Pourchet's theorem in action: decomposing univariate nonnegative polynomials as sums of five squares

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ABSTRACT

Pourchet proved in 1971 that every nonnegative univariate polynomial with rational coefficients is a sum of five or fewer squares. Nonetheless, there are no known algorithms for constructing such a decomposition. The sole purpose of the present paper is to present a set of algorithms that decompose a given nonnegative polynomial into a sum of six (five under some unproven conjecture or when allowing weights) squares of polynomials. Moreover, we prove that the binary complexity can be expressed polynomially in terms of classical operations of computer algebra and algorithmic number theory.

KEYWORDS

nonnegative univariate rational polynomial, rational polynomial, sums of squares decomposition, real algebraic geometry, norm equation

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1 INTRODUCTION

Let $\mathbb{Q}[x]$ and $\mathbb{R}[x]$ denote the sets of univariate polynomials with rational and real coefficients, respectively. Given a nonnegative polynomial $f \in \mathbb{Q}[x]$, we consider the problem of decomposing f as a sum of squares (SOS) of polynomials also lying in $\mathbb{Q}[x]$, possibly with rational positive weights.

This problem is not only of theoretical interest in the realm of real algebraic geometry but is also practically meaningful, for instance, to compute SOS-Lyapunov certificates to ensure the stability of a control system [37], certify polynomial approximations of transcendental functions evaluated within computer programs [7, 22], verify formally polynomial inequalities via proof assistants [1, 15, 21]. Since such proofs assistants have limited computational abilities, a typical approach is to rely on external tools providing SOS certificates of moderate bit size, so that further verification is

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© 2023 Association for Computing Machinery. ACM ISBN 978-x-xxxx-xxxx-x/YY/MM...\$15.00 https://doi.org/10.1145/nnnnnnnnnnnnnnnn not too time-consuming. Therefore, we are particularly motivated in designing algorithms that output SOS certificates of reasonable bit size, possibly with a bit complexity being polynomial in the input data.

Related works. It is well-known that every nonnegative univariate polynomial in $\mathbb{R}[x]$ can be decomposed as a sum of two squares. Very early research efforts have been focused on obtaining rational decompositions with the least number of needed squares. Landau proved in [19] that each nonnegative polynomial in $\mathbb{Q}[x]$ can be decomposed as a sum of at most eight polynomial squares in $\mathbb{Q}[x]$. This result has been improved by Pourchet in [34], where he proved that only five or fewer squares are needed. The proof of Pourchet's theorem heavily relies on the local-global principle, in particular the Hasse-Minkowski theorem, and at first glance, it is not straightforward to extract a constructive algorithm to output the desired SOS decomposition. For a presentation in English of Pourchet's theorem, we refer the reader to [36, Chapter 17]. Later, Schweighofer derived in [39] an algorithm to produce SOS decompositions of polynomials with coefficients lying in any subfield of \mathbb{R} . Here the number of output squares does not exceed the degree of the input polynomial. This recursive algorithm performs real root isolation and quadratic approximations of positive polynomials at each of the recursion steps, thus has an exponential bit complexity [25]. Another algorithm derived in [7, § 5.2] relies on approximating complex roots of perturbed positive polynomials. In contrast to Schweighofer's algorithm, the input must be rational, and three additional squares can appear in the resulting decompositions, but the bit complexity happens to be polynomial [25].

Alternatively, one can obtain an approximate rational SOS decomposition by checking the feasibility of a semidefinite program (SDP), namely by solving a convex problem involving linear equalities and linear matrix inequalities; see the seminal works by Parrilo [30] and Lasserre [20]. The modern development of floating-point SDP solvers ensures that this task can be efficiently done in practice, but a post-processing step is mandatory to obtain an exact decomposition, either based on rounding-projection schemes [31], or perturbation-compensation techniques [24]. One can also directly compute exact algebraic solutions to such SDP programs [16], but the related schemes have more limited scalability. Note that all such SDP-based frameworks can be generally applied to prove the existence of SOS decompositions in the multivariate case.

Such schemes based on rounding-projection or perturbation-compensation techniques have been extended to design and analyze algorithms producing positivity certificates for trigonometric polynomials [26],

sums of nonnegative circuits [29], and sums of arithmetic-geometric-exponentials [28]. Recently, some other generalizations have been studied; they include the case of univariate polynomials sharing common real roots with another univariate polynomial [17], or the case of multivariate polynomials whose gradient ideals are zero-dimensional and radical [27]. The two algorithms from [25] providing univariate rational SOS decompositions have been implemented in the RealCertify Maple library [23].

Contributions. To the best of our knowledge, no constructive algorithm implements Pourchet's theorem to decompose a given nonnegative univariate polynomial from $\mathbb{Q}[x]$ into a sum of five squares of polynomials in $\mathbb{Q}[x]$. We also investigate the bit complexity trade-off between decompositions involving a fixed number of squares (namely five) and the existing schemes analyzed in [25] where the decompositions involve a number of squares depending on the degree of the input. The central contribution of our work is to present and analyze an algorithm (Algorithm 8) to decompose a nonnegative univariate polynomial $f \in \mathbb{Q}[x]$ into a sum of six squares of polynomials, also with rational coefficients. When allowing positive rational weights, this algorithm can be used to obtain a weighted rational sum of five squares with rational coefficients. This main algorithm relies on other procedures related to the number of squares involved in the decomposition of the input.

- First, we focus in Section 2 on the case where f is a sum of two squares and design Algorithm 1 to implement the related decomposition. The main step of the algorithm consists of solving a norm equation involving the leading coefficient of f;
- Then, we handle in Section 3 the case when f is a sum of four squares and design Algorithm 5 to implement the related decomposition. This algorithm is based on decompositions of positive rational numbers as sums of four squares of rational numbers, Euler's identity, and another norm equation:
- Section 4 focuses on the reduction to the four square case. This reduction is performed in Algorithm 6 by examining the 2-adic valuations of the constant term and leading coefficient of *f* , and using a perturbation argument, similar to the one used in [7, § 5.2];
- Last but not least, let f be a nonnegative univariate rational polynomial of degree d with coefficients of maximal bitsize τ. We prove that Algorithm 8 computes a decomposition of f as a sum of 6 squares. The expected bit complexity of this computation can be expressed polynomially in terms of: (1) integer factorization, (2) factorization in Q[x], (3) computation of the unit group of a number field, (4) the size of a system of fundamental units of a number field, all applied on parameters of size polynomial in d, τ, and the minimal values of f and its reciprocal. We write expected as there are some Las Vegas subalgorithms in Section 3. The output will also have bitsize polynomial in the size of a system of fundamental units of a number field with the same parameters.

Complexity of ordinary operations in computer algebra and algorithmic number theory. The complexity analysis of our algorithms will be expressed in terms of the binary complexity of classical operations from computer algebra and algorithmic number theory. In doing so, we have to consider the height of any rational that appears as an input. The notion of height is defined as follows.

