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Graph isomorphism completeness for chordal bipartite graphs and strongly chordal graphs

Ryuhei Uehara[a,1], Seinosuke Toda[b], Takayuki Nagoya[c,*]

[a] Natural Science Faculty, Komazawa University, 1-23-1 Komazawa, Setagaya-Ku, Tokyo 154-8525, Japan
[b] Department of Computer Science and System Analysis, College of Humanities and Sciences, Nihon University, 3-25-40 Sakurajosui, Setagaya-Ku, Tokyo 156-8550, Japan
[c] Department of Mathematical Science, College of Sceince and Engineering, Tokyo Denki University, Hatoyama, Saitama 350-0394, Japan

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Abstract

This paper deals with the graph isomorphism (GI) problem for two graph classes: chordal bipartite graphs and strongly chordal graphs. It is known that GI problem is GI complete even for some special graph classes including regular graphs, bipartite graphs, chordal graphs, comparability graphs, split graphs, and k-trees with unbounded k. On the other hand, the relative complexity of the GI problem for the above classes was unknown. We prove that deciding isomorphism of the classes are GI complete.
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1. Introduction

The graph isomorphism (GI) problem is a well-known problem and exploring its precise complexity has become an important open question in computational complexity theory three decades ago (see [8]). Although the problem is trivially in NP, the problem is not known to be in P and not known to be NP-complete either (see [9,11]). It is very unlikely that the GI problem is NP-complete. As its evidence, the counting version of the GI problem is known to be polynomial time equivalent to the original GI problem (see [10]), while for almost all NP complete problems, their counting versions appear to be of much higher complexity than themselves. Furthermore, it is recently shown by Arvind and Kurur [1] that the GI problem is in SPP and hence the problem is low for any counting complexity classes defined via $\#P$ (or GapP) functions. On the other hand, any NP complete problems do not appear to reveal this property. It also seems unlikely that the GI problem is in P. Though there does not seem to be a good evidence supporting this unlikelihood, we may note that no efficient algorithm has been found even if we allowed to use some probabilistic method (or some quantum mechanism).

The current status on the computational complexity of the GI problem mentioned above has motivated us to introduce a notion of "GI completeness". A problem is *GI complete* if it is polynomial time equivalent to the GI problem. There are many GI complete problems (see [3,9]). In fact, the GI problem itself remains to be GI complete for several graph classes including regular graphs, bipartite graphs, chordal graphs, comparability graphs, split graphs, and k-trees with unbounded k (see [3] for

* Corresponding author
E-mail address: nagoya@r.dendai.ac.jp (T. Nagoya).
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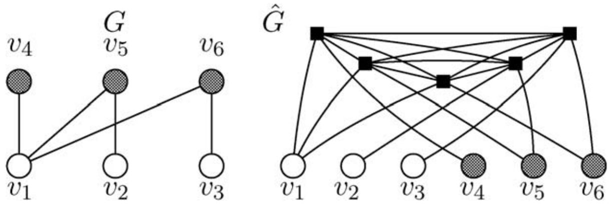

Fig. 1. Reduction by Babel et al.

a review). On the other hand, the GI problem is solvable in polynomial time when it is restricted to special graph classes, e.g., graphs of bounded degrees, planar graphs, interval graphs, permutation graphs, k-trees with fixed k (see [2] for reference), and convex graphs [5].

Recently, many graph classes have been proposed and widely investigated (see [4] for a comprehensive survey). However, relative complexity of the GI problem is not known for some graph classes. Among them, the classes of strongly chordal graphs and chordal bipartite graphs are on the border. We show that the GI problem for the graph classes is GI complete. These results solve the open questions noted by Spinrad [2,12].

2. Preliminaries

For a given graph $G=(V,E)$, $G[U]$ denotes the subgraph of G induced by $U \subseteq V$. Two graphs $G=(V,E)$ and $G'=(V',E')$ are *isomorphic* if and only if there is a one-to-one mapping $\phi : V \to V'$ such that $\{u,v\} \in E$ if and only if $\{\phi(u), \phi(v)\} \in E'$ for every pair of vertices $u, v \in V$. We denote by $G \sim G'$ if G and G' are isomorphic. The *graph isomorphism* (*GI*) *problem* is to determine if $G \sim G'$ for given graphs G and G'. An edge is a *chord* of a cycle if it joins two vertices of the cycle but is not itself an edge of the cycle. A graph is *chordal* if every cycle of length at least 4 has a chord. A graph is *chordal bipartite* if the graph is bipartite and every cycle of length at least 6 has a chord. A chord $\{x_i, x_j\}$ in a cycle $(x_1, x_2, \ldots, x_{2k}, x_1)$ of even length $2k$ is an *odd chord* if $|j-i| \equiv 1 \pmod 2$. A graph is *strongly chordal* if G is chordal and each cycle in G of even length at least 6 has an odd chord. I_n and K_n denote an independent set and a clique of size n, respectively. A graph $G=(V,E)$ is a *split graph* if V can be partitioned into two subsets X and Y such that $G[X] \sim K_{|X|}$ and $G[Y] \sim I_{|Y|}$.

