SIAM J. COMPUT.
Vol. 34, No. 5, pp. 1025–1106

THE BISIMULATION PROBLEM FOR EQUATIONAL GRAPHS OF FINITE OUT-DEGREE*

GÉRAUD SÉNIZERGUES†

Abstract. The *bisimulation* problem for equational graphs of finite out-degree is shown to be decidable. We reduce this problem to the η-bisimulation problem for deterministic rational (vectors of) boolean series on the alphabet of a deterministic pushdown automaton $\mathcal{M}$. We then exhibit a *complete formal system* for deducing equivalent pairs of such vectors.

Key words. bisimulation, equational graphs, deterministic pushdown automata, rational languages, finite dimensional vector spaces, matrix semigroups, complete formal systems

AMS subject classifications. Primary, 68Q70; Secondary, 03D05, 05C75, 20M35

DOI. 10.1137/S0097539700377256

1. Introduction.

1.1. Motivations.

Processes. In the context of concurrency theory, several notions of "behavior of a process" and "behavioral equivalence between processes" have been proposed. Among them, the notion of *bisimulation* equivalence seems to play a prominent role (see [26]). The question of whether this equivalence is *decidable* or not for various classes of infinite processes has been the subject of many works in the last fifteen years (see, for example, [1, 5, 18, 7, 14, 17, 8, 6, 45, 19, 37, 44, 22]).

The aim of this work is to show decidability of the bisimulation equivalence for the class of all processes defined by pushdown automata (pda) whose ϵ-transitions are deterministic and decreasing (of course, we assume that ϵ-transitions are *not* visible, which implies that the graphs of the processes considered here might have infinite in-degree). This problem was raised in [6] (see Problem 6.2 of this reference) and is a significant subcase of the problem raised in [45] (as the bisimulation problem for processes "of type -1").

Infinite graphs. A wide class of graphs enjoying interesting decidability properties has been defined in [11, 2, 3] (see [12] for a survey). In particular it is known that the problem

$$\textit{are } \Gamma, \Gamma' \textit{ isomorphic?}$$

is decidable for pairs Γ, Γ' of equational graphs. It seems quite natural to investigate whether the problem

$$\textit{are } \Gamma, \Gamma' \textit{ bisimilar?}$$

is decidable for pairs Γ, Γ' of equational graphs. We show here that this problem is decidable for equational graphs of finite out-degree.

*Received by the editors August 29, 2000; accepted for publication (in revised form) August 18, 2004; published electronically June 3, 2005. This work has been fully supported by the CNRS, at Bordeaux, in 1996–1998 and by the Humboldt Foundation, at Stuttgart, in 2004.
http://www.siam.org/journals/sicomp/34-5/37725.html
†LaBRI, Université de Bordeaux I, 351, Cours de la Libération 33405 Talence, France (ges@labri.fr).

Formal languages. Another classical equivalence relation between processes is the notion of *language* equivalence. The decidability of language equivalence for *deterministic* pushdown automata (dpda) has been established in [34, 40] (see also [36, 35] for shorter expositions of this result). It was first noticed in [1] that, in the case of deterministic processes, language equivalence and bisimulation equivalence are identical. Moreover dpda can always be normalized (with preservation of the language) in such a way that ϵ-transitions are all decreasing. Hence the main result of this work is a generalization of the decidability of the equivalence problem for dpda.

Mathematical generality. More precisely, the present work *extends the notions* developed in [34] so as to obtain a more general result. As a by-product of this extension, we obtain a deduction system which, in the deterministic case, seems *simpler* than the one presented in [34] (see system $\mathcal{B}_3$ in section 10).

The present work can also be seen as a common generalization of three different results: the results of [45, 19] establishing decidability of the bisimulation equivalence in two nondeterministic subclasses of the class considered here, and the result of [34] dealing only with deterministic pda (or processes).

Logics. Our solution consists in constructing a *complete* formal system, in the general sense taken by this word in mathematical logics; i.e., it consists of a set of well-formed assertions, a subset of basic assertions, the axioms, and a set of deduction rules allowing one to derive new assertions from assertions which are already generated. The well-formed assertions we are considering are pairs (S, T) of rational boolean series over the nonterminal alphabet V of some strict-deterministic grammar $G = \langle X, V, P\rangle$. Such an assertion is true when the two series S, T are bisimilar.

Several simple formal systems generating all the identities between boolean rational expressions have been the subject of several works (see [32, 4, 21]); the case of bisimilar rational expressions has been addressed in [25, 20].

A tableau proof-system generating all the bisimilar pairs of words with respect to a given context-free grammar in Greibach normal form was also given in [18].

Our complete formal systems can be seen as participating in this general research stream ([39] provides an overview of this subject in the context of equivalence problems for pda).

1.2. Results.

The main decidability result of this work is the following.

THEOREM 10.7. *The bisimulation problem for rooted equational* 1*-graphs of finite out-degree is decidable.*

The main structural result obtained here is as follows.

THEOREM 10.14. $\mathcal{B}_3$ *is a complete deduction system.*

Here, $\mathcal{B}_3$ is a formal system whose elementary rules just express the basic algebraic properties of bisimulation: the fact that it is an equivalence relation, that it is compatible with right and left (matricial) product, that Arden's lemma remains true modulo bisimulation and, at last, its link with one-step derivation (rule (R34)). Completeness means here that *all* pairs of bisimilar rational "deterministic" boolean series are generated by this formal system.

1.3. Overview and roadmap.

Overview: Large scale. Let us describe the principal flow of ideas which underpins our proof.

Step 1: *Reduction.* We reduce the bisimulation problem for vertices v, v' of an equational graph Γ of finite out-degree to an analogous problem, but with more algebraic flavor: given two deterministic rational row-vectors S, S', are they η-bisimilar?

The precise definition of what is a deterministic rational row-vector is given in section 3.1, the notion of η-bisimilarity for vectors is defined in section 3.2, and the reduction itself is stated in Lemma 3.25.

Step 2: *Logical system.* We define a formal system, named $\mathcal{B}_0$, whose assertions are the pairs of deterministic rational row-vectors (S, S'). Such an assertion is considered as true iff the vectors S, S' are η-bisimilar. The system $\mathcal{B}_0$ is defined in section 4.3 and its soundness is immediately proved there.

Step 3: *Strategies for* $\mathcal{B}_0$. We define some suitable notions of *strategies* for a formal system: a strategy is a map sending every partial proof P into a partial proof P' containing P. What we have in mind is to build, step by step, a proof for a given true assertion by iteratively applying a strategy. Section 7 is devoted to the construction of such strategies for the system $\mathcal{B}_0$.

Step 4: *Completeness of* $\mathcal{B}_0$. We proceed to a close analysis of the "proof-trees" produced by the above strategies in order to show that, starting from a true assertion (S, S'), a finite proof-tree is always reached. This succeeds in proving that $\mathcal{B}_0$ is a *sound* and *complete* logical system. Unfortunately, $\mathcal{B}_0$ cannot be easily shown to be *recursively enumerable*, due to one of its rules, namely, rule (R5).

Step 5: *Elimination.* It turns out that this problematic rule (R5) can be *eliminated* from $\mathcal{B}_0$, resulting in a smaller system $\mathcal{B}_1$ which is still sound, complete, and recursively enumerable. This logical result allows us to prove decidability of the η-bisimulation problem for deterministic rational row-vectors, and hence of the bisimulation problem for vertices of an equational graph of finite out-degree (by Step 1).

Step 6: *Simplifications.* This last step is useless for decidability purposes but sheds light on the *structure* of bisimulation equivalence. Successive elimination arguments lead us to a fairly simple logical system, named $\mathcal{B}_3$, which is still sound and complete. The rules of $\mathcal{B}_3$ are just consisting of the principal algebraic properties of vectors-η-bisimulation and a single rule expressing the precise grammar (or process) we are examining (rule (R34)).

Overview and roadmap: Medium scale. Let us describe now, section by section, the main successive technical contributions to the general flow described in the previous overview.[1]

Section 2.

2.1. We recall the notion of graph-bisimulation as classically stated in the literature.

2.2. As well, we recall the notion of pda.

2.3. We characterize the class of graphs under scrutiny, the "equational 1-graphs of finite out-degree," as the computation-graphs of normalized pda (a precise proof is delayed to the appendix).

2.4. We recall the definition of a deterministic context-free grammar.

2.5. We state the basic definitions allowing us to translate the usual notions of left-derivation with respect to (w.r.t.) (resp., language generated by) a context-free grammar G into a more algebraic framework: everything is formulated now within semirings of boolean series over finite sets of undeterminates,

[1] Sections 1 and 11 do not participate in the proof itself and therefore are left out of this overview. The appendix is also neglected here because of its marginality.

endowed with two right-actions $\odot, \bullet$: the first right-action expresses a one-step left-derivation, while the second right-action is just the well-known residual-action.

Section 3.

3.1. We recall the notion of *deterministic boolean series*, which plays a role analogous to that of "configuration" of a dpda. The advantage of this notion is that it allows extensions to vectors and matrices and supports several well-behaved operations.

3.2. We reduce the initial bisimulation problem over graphs to a bisimulation problem over deterministic rational row-vectors (we call this new kind of bisimulation the σ-η-bisimulation). For every pair (S, S') of deterministic series, we introduce a notion of word-η-bisimulation which is a kind of equivalence relation over words that "witnesses" the fact that (S, S') are indeed σ-η-bisimilar. Operations on such word-η-bisimulations are introduced in order to prove, later on, some algebraic properties of σ-η-bisimulation.

3.3. The usual notions of derivation and *stacking* derivation are translated in our framework. Some basic properties are demonstrated.

Section 4.

4.1. We define a very general notion of *deduction system*. Its only nonstandard aspect, as compared to the usual "Hilbert-style systems," is that the proofs can "loop": it may happen that a correct proof contains an assertion A, which is deduced, by means of finitely many steps, from a set of axioms and (though such a feature was not expected) A itself.

4.2. We introduce a general notion of *strategy* w.r.t. a given deduction system: it is just a map sending every partial proof P into a partial proof P' containing P. The desirable properties of such strategies are defined there.

4.3. We define here *the* principal deduction system $\mathcal{B}_0$ that we expect to be able to generate exactly the set of σ-η-bisimilar pairs of vectors (S, S'). We immediately establish that $\mathcal{B}_0$ is sound (Lemma 4.9). Some interesting algebraic corollaries are deduced.

4.4. We isolate a subsystem of $\mathcal{B}_0$, which we name $\mathcal{C}$, that is independent of any graph or automaton: the rules of $\mathcal{C}$ capture the essential algebraic properties relating the σ-η-bisimulation relation with the matricial product. We then state four useful general deductions within the system $\mathcal{C}$.

Section 5.

5.1. A set of row-vectors which is closed under linear combination is named a *d-space*. Such a d-space, endowed with the right-operation $\odot$, is the key algebraic structure we shall use later. We define a natural notion of *linear independence* for families of vectors and show that it enjoys one of the usual properties of linear independence: if a family is dependent, then one of the vectors of the family is a linear combination of the others.

5.2. Applying repeatedly the above property of dependent families, we exhibit a "triangulation process" for systems of equations: given a system $\mathcal{S}$ of n equations $\sum_{j=1}^{d} \alpha_{i,j} S_j \sim \sum_{j=1}^{d} \beta_{i,j} S_j$ (where $\sim$ denotes the σ-η-bisimulation) one can transform $\mathcal{S}$ in such a way that the last equation, which we call the *inverse equation*, combines the coefficients $\alpha_{i,\star}, \beta_{i,\star}$ but does not involve the S_j anymore. The exact relationship between $\mathcal{S}$ and the inverse equation, $\mathrm{INV}(\mathcal{S})$, is studied there.

In fact, the above description is accurate only when η is just the equality relation. In the general case we are lead to introduce the notion of an *oracle*: an oracle is a choice of word-η-bisimulation for every pair of η-bisimilar vectors. The "inverse" equation determined by $\mathcal{S}$ and by an oracle $\mathcal{O}$ is denoted by $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})$.

Section 6. Here are collected all the definitions of "constants" used throughout this article: these are all the integers, depending on a given initial automaton $\mathcal{M}$, an equivalence η, and an initial pair of vectors (S_0^-, S_0^+) which one would like to test for σ-η-bisimilarity.

Section 7. We define here strategies for the particular deduction system $\mathcal{B}_0$.

7.1. We define several substrategies based on both the algebraic properties established in section 4.3 and the triangulation process studied in section 5.2.

7.2. All the substrategies from section 7.1 are synthesized into a global strategy $\hat{\mathcal{S}}_{ABC}$.

Section 8. The proofs built by $\hat{\mathcal{S}}_{ABC}$ are naturally structured as "proof-trees." We examine here a hypothetical infinite branch belonging to some infinite proof built by $\hat{\mathcal{S}}_{ABC}$. In section 8.2 we prove that such an infinite branch must possess an infinite suffix, which is a *B-stacking sequence*: roughly speaking, it begins with a T_B-application and later on, each time some T_B^α is applied, the vector which is used by T_B^α for constructing the new vector (on side α) has a norm which is *large enough*.

We show carefully (Lemmas 8.4 to 8.10) that such a sequence of equations contains a subsequence of equations, on which the triangulation process (section 5) can be applied. Consequently the substrategy T_C could apply to one node of this suffix.

Section 9. From the technical result proved in section 8, we quickly deduce that $\hat{\mathcal{S}}_{ABC}$ cannot build an *infinite* proof-tree from any true assertion. Unfortunately two problems remain to be solved:

- we must show that $\hat{\mathcal{S}}_{ABC}$ really builds a finite proof-tree from every true assertion, i.e., that $\hat{\mathcal{S}}_{ABC}$ is *closed*;
- at this point we do not know whether $\mathcal{B}_0$ is recursively enumerable or not, because of metarule (R5): this rule specifies that the conclusion is a pair of ... σ-η-bisimilar vectors, which is somewhat circular.

Section 10.

10.1. We overcome here the two difficulties quoted above:
  - $\hat{\mathcal{S}}_{ABC}$ is shown to be *closed*;
  - we show that every proof-tree constructed by $\hat{\mathcal{S}}_{ABC}$ is in fact also a proof for the smaller system $\mathcal{B}_1$ obtained from $\mathcal{B}_0$ by removing rule (R5) (proof of Theorem 10.6).

  In other words we show that (R5) can be *eliminated* from $\mathcal{B}_0$, while preserving completeness. Hence $\mathcal{B}_1$ is a deduction system which is sound, complete, and recursively enumerable. The main decidability theorem (Theorem 10.7) follows: The bisimulation problem for rooted equational 1-graphs of finite out-degree is decidable.

10.2, 10.3. We perform successive simplifications of system $\mathcal{B}_1$. We finally obtain a system $\mathcal{B}_3$ whose elementary rules just express the basic algebraic properties of bisimulation: the fact that it is an equivalence relation, that it is compatible with right and left (matricial) products, that Arden's lemma remains true modulo bisimulation and, at last, its link with one-step derivation (rule (R34)).

We provide the reader with a roadmap which might help him in finding his way across the different sections (see Figure 1). The essential steps of the proof are placed in a column corresponding to the main theme to which they belong (Graphs, Algebra,

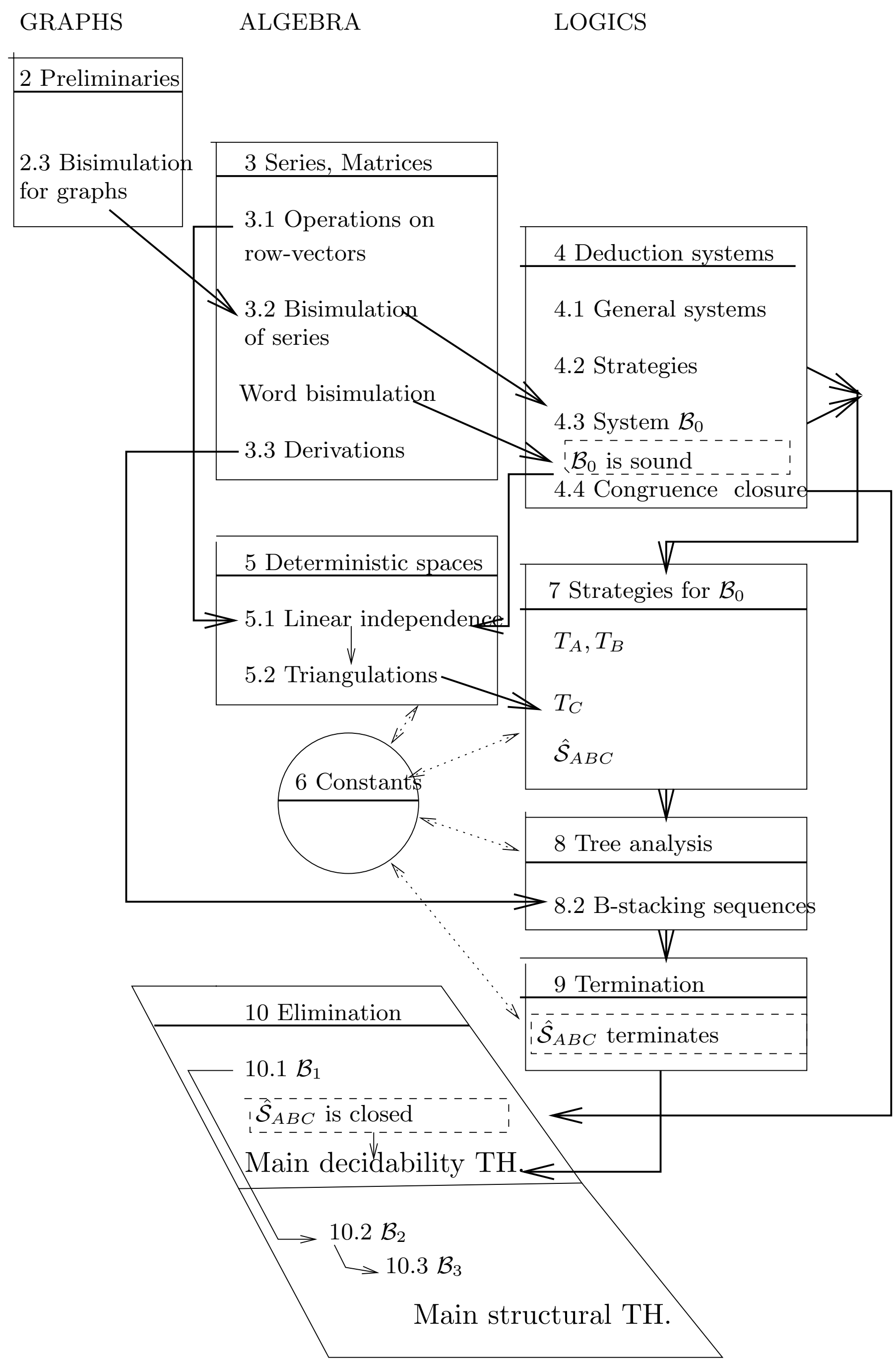

Fig. 1. *Roadmap.*

or Logics) and in a line corresponding to the physical linear ordering of the proof (from beginning to end). Arrows are added showing the main logical dependencies (they define a partial ordering over the steps that the written proof must extend into a total ordering).

Overview: Small scale. The proof exposed here is an updated version of the full proof given in [33] and presented concisely in [37]. This last reference can be used as a small-scale overview of our proof.

2. Preliminaries.

2.1. Graphs. Let X be a finite alphabet. We call *graph over* X any pair $\Gamma = (V_\Gamma, E_\Gamma)$ where V_Γ is a set and E_Γ is a subset of $V_\Gamma \times X \times V_\Gamma$. For every integer $n \in \mathbb{N}$, we call an n-graph every $n+2$-tuple $\Gamma = (V_\Gamma, E_\Gamma, v_1, \ldots, v_n)$ where (V_Γ, E_Γ) is a graph and $(v_1, \ldots, v_n)$ is a sequence of distinguished vertices: they are called the *sources* of Γ.

A 1-graph (V, E, v_1) is said to be *rooted* iff v_1 is a root of (V, E) and $V \neq \{v_1\}$. A 2-graph (V, E, v_1, v_2) is said *birooted* iff v_1 is a root, v_2 is a coroot of (V, E), $v_1 \neq v_2$, and there is no edge going out of v_2 (this last technical condition will be useful for reducing the bisimilarity notion for graphs to an analogous notion on series; see sections 2.1, 2.3, and 3.2).

The *equational* graphs are the least solutions (in a suitable sense) of the systems of (hyperedge) graph-equations (see [12] for precise definitions). Let us mention that the equational graphs of *finite degree* are exactly the *context-free* graphs defined in [27].

Bisimulations.

DEFINITION 2.1. *Let* $\Gamma = (V_\Gamma, E_\Gamma, v_1, \ldots, v_n)$, $\Gamma' = (V_{\Gamma'}, E_{\Gamma'}, v'_1, \ldots, v'_n)$ *be two* n*-graphs over an alphabet* X*. Let* $\mathcal{R}$ *be some binary relation* $\mathcal{R} \subseteq V_\Gamma \times V_{\Gamma'}$. $\mathcal{R}$ *is a* simulation *from* Γ *to* Γ' *iff*

(1) $\mathrm{dom}(\mathcal{R}) = V_\Gamma$,
(2) $\forall i \in [1, n]$, $(v_i, v'_i) \in \mathcal{R}$,
(3) $\forall v \in V_\Gamma$, $w \in V_\Gamma$, $v' \in V_{\Gamma'}$, $x \in X$, *such that* $(v, x, w) \in E_\Gamma$ *and* $v\mathcal{R}v'$,
there exists $w' \in V_{\Gamma'}$ *such that* $(v', x, w') \in E_{\Gamma'}$ *and* $w\mathcal{R}w'$.

$\mathcal{R}$ *is a* bisimulation *from* Γ *to* Γ' *iff* $\mathcal{R}$ *is a simulation from* Γ *to* Γ' *and* $\mathcal{R}^{-1}$ *is a simulation from* Γ' *to* Γ.

This definition corresponds to the standard one [29, 26, 6] in the case where $n = 0$. The n-graphs Γ, Γ' are said to be *bisimilar*, which we denote by $\Gamma \sim \Gamma'$, iff there exists a bisimulation $\mathcal{R}$ from Γ to Γ'.

Let us now extend this definition by means of a relational morphism between free monoids.

DEFINITION 2.2. *Let* X, X' *be two alphabets. A binary relation* $\eta \subseteq X^* \times X'^*$ *is called a strong relational morphism from* X^* *to* X'^* *iff*

(1) η *is a submonoid of* $X^* \times X'^*$,
(2) $\mathrm{dom}(\eta) = X^*$, $\mathrm{im}(\eta) = X'^*$,
(3) η *is generated (as a submonoid) by the subset* $\eta \cap (X \times X')$.

One can easily check that strong relational morphisms are preserved by inversion and composition and that any surjective map $\eta : X \to X'$ induces a strong relational morphism from X^* to X'^*. Let $\Gamma = (V_\Gamma, E_\Gamma, v_1, \ldots, v_n)$ be an n-graph over the alphabet X, and let $\Gamma' = (V_{\Gamma'}, E_{\Gamma'}, v'_1, \ldots, v'_n)$ be an n-graph over the alphabet X'. Let $\eta \subseteq X^* \times X'^*$ be some strong relational morphism, and let $\mathcal{R}$ be some binary relation $\mathcal{R} \subseteq V_\Gamma \times V_{\Gamma'}$.

DEFINITION 2.3. $\mathcal{R}$ *is an* η-simulation *from* Γ *to* Γ' *iff*

(1) $\mathrm{dom}(\mathcal{R}) = V_\Gamma$,
(2) $\forall i \in [1, n]$, $(v_i, v'_i) \in \mathcal{R}$,
(3) $\forall v, w \in V_\Gamma$, $v' \in V_{\Gamma'}$, $x \in X$, *such that* $(v, x, w) \in E_\Gamma$ *and* $v\mathcal{R}v'$,

$$\exists w' \in V_{\Gamma'},\ x' \in \eta(x) \text{ such that } (v', x', w') \in E_{\Gamma'} \text{ and } w\mathcal{R}w'.$$

$\mathcal{R}$ *is an* η-bisimulation *iff* $\mathcal{R}$ *is an* η*-simulation and* $\mathcal{R}^{-1}$ *is an* η^{-1}*-simulation.*

For every $v \in V_\Gamma$, $v' \in V_{\Gamma'}$, we denote by $v \sim v'$ the fact that there exists some η-bisimulation $\mathcal{R}$ from Γ to Γ' such that $(v, v') \in \mathcal{R}$. Throughout this work, the composition of binary relations is denoted by $\circ$ and defined by the following: if $\mathcal{R}_1 \subseteq E \times F$ and $\mathcal{R}_2 \subseteq F \times G$, then

$$\mathcal{R}_1 \circ \mathcal{R}_2 = \{(x, z) \in E \times G \mid \exists y \in F, (x, y) \in \mathcal{R}_1, (y, z) \in \mathcal{R}_2\}. \tag{2.1}$$

FACT 2.4.
(1) *If* $\mathcal{R}$ *is an* η*-bisimulation, then* $\mathcal{R}^{-1}$ *is an* η^{-1}*-bisimulation.*
(2) *If* $\mathcal{R}_1$ *is an* η_1*-bisimulation and* $\mathcal{R}_2$ *is an* η_2*-bisimulation, then* $\mathcal{R}_1 \circ \mathcal{R}_2$ *is an* $\eta_1 \circ \eta_2$*-bisimulation.*
(3) *If for every* $i \in I$*,* $\mathcal{R}_i$ *is an* η*-bisimulation, then* $\bigcup_{i \in I} \mathcal{R}_i$ *is an* η*-bisimulation.*

2.2. Pushdown automata. A *pushdown automaton* (pda) on the alphabet X is a 7-tuple $\mathcal{M} = \langle X, Z, Q, \delta, q_0, z_0, F\rangle$ where Z is the finite stack-alphabet, Q is the finite set of states, $q_0 \in Q$ is the initial state, z_0 is the initial stack-symbol, F is a finite subset of QZ^*, the set of *final* configurations, and δ, the transition function, is a mapping $\delta : QZ \times (X \cup \{\epsilon\}) \to \mathcal{P}_f(QZ^*)$.

Let $q, q' \in Q$, $\omega, \omega' \in Z^*$, $z \in Z$, $f \in X^*$, and $a \in X \cup \{\epsilon\}$; we note that $(qz\omega, af) \longmapsto_{\mathcal{M}} (q'\omega'\omega, f)$ if $q'\omega' \in \delta(qz, a)$. The binary relation $\overset{*}{\longmapsto}_{\mathcal{M}}$ is the reflexive and transitive closure of $\longmapsto_{\mathcal{M}}$.

For every $q\omega, q'\omega' \in QZ^*$ and $f \in X^*$, we note $q\omega \xrightarrow{f}_{\mathcal{M}} q'\omega'$ iff $(q\omega, f) \overset{*}{\longmapsto}_{\mathcal{M}} (q'\omega', \epsilon)$.

$\mathcal{M}$ is said to be *deterministic* iff, for every $z \in Z$, $q \in Q$, $x \in X$,

$$\mathrm{Card}(\delta(qz, \epsilon)) \in \{0, 1\}, \tag{2.2}$$

$$\mathrm{Card}(\delta(qz, \epsilon)) = 1 \Rightarrow \mathrm{Card}(\delta(qz, x)) = 0, \tag{2.3}$$

$$\mathrm{Card}(\delta(qz, \epsilon)) = 0 \Rightarrow \mathrm{Card}(\delta(qz, x)) \leq 1. \tag{2.4}$$

$\mathcal{M}$ is said to be *real-time* iff, for every $q \in Q$, $z \in Z$, $\mathrm{Card}(\delta(qz, \epsilon)) = 0$.

A configuration $q\omega$ of $\mathcal{M}$ is said to be ϵ-bound iff there exists a configuration $q'\omega'$ such that $(q\omega, \epsilon) \longmapsto_{\mathcal{M}} (q'\omega', \epsilon)$; $q\omega$ is said to be ϵ-free iff it is not ϵ-bound.

A pda $\mathcal{M}$ is said to be *normalized* iff it fulfills conditions (2.2), (2.3) above and (2.5), (2.6), (2.7):

$$q_0 z_0 \text{ is } \epsilon\text{-free} \tag{2.5}$$

and for every $q \in Q$, $z \in Z$, $x \in X$,

$$q'\omega' \in \delta(qz, x) \Rightarrow \mid \omega' \mid \leq 2, \tag{2.6}$$

$$q'\omega' \in \delta(qz, \epsilon) \Rightarrow \mid \omega' \mid = 0. \tag{2.7}$$

All the pda considered here are assumed to fulfill condition (2.5). A pda $\mathcal{M}$ is called *birooted* iff it fulfills (2.8) and (2.9):

$$\exists \bar{q} \in Q, F = \{\bar{q}\}, \tag{2.8}$$

$$\forall q \in Q, \omega \in Z^*, f \in X^*, \quad q_0 z_0 \xrightarrow{f}_{\mathcal{M}} q\omega \Rightarrow \exists g \in X^*, q\omega \xrightarrow{g}_{\mathcal{M}} \bar{q}. \tag{2.9}$$

The *language recognized* by $\mathcal{M}$ is

$$\mathrm{L}(\mathcal{M}) = \{w \in X^* \mid \exists c \in F, q_0 z_0 \xrightarrow{w}_{\mathcal{M}} c\}.$$

It is a "folklore" result that, given a dpda $\mathcal{M}$, one can effectively compute another dpda $\mathcal{M}'$ which is normalized and fulfills

$$\mathrm{L}(\mathcal{M}') = \mathrm{L}(\mathcal{M}) - \{\varepsilon\}.$$

2.3. Graphs and pda.

Equational graphs and pda. We call a *transition-graph* of a pda $\mathcal{M}$, denoted $\mathcal{T}(\mathcal{M})$, the 0-graph $\mathcal{T}(\mathcal{M}) = (V_{\mathcal{T}(\mathcal{M})}, E_{\mathcal{T}(\mathcal{M})})$, where $V_{\mathcal{T}(\mathcal{M})} = \{q\omega \mid q \in Q,\ \omega \in Z^*,$ $q\omega$ is ϵ-free$\}$ and

$$E_{\mathcal{T}(\mathcal{A})} = \{(c, x, c') \in V_{\mathcal{T}(\mathcal{M})} \times V_{\mathcal{T}(\mathcal{M})} \mid c \xrightarrow{x}_{\mathcal{M}} c'\}. \tag{2.10}$$

We call a *computation* 1-*graph* of the pda $\mathcal{M}$, denoted $(\mathcal{C}(\mathcal{M}), v_{\mathcal{M}})$, the subgraph of $\mathcal{T}(\mathcal{M})$ induced by the set of vertices which are accessible from the vertex $q_0 z_0$, together with the source $v_{\mathcal{M}} = q_0 z_0$. In the case where $\mathcal{M}$ is birooted, we call a *computation* 2-*graph* of the pda $\mathcal{M}$, denoted $(\mathcal{C}(\mathcal{M}), v_{\mathcal{M}}, \bar{v}_{\mathcal{M}})$, the graph $\mathcal{C}(\mathcal{M})$ defined just above, together with the sources $v_{\mathcal{M}} = q_0 z_0$, $\bar{v}_{\mathcal{M}} = \bar{q}$.

THEOREM 2.5. *Let $\Gamma = (\Gamma_0, v_0)$ be a rooted 1-graph over X. The following conditions are equivalent:*

(1) Γ *is equational and has finite out-degree.*

(2) Γ *is isomorphic to the computation* 1-*graph* $(\mathcal{C}(\mathcal{M}), v_{\mathcal{M}})$ *of some normalized pda $\mathcal{M}$.*

The formal proof of this theorem is quite technical and is omitted here. (See the appendix for a sketch of proof.)

COROLLARY 2.6. *Let $\Gamma = (\Gamma_0, v_0, \bar{v})$ be a birooted 2-graph over X. The following conditions are equivalent:*

(1) Γ *is equational and has finite out-degree.*

(2) Γ *is isomorphic to the computation* 2-*graph* $(\mathcal{C}(\mathcal{M}), v_{\mathcal{M}}, \bar{v}_{\mathcal{M}})$ *of some birooted normalized pda $\mathcal{M}$.*

Bisimulation for nondeterministic (versus deterministic) graphs. In this section, we reduce the classical notion of *bisimulation* for equational graphs to the notion of η-bisimulation for *deterministic* equational graphs, where η has been suitably chosen (see Definition 2.3).

LEMMA 2.7. *Let Γ_1 be some rooted equational* 1-*graph over a finite alphabet Y_1 and let $\#$ be a new letter $\# \notin Y_1$. Then one can construct an equational birooted* 2-*graph Γ over the alphabet $Y = Y_1 \cup \{\#\}$ such that*

(1) $V_{\Gamma_1} \subseteq V_{\Gamma}$,

(2) *for every $v, v' \in V_{\Gamma_1}$, (v, v' are bisimilar in Γ_1) iff (v, v' are bisimilar in Γ),*

(3) Γ_1 *has finite out-degree iff Γ has finite out-degree.*

Sketch of proof. Let us define Γ from Γ_1 by

$$V_{\Gamma} = V_{\Gamma_1} \cup \{\bar{v}\}, \quad E_{\Gamma} = E_{\Gamma_1} \cup \{(w, \#, \bar{v}) \mid w \in V_{\Gamma_1}\}, \quad \Gamma = (\Gamma_1, \bar{v}),$$

where $\bar{v}$ is a new vertex $\bar{v} \notin V_{\Gamma_1}$. One can easily check that Γ is equational iff Γ_1 is equational and that, provided Γ_1 is rooted, Γ is birooted. Points (1) and (3) of the lemma are clear. One can check that the mapping $\mathcal{R} \mapsto \mathcal{R} \cup \{(\bar{v}, \bar{v})\}$ is a bijection from the set of all the bisimulations over Γ_1 (i.e., from Γ_1 to Γ_1) to the set of all the bisimulations over Γ. Hence point (2) is true. $\square$

Let us consider finite alphabets X, Y, a length-preserving homomorphism $\psi : X^* \to Y^*$, and the strong relational morphism $\bar{\psi} = \psi \circ \psi^{-1} \subseteq X^* \times X^*$. An n-graph Γ over X will be called *$\bar{\psi}$-saturated* iff, for every $v \in V_\Gamma$, for every $(x, x') \in \bar{\psi}$,

$$(\exists v_1 \in V_\Gamma, (v, x, v_1) \in E_\Gamma) \Leftrightarrow (\exists v_1' \in V_\Gamma, (v, x', v_1') \in E_\Gamma).$$

LEMMA 2.8. *Let Γ_1 be an equational birooted 2-graph of finite out-degree over an alphabet Y. One can construct a finite alphabet X, a surjective length-preserving homomorphism $\psi : X^* \to Y^*$, and an equational, birooted 2-graph Γ over the alphabet X, such that*

(1) Γ *is deterministic,*

(2) Γ *is $\bar{\psi}$-saturated,*

(3) $V_{\Gamma_1} = V_\Gamma$*,*

(4) $\mathrm{Id} : V_\Gamma \to V_{\Gamma_1}$ *is a ψ-bisimulation from Γ to Γ_1.*

Sketch of proof. By Lemma 2.6, we can suppose that Γ_1 is the computation 2-graph $(\mathcal{C}(\mathcal{M}_1), v_{\mathcal{M}_1}, \bar{v}_{\mathcal{M}_1})$ of some birooted normalized pda $\mathcal{M}_1 = \langle Y, Z, Q, \delta_1, q_0, z_0, \{\bar{q}\}\rangle$. Let us consider the following integers: $\forall q \in Q$, $z \in Z$, $y \in Y$,

$$t_1(qz, y) = \mathrm{Card}(\delta_1(qz, y)), \quad \bar{t_1} = \max\{t_1(qz, y) \mid q \in Q, z \in Z, y \in Y\}.$$

Let $X = Y \times [1, \bar{t_1}]$ and let $\psi : X \to Y$ be the first projection. Let $\rho : QZ \times Y \times \mathbb{N} \to QZ^*$ such that $\mathrm{dom}(\rho) = \bigcup_{q \in Q, z \in Z, y \in Y} \{qz\} \times \{y\} \times [1, t_1(qz, y)]$ and

$$\rho(qz, y, \star) : \{qz\} \times \{y\} \times [1, t_1(qz, y)] \to \delta_1(qz, y)$$

is a bijection (for every triple (q, z, y)). We then define $\mathcal{M} = \langle X, Z, Q, \delta, q_0, z_0, \{\bar{q}\}\rangle$ by the following: for every $q \in Q$, $z \in Z$, $y \in Y$, $i \in [1, \bar{t_1}]$,

$$\delta(qz, \epsilon) = \delta_1(qz, \epsilon) \text{ if } qz \text{ is } \epsilon\text{-bound},$$

$$\delta(qz, (y, i)) = \{q'\omega'\} \text{ if}$$
$$\rho(qz, y, i) = q'\omega' \text{ or } [(1 \leq t_1(qz, y) < i \leq \bar{t_1} \text{ and } \rho(qz, y, 1) = q'\omega')].$$

The 2-graph $\Gamma = (\mathcal{C}(\mathcal{M}), v_\mathcal{M}, \bar{v}_\mathcal{M})$ fulfills the required properties. $\square$

Let us remark that, by point (4) and by composition of η-bisimulations, for every $v, v' \in V_\Gamma$, v, v' are $\bar{\psi}$-bisimilar (w.r.t. Γ) iff v, v' are bisimilar (w.r.t. Γ_1).

2.4. Deterministic context-free grammars. Let $\mathcal{M}$ be some dpda (we suppose here that $\mathcal{M}$ is normalized). The *variable* alphabet $V_\mathcal{M}$ associated to $\mathcal{M}$ is defined as

$$V_\mathcal{M} = \{[p, z, q] \mid p, q \in Q, z \in Z\}.$$

The *context-free* grammar $G_\mathcal{M}$ associated to $\mathcal{M}$ is then

$$G_\mathcal{M} = \langle X, V_\mathcal{M}, P_\mathcal{M}\rangle,$$

where $P_\mathcal{M}$ is the set of all the pairs of one of the following forms:

$$([p, z, q], x[p', z_1, p''][p'', z_2, q]), \tag{2.11}$$

where $p, q, p', p'' \in Q$, $x \in X$, $p'z_1z_2 \in \delta(pz, x)$,

$$([p, z, q], x[p', z', q]), \tag{2.12}$$

where $p, q, p' \in Q$, $x \in X$, $p'z' \in \delta(pz, x)$,

$$([p, z, q], a), \tag{2.13}$$

where $p, q, \in Q$, $a \in X \cup \{\epsilon\}$, $q \in \delta(pz, a)$. $G_{\mathcal{M}}$ is a *strict-deterministic* grammar (see Definition 3.7 below). A general theory of this class of grammars is presented in [15] and used in [16].

2.5. Free monoids acting on semirings.

Semiring $\mathsf{B}\langle\langle\ W\ \rangle\rangle$. Let $(\mathsf{B}, +, \cdot, 0, 1)$, where $\mathsf{B} = \{0, 1\}$ denotes the semiring of "booleans." Let W be some alphabet. By $(\mathsf{B}\langle\langle\ W\ \rangle\rangle, +, \cdot, \emptyset, \epsilon)$ we denote the semiring of *boolean series* over W: the set $\mathsf{B}\langle\langle\ W\ \rangle\rangle$ is defined as B^{W^*}; the sum and product are defined as usual; each word $w \in W^*$ can be identified with the element of B^{W^*} mapping the word w on 1 and every other word $w' \neq w$ on 0; every boolean series $S \in \mathsf{B}\langle\langle\ W\ \rangle\rangle$ can then be written in a unique way as

$$S = \sum_{w \in W^*} S_w \cdot w,$$

where, for every $w \in W^*$, $S_w \in \mathsf{B}$.

The *support* of S is the language

$$\mathrm{supp}(S) = \{w \in W^* \mid S_w \neq 0\}.$$

In the particular case where the semiring of coefficients is B (which is the only case considered in this article) we sometimes identify the series S with its support. A series $S \in \mathsf{B}\langle\langle\ W\ \rangle\rangle$ is called a boolean *polynomial* over W iff its support is *finite*. The set of all boolean polynomials over W is denoted by $\mathsf{B}\langle W\rangle$.

The usual ordering $\leq$ on B extends to $\mathsf{B}\langle\langle\ W\ \rangle\rangle$ by

$$S \leq S' \text{ iff } \forall w \in W^*,\ S_w \leq S'_w.$$

We recall that for every $S \in \mathsf{B}\langle\langle\ W\ \rangle\rangle$, S^* is the series defined by

$$S^* = \sum_{0 \leq n} S^n. \tag{2.14}$$

Given two alphabets W, W', a map $\psi : \mathsf{B}\langle\langle\ W\ \rangle\rangle \to \mathsf{B}\langle\langle\ W'\ \rangle\rangle$ is said to be σ-*additive* iff it fulfills the following condition: for every denumerable family $(S_i)_{i \in \mathbb{N}}$ of elements of $\mathsf{B}\langle\langle\ W\ \rangle\rangle$,

$$\psi\left(\sum_{i \in \mathbb{N}} S_i\right) = \sum_{i \in \mathbb{N}} \psi(S_i). \tag{2.15}$$

A map $\psi : \mathsf{B}\langle\langle\ W\ \rangle\rangle \to \mathsf{B}\langle\langle\ W'\ \rangle\rangle$ which is both a semiring homomorphism and a σ-additive map is usually called a *substitution*.

