar{\Gamma}\backslash\{x\}$ is disconnected. For a fixed x, we may suppose that $\Gamma\backslash\{x\}$ is disconnected. Then x is adjacent to all other vertices. (For, if not, then there is a path xuv of length 2; if w is joined to x but lies in a different component of $\Gamma\backslash\{x\}$ to u, then $wxuv$ is a path of length 3.) But then x is isolated in $\bar{\Gamma}\backslash\{x\}$.

(b) follows from the construction of $\mathscr{C}$ in (a). Note that there are equally many connected and disconnected N-free graphs on n vertices, up to isomorphism, if $n > 1$.

(c) The first part is clear; the second is proved by induction. Let Γ be an N-free graph. If Γ is disconnected, then each component is orientable to form an N-free poset. If Γ is connected, let $X_1, \ldots, X_n$ be the vertex sets of the connected components of $\bar{\Gamma}$; choose an N-free orientation of the induced subgraph on X_i for each i, and orient edges between these sets by the rule that $x \rightarrow y$ if $x \in X_i$, $y \in X_j$, and $i < j$.

Unfortunately, neither N-free graphs nor N-free posets has the amalgamation property. Consider, for example, the graphs F_1 and F_2 of Figure 7, with F_0 the induced subgraph on $\{1, 2, 3\}$. We cannot identify 4 and 5, because their relations to 1 and 3 differ. If we make 4 and 5

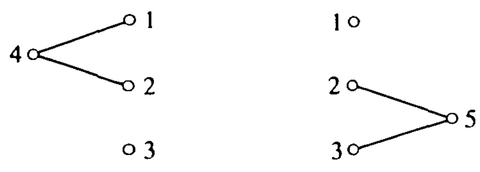

FIG. 7

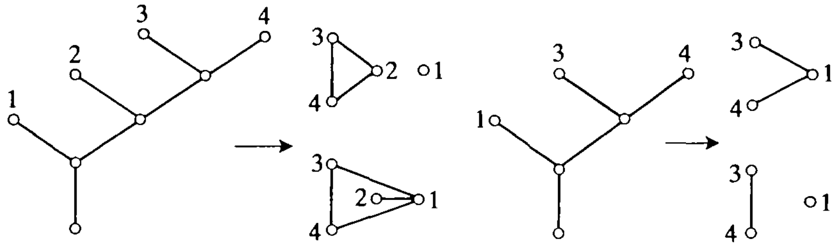

FIG. 8

non-adjacent, then 1425 is a path of length 3; if we joint them, then 1453 is such a path.

There are two, quite different, ways to salvage the amalgamation property. One is based on the observation that the sequence enumerating connected N-free graphs (or, equivalently, complementary pairs of N-free graphs) is Sloane 558, series-reduced planted trees, which we have already met: it is ∂T. As we noted, the structures counted by ∂T, namely, those induced on the leaves of a rooted tree, do have the amalgamation property. A bijection is established inductively as follows. Given an N-free graph Γ (up to complementation), we may suppose that Γ is disconnected. Now, for each component, we take a node attached to the root of a tree; at each such node, we attach a copy of the rooted tree already associated with the corresponding component. Of course, we cannot expect too much: this correspondence fails to preserve induced substructures (Figure 8).

Incidentally, the sequence enumerating all N-free graphs also appears, as Sloane 466 (series-parallel networks). A bijection is established inductively thus. To the graph with one vertex corresponds the network with a single element. If Γ is disconnected it corresponds to the networks associated with its components, connected in series; if it is connected, it corresponds to the networks associated with its induced subgraphs on the connected components of $\bar{\Gamma}$, connected in parallel. An example is shown in Figure 9. This sequence also counts orbits of a permutation group, as we will see in Section 7.

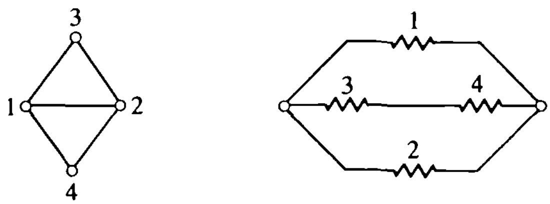

FIG. 9

The second method was discovered by Jacinta Covington **[10]**. It involves imposing extra structure so that the apparent failure of the amalgamation property does not occur, because the maps f_1, f_2 are no longer both embeddings. (For a similar but easier example: bipartite graphs fail to have the amalgamation property; but if we adjoin a new binary relation whose interpretation is "lie in the same bipartite block", the resulting structures do have the amalgamation property.) The structures defined here are identical to Covington's; but the presentation is different.

unordered binary trees

We start with the objects ∂T_3 counted by the Wedderburn–Etherington numbers, that is, binary trees with the left-right distinction ignored. As we have seen, the leaves of such a tree carry a ternary relation. Now suppose that the non-leaves of the tree are coloured with two colours, black and white. Given any two leaves, the paths from them to the root first meet at an internal vertex. Form a graph by joining the two leaves if and only if this vertex is coloured black. Considering the $2^3 = 8$ colourings of each of the two trees with four leaves, we see that the resulting graph is indeed N-free. Moreover, all N-free graphs arise in this way. The resulting structures with the two relations (graph and ternary relation) do have the amalgamation property. The orbit-counting sequence is 1, 2, 4, 44, 164, 616, ...

If we count labelled objects of this type, we obtain the numbers $n!C_n = 2^{n-1}(2n-3)!!$ again, as for the labelled version of the Catalan numbers. This is simply because the recurrence relation is the same: to add a new leaf to an n-element structure, we choose which of the $2n-1$ edges of the tree will be split by the new internal vertex; now, instead of deciding on which side of the edge to grow the new leaf, we decide which colour to apply to the newly-created vertex. (Although there is thus a bijection between labelled objects, there is no "natural" bijection which extends to unlabelled objects, since the numbers are different.)

In a structure of this sort, interchanging the colours black and white corresponds to replacing the graph by its complement; thus, regarding the colours as interchangeable, we see that the above sequence can be halved to obtain 1, 1, 2, 7, 22, 82, 308, . . . , also an orbit-counting sequence. As before, it corresponds to a group G_1 which contains the preceding group G_2 as a subgroup of index 2.