Definition 1. For $p, q \in \mathbb{Z}, q \neq 0$, we define $\operatorname{height}(p/q) = \max\left(\lg |p|, \lg |q|\right),$

where $\lg(\cdot) = \frac{\log(\cdot)}{\log(2)}$.

We will also use the following definition to estimate our complexities.

DEFINITION 2. For an integer k > 0 and a map $\phi : \mathbb{N}^k \to \mathbb{R}$, we write $\phi(x_1, \dots, x_k) = \operatorname{poly}(x_1, \dots, x_k)$ if there is a polynomial $P \in \mathbb{Q}[x_1, \dots, x_k]$ such that $\phi(x_1, \dots, x_k) = O(P(x_1, \dots, x_k))$.

We now present the complexity of the fundamental operations that we will use.

DEFINITION 3. We define IntFact(H) to be the binary cost of factoring an integer n of height $\leq H$.

Using classical algorithms, such as continued fractions or a general number field sieve, $\mathrm{IntFact}(H)$ is (sub)-exponential in H.

DEFINITION 4. We define PolyFact(d, H) to be the binary cost of factoring a polynomial in $\mathbb{Q}[x]$ of degree d whose coefficients are of heights $\leq H$.

Using classical algorithms (see, e.g., § 21 of [4]), PolyFact(d, H) is polynomial in d and H.

DEFINITION 5. We define UnitGroupComp(L, f) to be the binary cost of computing a presentation of the unit group of the number field $L = \mathbb{Q}[x]/(f)$ defined by the irreducible polynomial f. We then define UnitGroupComp(d, H) to be an upper bound on the binary cost of computing a presentation of the unit group of a number field L defined as a quotient $\mathbb{Q}[x]/(f)$ with an irreducible polynomial f of degree d with coefficients of heights $\leq H$.

This operation is a central and delicate task in algorithmic number theory. The state-of-the-art algorithms claim sub-exponential complexities under special assumptions or heuristics (see [2, 13, 14]). A polynomial quantum algorithm was developed in [3].

While special compact representations of fundamental units exist (see [41, 42]), bounding SizeFundUnits (L, f) or SizeFundUnits (d, H) is also a delicate question, depending on the discriminant and regulator of L. Taking representation with polynomials in $\mathbb{Q}[x]$, we can only assume that SizeFundUnits (d, H) is exponential in d and H.

Remark 7. As all these complexities are (sub)-exponential or polynomial with a high degree, we can safely assume that PolyFact(d, H), UnitGroupComp(d, H), SizeFundUnits(d, H) are superlinear in d and h, e.g., $PolyFact(d, H) + PolyFact(d', H) \leq PolyFact(d + d', H)$ for any positive d, d', H.

SOLVING SUMS OF TWO SOUARES

The following fact is well known. It is a special case of [36, Theorem 17.4]. We present it here for the sake of completeness as it provides an explicit, algorithmic method of deciding whether a given polynomial can be expressed as a sum of two squares.

Observation 8. A square-free polynomial f is a sum of two squares if and only if the following two conditions hold simultaneously:

- (1) lc(f) is a sum of two squares (in \mathbb{Q}),
- (2) -1 is a square in $\mathbb{Q}[x]/(p)$ for every irreducible factor p of f.

We are now ready to present an algorithm (see Algorithm 1) that decomposes a given polynomial into a sum of two squares.

PROPOSITION 9. Let $f \in \mathbb{Q}[x]$ be a polynomial which is a sum of two squares. Then Algorithm 1 outputs polynomials $a,b\in\mathbb{Q}[x]$ such that $a^2 + b^2 = f$.

Moreover, if the degree of f is d and the heights of its coefficients are bounded from above by H, some $H_1 = poly(d, H)$, the heights of the output are in $O(H_1)$ and the number of binary operations is *upper-bounded by* IntFact(H) + d PolyFact($2d, H_1$).

Algorithm 1 Computing a decomposition of a polynomial as a sum of two squares

Input: A polynomial $f \in \mathbb{Q}[x]$, which is a priori known to be a sum of two squares in $\mathbb{Q}[x]$.

Output: Polynomials $a, b \in \mathbb{Q}[x]$ such that $a^2 + b^2 = f$.

- 1: Construct the quadratic field extension $\mathbb{Q}(i)/\mathbb{Q}$.
- 2: Solve the norm equation

$$lc(f) = N_{\mathbb{Q}(i)/\mathbb{Q}}(x)$$

and denote a solution by $a + bi \in \mathbb{Q}(i)$.

3: Factor f into a product of monic irreducible polynomials

$$f = \operatorname{lc}(f) \cdot p_1^{e_1} \cdots p_k^{e_k}$$
.

- 4: **for** every factor p_i , such that the corresponding exponent e_i
- Factor p_i over $\mathbb{Q}(i)$ into a product $p_i = g_i \cdot h_i$ with $g_i, h_i \in$ $\mathbb{Q}(i)[x].$
- Set

$$a_j := \frac{1}{2} \cdot (g_j + h_j), \qquad b_j := \frac{1}{2i} \cdot (g_j - h_j).$$

Update *a* and *b* setting:

$$a := aa_j + bb_j$$
 and $b := ab_j - ba_j$

8: Update
$$a$$
 and b setting:
$$a := a \cdot \prod_{j \le k} p_j^{2 \lfloor \frac{e}{j}/2 \rfloor} \qquad \text{and} \qquad b := b \cdot \prod_{j \le k} p_j^{2 \lfloor \frac{e}{j}/2 \rfloor}.$$

9: **return** *a, b*.

Proof of correctness of Algorithm 1. Let $\xi = a + bi \in \mathbb{Q}(i)$ be the element constructed in line 2 of the algorithm. Then

$$a^2 + b^2 = N_{\mathbb{Q}(i)/\mathbb{Q}}(\xi) = \operatorname{lc}(f)$$

is a decomposition of the leading coefficient of f into a sum of two squares. By the previous observation, such decomposition exists. Hence the norm equation has a solution.

