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Discontinuous differential games and control systems with supremum cost

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Abstract

This paper concerns with the existence of a value for a zero sum two-player differential game with supremum cost of the form $C_{t_0,x_0}(u,v) = \sup_{\tau\in[t_0,T]} h(x(\tau;t_0,x_{0,}u,v))$ under Isaacs' condition. We characterize the value function as the unique solution—in a suitable sense—to a PDE, namely the Hamilton–Jacobi–Isaacs equation. As a byproduct, we obtain a PDE characterization of the value function for control system. © 2002 Elsevier Science (USA). All rights reserved.

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1. Introduction

We consider a differential game which dynamics is given by

$$x'(t) = f\big(x(t), u(t), v(t)\big), \tag{E}$$

where the state variable x belongs to $\mathbb{R}^N$ and the controls $u(\cdot):[0,T]\to U$ and $v(\cdot):[0,T]\to V$ are measurable functions.

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With any solution $x(\cdot\,; t_0, x_0, u(\cdot), v(\cdot))$ to (E) starting from $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$, we associate the following payoff:

$$C_{t_0,x_0}\big(u(\cdot), v(\cdot)\big) = \sup_{\tau\in[t_0,T]} h\big(x\big(\tau; t_0, x_0, u(\cdot), v(\cdot)\big)\big),$$

where $h : \mathbb{R}^N \to \mathbb{R}_+$ is a given bounded function.

We will study the differential game, namely the first player acting on the control u tries to minimize the cost while the other player acting on the control v tries to maximize it. These optimal behaviors of the two players are modelled in the framework of Varayia, Elliot, Kalton, and Roxin.

Here strategies (cf. [1]) that we recall now are Elliot–Kalton–Varayia strategies.

We denote by

$$\mathcal{U}(t_0) = \big\{u(\cdot) : [t_0, T] \to U,\ \text{measurable function}\big\},$$
$$\mathcal{V}(t_0) = \big\{v(\cdot) : [t_0, T] \to V,\ \text{measurable function}\big\}$$

the sets of time measurable controls, with values in U and V, two compact metric spaces. The nonanticipative strategies are defined in the following way:

Definition 1. We say that a map $\alpha : \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ is a nonanticipative strategy (for the first player) if it satisfies the following condition: For any $t \geqslant t_0$ and for any v_1 and v_2 belonging to $\mathcal{V}(t_0)$ such that v_1 and v_2 coincide almost everywhere on $[t_0, t]$, the images $\alpha(v_1)$ and $\alpha(v_2)$ coincide almost everywhere on $[t_0, t]$.

Nonanticipative strategies $\beta : \mathcal{U}(t_0) \to \mathcal{V}(t_0)$ (for the second player) are defined in the same way. Namely, for any $t \geqslant t_0$ and for any u_1 and u_2 belonging to $\mathcal{U}(t_0)$, such that u_1 and u_2 coincide almost everywhere on $[t_0, t]$, the images $\beta(u_1)$ and $\beta(u_2)$ coincide almost everywhere on $[t_0, t]$.

From now on $\Gamma(t_0)$ (respectively, $\Delta(t_0)$) stands for the set of all nonanticipative strategies of the first (respectively, the second) player, starting at time $t_0 \in [0, T]$.

With the previous notations we define:

- the lower value

$$V^-(t_0, x_0) = \inf_{\alpha\in\Gamma(t_0)} \sup_{v\in\mathcal{V}(t_0)} C_{t_0,x_0}\big(\alpha(v), v\big),$$

- the upper value

$$V^+(t_0, x_0) = \sup_{\beta\in\Delta(t_0)} \inf_{u\in\mathcal{U}(t_0)} C_{t_0,x_0}\big(u, \beta(u)\big)$$

for all $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$.

The main question we address here is the existence of a value of the game, namely the relation $V^+ = V^-$. The question is crucial in differential games (see [1–7,25–28]). This is a way to model the fact that the players act simultaneously and to show that V^+ and V^- solve certain nonlinear PDE in the viscosity sense introduced below. Some results for the well-known case of a Lipschitz function h are in [2]. Namely, under Isaacs condition

$$\max_{v\in V}\min_{u\in U}\big(p\cdot f(x,u,v)\big) = \min_{u\in U}\max_{v\in V}\big(p\cdot f(x,u,v)\big) \quad \text{for all } x\in\mathbb{R}^N \tag{CI}$$

the value exists and it is the unique Lipschitz viscosity solution for the following equation:

$$\begin{cases} \max\big[\big(\frac{\partial V}{\partial t}(t,x) + H\big(x, V(t,x), \frac{\partial V}{\partial x}(t,x)\big)\big); (h(x) - V(t,x))\big] = 0, \\ (t,x)\in[0,T)\times\mathbb{R}^N, \\ \text{with final condition } V(T,x) = h(x),\ x\in\mathbb{R}^N, \end{cases} \tag{HJI}$$

where the Hamiltonian $H:\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\to\overline{\mathbb{R}}$ is given by

$$H(x,r,p) = \begin{cases} \max_{v\in V}\min_{u\in U}(p\cdot f(x,u,v)), & h(x)\leqslant r, \\ +\infty, & h(x) > r. \end{cases}$$

Remark 2. By a change of variables the problem with the following cost:

$$C_{t_0,x_0}\big(u(\cdot),v(\cdot)\big) = \sup_{\tau\in[t_0,T]}\left\{ h\big(x(\tau;t_0,x_0,u,v)\big) + \int_{t_0}^{\tau} l\big(x(s;t_0,x_0,u,v)\big)\,ds \right\}$$

may be reduced to above problem.

Our second main interest is the supremum control problem; i.e., we investigate the following value function:

$$V(t_0,x_0) = \inf_{u\in\mathcal{U}(t_0)}\ \sup_{\tau\in[t_0,T]} h\big(x\big(\tau;t_0,x_0,u(\cdot)\big)\big),$$

where the dynamics is given by

$$\begin{cases} x'(t) = f(x(t),u(t)) \text{ for almost all } t\in[t_0,T], \\ x(t_0) = x_0, \end{cases} \tag{PC$_1$}$$

first considered in [8] for a continuous function h, when h is semicontinuous or only bounded.

Before explaining how we solve the discontinuous supremum problem for games, let us recall the approach of Evans and Souganidis [5] in the case of a Lipschitz continuous terminal cost. They prove that the value functions are the unique continuous viscosity solutions of two Hamilton–Jacobi–Isaacs equations.

Under the assumption that the two Hamiltonians coincide—the so-called Isaacs condition—these two value functions are hereby equal because of uniqueness.

In the context of discontinuous h the value function may be discontinuous and standard uniqueness results for viscosity solutions of PDE cannot be used. Semicontinuous solutions and semicontinuous value function have been introduced by Barron and Jensen [9] and Frankowska [10].

Here our approach consists in:

- proving the existence of a value of the upper discontinuous game, using a suitable definition for the value functions;
- characterizing the value as unique solution—in a suitable sense—of a PDE;[1]
- obtaining the results for the case of discontinuous value function for control systems.

We note that this problem was not studied for differential games (respectively, control) with discontinuous h (see [2,8,11] for the continuous case) and we will discuss separately the upper and lower case, because they are not analogous (like in [7] for Mayer problem).

This paper is organized as follows: in Section 2 we introduce some preliminaries, in Section 3 we present our main results concerning the discontinuous game problem, and Section 4 presents results concerning the discontinuous supremum control problem. Finally, in Appendix A, we summarize without proof the Lipschitz continuous case and we give a proof of a technical lemma.

