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NOTE

A SYNTACTIC CONGRUENCE FOR RATIONAL ω-LANGUAGES

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Abstract. Büchi has proved that if L is a rational ω-language, then there exists a finite congruence for which L is saturated in the following sense: $[u][v]^\omega \cap L \neq \emptyset \Rightarrow [u][v]^\omega \subset L$. Here, we define the syntactic congruence of L, which is the largest congruence having this property.

Introduction

It is well known that a language $L \subset A^*$ is rational (regular, or recognizable) iff there exists a finite congruence over A^* such that L is a union of congruence classes. Moreover, there exists a largest such congruence, the so-called syntactic congruence of L.

In [2], Büchi proved that if L is a rational ω-language, there exists a finite congruence $\sim$ over A^* such that the following two properties hold:

(i) $[u][v]^\omega \cap L \neq \emptyset \Rightarrow [u][v]^\omega \subset L$,

(ii) L is a finite union of sets $[u][v]^\omega$, where $[u]$ is the $\sim$-class of u.

Indeed, these two properties are always equivalent for finite congruences and, by analogy with the usual case of languages in A^*, we say that the finite congruence $\sim$ *recognizes* the ω-language L if the two above properties are satisfied. Therefore, the result of Büchi can be rephrased: every rational ω-language is recognizable, whilst the converse is a consequence of the celebrated Büchi's theorem [2].

Next we prove that if a ω-language L is rational (or recognizable), there exists a largest congruence which recognizes it and we give an explicit definition of the syntactic congruence for languages in A^ω; thus, we name this congruence the syntactic congruence of L.

This note contains two parts. In Section 1 we recall some preliminary notions and results, we introduce the notion of a recognizable ω-language and we prove Kleene's theorem for ω-languages. In Section 2 we define the syntactic congruence and we prove it to be the largest one which recognizes a rational ω-language.

1. Preliminaries

A congruence $\sim$ over A^* is an equivalence relation which satisfies: $\forall u, u', v, w \in A^*$, $u \sim u' \Rightarrow vuw \sim vu'w$. We denote by $[u]$ the $\sim$-class of u restricted to A^+ and we say that this congruence is finite if it has a finite number of classes.

It is well known [3] that if $\sim$ is a finite congruence over A^*, $[u] \subset A^+$ is a rational language.

The Büchi–McNaughton Theorem gives several equivalent definitions of a rational ω-language (see [3]). We single out the following one: $L \subset A^\omega$ is said to be rational if $L = \bigcup_{i=1}^n K_i L_i^\omega$, where K_i and L_i are rational languages in A^+.

Let $L \subset A^\omega$ be any ω-language and $\sim$ be any congruence over A^*. We say that

— $\sim$ saturates L if $[u][v]^\omega \cap L \neq \emptyset \Rightarrow [u][v]^\omega \subset L$;

— $\sim$ covers L if $L = \bigcup\{[u][v]^\omega \mid uv^\omega \in L\}$

Lemma 1.1. *If $\sim$ is a finite congruence, then $\sim$ saturates L iff $\sim$ covers L.*

Proof. The idea of this proof can be found in [2].

(1) Let us assume that $\sim$ saturates L. We get: $\forall u, v \in A^*, [u][v]^\omega \subset L$ or $[u][v]^\omega \subset A^\omega - L$. In order to prove that $\sim$ covers L, we first have to prove $A^\omega = \bigcup_{u,v \in A^*} [u][v]^\omega$. Let u be in A^ω; since $\sim$ is finite, by Ramsay's theorem (referred to in [2]), there exist $u_0, u_1, \ldots, u_n, \ldots \in A^+$, $v \in A^*$ such that $u = u_0 u_1 \ldots u_n \ldots$ and $v \sim u_1 \sim u_2 \sim \cdots \sim u_n \sim \cdots$, hence $u \in [u_0][v]^\omega$.

Hence, we get $L = \bigcup \{[u][v]^\omega \mid [u][v]^\omega \subset L\}$; but, since $\sim$ saturates L, $uv^\omega \in L$ iff $[u][v]^\omega \subset L$, and $L = \bigcup \{[u][v]^\omega \mid uv^\omega \in L\}$.

