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PETRI NETS AND LARGE FINITE SETS*

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Abstract. An upper bound is given for the complexity of the Karp–Miller decision procedure for the Finite Containment Problem for pairs of Petri nets; the procedure is shown to be primitive recursive in the Ackermann function. Bounds for the lengths of the searches involved are obtained in terms of *large* finite sets in the sense of Paris–Harrington and of Ketonen–Solovay.

Introduction

The starting point for this paper was a striking result of Mayr and Meyer, on the containment problems for the reachability sets of finite reachable Petri nets and vector addition systems (cf. [7]). On the one hand, Karp and Miller provided an algorithm for deciding whether two nets or two vector addition systems were each finite reachable and, if so, whether the reachability set of the first net contained that of the second. The work of Mayr and Meyer showed that this decision problem did not admit of a primitive recursive decision procecure.

What we do here is to analyze the complexity of the Karp–Miller procedure and show that it is *primitive recursive in the Ackermann function.* To that end, we prove a Main Lemma which serves to bound the size of the Karp–Miller primary coverability tree of a Petri net in terms of 'n-relatively large' finite sets. This notion is an extension of the notion of 'relatively large' set of Paris and Harrington [10] or 'dense set' of Paris [8]. It is directly related to the indicator for semi-regular initial segments of models of arithmetic of the Kirby–Paris paper [5] and can also be described in terms of the Ketonen–Solovay notion of α-large finite set. The point is that 'n-relatively large' sets replace the infinite sets that come up in the usual proof of the finiteness of the coverability tree (cf. [2] or [11]).

This paper is divided into four sections. Section 1 introduces 'n-relatively-large' sets and contains the Main Lemma. Section 2 gives applications of the Main Lemma to Petri nets including the upper-bound on the complexity of the Karp–Miller procedure and a result of Hack announced in [1], to the effect that functions weakly

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computable by Petri nets are primitive recursive. Section 3 applies the Main Lemma to obtain similar results for a class of generalized Petri nets. Section 4 finally treats the relationship of 'n-relatively large' sets with semi-regular cuts and with α-large finite sets.

1. A combinatorial lemma

For $V=(V_1,\ldots,V_k)$ a vector in N^k we set $|V|=\max(V_1,\ldots,V_k)$. If S is a sequence of such vectors, the length of S is denoted $\mathrm{lh}(S)$ and the terms of the sequence by $S(i)$, $i=0,1,\ldots,\mathrm{lh}(S)-1$. A sequence S of vectors in N^k is said to be *almost strictly decreasing* if for $s<t<\mathrm{lh}(S)$ we have $S(s)_p>S(t)_p$ for some coordinate p, p depending on s and t. If a, b are natural numbers, we write $[a,b]$ to denote the set $\{a,a+1,\ldots,b\}$.

The Main Lemma below will yield a bound for the lengths of almost strictly decreasing sequences in terms of the following notion of 'large' finite set.

The interval $[a,b]$ is *0-relatively large* if $b+1\geq 2a$; the interval $[a,b]$ is *$(n+1)$-relatively large* if for all non-decreasing sequences $a=x_0\leq x_1\leq\cdots\leq x_s=b$ with $s\leq a$, there exists a $i_0\leq s-1$ such that $[x_{i_0},x_{i_0}+1]$ is n-relatively large.

The notion of 0-relatively large set coincides with the Paris–Harrington notion of relatively large set [10]. The inductively defined notion of n-relatively large set is directly related to the Kirby–Paris indicator for semi-regular cuts in models of arithmetic [5] (for further comments, see Section 4).

To achieve desired generality, we extend the notion of n-relatively large set. From now on we use the following convention:

Let $f: N\to N$ be a non-decreasing total function which satisfies $f(x)\geq x$.

