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THE EVOLUTION OF...

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Bolzano, Cauchy, Epsilon, Delta

Walter Felscher

Introduction. In this article I describe how the notions of limit and continuity were explained and used between 1817 and 1823 by Bolzano and Cauchy. In a closing section I discuss the extent to which the technique of epsilon and delta serves as a tool to write finite proofs of statements which, involving limits and continuity, refer to infinite processes.

1. Today's terminology. The following definitions have been in common use for more than a hundred years.

Let f be a function defined in a neighborhood $V = \{x;\ |a - x| < q\}$ of a number a. A number b is said to be the *B-limit of f at a* if for every $\epsilon > 0$ there is a $\delta > 0$ (with $\delta < q$) such that for every $y, |a - y| < \delta$ implies that $|b - f(y)| < \epsilon$.

Let s be a sequence $\langle x(i) \mid \in \omega \rangle$ of numbers $x(i)$ indexed with the set ω of natural numbers. The sequence s is said to *converge to a*, and a is then called the *limit* of s, if for every $\epsilon > 0$ there is an $n \in \omega$ such that for every $i \in \omega$, $i > n$ implies that $|a - x(i)| < \epsilon$.

A number b is said to be the C-limit of f at a if for every sequence $\langle x(i) \mid i \in \omega \rangle$ with values in V the convergence of s to a implies the convergence of $f \circ s = \langle f(x(i)) \mid i \in \omega \rangle$ to b.

Lemma. *b is the B-limit of f at a if, and only if, b is the C-limit of f at a.*

In this connection I mention an observation by Sierpiński (1916): The implication that a B-limit is a C-limit requires, and is actually equivalent to, the following weak form of the Axiom of Choice: For each countable family of nonempty sets of real numbers there exists a choice function.

Let $\%$ be one of B and C. The function f is said to be *%-continuous* at a if $f(a)$ is the $\%$-limit at a.

Lemma. *f is B-continuous at a if, and only if, f is C-continuous at a.*

Again, the statement that a B-continuous function is C-continuous requires, and is actually equivalent to, the weak form of the Axiom of Choice mentioned earlier.

Methodological remark. The B-notions require quantifiers ranging over sets of numbers (namely over ϵ, δ, and y). The C-notions also require such quantifiers (namely ϵ, n, and i), but in addition they require a quantifier ranging over all sequences s. Sequences are notions of a higher order (the 2nd) than numbers (of 1st order). In this sense the logical complexity of the C-notions is higher than that of the B-notions. This fact is irrelevant in analysis because the notion of function is also of 2nd order.

In particular, B-continuity directly connects local properties at a and at $f(a)$, while C-continuity makes use of an intermediate, additional abstraction, that of sequences and their convergence (or their limits): limiting processes at a are put in correspondence with limiting processes at $f(a)$.

Terminological remark. Use of the letters ϵ and δ as in the preceding definition has become canonical throughout the literature (including publications using the Cyrillic alphabet). Specifically, the B-notions are denoted with the help of the letters ϵ and δ, and convergence of a sequence, and thus the C-limit, is denoted with the help of the letter ϵ and the letter n or N. Thus the C-notions do not involve the use of δ. Work with notions employing ϵ is sometimes referred to as "epsilontics", while references to epsilon-delta techniques indicate that a letter (and then a quantifier) for δ is also used, i.e., that B-notions are considered.

2. D'Alembert's program. Jean-Baptist le Rond d'Alembert (1717–1783) was, together with Euler and the brothers Bernoulli, one of the mathematicians representing the heroic age of calculus. Together with Denis Diderot he edited the *Encyclopédie ou Dictionnaire Raisonné des Sciences, des Arts et des Métiers*. In its 9th volume of 1765 he wrote in the article *Limite*:

> On dit qu'une grandeur est la limite d'une autre grandeur, quand la seconde peut approcher de la première plus près que d'une grandeur donnée, si petite qu'on la puisse supposer, sans pourtant que la grandeur qui approche, puisse jamais surpasser la grandeur dont elle approche; en sorte que la différence d'un pareille quantité à sa *limite* est absolument inassignable ...
>
> A proprement parler, la limite ne coincide jamais, ou ne devient jamais égale, à la quantité dont elle est la limite; mais celle-ci s'en approche toujours de plus en plus et peut en différer aussi peut qu'on voudra ...
>
> One says that a *grandeur* [quantity, magnitude; actually the notion refers to a 'point of the geometric continuum'] is the limit of another *grandeur* if the second can approach the first more closely than any given *grandeur*, as small as one may suppose, without, however, the approaching *grandeur* surpassing that which is being approached, such that the difference between such *grandeur* and its limit is absolutely indeterminable.
>
> Strictly speaking, the limit never coincides, and never becomes equal to, the quantity of which it is the limit; but the latter approaches it more and more and will differ from it as little as one wishes.

In its 4th volume (1754), in the article *Différentiel*, d'Alembert discussed differential quotients and, for the parabola $y^2 = ax$, determined geometrically the slope $a/2y$ of a tangent as the limit of the ratio $a/(2y + \zeta)$. He concluded the article by writing:

