ON SYSTEMS OF FIRST ORDER LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH L^p COEFFICIENTS

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Abstract. We establish the existence, the uniqueness, and the stability of the solution to the Cauchy problem associated to a general class of systems of first order linear partial differential equations under minimal regularity assumptions on their coefficients.

Contents

1. Introduction

A general system of first order linear partial differential equations over an open subset $\Omega \subset \mathbb{R}^{\mathrm{d}}$, $\mathrm{d} \geq 2$, has the following form:

$$\sum_{j=1}^{m} \sum_{i=1}^{\mathrm{d}} c_k^{ij} \frac{\partial y_j}{\partial x_i} = \sum_{j=1}^{m} b_k^j y_j + b_k, \tag{1.1}$$

where k varies in the set $\{1, 2, \ldots, r\}$ for some $r \in \mathbb{N}$, the unknowns being the functions $y_1, y_2, \ldots, y_m : \Omega \to \mathbb{R}$.

In this paper we are interested in systems of the simpler form:

$$\frac{\partial y_j}{\partial x_i} = \sum_{k=1}^{\ell} a_{ij}^k y_k + b_{ij}, \tag{1.2}$$

the unknowns being the functions $y_1, y_2, \ldots, y_\ell : \Omega \to \mathbb{R}$. This system in fact is not very different from the general system, since (at least pointwise) after some

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algebraic operations and the elimination of the eventual non essential unknowns, the system (1.1) reduces to a system of type (1.2). Of course, such a reduction is not always possible in the entire domain Ω since the algebraic operations and the essential unknowns may not be the same at each point of Ω. But even in this case, it is possible in many cases to reduce the general system (1.1) to several systems of type (1.2) by splitting the domain Ω into several parts.

A simple example of general system equivalent with a system of type (1.2) is a system (1.1) where $r = \mathrm{d}m$ and the matrix $(c_k^{ij}) \in \mathbb{M}^{\mathrm{d}m\times\mathrm{d}m}$ is invertible at each point of Ω (c_k^{ij} is the element at the $((j-1)\mathrm{d}+i)$-th row and k-th column).

To properly address the question of existence and uniqueness of solutions, we study here the Cauchy problem associated with the system (1.2), i.e., the problem of finding functions $(y_1, ..., y_\ell)$, defined over the connected open set Ω, that satisfy the problem

$$
\begin{aligned}
&\frac{\partial y_j}{\partial x_i} = \sum_{k=1}^{\ell} a_{ij}^k y_k + b_{ij},\\
&y_i(x^0) = y_i^0,
\end{aligned}
\tag{1.3}
$$

for some given point $x^0 \in \Omega$ and initial values $(y_1^0, y_2^0, \dots, y_\ell^0) \in \mathbb{R}^\ell$.

If we wish this Cauchy problem to have a unique solution, it is natural to require that all the partial derivatives of all the unknowns be represented exactly once in the system. Therefore the system in (1.3) will henceforth have $\mathrm{d}\ell$ equations, that is, it will be understood that the equations of (1.3) are satisfied for all $i \in \{1, ..., \mathrm{d}\}$ and $j \in \{1, ..., \ell\}$.

If we denote by Y the line matrix $(y_1\ y_2\ \dots\ y_\ell) \in \mathbb{M}^{1\times\ell}$, by A_i the square matrix $A_i := (a_{ij}^k) \in \mathbb{M}^{\ell\times\ell}$, where a_{ij}^k is the element at the k-th row and j-th column, and finally by B_i the line matrix $(b_{i1}\ b_{i2}\ \dots\ b_{i\ell}) \in \mathbb{M}^{1\times\ell}$, $i \in \{1, ..., \mathrm{d}\}$, then the Cauchy problem (1.3) can be written in the following matrix form:

$$
\begin{aligned}
&\frac{\partial Y}{\partial x_i} = Y A_i + B_i \quad \text{for all} \quad i \in \{1, 2, \dots, \mathrm{d}\},\\
&Y(x^0) = Y^0,
\end{aligned}
\tag{1.4}
$$

where Y^0 is the line matrix $(y_1^0\ y_2^0\ \dots\ y_\ell^0) \in \mathbb{M}^{1\times\ell}$.

The classical theory of existence and uniqueness of solutions for this system applies under the assumption that the coefficients A_i and B_i are (at least) of class $\mathcal{C}^1$ over Ω (see Cartan [5], Malliavin [11], Thomas [17]). This theory was subsequently generalised by Hartman and Wintner [9] to systems with coefficients of class $\mathcal{C}^0$ over Ω and finally by the author, in [12] to systems with coefficients of class L^∞_{loc} over Ω, and in [13] to systems with coefficients of class L^p_{loc}, $p > 2$, in the particular case where the domain Ω is two-dimensional.

Our objective here is to solve the above system in the general case of domains $\Omega \subset \mathbb{R}^\mathrm{d}$ of any dimension, the coefficients of (1.4) being then of class L^p_{loc}, with $p > \mathrm{d}$, and the equations being understood in the distributional sense.

As in the theory of first order linear ordinary differential equations, it is easy to see that solving the Cauchy problem (1.4) is equivalent with solving two simpler systems: If the matrix field $X : \Omega \to \mathbb{M}^{\ell\times\ell}$ satisfies the homogeneous system

$$\begin{aligned} &\frac{\partial X}{\partial x_i} = XA_i \quad \text{for all} \quad i \in \{1, 2, \ldots, \mathrm{d}\}, \\ &X(x^0) = I, \end{aligned} \tag{1.5}$$

where I is the identity matrix in $\mathbb{M}^{\ell\times\ell}$, and the matrix field $Z : \Omega \to \mathbb{M}^{1\times\ell}$ satisfies the system

$$\begin{aligned} &\partial_i Z = B_i X^{-1} \quad \text{for all} \quad i \in \{1, 2, \ldots, \mathrm{d}\}, \\ &Z(x^0) = Y^0, \end{aligned} \tag{1.6}$$

then a straightforward computation shows that the matrix field $Y := ZX$ satisfies the initial system (1.4).

Therefore our task in this paper is to study the existence and uniqueness of solutions to these last two systems. The second system (1.6) is relatively easy to solve and its resolution constitutes a generalization of the classical Poincaré theorem (see Theorem 6.5). The first system (1.5) is considerably more difficult to solve and its resolution occupies most of this paper (the final result is Theorem 6.4). Combined with Lemma 6.1 establishing the invertibility of the matrix field X, these two theorems establish the following existence and uniqueness result for the system (1.4):

Theorem 1.1. *Let Ω be a connected and simply connected open subset of $\mathbb{R}^{\mathrm{d}}$ and let $A_i \in L^p_{\mathrm{loc}}(\Omega; \mathbb{M}^{\ell\times\ell})$ and $B_i \in L^p_{\mathrm{loc}}(\Omega; \mathbb{M}^{1\times\ell})$, $p > \mathrm{d}$, be matrix fields that satisfy in the distributional sense the compatibility conditions*

$$\frac{\partial A_j}{\partial x_i} + A_i A_j = \frac{\partial A_i}{\partial x_j} + A_j A_i \quad \textit{for all} \quad i \neq j\,, \tag{1.7}$$

$$\frac{\partial B_j}{\partial x_i} + B_i A_j = \frac{\partial B_i}{\partial x_j} + B_j A_i \quad \textit{for all} \quad i \neq j\,. \tag{1.8}$$

Then the problem (1.4) *has a unique solution in the space $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{M}^{1\times\ell})$.*

Note that the first compatibility condition, which expresses the commutativity of the second order derivatives of the solution X, insures the existence of solutions to the system (1.5) (see Theorem 6.4), while the second compatibility condition, which expresses the commutativity of the second order derivatives of Z, insures the existence of solutions to the system (1.6); indeed, the second compatibility condition implies that

$$\frac{\partial (B_i X^{-1})}{\partial x_j} = \frac{\partial (B_j X^{-1})}{\partial x_i} \quad \text{for all} \quad i \neq j\,,$$

which are precisely the compatibility conditions allowing to apply the Poincaré Theorem 6.5 to the system (1.6).

We now describe in more detail how we solve in this paper the homogeneous system (1.5). In fact, we will consider the more general homogeneous system where the unknown is a field with values in any space of matrices, not necessarily square. More specifically, we henceforth study the system

$$\begin{aligned} &\frac{\partial Y}{\partial x_i} = YA_i \quad \text{for all} \quad i \in \{1, 2, \ldots, \mathrm{d}\}, \\ &Y(x^0) = Y^0, \end{aligned} \tag{1.9}$$

where $Y : \Omega \to \mathbb{M}^{q\times\ell}$ is the unknown, $A_i : \Omega \to \mathbb{M}^{\ell\times\ell}$ are matrix fields of class $L^p_{\mathrm{loc}}(\Omega)$, $p > \mathrm{d}$, $x^0 \in \Omega$, and $Y^0 \in \mathbb{M}^{q\times\ell}$. Such systems are called *Pfaff systems* in most of the mathematical literature, and appear notably in Differential Geometry to describe the covariant derivatives of vector fields or the equations of a moving frame along a differential manifold. From now on, we adopt this terminology whenever refering to systems of type (1.9).

We know how to solve a Pfaff system in the classical setting, that is, when $A_i \in \mathcal{C}^1(\Omega)$. The idea is to integrate the system along smooth curves (contained in Ω) joining x^0 with the other points x of the set Ω (the system becomes an ordinary differential equation along each curve). Thanks to the compatibility conditions and to the simple connectedness of Ω, it is then seen that the value at x of the solution to this ordinary differential equation does not depend on the curve joining x^0 to x. This invariance allows to unambiguously define a solution Y to the Cauchy problem (1.9) by taking $Y(x)$ to be the value at x of the solution to the ordinary differential equation associated with a curve (arbitrarily chosen) joining x^0 to x.

Note that this strategy cannot be applied in our setting, i.e., when $A_i \in L^p_{\mathrm{loc}}(\Omega)$, since the restriction of a L^p_{loc}-function to a curve is not well defined. Another difficulty in the resolution of our problem is that the compatibility conditions are nonlinear, since the product of matrices is not commutative (as long as the coefficients A_i are true matrix fields, not merely scalar fields). For instance, we cannot reduce the resolution of Pfaff systems with L^p_{loc}-coefficients to the resolution of Pfaff systems with $\mathcal{C}^1$-coefficients by mollifying its coefficients with a convolution kernel since the compatibility conditions are not satisfied by the regularized coefficients and therefore it is not possible to apply the classical theory to the regularized Pfaff systems. Nevertheless, we do prove in this paper that a Pfaff system with L^p_{loc}-coefficients can be approached locally with Pfaff systems whose coefficients are smooth and satisfy the compatibility conditions (1.7) (understood in the distributional sense), by a new method of approximation (devised in the proof of Theorem 5.1).

The paper is organised as follows. The next Section and the Appendix prepare the background for all the other sections; we list in particular several theorems concerning the Sobolev spaces and a regularity result for the solution to a variational equation that are needed in the subsequent sections. For the convenience of the reader, proofs are provided for those statements that are less known in the literature under the specific assumptions needed in this paper. Section 3 is a preliminary to Section 5 and may be skipped altogether if the main result of Section 5 concerning the regularization of Pfaff systems (stated in Theorem 5.1) is accepted without proof. Section 4 establishes the stability and the uniqueness of the solution to the Cauchy problem (1.9). It relies on Section 2 and is itself needed in Section 6. Finally, Section 6 establishes necessary and sufficient conditions for the existence of solutions to the Cauchy problem (1.9) and it is based on the results of Sections 4 and 5.

We end this section with a few remarks on the regularity of the coefficients. We consider that the coefficients of the Pfaff system belong to a Lebesgue space of type L^p_{loc}, $p \geq 1$, and wish to know which is the minimal value of p for which the Cauchy problem (1.9) has solutions.

The first observation is that the compatibility conditions are meaningful in the distributional setting if and only if $A_i \in L^2_{\mathrm{loc}}(\Omega)$. However, further regularity on coefficients in needed if we wish the system (1.9) to be well posed in a distributional setting.

To deduce the optimal regularity for the coefficients A_i, we consider the scalar case, where the coefficients A_i and the unknown Y are usual functions on Ω. In this case we have an explicit formula for the solution. Indeed, since the product is commutative in the scalar case, the compatibility conditions take the simpler form

$$\frac{\partial A_j}{\partial x_i} = \frac{\partial A_i}{\partial x_j} \quad \text{for all} \quad i \neq j\,.$$

Then the Poincaré Theorem (see Theorem 6.5) shows that there exists a primitive of (A_i), i.e., a function $a \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that

$$\frac{\partial a}{\partial x_i} = A_i \quad \text{for all} \quad i \in \{1, 2, \dots, \mathrm{d}\}.$$

In turn, this implies that the function $Y := Ce^a$ satisfies the system (1.9), the constant C being defined as the solution to the equation $Y(x^0) = Y^0$.

If $p < \mathrm{d}$, the problem (1.9) is not well posed in the distributional sense. It suffices to consider the example where Ω is the unit ball in $\mathbb{R}^{\mathrm{d}}$ and A_i are the partial derivatives of the function $a : \Omega \to \mathbb{R}$, defined by

$$a(x) = |x|^{-\alpha},$$

with $0 < \alpha < \mathrm{d}/p - 1$. It is easy to verify that $A_i \in L^p(\Omega)$. If the solution (in the sense of distributions) to the system (1.9) exists, then it must be of the form $Y = Ce^a$ for some constant C. Indeed, if Y is a solution to (1.9) in Ω, then it is also a solution in the smaller domain $\Omega \setminus \{0\}$. Since the coefficients A_i are of class C^∞ over $\Omega \setminus \{0\}$, we must have $Y = Ce^a$ in $\Omega \setminus \{0\}$, by the uniqueness of the solution to the Cauchy problem (1.9) posed over $\Omega \setminus \{0\}$ (which falls in the classical setting). But $e^a \notin L^1_{\mathrm{loc}}(\Omega)$, hence Y is not a distribution over Ω if $C \neq 0$.

If $p = \mathrm{d}$, we cannot define the initial value for Y in the Cauchy problem. At least not in any point. This is true even if we replace the initial condition $Y(x^0) = Y^0$ with the more general condition

$$\limsup_{r \searrow 0} \frac{\int_{B_r(x^0)} Y(x)\,dx}{|B_r(x^0)|} = Y^0,$$

in the eventuality of noncontinuous solutions. In the above formula, $B_r(x^0)$ denotes the ball of radius r centered at x^0 and $|B_r(x^0)|$ denotes its Lebesgue measure. Consider the following example: Ω is the ball of radius $1/2$ centered at the origin and A_i are the partial derivatives of the function $a : \Omega \to \mathbb{R}$, defined by

$$a(x) = \left(\log \frac{1}{|x|}\right)^{\alpha}$$

with $0 < \alpha < 1 - 1/\mathrm{d}$. It is easy to verify that A_i belong to the space $L^{\mathrm{d}}(\Omega)$. For the same reasons as above, a solution to the problem (1.9) must be of the form $Y = Ce^a$ for some constant C. Or we cannot define an initial condition at the origin since

$$\limsup_{r \searrow 0} \frac{\int_{B_r(0)} e^{a(x)}\,dx}{|B_r(0)|} = +\infty\,.$$

Nor can we allow Y^0 to be infinite in the Cauchy problem (1.9), since we would lose the uniqueness of the solution (only the sign of the constant C would be known, not its value).

If $p > \mathrm{d}$, then the primitive a of the coefficients A_i is continuous over Ω, since it belongs to $W^{1,p}_{\mathrm{loc}}(\Omega)$. Therefore the function e^a is continuous over Ω and so its value at any point is well-defined. Moreover, e^a belongs to the space $W^{1,p}_{\mathrm{loc}}(\Omega)$, which shows that the Cauchy problem (1.9) is well posed in the case where $p > \mathrm{d}$.

We conclude from the above discussion on the value of p that the minimal regularity assumption that leads to a well posed Cauchy problem (1.9) is $A_i \in L^p_{\mathrm{loc}}(\Omega)$, with $p > \mathrm{d}$.

Note that, in the case where $p < \mathrm{d}$, even the problem of finding non trivial solutions to the Pfaff system $\dfrac{\partial Y}{\partial x_i} = YA_i$ (the initial condition at x^0 is dropped) is not well posed in the distributional sense, since we may not have a solution in $L^1_{\mathrm{loc}}(\Omega)$.

On the contrary, in the case where $p = \mathrm{d}$, chances are that the Pfaff system alone (we drop again the initial condition at x^0) has nontrivial solutions. At least this is what the scalar case suggests. Indeed, if A_i are (scalar) functions in $L^{\mathrm{d}}_{\mathrm{loc}}(\Omega)$ that satisfy the compatibility conditions $\frac{\partial A_j}{\partial x_i} = \frac{\partial A_i}{\partial x_j}$, then their primitive a belongs to the space $W^{1,\mathrm{d}}_{\mathrm{loc}}(\Omega)$. Hence $e^a \in L^1_{\mathrm{loc}}(\Omega)$. In fact we even have that

$$e^{|a|^{\mathrm{d}/(\mathrm{d}-1)}} \in L^1_{\mathrm{loc}}(\Omega)$$

(see e.g Adams [1] or Gilbarg and Trudinger [8]). However, in the general case where the coefficients of the Pfaff system are (true) matrix fields of class $L^{\mathrm{d}}_{\mathrm{loc}}(\Omega)$, the question of whether there exists a nontrivial solution to the Pfaff system remains open.

2. Preliminaries

This section gathers the notation and the definitions used in this article, as well as various preliminary results that will be subsequently needed.

The notation $\mathbb{M}^{q\times\ell}$ and $\mathbb{M}^{\ell} := \mathbb{M}^{\ell\times\ell}$ respectively designate the space of all matrices with q rows and ℓ columns and the space of all square matrices of order ℓ. We equip these spaces with the norm

$$|A| := \sum_{i,j} |A_{ij}|$$

and with the inner product

$$A \cdot B := \sum_{i,j} A_{ij} B_{ij},$$

where $A = (A_{ij})$ and $B = (B_{ij})$.

The notation A^T designates the transposed of the matrix A. Hence, if $A \in \mathbb{M}^{q\times\ell}$, A^T is a matrix in $\mathbb{M}^{\ell\times q}$. If $A \in \mathbb{M}^{\ell}$, its cofactor matrix is defined as the matrix whose element at the i-th row and j-th column is the determinant of the matrix in $\mathbb{M}^{\ell-1}$ obtained from A by deleting its i-th row and j-th column. We recall the formula

$$A^{-1} = \frac{1}{\det A} (\mathrm{Cof} A)^T$$

for an invertible matrix $A \in \mathbb{M}^\ell$.

In this paper, all scalar, vector, or matrix-valued functions are defined on subsets of the Euclidean space $\mathbb{R}^{\mathrm{d}}$, thus equipped with the norm

$$|x| := \Big(\sum_i x_i^2\Big)^{1/2},$$

where $x = (x_1, ..., x_{\mathrm{d}})$ denotes a generic point in $\mathbb{R}^{\mathrm{d}}$. The distance between two subsets A, B of $\mathbb{R}^{\mathrm{d}}$ is defined by

$$\mathrm{dist}(A, B) = \inf_{\substack{a \in A \\ b \in B}} |a - b|,$$

the diameter of a subset $A \subset \mathbb{R}^{\mathrm{d}}$ is defined by

$$D_A := \sup_{x,y \in A} |x - y|,$$

and the Lebesgue measure of a measurable subset $A \subset \mathbb{R}^{\mathrm{d}}$ is denoted $|A|$. The closure in $\mathbb{R}^{\mathrm{d}}$ of a subset $\Omega \subset \mathbb{R}^{\mathrm{d}}$ is denoted $\overline{\Omega}$, the complement of $\Omega \subset \mathbb{R}^d$ is denoted $\Omega^c := \mathbb{R}^{\mathrm{d}} \setminus \Omega$, and an open ball with radius R centered at $x \in \mathbb{R}^{\mathrm{d}}$ is denoted $B_R(x)$, or B_R if its center is irrelevant in the subsequent analysis. If $\Omega \subset \mathbb{R}^{\mathrm{d}}$ is a domain with Lipschitz boundary, we denote by $\nu = (\nu_i)_{i=1}^{\mathrm{d}}$ the exterior unit normal to its boundary. In particular, ν is a vector field of class L^∞ on $\partial\Omega$.

The space of distributions over an open set $\Omega \subset \mathbb{R}^{\mathrm{d}}$ is denoted $\mathcal{D}'(\Omega)$ and partial derivatives of first and second order are denoted $\partial_i = \frac{\partial}{\partial x_i}$ and $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$. The usual Sobolev spaces are denoted $W^{m,p}(\Omega)$, or $H^m(\Omega) := W^{m,2}(\Omega)$ if $p = 2$, and

$$W^{m,p}_{\mathrm{loc}}(\Omega) := \{f \in \mathcal{D}'(\Omega); f \in W^{m,p}(U) \text{ for all open set } U \Subset \Omega\},$$

where the notation $U \Subset \Omega$ means that the closure of U in $\mathbb{R}^{\mathrm{d}}$ is a compact subset of Ω. If $p > \mathrm{d}$, the classes of functions in $W^{1,p}(\Omega)$ are identified with their continuous representatives, as in the Sobolev embedding theorem (see, e.g., Adams [1]). For matrix-valued and vector-valued function spaces, we shall use the notation $W^{m,p}(\Omega; \mathbb{M}^{q\times\ell})$, $W^{m,p}(\Omega; \mathbb{R}^\ell)$, etc.

The Lebesgue spaces $L^p(\Omega; \mathbb{R}^d)$ and $L^p(\Omega; \mathbb{M}^{q\times\ell})$ are equipped with the norms

$$\|\mathrm{v}\|_{L^p(\Omega)} = \sum_i \|v_i\|_{L^p(\Omega)}$$

and

$$\|A\|_{L^p(\Omega)} = \sum_{i,j} \|A_{ij}\|_{L^p(\Omega)},$$

and the Sobolev spaces $W^{1,p}(\Omega; \mathbb{M}^{q\times\ell})$ and $W^{2,p}(\Omega; \mathbb{M}^{q\times\ell})$ are equipped with the norms

$$\|Y\|_{W^{1,p}(\Omega)} = \|Y\|_{L^p(\Omega)} + \sum_i \|\partial_i Y\|_{L^p(\Omega)}$$

and

$$\|Y\|_{W^{2,p}(\Omega)} = \|Y\|_{W^{1,p}(\Omega)} + \sum_{i,j} \|\partial_{ij} Y\|_{L^p(\Omega)}.$$

The image of an application $f : X \to Y$ is defined by

$$\mathrm{Im}\, f = \{f(x); x \in X\}.$$

If X and Y are two topological vector spaces, we denote by $\mathrm{Isom}(X;Y)$ the space of isomorphisms between X and Y, i.e.

$$\mathrm{Isom}(X;Y) := \Big\{ f : X \to Y \ ;\ f \text{ is linear and bijective,} \\ f \text{ and } f^{-1} \text{ are continuous} \Big\}.$$

Finally, we make the following convention: all the cuboids ω considered in this paper have edges parallel with the axes of coordinates, i.e.,

$$\omega = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_{\mathrm{d}}, b_{\mathrm{d}})$$

for some $a_i, b_i \in \mathbb{R}$ such that $a_i < b_i$.

The remainder of this section gathers all the preliminary results that will be subsequently needed. Proofs are provided only for those statements that are less common in the literature in the present form. We begin with the classical implicit function theorem (for a proof see, e.g., Schwartz[15, Chap. 3, sect. 8] or Deimling [7, Theorem 15.1]):

Theorem 2.1 (implicit function theorem)**.** *Let there be given three Banach spaces X, Y and Z, an open subset Ω of the space $X \times Y$ containing a point (x_0, y_0), and a mapping $f \in \mathcal{C}^1(\Omega; Z)$ such that its derivative with respect to the second variable satisfies*

$$D_y f(x_0, y_0) \in \mathrm{Isom}(Y; Z).$$

Then there exist open balls $B_\delta(x_0) \subset X$ and $B_\varepsilon(x_0) \subset Y$ and exactly one map $g : B_\delta(x_0) \to B_\varepsilon(y_0)$ such that

$$f(x, g(x)) = f(x_0, y_0) \quad \text{for all } x \in B_\delta(x_0).$$

Furthermore, the map g is of class $\mathcal{C}^1$.

The next theorem establishes the Morrey inequality with an explicit constant in the particular case where the domain is convex. A more general Morrey inequality can be found in Maz'ja [14, Section 1.4.5]. However, the result below is sufficient for our analysis and has an elementary proof (the only prerequisite is the density of $\mathcal{C}^1(\overline{\Omega})$ in $W^{1,p}(\Omega)$), which is given here for the sake of completeness.

Theorem 2.2 (Morrey's inequality)**.** *Let $\Omega \subset \mathbb{R}^{\mathrm{d}}$ be a bounded open convex set of diameter $D_\Omega > 0$ and let $p > \mathrm{d}$. Then $W^{1,p}(\Omega) \subset \mathcal{C}(\overline{\Omega})$ and*

$$|u(x) - u(y)| \le \frac{2 D_\Omega}{(1 - \mathrm{d}/p)|\Omega|^{1/p}} \|\nabla u\|_{L^p(\Omega)} \tag{2.1}$$

for all $u \in W^{1,p}(\Omega)$ and all $x, y \in \overline{\Omega}$.

Proof. We first prove that (2.1) is valid for all $u \in C^1(\overline{\Omega})$. For $x, z \in \Omega$, let $\varphi : [0,1] \to \mathbb{R}$ be defined by $\varphi(t) = u(x + t(z - x))$. Then

$$u(z) - u(x) = \varphi(1) - \varphi(0) = \int_0^1 \varphi'(t)\, dt = \int_0^1 \nabla u(x + t(z-x)) \cdot (z - x)\, dt$$

By integrating with respect to z over Ω and dividing by $|\Omega|$, we obtain

$$\overline{u} - u(x) = \frac{1}{|\Omega|} \int_\Omega \int_0^1 \nabla u(x + t(z-x)) \cdot (z-x)\, dt\, dz\,,$$

where $\overline{u} := \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u\,dx$.

