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THE JET SCHEME OF A MONOMIAL SCHEME

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We explicitly compute the equations and components of the jet schemes of a monomial subscheme of affine space from an algebraic perspective.

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1. INTRODUCTION

Jet schemes have recently generated new interest in commutative algebra because of their appearance in Konstevich's theory of Motivic integration; see, for example, Kontsevich (1995), Blickle (2006), Denef and Loeser (2001), Looijenga (2002) and Mustaţă (2001). Still, little has been done in the way of explicit calculation of examples. In this note, we begin by explicitly calculating the very simple case of jet schemes of monomial schemes from an algebraic perspective, computing the components and the defining equations for the reduced subschemes of these jet schemes. Interestingly, although the jets schemes of a monomial ideal are not themselves monomial in a natural sense, their reduced subschemes are.

Let X be a scheme of finite type over a field k. Fix a non-negative integer m. An *m-jet* of X/k is a map of k-schemes

$$\psi : \operatorname{Spec} k[t]/\left(t^{m+1}\right) \longrightarrow X.$$

The collection of all m-jets on X forms a scheme in a natural way, called the mth *jet scheme* of X, and denoted by $\mathcal{J}_m(X)$. For background on jet schemes, see for example, Mustaţă (2001) or Blickle (2006).

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Let $X = \operatorname{Spec} k[x_1, \ldots, x_n]/I$ be an affine scheme. To explicitly compute the jet schemes $\mathcal{J}_m(X)$ of X, note that an m-jet is equivalent to a k-algebra homomorphism

$$\phi : k[x_1, \ldots, x_n]/I \longrightarrow k[t]/\big(t^{m+1}\big).$$

Fixing a set of generators $f_1, \ldots, f_r$ for the ideal I, the map ϕ is completely determined by where it sends the coordinates x_i,

$$\begin{aligned}
x_1 &\longmapsto x_1^{(0)} + x_1^{(1)}t + x_1^{(2)}t^2 + \cdots + x_1^{(m)}t^m \\
x_2 &\longmapsto x_2^{(0)} + x_2^{(1)}t + x_2^{(2)}t^2 + \cdots + x_2^{(m)}t^m \\
&\vdots \\
x_n &\longmapsto x_n^{(0)} + x_n^{(1)}t + x_n^{(2)}t^2 + \cdots + x_n^{(m)}t^m.
\end{aligned}$$

Of course, the relations

$$f_i(x_1^{(0)} + x_1^{(1)}t + \cdots + x_1^{(m)}t^m, \ldots, x_n^{(0)} + x_n^{(1)}t + \cdots + x_n^{(m)}t^m) = 0 \tag{1}$$

must hold for each f_i in our chosen set of generators. Write the polynomials (1) in the form

$$f_i^{(0)} + f_i^{(1)}t + f_i^{(2)}t^2 + \cdots + f_i^{(m)}t^m,$$

where the $f_i^{(j)}$ are polynomials in the $x_i^{(j)}$. Then the mth jet scheme $\mathcal{J}_m(X)$ is defined by the polynomials $f_k^{(l)}$ (where k ranges from 1 to r and l ranges from zero to m) in the coordinates $x_i^{(j)}$ (where i ranges from 1 to n and j ranges from zero to m). We will denote by $J_m(I)$ the ideal of this jet scheme, that is, $J_m(I)$ is the ideal generated by the $f_k^{(l)}$ in the polynomial ring $k[x_i^{(j)}]$.

Thus the question we ask is: What can be said about the ideal $J_m(I)$ when I is a monomial ideal?

Example 1. The first jet scheme of the scheme defined by the monomial xy is defined by the two equations $x^{(0)}y^{(0)}$ and $x^{(0)}y^{(1)} + x^{(1)}y^{(0)}$. This ideal is not monomial in the coordinates $x^{(0)}, x^{(1)}, y^{(0)}, y^{(1)}$. However, it is easy to check that its minimal primes are $(x^{(0)}, x^{(1)})$, $(x^{(0)}, y^{(0)})$, and $(y^{(0)}, y^{(1)})$, and therefore its radical is the monomial ideal $(x^{(0)}y^{(0)}, x^{(0)}y^{(1)}, x^{(1)}y^{(0)})$.

