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EQUALITY BETWEEN FUNCTIONALS REVISITED

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In this note we shall try to place Friedman's remarkable little paper ([3]) in the context of what we know today about the model theory of the typed λ-calculus. In order to do this, it is appropriate to survey recent work in this area, in so far as it touches on issues raised by Friedman. We don't intend a general survey; much will be left out. In particular, we shall omit discussion of unification, fragments of L.C.F., and polymorphic types, which are areas of interest to the author, and the specialist will surely find other omissions. However, we will try to show the reader how Friedman's paper lays the foundation for the general model theory of the typed λ-calculus.

The plan of this note is the following. We shall begin by a brief, informal introduction to the subject. The reader is then advised to read Friedman's paper. It is, after all, quite accessible and easy to read. We shall then proceed to consider the three major issues touched on in "Equality between functionals" as we view them today. These issues are

(1) completeness theorems,

(2) the solvability of higher type functional equations, and

(3) logical relations.

THE TYPED λ-CALCULUS

Let A be a nonempty set. Then we can form the function spaces $A^A, A^{(A^A)}, (A^A)^A, \ldots$ etc. Each function space will be assigned an expression (type) describing how it is constructed. 0 is assigned to A, $0 \to 0$ to A^A, $(0 \to 0) \to 0$ to $A^{(A^A)}$, $0 \to (0 \to 0)$ to $(A^A)^A$ etc. Each type τ can be written uniquely in the form $\tau(1) \to (\ldots(\tau(t) \to 0)\ldots)$. More generally a frame (for Friedman "pre structure") $\mathfrak{A}$ is a map from types to non-empty sets $\tau \to \mathfrak{A}^\tau$ such that

$$\mathfrak{A}^{\sigma\to\tau} \subseteq {\mathfrak{A}^{\tau}}^{\mathfrak{A}^{\sigma}}.$$

When $\mathfrak{A}^{\sigma\to\tau}$ is always ${\mathfrak{A}^{\tau}}^{\mathfrak{A}^{\sigma}}$, is the full type structure $\mathcal{S}_{\mathcal{U}}$

for $\mu = |\mathfrak{A}^0|$.

Functions of several arguments can be included by "Currying". For example, if $\Phi \in A^{A\times A}$ for $a \in A$ define Φ_a by $\Phi_a x = \Phi(a,x)$ and $\hat{\Phi}$ by $\hat{\Phi} a = \Phi_a$ so $\hat{\Phi} \in (A^A)^A$.

Let $\mathfrak{A}^*$ result from $\mathfrak{A}$ by adjoining infinitely many indeterminates to each $\mathfrak{A}^\tau$ just as one does in algebra. $\mathfrak{A}$ is called a <u>model</u> (for Friedman "structure") if for each Φ in $\mathfrak{A}^*$ and each indeterminate x there exists a Ψ in $\mathfrak{A}^*$ which does not depend on x (as a function depends on its arguments) such that $\Psi x = \Phi$. In other words each Φ can be made an explicit function of its parameters. Such a Ψ is denoted $\lambda x\, \Phi$. We say that models are frames closed under "explicit definition".

<u>Type theory</u> is a language for describing models. Its equational part is the <u>typed λ-calculus</u>. Terms are built up from some set Σ of constants of various types, and variables $x^\sigma, y^\tau, z^\rho, \ldots$ of types $\sigma, \tau, \rho, \ldots$ by means of the application symbol $(\ )$ and abstraction symbols $\lambda x^\sigma, \lambda y^\tau, \lambda z^\rho, \ldots$. Terms have the obvious meaning in models $\mathfrak{A}$, where $(\ )$ is the union overall σ and τ of the maps $(\Phi,\Psi) \to \Phi\Psi$ in $\mathfrak{A}^{\tau\,\mathfrak{A}^{\sigma\to\tau}\times\mathfrak{A}^\sigma}$, and λx^σ is the union overall τ of the maps $\Phi \to \lambda x \Phi$ in $\mathfrak{A}^{*\sigma-\tau\,\mathfrak{A}^{*\tau}}$. The resulting set of terms is denoted $\Lambda(\Sigma)$ with the set of terms of type τ denoted $\Lambda(\Sigma)^\tau$. $\overline{\Lambda}(\Sigma)$ denotes the set of closed terms, with the set of closed terms of type τ denoted $\overline{\Lambda}(\Sigma)^\tau$. We reserve the letters M and N to denote member of $\overline{\Lambda}$, where $\Lambda = \Lambda(\emptyset)$.

