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Polygons of Petrović and Fine, algebraic ODEs, and contemporary mathematics

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Abstract

In this paper, we study the genesis and evolution of geometric ideas and techniques in investigations of movable singularities of algebraic ordinary differential equations. This leads us to the work of Mihailo Petrović on algebraic differential equations (ODEs) and in particular the geometric ideas expressed in his polygon method from the final years of the nineteenth century, which have been left completely unnoticed by the experts. This concept, also developed independently and in a somewhat different direction by Henry Fine, generalizes the famous Newton–Puiseux polygonal method and applies to algebraic ODEs rather than algebraic equations. Although remarkable, the Petrović legacy has been practically neglected in the modern literature, although the situation is less severe in the case of results of Fine. Therefore, we study the development of the ideas of Petrović and Fine and their places in contemporary mathematics.

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1 Introduction

The study of the genesis and evolution of geometric ideas and techniques related to movable singularities of ordinary differential equations leads to the work of Mihailo Petrović on algebraic differential equations, and in particular the geometric ideas expressed in his polygonal method from the final years of the nineteenth century, which have been left completely unnoticed by experts. This concept, also developed independently by Henry Fine in a somewhat different direction, generalizes the famous Newton–Puiseux polygonal method (Newton 1670; Cramer 1750; Puiseux 1850), see also contemporary sources like Brieskorn and Knörrer (1986) and Ghys (2017). Kaveh and Khovanskii (2012) and references therein) and applies to algebraic ODEs rather than algebraic equations. Mihailo Petrović (1868–1943) was an extraordinary person and the leading Serbian mathematician of his time. His results, despite their significance, are practically unknown to mathematicians nowadays. The situation is less severe with Fine's results. Thus, we emphasize here the development of the ideas of Petrović and Fine from the point of view of contemporary mathematics.

In its essence, this is a story about two outstanding individuals, Henry Fine and Mihailo Petrović, and their fundamental contributions to the development from a modern mathematical point of view of an important branch of the analytic theory of differential equations. Both men also worked to elevate mathematical research in their native countries to a remarkable new level, and both left deep traces in the development of their own academic institutions as they transformed from a local college to a renewed university: Fine to American mathematics and Princeton University and Petrović to Serbian mathematics and the University of Belgrade. The list of striking similarities between the two scientists is not even closely exhausted here. Both mathematicians were sons of a theologian. Both were in love with their own two beautiful rivers. Both actively enjoyed music playing their favorite instruments. Both went abroad, to Western Europe, to get top mathematicians of their time as mentors to do their PhD theses. And both had a state official of the highest rank in their native countries as their closest friend. These friendships heavily shaped their lives. On the other hand, Fine published five scientific papers and had no known students, while Petrović published more than 300 papers and had more than 800 scientific descendants. It seems, surprisingly, that the two did not know each other and did not know about each other's work.

1.1 Mihailo Petrović

Mihailo Petrović defended his PhD thesis (Petrowitch 1894) at the Sorbonne in 1894. His advisors were Charles Hermite and Emil Picard. During his studies in Paris, he learned a lot from Paul Painlevé and Jules Tannery, and they become friends later. He was the founding father of the modern Serbian mathematical school and one of the first eight full professors of the newly formed University of Belgrade (see Petrowitch 2019). He became a member of the Serbian Academy of Sciences and Arts, a corresponding member of the Yugoslav Academy of Sciences and Arts, and a foreign member of several other academies. He was invited speaker at five international congresses of mathematicians: Rome 1908, Cambridge 1912, Toronto 1924, Bologna 1928, and Zurich 1932.

At the same time (see Petrowitch 2019), he was a world traveler who visited both North and South Poles, and a very talented travel writer. Petrović also received many awards as an inventor, including the gold medal at the World Exposition in Paris 1900. His nickname, Alas, referred to his second profession, a fisherman on the Sava and the Danube rivers. He was more proud of his fisherman's achievements than any others, and Alas became an integral part of his full name. He played the violin and founded a music band called Suz. He was also the personal teacher and mentor of the then-crown prince George of Serbia, with whom he remained friends for life. Their friendship became even stronger after George abdicated in favor of his younger brother and later king, Alexander I of Yugoslavia (see Karadjordjevic 2017).

Petrović's legacy (see Petrowitch 2019) included 11 PhD students and almost 900 PhD students of his PhD students and their students. He was a veteran of Balkan wars and the First World War. The Serbian and later Yugoslav army used his cryptography works for many years. In 1941, when the Second World War arrived in his country, he was mobilized as a reserve officer. After the Axis powers occupied his country, Petrović ended up as a prisoner of war in Nuremberg, at the age of 73. The former crown prince George made a plea to his aunt, the queen Elena of Italy, based on Petrović's illness, which secured his release from the prisoner-of-war camp, but he died soon afterward in Belgrade, his place of birth. A street in the downtown, an elementary school, a high school, and a fish restaurant in Belgrade are named after Mihailo Petrović Alas.

The main goal of this paper is to introduce modern readers to the results of Mihailo Petrović and to relate them to the results of modern theory of analytic differential equations. The main source of Petrović's results for us is his doctoral dissertation (Petrowitch 1894). It was written in French in 1894. (It was reprinted along with a translation in Serbian, edited by Academician Bogoljub Stanković in Volume 1 of Petrowitch (1999) in 1999.) The dissertation consists of two parts. The first part of the thesis mostly presents results related to the first-order algebraic ODEs, while the second part is related to algebraic ODEs of higher orders. At the beginning of the first part of the thesis, Petrović introduced a new method. We are going to call it *the method of Petrović polygons*. This method allows one to study the analytic properties of solutions of algebraic ODEs in a neighborhood of the nonsingular points of the equation. The method of Petrović polygons is a certain modification of the method of Newton–Puiseux, applicable to the study of solutions to algebraic equations. Further on, Petrović applies his method to study zeros and singularities of algebraic ODEs of the first order. He formulates and proves necessary and sufficient conditions for the nonexistence of movable zeros and poles of solutions (see Theorem 9).

Contrary to the necessary and sufficient conditions for the nonexistence of movable critical points of solutions of algebraic ODEs of the celebrated Theorem of Lazarus Fuchs (Fuchs 1885b) (provided below as Theorem 5), the conditions of Petrović's Theorem 9 do not require either the computation of solutions of the discriminant equation or to have the equation resolved with respect to the derivative. The conditions of Petrović's Theorem 9 can be checked easily and effectively using a simple construction of a geometric figure that corresponds to the given equation. The first part of the dissertation also contains the theorems which provide a classification of rational, first-order ODEs explicitly resolved with respect to the derivative and that have uniform (single-valued) solutions (see Petrović's Theorem 12). Later on, these results of Petrović were essentially improved by Axel Johannes Malmquist (Malquist 1913) (see Theorem 14). In addition, in the first part of the thesis, the class of binomial ODEs of the first order is studied, and the equations with solutions without movable singular points are described.

Petrović also characterized those binomial equations which possess uniform (single-valued) solutions. His results are very similar to those obtained by K. Yosida more than 30 years later (Yosida 1933) (see Theorem 19). The second part of the dissertation is devoted to the applications of the polygon method to the study of zeros and singularities of the algebraic higher-order ODEs. We will present some of the results related to higher-order ODEs a bit later (see Sect. 10 and Theorems 21 and 22).

Mihailo Petrović is one of the most respected and influential mathematicians in Serbia. Petrović's collected works in 15 volumes were published in 1999 (Petrowitch 1999). The year 2018 was the Year of Mihailo Petrović in Serbia on the occasion of his 150th anniversary. A monograph was published by the Serbian Academy of Sciences and Arts to celebrate his life and scientific results (see Petrowitch 2019). Nevertheless, it happened that none of his students and followers in Serbia continued to develop further the geometric ideas from his PhD thesis. The first two out of eleven of his direct PhD students, Dr. Mladen Berić and Dr. Sima Marković were involved in the topics closely related to their mentor's dissertation, (see Berić 1912; Marcovic 1913). However, the life produced unexpected turns: Dr. Berić was forced to leave the University of Belgrade at the beginning of 1920s because of issues related to his personal life. On the other hand, Dr. Marković became the first secretary of the Yugoslav Communist Party. When the Communist Party was prohibited by law in 1920, Marković immediately lost his position at the University of Belgrade. Later on he came into a dispute with the Party, and he was executed after a quick trial in Moscow in 1939; he was rehabilitated in 1958. These extraordinary circumstances can at least partially explain the lack of continuity within the school founded by Petrović in the field of analytic theory of differential equations, while his school maintained continuity in many other directions. These other directions were pursued by later students of Petrović, such as Pejović, Mitrinović, Kašanin, and Karamata. Serbian mathematicians who are active nowadays in the field of the analytic theory of differential equations [see, for example (Dragović and Shramchenko 2019; Dragović et al. 2018; Joshi and Radnović 2016, 2018)] neither methodologically nor according to their mathematical genealogy belong to the Petrović school. Certainly, some of the Petrović's results in that field were quite well known at the beginning of the twentieth century. Nevertheless, neither Golubev, who extensively used some other results from Petrović's thesis in his famous book (Golubev 1941), nor any other mathematician who used later analogous geometric methods in the study of the solutions of algebraic ODEs, ever quoted Petrović's foundational results in this field (see Cano 1993b; Grigor'ev and Singer 1991; Bruno 2004).

1.2 Henry Fine

A couple of years prior to Petrović, the American mathematician Henry Fine invented another modification of the Newton–Puiseux method for studying the formal solutions of algebraic ODEs (Fine 1889). Although Fine's construction is similar to Petrović's, they were not identical and the questions they were considering were very different. As we have said, it seems that Petrović and Fine did not know about each other's results. At the end of the twentieth century, the Fine method was developed further by J. Cano (Cano 1993a, b). As of today, contemporary methods based on different modifications of the Newton–Puiseux polygonal method allow wide classes of formal solutions to be computed for analytic differential equations and their systems (Bruno 2004), and to prove their convergence. They also allow to define Gevrey orders (the numbers that characterize the rates of growth of the coefficients) of formal series (Cano 1993b; Malgrange 1989; Ramis 1978; Sibuya 1990), which is needed to know for summation formal series.

Henry Burchard Fine (1858–1928) was Dean of Faculty and the first and only Dean of the departments of science at Princeton (O'Connor and Robertson n.d.). He was one of the few who did most to help Princeton develop from a college into a university. He made Princeton a leading center for mathematics and fostered the growth of creative work in other branches of science as well. Professor Oswald Veblen, in his memorial article (Veblen 1929) described Fine's contribution on the nationwide scale in his opening sentence by saying that "Dean Fine was one of the group of men who carried American mathematics forward from a state of approximate nullity to one verging on parity with the European nations."

In 1884, he traveled to Leipzig, Germany to attend lectures by Klein, Carl Neumann and others and to prepare his doctoral dissertation "On the singularities of curves of double curvature," a topic suggested by Study, and approved by Klein. Fine successfully defended his dissertation in May 1885 at the University of Leipzig. Upon his return from Germany, Fine was appointed assistant professor at Princeton where, despite great promise as a research mathematician, he moved very soon into other areas of academic life. He mainly devoted his time to teaching, administration, and the logical exposition of elementary mathematics. He wrote several books on elementary mathematics, including "Number system of algebra treated theoretically and historically," "A college algebra," "Coordinate geometry," and "Calculus."

His first research paper came out of his thesis, had the same title "On the singularities of curves of double curvature," and appeared in the *American Journal of Mathematics* in 1886. In the following year, he published a generalization of these results to n dimensions in the same journal. Two further papers "On the functions defined by differential equations with an extension of the Puiseux polygon construction to these equations," and "Singular solutions of ordinary differential equations" from 1889 and 1890 respectively, are of the utmost importance for our current presentation (see Fine 1889, 1890). His last research publication appeared in 1917: "On Newton's method of approximation."

Fine was one of the founders of the American Mathematical Society and served as the AMS president in 1911 and 1912.