Assume now that p_i is a monic irreducible factor of f of odd multiplicity. The preceding observation asserts that −1 is a square in $\mathbb{Q}[x]/(p_i)$. In other words, the place of $\mathbb{Q}(x)$ associated to p_j splits in $\mathbb{Q}(i, x)$. Indeed, since -1 is a square in $\mathbb{Q}[x]/(p_i)$, we have $-1 = s^2 + qp_i$ for some $q, s \in \mathbb{Q}[x]$, or equivalently $qp_i = s^2 + 1 = s^2 + qp_i$ (s-i)(s+i). The decomposition of s+i into irreducible factors on $\mathbb{Q}(i)[x]$ does not involve any real factor, only the complex ones since s + i has no real roots. Therefore, the decomposition of p_i in the same ring involves only non-real complex factors (arising from s+i) together with their conjugates. This implies that p_i factors in $\mathbb{Q}(i)[x]$ into a product of two conjugate irreducible polynomials $p_i = g_i \cdot h_i$, where

$$g_i = a_i + b_i i$$
 and $h_i = a_i - b_i i$

for some polynomials $a_j, b_j \in \mathbb{Q}[x]$. It is now clear that $p_j = a_j^2 + b_j^2$

$$a_j = \frac{1}{2} \cdot (g_j + h_j), \qquad b_j = \frac{1}{2i} (g_j - h_j).$$

Suppose for a moment that f is square-free, i.e., all the exponents e_i are equal to 1. A simple induction combined with the following well-known formula for the product of two sums of squares

$$(A^2 + B^2) \cdot (a^2 + b^2) = (Aa + Bb)^2 + (Ab - Ba)^2$$

shows that after exceeding the loop in line 4, the polynomials a and b satisfy the condition $a^2 + b^2 = f$.

Finally, let f be arbitrary. We can write it as a product $f = gh^2$, where

$$g \coloneqq \operatorname{lc}(f) \cdot \prod_{\substack{j \le k \\ 2 \nmid e_j}} p_j$$
 and $h \coloneqq \prod_{\substack{j \le k \\ 2 \nmid e_j}} p_j^{\lfloor e_j/2 \rfloor}$.

Then g is square-free and decomposes into a sum of two squares $g = a^2 + b^2$ by the preceding part of the proof. It follows that f = $(ah)^2 + (bh)^2$, and this proves the correctness of the algorithm. \Box

Complexity analysis of Algorithm 1. Let f be of degree dand with coefficients of heights upper-bounded by H. To solve the norm equation in line 2, it is enough to: (1) factor lc(f), (2) solve the norm equation for each prime factor, (3) combine the solutions using the Brahmagupta–Fibonacci identity, i.e., $(\alpha^2 + \beta^2)(\gamma^2 + \delta^2) =$ $(\alpha \gamma - \beta \delta)^2 + (\alpha \delta + \beta \gamma)^2 = (\alpha \gamma + \beta \delta)^2 + (\alpha \delta - \beta \gamma)^2$. The first item is in IntFact(H). Thanks to [43, p. 128] and [38], the second item is polynomial in H. The recombination is also polynomial in H and the output solutions are of heights upper-bounded by H.

Line 3 is in PolyFact(n, H). Line 5 is done in PolyFact $(2d, H_1)$ for some H_1 polynomial in H and d, by [8, § 3.6.2]. After that step, there are only arithmetic operations.

All in all, for some H_1 polynomial in H and d, the total complexity of Algorithm 1 is in $IntFact(H) + d PolyFact(2d, H_1)$.

Remark 10. To solve the norm equation in line 2 of Algorithm 1 (as well as in line 2 of Algorithm 3 below), it is possible to rely also on the methods described in [9-12, 40].

3 SOLVING SUMS OF FOUR SQUARES

The goal of this section is to design and analyze a procedure to decompose a nonnegative element of $\mathbb{Q}[x]$ into a sum of four squares in $\mathbb{Q}[x]$, assuming that such a decomposition exists. This is achieved via Algorithm 3 and Algorithm 4 below and requires computing a decomposition of -1 into a sum of two squares in the number field $K:=\mathbb{Q}[x]/(f)$. This basically boils down to solving a norm equation $-1=N_{L/K}x$, where $L=K(\sqrt{-1})$. To this end, one can use any of the methods described in [9–12, 40]. Nevertheless, to make it possible to perform the complexity analysis of Algorithm 3, we provide in Algorithm 2 a stripped-down version of Simon's algorithm for solving a norm equation (cf. [40]) which suffices to our purpose.

Recall that a field K is called *non-real* if -1 is a sum of squares in K. The minimal number of summands needed to express -1 as a sum of squares is called the *level* of K and denoted s(K).

Algorithm 2 Decomposing -1 as a sum of two squares over the number field $K := \mathbb{Q}[x]/(f)$.

Input: Number field K of level $s(K) \le 2$.

Output: $a, b \in K$ such that $a^2 + b^2 = -1$.

- 1: **if** there is $c \in K$ such that $c^2 = -1$ **then**
- 2: **return** a = c, b = 0.
- 3: Construct the quadratic field extension $L := K(\sqrt{-1})$.
- 4: Construct the unit groups U_K and U_L of K and L, respectively.
- 5: Construct the quotient groups $G_K := U_K/U_K^2$ and $G_L := U_L/U_L^2$. Let $\mathcal{K} := \{\kappa_1, \dots, \kappa_k\} \subset U_K$ and $\mathcal{L} := \{\lambda_1, \dots, \lambda_l\} \subset U_L$ be sets of elements forming bases of G_K and G_L , treated as \mathbb{F}_2 -vector spaces.
- 6: Let $V = (v_1, \dots, v_k)$ be the coordinates with respect to \mathcal{K} of the coset $-1 \cdot U_K^2$.
- 7: For every $i \leq l$ denote by (m_{i1}, \ldots, m_{ik}) the coordinates of $N_{L/K}(\lambda_i) \cdot U_K^2$ with respect to \mathcal{K} .
- 8: Solve the system of \mathbb{F}_2 -liner equations $M^T \cdot X = V$, where $M = (m_{ij})$. Denote a solution by $(\varepsilon_1, \dots, \varepsilon_l)$.
- 9: Set $\lambda:=\lambda_1^{\varepsilon_1}\cdots\lambda_l^{\varepsilon_l}$ and let $c\in K$ be such that $c^2=-N_{L/K}(\lambda)$.
- 10: Write $^{\lambda}/_{c}$ as $^{\lambda}/_{c} = a + b\sqrt{-1}$.
- 11: **return** *a, b*.

PROPOSITION 11. Let $K = \mathbb{Q}[x]/(f)$ be a number field of level $s(K) \leq 2$ and specified by its generating polynomial f. Then Algorithm 2 outputs $a, b \in K$ such that $a^2 + b^2 = -1$.

Moreover, if the degree of f is d and the heights of its coefficients are bounded from above by H, then a, b have binary-size polynomial in SizeFundUnits(2d, $2Hd + 8d(1 + \lg(d))$) and the number of binary operations is polynomial in UnitGroupComp(2d, $2Hd + 8d(1 + \lg(d))$) and SizeFundUnits(2d, $2Hd + 8d(1 + \lg(d))$).