We shall make the following notational conventions. We delete type superscripts when no confusion can result. We also delete application symbols associating to the left. Finally we write $\lambda x_1 \ldots x_n$ for $\lambda x_1 \lambda x_2 \ldots \lambda x_n$. Thus, for example, $\lambda x^{0\to(0\to0)} \lambda yz\ \ xz(yz)$ abbreviates the term $\lambda x^{0\to(0\to0)} \lambda y^{0\to0} \lambda z^0 ((x^{0\to(0\to0)} z^0)(y^{0\to0} z^0))$.

The language of type theory is built up from equations between terms by the usual propositional operations and typed quantifiers.

Now the equations

$$\lambda x X\ Y = [Y/x]X \qquad (\beta),$$

where $[Y/x]X$ is the result of substituting Y for x in X, and

$$\lambda x(Xx) = X \qquad (\eta),$$

for x not free in X, are clearly valid in all models. Let $\underset{\beta\eta}{=}$ be the congruence relation on terms generated by (β) and (η). Are there any other valid equations? Friedman's completeness theorems answer this question.

COMPLETENESS THEOREMS

It now seems fair to say that, with the 1-section theorem below, we have a satisfactory understanding of the completeness phenomenon. This was not the case in 1970. Friedman's <u>completeness theorems</u>

stand as the first contributions to the model theory of the subject. Although his techniques have been superceeded in this area, they make a direct contribution to the theory of logical relations which will be discussed later. In addition, Friedman's emphasis on the universal algebraic aspects of completeness motivated the author to look for something like the 1-section theorem.

If $\mathcal{V}$ is a class of models we say $\mathcal{V}$ is complete if $M \underset{\beta\eta}{=} N \Longleftrightarrow \forall \mathfrak{A} \in \mathcal{V}\ \mathfrak{A} \models M = N$. $\underset{\beta\eta}{=}$ is in some sense an equational theory (although, extensionality is required for models). Thus one expects that something like free models should exist. Assume that $\overline{\Lambda}(\Sigma)^{0} \neq \emptyset$. For $T_1, T_2 \in \overline{\Lambda}(\Sigma)^{\tau}$ define

$$T_1 \sim T_2 \Longleftrightarrow \forall U_1 \in \overline{\Lambda}(\Sigma)^{\tau(1)} \ldots \forall U_t \in \overline{\Lambda}(\Sigma)^{\tau(t)}$$
$$T_1 U_1 \ldots U_t \underset{\beta\eta}{=} T_2 U_1 \ldots U_t.$$

If $\sim$ is a congruence relation then we can build a model $\mathfrak{M}_\Sigma$, consisting of $\sim$ congruence classes, satisfying

$$\mathfrak{M}_\Sigma \models T_1 = T_2 \Longleftrightarrow T_1 \sim T_2.$$

Friedman's first completeness theorem is just this. If for all τ Σ^{τ} is infinite then $\mathfrak{M}_\Sigma$ exists and $\mathfrak{M}_\Sigma \models T_1 = T_2 \Longleftrightarrow T_1 \underset{\beta\eta}{=} T_2$. This generalizes to the

<u>Existence of Free Models</u> ([15])

> $\mathfrak{M}_\Sigma$ always exists. Moreover, only five local structures (see [2] pg. 496) are possible.