In this simple example, we can already see that the jet scheme of a monomial scheme is not defined by monomials (in the "obvious" coordinates). However, the corresponding reduced subscheme is monomial. Below we prove that this is a general phenomenon, computing the corresponding coordinate subspaces explicitly at least in simple cases. This suggests basic questions that we have not yet studied in detail: What are the multiplicities along the various components? Can one describe an explicit primary decomposition? We believe that these and other questions are worth investigating.

2. THE CASE OF A REDUCED MONOMIAL SCHEME

Although the arguments are similar, for the sake of clarity we treat first separately the case of a reduced monomial scheme. In this case, we also get a slightly sharper result.

Theorem 2.1. *Let I be an ideal generated by square-free monomials in coordinates $x_1, \ldots, x_n$. Then $\sqrt{J_m(I)}$ is a square-free monomial ideal in the coordinates $x_1^{(0)}, \ldots, x_1^{(m)}, x_2^{(0)} \ldots x_2^{(m)}, \ldots, x_n^{(0)} \ldots x_n^{(m)}$. The generators can be described as follows: for each monomial minimal generator of I, say $x_1 \ldots x_r$ after relabeling, the monomials*

$$x_1^{(i_1)} \ldots x_r^{(i_r)} \quad \text{where } \sum i_j \leq m$$

are minimal monomial generators of $\sqrt{J_m(I)}$. The collection of all such monomials as we range through the minimal monomial generators of I is a generating set for the radical of $J_m(I)$.

The following lemma reduces the proof of Theorem 2.1 to the hypersurface case.

Lemma 2.1. *If I and J are monomial ideals in a polynomial ring, then $\sqrt{(I+J)} = \sqrt{I} + \sqrt{J}$.*

Proof. Since $\sqrt{I} + \sqrt{J} \subset \sqrt{I+J}$ in general, it remains to check the reverse inclusion in the monomial case. But since $\sqrt{(I+J)} = \sqrt{(\sqrt{I} + \sqrt{J})}$ in general, it suffices to show that $\sqrt{(\sqrt{I} + \sqrt{J})} = \sqrt{I} + \sqrt{J}$ for monomial ideals. This follows because a monomial ideal is radical if and only if it is generated by square-free monomials. $\square$

Thus Theorem 2.1 follows from the following result.

Theorem 2.2. *Let I be the principal monomial ideal generated by $x_1 \ldots x_r$. Then:*

1. *The minimal primes P of $\sqrt{J_m(I)}$ are exactly the primes of the form*

$$P = \left(x_1^{(0)}, x_1^{(1)}, \ldots, x_1^{(t_1)}, x_2^{(0)}, x_2^{(1)}, \ldots, x_2^{(t_2)}, \ldots, x_r^{(0)}, \ldots, x_r^{(t_r)}\right), \tag{2}$$

   where $-1 \leq t_i \leq m$ and $\sum_{i=1}^{r} t_i = m + 1 - r$. (Here, we adopt the convention that the value $t_i = -1$ means the variable x_i doesn't appear at all.)
2. *The ideal $\sqrt{J_m(I)}$ is the monomial ideal generated by the monomials*

$$x_1^{(i_1)} \ldots x_r^{(i_r)}, \quad \text{where } i_j \in \mathbf{N} \text{ and } \sum i_j \leq m. \tag{3}$$

For future reference, we isolate the following simple calculation as a lemma.

