Chapter 4
C-Finite Sequences

Polynomial sequences obey certain linear recurrence equations with constant coefficients, as we have seen in the previous chapter. But in general, a sequence satisfying a linear recurrence equation with constant coefficients need not be a polynomial sequence. The solutions to such recurrence equations form a strictly larger class of sequences, the so-called C-finite sequences. This is the class we consider now.

4.1 Fibonacci Numbers

Fibonacci numbers are classically introduced as the total number of offspring a single pair of rabbits produces in n months, assuming that the initial pair is born in the first month, and that every pair which is two months or older produces another pair of rabbits per month (Fig. 4.1). The growth of the population under these assumptions is easily described: In the n-th month, we have all the pairs of rabbits that have already been there in the $(n-1)$-st month (rabbits don't die in our model), and all the new pairs of rabbits born in the n-th month. The number of pairs born in the n-th month is just the number of pairs who were already there in the $(n-2)$-nd month. The number of pairs of rabbits in the n-th month is therefore given by the n-th Fibonacci number, which is recursively defined via

$$F_{n+2} = F_{n+1} + F_n \quad (n \geq 0), \qquad F_0 = 0, F_1 = 1.$$

The first terms of this sequence are

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144, \ldots$$

Although this sequence has been studied at least one thousand years before it was popularized in the west in 1202 by Leonardo da Pisa, it still today attracts the attention of a large number of both professional and amateur mathematicians. This is partly because of the simple definition it has, partly because of the multitude of non-

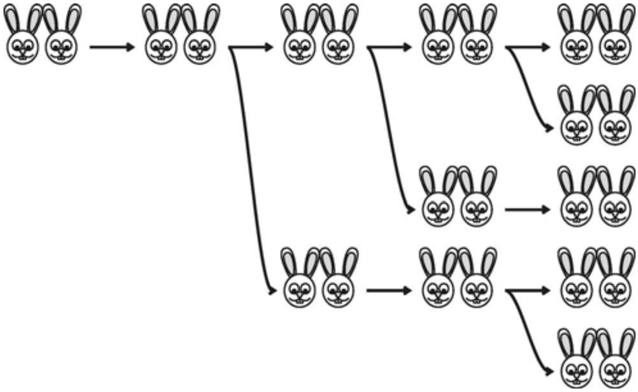

Fig. 4.1 Multiplying pairs of rabbits

mathematical contexts in which it arises, and partly because of some pretty peculiar mathematical properties it enjoys.

Lots of facts about Fibonacci numbers are available in classical textbooks on the subject or even on the web. We do not have the place here to give even a narrow overview over the material, and will only make some introductory remarks. The reader is encouraged to type "Fibonacci numbers" into a search engine and explore further properties on his or her own.

The recurrence equation quoted above allows us to compute F_n for every n recursively, by first computing F_{n-1} and F_{n-2} and then adding the two results to obtain F_n. If this is programmed naively, it leads to a horrible performance: If T_n is the number of additions needed to compute F_n in this way, we have $T_n = T_{n-1} + T_{n-2} + 1$ ($n \geq 2$), $T_0 = T_1 = 0$. A quick induction confirms that $T_n > F_n$ for $n \geq 2$, so computing

$$F_{100} = 354224848179261915075$$

in this way would take more than 10000 years even under the favorable assumption that we can do 10^9 additions per second. There must be a better way.

The problem is that there is an unnecessary amount of recomputation going on. For computing F_n, the naive program will first compute F_{n-1} recursively, then compute F_{n-2} recursively, and then return their sum. A clever implementation will take into account that F_{n-2} was already computed during the computation of F_{n-1}, and does not need to be computed again. The most easy way to avoid recomputation is to turn the second order recurrence into a first order matrix recurrence:

$$\begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} \qquad (n \geq 0).$$

By unfolding this recurrence in the obvious way, we can determine F_{n+1} and F_{n+2} simultaneously using n additions only. The computation of F_{100} is now a piece of cake.

But this is not the end. The unwinding of the recurrence is nothing else than a repeated multiplication of a fixed matrix to the initial values. Written in the form

$$\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} F_0 \\ F_1 \end{pmatrix} \qquad (n \geq 0),$$

we see that computing the n-th Fibonacci number is really not more than raising a certain matrix to the n-th power. Taking into account that $F_0 = 0$ and $F_1 = 1$, and setting $F_{-1} = 1$ (which is consistent with the recurrence), we can write this also as

$$\begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} F_{-1} & F_0 \\ F_0 & F_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \qquad (n \geq 0).$$

An immediate consequence of this equation is the relation

$$\begin{pmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{2n} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}^2 \qquad (n \geq 0),$$

which implies

$$\begin{aligned} F_{2n} &= F_n(F_{n+1} + F_{n-1}) = F_n(2F_{n+1} - F_n) \\ \text{and } F_{2n+1} &= F_n^2 + F_{n+1}^2 \end{aligned}$$

for $n \geq 0$. These formulas break any pair (F_n, F_{n+1}) down to (F_0, F_1) in just $\log_2(n)$ iterations. A single iteration requires two additions and four multiplications, making together $6\log_2(n)$ operations. For computing F_{100}, this scheme needs 40 operations, which may not seem like a big improvement compared to the 100 additions we needed before, but the difference becomes more and more significant as n grows: it certainly makes a difference whether $F_{1000000000}$ is computed with 180 operations or with 1000000000. On the other hand, counting only the number of operations is not really fair, because the computation time depends also on the length of the integers appearing as intermediate results. If these are properly taken into account, it turns out that both schemes have a linear runtime. But in finite domains, where numbers have a fixed size, the advantage of the logarithmic algorithm is striking.

The representation of Fibonacci numbers as a matrix power is not only of computational interest. It is also the source of several important identities. For example, taking determinants on both sides of

$$\begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \qquad (n \geq 0)$$

gives directly Cassini's identity

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \qquad (n \geq 0).$$

Another famous identity follows from diagonalization. The decomposition

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{pmatrix}}_{=:T} \underbrace{\begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix}}_{=:D} \underbrace{\begin{pmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{pmatrix}^{-1}}_{=T^{-1}}$$

with $\phi_1 = \frac{1}{2}(1+\sqrt{5})$ and $\phi_2 = \frac{1}{2}(1-\sqrt{5})$ implies

$$\begin{aligned}\begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1}\end{pmatrix} &= (TDT^{-1})^n = TD^nT^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ \phi_1 & \phi_2\end{pmatrix}\begin{pmatrix} \phi_1^n & 0 \\ 0 & \phi_2^n\end{pmatrix}\begin{pmatrix} 1 & 1 \\ \phi_1 & \phi_2\end{pmatrix}^{-1} \qquad (n \geq 0).\end{aligned}$$

Carrying out the matrix product leads to the Euler-Binet formula

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) \qquad (n \geq 0).$$

The number $\phi = \frac{1}{2}(1+\sqrt{5}) \approx 1.61803$ appearing here is known as the *golden ratio*. Because of $|\frac{1}{2}(1-\sqrt{5})| \approx |-0.61803| < \phi$, the term ϕ^n dominates the asymptotic behavior. We have

$$F_n \sim \frac{1}{\sqrt{5}}\phi^n \qquad (n \to \infty),$$

the error being so small that F_n is actually equal to the integer closest to $\frac{1}{\sqrt{5}}\phi^n$ for every $n \geq 0$. Another consequence of the asymptotic estimate is

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi.$$

4.2 Recurrences with Constant Coefficients

The Fibonacci numbers are a prototypical example for a sequence that satisfies a linear recurrence with constant coefficients. Such equations are called *C-finite recurrences*, and a sequence $(a_n)_{n=0}^{\infty}$ is called C-finite if it satisfies a C-finite equation.To be explicit, a sequence $(a_n)_{n=0}^{\infty}$ is called C-finite (of order r) if there are numbers $c_0, c_1, \ldots, c_{r-1} \in \mathbb{K}$ with $c_0 \neq 0$ such that

$$a_{n+r} + c_{r-1}a_{n+r-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0 \qquad (n \geq 0).$$

All the terms in a C-finite sequence of order r are uniquely determined by the coefficients $c_0, \ldots, c_{r-1}$ of the recurrence equation and the initial values $a_0, a_1, \ldots, a_{r-1}$. It is important to note that this is only a finite amount of data and can be stored and manipulated faithfully in a computer algebra system. (Hence the name C-finite, the C refers to the constant coefficients.)