Friedman's first theorem has a short of converse which resembles Hilbert-Post completeness.

<u>Consistency Theorem</u> ([14])

> $\underset{\beta\eta}{=}$ is the maximal non-trivial congruence on closed terms satisfying typical ambiguity.

Here typical ambiguity just means that 0 can be replaced by any other type.

Friedman's <u>second completeness theorem</u> concerns the full type structure $\mathcal{S}_\kappa$ over a ground domain of size κ. In short, if $\kappa \geq \aleph_0$ then $\{\mathcal{S}_\kappa\}$ is complete. He also pointed out that for no finite m is $\{\mathcal{S}_m\}$ complete. In [13] we refined this to the

<u>Finite Model Property</u>

> For each M there exists an m, recursive in M, such that for all N
>
> $$\mathcal{S}_m \models M = N \Longleftrightarrow M \underset{\beta\eta}{=} N.$$

Thus the typed λ-calculus exhibits a rather strong separation property with respect to its hereditarily finite full models.

If $\mathcal{O}$ is a model the 1-section of $\mathcal{O}$ is $\mathcal{O}^0 \cup \mathcal{O}^{0\to 0} \cup \mathcal{O}^{0\to(0\to 0)} \cup \ldots$ (i.e. the first-order part of $\mathcal{O}$). The next theorem says, somewhat surprisingly, that completeness depends only on 1-sections and not higher types. Let $\mathcal{J}$ be the free term algebra on a single constant and a single binary function symbol.

<u>1-Section Theorem ([13])</u>

> $\mathcal{V}$ is complete if and only if $\mathcal{J}$ can be embedded in the 1-section of some countable direct product of members of $\mathcal{V}$.

One consequence which Friedman observed of his second completeness theorem is that equality of λ definable functionals (i.e. those denoted by members of $\overline{\Lambda}$) in $\mathcal{S}_{\mathcal{M}}$ is decidable. Unfortunately, when $\mathcal{M} > 1$, there is no efficient decision procedure.

<u>Complexity of $\beta\eta$ Conversion ([12] and [13])</u>

> Any elementary recursive set of closed terms closed under $\underset{\beta\eta}{=}$ contains none or all of the terms of any given type.

SOLVING FUNCTIONAL EQUATIONS AT HIGHER TYPES

Much of the literature on the typed λ-calculus prior to 1970 concerned functional equations

$$E \equiv E(\vec{y},\vec{x}) \equiv M\,\vec{y}\,\vec{x} = N\,\vec{y}\,\vec{x}$$

which, given parameters $\vec{y}$, we wish to solve for $\vec{x}$. Roughly speaking the literature sorts itself into three topics; constructive solvability (e.g. Kleene [8], Kreisel [9], Gödel [4], Scott [11]), solvability in all models i.e. unification (e.g. Andrews [1], Gould [5], Guard [6]), and solvability of special classes of equations (e.g. Scott [11]). Friedman first recognized the significance of the <u>axiom of choice</u> (A.C.) in this area.

(A.C.) $\qquad \forall x\ \exists y\ \mathcal{F}(x,y) \to \exists z\ \forall x\ \mathcal{F}(x,zx)$

Functional equations are always closed under conjunction ([3],[13]). Under A.C. they are closed under negation. Let's call a sentence of type theory which asserts that a parameterless functional equation has a solution an $\exists E$ sentence. Friedman observed that there exist a (polynomial time) computable map $\mathcal{F} \mapsto \mathcal{F}^+$ from sentences to $\exists E$ sentences such that A.C. $\vdash \mathcal{F} \longleftrightarrow \mathcal{F}^+$. In [16] we refined this as follows. Let Δ be the sentence that asserts that the definition by cases functional (IF____THEN____ELSE____) of lowest type exists.

