

# The monadic theory of order

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## Abstract

We deal with the monadic (second-order) theory of order. We prove all known results in a unified way, show a general way of reduction, prove more results and show the limitation on extending them. We prove (CH) that the monadic theory of the real order is undecidable. Our methods are model-theoretic, and we do not use automaton theory.

## 0. Introduction

The monadic logic is first order logic when we add variables ranging over sets, and allow quantification over them. If pairing functions are available this is essentially second order logic. The monadic theory of a class  $K$  of  $L$ -models is  $\{\psi: \psi \text{ is a sentence in monadic logic, satisfied by any member of } K\}$ .

Here we shall investigate cases where the members of  $K$  are linear orders (with one-place predicates).

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Let us review the history. Ehrenfeucht [Eh 1] proved the decidability of the first-order theory of order. Gurevich [Gu 1] deduced from it the case of linear order with one-place predicates. Büchi [B 1] and Elgot [El 1] proved the decidability of the weak monadic theory (i.e., we can quantify over finite sets) of (the order of)  $\omega$ , using automaton theory. Büchi continued in this direction, in [B 2], showing that also the monadic theory (i.e., quantification is possible over arbitrary sets) of  $\omega$  is decidable; and in [B 4] he showed the decidability of the weak monadic theory of ordinals. In [B 5, 96] he proved

the decidability of the monadic theory of countable ordinals. Rabin [Ra 1] proved a very strong and difficult result, implying the decidability of the monadic theory of countable orders. Büchi [B 5] showed the decidability of the monadic theory of  $\omega_1$  and of  $\{\alpha: \alpha < \omega_2\}$ .

Meanwhile Laüchli [La 1], using methods of Ehrenfeucht [Eh 2] and Fraïssé [Fr 1] and continuing works of Galvin (unpublished) and Laüchli and Leonard [LL 1], proved the decidability of the weak monadic theory of order. He did not use automaton theory. Pinus [Pi 1] strengthened, somewhat, those results. Our results have been announced in [Sh 1], [Sh 2].

By our notation Laüchli used  $Th_k^n$  only for  $\bar{k} = \langle 1, 1, 1, \dots \rangle$  (changed for the quantification over finite sets).

*Remark.* We are not interested here in results without the axiom of choice. See Siefkes [Si 2] which shows that the result on  $\omega$  is provable in ZF. This holds also for  $\alpha < \omega^\omega$ . Litman [Li 1] pointed out some mistakes in [B 5, § 6] (theorems without AC); proved connected results, and showed in ZF that  $\omega_1$  is always characterizable by a sentence.

In Section 7 we prove (CH) the undecidability of the monadic theory of the real order and of the class of orders, and related problems. It can be read independently, and has a discussion on those problems. Gurevich finds that our proof works also for the lattice of subsets of a Cantor discontinuum, with the closure operation, and similar spaces. Hence Grzegorzcz's [Gr 1] question is answered (under CH).<sup>1</sup>

Our work continues [La 1], but for well ordering we use ideas of Büchi and Rabin. We reduce here the decision problem of the monadic theories of some (classes of) orders [e.g., well orderings; the orders which do not embed  $\omega_1$  nor  $\omega_1^*$ ] to problems more combinatorial in nature. So we get a direct proof for the decidability of countable orders (answering a question of Büchi [B 5, p. 35]). Our proof works for a wider class, thus showing that the countable orders cannot be characterized in monadic theory, thus answering a question of Rabin [Ra 1, p. 12]. Moreover, there are uncountable orders which have the same monadic theory as the rationals (e.g., dense Specker order; see [Je 1] for their existence; and also some uncountable subsets of the reals). We also show that the monadic theory of  $\{\alpha: \alpha < \lambda^+\}$  is recursive in that of  $\lambda$ , generalizing results of Büchi for  $\omega$  and  $\omega_1$ . Unfortunately, even the monadic theory of  $\omega_2$  contains a statement independent of ZFC. For a set  $A$  of ordinals, let  $F(A) = \{\alpha: \alpha \text{ is a limit ordinal of cofinality } > \omega, \alpha < \sup A, \text{ and } \alpha \cap A \text{ is a stationary subset of } \alpha\}$ .

<sup>1</sup> Gurevich meanwhile has proved more and has a paper in preparation.

Now Jensen [J 1] proved the following:

**THEOREM 0.1** ( $V = L$ ). *A regular cardinal  $\kappa$  is weakly compact if and only if for every stationary  $A \subseteq \kappa$ , such that  $(\forall \alpha \in A) [cf \alpha = \omega]$ ,  $F(A) \neq \emptyset$ .*

As the second part is expressible in the monadic theory of order, the Hanf number of the monadic theory of order is high. Clearly also the monadic theory of the ordinals depends on an axiom of large cardinals.

Now, Baumgartner [Ba 1] shows that if  $ZFC +$  (there is a weakly compact cardinal) is consistent, then it is consistent with  $ZFC$  that

(\*) for any stationary  $A \subseteq \omega_2$ , if  $(\forall \alpha \in A) [cf \alpha = \omega]$ , then  $F(A) \neq \emptyset$  (and in fact is stationary).

So  $ZFC$  does not determine the monadic theory of  $\omega_2$ . This partially answers [B 5, p. 34-43; p. 38, Problem 3].

We can still hope that the number of possible such theories is small, and each decidable, but this seems unlikely. We can also hope to find the sentences true in every model of  $ZFC$ . A more hopeful project is to find a decision procedure assuming  $V = L$ . We show that for this it suffices to prove only the following fact. Let  $D_{\omega_2}$  be the filter of closed unbounded subsets of  $\omega_2$ . (Magidore disproves (\*\*) in  $V = L$ , but it may still be consistent with  $ZFC$ .)

(\*\*) if  $A \subseteq \{\alpha < \omega_2: cf \alpha = \omega\}$ ,  $F(A) = B \cup C$ ,  $A$  is stationary, then there are  $A_1, A_2$ , such that  $A = A_1 \cup A_2$ ,  $A_1 \cap A_2 = \emptyset$ ,  $A_1, A_2$  are stationary and  $F(A_1) = B \pmod{D_{\omega_2}}$ ,  $F(A_2) = C \pmod{D_{\omega_2}}$ .

We prove, in fact, more: that the monadic theory of  $\omega_2$  and the first order theory of  $\langle P(\omega_2)/D_{\omega_2}, \cap, \cup, F \rangle$  are recursive one in the other.

**Conjecture 0.1** ( $V = L$ ). The monadic theory of  $\omega_2$  (and even  $\omega_n$ ) is decidable.

**Conjecture 0.2** ( $V = L +$  there is no weakly compact cardinal). The monadic theory of well orders is decidable.

Laüchli and Leonard [LL 1] define a family  $\underline{M}$  of orders as follows: It is the closure of  $\{1\}$  by

(1)  $M + N$ ,

(2)  $M \cdot \omega$  and  $M \cdot \omega^*$ ,

(3)  $\sum_{i < \omega} M_i$  which is  $\sum_{a \in Q} M_a$  and  $\{a \in Q: M_a = M_i\}$  is a dense subset of the rationals, and each  $M_a \in \{M_i: i < \omega\}$ .

(See Rosenstein [Ro 1] and Rubin [Ru 1] for generalization.)

Laüchli [La 1] proved that every sentence from the weak monadic language of order has a countable model if and only if it has a model in  $\underline{M}$ . Easy checking of Section 4 shows this holds also for the monadic language. On the other hand, looking at the definition of  $\underline{M}$ , we can easily see that for every

$M \in \underline{M}$  there is a monadic sentence  $\psi$  such that  $M \models \psi$ , and  $\|N\| \leq \aleph_0$ ,  $N \models \psi$  imply  $N \cong M$ .

In this way we have a direct characterization of  $\underline{M}$ .

**THEOREM 0.2.**  $M \in \underline{M}$  if and only if  $M$  is countable and satisfies some monadic sentence which is  $(\leq \aleph_0)$ -categorical.

Also for other classes whose decidability we prove, we can find subclasses analogous to  $\underline{M}$ . This theorem raises the following question:

**Conjecture 0.3.** For every  $N \in \underline{M}$  there is a monadic sentence  $\psi$  such that  $M \models \psi$  implies that  $M$  and  $N$  have the same monadic theory. (It suffices to prove this for the rational order.)

Related questions are:

**Conjecture 0.4.** There is a monadic sentence  $\psi$  such that  $R \models \psi$  and  $M \models \psi$  imply that  $M$  and  $R$  have the same monadic theory.<sup>2</sup>

**Conjecture 0.5.** There is an order  $M$  which has the same monadic theory as  $R$ , but is not isomorphic to  $R$ .<sup>3</sup>

**Conjecture 0.6.** There are orders with the same monadic theories, whose completions do not have the same monadic theories.<sup>4</sup>

The characterization of  $\underline{M}$  gives us also

**Conclusion 0.3.** The question whether a sentence in the first-order (or even monadic) theory of order is  $(\leq \aleph_0)$ -categorical (or  $\aleph_0$ -categorical) is decidable.

A natural question is whether the monadic theory of  $\mathfrak{N}$  is more “complex” than that of the ordinals (the orders in  $\mathfrak{N}$  are countable unions of scattered types; see Laver [Lv §3], which includes results of Galvin). To answer this, we have the

**Definition.** For a model  $M$  with relations only, let  $M^*$  be the following model:

- (i) its universe is the set of finite sequences of elements of  $M$ ;
- (ii) its relations are
  - (a)  $<$ , where  $\bar{a} < \bar{b}$  means  $\bar{a}$  is an initial segment of  $\bar{b}$ ,
  - (b) for each  $n$ -place predicate  $R$  from the language of  $M$ ,

$$R^{M^*} = \{ \langle \langle a_1, \dots, a_{m-1}, b^1 \rangle, \langle a_1, \dots, a_{m-1}, b^2 \rangle, \dots, \langle a_1, \dots, a_{m-1}, b^n \rangle \rangle : \\ a_i, b^i \text{ are elements of } M, \text{ and } M \models R[b^1, \dots, b^n] \}.$$

<sup>2</sup> Confirmed by Gurevich.

<sup>3</sup> Refuted by Gurevich.

<sup>4</sup> Confirmed by Gurevich.

The author suggested a generalization of Rabin's automaton from [Ra 1], proved the easy parts: the lemmas on union and intersection, and solved the emptiness problem. Then J. Stup elaborated those proofs, and proved the complementation lemma. Thus a generalization of the theorem and proof of [Ra 1] gives

**THEOREM 0.4.** *The monadic theory of  $M^\#$  is recursive in the monadic theory of  $M$ .*

Thus, using [Lv 1, § 3] notation, we get, e.g.,

*Conclusion 0.5.* The monadic theory of  $\{M: M \in \mathfrak{M}, \|M\| \leq \lambda\}$  is recursive in the monadic theory of  $\lambda$ .

Because by Section 2 the monadic theory of  $\sigma_{\lambda^+, \lambda^+}$  is recursive in the monadic theory of  $\lambda$ , by 0.4 the monadic theory of  $\eta_{\lambda^+, \lambda^+}$  is recursive in the monadic theory of  $\lambda$ , and so we finish, as by [Lv 3.2 (iv), 3.4]  $\eta_{\lambda^+, \lambda^+}$  is a universal member of  $\{M \in \mathfrak{M}: \|M\| \leq \lambda\}$ .

Also useful are the following (Le Tourneau [To 1] proved parts (1), (2) at least):<sup>5</sup>

**THEOREM 0.6.** *Let  $L$  be a language with one one-place function symbol equality and one place predicates.*

(1) *The monadic theory of  $L$  is decidable.*

(2) *If a monadic sentence  $\psi$  of  $L$  has a model, it has a model of cardinality  $\leq \aleph_0$ .*

(3) *In (2) we can find  $n = n(\psi) < \aleph_0$  and a model  $M$  such that  $|\{b \in |M|: f(b) = a\}| \leq n$  for any  $a \in |M|$ .*

This is because, if  $M_\lambda$  is the model whose universe is  $\lambda$ , and whose language contains equality only, in  $M_\lambda^\#$  we can interpret a universal  $L$ -model (see Rabin [Ra 1]). This implies (1). Note that all  $M_\lambda$  ( $\lambda$  an infinite cardinal) have the same monadic theory. This proves (2). For (3) note that if  $M_{\aleph_0} \models \psi$ , then for all big enough  $n$ ,  $M_n \models \psi$ .

*Remark (1).* Rabin [Ra 1] prove the decidability of the countable Boolean algebras, in first-order logic expanded by quantification over ideals. By the Stone representation theorem, each countable Boolean algebra can be represented as the Boolean algebra generated by the intervals of a countable order. By the method of Section 3 we can prove that the theory of countable linear orders in monadic logic expanded by quantification over such ideals, is decidable, thus reproving Rabin's result. (The only point is that the methods of Section 2 apply.)

<sup>5</sup> Le Tourneau only claimed the result. Lately also Routenberg and Vinner proved this theorem.

**Conjecture 0.7.** The monadic theory of orders of cardinality  $\leq \aleph_1$  is decidable when  $\aleph_1 < 2^{\aleph_0}$ .

**Conjecture 0.8.** The theory of Boolean algebras of cardinality  $< \lambda$  or in first-order logic expanded by allowing quantification over ideals is decidable, when  $\lambda \leq 2^{\aleph_0}$  ( $\lambda = \aleph_2 \leq 2^{\aleph_0}$ ).

**Remark.** We can prove Conclusion 0.5 by amalgamating the methods of Sections 4, 5, and 6.

### 1. Ramsey theorem for additive coloring

A *coloring* of a set  $I$  is a function  $f$  from the set of unordered pairs of distinct elements of  $I$ , into a finite set  $T$  of colors. We write  $f(x, y)$  instead of  $f(\{x, y\})$ , assuming usually that  $x < y$ . The coloring  $f$  is *additive* if for  $x_i < y_i < z_i \in I$  ( $i = 1, 2$ ),

$$f(x_1, y_1) = f(x_2, y_2), \quad f(y_1, z_1) = f(y_2, z_2)$$

imply  $f(x_1, z_1) = f(x_2, z_2)$ . In this case a (partial) operation  $+$  is defined on  $T$ , such that for  $x < y < z \in I$ ,  $f(x, z) = f(x, y) + f(y, z)$ . A set  $J \subseteq I$  is *homogeneous* (for  $f$ ) if there is a  $t_0 \in T$  such that for every  $x < y \in J$ ,  $f(x, y) = t_0$ .

Ramsey's theorem [Rm 1] states, in particular, that if we color an infinite set with a finite set of colors, then there is an infinite homogeneous subset. This theorem has many generalizations and applications. It was used in [B 2] for a coloring which was, in fact, additive. Using an idea of Rabin, Büchi [B 5, (12, p. 58)] offered an alternative proof (using, in fact, additivity) and in [B 5, (6.2, p. 111)] straightforwardly generalized it to  $\omega_1$  (the result for  $\omega_1$  is not true for colorings in general). We give the natural extension to arbitrary ordinals (which is immediate, and included for completeness) and a parallel theorem for dense orders.

**THEOREM 1.1.** *If  $\delta$  is a limit ordinal,  $f$  an additive coloring of  $\delta$  (by a set  $T$  of  $n$  colors), then there is an unbounded homogeneous subset  $J$  of  $\delta$ .*

**Remarks.** (1) If the cofinality of  $\delta$  is  $\geq \omega_1$  we can assume that if  $a, b < c'$ ,  $f(a, c') = f(b, c')$ , then  $a, b < c \in J$  implies  $f(a, c) = f(b, c)$ .

(2) Instead of  $|T| < \aleph_0$ , we need assume only  $|T| < cf(\delta)$ .

**Conclusion 1.2.** Under the condition of 1.1, there are a closed unbounded subset  $J$  of  $\delta$ , and  $J_k, J^l$ ,  $1 \leq k, l \leq |T|$  and  $t_k^l \in T$  such that  $J = \bigcup_k J_k = \bigcup_l J^l$ , the  $J_k$ 's are disjoint, the  $J^l$ 's are disjoint, and if  $a < b \in J$ ,  $a \in J_k$ ,  $b \in J^l$  then  $f(a, b) = t_k^l$ .

**THEOREM 1.3.** *If  $f$  is an additive coloring of a dense set  $I$ , by a finite*

set  $T$  of  $n$  colors, then there is an interval of  $I$  which has a dense homogeneous subset.

*Conclusion 1.4.* Under the hypothesis of 1.3, there is an interval  $(a, b)$  of  $I$ , and  $(a, b) = \bigcup_{k=1}^{|T|} J_k = \bigcup_{l=1}^{|T|} J^l$  and colors  $t_k^l \in T$  such that for  $x < y$ ,  $x \in J_k$ ,  $y \in J^l$ ,  $f(x, y) = t_k^l$ .

*Remark.* We can choose the  $J_0, J_k, J^l$ 's so that they are definable by first-order formulas with parameters in the structure  $(\delta, <, f)$  (or  $(I, <, f)$ ).

*Proof of Theorem 1.1.* Define: For  $x, y \in \delta$ ,  $x \sim y$  if there is a  $z$  such that  $x, y < z < \delta$ , and  $f(x, z) = f(y, z)$ ; clearly this implies by the additivity of  $f$  that for any  $z', z < z' < \delta$ ,  $f(x, z') = f(y, z')$ . It is easy to verify that  $\sim$  is an equivalence relation with  $\leq |T|$  equivalence classes. So there is at least one equivalence class  $I$ , which is an unbounded subset of  $\delta$ . Let  $x_0$  be the first element of  $I$ . Let, for  $t \in T$ ,  $I_t = \{y: x_0 \neq y \in I, f(x_0, y) = t\}$ . Clearly  $I - \{x_0\} = \bigcup_{t \in T} I_t$ , hence for some  $s$ ,  $I_s$  is an unbounded subset of  $\delta$ . Let  $\langle a_i: i < cf\delta \rangle$  be an increasing unbounded sequence of elements of  $\delta$ . Define by induction on  $i$  elements  $y_i \in I$ . If for all  $j < i$  ( $i < cf\delta$ ),  $y_j$  have been defined, let  $y_i < \delta$  be such that  $y_i > y_j$ ,  $y_i > a_j$ ,  $y_i > x_0$  and  $f(x_0, y_i) = f(y_j, y_i)$  for any  $j < i$ , and  $y_i \in I_s$ . Now  $J = \{y_i: i < cf\delta\}$  is the desired set. Clearly it is unbounded. If  $y_j < y_i$  (hence  $j < i$ ) then

$$f(y_j, y_i) = f(x_0, y_i) = s.$$

So  $J$  is homogeneous.

*Proof of Conclusion 1.2.* If the cofinality of  $\delta$  is  $\aleph_0$ , then the  $J$  from 1.1 is also closed (trivially). So assume  $cf\delta > \aleph_0$ , let  $T = \{t_1, \dots, t_n\}$ , and let  $J, y_j$  be as defined in the proof of 1.1; and let  $J^*$  be the closure of  $\{y_{j+1}: j < cf\delta\}$ . Then  $J^* = \{y^j: j < cf\delta\}$  is increasing, continuous, and  $y^{j+1} = y_{j+1}$ . Let  $J' = \{y^j: j \text{ is a limit ordinal}\}$ ,

$$J_k = \{y^j: j \text{ is a limit ordinal, } f(y^j, y^{j+1}) = t_k\},$$

$$J^l = \{y^j: j \text{ is a limit ordinal, and } (\forall i < j)(\exists \alpha) (i < \alpha < j \wedge f(y^{\alpha+1}, y^j) = t_l)\}$$

but this does not hold for any  $l' < l$ .