For a fixed C-finite recurrence, different initial values lead to different sequences. The set of all sequences that satisfy some fixed C-finite recurrence equation forms a vector space over $\mathbb{K}$, for if two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are such that

$$\begin{aligned} & a_{n+r} + c_{r-1}a_{n+r-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0 \qquad (n \geq 0) \\ \text{and} \quad & b_{n+r} + c_{r-1}b_{n+r-1} + \cdots + c_1 b_{n+1} + c_0 b_n = 0 \qquad (n \geq 0),\end{aligned}$$

then multiplying the first equation by $\alpha \in \mathbb{K}$, the second by $\beta \in \mathbb{K}$, and adding them together gives

$$(\alpha a_{n+r} + \beta b_{n+r}) + \cdots + c_1(\alpha a_{n+1} + \beta b_{n+1}) + c_0(\alpha a_n + \beta b_n) = 0 \qquad (n \geq 0),$$

so the linear combination $(\alpha a_n + \beta b_n)_{n=0}^{\infty}$ satisfies the same recurrence as $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$. The vector space of all the sequences that satisfy some fixed C-finite recurrence is called the *solution space* of that recurrence.

The solution space of a C-finite recurrence has only a finite dimension, in fact, its dimension equals the order r of the recurrence. To see that it has at least this dimension, it is sufficient to guarantee that there are r linearly independent solutions. This is easy because any choice of the r initial values can be extended in a unique way to a solution of the recurrence, so if for $i = 0, \ldots, r-1$ we let $(b_n^{(i)})_{n=0}^{\infty}$ be the solutions that start like

$$0, 0, \ldots, 0, \underset{\substack{\uparrow \\ \text{index } i}}{1}, 0, \ldots, 0, b_r^{(i)}, b_{r+1}^{(i)}, b_{r+2}^{(i)}, \ldots,$$

then these r sequences are clearly linearly independent over $\mathbb{K}$. This proves that the solution space has at least dimension r. To prove that its dimension cannot be more than r, it is sufficient to observe that every solution is uniquely determined by its first r values, all the further values being forced by the recurrence. So if $(a_n)_{n=0}^{\infty}$ is any solution to the recurrence, then

$$a_n = a_0 b_n^{(0)} + a_1 b_n^{(1)} + \cdots + a_{r-1} b_n^{(r-1)} \qquad (n \geq 0),$$

because both sides agree for $n = 0, \ldots, r-1$ and both sides are solutions of the recurrence. In short, what we have shown is that the operation of truncating a sequence $(a_n)_{n=0}^{\infty}$ to a vector $(a_n)_{n=0}^{r-1} \in \mathbb{K}^r$ induces a vector space isomorphism between the solution space of the recurrence and $\mathbb{K}^r$.

With the solutions $(b_n^{(i)})_{n=0}^{\infty}$ $(i = 0, \ldots, r-1)$, we already know a vector space basis of the solution space. But these solutions are not particularly handy. Our next goal is to construct a basis that consists of sequences which admit closed form expressions. We have already seen in the previous section that the Fibonacci numbers can be expressed as a linear combinations of the two *exponential sequences* $(\phi^n)_{n=0}^{\infty}$ and $((-\phi)^{-n})_{n=0}^{\infty}$ where $\phi = \frac{1}{2}(1+\sqrt{5})$ is the golden ratio. Indeed, these two sequences themselves form a basis of the solution space of the Fibonacci recurrence. Does this generalize to arbitrary C-finite sequences? Let's see. Suppose that the recurrence

$$a_{n+r} + c_{r-1} a_{n+r-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0 \qquad (n \geq 0)$$

has a solution $(u^n)_{n=0}^{\infty}$ for some $u \in \mathbb{K} \setminus \{0\}$. Then

$$u^{n+r} + c_{r-1} u^{n+r-1} + \cdots + c_1 u^{n+1} + c_0 u^n = 0 \qquad (n \geq 0),$$

and dividing both sides by u^n leads to

$$u^r + c_{r-1} u^{r-1} + \cdots + c_1 u + c_0 = 0,$$

a plain polynomial equation for u which no longer depends on n. The polynomial $x^r + c_{r-1}x^{r-1} + \cdots + c_1 x + c_0 \in \mathbb{K}[x]$ is called the *characteristic polynomial* of the recurrence. Every root u of the characteristic polynomial gives rise to a solution $(u^n)_{n=0}^{\infty}$ of the corresponding recurrence. For example, in the case of the Fibonacci recurrence

$$a_{n+2} - a_{n+1} - a_n = 0,$$

the characteristic polynomial is $x^2 - x - 1$, and its two roots are $\frac{1}{2}(1+\sqrt{5})$ and $\frac{1}{2}(1-\sqrt{5})$, in accordance with what we have found before.

The characteristic polynomial suggests the viewpoint of operators acting on sequences. If we define the mapping

$$\begin{aligned}&\bullet \colon \mathbb{K}[x] \times \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}\\ &(c_r x^r + c_{r-1}x^{r-1} + \cdots + c_0) \bullet (a_n)_{n=0}^{\infty} := (c_r a_{n+r} + \cdots + c_0 a_n)_{n=0}^{\infty},\end{aligned}$$

then a C-finite recurrence can be stated compactly in the form $c(x) \bullet (a_n)_{n=0}^{\infty} = 0$ where $c(x) \in \mathbb{K}[x]$ is the characteristic polynomial of the recurrence. In this setting, monomials x^i can be regarded as shift operators $n \mapsto n+i$. Polynomial arithmetic is compatible with the composition of operators in the sense that

$$\begin{aligned}(u(x)+v(x)) \bullet (a_n)_{n=0}^{\infty} &= u(x) \bullet (a_n)_{n=0}^{\infty} + v(x) \bullet (a_n)_{n=0}^{\infty}\\ \text{and } (u(x)v(x)) \bullet (a_n)_{n=0}^{\infty} &= u(x) \bullet (v(x) \bullet (a_n)_{n=0}^{\infty})\end{aligned}$$

for every $u(x), v(x) \in \mathbb{K}[x]$ and every sequence $(a_n)_{n=0}^{\infty}$. As a consequence, if $(a_n)_{n=0}^{\infty}$ is a solution of $v(x) \bullet (a_n)_{n=0}^{\infty} = 0$ for some $v(x) \in \mathbb{K}[x]$, then also $(u(x)v(x)) \bullet (a_n)_{n=0}^{\infty} = 0$, because

$$(u(x)v(x)) \bullet (a_n)_{n=0}^{\infty} = u(x) \bullet (v(x) \bullet (a_n)_{n=0}^{\infty}) = u(x) \bullet 0 = 0.$$

In terms of operators, finding exponential solutions $(u^n)_{n=0}^{\infty}$ of a C-finite recurrence amounts to finding linear factors $x-u$ of the characteristic polynomial, because $(u^n)_{n=0}^{\infty}$ is a solution of the recurrence $a_{n+1} - ua_n = 0$ which corresponds to the linear polynomial $x-u$.

If $\mathbb{K}$ is algebraically closed, then we know that a characteristic polynomial of degree r will split into r linear factors. Each linear factor gives rise to a solution of the recurrence, and if all the linear factors are different, then we have found r different solutions of the recurrence. These are linearly independent (Problem 4.7) and so they form a basis of the solution space. It remains to consider what happens when the characteristic polynomial has multiple roots. We have actually seen characteristic polynomials with multiple roots already in the previous chapter, where we observed that any polynomial sequence $(p(n))_{n=0}^{\infty}$ with $p(x) \in \mathbb{K}[x]$ of degree 2 satisfies the recurrence

$$\Delta^3 p(n) = p(n+3) - 3p(n+2) + 3p(n+1) - p(n) = 0 \qquad (n \geq 0).$$

The characteristic polynomial of this recurrence is $(x-1)^3$. The observation is that multiple roots of the characteristic polynomial induce polynomial factors in the solutions. In the following theorem, we give the general result.

Theorem 4.1 *Suppose that $c_0,\dots,c_{r-1}\in\mathbb{K}$ with $c_0\neq 0$ are such that*

$$x^r+c_{r-1}x^{r-1}+\dots+c_1x+c_0=(x-u_1)^{e_1}(x-u_2)^{e_2}\cdots(x-u_m)^{e_m}$$

for $e_1,\dots,e_m\in\mathbb{N}\setminus\{0\}$ and pairwise distinct $u_1,\dots,u_m\in\mathbb{K}$. Then the sequences $(n^iu_j^n)_{n=0}^\infty$ ($j=1,\dots,m$, $i=0,\dots,e_j-1$) form a basis of the $\mathbb{K}$-vector space of all solutions of the recurrence equation

$$a_{n+r}+c_{r-1}a_{n+r-1}+\dots+c_1a_{n+1}+c_0a_n=0 \qquad (n\geq 0).$$

Proof. There are two things to show: (i) all the sequences $(n^iu_j^n)_{n=0}^\infty$ are solutions of the recurrence, and (ii) all other solutions are linear combinations of those.