2. Preliminaries

2.1. Definitions, assumptions and notations

2.1.1. Hypotheses

We assume that

$$f:\mathbb{R}^N\times U\times V\to\mathbb{R}^N \text{ is continuous and satisfies}$$
$$\begin{cases}\|f(x,u,v)\|\leqslant a(1+\|x\|),\\ \|f(x,u,v)-f(y,u,v)\|\leqslant c_1\|x-y\|,\end{cases}$$
$$\forall x,y\in\mathbb{R}^N,\ u\in U,\ v\in V, \tag{H$_f$}$$

[1] The proposed definitions and our methods are inspired by [7] and is strongly related with Frankowska and Barron–Jensen semicontinuous solutions and Subbotin minimax solutions.

where $c_1, a > 0$ are constants; U, V are compact metric spaces;

$$\begin{cases} \forall x \in \mathbb{R}^N,\ \forall u \in U,\ \forall v \in V, \\ f(x,u,V) = \{f(x,u,v), v \in V\} \text{ and} \\ f(x,U,v) = \{f(x,u,v), u \in U\} \text{ are convex.} \end{cases} \tag{H_w}$$

It is well known that under (H_f), for every $(t_0, x_0) \in [0,T] \times \mathbb{R}^N$, the Cauchy problem (PC)

$$\begin{cases} x'(t) = f(x(t), u(t), v(t)) \text{ for almost every } t \in [t_0, T], \\ x(t_0) = x_0 \end{cases} \tag{PC}$$

has an unique absolutely continuous solution denoted by $x(\cdot\,; t_0, x_0, u(\cdot), v(\cdot)) \in W^{1,1}[0,T]$ (see [22]).

2.1.2. *Viscosity solutions*

To describe the upper (lower) value function as a unique solution to a corresponding HJI (Hamilton–Jacobi–Isaacs equation) we introduce the following concept of solutions related to Subbotin solutions [12].

Definition 3. Let $H : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \overline{\mathbb{R}}$, be an Hamiltonian. A function $(t,x) \to V(t,x)$ is a generalized solution to HJI if and only if for all $(t,x) \in [0,T] \times \mathbb{R}^N$

$$V(t,x) = \sup \left\{ \begin{array}{l} \varphi(t,x) \mid \varphi \text{ upper semicontinuous subsolution for HJI,} \\ \varphi(t,x) \geqslant h(x) \text{ for all } (t,x) \in [0,T] \times \mathbb{R}^N, \\ \varphi(T,x) \leqslant h(x) \text{ for all } x \in \mathbb{R}^N \end{array} \right\}$$

and

$$V(t,x) = \inf \left\{ \begin{array}{l} \psi(t,x) \mid \psi \text{ lower semicontinuous supersolution for HJI,} \\ \psi(t,x) \geqslant h(x) \text{ for all } (t,x) \in [0,T] \times \mathbb{R}^N, \\ \varphi(T,x) \geqslant h(x) \text{ for all } x \in \mathbb{R}^N \end{array} \right\}.$$

Here we recall (see [24]) that a *viscosity supersolution* for HJI is a lower semicontinuous function $\psi : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ such that

$$\begin{aligned} &\text{for any } \phi \in C^1 \text{ and } (t_0, x_0) \in \arg\operatorname{Min}(\psi - \phi), \\ &\max\left\{ \left(\frac{\partial \phi}{\partial t}(t_0, x_0) + H\left(x_0, \phi(t_0, x_0), \frac{\partial \phi}{\partial x}(t_0, x_0) \right) \right); \right. \\ &\qquad \left. \big(h(x_0) - \phi(t_0, x_0) \big) \right\} \leqslant 0, \end{aligned}$$

and a *viscosity subsolution* for HJI is an upper semicontinuous function $\varphi : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ such that

$$\text{for any } \phi \in C^1 \text{ and } (t_0, x_0) \in \arg\operatorname{Max}(\varphi - \phi),$$
$$\max\left\{\left(\frac{\partial\phi}{\partial t}(t_0, x_0) + H\left(x_0, \phi(t_0, x_0), \frac{\partial\phi}{\partial x}(t_0, x_0)\right)\right);\right.$$
$$\left.\big(h(x_0) - \phi(t_0, x_0)\big)\right\} \geqslant 0.$$

A *viscosity solution* of HJI is a function which is in the same time subsolution and supersolution so it is in particular continuous.

The definition of super- and subsolutions to HJI can be written equivalently in terms of subdifferentials (see [7]):

A *viscosity supersolution* for HJI is a l.s.c. function $\psi : (0, T) \times K \to \mathbb{R}$ such that

$$\text{for any } (t_0, x_0) \in (0, T) \times K \text{ and } (p_t, p_x) \in \partial_-\psi(t_0, x_0),$$
$$\max\Big\{\big(p_t + H\big(x_0, \phi(t_0, x_0), p_x\big)\big);\ \big(h(x_0) - \psi(t_0, x_0)\big)\Big\} \leqslant 0.$$

A *viscosity subsolution* for HJI is an u.s.c. function $\varphi : (0, T) \times K \to \mathbb{R}$ such that

$$\text{for any } (t_0, x_0) \in (0, T) \times K \text{ and } (p_t, p_x) \in \partial_+\varphi(t_0, x_0),$$
$$\max\Big\{\big(p_t + H\big(x_0, \phi(t_0, x_0), p_x\big)\big);\ \big(h(x_0) - \varphi(t_0, x_0)\big)\Big\} \geqslant 0.$$

Here for an extended l.s.c. function $\psi : \mathbb{R}^N \to \overline{\mathbb{R}}$ with $\psi(x_0) \neq \pm\infty$, the subdifferential of ψ at x_0 is given by

$$\partial_-\psi(x_0) = \left\{p \in \mathbb{R}^N \mid \liminf_{x \to x_0} \frac{\psi(x) - \psi(x_0) - \langle p, x - x_0\rangle}{\|x - x_0\|} \geqslant 0\right\},$$

and for an extended u.s.c. function $\varphi : \mathbb{R}^N \to \overline{\mathbb{R}}$ with $\varphi(x_0) \neq \pm\infty$ the superdifferential of φ at x_0 is given by

$$\partial_+\varphi(x_0) = \left\{p \in \mathbb{R}^N \mid \limsup_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - \langle p, x - x_0\rangle}{\|x - x_0\|} \leqslant 0\right\}.$$

The notion of generalized solution (cf. [13]) of HJI in the viscosity sense and our proofs are inspired by the comparison principle from [7] and by Barles stability result (see Theorem 2.3 of [14]) for viscosity super- and subsolutions. For the different definition of discontinuous viscosity solutions see Ishii solutions (exposed in [14]) based on semicontinuous envelopes of functions, Barron and Jensen semicontinuous solutions [9,14] for convex Hamiltonians, and envelope solutions [15] which are related to Subbotin minimax solution [12] (called bilateral solutions in [15]).

3. Discontinuous supremum problem for differential games

3.1. The upper semicontinuous case

We are interested to study our game under the assumption

$$\begin{cases} |h(x)| \leqslant c_2 \text{ for all } x \in \mathbb{R}^N, \\ h \text{ is upper semicontinuous in } \mathbb{R}^N. \end{cases} \tag{$\mathrm{H_{upp}}$}$$

If h is discontinuous then so are the value functions. This section is devoted to the characterization of the value functions by Hamilton–Jacobi–Isaacs equation (HJI). The main tool is viability theory (see [14,16]).

The discontinuous case has not been studied for differential games. The L^∞ problem with a Lipschitz continuous function h was analyzed in [2,11]. For the convenience of the reader we repeat, in Appendix A, the relevant material from [2] without proofs, thus making our exposition self-contained.

With the proposed cost we will also cover the discontinuous and only bounded case for supremum control problem (see Section 4).

The choice of h upper semicontinuous seems to be the best adapted to our problem because we will be able to establish that our game has a value and it is the generalized solution to HJI—in the sense of Definition 3 adapted from [13].

For a deeper discussion on the notions of generalized solutions of PDE's we refer the reader to [10,11,14]. The key point is an observation that the properties of upper semicontinuous functions and the solutions set for (E) can be used to "modify" the definition of value functions.

Note that in the lower semicontinuous case we can touch the lower value function from only one side (see (3) of Proposition 17).

3.1.1. First comparison result

We will look more closely at the definition of value function. Here the advantage lies in the fact that h is upper semicontinuous and the set $\{x(\tau; t_0, x_{0,} u, \beta(u))$, $\tau \in [t_0, T]\}$ is compact for all $u \in \mathcal{U}(t_0)$, $\beta \in \Delta(t_0)$.