Fine grew up between two major rivers, the St. Lawrence and the Mississippi river and was always astonished by them. Moreover, "He played the flute in the college orchestra, rowed on one of the crews, and served for 3 years as an editor of the Princetonian, where he began a life-long friendship with Woodrow Wilson (in) 1879, whom he succeeded as managing editor." (Leitch 1978) We quote a few more very illustrative parts from (Leitch 1978):

> "In 1903, shortly after he became president of the University, Wilson appointed Fine dean of the faculty, and Fine's energies were thereafter devoted chiefly to university administration. He worked shoulder to shoulder with Wilson in improving the curriculum and strengthening the faculty, and bore the onus of the student dismissals made inevitable by the raising of academic standards. In the controversies over the quad plan and the graduate college, Fine supported Wilson completely. After Wilson resigned to run for governor of New Jersey in 1910, Fine, as dean of the faculty, carried the chief burden of the university administration during an interregnum of 2 years; and when the trustees elected John Grier Hibben as Wilson's successor, Fine, who many had thought would receive the election, magnanimously pledged Hibben his wholehearted support. 'He was singularly free from petty prejudices and always had the courage of his convictions,' Hibben later recalled. 'Every word and act was absolutely in character, and he was completely dependable in every emergency.' …After his election as president of the United States, Wilson urged Fine to accept appointment as Ambassador to Germany and later as a member of the Federal Reserve Board, but Fine declined both appointments, saying quite simply that he preferred to remain at Princeton as a professor of mathematics. Fine also declined a call to the presidency of Johns Hopkins University and several to the presidency of Massachusetts Institute of Technology…. In the summer of 1928, he went to Europe, where he revisited old scenes and old friends, and recovered to some extent, in the distractions of travel, from the sorrows he had suffered in the recent death of his wife and the earlier deaths of two of his three children. Professor Veblen who talked with him soon after his return, reported later that Fine 'spoke with humorous appreciation of the change he had observed in the attitude of European mathematicians toward their American colleagues and with pride of the esteem in which he had found his own department to be held.'
>
> Tall and erect, Dean Fine was a familiar figure on his bicycle, which he rode to and from classes and used for long rides in the country. While riding his bicycle on the way to visit his brother at the Princeton Preparatory School late one December afternoon, he was struck from behind by an automobile whose driver had failed to see him in the uncertain light of dusk. He died the next morning …"

The Mathematics Department of the Princeton University is housed in the Fine Hall, the building named after its first chairman.

2 Historic context: the 1880s

Across the entire nineteenth century, there was a significant and constant attention to the study of analytic differential equations and their classification. The French school, from August Cauchy, Charles Briot, Jaun-Claude Bouquet, Joseph Liouville, Émile Picard, Charles Hermite, to Henri Poincaré, and Paul Painlevé made tremendous contributions. Of course, among those who gave a key contribution to the analytic theory of differential equations were scientists from other countries as well, such as Carl Friedrich Gauss, Bernhard Riemann, Carl Weierstrass, Lazarus Fuchs, and Sofya Vasilyevna Kovalevskaya. Nevertheless, the center of the attention to analytic theory of differential equations was undoubtedly in France especially in 1880s, when French mathematics contributed greatly to further development of complex analysis and to applications of its methods to the study of differential equations. So-called algebraic differential equations and systems of such equations occupied a special place.

Let us recall that an algebraic ODE has the following form

$$f(x, y, y', \dots, y^{(n)}) = 0, \tag{1}$$

$$f = \sum_{i=1}^{j} \varphi_i(x)\, y^{m_{0i}}\, y'^{m_{1i}} \cdots y^{(n)m_{ni}}, \tag{2}$$

$j \in \mathbb{N}$, where $m_{0i}, \dots, m_{ni} \in \mathbb{Z}_+$, and φ_i are algebraic or analytic functions.

Let us also recall the main notions of the analytic theory of differential equations [see for example (Golubev 1941)], which are necessary to formulate the results of Petrović and his predecessors. Here, it is typically assumed that the focus is on complex functions of a complex variable.

The points where a given solution of Eq. (1) is not analytic or is not defined are called *the singular points of the solution*. Simple examples of singular points of a solution are its poles, i.e., the points such that in their punctured neighborhoods the solution is presented by Laurent series with finite principal parts. Paul Painlevé suggested classifying isolated singular points of a function according to the number of values it takes while going around the singular points. This led to the division of singular points into *critical* and *noncritical* points. If the function changes its values while going around a singular point, the singular point is *critical*. Examples: the point $x = 0$ is a critical point for the function $\sqrt{x}$ and for the function $\ln x$. If, on the contrary, a function does not change its value while going around a singular point, the singular point is called *noncritical*. Examples: the point $x = 0$ is a noncritical singular point for functions $1/x$ and $e^{1/x}$. If going around a critical singular point $x = a$, the function takes finitely many values and has a limit in that point (as $x \to a$ inside any sector with the vertex in a and with a finite angle), then such a point is either an algebraic critical point or a critical pole. It is called *an algebraic critical point* if in its neighborhood the function has an expansion of the form

$$y = c_0 + c_1(x - a)^{1/s} + c_2(x - a)^{2/s} + \cdots$$

(i.e., the limit is finite). The point $x = a$ is called *a critical pole*, if in a punctured neighborhood of the point, the function has an expansion of the form:

$$y = c_{-m}(x-a)^{-m/s} + \cdots + c_{-1}(x-a)^{-1/s} + c_0 + c_1(x-a)^{1/s} + \cdots, \qquad s \geq 2$$

(i.e., the limit is infinite). The critical algebraic points, poles, and critical poles form the set of *algebraic singular points*.

In turn, as seen from the above definitions, *nonalgebraic critical points* are of two types: the first ones are such that the function takes an infinitely many values when going around a critical point (*transcendental points*; for example, the point $x = 0$ for the function $\ln x$), the second ones are those at which the function does not have a limit (*essential singular points*; for example, the point $x = 0$ for the function $\sin(1/\sqrt{x})$).

Here we do not consider nonisolated singular points of functions. The question of the classification of such singular points is more delicate. Some examples of the application of the above classification of isolated singular points to studying singular points forming *singular curves* can be found in (Golubev 1941, Chap. I, Sect. 7).

It is well known that for a linear ODE the singular points of the solutions form a subset of the set of singular points of the coefficients of the equation. For nonlinear ODEs, the points where the coefficients $\varphi_i(x)$ of Eq. (1) are zero or undefined, as a rule, are singular points of its solutions. We will call such points *the singular points of the equation*. But it is important to stress that not only the singular points of a nonlinear ODE can be singular points of its solutions. This property of nonlinear ODEs led L. Fuchs (Fuchs 1885a, b) to divide the singular points of the solutions of a nonlinear ODE into movable and fixed singular points. *A fixed singular point of a solution* of Eq. (1) is a singular point whose position does not depend on initial data that determine the solution, i.e., such a singular point is a common singular point for n-parameter family of solutions, or in yet another words, the common singular point for the general solution (the general solution is also called the integral) of the equation. Example: the point $x = 0$ is a fixed singular point of the general solution $y = (C - \ln x)^2$, where C is an arbitrary constant corresponding to the initial data, for the equation $x^2y'^2 - 4y = 0$; the point $x = 0$ is a singular point of the last equation. *A movable singular point of a solution* is such an its singular point, whose position depends of the constants of integration. For example, $x = -C$ where C is an arbitrary constant, is a movable singular point of the solution $y = 1/(x + C)$ of the equation $y' + y^2 = 0$. The last equation does not have singular points.

For nonlinear ODEs of the first order, as for linear ODEs of any order, it can be shown that fixed singular points are always singular points of the equation, see for example Chap. I, Sect. 8 of (Golubev 1941).

As it has already been said, Petrović was concerned about the conditions on algebraic ODEs, under which their solutions would or would not have movable singular points. He was also studying the nature of the solutions, whether they are single-valued, rational, or elliptic functions. The importance of movable critical points lies in the fact that their presence prevents the possibility for the given differential equation of constructing a unique Riemann surface, which could serve as the domain for all the solutions of the given equation as uniform (single-valued) functions.

Now we are going to list the results that had been widely known at the moment when Petrović arrived in Paris in 1889 to pursue his graduate education and which he systematically used in his doctoral dissertation.

Theorem 1 [Little Picard theorem (Picard 1879, 1880; Solomencev 1984)] *Outside the image of any nonconstant entire function there could be at most one complex number.*

Theorem 2 [Big Picard theorem (Picard 1879, 1880; Solomencev 1984), meromorphic version] *Let $\mathcal{M}$ be a Riemann surface and $\mathcal{S}$ the Riemann sphere, and $w \in \mathcal{M}$ an arbitrary point. Let $f : \mathcal{M}\backslash\{w\} \to \mathcal{S}$ be a holomorphic function with an essential singularity at the point w. Then, all points on the sphere $\mathcal{S}$ except at most two have infinitely many inverse images.*

Let us mention some important results about the first-order algebraic differential equations, used frequently by Petrović.

Theorem 3 [Hermite (Golubev 1941; Hermite 1873)] *Given a polynomial P. If solutions of the equation $P(y, y') = 0$ do not have movable critical points, then the genus of the curve $P(x, y) = 0$ is equal either 0 or 1. In such a case, the solutions are either rational functions or could be rationally expressed through exponential and elliptic functions.*

Theorem 4 [Poincaré, and Fuchs, 1884–5 (Fuchs 1885b; Golubev 1941; Poincaré 1885)] *Algebraic differential equations of the first order without movable critical points reduce to linear, Riccati [see (29)], or Weierstrass equations [see (40)].*

In 1884 Lazarus Fuchs provided necessary conditions for the absence of movable critical points of solutions of algebraic ODEs of the first order (Fuchs 1885b). In fact, his theorem provided also sufficient conditions for the absence of movable *algebraic* critical points of solutions.[1] But a result of Painlevé in 1887 allowed one to strengthen the assertion of Fuchs's Theorem into sufficient conditions for the absence of *any* movable critical points. (This result of Painlevé is also going to be stated a paragraph below as Theorem 6.) First, we formulate the corresponding Theorem of Fuchs, which is going to include the sufficient part as well.

Let

$$F(x, y, y') = A_0(x, y)(y')^s + A_1(x, y)(y')^{s-1} + \cdots + A_s(x, y), \tag{3}$$

where $A_0, \ldots, A_s$ are polynomials in y and analytic in x. *The discriminant equation* $D(x, y) = 0$ is the result of the elimination of y' from the equations $F(x, y, y') = 0$ and $\dfrac{\partial F(x, y, y')}{\partial y'} = 0$.

[1] Let us briefly mention basic biographic data about Lazarus Fuchs. He was born in 1833 in Moschina, the Grand Duchy of Posen of Kingdom of Prussia, nowadays Poland. He worked on his PhD in Berlin University with Kummer as his advisor, from 1854 till 1858, when he defended a thesis on the lines of curvature on surfaces. His interest in differential equations came from his association with Weierstrass (Gray 1984). In 1882 he returned to Berlin where he got a position of a full professor of the Berlin University. He was elected a member of Berlin Academy in 1884. From 1892 till his death, Fuchs served as the Editor-in-Chief of "Journal für die reine und angewandte Mathematik" (Crelle's journal). He died in Berlin in 1902.

Theorem 5 [Fuchs, 1884–5 (Fuchs 1885b; Golubev 1941)] *The solutions of the equation $F(x, y, y') = 0$, where F is defined in (3), have no movable critical points if and only if:*

- *The coefficient $A_0(x, y)$ does not depend on y.*
- *The degree of each of the polynomials $A_k(x, y)$ with respect to y does not exceed $2k$.*
- *The solutions $\phi(x)$ of the discriminant equation $D(x, y) = 0$ are integrals of the given equation.*
- *For each fixed x_0, the Puiseux expansion of y' with respect to y in a neighborhood of the point $y_0 = \phi(x_0)$ has the form*

$$y' = \phi'(x_0) + b_k(x_0)(y - y_0)^{k/m} + b_{k+1}(x_0)(y - y_0)^{(k+1)/m} + \cdots$$

with $k \geq m - 1$.

Let us give a proof of the "only if" part of Fuchs's Theorem [one can see a proof in Fuchs (1885b) or in Golubev (1941)].

First, we suppose that $A_0(x, y)$ contains y. We take an arbitrary value x_0 which differs from singular points of the equation and a value y_0 that satisfies $A_0(x_0, y_0) = 0$. As known from the course of analytic theory of differential equations [see, for example, Sect. 1, Chap. II in Golubev (1941)] near such points (x_0, y_0) the equation $F(x, y, y') = 0$ is resolved with respect to y' in the form

$$y' = \frac{1}{c_l(y - y_0)^l + c_{l+1}(y - y_0)^{l+1} + \cdots} + \alpha, \quad \text{if} \ \ D(x_0, y_0) \neq 0, \tag{4}$$

where $c_j \in \mathbb{C}\{x - x_0\}$, $\alpha \in \mathbb{C}$, or in the form

$$y' = \frac{1}{c_l(y - y_0)^{l/m} + c_{l+1}(y - y_0)^{(l+1)/m} + \cdots} + \alpha, \quad \text{if} \ \ D(x_0, y_0) = 0, \tag{5}$$

where $c_j \in \mathbb{C}\{x - x_0\}$, $\alpha \in \mathbb{C}$. In Eqs. (4) and (5) the coefficient $c_l(x_0) \neq 0$ due to the point $x = x_0$ is not a singular point of the equation.

We can apply the lemma below in Eq. (4) and, after the change of variable $(y - y_0)^{1/m} = u$, in Eq. (5).

