The boundedness and zero isolation problems for weighted automata over nonnegative rationals

Wojciech Czerwiński
University of Warsaw, Poland
wczerwin@mimuw.edu.pl

Engel Lefaucheux
Université de Lorraine, Inria, LORIA, Nancy, France
engel.lefaucheux@inria.fr

Filip Mazowiecki
Max Planck Institute for Software Systems, Saarland Informatics Campus, Saarbrücken, Germany
filipm@mpi-sws.org

David Purser
Max Planck Institute for Software Systems, Saarland Informatics Campus, Saarbrücken, Germany
dpurser@mpi-sws.org

Markus A. Whiteland
Max Planck Institute for Software Systems, Saarland Informatics Campus, Saarbrücken, Germany
University of Liège, Belgium
mwhiteland@uliege.be

ABSTRACT

We consider linear cost-register automata (equivalent to weighted automata) over the semiring of nonnegative rationals, which generalise probabilistic automata. The two problems of boundedness and zero isolation ask whether there is a sequence of words that converge to infinity and to zero, respectively. In the general model both problems are undecidable so we focus on the copyless linear restriction. There, we show that the boundedness problem is decidable.

As for the zero isolation problem we need to further restrict the class. We obtain a model, where zero isolation becomes equivalent to universal coverability of orthant vector addition systems (OVAS), a new model in the VAS family interesting on its own. In standard VAS runs are considered only in the positive orthant, while in OVAS every orthant has its own set of vectors that can be applied in that orthant. Assuming Schanuel's conjecture is true, we prove decidability of universal coverability for three-dimensional OVAS, which implies decidability of zero isolation in a model with at most three independent registers.

CCS CONCEPTS

• **Theory of computation** → **Formal languages and automata theory**; **Models of computation**.

KEYWORDS

Weighted automata, vector addition systems, boundedness problem, isolation problem

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1 INTRODUCTION

Weighted automata are a natural model of computation that generalise finite automata [17] and linear recursive sequences [3]. They have various equivalent presentations: e.g. finite automata, rational series, matrix representation [5, 34]; or recently linear cost-register automata (linear CRA) [2]. A typical example is a probabilistic automaton $\mathcal{A}$ that assigns to each word w its probability of acceptance, denoted $\mathcal{A}(w)$ [16, 19, 22, 32]. More generally, weighted automata are defined with respect to a semiring: a domain with two binary operations. In the example of probabilistic automata the domain is the nonnegative rationals (thus $\mathcal{A}(w) \in \mathbb{Q}_{\geqslant 0}$) with the usual operations: $+$ and $\cdot$.

Depending on the context, different semirings for weighted automata have been studied. For instance when considering learning, the semirings are usually fields, like the rationals or reals [4, 21]. Most results on learning weighted automata depend on Schützenberger's polynomial time algorithm deciding the equivalence problem of weighted automata over fields [34]. On the other hand, when considering regular expressions, weighted automata are usually studied over the tropical semiring, i.e. $\mathbb{N} \cup \{+\infty\}$ with the operations: min and $+$. The star height problem for regular languages can for instance be reduced to the boundedness problem of such automata. Hashiguchi showed that this problem is decidable [26]. Due to Hashiguchi's proof being difficult, many alternative proofs of this result appeared; among them: via Simon's factorisation trees [36]; and via games [8].

This paper is primarily interested in weighted automata over the semiring of nonnegative rationals with $+$ and $\cdot$, denoted $\mathbb{Q}_{\geqslant 0}(+, \cdot)$. This is the minimal weighted automata model that captures probabilistic automata, but does not impose any restrictions on the model.

Probabilistic automata assign only probabilities to words, i.e. values in the interval $[0, 1]$. This requires some restrictions, e.g. the transitions are defined by probabilistic distributions. Similar generalisations of probabilistic automata were studied e.g. in [10, 37].