Actions of monoids. Given a semiring $(\mathsf{S}, +, \cdot, 0, 1)$ and a monoid $(\mathsf{M}, \cdot, 1_M)$, a map $\circ : \mathsf{S} \times \mathsf{M} \to \mathsf{S}$ is called a *right-action* of the monoid M over the semiring S iff, for every $S, T \in \mathsf{S}$, $m, m' \in \mathsf{M}$,

$$0 \circ m = 0, \quad S \circ 1_M = S,$$
$$(S + T) \circ m = (S \circ m) + (T \circ m), \quad S \circ (m \cdot m') = (S \circ m) \circ m'. \tag{2.16}$$

In the particular case where $\mathsf{S} = \mathsf{B}\langle\langle\, W \,\rangle\rangle$, $\circ$ is said to be a σ-right-action if it fulfills the additional property that, for every denumerable family $(S_i)_{i\in\mathbb{N}}$ of elements of S and $m \in \mathsf{M}$,

$$\left(\sum_{i\in\mathbb{N}} S_i\right) \circ m = \sum_{i\in\mathbb{N}} (S_i \circ m). \tag{2.17}$$

The action of W^* on $\mathsf{B}\langle\langle\, W \,\rangle\rangle$. We recall the following classical σ-right-action $\bullet$ of the monoid W^* over the semiring $\mathsf{B}\langle\langle\, W \,\rangle\rangle$: $\forall S, S' \in \mathsf{B}\langle\langle\, W \,\rangle\rangle$, $u \in W^*$,

$$S \bullet u = S' \Leftrightarrow \forall w \in W^*, (S'_w = S_{u\cdot w})$$

(i.e., $S \bullet u$ is the *left-quotient* of S by u, or the *residual* of S by u).

For every $S \in \mathsf{B}\langle\langle\, W \,\rangle\rangle$ we denote by $\mathsf{Q}(S)$ the set of residuals of S:

$$\mathsf{Q}(S) = \{S \bullet u \mid u \in W^*\}.$$

We recall that S is said to be *rational* iff the set $\mathsf{Q}(S)$ is *finite*. We define the *norm* of a series $S \in \mathsf{B}\langle\langle\, W \,\rangle\rangle$, denoted $\|S\|$, by

$$\|S\| = \mathrm{Card}(\mathsf{Q}(S)) \in \mathbb{N} \cup \{\infty\}.$$

The reduced grammar G. The classical reduced and ϵ-free grammar associated with $G_{\mathcal{M}}$ is $G_0 = \langle X, V_0, P_0\rangle$, where

$$V_0 = \{v \in V_{\mathcal{M}} \mid \exists w \in X^+, v \xrightarrow{*}_{P_{\mathcal{M}}} w\}, \tag{2.18}$$

$$\varphi_0 : \mathsf{B}\langle\langle\, V \,\rangle\rangle \to \mathsf{B}\langle\langle\, V_0 \,\rangle\rangle$$

is the unique substitution such that, for every $v \in V$,

$$\varphi_0(v) = v \text{ (if } v \in V_0), \quad \varphi_0(v) = \epsilon \text{ (if } v \xrightarrow{*}_{P_{\mathcal{M}}} \epsilon), \quad \varphi_0(v) = \emptyset \text{ (otherwise)},$$

$$\begin{aligned} P_0 = \{(v, w') \in V_0 \times (X \cup V_0)^+ \mid \\ v \in V_0, \exists w \in (X \cup V_{\mathcal{M}})^*, (v, w) \in P_{\mathcal{M}}, w' = \varphi_0(w)\}. \end{aligned} \tag{2.19}$$

G_0 is the *reduced* and *ϵ-free* form of $G_{\mathcal{M}}$. It is well known that, $\forall v \in V_0$,

$$\exists w \in X^+, v \xrightarrow{*}_{P_0} w,$$

$$\{w \in X^*, v \xrightarrow{*}_{P_{\mathcal{M}}} w\} = \{w \in X^*, v \xrightarrow{*}_{P_0} w\}.$$

For technical reasons (which will be made clear in section 7), we introduce an alphabet of "marked variables" $\bar{V}_0$ together with a fixed bijection: $v \mapsto \bar{v}$ from V_0 to $\bar{V}_0$. Let

$V = V_0 \cup \bar{V}_0$. We denote by ρ_e (the letter e stands here for "erasing the marks") the literal morphism $V^* \to V_0^*$ defined by the following: for every $v \in V_0$,

$$\rho_e(v) = v, \quad \rho_e(\bar{v}) = v.$$

Similarly, $\bar{\rho}_e$ is the literal morphism $V^* \to \bar{V}_0^*$ defined by the following: for every $v \in V_0$,

$$\bar{\rho}_e(v) = \bar{v}, \quad \bar{\rho}_e(\bar{v}) = \bar{v}.$$

We denote also by $\rho_e, \bar{\rho}_e$ the unique substitutions extending these monoid homomorphisms.

At last, the grammar G is defined by $G = \langle X, V, P \rangle$, where

$$P = P_0 \cup \{(\bar{\rho}_e(v), \bar{\rho}_e(w)) \mid (v, w) \in P_0\}.$$

In other words, the rules of G consist of the rules of the usual proper and reduced grammar associated with $\mathcal{M}$ together with their marked copies.

The action of X^* on $\mathsf{B}\langle\langle\ V\ \rangle\rangle$. Let us now fix a deterministic (normalized) pda $\mathcal{M}$ and consider the associated grammar G. We define a σ-right-action $\odot$ of the monoid X^* over the semiring $\mathsf{B}\langle\langle\ V\ \rangle\rangle$ by the following: for every $v \in V$, $\beta \in V^*$, $x \in X$,

$$(v \cdot \beta) \odot x = \left(\sum_{(v,h) \in P} h \bullet x \right) \cdot \beta, \tag{2.20}$$

$$\epsilon \odot x = \emptyset. \tag{2.21}$$

Let us consider the unique substitution $\varphi : \mathsf{B}\langle\langle\ V\ \rangle\rangle \to \mathsf{B}\langle\langle\ X\ \rangle\rangle$ fulfilling the following: for every $v \in V$,

$$\varphi(v) = \{u \in X^* \mid v \xrightarrow{*}_P u\}$$

(in other words, φ maps every subset $L \subseteq V^*$ on the language generated by the grammar G from the set of axioms L).

LEMMA 2.9. *For every $S \in \mathsf{B}\langle\langle\ V\ \rangle\rangle$, $u \in X^*$, $\varphi(S \odot u) = \varphi(S) \bullet u$ (i.e., φ is a morphism of right-actions).*

Proof. Let $v \in V$, $\beta \in V^*$, $x \in X$. Recall that G is in Greibach normal form (i.e., $P \subseteq V \times X \cdot V^*$). One can then check with formulas (2.20), (2.21) that

$$\varphi(\epsilon \odot x) = \varphi(\epsilon) \bullet x \quad \text{and} \quad \varphi((v \cdot \beta) \odot x) = \varphi(v \cdot \beta) \bullet x.$$

By induction on $|w|$, it follows that, $\forall w \in V^*$,

$$\varphi(w \odot x) = \varphi(w) \bullet x.$$

By σ-additivity of φ, it follows that, $\forall S \in \mathsf{B}\langle\langle\ V\ \rangle\rangle$,

$$\varphi(S \odot x) = \varphi(S) \bullet x.$$

By induction on $|u|$, it follows that, $\forall u \in X^*$,

$$\varphi(S \odot u) = \varphi(S) \bullet u. \qquad \square$$

We denote by $\equiv$ the kernel of φ, i.e., for every $S, T \in \mathsf{B}\langle\langle\ V\ \rangle\rangle$,

$$S \equiv T \Leftrightarrow \varphi(S) = \varphi(T).$$

3. Series and matrices.

3.1. Deterministic series, vectors, and matrices. We introduce here a notion of *deterministic* series which, in the case of the alphabet V associated to a dpda $\mathcal{M}$, generalizes the classical notion of *configuration* of $\mathcal{M}$. The main advantage of this notion is that, unlike for configurations, we shall be able to define *nice algebraic operations* on these series (see, in particular, section 5.1). Let us consider a pair $(W, \smile)$ where W is an alphabet and $\smile$ is an equivalence relation over W. We call $(W, \smile)$ a *structured* alphabet. We have the following two examples in mind:

- the case where $W = V_{\mathcal{M}}$, the variable alphabet associated to $\mathcal{M}$ and $[p, z, q] \smile [p', z', q']$ iff $p = p'$ and $z = z'$ (see [15]);
- the case where $W = X$, the terminal alphabet of $\mathcal{M}$ and $x \smile y$ holds for every $x, y \in X$ (see [15]).

Definitions.

DEFINITION 3.1. *Let $S \in \mathsf{B}\langle\langle\, W \,\rangle\rangle$. S is said to be* left-deterministic *iff either*

(1) $S = \emptyset$ *or*

(2) $S = \epsilon$ *or*

(3) $\exists i_0 \in [1, m]$, $S_{i_0} \neq \emptyset$ *and* $\forall w, w' \in W^*$, $S_w = S_{w'} = 1 \Rightarrow [\exists A, A' \in W$, $w_1, w_1' \in W^*$, $A \smile A'$, $w = A \cdot w_1$, *and* $w' = A' \cdot w_1']$.

A left-deterministic series S is said to have the type $\emptyset$ (resp., ϵ, $[A]_\smile$) if case (1) (resp., (2), (3)) occurs.

DEFINITION 3.2. *Let $S \in \mathsf{B}\langle\langle\, W \,\rangle\rangle$. S is said to be* deterministic *iff, for every $u \in W^*$, $S \bullet u$ is left-deterministic.*

This notion is the straightforward extension to the infinite case of the notion of a (finite) *set of associates* defined in [16, Definition 3.2, p. 188].

We denote by $\mathsf{DB}\langle\langle\, W \,\rangle\rangle$ the subset of deterministic boolean series over W. Let us denote by $\mathsf{B}_{n,m}\langle\langle\, W \,\rangle\rangle$ the set of (n, m)-matrices with entries in the semiring $\mathsf{B}\langle\langle\, W \,\rangle\rangle$.

DEFINITION 3.3. *Let $m \in \mathbb{N}$, $S \in \mathsf{B}_{1,m}\langle\langle\, W \,\rangle\rangle : S = (S_1, \ldots, S_m)$. S is said to be* left-deterministic *iff either*

(1) $\forall i \in [1, m]$, $S_i = \emptyset$ *or*

(2) $\exists i_0 \in [1, m]$, $S_{i_0} = \epsilon$ *and* $\forall i \neq i_0$, $S_i = \emptyset$ *or*

(3) $\forall w, w' \in W^*$, $\forall i, j \in [1, m]$, $(S_i)_w = (S_j)_{w'} = 1 \Rightarrow [\exists A, A' \in W$, $w_1, w_1' \in V^*$, $A \smile A'$, $w = A \cdot w_1$, *and* $w' = A' \cdot w_1']$.

A left-deterministic row-vector S is said to have the type $\emptyset$ (resp., (ϵ, i_0), $[A]_\smile$) if case (1) (resp., (2), (3)) occurs.

The right-action $\bullet$ on $\mathsf{B}\langle\langle\, W \,\rangle\rangle$ is extended componentwise to $\mathsf{B}_{n,m}\langle\langle\, W \,\rangle\rangle$: for every $S = (s_{i,j})$, $u \in W^*$, the matrix $T = S \bullet u$ is defined by

$$t_{i,j} = s_{i,j} \bullet u.$$

The ordering $\leq$ on B is also extended componentwise to $\mathsf{B}_{n,m}\langle\langle\, W \,\rangle\rangle$.

DEFINITION 3.4. *Let $S \in \mathsf{B}_{1,m}\langle\langle\, W \,\rangle\rangle$. S is said to be* deterministic *iff, for every $u \in W^*$, $S \bullet u$ is left-deterministic.*

We denote by $\mathsf{DB}_{1,m}\langle\langle\, W \,\rangle\rangle$ the subset of deterministic row-vectors of dimension m over $\mathsf{B}\langle\langle\, W \,\rangle\rangle$.

DEFINITION 3.5. *Let $S \in \mathsf{B}_{n,m}\langle\langle\, W \,\rangle\rangle$. S is said to be* deterministic *iff, for every $i \in [1, n]$, $S_{i,.}$ is a deterministic row-vector.*

Let us note first some easy facts about deterministic matrices.

FACT 3.6. *Let $S \in \mathsf{DB}\langle\langle\, W\,\rangle\rangle$. For every $T \in \mathsf{B}\langle\langle\, W\,\rangle\rangle$, $u \in W^*$*
(1) $T \leq S \Rightarrow T \in \mathsf{DB}\langle\langle\, W\,\rangle\rangle$.
(2) $T = S \bullet u \Rightarrow T \in \mathsf{DB}\langle\langle\, W\,\rangle\rangle$.

Norm. Let us generalize the classical definition of *rationality* of series in $\mathsf{B}\langle\langle\, W\,\rangle\rangle$ to matrices. Given $M \in \mathsf{B}_{n,m}\langle\langle\, W\,\rangle\rangle$ we denote by $\mathsf{Q}(M)$ the set of *residuals* of M:

$$\mathsf{Q}(M) = \{M \bullet u \mid u \in W^*\}.$$

Similarly, we denote by $\mathsf{Q}_r(M)$ the set of *row-residuals* of M:

$$\mathsf{Q}_r(M) = \bigcup_{1 \leq i \leq n} \mathsf{Q}(M_{i,*}).$$

M is said to be *rational* iff the set $\mathsf{Q}(M)$ is finite. One can check that it is equivalent to the property that every coefficient $M_{i,j}$ is rational, or to the property that $\mathsf{Q}_r(M)$ is finite. We denote by $\mathsf{RB}_{n,m}\langle\langle\, W\,\rangle\rangle$ (resp., $\mathsf{DRB}_{n,m}\langle\langle\, W\,\rangle\rangle$) the set of rational (resp., deterministic, rational) matrices over $\mathsf{B}\langle\langle\, W\,\rangle\rangle$. For every $M \in \mathsf{RB}_{n,m}\langle\langle\, W\,\rangle\rangle$, we define the norm of M as

$$\|M\| = \mathrm{Card}(\mathsf{Q}_r(M)).$$

Grammars.

DEFINITION 3.7. *Let $G = \langle X, V, P\rangle$ be a context-free grammar in Greibach normal form. G is said to be* strict-deterministic *iff there exists an equivalence relation $\smile$ over V fulfilling the following condition: for every $E \in V$, $x \in X$, if $(E_k)_{1\leq k\leq m}$ is a bijection $[1,m] \to [E]_\smile$, and $H_k = \sum_{(E_k,h)\in P} h \bullet x$, then*

$$(H_1, H_2, \ldots, H_m) \textit{ is a deterministic vector.}$$

Any equivalence $\smile$ satisfying the above condition is said to be a strict equivalence *for the grammar G.*

This definition is a reformulation of [15, Definition 11.4.1, p. 347] adapted to the case of a Greibach normal form.

THEOREM 3.8. *Let $G_1 = \langle X, V_1, P_1\rangle$ be a strict-deterministic grammar. Then its reduced form $G_0 = \langle X, V_0, P_0\rangle$, as defined in formulas* (2.18), (2.19)*, is strict-deterministic too. Moreover, if $\smile$ is a strict equivalence for G_1, its restriction over V_0 is a strict equivalence for G_0.*

The proof would consist in slightly extending the proof of [15, Theorem 11.4.1, p. 350].

It is known that, given a dpda $\mathcal{M}$, its associated grammar $G_{\mathcal{M}}$ is strict-deterministic. By Theorem 3.8 G_0 is strict-deterministic too. Let us consider the minimal strict equivalence $\smile$ for G_0 and extend it to V by, $\forall v, v' \in V_0$,

$$\bar{v} \smile \bar{v}' \Leftrightarrow v \smile v'; \quad \bar{v} \not\smile v'.$$

Then $\smile$ is a strict equivalence for G (the grammar G is defined in section 2.5). This ensures that G is strict-deterministic.

Residuals.

LEMMA 3.9 (see [42, Lemma 37]). *Let $S \in \mathsf{DB}\langle\langle\, W\,\rangle\rangle$, $T \in \mathsf{B}\langle\langle\, W\,\rangle\rangle$, $u \in W^*$. If $S \bullet u \neq \emptyset$, then $(S \cdot T) \bullet u = (S \bullet u) \cdot T$.*

LEMMA 3.10 (see [42, Lemma 39]). *Let $S \in \mathsf{DB}_{1,m}\langle\langle\, W\,\rangle\rangle$, $T \in \mathsf{B}_{m,s}\langle\langle\, W\,\rangle\rangle$, $u \in W^*$, and $U = S \cdot T$. Exactly one of the following cases is true:*

(1) $\exists j,\ S_j \bullet u \notin \{\emptyset, \epsilon\}$. *In this case* $U \bullet u = (S \bullet u) \cdot T$.
(2) $\exists j_0,\ \exists u', u'',\ u = u' \cdot u'',\ S_{j_0} \bullet u' = \epsilon$. *In this case* $U \bullet u = T_{j_0} \bullet u''$.
(3) $\forall j,\ \forall u' \preceq u,\ S_j \bullet u = \emptyset,\ S_j \bullet u' \neq \epsilon$. *In this case* $U \bullet u = \emptyset = (S \bullet u) \cdot T$.

LEMMA 3.11. *For every* $S \in \mathsf{DB}_{n,m}\langle\langle\ W\ \rangle\rangle$, $T \in \mathsf{DB}_{m,s}\langle\langle\ W\ \rangle\rangle$, $S \cdot T \in \mathsf{DB}_{n,s}\langle\langle\ W\ \rangle\rangle$.

LEMMA 3.12. *Let* $A \in \mathsf{DB}_{n,m}\langle\langle\ W\ \rangle\rangle$, $B \in \mathsf{B}_{m,s}\langle\langle\ W\ \rangle\rangle$. *Then* $\|A \cdot B\| \leq \|A\| + \|B\|$.

The two above lemmas correspond to [42, Lemma 310].

$\boldsymbol{W = V}$**.** Let $(W, \smile)$ be the structured alphabet $(V, \smile)$ associated with $\mathcal{M}$ and let us consider a bijective numbering of the elements of Q: $(q_1, q_2, \ldots, q_{n_Q})$. Let us define here a handful of notations for some particular vectors or matrices. Let us use the *Kronecker symbol* $\delta_{i,j}$ to mean ϵ if $i = j$ and $\emptyset$ if $i \neq j$. For every $1 \leq n$, $1 \leq i \leq n$, we define the row-vector ϵ_i^n as

$$\epsilon_i^n = (\epsilon_{i,j}^n)_{1 \leq j \leq n}, \quad \text{where } \forall j,\ \epsilon_{i,j}^n = \delta_{i,j}.$$

We call any vector of the form ϵ_i^n a *unit row-vector*.

For every $1 \leq n$, we denote by $\emptyset^n \in \mathsf{DB}_{1,n}\langle\langle\ V\ \rangle\rangle$ the row-vector

$$\emptyset^n = (\emptyset, \ldots, \emptyset).$$

For every $\omega \in Z^*$, $p, q \in Q$, $[p\omega q]$ is the deterministic series defined inductively by

$$[p\epsilon q] = \emptyset \text{ if } p \neq q, \quad [p\epsilon q] = \epsilon \text{ if } p = q,$$

$$[p\omega q] = \sum_{r \in Q} [p, z, r] \cdot [r\omega' q] \text{ if } \omega = z \cdot \omega' \text{ for some } z \in Z,\ \omega' \in Z^*.$$

Let us define

$$K_0 = \max\{\|(E_1, E_2, \ldots, E_n) \odot x\| \mid (E_i)_{1 \leq i \leq n} \text{ is a bijective numbering of some class in } V/\smile,\ x \in X\}. \tag{3.1}$$

LEMMA 3.13 (see [42, Lemma 318]). *For every* $S \in \mathsf{DB}_{1,\lambda}\langle\langle\ V\ \rangle\rangle$, $u \in X^*$,
(1) $S \odot u \in \mathsf{DB}_{1,\lambda}\langle\langle\ V\ \rangle\rangle$,
(2) $\|S \odot u\| \leq \|S\| + K_0 \cdot |u|$.

LEMMA 3.14 (see [42, Lemma 319]). *Let* $\lambda \in \mathbb{N} - \{0\}$, $S \in \mathsf{DRB}_{1,\lambda}\langle\langle\ V\ \rangle\rangle$, $u \in X^*$. *One of the three following cases must occur:*
(1) $S \odot u = \emptyset^\lambda$.
(2) $S \odot u = \epsilon_j^\lambda$ *for some* $j \in [1, \lambda]$.
(3) $\exists u_1, u_2 \in X^*$, $v_1 \in V^*$, $q \in \mathbb{N}$, $E_1, \ldots, E_k, \ldots, E_q \in V$, $\Phi \in \mathsf{DRB}_{q,\lambda}\langle\langle\ V\ \rangle\rangle$ *such that*

$$u = u_1 \cdot u_2,\ \ S \odot u_1 = S \bullet v_1 = \sum_{k=1}^{q} E_k \cdot \Phi_k,\ \ S \odot u = \sum_{k=1}^{q} (E_k \odot u_2) \cdot \Phi_k,\ \ \textit{and}$$

$$\forall k \in [1, q], E_k \smile E_1, E_k \odot u_2 \notin \{\epsilon, \emptyset\}.$$

We now give an adaptation of Lemma 3.10 to the action $\odot$ in place of $\bullet$.

LEMMA 3.15 (see [42, Lemma 321]). *Let* $S \in \mathsf{DB}_{1,m}\langle\langle\ V\ \rangle\rangle$, $T \in \mathsf{B}_{m,s}\langle\langle\ V\ \rangle\rangle$, $u \in X^*$, *and* $U = S \cdot T$. *Exactly one of the following cases is true:*
(1) $S \odot u \notin \{\emptyset^m\} \cup \{\epsilon_j^m \mid 1 \leq j \leq m\}$. *In this case* $U \odot u = (S \odot u) \cdot T$.
(2) $\exists j_0,\ \exists u', u'',\ u = u' \cdot u'',\ S \odot u' = \epsilon_{j_0}^s$. *In this case* $U \odot u = T_{j_0} \odot u''$.
(3) $\forall j,\ \forall u' \preceq u,\ S \odot u = \emptyset^m$ *and* $S \odot u' \neq \epsilon_j^m$. *In this case* $U \odot u = \emptyset^s = (S \odot u) \cdot T$.

Marks. A word $w \in V^*$ is said to be *marked* iff $w \in V^* \cdot \bar{V}_0 \cdot V^*$; it is said to be *fully marked* iff $w \in \bar{V}_0^*$.

A series $S \in \mathsf{B}\langle\langle V \rangle\rangle$ is said to be *marked* iff $\exists w \in \mathrm{supp}(S)$, w is marked; it is said to be *fully marked* iff $\forall w \in \mathrm{supp}(S)$, w is fully marked. It is said to be *unmarked* iff it is *not* marked. A matrix $S \in \mathsf{B}_{m,n}\langle\langle V \rangle\rangle$ is said to be marked (resp., fully marked, unmarked) iff, for every $i \in [1, m]$, the series $\sum_{j=1}^{n} S_{i,j}$ is marked (resp., fully marked, unmarked).

DEFINITION 3.16. *Let $d \in \mathbb{N}$. A vector $S \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle$ is said to be d-marked iff there exists $q \in \mathbb{N}$, $\alpha \in \mathsf{DRB}_{1,q}\langle V \rangle$, $\Phi \in \mathsf{DRB}_{q,\lambda}\langle\langle V \rangle\rangle$ such that*

$$S = \sum_{k=1}^{q} \alpha_k \cdot \Phi_k \text{ and } \|\alpha\| \leq d,$$

and Φ is unmarked.

LEMMA 3.17. *For every $S \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle$,*
(1) *$\rho_e(S) \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle$,*
(2) *$\|\rho_e(S)\| \leq \|S\|$.*

Sketch of proof.

(1) Let us notice that the homomorphism $\rho_e : V^* \to V^*$ preserves the equivalence $\smile$: for every $v, v' \in V$, if $v \smile v'$, then $\rho_e(v) \smile \rho_e(v')$. It follows that the corresponding substitution ρ_e preserves determinism.

(2) Let $S \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle$. For every $v \in V_0$,

$$\rho_e(S) \bullet v = \rho_e(S \bullet v) \quad \text{or} \quad \rho_e(S) \bullet v = \rho_e(S \bullet \bar{v})$$

according to the fact that the leftmost letters of the monomials of S are in $[v]_\smile$ or in $[\bar{v}]_\smile$; both formulas are true when S is null or is a unit.

By induction on the length, it follows that, for every $w \in V_0^*$, there exists $w' \in V^*$ such that

$$\rho_e(w') = w \quad \text{and} \quad \rho_e(S) \bullet w = \rho_e(S \bullet w').$$

Moreover, for every $w \in V^* \bar{V}_0 V^*$,

$$\rho_e(S) \bullet w = \emptyset^\lambda,$$

but in this case, too, there exists some $w' \in V^*$ such that $\rho_e(S) \bullet w = \rho_e(S \bullet w')$.

The map $T \mapsto \rho_e(T)$ is then a surjective map from $\mathsf{Q}(S)$ onto $\mathsf{Q}(\rho_e(S))$, which proves that $\|\rho_e(S)\| \leq \|S\|$. ∎

Operations on row-vectors. Let us introduce two new operations on row-vectors and prove some technical lemmas about them.

Given $A, B \in \mathsf{B}_{1,m}\langle\langle W \rangle\rangle$ and $1 \leq j_0 \leq m$ we define the vector $C = A \nabla_{j_0} B$ as follows: if $A = (a_1, \ldots, a_j, \ldots, a_m)$, $B = (b_1, \ldots, b_j, \ldots, b_m)$, then $C = (c_1, \ldots, c_j, \ldots, c_m)$, where

$$c_j = a_j + a_{j_0} \cdot b_j \text{ if } j \neq j_0, \quad c_j = \emptyset \text{ if } j = j_0.$$

LEMMA 3.18 (see [42, Lemma 311]). *Let $A, B \in \mathsf{B}_{1,m}\langle\langle W \rangle\rangle$ and $1 \leq j_0 \leq m$.*
(1) *If A, B are deterministic, then $A \nabla_{j_0} B$ is deterministic.*
(2) *If A, B are deterministic, then $\|A \nabla_{j_0} B\| \leq \|A\| + \|B\|$.*

Given $A \in \mathsf{DB}_{1,m}\langle\langle\ W\ \rangle\rangle$ and $1 \leq j_0 \leq m$ we define the vector $A' = \nabla^*_{j_0}(A)$ as follows: if $A = (a_1, \dots, a_j, \dots, a_m)$, then $A' = (a'_1, \dots, a'_j, \dots, a'_m)$, where

$$a'_j = a^*_{j_0} \cdot a_j \text{ if } j \neq j_0, \quad a'_j = \emptyset \text{ if } j = j_0.$$

LEMMA 3.19 (see [42, Lemma 312]). *Let $A \in \mathsf{DB}_{1,m}\langle\langle\ W\ \rangle\rangle$ and $1 \leq j_0 \leq m$. Then $\nabla^*_{j_0}(A) \in \mathsf{DB}_{1,m}\langle\langle\ W\ \rangle\rangle$ and $\|\nabla^*_{j_0}(A)\| \leq \|A\|$.*

3.2. Bisimulation of series. Up to the end of this section, we consider the structured alphabet V associated with a dpda $\mathcal{M}$ over X. We suppose a strong relational morphism $\eta \subseteq X^* \times X^*$ is given (see Definition 2.2).

Series, words, and graphs. Let us first give a slight adaptation of Definition 2.1 to the n-graph $(\mathsf{DRB}_{1,n}\langle\langle\ V\ \rangle\rangle, \odot, (\epsilon^n_i)_{1\leq i\leq n})$.

DEFINITION 3.20. *Let $\mathcal{R}$ be some binary relation $\mathcal{R} \subseteq \mathsf{DRB}_{1,n}\langle\langle\ V\ \rangle\rangle \times \mathsf{DRB}_{1,n}\langle\langle\ V\ \rangle\rangle$. $\mathcal{R}$ is a σ-η-bisimulation iff*

(1) $\forall (S, S') \in \mathcal{R}$, $\forall x \in X$,

$$\exists x' \in \eta(x),\ (S \odot x, S' \odot x') \in \mathcal{R} \textit{ and } \exists x'' \in \eta^{-1}(x),\ (S \odot x'', S' \odot x) \in \mathcal{R},$$

(2) $\forall (S, S') \in \mathcal{R}$, $\forall i \in [1, n]$, $(S = \epsilon^n_i \Leftrightarrow S' = \epsilon^n_i)$.

We denote by $S \sim S'$ the fact that there exists some σ-η-bisimulation $\mathcal{R}$ such that $(S, S') \in \mathcal{R}$. One can notice that $\sim$ is the greatest σ-η-bisimulation (w.r.t. the inclusion ordering) over $\mathsf{DRB}_{1,n}\langle\langle\ V\ \rangle\rangle$. The σ-bisimulation relations can be conveniently expressed in terms of *word*-bisimulations.

DEFINITION 3.21. *Let $S, S' \in \mathsf{DRB}_{1,n}\langle\langle\ V\ \rangle\rangle$ and $\mathcal{R} \subseteq X^* \times X^*$. $\mathcal{R}$ is a w-η-bisimulation with respect to (S, S') iff $\mathcal{R} \subseteq \eta$ and*

(1) totality: $\mathrm{dom}(\mathcal{R}) = X^*$, $\mathrm{im}(\mathcal{R}) = X^*$*;*

(2) extension: $\forall (u, u') \in \mathcal{R}$, $\forall x \in X$,

$$\exists x' \in \eta(x),\ (u \cdot x, u' \cdot x') \in \mathcal{R} \textit{ and } \exists x'' \in \eta^{-1}(x),\ (u \cdot x'', u' \cdot x) \in \mathcal{R};$$

(3) coherence: $\forall (u, u') \in \mathcal{R}$, $\forall i \in [1, n]$, $(S \odot u = \epsilon^n_i) \Leftrightarrow (S' \odot u' = \epsilon^n_i)$*;*

(4) prefix: $\forall (u, u') \in X^* \times X^*$, $\forall (x, x') \in X \times X$, $(u \cdot x, u' \cdot x') \in \mathcal{R} \Rightarrow (u, u') \in \mathcal{R}$.

(Condition (1) can be equivalently replaced by "$(\epsilon, \epsilon) \in \mathcal{R}$.") $\mathcal{R}$ is said to be a w-η-bisimulation of *order* m w.r.t. (S, S') iff it fulfills conditions (3)–(4) above and the modified conditions

(1′) $\mathrm{dom}(\mathcal{R}) = X^{\leq m}$, $\mathrm{im}(\mathcal{R}) = X^{\leq m}$,

(2′) $\forall (u, u') \in \mathcal{R} \cap (X^{\leq m-1} \times X^{\leq m-1})$, $\forall x \in X$,

$$\exists x' \in \eta(x), (u \cdot x, u' \cdot x') \in \mathcal{R} \quad \text{and} \quad \exists x'' \in \eta^{-1}(x), (u \cdot x'', u' \cdot x) \in \mathcal{R}.$$

The w-η-bisimulations are also called w-η-bisimulations of *order* ∞. Lemmas 3.22 and 3.25 below relate the notions of w-η-bisimulation (on words), σ-η-bisimulation (on series), and η-bisimulation (on the vertices of the computation 2-graph of $\mathcal{M}$).

LEMMA 3.22. *Let $S, S' \in \mathsf{DRB}_{1,n}\langle\langle\ V\ \rangle\rangle$. The following properties are equivalent:*

(i) $S \sim S'$.

(ii) *There exists $\mathcal{R} \subseteq X^* \times X^*$ which is a w-η-bisimulation w.r.t. (S, S').*

(iii) $\forall m \in \mathbb{N}$*, there exists $\mathcal{R}_m \subseteq X^{\leq m} \times X^{\leq m}$ which is a w-η-bisimulation of order m w.r.t. (S, S').*

Proof. **(i) ⇒ (iii):** Suppose that $\mathcal{S}$ is a σ-η-bisimulation w.r.t. (S, S'). Let us prove by induction on the integer m the following property $\mathcal{P}(m)$: $\exists \mathcal{R}_m$, w-η-bisimulation of order m w.r.t. (S, S') such that

$$(3.2) \qquad \forall (u, u') \in \mathcal{R}_m, \quad (S \odot u, S' \odot u') \in \mathcal{S}.$$

$\boldsymbol{m = 0}$**:** Let $\mathcal{R}_0 = \{(\epsilon, \epsilon)\}$. $\mathcal{R}_0$ clearly fulfills points $(1')$, $(2')$, (4) of the above definition. Moreover, as $(S, S') \in \mathcal{S}$, where $\mathcal{S}$ fulfills condition (2) of Definition 3.20, $\mathcal{R}_0$ fulfills point (3) of Definition 3.21.

$\boldsymbol{m = m' + 1}$**:** Let $\mathcal{R}_{m'}$ be some w-η-bisimulation of order m' w.r.t. (S, S'). Let us define $\mathcal{R}_m = \mathcal{R}_{m'} \cup \{(u \cdot x, u' \cdot x') \mid (u, u') \in \mathcal{R}_{m'}, (S \odot ux, S' \odot u'x') \in \mathcal{S}$, and $(x, x') \in \eta\}$. Property (1) of $\mathcal{S}$ and property $(1')$ of $\mathcal{R}_{m'}$ imply that

$$(3.3) \qquad \mathrm{dom}(\mathcal{R}_m) = X^{\leq m}, \quad \mathrm{im}(\mathcal{R}_m) = X^{\leq m}.$$

Property (1) of $\mathcal{S}$ and property $(2')$ of $\mathcal{R}_{m'}$ imply that $\forall (u, u') \in \mathcal{R}_m \cap (X^{\leq m-1} \times X^{\leq m-1})$, $\forall x \in X$,

$$(3.4) \qquad \exists x' \in \eta(x), (u \cdot x, u' \cdot x') \in \mathcal{R}_m \quad \text{and} \quad \exists x'' \in \eta^{-1}(x), (u \cdot x'', u' \cdot x) \in \mathcal{R}_m.$$

Property (2) of $\mathcal{S}$ and property (3) of $\mathcal{R}_{m'}$ imply that

$$(3.5) \qquad \forall (u, u') \in \mathcal{R}_m, \forall i \in [1, n], \quad (S \odot u = \epsilon_i^n) \Leftrightarrow (S' \odot u' = \epsilon_i^n).$$

Property (4) of $\mathcal{R}_{m'}$ and the definition of $\mathcal{R}_m$ imply that

$$(3.6) \quad \forall (u, u') \in X^* \times X^*, \forall (x, x') \in X \times X, \quad (u \cdot x, u' \cdot x') \in \mathcal{R}_m \Rightarrow (u, u') \in \mathcal{R}_m.$$

Property (3.2) for $\mathcal{R}_{m'}$ and the definition of $\mathcal{R}_m$ imply that (3.2) is fulfilled by $\mathcal{R}_m$ too. Equations (3.3), (3.4), (3.5), (3.6) prove that $\mathcal{R}_m$ is a w-η-bisimulation of order m w.r.t. (S, S'), and hence $\mathcal{P}(m)$ is proved.

(iii) ⇒ (ii): Let us notice that, as the alphabet X is finite, for every w-η-bisimulation $\mathcal{R}$ of order m w.r.t. (S, S'),

$$\mathrm{Card}\{\mathcal{R}' \subseteq X^* \times X^* \mid \mathcal{R} \subseteq \mathcal{R}', \mathcal{R}' \text{ is a } w\text{-}\eta\text{-bisim. of ord. } m+1 \text{ w.r.t. } (S, S')\} < \infty.$$

Hence, by Koenig's lemma, if (iii) is true, then there exists an infinite sequence $(\mathcal{R}_m)_{m \in \mathbb{N}}$ such that for every $m \in \mathbb{N}$, $\mathcal{R}_m$ is a w-η-bisimulation of order m w.r.t. (S, S') and $\mathcal{R}_m \subseteq \mathcal{R}_{m+1}$. Let us then define

$$\mathcal{R} = \bigcup_{m \geq 0} \mathcal{R}_m.$$

$\mathcal{R}$ is a w-η-bisimulation of order ∞ w.r.t. (S, S').

(ii) ⇒ (i): Let $\mathcal{R}$ be a w-η-bisimulation of order ∞ w.r.t. (S, S'). Let us define a relation $\mathcal{S}$ by

$$\mathcal{S} = \{(S \odot u, S' \odot u') \mid (u, u') \in \mathcal{R}\}.$$

The totality property of $\mathcal{R}$ implies that $(S, S') \in \mathcal{S}$. The extension property of $\mathcal{R}$ implies that $\mathcal{S}$ fulfills condition (1) of Definition 3.20 and the coherence property of $\mathcal{R}$ implies that $\mathcal{S}$ fulfills condition (2). ∎

Lemma 3.22 leads naturally to the following definition. We denote by $\mathcal{B}_n(T,T')$ the set of all w-η-bisimulations of order n w.r.t. (T,T').

DEFINITION 3.23. *Let* $\lambda \in \mathbb{N} - \{0\}$, $S, S' \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V\, \rangle\rangle$. *We define the* divergence *between* S *and* S' *as*

$$\mathrm{Div}(S,S') = \inf\{n \in \mathbb{N} \mid \mathcal{B}_n(S,S') = \emptyset\}.$$

(It is understood that $\inf(\emptyset) = \infty$.*)*

Let us suppose that the dpda $\mathcal{M} = \langle X, Z, Q, \delta, q_0, z_0, \{\bar{q}\}\rangle$ is normalized and birooted. Let $\psi : X^* \to Y^*$ be a monoid homomorphism such that $\psi(X) \subseteq Y$ and let $\bar{\psi} = \psi \circ \psi^{-1}$ ($\bar{\psi}$, the kernel of ψ, is a strong relational morphism which is also an equivalence relation; this additional property will be used in what follows). Let Γ be the computation 2-graph of $\mathcal{M}$ and let us suppose Γ is $\bar{\psi}$-saturated.

Let $\theta : V_\Gamma \to \mathsf{DRB}\langle\langle\, V\, \rangle\rangle$ be the mapping defined by the following: $\forall q \in Q$, $\forall \omega \in Z^*$, such that $q\omega \in V_\Gamma$,

$$\theta(q\omega) = \varphi_0([q\omega\bar{q}]).$$

For every $q\omega \in V_\Gamma$, $S \in \mathsf{DRB}\langle\langle\, V\, \rangle\rangle$ we also define

$$\mathrm{L}(q\omega) = \{u \in X^*, q\omega \xrightarrow{u}_\Gamma \bar{q},\}; \quad \mathrm{L}(S) = \{u \in X^*, S \odot u = \epsilon\}.$$

LEMMA 3.24. *For every* $q\omega \in V_\Gamma$, $\mathrm{L}(q\omega) = \mathrm{L}(\varphi_0([q\omega\bar{q}]))$.

This lemma follows from the classical result that the language recognized by $\mathcal{M}$ with starting configuration $q\omega$ and final configuration $\bar{q}$ is exactly the language generated by $G_{\mathcal{M}}$ from the polynomial $[q\omega\bar{q}]$, which, in turn, is equal to the language generated by G_0 from the polynomial $\varphi_0([q\omega\bar{q}])$. At last, G and G_0 generate the same language from any given polynomial over V_0.

LEMMA 3.25. *Let* v, v' *be vertices of* Γ. *Then* $v \sim v'$, *in the sense of Definition* 2.1 *iff* $\theta(v) \sim \theta(v')$, *in the sense of Definition* 3.20.

Proof. In this proof we denote by $\odot_\Gamma$ the right-action of X^* over $V_\Gamma \cup \{\perp\}$ defined by the following: for every $v, v' \in V_\Gamma, u \in X^*$,

$$v \odot_\Gamma u = v' \text{ if } v \xrightarrow{u}_\Gamma v',$$
$$v \odot_\Gamma u = \perp \text{ if there is no } v'', \text{ such that } v \xrightarrow{u}_\Gamma v'',$$
$$\perp \odot_\Gamma u = \perp.$$

Step 1. Let us suppose that $(v, v') \in \mathcal{R}$, where $\mathcal{R}$ is some $\bar{\psi}$-bisimulation over Γ.

Let $\mathcal{S} = \{(\theta(v) \odot u, \theta(v') \odot u') \mid (u, u') \in \bar{\psi}, (v \odot_\Gamma u, v' \odot_\Gamma u') \in \mathcal{R}\} \cup \{(\emptyset, \emptyset)\}$. Let us show that $\mathcal{S}$ is a σ-$\bar{\psi}$-bisimulation.

Let us consider some pair of series in $\mathcal{S}$. If the given pair is $(\emptyset, \emptyset)$, points (1), (2) of Definition 3.20 are clearly fulfilled. Otherwise, it has the form $(\theta(v) \odot u, \theta(v') \odot u')$, where $(u, u') \in \bar{\psi}$ and $(v \odot_\Gamma u, v' \odot_\Gamma u') \in \mathcal{R}$.

Step 1.1. Let $x \in X$.