We will denote by

$$B_{\beta,h}(t_0, x_0) = \left\{ \begin{matrix} x = x(\tau_u; t_0, x_{0,} u, \beta(u)) \mid \\ h(x) = \sup_{[t_0,T]} h(x(\tau; t_0, x_{0,} u, \beta(u))),\ u \in \mathcal{U}(t_0) \end{matrix} \right\},$$
$$A_\alpha(t_0, x_0) = \left\{ x = x\big(\tau; t_0, x_0, \alpha(v), v\big) \mid v \in \mathcal{V}(t_0),\ \tau \in [t_0, T] \right\}$$

for all $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$, $\beta \in \Delta(t_0)$, $\alpha \in \Gamma(t_0)$. According to the above definition we have[2]

[2] This observation allows to replace the sequence "sup inf sup" in the definition of V^+, which is the difficulty of our problem.

$$V^-(t_0,x_0)=\inf_{\alpha\in\Gamma(t_0)}\sup\bigl\{h(x),\ x\in A_\alpha(t_0,x_0)\bigr\},$$
$$V^+(t_0,x_0)=\sup_{\beta\in\Delta(t_0)}\inf\bigl\{h(x),\ x\in B_{\beta,h}(t_0,x_0)\bigr\}$$

for all $(t_0,x_0)\in[0,T]\times\mathbb{R}^N$.

With the general definition we can now state one inequality, namely:

Proposition 4. *If* (H_f) *and* $(\mathrm{H}_{\mathrm{upp}})$ *hold true then*

$$V^+(t,x)\geqslant V^-(t,x)\quad \textit{for all } (t,x)\in[0,T]\times\mathbb{R}^N.$$

Proof. We will use a sup-convolution method. We define a sequence h_n:

$$h_n(x)=\sup_{y\in\mathbb{R}^N}\bigl(h(y)-n\|y-x\|\bigr),\quad n\in\mathbb{N}. \tag{8}$$

The functions h_n are Lipschitz, $h_n(x)\geqslant h_{n+1}(x)$ and $\lim_{n\to\infty}h_n(x)=h(x)$ for every $x\in\mathbb{R}^N$. Using the results for the Lipschitz continuous case we obtain that $V^+_{h_n}=V^-_{h_n}$ is a decreasing sequence of viscosity solutions for HJI which limit is denoted by $U:=\lim_{n\to\infty}V^\pm_{h_n}$.

On one hand $h_n\geqslant h$ yields $U\geqslant V^\pm_h$. On the other hand, U is a decreasing limit of subsolutions. In view of Theorem 4.1 in [14], U is a upper semicontinuous subsolution to HJI.

We will use in an adapted version the following result (proved in Appendix A) of [7]:

Lemma 5. *Assume that* (H_f) *and* (H_w) *hold true. If* φ *is a upper semicontinuous subsolution for HJI*$^+$ *with* $\varphi(t,x)\geqslant h(x)$ *for all* $(t,x)\in[0,T]\times\mathbb{R}^N$ *then for every* $(t_0,x_0)\in(0,T)\times\mathbb{R}^N$ *there exists a nonanticipative strategy* $\beta\in\Delta(t_0)$*, such that for all* $u\in\mathcal{U}(t_0)$ *the solution* $x(\cdot):=x(\cdot\,;t_0,x_0,u,\beta(u))$ *satisfies*: *for every* $t\in[0,T]$,

if for all $s\in[0,t]$ *we have* $\varphi(s,x(s))>h(x(s))$,
then $\varphi(t,x(t))\geqslant\varphi(t_0,x_0)$.

This means that the function $t\to(t,x(t),\varphi(t_0,x_0))$ remains in the hypograph of φ until it (possibly) reaches the set $[0,T]\times\mathrm{Hypo}(h)$. For control this phenomenon is known as the viability property with target (introduced in [17]). It follows immediately:

Corollary 6. *Under the assumption of Lemma* 5 *and* $\varphi(T,x)\leqslant h(x)$ *for all* $x\in\mathbb{R}^N$:

$$\varphi(t,x)\leqslant V^+_h(t,x)\quad \textit{for all } (t,x)\in[0,T]\times\mathbb{R}^N.$$

Now because $U(t,x) \geqslant h(x)$ on $[0,T] \times \mathbb{R}^N$ the inequality $U \leqslant V_h^+$ follows from Corollary 6. So $V^- \leqslant V^+ = U$ and Proposition 4 is proved. □

Remark 7. The above result says that for an upper semicontinuous function h, V^+ is an upper semicontinuous, bounded function and a subsolution to the HJI but it does not furnish other informations of this type for V^-, so we are not in a position to claim the existence of a value. Note that the proof is based on the fact that h is upper semicontinuous, so for the reverse inequality, our next objective is to use the properties of the solutions set for (E).

The previous observation and methods of [7] leads to

Definition 8. For all $(t_0, x_0) \in [0,T] \times \mathbb{R}^N$, $\beta \in \Delta(t_0)$, $\alpha \in \Gamma(t_0)$ we will denote by

$$\tilde{V}^-(t_0,x_0) = \inf_{\alpha \in \Gamma(t_0)} \sup\bigl\{h(x),\ x \in \operatorname{cl} A_\alpha(t_0,x_0)\bigr\},$$

$$\tilde{V}^+(t_0,x_0) = \sup_{\beta \in \Delta(t_0)} \inf\bigl\{h(x),\ x \in \operatorname{cl} B_{\beta,h}(t_0,x_0)\bigr\},$$

where cl means closure.

Remark 9. Because h is upper semicontinuous, we can easily check that

$$\sup_{\beta \in \Delta(t_0)} \inf\bigl\{h(x),\ x \in B_{\beta,h}(t_0,x_0)\bigr\} = \sup_{\beta \in \Delta(t_0)} \inf\bigl\{h(x),\ x \in \operatorname{cl} B_{\beta,h}(t_0,x_0)\bigr\}.$$

So, for h upper semicontinuous we have $V^+ = \tilde{V}^+$.

Remark 10. Let us notice that when h is continuous we can skip the closure in Definition 8 and we claim (see [2]) that for all $(t,x) \in [0,T] \times \mathbb{R}^N$

$$\tilde{V}_h^-(t,x) = \tilde{V}_h^+(t,x) = V_h^-(t,x) = V_h^+(t,x).$$

We are interested to use the properties of $B_{\beta,h}(t_0,x_0)$ and $A_\alpha(t_0,x_0)$. The point of Definition 8 is that it allows us to obtain the following result very important in the proof of the existence of a value.

Proposition 11. *Assume that* (H_f) *and* $(\mathrm{H}_{\mathrm{upp}})$ *hold true. Then*

$$\tilde{V}_h^+(t_0,x_0) \leqslant \tilde{V}_h^-(t_0,x_0) \quad \textit{for all } (t_0,x_0) \in [0,T] \times \mathbb{R}^N.$$

The proof is a direct conclusion from Lemma 12.

Lemma 12. *If* (H_f) *and* $(\mathrm{H}_{\mathrm{upp}})$ *hold true then for all* $(t_0,x_0) \in [0,T] \times \mathbb{R}^N$, $\beta \in \Delta(t_0)$, $\alpha \in \Gamma(t_0)$ *we have*

$$\operatorname{cl} B_{\beta,h}(t_0,x_0) \cap \operatorname{cl} A_\alpha(t_0,x_0) \neq \emptyset.$$

Proof. Suppose to the contrary that there exist $\beta \in \Delta(t_0)$, $\alpha \in \Gamma(t_0)$ such that

$$\operatorname{cl} B_{\beta,h}(t_0, x_0) \cap \operatorname{cl} A_\alpha(t_0, x_0) = \emptyset.$$

Then there exists a Lipschitz continuous function $l : \mathbb{R}^N \to [m, M]$,

$$l(x) = \begin{cases} m, & x \in \operatorname{cl} A_\alpha(t_0, x_0), \\ M, & x \in \operatorname{cl} B_{\beta,h}(t_0, x_0), \end{cases}$$

with $l(x) < M$ if $x \notin \operatorname{cl} B_{\beta,h}(t_0, x_0)$.