By using the Cauchy-Schwartz inequality and the fact that $|z-x| \le D_\Omega$ for all $z \in \Omega$, we next obtain

$$|\overline{u} - u(x)| \le \frac{D_\Omega}{|\Omega|}\int_\Omega \int_0^1 |\nabla u(x + t(z-x))|\,dt\,dz.$$

Furthermore, the Fubini formula and the inequality $\sqrt{v\cdot v} \le \sum_{i=1}^{\mathrm{d}} |v_i|$ for all $v = (v_i) \in \mathbb{R}^{\mathrm{d}}$ imply that

$$|\overline{u} - u(x)| \le \frac{D_\Omega}{|\Omega|}\int_0^1 \int_\Omega \sum_{i=1}^{\mathrm{d}} |\partial_i u(x + t(z-x))|\,dz\,dt.$$

Now, by making the change of variables $y = x + t(z-x)$, we obtain

$$|\overline{u} - u(x)| \le \frac{D_\Omega}{|\Omega|}\int_0^1 \left(\int_{\Omega_t(x)} \sum_{i=1}^{\mathrm{d}} t^{-\mathrm{d}} |\partial_i u(y)|\,dy\right) dt\,,$$

where

$$\Omega_t(x) = (1-t)x + tK := \{x + t(z-x)\ ;\ z \in K\}.$$

Note that $\Omega_t(x) \subset \Omega$, since Ω is convex, and that $|\Omega_t(x)| = t^d|\Omega|$.

Using now Hölder inequality with $\dfrac{1}{p} + \dfrac{1}{q} = 1$ in the last integral gives

$$\begin{aligned}
|\overline{u} - u(x)| &\le \frac{D_\Omega}{|\Omega|}\int_0^1 t^{-\mathrm{d}} \|\nabla u\|_{L^p(\Omega_t(x))} |\Omega_t(x)|^{\frac{1}{q}}\,dt \\
&\le \frac{D_\Omega}{|\Omega|}|\Omega|^{\frac{1}{q}} \int_0^1 t^{-\mathrm{d}+\frac{\mathrm{d}}{q}} \|\nabla u\|_{L^p(\Omega_t(x))}\,dt \\
&\le \frac{D_\Omega}{|\Omega|^{\frac{1}{p}}} \|\nabla u\|_{L^p(\Omega)} \int_0^1 t^{-\frac{\mathrm{d}}{p}}\,dt \\
&= \frac{D_\Omega}{|\Omega|^{\frac{1}{p}}\left(1 - \frac{\mathrm{d}}{p}\right)} \|\nabla u\|_{L^p(\Omega)}\,.
\end{aligned}$$

Finally, using the triangle inequality and the continuity of u on $\overline{\Omega}$, we obtain the inequality (2.1) for $u \in \mathcal{C}^1(\overline{\Omega})$ and $x, y \in \overline{\Omega}$.

Now let $u \in W^{1,p}(\Omega)$ and let a sequence $(u_n) \subset \mathcal{C}^1(\overline{\Omega})$ that converges to u in the $W^{1,p}$-norm. By passing if necessary to a subsequence, we can assume that $u_n \to u$ a.e. in Ω. Choose a point $x_0 \in \Omega$ such that $u_n(x_0) \to u(x_0)$. In particular, $\big(u_n(x_0)\big)$ is a Cauchy sequence in $\mathbb{R}$. By using inequality (2.1) for the function $(u_n - u_m)$ and $y = x_0$, we obtain that, for all $x \in \overline{\Omega}$,

$$|(u_n - u_m)(x)| \le |(u_n - u_m)(x_0)| + \frac{2D_\Omega}{\left(1 - \frac{\mathrm{d}}{p}\right)|\Omega|^{\frac{1}{p}}} \|\nabla(u_n - u_m)\|_{L^p(\Omega)}.$$

Consequently,

$$\|u_n - u_m\|_{L^\infty(\overline{\Omega})} \le |(u_n - u_m)(x_0)| + \frac{2D_\Omega}{\left(1 - \frac{\mathrm{d}}{p}\right)|\Omega|^{\frac{1}{p}}} \|\nabla(u_n - u_m)\|_{L^p(\Omega)}\,,$$

which shows that (u_n) is a Cauchy sequence with respect to the $L^\infty(\overline{\Omega})$-norm. Therefore the sequence (u_n) converges in $L^\infty(\overline{\Omega})$ and, by the uniqueness of the limit in the L^p-norm, we must have $u_n \to u$ in the $L^\infty(\overline{\Omega})$-norm. Since $u_n \in \mathcal{C}(\overline{\Omega})$, we deduce that $u \in \mathcal{C}(\overline{\Omega})$. The inequality (2.1) is then obtained by passing to the limit in the inequalities of the same type satisfied by u_n. $\square$

An immediate consequence of Theorem 2.2 is the following:

Corollary 2.3. *Let $\Omega \subset \mathbb{R}^{\mathrm{d}}$ be an bounded open convex set of diameter $D_\Omega > 0$ and let $p > \mathrm{d}$. Then, for all $u \in W^{1,p}(\Omega)$ and all $x, y \in \overline{\Omega}$,*

$$|u(x) - u(y)| \leq \frac{2}{\left(1 - \frac{\mathrm{d}}{p}\right)|\Omega_0|^{\frac{1}{p}}} D_\Omega^{1-\frac{\mathrm{d}}{p}} \|\nabla u\|_{L^p(\Omega)}, \tag{2.2}$$

where $\Omega_0 := \frac{1}{D_\Omega}\Omega$ (i.e., Ω_0 is a set of diameter 1 homothetic with Ω).

In particular, if $\Omega = B_R \subset \mathbb{R}^{\mathrm{d}}$ is an open ball of radius R, then there exists a constant $C_1 > 0$ depending only on (d, p) such that

$$|u(x) - u(y)| \leq C_1 R^{1-\frac{\mathrm{d}}{p}} \|\nabla u\|_{L^p(B_R)} \tag{2.3}$$

for all $u \in W^{1,p}(B_R)$ and all $x, y \in \overline{B}_R$.

The Sobolev Embedding Theorem and the corresponding Sobolev inequalities (see, e.g., Gilbarg and Trudinger [8]) will be used extensively in this paper. We will need in particular the following Sobolev inequality with an explicit constant (the dependence on R is obtained by a scaling argument):

Theorem 2.4 (Sobolev inequality)**.** *Let $\omega_R \subset \mathbb{R}^{\mathrm{d}}$ be an open cube with edges of length $R > 0$ and let $p > \mathrm{d}$. There exists a constant $C_2 > 0$ depending only on (d, p) such that*

$$\|u\|_{L^{2p/(p-2)}(\omega_R)} \leq C_2 R^{1-\mathrm{d}/p} \|\nabla u\|_{L^2(\omega_R)}$$

for all $u \in H^1(\omega_R)$ such that $u = 0$ on at least one lateral face of the cube ω_R.

In the remainder of this section, the set ω is an open cuboid in $\mathbb{R}^{\mathrm{d}}$, that is, a set of the form

$$\omega = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_{\mathrm{d}}, b_{\mathrm{d}}),$$

and Γ_i denotes the union of its two lateral faces that are perpendicular to the axis of coordinates x_i, that is,

$$\Gamma_i := (\partial\omega \cap \{x_i = a_i\}) \cup (\partial\omega \cap \{x_i = b_i\}).$$

The next result is a generalisation of the Poincaré theorem and it can be found in Bourgain, Brezis, Mironescu [4] for the case of smooth and simply connected sets. The proof in [4] remains valid for cuboids. For the sake of completeness, we reproduce it here:

Theorem 2.5. *Let there be given $f_1, f_2, ..., f_{\mathrm{d}} \in L^p(\omega)$, where $p \geq 1$, that satisfy in the distributional sense the compatibility conditions*

$$\partial_i f_j = \partial_j f_i \text{ in } \omega,$$

for all $i, j \in \{1, 2, ..., \mathrm{d}\}$. Then there exists $\psi \in W^{1,p}(\omega)$, unique up to a constant, such that

$$\partial_i \psi = f_i \text{ in } \omega$$

for all $i \in \{1, 2, ..., \mathrm{d}\}$.

Proof. It suffices to prove that for any open cuboid $\omega' \Subset \omega$, there exists a function $\psi \in W^{1,p}(\omega')$ such that

$$\partial_i \psi = f_i \text{ in } \omega'.$$

Indeed, we can construct from this result a function $\psi \in W^{1,p}_{\text{loc}}(\omega)$ that satisfy

$$\partial_i \psi = f_i \text{ in } \omega,$$

since the set ω can be written as a union of open cuboids, $\omega = \cup_{n\in\mathbb{N}} \omega_n$, such that $\omega_n \subset \omega_{n+1} \Subset \omega$ for all $n \in \mathbb{N}$. Since $f_i \in L^p(\omega)$ and the domain ω is a cuboid (thus its boundary is Lipschitz-continuous), the function ψ is in fact in $W^{1,p}(\omega)$ (see, e.g., Maz'ja [14], Corollary in Section 1.1.11).

Let $\omega' \Subset \omega$. By taking the convolution of functions f_i with a sequence of mollifiers, one can find sequences $(f_i^\varepsilon) \subset \mathcal{C}^\infty(\overline{\omega}')$ with the following properties:

$$\begin{aligned} &\partial_i f_j^\varepsilon = \partial_j f_i^\varepsilon \text{ in } \omega', \\ &f_i^\varepsilon \to f_i \text{ in } L^p(\omega') \text{ as } \varepsilon \to 0, \end{aligned}$$

for all $i, j \in \{1, 2, ..., \mathrm{d}\}$. Then the classical Poincaré theorem shows that, for each ε, there exists a function $\psi^\varepsilon \in \mathcal{C}^\infty(\overline{\omega}')$ whose derivatives are given by

$$\partial_i \psi^\varepsilon = f_i^\varepsilon \text{ in } \omega',$$

and that satisfies in addition

$$\int_{\omega'} \psi^\varepsilon(x)dx = 0.$$

Since (f_i^ε) are Cauchy sequences in $L^p(\omega')$, the Poincaré-Wirtinger inequality shows that (ψ^ε) is a Cauchy sequence in the space $W^{1,p}(\omega')$. Since this space is complete, there exists a function $\psi \in W^{1,p}(\omega')$ such that $\psi^\varepsilon \to \psi$ in $W^{1,p}(\omega')$ as $\varepsilon \to 0$. This function satisfies the conditions of the Theorem on ω'.

Finally, the uniqueness result is an obvious consequence of the general distribution theory: if $\psi, \tilde{\psi} \in W^{1,p}(\omega)$ satisfy $\partial_i \psi = \partial_i \tilde{\psi} = f_i$ in ω, then $\nabla(\psi - \tilde{\psi}) = 0$ in ω; hence $(\psi - \tilde{\psi})$ is a constant, since the set ω is connected. □

The following theorem gathers the density results for Sobolev spaces that are needed in this paper. We state them for cuboids in $\mathbb{R}^{\mathrm{d}}$ even though they hold true for a much larger class of open sets. However, we want to emphasize that in the case of cuboids these density results have an elementary proof based on extensions of functions in Sobolev spaces.

Theorem 2.6. *Let $p \in [1, +\infty)$, $m \geq 1$, and $i_1, \ldots, i_k \in \{1, \ldots, \mathrm{d}\}$. Then the following assertions hold true:*

(i) The space $\mathcal{C}^\infty(\overline{\omega})$ is dense in $W^{m,p}(\omega)$.

(ii) The space

$$\{u \in \mathcal{C}^\infty(\overline{\omega})\ ;\ u = 0 \text{ on } \Gamma_{i_1} \cup \cdots \cup \Gamma_{i_k}\}$$

is dense in the space

$$\{u \in W^{1,p}(\overline{\omega})\ ;\ u = 0 \text{ on } \Gamma_{i_1} \cup \cdots \cup \Gamma_{i_k}\}.$$

As a consequence of this Theorem, we now establish the following result about traces of functions in Sobolev spaces:

Corollary 2.7. *Let $u \in W^{2,p}(\omega)$, $p \geq 1$, such that $u = 0$ on Γ_i for some $i \in \{1, \ldots, \mathrm{d}\}$. Then $\partial_j u = 0$ on Γ_i for all $j \neq i$.*

Proof. By Theorem 2.6(i), there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{C}^2(\overline{\omega})$ such that $u_n \to u$ in $W^{2,p}(\omega)$. Then $u_{n|\partial\omega} \to u_{|\partial\omega}$ in $W^{1,p}(\partial\omega)$. In particular, $u_{n|\Gamma_i} \to u_{|\Gamma_i} = 0$ in $W^{1,p}(\Gamma_i)$, so that we have on one hand

$$\partial_j(u_{n|\Gamma_i}) \to 0 \text{ in } L^p(\Gamma_i) \text{ for all } j \neq i.$$

On the other hand, $\partial_j u_n \to \partial_j u$ in $W^{1,p}(\omega)$, hence $(\partial_j u_n)_{|\partial\omega} \to (\partial_j u)_{|\partial\omega}$ in $L^p(\partial\omega)$. In particular $(\partial_j u_n)_{|\Gamma_i} \to (\partial_j u)_{|\Gamma_i}$ in $L^p(\Gamma_i)$.

But $\partial_j(u_{n|\Gamma_i}) = (\partial_j u_n)_{|\Gamma_i}$ for all n and all $j \neq i$, since $u_n \in \mathcal{C}^2(\overline{\omega})$. Therefore, we obtain by passing to the limit $n \to \infty$ that

$$(\partial_j u)_{|\Gamma_i} = 0 \text{ for all } j \neq i.$$

□

Remark 2.8. In other words, we have proved that $(\partial_j u)_{|\Gamma_i} = \partial_j(u_{|\Gamma_i}) = 0$, and this thanks to the fact that we have enough regularity for u. While the second equality is obvious and is true even for $u \in W^{1,p}(\omega)$, the term $(\partial_j u)_{|\Gamma_i}$ is not well defined if u is only of class $W^{1,p}$ over ω.

We end this Section with a regularity result for the solution of a variational equation under assumptions that are relevant to our subsequent analysis. The proof is given in the Appendix (see Theorem 7.2 and Corollary 7.4).

Theorem 2.9. *Let $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and fix any indices $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, \mathrm{d}\}$. Let*

$$V_p(\omega) := \{u \in W^{1,p}(\omega);\ u = 0 \text{ on } \Gamma_{i_1} \cup \Gamma_{i_2} \cup \ldots \cup \Gamma_{i_k}\},$$
$$V_q(\omega) := \{v \in W^{1,q}(\omega);\ v = 0 \text{ on } \Gamma_{i_1} \cup \Gamma_{i_2} \cup \ldots \cup \Gamma_{i_k}\},$$

and let $f \in L^r(\omega)$, where $r \in (1, +\infty)$ satisfies $\frac{1}{r} + \frac{1}{q} \leq 1 + \frac{1}{\mathrm{d}}$.

If a function $u \in V_p(\omega)$ satisfies the variational equation

$$\int_\omega \nabla u \cdot \nabla v \, dx = \int_\omega f v \, dx \quad \text{for all} \quad v \in V_q(\omega),$$

then $u \in W^{2,r}(\omega)$ and satisfies the boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \omega, \\ u &= 0 \quad \text{on } \Gamma_{i_1} \cup \Gamma_{i_2} \cup \ldots \cup \Gamma_{i_k}, \\ \partial_j u &= 0 \quad \text{on } \Gamma_j \quad \text{for all} \quad j \notin \{i_1, i_2, \ldots, i_k\}. \end{aligned}$$

3. Matrix-valued differential forms

In this section, as well as in Section 5, we use the differential calculus on an open set $\Omega \subset \mathbb{R}^{\mathrm{d}}$ viewed as a (trivial) differential manifold endowed with the Euclidean metric. The differential manifold Ω being parametrized by the identity map, the covariant basis of the tangent space at any point of Ω is the canonical basis of $\mathbb{R}^{\mathrm{d}}$.

In order to simplify the notation, we make the convention that all the indices appearing in this section belong to the set $\{1, 2, \ldots, \mathrm{d}\}$. For instance, we will write $\displaystyle\sum_{i_1 < \cdots < i_k}$ instead of $\displaystyle\sum_{1 \leq i_1 < \cdots < i_k \leq \mathrm{d}}$. We will also write $\displaystyle\sum_{i_1 < \cdots < i_k} \Big(\sum_{j \neq i_r} \Big)$ instead of $\displaystyle\sum_{1 \leq i_1 < \cdots < i_k \leq \mathrm{d}} \Big(\sum_{j \notin \{i_1, \ldots, i_k\}} \Big)$.

All the differential forms in this paper are *matrix-valued*: a k-form, $k \geq 1$, is defined by

$$\alpha := \sum_{i_1<\cdots<i_k} A_{i_1\dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where the coefficients are matrix fields $A_{i_1\dots i_k} : \Omega \to \mathbb{M}^{q\times\ell}$, and a 0-form is a field $\phi : \Omega \to \mathbb{M}^{q\times\ell}$. Note that a form of degree $k > \mathrm{d}$ vanish identically in Ω. The space of all k-forms is denoted $\Lambda^k(\Omega; \mathbb{M}^{q\times\ell})$.

We say that a k-form is of class $W^{m,p}$ over Ω, and we denote $\alpha \in W^{m,p}(\Omega; \Lambda^k)$, if all its coordinates $A_{i_1\dots i_k}$ are in $W^{m,p}(\Omega; \mathbb{M}^{q\times\ell})$. We equip the space $W^{m,p}(\Omega; \Lambda^k)$ with the norm

$$\|\alpha\|_{W^{m,p}(\Omega;\Lambda^k)} := \sum_{i_1<\cdots<i_k} \|A_{i_1\dots i_k}\|_{W^{m,p}(\Omega;\mathbb{M}^{q\times\ell})} .$$

As usual, we use the notation $L^p(\Omega; \Lambda^k)$ instead of $W^{0,p}(\Omega; \Lambda^k)$ and $H^m(\Omega; \Lambda^k)$ instead of $W^{m,2}(\Omega; \Lambda^k)$. When the degree of a form is unambiguously deduced from the context, we will use the short notation $W^{m,p}(\Omega)$ instead of $W^{m,p}(\Omega; \Lambda^k)$.

Similar definitions and conventions are used for the spaces $W^{m,p}_{\mathrm{loc}}(\Omega; \Lambda^k)$, $\mathcal{C}^m(\Omega; \Lambda^k)$, and $\mathcal{C}^m(\overline{\Omega}; \Lambda^k)$.

In all that follows, we make the convention that the indices of the coefficients of a matrix-valued form are always written in increasing order, that $\epsilon_{i_1\dots i_r}$ is the sign of the permutation $\{i_1, \dots, i_r\}$ of the set $\{1, 2, \dots, r\}$, and that the indices of ϵ always form a permutation of the set of integers from 1 to the number of indices. The notation $A_{\dots\widehat{i_p}\dots}$ means that the indice i_p is missing.

On the space of all forms, we define several algebraic and differential operators. For the generic forms

$$\begin{aligned}
\alpha &:= \sum_{i_1<\cdots<i_k} A_{i_1\dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^k(\Omega; \mathbb{M}^{q\times\ell}),\\
\beta &:= \sum_{i_1<\cdots<i_k} B_{i_1\dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^k(\Omega; \mathbb{M}^{q\times\ell}),\\
\phi &:= \sum_{i_1<\cdots<i_h} \Phi_{i_1\dots i_h} dx^{i_1} \wedge \cdots \wedge dx^{i_h} \in \Lambda^h(\Omega; \mathbb{M}^{\ell\times m}),
\end{aligned}$$

where $k + h \leq \mathrm{d}$, we define

- the *Hodge star operator* $\ast : \Lambda^k(\Omega; \mathbb{M}^{q\times\ell}) \to \Lambda^{\mathrm{d}-k}(\Omega; \mathbb{M}^{q\times\ell})$,

$$\ast\, \alpha := \sum_{i_{k+1}<\cdots<i_{\mathrm{d}}} A_{i_1\dots i_k} \epsilon_{i_1\dots i_k i_{k+1}\dots i_{\mathrm{d}}} dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_{\mathrm{d}}},$$

- the *exterior product* $\wedge : \Lambda^k(\Omega; \mathbb{M}^{q\times\ell}) \times \Lambda^h(\Omega; \mathbb{M}^{\ell\times m}) \to \Lambda^{k+h}(\Omega; \mathbb{M}^{q\times m})$,

$$\alpha \wedge \phi := \sum_{i_1<\cdots<i_{k+h}} \Big(\sum_{1\leq j_1<\cdots<j_k\leq k+h} A_{i_{j_1}\dots i_{j_k}} \Phi_{i_{j_{k+1}}\dots i_{j_{k+h}}} \epsilon_{j_1\dots j_k j_{k+1}\dots j_{k+h}} \Big) dx^{i_1} \wedge \cdots \wedge dx^{i_{k+h}},$$

- the *scalar product* $\cdot : \Lambda^k(\Omega; \mathbb{M}^{q\times\ell}) \times \Lambda^k(\Omega; \mathbb{M}^{q\times\ell}) \to \Lambda^0(\Omega; \mathbb{R})$,

$$\alpha \cdot \beta := \sum_{i_1<\cdots<i_k} A_{i_1\dots i_k} \cdot B_{i_1\dots i_k},$$

- the *differential*, or the *exterior derivative* $d : \Lambda^k(\Omega; \mathbb{M}^{q\times\ell}) \to \Lambda^{k+1}(\Omega; \mathbb{M}^{q\times\ell})$, $k \leq \mathrm{d} - 1$,

$$d\alpha := \sum_{i_1<\cdots<i_{k+1}} \Big(\sum_{p=1}^{k+1} (-1)^{p-1} \partial_{i_p} A_{i_1\ldots\widehat{i_p}\ldots i_{k+1}} \Big) dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}},$$

- the *codifferential* $\delta : \Lambda^k(\Omega; \mathbb{M}^{q\times\ell}) \to \Lambda^{k-1}(\Omega; \mathbb{M}^{q\times\ell}),\ k \geq 1$,

$$\delta\alpha := \sum_{i_1<\cdots<i_{k-1}} \Big(\sum_{j\neq i_r} (-1)^p \partial_j A_{i_1\ldots i_{p-1} j i_p \ldots i_{k-1}} \Big) dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}},$$

- the *laplacean* $\Delta : \Lambda^k(\Omega; \mathbb{M}^{q\times\ell}) \to \Lambda^k(\Omega; \mathbb{M}^{q\times\ell})$,

$$\Delta\alpha := \sum_{i_1<\cdots<i_k} (\Delta A_{i_1\ldots i_k}) dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

- the *gradient* $\nabla : \Lambda^k(\Omega; \mathbb{M}^{q\times\ell}) \to \big(\Lambda^k(\Omega; \mathbb{M}^{q\times\ell})\big)^{\mathrm{d}}$,

$$\nabla\alpha := (\partial_i \alpha)_{i=1}^{\mathrm{d}}, \quad \text{where} \quad \partial_i\alpha := \sum_{i_1<\cdots<i_k} \partial_i A_{i_1\ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

and the *scalar product of gradients* ,

$$\nabla\alpha \cdot \nabla\beta := \sum_{i=1}^{\mathrm{d}} \partial_i\alpha \cdot \partial_i\beta .$$

Remarks 3.1. 1. Note that in general

$$\alpha \wedge \phi \neq (-1)^{kh} \phi \wedge \alpha$$

(even if α and β were square matrix-valued forms), in contrast with what happens in the case of scalar-valued forms.

2. With the exception of the scalar product, all the above operators have also intrinsic definitions (not involving coordinates). In Differential Geometry, it is customary to start with intrinsic definitions and only then deduce their equivalent in coordinates. For instance, the codifferential can be defined by means of the Hodge star operator and the exterior differential (both of which have intrinsic definitions) by $\delta := (-1)^{\mathrm{d}(k+1)+1} * d*$ (see e.g. Bleecker [3], Taylor [16]). The laplacean is actually defined by means of d and δ as $\Delta_{\delta,d} := d\delta + \delta d$. This leads sometimes to confusion since in a Euclidean space we have the formula (see e.g. Jost [10]) $d\delta + \delta d = -\Delta$, where Δ is the operator defined by taking the laplacean of all the coefficients of the form (as we did in this Section). In this paper, we will use exclusively this last definition of the laplacean (hence the formula $-\Delta = d\delta + \delta d$), thus avoiding the confusion with the operator $\Delta_{\delta,d}$.

3. The definition of the operators d and $\wedge$ translate verbatim in any other basis (associated with a chart parametrizing the manifold), while all the other operators translate into more complicated expressions involving the contravariant coordinates of the metric induced by the chart and the associated Christofell symbols.

From the definitions of the the operators d, δ and $\wedge$, it is easy to see that they satisfy the following properties, which will be useful henceforth.

Theorem 3.2. *The operators d and δ satisfy the identities*

$$dd = 0\ ,\ \delta\delta = 0\,,$$
$$d\delta + \delta d = -\Delta.$$

If α, β, γ are 1-forms whose coefficients are matrix fields, then

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma),$$
$$d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta.$$

If α is a k-form, we let

$$|\alpha| := \sum_{i_1 < \cdots < i_k} |A_{i_1 \dots i_k}|$$

and, if $\alpha \in W^{1,p}(\Omega; \Lambda^k)$, we define the L^p-norm of its gradient by letting

$$\|\nabla \alpha\|_{L^p(\Omega)} := \|\alpha\|_{W^{1,p}(\Omega)} - \|\alpha\|_{L^p(\Omega)} = \sum_{i=1}^{\mathrm{d}} \|\partial_i \alpha\|_{L^p(\Omega)}.$$

It is then easy to see that the follwing inequalities are satisfied

$$\begin{aligned} |\alpha \cdot \beta| &\leq \sqrt{\alpha \cdot \alpha}\, \sqrt{\beta \cdot \beta} \leq |\alpha||\beta|, \\ |\alpha \wedge \phi| &\leq |\alpha||\phi|, \\ \big\| |\alpha| \big\|_{L^p(\Omega)} &\leq \|\alpha\|_{L^p(\Omega;\Lambda^k)}, \\ \|d\alpha\|_{L^p(\Omega,\Lambda^{k+1})} &\leq \|\nabla \alpha\|_{L^p(\Omega)}, \\ \|\delta\alpha\|_{L^p(\Omega,\Lambda^{k-1})} &\leq \|\nabla \alpha\|_{L^p(\Omega)}\,. \end{aligned} \tag{3.1}$$

The remainder of this section gathers various theorems about Sobolev spaces and Stokes' type formulas in the framework of matrix-valued forms. All the forms appearing in these theorems are defined over an open cuboid in $\mathbb{R}^{\mathrm{d}}$, that is,

$$\omega = (a_1, b_1) \times \cdots \times (a_{\mathrm{d}}, b_{\mathrm{d}}),$$

and the notation Γ_i designates the union of the two lateral faces of ω that are perpendicular to the x_i-axis, i.e.,

$$\Gamma_i := (\partial\omega \cap \{x_i = a_i\}) \cup (\partial\omega \cap \{x_i = b_i\})\,.$$

We begin by translating the Poincaré Theorem 2.5 into the framework of matrix-valued forms:

Theorem 3.3. *Let $\alpha \in L^p(\omega; \Lambda^1)$ be a 1-form that satisfies $d\alpha = 0$ in the distributional sense. Then there exists $\phi \in W^{1,p}(\omega; \Lambda^0)$ such that $d\phi = \alpha$.*

If $\phi \in W^{1,p}(\omega; \Lambda^k)$ for some $p \geq 1$, we say that $\phi = 0$ on $\partial\omega$ if $i^*(\phi) = 0$, where i^* is the pull-back of the canonical inclusion $i : \partial\omega \hookrightarrow \overline{\omega}$. The following lemma characterises this boundary condition on a form in terms of boundary conditions on its coefficients.