Now clearly  $J' = \bigcup_k J_k = \bigcup_l J^l$ , and if  $x \in J_k$ ,  $z \in J^l$ ,  $x < z$  then  $x = y^i$ ,  $z = y^j$ ,  $i < j$ ,  $i, j$  are limit ordinals and there is an  $\alpha$ ,  $i < \alpha < j$ , such that  $f(y^{\alpha+1}, y^j) = t_l$ . Hence

$$\begin{aligned} f(x, z) &= f(y^i, y^j) = f(y^i, y^{i+1}) + f(y^{i+1}, y^{\alpha+1}) + f(y^{\alpha+1}, y^j) \\ &= t_k + f(y_{i+1}, y_{\alpha+1}) + t_l = t_k + s + t_l \stackrel{\text{def}}{=} t_k^l. \end{aligned}$$

Clearly all the demands are satisfied.

*Proof of Theorem 1.3.* Remember that  $J \subseteq I$  is dense in an interval  $(a, b)$

if for every  $x, y \in I$ ,  $a < x < y < b$ , there is a  $z \in J$  such that  $x < z < y$ . It is easy to see that if  $J \subseteq I$  is dense in an interval  $(a, b)$  and  $J = \bigcup_{k=1}^m J_k$  ( $m > 1$ ) then there are  $k$  and  $a', b'$  such that  $a < a' < b' < b$ ,  $1 \leq k \leq m$  and  $J_k$  is dense in  $(a', b')$ .

Define for any  $a \in I$ ,  $J \subseteq I$

$$F(a, J) = \{t: t \in T, (\forall x > a) (\exists y \in J)(a < y < x \wedge f(a, y) = t)\}.$$

Notice, that since  $T$  is finite, for any  $a \in I$ , and any  $J \subseteq I$  there is a  $b$ ,  $a < b \in I$  such that:

$$t \in F(a, J) \text{ if and only if there is a } y \in J, a < y < b, f(a, y) = t.$$

We define by induction on  $m \leq n2^n + 2$  intervals  $(a_m, b_m)$ , sets  $J_m$  dense in  $(a_m, b_m)$ , and (for  $m > 0$ ) sets  $D_m \subseteq T$ .

For  $m = 0$ , let  $(a_0, b_0)$  be any interval of  $I$ , and  $J_0 = \{x \in I: a_0 < x < b_0\}$ . Suppose  $(a_m, b_m)$ ,  $J_m$  are defined. For any  $D \subseteq T$  let  $J_m(D) = \{a \in J_m: F(a, J_m) = D\}$ . Clearly  $J_m = \bigcup_{D \subseteq T} J_m(D)$  and as there are only finitely many possible  $D$ 's ( $\leq 2^n$ ), there is an interval  $(a_{m+1}, b_{m+1})$  and  $D_{m+1} \subseteq T$  such that  $J_m(D_{m+1})$  is dense in  $(a_{m+1}, b_{m+1})$ , and  $a_m < a_{m+1} < b_{m+1} < b_m$ . Let  $J_{m+1} = (a_{m+1}, b_{m+1}) \cap J_m(D_{m+1})$ . Clearly  $J_m \supseteq J_{m+1}$ , and  $m > k$  implies  $J_k \supseteq J_m$ , and  $(a_m, b_m)$  is a sub-interval of  $(a_k, b_k)$ .

As there are only  $\leq 2^n$  possible  $D_m$ , there are a  $D \subseteq T$  and  $0 \leq m_0 < \dots < m_n \leq n2^n + 1$  such that  $D_{m_i+1} = D$ . Define, for  $0 \leq k \leq n$ ,  $a^k = a_{m_k}$ ,  $b^k = b_{m_k}$ ,  $J^k = J_{m_k}$ .<sup>6</sup>

It is easy to check that if  $0 \leq k < l \leq n$ ,  $x \in J^l$  then  $x \in J_{m_l} \subseteq J_{m_k+1}$ , hence  $F(x, J^k) = F(x, J_{m_k}) = D_{m_k+1} = D$ . It is clear that  $J^0 \supseteq J^1 \supseteq \dots \supseteq J^n$ .

Choose  $x_0 \in J^n$ . Then there is  $x_1$ ,  $x_0 < x_1 < b^n$ , such that  $x_0 < y < x_1$ ,  $y \in J^0$  implies  $f(x_0, y) \in F(x_0, J^0) = D$ . Hence  $t \in D$  if and only if there is  $y \in J^{n-1}$ ,  $x_0 < y < x_1$ ,  $f(x_0, y) = t$ , if and only if there is  $y \in J_0$ ,  $x_0 < y < x_1$ ,  $f(x_0, y) = t$ . Clearly

$$J^n \cap (x_0, x_1) = \bigcup_{t \in T} \{y: y \in J^n, x_0 < y < x_1, f(x_0, y) = t\}.$$

Hence there are  $a, b, t_0$  such that  $x_0 < a < b < x_1$  and

$$J^* = \{y: y \in J^n, a < y < b, f(x_0, y) = t_0\}$$

is dense in  $(a, b)$ . Clearly  $t_0 \in D$ .

It is easy to check that for  $t, s \in D$ ,  $t + s$  is defined and  $\in D$ , so for  $t \in D$ ,  $m \geq 1$  define  $mt \in T$ , by induction on  $m$ :  $1t = t$ ,  $(m+1)t = mt + t$ . As  $T$  has  $n$  elements,  $1t_0, 2t_0, \dots, (n+1)t_0$  cannot be pairwise distinct. So there are  $i, j$ ,  $1 \leq i < (i+j) \leq n+1$  such that  $it_0 = (i+j)t_0$ . Define

<sup>6</sup> In fact  $D_m(T) \supseteq D_n(T)$ , hence we can replace  $n2^n + 2$  by  $n^2 + 2$ .



$$J = \{y: a < y < b, f(x_0, y) = jt_0, y \in J^{n-j+1}\}.$$

We shall show that  $J$  is the desired set.

(I)  $J$  is dense in  $(a, b)$ .

Suppose  $a < a' < b' < b$ , and we shall find  $z \in J$ ,  $a' < z < b'$ . As  $J^*$  is dense in  $(a, b)$  there are  $z^n \in J^* \subseteq J^n$ ,  $a' < z^n < b'$ . We define by downward induction  $z^k$  for  $n - j + 1 \leq k \leq n$  such that  $z^k \in J^k$ ,  $a' < z^k < b'$ . For  $k = n$ ,  $z^k$  is defined. Suppose  $z^{k+1}$  is defined, then as  $z^{k+1} \in J^{k+1}$  it follows that  $F(z^{k+1}, J^k) = D$ . As  $t_0 \in D$  there is  $z^k \in J^k$ , such that  $z^{k+1} < z^k < b'$  and  $f(z^{k+1}, z^k) = t_0$ . Clearly

$$x < z^n < z^{n-1} < \dots < z^{n-j+1}, \\ f(x_0, z^n) = t_0, \quad f(z^{i+1}, z^i) = t_0.$$

Hence  $f(x_0, z^{n-j+1}) = t_0 + \dots + t_0 = jt_0$ , so  $z^{n-j+1} \in J$ ,  $a' < z^{n-j+1} < b'$ .

(II)  $J$  is homogeneous.

Suppose  $a < y < z < b$ ,  $y, z \in J$ . Then  $y \in J^{n-j+1}$ . Now define by downward induction  $y^k \in J^k$  for  $0 \leq k \leq i$ ,  $y \leq y^k < z$ . Let  $y^i = y$  ( $y^i \in J^i$  because  $y^i = y \in J^{n-j+1}$ , and as  $i + j \leq n + 1$ ,  $i \leq n - j + 1$  hence  $J^{n-j+1} \subseteq J^i$ ). If  $y^{k+1}$  is defined then  $F(y^{k+1}, J^k) = D$ , hence there are  $y^k \in J^k$ ,  $y^{k+1} < y^k < z$  such that  $f(y^{k+1}, y^k) = t_0$ . It follows that  $x_0 < y = y^i < y^{i-1} < \dots < y^0 < z$  and

$$f(y^k, y^{k-1}) = t_0.$$

Hence

$$f(y, y^0) = f(y^i, y^0) = it_0.$$

So

$$\begin{aligned} f(y, z) &= f(y, y^0) + f(y^0, z) = it_0 + f(y^0, z) \\ &= (i + j)t_0 + f(y^0, z) = jt_0 + it_0 + f(y^0, z) \\ &= f(x_0, y) + f(y, y^0) + f(y^0, z) = f(x_0, z) = jt_0. \end{aligned}$$

This proves the homogeneity of  $J$ .

*Proof of Conclusion 1.4.* Let  $(a, b)$ ,  $J$  and  $t_0$  be as in the proof of 1.3. Let  $T = \{t_1, \dots, t_n\}$ . Let

$$\begin{aligned} J_k &= \{y: y \in (a, b), t_k \in F(y, J), t_1, \dots, t_{k-1} \notin F(y, J)\}, \\ J^l &= \{y: y \in (a, b), t_l \in F'(y, J), t_1, \dots, t_{l-1} \notin F'(y, J)\} \end{aligned}$$

where  $F'$  is defined just as  $F$  is, but for the reversed order.

Clearly  $(a, b) = \bigcup_k J_k = \bigcup_l J^l$ . Suppose  $x < y$ ,  $x \in J_k$ ,  $y \in J_l$ . Then we can find  $x', y'$ ,  $x < x' < y' < y$ ,  $x', y' \in J$ , such that  $f(x, x') = t_k$ ,  $f(y', y) = t_l$ . Hence

$$f(x, y) = f(x, x') + f(x', y') + f(y', y) = t_k + t_0 + t_l \stackrel{\text{def}}{=} t_k^l.$$

## 2. The monadic theory of generalized sums

Feferman and Vaught [FV 1] proved that the first order theory of sum, product, and even generalized products of models depends only on the first-order theories of the models. Their theorem has generalizations to even more general products (see Olmann [Ol, 1]) and to suitable infinitary languages ( $L_\alpha$ , see Malitz [Ma 1]). On the other hand, it is well-known that for second order theory this is false even for sum (as there is a sentence true in the sum of two models if and only if they are isomorphic, for fixed finite language, of course). Also for monadic (second-order) theory this is false for products of models (there is a sentence true in a direct product of two models of the theory of linear order if and only if the orders are isomorphic). We notice here that the monadic theory of generalized sum depends only on the monadic theories of the summands and notice also generalization of known refinement (see Fraïssé [Fr 1]). We can prove them using natural generalizations of Ehrenfeucht games (see [Eh 1]). Laüchli [La 1] uses some particular cases of those theorems for the weak monadic theory. As there is no new point in the proofs, we skip them. We should notice only that a subset of sum of models is the union of subsets of the summands. The results of [FV] can be applied directly by replacing  $M$  by  $(|M| \cup \underline{P}(M), M, \in)$ .

*Notation.*  $L$  will be first-order language with a finite number of symbols,  $L^M$  the corresponding monadic language,  $L(M)$  the first-order, language corresponding to the model  $M$ , the universe of  $M$ , is  $|M|$ . Let  $x, y, z$  be individual variables;  $X, Y, Z$  set variables;  $a, b, c$  elements;  $P, Q$  sets;  $\underline{P}(M) = \{P: P \subseteq |M|\}$ . Bar denotes that this is a finite sequence, e.g.,  $\bar{a}; l(\bar{a})$  its length,  $\bar{a} = \langle \dots, a_i, \dots \rangle_{i < l(\bar{a})}$ , and let  $\bar{a}(i) = a_i$ . We write  $\bar{a} \in A$  instead of  $a_i \in A$  and  $\bar{a} \in M$  instead of  $\bar{a} \in |M|$ .  $K$  is a class of  $L(K)$  models ( $L(K) = L(M)$  for any  $M \in K$ ). Let

$$K^m = \{(M, \bar{P}): \bar{P} \in \underline{P}(M)^m\}, K^\infty = \bigcup_{m < \omega} K^m.$$

Let  $k, l, m, n, p, q, r$  denote natural numbers.

*Definition 2.1.* For any  $L$ -model  $M$ ,  $\bar{P} \in \underline{P}(M)$ ,  $\bar{a} \in |M|$ ,  $\Phi$  a finite set of formulas  $\varphi(X_1, \dots, x_1, \dots) \in L$ , a natural number  $n$ , and a sequence of natural numbers  $\bar{k}$  of length  $\geq n$ , define

$$t = th_{\bar{k}}^n((M, \bar{P}, \bar{a}), \Phi)$$

by induction on  $n$ :

For  $n = 0$ :

$$t = \{\varphi(X_{i_1}, \dots, x_{j_1}, \dots): \varphi(X_1, \dots, x_1, \dots) \in \Phi, M \models \varphi[P_{i_1}, \dots, a_{j_1}, \dots]\}.$$

For  $n = m + 1$ :

$$t = \{th_k^m(M, \bar{P}, \bar{a} \wedge \bar{b}) : \bar{b} \in |M|^{\bar{k}(m)}\}.$$

**Definition 2.2.** For any  $L$ -model  $M$ ,  $\bar{P} \in \underline{P}(M)$ , a finite set  $\Phi$  of formulas  $\varphi(X_1, \dots, x_1, \dots) \in L$ ,  $n$ ,  $\bar{k}$  of length  $\geq n + 1$ , define  $T = Th_k^n((M, \bar{P}), \Phi)$  by induction on  $n$ :

For  $n = 0$ :

$$T = th_k^1((M, \bar{P}), \Phi).$$

For  $n = m + 1$ :

$$T = \{Th_k^m((M, \bar{P} \wedge \bar{Q}), \Phi) : \bar{Q} \in \underline{P}(M)^{\bar{k}(n)}\}.$$

**Remarks.** (1) If  $\Phi$  is the set of atomic formulas we shall omit it and write  $Th_k^n(M, \bar{P})$ .

(2) We always assume  $\bar{k}(i) \geq 1$  for any  $i < l(\bar{k})$ , and  $\bar{k}(0) \geq m_R$  if  $R \in L(M)$  is  $m_R$ -place.

(3) If we write  $\bar{k}(i)$  for  $i \geq l(\bar{k})$ , then we mean 1, and when we omit  $\bar{k}$  we mean  $\langle \max m_R, 1, \dots \rangle$ .

(4) We could have mixed Definitions 2.1, and 2.2, and obtained a similar theorem which would be more refined.

**LEMMA 2.1.** (A) For every formula  $\psi(\bar{X}) \in L^M(M)$  there is an  $n$  such that from  $Th_k^n(M, \bar{P})$  we can find effectively whether  $M \models \psi[\bar{P}]$ .

(B) For every  $L, \bar{k}, n, \Phi \subseteq L$ , and  $m$  there is a set  $\Psi = \{\psi_l(\bar{X}) : l < l_0(<\omega), l(\bar{X}) = m\}$  ( $\psi_l \in L^M$ ) such that for any  $L$ -models  $M, N$  and  $\bar{P} \in \underline{P}(M)^m$ ,  $\bar{Q} \in \underline{P}(N)^m$  the following hold:

(1)  $Th_k^n((N, \bar{Q}), \Phi)$  can be computed from  $\{l < l_0 : N \models \psi_l[\bar{Q}]\}$ .

(2)  $Th_k^n((N, \bar{Q}), \Phi) = Th_k^n((M, \bar{P}), \Phi)$  if and only if for any  $l < l_0$ ,  $M \models \psi_l[\bar{P}] \Leftrightarrow N \models \psi_l[\bar{Q}]$ .

**Proof.** Immediate. In (A) it suffices to take for  $n$  the quantifier depth of  $\psi$ .

**LEMMA 2.2.** (A) For given  $L, n, m, \bar{k}$ , each  $Th_k^n(M, \bar{P})$  is hereditarily finite, and we can compute the set of formally possible  $Th_k^n(M, \bar{P}), l(\bar{P}) = m$ ,  $M$  an  $L$ -model. The same holds for  $\Phi$ .

(B) If  $\bar{l}(0) \geq \bar{k}(0)$ ,  $1 = p_0 < p_1 < p_2 < \dots < p_n \leq m$  and for  $1 \leq i \leq n$ ,  $\bar{l}(i) \leq \sum_{p_{i-1} \leq j \leq p_i} \bar{l}(j)$  then from  $Th_i^n((M, \bar{P}), \Phi)$  we can effectively compute  $Th_k^n((M, \bar{P}), \Phi)$ .

(C) For every  $n, \bar{k}, \bar{l}$  we can compute  $m$  such that from  $Th_i^n((M, \bar{P}), \Phi)$  we can effectively compute  $Th_k^n((M, \bar{P}), \Phi)$ .

(D) Suppose in Definition 2.2 we make the following changes: We

restrict ourselves to partition  $\bar{P}$ , and let  $\bar{Q}$  be a partition refining  $\bar{P}$ , which divides each  $P_i$  to  $2^{k(m)}$  parts. What we get we call  $pTh_k^n((M, \bar{P}), \Phi)$ . Then from  $pTh_k^n((M, \bar{P}), \Phi)$  we can effectively compute  $Th_k^n((M, \bar{P}), \Phi)$ , and vice-versa.

(E) Let  $K, n, \Phi$  be given. If for every  $\bar{k}$  there is an  $\bar{l}$  such that for every  $m, M, N \in K^m$ ,

$$Th_l^n(M, \Phi) = Th_l^n(N, \Phi) \implies Th_{\bar{k}}^{n+1}(M, \Phi) = Th_{\bar{k}}^{n+1}(N, \Phi)$$

then for every  $m, \bar{k}$  there is an  $\bar{l}$  such that for any  $m, M, N \in K^m$

$$Th_l^n(M, \Phi) = Th_l^n(N, \Phi) \implies Th_{\bar{k}}^m(M, \Phi) = Th_{\bar{k}}^m(N, \Phi).$$

*Remark.* This is parallel to elimination of quantifiers.

(F) In (E), if in the hypothesis  $\bar{l}$  can be found effectively from  $\bar{k}$  then in the conclusion,  $\bar{l}$  can be found effectively from  $m, \bar{k}$ . If in addition  $\{Th_{\bar{k}}^n(M, \Phi): M \in K^m\}$  is recursive in  $\bar{k}, m$  then  $\{Th_{\bar{k}}^n(M, \Phi): M \in K\}$  is recursive in  $p, \bar{k}$ .

*Proof.* Immediate.

The following generalizes the ordered sum of ordered sets (which will be our main interest) to the notion of a generalized sum of models. (Parts (1), (2), (3) of the definition are technical preliminaries.)

*Definition 2.3.* Let  $L_1, L_2, L_3$  be first-order languages,  $M_i$  an  $L_1$ -model (for  $i \in |N|$ ),  $N$  an  $L_2$ -model, and we shall define the  $L_3$ -model  $M = \sum_{i \in |N|}^{\sigma} M_i$  (the generalized sum of the  $M_i$ 's relative to  $\sigma$ ).<sup>7</sup>

(1) An  $n$ -condition  $\tau$  is a triple  $\langle E, \Phi, \Psi \rangle$  where:

(A)  $E$  is an equivalence relation on  $\{0, 1, \dots, n-1\}$ .

(B)  $\Phi$  is a finite set of formulas of the form  $\varphi(x_{j_1}, \dots, x_{j_k})$  where  $j_1, \dots, j_k$  are  $E$ -equivalent and  $< n$ ; and  $\varphi \in L_1$ .

(C)  $\Psi$  is a finite set of formulas of the form  $\psi(x_{j_1}, \dots, x_{j_k})$  where  $j_1, \dots, j_k < n$ ,  $\psi \in L_2$ .

