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PROOF OF THE INDEPENDENCE OF THE PRIMITIVE SYMBOLS OF HEYTING'S CALCULUS OF PROPOSITIONS

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In this paper I shall show that no one of the four primitive symbols of Heyting's calculus of propositions[1] is definable in terms of the other three. So as to make the paper self-contained, I begin by stating the rules and primitive sentences given by Heyting.

The primitive symbols of the calculus are "$\neg$", "$\vee$", "$\wedge$", and "$\supset$", which may be read, respectively, as "not," "either...or," "and," and "if...then." The symbol "$\supset\subset$", which may be read "if and only if," is defined in terms of these as follows:

$$a \supset\subset b \,.\, \overset{\mathrm{D}}{=} \,.\, a \supset b \,.\, \wedge \,.\, b \supset a.$$

The rule of substitution is assumed, and the rule that S_2 follows from S_1 and $S_1 \supset S_2$; in addition it is assumed that $S_1 \wedge S_2$ follows from S_1 and S_2. The primitive sentences are as follows:

2.1 $\vdash\vdash a \supset a \wedge a.$
2.11 $\vdash\vdash a \wedge b \supset b \wedge a.$
2.12 $\vdash\vdash \,.\, a \supset b \,.\, \supset \,.\, a \wedge c \supset b \wedge c.$
2.13 $\vdash\vdash \,.\, a \supset b \,.\, \wedge \,.\, b \supset c \,.\, \supset \,.\, a \supset c.$
2.14 $\vdash\vdash \,.\, b \supset \,.\, a \supset b.$
2.15 $\vdash\vdash \,.\, a \wedge \,.\, a \supset b \,.\, \supset b.$
3.1 $\vdash\vdash \,.\, a \supset a \vee b.$
3.11 $\vdash\vdash \,.\, a \vee b \supset b \vee a.$
3.12 $\vdash\vdash \,.\, a \supset c \,.\, \wedge \,.\, b \supset c \,.\, \supset \,.\, a \vee b \supset c.$
4.1 $\vdash\vdash \,.\, \neg a \supset \,.\, a \supset b.$
4.11 $\vdash\vdash \,.\, a \supset b \,.\, \wedge \,.\, a \supset \neg b \,.\, \supset \neg a.$

With regard to the primitive symbols of this calculus, Heyting states[2] that no one of them is definable in terms of the others, but he does not offer a proof of this fact. It is true that the three-element system given by Heyting shows that it would be impossible to define "$\vee$", "$\wedge$", or "$\supset$" by any of the usual definitions; his table, for example, does not satisfy the formula "$a \vee b \,.\, \supset\subset \,.\, \neg a \supset b$", so that this is not a provable formula. But this does not eliminate the possibility, that there might be some other formula S_1, not involving the mark "$\vee$", such that "$a \vee b \,.\, \supset\subset \,.\, S_1$" was provable;[3] and, in such a case, the symbol "$\vee$"

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[1] See A. Heyting, *Die formalen Regeln der intuitionistischen Logik*, ***Sitzungsberichte der Preussischen Akademie der Wissenschaften***, Physikalisch-mathematische Klasse, 1930, pp. 42–56.

[2] Loc. cit., page 44.

[3] As a matter of fact, the formula

$$\text{“}\{a \vee b\} \supset\subset \{[(a \supset b) \supset b] \wedge [(b \supset a) \supset a]\}\text{”}$$

is satisfied by Heyting's three-element matrix.

would be definable. In view of the importance of Heyting's calculus, I am therefore offering a proof, by means of tables, that the primitive symbols are mutually independent.

At the end of the present paper will be found three matrices each of which satisfies all of Heyting's primitive sentences and rules. Matrix I is the same as the three-element matrix given by Heyting, except that I have written "1" where Heyting writes "0", and "3" where Heyting writes "1".

I now give the proof that the symbol "$\urcorner$" is not definable. We notice that, in Matrix I, we have $1 \supset 1 = 1 \wedge 1 = 1 \vee 1 = 1$, and that $\urcorner 1 = 3$. Suppose now, if possible, that the "$\urcorner$" symbol were definable. Then there would exist a sentence S_1, which did not involve the mark "$\urcorner$", and such that

$$\text{“}\urcorner a \supset\subset S_1\text{”}$$

was a provable formula of the Heyting calculus. Suppose, however, we were to replace every propositional variable in the above by "1". It is clear that S_1 would have to reduce to "1". Then we should have $\urcorner 1 \supset\subset 1$, or $3 \supset\subset 1$. But $3 \supset\subset 1$ has the value 3. Hence there exists no such S_1, and the symbol "$\urcorner$" is not definable in terms of the other symbols.

More generally, we can see, that if three of the operations we are considering are class-closing on some proper sub-class of the elements of a matrix, while the fourth operation is not class-closing on this proper sub-class, then the fourth operation is not definable in terms of the other three. I shall use this same method[4] to show the independence of "$\vee$", "$\wedge$", and "$\supset$".