Case 1.1.1. $\theta(v) \odot ux \neq \emptyset$.

$$\mathrm{L}(\theta(v) \odot ux) \neq \emptyset$$

(because the grammar G is reduced); hence, using Lemma 3.24,

$$\mathrm{L}(v \odot_\Gamma ux) = \mathrm{L}(v) \bullet ux = \mathrm{L}(\theta(v)) \bullet ux \neq \emptyset.$$

It follows that

$$v \odot_\Gamma ux \neq \perp.$$

As $\mathcal{R}$ is a $\bar{\psi}$-simulation, there must exist some $x' \in \bar{\psi}(x)$ such that

$$(v \odot_\Gamma ux, v' \odot_\Gamma u'x') \in \mathcal{R}.$$

Hence

$$(\theta(v) \odot ux, \theta(v') \odot u'x') \in \mathcal{S}.$$

Case 1.1.2. $\theta(v) \odot ux = \emptyset$. In this case, by Lemma 3.24 and the fact that Γ is birooted, $v \odot_\Gamma ux$ must be equal to $\perp$. As Γ is $\bar{\psi}$-saturated, it follows that

$$\forall x' \in \bar{\psi}(x), \quad v \odot_\Gamma ux' = \perp.$$

As $\mathcal{R}^{-1}$ is a $\bar{\psi}^{-1}$-simulation, it must also be true that

$$\forall x' \in \bar{\psi}(x), \quad v' \odot_\Gamma u'x' = \perp.$$

Choosing some particular $x' \in \bar{\psi}(x)$, and again using Lemma 3.24, we obtain

$$\theta(v') \odot u'x' = \emptyset.$$

In both cases, as v, v' are playing symmetric roles, property (1) of Definition 3.20 has been verified. If the starting pair in $\mathcal{S}$ is $(\emptyset, \emptyset)$, property (1) is again verified.

Step 1.2. Let us suppose that $\theta(v) \odot u = \epsilon$. This means that

$$\mathrm{L}(\theta(v)) \bullet u = \epsilon,$$

and hence, using Lemma 3.24, that

$$\mathrm{L}(v \odot_\Gamma u) = \epsilon;$$

hence

$$v \odot_\Gamma u = \bar{q}.$$

As Γ is birooted, $\bar{q}$ is the only vertex having no outgoing edge (see section 2.1). As $\mathcal{R}$ is a $\bar{\psi}$-bisimulation, $v' \odot_\Gamma u'$, we must also have no outgoing edge; hence

$$v' \odot_\Gamma u' = \bar{q},$$

and by the same arguments, used backwards now,

$$\mathrm{L}(\theta(v')) \bullet u' = \epsilon,$$

which, as the grammar G is proper and reduced, implies

$$\theta(v') \odot u' = \epsilon.$$

As (v, v') are playing symmetric roles, property (2) of Definition 3.20 has been verified.

Step 2. Let us suppose that $(\theta(v), \theta(v')) \in \mathcal{S}$, where $\mathcal{S}$ is some σ-$\bar{\psi}$-bisimulation.

Let $\mathcal{R} = \{(v \odot_\Gamma u, v' \odot_\Gamma u') \mid (u, u') \in \bar{\psi}, (\theta(v) \odot u, \theta(v') \odot u') \in \mathcal{S} - \{(\emptyset, \emptyset)\}\} \cup \{(c, c) \mid c \in V_\Gamma\}$. We show that $\mathcal{R}$ is a $\bar{\psi}$-bisimulation over Γ.

Step 2.1. Using Lemma 3.24, we obtain

$$\theta(v) \odot u \neq \emptyset \Rightarrow v \odot_\Gamma u \neq \perp.$$

Hence

$$\mathrm{dom}(\mathcal{R}) \subseteq V_\Gamma.$$

Conversely, due to the term $\{(c, c) \mid c \in V_\Gamma\}$,

$$\mathrm{dom}(\mathcal{R}) \supseteq V_\Gamma.$$

Finally, point (1) of Definition 2.1 is fulfilled.

Step 2.2. Due to the term $\{(c, c) \mid c \in V_\Gamma\}$, point (2) of Definition 2.1 is fulfilled.

Step 2.3. Let us consider some pair of configurations in $\mathcal{R}$. It must have the form $(v \odot_\Gamma u, v' \odot_\Gamma u')$, where $(u, u') \in \bar{\psi}$ and $(\theta(v) \odot u, \theta(v') \odot u') \in \mathcal{S} - \{(\emptyset, \emptyset)\}$.

By the same arguments as in Case 1.1.1 above, one can show that, for every $x \in X$, such that

$$v \odot_\Gamma ux \neq \perp,$$

there exists some $x' \in \bar{\psi}(x)$ such that

$$v' \odot_\Gamma u'x' \neq \perp.$$

Hence $\mathcal{R}$ fulfills the three points of Definition 2.1. By the same means, $\mathcal{R}^{-1}$ fulfills them too, so that $\mathcal{R}$ is a $\bar{\psi}$-bisimulation over the graph Γ. $\square$

Extension to matrices. Let $\delta, \lambda \in \mathbb{N} - \{0\}$. We extend the binary relation $\sim$ from vectors in $\mathsf{DRB}_{1,\lambda}\langle\langle\, V\,\rangle\rangle$ to matrices in $\mathsf{DRB}_{\delta,\lambda}\langle\langle\, V\,\rangle\rangle$ as follows: for every $T, T' \in \mathsf{DRB}_{\delta,\lambda}\langle\langle\, V\,\rangle\rangle$,

$$T \sim T' \Leftrightarrow \forall i \in [1, \delta], T_{i,*} \sim T'_{i,*}. \tag{3.7}$$

We call a *w-η-bisimulation* of order $n \in \mathbb{N} \cup \{\infty\}$ with respect to (T, T') every

$$\mathcal{R} = (\mathcal{R}_i)_{i \in [1,\delta]} \text{ such that } \forall i \in [1, \delta],\ \mathcal{R}_i \in \mathcal{B}_n(T_{i,*}, T'_{i,*}).$$

We denote by $\mathcal{B}_n(T, T')$ the set of w-η-bisimulations of order n w.r.t. (T, T').

Some algebraic properties of this extended relation $\sim$ will be established in Corollary 4.10.

Operations on w-bisimulations. The following operations on word-$\bar{\psi}$-bisimulations turn out to be useful.

Right-product. Let $\delta, \lambda \in \mathbb{N} - \{0\}$, $S, S' \in \mathsf{DRB}_{1,\delta}\langle\langle\, V\,\rangle\rangle$, $T \in \mathsf{DRB}_{\delta,\lambda}\langle\langle\, V\,\rangle\rangle$. For every $n \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{R} \in \mathcal{B}_n(S, S')$ we define

$$\begin{aligned} \langle S|\mathcal{R}\rangle = [\{(u, u') \in \mathcal{R} \mid \forall v \preceq u, \forall i \in [1, \delta], S \odot v \neq \epsilon_i^\delta\} \\ \cup \{(u \cdot w, u' \cdot w) \mid (u, u') \in \mathcal{R}, w \in X^*, \exists i \in [1, \delta], S \odot u = \epsilon_i^\delta\}] \\ \cap X^{\leq n} \times X^{\leq n}. \end{aligned} \tag{3.8}$$

One can check that $\langle S|\mathcal{R}\rangle \in \mathcal{B}_n(S \cdot T, S' \cdot T)$.

Left-product. Let $\delta, \lambda \in \mathbb{N} - \{0\}$, $S \in \mathsf{DRB}_{1,\delta}\langle\langle V \rangle\rangle$, $T, T' \in \mathsf{DRB}_{\delta,\lambda}\langle\langle V \rangle\rangle$. For every $n \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{R} \in \mathcal{B}_n(T, T')$ we define

$$\begin{aligned}\langle S, \mathcal{R}\rangle = [&\{(u,u) \mid u \in X^*, \forall v \preceq u, \forall i \in [1,\delta], S \odot v \neq \epsilon_i^\delta\} \\ &\cup \{(u \cdot w, u \cdot w') \mid u \in X^*, \exists i \in [1,\delta], S \odot u = \epsilon_i^\delta, (w, w') \in \mathcal{R}_i\}] \\ &\cap X^{\leq n} \times X^{\leq n}.\end{aligned} \tag{3.9}$$

One can check that $\langle S, \mathcal{R}\rangle \in \mathcal{B}_n(S \cdot T, S \cdot T')$.

Star. Let $\lambda \in \mathbb{N} - \{0\}$, $S_1 \in \mathsf{DRB}_{1,1}\langle\langle V \rangle\rangle$, $S_1 \neq \epsilon$, $(S_1, S) \in \mathsf{DRB}_{1,\lambda+1}\langle\langle V \rangle\rangle$, $T \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$. For every $n \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{R} \in \mathcal{B}_n(S_1 \cdot T + S, T)$ we define

$$\mathcal{R}_0 = \mathcal{R}, \tag{3.10}$$

$$\mathcal{S}_0 = \begin{pmatrix} \mathcal{R}_0 \\ \vdots \\ \mathcal{R}_0 \end{pmatrix}, \tag{3.11}$$

$$\forall k \geq 0, \mathcal{R}_{k+1} = \langle (S_1, S), \mathcal{S}_k\rangle \circ \mathcal{R}_0, \tag{3.12}$$

$$\mathcal{S}_k = \begin{pmatrix} \mathcal{R}_k \\ \vdots \\ \mathcal{R}_k \end{pmatrix}, \tag{3.13}$$

and finally

$$\mathcal{R}^{\langle S_1, *\rangle} = \bigcup_{k \geq 0} \mathcal{R}_k \cap X^{\leq k} \times X^{\leq k}. \tag{3.14}$$

One can check that, for every $k \geq 0$,

$$\mathcal{R}_k \in \mathcal{B}_n\left(S_1^{k+1} + \sum_{i=0}^{k} S_1^i \cdot S, T\right), \tag{3.15}$$

$$\mathcal{S}_k \in \mathcal{B}_n\left(\begin{pmatrix} S_1^{k+1} + \sum_{i=0}^{k} S_1^i \cdot S \\ \mathrm{I}_\lambda \end{pmatrix}, \begin{pmatrix} T \\ \mathrm{I}_\lambda \end{pmatrix}\right), \tag{3.16}$$

and finally $\mathcal{R}^{\langle S_1, *\rangle} \in \mathcal{B}_n(S_1^* \cdot S, T)$.

REMARK 3.26. *In fact operations could be more adequately defined on "pointed" w-bisimulations, i.e., on binary relations with sets of "terminal pairs of words" of type $i \in [1,\delta]$ corresponding to the pairs (u, u') such that $S \odot u = \epsilon_i^\delta$, $S' \odot u' = \epsilon_i^\delta$. The two different external operations $\langle S, \mathcal{R}\rangle, \langle S|\mathcal{R}\rangle$ could then be replaced by only one binary operation $\langle \mathcal{R}_1, \mathcal{R}_2\rangle$ over "pointed" w-bisimulations.*

3.3. Derivations. For every $u \in X^*$ we define the binary relation $\uparrow(u)$ over $\mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle$ by the following: for every $S, S' \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle$, $S \uparrow (u) S' \Leftrightarrow \exists q \in \mathbb{N}$, $\exists E_1, \ldots, E_k, \ldots, E_q \in V$, $\Phi \in \mathsf{DB}_{q,\lambda}\langle\langle V \rangle\rangle$ such that

$$S = \sum_{k=1}^{q} E_k \cdot \Phi_k, \qquad S' = \sum_{k=1}^{q} (E_k \odot u) \cdot \Phi_k,$$

and $\forall k \in [1,q]$, $E_1 \smile E_k$, $E_k \odot u \notin \{\emptyset, \epsilon\}$.

It is clear that if $S \uparrow (u)S'$, then $S \odot u = S'$ and that the converse is not true in general. A sequence of deterministic row-vectors $S_0, S_1, \ldots, S_n$ is a *derivation* iff there exist $x_1, \ldots, x_n \in X$ such that $S_0 \odot x_1 = S_1, \ldots, S_{n-1} \odot x_n = S_n$. The *length* of this derivation is n. If $u = x_1 \cdot x_2 \cdot \ldots \cdot x_n$, we call $S_0, S_1, \ldots, S_n$ the derivation *associated* with (S, u). We denote this derivation by $S_0 \xrightarrow{u} S_n$.

A derivation $S_0, S_1, \ldots, S_n$ is said to be *stacking* iff it is the derivation associated to a pair (S, u) such that $S = S_0$ and $S_0 \uparrow (u)S_n$. A derivation $S_0, S_1, \ldots, S_n$ is said to be a *subderivation* of a derivation $S'_0, S'_1, \ldots, S'_m$ iff there exists some $i \in [0, m]$ such that, $\forall j \in [1, n]$, $S_j = S'_{i+j}$.

DEFINITION 3.27. *A vector $S \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle$ is said to be* loop-free *iff for every $v \in V^+$, $S \bullet v \neq S$.*

Let us note that every polynomial is loop-free. The following two lemmas give other examples of loop-free vectors.

LEMMA 3.28. *Let $\alpha \in \mathsf{DB}_{1,n}\langle V\rangle$, $\Phi \in \mathsf{B}_{n,\lambda}\langle\langle\, V \,\rangle\rangle$, such that $\infty > \|\alpha \cdot \Phi\| > \|\Phi\|$. Then $\alpha \cdot \Phi$ is loop-free.*

Proof. Let α, Φ fulfill the hypothesis of the lemma and suppose, for the sake of contradiction, that there exists some $v \in V^+$ such that

$$(\alpha \cdot \Phi) \bullet v = \alpha \cdot \Phi.$$

By induction, for every $n \geq 0$,

$$(\alpha \cdot \Phi) \bullet v^n = \alpha \cdot \Phi. \tag{3.17}$$

As α is a polynomial, there exists some $n_0 \geq 0$ such that $|v^{n_0}|$ is greater than the greatest length of a monomial of α. Using Lemma 3.10, equality (3.17) for such an integer n_0 means that there exists some $k \in [1, n]$, v'' suffix of v^{n_0} such that

$$\Phi_k \bullet v'' = \alpha \cdot \Phi. \tag{3.18}$$

Using the hypothesis of the lemma, we conclude that

$$\|\Phi\| \geq \|\Phi_k \bullet v''\| = \|\alpha \cdot \Phi\| > \|\Phi\|,$$

which is contradictory. $\square$

LEMMA 3.29. *Let $S \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle$, $u \in X^*$, such that $\|S \odot u\| > \|S\|$. Then $S \odot u$ is loop-free.*

Proof. Let us consider S, u fulfilling the hypothesis of the lemma and let us consider the three possible forms of $S \odot u$ proposed by Lemma 3.14. The forms (1) or (2) are incompatible with the inequality $\|S \odot u\| > \|S\|$. Hence $S \odot u$ has the form (3):

$$u = u_1 \cdot u_2, \quad S \odot u_1 = S \bullet v_1 = \sum_{k=1}^{q} E_k \cdot \Phi_k, \quad S \odot u = \sum_{k=1}^{q} (E_k \odot u_2) \cdot \Phi_k, \quad \text{and}$$

$$\forall k \in [1, q], E_k \smile E_1, E_k \odot u_2 \notin \{\epsilon, \emptyset\}.$$

Hence $S \odot u = \alpha \cdot \Phi$ for some polynomial $\alpha \in \mathsf{DRB}_{1,q}\langle V\rangle$. As for every k, $\Phi_k = S \bullet (v_1 E_k)$, we obtain that $\|S\| \geq \|\Phi\|$. Finally

$$\infty > \|S \odot u\| = \|\alpha \cdot \Phi\| > \|S\| \geq \|\Phi\|,$$

and by Lemma 3.28, $S \odot u$ is loop-free. $\square$

LEMMA 3.30. *Let* $S \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$, $w \in X^*$, *such that*

(1) S *is loop-free,*

(2) $\forall u \preceq w$, $\|S \odot u\| \geq \|S\|$.

Then the derivation $S \xrightarrow{w} S \odot w$ *is stacking.*

Proof. S is left-deterministic. If it has type $\emptyset$ or (ϵ, j), the lemma is trivially true. Otherwise

$$S = \sum_{k=1}^{q} E_k \cdot \Phi_k$$

for some class of letter $[E_1]_{\smile} = \{E_1, \ldots, E_q\}$ and some matrix $\Phi \in \mathsf{DRB}_{q,\lambda}\langle\langle V \rangle\rangle$. Suppose that for some prefix $u \preceq w$ and $k \in [1, q]$,

$$E_k \odot u = \epsilon. \tag{3.19}$$

Then $S \odot u = \Phi_k$ so that $\|S \odot u\| \leq \|\Phi\| \leq \|S\|$, which shows that $S = S \odot u$ while $u \neq \epsilon$. This would contradict the hypothesis that S is loop-free; hence (3.19) is impossible.

Let us apply now Lemma 3.15 to the expression $(E \cdot \Phi) \odot w$: case (2) is impossible, and hence

$$(E \cdot \Phi) \odot w = (E \odot w) \cdot \Phi,$$

which is equivalent to

$$S \uparrow (w) S \odot w. \qquad \square$$

LEMMA 3.31. *Let* $S \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$, $w \in X^*$, $k \in \mathbb{N}$, *such that*

$$\|S \odot w\| \geq \|S\| + k \cdot K_0 + 1.$$

Then the derivation $S \xrightarrow{w} S \odot w$ *contains some stacking subderivation of length* k.

Sketch of proof. Let $S = S_0, \ldots, S_i, \ldots, S_n$ be the derivation associated to (S, w). Let $i_0 = \max\{i \in [0, n] \mid \|S_i\| = \min\{\|S_j\| \mid 0 \leq j \leq n\}\}$ and $i_1 = \max\{i \in [i_0 + 1, n] \mid \|S_i\| = \min\{\|S_j\| \mid i_0 + 1 \leq j \leq n\}\}$. Let $w = w_0 w_1 w'$, where $|w_0| = i_0$, $|w_0 w_1| = i_1$.

As $\|S \odot w_0 w_1\| > \|S \odot w_0\|$, by Lemma 3.29 $S \odot w_0 w_1 = S_{i_1}$ is loop-free. Using Lemma 3.13,

$$\|S_n\| - \|S_{i_1}\| \geq \|S_n\| - \|S_{i_0}\| - (\|S_{i_1}\| - \|S_0\|) \geq (k - 1) \cdot K_0 + 1.$$

Using Lemma 3.13 we must have $|w'| \geq k$. Let $w' = w_2 w_3$ with $|w_2| = k$. By definition of i_1, $\forall i \in [i_1 + 1, i_1 + k]$, $\|S_i\| \geq \|S_{i_1}\| + 1$.

By Lemma 3.30, the subderivation $S_{i_1}, \ldots, S_{i_1+k}$ (associated to (S_{i_1}, w_2)) is stacking. $\square$

LEMMA 3.32. *Let* $S, S' \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$, $w \in X^*$, $k, d, d' \in \mathbb{N}$, *such that* S *is* d*-marked and*

(1) *the derivation* $S \xrightarrow{w} S'$ *contains no stacking subderivation of length* k,

(2) $|w| \geq d \cdot k$.

Then S' *is unmarked.*

Proof. By the hypothesis,

$$S = \sum_{k=1}^{q} \alpha_k \cdot \Phi_k$$

for some $\alpha \in \mathsf{DRB}_{1,q}\langle V \rangle$, $\Phi \in \mathsf{DRB}_{q,\lambda}\langle\langle V \rangle\rangle$, $\|\alpha\| \leq d$, Φ unmarked.

Let $S \xrightarrow{w} S' = (S_0, \ldots, S_n)$. By induction on ℓ, using hypothesis (1) and Lemma 3.30 (on polynomials, which are particular cases of loop-free series) one can show that for every $\ell \in [0, d]$, there exists some prefix w_ℓ of w, with length $|w_\ell| \le k \cdot \ell$, such that either

$$S \odot w_\ell = \sum_{k=1}^{q} (\alpha_k \odot w_\ell) \cdot \Phi_k \text{ with } \|\alpha_\odot w_\ell\| < \|\alpha\| - \ell \tag{3.20}$$

or there exists an integer $k \in [1, q]$ such that

$$S \odot w_\ell = \Phi_k. \tag{3.21}$$

Let us apply this property to $\ell = d$: inequality (3.20) is not possible for this value of ℓ because, by hypothesis (2) of the lemma, $\|\alpha\| - \ell \le 0$. Hence (3.21) is true and, as Φ is unmarked, Φ_k is unmarked, so that $S \odot w$ is unmarked. ▯

4. Deduction systems.

4.1. General formal systems.

We follow here the general philosophy of [16, 9]. Let us call a *formal system* any triple $\mathcal{D} = \langle \mathcal{A}, H, \vdash\!\!- \rangle$ where $\mathcal{A}$ is a denumerable set called the *set of assertions*, H, the *cost function*, is a mapping $\mathcal{A} \to \mathbb{N} \cup \{\infty\}$, and $\vdash\!\!-$, the *deduction relation*, is a subset of $\mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$; $\mathcal{A}$ is given with a fixed bijection with $\mathbb{N}$ (an "encoding" or "Gödel numbering") so that the notions of recursive subset, recursively enumerable subset, recursive function, over $\mathcal{A}, \mathcal{P}_f(\mathcal{A})$, are defined, up to this fixed bijection; we assume that $\mathcal{D}$ satisfies the following axioms:

(A1) $\forall (P, A) \in \vdash\!\!-$, $(\min\{H(p), p \in P\} < H(A))$ or $(H(A) = \infty)$.

(We let $\min(\emptyset) = \infty$.) We call $\mathcal{D}$ a *deduction system* iff $\mathcal{D}$ is a formal system satisfying the additional axiom:

(A2) $\vdash\!\!-$ is recursively enumerable.

In what follows we use the notation $P \vdash\!\!- A$ for $(P, A) \in \vdash\!\!-$. We call a *proof* in the system $\mathcal{D}$, *relative to the set of hypotheses* $\mathcal{H} \subseteq \mathcal{A}$, any subset $P \subseteq \mathcal{A}$ fulfilling

$$\forall p \in P,\ (\exists Q \subseteq P, Q \vdash\!\!- p) \text{ or } (p \in \mathcal{H}).$$

We call P a *proof* iff

$$\forall p \in P, (\exists Q \subseteq P, Q \vdash\!\!- p)$$

(i.e., iff P is a proof relative to $\emptyset$).

Let us define the total map $\chi : \mathcal{A} \to \{0, 1\}$ and the partial map $\overline{\chi} : \mathcal{A} \to \{0, 1\}$ by

$$\chi(A) = 1 \text{ if } H(A) = \infty, \quad \chi(A) = 0 \text{ if } H(A) < \infty,$$

$$\overline{\chi}(A) = 1 \text{ if } H(A) = \infty, \quad \overline{\chi} \text{ is undefined if } H(A) < \infty.$$

(χ is the "truth-value function"; $\overline{\chi}$ is the "1-value function.")

LEMMA 4.1. *Let P be a proof relative to $\mathcal{H} \subseteq H^{-1}(\infty)$ and $A \in P$. Then $\chi(A) = 1$.*

In other words, if an assertion is provable from true hypotheses, then it is true.

Proof. Let P be a proof. We prove by induction on n that

$$\mathcal{P}(n) : \forall p \in P, H(p) \ge n.$$

It is clear that, $\forall p \in P$, $H(p) \geq 0$. Suppose that $\mathcal{P}(n)$ is true. Let $p \in P - \mathcal{H} : \exists Q \subseteq P$, $Q \vdash p$. By the induction hypothesis, $\forall q \in Q$, $H(q) \geq n$ and by (A1), $H(p) \geq n+1$. It follows that, $\forall p \in P - \mathcal{H}$, $H(p) = \infty$. But by the hypothesis, $\forall p \in \mathcal{H}$, $H(p) = \infty$. $\square$

A formal system $\mathcal{D}$ will be *complete* iff, conversely, $\forall A \in \mathcal{A}$, $\chi(A) = 1 \Longrightarrow$ there exists some *finite* proof P such that $A \in P$. (In other words, $\mathcal{D}$ is complete iff every true assertion is "finitely" provable.)

LEMMA 4.2. *If $\mathcal{D}$ is a complete deduction system, $\overline{\chi}$ is a recursive partial map.*

Proof. Let $i \mapsto P_i$ be some recursive function whose domain is $\mathbb{N}$ and whose image is $\mathcal{P}_f(\mathcal{A})$. Let $h : (\mathcal{P}_f(\mathcal{A}) \times \mathcal{A} \times \mathbb{N}) \to \{0,1\}$ be a total recursive function such that

$$P \vdash A \text{ iff } \exists n \in \mathbb{N},\, h(P, A, n) = 1$$

(such an h exists, because the recursively enumerable sets are the projections of the recursive sets; see [30]).

The following (informal) semialgorithm computes $\overline{\chi}$ on the assertion A:

1. i `:=` 0 `;` n `:=` 0 `;` s `:=` $i+n$`;`
2. P `:=` P_i`;`
3. b `:=` $\min_{p \in P}\{\max_{Q \subseteq P}\{h(Q, p, n)\}\}$`;`
4. c `:=` $(A \in P)$`;`
5. `if` $(b \wedge c)$ `then` $(\overline{\chi}(A) = 1$ `; stop);`
6. `if` $i = 0$ `then` $(i := s+1$ `;` $n := 0$`;` $s := i+n)$
   `else` $(i := i-1$ `;` $n := n+1)$`;`
7. `goto 2;` $\square$

In order to define deduction relations from more elementary ones, we set the following definitions.

Let $\vdash \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$. For every $P, Q \in \mathcal{P}_f(\mathcal{A})$ we set

- $P \overset{[0]}{\vdash} Q$ iff $P \supseteq Q$,
- $P \overset{[1]}{\vdash} Q$ iff $\forall q \in Q$, $\exists R \subseteq P$, $R \vdash q$,
- $P \overset{\langle 0 \rangle}{\vdash} Q$ iff $P \overset{[0]}{\vdash} Q$,
- $P \overset{\langle 1 \rangle}{\vdash} Q$ iff $\forall q \in Q$, $(\exists R \subseteq P,\, R \vdash q)$ or $(q \in P)$,
- $P \overset{\langle n+1 \rangle}{\vdash} Q$ iff $\exists R \in \mathcal{P}_f(A)$, $P \overset{\langle 1 \rangle}{\vdash} R$ and $R \overset{\langle n \rangle}{\vdash} Q$ (for every $n \geq 0$),
- $\overset{\langle * \rangle}{\vdash} = \bigcup_{n \geq 0} \overset{\langle n \rangle}{\vdash}$.

Given $\vdash_1, \vdash_2 \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{P}_f(\mathcal{A})$, for every $P, Q \in \mathcal{P}_f(\mathcal{A})$ we set

$$P(\vdash_1 \circ \vdash_2)Q \text{ iff } \exists R \subseteq \mathcal{A},\, (P \vdash_1 R) \wedge (R \vdash_2 Q).$$

4.2. Strategies. We define here a notion of strategy which consists of a map allowing us to build proofs within a formal system $\mathcal{D}$. This notion will be an essential tool for proving that a formal system is *complete*. Let $\mathcal{D} = \langle \mathcal{A}, H, \vdash \rangle$ be a formal system. We call a *strategy* for $\mathcal{D}$ any map $\mathcal{S} : \mathcal{A}^+ \to \mathcal{P}(\mathcal{A}^*)$ such that

(S1) if $B_1 \cdots B_m \in \mathcal{S}(A_1 A_2 \cdots A_n)$, then $\exists Q \subseteq \{A_i \mid 1 \leq i \leq n-1\}$ such that

$$\{B_j \mid 1 \leq j \leq m\} \cup Q \vdash A_n;$$

(S2) if $B_1 \cdots B_m \in \mathcal{S}(A_1 A_2 \cdots A_n)$, then

$$\min\{H(A_i) \mid 1 \leq i \leq n\} = \infty \Longrightarrow \min\{H(B_j) \mid 1 \leq j \leq m\} = \infty.$$

REMARK 4.3. *It may happen that* $\epsilon \in \mathcal{S}(A_1 A_2 \cdots A_n)$ *(and, correspondingly, that* $m = 0$ *in the above conditions): it just means that* $\{A_1, \ldots, A_{n-1}\} \mid\!\!-\!\!- A_n$. *It may also happen that* $\mathcal{S}(A_1 A_2 \cdots A_n) = \emptyset$*: this means, intuitively, that* $\mathcal{S}$ *"does not know" how to extend a proof (with hypothesis), with the only information being that the given proof contains the assertions* $A_1, A_2, \ldots, A_n$.

REMARK 4.4. *Axiom* (A1) *on systems is similar to the "monotonicity" condition of* [16] *or axiom* (2.4.2′) *of* [9]. *Axiom* (S2) *on strategies is similar to the "validity" condition of* [16] *or property* (2.4.1′) *of* [9].

Given a strategy $\mathcal{S}$, we define $\mathcal{T}(\mathcal{S}, A)$, the set of proof-trees associated to the strategy $\mathcal{S}$ and the assertion A, as the set of all trees t fulfilling the following properties:

$$\varepsilon \in \mathrm{dom}(t), \quad t(\varepsilon) = A, \tag{4.1}$$

and, for every path $x_0 x_1, \ldots, x_{n-1}$ in t, with labels $t(x_i) = A_{i+1}$ (for $0 \le i \le n-1$), if x_{n-1} has m sons $x_{n-1} \cdot 1, \ldots, x_{n-1} \cdot m \in \mathrm{dom}(t)$ with labels $t(x_{n-1} \cdot j) = B_j$ (for $1 \le j \le m$), then

$$(B_1 \cdots B_m) \in \mathcal{S}(A_1 \cdots A_n) \quad \text{or} \quad m = 0. \tag{4.2}$$

The proof-tree t is said to be *closed* iff it fulfills the following additional condition: for every path $x_0 x_1, \ldots, x_{n-1}$ in t, with labels $t(x_i) = A_{i+1}$ (for $0 \le i \le n-1$), if x_{n-1} has m sons $x_{n-1} \cdot 1, \ldots, x_{n-1} \cdot m \in \mathrm{dom}(t)$ with labels $t(x_{n-1} \cdot j) = B_j$ (for $1 \le j \le m$), then

$$m = 0 \Rightarrow ((\exists i \in [1, n-1],\, A_i = A_n) \text{ or } (\varepsilon \in \mathcal{S}(A_1 \cdots A_n))). \tag{4.3}$$

A node $x \in \mathrm{dom}(t)$ is said to be *closed* iff it is an internal node or it is a leaf fulfilling property (4.3) above.

The proof-tree t is said to be *repetition-free* iff, for every $x, x' \in \mathrm{dom}(t)$,

$$[x \preceq x' \text{ and } t(x) = t(x')] \quad \Rightarrow \quad x = x' \text{ or } x' \text{ is a leaf.}$$

For every tree t let us define

$$\mathcal{L}(t) = \{t(x) \mid \forall y \in \mathrm{dom}(t), x \preceq y \Rightarrow x = y\}, \quad \mathcal{I}(t) = \{t(x) \mid \exists y \in \mathrm{dom}(t), x \prec y\}.$$

(Here $\mathcal{L}$ stands for "leaves" and $\mathcal{I}$ stands for "internal nodes.")

LEMMA 4.5. *If* $\mathcal{S}$ *is a strategy for the deduction-system* $\mathcal{D}$*, then, for every true assertion* A *and every* $t \in \mathcal{T}(\mathcal{S}, A)$*,*

(1) *the set of labels of* t *is a* $\mathcal{D}$*-proof, relative to the set* $\mathcal{L}(t) - \mathcal{I}(t)$*;*

(2) *every label of a leaf is true.*

Proof. Let us suppose that $H(A) = \infty$. Let $t \in \mathcal{T}(\mathcal{S}, A)$, $P = \mathrm{im}(t)$ (the set of labels of t), $\mathcal{H} = \mathcal{L}(t) - \mathcal{I}(t)$.

Using (S2), one can prove by induction on the depth of $x \in \mathrm{dom}(t)$ that $H(t(x)) = \infty$. Point (2) is then proved. Let x be an internal node of t, with sons $x \cdot 1, x \cdot 2, \ldots, x \cdot m$ ($m \ge 1$), and with ancestors $y_1, y_2, \ldots, y_{n-1}, y_n = x$ ($n \ge 1$), such that

$$t(y_1) \cdots t(y_n) = A_1 \cdots A_n, \quad t(x_1) \cdots t(x_m) = B_1 \cdots B_m.$$

By definition of $\mathcal{T}(\mathcal{S}, A)$,

$$B_1 \cdots B_m \in \mathcal{S}(A_1 \cdots A_n),$$

and by condition (S1),

$$\exists Q \subseteq \{A_i \mid i \leq n-1\}, \text{ such that } \{B_j \mid 1 \leq j \leq m\} \cup Q \mathrel{|\!\!-\!\!-} A_n.$$

It follows that for every $p \notin \mathcal{H}$, $\exists R \subseteq P$, $R \mathrel{|\!\!-\!\!-} p$; hence

$$\forall p \in P, \quad (\exists R \subseteq P,\ R \mathrel{|\!\!-\!\!-} p) \text{ or } p \in \mathcal{H}.$$

Point (1) is proved. ∎

For every $\mathcal{D}$-strategy $\mathcal{S}$, we use the notation

$$\mathcal{T}(\mathcal{S}) = \bigcup_{A \in H^{-1}(\infty)} \mathcal{T}(\mathcal{S}, A).$$

We call a *global strategy* w.r.t. $\mathcal{S}$ any total map $\hat{\mathcal{S}} : \mathcal{T}(\mathcal{S}) \to \mathcal{T}(\mathcal{S})$ such that

$$\forall t \in \mathcal{T}(\mathcal{S}), \quad t \preceq \hat{\mathcal{S}}(t). \tag{4.4}$$

$\hat{\mathcal{S}}$ is a *terminating* global strategy iff

$$\forall A_0 \in H^{-1}(\infty),\ \exists n_0 \in \mathbb{N}, \quad \hat{\mathcal{S}}^{n_0}(A_0) = \hat{\mathcal{S}}^{n_0+1}(A_0), \tag{4.5}$$

and $\hat{\mathcal{S}}$ is a *closed* global strategy iff

$$\forall A_0 \in H^{-1}(\infty), \forall n \in \mathbb{N}, \quad \hat{\mathcal{S}}^{n}(A_0) \text{ is closed} \iff \hat{\mathcal{S}}^{n}(A_0) = \hat{\mathcal{S}}^{n+1}(A_0) \tag{4.6}$$

(where the assertion A_0 is identified with the tree reduced to one node whose label is A_0).

LEMMA 4.6. *Let $\mathcal{D}$ be a formal system, $\mathcal{S}$ a strategy for $\mathcal{D}$, and $\hat{\mathcal{S}}$ a global strategy w.r.t. $\mathcal{S}$. If $\hat{\mathcal{S}}$ is terminating and $\hat{\mathcal{S}}$ is closed, then $\mathcal{D}$ is complete.*

Proof. Let $A_0 \in \mathcal{A}$. Under the hypothesis of the lemma, $\exists n_0 \in \mathbb{N}$ such that (4.5) and (4.6) are both true. Hence $t_\infty = \hat{\mathcal{S}}^{n_0}(A_0)$ is a closed proof-tree for $\mathcal{S}$. By Lemma 4.5 $\mathrm{im}(t_\infty)$ is a $\mathcal{D}$-proof relative to the set $\mathcal{L}(t_\infty) - \mathcal{I}(t_\infty)$. Let x be a leaf such that $t_\infty(x) \in \mathcal{L}(t_\infty) - \mathcal{I}(t_\infty)$. Let $A_0, A_1, \ldots, A_n = t_\infty(x)$ be the word labeling the path from the root to x. As x is closed and $t_\infty(x) \in \mathcal{L}(t_\infty) - \mathcal{I}(t_\infty)$ by (4.3), $\varepsilon \in \mathcal{S}(A_1 \cdots A_n)$; hence $\{A_1, \ldots, A_{n-1}\} \mathrel{|\!\!-\!\!-} t_\infty(x)$. It follows that $\mathrm{im}(t_\infty)$ is a $\mathcal{D}$-proof. ∎

4.3. System $\mathcal{B}_0$. Let us define here a particular formal system $\mathcal{B}_0$ "tailored for the σ-$\bar{\psi}$-bisimulation problem for deterministic series."

Let us fix two finite alphabets X, Y, a surjection $\psi : X \to Y$ (which induces a surjection $X^* \to Y^*$ denoted by the same symbol ψ), and its kernel $\bar{\psi} = \mathrm{Ker}\,\psi \subseteq X^* \times X^*$ (see section 3.2). We also fix a dpda $\mathcal{M}$ over the terminal alphabet X and consider the variable alphabet V associated to $\mathcal{M}$ (see section 3.1) and the sets $\mathsf{DRB}_{\delta,\lambda}\langle\langle\, V \,\rangle\rangle$ (the sets of deterministic rational boolean matrices over V^*, with δ rows and λ columns). The set of assertions is defined by

$$\mathcal{A} = \bigcup_{\lambda \geq 1} \mathbb{N} \times \mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle \times \mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle;$$

i.e., an assertion is here a *weighted equation* over $\mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle$ for some integer λ.

For every $n \geq 0$ we define

$$\bar{\mathcal{B}}_n = \{\mathcal{R} \subseteq \bar{\psi} \mid \mathcal{R} \text{ fulfills conditions } (1'), (2'), \text{ and } (4) \text{ of Definition 3.21}\}. \tag{4.7}$$

We call the elements of $\bar{\mathcal{B}}_n$ the *admissible* relations of order n over $X^* \times X^*$. For every pair $(S, S') \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle \times \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$, and $n \in \mathbb{N} \cup \{\infty\}$, we define

$$\mathcal{B}_n(S, S') = \{\mathcal{R} \subseteq \bar{\psi} \mid \mathcal{R} \text{ is a } w\text{-}\bar{\psi}\text{-bisimulation of order } n \text{ w.r.t. } (S, S')\}. \tag{4.8}$$

The "cost-function" $H : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ is defined by

$$H(n, S, S') = n + 2 \cdot \mathrm{Div}(S, S'),$$

where $\mathrm{Div}(S, S')$ is the *divergence* between S and S' (Definition 3.23). We recall that it is defined by

$$\mathrm{Div}(S, S') = \inf\{n \in \mathbb{N} \mid \mathcal{B}_n(S, S') = \emptyset\}.$$

(We recall $\inf(\emptyset) = \infty$.)

Let us note that, by Lemma 3.22,

$$\chi(n, S, S') = 1 \Longleftrightarrow S \sim S'.$$

We define a binary relation $|\!\!\vdash\!\!- \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$, the *elementary deduction relation*, as the set of all the pairs having one of the following forms:

(R0)

$$\{(p, S, T)\} \,|\!\!\vdash\!\!-\, (p+1, S, T)$$

for $p \in \mathbb{N}$, $\lambda \in \mathbb{N} - \{0\}$, $S, T \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$;

(R1)

$$\{(p, S, T)\} \,|\!\!\vdash\!\!-\, (p, T, S)$$

for $p \in \mathbb{N}$, $\lambda \in \mathbb{N} - \{0\}$, $S, T \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$;

(R2)

$$\{(p, S, S'), (p, S', S'')\} \,|\!\!\vdash\!\!-\, (p, S, S'')$$

for $p \in \mathbb{N}$, $\lambda \in \mathbb{N} - \{0\}$, $S, T \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$;

(R3)

$$\emptyset \,|\!\!\vdash\!\!-\, (0, S, S)$$

for $S \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$;

(R′3)

$$\emptyset \,|\!\!\vdash\!\!-\, (0, S, \rho_e(S))$$

for $S \in \mathsf{DRB}_{1,1}\langle\langle V \rangle\rangle$;

(R4)

$$\{(p+1, S \odot x, T \odot x') \mid (x, x') \in \mathcal{R}_1\} \,|\!\!\vdash\!\!-\, (p, S, T)$$

for $p \in \mathbb{N}$, $\lambda \in \mathbb{N} - \{0\}$, $S, T \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$, $(S \neq \epsilon \wedge T \neq \epsilon)$, and $\mathcal{R}_1 \in \bar{\mathcal{B}}_1$;

(R5)

$$\{(p, S, S')\} \,|\!\!\vdash\!\!-\, (p+2, S \odot x, S' \odot x')$$

for $p \in \mathbb{N}$, $\lambda \in \mathbb{N} - \{0\}$, $S, T \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$, $(x, x') \in \bar{\psi}$, $S \sim S' \wedge S \odot x \sim S' \odot x'$;

(R6)

$$\{(p, S_1 \cdot T + S, T)\} \,||\!\!-\!\!- (p, S_1^* \cdot S, T)$$

for $p \in \mathbb{N}$, $\lambda \in \mathbb{N}-\{0\}$, $S_1 \in \mathsf{DRB}_{1,1}\langle\langle V \rangle\rangle$, $S_1 \neq \epsilon$, $(S_1, S) \in \mathsf{DRB}_{1,\lambda+1}\langle\langle V \rangle\rangle$, $T \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$;

(R7)

$$\{(p, S, S')\} \,||\!\!-\!\!- (p, S \cdot T, S' \cdot T)$$

for $p \in \mathbb{N}$, $\delta, \lambda \in \mathbb{N} - \{0\}$, $S, S' \in \mathsf{DRB}_{1,\delta}\langle\langle V \rangle\rangle$, $T \in \mathsf{DRB}_{\delta,\lambda}\langle\langle V \rangle\rangle$;

(R8)

$$\{(p, T_{i,*}, T'_{i,*}) \mid 1 \leq i \leq \delta\} \,||\!\!-\!\!- (p, S \cdot T, S \cdot T')$$

for $p \in \mathbb{N}$, $\delta, \lambda \in \mathbb{N} - \{0\}$, $S \in \mathsf{DRB}_{1,\delta}\langle\langle V \rangle\rangle$, $T, T' \in \mathsf{DRB}_{\delta,\lambda}\langle\langle V \rangle\rangle$.

REMARK 4.7. *We do not claim that this formal system is* recursively enumerable*: due to rule* (R5)*, establishing this property is as difficult as solving the general bisimulation problem for equational graphs of finite out-degree. This difficulty will be overcome in section* 10 *by an* elimination *lemma.*

LEMMA 4.8. *Let $P \in \mathcal{P}_f(\mathcal{A})$, $A \in \mathcal{A}$, such that $P \,||\!\!-\!\!- A$. Then* $\min\{H(p) \mid p \in P\} \leq H(A)$.