The construction can be generalized, by using more than two colours. We can regard these structures as having the edges of the complete graph coloured with r colours in such a way that

(i) each monochromatic subgraph is N-free;

(ii) each triangle involves at most two colours. The set also carries a ternary relation of the type derived from a binary tree. Furthermore, the relation and the colouring are connected by the rule that $a|bc$ holds in

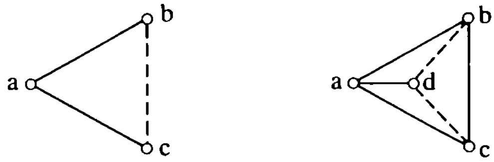

FIG. 10

either situation in Figure 10. In fact, given any colouring satisfying (i) and (ii), there is a ∂T_3 relation connected to it in this way.

In general, we use the suffix (r) to denote the structure obtained from any of the previous rooted structures by colouring the internal vertices with r colours, and partitioning the set of pairs of leaves into r classes in the way described above. (Thus Covington's structures are $\partial T_3(2)$.) A suffix (r^*) denotes the corresponding structure when the colours are regarded as interchangeable. Analogously, a suffix (r) added to an unrooted structure denotes the result of colouring the internal vertices with r colours; this time, we partition the set of triples of leaves, since any triple determines a unique internal vertex where the connecting paths meet. Now $T_3(2)$ is the class of pentagon-free two-graphs carrying a quaternary relation of type T_3, related in the appropriate way.

(A *two-graph* Δ, see [**20**], is a collection of 3-subsets of a set X having the property that, in any 4-subset of X, an even number of the 3-subsets belong to Δ. In this context, a pentagon is a two-graph on five points isomorphic to

$$\{123, 234, 345, 451, 512\}.)$$

Again, (r^*) denotes the modification where the colours are interchangeable. Our observation above about labelled objects asserts that the sequences LPT_3 and $LT_3(2)$ are identical.

We conclude this section with another class of N-free objects. Consider the structures ∂PT. As we have seen, a ∂T-structure corresponds to a complementary pair of N-free graphs. The assumption of embedding in the plane means that there is a natural total order on the vertices (the tree-search order). It is easily seen that, if we use this order to direct the edges of each N-free graph in the pair, we obtain a complementary pair of N-free posets; that is, any pair of points is comparable in just one of the posets, and both are N-free. Conversely, given such a complementary pair of N-free posets, the union of the two partial orders is a total order, and the tree corresponding to the pair of comparability graphs can be embedded in the plane in such a way as to yield the order. So ∂PT enumerates complementary pairs of N-free posets. (I do not know of any groups for which the orbit-counting function enumerates N-free posets, or connected or disconnected N-free posets.)

6. Some wreath products

We require in this section a special case of Philip Hall's wreath power of a permutation group indexed by a totally ordered set [12]. For wreath products over partially ordered sets, see [4].

Let Λ be a totally ordered set. For each $\lambda \in \Lambda$, let Ω_λ be a set with a distinguished element o_λ, and G_λ a transitive group of permutations of Ω_λ. Let $\Omega[\Lambda]$ be the set of functions from Λ to $\bigcup \Omega_\lambda$ to which satisfy

(i) for all $\lambda \in \Lambda$, $f(\lambda) \in \Omega_\lambda$;

(ii) for all but finitely many $\lambda \in \Lambda$, $f(\lambda) = o_\lambda$.

For each triple (λ, e, g), where $\lambda \in \Lambda$, e is a function from $(-\infty, \lambda) = \{\mu \in \Lambda \mid \mu < \lambda\}$ which satisfies (i) and (ii), and $g \in G_\lambda$, define a permutation $x(\lambda, e, g)$ as follows:

$$f \cdot x(\lambda, e, g)(\mu) = \begin{cases} (f, g)(\lambda) & \text{if} \quad \mu = \lambda \text{ and } f \mid (-\infty, \lambda) \\ f(\mu) \text{ otherwise.} \end{cases}$$

Now the generalised wreath product $G(\Lambda)$ is generated by all such permutations.

We specialise still further, to the case where, for all $\lambda \in \Lambda$, we have $\Omega_\lambda = \Omega$, $o_\lambda = o$, $G_\lambda = G$, for a fixed triple (Ω, o, G). Let A be the group of order-automorphisms of Λ. Then A normalises $G[\Lambda]$, so the product $X[\Lambda] = G[\Lambda]$, A is a group.

LEMMA 6.1. *If A is transitive on Λ and $|\Lambda| > 1$, $|\Omega| > 1$, then $X[\Lambda]$ is a primitive Jordan group with rank equal to that of G.*

(A *Jordan group* is a group G containing a non-identity subgroup H which acts transitively on the set of points it does not fix, where the degree of transitivity of G is smaller than the number of points fixed by H. A permutation group G is *primitive* on X if there is no G-invariant equivalence relation on X apart from the two trivial cases, equality and the universal relation.)

Proof. Given $f \in \Omega[\Lambda]$, let $\lambda_1, \ldots, \lambda_n$ be the values of λ for which $f(\lambda) \neq o$; choose $g_i \in G_{\lambda_i}$ with $f(\lambda_i)g_i = o$. Then $f \cdot g_n \cdots g_1$ is the all-o function. This establishes the transitivity of $G[\Lambda]$. The same argument shows that, given any $f_1, f_2 \in \Omega[\Lambda]$, if λ is the smallest element of Λ where f_1 and f_2 differ, then (applying a suitable element of $G[\Lambda]$) we may assume that $f_1(\mu) = f_2(\mu) = o$ for all $\mu \neq \lambda$. Applying an element of A, we can map λ to any preassigned element of Λ. So the number of $X[\Lambda]$-orbits on ordered pairs is the same as for G. Let O_1 and O_2 be G-orbits on pairs, with $(o, \alpha_i) \in O_i$ for $i = 1, 2$. Choose $\lambda, \mu \in \Lambda$ with

$\lambda < \mu$, and define $f_1, f_2, f_3 \in \Omega[\Lambda]$ as follows:

$$f_1(\nu) = o \quad \text{for all } \nu \in \Lambda;$$
$$f_2(\lambda) = \alpha_1, \qquad f_2(\nu) = o \quad \text{for all} \quad \nu \neq \lambda;$$
$$f_3(\mu) = \alpha_2, \qquad f_3(\nu) = o \quad \text{for all} \quad \nu \neq \mu.$$

It is readily checked that (f_1, f_2) and (f_3, f_2) lie in the orbit corresponding to O_1, while (f_1, f_3) lies in the orbit corresponding to O_2. This establishes primitivity. Finally, for any $\lambda \in \Lambda$, the subgroup of $G[\Lambda]$ generated by all $x(\mu, e, g)$ for which $\mu \geq \lambda$ and e has prescribed restriction to $(-\infty, \lambda)$, is transitive on the points of $\Omega[\Lambda]$ which it moves.