(2) If  $a_0, \dots, a_{n-1} \in \bigcup_{i \in |N|} M_i$ ,  $\tau = \langle E, \Phi, \Psi \rangle$  is an  $n$ -condition,  $a_l \in M_{i(l)}$ , then we say  $\langle a_0, \dots, a_{n-1} \rangle$  satisfies  $\tau$  if:

(A)  $i(l) = i(m) \iff lEm$ ;

(B)  $\varphi(x_{j_1}, \dots, x_{j_k}) \in \Phi \implies M_{i(j_1)} \models \varphi[a_{j_1}, \dots, a_{j_k}]$ ;

(C)  $\psi(x_{j_1}, \dots, x_{j_k}) \in \Psi \implies N \models \psi[i(j_1), \dots, i(j_k)]$ .

(3) The rule,  $\sigma$ , is  $\langle L_1, L_2, L_3, \sigma^* \rangle$  where  $\sigma^*$  is a function whose domain is the set of predicates of  $L_3$ ; if  $R$  is an  $n$ -place predicate in  $L_3$ ,  $\sigma^*(R)$  will be a finite set of  $n$ -conditions.

<sup>7</sup> We assume, of course, that the  $M_i$ 's are pairwise disjoint.

(4)  $M = \sum_{i \in |N|}^{\sigma} M_i$  is an  $L_3$ -model, whose universe is  $\bigcup_{i \in |N|} |M_i|$ , and for every predicate  $R \in L_3$ ,  $R^M = \{\langle a_0, \dots, a_{n-1} \rangle : \langle a_0, \dots, a_{n-1} \rangle \text{ satisfies some } \tau \in \sigma^*(R)\}$ .

Let  $\Phi(\sigma)(\Psi(\sigma))$  be the set of all formulas  $\varphi_j \in L_1(\sigma)$  ( $\psi_p \in L_2(\sigma)$ ) appearing in the  $\sigma(R)$ 's,  $R \in L_3(\sigma)$ , and the equality.

*Remarks.* (1) We use the convention that

$$\sum_{i \in N}^{\sigma} (M_i, \bar{P}^i) = (\sum_{i \in N}^{\sigma} M_i, \bigcup_{i \in N} \bar{P}^i)$$

where for  $\bar{P}^i = \langle P_1^i, \dots, P_m^i \rangle$ ,  $\bigcup_i \bar{P}^i = \langle \bigcup_i P_1^i, \dots, \bigcup_i P_m^i \rangle$ .

(2) We could have defined the sum more generally, by allowing the universe and the equality to be defined just as the other relations.

LEMMA 2.3. For any  $\sigma, n, m, \bar{k}$ , if for  $l = 1, 2$ ,  $\bar{P}_l^i \in \underline{P}(M_l^i)^m$  and for every  $i \in N$ ,

$$Th_{\bar{k}}^n((M_l^i, \bar{P}_l^i), \Phi(\sigma)) = Th_{\bar{k}}^n((M_l^i, \bar{P}_l^2), \Phi(\sigma)),$$

then

$$Th_{\bar{k}}^n(\sum_{i \in N}^{\sigma} (M_l^i, \bar{P}_l^i)) = Th_{\bar{k}}^n(\sum_{i \in N}^{\sigma} (M_l^i, \bar{P}_l^2)),$$

THEOREM 2.4. For any  $\sigma, n, m, \bar{k}$  we can find an  $\bar{r}$  such that: if  $M = \sum_{i \in N}^{\sigma} M_i$ ,  $t_i = Th_{\bar{k}}^n((M_i, \bar{P}_i), \Phi(\sigma))$ , and  $Q_t = \{i \in N : t_i = t\}$ ,  $l(\bar{P}_i) = m$ , then from  $Th_{\bar{r}}^n((N, \dots, Q_i, \dots), \Psi(\sigma))$  we can effectively compute  $Th_{\bar{k}}^n(M, \bigcup_i \bar{P}_i)$  (which is uniquely determined).

Definition 2.4. (1) For a class  $K$  of models

$$Th_{\bar{k}}^n(K, \Phi) = \{Th_{\bar{k}}^n(M, \Phi) : M \in K\}.$$

(2) The monadic theory of  $K$  is the set of monadic sentences true in every model in  $K$ .

(3) For any  $\bar{\sigma}$ ,  $K_1, K_2$ , let  $Cl^{\bar{\sigma}}(K_1, K_2)$  be the minimal class  $K$  such that

(A)  $K_1 \subseteq K$ ,

(B) if  $j < l(\bar{\sigma})$ ,  $M_i \in K$ ,  $N \in K_2$  then  $\sum_{i \in |N|}^{\bar{\sigma}(i)} M_i \in K$ .

Conclusion 2.5. Suppose  $\bar{\sigma}, n, \bar{k}, m$  are given.  $L_1(\sigma_i) = L_3(\sigma_i) = L$ ,  $L_2(\sigma_i) = L_2$ ;  $L, L_2$  are finite and each  $\Phi(\sigma_i), \Psi(\sigma_i)$  is a set of atomic formulas. There is an  $\bar{r}$  such that for every  $K_1, K_2$ , from  $Th_{\bar{r}}^n(K_2^{\bar{r}(n+1)}), Th_{\bar{k}}^n(K_1^m)$  we can effectively compute  $Th_{\bar{k}}^n(K^m)$  where  $K = Cl^{\bar{\sigma}}(K_1, K_2)$  (remember  $K_1^m = \{(M, \bar{P}) : M \in K_1, \bar{P} \in \underline{P}(M)^m\}$  ( $K_1$  should be a class of  $L$ -models,  $K_2$  a class of  $L_2$ -models).

*Proof.* For every  $j < l(\bar{\sigma})$  let  $\bar{r}^j$  relate to  $\bar{\sigma}(j)$ ,  $n, \bar{k}, m$  just as  $\bar{r}$  relates to  $\sigma, n, k, m$  in Theorem 2.4. Now choose an  $\bar{r}$  such that for every  $l \leq n$ ,  $\bar{r}(l) \geq \bar{r}^j(l)$ .

Let  $T$  be the set of formally possible  $Th_{\bar{k}}^n(M, \bar{P})$ , for  $M$  an  $L$ -model,

$l(\bar{P}) = m$ , and we can define  $r(n+1) = |T|$ . Let  $T = \{t(0), \dots, t(p-1)\}$  (so  $p = |T| = r(n+1)$ ).

Clearly, by the definition of  $\bar{r}^j$ , and by (a trivial case of) 2.2 (B), if  $M = \sum_{i \in N}^{(j)} M_i$ ,  $t_i = Th_k^n(M_i, \bar{P}_i)$ ,  $Q_i = \{i \in N: t_i = t(l)\}$ ,  $l(\bar{P}_i) = m$ , then from  $t = Th_{\bar{r}}^n(N, \dots, Q_i, \dots)_{i < p}$  we can effectively compute  $Th_k^n(M, \bigcup_i \bar{P}_i)$ , and denote it by  $g(t)$ .

Now define by induction on  $l$ ,  $T_l \subseteq T$ .

Let  $T_0 = Th_k^n(K_1^m)$ , and if  $T_q$  is defined let  $T_{q+1}$  be the union of  $T_q$  with the set of  $t \in T$  satisfying the following condition:

(\*) There is a  $t^* \in Th_{\bar{r}}^n(K_2^{r(n+1)})$  such that  $t = g(t^*)$ , and if  $t^*$  implies that  $Q_i$  is not empty, then  $t(l) \in T_q$ .

*Remark.* Clearly if  $t^* = Th_{\bar{r}}^n(N, \dots, Q_i, \dots)$  then from  $t^*$  we can compute  $Th_{\bar{r}}^0(N, \dots, Q_i, \dots)$  and hence know whether  $Q_i \neq \emptyset$ .

Clearly  $T_0 \subseteq T_1 \subseteq T_2, \dots \subseteq T$  so, as  $|T| = p$ , for some  $q \leq p$ ,  $T_q = T_{q+1}$ .

Now let

$$K_* = \{M \in K: \text{for every } \bar{P} \in (\underline{P}(|M|)^m) Th_k^n(M, \bar{P}) \in T_q\}.$$

Clearly  $Th_k^n(K_*^m) \subseteq T_q$ , and we can effectively find  $T_q$ . Now if  $N \in K_2$ ,  $M_i \in K_*$  for  $i \in N$ , and  $M = \sum_{i \in N}^{(j)} M_i$ , then for any  $\underline{P} \in \bar{P}(|M|)^m$ ,  $Th_k^n(M, \underline{P}) \in T_{q+1} = T_q$  by the definition of  $T_{q+1}$ , and  $M \in K$  by the definition of  $K$ , hence  $M \in K_*$ . As clearly  $K_1 \subseteq K_* \subseteq K$ , by the definition of  $K = Cl^{\bar{r}}(K_1, K_2)$  necessarily  $K_* = K$ . So it suffices to prove that  $Th_k^n(K_*^m) \supseteq T_l$ . (Take  $l = q$ .) This is done by induction on  $l$ .

**LEMMA 2.6.** *If  $M$  is a finite model, then for any  $\Phi, n, \bar{k}$  we can effectively compute  $Th_k^n(M, \Phi)$  from  $M$ .*

*Remark 2.7.* Naturally we can ask whether we can add to (or replace the) monadic quantifiers (by) other quantifiers, without essentially changing the conclusions of this section. It is easily seen that, e.g., the following quantifiers are suitable:

- (1)  $(\exists^f X)$  —there is a finite set  $X$
- (2)  $(\exists^\lambda X)$  —there is a set  $X$ ,  $|X| < \lambda$  ( $\lambda$  a regular cardinal). When dealing with ordered sums of linear order, also
- (3)  $(\exists^{w.o.} X)$  —there is a well-ordered set  $X$
- (4)  $(\exists_\lambda X)$  —there is a set  $X$ , with no increasing nor decreasing sequence in it of length  $\lambda$  ( $\lambda$  a regular cardinal).

If we add some of those quantifiers, we should, in the definition of  $Th_n^0((M, \bar{P}), \Phi)$  state which Boolean combinations of the  $P_i$ 's are in the range of which quantifiers. If we e.g., replace the monadic quantifier by  $(\exists^f X)$ , we

should restrict the  $P$ 's to sets of cardinality  $< \lambda$ .

Another possible generalization is to generalized products. Let  $M = \prod_{i \in N}^\sigma M_i$  (where  $L(M_i) = L_1(\sigma)$ ,  $L(N) = L_2(\sigma)$ ,  $L(M) = L_3(\sigma)$ ) means:  $|M| = \prod_{i \in N} |M_i|$ , and if  $f_1, \dots, f_n \in M$ ,  $M \models R[f_1, \dots, f_n]$  if and only if  $N \models \psi_R[\dots, P_i, \dots]$  where

$$P_i = \{i \in N: M_i \models \varphi_i^R[f_1(i), \dots, f_n(i)]\}$$

(and  $\varphi_i$  is a first order sentence from  $L_1(\sigma)$ ,  $\psi_R$  a monadic sentence from  $L_3(\sigma)$ ). Then, of course, we use  $Th_k^n(N, \bar{P})$ ,  $th_k^n(M_i, \bar{a})$ , and  $th_k^n(M, \bar{a})$ . All our theorems generalize easily, but still no application was found.

If not specified otherwise, we restrict ourselves to the class  $K_{\text{ord}}$  of models of the theory of order (sometimes with one-place relations which will be denoted, e.g.,  $(M, \bar{P})$ ).  $\sigma = \sigma_{\text{ord}}$  is the ordered sum of ordered sets and is omitted. Therefore  $\psi(\sigma)$  and  $\Phi(\sigma)$  are the set of atomic formulas. For the sum of two orders we write  $M_1 + M_2$ . The ordinals, the reals  $\mathbf{R}$ , and the rationals  $\mathbf{Q}$  have their natural orders. If  $M = \sum_{i \in |N|} M_i$  we write  $Th_k^n(M, \bar{P}) = \sum_{i \in |N|} Th_k^n(M_i, \bar{P}_i)$  where  $\bar{P} = \bigcup_i \bar{P}_i$ . Let  $T(n, m, \bar{k})$  be the set of formally possible  $Th_k^n(M, \bar{P})$ ,  $M$  an order,  $l(\bar{P}) = m$ .

**COROLLARY 2.8.** *For any  $n, m, \bar{k}$  there is  $\bar{r} = \bar{r}(n, m, \bar{k})$  such that if  $P_t = \{i \in N: t_i = t\}$  for  $t \in T(n, m, \bar{k})$  then  $\sum_{i \in N} t_i$  can be effectively computed from  $Th_{\bar{r}}^n(N, \dots, P_i, \dots)$ .*

### 3. Simple application for decidability

Using Section 2 we shall prove here some theorems, most of them known. We prove the decidability of the theories of the finite orders, the countable ordinals ([B 5]), and show that from the monadic theory of  $\lambda$  we can compute effectively the monadic theory of  $K = \{\alpha: \alpha < \lambda^+\}$  (this was shown for  $\lambda = \omega$ ,  $\lambda = \omega_1$  in [B 5]). We do not try to prove the results on definability and elimination of quantifiers. For finite orders this can be done and the method becomes similar to that of automaton theory. For  $\omega$ ,  $\{\alpha: \alpha < \omega\}$ ,  $\omega_1$  this can be done by using the previous cases (e.g., for  $\omega$  using the result on the finite orders). We can prove the decidability of the weak monadic theory (with  $\exists^1$  only) of the  $n$ -successors theory by the method of this section (Doner [D 1] proved it). It would be very interesting if we could have proved in this way that the monadic theory of the 2-successor theory is decidable (Rabin [Ra 1] proved it).

In order to use Section 1 we should note

**LEMMA 3.1.** *For any  $n, \bar{k}$ ,  $(N, \bar{P})$ , the coloring  $f_k^n$  on  $N$  is additive where*

$$f_k^n(a, b) = Th_k^n((N, \bar{P}) \upharpoonright [a, b]),$$

where  $(N, \bar{P}) \upharpoonright [a, b]$  is a submodel of  $(N, \bar{P})$  with the universe  $[a, b] = \{x \in N: a \leq x < b\}$ .

*Proof.* By Lemma 2.3.

Let us list some immediate claims.

LEMMA 3.2. (A) *If for any  $n, \bar{k}$  we can compute effectively  $Th_k^n(K)$ , then the monadic theory of  $K$  is decidable; and vice-versa.*

(B) *If the monadic theory of  $K$  is decidable then so is the monadic theory of  $K'$  where  $K'$  is the class of:*

- (i) *submodels of models of  $K$ ,*
- (ii) *initial segments of orders from  $K$ ,*
- (iii) *orders which we get by adding (deleting) first (last) elements from orders of  $K$ ,*
- (iv) *converses of orders from  $K$ ,*
- (v)  *$(M, \bar{P}), M \in K, \bar{P} \in \underline{P}(M)^m$ .*

*Proof.* Immediate.

THEOREM 3.3. *The monadic theory of the class  $K_{\text{fin}}$  of finite orders is decidable.*

*Proof.* Let  $K_n$  be the class of orders of cardinality  $n$ ; up to isomorphism  $K_n$  has only one element,  $n$ . Hence by Lemma 2.6 we can compute  $Th_k^n(K_i)$ . Hence by Conclusion 2.5, for every  $n, \bar{k}$  we can compute  $Th_k^n(K)$  where  $K = \text{Cl}(K_1, K_2)$ . But clearly  $K$  is the class of finite orders. So by 3.2 (A) we finish.

THEOREM 3.4. *The monadic theory of  $\omega$  is decidable.*

*Proof.* We shall compute  $\{Th_k^n(\omega, \bar{P}): \bar{P} \in \underline{P}(\omega)^m\}$  by induction on  $n$ , for every  $\bar{k}, m$  simultaneously.

For  $n = 0$  it is easy.

Suppose we have done it for  $n - 1$  and we shall do it for  $n, m, \bar{k}$ . By the induction hypothesis we can compute  $Th_k^n(\omega)$  for every  $\bar{l}$ , in particular for  $\bar{r} = \bar{r}(n, m, \bar{k})$  (see 2.8). Now for any  $M = (\omega, P_1, \dots, P_m)$ , by 1.1 we can find an  $f_k^n$ -homogeneous set  $\{a_i: i < \omega\}$  ( $a_i < a_{i+1}$ ). So letting

$$t = Th_k^n((\omega, \bar{P}) \upharpoonright [0, a_0]), \quad s = Th_k^n((\omega, \bar{P}) \upharpoonright [a_i, a_j]) \quad \text{for } i < j;$$

we have

$$\begin{aligned} Th_k^n(\omega, \bar{P}) &= Th_k^n((\omega, \bar{P}) \upharpoonright [0, a_0]) + \sum_{i < \omega} Th_k^n((\omega, \bar{P}) \upharpoonright [a_i, a_{i+1})) \\ &= t + \sum_{i < \omega} s. \end{aligned}$$

As  $Th_r^n(\omega)$  is known, by 2.8, we can compute  $Th_k^n(M, \bar{P})$  from  $s, t$ . Now



for any  $t, s \in Th_k^n(K_{fin}^m)$ ,  $s \neq Th_k^n(0, \bar{P})$ ,  $\bar{P} \in \underline{P}(\emptyset)^m$ , there is an  $(\omega, \bar{P})$  such that  $Th_k^n(\omega, \bar{P}) = t + \sum_{i < \omega} s$ .

As we know  $Th_k^n(K_{fin}^m)$  by 3.3, and can easily find whether  $s \in Th_k^n(K_{fin}^m) - Th_k^n(\{0\})$ , we finish.

**THEOREM 3.5.** (A) *From the monadic theory of  $\lambda$  ( $\lambda$  a cardinal) we can compute effectively the monadic theory of  $K = \{\alpha: \alpha < \lambda^+\}$ .*

(B) *Moreover every monadic sentence which has model  $\alpha < \lambda^+$ , has a model  $\beta < \lambda^\omega$ .*

(C) (i) *For every  $\alpha < \lambda^+$  there is a  $\beta < \lambda^{\omega+1} + \lambda^\omega$  which has the same monadic theory*

(ii) *If  $\mu \leq \lambda$  and for every regular  $\chi \leq \lambda$  there is a  $\chi' \leq \mu$  such that  $\chi, \chi'$  have the same monadic theory, then we can chose  $\beta < \lambda^\omega \mu + \lambda^\omega$ .<sup>8</sup>*

(iii) *If we could always find  $\chi < \mu$  then  $\beta < \lambda^\omega \mu$ , and if  $\lambda = \omega$ ,  $\beta < \lambda^\omega + \lambda^\omega$ .<sup>9</sup>*

(iv) *Also, for every  $\alpha < \lambda^+$ , there are  $n < \omega$ ,  $\lambda_1, \dots, \lambda_n \leq \lambda$ , such that the monadic theory of  $\alpha$  is recursive in the monadic theories of  $\lambda_1, \dots, \lambda_n$ , and  $\lambda_i$  is a regular cardinal.*

(D) *In general, the bounds in (B), (C) cannot be improved.*

*Remark.* Büchi [B 5] already proved (B), (C) for  $\lambda = \omega$  and (B) for  $\lambda = \omega_1$ .

*Proof.* (A) Define  $K_1 = K_2 = \{\alpha: \alpha \leq \lambda\}$ ; by 3.2 (B) (i) and 3.2 (A) we can compute  $Th_k^n(K_i)$  for every  $n, \bar{k}$  and  $i = 1, 2$  (from the monadic theory of  $\lambda$ , of course). Hence by 2.5 we can compute  $Th_k^n(K')$  for every  $n, \bar{k}$ , where  $K' = Cl(K_1, K_2)$ . Clearly every member of  $K'$  is well-ordered and has cardinality  $\leq \lambda$ . So up to isomorphism  $K' \subseteq K$ . We should prove now only that equality holds. If not, let  $\alpha$  be the first ordinal not in  $K'$ , and  $\alpha < \lambda^+$ . If  $\alpha$  is a successor ordinal,  $\alpha - 1 \in K'$ ;  $1, 2 \in K$  hence  $\alpha = (\alpha - 1) + 1 \in K'$ , a contradiction. If  $\alpha$  is a limit ordinal, its cofinality is  $\leq \lambda$ . Let  $\alpha = \sum_{i < i_0} \alpha_i$ ,  $i_0 \leq \lambda$ ,  $\alpha_i < \alpha$ ; then  $i_0, \alpha_i \in K'$  so  $\alpha \in K'$ , a contradiction.