> Celui-ci nous paroît suffire pour faire entendre aux commençans la vraie métaphysique du calcul différentiel. Quand une fois on l'aura bien comprise, on sentira que la supposition que l'on y fait de quantitées infiniment petites, n'est que pour abrèger & simplifier les raisonnements; mais que dans le fond le calcul différentiel ne suppose point nécessairement l'existence de ces quantitées; que ce calcul ne consiste qu'à *déterminer algébriquement la limite d'un rapport*...
>
> Il ne s'agit point, comme on le dit encore ordinairement, de quantitées infiniment petites dans le calcul différentiel; il s'agit uniquement de limites de quantitées finites. Ainsi la métaphysique & des quantitées infiniment petites plus grandes ou plus petites les unes que les autres, est totalement inutile au calcul différentiel. On ne se sert du terme *d'infiniment petit*, que pour abrèger les expressions. Nous ne dirons donc pas avec bien des géometres qu'une quantité est infiniment petite, non avant qu'elle s'évanoüisse, non après qu'elle est évanoüie, mais dans l'instant même où elle s'évanoüit; car que veut dire une définition si fausse, cent fois plus obscure que ce qu'on veut définir? Nous dirons qu'il n'y a point dans le calcul différentiel de quantitées infiniment petites.
>
> Mais les inventeurs cherchent à mettre le plus de mystère qu'ils peuvent dans leurs découvertes; & en général les hommes ne haïssant point l'obscurité, pourvû qu'il en résulte quelque chose merveilleux. Charlatanerie que tout cela! La verité est simple, & peut être toujours mise à portée de tout le monde, quand on veut en prendre la peine.
>
> > This seems to suffice to make the beginner understand the true metaphysics of the differential calculus. Once this has been well understood, it will become clear that the assumption made of infinitely small quantities serves only to abbreviate and to simplify the reasoning; that, basically, the differential calculus does not necessarily assume the existence of these quantities, and that what this calculus reduces to is none other than the algebraic determination of the limit of a quotient. ...
> >
> > It is not, as one ordinarily says, infinitely small quantities that matter in the differential calculus, what matters is only limits of finite quantities. Thus the metaphysics and the infinitely small quantities, whether larger or smaller than one another, are totally useless in the differential calculus. One employs the term of an infinitely small only as an abbreviation. We shall not say, as many geometers do, that a quantity is infinitely small before it vanishes or after it has vanished, but at the very moment when it vanishes, for what would so spurious a definition mean which is a hundred times more obscure than what it wants to explain? We shall say that there are no infinitely small quantities in the differential calculus.
> >
> > But the inventors try to put as much mystery into their discoveries as they can, and in general people do not hate obscurity at all, provided it results in something marvellous. All this is quackery! The truth is simple, and can be made accessible to the entire world if only one wants to make the effort.

Reading these words today we may get the impression that they were written at the time of Weierstrass or Cantor, or even by a contemporary mathematician. But when they were actually written, the details of how to work mathematically with limits had not yet been worked out in a conceptual manner, and what d'Alembert wrote was less a description of the actual state of affairs than a program to be carried out in the future. It *was* carried out, with the decisive steps taken by Bolzano and Cauchy.

3. Bolzano. Bernhard Bolzano (1781–1848) studied mathematics in Prague with Stanislav Vydra and Franz-Josef Gerstner. At the same time he also studied theology, and in 1805, only days before receiving his doctorate in mathematics, he was ordained as a (secular) priest. As early as 1807 he obtained a chair in "Religionslehre" in Prague. But he continued to cultivate his mathematical interests, and in 1817 he published a small book **[1]** devoted to a proof of the intermediate value theorem.

On p. 11 of this book he gave a definition of continuity of a function f at an argument x:

> Nach einer richtigen Erklärung nähmlich versteht man unter der Redensart, dass eine Function $f(x)$ für alle Werthe von x, die inner- oder ausserhalb gewisser Grenzen liegen, nach dem Gesetze der Stetigkeit sich ändere, nur so viel, dass, wenn x irgend ein solcher Werth ist, der Unterschied $f(x+\omega) - f(x)$ kleiner als jede gegebene Grösse gemacht werden könne, wenn man nur ω so klein, als man nur immer will, annehmen kann.
>
> According to a correct explanation, the phrase that a function $f(x)$ changes by the law of continuity for all values of x inside or outside certain bounds means just this: if x is such a value, then the difference $f(x+\omega) - f(x)$ can be made smaller than any given magnitude if only ω may be assumed to be as small as one wishes.

Thus given $\epsilon > 0, |f(x+\omega) - f(x)| < \epsilon$ provided that ω is 'as small as one wishes'.

There are only two places where Bolzano uses his definition. The first appears on p. 52, where it is shown that if f and φ are functions defined and continuous in a neighborhood of a number α and $f(\alpha) < \varphi(\alpha)$, then $f(\alpha+\omega) < \varphi(\alpha+\omega)$ provided that ω is sufficiently small (e.g., there is a δ such that $|\omega| < \delta$ implies that $f(\alpha+\omega) < \varphi(\alpha+\omega)$).

I reproduce Bolzano's proof using his notation. I do not translate it word for word but faithfully restate his argument. Put $A = \varphi(\alpha) - f(\alpha)$. For any positive Ω and Ω' we have, by continuity,

$$
\begin{array}{llll}
(1) & \varphi(\alpha) - \Omega' & < \varphi(\alpha+\omega) & < \varphi(\alpha) + \Omega' \\
(2) & f(\alpha) - \Omega & < f(\alpha+\omega) & < f(\alpha) + \Omega, \quad \text{hence} \\
(3) & -f(\alpha) - \Omega & < -f(\alpha+\omega) & < -f(\alpha) + \Omega
\end{array}
$$

provided that ω is sufficiently small (i.e., there are δ_1 and δ_2 such that $|\omega| < \delta_1$ implies that $|\mathrm{f}(\alpha) - \mathrm{f}(\alpha+\omega)| < \Omega$ and $|\omega| < \delta_2$ implies that $|\varphi(\alpha) - \varphi(\alpha+\omega)| < \Omega'$, hence (1) and (2) hold provided that $|\omega| < \delta$, where δ is the smaller of δ_1 and δ_2). Adding the left sides of (1) and (3) yields

$$\varphi(\alpha) - \Omega' + (-f(\alpha) - \Omega) < \varphi(\alpha+\omega) - f(\alpha+\omega)$$

provided that ω is sufficiently small. But here the left side is $A - (\Omega + \Omega')$, and this stays positive if Ω and Ω' are chosen sufficiently small (i.e., chosen as $A/3$).

The second use Bolzano makes of his definition occurs on pp. 57–58, where it is shown that the function given by a polynomial $q(x) = a + b \cdot x^m + c \cdot x^n + \cdots + p \cdot x^r$ is continuous. Again, I reproduce Bolzano's proof using his notation, and, while I do not translate it word for word, I faithfully restate his argument. First, the difference $q(x + \omega) - q(x)$ is expressed as

$$b \cdot \left((x + \omega)^m - x^m\right) + c \cdot \left((x + \omega)^n - x^n\right) + \cdots + p \cdot \left((x + \omega)^r - x^r\right).$$

Making use of the binomial theorem, he rewrites this as $\omega \cdot T(\omega)$ where, x being kept fixed, $T(\omega)$ is the expression

$$\left(\begin{array}{llll} m \cdot b \cdot x^{m-1} & +m \cdot (m-1)/2b \cdot \omega \cdot x^{m-2} & + \cdots & +b \cdot \omega^{m-1} \\ +n \cdot c \cdot x^{n-1} & +n \cdot (n-1)/2c \cdot \omega \cdot x^{n-2} & + \cdots & +c \cdot \omega^{n-1} \\ + \cdots & & & \\ +r \cdot p \cdot x^{r-1} & +r \cdot (r-1)/2p \cdot \omega \cdot x^{r-2} & + \cdots & +p \cdot \omega^{r-1} \end{array}\right).$$

(There is an irrelevant misprint in Bolzano's text; the factors $b, c, \ldots, p$ were omitted in the last column). Now choose a fixed positive ω_1, and let S be the nonnegative number obtained from the sum $T(\omega_1)$ if each of its summands is replaced by its absolute value. If $0 < \omega < \omega_1$, then $|T(\omega)| < |T(\omega_1)| \leq S$, hence $|q(x + \omega) - q(x)| = \omega \cdot |T(\omega)| < \omega \cdot S$. Thus if D is positive and ω is smaller than both ω_1 and D/S, then $|q(x + \omega) - q(x)| < D$.