Let $a\leq b$ be natural numbers. The interval $[a,b]$ is *$(0-f)$-relatively large* if $f(a)+a\leq b+1$; the interval $[a,b]$ is *$(n+1-f)$-relatively large* if for all non-decreasing sequences $a=x_0\leq x_1\leq\cdots\leq x_s=b$ with $s\leq a$ there exists a $i_0\leq s-1$ such that $[x_{i_0},x_{i_0+1}]$ is $(n-f)$-relatively large.

The existence of n-relatively large sets $[a,b]$ is easily established. First we define a hierarchy of primitive recursive functions following Ketonen and Solovay [4]:

$$F_1(x)=2x+1,\qquad F_{n+1}(x)=F_n^{(x+1)}(x),$$

where, in general, $h^0(x)=x$, $h^{m+1}(x)=h(h^m(x))$.

And, for f as above we define functions $F_n^f(x)$ as follows:

$$F_1^f(x)=x+f(x)+1,\qquad F_{n+1}^f(x)=(F_n^f)^{(x+1)}(x).$$

We have by the 'pigeon-hole principle' and induction on n, the following lemma.

Lemma 1.1. (i) *The functions $F_n(x)$ are primitive recursive and the interval $[a,F_{n+1}(a)]$ is an n-relatively large set.*

(ii) *The functions $F_n^f(x)$ are primitive recursive in f and the interval $[a, F_{n+1}^f(a)]$ is an $(n-f)$-relatively large set.*

We define

$$G_n(x) = \mu y([x, y] \text{ is } n\text{-relatively large}),$$

$$G_n^f(x) = \mu y([x, y] \text{ is } (n-f)\text{-relatively large}).$$

Since $G_n(x) \leq F_{n+1}(x)$ and $G_n^f(x) \leq F_{n+1}^f(x)$, we have the following lemma.

Lemma 1.2. (i) *The functions G_n are primitive recursive.*
(ii) *The functions G_n^f are primitive recursive in f.*

In Section 4 we further comment on the rate of growth of the functions G_n; it turns out that every primitive recursive function is majorized by some G_n.

Lemma 1.3. (i) *If $a \leq a' < b' \leq b$ and $[a', b']$ is $(n-f)$-relatively large, so is $[a, b]$.*
(ii) *If $[a, b]$ is $(n+1-f)$-relatively large, then $[a, b]$ is $(n-f)$-relatively large.*

Lemma 1.4. *For $a \geq 4$, if $[a, b]$ is 1-relatively large, then $b \geq 2^a$.*

Proof. Define $x_0 = a$, $x_{i+1} = 2x_i - 2$ for $i \leq a-2$ and $x_{a-1} = b$. No interval $[x_i, x_{i+1}]$ with $i \leq a-3$ is 0-relatively large, and so $[x_{a-2}, b]$ must be 0-relatively large. Thus $b \geq 2x_{a-2} - 1$; but since $a \geq 4$, we have $b \geq 2^a$. □

Define, for $n \geq 1$, $b^{[f,n]}$ as the least a such that $[a, b]$ is *not* $(n-f)$-relatively large. Define C_i^k to be $k!/i!(k-i)!$.

Main Lemma. *Let S be an almost strictly decreasing sequence of vectors in N^k, $k \geq 2$. Let $f: N \to N$ be such that $f(t) \geq S(t')$ for all $t' < t < \text{lh}(S)$ and let $C = f(0)^2 \cdot C_0^k \cdot C_1^k \cdot \ldots \cdot C_{k-1}^k \cdot k!$. Then if $a > C$ and $[a, b]$ is $(k-f)$-relatively large, we have $\text{lh}(S) < b - b^{[f,k-1][f,k-2]\ldots[f,1]}$; thus $\text{lh}(S) < G_k^f(C+1)$.*