Lemma 2.2. *The polynomials defining the mth jet scheme of the scheme defined by the monomial $x_1 \ldots x_r$ are*

$$g_k = \sum_{\sum i_j = k} x_1^{(i_1)} x_2^{(i_2)} \ldots x_r^{(i_r)}$$

where $0 \leq i_j \leq m$ and $0 \leq k \leq m$.

Proof. This follows easily by expanding the products

$$(x_1^{(0)} + x_1^{(1)}t + \cdots + x_1^{(m)}t^m) \cdots (x_r^{(0)} + x_r^{(1)}t + \cdots + x_r^{(m)}t^m)$$

and examining the coefficient of t^k for each $k = 0, \ldots, m$. □

Proof of Theorem 2.2. To prove statement 1, we induce on m. Suppose $m = 0$. Then we have $J_0 = (x_1^{(0)} x_2^{(0)} \ldots x_r^{(0)})$ and so $\sqrt{J_0} = \bigcap_{i=1}^N (x_i^{(0)})$.

Now suppose statement 1 is true for $m - 1$. Then the minimal primes of $\sqrt{J_{m-1}}$ are of the form (2), where $-1 \le t_i \le m - 1$ and $\sum t_i = m - r$. Let Q be a prime containing J_m. Since $J_{m-1} \subset J_m$, Q must contain a minimal prime of J_{m-1}. By induction, then, Q contains at least one prime ideal P of the form (5) above where $-1 \le t_i \le m - 1$ and $\sum t_i = m - r$. Fix the indices $t_1, \ldots, t_r$ corresponding to this prime P.

With notation as in Lemma 2.2, the generators $g_0, \ldots, g_{m-1}$ for J_{m-1} are in P and hence in Q. The only remaining generator for $J_m(I)$ is the polynomial $g_m = \sum_{\sum i_j = m} x_1^{(i_1)} x_2^{(i_2)} \ldots x_r^{(i_r)}$. If some term of g_m fails to be in P, then it is of the form $x_1^{(i_1)} x_2^{(i_2)} \ldots x_r^{(i_r)}$ where each $i_j \ge t_j + 1$, for $1 \le j \le r$. This implies $m = \sum_{j=1}^r i_j \ge \sum_{j=1}^r (t_j + 1) = \sum_{j=1}^r t_j + r$, which is equal to m by our assumption above. The only way this can happen is that each i_j is equal to $t_j + 1$, for $1 \le j \le r$. Therefore, every term of g_m is in P except the one term

$$x_1^{(t_1+1)} x_2^{(t_2+1)} \ldots x_r^{(t_r+1)}. \tag{4}$$

Since $g_m \in Q$ and $P \subset Q$, it follows that the term (4) is in the prime ideal Q. Therefore, Q must contain $x_j^{(t_j+1)}$ for some j between 1 and r. In particular, a minimal prime Q of J_m must therefore be of the form $P + (x_j^{(t_j+1)})$ for some j. This shows that Q has the desired form and also that each of the ideals of this form is a minimal prime of $\sqrt{J_m(I)}$.

To prove statement 2, we first recall that the radical of any ideal is equal to the intersection of its minimal primes; thus statement 1 implies that $\sqrt{J_m(I)}$ is a monomial ideal. Now note that the monomials $x_1^{(i_1)} \cdots x_r^{(i_r)}$ such that $\sum i_j \le m$ are precisely the terms of the generators g_k for $J_m(I)$. Since every monomial ideal containing $g_1, \ldots, g_k$ must contain all these terms, it follows that the monomials of the form (3) are all contained in $\sqrt{(g_1, \ldots, g_k)} = \sqrt{J_m(I)}$. On the other hand, since these monomials are all square free, they generate a radical ideal containing $J_m(I)$. Thus this is the smallest radical ideal containing $J_m(I)$, and hence must be $\sqrt{J_m(I)}$ exactly. The theorem is proven. □

It follows that the reduced subscheme of the mth jet scheme of a reduced monomial hypersurface defined by $x_1 \cdots x_r$ in affine n-space is equidimensional of codimension $m + 1$ in affine $n(m + 1)$ space. One checks that the number of its components is

$$\binom{m+r}{m+1}.$$

Indeed, the number of its components is the same as the number of ways to choose r numbers between 1 and $m+1$ whose sum is $m+1$. But because this is the same as the coefficient of x^{m+1} in $(1+x+x^2+\cdots)^r$, the desired formula follows from the (formal) binomial theorem after substituting the expression $\frac{1}{1-x}$ for the power series $1+x+x^2+\cdots$.