We see that while, in the first application, δ_1 and δ_2, associated with Ω and Ω', respectively, are not explicitly mentioned, in the second application, the δ associated with $\epsilon = D$ is explicitly determined as the smaller of ω_1 and D/S. Thus in both cases Bolzano uses his definition precisely in today's sense of B-continuity.

Bolzano's concepts, presented with unambiguous clearness, appear at first sight to have sprung from his head as Athena sprang from the head of Zeus.

Bolzano seems not to have had contact with mathematicians apart from his acquaintances in Prague. As a result, his mathematical work remained completely unknown and came to the notice of the mathematical community only thirty years after his death. In particular, Bolzano's writings had no influence on the rediscovery of B-continuity by Weierstrass and others.

4. Cauchy

4A. Cauchy on variables and their limits. Today's mathematical textbooks still use the word 'variable', but they do not define a mathematical object named by this word. A 'variable' today is a linguistic object, a letter used for denoting, and it is only implicitly that the student learns to form sentences involving this word. This conforms to the tendency to view mathematics as a subject dealing only with concepts and to disregard connections with the language used to speak about these concepts. In modern mathematics this elimination of linguistic features has been carried out with remarkable (and sometimes regrettable) success.

Augustin Louis Cauchy (1789–1857) implemented d'Alembert's program in his two textbooks of analysis [2] and [3].

In both books Cauchy begins by attempting to give a mathematical description of what is (spoken about as if it were a mental experiment, and in this vein then) conceived as a limiting process. On p. 4 of [2] (Œuvres 3, p. 19) he explains the term 'variable quantity':

> On nomme quantité variable celle que l'on considère comme devant recevoir successivement plusieurs valeurs différentes les unes après des autres. On désigne une semblable quantité par une lettre prise parmi les dernières de l'alphabet. ...
>
> > The name variable is given to a quantity of which one assumes that it can take on successively several different values, one after the other. One denotes such a quantity by a letter taken from the last letters of the alphabet.

He proceeds to explain limits of such assignments:

> Lorsque les valeurs successivement attribuées à une mûeme variable s'approchent indéfiniment d'une valuer fixe, de manière à finir par en différer aussi petit que l'on voudre, cette dernière est appelée la limite de toutes les autres. ...
>
> > If the values assigned successively to the same variable approach indefinitely a fixed value, so that they differ from it as little as one wishes, than the latter (value) is called their limit.

In particular, there are variables with assignments that have the limit zero:

> Lorsque les valeurs numériques successives d'une mûeme variable décroissant indéfiniment, de manière à s'abaisser au-dessous de tout nombre donné, cette variable devient ce qu'on nomme un infiniment petit ou une quantité infiniment petite. Une variable de cette espèce a zéro pour limite.
>
> > If the successive numerical values of a variable decrease indefinitely, so that they diminish below every given number, then this variable becomes what one calls an *infinitely small* or an infinitely small quantity. A variable of this kind has zero as its limit.

The same definitions appear in [3] (Œuvres 4, p. 16). When these definitions are referred to in Cauchy's subsequent text, the phrase 'aussi petit que l'on voudre' is usually expressed by saying that the difference between the *valeurs* of an assignment to the variable and the *valeur fixe* can be made smaller than any given positive number.

4B. Comments. In Cauchy's definition a variable *is not* a letter, but a mathematical concept, the concept of a form to be filled by *assignments* A of values. In this connection I make the following three observations:

1. Cauchy writes that values are assigned to a variable; he does not use the substantivated notion of an assignment. Nor does he distinguish notationally between (1) a variable, for which he may write x, (2) an assignment, and (3) the values of the assignment. Thus, quite frequently, x also stands for an assigned value.

2. The domains L of such assignments A remain unspecified; apparently they were supposed, at the very least, to be subject to an order relation (so that one could speak of *valeurs les unes après des autres*). In particular, assignments *may* be conceived as sequences defined on ω, but they may also be more general assignments, e.g., the identity map on a whole interval (recall the modern notion of a Moore-Smith sequence). The ranges of assignments are not specified either, but it seems clear from Cauchy's words that they were to be *numériques*, i.e., to consist of real numbers.

3. Observe that every half-line $(z, \rightarrow)$ contains a cofinal subset consisting of natural numbers above a certain number N. Thus the situations involving limits of assignments A considered by Cauchy can be subsumed under the following cases:

(c1) if a is finite and L is a cofinal subset of $(z, \rightarrow)$ with ascending order (particular case: L is the set ω of natural numbers), then given ϵ there is a y in L such that $l \in L$ and $y > l$ imply that $|a - A(l)| < \epsilon$;

(c2) if a is finite and L is a cofinal subset of $(\leftarrow, z)$ with ascending order (particular case: L is the set of all $z - 1/n$ with $n \in \omega$), then given ϵ there is a δ in L such that $l \in L$ and $z - \delta < l$ imply that $|a - A(l)| < \epsilon$;

(c3) if a is infinite and L is a cofinal subset of $(z, \rightarrow)$ with ascending order (particular case: L is the set ω of natural numbers), then given n there is $y \in L$ such that $l \in L$ and $l > y$ imply that $n < A(l)$;

(c4) if a is infinite and L is a cofinal subset of $(\leftarrow, z)$ with ascending order (particular case: L is the set of all $z - 1/n$ with $n \in \omega$), then given n there is a $\delta \in L$ such that $l \in L$ and $z - \delta < l$ imply that $n < A(l)$.