Proof of correctness of Algorithm 2. If the algorithm terminates in line 1, then the correctness of its output is trivial. Hence, without loss of generality, we may assume that $-1 \notin K^{\times 2}$. In particular, this means that $L = K(\sqrt{-1})$ is a proper quadratic extension of K. By assumption -1 is a sum of two squares in K, say -1 =

 $a^2 + b^2$ with $a, b \in K^{\times}$. Then $-1 = N_{L/K}(\mu)$, where $\mu = a + b\sqrt{-1}$. Observe that μ must be a unit in L since

$$N_{L/\mathbb{O}}(\mu) = (N_{K/\mathbb{O}} \circ N_{L/K})(\mu) = N_{K/\mathbb{O}}(-1) = (-1)^{(K:\mathbb{Q})}.$$

It follows that we can write the element μ as

$$\mu = \lambda_1^{\varepsilon_1} \cdots \lambda_l^{\varepsilon_l} \cdot \nu^2,$$

where $v \in U_L$ and $\varepsilon_1, \ldots, \varepsilon_l \in \{0, 1\}$ are the coordinates of the coset $\mu \cdot U_L^2$ with respect to the basis $\mathcal{L} = \{\lambda_1, \ldots, \lambda_l\}$ of $G_l = U_L/U_L^2$. Computing the norms of both sides of the above equation we obtain

$$-1 = N_{L/K}(\mu) = N_{L/K}(\lambda_1)^{\varepsilon_1} \cdots N_{L/K}(\lambda_l)^{\varepsilon_l} \cdot N_{L/K}(\nu)^2.$$

Now, let m_{ij} and v_j with $i \le l$, $j \le k$ be as in steps (6–7) of the algorithm. In the quotient group $G_K = U_K/U_K^2$ we have

$$\left(\kappa_1^{v_1}\cdots\kappa_k^{v_k}\right)\cdot U_K^2=\prod_{i\leq l}\prod_{j\leq k}\kappa_j^{m_{ij}\varepsilon_i}\cdot U_K^2.$$

The cosets of $\kappa_1, \ldots, \kappa_k$ form a basis of G_K . In particular they are linearly independent. Therefore for every $j \le k$ we have

$$v_j = m_{1j}\varepsilon_1 + \cdots + m_{lj}\varepsilon_l.$$

This way we have proved that the system of \mathbb{F}_2 -linear equations considered in line 8 has a solution.

Conversely, let $\varepsilon_1,\ldots,\varepsilon_l\in\{0,1\}$ form a solution to the system $M^T\cdot X=V$ and let $\lambda=\lambda_1^{\varepsilon_1}\cdots\lambda_l^{\varepsilon_l}$. It follows from the preceding part of the proof that the cosets $-1\cdot U_K^2$ and $N_{L/K}(\lambda)\cdot U_K^2$ coincide. Therefore, there is a unit $c\in U_K$ such that $-c^2=N_{L/K}(\lambda)$. Consequently $-1=N_{L/K}(\lambda/c)$ and so if $\lambda/c=a+b\sqrt{-1}$, we obtain the sought decomposition $-1=a^2+b^2$.

Proposition 12. Let $f \in \mathbb{Q}[x]$ be an irreducible polynomial of degree d with coefficients of heights upper-bounded by H, and known to be a sum of 3 or 4 squares in $\mathbb{Q}[x]$. Then Algorithm 3 computes polynomials h and g_1, \ldots, g_4 in $\mathbb{Q}[x]$, such that $\deg h \leq \deg f - 2$ and $fh = g_1^2 + \cdots + g_4^2$.

The binary complexity is polynomial in UnitGroupComp(2d, $2Hd+8d(1+\lg(d))$) and SizeFundUnits(2d, $2Hd+8d(1+\lg(d))$). The outputs are of binary size polynomial in SizeFundUnits(2d, $2Hd+8d(1+\lg(d))$).

Algorithm 3 Initial solution: modular sum of squares

Input: An irreducible polynomial $f \in \mathbb{Q}[x]$, which is a priori known to be a sum of 3 or 4 squares.

Output: Polynomials h and g_1, \ldots, g_4 in $\mathbb{Q}[x]$, such that deg $h \leq$ $\deg f - 2$ and $fh = g_1^2 + \dots + g_4^2$.

1: Construct the number fields:

$$K := \mathbb{Q}[x]/(f)$$
 and $L := K(i)$.

2: Solve the norm equation

$$-1 = N_{L/K}(x)$$

and denote the solution by $\xi = \overline{g}_1 + \overline{g}_2 i$, where $g_1, g_2 \in \mathbb{Q}[x]$ are polynomials of degree strictly less than deg f and \overline{g}_i denotes the image of g_i under the canonical epimorphism $\mathbb{Q}[x] \twoheadrightarrow K$.

- 3: Set $g_3 := 1$, $g_4 := 0$ and let $h := (g_1^2 + \dots + g_4^2)/f$.
- 4: **return** h, g_1, g_2, g_3, g_4 .

Proof of correctness of Algorithm 3. By assumption f is a sum of 4 or fewer squares of polynomials, say

$$f = p_1^2 + p_2^2 + p_3^2 + p_4^2.$$

Not all p_i are zeros. Assume that $p_1 \neq 0$. Then in K we have

$$-1 = \left(\frac{\overline{p}_2}{\overline{p}_1}\right)^2 + \left(\frac{\overline{p}_3}{\overline{p}_1}\right)^2 + \left(\frac{\overline{p}_4}{\overline{p}_1}\right)^2.$$

Here again, \overline{p}_i denotes the image of p_i in K. It follows that K is a non-real field, and its level does not exceed 3. But it is well known (see, e.g., [18, Theorem XI.2.2]) that the level of a non-real field is always a power of 2. Therefore, -1 is a sum of 2 squares in K. Thus, there are polynomials $g_1, g_2 \in \mathbb{Q}[x]$ such that $\deg g_i < \deg f$ and

$$\overline{g}_1^2 + \overline{g}_2^2 = N_{L/K}(\overline{g}_1 + \overline{g}_2 i) = -1.$$

It is now clear that $g_1^2+g_2^2+1$ is divisible by f. Denoting the quotient by h we obtain $fh = g_1^2 + g_2^2 + 1$. Moreover, we have $\deg g \le \deg f - 1$ and so deg h cannot exceed deg f - 2, as expected.

Complexity analysis of Algorithm 3. By the analysis of Algorithm 2, if f is of degree d and its coefficients of heights $\leq H$, then the binary complexity of Algorithm 3 is polynomial in UnitGroupComp(2d, 2Hd+ $8d(1+\lg(d))$) and SizeFundUnits $(2d, 2Hd+8d(1+\lg(d)))$. The outputs are also of binary size polynomial in SizeFundUnits(2d, 2Hd+ $8d(1 + \lg(d))).$

We are now ready to present a method (see Algorithm 4) to decompose an irreducible rational polynomial as a sum of 3 or 4 squares in $\mathbb{Q}[x]$, provided that such a decomposition exists. We need to fix which of the different variants of Euler's identity will be used for multiplying sums of four squares. The following one ensures that the succeeding algorithm will output polynomials.

