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# POINTLIKE SETS: THE FINEST APERIODIC COVER OF A FINITE SEMIGROUP

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The research in this paper is motivated by the open question: "Is the complexity of a finite semigroup $S$ decidable?" Following the lead of the Presentation Lemma (Rhodes), we describe the finest cover on $S$ that can be computed using an aperiodic semigroup and give an explicit relation. The central idea of the proof is that an aperiodic computation can be described by a new 'blow-up operator' $H^{\omega}$. The proof also relies on the Rhodes expansion of $S$ and on Zeiger coding.

## Guide to the paper

*Chapter* 1. *Elementary definitions and notation* should be omitted on first reading and used as a reference as needed.

*Chapter* 2. *The* Pl-*functor* defines pointlike sets in a general setting and shows by an abstract compactness argument that Pl($S$) can be computed by an aperiodic semigroup.

*Chapter* 3. *Definition of* $C^{\omega}(S)$ *and* $H^{\omega}$ defines $C^{\omega}(S)$, a collection of pointlike sets, in a constructive manner. $H^{\omega}$ is the 'blow-up-operator' that we will use in Chapter 5 to show $C^{\omega}(S)=\mathrm{Pl}(S)$. It has some examples in the end.

*Chapter* 4. *The Rhodes-expansion* defines the tools needed in Chapter 5.

*Chapter* 5. $C^{\omega}(S)=\mathrm{Pl}(S)$ shows the main result by actually constructing a relation $S \overset{R}{—} \mathrm{CP}(S)$ computing $C^{\omega}(S)$ with $\mathrm{CP}(S)$ aperiodic. It uses $H^{\omega}$, generalized to $\hat{H}^{\omega}$ on $\hat{C}^{\omega}(S)$ 'to get rid of groups by blowing up'.

## 1. Elementary definitions and notation

In the following we list definitions and results used throughout Chapters 1–5. Most of this material is standard and is included for the convenience of the reader; for a more thorough treatment and additional background see [4], and especially for relations, Tilson [8]. We assume the reader is familiar with basic semigroup theory.

All semigroups considered are finite, unless otherwise stated.

**Definition 1.1.** A *semigroup* is a finite set $S$ with an associative multiplication.

**Definition 1.2.** $S \times T$ denotes the Cartesian product of $S$ and $T$. $S \leq T$ means $S$ is a subsemigroup of $T$ (i.e., a subset closed under multiplication). $S \leq\leq T_1 \times T_2$, or '$S$ is a subdirect product of $T_1$ and $T_2$', means $S \leq T_1 \times T_2$ and the projection maps

$$p_1 : \begin{cases} T_1 \times T_2 \twoheadrightarrow T_1 \\ (t_1, t_2) \twoheadrightarrow t_1 \end{cases} \quad \text{and} \quad p_2 : \begin{cases} T_1 \times T_2 \twoheadrightarrow T_2 \\ (t_1, t_2) \twoheadrightarrow t_2 \end{cases}$$

project $S$ fully onto $T_1$ and $T_2 : p_1(S) = T_1$, $p_2(S) = T_2$. If $S \subseteq T$, then $\langle S \rangle$ is the semigroup generated by $S$ in $T$.

**Definition 1.3.** $R$ is a *relation* between $S$ and $T$, or $R : S - T$ or $S \overset{R}{—} T$ if and only if $R \leq\leq S \times T$.

(Note: We always assume $R$ to be *onto*. In the literature this is usually not required. If $R$ has only $p_1(R) = S$, we write $S \xhookrightarrow[\text{into}]{R} T$. We will not consider 'partial' relations, where $p_1(R) \subsetneq S$.)

**Definition 1.4.** If $R$ is a function (i.e., $R(s) = \{t\}$ is one point), it is called a *surmorphism* (or *epimorphism*).

**Note.** We usually call $R$ also a homomorphism, although in the literature 'homomorphism' does not always mean $R$ is onto (i.e., $p_2(R) = T$).

**Note.** We usually denote homomorphisms with small Greek letters, like $\phi : S \twoheadrightarrow T$.

**Note.** Definitions 1.3 and 1.4 are 'elegant' definitions. In more concrete terms, $R$ is a relation if and only if $R(s_1) \cdot R(s_2) \subseteq R(s_1 s_2)$ for all $s_1, s_2 \in S$. $\phi$ is a homomorphism if and only if $\phi(s_1) \cdot \phi(s_2) = \phi(s_1 s_2)$ for all $s_1, s_2 \in S$. Yet another form is $R$ is a relation if and only if $s_1 R t_1$, $s_2 R t_2$ implies $s_1 s_2 R t_1 t_2$.

**Fact 1.5.** *$R$ is a relation if and only if $R^{-1}$ is a relation, where $xR^{-1}y$ if and only if $yRx$. So $S \overset{R}{—} T$ and $T \overset{R^{-1}}{—} S$.*

Often relations arise as a 'lift and push diagram' of homomorphisms:

$$(*) \qquad \begin{tikzcd} R \arrow[r, two heads, "\theta"] \arrow[d, two heads, "\phi"'] & T \\ S & \end{tikzcd}$$

Given $(*)$, one may define the relation $\phi^{-1}\theta : S - T$. Then

$$\begin{tikzcd}
R \arrow[rr, two heads, "\theta"] \arrow[dr, "\psi"'] \arrow[dd, "\phi"'] & & T \\
& \phi^{-1}\theta \arrow[ur, "p_2"'] \arrow[dl, "p_1"] & \\
S & &
\end{tikzcd}$$

commutes, with $\psi(r) = \{(s,t) \mid \phi(r) = s,\ \theta(r) = t\}$ and $p_1^{-1}p_2 = \phi^{-1}\theta \leq\leq S \times T$.

**Notation 1.6.** Let

$$S^1 := \begin{cases} S \cup \{1\} & \text{if } S \text{ is not a monoid } (1 = \text{identity of } S), \\ S & \text{if } S \text{ is a monoid;} \end{cases}$$

$$S^0 := \begin{cases} S \cup \{0\} & \text{if } S \text{ does not have a zero } 0, \\ S & \text{if } S \text{ has a zero.} \end{cases}$$

$|A|$ denotes the cardinality of $A$;
$\sum S$ denotes the free semigroup on letters $s \in S$, i.e., all strings $(s_1, \ldots, s_n)$ with $s_i \in S$ and concatenation as multiplication.
We write:

$$\begin{aligned}
s_1 \mathscr{L} s_2 &\quad \text{if and only if} \quad S^1 s_1 = S^1 s_2, \\
s_1 \mathscr{R} s_2 &\quad \text{if and only if} \quad s_1 S^1 = s_2 S^1, \\
s_1 \mathscr{H} s_2 &\quad \text{if and only if} \quad s_1 \mathscr{L} s_2 \text{ and } s_1 \mathscr{R} s_2, \\
s_1 \mathscr{J} s_2 &\quad \text{if and only if} \quad S^1 s_1 S^1 = S^1 s_2 S^1,
\end{aligned}$$

and

$$\begin{aligned}
s_1 \leq_{\mathscr{L}} s_2 &\quad \text{if and only if} \quad S^1 s_1 \subseteq S^1 s_2, \\
s_1 \leq_{\mathscr{R}} s_2 &\quad \text{if and only if} \quad s_1 S^1 \subseteq s_2 S^1, \\
s_1 \leq_{\mathscr{J}} s_2 &\quad \text{if and only if} \quad S^1 s_1 S^1 \subseteq S^1 s_2 S^1, \\
s_1 \leq_{\mathscr{H}} s_2 &\quad \text{if and only if} \quad s_1 \leq_{\mathscr{L}} s_2 \text{ and } s_1 \leq_{\mathscr{R}} s_2.
\end{aligned}$$

If $f : X \to Y$ is a function, we usually write $f(x)$ for the value of $f$ at $x \in X$. $F_R(X)$ denotes the semigroup of all functions $f : X \to X$ with multiplication $[f \circ g](x) = g(f(x))$.

Because of the apparent reversal of symbols, often it is more natural to write the functions on the right, i.e. $xf$ and $x(f \circ g) = (xf)g$. We will also use this notation if the context makes it clear.

If $s = (a, g, b) \in \mathscr{J}$ with respect to a fixed Rees-matrix coordinazation $\mathscr{J}^0 \cong$

$\mathcal{M}^0(A,B,G,C)$ (see Definition 1.7), we write
$g(s)=g(a,g,b)=g$ is the *group-coordinate* of $s$,
$b(s)=b(a,g,b)=b$ is the *B-coordinate* of $s$.
We say $S$ is *aperiodic* if and only if maximal subgroups of $S$ are one point.

An important way to construct semigroups is via a 'Rees-matrix construction':

**Definition 1.7.** Let $S$ be a semigroup with zero 0, $A,B$ finite sets, $C$ a $B\times A$-matrix with entries in $S$. Then

$$\mathcal{M}^0(A,B,S,C):=\{(a,s,b)\mid a\in A,\ b\in B,\ s\in S\}$$

with multiplication given by

$$(a,s,b)*(a',s',b'):=(a,sC(b,a')s',b'),$$

and all elements $(a,0,b)$ identified as zero.

**Note 1.8.** Another way to construct semigroups is to 'add a group of units $G$', i.e., given $S$, form $S\cup G$, $G$ a group acting as permutations on the left and right of $S$, and $\perp\in G$ the identity of $S$.

Rhodes [1] has shown that every semigroup can be constructed alternating the 'Rees-matrix construction' and 'adding a group of units' (up to a very nice division). The $\mathcal{J}$-class structure after two iterations (i.e., for 3-$\mathcal{J}$-class semigroups) is worked out in Austin and Rhodes (unpublished). We use this method to construct Example 3 in Chapter 2.

**Note 1.9.** It is well known that every $\mathcal{J}$-class of $S$ can be described in this way, with $S=G^0$, $G=$ Schützenberger-group. $C$ is called the structure-matrix, $\mathcal{A}=$ the set of $\mathcal{R}$-classes, $\mathcal{B}=$ the set of $\mathcal{L}$-classes (see [4]).

**Definition 1.10.** A $\mathcal{J}$-class of $S$ is called '$H$-trivial' if and only if the Schützenberger-group of $\mathcal{J}$ is trivial, i.e., one point.

**Note 1.11** (Schützenberger-groups). Let $H$ be an $H$-class of $S$. The *Schützenberger-group* $G$ of $H$ is $\{s\in S\mid H\cdot s=H\}$, i.e., all permutations of $H$, made faithful on $H$. (It is therefore not necessarily itself a subgroup of $S$.)

We have a homomorphism of the right idealizer of $H$ onto the Schützenberger-group. Let $G'$ be any subgroup of the right idealizer mapping onto $G$. Since $G$ is transitive, $h\cdot G'=h*G=H$ for any $h\in H$. Now $G'$ is a subgroup of $S$, and in the future we will call $G'$, abusing notation, also 'the Schützenberger group'. Really all we are interested in is

**Fact 1.12.** *If $H$ is an $H$-class of $S$, then there exist groups $G',G''$ of $S$ such that $H=h\cdot G=G'h$ for any $h\in H$.*

*The powerset-functor and the union-map*

**Definition 1.13.** Let $P(S) := \{A \mid A \subseteq S\}$, the set of all subsets of $S$. $P(S)$ carries a natural multiplication, induced by multiplication in $S$, via

$$A \cdot B = \{a \cdot b \mid a \in A,\ b \in B\} \quad (A, B \in P(S)).$$

**Definition 1.14.** Given a homomorphism $\phi : S \twoheadrightarrow T$, define a map $P(\phi) : P(S) \twoheadrightarrow P(T)$ by

$$[P(\phi)](A) := \{\phi(a) \mid a \in A\}.$$

$P(\phi)$ is a homomorphism, so $P$ is a *functor* $\mathcal{SG} \to \mathcal{SG}$ ($\mathcal{SG}$ = category of finite semigroups).

**Fact 1.15.** *$S \leq P(S)$ via the embedding $S \overset{i}{\hookrightarrow} P(S)$, $i(s) = \{s\}$ (singleton-set, containing $s$ as only element). Abusing notation, we often write $s$ for $\{s\}$.*

**Fact 1.16.** *$S \leq T$ implies $P(S) \leq P(T)$.*

**Definition 1.17.** Define $P^0(S) := S$, $P^{n+1}(S) := P(P^n(S))$, $n \geq 0$.

**Fact 1.18.** *$P^n(S) \leq P^{n+1}(S)$, $n \geq 0$.*

**Definition 1.19.** An important homomorphism is given by the familiar (set-theoretic) union-map $\bigcup : P^2(S) \twoheadrightarrow P(S)$, namely if $A \in P^2(S)$, $A = \{A_1, \ldots, A_n\}$ with $A_i \subseteq S$, then $\bigcup A = A_1 \cup \cdots \cup A_n \in P(S)$. This map is easily seen to be a homomorphism.

**Note.** Thus $\bigcup : P^{n+1}(S) \twoheadrightarrow P^n(S)$, $n \geq 1$ is defined, but *not* $P(S) \twoheadrightarrow S$.

**Fact 1.20.** *$\bigcup$ is a* retract, *i.e., $P(S) \leq P^2(S)$, $P(S)$ (as considered $\leq P^2(S)$) maps onto $P(S)$, and $\bigcup$ is the inverse of the embedding $A \hookrightarrow i(A) = \{A\}$, i.e., $\bigcup \{A\} = A$.*

**Definition 1.21.** $P(S)$ carries a natural partial order, namely $A \subseteq B$ (set-theoretic inclusion).

**Fact 1.22.** *If $A \subseteq B$ and $C \in P(S)$, then $CA \subseteq CB$ and $AC \subseteq BC$.*

**Fact 1.23.** *If $A \subseteq B$, $A, B \in P^2(S)$, then $\bigcup A \subseteq \bigcup B$.*

**Note 1.24.** If it is clear from the context, we write $s$ for $\{s\}$ and $\{\{s\}\}$, etc. For example, if $A = \{A_1, \ldots, A_n\} \in P^2(S)$, then $\bigcup s \cdot A = s \cdot \bigcup A$, which really means $\bigcup [\{\{s\}\} \cdot A] = \{s\} \cdot \bigcup A$ (since $\cup \{\{s\}\} = \{s\}$).

**Definition 1.25.** Let $A \in P^2(S)$. Define $\text{sub}(A) = \{A_0 \in P(S) \mid A_0 \subseteq A_1$ for some $A_1 \in A$ and $A_0 \notin \emptyset\}$, i.e. $\text{sub}(A) = A$ and all subsets of members of $A$ (excluding the empty set).

*Relations and covers*

**Definition 1.26.** A *cover* on a set $X$ is a set $D \in P^2(X)$ such that $\bigcup D = X$ and $\emptyset \notin D$ ($\emptyset$ = the empty set).

**Note.** Given a cover $D$, $\text{sub}(D)$ is also a cover. In applications one usually studies $\text{sub}(D)$.

**Definition 1.27.** A cover $D$ is called 'full' if and only if $\text{sub}(D) = D$.

**Definition 1.28.** We can put a partial ordering on covers by $C_1 \leq C_2$ if and only if for every $X \in C_1$ there exists a $Y \in C_2$ such that $X \subseteq Y$.

**Note.** $C_1, C_2 \in P^2(S) = P(P(S))$ are naturally partially ordered by inclusion: $C_1 \subseteq C_2$ if and only if $X \in C_1$ implies $X \in C_2$.

**Fact 1.29.** *$C_1 \subseteq C_2$ implies $C_1 \leq C_2$ and if $C_2$ is full, $C_1 \leq C_2$ is equivalent to $C_1 \subseteq C_2$. In fact, $C_1 \leq C_2$ could be defined as $C_1 \leq C_2$ if and only if $C_1 \subseteq \text{sub}(C_2)$.*

**Fact 1.30.** *Both partial orders have a G.L.B., namely*

$$C_1 \wedge C_2 := \{X \mid X = X_1 \cap X_2 \textit{ with } X_1 \in C_1,\ X_2 \in C_2 \textit{ and } X \neq \emptyset\}$$

*for $\leq$ and*

$$C_1 \cap C_2 = \{X \mid X \in C_1 \textit{ and } X \in C_2 \textit{ and } X \neq \emptyset\}$$

*for $\subseteq$.*

**Fact 1.31.** *If $C_1, C_2$ are full, $C_1 \wedge C_2 = C_1 \cap C_2$.*

**Definition 1.32.** A cover $D$ on a semigroup is called *weakly preserved* (*preserved*) if and only if for every $X \in D$, $s \in S$, $X \cdot s \in \text{sub}(D)$ (for every $X, Y \in D$, $X \cdot Y \in \text{sub}(D)$).

**Fact 1.33.** *If $D$ is full, $D$ is preserved if and only if $D \leq P(S)$.*

**Definition 1.34.** A *cover-semigroup on $S$* is a cover $D$ on $S$ with
(i) $D$ is full;
(ii) $D$ is preserved, i.e., $D \leq P(S)$.

**Definition 1.35.** Given a cover $D$ on $S$, one may always construct the cover semigroup $\bar{D} := \text{sub}\langle D \rangle$.

**Fact 1.36.** $\mathrm{sub}\langle D\rangle = \mathrm{sub}\langle \mathrm{sub}(D)\rangle$.

**Definition 1.37.** Given a relation $S \overset{R}{-\!\!-} T$, define the cover $C(R) := \{R^{-1}(t) \mid t \in T\}$. $C(R)$ is preserved, hence $\mathrm{sub}(C(R))$ is a cover-semigroup, denoted by $\mathscr{C}(R)$. In the sequel we will usually deal with $\mathscr{C}(R)$.

**Definition 1.38.** We say "$T$ computes the cover semigroup $D$ on $S$ (via $R$)" if and only if $\mathscr{C}(R) = D$.

**Definition 1.39.** We say "$D$ is computable by an aperiodic semigroup" if and only if there exists a relation $S \overset{R}{-\!\!-} C$ with $C$ aperiodic such that $C$ computes $D$ (via $R$).