Proof. The case $f(0) = 0$ is immediate as is the case where $f(0) = 1$ and $k = 2$. So we assume $f(0) > 0$ and $C \geq 16$. To begin with we pose $a_0 = a$, $b_0 = b$, $L_0 = \{0\}$; we inductively construct a decreasing sequence of intervals

$$[a_0, b_0] \supset [a_1, b_1] \supset \cdots \supset [a_{k-1}, b_{k-1}]$$

and an increasing sequence of sets

$$L_0 \subset L_1 \subset \cdots \subset L_{k-1}$$

such that, for $1 \leq i \leq k-1$,

(a) $[a_i, b_i]$ is $((k-i-f)$-relatively large,

(b) $\quad (a_i, b_i) \cap (L_i \cup f''L_i \cup (f^2)''L_i \cup \{a_{i-1}^2, Ca_{i-1}^2, b^{|f,k-1|\dots|f,k-i|}\}) = \emptyset$

(where $f^2(x) = (f(x))^2$),

(c) $\quad L_i \subset [0, \sqrt{a_1}]$ and $|L_i| < a_{i-1}/\mu\sqrt{C}\,c$,

(d) $\quad f''L_i \subset [0, \sqrt{a_i}]$,

(e) $\quad b_i < b_{i-1}^{|f,k-i|}$,

(f) $\quad a_{i-1}^2 \leq a_i$ and $Ca_{i-1}^2 \leq a_i$,

(g) for $t \in (a_i, b_i)$ there are $t' \in L_i$, $X \subset \{1, \dots, k\}$, $|X| = i$ such that $S(t)_p = S(t')_p$ for all $p \in X$.

Let $0 \leq i < k-1$ and suppose $L_0, \dots, L_i$, $a_0, \dots, a_i$, $b_0, \dots, b_i$ have been constructed.

For each $t \in L_i$, for each $p \in \{1, \dots, k\}$, for each i-element subset X of $\{1, \dots, p-1, p+1, \dots, k\}$ and for each $x < S(t)$, let $\lambda(x, t, p, X)$ be the least $\bar{t} < b_i$ such that

$$S(\bar{t})_p = x, \qquad S(\bar{t})_q = S(t)_q \quad \text{for } q \in X.$$

If no such $\bar{t}$ exists, $\lambda(x, t, p, X)$ is set equal to 0. Set $\bar{L}_{i+1}$ to be the set of all $\lambda(x, t, p, X)$.

If $i = 0$, by inspection $|\bar{L}_{i+1}| < a_0/\sqrt{C}$; if $i > 0$, by the induction hypothesis, $|\bar{L}_{i+1}| < |L_i| \cdot C_i^k \cdot (k-i) \cdot \sqrt{a_i}$ and $|L_i| < a_{i-1}/\sqrt{C}$; and so $|\bar{L}_{i+1}| < a_{i-1} \cdot (C_i^k/\sqrt{C}) \cdot (k-i) \cdot \sqrt{a_i}$; again by the induction hypothesis, $a_{i-1}^2 \cdot C \leq a_i$ and so $|\bar{L}_{i+1}| < a_i/\sqrt{C}$. Furthermore, since $C \geq 16$ and $a_0 > C$ we have $a_i > 3|\bar{L}_{i+1}| + 4$. So let $[a_{i+1}, b_{i+1}] \subset [a_i, b_i]$ be such that

$$(a_{i+1}, b_{i+1}) \cap (\bar{L}_{i+1} \cup f''L_{i+1} \cup (f^2)''L_{i+1} \cup \{a_i^2, Ca_i^2, b_i^{|f,k-(i+1)|}\}) = \emptyset,$$

$$[a_{i+1}, b_{i+1}] \text{ is } (k-(i+1)-f)\text{-relatively large},$$

$$f(a_{i+1}) < b_{i+1}.$$

Then set $L_{i+1} = \bar{L}_{i+1} \cap [0, a_{i+1}]$. Note that $L_{i+1} \supseteq L_i$.