One of the most natural questions for such automata are the threshold problems: i.e. given an automaton $\mathcal{A}$ and a constant c, decide whether $\mathcal{A}(w) \leqslant c$ or whether $\mathcal{A}(w) \geqslant c$ for all words w. We study existential variants of these problems, where only $\mathcal{A}$ is given in the input: the *zero isolation* asks whether there exists $c > 0$ such that for all words w it holds $\mathcal{A}(w) \geqslant c$ and *boundedness* asks whether there exists $c < +\infty$ such that for all words w it holds $\mathcal{A}(w) \leqslant c$. More intuitively, the complements of the two problems ask whether there exist a sequence of words $w_1, w_2, \ldots$ such that $\lim_{i\to+\infty} \mathcal{A}(w_i)$ equals 0 and $+\infty$, respectively.

Notice that most of the mentioned problems are well-defined already for probabilistic automata. Moreover, since probabilistic automata are known to be closed under complement (it is easy to define $\mathcal{B}(w) = 1-\mathcal{A}(w)$) the two threshold problems are equivalent and undecidable [32]. In probabilistic automata the zero isolation problem, due to complementation, is equivalent to the value-1 problem: this is also undecidable [22], but decidable for the special class of *leaktight* probabilistic automata [18]. The boundedness problem is not interesting for probabilistic automata (since the output is always bounded by 1), but a folklore argument shows that it is undecidable for $\mathbb{Q}_{\geqslant 0}(+, \cdot)$ (see Section 3).

Since the above problems are undecidable in general, we are interested in these problems on subclasses of weighted automata. A common restriction is bounding the ambiguity, i.e. the number of accepting runs. The two most interesting classes are finitely-ambiguous and polynomially-ambiguous automata; when the number of accepting runs is bounded by: a constant (universal for all words), and by a polynomial (in the size of the input word), respectively. Both classes have nice characterisations, by excluding some simple patterns in the automata [39]. In particular, it is easy to check if an automaton is finitely-ambiguous or polynomially-ambiguous.

Both threshold problems are undecidable for polynomially-ambiguous probabilistic automata [16, 19]. In the finitely-ambiguous case they are decidable [19], and one can infer that they remain decidable in the general setting of finitely-ambiguous weighted automata over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$ [16]. Unlike for probabilistic automata, the two threshold problems are different (the closure under complement is not true in general over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$), and while one of the inequalities is trivial to decide, the other one is known to be decidable [16] only assuming Schanuel's conjecture [27]. Similarly, for boundedness and zero isolation, even though one could suspect they are equivalent problems, we also see a difference. One can show that for finitely-ambiguous weighted automata over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$ the boundedness problem is trivially decidable; and exploiting [11] we show that zero isolation is decidable subject to Schanuel's conjecture (see Section 3). The argument in the latter case is more involved. The aforementioned decidability results for zero isolation on leaktight probabilistic automata do not hold over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$.

The decidability border between the finitely-ambiguous and polynomially-ambiguous classes is not surprising. It is often the case that undecidable problems for weighted automata are decidable for the finitely-ambiguous class [20]; and remain undecidable even for very restricted variants of polynomially-ambiguous automata, e.g. *copyless* linear CRA [1]. However, it is not always the case, for example the ϵ-gap threshold problem is decidable for polynomially-ambiguous probabilistic automata [16], and undecidable in general [12]. For zero isolation and boundedness the undecidability reductions do not work for polynomially-ambiguous automata, which is the starting point of our paper.

Our contributions and techniques. We study boundedness and zero isolation for *copyless linear CRA*, introduced in [2], and known to be strictly contained in polynomially-ambiguous weighted automata [1]. We show that boundedness is decidable for copyless linear CRA. Our proof shows that unboundedness can be detected with simple patterns in the style of patterns for finitely-ambiguous and polynomially-ambiguous automata in [39]. Intuitively, an automaton is unbounded if and only if either there is a loop of value larger than 1 or there is a pattern that generates unboundedly many runs of the same value. Like in [39] the patterns are easy to detect even in polynomial time, the difficulty is to prove correctness of the characterisation. Similarly, as in one of the mentioned proofs of Hashiguchi's theorem [36], we find a way to abstract the set of generated matrices into a finite monoid, that allows us to exploit Simon's factorisation trees. Otherwise, the proof is rather different from [36], as we need to exploit the particular shapes of the matrices (imposed by the copyless restriction), while the proof in [36] works for the general class of matrices. We conjecture that our pattern characterisation works for the whole class of polynomially-ambiguous automata.