3. Main results

In [2], Babel et al. gives the following reduction from a bipartite graph to a directed path (DP) graph such that two given bipartite graphs are isomorphic if and only if the reduced DP graphs are isomorphic: given bipartite graph $G=(X,Y,E)$ with $|X \cup Y| = n$ and $|E| = m$, the edge set $\hat{E}$ of the reduced graph $\hat{G} = (X \cup Y \cup E, \hat{E})$ contains $\{e, e'\}$ for all $e, e' \in E$, and $\{x, e\}$ and $\{y, e\}$ for each $e = \{x, y\} \in E$ (Fig. 1). Our starting point is the DP graph $\hat{G}$ that is a split graph having the following properties: (a) $\hat{G}[E] \sim K_m$, (b) $\hat{G}[X \cup Y] \sim I_n$, and (c) each $e \in E$ has exactly one neighbor in X and another one in Y (thus $d(e) = m+1$). Without loss of generality, we also assume that (d) $m > 1$ and (e) $|X| > |Y| > 1$ (if $|X| = |Y|$, construct a new graph $(X_1 \cup Y_2 \cup \{v\}, X_2 \cup Y_1, E')$ from (X, Y, E) as follows: for each $e = \{x, y\} \in E$, $x_i \in X_i$, $y_i \in Y_i$, and $\{x_i, y_i\} \in E'$ for $i = 1, 2$, and for every $u \in X_2 \cup Y_1$, $\{v, u\} \in E'$).

We reduce the split graph $\hat{G} = (X \cup Y \cup E, \hat{E})$ to a graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$. We set $\mathscr{V} = X \cup Y \cup E \cup E' \cup B \cup W$ such that each vertex $e \in E$ corresponds to three vertices $e' \in E'$, $e_b \in B$, and $e_w \in W$, respectively (hence $|E| = |E'| = |B| = |W| = m$). Vertices are connected as follows: (1) for each $e \in E$, $\{e, e'\}$, $\{e', e_b\}$, $\{e_b, e_w\}$, $\{e, e_w\} \in \mathscr{E}$, (2) for each $e_1, e_2 \in E$, $\{e_1, e_2'\}$, $\{e_1', e_2\} \in \mathscr{E}$ (thus $\mathscr{G}[E \cup E'] \sim K_{m,m}$),and (3) for each vertex $x \in X$, $\{x, e\} \in \mathscr{E}$ if $\{x, e\} \in \hat{E}$, and for each vertex $y \in Y$, $\{y, e'\} \in \mathscr{E}$ if $\{y, e\} \in \hat{E}$. The reduced graph $\mathscr{G}$ for $\hat{G}$ in Fig. 1 is shown in Fig. 2. The reduction can be done in polynomial time.

Lemma 1. *$\mathscr{G}$ is chordal bipartite.*

Proof. Dividing $\mathscr{V}$ into $\mathscr{V}^w = X \cup E' \cup W$ and $\mathscr{V}^b = Y \cup E \cup B$, $\mathscr{G}$ is bipartite. In Fig. 2 the vertices of $\mathscr{V}^w$ are colored white and the vertices of $\mathscr{V}^b$ are colored black. To show the chordal property, let C be a cycle of length at least 6. If C contains at least one vertex in $B \cup W$, then we have four consecutive vertices v_0, v_1, v_2, and v_3 on C such that both v_1 and v_2 are in $B \cup W$ and both v_0 and v_3 are in $E \cup E'$. It is obvious that $\{v_0, v_3\}$ is a chord of C. We next suppose that all vertices of C are in $X \cup Y \cup E \cup E'$. Let v_0, v_1, v_2, v_3 be consecutive vertices on C. If they are all in $E \cup E'$, then we have $\{v_0, v_3\} \in \mathscr{E}$ since

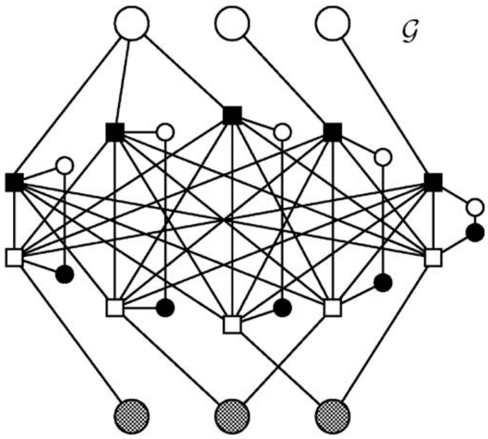

Fig. 2. Reduction to chordal bipartite graph.

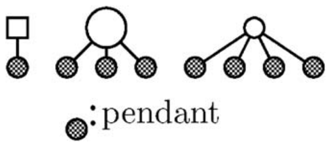

Fig. 3. Pendant vertices.