Conditions (a), (b) and (c) are clearly verified. To check (d), note that $f(a_{i+1}) < b_{i+1}$ since $[a_{i+1}, b_{i+1}]$ is $(1-f)$-relatively large. As f is non-decreasing, for $t \in L_{i+1}$ we have $f(t) < a_{i+1}$ for otherwise we would have $b_i \leq f(t) \leq f(a_{i+1})$. Therefore, for $t \in L_{i+1}$ we also have $f(t)^2 \leq a_{i+1}$ since $b_{i+1} - a_{i+1} > a_{i+1}^2 \geq f(t)^2$ by Lemma 1.4 and the fact that $[a_{i+1}, b_{i+1}]$ is 1-relatively large. Similarly, for (e), $a_{i+1} < a_i^2$ would imply $b_{i+1} < a_{i+1}^2$, again a contradiction, and since $a_i^2 \leq a_{i+1}$, $Ca_i^2 > a_{i+1}$ would imply $b_{i+1} \leq Ca_i^2 < Ca_{i+1} < a_{i+1}^2$, which is impossible. As for (f), $b_{i+1} > b_i^{|f,k-(i+1)|}$ would imply $a_{i+1} > b_i^{|f,k-(i+1)|}$, contradicting the fact that $[a_{i+1}, b_{i+1}]$ is a $((k-(i+1))-f)$-relatively large subset of $[a_i, b_i]$. To verify (g), suppose $a_{i+1} < t < b_{i+1}$; let $t' \in L_i$ and let X be an i-element subset of $\{1, \dots, k\}$ such that $S(t)_q = S(t')_q$ for all $q \in X$ (t' and X exist vacuously if $i = 0$, by the induction hypothesis if $i > 0$). Since S is almost strictly decreasing, for some $p \notin X$ and some $x < S(t')_p$, we have $S(t)_p = x$. We claim that $\lambda(x, t', p, X) \leq a_{i+1}$ and so $\lambda(x, t, p, X) \in L_{i+1}$: for otherwise $b_{i+1} \leq \lambda(x, t', p, X)$ which contradicts the choice of $\lambda(x, t', p, X)$.

With the construction completed, we have $[a_{k-1}, b_{k-1}]$ which is $(1-f)$-relatively large and $L_{k-1} \subset [0, a_{k-1}]$ with properties (a)–(e) satisfied. We now define L_k in strict analogy with the definitions of L_i, $i<k$. For $t \in L_{k-1}$ and for $p \in \{1, \ldots, k\}$, for $X = \{1, \ldots, p-1, p+1, \ldots, k\}$ and for $x < S(t)_p$, let $\lambda(x, t, p, X)$ be the least $\bar{t} < b_{k-1}$ such that $S(\bar{t})_p = x$ and $S(\bar{t})_q = S(t)_q$ for $q \neq p$. Hence

$$|\bar{L}_k| < |L_{k-1}| \cdot C^k_{k-1} \cdot \sqrt{a_{k-1}} < a_{k-1}.$$

But since $b_{k-1} - a_{k-1} > a_{k-1}$, there is a $t \in (a_{k-1}, b_{k-1})$ such that $t \notin L_k$; but then there is a $\bar{t} < t$ such that $S(\bar{t}) = S(t)$ which contradicts the fact that S is almost strictly decreasing. □

The following is a corollary to the *proof* of the Main Lemma.

Main Corollary. *Let S be an almost strictly decreasing sequence of vectors in N^k, $k \geq 2$. Suppose $f: N \to N$ is such that $f(t) \geq |S(t')|$ for all $t' < t < \mathrm{lh}(S)$ and suppose that $f(x) < 2^x$ for $x \geq c$. Let $C = \max(c, f(0)^2 \cdot C^k_0 \ldots C^k_{k-1} \cdot k!)$. Then if $a > C$ and if $[a, b]$ is k-relatively large, we have $\mathrm{lh}(S) < b$; thus $\mathrm{lh}(S) < G_k(a)$.*