(i) Take an arbitrary $j\in\{1,\dots,m\}$ and $i\in\{0,\dots,e_j-1\}$. We show that $(n^iu_j^n)_{n=0}^\infty$ is a solution of the recurrence. As $(x-u_j)^{i+1}$ is a factor of the characteristic polynomial, it suffices to show that $(x-u_j)^{i+1}\bullet(n^iu_j^n)_{n=0}^\infty=0$, which is easily done by induction on i.

(ii) Since the characteristic polynomial has degree r, we have $e_1+\dots+e_m=r$, so the number of solutions we are considering agrees with the dimension of the solution space. What remains to be shown is that the solutions stated in the theorem are linearly independent. Suppose this is not the case. Then there is a nontrivial $\mathbb{K}$-linear combination of them which is identically zero. By grouping sequences of like exponential parts together, we can write this linear combination in the form

$$p_1(n)u_1^n+p_2(n)u_2^n+\dots+p_m(n)u_m^n=0 \qquad (n\geq 0),$$

with certain $p_1(x),\dots,p_m(x)\in\mathbb{K}[x]$ not all of which are zero. Our plan is to take such a linear combination which is minimal in a certain sense, and then construct a smaller one. This gives the desired contradiction.

Although not all of $p_1(x),\dots,p_m(x)\in\mathbb{K}[x]$ are zero, some of them may be. For every relation, there will be an index i such that $p_1(x)=\dots=p_{i-1}(x)=0\neq p_i(x)$. From all the assumed relations, select those where this index i is as large as possible. Among those, we may consider one in which $\deg p_i(x)$ is as small as possible. Then we have

$$p_i(n)u_i^n+p_{i+1}(n)u_{i+1}^n+\dots+p_m(n)u_m^n=0 \qquad (n\geq 0).$$

This equation implies both

$$p_i(n+1)u_i^{n+1}+p_{i+1}(n+1)u_{i+1}^{n+1}+\dots+p_m(n+1)u_m^{n+1}=0 \quad (\text{shift } n\mapsto n+1)$$

and

$$p_i(n)u_i^{n+1}+p_{i+1}(n)u_iu_{i+1}^n+\dots+p_m(n)u_iu_m^n=0 \qquad (\text{multiply by } u_i).$$

Subtracting the two equations gives

$$(\Delta p_i)(n)u_iu_i^n+\tilde{p}_{i+1}(n)u_{i+1}^n+\dots+\tilde{p}_m(n)u_m^n=0 \qquad (n\geq 0),$$

an equation that violates our minimality assumptions, for either $\Delta p_i(x)=0$, then i was not maximal, or $\Delta p_i(x)\neq 0$, then $\deg\Delta p_i(x)=\deg p_i(x)-1$ and so $\deg p_i(x)$ was not minimal. □

This theorem allows us to express any C-finite sequence as a linear combination of terms of the form $n^d u^n$. All we need to do is to determine the roots of the characteristic polynomials along with their multiplicities and then find a linear combination of the terms predicted in the theorem above that matches the initial values of the sequence at hand. For example, for the sequence $(a_n)_{n=0}^{\infty}$ defined via

$$a_{n+4} - 4a_{n+3} + 2a_{n+2} + 4a_{n+1} - 3a_n = 0 \qquad (n \geq 0)$$
$$\text{and } a_0 = 1, \quad a_1 = 3, \quad a_2 = -4, \quad a_3 = 0,$$

we find first that

$$x^4 - 4x^3 + 2x^2 + 4x - 3 = (x-1)^2(x+1)(x-3).$$

By the theorem there must be some constants c_1, c_2, c_3, c_4 such that

$$a_n = c_1 + c_2 n + c_3(-1)^n + c_4 3^n \qquad (n \geq 0).$$

These constants are quickly determined by setting $n = 0,1,2,3,4$, which gives a linear system for the c_i. It is a consequence of Theorem 4.1 that this system must have a unique solution. In our case, we find $c_1 = \frac{13}{4}$, $c_2 = -3$, $c_3 = -\frac{19}{8}$, $c_4 = \frac{1}{8}$, so the final result is

$$a_n = \tfrac{13}{4} - 3n - \tfrac{19}{8}(-1)^n + \tfrac{1}{8}3^n \qquad (n \geq 0).$$

4.3 Closure Properties

There are C-finite sequences that arise from some specific combinatorial or number theoretic considerations. The most famous one is certainly the Fibonacci sequence. Another one is the sequence of *Lucas numbers* $(L_n)_{n=0}^{\infty}$ defined via

$$L_{n+2} - L_{n+1} - L_n = 0 \quad (n \geq 0), \qquad L_0 = 2, L_1 = 1.$$

This sequence behaves in many ways similarly to the Fibonacci numbers.

Another C-finite sequence is the Perrin sequence $(P_n)_{n=0}^{\infty}$, which is defined by

$$P_{n+3} - P_{n+1} - P_n = 0 \quad (n \geq 0), \qquad P_0 = 3, P_1 = 0, P_2 = 2.$$

The first numbers in this sequence are

$$3,\ 0,\ 2,\ 3,\ 2,\ 5,\ 5,\ 7,\ 10,\ 12,\ 17, \ldots$$

Perrin numbers are known for a curious number theoretic property they have. It appears that $n \in \mathbb{N}$ is prime “almost if and only if” $P_n \bmod n = 0$. For small n, we have

n	2	3	4	5	6	7	8	9	10	11	12
$P_n \bmod n$	0	0	2	0	5	0	2	3	7	0	5

The first mismatch is at $n = 271441$, for which $P_n \bmod n = 0$ although $271441 = 521^2$ is composite. On the other hand, it can be shown that if n is prime, then definitely $P_n \bmod n = 0$, and therefore, the Perrin numbers provide a strong necessary condition for a natural number to be a prime.

We know from the previous section that the Fibonacci and the Lucas numbers satisfy the same linear recurrence, so any linear combination of them will satisfy that recurrence as well. In particular, we find that the numbers $F_n + L_n$ are C-finite. What about the numbers $F_n + P_n$? Do they also form a C-finite sequence? Yes, they do, and it is not hard to come up with a recurrence satisfied by them. We just need to take the recurrence whose characteristic polynomial is the product of the two respective characteristic polynomials for F_n and P_n. This recurrence will have $(F_n)_{n=0}^{\infty}$ as a solution, because its characteristic polynomial is a multiple of $x^2 - x - 1$, and it will have $(P_n)_{n=0}^{\infty}$ as a solution, because its characteristic polynomial is a multiple of $x^3 - x - 1$. Consequently, also $(F_n + P_n)_{n=0}^{\infty}$ will be a solution of this recurrence.

By the same reasoning, it follows that whenever $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are C-finite sequences then so is their sum $(a_n + b_n)_{n=0}^{\infty}$. We say that the class of C-finite sequences is *closed under addition.* There are several other closure properties by which new C-finite sequences can be obtained from known ones. The following theorem provides an overview.

Theorem 4.2 *Let $(u_n)_{n=0}^{\infty}$ and $(v_n)_{n=0}^{\infty}$ be C-finite sequences in $\mathbb{K}$ of order r and s, respectively, and let $m \in \mathbb{N}$, $m \geq 1$. Then:*

1. *$(u_n + v_n)_{n=0}^{\infty}$ is C-finite of order at most $r + s$,*
2. *$(u_n v_n)_{n=0}^{\infty}$ is C-finite of order at most rs,*
3. *$(\sum_{k=0}^{n} u_k)_{n=0}^{\infty}$ is C-finite of order at most $r + 1$,*
4. *$(u_{mn})_{n=0}^{\infty}$ is C-finite of order at most r,*
5. *$(u_{\lfloor n/m \rfloor})_{n=0}^{\infty}$ is C-finite of order at most mr.*

Proof. If $c_0, \dots, c_{r-1} \in \mathbb{K}$ are such that

$$u_{n+r} + c_{r-1} u_{n+r-1} + \cdots + c_1 u_{n+1} + c_0 u_n = 0 \qquad (n \geq 0),$$

then also

$$u_{n+i} + c_{r-1} u_{n+i-1} + \cdots + c_1 u_{n+i-r+1} + c_0 u_{n+i-r} = 0 \qquad (n \geq 0)$$

for every fixed $i \geq r$. Repeated use of the recurrence shows that $(u_{n+i})_{n=0}^{\infty}$ ($i \geq 0$ fixed) is a linear combination of $(u_n)_{n=0}^{\infty}, (u_{n+1})_{n=0}^{\infty}, \dots, (u_{n+r-1})_{n=0}^{\infty}$. In other words, the shifted sequences $(u_{n+i})_{n=0}^{\infty}$ ($i \geq 0$ fixed) all belong to the $\mathbb{K}$-vector space U of dimension at most r generated by the sequences $(u_n)_{n=0}^{\infty}, \dots, (u_{n+r-1})_{n=0}^{\infty}$. Analogously, the sequences $(v_{n+i})_{n=0}^{\infty}$ ($i \geq 0$ fixed) all belong to the $\mathbb{K}$-vector space V of dimension (at most) s generated by the sequences $(v_n)_{n=0}^{\infty}, \dots, (v_{n+s-1})_{n=0}^{\infty}$.