Cauchy writes that a variable with an assignment converging to zero is said "to become" a "quantité infiniment petite". Since Cauchy had earlier introduced a variable as a "quantité variable", the word "quantité" appears here with a more abstract meaning than that of number or of magnitude in the geometric continuum. (Of course, it is left open whether a quantité variable, with an assignment converging to zero, actually *is* or only *becomes* a quantité infiniment petite.) Thus a *quantité infiniment petite* belongs to quite a different species than numbers (the *valeurs numériques* of variables) or, equivalently, geometric magnitudes.

Cauchy's reason for the introduction of the *quantitées infiniment petites* is stated in [**3**, p. 9]:

> Mon but principal a été de concilier la rigeur, dont je m'étais fait une loi dans mon Cours d'analyse, avec le simplicité qui résulte de la considération directe des quantitées infiniment petites.
>
> My principal aim has been to reconcile the rigor which I respected as a law in my Cours d'analyse with the simplicity which results from the direct consideration of *quantitées infiniment petites*.

When dealing with *quantitées infiniment petites*, Cauchy's mathematical approach often rests on the observation that a number c is the limit of the assignment of values $c + j$ to the variable $x + j$ if, and only if, 0 is the limit of the assignment of values j to the variable x, i.e., this variable becomes *infiniment petite* under that assignment. On this basis, statements about limits can be translated into statements about *quantitées infiniment petites*.

As an instructive example, let me quote from [3] (Œuvres 4, p. 18):

> Cela posé, si la variable y est exprimée en fonction de la variable x par la équation
>
> (1) $$y = f(x),$$
>
> Δy, ou l'accroissement de y correspondent à l'accroissement Δx de la variable x, sera déterminé par la formule
>
> (3) $$y + \Delta y = f(x + \Delta x).$$
>
> [p. 19] ... Il est bon d'observer que, des équations (1) et (2) réunies, on conclut
>
> (5) $$\Delta y = f(x + \Delta x) - f(x).$$
>
> Soient maintenant h et i deux quantitées distinctes, la première finie, la seconde infiniment petite, et $\alpha = i/h$ le rapport infiniment petit de ces deux quantitées. Si l'on attribue à Δx la valeur finie h, la valeur de Δy, donnée par l'equation (5), deviendra ce qu'on appelle la différence de la fonction $f(x)$, et sera ordinairement une quantité finie. Si, au contraire, l'on attribue à Δx une valeur infiniment petite, si l'on fait par example
>
> $$\Delta x = i = \alpha \cdot h,$$
>
> la valeur de Δy, savoir
>
> $$f(x + i) - f(x) \quad \text{ou} \quad f(x + \alpha \cdot h) - f(x)$$
>
> sera ordinairement une quantité infiniment petite. C'est ce que l'on verifiera aisément à l'égard des fonctions
>
> $$A^x, \ldots$$
>
> auxquelles correspondent les différences
>
> $$A^{x+i} - A^x = (A^i - 1) \cdot A^x, \ldots$$
>
> dont chacune renferme un facteur $A^i - 1$ ou ... qui converge indéfiniment avec i vers zero.

Thus Cauchy considers here a *quantité infiniment petite* i and a *quantité finie* h (of which we are not certain whether it is to be thought of as as a positive number or as a variable with an assignment converging to something different from zero). He then writes $\alpha = i/h$, which is (a variable with) an assigment having as values the quotients of the values of i and the value(s) of h (notational confusion arises from denoting both the variable i and its values by the same letter). If now Δx is $i = \alpha \cdot h$, then Δy is $f(x + i) - f(x)$. This he illustrates by the exponential function $f(x) = A^x$, where

$$\Delta y\colon A^{x+i} - A^x = (A^i - 1) \cdot A^x.$$

Thus here Δy is a *quantité infiniment petite*, represented by the *quantité infiniment petite* B with values $A^i - 1$ and the constant number A^x.

This example shows that the three *quantitées infiniment petites* i, Δy, and B have as values ordinary real numbers (and all converge to zero). No 'infinitesimal' non-Archimedean numbers are ever used by Cauchy for his *quantitées infiniment petites*.

4C. D'Alembert versus Cauchy. The definitions of a limit by d'Alembert:

> On dit qu'une grandeur est la limite d'une autre grandeur, quand la seconde peut approcher de la première plus près que d'une grandeur donnée, si petite qu'on la puisse supposer, ...

and by Cauchy:

> Lorsque les valeurs successivement attribués à une mûeme variable s'approchent indéfiniment d'une valuer fixe, de manière à finir par en différer aussi petit que l'on voudre, cette dernière est appelée la limite de toutes les autres.

have the same content. Both contain the clause "for every ϵ" (in d'Alembert's case in the form of

> approcher ... plus près que d'une grandeur donnée, si petite qu'on la puisse supposer

and in Cauchy's case in the form of

> différer aussi petit que l'on voudre)

but neither contains the clause "there exists δ"—presumably because the two authors considered it as obvious that an approximation, once achieved, would inevitably improve. One difference between the two authors is the use they make of their definitions. D'Alembert wrote necessarily short articles for an encyclopedia and his writing remained descriptive, while Cauchy, who wrote a textbook, applied his notions to prove nontrivial mathematical propositions. But the main difference between the two was the extent of conceptual elaboration.

Both authors started with the notion of a limit obtained by approximation. Approximation, sometimes conceived as a process taking place in time or even in motion, appears in mathematical practice as a process of computations or constructions, with stages counted either by discrete numbers or by a piece of the continuum. Cauchy's conceptual progress consisted in the mathematical reduction of the approximation process to the notion of a *variable plus an assignment* of values.

Using today's terminology, one would describe Cauchy's forms to be filled by assignments as functions, but in order to distinguish them from the actual functions subsequently considered by Cauchy, one might call them *functional germs*. In its development since (at least) the time of Weierstrass, mathematics has eliminated forms and assignments and has replaced them by *sequences*, be they ω-sequences or Moore-Smith-sequences, or has used the explicit definition of B-limits.

While Cauchy's *definition* of limits employed variables and assignments, his *use* of the term and the concept in his books includes the characterizations listed in Section 4B under (c1)–(c4), and hence also the B-limits obtained if the assignments in them are taken as the identity on L. In her three articles **[5]**, **[6]**, **[7]** J. V. Grabiner has pointed out that Cauchy defined the differential quotients, which he investigated in the Théorème in [3] (Œuvres 4, p. 44), as limits, but that the property which he employed was that of a B-limit. As a matter of fact, Cauchy there used the letters ϵ and δ as we do today.