We make a final specialisation: assume now that $\Lambda = \mathbb{Q}$, and that G is the symmetric group on Ω where Ω is finite or countable. Set $X = X[\Lambda]$. By Lemma 6.1. X is doubly transitive.

Let F be a finite subset of $\Omega[\Lambda]$. For $q \in \mathbb{Q}$, two functions $f_1, f_2 \in F$ *agree up to* q if $f_1 \mid (-\infty, q) = f_2 \mid (-\infty, q)$. This is an equivalence relation on F. We say that q is an *event* for F if the agreement relation at q is strictly coarser than at any $q' > q$; that is, if some pair of functions in F agree to the left of q but disagree at q. The finitely many events $q_1, \ldots, q_m$ divide $\mathbb{Q}$ into intervals $(-\infty, q_1)$, $(q_1, q_2), \ldots, (q_m, \infty)$; the agreement relation is constant on each interval. Now construct a tree as follows: the vertex set is the union of the sets of agreement classes on all intervals; two vertices are adjacent if the classes are defined on adjacent intervals and the smaller class refines the larger. (This definition automatically suppresses divalent vertices.) The root of the tree is the single class on $(-\infty, q_1)$; the leaves are the singleton classes on (q_m, ∞).

It is readily shown that, if two n-tuples of elements of $\Omega[\Lambda]$ lie in the same orbit of X, then the corresponding rooted trees coincide. Hence X preserves the ternary relation derived from the trees, and is a subgroup of ∂T (if Ω is infinite) or of ∂T_n (if $|\Omega| = n - 1$).

The additional structure preserved by X results from the fact that the internal vertices of the tree are indexed by rational numbers (the events), and so carry a preorder (a reflexive transitive binary relation τ such that, for any $x \neq y$, either $\tau(x, y)$ or $\tau(y, x)$ holds). Figure 11 shows some 5-tuples giving the same tree but lying in different X-orbits.

In any rooted tree, the internal vertices carry a natural partial order (x precedes y if x lies on the path from the root to y). The preorder extends this partial order. Any preorder extending the natural partial order can be realised in this way. The preorder on the internal vertices can be described by a quaternary relation on the leaves, since any two leaves determine an internal vertex and any internal vertex is so determined. Two n-tuples in $\Omega[\Lambda]$ lie in the same X-orbit if and only if both tree and preorder coincide. We denote these classes of relational structures by

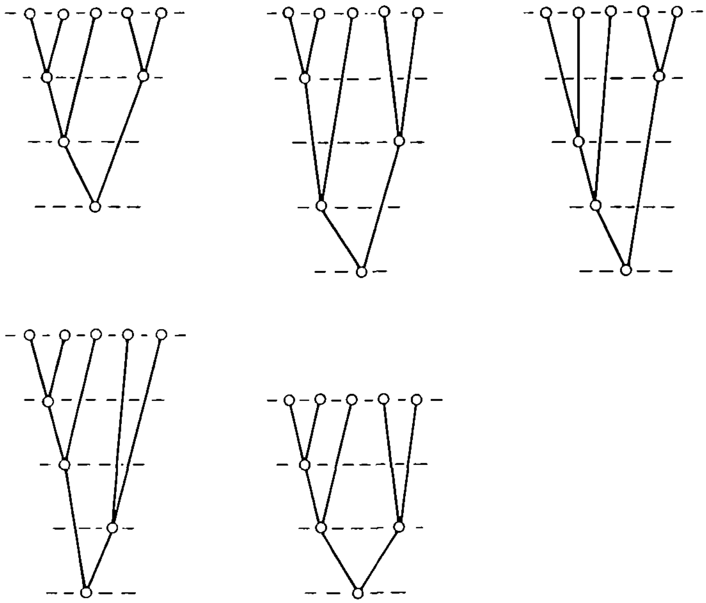

FIG. 11

$\partial T(<)$ or $\partial T_n(<)$, where $<$ suggests the preorder. Further structures may be obtained by colouring the internal vertices. Note that $\partial T_3(<)$, like ∂T_3, is transitive on unordered 3-sets.

I do not know how to construct an object which would carry the name $T(<)$ or $T_n(<)$!

From the construction of the trees, we see that the number $F(n)$ of (labelled) structures on n points in the class $L\,\partial T(<)$ (the number of orbits of $X[\Lambda]$ on ordered n-tuples) is equal to the number of paths in the lattice of partitions of an n-set, a sequence which satisfies the recurrence relation

$$F(n)=\sum_{k=1}^{n-1} S(n,k)F(k),$$ is it CDA?

where $S(n, k)$ is the Stirling number of the second kind. This sequence has recently been studied by Lengyel [**16**].

In [**6** IV], a different construction was given for the structures T_3 and ∂T_3. We now derive these constructions from the generalised wreath product construction given above.