(2) Let us assume that $\sim$ covers L. Since $\sim$ is finite, L is a finite union of sets $[u][v]^\omega$ where $[u]$ and $[v]$ are rational languages in A^+, hence L is rational. Let u and v be such that $[u][v]^\omega \cap L$ is not empty. Since $[u][v]^\omega \cap L$ is a rational, nonempty ω-language, by definition, it contains an ultimately periodic word xy^ω; then there exist $p, p', q, q' \geq 0$, $y_1, y_2 \in A^*$ such that $y = y_1 y_2$, $xy^{p'} y_1 \in [u][v]^p$, $y_2 y^{q'} y_1 \in [v]^q$. Let $w_1 = xy^{p'} y_1$, $w_2 = y_2 y^{q'} y_1$. We have $w_1 w_2^\omega = xy^\omega \in L$, hence, $[w_1][w_2]^\omega \subset L$ and, since $\sim$ is a congruence, $w_1 \in [u][v]^p \Rightarrow [u][v]^p \subset [w_1]$, $w_2 \in [v]^q \Rightarrow [v]^q \subset [w_2]$, hence, $[u][v]^\omega \subset [w_1][w_2]^\omega \subset L$ and $\sim$ saturates L. □

Thus, we say that a finite congruence recognizes an ω-language L if it saturates/covers this language and we say that an ω-language is recognizable if it is recognized by a finite congruence (see also [4]).

Theorem 1.2. *An ω-language is recognizable iff it is rational.*

Proof. Büchi has proved [2] that every rational ω-language is saturated by a finite congruence. Conversely, if L is covered by a finite congruence, it is rational. □

2. The syntactic congruence

Let L be any ω-language. Let us define the relation $\approx$ over A^* by $w \approx w'$ iff $\forall u, v_1, v_2 \in A^*$, $u(v_1wv_2)^\omega \in L$ iff $u(v_1w'v_2)^\omega \in L$ and $v_1wv_2u^\omega \in L$ iff $v_1w'v_2u^\omega \in L$.

It is clear that $\approx$ is a congruence. Let us call it the syntactic congruence of L and let us denote by $[\![u]\!]$ the $\approx$-class of u restricted to A^+.

Lemma 2.1. *The syntactic congruence of L is larger than any congruence which saturates L.*

Proof. We have to prove that if $\sim$ saturates L, then $w \sim w'$ implies $w \approx w'$. Let us assume $w \sim w'$; then $[v_1wv_2] = [v_1w'v_2]$. If $u(v_1wv_2)^\omega \in L$, then, since $\sim$ saturates L, $[u][v_1wv_2]^\omega \subset L$, hence $[u][v_1w'v_2]^\omega \subset L$ and $u(v_1w'v_2)^\omega \in L$. Similarly, $v_1wv_2u^\omega \in L$ implies $v_1w'v_2u^\omega \in L$. It follows that $w \approx w'$. □

Lemma 2.2. *If L is rational, its syntactic congruence is finite and recognizes L.*

Proof. If L is rational, it is saturated by a finite congruence and, by Lemma 2.1, $\approx$ is finite.

Let us assume that the rational ω-language $[\![u]\!][\![v]\!]^\omega \cap L$ is not empty. Like in the proof of part (2) of Lemma 1.1, we get that there exists $w_1w_2^\omega \in L$ such that $[\![u]\!][\![v]\!]^\omega \subset [\![w_1]\!][\![w_2]\!]^\omega$. Thus, it remains to prove, in order to obtain that $\approx$ saturates L, that $uv^\omega \in L \Rightarrow [\![u]\!][\![v]\!]^\omega \subset L$. If this is not the case, there exists $xy^\omega \in [\![u]\!][\![v]\!]^\omega - L$. The infinite word xy^ω can be written $u_0v_1 \dots v_p(v'_1 \dots v'_q)^\omega$ with $u_0 \approx u$, $v_i \approx v'_j \approx v$, hence, $u_0v_1 \dots v_p \approx uv^P$, $v'_1 \dots v'_q \approx v^q$, and $uv^\omega \in L$ iff $uv^P(v^q)^\omega \in L$ iff $u_0v_1 \dots v_p(v^q)^\omega \in L$ iff $u_0v_1 \dots v_p(v'_1 \dots v'_q)^\omega \in L$, a contradiction. □

From Lemmata 2.1 and 2.2 we have the following theorem.

Theorem 2.3. *The syntactic congruence of a rational ω-language is finite and is the largest one which recognizes this language.*

Note. From Lemma 2.2 it follows that if L is rational, then its syntactic congruence is finite, but the converse is not true: two ω-languages which have the same set of ultimately periodic words will have the same syntactic congruence, thus, even if a language has a finite syntactic congruence, it cannot be recognized by this congruence.

References

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