Lemma 1 [Lemma about a critical point (Golubev 1941)] *Let $f(x_0, y_0) = \infty$ and $1/f(x, y)$ be holomorphic in a neighborhood of the point (x_0, y_0). Then, x_0 is a movable critical algebraic point of the equation $y' = f(x, y)$. More precisely, the integral determined by the initial data (x_0, y_0) has an expansion*

$$y = y_0 + a_1(x - x_0)^{1/k} + a_2(x - x_0)^{2/k} + \cdots$$

near the point x_0, where $k = 1 + \mathrm{ord}_{y=y_0}(1/f(x_0, y))$.

According to this lemma, both equations have movable critical points of their solutions. Hence, if a solution of the equation $F(x, y, y') = 0$ has only fixed critical points then the coefficient $A_0(x, y)$ does not contain y.

We now prove the necessity of the first condition of the Fuchs theorem.

Further we write the equation $F(x, y, y') = 0$ in the form

$$(y')^s + A_1(x, y)(y')^{s-1} + \cdots + A_s(x, y) = 0, \tag{6}$$

where $A_1, \ldots, A_s$ are polynomials in y and analytic in x.

Let p_j be the degree of the polynomial A_j with respect to y. The coefficients

$$A_j(x, y) = A_j(x, 1/w) = \frac{B_j(x, w)}{w^{p_j}},$$

where $B_j(x, w)$ is a polynomial with respect to w. So by means of the transformation $y = 1/w$, Eq. (6) reduces to the equation

$$(w')^s - \frac{B_1(x, w)}{w^{p_1-2}}(w')^{s-1} + \frac{B_2(x, w)}{w^{p_2-4}}(w')^{s-2} - \cdots + (-1)^s \frac{B_s(x, w)}{w^{p_s-2s}} = 0. \tag{7}$$

As a solution of Eq. (7) has no movable critical points, this equation does not contain w in the denominators of the coefficients $\dfrac{B_j(x, w)}{w^{p_j-2j}}$. Thus, we prove the necessity of the second condition of the Fuchs theorem, that the equation $F(x, y, y') = 0$ must be of the form (6) with the polynomials A_j of degree $p_j \leq 2j$.

Equation (6) with the polynomials A_j of degree $p_j \leq 2j$ can be resolved with respect to y' in the form

$$y' = s_0 + b_k(y - v)^{k/m} + b_{k+1}(y - v)^{(k+1)/m} + \cdots, \tag{8}$$

where $F(x, v, s_0) = 0$, v is an arbitrary function with respect to x, the functions b_j are holomorphic near x_0, $k,\ m \in \mathbb{N}$, $b_k \neq 0$. If $D(x, v) \neq 0$ then $m = 1$. If $D(x, v) = 0$ then $m > 1$.

Making the change of variable $y - v = w^m$ (whence $y' = mw^{m-1}w' + v'$) in Eq. (8), we get the equation

$$w' = \frac{s_0 - v' + b_k w^k + b_{k+1} w^{k+1} + \cdots}{m w^{m-1}}. \tag{9}$$

First, we consider the case $s_0 \not\equiv v'$. If $s_0(x_0) - v'(x_0) \neq 0$, then according to Lemma 1 Eq. (9) with the initial data $w(x_0) = 0$ has the solution

$$w = a_1(x - x_0)^{1/m} + a_2(x - x_0)^{2/m} + \cdots, \quad a_1 \neq 0.$$

If $s_0(x_0) = v'(x_0)$ (say, $s_0 - v' = (x - x_0)^n \varphi(x)$, $\varphi(x_0) \neq 0$), then changing the variable $w = (x - x_0)^n u$ in Eq. (9) and applying the ideas above in the modified equation we get that in this case Eq. (9) has solutions

$$w = a_l (x - x_0)^{l/m} + a_{l+1} (x - x_0)^{(l+1)/m} + \cdots, \quad a_l \neq 0.$$

Taking into account the change of variable $(y - v)^{1/m} = w$ and Eq. (8), we get

$$y' = s_0 + b_k a_1^k (x - x_0)^{k/m} + \cdots .$$

Integrating the last equality, we find that if k is not divided by m then the solution y has an algebraic critical point x_0.

So for the absence of critical movable points, it is necessary for k/m to be integer. In this way, we can prove that all other terms of the series in the right part of Eq. (8) contain $y - v$ in integer powers. Therefore, when Eq. (8) has fractional power exponents of $y - v$ (that is, when $D(x, v) = 0$) for $s_0 - v' \not\equiv 0$, Eq. (6) has integrals with movable algebraic critical points. This means that a solution v of the discriminant equation $D(x, v) = 0$ should satisfy $v' \equiv s_0$, that is, $F(x, v, v') = 0$.

Hence, we prove the necessity of the third condition of Fuchs's theorem; namely, if Eq. (6) has only fixed critical points, then all solutions of the discriminant equation are solutions of this equation too.

In the last part of the proof, we have that $s_0 = v'$. Equation (9) now has the form

$$w' = \frac{b_k w^k + b_{k+1} w^{k+1} + \cdots}{m w^{m-1}}. \tag{10}$$

If $k \geq m - 1$, then Eq. (10) and also Eq. (8) have holomorphic integral with initial data $w(x_0) = 0$.

In the case $k < m - 1$, Eq. (10) can be written in the form

$$w' = \frac{b_k + b_{k+1} w + \cdots}{m w^{m-k-1}}. \tag{11}$$

Its integral that vanishes at the point $x = x_0$ has the form

$$w = a_1 (x - x_0)^{1/(m-k)} + a_2 (x - x_0)^{2/(m-k)} + \cdots .$$

From this for the right part of Eq. (8), we get an expansion

$$y' = s_0 + b_k a_1^k (x - x_0)^{k/(m-k)} + \cdots, \quad b_k(x_0),\ a_1(x_0) \neq 0.$$

Integrating the last equation, we get that in this case a solution of Eq. (8) has an algebraic critical movable point. Thus, we prove the last condition of Fuchs's theorem: in the expansion of y' in fractional powers of $y - \phi(x)$, where $\phi(x)$ is a solution of the equation $D(x, y) = 0$, one has

$$y' = \phi'(x) + b_k (y - \phi(x))^{k/m} + b_{k+1} (y - \phi(x))^{(k+1)/m} + \cdots$$

with $k \geq m - 1$. The proof of the "only if" part of Fuchs's theorem is finished.

As was mentioned in Introduction, in his doctoral dissertation (Painlevé 1887) in 1887 Painlevé formulated and proved a remarkable result, which inspired many to further investigations of solutions of algebraic ODEs. Painlevé defended his dissertation on 10th of June of 1887 in front of the committee chaired by Hermite, with Appell and Picard as the members. The thesis was dedicated to Picard, the mentor of Painlevé. The dissertation was published as a journal paper a year later, see (Painlevé 1888).[2]

The central result of the dissertation is the following:

Theorem 6 (Painlevé 1887; Golubev 1941; Painlevé 1887, 1888) *Differential equations of the first order, algebraic with respect to a unknown function and its derivative, do not possess movable nonalgebraic singularities.*

The proof uses the following result of Cauchy about the differential equation

$$w' = f(z, w),$$

where f is holomorphic within the disks $|w| \leq \rho > 0$ and $|z| \leq r > 0$. For any ρ_1, r_1 such that $0 < \rho_1 < \rho$ and $0 < r_1 < r$, there exists M such that $|f(z, w)| < M$ for all w, z with $|w| = \rho_1$ and $|z| = r_1$. Then, there exists a solution of the above differential equation $w_1 = w_1(z)$, such that $w_1(0) = 0$ and w_1 is holomorphic within the disk

$$|z| < r_1(1 - \exp(-\rho_1/(2Mr_1))). \tag{12}$$

The second ingredient of the proof is Painlevé's Theorem 3, from p. 36 of the Painlevé dissertation (Painlevé 1887), which states: *If for all points z_0 of a region S, the function $f(z_0, u)$ has at most a discrete set of essential points $a_1, a_2, \ldots, a_m, \ldots$ with the coordinates depending on z_0 analytically, the root $u(z)$ of the equation $f(z, u) = 0$ is single-valued (or n-valued) in a region S' with the boundary σ, a subset of the interior of S, with a continuation across σ, with the poles (or critical algebraic points) as the singularities in S'.*

Then, on p. 41 of (Painlevé 1887), Painlevé formulated the following statement: *Given a differential equation*

$$\frac{\mathrm{d}u}{\mathrm{d}z} = f(z, u),$$

where f is a single-valued function when z varies in S and u in the complex plain. If for all points z_0 of a region S, the function $f(z_0, u)$ has at most a discrete set of points where it is not defined, with the coordinates depending on z_0 analytically, all integrals $u(z)$ are single-valued (or n-valued) in a region S' with the boundary σ, a subset of

[2] Paul Painlevé was born in Paris in 1863. He graduated from the École Normale in 1877 and went on to become a full professor of the École Normale and Sorbonne. He was an elected member of the French Academy since 1900. After 1910, and election to the national parliament, Painlevé shifted his focus from science to politics. He was a minister of several French governments, including the post of the Minster of War during the World War I. Painlevé served as the Prime Minister of France two times: September 12–November 16, 1917, and April 17–November 28, 1925. Painlevé died in Paris in 1933.

the interior of S, with a continuation across σ, with the poles (or critical algebraic points) as the singularities in S'.

As an important class of examples, Painlevé gives the equations of the form:

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{P(w,z)}{Q(w,z)}, \tag{13}$$

where P, Q are polynomials with respect to w. We will provide more details of the proof of the Painlevé theorem in this case, following (Golubev 1941). First, let M_1 denote the set of points which are singularities in z of the coefficients of P, Q as polynomials in w and the common zeros of all the coefficients of Q. Let M_2 denote the set of z points for which $P = 0$ and $Q = 0$ has common solutions. Similarly, changing the variable $w_1 = 1/w$ we get the equation

$$\frac{\mathrm{d}w_1}{\mathrm{d}z} = \frac{P_1(w_1,z)}{Q_1(w_1,z)}.$$

Let M_3 denote the common zeros of $P_1(0,z) = 0$ and $Q_1(0,z) = 0$. Changing the variable $z_1 = 1/z$, we get the equation

$$\frac{\mathrm{d}w}{\mathrm{d}z_1} = \frac{P_2(w,z_1)}{Q_2(w,z_1)}.$$

If $z_1 = 0$ contributes to one of the sets M_1, M_2, or M_3, we add it as well and then the union of these sets we denote as $\mathcal{M}$. Let us assume that a point z_0 outside $\mathcal{M}$ is a singular nonalgebraic point of a solution $w = w(z)$ of Eq. (13). There are two options: either (i) z_0 is a transcendental point of $w(z)$ or (ii) z_0 is an essential singularity of $w(z)$.

Let us consider first (i). Then, $w(z)$ either (ia) has a finite limit which can either (ia1) be a zero of the equation

$$Q(w,z_0) = 0$$

or (ia2) is not a zero of that equation, or (ib) tends to infinity. In (ia2) $Q(w_0,z_0) \neq 0$ and according to the Cauchy Theorem, there is a unique integral $w_0(z)$ determined by the initial condition (w_0,z_0) and it is analytic in the neigborhood of z_0. Thus, $w(z)$ coincides with $w_0(z)$ and is analytic at z_0.

In (ia1) case $Q(w_0,z_0) = 0$ and thus $P(w_0,z_0) \neq 0$, since z_0 is not in $\mathcal{M}$. According to Lemma 1 about a critical point, z_0 is a critical algebraic point of the solution $w(z)$ of Eq. (13).

The case (ib) can be treated similarly, by changing variables to $w_1 = 1/w$.

Now, let us consider (ii): we assume that z_0 is an essential singularity of the integral $w(z)$. Denote by W_j the zeros of the polynomial $Q(w,z_0) = 0$. Let D denote the region of indefiniteness of the integral $w(z)$ at z_0, as the set of values the integral attains when the argument approaches the essential singularity. Let A be the point of D closest to the origin (which can be the origin itself). Let $\bar{w}$ be an arbitrary point of D and d_j

the distances from $\bar{w}$ to W_j. There are finitely many d_js, they are all strictly positive, and thus, their minimum d is strictly positive. Take any ρ such that $0 < \rho < d$ and construct circles C_j centered at W_j with radius ρ. Also take R, $R > |A|$ and construct circle Γ centered at the origin with radius R. Let D_0 be the part of D inside Γ and outside all C_j. Construct circles C'_j concentric with C_j with radius $\rho/3$. Select r small enough that for all z from the disk S centered at z_0 with radius r, the solutions of the equation $Q(w, z) = 0$ are inside C'_j. Let us also construct circles C''_j concentric with C_j with radius $\rho/2$ and circles Γ', Γ'' concentric with Γ with radii $R + \rho$ and $R + \rho/2$, respectively. Denote by D_1 the part of D bounded by Γ'' and all C''_j. There exists $M > 0$ such that

$$\left|\frac{P(w, z)}{Q(w, z)}\right| < M \tag{14}$$

on $S \times D_1$. Thus, for all w_1 in D_0 and z_1 in the disk S_1 concentric with S and radius $r/2$, the disk C of radius $\rho/2$ and center w_1 and the disk γ with radius $r/2$ and center z_1 have the property that the function $P(w, z)/Q(w, z)$ is holomorphic on $\gamma \times C$ and satisfies the inequality (14).