By definition, $\tilde{V}_l^-(t_0, x_0) \leqslant m$. We claim that

$$\operatorname{cl} B_{\beta,h}(t_0, x_0) = \operatorname{cl} B_{\beta,l}(t_0, x_0).$$

Indeed, if $y \in B_{\beta,h}(t_0, x_0)$,

$$y = x\bigl(\tau; t_0, x_{0,} u, \beta(u)\bigr) \quad \text{and} \quad h(y) = \sup_{\tau \in [t_0, T]} h\bigl(x\bigl(\tau; t_0, x_{0,} u, \beta(u)\bigr)\bigr);$$

so

$$l(y) = M = l\bigl(x\bigl(\tau; t_0, x_{0,} u, \beta(u)\bigr)\bigr) \leqslant \sup_{\tau \in [t_0, T]} l\bigl(x\bigl(\tau; t_0, x_{0,} u, \beta(u)\bigr)\bigr) \leqslant M$$

and

$$l(y) = M = l\bigl(x\bigl(\tau; t_0, x_{0,} u, \beta(u)\bigr)\bigr) = \sup_{\tau \in [t_0, T]} l\bigl(x\bigl(\tau; t_0, x_{0,} u, \beta(u)\bigr)\bigr).$$

By definition of $B_{\beta,l}(t_0, x_0)$, we obtain $y \in B_{\beta,l}(t_0, x_0)$ and obviously

$$\operatorname{cl} B_{\beta,h}(t_0, x_0) \subset \operatorname{cl} B_{\beta,l}(t_0, x_0).$$

For the reverse inclusion, we consider $y \in B_{\beta,l}(t_0, x_0)$, where

$$y = x\bigl(\tau_l; t_0, x_{0,} u, \beta(u)\bigr) \quad \text{and} \quad l(y) = \sup_{\tau \in [t_0, T]} l\bigl(x\bigl(\tau; t_0, x_{0,} u, \beta(u)\bigr)\bigr).$$

Since h is upper semicontinuous and $\operatorname{Im} x(\tau; t_0, x_{0,} u, \beta(u))$ is compact, there exists $z = x(\tau_h; t_0, x_{0,} u, \beta(u))$ such that

$$h(z) = \sup_{\tau \in [t_0, T]} h\bigl(x\bigl(\tau; t_0, x_{0,} u, \beta(u)\bigr)\bigr) = h\bigl(x\bigl(\tau_h; t_0, x_{0,} u, \beta(u)\bigr)\bigr).$$

If $l(y) < M$ we have $\tau_l \neq \tau_h$ and

$$\begin{aligned} M > l\bigl(x\bigl(\tau_l; t_0, x_{0,} u, \beta(u)\bigr)\bigr) &= \sup_{\tau \in [t_0, T]} l\bigl(x\bigl(\tau; t_0, x_{0,} u, \beta(u)\bigr)\bigr) \\ &\geqslant l\bigl(x\bigl(\tau_h; t_0, x_{0,} u, \beta(u)\bigr)\bigr) = M, \end{aligned}$$

because $z = x(\tau_h; t_0, x_{0,} u, \beta(u)) \in B_{\beta,h}(t_0, x_0)$. This contradiction gives $l(y) = M$ and by definition of l, $y \in \operatorname{cl} B_{\beta,h}(t_0, x_0)$. Finally we have the reverse inclusion: $\operatorname{cl} B_{\beta,l}(t_0, x_0) \subset \operatorname{cl} B_{\beta,h}(t_0, x_0)$. According to Definition 8,

$$\begin{aligned}\tilde{V}_l^+(t_0,x_0) &= \sup_{\beta\in\Delta(t_0)} \inf\bigl\{l(x),\ x\in \operatorname{cl} B_{\beta,l}(t_0,x_0)\bigr\}\\ &\geqslant \inf\bigl\{l(x),\ x\in \operatorname{cl} B_{\beta,l}(t_0,x_0)\bigr\} = \inf\bigl\{l(x),\ x\in \operatorname{cl} B_{\beta,h}(t_0,x_0)\bigr\}\\ &= M;\end{aligned}$$

hence

$$\tilde{V}_l^-(t_0,x_0) \leqslant m < M \leqslant \tilde{V}_l^+(t_0,x_0).$$

This is a contradiction because Theorem 37 (see Appendix A) states that if l is Lipschitz continuous then the upper value is equal to the lower value. □

3.1.2. Main result for the upper semicontinuous case

The interest of this result is that it proves the existence of a value and it allows to characterize the value function as a solution of HJI.

Theorem 13. *Assume that* (H_f), (H_w), $(\mathrm{H}_{\mathrm{upp}})$ *and* (CI) *hold true. Then*:

(1) *the game has a value*; *i.e.,*

$$\tilde{V}^+(t_0,x_0) = \tilde{V}^-(t_0,x_0) \quad \text{for all } (t_0,x_0)\in[0,T]\times\mathbb{R}^N.$$

(2) *$\tilde{V}^+ = \tilde{V}^-$ is an upper semicontinuous, bounded function and a subsolution to the HJI with $\tilde{V}^\pm(T,x) = h(x)$ for every $x\in\mathbb{R}^N$.*

(3) *Moreover,*

$$\tilde{V}^\pm = \max\left\{\begin{array}{l}\varphi \mid \varphi \text{ upper semicontinuous subsolution for HJI},\\ \varphi(t,x)\geqslant h(x) \text{ for all } (t_0,x_0)\in[0,T]\times\mathbb{R}^N,\\ \varphi(T,x)\leqslant h(x) \text{ for all } x\in\mathbb{R}^N\end{array}\right\},$$

and for every $(t_0,x_0)\in[0,T]\times\mathbb{R}^N$

(4)
$$\tilde{V}^-(t_0,x_0) = \inf\left\{\begin{array}{c}\psi(t_0,x_0) \mid \psi \text{ lower semicontinuous}\\ \text{supersolution for HJI},\\ \psi(t,x)\geqslant h(x) \text{ for all } (t,x)\in[0,T]\times\mathbb{R}^N\end{array}\right\}.$$

Remark 14. According to (3) and (4) we conclude that $\tilde{V}^\pm$ is a generalized solution to HJI in the sense of Definition 3.

The proof of Theorem 13 will be divided into 3 lemmas.

Lemma 15. *If* $(\mathrm{H}_{\mathrm{upp}})$ *holds true then*

$$\tilde{V}^- \leqslant \tilde{V}^+ \quad \text{in } [0,T]\times\mathbb{R}^N.$$

Proof. In the same way, we will use a sup-convolution method. We define a sequence $h_n:\mathbb{R}^N\to\mathbb{R}$ by formula (1). From the results of the Lipschitz continuous case it follows that $V_{h_n}^+ = V_{h_n}^- = \tilde V_{h_n}^+ = \tilde V_{h_n}^-$ is a decreasing sequence of viscosity solutions for HJI which limit is denoted by $U := \lim_{n\to\infty} V_{h_n}^{\pm}$.

On one hand $h_n \geqslant h$ yields $U \geqslant \tilde V_h^{\pm}$. On the other hand, U is a decreasing limit of subsolutions. In view of Theorem 4.1 in [14], U is a upper semicontinuous subsolution to HJI.

Because $U(t,x) \geqslant h(x)$ and $U(T,x) = h(x)$, $U \leqslant \tilde V_h^+ = V_h^+$ follows from Corollary 6.

So we obtain $\tilde V^- \leqslant \tilde V^+ = U$.

Obviously, we have $\tilde V^- \geqslant \tilde V^+$ by Proposition 11, so (1)–(3) of Theorem 13 are true.

It remains to prove (4). Fix $(t_0,x_0) \in (0,T)\times\mathbb{R}^N$.

On one hand, by Lemma 18, for any lower semicontinuous supersolution for HJI, ψ, with $\psi(t,x) \geqslant h(x)$ for all $(t,x) \in [0,T]\times\mathbb{R}^N$, there exists a non-anticipative strategy $\alpha \in \Gamma(t_0)$, such that

$$\psi(t_0,x_0) \geqslant \psi\bigl(t, x\bigl(t; t_0, x_0, \alpha(v), v\bigr)\bigr)$$

for every $v \in \mathcal{V}(t_0)$ and for all $t \in [t_0,T]$. We claim that $\psi(t_0,x_0) \geqslant h(y)$ for all $y \in \operatorname{cl} A_\alpha(t_0,x_0)$.