For the zero isolation problem we have to further restrict the class of copyless linear CRA to a class in which the registers do not interact, that we call Independent-CRA. A similar model of CRA with independent registers was already defined in [15]. We start with a chain of reductions to equivalent problems. Firstly, we show that zero-isolation over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$ is essentially equivalent to the boundedness problem over the semiring $\mathbb{Z}(\min, +)$, i.e. the same problem as in Hashiguchi's theorem with the exception that the domain includes negative numbers. This problem is known to be undecidable for the full class of weighted automata [1], but for polynomially-ambiguous, or even copyless linear CRA, decidability was left as an open problem in the same paper. Secondly, we further reduce this problem to a variant of the coverability problem for a new class of *orthant vector addition systems* (OVAS).

The OVAS class lies between the standard VAS [13] and its integer relaxation [25]. Intuitively, in the standard VAS runs are considered only in the positive orthant, while in the integer relaxation runs go through the whole space. In OVAS every orthant has its own set of vectors that can be applied in that orthant. The *universal coverability* problem asks whether from any starting point the positive orthant can be reached. We prove that universal coverability is decidable in dimension 3. The proof is nontrivial and relies on a notion of a *separator* between the reachability set and the positive orthant that can be expressed in the first order logic over the reals. Depending on the encoding of the numbers, we can either rely on Tarski's theorem [23], or the formula might require the exponential function. In the latter case decidability depends on Schanuel's conjecture [27]. Since most of the proof works in any dimension, we believe that this is an important step to prove the theorem for arbitrary dimensions. Interestingly, the proof relies on results about reachability for continuous VAS [7]. From universal coverability we infer decidability of zero isolation for copyless linear CRA with 3 independent registers. More importantly, we establish a nontrivial connection between: zero isolation over $\mathbb{Q}_{\geqslant 0}(+,\cdot)$; boundedness over $\mathbb{Z}(\min,+)$; and our new model OVAS. We are convinced that the latter model is of independent interest. Interestingly, we show that the usual coverability problem (with a fixed initial point) in undecidable.

We leave as an open problem decidability of zero isolation for polynomially ambiguous weighted automata over $\mathbb{Q}_{\geqslant 0}(+,\cdot)$. Nevertheless we show that the problem is undecidable for copyless CRA (nonlinear). The latter class is known to be: strictly between the finitely-ambiguous and the full class of weighted automata [29]; and incomparable with the polynomially-ambiguous class [30, 31].

The closest results to our work are presented in [16] and [10, 11]. In the first mentioned paper the authors study the containment problem for finitely-ambiguous probabilistic automata, where one of the automata is unambiguous. The latter restriction makes the problem essentially equivalent to the threshold problems for the general class of weighted automata over $\mathbb{Q}_{\geqslant 0}(+,\cdot)$. The papers [10, 11] deal with the Big-O problem for finitely-ambiguous weighted automata over $\mathbb{Q}_{\geqslant 0}(+,\cdot)$, which given two automata asks if there is a constant $C > 0$ such that $\mathcal{A}(w) \leqslant C \cdot \mathcal{B}(w)$ for all words w. By fixing $\mathcal{A}$ or $\mathcal{B}$ to a positive constant we get the zero isolation and boundedness problems, respectively. Boundedness is sometimes also called limitedness, but should not be confused with the finiteness problem. Finiteness asks whether the range of a weighted automaton over the rationals is finite, and is known to be decidable [9, 28].

We state and organise our results in Section 3, after formally defining the setting in Section 2.