Let us introduce the following notation: for every $n \in \mathbb{N} \cup \{\infty\}$, $\lambda \in \mathbb{N} - \{0\}$, $S, S' \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$,

$$S \sim_n S' \Leftrightarrow \mathcal{B}_n(S, S') \neq \emptyset.$$

Proof. Let us check this property for every type of rule.

(R0) $p + 2 \cdot \mathrm{Div}(S,T) \leq p + 1 + 2 \cdot \mathrm{Div}(S,T)$.

(R1) $p + 2 \cdot \mathrm{Div}(S,T) = p + 2 \cdot \mathrm{Div}(T,S)$.

(R2) As the weight p is the same in all the considered equations, we are reduced to proving that $\forall n \in \mathbb{N}$, $S \sim_n S' \wedge S' \sim_n S'' \Longrightarrow S \sim_n S''$. This is true because, if $\mathcal{R} \in \mathcal{B}_n(S,S')$ and $\mathcal{R}' \in \mathcal{B}_n(S',S'')$, then $\mathcal{R} \circ \mathcal{R}' \in \mathcal{B}_n(S,S'')$.

(R3) Let us note that $\mathrm{Id}_{X^*} \subseteq \bar{\psi}$. It follows that $\infty = \mathrm{Div}(S,S)$.

(R′3) The definition of G from G_0 is such that $S \equiv \rho_e(S)$; hence $S \sim \rho_e(S)$ and $\infty = \mathrm{Div}(S, \rho_e(S))$.

(R4) Let $n \in \mathbb{N}$ such that

$$\forall (x,x') \in \mathcal{R}_1, \quad n \leq \mathrm{Div}(S \odot x, S' \odot x').$$

Let us choose, for every $(x,x') \in \mathcal{R}_1$, some $\mathcal{R}_{x,x'} \in \mathcal{B}_n(S \odot x, S' \odot x')$. Let us then define

$$\mathcal{R} = \bigcup_{(x,x') \in \mathcal{R}_1} (x,x') \cdot \mathcal{R}_{x,x'}.$$

$\mathcal{R}$ belongs to $\mathcal{B}_{n+1}(S,S')$. It follows that

$$\min\{\mathrm{Div}(S \odot x, S' \odot x') \mid (x,x') \in \mathcal{R}_1\} + 1 \leq \mathrm{Div}(S,S'),$$

and hence that

$$\min\{H(p+1, S \odot x, S' \odot x') \mid (x,x') \in \mathcal{R}_1\} \leq H(p,S,S') - 1.$$

(R5) By the hypothesis, $H(p+2, S \odot x, S' \odot x') = \infty$.

(R6) Let $n \in \mathbb{N}$ such that

$$n \leq \mathrm{Div}(S_1 \cdot T + S, T).$$

Let $\mathcal{R} \in \mathcal{B}_n(S_1 \cdot T + S, T)$. Let $\mathcal{R}' = \mathcal{R}^{\langle S_1, * \rangle}$ (see definition (3.14) in section 3.2). Since we have

$$\mathcal{R}' \in \mathcal{B}_n(S_1^* \cdot S, T),$$

we get the inequality $\mathrm{Div}(S_1 \cdot T + S, T) \leq \mathrm{Div}(S_1^* \cdot S, T)$.

(R7) Let $n \leq \mathrm{Div}(S, S')$ and $\mathcal{R} \in \mathcal{B}_n(S, S')$. Let us consider $\mathcal{R}' = \langle S | \mathcal{R} \rangle$ (see definition (3.8) in section 3.2). Since we have $\mathcal{R}' \in \mathcal{B}_n(S \cdot T, S' \cdot T)$, the required inequality is proved.

(R8) Let $n \leq \min\{\mathrm{Div}(T_{i,*}, T'_{i,*}) \mid 1 \leq i \leq \delta\}$ and, for every $i \in [1, \delta]$, let $\mathcal{R}_i \in \mathcal{B}_n(T_{i,*}, T'_{i,*})$. Let us consider $\mathcal{R}' = \langle S, \mathcal{R} \rangle$ (see definition (3.9) in section 3.2). Since we know that

$$\mathcal{R}' \in \mathcal{B}_n(S \cdot T, S \cdot T'),$$

the required inequality is proved. ◻

Let us define $\vdash\!\!-$ as follows: for every $P \in \mathcal{P}_f(\mathcal{A})$, $A \in \mathcal{A}$,

$$P \vdash\!\!- A \Longleftrightarrow P \overset{\langle * \rangle}{||\!\!-\!\!-} \circ \overset{[1]}{||\!\!-\!\!-}_{0,3,4} \circ \overset{\langle * \rangle}{||\!\!-\!\!-} \{A\},$$

where $||\!\!-\!\!-_{0,3,4}$ is the relation defined by R_0, R_3, R'_3, R_4 only. We let

$$\mathcal{B}_0 = \langle \mathcal{A}, H, \vdash\!\!- \rangle.$$

LEMMA 4.9. *$\mathcal{B}_0$ is a formal system.*

Proof. Using Lemma 4.8, one can show by induction on n that

$$P \overset{\langle n \rangle}{||\!\!-\!\!-} Q \Longrightarrow \forall q \in Q, \quad \min\{H(A) \mid A \in P\} \leq H(q).$$

The proof of Lemma 4.8 also reveals that

$$P \,||\!\!-\!\!-_{\{0,3,4\}} q \Longrightarrow (\min\{H(p) \mid p \in P\} < H(q)) \quad \text{or} \quad H(q) = \infty.$$

It follows that, for every $m, n \geq 0$,

$$P \overset{\langle n \rangle}{||\!\!-\!\!-} Q \overset{[1]}{||\!\!-\!\!-}_{0,3,4} R \overset{\langle m \rangle}{||\!\!-\!\!-} q \Longrightarrow (\min\{H(p) \mid p \in P\} < H(q)) \quad \text{or} \quad H(q) = \infty.$$

Hence axiom (A1) is fulfilled. ◻

Let us note the following algebraic corollaries of Lemma 4.8.

COROLLARY 4.10.

(C1) $\forall \lambda \in \mathbb{N} - \{0\}$, $S_1 \in \mathsf{DRB}_{1,1}\langle\langle\, V \,\rangle\rangle$, $S_1 \not\equiv \epsilon$, $(S_1, S) \in \mathsf{DRB}_{1,\lambda+1}\langle\langle\, V \,\rangle\rangle$, $T \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle$,

$$S_1 \cdot T + S \sim T \Longrightarrow S_1^* \cdot S \sim T.$$

(C2) $\forall \delta, \lambda \in \mathbb{N} - \{0\}$, $S, S' \in \mathsf{DRB}_{1,\delta}\langle\langle\, V \,\rangle\rangle$, $T \in \mathsf{DRB}_{\delta,\lambda}\langle\langle\, V \,\rangle\rangle$,

$$S \sim S' \Longrightarrow S \cdot T \sim S' \cdot T.$$

(C3) $\forall \lambda \in \mathbb{N} - \{0\},\ S, S' \in \mathsf{DRB}_{1,1}\langle\langle\ V\ \rangle\rangle,\ T \in \mathsf{DRB}_{1,\lambda}\langle\langle\ V\ \rangle\rangle,$

$$[S \cdot T \sim S' \cdot T \ \mathit{and}\ T \neq \emptyset^{\lambda}] \Longrightarrow S \sim S'.$$

(C4) $\forall \delta, \lambda \in \mathbb{N} - \{0\},\ S \in \mathsf{DRB}_{1,\delta}\langle\langle\ V\ \rangle\rangle,\ T, T' \in \mathsf{DRB}_{\delta,\lambda}\langle\langle\ V\ \rangle\rangle,$

$$T \sim T' \Longrightarrow S \cdot T \sim S \cdot T'.$$

Proof. Statement (Ci) (for $i \in \{1, 2, 4\}$) is a direct corollary of the fact that the value of H on the left-hand side of some rule (Rj) is smaller than or equal to the value of H on the right-hand side of rule (Rj): (C1) is justified by (R6), (C2) by (R7), (C4) by (R8).

Let us prove (C3): Suppose that $\lambda \in \mathbb{N} - \{0\}$, $S, S' \in \mathsf{DRB}_{1,1}\langle\langle\ V\ \rangle\rangle$, $T \in \mathsf{DRB}_{1,\lambda}\langle\langle\ V\ \rangle\rangle$, and

$$S \cdot T \sim S' \cdot T \quad \text{and} \quad S \not\sim S'. \tag{4.9}$$

Let $\mathcal{R} \in \mathcal{B}_{\infty}(S \cdot T, S' \cdot T)$ and let

$$(u, u') = \min\{(v, v') \in \mathcal{R} \mid (S \odot v = \epsilon) \Leftrightarrow (S' \odot v' \neq \epsilon)\}.$$

From the hypothesis that $\mathcal{R} \in \mathcal{B}_{\infty}(S \cdot T, S' \cdot T)$, we get that

$$\forall (v, v') \in \mathcal{R}, \quad (S \cdot T) \odot v \sim (S' \cdot T) \odot v',$$

and by the choice of (u, u') we obtain that

$$T \sim (S' \odot u') \cdot T \quad \text{or} \quad (S \odot u) \cdot T \sim T,$$

which, by (C1), implies

$$T \sim (S' \odot u')^{*} \cdot \emptyset^{\lambda} \quad \text{or} \quad (S \odot u)^{*} \cdot \emptyset^{\lambda} \sim T,$$

i.e., $T \sim \emptyset^{\lambda}$, which implies (because G is a reduced grammar) that

$$T = \emptyset^{\lambda}. \tag{4.10}$$

We have proved that (4.9) implies (4.10), and hence (C3). $\square$

4.4. Congruence closure. Let us consider the subset $\mathcal{C}$ of the rules of $\mathcal{B}_0$, consisting of all the instances of the metarules (R0), (R1), (R2), (R3), (R$'$3), (R6), (R7), (R8). We also denote by $\Vdash\!\!-_{\mathcal{C}} \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$ the set of all instances of these metarules. We are interested here (and later in section 10.1) in special subsets of $\mathcal{A}$ which express an ordinary weighted equation (p, S, S') together with an admissible binary relation $\mathcal{R}$ of finite order (which is a *candidate* to be a w-$\bar{\psi}$-bisimulation w.r.t. (S, S')).

For every $p, n \in \mathbb{N}$, $\lambda \in \mathbb{N} - \{0\}$, $S, S' \in \mathsf{DRB}_{1,\lambda}\langle\langle\ V\ \rangle\rangle$, $\mathcal{R} \in \bar{\mathcal{B}}_n$, we use the notation

$$[p, S, S', \mathcal{R}] = \{(p + |u|, S \odot u, S' \odot u') \mid (u, u') \in \mathcal{R}\}. \tag{4.11}$$

One can check the following properties.

Composition. For every $p, n \in \mathbb{N}$, $\lambda \in \mathbb{N} - \{0\}$, $S, T \in \mathsf{DRB}_{1,\lambda}\langle\langle\ V\ \rangle\rangle$, $\mathcal{R}_1, \mathcal{R}_2 \in \bar{\mathcal{B}}_n$,

$$[p, S, S', \mathcal{R}_1] \cup [p, S', S'', \mathcal{R}_2] \ \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathcal{C}} [p, S, S'', \mathcal{R}_1 \circ \mathcal{R}_2].$$

Star. For every $p, n \in \mathbb{N}$, $\lambda \in \mathbb{N} - \{0\}$, $S_1 \in \mathsf{DRB}_{1,1}\langle\langle\, V\,\rangle\rangle$, $S_1 \not\equiv \epsilon$, $(S_1, S) \in \mathsf{DRB}_{1,\lambda+1}\langle\langle\, V\,\rangle\rangle$, $T \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V\,\rangle\rangle$, $\mathcal{R} \in \bar{\mathcal{B}}_n$,

$$[p, S_1 \cdot T + S, T, \mathcal{R}] \ \overset{\langle * \rangle}{|\!\!\vdash\!\!-}_{\mathcal{C}} [p, S_1^* \cdot S, T, \mathcal{R}^{\langle S_1, * \rangle}].$$

Right-product. For every $p, n \in \mathbb{N}$, $\delta, \lambda \in \mathbb{N} - \{0\}$, $S, S' \in \mathsf{DRB}_{1,\delta}\langle\langle\, V\,\rangle\rangle$, $T \in \mathsf{DRB}_{\delta,\lambda}\langle\langle\, V\,\rangle\rangle$, $\mathcal{R} \in \bar{\mathcal{B}}_n$,

$$[p, S, S', \mathcal{R}] \ \overset{\langle * \rangle}{|\!\!\vdash\!\!-}_{\mathcal{C}} [p, S \cdot T, S' \cdot T, \langle S | \mathcal{R} \rangle].$$

Left-product. For every $p, n \in \mathbb{N}$, $\delta, \lambda \in \mathbb{N} - \{0\}$, $S \in \mathsf{DRB}_{1,\delta}\langle\langle\, V\,\rangle\rangle$, $T, T' \in \mathsf{DRB}_{\delta,\lambda}\langle\langle\, V\,\rangle\rangle$, $\mathcal{R}_1, \ldots, \mathcal{R}_\delta \in \bar{\mathcal{B}}_n$,

$$\bigcup_{1 \le i \le \delta} [p, T_{i,*}, T'_{i,*}, \mathcal{R}_i] \ \overset{\langle * \rangle}{|\!\!\vdash\!\!-}_{\mathcal{C}} [p, S \cdot T, S \cdot T', \langle S, \mathcal{R} \rangle].$$

Given a subset $P \in \mathcal{P}_f(\mathcal{A})$, we define the *congruence closure* of P, denoted by $\mathrm{Cong}(P)$, to be the set

$$\mathrm{Cong}(P) = \{A \in \mathcal{A} \mid P \ \overset{\langle * \rangle}{|\!\!\vdash\!\!-}_{\mathcal{C}} \{A\}\}. \tag{4.12}$$

Also, for every integer $q \geq 0$ we define

$$\mathrm{Cong}_q(P) = \{A \in \mathcal{A} \mid P \ \overset{\langle q \rangle}{|\!\!\vdash\!\!-}_{\mathcal{C}} \{A\}\}. \tag{4.13}$$

5. Deterministic spaces.

5.1. Linear independence.

We adapt here the key idea of [23, 24] to bisimulation of vectors.

Definitions. Let $(W, \sim)$ be some structured alphabet. A vector $U = \sum_{i=1}^n \gamma_i \cdot U_i$ where $\vec{\gamma} \in \mathsf{DRB}_{1,n}\langle\langle\, W\,\rangle\rangle$, $U_i \in \mathsf{DRB}_{1,\lambda}\langle\langle\, W\,\rangle\rangle$ is called a *linear combination* of the U_i's. We define as a *deterministic space* of rational vectors (d-space for short) any subset V of $\mathsf{DRB}_{1,\lambda}\langle\langle\, W\,\rangle\rangle$ which is closed under finite linear combinations. Given any set $\mathcal{G} = \{U_i \mid i \in I\} \subseteq \mathsf{DRB}_{1,\lambda}\langle\langle\, W\,\rangle\rangle$, one can check that the set V of all (finite) linear combinations of elements of $\mathcal{G}$ is a d-space (by Lemma 3.11) and that it is the smallest d-space containing $\mathcal{G}$. Therefore we call V the d-space *generated* by $\mathcal{G}$ and we call $\mathcal{G}$ a *generating set* of V (we note $\mathsf{V} = \mathsf{V}(\{U_i \mid i \in I\})$). (Similar definitions can be given for *families* of vectors.)

Linear independence. We now let $W = V$. Following an analogy with classical linear algebra, we now develop a notion corresponding to a kind of *linear independence* of the classes (mod $\sim$) of the given vectors. Let us extend the equivalence relation $\sim$ to d-spaces, as follows: if $\mathsf{V}_1, \mathsf{V}_2$ are d-spaces,

$$\mathsf{V}_1 \sim \mathsf{V}_2 \Leftrightarrow \forall i, j \in \{1, 2\}, \forall S \in \mathsf{V}_i, \exists S' \in \mathsf{V}_j, S \sim S'.$$

LEMMA 5.1. *Let $S_1, \ldots, S_j, \ldots, S_m \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V\,\rangle\rangle$. The following are equivalent:*

(1) $\exists \vec{\alpha}, \vec{\beta} \in \mathsf{DRB}_{1,m}\langle\langle\, V\,\rangle\rangle$, $\vec{\alpha} \not\sim \vec{\beta}$, *such that*

$$\sum_{j=1}^m \alpha_j \cdot S_j \sim \sum_{j=1}^m \beta_j \cdot S_j.$$

(2) $\exists j_0 \in [1,m]$, $\exists \vec{\gamma} \in \mathsf{DRB}_{1,m}\langle\langle\ V\ \rangle\rangle$, $\vec{\gamma} \not\sim \epsilon^m_{j_0}$, *such that*

$$S_{j_0} \sim \sum_{j=1}^{m} \gamma_j \cdot S_j.$$

(3) $\exists j_0 \in [1,m]$, $\exists \vec{\gamma}' \in \mathsf{DRB}_{1,m}\langle\langle\ V\ \rangle\rangle$, $\gamma'_{j_0} \sim \emptyset$, *such that*

$$S_{j_0} \sim \sum_{j=1}^{m} \gamma'_j \cdot S_j.$$

(4) $\exists j_0 \in [1,m]$, *such that*

$$\mathsf{V}((S_j)_{1\le j\le m}) \sim \mathsf{V}((S_j)_{1\le j\le m, j\ne j_0}).$$

The equivalence between (1), (2), and (3) was first proved in [23, 24] in the case where the S_j's are configurations $q_j\omega$, with the same ω and $\bar{\psi} = \mathrm{Id}_{X^*}$, and hence $\sim$ is just the language equivalence relation $\equiv$. This is the key idea around which we have developed the notion of d-spaces.

Proof. **(1) ⇒ (2):** Let us consider $\mathcal{R} \in \mathcal{B}_\infty(\vec{\alpha}\cdot S, \vec{\beta}\cdot S)$, $\nu = \mathrm{Div}(\vec{\alpha}, \vec{\beta})$, and

$$(u,v) = \min\{(w,w') \in \mathcal{R} \cap X^{\le\nu} \times X^{\le\nu} \mid \exists j \in [1,m], \tag{5.1}$$
$$(\vec{\alpha}\odot w = \epsilon^m_j) \Leftrightarrow (\vec{\beta}\odot w' \ne \epsilon^m_j)\}.$$

Let us suppose, for example, that $\vec{\alpha}\odot u = \epsilon^m_{j_0}$ while $\vec{\beta}\odot v \ne \epsilon^m_{j_0}$ and let $\vec{\gamma} = \vec{\beta}\odot u$. As $(u,v) \in \mathcal{R} \in \mathcal{B}_\infty(\vec{\alpha}\cdot S, \vec{\beta}\cdot S)$,

$$(\vec{\alpha}\cdot S)\odot u \sim (\vec{\beta}\cdot S)\odot v. \tag{5.2}$$

Using Lemma 3.15 we obtain

$$(\vec{\alpha}\cdot S)\odot u = S_{j_0}. \tag{5.3}$$

Let us examine now the right-hand side of equality (5.2). Let $(u',v') \prec (u,v)$ with $|u'| = |v'|$. By condition (4) in Definition 3.21 $(u',v') \in \mathcal{R}$ (here is the main place where this condition (4) is used) and by minimality of v, $\vec{\beta}\odot v'$ is a unit iff $\vec{\alpha}\odot u'$ is a unit. But if $\vec{\alpha}\odot u'$ is a unit, then $\vec{\alpha}\odot u = \emptyset$, which is false. Hence $\vec{\beta}\odot v'$ is not a unit. Hence, $\forall v' \prec v$, $\vec{\beta}\odot v'$ is not a unit. By Lemma 3.15

$$(\vec{\beta}\cdot S)\odot v = (\vec{\beta}\odot v)\cdot S. \tag{5.4}$$

Let us plug equalities (5.3) and (5.4) into equivalence (5.2) and let us define $\vec{\gamma} = \vec{\beta}\odot v$. We obtain

$$S_{j_0} \sim \vec{\gamma}\cdot S, \qquad \vec{\gamma} \ne \epsilon^m_{j_0}.$$

(2) ⇒ (3):

$$S_{j_0} \sim \gamma_{j_0}\cdot S_{j_0} + \left(\sum_{j\ne j_0} \gamma_j \cdot S_j\right), \quad \gamma_{j_0} \ne \epsilon.$$

By Corollary 4.10, point (C1), we can deduce that

$$S_{j_0} \sim \sum_{j \neq j_0} \gamma_{j_0}^* \gamma_j \cdot S_j = \nabla_{j_0}^*(\gamma) \cdot S.$$

Taking $\gamma' = \nabla_{j_0}^*(\gamma)$ we obtain

$$S_{j_0} \sim \gamma' \cdot S, \text{ where } \gamma'_{j_0} = \emptyset.$$

(3) $\Rightarrow$ (4): Let us denote by $\hat{S}$ the vector $(S_1, \dots, S_{j_0-1}, \emptyset, S_{j_0+1} \dots, S_m) \in \mathsf{DB}_{m,1}\langle\langle\ V\ \rangle\rangle$. If $T = \vec{\alpha} \cdot S$, then $T \sim (\vec{\alpha} \nabla_{j_0} \vec{\gamma'}) \cdot \hat{S}$.

(4) $\Rightarrow$ (1): Let us suppose (4) is true for some integer j_0. The element S_{j_0} is clearly equivalent (mod $\sim$) to two linear combinations of the S_j's with nonequivalent vectors of coefficients (mod $\sim$). Hence (1) is true. □

5.2. Triangulations. Let $S_1, S_2, \dots, S_d$ be a family of deterministic row-vectors over the structured alphabet V (i.e., $S_i \in \mathsf{DRB}_{1,\lambda}\langle\langle\ V\ \rangle\rangle$, where $\lambda \in \mathbb{N} - \{0\}$). We recall that V is the alphabet associated with some dpda $\mathcal{M}$ as defined in section 2.4.

Let us consider a sequence $\mathcal{S}$ of n "weighted" linear equations:

$$(\mathcal{E}_i) : p_i, \sum_{j=1}^{d} \alpha_{i,j} S_j, \sum_{j=1}^{d} \beta_{i,j} S_j, \tag{5.5}$$

where $p_i \in \mathbb{N} - \{0\}$, and $A = (\alpha_{i,j})$, $B = (\beta_{i,j})$ are deterministic rational matrices of dimension (n, d), with indices $m \leq i \leq m + n - 1$, $1 \leq j \leq d$.

For any weighted equation, $\mathcal{E} = (p, S, S')$, we recall that the "cost" of this equation is $H(\mathcal{E}) = p + 2 \cdot \mathrm{Div}(S, S')$.

Let us define an *oracle* on deterministic vectors as a mapping $\mathcal{O} : \bigcup_{\lambda \geq 1} \mathsf{DRB}_{1,\lambda}\langle\langle\ V\ \rangle\rangle \times \mathsf{DRB}_{1,\lambda}\langle\langle\ V\ \rangle\rangle \to \mathcal{P}(X^* \times X^*)$ such that

$$\forall (S, S') \in \mathsf{DRB}_{1,\lambda}\langle\langle\ V\ \rangle\rangle \times \mathsf{DRB}_{1,\lambda}\langle\langle\ V\ \rangle\rangle, \quad S \sim S' \Rightarrow \mathcal{O}(S, S') \in \mathcal{B}_\infty(S, S').$$

In other words, an oracle is a *choice* of w-$\bar{\psi}$-bisimulation for every pair of equivalent vectors (modulo $\sim$). Let us denote by Ω the set of all oracles. Let us fix an oracle $\mathcal{O}$ throughout this section.

We associate to every system (5.5) another equation, $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})$, which "translates the equations of $\mathcal{S}$ into equations over the coefficients $(\alpha_{i,j}, \beta_{i,j})$ only."[2] The general idea of the construction of $\mathrm{INV}^{(\mathcal{O})}$ consists in iterating the transformation used in the proof of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) in Lemma 5.1, i.e., the classical idea of *triangulating* a system of linear equations. Of course we must deal with the weights and relate the construction with the deduction system $\mathcal{B}_0$.

We assume here that

$$\forall j \in [1, d], \quad S_j \neq \emptyset^\lambda. \tag{5.6}$$

Let us define $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}), \mathrm{W}^{(\mathcal{O})}(\mathcal{S}) \in \mathbb{N} \cup \{\perp\}$, $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) \in \mathbb{N}$ by induction on n. $\mathrm{W}^{(\mathcal{O})}(\mathcal{S})$ is the *weight* of $\mathcal{S}$. $\mathrm{D}^{(\mathcal{O})}(\mathcal{S})$ is the *weak codimension* of $\mathcal{S}$.

[2]The function INV defined in [34] was an "elaborated version" of the *inverse* systems defined in [23, 24] in the case of a single equation. We consider here a *relativization* of this notion to some oracle $\mathcal{O}$.

Case 1. $\alpha_{m,*} \sim \beta_{m,*}$.

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) = (\mathrm{W}^{(\mathcal{O})}(\mathcal{S}), \alpha_{m,*}, \beta_{m,*}), \quad \mathrm{W}^{(\mathcal{O})}(\mathcal{S}) = p_m - 1, \quad \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = 0.$$

Case 2. $\alpha_{m,*} \not\sim \beta_{m,*}$, $n \geq 2$, $p_{m+1} - p_m \geq 2 \cdot \mathrm{Div}(\alpha_{m,*}, \beta_{m,*}) + 1$. Let us consider $\mathcal{R} = \mathcal{O}(\sum_{j=1}^{d} \alpha_{m,j} S_j, \sum_{j=1}^{d} \beta_{m,j} S_j)$, $\nu = \mathrm{Div}(\alpha_{m,*}, \beta_{m,*})$, and

$$(u, u') = \min\{(v, v') \in \mathcal{R} \cap X^{\leq \nu} \times X^{\leq \nu} \mid \exists j \in [1, d], \tag{5.7}$$
$$(\alpha_{m,*} \odot v = \epsilon_j^{\lambda}) \Leftrightarrow (\beta_{m,*} \odot v' \neq \epsilon_j^{\lambda})\}.$$

Let us consider the integer $j_0 \in [1, d]$ such that $(\alpha_{m,*} \odot u = \epsilon_{j_0}^{\lambda}) \Leftrightarrow (\beta_{m,*} \odot u' \neq \epsilon_{j_0}^{\lambda})$.

Subcase 1. $\alpha_{m,j_0} \odot u = \varepsilon$, $\beta_{m,j_0} \odot u' \neq \varepsilon$. Let us consider the equation

$$(\mathcal{E}'_m) : p_m + 2 \cdot |u|,\ S_{j_0},\ \sum_{\substack{j=1 \\ j \neq j_0}}^{d} (\beta_{m,j_0} \odot u')^* (\beta_{m,j} \odot u') S_j$$

and define a new system of weighted equations $\mathcal{S}' = (\mathcal{E}'_i)_{m+1 \leq i \leq m+n-1}$ by

$$(\mathcal{E}'_i) : p_i,\ \sum_{j \neq j_0} [(\alpha_{i,j} + \alpha_{i,j_0} (\beta_{m,j_0} \odot u')^* (\beta_{m,j} \odot u')] S_j, \tag{5.8}$$
$$\sum_{j \neq j_0} [(\beta_{i,j} + \beta_{i,j_0} (\beta_{m,j_0} \odot u')^* (\beta_{m,j} \odot u')] S_j,$$

where the above equation is seen as an equation between two linear combinations of the S_i's, $1 \leq i \leq d$, where the j_0th coefficient is $\emptyset$ on both sides. We then define

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) = \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}'), \quad \mathrm{W}^{(\mathcal{O})}(\mathcal{S}) = \mathrm{W}^{(\mathcal{O})}(\mathcal{S}'), \quad \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = \mathrm{D}^{(\mathcal{O})}(\mathcal{S}') + 1.$$

Subcase 2. $\alpha_{m,j_0} \odot u \neq \varepsilon$, $\beta_{m,j_0} \odot u' = \varepsilon$ (analogous to Subcase 1).

Case 3. $\alpha_{m,*} \not\sim \beta_{m,*}$, $n = 1$. We then define

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) = \perp, \quad \mathrm{W}^{(\mathcal{O})}(\mathcal{S}) = \perp, \quad \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = 0,$$

where $\perp$ is a special symbol which can be understood as meaning "undefined."

Case 4. $\alpha_{m,*} \not\sim \beta_{m,*}$, $n \geq 2$, $p_{m+1} - p_m \leq 2 \cdot \mathrm{Div}(\alpha_{m,*}, \beta_{m,*})$. We then define

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) = \perp, \quad \mathrm{W}^{(\mathcal{O})}(\mathcal{S}) = \perp, \quad \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = 0.$$

LEMMA 5.2. *Let $\mathcal{S}$ be a system of weighted linear equations with deterministic rational coefficients. If $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp$, then $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})$ is a weighted linear equation with deterministic rational coefficients.*

Proof. The proof follows from Lemmas 3.18 and 3.19 and the formula defining $\mathcal{S}'$ from $\mathcal{S}$. □

From now on, and up to the end of this section, we simply write "linear equation" to mean "weighted linear equations with deterministic rational coefficients."

LEMMA 5.3. *Let $\mathcal{S}$ be a system of linear equations. If $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp$, then*

(1) $\{\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})\} \cup \{\mathcal{E}_i \mid m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) - 1\} \mid\!\!-\!\!- \mathcal{E}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}$;

(2) $\min\{H(\mathcal{E}_i) \mid m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})\} = \infty \Longrightarrow H(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})) = \infty$.

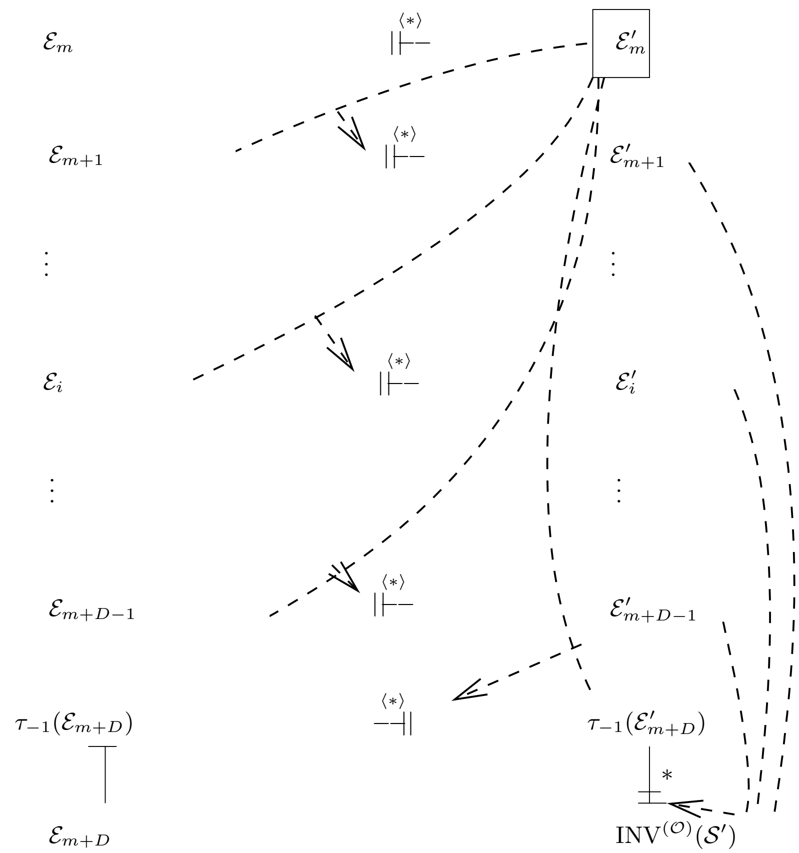

Fig. 2. *Proof of Lemma* 5.2.

Proof. See in Figure 2 the "graph of the deductions" we use for proving point (1). Let us prove by induction on $\mathrm{D}^{(\mathcal{O})}(\mathcal{S})$ the following strengthened version of point (1):

$$\{\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})\} \cup \{\mathcal{E}_i \mid m \le i \le m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) - 1\} \overset{\langle * \rangle}{|\!\!\vdash\!\!-} \tau_{-1}(\mathcal{E}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}), \tag{5.9}$$

where, for every integer $k \in \mathbb{Z}$, $\tau_k : \{(p, S, S') \in \mathcal{A} \mid p \ge -k\} \to \mathcal{A}$ is the *translation* map on the weights $\tau_k(p, S, S') = (p + k, S, S')$.

If $\mathbf{D}^{(\mathcal{O})}(\mathcal{S}) = 0$: As $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \ne \perp$, $\mathcal{S}$ must fulfill the hypothesis of Case 1.

$$\mathcal{E}_m = \left(p_m, \sum_{j=1}^{d} \alpha_{m,j} S_j, \sum_{j=1}^{d} \beta_{m,j} S_j\right) = \mathcal{E}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})},$$

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) = (p_m - 1, \alpha_{m,*}, \beta_{m,*}).$$

Using rule (R7) we obtain

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \overset{\langle * \rangle}{|\!\!\vdash\!\!-} \left(p_m - 1, \sum_{j=1}^{d} \alpha_{m,j} S_j, \sum_{j=1}^{d} \beta_{m,j} S_j\right) = \tau_{-1}(\mathcal{E}_m).$$

If $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = n+1$, $n \geq 0$: $\mathcal{S}$ must fulfill Case 2.

• Suppose Case 2, Subcase 1 occurs.

As the relation $\mathcal{R}$ used in the construction of $\mathcal{E}'_m$ from $\mathcal{E}_m$ is a w-$\bar{\psi}$-bisimulation w.r.t. the pair of sides of equation $\mathcal{E}_m$, using (R5) and then (R6) (this is possible because $\beta_{m,j_0} \odot u' \neq \epsilon$), we obtain a deduction:

$$\mathcal{E}_m \overset{\langle 2\cdot|u|+1\rangle}{||\!\!-\!\!-} \mathcal{E}'_m. \tag{5.10}$$

Using (R2), (R8) we get that, for every $i \in [m+1, m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})]$,

$$\{\mathcal{E}_i, \mathcal{E}'_m\} \overset{\langle * \rangle}{||\!\!-\!\!-}$$

$$\left(\max\{p_i, p_m + 2\,|\,u\,|\}, \sum_{j \neq j_0} (\alpha_{i,j} + \alpha_{i,j_0}(\beta_{m,j} \odot u'))S_j, \sum_{j \neq j_0} (\beta_{i,j} + \beta_{i,j_0}(\beta_{m,j} \odot u'))S_j\right)$$

but the hypothesis of Case 2 implies that $\max\{p_{m+1}, p_m + 2\,|\,u\,|\} = p_{m+1}$, and the fact that $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}')$ is defined implies that $\forall i \in [m+1, m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})]$, $p_i \geq p_{m+1}$; hence, $\max\{p_i, p_m + 2\,|\,u\,|\} = p_i$ and the right-hand side of the above deduction is exactly $\mathcal{E}'_i$. Therefore,

$$\forall i \in [m+1, m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})], \quad \{\mathcal{E}_i, \mathcal{E}'_m\} \overset{\langle * \rangle}{||\!\!-\!\!-} \mathcal{E}'_i. \tag{5.11}$$

Using deductions (5.10) and (5.11), we obtain that

$$\{\mathcal{E}_i \mid m \leq i \leq m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})-1\} \overset{\langle * \rangle}{||\!\!-\!\!-} \{\mathcal{E}'_i \mid m \leq i \leq m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})-1\}. \tag{5.12}$$

By the induction hypothesis,

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}') \cup \{\mathcal{E}'_i \mid m+1 \leq i \leq m+1+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}')-1\} \overset{\langle * \rangle}{||\!\!-\!\!-} \tau_{-1}(\mathcal{E}'_{m+1+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}')}),$$

which is equivalent to

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \cup \{\mathcal{E}'_i \mid m+1 \leq i \leq m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})-1\} \overset{\langle * \rangle}{||\!\!-\!\!-} \tau_{-1}(\mathcal{E}'_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}). \tag{5.13}$$

As $p_m + 2\cdot|u| \leq p_{m+1} - 1 \leq p_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})} - 1$, we have also the following inverse deduction (which is similar to deduction (5.11)):

$$\{\mathcal{E}'_m, \tau_{-1}(\mathcal{E}'_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})})\} \overset{\langle * \rangle}{||\!\!-\!\!-} \tau_{-1}(\mathcal{E}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}). \tag{5.14}$$

Combining deductions (5.12), (5.13), and (5.14), we have proved (5.9). Using rule (R0), this last deduction leads to point (1) of the lemma.

• Suppose now that Case 2, Subcase 2 occurs.

This case can be treated in the same way as Subcase 1, by simply exchanging the roles of α, β.

Let us prove statement (2) of the lemma.

We prove by induction on $\mathrm{D}^{(\mathcal{O})}(\mathcal{S})$ the statement

$$\min\{H(\mathcal{E}_i) \mid m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})\} = \infty \Longrightarrow H(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})) = \infty. \tag{5.15}$$

If $\mathbf{D}^{(\mathcal{O})}(\mathcal{S}) = 0$: As $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp$, Case 1 must occur. $\alpha_{m,*} \sim \beta_{m,*}$ implies that $H(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})) = \infty$, and hence the statement is true.

If $\mathbf{D}^{(\mathcal{O})}(\mathcal{S}) = p + 1,\ p \geq 0$: As $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) \geq 1$ and $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp$, Case 2 must occur.

Using deductions (5.10) and (5.11) established above we obtain that

$$\{\mathcal{E}_i \mid m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})\} \,||\!\!\overset{\langle * \rangle}{-\!\!-}\, \{\mathcal{E}'_i \mid m + 1 \leq i \leq m + 1 + \mathrm{D}^{(\mathcal{O})}(\mathcal{S}')\},$$

which proves that

$$\min\{H(\mathcal{E}_i) \mid m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})\} \leq \min\{H(\mathcal{E}'_i) \mid m + 1 \leq i \leq m + 1 + \mathrm{D}^{(\mathcal{O})}(\mathcal{S}')\}. \tag{5.16}$$

Since $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}') = \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) - 1$, we can use the induction hypothesis

$$\min\{H(\mathcal{E}'_i) \mid m + 1 \leq i \leq m + 1 + D(\mathcal{S}')\} = \infty \Longrightarrow H(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}')) = \infty. \tag{5.17}$$

Since $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) = \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}')$, (5.16), (5.17) imply statement (5.15). ∎

LEMMA 5.4. *Let $\mathcal{S}$ be a system of linear equations satisfying the hypothesis of Case 2. Then, $\forall i \in [m + 1, m + n - 1]$,*

$$\| \alpha'_{i,*} \| \leq \| \alpha_{i,*} \| + \| \beta_{m,*} \| + K_0 \mid u \mid,$$
$$\| \beta'_{i,*} \| \leq \| \beta_{i,*} \| + \| \beta_{m,*} \| + K_0 \mid u \mid.$$

Proof. The formula defining $\mathcal{S}'$ from $\mathcal{S}$ shows that

$$\alpha'_{i,*} = \alpha_{i,*} \nabla_{j_0}(\nabla^*_{j_0}(\beta_{m,*} \odot u')),$$
$$\beta'_{i,*} = \beta_{i,*} \nabla_{j_0}(\nabla^*_{j_0}(\beta_{m,*} \odot u')).$$

From these equalities and Lemmas 3.18, 3.19, and 3.13, the inequalities on the norm follow. ∎

Let us consider the function F defined by

$$F(d, n) = \max\{\mathrm{Div}(A, B) \mid A, B \in \mathsf{DRB}_{1,d}\langle\langle\, V\, \rangle\rangle, \| A \| \leq n, \| B \| \leq n, A \not\sim B\}.$$

For every integer parameter $K_0, K_1, K_2, K_3, K_4 \in \mathbb{N} - \{0\}$, we define integer sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \Sigma_i)_{m \leq i \leq m+n-1}$ by

$$\delta_m = 0,\ \ell_m = 0,\ L_m = K_2,\ s_m = K_3 \cdot K_2 + K_4,\ S_m = 0,\ \Sigma_m = 0, \tag{5.18}$$

$$\begin{cases} \delta_{i+1} = 2 \cdot F(d, s_i + \Sigma_i) + 1, \\ \ell_{i+1} = 2 \cdot \delta_{i+1} + 3, \\ L_{i+1} = K_1 \cdot (L_i + \ell_{i+1}) + K_2, \\ s_{i+1} = K_3 \cdot L_{i+1} + K_4, \\ S_{i+1} = s_i + \Sigma_i + K_0 F(d, s_i + \Sigma_i), \\ \Sigma_{i+1} = \Sigma_i + S_{i+1} \end{cases} \tag{5.19}$$

for $m \le i \le m + n - 2$.

These sequences are intended to have the following meanings when K_0, K_1, K_2, K_3, K_4 are chosen to be the constants defined in section 6 and the equations $(\mathcal{E}_i)$ are labeling nodes of a B-stacking sequence (see section 8.2):

- $\delta_{i+1} \le$ increase of weight between $\mathcal{E}_i, \mathcal{E}_{i+1}$.
- $\ell_{i+1} \ge$ increase of depth between $\mathcal{E}_i, \mathcal{E}_{i+1}$.
- $L_{i+1} \ge$ increase of depth between $\mathcal{E}_m, \mathcal{E}_{i+1}$.
- $s_{i+1} \ge$ size of the coefficients of $\mathcal{E}_{i+1}$.
- $S_{i+1} \ge$ size of the coefficients of $\mathcal{E}_{i+1}^{(i+1-m)}$ (these systems are introduced below in the proof of Lemma 5.5).
- $\Sigma_{i+1} \ge$ increase of the coefficients between $\mathcal{E}_k^{(i-m)}$, $\mathcal{E}_k^{(i+1-m)}$ (for $k \ge i+1$).