Lemma 3.4. *Let $\phi = \sum_{i_1 < \cdots < i_k} \Phi_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ be a k-form.*

(i) If $\phi \in W^{1,p}(\omega; \Lambda^k)$, $p \geq 1$, then

$$*\phi = 0 \text{ on } \partial\omega \iff \Phi_{i_1 \dots i_k} = 0 \text{ on } \Gamma_{i_1} \cup \cdots \cup \Gamma_{i_k} \quad \text{for all } i_1 < \cdots < i_k.$$

(ii) If $\phi \in W^{2,p}(\omega)$, $p \geq 1$, then

$$\begin{cases} *\phi = 0 & \text{on } \partial\omega, \\ *d\phi = 0 & \text{on } \partial\omega, \end{cases} \iff \begin{cases} \Phi_{i_1 \dots i_k} = 0 & \text{on } \Gamma_{i_1} \cup \cdots \cup \Gamma_{i_k}, \\ \partial_j \Phi_{i_1 \dots i_k} = 0 & \text{on } \Gamma_j \text{ for all } j \notin \{i_1, \dots, i_k\}, \end{cases}$$
$$\iff \begin{cases} \Phi_{i_1 \dots i_k} = 0 & \text{on } \Gamma_{i_1} \cup \cdots \cup \Gamma_{i_k}, \\ \partial_\nu \Phi_{i_1 \dots i_k} = 0 & \text{on } \partial\omega \setminus \big(\Gamma_{i_1} \cup \cdots \cup \Gamma_{i_k}\big). \end{cases}$$

Proof. We first recall that, for all $h \neq j$, $i^*(dx^h) = dx^h$ and $i^*(dx^j) = 0$ on Γ_j (see e.g. Barden and Thomas [2]).

(i) This property follows immediately from the definition of the star operator, the linearity of the pull-back i^*, and the formula

$$i^*(dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_{\mathrm{d}}}) = i^*(dx^{i_{k+1}}) \wedge \cdots \wedge i^*(dx^{i_{\mathrm{d}}}).$$

(ii) Thanks to the part (i), we have

$$\begin{cases} *\phi = 0 \text{ on } \partial\omega, \\ *d\phi = 0 \text{ on } \partial\omega, \end{cases} \iff \begin{cases} \Phi_{i_1\dots i_k} = 0 \text{ on } \Gamma_{i_1} \cup \cdots \cup \Gamma_{i_k}, \\ \displaystyle\sum_{p=1}^{k+1} (-1)^{p-1} \partial_{i_p} \Phi_{i_1\dots\widehat{i_p}\dots i_{k+1}} = 0 \text{ on } \Gamma_{i_1} \cup \cdots \cup \Gamma_{i_{k+1}}. \end{cases}$$

Fix a $(k+1)$-tuple $i_1 < i_2 < \cdots < i_{k+1}$ and a number $h \in \{1, 2, \dots, k+1\}$. By applying Corollary 2.7 to each of the $q\ell$ components of the matrix fields $\Phi_{i_1\dots\widehat{i_p}\dots i_{k+1}}$, we obtain that $\partial_{i_p}\Phi_{i_1\dots\widehat{i_p}\dots i_{k+1}} = 0$ on Γ_{i_h} for all $p \neq h$. Therefore

$$\sum_{p=1}^{k+1} (-1)^{p-1} \partial_{i_p} \Phi_{i_1\dots\widehat{i_p}\dots i_{k+1}} = (-1)^{h-1} \partial_{i_h} \Phi_{i_1\dots\widehat{i_h}\dots i_{k+1}} = 0 \quad \text{on } \Gamma_{i_h}.$$

Hence $\partial_{i_h}\Phi_{i_1\dots\widehat{i_h}\dots i_{k+1}} = 0$ on Γ_{i_h} for all $(k+1)$-tuple $i_1 < i_2 < \cdots < i_{k+1}$ and all $h \in \{1, 2, \dots, k+1\}$. Or this is equivalent with $\partial_j \Phi_{i_1\dots i_k} = 0$ on Γ_j for all $j \notin \{i_1, \dots, i_k\}$. □

For any $1 \leq p \leq +\infty$, we define the spaces

$$W^{1,p}_*(\omega; \Lambda^k) := \{\phi \in W^{1,p}(\omega) \ ; \ *\phi = 0 \ \text{ on } \partial\omega\},$$
$$W^{2,p}_*(\omega; \Lambda^k) := \{\phi \in W^{2,p}(\omega) \ ; \ *\phi = 0 \ \text{ and } \ * d\phi = 0 \ \text{ on } \ \partial\omega\},$$

and we use the convention of notation

$$H^1_*(\omega; \Lambda^k) := W^{1,2}_*(\omega; \Lambda^k).$$

Remark 3.5. With the notation of Theorem 2.9, Lemma 3.4(i) states that $\phi := \sum_{i_1 < \cdots < i_k} \Phi_{i_1\dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in W^{1,p}_*(\omega; \Lambda^k)$ if and only if all the $q\ell$ components of each coordinate $\Phi_{i_1\dots i_k}$ belong to the space $V_p(\omega)$ associated with the indices $\{i_1, \dots, i_k\}$.

Note that for $k \geq 1$, the space $H^1_*(\omega; \Lambda^k)$ endowed with the scalar product

$$(\alpha, \beta) \mapsto \int_\omega \nabla\alpha \cdot \nabla\beta \, dx$$

is a Hilbert space. The norm associated with this scalar product, which is denoted $\|\cdot\|_{H^1_*(\omega;\Lambda^k)}$, is equivalent with the L^2-norm of the gradient, since we have, for all $\alpha \in H^1_*(\omega; \Lambda^k)$,

$$\|\alpha\|_{H^1_*(\omega;\Lambda^k)} \leq \|\nabla\alpha\|_{L^2(\omega;\Lambda^k)} \leq \sqrt{\mathrm{d}q\ell \binom{\mathrm{d}}{k}} \, \|\alpha\|_{H^1_*(\omega;\Lambda^k)} \tag{3.2}$$

where $\binom{\mathrm{d}}{k} := \frac{\mathrm{d}!}{k!(\mathrm{d}-k)!}$.

For any $1 \leq p < +\infty$, we also define the space $W_p(\omega; \Lambda^k)$ as the closure of $\mathcal{C}^1(\overline{\omega}; \Lambda^k)$ in the space $\{\phi \in L^p(\omega) \ ; \ d\phi \in L^p(\omega)\}$ with respect to the norm

$$\|\phi\| := \|\phi\|_{L^p(\omega)} + \|d\phi\|_{L^p(\omega)}.$$

Remark 3.6. Using the fact that ω is a cuboid, one can show that

$$W_p(\omega;\Lambda^k) = \{\phi \in L^p(\omega)\ ;\ d\phi \in L^p(\omega)\}.$$

Note that in the case of a general domain ω, the space $W_p(\omega;\Lambda^k)$ may be strictly contained in the space $\{\phi \in L^p(\omega)\ ;\ d\phi \in L^p(\omega)\}$.

As previously convened for spaces of forms, we will use the shorter notation $W_*^{1,p}(\omega)$ and $W_p(\omega)$ when the degree of the forms in these spaces is unambiguously deduced from the context. Note that, by Theorem 2.6(i), $W_*^{1,p}(\omega) \subset W^{1,p}(\omega) \subset W_p(\omega)$ and $W^{1,p}(\omega;\Lambda^0) = W_p(\omega;\Lambda^0)$.

The next theorem establishes an inequality of Sobolev type for forms in the space $H_*^1(\omega;\Lambda^k)$, the proof of which is deduced from the corresponding inequality on functions (see Theorem 2.4) by applying the latter to each component of each coordinate (which is a matrix field) of any k-form α in $H_*^1(\omega;\Lambda^k)$:

Theorem 3.7. *Let $p > \mathrm{d}$ and $k \geq 1$ and let $\omega \subset \mathbb{R}^{\mathrm{d}}$ be an open cube with edges of length $R > 0$. There exists a constant $C_2 > 0$ depending only on (d, p) such that*

$$\|\alpha\|_{L^{2p/(p-2)}(\omega)} \leq C_2 R^{1-\mathrm{d}/p} \|\nabla \alpha\|_{L^2(\omega)}$$

for all $\alpha \in H_*^1(\omega;\Lambda^k)$.

In the remainder of this section, k is a fixed, but otherwise arbitrary, integer in $\{1, 2, \dots, \mathrm{d}\}$.

The following two theorems and corollary establish basic formulas of integration in the framework of differential forms. We start with a formula of integration by parts, whose expression explains in particular why the operator δ is called *codifferential.* Note that this formula holds true in the particular, but important, case where one of forms involved has compact support in Ω.

Theorem 3.8. *Let $q > 1$ and $1 \leq p, r < +\infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and*

$$\begin{aligned} \frac{1}{r} + \frac{1}{q} \leq 1 + \frac{1}{\mathrm{d}} \qquad & \textit{if} \quad q \neq \mathrm{d}, \\ r > 1 \qquad & \textit{if} \quad q = \mathrm{d}. \end{aligned}$$

Then, for all $\alpha \in W^{1,r}(\omega;\Lambda^{k-1}) \cup W_p(\omega;\Lambda^{k-1})$ and all $\beta \in W_*^{1,q}(\omega;\Lambda^k)$,

$$\int_\omega d\alpha \cdot \beta \, dx = \int_\omega \alpha \cdot \delta\beta \, dx\,. \tag{3.3}$$

Proof. Note that the integrals in (3.3) are well defined in the case where $\alpha \in W^{1,r}(\omega;\Lambda^{k-1})$, thanks to the appropriate Sobolev embeddings applied to each component of each coordinate of the two forms.

It suffices to prove formula (3.3) for $\alpha \in \mathcal{C}^1(\overline{\omega};\Lambda^{k-1})$ and $\beta \in W_*^{1,q}(\omega;\Lambda^k)$, since the space $\mathcal{C}^1(\overline{\omega};\Lambda^{k-1})$ is dense in both spaces $W_p(\omega;\Lambda^{k-1})$ and $W^{1,r}(\omega;\Lambda^{k-1})$.

Using the definition of $d\alpha$ and the Stokes formula, we have on one hand

$$\begin{aligned}\int_\omega d\alpha\cdot\beta\,dx &= \int_\omega \sum_{i_1<\dots<i_k}\left(\sum_{p=1}^{k}(-1)^{p-1}\partial_{i_p}A_{i_1\dots\widehat{i_p}\dots i_k}\right)\cdot B_{i_1\dots i_k}\,dx\\ &= \sum_{i_1<\dots<i_k}\sum_{p=1}^{k}\int_\omega(-1)^{p-1}\partial_{i_p}A_{i_1\dots\widehat{i_p}\dots i_k}\cdot B_{i_1\dots i_k}\,dx\\ &= \sum_{i_1<\dots<i_k}\sum_{p=1}^{k}\Bigg(\int_\omega(-1)^{p}A_{i_1\dots\widehat{i_p}\dots i_k}\cdot\partial_{i_p}B_{i_1\dots i_k}\,dx\\ &\quad+\int_{\partial\omega}(-1)^{p-1}A_{i_1\dots\widehat{i_p}\dots i_k}\cdot B_{i_1\dots i_k}\nu_{i_p}\,d\sigma\Bigg)\\ &= \int_\omega\sum_{i_1<\dots<i_k}\sum_{p=1}^{k}(-1)^{p}A_{i_1\dots\widehat{i_p}\dots i_k}\cdot\partial_{i_p}B_{i_1\dots i_k}\,dx\\ &\quad+\sum_{i_1<\dots<i_k}\sum_{p=1}^{k}\int_{\partial\omega}(-1)^{p-1}A_{i_1\dots\widehat{i_p}\dots i_k}\cdot B_{i_1\dots i_k}\nu_{i_p}\,d\sigma.\end{aligned}$$

On the other hand, the integrand of the first term of the right hand side satisfies

$$\begin{aligned}&\sum_{i_1<\dots<i_k}\sum_{p=1}^{k}(-1)^{p}A_{i_1\dots\widehat{i_p}\dots i_k}\cdot\partial_{i_p}B_{i_1\dots i_k}\\ &\qquad=\sum_{i_1<\dots<i_{k-1}}\Big(A_{i_1\dots i_{k-1}}\cdot\sum_{j\neq i_r}(-1)^p\partial_jB_{i_1\dots i_{p-1}ji_p\dots i_{k-1}}\Big)=\alpha\cdot\delta\beta\,,\end{aligned}$$

(in the last sum, p is the unique index that satisfies $i_{p-1}<j<i_p$) and the second term satisfies

$$\begin{aligned}&\sum_{i_1<\dots<i_k}\sum_{p=1}^{k}\int_{\partial\omega}(-1)^{p-1}A_{i_1\dots\widehat{i_p}\dots i_k}\cdot B_{i_1\dots i_k}\nu_{i_p}\,d\sigma\\ &\qquad=\sum_{i_1<\dots<i_k}\sum_{p=1}^{k}(-1)^{p-1}\int_{\Gamma_{i_p}}A_{i_1\dots\widehat{i_p}\dots i_k}\cdot B_{i_1\dots i_k}\nu_{i_p}\,d\sigma=0,\end{aligned}$$

since $\nu_{i_p}=0$ on $\partial\omega\setminus\Gamma_{i_p}$ and $B_{i_1\dots i_k}=0$ on Γ_{i_p} (by Lemma 3.4(*i*)). Therefore the equality previously obtained by Stokes formula shows that the formula (3.3) is true for all $\alpha\in\mathcal{C}^1(\overline{\omega};\Lambda^{k-1})$ and $\beta\in W^{1,q}_*(\omega;\Lambda^k)$. □

Remarks 3.9. 1. The formula (3.3) is well-known in the Hodge Theory in the case of compactly supported forms.

2. Another proof of Theorem 3.8 uses the formula

$$\int_\omega\alpha\cdot\delta\beta\,dx=\int_\omega d\alpha\cdot\beta\,dx-\int_{\partial\omega}i^*(\alpha)\wedge i^*(*\beta)$$

for scalar forms (the formula (3.3) for matrix-valued forms is then simply obtained by summation), itself a consequence of the Stokes formula on manifolds combined with the following formulas

$$\begin{aligned}
** &= (-1)^{k(\mathrm{d}-k)}\mathrm{Id} \quad \text{on} \quad \Lambda^k, \\
\delta &= (-1)^{\mathrm{d}(k+1)+1} * d * \quad \text{on} \quad \Lambda^k, \\
\phi \cdot \eta &= \phi \wedge *\eta,
\end{aligned}$$

where Id is the identity mapping on Λ^k and i^* is the pull-back of the canonical inclusion $i : \partial\omega \hookrightarrow \overline{\omega}$ (see, e.g., Bleecker [3] for further details).

The next theorem establishes a formula that will be crucial in proving the last theorem of this Section, which in turn will be used extensively in the proof of the Theorem 5.2.

Theorem 3.10. *Let $\alpha \in W^{1,p}_*(\omega;\Lambda^k)$ and $\beta \in W^{1,q}_*(\omega;\Lambda^k)$, where $p,q \geq 1$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then*

$$\int_\omega \nabla\alpha \cdot \nabla\beta \, dx = \int_\omega (d\alpha \cdot d\beta + \delta\alpha \cdot \delta\beta) \, dx. \tag{3.4}$$

Proof. Since $\frac{1}{p} + \frac{1}{q} = 1$, we have either $p < +\infty$ or $q < +\infty$. Suppose $p < +\infty$. In view of Theorem 2.6(ii) and Lemma 3.4(i), we can approximate the form α (with respect to the $W^{1,p}(\omega)$-norm) by a sequence of forms $\alpha^n \in \mathcal{C}^2(\overline{\omega})$ such that $*\alpha^n = 0$ on $\partial\omega$. Hence it suffices to prove the formula (3.4) for forms $\alpha = \sum_{i_1<\cdots<i_k} A_{i_1\ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $\beta = \sum_{i_1<\cdots<i_k} B_{i_1\ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, where $A_{i_1\ldots i_k} \in \mathcal{C}^2(\overline{\omega})$, $B_{i_1\ldots i_k} \in W^{1,q}(\omega)$, and

$$A_{i_1\ldots i_k} = B_{i_1\ldots i_k} = 0 \quad \text{on} \;\; \Gamma_{i_1} \cup \cdots \cup \Gamma_{i_k}. \tag{3.5}$$

From the definition of the operators d and δ, we first have

$$\begin{aligned}
&\int_\omega d\alpha \cdot d\beta \, dx \\
&= \int_\omega \sum_{i_1<\cdots<i_{k+1}} \Big(\sum_{p=1}^{k+1} (-1)^{p-1} \partial_{i_p} A_{i_1\ldots\widehat{i_p}\ldots i_{k+1}} \Big) \cdot \Big(\sum_{s=1}^{k+1} (-1)^{s-1} \partial_{i_s} B_{i_1\ldots\widehat{i_s}\ldots i_{k+1}} \Big) \, dx, \\
&\int_\omega \delta\alpha \cdot \delta\beta \, dx \\
&= \int_\omega \sum_{j_1<\cdots<j_{k-1}} \Big(\sum_{t \neq j_r} (-1)^{p'} \partial_t A_{j_1\ldots j_{p'-1} t j_{p'} \ldots j_{k-1}} \Big) \cdot \Big(\sum_{l \neq j_r} (-1)^{s'} \partial_l B_{j_1\ldots j_{s'-1} l j_{s'} \ldots j_{k-1}} \Big) \, dx.
\end{aligned}$$

By ordering the sums of products appearing in the above two formulas according to the fact that $p = s$ or not, and respectively $t = l$ or not, we obtain

(3.6)
$$\begin{aligned}
&\int_\omega d\alpha \cdot d\beta \, dx \\
&\qquad = \int_\omega \sum_{i_1<\cdots<i_{k+1}} \Big(\sum_{p=1}^{k+1} \partial_{i_p} A_{i_1\ldots\widehat{i_p}\ldots i_{k+1}} \cdot \partial_{i_p} B_{i_1\ldots\widehat{i_p}\ldots i_{k+1}} \Big) \, dx \\
&\qquad + \int_\omega \sum_{i_1<\cdots<i_{k+1}} \Big(\sum_{p\neq s} (-1)^{p+s} \partial_{i_p} A_{i_1\ldots\widehat{i_p}\ldots i_{k+1}} \cdot \partial_{i_s} B_{i_1\ldots\widehat{i_s}\ldots i_{k+1}} \Big) \, dx,
\end{aligned}$$

(3.7)
$$\begin{aligned}
&\int_\omega \delta\alpha \cdot \delta\beta \, dx \\
&\qquad = \int_\omega \sum_{j_1<\cdots<j_{k-1}} \Big(\sum_{t\neq j_r} \partial_t A_{j_1\ldots j_{p'-1} t j_{p'} \ldots j_{k-1}} \cdot \partial_t B_{j_1\ldots j_{p'-1} t j_{p'} \ldots j_{k-1}} \Big) \\
&\qquad + \int_\omega \sum_{j_1<\cdots<j_{k-1}} \Big(\sum_{\substack{t,l\neq j_r \\ t\neq l}} (-1)^{p'+s'} \partial_t A_{j_1\ldots j_{p'-1} t j_{p'} \ldots j_{k-1}} \cdot \partial_l B_{j_1\ldots j_{s'-1} l j_{s'} \ldots j_{k-1}} \Big) \, dx.
\end{aligned}$$

Note that the sum of the first integrals of the right hand sides of these last two formulas is exactly $\int_\omega \nabla\alpha \cdot \nabla\beta \, dx$. So in order to prove the Theorem, it remains to prove that the sum of the second integrals of the right hand sides vanishes.

Let us fix $i_1 < \cdots < i_{k+1}$ and $p, s \in \{1, \ldots, k+1\}$, $p \neq s$. By applying the Stokes theorem twice, we have

$$\begin{aligned}
&\int_\omega \partial_{i_p} A_{i_1\ldots\widehat{i_p}\ldots i_{k+1}} \cdot \partial_{i_s} B_{i_1\ldots\widehat{i_s}\ldots i_{k+1}} \, dx \\
&\qquad = -\int_\omega \partial_{i_p i_s} A_{i_1\ldots\widehat{i_p}\ldots i_{k+1}} \cdot B_{i_1\ldots\widehat{i_s}\ldots i_{k+1}} \, dx \\
&\qquad\quad + \int_{\partial\omega} \partial_{i_p} A_{i_1\ldots\widehat{i_p}\ldots i_{k+1}} \cdot B_{i_1\ldots\widehat{i_s}\ldots i_{k+1}} \nu_{i_s} \, d\sigma \\
&\qquad = \int_\omega \partial_{i_s} A_{i_1\ldots\widehat{i_p}\ldots i_{k+1}} \cdot \partial_{i_p} B_{i_1\ldots\widehat{i_s}\ldots i_{k+1}} \, dx \\
&\qquad\quad - \int_{\partial\omega} \partial_{i_s} A_{i_1\ldots\widehat{i_p}\ldots i_{k+1}} \cdot B_{i_1\ldots\widehat{i_s}\ldots i_{k+1}} \nu_{i_p} \, d\sigma \\
&\qquad\quad + \int_{\partial\omega} \partial_{i_p} A_{i_1\ldots\widehat{i_p}\ldots i_{k+1}} \cdot B_{i_1\ldots\widehat{i_s}\ldots i_{k+1}} \nu_{i_s} \, d\sigma.
\end{aligned}$$

But

$$\begin{aligned}
&\int_{\partial\omega} \partial_{i_s} A_{i_1\ldots\widehat{i_p}\ldots i_{k+1}} \cdot B_{i_1\ldots\widehat{i_s}\ldots i_{k+1}} \nu_{i_p} \, d\sigma \\
&\qquad = \int_{\Gamma_{i_p}} \partial_{i_s} A_{i_1\ldots\widehat{i_p}\ldots i_{k+1}} \cdot B_{i_1\ldots\widehat{i_s}\ldots i_{k+1}} \nu_{i_p} \, d\sigma = 0,
\end{aligned}$$

since $\nu_{i_p} = 0$ on $\partial\omega \setminus \Gamma_{i_p}$ and $B_{i_1\dots\widehat{i_s}\dots i_{k+1}} = 0$ on Γ_{i_p}, and

$$\int_{\partial\omega} \partial_{i_p} A_{i_1\dots\widehat{i_p}\dots i_{k+1}} \cdot B_{i_1\dots\widehat{i_s}\dots i_{k+1}} \nu_{i_s}\, d\sigma = \int_{\Gamma_{i_s}} \partial_{i_p} A_{i_1\dots\widehat{i_p}\dots i_{k+1}} \cdot B_{i_1\dots\widehat{i_s}\dots i_{k+1}} \nu_{i_s}\, d\sigma = 0,$$

since $\nu_{i_s} = 0$ on $\partial\omega \setminus \Gamma_{i_s}$ and $\partial_{i_p} A_{i_1\dots\widehat{i_p}\dots i_{k+1}} = 0$ on Γ_{i_s}. That $B_{i_1\dots\widehat{i_s}\dots i_{k+1}} = 0$ on Γ_{i_p} is a consequence of (3.5), since $i_p \in \{i_1, \dots, i_{k+1}\} \setminus \{i_s\}$ (remember that we have fixed $p \neq s$). For similar reasons, $A_{i_1\dots\widehat{i_p}\dots i_{k+1}} = 0$ on Γ_{i_s}, which implies that $\partial_j A_{i_1\dots\widehat{i_p}\dots i_{k+1}} = 0$ on Γ_{i_s} for all $j \neq i_s$, and in particular for $j = i_p$ as was needed to obtain the second formula.

Summing up the above arguments, we have proved that

$$\int_\omega \partial_{i_p} A_{i_1\dots\widehat{i_p}\dots i_{k+1}} \cdot \partial_{i_s} B_{i_1\dots\widehat{i_s}\dots i_{k+1}}\, dx = \int_\omega \partial_{i_s} A_{i_1\dots\widehat{i_p}\dots i_{k+1}} \cdot \partial_{i_p} B_{i_1\dots\widehat{i_s}\dots i_{k+1}}\, dx\,.$$

Now let $t := i_s$, $l := i_p$, and $j_1 < \cdots < j_{k-1}$ such that $\{j_1, \dots, j_{k-1}\} = \{i_1, \dots, i_{k+1}\} \setminus \{i_s, i_p\}$. Using this notation in the right-hand side of the above equation, we deduce that

$$\int_\omega \partial_{i_p} A_{i_1\dots\widehat{i_p}\dots i_{k+1}} \cdot \partial_{i_s} B_{i_1\dots\widehat{i_s}\dots i_{k+1}}\, dx = \int_\omega \partial_t A_{j_1\dots j_{p'-1} t j_{p'} \dots j_{k-1}} \cdot \partial_l B_{j_1 \dots j_{s'-1} l j_{s'} \dots j_{k-1}}\, dx,$$

where p' and s' are the indices for which $j_{p'-1} < t < j_{p'}$ and $j_{s'-1} < l < j_{s'}$.