(B) Let us first show that

(\*) For every  $n, \bar{k}$  there is  $q = q(n, \bar{k}) < \omega$  such that if  $\alpha, \beta < \lambda$ ,  $cf(\alpha) = cf(\beta)$ , and  $\alpha, \beta$  are divisible by  $\lambda^q$ , then  $Th_k^n(\alpha) = Th_k^n(\beta)$ .

For  $n = 0$  it is immediate, and we prove it for  $n$ . By the pigeon-hole principle there are  $1 < l < p \leq 2 \mid T(n, 0, \bar{k}) \mid + 1$  such that  $Th_k^n(\lambda^l) = Th_k^n(\lambda^p)$ . Clearly,

$$\lambda^{l+2} = \sum_{i < \lambda} (\lambda^{l+1} + \lambda^l).$$

Hence

<sup>8</sup> In fact,  $\beta < M^{\omega+1} + M^\omega$ .

<sup>9</sup> In the first case  $\beta < M$ .

$$\begin{aligned}
Th_k^n(\lambda^{l+2}) &= Th_k^n[\sum_{i < \lambda} (\lambda^{l+1} + \lambda^i)] = \sum_{i < \lambda} Th_k^n(\lambda^{l+1} + \lambda^i) \\
&= \sum_{i < \lambda} [Th_k^n(\lambda^{l+1}) + Th_k^n(\lambda^i)] = \sum_{i < \lambda} [Th_k^n(\lambda^{l+1}) \\
&\quad + Th_k^n(\lambda^p)] = \sum_{i < \lambda} Th_k^n(\lambda^{l+1} + \lambda^p) = \sum_{i < \lambda} Th_k^n(\lambda^p) \\
&= \sum_{i < \lambda} Th_k^n(\lambda^i) = Th_k^n(\sum_{i < \lambda} \lambda^i) = Th_k^n(\lambda^{l+1}) .
\end{aligned}$$

Hence we prove by induction on  $m$ ,  $l < m < \omega$  that  $Th_k^n(\lambda^m) = Th_k^n(\lambda^{l+1})$ ; choose  $q = q(n, \bar{k}) = l + 1$ . Let  $\alpha, \beta < \lambda^+$  be divisible by  $\lambda^q$  and have the same cofinality, and we shall prove  $Th_k^n(\alpha) = Th_k^n(\beta)$ . Clearly it suffices to prove  $Th_k^n(\alpha) = Th_k^n(\lambda^q \mu)$  where  $\mu = cf(\alpha)$ . Let us prove it by induction on  $\alpha$ , and let  $\alpha = \lambda^q \gamma$ . If  $\gamma = \gamma_1 + 1$ , then for  $\gamma_1 = 0$  it is trivial, and for  $\gamma_1 > 0$

$$\begin{aligned}
Th_k^n(\alpha) &= Th_k^n(\lambda^q \gamma_1 + \lambda^q) = Th_k^n(\lambda^q \gamma_1) + Th_k^n(\lambda^q) \\
&= Th_k^n[\lambda^q \circ cf(\lambda^q \gamma_1)] + Th_k^n(\lambda^{q+2}) \\
&= Th_k^n[\lambda^q \circ cf(\lambda^q \gamma_1) + \lambda^{q+2}] = Th_k^n(\lambda^{q+2}) = Th_k^n(\lambda^q \circ \lambda) \\
&= Th_k^n[\lambda^q \circ cf(\alpha)] .
\end{aligned}$$

If  $\gamma$  is a limit ordinal  $\gamma = \sum_{i < cf(\gamma)} \gamma_i$ ,  $\gamma_i < \gamma$  a successor,

$$\begin{aligned}
Th_k^n(\alpha) &= Th_k^n[\lambda^q(\sum_{i < cf(\gamma)} \gamma_i)] = Th_k^n(\sum_{i < cf(\gamma)} \lambda^q \gamma_i) \\
&= \sum_{i < cf(\gamma)} Th_k^n(\lambda^q \gamma_i) \\
&= \sum_{i < cf(\gamma)} Th_k^n[\lambda^q \circ cf(\lambda^q \gamma_i)] \\
&= \sum_{i < cf(\gamma)} Th_k^n(\lambda^{q+1}) = \sum_{i < cf(\gamma)} Th_k^n(\lambda^q) \\
&= Th_k^n[\lambda^q \circ cf(\gamma)] .
\end{aligned}$$

So we have proved (\*). Let us prove (B). Let  $\alpha < \lambda^+$  be a model of a sentence  $\psi$ . Choose by 2.1 (A),  $n, \bar{k}$  such that from  $Th_k^n(\beta)$  we know whether  $\beta \models \psi$ , and let  $q = q(n, \bar{k})$ , and let  $\alpha = \lambda^q \beta + \gamma$ ,  $\gamma < \lambda^q$ . Then

$$Th_k^n(\alpha) = Th_k^n[\lambda^q \circ cf(\lambda^q \beta) + \gamma] , \quad \text{and} \quad \lambda^q \circ cf(\lambda^q \beta) + \gamma < \lambda^{q+2} .$$

(C) Divide  $\alpha$  by  $\lambda^\omega$  so  $\alpha = \lambda^\omega \alpha_1 + \alpha_2$ ,  $\alpha_2 < \lambda^\omega$ . Let  $\alpha'_1$  be 1 if  $\alpha_1$  is a successor, and  $cf \alpha_1$  otherwise. Then  $\lambda^\omega \alpha_1, \lambda^\omega \alpha'_1$  are divisible by  $\lambda^{q(n, \bar{k})}$  for every  $n, \bar{k}$  and have equal cofinality. So by the proof of (B), for every  $n, \bar{k}$ ,  $Th_k^n(\lambda^\omega \alpha_1) = Th_k^n(\lambda^\omega \alpha'_1)$ . Hence  $\lambda^\omega \alpha_1 + \alpha_2, \lambda^\omega \alpha'_1 + \alpha_2$  has the same monadic theory, and  $\lambda^\omega \alpha'_1 + \alpha_2 < \lambda^\omega \lambda + \lambda^\omega = \lambda^{\omega+1} + \lambda^\omega$ . This proves (C) (i).

If  $\chi' \leq \mu$  has the same monadic theory as  $\alpha'_1$  then  $\lambda^\omega \alpha_1 + \alpha_2, \lambda^\omega \alpha'_1 + \alpha_2$  and  $\lambda^\omega \chi' + \alpha_2$  (which is  $< \lambda^\omega \mu + \lambda^\omega$ ) have the same monadic theories. If  $\chi' < \mu$  clearly  $\lambda^\omega \chi' + \alpha_2 < \lambda^\omega \mu$ .

If  $\lambda = \omega$  then  $cf(\lambda^\omega \alpha_1) = \omega$  in any case, hence  $\alpha = \omega^\omega \alpha_1 + \alpha_2$ , and  $\omega^\omega + \alpha_1 < \omega^\omega + \omega^\omega$  has the same monadic theory. Every  $\alpha < \lambda^+$  we can uniquely represent as

$$\alpha = \lambda^\omega \alpha' + \lambda^n \alpha_n + \dots + \lambda^1 \alpha_1 + \alpha_0; \alpha_i < \lambda .$$

The monadic theory of  $\alpha$  is recursive in the monadic theories of  $\lambda$ ,  $cf(\lambda^\omega \alpha')$ ,  $\alpha_n$ ,  $\dots$ ,  $\alpha_0$ . So we can prove inductively (C) (iv).

(D) Suppose  $\lambda > \omega$ ,  $\lambda$  is regular, and there is a sentence  $\psi$  such that  $\alpha \models \psi$  if and only if  $\alpha = \lambda$ . Then there are sentences  $\psi_n$  such that  $\alpha \models \psi_n$  if and only if  $\alpha = \lambda^n$ , sentences  $\varphi_n$  such that  $\alpha \models \varphi_n$  if and only if  $\alpha$  is divisible by  $\lambda^n$ , and sentence  $\varphi$  such that  $\alpha \models \varphi$  if and only if  $cf\alpha = \lambda$ . Then  $\lambda^{\omega+1}$  is a model of  $\{\varphi, \varphi_n: n < \omega\}$ . If  $\alpha$  is also a model of  $\{\varphi, \varphi_n: n < \omega\}$  then  $\lambda^n$  divides  $\alpha$  for every  $n$ , hence  $\lambda^\omega$  divides  $\alpha$ , so  $\alpha = \lambda^\omega \beta$ . If  $\beta$  is a successor,  $cf(\alpha) = \omega$  but  $\alpha \models \varphi$  so  $\beta$  is a limit hence  $cf(\alpha) = cf(\beta)$ , so  $cf\beta = \lambda$ , so  $\beta \geq \lambda$  hence  $\alpha \geq \lambda^\omega \circ \lambda = \lambda^{\omega+1}$ . Similarly  $\lambda^{\omega+1} + \lambda^n$  is the smallest model of its monadic theory.

LEMMA 3.6. (A) *In 3.5 (A) it suffices to know the monadic theory of  $\{\mu: \mu \text{ a regular cardinal } \leq \lambda\}$ . So if  $\lambda$  is singular it suffices to know the monadic theory of  $\{\alpha: \alpha < \lambda\}$ .*

(B) *For every sentence  $\psi$ , (1) there is a sentence  $\varphi$  (all in the monadic theory of order) such that  $\alpha \models \varphi$  if and only if  $\alpha$  is a limit and  $cf(\alpha) \models \psi$ , (2) there is a sentence characterizing the first ordinal which satisfies  $\psi$  and (3) for every  $n < \omega$  there is  $\varphi_n$  such that  $\alpha \models \varphi_n$  if and only if  $\alpha$  is the  $n^{\text{th}}$  regular cardinal satisfying  $\psi$ .*

(C) *There are monadic sentences  $\varphi_n$  such that  $\alpha \models \varphi_n$  if and only if  $\alpha = \omega_n$ . If  $V = L$  there are monadic sentences  $\varphi_n^1$  such that  $\alpha \models \varphi_n^1$  if and only if  $\alpha$  is the  $n^{\text{th}}$  weakly compact cardinal.*

*Proof.* (A) Immediate by 3.5 (C) (iv).

(B) (1) Let  $\varphi$  say that there is no last element, and for any unbounded  $P$  there is an unbounded  $Q \subseteq P$  which satisfies  $\psi$  (if  $cf\alpha \models \neg\psi$  we can choose  $Q$  as a set of order-type  $cf\alpha$ ; so  $\alpha \models \varphi$ . If  $cf\alpha \models \neg\psi$ , let  $P$  be a subset of  $\alpha$  of order-type  $cf\alpha$ ; hence any unbounded  $Q \subseteq P$  has order-type  $cf\alpha$ , so  $\alpha \models \neg\varphi$ ).

(2) Immediate.

(3) We use (1) and (2) to define  $\varphi_n$  inductively. Let  $\varphi_0$  say that  $\alpha$  is the first ordinal whose cofinality satisfies  $\psi$ . Let  $\varphi_{n+1}$  say that  $\alpha$  is the first ordinal whose cofinality satisfies  $\psi \wedge \neg\varphi_0 \wedge \dots \wedge \neg\varphi_n$ .

(C) For  $\varphi_n$  use (B) (3) for  $\psi$  saying  $\alpha$  is an infinite ordinal. For  $\varphi_n^1$  use (B) (3) and Theorem 0.1 (of Jensen).

#### 4. The monadic theory of well-orderings

If  $a \in (M, \bar{P})$  let

$$th(a, \bar{P}) = \{x \in X_i: a \in P_i\} \cup \{x \notin X_i: a \notin P_i\}$$

(so it is a set of formulas).

Let  $D_\alpha$  denote the filter of (generated by) the closed unbounded subsets of  $\alpha$ ,  $cf\alpha > \omega$ .

**LEMMA 4.1.** *If the cofinality of  $\alpha$  is  $> \omega$ , then for every  $\bar{P} \in \underline{P}(\alpha)^m$  there is a closed unbounded subset  $J$  of  $\alpha$  such that: for each  $\beta < \alpha$ , all the models*

$$\{(\alpha, \bar{P}) \upharpoonright [\beta, \gamma) : \gamma \in J, cf(\gamma) = \omega, \gamma > \beta\}$$

*have the same monadic theory.*

*Remark.* Büchi [B 5, 6.1, p. 110] proved Lemma 4.1 for  $\alpha = \omega_1$ , by a different method.

*Proof.* For every  $n, \bar{k}$  there is, by 1.1, 3.1 a homogeneous unbounded  $I_{\bar{k}}^n \subseteq \alpha$ , by the coloring  $f_{\bar{k}}^n$  of  $(\alpha, \bar{P})$ , so there is  $t_{\bar{k}}^n$  such that for every  $\beta < \gamma \in I_{\bar{k}}^n$ ,  $Th_{\bar{k}}^n((\alpha, \bar{P}) \upharpoonright [\beta, \gamma)) = t_{\bar{k}}^n$ . Let  $J_{\bar{k}}^n$  be the set of accumulation points of  $I_{\bar{k}}^n$ , and  $J = \bigcap_{n, \bar{k}} J_{\bar{k}}^n$ . Clearly  $J$  is a closed and unbounded subset of  $\alpha$ .

Let  $\beta < \alpha$ , and  $\beta_{\bar{k}}^n$  be the first ordinal  $> \beta$  in  $I_{\bar{k}}^n$ . Then for any  $\gamma \in J$ ,  $\gamma > \beta$ ,  $cf(\gamma) = \omega$ , and for every  $n, \bar{k}$  we can find  $\gamma_l \in I_{\bar{k}}^n$ ,  $\gamma_l < \gamma_{l+1}$ ,  $\lim_{l \rightarrow \omega} \gamma_l = \gamma$  and  $\gamma_0 = \beta_{\bar{k}}^n$ . Therefore

$$\begin{aligned} Th_{\bar{k}}^n((\alpha, \bar{P}) \upharpoonright [\beta, \gamma)) &= Th_{\bar{k}}^n((\alpha, \bar{P}) \upharpoonright [\beta, \beta_{\bar{k}}^n)) + \sum_{l < \omega} Th_{\bar{k}}^n((\alpha, \bar{P}) \upharpoonright [\gamma_l, \gamma_{l+1})) \\ &= Th_{\bar{k}}^n((\alpha, \bar{P}) \upharpoonright [\beta, \beta_{\bar{k}}^n)) + \sum_{l < \omega} t_{\bar{k}}^n. \end{aligned}$$

So,  $Th_{\bar{k}}^n((\alpha, \bar{P}) \upharpoonright [\beta, \gamma))$  does not depend on the particular  $\gamma$ .

**Definition 4.1.**  $A_{Th_{\bar{k}}^n}(\beta, (\alpha, \bar{P}))$  for  $\beta < \alpha$ ,  $\alpha$  a limit ordinal of cofinality  $> \omega$  is  $Th_{\bar{k}}^n((\alpha, \bar{P}) \upharpoonright [\beta, \gamma))$  for every  $\gamma \in J$ ,  $\gamma > \beta$ ,  $cf(\gamma) = \omega$ ; where  $J$  is from Lemma 4.1.

*Remark.* As  $D_\alpha$  is a filter, this definition does not depend on the choice of  $J$ .

**Definition 4.2.** We define  $WTh_{\bar{k}}^n(\alpha, \bar{P})$ :

- (1) if  $\alpha$  is a successor or has cofinality  $\omega$ , it is  $\emptyset$ ,
- (2) otherwise we define it by induction on  $n$ :

**for  $n = 0$ :**  $WTh_{\bar{k}}^n(\alpha, \bar{P}) = \{t : \{\beta < \alpha : th(\beta, \bar{P}) = t\} \text{ is a stationary subset of } \alpha\}$ ,

**for  $n + 1$ :** let  $WTh_{\bar{k}}^{n+1}(\alpha, \bar{P}) = \{\langle S_1(\bar{Q}), S_2(\bar{Q}) \rangle : \bar{Q} \in \underline{P}(\alpha)^{\bar{k}(n+1)}\}$

where

$$S_1(\bar{Q}) = WTh_{\bar{k}}^n(\alpha, \bar{P}, \bar{Q}),$$

$$S_2(\bar{Q}) = \{\langle t, s \rangle : \{\beta < \alpha : WTh_{\bar{k}}^n((\alpha, \bar{P}, \bar{Q}) \upharpoonright \beta) = t, th(\beta, \bar{P} \wedge \bar{Q}) = s\}$$

is a stationary subset of  $\alpha\}$ .

*Remark.* Clearly, if we replace  $(\alpha, \bar{P})$  by a submodel whose universe is

a close unbounded subset of  $\alpha$ ,  $WTh_k^n(\alpha, \bar{P})$  will not change. Of course  $WTh_k^n(M)$  is well defined for every well-ordered model.

*Definition 4.3.* Let  $cf(\alpha) > \omega$ ,  $M = (\alpha, \bar{P})$  and we define the model  $g_k^n(M) = (\alpha, g_k^n(\bar{P}))$ .

Let

$$(g_k^n(\bar{P}))_s = \{\beta < \alpha : s = Ath_k^n(\beta, M)\}$$

and (when  $m = l(\bar{P})$ )

$$g_k^n(\bar{P}) = \langle \dots, (g_k^n(\bar{P}))_s, \dots \rangle_{s \in T(n, m, \bar{k})}.$$

*Remark.* (1) In  $g_k^n(\bar{P})$  we unjustly omit  $\alpha$ , but there will be no confusion.

(2) Remember  $T(n, m, \bar{k})$  is the set of formally possible  $Th_k^n(M, \bar{P})$ ,  $l(\bar{P}) = m$ .

LEMMA 4.2. (A)  $g_k^n(\bar{P})$  is a partition of  $\alpha$ .

(B)  $g_k^n(\bar{P} \wedge \bar{Q})$  is a refinement of  $g_k^n(\bar{P})$  and we can effectively correlate the parts.

(C)  $g_k^{n+1}(\bar{P})$  is a refinement of  $g_k^n(\bar{P})$  and we can effectively correlate the parts.

(D) The parallels of Lemma 2.2 for  $Th$ ,  $pTh$ , hold for  $WTh$ ,  $pWTh$ .

*Proof.* Immediate.

THEOREM 4.3. For every  $n, m, \bar{k}$  we can effectively find  $\bar{r} = \bar{r}_1(n, m, \bar{k})$  such that: If  $cf(\alpha^i) > \omega$ ,  $M_i = (\alpha^i, \bar{P}^i)$ ,  $l(\bar{P}^i) = m$  for  $i = 1, 2$  and  $Ath_k^n(0, M_1) = Ath_k^n(0, M_2)$  and  $WTh_r^n(g_k^n(M_1)) = WTh_r^n(g_k^n(M_2))$  then  $Th_k^n(M_1) = Th_k^n(M_2)$ .

*Proof.* We prove by induction on  $n$ .

For  $n = 0$ , it is easy to check that  $Th_k^n(M_i) = Ath_k^n(0, M_i)$  hence the theorem is trivial.

