Formalizing forcing arguments in subsystems of second-order arithmetic [1]

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Abstract

We show that certain model-theoretic forcing arguments involving subsystems of second-order arithmetic can be formalized in the base theory, thereby converting them to effective proof-theoretic arguments. We use this method to sharpen the conservation theorems of Harrington and Brown–Simpson, giving an effective proof that $WKL+_0$ is conservative over RCA_0 with no significant increase in the lengths of proofs.

1. Introduction

Although forcing is usually considered to be a model-theoretic technique it has a proof-theoretic side as well, in that forcing notions can usually be expressed syntactically in the base theory. One can often use this fact to convert a model-theoretic forcing argument to a proof-theoretic one, and in doing so obtain a sharper and more effective version of the theorem being proved. We illustrate this approach by formalizing two conservation results, one due to Harrington and the other due to Brown and Simpson, involving subsystems of second-order arithmetic. These results orginally appeared in the author's dissertation [1], where more details can be found.

RCA_0 denotes the weak base theory in the language of second-order arithmetic consisting of the quantifier-free defining axioms for the operations S, $+$, and $\times$; induction is restricted to Σ_1^0 formulas, possibly with set parameters, and a second-order axiom comprehension scheme

$$(RCA) \qquad \forall x(\varphi(x) \leftrightarrow \psi(x)) \rightarrow \exists Y \forall x(x \in Y \leftrightarrow \varphi(x)),$$

where φ and ψ are Σ_1^0 and Π_1^0 formulas, respectively. Intuitively, this axiom asserts the existence of sets that are Δ_1^0-definable, or recursive, in parameters from the model.

The theory WKL_0 adds to this a second-order axiom formalizing a weak version of König's lemma,

(WKL) T an infinite binary tree $\rightarrow \exists X(X$ is a path through $T)$,

stating that there is a path through any infinite binary tree. Here a binary tree T is a set of (codes of) binary sequences that is closed under initial segments, and a path through T is a set X such that every initial segment of the characteristic function of X is in T. For more information, see e.g. [12–14].

Both RCA_0 and WKL_0 were first defined by Harvey Friedman, who gave a model-theoretic proof that in fact WKL_0 is conservative over PRA (primitive recursive arithmetic) for Π_2^0 sentences, in the following sense: if WKL_0 proves $\forall x \exists y \varphi(x, y)$ for some quantifier-free formula φ, then in fact PRA proves $\varphi(x, f(x))$ for some term f. Using a model-theoretic forcing argument Harrington was able to strengthen Friedman's result by showing that in fact WKL_0 is conservative over RCA_0 for Π_1^1 formulas. (RCA_0 is interpretable in the fragment of Peano arithmetic $I\Sigma_1$, and the conservation of the latter system over PRA for Π_2^0 sentences has long been known.) Sieg [10, 11] later gave an effective version of Friedman's result using cut-elimination, a Herbrand analysis, and Howard's notion of hereditary majorizability for primitive recursive terms.

Because Sieg's proof uses cut-elimination, it allows for the possibility of a super-exponential increase in the lengths of proofs between the two systems. In fact, one can show that though there is no significant increase in the lengths of proofs between RCA_0 and $I\Sigma_1$, there *is* a superexponential increase between $I\Sigma_1$ and PRA [6]. This leaves open the question of whether one can obtain a conservation result for WKL_0 over RCA_0 with a more restrictive bound on the increase in the lengths of proofs.

Meanwhile, Brown and Simpson [3] used forcing methods again to extend Harrington's conservation result to the theory $WKL+_0$, which adds to WKL_0 an additional axiom scheme

(BCT) $\forall n \forall \sigma \exists \tau \supseteq \sigma \varphi(n, \tau) \rightarrow \exists f \forall n \exists m \, \varphi(n, f[m])$,

where σ and τ are taken to range over binary sequences, $\tau \supseteq \sigma$ means that τ extends σ, f is assumed to be a function with range $\{0, 1\}$, and $f[m]$ denotes the sequence consisting of the first m values of f. Intuitively, this says that if $\{\varphi_n\}$ defines a sequence of open dense subsets of 2^ω, then there is a point lying in their intersection. Brown and Simpson [3] use this axiom to prove a version of the usual Baire Category Theorem which is not derivable in RCA_0 as well and the Open Mapping Theorem for separable Banach spaces.

Because the axioms (BCT) deal with formulas of arbitrary complexity, Sieg's methods cannot be applied, leaving open the question of whether there is an effective proof of their result.

In this paper we give an affirmative answer to both questions, by showing that the Harrington and Brown–Simpson arguments can be formalized within RCA_0 to yield the following result:

Theorem 1.1. *There is a recursive function f and a polynomial p such that the following holds: if d codes a proof in $WKL+_0$ of a Π_1^1 formula φ, then $f(d)$ codes a proof of φ in RCA_0, and the length of $f(d)$ is less than $p(length\ of\ d)$. Here the "length" of a proof is measured by the number of symbols.*

In other words, one can translate $WKL+_0$ proofs to RCA_0 proofs without a significant increase in size. The rest of this paper will outline a proof of this theorem. For expository reasons we focus on proving the result for WKL_0, while Section 9 will discuss the modifications necessary to handle $WKL+_0$.

2. The model-theoretic conservation result

To our knowledge, an account of Harrington's argument for the conservation of WKL_0 over RCA_0 has never been published, so we describe it here briefly. The result has its origins in the low-basis theorem of [7] (see also [15] or [4]), but it can be phrased as a forcing argument, as is done in [12].

Let L_2 be the language of second-order arithmetic. An L_2 structure M consists of a first-order part, henceforth denoted by $|M|$, and a second-order part, which we denote by S_M. M_1 is an ω-submodel of M_2, written $M_1 \subseteq_\omega M_2$, if $|M_1| = |M_2|$ and $S_{M_1} \subseteq S_{M_2}$; in other words, M_2 has the same first-order part as M_1 but possibly more sets. Note that in this latter definition $|M_1| = |M_2|$ may still be nonstandard. If M is an L_2 structure and X is a subset of $|M|$, we use $M \cup \{X\}$ to denote the structure whose first-order part is $|M|$ and second-order part is $S_M \cup \{X\}$.

If M is a model of RCA_0 then, following [12], we use T_M to denote the set of $T \in S_M$ such that $M \models$ "T is an infinite binary tree."

We use $I\Sigma_1$ to denote the theory based on the axioms and rules of second-order logic that consists of the quantifier-free axioms of arithmetic and induction restricted to Σ_1^0 formulas, possibly involving set parameters.[2] Harrington's method involves showing that any countable model of RCA_0 can be expanded to a model of WKL_0 by adding sets to the second-order part and leaving the first-order part alone. One first proves the following two lemmas:

Lemma 2.1. *Let M be a countable model of $I\Sigma_1$. Then M can be expanded to a model of RCA_0 by adding countably many sets to its second-order part.*

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[2] This is really an abuse of notation since $I\Sigma_1$ is generally used to denote the corresponding first-order theory, but this should cause no confusion here.

Proof. Define M' to be the structure whose first-order part is the same as that of M, and whose second-order part consists of all sets Δ_1^0-definable in M with parameters from $|M| \cup S_M$. Since M' has the same first-order part as M the first-order axioms of RCA_0 hold, and we have added enough sets to handle recursive comprehension. Finally, Σ_1^0-induction in M' reduces to that in M once parameters are replaced by their Δ_1^0-definitions. □

Lemma 2.2. *Let M be model of RCA_0, and let $T \in T_M$. Then there is a subset G of $|M|$ such that*

(1) $M \cup \{G\} \models$ *"G is a path through T"*

(2) $M \cup \{G\} \models I\Sigma_1$.

This lemma, which says that one can add paths through trees to models of RCA_0 and still maintain Σ_1^0 induction, forms the heart of the argument. Its proof will be discussed below. Putting the two results together, we see that given any countable model M of RCA_0 and any $T \in T_M$, we can expand M to another countable model M' of RCA_0 such that $M \subseteq_\omega M'$ and $M' \models$ "there is a path through T." Iterating this process gives us the main theorem:

Theorem 2.3. *Let M be a countable model of RCA_0. Then M can be expanded to a model M' of WKL_0 such that $M \subseteq_\omega M'$.*

Proof. Use the previous two lemmas to form a sequence of models

$$M = M_0 \subseteq_\omega M_1 \subseteq_\omega M_2 \subseteq_\omega \cdots \subseteq_\omega M_i \subseteq_\omega \cdots,$$

where each M_i is a model of RCA_0 and for every $T \in T_{M_i}$ there is a $j > i$ such that $M_j \models$ "there exists a path through T." If we let $M_\omega = \bigcup M_i$, this M_ω satisfies the conclusion of the theorem. □

This theorem implies the conservation result:

Corollary 2.4. *WKL_0 is conservative over RCA_0 for Π_1^1 sentences.*

Proof. If RCA_0 does not prove φ for some Π_1^1 formula φ, by completeness there is a model M of $RCA_0 \cup \{\neg\varphi\}$. By the Skolem–Löwenheim theorem we can assume that this model is countable. Expand M to a model M' of WKL_0 as in Theorem 2.3. Since $\neg\varphi$ is Σ_1^1 and M' contains all the sets of M, M' models $\neg\varphi$ as well. So WKL_0 does not prove φ either. □

We now turn to a discussion of the proof of Lemma 2.2. One proves this using forcing, where the forcing conditions are in fact members of T_M ordered under inclusion. The "generic" path G which is then added to M is the intersection of a generic set of members of T_M. As usual, a formula φ making reference to the new generic G is "forced" by a condition T if φ is true whenever G is interpreted as a

generic path through T. One then shows that if $T \in T_M$ and G is a generic path through T, then properties (1) and (2) hold. The first is immediate from the definition of G, while the second requires showing that certain references to forcing are expressible in M by first-order formulas of low complexity. Though we omit the details here, the syntactic analog of the lemma can be found in Section 5.

