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On the Density of Families of Sets

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If $\mathscr{F}$ is a family of sets and A some set we denote by $\mathscr{F} \cap A$ the following family of subsets of A: $\mathscr{F} \cap A = \{F \cap A; F \in \mathscr{F}\}$. P. Erdös (oral communication) transmitted to me in Nice the following question: Is it true that if $\mathscr{F}$ is a family of subsets of some infinite set S then either there exists to each number n a set $A \subset S$ with $|A| = n$ such that $|\mathscr{F} \cap A| = 2^n$ or there exists some number N such that $|\mathscr{F} \cap A| \leqslant |A|^c$ for each $A \subset S$ with $|A| \geqslant N$ and some constant c? In this paper we will answer this question in the affirmative by determining the exact upper bound. (Theorem 2).[1]

DEFINITIONS. The *density* of a family $\mathscr{F}$ of sets is the largest number n such that there exists a set A with $|A| = n$ and $|\mathscr{F} \cap A| = 2^n$. If such an n does not exist we say that the density of $\mathscr{F}$ is ∞. We observe that the density of $\mathscr{F}$ can only be 0 if $|\mathscr{F}| \leqslant 1$. If $\mathscr{F}$ is a family of subsets of some set S with p in S then $\mathscr{F}_p = \mathscr{F} \cap \{S - p\}$. $\mathscr{F}$ has a *pair* (A, B) at p if there exist two sets $A, B \in \mathscr{F}$ such that $A - B = p$ and $B \subset A$. $P_1(\mathscr{F}, p) = \{A \in \mathscr{F}; (A, B)$ is a pair at p for some $B \in \mathscr{F}\}$ and $P_2(\mathscr{F}, p) = \{B \in \mathscr{F}; (A, B)$ is a pair at p for some $A \in \mathscr{F}\}$. We observe that, if (A, B_1), (A, B_2), (A_1, B), and (A_2, B) are pairs at p, then $B_1 = B_2$ and $A_1 = A_2$. Therefore $|P_1(\mathscr{F}, p)| = |P_2(\mathscr{F}, p)|$.

THEOREM 1. *If the density of the family $\mathscr{F}$ of subsets of a set S with $|S| = m$ is less than n then*

$$|\mathscr{F}| \leqslant \sum_{i=0}^{n-1} \binom{m}{i}.$$

There exists a family $\mathscr{F}$ of subsets of S with $|\mathscr{F}| = \sum_{i=0}^{n-1} \binom{m}{i}$ such that the density of $\mathscr{F}$ is $n - 1$ $(m \geqslant n \geqslant 1)$.

[1] The referee of this paper wrote that these results have also been established by S. Shelah [1, 2].

In order to prove the theorem we need the following two lemmas.

LEMMA 1. *If $\mathscr{F}$ is a family of subsets of the finite set S and $p \in S$ then* $|\mathscr{F}| - |\mathscr{F}_p| = |P_2(\mathscr{F}, p)|$.

$$\mathscr{F}_p = \{F \in \mathscr{F}; p \notin F\} \cup \{H - p; H \in \mathscr{F} \text{ and } p \in H\}.$$

$$\begin{aligned} |\mathscr{F}_p| &= |\{F \in \mathscr{F}; p \notin F\}| + |\{H - p; H \in \mathscr{F} \text{ and } p \in H\}| \\ &\quad - |\{F \in \mathscr{F}; p \notin F\} \cap \{H - p; H \in \mathscr{F} \text{ and } o \in H\}| \\ &= |\{F \in \mathscr{F}; p \notin F\}| + |\{H; H \in \mathscr{F} \text{ and } p \in H\}| \\ &\quad - |P_2(\mathscr{F}, p)| = |\mathscr{F}| - |P_2(\mathscr{F}, p)|. \end{aligned}$$

LEMMA 2. *If $P_2(\mathscr{F}, p)$ has density $n - 1$ in $S - p$ then $\mathscr{F}$ has density n.*

We will prove that if $P_2(\mathscr{F}, p)$ has density $n - 1$ then

$$G = P_1(\mathscr{F}, p) \cup P_2(\mathscr{F}, p)$$

has density n.