Suppose we have proved the theorem for  $n$ , and we shall prove it for  $n + 1$ . Suppose  $\bar{Q}^1 \in \underline{P}(\alpha^1)^{\bar{k}(n+1)}$ , and we shall find  $\bar{Q}^2 \in \underline{P}(\alpha^2)^{\bar{k}(n+1)}$  such that  $Th_k^n(\alpha^1, \bar{P}^1, \bar{Q}^1) = Th_k^n(\alpha^2, \bar{P}^2, \bar{Q}^2)$ ; by the symmetry in the hypothesis this is sufficient. Let  $g_k^n(\bar{P}^1 \wedge \bar{Q}^1) = \bar{Q}^{*1}$ ,  $g_k^{n+1}(\bar{P}^1) = \bar{P}^{*1}$ ,  $g_k^{n+1}(\bar{P}^2) = \bar{P}^{*2}$ . Define  $\bar{r}(n + 1) = l(g_k^n(\bar{P}^1 \wedge \bar{Q}^1)) = l(\bar{Q}^{*1})$  and  $\bar{r} \upharpoonright (n + 1) = r_1(n, m + l(\bar{P}^1), \bar{k})$ .

By the assumptions and Definition 4.2, there is  $\bar{Q}^{*2} \in \underline{P}(\alpha^2)^{\bar{k}(n+1)}$  such that (for our  $n, \bar{r}$  and  $\alpha^2, \bar{P}^{*2}; \alpha^1, \bar{P}^{*1}$ ),  $S_l(\bar{Q}^{*1}) = S_l(\bar{Q}^{*2})$  for  $l = 1, 2$ . (The notation is inaccurate, but should be clear.) So, for  $l = 1$ , we get  $WTh_r^n(\alpha^1, \bar{P}^{*1}, \bar{Q}^{*1}) = WTh_r^n(\alpha^2, \bar{P}^{*2}, \bar{Q}^{*2})$ , and without loss of generality  $0 \in Q_s^{*1} \leftrightarrow 0 \in Q_s^{*2}$ . (From now on we can replace  $\bar{r}$  by  $\bar{r} \upharpoonright (n + 1)$ .) So by Lemma 4.2, for  $l = 1, 2$ ,  $\bar{Q}^{*l}$  is a partition of  $\alpha^l$  refining  $\bar{P}^{*l}$ , hence for every  $\beta < \alpha^l$  there is a unique  $s_l(\beta)$  such that  $\beta \in Q_{s_l(\beta)}^{*l}$ .

Now, for  $l = 1, 2$ , choose a closed unbounded subset  $J_l$  of  $\alpha^l$  such that:

(0) every member of  $J_l$  which is not an accumulation point of  $J_l$ , has cofinality  $\omega$ ,

(1) for any  $s$ , if  $Q_s^{*l}$  is not a stationary subset of  $\alpha^l$  then  $Q_s^* \cap J_l = \emptyset$ ,

(2) if  $\beta < \gamma < \alpha^l$ ;  $\beta, \gamma \in J_l$ ;  $cf(\gamma) = \omega$  then

$$Th_k^{n+1}((\alpha^l, \bar{P}^l) \upharpoonright [\beta, \gamma)) = ATh_k^{n+1}(\beta, (\alpha^l, \bar{P}^l)) \quad (\text{use Lemma 4.1}),$$

(3) for every  $\gamma \in J_l$ ,  $cf(\gamma) = \omega$ ,

$$Th_k^{n+1}((\alpha^l, \bar{P}^l) \upharpoonright [0, \gamma)) = ATh_k^{n+1}(0, (\alpha^l, \bar{P}^l)),$$

(4) if  $Q_s^{*l} \cap J_l \neq \emptyset$ ,  $\beta \in J_l$

then there are  $\gamma \in J_l$ ,  $\gamma > \beta$ ,  $s_l(\gamma) = s$  such that  $\{\xi \in J_l: \beta \leq \xi \leq \gamma\}$  is finite,

(5) for any  $s, t$ , if  $\{\beta < \alpha^l: t = WTh_r^n((\alpha^l, \bar{Q}^{*l}) \upharpoonright \beta), s = Th(\beta, \bar{Q}^{*l})\}$  is not a stationary subset of  $J_l$ , then it is disjoint to  $J_l$ .

*Remark.* Note that (5) just strengthens (1).

Now we define  $\bar{Q}^2$  by parts. That is, for every  $\beta < \gamma \in J_2 \cup \{0\}$ ,  $\gamma$  is the successor of  $\beta$  in  $J_2$ , we define  $\bar{Q}^2 \upharpoonright [\beta, \gamma)$  such that

$$s_2(\beta) = Th_k^n((\alpha^2, \bar{P}^{2*} \bar{Q}^2) \upharpoonright [\beta, \gamma)).$$

This is possible as by definition of  $s_2(\beta)$ ,  $\beta \in Q_{s_1(\beta)}^{*2}$ , hence

$$s_i(\beta) \in ATh_k^{n+1}(\beta, (\alpha^2, \bar{P}^{2*})).$$

We now prove

(\*) if  $\beta < \gamma \in J_2 \cup \{0\}$ ,  $cf(\gamma) = \omega$ , then

$$s_2(\beta) = Th_k^n((\alpha^2, \bar{P}^2, \bar{Q}^2) \upharpoonright [\beta, \gamma)).$$

We prove it by induction on  $\gamma$  for all  $\beta$ .

(i) By (0) the first  $\gamma > \beta$ ,  $\gamma \in J_2$  has cofinality  $\omega$ , and by the definition of  $\bar{Q}^2$  (\*) is satisfied.

(ii) Let  $\beta < \xi < \gamma$ ,  $\xi \in J_2$ , for no  $\zeta \in J_2$ ,  $\xi < \zeta < \gamma$ , and  $\xi$  has cofinality  $\omega$ . Then by the induction hypothesis  $Th_k^n((\alpha^2, \bar{P}^2, \bar{Q}^2) \upharpoonright [\beta, \xi)) = s_2(\beta)$  and

$$Th_k^n((\alpha^2, \bar{P}^2, \bar{Q}^2) \upharpoonright [\xi, \gamma)) = s_2(\xi).$$

We should show that  $s_2(\beta) + s_2(\xi) = s_2(\gamma)$ . So it suffices to find  $\beta' < \xi' < \gamma' \in J_1$ ,  $s_1(\beta') = s_2(\beta)$ ,  $cf \xi' = \omega = cf \gamma'$ ,  $s_1(\xi') = s_2(\xi)$ ; and by the definition of  $S_2$  in 4.2 this is possible. As  $\beta \in J_2$ ,  $Q_{s_2(\beta)}^{*2}$  is a stationary subset of  $\alpha^2$ ,  $Q_{s_2(\beta)}^{*1}$  is a stationary subset of  $\alpha^1$ , hence for some  $\beta' \in J_1$ ,  $\beta' \in Q_{s_2(\beta)}^{*1}$  hence  $s_2(\beta') = s_2(\beta)$ . As  $\xi \in J_2$ ,

$$\{\zeta \in Q_{s_2(\xi)}^{*2}: WTh_r^n(\alpha^2, \bar{P}^{*2}, \bar{Q}^{*2}) = \emptyset\}$$

is stationary, hence we can find  $\xi' \in J_1$ ,  $ch(\xi') = \omega$ ,  $s_2(\xi') = s_2(\xi)$ .

(iii) If  $\gamma$  is an accumulation point of  $J_2$  the proof is similar to that of (ii). Choose  $\xi_m$ ,  $m < \omega$ ,  $\beta < \xi_m < \xi_{m+1} < \gamma$ ,  $\lim_m \xi_m = \gamma$ ,  $cf(\xi_m) = \omega$ , and  $s_2(\xi_m) = s_2(\xi_{m+1})$  (use (4)). Then

$$\begin{aligned} Th_k^n((\alpha^2, \bar{P}^2, \bar{Q}^2) \upharpoonright [\beta, \gamma)) &= Th_k^n((\alpha^2, \bar{P}^2, \bar{Q}^2) \upharpoonright [\beta, \xi_0)) \\ &\quad + \sum_{m < \omega} Th_k^n((\alpha^2, \bar{P}^2, \bar{Q}^2) \upharpoonright [\xi_m, \xi_{m+1})) \\ &= s_2(\beta) + \sum_{m < \omega} s_2(\xi_0). \end{aligned}$$

We should prove this sum is  $s_2(\beta)$ , and this is done as in (ii).

(iv) There are  $\xi \in J$ ,  $\beta < \xi < \gamma$ ,  $\gamma$  the successor of  $\xi$  in  $J_2$  and  $cf(\xi) > \omega$ .

As before we can find  $\beta' < \xi' < \gamma' \in J_1$ ,  $s_1(\beta') = s_2(\beta)$ ,  $WTh_r^n((\alpha^1, \bar{P}^{1*}) \upharpoonright \xi') = WTh_r^n((\alpha^2, \bar{P}^{*2}) \upharpoonright \xi)$ ,  $s_1(\xi') = s_2(\xi)$ ,  $cf \xi' > \omega$ ,  $cf \gamma' = \omega$ . So clearly

$$Th_k^n((\alpha^2, \bar{P}^2, \bar{Q}^2) \upharpoonright [\xi, \gamma)) = s_2(\xi) = s_2(\xi') = Th_k^n((\alpha^1, \bar{P}^1, \bar{Q}^1) \upharpoonright [\xi', \gamma')).$$

Now also

$$Th_k^n((\alpha^2, \bar{P}^2, \bar{Q}^2) \upharpoonright [\beta, \xi)) = Th_k^n((\alpha^1, \bar{P}^1, \bar{Q}^1) \upharpoonright [\beta', \xi'))$$

by the induction hypothesis on  $n$  and on  $\gamma$ .

So we have proved (\*) and  $g_k^n((\alpha^2, \bar{P}^2, \bar{Q}^2)) = (\alpha^2, \bar{Q}^{*2})$ .

Now by the induction hypothesis on  $n$  it follows that  $Th_k^n(\alpha^1, \bar{P}^1, \bar{Q}^1) = Th_k^n(\alpha^2, \bar{P}^2, \bar{Q}^2)$ .

**THEOREM 4.4.** *If  $cf(\alpha) > \omega$ ,*

$$t_1 = WTh_r^n(g_k^n(\bar{P})), t_2 = ATh_k^n(0, (\alpha, \bar{P})), \bar{r} = \bar{r}_1(n, l(\bar{P}), \bar{k})$$

*then we can effectively compute  $Th_k^n(\alpha, \bar{P})$  from  $t_1, t_2$ .*

*Proof.* The proof is similar to that of 4.3.

**Conclusion 4.5.** If  $\lambda$  is a regular cardinal, and we know  $ATH_k^n(0, \lambda)$ ,  $WTh_r^n(\lambda)$  ( $\bar{r} = r_1(n, 0, \bar{k})$ ), then we can compute  $Th_k^n(\lambda)$ .

**LEMMA 4.6.** *If  $\lambda$  is a regular cardinal  $> \omega$ ,  $\bar{r} = r(n, 0, \bar{k})$ , then, letting  $T_1 = \{Th_r^n(\mu): \omega < \mu < \lambda, \mu \text{ a regular cardinal}\}$ ,  $T_2 = \{Th_r^n(\alpha): \alpha < \lambda\}$ , we can compute effectively  $ATH_k^n(0, \lambda)$  from  $T_1$ ; and we can compute  $T_1$  effectively from  $T_2$ .*

*Proof.* Let  $T = \{t_1, \dots, t_n\}$ , and if  $t_i = Th_r^n(\mu)$  let  $t'_i = Th_k^n(\mu^q)$ ,  $q = q(n, \bar{k})$  (we can compute it effectively: see the proof of 3.5 (B) for the definition of  $q(n, \bar{k})$ ) and let  $t = t'_1 + \dots + t'_n$ , then

$$\sum_{m < \omega} t = t\omega = ATH_k^n(0, \lambda).^{10}$$

**Conclusion 4.7.** Let  $\lambda$  be a regular cardinal. If the monadic theory of  $\{\alpha: \alpha < \lambda\}$ , and  $\{WTh_k^n(\lambda): n, \bar{k}\}$  are given then we can compute effectively the monadic theory of  $\lambda$ .

<sup>10</sup> The second phrase is immediate by 3.6 B(3).

LEMMA 4.8. For a regular  $\lambda$ ,  $\{WTh^n(\lambda): n < \omega\}$  and the first-order theory of  $M^\lambda = (\underline{P}(\lambda)/D_\lambda, \cup, \cap, -, \emptyset, 1, \dots, R_t^\lambda, \dots)$  are recursive one in the other, where  $R_t^\lambda(P, \bar{Q})$  holds if and only if

$$\{\beta < \lambda: \beta \in P, \text{ and for some } n, t = WTh^n((\lambda, \bar{Q}) \upharpoonright \beta)\} \neq \emptyset \pmod{D_\lambda}.$$

*Remark.* Note that for every  $t$  there is at most one possible  $n$ .

*Proof.* Immediate, similar to the proof of Lemma 2.1.

Conclusion 4.9. If the monadic theory of  $\{\alpha: \alpha < \lambda\}$  and the first-order theory of  $M^\lambda$  are decidable, then so is the monadic theory of  $\lambda$ .

Using 4.9 we can try to prove the decidability of the monadic theory of  $\lambda$  by induction on  $\lambda$ .

For  $\lambda = \omega$  we know it by 3.4.

For  $\lambda = \omega_1$  the  $R_t^{\omega_1}$ 's are trivial, (because each  $\beta < \omega_1$  is a successor or  $cf\beta = \aleph_0$ , hence by Definition 4.3 (1),  $R_t^{\aleph_1}(P, \bar{Q})$  holds if and only if  $t = \emptyset$ ). So it suffices to prove the decidability of  $(\underline{P}(\omega_1)/D_{\omega_1}, \cap, \cup, -, \emptyset, 1)$ . But by Ulam [U 1] this is an atomless Boolean algebra, so its theory is decidable. Hence we reprove the theorem of Büchi [B 5].

Conclusion 4.10. The monadic theory of  $\omega_1$  is decidable.

Now we can proceed to  $\lambda = \omega_2$ . Looking more closely at the proof for  $\omega_1$ , we see that  $WTh_k^n(\omega_1, \bar{P})$  can be computed from the set of atoms in the Boolean algebra generated by the  $P_i$  which are stationary subsets of  $\omega_1$ ; and we can replace  $\omega_1$  by any ordinal of cofinality  $\omega_1$ . So all the  $R_t^{\omega_2}$  can be defined by the function  $F/D_{\omega_2}$ ,

$$F(I) = \{\alpha < \omega_2: cf(\alpha) = \omega_1, \alpha - I \cap \omega_2 \notin D_\alpha\}.$$

Conclusion 4.11. The first order theory of

$$M_1^{\omega_2} = (\underline{P}(\omega_2)/D_{\omega_2}, \cap, \cup, -, \emptyset, 1, F/D_{\omega_2})$$

is decidable if and only if the monadic theory of  $\omega_2$  is decidable.

Notice that  $F(I \cup J) = F(I) \cup F(J)$ , and that for  $M_1^{\omega_2}$  to have a decidable theory, it suffices that it have elimination of quantifiers. For this it suffices

(\*) for any stationary  $A \subseteq \{\alpha < \omega_2: cf(\alpha) = \omega\}$  and  $B, C$  such that  $F(A) = B \cup C$  there are stationary  $A', B', A = A' \cup B', A' \cap B' = \emptyset, F(A') = A \pmod{D_{\omega_2}}$  and  $F(B') = B \pmod{D_{\omega_2}}$ .

Conjecture 4 (A). (\*) is consistent with ZFC.

## 5. From orders to uniform orders

An equivalence relation  $E$  on ordered set  $N$  is *convex* if  $x E y, x < z < y \in N$ , implies  $x E z$ , i.e., every equivalence class is convex. On  $N/E = \{a/E$ :



$a \in N$ ) a natural ordering is defined. If  $J$  is a convex subset of a model  $(M, \bar{P})$  then  $th(J, \bar{P})$  is  $\langle l, s_1, s_2 \rangle$  such that if there is no last (first) element in  $J$ ,  $s_2 = 1$  ( $s_1 = 1$ ), if  $b$  is the last (first) element,  $s_2 = th(b, \bar{P})$  ( $s_1 = th(b, \bar{P})$ ) (for definition, see the beginning of Section 4) and  $l = \min(|J|, 2)$ .

*Definition 5.1.* (1)  $\kappa(M)$  is the first cardinal  $\kappa$ , such that neither  $\kappa$  nor  $\kappa^*$  is embeddable in  $M$ .

(2)  $\kappa(K)$  is l.u.b.  $\{\kappa(M) : M \in K\}$ .

*Definition 5.2.* We define for every  $n, \bar{k}$ , the class  $U_k^n$  and  $UTh_k^n((M, \bar{P}))$  for  $M \in U_k^n$

(1)  $U_k^n = \{(M, \bar{P}) : M \text{ is dense order with no first nor last element and there are } t_0 \text{ and a dense } I \subseteq |M| \text{ such that for every } a < b \in I:$

$$t_0 = Th_k^n((M, \bar{P}) \upharpoonright (a, b)) \text{ and } th(a, \bar{P}) = th(b, \bar{P})\}.$$

Now we define  $UTh_k^n(M, \bar{P})$  by induction on  $n$ .

(2)  $UTh_k^0(M, \bar{P}) = Th_k^0(M, \bar{P})$ .

(3)  $UTh_k^{n+1}(M, \bar{P}) = \langle S_1, S_2, \text{com} \rangle$  where

(A)  $S_1 = \{UTh_k^n((M, \bar{P}, \bar{Q})) : \bar{Q} \in \underline{P}(M)^{\bar{k}(n+1)}, (M, \bar{P}, \bar{Q}) \in U_k^n\}$ ,

(B) Before we define  $S_2$ , we make some conventions:

- ( $\alpha$ )  $T_1[T_2]$  is the set of formally possible  $th(J, \bar{P}^1)$ ,  $J \neq \emptyset$ , and  $l(\bar{P}^1) = l(\bar{P})$ ,  $(l(\bar{P}^1) = l(\bar{P}) + \bar{k}(n+1))$ ;
- ( $\beta$ )  $T_3 = \{\langle l, s_1, t, s_2 \rangle : \langle l, s_1, s_2 \rangle \in T_2, t \in T(n, l(\bar{P}) + \bar{k}(n+1), \bar{k}) \text{ and } l = 1 \text{ if and only if } t \text{ is the "theory" of the empty model}\}$ ;
- ( $\gamma$ ) If  $\langle l, s_1, s_2 \rangle \in T_1$ ,  $\langle l', s'_1, t, s'_2 \rangle \in T_3$  then  $\langle l, s_1, s_2 \rangle \leq \langle l', s'_1, t, s'_2 \rangle$  when:  $l = l'$  and  $s_1 = 1 \Leftrightarrow s'_1 = 1$ ,  $s_2 = 1 \Leftrightarrow s'_2 = 1$  and  $s_1 \neq 1 \Rightarrow s_1 \subseteq s'_1$ ,  $s_2 \neq 1 \Rightarrow s_2 \subseteq s'_2$ ;
- ( $\delta$ ) At last let  $\bar{r} = \bar{r}(n, l(\bar{P}), \bar{k})$  be from 2.8,  $S_2 = \{UTh_{\bar{r}}^n(M/E, \bar{P}^*, \bar{Q}^*) : E \text{ a non-trivial convex equivalence relation over } |M|, (M/E, \bar{P}^*, \bar{Q}^*) \in U_{\bar{r}}^n, \bar{P}^* = \langle \dots, P_t^*, \dots \rangle_{t \in T_1}, \text{ where } P_t^* = \{a/E : a \in |M|, th(a/E, \bar{P}) = t\} \text{ and } Q^* = \langle \dots, Q_t^*, \dots \rangle_{t \in T_3} \text{ is a partition of } |M|/E \text{ refining } \bar{P}^* \text{ and } \phi \neq Q_{t(1)}^* \subseteq P_t^* \text{ implies } t(1) \leq t\}$ .