In Matrix II, it will be observed that, when a and b are any elements of the subset $\{1, 2, 3, 5\}$, then $a \supset b$ is an element of the subset $\{1, 2, 3, 5\}$; indeed, the only case where $a \supset b$ is 4, is when $a = 1$ and $b = 4$. Similarly, $a \wedge b$ is in $\{1, 2, 3, 5\}$ when a and b are in $\{1, 2, 3, 5\}$. We also observe, that $\urcorner a$ is never equal to 4. Thus the three operations "$\supset$", "$\wedge$", and "$\urcorner$" are class-closing on the class $\{1, 2, 3, 5\}$. On the other hand, we observe that $2 \vee 3 = 4$. Suppose now that the "$\vee$" were definable. Then there would exist a sentence S_1, in which the symbol "$\vee$" did not occur, and such that

$$\text{“}a \vee b \supset\subset S_1\text{”}$$

was provable. Let S_2 be the expression that results from S_1 by replacing "a" by "2", "b" by "3", and all the other variables (if any) in S_1 by "3". Then

[4] It will be observed that this method of showing the independence of undefined notions is altogether different from the method of Padoa (*Essai d'une théorie algébrique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque*, ***Bibliothèque du Congrès International de Philosophie***, vol. 3 (1901), pp. 309–365). For Padoa's method would require two distinct realizations in order to show the independence of a given primitive, while the present method requires but one. Properly speaking, we have here, not just a difference in method, but rather a difference in the thing proved. When Padoa's method can be applied, to show the independence of a primitive operation for example, we can conclude that the values of this operation are not determined when values are assigned to the other notions. When the present method is applied, however, we can conclude merely, that the operation whose independence we are proving, cannot be expressed by means of a formula involving only free variables and the other notions.

2∨3 $\supset\subset$ S_2 should give the value 1 under Matrix II. But we see from the above that S_2 must have a value different from 4, while **2∨3** has the value 4. Since the table for "$\supset\subset$" in Matrix II never gives the value 1 when a and b are distinct, this is impossible. Hence the "**∨**" cannot be defined in terms of the other symbols.

In Matrix III, we now verify that the three operations "$\neg$", "**∨**", and "**∧**" are class-closing on the class {11, 12, 22, 33}, while 12$\supset$22 is 21. Hence, by an argument similar to that given above, the "$\supset$" cannot be defined in terms of the other symbols.

Finally, we now verify that in Matrix III the three operations "$\neg$", "**∨**", and "$\supset$" are class-closing on the class {11, 12, 13, 31, 33}, while **12∧31** is 32. Hence the "**∧**" cannot be defined in terms of the other symbols.

This completes the proof that no one of the primitive symbols can be defined in terms of the others.

Matrix I ("designated" element: 1)

$\supset$	1	2	3
1	1	2	3
2	1	1	3
3	1	1	1

∧	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

∨	1	2	3
1	1	1	1
2	1	2	2
3	1	2	3

a	$\neg a$
1	3
2	3
3	1

$\supset\subset$	1	2	3
1	1	2	3
2	2	1	3
3	3	3	1

Matrix II ("designated" element: 1)

$\supset$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	1	3
3	1	2	1	1	2
4	1	2	3	1	5
5	1	1	1	1	1

∧	1	2	3	4	5
1	1	2	3	4	5
2	2	2	5	2	5
3	3	5	3	3	5
4	4	2	3	4	5
5	5	5	5	5	5

∨	1	2	3	4	5
1	1	1	1	1	1
2	1	2	4	4	2
3	1	4	3	4	3
4	1	4	4	4	4
5	1	2	3	4	5

a	$\neg a$
1	5
2	3
3	2
4	5
5	1

$\supset\subset$	1	2	3	4	5
1	1	2	3	4	5
2	2	1	5	2	3
3	3	5	1	3	2
4	4	2	3	1	5
5	5	3	2	5	1

Matrix III ("designated" element: 11)

The elements of this matrix are the following set of numbers: {11, 12, 13, 21, 22, 23, 31, 32, 33}, and 11 is the "designated" element. The various operations are defined in such a way, that this matrix is the direct product[5] of Matrix I by itself. Thus, for example, 12$\supset$32 is 31; since 1$\supset$3 is 3 in Matrix I, and 2$\supset$2 is 1 in Matrix I. Similarly, **22∧31** is 32; since **2∧3** is 3 in Matrix I, and **2∧1** is 2

[5] The idea of forming direct products of matrices in this way is suggested by Jaśkowski, *Recherches sur le système de la logique intuitioniste*, ***Actes du Congrès International de Philosophie Scientifique,*** Paris 1936, part VI, pp. 58–61. This author also describes an operation "Γ" which is such, that if M is an n-element matrix satisfying Heyting's postulates, then $\Gamma(M)$ will be an $(n + 1)$-element matrix also satisfying these postulates. It will perhaps be of some interest to those familiar with Jaśkowski's paper, to remark that my Matrix II is the same as $\Gamma(B_4)$, where B_4 is a four-element Boolean algebra. Also that Matrix I is the same as $\Gamma(B_2)$, where B_2 is a two-element Boolean algebra (i.e., the usual matrix for classical logic). It is of course well known that B_4 is the direct product of B_2 by itself.

in Matrix I. Similarly, $\neg 13$ is 31; since $\neg 1$ is 3, and $\neg 3$ is 1. It can thus be seen, that every formula satisfied by Matrix I will also be satisfied by Matrix III; for each component of the designated element of Matrix III is the designated element of Matrix I.

Note added November 2, 1939. Since writing this paper, I have discovered that the same problem was solved by Mordchaj Wajsberg, in his *Untersuchungen über den Aussagenkalkül von A. Heyting, Wiadomości matematyczne,* vol. 46 (1938), pp. 45–101. Wajsberg's method of proof, which is quite unlike mine, involves application of an Entscheidungsverfahren for the Heyting calculus.

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