EULER'S IDENTITY.

$$(A^{2} + B^{2} + C^{2} + D^{2})(a^{2} + b^{2} + c^{2} + d^{2}) =$$

$$= (Aa + Bb + Cc + Dd)^{2} + (-Ab + Ba - Cd + Dc)^{2} + (-Ac + Bd + Ca - Db)^{2} + (-Ad - Bc + Cb + Da)^{2}.$$

Algorithm 4 Decomposition of a monic irreducible polynomial into a sum of 3 or 4 squares

Input: A monic irreducible polynomial $f \in \mathbb{Q}[x]$, which is a priori known to be a sum of 3 or 4 squares.

Output: Polynomials $f_1, \ldots, f_4 \in \mathbb{Q}[x]$ such that $f_1^2 + \cdots + f_4^2 = f$.

- 1: Execute Algorithm 3 to construct polynomials h, g_1, \ldots, g_4 , such that $fh = g_1^2 + \dots + g_4^2$ and $\deg h \le \deg f - 2$.
- **while** $\deg h > 0$ **do**
- Compute the remainders $r_i := (g_i \mod h)$ of g_1, \ldots, g_4 mod-
- Using Euler's identity, express the product

$$\left(g_1^2+g_2^2+g_3^2+g_4^2\right)\left(r_1^2+r_2^2+r_3^2+r_4^2\right)$$

as a sum of four squares. Denote the result by g'_1, \ldots, g'_4 .

- Update $g_1, ..., g_4$ setting $g_j := g_j/h$ for $j \in \{1, 2, 3, 4\}$.
- Update h setting $h := (g_1^2 + \dots + g_4^2)/f$.
- 7: Decompose h into a sum of fours squares of rational numbers $h = a_1^2 + \cdots + a_4^2$, where $a_1, \ldots, a_4 \in \mathbb{Q}$.
- 8: Use Euler's identity to express the product

$$\left(g_1^2 + \dots + g_4^2\right) \cdot \left(\left(\frac{a_1}{h}\right)^2 + \dots + \left(\frac{a_4}{h}\right)^2\right)$$

as a sum of four squares. Store the result in f_1, \ldots, f_4 .

9: **return** $f_1, ..., f_4$.

PROPOSITION 13. Let $f \in \mathbb{Q}[x]$ be an irreducible polynomial of degree d with coefficients of heights upper-bounded by H, and known to be a sum of 3 or 4 squares in $\mathbb{Q}[x]$. Then Algorithm 4 computes polynomials g_1, \ldots, g_4 in $\mathbb{Q}[x]$, such that $f = g_1^2 + \cdots + g_4^2$.

 $The \ binary\ complexity\ is\ polynomial\ in\ Unit Group Comp(2d, 2Hd+1)$ $8d(1+\lg(d)))$ and SizeFundUnits $(2d, 2Hd+8d(1+\lg(d)))$. The output is of binary size polynomial in SizeFundUnits (2d, 2Hd + 8d(1 + $\lg(d))$.

PROOF OF CORRECTNESS OF ALGORITHM 4. We shall denote the values of the variables h, g_j , g'_j and r_j after the k-th iteration of the main loop by h_k , g_{kj} , g'_{kj} and r_{kj} , respectively. In particular, the initial values computed in line 1 will be denoted by h_0 and

By the means of line 6, for every $k \ge 0$, we have $fh_k = g_{k1}^2 +$ $\cdots + g_{k_A}^2$. We shall prove that all h_k 's are polynomials and their degrees form a strictly decreasing sequence and likewise that all g_{k1}, \ldots, g_{k4} are polynomials, too. Of course, h_0 and g_{01}, \ldots, g_{04} are polynomials. Let us examine the product of two sums of squares that appears in line 4 of the algorithm:

$$\left((g'_{k+1,1})^2 + \dots + (g'_{k+1,4})^2 \right) = \left(g_1^2 + g_2^2 + g_3^2 + g_4^2 \right) \left(r_1^2 + r_2^2 + r_3^2 + r_4^2 \right).$$

Here for every $j \le 4$ we have $r_{kj} = g_{kj} - q_{kj}h_k$ for some polynomial $q_{kj} \in \mathbb{Q}[x]$ and $\deg r_{kj} < \deg h_k$. Euler's identity yields

$$\begin{split} g'_{k+1,j} &= g_{k1}r_{k1} + g_{k2}r_{k2} + g_{k3}r_{k3} + g_{k4}r_{k4} \\ &= g_{k1} \cdot (g_{k1} - q_{k1}h_k) + g_{k2} \cdot (g_{k2} - q_{k2}h_k) \\ &\quad + g_{k3} \cdot (g_{k3} - q_{k3}h_k) + g_{k3} \cdot (g_{k3} - q_{k3}h_k) \\ &= \left(g_{k1}^2 + g_{k2}^2 + g_{k3}^2 + g_{k4}^2\right) \\ &\quad - \left(g_{k1}q_{k1} + g_{k2}q_{k2} + g_{k3}q_{k3} + g_{k4}q_{k4}\right) \cdot h_k \\ &= \left(f - g_{k1}q_{k1} - g_{k2}q_{k2} - g_{k3}q_{k3} - g_{k4}q_{k4}\right) \cdot h_k. \end{split}$$

This shows that $g'_{k+1,1}$ is divisible by h_k , hence $g_{k+1,1} = g'_{k+1,1}/h_k$ is indeed a polynomial. Likewise

$$\begin{split} g_{k+1,2}' &= -g_{k1}r_{k2} + g_{k2}r_{k1} - g_{k3}r_{k4} + g_{k4}r_{k3} \\ &= -g_{k1} \cdot (g_{k2} - q_{k2}h_k) + g_{k2} \cdot (g_{k1} - q_{k1}h_k) \\ &- g_{k3} \cdot (g_{k4} - q_{k4}h_k) + g_{k4} \cdot (g_{k3} - q_{k3}h_k) \\ &= (g_{k1}q_{k2} - g_{k2}q_{k1} + g_{k3}q_{k4} - g_{k4}q_{k3}) \cdot h_k. \end{split}$$

Therefore, $g_{k+1,2} = g_{k+1,2}/h_k$ is a polynomial, too. Analogous arguments apply to $g_{k+1,3}$ and $g_{k+1,4}$, as well.