Indeed, if $y = \lim x(t_n; t_0, x_0, \alpha(v_n), v_n)$ then

$$\psi(t_0,x_0) \geqslant \psi\bigl(t_n, x\bigl(t_n; t_0, x_0, \alpha(v_n), v_n\bigr)\bigr) \quad \text{for every } n \in \mathbb{N}.$$

Hence ψ is lower semicontinuous,

$$\begin{aligned}\psi(t_0,x_0) &\geqslant \liminf \psi\bigl(t_n, x\bigl(t_n; t_0, x_0, \alpha(v_n), v_n\bigr)\bigr)\\ &\geqslant \psi\bigl(\liminf\bigl(t_n, x\bigl(t_n; t_0, x_0, \alpha(v_n), v_n\bigr)\bigr)\bigr)\\ &\geqslant \psi(\liminf t_n, y) \geqslant h(y), \quad \forall y \in \operatorname{cl} A_\alpha(t_0,x_0);\end{aligned}$$

so

$$\begin{aligned}&\psi(t_0,x_0) \geqslant h(y), \quad \forall y \in \operatorname{cl} A_\alpha(t_0,x_0),\\ &\psi(t_0,x_0) \geqslant \sup\bigl\{h(x), x \in \operatorname{cl} A_\alpha(t_0,x_0)\bigr\},\end{aligned}$$

and by the very definition of the value function

$$\psi(t_0,x_0) \geqslant \tilde V_h^-(t_0,x_0) \quad \text{for every } (t_0,x_0) \in [0,T]\times\mathbb{R}^N.$$

On the other hand, let us recall that

$$\tilde V_h^-(t_0,x_0) = \inf_{\alpha\in\Gamma(t_0)} \sup\bigl\{h(x),\ x \in \operatorname{cl} A_\alpha(t_0,x_0)\bigr\}.$$

If $\varepsilon > 0$ and $M > \max\{c_2,\ V_h^-(t_0,x_0) + \varepsilon/2\}$, then there exist $\alpha_\varepsilon \in \Gamma(t_0)$ with

$$M > \tilde V_h^-(t_0,x_0) + \frac{\varepsilon}{2} > \sup\bigl\{h(x),\ x \in \operatorname{cl} A_{\alpha_\varepsilon}(t_0,x_0)\bigr\}.$$

We define

$$l_\varepsilon(x) = \begin{cases} \tilde{V}_h^-(t_0, x_0) + \varepsilon/2, & x \in \operatorname{cl} A_{\alpha_\varepsilon}(t_0, x_0), \\ M, & x \notin \operatorname{cl} A_{\alpha_\varepsilon}(t_0, x_0). \end{cases}$$

Obviously l_ε is lower semicontinuous and $l_\varepsilon \geqslant h$. By Proposition 17, $V_{l_\varepsilon}^- \geqslant h$ is a lower semicontinuous supersolution to HJI.

Using the definition, $V_{l_\varepsilon}^-(t_0, x_0) \leqslant \tilde{V}_h^-(t_0, x_0) + \varepsilon/2 < \tilde{V}_h^-(t_0, x_0) + \varepsilon$. So (4) may be concluded. □

Remark 16. For h continuous we have that the L^∞ problem is the same with supremum problem (see [2]) and the value is the max of subsolutions and the min of supersolutions, so it is a continuous function.

3.2. The lower semicontinuous case

Consider $h : \mathbb{R}^N \to \mathbb{R}_+$ with

$$\begin{cases} |h(x)| \leqslant c_2 \text{ for all } x \in \mathbb{R}^N, \\ h \text{ is lower semicontinuous in } \mathbb{R}^N. \end{cases} \tag{H$_{\text{low}}$}$$

It would be desirable to establish the existence of a value but we have not been able to do this because for h lower semicontinuous the method of the case h upper semicontinuous breaks down.

The point of this section is to prove the following

Proposition 17. *Assume that* (H$_f$), (H$_{\text{low}}$), (CI) *hold true. Then*:

(1) $V^+(t_0, x_0) \geqslant V^-(t_0, x_0)$ *for all* $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$.
(2) V^- *is a lower semicontinuous, bounded function and a supersolution to the HJI with* $V^-(T, x) = h(x)$ *for all* $x \in \mathbb{R}^N$.
(3) *Moreover,*

$$V^- = \min \left\{ \begin{array}{l} \varphi \mid \varphi \text{ lower semicontinuous supersolution for HJI}, \\ \varphi(t, x) \geqslant h(x) \text{ for all } (t, x) \in [0, T] \times \mathbb{R}^N \end{array} \right\}.$$

The proof of Proposition 17 is divided in several lemmas.

Let us recall in a adapted version the following result, which plays an important role in the proof of Lemma 20:

Lemma 18 (cf. Proposition 6 in [7]). *Assume that* (H$_f$) *and* (H$_w$) *hold true. If* ψ *is a lower semicontinuous supersolution for HJI*$^-$ *with* $\psi(t, x) \geqslant h(x)$ *for all* $(t, x) \in [0, T] \times \mathbb{R}^N$, *then for every* $(t_0, x_0) \in (0, T) \times \mathbb{R}^N$ *there exists a nonanticipative strategy* $\alpha \in \Gamma(t_0)$, *such that*

$$\psi(t_0, x_0) \geqslant \psi\big(t, x\big(t; t_0, x_0, \alpha(v), v\big)\big), \quad t_0 \leqslant t \leqslant T,$$

for every $v \in \mathcal{V}(t_0)$ *and* $t \in [t_0, T]$.

The proof of this lemma can be deduced from Proposition 6 in [7]. An important consequence of this result is

Corollary 19. *Under the assumption of Lemma* 18 *we obtain*

$$\psi(t_0, x_0) \geqslant V^-(t_0, x_0) \quad \textit{for all } (t_0, x_0) \in [0, T] \times \mathbb{R}^N.$$

Proof. By Lemma 18 we obtain for fixed $\alpha \in \Gamma(t_0)$ for all $v \in \mathcal{V}(t_0)$ that

$$\psi(t_0, x_0) \geqslant \psi\bigl(t, x\bigl(t; t_0, x_0, \alpha(v), v\bigr)\bigr) \geqslant h\bigl(x\bigl(t; t_0, x_0, \alpha(v), v\bigr)\bigr)$$

for all $t \in [t_0, T]$; so, passing to the supremum,

$$\psi(t_0, x_0) \geqslant \sup_{v \in \mathcal{V}(t_0)} \sup_{\tau \in [t_0, T]} h\bigl(x\bigl(\tau; t_0, x_0, \alpha(v), v\bigr)\bigr) \geqslant V^-(t_0, x_0)$$

which is the desired inequality. □

Lemma 20. *If* $(\mathrm{H}_{\mathrm{low}})$ *holds true then*

$$V^-(t, x) \leqslant V^+(t, x) \quad \textit{for all } (t, x) \in [0, T] \times \mathbb{R}^N.$$

Proof. The main idea of the proof is a inf-convolution method. We define the following sequence

$$h_n(x) = \inf_{y \in \mathbb{R}^N} \bigl(h(y) + n\|y - x\|\bigr), \quad n \in \mathbb{N}, \tag{2}$$

of Lipschitz functions; $h_n(x) \leqslant h_{n+1}(x)$ and $\lim_{n\to\infty} h_n(x) = h(x)$ for every $x \in R^N$.

Using the results for the Lipschitz continuous case (see Theorem 37 in Appendix A) we obtain that $V^+_{h_n} = V^-_{h_n}$ is an increasing sequence of viscosity solutions for HJI. Observe that

$$U := \lim_{n\to\infty} V^+_{h_n} = \lim_{n\to\infty} V^-_{h_n}$$

is a lower semicontinuous supersolution of HJI (cf. Theorem 4.1 in [14]).

Obviously, $U \leqslant V^{\pm}$, $U(T, x) = \lim_{n\to\infty} h_n(x) = h(x)$ for all $x \in \mathbb{R}^N$ and $U(t, x) \geqslant h(x)$ for all $x \in [0, T] \times \mathbb{R}^N$.

By Corollary 19, $U \geqslant V^-$; hence $U = V^- \leqslant V^+$. Finally we have

$$V^- = \min \left\{ \begin{matrix} \varphi \mid \varphi \text{ lower semicontinuous supersolution for HJI,} \\ \varphi(t, x) \geqslant h(x) \text{ for every } (t, x) \in [0, T] \times \mathbb{R}^N \end{matrix} \right\}$$

and the proof of Proposition 17 is complete. □

Remark 21. Since $V^{\pm} \geqslant h$ in $[0, T] \times \mathbb{R}^N$, we want to characterize the value function by the super- and subsolutions of HJI greater or equal then h.