(C) Com is  $+$  if  $M$  is a complete order, and  $-$  otherwise.

LEMMA 5.1. (A) From  $Th_k^{n+2}(M, \bar{P})$  we can check whether  $(M, \bar{P}) \in U_k^n$  and compute  $UTh_k^n(M, \bar{P})$ .

(B) Also the parallel to 2.2 holds.

LEMMA 5.2. For every dense  $N \in K$ ,  $\|N\| > 1$ ,  $n, \bar{k}$ , there is a convex submodel  $M$  of  $N$  which belongs to  $U_k^n$ ,  $\|M\| > 2$ .

*Proof.* By Theorem 1.3, and 4.1 (A).

LEMMA 5.3. Suppose  $N$  is a dense order,  $\kappa(N) \leq \aleph_1$ ;  $I \subseteq |N|$  is a dense subset, and for every  $a < b \in I$ ,  $t_0 = Th_k^n((N, P) \upharpoonright [a, b])$ . Then there is  $t_1$

such that

(1) for every  $a < b \in |N|$ ,  $t_1 = Th_k^n((N, \bar{P}) \upharpoonright (a, b))$ .

(2) Moreover for every convex  $J \subseteq |N|$ , with no first nor last element,  $t_1 = Th_k^n((N, \bar{P}) \upharpoonright J)$ .

*Proof.* Clearly it suffices to prove (2). Choose  $a_0 \in J \cap I$ . Now define  $a_n$ ,  $0 < n < \omega$  such that  $a_n \in J \cap I$ ,  $a_n < a_{n+1}$  and  $\{a_n : n < \omega\}$  is unbounded in  $J$  (this is possible as  $\kappa(N) \leq \aleph_1$ ). Now define similarly,  $a_n \in J \cap I$ ,  $n$  a negative integer so that  $a_{n-1} < a_n < a_0$  and  $\{a_n : n \text{ is a negative integer}\}$  is unbounded from below in  $J$ .

So, letting  $\mathbf{Z}$  be the integers,

$$Th_k^n((N, \bar{P}) \upharpoonright J) = \sum_{n \in \mathbf{Z}} Th_k^n((N, \bar{P}) \upharpoonright [a_n, a_{n+1})) = \sum_{n \in \mathbf{Z}} t_0 \stackrel{\text{def}}{=} t_1.$$

**THEOREM 5.4.** Let  $M$  be an order,  $\kappa(M) \leq \aleph_1$ .

(A) Knowing  $t$  and that  $t = UTh_k^n(M, \bar{P})$ ,  $(M, \bar{P}) \in U_k^n$  we can effectively compute  $F(t) = Th_k^n(M, \bar{P})$ .

(B) If  $(M^2, \bar{P}^i) \in U_k^n$  for  $i = 1, 2$ , and  $UTh_k^n(M^1, \bar{P}^1) = UTh_k^n(M^2, \bar{P}^2)$  then  $Th_k^n(M^1, \bar{P}^1) = Th_k^n(M^2, \bar{P}^2)$ .

*Proof.* Clearly (A) implies (B). So we prove (A) by induction on  $n$ .

For  $n = 0$  it is trivial.

Suppose we have proved the theorem for  $n$ , and we shall prove it for  $n + 1$ .

Let  $UTh_k^{n+1}(M, \bar{P}) = \langle S_1, S_2, \text{com} \rangle$ . We should find

$$T = \{Th_k^n(M, \bar{P}, \bar{Q}) : \bar{Q} \in \underline{P}(M)^{\bar{k}(n+1)}\}.$$

If  $t \in S_1$ , then for some  $\bar{Q} \in \underline{P}(M)^{\bar{k}(n+1)}$ ,  $(M, \bar{P}, \bar{Q}) \in U_k^n$  and  $t = UTh_k^n(M, \bar{P}, \bar{Q})$ , hence, by the induction hypothesis  $F(t) = Th_k^n(M, \bar{P}, \bar{Q})$ , so  $F(t) \in T$ . We can conclude that  $T' = \{F(t) : t \in S_1\} \subseteq T$ .

Now if  $t^* \in S_2$ , then there is a convex equivalence relation  $E$  on  $M$ , such that  $t^* = UTh_k^n(M/E, \bar{P}^*, \bar{Q}^*)$  where the conditions of  $S_2$  are satisfied. If  $Q_{\langle l, s_1, t, s_2 \rangle}^* \neq \emptyset$ , and  $l > 1$  implies  $t \in T$  then we can define  $\bar{Q} \in \underline{P}(M)$  such that for  $a/E \in Q_{\langle l, s_1, t, s_2 \rangle}^*$ :

(1)  $UTh_k^n((M, \bar{P}, \bar{Q}) \upharpoonright \text{int}(a/E)) = t$ ,

(2)  $th(a/E, \bar{Q}) = \langle l, s_1, s_2 \rangle$ .

*Remark.* (1) can be done because by Lemma 4.3 (2), if  $\text{int}(a/E) \neq \emptyset$  then

$$Th_k^{n+1}((M, \bar{P}) \upharpoonright \text{Int}(a/E)) = Th_k^{n+1}(M, \bar{P}) = T.$$

Now clearly knowing  $t^*$  we can compute

$$S(t^*) = \{t : Q_{\langle l, s_1, t, s_2 \rangle}^* \neq \emptyset, t \neq Th_k^n(\emptyset), \text{ for some } s_1, s_2\}$$

where  $\bar{Q}^*$  is as above. We can also compute  $G(t) = Th_k^n(M, \bar{P}, \bar{Q})$ . We know that  $t \in S_2$ ,  $S(t) \subseteq T$ , imply  $G(t) \in T$ .

We know also that if (i)  $t = Th_k^n((M, \bar{P}) \upharpoonright \{a\})$  for some  $a \in M$ , and (ii)  $t_1, t_2 \in T$ , then:  $\sum_{0 \leq n} (t_1 + t) \in T$  and  $\sum_{\substack{n < \omega \\ n \in \mathbb{Z}}} (t + t_2) \in T$ ,  $t_1 + t + t_2 \in T$  and if  $\text{com}$  is  $-$ ,  $t_1 + t_2 \in T$  (where  $\mathbb{Z}$  is the set of integers) (we use the facts that  $M$  is dense,  $\kappa(M) \leq \aleph_1$ ).

Now let  $T^*$  be the minimal subset of  $T(n, l(\bar{P}), \bar{k})$  such that

- (a)  $T^* \supseteq T'$ ,
- (b)  $t \in S_2$ ,  $S(t) \subseteq T^*$  imply  $G(t) \in T^*$ ,
- (c) if  $t_1, t_2 \in T^*$ ,  $t = Th_k^n((M, P) \upharpoonright \{a\})$  then  $t_1 + t + t_2 \in T^*$ ;
- (d) if  $t_2 \in T^*$ ,  $t_1 = Th_k^n((M, \bar{P}) \upharpoonright \{a\})$  for some  $a \in M$  then

$$\sum_{0 \leq n < \omega} (t_2 + t_1) \in T^*, \sum_{\substack{n \leq 0 \\ n \in \mathbb{Z}}} (t_1 + t_2) \in T^* ;$$

- (e) if  $t_1, t_2 \in T_2$ ,  $\text{com}$  is  $-$  then  $t_1 + t_2 \in T^*$ .

It is easy to see that as  $S_1, S_2$  are given and  $T(n, l(\bar{P}), \bar{k})$  is (hereditarily) finite and known, we can effectively compute  $T^*$ . So it suffices to prove that  $T = T^*$  but as clearly  $T^* \subseteq T$  it suffices to prove:

$$t \in T \implies t \in T^* .$$

As  $t \in T$ , there is  $\bar{Q} \in \underline{P}(M)^{\bar{k}(n+1)}$  such that  $t = Th_k^n(M, \bar{P}, \bar{Q})$ . Define the equivalence relation  $E$  on  $M$ :  $a E b$  if and only if  $a = b$  or, without loss of generality we assume that  $a < b$ , for every  $a', b' \in M$ ,  $a \leq a' < b' \leq b$ ,  $Th_k^n((M, \bar{P}, \bar{Q}) \upharpoonright (a', b')) \in T^*$ . It is easy to check that  $E$  is a convex equivalence relation over  $M$ . Now we shall show that if  $a \in M$ ,  $\text{int}(a/E) \neq \emptyset$  then  $Th_k^n((M, \bar{P}, \bar{Q}) \upharpoonright \text{int}(a/E))$  belongs to  $T^*$ . Choose  $a_0 \in a/E$ , and then define  $a_n$ ,  $n \geq 0$  such that  $a_n < a_{n+1}$ ,  $\{a_n: 0 \leq n < \omega\}$  is unbounded in  $\text{int}(a/E)$ . Without loss of generality  $th(a_n, \bar{P} \wedge \bar{Q}) = s_0$  for every  $n > 0$ . Hence

$$\begin{aligned} & Th_k^n((M, \bar{P} \wedge \bar{Q}) \upharpoonright \{x \in \text{int}(a/E): a_0 < x\}) \\ &= \sum_{0 \leq n < \omega} [Th_k^n((M, \bar{P}, \bar{Q}) \upharpoonright (a_n, a_{n+1})) + Th_k^n((M, \bar{P}, \bar{Q}) \upharpoonright \{a_{n+1}\})] . \end{aligned}$$

By the definition of  $E$ ,  $Th_k^n((M, \bar{P}, \bar{Q}) \upharpoonright (a_n, a_{n+1})) \in T^*$ , hence by (d),

$$Th_k^n((M, \bar{P}, \bar{Q}) \upharpoonright \{x \in \text{int}(a/E): a_0 < x\}) \in T^* .$$

Similarly,

$$Th_k^n((M, \bar{P}, \bar{Q}) \upharpoonright \{x \in \text{int}(a/E): x < a_0\}) \in T^* .$$

So by (c),

$$Th_k^n((M, \bar{P}, \bar{Q}) \upharpoonright \text{int}(a/E)) \in T^* .$$

Similarly, by (c), (e) in  $M/E$  there are no two successive elements, so  $M/E$  is a dense order.

Define  $\bar{P}^* = \langle \dots, P_{\langle l, s_1, s_2 \rangle}, \dots \rangle$ ,  $\bar{Q}^* = \langle \dots, Q_{\langle l, s_1, t, s_2 \rangle}^*, \dots \rangle$  such that

(1)  $a/E \in P_{\langle l, s_1, s_2 \rangle}$  if and only if  $th(a/E, \bar{P}) = \langle l, s_1, s_2 \rangle$ ,

(2)  $a/E \in Q_{\langle l, s_1, t, s_2 \rangle}^*$  if and only if  $Th_k^n((M, \bar{P}, \bar{Q}) \upharpoonright \text{int}(a/E)) = t$ ; and  $th(a/E, \bar{P} \wedge \bar{Q}) = \langle l, s_1, s_2 \rangle$ .

By Lemma 5.2,  $(M/E, \bar{P}^*, \bar{Q}^*)$  either has only one element or it has an interval  $(a/E, b/E) \neq \emptyset$  such that  $(M/E, \bar{P}^*, \bar{Q}^*) \upharpoonright (a/E, b/E) \in U_{\bar{r}}^n$ .

Now we prove  $a E b$  and so show that this case does not occur and  $E$  has one equivalence relation, hence  $Th_k^n(M, \bar{P}, \bar{Q}) \in T^*$  and so we shall finish.

Let  $a \leq a' < b' \leq b$ , then let

$$J_2 = \{c \in M: a'/E < c/E < b'/E\},$$

$$J_1 = \{c \in M: a' < c \in \text{int } a'/E\},$$

$$J_3 = \{c \in M: b' > c \in \text{int } (b'/E)\}.$$

By (b),  $Th_k^n((M, \bar{P}, \bar{Q}) \upharpoonright J_2) \in T^*$ ; by (d)  $Th_k^n((M, \bar{P}, \bar{Q}) \upharpoonright J_i) \in T^*$  for  $i = 1, 3$ . Hence by (c) and (e)  $Th_k^n((M, \bar{P}, \bar{Q}) \upharpoonright (a', b')) \in T^*$ . So  $a E b$ , and we finish.

**THEOREM 5.5.** (A) If  $\kappa(K) \leq \aleph_1$ , and for every  $M \in K$ , there is  $N \in K \cap U^{n+1}$  extending  $M$ , then from  $UTH_k^{n+1}(K) = \{UTH_k^{n+1}(M): M \in K \cap U_k^{n+1}\}$ , we can compute  $Th_k^n(K)$ . Hence if  $UTH^n(K)$  is recursive in  $n$ , then the monadic theory of  $K$  is decidable.

(B) Suppose  $\kappa(K) \leq \aleph_1$ ,  $K$  is closed under  $M + N$ ,  $\sum_{n < \omega} M_i$ ,  $\sum_{\substack{n \in \mathbb{Z} \\ n \leq 0}} M_n$ ,  $\sum_{i \in \mathbb{Q}} M_i$  are convex submodels and division by convex equivalence relations. Then from  $UTH_{\bar{r}}^n(K)$  ( $\bar{r} = r(n, 0, \bar{k})$ ) we can compute  $Th_k^n(K)$ . Hence if  $UTH^n(K)$  is recursive in  $n$ , then the monadic theory of  $K$  is decidable.

*Proof.* (A) Immediate.

(B) Essentially the same as the proof of 5.4.

*Remark.* Of course there are other versions of (B), e.g., for a class of complete orders.

## 6. Applications of Section 5 to dense orders

**Definition 6.1.**  $K_S$  is the class of orders  $M$  such that no submodel of  $M$  is isomorphic to  $\omega_1$  or  $\omega_1^*$  or an uncountable subset of the reals.<sup>11</sup>

**LEMMA 6.1.** (A)  $K_S$  satisfies the hypothesis of 5.5 (B). Also no member of  $K_S$  is complete, except the finite ones.

(B)  $K_S$  has uncountable members, but  $M \in K_S$  implies  $\|M\| \leq \aleph_1$ .

*Proof.* (A) Immediate.

<sup>11</sup> Those are the Specker orders; we get them from Aronszajn trees.

(B) The Specker orders. See e.g., [Je 1]<sup>12</sup> for existence.

**THEOREM 6.2.** (A) *The monadic theory of  $K_s$  is decidable.*

(B) *All dense order from  $K_s$ , with no first nor last element, have the same monadic theory.*

*Proof.* We shall show that for  $(M, \bar{P}) \in U^0(K)$ ,  $\bar{P}$  a partition,  $pUTH^1(M, \bar{P})$  can be computed from  $pUTH^0(M, \bar{P})$  (hence the former uniquely determine the latter). Then by the parallel to Lemma 2.2, (B) follows immediately and (A) follows by 5.6.

So let  $t = pUTH^0(M, \bar{P})$  be given; that is, we know that  $\bar{P}$  is a partition of  $M$  to dense or empty subsets,  $M \in U^0$ , hence  $M$  is dense with no first and no last element,  $M \in K$ , and we know  $\{i: P_i \neq \emptyset\}$ . So without loss of generality.  $P_i \neq \emptyset$  for every  $i$  and also  $M \neq \emptyset$ ,  $P_i$  is dense. Let  $pUTH^1(M, \bar{P}) = \langle S_1, S_2, \text{com} \rangle$ , so we should compute  $\text{com}$ ,  $S_1$ ,  $S_2$ .

**Part (1) com:** As  $M \in K$ , and as clearly the rational order is embeddable in  $M$ ,  $M$  cannot be complete.

**Part (2)  $S_1$ :** It suffices to prove that any dense subset  $P$  of  $M$  can be split into two disjoint dense subsets of  $M$ .

So we shall prove more.

(\*) If  $M$  is a dense order,  $I \subseteq M$  is a dense subset, then we can partition  $I$  to two dense subsets of  $M$ . That is, there are  $J_1, J_2$ ,  $I = J_1 \cup J_2$ ,  $J_1 \cap J_2 = \emptyset$  and  $J_1, J_2$  are dense subsets of  $M$ .

We define an equivalence relation  $E$  on  $I$ :  $a E b$  if,  $a = b$  or there are  $a_0 < a$ ,  $b < b_0$  and  $a_0 < a' < b' < b_0$  implies  $|\{c \in I: a' < c < b'\}| = |\{c \in I: a < c < b\}|$  (and they are infinite by assumption). Now for every  $E$ -equivalence class  $a/E$  with more than one element, let  $\lambda = |\{a \in I: b' < a < c'\}|$  for every  $b' \leq c' \in a/E$ .

**Case I.**  $|a/E| = \lambda > 0$ .

Then let  $\{\langle b_i, c_i \rangle: i < \lambda\}$  be an enumeration of all pairs  $\langle b, c \rangle$  such that  $b, c \in a/E$ ,  $b < c$ . Define by induction on  $i < \lambda$ ,  $a_i^1, a_i^2 \in a/E$ . If we have defined them for  $j < i$ , choose

$$\begin{aligned} a_i^1 &\in \{d \in I: b_i < d < c_i\} - \{a_j^2: j < i\}, \\ a_i^2 &\in \{d \in I: b_i < d < c_i\} - \{a_j^1: j \leq i\}. \end{aligned}$$

By cardinality considerations this is possible. Define  $J_1(a/E) = \{a_i^1: i < \lambda\}$ .

**Case II.**  $\lambda < |a/E|$ .

Then clearly  $|a/E| = \lambda^+$ , and we can partition  $a/E$  into  $\lambda^+$  convex subsets  $A_i$ ,  $i < \lambda^+$ , each of power  $\lambda$ . So on each we can define  $J_1(A_i)$  such that  $J_1(A_i)$ ,  $A_i - J_1(A_i)$  are dense subsets of  $A_i$ . Let  $J_1(a/E) = \bigcup_{i < \lambda^+} J_1(A_i)$ .

<sup>12</sup> There is some overlapping between  $S_1$  and  $S_2$ .

Case III.  $\lambda = 0$ , so  $|a/E| = 1$ .

Let  $J_1(a/E) = \emptyset$ . Let  $J_1 = \bigcup_{a \in I} J_1(a/E)$ ,  $J_2 = I - J_1$ .

It is easy to check that  $J_1, J_2$  are the desired subsets.

Part (3)  $S_2$ : By (2) it suffices to find the possible  $UTh^0(M/E, \bar{P}^*)$ , where  $\bar{P}^* = \langle \dots, P_{\langle l, s_1, s_2 \rangle}^*, \dots \rangle$ ,  $P_{\langle l, s_1, s_2 \rangle}^* = \{a/E; th(a/E, \bar{P}) = \langle l, s_1, s_2 \rangle\}$ , and  $(M/E, \bar{P}^*) \in U^0(K)$ ; so  $W_E = \{\langle l, s_1, s_2 \rangle: P_{\langle l, s_1, s_2 \rangle}^* \neq \emptyset\}$  contain all relevant information. Clearly  $W_E \neq \emptyset$  and  $\langle l, s_1, s_2 \rangle \in W_E \Rightarrow l > 0$  and we can also discard the case  $\langle l, s_1, s_2 \rangle \in W_E \Rightarrow l = 1$ . Also if  $\langle l, s_1, s_2 \rangle \in W_E$ , then  $\langle l, s_1, s_2 \rangle$  is formally possible.