We consider the case where $|\Omega|=2$; say $\Omega=\{o,x\}$. We want to remove the extra structure on $\Omega[\mathbb{Q}]$. Any function from $\mathbb{Q}$ to $\{o,x\}$ which takes the value x only finitely often can be uniquely represented by the (possibly empty) finite sequence $\{q_1,\ldots,q_n\}$ of elements of $\mathbb{Q}$, where $q_0 < \cdots < q_n$ are the points where the function takes the value x. Now we define a bijection between the set of finite increasing sequences of rationals and the set of all finite sequences of rationals, as follows: for each $q \in Q$, let θ_q be an order-preserving bijection from (q, ∞) to $\mathbb{Q}$; then map the sequence $q_1, \ldots, q_n$ to the sequence $r_1, \ldots, r_n$, where $r_1 = q_1$ and, for $i > 1$, $r_i = \theta_{q_{i-1}}(q_i)$. The image of the ∂T_3 ternary relation already defined, under this bijection, is easily checked to be the ∂T_3 relation on the set of finite sequences of rationals defined in [**6** IV]. For convenience, we repeat the specification of the relation. Let X be the set of finite sequences of rationals. For all $w, x, y, z \in X$ and all $q, r, s \in \mathbb{Q}$, we have:

$$\begin{gathered} x \mid xqy,\ xqz; \\ xqy \mid xrz, \quad xrw \quad \text{if} \quad q \neq r; \\ xqy \mid x, \quad xrz \quad \text{if} \quad q < r; \\ xqy \mid xrz,\ xsw \quad \text{if} \quad q < r < s; \\ \text{and} \quad x \mid y, z \quad \text{implies} \quad x \mid z, y. \end{gathered}$$

Moreover, the quaternary relation of type T_3 can be defined on $X \cup \{\infty\}$ by the rules

$$\begin{gathered} \infty x \mid yz \quad \text{if} \quad x \mid yz; \\ xy \mid zw \quad \text{if} \quad x \mid yz, \quad x \mid yw, \quad x \mid zw \quad \text{and} \quad y \mid zw \\ \text{or if} \quad x \mid zw; \quad y \mid zw, \quad z \mid xy \quad \text{and} \quad w \mid xy; \\ xy \mid zw \quad \text{implies} \quad yx \mid zw \quad \text{and} \quad zw \mid xy. \end{gathered}$$

Now let $A_1, \ldots, A_r$ be pairwise disjoint dense sets of rational numbers whose union is $\mathbb{Q}$. (Any two such r-tuples are equivalent under the group of order-automorphisms of $\mathbb{Q}$.) Define symmetric binary relations $\sigma_1, \ldots, \sigma_r$ on X by the rules

$$\begin{gathered} (x, xqy) \in \sigma_i \quad \text{if} \quad q \in A_i; \\ (xqy, xrz) \in \sigma_i \quad \text{if} \quad q \neq r \quad \text{and} \quad \min(q, r) \in A_i. \end{gathered}$$

The resulting object is $\partial T_3(r)$. Furthermore, if we define symmetric ternary relations $\tau_1, \ldots, \tau_r$ on $X \cup \{\infty\}$ by the rules

$$\begin{gathered} (\infty, x, y) \in \tau_i \quad \text{if} \quad (x, y) \in \sigma_i; \\ (x, y, z) \in \tau_i \text{ if an odd number of} \\ (x, y),\ (y, z),\ (z, x) \text{ belong to } \sigma_i; \end{gathered}$$

then we obtain $T_3(r)$. (Note that in $\partial T_3(r)$, the three sides of a triangle belong to at most two of the relations σ_i; so the last rule above assigns each triple of distinct elements of X to exactly one τ_i.)

7. Wreath products again

We now turn to the (ordinary) wreath product of two permutation groups. The generalised wreath product of the preceding section, over the ordered set $\{1,2\}$ (with $1<2$), is what is usually called the *restricted wreath product* of G_2 with G_1, written $G_2 \operatorname{wr} G_1$; it acts on $\Omega_1 \times \Omega_2$, and is generated by permutations of two types:

(i) for each $\omega \in \Omega_1$ and each $g \in G_2$, the map

$$(\alpha, \beta) \mapsto \begin{cases} (\alpha, \beta g) & \text{if } \alpha = \omega \\ (\alpha, \beta) & \text{otherwise} \end{cases}$$

(these permutations generate the *base group*);

(ii) for each $g \in G_1$, the map $(\alpha, \beta) \mapsto (\alpha g, \beta)$.

There is also an *unrestricted wreath product* $G_2 \operatorname{Wr} G_1$, where the base group consists of all permutations which fix the first component of any pair, and act on the second components of pairs with a given first component as an element of G_2 (that is, the cartesian product of $|\Omega_1|$ copies of G_2). It, rather than the restricted product, will be the full automorphism group of the structures we construct.

If G and H are automorphism groups of homogeneous structures on X and Y respectively, then there is a homogeneous structure on $X \times Y$ with group $G \operatorname{wr} H$, as follows. Take the (disjoint) union of the languages L_X and L_Y of X and Y, and adjoin a single binary relation symbol ω. We specify that, in the target structure, the interpretation of ω will be an equivalence relation whose equivalence classes are the "fibres" $\{(x, y) \mid x \in X\}$ of $X \times Y$; each fibre carries a structure X; and the whole set carries a structure for which ω is a congruence, the quotient being Y Thus, the finite substructures F of $X \times Y$ have the following shape:

(i) ω is an equivalence relation on F;

(ii) ω is a congruence for all the relations in L_Y (that is, substitution of ω-equivalent elements leaves the truth values of the relations unchanged), and the quotient structure is in $\mathcal{F}(Y)$;

(iii) relations in L_X hold only if all arguments are ω-equivalent, and the restriction of the L_X-structure to each equivalence class is in $\mathcal{F}(X)$.

We will use the symbol S to denote the countable symmetric group, corresponding to structures over an empty language.

For a first example, note that, if the orbit-counting function for a structure Σ enumerates the connected graphs in some class $\mathscr{C}$ closed under disjoint unions, then the orbit-counting function for $\Sigma \operatorname{wr} S$ enumerates arbitrary graphs in the class. (The equivalence classes of ω are the connected components.) We have seen that ∂T enumerates the connected N-free graphs (Sloane 558); so $\partial T \operatorname{wr} S$ enumerates arbitrary N-free graphs (Sloane 466). Note that terms (after the first) in the latter series are just twice those in the first series, because complementation is a bijection between connected and disconnected N-free graphs.