For every linear equation $\mathcal{E} = (p, \sum_{j=1}^{d} \alpha_j S_j, \sum_{j=1}^{d} \beta_j S_j)$, we define

$$\| \mathcal{E} \| = \max\{\| (\alpha_1, \ldots, \alpha_d) \|, \| (\beta_1, \ldots, \beta_d) \|\}.$$

LEMMA 5.5. *Let $\mathcal{S} = (\mathcal{E}_i)_{m \le i \le m+d-1}$ be a system of d linear equations such that $H(\mathcal{E}_i) = \infty$ (for every i) and*

(1) $\forall i \in [m, m+d-1]$, $\| \mathcal{E}_i \| \le s_i$,

(2) $\forall i \in [m, m+d-2]$, $\mathrm{W}(\mathcal{E}_{i+1}) - \mathrm{W}(\mathcal{E}_i) \ge \delta_{i+1}$.

Then

(3) $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \ne \perp$,

(4) $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) \le d - 1$,

(5) $\| \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \| \le \Sigma_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})} + s_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}$.

Proof. (Figure 3 might help the reader to follow the definitions below.) Let us define a sequence of systems $\mathcal{S}^{(i-m)} = (\mathcal{E}_k^{(i-m)})_{m \le i \le k \le m+d-1}$, where $i \in [m, m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})]$; by induction

- $\mathcal{E}_k^{(0)} = \mathcal{E}_k$ for $m \le k \le m+d-1$;
- if Case 1 or Case 3 or Case 4 is realized, $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = 0$; hence $\mathcal{S}^{(i-m)}$ is well-defined for $m \le i \le m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})$;
- if Case 2 is realized, then we set, $\forall i \ge m+1$, $\mathcal{E}_k^{(i-m)} = (\mathcal{E}'_k)^{(i-m-1)}$ for $m+1 \le k \le m+d-1$.

Let us prove by induction on $i \in [m, m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})]$ that, $\forall k \in [i, m+d-1]$,

$$\| \mathcal{E}_k^{(i-m)} \| \le s_k + \Sigma_i. \tag{5.20}$$

$\boldsymbol{i = m}$**:** In this case

$$\| \mathcal{E}_k^{(i-m)} \| = \| \mathcal{E}_k \| \le s_k = s_k + \Sigma_m.$$

$\boldsymbol{i + 1 \le m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})}$**:** In this case, by Lemma 5.4,

$$\| \mathcal{E}_k^{(i+1-m)} \| \le \| \mathcal{E}_k^{(i-m)} \| + \| \mathcal{E}_i^{(i-m)} \| + K_0 \mid u \mid,$$

$$
\begin{array}{lllll}
\mathcal{E}_m = \mathcal{E}_m^{(0)} & \boxed{(\mathcal{E}_m^{(0)})'} & & & \\[2em]
\mathcal{E}_{m+1} = \mathcal{E}_{m+1}^{(0)} & \mathcal{E}_{m+1}^{(1)} & \ddots & & \\[2em]
\vdots & \vdots & & \boxed{(\mathcal{E}_{i-1}^{(i-1-m)})'} & \\[2em]
\mathcal{E}_i = \mathcal{E}_i^{(0)} & \mathcal{E}_i^{(1)} & \cdots & \mathcal{E}_i^{(i-m)} & \ddots \\[3em]
 & & & & \boxed{(\mathcal{E}_{m-1+D}^{(D-1)})'} \\
\vdots & \vdots & & \vdots & \vdots \\[3em]
\mathcal{E}_{m+d-1} = \mathcal{E}_{m+d-1}^{(0)} & \mathcal{E}_{m+d-1}^{(1)} & \cdots & \mathcal{E}_{m+d-1}^{(i-m)} & \mathcal{E}_{m+d-1}^{(D)}
\end{array}
$$

FIG. 3. *Proof of Lemma* 5.4.

where $\mathcal{R} = \mathcal{O}(\sum_{j=1}^{d} \alpha_{i,j}^{(i-m)} S_j, \sum_{j=1}^{d} \beta_{i,j}^{(i-m)} S_j)$, $\nu = \mathrm{Div}(\alpha_{i,*}^{(i-m)}, \beta_{i,*}^{(i-m)})$, and

$$
\begin{aligned}
(u, u') = \min\{(v, v') \in \mathcal{R} \cap X^{\leq \nu} \times X^{\leq \nu} \mid \exists j \in [1, d],& \\
(\alpha_{i,*}^{(i-m)} \odot v = \epsilon_j^{\lambda}) \Leftrightarrow (\beta_{i,*}^{(i-m)} \odot v' \neq \epsilon_j^{\lambda})\}.&
\end{aligned}
$$

By definition of F and the induction hypothesis,

$$
\mid u \mid \leq F(d, \||\, \mathcal{E}_i^{(i-m)} \,\||) \leq F(d, s_i + \Sigma_i).
$$

Hence

$$
\begin{aligned}
\||\, \mathcal{E}_k^{(i+1-m)} \,\|| \leq (s_k + \Sigma_i) + (s_i + \Sigma_i) + K_0 F(d, s_i + \Sigma_i) &= (s_k + \Sigma_i) + S_{i+1} \\
&= s_k + \Sigma_{i+1}.
\end{aligned}
$$

Let us note that $\mathrm{D}^{(\mathcal{O})}(\mathcal{S})$ is always an integer and that this proof is valid for $m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})$, $i \leq k \leq m + d - 1$.

We now prove that $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp$. Let us consider the system

$$
(\mathcal{E}_k^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))})_{m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) \leq k \leq m+d-1}.
$$

If $\mathbf{D}^{(\mathcal{O})}(\mathcal{S}) = d - 1$: $(\mathcal{E}^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))})$ fulfills either Case 1 or Case 3 of the definition of $\mathrm{INV}^{(\mathcal{O})}$ (just because this system consists of a single equation).

Using the successive deductions (5.10), (5.11) established in the proof of Lemma 5.3 we get that

$$\{\mathcal{E}_i \mid m \leq i \leq m+d-1\} \ \overset{\langle * \rangle}{|\!\vdash\!-} \ \{\mathcal{E}^{(d-1)}_{m+d-1}\}.$$

Using now the hypothesis that $H(\mathcal{E}_i) = \infty$ (for $m \leq i \leq m+d-1$), we obtain

$$H(\mathcal{E}^{(d-1)}_{m+d-1}) = \infty. \tag{5.21}$$

For any system of equations $\mathcal{S}$, let us define the *support* of the system as

$$\mathrm{supp}(\mathcal{S}) = \left\{ j \in [1,d] \,\middle|\, \sum_{i=m}^{m+n-1} \alpha_{i,j} + \beta_{i,j} \neq \emptyset \right\}.$$

Let us consider $\delta = \mathrm{Card}(\mathrm{supp}(\mathcal{S}^{(d-1)})$. One can prove by induction on i that

$$\mathrm{Card}(\mathrm{supp}(\mathcal{S}^{(i-m)}) \leq d - i + m,$$

and hence

$$\delta = \mathrm{Card}(\mathrm{supp}(\mathcal{S}^{(d-1)}) \leq d - (d-1) = 1.$$

- If $\delta = 1$, $\mathrm{supp}(\mathcal{S}^{(d-1)}) = \{j_0\}$ for some $j_0 \in [1,d]$.
  By Corollary 4.10, point (C3), and by hypothesis (5.6), the implication

  $$[(\alpha^{(d-1)}_{m+d-1,j_0} S_{j_0} \sim \beta^{(d-1)}_{m+d-1,j_0} S_{j_0}) \Longrightarrow \alpha^{(d-1)}_{m+d-1,j_0} \sim \beta^{(d-1)}_{m+d-1,j_0}]$$

  holds. Hence, by (5.21), $\alpha^{(d-1)}_{m+d-1,j_0} \sim \beta^{(d-1)}_{m+d-1,j_0}$; i.e., $\mathcal{S}^{(d-1)}$ fulfills Case 1, so that

  $$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) = \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}^{(d-1)}) \neq \perp.$$

- If $\delta = 0$, $\mathrm{supp}(\mathcal{S}) = \emptyset$.
  Then $\alpha^{(d-1)}_{m+d-1,*} = \beta^{(d-1)}_{m+d-1,*} = \emptyset^d$. Here also $\mathcal{S}^{(d-1)}$ fulfills Case 1.

If $\mathbf{D}^{(\mathcal{O})}(\mathcal{S}) < d-1$: By the hypothesis,

$$\begin{aligned}\mathrm{W}(\mathcal{E}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})+1}) - \mathrm{W}(\mathcal{E}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}) &\geq \delta_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})+1} \\ &= 2F(d, s_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})} + \Sigma_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}) + 1.\end{aligned}$$

If $\alpha^{\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}),*} \sim \beta^{\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}),*}$, then $\mathcal{E}^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}$ fulfills Case 1 of the definition of $\mathrm{INV}^{(\mathcal{O})}$; hence $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp$.

Otherwise, let us consider

$$\mathcal{R} = \mathcal{O}\left(\sum_{j=1}^{d} \alpha^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}),j} S_j, \sum_{j=1}^{d} \beta^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}),j} S_j \right),$$

$$\nu = \mathrm{Div}\left(\alpha^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}),*}, \beta^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}),*} \right), \text{ and}$$

$$\begin{aligned}(u,u') = \min\Big\{ (v,v') \in \mathcal{R} \cap X^{\leq \nu} \times X^{\leq \nu} \,\Big|\, \exists j \in [1,d], \Big(\alpha^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}),*} \odot v = \epsilon^{\lambda}_j\Big) \Leftrightarrow \\ \Big(\beta^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}),*} \odot v' \neq \epsilon^{\lambda}_j\Big)\Big\}.\end{aligned}$$

By the definition of F and inequality (5.20),

$$| u | \leq F\left(d, \left\|\left|\mathcal{E}^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}\right|\right\|\right) \leq F(d, s_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})} + \Sigma_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}).$$

Hence $p_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})+1} - p_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})} \geq 2|u| + 1$; i.e., the hypothesis of Case 2 is realized. This proves that $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}) \geq 1$, while in fact $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}) = 0$. This contradiction shows that this last case ($\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) < d-1$ and $\mathcal{E}^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}$ not fulfilling Case 1 of definition of $\mathrm{INV}^{(\mathcal{O})}$) is impossible. We have proved point (3) of the lemma. ∎

6. Constants. Let us *fix* a birooted dpda $\mathcal{M}$, a strong relational morphism $\bar{\psi}$, and an initial equation $A_0 = (\Pi_0, S_0^-, S_0^+) \in \mathbb{N} \times \mathsf{DRB}_{1,\lambda_0}\langle\langle\ V\ \rangle\rangle \times \mathsf{DRB}_{1,\lambda_0}\langle\langle\ V\ \rangle\rangle$ in the corresponding set of assertions. This short section is devoted to the definition of some integer *constants*: these integers are constant in the sense that they depend only on this triple: $(\mathcal{M}, \bar{\psi}, A_0)$. The *motivation* of each of these definitions will appear later on, in different places for the different constants. The equations below provide merely an overview of the dependencies between these constants and allow us to check that the definitions are sound (i.e., there is no hidden loop in the dependencies).

$$k_0 = \max\{\nu(v) \mid v \in V\}, \quad k_1 = \max\{2k_0 + 1, 3\}, \tag{6.1}$$

$$K_0 = \max\{\|(E_1, E_2, \ldots, E_n) \odot x\| \mid (E_i)_{1\leq i\leq n} \text{ is a bijective numbering of some class in } V/\smile,\ x \in X\}. \tag{6.2}$$

K_0 serves as an upper-bound on the possible increase of norm under the right-action of a single letter $x \in X$; see Lemma 3.13.

$$D_1 = k_0 \cdot K_0 + |Q| + 2, \quad k_2 = D_1 \cdot k_1 \cdot K_0 + 2 \cdot k_1 \cdot K_0 + K_0. \tag{6.3}$$

k_1 is used in the definition of strategy T_B (section 7), D_1 appears as an upper-bound on the marked part of series, and k_2 is used in Lemma 8.4.

$$k_3 = k_2 + k_1 \cdot K_0, \quad k_4 = (k_3 + 1) \cdot K_0 + k_1. \tag{6.4}$$

k_3 appears in in Lemma 8.5, and k_4 is used in the definition (8.15) of the d-space V_0.

$$\begin{aligned} K_1 &= k_1 \cdot K_0 + 1, \\ K_2 &= k_1^2 \cdot D_1 \cdot K_0 + k_1^2 \cdot K_0 + 2 \cdot k_1 \cdot K_0 + D_1 \cdot k_1 + 2 \cdot k_1 + 4. \end{aligned} \tag{6.5}$$

These constants K_1, K_2 appear in Lemma 8.7.

$$K_3 = k_0|Q|, \quad K_4 = D_1. \tag{6.6}$$

These constants K_3, K_4 appear in Lemma 8.8.

$$d_0 = \mathrm{Card}(X^{\leq k_4}). \tag{6.7}$$

d_0 appears as an upper-bound on the dimension of the d-space V_0 defined by equation (8.15) and used in Lemma 8.7. We consider now the integer sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \Sigma_i)_{m\leq i\leq m+n-1}$ defined by the relations (5.19) of section 5.2 where the parameters

$K_0, K_1, \ldots, K_4$ are chosen to be the above constants and $m = 1$, $n = d = d_0$. Equivalently, they are defined by

$$\delta_1 = 0,\ \ell_1 = 0,\ L_1 = K_2,\ s_1 = K_3 \cdot K_2 + K_4,\ S_1 = 0,\ \Sigma_1 = 0, \tag{6.8}$$

$$\begin{cases} \delta_{i+1} = 2 \cdot F(d_0, s_i + \Sigma_i) + 1, \\ \ell_{i+1} = 2 \cdot \delta_{i+1} + 3, \\ L_{i+1} = K_1 \cdot (L_i + \ell_{i+1}) + K_2, \\ s_{i+1} = K_3 \cdot L_{i+1} + K_4, \\ S_{i+1} = s_i + \Sigma_i + K_0 \cdot F(d_0, s_i + \Sigma_i), \\ \Sigma_{i+1} = \Sigma_i + S_{i+1} \end{cases} \tag{6.9}$$

for $1 \leq i \leq d_0 - 1$. The function F is defined in section 5.2 and depends on the pair $(\mathcal{M}, \bar{\psi})$ only.

$$D_2 = \max\{\Sigma_{d_0} + s_{d_0}, \|S_0^-\|, \|S_0^+\|\}. \tag{6.10}$$

$\Sigma_{d_0} + s_{d_0}$ appears in the conclusion of Lemma 5.5 when we take $d = d_0$ in the hypothesis and suppose that $\mathrm{D}^{(\mathcal{O})}(\mathcal{S})$ has its maximal possible value, i.e., $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = d_0 - 1$. It is used as an upper-bound on the norm of vectors at the root of the trees τ analyzed in section 8 (inequality (8.1)).

$$\lambda_2 = \max\{\lambda_0, d_0\}. \tag{6.11}$$

The integer λ_2 is used as an upper-bound on the length of vectors at the root of the trees τ analyzed in section 8 (inequality (8.2)).

$$N_0 = 1 + k_3 + D_2. \tag{6.12}$$

N_0 appears as a lower-bound for the norm in the definition of a B-stacking sequence (section 8.2, condition (8.5)).

7. Strategies for $\mathcal{B}_0$.

Let us define strategies for the particular system $\mathcal{B}_0$.

7.1. Strategies.

We shall first define auxiliary strategies $T_{cut}, T_\emptyset, T_\varepsilon$; and then for every oracle $\mathcal{O} \in \Omega$ auxiliary strategies $T_A^{(\mathcal{O})}, T_B^{(\mathcal{O})}, T_C^{(\mathcal{O})}$, we define the strategies T_A, T_B, T_C and finally the "compound" strategies $\mathcal{S}_{AB}^{(\mathcal{O})}, \mathcal{S}_{ABC}^{(\mathcal{O})}, \mathcal{S}_{AB}, \mathcal{S}_{ABC}$. Let us fix here some total ordering on $X : x_1 < x_2 < \cdots < x_\alpha$.

- T_{cut}:
  $B_1 \cdots B_m \in T_{cut}(A_1 \cdots A_n)$ iff $\exists i \in [1, n-1]$, $\exists S, T$,
  $$A_i = (p_i, S, T),\ A_n = (p_n, S, T),\ p_i < p_n, \text{ and } m = 0.^3$$
- $T_\emptyset$:
  $B_1 \cdots B_m \in T_\emptyset(A_1 A_2 \cdots A_n)$ iff $\exists S, T$,
  $$A_n = (p, S, T),\ p \geq 0,\ S = T = \emptyset^\lambda, \text{ and } m = 0.$$
- T_ε:
  $B_1 \cdots B_m \in T_\varepsilon(A_1 \cdots A_n)$ iff
  $$A_n = (p, S, T),\ p \geq 0,\ S = T = \varepsilon_i^\lambda \text{ (for some } i \in [1, \lambda]), \text{ and } m = 0.$$

[3] That is, $B_1 \cdots B_m = \epsilon$.

Let us consider an oracle $\mathcal{O} \in \Omega$.

- $T_A^{(\mathcal{O})}$:
  $B_1 \cdots B_m \in T_A^{(\mathcal{O})}(A_1 \cdots A_n)$ iff
  $$A_n = (p, S, T), \quad \mid X \mid \leq m \leq \mid X \mid^2,$$
  $$B_1 = (p+1, S \odot x_1, T \odot x'_1), \ldots, B_m = (p+1, S \odot x_m, T \odot x'_m),$$
  where $S \not\equiv \varepsilon$, $T \not\equiv \varepsilon$, $\mathcal{O}(S,T) \cap X \times X = \{(x_1, x'_1), \ldots, (x_i, x'_i), \ldots, (x_m, x'_m)\}$.
- $T_B^{(\mathcal{O}),+}$:
  $B_1 \cdots B_m \in T_B^{+}(A_1 \cdots A_n)$ iff $n \geq k_1 + 1$, $A_{n-k_1} = (\pi, \overline{U}, U')$ (where $\overline{U}$ is unmarked)
  $$U' = \sum_{k=1}^{q} E_k \cdot \Phi_k \quad \text{for some } q \in \mathbb{N},\ E_k \in V,$$
  $(E_k)_{1 \leq k \leq q}$ is the bijective numbering of a class in $V/\smile$, $\Phi_k \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle$, $A_i = (\pi + k_1 + i - n, U_i, U'_i)$ for $n - k_1 \leq i \leq n$, $(U_i)_{n-k_1 \leq i \leq n}$ is a derivation, $(U'_i)_{n-k_1 \leq i \leq n}$ is a "stacking derivation" (see definitions in section 3.3),
  $$U'_n = \sum_{k=1}^{q} (E_k \odot u) \cdot \Phi_k \quad \text{for some } u \in X^*,$$
  $m = 1$, $B_1 = (\pi + k_1 - 1, V, V')$, $V = U_n$,
  $$V' = \sum_{k=1}^{q} \bar{\rho}_e(E_k \odot u) \cdot (\overline{U} \odot u_k),$$
  where $\forall k \in [1,q]$, $u'_k = \min(\varphi(E_k))$, and if $\mathcal{R} = \mathcal{O}(S,T)$, $\forall k \in [1,q]$, $u_k = \min\{\mathcal{R}^{-1}(u'_k)\}$.
- $T_B^{(\mathcal{O}),-}$:
  $T_B^{(\mathcal{O}),-}$ is defined in the same way as $T_B^{(\mathcal{O}),+}$ by exchanging the left series (S^-) and right series (S^+) in every assertion (p, S^-, S^+).
- $T_C^{(\mathcal{O})}$:
  $B_1 \cdots B_m \in T_C^{(\mathcal{O})}(A_1 \cdots A_n)$ iff there exists $d \in [1, d_0]$, $D \in [0, d-1]$, $\lambda \in \mathbb{N} - \{0\}$, $S_1, S_2, \ldots, S_d \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle - \{\emptyset^\lambda\}$, $1 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_{D+1} = n$, such that
  - (C1) every equation $\mathcal{E}_i = A_{\kappa_i} = (p_{\kappa_i} S^-_{p_{\kappa_i}}, S^+_{p_{\kappa_i}})$ is a weighted equation over $S_1, S_2, \ldots, S_d$, with $p_{\kappa_i} \geq 1$;
  - (C2) $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = D$ (where $\mathcal{S} = (\mathcal{E}_i)_{1 \leq i \leq D+1}$);
  - (C3) $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp$, $|||\ \mathrm{INV}^{(\mathcal{O})}(\mathcal{S})\ ||| \leq \Sigma_{d_0} + s_{d_0}$;
  - (C4) $m = 1$ and $B_1 = \rho_e(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}))$ (where ρ_e is the obvious extension of ρ_e to weighted pairs of deterministic row-vectors; in other words the result of $T_C^{(\mathcal{O})}$ is $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})$, where the marks have been removed).

We then set, for every $W \in \mathcal{A}^+$,

$$T_A(W) = \bigcup_{\mathcal{O} \in \Omega} T_A^{(\mathcal{O})}(W),$$

$$T_B^+(W) = \bigcup_{\mathcal{O} \in \Omega} T_B^{(\mathcal{O}),+}(W), \quad T_B^-(W) = \bigcup_{\mathcal{O} \in \Omega} T_B^{(\mathcal{O}),-}(W),$$

$$T_C(W) = \bigcup_{\mathcal{O} \in \Omega} T_C^{(\mathcal{O})}(W).$$

LEMMA 7.1. *$T_{cut}, T_\emptyset, T_\varepsilon, T_A$ are $\mathcal{B}_0$-strategies.*

Proof.

T_{cut}: (S1) is true by rule (R0). (S2) is trivially true.

$T_\emptyset$: (S1) is true by rule (R$'$3). (S2) is trivially true.

T_ε: (S1) is true by rule (R$'$3). (S2) is trivially true.

T_A: By rule (R4), $\{B_j \mid 1 \le j \le m\} \mid\!\!\vdash\!\!-_4 A_n$, which proves (S1). Suppose $H(A_n) = \infty$, i.e., $S \sim T$. Then, $\forall j \in [1, m]$, $S \odot x_j \sim T \odot x'_j$, so that $\min\{H(B_j) \mid 1 \le j \le m\} = \infty$. (S2) is proved. $\square$

LEMMA 7.2. *T_B^+, T_B^- are $\mathcal{B}_0$-strategies.*

Proof. Let us show that T_B^+ is a $\mathcal{B}_0$-strategy.

We use the notation of the definition of $T_B^{(\mathcal{O}),+}$. Let $\mathcal{H} = \{(\pi, \overline{U}, U'), (\pi + k_1 - 1, V, V')\}$. Let us show that

$$\mathcal{H} \overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{\mathcal{B}_0} (\pi + k_1 - 1, U_n, U'_n). \tag{7.1}$$

Using rule (R5) we obtain, $\forall k \in [1, q]$,

$$\begin{aligned}\{(\pi, \overline{U}, U')\} = \left\{ \left(\pi, \overline{U}, \sum_{j=1}^{q} E_j \cdot \Phi_j \right) \right\} &\overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{R5} (\pi + 2 \cdot \mid u_k \mid, \overline{U} \odot u_k, U' \odot u'_k) \\ &\overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{R0} (\pi + 2 \cdot k_0, \overline{U} \odot u_k, U' \odot u'_k) \\ &= \quad (\pi + 2 \cdot k_0, \overline{U} \odot u_k, \Phi_k). \end{aligned} \tag{7.2}$$

Using rule (R$'$3),

$$\emptyset \mid\!\!\vdash\!\!-_{R'3} (0, (\rho_e(E_1 \odot u), \ldots, \rho_e(E_q \odot u)), (E_1, \ldots, E_q)). \tag{7.3}$$

Using (7.3), (7.2) and rules (R3), (R7), (R8), we obtain

$$\begin{aligned}\{(\pi, \overline{U}, U')\} &\overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{\mathcal{B}_0} \left(\pi + 2k_0, \sum_{k=1}^{q} (E_k \odot u) \cdot \Phi_k, \sum_{k=1}^{q} \rho_e(E_k \odot u) \cdot (\bar{U} \odot u_k) \right) \\ &= \quad \{(\pi, \overline{U}, U')\} \overset{\langle * \rangle}{\mid\!\!\vdash\!\!-} (\pi + 2k_0, U'_n, V'). \end{aligned} \tag{7.4}$$

Let us recall that $U_n = V$. Hence, by (R0), (R1), (R2)

$$\{(\pi + k_1 - 1, V, V'), (\pi + 2k_0, U'_n, V')\} \overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{\mathcal{C}} (\pi + k_1 - 1, U_n, U'_n). \tag{7.5}$$

By (7.4) and (7.5), (7.1) is proved. Now, using (7.1) and rule (R0), we obtain

$$\mathcal{H} \overset{\langle * \rangle}{|\!\vdash\!\!-}_{\mathcal{B}_0} (\pi + k_1 - 1, U_n, U'_n) \vdash\!\!-_{R0} (\pi + k_1, U_n, U'_n), \tag{7.6}$$

i.e., T_B^+ fulfills (S1).

Let us suppose now that $\forall i \in [n - k_1, n]$, $U_i \sim U'_i$. Then, by (7.4), $U'_n \sim V'$ and by the hypothesis, $V = U_n \sim U'_n$. Hence $V \sim V'$. This shows that T_B^+ fulfills (S2).

An analogous proof can obviously be written for T_B^-. ☐

LEMMA 7.3. *Let (p, S, S') be a weighted equation, i.e., $p \in \mathbb{N}$, $\lambda \in \mathbb{N} - \{0\}$, $S, S' \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$. Then*

$$\{(p, S, S')\} \overset{\langle * \rangle}{|\!\vdash\!\!-}_{\mathcal{C}} \{(p, \rho_e(S), \rho_e(S'))\},$$

$$\{(p, \rho_e(S), \rho_e(S'))\} \overset{\langle * \rangle}{|\!\vdash\!\!-}_{\mathcal{C}} \{(p, S, S')\}.$$

Proof. The proof follows easily from (R1), (R2), (R′3). ☐

LEMMA 7.4. *For every $\mathcal{O} \in \Omega$, $T_C^{\mathcal{O}}$ is a $\mathcal{B}_0$-strategy.*

Proof. By Lemma 5.3, point (1), combined with Lemma 7.3, (S1) is proved. By Lemma 5.3, point (2), combined with Lemma 7.3, (S2) is proved. ☐

Let us define the strategy $\mathcal{S}_{ABC}$ by the following: for every $W = A_1 A_2 \cdots A_n$,

(0) if $T_{cut}(W) \neq \emptyset$, then $\mathcal{S}_{ABC}(W) = T_{cut}(W)$;
(1) else if $T_\emptyset(W) \neq \emptyset$, then $\mathcal{S}_{ABC}(W) = T_\emptyset(W)$;
(2) else if $T_\varepsilon(W) \neq \emptyset$, then $\mathcal{S}_{ABC}(W) = T_\varepsilon(W)$;
(3) else if $T_B^+(W) \cup T_B^-(W) \neq \emptyset$, then $\mathcal{S}_{ABC}(W) = T_B^+(W) \cup T_B^-(W) \cup T_C(W)$;
(4) else $\mathcal{S}_{ABC}(W) = T_A(W) \cup T_C(W)$.

The strategy $\mathcal{S}_{AB}$ is obtained from $\mathcal{S}_{ABC}$ by removing the occurrence of T_C in cases (3) and (4).

7.2. Global strategy. Let us define a global strategy $\hat{\mathcal{S}}_{ABC}$ w.r.t. the strategy $\mathcal{S}_{ABC}$. Let us fix (until the end of this article) a total well-ordering $\sqsubseteq$ over the set of oracles Ω. We need now three technical definitions.

DEFINITION 7.5. *Let $P \in \mathcal{P}_f(\mathcal{A})$, $\mathcal{O} \in \Omega$, and $\bar{\pi} \in \mathbb{N} \cup \{\infty\}$. $\mathcal{O}$ is* $\bar{\pi}$-consistent *with P iff, for every $(\pi, S, S') \in \mathrm{Cong}(P)$ and every $n \in \mathbb{N}$, if*

$$\pi + n - 1 < \bar{\pi},$$

then the binary relation $\mathcal{R}_n = \mathcal{O}(S, S') \cap X^{\leq n} \times X^{\leq n}$ fulfills

$$[\pi, S, S', \mathcal{R}_n] \subseteq \mathrm{Cong}(P).$$

We use the notation

$$\Omega(\bar{\pi}, P) = \{\mathcal{O} \in \Omega \mid \mathcal{O} \text{ is } \bar{\pi}\text{-consistent with } P\}.$$

DEFINITION 7.6. *Let P be a finite subset of $\mathcal{A}$, and let $\bar{\pi} \in \mathbb{N} \cup \{\infty\}$. P is said to be* $\bar{\pi}$-consistent *iff there exists some oracle $\mathcal{O} \in \Omega$, which is $\bar{\pi}$-consistent with P.*

For every proof-tree $t \in \mathcal{T}(\mathcal{S}_{ABC})$, we denote by $\bar{\Pi}(t)$ the integer

$$\bar{\Pi}(t) = \min\{\pi \in \mathbb{N} \mid \exists x \in \mathrm{dom}(t),\ x \text{ is not closed for } \mathcal{S}_{ABC},\ \exists S, S',\ t(x) = (\pi, S, S')\}. \tag{7.7}$$

(We admit here that $\min(\emptyset) = \infty$.)

DEFINITION 7.7. *Let t be a finite proof-tree for the strategy $\mathcal{S}_{ABC}$, $t \in \mathcal{T}(\mathcal{S}_{ABC})$. t is* consistent *iff* $\mathrm{im}(t)$ *is* $\bar{\Pi}(t)$*-consistent.*

Let us consider some tree $t \in \mathcal{T}(\mathcal{S}_{ABC})$ which is consistent and not closed. Let $\bar{\pi} = \bar{\Pi}(t)$, and let x be the smallest unclosed node of weight $\bar{\pi}$. Let

$$W = A_1 \cdots A_n \tag{7.8}$$

be the word labeling the path from the root to x in t. (One can notice that, as x is not closed, $T_{cut}(W) \cup T_{\emptyset}(W) \cup T_{\varepsilon}(W) = \emptyset$.) We define a tree of height one, $\hat{\Delta}(t)$, as follows:

(0) If $\exists \mathcal{O} \in \Omega(\bar{\pi}, \mathrm{im}(t))$, $T_C^{(\mathcal{O})}(W) \neq \emptyset$, then

$$\mathcal{O}_0 = \min\{\mathcal{O} \in \Omega(\bar{\pi}, \mathrm{im}(t)), T_C^{(\mathcal{O})}(W) \neq \emptyset\}, \quad \hat{\Delta}(t) = A_n(T_C^{(\mathcal{O}_0)}(W));$$

(1) else if $T_B^{+}(W) \neq \emptyset$, then

$$\mathcal{O}_0 = \min(\Omega(\bar{\pi}, \mathrm{im}(t))), \quad \hat{\Delta}(t) = A_n(T_B^{(\mathcal{O}_0),+}(W));$$

(2) else if $T_B^{-}(W) \neq \emptyset$, then

$$\mathcal{O}_0 = \min(\Omega(\bar{\pi}, \mathrm{im}(t))), \quad \hat{\Delta}(t) = A_n(T_B^{(\mathcal{O}_0),-}(W));$$

(3) else

$$\mathcal{O}_0 = \min(\Omega(\bar{\pi}, \mathrm{im}(t))), \quad \hat{\Delta}(t) = A_n(T_A^{(\mathcal{O}_0)}(W)).$$

(In the above definition, by $A(W')$, where $A \in \mathcal{A}$, $W' \in \mathcal{A}^+$, we mean the tree of height one with root labeled by A and whose sequence of leaves is the word W'.)

$$\hat{\mathcal{S}}_{ABC}(t) = t[\hat{\Delta}(t)/x], \tag{7.9}$$

i.e., $\hat{\mathcal{S}}_{ABC}(t)$ is obtained from t by substituting $\hat{\Delta}(t)$ at the leaf x.

LEMMA 7.8. *For every $t \in \mathcal{T}(\mathcal{S}_{ABC})$, if t is consistent, then $\hat{\Delta}(t)$ is defined.*

Proof. By the definition of consistency the oracle $\mathcal{O}_0$ is always defined (that is, $\Omega(\bar{\pi}, \mathrm{im}(t)) \neq \emptyset$), and for the word W defined above, $T_{\varepsilon}(W) = \emptyset \Rightarrow \forall \mathcal{O} \in \Omega$, $T_A^{(\mathcal{O})}(W) \neq \emptyset$; hence one of cases (0)–(3) must occur. □

If t is not consistent or is closed, then we define

$$\hat{\mathcal{S}}_{ABC}(t) = t. \tag{7.10}$$

LEMMA 7.9. *$\hat{\mathcal{S}}_{ABC}$ is a global strategy for $\mathcal{S}_{ABC}$.*

Sketch of proof. By Lemma 7.8 $\hat{\mathcal{S}}_{ABC}$ is defined on every $t \in \mathcal{T}(\mathcal{S}_{ABC})$. It suffices to check that, in every case, the word constituted by the leaves of $\hat{\Delta}(t)$ belongs to $\mathcal{S}_{ABC}(W)$ (where W is the word considered in (7.8)). □

8. Tree analysis. This section is devoted to the analysis of the proof-trees τ produced by the strategy $\mathcal{S}_{AB}$ defined in section 7. The main results are Lemmas 8.9 and 8.10, whose combination asserts that if some branch of τ is infinite, then there exists some finite prefix on which T_C has a nonempty value. This key technical result will ensure termination of the global strategy $\hat{\mathcal{S}}_{ABC}$ (see section 9).

We fix throughout this section a tree $\tau \in \mathcal{T}(\mathcal{S}_{AB}, (\pi_0, U_0^-, U_0^+))$ (i.e., τ is a proof-tree associated to the assertion $(\pi_0, U_0^-, U_0^+))$ by the strategy $\mathcal{S}_{AB}$). We suppose that

$$\|U_0^-\| \leq D_2, \quad \|U_0^+\| \leq D_2, \quad U_0^-, U_0^+ \text{ are both unmarked,} \tag{8.1}$$

$$U_0^-, U_0^+ \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle \text{ with } \lambda \leq \lambda_2, \tag{8.2}$$

$$U_0^- \equiv U_0^+. \tag{8.3}$$

We recall that, formally, τ is a map $\mathrm{dom}(\tau) \to \mathbb{N} \times \mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle \times \mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle$ such that $\mathrm{dom}(\tau) \subseteq \{1, \ldots, |X|^2\}^*$ is closed under prefix and under "left-brother" (i.e., $w \cdot (i+1) \in \mathrm{dom}(\tau) \Rightarrow w \cdot i \in \mathrm{dom}(\tau)$). We denote by $pr_{2,3} : \mathbb{N} \times \mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle \times \mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle \to \mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle \times \mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle$ the projection $(\pi, U, U') \mapsto (U, U')$. By τ_s we denote the tree obtained from τ by forgetting the weights $\tau_s = \tau \circ pr_{2,3}$.

8.1. Depth and weight. In this paragraph we check that the *weight* and the *depth* of a given node are closely related. Let us say that the strategy T "occurs at" node x iff

$$\tau(x) \in T(\tau(x[0]) \cdot \tau(x[1]) \cdots \tau(x[|x| - 1]));$$

i.e., the label of x belongs to the image of the path from ϵ (included) to x (excluded) by the strategy T.

LEMMA 8.1. *Let $\alpha \in \{-, +\}$, $A_1, \ldots, A_n \in \mathcal{A}$ such that $T_B^\alpha(A_1 \cdots A_n) \neq \emptyset$. Then, $\forall i \in [n - k_1 + 1, n]$, $A_i \notin T_B(A_1 \cdots A_{i-1})$.*

In other words, if T_B occurs at node x of τ, it cannot occur at any of its k_1 above immediate ancestors.

Proof. Suppose that $\exists i \in [n - k_1 + 1, n]$, $A_i \in T_B(A_1 \cdots A_{i-1})$. Hence $\pi_i = \pi_{i-1} - 1 < \pi_{n-k_1} + i$, contradicting one of the hypotheses under which $T_B(A_1 \cdots A_n)$ is not empty. ∎

Lemma 8.1 ensures that, in every branch $(x_i)_{i \in I}$ and for every interval $[n + 1, n + 4] \subseteq I$, at most one integer j is such that T_B occurs at j.

LEMMA 8.2. *Let τ be a proof-tree associated to the strategy $\mathcal{S}_{AB}$. Let $x, x' \in \mathrm{dom}(\tau)$, $x \preceq x'$. Then $\mid W(x') - W(x) \mid \leq |x'| - |x| \leq 2 \cdot (W(x') - W(x)) + 3$.*

(We recall the *depth* of a node x is just its length $|x|$.) We denote by $W(x)$ the weight of x which we define as the first component of $\tau(x)$, i.e., the weight of the equation labeling x.

Proof. Let x, x' be such that $|x'| = |x| + 1$. Then $W(x') - W(x) \in \{-1, +1\}$, and hence the inequality $\mid W(x') - W(x) \mid \leq |x'| - |x|$ is fulfilled by such nodes. The general case follows by induction on $(|x'| - |x|)$.

Let us prove now the other inequality. We distinguish two cases.

Case 1. $|x'| - |x| \leq 3$. Then $|x'| - |x| \leq 2 \cdot (W(x') - W(x)) + 3$ (because there is at most one T_B step in a sequence of length ≤ 3).

Case 2. $|x'| - |x| \geq 4$. Let $x = x_0, x_1, \ldots, x_q, x'$ be the sequence of nodes such that $|x'| - |x| = 4 \cdot q + r$, $0 \leq r < 4$, and $\forall i \in [0, q - 1]$, $|x_{i+1}| - |x_i| = 4$.

By Lemma 8.1, in every set $\{y \in \mathrm{dom}(\tau) \mid x_i \prec y \preceq x_{i+1}\}$ at most one node z is such that T_B occurs at z. Hence $W(x_{i+1}) - W(x_i) \geq 2$.

It follows that

$$\begin{aligned}|x'| - |x| &= \sum_{i=0}^{q-1}[|x_{i+1}| - |x_i|] + |x'| - |x_q| \\ &\leq \sum_{i=0}^{q-1} 2(W(x_{i+1} - W(x_i)) + |x'| - |x_q| \\ &\leq 2(W(x_q) - W(x)) + 2(W(x') - W(x_q)) + 3 \ \text{(by the first case)} \\ &\leq 2(W(x') - W(x)) + 3. \qquad \square\end{aligned}$$

Let us recall the values of some constants (defined in section 6):

$$k_0 = \max\{\nu(v) \mid v \in V\}, \quad k_1 = \max\{2k_0 + 1, 3\}, \quad D_1 = k_0 \cdot K_0 + |Q| + 2,$$

$$k_2 = D_1 \cdot k_1 \cdot K_0 + 2 \cdot k_1 \cdot K_0 + K_0, \quad k_3 = k_2 + k_1 \cdot K_0,$$

$$k_4 = (k_3 + 1) \cdot K_0 + k_1, \quad d_0 = \mathrm{Card}(X^{\leq k_4}), \quad N_0 = 1 + k_3 + D_2.$$

8.2. B-stacking sequences. We establish here that every infinite branch must contain an infinite suffix (a "B-stacking sequence") where at least d_0 labels (U, U') belong to the same d-space V_0 of dimension $\leq d_0$ with coordinates not greater than s_{d_0} (over some fixed generating family of cardinality $\leq d_0$).

Let $\sigma = (x_i)_{i \in I}$ be a path in τ, where $I = [i_0, \infty[$ and let $(x_i)_{i \geq 0}$ be the unique branch of τ containing σ. Let us note $\tau(x_i) = (\pi_i, U_i^-, U_i^+)$.

We call σ a *B-stacking sequence* iff there exists some $\alpha_0 \in \{-, +\}$ such that

$$T_B^{\alpha_0} \text{ occurs at } x_{i_0+k_1+1}, \tag{8.4}$$

and, for every $i \in I, \alpha \in \{-, +\}$, if T_B^{α} occurs at x_{i+k_1+1}, then

$$\|U_i^{-\alpha}\| \geq \|U_{i_0}^{-\alpha_0}\| \geq N_0. \tag{8.5}$$

From now on and until Lemma 8.10, we fix a B-stacking sequence $\sigma = (x_i)_{i \in I}$ and denote by S_0 the series $U_{i_0}^{-\alpha_0}$.

LEMMA 8.3. *There exists some word $u_0 \in X^*$ and some sign $\alpha'_0 \in \{-, +\}$ such that $S_0 = U_0^{\alpha'_0} \odot u_0$.*

Proof. One can prove by induction on $i \in \mathbb{N}$ that, for every $\alpha \in \{-, +\}$, U_i^{α} has one of the two following forms:

(1) $U_i^{\alpha} = U_0^{\alpha'} \odot u$ for some $\alpha' \in \{-, +\}$, $|u| \leq i$;

(2) $U_i^{\alpha} = \sum_{k=1}^{q} \beta_k \cdot (U_0^{\alpha'} \odot uu_k)$

for some deterministic rational vector β, $\alpha' \in \{-, +\}$, $|u \cdot u_k| \leq i$, $|u_k| \leq k_0$. $\square$

LEMMA 8.4. *Suppose that $i_0 \leq j < i$, no T_B occurs in $[j + 1, i]$, $U_j^{-\alpha}$ is D_1-marked, and U_j^{α} is unmarked. Then, for every $j' \in [j, i]$, $\|U_{j'}^{\alpha}\| \geq \|U_i^{\alpha}\| - k_2$.*

Proof. Let i, j fulfill the hypothesis of the lemma.