If $p < s$, then $i_p \in \{i_1, \dots, i_{s-1}\}$ and $i_s \notin \{i_1, \dots, i_p\}$; therefore we have in that case $p' = s - 1$ and $s' = p$. Otherwise, if $p > s$, then $i_p > i_s$, and we have in that case $p' = s$ and $s' = p - 1$.

Whatever the case, we have $p + s = p' + s' + 1$ and the last relation implies that

$$(-1)^{p+s} \int_\omega \partial_{i_p} A_{i_1\dots\widehat{i_p}\dots i_{k+1}} \cdot \partial_{i_s} B_{i_1\dots\widehat{i_s}\dots i_{k+1}}\, dx = -(-1)^{p'+s'} \int_\omega \partial_t A_{j_1\dots j_{p'-1} t j_{p'} \dots j_{k-1}} \cdot \partial_l B_{j_1 \dots j_{s'-1} l j_{s'} \dots j_{k-1}}\, dx.$$

Note that summing with respect to $i_1 < \cdots < i_{k+1}$ and $p \neq s$ in the left hand side of this equation corresponds to summing with respect to $j_1 < \cdots < j_{k-1}$ and $t, l \notin \{j_1, \dots, j_{k-1}\}$, $t \neq l$, in the right hand side. This observation shows that the sum of the second integrals of the right hand sides of the equations (3.6) and (3.7) vanishes. This completes the proof of the theorem.

□

Remark 3.11. The proof of Theorems 3.8 and 3.10 shows that these theorems still hold true if one of the forms belongs to $\mathcal{D}(\omega)$ (the space of $\mathcal{C}^\infty$-forms with compact support in ω) while the other form belongs to $W^{1,1}_{\text{loc}}(\omega)$. Therefore, if $\alpha \in W^{2,1}_{\text{loc}}(\omega; \Lambda^k)$, we have

$$\int_\omega (-\Delta\alpha)\cdot\beta\, dx = \int_\omega \nabla\alpha\cdot\nabla\beta\, dx = \int_\omega (d\alpha\cdot d\beta + \delta\alpha\cdot\delta\beta)\, dx = \int_\omega (\delta d + d\delta)\alpha\cdot\beta\, dx \tag{3.8}$$

for all $\beta \in \mathcal{D}(\omega; \Lambda^k)$, hence that

$$-\Delta = \delta d + d\delta \text{ on } W^{2,1}_{\text{loc}}(\omega; \Lambda^k).$$

In this way, we retrieved the equality $-\Delta = \delta d + d\delta$ of Theorem 3.2, which otherwise can be established solely by algebraic calculus (see, e.g., Jost [10]).

The next Corollary asserts that the first equality of (3.8) still holds if the two forms belong to $W^{m,p}_*(\omega; \Lambda^k)$-type spaces. The proof given below is based on the the identity $-\Delta = \delta d + d\delta$ (derived algebraically) and on the Theorems 3.8 and 3.10. A more elementary proof uses Lemma 3.4 to derive the corresponding formula for the coefficients of the two forms (the sought equality is then simply obtained by summation). This second proof is not given here.

Corollary 3.12. *Let $1 < r, q < +\infty$ such that $\frac{1}{r} + \frac{1}{q} \leq 1 + \frac{1}{\mathrm{d}}$. Then, for all $\alpha \in W^{2,r}_*(\omega; \Lambda^k)$ and all $\beta \in W^{1,q}_*(\omega; \Lambda^k)$,*

$$\int_\omega (-\Delta\alpha) \cdot \beta \, dx = \int_\omega \nabla\alpha \cdot \nabla\beta \, dx \, .$$

Proof. By using the formula $-\Delta = d\delta + \delta d$ and the Theorems 3.8 and 3.10, we deduce that

$$\begin{aligned}\int_\omega (-\Delta\alpha) \cdot \beta \, dx &= \int_\omega d\delta\alpha \cdot \beta \, dx + \int_\omega \delta d\alpha \cdot \beta \, dx \\ &= \int_\omega \delta\alpha \cdot \delta\beta \, dx + \int_\omega d\alpha \cdot d\beta \, dx \\ &= \int_\omega \nabla\alpha \cdot \nabla\beta \, dx.\end{aligned}$$

The second equality was obtained by applying Theorem 3.8 twice, first for the couple of forms $(\delta\alpha, \beta) \in W^{1,r}(\omega; \Lambda^{k-1}) \times W^{1,q}_*(\omega; \Lambda^k)$, then for $(\beta, d\alpha) \in W^{1,q}(\omega; \Lambda^k) \times W^{1,r}_*(\omega; \Lambda^{k+1})$, while the third equality was obtained by applying Theorem 3.10 for $(\alpha, \beta) \in W^{1,p}_*(\omega; \Lambda^k) \times W^{1,q}_*(\omega; \Lambda^k)$. Note that the form α belongs to the space $W^{1,p}_*(\omega; \Lambda^k)$ thanks to the Sobolev embedding $W^{2,r}(\omega) \subset W^{1,p}(\omega)$, where p is defined by $\frac{1}{p} + \frac{1}{q} = 1$. □

We end this section with the following theorem which establishes two remarkable properties of the solution to a variational equation defined over spaces of forms:

Theorem 3.13. *Let $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $\alpha \in W^{1,p}_*(\omega; \Lambda^k)$. Let there be given a k-form $\phi \in L^r(\omega; \Lambda^k)$, where $r \in (1, +\infty)$ satisfies $\frac{1}{r} + \frac{1}{q} \leq 1 + \frac{1}{\mathrm{d}}$. Suppose that the form α satisfies*

$$\int_\omega \nabla\alpha \cdot \nabla\beta \, dx = \int_\omega \phi \cdot \beta \, dx \quad \text{for all } \beta \in W^{1,q}_*(\omega; \Lambda^k) \, . \tag{3.9}$$

Then $\alpha \in W^{2,r}_*(\omega; \Lambda^k)$.

Furthermore, if $\phi \in W_r(\omega; \Lambda^k)$, then the form $d\alpha$, which belongs to the space $W^{1,r}_*(\omega; \Lambda^{k+1})$, satisfies the variational equation:

$$\int_\omega \nabla(d\alpha) \cdot \nabla\beta \, dx = \int_\omega d\phi \cdot \beta \, dx \quad \text{for all } \beta \in W^{1,s}_*(\omega; \Lambda^{k+1}) \, , \tag{3.10}$$

where s is defined by $\dfrac{1}{r} + \dfrac{1}{s} = 1$.

Proof. It is clear that the form α satisfies equation (3.9) if and only if each component of each coordinate $A_{i_1\dots i_k}$ of α satisfies the same type of equation, but where β is replaced with a function in $W^{1,q}(\omega)$ that vanishes on $\Gamma_{i_1}\cup\dots\cup\Gamma_{i_k}$ and the scalar product is replaced by the usual product in $\mathbb{R}$. More specifically, if u denotes the component of $A_{i_1\dots i_k}$ at the i-th row and j-th column, if f denotes the component of $\Phi_{i_1\dots i_k}$ at the i-th row and j-th column, then $u\in V_p(\omega)$ (see Remark 3.5) and

$$\int_\omega \nabla u\cdot\nabla v\,dx=\int_\omega fv\,dx \quad \text{ for all }\ v\in V_q(\omega)\,,$$

where the spaces $V_p(\omega)$ and $V_q(\omega)$ are those defined in the statement of Theorem 2.9 for the choice of indices $\{i_1,\dots,i_k\}$.

By applying Theorem 2.9 to this equation, we deduce that $\alpha\in W^{2,r}(\omega;\Lambda^k)$ and that

$$\partial_j A_{i_1\dots i_k}=0 \ \text{ on }\ \Gamma_j \quad \text{ for all }\ j\notin\{i_1,\dots,i_k\}. \tag{3.11}$$

Since we know that

$$A_{i_1\dots i_k}=0 \ \text{ on } \Gamma_{i_1}\cup\dots\cup\Gamma_{i_k},$$

Lemma 3.4(ii) shows that $*d\alpha=0$ on ω. Hence $\alpha\in W^{2,r}_*(\omega;\Lambda^k)$.

We now prove that $d\alpha$ satisfies the variational equation (3.10). By the the same density arguments as those used in the proof of Theorem 3.10, it suffices to prove that $d\alpha$ satisfies the variational equation (3.10) for all forms

$$\beta=\sum_{i_1<\dots<i_{k+1}} B_{i_1\dots i_{k+1}}dx^{i_1}\wedge\dots\wedge dx^{i_{k+1}}\in\mathcal{C}^2(\overline{\omega};\Lambda^{k+1})$$

that satisfy $*\beta=0$ on $\partial\omega$.

The first observation is that $\delta\beta\in\mathcal{C}^1(\overline{\omega},\Lambda^k)$ and $*\delta\beta=0$ on $\partial\omega$. Indeed, for any fixed indices $i_1<\dots<i_k$ and $l\in\{1,\dots,k\}$, we have that $B_{i_1\dots i_{r-1}ji_r\dots i_k}=0$ on Γ_{i_l} for all $j\notin\{i_1,\dots,i_k\}$. Hence $\partial_t B_{i_1\dots i_{r-1}ji_r\dots i_k}=0$ on Γ_{i_l} for all $t\neq i_l$, and in particular for all $t=j\notin\{i_1,\dots,i_k\}$. Therefore,

$$\sum_{j\neq i_s}(-1)^r\partial_j B_{i_1\dots i_{r-1}ji_r\dots i_k}=0 \quad \text{ on }\ \Gamma_{i_l}\,,$$

which means that $*\delta\beta=0$ on ω (see Lemma 3.4(i)). In particular, $\delta\beta\in W^{1,q}_*(\omega;\Lambda^k)\cap W^{1,s}_*(\omega;\Lambda^k)$ and $\beta\in W^{1,s}_*(\omega;\Lambda^{k+1})$.

This observation allows us to apply the Theorems 3.8 and 3.10, together with the identities $dd=0$ and $\delta\delta=0$, to obtain the following relations:

$$\begin{aligned}\int_\omega \nabla(d\alpha)\cdot\nabla\beta\,dx&=\int_\omega(dd\alpha\cdot d\beta+\delta d\alpha\cdot\delta\beta)\,dx=\int_\omega\delta d\alpha\cdot\delta\beta\,dx\\ &=\int_\omega d\alpha\cdot d\delta\beta\,dx=\int_\omega(d\alpha\cdot d\delta\beta+\delta\alpha\cdot\delta\delta\beta)\,dx\\ &=\int_\omega\nabla\alpha\cdot\nabla(\delta\beta)\,dx.\end{aligned}$$

But $\delta\beta$ belongs to the space $W^{1,q}_*(\omega;\Lambda^k)$, so it can be used in the variational equation (3.9) satisfied by α. Therefore the last equality implies that

$$\int_\omega\nabla(d\alpha)\cdot\nabla\beta\,dx=\int_\omega\phi\cdot\delta\beta\,dx.$$

Using again Theorem 3.8 (which can be applied to the last integral of the above inequality since $\phi \in W_r(\omega; \Lambda^k)$ by assumption), we finally deduce that

$$\int_\omega \nabla(d\alpha) \cdot \nabla\beta \, dx = \int_\omega d\phi \cdot \beta \, dx \, .$$

$\square$

Remark 3.14. Note that the hypotheses of Theorem 3.13 are satisfied if $1 < p \leq r < +\infty$ or if $p = 2$ and $1 \leq \frac{\mathrm{d}}{2} < r < +\infty$. These two special cases occur in Section 5.

4. Stability and uniqueness

Besides its interest on its own, the stability and uniqueness results of this section, which were first established in [13], are needed to establish the existence of a solution to the Pfaff system (see Section 6). We first show that small perturbations in the L^p-norm of the coefficients of the Pfaff system and of its "initial data" induce small perturbations of its solution (in the Fréchet space $W^{1,p}_{\mathrm{loc}}$):

Theorem 4.1. *Let Ω be a connected open subset of $\mathbb{R}^{\mathrm{d}}$, let $p > \mathrm{d}$, and let there be given sequences of matrix fields $\big(A_i^n\big)_{n\in\mathbb{N}} \subset L^p(\Omega; \mathbb{M}^\ell)$ for $i \in \{1, 2, \dots, \mathrm{d}\}$, and $\big(Y^n\big)_{n\in\mathbb{N}} \subset W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^{q\times\ell})$ that satisfy the Pfaff systems*

$$\partial_i Y^n = Y^n A_i^n \quad \text{in } \Omega \ \text{ for all } i$$

in the distributional sense. Fix a point $x^0 \in \Omega$ and assume that there exists a constant $M > 0$ such that

$$|Y^n(x^0)| + \sum_i \|A_i^n\|_{L^p(\Omega)} \leq M \ \text{ for all } n \in \mathbb{N}.$$

Then, for any open set $K \Subset \Omega$, there exist a constant $C > 0$ (depending on M) such that

$$\|Y^n - Y^m\|_{W^{1,p}(K)} \leq C \left\{ |Y^n(x^0) - Y^m(x^0)| + \sum_i \|A_i^n - A_i^m\|_{L^p(\Omega)} \right\}$$

for all $n, m \in \mathbb{N}$.

Proof. Without losing in generality, we may consider that K is a connected set containing x^0. To see this, it suffices to prove that, for any open set $K \Subset \Omega$, there exists a connected open set Ω' such that $\overline{K} \cup \{x^0\} \subset \Omega' \Subset \Omega$.

In order to construct such a set Ω', we consider a countable covering with open balls of Ω, i.e.,

$$\Omega = \bigcup_{k=1}^{+\infty} B^k,$$

such that $\overline{B^k} \subset \Omega$ for all k, where $B^k := B^k(x^k)$ is an open ball centered at the point $x^k \in \Omega$. Such a covering exists thanks to the Lindelöf Theorem, which states that in a topological space having a countable basis for its topology, any open covering of a set contains a countable subcovering.

Then we construct a sequence of connected sets $\Omega_k \subset \Omega_{k+1} \Subset \Omega$ such that $\Omega = \cup_{k=1}^{+\infty} \Omega_k$ in the following way:

We start with $\Omega_1 := B^1$. Then we fix a continuous path $\gamma \subset \Omega$ joining x^1 with x^2 and a finite covering of γ with balls B^k (covering which always exists since γ is compact), say $\gamma \subset \cup_{j\in J} B^j$, where J is a finite subset of $\mathbb{N}$, such that $\gamma \cap B^j \neq \emptyset$ for all $j \in J$. Then we define

$$\Omega_2 := \Omega_1 \cup \Big(\bigcup_{j\in J} B^j\Big) \cup B^2.$$

By repeating this process, we construct a sequence of connected open sets Ω_k, defined as finite unions of some balls B^j, that satisfy $B^k \subset \Omega_k \subset \Omega_{k+1}$. Hence $\overline{\Omega}_k \subset \Omega$ and $\Omega = \cup_{k=1}^{+\infty} \Omega_k$.

It is now clear how to construct the set Ω': since $\overline{K} \cup \{x^0\}$ is a compact subset of Ω, there exists a number $k_0 \in \mathbb{N}$ such that $\overline{K} \cup \{x^0\} \subset \Omega_{k_0}$; hence we can choose $\Omega' = \Omega_{k_0}$.

In the remainder of the proof, K is a connected open set containig x^0. Let $R > 0$ be fixed such that

$$R < \mathrm{dist}(K, \Omega^c) \quad \text{and} \quad C_1 R^{1-\mathrm{d}/p} \leq \frac{1}{2M},$$

where C_1 is the constant appearing in the inequality (2.3) of Corollary 2.3.

Fix any open ball $B_R = B_R(x) \Subset \Omega$ with center $x \in \overline{K}$. We begin by establishing an upper bound for the L^∞-norm of Y^m over B_R. Using Morrey's inequality (see Corollary 2.3), we have

$$\begin{aligned}
\|Y^m\|_{L^\infty(B_R)} &\leq |Y^m(x)| + C_1 R^{1-\mathrm{d}/p} \sum_i \|\partial_i Y^m\|_{L^p(B_R)} \\
&\leq |Y^m(x)| + \frac{1}{2M} \sum_i \|Y^m A_i^m\|_{L^p(B_R)} \\
&\leq |Y^m(x)| + \frac{1}{2} \|Y^m\|_{L^\infty(B_R)},
\end{aligned}$$

from which we deduce that

$$\|Y^m\|_{L^\infty(B_R)} \leq 2|Y^m(x)|.$$

It is clear that any point $x \in \overline{K}$ can be joined to x^0 by a broken line formed by maximum N segments with ends in $\overline{K}$ and lengths $< R$, where the number N depends only on x^0, K and R. Therefore, using the previous inequality several times shows that

$$\|Y^m\|_{L^\infty(B_R)} \leq 2^{N+1} |Y^m(x^0)| \leq 2^{N+1} M. \tag{4.1}$$

We now estimate the L^∞-norm of $(Y^n - Y^m)$ and the L^p-norm of $\partial_i(Y^n - Y^m)$ over B_R. From the relation

$$\partial_i(Y^n - Y^m) = (Y^n - Y^m)A_i^n + Y^m(A_i^n - A_i^m),$$

we first obtain

$$
\begin{aligned}
&\sum_i \|\partial_i(Y^n - Y^m)\|_{L^p(B_R)} \\
&\qquad \le M\|Y^n - Y^m\|_{L^\infty(B_R)} + \|Y^m\|_{L^\infty(B_R)} \sum_i \|A_i^n - A_i^m\|_{L^p(B_R)} \\
&\qquad \le M\|Y^n - Y^m\|_{L^\infty(B_R)} + 2^{N+1} M \sum_i \|A_i^n - A_i^m\|_{L^p(B_R)}.
\end{aligned} \tag{4.2}
$$

But Morrey's inequality (see Corollary 2.3) and the choice of the radius R show that

$$
\|Y^n - Y^m\|_{L^\infty(B_R)} \le |(Y^n - Y^m)(x)| + \frac{1}{2M} \sum_i \|\partial_i(Y^n - Y^m)\|_{L^p(B_R)}. \tag{4.3}
$$

Therefore, the inequality (4.2) yields

$$
\begin{aligned}
&\sum_i \|\partial_i(Y^n - Y^m)\|_{L^p(B_R)} \\
&\qquad\qquad \le 2M|(Y^n - Y^m)(x)| + 2^{N+2} M \sum_i \|A_i^n - A_i^m\|_{L^p(B_R)},
\end{aligned} \tag{4.4}
$$

which combined with the inequality (4.3) next yields

$$
\|Y^n - Y^m\|_{L^\infty(B_R)} \le 2|(Y^n - Y^m)(x)| + 2^{N+1} \sum_i \|A_i^n - A_i^m\|_{L^p(B_R)}. \tag{4.5}
$$

Let now the open set $K \Subset \Omega$ be covered with a finite number of balls of radius R centered in points that belong to $\overline{K}$. Let the center, say x, of any such ball be joined to the given point x^0 with a broken line formed by maximum N segments with ends in $\overline{K}$ and lengths $< R$ (the number N depends only on x^0, K and R). Then a recursion argument based on the inequality (4.5) shows that

$$
\|Y^n - Y^m\|_{L^\infty(B_R(x))} \le 2^{N+1}|(Y^n - Y^m)(x^0)| + (4^{N+1} - 2^{N+1}) \sum_i \|A_i^n - A_i^m\|_{L^p(\Omega)}.
$$

Combined with (4.4), this inequality further implies that

$$
\begin{aligned}
&\sum_i \|\partial_i(Y^n - Y^m)\|_{L^p(B_R)} \\
&\qquad\qquad \le 2^{N+2} M|(Y^n - Y^m)(x^0)| + 2^{2N+3} M \sum_i \|A_i^n - A_i^m\|_{L^p(\Omega)}.
\end{aligned}
$$

The last two inequalities show that there exists a constant $C > 0$, independent of n, m, such that

$$
\|Y^n - Y^m\|_{W^{1,p}(B_R(x))} \le C\Big(|(Y^n - Y^m)(x^0)| + \sum_i \|A_i^n - A_i^m\|_{L^p(\Omega)}\Big).
$$

Since this inequality is valid for any ball $B_R(x)$ in the chosen covering of K, summing all these inequalities yields the desired inequality. $\square$

An immediate consequence of the stability result of Theorem 4.1 is the following uniqueness result for Pfaff systems:

Theorem 4.2. *Let Ω be a connected open subset of $\mathbb{R}^{\mathrm{d}}$ and let $A_i \in L^p_{\mathrm{loc}}(\Omega; \mathbb{M}^\ell)$, $p > \mathrm{d}$. Then the Pfaff system*

$$\partial_i Y = Y A_i \quad \text{in } \Omega \quad \text{for all } i$$

has at most one solution in $W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^{q\times\ell})$ satisfying $Y(x^0) = Y^0$ for some $x^0 \in \Omega$ and $Y^0 \in \mathbb{M}^{q\times\ell}$.

5. Local regularization of Pfaff systems

This section is crucial towards establishing the existence of solutions to Pfaff systems with coefficients that are only of class L^p (see Section 6). It consists in establishing that a Pfaff system with L^p-coefficients can be approched locally with a sequence of Pfaff systems with smooth coefficients. More specifically, we have the following

Theorem 5.1. *Let Ω be an open subset of $\mathbb{R}^{\mathrm{d}}$ and let there be given d matrix fields $A_i \in L^p(\Omega; \mathbb{M}^\ell)$, $p > \mathrm{d}$, such that*

$$\partial_j A_i - \partial_i A_j = A_i A_j - A_j A_i \quad \text{for all } i, j \in \{1, 2, ..., \mathrm{d}\} \tag{5.1}$$

in the space of distributions $\mathcal{D}'(\Omega; \mathbb{M}^\ell)$.

Then there exists $R_0 > 0$, depending only on p, d, ℓ and $\|A_i\|_{L^p(\Omega)}$, with the following property: For any open cube $\omega \Subset \Omega$ whose edges have lengths $< R_0$, there exist sequences of matrix fields (A_i^ε) in $\mathcal{C}^\infty(\omega; \mathbb{M}^\ell) \cap L^p(\omega; \mathbb{M}^\ell)$ that satisfy the relations

$$\begin{aligned}
&\partial_j A_i^\varepsilon - \partial_i A_j^\varepsilon = A_i^\varepsilon A_j^\varepsilon - A_j^\varepsilon A_i^\varepsilon \quad \text{in } \omega \quad \text{for all } i, j,\\
&A_i^\varepsilon \to A_i \text{ in } L^p(\omega; \mathbb{M}^\ell) \text{ as } \varepsilon \to 0.
\end{aligned}$$

The proof of this theorem is best described in the framework of matrix-valued differential forms, of which a short presentation is given in Section 3. In this setting, Theorem 5.1 is expressed in the following equivalent form:

Theorem 5.2. *Let Ω be an open subset of $\mathbb{R}^{\mathrm{d}}$ and let $\alpha \in L^p(\Omega; \Lambda^1)$, $p > \mathrm{d}$, be a 1-form with matrix coefficients that satisfies in the distributional sense*

$$d\alpha + \alpha \wedge \alpha = 0 \quad \text{in } \omega. \tag{5.2}$$

Then there exists $R_0 > 0$, depending only on p, d, ℓ and $\|\alpha\|_{L^p(\Omega)}$, with the following property: For any open cube $\omega \Subset \Omega$ whose edges have lengths $< R_0$, there exists a sequence of 1-forms (α^ε) in $\mathcal{C}^\infty(\omega; \Lambda^1) \cap L^p(\omega; \Lambda^1)$ such that

$$\begin{aligned}
&d\alpha^\varepsilon + \alpha^\varepsilon \wedge \alpha^\varepsilon = 0 \quad \text{in } \omega\\
&\alpha^\varepsilon \to \alpha \text{ in } L^p(\omega) \text{ as } \varepsilon \to 0.
\end{aligned}$$

Proof. The proof uses the notions, notation and results of Section 3, especially Theorem 3.13. The notation Λ^k designates the space of all k-forms with matrix coefficients in $\mathbb{M}^\ell$ and the spaces $W^{1,p}_*(\omega; \Lambda^k)$, $W^{2,p}_*(\omega; \Lambda^k)$, $H^1_*(\omega; \Lambda^k)$, and $W_p(\omega; \Lambda^k)$ are those defined in Section 3.

Let $\alpha := A_i dx^i$, where $A_i \in L^p(\Omega; \mathbb{M}^\ell)$, be a 1-form that satisfies

$$d\alpha + \alpha \wedge \alpha = 0 \text{ in } \Omega.$$

Let there be given a cube

$$\omega := (a_1, b_1) \times (a_2, b_2) \times ... \times (a_{\mathrm{d}}, b_{\mathrm{d}})$$

that satisfies the assumptions of the Theorem (the length of the edges of ω, hereafter denoted R, satisfies $R < R_0$) and let

$$\Gamma_k := \partial\omega \cap \{x = (x_i) \in \mathbb{R}^{\mathrm{d}}; x_k = a_k \text{ or } x_k = b_k\}$$

denote the union of the two lateral faces of ω that are orthogonal to the x_k-axis.

The proof, whose general structure is given in the outline below, is broken into six steps.

Outline. Our aim is to find a sequence of smooth matrix-valued 1-forms $\alpha^\varepsilon := A_i^\varepsilon dx^i$, defined over ω, such that

$$\begin{aligned} &d\alpha^\varepsilon + \alpha^\varepsilon \wedge \alpha^\varepsilon = 0 \text{ in } \omega, \\ &\alpha^\varepsilon \to \alpha \text{ in the } L^p(\omega)\text{-norm.} \end{aligned}$$

The idea is to decompose the 1-form α into two parts, namely

$$\alpha = d\phi + \delta\psi \quad \text{in } \omega,$$

where ϕ is a 0-form of class $W^{1,p}(\omega)$ and ψ is a *closed* (i.e., $d\psi = 0$) 2-form of class $W^{2,p/2}(\omega)$ in such a way that ϕ is not subjected to any other constraint while ψ is determined by the relation $d\alpha + \alpha \wedge \alpha = 0$. To insure that the form ψ determined in this fashion is closed, we impose some boundary conditions on ψ: for reasons that will be made clear afterwards, we require ψ to satisfy the (mixed) boundary conditions

$$*\psi = 0 \text{ and } *d\psi = 0 \text{ on } \partial\omega\,.$$

Note that in coordinates, these boundary conditions mean that (Lemma 3.4 can be applied since ω is a cube), for all $i < j$,

$$\begin{aligned} &\Psi_{ij} = 0 \text{ on } \Gamma_i \cup \Gamma_j\,, \\ &\partial_\nu \Psi_{ij} = 0 \text{ on } \partial\omega \setminus (\Gamma_i \cup \Gamma_j)\,, \end{aligned}$$

where $\psi = \sum_{i<j} \Psi_{ij} dx^i \wedge dx^j$. That such a decomposition does exist is proven in the step (ii).