In the following, we wish to determine the finest cover-semigroup $D$ on a given set $S$, such that $D$ is computable aperiodically. We will give an effective procedure to determine $D$ $(= C^{\omega}(S))$ and a specific relation $S \overset{R}{-\!\!-} C$ such that $\mathscr{C}(R) = D$ $(C = \mathrm{CP}(S)$ is aperiodic).

**Definition 1.40.** Given two relations $R_1 : S - D_1$ and $R_2 : S - D_2$ (i.e., $R_1 \leq\leq S \times D_1$ and $R_2 \leq\leq S \times D_2$), we can form $R_1 \times R_2 \leq (S \times D_1) \times (S \times D_2)$ and 'take the diagonal', i.e., define $\Delta(R_1, R_2) := \{(s, x_1, x_2) \mid sR_1x_1$ and $sR_2x_2\}$. Letting $\Delta(D_1, D_2) := \{(x_1, x_2) \mid x_i \in D_i$ and $R_1^{-1}(x_1) \cap R_2^{-1}(x_2) \neq \emptyset\}$, we see $\Delta(R_1, R_2) \leq\leq S \times \Delta(D_1, D_2)$, or $\Delta(R_1, R_2)$ is a relation $S - \Delta(D_1, D_2)$.

**Definition 1.41.** Extend Definition 1.40 by $\Delta(R_1, R_2, R_3) := \Delta(\Delta(R_1, R_2), R_3)$ and so on.

**Fact 1.42.** *The cover defined by* $\Delta(R_1, R_2)$ *is* $C(R_1) \wedge C(R_2)$.

**Fact 1.43.** $\mathscr{C}(\Delta(R_1, R_2)) = \mathscr{C}(R_1) \cap \mathscr{C}(R_2)$ *and* $\mathscr{C}(R_i) \subsetneq \mathscr{C}(\Delta(R_1, R_2))$ *follows from Facts* 1.31 *and* 1.42.

## 2. The Pl-functor

**Notation 2.1.** Let $S$ be a semigroup, $K$ a collection of relations $S \overset{R}{-\!\!-} D(R)$, and let $D(K) := \{D(R) \mid R \in K\}$ be the set of ranges of the relations $R \in K$.

**Definition 2.2.** $X \subseteq S$ is pointlike with respect to $K$ if and only if for each $R \in K$ there exists a $c_0 \in D(R)$ such that $A \subseteq R^{-1}(c_0)$, or in short notation (see Chapter 1), $X \in \mathscr{C}(R)$.

**Notation 2.3.** Let $\mathrm{Pl}_K(S) := \{X \subseteq S \mid X$ is pointlike with respect to $K\}$.

**Fact 2.4.** $\mathrm{Pl}_K(S)=\bigcap_{R\in K}\mathscr{C}(R)$.

(Notice that this intersection is in fact finite, for there are only finitely many cover-semigroups on $S$.)

**Proof.** $X\in\mathrm{Pl}_K(S)$ if and only if $X\in\mathscr{C}(R)$ for every $R\in K$, if and only if $X\in\bigcap_{R\in K}\mathscr{C}(R)$. $\square$

From now on, let $K$ be countable. (This is always guaranteed, if we restrict attention to finite semigroups.)

**Fact 2.5.** $\mathrm{Pl}_K(S)$ *is computed by* $\Delta(R_1,\ldots,R_N)$ *for a finite number* $N$ *of relations* $R_i\in K$.

**Proof.** $\mathrm{Pl}_K(S)=\bigcap_{R\in K}\mathscr{C}(R)$. Enumerate $R\in K$ as $K=\{R_i\}_{i\in\mathbb{N}}$. Then

$$(*)\qquad \mathscr{C}(R_1)\supseteq\mathscr{C}(R_1)\cap\mathscr{C}(R_2)\supseteq\cdots\supseteq\bigcap_{1\le i\le n}\mathscr{C}(R_i)\supseteq\cdots$$

and $\bigcap_{1\le i\le n}\mathscr{C}(R_i)$ is computed by $\Delta(R_1,\ldots,R_n)$. Now some cover $C=\bigcap_{1\le i\le N}\mathscr{C}(R_i)$ occurs infinitely often in $(*)$, so $\bigcap_{R\in K}\mathscr{C}(R)=\bigcap_{1\le i\le N}\mathscr{C}(R_i)$ for some $N$. $\square$

**Fact 2.6.** *Assume that* $K$ *is closed under* $\Delta$ *(i.e.,* $R_1,R_2\in K\Rightarrow\Delta(R_1,R_2)\in K$*). Then* $\mathrm{Pl}_K(S)=\mathscr{C}(R)$ *for some* $R\in K$.

**Proof.** Follows from Fact 2.5. $\square$

In the following, we will place some more restrictions on $K$.

**Definition 2.7.** Let $\mathscr{D}$ be any collection of (finite) semigroups closed under $\le\le$ (i.e., if $D_1,D_2\in\mathscr{D}$, $X\le\le D_1\times D_2$, then $X\in\mathscr{D}$). Then let $K(\mathscr{D})$ be the set of *all* relations $S\overset{R}{\text{---}}C$ with $C\in\mathscr{D}$. Obviously, $K(\mathscr{D})$ is closed under $\Delta$.

**Definition 2.8.** Let $\mathscr{C}:=$ the set of all aperiodic semigroups. We write $\mathrm{Pl}(S)$ for $\mathrm{Pl}_{K(\mathscr{C})}(S)$ in the following, and Fact 2.6 reads now (with $K=K(\mathscr{C})$):

**Fact 2.9.** $\mathrm{Pl}(S)=\mathscr{C}(R)$ *for some* $S\overset{R}{\text{---}}C$ *with* $C$ *an aperiodic semigroup, or* $\mathrm{Pl}(S)$ *is computable by an aperiodic semigroup.*

**Note.** Fact 2.5 does not assure computability of $\mathrm{Pl}(S)$. Even though the intersection $\bigcap_{R\in K(\mathscr{C})}\mathscr{C}(R)$ is finite, we do not know when to stop forming $C(R_1)\supseteq\cdots\supseteq\bigcap_{1\le i\le n}C(R_i)\supseteq\cdots$. On the other hand, Fact 2.5 gives a proof-scheme for proving that a certain cover-semigroup $C$ equals $\mathrm{Pl}(S)$:

**Proof-scheme 2.10.** *Let $D$ be a cover-semigroup on $S$. Then $D=\mathrm{Pl}(S)$, if $D$ satisfies*

(i) *The elements of $D$ are pointlike, i.e., $D \leq \mathrm{Pl}(S)$;*

(ii) *$D$ can be computed combinatorially, i.e., $D=\mathscr{C}(R)$ with $S \overset{R}{—} C$ and $C$ combinatorial.*

**Proof.** (i) assures $D \leq \mathrm{Pl}(S)$.

(ii) shows $\mathrm{Pl}(S) \leq D$ (since $\mathrm{Pl}(S)=\bigcap_{R \in K(\mathscr{C})} \mathscr{C}(R)$, and $D$ occurs in the intersection).

Together (i) and (ii) show $D=\mathrm{Pl}(S)$. $\square$

**Note.** We will define $D:=C^{\omega}(S)$ in Chapter 3, and Chapter 5 will construct $S \overset{R}{—} \mathrm{CP}(S)$ with $\mathscr{C}(R)=C^{\omega}(S)$ and $\mathrm{CP}(S)$ aperiodic.

We will postpone exploration of the functorial and other properties until later [2], so that we may then use $\mathrm{Pl}(S)=C^{\omega}(S)$ to facilitate some proofs. However, most 'general' properties of $\mathrm{Pl}(S)$ are independent of that fact and could be proved earlier, just using the 'universal' definition of $\mathrm{Pl}(S)$ as $\mathrm{Pl}_{K(\mathscr{C})}$.

*Excursion: transformation-semigroups*

**Definition 2.11.** A *finite transformation-semigroup* (*TSG*) is a pair $(X,S)$ with $X$ a finite, non-empty set and $S \leq F_R(X)$ a subsemigroup of the semigroup $F_R(X)$ of all functions on $X$, written on the right.

**Note.** In the literature sometimes a more general approach is taken, allowing $S$ to be not faithful on $X$. However, this definition is general enough for our purpose.

All the notions defined in Chapter 1 can be generalized to TSG in the obvious way, for example

**Definition 2.12.** A *relation of TSG's* $(X,S) \overset{R}{—} (Z,T)$ is a pair of relations $X \overset{R_1}{—} Z$, $S \overset{R_2}{—} T$ such that

(i) $S \overset{R_2}{—} T$ is an SG-relation;

(ii) $(xR_1)\cdot(s_1R_2) \subseteq (x \cdot s_1)R_1$ for all $x \in X$, $s_1 \in S$ (compatibility).

**Definition 2.13.** The relation $R=(R_1,R_2)$ defines a cover semigroup $\mathscr{C}(R):=\mathscr{C}(R_2)$ on $S$ as in Chapter 1. In addition, we get a *'cover on the letters'*, defined as

$$C^X(R):=\mathrm{sub}(\{R_1^{-1}(z) \mid z \in \mathbb{Z}\}).$$

**Fact 2.14.** *Let $S \overset{R}{—} T$ be a relation. Extend $R$ to $S^1, T^1$ by $1R1$ (this is again a relation). Then $(S^1,S) \overset{(R,R)}{—} (T^1,T)$ is a TSG-relation.*

**Definition 2.15.** Let $\bar{K}$ be a collection of TSG-relations $(X,S) \overset{R}{\text{---}} (Y,T)$, $R=(R_1,R_2)$. Then let

$$\mathrm{Pl}_{\bar{K}} := \{S_0 \subseteq S \mid S_0 \in \mathscr{C}(R) \text{ for all } R \in \bar{K}\},$$
$$\mathrm{Pl}_{\bar{K}}^X := \{X_0 \subseteq X \mid X_0 \in \mathscr{C}^X(R) \text{ for all } R \in \bar{K}\}.$$

With these definitions, all the facts of Chapter 2 extend to TSG. We list (without proof):

**Fact 2.16.**
$$\mathrm{Pl}_{\bar{K}}(X,S) = \bigcap_{R\in\bar{K}} \mathscr{C}(R), \qquad \mathrm{Pl}_{\bar{K}}^X(X,S) = \bigcap_{R\in\bar{K}} \mathscr{C}^X(R).$$

**Fact 2.17.** *$\bar{K}$ countable, $\mathrm{Pl}_{\bar{K}}(X,S) = \mathscr{C}(\Delta(R_1,\dots,R_N))$, $\mathrm{Pl}_{\bar{K}}^X(X,S) = \mathscr{C}^X(\Delta(R_1,\dots,R_N))$ for a finite number $N$ of relations $R_i \in \bar{K}$.*

**Fact 2.18.** *If $\bar{K}$ is closed under $\Delta$, then $\mathrm{Pl}_{\bar{K}}(X,S) = \mathscr{C}(R)$ and $\mathrm{Pl}_{\bar{K}}^X(X,S) = \mathscr{C}^X(R)$ for some $R \in \bar{K}$.*

**Definition 2.19.** Let $\bar{\mathscr{D}}$ be any collection of TSG's closed under $\leq\leq$. Let $\bar{K}(\bar{\mathscr{D}})$ be the set of all relations $(X,S) \overset{R}{\text{---}} (Y,C)$ with $(Y,C) \in \bar{D}$.

**Proof-scheme 2.20.** *Let $(X,S)$ be a TSG. Let $D$ be a cover on $X$. Then $D=\mathrm{Pl}^X(X,S)$, if $D$ satisfies*

(i) *The elements of $D$ are pointlike, i.e., $D \subseteq \mathrm{Pl}^X(X,S)$;*

(ii) *$D$ can be computed by an aperiodic semigroup, i.e., $D=\mathscr{C}^X(R)$ with $(X,S) \overset{R}{\text{---}} (Z,C)$ and $C$ aperiodic.*

Now the following connection exists between $\mathrm{Pl}_{\bar{K}}$ and $\mathrm{Pl}_K$:

**Definition 2.21.** Let $\bar{K}$ be a collection of TSG relations $(X,S) \overset{(R_1,R_2)}{\text{---}} (Y,T)$, $K$ be a collection of SG relations $S \overset{R}{\text{---}} S'$. $\bar{K}$ and $K$ are *connected* if and only if

(i) For every $R=(R_1,R_2) \in \bar{K}$, $R_2 \in K$;

(ii) For every $R \in K$, there exists $\bar{R}$ such that $\bar{R}=(R_1,R)$ and $\bar{R} \in \bar{K}$ (in most applications $\bar{R}=(R,R):(S^1,S)$ — $(T^1,T)$ will do).

**Fact 2.22.** *If $K$ and $\bar{K}$ are connected, then $\mathrm{Pl}_K(S) = \mathrm{Pl}_{\bar{K}}(X,S)$.*

**Proof.**
$$\mathrm{Pl}_K(S) = \bigcap_{R\in K} \mathscr{C}(R) \supseteq \bigcap_{R\in\bar{K}} \mathscr{C}(R) \quad \text{by (ii).}$$
$$\mathrm{Pl}_{\bar{K}}(X,S) = \bigcap_{R\in\bar{K}} \mathscr{C}(R) = \bigcap_{(R_1,R_2)\in\bar{K}} \mathscr{C}(R_2) \supseteq \bigcap_{R\in K} \mathscr{C}(R) \quad \text{by (i).} \qquad \square$$

**Fact 2.23.** *Let $S$ be a monoid, $K$ be a collection of relations on $S$ closed under $\leq\leq$, and $\bar{K}=\{\bar{R}\mid(X,S)\xrightarrow{\bar{R}=(R_1,R_2)}(Y,T)$ and $R_2\in K\}$ (i.e., all extensions of $R\in K$ to a TSG relation $\bar{R}$). Then $\mathrm{Pl}_{\bar{K}}^X(X,S)=X\cdot\mathrm{Pl}_K(S)$ where $X\cdot\mathrm{Pl}_K(S)$ is short for $\{x_0\cdot A\mid x_0\in X,\ A\in\mathrm{Pl}_K(S)\}$.*

**Proof.** Since $K$ and $\bar{K}$ are connected (Fact 2.14), $\mathrm{Pl}_{\bar{K}}(X,S)=\mathrm{Pl}_K(S)$. So $X\cdot\mathrm{Pl}_K(S)=X\cdot\mathrm{Pl}_{\bar{K}}(X,S)\subseteq\mathrm{Pl}_{\bar{K}}^X(X,S)$.

For the converse let $R$ be the relation computing $\mathrm{Pl}_K(S)$ (which exists by Fact 2.18), $S\xrightarrow{R}T$ and $\mathscr{C}(R)=\mathrm{Pl}_K(S)$. We will construct $(X,S)\xrightarrow{\bar{R}}(Z,T)$, $\bar{R}=(\bar{R}_1,R)$ with $\mathscr{C}^X(\bar{R})=X\cdot\mathrm{Pl}_K(S)$, proving $\mathrm{Pl}_{\bar{K}}^X(X,S)\subseteq X\cdot\mathrm{Pl}_K(S)$.

So let $Z:=X\times T$, and let $T$ act on $Z$ by $(x,t)*t'=(x,tt')$. Define $X\xrightarrow{\bar{R}_1}X\times T$ by $\bar{R}_1(x_0)=\{(x,t)\mid x_0\in x\cdot R^{-1}(t)\}$. (Here we use the assumption that $S$ is a monoid. It guarantees $\bar{R}_1(x_0)\neq\emptyset$, since $x_0\in x_0\cdot R^{-1}(R(1))$.)

Then $\bar{R}$ is a TSG-relation, and $\bar{R}_1^{-1}(x,t)=x\cdot R^{-1}(t)$, so $\mathscr{C}^X(\bar{R})=X\cdot\mathrm{Pl}_K(S)$ by Proof-scheme 2.20. $\square$

**Definition 2.24.** Define $\mathrm{Pl}^X(X,S)=\mathrm{Pl}_{\bar{K}}^X(X,S)$ and $\mathrm{Pl}(X,S)=\mathrm{Pl}_{\bar{K}}(X,S)$ with $\bar{K}=$ all TSG-relations $(X,S)\xrightarrow{\bar{R}}(Y,T)$ with $T$ aperiodic.

**Corollary 2.25.** *Let $S$ be a monoid. Then $\mathrm{Pl}_X(X,S)=X\cdot\mathrm{Pl}(S)$.*

**Proof.** Definition 2.19 for $K=K(\mathscr{C})$ (all relations with aperiodic SG) and $\bar{K}=$ all relations $(X,S)\xrightarrow{\bar{R}}(Y,T)$ with $T$ aperiodic. $\square$

**Note.** Corollary 2.25 is the motivation for restricting the study to semigroups and the Pl-functor.

The proof schemes 2.10 and 2.20 are due to Rhodes (unpublished).

## 3. Definition of $C^\omega(S)$ and $H^\omega$

In this chapter we define the crucial 'blow-up-operator $H^\omega$' and $C^\omega(S)$, the semigroup of 'constructible' pointlike sets of $S$. Three instructive examples are given at the end of the chapter.

**Definition 3.1.** Define a map $H:S\xhookrightarrow{\text{into}}P(S)$ by $H(s):=\{t\mid t\mathscr{H}s\}=$ the $\mathscr{H}$-class of $s\in S$. If clarity requires, we write $H^S(s)$ to indicate that the $\mathscr{H}$-classes are being taken with respect to $S$.

**Note.** $H$ is just a map, not a homomorphism.

**Fact 3.2.** *If $S\leq T$, then $H^S(s)\subseteq H^T(s)$.*

**Definition 3.3.** Given $T \le P(S)$, define the important map $\bar{H}^T = H^T \circ \bigcup$. Since $T \subseteq P(S)$, $H^T(T) \subseteq P^2(S)$, so $H^T \circ \bigcup(T) \subseteq P(S)$. $\bar{H}^T : T \xhookrightarrow{\text{into}} P(S)$ is called '*H*-amalgamation'. It takes $A \in T$ (i.e., $A \subseteq S$), forms the $\mathscr{H}$-class of $A$ (with respect to $T$) $H(A) = \{A, A_1, \ldots, A_n \mid A_i \mathscr{H} A\}$ and unions $\bar{H}^T(A) = A \cup A_1 \cup \cdots \cup A_n$. Hence

**Definition 3.4.** Let $C^\omega(S) :=$ unique smallest semigroup $T$ such that
(i) $S \le T \le P(S)$;
(ii) $T$ is closed under subsets (i.e., $\mathrm{sub}(T) = T$);
(iii) $T$ is closed under $H$-amalgamation (with respect to $T$) (i.e., if $P \in T$, then $\bar{H}^T(P) \in T$).