2 PRELIMINARIES

We write $\mathbb{Q}$, $\mathbb{Q}_{\geqslant 0}$, $\mathbb{Q}_{>0}$, $\mathbb{Q}_{\leqslant 0}$, $\mathbb{Q}_{<0}$ for the sets of rationals, nonnegative rationals, etc; and we use similar notation for other domains. Throughout the paper we assume that the base of the logarithm is 2 unless otherwise stated. By $\mathbb{LogQ}$ we denote the set of logarithms of positive rational numbers: $\mathbb{LogQ} = \{\log(q) \mid q \in \mathbb{Q}_{>0}\}$. Observe that $\mathbb{Q} \subseteq \mathbb{LogQ}$ and that $\mathbb{LogQ}$ is closed under addition. For $a, b \in \mathbb{N}$, $a \leqslant b$ we write $[a,b]$ as a shorthand for $\{a,\ldots,b\}$. Given a vector $\mathbf{v} = (v_1,\ldots,v_d) \in \mathbb{R}^d$ we write $\mathbf{v}[i] = v_i$ for every $i \in [1,d]$. For $\mathbf{v},\mathbf{w} \in \mathbb{R}^d$ we write $\mathbf{v} \leqslant \mathbf{w}$ if $\mathbf{v}[i] \leqslant \mathbf{w}[i]$ for all $i \in [1,d]$. The norm of a vector $\mathbf{v}$ is defined as $\|\mathbf{v}\| = \max_{i\in[1,d]} |\mathbf{v}[i]|$. Given a finite set Q, where $|Q| = d$ sometimes we consider vectors in $\mathbb{R}^Q$ understood as vectors in $\mathbb{R}^d$ for some implicit bijection between Q and $[1,d]$.

Let $\mathbb{S}(\oplus,\odot)$ be a commutative semiring with the sum $\oplus$ and product operations $\odot$. We will use $\mathbb{S}$ to denote the domain of the semiring $\mathbb{S}(\oplus,\odot)$. In this paper most of the time we will consider two types of semirings. The *standard semiring*, where the domain is nonnegative rational numbers $\mathbb{Q}_{\geqslant 0}(+,\cdot)$ with the standard sum and product operations. The *tropical semirings* $\mathbb{Z}(\min,+)$ and $\mathbb{LogQ}(\min,+)$ with domains $\mathbb{Z}\cup\{+\infty\}$ and $\mathbb{LogQ}\cup\{+\infty\}$, respectively, where $\oplus$ is min and $\odot$ is $+$. Whenever the semiring is not specified we write $\mathbb{0}$ and $\mathbb{1}$ for the zero and one of the semiring. Over $\mathbb{Q}_{\geqslant 0}(+,\cdot)$ these are as expected $\mathbb{0} = 0$ and $\mathbb{1} = 1$; but over $\mathbb{LogQ}(\min,+)$ and $\mathbb{Z}(\min,+)$ these are $\mathbb{0} = +\infty$ and $\mathbb{1} = 0$.

2.1 Weighted automata

A *weighted automaton* (WA) over a semiring $\mathbb{S}(\oplus,\odot)$ is a tuple $\mathcal{A} = (\Sigma, I, F, (M_a)_{a\in\Sigma})$, where: Σ is a finite alphabet; $I, F \in \mathbb{S}^d$ are d-dimensional vectors; and $M_a \in \mathbb{S}^{d\times d}$ are d-dimensional square matrices for some fixed $d \in \mathbb{N}$. For every word $w = w_1\cdots w_n \in \Sigma^*$ we define the matrix $M_w = M_{w_1}M_{w_2}\cdots M_{w_n}$, where the matrices are multiplied with respect to the sum and the product of $\mathbb{S}$. If w is the empty word then M is the identity matrix. For every word $w \in \Sigma$ the automaton outputs $\mathcal{A}(w) = I^{\mathsf{T}} M_w F \in \mathbb{S}$. Thus $\mathcal{A}$ can be seen as a function $\Sigma^* \to \mathbb{S}$. Whilst formally $\mathcal{A}$ does not have states, one can think that coordinates in I, F, and the matrices $(M_a)_{a\in\Sigma}$ are indexed by states rather than natural numbers. In which case, we write $q \xrightarrow{w|r} q'$ if $M_w[q,q'] = r$ (regardless of whether w is a word or character). We also say that $I[q]$ and $F[q]$ are the initial and the final value of q for every state q.

