Unambiguity-Preserving Operation on Tree Languages That Lifts Topological Complexity

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HIGHLIGHTS 2013

- infinite trees (binary)

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- parity automata

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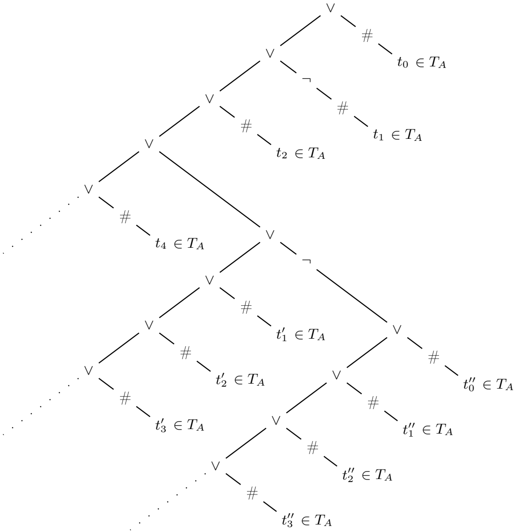
  $=$ reductions by continuous functions

Sigma Operation

Given language $L \subseteq T_A$, we construct $\varsigma(L)$ of trees like this:



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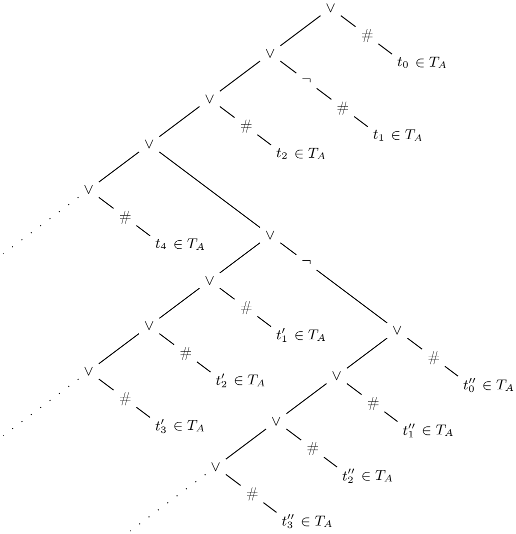
Trees represent formulas:

- countable disjunction
- negation
- $t \in T_A$ as atoms



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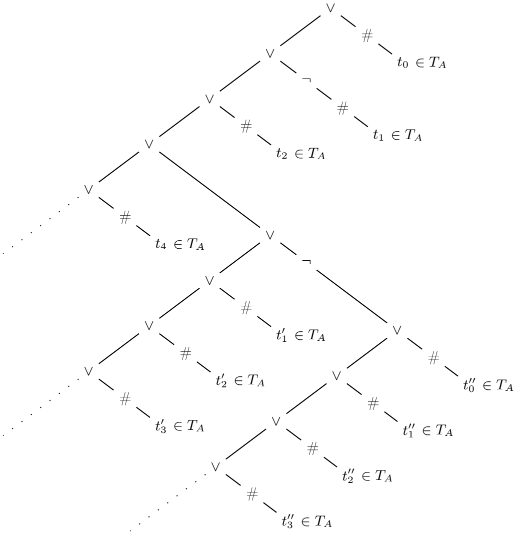

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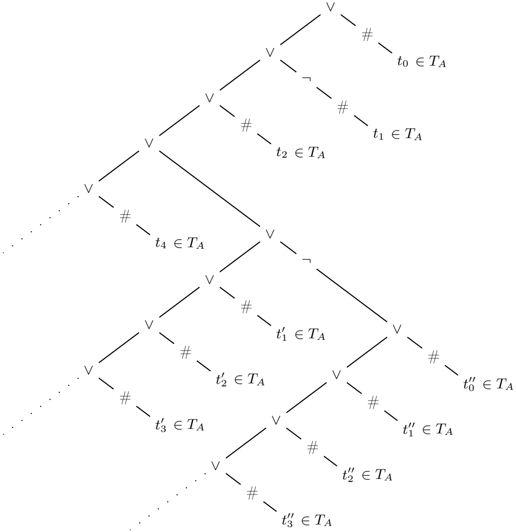