Proof. We can suppose $f(0) > 1$ and $a > 4$. By Lemma 1.4, if $a < u$ and $[u, v]$ is 1-relatively large, we have $f(u) < v$. In the proof of the Main Lemma we use the fact that $[a_i, b_i]$ being $((k-i)-f)$-relatively large implies $b_i > f(a_i)$ only for $i = 0, \ldots, k-1$. With our current hypotheses, this condition is verified if $[a_i, b_i]$ is $(k-i)$-relatively large, $i \leq k-1$. The final partition of $[a_{k-1}, b_{k-1}]$ is done into fewer than a_{k-1} pieces and we only need at this point in the argument to have $b_{k-1} - a_{k-1} > a_{k-1}$, which is again verified if $[a_{k-1}, b_{k-1}]$ is simply 1-relatively large. □

2. Main result

Theorem 2.1. *Let N be a k-place Petri net with initial marking M_0. Let K be a constant such that for all markings M and all transitions t we have $|t \upharpoonright M| \leq |M| + K$. Then the height of the Karp–Miller primary coverability tree for N is bounded by $G_k(\max(2^{k^2} \cdot k! m_0, K + 2\sqrt{m_0}))$, where $m_0 = |M_0|$.*

Proof. The case $k = 1$ is immediate so suppose $k \geq 2$. If M is derived from M_0 by the firing of x-transitions we have $|M| \leq |M_0| + Kx$. Let S be a branch of the Karp–Miller primary coverability tree up to but not including the top element. Then S is an almost decreasing sequence of vectors in N^k. Moreover, $S(x) \leq |M_0| + Kx$. Setting $f(x) = |M_0| + Kx$ and $C = f(0)^2 \cdot C^k_1 \cdot \ldots \cdot C^k_{k-1} \cdot k!$ we apply the Main Corollary to obtain

$$\mathrm{lh}(S) < G_k(\max\{C+1, K + 2\sqrt{m_0}\}).$$

Since $C + 1 < m_0 \cdot 2^{k^2} \cdot k!$, the result follows. □

Corollary 2.2. *Suppose $g: N \to N$ is weakly computable by a k-place Petri net and suppose K is such that for all transitions t and all markings M we have $|t \upharpoonright M| < K + |M|$. Let M_0 be the initial marking and let q be a place such that $g(x)$ is obtained as the maximum length of all firing sequences when $(M_0)_q$ is set equal to x; set $m_0(x) = \max\{x, |M_0|\}$. Then we have $g(x_0) < m_0(x_0) + K \cdot G_k(\max\{2^{k^2} \cdot k! \cdot m_0(x_0), K + 2\sqrt{m_0(x_0)}\})$.*

Proof. For each argument x_0, we have

$$g(x_0) \leq m_0(x_0) + K \cdot h,$$

where h is the height of the coverability tree of the net obtained when $(M_0)_q = x_0$. □

The following result is one which is announced in [2].

Corollary 2.3. *Weakly computable functions are primitive recursive.*

Proof. The algorithm to compute such a function is to generate the Karp–Miller primary coverability tree, to list all reachable markings and to find the greatest value reached in the designated place. By virtue of Theorem 2.1, the lengths of the various searches required are majorized by primitive recursive bounds. □

For the same reasons, we have the following corollary.

Corollary 2.4. *For each $k \geq 1$, the Karp–Miller procedure for the finite containment problem for pairs of finite reachable k-place Petri nets is primitive recursive.*

And passing to the limit, we have the following corollary.

Corollary 2.5. *The Karp–Miller procedure for the finite containment problem for pairs of finite reachable Petri nets is primitive recursive in the Ackermann function.*

Proof. We use a metamathematical argument. Let $I\Sigma_1$ be the fragment of Peano arithmetic obtained by restricting the induction scheme to Σ_1-formulas—i.e., to formulas of the form $\exists x B(x, x)$ where $B(x, x)$ has no quantifiers except possibly bounded quantifiers. The probably recursive functions of $I\Sigma_1$ are precisely the primitive recursive functions (cf. [5]). For a given pair N_1, N_2 of Petri nets of k-places, one can prove in $I\Sigma_1$ that the Karp–Miller procedure converges:

$I\Sigma_1 \vdash \exists c$($c$ is the Gödel number of a computation of the Karp–Miller algorithm to decide whether $\bar{N}_1$ and $\bar{N}_2$ are finite reachable and, if so, whether the reachability set of $\bar{N}_1$ contains that of $\bar{N}_2$.)