**Note.** $C^\omega(S)$ is well defined, since $P(S)$ satisfies (i)–(iii), and if the $S_i$ ($i \in I$) satisfy (i)–(iii), then $\bigcap_{i \in I} S_i$ satisfies (i)–(iii).

**Fact 3.5.** (iii) *in Definition* 3.4 *may be replaced by*
(iii)′ *For every* $P \in G \le T$, $G$ *a subgroup of* $T$, $\bar{H}^T(P) \in T$.

**Proof.** (iii) implies (iii)′. Conversely, if $T$ satisfies (i), (ii), (iii)′, and $P \in T$, then $H(P) = P \cdot G$ ($G =$ Schützenberger-group), so $\bigcup H(P) = \bigcup (P \cdot G) = P \cdot \bigcup G \in T$ by (iii)′. □

An inductive definition of $C^\omega(S)$ can be given as follows:

**Definition 3.6.** $C^0(S) := S$, $C^n(S) := \mathrm{sub}(\langle \{\bar{H}^{C^{n-1}}(X) \mid X \in C^{n-1}\} \cup C^{n-1} \rangle)$ (where $\langle A \rangle$ denotes the semigroup generated by $A$), i.e., $C^n(S) =$ the semigroup generated by $C^{n-1}$ and all its $\mathscr{H}$-amalgams, closed under subsets. Then $S \le C^i(S) \le C^{i+1} \le P(S)$, so there exists an $N$ such that $C^N(S) = C^{N+1}(S)$. Then $C^\omega(S) = C^N(S)$. This shows

**Fact 3.7.** $C^\omega(S)$ *can be effectively determined* (*is computable*).

**Proof.** The iteration of Definition 3.6 has to stop in less than $2^{|S|}$ steps. (That the iteration may take more than 1 step is shown in Example 3.) □

**Fact 3.8.** $C^\omega(S) \le \mathrm{Pl}(S)$.

**Proof.** Look at the relation $R : S \twoheadrightarrow C$ with $\mathscr{C}(R) = \mathrm{Pl}(S)$, which exists by Fact 2.9. $R$ extends in the obvious way to $\bar{R} : \mathrm{Pl}(S) \twoheadrightarrow C$ (with $\bar{R}(A) = \{c_0 \in C \mid A \subseteq R^{-1}(c_0)\}$). Let $A \in \mathrm{Pl}(S)$, $A \in G \le \mathrm{Pl}(S)$, $G$ a subgroup of $\mathrm{Pl}(S)$. So in

$$\begin{array}{ccc} \bar{R} & \xrightarrow{p_2} & C \\ {\scriptstyle p_1}\downarrow & & \\ \mathrm{Pl}(S) & & \end{array}$$

we can lift $G \leq \mathrm{Pl}(S)$ to $\bar{G} \leq \bar{R}$ (such that $p_1(\bar{G}) = G$). $p_2(\bar{G}) \leq C$ is a subgroup, hence $p_2(\bar{G}) = c_0$ for some $c_0 \in C$, and $G \subseteq \bar{R}^{-1}(c_0)$. Since $\bar{R}^{-1}(c_0) = \{X \in \mathrm{Pl}(S) \mid X \subseteq R^{-1}(c_0)\}$, $\bigcup G \subseteq \bigcup \bar{R}^{-1}(c_0) \subseteq R^{-1}(c_0)$, hence $G \in \mathrm{Pl}(S)$. This shows $\mathrm{Pl}(S)$ satisfies (iii)$'$. It also satisfies (i), (ii) in Definition 3.4, hence $C^{\omega}(S) \leq \mathrm{Pl}(S)$. □

**Note.** The converse of Fact 3.8, namely $\mathrm{Pl}(S) \leq C^{\omega}(S)$, is the subject of Chapter 5.

*The $\bar{H}$-operator (H-amalgamation)*

**Notation 3.9.** From now on, we will work in $C^{\omega}(S) \leq P(S)$; all $\mathscr{H}$-classes are taken with respect to $C^{\omega}(S)$, so we write $H, \bar{H}$ sortly for $H^{C^{\omega}(S)}, \bar{H}^{C^{\omega}(S)}$. $A, B, C, \ldots$ denote elements of $C^{\omega}(S)$.

**Fact 3.10** (Properties of the $\bar{H}$-operator). (i) $\bar{H}(A) \supseteq A$; *equality holds if and only if the $\mathscr{J}$-class of $A$ is H-trivial.*

(ii) *($\bar{H}$ is local). There exist idempotents $e(A), e'(A)$ such that $\bar{H}(A) = A \cdot e(A) = e'(A) \cdot A$. Moreover, $e(A) = \bar{H}(G)$ with $G =$ right Schützenberger group of $A$, $e'(A) = \bar{H}(G')$ with $G' =$ left Schützenberger group of $A$ and if $1 \in G$, $1' \in G'$ are the identities of $G, G'$, then $A \cdot 1 = A$, $1' \cdot A = A$.*

(iii) $\bar{H}(A) \geq_{\mathscr{H}} A$; *$\bar{H}(A) = A$ if and only if the $\mathscr{J}$-class of $A$ is H-trivial. (So if the $\mathscr{J}$-class of $A$ is not H-trivial, $\bar{H}(A) \lneq A$ in the $\mathscr{L}$- and $\mathscr{R}$-order. This fact, together with* (i), *is of crucial importance in this paper*).

(iv) *If $A \mathscr{L} B$ via $X \cdot A = B$, $Y \cdot B = A$, then $\bar{H}(A) \mathscr{L} \bar{H}(B)$, via $X \cdot \bar{H}(A) = \bar{H}(B)$, $Y \cdot \bar{H}(B) = \bar{H}(A)$. Similar results are true for $\mathscr{R}$- and $\mathscr{J}$-equivalence.*

(v) *$A \mathscr{H} B$ implies $\bar{H}(A) = \bar{H}(B)$.*

(vi) *$Ax = A'x$ implies $A\bar{H}(x) = A'\bar{H}(x)$.*

(vii) *$Ax \mathscr{L} A'x$ implies $A\bar{H}(x) \mathscr{L} A'\bar{H}(x)$, $xA \mathscr{R} xA'$ implies $\bar{H}(x)A \mathscr{R} \bar{H}(x)A'$.*

**Negative results:**

(viii) (v) *has no converse*: *$\bar{H}(A) = \bar{H}(B)$, even though $A$ is not H-equivalent to $B$. Even adding $A \mathscr{J} B$ to the assumption does not help. See Example* 1.

(ix) *$\bar{H} : \mathscr{J}_1 \xrightarrow{\text{into}} \mathscr{J}_2$ need not be onto $\mathscr{J}_2$, see Example* 2.

(x) *$\bar{H}(A)$ is not necessarily H-trivial, and $\bar{H}^2 \neq \bar{H}$. (This leads to the definition of $H^{\omega}$.) See Example* 1.

(xi) *$A \subseteq B$ does not imply $\bar{H}(A) \subseteq \bar{H}(B)$, see Example* 1.

(xii) *$A \leq_{\mathscr{J}} B$ does not imply $\bar{H}(A) \leq_{\mathscr{J}} \bar{H}(B)$, see Example* 1.

**Proof.** (i) $\bar{H}(A) \supseteq A$ follows directly from the definition. Let $\bar{H}(A) = A$; $A_1 H A$; $A_1 \neq A$; let $G =$ right Schützenberger group of $H(A)$. With no loss of generality assume $A_1 = A \cdot g$, $A = A \cdot e$, $g \in G$, $g \neq e$, $e =$ identity of $G$. Then $A_1 \cup A = A$, so $A_1 \subseteq A$; but $A_1 \neq A$, so $A_1 = A \cdot g \subsetneq A \cdot e = A$. Multiplying by $g^{-1} \in G$ (which is 1-1 on $H(A)$) $A \subsetneq Ag$. Contradiction.

(ii) Since $H(A)$ can be written as $A\cdot G$ or $G'\cdot A$ with $G, G'\leq C^{\omega}(S)$ subgroups, $\bar{H}(A)=\bigcup H(A)=\bigcup(A\cdot G)=A\cdot\bigcup G$ and $\bar{H}(A)=\bigcup(G'\cdot A)=\bigcup G'\cdot A$. So let $e(A)=\bigcup G$, $e'(A)=\bigcup G'$. $\bigcup G$ is an idempotent, since $\bigcup G\cdot\bigcup G=\bigcup(G\cdot G)=\bigcup G$. Also $\bigcup G\in C^{\omega}(S)$, since $\bigcup G=\bigcup H(g_0)$, $g_0\in G$.

(iii) The first part of (iii) follows from (ii). If $H(A)=\{A\}$, then $\bar{H}(A)=\bigcup\{A\}=A$ trivially. Let $|H(A)|\geq 2$. By (ii), $\bar{H}(A)\leq A$ in $\mathscr{L}$ and $\mathscr{R}$. The following argument is of critical importance in this paper:

Let $H(A)=A\cdot G$. If $\bar{H}(A)=A\cdot\bigcup G$ was *$\mathscr{R}$-equivalent to* $A$, then the map given by multiplying with $\bigcup G$ on the right would be 1-1 on $H(A)$. But $A\cdot g\cdot\bigcup G=A\cdot\bigcup g\cdot G=A\cdot\bigcup G$ for all $g\in G$, so $\bigcup G$ is not 1-1 on $H(A)$ if $|G|\geq 2$.

The dual argument shows $\bar{H}(A)<A$ in $\mathscr{R}$.

(iv) The maps $X\circ-$ and $Y\circ-$ are 1-1 on $H(A)$ and $H(B)$, so $X\cdot H(A)=H(B)$, $Y\cdot H(B)=H(A)$. Hence $X\cdot\bar{H}(A)=X\cdot\bigcup H(A)=\bigcup H(B)=\bar{H}(B)$ and so on.

(v) Clear.

(vi) $A\bar{H}(X)=A\cdot X\cdot e(X)=A'Xe(X)=A'\bar{H}(X)$.

(vii) $A\bar{H}(X)=AXe(X)\mathscr{L}A'Xe(X)=A'\bar{H}(X)$, dually for $\mathscr{R}$. □

Fact 3.10(viii) suggests the following definition:

**Definition 3.11.** $H^{\omega}(A):=\bar{H}^{n}(A)$ with $n$ large enough so that $\bar{H}^{n+1}(A)=\bar{H}^{n}(A)$. This defines a map $H^{\omega}: C^{\omega}(S)\xrightarrow{\text{into}} C^{\omega}(S)$.

**Fact 3.12.** *$H^{\omega}$ has the following properties:*

(i) *$H^{\omega}(A)\supseteq A$, $H^{\omega}(A)=A$ if and only if the $\mathscr{J}$-class of $A$ is H-trivial.*

(ii) *($H^{\omega}$ is local). $A\in C^{\omega}(S)$, then there exist idempotents $E_i$ ($i=1,\ldots,n$ with $n\leq 2^{|S|}$) such that $H^{\omega}(A)=A\cdot E_1\cdot\ldots\cdot E_n$. Similarly $H^{\omega}(A)=F_m\cdot\ldots\cdot F_1A$ ($F_j^2=F_j$). Moreover, $E_i=\bigcup G_i$ for some group $G_i\leq C^{\omega}(S)$, and with $1_{G_i}\in G_i$ the identity $(AE_1\cdot\ldots\cdot E_{i-1})\cdot 1_{G_i}=AE_1\cdots E_{i-1}$ and $(AE_1\cdots E_{i-1})\subseteq AE_1\cdots E_i$.*

(iii) *$H^{\omega}(A)\geq_{\mathscr{H}}A$; $H^{\omega}(A)=A$ if and only if the $\mathscr{J}$-class of $A$ is $\mathscr{H}$-trivial. (So if the $\mathscr{J}$-class of $A$ is not H-trivial, $H^{\omega}(A)<_{\mathscr{H}}A$ is strict.)*

(iv) *If $A\mathscr{L}B$ via $X\cdot A=B$, $Y\cdot B=A$, then $H^{\omega}(A)\mathscr{L}H^{\omega}(B)$ via $X\cdot H^{\omega}(A)=H^{\omega}(B)$, $Y\cdot H^{\omega}(B)=H^{\omega}(A)$. Similar results are true for $\mathscr{R}$ and $\mathscr{J}$-equivalences.*

(v) *$A\mathscr{H}B$ implies $H^{\omega}(A)=H^{\omega}(B)$.*

(vi) *$H^{\omega}(A)$ is H-trivial.*

(vii) *$(H^{\omega})^2(A)=H^{\omega}(A)$.*

**Negative results:**

*Same as for $\bar{H}$* (Fact 3.10), *except now we have $(H^{\omega})^2=H^{\omega}$.*

**Proof.** (i)–(v) follow by iteration from Fact 3.10. (vii) is a consequence of (vi).

(vi) Assume $H^{\omega}(A)$ not $H$-trivial; then by (iii) $H^{\omega}(H^{\omega}(A))<H^{\omega}(A)$, a contradiction to the definition of $H^{\omega}$. □

The same examples as for $\bar{H}$ apply to $H^{\omega}$ to show negative results.

**Example 1.** Let $G$ be a group. Then $\mathrm{Pl}(G)=P(G)$. We will investigate the structure of $P(G)$ somewhat, to get interesting examples for the behavior of $\bar{H}$ and $H^{\omega}$.

*Definition* 1. Let $|A|$ denote the cardinality of $A\leq G$.

*Fact* 2. $|A\cdot B|\geq\max(|A|,|B|)$, $A,B\in P(G)$.

*Proof.* Follows from the fact that every map $a\circ -$ (multiplication by $a\in G$) is a 1-1 map.

*Fact* 3. $A\mathcal{J}B$ implies $|A|=|B|$. Follows from (2).

*Fact* 4. $AX=B$ and $|A|=|B|$, then $Ax_0=B$ for any $x_0\in X$.

*Proof.* Since $-\circ x_0$ is 1-1 and $|B|=|A|$, it is onto.

*Fact* 5. $A\mathcal{J}B$, then there exist $g_0,g_1\in G$ such that $g_0Ag_1=B$.

*Proof.* Follows from Facts 3 and 4.

So we see, that in order to show $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$-equivalence of elements in $P(G)$, one only has to look in $G$.

*Notation* 6. We write '$G_0\lhd G$' for $G_0$ is a normal subgroup of $G$ (i.e., $gG_0g^{-1}=G_0$ for all $g\in G$).

*Definition* 7. Let $G_0\leq G$ be a subgroup of $G$. Then let $N(G_0):=$ maximal subgroup $N\leq G$ such that $G_0\lhd N$.

It is well known that there is such a maximal subgroup called the normalizer of $G_0$ in $G$, and the following holds:

*Fact* 8. $N(G_0)=\{g\in G\mid gG_0=G_0g\}$.

*Fact* 9. $H(G_0)=\{G_0g\mid g\in N(G_0)\}=$ all cosets of $G_0$ in its normalizer.

*Proof.* If $X\mathcal{H}G_0$, then by Fact 5, $X=\alpha G_0=G_0\beta$, $\alpha,\beta\in G$. Then $\beta=\alpha n$ for some $n\in G_0$, so $\alpha G_0=G_0\beta=G_0\alpha n$, hence $\alpha G_0n^{-1}=\alpha G_0=G_0\alpha=X$, $\alpha\in N(G_0)$ and $X=G_0\alpha$.

Conversely, if $g\in N(G_0)$, $G_0g=gG_0$ and $G_0gg^{-1}=g^{-1}gG_0=G_0$, so $G_0g\mathcal{H}G_0$.

*Fact* 10. $\bar{H}(G_0)=N(G_0)$.

*Proof.* $\bar{H}(G_0)=\bigcup\{G_0g\mid g\in N(G_0)\}=N(G_0)$.

Now assume that $N(G_0)\lhd G$, but $N(G_0)\neq G$, $N(G_0)\neq G_0$ (so we have $G_0 \mathrel{\underset{\neq}{\lhd}} N(G_0)\mathrel{\underset{\neq}{\lhd}} G$, but $G_0$ *not normal* in $G$). Then $\bar{H}(G_0)=N(G_0)$, $\bar{H}(N(G_0))=G$, so $\bar{H}\neq\bar{H}^2$ and especially $H^{\omega}\neq\bar{H}$.

Now assume also that $|G|/|N(G_0)|=2$, so $G=N(G_0)\cup N(G_0)x$ for any $x\notin N(G_0)$ and $N(G_0)\cdot x=x\cdot N(G_0)$. Then $G_0\mathcal{R}G_0\cdot x$ and $x\cdot G_0\mathcal{L}G_0$, so $G_0x\mathcal{J}x\cdot G_0$ and $\bar{H}(G_0\cdot x)=[\bar{H}(G_0)]\cdot x=N(G_0)\cdot x$, $\bar{H}(x\cdot G_0)=x\cdot\bar{H}(G_0)=x\cdot N(G_0)$.

Also notice that if $G_0=N(G_0)$, $\bar{H}(G_0)=G_0$. But let $1\in G$ be the identity; then $\{1\}\subseteq G_0$, but $\bar{H}\{1\}=G\geq G_0=\bar{H}(G_0)$. Also $G_0\leq_{\mathcal{L}}\{1\}$, but $G_0\geq_{\mathcal{L}}G=\bar{H}(\{1\})$.

**Example 2.** Let $\mathbb{Z}_4$ with generator $z$ act on $\mathcal{M}$ $(A=\{a_1,a_2\},\ B=\{b_1,b_2\},\ \mathbb{Z}_2=\{\pm 1\},$ $C=\left(\begin{smallmatrix}1&1\\1&-1\end{smallmatrix}\right))$ by $(g,b_1)\cdot z=(-g,b_2)$, $(g,b_2)\cdot z=(g,b_1)$. (This action is linked with $z\cdot(a_1,g)=(a_2,-g)$, $z\cdot(a_2,g)=(a_1,g)$.)