1. Let $(w_n)_{n=0}^\infty = (u_n)_{n=0}^\infty + (v_n)_{n=0}^\infty$. All shifted sequences $(w_{n+i})_{n=0}^\infty$ ($i \geq 0$ fixed) belong to the $\mathbb{K}$-vector space $W := U + V$ generated by $(u_n)_{n=0}^\infty, \dots, (u_{n+r-1})_{n=0}^\infty$ and $(v_n)_{n=0}^\infty, \dots, (v_{n+s-1})_{n=0}^\infty$, because this vector space contains all the shifted sequences $(u_{n+i})_{n=0}^\infty$ and all the shifted sequences $(v_{n+i})_{n=0}^\infty$. The dimension of W is at most $r+s$, so any $r+s+1$ sequences $(w_{n+i})_{n=0}^\infty$ ($i \geq 0$ fixed) must be linearly dependent. In particular, the sequences
$$(w_n)_{n=0}^\infty,\ (w_{n+1})_{n=0}^\infty,\ \dots, (w_{n+r+s})_{n=0}^\infty \in W$$
must be linearly dependent over $\mathbb{K}$. This means there are constants $d_0, \dots, d_{r+s} \in \mathbb{K}$, not all zero, such that
$$d_0 w_n + d_1 w_{n+1} + \cdots + d_{r+s} w_{n+r+s} = 0 \qquad (n \geq 0),$$
and therefore $(w_n)_{n=0}^\infty$ is C-finite of order at most $r+s$.
2. Let $(w_n)_{n=0}^\infty = (u_n)_{n=0}^\infty \odot (v_n)_{n=0}^\infty = (u_n v_n)_{n=0}^\infty$. All shifted sequences $(w_{n+i})_{n=0}^\infty$ ($i \geq 0$ fixed) belong to the $\mathbb{K}$-vector space $W = U \otimes V$ generated by the mutual products
$$\begin{aligned}
&(u_n v_n)_{n=0}^\infty, (u_{n+1} v_n)_{n=0}^\infty, \dots, (u_{n+r-1} v_n)_{n=0}^\infty,\\
&(u_n v_{n+1})_{n=0}^\infty, (u_{n+1} v_{n+1})_{n=0}^\infty, \dots, (u_{n+r-1} v_{n+1})_{n=0}^\infty,\\
&\vdots\\
&(u_n v_{n+s-1})_{n=0}^\infty, (u_{n+1} v_{n+s-1})_{n=0}^\infty, \dots, (u_{n+r-1} v_{n+s-1})_{n=0}^\infty.
\end{aligned}$$
Arguing as before, since W has dimension at most rs, the sequence $(w_n)_{n=0}^\infty$ must satisfy a recurrence with constant coefficients of order at most rs.
3. Now let $(w_n)_{n=0}^\infty = (\sum_{k=0}^n u_k)_{n=0}^\infty$. Because of
$$\sum_{k=0}^{n+i} u_k = \left(\sum_{k=0}^{n} u_k\right) + u_{n+1} + u_{n+2} + \cdots + u_{n+i} \qquad (n \geq 0)$$
for every fixed $i \geq 0$, all the shifted sequences $(w_{n+i})_{n=0}^\infty$ ($i \geq 0$ fixed) belong to the vector space W generated by $(w_n)_{n=0}^\infty$ and $(u_n)_{n=0}^\infty, \dots, (u_{n+r-1})_{n=0}^\infty$. Since W has dimension at most $r+1$, the claim follows.
4. and 5. Problem 4.5. □

Theorem 4.2 is constructive in the sense that recurrences for the sequences proved C-finite can be obtained computationally from recurrences for of the given sequences by making the linear algebra reasoning in the proof explicit. For example, in order to find a recurrence for $(F_n^2)_{n=0}^\infty$, use the Fibonacci recurrence to get the representations
$$\begin{aligned}
F_n^2 &= 1F_n^2 + 0F_nF_{n+1} + 0F_{n+1}^2,\\
F_{n+1}^2 &= 0F_n^2 + 0F_nF_{n+1} + 1F_{n+1}^2,\\
F_{n+2}^2 &= 1F_n^2 + 2F_nF_{n+1} + 1F_{n+1}^2,\\
F_{n+3}^2 &= 1F_n^2 + 4F_nF_{n+1} + 4F_{n+1}^2,\\
F_{n+4}^2 &= 4F_n^2 + 12F_nF_{n+1} + 9F_{n+1}^2.
\end{aligned}$$

The coefficients c_0, c_1, c_2, c_3, c_4 of the desired recurrence appear in the nullspace vectors of the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 4 \\ 0 & 0 & 2 & 4 & 12 \\ 0 & 1 & 1 & 4 & 9 \end{pmatrix}.$$

The nullspace here is generated by $(1,-2,-2,1,0)$ and $(2,-3,-6,0,1)$, corresponding to the two recurrence equations

$$F_n^2 - 2F_{n+1}^2 - F_{n+2}^2 + F_{n+3}^2 = 0 \qquad (n \geq 0)$$

$$\text{and} \quad 2F_n^2 - 3F_{n+1}^2 - 6F_{n+2}^2 + F_{n+4}^2 = 0 \qquad (n \geq 0).$$

Closure properties are extremely useful for systematically proving identities. For example, in order to prove the index duplication formula for Fibonacci numbers,

$$F_{2n} = 2F_nF_{n+1} - F_n^2 \qquad (n \geq 0),$$

all we need to do is to obtain a recurrence for the sequence $(a_n)_{n=0}^{\infty}$ where

$$a_n := F_{2n} - 2F_nF_{n+1} + F_n^2 \qquad (n \geq 0).$$

With the help of Theorem 4.2 (and appropriate computer algebra software), this is easily done. We may find that the sequence $(a_n)_{n=0}^{\infty}$ satisfies the recurrence

$$a_{n+3} = -a_n + 2a_{n+1} + 2a_{n+2} \qquad (n \geq 0),$$

and so in order to show that $a_n = 0$ for all n, it suffices to check that $a_0 = a_1 = a_2 = 0$ and resort to induction on n.

Even better, we do not need to compute the recurrence equations explicitly, but just refer to the bounds on their order that are provided in Theorem 4.2. Given that the Fibonacci numbers F_n satisfy a recurrence of order two, we find without any computation that:

- $(F_{2n})_{n=0}^{\infty}$ satisfies a recurrence of order at most 2,
- $(F_nF_{n+1})_{n=0}^{\infty}$ satisfies a recurrence of order at most 4,
- $(F_n^2)_{n=0}^{\infty}$ satisfies a recurrence of order at most 4,
- Therefore $(2F_nF_{n+1} + F_n^2)_{n=0}^{\infty}$ satisfies a recurrence of order at most 8,
- Therefore $(F_{2n} - 2F_nF_{n+1} - F_n^2)_{n=0}^{\infty}$ satisfies a recurrence of order at most 10.

The index duplication formula for the Fibonacci numbers is therefore proven as soon as it is verified for the finitely many indices $n = 0, 1, \ldots, 9$.

The important feature of this method is that virtually any identity about any C-finite sequences can be reduced mechanically to checking the identity for a finite number of values.

4.4 The Tetrahedron for C-finite Sequences

The most immediate vertex in the Concrete Tetrahedron for C-finite sequences is the recurrence corner, for the simple reason that C-finite sequences are by definition those that satisfy a linear recurrence with constant coefficients.