The proof of the Brown–Simpson result is similar, and involves showing that one can also add Cohen reals to expand M to a model of (BCT). Here the forcing conditions are finite binary sequences, where the condition σ extends the condition τ if σ extends τ in the usual sense. A full proof of this result appears in [3] and is discussed in Section 9.

3. An overview of the formalization

The major goal of this paper is to formalize the arguments of Section 2. To do so, we define notions of forcing and generic validity from within RCA_0, where "φ is generically valid" corresponds roughly to the statement "φ is true in the model M_ω." We show that, with these definitions, RCA_0 proves that the axioms of WKL_0 are generically valid, and that generic validity is preserved under rules of inference. Furthermore, for arithmetic φ, RCA_0 will prove that φ is generically valid if and only if it is true, and that the sets of M_ω "contain" the sets of the original universe. With these notions in hand, our proof will run as follows:

> Suppose WKL_0 proves that some arithmetic φ is true of all sets. Then RCA_0 proves that φ is generically valid. And so RCA_0 proves that φ is true of all sets in the original universe.

As it turns out, the formalization is not entirely routine. We summarize the main components here.

In Section 4 we present the syntactic definition of a general forcing relation, modulo the specific choice of poset, names, and atomic forcing notions. We need to define both "strong" and "weak" forcing relations: though weak forcing has a clearer semantic interpretation, complexity considerations require us to use strong forcing for the Brown–Simpson result.

In Section 5 we use this framework to define the notions of "1-forcing" and "1-generic validity" to refer to truth in the generic extension M_1, in which we have added a generic path through some binary tree. In this case the forcing conditions, infinite binary trees, form a definable class in the base model rather than a set, but this does not pose any immediate difficulties. More problematic is the fact that we have to name sets which are Δ_1^0-definable from parameters in the original model and the new generic, where the notion of Δ_1^0-definability depends on the generic itself. As Lemmas 2.1 and 2.2 suggest, we need to proceed in two stages: first we define an intermediate "$\frac{1}{2}$-forcing" to describe a model with a single new generic, and then add the sets Δ_1^0-definable from that.

Section 6 shows how to iterate the forcing and define "n-forcing" and "n-generic validity" for each standard natural number n. In Section 7 we show that these definitions can be given *uniformly*, so that n is a parameter. This uniformity does not come easily: a priori the complexity of the forcing definitions seems to increase with each iteration, and we have to take care to use low-complexity equivalents.

Using the uniform definitions we can finally define "ω-forcing" in Section 8, which allows us to iterate the forcing "generically" and add paths through every infinite binary tree in the final model. The last two sections discuss briefly length-of-proof considerations as well as the modifications necessary to handle $WKL+_0$.

4. Formalizing forcing arguments

In this section we describe two kinds of forcing relations, namely "strong" and "weak" forcing. In formalizing Harrington's argument, we will define a number of weak forcing relations; it is only with the Brown–Simpson argument that we have to use strong forcing. In any event, in defining these relations the general pattern will be the same: from within the base theory we first identify the conditions and the names, and give the definition of forcing for atomic formulas. Then we extend the definition to arbitrary formulas by induction on their logical complexity. Although the conditions, names, and atomic clauses will vary, this last part remains the same, and so we give the general framework here. (Much of our approach follows the presentation in [2]; see also [8].)

Before we define any forcing notion, we need to identify from within the base theory what it means for P to be a *condition*, which we write $Condition(P)$, and what it means for a condition P to extend a condition Q, which we write $P \leqslant Q$.[3] We always chose our conditions so that the base theory proves that they form a partial order with a greatest element $\emptyset$. Note that the conditions can be either first- or second-order objects. In the latter case it makes no sense to talk about "the set of all conditions," but in neither case do we need to. All we need is that the class of conditions is defined by some formula.

Next, we need names for the first- and second-order objects in our forcing extensions. In our applications the first-order objects will always name themselves, while sets in the forcing extension will be named by other sets in the base model. Whether or not an object is a valid name may depend on an associated condition; so in fact we need to specify what it means for a condition P to force that an object X is a valid name, which we denote as $P \Vdash^s Name(X)$. In all our applications the second-order universe of the base theory is intended to be a subset of that of the forcing extension, so for each set X our base theory will have a canonical name for it, $\hat{X}$, with properties that will be discussed later on.

Finally, we need to define what it means for a condition P to (strongly) force an atomic formula φ, written $P \Vdash^s \varphi$. The definitions of $P \Vdash^s Name(X)$ and $P \Vdash^s \varphi$ should satisfy some basic requirements.

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[3] In [1] we used $\geqslant$ for this purpose.

Definition 4.1. Assume the notions "*Condition(P)*" and "$P \leqslant Q$" have been defined so that the base theory proves that the class of conditions under $\leqslant$ forms a partial order with least element $\emptyset$. The definition of "$P \Vdash^s Name(X)$" and the association of a formula "$P \Vdash^s \varphi$" to every atomic formula φ will be called a *good strong forcing notion* if the following are satisfied:

(1) The free variables of $P \Vdash^s \varphi$ are the same as those of φ, plus an additional variable for P; the free variables of $P \Vdash^s Name(X)$ are P and X.

(2) Monotonicity: the base theory proves that if P and Q are conditions such that $P \Vdash^s \varphi$ and $Q \leqslant P$, then $Q \Vdash^s \varphi$ as well; likewise for $P \Vdash^s Name(X)$.

(3) Substitution: for each term t

$$P \Vdash^s (\varphi(x/t)) \leftrightarrow (P \Vdash^s \varphi)(x/t).$$

Assuming, then, that the above are in place – a partial order and a good strong forcing notion – we now indicate how to extend the forcing definition to arbitrary formulas. We adopt the following convention: variables P, Q, and R are intended to range over conditions, so quantifiers ranging over these variables should be taken to be so relativized. In other words, a formula written $\forall P \ldots$ should be taken as shorthand for the formula $\forall P(Condition(P) \rightarrow \ldots)$ and a formula written $\exists P \ldots$ should be taken to mean $\exists P(Condition(P) \wedge \ldots)$.

Definition 4.2. Given a good strong forcing notion $P \Vdash^s \varphi$ for atomic φ we extend the definition to arbitary formulas inductively as follows:

(1) $P \Vdash^s \neg\varphi \equiv_{\text{def}} \forall Q \leqslant P(Q \Vdash^s \varphi)$,

(2) $P \Vdash^s \varphi \wedge \psi \equiv_{\text{def}} (P \Vdash^s \varphi) \wedge (P \Vdash^s \psi)$,

(3) $P \Vdash^s \varphi \vee \psi \equiv_{\text{def}} (P \Vdash^s \varphi) \vee (P \Vdash^s \psi)$,

(4) $P \Vdash^s \varphi \rightarrow \psi \equiv_{\text{def}} \forall Q \leqslant P(Q \Vdash^s \varphi \rightarrow \exists R \leqslant Q(R \Vdash^s \psi))$,

(5) $P \Vdash^s \exists x \varphi \equiv_{\text{def}} \exists x(P \Vdash^s \varphi)$,

(6) $P \Vdash^s \forall x \varphi \equiv_{\text{def}} \forall x \forall Q \leqslant P \exists R \leqslant Q(R \Vdash^s \varphi)$,

(7) $P \Vdash^s \exists X \varphi \equiv_{\text{def}} \exists X(P \Vdash^s Name(X) \wedge \varphi)$,

(8) $P \Vdash^s \forall X \varphi \equiv_{\text{def}} \forall X \forall Q \leqslant P \exists R \leqslant Q(R \Vdash^s Name(X) \rightarrow \varphi)$.

If $\emptyset \Vdash^s \varphi$ we write $\Vdash^s \varphi$ and say "φ is strongly forced."

One can check by induction on the complexity of φ that the conditions of Definition 4.1 hold for all formulas:

Lemma 4.3. *If one begins with a good strong forcing notion and extends it to all formulas as in the previous definition, the resulting forcing relation satisfies monotonicity and substitution as well.*

Strong forcing, as we have just defined, is necessary in many applications where the complexity of the forcing definition is of key concern, but its semantic interpretation is problematic: for example, the basic axioms of logic are not strongly forced, and it is possible to have $P \Vdash^s \varphi$ without $P \Vdash^s \neg\neg\varphi$ as well. As a result, the notion of weak

forcing which we are about to define is a more natural one. The semantic interpretation of "P weakly forces φ," which we write $P \Vdash \varphi$, is that φ is true in every generic extension containing P.

