

# A NULLSTELLENSATZ FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

J. CIMPRIC

ABSTRACT. In this paper an *equation* means a homogeneous linear partial differential equation in  $n$  unknown functions of  $m$  variables which has real or complex polynomial coefficients. The *solution set* consists of all  $n$ -tuples of real or complex analytic functions that satisfy the equation. For a given system of equations we would like to characterize its *Weyl closure*, i.e. the *set of all equations that vanish on the solution set of the given system*. It is well-known that in many special cases the Weyl closure is equal to  $B_m(\mathbb{F})N \cap A_m(\mathbb{F})^n$  where  $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ , the algebra  $A_m(\mathbb{F})$  (respectively  $B_m(\mathbb{F})$ ) consists of all linear partial differential operators with coefficients in  $\mathbb{F}[x_1, \dots, x_m]$  (respectively  $\mathbb{F}(x_1, \dots, x_m)$ ) and  $N$  is the submodule of  $A_m(\mathbb{F})^n$  generated by the given system. Our main result is that this formula holds in general. In particular, we do not assume that the module  $A_m(\mathbb{F})^n/N$  has finite rank which used to be a standard assumption. Our approach works also for the real case which was not possible with previous methods. Moreover, our proof is constructive as it depends only on the Riquier-Janet theory.

## 1. INTRODUCTION

Let  $\mathbb{F}$  be either  $\mathbb{R}$  or  $\mathbb{C}$  and let  $m$  and  $n$  be integers. A *homogeneous linear partial differential equation with polynomial coefficients* in  $n$  unknown functions  $u^1, \dots, u^n$  of  $m$  variables  $x_1, \dots, x_m$  can be written as

$$p_1[u^1] + \dots + p_n[u^n] = 0$$

where linear partial differential operators  $p_1, \dots, p_n$  have polynomial coefficients; in other words,  $p_1, \dots, p_n$  belong to the Weyl algebra  $A_m(\mathbb{F})$  which is generated by  $x_1, \dots, x_m$  and  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ . Its solution at point  $(x_1^0, \dots, x_m^0) \in \mathbb{F}^m$  is an  $n$ -tuple of convergent power series in  $x_1 - x_1^0, \dots, x_m - x_m^0$  that satisfy the equation. The solution set consists of all solutions at all points of  $\mathbb{F}^m$ .

The aim of this paper is to prove a nullstellensatz type result for such equations. Consider a system of  $k$  equations

$$(1) \quad \begin{aligned} p_{11}[u^1] + \dots + p_{1n}[u^n] &= 0 \\ &\vdots \\ p_{k1}[u^1] + \dots + p_{kn}[u^n] &= 0 \end{aligned}$$

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Address: University of Ljubljana, Faculty of Mathematics and Physics, Department of Mathematics, Jadranska 21, SI-1000 Ljubljana, Slovenia.

E-mail: cimpric@fmf.uni-lj.si.

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We would like to determine when another equation

$$(2) \quad q_1[u^1] + \dots + q_n[u^n] = 0$$

vanishes on the solution set of (1). Our main result is that this happens if and only if there exists a nonzero polynomial  $w \in \mathbb{F}[x_1, \dots, x_m]$  and a  $k$ -tuple of linear partial differential operators  $(h_1, \dots, h_k) \in A_m(\mathbb{F})^k$  such that the following matrix equation is true

$$(3) \quad w \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix} = \begin{bmatrix} h_1 & \dots & h_k \end{bmatrix} \begin{bmatrix} p_{11} & \dots & p_{1n} \\ \vdots & & \vdots \\ p_{k1} & \dots & p_{kn} \end{bmatrix}$$

The set of all equations (2) that vanish on the solution set of the system (1) is usually called the **Weyl closure** of (1). Let  $N$  be the submodule of  $A_m(\mathbb{F})^n$  that is generated by the rows of the  $p_{ij}$  matrix. Our result can be rephrased as follows: the Weyl closure of the system (1) is equal to

$$\mathbb{F}(x_1, \dots, x_m)N \cap A_m(\mathbb{F})^n.$$

For constant coefficients our main result follows from [8, Examples 1.13 and 1.13 (real), Assumption 2.55, Theorems 2.61 and 4.54]. Note that [8] also covers other notions of solution which is further developed in [17]. For holonomic systems (with  $\mathbb{F} = \mathbb{C}$ ) our main result follows from [20, Proposition 2.1.9]. This result uses global solutions instead of our local solutions. We will discuss it in subsection 5.2.

The proof of our main result uses **Riquier-Janet theory**. **Riquier bases are Weyl algebra analogues of Gröbner bases while Janet's algorithm is an analogue of Buchberger's algorithm.** **Riquier existence theorems are generalizations of the Cauchy-Kovalevskaya theorem.** For a recent survey of this theory, see [16, Chapter 4].

## 2. PRELIMINARIES

Let  $\mathbb{F}$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . For every  $m \in \mathbb{N}$ ,<sup>1</sup> the *Weyl algebra*  $A_m(\mathbb{F})$  is the  $\mathbb{F}$ -algebra with generators  $x_1, \dots, x_m, D_1, \dots, D_m$  and relations  $x_i x_j = x_j x_i$ ,  $D_i D_j = D_j D_i$  and  $D_j x_i - x_i D_j = \varepsilon_{ij} \cdot 1$  for all  $i, j = 1, \dots, m$ , where  $\varepsilon_{ij} = 1$  if  $i = j$  and  $\varepsilon_{ij} = 0$  if  $i \neq j$ . Clearly,  $A_m(\mathbb{F})$  is a left module over  $\mathbb{F}[\mathbf{x}] := \mathbb{F}[x_1, \dots, x_m]$ . We will also need its localization  $B_m(\mathbb{F}) := (\mathbb{F}[\mathbf{x}] \setminus \{0\})^{-1} A_m(\mathbb{F})$  which is a left vector space over  $\mathbb{F}(\mathbf{x}) := \mathbb{F}(x_1, \dots, x_m)$ . It is well-known that  $A_m(\mathbb{F})$  and  $B_m(\mathbb{F})$  are Noetherian domains, see e.g. [6, pp. 19–20], (which implies the Ore property by [6, pp. 46–47]). For every  $n \in \mathbb{N}$ , the left  $A_m(\mathbb{F})$ -module  $A_m(\mathbb{F})^n$  and the left  $B_m(\mathbb{F})$ -module  $B_m(\mathbb{F})^n$  are also Noetherian. For additional ring-theoretic information on  $A_m(\mathbb{F})$  and  $B_m(\mathbb{F})$  see [18, 9].