The special form of a C-finite recurrence implies also a special form of the generating function of a C-finite sequence. Starting from a recurrence equation

$$a_{n+r} + c_{r-1}a_{n+r-1} + \cdots + c_0 a_n = 0 \qquad (n \geq 0),$$

multiplying by x^n and summing over all $n \geq 0$ gives

$$\sum_{n=0}^{\infty} a_{n+r}x^n + c_{r-1}\sum_{n=0}^{\infty} a_{n+r-1}x^n + \cdots + c_0 \sum_{n=0}^{\infty} a_n x^n = 0.$$

In terms of the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$ of some solution $(a_n)_{n=0}^{\infty}$, this equation turns into

$$\begin{aligned}
&\big(a(x) - (a_0 + a_1 x + \cdots a_{r-1}x^{r-1})\big)/x^r \\
&+ c_{r-1}\big(a(x) - (a_0 + a_1 x + \cdots + a_{r-2}x^{r-2})\big)/x^{r-1} \\
&\quad + \cdots\cdots \\
&\quad + c_1\big(a(x) - a_0\big)/x \\
&\quad + c_0 a(x) = 0,
\end{aligned}$$

which is of the form

$$a(x) + c_{r-1}xa(x) + \cdots + c_1 x^{r-1}a(x) + c_0 x^r a(x) = p(x)$$

for some polynomial $p(x)$ of degree at most $r-1$ that depends on the initial values $a_0, \ldots, a_{r-1}$ of the sequence $(a_n)_{n=0}^{\infty}$ at hand. We have found that the generating function $a(x)$ is just a rational function

$$a(x) = \frac{p(x)}{1 + c_{r-1}x + \cdots + c_1 x^{r-1} + c_0 x^r}$$

where the coefficients of the denominator polynomial agree with the coefficients in the original C-finite recurrence. Going through the steps of the argument in reverse order shows that every rational function $p(x)/q(x)$ with $\deg p(x) < \deg q(x)$ is the generating function of some C-finite recurrence:

Theorem 4.3 *A sequence $(a_n)_{n=0}^{\infty}$ in $\mathbb{K}$ satisfies a C-finite recurrence*

$$a_{n+r} + c_{r-1}a_{n+r-1} + \cdots + c_0 a_n = 0 \qquad (n \geq 0)$$

with $c_0, \ldots, c_r \in \mathbb{K}$, $c_0 \neq 0$, if and only if

$$\sum_{n=0}^{\infty} a_n x^n = \frac{p(x)}{1 + c_{r-1}x + \cdots + c_1 x^{r-1} + c_0 x^r}$$

for some polynomial $p(x) \in \mathbb{K}[x]$ of degree at most $r-1$.

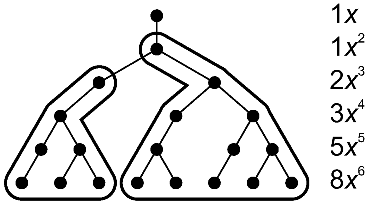

Fig. 4.2 Self-similarity in the Fibonacci tree

A pictorial interpretation for the generating function $F(x)$ of the Fibonacci numbers is given in Fig. 4.2: as the picture indicates, the family tree of the fertile pairs of rabbits is *self-similar* in the sense that chopping off the root of the tree (corresponding to the linear term x) leaves us with two copies of the family tree, one rooted at level 1 (corresponding to $xF(x)$) and one rooted at level 2 (corresponding to $x^2F(x)$). This motivates the identity

$$F(x) = x + xF(x) + x^2F(x),$$

from which we get

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}.$$

Theorem 4.3 also confirms independently some of the closure properties we have proved in the previous section. For instance, the closure of C-finite sequences under addition follows readily from the closure of rational functions under addition and the closure under summation is a direct consequence of the closure of rational functions under multiplication by $1/(1-x)$.

In the case where $\mathbb{K}$ is algebraically closed, also the result about the representation of C-finite sequences in terms of polynomials and exponentials is accessible from the generating function point of view. We know from the previous chapter that the generating function of a polynomial sequence $(p(n))_{n=0}^{\infty}$ is a rational function $q(x)/(1-x)^d$ with $\deg q(x) < d$. The substitution $x \mapsto ux$ for fixed $u \in \mathbb{K}$ reveals that then $q(ux)/(1-ux)^d$ is the generating function of $(p(n)u^n)_{n=0}^{\infty}$. From the fact that every rational function $a(x)/b(x) \in \mathbb{K}(x)$ with $b(0) \neq 0$ and $\deg a(x) < \deg b(x)$ can be brought to the form

$$\frac{a(x)}{b(x)} = \frac{q_1(x)}{(1-u_1x)^{d_1}} + \frac{q_2(x)}{(1-u_2x)^{d_2}} + \cdots + \frac{q_m(x)}{(1-u_mx)^{d_m}}$$

for distinct $u_1,\ldots,u_m \in \mathbb{K}\setminus\{0\}$ such that $1/u_i$ is a d_i-th root of $b(x)$ and $q_i(x) \in \mathbb{K}[x]$ is of degree less than d_i (partial fraction decomposition), we obtain an alternative proof of Theorem 4.1.

In the case where $\mathbb{K} \subseteq \mathbb{C}$, the asymptotic behavior of a C-finite sequence can be read off from the partial fraction decomposition of the generating function, or, for that matter, from the closed form representation of the sequence. If we sort exponential terms via

$$n^{d_1}a_1^n \prec n^{d_2}a_2^n \quad :\iff \quad |a_1| < |a_2| \text{ or } \big(|a_1| = |a_2| \text{ and } d_1 < d_2\big)$$

then the asymptotics of a C-finite sequence is determined by the terms in the closed form representation which are maximal with respect to this order. For example, for $(a_n)_{n=0}^{\infty}$ defined via

$$\begin{aligned} a_{n+5} - \tfrac{1}{2}a_{n+4} - 2a_{n+3} + a_{n+2} + a_{n+1} - \tfrac{1}{2}a_n = 0 \qquad (n \geq 0),\\ \text{and } a_0 = 0, a_1 = 1, a_2 = -1, a_3 = 5, a_4 = 0, \end{aligned}$$

we have

$$a_n = -\tfrac{7}{2} - \tfrac{1}{18}(-1)^n + \tfrac{32}{9}\big(\tfrac{1}{2}\big)^n + \tfrac{7}{4}n - \tfrac{11}{12}n(-1)^n \qquad (n \geq 0),$$

and therefore

$$a_n \sim n\big(\tfrac{7}{4} - \tfrac{11}{12}(-1)^n\big) \qquad (n \to \infty).$$

The exponential terms 1^n and $(-1)^n$ in this asymptotic estimate reflect the fact that 1 and -1 are the poles of the generating function (considered as an analytic function) which are closest to the origin. The polynomial factor n reflects the fact that these poles have multiplicity two.

The closed form representation also gives an immediate access to the summation problem. By the finite version of the geometric series, we have

$$\sum_{k=0}^{n} u^k = \frac{1-u^{n+1}}{1-u} \qquad (n \geq 0, u \neq 1),$$

and therefore the sum over an exponential term is essentially again an exponential term. The generating function technique from Sect. 3.3 for summation of polynomial sequences $(p(n))_{n=0}^{\infty}$ admits a straight-forward extension to sequences of the form $(p(n)u^n)_{n=0}^{\infty}$, which provides yet another proof that C-finite sequences are closed under summation.

While this may be acceptable as a theoretical argument, we won't usually proceed along these lines for solving particular summation problems. Most often it will be more interesting to ask whether a given sum over a C-finite sequence can be expressed in terms of the summand sequence itself, like in the identity

$$\sum_{k=0}^{n} F_k = F_n + F_{n+1} - 1 \qquad (n \geq 0).$$

This closed form evaluation is certainly a more comfortable result than some mess of exponentials involving the golden ratio.

Such representations always exist, and it is not hard to write a program that finds them. In order to evaluate a sum $\sum_{k=0}^{n} a_k$ in terms of $a_n, a_{n+1}, \dots, a_{n+r-1}$, where $(a_n)_{n=0}^{\infty}$ satisfies the recurrence

$$a_{n+r} + c_{r-1}a_{n+r-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0 \qquad (n \geq 0),$$

all we need to do is to find a linear combination $(b_n)_{n=0}^\infty$ of $(a_n)_{n=0}^\infty, \dots, (a_{n+r-1})_{n=0}^\infty$ such that $b_{n+1} - b_n = a_n$ for all n, and then apply telescoping. In the example of the Fibonacci identity above, the key step is the observation

$$(F_{n+1} + F_{n+2}) - (F_n + F_{n+1}) = F_{n+1} \qquad (n \geq 0).$$

If the characteristic polynomial $c(x) = x^r + c_{r-1}x^{r-1} + \cdots + c_1 x + c_0$ is such that $c(1) \neq 0$, then we can find by division with remainder a polynomial $q(x) = x^{r-1} + q_{r-2}x^{r-2} + \cdots + q_1 x + q_0 \in \mathbb{K}[x]$ with

$$c(x) = (x-1)q(x) + c(1).$$

Setting

$$(b_n)_{n=0}^\infty := q(x) \bullet (a_n)_{n=0}^\infty = (a_{n+r-1} + q_{r-2}a_{n+r-2} + \cdots + q_1 a_{n+1} + q_0 a_n)_{n=0}^\infty,$$

the equation $c(x) = (x-1)q(x) + c(1)$ translates into

$$b_{n+1} - b_n + c(1)a_n = 0 \qquad (n \geq 0),$$

and telescoping gives

$$\sum_{k=0}^n a_k = -\frac{1}{c(1)}\big(b_{n+1} - b_0\big) \qquad (n \geq 0),$$

which is of the desired form.