Moreover, there is a primitive recursive map which associates this proof in $I\Sigma_1$ with the pair N_1, N_2. Therefore, the decision procedure in question can be shown to be total if we know that all Σ_1-sentences provable in $I\Sigma_1$ are, in fact, true. In other words, the decision procedure is provably total in the theory $I\Sigma_1 +$ 1-Consistency($I\Sigma_1$). The result follows now from the *claim* that all functions provably total in $I\Sigma_1 +$ 1-Consistency($I\Sigma_1$) are primitive recursive in the Ackermann function. To verify this claim we proceed as follows: We define functions $B_n(x)$ by

$$B_0(x) = A_x(x),$$

where $A_m(n)$ is the Ackermann function,

$$B_{m+1}(x) = B_m^{(x+2)}(x).$$

Define $Y(a, b) = \mu c(B_c(a) \geq b)$. By adapting arguments of [5] it is seen that Y is an indicator for semi-regular initial segments of models of Peano arithmetic which are closed under Ackermann's function. By [9, Corollary 40] such initial segments are models of $I\Sigma_1 +$ 1-Consistency($I\Sigma_1$); conversely, by indicator theory, the class of initial segments which are models of $I\Sigma_1 +$ 1-Consistency($I\Sigma_1$) is symbiotic with that of semi-regular cuts (and even regular cuts) which are models of $I\Sigma_1 +$ 1-Consistency($I\Sigma_1$). Therefore, the Karp–Miller procedure is total in all such initial segments and the function which associates with a pair of nets the Gödel number c of the computation of the Karp–Miller algorithm for the pair is total in all such initial segments. By [9, Theorem 0] this function is bounded by some $B_n(x)$ and so the decision procedure itself is primitive recursive in $B_n(x)$ and thus in the Ackermann function. □

To conclude this section, let us note, as R. Solovay has remarked, that the Main Lemma can be used to bound the complexity of the Karp–Miller procedure more explicitly in terms of the Ackermann function which is (approximately) the map $x \mapsto F_x(x)$; this point and other applications of the techniques of this paper will appear in forthcoming work.

3. Generalizations

In this section we consider a generalized notion of Petri net to which we apply the Main Lemma to obtain a Karp–Miller type result.

A *generalized Petri net* is given by a set $P = \{p_1, \ldots, p_k\}$ of places, an initial marking M_0 and a set $T = \{t^1, \ldots, t^s\}$ of transitions which are maps from N^k to N^k and an entry function $E : T \times P \to N$; we require that each transition t be of the form $t(x_1, \ldots, x_k) = (t_1(x_1), \ldots, t_k(x_k))$ where each t_i is a non-decreasing function from N to N. We say that t can be *fired* at a marking M if $M(p) \geq E(t, p)$ for all places p; we then set $t \upharpoonright M = M'$ where $M'_p = t_p(M_p - E(t, p))$.

Remark. The usual Petri net is the case where the transitions t are of the form $t(x_1, \ldots, x_k) = (x_1 + C^t_1, \ldots, x_k + C^t_k)$.

With a generalized Petri net P we associate a *growth function* $\Gamma = \Gamma_p$ where

$$\Gamma(0) = |M_0|,$$

$$\Gamma(n+1) = \max_{t=(t_1,\ldots,t_k)\in T, 1\leq i\leq k} |t_i(\Gamma(n))|.$$

Thus for all firing sequences $M_0 \to M_1 \to \cdots \to M_s$ we have $|M_s| \leq \Gamma(s)$.

We can associate with a generalized Petri net a Karp–Miller primary coverability tree just as in the standard case. We then have as a direct consequence of the Main Lemma the following theorem.