Definition 4.4. A good strong forcing notion $\Vdash$ is also a *good weak forcing notion* if it satisfies

$$P \Vdash \varphi \leftrightarrow P \Vdash \neg\neg\varphi$$

for all atomic formulas φ as well as the predicate *Name(X)*. In other words, for all such φ,

$$P \Vdash \varphi \leftrightarrow \forall Q \leqslant P \exists R \leqslant Q (R \Vdash \varphi).$$

As in the case of strong forcing we can extend the definition to arbitrary formulas, using the clauses given in the following definition.

Definition 4.5. Given a good weak forcing notion $P \Vdash \varphi$ for atomic φ we extend the definition to arbitrary formulas inductively as follows:

(1) $P \Vdash \neg\varphi \equiv \forall Q \leqslant P(Q \nVdash \varphi)$,

(2) $P \Vdash \varphi \wedge \psi \equiv (P \Vdash \varphi) \wedge (P \Vdash \psi)$,

(3) $P \Vdash \varphi \vee \psi \equiv \forall P \leqslant Q \exists R \leqslant Q((R \Vdash \varphi) \vee (R \Vdash \psi))$,

(4) $P \Vdash \varphi \rightarrow \psi \equiv \forall Q \leqslant P(Q \Vdash \varphi \rightarrow \exists R \leqslant Q(R \Vdash \psi))$,

(5) $P \Vdash \exists x \varphi \equiv \forall Q \leqslant P \exists R \leqslant Q \exists x(R \Vdash \varphi)$,

(6) $P \Vdash \forall x \varphi \equiv \forall x(P \Vdash \varphi)$,

(7) $P \Vdash \exists X \varphi \equiv \forall Q \leqslant P \exists R \leqslant Q \exists X(R \Vdash Name(X) \wedge \varphi)$,

(8) $P \Vdash \forall X \varphi \equiv \forall X(P \Vdash Name(X) \rightarrow \varphi)$.

If $\emptyset \Vdash \varphi$ we will write $\Vdash \varphi$ and say "φ is weakly forced" or "φ is generically valid."

Using induction on formula complexity one can show

Lemma 4.6. *If one begins with a good weak forcing notion and extends it to all formulas as in the previous definition, the resulting forcing relation satisfies monotonicity and substitution, as well as*

$$P \Vdash \varphi \leftrightarrow P \Vdash \neg\neg\varphi$$

for all formulas φ; in other words,

$$P \Vdash \varphi \leftrightarrow \forall Q \leqslant P \exists R \leqslant Q(R \Vdash \varphi).$$

The connection with strong forcing and an explanation of the "strong"/"weak" terminology is given by the following

Proposition 4.7. *Suppose $\Vdash^s$ is a good strong forcing notion. Then the relation $\Vdash$ given by*

$$P \Vdash \varphi \equiv_{\mathrm{def}} P \Vdash^s \neg\neg\varphi$$

for atomic φ and the predicate Name(X) is a good weak forcing notion, and further-more, the above equivalence holds for all formulas φ.

So one can define a good strong forcing notion $\Vdash^s \varphi$ and work with that, with the knowledge that $\Vdash^s \neg\neg\varphi$ is a good weak forcing notion which satisfies the clauses of Definition 4.5. Below we drop the adjectives "strong" and "weak" and say "*P* forces φ" when it is clear which type of relation we mean.

That weak forcing is a natural concept is guaranteed by the following proposition, which states that the laws and rules of second-order logic are generically valid.

Proposition 4.8. *Let φ be an axiom of second-order predicate logic with free set variables $X_1 \ldots X_n$. Then the base theory proves*

$$\Vdash Name(X_1) \wedge \cdots \wedge Name(X_n) \rightarrow \varphi.$$

Furthermore, if φ follows from premises $\psi_1 \ldots \psi_n$ by a logical rule of inference, then the base theory proves

$$(P \Vdash \psi_1) \wedge \cdots \wedge (P \Vdash \psi_n) \rightarrow (P \Vdash \varphi).$$

The proof involves picking your favorite set of axioms and rules and unwinding defi-nitions (see [2]).

5. Defining 1-forcing

We are now ready to apply the tools of the previous section to the task at hand. Working in the base theory RCA_0 we want to define a forcing relation that will capture the notion of truth in a model M_1 which is again a model of RCA_0 and contains a new generic path through some infinite tree. Referring back to Lemma 2.2 we see that in the semantic argument this was done in two steps: first we added a single generic to our base model M to obtain the model $M_{\frac{1}{2}} = M[G]$, and then we expanded $M_{\frac{1}{2}}$ to the model M_1 by adding all sets Δ_1^0-definable from parameters. Our syntactic development will be the clearest if we emulate these two steps. First we define a forcing notion $\Vdash_{\frac{1}{2}}$ to describe truth in $M_{\frac{1}{2}}$, using names for the sets of the base model plus a name for the new generic. Then we define a forcing notion $\Vdash_1$ to describe truth in the model M_1, using names for the sets Δ_1^0-definable with parameters from $M_{\frac{1}{2}}$. Both will be weak forcing relations.

$\frac{1}{2}$-conditions are defined to be infinite binary trees. In other words, we define $\frac{1}{2}$-*Condition*(P) to mean

$$P \text{ is a binary tree } \wedge \forall n \exists \sigma(\sigma \in P \wedge length(\sigma) = n).$$

Since there are only 2^n binary sequences of length n, the length of σ can be bounded by a primitive recursive term, and so RCA_0 proves that this is equivalent to a Π_1^0 formula. We then define $P \leqslant_{\frac{1}{2}} Q$ to mean $P \subseteq Q$.

We will have two types of $\frac{1}{2}$-names: names for the sets of base model, say $\hat{X} = \{\langle 0, x \rangle \mid x \in X\}$, and a name for the new generic, say $\hat{G} = \{\langle 1, 0 \rangle\}$. The predicate $P \Vdash_{\frac{1}{2}} Name(X)$ will simply assert that X has one of these forms; note that this assertion does not depend on P.

Finally, we have to define $P \Vdash_{\frac{1}{2}} \varphi$ for atomic φ. We can take $P \Vdash_{\frac{1}{2}} t_1 = t_2$ to be simply $t_1 = t_2$, where t_1 and t_2 are terms in the language of arithmetic. Similarly, we can take $P \Vdash_{\frac{1}{2}} t \in \hat{X}$ to mean $t \in X$ for sets X of the base model. How should we define, then, $P \Vdash_{\frac{1}{2}} t \in \hat{G}$? Recalling that G is supposed to represent an infinite path through P, it makes sense to say that P forces that t is in $\hat{G}$ if and only if every infinite path through P necessarily contains t; this will be true if and only if only finitely many nodes of P do not contain t. In other words, we define $P \Vdash_{\frac{1}{2}} t \in \hat{G}$ to mean

$$\{\sigma \in P \mid \sigma_t = 0\} \text{ is finite.}$$

Since P is a tree, note that this is equivalent to a Σ_1^0 statement, namely

$$\exists n \forall \sigma (\sigma \in P \wedge length(\sigma) = n \rightarrow \sigma_t = 1),$$

where we can again bound the universal quantifier over σ.

Our first order of business is to verify the following

Lemma 5.1. *The $\frac{1}{2}$-forcing notion just described is a good weak forcing notion.*

Proof. Monotonicity and substitution are easy to verify. The only part that requires any work is showing that

$$P \Vdash_{\frac{1}{2}} t \in \hat{G} \leftrightarrow P \Vdash_{\frac{1}{2}} \neg\neg(t \in \hat{G})$$

for every condition P. First note that a condition Q $\frac{1}{2}$-forces $\neg(t \in \hat{G})$ if and only if (by definition) no extension R $\frac{1}{2}$-forces $t \in \hat{G}$; which is true if and only if every extension R of Q has infinitely many nodes σ such that $\sigma_t = 0$; which is true if and only if only finitely many nodes σ of Q have $\sigma_t = 1$ (because otherwise $R = Q - \{\sigma \in Q \mid \sigma_t = 0\}$ would be a condition extending Q with only finitely many (in fact zero) nodes σ such that $\sigma_t = 0$). In other words, for any condition Q, $Q \Vdash_{\frac{1}{2}} \neg(t \in \hat{G})$ if and only if

$$\{\sigma \in Q \mid \sigma_t = 1\} \text{ is finite.}$$

Repeating the argument, P $\frac{1}{2}$-forces $\neg\neg(t \in \hat{G})$ if and only if (by definition) no extension Q $\frac{1}{2}$-forces $\neg(t \in \hat{G})$; this is true if and only if every extension Q of P has infinitely many nodes σ such that $\sigma_t = 1$, which in turn is true if and only if P has only finitely many nodes σ such that $\sigma_t = 0$; this is true, by definition, if and only if $P \Vdash t \in \hat{G}$. □

The next lemma states that an arithmetic formula with parameters from the base model is $\frac{1}{2}$-forced if and only if it is true in the original model. From the semantic

point of view, this should be clear; such a formula makes no reference to the new generic set.