If $c(1) = 0$, we can still write $c(x) = (x-1)^m \bar{c}(x)$ with $m \geq 1$ and $\bar{c}(x) \in \mathbb{K}[x]$ such that $\bar{c}(1) \neq 0$, and we will find some $\bar{q}(x) = x^{r-m-1} + \bar{q}_{r-m-2}x^{r-m-2} + \cdots + \bar{q}_0 \in \mathbb{K}[x]$ with

$$c(x) = (x-1)^m\big((x-1)\bar{q}(x) + \bar{c}(1)\big).$$

Defining $(b_n)_{n=0}^\infty := \bar{q}(x) \bullet (a_n)_{n=0}^\infty$, this equation translates into

$$\Delta^m\big(b_{n+1} - b_n + \bar{c}(1)a_n\big) = 0 \qquad (n \geq 0),$$

where Δ is the forward difference operator. These m difference operators can be undone by m repeated summations. After a first summation, we find that the sequence $\Delta^{m-1}(b_{n+1} - b_n + \bar{c}(1)a_n)_{n=0}^\infty$ is constant. Summing this constant sequence once more, we find that the sequence $\Delta^{m-2}(b_{n+1} - b_n + \bar{c}(1)a_n)_{n=0}^\infty$ must be a polynomial sequence of degree one, and so on. After m summations, we eventually find that

$$b_{n+1} - b_n + \bar{c}(1)a_n = \bar{p}(n) \qquad (n \geq 0)$$

for some polynomial $\bar{p}(x) \in \mathbb{K}[x]$ of degree at most $m-1$. One final summation then gives the representation

$$\sum_{k=0}^n a_k = -\frac{1}{\bar{c}(1)} b_{n+1} + p(n) \qquad (n \geq 0)$$

for some polynomial $p(x) \in \mathbb{K}[x]$ of degree at most m.

4.5 Systems of C-finite Recurrences

A C-finite system of recurrence equations is a simultaneous equation for several sequences $(a_n^{(1)})_{n=0}^\infty, \dots, (a_n^{(s)})_{n=0}^\infty$ of the form

$$\begin{pmatrix} a_{n+r}^{(1)} \\ a_{n+r}^{(2)} \\ \vdots \\ a_{n+r}^{(s)} \end{pmatrix} + C_{r-1} \begin{pmatrix} a_{n+r-1}^{(1)} \\ a_{n+r-1}^{(2)} \\ \vdots \\ a_{n+r-1}^{(s)} \end{pmatrix} + \dots + C_1 \begin{pmatrix} a_{n+1}^{(1)} \\ a_{n+1}^{(2)} \\ \vdots \\ a_{n+1}^{(s)} \end{pmatrix} + C_0 \begin{pmatrix} a_{n}^{(1)} \\ a_{n}^{(2)} \\ \vdots \\ a_{n}^{(s)} \end{pmatrix} = 0 \quad (n \geq 0)$$

where $C_0, \dots, C_{r-1} \in \mathbb{K}^{s \times s}$ are matrices and C_0 is invertible. Clearly, each C-finite sequence is also a solution of such a system, as the special case $s = 1$ reduces to the usual C-finite recurrences. But for $s > 0$, the system may relate the values of the s sequences mutually to each other. For example, the system

$$\begin{aligned} u_{n+1} &= u_n + 3v_n \qquad (n \geq 0), \\ v_{n+1} &= 2u_n + 2v_n \quad (n \geq 0), \end{aligned}$$

is of the form we are considering. (For clarity, the matrix-vector multiplication is explicitly spelled out here.) It is natural to ask whether this additional freedom allows for defining sequences that are not C-finite in the usual sense, that is, whether the solutions of C-finite systems form a larger class of sequences than the solutions of (scalar) C-finite recurrences do.

It turns out that this is not the case. To see this, observe first that we can restrict ourselves without loss of generality to first order systems, because a system of order r is equivalent to the first order system

$$\begin{pmatrix} A_{n+1} \\ A_{n+2} \\ \vdots \\ \vdots \\ A_{n+r} \end{pmatrix} + \begin{pmatrix} 0 & I_s & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & I_s \\ C_0 & C_1 & \cdots & \cdots & C_{r-1} \end{pmatrix} \begin{pmatrix} A_n \\ A_{n+1} \\ \vdots \\ \vdots \\ A_{n+r-1} \end{pmatrix} = 0 \qquad (n \geq 0),$$

where I_s refers to the $s \times s$ unit matrix and A_{n+i} refers to the vector $(a_{n+i}^{(1)}, \dots, a_{n+i}^{(s)})$. So let us consider a first order system

$$A_{n+1} = MA_n \qquad (n \geq 0),$$

where M is some invertible $s \times s$ matrix over $\mathbb{K}$. If we unfold this recurrence like a scalar recurrence, we get

$$A_n = M^n A_0 \qquad (n \geq 0),$$

where M^n is the n-fold matrix multiplication of M with itself.

By the theorem of Cayley-Hamilton, the characteristic polynomial $c(x) := \det(M - xI_s)$ of the matrix M has the property that $c(M) = 0$, therefore, if $c(x) = c_0 + c_1x + \cdots + c_sx^s$, then

$$c_sM^s + c_{s-1}M^{s-1} + \cdots + c_1M + c_0I_s = 0.$$

Multiplying this from the left to M^nA_0, we find

$$c_sA_{n+s} + c_{s-1}A_{n+s-1} + \cdots + c_1A_{n+1} + c_0A_n = 0 \quad (n \geq 0).$$

This means that all the coordinate sequences $(a_n^{(i)})_{n=0}^{\infty}$ satisfy the same recurrence equation

$$c_sa_{n+s}^{(i)} + c_{s-1}a_{n+s-1}^{(i)} + \cdots + c_1a_{n+1}^{(i)} + c_0a_n^{(i)} = 0 \quad (n \geq 0),$$

so in particular they are C-finite.

Let us use this insight to determine the solutions of the example system

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \qquad (n \geq 0).$$

The characteristic polynomial of the matrix M in this example is $(x-4)(x+1)$, and therefore any solution (u_n, v_n) will have the form

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} c_{1,1} \\ c_{2,1} \end{pmatrix} 4^n + \begin{pmatrix} c_{1,2} \\ c_{2,2} \end{pmatrix} (-1)^n \qquad (n \geq 0)$$

for certain constants $c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}$. But not every choice of constants gives rise to solutions of the system. To find out which ones do, we can simply substitute the general form of the solution into the original system and compare the coefficients of 4^n and $(-1)^n$ to zero. This gives constraints on the $c_{i,j}$. The requirement

$$\begin{pmatrix} c_{1,1} \\ c_{2,1} \end{pmatrix} 4^{n+1} + \begin{pmatrix} c_{1,2} \\ c_{2,2} \end{pmatrix} (-1)^{n+1} \stackrel{!}{=} \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \left(\begin{pmatrix} c_{1,1} \\ c_{2,1} \end{pmatrix} 4^n + \begin{pmatrix} c_{1,2} \\ c_{2,2} \end{pmatrix} (-1)^n \right)$$

implies the condition

$$\begin{pmatrix} 3c_{1,1} - 3c_{2,1} \\ -2c_{1,1} + 2c_{2,1} \end{pmatrix} 4^n + \begin{pmatrix} -2c_{1,2} - 3c_{2,2} \\ -2c_{1,2} - 3c_{2,2} \end{pmatrix} (-1)^n \stackrel{!}{=} 0.$$

Forcing the coefficient vectors to zero gives the linear system

$$\begin{pmatrix} 3 & 0 & -3 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & -3 \\ 0 & -2 & 0 & -3 \end{pmatrix} \begin{pmatrix} c_{1,1} \\ c_{1,2} \\ c_{2,1} \\ c_{2,2} \end{pmatrix} = 0$$

whose solution space is generated by the vectors $(1,0,1,0)$ and $(0,-3,0,2)$. Consequently, the solution space of the original C-finite recurrence system consists precisely of the $\mathbb{K}$-linear combinations of

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} 4^n \quad \text{and} \quad \begin{pmatrix} -3 \\ 2 \end{pmatrix} (-1)^n.$$

The vectors $(1,1)$ and $(-3,2)$ appearing here are eigenvectors of M for the eigenvalues 4 and -1, respectively. Of course, this is not a coincidence but a rule.