An element of  $B_m(\mathbb{F})^n$  is a *derivative* if it is of the form  $\delta_\alpha^i := D^\alpha \mathbf{e}_i$  where  $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ ,  $D^\alpha := D_1^{\alpha_1} \dots D_m^{\alpha_m}$  and  $\mathbf{e}_i = (\varepsilon_{i1}, \dots, \varepsilon_{im})$  is the  $i$ -th standard basis vector of  $B_m(\mathbb{F})^n$ . The set of all derivatives will be denoted by  $\Delta$ . Every element  $\mathbf{p} \in B_m(\mathbb{F})^n$  can be converted into a *standard form*, i.e. it can be expressed uniquely as a left  $\mathbb{F}(\mathbf{x})$ -linear combination of different derivatives. We write  $\text{cf}(\mathbf{p})(\delta)$  for the coefficient of  $\mathbf{p}$  at  $\delta \in \Delta$ , so  $\mathbf{p} = \sum_{\delta \in \Delta} \text{cf}(\mathbf{p})(\delta) \delta$ . The *standard ranking* is a linear ordering  $\prec$  of the set  $\Delta$  which is defined by

$$\delta_\alpha^i \prec \delta_\beta^j \Leftrightarrow (|\alpha|, i, \alpha_1, \dots, \alpha_{m-1}) \leq_{\text{lex}} (|\beta|, j, \beta_1, \dots, \beta_{m-1})$$

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<sup>1</sup> $\mathbb{N} = \{0, 1, 2, \dots\}$  is the set of natural numbers.

where  $|\alpha| := \alpha_1 + \dots + \alpha_m$  and  $\leq_{\text{lex}}$  is the usual lexicographic ordering. It determines the notions of the leading coefficient  $\text{hc } \mathbf{p}$  and the highest derivative  $\text{hd } \mathbf{p}$  of an element  $\mathbf{p} \in B_m(\mathbb{F})$ . If  $\text{hd } \mathbf{p} = \delta_\alpha^i$  we define the *degree* of  $\mathbf{p}$  by  $\deg \mathbf{p} := |\alpha|$ . The standard ranking satisfies the following property (which defines a *ranking*): if  $\delta_\alpha^i \prec \delta_\beta^j$  for some  $\alpha, \beta \in \mathbb{N}^m$  and  $i, j = 1, \dots, n$ , then  $\delta_{\alpha+\gamma}^i \prec \delta_{\beta+\gamma}^j$  for all  $\gamma \in \mathbb{N}^m$ . The standard ranking belongs to several interesting classes of rankings that appear in the literature (positive rankings, orderly rankings, Riquier rankings); see [15]. Similar remarks apply to elements of  $A_m(\mathbb{F})^n$ .

For a given point  $\mathbf{x}^0 = (x_1^0, \dots, x_m^0) \in \mathbb{F}^m$  we will write  $\mathbb{F}[[\mathbf{x} - \mathbf{x}^0]]$  for the set of all formal power series in  $x_1 - x_1^0, \dots, x_m - x_m^0$ . We say that a formal power series is *convergent* if it has a nonzero convergence radius. In this case it defines an analytic function on a ball around  $\mathbf{x}^0$ . Every element  $p \in B_m(\mathbb{F})$  which is defined (i.e. whose coefficients are defined) at  $\mathbf{x}^0$  induces in a natural way a mapping  $u \mapsto p[u]$  from  $\mathbb{F}[[\mathbf{x} - \mathbf{x}^0]]$  to itself which respects convergence. Similarly, every element  $\mathbf{p} = (p_1, \dots, p_n) \in B_m(\mathbb{F})^n$  which is defined at  $\mathbf{x}^0$  induces a mapping from  $\mathbb{F}[[\mathbf{x} - \mathbf{x}^0]]^n$  to  $\mathbb{F}[[\mathbf{x} - \mathbf{x}^0]]$  by  $\mathbf{u} = (u^1, \dots, u^n) \mapsto \mathbf{p}[\mathbf{u}] := p_1[u^1] + \dots + p_n[u^n]$ .

For every finite subset  $\{\mathbf{p}_1, \dots, \mathbf{p}_k\}$  of  $B_m(\mathbb{F})^n$ , we have a *system*

$$(4) \quad \mathbf{p}_1[\mathbf{u}] = \dots = \mathbf{p}_k[\mathbf{u}] = 0$$

of partial differential equations corresponding to it. We say that an element  $\mathbf{u} \in \mathbb{F}[[\mathbf{x} - \mathbf{x}^0]]^n$  is a *formal solution* of system (4) at point  $\mathbf{x}^0 \in \mathbb{F}^m$  if all  $\mathbf{p}_1, \dots, \mathbf{p}_k$  are defined at  $\mathbf{x}^0$  and  $\mathbf{u}$  satisfies (4) in  $\mathbb{F}[[\mathbf{x} - \mathbf{x}^0]]$ . If a formal solution at  $\mathbf{x}^0$  is convergent, then the corresponding analytic function solves the system on a ball around  $\mathbf{x}^0$ . If two finite subsets of  $B_m(\mathbb{F})^n$  generate the same submodule of  $B_m(\mathbb{F})^n$  then the corresponding systems are *equivalent*, i.e. they have the same formal and the same analytic solutions at every point  $\mathbf{x}^0$  from some open dense subset of  $\mathbb{F}^m$ .