Lemma 5.2. *Let $\varphi(X_1, \ldots, X_n)$ be arithmetic with only the set parameters shown. Then RCA_0 proves that for all sets $X_1, \ldots, X_n$, the following are equivalent:*

(1) $\forall P(P \Vdash_{\frac{1}{2}} \varphi(\hat{X}_1, \ldots, \hat{X}_n))$,

(2) $\exists P(P \Vdash_{\frac{1}{2}} \varphi(\hat{X}_1, \ldots, \hat{X}_n))$,

(3) $\varphi(X_1, \ldots, X_n)$.

Proof. Routine, using induction on the complexity of φ. □

The next lemma states something that again should seem intuitively clear: a condition P forces that a binary string σ codes an initial segment of the characteristic function of the generic if and only if finitely many nodes of P disagree with σ. We use the notation $\sigma \subset X$ to mean that the binary sequence σ is an initial segment of the characteristic function of X, and we use the notation $\sigma \perp \tau$ to mean that the binary sequences σ and τ are incompatible, i.e. disagree at some position in both their domains.

Lemma 5.3. *$P \frac{1}{2}$-forces $\sigma \subset \hat{G}$ if and only if*

$$\{\tau \in P \mid \tau \perp \sigma\}$$

is finite. In particular, if $P \frac{1}{2}$-forces $\sigma \in \hat{G}$, then $\sigma \in P$ as well.

Proof. "$\sigma \subset \hat{G}$" is an abbreviation for the assertion

$$\forall i < length(\sigma)(\sigma_i = 0 \rightarrow i \notin \hat{G} \wedge \sigma_i = 1 \rightarrow i \in \hat{G}).$$

Let $S_i = \{\tau \in P \mid \tau_i \neq \sigma_i\}$, so that S_i represents the nodes of P that disagree with σ at the ith place. Then by expanding out the forcing definition one can show the following equivalences:

$$P \Vdash_{\frac{1}{2}} \sigma \subset G \leftrightarrow \forall i < length(\sigma)P \Vdash_{\frac{1}{2}} (\sigma_i = 0 \rightarrow i \notin \hat{G} \wedge \sigma_i = 1 \rightarrow i \in \hat{G})$$

$$\leftrightarrow \forall i < length(\sigma)(S_i \text{ is finite})$$

$$\leftrightarrow \bigcup_{i < length(\sigma)} S_i \text{ is finite}$$

$$\leftrightarrow \{\tau \in P \mid \tau \perp \sigma\} \text{ is finite.} □$$

The heart of Harrington's forcing argument involves showing that the statement $P \Vdash_{\frac{1}{2}} \varphi$ for Σ_1^0 formulas φ is equivalent to a Σ_1^0 assertion. To state this more precisely we need the following well-known fact, which states that a Σ_1^0 formula involving a set parameter can be put into a nice normal form.

Lemma 5.4 (Normal Form Lemma). *Let $\varphi(X)$ be a Σ_1^0 formula with set parameter X. Then there is a Δ_0^0 formula $\theta_\varphi(\sigma)$ such that the quantifier-free axioms of*

arithmetic prove

$$\varphi(X) \leftrightarrow \exists \sigma \subset X \theta_\varphi(\sigma)$$

and furthermore

$$\theta(\sigma) \wedge \tau \supset \sigma \rightarrow \theta_\varphi(\tau).$$

Intuitively, this lemma states that a Σ_1^0 formula in X is true if and only if it is made true by a finite part of X. The proof for Δ_0^0 formulas proceeds by induction on their complexity. The claim can then be extended to formulas with an existential quantifier by noting that, if $\alpha(x, X)$ is Δ_0^0, we have

$$\exists x \, \alpha(x, X) \leftrightarrow \exists x \, \exists \sigma \subset X \theta_\alpha(x, \sigma)$$

$$\leftrightarrow \exists \sigma \subset X \exists x < length(\sigma) \theta_\alpha(x, \sigma),$$

where the last formula in the equivalence has the desired form.

With this behind us, we can now state

Lemma 5.5. *Suppose $\varphi(X_1, \ldots, X_n, G)$ is a Σ_1^0 formula with only the free set variables shown. Then RCA_0 proves the following: For any $X_1, \ldots, X_n$, the statement $P \Vdash_{\frac{1}{2}} \varphi(\hat{X}_1, \ldots, \hat{X}_n, \hat{G})$ is equivalent to the statement*

$$\{\sigma \in P \,|\, \neg\theta_\varphi(X_1, \ldots, X_n, \sigma)\} \text{ is finite,}$$

where θ_φ is as in the statement of the Normal Form Lemma 5.4.

Proof. By Lemma 5.2 we know that the quantifier-free axioms of arithmetic are $\frac{1}{2}$-generically valid, so that the formula $P \Vdash_{\frac{1}{2}} \varphi$ is equivalent to

$$P \Vdash_{\frac{1}{2}} \exists \sigma \subset \hat{G} \theta_\varphi(\hat{X}_1, \ldots, \hat{X}_n, \sigma) \tag{1}$$

in RCA_0.[4] From here on we argue in RCA_0. Unwinding clauses of the forcing definition, (1) is equivalent to

$$\forall Q \leqslant_{\frac{1}{2}} P \exists R \leqslant_{\frac{1}{2}} Q \exists \sigma (R \Vdash_{\frac{1}{2}} \sigma \subset \hat{G} \wedge \theta_\varphi(\hat{X}_1, \ldots, \hat{X}_n, \sigma)).$$

---

[4] Ulrich Kohlenbach has pointed out the reduction of φ to normal form implicitly requires the fact that

$$\forall x \exists \sigma (length(\sigma) = x \wedge \sigma \subset \hat{G}).$$

To see that this is $\frac{1}{2}$-forced, observe that using Σ_1^0-collection RCA_0 can prove

$$\forall x \forall P \exists Q \leqslant P(Q \text{ has exactly one sequence } \sigma \text{ of length } x)$$

and such a condition Q $\frac{1}{2}$-forces

$$length(\sigma) = x \wedge \sigma \subset \hat{G}.$$

Applying Lemmas 5.2 and 5.3 this is equivalent to

$$\forall Q \leqslant_{\frac{1}{2}} P \exists R \leqslant_{\frac{1}{2}} Q \exists \sigma(\{\tau \in R \,|\, \tau \perp \sigma\} \text{ is finite} \wedge \theta_\varphi(X_1, \ldots, X_n, \sigma)).$$

Letting

$$S = \{\sigma \in P \,|\, \neg\theta_\varphi(X_1, \ldots, X_n, \sigma)\}, \tag{2}$$

we claim (2) is equivalent to the assertion "S is finite," as required. To show one direction of the equivalence, suppose S is finite and let Q be any condition extending P. Since S is finite, there is some n so that every sequence of length n in Q satisfies $\theta_\varphi(X_1, \ldots, X_n, \sigma)$. Since Q is infinite, by Σ_1^0 collection (provable from Σ_1^0 induction in RCA_0, as in [4]) some sequence σ of length n has infinitely many descendents. Letting

$$R = \{\tau \in P \,|\, \tau \text{ is compatible with } \sigma\},$$

we have

$$\{\tau \in R \,|\, \tau \perp \sigma\} \text{ is finite} \wedge \theta_\varphi(X_1, \ldots, X_n, \sigma).$$

Conversely, suppose S is infinite. Then S itself is a condition extending P, and for every σ in S we have $\neg\theta_\varphi(X_1, \ldots, X_n, \sigma)$. Then there is no condition R extending S and string σ such that

$$\{\tau \in R \,|\, \tau \perp \sigma\} \text{ is finite} \wedge \theta_\varphi(X_1, \ldots, X_n, \sigma),$$

since whenever $\theta_\varphi(X_1, \ldots, X_n, \sigma)$ holds, σ is not in S and hence incompatible with infinitely many elements of R. □

Corollary 5.6. *Suppose φ is a Σ_1^0 (resp. Π_2^0) formula. Then RCA_0 proves that the formula $P \Vdash_{\frac{1}{2}} \varphi$ is equivalent to another Σ_1^0 (resp. Π_2^0) formula.*

Proof. For Σ_1^0 formulas this follows immediately from the previous lemma, since saying

$$\{\sigma \in P \,|\, \neg\theta_\varphi(X_1, \ldots, X_n)\} \text{ is finite}$$

amounts to saying that there exists an m such that all the sequences of length m in P satisfy $\theta_\varphi(X_1, \ldots, X_n)$.