Epsilon-delta arguments are arguments about inequalities. Today they are written using ϵ and δ. But epsilon-delta arguments (usually not involving these particular letters) were used decades before Bolzano and Cauchy, when there were no epsilon-delta definitions. In a communication on the e-mail list *Historia-Matematica* on February 2, 2000, João Filipe Queiró noted that the Portuguese work [4] contains a proposition (Proposition 1 in Book XV, pp. 197–198) which states that if the variable x tends to zero, then so does the polynomial $ax + bx^2 + \cdots + gx^n$. Indeed (so goes the argument), let Q be given. Then, if $x < 1$ and $x < Q/nP$, where P is larger than each of the coefficients, $dx^k < Q/n$ for each of the summands dx^k. But then Q is larger than their sum. (See also [10].)

Grabiner notes that in [8] Lagrange used an epsilon-delta argument in proving a theorem about derivatives. Specifically, he wrote (Œuvres p. 87):

> Soit D une quantité donnée qu'on pourra prendre aussi petite qu'on voudra; on pourra donc toujours donner à i une valeur assez petite pourque la valeur de V soit renfermée entre des limites D et $-D$; ...
>
> > Let D be a given quantity that can be taken as small as one wishes; then one can always assign to i a value small enough for the value of V to be between the limits D and $-D$; ...

Grabiner's suggestion that both Cauchy and Bolzano may have learned to argue the way they did from this source is supported by the fact that both quoted Lagrange's treatise.

4D. Continuity. Cauchy's definition of continuity in [2, pp. 34–35] (Œuvres p. 43) reads:

> Cela posé, la fonction $f(x)$ sera, entre les deux limites assignés à la variable x, fonction continue de cette variable, si, pour chaque valeur de x intermédiaire entre ces limites, la valeur numérique de la différence $f(x + \alpha) - f(x)$ décroît indéfiniment avec celle de α. En d'autres termes, la fonction $f(x)$ restera continue par rapport à x entre les limites donnés, si, entre ces limites, un accroissement infiniment petit de la variable produit toujours un accroissement infiniment petit de la fonction elle-mûeme.
>
> > This having been stated, the function $f(x)$, between two bounds of the variable x, will be continuous in this variable if, for any value of x between these bounds, the numerical value of the difference $f(x + \alpha) - f(x)$ decreases indefinitely together with that of α. In other words, the function $f(x)$ remains continuous with respect to x between these given bounds if, between these bounds, an infinitely small increment of the variable always produces an infinitely small increment of the function itself.

Here, actually, there are two formulations. In [3] only the second one appears in the definition of continuity.

Each of the two sentences is a complete definition and the phrase "en d'autres termes" suggests that they are equivalent.

In the first formulation the variable x is kept fixed and the letter α now denotes a variable. This formulation says that $|f(x+\alpha)-f(x)|$ decreases indefinitely "together" with α decreasing indefinitely. Here "avec celle de α" may seem symmetrical; if it were understood as a one-sided conditional "si celle de a décroît indéfiniment", then it would express C- or B-continuity (depending on whether the variable α is viewed as decreasing in a sequence or continuously). There is no reason to assume that Cauchy, if pressured, would not have chosen the conditional "si".

The clause "en d'autres termes" expresses the following meaning: if the assignment with values α is a *quantité infiniment petite*, then so is the assignment with values $|f(x+\alpha)-f(x)|$. But if x is fixed and α is a variable, then α converges to zero if, and only if, $x+\alpha$ converges to x, and, since $f(x)$ is fixed as well, $|f(x+\alpha)-f(x)|$ converges to zero if, and only if, $f(x+\alpha)$ converges to $f(x)$. So here Cauchy again defines C- or B-continuity.

So far as I can see, Cauchy did not use the first formulation in any of his examples. Among other things, he used his second formulation to characterize the continuous solution of his functional equation ([**2**, p. 103 ff]), and later to prove the intermediate value theorem ([**2**, p. 462]). A particularly striking application of C-continuity occurs in the latter proof in the form of the observation that if two variables converge to the same value a, then, "puisque la fonction $f(x)$ reste continue" (since the function $f(x)$ remains continuous), the associated values of f also converge to $f(a)$.

In her articles [**5**], [**6**], [**7**], Grabiner notes that in [**9**] Lagrange gave an almost identical description (not definition!) of continuity when he considered a function written as $i \cdot P$ (where P may further depend on i) and wrote (Œuvres, sér. 2, vol. 9, p. 28):

> il s'ensuit que, en considérant la courbe dont i serait l'abscisse et l'une de ces fonctions $[i \cdot P]$ l'ordonnée, cette courbe coupera l'axe à l'origine des abscisses, et ... le cours de la courbe sera nécessairement continue depuis ce point; donc elle s'approchera peu à peu de l'axe avant de le couper et s'en approchera, par conséquent, d'une quantité moindre qu'aucune quantité donnée, de sorte qu'on pourra toujours trouver une abscisse i correspondant à une ordonnée moindre qu'une quantité donnée, et alors toute valeur plus petite de i répondra aussi dès ordonnées moindres que la quantité donnée.
>
> it follows that, in considering the curve for which i is the abscissa and one of the functions $[i \cdot P]$ is the ordinate, this curve will cut the axis at the origin of abscissas, and ... the course of the curve will necessrily continue beyond this point; hence it will indefinitely approch the axis before cutting it, and consequently will become less than any given quantity, so we can always find an abscissa i for which the corresponding ordinate is less than a given quantity, and then every value smaller than i will also give an ordinate less than the given quantity.

5. Bolzano versus Cauchy. Both Bolzano and Cauchy gave definitions of continuity which express today's B- and C-continuity. Both made their definitions precise and used them in today's sense; both employed them by comparing numbers and their distances with the help of inequalities in order to prove important theorems in analysis. However, Cauchy defined and used the notion of limit, whereas Bolzano did not.

For limits and continuity Cauchy preferred definitions which used an intermediate notion describing a process of approximation: the notion of functional germs, of variables with assignments. Bolzano, on the other hand, did not concern himself with what mathematicians imagine themselves to be doing when speaking of infinite processes: his definition of continuity reached down to the very basics, as laid bare in mathematical practise. The same is true of Cauchy's first definition, but it remained unused, possibly for didactical reasons.