We will now summarize the Riquier-Janet theory. Let  $N$  be a submodule of  $B_m(\mathbb{F})^n$  and let  $\mathcal{N}$  be a finite generating set of  $N$ . A procedure called the *Janet's algorithm*<sup>2</sup> transforms  $\mathcal{N}$  into a better finite generating set  $\mathcal{M}$  that we call a *Riquier basis*. The idea is to transform each element  $a\delta + L \in \mathcal{N}$  (where  $a \in \mathbb{F}(\mathbf{x})$ ,  $\delta \in \Delta$  and  $\text{hd } L \prec \delta$ ) into a substitution rule  $\delta \mapsto -a^{-1}L$  that is used to reduce other elements of  $\mathcal{N}$ . We must also ensure that by differentiating the substitution rules for  $\delta_\alpha^i$  and  $\delta_\beta^j$  (when they exist) we get only one substitution rule for  $\delta_{\alpha+\beta}^i$ . By definition, all elements of  $\mathcal{M}$  are monic. The system corresponding to  $\mathcal{M}$  is equivalent to the system corresponding to  $\mathcal{N}$  but it is much easier to solve.

The procedure to formally solve the system corresponding to  $\mathcal{M}$  is given by the Formal Riquier Existence Theorem. The idea is to split the set  $\Delta$  into two parts, the set of *principal derivatives*  $\text{Prin } \mathcal{M}$  which is defined by

$$\text{Prin } \mathcal{M} := \{\delta \in \Delta \mid \delta = D^\alpha \text{hd } \mathbf{f} \text{ for some } \alpha \in \mathbb{N}^m \text{ and some } \mathbf{f} \in \mathcal{M}\}$$

and the set of *parametric derivatives*  $\text{Par } \mathcal{M} := \Delta \setminus \text{Prin } \mathcal{M}$ . Pick a point  $\mathbf{x}^0$  in which all elements of  $\mathcal{M}$  are defined. For each parametric derivative, we can specify an initial condition in  $\mathbf{x}^0$ . We then use the equations from  $\mathcal{M}$  to (uniquely) compute the values of principal derivatives at  $\mathbf{x}^0$  and thus obtain a formal solution of the system corresponding to  $\mathcal{M}$ . If the set  $\text{Par } \mathcal{M}$  is empty, then the system corresponding to  $\mathcal{M}$  has only the trivial solution. We refer the reader to [14, Theorem 2] or to [15] for the details, including the details about Riquier bases.

<sup>2</sup>The original reference is [3]. A recent monography is [13, Section 2.1]. We use the terminology from [15, chapter 5].

Finally, the Analytic Riquier Existence Theorem states that the formal solution of the system defined by  $\mathcal{M}$  is convergent if all initial determinations are convergent. Recall that for each  $i = 1, \dots, n$  the *initial determination* of  $u^i$  is the formal power series with support  $\{\alpha \in \mathbb{N}^m \mid \delta_\alpha^i \in \text{Par } \mathcal{M}\}$  and with coefficients determined by the initial conditions. We refer the reader to [12, Chapter VIII] for the proof. The original reference is [11]. We do not use the full generality of this result since we only work with linear partial differential equations. Reference [15] claims a generalization of the original result from Riquier to orderly rankings but this has been disputed in [5]. This is not a problem for us because the standard ranking is a Riquier ranking.

### 3. A TECHNICAL RESULT

The aim of this section is to prove the following technical result. For every integer  $s$  we write  $I_s = \{\alpha \in \mathbb{N}^m \mid |\alpha| \leq s\}$  and  $\Delta_s = \{\delta_\alpha^i \in \Delta \mid \alpha \in I_s, i = 1, \dots, n\}$ .

**Proposition 1.** *Let  $\mathcal{M}$  be a Riquier basis in  $B_m(\mathbb{F})^n$ . Let  $s_0$  be the maximum of degrees of all elements from  $\mathcal{M}$ . (Recall that degrees are defined with respect to the standard ranking.) We claim that for every integer  $s \geq s_0$ , every point  $\mathbf{x}^0 \in \mathbb{F}^m$  in which all elements of  $\mathcal{M}$  are defined (note that all  $D^\beta \mathbf{p}$  are defined in every point in which  $\mathbf{p}$  is defined) and every  $c \in \mathbb{F}^{\Delta_s}$  the following are equivalent.*

- (1) *There exists a convergent  $\mathbf{u} \in \mathbb{F}[[\mathbf{x} - \mathbf{x}^0]]^n$  such that*
  - (a)  $\mathbf{p}[\mathbf{u}] = 0$  for every  $\mathbf{p} \in \mathcal{M}$  and
  - (b)  $\delta[\mathbf{u}](\mathbf{x}^0) = c(\delta)$  for every  $\delta \in \Delta_s$ .
- (2) *For every  $\mathbf{p} \in \mathcal{M}$  and every  $\beta \in I_{s-\deg \mathbf{p}}$ , we have that*

$$\sum_{\delta \in \Delta_s} \text{cf}(D^\beta \mathbf{p})(\delta) \big|_{\mathbf{x}^0} c(\delta) = 0.$$

*Proof.* To prove that (1) implies (2) we multiply (a) with  $D^\beta$ , convert into standard form, insert  $\mathbf{x}^0$  and finally apply (b). Suppose now that (2) is true. If  $\text{Par } \mathcal{M}$  is empty, then  $\mathcal{M}$  must contain elements with highest derivatives  $\delta_0^i = \mathbf{e}_i$  for all  $i$ . Then assumption (2) implies that  $c(\delta) = 0$  for every  $\delta \in \Delta_s$ . Now the trivial solution satisfies (1). If  $\text{Par } \mathcal{M}$  is nonempty, we can proceed as in the Formal Riquier Existence Theorem. We compute the formal solution  $\mathbf{u} = (u^1, \dots, u^n)$  of the system defined by  $\mathcal{M}$  that satisfies the following initial conditions

$$\delta[\mathbf{u}](\mathbf{x}^0) := \begin{cases} c(\delta) & \text{if } \delta \in \text{Par } \mathcal{M} \cap \Delta_s \\ 0 & \text{if } \delta \in \text{Par } \mathcal{M} \setminus \Delta_s \end{cases}$$