The corresponding statement for Π_2^0 formulas follows by adding a universal quantifier to φ and applying the appropriate clause of Definition 4.5. □

Lemma 5.7. *RCA_0 proves that Σ_1^0-induction is $\frac{1}{2}$-generically valid.*

Proof. A typical instance of Σ_1^0-induction is of the form

$$\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x\varphi(x),$$

where φ is Σ_1^0, possibly with set parameters. Suppose P $\frac{1}{2}$-forces $\varphi(0)$ and $\varphi(x) \to \varphi(x+1)$ for every x; we want to show that there is a condition Q extending P such

that Q $\frac{1}{2}$-forces $\forall x \varphi(x)$. Since by Corollary 5.6 the statement $P \Vdash_{\frac{1}{2}} \varphi(x)$ is Σ_1^0, so we can use the hypotheses and Σ_1^0-induction in the base model to prove that for every x we have $P \Vdash_{\frac{1}{2}} \varphi(x)$. By Definition 4.5 this means that P itself $\frac{1}{2}$-forces $\forall x \varphi(x)$ and we are done. $\square$

So now we have got a truth definition for a model $M_{\frac{1}{2}}$ which satisfies all the axioms of $I\Sigma_1$. We can use this to define a 1-forcing relation that is very similar to the $\frac{1}{2}$-forcing relation, except that it allows us to name sets that are recursive in parameters from the model $M_{\frac{1}{2}}$.

1-conditions are the same as $\frac{1}{2}$-conditions, namely infinite trees of the base model, and we use the same ordering as before.

We want our 1-names to code sets that are Δ_1^0 in the generic and a parameter from the original model (thanks to pairing, one parameter is enough). So we take our 1-names to be triples $\langle X, \psi, \chi \rangle$ where $\psi(x, X, G)$ and $\chi(x, X, G)$ are (codes of) the Σ_1^0 and Π_1^0 formulas that will code the recursive set. Letting $Tr_{\Sigma_1^0}$ and $Tr_{\Pi_1^0}$ be appropriate truth predicates, we define $P \Vdash_1 Name(\langle X, \psi, \chi \rangle)$ to mean

$$P \Vdash_{\frac{1}{2}} \forall x (Tr_{\Sigma_1^0}(\psi, x, \hat{X}, \hat{G}) \leftrightarrow Tr_{\Pi_1^0}(\chi, x, \hat{X}, \hat{G}));$$

in other words, P forces that the formulas ψ and χ represent a valid Δ_1^0-definition. Note that by Corollary 5.6 this definition has complexity Π_2^0.

Finally, the definition of $P \Vdash_1 t \in \langle X, \psi, \chi \rangle$ is given by

$$P \Vdash_{\frac{1}{2}} Tr_{\Sigma_1^0}(\psi, t, \hat{X}, \hat{G}).$$

Of course, if $P \Vdash_1 Name(\langle X, \psi, \chi \rangle)$ then this is equivalent to

$$P \Vdash_{\frac{1}{2}} Tr_{\Pi_1^0}(\chi, t, \hat{X}, \hat{G}).$$

It is not difficult to show that 1-forcing is also a good weak forcing notion. We can again define canonical 1-names for sets in the base model and the generic, taking, for example, $\hat{X}$ to be

$$\langle X, x \in X, x \in X \rangle$$

and $\hat{G}$ to be

$$\langle \phi, x \in G, x \in G \rangle.$$

With a little bit of work (see [1]), we can show that most of the properties of $\frac{1}{2}$-forcing are true of 1-forcing as well:

Lemma 5.8. *Let $\varphi(X_1, \ldots, X_n)$ be arithmetic with only the set parameters shown. Then RCA_0 proves that for all sets $X_1 \ldots X_n$, the following are equivalent:*
    (1) $\forall P(P \Vdash_1 \varphi(\hat{X}_1, \ldots, \hat{X}_n))$,
    (2) $\exists P(P \Vdash_1 \varphi(\hat{X}_1, \ldots, \hat{X}_n))$,
    (3) $\varphi(X_1, \ldots, X_n)$.

Lemma 5.9. *RCA_0 proves that P 1-forces $\sigma \subset \hat{G}$ if and only if $\{\tau \in P \mid \tau \perp \sigma\}$ is finite.*

Lemma 5.10. *Suppose φ is a Σ_1^0 (resp. Π_2^0) formula. Then RCA_0 proves that the formula $P \Vdash_1 \varphi$ is equivalent to another Σ_1^0 (resp. Π_2^0) formula.*

Lemma 5.11. *RCA_0 proves that Σ_1^0-induction is 1-generically valid.*

In fact, we can show

Lemma 5.12. *RCA_0 proves that each axiom of RCA_0 is 1-generically valid.*

Proof. We have already handled the axioms of $I\Sigma_1$, and our naming scheme handles (RCA). □

We may not seem to have accomplished much so far: we have taken great pains to produce, in RCA_0, an interpretation of the phrase "φ is true in M_1" where M_1 is another model of RCA_0. Our major gain, though, is represented by the following

Lemma 5.13. *RCA_0 proves that if P is a 1-condition then*

$$P \Vdash_1 \exists X(X \text{ is a path through } \hat{P}).$$

In other words,

$$P \Vdash_1 \exists X \forall \sigma(\sigma \subset X \rightarrow \sigma \in \hat{P}).$$

Proof. In fact, Lemma 5.9 implies that

$$P \Vdash_1 \forall \sigma(\sigma \in \hat{G} \rightarrow \sigma \in \hat{P}). \quad □$$

This is a far cry from showing that (WKL) is 1-forced: semantically speaking all we can say is that if P, an infinite binary tree in the base model, happens to be one of the conditions of the generic, then $M[G]$ has a path through P. But as we see we can handle more trees by iterating the forcing, and in fact, if we iterate the forcing "generically" we can get them all.

6. Iterating a forcing definition

Suppose we begin with a theory T and define within T a forcing relation in which the axioms of T are again generically valid. Then we can use Proposition 4.8 to iterate the process and show that T proves that it is generically valid that the axioms of T are generically valid. Having added one generic to obtain a model $M[G_1]$ of T, we are essentially adding another generic to obtain $M[G_1][G_2]$. The conditions of the second

forcing relation are conditions of $M[G_1]$, sets in $M[G_1][G_2]$ are named by sets of $M[G_1]$, and so on. Iterating the process we obtain the models M_n for each standard n.

To carry out the iteration uniformly, however, we need to find, for each n, a single forcing relation that will add n generics at once. We present such a relation below. Proposition 6.3 then shows that adding all the generics at once is equivalent to adding them one at a time.

Definition 6.1. Once n-forcing notions have been defined, $n + 1$-forcing notions are defined as follows.

(1) An $n + 1$-condition is defined to be a pair $\langle P, P' \rangle$ such that

$$n\text{-}Condition(P) \wedge P \Vdash_n (Name(P') \wedge 1\text{-}Condition(P')).$$

(2) If $\langle P, P' \rangle$ and $\langle Q, Q' \rangle$ are $n + 1$-conditions, we say $\langle P, P' \rangle \leqslant_{n+1} \langle Q, Q' \rangle$ if

$$P \leqslant_n Q \wedge P \Vdash_n (P' \leqslant_1 Q').$$

(3) $\langle P, P' \rangle \Vdash_{n+1} Name(X)$ is defined to mean

$$P \Vdash_n (P' \Vdash_1 Name(X)).$$

(4) If φ is atomic, $\langle P, P' \rangle \Vdash_{n+1} \varphi$ is defined to mean

$$P \Vdash_n (P' \Vdash_1 \varphi).$$

(5) Finally, we extend the definition of weak $n + 1$-forcing to arbitrary formulas φ using the clauses of Definition 4.5.

Proving the following propositions involves unwinding the forcing definitions. Details can be found in [1].

Proposition 6.2. *Suppose T proves that 1-forcing is a good weak forcing notion and that the axioms of T are 1-generically valid. If n-forcing is a good weak forcing notion for some standard n, then so is $n + 1$-forcing.*

Proposition 6.3. *Let T be as above and suppose n-forcing is a good weak forcing notion. Then for each formula φ, T proves that*

$$(\langle P, P' \rangle \Vdash_{n+1} \varphi) \leftrightarrow (P \Vdash_n (P' \Vdash_1 \varphi)).$$

Proposition 6.4. *Let T be as above. Then for each standard n, if T proves that the axioms of T are n-generically valid, then it proves that they are $n + 1$-generically valid as well.*

Of course, these propositions imply that for each standard integer n, T proves that the axioms of T are n-generically valid. The reason behind our strange wording will become clear later on.