The works of Bolzano and Cauchy discussed here were written for different purposes. Cauchy wrote textbooks intended as aids for his lectures on analysis. The purpose of lectures is to teach new things, and it is a didactical technique also to use motivations from known things and from intuition. What Bolzano wrote was not a textbook but an essay laying the conceptual foundation for a particular, basic, notion of analysis, namely continuity. What for Cauchy were tools for things to come, for Bolzano was the object of the investigation itself.

In his textbooks Cauchy covered many more theorems than Bolzano in his foundational essay. (There are extensive manuscripts on analysis by Bolzano published from his Nachlass, but they date from 1830.) Cauchy was read by contemporaries and the following generations, Bolzano was not. Thus Bolzano's influence was negligible compared to Cauchy's.

Cauchy used the letters ϵ and δ once in connection with limits; Bolzano used D and ω in connection with continuity; he used the letters ϵ and δ, albeit in a different connection, in the proof on pp. 47–48 of [**1**].

But the history of mathematics is not one of notation, it is a history of inventions and concepts. And there 1817 precedes 1821 by four years. Of course, there seems to be no reason to assume that Cauchy was aware of Bolzano's work.

Apart from mathematics, the fates of both men were decisively determined by the political developments of their times.

An extensive biographical notice on Bolzano can be found in the "Biographisches Bibel-Lexikon", accessible online at www.bautz.de/bbkl. His philosophical and religious views reflected the Josephine enlightenment. The chair he was called to in 1807 had been established for the purpose of counteracting the liberal mindset flowing from France. With his lectures to the university community Bolzano was most successful in spreading awareness of the social components of Catholic ethics. But until 1811 he had been under orders to use a textbook written by a Viennese theologian. He objected and succeeded in the end, but, as a result, made influential personal enemies. When in 1819 a friend of his became involved with a local incident of student unrest, his enemies used the occasion to denounce him as a dangerous radical, and in 1820 he was dismissed from his chair and forbidden to teach. There ensued an internal Church review, aimed at withdrawing his missio canonica. It failed in 1825 as a result of the support of his bishops. After his dismissal, Bolzano eked out a living from occasional work as a tutor and as a substitute priest and from assistance of personal friends.

In 1792 Lazare Carnot had been one of the *conventionels régicides*; later he was Napoléon's last minister of the interior. In 1815 he fled France never to return, and in 1816 was struck from the list of members of the Académie. Cauchy, not yet

27 years old, was appointed his successor. At the Polytechnique, Cauchy held his chair until 1830. In July of that year the French liberals, mobilizing the Paris mob, forced king Charles X to abdicate, thus ending the house of Bourbon. They set up an Orléans, Louis-Philippe, son of the infamous Philippe Égalité, as their kinglet. Cauchy refused to swear allegiance to someone he considered a usurper and, as a result, lost his position. However, he remained a member of the Académie. He left France, taught in Torino and Prague, and returned to Paris only in 1838. In 1839 the ministry vetoed the vote of his colleagues who had elected him to a seat at the Collége de France. In the same year the members of the Bureau des Longitudes elected him to their body, but after he had served there for four years, the ministry had the election annulled. In 1848 the republic appointed Cauchy to a chair at the Faculté des Sciences; in June 1852 he had to give it up when an oath of allegiance was again required, this time by Louis Napoléon; in 1854 he was finally granted the chair by special privilege and *sans condition.*

6. Bestiarium infinitesimale. In the preceding sections Bolzano's formulations required no interpretation to fit into today's terminology. Cauchy, on the other hand, wrote about *quantitées infiniment petites,* a term no longer in use today. He did so, however, only after introducing limits in a terminology needing no interpretation either, and he defined the *quantitées infiniment petites* as a special case of limits. Thus, together with the term limit, the term *quantité infiniment petite* can also receive an interpretation in today's terminology that one may view as the standard interpretation. Thus Cauchy used the *quantitées infiniment petites* as a (rigorous) *façon de parler*, a device that made it possible to keep a formal connection with the past in which infinitesimals were conceived as numbers from a lower class of Archimedicity.

Abraham Robinson embedded the real numbers R in a non-Archimedean field S in which all L-sentences true in R are also true in S. L is a language with names for all real numbers, predicate symbols for all sets and relations of and between real numbers, and with function symbols for all real functions. This metamathematical connection between R and S can be expressed by a simpler-looking 'solution set' condition as found in J. Keisler's textbook.

It then becomes possible to extend the notions of 'standard' analysis from R to a 'nonstandard' analysis on S, where now the presence of infinitesimal numbers, i.e., numbers below all $1/n$ with natural n, permits us to phrase certain arguments in a new and rigorous way that Leibniz and his contemporaries had to leave vague.

In [12] Robinson quoted Cauchy's definitions on pp. 269–270 and then continued

> We gather from the above passages that infinitely small quantities are fundamental in Cauchy' approach to Analysis. However, these quantities are not numbers but variables, or rather, states of variables whose limit is zero. ...
>
> Whatever the precise picture of an infinitely small quantity may have been in Cauchy's mind, we may examine his subsequent definitions and see what they amount to if we interpret the infinitely small and infinitely large quantities mentioned in them in the sense of Non-standard Analysis. For the notion of continuity, Cauchy's definition may thus be interpreted as stating that for $f(x)$ defined in the interval $a < x < b$, $f(x)$ is continuous in that

interval if, for infinitesimal α, the difference $f(x+\alpha)-f(x)$ is always (toujours) infinitesimal. If now we interpret 'always' as meaning 'for all standard x' then we obtain ordinary continuity in the interval, but if by 'always' we mean 'for all x', then we obtain uniform continuity.

With the latter paragraph Robinson introduced what I choose to call the Robinson interpretation:

(a) where Cauchy writes about *quantitées infiniment petites*, assume, whenever possible, that they are infinitesimal numbers in the sense of nonstandard analysis, and

(b) read Cauchy's subsequent developments assuming (a) together with today's (or Weierstrass') standard knowledge of distinctions such as continuity versus uniform continuity, etc.

This interpretation resulted in beautiful mathematical insights. In particular, certain erroneous statements of Cauchy's (on series of continuous functions, on continuity in the case of several variables ...) could, in this way, be read as correct statements in nonstandard analysis. In this connection I mention John P. Cleave's informative articles **[13]** and **[14]**.