In $C^{\omega}(S)$, we have the element $\alpha=\{(a_1,1,b_1),(a_1,-1,b_1)\}=(a_1,\pm 1,b_1)=\bar{H}(a_1,1,b_1)$. $(a_1,\pm 1,b_1)\cdot\mathbb{Z}_4=\{a_1,\pm 1,b_1 \text{ or } b_2\}=\beta$ since $\alpha\cdot\mathbb{Z}_4=\beta$, $\beta\cdot\alpha=\alpha$, $\alpha$ and $\beta$ are $\mathcal{R}$-related.

However, $\beta$ is not an $H$-amalgam of any $x\mathcal{J}(a_1,1,b_1)$, $\beta\neq\bar{H}(x)$ for all $x\in C^{\omega}(S)$, $x\mathcal{J}(a_1,1,b_1)$. If it were, $\beta=\beta_1\cap\beta_2$ with $\{\beta_1,\beta_2\}$ forming an $H$-class.

Then (i), (ii) or (iii).

(i) $\beta_1=(a_1,1,b_1 \text{ or } b_2)$, $\beta_2=(a_1,-1,b_1 \text{ or } b_2)$.

(ii) $\beta_1=\{(a_1,1,b_1),(a_1,-1,b_2)\}$, $\beta_2=\{(a_1,-1,b_1),(a_1,1,b_2)\}$.

(iii) $\beta_1=\{(a_1,1,b_1)(a_1,-1,b_1)\}$, $\beta_2=\{(a_1,1,b_2),(a_1,-1,b_2)\}$.

Case (iii) is ruled out, because $\beta_1$ and $\beta_2$ are not $\mathcal{L}$-equivalent. Case (i) and case (ii) are ruled out, since $(b_1)z=1$, $(b_2)z=-1$.

**Example 3.** Let

$$S^{(0)}:=\mathcal{M}^0\left(A=\{a,b\},\ B=\{a,b\},\ G=\{1\},\ C=\begin{array}{c} \\ a\\ b\end{array}\!\!\begin{array}{c}\begin{array}{cc}a & b\end{array}\\ \begin{pmatrix}1&0\\0&1\end{pmatrix}\end{array}\right).$$

Add an identity $\bar{1}$ to $S^{(0)}$ forming $S^{(1)}:=S^{(0)}+\bar{1}$. Form a Rees-matrix semigroup with coefficients in $S^{(1)}$ by

$$S^{(2)}:=\mathcal{M}^0\left(A'=\{1,2\},\ B'=\{1,2\},\ S^{(1)}, C'=\begin{pmatrix}1_{ab} & 1_{ba}\\ 1_{ba} & 1_{ab}\end{pmatrix}\right).$$

Now $\mathbb{Z}_2$ acts on $C'$ via $1\cdot z=2$, $2\cdot z=1$ ($z$ generator) and this action is linked. So add $\mathbb{Z}_2$ to $S^{(2)}$ forming $S^{(3)}:=S^{(2)}+\mathbb{Z}_2$. As can be easily seen, the semigroup-structure of $S^{(3)}$ is now

$\mathcal{J}_1$ $\quad\mathbb{Z}_2$;

$\mathcal{J}_2$

$$\begin{array}{c|c|c|}
 & 1 & 2\\ \hline
1 & 0 & 0\\ \hline
2 & 0 & 0\\ \hline
\end{array}$$

null $\mathcal{J}$-class elements are $(x,\bar{1},y)$, $x,y\in\{1,2\}$;

$\mathcal{J}_3$

$$\begin{array}{cc|cc|}
 & & \overbrace{ab}^{1} & \overbrace{ab}^{2}\\ \hline
1 & a & 01 & 00\\
 & b & 00 & 10\\
2 & a & 00 & 01\\
 & b & 10 & 00\\ \hline
\end{array}$$

regular, $G=\{1\}$;

$\mathcal{J}_4$ $\quad 0$ $\quad$ zero.

Now consider

$$x := \mathbb{Z}_2 \cdot (1, \bar{1}, 1) \cdot \mathbb{Z}_2, \quad \text{an element of } C^1(S);$$

$$x^2 = \{(x, (a, 1, b), y)\} \cup \{x, (b, 1, a), y\} \quad (x, y \in \{1, 2\});$$

$$x^3 = \{(x, (a, 1, a), y)\} \cup \{x, (b, 1, b), y\};$$

$$x^4 = x^2.$$

So $\{x^2, x^3\}$ is a group in $C^1(S)$, hence $\bar{H}(x^2) = x^2 \cup x^3 = \{(x, \alpha, 1, \beta, y) \mid x, y \in \{1, 2\}, \alpha, \beta \in \{a, b\}\}$ is an element of $C^2(S)$.

However, $\bar{H}(x^2)$ is not in $C^1(S)$ already, as can be easily checked. This is an example for $C^2(S) \neq C^1(S)$. It shows, moreover, that there are groups in $C^\omega(S)$ that are not 'derived from' groups of $S$, i.e., whose elements are not singletons (or subsets of groups). The group showing up here is a complicated subset of the bottom $\mathcal{J}$-class $\mathcal{J}_3$, created by the 'falling' of the middle-null-$H$ classes, after they were 'spread out' by the top group. One gets a good feel that $C^\omega(S)$ is really a definition tied to the *global* structure of $S$.

## 4. The Rhodes-expansion

The Rhodes-expansion, due to Rhodes, is an extremely useful tool in finite semigroup theory. In Subsection 4.1 we give a short definition and list some properties; for an extensive treatment of proofs, see [8]. In Subsection 4.2 we embed the Rhodes-expansion into a wreath-product, using a method originally due to Zeiger [9] and adapted to $\hat{S}$ by Rhodes [6] ('Zeiger-coding'). This embedding, extended to the infinite case, is also studied in [3].

### *4.1. Definition and elementary properties of the Rhodes-expansion*

**Note 4.1.** Throughout this chapter $\leq$ means $\leq_{\mathcal{L}}$ as defined in Notation 1.6.

**Definition 4.2.** $\hat{S} := \{(j_n, \dots, j_1) \mid j_n < \cdots < j_1,\ j_i \in S^1,\ n \geq 1\}$.

We will write suggestively $j_n < \cdots < j_1$ if $(j_n, \dots, j_1) \in \hat{S}$ (see Note 4.14 below concerning notation).

Define a canonical way to reduce $\leq$-chains to elements of $\hat{S}$ in the following way:

**Definition 4.3.** Let $x_n \leq \cdots \leq x_1$ be a $\leq$-chain, $x_i \in S^1$, $n \geq 1$. Then the min-operator goes down the chain from right to left, striking all elements in a run of $\mathcal{L}$-equivalent elements but the last one.

To be more precise, let $k = k(x_n \leq \cdots \leq x_1)$ be the number of $\mathcal{L}$-distinct elements in $x_n \leq \cdots \leq x_1$. Define

$$\alpha_1 := \max \alpha \text{ such that } x_\alpha \mathscr{L} x_1 \text{ and } x_{\alpha+1} <_{\mathscr{L}} x_\alpha,$$

$$\alpha_{n+1} := \max \alpha \text{ such that } x_\alpha \mathscr{L} x_{\alpha_n+1} \text{ and } x_{\alpha+1} <_{\mathscr{L}} x_\alpha$$

$(n \leq k-1)$. Note that $\alpha_k = n$.

Then

$$\min(x_n \leq \cdots \leq x_1) := x_{\alpha_k} < x_{\alpha_{k-1}} < \cdots < x_{\alpha_1}.$$

In other words: if $x_n \leq \cdots \leq x_1$ is of the form

$$(x_n = x_{\alpha_k} < \cdots < x_{\alpha_{k-1}} < \cdots < x_{\alpha_2} \mathscr{L} \cdots \mathscr{L} x_{\alpha_1+1} < x_{\alpha_1} \mathscr{L} \cdots \mathscr{L} x_1),$$

then $\min(x_n \leq \cdots \leq x_1)$ is as above.

**Definition 4.4.** Define a product on $\hat{S}$ by

$$(j_n < \cdots < j_1) \cdot (j'_m < \cdots < j'_1) = \min(j_n j'_m \leq \cdots \leq j_1 j'_m < \cdots < j'_1).$$

It is easy to see that this product is associative. Some of the properties of $\hat{S}$ are the following:

**Fact 4.5.** *$\hat{S}$ is generated by strings of length one.*

**Definition 4.6.** Let $\eta_S : \hat{S} \twoheadrightarrow S^1$ be defined by

$$\eta_S(j_n < \cdots < j_1) = j_n.$$

($\eta_S$ is called the 'canonical map'). Then

**Fact 4.7.** *$\eta_S$ is a $\gamma$-map.*

**Fact 4.8.** *Let $\hat{s} = j_n < \cdots < j_1$, $\hat{t} = j'_m < \cdots < j'_1$ be elements of $\hat{S}$. Then*

$$\hat{s} \leq \hat{t} \ (\text{in } \hat{S}) \quad \text{iff} \quad m \leq n,\ j_i = j'_i \ (i = 1, \ldots, m-1) \text{ and } j_m \mathscr{L} j'_m \ (\text{in } S).$$

**Fact 4.9.**

$$\hat{s} \mathscr{L} \hat{t} \ (\text{in } \hat{S}) \quad \text{iff} \quad m = n,\ j_i = j'_i \ (i = 1, \ldots, m-1) \text{ and } j_n \mathscr{L} j'_n \ (\text{in } S).$$

**Fact 4.10.** *$(\hat{S}, \hat{S})$ is faithful.*

**Proof.** 1 acts as left (but not right) identity:

$$(1)(j_n < \cdots < j_1) = \min(1 \cdot j_n \leq j_n < \cdots < j_1) = (j_n < \cdots < j_1). \qquad \square$$

**Notation 4.11.** Every $j_n < \cdots < j_1 \in \hat{S}$ can be given a representation (which is *not* unique) as follows:

$$j_n < \cdots < j_1 = (s_n \cdot \ldots \cdot s_1 < s_{n-1} \cdot \ldots \cdot s_1 < \cdots < s_1)$$

with $s_i \in S^1$ or inductively,

$$j_1 = s_1, \qquad j_{k+1} = s_{k+1} \cdot j_k \quad (k+1 \leq n).$$

**Fact 4.12.** *Let $s \in S^1$, $k_0$ be the first $k$ such that $j_k s \mathcal{R} j_k$ (i.e., $j_e s <_{\mathcal{R}} j_e$ for $e < k$). Then $j_{k_0+\alpha} s \mathcal{R} j_{k_0+\alpha}$, $\alpha = 1, \ldots, n-k_0$.*

**Proof.** By induction on $x$;

$$j_{\alpha+x} s = s_{\alpha+x} j_{\alpha+x-1} \cdot s \mathcal{R} s_{\alpha+x} j_{\alpha+x-1} = j_{\alpha+x+1}$$

since $j_{\alpha+x-1} \cdot s \mathcal{R} j_{\alpha+x-1}$ by inductive assumption, and $\mathcal{R}$ is a left congruence. $\square$

**Notation 4.13.** Let

$$(s_n \ldots s_1 < \cdots < s_1) \cdot s = \min(s_n \ldots s_1 s \leq \cdots \leq s_1 s \leq s) = (s'_m \ldots s'_1 s < \cdots < s'_1 s)$$

notationally. Then the $s'_i$ can be chosen as follows: If $s'_\beta \ldots s'_1 s = s_{k_0} \ldots s_1 s$, then

$$s'_{\beta+\alpha} = s_{k_0+\alpha}, \quad \alpha = 1, \ldots, n-k_0.$$

(Follows from Fact 4.12).

**Note 4.14.** The notation $j_n < \cdots < j_1$ for an element in $\hat{S}$ is bad, but $s_n \ldots s_1 < \cdots < s_2 s_1 < s_1$ is even worse. The reason is that an input sequence $(s_1, \ldots, s_n)$, applied to the string (1), yields successively

$$(1) \xrightarrow{s_1} \min(s_1 \leq 1) \xrightarrow{s_2} \min(s_1 s_2 \leq s_2 \leq 1) \rightarrow \cdots$$

$$\xrightarrow{s_n} \min(s_1 \ldots s_n \leq s_2 \ldots s_n \leq \cdots \leq s_n \leq 1).$$

(This is the famous 'reversal of time' connected with $\hat{S}$, see [3].) However, since it has been used in the literature, we are going to stick with it for the time being.

### 4.2. *$(\hat{S}, \hat{S})$ is in a canonical wreath product*

(See also [3] for an expanded version.)

**Note 4.15.** We now want to show that $(\hat{S}, \hat{S})$ can be embedded into a certain canonical wreath product. The embedding is done in three steps:

(i) We pass to an isomorphic semigroup $(\mathrm{Code}(\hat{S}), Z(S))$ using a 1-1 map 'Code' ('Zeiger-coding').

(ii) We define a coordinate map $\mathrm{Exp}: \mathrm{Code}(\hat{S}) \xrightarrow[\text{into}]{\text{1-1}} \Pi$ ('expansion into coordinates', $\Pi = \bar{\mathscr{J}}_N \times \cdots \times \bar{\mathscr{J}}_1$ is roughly the Cartesian product of the $\mathscr{J}$-classes of $S$) and use it to construct an isomorphism

$$(\mathrm{Code}(\hat{S}), Z(S)) \cong (\mathrm{Exp}(\mathrm{Code}(\hat{S})), \mathrm{Exp}^{-1} Z(S)\mathrm{Exp}).$$

(iii) We show that each element $\alpha \in \mathrm{Exp}^{-1} Z(S)\mathrm{Exp}$ can be extended to a triangular map on all of $\Pi \supseteq \mathrm{Exp}(\mathrm{Code}(\hat{S}))$ with the correct component action, completing the embedding.

*4.2.1. Zeiger-coding*

The original idea of Zeiger-coding is due to Zeiger. Rhodes refined Zeiger's idea and adapted it to $\hat{S}$. We will present it here in an algebraic form, although it really is more a computer science idea on how to code nests $X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n$.

**Definition 4.16.** Let $\vec{S} := \{x_m <_{\mathscr{J}} \cdots <_{\mathscr{J}} x_1 \mid x_i \in S^1,\ m \geq 1\}$ be the set of all (strict) $\mathscr{J}$-chains on $S$.

**Definition 4.17.** Define a code-map $\mathrm{Code} : \hat{S} \xhookrightarrow[\text{into}]{\text{1-1}} \vec{S}$ as follows: For each $j = (a, g, b)$ let $t(j), \bar{t}(j)$ be arbitrary fixed elements such that

$$j \cdot t(j) = (a, 1, 1) \quad \text{and} \quad (a, 1, 1)\bar{t}(j) = j, \tag{4.18}$$

so

$$j \cdot t(j) \cdot \bar{t}(j) = j.$$

Note that

$$j \mathscr{R} x \quad \text{implies} \quad jt(j) = x \cdot t(x). \tag{4.19}$$

Let $\hat{j} = j_n < \cdots < j_1 \in \hat{S}$. Define

$$x_1 := j_1, \qquad x_n := j_n \cdot t(x_1)t(x_2) \ldots t(x_{n-1}) \tag{4.20}$$

and

$$\mathrm{Code}(\hat{j}) = (x_n <_{\mathscr{J}} \cdots <_{\mathscr{J}} x_1) \in \vec{S}. \tag{4.21}$$

**Note 4.22.** To understand this definition better, take a representation of $\hat{j} = s_n \ldots s_1 < \cdots < s_2 s_1 < s_1$ ($s_i \in S^1$). Then

$$\begin{aligned}
\mathrm{Code}(\hat{j}) = \quad & s_1 && = x_1 \\
& s_2 s_1 t(s_1) && = x_2 \\
& \underbrace{s_3 s_2 s_1 t(s_1)}_{x_2}\ \underbrace{t(s_2 s_1 t(s_1))}_{t(x_2)} && = x_3 \\
& \quad \vdots && \quad \vdots
\end{aligned}$$

or in general we have the recursion formulas

(4.23) representation of $\hat{j}$ code of $\hat{j}$

$$\begin{array}{ll} j_1 := s_1 & x_1 := s_1 \\ j_n := s_n \cdot j_{n-1} & x_n := s_n \cdot x_{n-1} \cdot t(x_{n-1}) \end{array}$$

In words: $j_n$ comes from $j_{n-1}$ via hitting by $s_n$ on the left, $x_n$ comes from $x_{n-1}$, first *adjusted to standard element* $x_{n-1} \cdot t(x_{n-1}) = (a, 1, 1)$, via hitting by $s_n$ on the left. (This is the reason for Fact 4.29(ii) below.)

Clearly Code is 1-1 and maps into $\vec{S}$ (see Fact 4.26). In fact the inverse Decode : $\mathrm{Code}(\hat{S}) \xrightarrow[\text{onto}]{1\text{-}1} \hat{S}$ is given by

**Definition 4.24.** Let $(x_n, \ldots, x_1) \in \mathrm{Code}(\hat{S})$. Define

$$j_1 := x_1, \qquad j_n := x_n \cdot \bar{t}(x_{n-1}) \ldots \bar{t}(x_1).$$

Let $\mathrm{Decode}(x_n, \ldots, x_1) = (j_n < \cdots < j_1)$.