Theorem 3.1. *Let P be a generalized Petri net with initial marking M_0 and with growth function Γ. The height of the Karp–Miller primary coverability tree of P is bounded by $G_k^\Gamma(m_0 \cdot k \cdot 2^k \cdot k!)$ where $m_0 = |M_0|$.*

The Karp–Miller procedure for testing the finiteness of the reachability set of a Petri net extends mutatis mutandis to generalized nets in our sense. There is also the obvious extension of the notion of weakly computable function. We then have the following result.

Corollary 3.2. *If $f : N \to N$ is weakly computable by means of a generalized k-place Petri net with transitions $t^1 = (t_1^1, \ldots, t_k^1), \ldots, t^s = (t_1^s, \ldots, t_k^s)$, then f is primitive recursive in the functions $t_1^1, t_2^1, \ldots, t_k^s$.*

Proof. The growth function Γ of the net is primitive recursive in $t_1^1, t_2^1, \ldots, t_k^s$. The result then follows from the proof of the Main Lemma and the Karp–Miller algorithm. □

4. Large finite sets

In [4], Ketonen and Solovay introduce the notion of a 'large' set which involves ordinals less than ε_0; ε_0 has the property that $\alpha < \varepsilon_0 \Rightarrow \omega^\alpha < \varepsilon_0$ and it is the least such ordinal >0. With each non-zero limit ordinal $\lambda < \varepsilon_0$ is associated a monotone increasing sequence $\{\lambda\}(n)$ which has λ as its limit. Since $\lambda < \varepsilon_0$, there are two cases:

Case 1. $\lambda = \omega^{\alpha+1}(\beta+1)$. Then $\{\lambda\}(n) = \omega^{\alpha+1} \cdot \beta + \omega^\alpha \cdot n$.

Case 2. $\lambda = \omega^\gamma \cdot (\beta+1)$ and $\gamma < \lambda$ is a limit ordinal. Then $\{\lambda\}(n) = \omega^\gamma \cdot \beta + \omega^{\{\gamma\}(n)}$.

From Case 1 we have $\{\omega\}(n) = n$. We also set $\{\alpha+1\}(n) = \alpha$ and $\{0\}(n) = 0$. Let $X = \{x_0, \ldots, x_l\}$ be a finite set of natural numbers in increasing order. With $\alpha < \varepsilon_0$, we associate a sequence of ordinals $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_l$ by setting

$$\alpha_0 = \{\alpha\}(x_0), \qquad \alpha_{i+1} = \{\alpha_i\}(x_{i+1}).$$

Then X is said to be *α-large* if $\alpha_l = 0$.

Ketonen and Solovay define functions H_α, $\alpha < \varepsilon_0$, by

$$H_\alpha(x) = \mu y([x, y] \text{ is } \alpha\text{-large}),$$

and they show [4, Theorem 4.8] that

$$H_{\omega^k}(n+1) \geq F_k(n), \qquad F_k(n+1) \geq H_{\omega^k}(n),$$

where the F_k's are the functions introduced in Section 1. We then have the following.

Proposition 4.1. *If $[n+1, m]$ is ω^{k+1}-large, then $[n, m]$ is k-relatively large. Hence, in Theorem 2.1, we can replace the bound in terms of G_k by $H_{\omega^{k+1}}(\max(2^{k^2} \cdot k! \cdot m_0, K + 2\sqrt{m_0}) + 1)$.*

Following [6] we can extend the notion of 'α-large' to that of 'α-f-large': Given $X = \{x_0, \ldots, x_t\}$, and $\alpha < \varepsilon_0$ define a sequence $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_t$ by

$$\alpha_0 = \{\alpha\}(f(x_0)), \qquad \alpha_{i+1} = \{\alpha_i\}(f(x_i)).$$

The set X is said to be *$(\alpha - f)$-large* if $\alpha_t = 0$. To describe $(n-f)$-relatively large sets in terms of $(\alpha - f)$-large sets, we shall use a partition result.