From the Cauchy's result stated above, we know that the integral of Eq. (13), defined by the initial condition (w_1, z_1) is holomorphic within the disk centered at z_1 with radius

$$\lambda = \frac{r}{2}(1 - \exp(-\rho/(2Mr)))$$

according to (12).

Now we can conclude the proof of the Painlevé Theorem in the case of the equations of the form (13). Indeed, we can construct a disk σ, centered at z_0 with radius λ. Then, within σ there exists a point z_1 such that $w(z_1) = w_1$ belongs to D_0. Using the facts derived above, we see that the integral, analytic at z_1 and defined with (w_1, z_1) as the initial conditions, is also holomorphic at z_0, since

$$|z_1 - z_0| < \lambda.$$

The contradiction just obtained concludes the proof.

Let us observe that, in the process of proving Theorem 6, Painlevé also verified that the conditions of Fuchs's Theorem are also sufficient, as stated above, see Theorem 5. This remark about Fuchs's Theorem and the related Poincaré result is contained in Sect. 8 of Chapter II of the First Part of the dissertation, on p. 57, see (Painlevé 1887, 1888).

Let us conclude this section by mentioning two important papers from 1889, the year when Petrović came to Paris (Kowalevski 1889a; Picard 1889). The Kovalevskaya[3] paper (Kowalevski 1889a) has become one of the most celebrated papers in the history of mathematics. Kovalevskaya successfully developed some of the above ideas and concepts further and applied them to the study of a system of algebraic equations, the so-called Euler–Poisson equations of motion of a heavy rigid body rotating around a

3 Kovalevskaya's name on her published papers is given as Kowalevski but she is often called Kovalevskaya in the current literature, following English transcription of her Russian family name.

fixed point. He investigated the possibility that such a system has a general solution with poles as only possible movable singularities. In other words, she was looking for the cases where the general solution as a function of complex time is single-valued. As a result, she found what became to be known as the Kovalevskaya top. She integrated the equations of motion in that case explicitly using genus 2 theta functions and proved that her case indeed satisfied the initial analytic assumption. For these results, Kovalevskaya received the famous Prix Bordin of the French Academy of Science, which was augmented from 3000 franks to 5000 franks. For the second paper on the rotation of a rigid body (Kowalevski 1889b), Kovalevskaya got a prize of the Swedish Royal Academy. Together with the later work of Painlevé and his students on the second-order equations (see Sect. 10.1), these ideas of Kovalevskaya laid the foundations of the so-called Kovalevskaya–Painlevé analysis, which is also known as a test of integrability. More about Kovalevskaya one can find, for example, in Cooke (1984). With this we conclude a brief description of the scientific atmosphere in which Petrović started the work on his PhD thesis.

3 Petrović polygons

The key assumption of Petrović's main construction is that a given point x_0 is a nonsingular point of Eq. (1). In such a case, to each term in the sum (2) with the coefficient $\varphi_i(x_0) = \text{const} \neq 0$ one assigns a point in the MON plane, according to the following formulae

$$Q_i = (M_i,\ N_i), \quad M_i = m_{0i} + \cdots + m_{ni}, \quad N_i = m_{1i} + 2m_{2i} + \cdots + nm_{ni}.$$

It should be noticed that one and the same point in the plane can correspond to one or more terms in the sum (2). Let

$$S = \{Q_i, i = 1 \dots s,\ s \leq j\}$$

be the set of all points obtained in such a assignment. We can draw these points in the MON plane. In the Petrović dissertation, the set S is extended by two segments T_l and T_r which are orthogonal to the axis OM and connect the leftmost and the rightmost point of the set S, respectively, with their projections on the OM axis. The boundary of the convex hull of the set $S \bigcup T_l \bigcup T_r$ is a polygon. Both that polygon and the concave part of the boundary of the convex hull of the set S will be called *the Petrović polygon.* We will denote the Petrović polygon by $\mathcal{N}$.

Let us point out that neither the vertical sides nor the horizontal side which lies at the OM-axis play any role in the applications of Petrović method. They do not correspond to any solution of the equation and in that sense their inclusion can be treated as artificial and unnecessary. However, they were included in Petrović's original definition *not only* for purely formal or aesthetic reasons. They are indeed needed for methodological reasons as well, to allow a precise and elegant derivation of the main properties of the Petrović polygons. Thus, in our considerations, we will at the beginning, use the polygon as Petrović did, but later on in applications and computations, for simplicity we will not add these vertical segments anymore and we will operate with the concave part of the convex hull of the set S only. In our further deliberations, the Petrović polygon is an *irregular zigzag line*. It is important to stress that, nevertheless, the conclusions from the considerations of the zigzag line are identical to those coming from the entire polygon. [Let us recall that Newton himself used irregular zigzag lines, not polygons (Newton 1670), see also Brieskorn and Knörrer (1986) and Ghys (2017).]

Let Eq. (1) has a solution $y = y(x)$ which, in a neighborhood of a point $x = x_0$, where x_0 is an arbitrary constant distinct from the singular points of the equation, can be presented in the form of a power series with fractional exponents, i.e., in a form of the Puiseux series:

$$y = \sum_{k=0}^{\infty} c_k (x - x_0)^{\lambda + k\sigma}, \tag{15}$$

$\lambda = l\sigma$, $\sigma \in \mathbb{Q}$, $\sigma > 0$, $l \in \mathbb{Z}$, $c_k \in \mathbb{C}$, $c_0 \neq 0$. The main idea of Petrović was to use the polygon $\mathcal{N}$ to keep those terms $\varphi_i(x_0)\, y^{m_{0i}}\, y'^{m_{1i}} \ldots y^{(n)m_{ni}}$ of Eq. (1) (called *the leading terms*) which form an equation (called *approximate equation*) having $\chi = c_0(x - x_0)^{\lambda}$ as its solution. Thus, χ would be the asymptotic expansion of the solution (15) in a neighborhood of the point $x = x_0$. In that way we would effectively find such an asymptotic expansion. As we see, these ideas of Petrović resemble the main idea behind the Newton–Puiseux polygons in finding the asymptotic expansions of solutions of algebraic equations. We are going to describe the ideas of Petrović in detail, paying attention to the specifics of the case of ODEs.

Let us plug the formal series (15) into Eq. (1). The solution (15) and its derivatives can be rewritten in the form

$$\begin{aligned}
y \;\; &= (c_0 + o(1))\; (x - x_0)^{\lambda},\\
y' \;\; &= (c_0\, \lambda + o(1))\; (x - x_0)^{\lambda - 1},\\
&\ldots\\
y^{(n)} &= (c_0\, \lambda \cdot \ldots \cdot (\lambda - n + 1) + o(1))\; (x - x_0)^{\lambda - n}.
\end{aligned}$$

Their powers have the following form:

$$\begin{aligned}
y^{m_{0i}} \quad &= \left(c_0^{m_{0i}} + o(1)\right)\; (x - x_0)^{\lambda m_{0i}},\\
y'^{m_{1i}} \quad &= \left(c_0^{m_{1i}}\; \lambda^{m_{1i}} + o(1)\right)\; (x - x_0)^{\lambda m_{1i} - m_{1i}},\\
&\ldots\\
y^{(n)m_{ni}} &= \left(c_0^{m_{ni}}\; (\lambda \cdot \ldots \cdot (\lambda - n + 1))^{m_{ni}} + o(1)\right)\; (x - x_0)^{\lambda m_{ni} - n m_{ni}}.
\end{aligned}$$

For every term of $\varphi_i(x)\, y^{m_{0i}}\, y'^{m_{1i}} \ldots y^{(n)m_{ni}}$ for the formal series (15), we get the following expressions:

$$\begin{aligned}
\varphi_i(x)\, y^{m_{0i}}\, y'^{m_{1i}} \ldots y^{(n)m_{ni}} &= \varphi_i(x)\, (A_i(\lambda)\, c_0^{M_i} + o(1))\, (x - x_0)^{\lambda M_i - N_i}\\
&= (A_i(\lambda)\, c_0^{M_i}\, \varphi_i(x_0) + o(1))\, (x - x_0)^{\langle R,\, Q_i \rangle},
\end{aligned}$$

where

$$A_i(\lambda) = \lambda^{\gamma_{1i}}(\lambda-1)^{\gamma_{2i}}\dots(\lambda-n+1)^{\gamma_{ni}}, \tag{16}$$

$$\begin{aligned}\gamma_{1i} &= m_{1i} + m_{2i} + \dots + m_{ni},\\ \gamma_{2i} &= m_{2i} + m_{3i} + \dots + m_{ni},\\ &\dots\\ \gamma_{ni} &= m_{ni}.\end{aligned}$$

We will use the vector $R = (\lambda, -1)$ and denote as $\langle\rangle$ the dot product.

Thus, by substituting the series (15) into (1) we get the formula

$$(C_0 + o(1))(x - x_0)^{\gamma_\lambda} = 0, \tag{17}$$

where the coefficient is given by

$$C_0 = \sum_{i:\,\langle R, Q_i\rangle = \gamma_\lambda} A_i(\lambda)\, c_0^{M_i}\varphi_i(x_0),$$

where

$$\gamma_\lambda = \min_{i=1,\dots,s}\langle R, Q_i\rangle.$$

By a further analysis, formula (17) can be rewritten in a more precise form as

$$C_k(c_0,\dots,c_k)(x-x_0)^{\gamma_\lambda + k\sigma} = 0. \tag{18}$$

Here, C_k are polynomials of their arguments. Since the series (15) satisfies Eq. (1), Eq. (18) is satisfied identically, meaning that the coefficients C_k in (18) should all be zero. Consequently, if the solution of Eq. (1) exists in the form (15), then by solving the equations $C_k(c_0,\dots,c_k) = 0$, $k \in \mathbb{Z}_+$ we are getting the coefficients c_k for which the series (15) gives the solution (1). The *leading terms* of Eq. (1) are those

$$\varphi_i(x_0)\, y^{m_{0i}} \cdots y^{(n)m_{ni}}$$

corresponding to the points Q_i, for which the dot product is *minimal*:

$$\langle R,\ Q_i\rangle = \langle(\lambda, -1),\ (M_i, N_i)\rangle = \lambda M_i - N_i = \gamma_\lambda.$$

At the beginning of Chapter 1 of Part 1 of his thesis, Petrović proved the following statement.

Proposition 1 *If $x = x_0$ is a nonsingular point of Eq.* (1)*, and if $y = y(x)$ is a solution of the equation with an expansion* (15) *with initial conditions $y(x_0) = 0$ or $y(x_0) = \infty$, then the first term $c_0(x-x_0)^\lambda$ of the expansion into a Puiseux series of*

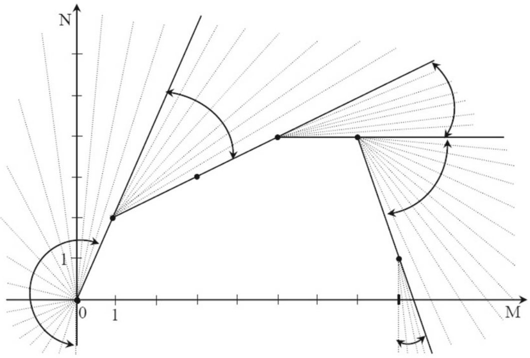

Fig. 1 Construction of a Petrović polygon

the solution (15) *is a solution of the approximate equation, which corresponds either to a vertex or to a slanted edge of the polygon* $\mathcal{N}$ *of Eq.* (1).

Consider the line $\lambda M - N = \alpha$, $\alpha \in \mathbb{R}$, in the MON plane. When the number α increases, the line $\lambda M - N = \alpha$ moves along the vector R, directed inside the polygon. Thus, for the points Q_i inside the polygon $\mathcal{N}$ there is the relation for the dot product

$$\langle R, Q_i \rangle > \gamma_\lambda,$$

and its minimum is attained at the boundary of the polygon: for some points Q_i lying at the boundary of the polygon there is the following relation for the dot product

$$\langle R, Q_i \rangle = \gamma_\lambda.$$

A point Q_i, such that $\langle R, Q_i \rangle = \gamma_\lambda$, either is a vertex of the polygon or lies on one of its edges. Thus, the leading term of Eq. (1) corresponds either to a point which is a vertex of the polygon or to the points lying on an edge of the polygon. It follows from the above constructions that there are finitely many values of λ, for which the minimum of the dot product γ_λ is attained on the edges, while there are infinitely many (continuum many) values of λ, for which the minimum of γ_λ is attained on vertices. The upper half-plane of the plane MON can be decomposed on rays with the angular coefficients λ and the open angular sectors, containing rays with angular coefficients λ, where λ are all the values between the angular coefficients of the edges meeting at the given vertex, see Fig. 1.