The next step consists in choosing a sequence of smooth fields ϕ^ε that converges to ϕ in the $W^{1,p}(\omega)$-norm and to define ψ^ε as a function of ϕ^ε, determined by the relation $d\alpha^\varepsilon + \alpha^\varepsilon \wedge \alpha^\varepsilon = 0$ with $\alpha^\varepsilon = d\phi^\varepsilon + \delta\psi^\varepsilon$. This is done in steps (iii)-(v).

The final step is to show that the sequence of 1-forms defined by

$$\alpha^\varepsilon := d\phi^\varepsilon + \delta\psi^\varepsilon,$$

satisfies the conditions of the theorem.

(i) *Preliminary result:* $\alpha \wedge \alpha \in W_{\frac{p}{2}}(\omega; \Lambda^2)$. It is clear that $\alpha \wedge \alpha \in L^{\frac{p}{2}}(\omega; \Lambda^2)$ and, since

$$d(\alpha \wedge \alpha) = d(-d\alpha) = -dd\alpha = 0,$$

we have in particular $d(\alpha \wedge \alpha) \in L^{\frac{p}{2}}(\omega; \Lambda^2)$.

On the other hand, we know that $\alpha \wedge \alpha = -d\alpha$ in the entire domain Ω, so $d(\alpha \wedge \alpha) = 0$ in Ω. Regularizing by convolution the coordinates of $\alpha \wedge \alpha$ yields a sequence of 2-forms $\gamma^n \in \mathcal{C}^\infty(\overline{\omega}; \Lambda^2)$ such that

$$\gamma^n \to \alpha \wedge \alpha \quad \text{in } L^{\frac{p}{2}}(\omega; \Lambda^2) \quad \text{and} \quad d\gamma^n = 0 \quad \text{on } \omega$$

(the second relation holds since $d(\alpha \wedge \alpha) = 0$ in Ω and d is a linear operator). In particular, $d\gamma^n \to d(\alpha \wedge \alpha)$ in $L^{\frac{p}{2}}(\omega; \Lambda^2)$. This shows that $\alpha \wedge \alpha \in W_{\frac{p}{2}}(\omega; \Lambda^2)$.

(ii) *Decomposition of the* 1*-form* α. We define the 2-form ψ as the unique solution in the space $H^1_*(\omega; \Lambda^2)$ to the variational equation

$$\int_\omega \nabla\psi \cdot \nabla\beta \, dx = -\int_\omega (\alpha \wedge \alpha) \cdot \beta \, dx \tag{5.3}$$

for all $\beta \in H^1_*(\omega; \Lambda^2)$. That this equation possesses a unique solution is proved by the Lax-Milgram Theorem. We only need to prove that the linear form appearing in the right-hand side of (5.3) is continuous over the Hilbert space $H^1_*(\omega; \Lambda^2)$.

Since the 2-form $\alpha \wedge \alpha$ belongs to the space $L^{p/2}(\omega; \Lambda^2)$, we have on one hand (see inequalities (3.1))

$$\Big| \int_\omega (\alpha \wedge \alpha) \cdot \beta \, dx \Big| \le \|\alpha \wedge \alpha\|_{L^{p/2}(\omega)} \|\beta\|_{L^{\frac{p}{p-2}}(\omega)}$$

for all 2-form $\beta \in L^{\frac{p}{p-2}}(\omega; \Lambda^2)$. On the other hand, the Sobolev inequality established in Theorem 3.7 combined with inequality (3.2) show that, for some constants $C > 0$,

$$\|\beta\|_{L^{\frac{p}{p-2}}(\omega)} \le C\|\beta\|_{L^{\frac{2p}{p-2}}(\omega)} \le C\|\nabla\beta\|_{L^2(\omega)} \le C\|\beta\|_{H^1_*(\omega)}$$

for all 2-form $\beta \in H^1_*(\omega; \Lambda^2)$. Combining these last two inequalities shows that the linear form appearing in the right-hand side of (5.3) is indeed continuous over the Hilbert space $H^1_*(\omega; \Lambda^2)$.

We have proved that the equation (5.3) has a solution in $H^1_*(\omega; \Lambda^2)$. Since $\alpha \wedge \alpha$ belongs to the space $L^{p/2}(\omega; \Lambda^2)$, the regularity result established in Theorem 3.13 (see also Remark 3.14) shows that

$$\psi \in W^{2,p/2}(\omega; \Lambda^2),$$

and that it satisfies the boundary value problem

$$\begin{aligned} \Delta\psi &= \alpha \wedge \alpha && \text{in } \omega, \\ *\,\psi = 0 \text{ and } *\,d\psi &= 0 && \text{on } \partial\omega. \end{aligned} \tag{5.4}$$

Hence we have found a form ψ belonging to the space $W^{2,p/2}_*(\omega; \Lambda^2)$ that satisfies the variational equation (5.3). By step (i), we know that $\alpha \wedge \alpha \in W_{p/2}(\omega; \Lambda^2)$. Therefore the second assertion of Theorem 3.13 shows that $d\psi$ belongs to the space $W^{1,p/2}_*(\omega; \Lambda^3)$ and satisfies the variational equation

$$\int_\omega \nabla(d\psi) \cdot \nabla\beta \, dx = 0$$

for all $\beta \in W^{1,\frac{p}{p-2}}_*(\omega; \Lambda^3)$. But the regularity result of Theorem 3.13 shows that $d\psi$ belongs to the space $H^1_*(\omega; \Lambda^3)$ (it suffices to take $r = \max\{2, \frac{p}{2}\}$ in Theorem 3.13; see also Remark 3.14), which next implies that the variational equation above is also satisfied for all $\beta \in H^1_*(\omega; \Lambda^3)$, since the space

$$\{\gamma \in \mathcal{C}^\infty(\overline{\omega}; \Lambda^3) \ ; \ *\gamma = 0 \ \text{ on } \ \partial\omega\} \subset W^{1,\frac{p}{p-2}}_*(\omega; \Lambda^3)$$

is dense in $H^1_*(\omega; \Lambda^3)$ by Theorem 2.6(ii). By taking $\beta = d\psi$ in the variational equation above, we immediately deduce that

$$d\psi = 0 \text{ in } \omega,$$

which means that the 2-form ψ is closed.

Note that since $p > \mathrm{d}$, $\psi \in W^{1,p}(\omega; \Lambda^2)$ by the Sobolev embedding $W^{2,p/2}(\omega) \subset W^{1,p}(\omega)$. In particular $\delta\psi \in L^p(\omega; \Lambda^1)$, so that $\alpha - \delta\psi \in L^p(\omega; \Lambda^1)$.

We now infer from the relation $d\psi = 0$ and from the identity

$$-\Delta = d\delta + \delta d,$$

that

$$d(\alpha - \delta\psi) = d\alpha - d\delta\psi = d\alpha - (d\delta + \delta d)\psi = d\alpha + \Delta\psi = -\alpha \wedge \alpha + \Delta\psi = 0.$$

Therefore the 1-form $(\alpha - \delta\psi)$ is closed, which next implies, by the Poincaré Theorem 3.3, that there exists a 0-form $\phi \in W^{1,p}(\omega; \Lambda^0)$ such that

$$\alpha - \delta\psi = d\phi.$$

(iii) *Regularization of ϕ and ψ.* Since the space $\mathcal{C}^\infty(\overline{\omega})$ is dense in $W^{1,p}(\omega)$, there exists a sequence of 0-forms ϕ^ε in $\mathcal{C}^\infty(\overline{\omega}; \Lambda^0)$ such that, as $\varepsilon \to 0$,

$$\phi^\varepsilon \to \phi \text{ in the } W^{1,p}(\omega)\text{-norm}.$$

Next, for each ε, we define a 2-form ψ^ε by solving the nonlinear problem (to be compared with (5.4)):

$$\begin{aligned} &-\Delta\psi^\varepsilon + (d\phi^\varepsilon + \delta\psi^\varepsilon) \wedge (d\phi^\varepsilon + \delta\psi^\varepsilon) = 0 \text{ in } \omega, \\ &*\,\psi^\varepsilon = 0 \text{ and } *\,d\psi^\varepsilon = 0 \text{ on } \partial\omega. \end{aligned} \tag{5.5}$$

That this system has a solution is proven by the Implicit Function Theorem. To this end, define the spaces

$$\begin{aligned} E &:= W^{1,p}(\omega; \Lambda^0), \\ F &:= W^{2,p/2}_*(\omega; \Lambda^2), \\ G &:= L^{p/2}(\omega; \Lambda^2), \end{aligned}$$

and define the mapping $f : E \times F \to G$ by letting

$$f(\sigma, \tau) = -\Delta\tau + (d\sigma + \delta\tau) \wedge (d\sigma + \delta\tau).$$

It is clear that this mapping is well defined and that it is of class $\mathcal{C}^\infty$ (since its differential is a continuous affine mapping, i.e., a translation of a continuous linear operator). Its derivative with respect to the second variable at (ϕ, ψ) is the linear map $D_\tau f(\phi, \psi) : F \to G$ defined by

$$D_\tau f(\phi, \psi)(\theta) = -\Delta\theta + \alpha \wedge \delta\theta + \delta\theta \wedge \alpha,$$

where (see the step (ii))

$$\alpha = d\phi + \delta\psi.$$

Now we show that the mapping $D_\tau f(\phi, \psi)$ is an isomorphism by using several properties of second order linear elliptic equations. For each $\xi \in G$, we first prove that the linear system

$$-\Delta\theta + \alpha \wedge \delta\theta + \delta\theta \wedge \alpha = \xi, \tag{5.6}$$

has one and only one solution in the space F. To this end, we apply the Lax-Milgram lemma to the following variational equation: Find $\theta \in H^1_*(\omega; \Lambda^2)$ such that

$$\int_\omega \nabla\theta \cdot \nabla\beta \, dx + \int_\omega (\alpha \wedge \delta\theta + \delta\theta \wedge \alpha) \cdot \beta \, dx = \int_\omega \xi \cdot \beta \, dx \tag{5.7}$$

for all $\beta \in H^1_*(\omega; \Lambda^2)$.

The assumptions of the Lax-Milgram lemma are satisfied because, on one hand, the linear form appearing in the right hand side of (5.7) is continuos over $H^1_*(\omega;\Lambda^2)$ (the proof is the same as in step (ii)) and, on the other hand, the bilinear form appearing in the left-hand side of the same equation is continuous and coercive over $H^1_*(\omega;\Lambda^2)$, as we now show. First, using succesively the Hölder inequality, Theorem 3.7, and the inequalities (3.1) and (3.2), we deduce that, for all $\theta,\beta \in H^1_*(\omega;\Lambda^2)$,

$$\begin{aligned}\left|\int_\omega \nabla\theta\cdot\nabla\beta\,dx + \int_\omega (\alpha\wedge\delta\theta+\delta\theta\wedge\alpha)\cdot\beta dx\right| \\ &\leq \|\theta\|_{H^1_*(\omega)}\|\beta\|_{H^1_*(\omega)} + 2\|\alpha\|_{L^p(\omega)}\|\delta\theta\|_{L^2(\omega)}\|\beta\|_{L^{2p/(p-2)}(\omega)} \\ &\leq \|\theta\|_{H^1_*(\omega)}\|\beta\|_{H^1_*(\omega)} + C\|\alpha\|_{L^p(\omega)}\|\delta\theta\|_{L^2(\omega)}\|\nabla\beta\|_{L^2(\omega)} \\ &\leq C\|\theta\|_{H^1_*(\omega)}\|\beta\|_{H^1_*(\omega)},\end{aligned}$$

where the last constant C depends on $\|\alpha\|_{L^p(\omega)}$, but is independent on θ and β. Thus the bilinear form is continuous. Using the same ingredients, we have that, for all $\theta \in H^1_*(\omega;\Lambda^2)$,

$$\begin{aligned}\left|\int_\omega (\alpha\wedge\delta\theta+\delta\theta\wedge\alpha)\cdot\theta dx\right| &\leq 2\|\alpha\|_{L^p(\omega)}\|\delta\theta\|_{L^2(\omega)}\|\theta\|_{L^{2p/(p-2)}(\omega)} \\ &\leq CR^{1-\mathrm{d}/p}\|\alpha\|_{L^p(\Omega)}\|\delta\theta\|_{L^2(\omega)}\|\nabla\theta\|_{L^2(\omega)} \\ &\leq CR^{1-\mathrm{d}/p}\|\alpha\|_{L^p(\Omega)}\|\nabla\theta\|^2_{L^2(\omega)} \\ &\leq CR^{1-\mathrm{d}/p}\|\alpha\|_{L^p(\Omega)}\|\theta\|^2_{H^1_*(\omega)},\end{aligned}$$

for some constants C, the last one depending only on p,d and ℓ. It is clear then that for small R, i.e., for those R that are smaller than a certain number R_0 depending only on p,d,ℓ and $\|\alpha\|_{L^p(\Omega)}$ (for an explicit value of R_0, see Remark 5.4), we have

$$\left|\int_\omega (\alpha\wedge\delta\theta+\delta\theta\wedge\alpha)\cdot\theta dx\right| \leq a\|\theta\|^2_{H^1_*(\omega)},$$

for some constant $a<1$, which shows that the bilinear form associated with (5.7) is coercive over the space $H^1_*(\omega;\Lambda^2)$.

We now show by a bootstraap argument that the solution to the system (5.7) belongs to the space $W^{2,p/2}_*(\omega;\Lambda^2)$. In turn, this implies that the equation (5.6) is equivalent with the variational equation (5.7), and therefore the equation (5.6) has a unique solution if and only if the equation (5.7) has a unique solution. To prove this equivalence, we first observe that, if the solution θ to the variational equation (5.7) belongs to the space $W^{2,p/2}_*(\omega;\Lambda^2)$, then θ is a strong solution to equation (5.6). In the opposite sense, if θ is solution to (5.6), then θ is solution to the variational equation (5.7), since

$$\int_\omega (-\Delta\theta)\cdot\beta\,dx = \int_\omega \nabla\theta\cdot\nabla\beta\,dx$$

for all $\beta\in H^1_*(\omega;\Lambda^2)$ by Corollary 3.12 (the hypotheses of the Corollary are satisfied since $\frac{1}{2}+\frac{2}{p}<1+\frac{1}{\mathrm{d}}$ for $p>\mathrm{d}\geq 2$).

In order to prove that θ does belong to the space $W^{2,p/2}_*(\omega;\Lambda^2)$, it suffices (in view of Theorem 3.13 and Remark 3.14) to prove that

$$\xi - (\alpha\wedge\delta\theta+\delta\theta\wedge\alpha) \in L^{p/2}(\omega;\Lambda^2).$$

In fact, since $\xi \in L^{p/2}(\omega;\Lambda^2)$ and $\alpha \in L^p(\omega;\Lambda^1)$, it suffices to prove that

$$\delta\theta \in L^p(\omega;\Lambda^1).$$

We first prove by a recursion argument that there exists a number $r \geq \overline{r} := \min\{p, \frac{\mathrm{d}p}{p-\mathrm{d}}\}$ such that $\delta\theta \in L^r(\omega;\Lambda^1)$. To this end, we define recursively a sequence of numbers $r_0 < r_1 < r_2 < \dots$ such that $\delta\theta \in L^{r_k}(\omega;\Lambda^1)$: set $r_0 = 2$ and suppose we have defined $r_0 < r_1 < \cdots < r_k < \overline{r}$. Then

$$(\alpha \wedge \delta\theta + \delta\theta \wedge \alpha) - \xi \in L^{s_k}(\omega;\Lambda^2),$$

where $\frac{1}{s_k} = \frac{1}{r_k} + \frac{1}{p}$, which next implies that

$$\theta \in W^{2,s_k}(\omega;\Lambda^2),$$

by applying Theorem 3.13 to the equation (5.7). Furthermore, the Sobolev Embedding Theorem (note that $s_k < \mathrm{d}$) shows that

$$\delta\theta \in W^{1,s_k}(\omega;\Lambda^1) \subset L^{r_{k+1}}(\omega;\Lambda^1),$$

where

$$\frac{1}{r_{k+1}} = \frac{1}{s_k} - \frac{1}{\mathrm{d}} = \frac{1}{r_k} - \Big(\frac{1}{\mathrm{d}} - \frac{1}{p}\Big).$$

The last relation shows that $r_k < r_{k+1}$ and that after a finite number of iterations we will find $r = r_h \geq \overline{r}$.

If $r \geq p$, then $\delta\theta \in L^r(\omega;\Lambda^1) \subset L^p(\omega;\Lambda^1)$. If $\frac{\mathrm{d}p}{p-\mathrm{d}} \leq r < p$, then we deduce by the same argument as above that $\theta \in W^{2,s}(\omega;\Lambda^2)$, where $\frac{1}{s} = \frac{1}{r} + \frac{1}{p} \leq \frac{1}{\mathrm{d}}$. Hence $s \geq \mathrm{d}$ and the Sobolev Embedding Theorem shows that

$$\delta\theta \in W^{1,s}(\omega;\Lambda^1) \subset L^p(\omega;\Lambda^1).$$

We already proved that the linear mapping $D_\beta f(\phi,\psi)$ is a bijection. Since it is also continuous, the closed graph theorem shows that it is an isomorphism. Consequently, the mapping f satisfies the assumptions of the implicit function theorem (see Theorem 2.1). Therefore there exist open balls $B_\delta(\phi) \subset E$, $B_r(\psi) \subset F$ and a mapping $g : B_\delta(\phi) \to B_r(\psi)$ of class $\mathcal{C}^1$ such that $g(\phi) = \psi$ and

$$f(\sigma, g(\sigma)) = 0 \text{ for all } \sigma \in B_\delta(\phi).$$

In particular, for $\sigma = \phi^\varepsilon$ there exists $\psi^\varepsilon := g(\phi^\varepsilon)$ such that $f(\phi^\varepsilon, \psi^\varepsilon) = 0$. Moreover, since g is continuous and since $\phi^\varepsilon \to \phi$ in $W^{1,p}(\omega;\Lambda^0)$, we have $\psi^\varepsilon \to \psi$ in $W^{2,p/2}(\omega;\Lambda^2)$. Hence

$$\psi^\varepsilon \to \psi \text{ in } W^{1,p}(\omega;\Lambda^2)$$

thanks to the Sobolev embedding $W^{2,p/2}(\omega) \subset W^{1,p}(\omega)$, which is valid because $p \geq \mathrm{d}$.

(iv) *The 2-form ψ^ε belongs to the space $W^{2,r}(\omega;\Lambda^2) \cap \mathcal{C}^\infty(\omega)$ for all $r \in [1,\infty)$.* We proved in step (iii) that $\psi^\varepsilon \in W^{2,p/2}_*(\omega;\Lambda^2)$ and that it satisfies the boundary value problem (see (5.5)):

$$\begin{aligned} &-\Delta\psi^\varepsilon + (d\phi^\varepsilon + \delta\psi^\varepsilon) \wedge (d\phi^\varepsilon + \delta\psi^\varepsilon) = 0 && \text{in } \omega, \\ &*\,\psi^\varepsilon = 0 \text{ and } *\,d\psi^\varepsilon = 0 && \text{on } \partial\omega, \end{aligned}$$

where $\phi^\varepsilon \in \mathcal{C}^\infty(\overline{\omega})$. We now prove that $\psi^\varepsilon \in W^{2,r}(\omega;\Lambda^2)$ for all $r \in [1,\infty)$.

By multiplying the equation satisfied by ψ^ε with a test function $\beta \in W_*^{1,\frac{p}{p-2}}(\omega;\Lambda^2)$, which is possible since the space $L^{\frac{p}{p-2}}(\omega)$ is the dual of the space $L^{\frac{p}{2}}(\omega)$, we first deduce that

$$-\int_\omega \Delta\psi^\varepsilon \cdot \beta\, dx = -\int_\omega ((d\phi^\varepsilon + \delta\psi^\varepsilon) \wedge (d\phi^\varepsilon + \delta\psi^\varepsilon)) \cdot \beta\, dx\,, \tag{5.8}$$

hence, by Corollary 3.12,

$$\int_\omega \nabla\psi^\varepsilon \cdot \nabla\beta\, dx = -\int_\omega ((d\phi^\varepsilon + \delta\psi^\varepsilon) \wedge (d\phi^\varepsilon + \delta\psi^\varepsilon)) \cdot \beta\, dx \tag{5.9}$$

for all $\beta \in W_*^{1,\frac{p}{p-2}}(\omega;\Lambda^2)$.

We prove that $\psi^\varepsilon \in W^{2,r}(\omega;\Lambda^2)$ for all $r \in [1,\infty)$ by applying several times the regularity result established in Theorem 3.13. To this end, we first prove by a recursion argument that there exists a number $\overline{p} \geq 2\mathrm{d}$ such that $\psi^\varepsilon \in W^{2,\frac{\overline{p}}{2}}(\omega;\Lambda^2)$.

We know that $\psi^\varepsilon \in W^{2,\frac{p}{2}}(\omega;\Lambda^2)$. If $p \geq 2\mathrm{d}$, we set $\overline{p} = p$. If $p < 2\mathrm{d}$, we define recursively a sequence of numbers $p_0 < p_1 < p_2 < ...$ such that

$$\psi^\varepsilon \in W^{2,\frac{p_i}{2}}(\omega;\Lambda^2),\ i = 0, 1, ...$$

until we reach a number $p_k \geq 2\mathrm{d}$. For, let $p_0 = p$ and assume that we already defined the numbers $p_0 < p_1 < ... < p_i < 2\mathrm{d}$. Since $\psi^\varepsilon \in W^{2,\frac{p_i}{2}}(\omega;\Lambda^2)$, we deduce that

$$(d\phi^\varepsilon + \delta\psi^\varepsilon) \wedge (d\phi^\varepsilon + \delta\psi^\varepsilon) \in L^{\frac{p_{i+1}}{2}}(\omega;\Lambda^2),$$

where

$$\frac{1}{p_{i+1}} = \frac{2}{p_i} - \frac{1}{\mathrm{d}}, \tag{5.10}$$

by using the Sobolev embedding

$$W^{1,\frac{p_i}{2}}(\omega) \subset L^{p_{i+1}}(\omega).$$

Then, by applying Theorem 3.13 (see also Remark 3.14) to the variational equation (5.9), we deduce that

$$\psi^\varepsilon \in W^{2,\frac{p_{i+1}}{2}}(\omega;\Lambda^2).$$

Moreover, $p_{i+1} > p_i$ since

$$\frac{1}{p_{i+1}} = \frac{1}{p_i} - \Big(\frac{1}{\mathrm{d}} - \frac{1}{p_i}\Big) \leq \frac{1}{p_i} - \Big(\frac{1}{\mathrm{d}} - \frac{1}{p}\Big).$$

This shows that relation (5.10) correctly defines the sequence p_i. But the last inequality also shows that, after a finite number of iterations, we obtain $p_{i+1} \geq 2\mathrm{d}$. We thus set $\overline{p} = p_{i+1}$ in the case where $p < 2\mathrm{d}$.

We found a number $\overline{p} \geq 2\mathrm{d}$ such that $\psi^\varepsilon \in W^{2,\frac{\overline{p}}{2}}(\omega;\Lambda^2)$. In addition, the Sobolev Embedding Theorem shows that

$$W^{1,\frac{\overline{p}}{2}}(\omega) \subset L^r(\omega) \text{ for all } r \geq 1.$$

Combined with the fact that $\phi^\varepsilon \in \mathcal{C}^\infty(\overline{\omega};\mathbb{M}^\ell)$, this implies that

$$(d\phi^\varepsilon + \delta\psi^\varepsilon) \wedge (d\phi^\varepsilon + \delta\psi^\varepsilon) \in L^r(\omega;\Lambda^2) \text{ for all } r \geq 1,$$

so that we can apply once again Theorem 3.13 to the variational equation (5.9) and thus obtain

$$\psi^\varepsilon \in W^{2,r}(\omega;\Lambda^2) \text{ for all } r \geq 1.$$

We now prove that the 2-forms ψ^ε are of class $\mathcal{C}^\infty$ in ω by using the interior regularity of the Laplacian combined with a bootstraap argument.

Since $\phi^\varepsilon \in \mathcal{C}^\infty(\overline{\omega};\mathbb{M}^\ell)$ and since $\psi^\varepsilon \in W^{2,r}(\omega;\Lambda^2)$ for all $r \geq 1$, we have (see (5.5))

$$\Delta\psi^\varepsilon = (d\phi^\varepsilon + \delta\psi^\varepsilon) \wedge (d\phi^\varepsilon + \delta\psi^\varepsilon) \in W^{1,\frac{r}{2}}(\omega;\Lambda^2) \text{ for all } r > 2.$$

Then the interior regularity of the Laplacian (see, e.g., Dautray and Lions [6]) shows that

$$\psi^\varepsilon \in W^{3,\frac{r}{2}}_{\mathrm{loc}}(\omega;\Lambda^2) \text{ for all } r > 2.$$

In turn, this implies that

$$\Delta\psi^\varepsilon = (d\phi^\varepsilon + \delta\psi^\varepsilon) \wedge (d\phi^\varepsilon + \delta\psi^\varepsilon) \in W^{2,\frac{r}{2^2}}_{\mathrm{loc}}(\omega;\Lambda^2) \text{ for all } r > 2^2,$$

so that we have, again by the interior regularity of the Laplacian,

$$\psi^\varepsilon \in W^{4,\frac{r}{2^2}}_{\mathrm{loc}}(\omega;\Lambda^2) \text{ for all } r > 2^2.$$

By applying this argument recursively, we obtain that

$$\psi^\varepsilon \in W^{m,r}_{\mathrm{loc}}(\omega;\Lambda^2) \text{ for all } m \geq 2, r > 1.$$

Hence $\psi^\varepsilon \in \mathcal{C}^\infty(\omega;\Lambda^2)$ by the Sobolev Embedding Theorem.