For a slightly more elaborate example, consider Covington's structures $\partial T_3(2)$. As we noted earlier, such structures consist of an N-free graph with extra structure given by the ∂T_3 ternary relation. It is straightforward to check that this relation has the following property: if the arguments do not all belong to the same connected component, then the relation is unaffected by replacing an argument by another in the same component. Thus the structure is determined by

(i) its restriction to each connected component; and

(ii) a ∂T_3 structure on the set of connected components. Furthermore, a $\partial T_3(2^*)$ structure consists of an (interchangeable) complementary pair of N-free graphs, with a ∂T_3 relation; and, of a complementary pair of N-free graphs on more than one vertex, just one is connected. Thus the number of $\partial T_3(2)$ structures for which the graph is connected is equal to the number of $T_3(2^*)$ structures. We conclude that the orbit-counting sequences for the groups $\partial T_3(2)$ and $\partial T_3(2^*) \text{ wr } \partial T_3$ are identical. This answers a question raised in [**5**], by producing an example of a primitive and an imprimitive group with the same number of orbits on n-sets for all n.

We have, incidentally, produced two solutions to the following curious question:

Problem. For which (infinite transitive) permutation groups G and H is it true that, for all $n > 1$, $H \text{ wr } G$ has twice as many orbits on n-sets as H does?

(We have $(G, H) = (S, \partial T)$ or $(\partial T_3, \partial T_3(2^*))$.)

Similarly, if G is the group $\partial T(<)$ of the preceding section, then $S \text{ wr } G$ has twice as many orbits on n-tuples of distinct elements as does G for $n > 1$. (Cameron and Taylor [**7**] show that if G has F_n orbits on ordered n-tuples, then $S \text{ wr } G$ has

$$\sum_{k=1}^{n} S(n, k) F_k$$

orbits on n-tuples, where $S(n, k)$ is the Stirling number of the second kind; the result now follows from the recurrence relation of Lengyel [**16**] quoted in the last section. Note also that the number of orbits of $S \text{ wr } G$ on n-tuples of distinct elements is equal to the number of orbits of G on all n-tuples, with repetitions allowed; see [**7**].)

8. Circular structures

We begin this section with a class of structures unrelated to trees but also having exponential growth rate. These are then combined with the treelike objects to produce new examples. The circular structures were found independently by H. D. Macpherson, who also suggested the possibility of the "combined" structures.

A member of the class C_r is a set carrying a circular order γ and r binary relations $\sigma_0, \ldots, \sigma_{r-1}$ with the properties

(i) if $(x, y) \in \sigma_i$, then $x \neq y$;

(ii) if $x \neq y$, then $(x, y) \in \sigma_i$ for a unique value of i;

(iii) if $(x, y) \in \sigma_i$, then $(y, x) \in \sigma_{r-i-1}$;

(iv) if $(x, y) \in \sigma_i$, $(y, z) \in \sigma_j$, and $(x, y, z) \in \gamma$, then $(x, z) \in \sigma_{i+j}$ (where subscripts are taken mod r). Note that, if $(x, y) \in \sigma_i$, $(y, z) \in \sigma_j$, and $(x, y, z) \notin \gamma$, then $(x, z) \in \sigma_{i+j+1}$. In the case $r = 2$, the relation σ_0 is a local order in the sense of [6 II], that is, a tournament in which no point dominates or is dominated by a 3-cycle.

Rather than verify the amalgamation property directly, I give a construction of the countable homogeneous structure, and then state a stronger version of the amalgamation property.

Take a countable dense set S of points on the unit circle, with the property that $z \in S$ implies $ze^{2\pi i/r} \in S$ (for example, the roots of unity.) Now S is a union of orbits under the multiplicative group of rth roots of unity, and the set of equivalence classes inherits a circular order. Select one member of each equivalence class in such a way that the resulting set is still dense (this will occur with probability 1 if the choices are made independently at random with probability $1/r$ for each outcome). Let S' be the resulting set. Now take γ to be the circular order on S' induced by the order on the equivalence classes (note that this is not the order induced from the circle), and put $(x, y) \in \sigma_j$ if $2\pi j/r < \arg(y/x) < 2\pi(j+1)/r$. This is the countable homogeneous structure.

Lemma 8.1. *Given C_r-structures X_0, X_1, X_2 and embeddings f_i: $X_0 \rightarrow X_i$ $(i = 1, 2)$, for any circular order X_3 with embeddings g_i: $X_i \rightarrow X_3$ forming an amalgam of circular orders, it is possible to define a C_r-structure on X_3 so as to obtain an amalgam of C_r-structures.*

This can be most easily proved by working within the countable homogeneous structure.

The sequence enumerating C_2-structures (local orders) is Sloane 122: shift register sequences; see [5] for discussion.

Now let Σ be a class of structures having the amalgamation property, each Σ-structure carrying (explicitly or implicitly) a circular order. Let $\Sigma \cdot C_r$ denote the class of sets carrying both a Σ-structure and a C_r-structure in such a way that the circular orders coincide. Then, from the Lemma, $\Sigma \cdot C_r$ also has the amalgamation property. In particular, this can be applied to the classes PT and PT_3 to produce new examples. All these have exponential growth.

9. Internal structures

It is also possible to define relational structures on arbitrary subsets of the vertex set of a tree so that the amalgamation property holds. (This was outlined in [6].) We require a quaternary relation $xy \mid zw$ as for our earlier structure T, and also a ternary relation x: yz ("betweenness") which holds whenever x lies on the path joining y to z in the tree. Let us call these structures, or the corresponding homogeneous structure, U. If X is a U-structure and Y a subset of X, the induced U-structure on Y is derived from the three obtained by taking all paths (in the tree of X) joining pairs of vertices in Y.

We outline the proof of the amalgamation property, following the argument of Section 3. First we show that the relational structure determines the tree. A point is a leaf if and only if it never occurs "in the middle" in an instance of the betweenness relation. So the set of leaves is determined. By Proposition 3.1, the quaternary relation on this set determines the tree up to divalent vertices. Each further point must be placed at a vertex or in an edge of the tree. The relevant vertex or edge is determined by instances of the betweenness relation involving that point and the leaves. Finally, if several points lie in the same edge, the order in which they come is determined by their betweenness relation with one further point.