Obviously, for any value of λ there is a ray with the angular coefficient λ, which either corresponds to an edge or to a vertex. Thus, if Eq. (1) has a solution $y(x)$, represented in a neighborhood of the point $x = x_0$ in the form of the series (15), then this solution has to correspond either to a vertex or to an edge of the polygon. Moreover, if $\lambda < 0$, which means that $x = x_0$ is a pole of $y(x)$, then these values correspond either to edges or vertices of *the right part* of the polygon $\mathcal{N}$. Similarly, if $\lambda > 0$, i.e., $x = x_0$ is a zero of the solution $y(x)$, then corresponding edges and vertices belong to *the left part* of the polygon $\mathcal{N}$.

In our clarification of the construction of the Petrović polygon, we used the Puiseux series. If, instead of the Puiseux series, one considers generalized power series, i.e., the series with complex exponents (the complex exponents belong to a finitely generated semi-group), then the idea and the mode of the construction of the polygon remains the same. Let us notice that Petrović in his thesis considered complex exponents of the asymptotic expansion of the solutions in a neighborhood of nonsingular points of an algebraic ODE.

Let us clarify how to detect the complex exponents by using the Petrović polygon. First, observe that since the given equation is algebraic, the edges have rational slopes, i.e., the minimum of the dot product on edges is attained only for the rational exponents, while on vertices one can have complex exponents of asymptotics. Let us consider now the general case, when one vertex $Q_1 = (\alpha, \beta)$ of the polygon $\mathcal{N}$ corresponds to more than one term in Eq. (1), i.e., in the sum

$$\sum_i \varphi_i(x)\, y^{m_{0i}}\, y'^{m_{1i}}\, y''^{m_{2i}} \ldots y^{(n)m_{ni}},$$
$$\varphi_i(x_0) = \mathrm{const} \neq 0,\ \alpha = m_{0i} + m_{1i} + m_{2i} + \cdots + m_{ni},\ \beta = m_{1i} + 2m_{2i} + \cdots + nm_{ni}. \tag{19}$$

We expand the sum (19) in series in powers of $x - x_0$ and then into that we substitute the expression $y = (c_0 + o(1))(x - x_0)^\lambda$ (here $c_0 \neq 0$ is an arbitrary constant and λ is a sought exponent), to get the expression

$$\left(c_0^\alpha \sum_i \varphi_i(x_0)\, \lambda^{m_{1i}} \cdots (\lambda(\lambda-1)\cdots(\lambda-n+1))^{m_{ni}} + o(1) \right) (x - x_0)^{\lambda\alpha-\beta}.$$

The leading terms in this sum are the following ones:

$$c_0^\alpha \sum_i \varphi_i(x_0)\, \lambda^{m_{1i}} \cdots (\lambda(\lambda-1)\cdots(\lambda-n+1))^{m_{ni}}\ (x - x_0)^{\lambda\alpha-\beta}. \tag{20}$$

As in (16), we introduce the following notation

$$A_i(\lambda) = \lambda^{\gamma_{1i}} (\lambda-1)^{\gamma_{2i}} \cdots (\lambda-n+1)^{\gamma_{ni}},$$

where

$$\begin{aligned} \gamma_{1i} &= m_{1i} + m_{2i} + \cdots + m_{ni}, \\ \gamma_{2i} &= m_{2i} + m_{3i} + \cdots + m_{ni}, \\ &\ldots \\ \gamma_{ni} &= m_{ni}, \end{aligned}$$

and rewrite the expression (20) in the form

$$c_0^\alpha (x - x_0)^{\lambda\alpha-\beta} \sum_i A_i(\lambda)\varphi_i(x_0).$$

This sum is equal to zero only if the polynomial

$$\sum_i A_i(\lambda)\, \varphi_i(x_0) = a_\beta \lambda^\beta + \cdots + a_1\lambda + a_0$$

is zero.

Definition 1 (Petrović, 1894) The equation

$$a_\beta \lambda^\beta + \cdots + a_1\lambda + a_0 = 0 \tag{21}$$

will be called *the characteristic equation* of a given vertex.

Obviously, if $\lambda \in \mathbb{C}$ satisfies the characteristic Eq. (21) and if it is the exponent of the first term of the expansion of a solution of the given ODE, then the minimum of the dot products $\min_i \langle (\operatorname{Re}\lambda, -1), Q_i\rangle = \gamma_\lambda$ is attained only at the vertex Q_1, i.e.,

$$\langle (\operatorname{Re}\lambda, -1), Q_1\rangle = \gamma_\lambda.$$

For an algebraic ODE of the first order, according to the Painlevé Theorem 6, its solutions could possess only algebraic movable singular points, i.e., every solution $y = y(x)$ in a neighborhood of a nonsingular point $x = x_0$ of the equation presents in a form of a power series (15) in a general case with fractional exponents and uniquely defined coefficients. By using a generalized method of Newton–Puiseux polygons, the method of polygons of Petrović or of Fine, one can completely determine the expansions of the solutions of an algebraic ODE of the first order around a point which is nonsingular for the equation. Because of that, Petrović was able to fully resolve the question about the conditions under which the solutions of an algebraic ODE of the first order have or not have movable zeros or movable poles, see Theorems 9–11.

We should make an important observation. In his studies of algebraic ODEs of the first order, Petrović considered only the slanted edges of a polygon. If a solution under consideration has the initial data $y(x_0) = 0$ or $y(x_0) = \infty$, then, obviously, it corresponds to slanted edges of the Petrović polygon, left or right respectively. The vertices and horizontal edges of the Petrović polygon correspond to the solutions of an algebraic ODE with the initial data $y(x_0) = C = \text{const} \neq 0$. In this case, the point $x = x_0$ is not a zero. By making a change of dependent variable $y = C + u$ in the initial equation, and then, by applying the method of Petrović polygon to the transformed equation, one can determine the order of the zero of the solution $u = u(x)$, $u(x_0) = 0$.

Let us notice that if $x = x_0$ is not a singular point of a given algebraic ODE of the first order, then it can be:

- either a nonsingular point of the solutions, and in that case there is a one-parameter family in C of holomorphic solutions in a neighborhood of that point; in other words the assumptions of the implicit function theorem and Cauchy theorem are met;
- or a movable algebraic singular point of those solutions; in this case all the coefficients of the corresponding expansions in Puiseux series with a finite principal part are uniquely determined.

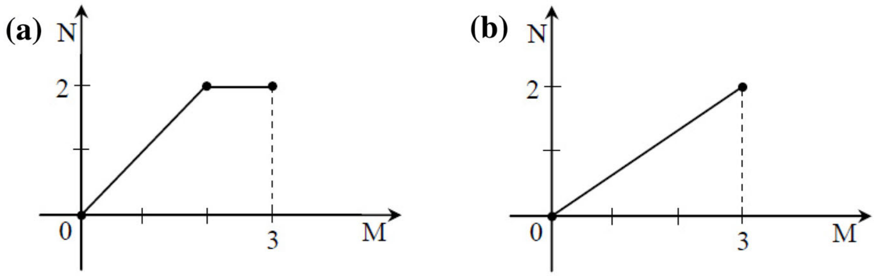

Fig. 2 The Petrović polygons of **a** Eq. (22); **b** Eq. (24)

Example 1 Let us consider the equation

$$f(y', y) = y'^2(y-1) + 1 = 0. \tag{22}$$

It possesses two general solutions

$$y = \left(\frac{C \pm 3x}{2}\right)^{2/3} + 1.$$

Without calculating the solutions of Eq. (22), one can observe that the solutions have no fixed singular points. This is a consequence of the fact that the coefficients of the equation have no zeros. Moreover, for an arbitrary $x_0 \in \mathbb{C}$, solutions with the initial condition $y(x_0) \neq 1$ are holomorphic in a neighborhood of the point $x = x_0$, and solutions with the initial condition $y(x_0) = 1$ (which correspond to integral constants $C = \pm 3x_0$) have x_0 as a movable algebraic critical singular point.

Let us investigate Eq. (22) by use of the Petrović polygon. The three terms y'^2y, $-y'^2$, 1 of Eq. (22) correspond to the points (3, 2), (2, 2), (0, 0). The polygon of Eq. (22) (see Fig. 2a) has two edges: the horizontal one [(2, 2), (3, 2)] and a left-sloped one [(2, 2), (0, 0)] with the angular coefficient equal to 1. There are three vertices: (3, 2), (2, 2), (0, 0).

The vertices (3, 2) and (2, 2) correspond to the approximate equations $y'^2y = 0$ and $y'^2 = 0$ respectively. The approximate equations correspond to the solutions $y = \text{const} \neq 0$, i. e. the exponent is $\lambda = 0$. When we substitute into the equation an expansion of the solution, which begins with a constant, we observe that the minimum of the dot products of the exponents

$$\min_i \langle(\lambda, -1), Q_i\rangle = \min\left(\langle(0,-1),(3,2)\rangle,\ \langle(0,-1),(2,2)\rangle,\ \langle(0,-1),(0,0)\rangle\right) = -2$$

is attained not at a single vertex, (3, 2) or (2, 2) but along the entire horizontal edge [(2, 2), (3, 2)], i.e., $\langle(0,-1),(3,2)\rangle = -2$ and $\langle(0,-1),(2,2)\rangle = -2$. Thus, there are no solutions here that would correspond to the vertices.

The solution which begins with a constant corresponds to the horizontal edge [(2, 2), (3, 2)] with the approximate equation $y'^2(y-1) = 0$, which possesses two solutions $y = C_0 = \text{const} \neq 0$ and $y = 1$. By changing the dependent variable $y = C_0 + u,\ C_0 \neq 1$, in Eq. (22) we get the equation $u'^2(u + C_0 - 1) + 1 = 0$ with the same polygon as Eq. (22). However, there is now an additional condition $u(x_0) = 0$. We should not consider the vertices of the polygon in this new situation because the approximate equations which correspond to them lead to constant solutions, in other words $u(x_0) \neq 0$.

Because the expansion of a solution is given in increasing order of powers of $x - x_0$, we are interested in edges with positive slopes only, these are the left-sloped edges. In the new polygon, this is the edge with the slope 1. This edge corresponds to the approximate equation $u'^2(C_0 - 1) + 1 = 0$ with the solution $u = \pm(x - x_0)/\sqrt{-C_0 + 1}$. Obviously, according to the Cauchy theorem the solution

$$y = C_0 \pm (x - x_0)/\sqrt{-C_0 + 1} + \cdots, \qquad C_0 \neq 1, \tag{23}$$

of Eq. (22) coincides with the holomorphic solution that satisfies the initial condition $y(x_0) = C_0$. If we now change the dependent variable $y = 1 + u$, $u(x_0) = 0$ in Eq. (22), we come to the equation

$$u'^2 u + 1 = 0. \tag{24}$$

The polygon of this equation consists of only one, slanted edge $[(3, 2),\ (0, 0)]$ with the positive slope equal to $2/3$, and two vertices $(3, 2)$ and $(0, 0)$ (see Fig. 2b). Again, we do not consider the vertices of the new polygon, since they correspond only to solutions of the new equation with expansions with $u(x_0) \neq 0$. The edge $[(3, 2),\ (0, 0)]$ corresponds to the approximate equation $u'^2 u + 1 = 0$ with the solution $u = \pm(3/2(x - x_0))^{2/3}$. Obviously, with x_0 fixed, we get uniquely determined solution

$$y = 1 \pm (3/2(x - x_0))^{2/3},$$

which is a particular solution of Eq. (22) with the initial condition $y(x_0) = 1$. Here the point x_0 is a movable critical algebraic singular point of the solutions. But, it is not a zero of those solutions.

There is one more element of the polygon of Eq. (22), which remains to be considered: the slanted edge $[(2, 2),\ (0, 0)]$ with the angular coefficient 1. This edge corresponds to the approximate equation $-y'^2 + 1 = 0$ with the solution $y = \pm(x - x_0)$. Clearly, the solution $y = \pm(x - x_0) + \cdots$ of Eq. (22) is holomorphic according to the Cauchy theorem and it is a particular solution of the family of solutions (23) for $C_0 = 0$. We conclude this simple example which illustrates the method of investigation of the singularities of the solutions of an algebraic ODE based on the use of the Petrović polygon.

3.1 Generalized homogeneity and some limitations concerning higher-order ODEs

Let us recall that an approximate equation is called *generalized homogeneous* in $x - x_0$ (or in y) if it is invariant with respect to the change of $x - x_0$ to $k(x - x_0)$ (of y to

$ky)$, $k \in \mathbb{C}$. Approximate equations which are generalized homogenous often can be solved explicitly. An important feature of the approximate equations obtained through the Petrović polygons is their *generalized homogeneity*. This means that if there exists an approximate equation

$$h(x_0, y, \ldots, y^{(n)}) = 0,$$

corresponding to a slanted edge with the angular coefficient equal to λ, then after the transformation $y = (x-x_0)^\lambda u$ the approximate equation transforms to a new equation $(x-x_0)^\lambda \widetilde{h}(x_0, u, (x-x_0)u', \ldots, (x-x_0)^n u^{(n)}) = 0$, where $\widetilde{h}$ is a polynomial function of its arguments and a generalized homogeneous function in $x - x_0$.