(1) Let us first treat the case where $j' = j$. If $(i - j) \leq (D_1 + 1)k_1$, then, by Lemma 3.13,

$$\|U_i^{\alpha}\| \leq \|U_j^{\alpha}\| + (D_1 + 1) \cdot k_1 \cdot K_0 \leq k_2;$$

hence the lemma is true.

Let us suppose now that $(i - j) \geq (D_1 + 1)k_1 + 1$. We can then define the integers $j < i_1 < i_2 < i$ by

$$i_1 = j + D_1 \cdot k_1, \ i_2 = i - k_1 - 1.$$

By Lemma 3.13 we know that

$$\|U^{\alpha}_{i_1}\| \leq \|U^{\alpha}_{j}\| + D_1 \cdot k_1 \cdot K_0 \quad \text{and} \quad \|U^{\alpha}_{i}\| \leq \|U^{\alpha}_{i_2}\| + (k_1+1) \cdot K_0. \tag{8.6}$$

If there were some stacking subderivation of length k_1 in $U^{-\alpha}_{j} \to U^{-\alpha}_{i_1}$, as all the U^{α}_{k} (for $k \in [j, i]$) are unmarked, T_B would occur at some integer in $[j + k_1 + 1, i_1 + 1]$, which is untrue. Hence there is no such stacking subderivation, and by Lemma 3.32 $U^{-\alpha}_{i_1}$ is unmarked.

If there were some stacking subderivation of length k_1 in $U^{\alpha}_{i_1} \to U^{\alpha}_{i_2}$, as all the $U^{-\alpha}_{k}$ (for $k \in [i_1, i]$) are unmarked, T_B would occur at some integer in $[i_1 + k_1 + 1, i]$, which is untrue. Hence there is no such stacking subderivation, and by Lemma 3.31

$$\|U^{\alpha}_{i_2}\| \leq \|U^{\alpha}_{i_1}\| + k_1 \cdot K_0. \tag{8.7}$$

Adding inequalities (8.6), (8.7) we obtain

$$\|U^{\alpha}_{i}\| \leq \|U^{\alpha}_{j}\| + (D_1 \cdot k_1 + 2 \cdot k_1 + 1) \cdot K_0 = \|U^{\alpha}_{j}\| + k_2,$$

which was to be proved.

(2) Let us suppose now that $j \leq j' \leq i$. If $(i-j) \leq (D_1+1)k_1$, the same inequality is true for $i - j'$ and the conclusion is true for j'.

Otherwise, if $j' \leq i_1$, then (8.6), (8.7) are still true for j' instead of j, and hence the conclusion too.

Otherwise, by the arguments of part (1), $U^{-\alpha}_{j'}, U^{\alpha}_{j'}$ are both unmarked. Therefore the hypotheses of part (1) are met by (j', i) instead of (j, i), and hence the conclusion is met too. (We illustrate our argument in Figure 4.) ∎

LEMMA 8.5. *Let $i \in I$, $\alpha \in \{-, +\}$ such that T^{α}_{B} occurs at $i + k_1 + 1$. Then there exists $u \in X^*$, $|u| \leq (i - i_0)$, $U^{-\alpha}_{i} = S_0 \odot u$, and, for every prefix $w \preceq u$,*

$$\|S_0 \odot w\| \geq \|S_0\| - k_3.$$

Proof. We prove the lemma by induction on $i \in [i_0, \infty[$.

Basis: $i = i_0$.

Choosing $u = \epsilon$, the lemma is true.

Induction step: $i_0 \leq i' < i$, $T^{\alpha'}_{B}$ occurs at $i' + k_1 + 1$, T^{α}_{B} occurs at $i + k_1 + 1$, and T_B does not occur in $[i' + k_1 + 2, i + k_1]$.

By the induction hypothesis, there exists some $u' \in X^*$, $|u'| \leq (i' - i_0)$, fulfilling

$$U^{-\alpha'}_{i'} = S_0 \odot u', \tag{8.8}$$

$$\forall w' \preceq u', \quad \|S_0 \odot w'\| \geq \|S_0\| - k_3. \tag{8.9}$$

Let us define $j = i' + k_1 + 1$.

Let $\bar{u} \in X^*$ be the word such that

$$U^{-\alpha}_{j} \xrightarrow{\bar{u}} U^{-\alpha}_{i} \tag{8.10}$$

is the derivation described by the $-\alpha$ component of the path from x_j to x_i.

Case 1. $\alpha' = \alpha$.

$$U^{-\alpha}_{j} = U^{-\alpha'}_{i'} \odot u_1$$

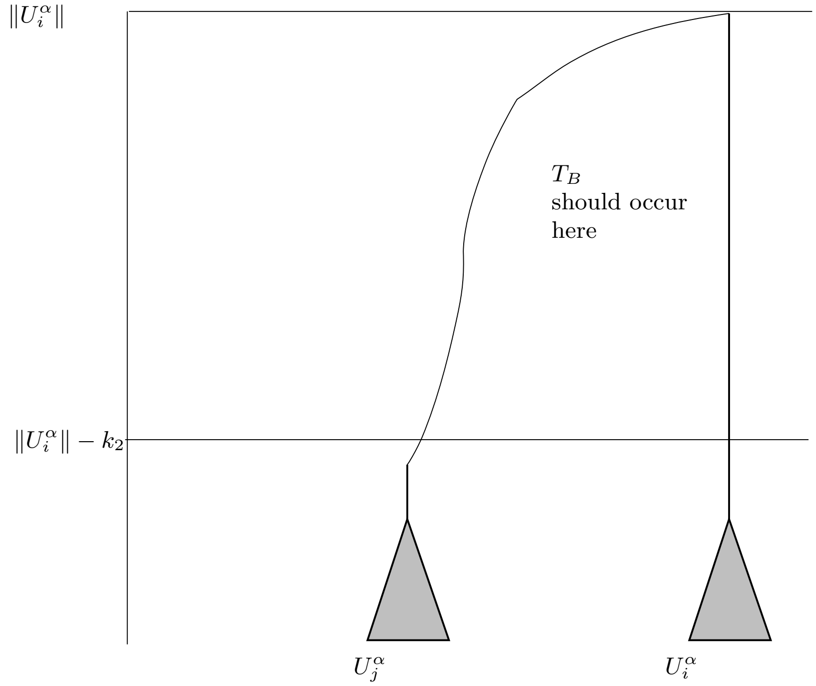

FIG. 4. $\|U_j^\alpha\|$ *too small is impossible.*

for some $u_1 \in X^*$, $|u_1| = k_1$, and U_j^α is D_1-marked. Let us choose $u = u' \cdot u_1 \cdot \bar{u}$. Hence

$$U_i^{-\alpha} = S_0 \odot u. \tag{8.11}$$

Let us consider some prefix w of u.

Subcase 1. $w \preceq u'$. By (8.9) we know that $\|S_0 \odot w\| \geq \|S_0\| - k_3$.

Subcase 2. $w = u' \cdot u_1 \cdot u''$ for some $u'' \preceq \bar{u}$. By Lemma 8.4 we know that $\|S_0 \cdot w\| \geq \|U_i^\alpha\| - k_2$, and by the definition of a B-stacking sequence we also know that $\|U_i^\alpha\| \geq \|S_0\|$. Hence

$$\|S_0 \odot w\| \geq \|S_0\| - k_2.$$

Subcase 3. $w = u' \cdot u_1'$, where u_1' is a prefix of u_1. Then, by Lemma 3.13 and the above inequality, we get

$$\|S_0 \odot w\| \geq \|S_0 \odot u' u_1\| - k_1 \cdot K_0 \geq \|S_0\| - k_3.$$

Case 2. $-\alpha' = \alpha$.

$$U_j^{-\alpha} = \sum_{k=1}^{q} \beta_k \cdot (U_{i'}^\alpha \odot u_k),$$

where β is a polynomial which is fully marked and every $|u_k| \leq k_0$.

By Lemma 3.15 either $U_i^{-\alpha} = \sum_{k=1}^{q} (\beta_k \odot \bar{u}) \cdot (U_{i'}^\alpha \odot u_k)$ or there exists a decomposition

$$\bar{u} = \bar{u}_1 \cdot \bar{u}_2 \tag{8.12}$$

and an integer $k \in [1, q]$ such that

$$U_i^{-\alpha} = U_{i'}^{\alpha} \odot u_k \bar{u}_2. \tag{8.13}$$

But, as $U_i^{-\alpha}$ is unmarked (by definition of T_B^{α}), the first formula is impossible unless $\beta \odot \bar{u}$ is unitary or null. Hence (8.12), (8.13) is the only possibility.

Let us choose $u = u' \cdot u_k \cdot \bar{u}_2$. It is clear from (8.13) that $U_i^{-\alpha} = S_0 \odot u$.

Let us consider some prefix w of u.

Subcase 1. $w \preceq u'$. We use the same arguments as in Case 1, Subcase 1.

Subcase 2. $w = u' \cdot u_k \cdot u''$ for some $u'' \preceq \bar{u}_2$. By Lemma 8.4 applied to the interval $[j + |\bar{u}_1| + 1, i]$, we can conclude that

$$\|S_0 \odot w\| \geq \|S_0\| - k_3.$$

Subcase 3. $w = u' \cdot u'_k$, where u'_k is a prefix of u_k. We use the same arguments as in Case 1, Subcase 3. ∎

Let us now define the following set of vectors and d-spaces:

$$\mathcal{G}_0 = \{S_0 \odot u \mid u \in X^*, |u| \leq k_4\}, \tag{8.14}$$

$$V_0 = \mathsf{V}(\mathcal{G}_0). \tag{8.15}$$

LEMMA 8.6. *Let $i \geq i_0$ such that T_B occurs at i. Then $U_i^-, U_i^+ \in V_0$.*

Proof. Let us suppose that T_B^{α} occurs at i. By Lemma 8.5, $U_{i-k_1-1}^{-\alpha} = S_0 \odot u$ and, for every prefix $w \preceq u$,

$$\|S_0 \odot w\| \geq \|S_0\| - k_3.$$

By Lemma 3.14, $\exists u_1, u_2 \in X^*$, $v_1 \in V^*$, $E_1, \ldots, E_k \in V$, $E_1 \smile E_2 \smile \cdots \smile E_k$, $\Phi \in \mathsf{DRB}_{q,\lambda}\langle\langle\, V\, \rangle\rangle$, such that $u = u_1 \cdot u_2$,

$$S_0 \odot u_1 = S_0 \bullet v_1 = \sum_{k=1}^{q} E_k \cdot \Phi_k, \tag{8.16}$$

$$S_0 \odot u = \sum_{k=1}^{q} (E_k \odot u_2) \cdot \Phi_k. \tag{8.17}$$

Without loss of generality, we can suppose that v_1 is a minimal word realizing the equality (8.16). Let us notice that, as G is a reduced grammar, for every $v \preceq v_1$, there exists some $\bar{v} \in X^*$, such that $S_0 \bullet v = S_0 \odot \bar{v}$. Hence, for every $v \preceq v_1$,

$$S_0 \bullet v = U_0^{\alpha'_0} \odot u_0 \cdot \bar{v} \quad \text{and} \quad \|U_0^{\alpha'_0} \odot u_0 \cdot \bar{v}\| \geq \|S_0 \odot u_1\| > D_2 = \|U_0^{\alpha'_0}\|.$$

By Lemma 3.29, all the vectors $S_0 \bullet v$ for $v \preceq v_1$ are loop-free. It follows that, for every $v \preceq v' \preceq v_1$,

$$v \prec v' \Rightarrow \|S_0 \bullet v\| > \|S_0 \bullet v'\|,$$

and hence

$$|v_1| \leq \|S_0\| - \|S_0 \bullet v_1\| \leq k_3.$$

The formula (8.17) can be rewritten

$$U^{-\alpha}_{i-k_1-1} = \sum_{k=1}^{q}(E_k \odot u_2) \cdot (S_0 \bullet v_1 E_k) = \sum_{k=1}^{q}(E_k \odot u_2) \cdot (S_0 \odot \bar{u}_k),$$

where $\bar{u}_k \in X^*$, $|\bar{u}_k| \leq (k_3 + 1) \cdot K_0$.

Using Lemmas 3.15 and 3.11 we can deduce from the above form of $U^{-\alpha}_{i-k_1-1}$ that

$$U_i^{\alpha} \in \mathsf{V}(\{S_0 \odot w \mid w \in X^*, |w| \leq (k_3+1) \cdot K_0 + k_0\}),$$
$$U_i^{-\alpha} \in \mathsf{V}(\{S_0 \odot w \mid w \in X^*, |w| \leq (k_3+1) \cdot K_0 + k_1\}),$$

and hence that both $U_i^{-\alpha}, U_i^{\alpha}$ belong to V_0. ∎

We recall that

$$K_1 = k_1 \cdot K_0 + 1, \quad K_2 = k_1^2 \cdot D_1 \cdot K_0 + k_1^2 \cdot K_0 + 2 \cdot k_1 \cdot K_0 + D_1 \cdot k_1 + 2 \cdot k_1 + 4.$$

LEMMA 8.7. *For every $L \geq 0$ there exists $i \in [i_0 + L, i_0 + K_1 \cdot L + K_2]$ such that $U_i^-, U_i^+ \in V_0$.*

Proof. Let us establish that

(8.18)
$$\exists i \in [i_0 + L, i_0 + K_1 \cdot L + K_2 - k_1 - 1], \exists \alpha \in \{-,+\}, \quad T_B^{\alpha} \text{ occurs at } i + k_1 + 1.$$

Let $L \geq 0$ and let $i' \geq i_0$ be the greatest integer in $[i_0, i_0 + L]$ such that T_B occurs at $i' + k_1 + 1$. Let $j = i' + k_1 + 1$. We then have

$$U_j^{\alpha'} = \sum_{k=1}^{q} \beta_k \cdot (U_{i'}^{-\alpha'} \odot u_k),$$

where $\|\beta\| \leq D_1$ and $U_j^{-\alpha'}$ is unmarked.

Case 1. *There exists $i \in [j, j + k_1 \cdot D_1]$, such that T_B occurs at $i + k_1 + 1$.* In this case the small constants $K_1 = 0$, $K_2 = k_1 \cdot D_1 + k_1 + 1$ would be sufficient to satisfy (8.18). A fortiori the given constants satisfy (8.18).

Case 2. *There exists no $i \in [j, j + k_1 \cdot D_1]$, such that T_B occurs at $i + k_1 + 1$.* Then there is no stacking subderivation of length k_1 in $U_j^{\alpha'} \longrightarrow U^{\alpha'}_{j+k_1 \cdot D_1}$. By Lemma 3.32 it follows that both $U^{\alpha}_{j+D_1 \cdot k_1}$ are unmarked.

(1) Let $j_1 = j + D_1 \cdot k_1$ and let us show that there exists some $i \geq j_1$ such that T_B occurs at $i + k_1 + 1$.

If such an i does not exist, then for every $\alpha \in \{-,+\}$, the infinite derivation

$$U^{\alpha}_{j_1} \longrightarrow U^{\alpha}_{j_1+1} \longrightarrow \cdots$$

does not contain any stacking sequence of length k_1. By Lemma 3.31 we would have

$$\forall k \geq j_1, \quad \|U_k^{\alpha}\| \leq \|U^{\alpha}_{j_1}\| + k_1 \cdot K_0.$$

As the set $\{\|U_k^{\alpha}\|, \, k \geq j_1, \, \alpha \in \{-,+\}\}$ is finite, there would be a repetition

$$(U_k^-, U_k^+) = (U_{k'}^-, U_{k'}^+) \quad \text{with } j_1 \leq k < k' \text{ and } \pi_k < \pi_{k'},$$

so that T_{cut} would have been defined on some finite prefix of the branch, contradicting the hypothesis that the branch is infinite.

(2) Let $i > i'$ be the smallest integer (in $[j_1, \infty[$) fulfilling point (1) above, and suppose that T_B^α occurs at $i + k_1 + 1$.

By Lemma 8.4,

$$\forall \ell \in [j_1, i], \quad \|U_\ell^{-\alpha}\| \geq N_0 - k_2 > D_2.$$

Using Lemma 8.3, Lemma 3.29, and inequality (8.1) we conclude that

$$\forall \ell \in [j_1, i], \quad U_\ell^{-\alpha} \text{ is loop-free.}$$

By an argument analogous to that used in Lemma 8.3 we see that $U_{j_1}^{-\alpha} = S_0 \odot u$ for some $|u| \leq (j_1 - i_0)$, and by Lemma 3.13 we get

$$\|U_{j_1}^{-\alpha}\| \leq (j_1 - i_0) \cdot K_0 + \|S_0\|. \tag{8.19}$$

We also know that

$$\|S_0\| \leq \|U_i^{-\alpha}\| \leq \|U_{i-1}^{-\alpha}\| + K_0. \tag{8.20}$$

As the derivation $U_{j_1}^{-\alpha} \longrightarrow U_{i-1}^{-\alpha}$ contains no stacking subderivation of length k_1 and consists of loop-free series only, by Lemma 3.30 we obtain

$$\|U_{i-1}^{-\alpha}\| \leq \|U_{j_1}^{-\alpha}\| - (i - j_1 - 2)/k_1. \tag{8.21}$$

Combining the three inequalities (8.19), (8.20), (8.21) we get successively

$$\|S_0\| \leq \|S_0\| + (j_1 - i_0 + 1) \cdot K_0 - (i - j_1 - 2)/k_1,$$

$$(i - j_1 - 2) \leq (j_1 - i_0 + 1) \cdot k_1 K_0,$$

(8.22)
$$\begin{aligned}(i - i') &= (i - j_1 - 2) + (j_1 - i' + 2) \leq (j_1 - i_0 + 1) \cdot k_1 \cdot K_0 + D_1 \cdot k_1 + k_1 + 3\\ &= (i' - i_0) \cdot k_1 \cdot K_0 + k_1^2 \cdot D_1 \cdot K_0 + k_1^2 \cdot K_0 + 2 \cdot k_1 \cdot K_0 + D_1 \cdot k_1 + k_1 + 3\\ &= (K_1 - 1)(i' - i_0) + K_2 - k_1 - 1.\end{aligned}$$

(3) By the choice of i', i, we know that $i' \leq i_0 + L \leq i$. Using (8.22) we obtain

$$i \leq i' + (K_1 - 1)(i' - i_0) + K_2 - k_1 - 1,$$

$$i \leq i_0 + K_1 \cdot L + K_2 - k_1 - 1.$$

Assertion (8.18) is now established for Case 2 as well as for Case 1.

From (8.18) and Lemma 8.6 the lemma follows.

(We illustrate our argument in Figure 5.) □

Let us give now a stronger version of Lemma 8.7 in which we analyze the *size of the coefficients* of the linear combinations whose existence is proved in Lemma 8.7. We recall that

$$K_3 = K_0|Q|, \quad K_4 = D_1.$$

Let us fix a total ordering on $\mathcal{G}_0$:

$$\mathcal{G}_0 = \{\theta_1, \theta_2, \ldots, \theta_d\}, \text{ where } d = \text{Card}(\mathcal{G}_0).$$

Let us remark that $d \leq \text{Card}(X^{\leq k_4}) = d_0$.

LEMMA 8.8. *Let $L \geq 0$. There exists $i \in [i_0 + L, i_0 + K_1 \cdot L + K_2]$ and, for every $\alpha \in \{-, +\}$, there exists a deterministic rational family $(\beta_{i,j}^\alpha)_{1 \leq j \leq d}$ fulfilling*

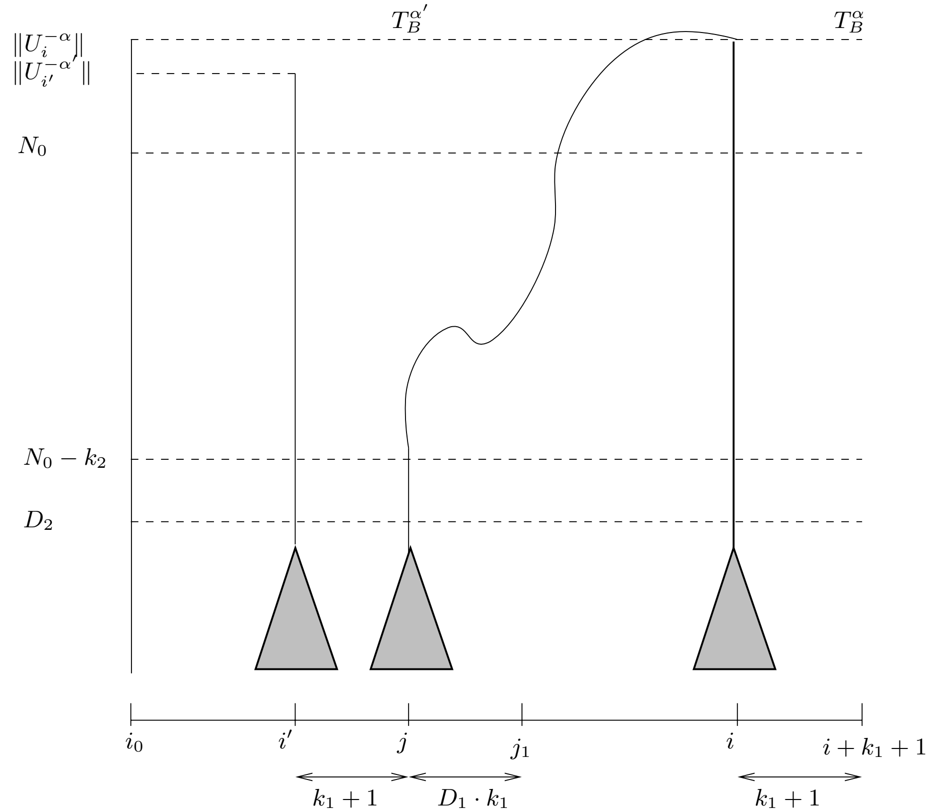

FIG. 5. *Two successive T_B.*

(1) $U_i^{\alpha} = \sum_{j=1}^{d} \beta_{i,j}^{\alpha} \cdot \theta_j$,
(2) $\|\beta_{i,*}^{\alpha}\| \leq K_3 \cdot (i - i_0) + K_4$.

Proof. By Lemma 8.7 there exists $i \in [i_0 + L, i_0 + K_1 \cdot L + K_2]$ and $\alpha \in \{-,+\}$ such that T_B^{α} occurs at i. Let us use the notation of the proof of Lemma 8.6 and compute upper-bounds on the coefficients of $U_i^{-\alpha}, U_i^{\alpha}$ expressed as linear combinations of the vectors of $\mathcal{G}_0$.

Coefficients of $U_i^{-\alpha}$. $U_i^{-\alpha} = U_{i-k_1-1}^{-\alpha} \odot u'$ for some $u' \in X^*$, $|u'| = k_1$. By Lemma 3.15, $U_i^{-\alpha}$ can be expressed in one of the two following forms:

$$U_i^{-\alpha} = S_0 \odot (\bar{u}_k u''), \text{ where } u'' \text{ is a suffix of } u', \tag{8.23}$$

$$U_i^{-\alpha} = \sum_{k=1}^{q} (E_k \odot u_2 u') \cdot (S_0 \odot \bar{u}_k). \tag{8.24}$$

In case (8.23) we can choose as vector of coordinates: $\beta_{i,\star}^{-\alpha} = \epsilon_{j_0}^{d}$. We then have $\|\beta_{i,\star}\| = 2 \leq K_4$.

In case (8.24), we can choose $\beta_{i,\star}^{-\alpha} = E \odot u_2 u'$ (completed with $\emptyset$ in all the columns j not corresponding to some vector $S_0 \odot \bar{u}_k$ of $\mathcal{G}_0$). We then have

$$\|\beta_{i,\star}\| = \|E \odot u_2 u'\| \leq K_0 \cdot (i - i_0) \leq K_3 \cdot (i - i_0).$$

Coefficients of U_i^α. By the definition of T_B^α

$$U_i^\alpha = \sum_{\ell=1}^{r} \tau_\ell \cdot (U_{i-k_1-1}^{-\alpha} \odot \bar{w}_\ell), \tag{8.25}$$

where $\|\tau\| \leq D_1$, $|\bar{w}_\ell| \leq k_0$.

Replacing u' by $\bar{w}_\ell$ in the above analysis, we get

$$\forall \ell \in [1, r], \quad U_{i-k_1-1}^{-\alpha} \odot \bar{w}_\ell = \sum_{j=1}^{d} \gamma_{\ell,j} \cdot \theta_j, \tag{8.26}$$

with $\|\gamma_{\ell,\star}\| \leq K_0 \cdot (i - i_0)$.

Equalities (8.25), (8.26) show that

$$U_i^\alpha = \tau \cdot \gamma \cdot \theta,$$

where τ, γ, θ are deterministic rational matrices of dimensions $(1, r)$, (r, d), $(d, 1)$, respectively. Let us choose $\beta_{i,\star} = (\tau \cdot \gamma)$.

$$\begin{aligned} \|\beta_{i,\star}\| &\leq \|\tau\| + \|\gamma\| \leq D_1 + r \cdot K_0 \cdot (i - i_0) \\ &\leq D_1 + |Q| \cdot K_0 \cdot (i - i_0) = K_3 \cdot (i - i_0) + K_4. \end{aligned} \qquad \square$$

Lemma 8.9. *There exists* $i_0 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_d$ *and deterministic rational vectors* $(\beta_{i,j}^\alpha)_{1 \leq j \leq d}$ *(for every* $i \in [1, d]$*) such that*

(0) $W(\kappa_1) \geq 1$,

(1) $\forall i, \forall \alpha, U_{\kappa_i}^\alpha = \sum_{j=1}^{d} \beta_{i,j}^\alpha \theta_j \in V_0$,

(2) $\forall i, \forall \alpha, \|\beta_{i,*}^\alpha\| \leq s_i$,

(3) $\forall i, W(\kappa_{i+1}) - W(\kappa_i) \geq \delta_{i+1}$,

where the sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \sigma_i)$ *are those defined by relations* (6.8), (6.9) *in section* 6.

Proof. Let us consider the additional property

(4) $\kappa_i - i_0 \leq L_i$.

We prove by induction on i the conjunction (1) $\wedge$ (2) $\wedge$ (3) $\wedge$ (4).

$\boldsymbol{i = 1}$**:** By Lemma 8.8, there exists $\kappa_1 \in [i_0, i_0 + K_2]$ such that $\forall \alpha \in \{-, +\}$, there exists a deterministic vector $(\beta_{1,j}^\alpha)_{1 \leq j \leq d}$, such that

$$U_{\kappa_1}^\alpha = \sum_{j=1}^{d} \beta_{1,j}^\alpha \theta_j,$$

and in addition $\|\beta_{1,*}^\alpha\| \leq K_3 K_2 + K_4 = s_1$.

$\boldsymbol{i \to i + 1}$**:** Suppose that $\kappa_1 < \kappa_2 < \cdots < \kappa_i$ fulfill (1) $\wedge$ (2) $\wedge$ (3) $\wedge$ (4). By Lemma 8.8, there exists $\kappa_{i+1} \in [i_0 + L_i + \ell_{i+1}, i_0 + K_1(L_i + \ell_{i+1}) + K_2]$ such that $\forall \alpha \in \{-, +\}$, there exists a deterministic vector $(\beta_{i+1,j}^\alpha)_{1 \leq j \leq d}$, such that

$$U_{\kappa_{i+1}}^\alpha = \sum_{j=1}^{d} \beta_{i+1,j}^\alpha \theta_j, \tag{8.27}$$

and in addition

$$\begin{aligned} \|\beta_{i+1,*}^\alpha\| \leq K_3(K_1(L_i + \ell_{i+1}) + K_2) + K_4 &= K_3 L_{i+1} + K_4 \\ &= s_{i+1}. \end{aligned} \tag{8.28}$$

By Lemma 8.2

$$2(W(\kappa_{i+1}) - W(\kappa_i)) + 3 \geq \kappa_{i+1} - \kappa_i \geq \ell_{i+1} = 2\delta_{i+1} + 3,$$

and hence

$$W(\kappa_{i+1}) - W(\kappa_i) \geq \delta_{i+1}. \tag{8.29}$$

Finally,

$$\kappa_{i+1} - i_0 \leq K_1(L_i + l_{i+1}) + K_2 = L_{i+1}. \tag{8.30}$$

The above properties (8.27)–(8.30) prove the required conjunction.

It remains to prove point (0): the integer κ_1 introduced by Lemma 8.8 is such that T_B occurs at κ_1, and hence

$$\begin{aligned} W(\kappa_1) &= W(\kappa_1 - k_1 - 1) + k_1 - 1 \\ &\geq W(\kappa_1 - k_1 - 1) + 2 \geq 1. \qquad \square \end{aligned}$$

LEMMA 8.10. *Let $(x_i)_{i\in\mathbb{N}}$ be an infinite branch of τ. Then there exist some $i_0 \in \mathbb{N}$ such that $(x_i)_{i\geq i_0}$ is a B-stacking sequence.*

Proof. Let us distinguish, a priori, several cases, and see that only the case where τ admits a B-stacking sequence is possible.

Case 1. *T_B occurs finitely often on τ.* Let j be the largest integer such that T_B occurs at j. By the arguments used in the proof of Lemma 8.7, Case 2, we know that $U^-_{j+k_1\cdot D_1}, U^+_{j+k_1\cdot D_1}$ are both unmarked, and that

$$\forall k \geq j + k_1 \cdot D_1, \forall \alpha \in \{-,+\}, \quad \|U^\alpha_k\| \leq \|U^\alpha_{j+k_1\cdot D_1}\| + k_1 \cdot K_0.$$

This would imply that the branch contains a finite prefix on which T_{cut} is defined, which is impossible on an infinite branch.

Case 2. *For some sign α, there are infinitely many integers i such that $[T^\alpha_B$ occurs at $i + k_1 + 1$ and $\|U^{-\alpha}_i\| < N_0]$.* In this case there would exist an infinite sequence of integers $i_1 < i_2 < \cdots < i_\ell <$ such that

$$\forall \ell \geq 0, \quad U^{-\alpha}_{i_1} = U^{-\alpha}_{i_\ell}.$$

For a given $U^{-\alpha}_i$, only a finite number of values are possible for the pair $(U^-_{i+k_1+1}, U^+_{i+k_1+1})$. Hence there exist integers $\ell < \ell'$ such that

$$\ell < \ell', \quad \pi_\ell < \pi_{\ell'}, \quad \text{and} \quad (U^-_{\ell+k_1+1}, U^+_{\ell+k_1+1}) = (U^-_{\ell'+k_1+1}, U^+_{\ell'+k_1+1}).$$

Here again T_{cut} would have a nonempty value on some prefix of τ, which is impossible.

Case 3. *T_B occurs infinitely often on τ and, for every sign α, there are only finitely many integers i such that $[T^\alpha_B$ occurs at $i + k_1 + 1$ and $\|U^{-\alpha}_i\| < N_0]$.*

Let us consider the set I_0 of the integers i such that there exists a sign α_i such that

$$[T^{\alpha_i}_B \text{ occurs at } i + k_1 + 1 \text{ and } \|U^{-\alpha_i}_i\| \geq N_0].$$

By the hypothesis of Case 3, $I_0 \neq \emptyset$. Let i_0 be such that

$$\|U^{-\alpha_{i_0}}_{i_0}\| = \min\{\|U^{-\alpha_i}_i\| \mid i \in I_0\}.$$

Then $(x_i)_{i\geq i_0}$ is a B-stacking sequence. $\square$

9. Termination.

LEMMA 9.1. *$\hat{\mathcal{S}}_{ABC}$ is terminating on every unmarked assertion A_0: if $A_0 \in \mathcal{A}$ is unmarked, then $\exists n_0 \in \mathbb{N}$, $\hat{\mathcal{S}}_{ABC}^{n_0+1}(A_0) = \hat{\mathcal{S}}_{ABC}^{n_0}(A_0)$.*

Proof. Suppose $A_0 \in \mathcal{A}$, A_0 is true, A_0 is unmarked, and

$$\forall n \in \mathbb{N}, \quad \hat{\mathcal{S}}_{ABC}^{n}(A_0) \prec \hat{\mathcal{S}}_{ABC}^{n+1}(A_0). \tag{9.1}$$

Let us consider all the constants associated to this precise A_0, the equivalence $\bar{\psi}$, and the dpda $\mathcal{M}$ in section 6. Let us note that $t_n = \hat{\mathcal{S}}_{ABC}^{n}(A_0)$ (for every $n \in \mathbb{N}$) and let

$$t_\infty = \text{least upper-bound}\{t_n \mid n \in \mathbb{N}\}.$$

Let us note that, by definition (7.10), the strict inequality (9.1) implies that

$$\forall n \in \mathbb{N}, \quad t_n \text{ is consistent.} \tag{9.2}$$

Let us denote by x_n the node of t_n such that $t_{n+1} = t_n[\hat{\Delta}(t_n)/x_n]$. Let us notice that as every x_n is unclosed in t_n, one can prove by induction that every t_n is repetition-free. Hence

$$t_\infty \text{ is repetition-free.} \tag{9.3}$$

By Koenig's lemma, t_∞ contains an infinite branch $y_0 y_1 \cdots y_s \cdots$ whose (infinite) labeling word is $A_0 A_1 \cdots A_s \cdots$ (where $A_s = t_\infty(y_s)$).

Condition (C3) in the definition of $T_C^{(\mathcal{O})}$, combined with Lemma 3.17, shows that every equation (π, T, U) produced by T_C has size

$$\max\{\|T\|, \|U\|\} \le D_2, \tag{9.4}$$

and hence that the number of possible unweighted equations produced by T_C is *finite*. Hence T_C occurs only a finite number of times on this branch (because t_∞ is repetition-free (9.3) and T_{cut} cannot occur on an infinite branch). Let n_0 be the last point where T_C occurs (or $n_0 = 0$ if T_C never occurs on this branch). $(y_{n_0+i})_{i\ge 0}$ is a branch of a tree $t' \in \mathcal{T}(\mathcal{S}_{AB}, A_{n_0})$. Let us notice also that

$$\text{every equation produced by } T_C \text{ is unmarked} \tag{9.5}$$

(by condition (C4) in the definition of $T_C^{(\mathcal{O})}$; see section 7), and

$$\text{every equation produced by } T_C \text{ has a length } \lambda \le \lambda_2, \tag{9.6}$$

because it has a length $\le d_0$ and $d_0 \le \lambda_2$ by definition (6.11) in section 6. Moreover, the root A_0 of t_∞ must have a size $\le D_2$ (by definition (6.10) in section 6), must be unmarked (by the hypothesis of the lemma), and must have a length $\lambda_0 \le \lambda_2$ (by definition (6.11) in section 6). Hence, in either case, t' fulfills the hypotheses (8.1) and (8.2) stated in section 3.3 and assumed in section 8.

As $\mathcal{S}_{ABC}$ is a strategy for $\mathcal{B}_0$ and A_0 is true, A_{n_0} is also true, and hence hypothesis (8.3) assumed in section 8 is fulfilled. We may now apply the results obtained in section 8.2.

By Lemma 8.10, the branch $(y_{n_0+i})_{i\ge 0}$ must contain an infinite B-stacking sequence. Let us remark that, as $T_\emptyset$ does not occur (otherwise the branch would be finite), every equation (π, U^-, U^+) labeling this branch is such that $U^- \ne \emptyset$, $U^+ \ne \emptyset$.

By Lemma 8.9 such a B-stacking sequence contains a subsequence $(A_{\kappa_1}, A_{\kappa_2}, \ldots, A_{\kappa_d})$ with $d \leq d_0$, fulfilling hypotheses (1), (2) of Lemma 5.5, and by the above remark it fulfills hypothesis (5.6) of section 5.2 too. Let $n_i \in \mathbb{N}$ such that $x_{n_i} = y_{\kappa_i}$ for $1 \leq i \leq d$. By (9.2), $\Omega(\bar{\Pi}(t_{n_d}), \mathrm{im}(t_{n_d})) \neq \emptyset$. Let us consider some

$$\mathcal{O} \in \Omega(\bar{\Pi}(t_{n_d}), \mathrm{im}(t_{n_d})).$$

Let $\mathcal{S}_d = (A_{\kappa_i})_{1 \leq i \leq d}$ and $D = \mathrm{D}^{(\mathcal{O})}(\mathcal{S}_d)$. By Lemma 5.5,

$$\text{(9.7)} \qquad \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}_d) \neq \perp, \quad D \in [0, d-1], \quad \text{and} \quad \||\, \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}_d) \,\|| \leq \Sigma_{d_0} + s_{d_0}.$$

Let $\mathcal{S}_{D+1} = (A_{\kappa_i})_{1 \leq i \leq D+1}$. By hypothesis (2) of Lemma 5.5 (we established that this hypothesis is true),

$$\bar{\Pi}(t_{n_{D+1}}) \leq \bar{\Pi}(t_{n_d}),$$

and it is straightforward that

$$\mathrm{im}(t_{n_{D+1}}) \subseteq \mathrm{im}(t_{n_d});$$

hence,

$$\text{(9.8)} \qquad \mathcal{O} \in \Omega(\bar{\Pi}(t_{n_{D+1}}), \mathrm{im}(t_{n_{D+1}})).$$

Let $W_{D+1} = A_0 \cdot A_1 \cdots A_{\kappa_1} \cdots A_{\kappa_{D+1}}$ (the word from the root to $y_{\kappa_{D+1}}$). Let us notice that

$$\text{(9.9)} \qquad \mathrm{D}^{(\mathcal{O})}(\mathcal{S}_{D+1}) = \mathrm{D}^{(\mathcal{O})}(\mathcal{S}_d) = D, \quad \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}_{D+1}) = \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}_d).$$

By (9.7), (9.9),

$$\text{(9.10)} \qquad \rho_e(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}_{D+1})) \in T_C^{(\mathcal{O})}(W_{D+1}).$$

By (9.8), (9.10), the set $\{\mathcal{O} \in \Omega(\bar{\Pi}(t_{n_{D+1}}), \mathrm{im}(t_{n_{D+1}})), T_C^{(\mathcal{O})}(W_{D+1}) \neq \emptyset\}$ is not empty, so that case (0) of the definition of $\hat{\Delta}(t)$ (see section 7) is fulfilled and

$$\hat{\Delta}(t_{n_{D+1}}) = A_{n_{D+1}}(T_C^{(\mathcal{O}_0)}(W_{D+1})),$$

i.e., T_C occurs at $y_{\kappa_{D+1}+1}$. This is a contradiction with the minimality of n_0. We have proved that hypothesis (9.1) is impossible. Hence the lemma is proved. □

10. Elimination.

10.1. System $\mathcal{B}_1$.

We prove here that the new formal system $\mathcal{B}_1$ obtained by *elimination* of metarule (R5) in $\mathcal{B}_0$ is recursively enumerable and complete. The decidability of the bisimulation problem follows.

Let $\mathcal{B}_1 = \langle \mathcal{A}, H, \models\!\!-_{\mathcal{B}_1} \rangle$, where $\mathcal{A}, H$ are the same as in $\mathcal{B}_0$, but the *elementary deduction relation* $|\!\models\!\!-_{\mathcal{B}_1}$ is the relation generated by the subset of metarules (R0), (R1), (R2), (R3), (R′3), (R4), (R6), (R7), (R8), i.e., all the metarules of $\mathcal{B}_0$ except (R5). The deduction relation $\models\!\!-_{\mathcal{B}_1}$ is now defined by

$$\models\!\!-_{\mathcal{B}_1} = \overset{\langle * \rangle}{|\!\models\!\!-}_{\mathcal{B}_1} \circ \overset{[1]}{|\!\models\!\!-}_{R0,R3,R'3,R4} \circ \overset{\langle * \rangle}{|\!\models\!\!-}_{\mathcal{B}_1}.$$

LEMMA 10.1. *$\mathcal{B}_1$ is a deduction system.*

Sketch of proof. As $\models\!\!-_{\mathcal{B}_1} \subseteq \models\!\!-_{\mathcal{B}_0}$, property (A1) is fulfilled by $\models\!\!-_{\mathcal{B}_1}$.

By the well-known decidability properties for finite-automata, rules (R0), (R1), (R2), (R3), (R′3), (R4), (R6), (R7), (R8) are recursively enumerable. Hence property (A2) is fulfilled by $\mathcal{B}_1$. □

Completeness.

Definition 10.2. *Let P be a finite subset of $\mathcal{A}$ and let $\bar{\pi} \in \mathbb{N}$. P is said to be* locally $\bar{\pi}$-consistent *iff, for every $(\pi, S, S') \in P$, if*

$$\pi < \bar{\pi},$$

then there exists $\mathcal{R}_1 \in \bar{\mathcal{B}}_1$ such that

$$[\pi, S, S', \mathcal{R}_1] \subseteq \mathrm{Cong}(P).$$

Lemma 10.3. *Let P be a finite subset of $\mathcal{A}$ and let $\bar{\pi} \in \mathbb{N}$. If P is locally $\bar{\pi}$-consistent, then P is $\bar{\pi}$-consistent.*

Proof. Let us consider, for every integer $n \geq 0$ $p \geq 0$, the following property $\mathcal{Q}(n,p)$: $\forall \pi \in \mathbb{N}$, $\lambda \in \mathbb{N} - \{0\}$, $S, S' \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V\, \rangle\rangle$,

$$(\pi, S, S') \in \mathrm{Cong}_p(P) \quad \text{and} \quad \pi + n - 1 < \bar{\pi} \Rightarrow$$
$$\exists \mathcal{R}_n \in \mathcal{B}_n(S, S'), \;\; [\pi, S, S', \mathcal{R}_n] \subseteq \mathrm{Cong}(P). \tag{10.1}$$

Let us prove by lexicographic induction on (n, p) that

$$\forall (n,p) \in \mathbb{N} \times \mathbb{N}, \quad \mathcal{Q}(n,p). \tag{10.2}$$

$\boldsymbol{n = 0,\ p = 0}$**:** The only possible value of $\mathcal{R}_0 \in \mathcal{B}_0(S, S')$ is $\mathcal{R}_0 = \{(\epsilon, \epsilon)\}$, and $[\pi, S, S', \mathcal{R}_0] = \{(\pi, S, S')\} \subseteq \mathrm{Cong}_0(P)$.