But it must be emphasized that Robinson's interpretation is in no way an explanation of the historical content of Cauchy's writings, and it is clear from Robinson's words above that it does not purport to be one. Cauchy wrote explicitly that his *quantitées infiniment petites* have *valeurs numériques*, they are real numbers:

Lorsque les valeurs numériques successives d'une même variable ....

When the successive numerical values of the same variable ....

Of course, the non-Archimedean, infinitesimal numbers of Pascal, say, were still in the back of Cauchy's and his contemporaries' minds. But for his *quantitées infiniment petites* Cauchy purposefully stipulated that they were assignments to variables the values of which were real numbers. There is no mention of infinitesimals anywhere in Cauchy's writings. Nonetheless, there appear from time to time articles whose authors disregard Cauchy's text and try to persuade themselves that the Robinson interpretation explains what Cauchy "really meant". Here I mention Gordon Fisher's paper **[15]**, where the possibility that the Robinson interpretation is historically correct is repeatedly asserted as based on the (mistaken [W. F.]) claim that Cauchy does *not* expressly exclude infinitesimal values for his *quantitées infiniment petites*:

"When the successive numerical values of the same variable decrease indefinitely, in such a way as to fall below any given number, this variable becomes what one calls an *infinitely small* (un infiniment petit), or an infinitely small quantity. A variable of this kind has zero for limit." The first sentence *could* mean that when the numerical (i.e., absolute) values of a variable decrease in such a way as to be less than any positive number, then the variable takes on infinitesimal values (bottom of p. 316).

> We have already observed that Cauchy's definition of a variable does not exclude infinitesimal values (para. 3, p. 318).
>
> ... might be taken as ... in which α is not permitted to take on non-zero infinitesimal values. But there is no necessity to make such restriction, nor does Cauchy say that this would be desirable or even possible (para. 2, p. 319).

In this connection one must also mention certain articles and books by D. Laugwitz, in which, while avoiding nonstandard analysis as a detracting invention of logicians, he develops his own 'mathematics of the infinitesimal' and uses it to interpret skillfully various aspects of the mathematics of the period from Euler to Cauchy.

And so we have glanced at the bestiarium infinitesimale.

7. The workings of finitarization. Mathematical analysis is marvellous.

It is not surprising that man can prove insights about triangles or about numbers, for example the decomposition into prime factors and its uniqueness. But analysis deals with infinite processes—or at least what we imagine to be such—and yet we manage to prove our statements by writing only a finite number of lines.

We speak of a function or a variable approaching some value *indéfiniment* (indefinitely); we imagine a limiting process. Bolzano's and Cauchy's analysis of the notions of limit and of continuity started from ϵ as a measure of approximation and proceeded to δ on which ϵ depends. Expressed in terms of ϵ and δ, the conception of a limiting process appears to have been described in a finitary manner—at least as far as the mathematical use is concerned.

Thus far, ϵ and δ (and in case of sequences also n and N) appear as handles affixed to the stages of those infinite processes. It seems that if appropriately handled in our mental exercises, they enable us to use finitely many arguments to prove statements that, in the end, speak about all the stages of the infinite process. Of course, mental exercises are exercises in logic. Let me illustrate their use in a simple proof from analysis.

I want to show that the function $f(x) = a^x$ with $a > 1$ for positive rational x has the B-limit 0 at 0. I abbreviate implication and conjunction by $\Rightarrow$ and $\wedge$ and the quantifiers "for all x:" and "there exists x:" by $\forall x$ and $\exists x$. I use the Fregean quantifier rules

$$\begin{array}{llrll}
(RE) & \text{From} & B \Rightarrow C(t) & \text{conclude} & B \Rightarrow \exists x C(x) & \\
(LE) & \text{From} & B(y) \Rightarrow C & \text{conclude} & \exists x B(x) \Rightarrow C & (*) \\
(RA) & \text{From} & C \Rightarrow B(y) & \text{conclude} & C \Rightarrow \forall x B(x) & (*) \\
(LA) & \text{From} & B(t) \Rightarrow C & \text{conclude} & \forall x B(x) \Rightarrow C &
\end{array}$$

with the stipulations in (LE) and in (RA) that

$(*)$ the letter y is a free variable not occurring in C.

Actually, I make no use of (LA) here. The first five steps of the proof are familiar computations with inequalities. From the definition of f there follows

(0) $\qquad x < y \Rightarrow f(x) < f(y)$

Now $h = f(1/n) - 1$ is positive, and Bernoulli's inequality states

(1) $\quad h > 0 \Rightarrow 1 + nh < (1+h)^n$, hence

(2) $\quad 1 + n \cdot (f(1/n) - 1) < a$, hence

(3) $\quad f(1/n) < (a-1)/n$, hence

(4) $\quad 1/n < \epsilon/(a-1) \Rightarrow f(1/n) < \epsilon$.

It follows from the principle of mathematical induction, used to derive Bernoulli's inequality (1), that there the letter n is a free variable; hence also in (4) both ϵ and n are free. My first aim is to prove

(5) $\quad \exists n\,(1/n < \epsilon/(a-1)) \Rightarrow \exists n(1/m < 1/n \Rightarrow f(1/m) < \epsilon)$.

This follows with the rule (LE) from

$$1/n < \epsilon/(a-1) \Rightarrow \exists n(1/m < 1/n \Rightarrow f(1/m) < \epsilon),$$

where n on the left is free, which follows with the rule (RE) from

$$1/n < \epsilon/(a-1) \Rightarrow \quad (1/m < 1/n \Rightarrow f(1/m) < \epsilon).$$

This follows from (0) and (4), and usually we do not spell out such arguments. However, to show that no quantifiers are used for this I indicate the (propositional) details:

$a.$	$1/m < 1/n \Rightarrow f(1/m) < f(1/n)$	from (0)
$b.$	$f(1/n) < \epsilon \ \wedge f(1/m) < f(1/n) \Rightarrow f(1/m) < \epsilon$	transitivity
$c.$	$(1/n) < \epsilon \ \wedge 1/m < 1/n \Rightarrow f(1/m) < \epsilon$	from b and a
$d.$	$1/n < \epsilon/(a-1) \wedge 1/m < 1/n \Rightarrow f(1/m) < \epsilon$	from c and (4)
	$1/n < \epsilon/(a-1) \Rightarrow (1/m < 1/n \Rightarrow f(1/m) < \epsilon)$	from d.