Now, assume that we have proved that h_k is a polynomial for some $k \ge 0$. We have

$$fh_k = g_{k1}^2 + \dots + g_{k4}^2 \tag{1}$$

and $r_{kj} = (g_{kj} \mod h_k)$. We deduce that

$$r_{k1}^2 + \dots + r_{k4}^2 \equiv 0 \pmod{h_k}.$$

Hence, there exists some h'_{k} such that

$$h_k h_k' = r_{k1}^2 + \dots + r_{k4}^2.$$
 (2)

Now, $\deg r_{kj} < \deg h_k$ for all $j \le 4$ and so $\deg h_k' \le \deg h_k - 2$. Combining (1) with (2) we obtain

$$f \cdot h_k^2 \cdot h_k' = (g_{k1}^2 + \dots + g_{k4}^2)(r_{k1}^2 + \dots + t_{k4}^2) = (g_{k+1,1}')^2 + \dots + (g_{k+1,k}')^2.$$

This yields $fh'_k = g^2_{k+1,1} + \cdots + g^2_{k+1,4}$. It follows that

$$h'_{k} = \frac{g_{k+1,1}^{2} + \dots + g_{k+1,4}^{2}}{f} = h_{k+1}.$$

It shows that h_{k+1} is a polynomial of degree $\leq \deg h_k - 2$, proving our claim. Consequently, after finitely many steps the degree of h_k will eventually drop to zero and so the algorithm will terminate.

Now, let h_k be such that deg $h_k = 0$. We know that

$$fh_k = g_{k1}^2 + \dots + g_{k4}^2.$$

By assumption, f itself is a sum of squares. Therefore h_k must be a positive rational number, hence a sum of four (or fewer) squares. Write $h_k = a_1^2 + \cdots + a_4^2$ for some rational numbers $a_1, \ldots, a_4 \in \mathbb{Q}$. It follows that the product in line 8 of the algorithm is a sum of four squares that equals f. This proves the correctness of the presented algorithm.

COMPLEXITY ANALYSIS OF ALGORITHM 4. Thanks to the analysis of Algorithm 3, if the polynomial f is of degree d and the heights of its coefficients are bounded from above by H, then line 1 has a binary complexity polynomial in UnitGroupComp(2d, 2Hd + 8d(1 + $\lg(d)$) and SizeFundUnits $(2d, 2Hd + 8d(1 + \lg(d)))$. The outputs are of binary size polynomial in SizeFundUnits $(2d, 2Hd+8d(1+\lg(d)))$. The **while** will be executed less than *d*-times and thus involves only a polynomial in d amount of arithmetic operations.

Line 7 has an expected (Las Vegas) binary complexity polynomial in the height of h, by [33, 35]. All in all, the algorithm has an expected binary complexity polynomial in UnitGroupComp(2d, 2Hd+ $8d(1+\lg(d))$) and SizeFundUnits $(2d, 2Hd + 8d(1+\lg(d)))$ and the outputs are still of binary size polynomial in SizeFundUnits (2d, 2Hd+ $8d(1 + \lg(d))$.

A decomposition of an arbitrary polynomial into a sum of four squares (provided that such decomposition exists) is now straightforward. For the sake of completeness, we present an explicit algorithm below.

Algorithm 5 Decomposition into a sum of 3 or 4 squares

Input: A polynomial $f \in \mathbb{Q}[x]$, which is a priori known to be a sum of 3 or 4 squares.

Output: Polynomials $f_1, \ldots, f_4 \in \mathbb{Q}[x]$ such that $f_1^2 + \cdots + f_4^2 = f$.

- 1: Using square-free decomposition, construct $q, h \in \mathbb{Q}[x]$ such that $f = lc(f) \cdot g \cdot h^2$, where g is monic and square-free.
- Decompose lc(f) into a sum of four squares of rational numbers and initialize f_1, \ldots, f_4 with the result.
- 3: Factor g into a product of monic irreducible polynomials: g = $p_1 \cdots p_k$.
- 4: for each p_i do
- Execute Algorithm 4 to express p_i as a sum of 4 squares

$$p_j = g_1^2 + \dots + g_4^2.$$

- Use Euler's identity to express the product $(f_1^2 + \cdots + f_4^2)(g_1^2 + \cdots + f_4^2)$ $\cdots + g_4^2$) as a sum of 4 squares. Store the result again in f_1, \ldots, f_4 .
 7: **return** f_1, \ldots, f_4 .

Proposition 14. Let $f \in \mathbb{Q}[x]$ be a polynomial of degree d with coefficients of heights upper-bounded by H, and known to be a sum of 3 or 4 squares. Then Algorithm 5 computes polynomials g_1, \ldots, g_4 in $\mathbb{Q}[x]$, such that $f = g_1^2 + \cdots + g_4^2$.

The expected binary complexity is polynomial in UnitGroupComp(2d, 2Hd+ $8d(1 + \lg(d))$ and SizeFundUnits $(2d, 2Hd + 8d(1 + \lg(d)))$. The output is of binary size polynomial in SizeFundUnits(2d, 2Hd+ $8d(1 + \lg(d))).$

In view of Algorithm 4, the correctness of Algorithm 5 is obvious. Hence, we restrict ourselves to the complexity analysis.

COMPLEXITY ANALYSIS OF ALGORITHM 5. The algorithm relies on polynomial factorization, application of Algorithm 4 for some factors and recombination using Euler's identity.

Thanks to our super-linearity assumption, we can upper-bound the sum of the costs of, e.g., the UnitGroupComp's of the factors by UnitGroupComp $(2d, 2Hd + 8d(1 + \lg(d)))$. All in all, the algorithm has an expected binary complexity polynomial in PolyFact(d, H), UnitGroupComp $(2d, 2Hd+8d(1+\lg(d)))$ and SizeFundUnits(2d, 2Hd+ $8d(1 + \lg(d))$ and the outputs are still of binary size polynomial in SizeFundUnits $(2d, 2Hd + 8d(1 + \lg(d)))$.

Remark 15. In order to decompose a natural number into a sum of four squares (see line 7 of Algorithm 4 and line 2 of Algorithm 5) one can use, e.g., [5, 32, 33, 35]. If one needs a deterministic algorithm, one can (1) Factor the entry z over \mathbb{Z} , (2) For each prime factor p, compute a decomposition of p as a sum of four squares using [5, §4.3] and then [35, §3] (3) Recombine using Euler's identity. The complexity is then polynomial in IntFact(height(z)), and height(z).

4 REDUCTION TO FOUR SQUARES

In this Section we introduce Algorithms 6 and 7 that reduce our main problem of decomposing a positive polynomial into a sum of squares to the problem of decomposing a polynomial into a sum of four squares, which we already tackled in the previous section. Given $f = c_0 + c_1x + \cdots + c_dx^d \in \mathbb{Q}[x]$, we define the associated reciprocal polynomial $f_* := c_d + c_{d-1}x + \cdots + c_0x^d \in \mathbb{Q}[x]$. First we focus on the case where the 2-adic valuation of c_d is odd, in which case we rely on Algorithm 6, and next we handle the general case via Algorithm 7.

Algorithm 6 Reduction to a sum of 4 squares: odd valuation case

Input: A positive square-free polynomial $f = c_0 + c_1x + \cdots + c_dx^d \in \mathbb{Q}[x]$. The 2-adic valuations of the coefficients of f are $k_j := \operatorname{ord}_2 c_j$ for $0 \le j \le d$. Ensure k_d is odd. It is assumed that f is not a sum of 4 squares.