$\boldsymbol{p > 0}$**:** There exists a subset $Q \subseteq \mathcal{P}_f(\mathcal{A})$, such that

$$P \overset{\langle p-1 \rangle}{|\!\!\vdash\!\!-}_{\mathcal{C}} Q \quad \text{and} \quad Q \overset{\langle 1 \rangle}{|\!\!\vdash\!\!-}_{\mathcal{C}} \{(\pi, S, S')\}.$$

As every rule of $\mathcal{B}_0$ increases the weight, we can suppose that every assertion of Q has a weight $\leq \pi$. Hence, by the induction hypothesis,

$$\forall (\pi', T, T') \in Q, \; \exists \mathcal{R}_n \in \mathcal{B}_n(T, T'), \quad [\pi', T, T', \mathcal{R}_n] \subseteq \mathrm{Cong}(P). \tag{10.3}$$

Let us consider the type of rule used in the last step, $Q \overset{\langle 1 \rangle}{|\!\!\vdash\!\!-}_{\mathcal{C}} \{(\pi, S, S')\}$, of the above deduction.

(R0): $(\pi - 1, S, S') \in Q$. By (10.3), $\exists \mathcal{R}_n \in \mathcal{B}_n(S, S')$,

$$[\pi - 1, S, S', \mathcal{R}_n] \subseteq \mathrm{Cong}(P).$$

As $[\pi - 1, S, S', \mathcal{R}_n] \overset{\langle 1 \rangle}{|\!\!\vdash\!\!-}_{\mathcal{C}} [\pi, S, S', \mathcal{R}_n]$,

$$[\pi, S, S', \mathcal{R}_n] \subseteq \mathrm{Cong}(P).$$

(R1): $(\pi, S', S) \in Q$ (analogous to the above case).

(R2): $(\pi, S, T), (\pi, T, S') \in Q$. By (10.3), $\exists \mathcal{R}_n \in \mathcal{B}_n(S, T)$, $\mathcal{R}'_n \in \mathcal{B}_n(T, S')$,

$$[\pi, S, T, \mathcal{R}_n] \subseteq \mathrm{Cong}(P), [\pi, T, S', \mathcal{R}'_n] \subseteq \mathrm{Cong}(P).$$

Using the properties mentioned in section 4.4, we get that

$$[\pi, S, S', \mathcal{R}_n \circ \mathcal{R}'_n] \subseteq \mathrm{Cong}(P).$$

(R3): In this case, $\mathcal{R}_n = \mathrm{Id} \cap X^{\leq n} \times X^{\leq n} \in \mathcal{B}_n(S, S')$, and

$$\begin{aligned}[\pi, S, S', \mathcal{R}_n] &\subseteq \{(\pi, S, S')\} \cup \{(\pi + k, T, T), 1 \leq k \leq n, T \in \mathsf{DRB}_{1,\lambda}\langle\langle\ V\ \rangle\rangle\} \\ &\subseteq \mathrm{Cong}(P).\end{aligned}$$

(R′3): In this case, $\mathcal{R}_n = \mathrm{Id} \cap X^{\leq n} \times X^{\leq n} \in \mathcal{B}_n(S, S')$, and

$$[\pi, S, S', \mathcal{R}_n] = \{(\pi + k, S \odot u, \rho_e(S) \odot u) \mid 0 \leq k \leq n, u \in X^k\} \subseteq \mathrm{Cong}(P)$$

(because $\rho_e(S) \odot u = \rho_e(S \odot u)$).

(R6): $(\pi, S_1 \cdot S' + U, S') \in Q$, $S = S_1^* \cdot U$. By (10.3), $\exists \mathcal{R}_n \in \mathcal{B}_n(S_1 \cdot S' + U, S')$,

$$[\pi, S_1 \cdot S' + U, S', \mathcal{R}_n] \subseteq \mathrm{Cong}(P).$$

Using the properties mentioned in section 4.4, we get that

$$\begin{aligned}[\pi, S, S', \mathcal{R}_n^{\langle S_1,*\rangle}] &= [\pi, S_1^* \cdot U, S', \mathcal{R}_n^{\langle S_1,*\rangle}] \\ &\subseteq \mathrm{Cong}[\pi, S_1 \cdot S' + U, S', \mathcal{R}_n] \\ &\subseteq \mathrm{Cong}(Q) \subseteq \mathrm{Cong}(P).\end{aligned}$$

(R7): $(\pi, S_1, S_1') \in Q$, $S = S_1 \cdot T$, $S' = S_1' \cdot T$. By (10.3), $\exists \mathcal{R}_n \in \mathcal{B}_n(S_1, S_1')$, $[\pi, S_1, S_1', \mathcal{R}_n] \subseteq \mathrm{Cong}(P)$. Using the properties mentioned in section 4.4, we get that

$$\begin{aligned}[\pi, S, S', \langle S_1 | \mathcal{R}_n\rangle] &= [\pi, S_1 \cdot T, S_1' \cdot T, \langle S_1 | \mathcal{R}_n\rangle] \\ &\subseteq \mathrm{Cong}([\pi, S_1, S_1', \mathcal{R}_n]) \\ &\subseteq \mathrm{Cong}(Q) \subseteq \mathrm{Cong}(P).\end{aligned}$$

(R8): $\forall i \in [1, \delta]$, $(\pi, T_{i,*}, T'_{i,*}) \in Q$, $S = S_1 \cdot T$, $S' = S_1 \cdot T'$. By (10.3), $\exists \mathcal{R}_{1,n}, \ldots, \mathcal{R}_{\delta,n} \in \mathcal{B}_n(T_{i,*}, T'_{i,*})$, such that

$$[\pi, T_{i,*}, T'_{i,*}, \mathcal{R}_{i,n}] \subseteq \mathrm{Cong}(P).$$

Using the properties mentioned in section 4.4, we get that

$$\begin{aligned}[\pi, S, S', \langle S, \mathcal{R}_{*,n}\rangle] &= [\pi, S_1 \cdot T, S_1 \cdot T', \langle S, \mathcal{R}_{*,n}\rangle] \\ &\subseteq \mathrm{Cong}\left(\bigcup_{1 \leq i \leq \delta} [\pi, T_{i,*}, T'_{i,*}, \mathcal{R}_{i,n}]\right) \\ &\subseteq \mathrm{Cong}(Q) \subseteq \mathrm{Cong}(P).\end{aligned}$$

In all cases $\mathcal{Q}(n, p)$ has been established.

***n* > 0, *p* = 0:** $(\pi, S, S') \in P$. As P is locally $\bar{\pi}$-consistent and $\pi \leq \pi + n - 1 < \bar{\pi}$, there exist $\mathcal{R}_1 \in \mathcal{B}_1(S, S')$, $q \in \mathbb{N}$ such that

$$[\pi, S, S', \mathcal{R}_1] \subseteq \mathrm{Cong}_q(P). \tag{10.4}$$

As $(n-1, q) < (n, 0)$, by the induction hypothesis, $\forall (x, x') \in \mathcal{R}_1 \cap X \times X$, $\exists \mathcal{R}_{x,x',n-1} \in \mathcal{B}_{n-1}(S \odot x, S' \odot x')$ such that

$$[\pi + 1, S \odot x, S' \odot x', \mathcal{R}_{x,x',n-1}] \subseteq \mathrm{Cong}(P). \tag{10.5}$$

Let us consider $\mathcal{R}_n = \{(\epsilon,\epsilon)\}\bigcup_{(x,x')\in\mathcal{R}_1\cap X\times X}\{(x,x')\}\cdot\mathcal{R}_{x,x',n-1}$. One can check that $\mathcal{R}_n \in \mathcal{B}_n(S,S')$ and, by (10.4), (10.5), we obtain

$$[\pi, S, S', \mathcal{R}_n] = \{(\pi, S, S')\} \bigcup_{(x,x')\in\mathcal{R}_1\cap X\times X} [S\odot x, S'\odot x', \mathcal{R}_{x,x',n-1}] \subseteq \mathrm{Cong}(P).$$

Let us define now an oracle $\mathcal{O}\in\Omega$ which is $\bar{\pi}$-consistent with P. For every $(S,S')\in \bigcup_{\lambda\geq 1}\mathsf{DRB}_{1,\lambda}\langle\langle\ V\ \rangle\rangle \times \mathsf{DRB}_{1,\lambda}\langle\langle\ V\ \rangle\rangle$ occurring in $\mathrm{Cong}(P)$ (i.e., as the last two components of an assertion in $\mathrm{Cong}(P)$), let us note the following:

$$\begin{aligned} W(S,S') &= \min\{\pi\in\mathbb{N} \mid (\pi,S,S')\in\mathrm{Cong}(P)\},\\ D(S,S') &= \max\{\bar{\pi} - W(S,S'), 0\},\\ C(S,S') &= \min\{\mathcal{R}\in\mathcal{B}_{D(S,S')}(S,S') \mid [W(S,S'),S,S',\mathcal{R}]\subseteq\mathrm{Cong}(P)\}.\end{aligned}$$

Notice that $C(S,S')$ is well-defined, owing to property (10.2). We then define $\mathcal{O}$ as follows: for every (S,S') occurring in $\mathrm{Cong}(P)$,

$$\mathcal{O}(S,S') = \min\{\mathcal{R}\in\mathcal{B}_\infty(S,S') \mid C(S,S') = \mathcal{R}\cap(X^{\leq D(S,S')}\times X^{\leq D(S,S')})\}, \tag{10.6}$$

and for every (S,S') not occurring in $\mathrm{Cong}(P)$,

$$\mathcal{O}(S,S') = \min\{\mathcal{R}\in\mathcal{B}_\infty(S,S')\} \ \ (\text{if } S\sim S'),$$
$$\mathcal{O}(S,S') = \mathrm{Id}_{X^*} \ \ (\text{if } S\not\sim S'). \tag{10.7}$$

One can check that, by the choice of $C(S,S')$, $\mathcal{O}$ is $\bar{\pi}$-consistent with P. □

LEMMA 10.4. *Let $A_0\in\mathcal{A}$ such that $H(A_0)=\infty$. Let us consider the sequence of trees $t_n = \hat{\mathcal{S}}^n_{ABC}(A_0)$. For every integer $n\geq 0$, t_n is consistent.*

Let us say that the strategy T "applies to" node x iff x has exactly m sons $x\cdot 1, x\cdot 2,\ldots,x\cdot m$ and

$$\tau(x1)\cdot\tau(x\cdot 2)\cdots\tau(x\cdot m)\in T(\tau(x[0])\cdot\tau(x[1])\cdots\tau(x[|x|]));$$

i.e., the word consisting of the labels of the sons of x belongs to the image of the path from ϵ (included) to x (included) by the strategy T.

Proof. For every $k\in\mathbb{N}$ we define

$$\bar{\pi}_k = \bar{\Pi}(t_k).$$

We prove by induction on (n,π) the following property $\mathcal{R}(n,\pi)$:

$$\forall x\in\mathrm{dom}(t_n), \text{ if } t_n(x) = (\pi,S,S') \text{ with } \pi<\bar{\pi}_n, \text{ then} \tag{10.8}$$
$$\exists\mathcal{R}_1\in\mathcal{B}_1(S,S'), [\pi,S,S',\mathcal{R}_1]\subseteq\mathrm{Cong}(\mathrm{im}(t_n)). \tag{10.9}$$

At every step of our proof by induction, we consider some node x of t_n fulfilling hypothesis (10.8) and show that it must fulfill (10.9). Let us notice that if x is not closed, then hypothesis (10.8) cannot be true, by minimality of $\bar{\pi}_n$. Let us notice also that if x is closed, but there is some $x'\prec x$ such that $t_n(x')=t_n(x)$, then (10.9) on x is the same property as (10.9) for x'. Hence, in what follows, we can suppose that x is closed and that it is minimal (w.r.t. $\preceq$):

$$x = \min\nolimits_{\preceq}\{y\in\mathrm{dom}(t_n) \mid t_n(y) = t_n(x)\}. \tag{10.10}$$

$\boldsymbol{n=0,\ \pi=0}$**:** $\mathrm{dom}(t_0)=\{\epsilon\}$, $t_0(\epsilon)=A_0$. If ϵ is not closed, then $\bar{\pi}_0=\pi=0$, and hence there is no node x fulfilling hypothesis (10.8). Otherwise, $\bar{\pi}_0=\infty$ and $x=\epsilon$ is closed: either $T_\emptyset(A_0)=\{\epsilon\}$ or $T_\varepsilon(A_0)=\{\epsilon\}$. Let us choose

$$\mathcal{R}_1=\mathrm{Id}_{X^*}\cap X^{\leq 1}\times X^{\leq 1}. \tag{10.11}$$

If we note $A_0=(\pi,S_0^-,S_0^+)$, then

$$[\pi,S_0^-,S_0^+,\mathcal{R}_1]=\{(\pi,S_0^-,S_0^+)\}\cup\{(\pi+1,S_0^-\odot x,S_0^+\odot x)\mid x\in X\},$$

where, $\forall x\in X$, $S_0^-\odot x\equiv S_0^+\odot x\equiv\emptyset$. Using rule (R$'$3), we see that

$$[\pi,S,S',\mathcal{R}_1]\subseteq\mathrm{Cong}(\emptyset)\subseteq\mathrm{Cong}(\mathrm{im}(t_n)). \tag{10.12}$$

$\boldsymbol{n>0,\ \pi=0}$**:** Let x be some node of t_n such that $\exists S,S'$, $t_n(x)=(\pi,S,S')$ and $\pi<\bar{\pi}_n$. Let us denote by W_x the word labeling the path from the root of t_n (included) to x (included).

Case 1. $\exists x'\in\mathrm{dom}(t_n)$*, x' internal node, such that* $t_n(x')=t_n(x)$. As $\pi=0$, the sons $x'\cdot 1,x'\cdot 2,\dots,x'\cdot m$ of x' are such that $t_n(x'\cdot 1)\cdot t_n(x'\cdot 2)\cdots t_n(x'\cdot m)\in T_A^{(\mathcal{O})}(W_{x'})$ for some oracle $\mathcal{O}$. Let us choose

$$\mathcal{R}_1=\mathcal{O}(S,S')\cap X^{\leq 1}\times X^{\leq 1}. \tag{10.13}$$

Then

$$[\pi,S,S',\mathcal{R}_1]\subseteq\mathrm{im}(t_n). \tag{10.14}$$

Case 2. $T_\emptyset(W_x)=\{\epsilon\}$ *or* $T_\varepsilon(W_x)=\{\epsilon\}$. In this case the choice $\mathcal{R}_1=\mathrm{Id}_{X^*}\cap X^{\leq 1}\times X^{\leq 1}$ again satisfies (10.12).

$\boldsymbol{\pi>0}$**:** Let x fulfill hypothesis (10.8). As t_n is a proof-tree for $\mathcal{S}_{ABC}$, and as we suppose x is closed and minimal (10.10), one of the following cases must occur.

Case 1. T_{cut} *applies to* x. There exists $x'\in\mathrm{dom}(t_n)$, $\exists\pi'\in\mathbb{N}$, such that

$$t_n(x')=(\pi',S,S')\quad\text{and}\quad\pi'<\pi.$$

By the induction hypothesis

$$\exists\mathcal{R}_1\in\mathcal{B}_1(S,S'),\quad[\pi',S,S',\mathcal{R}_1]\subseteq\mathrm{Cong}(\mathrm{im}(t_n)),$$

and by means of rule (R0),

$$[\pi,S,S',\mathcal{R}_1]\subseteq\mathrm{Cong}([\pi',S,S',\mathcal{R}_1]).$$

Hence (10.9) is true.

Case 2. $T_\emptyset$ *or* T_ϵ *applies to* x. Here again, the choice (10.11) fulfills property (10.12).

In the remaining cases we use the following notation: for every $k\in\mathbb{N}$ such that t_k is not closed,

$$x_k=\min\{x\in\mathrm{dom}(t_k),\ x\text{ is not closed for }\mathcal{S}_{ABC}\text{ and }\exists S,S',\ t(x)=(\bar{\pi}_k,S,S')\}.$$

If $\exists k<n\mid t_k$ is not consistent or is closed, then by (7.10), $t_k=t_{k+1}=\cdots=t_n$; hence $\mathcal{R}(n,\pi)\Leftrightarrow\mathcal{R}(k,\pi)$, and this last property is true by the induction hypothesis.

Let us suppose now that $\forall k < n$, t_k is consistent and unclosed. According to formula (7.9),

$$t_{k+1} = t_k[e_{k+1}/x_k]$$

for some tree of depth one, e_{k+1}.

Let $k \in [0, n-1]$, $x = x_k$, $\pi = \bar{\pi}_k$ (such a k must exist because x is internal). Let $x \cdot 1, \ldots, x \cdot \mu$ be the sequence of sons of x.

Case 3. *T_A applies to x*. Hence there exists some oracle $\mathcal{O}$ such that $T_A^{(\mathcal{O})}$ applies to x. The choice (10.13) fulfills property (10.14).

Case 4. *T_B^α applies to x (for some $\alpha \in \{-,+\}$)*. Let us suppose $\alpha = +$. Let $x' = x(|x| - k_1)$ (the prefix of x having length $|x| - k_1$), $t_n(x') = (\pi', \bar{U}, U')$. By definition of $\hat{\mathcal{S}}_{ABC}$, there exists some oracle $\mathcal{O}$ which is $\bar{\pi}_k$-consistent with $\mathrm{im}(t_k)$ and such that

$$\mu = 1 \quad \text{and} \quad t_n(x \cdot 1) = T_B^{(\mathcal{O}),+}(W_x).$$

Let us look at the proof of Lemma 7.2 in the particular case of this oracle $\mathcal{O}$: as the pairs (u_ℓ, u'_ℓ) belong to $\mathcal{O}(\bar{U}, U')$ (for every $\ell \in [1, q]$) and $\pi' + |u_\ell| - 1 < \pi' + k_0 \leq \pi' + 2 \cdot k_0 < \bar{\pi}_k$, deduction (7.2) can be obtained just by using rules in $\mathcal{C}$. As deduction (7.2) is the only one (in the proof of Lemma 7.2) using rules in $\mathcal{B}_0 - \mathcal{C}$ we conclude that deduction (7.1) can be replaced by

$$\{t_n(x'), t_n(x \cdot 1)\} \cup \mathrm{im}(t_k) \;\; |\!\overset{\langle * \rangle}{\vdash\!\!-}_{\mathcal{C}} \; \tau_{-1}(t_n(x)). \tag{10.15}$$

(We recall that τ_{-1} consists in replacing the weight of a given weighted equation into its predecessor.) Deduction (10.15) implies that

$$\exists p \in \mathbb{N}, \quad (\pi - 1, S, S') \in \mathrm{Cong}_p(\mathrm{im}(t_n)). \tag{10.16}$$

By the induction hypothesis, as $\pi - 1 < \bar{\pi}_n$, $\mathrm{im}(t_n)$ is locally $\pi - 1$-consistent, and hence, by Lemma 10.3, $\mathrm{im}(t_n)$ is $\pi - 1$-consistent. Hypothesis (10.16) implies that

$$\exists \mathcal{R}_1 \in \mathcal{B}_1(S, S'), \quad [\pi - 1, S, S', \mathcal{R}_1] \subseteq \mathrm{Cong}(\mathrm{im}(t_n)),$$

and hence, using (R0), that

$$\exists \mathcal{R}_1 \in \mathcal{B}_1(S, S'), \quad [\pi, S, S', \mathcal{R}_1] \subseteq \mathrm{Cong}(\mathrm{im}(t_n)).$$

Case 5. *T_C applies to x*. By definition of $\hat{\mathcal{S}}_{ABC}$, there exists some oracle $\mathcal{O}$ which is $\bar{\pi}_k$-consistent with $\mathrm{im}(t_k)$ and such that

$$\mu = 1 \quad \text{and} \quad t_n(x \cdot 1) = T_C^{(\mathcal{O})}(W_x).$$

Let $W_x = A_1 \cdots A_\ell \cdots A_{|x|+1}$, $\kappa_1 < \cdots < \kappa_i < \kappa_{i+1} < \cdots \kappa_{D+1} = |x| + 1$, $\mathcal{S} = (\mathcal{E}_i)_{1 \leq i \leq D+1}$, where, for every $1 \leq i \leq d$,

$$\mathcal{E}_i = A_{\kappa_i} = \left(\pi_i, \sum_{j=1}^{d} \alpha_{i,j} S_j, \sum_{j=1}^{d} \beta_{i,j} S_j \right)$$

and

$$T_C^{(\mathcal{O})}(W_x) = \rho_e(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})), \quad \mathrm{W}^{(\mathcal{O})}(\mathcal{S}) \neq \bot, \quad \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = D \leq d - 1.$$

Let us look at the proof of Lemma 5.3 in the particular case of this oracle $\mathcal{O}$: the only place where a rule in $\mathcal{B}_0 - \mathcal{C}$ is used is in deduction (5.10), when Case 2, Subcase 1 (or Case 2, Subcase 2) of the recursive definition of $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})$ occurs. Let us recall that the pair (u, u') chosen by the oracle $\mathcal{O}$ is such that

$$\mathcal{R} = \mathcal{O}\left(\sum_{j=1}^{d} \alpha_{1,j} S_j, \sum_{j=1}^{d} \beta_{1,j} S_j\right),$$

$$\nu = \mathrm{Div}(\alpha_{1,*}, \beta_{1,*}), \quad \mathcal{R}_\nu = \mathcal{R} \cap X^{\leq \nu} \times X^{\leq \nu}, \quad (u, u') \in \mathcal{R}_\nu.$$

Note that $\pi_1 + \nu - 1 < \pi_1 + 2 \cdot \nu < \pi_2 \leq \mathrm{W}^{(\mathcal{O})}(\mathcal{S}) + 1 = \pi = \bar{\pi}_k$. As $\mathcal{O}$ is $\bar{\pi}_k$-consistent with $\mathrm{im}(t_k)$, we conclude that

$$\left(\pi_1 + |u|, \left(\sum_{j=1}^{d} \alpha_{i,j} S_j\right) \odot u, \left(\sum_{j=1}^{d} \beta_{i,j} S_j\right) \odot u'\right) \in \left[\pi_1, \sum_{j=1}^{d} \alpha_{i,j} S_j, \sum_{j=1}^{d} \beta_{i,j} S_j, \mathcal{R}_\nu\right] \subseteq \mathrm{Cong}(\mathrm{im}(t_k)).$$

Hence deduction (5.10) can be replaced by

$$\mathcal{E}'_1 \in \mathrm{Cong}(\mathrm{im}(t_k)). \tag{10.17}$$

Similarly, for every $i \in [2, D]$, as $\pi_i + 2 \cdot \mathrm{Div}(\alpha_{i,*}^{(i-1)}, \beta_{i,*}^{(i-1)}) < \pi_{i+1} \leq \mathrm{W}^{(\mathcal{O})}(\mathcal{S}) + 1 = \pi = \bar{\pi}_k$, and $\mathcal{E}_i^{(i-1)} \in \mathrm{Cong}(\mathrm{im}(t_k))$,

$$(\mathcal{E}_i^{(i-1)})' \in \mathrm{Cong}(\mathrm{im}(t_k)). \tag{10.18}$$

It follows that deduction (5.9) can be replaced by

$$\{\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})\} \cup \mathrm{im}(t_k) \ \overset{\langle * \rangle}{|\!\!\vdash\!\!-}_{\mathcal{C}} \ \tau_{-1}(t_n(x)). \tag{10.19}$$

Using the facts that $\rho_e(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})) \ \overset{\langle * \rangle}{|\!\!\vdash\!\!-}_{\mathcal{C}} \ \mathrm{INV}^{(\mathcal{O})}(\mathcal{S})$ and $\mathrm{im}(t_k) \subseteq \mathrm{im}(t_n)$ we may conclude that

$$\{t_n(x \cdot 1)\} \cup \mathrm{im}(t_n) \ \overset{\langle * \rangle}{|\!\!\vdash\!\!-}_{\mathcal{C}} \ \tau_{-1}(t_n(x)) = (\pi - 1, S, S'). \tag{10.20}$$

From (10.20) and the induction hypothesis, we can conclude, as in Case 4, that

$$\exists \mathcal{R}_1 \in \mathcal{B}_1(S, S'), \quad [\pi, S, S', \mathcal{R}_1] \subseteq \mathrm{Cong}(\mathrm{im}(t_n)).$$

(This ends the induction.)

By the above induction, for every $n \in \mathbb{N}$, $\mathrm{im}(t_n)$ is $\bar{\pi}_n$-consistent, i.e., t_n is consistent. ∎

LEMMA 10.5. *$\hat{\mathcal{S}}_{ABC}$ is closed.*

Proof. Let $A_0 \in \mathcal{A}$. By Lemma 10.4, $\forall n \in \mathbb{N}$, $\hat{\mathcal{S}}^n_{ABC}(A_0)$ is consistent.

If $\hat{\mathcal{S}}^n_{ABC}(A_0)$ is consistent and is not closed, then, by definition (7.9),

$$\hat{\mathcal{S}}^n_{ABC}(A_0) \neq \hat{\mathcal{S}}^{n+1}_{ABC}(A_0);$$

if $\hat{\mathcal{S}}^n_{ABC}(A_0)$ is consistent and is closed, then, by definition (7.10),

$$\hat{\mathcal{S}}^n_{ABC}(A_0) = \hat{\mathcal{S}}^{n+1}_{ABC}(A_0).$$

Hence the equivalence (4.6), which defines the notion of closed global strategy, is fulfilled by $\hat{\mathcal{S}}_{ABC}$. ∎

THEOREM 10.6. *The formal systems $\mathcal{B}_0, \mathcal{B}_1$ are complete.*

Proof. By Lemma 9.1 $\hat{\mathcal{S}}_{ABC}$ is terminating on every unmarked assertion, and by Lemma 10.5 $\hat{\mathcal{S}}_{ABC}$ is closed. Let A_0 be some unmarked true assertion. According to the proof of Lemma 4.6, $\exists n_0 \in \mathbb{N}$ such that $t_\infty = \hat{\mathcal{S}}^{n_0}(A_0)$ is a proof-tree which is closed, and hence such that $\bar{\Pi}(t_\infty) = \infty$. By Lemma 10.5, t_∞ is consistent, i.e., $\mathrm{im}(t_\infty)$ is ∞-consistent: $\forall(\pi, S, S') \in \mathrm{im}(t_\infty)$,

$$\exists \mathcal{R}_1 \in \mathcal{B}_1(S, S'), \quad [\pi, S, S', \mathcal{R}_1] \subseteq \mathrm{Cong}(\mathrm{im}(t_\infty));$$

hence,

$$\mathrm{im}(t_\infty) \;\; |\overset{\langle * \rangle}{\vdash\!\!-}_{\mathcal{C}} [\pi, S, S', \mathcal{R}_1] \vdash\!\!-_{R4}(\pi, S, S'). \tag{10.21}$$

As the rules of $\mathcal{C}$ and (R4) are rules of $\mathcal{B}_1$, deduction (10.21) shows that

$$\mathrm{im}(t_\infty) \vdash\!\!-_{\mathcal{B}_1}(\pi, S, S'); \tag{10.22}$$

i.e., $\mathrm{im}(t_\infty)$ is a $\mathcal{B}_1$-proof.

In the general case where $A_0 = (\pi_0, U_0^-, U_0^+)$ might be marked, we observe that, owing to rules (R1), (R2), (R′3),

$$\{\rho_e(A_0)\} \;\; |\overset{\langle * \rangle}{\vdash\!\!-}_{\mathcal{C}} \{A_0\}.$$

This deduction, combined with some $\mathcal{B}_1$-proof of $\rho_e(A_0)$, gives a $\mathcal{B}_1$-proof of A_0. ∎

THEOREM 10.7. *The bisimulation problem for rooted equational 1-graphs of finite out-degree is decidable.*

Proof. Let us consider the sequence of statements: Lemma 2.7, Lemma 2.8, Corollary 2.6 and Lemma 3.25. By means of the above statements, the bisimulation problem for rooted equational 1-graphs of finite out-degree reduces to the following decision problem (we call it the bisimulation problem for deterministic vectors):

Instance: a birooted, normalized dpda $\mathcal{M}$, its terminal alphabet X, a surjective literal morphism $\psi : X^* \to Y^*$ (we denote its kernel by $\bar{\psi}$), and $\lambda \in \mathbb{N} - \{0\}$, $S, S' \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle$ (where V is the structured alphabet associated with $\mathcal{M}$).

Question: $S \sim S'$? (where $\sim$ is the $\bar{\psi}$-bisimulation relation).

Let us consider $\mathcal{M}, X, V, \bar{\psi}$ given by some instance.

The equivalence relation $\sim$ on $\mathsf{DRB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle$ has a recursively enumerable complement (this is well known). By Theorem 10.6 and Lemma 4.2, relation $\sim$ is recursively enumerable too. Hence $\sim$ is recursive.

But the function associating to every $\mathcal{M}, X, V, \bar{\psi}$ the corresponding deduction system $\mathcal{B}_1$ is recursive. Hence the bisimulation problem for deterministic vectors is decidable. ∎

10.2. System $\mathcal{B}_2$. We exhibit here a deduction system $\mathcal{B}_2$ which is simpler than $\mathcal{B}_1$ and is still complete.

Elementary rules. Let us *eliminate* the weights in the rules of $\mathcal{B}_1$: we define a new set of assertions, $\mathcal{A}_2$, by

$$\mathcal{A}_2 = \bigcup_{\lambda \in \mathbb{N}-\{0\}} \mathsf{DRB}_{1,\lambda}\langle\langle\, V\,\rangle\rangle \times \mathsf{DRB}_{1,\lambda}\langle\langle\, V\,\rangle\rangle.$$

We define a binary relation $|\!\!\vdash\!\!- \subseteq \mathcal{P}_f(\mathcal{A}_2) \times \mathcal{A}_2$, the *elementary deduction relation*, as the set of all the pairs having one of the following forms:

(R21)

$$\{(S,T)\} \,|\!\!\vdash\!\!-\, (T,S)$$

for $\lambda \in \mathbb{N}-\{0\}$, $S,T \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V\,\rangle\rangle$;

(R22)

$$\{(S,S'),(S',S'')\} \,|\!\!\vdash\!\!-\, (S,S'')$$

for $\lambda \in \mathbb{N}-\{0\}$, $S,T \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V\,\rangle\rangle$;

(R23)

$$\emptyset \,|\!\!\vdash\!\!-\, (S,S)$$

for $S \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V\,\rangle\rangle$;

(R′23)

$$\emptyset \,|\!\!\vdash\!\!-\, (S,\rho_e(S))$$

for $S \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V\,\rangle\rangle$;

(R24)

$$\{(S\odot x, T\odot x') \mid (x,x') \in \mathcal{R}_1\} \,|\!\!\vdash\!\!-\, (S,T)$$

for $\lambda \in \mathbb{N}-\{0\}$, $S,T \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V\,\rangle\rangle$, $(S \not\equiv \epsilon \wedge T \not\equiv \epsilon)$, and $\mathcal{R}_1 \in \bar{\mathcal{B}}_1$;

(R25)

$$\{(S_1\cdot T + S, T)\} \,|\!\!\vdash\!\!-\, (S_1^*\cdot S, T)$$

for $\lambda \in \mathbb{N}-\{0\}$, $S_1 \in \mathsf{DRB}_{1,1}\langle\langle\, V\,\rangle\rangle$, $S_1 \not\equiv \epsilon$, $(S_1,S) \in \mathsf{DRB}_{1,\lambda+1}\langle\langle\, V\,\rangle\rangle$, $T \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V\,\rangle\rangle$;

(R26)

$$\{(S,S')\} \,|\!\!\vdash\!\!-\, (S\cdot T, S'\cdot T)$$

for $\delta,\lambda \in \mathbb{N}-\{0\}$, $S,S' \in \mathsf{DRB}_{1,\delta}\langle\langle\, V\,\rangle\rangle$, $T \in \mathsf{DRB}_{\delta,\lambda}\langle\langle\, V\,\rangle\rangle$;

(R27)

$$\{(T_{i,*},T'_{i,*}) \mid 1 \le i \le \delta\} \,|\!\!\vdash\!\!-\, (S\cdot T, S\cdot T')$$

for $\delta,\lambda \in \mathbb{N}-\{0\}$, $S \in \mathsf{DRB}_{1,\delta}\langle\langle\, V\,\rangle\rangle$, $T,T' \in \mathsf{DRB}_{\delta,\lambda}\langle\langle\, V\,\rangle\rangle$.

We define $\vdash\!\!-_{\mathcal{B}_2}$ as follows: for every $P \in \mathcal{P}_f(\mathcal{A}_2)$, $A \in \mathcal{A}_2$,

$$P \vdash\!\!- A \Longleftrightarrow P \;\overset{\langle * \rangle}{|\!\!\vdash\!\!-}\; \circ \overset{[1]}{|\!\!\vdash\!\!-}_{23,24} \circ \;\overset{\langle * \rangle}{|\!\!\vdash\!\!-}\; \{A\},$$

where $|\!\!\vdash\!\!-_{23,24}$ is the relation defined by (R23), (R′23), (R24) only.

We define a simpler cost function $H_2 : \mathcal{A}_2 \to \mathbb{N} \cup \{\infty\}$ by

$$\forall (S, S') \in \mathcal{A}_2, \quad H_2(S, S') = \mathrm{Div}(S, S').$$

We let

$$\mathcal{B}_2 = \langle \mathcal{A}_2, H_2, \models\!\!-_{\mathcal{B}_2} \rangle.$$

LEMMA 10.8. *$\mathcal{B}_2$ is a deduction system.*

Completeness. Let us denote by $\mathcal{C}_2$ the subset of rules of $\mathcal{B}_2$ obtained by removing the weights in the rules of $\mathcal{C}$.

DEFINITION 10.9. *Let $P \in \mathcal{P}_f(\mathcal{A}_2)$. P is said to be* self-generating *iff, for every $(S, S') \in P$,*

1. *either $S = S' = \epsilon$, or*
2. *$\exists \mathcal{R}_1 \in \bar{\mathcal{B}}_1(S, S')$, $\forall (x, x') \in \mathcal{R}_1$, $P \mid\!\!\models\!\!\overset{\langle * \rangle}{-}_{\mathcal{C}_2} (S \odot x, S' \odot x')$.*

(See Remark 10.12 below for the origins of this notion.)

LEMMA 10.10. *Let $A \in \mathcal{A}_2$ such that A is unmarked. Then $H(A) = \infty$ iff there exists a finite self-generating set $P \subseteq \mathcal{A}_2$ such that $A \in P$.*

Proof. Owing to metarules (R23), (R24) it is clear that every self-generating set $P \in \mathcal{P}_f(\mathcal{A}_2)$ is a $\mathcal{B}_2$-proof. Hence, if A belongs to some self-generating set, then $H(A) = \infty$.

Let us suppose now that $H_2(A) = \infty$. Let us consider the closed proof-tree t_∞ obtained by applying the global strategy $\hat{S}_{ABC}$ on the assertion $(0, A)$. By Lemma 9.1 t_∞ is finite, and by Lemma 10.5 t_∞ is consistent, which means that $\mathrm{im}(t_\infty)$ is ∞-consistent. Let

$$P = pr_{2,3}(\mathrm{im}(t_\infty))$$

(where $pr_{2,3} : \mathcal{A} \to \mathcal{A}_2$ is the map erasing the weights).

As $\mathrm{im}(t_\infty)$ is ∞-consistent, P is self-generating and $A \in P$. $\square$

THEOREM 10.11. *$\mathcal{B}_2$ is a complete deduction system.*

Proof. We already noticed that every self-generating set is a $\mathcal{B}_2$-proof. Hence Lemma 10.10 proves that every true, unmarked assertion possesses some finite $\mathcal{B}_2$-proof.

Let A be any true assertion. $\rho_e(A)$ has a finite proof P. Owing to rules (R1), (R2), (R′3), $Q = P \cup \{A\}$ is a $\mathcal{B}_2$-proof of A. $\square$

10.3. System $\boldsymbol{\mathcal{B}_3}$. We exhibit here a deduction system $\mathcal{B}_3$ which is even simpler than $\mathcal{B}_2$ and is still complete. Let us consider $\mathcal{B}_3 = \langle \mathcal{A}_3, H_3, \models\!\!-_{\mathcal{B}_3} \rangle$, where

$$\mathcal{A}_3 = \bigcup_{\lambda \in \mathbb{N} - \{0\}} \mathsf{DRB}_{1,\lambda}\langle\langle\, V_0 \,\rangle\rangle \times \mathsf{DRB}_{1,\lambda}\langle\langle\, V_0 \,\rangle\rangle.$$

$H_3 = H_{2|\mathcal{A}_3}$ and $\models\!\!-_{\mathcal{B}_3}$ is defined below: the metarules of $\mathcal{B}_3$ are essentially those of $\mathcal{B}_2$, but restricted to the unmarked vectors.

(R31)

$$\{(S, T)\} \mid\!\!\models\!\!- (T, S)$$

for $\lambda \in \mathbb{N} - \{0\}$, $S, T \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V_0 \,\rangle\rangle$;

(R32)

$$\{(S, S'), (S', S'')\} \mid\!\vdash\!\!- (S, S'')$$

for $\lambda \in \mathbb{N} - \{0\}$, $S, T \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V_0 \,\rangle\rangle$;

(R33)

$$\emptyset \mid\!\vdash\!\!- (S, S)$$

for $S \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V_0 \,\rangle\rangle$;

(R34)

$$\{(S \odot x, T \odot x') \mid (x, x') \in \mathcal{R}_1\} \mid\!\vdash\!\!- (S, T)$$

for $\lambda \in \mathbb{N} - \{0\}$, $S, T \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V_0 \,\rangle\rangle$, $(S \not\equiv \epsilon \wedge T \not\equiv \epsilon)$, and $\mathcal{R}_1 \in \bar{\mathcal{B}}_1$;

(R35)

$$\{(S_1 \cdot T + S, T)\} \mid\!\vdash\!\!- (S_1^* \cdot S, T)$$

for $\lambda \in \mathbb{N} - \{0\}$, $S_1 \in \mathsf{DRB}_{1,1}\langle\langle\, V_0 \,\rangle\rangle$, $S_1 \not\equiv \epsilon$, $(S_1, S) \in \mathsf{DRB}_{1,\lambda+1}\langle\langle\, V_0 \,\rangle\rangle$, $T \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V_0 \,\rangle\rangle$;

(R36)

$$\{(S, S')\} \mid\!\vdash\!\!- (S \cdot T, S' \cdot T)$$

for $\delta, \lambda \in \mathbb{N} - \{0\}$, $S, S' \in \mathsf{DRB}_{1,\delta}\langle\langle\, V_0 \,\rangle\rangle$, $T \in \mathsf{DRB}_{\delta,\lambda}\langle\langle\, V_0 \,\rangle\rangle$;

(R37)

$$\{(T_{i,*}, T'_{i,*}) \mid 1 \le i \le \delta\} \mid\!\vdash\!\!- (S \cdot T, S \cdot T')$$

for $\delta, \lambda \in \mathbb{N} - \{0\}$, $S \in \mathsf{DRB}_{1,\delta}\langle\langle\, V_0 \,\rangle\rangle$, $T, T' \in \mathsf{DRB}_{\delta,\lambda}\langle\langle\, V_0 \,\rangle\rangle$.

We then define $\vdash\!\!-_{\mathcal{B}_3}$ as follows: for every $P \in \mathcal{P}_f(\mathcal{A}_3)$, $A \in \mathcal{A}_3$,

$$P \vdash\!\!-_{\mathcal{B}_3} A \Longleftrightarrow P \overset{\langle * \rangle}{\mid\!\vdash\!\!-}_{\mathcal{B}_3} \circ \overset{[1]}{\mid\!\vdash\!\!-}_{33,34} \circ \overset{\langle * \rangle}{\mid\!\vdash\!\!-}_{\mathcal{B}_3} \{A\},$$

where $\mid\!\vdash\!\!-_{33,34}$ is now the relation defined by (R33), (R34) only.

As $\vdash\!\!-_{\mathcal{B}_3} \subseteq \vdash\!\!-_{\mathcal{B}_2}$, $H_3 = H_2$, it is clear that $\mathcal{B}_3$ is a deduction system.

Completeness. Let us call $\mathcal{C}_3$ the intersection of the set of the rules of $\mathcal{C}$ with the set of the rules of $\mathcal{B}_3$ (it is also equal to the set of instances of (R31), (R32), (R33), (R35), (R36), (R37)). Let us now call $P \in \mathcal{P}_f(\mathcal{A}_3)$ a $\mathcal{C}_2$-*self-generating* set iff it fulfills Definition 10.9 and a *self-generating* set iff it fulfills Definition 10.9 but where $\mathcal{C}_2$ is replaced by $\mathcal{C}_3$.

REMARK 10.12.

1. *This notion of a "self-generating set (of pairs)" is a straightforward adaptation to our d-space of vectors of the notion of the "self-proving set of pairs" defined in* [10, *p.* 162] *for the magma* $M(F \cup \Phi, V)$.

2. *The notion of "self-bisimulation"* (*introduced in* [5] *and also used in* [18, 17]) *was also such an adaptation, but in the context of a monoid-structure. The notion we use in this work can be seen, as well, as a generalization of this notion of self-bisimulation: when every class in* $V_0/\smile$ *has just one element, the only "rational deterministic boolean series" over* V_0 *are the words; in this case the self-bisimulations are exactly the self-generating sets.*

LEMMA 10.13. *Let $A \in \mathcal{A}_3$. Then $H_3(A) = \infty$ iff there exists a finite self-generating set $P \subseteq \mathcal{A}_3$ such that $A \in P$.*

Proof. Owing to metarules (R33) and (R34), every self-generating set is a $\mathcal{B}_3$-proof.