(v) *The 2-form ψ^ε is closed for ε small enough.* We may assume that the dimension of Ω satisfies $\mathrm{d} \geq 3$ since in the case $\mathrm{d} = 2$ we have $d\psi^\varepsilon = 0$ simply because $d\psi^\varepsilon$ is a form of degree > 2.

Let $\alpha^\varepsilon := (d\phi^\varepsilon + \delta\psi^\varepsilon)$. Note that $\alpha^\varepsilon \to \alpha$ in $L^p(\omega;\Lambda^1)$ since $\phi^\varepsilon \to \phi$ in $W^{1,p}(\omega;\Lambda^0)$ and $\psi^\varepsilon \to \psi$ in $W^{1,p}(\omega;\Lambda^2)$. In particular

$$\|\alpha^\varepsilon\|_{L^p(\omega)} \leq \|\alpha\|_{L^p(\omega)} + 1 \leq \|\alpha\|_{L^p(\Omega)} + 1$$

for ε small enough, that is, smaller than a certain value ε_0. We will prove that, for all $\varepsilon < \varepsilon_0$, the 2-forms ψ^ε are closed.

We wish to prove that $d\psi^\varepsilon = 0$ by applying the second assertion of Theorem 3.13 to the variational equation (see (5.9))

$$\int_\omega \nabla\psi^\varepsilon \cdot \nabla\beta \, dx = -\int_\omega (\alpha^\varepsilon \wedge \alpha^\varepsilon) \cdot \beta \, dx \tag{5.11}$$

for all $\beta \in W^{1,\frac{p}{p-2}}_*(\omega;\Lambda^2)$. Since by the step (iv),

$$\alpha^\varepsilon \wedge \alpha^\varepsilon \in W^{1,r}(\omega;\Lambda^2) \subset W_r(\omega;\Lambda^2) \text{ for all } r \geq 1,$$

and in particular for $r = \max\{2, \frac{p}{2}\}$, we obtain by Theorem 3.13 that $d\psi^\varepsilon \in W^{1,r}_*(\omega;\Lambda^3) \subset H^1_*(\omega;\Lambda^3)$ satisfies

$$\int_\omega \nabla(d\psi^\varepsilon) \cdot \nabla\beta \, dx = -\int_\omega d(\alpha^\varepsilon \wedge \alpha^\varepsilon) \cdot \beta \, dx \tag{5.12}$$

for all $\beta \in H^1_*(\omega;\Lambda^3)$.

On the other hand, we have

$$d(\alpha^\varepsilon \wedge \alpha^\varepsilon) = (d\delta\psi^\varepsilon) \wedge \alpha^\varepsilon - \alpha^\varepsilon \wedge (d\delta\psi^\varepsilon).$$

But the 2-form $(d\delta\psi^\varepsilon)$ satisfies

$$d\delta\psi^\varepsilon = -\Delta\psi^\varepsilon - \delta(d\psi^\varepsilon) = -\alpha^\varepsilon \wedge \alpha^\varepsilon - \delta(d\psi^\varepsilon),$$

which combined with the previous equation gives

$$d(\alpha^\varepsilon \wedge \alpha^\varepsilon) = -\delta(d\psi^\varepsilon) \wedge \alpha^\varepsilon + \alpha^\varepsilon \wedge \delta(d\psi^\varepsilon).$$

Using this expression in the right hand side of (5.12), we deduce that

$$\int_\omega \nabla(d\psi^\varepsilon) \cdot \nabla\beta \, dx = \int_\omega (\delta(d\psi^\varepsilon) \wedge \alpha^\varepsilon) \cdot \beta - (\alpha^\varepsilon \wedge \delta(d\psi^\varepsilon)) \cdot \beta \, dx \tag{5.13}$$

for all $\beta \in H^1_*(\omega;\Lambda^3)$.

In order to prove that $d\psi^\varepsilon = 0$, we let $\beta = d\psi^\varepsilon$ in the variational equation (5.13) and we obtain the following estimate (we use in particular the inequalities (3.1)):

$$\begin{aligned}\left|\int_\omega \nabla d\psi^\varepsilon \cdot \nabla d\psi^\varepsilon\right| &\le 2\|\alpha^\varepsilon\|_{L^p(\omega)} \|\nabla(d\psi^\varepsilon)\|_{L^2(\omega)} \|d\psi^\varepsilon\|_{L^{2p/(p-2)}(\omega)} \\ &\le 2\big(\|\alpha\|_{L^p(\Omega)} + 1\big) \|\nabla(d\psi^\varepsilon)\|_{L^2(\omega)} \|d\psi^\varepsilon\|_{L^{2p/(p-2)}(\omega)},\end{aligned}$$

which combined with the inequalities (3.2) and with the Sobolev inequality of Theorem 3.7 shows that

$$\|\nabla(d\psi^\varepsilon)\|^2_{L^2(\omega)} \le CR^{1-\mathrm{d}/p}\big(\|\alpha\|_{L^p(\Omega)} + 1\big) \|\nabla(d\psi^\varepsilon)\|^2_{L^2(\omega)},$$

the constant C depending only on p, d and ℓ. Thus, for R small enough, i.e., for R smaller than a certain number R_0 depending only on p, d, ℓ and $\|\alpha\|_{L^p(\Omega)}$ (see Remark 5.4 for an explicit value of R_0), the left-hand side of the previous inequality must vanish. Combined with the null boundary conditions satisfyed by $d\psi^\varepsilon$, this implies that $d\psi^\varepsilon = 0$.

(vi) *Approximation of the* 1*-form* α. We have already seen that

$$\alpha^\varepsilon := d\phi^\varepsilon + \delta\psi^\varepsilon \to \alpha \ \text{ in } L^p(\omega;\Lambda^1).$$

In addition, α^ε is of class $\mathcal{C}^\infty$ in ω since ϕ^ε and ψ^ε are of class $\mathcal{C}^\infty(\omega)$. On the other hand $d\psi^\varepsilon = 0$ for ε small enough, as we have seen in the previous step. Therefore

$$d(d\phi^\varepsilon + \delta\psi^\varepsilon) = d\delta\psi^\varepsilon = (d\delta + \delta d)\psi^\varepsilon = -\Delta\psi^\varepsilon = -(d\phi^\varepsilon + \delta\psi^\varepsilon) \wedge (d\phi^\varepsilon + \delta\psi^\varepsilon),$$

from which we deduce that the 1-form α^ε (with ε small enough) satisfies the relation

$$d\alpha^\varepsilon + \alpha^\varepsilon \wedge \alpha^\varepsilon = 0.$$

$\square$

Remark 5.3. The forms α^ε constructed in the proof of Theorem 5.2 belong to the space $\mathcal{C}^\infty(\omega;\Lambda^1) \cap \left(\bigcap_{1\le r<+\infty} W^{1,r}(\omega;\Lambda^1)\right)$.

Remark 5.4 (an explicit formula for R_0)**.** It is clear that we can choose the number R_0 appearing in the statement of Theorems 5.1 and 5.2 as the minimum between the constants (still denoted R_0) appearing in the proofs of the above steps (iii) and (v). Following the inequalities therein, we can take

$$R_0 := \min\left\{ \Big(C_2\ell^2\mathrm{d}^2(\mathrm{d}-1)\|\alpha\|_{L^p(\Omega)}\Big)^{\frac{p}{\mathrm{d}-p}}, \left(\frac{C_2\ell^2\mathrm{d}^2(\mathrm{d}-1)(\mathrm{d}-2)\big(\|\alpha\|_{L^p(\Omega)}+1\big)}{3}\right)^{\frac{p}{\mathrm{d}-p}} \right\}, \tag{5.14}$$

where C_2 is the constant of Theorem 3.7.

An immediate consequence of Theorem 5.2 is the following

Corollary 5.5. *Let $\tilde{\Omega}$ be an open subset of $\mathbb{R}^{\mathrm{d}}$ and let $\alpha \in L^p_{\mathrm{loc}}(\tilde{\Omega}; \Lambda^1)$, $p > \mathrm{d}$, be a 1-form with matrix coefficients that satisfies in the distributional sense*

$$d\alpha + \alpha \wedge \alpha = 0 \quad in \ \ \tilde{\Omega}.$$

Then there exists a non-increasing function $R_0 : \{\Omega \Subset \tilde{\Omega} \ ; \ \Omega \text{ open}\} \to (0, \infty]$ (i.e., $\Omega_1 \subset \Omega_2 \Subset \tilde{\Omega} \Rightarrow R_0(\Omega_1) \geq R_0(\Omega_2)$) with the following property: given any open subset $\Omega \Subset \tilde{\Omega}$ and any open cube $\omega \Subset \Omega$ whose edges have lengths $< R_0(\Omega)$, there exists a sequence of 1-forms (α^ε) in $\mathcal{C}^\infty(\omega; \Lambda^1) \cap L^p(\omega; \Lambda^1)$ such that

$$\begin{aligned} &d\alpha^\varepsilon + \alpha^\varepsilon \wedge \alpha^\varepsilon = 0 \quad in \ \ \omega \\ &\alpha^\varepsilon \to \alpha \ \ in \ L^p(\omega) \ \ as \ \varepsilon \to 0. \end{aligned} \tag{5.15}$$

Proof. For any open subset $\Omega \Subset \tilde{\Omega}$, the form $\alpha|_\Omega$ satisfies the hypotheses of Theorem 5.2. Hence there exists $R_0 \in (0, \infty]$ (depending on p, d, ℓ and $\|\alpha\|_{L^p(\Omega)}$) such that the form $\alpha|_\Omega$ can be approached with smooth forms satisfying the conditions (5.15) over any open cube $\omega \Subset \Omega$ whose edges have lengths $< R_0$.

Among all the constants $R_0 \in (0, \infty]$ satisfying the above property, we select one such that the mapping $\Omega \mapsto R_0$ be non-increasing.

To do so, it suffices to define R_0 by the formula (5.14). Another possibility is to define R_0 as the largest admissible value in the statement of Theorem 5.2 (it is easy to see that such a value does exist). ☐

6. Existence theory

6.1. **Necessary conditions.** This section provides the necessary conditions that the coefficients A_i must satisfy in order that the Pfaff system

$$\partial_i Y = Y A_i \quad \text{in } \Omega \ \ ; \ \ Y(x^0) = Y^0$$

has at least a solution for any given matrix $Y^0 \in \mathbb{M}^{q\times\ell}$. We begin with a preliminary result:

Lemma 6.1. *Let $X \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^\ell)$ be a solution to the Pfaff system*

$$\partial_i X = X A_i \quad in \ \ \Omega \ \ for \ all \quad i,$$

where Ω is a connected open subset of $\mathbb{R}^{\mathrm{d}}$, p is a number $> \mathrm{d}$, and $A_i \in L^p_{\mathrm{loc}}(\Omega; \mathbb{M}^\ell)$, $i \in \{1, 2, ..., \mathrm{d}\}$ are matrix fields. Assume that there exists a point $x^0 \in \Omega$ such that the matrix $X(x^0)$ is invertible. Then

(i) the matrix $X(x)$ is invertible at each point $x \in \Omega$.

(ii) the matrix field $X^{-1} : \Omega \to \mathbb{M}^\ell$, defined by $X^{-1}(x) = (X(x))^{-1}$ for all $x \in \Omega$, belongs to the space $W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^\ell)$ and satisfies the system

$$\partial_i(X^{-1}) = -A_i X^{-1} \quad in \ \ \Omega \ \ for \ all \quad i.$$

Proof. (i) Assume on the contrary that there exists $x^1 \in \Omega$ such that the matrix $X(x^1)$ is not invertible. Hence $\left[X(x^1)\right]^T$ is not invertible either and therefore 0 is one of its eigenvalues. This implies the existence of a vector $\mathrm{v} \neq 0$ in $\mathbb{R}^\ell$ satisfying

$$\left[X(x^1)\right]^T \mathrm{v} = 0 \text{ in } \mathbb{R}^\ell.$$

Since X satisfies $\partial_i X = X A_i$, it follows that the vector field $(\mathrm{v}^T X) : \Omega \to \mathbb{M}^{1\times\ell}$, defined by

$$(\mathrm{v}^T X)(x) = \mathrm{v}^T X(x) \ \forall x \in \Omega,$$

satisfies the system

$$\begin{aligned}&\partial_i(\mathrm{v}^T X) = (\mathrm{v}^T X)A_i \quad \text{for all } i,\\ &(\mathrm{v}^T X)(x^1) = 0.\end{aligned}$$

By the uniqueness result of Theorem 4.2, the solution of this system must vanish in the entire Ω. In particular, $(\mathrm{v}^T X)(x^0) = 0$. But this clearly contradicts the assumption that $X(x^0)$ is invertible (remerber that $\mathrm{v} \neq 0$). Hence the first part of the Theorem is established.

(ii) Since the matrix field X is continuous and invertible in a connected open set, its determinant $\det X : \Omega \to \mathbb{R}$, which is a continuous function, must be either positive at every point of Ω, or negative at every point of Ω. Without losing in generality, we may assume the former. Let $K \Subset \Omega$. Then there exists a constant $\varepsilon > 0$, depending on K, such that

$$\det X(x) \geq \varepsilon \ \forall x \in \overline{K}.$$

Since $W^{1,p}_{\mathrm{loc}}(\Omega)$ is an algebra, the function $\det X$ and the matrix field $\mathrm{Cof}X$ belong respectively to $W^{1,p}_{\mathrm{loc}}(\Omega)$ and $W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^\ell)$. Together with the inequality above, this implies that the field X^{-1} also belongs to $W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^\ell)$, since

$$X^{-1} = \frac{1}{\det X}(\mathrm{Cof}X)^T.$$

This allows to compute the partial derivatives (in the distributional sense) of the relation $XX^{-1} = I$, where $I : \Omega \to \mathbb{M}^\ell$ is the constant matrix field whose value is the identity matrix. Hence

$$(\partial_i X)X^{-1} + X(\partial_i X^{-1}) = 0,$$

which next implies that

$$X(A_i X^{-1} + \partial_i X^{-1}) = 0 \text{ in } L^p(\Omega).$$

Multiplying this relation by X^{-1} to the left, we finally obtain that X^{-1} satisfies the system of the part (ii) of the Lemma. □

We are now in a position to achieve the objective of this section:

Theorem 6.2. *Let Ω be an open and connected subset of $\mathbb{R}^{\mathrm{d}}$, let $x^0 \in \Omega$, and let $A_i \in L^p_{\mathrm{loc}}(\Omega; \mathbb{M}^\ell)$, where $p \geq \mathrm{d}$. If for any given matrix $Y^0 \in \mathbb{M}^{q\times\ell}$, the system*

$$\begin{aligned}&\partial_i Y = Y A_i \quad \text{in } \ \Omega \ \text{ for all } \ i,\\ &Y(x^0) = Y^0\end{aligned} \tag{6.1}$$

has at least a solution in $W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^{q\times\ell})$, then its coefficients satisfy in the distributional sense the compatibility conditions

$$\partial_j A_i - \partial_i A_j = A_i A_j - A_j A_i \quad \text{in } \ \Omega,$$

for all $i, j \in \{1, 2, \ldots, \mathrm{d}\}$.

Proof. For each $k \in \{1, 2, ..., \ell\}$, let $Y^0(k)$ denote the matrix in $\mathbb{M}^{q\times\ell}$ whose only non zero element is that situated at the first row and k-th column. By assumption, we know that each system

$$\begin{aligned}&\partial_i Y = Y A_i,\\ &Y(x^0) = Y^0(k),\end{aligned}$$

has at least a solution, say $Y(k) \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^{q\times\ell})$.

Let $X : \Omega \to \mathbb{M}^\ell$ be the matrix field whose k-th row is the first row of the matrix field $Y(k)$. It is then clear from the above system that the field X, which belongs to the space $W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^\ell)$, satisfies the system

$$\partial_i X = X A_i,$$
$$X(x^0) = I,$$

where $I : \Omega \to \mathbb{M}^\ell$ is the identity field. Now, since the second derivatives of X commute, it follows that, for all i, j,

$$\partial_j(XA_i) = \partial_i(XA_j),$$

which in turn implies that, in the distributional sense,

$$X(A_j A_i + \partial_j A_i) = X(A_i A_j + \partial_i A_j).$$

But we have shown in Lemma 6.1 that the matrices $X(x)$ are invertible at any point $x \in \Omega$ and that $X^{-1} \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^\ell)$. Noting that it make sense to multiply the above (distributional) equation with X^{-1} to the left, we deduce that

$$A_j A_i + \partial_j A_i = A_i A_j + \partial_i A_j$$

in the space of distributions. The proof is now complete.

□

6.2. **Sufficient conditions.** We study here the existence of solutions to a Pfaff system whose coefficients are matrix fields of class L^p_{loc}, $p > \mathrm{d}$, in an open set $\Omega \subset \mathbb{R}^{\mathrm{d}}$. We prove that this system has locally a solution if its coefficients satisfy the necessary compatibility conditions of Theorem 6.2. If in addition the set Ω is simply connected, we prove that the Pfaff system has in fact a global solution of class $W^{1,p}_{\mathrm{loc}}(\Omega)$. To do so, we proceed in two steps: first we deduce from the local existence result that the Cauchy problem associated with the Pfaff system has a solution in every small enough ball contained in Ω, then we prove that these local solutions can be glued together to obtain a global solution.

The key tool in the proof of the local existence result is the approximation result for Pfaff systems established in Section 5 (the balls have to be small enough so as to fit in a cube where the coefficients of the Pfaff system can be approximated according to Theorem 5.1):

Theorem 6.3. *Let $\tilde{\Omega}$ be an open subset of $\mathbb{R}^{\mathrm{d}}$ and let there be given* d *matrix fields $A_i \in L^p_{\mathrm{loc}}(\tilde{\Omega}; \mathbb{M}^\ell)$, $p > \mathrm{d}$, that satisfy the relations (for all i, j)*

$$\partial_j A_i - \partial_i A_j = A_i A_j - A_j A_i$$

in the space of distributions $\mathcal{D}'(\tilde{\Omega}; \mathbb{M}^\ell)$. Let there be given an open subset $\Omega \Subset \tilde{\Omega}$, a matrix $Y^0 \in \mathbb{M}^{q\times\ell}$, and an open ball $B_r = B_r(x^0)$ centered at $x^0 \in \Omega$ and with radius

$$r < \min\left(\frac{1}{\mathrm{d}}\mathrm{dist}(x^0, \Omega^c), R(\Omega)\right)$$

where $R(\Omega) := \dfrac{R_0(\Omega)}{2}$, $R_0(\Omega)$ being the number defined in Corollary 5.5 (hence the function $R : \{\Omega' \Subset \tilde{\Omega}\ ;\ \Omega' \text{ open}\} \to (0, +\infty]$ is non-increasing, i.e., if $\Omega_1 \subset \Omega_2 \Subset \tilde{\Omega}$, then $R(\Omega_1) \geq R(\Omega_2)$).

Then the Pfaff system

$$\begin{aligned} &\partial_i Y = Y A_i \ \textit{in}\ \mathcal{D}'(B_r; \mathbb{M}^{q\times\ell}) \ \ \textit{for all}\ \ i, \\ &Y(x^0) = Y^0 \end{aligned} \tag{6.2}$$

has a solution $Y \in W^{1,p}(B_r; \mathbb{M}^{q\times\ell})$.

Proof. It is based on an approximating method, the solution of the system (6.2) being found as a limit of solutions to a sequence of Pfaff systems with smooth coefficients.

Choose a number R such that $r < R < \min\big(\frac{1}{\mathrm{d}}\mathrm{dist}(x^0, \Omega^c), R(\Omega)\big)$. Then $B_{\mathrm{d}R}(x^0) \Subset \Omega$. Since the open cube ω_{2R} centered at x^0 with edges of length $2R$ is contained in $B_{\mathrm{d}R}(x^0)$, Theorem 5.1 shows that there exist sequences of matrix fields $A_i^n \in \mathcal{C}^\infty(\omega_{2R}; \mathbb{M}^\ell) \cap L^p(\omega_{2R}; \mathbb{M}^\ell)$ that satisfy

$$\begin{aligned} &\partial_j A_i^n - \partial_i A_j^n = A_i^n A_j^n - A_j^n A_i^n \text{ in } \omega_{2R}, \\ &A_i^n \to A_i \text{ in } L^p(\omega_{2R}; \mathbb{M}^\ell) \text{ as } n \to \infty. \end{aligned}$$

Since the coefficients A_i^n are smooth, the classical existence result on Pfaff systems (see, e.g., Thomas [17]) shows that there exists a matrix field $Y^n \in \mathcal{C}^\infty(\omega_{2R}; \mathbb{M}^{q\times\ell})$ that satisfies

$$\begin{aligned} &\partial_i Y^n = Y^n A_i^n \text{ in } \omega_{2R}, \\ &Y^n(x^0) = Y^0. \end{aligned} \tag{6.3}$$

By the stability result of Theorem 4.1, there exists a constant $C > 0$ such that

$$\|Y^n - Y^m\|_{W^{1,p}(B_r)} \le C \sum_i \|A_i^n - A_i^m\|_{L^p(\omega_{2R})},$$

which means that (Y^n) is a Cauchy sequence in the space $W^{1,p}(B_r; \mathbb{M}^{q\times\ell})$. Since this space is complete, there exists a field $Y \in W^{1,p}(B_r; \mathbb{M}^{q\times\ell})$ such that

$$Y^n \to Y \text{ in } W^{1,p}(B_r; \mathbb{M}^{q\times\ell}) \text{ as } n \to \infty.$$

In addition, the Sobolev continuous embedding $W^{1,p}(B_r; \mathbb{M}^{q\times\ell}) \subset \mathcal{C}^0(\overline{B}_r; \mathbb{M}^{q\times\ell})$ shows that

$$Y^n(x^0) \to Y(x^0) \text{ in } \mathbb{M}^{q\times\ell} \text{ as } n \to \infty.$$

By passing to the limit $n \to \infty$ in the equations of the system (6.3), we deduce that the field Y satisfies the Pfaff system (6.2). $\square$

If in addition to the assumptions of the previous Theorem, the set Ω is connected and simply-connected, then there exists a unique global solution (i.e., a solution defined over the entire set Ω) to the Pfaff system. More specifically, we have the following global existence result:

Theorem 6.4. *Let* $\Omega \subset \mathbb{R}^{\mathrm{d}}$ *be a connected and simply connected open set, let* $x^0 \in \Omega$*, and let* $Y^0 \in \mathbb{M}^{q\times\ell}$*. Let there be given* d *matrix fields* $A_i \in L^p_{\mathrm{loc}}(\Omega; \mathbb{M}^\ell)$*,* $p > \mathrm{d}$*, satisfying the relations (for all* i, j*)*

$$\partial_j A_i - \partial_i A_j = A_i A_j - A_j A_i$$

in the space of distributions $\mathcal{D}'(\Omega; \mathbb{M}^\ell)$*. Then the Pfaff system*

$$\begin{aligned} &\partial_i Y = Y A_i \ \textit{in}\ \mathcal{D}'(\Omega; \mathbb{M}^{q\times\ell}) \ \ \textit{for all}\ \ i, \\ &Y(x^0) = Y^0 \end{aligned} \tag{6.4}$$

has one and only one solution $Y \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{M}^{q\times\ell})$.

Proof. The uniqueness of the solution to the Pfaff system (6.4) being given by Theorem 4.2, it suffices to establish the existence of a solution to such a system.

Outline. We follow the proof of Theorem 3.1 of [12]. The idea is to define a field $Y : \Omega \to \mathbb{M}^{q\times\ell}$ by glueing together some sequences of local solutions, whose existence is established by Theorem 6.3, along curves starting from the given point x^0 (see step (i) below). We shall prove that this definition is unambiguous thanks to the uniqueness result proved in Theorem 4.2 and to the simple connectedness of the set Ω (see steps (ii)-(iv)). Finally, we show in step (v) that the field Y defined in this fashion satisfies the system (6.4).

(i) *Definition of a global solution to the Pfaff system* (6.4) *from local solutions.* With any point $x \in \Omega$, we associate a path $\gamma \in \mathcal{C}^0([0,1];\Omega)$ joining x^0 to x (i.e., $\gamma(0) = x^0$ and $\gamma(1) = x$), a positive number R, and a division $\Delta = \{t_0, t_1, t_2, ..., t_N\}$ of the interval $[0,1]$ such that

$$(6.5)\qquad R < \min\left(\frac{1}{\mathrm{d}}\mathrm{dist}(\mathrm{Im}\,\gamma, \Omega^c_\gamma), R(\Omega_\gamma)\right),$$

$$\gamma(t) \in B_R(x^i) \text{ for all } t \in [t_{i-1}, t_{i+1}] \text{ and all } i \in \{0, 1, ..., N\},$$

where

$$0 = t_{-1} = t_0 < t_1 < t_2 < ... < t_N = t_{N+1} = 1 \text{ and } x^i := \gamma(t_i),$$

$$R(\Omega_\gamma) \text{ is the number defined in Theorem 6.3,}$$

$$(6.6)\qquad \Omega_\gamma := \{x \in \mathbb{R}^{\mathrm{d}}\ ;\ \mathrm{dist}(x, \mathrm{Im}\,\gamma) < \zeta(\gamma, \Omega)\},$$

the number $\zeta(\gamma, \Omega)$ being defined by

$$(6.7)\qquad \zeta(\gamma, \Omega) = \begin{cases} \dfrac{1}{2}\mathrm{dist}(\mathrm{Im}\,\gamma, \Omega^c) & \text{if } \Omega \neq \mathbb{R}^{\mathrm{d}}, \\ 1 & \text{if } \Omega = \mathbb{R}^{\mathrm{d}}. \end{cases}$$

Note that $\Omega_\gamma \Subset \Omega$ since $\mathrm{dist}(\Omega_\gamma, \Omega^c) \geq \frac{1}{2}\mathrm{dist}(\mathrm{Im}\,\gamma, \Omega^c)$ in the case $\Omega \neq \mathbb{R}^{\mathrm{d}}$ and Ω_γ is bounded in the case $\Omega = \mathbb{R}^{\mathrm{d}}$. Therefore $A_i \in L^p(\Omega_\gamma; \mathbb{M}^\ell)$ and thus the number $R(\Omega_\gamma) > 0$ is well defined. Note also that $\mathrm{dist}(\mathrm{Im}\,\gamma, \Omega^c_\gamma) \geq \zeta(\gamma, \Omega)$ and $R < \dfrac{1}{\mathrm{d}}\mathrm{dist}(x, \Omega^c_\gamma)$ for all $x \in \mathrm{Im}\,\gamma$, inequalities which will be useful in the subsequent analysis.