By construction,  $\mathbf{u}$  satisfies (a). Let us show now that  $\mathbf{u}$  is analytic. For each  $i = 1, \dots, n$ , the initial determination of  $u^i$ , i.e. the formal power series

$$\sum_{\substack{\alpha \in \mathbb{N}^m \\ \delta_\alpha^i \in \text{Par } \mathcal{N}}} \frac{D^\alpha u^i(\mathbf{x}^0)}{\alpha!} (\mathbf{x} - \mathbf{x}^0)^\alpha = \sum_{\substack{\alpha \in \mathbb{N}^m \\ \delta_\alpha^i \in \text{Par } \mathcal{N}}} \frac{\delta_\alpha^i[\mathbf{u}](\mathbf{x}^0)}{\alpha!} (\mathbf{x} - \mathbf{x}^0)^\alpha = \sum_{\substack{\alpha \in \mathbb{N}^m \\ \delta_\alpha^i \in \text{Par } \mathcal{N} \cap \Delta_s}} \frac{c(\delta_\alpha^i)}{\alpha!} (\mathbf{x} - \mathbf{x}^0)^\alpha$$

is a polynomial. By the Analytic Riquier Existence Theorem<sup>3</sup> it follows that the formal power series for  $\mathbf{u}$  is convergent. It remains to show that  $\mathbf{u}$  satisfies (b). By construction, we already know that

$$(5) \quad \delta[\mathbf{u}](\mathbf{x}^0) = c(\delta)$$

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<sup>3</sup>See the last paragraph of Section 2.

holds for every  $\delta \in \text{Par}\mathcal{M} \cap \Delta_s$ . We claim that (5) also holds for every  $\delta \in \text{Prin}\mathcal{M} \cap \Delta_s$ . We will prove this claim by induction. Pick any  $\delta_\alpha^i \in \text{Prin}\mathcal{M} \cap \Delta_s$  and assume that (5) holds for all  $\delta \prec \delta_\alpha^i$ . By the definition of  $\text{Prin}\mathcal{M}$  there exists  $\mathbf{p} \in \mathcal{M}$  and  $\beta \in \mathbb{N}^m$  such that  $\delta_\alpha^i = D^\beta \text{hd } \mathbf{p}$ . Now assumption (2) implies that

$$\sum_{\delta \prec \delta_\alpha^i} \text{cf}(D^\beta \mathbf{p})(\delta)|_{\mathbf{x}^0} c(\delta) + \text{cf}(D^\beta \mathbf{p})(\delta_\alpha^i)|_{\mathbf{x}^0} c(\delta_\alpha^i) = 0.$$

On the other hand, by multiplying the equation  $\mathbf{p}[\mathbf{u}] = 0$  with  $D^\beta$ , converting into the standard form and inserting  $\mathbf{x}^0$  we obtain that

$$\sum_{\delta \prec \delta_\alpha^i} \text{cf}(D^\beta \mathbf{p})(\delta)|_{\mathbf{x}^0} \delta[\mathbf{u}](\mathbf{x}^0) + \text{cf}(D^\beta \mathbf{p})(\delta_\alpha^i)|_{\mathbf{x}^0} \delta_\alpha^i[\mathbf{u}](\mathbf{x}^0) = 0.$$

Now, the induction hypothesis implies that

$$\text{cf}(D^\beta \mathbf{p})(\delta_\alpha^i)|_{\mathbf{x}^0} c(\delta_\alpha^i) = \text{cf}(D^\beta \mathbf{p})(\delta_\alpha^i)|_{\mathbf{x}^0} \delta_\alpha^i[\mathbf{u}](\mathbf{x}^0)$$

The fact that all elements of  $\mathcal{M}$  are monic implies that  $\text{cf}(D^\beta \mathbf{p})(\delta_\alpha^i)|_{\mathbf{x}^0} = 1$ , so

$$c(\delta_\alpha^i) = \delta_\alpha^i[\mathbf{u}](\mathbf{x}^0)$$

which completes our induction and proves the claim.  $\square$

#### 4. PROOF OF THE MAIN RESULT

We will prove a slight generalization of the promised result. Namely, that for every nonempty open set  $U \subseteq \mathbb{F}^m$  we can restrict our solution set from a subset of  $\bigcup_{\mathbf{x}^0 \in \mathbb{F}^m} \mathbb{F}[[\mathbf{x} - \mathbf{x}^0]]$  to a subset of  $\bigcup_{\mathbf{x}^0 \in U} \mathbb{F}[[\mathbf{x} - \mathbf{x}^0]]$ . We will use several times that a nonzero polynomial from  $\mathbb{F}[\mathbf{x}]$  cannot vanish on a nonempty open subset of  $\mathbb{F}^m$ . It follows that the zero set of a nonzero polynomial has the property that its relative complement in any nonempty open subset of  $\mathbb{F}^m$  is dense in that subset.