Remark 22. We did not succeed to prove that the value exist and that it is a generalized (see Definition 3 in the Introduction) solution to HJI, that is possible for the case h upper semicontinuous; so the reverse inequality remains an open problem. The main result for h upper semicontinuous is more complete than the result in the lower semicontinuous case.

4. Supremum control problem

We follow in this section [13] and we obtain a result which is stronger that in game case. Recall that we investigate the supremum problem:

$$\text{minimize} \sup_{\tau\in[t_0,T]} h\bigl(x\bigl(\tau; t_0, x_0, u(\cdot)\bigr)\bigr)$$

over solutions of

$$\begin{cases} x'(t) = f(x(t), u(t)) \text{ for almost all } t \in [t_0, T], \\ x(t_0) = x_0, \end{cases} \tag{PC$_1$}$$

where $u \in \mathcal{U}(t_0)$ and $h : \mathbb{R}^N \to \mathbb{R}_+$ is a bounded function.

The value function corresponding to the above optimal control problem is

$$V(t_0, x_0) = \inf_{u\in\mathcal{U}(t_0)} \sup_{\tau\in[t_0,T]} h\bigl(x\bigl(\tau; t_0, x_0, u(\cdot)\bigr)\bigr).$$

If the value is a Lipschitz function then it is viscosity Lipschitz solution to the corresponding Hamilton–Jacobi–Bellman equation:

$$\begin{cases} \max\bigl[\bigl(\frac{\partial V}{\partial t}(t,x) + H\bigl(x, V(t,x), \frac{\partial V}{\partial x}(t,x)\bigr)\bigr); (h(x) - V(t,x))\bigr] = 0 \\ \text{for all } (t,x) \in [0,T) \times \mathbb{R}^N, \\ \text{with the final condition } V(T,x) = h(x) \text{ for every } x \in \mathbb{R}^N, \end{cases} \tag{HJB}$$

with the Hamiltonian given by

$$H(x, r, p) = \begin{cases} \min_{u\in U}(p \cdot f(x,u)), & h(x) \leqslant r, \\ +\infty, & h(x) > r. \end{cases}$$

Moreover, in [11] under the assumption that h is a continuous function, we can see that V is the unique lower semicontinuous viscosity solution of HJB. The basic idea (see [9]) is that for concave Hamiltonians a continuous function is viscosity solution if and only if it is a viscosity solution in the sense of lower semicontinuous solutions.

In this section our main aim is to characterize the value function V as the unique solution to the corresponding PDE, but for an arbitrary upper semicontinuous h. The question is also how to use the viscosity theory for describe V (for only bounded h), so it is natural to establish a relation between our value and the viscosity sub- or supersolutions for HJB.

Let us state the following

Lemma 23. *Suppose that* (H_f), (H_w) *and* $(\mathrm{H}_{\mathrm{low}})$ *hold true. Then for every* $(t,x)\in[0,T]\times\mathbb{R}^N$

$$V_h=\min\left\{\begin{array}{l}\psi\mid\psi \textit{ lower semicontinuous supersolution for HJB},\\ \psi(t,x)\geqslant h(x) \textit{ for all } (t,x)\in[0,T]\times\mathbb{R}^N\end{array}\right\}.$$

Proof. We use a inf-convolution method. We define a sequence of functions h_n by formula (2). The results for the Lipschitz continuous case yields that V_{h_n} is an increasing sequence of viscosity solutions for HJB, so

$$U:=\lim_{n\to\infty}V_{h_n}\leqslant V_h$$

is a lower semicontinuous supersolution of HJI (cf. Theorem 4.1 in [14]).

Let us recall in a adapted version the following result, which plays an important role in the proof of Theorem 25:

Lemma 24 (cf. [7,10]). *Assume that* (H_f) *and* (H_w) *hold true. If* ψ *is a lower semicontinuous supersolution for HJB with* $\psi(t,x)\geqslant h(x)$ *for all* $(t,x)\in[0,T]\times\mathbb{R}^N$, *then for every* $(t_0,x_0)\in(0,T)\times\mathbb{R}^N$ *there exists* $u\in\mathcal{U}(t_0)$ *such that*

$$\psi(t_0,x_0)\geqslant\psi\big(t,x(t;t_0,x_0,u)\big),\quad t_0\leqslant t\leqslant T,$$

for every $t\in[t_0,T]$.

This lemma can be deduced from Lemma 18 and it also has been proved in [10] (see Theorem 2.3).

We can easily see that

$$\psi(t_0,x_0)\geqslant h\big(x(t;t_0,x_0,u)\big),\quad t_0\leqslant t\leqslant T,$$

so by definition of the value function and above lemma we obtain

$$\psi\geqslant V\quad\text{in } [0,T]\times\mathbb{R}^N.$$

Obviously, $U(T,x)=\lim_{n\to\infty}h_n(x)=h(x)$ for all $x\in\mathbb{R}^N$ and $U(t,x)\geqslant h(x)$ for all $x\in[0,T]\times\mathbb{R}^N$, so $U\geqslant V$. Hence, $U=V$ in $[0,T]\times\mathbb{R}^N$, and finally

$$V^-=\min\left\{\begin{array}{l}\varphi\mid\varphi\text{ lower semicontinuous supersolution for HJB},\\ \varphi(t,x)\geqslant h(x)\text{ for every }(t,x)\in[0,T]\times\mathbb{R}^N\end{array}\right\}$$

and the proof of Lemma 23 is complete. □

Theorem 25. *Suppose that* (H_f), (H_w) *hold true and* h *is a bounded function. Then for every* $(t_0,x_0)\in[0,T]\times\mathbb{R}^N$

$$V_h(t_0,x_0)=\inf\left\{\begin{array}{l}\psi(t_0,x_0)\mid\psi \textit{ lower semicontinuous}\\ \qquad\qquad\textit{supersolution for HJB},\\ \psi(t,x)\geqslant h(x) \textit{ for all } (t,x)\in[0,T]\times\mathbb{R}^N\end{array}\right\}.$$

Proof. By Lemma 24 we have

$$\psi \geqslant V \quad \text{in } [0,T]\times\mathbb{R}^N,$$

and by definition of V there exists $u_n(\cdot)\in\mathcal{U}(t_0)$ such that

$$\lim_{n\to\infty}\sup_{\tau\in[t_0,T]} h\bigl(x\bigl(\tau;t_0,x_0,u_n(\cdot)\bigr)\bigr)=V_h(t_0,x_0).$$

For $M>c_2$ we define the function $l_n:\mathbb{R}^N\to\mathbb{R}_+$ by

$$l_n(x)=\begin{cases}\sup_{\tau\in[t_0,T]} h(x(\tau;t_0,x_0,u_n(\cdot))),\\ \qquad\qquad x\in\{x(\tau;t_0,x_0,u_n(\cdot)),\ \tau\in[t_0,T]\},\\ M, \quad x\notin\{x(\tau;t_0,x_0,u_n(\cdot)),\ \tau\in[t_0,T]\}.\end{cases}$$

Obviously $l_n\geqslant h$, l_n is lower semicontinuous because $\{x(\tau;t_0,x_0,u_n(\cdot)),\ \tau\in[t_0,T]\}$ is closed; so, V_{l_n} is a lower semicontinuous supersolution for HJB with $V_{l_n}\geqslant l_n\geqslant h$ and $V_{l_n}(t_0,x_0)=\sup_{\tau\in[t_0,T]} h(x(\tau;t_0,x_0,u_n(\cdot)))$ for all $n\in\mathbb{N}$.

The proof is complete if we recall that

$$\lim_{n\to\infty} V_{l_n}(t_0,x_0)=\lim_{n\to\infty}\sup_{\tau\in[t_0,T]} h\bigl(x\bigl(\tau;t_0,x_0,u_n(\cdot)\bigr)\bigr)=V_h(t_0,x_0). \qquad \square$$

Using Theorems 13 and 25 and the fact that $V^+=\tilde{V}^+=V$ (if $f=f(x,u)$), we can easily deduce the following

Proposition 26. *Assume that* (H_f), (H_w) *and* $(\mathrm{H}_{\mathrm{upp}})$*, hold true. Then*:

(1) *V is an upper semicontinuous, bounded function and a subsolution to the HJI with $V(T,x)=h(x)$ for every $x\in\mathbb{R}^N$.*
(2) *Moreover,*

$$V=\max\left\{\begin{array}{l}\varphi \mid \varphi \text{ upper semicontinuous subsolution for HJI},\\ \varphi(t,x)\geqslant h(x) \text{ for all } (t,x)\in[0,T]\times\mathbb{R}^N,\\ \varphi(T,x)\leqslant h(x) \text{ for all } x\in\mathbb{R}^N\end{array}\right\},$$

and for every $(t_0,x_0)\in[0,T]\times\mathbb{R}^N$

(3) $$V(t_0,x_0)=\inf\left\{\begin{array}{c}\psi(t_0,x_0)\mid \psi \text{ lower semicontinuous}\\ \text{supersolution for HJI},\\ \psi(t,x)\geqslant h(x) \text{ for all } (t,x)\in[0,T]\times\mathbb{R}^N\end{array}\right\}.$$

Remark 27. According to (2) and (3) we conclude that V is a generalized solution to HJB in the sense of Definition 3.