3. THE GENERAL CASE

Theorem 3.1. *If I is generated by monomials in coordinates $x_1, \ldots, x_n$, then $\sqrt{J_m(I)}$ is a square-free monomial ideal in the coordinates $x_i^{(j)}$ where $1 \leq i \leq n$ and $0 \leq j \leq m$. The generators can be described as follows: for each monomial minimal generator of I, say $x_1^{a_1} \ldots x_r^{a_r}$ after relabeling, the monomials*

$$\sqrt{x_1^{(i_1)} x_1^{(i_2)} \ldots x_1^{(i_{a_1})} x_2^{(i_{a_1+1})} \ldots x_2^{(i_{a_1+a_2})} x_3^{(i_{a_1+a_2+1})} \ldots x_r^{(i_{a_1+\cdots+a_r})}} \quad \text{where } \sum i_j \leq m$$

are monomial generators of $\sqrt{J_m(I)}$. The collection of all such monomials as we range through the minimal monomial generators of I is a generating set for $\sqrt{J_m(I)}$.

Remark. It is not true that $\sqrt{J}_m(I) = \sqrt{J}_m(\sqrt{I})$. See Example 2 below.

As in the square-free case, Theorem 3.1 follows from the following.

Theorem 3.2. *Let I be a monomial ideal generated by $x_1^{a_1} \ldots x_r^{a_r}$. Then the minimal primes of $J_m(I)$ are precisely the minimal members of the set of ideals*

$$\left(x_1^{(0)}, x_1^{(1)}, \ldots, x_1^{(t_1)}, x_2^{(0)}, \ldots, x_2^{(t_2)}, \ldots, x_r^{(0)}, \ldots, x_r^{(t_r)}\right), \tag{5}$$

where $-1 \leq t_i \leq m$ (with the convention that $t_i = -1$ means that the variable x_i doesn't appear), and $\sum a_i(t_i+1) \geq m+1$.

Proof. We again induce on m. The result being easy to verify when $m=0$, we assume it holds for $m-1$ and consider $J_m(I)$. Its generators are the polynomials $g_0, g_1, \ldots, g_m$, where

$$g_h = \sum_{\sum i_k = h} x_1^{(i_1)} x_1^{(i_2)} \ldots x_1^{(i_{a_1})} x_2^{(i_{a_1+1})} \ldots x_2^{(i_{a_1+a_2})} x_3^{(i_{a_1+a_2+1})} \ldots x_r^{(i_{a_1+\cdots+a_r})};$$

here the sum is taken over all possible choices of the indices $(i_1, \ldots, i_{a_1+\cdots+a_r})$ with each i_k non-negative and all the i_k summing to h. This is proven in exactly the same way as Lemma 2.2.

Fix a minimal prime Q containing $J_m(I)$. Since $J_{m-1}(I) \subset J_m(I)$, we know that some minimal prime P of $J_{m-1}(I)$ is contained in Q. By induction, this prime has the form

$$\left(x_1^{(0)}, x_1^{(1)}, \ldots, x_1^{(t_1)}, x_2^{(0)}, \ldots, x_2^{(t_2)}, \ldots, x_r^{(0)}, \ldots, x_r^{(t_r)}\right),$$

where $-1 \leq t_i \leq m-1$ and $\sum a_i(t_i+1) \geq m$. Fix the indices $t_1, \ldots, t_r$ corresponding to this prime P.