We will need the following auxiliary observation:

**Lemma 2.** *Pick  $t \in \mathbb{N}$  and let  $\langle \cdot, \cdot \rangle$  be the standard inner product on  $\mathbb{F}^t$ . We claim that for every  $\mathbf{g}_1, \dots, \mathbf{g}_k, \mathbf{f} \in \mathbb{F}(\mathbf{x})^t$  the following are equivalent:*

- (1) *There exists a nonempty open subset  $W \subseteq \mathbb{F}^m$  on which all  $\mathbf{g}_1, \dots, \mathbf{g}_k, \mathbf{f}$  are defined such that for every  $\mathbf{x}^0 \in W$  and for every  $\mathbf{c} \in \mathbb{F}^t$  which satisfy  $\langle \mathbf{g}_1(\mathbf{x}^0), \mathbf{c} \rangle = \dots = \langle \mathbf{g}_k(\mathbf{x}^0), \mathbf{c} \rangle = 0$  we have that  $\langle \mathbf{f}(\mathbf{x}^0), \mathbf{c} \rangle = 0$ .*
- (2)  *$\mathbf{f} \in \mathbb{F}(\mathbf{x})\mathbf{g}_1 + \dots + \mathbb{F}(\mathbf{x})\mathbf{g}_k$ .*

*Proof.* If (2) is true then  $\mathbf{f} = \sum_{j=1}^k h_j \mathbf{g}_j$  for some  $h_j \in \mathbb{F}(\mathbf{x})$ . Let  $p$  be the product of denominators of all  $h_j$  and of all components of  $\mathbf{f}$  and of all components of all  $\mathbf{g}_j$ . The set  $W := \{\mathbf{x}^0 \in \mathbb{F}^m \mid p(\mathbf{x}^0) \neq 0\}$  is an open subset of  $\mathbb{F}^m$  on which  $\mathbf{f}$  and all  $\mathbf{g}_j$  are defined. Pick any  $\mathbf{x}^0 \in W$  and any  $\mathbf{c} \in \mathbb{F}^t$  such that  $\langle \mathbf{g}_j(\mathbf{x}^0), \mathbf{c} \rangle = 0$  for all  $j = 1, \dots, k$  and note that  $\langle \mathbf{f}(\mathbf{x}^0), \mathbf{c} \rangle = \sum_{j=1}^k h_j(\mathbf{x}^0) \langle \mathbf{g}_j(\mathbf{x}^0), \mathbf{c} \rangle = 0$ . So, (1) is true.

Suppose now that (1) is true. Let  $G$  be the matrix with rows  $\mathbf{g}_1, \dots, \mathbf{g}_k$  and let  $\mathbf{v} \in \mathbb{F}(\mathbf{x})^t$  be a column vector such that  $G\mathbf{v} = 0$ . We claim that  $\mathbf{f}\mathbf{v} = 0$ . We may assume that  $\mathbf{v} \in \mathbb{F}[\mathbf{x}]^t$ . Pick any  $\mathbf{x}^0 \in W$ , write  $\mathbf{c} = \overline{\mathbf{v}(\mathbf{x}^0)}^T$  and note that  $\langle \mathbf{g}_j(\mathbf{x}^0), \mathbf{c} \rangle = 0$  for all  $j = 1, \dots, k$ . By (1), it follows that  $\mathbf{f}(\mathbf{x}^0)\mathbf{v}(\mathbf{x}^0) = \langle \mathbf{f}(\mathbf{x}^0), \mathbf{c} \rangle = 0$ . We proved that  $\mathbf{f}\mathbf{v}$  vanishes on  $W$ . As  $\mathbf{f}$  is defined on  $W$ , it follows that the

numerator of  $\mathbf{f} \mathbf{v}$  vanishes on  $W$ . Thus  $\mathbf{f} \mathbf{v} = 0$  in  $\mathbb{F}(\mathbf{x})$ . Now we use a standard linear algebra trick. We define a  $\mathbb{F}(\mathbf{x})$ -linear function

$$\phi: \mathbb{F}(\mathbf{x})^k \rightarrow \mathbb{F}(\mathbf{x}), \quad \phi(\mathbf{u}) = \begin{cases} \mathbf{f} \mathbf{v} & \text{if } \mathbf{u} = G \mathbf{v} \text{ for some } \mathbf{v} \in \mathbb{F}[\mathbf{x}]^t \\ 0 & \text{if } \mathbf{u} \neq G \mathbf{v} \text{ for every } \mathbf{v} \in \mathbb{F}[\mathbf{x}]^t \end{cases}$$

Since  $G \mathbf{v} = 0$  implies  $\mathbf{f} \mathbf{v} = 0$ ,  $\phi$  is well-defined. By construction, we have that  $\phi(G \mathbf{v}) = \mathbf{f} \mathbf{v}$  for every  $\mathbf{v} \in \mathbb{F}(\mathbf{x})^t$ . It follows that  $\mathbf{f} = \sum_{j=1}^k \phi(\mathbf{e}_j) \mathbf{g}_j$  where  $\mathbf{e}_j$  is the  $j$ -th standard basis vector of  $\mathbb{F}(\mathbf{x})^k$ . So, (2) is true.  $\square$

We are now ready for the proof of our main result.

**Theorem 3.** *Let  $U$  be a nonempty open subset of  $\mathbb{F}^m$ . For every submodule  $N$  of  $A_m(\mathbb{F})^n$  and every element  $\mathbf{q} \in A_m(\mathbb{F})^n$ , the following are equivalent:*

- (1) *Every convergent  $\mathbf{u} \in \bigcup_{\mathbf{x}^0 \in \mathbb{F}^m} \mathbb{F}[[\mathbf{x} - \mathbf{x}^0]]^n$  which solves  $\mathbf{p}[\mathbf{u}] = 0$  for all  $\mathbf{p} \in N$ , also solves  $\mathbf{q}[\mathbf{u}] = 0$ .*
- (1') *Every convergent  $\mathbf{u} \in \bigcup_{\mathbf{x}^0 \in U} \mathbb{F}[[\mathbf{x} - \mathbf{x}^0]]^n$  which solves  $\mathbf{p}[\mathbf{u}] = 0$  for all  $\mathbf{p} \in N$ , also solves  $\mathbf{q}[\mathbf{u}] = 0$ .*
- (2) *There exists a nonzero  $w \in \mathbb{F}[\mathbf{x}]$  such that  $w \mathbf{q} \in N$ .*

*Proof.* Clearly, (1) implies (1').