A run ρ over a word $w = w_1\cdots w_n$ in $\mathcal{A}$ is a sequence of states interleaved with values: $q_0, v_1, q_1, \ldots, v_n, q_n$ such that $q_{i-1} \xrightarrow{w_i|v_i} q_i$ for $i \in \{1,\ldots,n\}$. We then associate the *value of the run* $\mathrm{val}(\rho) = I[q_0] \odot v_1 \odot \ldots \odot v_n \odot F[q_n]$. We say that ρ is an *accepting run* if $\mathrm{val}(\rho) \neq \mathbb{0}$. Equivalently all elements in the product $I[q_0]$, v_1, ..., v_n, $F[q_n]$ are different from $\mathbb{0}$ (for the semirings in this paper). We denote the set of all accepting runs of $\mathcal{A}$ over w by $Acc(\mathcal{A},w)$. Then $\mathcal{A}(w) = \bigoplus_{\rho_i\in Acc(\mathcal{A},w)} \mathrm{val}(\rho_i)$. The equivalence with the matrix definition is clear for all commutative semirings since runs that are not accepting contribute $\mathbb{0}$ to the sum.

Consider a weighted automaton $\mathcal{A}$. We write that $\mathcal{A}$ is:

- *unambiguous* if $|Acc(\mathcal{A},w)| \leqslant 1$ for all $w \in \Sigma^*$;
- *finitely-ambiguous* if there exists $k \in \mathbb{N}$ such that $|Acc(\mathcal{A},w)| \leqslant k$ for all $w \in \Sigma^*$;
- *polynomially-ambiguous* if there exists a polynomial function p such that $|Acc(\mathcal{A},w)| \leqslant p(|w|)$ for all $w \in \Sigma^*$. If p is linear we also say that $\mathcal{A}$ is linearly-ambiguous.

Below we show two examples of weighted automata over the semiring $\mathbb{Q}_{\geqslant 0}(+,\cdot)$.

Example 2.1. Consider $\mathcal{A} = (\{a\}, I, F, M_a)$, where $I = (1,0)$, $F = (0,1)$, and $M_a = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Then $\mathcal{A}(a^n) = n \mod 2$. The automaton $\mathcal{A}$ is unambiguous (see Figure 1).

Example 2.2. Consider $\mathcal{B} = (\{a\}, I, F, M'_a)$, the same as $\mathcal{A}$ except that $M'_a = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Then $\mathcal{B}(a^n) = n$. The automaton $\mathcal{B}$ is linearly-ambiguous (see Figure 1). Moreover, it can be shown that the function defined by $\mathcal{B}$ cannot be defined by a finitely-ambiguous automaton (see e.g. [3, Lemma 12]).

2.2 Cost-register automata and its restrictions

For a semiring $\mathbb{S}$ and a set of registers $\mathcal{X}$ we write affine$(\mathbb{S},\mathcal{X})$ for the set of affine expressions, i.e., expressions of the form $c \oplus \bigoplus_{x\in\mathcal{X}}(s_x \odot x)$, where $s_x, c \in \mathbb{S}$ and $x \in \mathcal{X}$. A different presentation of weighted automata are *linear cost-register automata* (linear CRA). A linear CRA $\mathcal{B}$ is defined as a tuple $(\Sigma, Q, q_0, I, F, \mathcal{X}, \delta)$, where: Σ is a finite alphabet; Q is a finite set of states, with $q_0 \in Q$ the designated initial state; $\mathcal{X}$ is a finite set of registers; $I : \mathcal{X} \to \mathbb{S}$ and $F : Q \times \mathcal{X} \to \mathbb{S}$ are, respectively, the initial values and final