Output: A polynomial $h \in \mathbb{Q}[x]$ such that $f - h^2$ is a sum of 4 (or fewer) squares.

1: Find a positive number ε such that

$$\varepsilon < \inf\{f(x) \mid x \in \mathbb{R}\}.$$

- 2: Set $l_1 := \left[-\frac{1}{2} \cdot \lg \varepsilon \right]$.
- 3: Set $l_2 := \lceil -k_0/2 \rceil + 1$.
- 4: Set

$$l_3 := \left[\max \left\{ \frac{jk_d - dk_j}{2d - 2j} \mid 0 < j < d \right\} \right].$$

- 5: Initialize $l := \max\{l_1, l_2, l_3\}$.
- 6: **while** $gcd(d, 2l + k_d) \neq 1$ **do**
- 7: l := l + 1.
- 8: **return** $h := 2^{-l}$

PROPOSITION 16. Let f be a positive square-free polynomial in $\mathbb{Q}[x]$ of degree d with coefficients of heights $\leq H$. Let us assume that f is not a sum of 4 (or fewer) squares. Then Algorithm 7, relying on Algorithm 6, computes polynomials $g_1, g_2 \in \mathbb{Q}[x]$ such that $f - g_1^2 - g_2^2$ is a sum of 4 (or fewer) squares.

The binary complexity and the bitsize of the outputs are polynomials in d, H, $lg(\inf f)$ and $lg(\inf f_*)$.

PROOF OF CORRECTNESS OF ALGORITHM 6. We have assumed that f is not a sum of four squares. Let us first observe that the loop in line 6 terminates. Indeed, k_d is odd by assumption so by Dirichlet's prime number theorem the arithmetic progression $(k_d + 2l \mid l \geq \max\{l_1, l_2, l_3\})$ contains infinitely many prime numbers. In particular it must contain a number relatively prime to the degree d. Moreover, $(k_d + 2l \mod d)$ has period $\leq d$ so the Loop in Line 6 is executed at most d times and the Algorithm terminates.

Algorithm 7 Reduction to a sum of 4 squares: general case

Input: A positive square-free polynomial $f = c_0 + c_1 x + \cdots + c_d x^d \in \mathbb{Q}[x]$, that is a priori known not to be a sum of 4 squares.

Output: Polynomials $g_1, g_2 \in \mathbb{Q}[x]$ such that $f - g_1^2 - g_2^2$ is a sum of 4 (or fewer) squares.

- 1: Denote the 2-adic valuations of the constant term and the leading coefficient of f by $k_0 := \operatorname{ord}_2 c_0$ and $k_d := \operatorname{ord}_2 c_d$, respectively.
- 2: **if** k_d is odd **then**
- 3: Execute Algorithm 6. Denote its output by h.
- 4: **return** $q_1 := h, q_2 := 0.$
- 5: **else if** k_0 is odd **then**
- 6: Set $f_* := c_d + c_{d-1}x + \dots + c_0x^d$.
- 7: Execute Algorithm 6 for f_* and denote its output by h.
- 8: **return** $q_1 := x^{d/2} \cdot h(1/x), q_2 := 0.$
- 9: else
- 10: Execute Algorithm 6 for 2f and denote its output by h.
- 11: **return** $g_1 := h/2, g_2 := h/2.$

Let h be the polynomial constructed by the algorithm. We claim that $f - h^2$ is irreducible in $\mathbb{Q}_2[x]$. We have $l \ge l_2 > -\frac{k_0}{2}$, hence the 2-adic valuation of the constant term of $f - h^2$ is

$$\operatorname{ord}_2(c_0 - 2^{-2l}) = \min\{k_0, -2l\} = -2l.$$

All the other coefficients of $f-h^2$ coincide with the corresponding coefficients of f. The condition $l \geq l_3$ implies that for every $j \in \{1,\ldots,d-1\}$ the point (j,k_j) lies on or above the line segment with the endpoints (0,-2l) and (d,k_d) . Thus, the Newton polygon of $f-h^2$ consists of just this single line segment, whose slope is $({}^{2l+k_d})/_d$. Now, d and $2l+k_d$ are relatively prime, hence $f-h^2$ is irreducible in $\mathbb{Q}_2[x]$ by the well known property of the Newton polygon (see, e.g., [6, Lemma 3.5]). This proves our claim.

Finally, the polynomial f being positive and square-free cannot have any real roots. Hence it is separated from zero, which means that one can find $\varepsilon > 0$ as in line 1. Now, the condition $l \ge l_1 \ge \lceil -1/2 \cdot \lg \varepsilon \rceil$ implies that $f - h^2 = f - 2^{-2l}$ is positive, too. Consequently, [36, Theorem 17.2] asserts that $f - h^2$, being positive and irreducible in $\mathbb{Q}_2[x]$, is a sum of four (or fewer) squares. \square

PROOF OF CORRECTNESS OF ALGORITHM 7. First, observe that f, being positive, cannot have any real roots. In particular, its constant term c_0 must be nonzero. Now, if k_d is odd then the correctness of line 4 follows from the already proven correctness of Algorithm 6. Now, suppose that it is the constant term of f that has an odd 2-adic valuation. The reciprocal polynomial f_* constructed in line 6 satisfies the identity

$$f_*(x) = x^d \cdot f(1/x).$$

Thus, it is positive hence a sum of squares, and its leading coefficient is c_0 , which has an odd 2-adic valuation. Consequently, using Algorithm 6 we can find $h \in \mathbb{Q}[x]$ such that $g := f_* - h^2$ is a sum of 4 squares. Say $g = a_1^2 + \cdots + a_4^2$. Let g_1 be given as in line 8. We

have

$$f - g_1^2 = x^d \cdot f_* \left(\frac{1}{x}\right) - \left(x^{\frac{d}{2}} \cdot h\left(\frac{1}{x}\right)\right)^2 = x^d \cdot g\left(\frac{1}{x}\right)$$
$$= \left(x^{\frac{d}{2}} \cdot a_1\left(\frac{1}{x}\right)\right)^2 + \dots + \left(x^{\frac{d}{2}} \cdot a_4\left(\frac{1}{x}\right)\right)^2$$

is a sum of four squares, as well. This proves the correctness of line 8. The correctness of the last case, when both k_0 and k_d are even, is trivial.

Complexity analysis of Algorithm 6 and Algorithm 7. For Algorithm 6, the desired ε can be computed by checking whether $f-2^{2^s}$ has any real roots, for increasing s. This can be done using classical methods like Descartes' rule of sign or the computation of signatures of special Hankel matrices defined by the coefficients of $f-2^{2^s}$. This is polynomial in d, the heights of the c_i 's and $\lg(\inf f)$. The same is true for the remaining operations. The output h has size polynomial in these quantities. Then, it is clear that the complexity of Algorithm 7 is completely given by that of Algorithm 6.