Thus (5) has been proved. By Archimedicity the premise of (5) is provable:

(6) $\quad \exists n\, 1/n < \epsilon/(a-1)$, hence

(7) $\quad \exists n\,(1/m < 1/n \Rightarrow f(1/m) < \epsilon)$.

My second aim now is to prove

(8) $\quad \exists n\,(1/m < 1/n \Rightarrow f(1/m) < \epsilon) \Rightarrow \exists\delta\ \forall y(y < \delta \Rightarrow f(y) < \epsilon)$,

which follows with the rule (LE) from

$$(1/m < 1/n \Rightarrow f(1/m) < \epsilon) \Rightarrow \exists\delta\ \forall y(y < \delta \Rightarrow f(y) < \epsilon),$$

where n on the left is free, which follows with the rule (RE) from

$$(1/m < 1/n \Rightarrow f(1/m) < \epsilon) \Rightarrow \forall y(y < 1/(n+1) \Rightarrow f(y) < \epsilon),$$

which follows with the rule (RA) from

$$(1/m < 1/n \Rightarrow f(1/m) < \epsilon) \Rightarrow (y < 1/(n+1) \Rightarrow f(y) < \epsilon),$$

where n and y are free. Again, we usually do not spell this out; but to show that no quantifiers are used, I indicate the (propositional) details:

$a.\ y < 1/(n+1) \Rightarrow f(y) < f(1/(n+1))$ from (0)

$b.\ (1/m < 1/n \Rightarrow f(1/m) < \epsilon) \Rightarrow f(1/(n+1)) < \epsilon$

$c.\ (1/m < 1/n \Rightarrow f(1/m) < \epsilon) \wedge y < 1/(n+1) \Rightarrow$
$f(y) < f(1/(n+1)) \wedge f(1/(n+1)) < \epsilon$ from a and b

$d.\ f(y) < f(1/n+1) \wedge f(1/n+1) < \epsilon \Rightarrow f(y) < \epsilon$ transitivity

$e.\ (1/m < 1/n \Rightarrow f(1/m) < \epsilon) \wedge y < 1/(n+1) \Rightarrow f(y) < \epsilon$ from c and d

$(1/m < 1/n \Rightarrow f(1/m) < \epsilon) \Rightarrow (y < 1/(n+1) \Rightarrow f(y) < \epsilon)$ from ϵ.

Thus (8) has been proved. By (7) the premise of (8) is provable, hence

$$(9) \qquad \exists\delta\ \forall y(y < \delta \Rightarrow f(y) < \epsilon).$$

As ϵ here is free, the rule (RA) gives

$$(10) \qquad \forall\epsilon\ \exists\delta\ \forall y(y < \delta \Rightarrow f(y) < \epsilon).$$

So f has the limit 0 at 0. Setting $f(0) = 0$, there now follows, as usual, the continuity of f for non-negative rational arguments.

Statement (10) is typical of statements considered in analysis. Its proof consisted first of transformations of inequalities, with the free variables y, n, ϵ; the quantifier rules were applied only towards the end. The formulas formed from combinations of inequalities without quantifiers were provable for the infinitely many numbers because they were provable for free variables (a particular feature of Bernoulli's inequality is that the induction rule can be applied in a form yielding n as a free variable). Thus the first step to capture infinity was the use of free variables.

The second step was the use of the rules (LE) and (RA) with their stipulation $(*)$ on free variables. Trying to secure

$$C \Rightarrow \forall x\, B(x), \tag{a1}$$

a reader motivated by the semantical interpretation of "for all x" might try to do so by securing for every number constant $1, 2, \ldots, r, \ldots$

$$C \Rightarrow B(1), C \Rightarrow B(2), \ldots, C \Rightarrow B(r), \ldots \tag{a2}$$

which could become an endless attempt. Instead, securing as premise of (LR)

$$C \Rightarrow B(x), \tag{a3}$$

where x is a free variable that does not occur in C, the proof of $C \Rightarrow B(x)$ contains the variable x as a parameter that can be replaced by every number constant r, and thus for each formula in (a2) it produces a proof as its copy with r replacing x. For this reason, a proof of (a3) may be viewed as a *uniform* proof for the infinitely many formulas (a2).

The same situation prevails for the rule (LE), where

$$\exists x\, B(x) \Rightarrow C \tag{b1}$$

is not secured by establishing one of the infinitely many formulas

$$B(1) \Rightarrow C, B(2) \Rightarrow C, \ldots, B(r) \Rightarrow C, \ldots, \tag{b2}$$

but by establishing as premise of (LE)

$$B(x) \Rightarrow C, \tag{b3}$$

where x is a free variable that does not occur in C.

Again, a proof of (b3) may be viewed as a *uniform* proof for one of the infinitely many formulas (b2).

The reader will note that (LE) also describes the argument used in order to prove a statement C from the hypothesis that a set B is not empty: choose an arbitrary element x in B and derive C from the hypothesis $B(x)$ that x is in B. Conscientious mathematicians, such as Erich Kamke and Peter Hilton, when analyzing their argumentations, believed that in this situation they needed a special axiom to *choose* the arbitrary x in the nonempty set B.

So ϵ and δ were not the principal tools which enabled us to write our proof in finitely many lines. Rather, these tools were the quantifier rules (RA) and (LE) with their use of free variable premisses under the stipulation $(*)$. The ϵ and δ were only those particular free variables themselves, of course strategically chosen in order to prepare the quantifierless premisses to which the quantifier rules were then applied.

The quantifier rules (LE) and (RA) with $(*)$ were discovered in 1879 by Gottlob Frege (1848–1925) who in this way laid the foundations of mathematical logic.

Of course, the contemporary mathematician, who thinks of mathematical objects as accumulations of set formations, may not like the fact that the only way to prove "for every $x : B(x)$" is to consider nonconceptual, real things, namely to prove first $B(y)$ with a *letter* y in the role of a free variable. But when he inspects the proofs he actually carries out, he will notice that this is precisely what he *always* does, rather than carry out an infinity of individual proofs of instances $B(r)$ for each of infinitely many numbers r.

And so the quantifier rules belong to the few places in the mathematical teddy bear's fur where the stitches become visible. They are the seams that hide the linguistic sawdust and splinters, the letters serving as variables, say, which make up its body. These linguistic tools, the recourse to the plain facts of language, enable us to finitarize our handling of what we conceive of as infinite processes.

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