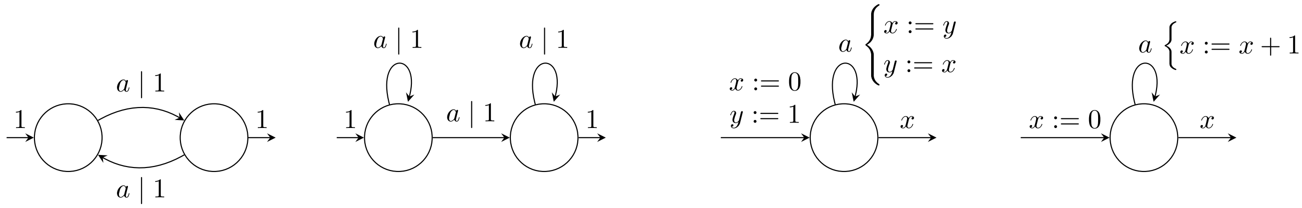


Figure 1: On the left: pictures for Example 2.1 and Example 2.2, respectively. Vectors I and F are presented with ingoing and outgoing edges, respectively; and the elements of M_a by edges between states. In all cases 0 values are omitted. It is easy to see that the left automaton is unambiguous (even deterministic) and that the automaton on the right has n accepting runs on a^n, and thus is linearly-ambiguous. On the right: pictures of stateless CRA equivalent to the ones on the left. The initial and final coefficients are presented by the ingoing and outgoing arrows, respectively. For example: on the left $F(x) = 1$, $F(y) = 0$; and on the right $F(x) = 1$. Both CRA are linear and copyless.

coefficients of registers; and $\delta\colon Q \times \Sigma \to Q \times (\mathcal{X} \to \text{affine}(\mathbb{S}, \mathcal{X}))$ is a deterministic transition function.[1]

A configuration of a CRA is a pair (q, σ), consisting of a state and a valuation of the registers $\sigma \in \mathbb{S}^{\mathcal{X}}$. The initial configuration is (q_0, I). For every letter $a \in \Sigma$ we write $(q, \sigma) \xrightarrow{a} (q', \sigma')$ if $\delta(q, a) = (q', \sigma_{q,a})$ and $\sigma'(x) = \sigma_{q,a}(x)(\sigma)$, that is $\sigma_{q,a}(x)$, where every register is substituted with its previous valuation σ. Since $\mathcal{B}$ is deterministic for every input word $w = w_1 \cdots w_n$ there is a unique *run*, defined as a sequence of configurations: $(q_0, \sigma_0), \ldots, (q_n, \sigma_n)$, where (q_0, σ_0) is the initial configuration and $(q_{i-1}, \sigma_{i-1}) \xrightarrow{w_i} (q_i, \sigma_i)$ for every $i \in \{1, \ldots, n\}$. In such a case, we write $(q_0, \sigma_0) \xrightarrow{w} (q_n, \sigma_n)$. Finally, if the configuration after reading w is (q, σ_w) then the output of $\mathcal{B}(w)$ is $\bigoplus_{x \in \mathcal{X}} F[q, x] \odot \sigma_w(x)$. Thus $\mathcal{B}$ is a function $\Sigma^* \to \mathbb{S}$.

Our linear CRA are defined with states Q, as per their first introduction [2]. However, in the general case it is not hard to see that the *stateless* (or single state) model is equivalent. Indeed, it suffices to encode states into registers and consider $Q \times \mathcal{X}$ as registers. Note that this construction does not have to hold for restricted CRAs. Thus a stateless linear CRA is defined as a tuple $\mathcal{B} = (\Sigma, I, F, \mathcal{X}, \delta)$. All the notations are the same as for CRAs, but we will omit states in the stateless case, e.g. a configuration of $\mathcal{B}$ is a valuation of the registers $\sigma : \mathcal{X} \to \mathbb{S}$.