If an edge is horizontal, then the corresponding approximate equation defines a generalized homogeneous function in $x - x_0$. If an edge is vertical, the corresponding equation is a generalized homogeneous in y. An approximate equation which corresponds to a vertex is generalized homogeneous both in $x - x_0$ and in y.

As a benefit from the Painlevé Theorem 6, Petrović did not need to consider a higher-dimensional constructions—the polyhedra of the algebraic ODE of the first order—in order to investigate fully the movable singularities of their solutions. According to the Painlevé Theorem, all movable singularities of such equations are algebraic and the planar polygon captures all such singularities.

It is, however, very important to stress that Petrović in his dissertation clearly pointed out the limitations of the applications of his polygonal method to the algebraic ODEs of higher orders. He showed that the method of planar polygons could be successfully applied to higher-order algebraic ODEs to study some types of movable singularities of the solutions. However, due to the lack of a Painlevé-type result for higher-order equations, Petrović understood that his method was powerless for proving the absence of other types of movable singularities. In other words, the algebraic ODEs starting from order two can have movable singularities that are not algebraic. Moreover, the algebraic ODEs starting from order three can even have nonisolated movable singularities. If we pass from the Petrović polygons to higher-dimensional polyhedra with the aim of studying nonalgebraic movable singular points of algebraic ODEs of higher order, we may at first hope to use the results of the modern theory of Newton polyhedra. However, the approximate equations obtained through the polyhedra are quite complex, and they do not possess the property of generalized homogeneity and often are not exactly solvable.

4 Fine polygons

Fine also generalized the polygonal method of Newton and Puiseux. He used his generalization to study the formal asymptotics of the solutions of algebraic ODEs (1) at the point $x = 0$. In his considerations, he included both cases, when the point $x = 0$ is a singular point of the equation and also when it is not a singular point of the equation. In his papers (Fine 1889, 1890), Fine used results by Puiseux (Puiseux 1850) and Briot–Bouquet (Briot and Bouquet 1856) and generalized them. The Fine and Petrović methods of construction of approximate equations are based on the same principles. Therefore, it is natural that the Fine method matches the steps of the Petrović method. In the construction of Fine polygons, we assign a point to every term of the equation of the type

$$c\, x^{l_{it}}\, y^{m_{0i}}\, y'^{m_{1i}} \cdots y^{(n)m_{ni}}, \qquad c \in \mathbb{C},$$

where the point (N_{it}, M_i), is determined by the formula $N_{it} = l_{it} - N_i$, where N_i and M_i are defined in the same way as in Petrović's construction above. If the points (N_{it}, M_i) are depicted in the plane and if we consider the boundary of the convex hull of all the points (N_{it}, M_i), then the left part of that boundary (consisting of the edges and vertices where the external normal is pointed to the left) captures the behavior of the solutions in a neighborhood of the point $x = 0$. We will call this left part of the boundary *the Fine polygon*. The vertices and edges of the Fine polygon correspond to the leading terms of the equation, i.e., those terms of Eq. (1) which can form approximate equations. The candidates for the role of the asymptotic expansions of the true solutions of the original equation lie among the solutions of the approximate equations. Let us observe that the Fine polygon takes into account the exponents l_{it} of the independent variable x in the coefficients $\varphi_i(x)$ of Eq. (1), because here $x = 0$ can be a singular point for Eq. (1), i. e. $\varphi_i(0)$ could be zero or be undefined.

Let us also observe that by using the change $x = z + x_0$, the problem of analysis of the solutions in a neighborhood of an arbitrary point $x = x_0$ reduces to the problem of the analysis of solutions in a neighborhood of the point $z = 0$. Thus, we proved the following:

Theorem 7 *The Fine polygon of the equation* $f(z + x_0, y, y', \ldots, y^{(n)}) = 0$*, where* x_0 *is not a singular point of Eq.* (1)*, coincides with the Petrović polygon of Eq.* (1) *rotated by* $\pi/2$ *in the counterclockwise direction.*

Fine's paper (Fine 1889) is mostly devoted to the question of calculations of terms in the expansion of formal solutions, (which have a form of Puiseux series) of algebraic ODEs in a neighborhood of the point $x = 0$. Fine also treated the question of the convergence of formal series. Fine proved the following result:

Theorem 8 (Fine 1889) *If every term of an algebraic ODE contains the dependent variable and its derivatives of all orders, i.e., if every term*

$$\varphi_i(x)\, y^{m_{0i}}\, y'^{m_{1i}} \cdots y^{(n)m_{ni}}$$

of (1) *and* (2) *satisfies* $m_{0i}, \ldots, m_{ni} > 0$*, then all the formal Puiseux series which formally satisfy the given equation converge.*

5 Further development of the methods of polygons of Petrović and Fine

A century after Fine's result, Malgrange (Malgrange 1989) gave sufficient conditions for the convergence of a formal power solution that solves a given analytic ODE.

As mentioned in Introduction, J. Cano developed the method of Fine polygons further in Cano (1993a, b), Aroca et al. (2003). He applied these methods in calculations of formal solutions of ODEs and partial differential equations and to prove the convergence of the formal solutions.

Not being aware of the works of other authors—the predecessors Fine and Petrović, and his own contemporaries, Cano, Grigor'ev, and Singer (Grigor'ev and Singer 1991)—A. D. Bruno suggested methods for calculating formal solutions of algebraic differential equations and systems of equations (see Bruno 2000, 2004). These methods are also based on generalizations of the Newton–Puiseux polygons and they repeat the ideas of Petrović and Fine, enriching them with some additions and extensions. These additions and extensions are, essentially, related to calculations of finitely generated semi-groups of exponents of terms of generalized formal series, which formally satisfy a given algebraic ODE and also to extensions of the classes of the considered formal solutions.

To explain these ideas, let us introduce the notion of the order $p(f)$ of function $f(x)$. If there exists the limit

$$\lim_{x\to 0, x\in\mathcal{D}} \frac{\ln|f(x)|}{\ln|x|} = \lim_{x\to\infty, x\in\mathcal{D}} \frac{\ln|f(x)|}{\ln|x|} = p(f,\mathcal{D}) \in \mathbb{R}\bigcup\{\pm\infty\},$$

then the value $p(f,\mathcal{D})$ will be called *the order of the function* $f(x)$ *with respect to a domain* $\mathcal{D}\subset\mathbb{C}$, where the closure of $\mathcal{D}$ contains the points 0 and ∞. Similar definition of the order of a function can be found in the work of Nevanlinna (Nevanlinna 1936). Notice that the power functions, logarithms, and the products of such functions have finite orders. Moreover, the order of the derivatives of these functions, if the derivatives are not identically equal to zero, with each differentiation decreases its order by 1. For example, $p(x^2\ln\ln x) = 2$, and $p((x^2\ln\ln x)') = 1$, $x\in\mathbb{C}$. The same rule does not work for the functions $\cos x$, $x\in\mathbb{C}$, $0<|\operatorname{Im} x|<A$, since: $p(\cos x) = 0$ è $p((\cos x)') = 0$. The constructions of polygons of Petrović and Fine take into account the finiteness of the order and the rule of change of the order by 1 with each differentiation of a power function or of a formal series. We will use the sign $\leftrightarrow$ to denote the correspondence between a term of a given algebraic ODE and a point of its polygon in the plane. For simplicity, let us consider the terms $y,\ y',\ \ldots,\ y^{(n)}$. Indeed, Petrović had used the following correspondence:

$$y\leftrightarrow(1,0),\ \ y'\leftrightarrow(1,1),\ldots,y^{(n)}\leftrightarrow(1,n),$$

while Fine's choice was:

$$y\leftrightarrow(0,1),\ \ y'\leftrightarrow(-1,1),\ldots,y^{(n)}\leftrightarrow(-n,1).$$

Taking into account various classes in which one searches for formal solutions of ODEs, it is possible to generalize the polygons of Petrović and Fine. One of those was accomplished in (Bruno and Goryuchkina 2008). See also Sect. 10.1 for some further considerations.

6 On movable singularities of algebraic ODEs of the first order

In the first part of his dissertation, Petrović considered algebraic ODEs of the first order of a general type:

$$F(x, y, y') = 0, \qquad F = \sum_{i=1}^{j} \varphi_i(x) y^{m_{0i}} y'^{m_{1i}}. \tag{25}$$

In the sequel, the notions of zeros and poles will include both ordinary and critical zeros and poles.

Consider a point $x = x_0$, where x_0 is not a singular point of Eq. (25). Assume that Eq. (25) has a solution $y = y(x)$, such that $y(x_0) = 0$ or $y(x_0) = \infty$. According to the Painlevé Theorem 6, the expansion of the solution $y = y(x)$ in a neighborhood of the point $x = x_0$ is a power series, in a general case, with fractional exponents. All such solutions are detectable through the method of Petrović a polygon whose distinctive feature in the case of an algebraic ODE of the first order is that each its vertex corresponds to the exactly one monomial $\varphi_i(x_0) y^{m_{0i}} y'^{m_{1i}}$.

At the beginning of Chapter 1 of Part 1 of his thesis, after Proposition 1, Petrović formulated and proved necessary and sufficient conditions for the absence of movable zeros and poles of the solutions of Eq. (25).

Theorem 9 (Petrović 1894) *The necessary and sufficient condition for poles (zeros) of the general solution of a given algebraic ODE of the first order* (25) *not to depend on the constants of integration is that the polygon $\mathcal{N}$ of Eq.* (25) *does not contain right (left) slanted edges.*

Sufficiency in Theorem 9 follows from Proposition 1 whereas necessity is proved by the methods of analytic theory of differential equations. In particular, Petrović used some techniques of L. Fuchs from (Fuchs 1885b) from the proof of what we presented as Theorem 5.

Thus, the situation in which both zeros and poles of the general solution of the algebraic differential equation of the first order (25) are not movable, can arise if the polygon $\mathcal{N}$ of Eq. (25) is a horizontal segment or more generally a part of a rectangle.

Let us underline that the conditions of Petrović Theorem 9 can be easily verified just by a simple construction of the polygon of the equation, while the verification of the conditions of the Theorem of Fuchs require much more elaborate work which includes the discriminant equation to be solved, then the equation $F(x, y, y') = 0$ to be resolved with respect to the derivative, and finally an expansion of the derivative in a series in a neighborhood of a discriminant solution is needed. However, Petrović's Theorem allows one to promptly check the existence or the absence of movable zeros or poles, while a situation of the absence of movable critical singularities which are not zeros, could be detected either by the use of the Fuchs Theorem or by repeated use of Petrović's Theorem, see Example 1 and Examples 2, 3 below.

Further on in Chapter 1 of Part 1 of the thesis, Petrović continued to study the existence of singular points of the general solution of Eq. (25). He formulated and proved a few theorems.

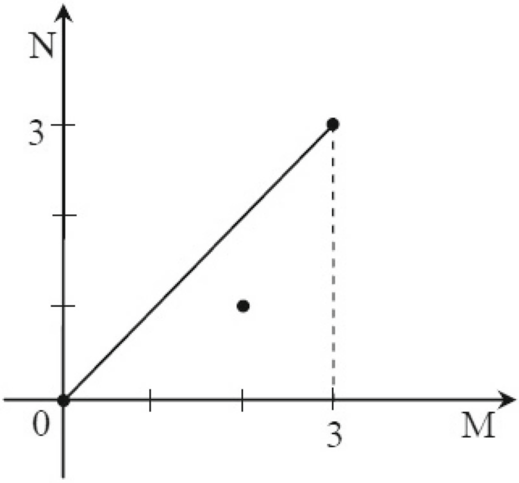

Fig. 3 The Petrović polygon of Eq. (26)

Theorem 10 *The necessary and sufficient conditions for the general solution of a given algebraic ODE of the first order* (25) *to possess a movable zero (or a pole) of order* λ *is that the polygon* $\mathcal{N}$ *of Eq.* (25) *contains an edge with the angular coefficient* λ *(or* $-\lambda$*).*

Theorem 11 *If the general solution of a given algebraic ODE of the first order (25) possesses poles which are independent of the constants of integration, then there are finitely many such poles.*

Example 2 Consider the equation

$$xy'^3 + yy' - 1 = 0. \tag{26}$$

This equation can be solved implicitly, for example with the assistance of computer algebra. Obviously, there are three general solutions since the left hand side of Eq. (26) is a polynomial of degree three in y'. These solutions are cumbersome and because of that we are not going to list them here. Let us check the conditions of the Fuchs Theorem for Eq. (26). As we see, the first two conditions of Fuchs's Theorem 5 are satisfied. The discriminant equation $y^3 + 27x/4 = 0$ has a multi-valued solution $y = \sqrt[3]{-27x/4}$, which is not a solution of Eq. (26). The third condition of the Fuchs's Theorem is not satisfied. Thus, solutions of Eq. (26) possess movable critical points.