Now consider the problem of amalgamating structures X_1 and X_2 over a common substructure X_0. First construct the tree corresponding to X_0. Some of the points of X_1 and X_2 may have to be added to this tree. The remaining points belong to a number of branches which are added as in Proposition 3.2.

Note that U does not have the strong amalgamation property. For, in the above argument, X_1 and X_2 may contain points which have to be added at the same vertex of the tree corresponding to X_0; these points must be identified in the amalgam (Figure 12). Hence we cannot construct a homogeneous structure LU. Most of the other variations work in this case, however: these include U_n, ∂U, PU, etc. Note that U_2 is just the usual betweenness relation on the set of rational numbers; its

FIG. 12

FIG. 13

automorphism group acts transitively on n-sets for all n. However, PU_n fails the amalgamation property even for $n = 3$: we might be required to add branches at an existing divalent vertex but on opposite sides (Figure 13).

Considering trees with a distinguished vertex, we see that $\partial U = V \operatorname{wr} S$, where V describes subsets of the vertex set of a tree rooted at a leaf. In other words, a V structure carries both a *semilinear order* (a partial order with the property that the set of predecessors of any point is totally ordered) and a ternary relation xyz which holds for certain incomparable triples. In a similar manner, $\partial U_n = V_n \operatorname{wr} S_{n-1}$, where S_{n-1} is the symmetric group of degree $n-1$, and V_n arises from trees with valency not exceeding n (that is, semilinear orders which are at most $(n-1)$-branching). Adeleke **[1]** has given a nice treatment of some of the material included here, starting from the semilinear orders. (In the case $n = 3$, a description in terms of finite sequences of rational numbers is possible, similar to that given for T_3 in the previous section.)

10. Cycle index

A small modification of the cycle index function of Polya theory is useful in counting the orbits of a permutation group on ordered or unordered n-sets. In this section, we repeat briefly the main results of **[6** III**]**, slightly adapted to the present setting.

If G is a permutation group on a finite set X, the *cycle index* of G is the polynomial in the indeterminates $s_1, \ldots, s_n$ (where $n = |X|$) given by

$$Z(G; s_1, \ldots, s_n) = \frac{1}{|G|} \sum_{g \in G} s_1^{j_1(g)} \cdots s_n^{j_n(g)}$$

where $j_m(g)$ is the number of m-cycles in the cycle decomposition of g.

If G is a permutation group on an arbitrary set X, having only finitely many orbits on n-element subsets of X for all n, then the *modified cycle index* of G is the formal power series $\tilde{Z}(G; s_1, s_2, \ldots)$ obtained by choosing representatives $X_1, X_2, \ldots$ for the G-orbits on finite sets and summing the cycle indices of $G_1, G_2, \ldots$, where G_i is the permutation group induced on X_i by its setwise stabiliser in G. (We adopt the convention that the empty set forms a single orbit for which the cycle index is 1.) We note in passing that, if X is finite, then

$$\tilde{Z}(G; s_1, s_2, \ldots) = Z(G; s_1 + 1, s_2 + 1, \ldots).$$

Let $\mathscr{C}$ be a class of finite relational structures satisfying the hypotheses of Theorem 2.1 and the further condition that there are only finitely many members of $\mathscr{C}$ on n points, up to isomorphism, for every n. The modified cycle index of the automorphism group of the associated homogeneous structure X can conveniently be computed directly from $\mathscr{C}$: take a set of representatives of the isomorphism classes of $\mathscr{C}$-structures, and sum the cycle indices of their automorphism groups. (This usage is consistent with that of Joyal **[14]**.) In a similar manner, the number f_n of orbits of $\mathrm{Aut}\,(X)$ on n-element subsets is equal to the number of (unlabelled) n-element structures in $\mathscr{C}$ (up to isomorphism); and the number F_n of orbits of $\mathrm{Aut}\,(X)$ on ordered n-tuples of distinct elements is equal to the number of labelled n-element structures in $\mathscr{C}$, that is, the number of different $\mathscr{C}$-structures on the set $\{1, \ldots, n\}$. We will consider the ordinary generating function $f(t) = \Sigma f_n t^n$ and the exponential generating function $F(t) = \Sigma F_n t^n/n!$. Both of these are specialisations of the modified cycle index (see **[6** III**]**):

PROPOSITION 10.1. (i) $f(\mathscr{C}, t) = \tilde{Z}(\mathscr{C}; t, t^2, t^3, \ldots)$.
(ii) $F(\mathscr{C}, t) = \tilde{Z}(\mathscr{C}; t, 0, 0, \ldots)$.

(Here and subsequently, we use the symbol for a class of structures in place of a symbol for a group.)

We also take, from **[6** III**]**, formulae for the behaviour of the modified cycle index under some group-theoretic constructions.

PROPOSITION 10.2. (i) $\tilde{Z}(G \times H) = \tilde{Z}(G)\tilde{Z}(H)$.
(ii) $\tilde{Z}(G \mathrm{\ wr\ } H) = \tilde{Z}(H; \tilde{Z}(G) - 1)$.
(iii) $\Sigma \tilde{Z}(G_x) = \dfrac{\partial}{\partial s_1} \tilde{Z}(G)$, *where the summation is over a set of representatives for the G-orbits, and G_x denotes the stabiliser of x, acting on $X \backslash \{x\}$.*

(The substitution in (ii) is defined as follows. If A and B are formal power series in $s_1, s_2, \ldots$, then $A(B)$ means the series

$$A(B(s_1, s_2, \ldots), B(s_2, s_4, \ldots), B(s_3, s_6, \ldots), \ldots).$$

A particular consequence of (iii) is a "justification" of our use of the notation ∂: if the class $\mathscr{C}$ satisfies our hypothesis and has $f_1(\mathscr{C}) = 1$ (only one 1-element structure, up to isomorphism), then

$$\tilde{Z}(\partial\mathscr{C}) = \frac{\partial}{\partial s_1}\tilde{Z}(\mathscr{C}).$$

We record some useful specialisations of Propositions 10.2.

