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A note on the commutative closure of star-free languages [1]

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Abstract

We show that the commutative closure of a star-free language is either star-free or not regular anymore. Actually, this property is shown to hold exactly for the closure with respect to a partial commutation corresponding to a transitive dependence relation. Moreover, the question whether the closure of a star-free language remains star-free is decidable precisely for transitive partial commutation relations.

Keywords: Formal languages

It is well-known that the commutative closure of regular languages need not be regular. A typical example is the star-free(!) language $(ab)^*$, the commutative closure of which consists of all words over $\{a, b\}$ with equal number of a and b. Moreover, as soon as the closure of languages under partial commutation is considered the question whether the closure of a given regular language remains regular or not is in general undecidable [9]. On the other hand, the question whether the closure of the star of a closed regular language remains regular or not is still an open question for partial commutations. Recently, some progress has been achieved towards a solution for this problem [6,8] (for a survey and further references see also [2, Chapter 6]).

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The aim of this paper is to clarify another aspect concerning the still mysterious behaviour of recognizable languages with respect to closure operations, namely the relationship between star-freeness and closure under commutation. We show that surprisingly there are no star-free languages, where the commutative closure is regular, but not star-free. In fact, we can show a more general result concerning the closure of star-free languages under partial commutations.

Definition. Let Σ be a finite alphabet and $I \subseteq \Sigma \times \Sigma$ be a symmetric relation. The closure of a language $L \subseteq \Sigma^*$ with respect to the partial commutation relation I is denoted by $[L]_I$. It is defined as the least language of Σ^* containing L, which satisfies $uabv \in [L]_I \Leftrightarrow ubav \in [L]_I$ for all $u, v \in \Sigma^*$, $(a, b) \in I$.

Note that if $I = \Sigma \times \Sigma$, then $[L]_I$ is the commutative closure, whereas in the other extreme, $[L]_\emptyset = L$.

Recall that the syntactic congruence $\sim_L$ of a language $L \subseteq \Sigma^*$ is given by $u \sim_L v$ if and only if $xuy \in L \Leftrightarrow xvy \in L$, for all $x, y \in \Sigma^*$. A language

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$L \subseteq \Sigma^*$ is *aperiodic*, if both the syntactic congruence $\sim_L$ is of finite index and the syntactic monoid of L, $\Sigma^*/\sim_L$, satisfies the equation $x^n = x^{n+1}$ for some integer $n \geqslant 0$. Equivalently, aperiodic languages are precisely the star-free resp. first-order definable languages (see [10,5,7]).

Assume that $D = (\Sigma \times \Sigma) \setminus I$ is transitive and let $\equiv_I \subseteq \Sigma^* \times \Sigma^*$ be the congruence generated by the set $\{(ab, ba) \mid (a,b) \in I\}$. Then the quotient monoid $\mathbb{M}(\Sigma, I) := \Sigma^*/\equiv_I$ is a direct product of free monoids, i.e., $\mathbb{M}(\Sigma, I) = \prod_{i=1}^k \Sigma_i^*$ for the partition $\Sigma = \dot{\bigcup}_{i=1}^k \Sigma_i$ of the alphabet corresponding to the connected components of (Σ, D). By abuse of notation we will identify in this case the closure $[L]_I \subseteq \Sigma^*$ of a language $L \subseteq \Sigma^*$ with the subset $L/\equiv_I$ of $\mathbb{M}(\Sigma, I)$. Hence in this case we will consider $[L]_I$ as a subset of $\prod_{i=1}^k \Sigma_i^*$. By $\pi_i : \Sigma^* \to \Sigma_i^*$ for $1 \leqslant i \leqslant k$ we denote the canonical projections.

Theorem 1. *Let Σ be a finite alphabet, $I \subseteq \Sigma \times \Sigma$ a partial commutation and $D = (\Sigma \times \Sigma) \setminus I$ the complementary relation. Then the following assertions hold*:

If D is transitive, then the closure $[L]_I$ of a star-free language $L \subseteq \Sigma^*$ is either star-free or not regular.

Conversely, if D is not transitive, then there exist star-free languages $L \subseteq \Sigma^*$ such that $[L]_I$ is regular, but not star-free.

Proof. Let us first assume that $D \subseteq \Sigma \times \Sigma$ is transitive and the closure $[L]_I$ of the star-free language L is regular. We have to show the existence of some integer $n \geqslant 0$ such that $xv^n y \in [L]_I \Rightarrow xv^{n+\delta} y \in [L]_I$, for all $x, v, y \in \Sigma^*$, $\delta \in \{-1, +1\}$.