Remark 28. For the lower semicontinuous case we do not succeed to prove that the value is an generalized viscosity solution of HJB because we have $\varphi(T,x)=h(x)$ for all $x\in\mathbb{R}^N$, where φ is upper semicontinuous and this clearly forces h continuous. By the same argument for a bounded h, the precedent equality says that h must be an upper semicontinuous function.

Appendix A

A.1. Basic results on the case h Lipschitz

In this section we recall the results of [2]. The results hold true also for $h = h(x,u,v)$. In this paper for the discontinuous problem we take $h = h(x)$ because if, for instance, we have $h(x,u,v) = u$ and $u_1 = u_2 = 0$ on $(0,T]$, $u_1(0) = 1$, $u_2(0) = 0$, $1 = \sup h(x_1) \neq \sup h(x_2) = 0$ with $u_1 = u_2$ in the class of measurable function.

Let $h:\mathbb{R}^N \to \mathbb{R}_+$ be a bounded and Lipschitz function; then the value functions are bounded and Lipschitz and we have the dynamic programming principle on which our derivation of the Isaacs equation is based. Proposition 31 is modelled on the classical assertion in [5] and we can use the following for a shorter proof:[3]

Lemma 29. *Let $u(\cdot) \in \mathcal{U}(t_0)$, $\beta \in \Delta(t_0)$, $v \in \mathcal{V}(t_0)$ and $\alpha \in \Gamma(t_0)$. Then*

$$r(t) = \max\Big[\inf_{\bar u\in\mathcal{U}(t)} \sup_{\tau\in[t,T]} h\big(\bar x\big(\tau; t, x(t), \bar u, \beta(\bar u)\big)\big);\ \sup_{\tau\in[t_0,t]} h\big(x\big(\tau; t_0, x_0, , u, \beta(u)\big)\big)\Big],$$

respectively

$$s(t) = \max\Big[\sup_{\bar v\in\mathcal{V}(t)} \sup_{\tau\in[t,T]} h\big(x\big(\tau; t, x(t), \alpha(\bar v), \bar v\big)\big);\ \sup_{\tau\in[t_0,t]} h\big(x\big(\tau; t_0, x_0, \alpha(v), v\big)\big)\Big],$$

is increasing, respectively decreasing, in $[0,T]$.

Remark 30. The monotonicity properties of the value functions along trajectories [11], or equivalently invariance or viability of the epigraph and/or hypograph studied in [16] and [18], was used to define a notion of solution to HJI by Frankowska [10].

Proposition 31 (Dynamic programming principle). *Let $h:\mathbb{R}^N \to \mathbb{R}_+$ be a bounded function and suppose that* (H_f) *holds true. Then for all* $(t_0,x_0) \in [0,T]\times\mathbb{R}^N$

$$\begin{aligned} &V^-(t_0,x_0) \\ &\quad= \inf_{\alpha\in\Gamma(t_0)} \sup_{v\in\mathcal{V}(t_0)} \Big\{\max\Big[V^-\big(t_0+h, x\big(t_0+h; t_0, x_0, \alpha(v)(\cdot), v(\cdot)\big)\big);\\ &\qquad\qquad \sup_{\tau\in[t_0,t_0+h]} h\big(x\big(\tau; t_0, x_0, \alpha(v)(\cdot), v(\cdot)\big)\big)\Big]\Big\}, \end{aligned} \tag{A}$$

[3] See [11] for the control problem.

$$
\begin{aligned}
&V^+(t_0,x_0)\\
&\quad=\sup_{\beta\in\Delta(t_0)}\inf_{u\in\mathcal{U}(t_0)}\Big\{\max\Big[V^+\big(t_0+h,x\big(t_0+h;t_0,x_0,u(\cdot),\beta(u)(\cdot)\big)\big);\\
&\qquad\qquad\sup_{\tau\in[t_0,t_0+h]}h\big(x\big(\tau;t_0,x_0,u(\cdot),\beta(u)(\cdot)\big)\big)\Big]\Big\}.
\end{aligned}
\tag{B}
$$

The Bellman dynamic programming gives the form of Isaacs equation for our problem. It is established (see [2]) that the value functions are viscosity solutions of Hamilton–Jacobi–Isaacs equation HJI$^\pm$ involving $H^\pm$ as Hamiltonians:[4]

$$
\begin{cases}
\max\big[\big(\frac{\partial V}{\partial t}(t,x)+H^\pm\big(x,V(t,x),\frac{\partial V}{\partial x}(t,x)\big)\big);(h(x)-V(t,x))\big]=0\\
\text{for all } (t,x)\in[0,T)\times\mathbb{R}^N,\\
\text{with final condition } V(T,x)=h(x) \text{ for all } x\in\mathbb{R}^N,
\end{cases}
\tag{HJI$^\pm$}
$$

where $H^+, H^-:\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\to\bar{\mathbb{R}}$, and

$$
H^-(x,r,p)=\begin{cases}\max_{v\in V}\min_{u\in U}(p\cdot f(x,u,v)), & h(x)\leqslant r,\\ +\infty, & h(x)>r,\end{cases}
$$

$$
H^+(x,r,p)=\begin{cases}\min_{u\in U}\max_{v\in V}(p\cdot f(x,u,v)), & h(x)\leqslant r,\\ +\infty, & h(x)>r.\end{cases}
$$

Remark 32. The assumption of Lipschitz continuity and Rademacher's theorem tell us that the value functions are differentiable almost everywhere; therefore V^+ and V^- satisfies the corresponding equation almost everywhere. Nevertheless the difficult problem is the uniqueness of solution for PDE. However, if we merely assumed uniform continuity for V^+ and V^-, then the value functions would still satisfy the appropriate equation in the viscosity sense.

Remark 33. We recall Wazewski's lemma which simplifies the proof of Theorem 2.4 in [2]:

Lemma 34. *Let K be a compact subset of $\mathbb{R}^N$, $u\in\mathcal{U}(t_0)$. If* (H_f) *and* (H_w) *hold true, then the set* $\{x(\cdot\,;t_0,x_0,u(\cdot),v(\cdot));\ x_0\in K,\ v\in\mathcal{V}(t_0)\}$ *is compact in* $C[(t_0,T)]$ *with uniform convergence topology.*

A.2. The existence of a value

Under Isaacs min–max condition (CI) the existence of a value is an immediate consequence of the following results:

[4] We denote by HJI$^+$ (respectively, HJI$^-$) the system having the Hamiltonian H^+ (respectively, H^-).

Proposition 35 (Comparison result (adapted from [14] and [19])). *Assume that* (H_f) *holds true and* h *is a Lipschitz function. If* $u, v \in BUC([0,T] \times \mathbb{R}^N)$ *are a lower semicontinuous supersolution and respectively an upper semicontinuous subsolution to* $HJI^{\pm}$ *and* $u(t,x), v(t,x) \geqslant h(x)$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^N$, $u(T,x) \geqslant v(T,x)$ *for all* $x \in \mathbb{R}^N$, *then* $u \geqslant v$ *in* $[0,T] \times \mathbb{R}^N$.

Theorem 36 (Uniqueness result (adapted from [2] and [19])[5]). *Assume that* (H_f) *holds true and* h *is a Lipschitz function. There exist at most one viscosity solution* $u \in BUC([0,T] \times \mathbb{R}^N)$ *of* HJI^+ *(respectively,* HJI^-*) satisfying* $u(t,x) \geqslant h(x)$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^N$ *and* $u(T,x) = h(x)$ *for all* $x \in \mathbb{R}^N$.