COROLLARY 10.3. (i) $f(G \times H; t) = f(G; t)f(H; t)$.
(ii) $F(G \times H; t) = F(G; t)F(H; t)$.
(iii) $F(G \operatorname{Wr} H; t) = F(H; F(G; t) - 1)$.
(iv) *If G is transitive then* $F(G_x; t) = \dfrac{\mathrm{d}}{\mathrm{d}t}F(G; t)$.

There can be no formula for $f(G_x; t)$ in terms of $f(G; t)$, or for $f(G \operatorname{wr} H; t)$ in terms of $f(G; t)$ and $f(H; t)$—simple examples show that these series are not uniquely determined by the given information.

11. Some growth rates

For almost all classes $\mathscr{C}$ of structures described earlier (the exceptions being labelled structures and those of Section 6 involving a preorder), the number $f_n(\mathscr{C})$ of unlabelled structures is bounded above by an exponential function c^n, for some constant c; in other words, the power series $f(\mathscr{C}, t)$ has positive radius of convergence. Adopting the principle that "groups with fewest orbits are the most interesting", these examples are notable in view of Macpherson's theorem [**18**]: there is a constant $c > 1$ such that, if the group G is primitive on the infinite set X, then either G is *highly homogeneous* ($f_n(G) = 1$ for all n), or else $f_n(G) > c^n$ for all sufficiently large n.

It is not my purpose here to provide accurate asymptotic estimates for $f_n(\mathscr{C})$ for all the classes $\mathscr{C}$ we have considered, though the available techniques (see [**13**], for example) make such a program feasible. I will simply treat a few cases in just enough detail to find estimates for $\limsup (f_n(\mathscr{C}))^{1/n}$, the reciprocal of the radius of convergence of $f(\mathscr{C}, t)$. Note that, since $(f_n(\mathscr{C}))$ is a sequence of positive (indeed nondecreasing) terms, a singularity of least absolute value will occur on the positive t-axis, and our question is answered by locating this singularity. More detailed results require consideration of its nature.

These results may be regarded as a contribution to the solution of the following problem:

For which positive real numbers c is there a (primitive) permutation group G with $\limsup (f_n(G))^{1/n} = c$? This is itself a very special case of a much more general question:

For which sequences (f_n) is there a (primitive) permutation group G

with $f_n(G)=f_n$ for all n? The latter question is a long way from a satisfactory answer although it is known that $2^{\aleph_0}$ sequences are realisable in this way [5], [17].

We begin with the observation that, for any class $\mathscr{C}$ of structures which involve a linear order, either explicitly (e.g. labelled structures), or implicitly (e.g. ∂PT or ∂PT_3), the group induced on any finite set by its setwise stabiliser is trivial, and so the modified cycle index only involves the indeterminate s_1; we have

$$Z(\mathscr{C}; s_1, \ldots) = F(\mathscr{C}; s_1) = f(\mathscr{C}; s_1).$$

For example, a Catalan structure (∂PT_3) on more than one point has a uniquely determined left and right substructure, and is a direct product of these substructures; we have, with $u(t) = f(\partial PT_3; t) - 1$,

$$u(t) = t + u(t)^2,$$

with solution $u(t) = \frac{1}{2}(1-\sqrt{1-4t})$, taking the negative sign for the square root to satisfy $u(0)=0$. This function has a branch-point at $t=\frac{1}{4}$, so the radius of convergence is $\frac{1}{4}$, and the exponential constant is 4. (Of course, the much more precise information that $f_n(\partial PT_3) = \binom{2n-2}{n-1}\Big/n$ can be obtained from the Taylor series expansion of $u(t)$.)

Similarly, the recurrence for $\partial PT_3(r)$ leads to the equation

$$u(t) = t + ru(t)^2,$$

giving exponential constant $4r$.

In the same way, for ∂PT, a structure with more than one point is a union of an ordered sequence of structures; so, with $u(t) = f(\partial PT; t) - 1$ and A the group of order-preserving permutations of $\mathbb{Q}$, we obtain

$$u(t) = t + \tilde{Z}(A; u(t)) - 1 - u(t) = t + u(t)^2/(1-u(t)),$$

from which we obtain

$$u(t) = \tfrac{1}{4}(1 + t - \sqrt{1 - 6t + t^2}).$$

The branch-points are at $3 \pm 2\sqrt{2}$, so the radius of convergence is $3-2\sqrt{2}$, and the exponential constant $3+2\sqrt{2} = 5.8284271$. In fact, the binomial theorem gives the explicit formula

$$u_n = \sum_{m=\lceil n/2 \rceil}^{n} \frac{(2m-2)!\,3^{2m-n}(-1)^{n-m}}{(m-1)!\,(2m-n)!\,(n-m)!\,2^{n+1}}, \qquad n \geq 2.$$

Note that:

(i) If H has finite index in G, then

$$f_n(G) \leq f_n(H) \leq |G:H|\, f_n(G),$$

so the radii of convergence of $f(G;t)$ and $f(H;t)$ are equal.

(ii) $f_{n+1}(G) \leq f_n(G_x) \leq (n+1)f_{n+1}(G)$, where G_x is the stabiliser of x, so the radii of convergence of $f(G;t)$ and $f(G_x;t)$ are equal.

Thus we have determined the exponential constants for PT, ∂P^*T, ∂P^*T, and for PT_3, etc. also, in view of our identification of ∂PT with complementary pairs of N-free posets, and of ∂T with complementary pairs of N-free graphs, we have obtained upper bounds for the exponential constants for T (about which more later) and for N-free posets (about which I have little further information).

Now we turn to the non-plane objects T, T_n, etc. For ∂T and ∂T_n, an object with more than one point is, in a unique way, the disjoint union is smaller objects of the same type, with the proviso that for ∂T_n there are at most $n-1$ objects. Thus, we have:

$$\tilde{Z}(\partial T_n) = 1 + s_1 + \sum_{k=2}^{n-1} Z(S_k; \tilde{Z}(\partial T_n) - 1),$$

and

$$\tilde{Z}(\partial T) = 1 - s_1 + \sum_{k=2}^{\infty} Z(S_k; \tilde{Z}(\partial T) - 1);$$

in other words,

$$\tilde{Z}(\partial T_n \text{ wr } S_{n+1}) = 2\tilde{z}(\partial T_n) - s_1 - 1,$$

while

$$\tilde{Z}(\partial T \text{ wr } S) = 2\tilde{Z}(\partial T) - s_1 - 1.$$

(The last formula expresses our observation that each ∂T-object on more than one point corresponds naturally to a pair of objects of type ∂T wr S.)