By Mezei's theorem [1] we can express $[L]_I$ as a finite union $[L]_I = \bigcup_{\text{fin}} \prod_{i=1}^k L_i$, where every $L_i \subseteq \Sigma_i^*$ is recognizable. Moreover, different $L_i, L_i' \subseteq \Sigma_i^*$ in this union are either disjoint or equal. Let S be the set of all languages L_i occurring in the above representation of $[L]_I$. Let $m > 0$ denote the integer $\max_{L' \in S}(\min_{w \in L'} |w|)$. Further an integer $p > 0$ is chosen such that $u^p \sim_L u^{p+\delta}$ for all u and $\delta \in \{-1, +1\}$ (recall, L is aperiodic). We let $n = [(k-1)m + 1]p$ and show that this value suffices in order to obtain the desired property for $[L]_I$.

Consider $xv^n y \in [L]_I$, $\delta \in \{-1, +1\}$, with $xv^n y = (w_1, \ldots, w_k)$ and let $xv^{n+\delta} y = (w_1', \ldots, w_k')$. Suppose, we have already shown that $(w_1', \ldots, w_{i-1}', w_i, \ldots, w_k) \in [L]_I$ for $i \geqslant 1$. Therefore we have for some $\prod_{i=1}^k L_i$ in the above representation of $[L]_I$: $(w_1', \ldots, w_{i-1}', w_i, \ldots, w_k) \in \prod_{i=1}^k L_i$. Let $z_i = w_i$ and consider $z_j \in \Sigma_j^*$, $1 \leqslant j \leqslant k$, $j \neq i$, of minimal length satisfying $z_j \in L_j$. Note that we also have $(z_1, \ldots, z_k) \in [L]_I$. Hence, there exists some $z \in L$ with $\pi_i(z) = z_i$, for every $1 \leqslant i \leqslant k$.

Let $q = \sum_{j \neq i} |z_j| \leqslant (k-1)m$, then we can factorize z such that

$$z = u_0 x_1 u_1 \cdots x_q u_q$$

with $u_l \in \Sigma_i^*$ and $x_l \in \Sigma \setminus \Sigma_i$ for all l. Due to the choice of n, there remain at least $[(k-1)m + 1](p-1) + 1$ occurrences of v_i being factors of at most $(k-1)m+1$ words u_l, $0 \leqslant l \leqslant q$. Applying the pigeon-hole principle, we obtain that at least one u_l can be decomposed as $u_l = x' v_i^p y'$, for suitable $x', y' \in \Sigma_i^*$. This yields inevitably the word factor v_i^p in z. Hence, $z = x'' v_i^p y''$ and we have $x'' v_i^{p+\delta} y'' \in L$. Thus, $(z_1, \ldots, z_{i-1}, z_i', z_{i+1}, \ldots, z_k) \in [L]_I$, where $z_i' = w_i'$. It follows that $z_j \in L_j'$, $j \neq i$, and $z_i' \in L_i'$ hold for some $\prod_{i=1}^k L_i'$ in the finite union representing $[L]_I$. For $j \neq i$ we have from $L_j \cap L_j' \neq \emptyset$ also $L_j' = L_j$. This is due to the property of the representation given by Mezei's theorem noticed above. Hence, we obtain $(w_1', \ldots, w_{i-1}', w_i', w_{i+1}, \ldots, w_k) \in \prod_{i=1}^k L_i'$. In particular, it follows $(w_1', \ldots, w_{i-1}', w_i', w_{i+1}, \ldots, w_k) \in [L]_I$.

For the converse, consider a monoid $\mathbb{M}(\Sigma, I)$ where $D = (\Sigma \times \Sigma) \setminus I$ is non-transitive. Hence, there are distinct letters $a, b, c \in \Sigma$ with $\{(a,c), (b,c)\} \subseteq D$, but $(a,b) \in I$. Now, it is easy to check that the language $(abcbac)^*$ is aperiodic (star-free):

$$\begin{aligned}(abcbac)^* &= (abc\Sigma^* \cap \Sigma^* bac) \setminus (\Sigma^* abc(\Sigma^* \setminus bac\Sigma^*) \\ &\qquad \cup \Sigma^* bac(\Sigma^* \setminus abc\Sigma^*)).\end{aligned}$$

However, the closure $K := [(abcbac)^*] = [(ab + ba)c(ab+ba)c]^*$ is obviously not aperiodic since for every $n > 0$ we have $(abc)^n \not\sim_K (abc)^{n+1}$. $\square$

Remark. Note that this result cannot be extended to languages over Σ^ω. As soon as we have a pair $(a,b) \in I$ we can exhibit the star-free language $(aab)^* b^\omega$, where the closure $\{w \in \{a,b\}^\omega \mid w$ contains an even number of a's and an infinite number of b's } is regular, but not star-free anymore.