To show that (2) implies (1), one must show that  $(w \mathbf{q})[\mathbf{u}] = 0$  implies  $\mathbf{q}[\mathbf{u}] = 0$ . This follows from continuity of analytic functions (and their derivatives) and the fact that the complement of the zero set of  $w$  is dense in any ball around  $\mathbf{x}^0$ .

To show that (1') implies (2), note first that  $\mathbb{F}(\mathbf{x})N$  is a submodule of  $B_m(\mathbb{F})^n$ . Pick a Riquier basis  $\mathbf{p}_1, \dots, \mathbf{p}_k$  of  $\mathbb{F}(\mathbf{x})N$  and write

$$s = \max\{\deg \mathbf{q}, \deg \mathbf{p}_1, \dots, \deg \mathbf{p}_k\}, \quad t = \text{card } \Delta_s.$$

The standard ranking identifies  $\Delta_s$  with  $\{1, \dots, t\}$ ,  $\mathbb{F}^{\Delta_s}$  with  $\mathbb{F}^t$  and  $\mathbb{F}(\mathbf{x})^{\Delta_s}$  with  $\mathbb{F}(\mathbf{x})^t$ . Let  $\text{cf}_s: B_m(\mathbb{F})^n \rightarrow \mathbb{F}(\mathbf{x})^{\Delta_s}$  be the compositum of  $\text{cf}: B_m(\mathbb{F}) \rightarrow \mathbb{F}(\mathbf{x})^\Delta$  with the restriction map  $\mathbb{F}(\mathbf{x})^\Delta \rightarrow \mathbb{F}(\mathbf{x})^{\Delta_s}$ .

We claim that elements  $\mathbf{f} := \text{cf}_s(\mathbf{q})$  and  $\mathbf{g}_{j,\beta} := \text{cf}_s(D^\beta \mathbf{p}_j)$  (for  $j = 1, \dots, k$  and  $\beta \in I_{s-\deg \mathbf{p}_j}$ ) satisfy part (1) of Lemma 2. The set  $W$  of all  $\mathbf{x}^0 \in U$  in which  $\mathbf{f}$  and all  $\mathbf{g}_{j,\beta}$  are defined is clearly nonempty and open. Pick any  $\mathbf{x}^0 \in W$  and any  $\mathbf{c} = (c(\delta))_{\delta \in \Delta_s} \in \mathbb{F}^{\Delta_s}$  such that  $\langle \mathbf{g}_{j,\beta}(\mathbf{x}^0), \mathbf{c} \rangle = 0$  for all  $j$  and  $\beta$ . Note that part (2) of Proposition 1 is satisfied since  $\sum_{\delta \in \Delta_s} \text{cf}(D^\beta \mathbf{p}_j)(\delta)|_{\mathbf{x}^0} c(\delta) = \sum_{\delta \in \Delta_s} \mathbf{g}_{j,\beta}(\delta)|_{\mathbf{x}^0} \overline{c(\delta)} = \langle \mathbf{g}_{j,\beta}|_{\mathbf{x}^0}, \mathbf{c} \rangle = 0$  for every  $\mathbf{p}_j$  and every  $\beta \in I_{s-\deg \mathbf{p}_j}$ . By part (1) of Proposition 1, there exists a convergent  $\mathbf{u} \in \mathbb{F}[[\mathbf{x} - \mathbf{x}^0]]^n$  such that  $\mathbf{p}_j[\mathbf{u}] = 0$  for every  $j = 1, \dots, k$  and  $\delta[\mathbf{u}](\mathbf{x}^0) = c(\delta)$  for every  $\delta \in \Delta_s$ . It follows that  $\mathbf{p}[\mathbf{u}] = 0$  for every  $\mathbf{p} \in N$ . (This requires a continuity argument as above, as  $\mathbf{p} \in \sum_{j=1}^k B_m(\mathbb{F})\mathbf{p}_j$  implies only that  $(z\mathbf{p})[\mathbf{u}] = 0$  for some nonzero  $z \in \mathbb{F}[\mathbf{x}]$ .) By assumption (1') it follows that  $\mathbf{q}[\mathbf{u}] = 0$ . If we insert  $\mathbf{x}^0$  and use that  $\delta[\mathbf{u}](\mathbf{x}^0) = c(\delta)$ , we get that  $\langle \mathbf{f}(\mathbf{x}^0), \mathbf{c} \rangle = 0$ . This proves the claim. Now, Lemma 2 implies that

$$\mathbf{f} \in \sum_{j=1}^k \sum_{\beta \in I_{s-\deg \mathbf{p}_j}} \mathbb{F}(\mathbf{x}) \mathbf{g}_{j,\beta}.$$

Since  $\sum_{\delta \in \Delta_s} \mathbf{f}(\delta) \delta = \mathbf{q}$  and  $\sum_{\delta \in \Delta_s} \mathbf{g}_{j,\beta}(\delta) \delta = D^\beta \mathbf{p}_j$ , we obtain

$$\mathbf{q} \in \sum_{j=1}^k \sum_{\beta \in I_{s-\deg \mathbf{p}_j}} \mathbb{F}(\mathbf{x}) D^\beta \mathbf{p}_j \subset \sum_{j=1}^k B_m(\mathbb{F}) \mathbf{p}_j = \mathbb{F}(\mathbf{x}) \cdot N$$