Lemma 4.2. *For $k \geq 2$, if X is $(\omega^k - f)$-large with $0 < \min X$ and if $m_0 \leq m_1 \leq \cdots \leq m_s$ with $s \leq \min X$ is such that $m_0 \leq \min X$ and $m_s \geq \max X$, then, for some $i < s$, $[m_i, m_{i+1}] \cap X$ is $(\omega^{k-1} - f)$-large.*

For the proof of the above lemma we refer the reader to [9, Lemma 10], where it is proved for the α-large case, i.e., the case where f is the identity function; the extension to the $(\alpha - f)$-large case is straightforward.

Proposition 4.3. *If X is $(\omega^{k+1} - f)$-large and $0 < \min X$, then X is $(k - f)$-relatively large.*

Proof. The proof follows by induction on k. For $k = 0$, the result follows from the fact that X is $(\omega - f)$-large if and only if $|X| > f(\min X)$. For $k > 0$, the result follows by virtue of the previous lemma. □

The above proposition implies the result of Proposition 4.1. For working with intervals, one could also introduce functions H^f_α and establish the relation between H^f_α and F^f_α.

Converses to Propositions 4.1 and 4.3 can be obtained by the methods of Kirby and Paris [5]. By the same proof as in [5, Proposition 1], the function

$$Z(a, b) = \mu n[G_n(a+2) > b]$$

is an indicator for semi-regular initial segments of models of arithmetic. The +2 appears because $G_n(1) = 2$ the way we have defined n-relatively large. Moreover, if we extend the language of arithmetic to include a function symbol for $f: N \to N$, then

$$Z_f(a, b) = \mu n[G^f_n(a+2) > b]$$

is an indicator in elementary extensions of $(N, 0, 1+, x, f)$ for semi-regular initial segments closed under f. Then, by [9, Theorem 0] and its relativization to f, we have the following.

Proposition 4.4. (i) *Every primitive recursive function is majorized by $G_n(x+2)$ for some n.*

(ii) *Every function primitive recursive in f is majorized by $G^f_n(x+2)$ for some n.*

Thus each of the functions $F_n(x)$ and $H_{\omega^n}(x)$ is also majorized by some $G_{n'}(x+2)$. One can see that the G_n hierarchy grows somewhat more slowly than the F_{n+1} hierarchy but n' as a function of n remains to be explicitly determined.

Finally we have the following.

Proposition 4.5. (i) *There is a function $\psi: N \to N$ such that if X is $\psi(n)$-relatively large, then X is ω^n-large.*

(ii) *There is a function $\psi_f: N \to N$ such that if X is $(\psi_f(n) - f)$-relatively large, then X is $(\omega^n - f)$-large.*

Proof. Part (ii) clearly includes part (i). The proof is by a compactness argument and the indicator methods of [5]. If the integer $\psi_f(n)$ did not exist, there would be a countable non-standard model $M = (M, 0, 1, +, x, f)$ which is an elementary extension of $(N, 0, 1, +, x, f)$ and a set X coded in M, finite in the sense of M such that

$$M \models X \text{ is not } (\omega^n - f)\text{-relatively large},$$

while, for some infinite $a \in M$,

$$M \models X \text{ is } (a - f)\text{-relatively large}.$$

But then by the argument of [5, Proposition 1] there is an initial segment $I < M$ closed under f such that I is semi-regular and $X \cap I$ is unbounded in I. Now using the argument of [4, Theorem 4.8] one verifies that

$$M \models X \text{ is } (\omega^t - f)\text{-large}$$

for all standard t and thus for $t = n$ which is a contradiction. $\square$

Finally, let us remark that our original proof of the fact that, for each dimension k, the Karp–Miller procedure for the finite containment problem is primitive recursive used a 'models-of-arithmetic' argument and the conservation results of [9]; what has to be shown is that, for fixed standard k, regular cuts satisfy the proposition that the Karp-Miller procedure for k-place Petri nets converges.

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