The triple (γ, R, Δ) being chosen as before, we successively define the field $Y^i := Y^i(\gamma, R, \Delta) \in W^{1,p}(B_i; \mathbb{M}^{q\times\ell})$, $i = 0, 1, 2, ..., N$, where $B_i := B_R(x^i)$, as the solutions to the systems

$$(6.8)\qquad \begin{aligned} \partial_j Y^i &= Y^i A_j \text{ in } \mathcal{D}'(B_i; \mathbb{M}^{q\times\ell}), \\ Y^i(x^i) &= Y^{i-1}(x^i), \end{aligned}$$

with the convention that $Y^{-1}(x^0) := Y^0$. Note that this definition is correct thanks to Theorems 6.3 and 4.2 which establish the existence and uniqueness of such solutions, the assumption $R < \min\left(\dfrac{1}{\mathrm{d}}\mathrm{dist}(x^i, \Omega^c_\gamma), R(\Omega_\gamma)\right)$ needed in the first theorem being clearly satisfied.

Finally, we define the matrix field Y by letting

$$(6.9)\qquad Y(x) := Y^N(x).$$

We shall prove in the steps (ii)-(iv) below that this definition is unambiguous, that is, it does not depend on the choice of γ, R, and Δ.

(ii) *The definition* (6.9) *of* $Y(x)$ *does not depend on the division* Δ. Let a triple (γ, R, Δ) be given as in the step (i), let $t^* \in (t_k, t_{k+1})$, and let

$$\Delta^* = \{t_0, t_1, ..., t_k, t^*, t_{k+1}, ..., t_N\}.$$

With the triple (γ, R, Δ), we associate the functions Y^i, $i \in \{0, 1, 2, ..., N\}$, solutions to the systems (6.8). In the same way, with the triple (γ, R, Δ^*), we associate the functions Y_*^0, Y_*^1,..., Y_*^k, Y_*, Y_*^{k+1},...,Y_*^N. We wish to show that $Y^N(x) = Y_*^N(x)$.

By the uniqueness of the solution to the system (6.8) with $i = 0, 1, 2, ..., k$ (see Theorem 4.2), we deduce that

$$Y_*^i = Y^i \text{ in } B_i \text{ for all } i = 0, 1, 2, ..., k.$$

Let $x^* := \gamma(t^*)$ and $B_* := B_R(x^*)$. Since Y_* and Y^k satisfy

$$\begin{aligned} &\partial_j Y_* = Y_* A_j \text{ in } \mathcal{D}'(B_*; \mathbb{M}^{q\times\ell}), \\ &\partial_j Y^k = Y^k A_j \text{ in } \mathcal{D}'(B_k; \mathbb{M}^{q\times\ell}), \\ &Y_*(x^*) = Y_*^k(x^*) = Y^k(x^*), \end{aligned}$$

we infer from Theorem 4.2 that $Y_* = Y^k$ in $B_* \cap B_k$. In particular, $Y_*(x^{k+1}) = Y^k(x^{k+1})$, which next implies that $Y_*^{k+1}(x^{k+1}) = Y^{k+1}(x^{k+1})$. Since Y_*^{k+1} and Y^{k+1} satisfy in addition the relations

$$\begin{aligned} &\partial_j Y_*^{k+1} = Y_*^{k+1} A_j \text{ in } \mathcal{D}'(B_{k+1}; \mathbb{M}^{q\times\ell}), \\ &\partial_j Y^{k+1} = Y^{k+1} A_j \text{ in } \mathcal{D}'(B_{k+1}; \mathbb{M}^{q\times\ell}), \end{aligned}$$

we infer again from Theorem 4.2 that

$$Y_*^{k+1} = Y^{k+1} \text{ in } B_{k+1}.$$

By the uniqueness of the solution to the system (6.8) with $i = k+2, k+3, ..., N$, we finally obtain that

$$Y_*^i = Y^i \text{ in } B_i \text{ for all } i = k+2, k+3, ..., N.$$

In particular, $Y_*^N = Y^N$ in $B_R(x)$.

Now, let $\Delta = \{t_0, t_1, t_2, ..., t_N\}$ and $\Delta' = \{t'_0, t'_1, t'_2, ..., t'_M\}$ be two divisions of the interval $[0, 1]$ satisfying the conditions of step (i). Let $\Delta \cup \Delta' = \{s_0, s_1, s_2, ..., s_P\}$. Beginning with Δ and "refining" it to $\Delta \cup \Delta'$, we show by using the previous argument a finite number of times that

$$Y^P(\gamma, R, \Delta \cup \Delta') = Y^N(\gamma, R, \Delta) \text{ in } B_R(x).$$

In the same way, but "refining" Δ' to $\Delta \cup \Delta'$, we also find that

$$Y^P(\gamma, R, \Delta \cup \Delta') = Y^M(\gamma, R, \Delta') \text{ in } B_R(x).$$

Therefore, $Y^N(\gamma, R, \Delta) = Y^M(\gamma, R, \Delta')$ in $B_R(x)$. This implies that the definition (6.9) of $Y(x)$ does not depend on the division Δ.

(iii) *The definition* (6.9) *of* $Y(x)$ *does not depend on the number* R. Let $Y(x)$ be defined as in step (i) and let $\tilde{R} > R$ be a number that satisfies the inequality

(6.5). For each $i \in \{0, 1, 2, ..., N\}$, let $\tilde{Y}^i \in W^{1,p}(B_{\tilde{R}}(x^i); \mathbb{M}^{q\times\ell})$ be the solution to the system

$$\partial_j \tilde{Y}^i = \tilde{Y}^i A_j \text{ in } \mathcal{D}'(B_{\tilde{R}}(x^i); \mathbb{M}^{q\times\ell}),$$
$$\tilde{Y}^i(x^i) = \tilde{Y}^{i-1}(x^i),$$

where $\tilde{Y}^{-1}(x^0) := Y^0$. We now prove by a recursion argument that $\tilde{Y}^i = Y^i$ in $B_R(x^i)$.

For $i = 0$ this is a consequence of Theorem 4.2. Assume now that $\tilde{Y}^k = Y^k$ in $B_R(x^k)$ for a fixed $k \in \{0, 1, 2, ..., N-1\}$. Since $x^{k+1} \in B_R(x^k) \subset B_{\tilde{R}}(x^k)$, it follows that $\tilde{Y}^k(x^{k+1}) = Y^k(x^{k+1})$. By the uniqueness of the solution to the system (6.8) with $i := k+1$, this implies that $\tilde{Y}^{k+1} = Y^{k+1}$ in $B_R(x^{k+1})$. After a finite number of iterations, we obtain that $\tilde{Y}^N = Y^N$ in $B_R(x)$. Hence the definition (6.9) of $Y(x)$ does not depend on the number R.

(iv) *The definition* (6.9) *of* $Y(x)$ *does not depend on the path* γ. Let there be given two paths $\gamma, \tilde{\gamma} \in \mathcal{C}^0([0,1]; \Omega)$ joining x^0 to x. Since Ω is simply connected, there exists a homotopy $\boldsymbol{\varphi} \in C^0([0,1]\times[0,1]; \Omega)$ such that

$$\boldsymbol{\varphi}(t,0) = \gamma(t),\ \boldsymbol{\varphi}(t,1) = \tilde{\gamma}(t),$$
$$\boldsymbol{\varphi}(0,s) = x^0, \boldsymbol{\varphi}(1,s) = x.$$

Let a number $R > 0$ be fixed that satisfies

$$R < \min\left(\frac{1}{\mathrm{d}} \inf_{s\in[0,1]} \Big\{\mathrm{dist}\big(\mathrm{Im}\,\boldsymbol{\varphi}(\cdot,s), \Omega^c_{\boldsymbol{\varphi}(\cdot,s)}\big)\Big\}, R(\Omega_{\boldsymbol{\varphi}})\right), \tag{6.10}$$

where $R(\Omega_{\boldsymbol{\varphi}})$ is the number defined in Theorem 6.3 and

$$\Omega_{\boldsymbol{\varphi}} := \bigcup_{s\in[0,1]} \Omega_{\boldsymbol{\varphi}(\cdot,s)},$$

where $\Omega_{\boldsymbol{\varphi}(\cdot,s)}$ is given by (6.6) with $\gamma = \boldsymbol{\varphi}(\cdot,s)$. Note that such a number R exists since, on one hand, $\mathrm{Im}\,\boldsymbol{\varphi} \Subset \Omega$ and $\Omega_{\boldsymbol{\varphi}} \Subset \Omega$ $\big($the last inclusion being a consequence of the inequality $\mathrm{dist}(\Omega_{\boldsymbol{\varphi}}, \Omega^c) \geq \frac{1}{2}\mathrm{dist}(\mathrm{Im}\,\boldsymbol{\varphi}, \Omega^c)$ in the case $\Omega \neq \mathbb{R}^{\mathrm{d}}$ and of the fact that $\Omega_{\boldsymbol{\varphi}}$ is bounded in the case $\Omega = \mathbb{R}^{\mathrm{d}}\big)$ and, on the other hand, for all $s \in [0,1]$ we have that

$$\mathrm{dist}\big(\mathrm{Im}\,\boldsymbol{\varphi}(\cdot,s), \Omega^c_{\boldsymbol{\varphi}(\cdot,s)}\big) \geq \frac{1}{2}\mathrm{dist}\big(\mathrm{Im}\,\boldsymbol{\varphi}(\cdot,s), \Omega^c\big) \geq \frac{1}{2}\mathrm{dist}(\mathrm{Im}\,\boldsymbol{\varphi}, \Omega^c) \qquad \text{if } \Omega \neq \mathbb{R}^{\mathrm{d}},$$
$$\mathrm{dist}\big(\mathrm{Im}\,\boldsymbol{\varphi}(\cdot,s), \Omega^c_{\boldsymbol{\varphi}(\cdot,s)}\big) \geq 1 \qquad \text{if } \Omega = \mathbb{R}^{\mathrm{d}}.$$

Since $\boldsymbol{\varphi}$ is uniformly continuous over the compact set $[0,1]\times[0,1]$, there exists an integer $N > 0$ such that

$$|\boldsymbol{\varphi}(t,s) - \boldsymbol{\varphi}(t',s')| < R \tag{6.11}$$

for all $(t,s), (t',s') \in [0,1]\times[0,1]$ that satisfy $\{|t-t'|^2 + |s-s'|^2\}^{1/2} \leq \sqrt{2}/N$. Let $t_k = s_k := k/N$ for all $k = 0, 1, ..., N$. Then one can see that $Y(x)$ can be defined as in step (i) by means of the path $\gamma^k := \boldsymbol{\varphi}(\cdot, s_k)$, of the number R, and of the division $\Delta := \{t_0, t_1, t_2, ..., t_N\}$.

In order to prove that the definition (6.9) of $Y(x)$ does not depend on the choice of the path joining x^0 to x, it suffices to prove that, for a given $k \in \{1, 2, ..., N\}$, the definition (6.9) based on the triple $(\gamma^{k-1}, R, \Delta)$ coincides with the definition (6.9) based on the triple (γ^k, R, Δ).

Let $x^{k,i} := \gamma^k(t_i)$ and $B_{k,i} := B_R(x^{k,i})$. For each $i \in \{0,1,2,...,N\}$, let $Y^{k,i} \in W^{1,p}(B_{k,i};\mathbb{M}^{q\times\ell})$ be the solution to the system (see (6.8))

$$\begin{aligned} &\partial_j Y^{k,i} = Y^{k,i}A_j \text{ in } \mathcal{D}'(B_{k,i};\mathbb{M}^{q\times\ell}),\\ &Y^{k,i}(x^{k,i}) = Y^{k,i-1}(x^{k,i}), \end{aligned} \tag{6.12}$$

where $Y^{k,-1}(x^{k,0}) := Y^0$. We wish to prove that $Y^{k-1,N}(x) = Y^{k,N}(x)$.

First, notice that $Y^{k-1,0} = Y^{k,0}$ in $B_{k-1,0}\cap B_{k,0}$ thanks to the uniqueness of the solution to the system (6.12) with $i = 0$ (see Theorem 4.2). Assume that for a fixed $i \in \{0,1,2,...,N-1\}$ we have $Y^{k-1,i} = Y^{k,i}$ in $B_{k-1,i}\cap B_{k,i}$. In particular, $Y^{k-1,i}(x^{k-1,i+1}) = Y^{k,i}(x^{k-1,i+1})$, which implies on the one hand that

$$Y^{k-1,i+1}(x^{k-1,i+1}) = Y^{k,i}(x^{k-1,i+1}).$$

On the other hand, Theorem 4.2 implies that $Y^{k,i+1} = Y^{k,i}$ in the set $B_{k,i+1}\cap B_{k,i}$. Since $x^{k-1,i+1}$ belongs to this set, we obtain in particular that

$$Y^{k,i+1}(x^{k-1,i+1}) = Y^{k,i}(x^{k-1,i+1}).$$

Combining the two relations above gives that

$$Y^{k-1,i+1}(x^{k-1,i+1}) = Y^{k,i+1}(x^{k-1,i+1}).$$

We then infer from Theorem 4.2 that $Y^{k-1,i+1} = Y^{k,i+1}$ in $B_{k-1,i+1}\cap B_{k,i+1}$. After N iterations, we eventually find that $Y^{k-1,N} = Y^{k,N}$ in $B_{k-1,N}\cap B_{k,N} = B_R(x)$, which implies in particular that $Y^{k-1,N}(x) = Y^{k,N}(x)$. Therefore, the definition (6.9) of $Y(x)$ does not depend on the choice of the path γ joining x^0 to x.

(v) *The field Y defined in step (i) satisfies the Pfaff system* (6.4). For any given point $x \in \Omega$, let $\gamma \in \mathcal{C}^0([0,1];\Omega)$ be a path joining x^0 to x, let

$$\Omega_\gamma^\sharp := \left\{x \in \mathbb{R}^{\mathrm{d}}\ ;\ \mathrm{dist}(x;\operatorname{Im}\gamma) < \frac{3}{2}\zeta(\gamma,\Omega)\right\}\ ,$$

let $R > 0$ be a number that satisfies

$$R < \min\left(\frac{2}{2\mathrm{d}+1}\zeta(\gamma,\Omega), R(\Omega_\gamma^\sharp)\right),$$

and let a division $\Delta := \{t_0,t_1,\dots,t_N\}$ of the interval $[0,1]$ that satisfies

$$\gamma(t) \in B_R(x^i) \text{ for all } t \in [t_{i-1},t_{i+1}] \text{ and all } i \in \{0,1,...,N\},$$

where $0 = t_{-1} = t_0 < t_1 < t_2 < ... < t_N = t_{N+1} = 1$ and $x^i := \gamma(t_i)$. Remember that $\zeta(\gamma,\Omega)$ is the number defined by (6.7), $R(\Omega_\gamma^\sharp)$ is the number defined in Theorem 6.3, and note that $A_i \in L^p(\Omega_\gamma^\sharp;\mathbb{M}^\ell)$ since $\Omega_\gamma^\sharp \Subset \Omega$ for reasons similar to those used in step (i) to prove that $\Omega_\gamma \Subset \Omega$.

It is clear that $\Omega_\gamma \subset \Omega_\gamma^\sharp$ and $R < \frac{1}{\mathrm{d}}\zeta(\gamma,\Omega) \le \frac{1}{\mathrm{d}}\mathrm{dist}(\operatorname{Im}\gamma,\Omega_\gamma^c)$, hence that $R < \min\left(\frac{1}{\mathrm{d}}\mathrm{dist}(\operatorname{Im}\gamma,\Omega_\gamma^c), R(\Omega_\gamma)\right)$. This allows us to define the value of Y at the point x by letting $Y(x) = Y^N(x)$, where Y^N is defined as in step (i) by means of γ, R and Δ. Now, we prove that in fact $Y = Y^N$ over the entire ball $B_R(x)$.

Let $\tilde{x}$ be a fixed, but otherwise arbitrary, point in $B_R(x)$, let the path $\tilde{\gamma}$ be defined by (note that $\tilde{\gamma}$ is a path joining x^0 to $\tilde{x}$)

$$\tilde{\gamma}(\tilde{t}) = \begin{cases} \gamma(2\tilde{t}) & \text{for all } \tilde{t} \in [0,1/2]\,,\\ (2-2\tilde{t})x + (2\tilde{t}-1)\tilde{x} & \text{for all } \tilde{t} \in (1/2,1]\,, \end{cases}$$

and let $\tilde{\Delta} := \{\tilde{t}_0, \tilde{t}_1, \ldots, \tilde{t}_N, \tilde{t}_{N+1}\}$, where $\tilde{t}_i = t_i/2$ for all $i \in \{0, 1, 2, \ldots, N\}$ and $\tilde{t}_{N+1} = 1$. Using the inequalities $\text{dist}(z, \text{Im}\,\tilde{\gamma}) \leq \text{dist}(z, \text{Im}\,\gamma) \leq \text{dist}(z, \text{Im}\,\tilde{\gamma}) + R$, which are valid for all $z \in \mathbb{R}^{\text{d}}$, one can prove that $\Omega_{\tilde{\gamma}} \subset \Omega_\gamma^\sharp$ and that $\text{dist}(\text{Im}\,\tilde{\gamma}, \Omega_{\tilde{\gamma}}^c) \geq \zeta(\tilde{\gamma}, \Omega) \geq \zeta(\gamma, \Omega) - \frac{R}{2}$, hence

$$R < \min\left(\frac{1}{\text{d}}\text{dist}(\text{Im}\,\tilde{\gamma}, \Omega_{\tilde{\gamma}}^c), R(\Omega_{\tilde{\gamma}})\right).$$

Therefore we can define $Y(\tilde{x})$ by means of $\tilde{\gamma}$, R and $\tilde{\Delta}$ in the same way that $Y(x)$ is defined by means of γ, R and Δ. More specifically, let $\tilde{x}^i := \tilde{\gamma}(\tilde{t}_i)$. Then

$$Y(\tilde{x}) := \tilde{Y}^{N+1}(\tilde{x}),$$

where, for all $i = 0, 1, 2, ..., N+1$, $\tilde{Y}^i = \tilde{Y}^i(\tilde{\gamma}, R, \tilde{\Delta}) \in W^{1,p}(B_R(\tilde{x}^i); \mathbb{M}^{q\times\ell})$ is the solution to the system

$$\begin{aligned} &\partial_j \tilde{Y}^i = \tilde{Y}^i A_j \text{ in } \mathcal{D}'(B_R(\tilde{x}^i); \mathbb{M}^{q\times\ell}), \\ &\tilde{Y}^i(\tilde{x}^i) = \tilde{Y}^{i-1}(\tilde{x}^i), \end{aligned} \tag{6.13}$$

where $\tilde{Y}^{-1}(\tilde{x}^0) := Y^0$.

Since $x^i = \tilde{x}^i$ for all $i \in \{0, 1, 2, ..., N\}$, we infer from the uniqueness result of Theorem 4.2 that $\tilde{Y}^i = Y^i$ in $B_R(x^i)$ for all $i \in \{0, 1, 2, ..., N\}$. Hence $\tilde{Y}^N = Y^N$ in $B_R(x^N)$, which implies in particular that $\tilde{Y}^N(\tilde{x}) = Y^N(\tilde{x})$. On the other hand, since $\tilde{x} = \tilde{x}^{N+1}$, we have $\tilde{Y}^{N+1}(\tilde{x}) = \tilde{Y}^N(\tilde{x})$ by the second equation of (6.13). By combining these relations, we finally obtain

$$Y(\tilde{x}) = \tilde{Y}^{N+1}(\tilde{x}) = \tilde{Y}^N(\tilde{x}) = Y^N(\tilde{x}).$$

We have proved that the matrix field $Y : \Omega \to \mathbb{M}^{q\times\ell}$ satisfies $Y = Y^N$ in the open ball $B_R(x)$. Since Y^N belongs to the space $W^{1,p}(B(x, R); \mathbb{M}^{q\times\ell})$ and satisfies the system

$$\partial_j Y^N = Y^N A_j \text{ in } \mathcal{D}'(B_R(x); \mathbb{M}^{q\times\ell}),$$

letting x vary in the set Ω shows that Y belongs to the space $W^{1,p}_{\text{loc}}(\Omega; \mathbb{M}^{q\times\ell})$ and satisfies the system

$$\begin{aligned} &\partial_j Y = Y A_j \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^{q\times\ell}), \\ &Y(x^0) = Y^0. \end{aligned}$$

This completes the proof of the Theorem.

□

The proof of Theorem 6.4, which shows how to obtain a global existence result from a local existence result and a uniqueness result, is based on a strategy that can be easily adapted to other systems of partial differential equations. In particular, this strategy can be used to prove the following generalized Poincaré theorem in simply connected sets by using the Poincaré theorem in a cube, the latter being established by Theorem 2.5. Note that this generalised Poincaré theorem was mentioned in the Introduction in relation to the solvability of a nonhomogeneous system of first order partial differential equations of type (1.4) (see Theorem 1.1).

Theorem 6.5. *Let Ω be a connected and simply connected open subset of $\mathbb{R}^{\text{d}}$. Let $f_1, f_2, ..., f_{\text{d}} \in L^p_{\text{loc}}(\Omega)$, where $p \geq 1$, satisfy in the distributional sense the compatibility conditions*

$$\partial_i f_j = \partial_j f_i \text{ in } \Omega$$

for all $i, j \in \{1, 2, ..., \mathrm{d}\}$*. Then there exists* $\psi \in W^{1,p}_{\mathrm{loc}}(\Omega)$*, unique up to a constant, such that*

$$\partial_i \psi = f_i \ \text{in } \Omega$$

for all $i \in \{1, 2, ..., \mathrm{d}\}$*.*

We end this section with a regularity result "up to the boundary" for Pfaff systems. As expected, this will require global L^p-regularity for the matrix fields A_i, but also some regularity condition for the domain Ω.

Definition 6.6. We say that Ω satisfies the uniform interior cone condition if there exists a fixed open cone ω such that, for every $x \in \Omega$, there exists a cone ω_x congruent with ω, with vertex x, such that $\overline{\omega}_x \subset \Omega$.

Remark 6.7. All bounded open sets with Lipschitz boundary satisfy the uniform interior cone condition, but the class of open sets with this property is strictly larger. For example, the set $B_R(x) \setminus \{x\}$ satisfy the uniform interior cone condition. The property also allows for sets with inward cusps.

Theorem 6.8. *Let* $\Omega \subset \mathbb{R}^{\mathrm{d}}$ *be a connected and simply connected bounded open set that satisfies the uniform interior cone condition. Let there be given a point* $x^0 \in \Omega$*,* $p > \mathrm{d}$*, and* $Y^0 \in \mathbb{M}^{q\times\ell}$*. Let* $A_i \in L^p(\Omega; \mathbb{M}^\ell)$ *be matrix fields that satisfy the relations (for all* i, j*)*

$$\partial_j A_i - \partial_i A_j = A_i A_j - A_j A_i$$

in the space of distributions $\mathcal{D}'(\Omega; \mathbb{M}^\ell)$*. Then the Pfaff system*

$$\begin{aligned} &\partial_i Y = Y A_i \ \text{ in } \mathcal{D}'(\Omega; \mathbb{M}^{q\times\ell}) \ \text{ for all } \ i, \\ &Y(x^0) = Y^0 \end{aligned} \tag{6.14}$$

has one and only one solution $Y \in W^{1,p}(\Omega; \mathbb{M}^{q\times\ell})$*.*

Proof. In this proof we drop the matrix spaces from notation such as $L^p(\Omega; \mathbb{M}^\ell)$, $W^{1,p}(\Omega; \mathbb{M}^{q\times\ell})$, etc.

By Theorem 6.4, we know that there exists a unique solution $Y \in W^{1,p}_{\mathrm{loc}}(\Omega)$ to the problem (6.14). We only have to prove that Y belongs in fact to the space $W^{1,p}(\Omega)$.

Let ω be the cone of the uniform interior cone condition satisfied by Ω and let a point $y \in \omega$ be given once and for all. Let $\varepsilon := \mathrm{dist}(y, \omega^c)$ and note that $\varepsilon > 0$.

Now let a point $x \in \Omega$ and choose a cone ω_x with vertex x that is congruent with ω and satisfies $\overline{\omega}_x \subset \Omega$. Since $\omega_x \Subset \Omega$, we have that $Y \in W^{1,p}(\omega_x)$. Let $y_x \in \omega_x$ be the point corresponding to $y \in \omega$ through the congruence of ω_x with ω. By the inequality (2.2) of Corollary 2.3, applied for each component of the matrix field Y, there exists a constant C depending only on d, p and $|\omega_0|$ (ω_0 is the cone of diameter 1 homothetic with ω) such that, for all $z \in \overline{\omega}_x$,

$$\begin{aligned}|Y(z)| &\leq |Y(y_x)| + CD_\omega^{1-\mathrm{d}/p} \sum_{i=1}^{\mathrm{d}} \|\partial_i Y\|_{L^p(\omega_x)} \\ &= |Y(y_x)| + CD_\omega^{1-\mathrm{d}/p} \sum_{i=1}^{\mathrm{d}} \|YA_i\|_{L^p(\omega_x)} \\ &\leq |Y(y_x)| + CD_\omega^{1-\mathrm{d}/p} \sum_{i=1}^{\mathrm{d}} \|Y\|_{L^\infty(\overline{\omega}_x)} \|A_i\|_{L^p(\omega_x)} \\ &\leq |Y(y_x)| + CD_\omega^{1-\mathrm{d}/p} \|Y\|_{L^\infty(\overline{\omega}_x)} \sum_{i=1}^{\mathrm{d}} \|A_i\|_{L^p(\Omega)},\end{aligned}$$

with the notation of Corollary 2.3. Therefore

$$\Big(1 - CD_\omega^{1-\mathrm{d}/p} \sum_{i=1}^{\mathrm{d}} \|A_i\|_{L^p(\Omega)}\Big) \|Y\|_{L^\infty(\overline{\omega}_x)} \leq |Y(y_x)| \leq \|Y\|_{L^\infty(\Omega_\varepsilon)},$$

where $\Omega_\varepsilon := \{x \in \Omega\ ;\ \mathrm{dist}(x, \Omega^c) > \varepsilon\}$. To obtain the last inequality, we used the fact that $y_x \in \Omega_\varepsilon$ (since $\mathrm{dist}(y_x, \Omega^c) > \mathrm{dist}(y_x, \omega_x^c) = \varepsilon$) and the inclusion $\Omega_\varepsilon \Subset \Omega$, which implies that $Y \in L^\infty(\Omega_\varepsilon)$.