which implies (2).  $\square$

## 5. COMMENTS AND EXAMPLES

**5.1. Simplifications in the  $m = n = 1$  case.** If  $m = 1$  then  $B_m(\mathbb{F})$  is a principal left ideal domain by [6, Theorem 1.5.9 (ii)]. If  $n = 1$  then every submodule of  $B_m(\mathbb{F})^n$  is a left ideal of  $B_m(\mathbb{F})$ . Therefore, if  $m = n = 1$  then every submodule of  $B_m(\mathbb{F})^n$  is principal. Let  $I$  be a left ideal of  $B_1(\mathbb{F})$  and let  $p = \sum_{i=0}^{s_0} p_i(x)D^i$ , where  $p_{s_0} = 1$ , be its principal generator. The set  $\mathcal{I} = \{p\}$  is then a Riquier basis of  $I$ . We have that  $\Delta = \{D^n, n \in \mathbb{N}\}$  and its standard ranking comes from the usual ordering of  $\mathbb{N}$ . We can decompose  $\Delta$  into  $\text{Par}\mathcal{I} = \{D^n \mid n = 0, \dots, s_0 - 1\}$  and  $\text{Prin}\mathcal{I} = \{D^n \mid n \geq s_0\}$ . Pick a point  $\mathbf{x}^0$  in which all coefficients of  $p$  are defined. The Analytic Riquier Existence Theorem reduces to the well-known fact that the initial value problem  $\sum_{i=0}^{s_0} p_i(x)u^{(i)}(x) = 0$ ,  $(u(x^0), u'(x^0), \dots, u^{(s_0-1)}(x^0)) = \mathbf{c}$  has a unique convergent power series solution for each  $\mathbf{c} \in \mathbb{F}^{s_0}$ . Apart from these simplifications the length of the proof of Theorem 3 in the  $m = n = 1$  case remains the same as in the general case.

**5.2. Nonsingular points.** We define a singular point of system (1) as a point (in  $\mathbb{F}^m$ ) that belongs to its singular locus, see [20, Definition 2.1.3]. If  $m = n = 1$  this coincides with the usual definition. System (1) has a nonsingular point if and only if the module  $A_m(\mathbb{F})^n/N$ , where submodule  $N$  is generated by the rows of the  $[p_{ij}]$  matrix, has finite rank, see [20, Lemma 2.1.5]. Note that the set of all nonsingular points is open in  $\mathbb{F}^n$ . Proposition 4 strengthens Theorem 3 in a special case.

**Proposition 4.** *Let  $N$  be as above. Suppose that  $\mathbb{F} = \mathbb{C}$  and system (1) has a nonsingular point  $\mathbf{x}^0$ . Let  $U$  be a nonempty simply connected open subset of the set of all nonsingular points. Then the following are equivalent for every  $\mathbf{q} \in A_m(\mathbb{F})^n$ :*

- (1'') *Every convergent  $\mathbf{u} \in \mathbb{F}[[\mathbf{x} - \mathbf{x}^0]]^n$  which solves  $\mathbf{p}[\mathbf{u}] = 0$  for all  $\mathbf{p} \in N$ , also solves  $\mathbf{q}[\mathbf{u}] = 0$ .*
- (2) *There exists a nonzero  $w \in \mathbb{F}[\mathbf{x}]$  such that  $w\mathbf{q} \in N$ .*
- (3) *Every  $n$ -tuple  $\mathbf{u}$  of analytic functions on  $U$  which solves  $\mathbf{p}[\mathbf{u}] = 0$  for all  $\mathbf{p} \in N$ , also solves  $\mathbf{q}[\mathbf{u}] = 0$ .*

*Proof.* Pick an open ball  $B$  around  $\mathbf{x}^0$  in the set of all nonsingular points. By the Cauchy-Kovalevskaya-Kashiwara theorem<sup>4</sup>, the dimension of the space of all analytic solutions on  $B$  is finite and equal to the rank of  $A_m(\mathbb{F})^n/N$ . It follows that every convergent power series solution at  $\mathbf{x}^0$  comes from some analytic solution on  $B$ . Therefore, the equivalence of (2) and (1'') follows from the equivalence of (2) and (3). The equivalence of (2) and (3) is a reformulation of [20, Proposition 2.1.9] (which is also a corollary of the Cauchy-Kovalevskaya-Kashiwara theorem).  $\square$

Proposition 4 also holds for some singular points  $\mathbf{x}^0$  and some open  $U$  that are not simply connected (see Example 5) but not for all of them (see Example 6).

**Example 5.** Take  $\mathbb{F} = \mathbb{C}$ ,  $U = \mathbb{F} \setminus \{0\}$ ,  $x^0 = 0$  and  $p = x^2D^2 - 2xD + 2$ . Clearly  $x^0$  is a singular point of  $p$  and  $U$  is not simply connected. We claim that (1''), (2) and (3) are equivalent for every  $q \in A_1(\mathbb{F})$ . Suppose that  $q \in A_1(\mathbb{F})$  satisfies either (1'') or (3). Every convergent power series solution at  $x^0$  and every analytic solution on

<sup>4</sup>This version is from [20, Theorem 2.1.8] or [7, Section 4]. The original reference is Kashiwara's master's thesis [4, Theorem 2.3.1].

$U$  of  $p[u] = 0$  are of the form  $u = c_1x + c_2x^2$ . Therefore,  $q[x] = q[x^2] = 0$ . It follows that  $q$  also satisfies (1') of Theorem 3 and so (2) is true. The converse is clear.

**Example 6.** Take  $U = \mathbb{F} \setminus \{0\}$ ,  $x^0 = 0$  and  $p = x^2D^2 - xD + \frac{3}{4}$  then a general solution of  $p[u] = 0$  is  $u = c_1\sqrt{x} + c_2x\sqrt{x}$ . Therefore,  $p[u] = 0$  has no convergent power series solution at  $x^0$  and no analytic solution on  $U$  which implies that (1'') and (3) are trivially true for all  $q$ . On the other hand, (2) is false for some  $q$ .

**5.3. Generic solution.** Let  $N$  be submodule of  $A_m(\mathbb{F})^n$  generated by the rows of the  $[p_{ij}]$  matrix of system (1) and let  $M = A_m(\mathbb{F})^n/N$ . Let  $\pi: A_m(\mathbb{F})^n \rightarrow M$  be the canonical projection and let  $y_i = \pi(\mathbf{e}_i)$ ,  $i = 1, \dots, n$  be the projections of the standard basis of  $A_m(\mathbb{F})^n$ . We will call  $(y_1, \dots, y_n)$  the *generic solution* of system (1), see [10, Definition 3.5.1 and Example 3.5.2]. To show that the generic solution is indeed a solution, note that by the definition of  $N$ ,  $\sum_{j=1}^n p_{ij}\mathbf{e}_j \in N$  for every  $i = 1, \dots, k$ . It follows that  $\sum_{j=1}^n p_{ij}y_j = \sum_{j=1}^n p_{ij}\pi(\mathbf{e}_j) = \pi(\sum_{j=1}^n p_{ij}\mathbf{e}_j) = 0$  for every  $i = 1, \dots, k$  as desired.