<u>Normal Form Theorem part 1</u>

> There exists a polynomial time computable map $\mathcal{F} \mapsto \mathcal{F}^*$ from sentences to $\exists E$ sentences s.t.

$$\text{A.C.} \vdash \mathcal{F} \longleftrightarrow \mathcal{F}^*$$
$$\Delta \vdash \mathcal{F}^* \longrightarrow \mathcal{F}$$

Models of Δ are just models of the $\lambda\delta$ calculus where

$$\delta xyuv = u \quad \text{if} \quad x = y$$
$$= v \quad \text{if} \quad x \neq y.$$

The normal form theorem extends to theories.

Normal Form Theorem part 2

There exists a polynomial time computable set of $\exists E$ sentences $A.C.^{\exists E}$ such that if $\mathcal{T}$ is any theory at least as strong as A.C.

$$\mathcal{T} \text{ is equivalent to } \mathcal{T}^* + A.C.^{\exists E} + \Delta$$
$$\mathcal{T} \vdash \mathcal{F} \Longleftrightarrow \mathcal{T}^* + A.C.^{\exists E} \vdash \mathcal{F}^*$$

One consequence of Friedman's observation is that if a functional equation is not solvable in a model of A.C. then it is not solvable in any extension (in the obvious sense) of that model. This corollary motivated the author to search for a result which equates the unsolvability of E in every extension of $\mathfrak{A}$ with the solvability of some other $\tilde{E}$ in , for general models $\mathfrak{A}$. Given E as above define $\tilde{E}_n \equiv \tilde{E}_n(z_0 z_1 \vec{y},\vec{u})$ for $z_o, z_1 \in 0$ to be

$$\lambda\vec{x}\ z_o = \lambda\vec{x}\ u_1\vec{x}\ (M\vec{y}\ \vec{x})(N\vec{y}\ \vec{x}) \ \wedge$$
$$\lambda\vec{x}\ u_1\vec{x}\ (N\vec{y}\ \vec{x})(M\vec{y}\ \vec{x}) = \lambda\vec{x}\ u_2\vec{x}\ (M\vec{y}\ \vec{x})(N\vec{y}\ \vec{x}) \ \wedge$$
$$\vdots$$
$$\lambda\vec{x}\ u_n\vec{x}\ (N\vec{y}\ \vec{x})(M\vec{y}\ \vec{x}) = \lambda\vec{x}\ \vec{z}_1$$

No Counterexample Theorem ([18])

$E(\vec{\Phi},\vec{x})$ is solvable in an extension of $\mathfrak{A}$ if and only if for each n and $a,b \in \mathfrak{A}^*$ s.t. $a \neq b$ $\tilde{E}_n(ab\vec{\Phi},\vec{u})$ is not solvable in $\mathfrak{A}$.

Friedman used his normal form theorem to conclude that the problem of determining if a funtional equation has a solution in all models of A.C. is undecidable (Σ^0_1 complete). It also follows from his normal form theorem that the problem of determining if a functional equation has a solution in some (general) model is Π^0_1 complete.

LOGICAL RELATIONS

Examples of logical relations occur throughout the literature on the typed λ-calculus. These are hereditarily defined classes of functionals of which Howard's hereditarily majorizable functionals ([7]) is typical. Friedman's notion of partial homomorphism, central to his second completeness proof, is easily cast in the form of a logical relation on the product of models. In [10] Plotkin introduced a general notion of logical relation in a model theoretic context. However, perhaps the most striking use of relations of this type is Tait's proof ([20]) of the normalization theorem, and his proof is purely syntactic. In addition, as Friedman pointed out, his notion makes perfectly good sense for frames ("pre structures") where λ-terms can fail to denote. Our theory of logical relations is designed to cover each of these cases. The form of words "logical relation" is particularly appropriate, since the principal properties of such relations are closure under existential quantifications, infinite conjunctions, and in the context of models, infinite disjunctions, when these are suitably defined.