**Fact 4.25.** Decode *is the inverse of* Code.

**Proof.** By induction,

$$\begin{aligned} x_k \cdot \bar{t}(x_{k-1}) \ldots \bar{t}(x_1) &= s_k \cdot x_{k-1} \cdot t(x_{k-1})\bar{t}(x_{k-1}) \ldots \bar{t}(x_1) \quad \text{(by (4.23))} \\ &= s_k \cdot x_{k-1} \cdot \bar{t}(x_{k-2}) \ldots \bar{t}(x_1) \quad \text{(by (4.18))} \\ &= s_k \cdot j_{k-1} \quad \text{(by induction hypothesis)} \\ &= j_k. \end{aligned}$$

□

**Fact 4.26.** *With notation as above,*

(i) $j_i \mathscr{R} x_i$ $(1 \leq i \leq n)$;

(ii) $x_n <_{\mathscr{J}} \cdots <_{\mathscr{J}} x_1$ *is strict.*

**Proof.** (i) By induction

$$x_k t(x_k) \mathscr{R} x_k \mathscr{R} j_k \quad \text{(by induction assumption)}$$

so

$$x_{k+1} = s_{k+1} x_k t(x_k) \mathscr{R} s_{k+1} j_k = j_{k+1}.$$

(ii) $x_n \leq_{\mathscr{J}} \cdots \leq_{\mathscr{J}} x_1$ by definition. (i) and $j_n <_{\mathscr{L}} \cdots <_{\mathscr{L}} j_1$ show (ii). □

**Definition 4.27.** Using the 1-1 map Code : $\hat{S} \xrightarrow[\text{onto}]{1\text{-}1} \mathrm{Code}(\hat{S}) \leq \vec{S}$, we get an isomorphism in the usual way, i.e., let

$$Z(S) := \{\mathrm{Decode} \circ \hat{s} \circ \mathrm{Code} \mid \hat{s} \in \hat{S}\}.$$

Then $Z(S)$ acts on Code$(\hat{S})$, and

$$(\hat{S}, \hat{S}) \cong (\mathrm{Code}(\hat{S}), Z(S)).$$

Notationally, the action of $Z(S)$ on Code$(\hat{S})$ can be described as follows (Note: $Z(S)$ is generated by Decode $\circ\,(s)\circ$ Code, $s \in S^1$):

**Notation 4.28.** Let $s \in S^1$ and let

$$(s_n \dots s_1 < \dots < s_1) \cdot s = (s'_m \dots s'_1 s < \dots < s'_1 s)$$

(see Notation 4.13). Let

$$x'_1 = s'_1 s, \qquad x'_\alpha = s'_\alpha x'_{\alpha-1} t(x'_{\alpha-1}) \quad (\alpha \le m).$$

Then, notationally,

$$(x_n <_{\mathscr{J}} \dots <_{\mathscr{J}} x_1) \cdot s = x'_m < \dots < x'_1.$$

**Fact 4.29.** (i) *If $j_n s <_{\mathscr{R}} j_n$, then $x'_m < x_n$.*

(ii) *Assume $x_n \mathscr{R} x'_m$. Then*

$$(x_{n+1} <_{\mathscr{J}} \dots <_{\mathscr{J}} x_1) \cdot s = (x'_{m+1} <_{\mathscr{J}} x'_m <_{\mathscr{J}} \dots <_{\mathscr{J}} x'_1)$$

*with $x'_{m+1} = x_{n+1}$.*

**Proof.** (i) $x'_m \mathscr{R} s_n \dots s_1 s < s_n \dots s_1 \mathscr{R} x_n$.

(ii) Let Decode$(x_{n+1} <_{\mathscr{J}} \dots <_{\mathscr{J}} x_1) = (s_{n+1} s_n \dots s_1 < \dots < s_1)$. $x_n \mathscr{R} x'_m$ means $s_n \dots s_1 s \mathscr{R} s_n \dots s_1$, and hence $(s_{n+1} s_n \dots s_1 < \dots < s_1) \cdot s = s_{n+1} \dots s_1 s < s_n \dots s_1 s < s'_{m-1} \dots s'_1 s < \dots < s'_1 s$ (with $s'_m \dots s'_1 s = s_n \dots s_1 s$, see Notation 4.13). Hence

$$\begin{aligned} x'_{m+1} &= s_{n+1} x'_m t(x'_m) \\ &= s_{n+1} x_n t(x_n) \quad \text{(by (4.19))} \\ &= x_{n+1} \quad \text{(by (4.23))}. \quad \square \end{aligned}$$

**Fact 4.30.** *Let $(x_n < \dots < x_1) \cdot s = (x'_m < \dots < x'_1)$. Let $k_0$ be the first $k$ such that $x_k \mathscr{R} x'_\alpha$ (where $(x_k < \dots < x_1) \cdot s = (x'_\alpha < \dots < x'_1)$). Then $n - k = m' - \alpha$ and $x_{k+x} = x_{\alpha+x}$, $x = 1, \dots, n-k$.*

**Proof.** Follows from Fact 4.29(ii). $\square$

*4.2.2. Expansion into coordinates*

We now define the embedding of $(\hat{S}, \hat{S})$ into a wreath-product, using the following Exp map (Expansion into coordinates).

**Definition 4.31.** In the following choose an arbitrary fixed ordering of the $\mathscr{J}$-classes of $S^1$, compatible with the $\mathcal{J}$-order, i.e., enumerate the $\mathcal{J}$-classes of $S^1$ so that $J_i <_{\mathcal{J}} J_k$ implies $k < i$. Also for each $B_i, A_i$ choose an ordering $B_i = \{b_1^{(i)}, \dots, b_n^{(i)}\}$, $A_i = \{a_1^{(i)}, \dots, a_m^{(i)}\}$. If the $i$ is clear from the context, we denote $b_1^{(i)}$ and $a_1^{(i)}$ shortly by 1. So $J_1 = G$, the group of units of $S^1$, and if $N$ is the number of $\mathcal{J}$-classes of $S^1$, $J_N = K(S^1)$ is the kernel of $S$.

**Note 4.32.** It follows from the definition that if $j \in J_k$, $s \in S^1$, $j \cdot s \notin J_k$, then $j \cdot s \in J_i$ for some $\mathcal{J}$-class $J_i$ with $i > k$.

**Definition 4.33.** Give every $\mathcal{J}$-class $J_i$ of $S^1$ an arbitrary fixed Rees-matrix representation, so that

$$J_i^0 \cong \mathcal{M}^0(A_i, B_i, G_i, C_i) \quad (i = 1, \dots, N)$$

(with $A_i \leftrightarrow$ the set of $\mathcal{R}$-classes of $J_i$, $B_i \leftrightarrow$ the set of $\mathcal{L}$-classes of $J_i$, $G_i \cong$ the Schützenberger-group of $J_i$, $C_i =$ structure-matrix of $J_i$). Choose all $B_i, A_i$ disjoint. Let $b$ be a separate symbol, and define

**Definition 4.34.**

$$\bar{B}_1 := B_1 \cup \cdots \cup B_N, \qquad \bar{G}_1 := (A_1 \times G_1) \cup \{b\},$$

$$\bar{B}_k := \bigcup_{N \geq i \geq k} B_i \cup \{b\}, \qquad \bar{G}_k := (A_k \times G_k) \cup \{b\}.$$

Call the $\bar{B}_i$ '(generalized) $B$-coordinates' and $\bar{G}_k$ '(generalized) $G$-coordinates'.

$$\bar{J}_k := \bar{G}_k \times \bar{B}_k, \qquad \Pi := \bar{J}_N \times \cdots \times \bar{J}_1.$$

If necessary, we write $\Pi(S)$ to indicate the semigroup whose $\mathcal{J}$-classes are used in $\bar{J}_i$.

**Notation 4.35.** We denote an element of $\Pi$ by

$$\vec{y} = (y_N, \dots, y_1).$$

Then $y_i = y_i^{(g)}, y_i^{(b)}$ with $y_i^{(g)} \in \bar{G}_i$, $y_i^{(b)} \in \bar{B}_i$. (We will also denote initial substrings of an element of $\Pi$ by $\vec{y}$.)

Let $x_i = (a_i, g_i, b_i) \in J_{n_i}$ $(i = 1, \dots, k)$ in the above coordinatization and $\mathcal{J}$-class ordering. So $n_1 < n_2 < \cdots < n_k$.

With every element $\vec{x} = (x_n <_{\mathcal{J}} \cdots <_{\mathcal{J}} x_1) \in \vec{S}$ we want to associate an element of $\Pi$, called the 'coordinates of $\vec{x}$'.

**Definition 4.36.** Let $n_0 := 0$, $1 \leq l \leq k$.

$$C_\alpha(\vec{x}) = \begin{cases} ((a_l, g_l), b_l) & \text{for } \alpha = n_l, \\ (b, b_l) & \text{for } n_{l-1} < \alpha < n_l, \\ (b, b) & \text{for } \alpha > n_k. \end{cases}$$

**Definition 4.37.** Define $\operatorname{Exp}(\vec{x})$, the 'expansion of $\vec{x}$ (into coordinates)' by

$$\operatorname{Exp}(\vec{x}) = C_N(\vec{x}), \ldots, C_1(\vec{x}).$$

Then Exp is a map $\operatorname{Exp} : \vec{S} \xhookrightarrow{\text{into}} \Pi$, which is clearly 1-1. The 'picture' is:

| | | | |
|---|---|---|---|
| $\mathscr{J}$-classes listed compatible with $\mathscr{J}$-order $\downarrow$ | $\mathscr{J}_1$ | $b_1$ | put $b_1$ in $\bar{B}$-spot indicating that $b_1$ |
| | | $b$ | is 'coming up'; $b$ stands for blank. |
| | $\vdots$ | $b_1$ | |
| | | $b$ | |
| | | $\vdots$ | |
| | | $b_1$ | put $(a_1, g_1), b_1$ into $J_{n_1}$-slot. |
| | $\mathscr{J}_{n_1}$ | $\left.\begin{matrix} g_1 \\ a_1 \end{matrix}\right\}$ | |
| | $\vdots$ | $b_2$ | put 'upcoming' $b_2$. |
| | | $b$ | |
| | | $b_2$ | |
| | $\mathscr{J}_{n_2}$ | $\left.\begin{matrix} g_2 \\ a_2 \end{matrix}\right\}$ | |
| | $\vdots$ | $\vdots$ | |
| | $\mathscr{J}_N$ | $b$ | |
| | | $b$ | |

Intuitively, $b$ stands for blank; $a_i, g_i, b_i \in J_{n_i}$ are put in the $J_{n_i}$-slot, and the 'upcoming' $b_i \in J_{n_i}$ are 'slid-up' or 'advanced' to the highest spot possible (i.e., not occupied by a 'real' $a_{i-1} g_{i-1} b_{i-1}$).

**Remark 4.38.**

$$\begin{tikzcd}
(\hat{S}, \hat{S}) \arrow[rr, leftrightarrow, "\text{Code}"] \arrow[d, leftrightarrow, "\text{Exp}"'] & & (\operatorname{Code}(\hat{S}), Z(S)) \arrow[d, leftrightarrow, "\text{Exp}"] \\
(\Pi, \omega) \geq (\operatorname{Exp}(\hat{S}), \operatorname{Exp}^{-1} \hat{S} \operatorname{Exp}) \arrow[rr, leftrightarrow, "\operatorname{Exp}^{-1} \operatorname{Code} \operatorname{Exp}"'] & & (\operatorname{Exp} \operatorname{Code}(\hat{S}), \operatorname{Exp}^{-1} Z(\hat{S}) \operatorname{Exp}) \\
& & \leq (\Pi, \omega)
\end{tikzcd}$$

($\omega$ = wreath product of $F_R(\bar{J}_i)$'s).

The map $\overline{\operatorname{Code}} = \operatorname{Exp}^{-1} \operatorname{Code} \operatorname{Exp} : \operatorname{Exp}(\hat{S}) \to \operatorname{Exp} \operatorname{Code}(\hat{S})$ is 1-1, onto (composition of 1-1 onto maps).

Moreover, it is easy to see from the definition of Code that $\overline{\operatorname{Code}}$ is triangular (on $\operatorname{Exp}(\hat{S})$).

It is also easy to see that $\mathrm{Exp}^{-1}\hat{S}\,\mathrm{Exp}$ is triangular, even without Advance $b_i$; for meaning of Advance $b_i$ see Remark 4.54(ii) below. Hence $\mathrm{Exp}^{-1}Z(S)\mathrm{Exp}$ is triangular.

We will now prove that the component action of $\alpha \in \mathrm{Exp}^{-1}Z(S)\mathrm{Exp}$ is remarkably simple, and that every element $\alpha \in \mathrm{Exp}^{-1}Z(S)\mathrm{Exp}$ can be extended to all of $\Pi$ in the obvious way, keeping the same simple component action.

*4.2.3. Extending* $\mathrm{Exp}^{-1}Z(S)\mathrm{Exp}$ *to all of* $\Pi$

**Definition 4.39.** Putting Code and Exp together, we get a map

$$\alpha = \mathrm{Code}\circ\mathrm{Exp}: \left\{\begin{array}{ccccc} \mathscr{L}\text{-chains} & & \mathscr{J}\text{-chains} & & \text{coordinates} \\ \hat{S} & \xrightarrow[\text{into}]{\text{1-1}} & \vec{S} & \xrightarrow[\text{into}]{\text{1-1}} & \Pi \end{array}\right.$$

Using $\alpha$, $(\hat{S},\hat{S})$ can be transformed to a *TSG* on $\alpha(\hat{S}) \subseteq \Pi$ by letting $\hat{s} \in \hat{S}$ act as $\alpha^{-1}\hat{s}\alpha$, i.e., let $p \in \alpha(\hat{S})$, $p = \alpha(\hat{t}) = \hat{t}\alpha$, define an action $*$ of $\hat{S}$ on $\alpha(\hat{S})$ by

$$p * \hat{s} = p\alpha^{-1}\hat{s}\alpha = (\hat{t}\alpha)\alpha^{-1}\hat{s}\alpha = \hat{t}\hat{s}\alpha.$$

So we get a TSG $(\hat{S}\alpha, \hat{S}) \leq (\Pi, F_R(\Pi))$ (via $\alpha$).

**Remark 4.40.** Of course, $(\hat{S}\alpha, \hat{S})$ may also be viewed as isomorphic to $(\mathrm{Code}(\hat{S}), Z(S))$ via Exp. The crucial fact is then Fact 4.30, which translates into Remark 4.53 on $\Pi$.

**Fact 4.41.** *Let* $s \in S^1$, $\hat{s} = (s)$, *the string of length one containing s. (Remember Fact 4.5:* $\hat{S}$ *is generated by* $\hat{s}$.*) Every* $\alpha^{-1}\hat{s}\alpha : \hat{S}\alpha \to \hat{S}\alpha \subseteq \Pi$ *can be extended to a map* $\bar{s} : \Pi \xhookrightarrow{\text{into}} \Pi$ *such that*

(o) $\bar{s}$ *restricted to* $\hat{S}\alpha = \alpha^{-1}\hat{s}\alpha$ *(extension)*;

(i) $\bar{s}$ *is triangular on* $\Pi$;

(ii) *The component action of* $\bar{s}$ *is as follows*:

(a) *on B-coordinates* $\bar{B}_i : (\bar{B}_i, 1^*)$,

(b) *on G-coordinates* $\bar{G}_i : (\bar{G}_i, (1 \times G_i)^*)$ *(where* $(1 \times G_i)$ *is extended to b by* $b \cdot g = b$*).*

*Here the component action is permutation reset, with non-trivial groups only in G-coordinates.*

**Note 4.42.** This gives another constructive proof of the prime decomposition theorem, by $S \twoheadleftarrow \hat{S} \leq (\Pi, (1 \times G_N)^* \circ 1^* \circ \cdots \circ (1 \times G_1)^* \circ 1^*)$.

**Proof of Fact 4.41.** We will construct $\bar{s}$ by induction and simultaneously show (o)–(ii).

**Notation 4.43.** In the proof we will use as standard notation:

Representation

$$\hat{j} = \begin{array}{c} j_1 \\ < \\ \vdots \\ < \\ j_n \\ < \\ j_{n+1} \end{array} = \begin{array}{c} s_1 \\ < \\ s_2 s_1 \\ < \\ \vdots \\ s_n \dots s_1 \\ < \\ s_{n+1} s_n \dots s_1 \end{array} \xrightarrow{\text{action} - \circ s} \begin{array}{c} s_1' s \\ < \\ \vdots \\ < \\ s_m' \dots s_1' s \end{array} \quad (= s_{n+1} s_n \dots s_1 s)$$

$$\begin{array}{ccc} \big\updownarrow \text{Code} & \qquad\qquad & \big\updownarrow \end{array}$$

$$\vec{x} = \begin{array}{c} x_1 \\ < \\ \vdots \\ < \\ x_n \\ < \\ x_{n+1} \end{array} = \mathrm{Code}(\hat{j}) \xrightarrow{\text{induced action of } s} \begin{array}{c} x_1' \\ < \\ \vdots \\ < \\ x_m' \end{array} = \mathrm{Code}(\hat{j} \cdot s)$$

$$\begin{array}{ccc} \big\updownarrow \text{Exp} & \qquad\qquad & \big\updownarrow \text{Exp} \end{array}$$

$$\begin{array}{c} y_1 \\ \vdots \\ y_N \end{array} \xrightarrow{\alpha^{-1} s \alpha} \begin{array}{c} y_1' \\ \vdots \\ y_N' \end{array}$$

**Note 4.44.**

$$x_{n+1} \xrightarrow{\text{Code}^{-1}} x_{n+1} \bar{t}(x_n) \dots \bar{t}(x_1) = j_{n+1} \xrightarrow{s} j_{n+1} s$$
$$= x_{n+1} \bar{t}(x_n) \dots \bar{t}(x_1) \cdot s \xrightarrow{\text{Code}} x_{n+1} \bar{t}(x_n) \dots \bar{t}(x_1) \cdot s \cdot t(x_1') \dots t(x_{m-1}').$$

**Definition 4.45.** Let $s(\vec{x}) = \bar{t}(x_n) \dots \bar{t}(x_1) s t(x_1') \dots t(x_{m-1}')$ for $\vec{x} = x_n < \cdots < x_1$.

**Note 4.46.** Note 4.44 shows that

$$b(x_m') = b(x_n) \cdot s(\vec{x}), \qquad g(x_m') = g(x_n) \cdot (b(x_n))[s(\vec{x})]_G .$$

**Definition 4.47.** Call $\vec{y} \in \hat{S}\alpha \leq \Pi$ a 'legal' string. Also call any initial substring of a legal string legal. Call a string illegal if it is not legal.