Remark 17. Algorithm 6 always outputs a constant polynomial, hence in line 8 above we actually have $q_1 = x^{d/2} \cdot h(1/x) = x^{d/2} \cdot h$. We use the "baroque" notation h(1/x) to allow one to substitute Algorithm 6 with any other algorithm having the same input and output specification.

FINAL ALGORITHM: SUMS OF SIX **SOUARES**

We are now ready to present our main algorithm that decomposes any nonnegative univariate polynomial in $\mathbb{Q}[x]$ into a sum of six squares of polynomials in $\mathbb{Q}[x]$. Its correctness is clear thanks to the correctness of the previous algorithms.

Theorem 18. If $f \in \mathbb{Q}[x]$ is a nonnegative univariate polynomial of degree d with coefficients of heights $\leq H$, then Algorithm 8 computes a decomposition of f as a sum of at most 6 squares. There exists some $H_1 = \text{poly}(d, H, \inf(f), \inf(f_*))$ such that the expected binary complexity is polynomial in IntFact(H), $PolyFact(d, H_1)$, UnitGroupComp $(2d, H_1)$, and SizeFundUnits $(2d, H_1)$. The output size is polynomial in SizeFundUnits $(2d, H_1)$.

COMPLEXITY ANALYSIS OF ALGORITHM 8. First of all, we should remark that checking whether whether f is a sum of 2 or 4 squares (by Observation 8 or [36, Theorem 17.2] respectively) is at worse as costly as computing such a decomposition. Hence the total complexity will be given by the cost of a decomposition into a sum of 2 squares, a sum of 4 squares and a polynomial factorization.

If f is of degree d with coefficients of heights $\leq H$, then there is some $H_1 = \text{poly}(d, H, \inf(f), \inf(f_*))$, given by Algorithm 1, Algorithm 7 and the $u \mapsto 2ud + 8d(1 + \lg(d))$ mapping such that the total expected binary cost is polynomial in IntFact(H), $PolyFact(d, H_1)$, UnitGroupComp $(2d, H_1)$ and SizeFundUnits $(2d, H_1)$. The output size is polynomial in SizeFundUnits($2d, H_1$).

Remark 19. In Algorithm 7, either $g_2 = 0$ or $g_1 = g_2$. It means that for some $a_5 = 1$ or 2, the output of Algorithm 8 is of the form

$$f = f_1^2 + f_2^2 + f_3^2 + f_4^2 + a_5 f_5^2$$
.

Algorithm 8 Decomposition of a nonnegative univariate rational polynomial into a sum of 6 squares

Input: A nonnegative polynomial $f \in \mathbb{Q}[x]$.

Output: Polynomials $f_1, \ldots, f_6 \in \mathbb{Q}[x]$ such that $f_1^2 + \cdots + f_6^2 = f$.

- 1: **if** f is a square **then**
- **return** $f_1 := \sqrt{f}, f_2 := \cdots f_6 := 0.$
- 3: **if** f is a sum of 2 squares {Use Observation 8 to check it} **then**
- Execute Algorithm 1 to obtain $f_1, f_2 \in \mathbb{Q}[x]$ such that f_1^2 + $f_2^2 = f$. **return** f_1, f_2 and $f_3 := \cdots f_6 := 0$.
- 6: **if** f is a sum of 4 squares {Use [36, Theorem 17.2] to check it}
- Execute Algorithm 5, to obtain $f_1, \ldots, f_4 \in \mathbb{Q}[x]$ such that $f_1^2 + \dots + f_4^2 = f$. **return** f_1, \dots, f_4 and $f_5 := f_6 := 0$
- 9: Compute the square-free decomposition of $f = q \cdot h^2$, where $q, h \in \mathbb{Q}[x]$ and q is square-free.
- 10: Execute Algorithm 7 with g as an input to obtain $g_1, g_2 \in \mathbb{Q}[x]$ such that $g - g_1^2 - g_2^2$ is a sum of 4 squares in $\mathbb{Q}[x]$.
- Execute Algorithm 5 to decompose $g g_1^2 g_2^2$ into a sum of 4 squares in $\mathbb{Q}[x]$. Denote the output by g_3, \ldots, g_6 .
- 12: **return** $f_1 := g_1 h, \ldots, f_6 := g_6 h.$

In other words, the output is a weighted sum of 5 squares.

Remark 20. It is possible to adapt Algorithms 1 and 5 to output weighted sum of squares of the form $f = lc(f)f_1^2 + lc(f)f_2^2$ or f = $\mathrm{lc}(f)f_1^2+\cdots+\mathrm{lc}(f)f_4^2$. Using the previous remark, one can then adapt Algorithm 8 to output a weighted sum of 5 squares

$$f = a_1 f_1^2 + \dots + a_5 f_5^2$$

with the a_i 's being nonnegative integers. One major benefit if we do so is that the adapted Algorithm 1 will then need no integer factorization of lc(f), and we then need no reference to IntFact.

Algorithm 8 is sub-optimal in the sense that it produces six squares, while it is known that only **five** are really needed. Below, we present Algorithm 9, which is a variant of Algorithm 6.

We verified Algorithm 9 empirically on over 20 000 random nonnegative univariate polynomials. For all of them, it worked fine. Yet still, we don't know how to prove that it eventually terminates, except when either ord 2 c_0 or ord 2 c_d is odd, in which cases it reduces to Algorithm 6. On the other hand, the correctness of the output of the algorithm is immediate. Hence, if we can prove that it stops, we can use it to get the desired decomposition of any nonnegative polynomial into a sum of five (instead of six) squares. This is the first further research direction that is left open by this work. We also intend to compare our algorithms with the other ones analyzed in [25].

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Algorithm 9 Reduction to a sum of 4 squares

l := l + 1.

11:

```
Input: A positive square-free polynomial f = c_0 + c_1 x + \cdots + c_d x^d \in \mathbb{Q}[x].
```

Output: A polynomial $h \in \mathbb{Q}[x]$ such that $f - h^2$ is a sum of 4 (or fewer) squares.

```
1: if f is a sum of 4 squares then

2: return h := 0.

3: Set f_* := c_d + c_{d-1}x + \cdots + c_0x^d.

4: Find a positive number \varepsilon such that \varepsilon < \inf\{f(x) \mid x \in \mathbb{R}\} and \varepsilon < \inf\{f_*(x) \mid x \in \mathbb{R}\}.

5: Initialize l := \lceil -1/2 \cdot \lg \varepsilon \rceil.

6: while True do

7: if f - 2^{-2l} is irreducible in \mathbb{Q}_2[x] then

8: return h := 2^{-l}.

9: if f - 2^{-2l}x^d is irreducible in \mathbb{Q}_2[x] then

10: return h := 2^{-l}x^{d/2}.
```

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