Suppose  $W$  satisfies all those conditions, and we shall find a suitable  $E$  such that  $W_E = W$ . Let  $W = \{\langle l^i, s_1^i, s_2^i \rangle: i < q < \omega\}$ . Choose a  $J \subseteq |M|$ , countably dense in itself, unbounded in  $M$  from above and from below, such that each  $P_j \cap J$  is a dense subset of  $J$ , and for no  $a \in |M| - J$  is there a first (last) element in  $\{b \in J: b > a\}$  ( $\{b \in J: b < a\}$ ).  $J$  defines  $2^{\aleph_0}$  Dedekind cuts, but as  $M \in K$ , only  $\leq \aleph_0$  of them are realized. Let  $\{a_n: n < \omega\}$  be a set of representatives from those cuts (that is, for every  $a \in |M| - J$  there is  $n < \omega$  such that  $[a, a_n]$  or  $[a_n, a]$  is disjoint to  $J$ ). Let  $J = \{b_n: n < \omega\}$ . Now we define by induction on  $n$  a set  $H_n$  of convex disjoint subsets of  $M$ , such that:

(a)  $H_n \subseteq H_{n+1}$ ;  $H_n$  is finite.

(b) If  $I_1 \neq I_2 \in H_n$  then  $I_1 < I_2$  or  $I_2 < I_1$  and between them there are infinitely many members of  $J$ .

(c) If  $I \in H_n$ ,  $I$  has no last element, then for every  $a \in |M| - J$ ,  $a > I$ , there is  $b \in J$ ,  $I < b < a$ , and also  $J \cap I$  is unbounded in  $I$ .

(d) The same holds for the converse order.

(e) If  $I_1 < I_2 \in H_n$ ,  $i < q$  then there are  $I \in H_{n+1}$ ,  $th(I, \bar{P}) = \langle l^i, s_1^i, s_2^i \rangle$ .<sup>13</sup>

(f)  $a_n, b_n \in \bigcup \{I: I \in H_n\}$ .

(g) If  $I \in H_n$  has a first (last) element then this element belongs to  $J$ . It is not hard to define the  $H_n$ 's. Clearly  $\bigcup_n \bigcup_{I \in H_n} I = |M|$ . So define  $E$  as follows:

$a E b$  if and only if  $a = b$  or for some  $n < \omega$ ,  $I \in H_n$ ,  $a, b \in I$ .

It is not hard to check that  $W_E = W$ . So we finish the proof.

Along similar lines we can prove

**THEOREM 6.3.** Suppose  $M$  is a dense order with no first nor last elements,  $M$  is a submodel of the reals, and for every perfect set  $P$  of reals,  $P \cap |M|$  is countable, or even  $< 2^{\aleph_0}$ . Then the monadic theory of  $M$  is the monadic theory of rationals.

<sup>13</sup> Also,  $I_1 < I < I_2$ , and  $I_0 \in H_n$  implies  $th(I_0, \bar{P}) \in W$ .

*Remark 1.* We can integrate the results of 6.2, 6.3. Always some  $M$  satisfies the hypothesis of 6.3. If  $2^{\aleph_0} > \aleph_1$ , any dense  $M \subseteq R$ ,  $|M| < 2^{\aleph_0}$ , and if  $2^{\aleph_0} = \aleph_1$ , the existence can be proved.

*Remark 2.* In 6.3 we can demand less of  $|M|$ : For all countable, disjoint and dense sets  $Y_1, \dots, Y_n$  ( $n < \omega$ ) there is a perfect set  $P$  of reals such that  $Y_i$  is dense in  $P$  for  $1 \leq i \leq n$  and  $P \cap |M|$  is  $< 2^{\aleph_0}$  (see Section 7 for definition).

The proof of 5.6 is easily applied to the monadic theory of the reals. (We should only notice that  $R$  is complete.)

*Conclusion 6.4:* If we can compute the  $UTh^n(R)$  for  $n < \omega$  then the monadic theory of the real order is decidable.

*Remark:* Similar conclusions hold if we add to the monadic quantifier (or replace it by)  $(\exists^{<\aleph_1} X)$  (i.e., there is a countable  $X$ ). Notice that if  $E$  is a convex equivalence relation over  $R$ , then  $\{a/E: |a/E| > 1\}$  is countable.

Grzegorzczuk [Gr 1] asked whether the lattice of subsets of the reals with the closure operation has a decidable theory. One of the corollaries of Rabin [Ra 1] is that the theory of the reals with quantification over closed sets, and quantification over  $F_\sigma$  sets is decidable.

By our methods we can easily prove

**THEOREM 6.5.** *The reals, with quantifications over countable sets, has a decidable theory. (We can replace " $X$  countable" by " $|X| < 2^{\aleph_0}$ " or " $(\forall P) (P \text{ closed nowhere dense} \rightarrow |P \cap X| < 2^{\aleph_0})$ ".)*

As every closed set is a closure of a countable set, this proves again the result of Rabin [Ra 1] concerning Grzegorzczuk's question. We can also prove by our method Rabin's stronger results, but with more technical difficulties.

## 7. Undecidability of the monadic theory of the real order

Our main theorem here is

**THEOREM 7.** (A) (CH) *The monadic theory of the real order is undecidable.*

(B) (CH) *The monadic theory of order is undecidable.*

The method gives some variation of this result.

**THEOREM 7.2.** (CH) *The monadic theory of  $K_n = \{(R, Q_1, \dots, Q_n): Q_i \subseteq R\}$ , where the set quantifier ranges over countable sets,  $1 \leq n$ , is undecidable. (We can even restrict ourselves to sets of rationals.)*

Let  $2^{\leq \omega}$  be the set of sequences of ones and zeros of length  $\leq \omega$ ; let  $\leq$  be a partial ordering of  $2^{\leq \omega}$  meaning that it is an initial segment,  $<$  the lexicographic order.

THEOREM 7.3. (A) (CH) *The monadic theory of  $(2^{\leq \omega}, \leq, <)$  is undecidable.*

(B) (CH) *The monadic theory of  $K_n = \{(2^{\leq \omega}, \leq, <, Q_1, \dots, Q_n): Q_i \subseteq 2^{\leq \omega}\}$ , where the set quantifier ranges over countable sets,  $1 \leq n$ , is undecidable. (We can even restrict ourselves to subsets of  $2^{<\omega}$ ).*

Instead of the continuum hypothesis, we can assume only:

(\*) "The union of  $< 2^{\aleph_0}$  sets of the first category is not  $R$ ".

This is a consequence of Martin's axiom (see [Mr 1]) hence weaker than CH, but also its negation is consistent, see Hechler [He 1] and Mathias [Mat 1] and Solovay [So 1]). Aside from countable sets, we can use only a set constructible from any well-ordering of the reals. Remember that by Rabin [Ra 1] quantification over closed and  $F_\sigma$  sets gives us still a decidable theory.

Conjecture 7A. The monadic theory of  $(2^{\leq \omega}, \leq, <)$ , where the set quantifier ranges over Borel sets only, is decidable.

This should be connected to the conjecture on Borel determinacy (see Davis [Da 1], Martin [Mr 1] and Paris [Pa 1]).<sup>14</sup> This conjecture implies

Conjecture 7B. The monadic theory of the reals, where the set quantifier ranges over Borel sets, is decidable (by Rabin [Ra 1]).

Conjecture 7C. We can prove 7.1–7.3 in ZFC.

Theorems 7.1(A), (B), 7.3(A) answer well known problems (see e.g., Büchi [B 5, p. 38, Problem 1, 2a, 2b, 4a]). Theorem 7.3(B) answers a question of Rabin and the author.

Unless mentioned otherwise, we shall use CH or (\*).

*Notation.*  $\mathbf{R}$  denotes the reals. A *perfect* set is a closed, nowhere dense set of reals, with no isolated points and at least two points (this is a somewhat deviant definition). We use  $P$  to denote perfect sets. Let  $x$  be an inner point of  $P$  if  $x \in P$ , and for every  $\varepsilon > 0$ ,  $(x - \varepsilon, x) \cap P \neq \emptyset$ ,  $(x, x + \varepsilon) \cap P \neq \emptyset$ . Let  $D \subseteq P$  be dense in  $P$  if for every inner point  $x < y$  of  $P$ , there is an inner  $z \in P \cap D$ ,  $x < z < y$ . Note that if  $D$  is dense in  $P$ ,  $P$  is the closure of  $P \cap D$ . Real intervals will be denoted by  $(a, b)$  where  $a < b$ , or by  $I$ ;  $(a, b)$  is an interval of  $P$  if in addition  $a, b$  are inner points of  $P$ .

LEMMA 7.4. *Let  $J$  be an index-set, the  $D_i$  ( $i \in J$ ) countable dense subsets of  $\mathbf{R}$ , and  $D = \bigcup_{i \in J} D_i$ ; and for every  $P$ ,  $|D \cap P| < 2^{\aleph_0}$ . Then there is  $Q \subseteq \mathbf{R} - D$ ,  $Q = Q\{D_i: i \in J\}$ , such that*

(A) *if  $P \cap D \subseteq D_i$  ( $i \in J$ ) and  $D_i$  is dense in  $P$  ( $P$  is, of course, perfect)*

<sup>14</sup> Meanwhile Martin proved the Borel determinacy.



then  $|P \cap Q| < 2^{\aleph_0}$ ;

(B) if for no (interval)  $I$  of  $P$ , and  $i \in J$ ,  $P \cap D \cap I \subseteq D_i$  but  $D$  is dense in  $P$  then  $P \cap Q \neq \emptyset$ .

*Proof.* Let  $\{P_\alpha: 0 < \alpha < 2^{\aleph_0}\}$  be any enumeration of the perfect sets. We define  $x_\alpha$ ,  $\alpha < 2^{\aleph_0}$  by induction on  $\alpha$ .

For  $\alpha = 0$ ,  $x_\alpha \in \mathbf{R}$  is arbitrary.

For any  $\alpha > 0$ , if  $P_\alpha$  does not satisfy the assumptions of (B) then let  $x_\alpha = x_0$  and if  $P$  satisfies the assumptions of (B) let  $x_\alpha \in P_\alpha - \bigcup \{P_\beta: \beta < \alpha, (\exists i \in J)(P_\beta \cap D \subseteq D_i \text{ and } D \text{ is dense in } P_\beta)\} = D$ .

This is possible because for any  $\beta$ ,  $i$ , if  $P_\beta \cap D \subseteq D_i$ ,  $D$  is dense in  $P_\beta$ ,  $P_\beta \cap P_\alpha$  is a closed nowhere dense subset of  $P_\alpha$ . As otherwise for some interval  $I$  of  $P_\alpha$ ,  $P_\beta \cap P_\alpha$  is dense in  $P_\alpha$ , so by the closedness of  $P_\beta \cap P_\alpha$ ,  $P_\beta \cap P_\alpha \cap I = P_\alpha \cap I$ ; therefore

$$D_i \supseteq P_\beta \cap D \supseteq P_\alpha \cap I \cap D,$$

a contradiction of the assumption on  $P_\alpha$ . So by (\*) and the hypothesis  $|P_\alpha \cap D| < 2^{\aleph_0}$  there exist such  $x_\alpha$ .

Now let  $Q = \{x_\alpha: \alpha < 2^{\aleph_0}\}$ . If  $P$  satisfies the assumption of (A), then  $P \in \{P_\alpha: 0 < \alpha < 2^{\aleph_0}\}$ . Hence for some  $\alpha$ ,  $P = P_\alpha$ , hence  $P \cap D \subseteq \{x_\beta: \beta < \alpha\}$ , so  $|P \cap D| < 2^{\aleph_0}$ . If  $P = P_\alpha$  satisfies the assumption of (B) then  $x_\alpha \in P_\alpha$ ,  $x_\alpha \in Q$ , hence  $P_\alpha \cap Q \neq \emptyset$ . So we have proved the lemma.

LEMMA 7.5. *There is a dense  $D \subseteq \mathbf{R}$  and  $\{D_i: i \in J\}$ ,  $|J| = 2^{\aleph_0}$  such that*

(1)  $|D \cap P| < 2^{\aleph_0}$  for every perfect  $P$ .

(2) The  $D_i$  are pairwise disjoint.

(3)  $D_i \subseteq D$ ,  $D_i$  is dense.

*Proof.* Let  $\{P_\alpha: \alpha < 2^{\aleph_0}\}$  enumerate the perfect subsets of  $\mathbf{R}$ , and let  $\{I_n: n < \omega\}$  enumerate the rational intervals of  $\mathbf{R}$ , and if  $\alpha = \delta + n$  ( $n < \omega$ ,  $\delta$  a limit ordinal) choose  $x_\alpha \in I_n - \bigcup_{\beta < \alpha} P_\beta - \{x_\beta: \beta < \alpha\}$  and let  $D = \{x_\beta: \beta < 2^{\aleph_0}\}$ ,  $D_\alpha = \{x_{\omega\alpha+n}: n < \omega\}$ .

*Notation.*  $J$  will be an index set;  $[J]^n = \{U: U \subseteq J, |U| = n\}$ , and if  $D_i$  is defined for  $i \in J$ , let  $D_U = \bigcup_{i \in U} D_i$ . Subsets of  $[J]^n$ , i.e., symmetric  $n$ -place relations over  $J$ , are denoted by  $S$ ; and if we know  $\{D_i: i \in J\}$ ,  $Q_S$  will be  $Q\{D_U: U \in S \cup [J]^{n-1}\}$  from 7.4.

*Definition 7.1.* Let  $\varphi_n(X, D, Q, I^*)$  be the monadic formula saying

(A)  $X$  is a dense set in  $I^*$  and  $X \subseteq D$ .

(B) For every interval  $I \subseteq I^*$ , and sets  $Y_i$ ,  $i = 1, n+1$ , if  $Y_i \cap I \subseteq X$  and the  $Y_i$  are pairwise disjoint and each  $Y_i$  is dense in  $I$  then there is a perfect set  $P$ ,  $P \cap Q = \emptyset$ , and each  $Y_i \cap I$  is dense in  $P$ .

*Remark.* We can represent the interval  $I_0$  as a convex set.

LEMMA 7.6. *Let  $D$ ,  $\{D_i: i \in J\}$  be as in 7.5,  $I^*$  an interval,  $S \subseteq [J]^n$ ,  $Q_s = Q\{D_U: U \in S \cup [J]^{n-1}\}$  as in 7.4. Then for any set  $X \subseteq R$ ,  $R \models \varphi_n[X, D, Q_s, I^*]$  if and only if*

(A)  *$X$  is dense in  $I^*$ ,  $X \subseteq D$ ,*

(B) *for any interval  $I \subseteq I^*$  there is a subinterval  $I_1$  and  $U \in S \cup [J]^{n-1}$  such that  $X \cap I_1 \subseteq D_U$ .*

*Proof.* (I) Suppose  $R \models \varphi_n[X, D, Q_s, I^*]$ . Then by (A) from Definition 7.1,  $X$  is dense in  $I^*$ ,  $X \subseteq D$  so (A) from here is satisfied. To prove (B) let  $I \subseteq I^*$  be an interval, and suppose that for no subinterval  $I_1$  of  $I$  and for no  $U \in S \cup [J]^{n-1}$ , does  $X \cap I_1 \subseteq D_U$  hold, and we shall get a contradiction. Now we define by induction on  $l$ ,  $1 \leq l \leq n+1$ , distinct  $i(l) \in J$  and intervals  $I^l$ ,  $0 \leq l \leq n$  so that  $I^0 = I$ ,  $I^{l+1} \subseteq I^l$ , and  $X \cap D_{i(l)} \cap I^l$  is dense in  $I^l$ .

If we succeed, in Definition 7.1(B), choose  $I^{n+1}$  as  $I$ , and  $X \cap D_{i(l)} \cap I^{n+1}$  as  $Y_l$ . So necessarily by  $\varphi_n$ 's definition there is a perfect  $P$  such that  $X \cap D_{i(l)} \cap I^{l+1}$  is dense in  $P$  for  $l = 1, n+1$ , and  $P \cap Q_s = \emptyset$ . But this contradicts Lemma 7.4(B) by the definition of  $Q_s$ . So for some  $l < n+1$  we cannot find appropriate  $i(l+1)$ ,  $I^{l+1}$ . So if we let  $Y = (X - \bigcup_{k \leq l} D_{i(k)}) \cap I^l$ , for no  $I^+ \subseteq I^l$  and no  $i \in J$  is  $Y \cap D_i \cap I^+$  dense; i.e., for every  $i \in J$ ,  $Y \cap D_i$  is nowhere dense.

If  $l = n$ , but  $\{i(1), \dots, i(n)\} \notin S$  let  $D_{i(n)} \cap X \cap I^l = Y_n^1 \cup Y_{n+1}^1$ , where  $Y_n^1, Y_{n+1}^1$  are dense subsets of  $I^l$ , and  $Y_k^1 = X \cap D_{i(k)} \cap I^l$ , and we get a contradiction as before.

If  $Y$  is not dense in  $I^l$ , it is disjoint to some  $I^+ \subseteq I^l$ , so  $X \cap I^+ \subseteq \bigcup_{k < l} D_{i(k)}$ . So  $U = \{i(0), \dots, i(l)\} \in S \cup [J]^{n-1}$ ,  $X \cap I^+ \subseteq D_U$ , contradicting an assumption we made in the beginning of the proof. Hence  $Y$  is dense in  $I^l$ .

As  $(\forall i \in J) Y \cap D_i$  is nowhere dense also for every finite  $U \subseteq J$ ,  $Y \cap D_U$  is nowhere dense. So we can chose inductively distinct  $i_m \in J$  and distinct  $x_m \in Y \cap D_{i_m}$  such that  $\{x_{(n+3)m+k}: m < \omega\}$  are dense subsets of  $I^l$ , for  $0 \leq k < n+2$ . If we let  $Y_k^2 = \{x_{(n+3)m+k}: m < \omega\}$  for  $k \leq n+1$ , by Definition 7.1 there is a perfect  $P$ , such that  $Y_k^2$  is dense in  $P$ ,  $P \cap Q = \emptyset$ , and we get a contradiction by 7.4(B) and the choice of the  $x_m$ 's.

As all the ways give a contradiction, we finish one implication.

(II) Now we want to prove that  $R \models \varphi_n[X, D, Q, I^*]$  assuming the other side.

Clearly  $X \subseteq D$ , and  $X$  is dense in  $I^*$  (by condition (A) of Lemma 7.6). So condition (A) in Definition 7.1 holds. For condition (B) of that definition let  $I \subseteq I^*$  be an interval,  $Y_k \cap I \subseteq X$ ,  $Y_k$  dense in  $I$  for  $k = 1, n+1$  and

$k \neq l \Rightarrow Y_k \cap Y_l = \emptyset$ . We should find a perfect  $P$  such that  $P \cap Y_k$  is dense in  $P$  and  $P \cap Q = \emptyset$ . We can choose a  $U \in S \cup [J]^{n-1}$  and  $I_1 \subseteq I$  so that  $X \cap I_1 \subseteq D_U$  (by the hypothesis). Choose a perfect  $P$  such that each  $Y_k$  is dense in  $P$ . As  $D$  is as in Definition 7.4, either case gives  $|P \cap D| < 2^{\aleph_0}$ .

(\*) Now we can find perfect  $P_\alpha$  ( $\alpha < 2^{\aleph_0}$ ) such that each  $Y_k$  ( $1 \leq k \leq n+1$ ) is dense in  $P_\alpha$  and  $\alpha \neq \beta$  implies  $P_\alpha \cap P_\beta \subseteq \bigcup_{k=1}^{n+1} Y_k$ .