**Definition 4.48.** If $\vec{y}=(y_k,\dots,y_1)$ is legal, there exists a maximum $n$ such that $y_n^{(g)}\neq b$ ($n$ may be zero). Call this $n=n(\vec{y})$ the 'length of the entire part'. Let

$$(y_n,\dots,y_1):=\text{'the entire part of }\vec{y}\text{'},$$

$$(y_k,\dots,y_{n+1}):=\text{'the rest (of }\vec{y}\text{)'}.$$

**Fact 4.49.** *There exists a unique* $j_n<\cdots<j_1\in\hat{S}$ *such that* $\operatorname{Exp}\operatorname{Code}(j_n<\cdots<j_1)=(y_n,\dots,y_1)$. *'The rest' is either* $y_{n+x}=(\bar{b},b)$ *for* $x=1,\dots,k-m$ *or* $y_{n+x}=(b,b)$ *with* $\bar{b}\in\bar{B}_{k+\alpha}$ *for some* $\alpha\geq 1$.

**Proof.** Follows from definition of Exp. □

We now proceed with the definition of $\bar{s}$ by induction:

$k=1$. Remember $J_1^0\cong(\mathcal{M}^0(1,1,G,1))$ with $G$ the group of units. Let $s=(a_0,g_0,b_0)$. Then $\bar{s}(y_1^{(b)})=b_0$.

For $s\notin G$, set

$$\bar{s}(y_1^{(g)},y_1^{(b)})=b.$$

For $s\in G$, set

$$\bar{s}(y_1^{(g)},y_1^{(b)})=\begin{cases} b & \text{if } y_1^{(b)}\neq 1\in B_1,\\ y_1^{(g)}(y_1^{(b)})s_G & \text{if } y_1^{(b)}=1.\end{cases}$$

This clearly satisfies (i) and (ii), and since $(s_n<\cdots<s_1)s=\min(s_ns\leq\cdots\leq s_1s\leq s)=(s_ns<\cdots<s_ks)$ with $s_ks\mathscr{L}s$, $\bar{s}$ extends $\alpha^{-1}s\alpha$, i.e. (o).

**Induction assumption 4.50.** Assume $\bar{s}$ has been defined for $1\leq i\leq k$, satisfying (o)–(ii) on $\bar{J}_k\times\cdots\times\bar{J}_1$.

*Case* 1. $\vec{y}=(y_k,\dots,y_1)$ illegal. Then let $\bar{s}(y_{k+1},\dots,y_1)=b$. This clearly satisfies (i) and (ii), and (o) automatically.

*Case* 2. $\vec{y}=(y_k,\dots,y_1)$ is legal. Assume $(y_n,\dots,y_1)$, the entire part of $\vec{y}$, is $\operatorname{Exp}\operatorname{Code}(j_n<\cdots<j_1)=\operatorname{Exp}(x_n<_{\mathscr{J}}\cdots<_{\mathscr{J}}x_1)$ (see Notation 4.43).

(i) Assume further $j_is\mathscr{R}j_i$ for some $1\leq i\leq n$. Then Fact 4.30 shows that $\alpha^{-1}\hat{s}\alpha$ fixes all coordinates after $n_i$ ($j_i\in J_{n_i}$), so we may extend it to $\bar{s}(y_{k+1},\dots,y_1)=y_{k+1}$ and satisfy (o)–(ii).

(ii) Assume now that $j_ns<_{\mathscr{R}}j_n$.

**Notation 4.51.** Let

$$L(k):=\max_l\{j_l\cdot s\in J_k\},$$

$$L(k)=0\quad\text{if no such } l \text{ exist.}$$

($L(k)=0$ means no $j_l$ above $J_k$ falls into $J_k$ under $s$; $L(k)=l_0$ means some $j_l$ above $J_k$ falls into $J_k$, and among all such $j_l, j_{l_0}$ is the last one to do so.)

Let

$$M(k) := \min_{m \geq k} \text{ such that } L(m) \neq 0,$$

$$M(k) = 0 \quad \text{if no such } m \text{ exists.}$$

($M(k)=0$ means nothing falls into $J_k$ from above, and nothing falls by $J_k$, landing somewhere below. $M(k)=k$ means something falls into $J_k$. $M(k)=m_0>k$ means something falls into a $\mathcal{J}$-class below $k$, i.e., 'passes $k$ by'.)

(α) $M(k+1) \neq 0$ (i.e., something falls into or past $J_{k+1}$). Put

$$\bar{s}(y_{k+1}^{(b)}, \vec{y}) = b(j_{L(k+1)} \cdot s \cdot t(x_1') \dots t(x_{\alpha-1}'))$$

with $\alpha$ such that $x_\alpha' \in J_{k+1}$ and

(a) If $y_{k+1}^{(b)} \notin B_{k+1}$ or $j_{n+1} s <_{\mathcal{R}} j_{n+1}$, put

$$\bar{s}(y_{k+1}^{(g)}, y_{k+1}^{(b)}, \vec{y}) = g(j_{L(k+1)} \cdot s \cdot t(x_1') \dots t(x_{\alpha-1}'))$$

with $\alpha$ such that $x_\alpha' \in J_{k+1}$.

(b) If $y_{k+1}^{(b)} \in B_{k+1}$, and $j_{n+1} s \mathcal{R} j_{n+1}$, let $y_{k+1}^{(b)} = (a_{k+1}, g_{k+1})$ and put

$$\bar{s}(y_{k+1}^{(g)}, y_{k+1}^{(b)}, \vec{y}) = (a_{k+1}, g_{k+1} \cdot (y_{k+1}^{(b)})[s(\vec{x})]_G,$$

see Note 4.46.

This clearly satisfies (o), (i) and (ii).

(β) $M(k+1)=0$ (i.e., nothing falls into or past $J_{k+1}$).

(a) If $\operatorname{Rest}(\vec{y})$ is all $(b,b)$'s (i.e., $y_{m+x}=(b,b)$), let

$$\bar{s}(y_{k+1}^{(b)}, \vec{y}) = b, \qquad \bar{s}(y_{k+1}^{(g)}, y_{k+1}^{(b)}, \vec{y}) = b.$$

(b) If $\operatorname{Rest}(\vec{y})$ is all $(b_{n+1}, b)$ (i.e., $y_{m+x}=(b_{n+1},b)$), we have two cases:

(i) $b_{n+1} \notin J_{k+1}$. Then

$$\bar{s}(y_{k+1}^{(b)}, \vec{y}) = b_{n+1} \cdot s(\vec{x}) \quad \text{and} \quad \bar{s}(y_{k+1}^{(g)}, y_{k+1}^{(b)}, \vec{y}) = b;$$

(ii) $b_{n+1} \in J_{k+1}$. Then

$$\bar{s}(y_{k+1}^{(b)}, \vec{y}) = b_{n+1} \cdot s(\vec{x}).$$

(Notice this is reset.)

Let $y_{k+1}^{(g)} = (a_{k+1}, g_{k+1})$ and let $y_{k+1}^{(b)}, \vec{y}$ be legal. Then put $\bar{s}(y_{k+1}^{(g)}, y_{k+1}^{(b)}, \vec{y}) = (a_{k+1}, g_{k+1} \cdot (y_{k+1}^{(b)})[s(x)]_G$ (and $\bar{s}(b, y_{k+1}^{(b)}, \vec{y}) = b$). Also $\bar{s}(y_{k+1}^{(g)}, y_{k+1}^{(b)}, x) = y_{k+1}^{(g)}$ for $y_{k+1}^{(b)}, \vec{y}$ illegal. Again this satisfies (o)–(ii).

If $\operatorname{Rest}(\vec{y}) = \emptyset$, then $j_n \in J_k$, so since $j_n \cdot s <_{\mathcal{R}} j_n$ by assumption, $j_n \cdot s \in J_l$ with $l \geq k+1$. Hence $M(k+1) \neq 0$, which is case (α).

So for each generator $\hat{s}\in\hat{S}$ we have defined $\bar{s}:\Pi\to\Pi$ in triangular form with indicated component action, extending $\alpha^{-1}s\alpha:\hat{S}\alpha\to\hat{S}\alpha$. Letting $Y:=\hat{S}\alpha\subseteq\Pi$, $\theta:Y\to\hat{S}$, $\theta=\alpha^{-1}$, then $(\vec{y}\bar{s})\theta=(\vec{y}\theta)\hat{s}$ for $\vec{y}\in Y$, $\hat{s}\in\hat{S}$ generator and so by standard proof-scheme (see [7])

$$(\hat{S},\hat{S})\mid(\Pi,(1\times G_N)^*\circ 1^*\circ\cdots\circ(1\times G_1)^*\circ 1^*)$$

and, in fact,

$$(\hat{S},\hat{S})\leq(\Pi,(1\times G_N)^*\circ 1^*\circ\cdots\circ(1\times G_1)^*\circ 1^*). \tag{4.52}$$

**Remark 4.53.** Fact 4.30 shows that $\alpha^{-1}\hat{s}\alpha:\hat{S}\alpha\to\hat{S}\alpha\leq\Pi$ has the 'Zeiger-property': "once $\alpha^{-1}\hat{s}\alpha$ permutes, it is identity further down".

We do not need this property in the proof of Chapter 5, but using additional care in the definition of the extension, $\bar{s}$ (as in Fact 4.41) can be made to have the additional property:

(iii) (Zeiger-property) Let $(y_n,\dots,y_1)\in\Pi$, $s\in S$. Then there exists a $k_0$ such that
 (i) $x\to\bar{s}(x,y_\alpha,\dots,y_1)$ is a reset for $\alpha<k_0$;
 (ii) $x\to\bar{s}(x,y_{k_0}^{(b)},y_{k_0-1},\dots,y_1)$ is a permutation;
 (iii) $x\to\bar{s}(x,y_{k_0+\alpha},\dots,y_1)$ is identity for $1\leq\alpha\leq N-k_0$.

For details see [3].

**Remark 4.54** (Discussion of the proof of Fact 4.41). The only interesting ideas in the proof can be shortly described as this:

(i) Fact 4.30 shows that the component action of $\alpha^{-1}s\alpha$ will be reset for a while, then action of $s$ on $\bar{J}_k$ (which is in triangular form by the familiar $(a,g,b)\cdot s=(a,g(b)s_G,b\cdot s)$ with component action $(1\times G_k)$ on the $A\times G$-coordinate), and identity thereafter. It is clear how to extend to $\bar{s}$ within given component action. This leaves $(B,\mathrm{RLM}(S))$ as component action on $\bar{B}_k$.

(ii) The 'Advance' $b_i$ are put in to reduce the component action on $\bar{B}_k$. The idea is that if $j_k s\mathcal{R}j_k$ and $j_{k-1}s<_{\mathcal{R}}j_{k-1}$, $j_{k-1}\in J_{n_{k-1}}$, $j_k\in J_{n_k}$, then either

(α) $n_{k-1}=n_k-1$, and so $j_{k-1}\cdot s$ falls into $J_{n_k}$, so that $b(j_ks)=b(j_{k-1}\cdot s)$ can be read off from above, or

(β) $n_k-n_{k-1}\geq 2$. Then in the $n_{k-1}+1$-spot we will have $(b(j_k),b)$, so again we can read off $b(j_ks)=b(j_k)\cdot s$ from above (and in the $n_{k-1}+1$-spot, $b(j_k)\cdot s$ gets *reset* to $b(j_{k-1})\cdot s$, since $j_{k-1}s<j_{k-1}$).

## 5. $C^\omega(S)=\mathrm{Pl}(S)$

We now have the tools to prove $C^\omega(S)=\mathrm{Pl}(S)$. We show this by constructing a relation $S\overset{R}{—}\mathrm{CP}(S)$ such that $\mathscr{C}(R)\leq C^\omega(S)$ (see Proof-scheme 2.10).

In Subsection 5.1 we define $\hat{H}^\omega$, an adaptation of $H^\omega$ to $(C^\omega(S))\hat{}$, and show that $\alpha^{-1}\hat{H}^\omega\alpha$ (see Definition 4.39) can be extended to $\mathscr{H}$, a triangular map on all of $\Pi$ with very restrictive component action.

In Subsection 5.2 we combine the action of $(C^\omega(S))\hat{}$ on $\Pi$ with $\mathscr{H}$. The component action of $\mathscr{H}$ is such that the combined component action of $\mathscr{H}s\mathscr{H}$ is aperiodic in every component (i.e., $\mathscr{H}$ 'gets rid of the groups'). As a result we can construct the desired relation $R$.

### *5.1. $\widehat{H^\omega}$ and $\mathscr{H}$*

**Definition 5.1.** Let $E(A)$ be such that $H^\omega(A)=A\cdot E(A)$ (see Fact 3.12(ii)).

**Definition 5.2.** Use the map $H^\omega : C^\omega(S)\to C^\omega(S)$ to define $\widehat{H^\omega} : (C^\omega(S))\hat{} \to (C^\omega(S))\hat{}$ as follows:

(i) Extend $H^\omega$ to $C^\omega(S)^1$ by $H^\omega(1)=1$ (if $C^\omega(S)$ is not a monoid already. *Note*: If we use $S^1$ throughout, then $\{1\}$ is the identity on $C^\omega(S)$.)

(ii) Let $\hat{A}\in(C^\omega(S))\hat{}$, $\hat{A}=j_n<\cdots<j_1$ $(j_i\in C^{\omega^1}(S))$ and $A_n\dots A_1<\cdots<A_1$ be some representation.

**5.3.** Define

$$E_1 := E(j_1), \qquad E_k := E(j_kE_1\dots E_{k-1}), \quad k\le n.$$

**5.4.** Define

$$j_1'=j_1E_1, \qquad j_k'=j_kE_1\dots E_{k-1}E_k.$$

Then

$$j_n'\le_{\mathscr{L}}\cdots\le_{\mathscr{L}}j_1'$$

since

$$j_k'=j_kE_1\dots E_{k-1}E_k=A_k(j_{k-1}E_1\dots E_{k-1})E_k=A_kj_{k-1}'E_k$$

and

$$\begin{aligned} j_{k-1}'\ge_{\mathscr{L}}A_kj_{k-1}'\ge_{\mathscr{L}}H^\omega(A_kj_{k-1}') &= H^\omega(A_kj_{k-1}E_1\dots E_{k-1})\\ &=A_kE_1\dots E_{k-1}E_k=j_k' \end{aligned}$$

and so let

$$\widehat{H^\omega}(j_n<\cdots<j_1):=\min(j_n'\le_{\mathscr{L}}\cdots\le_{\mathscr{L}}j_1'). \tag{5.5}$$

The picture is this:

(5.6)
$$
\begin{tikzcd}
A_1 \arrow[r, dash, "\geq_{\mathscr{H}}"] \arrow[dd, dash, "<_{\mathscr{L}}"] & \boxed{A_1E_1} = H^{\omega}(A_1) \arrow[d, dash, "\leq_{\mathscr{L}}"] \\
& A_2A_1E_1 \arrow[dl, dash, "\geq_{\mathscr{R}}"'] \arrow[d, dash, "\leq_{\mathscr{H}}"] \\
A_2\ \ A_1 \arrow[r, dash, "\geq_{\mathscr{R}}"'] \arrow[dd, dash] & \boxed{A_2A_1E_1E_2} = H^{\omega}(A_2A_1E_1) = H^{\omega}(A_2H^{\omega}(A_1)) \arrow[d, dash] \\
& A_3A_2A_1E_1E_2 \arrow[d, dash] \\
A_3\ \ A_2\ \ A_1 \arrow[r, dash, "\geq_{\mathscr{R}}"'] & \boxed{A_3A_2A_1E_1E_2E_3}
\end{tikzcd}
$$

and take the min over the boxed expressions.

**Note 5.7.**

$$
\begin{aligned}
j'_k &= j_kE_1 \dots E_{k-1}E_k = A_k(j_{k-1}E_1 \dots E_{k-1})E_k \\
&= A_kj'_{k-1}E(A_kj'_{k-1}).
\end{aligned}
$$

would give another recursion formula.

**Fact 5.8.** (i) $j_k \geq_{\mathscr{R}} j'_k$, $k=1,\dots,n$.

(ii) $j_k \subseteq j'_k$, $k=1,\dots,n$.

(iii) *If* $j_k \in J$ *with* $J$ *not* $H$*-trivial, then* $j'_k <_{\mathscr{R}} j_k$.

(iv) *If* $j_k \mathscr{R} j'_k$ *and* $j_{k+1} = A_{k+1}j_k \in J$ *with* $J$ $H$*-trivial, then* $j'_{k+1} = A_{k+1}j'_k$.

**Proof.** (i) trivial.

(ii) $A_1 \subseteq A_1E_1 = H^{\omega}(A_1)$ and if $j_k \subseteq j'_k$, then $j_{k+1} = A_{k+1}j_k \subseteq A_{k+1}j'_k \subseteq A_{k+1}j'_k E(A_{k+1}j'_k) = j'_{k+1}$.

(iii) $j_k \geq_{\mathscr{R}} j'_k$ by (i), and if $j_k \mathscr{R} j'_k$, then $H^{\omega}(j'_k) <_{\mathscr{R}} j'_k$ (see Definition 5.2). Contradiction.

(iv) $j'_{k+1} = A_{k+1}j'_kE(A_{k+1}j'_k)$. But $j'_k \mathscr{R} j_k$ implies $A_{k+1}j'_k \mathscr{R} A_{k+1}j_k \in J$, so $E(A_{k+1}j'_k) = 1$ is trivial. $\square$

**Note 5.9.** $A_2A_1 <_{\mathscr{L}} A_1$ does *not* imply $H^\omega(A_2H^\omega(A_1)) < H^\omega(A_1)$ (so taking a min is really necessary). As an example look at Example 1, Section 3. If $1 \underset{\neq}{\triangleleft} G_0 \underset{\neq}{\triangleleft} G$ is a nontrivial subgroup, $G_0 <_{\mathscr{L}} \{1\}$. But $H^\omega(\{1\}) = G$, and $H^\omega(G_0 \cdot H^\omega(\{1\})) = H^\omega(G_0 \cdot G) = H^\omega(G) = G$ as well.