It follows immediately:

Theorem 37. *Assume that* (H_f), (H_w) *hold true and* h *is a Lipschitz function. If* f *satisfy the Isaacs condition* (CI),[6] *i.e.,*

$$H^-(x,r,p) = H^+(x,r,p) \quad \text{for every } (x,r,p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N, \tag{CI}$$

then $V^+ = V^-$; *i.e., the game has a value, namely the unique solution to HJI equation with the final condition* $V^{\pm}(T,x) = h(x)$ *for all* $x \in \mathbb{R}^N$.

A.3. Proof of Lemma 5

This section is devoted to the proof of Lemma 5. This proof is based on the fact that φ viscosity subsolution for (HJI) gives some invariance properties for the hypograph of φ which definition is

$$\mathrm{Hypo}(\varphi) = \left\{(t,x,r) \in [0,T] \times \mathbb{R}^N \times \mathbb{R} \mid \varphi(t,x) \geqslant r\right\}.$$

For the proof of Lemma 5 we recall the following lemma adapted from [20, Theorem 2.3] (see [21,23] for a more general form):

Lemma 38 (cf. Theorem 2.3 in [20]). *We assume that* D *and* E *are closed sets,* $g : \mathbb{R}^N \times U \times V \to \mathbb{R}^N$ *is continuous and satisfies the same hypotheses than* f. *Then the two following assertions are equivalent*:

(i) $$\forall x \in D \backslash E,\ \forall p \in NP_D(x), \quad \min_u \max_v \langle g(x,u,v), p\rangle \leqslant 0, \tag{3}$$

where $NP_D(x)$ *denotes the set of proximal normal to* D *at* x, *i.e., the set of* $p \in \mathbb{R}^N$ *such that the distance of* $x + p$ *to* D *is equal to* $\|p\|$.

---

[5] Here we deal with discontinuous, nonconvex Hamiltonians.

[6] Under (CI), $V^{\pm}$ are viscosity solutions for $\mathrm{HJI}^+ = \mathrm{HJI}^- = \mathrm{HJI}$.

(ii) $\exists\beta \in \Delta(t_0)$, *such that* $\forall u \in \mathcal{U}(t_0)$ *the solution of*

$$\begin{cases} x'(t) = g(x(t), u(t), \beta(u(t))) \text{ for almost every } t \in [t_0, T], \\ x(t_0) = x_0 \end{cases}$$

remains in D *as long as it does not reach* E.

We also recall that for $x \in K$ we define by

$$T_K(x) = \left\{ v \in \mathbb{R}^N \mid \liminf_{h \to 0^+} d_K(x + hv)/h = 0 \right\}$$

the tangent cone to K at x and by

$$N_K(x) = T_K(x)^- = \left\{ p \in \mathbb{R}^N \mid \langle p, v\rangle \leqslant 0 \ \forall v \in T_K(x) \right\}$$

the normal cone to K at x.

It is well known that $T_K(x)$ is a closed cone, $N_K(x)$ is a closed convex cone and $NP_K(x) \subset N_K(x)$ for all $x \in K$. We also have the followings equivalences:

$$p \in \partial_- w(x_0) \quad \Leftrightarrow \quad (p, -1) \in \left[T_{\mathrm{Epi}(w)}\big(x_0, w(x_0)\big)\right]^-,$$
$$p \in \partial_+ w(x_0) \quad \Leftrightarrow \quad (-p, 1) \in \left[T_{\mathrm{Hypo}(w)}\big(x_0, w(x_0)\big)\right]^-,$$

where Epi stands for the epigraph and Hypo stands for the hypograph.

Proof of Lemma 6. Fix $t_0 \in (0, T)$. We set

$$D_\varphi = \mathrm{cl}\big(\{(t, x, r)\colon t \in (0, T],\ x \in \mathbb{R}^N,\ r \leqslant \varphi(t, x)\}\big) \cup [T, \infty) \times \mathbb{R}^N \times \mathbb{R},$$

$$\tilde{f}(t, x, u, v, r) = \begin{cases} 0 & \text{if } t < 0, \\ \frac{t}{t_0}(1, f(x, u, v), 0) & \text{if } t \in [0, t_0], \\ (1, f(x, u, v), 0) & \text{if } t \in [t_0, T], \\ (1, f(x, u, v), 0) & \text{if } t > T. \end{cases}$$

We show that (3) holds true for $\tilde{f}$, D_φ and $[0, T] \times \mathrm{Hypo}(h)$. Let $z_0 = (s_0, x_0, r_0 := \varphi(t_0, x_0)) \in D_\varphi \backslash \{[0, T] \times \mathrm{Hypo}(h)\}$. If $s_0 = 0$, then $\tilde{f} = 0$. Obviously (3) holds true.

If $s_0 \geqslant T$ and $(p_s, p_x, p_r) \in N_{D_\varphi}(s_0, x_0, r_0)$, then $p_s \leqslant 0$, $p_x = 0$, $p_r = 0$. Hence (3) holds true.

It remains to consider the case $s_0 \in (0, T)$. We have

$$N_{D_\varphi}(s_0, x_0, r_0) \subset N_{D_\varphi}\big(s_0, x_0, \varphi(s_0, x_0)\big).$$

Let $(p_s, p_x, p_r) \in N_{D_\varphi}(s_0, x_0, \varphi(s_0, x_0))$.

If $p_r > 0$, then (see Proposition 4.1 in [10]) $(-p_s/p_r, -p_x/p_r) \in \partial_+\varphi(s_0, x_0))$. Since φ is a *supersolution* of (HJI) we have that

$$\max\Big\{\big(-p_s/p_r + H\big(x_0, \phi(t_0, x_0), -p_x/p_r\big)\big); \big(h(x_0) - \varphi(t_0, x_0)\big)\Big\} \geqslant 0,$$

and using the fact that $\varphi(s_0, x_0) > h(x_0)$ we obtain

$$\frac{p_s}{-p_r} + \min_v \max_u \left\langle f(x, u, v), \frac{p_x}{-p_r} \right\rangle \geqslant 0;$$

so,

$$p_s + \min_u \max_v \langle f(x, u, v),\ p_x \rangle \leqslant 0.$$

Hence

$$\min_u \max_v \langle \tilde{f}(x, u, v), (p_s, p_x, p_r) \rangle \leqslant 0.$$

Now we consider the case $p_r = 0$. By a Rockafellar's lemma (see Lemma 4.2 in [10] and Lemma 5.1 in [11]) there exists a sequence

$$s_n \to s_0, \qquad x_n \to x_0$$

such that

$$\varphi(s_n, x_n) \to \varphi(s_0, x_0)$$

and

$$p_{s_n} \to p_s, \qquad p_{x_n} \to p_x, \qquad p_{r_n} \to 0, \qquad p_{r_n} > 0$$

such that

$$(p_{s_n}, p_{x_n}, p_{r_n}) \in N_{D_\varphi}(p_{s_n}, p_{x_n}, p_{r_n}).$$

Since $p_{r_n} > 0$ and using the fact that φ and h are u.s.c., there exists a subsequence $(s_{n_k}, x_{n_{kk}})$ of (s_n, x_n) such that $\varphi(s_{n_k}, x_{n_{kk}}) > h(x_{n_k})$, from the previous case we obtain

$$\min_u \max_v \langle \tilde{f}(x_{n_k}, u, v), (p_{s_{n_k}}, p_{x_{nk}}, p_{r_{nk}}) \rangle \leqslant 0.$$

Since $\tilde{f}$ is Marchaud we obtain

$$\min_u \max_v \langle \tilde{f}(x, u, v), (p_s, p_x, p_r) \rangle \leqslant 0.$$

In view of the above lemma we have there exists a nonanticipative strategy $\beta \in \Delta(t_0)$ such that for all $u \in \mathcal{U}(t_0)$ the solution $z(\cdot\,; t_0, x_0, u, \beta(u))$ to the Cauchy problem

$$z'(s) \in \tilde{f}\big(z(s)\big), \qquad z(t_0) = z_0,$$

$z(s) = (t(s), x(s), r(s))$ with

$$t(s) = s, \qquad r(s) = r_0 = \varphi(t_0, x_0),$$

stays in D_φ until it reaches $[0, T] \times \mathrm{Hypo}(h)$ and the proof is complete. □

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