From these, we obtain functional equations for the functions $F(\partial T; t)$ and $F(\partial T_n; t)$. With $u(t) = F(\partial T; t) - 1$ and $u_n(t) = F(\partial T_n; t) - 1$, we have

$$e^{u(t)} = 1 + 2u(t) - t,$$

and

$$u_n(t) + \cdots + u_n(t)^{n-1}/(n-1)! = 2u_n(t) - t.$$

In each case, we have an explicit form for the inverse function (t as a function of u or u_n). We can now find the radius of convergence of the original function simply by seeking the first stationary value of the inverse function for positive values of the argument. For example, we have

$$t = 1 + 2u - e^u,$$
$$0 = dt/du = 2 - e^u,$$
$$u = \log 2,$$
$$t = 2\log 2 - 1 = 0.3863\ldots.$$

In a similar way, the radius of convergence of $u_3(t)$ is $\frac{1}{2} = 0.5$, and that of $u_4(t)$ is $\sqrt{3} - \frac{4}{3} = 0.3988\cdots$.

However, we are more interested in the series $f(\partial T; t)$ and $f(\partial T_n; t)$; for these, the calculation is less straightforward. Let $v(t) = f(\partial T; t) - 1$ and $v_n(t) = f(\partial T_n; t) - 1$. We have

$$v_n(t) = t + \sum_{k=2}^{n-1} Z(S_k; v_n(t)).$$

For example,

$$v_3(t) = t + \tfrac{1}{2}(v_3(t)^2 + v_3(t^2)).$$

To evaluate the radius of convergence r, note that $r < 1$, so that if t is smaller than r, then t^2 is much smaller than r, and the Taylor series for $v_3(t^2)$ converges reasonably rapidly. So, by evaluating the first few terms of the sequence $(f_n(\partial T_3))$, we can regard $v_3(t^2)$ as known, and we have a quadratic equation for v_3, whose solution is

$$v_3(t) = 1 - \sqrt{1 - 2t - v_3(t^2)}.$$

The smallest singularity is a branchpoint, occurring when the discriminant vanishes, that is, when $v_3(t^2) = 1 - 2t$. This equation can be solved numerically (approximating $v_3(t^2)$ by the calculated part of its Taylor series). We find that the exponential constant for ∂T_3 (or for T_3) is $2.4832535\cdots$ (cf. Comtet [8]).

Similarity, for v_4, we have a cubic; in terms of $x = 1 + v_4$, it is

$$x^3 + 3(v_4(t^2) - 3)x + 2(v_4(t^3) + 3t + 4) = 0,$$

with discriminant

$$(v_4(t^2) - 3)^3 + (v_4(t^3) + 3t + 4)^2,$$

giving exponential constant $3.3099461\cdots$.

I do not have a method for finding a good approximation to the exponential constant for T, which is computationally feasible. Methods of the above kind give a lower bound of $3.416\ldots$, which may not be too far from the truth.

The same arguments can be used to deal with $\partial T_n(r)$. For example, with $v(t) = f(\partial T_3(r)) - 1$, we obtain the equation

$\partial_t(v(t^2)) = \partial v(t^2) \cdot 2t$

$$v(t) = t + (r/2)(v(t)^2 + v(t^2)),$$

and the branch-point satisfies

$\partial v = 1 + \frac{r}{2}(2v\,\partial v + 2t\,\partial v(t^2)$

$$r^2 v(t^2) = 1 - 2rt$$

and is calculated as before. For $r = 2, 3, 4$, we obtain $4.8925511\ldots$, $7.3054928\ldots$, $9.7191203\ldots$; the exponential constant is asymptotically $(1 + \sqrt{2})r$ as $r \to \infty$.

The circular structure C_r has exponential constant r; indeed, it is clear that $f_n(\partial C_r) = r^n$. Now methods like those already described can be used to deal with $PT \cdot C_r$ or $PT_3 \cdot C_r$. For example, consider $w(t) = f(\partial PT_3 \cdot C_2; t) - 1$. We have

$$w(t) = rt + w(t)^2,$$

giving $w(t) = \frac{1}{2}(1 - \sqrt{1 - 4rt})$, and exponential constant $4r$.

Finally, the internal structures can be treated similarly. As an example, consider $x(t) = f(V_3; t) - 1$. We have

$$x(t) = t + tx(t) + (1 + t)Z(S_2; x(t)),$$

since a V_3-structure with more than one point has, immediately above the root, one of the following: a divalent node; a trivalent node; or a junction not marked by a node. This gives the quadratic

$$(1 + t)x(t)^2 - 2(1 - t)x(t) + ((1 + t)x(t^2) + 2t) = 0,$$

with solution

$$x(t) = (1/(1 + t))(1 - t - \sqrt{(1 - t)^2 - (1 + t)^2 x(t^2) - 2t(1 + t)}),$$

having a branch-point when

$$x(t^2) = (1 - 4t - t^2)/(1 + t)^2.$$

Using the first few terms

$$1, 2, 5, 15, 48, 166, 596, 2221, 8472, \ldots$$

of the sequence, we can solve this equation numerically, finding the radius of convergence to be $4.57798 \cdots$.

Similarly, for $y(t) = f(PV, t) - 1$, we have

$$y(t) = t + ty(t)/(1 - y(t)) + y(t)^2/(1 - y(t)),$$

or

$$2y^2 - y + t = 0;$$

so $y = \frac{1}{4}(1 - \sqrt{1 - 8t})$, and the exponential constant is 8. The nth term of the sequence is $2^{n-1}C_n$, where C_n is the nth Catalan number. This is the same as the nth term of the sequence for $PT_3(2)$. I do not know a direct proof of this fact.

The calculations reported here were performed with the help of a Sinclair ZX Spectrum personal computer.

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