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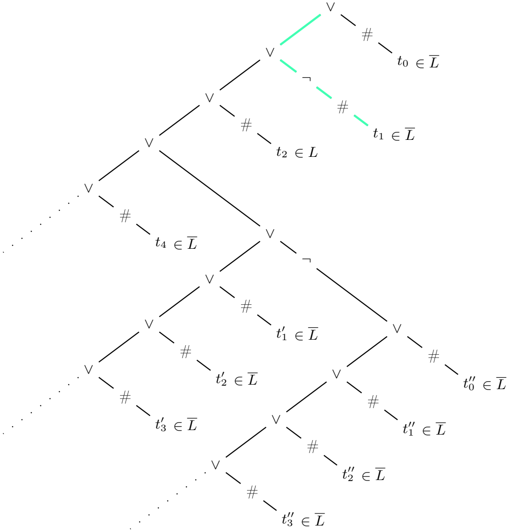
$\varsigma(L) \rightsquigarrow$ formulas evaluating to TRUE



Automata theoretic:

Theorem

L and $\overline{L}$ unambiguous $\implies$ $\varsigma(L)$ and $\overline{\varsigma(L)}$ unambiguous.

- $\varsigma(L)$ — rightmost witness of TRUE
- $\overline{\varsigma(L)}$ — rightmost witness of ill-foundedness

Properties of Sigma Operation

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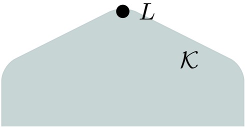
L hard for class $\mathcal{K}$ $\implies$ $\varsigma(L)$ hard for class $\sigma(\mathcal{K})$.



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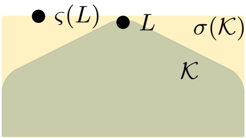

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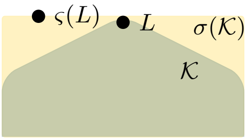
L stretchable $\implies \varsigma(L)$ stretchable.

Lemma (Arnold,Niwiński '07)

L stretchable $\implies \overline{L} \nleq_W L$.

Corollary

L stretchable $\implies \varsigma(L) \notin \sigma(\mathcal{K})$.



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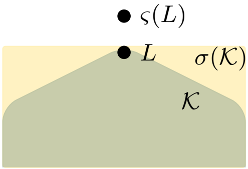

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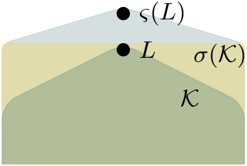

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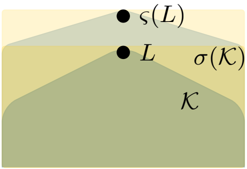

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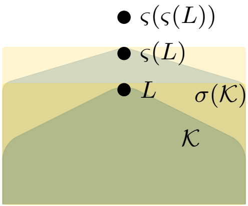

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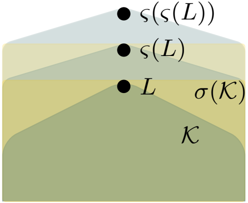

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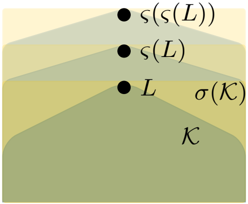

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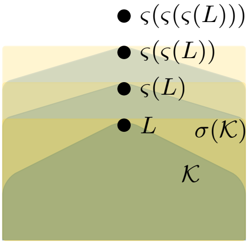

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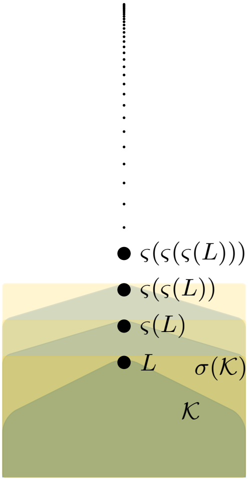

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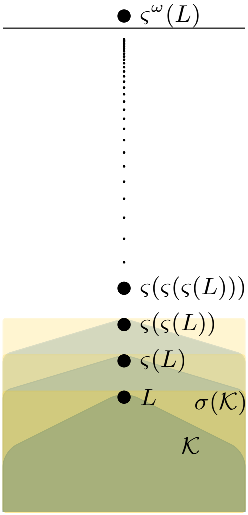

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- limit step:
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- alphabet extension:
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  - stretching
- Szczepan Hummel. *Unambiguous tree languages are topologically harder than deterministic ones.* GandALF 2012.
  - sets G and $\sigma(G)$ — the best lower topological complexity bounds for unambiguous languages known so far

$$\begin{array}{rclcl} G & \equiv_W & \overline{\varsigma(\emptyset)} & \in & \mathbf{\Sigma}^1_1\text{-complete (non-Borel)} \\ \sigma(G) & \equiv_W & \varsigma(\varsigma(\emptyset)) & \notin & \sigma(\mathbf{\Sigma}^1_1) \end{array}$$