Since $J_{m-1}(I) \subset P$, we know that P already contains all the generators g_i for $J_m(I)$ except possibly g_m. Consider the polynomial g_m

$$\sum_{\sum i_k = m} x_1^{(i_1)} x_1^{(i_2)} \dots x_1^{(i_{a_1})} x_2^{(i_{a_1}+1)} \dots x_2^{(i_{a_1+a_2})} x_3^{(i_{a_1+a_2+1})} \dots x_r^{(i_{a_1+\cdots+a_r})}. \tag{6}$$

If some term of g_m fails to be in P, then it is of the form

$$x_1^{(j_1)} x_1^{(j_2)} \dots x_1^{(j_{a_1})} x_2^{(j_{a_1}+1)} \dots x_2^{(j_{a_1+a_2})} x_3^{(j_{a_1+a_2+1})} \dots x_r^{(j_{a_1+\cdots+a_r})} \tag{7}$$

for some fixed indices $j_1, \dots, j_{a_1+\cdots+a_r}$ summing to m. Permuting the factors if necessary, we may assume that each j_{a_k} is minimal among the other superscripts on x_k in this expression. Then the failure of the term (7) to be in P is equivalent to each $j_{a_k} \geq t_k + 1$, for $k = 1, \dots, r$. In this case we compute that

$$m = \sum_{i=1}^{a_1+\cdots+a_r} j_i \geq \sum_{k=1}^{r} a_k(t_k + 1)$$

which is greater than or equal to m by our inductive assumption above. This can happen if and only if we have equality all along—that is, each superscript j_i attached to each x_k is equal to $j_{a_k} = t_k + 1$. Therefore, every term of g_m is in P except possibly one term of the form

$$\left(x_1^{(t_1+1)}\right)^{a_1} \left(x_2^{(t_2+1)}\right)^{a_2} \cdots (x_r^{(t_r+1)})^{a_r}; \tag{8}$$

Note that this monomial is a term of g_m if and only if $\sum a_k(t_k + 1) = m$. Indeed, although certain monomials in the sum (6) appear more than once, a monomial of the form (8) appears at most once and hence does not cancel in any characteristic.

Now, if $\sum a_k(t_k + 1) > m$, then no term of the form (8) appears in g_m and every term of g_m is in P; thus $Q = P$ is a minimal prime of $J_m(I)$ as well as $J_{m-1}(I)$. Clearly, then Q has the desired form (5). Conversely, in this case, such a P clearly contains all the terms of the generators of $J_m(I)$ and so is a prime containing $J_m(I)$.

Finally, if $\sum a_k(t_k + 1) = m$, then the calculation above shows that ideal P already contains all the terms of all the generators of $J_m(I)$ except for $(x_1^{(t_1+1)})^{a_1}(x_2^{(t_2+1)})^{a_2} \cdots (x_r^{(t_r+1)})^{a_r}$. Therefore Q must be of the form $P + (x_k^{(t_k+1)})$ for some k between 1 and r, and so has the desired form. Conversely, every ideal of this form is a prime ideal containing $J_m(I)$. This completes the proof. □

Example 2. Let $I = (x^2 y)$. Then one computes that the first few defining equations of the jets schemes are

$$\begin{aligned}
g_0 &= x_0^2 y_0 \\
g_1 &= x_0^2 y_1 + 2x_0 x_1 y_0 \\
g_2 &= x_0^2 y_2 + 2x_0 x_1 y_1 + 2x_0 x_2 y_0 + x_1^2 y_0 \\
g_3 &= x_0^2 y_3 + 2x_0 x_1 y_2 + x_1^2 y_1 + 2x_0 x_2 y_1 + 2x_1 x_2 y_0 + 2x_0 x_3 y_0,
\end{aligned}$$

where for the sake of sanity we have used subscripts instead of superscripts here, i.e,, $x_i = x^{(i)}$. Then

$$\begin{aligned}
\sqrt{J_0(I)} &= (x_0) \cap (y_0) \\
\sqrt{J_1(I)} &= (x_0) \cap (y_0, y_1) \\
\sqrt{J_2(I)} &= (x_0, y_0) \cap (x_0, x_1) \cap (y_0, y_1, y_2) \\
\sqrt{J_3(I)} &= (x_0, y_0, y_1) \cap (x_0, x_1) \cap (y_0, y_1, y_2, y_3).
\end{aligned}$$