Let us now turn to Petrović's Theorems 9 and 10. The points $Q_1 = (3, 3)$, $Q_2 = (2, 1)$, and $Q_3 = (0, 0)$ correspond to Eq. (26). The Petrović polygon (see Fig. 3) consists of one edge $[(3, 3), (0, 0)]$ with the angular coefficient equal to 1. According to Theorem 10, this edge corresponds to the solutions of Eq. (26) with movable zeros of order 1. Solutions with movable zeros of order greater than 1 do not exist.

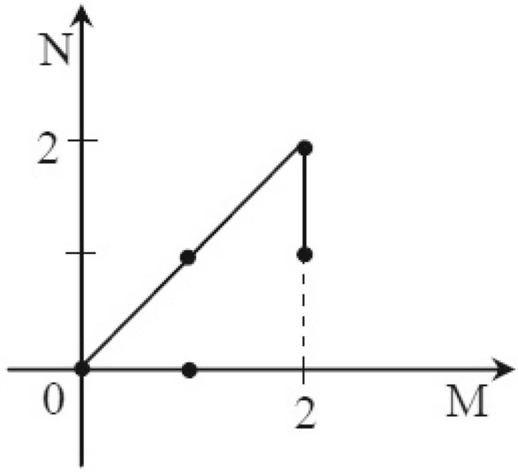

Fig. 4 The Petrović polygon of Eq. (28)

According to the Cauchy theorem in a neighborhood of a nonsingular point $x = x_0$ ($x_0 \neq 0, \infty$) of Eq. (26) any of its solutions having zero of order one at that point, $y(x_0) = 0$, is presented by a uniformly convergent series

$$y = c_1(x - x_0) + \sum_{k=2}^{\infty} c_k (x - x_0)^k, \tag{27}$$

where $c_1 = \sqrt[3]{1/x_0}$, and the consecutive complex coefficients c_k are uniquely determined coefficients. Thus, all movable zeros of Eq. (26) are noncritical, and the existence of movable critical points follows from the Fuchs Theorem.

Example 3 Consider the equation

$$f(x, y, y') = y'^2 - (y' - 1)(y - 1) + x = 0. \tag{28}$$

It has two singular points $x = 1$ and $x = \infty$. It solves implicitly. It has two general solutions since $f(x, y, y')$ is a polynomial of the second degree in y'. These solutions are cumbersome and because of that we are not going to list them here. Let us check the conditions of the Fuchs's Theorem for Eq. (28). The first condition of Fuchs Theorem 5 is satisfied. The second condition is not satisfied. Thus, solutions of Eq. (28) possess movable critical points.

We want to use Petrović's Theorems 9 and 10. The points $Q_1 = (2, 2)$, $Q_2 = (2, 1)$, $Q_3 = (1, 1)$, $Q_4 = (1, 0)$, $Q_5 = (0, 0)$ correspond to Eq. (28). The Petrović polygon (see Fig. 4) consists of one slanted edge $[(2, 2), (1, 1), (0, 0)]$ with the angular coefficient equal to 1. According to Theorem 10, this edge corresponds to the solutions of Eq. (28) with movable zeros of order 1, and movable zeros with order higher than 1 are absent.

Observe that if $y(x_0) = 0$, where x_0 is a nonsingular point of the equation, i.e., $x_0 \neq 1, \infty$, then $y'(x_0) \neq 0$.

In the case $x_0 \neq 1, \frac{5}{4}, \infty, y(x_0) = 0$, the Cauchy theorem can be applied to Eq. (28), and thus, there exists a solution which is represented by a uniformly convergent series

$$y = c_1(x - x_0) + \sum_{k=2}^{\infty} c_k (x - x_0)^k,$$

where

$$c_1 = -\frac{1 \pm \sqrt{5 - 4x_0}}{2},$$

and other complex coefficients c_k are uniquely determined coefficients.

However, the Cauchy theorem is not applicable in the case $x_0 = \frac{5}{4}$, $y(\frac{5}{4}) = 0$, $y'(\frac{5}{4}) = -\frac{1}{2}$ since $f(\frac{5}{4}, 0, -\frac{1}{2}) = 0$, $\frac{\partial f}{\partial y'}(\frac{5}{4}, 0, -\frac{1}{2}) = 0$. In the neighborhood of the point $x_0 = \frac{5}{4}$, there exists a solution of Eq. (28), which can be presented by the Puiseux series

$$y = -\frac{1}{2}\left(x - \frac{5}{4}\right) + \sum_{k=2}^{\infty} c_k \left(x - \frac{5}{4}\right)^{(k+1)/2}.$$

The Petrović polygon method is, in that sense, more universal than the Cauchy theorem. It is able to investigate all solutions with power asymptotics in a neighborhood of a point which is nonsingular for the equation. In this way, there is a movable critical zero of the solution in a neighborhood of a point ($x_0 = 5/4$) which is not singular for the equation.

7 On single-valued solutions of algebraic ODEs of the first order explicitly resolved w.r.t. the derivative. Generalized Riccati equations

Let us recall that the Riccati equations are of the form

$$w' = a_0(z)w^2(z) + a_1(z)w(z) + a_2(z), \tag{29}$$

where $a_0 \neq 0$, $a_2 \neq 0$, a_i are meromorphic functions. Their well-known properties include:

- The Riccati equations reduce to a linear second-order equation.
- The solutions of Riccati equations do not posses movable critical points.
- If one particular solution is known, the Riccati equation reduces to a linear first-order equation.
- If three particular solutions w_1, w_2, w_3 are known, then the cross-ratio

$$\big(w(z) : w_1(z) : w_2(z) : w_3(z)\big)$$

is constant along any solution w. There exists a rational function R, such that $w = R(w_1, w_2, w_3)$.

In his PhD thesis in 1894, Petrović had considered the following rational ODEs:

$$w' = \frac{P(w, z)}{Q(w, z)}, \tag{30}$$

where P, Q are polynomials in w, z, and he proved the following theorem:

Theorem 12 (Petrović 1894) *Such an equation cannot have more than three single-valued solutions which present essentially distinct transcendent functions.*

This result of Petrović caught the immediate attention of his contemporaries and Theorem 12 was quoted, for example, in Picard (1908), Golubev (1911) and Golubev (1941).

Let us outline a draft of the proof. As the first step, Petrović proves that the equations (30) could be reduced to the generalized Riccati equations of the form

$$w' = \frac{P_{n+2}(w, z)}{Q_n(w, z)}, \tag{31}$$

where P_{n+2}, Q_n are polynomials in w, z, of degree $n+2$, n respectively as polynomials in w. This transformation can be done by a change of variables. Then, he considered four cases:

1. Q_n has more than two distinct roots: he proved that then all single-valued solutions are rational.
2. Q_n has exactly two distinct roots: then all single-valued solutions reduce to at most one transcendental function.
3. Q_n has only one root: then all single-valued solutions reduce to at most two essentially distinct transcendental functions.
4. Q_n does not contain w, and thus, the equation corresponds to the Riccati equations: then all single-valued solutions reduce to at most three essentially distinct transcendental functions.

For an illustration, let us show how Petrović treated the first of the above four cases.

Let $f_1(z)$, $f_2(z)$, $f_3(z)$ be the roots of the polynomial Q_n understood as a polynomial in w with a parameter z, i.e., the solutions of $Q_n(w, z) = 0$ in $\mathbb{C}(z)[w]$, and w be a single-valued solution of the differential equation. Then, one can consider $\theta(z) = (w(z) : f_1(z) : f_2(z) : f_3(z))$ which is an algebraic function, since it does not have essential singularities by the Big Picard Theorem and the lemma about critical points (see Lemma 1 above). Thus, w is a rational combination of algebraic functions and is single-valued, and it is a rational function.

Let us mention two variations of the Petrović Theorem 12.

Theorem 13 (Golubev 1911) *If the above equation under the conditions that P, Q are polynomials in w with finitely many isolated singularities in the coefficients, has three rational solutions, then every single-valued solution is rational.*

A far-reaching generalization of the Petrović Theorem 12 was obtained by Malquist (1913). Using a very subtle analytic arguments coming from Boutroux, he managed to get a very elegant conclusion about the first three items of the above considerations.

Theorem 14 (Malquist 1913) *If Eq. (30) is not a Riccati equation, then all its single-valued solutions are rational functions.*

A similar result was reproved by Yosida (1933) using then-new Nevanlinna theory (Nevanlinna 1925, 1936), see also Theorem 19 below. As a matter of fact, Malmquist originally proved a much deeper result:

Theorem 15 (Malquist 1913) *If Eq.* (30) *with* P, Q *being polynomial in* w *with rational coefficients in* z *has at least one nonalgebraic solution which is algebraic over the field of meromorphic functions, then it can be transformed to a Riccati Eq.* (29) *with rational coefficients, by a transformation of the form:*

$$v = \frac{P_n(w, z)}{Q_{n-1}(w, z)}, \tag{32}$$

where P_n, Q_{n-1} *are monic polynomials in* w, *of degree* $n, n-1$ *respectively with coefficients rational in* z.

Further results of Malmquist are contained in Malquist (1920, 1923, 1941). Hille (1979) presents a very nice modern survey of the field. While Golubev and Malmquist quoted Petrović's result, Hille (1979) did not mention that result. For the most recent developments of this subject, see (Kecker 2014) and references therein.

8 On single-valued transcendental solutions of binomial ODEs of the first order

In Part 1 of his thesis (Petrowitch 1894), Petrović also studied the binomial equations

$$(y')^m = \frac{P(x, X, y)}{Q(x, X, y)}, \tag{33}$$

where $m \in \mathbb{N}$, $m \geq 2$, P, Q are polynomials in x, X and y; the variables x and X are assumed to be connected through an algebraic relation $G(x, X) = 0$. Consider the case $m = 2$. Then, Eq. (33) can be rewritten in the form

$$y' = \frac{B(x, X, y)\sqrt{R(x, X, y)}}{C(x, X, y)}. \tag{34}$$

Petrović proved the following theorems.

Theorem 16 *If in Eq.* (34) *the number of distinct nonconstant functions* $y_i = \varphi_i(x, X)$ *which are the roots either of the polynomial* C *or the polynomial* R *is greater than two, then all solutions of this equation that are single-valued in* x *and* X *are rational.*

Theorem 17 *If in Eq.* (34) *the polynomial* R *has one or two nonconstant roots, then this equation does not possess transcendental single-valued solutions.*

Theorem 18 *In order that Eq.* (34) *has single-valued transcendental solutions, it is necessary that the equation has the form*

$$y' = \frac{B(x, X, y)\sqrt{\rho(y)}}{(y - \varphi_1)^{k_1}(y - \varphi_2)^{k_2}}, \tag{35}$$

where the polynomial B *has the degree* $k_1 + k_2$ *in* y*, and* $\rho(y)$ *is a polynomial of degree four.*

Theorems 16–18 were proven analytically, and they could be considered as generalizations of Theorem 12. Petrović did not consider Eq. (33) with $m > 2$ in his thesis. The study of equations (33) in the case $m > 2$ is technically more involved, but he observed that statements and proofs in these cases are still similar to Theorems 16–18 and their proofs for the case $m = 2$.

Similar results were obtained 38 years later, by Yosida in 1932.

Theorem 19 (Yoshida 1932, 1933) *If algebraic ODE of the form*

$$y'^m = R(x, y),\ \ m \in \mathbb{N},$$

where R *is a polynomial in* y*, has a transcendental meromorphic solution, then the degree of the polynomial* R *is not greater than* $2m$*.*

Yosida in his paper (Yosida 1933) quoted the work of Malmquist, but he did not mention the dissertation of Petrović.

9 About solutions with fixed singular points of binomial ODEs of the first order

Petrowitch (1894) also characterized those binomial equations without movable singular points.

Theorem 20 *Among all the equations from the class*

$$y'^m = R(x, y), \quad m \in \mathbb{N}, \tag{36}$$

where R *is a function rational in* y*, only linear ODEs and the equations of the following two types*

$$y'^m = \chi(x)(y - a)^{m-1}, \tag{37}$$

$$y'^2 = \chi(x)(y - a)(y - b), \tag{38}$$

$a, b \in \mathbb{C}$*, are such that all solutions have fixed singular points only, i.e., exactly these are the equations for which the set of singular points of the equation coincides with the set of singular points of solutions.*

In the first part of the proof, Petrović proved that R has to be a polynomial if (36) has solutions with fixed singularities. In the case where the degree of R is not less than m, Petrović reduced Eq. (36) to a linear equation $y' = \sqrt[m]{\chi(x)}(y - \eta(x))$. In the case where the degree of R is less than m, Eq. (36) reduces to

$$\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^m = S(y), \tag{39}$$

where z is a multi-valued function with respect to x and $S(y)$ is a polynomial in y. Petrović then skillfully applied Hermite's Theorem 3 and the results of Briot and Bouquet (Briot and Bouquet 1875) to narrow down the class of equations (39) and get at the end only those with single-valued solutions and fixed singular points.