**Key Fact 5.10.** $\eta(\hat{A}) \subseteq \eta(\widehat{H^\omega}(\hat{A}))$.

**Proof.** Fact 5.8(ii) for $k = n$. $\square$

**Fact 5.11.** $(\widehat{H^\omega})^2 = \widehat{H^\omega}$.

**Proof.** Follows from $(H^\omega)^2 = H^\omega$ and $j'_k = H^\omega(A_k j'_{k-1})$. $\square$

**Definition 5.12.** Let $H\text{-}T(\hat{S}) := \{(s_n < \cdots < s_1) \mid s_i \in J_{k_i}\ \mathscr{H}\text{-trivial}\}$.

**Fact 5.13.** *Then* $\widehat{H^\omega} : (C^\omega(S))\hat{} \twoheadrightarrow H\text{-}T((C^\omega(S))\hat{})$ *and* $\widehat{H^\omega}$ *is identity on* $H\text{-}T((C^\omega(S))\hat{})$.

**Proof.** Trivial. $\square$

**Note 5.14.** We now want to show that $\alpha^{-1}\widehat{H^\omega}\alpha$ can be extended to $\mathscr{H}$ on all of $\Pi$ with certain restrictive component action. Just as the same fact for $\alpha^{-1}s\alpha$ is a consequence of $(a, g, b)s = (a, g(b)s_G, b \cdot s)$ being in triangular form, Fact 5.15 gives rise to $\alpha^{-1}\widehat{H^\omega}\alpha$ being triangular.

**Fact 5.15.** *Let $J$ be a $\mathscr{J}$-class of $C^\omega(S)$. Then there exist functions $\phi, \psi$ such that*

$$H^\omega(a, g, b) = (\phi(a), 1, \psi(b))$$

*for all* $(a, g, b) \in J$.

**Proof.** Follows from Fact 3.12. Also note that $\psi(b)$ determines the $\mathscr{J}$-class $J'$ of $H^\omega(a, g, b)$, and hence the group-coordinate $1 \in J'$. $\square$

We can therefore write shortly

**Definition 5.16.**

$$H^\omega(b) := b(H^\omega(a, g, b))$$

which is independent of $a, g$ by Fact 5.15.

$$H^{\omega}(a) := a(H^{\omega}(a, g, b)),$$

$$H^{\omega}_{(b)}(g) := 1 \in J' \quad \text{where } J' \ni H^{\omega}(b).$$

**Definition 5.17.** Let $\mathrm{Down}(\bar{B}_i) \leq F_R(\bar{B}_i)$ be defined as follows:

$$\mathrm{Down}(\bar{B}_i) := \{f : \bar{B}_i \to \bar{B}_i \mid f \text{ satisfies (i) and (ii) below}\}$$

with

(i) If $j \geq i$, $|G_j| \geq 2$, then $f : B_j \to B_k$ with $k > j$ and $B_k$ $H$-trivial;

(ii) If $j \geq i$, $|G_j| = 1$, then $f : B_j \to \bar{B}_j$ and either

(α) $B_j$ can be written disjointly as $B_j = B_j^{(0)} \cup B_j^{(1)}$ and $f : B_j^{(0)} \to B_k$ with $k > j$ and $B_k$ $H$-trivial. $f$ is identity on $B_j^{(1)}$; or

(β) $f$ is a reset to $\bar{b}_j \in \bar{B}_j$;

(iii) $f(b) = b$.

In words: On the non-$H$-trivial $B_j$'s in $\bar{B}_i$, $f$ maps into $B_k$ 'further down'; on the $H$-trivial $B_j$'s in $\bar{B}_i$, $f$ is either a reset or $f$ is partial identity and maps 'further down' where it is not identity.

**Fact 5.18.** $\mathrm{Down}(\bar{B}_i)$ *is aperiodic.*

**Proof.** Obvious. □

**Fact 5.19.** *Let $H^{\omega}$ act on $\bar{B}_i$ by Definition* 5.16 (*and* $H^{\omega}(b) = b$). *Then* $H^{\omega} \in \mathrm{Down}(\bar{B}_i)$.

**Proof.** (i) follows from Fact 3.12.

(ii) If $B_j$ $H$-trivial, $H^{\omega}$ is identity on $B_j$ by Fact 3.12. □

For definition of $\alpha$ see Definition 4.39.

**Fact 5.20.** $\alpha^{-1}\widehat{H^{\omega}}\alpha$ *can be extended to a triangular map* $\mathscr{H}$ *on* $\Pi \bar{J}_i$ *such that the component action of* $\mathscr{H}$ *is as follows*:

(i) *On B-coordinates* $(\bar{B}_i, \mathrm{Down}(\bar{B}_i)^*)$;

(ii) *On G-coordinates*

(a) *If* $|G_i| \geq 2$, $(\bar{G}_i, \mathrm{Reset}(b))$,

(b) *If* $|G_i| = 1$, $(\bar{G}_i, 1^*)$.

We will use in the proof the following standard notation:

**Notation 5.21.**

Representation

$$
\begin{array}{ccccc}
j_1 & & A_1 & & j_1' \\
< & & < & & < \\
\vdots & & \vdots & & \vdots \\
< & = & < & \xrightarrow{\widehat{H^\omega}} & < \\
j_n & & A_n \dots A_1 & & j_{m-1}' \\
< & & < & & < \\
j_{n+1} & & A_{n+1}A_n \dots A_1 & & j_m' \quad (=A_{n+1} \dots A_1 E_1 \dots E_n E_{n+1}) \\
\Big\uparrow \text{Decode} & & & & \Big\downarrow \text{Code} \\
x_1 & & & & x_1' \\
\vdots & & \xrightarrow{\text{Decode } \widehat{H^\omega} \text{ Code}} & & \vdots \\
x_n & & & & \\
x_{n+1} & & & & x_m' \\
\Big\uparrow \text{Exp}^{-1} & & & & \Big\downarrow \text{Exp} \\
y_1 & & & & y_1' \\
\vdots & & & & \vdots \\
y_k & & \xrightarrow{\alpha^{-1}\widehat{H^\omega}\alpha} & & y_k' \\
y_{k+1} & & & & \vdots \\
\vdots & & & & \\
y_N & & & & y_N'
\end{array}
$$

Notice that

$$
\begin{aligned}
x_{n+1} \xrightarrow{\text{Decode}} x_{n+1}\bar{t}(x_n) \dots \bar{t}(x_1) = j_{n+1} &\xrightarrow{\widehat{H^\omega}} j_m' = j_{n+1} \cdot E_1 \dots E_{n+1} \\
&= x_{n+1}\bar{t}(x_n) \dots \bar{t}(x_1) E_1 \dots E_{n+1} \\
\xrightarrow{\text{Code}} x_m' = j_m' t(x_1') \dots t(x_{m-1}'),
\end{aligned}
$$

so all in all

$$x_{n+1} \to x_{n+1}\bar{t}(x_n) \dots \bar{t}(x_1) E_1 \dots E_{n+1} t(x_1') \dots t(x_{m-1}')$$

(compare with Note 4.44).

**Proof.** By induction.

(i) $J_1 = \mathcal{J}$-class of the identity of $C^\omega(S)^1$. Then let $\mathcal{H}(y_1^{(b)}) = H^\omega(y_1^{(b)})$.

(ii) Let

$$\mathcal{H}(y_1^{(g)}, y_1^{(b)}) := \begin{cases} b & \text{if } y_1^{(b)} \in \bar{B}_2, \\ H_{\bar{b}}^\omega(y_1^{(g)}) \text{ with } \bar{b} = y_1^{(b)} & \text{if } y_1^{(b)} \in B_1. \end{cases}$$

This extends $\alpha^{-1}\widehat{H^{\omega}}\alpha$ and satisfies (i) and (ii).

Now assume $\mathscr{H}$ has been defined on $\bar{J}_k \times \cdots \times \bar{J}_1$, extending $\alpha^{-1}\widehat{H^{\omega}}\alpha$, with component action as given.

*Case* 1. $(y_k, \dots, y_1) \in \bar{J}_k \times \cdots \times \bar{J}_1$ is illegal (see Definition 4.47). Let $\mathscr{H}(y_{k+1}, \dots, y_1) = \mathrm{Reset}(b, b)$. This satisfies (i) and (ii).

*Case* 2. $\vec{y} = (y_k, \dots, y_1)$ is legal.

As before, define

$$(5.22)\qquad \begin{aligned} L(k) &:= \begin{cases} \max\limits_{l \le k}\{j'_l \in J_k\}, \\ 0 \quad \text{if no such } l \text{ exists.} \end{cases} \\ M(k) &:= \begin{cases} \min\limits_{m \ge k}\{L(k) \ne 0\}, \\ 0 \quad \text{if no such } m \text{ exists.} \end{cases} \end{aligned}$$

(i) Assume $M(k+1) \ne 0$ (i.e., under $\widehat{H^{\omega}}$ something falls by $J_{k+1}$ from above).

$$\mathscr{H}(y^{(b)}_{k+1}, \vec{y}) = b(j'_{L(k+1)}(t(x'_1) \dots t(x'_{\alpha-1})))$$

with $\alpha$ such that $x'_\alpha \in \mathscr{J}_{k+1}$

$$\mathscr{H}(y^{(g)}_{k+1}, y^{(b)}_{k+1}, \vec{y}) =$$

$$= \begin{cases} \text{(1) If } x_{n+1} \mathscr{R} \overbrace{x_{n+1}\bar{t}(x_n) \dots \bar{t}(x_1)E_1 \dots E_{n+1}t(x'_1) \dots t(x'_{m-1})}^{:= \bar{x}_{n+1}}, \\ \text{then} \qquad = y^{(g)}_{k+1} \text{ (note that in this case } J_{k+1} \text{ is aperiodic and} \\ \qquad\qquad\quad A\text{-coordinate does not change);} \\ \text{(2) If } x_{n+1} >_{\mathscr{R}} \bar{x}_{n+1}, \\ \text{then} \qquad = \begin{cases} \text{(a) } g(j'_{L(k+1)} \cdot t(x'_1) \dots t(x'_\alpha) & \text{if } M(k+1) = k+1, \\ \text{(b) } b & \text{if } M(k+1) > k+1. \end{cases} \end{cases}$$

So component action is either identity or reset. This extends $\alpha^{-1}\widehat{H^{\omega}}\alpha$, satisfying (i) and (ii).

(ii) Now assume $M(k+1) = 0$, i.e., nothing falls from above by or into $J_{k+1}$.

*Case* $\alpha$. $\mathrm{Rest}(\vec{y})$ has $y_{m+x} = (b, b)$. Then let $\mathscr{H}(y_{k+1}, \vec{y}) = (b, b)$.

*Case* $\beta$. $\mathrm{Rest}(\vec{y})$ has $y_{m+x} = (\bar{b}, b)$. Then $\bar{b} = b(x_{n+1})$ in Notation 4.43. So let

$$\mathscr{H}(y^{(b)}_{k+1}, \vec{y}) = H(\bar{b} \cdot \bar{t}(x_n) \dots \bar{t}(x_1)E_1 \dots E_n) \cdot t(x'_1) \dots t(x'_{m-1}).$$

(This is a reset.)

$$\mathscr{H}(y^{(g)}_{k+1}, y^{(b)}_{k+1}, \vec{y}) = \begin{cases} b & \text{if } \bar{b} \notin J_{k+1} \text{ or } y^{(b)}_{k+1}, \vec{y} \text{ illegal or } |G_{k+1}| \ge 2, \\ y^{(g)}_{k+1} & \text{if } \bar{b} = y^{(b)}_{k+1} \in J_{k+1} \text{ and } |G_{k+1}| = 1. \end{cases}$$

Again this extends $\alpha^{-1}\widehat{H^{\omega}}\alpha$, satisfying (i) and (ii).

*Case* $\gamma$ (critical case). Assume Rest $=\emptyset$, and also $j_n\mathcal{R}j'_{m-1}\in J_k$ (otherwise we are in (i)). Then $J_k$ is $H$-trivial.

For $y^{(b)}_{k+1}\in B_j$ not $H$-trivial, put

$$\mathcal{H}(y^{(b)}_{k+1},\vec{y})=H(y^{(b)}_{k+1}\bar{t}(x_n)\dots\bar{t}(x_1)E_1\dots E_n)t(x'_1)\dots t(x'_m)$$

which is in $B_{j'}<B_j$.

If $y^{(b)}_{k+1}\in B_j$, $j\geq k+1$, $B_j$ $H$-trivial, then by 4.23

$$j'_m=A_{n+1}j'_{m-1}$$

so

$$x'_m=A_{n+1}x'_{m-1}t(x'_{m-1})=A_{n+1}x_n t(x_n)=x_{n+1}.$$

Hence we can put

$$\mathcal{H}(y^{(b)}_{k+1},\vec{y})=y^{(b)}_{k+1}$$

for $y^{(b)}_{k+1}\in B_j$ $H$-trivial $\cup\{b\}$ and

$$\mathcal{H}(y^{(g)}_{k+1},y^{(b)}_{k+1},\vec{y})=\begin{cases}y^{(g)}_{k+1} & \text{if } y^{(b)}_{k+1}\in J_{k+1}\ H\text{-trivial},\\ b & \text{otherwise.}\end{cases}$$

This extends $\alpha^{-1}\widehat{H^{\omega}}\alpha$, and component action on $\bar{B}_{k+1}$ is partial identity, rest down and on $\bar{G}_{k+1}$ identity or reset.

So we have defined $\mathcal{H}$ with indicated component action in triangular form on $\Pi$, extending $\alpha^{-1}\widehat{H^{\omega}}\alpha$.

*5.2.* CP($S$)

Define a TSG CP($S$) $=(X,C)$ ('Computes Pointlikes').

**Definition 5.23.**

$$X:=(C^{\omega}(S))\hat{}\,,$$
$$C:=\{s_1\widehat{H^{\omega}}\dots\widehat{H^{\omega}}s_n\widehat{H^{\omega}}\mid s_i\in S,\ n\geq 1\}$$

(identifying $s\in S$ with $(\{s\})\in(C^{\omega}(S))\hat{}\,$).

**Note.** We will use CP($S$) also for the SG $C$ if it is clear from the context.

**Fact 5.24.** CP($S$) *is aperiodic.*

**Proof.** Using the map $\alpha:(C^{\omega}(S))\hat{}\xrightarrow[\text{into}]{\text{Code}}\vec{C^{\omega}}(S)\xrightarrow[\text{into}]{\text{Exp}}\Pi$ we get an isomorphism

$$(X,C)\cong(X\alpha,\{\alpha^{-1}c\alpha\mid c\in C\}).$$

Now

$$\alpha^{-1}s_1 \ldots \widehat{H^{\omega}}\alpha = (\alpha^{-1}s_1\alpha)\alpha^{-1} \ldots (\alpha^{-1}\widehat{H^{\omega}}\alpha)$$

and using the extensions $\mathscr{H}$ for $\alpha^{-1}\widehat{H^{\omega}}\alpha$ and $\bar{s}$ for $\alpha^{-1}s\alpha$ we get

$$\begin{aligned}(X, C) &\cong (\Pi, \{\bar{s}_1 \ldots \mathscr{H}\bar{s}_n\mathscr{H} \mid s_i \in S\}) \\ &\leq (\Pi, F_R(\bar{G}_N) \circ \cdots \circ F_R(\bar{G}_1) \circ F_R(\bar{B}_1)).\end{aligned}$$

Let $\mathrm{CA}\,\mathscr{H}(Y_k)$ denote the component action of $\mathscr{H}$ at $Y_k$, $\mathrm{CA}\,\bar{s}(Y_k)$ denote the component action of $\bar{s}$ at $Y_k$. Then the component action of $C$ at $Y_k$ is given by

$$C(Y_k) = \langle \mathrm{CA}\,\bar{s}(Y_k)\mathrm{CA}\,\mathscr{H}(Y_k)\rangle.$$

We claim this to be aperiodic for any $Y_k$.

*Case* $\alpha$. For general group coordinates $Y_k^{(g)}$ we have

(i) If $|G_k| \geq 2$, then $\mathrm{CA}\,\mathscr{H}(Y_k^{(g)}) = \{\mathrm{Reset}(b)\}$, so $C(Y_k^{(g)}) = \{\mathrm{Reset}(b)\}$.

(ii) If $|G_k| = 1$, $\mathrm{CA}\,\mathscr{H} = \mathrm{CA}\,\bar{s} = 1^*$.

*Case* $\beta$. For general $B$-coordinates $Y_k^{(b)}$ we have

$$\mathrm{CA}\,\mathscr{H}(Y_k^{(b)}) = \mathrm{Down}(\bar{B}_k), \qquad \mathrm{CA}\,\bar{s}(Y_k^{(b)}) = 1^*$$

so

$$C(Y_k^{(b)}) \leq \mathrm{Down}(\bar{B}_k) \cup \{1\}^*.$$

In any case $C(Y_k)$ is aperiodic, hence $(\Pi, \{\bar{s}_1 \ldots \mathscr{H}\bar{s}_n\mathscr{H} \mid s_i \in S\}) \cong (X, C)$ is aperiodic.

**Definition 5.25.** Define a TSG-relation $(S^1, S) \xrightarrow{R = \theta, \phi} (X, C)$ by

$$\phi(s) := \{s_1 \ldots \widehat{H^{\omega}}s_n\widehat{H^{\omega}} \mid s_1 \cdot \ldots \cdot s_n = s\},$$

$$\theta(s) := \{A_n < \cdots < A_1 \mid s \in A_n,\ A_i \in C^{\omega}(S)\} \quad \text{for } s \in S,$$

$\theta(1) := (1)$ (string of length one, $1 =$ identity of $C^{\omega}(S)$ if $S$ does not have an identity, and hence $C^{\omega}(S)$ does not have one either)

**Fact 5.26.** (i) $\phi$ *is onto.*

(ii) $\theta$ *is onto,* $\theta(s) \neq \emptyset$.

(iii) $\phi$ *is an SG-relation.*

(iv) $(\theta, \phi)$ *is a TSG-relation.*

**Proof.** (i) $s_1 \ldots \widehat{H^{\omega}}s_n\widehat{H^{\omega}} \in \phi(s_1 \cdot \ldots \cdot s_n)$.

(ii) Obvious.