Before concluding this section, we still want to mention that C-finite systems can also be approached by generating functions. For $i = 1,\dots,s$, let $a^{(i)}(x) := \sum_{n=0}^{\infty} a_n^{(i)} x^n$ be the generating functions of the sequences arising in a system $A_{n+1} = MA_n$ as above. The various recurrence equations

$$a_{n+1}^{(i)} = m_{i,1}a_n^{(1)} + m_{i,2}a_n^{(2)} + \cdots + m_{i,s}a_n^{(s)} \qquad (n \geq 0, i = 1,\dots,s)$$

that make up this system correspond to equations

$$a^{(i)}(x) - a^{(i)}(0) = x m_{i,1}\, a^{(1)}(x) + x m_{i,2}\, a^{(2)}(x) + \cdots + x m_{i,s}\, a^{(s)}(x) \qquad (i = 1,\dots,s)$$

for the generating functions $a^{(1)}(x),\dots,a^{(s)}(x)$. These equations form a linear system of equations over $\mathbb{K}(x)$ which has a unique solution of the form

$$\begin{pmatrix} a^{(1)}(x) \\ \vdots \\ a^{(s)}(x) \end{pmatrix} = \frac{1}{c_s + c_{s+1}x + \cdots + c_0 x^s} \begin{pmatrix} q^{(1)}(x) \\ \vdots \\ q^{(s)}(x) \end{pmatrix}$$

where $c_0 + c_1 x + \cdots + c_s x^s = \det(M - xI_s) \in \mathbb{K}[x]$ is the characteristic polynomial and $q^{(1)}(x),\dots,q^{(s)}(x) \in \mathbb{K}[x]$ are some polynomials of degree less than s.

4.6 Applications

Regular Languages

In the theory of formal languages, a *language* is a set of *words* consisting of some *letters* chosen from a fixed *alphabet*. Words are simply tuples of letters, for example, if the alphabet is $\Sigma = \{\mathtt{a},\mathtt{b},\mathtt{c}\}$, then the language Σ^* of all the words is

$$\Sigma^* = \{\varepsilon, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{aa}, \mathtt{ab}, \mathtt{ac}, \mathtt{ba}, \mathtt{bb}, \mathtt{bc}, \mathtt{ca}, \mathtt{cb}, \mathtt{cc}, \mathtt{aaa}, \mathtt{aab}, \mathtt{aac}, \dots\},$$

where ε refers to the empty word, a word that has no letters at all. An example for a nontrivial language is the language $L \subseteq \Sigma^*$ of all words that do not contain aa or bb or cc as subwords:

$$L = \{\varepsilon, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{ab}, \mathtt{ac}, \mathtt{ba}, \mathtt{bc}, \mathtt{ca}, \mathtt{cb}, \mathtt{aba}, \mathtt{abc}, \mathtt{aca}, \mathtt{acb}, \mathtt{bab}, \dots\} \subseteq \Sigma^*.$$

Language theorists have come up with a hierarchy of languages that allows to rank formal languages according to how difficult they are. One of the most elementary types of languages is the class of regular languages. These are languages that can be constructed inductively, by saying that:

- The empty language $\varnothing$ and the languages $\{\omega\}$ containing a single word $\omega \in \Sigma^*$ are regular languages.
- If A and B are regular languages, then so are their union $A \cup B$ as well as

$$A.B := \{ab : a \in A, b \in B\}$$
$$\text{and} \quad A^* := \{\varepsilon\} \cup A \cup A.A \cup A.A.A \cup A.A.A.A \cup \cdots,$$

where ab refers to the word obtained from a and b by joining them together.

Regular languages are used in computer science for describing the most elementary syntactic entities of a programming language. For example, if floating point numbers are written according to the usual rules as words over the alphabet $\{\texttt{+}, \texttt{-}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}, \texttt{.}, \texttt{e}\}$, then the set of words that are syntactically correct encodings of floating point numbers forms a regular language. It is less obvious, but also true, that the language L defined before is also a regular language.

The relation of regular languages to C-finite sequences is that the number of words with exactly n letters belonging to a given regular language is always a C-finite sequence. The language L, for example, contains exactly $3 \cdot 2^n$ words of length $n \geq 1$.

Unrestricted Lattice Walks

Multivariate recurrence equations with constant coefficients arise for instance in enumeration problems related to lattice walks. As an example, consider walks in $\mathbb{Z}^2$ starting in the origin $(0,0)$ and consisting of n steps where each single step can go either north-east ($\nearrow$) or south ($\downarrow$) or west ($\leftarrow$). One such walk with $n = 10$ steps is shown in Fig. 4.3.

We are interested in the number $a_{n,i,j}$ of walks with n steps that end at a given point $(i, j) \in \mathbb{Z}^2$. Obviously, we can compute this number recursively via

$$a_{n+1,i,j} = a_{n,i-1,j-1} + a_{n,i+1,j} + a_{n,i,j+1} \qquad (n \geq 0;\, i, j \in \mathbb{Z}),$$

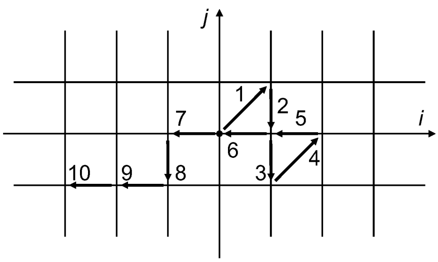

Fig. 4.3 A walk in the plane consisting of ten steps

the initial conditions being given by

$$a_{0,i,j} = \begin{cases} 1 & \text{if } i = j = 0 \\ 0 & \text{otherwise} \end{cases}.$$

It is clear that by making n steps we cannot reach any point (i, j) with $|i| > n$ or $|j| > n$, so we have $a_{n,i,j} = 0$ for these i, j. Therefore, for every fixed n, the doubly infinite series

$$\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_{n,i,j} x^i y^j \qquad (n \in \mathbb{N} \text{ fixed})$$

is actually just a rational function in x and y. The generating function

$$a(t,x,y) := \sum_{n=0}^{\infty} \left(\sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} a_{n,i,j} x^i y^j \right) t^n$$

can therefore be considered as an element of $\mathbb{Q}(x,y)[[t]]$.

A straight-forward calculation starting from the multivariate recurrence equation for the $a_{n,i,j}$ leads to the equation

$$\frac{1}{t}\big(a(t,x,y) - 1\big) = xya(t,x,y) + \frac{1}{x}a(t,x,y) + \frac{1}{y}a(t,x,y)$$

for the generating function, from which we directly obtain

$$\begin{aligned} a(t,x,y) &= \frac{1}{1 - t(xy + \frac{1}{x} + \frac{1}{y})} = \sum_{n=0}^{\infty} \left(xy + \frac{1}{x} + \frac{1}{y} \right)^n t^n \\ &= 1 + \big(xy + x^{-1} + y^{-1}\big)t + \big(x^{-2} + 2x + y^{-2} + 2x^{-1}y^{-1} + 2y + x^2y^2\big)t^2 + \cdots \end{aligned}$$

For example, the coefficient 2 in front of the term xt^2 $(= x^1y^0t^2)$ means that there are precisely two ways to get in two steps from the origin to the point $(1,0)$. Observe also that setting $x = y = 1$ (which in this special situation is allowed) gives the geometric series $1/(1-3t)$, in accordance with the fact that there are in total 3^n walks with n steps and arbitrary endpoint.

Chebyshev Polynomials

The Chebyshev polynomials $T_n(x)$ were introduced in Sect. 3.5 as the polynomials appearing in the series expansion of the rational function $(1-xy)/(1-2xy+y^2)$ with respect to y, hence they are C-finite. We have the recurrence relation

$$T_{n+2}(x) - 2xT_{n+1}(x) + T_n(x) = 0 \qquad (n \geq 0).$$

Chebyshev polynomials appear in the repeated application of the trigonometric addition theorem for cosine: We have $\cos(nt) = T_n(\cos t)$ $(n \geq 0)$, because this is true for $n = 0$ and $n = 1$, and the recurrence holds because of

$$\cos((n+1)t+t) + \cos((n+1)t-t) = 2\cos(t)\cos((n+1)t) = 0 \qquad (n \geq 0)$$

by applying the addition theorem for cosine twice.

The substitution $x = \cos t$ also allows for a quick proof of the orthogonality relation of Chebyshev polynomials (Sect. 3.5): Applying the substituting $x = \cos t$, $dx = -\sin t\, dt$ to the integral

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx$$

gives

$$\int_{\pi}^{0} \frac{T_m(\cos t)T_n(\cos t)}{\sin t}(-\sin t)dt = \int_{0}^{\pi} \cos(mt)\cos(nt)\,dt.$$

For this integral, the desired result is easily obtained by repeated integration by parts.