10 On singularities of algebraic ODEs of higher orders

We conclude this paper with considerations of singular points of higher-order algebraic ODEs. Contrary to the first-order case, which, as we mentioned above, was completely resolved by Petrović and his predecessors, the higher-order case is still wide open even now, more than 120 years later. There are, however, some important subcases which were successfully studied, and we are going to list some of them below. As we have already said, see Sect. 3.1, Petrović was fully aware of the obstacles preventing his polygonal method from producing complete results in higher orders and he listed them clearly. Nevertheless, the Petrović polygonal method can be successfully applied to get some partial answers about higher-order equations and to consider certain types of singularities. Petrović observed that his method could be applied to determine the poles of the solutions in the case of equations that do not depend explicitly on independent variables.

In order to motivate the next question posed by Petrović, let us go back to the first-order case and recall that the Weierstrass equation

$$y'^2 = P_3(y), \tag{40}$$

where P_3 is a degree-three polynomial without multiple zeros, does not depend explicitly on the independent variable and has the Weierstrass $\wp$-function as the solution. Probably motivated by Hermite's Theorem 3 as well, Petrović applied his polygonal method to study elliptic solutions of higher-order algebraic equations, not depending explicitly on independent variables in Petrovitch (1899) [for a recent English translation, see (Petrovitch 2018)].

Petrović singled out the following property.

Property I. *The polygonal line has at least one edge with a negative integer angular coefficient or it has at least a multiple vertex such that the corresponding characteristic equation has one or several negative integer roots, lying between the values of the angular coefficients of the two edges that form the multiple vertex.*

Theorem 21 (Petrovitch 1899, 2018) *If the equation*

$$Q(y, y', \dots, y^{(n)}) = 0, \tag{41}$$

where Q is a polynomial, has an elliptic solution, then its polygon has the property I.

Petrović also considered transformations of solutions. Let R be a rational function and $z = R(y, y', \dots, y^{(q)})$. Let $\Psi(z, z', \dots, z^{(q)}) = 0$ be a transformation of the equation (41). He derived the following result.

Theorem 22 (Petrovitch 1899, 2018) *If the polygon, corresponding to the equation $\Psi(z, z', \dots, z^{(q)}) = 0$ does not possess the property I, then the equation*

$$R(y, y', \dots, y^{(q)}) = \text{const} \tag{42}$$

plays the role of a partial first integral along doubly periodic solutions of Eq. (41), i.e., all solutions of Eq. (41) of a double-periodic nature also satisfy (42).

These partial first integrals could serve to reduce the order to get eventually an equation of the form $Q_1(y, y') = 0$, and to treat it further along the lines indicated by Briot–Bouquet (see Ince 1954, part 2, Chap. XIII).

Example 4 (Petrovitch 1899, 2018) As an example, Petrović considered the equations of the form

$$P_m(y'') = Q_n(y), \tag{43}$$

where P_m, Q_n are given polynomials of degrees m, n respectively. The polygon $\mathcal{N}$ is a triangle with vertices $A(0, 0)$, $B(n, 0)$, $C(m, 2m)$, see Fig. 5. In order to satisfy the Property I, the triangle $\triangle ABC$ has to be acute. The only edge with the negative angular coefficient is BC, provided $n > m$. The angular coefficient is equal to

$$\frac{2m}{m-n} \in \mathbb{Z}.$$

With $n > m$, the examples of (m, n) such that $2m/(m-n) \in \mathbb{Z}$ include $(m, n) \in \{(1, 2), (1, 3), (2, 4), (2, 6)\}$.

10.1 Instead of a conclusion

A year after this paper of Petrović appeared, Painlevé published his seminal paper (Painlevé 1900) followed by Painlevé (1902), which marked not only the turn of the centuries, but opened a new era in the analytic theory of differential equations of higher order. On the level of ideas, Painlevé put forward the Kowalevski position from Kowalevski (1889a) and set out a program to investigate the second-order ODEs of the form

$$y'' = Q(x, y, y'),$$

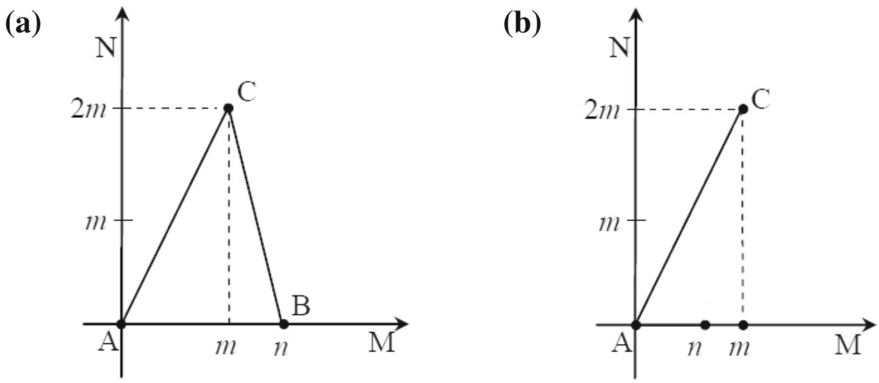

Fig. 5 The Petrović polygon of Eq. (43): **a** $n > m$; **b** $n < m$

where Q is a function rational in y, y' and meromorphic in x, with the following property:

(P)—*all solutions are single-valued around all movable singular points.*

This property is called *the Painlevé property*. Since there were some gaps and errors in Painlevé's calculations in Painlevé (1900, 1902), the program was completed by the students of Painlevé, Gambier in Gambier (1910) and Fuchs (Richard, a son of Lazarus) in Fuchs (1906). As the outcome, they provided 50 equations from which any equation with the Painlevé property can be obtained by gauge transformations based on Möbius transformations. Among these 50 equations, there are six (some of which are multi-parameter families) which cannot be solved in terms of existing classical functions (solutions of linear equations or elliptic functions). These equations are known as *the Painlevé equations I–VI*. For a general set of parameters, the solutions are new transcendental functions not expressible in terms of formerly known functions, called *the Painlevé transcendents*.

The question whether indeed the Painlevé equations possess the Painlevé property was immediately addressed by Painlevé for the Painlevé I equation. His considerations were not quite complete. These questions for all six Painlevé equations have occupied attention of scientists for more than a century. For example, Golubev in Golubev (1912) showed that the solutions of the equations Painlevé I–V possess the Painlevé property. His method was analytic, with a few incomplete spots, which could be easily restored. In early 1980s Jimbo and Miwa in Jimbo et al. (1981) proved the Painlevé property for the Painlevé VI equations using the connection with the Schlesinger equations. Malgrange obtained a similar result about the same time in Malgrange (1983).

A full published answer to this question for all six families of Painlevé equations appeared in the work of Shimomura (Shimomura 2003), see also (Hinkkanen and Laine 2004).

Although a close friend of Painlevé and very active in the period of early 1900's, Petrović did not pay enough attention to the new program of Painlevé. Neither he nor his students made any contribution in that direction.

Nevertheless, the ideas of the Petrović polygonal method have quite recently been applied to the Painlevé equations in Bruno and Goryuchkina (2010); Bruno and Parusnikova (2012) and Bruno and Goryuchkina (2008), through a recently established mathematical discipline, Power Geometry, as a modern reincarnation of Fine and Petrović's method. As we have already mentioned, at that time the authors were unaware of the results of Petrović and Fine.

For example, in (Bruno and Goryuchkina 2008) a method was suggested which allowed to compute the elliptic asymptotics of formal solutions of the equations Painlevé I–IV. The method is based on the extraction of approximate equations, with the use of the polyhedra associated with systems of ODEs of the first order. The main idea consists in transformations of the equations Painlevé I–IV, which all have a form $y'' = f(x, y, y')$, where f is rational in $y,\ y'$ and meromorphic in x, by use of power transformations of the form

$$y = x^{\alpha} v,\ u = x^{\beta},\ \beta > 0, \tag{44}$$

which lead to the equations of the form

$$\ddot{v} = h_0(v, \dot{v}) + \sum_{i=1}^{m} u^{-\gamma_i} h_i(v, \dot{v}), \quad \gamma_i > 0. \tag{45}$$

Here, the choice of numbers α and β in the transformation (45) is related to the coordinates of the external normals to the faces of the polyhedra of the systems of ODEs corresponding to the equations Painlevé I–IV. If the approximate equation $\ddot{v} = h_0(v, \dot{v})$ has a solution $\varphi_0(u)$, which is periodic or doubly periodic with a singular point $u = \infty$ [which obviously is also a singular point for Eq. (45)], then this function is a candidate for the first term of the series

$$\sum_{j=0}^{\infty} \varphi_j(u) u^{-j},$$

where $\varphi_j(u)$ are periodic or double-periodic functions with a singular point $u = \infty$, which formally satisfies Eq. (45). It should be mentioned that Boutroux found elliptic asymptotics of the formal solutions of the Painlevé I and II equations in Boutroux (1913).

Applying the same methods to the Painlevé VI equations, Bruno and his collaborator and student I. Goryuchkina found all formal solutions for all values of parameters in neighborhoods of all singular and nonsingular points of the equations. These formal solutions are not reduced to power series only. Among them, there are generalized power series (power series with complex exponents), Dulac series (integer power series with polynomial in logarithmic coefficients), exotic series (integer power series with coefficients that are meromorphic functions in $x^{i\gamma}$, where i is the imaginary unit and we assume $\gamma \neq 0$), composite series (integer power series with coefficients, which are formal Laurent series with a finite principal part in powers of $\ln^{-1} x$). The formal solutions of the Painlevé VI equations of the first three types are uniformly convergent in some open sectors with the vertex in the considered point. For the Painlevé V equations, all the solutions of given types were found; however, the formal solutions of the Painlevé V equations are not exhausted through these types. They require further study and interpretation.

Let us mention that Painlevé-type program has not yet been completed for the second-order ODEs if they are not explicitly resolved in terms of the second derivative. For higher-order ODEs, the similar questions are meaningful although very complex. Only some partial answers are known so far. Chazy was the first to consider the order three case in Chazy (1911). Very interesting results for the orders four and five with special forms of Q (for example Q being a polynomial) were obtained quite recently, see (Cosgrove 2000a, b, 2006) and references therein. There are also very recent extensions of the Painlevé program to the so-called *quasi-Painlevé property* which assumes the study of movable algebraic singularities and which has been performed for some classes of second-order algebraic ODEs in Shimomura (2007, 2008), Filipuk and Halburd (2009a, b, c), Kecker (2012, 2014), and references therein.

Along with the Painlevé equations, the generalized polygons of Petrović and Fine have been intensively applied in the studies of solutions of algebraic partial differential equations (Aroca et al. 2003; Bruno and Shadrina 2007), and also to the studies of solutions of Pfaff systems (Cano 1993a), solutions of q-difference equations (Cano and Fortuny Ayuso 2012). The polygons have even been used in the proof of Maillet–Malgrange Theorem (Cano 1993b), which provides estimates on the growth of the coefficients of power series, which is an important ingredient in the selection of summation methods. It is clear that the methods based on polygons of Petrović and Fine have wide applications and they continue to develop.

We hope we were able to bring to the attention of the specialists in this actively developing field of mathematics and also to a more general audience the gems almost buried in the past not only to restore the historic justice which these beautiful pioneering results and their outstanding authors deserve but even more—to propel these powerful ideas and put them in the synergy with modern techniques and questions.

A retrospective of the chronology The first idea of using the Newton–Puiseux methods in the theory of differential equations probably goes back to Briot and Bouquet, see (Briot and Bouquet 1856). Henry Fine proposed his polygons in 1889 in Fine (1889). Petrović independently constructed his polygons in his dissertation in 1894 (Petrowitch 1894). He developed some of his geometric ideas and his polygonal method further in his *Acta Mathematica* paper in 1899 (Petrovitch 1899). A student of Petrović, Berić defended his doctoral thesis on the polygonal method at the University of Belgrade in 1912 (Berić 1912). Starting in 1990s Cano implemented the ideas of Newton–Puiseux polygons to differential equations, see Cano (1993a, b). At the same time, Bruno developed the ideas of Newton–Puiseux polygons in differential equations within the field he named Power Geometry, see Bruno (2000). Cano and his collaborators, as well as Bruno and his collaborators, are still active in the field. The present work is, to the best of our knowledge, the first to consider all these important developments from a uniform perspective.

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