Let Σ_i for $1 \leq i \leq n$ be sets of constants. A logical relation $\mathcal{R}$ is a map $\tau \to \mathcal{R}_\tau \subseteq \Lambda(\Sigma_1)^\tau \times \ldots \times \Lambda(\Sigma_n)^\tau$ satisfying

$$\mathcal{R}_{\sigma\to\tau}(X_1,\ldots,X_n) \iff \forall Y_1\ldots Y_n \mathcal{R}_\sigma(Y_1,\ldots,Y_n) \longrightarrow \mathcal{R}_\tau(X_1Y_1,\ldots,X_nY_n).$$

Such an $\mathcal{R}$ is called admissible if it is closed under coordinatewise head expansions ([2] pg. 169) at type 0. It is called normal if it is closed under coordinatewise $=_{\beta\eta}$ at type 0.

Let θ_i range over substitutions such that $\forall x\ \theta_i x \in \Lambda(\Sigma_i)$. Define

$$\mathcal{R}^*(X_1,\ldots,X_n) \iff \forall\theta_1\ldots\theta_n\ \forall x \mathcal{R}(\theta_1 x,\ldots,\theta_n x) \longrightarrow \mathcal{R}(\theta_1 X_1,\ldots,\theta_n X_n)$$

and

$$\exists\mathcal{R}(X_2,\ldots,X_n) \iff \exists X_1 \mathcal{R}^*(X_1,\ldots,X_n).$$

If $\mathcal{S}$ is a set of logical relations define $\wedge\mathcal{S} = \bigcap_{\mathcal{R}\in\mathcal{S}} \mathcal{R}^*$, and $\vee\mathcal{S}$ = the applicative closure of $\bigcup_{\mathcal{R}\in\mathcal{S}} \mathcal{R}^*$.

<u>Closure Theorem ([17])</u>

> Suppose $\mathcal{R}$ is admissible and so is each member of $\mathcal{S}$. Then $\mathcal{R}^*, \exists\mathcal{R}$ and $\wedge\mathcal{S}$ are logical, admissible and fixed by $*$. Moreover, if each member of $\mathcal{S}$ is normal then $\vee\mathcal{S}$ is logical, normal and fixed by $*$.

<u>Fundamental Theorem of Logical Relations ([17])</u>

> If $\mathcal{R}$ is admissible and $X_1,\ldots,X_n \in \Lambda$ then $X_1 =_{\beta\eta} \ldots =_{\beta\eta} X_n \Rightarrow \mathcal{R}^*(X_1,\ldots,X_n)$.

The standard syntactic results about the typed λ-calculus such as the Church-Rosser property, normalization and strong normalization, the standardization theorem and η postponement follow quite easily from these theorems. Moreover, the above definitions carry over to frames in the obvious way. We have:

<u>Characterization Theorem ([17])</u>

1. Suppose $\mathfrak{A}$ is a frame. Then $\mathfrak{A}$ is a model if and only if for each frame $\mathfrak{B}$ and logical $\mathcal{R}$ on $\mathfrak{A}\times\mathfrak{B}$ $\exists\mathcal{R}$ is logical.
2. Suppose $\mathfrak{A}$ is a model. Then $\Phi \in \mathfrak{A}$ is λ-definable if and only if for each $\mathfrak{B} \supseteq \mathfrak{A}$ and each logical $\mathcal{R}$ on $\mathfrak{B}$, $\mathcal{R}(\Phi)$.

A similar characterization of λ-definability related to 2. is the

<u>Definability Theorem ([14])</u>

> In any model the λ-definable functionals are precisely those which are stable solutions of λ-free systems of functional equations.

Here stable solutions are roughly those which are unique and remain so under "perturbations" of the model by <u>partial homomorphisms</u>.

The outstanding open problem in the model theory of the typed λ-calculus is to find an effective characterization of λ-definability for the $\mathcal{P}_m$.

<u>λ-Definability Conjecture</u>

> λ-definability in the $\mathcal{P}_m$ is decidable.

For some consequence of this conjecture we refer the reader to [13].

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