It is natural to look for a decision procedure for the question, whether the closure of a given star-free language remains star-free. We can show that this problem is in general undecidable following [4,9]. We follow directly the proof given in [9] for the undecidability of the question whether the closure of a regular language is still regular. We included the proof below in order to examine the languages involved in the construction and show that they are star-free.

Let B and C be disjoint alphabets. An instance of Post's correspondence problem (PCP) will be encoded by two homomorphisms $g, h : B^* \to C^*$. A solution for this instance is a word $w \in B^+$ such that $g(w) = h(w)$.

Consider now the language W_g (and analogously, W_h) which has been used in the reduction given in [9]:

$$W_g = \{(wg(w), c^{|g(w)|}) \mid w \in B^+\}.$$

Let our commutation relation be $I = \{(x,c),(c,x) \mid x \in B \cup C\}$. We define below a star-free language L_g such that $[L_g]_I = \overline{W_g}$ (analogously for L_h).

The following technical remark yields a concise star-free description of L_g (and L_h).

Remark. Let Σ be an alphabet, $X \subseteq \Sigma$ a set of symbols, $c \neq\in X$, and $M \subset X\{c\}^*$ a finite set. Then M^* is a star-free language.

The last remark is justified by the fact that M is a very pure code and therefore M^+ is locally testable (i.e. it can be characterized by its suffixes, prefixes, and factors up to some given length) [7, p. 120].

Now, L_g is formed as a union of sets described in the following. The first set takes care of the case that a symbol from C precedes a symbol from B, which is easily seen to be star-free. In the following sets we consider four ways in which the number of c's can differ from the correct one. (There could be strictly less/more c's than in the g-images of all symbols from B, respectively strictly less/more c's than symbols from C. It is not difficult to form the corresponding sets using the remark above.) Finally, the last set describes – under the assumption that the number of c's agrees with both numbers mentioned above – those words that fail to be encodings of solutions because some symbol x is not properly mapped to its image $g(x)$. This set is defined by the following expression (again using the remark):

$$B^*\Big(\bigcup_{x\in B} xc^{|g(x)|}\{yc^{|g(y)|} \mid y \in B\}^*\{zc \mid z \in C\}^* \{w \mid w \neq g(x), |w| = |g(x)|\}\Big)C^*.$$

Since star-free languages are by definition closed with respect to union we have shown that L_g (and L_h) are star-free languages.

Theorem 2. *Let Σ be a finite alphabet and $I \subseteq \Sigma \times \Sigma$ a partial commutation.*

It is decidable whether the closure $[L]_I$ of a star-free language L is star-free if and only if I is transitive.

Proof. With the notations introduced above note that the language $[L_g \cup L_h]_I = \overline{W}_g \cup \overline{W}_h$ is equal to $(B \cup C)^* \times c^*$ (and hence star-free) if the PCP encoded by g, h has no solution. On the other hand, if the PCP instance has a solution consider some $w \in B^+$ such that $g(w) = h(w)$. Then

$$\begin{aligned}&(B \cup C)^* \times c^* \setminus [L_g \cup L_h]_I) \cap w^*g(w)^* \times c^* \\ &\quad = \{(w^n g(w)^n, c^{n|g(w)|}) \mid n \geqslant 1\}\end{aligned}$$

is not recognizable, hence $[L_g \cup L_h]_I$ is not recognizable (and thus not star-free).

Finally, we note that B and C can be encoded as usual by two letters. Moreover, if $I \subseteq \Sigma \times \Sigma$ is not transitive, then there exist different letters a, b, c in Σ such that $(a,c) \in I$, $(b,c) \in I$, but $(a,b) \in D$. This yields the first part of the claim, i.e., the undecidability of the question for I non-transitive.

For the second part we note that the closure $[L]_I$ of a regular language $L \subseteq \Sigma^*$ can be identified with a rational subset of the monoid $\mathbb{M}(\Sigma, I)$ (i.e., a subset generated from $\emptyset$ and the singleton sets using the operations concatenation, union and Kleene star). If I is transitive, then $\mathbb{M}(\Sigma, I)$ is a free product of free commutative monoids. Moreover, the question whether a rational subset of a free product of free commutative monoids is recognizable (i.e., saturated by a congruence of finite index) is effectively decidable [9]. More precisely, given a rational expression for $[L]_I$ a finite automaton can be computed, which recognizes exactly $[L]_I$ whenever $[L]_I$ is recognizable, which in turn is decidable. If the answer is positive, we can compute the syntactic monoid and decide whether $[L]_I$ is star-free or not. □

Remark. Concerning the different assumptions in Theorems 1 and 2 note that both I and $D = (\Sigma \times \Sigma) \setminus I$ are transitive if and only if the quotient monoid $\mathbb{M}(\Sigma, I)$ is either free or free commutative.

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