If $P_2(\mathscr{F}, p)$ has density $n - 1$ there exists a set $A \subset (S - p)$ with $|A| = n - 1$ such that $P_2(\mathscr{F}, p) \cap A = 2^A$. We have to prove that $G \cap (A \cup p) = 2^{(A \cup p)}$. Let us assume to the contrary, that there exists a set $H \subset (A \cup p)$ with $H \notin G \cap (A \cup p)$. $p \in H$ because otherwise $H \subset A$ and then $H \in P_2(\mathscr{F}, p) \cap A \subset G \cap A$ and $H \in G \cap (A \cup p)$. If $p \in H$ then $H - p \subset A$ and $H - p \in P_2(\mathscr{F}, p) \cap A$. Let $L \in P_2(\mathscr{F}, p)$ be the set such that $L \cap A = H - p$. Because $L \in P_2(\mathscr{F}, p)$ there exists a set $M \in P_1(\mathscr{F}, p)$ such that (M, L) is a pair at p.

$$M \cap (A \cup p) = (L \cup p) \cap (A \cup p) = p \cup (L \cap A) = p \cup (H - p) = H.$$

This implies that $H \in P_2(\mathscr{F}, p) \cap (A \cup p) \subset G \cap (A \cup p)$.

Proof of Theorem 1. The proof is by induction on $m - n$ and n. We observe that if $n = 1$ or $m = n$ then the theorem is true. Let us now assume that $|\mathscr{F}| > \sum_{i=0}^{n-1} \binom{m}{i}$. We will prove that the density of $\mathscr{F}$ is at least n. With $p \in S$ let us consider the family $\mathscr{F}_p$ in $S - p$. If

$$|\mathscr{F}_p| > \sum_{i=0}^{n-1} \binom{m-1}{i}$$

then we conclude by induction on $m - n$ that $\mathscr{F}_p$ has density of at least n in $S - p$ and therefore $\mathscr{F}$ has density of at least n in S. So, if we assume now that $|\mathscr{F}_p| \leqslant \sum_{i=0}^{n-1} \binom{m-1}{i}$, we get from Lemma 1:

$$|P_2(\mathscr{F}, p)| = |\mathscr{F}| - |\mathscr{F}_p| > \sum_{i=0}^{n-1} \binom{m}{i} - \sum_{i=0}^{n-1} \binom{m-1}{i} = \sum_{i=0}^{n-2} \binom{m-1}{i}.$$

This means because of the induction on n that $P_2(\mathscr{F}, p)$ has density of at least $n - 1$ in $S - p$. From Lemma 2 follows now that $\mathscr{F}$ has a density of at least n.

If the family $\mathscr{F}$ of subsets of S consists of the null-set together with all singletons and all pairs and all triples and $\cdots$ and all $n - 1$ tuples then $| \mathscr{F} | = \sum_{i=0}^{n-1} \binom{m}{i}$ but the density of $\mathscr{F}$ is $n - 1$. This proves the second part of the theorem.

THEOREM 2. *If $\mathscr{F}$ is a family of subsets of some infinite set S then the density of $\mathscr{F}$ is either ∞ or there exists a number n such that for all sets $A \subset S$ with $| A | \geqslant n$,*

$$| \mathscr{F} \cap A | \leqslant \sum_{i=0}^{n-1} \binom{| A |}{i}.$$

If there exists to each number n a set $A \subset S$ with $| A | = n$ and $| \mathscr{F} \cap A | = 2^n$ then the density of $\mathscr{F}$ is ∞; otherwise there is a number n, which is the density of $\mathscr{F}$. If

$$| \mathscr{F} \cap A | > \sum_{i=0}^{n-1} \binom{| A |}{i}$$

then $\mathscr{F} \cap A$ has a larger density than n. Because the density of $\mathscr{F}$ is larger than or equal to the density of $\mathscr{F} \cap A$ the density of $\mathscr{F}$ would be larger than n. ($| A | \geqslant n$).

REFERENCES

1. S. SHELAH, Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory (to appear in *Annuals of Math. Log.*).
2. S. SHELAH, A combinatorial problem; stability and order for models and theories in infinitary languages (to appear in *Pacific Journal of Mathematics*).

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