Note that unlike the case of a square-free monomial ideal, some coordinate subspaces can appear as components of the mth jet scheme for different m. Likewise, we see that the jet schemes of a nonreduced monomial scheme are not typically equidimensional. Finally, we can write down the defining equations of these reduced jet schemes:

$$\begin{aligned}
\sqrt{J_0(I)} &= (x_0 y_0) \\
\sqrt{J_1(I)} &= (x_0 y_0, x_0 y_1) \\
\sqrt{J_2(I)} &= (x_0 y_0, x_0 y_1, x_1 y_0, x_0 y_2) \\
\sqrt{J_3(I)} &= (x_0 y_0, x_0 y_1, x_1 y_0, x_0 y_2, x_0 y_3, x_1 y_1).
\end{aligned}$$

It is interesting to see explicitly that even in characteristic two, the generators for $\sqrt{J_m(I)}$ are simply the monomial terms of the generators for $J_m(I)$. This mystery is resolved by noticing that any term of g_h that cancels in characteristic two is a multiple of a (noncanceling) term in some earlier g_i.

Remark 3.3. One can also look at the irreducible components of the jet schemes of a monomial ideal from a more combinatorial point of view as follows. After extending scalars we may assume the field is algebraically closed. If one is interested only in the reduced scheme structure of the jet schemes, it is enough to understand when an m-jet on the ambient affine space lies in the mth jet scheme of the scheme defined by the monomial ideal I. Recall that the Newton polyhedron P_I of I is the convex hull of all those exponents u in $\mathbf{N}^n$ such that X^u is in I. Denote by Q_I the polyhedron

$$Q_I = \left\{ w \in Q_+^n \mid \sum_i u_i w_i \geq 1 \text{ for all } u \in P_I \right\}.$$

If an m-jet vanishes with order a_i along the hyperplane $X_i = 0$, then the condition for that jet to lie over the mth jet scheme of I is as follows: for every monomial X^u in I, we have $\sum_i a_i u_i \geq m + 1$. This shows that the set of m-jets γ on the affine space lying over the mth jet scheme of I is equal to $\bigcup_{a \in \mathbf{N}^n} C_a$, where

$$C_a = \{\gamma \mid \operatorname{ord} \gamma^*(X_i) \geq a_i\},$$

and where the union is taken over all $a \in \mathbf{N}^n$ such that a lies in $(m+1)Q_I$. Since every C_a is irreducible, and $C_a \subset C_b$ if and only if $b_i \leq a_i$ for every i, it follows that the irreducible components of the mth jet scheme of I correspond precisely to those a such that a is in $(m+1)Q_I$ and a is minimal with respect to this property.

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REFERENCES

Blickle, M. (2006). A Short Course on Geometric Motivic Integration. Notes available at www.mabli.org/jet.html. Website accessed March 14, 2006.

Denef, J., Loeser, F. (2001). Geometry on arc spaces of algebraic varieties. In: *European Congress of Mathematics*. Vol. I. (Barcelona, 2000). Progr. Math. 201. Basel: Birkhuser, pp. 327–348,

Kontsevich, M. (1995). *Motivic Integration*. Lecture at Orsay.

Looijenga, E. (2002). Motivic measures. Séminaire Bourbaki, Vol. 1999/2000. Astérisque 276:267–297.

Mustaţă, M. (2001). Jet schemes of locally complete intersection canonical singularities. *Invent. Math.* 145(3):397–424.