*Proof of (\*).* For  $\eta$  a finite sequence of ones and zeros  $X_\eta$  will be a set of closed-open intervals and singletons with endpoints in  $\bigcup_{k=1}^{n+1} Y_k$ , which are pairwise disjoint. We define  $X_\eta$  by induction on  $l(\eta)$ . Let  $X_{\langle \rangle} = \{[a, b)\}$ , where  $a, b \in Y_1$ , and if  $X_\eta$  is defined, for each interval  $[a, b) \in X_\eta$ , choose a decreasing sequence  $x_i^a$  ( $i < \omega$ ) whose limit is  $a$ , and  $x_0^a < b$  and  $x_i^a \in Y_k$  if and only if  $l(\eta) = k \bmod n+1$ ,  $1 \leq k \leq n+1$ . Let, for  $m = 0, 1$ ,

$$X_{\eta \wedge \langle m \rangle} = \{(x_{i+1}^a, x_i^a) : \text{for some } b, [a, b) \in X_\eta \text{ and } i = m \bmod 2\} \\ \cup \{[a) : \text{for some } b, [a, b) \in X_\eta, \text{ or } \{a\} \in X_\eta\}.$$

For  $\eta$  a sequence of ones and zeros of length  $\omega$ ,  $P_\eta = \bigcap_{l < \omega} (\bigcup X_{\eta|_l})$ .

Because  $|P \cap D| < 2^{\aleph_0}$  for some  $\alpha$ ,  $P_\alpha \cap D \subseteq \bigcup_{k=1}^{n+1} Y_k$ ; so by 7.4 (and the choice of  $Q$ 's),  $|P_\alpha \cap Q_S| < 2^{\aleph_0}$ . We can find  $P_\beta$  ( $\beta < 2^{\aleph_0}$ ) such that each  $Y_k$  is dense in  $P_\beta$  and  $\beta \neq \gamma \Rightarrow P_\beta \cap P_\gamma \subseteq \bigcup_{k=1}^{n+1} Y_k$ . So for some  $\beta$ ,  $P_\beta \cap Q \subseteq \bigcup_{k=1}^{n+1} Y_k \subseteq D$ , but  $Q \subseteq R - D$  hence  $P_\beta \cap Q = \emptyset$ , and we finish.

*Definition 7.2.* Let  $\psi_n(X, D, Q, I^*)$  be the monadic formula saying

(A)  $\varphi_n(X, D, Q, I^*)$ ,

(B) for any interval  $I_1 \subseteq I^*$ , if  $Y$  is disjoint to  $X$  and dense in  $I_1$  then  $\neg \varphi_n(X \cup Y, D, Q, I_1)$ .

**LEMMA 7.7.** Let  $D, J, D_i, S, Q_S$  be as in 7.6. Then for any  $X \subseteq R$   $R \models \psi_n[X, D, Q_S, I^*]$  if and only if

(A)  $X$  is dense in  $I^*$ ,  $X \subseteq D$ ,

(B) for any interval  $I \subseteq I^*$  there is a subinterval  $I_1$  and  $U \in S \cup \{V \in [J]^{n-1} : (\forall i \in J)(V \cup \{i\} \notin S)\}$  such that  $X \cap I_1 = D_U \cap I_1$ .

*Proof.* (I) Suppose  $R \models \psi_n[X, D, Q_S, I^*]$ , then clearly condition (A) holds. For condition (B) let  $I \subseteq I^*$  be an interval. By Definition 7.2(A),  $R \models \varphi_n[X, D, Q_S, I^*]$ , hence by Lemma 7.6(1)(B),  $I$  has a subinterval  $I_0$  such that  $X \cap I_0 \subseteq D_U$  where  $U \in S \cup [J]^{n-1}$ . If  $(D_U - X) \cap I_0$  is somewhere dense, let it be dense in  $I_1 \subseteq I_0$ , and let  $Y = (D_U - X) \cap I_1$ , which gives us a contradiction to Definition 7.2(1)(B). If  $U \in [J]^{n-1}$ , and for some  $i \in J$ ,  $V = U \cup \{i\} \in S$ , we can get a similar contradiction by  $Y = (D_V - X) \cap I_0$  in the interval  $I_0$  (as  $D_i \subseteq D_V - X$ ,  $Y$  is dense). We can conclude that :  $U \in S$  or  $U \in [J]^{n-1}$  and  $U \cup \{i\} \notin S$  for

every  $i \in J$  and that  $(D_v - X) \cap I_0$  is nowhere dense. Hence for some  $I_1 \subseteq I_0$ ,  $(D_v - X) \cap I_1 = \emptyset$  hence  $X \cap I_1 = D_v \cap I_1$ .

(II) Now suppose that conditions (A), (B) hold; by Lemma 7.6 it is easy to see that  $R \models \psi_n[X, D, Q_s, I^*]$ .

*Definition 7.3.* Let  $\chi_i(D, Q, I^*)$  be the monadic formula saying:

- (A)  $D$  is dense in  $I^*$ ,  $I^*$  an interval;
- (B) if  $I \subseteq I^*$ ,  $X, Y$  are dense in  $I$  and

$$R \models \psi_1[X, D, Q, I] \wedge \psi_1[Y, D, Q, I]$$

then for some  $I_1 \subseteq I$ ,

$$X \cap Y \cap I = \emptyset \text{ or } X \cap I_1 = Y \cap I_1.$$

LEMMA 7.8. (A) If  $D, \{D_i: i \in J\}$ , are as in 7.5 then for any interval  $I^*$ ,  $R \models \chi_1[D, Q_J, I^*]$ .

(B) If  $R \models \chi_1[D, Q, I^*]$  then we can find  $I \subseteq I^*$ , and  $X_i$ ,  $i < \alpha_0$  such that

- 1) each  $X_i$  is a dense subset of  $I$  and  $R \models \psi_1[X_i, D, Q, I]$ ,
- 2)  $i \neq j \Rightarrow X_i \cap X_j$  is nowhere dense,
- 3) if  $I_0 \subseteq I$ , and  $X \subseteq I_0$  is dense in  $I_0$  and  $R \models \psi_1[X, D, Q, I_0]$

then there are  $i < \alpha$  and  $I_1 \subseteq I_0$  such that  $X \cap I_1 = X_i \cap I_1$ .

(C) In (B),  $|\alpha_0|$  is uniquely defined by  $D, Q, I$ .

*Proof.* (A) By 7.7 it is immediate.

(B) Let  $\{X_i: i < \alpha\}$  be a maximal family satisfying (1) and (2) for  $I = I^*$ . If for some interval  $I$  there are no subintervals  $I^1$  and dense  $X^* \subseteq X \cap I^1$  such that  $(\forall i < \alpha_0) (X_i \cap X^* \text{ is nowhere dense})^{15}$  we are finished. Otherwise we can choose inductively on  $n$  intervals  $I^n \subseteq I^*$  disjoint to  $\bigcup_{l < n} I^l$  and  $X_n^* \subseteq X \cap I^n$  such that  $(\forall i < \alpha_0)$ ,  $X_i \cap X_n^*$  is nowhere dense<sup>16</sup>, and such that  $\bigcup_{n < \omega} I^n$  is dense in  $I$ . Then we could have defined  $X_{\alpha_0} = \bigcup_{n < \omega} X_n^*$ , a contradiction.

(C) Easy.

*Definition 7.4.* Let  $\chi^n(Q_1, D, Q, I^*)$  be the monadic formula saying

- (A)  $D$  is dense in  $I^*$ , which is an interval.
- (B) Suppose  $I_0 \subseteq I^*$ ,  $X_l \subseteq I_0$  ( $l < n$ ) and  $R \models \bigwedge_{l < n} \psi_1(X_l, D, Q, I_0)$ . Then there is  $I_1 \subseteq I_0$  such that for all  $I_2 \subseteq I_1$

$$R \models \psi_n(\bigcup_{l < n} X_l, D, Q_1, I_1) \equiv \psi_n(\bigcup_{l < n} X_l, D, Q_1, I_2).$$

LEMMA 7.9. If  $D, \{D_i: i \in J\}$  are as in Lemma 7.5,  $S \subseteq [J]^n$  then for any interval  $I^*$ ,  $R \models \chi^n[Q_s, D, Q, I^*]$ .

<sup>15</sup> and  $R \models \psi_1[X^*, D, Q, I^1]$ .

<sup>16</sup> and  $R \models \psi_1[X^*, D, Q, I^n]$ .

*Proof.* Immediate.

**THEOREM 7.10.** *The set  $A_r$  is recursive in the monadic theory of order; where  $A_r = \{\theta: \theta \text{ is a first order sentence which has an } \omega\text{-model i.e., a model } M \text{ such that } (|M|, R_1) \text{ is isomorphic to } (\omega, x + 1 = y)\}$ .*

**Conclusion 7.11.** True first order arithmetic is recursive in the monadic theory of order.

*Proof.* It suffices to define for every first order sentence  $\theta$ , a monadic sentence  $G(\theta)$  so that  $R \models G(\theta)$  if and only if  $\theta$  has an  $\omega$ -model.

By using Skolem-functions and then encoding them by relations, we can define effectively the sentence  $G_1(\theta)$  such that  $\theta$  has an  $\omega$ -model if and only if  $G_1(\theta)$  has an  $\omega$ -model and

$$G_1(\theta) = (\forall x_1, \dots, x_{n(0)})(\exists x_{n(0)+1}, \dots, x_{n(1)}) (\bigvee_i \bigwedge_j \theta_{ij}),$$

$\theta_{ij}$  is an atomic, or a negation of an atomic, formula; only the relations  $R_0, \dots, R_{n(2)}$  appear in it;  $R_0$  is the equality; and  $R_i$  has  $m(i)$ -places.

Define (where  $X, Y, D, Q$  are variables ranging over sets,  $I$  is a variable ranging over intervals and  $x, y$  are individual variables):

$$(0) \quad G_2(X_k = X_l) = (\forall I^1 \subseteq I^*)(\exists I^2 \subseteq I^1)(X_k \cap I^2 = X_l \cap I^2),$$

$$(1) \quad G_2[R_l(X_{k(1)}, \dots, X_{k(m(l))})] = (\exists Y)(Y \subseteq D - D^* \wedge \bigwedge_{i=1}^{m(l)} \psi_2(X_{k(i)} \cup Y, D, Q_i^l, I^*) \text{ (for } l < 0),$$

$$(2) \quad G_2[\neg R_l(X_{k(1)}, \dots, X_{k(m(l))})] = \neg G_2[R_l(X_{k(1)}, \dots, X_{k(m(l))})] \text{ (for } l \geq 0),$$

$$(3) \quad G_3(\theta) = (\forall X_1, \dots, X_{n(0)})(\exists X_{n(0)+1}, \dots, X_{n(1)})$$

$$(\forall I^0 \subseteq I)(\exists I^* \subseteq I^0)[\bigwedge_{i=1}^{n(0)} \psi_1(X_i, D, Q^*, I^*) \wedge \bigwedge_{i=1}^{n(0)} X_i \subseteq D^*$$

$$\longrightarrow \bigwedge_{i=n(0)+1}^{n(1)} X_i \subseteq D^* \cap \bigwedge_{i=n(0)+1}^{n(1)} \psi_1(X_i, D, Q^*, I^*) \wedge \bigwedge_i \bigvee_j G_2(\theta_{ij})].$$

(4) Let  $\chi^*$  be the conjunction of the following formulas:

$$(\alpha) \quad D, D^* \text{ are dense in } I, D^* \subseteq D,$$

$$(\beta) \quad \chi_1(D, Q^*, I),$$

$$(\gamma) \quad \chi^2(Q_i^1, D, Q^*, I).$$

Let us denote

$$\tilde{R}_1(X, Y, Q_1^1, Q_1^2, I') = (X \subseteq D^* \wedge Y \subseteq D^* \wedge X \cap Y = \emptyset \wedge$$

$$\psi_1(X, D, Q^*, I') \wedge \psi_1(Y, D, Q^* \wedge, I') \wedge (\exists Z)[Z \subseteq D - D^* \wedge \psi_1(Z, D, Q^*, I') \wedge$$

$$\psi_2(X \cup Z, D, Q_1^1, I') \wedge \psi_2(Y \cup Z, D, Q_1^2, I')]$$

and

$$(\delta) \quad \psi_1(X_0, D, Q^*, I) \wedge X_0 \subseteq D^* \wedge (\forall Y)[\psi_1(Y, D, Q^*, I) \wedge Y \subseteq D^* \longrightarrow (\exists Y_1)\tilde{R}_1(Y, Y_1)] \wedge (\forall I' \subseteq I)(\forall Y) \neg \tilde{R}_1(Y, X_0, Q_1^1, Q_1^2, I') \wedge$$

$$(\forall Y_1 Y_2 Y_3)(\forall I^0 \subseteq I)[\tilde{R}_1(Y_1, Y_2, Q_1^1, Q_1^2, I^0) \wedge \tilde{R}_1(Y_1, Y_3, Q_1^1, Q_1^2, I^0)$$

$$\longrightarrow (\forall I^1 \subseteq I^0)(\exists I^2 \subseteq I^1) Y_2 \cap I^2 = Y_3 \cap I^2].$$

( $\varepsilon$ ) The formula saying that if ( $\delta$ ) holds when we replace  $Q_i^1, Q_i^2$  by  $\tilde{Q}_i^1, \tilde{Q}_i^2$  resp. then

$$(\forall X)(\forall Y)(\forall I' \subseteq I)[\tilde{R}_1(X, Y, Q_i^1, Q_i^2, I') \longrightarrow \tilde{R}_1(X, Y, \tilde{Q}_i^1, \tilde{Q}_i^2, I')] .$$

$$(5) \quad G(\theta) = (\exists Q^*, D, D^*, X_0, \dots, Q_i, \dots)(\forall I)[\chi^* \wedge G_3(\theta)] .$$

Now we should prove only that  $\theta$  has an  $\omega$ -model if and only if  $R \models G(\theta)$ .

(I) Suppose  $M$  is an  $\omega$ -model of  $\theta$ , and without loss of generality  $|M| = \omega$ . Let  $J = \omega + \omega$ ,  $D_i (i < \omega + \omega)$  be countable, pairwise disjoint, dense subsets of  $R$ . Choose symmetric and reflexive relations  $S_i^j$  on  $\omega + \omega$  so that

$$M \models R_i(x_1, \dots, x_{k(i)}) \iff (\exists y \in \omega + \omega) \bigwedge_{i=1}^{k(i)} \langle y, x_i \rangle \in S_i^i \wedge y \notin \omega .$$

To prove  $R \models G(\theta)$ , let  $D = \bigcup_{i < \omega + \omega} D_i$ ,  $D^* = \bigcup_{i < \omega} D_i$ ,  $Q_i^i = Q_{(S_i^i)}$ ,  $X_0 = D_0$ , and  $Q^* = Q_{\omega + \omega}$ . Let  $I$  be any interval. It is not hard to check that under those assignments  $R \models \chi^* \wedge G_3(\theta)$ .

(II) Now suppose  $R \models G(\theta)$ . Let  $Q^*, D, D^*, X_0, Q_i^i$  be such that  $R \models (\forall I)(\chi^* \wedge G_3(\theta))$ . By (4) ( $\beta$ ), clearly  $R \models (\forall I)\chi_i(D, Q^*, I)$ . Hence by Lemma 7.8(B) there are  $I$  and  $D_i$ ,  $i < \alpha$  satisfying (1), (2), (3) from 7.8(B). As  $R \models (\forall I)(\chi^* \cap G_3(\theta))$ , then in particular  $R \models \chi^* \wedge G_3(\theta)$ . By (4) ( $\delta$ ),  $R \models \psi_1(X_0, D, Q^*, I)$ , so we can choose  $D_0 = X_0$ . (See the proof of 7.8.) By (4) ( $\delta$ ) we can also assume that  $R \models \tilde{R}_1(D_n, D_{n+1})$  for  $n < \omega$ . By (4) ( $\varepsilon$ ) necessarily  $D_i \subseteq D^* \iff i < \omega$ .

Let  $\{\bar{j}_l : l < \omega\}$  enumerate all sequences  $j = \langle j(1), \dots, j(n(0)) \rangle$  of natural numbers. As  $R \models G_3(\theta)$  for every  $\bar{j}_l$  we can choose  $X_i = D_{j_l(i)}$ , and so there is an assignment  $X_i \rightarrow D^{l,i}$  for  $n(0) < i \leq n(1)$  showing that  $R \models G_3(\theta)$ . So we can define by induction on  $n < \omega$  intervals  $I_n$  so that:  $I_{n+1} \subseteq I_n$ ,  $I_0 \subseteq I$ , and for every  $n(0) < i \leq n(1)$  for some  $j_n(i) < \alpha_0$ ,  $D^{l,i} \cap I_{n+1} = D_{j_n(i)} \cap I_{n+1}$ .

Now we define a model  $M$ :  $|M| = \omega$ , and  $M \models R_i[j(1), \dots, j(m(l))] \iff$  for some  $n$ ,  $R \models (\exists Y)[Y \subseteq D - D^* \bigwedge \bigwedge_{i=1}^{m(l)} \psi_2(D_{j(i)} \cap Y, D, Q^i, I_n)]$ .

It is easy to check that  $R \models \theta$ .

*Remark.* By some elaboration, we can add to the definition of  $A_r$  also the demand

“ $R_2$  is a well-founded two-place relation”

(also for uncountable structures). Thus, e.g., there are sentences  $\theta_n$ , such that MA implies:  $R \models \theta$  if and only if  $2^{\aleph_0} = \aleph_n$ .

**THEOREM 7.11.** *The set of first-order sentences which has a model, is recursive in the monadic theory of  $\{(R, Q) : Q \subseteq R\}$  where the set-variables range over subsets of the rationals.*

*Remark.* Notice that a quantification over  $P$  such that  $D$  is dense in  $P$

can be interpreted by a quantification over  $P \cap D$ , as the property “ $x$  in the closure of  $X$ ” is first-order. Hence  $\varphi_n, \psi_n$  are, in fact, in our restricted monadic theory.

By 7.10, 7.11, Theorems 7.1, 7.2, and 7.3 are in fact immediate. Theorem 7.1(B) can also be proved by the following observation of Litman [Li 1], which is similar to 3.6(B) (1):

LEMMA 7.12. *The monadic theory of the real order is recursive in the monadic theory of order.*

*Proof.* For every monadic sentence  $\theta$  let  $G(\theta)$  be the monadic sentence saying:

“If the set  $X$  is completely ordered, is dense and has no first nor last elements then some  $Y \subseteq X$  has those properties and in addition  $(Y, <) \models \theta$ .”

As every complete dense order contains a subset isomorphic to  $\mathbf{R}$ , and any complete dense order  $\subseteq \mathbf{R}$  with no first nor last element is isomorphic to  $\mathbf{R}$ , clearly  $\mathbf{R} \models G(\theta)$  if and only if  $\theta$  is satisfied by all orders so our result is immediate.

Conjecture 7D. The monadic theory of  $\mathbf{R}$  and the (pure) second-order theory of  $2^{\aleph_0}$  are recursive in each other.<sup>17</sup>

Conjecture 7E. The monadic theory of  $\{\mathbf{R}, \mathbf{Q}: \mathbf{Q} \subseteq \mathbf{R}\}$  with the set-quantifiers ranging over subsets of the rationals; and the (pure) second-order theory of  $\aleph_0$  are recursive in each other. Gurevich notes that if  $V = L$  the intersection of 7D, E holds.

Conjecture 7F. The monadic theory of order and the (pure) second-order theory, are recursive in each other.

In conjectures 7D, E, F use (\*) or CH if necessary.

Conjecture 7G. If  $D_l$  is a dense subset of  $R$ , and for every  $P$ ,  $|P \cap D_l| < 2^{\aleph_0}$ , for  $l = 1, 2$  then  $(R, D_1), (R, D_2)$  have the same monadic theory.<sup>18</sup>

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<sup>17</sup> Gurevich proved it when  $V = L$ .

<sup>18</sup> Gurevich disproved it.

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