$\mathscr{G}[E \cup E']$ is $K_{m,m}$. Thus, we suppose that at least one of them is in $X \cup Y$. Without loss of generality, we may assume that $v_1 \in X$ and hence $v_0, v_2 \in E$. Then, by (c), no other vertex in X is incident to v_0 and v_2. Since no vertices in Y are incident to E, we have $v_3 \in E'$. This implies $\{v_0, v_3\} \in \mathscr{E}$. $\square$

Lemma 2. *Given bipartite graphs G_1 and G_2, $G_1 \sim G_2$ if and only if $\mathscr{G}_1 \sim \mathscr{G}_2$.*

Proof. We first note that for each edge $e = \{u, v\}$ with $u, v \in B \cup W$, there exists only one cycle of length four which contains e. We call such a cycle a *handle*. Then, it is easy to see that any isomorphism between $\mathscr{G}_1$ and $\mathscr{G}_2$ maps the handles in $\mathscr{G}_1$ to those in $\mathscr{G}_2$. Furthermore, it is easy to see that for $i = 1, 2$, an isomorphic copy of $\hat{G}_i$ can be obtained by contracting each of the handles in $\mathscr{G}_i$ into a vertex. These facts immediately imply that $\hat{G}_1 \sim \hat{G}_2$ if $\mathscr{G}_1 \sim \mathscr{G}_2$. The other direction is obvious. $\square$

Theorem 3. *The GI problem is GI complete for chordal bipartite graphs and strongly chordal graphs.*

Proof. Lemmas 1 and 2 imply the claim for chordal bipartite graphs. To show the claim for strongly chordal graphs, we show below how to reduce a chordal bipartite graph to a strongly chordal graph.

Let $\mathscr{G} = (\mathscr{V}^w, \mathscr{V}^b, \mathscr{E})$ be a chordal bipartite graph constructed above. Then, for each $u, v \in \mathscr{V}^w$, we add an edge $e = \{u, v\}$ to $\mathscr{G}$ in order to change $\mathscr{V}^w$ into a clique. We further attach *pendant* vertices (pendant for short) to each vertex in $\mathscr{V}^w$ as follows (see Fig. 3): (1) for each vertex $e' \in E'$, we add a pendant vertex and an edge between the pendant and e', (2) for each vertex x in X, we add three pendant vertices and edges between those vertices and x, and (3) for each vertex w in W, we add four pendant vertices and edges between those vertices and w. Then, the resultant graph is strongly chordal if $\mathscr{G}$ is chordal bipartite ([4], Theorem 3.4.3).

As mentioned in the first paragraph of this section, we assume $|\mathscr{V}^w| > |\mathscr{V}^b|$ and $|E| > 1$. Then, it follows from these assumptions that $\mathscr{V}^w$ become the unique maximum clique in the resultant (strongly chordal) graph. This implies that any isomorphism between two such graphs maps the clique in one graph to that of the other. In other words, the isomorphism preserves the vertex colors. We further note that each of X, E', and W, which are subsets of $\mathscr{V}^w$, can be identified in the resultant graph via the number of pendants. Precisely speaking, X is the set of vertices in $\mathscr{V}^w$ which own three pendants, E' is the set of vertices in $\mathscr{V}^w$ which own one or two pendants where, in the latter case, one of the two pendants originally came from Y, and W is the set of vertices in $\mathscr{V}^w$ which own four pendants. Then, we can easily see that any isomorphism between two such graphs does not only preserve the vertex colors but also preserves each of the sets X, E', and W. From this, we can see that for any isomorphism between two such graphs, its restriction to the original (chordal bipartite) graphs is an isomorphism between the two original graphs. From this, we have our claim for strongly chordal graphs. □

4. Concluding remarks

The class of strongly chordal graphs is between chordal graphs and interval graphs. Babel et al. show that the GI problem for directed path (DP) graphs is GI complete, while the GI problem for rooted directed path (RDP) graphs is polynomial time solvable in Ref. [2]. The class of the RDP graphs is between the strongly chordal graphs and interval graphs, although the class of the DP graphs is incomparable to strongly chordal graphs. In the paper, we draw a line between the RDP graphs and strongly chordal graphs for GI completeness, which answers the open problem stated in Ref. [2].

The class of chordal bipartite graphs is between bipartite graphs and interval bigraphs. Recently, Hell and Huang showed that any interval bigraph is the complement of a circular arc graph [6]. Thus, combining the result by Hsu [7], we can see that the GI problem for interval bigraphs can be solved in polynomial time. Therefore, we draw a line between the interval bigraphs and chordal bipartite graphs for GI completeness, which improves the GI completeness results.

As mentioned in the introduction, we have many graph classes, which are proposed recently, and we do not know whether the GI problem is GI complete or polynomial time solvable on some classes. In order to clarify the complexity of the GI problem, considering the GI problem on such graph classes is future work. For example, trapezoid graphs are the natural and classic graph class such that the complexity of the GI problems is still unknown, which is mentioned by Spinrad [13].

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