We now apply these results to the forcing argument at hand. Using RCA_0 as our base theory and our particular definition of 1-forcing we get

Lemma 6.5. *Given the definition of 1-forcing in Section 5 and the corresponding definitions for weak n-forcing, if RCA_0 proves that the axioms of RCA_0 are n-generically valid for some standard n, then it proves that they are $n + 1$-generically valid.*

Again this implies that the axioms of RCA_0 are n-generically valid for each standard n. We also have the generalization of Lemma 5.13:

Lemma 6.6. *Suppose RCA_0 proves that the axioms of RCA_0 are n-generically valid. Then it also proves the following: If X is an n-name and P is an n-condition such that $P \Vdash_n (X$ is an infinite binary tree), then*

$$\langle P, X \rangle \Vdash_{n+1} \exists Y(Y \text{ is a path through } X).$$

Proof. By Proposition 6.3, RCA_0 proves that

$$\langle P, X \rangle \Vdash_{n+1} \exists Y(Y \text{ is a path through } X)$$

is equivalent to

$$P \Vdash_n (X \Vdash_1 \exists Y(Y \text{ is a path through } X)).$$

Since RCA_0 proves that axioms of RCA_0 are n-generically valid, it proves this latter formula by Lemma 5.13.   □

The final lemma of this section shows that for formulas φ of low complexity, the notion $P \Vdash_n \varphi$ remains of low complexity for every standard n. Recall that Lemma 5.10 showed that this is true when $n = 1$. Using Proposition 6.3 we get:

Lemma 6.7. *Suppose for some n RCA_0 proves that the axioms of RCA_0 are n-generically valid. Then if for every Σ_1^0 (resp. Π_2^0) formula φ RCA_0 proves that the assertion $P \Vdash_n \varphi$ is again of Σ_1^0 (resp. Π_2^0) complexity, then the same holds true when n is replaced by $n + 1$.*

Again, this conjoined with Lemma 6.5 implies that the complexity of $P \Vdash_n \varphi$ remains Σ_1^0 (resp. Π_2^0) for each standard n; and, of course, at the moment these are the only n's for which we have a meaningful definition of n-forcing. But the reason for our strange phrasing will become clear in the next section when we define n-forcing uniformly.

7. Defining *n*-forcing uniformly

At this stage we have shown how to define n-forcing for each standard n, but that is not quite sufficient for our purposes. What we really want is a uniform

definition of *n*-forcing, i.e. definitions of the notions "*P* is an *n*-condition," "*P* $\leqslant_n Q$," "*P* $\Vdash_n Name(X)$," and for each φ "*P* $\Vdash_n \varphi$," where *n* is a *parameter*. Furthermore, we would like to show within RCA_0 that, with our new definition, for every *n* the axioms of RCA_0 are *n*-generically valid. This last goal, however, turns out to be a little bit too ambitious, and more than what we need. In fact, it will be sufficient to show that our new forcing definition has this property for "enough" *n*, that is, for a set of numbers forming a *definable cut*.

Definition 7.1. Let *T* be a theory in a language that includes the language of arithmetic. Then a *definable cut* (with respect to *T*) is a formula *J* such that *T* proves

$$J(0) \wedge \forall n(J(n) \to J(n+1)).$$

Referring back to the definitions in Section 5 we see that our basic 1-forcing notions are of low complexity. "1-*Condition*(*P*)" is Π_2^0 in *P*; "*P* $\leqslant_1 Q$" is Δ_1^0 in *P* and *Q*; "*P* $\Vdash_1 Name(X)$" is Π_2^0 in *P* and *X*; and finally, the definition of "*P* $\Vdash_1 t \in X$" is Σ_1^0 in *P* and *X*. Moreover, repeated application of Lemmas 6.5 and 6.7 will reveal that for each standard *n*, RCA_0 proves that the basic *n*-forcing notions have the same complexity as their 1-forcing counterparts. Finally, we note that the transformations are uniform; fixing appropriate Π_2^0 truth predicates, there are primitive recursive functions *Condition*(*n*), *LessThanEq*(*n*), *Name*(*n*), and *ElementOf*(*n*) such that for each standard *n*, RCA_0 proves the following:

$$n\text{-}Condition(P) \leftrightarrow Tr_{\Pi_2^0}(Condition(n), P),$$

$$P \leqslant_n Q \leftrightarrow Tr_{\Pi_2^0}(LessThanEq(n), P, Q),$$

$$P \Vdash_n Name(X) \leftrightarrow Tr_{\Pi_2^0}(Name(n), P, X),$$

$$P \Vdash_n t \in X \leftrightarrow Tr_{\Pi_2^0}(ElementOf(n), t, P, X).$$

It then makes sense to use these functions for our uniform definition of *n*-forcing. In other words, we define a new forcing relation $\Vdash_n$ where *n* is now a parameter; *n*-conditions, $\leqslant_n$, *n*-names, and *n*-forcing for atomic formulas are now defined using the right-hand side of the above equivalences. The weak forcing is then exended to formulas of arbitrary complexity in the usual way.

Again we would like to emphasize that rather than using the clauses of Definition 6.1 to define $n + 1$-forcing from *n*-forcing, we are using their low-complexity equivalents guaranteed by Lemma 6.7. If our primitive recursive functions have been reasonably chosen we can prove these equivalences and show, from within RCA_0, that our uniform definition agrees with Definition 6.1:

Lemma 7.2. RCA_0 *proves the following: Suppose the axioms of* RCA_0 *are n-generically valid. Then the following hold:*

   (1) $n + 1$-*Condition*($\langle P, P' \rangle$) $\leftrightarrow$ *n*-*Condition*(*P*) $\wedge$ *P* $\Vdash_n$ (*Name*(*P'*) $\wedge$ 1-*Condition*(*P'*)),

(2) $\langle P, P' \rangle \leqslant_{n+1} \langle Q, Q' \rangle \leftrightarrow P \leqslant_n Q \wedge P \Vdash_n (P' \leqslant_1 Q')$,

(3) $\langle P, P' \rangle \Vdash_{n+1} Name(X) \leftrightarrow P \Vdash_n (P' \Vdash_1 Name(X))$,

(4) $\langle P, P' \rangle \Vdash_{n+1} t \in X \leftrightarrow P \Vdash_n (P' \Vdash_1 t \in X)$.

In the statements above, all the n- and n + 1-notions refer to our new definitions, whereas all mentions of 1-forcing refer to the original definitions given in Section 5.

Let $J(n)$ be the statement "all the axioms of RCA_0 are n-forced;" i.e.

$$J(n) \equiv_{def} \bigwedge \{ \Vdash_n \varphi \mid \varphi \text{ is an axiom of } RCA_0 \}.$$

Here we are relying on the fact that RCA_0 is finitely axiomatizable using truth predicates. It would be nice if we could prove from within our base theory that $J(n)$ holds for all n. The bad news is, since the statement $J(n)$ has second-order quantifiers, proving this claim requires more induction than is available to us in RCA_0. The good news is that we never really need to know that the predicate J is true of all numbers. In the next section we will see that it is enough to know that

Lemma 7.3. *$J(n)$ is a definable cut, that is, RCA_0 proves that for all n, $J(n) \rightarrow J(n + 1)$.*

Proof. The proof is virtually the same as the proof of Lemma 6.5, except that we use the new n-forcing definitions throughout.   □

Repeating the proof of Proposition 6.3 with our new definitions we get

Lemma 7.4. *With the new uniform definitions of n-forcing, for each formula φ RCA_0 proves the following: if $J(n)$ holds then*

$$(\langle P, P' \rangle \Vdash_{n+1} \varphi) \leftrightarrow (P \Vdash_n (P' \Vdash_1 \varphi)).$$

Once again, we want to emphasize that the key difference is that now n appears as a *parameter* in the statement of the lemma. In the same way, we can prove the analog of Lemma 6.6:

Lemma 7.5. *RCA_0 proves the following: Suppose $J(n)$ and*

$$P \Vdash_n (X \text{ is an infinite binary tree}).$$

Then

$$\langle P, X \rangle \Vdash_{n+1} \exists Y (Y \text{ is a path through } X).$$

Now, with these uniform forcing definitions in hand, we are finally ready to define a forcing relation that will capture the notion of truth in the model M_ω.

8. Putting it all together: the general forcing definition

The model M_ω we describe should first of all be a model in which ω-many generic sets have been added. But we want more than that: the sets should themselves be added *generically*, so that whenever an infinite binary tree gets added to the model, a path through it gets added at some later stage. This section will describe forcing notions that do the job. The conditions of our forcing relation will be n-conditions for varying n. Intuitively, an n-condition specifies information about the first n-generics, so we say that an n-condition P extends an n'-condition Q if and only if $n \leq n'$ and P extends Q with respect to the information they both specify. We use such a poset to define a new weak forcing argument $\Vdash_\omega$ and then show that it is sufficient for our purposes.

If Q is an n-condition and $m \geq n$ we can identify Q with an m-condition

$$\langle \ldots \langle \langle Q, \hat{\emptyset} \rangle, \hat{\emptyset} \rangle, \ldots, \hat{\emptyset} \rangle$$

by padding it with canonical names for the greatest 1-condition. Similarly, if X is an n-name and $m \geq n$ we can identify X with an m-name X' so that

$$(P \Vdash_n t \in X) \leftrightarrow (P \Vdash_m t \in X');$$

for example, if $m = n + 1$ we take $X' = \hat{X}$, the canonical name for X, and so on. So in an expression like $P \Vdash_k Name(X)$, where P is an n-condition and X is an m-name and $k \geq \max(m, n)$, we are implicitly identifying P and X with the corresponding k-condition and k-name, respectively. With this convention in place, we are ready to define ω-forcing.