(iii)

$$\begin{aligned}\phi(s) \cdot \phi(t) &= \{s_1 \ldots \widehat{H^{\omega}}s_n\widehat{H^{\omega}}t_1 \ldots \widehat{H^{\omega}}t_m\widehat{H^{\omega}} \mid s_1 \ldots s_n = s,\ t_1 \ldots t_m = t\} \\ &\subseteq \phi(st) \quad \text{since } s_1 \ldots s_n t_1 \ldots t_m = st.\end{aligned}$$

(iv)

$$\theta(s)\cdot\phi(t)=\{(A_n<\cdots<A_1)t_1\ldots\widehat{H^\omega}t_m\widehat{H^\omega}\mid s\in A_n,\ t_1\cdot\ldots\cdot t_n=t\}.$$

But $s\cdot t_1\ldots t_n=st\in\eta(\alpha)$ for all $\alpha\in\theta(s)\cdot\phi(t)$, so $\theta(s)\cdot\phi(t)\subseteq\theta(st)$. For 1, $\theta(1)\cdot\phi(t)=\{(1)t_1\ldots\widehat{H^\omega}t_m\widehat{H^\omega}\}$ and $t\in\eta(\alpha)$ for all $\alpha\in\theta(1)\cdot\phi(t)$. Hence $\theta(1\cdot t)=\theta(t)\supseteq\theta(1)\cdot\phi(t)$. $\square$

**Definition 5.27.** Let $\sigma=s_1\ldots\widehat{H^\omega}s_n\widehat{H^\omega}\in C$. Define

$$\mathrm{Pl}(\sigma):=\eta((1)\cdot\sigma).$$

**Fact 5.28.** $s_1\cdot\ldots\cdot s_n\in\mathrm{Pl}(\sigma)$.

**Proof.** By induction

$$(1)\cdot\{s_1\}\widehat{H^\omega}=\widehat{H^\omega}(\{s_1\}<1)=H^\omega\{s_1\}<1$$

and $s_1\in H^\omega\{s_1\}$. Let $\sigma_i:=s_1\widehat{H^\omega}\ldots\widehat{H^\omega}s_i\widehat{H^\omega}$. Assume $s_1\ldots s_i\in\mathrm{Pl}(\sigma_i)$. Then

$$\eta((1)\cdot\sigma_{i+1})=\eta((1)\cdot\sigma_i s_{i+1}\widehat{H^\omega})$$

since

$$\eta((1)\sigma_i)\ni s_1\ldots s_i,\qquad \eta((1)\sigma_i s_{i+1})\ni s_1\ldots s_i s_{i+1},$$

and

$$\eta((1)\sigma_i s_{i+1}\widehat{H^\omega})\supseteq\eta((1)\sigma_i\sigma_{i+1})\ni s_1\ldots s_i s_{i+1}.\qquad\square$$

**Fact 5.29.** $s\phi\sigma\ \Rightarrow\ s\in\mathrm{Pl}(\sigma)$ *follows from Fact* 5.28.

**Fact 5.30.** $\{\phi^{-1}(\sigma)\mid\sigma\in C\}\subseteq C^\omega(S)$.

**Proof.** $\phi^{-1}(\sigma)\subseteq\mathrm{Pl}(\sigma)$ and $\mathrm{Pl}(\sigma)=\eta((1)\cdot\sigma)\in C^\omega(S)$. Hence $C(\phi)\le C^\omega(S)\le\mathrm{Pl}(S)$. Since $C$ is aperiodic, $C(\phi)\supseteq\mathrm{Pl}(S)$ (see Fact 2.4), so we have

**Fact 5.31.** $\mathscr{C}(\phi)=C^\omega(S)=\mathrm{Pl}(S)$.

**Note.** The computation of $\mathrm{Pl}(S)$ via $\mathrm{CP}(S)$ is special in the following sense: If $A\theta$ $(A<A_n<\cdots<A_1)$, $s\phi s\widehat{H^\omega}$, we have: There exists a sequence of pointlikes $E_1\ldots E_n$ such that $\eta((A<\cdots<A_1)\cdot s\cdot\widehat{H^\omega})=A\cdot s\cdot E_1\ldots E_n$. Furthermore, the $E_i=\bigcup G_i$ with $1_{G_i}$ fixing $A\cdot s\cdot E_1\ldots E_{i-1}$.

It was at first thought that this condition ('local computation') would greatly help in the decidability of $\#_G(S)$.

**Note 5.32.** A rough estimate (with $|\hat{S}|\le 2^{|S|}$, $|F_R(X)|\le|X|^{|X|}$, $|C^\omega(S)|\le|P(S)|=2^{|S|}$) shows $|\mathrm{CP}(S)|\le\alpha^\alpha$ with $\alpha=|C^{\hat\omega}(S)|\le 2^{(2^{|S|})}$, which is quite large.

In terms of 'multiplicities', $A \in C^{\omega}(S)$ is represented by

$$\bar{A} := \{A < \cdots < A_1 \mid A_i \in C^{\omega}(S)\},$$

and roughly $|\bar{A}| \le |(C^{\omega}(S))\hat{}\,| \le 2^{(2^{|S|})}$, so we have large multiplicities.

The construction in Fact 2.23 shows that $\mathrm{Pl}^X(X,S)$ can be computed with $(X \times \mathrm{CP}(S), \mathrm{CP}(S))$. If we assume that there exists an $x_0 \in X$ such that for every $x \in X$ there exists an $A \in \mathrm{Pl}(S)$ with $x \in x_0 \cdot A$, this can be slightly reduced: $(C^{\hat{\omega}}(S), \mathrm{CP}(S))$ then computes $\mathrm{Pl}^X(X,S)$ as well by letting $\theta : X \to C^{\hat{\omega}}(S)$ be defined by

$$\theta(x) := \{A_n < \cdots < A_1 \mid x \in x_0 \cdot A_n\},$$

$\phi : S \to \mathrm{CP}(S)$ as before.

Then again $Y \in \mathrm{Pl}^X(X,S)$ is represented by $\bar{Y} := \{A_n < \cdots < A_1 \mid x_0 \cdot A_n = Y\}$ and roughly the multiplicities are $|\bar{Y}| \le 2^{(2^{|S|})}$.

One may ask if these multiplicities are really necessary to compute $\mathrm{Pl}^X(X,S)$. We believe that the above bound may still be improved considerably.

The following example shows that *some* multiplicities may be necessary, though, to compute $\mathrm{Pl}^X(X,S)$ by an aperiodic semigroup.

**Example 4.** Let $X := \{0,1,2,3,4,1',1'',4',4''\}$ and $S \le F_R(X)$ be generated by

$$1 = \text{identity on } X,$$

$$\alpha : \begin{cases} 1 \to 2 \\ 2 \to 3 \\ 3 \to 4 \\ 4' \to 4'' \end{cases} \qquad \beta : \begin{cases} 2 \to 3 \\ 3 \to 4 \\ 4 \to 1 \\ 1' \to 1'' \end{cases} \qquad \gamma : \begin{cases} 1 \to 1' \\ 1'' \to 2 \end{cases} \qquad \delta : \begin{cases} 4 \to 4' \\ 4'' \to 1 \end{cases}$$

and $\alpha, \beta, \gamma, \delta : x \to 0$ unless otherwise indicated.

*Fact* 1. The maximal pointlikes in $\mathrm{Pl}^X(X,S)$ are $\{1,3,0\}$, $\{2,4,0\}$ and $\{x,0\}$ with $x \in \{1',1'',4',4''\}$.

*Proof.* (a) The above sets are indeed pointlike, since $1 \xrightarrow{\alpha\beta} 3 \xrightarrow{\alpha\beta} 1$, so $\{1,3\}$ is pointlike; $\{1,3\} \xrightarrow{\beta} \{4,0\} \xrightarrow{\beta} \{1,0\}$ shows $\{4,0\}$ and $\{1,0\}$ are pointlike; $\{1,0\} \xrightarrow{\alpha\beta} \{3,0\} \xrightarrow{\alpha\beta} \{1,0\}$ shows $\{1,3,0\}$ is pointlike. Similarly $\{2,4,0\}$ is pointlike. $\{1,3,0\} \xrightarrow{\alpha} \{2,4,0\}$ proves $\{2,4,0\}$ to be pointlike, $\{1,0\} \xrightarrow{\gamma} \{1',0\}$ etc. show $\{x,0\}$, $x \in \{1',1'',4',4''\}$ to be pointlike.

(b) To show that they are maximal, we could compute $C^{\omega}(S)$ and $X \cdot C^{\omega}(S) = \mathrm{Pl}^X(X,S)$ by Corollary 2.25. It is easier to use Proof-scheme 2.20, though, and construct a relation that actually computes them using an aperiodic semigroup:

Let $\bar{X} := X \cup \{\{1,3\},\{2,4\},(\delta,1),(\gamma,2)\}$. Let $\bar{S} \le F_R(\bar{X})$ be generated by

$\bar{1}$ = identity on $\bar{X}$

$$\bar{\alpha}: \begin{cases} 1 \to \{2,4\} \\ (\delta,1) \to 2 \\ 2 \to 3 \\ (\gamma,2) \to 3 \\ 3 \to 4 \\ 4' \to 4'' \\ \{1,3\} \to \{2,4\} \\ \{2,4\} \to 3 \end{cases} \qquad \bar{\beta}: \begin{cases} 2 \to \{1,3\} \\ (\gamma,2) \to 3 \\ 3 \to 4 \\ 4 \to 1 \\ 1' \to 1'' \\ \{2,4\} \to \{1,3\} \\ \{1,3\} \to 4 \end{cases}$$

$$\bar{\gamma}: \begin{cases} 1 \to 1' \\ (\delta,1) \to 1' \\ \{1,3\} \to 1' \\ 1'' \to (\gamma,2) \end{cases} \qquad \bar{\delta}: \begin{cases} 4 \to 4' \\ \{2,4\} \to 4' \\ 4'' \to (\delta,1) \end{cases}$$

(and $x \to 0$ where not specified).

Notice the 'multiplicities': $\{1\}$ is represented as $\{1\}$ *and* as $(\delta,1)$, $\{2\}$ as $\{2\}$ *and* $(\gamma,2)$, i.e., we 'remember' if $\delta$ or $\gamma$ 'was used last'.

Also note the choice of 'blow-ups': for $\bar{\alpha}$ it occurs at $1 \to \{2,4\}$, for $\bar{\beta}$ at $2 \to \{1,3\}$.

Define $(X,S) \xrightarrow{R=(\theta,\phi)} (\bar{X},\bar{S})$ by

$$\phi(\sigma) = \{\bar{\sigma}_1 \ldots \bar{\sigma}_k \mid \sigma_1 \ldots \sigma_k = \sigma\} \quad (\sigma \in S),$$

$$\theta(x) = \{y \in \bar{X} \mid x \bar{\in} y\} \quad (x \neq 0),$$

$$\theta(0) = \bar{X}$$

(where $x \bar{\in} y$ if $x = y$ or $x \in y$ and also $1 \bar{\in} (\delta,1)$, $2 \bar{\in} (\gamma,2)$). It is easy to see that this is indeed a TSG-relation and that it computes the above sets.

To see that $\bar{S}$ is aperiodic observe that

(i) All generators ($\neq \bar{1}$) have $\bar{\sigma}^n$ = constant 0 for $n$ large enough.

(ii) $(x)\bar{\beta}\bar{\alpha} = \{2,4\}$ or 0 for all $x \in \bar{X}$.

(iii) By (i) and (ii) it follows that if $\bar{\sigma} = \bar{\sigma}_1 \ldots \bar{\sigma}_n$ is a non-trivial group-element of $\bar{S}$ (i.e., $\bar{\sigma}^2 \neq \bar{\sigma}$), it must contain $\bar{\delta}$ or $\bar{\gamma}$ in its decomposition. Assume $\bar{\sigma}_i = \bar{\gamma}$ and $i$ is the first index such that $s \cdot th \cdot \bar{\sigma}_i = \bar{\gamma}$, (the argument for $\bar{\delta}$ is similar), and that $\bar{x} \xrightarrow{\bar{\sigma}} \bar{x}^{(1)} \longrightarrow \cdots \xrightarrow{\bar{\sigma}} \bar{x}$ is a non-trivial cycle. Then $\bar{x} \xrightarrow{\bar{\sigma}_1} \cdots \xrightarrow{\bar{\sigma}_{i-1}} 1$, and we can assume without loss of generality that $\bar{x} = 1$ and $\bar{\sigma}_1 = \bar{\gamma}$. $1\bar{\gamma} = 1'$ implies $\bar{\sigma}_2 = \bar{\beta}$ and $\bar{\sigma}_3 = \bar{\gamma}$, so $\bar{\sigma} = \bar{\gamma}\bar{\beta}\bar{\gamma}\bar{T}$ with $\bar{T}: 2 \to 1''$. (Since $\bar{x} \xrightarrow{\bar{\sigma}} \bar{x}^{(1)} \to \cdots \to \bar{x}$ is non-trivial.) But then $1 \xrightarrow{\bar{\sigma}} 1'' \xrightarrow{\bar{\gamma}\bar{\beta}\bar{\delta}} 0$, a contradiction. Therefore a non-trivial group-element $\bar{\sigma}$ cannot exist.

*Fact* 2. Let $\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta}: \mathrm{Pl}^X(X,S) \to \mathrm{Pl}^X(X,S)$ be defined so that $Y\bar{\sigma} \supseteq Y \cdot \sigma$ for all $Y \in \mathrm{Pl}^X(X,S)$, $\sigma \in \{\alpha,\beta,\gamma,\delta\}$. Then $\bar{S} := \langle \bar{1},\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta} \rangle$ is not aperiodic. (In other words, it is impossible to act with $\sigma$ on $\mathrm{Pl}^X(X,S)$, computing $\mathrm{Pl}^X(X,S)$ with an aperiodic semigroup. Multiplicities have to be introduced to do so - e.g. $\{1\}$ is represented by 1 *and* $(\delta,1)$ in our computation.)

*Proof.* We omit the 0 in the following argument, e.g. writing $\{1,3\}$ for both $\{1,3,0\}$ and $\{1,3\}$ and 2 for $\{2,0\}$ and $\{2\}$.

(i) $4'' \xrightarrow{\bar\delta} 1$ is forced. Otherwise $4'' \xrightarrow{\bar\delta} \{1,3\} \xrightarrow{\bar\alpha} \{2,4\} \xrightarrow{\bar\delta} 4' \xrightarrow{\bar\alpha} 4''$ makes $\{4'',2,4\}$ pointlike.

Similarly $1'' \xrightarrow{\bar\gamma} \{2,4\} \xrightarrow{\bar\beta} \{3,1\} \xrightarrow{\bar\gamma} 1' \xrightarrow{\bar\beta} 1''$ shows

(ii) $1'' \xrightarrow{\bar\gamma} 2$ is forced.

(iii) $1 \xrightarrow{\bar\alpha} 2$ is forced. Otherwise $1 \xrightarrow{\bar\alpha} \{2,4\} \xrightarrow{\bar\delta} 4' \xrightarrow{\bar\alpha} 4'' \xrightarrow{\bar\delta} 1$ (using (i)) makes $\{1,4'\}$ pointlike.

Similarly $2 \xrightarrow{\bar\beta} \{1,3\} \xrightarrow{\bar\gamma} 1' \xrightarrow{\bar\beta} 1'' \xrightarrow{\bar\gamma} 2$ shows

(iv) $2 \xrightarrow{\bar\beta} 3$ is forced.

(v) $\{2,4\} \xrightarrow{\bar\alpha} 3$ is forced. Otherwise $\{2,4\} \xrightarrow{\bar\alpha} \{3,1\} \xrightarrow{\bar\alpha} \{2,4\}$ makes $\{1,2,3,4\}$ pointlike.

Similarly $\{1,3\} \xrightarrow{\bar\beta} \{2,4\} \xrightarrow{\bar\beta} \{1,3\}$ shows

(vi) $\{1,3\} \xrightarrow{\bar\beta} 4$ is forced.

(vii) $3 \xrightarrow{\bar\alpha} 4$ is forced. Otherwise $3 \xrightarrow{\bar\alpha} \{4,2\} \xrightarrow{\bar\alpha} 3$ makes $\{2,3,4\}$ pointlike.

Similarly, using $4 \xrightarrow{\bar\beta} \{1,3\} \xrightarrow{\bar\beta} 4$

(viii) $4 \xrightarrow{\bar\beta} 1$ is forced.

But with (iii), (iv), (vii), (viii),

$$\bar\alpha\bar\beta : \begin{cases} 1 \to 3 \\ 3 \to 1 \end{cases}$$

becomes a non-trivial element, and $1 \xrightarrow{\bar\alpha\bar\beta} 3 \xrightarrow{\bar\alpha\bar\beta} 1$ a non-trivial cycle.

## Afterword

Assume $\#_G(S)=1$ and $D$ is a way of 'putting $S$ up into a wreath product at complexity', with

$$D: \quad \begin{tikzcd} C\circ G\circ C \arrow[r, two heads] & G\circ C \arrow[r, two heads] & C \\ \leq & \leq & \leq \\ T_1 \arrow[r, two heads, "\gamma_1"'] \arrow[d, two heads, "\theta"'] & T_2 \arrow[r, two heads, "\gamma_2"'] & T_3 \\ S & (\bar R = \theta^{-1}\gamma_1\gamma_2). & \end{tikzcd}$$

Even though we have determined $\mathrm{Pl}(S)$ and exhibited a particular way $S \overset{R}{-\!\!-\!\!-} \mathrm{CP}(S)$ of computing it, and $S \overset{\bar R}{-\!\!-\!\!-} T_3$ will have $\mathrm{Pl}(S) \leq C(\bar R)$ *as covers*, there still might not be any way of using $\mathrm{CP}(S)$ alone to put $S$ up at complexity, i.e. $S \twoheadleftarrow T \leq C\circ G\circ \mathrm{CP}(S)$ might not exist...!

$T_3$ might use multiplicities 'designed to make adding on a group easy'.

We will explore these questions in a forthcoming paper [2] (where we give an example that $\mathrm{CP}(S)$ is indeed 'not necessarily good enough to put $S$ up at complexity'.

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## References

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