All solutions of system (1) can be obtained by specializing the generic solution, see [10, Theorem 1.1.1]. Let us explain the details. For every  $\mathbf{x}^0 \in \mathbb{F}^n$  write  $\mathcal{F}_{\mathbf{x}^0}$  for the abelian group of all convergent power series in  $\mathbb{F}[[\mathbf{x} - \mathbf{x}^0]]$ . Note that  $\mathcal{F}_{\mathbf{x}^0}$  has the structure of a left  $A_m(\mathbb{F})$ -module in the obvious way. Let  $\text{Hom}(M, \mathcal{F}_{\mathbf{x}^0})$  be the set of all  $A_m(\mathbb{F})$ -module homomorphisms from  $M$  to  $\mathcal{F}_{\mathbf{x}^0}$ . For every  $\varphi \in \text{Hom}(M, \mathcal{F}_{\mathbf{x}^0})$ ,  $(\varphi(y_1), \dots, \varphi(y_n))$  is a solution of system (1) at point  $\mathbf{x}^0$  and every solution can be obtained this way.

We will now rephrase Theorem 3 in this new terminology. Note that every element  $m \in M$  is of the form  $m = \pi(\mathbf{q}) = q_1y_1 + \dots + q_ny_n$  for some  $\mathbf{q} = (q_1, \dots, q_n) \in A_m(\mathbb{F})^n$  where  $(y_1, \dots, y_n)$  is the generic solution.

**Corollary 7.** *Let  $U$  be a nonempty open subset of  $\mathbb{F}^m$  and let  $M$  be as above. For every element  $m \in M$ , the following are equivalent:*

- (1) *For every  $\mathbf{x}^0 \in \mathbb{F}^m$  and every  $\varphi \in \text{Hom}(M, \mathcal{F}_{\mathbf{x}^0})$  we have that  $\varphi(m) = 0$ .*
- (1') *For every  $\mathbf{x}^0 \in U$  and every  $\varphi \in \text{Hom}(M, \mathcal{F}_{\mathbf{x}^0})$  we have that  $\varphi(m) = 0$ .*
- (2) *There exists a nonzero  $w \in \mathbb{F}[\mathbf{x}]$  such that  $wm = 0$ .*

Proposition 4 can be rephrased similarly.

**5.4. Rapidly decreasing solutions.** Recall that a function is rapidly decreasing if it belongs to  $\mathcal{S} := \{f \in \mathcal{C}^{(\infty)}(\mathbb{R}^m) \mid \sup_{x \in \mathbb{R}^m} |x^\alpha D^\beta f(x)| < \infty \text{ for all } \alpha, \beta \in \mathbb{N}^m\}$ . We define the  $\mathcal{S}$ -closure of system (1) as the set of all equations (2) that vanish on every rapidly decreasing solution of system (1). The  $\mathcal{S}$ -closure behaves very differently from the Weyl closure as the following example shows:

**Example 8.** Let  $q = D+x \in A_1(\mathbb{R})$  and  $p = q^*q = (-D+x)(D+x) = -D^2+x^2-1$ . (Recall that the standard involution on  $A_m(\mathbb{F})$  is defined by  $D_i^* = -D_i$ ,  $x_j^* = x_j$  for every  $i, j = 1, \dots, m$  and  $\alpha^* = \bar{\alpha}$  for every  $\alpha \in \mathbb{F}$ .) We claim that  $q$  belongs to the  $\mathcal{S}$ -closure of  $p$  but it does not belong to the Weyl closure of  $p$ . The general solution of  $q[u] = 0$  is  $u = ce^{-x^2/2}$  which is rapidly decreasing and the general solution of  $p[u] = 0$  is  $u = c_1e^{-x^2/2} + c_2v$  where  $v = e^{-x^2/2} \int e^{x^2} dx$  is not rapidly decreasing. It follows that  $q$  belongs to the  $\mathcal{S}$ -closure of  $p$ . Since  $q[v] = e^{x^2/2} \neq 0$ , we have that  $q$  does not belong to the Weyl closure of  $p$ .

An important advantage of  $\mathcal{S}$  is that it has an inner product and that  $q^*$  is the adjoint of  $q$  with respect to this inner product. A disadvantage is that the  $\mathcal{S}$ -closure



is often equal to  $A_m(\mathbb{F})^n$  because often there is no rapidly decreasing solution. Let  $N'$  be the  $\mathcal{S}$ -closure of a submodule  $N$  of  $A_m(\mathbb{F})^n$ . By using the inner product one can show that  $N'$  is a real submodule of  $A_m(\mathbb{F})^n$  in the sense that if

$$\sum_i \mathbf{p}_i^* \mathbf{p}_i = \sum_j (\mathbf{h}_j^* \mathbf{q}_j + \mathbf{q}_j^* \mathbf{h}_j) \quad (\text{in } A_m(\mathbb{F})^{n \times n})$$

for some  $\mathbf{p}_i, \mathbf{h}_j \in A_m(\mathbb{F})^n$  and  $\mathbf{q}_j \in N'$  then  $\mathbf{p}_i \in N'$  for all  $i$ . From the perspective of noncommutative real algebraic geometry (see [1, Example 1.3 and Theorem 1.6] and [2, Theorem 2]) it would be interesting to know when  $N'$  is the smallest real submodule of  $A_m(\mathbb{F})^n$  which contains  $N$  (i.e. when  $N'$  is the real radical of  $N$ ).

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