Chebyshev polynomials are relevant in numerical analysis. Here, instead of expanding an analytic function $f\colon \mathbb{C} \to \mathbb{C}$ in a Taylor series about the origin, one considers expansions of f as a series in terms of Chebyshev polynomials:

$$f(z) = \sum_{n=0}^{\infty} a_n T_n(z).$$

The coefficients in this expansion are easily determined thanks to the orthogonality relation:

$$\begin{aligned} \int_{-1}^{1} \frac{f(z)T_n(z)}{\sqrt{1-z^2}} dz &= \int_{-1}^{1} \sum_{k=0}^{\infty} a_k T_k(z) T_n(z) \frac{dz}{\sqrt{1-z^2}} \\ &= \sum_{k=0}^{\infty} a_k \int_{-1}^{1} T_k(z) T_n(z) \frac{dz}{\sqrt{1-z^2}} = a_n \pi \end{aligned}$$

for $n \geq 1$. (For $n = 0$, the integral exceptionally evaluates to $a_n\pi/2$.) This works analogously for any family of orthogonal polynomials.

The numerical interest in Chebyshev expansions is rooted in the feature that the polynomial $p_n(x) = \sum_{k=0}^{n} a_k T_k(x)$ obtained by truncating the series after the n-th term is such that

$$\max_{z \in [-1,1]} |f(z) - p_n(z)|$$

becomes almost as small as it can possibly get for a polynomial of degree n. See [48] for precise estimation statements and proofs.

For example, a pretty good approximation of the exponential function is $p_2(x) = a_0 + a_1 T_1(x) + a_2 T_2(x)$, where

$$a_0 = \frac{2}{\pi}\int_{-1}^{1} \frac{\exp(z)T_0(z)}{\sqrt{1-z^2}}dz \approx 1.26607,$$
$$a_1 = \frac{1}{\pi}\int_{-1}^{1} \frac{\exp(z)T_1(z)}{\sqrt{1-z^2}}dz \approx 1.13032,$$
$$a_2 = \frac{1}{\pi}\int_{-1}^{1} \frac{\exp(z)T_2(z)}{\sqrt{1-z^2}}dz \approx 0.271495.$$

The polynomial $p_2(z)$ is plotted together with $\exp(z)$ in Fig. 4.4. For the next polynomial $p_3(z)$, a difference to the curve of $\exp(z)$ would already be hardly visible.

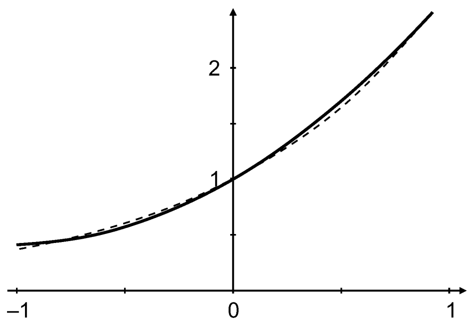

Fig. 4.4 $\exp(z)$ (dashed) and the quadratic polynomial (solid) that comes closest to it on $[-1,1]$

4.7 Problems

Problem 4.1 Determine the first ten decimal digits of $F_{2^{1000}}$.

Problem 4.2 Determine the last ten decimal digits of $F_{2^{1000}}$.

Problem 4.3 Express $\displaystyle\sum_{k=0}^{n} F_{n+k}$ in terms of Fibonacci numbers.

Problem 4.4 Prove the identity $\displaystyle\sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1}$ $(n \geq 0)$.

(*Hint:* Recall that $1/(1-x-xy) = \sum_{n,k=0}^{\infty} \binom{n}{k} x^n y^k$ and consider the substitution $y \mapsto x$ in the sense of power series composition.)

Problem 4.5 Prove parts 4 and 5 of Theorem 4.2.

Problem 4.6 Consider the continued fraction

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}.$$

What is the value of the finite continued fraction obtained by truncating at the n-th level? What is the value of the infinite continued fraction?

Problem 4.7 (Vandermonde's determinant) For $a_1,\dots,a_m \in \mathbb{K}$, consider

$$A := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_m \\ \vdots & \vdots & & \vdots \\ a_1^{m-1} & a_2^{m-1} & \cdots & a_m^{m-1} \end{pmatrix} \in \mathbb{K}^{m\times m}.$$

Use the determinant formula

$$\det A = \prod_{1\le i<j\le m} (a_i - a_j)$$

to show:

1. For any pairwise different numbers $a_i \in \mathbb{K}$ $(i = 1,\dots,m)$, the sequences $(a_i^n)_{n=0}^{\infty}$ are linearly independent over $\mathbb{K}$.
2. For any pairwise different numbers $a_i \in \mathbb{K}$ $(i = 1,\dots,m)$ and any numbers $b_i \in \mathbb{K}$ $(i = 1,\dots,m)$, there exists precisely one polynomial $p(x) \in \mathbb{K}[x]$ of degree $m-1$ (or less) such that $p(a_i) = b_i$ $(i = 1,\dots,m)$.

Problem 4.8 Let $(a_n)_{n=0}^{\infty}$ be a C-finite sequence and suppose that $(u_n)_{n=0}^{\infty}$ satisfies the inhomogeneous linear recurrence

$$u_{n+r} + c_{r-1}u_{n+r-1} + \cdots + c_1 u_{n+1} + c_0 u_n = a_n \qquad (n \ge 0).$$

Prove that $(u_n)_{n=0}^{\infty}$ is C-finite.

Problem 4.9 Show that $(H_n)_{n=0}^{\infty}$ is not C-finite.

Problem 4.10 Show that $(S_2(n,k))_{n=0}^{\infty}$ is C-finite for every fixed $k \in \mathbb{N}$.

Problem 4.11 The goal of this problem is to show that for a polynomial $p(x,y) \in \mathbb{Q}[x,y]$ we have $p(F_n, F_{n+1}) = 0$ for all $n \geq 0$ if and only if $p(x,y)$ is a multiple of $u(x,y)^2 - 1$ where $u(x,y) = x^2 - xy - y^2$.

1. Show that $u(F_{2n}, F_{2n+1}) = 1$ and $u(F_{2n+1}, F_{2n}) = -1$ for all $n \in \mathbb{N}$. Conclude the implication from right to left.
2. Show that for every polynomial $p(x,y)$ there is a polynomial $a(x,y)$ such that $q(x,y) := p(x,y) + (u(x,y) - 1)a(x,y)$ is linear in y. Conclude that $p(F_{2n}, F_{2n+1}) = 0$ for all $n \in \mathbb{N}$ if and only if $q(F_{2n}, F_{2n+1}) = 0$ for all $n \in \mathbb{N}$.
3. Show by an asymptotic argument that if two polynomials $q_0(x), q_1(x) \in \mathbb{Q}[x]$ are such that $q_0(F_{2n}) + q_1(F_{2n})F_{2n+1} = 0$ for all $n \in \mathbb{N}$ then $q_0(x) = q_1(x) = 0$. Conclude that $p(F_{2n}, F_{2n+1}) = 0$ for all $n \in \mathbb{N}$ if and only if $p(x,y)$ is a multiple of $u(x,y) - 1$.
4. Show that $p(F_{2n+1}, F_{2n}) = 0$ for all $n \in \mathbb{N}$ if and only if $p(x,y)$ is a multiple of $u(x,y) + 1$ by an analogous argument. Conclude that $p(F_n, F_{n+1}) = 0$ for all $n \in \mathbb{N}$ if and only if $p(x,y)$ is a multiple of $u(x,y)^2 - 1$.

Problem 4.12 We have observed that the solution $(a_n)_{n=0}^{\infty}$ of a C-finite recurrence of order r is uniquely determined by the initial values $a_0, \dots, a_{r-1}$. More generally, let $i_1, \dots, i_r \in \mathbb{N}$ be pairwise distinct. Is the solution $(a_n)_{n=0}^{\infty}$ uniquely determined by the r values $a_{i_1}, \dots, a_{i_r}$?

Problem 4.13 The *exponential generating function* $\bar{a}(x)$ of a sequence $(a_n)_{n=0}^{\infty}$ is defined as the formal power series $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$. Show that a sequence satisfies a linear recurrence with constant coefficients if and only if its exponential generating function satisfies a linear differential equation with constant coefficients.

Problem 4.14 Suppose that the characteristic polynomial of the C-finite recurrence

$$a_{n+r} - c_{r-1}a_{n+r-1} - \cdots - c_0 a_n = 0 \qquad (n \geq 0)$$

has r pairwise different roots $u_1, \dots, u_r \in \mathbb{K} \setminus \{0\}$, and let (like in Sect. 4.5 with $s = 1$)

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & \cdots & \cdots & c_{r-1} \end{pmatrix}.$$

Show that

$$M = \begin{pmatrix} 1 & \cdots & 1 \\ u_1 & \cdots & u_r \\ \vdots & & \vdots \\ u_1^{r-1} & \cdots & u_r^{r-1} \end{pmatrix} \begin{pmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & u_r \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \\ u_1 & \cdots & u_r \\ \vdots & & \vdots \\ u_1^{r-1} & \cdots & u_r^{r-1} \end{pmatrix}^{-1}$$

and use this result to obtain an alternative proof of Theorem 4.1 for the special case under consideration.