Let $A \in \mathcal{A}_3$ such that $H_3(A) = \infty$. By Lemma 10.10, there exists some $\mathcal{C}_2$-self-generating set P such that $A \in P$.

Let us consider $Q = \{\rho_e(B) \mid B \in P\}$.

One can check that ρ_e maps the set of rules of $\mathcal{C}_2$ into the set of rules of $\mathcal{C}_3$. One can also check that ρ_e and $\odot$ are commuting (i.e., $\rho_e(S \odot u) = \rho_e(S) \odot u$). Hence Q is such that, for every $(S, S') \in Q$,

1. either $S = S' = \epsilon$, or
2. $\exists \mathcal{R}_1 \in \bar{\mathcal{B}}_1(S, S')$, $\forall (x, x') \in \mathcal{R}_1$, $Q \overset{\langle * \rangle}{||\!\!-\!\!-}_{\mathcal{C}_3} (S \odot x, S' \odot x')$.

That is, Q is self-generating. ∎

THEOREM 10.14. *$\mathcal{B}_3$ is a complete deduction system.*

Proof. Lemma 10.13 implies the completeness property. ∎

10.4. System $\mathcal{B}_4$. We exhibit here a formal system $\mathcal{B}_4$ whose elementary rules can be considered as more natural than those of $\mathcal{B}_3$: they just consist in the rules expressing the basic algebraic properties of the bisimulation equivalence $\sim$ augmented with the grammatical rules (i.e., the rules of grammar G_0). This system $\mathcal{B}_4$ is still complete. Below, we just sketch the completeness proof, which is just a slight modification of the above completeness proofs.

Let us consider the alphabet $V_4 = V_0 \cup X$. The equivalence relation $\smile$ on V_0 is extended to V_4 as follows: for every $v, v' \in V_4$, $v \sim v'$ iff $[v \in V,\ v' \in V$, and there are equivalent in the sense used before$]$ or $[v \in X,\ v' \in X]$. (This equivalence is the one considered in [15].) The right-action $\odot$ is extended to $\mathsf{B}\langle\langle\ V_4\ \rangle\rangle \times X^*$ as follows: for every $x \in X$, $\beta \in V_4^*$, $x' \in X$,

$$(x \cdot \beta) \odot x' = (x \cdot \beta) \bullet x', \tag{10.23}$$

$$\mathcal{A}_4 = \bigcup_{\lambda \in \mathbb{N} - \{0\}} \mathsf{DRB}_{1,\lambda}\langle\langle\ V_4\ \rangle\rangle \times \mathsf{DRB}_{1,\lambda}\langle\langle\ V_4\ \rangle\rangle.$$

This extension of $\odot$ leads naturally to extensions of the relation $\sim$ and of the notion of divergence. The cost H_4 is still defined by

$$\forall (S, S') \in \mathcal{A}_4, \quad H_4(S, S') = \mathrm{Div}(S, S').$$

Elementary rules.

(R41)

$$\{(S, T)\} \,||\!\!-\!\!-\, (T, S)$$

for $\lambda \in \mathbb{N} - \{0\}$, $S, T \in \mathsf{DRB}_{1,\lambda}\langle\langle\ V_4\ \rangle\rangle$;

(R42)

$$\{(S, S'), (S', S'')\} \,||\!\!-\!\!-\, (S, S'')$$

for $\lambda \in \mathbb{N} - \{0\}$, $S, T \in \mathsf{DRB}_{1,\lambda}\langle\langle\ V_4\ \rangle\rangle$;

(R43)

$$\emptyset \,||\!\!-\!\!-\, (S, S)$$

for $S \in \mathsf{DRB}_{1,\lambda}\langle\langle\ V_4\ \rangle\rangle$;

(R′44)

$$\emptyset \mid\!\vdash\!- ((E_1, \ldots, E_i, \ldots, E_\lambda), (P_1, \ldots, P_i, \ldots, P_\lambda))$$

for $\lambda \in \mathbb{N} - \{0\}$, $(E_i)_{1 \le i \le \lambda}$, bijective numbering of some class in $V/\smile$ and P_i equal to the right-hand side of E_i in grammar G_0;

(R″44)

$$\{(S_x, T_{x'}) \mid (x, x') \in \mathcal{R}_1\} \mid\!\vdash\!- \left(\sum_{x \in X} x \cdot S_x, \sum_{x \in X} x \cdot T_x\right)$$

for $\lambda \in \mathbb{N} - \{0\}$, $S_x, T_x \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V_4 \,\rangle\rangle$, and $\mathcal{R}_1 \in \bar{\mathcal{B}}_1$;

(R45)

$$\{(S_1 \cdot T + S, T)\} \mid\!\vdash\!- (S_1^* \cdot S, T)$$

for $\lambda \in \mathbb{N} - \{0\}$, $S_1 \in \mathsf{DRB}_{1,1}\langle\langle\, V_4 \,\rangle\rangle$, $S_1 \not\equiv \epsilon$, $(S_1, S) \in \mathsf{DRB}_{1,\lambda+1}\langle\langle\, V_4 \,\rangle\rangle$, $T \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V_4 \,\rangle\rangle$;

(R46)

$$\{(S, S')\} \mid\!\vdash\!- (S \cdot T, S' \cdot T)$$

for $\delta, \lambda \in \mathbb{N} - \{0\}$, $S, S' \in \mathsf{DRB}_{1,\delta}\langle\langle\, V_4 \,\rangle\rangle$, $T \in \mathsf{DRB}_{\delta,\lambda}\langle\langle\, V_4 \,\rangle\rangle$;

(R47)

$$\{(T_{i,*}, T'_{i,*}) \mid 1 \le i \le \delta\} \mid\!\vdash\!- (S \cdot T, S \cdot T')$$

for $\delta, \lambda \in \mathbb{N} - \{0\}$, $S \in \mathsf{DRB}_{1,\delta}\langle\langle\, V_4 \,\rangle\rangle$, $T, T' \in \mathsf{DRB}_{\delta,\lambda}\langle\langle\, V_4 \,\rangle\rangle$.

Completeness.

PROPOSITION 10.15. *$\mathcal{B}_4$ is a complete deduction system.*

Sketch of proof. Let $(S, S') \in \mathcal{A}_4$ such that $S \sim S'$. Lemma 10.13 could be proved for assertions in $\mathcal{A}_4$ in the same way that it has been proved for assertions in $\mathcal{A}_3$. Hence there exists some self-generating set P containing (S, S').

In order to prove that P is a $\mathcal{B}_4$-proof, we just have to check that every instance of (R34) belongs to $\vdash\!-_{\mathcal{B}_4}$.

Let $T, T' \in \mathsf{DRB}_{1,\lambda}\langle\langle\, V_4 \,\rangle\rangle$ (for some $\lambda \ge 1$). Using metarule (R″44) we get

(10.24)

$$\{(T \odot x, T' \odot x') \mid (x, x') \in \mathcal{R}_1\} \vdash\!-_{\mathcal{B}_4} \left\{\left(\sum_{x \in X} x \cdot (T \odot x), \sum_{x \in X} x \cdot (T' \odot x)\right)\right\}.$$

Using metarule (R′44) (as well as other auxiliary rules) we get

$$\emptyset \vdash\!-_{\mathcal{B}_4} \left\{\left(T, \sum_{x \in X} x \cdot (T \odot x)\right)\right\}; \quad \emptyset \vdash\!-_{\mathcal{B}_4} \left\{\left(T', \sum_{x \in X} x \cdot (T' \odot x)\right)\right\}. \tag{10.25}$$

From deductions (10.24), (10.25) and metarules (R41), (R42) one derives

$$\{(T \odot x, T' \odot x') \mid (x, x') \in \mathcal{R}_1\} \vdash\!-_{\mathcal{B}_4} \{(T, T')\}. \qquad \square$$

11. Comparisons and perspectives.

11.1. Old tools. We have reused here the notions developed in [34]:

- the *deduction systems* (which were in turn inspired by [9]);
- the *deterministic boolean series* (which were in turn inspired by [16]);
- the *deterministic spaces* (which were elaborated around the Meitus notion of linear independence [23, 24]);
- the *analysis* of the proof-trees generated by a suitable strategy (which was somehow similar to the analysis of the parallel computations, interspersed with replacement moves, done in [49, 31, 28]).

Some simplifications of [34] found by Stirling [46] were taken into account in this proof too:

- the technical notion of the "N-stacking sequence" is replaced by the slightly simpler notion of the "B-stacking sequence" (see section 8.2);
- the analysis of section 8 uses a choice of "generating set" which is simpler than the choice given in [33, 37];
- a main simplification linked with this more clever choice is that we can restrict ourselves to the case of a *proper, reduced* strict-deterministic grammar (as is done in [46]), while in [33, 37] we could not assume this restriction.

11.2. New tools. We also have introduced new ideas:

1. the notion of η*-bisimulation* over deterministic row-vectors of boolean series (which, in some sense, translates the usual notion of bisimulation to the d-space of row-vectors of series);
2. the notion of *oracle*, which is a choice of bisimulation for every pair of bisimilar vectors; the notion of triangulation of systems of linear equations is now "parametrized" by such an oracle $\mathcal{O}$ (see section 5.2); the strategies are now parametrized by an oracle too (see section 7);
3. the *elimination* argument: roughly speaking, this argument shows that, in a proof-tree t, if we take into account not only the *branch* ending at a node x, but also the *whole* proof-tree, then the metarule (R5)

$$\{(p, S, S')\} \mid\mid\!\!-\!\!- (p+2, S \odot x, S' \odot x')$$

is not needed to show that $\mathrm{im}(t) \mid\!\!-\!\!- \{t(x)\}$ (see section 10.1); a nice (and unexpected) by-product of this elimination is that the *weights* can be removed from the equations (see sections 10.2, 10.3).

11.3. Perspectives and related works. In view of the above main result (and of other closely related results), many directions for future work naturally arise:

- Whether the bisimulation problem is decidable or not, for equational graphs of arbitrary out-degree, which was raised by [6], remains an open problem.
- The class of processes of type -1 was introduced in [45]: they correspond to the computation-graphs of pda with only decreasing ε-moves. Whether the bisimulation problem is decidable or not for such processes is a natural challenge: the present article gives hope that it is decidable, while the negative results from [44, 22] suggest it might be undecidable.
- Once we know that the bisimulation problem is decidable for equational 1-graphs, it is natural to ask what the intrinsic complexity of this problem is. It is proved in [48] that the equivalence problem for *deterministic pda* is primitive recursive. It is tempting to examine what techniques could be extracted from this work and adapted to the above complexity problem (we

discuss in section 11.4 below the difficulty of such adaptations). In [43] a general algebraic tool (which generalizes the "extension-theorem" from [48]), called the "subwords lemma," is introduced for deterministic dpda. It is also tempting to try to adapt this tool to nondeterministic pda (we discuss in section 11.4 below the difficulty of such an adaptation).

- The intrinsic complexity of the bisimulation problem for some subclasses of graphs would be interesting too: let us quote the *context-free graphs* and the computation-graphs of *finite-turn* pda.
- The equivalence problem for *deterministic pushdown transducers* from a free monoid into a free group (or a linear group) is shown decidable in [38]. We strongly believe that this result can be unified with the result proved here (Theorem 10.7) into a more general statement saying that "the bisimulation problem is decidable for nondeterministic pushdown transducers, with outputs in a linear group and with only deterministic, decreasing, ε-moves" (i.e., whose underlying pda fulfills the hypotheses of the present article). Of course the notion of bisimulation for transducers has to be defined carefully.
- Let us recall that, given two directed graphs G, G', labeled over the same alphabet X, G' is a *quotient* of G iff there exists a functional bisimulation from G to G'. It is proved in [41] that the quotient problem is decidable for deterministic equational 1-graphs. The same decision problem remains open for the equational 1-graphs of finite out-degree. It is open for the subclass of context-free graphs too.

11.4. Language equivalence versus bisimulation. Let us stress here some differences between the behavior of the equivalence $\equiv$ (on one hand) and the behavior of $\sim$ (on the other hand), w.r.t. important algebraic notions. These differences explain some otherwise "odd" aspects of the present article, as well as point to difficulties that must be overcome for reaching the above-mentioned perspectives. For illustrating these differences, we shall always refer to the following example.

EXAMPLE 11.1. *Let* $G = \langle X, V, P\rangle$, *where*

$$X = \{a_i \mid 1 \le i \le 6\} \cup \{b_i \mid 1 \le i \le 6\} \cup \{h, h', h''\},$$

$$V = \{A_i \mid 1 \le i \le 6\} \cup \{B_i \mid 1 \le i \le 6\} \cup \{H, H', H''\},$$

$$P = \{(A_i, a_i) \mid 1 \le i \le 6\} \cup \{(B_i, b_i) \mid 1 \le i \le 6\} \cup \{(H, h), (H', h'), (H'', h'')\}.$$

We define the equivalence relation $\smile$ *on* V *as the coarsest one:*

$$\forall v \in V, \quad [v]_\smile = V.$$

The grammar G *is strict-deterministic and admits the partition* $\smile$. *Let* $Y = \{a, b, h, h, h''\}$ *and let* $\Psi : X^* \to Y^*$ *be the strict-alphabetical homomorphism defined by*

$$\Psi(a_i) = a, \quad \Psi(b_i) = b, \quad \Psi(h) = h, \quad \Psi(h') = h', \quad \Psi(h'') = h''.$$

We define η *as the kernel of* Ψ*:*

$$\eta = \{(w, w') \in X^* \mid \Psi(w) = \Psi(w')\}.$$

Direct product. We show here that, unlike for the equivalence $\equiv$, relation $\sim$ over vectors does not easily reduce to the same relation over series, because it is *not* *compatible with direct product*. This explains the need for some *row-vectors* in the assertions as well as for some *matricial* rules in the systems $\mathcal{B}_i$ for $1 \leq i \leq 3$.

More precisely, it can be shown that, for every $(S_1, S_2), (T_1, T_2) \in \mathsf{DB}_{1,2}\langle\langle V \rangle\rangle$,

$$[(S_1, S_2) \equiv (T_1, T_2)] \Leftrightarrow [S_1 \equiv T_1 \text{ and } S_2 \equiv T_2], \tag{11.1}$$

while for the chosen series (see below)

$$[(S_1, S_2) \not\sim (T_1, T_2)] \quad \text{and} \quad [S_1 \sim T_1 \text{ and } S_2 \sim T_2]. \tag{11.2}$$

Let us choose:

$$S_1 = A_1A_2; \quad S_2 = A_1A_3A_3; \quad T_1 = A_4A_6; \quad T_2 = A_5A_6A_6.$$

The following binary relation $\mathcal{R}_1$ (resp., $\mathcal{R}_2$) over X^* is a word-η-bisimulation for (S_1, T_1) (resp., (S_2, T_2)):

$$\begin{aligned}
\mathcal{R}_1 = {} & \{(\varepsilon, \varepsilon)\} \cup \{(a_1, a_4)\} \cup \{(a_4u, a_1u) \mid u \in X^*\} \cup \{(a_iu, a_iu) \mid i \in \{2,3,5,6\}, u \in X^*\} \\
& \cup \{(a_1a_2u, a_4a_6u) \mid u \in X^*\} \cup \{(a_1a_6u, a_4a_2u) \mid u \in X^*\} \\
& \cup \{(a_1a_iu, a_4a_iu) \mid i \in \{1,3,4,5\}, u \in X^*\}, \\
\mathcal{R}_2 = {} & \{(\varepsilon, \varepsilon)\} \cup \{(a_1, a_5\}) \cup \{(a_5u, a_1u) \mid u \in X^*\} \cup \{(a_iu, a_iu) \mid i \in \{2,3,4,6\}, u \in X^*\} \\
& \cup \{(a_1a_3, a_5a_6)\} \cup \{(a_1a_6u, a_5a_3u) \mid u \in X^*\} \cup \{a_1a_iu, a_5a_iu \mid i \in \{1,2,4,5\}, u \in X^*\} \\
& \cup \{(a_1a_3a_3u, a_5a_6a_6u) \mid u \in X^*\} \cup \{(a_1a_3a_6u, a_5a_6a_3u) \mid u \in X^*\} \\
& \cup \{(a_1a_3a_iu, a_5a_6a_iu) \mid i \in \{1,2,4,5\}, u \in X^*\}.
\end{aligned}$$

Let us now check that

$$S_1 + S_2 \not\sim T_1 + T_2. \tag{11.3}$$

Let us consider some binary relation $\mathcal{R} \subseteq X^* \times X^*$ and show that it cannot be a word-η-bisimulation for $(S_1 + S_2, T_1 + T_2)$.

If $\mathcal{R}$ does not possess a pair with first component a_1, then it does not fulfill the totality condition.

If $(a_1, a_i) \in \mathcal{R}$, with $i \notin \{4, 5\}$, as $(S_1+S_2) \odot a_1 = A_2 + A_3A_3$ while $(T_1+T_2) \odot a_i = \emptyset$, then $\mathcal{R}$ cannot be a word-η-bisimulation.

If $(a_1, a_4) \in \mathcal{R}$, as $(T_1 + T_2) \odot a_4 = A_6$, and the set of lengths of the words generated from $A_2 + A_3A_3$ is $\{1, 2\}$ while the set of lengths of the words generated from A_6 is $\{1\}$, again $\mathcal{R}$ cannot be a word-η-bisimulation.

If $(a_1, a_5) \in \mathcal{R}$, as $A_2 + A_3A_3 \not\sim A_6A_6$ (apply the same argument on the set of lengths), then $\mathcal{R}$ cannot be a word-η-bisimulation.

We have proved (11.3).

But the properties established in section 4.3, namely, the soundness of rule (R8), applied with $T = T'$ equal to the column vector with two lines with entry ε show that we must then have

$$(S_1, S_2) \not\sim (T_1, T_2).$$

This ends the proof of (11.2). Let us notice that, of course, the implication from left to right, in (11.1), also holds for $\sim$. It is the implication from right to left which may fail for η-bisimulation.

Extension theorem. We show here that the "extension theorem," which is the main new tool introduced in [48], does not hold for η-bisimulation.

The vectors. Let us consider the deterministic vectors and matrices $S, T \in \mathsf{DB}_{2,1}\langle\langle\ V\ \rangle\rangle$, $A \in \mathsf{DB}_{2,2}\langle\langle\ V\ \rangle\rangle$, $\alpha, \beta \in \mathsf{DB}_{1,2}\langle\langle\ V\ \rangle\rangle$:

$$S = \begin{pmatrix} H \\ \varepsilon \end{pmatrix}, \quad T = \begin{pmatrix} H' \\ \varepsilon \end{pmatrix}, \quad A = \begin{pmatrix} H'' & 0 \\ 0 & H \end{pmatrix},$$

$$\alpha = (\alpha_1, \alpha_2), \quad \beta = (\beta_1, \beta_2) \quad \text{with}$$

$$\begin{aligned}
\alpha_1 &= A_1,\\
\alpha_2 &= A_2H + A_3H' + A_4H' + A_5H'' + A_6H''\\
&\quad + B_1H + B_2H + B_3H' + B_4H' + B_5H'' + B_6H'',\\
\beta_1 &= B_1,\\
\beta_2 &= B_2H + B_3H' + B_4H' + B_5H'' + B_6H''\\
&\quad + A_1H + A_2H + A_3H' + A_4H' + A_5H'' + A_6H''.
\end{aligned}$$

The equations. The following equations are true:

$$\alpha \cdot S \sim \beta \cdot S, \tag{11.4}$$

$$\alpha \cdot AS \sim \beta \cdot AS, \tag{11.5}$$

$$\alpha \cdot T \sim \beta \cdot T. \tag{11.6}$$

The reason for these three equivalences is that

$$\alpha_1 \cdot H + \alpha_2 \sim \beta_1 \cdot H + \beta_2$$

(this equation is even true for language equivalence),

$$\alpha_1 \cdot H' + \alpha_2 \sim \beta_1 \cdot H' + \beta_2 \quad \text{and} \quad \alpha_1 \cdot H'' + \alpha_2 \sim \beta_1 \cdot H'' + \beta_2;$$

these last two equivalences are due to the fact that the number of occurrences of some A_iH (resp., A_iH', A_iH'', A_i0, B_iH, B_iH', B_iH'', B_i0) in a deterministic sum does not modify its class modulo $\sim$, provided that this number remains nonnull (or remains null). We assert now that

$$\alpha \cdot AT \not\sim \beta \cdot AT. \tag{11.7}$$

Let us prove inequality (11.7). It reduces to

$$\alpha_1 \cdot H''H' + \alpha_2 \cdot H \not\sim \beta_1 \cdot H''H' + \beta_2 \cdot H,$$

but there is no word-η-bisimulation of depth 3 for the pair above, since

$$(\alpha_1 \cdot H''H' + \alpha_2 \cdot H) \odot a_1h''h' = \varepsilon,$$

while there is no pair $(a_1h''h', w) \in \eta$ such that

$$(\beta_1 \cdot H''H' + \beta_2 \cdot H) \odot w = \varepsilon.$$

The extension theorem. The "extension theorem" stated in [48, Theorem 1, p. 828] asserts that when $\sim$ is taken as the language equivalence relation, in the case where $n = 1$, $k = 0$, $m = \infty$, the conjunction of (11.4), (11.5), (11.6) implies

$$\alpha \cdot AT \sim \beta \cdot AT. \tag{11.8}$$

The above example shows that this theorem no longer holds when the equivalence relation $\sim$ is the η-bisimulation relation. (The same example also holds for depth $m = 3$ by the proof above.)

Subwords lemma. We show here that the "subwords lemma," which is the main new tool introduced in [43], does not hold for η-bisimulation.

The "subwords lemma" stated in [43, pp. 478–489] asserts that when $\sim$ is taken as the language equivalence relation, in the case where $\lambda = 2$, $n = \infty$, the conjunction of

$$\begin{aligned} \alpha \cdot S &\sim \beta \cdot S, \\ \alpha \cdot AS &\sim \beta \cdot AS, \\ \alpha \cdot BS &\sim \beta \cdot BS \end{aligned} \tag{11.9}$$

implies

$$\alpha \cdot ABS \sim \beta \cdot ABS. \tag{11.10}$$

But if we take

$$B = \begin{pmatrix} H' & 0 \\ 0 & H \end{pmatrix}$$

in the above example, the three premises (11.9) are valid, while (11.10) is equivalent to

$$\alpha_1 \cdot H''H'H + \alpha_2 \cdot HH \sim \beta_1 \cdot H''H'H + \beta_2 \cdot HH.$$

As the equivalence $\sim$ is right-cancellative (by Corollary 4.10, point (C3)), this equation is equivalent to

$$\alpha_1 \cdot H''H' + \alpha_2 \cdot H \sim \beta_1 \cdot H''H' + \beta_2 \cdot H,$$

which has been shown nonvalid in section 11.4.

Unifiers. We show here that, unlike for the equivalence $\equiv$, there is no unique most general unifier (modulo $\sim$) for a given finite system of linear equations (modulo $\sim$). This explains the need for some *oracle* in the definition of the triangulation process and, consequently, in the definition of strategy T_C.

It can be deduced from the type of reasoning used in [34], or even in the earlier paper [24], that any equation of the form

$$\alpha S \equiv \beta S \tag{11.11}$$

(where α, β are deterministic row-vectors in $\mathsf{DB}_{1,\lambda}\langle\langle\, V \,\rangle\rangle$) has a single "most-general unifier" up to $\equiv$, i.e., there exists a single column vector $S \in \mathsf{DB}_{\lambda,1}\langle\langle\, V \cup \mathcal{U} \,\rangle\rangle$, fulfilling (11.11) and such that all the solutions of (11.11) are obtained from S by substituting an arbitrary element of $\mathsf{DB}_{1,1}\langle\langle\, V \,\rangle\rangle$ to every variable in $\mathcal{U}$ (here $\mathcal{U}$ is a new alphabet of variables, disjoint from V and with cardinality $\leq \lambda$).

We have exhibited such an equation above, and we have seen that it admits at least three minimal unifiers,

$$\begin{pmatrix} HU \\ U \end{pmatrix}, \quad \begin{pmatrix} H'U \\ U \end{pmatrix}, \quad \begin{pmatrix} H''U \\ U \end{pmatrix},$$

where $U \in \mathcal{U}$ is a free variable. We could modify the example in such a way that the number of minimal unifiers matches an arbitrary value p. We can conclude that, in general, a left-linear equation modulo $\sim$ admits a finite number of minimal unifiers, but not a single one. This fact makes more difficult the treatment of systems of equations modulo bisimulation as compared to systems of equations modulo language equivalence.

Appendix. Let us sketch here a proof of Theorem 2.5.

LEMMA A.1. *Let $\Gamma = (\Gamma_0, v_0)$ be the computation 1-graph $(\mathcal{C}(\mathcal{M}), v_{\mathcal{M}})$ of some normalized pda $\mathcal{M}$. Then Γ is equational and has finite out-degree.*

Proof. Let $\mathcal{M} = \langle X, Z, Q, \delta, q_0, z_0, F\rangle$ be a normalized pda. Let us consider a new letter $e \notin X$ and build the real-time pda $\mathcal{M}_e = \langle X \cup \{e\}, Z, Q, \delta_e, q_0, z_0, F\rangle$ obtained by setting that, for every $x \in X$ and $q \in Q$, $z \in Z$,

$$\delta_e(qz, x) = \delta(qz, x); \quad \delta_e(qz, e) = \delta(qz, \epsilon).$$

By [27, Theorem 2.6, p. 62], the computation-graph $\mathcal{C}(\mathcal{M}_e)$ is context-free, and by [3, Theorem 6.3, p. 187] every context-free graph is equational. Hence $\mathcal{C}(\mathcal{M}_e)$ is equational. Let us remark that $\mathcal{C}(\mathcal{M})$ is obtained from this graph just by contracting all the edges labeled by e. Let us contract the edges labeled by e in some system of equations S_e defining $\mathcal{C}(\mathcal{M}_e)$: we obtain a system of equations S defining $\mathcal{C}(\mathcal{M})$. □

We now use the notation of [13]. Given a system of graph equations $S = \langle u_i = H_i;\ i \in [1, n]\rangle$, by $\mathcal{G}(S, u_i)$ we denote the ith component of the canonical solution of S.

DEFINITION A.2. *Let $S = \langle u_i = H_i;\ i \in [1, n]\rangle$ be a system of graph equations. It is* standard *iff it fulfills the following conditions:*

(1) *for every $i \in [1, n]$ and every distinct integer $k, \ell \in [1, \tau(H_i)]$, the sources $src(H_i, k)$, $src(H_i, \ell)$ are distinct vertices of H_i;*
(2) *for every $i \in [1, n]$ and every hyperedge h of H_i which is labeled by some unknown, all the vertices of h are distinct;*
(3) *for every $i \in [1, n]$, $k \in [1, \tau(u_i)]$, $\lambda \in \mathbb{N}$, if there exist λ edges going out of $src(\mathcal{G}(S, u_i), k)$, inside the graph $\mathcal{G}(S, u_i)$, then there exist also λ edges going out of $src(H_i, k)$, inside the graph H_i.*

LEMMA A.3. *Let $S = \langle u_i = H_i;\ i \in [1, n]\rangle$ be a system of graph equations where the unknown u_1 has type 1. One can compute from S a* standard *system of graph equations $S' = \langle u'_i = H'_i;\ i \in [1, n']\rangle$ such that the canonical solution of S' has a first component $\mathcal{G}(S', u'_1) = \mathcal{G}(S, u_1)$.*

Proof. From S one can construct a first system S_1 which generates the same first component $\mathcal{G}(S_1, u_1) = \mathcal{G}(S, u_1)$ and such that restrictions (1) and (2) of the lemma are fulfilled: this follows from [13, Proposition 2.10, p. 209] (notice that the condition "separated" in this reference is exactly the conjunction (1) $\wedge$ (2)).

Let $S_1 = \langle v_i = K_i;\ i \in [1, m]\rangle$. Let us replace every right-hand side K_i by a finite hypergraph L_i obtained by unfolding the graph K_i, according to the rules $v_j \to K_j$, as many times as necessary in order that every source $src(K_i, k)$ gets as many outgoing edges in L_i as in the "complete unfolded graph" $\mathcal{G}(S_1, v_i)$. The new system $S' = \langle v_i = L_i;\ i \in [1, m]\rangle$ still fulfills conditions (1) and (2), it also fulfills condition (3), and for every $i \in [1, m]$, $\mathcal{G}(S_1, v_i) = \mathcal{G}(S', v_i)$. Hence S' satisfies the conclusion of the lemma. □

LEMMA A.4. *Let $\Gamma = (\Gamma_0, v_0)$ be a rooted 1-graph over X which is the first component of the canonical solution of some standard system of graph equations. Then Γ is isomorphic to the computation 1-graph $(\mathcal{C}(\mathcal{M}), v_{\mathcal{M}})$ of some normalized pda $\mathcal{M}$.*

Sketch of proof. Let $S = \langle u_i = H_i;\ i \in [1, n]\rangle$ be a standard system of graph equations such that $\Gamma = \mathcal{G}(S, u_1)$.

Let us define $\mathcal{M} = \langle X, Z, Q, \delta, q_0, z_0, F\rangle$ as follows. In every right-hand side H_i we number bijectively all the unknown hyperedges, $\{h_{1,i}, \ldots, h_{j,i}, \ldots, h_{n_i,i}\}$, and all the vertices, $\{v_{1,i}, \ldots, v_{q,i}, \ldots, v_{N_i,i}\}$. Let us denote by $\beta(j,i)$ the label of $h_{j,i}$.

$$Z = \{[j,i] \mid 1 \leq i \leq n, 1 \leq j \leq n_i\} \cup \{[1,0]\}.$$

(We extend β by defining $\beta(1,0) = 1$.)

Intuitively every symbol $[j,i]$ describes the situation of a vertex which belongs to a component which has been glued to the jth unknown hyperedge of H_i.

Let $Q = [1, N]$, where N is the maximum number of vertices in the graphs H_i. Intuitively, the transitions of $\mathcal{M}$ starting from a mode $q[j,i]$ describe the edges starting from the qth vertex of $H_{\beta(j,i)}$. Let us define precisely the transitions starting from a mode $q[j,i]$:

Case 1. *q is strictly larger than the number of vertices of $H_{\beta(j,i)}$.* Then there is no transition starting from $q[j,i]$.

Case 2. *The vertex number q of $H_{\beta(j,i)}$ is a source of $H_{\beta(j,i)}$ and $i \neq 0$.* Then

$$q[j,i] \xrightarrow{\varepsilon} q',$$

where q' is the number of the vertex of H_i to which it is glued (it is some vertex of $h_{j,i}$).

Case 3. *The vertex number q of $H_{\beta(j,i)}$ is not a source of $H_{\beta(j,i)}$ or $i = 0$.*

Internal edges. For every edge $(v_{q,\beta(j,i)}, x, v_{q',\beta(j,i)})$, we add the transition

$$q[j,i] \xrightarrow{x} q'[j,i].$$

External edges. Let $k = \beta(j,i)$. For every ℓ such that $v_{q,\beta(j,i)}$ is a vertex of $h_{\ell,k}$ and every edge $(v_{r,\beta(\ell,k)}, x, v_{q',\beta(\ell,k)})$ where the vertex $v_{r,\beta(\ell,k)}$ of $H_{\beta(\ell,k)}$ is glued to the vertex $v_{q,\beta(j,i)}$ by the rewriting rule $u_{\beta(\ell,k)} \to H_{\beta(\ell,k)}$, we add the transition

$$q[j,i] \xrightarrow{x} q'[\ell,k][j,i].$$

The starting configuration is $1[1,0]$ (i.e., $q_0 = 1$, $z_0 = [1,0]$).

This pda is normalized (this is easy to check) and has a computation graph whose isomorphism class is exactly $\mathcal{G}(S, u_1)$ (this would be much more tedious to prove formally). □

Theorem 2.5 clearly follows from these three lemmas.

Acknowledgments. I thank O. Burkart, D. Caucal, P. Jancar, F. Moller, and C. Stirling for useful discussion and information about the subject treated in this work. This work has also benefited from the incisive questions, comments, and criticism of O. Burkart and C. Stirling about [34].

REFERENCES

[1] J. Baeten, J. Bergstra, and J. Klop, *Decidability of bisimulation equivalence for processes generating context-free languages*, in Proceedings of PARLE 87, Lecture Notes in Comput. Sci. 259, Springer-Verlag, Berlin, 1987, pp. 94–111.

[2] M. Bauderon, *Infinite hypergraph* I. *Basic properties (fundamental study)*, Theoret. Comput. Sci., 82 (1991), pp. 177–214.

[3] M. Bauderon, *Infinite hypergraph* II. *Systems of recursive equations*, Theoret. Comput. Sci., 103 (1992), pp. 165–190.

[4] M. Boffa, *Une remarque sur les systèmes complets d'identités rationnelles*, RAIRO Inform. Théor. Appl., 24 (1990), pp. 419–423.

[5] D. Caucal, *Graphes canoniques des graphes algébriques*, RAIRO Inform. Théor. Appl., 24 (1990), pp. 339–352.

[6] D. Caucal, *Bisimulation of context-free grammars and of pushdown automata*, in Modal Logic and Process Algebra, CSLI Lecture Notes 53, CSLI Publications, Stanford, CA, 1995, pp. 85–106.

[7] S. Christensen, Y. Hirshfeld, and F. Moller, *Bisimulation is decidable for basic parallel processes*, in Proceedings of CONCUR'93, Lecture Notes in Comput. Sci. 715, Springer-Verlag, Berlin, 1993, pp. 143–157.

[8] S. Christensen, H. Hüttel, and C. Stirling, *Bisimulation equivalence is decidable for all context-free processes*, Inform. and Comput., 121 (1995), pp. 143–148.

[9] B. Courcelle, *An axiomatic approach to the Korenjac-Hopcroft algorithms*, Math. Systems Theory, 16 (1983), pp. 191–231.

[10] B. Courcelle, *Fundamental properties of infinite trees*, Theoret. Comput. Sci., 25 (1983), pp. 95–169.

[11] B. Courcelle, *The monadic second-order logic of graphs* II. *Infinite graphs of bounded width*, Math. Systems Theory, 21 (1989), pp. 187–221.

[12] B. Courcelle, *Graph rewriting: An algebraic and logic approach*, in Handbook of Theoretical Computer Science, Vol. B, J. van Leeuwen, ed., MIT Press, Cambridge, MA, 1990, pp. 193–242.

[13] B. Courcelle, *The monadic second-order logic of graphs* IV. *Definability properties of equational graphs*, Ann. Pure Appl. Logic, 49 (1990), pp. 193–255.

[14] J. Groote and H. Hüttel, *Undecidable equivalences for basic process algebra*, Inform. and Comput., 115 (1994), pp. 354–371.

[15] M. Harrison, *Introduction to Formal Language Theory*, Addison-Wesley, Reading, MA, 1978.

[16] M. Harrison, I. Havel, and A. Yehudai, *On equivalence of grammars through transformation trees*, Theoret. Comput. Sci., 9 (1979), pp. 173–205.

[17] Y. Hirshfeld, M. Jerrum, and F. Moller, *A polynomial algorithm for deciding equivalence of normed context-free processes*, in Proceedings of FOCS'94, IEEE, 1994, pp. 623–631.

[18] H. Hüttel and C. Stirling, *Actions speak louder than words: Proving bisimilarity for context-free processes*, in Proceedings of LICS'91, IEEE, 1991, pp. 376–385.

[19] P. Jancar, *Bisimulation is decidable for one-counter processes*, in Proceedings of ICALP 97, Springer-Verlag, Berlin, 1997, pp. 549–559.

[20] D. Kozen, *A completeness theorem for Kleene algebras and the algebra of regular events*, in Proceedings of LICS'91, IEEE, 1991, pp. 214–225.

[21] D. Krob, *Complete systems of $\mathcal{B}$-rational identities*, Theoret. Comput. Sci., 89 (1991), pp. 207–343.

[22] R. Mayr, *Undecidability of weak bisimulation equivalence for* 1*-counter processes*, in Proceedings of ICALP 03, Lecture Notes in Comput. Sci. 2719, Springer-Verlag, Berlin, 2003, pp. 570–583.

[23] Y. Meitus, *The equivalence problem for real-time strict deterministic pushdown automata*, Kibernetika, 5 (1989), pp. 14–25 (in Russian; English translation in Cybernetics and Systems Analysis).

[24] Y. Meitus, *Decidability of the equivalence problem for deterministic pushdown automata*, Kibernetika, 5 (1992), pp. 20–45 (in Russian; English translation in Cybernetics and Systems Analysis).

[25] R. Milner, *A complete inference system for a class of regular behaviours*, J. Comput. System Sci., 28 (1984), pp. 439–466.

[26] R. Milner, *Communication and Concurrency*, Prentice-Hall, New York, 1989.

[27] D. Muller and P. Schupp, *The theory of ends, pushdown automata and second-order logic*, Theoret. Comput. Sci., 37 (1985), pp. 51–75.

[28] M. Oyamaguchi, *The equivalence problem for real-time d.p.d.a's*, J. Assoc. Comput. Mach., 34 (1987), pp. 731–760.

[29] D. Park, *Concurrency and automata on infinite sequences*, in Theoretical Computer Science, Lecture Notes in Comput. Sci. 104, Springer-Verlag, Berlin, 1981, pp. 167–183.

[30] H. Rogers, *Theory of Recursive Functions and Effective Calculability*, Series in Higher Mathematics, McGraw-Hill, New York, 1967.

[31] V. Romanovskii, *Equivalence problem for real-time deterministic pushdown automata*, Kibernetika, 2 (1985), pp. 13–23.

[32] A. Salomaa, *Two complete axiom systems for the algebra of regular events*, J. Assoc. Comput. Mach., 13 (1966), pp. 158–169.

[33] G. Sénizergues, $\Gamma(A) \sim \Gamma(B)$?, tech. report, nr1183-97, LaBRI, Université Bordeaux I, Talence, France, 1997; available online from http://www.labri.u-bordeaux.fr/~ges/.
[34] G. Sénizergues, $L(A) = L(B)$?, tech. report, nr1161-97, LaBRI, Université Bordeaux I, Talence, France, 1997.
[35] G. Sénizergues, $L(A) = L(B)$?, in Proceedings of INFINITY 97, Electronic Notes in Theoretical Computer Science 9, Elsevier, Amsterdam, 1997, pp. 1–26; available online from http://www.elsevier.nl/locate/entcs/volume9.html.
[36] G. Sénizergues, *The equivalence problem for deterministic pushdown automata is decidable*, in Proceedings ICALP 97, Lecture Notes in Comput. Sci. 1256, Springer-Verlag, Berlin, 1997, pp. 671–681.
[37] G. Sénizergues, *Decidability of bisimulation equivalence for equational graphs of finite out-degree*, in Proceedings of FOCS'98, R. Motwani, ed., IEEE Computer Society Press, Los Alamitos, CA, 1998, pp. 120–129.
[38] G. Sénizergues, $T(A) = T(B)$?, in Proceedings of ICALP 99, Lecture Notes in Comput. Sci. 1644, Springer-Verlag, Berlin, 1999, pp. 665–675. Full proofs in $T(A) = T(B)$?, tech. report 1209-99, LaBRI, Université Bordeaux I, Talence, France, 1999, pp. 1–61.
[39] G. Sénizergues, *Complete formal systems for equivalence problems*, Theoret. Comput. Sci., 231 (2000), pp. 309–334.
[40] G. Sénizergues, $L(A) = L(B)$? *Decidability results from complete formal systems*, Theoret. Comput. Sci., 251 (2001), pp. 1–166.
[41] G. Sénizergues, *Some applications of the decidability of dpda's equivalence*, in Proceedings of MCU'01, Lecture Notes in Comput. Sci. 2055, Springer-Verlag, Berlin, 2001, pp. 114–132.
[42] G. Sénizergues, $L(A) = L(B)$? *A simplified decidability proof*, Theoret. Comput. Sci., 281 (2002), pp. 555–608.
[43] G. Sénizergues, *The equivalence problem for t-turn dpda is co-NP*, in Proceedings of ICALP 03, Lecture Notes in Comput. Sci. 2719, Springer-Verlag, Berlin, 2003, pp. 478–489.
[44] J. Srba, *Undecidability of weak bisimilarity for pushdown processes*, in Proceedings of CONCUR 02, Lecture Notes in Comput. Sci. 2380, Springer-Verlag, Berlin, 2002, pp. 579–593.
[45] C. Stirling, *Decidability of bisimulation equivalence for normed pushdown processes*, in Proceedings of CONCUR 96, Lecture Notes in Comput. Sci. 1119, Springer-Verlag, Berlin, 1996, pp. 217–232.
[46] C. Stirling, *Decidability of DPDA Equivalence*, tech. report, Edinburgh ECS-LFCS-99-411, 1999, pp. 1–25; available online from http://www.lfcs.inf.ed.ac.uk/reports/99/ECS-LFCS-99-411/.
[47] C. Stirling, *Decidability of DPDA equivalence*, Theoret. Comput. Sci., 255 (2001), pp. 1–31.
[48] C. Stirling, *Deciding DPDA equivalence is primitive recursive*, in Proceedings of ICALP 02, Lecture Notes in Comput. Sci. 2380, Springer-Verlag, Berlin, 2002, pp. 821–832.
[49] L. Valiant, *The equivalence problem for deterministic finite-turn pushdown automata*, Inform. and Control, 25 (1974), pp. 123–133.