It is obvious that if Ω satisfies the uniform interior cone condition with a fixed cone ω, then it satisfies the same condition with any smaller cone homothetic with ω. Therefore we can choose a cone ω', smaller than ω, whose diameter $D_{\omega'}$ satisfies

$$CD_{\omega'}^{1-\mathrm{d}/p} \sum_{i=1}^{\mathrm{d}} \|A_i\|_{L^p(\Omega)} \leq \frac{1}{2}\,.$$

Then we deduce from the previous inequality that for all $x \in \Omega$,

$$\|Y\|_{L^\infty(\overline{\omega'_x})} \leq 2\|Y\|_{L^\infty(\Omega_{\varepsilon'})},$$

where $\varepsilon' = \mathrm{dist}(y', {\omega'}^c)$, y' being the point in ω' that was fixed once and for all. In particular,

$$|Y(x)| \leq 2\|Y\|_{L^\infty(\Omega_{\varepsilon'})}$$

for all $x \in \Omega$, hence $Y \in L^\infty(\Omega)$ and $\|Y\|_{L^\infty(\Omega)} \leq 2\|Y\|_{L^\infty(\Omega_{\varepsilon'})}$.

Since $\partial_i Y = YA_i$ for all $i \in \{1, \ldots, \mathrm{d}\}$ and $A_i \in L^p(\Omega)$, we obtain $\partial_i Y \in L^p(\Omega)$ for all $i \in \{1, \ldots, \mathrm{d}\}$. Therefore $Y \in W^{1,p}(\Omega)$.

□

Remark 6.9. Since the domain ω that appears in the statement of Corollary 2.3 only needs to be a convex set, not necessarily a cone, we can replace the "uniform interior cone condition" with an apparently weaker "uniform interior convex set condition", defined as in Definition 6.6 but with the fixed cone replaced by a fixed bounded open convex set ω and with the condition that x is the vertex of ω_x replaced by $x \in \overline{\omega}_x$. However, a closer look shows that these properties are equivalent, since any bounded open convex set satisfies the uniform interior cone condition.

7. Appendix

For completeness, we give here a simple proof of the regularity result for elliptic equations with mixed boundary conditions used in this paper. Throughout the Appendix the number d is an integer ≥ 2 given once and for all.

We begin with the following preliminary result about extensions:

Lemma 7.1. *Let $\Omega := (-L, L) \times \omega$ and $\tilde{\Omega} := (-3L, 3L) \times \omega$, where ω is an open set in $\mathbb{R}^{\mathrm{d}-1}$, and let $\Gamma_0 \subset \partial\omega$ be a relatively open subset of the boundary of ω. For any $s \geq 1$, define the spaces*

$$\begin{aligned}
X_s(\Omega) &:= \{v \in W^{1,s}(\Omega);\ v = 0 \text{ on } (-L, L) \times \Gamma_0\},\\
X_s(\tilde{\Omega}) &:= \{\tilde{v} \in W^{1,s}(\tilde{\Omega});\ \tilde{v} = 0 \text{ on } (-3L, 3L) \times \Gamma_0\},\\
Y_s(\Omega) &:= \{v \in W^{1,s}(\Omega);\ v = 0 \text{ on } ((-L, L) \times \Gamma_0) \cup (\{-L\} \times \omega) \cup (\{L\} \times \omega)\},\\
Y_s(\tilde{\Omega}) &:= \{\tilde{v} \in W^{1,s}(\tilde{\Omega});\ \tilde{v} = 0 \text{ on } ((-3L, 3L) \times \Gamma_0) \cup (\{-3L\} \times \omega) \cup (\{3L\} \times \omega)\},
\end{aligned}$$

and define the operators $S : L^s(\Omega) \to L^s(\tilde{\Omega})$ and $A : L^s(\Omega) \to L^s(\tilde{\Omega})$ by letting, for all $f \in L^s(\Omega)$,

$$(Sf)(x_1, x') = \begin{cases} f(x_1, x') & \text{if } (x_1, x') \in (-L, L) \times \omega,\\ f(2L - x_1, x') & \text{if } (x_1, x') \in (L, 3L) \times \omega,\\ f(-2L - x_1, x') & \text{if } (x_1, x') \in (-3L, -L) \times \omega, \end{cases}$$

and

$$(Af)(x_1, x') = \begin{cases} f(x_1, x') & \text{if } (x_1, x') \in (-L, L) \times \omega,\\ -f(2L - x_1, x') & \text{if } (x_1, x') \in (L, 3L) \times \omega,\\ -f(-2L - x_1, x') & \text{if } (x_1, x') \in (-3L, -L) \times \omega. \end{cases}$$

Let $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L^r(\Omega)$, where $r \geq 1$ and satisfies

$$\begin{aligned}
&\frac{1}{r} + \frac{1}{q} \leq 1 + \frac{1}{\mathrm{d}} \qquad && \text{if } \ q \neq \mathrm{d},\\
&r > 1 && \text{if } \ q = \mathrm{d}.
\end{aligned}$$

(i) If $u \in X_p(\Omega)$ satisfies the equation

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \text{ for all } \quad v \in X_q(\Omega),$$

then (Su) belongs to the space $X_p(\tilde{\Omega})$ and satisfies the equation

$$\int_{\tilde{\Omega}} \nabla(Su) \cdot \nabla \tilde{v} \, dx = \int_{\tilde{\Omega}} (Sf) \tilde{v} \, dx \quad \text{ for all } \quad \tilde{v} \in X_q(\tilde{\Omega}).$$

(ii) If $u \in Y_p(\Omega)$ satisfies the equation

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \text{ for all } \quad v \in Y_q(\Omega),$$

then (Au) belongs to the space $Y_p(\tilde{\Omega})$ and satisfies the equation

$$\int_{\tilde{\Omega}} \nabla(Au) \cdot \nabla \tilde{v} \, dx = \int_{\tilde{\Omega}} (Af) \tilde{v} \, dx \quad \text{ for all } \quad \tilde{v} \in Y_q(\tilde{\Omega}).$$

Proof. First note that the integrals appearing in the right hand side of the variational equations above are well defined since the test functions v and $\tilde{v}$ are of class $L^{\frac{r}{r-1}}$ (the dual of L^r) thanks to the Sobolev embedding $W^{1,q} \subset L^{\frac{r}{r-1}}$.

(i) It is clear from the definition of partial derivatives in the distributional sense that $(Su) \in W^{1,p}(\tilde{\Omega})$, which next implies that $(Su) \in X_p(\tilde{\Omega})$. For each $\tilde{v} \in X_q(\tilde{\Omega})$, let $v, w : \Omega \to \mathbb{R}$ be defined by

$$\begin{aligned} v(x_1, x') &= \tilde{v}(2L - x_1, x') \qquad \text{for all } \ (x_1, x') \in \Omega, \\ w(x_1, x') &= \tilde{v}(-2L - x_1, x') \quad \ \text{for all } \ (x_1, x') \in \Omega. \end{aligned}$$

Then $v, w \in W^{1,q}(\Omega)$ and, by changing variables in the integrals over the sets $(L, 3L) \times \omega$ and $(-3L, -L) \times \omega$, we have

$$\int_{\tilde{\Omega}} \nabla(Su) \cdot \nabla\tilde{v}\, dx = \int_{\Omega} \nabla u \cdot \nabla\tilde{v}\, dx + \int_{\Omega} \nabla u \cdot \nabla v\, dx + \int_{\Omega} \nabla u \cdot \nabla w\, dx.$$

Since $(\tilde{v}+v+w) \in X_q(\Omega)$, we next have, again by changing variables in the integral over the sets $(L, 3L) \times \omega$ and $(-3L, -L) \times \omega$, that

$$\int_{\tilde{\Omega}} \nabla(Su) \cdot \nabla\tilde{v}\, dx = \int_{\Omega} f(\tilde{v} + v + w)\, dx = \int_{\tilde{\Omega}} (Sf)\tilde{v}\, dx.$$

(ii) Since $u = 0$ on $\{-L, L\} \times \omega$, the definition of the partial derivatives of u in the distributional sense shows that $(Au) \in W^{1,p}(\tilde{\Omega})$, which next implies that $(Au) \in Y_p(\tilde{\Omega})$. For each $\tilde{v} \in Y_q(\tilde{\Omega})$, let $v, w : \Omega \to \mathbb{R}$ be defined by

$$\begin{aligned} v(x_1, x') &= -\tilde{v}(2L - x_1, x') \qquad \text{for all } \ (x_1, x') \in \Omega, \\ w(x_1, x') &= -\tilde{v}(-2L - x_1, x') \quad \ \text{for all } \ (x_1, x') \in \Omega. \end{aligned}$$

Then $v, w \in W^{1,q}(\Omega)$ and, by using a change of variables, we have

$$\int_{\tilde{\Omega}} \nabla(Au) \cdot \nabla\tilde{v}\, dx = \int_{\Omega} \nabla u \cdot \nabla\tilde{v}\, dx + \int_{\Omega} \nabla u \cdot \nabla v\, dx + \int_{\Omega} \nabla u \cdot \nabla w\, dx.$$

Since $(\tilde{v} + v + w) \in Y_q(\Omega)$ (in particular, $(\tilde{v} + v + w)$ vanishes on $\{-L, L\} \times \omega$), we next have, by using a change of variables,

$$\int_{\tilde{\Omega}} \nabla(Au) \cdot \nabla\tilde{v}\, dx = \int_{\Omega} f(\tilde{v} + v + w)\, dx = \int_{\tilde{\Omega}} (Af)\tilde{v}\, dx.$$

□

The remainder of Appendix is dedicated to the proof of Theorem 2.9. We begin by specifying the notation used henceforth.

If $\Omega \subset \mathbb{R}^{\mathrm{d}}$ is a generic cuboid, i.e.,

$$\Omega := (-a_1, a_1) \times (-a_2, a_2) \times ... \times (-a_{\mathrm{d}}, a_{\mathrm{d}}),$$

we denote Γ_j, where j is any number in the set $\{1, 2, ..., \mathrm{d}\}$, the union of its two lateral faces that are orthogonal to the x_j-axis, namely

$$\Gamma_j := (-a_1, a_1) \times ... \times (-a_{j-1}, a_{j-1}) \times \{-a_j, a_j\} \times (-a_{j+1}, a_{j+1}) \times ... \times (-a_{\mathrm{d}}, a_{\mathrm{d}}).$$

Note that the boundary of Ω satisfies

$$\partial\Omega = \bigcup_{j=1}^{\mathrm{d}} \overline{\Gamma}_j.$$

For any real number $s \geq 1$ and for any fixed subset of indices $\{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., \mathrm{d}\}$, we define the space

$$V_s(\Omega) := \{v \in W^{1,s}(\Omega);\ v = 0 \text{ on } \Gamma_{i_1} \cup \Gamma_{i_2} \cup ... \cup \Gamma_{i_k}\}. \tag{7.1}$$

Then we have the following regularity result for variational equations:

Theorem 7.2. *Let ω be an open cuboid in $\mathbb{R}^{\mathrm{d}}$ whose edges are parallel to the axes of coordinates and let $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, ..., \mathrm{d}\}$. Let $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L^r(\omega)$, where $r \in (1, +\infty)$ satisfies $\frac{1}{r} + \frac{1}{q} \leq 1 + \frac{1}{\mathrm{d}}$. If $u \in V_p(\omega)$ satisfies the variational equation*

$$\int_\omega \nabla u \cdot \nabla v \, dx = \int_\omega f v \, dx \quad \textit{for all} \quad v \in V_q(\omega), \tag{7.2}$$

then $u \in W^{2,r}(\omega)$.

Proof. Without losing in generality, we may assume that the indices appearing in the definition of $V_p(\omega)$ and $V_q(\omega)$ are $i_1 = 1$, $i_2 = 2$, ... , $i_k = k$. We apply several times the previous Lemma.

First, we define the extensions $u^1, f^1 : \omega^1 \to \mathbb{R}$ of $u, f : \omega \to \mathbb{R}$, where

$$\omega^1 := (-3a_1, 3a_1) \times (-a_2, a_2) \times ... \times (-a_{\mathrm{d}}, a_{\mathrm{d}}),$$

by letting, for all $x'' := (x_2, ..., x_{\mathrm{d}}) \in (-a_2, a_2) \times ... \times (-a_{\mathrm{d}}, a_{\mathrm{d}})$,

$$u^1(x_1, x'') = \begin{cases} u(x_1, x'') & \text{if } x_1 \in (-a_1, a_1), \\ -u(2a_1 - x_1, x'') & \text{if } x_1 \in (a_1, 3a_1), \\ -u(-2a_1 - x_1, x'') & \text{if } x_1 \in (-3a_1, -a_1) \end{cases}$$

and

$$f^1(x_1, x'') = \begin{cases} f(x_1, x'') & \text{if } x_1 \in (-a_1, a_1), \\ -f(2a_1 - x_1, x'') & \text{if } x_1 \in (a_1, 3a_1), \\ -f(-2a_1 - x_1, x'') & \text{if } x_1 \in (-3a_1, -a_1). \end{cases}$$

Then Lemma 7.1(ii) shows that u^1 belongs to the space $V_p(\omega^1)$ and satisfies the variational equation

$$\int_{\omega^1} \nabla u^1 \cdot \nabla v \, dx = \int_{\omega^1} f^1 v \, dx \quad \text{for all} \quad v \in V_q(\omega^1).$$

Second, we define the extensions $u^2, f^2 : \omega^2 \to \mathbb{R}$ of $u^1, f^1 : \omega^1 \to \mathbb{R}$, where

$$\omega^2 := (-3a_1, 3a_1) \times (-3a_2, 3a_2) \times (-a_3, a_3) ... \times (-a_{\mathrm{d}}, a_{\mathrm{d}}),$$

by letting, for all $x_1 \in (-3a_1, 3a_1)$ and $x'' := (x_3, ..., x_{\mathrm{d}}) \in (-a_3, a_3) \times ... \times (-a_{\mathrm{d}}, a_{\mathrm{d}})$,

$$u^2(x_1, x_2, x'') = \begin{cases} u^1(x_1, x_2, x'') & \text{if } x_2 \in (-a_2, a_2), \\ -u^1(x_1, 2a_2 - x_2, x'') & \text{if } x_2 \in (a_2, 3a_2), \\ -u^1(x_1, -2a_2 - x_2, x'') & \text{if } x_2 \in (-3a_2, -a_2) \end{cases}$$

and

$$f^2(x_1, x_2, x'') = \begin{cases} f^1(x_1, x_2, x'') & \text{if } x_2 \in (-a_2, a_2), \\ -f^1(x_1, 2a_2 - x_2, x'') & \text{if } x_2 \in (a_2, 3a_2), \\ -f^1(x_1, -2a_2 - x_2, x'') & \text{if } x_2 \in (-3a_2, -a_2). \end{cases}$$

Then Lemma 7.1(ii) shows that u^2 belongs to the space $V_p(\omega^2)$ and satisfies the variational equation

$$\int_{\omega^2} \nabla u^2 \cdot \nabla v \, dx = \int_{\omega^2} f^2 v \, dx \quad \text{for all} \quad v \in V_q(\omega^2).$$

Then, after k iterations of this argument, we define the extensions $u^k, f^k : \omega^k \to \mathbb{R}$ of $u^{k-1}, f^{k-1} : \omega^{k-1} \to \mathbb{R}$, where

$$\omega^k := (-3a_1, 3a_1) \times ... \times (-3a_k, 3a_k) \times (-a_{k+1}, a_{k+1}) \times ... \times (-a_\mathrm{d}, a_\mathrm{d}),$$

by letting, for all $x' := (x_1, ..., x_{k-1}) \in (-3a_1, 3a_1) \times ... \times (-3a_{k-1}, 3a_{k-1})$ and $x'' := (x_{k+1}, ..., x_\mathrm{d}) \in (-a_{k+1}, a_{k+1}) \times ... \times (-a_\mathrm{d}, a_\mathrm{d})$,

$$u^k(x', x_k, x'') = \begin{cases} u^{k-1}(x', x_k, x'') & \text{if } x_k \in (-a_k, a_k), \\ -u^{k-1}(x', 2a_k - x_k, x'') & \text{if } x_k \in (a_k, 3a_k), \\ -u^{k-1}(x', -2a_k - x_k, x'') & \text{if } x_k \in (-3a_k, -a_k) \end{cases}$$

and

$$f^k(x', x_k, x'') = \begin{cases} f^{k-1}(x', x_k, x'') & \text{if } x_k \in (-a_k, a_k), \\ -f^{k-1}(x', 2a_k - x_k, x'') & \text{if } x_k \in (a_k, 3a_k), \\ -f^{k-1}(x', -2a_k - x_k, x'') & \text{if } x_k \in (-3a_k, -a_k). \end{cases}$$

Then Lemma 7.1(ii) shows that u^k belongs to the space $V_p(\omega^k)$ and satisfies the variational equation

$$\int_{\omega^k} \nabla u^k \cdot \nabla v \, dx = \int_{\omega^k} f^k v \, dx \quad \text{for all} \quad v \in V_q(\omega^k).$$

Next, we define the extensions $u^{k+1}, f^{k+1} : \omega^{k+1} \to \mathbb{R}$ of $u^k, f^k : \omega^k \to \mathbb{R}$, where

$$\omega^{k+1} := (-3a_1, 3a_1) \times ... \times (-3a_{k+1}, 3a_{k+1}) \times (-a_{k+2}, a_{k+2}) \times ... \times (-a_\mathrm{d}, a_\mathrm{d}),$$

by letting, for all $x' := (x_1, ..., x_k) \in (-3a_1, 3a_1) \times ... \times (-3a_k, 3a_k)$ and $x'' := (x_{k+2}, ..., x_\mathrm{d}) \in (-a_{k+2}, a_{k+2}) \times ... \times (-a_\mathrm{d}, a_\mathrm{d})$,

$$u^{k+1}(x', x_{k+1}, x'') = \begin{cases} u^k(x', x_{k+1}, x'') & \text{if } x_{k+1} \in (-a_{k+1}, a_{k+1}), \\ u^k(x', 2a_{k+1} - x_{k+1}, x'') & \text{if } x_{k+1} \in (a_{k+1}, 3a_{k+1}), \\ u^k(x', -2a_{k+1} - x_{k+1}, x'') & \text{if } x_{k+1} \in (-3a_{k+1}, -a_{k+1}) \end{cases}$$

and

$$f^{k+1}(x', x_{k+1}, x'') = \begin{cases} f^k(x', x_{k+1}, x'') & \text{if } x_{k+1} \in (-a_{k+1}, a_{k+1}), \\ f^k(x', 2a_{k+1} - x_{k+1}, x'') & \text{if } x_{k+1} \in (a_{k+1}, 3a_{k+1}), \\ f^k(x', -2a_{k+1} - x_{k+1}, x'') & \text{if } x_{k+1} \in (-3a_{k+1}, -a_{k+1}). \end{cases}$$

Then Lemma 7.1(i) shows that u^{k+1} belongs to the space $V_p(\omega^{k+1})$ and satisfies the variational equation

$$\int_{\omega^{k+1}} \nabla u^{k+1} \cdot \nabla v \, dx = \int_{\omega^{k+1}} f^{k+1} v \, dx \quad \text{for all} \quad v \in V_q(\omega^{k+1}).$$

Finally, after $(\mathrm{d}-k)$ iterations of this argument, we define the extensions $u^\mathrm{d}, f^\mathrm{d} : \omega^\mathrm{d} \to \mathbb{R}$ of $u^{\mathrm{d}-1}, f^{\mathrm{d}-1} : \omega^{\mathrm{d}-1} \to \mathbb{R}$, where

$$\omega^\mathrm{d} := (-3a_1, 3a_1) \times ... \times (-3a_\mathrm{d}, 3a_\mathrm{d}),$$

by letting, for all $x' := (x_1, ..., x_{\mathrm{d}-1}) \in (-3a_1, 3a_1) \times ... \times (-3a_{\mathrm{d}-1}, 3a_{\mathrm{d}-1})$,

$$u^{\mathrm{d}}(x', x_{\mathrm{d}}) = \begin{cases} u^{\mathrm{d}-1}(x', x_{\mathrm{d}}) & \text{if } x_{\mathrm{d}} \in (-a_{\mathrm{d}}, a_{\mathrm{d}}), \\ u^{\mathrm{d}-1}(x', 2a_{\mathrm{d}} - x_{\mathrm{d}}) & \text{if } x_{\mathrm{d}} \in (a_{\mathrm{d}}, 3a_{\mathrm{d}}), \\ u^{\mathrm{d}-1}(x', -2a_{\mathrm{d}} - x_{\mathrm{d}}) & \text{if } x_{\mathrm{d}} \in (-3a_{\mathrm{d}}, -a_{\mathrm{d}}) \end{cases}$$

and

$$f^{\mathrm{d}}(x', x_{\mathrm{d}}) = \begin{cases} f^{\mathrm{d}-1}(x', x_{\mathrm{d}}) & \text{if } x_{\mathrm{d}} \in (-a_{\mathrm{d}}, a_{\mathrm{d}}), \\ f^{\mathrm{d}-1}(x', 2a_{\mathrm{d}} - x_{\mathrm{d}}) & \text{if } x_{\mathrm{d}} \in (a_{\mathrm{d}}, 3a_{\mathrm{d}}), \\ f^{\mathrm{d}-1}(x', -2a_{\mathrm{d}} - x_{\mathrm{d}}) & \text{if } x_{\mathrm{d}} \in (-3a_{\mathrm{d}}, -a_{\mathrm{d}}). \end{cases}$$

Then Lemma 7.1(i) shows that u^{d} belongs to the space $V_p(\omega^{\mathrm{d}})$ and satisfies the variational equation

$$\int_{\omega^{\mathrm{d}}} \nabla u^{\mathrm{d}} \cdot \nabla v \, dx = \int_{\omega^{\mathrm{d}}} f^{\mathrm{d}} v \, dx \quad \text{for all} \quad v \in V_q(\omega^{\mathrm{d}}).$$

This last equation implies that the function u^{d}, which belongs to $W^{1,p}(\omega^{\mathrm{d}})$, satisfies

$$-\Delta u^{\mathrm{d}} = f^{\mathrm{d}} \text{ in } \mathcal{D}'(\omega^{\mathrm{d}}).$$

Since $f^{\mathrm{d}} \in L^r(\omega^{\mathrm{d}})$, $r \in (1, +\infty)$, and since $\omega \Subset \omega^{\mathrm{d}}$ (i.e., the closure of ω is a compact subset of ω^{d}), the interior regularity of the Laplacian (see, e.g., Dautray and Lions [6]) then shows that $u = u^{\mathrm{d}}|_\omega \in W^{2,r}(\omega)$. The proof is now complete. □

Remark 7.3. Under the hypotheses of Theorem 7.2, one can show that the function u also satisfies the inequality

$$\|u\|_{W^{2,r}(\omega)} \le C \|f\|_{L^r(\omega)}$$

for some constant C.

An immediate consequence of Theorem 7.2 is that the solution of the variational equation (7.2) satisfies a Laplace equation supplemented with mixed boundary conditions:

Corollary 7.4. *Under the assumptions of Theorem 7.2, if u satisfies the variational equation* (7.2)*, then u is a strong solution to the boundary value problem*

$$\begin{aligned} -\Delta u &= f \quad \text{in } \omega, \\ u &= 0 \quad \text{on } \Gamma_{i_1} \cup ... \cup \Gamma_{i_k}, \\ \partial_j u &= 0 \quad \text{on } \Gamma_j \quad \text{for all} \quad j \notin \{i_1, ..., i_k\}. \end{aligned} \tag{7.3}$$

Proof. We know by Theorem 7.2 that $u \in W^{2,r}(\omega)$. By letting $v \in \mathcal{D}(\omega)$ and integrating by parts in the variational equation (7.2), we deduce that

$$-\Delta u = f$$

first in the distributional sense, then a.e. in ω since f and Δu belong to the space $L^r(\omega)$.

Since $u \in V_p(\omega)$, we also have

$$u = 0 \text{ on } \Gamma_{i_1} \cup ... \cup \Gamma_{i_k}.$$

We now prove that the normal derivative of u on Γ_j vanishes for all $j \notin \{i_1, ..., i_k\}$. Note first that this derivative is $\nu_j \partial_j u$ and that it is well defined in $W^{1-1/r,r}(\Gamma_j)$, since $u \in W^{2,r}(\omega)$. For all $v \in \mathcal{C}^\infty(\overline{\omega})$ that vanishes in a neighborhood of the set $\partial\omega \setminus \Gamma_j$, we have by Stokes formula that

$$\int_{\Gamma_j} v(\partial_j u)\nu_j \, dx_1 ... dx_{j-1} dx_{j+1} ... dx_{\mathrm{d}} = \int_\omega \nabla u \cdot \nabla v \, dx + \int_\omega (\Delta u) v \, dx.$$

Therefore

$$\int_{\Gamma_j} v(\partial_j u)\nu_j \, dx_1 ... dx_{j-1} dx_{j+1} ... dx_{\mathrm{d}} = 0$$

since u satisfies the variational equation (7.2) and $v \in V_q(\omega)$. This shows that

$$\nu_j \partial_j u = 0 \quad \text{on} \quad \Gamma_j$$

in the distributional sense, but also in $W^{1-1/r,r}(\Gamma_j)$ since $\nu_j \partial_j u$ belongs to this space. Thus $\partial_j u = 0$ on Γ_j since $\nu_j = \pm 1$ on Γ_j. Therefore u satisfies all the equations of the boundary value problem (7.3). □

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