Definition 8.1. From within RCA_0 we make the following definitions:

(1) An ω-condition is an n-condition for some n such that $J(n)$. That is, we define ω-Condition(P) to mean

$$\exists n(J(n) \wedge n\text{-}Condition(P)).$$

(2) If P is an m-condition and Q is an n-condition, say $P \leq_\omega Q$ if $m \geq n$ and $P \leq_m Q$.

(3) If P is an n-condition and X is an m-name such that $J(m)$, we say that $P \Vdash_\omega Name(X)$ if $P \Vdash_k Name(X)$ where $k = \max(m, n)$.

(4) Similarly, if P is an n-condition and X is an m-name, define $P \Vdash_\omega t \in X$ to mean $P \Vdash_k t \in X$ where $k = \max(m, n)$.

(5) Finally, we extend the definition to arbitrary formulas φ using the clauses of Definition 4.5.

It is straightforward to check that ω-forcing is a good weak forcing notion. If P is an n-condition and $m < n$, we define $P \upharpoonright m$ ("P restricted to m") to be the m-condition corresponding to an initial segment of P.

We now have a tentative truth definition for the model M_ω. We can think of the models M_n as being the ω-submodels of M_ω containing only the first n-generics (and

sets recursive in them). Now consider an arithmetic formula $\varphi(X_1, \ldots, X_j)$ with only the set parameters shown, and suppose $X_1, \ldots, X_j$ are all n-names for some n. Then the truth of φ in M_ω should be equivalent to its truth in M_n, since φ makes no reference to anything outside the M_n's universe. The following lemma, the statement of which relies heavily on our uniform definition of n-forcing, makes this precise. It is very much analagous to Lemma 5.8.

Lemma 8.2. *For any arithmetic $\varphi(X_1, \ldots, X_j)$ with only the set parameters shown, RCA_0 proves the following: Suppose $X_1, \ldots, X_j$ are n-names and P is an ω-condition. Then*

$$(P \Vdash_\omega \varphi(X_1, \ldots, X_j)) \leftrightarrow (P \upharpoonright n \Vdash_n \varphi(X_1, \ldots, X_j)).$$

Proof. The proof is by an easy induction on the logical complexity of φ. $\square$

Using this lemma to reduce ω-forcing to n-forcing for appropriate n, we can show

Lemma 8.3. *RCA_0 proves that each basic axiom of RCA_0 is ω-generically valid.*

Finally, we come to the point of the whole affair:

Lemma 8.4. *RCA_0 proves that (WKL) is ω-generically valid.*

Proof. Argue in RCA_0. Suppose P is an ω-condition and X is an ω-name such that

$$P \Vdash_\omega Name(X) \wedge (X \text{ is an infinite tree}).$$

We need to show that there is an ω-condition Q extending P such that

$$Q \Vdash_\omega \exists Y(Y \text{ is a path through } X).$$

Since P is an ω-condition and $P \Vdash_\omega Name(X)$ there is an n such that: $J(n)$ holds, P can be identified with an n-condition, and X can be identified with an n-name such that $P \Vdash_n Name(X)$. Hence by Lemma 8.2 we have that

$$P \Vdash_n (X \text{ is an infinite tree}).$$

Then $\langle P, X \rangle$ is an $n + 1$-condition and by Lemma 7.5 we have that

$$\langle P, X \rangle \Vdash_{n+1} \exists Y(Y \text{ is a path through } X).$$

But $J(n)$ implies $J(n + 1)$, so $\langle P, X \rangle$ is the desired ω-condition. $\square$

This yields our main result:

Theorem 8.5. *WKL_0 is conservative over RCA_0 for Π^1_1 sentences.*

Proof. Suppose WKL_0 proves $\forall X \varphi(X)$ for some arithmetic φ. Since RCA_0 proves that each axiom of WKL_0 is ω-generically valid and that ω-generic validity is maintained

under rules of inference, it also proves that $\forall X \varphi(X)$ is ω-generically valid, and hence

$$\forall X \;\Vdash_\omega (Name(X) \to \varphi(X)).$$

Now, arguing in RCA_0, let X be any set and let $\hat{X}$ be its canonical 1-name. Then $\Vdash_\omega Name(\hat{X})$ and so $\Vdash_\omega \varphi(\hat{X})$. By Lemma 8.2 we have $\Vdash_1 \varphi(\hat{X})$, and hence by Lemma 5.8 we can conclude $\varphi(X)$. $\square$

In the next section we will discuss what is necessary to extend this result to $WKL+_0$, and after that we will explore briefly what happens to the lengths of the proofs in the translation.

9. Handling $WKL+_0$

We will now consider what is needed to modify the whole argument to handle the scheme (BCT) given in Section 1, namely

$$\forall n \forall \sigma \exists \tau \supseteq \sigma \varphi(n, \tau) \to \exists f \forall n \exists m \varphi(n, f[m]).$$

Suppose for the moment we want to prove a conservation result for $RCA+_0$, i.e. RCA_0 conjoined with the axiom scheme (BCT). The semantic forcing argument, which appears in [3], is a simple instance of Cohen forcing in which the conditions are finite segments of 0's and 1's representing initial segments of the characteristic function of the generic set being added. Translating the argument to an effective one using the methods described here poses little difficulty. As in Section 5 we start by defining a "$\frac{1}{2}$-forcing" relation. The key difference is that due to complexity considerations we have to work with the strong version of the forcing relation as well.

Definition 9.1. Within RCA_0, we define new notions of $\frac{1}{2}$-forcing as follows.
 (1) A $\frac{1}{2}$-condition p is (a code for) a finite sequence of 0's and 1's.
 (2) If p and q are $\frac{1}{2}$-conditions, $p \leqslant_{\frac{1}{2}} q$ if p extends q in the usual sense.
 (3) $p \Vdash^s_{\frac{1}{2}} Name(X)$ if X is a canonical name for a set in the base model or the new generic, as in Section 5.
 (4) Define $p \Vdash^s_{\frac{1}{2}} t \in \hat{G}$ to mean $p_t = 1$.
 (5) We extend the relation $p \Vdash^s_{\frac{1}{2}} \varphi$ to arbitrary formulas φ and define the corresponding weak forcing notions as described in Section 5.

Note that $p \Vdash_{\frac{1}{2}} t \in \hat{G}$ will hold if and only if $p \Vdash^s_{\frac{1}{2}} t \in \hat{G}$, so for atomic formulas the strong and weak forcing notions agree. For arbitrary formulas the two notions diverge, though they are related by Proposition 4.7.
 It is easy to check that Lemma 5.2, which states that arithmetic formulas are forced if and only if they are true, still holds:

Lemma 9.2. *Let $\varphi(X_1,\ldots,X_n)$ be arithmetic with only the set parameters shown. Then RCA_0 proves that for all sets $X_1,\ldots,X_n$, the following are equivalent:*

(1) $\forall p(p \Vdash^s_{\frac{1}{2}} \varphi(\hat{X}_1,\ldots,\hat{X}_n))$,

(2) $\exists p(p \Vdash^s_{\frac{1}{2}} \varphi(\hat{X}_1,\ldots,\hat{X}_n))$,

(3) $\forall p(p \Vdash_{\frac{1}{2}} \varphi(\hat{X}_1,\ldots,\hat{X}_n))$,

(4) $\exists p(p \Vdash_{\frac{1}{2}} \varphi(\hat{X}_1,\ldots,\hat{X}_n))$,

(5) $\varphi(X_1,\ldots,X_n)$.

Furthermore, we have the following analog of Lemma 5.3:

Lemma 9.3. *RCA_0 proves that the following are equivalent:*

(1) $p \Vdash^s \sigma \subset \hat{G}$,

(2) $p \Vdash^s \sigma \subset \hat{G}$,

(3) $\sigma \subset p$.

Unfortunately, Corollary 5.6 is somewhat weakened, in that we can no longer make the same claim about Σ^0_1 formulas.

Lemma 9.4. *Suppose φ is a Π^0_2 formula. Then RCA_0 proves that the formula $P \Vdash_{\frac{1}{2}} \varphi$ is equivalent to another Π^0_2 formula.*

Proof. This is easy to check using the clauses of Definition 4.5 and noting that $\frac{1}{2}$-forcing for atomic formulas is Δ^0_1. $\square$

Because forcing for Σ^0_1 formulas is no longer Σ^0_1 we cannot use the same proof we gave in Section 5 to show that Σ^0_1 induction is 1-generically valid. However, *strong* forcing for Σ^0_1 formulas *does* remain Σ^0_1, and we can use that fact to push the proof through. Because strong forcing does not have the nice semantic properties that weak forcing does, it is harder to understand what exactly is going on in the following proof. But in both instances handling induction amounts to showing that certain forcing notions are of sufficiently low complexity to use induction in the base theory.

Lemma 9.5. *RCA_0 proves that Σ^0_1 induction is $\frac{1}{2}$-generically valid.*

Proof. Suppose φ is Σ^0_1. We need to show that

$$\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x\,\varphi(x)$$

is $\frac{1}{2}$-generically valid. In fact, we claim that it is strongly $\frac{1}{2}$-forced by the empty condition. Suppose p strongly $\frac{1}{2}$-forces the inductive hypothesis, so that

$$p \Vdash^s_{\frac{1}{2}} \varphi(0)$$

and

$$\forall x \forall q \leqslant_{\frac{1}{2}} p \exists r \leqslant_{\frac{1}{2}} q(r \Vdash^s_{\frac{1}{2}} \varphi(x) \to \varphi(x+1)).$$

We claim that p strongly $\frac{1}{2}$-forces the conclusion $\forall x \varphi(x)$; i.e.

$$\forall x \forall q \leqslant_{\frac{1}{2}} p \exists r \leqslant_{\frac{1}{2}} q(q \Vdash^s_{\frac{1}{2}} \varphi(x)).$$

Switching the first two quantifiers to get

$$\forall q \leqslant_{\frac{1}{2}} p \forall x \exists r \leqslant_{\frac{1}{2}} q(q \Vdash^s_{\frac{1}{2}} \varphi(x))$$

we now fix $q \leqslant_{\frac{1}{2}} p$ and use induction on x in the formula

$$\exists r \leqslant_{\frac{1}{2}} q(r \Vdash^s_{\frac{1}{2}} \varphi(x)).$$

The fact that $p \Vdash^s_{\frac{1}{2}} \varphi(0)$ gives the base case, and the fact that for every x and $q \leqslant_{\frac{1}{2}} p$ there is $r \leqslant_{\frac{1}{2}} q$ such $r \Vdash^s_{\frac{1}{2}} (\varphi(x) \to \varphi(x+1))$ yields the induction hypothesis. So by Σ^0_1-induction in our base theory we are done. $\square$

Now we can go on to develop the corresponding 1-forcing notions as was done in Section 5. As before, then, we can show that Σ^0_1-induction and recursive comprehension, and hence all the axioms of RCA_0, are 1-generically valid. Our gain is then represented by the following

Lemma 9.6. *Let* $\varphi(n, \sigma, X_1, \ldots, X_n)$ *be arithmetic with the set parameters shown. Then* RCA_0 *proves the following: for any sets* $X_1, \ldots, X_n$

$$\forall n \forall \sigma \exists \tau \supset \sigma \varphi(n, \tau, \hat{X}_1, \ldots, \hat{X}_n) \to \exists f \forall n \exists m \varphi(n, f[m], \hat{X}_1, \ldots, \hat{X}_n)$$

is 1-*generically valid.*

Proof. Argue in RCA_0. Suppose a condition p forces the left hand side. Then by Lemma 9.2 we have that

$$\forall n \forall \sigma \exists \tau \supset \sigma \varphi(n, \tau, X_1, \ldots, X_n)$$

is true. We claim that this implies that

$$\Vdash_{\frac{1}{2}} \forall n \exists m \varphi(n, \hat{G}[m], \hat{X}_1, \ldots, \hat{X}_n),$$

which suffices to prove the lemma. Working backwards, this latter formula is equivalent to

$$\Vdash_{\frac{1}{2}} \forall n \exists m, \tau(\tau = \hat{G}[m] \wedge \varphi(n, \tau, \hat{X}, \ldots, \hat{X}_n)).$$

Using the clauses of Definition 4.5 we see that this is equivalent to

$$\forall n \forall p \exists q \supset p \exists m(q \Vdash_{\frac{1}{2}} (\tau = \hat{G}[m] \wedge \varphi(n, \tau, \hat{X}_1, \ldots, \hat{X}_n))).$$

But it is not hard to show that $q \Vdash_{\frac{1}{2}} \tau = \hat{G}[m]$ is true if and only if $\tau = q[m]$, and so again by Lemma 9.2 we get

$$\forall n \forall p \exists q \supset p \exists \tau, m(\tau = q[m] \wedge \varphi(n, \tau, X_1, \ldots, X_n)),$$

which is equivalent to

$$\forall n \, \forall p \, \exists q \supset p \exists m(\varphi(n, q[m], X_1, \ldots, X_n)).$$

But it is easy to see that this is implied by

$$\forall n \, \forall \sigma \, \exists \tau \supset \sigma \varphi(n, \tau, X_1, \ldots, X_n)$$

as desired. $\square$

This lemma falls short of (BCT), since only parameters from the base model are allowed. However, when we iterate the forcing, we have that if φ satisfies the hypothesis of (BCT) with n-names as parameters, then the $n + 1$st generic will verify the conclusion. All our forcing notions are of low complexity: $1\text{-}Condition(p)$ is Δ_1^0, as is $p \leqslant_1 q$; $p \Vdash_1 Name(X)$ and $p \Vdash_1 t \in X$ are both Π_2^0, and Lemma 9.4 guarantees that this complexity is maintained when the forcing is iterated. So, repeating the constructions of Sections 6–8 yields the desired conservation result.

Finally, to handle both (WKL) and (BCT) at the same time, one has to define an n-forcing that alternates between the two forcing types, for example, using Harrington forcing at the odd stages and Brown–Simpson forcing at the even stages. Though the details may be tedious they present no conceptual difficulties, yielding:

Theorem 9.7. *$WKL+_0$ is conservative over RCA_0 for Π_1^1 sentences.*

10. The lengths of the proofs

Theorem 9.7 is not new, though the effective proof presented here is. We now take advantage of this effective proof to show that $WKL+_0$ does not have any significant speedup over RCA_0. That is, we show that when a proof in $WKL+_0$ is translated to one in RCA_0, there is at most a polynomial increase in length.

We assume that our proof systems are standard Hilbert-style proof systems. For the "length of a proof" we employ the measure used in [9] which counts the number of symbols involved.

Suppose one has proven a Π_1^1 sentence $\forall X \, \varphi(X)$ in $WKL+_0$ (we assume for simplicity that there is only one universal second-order quantifier). We examine the steps necessary to translating the proof p to one in RCA_0. The proof p is of the form

$$\varphi_1, \varphi_2, \ldots, \varphi_n(= \forall X \, \varphi(X)),$$

where each φ_i (1) is a logical axiom, (2) is an axiom of RCA_0, or (3) follows from previous φ_j by a deduction rule. The proof p will be replaced by a proof in RCA_0 having the following general form:

"φ_1 is ω-generically valid," "φ_2 is ω-generically valid," ...,

"φ_n is ω-generically valid."

This last line translates to

$$\forall X(\Vdash_\omega Name(X) \to \varphi(X)).$$

Now, arguing in RCA_0 let X be any set, and let $\hat{X}$ be its canonical name. Then we have

$$\Vdash_\omega \varphi(\hat{X})$$

followed by the conclusion

"$\varphi(X)$ is true."

That the logical axioms are ω-generically valid and that ω-generic validity is maintained by deductive inferences is guaranteed by Proposition 4.8, and the lengths of the proofs described there are polynomially related to the lengths of the relevant formulas. We can take RCA_0 (and hence WKL_0) to be finitely axiomatized (with only a polynomial increase in the length of proof over the more standard axiomatization); so proving that the axioms of WKL_0 are ω-generically valid can be done with a constant length of proof. Each instance of the Baire Category Theorem used in p also has to be shown to be ω-generically valid. It is straightforward to check that the lengths of proofs obtained from Lemma 5.8 (which states that an arithmetic formula is 1-forced if and only if it is true) and Lemma 8.2 (which states that an arithmetic formula involving only n-names if ω-forced if and only if it is n-forced) are polynomially bounded in the lengths of the formulas involved, as is then verifying that the length of the proof that an instance of (BCT) is ω-generically valid is also bounded by a polynomial in the length of the original axiom.

So, with a length of proof polynomial in the original, we can show that $\forall X\, \varphi(X)$ is ω-generically valid. Letting X be any set, RCA_0 can show that its canonical name $\hat{X}$ satisfies $\Vdash_\omega Name(X)$ in a constant-length proof, and hence conclude $\Vdash_\omega \varphi(\hat{X})$. Once again the application of Lemmas 5.8 and 8.2, which allow us to conclude that $\varphi(X)$ is true, can be done with a length-of-proof polynomial in the length of φ.

This gives us the main result of this paper, namely, Theorem 1.1.

11. Final comments

The work here was done independently of [5], in which P. Hájek shows that WKL_0 has no significant speedup over $I\Sigma_1$ in proving arithmetic sentences. Working with the language of recursion theory, he shows that one is able to abandon much of the forcing formalism and simply define an interpretation for the second-order sets of WKL_0 within $I\Sigma_1$. Though his arguments are streamlined and elegant, the methods here have some advantages:
  (1) The sets of our "model" M_ω include the sets of the base model, thus answering one of the questions posed at the end of [5].

(2) These methods work for the WKL_0+ as well; until this point, there was no way of proving the conservation result for the Baire Category Theorem effectively, even allowing for a nonpolynomial speedup.

(3) They can perhaps be adapted to other forcing arguments as well.

With regard to this last item, we remark that the methods of this paper can be applied to any argument involving iterated forcing over models of second-order arithmetic in which the basic forcing notions are of a complexity that does not grow in the iteration.

Acknowledgements

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