We say that two automata are equivalent if they define the same function $\Sigma^* \to \mathbb{S}$. Note that for every weighted automaton $\mathcal{A}$ there exists an equivalent (stateless) linear cost-register automaton $\mathcal{B}$, and conversely too. Indeed, it suffices to identify the dimension d in weighted automata with the size $|\mathcal{X}|$. Then $\sigma_a(x)$ in δ can be seen as rows of matrices in $\mathbb{S}^{|\mathcal{X}| \times |\mathcal{X}|}$. Formally, this is proved, e.g. in [2, Theorem 9]. In the remainder of the paper we will work with linear CRA and its subclasses.

We are also interested in automata where the linear update functions are *copyless*. A valuation $\sigma : \mathcal{X} \to \text{affine}(\mathbb{S}, \mathcal{X})$ is copyless if every register $x \in \mathcal{X}$ occurs at most once across all affine expressions. Formally, using the notation $\sigma(x) = \bigoplus_{z \in \mathcal{X}} (s_{x,z} \odot z) \oplus c_{x,z}$, for every $z \in \mathcal{X}$ at most one $s_{x,z}$ is different from $\mathbb{0}$. A CRA $\mathcal{B}$ is *copyless* if $\sigma_{q,a}$ is copyless for every transition $\delta(q, a) = (q', \sigma_{q,a})$.

Example 2.3. In Figure 1 we show that for both $\mathcal{A}(a^n) = n \bmod 2$ in Example 2.1 and $\mathcal{B}(a^n) = n$ in Example 2.2 there are equivalent stateless copyless linear CRA.

In general it is known that the classes are related as follows: copyless linear CRA are contained in copyless CRA, and the latter is contained in linear CRA [29]. Moreover, copyless linear CRA are contained in the class of linearly-ambiguous weighted automata [1, Remark 4]. A detailed presentation showing how CRA and weighted automata compare in terms of expressiveness is in Figure 2.

2.3 Independent-CRA, a more restricted CRA

We say that $\mathcal{B} = (\Sigma, I, F, \mathcal{X}, \delta)$ is an *Independent-CRA* if $\mathcal{B}$ is a stateless linear CRA such that $\sigma_a(x) = (c_{a,x} \odot x) \oplus d_{a,x}$, where $c_{a,x}, d_{a,x} \in \mathbb{S}$ for every $\delta(a) = \sigma_a$. In other words the new value of every register does not depend on other registers. Observe that Independent-CRA are a subclass of stateless copyless linear CRA. A similar model was studied in [15].

Example 2.4. The right automaton in Figure 1 is an example Independent-CRA. It is not hard to show that there is no Independent-CRA that is equivalent to the automaton $\mathcal{A}(a^n) = n \bmod 2$. Indeed, it follows immediately from the definition that if C is an Independent-CRA and $C(\epsilon) = C(a^2) = 0$ then $C(a^n) = 0$ for all $n \in \mathbb{N}$.

2.4 Decision problems

We define decision problems with respect to semirings. The problems are well-defined for functions $\Sigma^* \to \mathbb{S}$, in particular for weighted automata, linear CRA and Independent-CRA. The decision problems are well-defined with respect to all considered semirings, but we will mostly focus on the semiring $\mathbb{Q}_{\geqslant 0}(+, \cdot)$

The $\leqslant$*-threshold problem*: given an automaton $\mathcal{A}$ and a number c (from the domain of the semiring) is it the case that $\mathcal{A}(w) \leqslant c$ for all $w \in \Sigma^*$. The $\geqslant$*-threshold problem* is defined similarly, where $\mathcal{A}(w) \leqslant c$ is replaced with $\mathcal{A}(w) \geqslant c$.

[1] Linear CRA were originally defined with linear updates (rather than affine). Affine updates can be simulated by linear updates by introducing one extra register with value fixed to $\mathbb{1}$. We use affine updates because the register constraints we introduce later do not apply to this special register.

The *boundedness* problem: given an automaton $\mathcal{A}$ does there exist a finite number c such that $\mathcal{A}(w) \leqslant c$ for all $w \in \Sigma^*$. A sequence $w_1, w_2, \ldots$ of words with $\lim_{i\to\infty} \mathcal{A}(w_i) = +\infty$ is a witness of unboundedness.

The *zero isolation* problem: given an automaton $\mathcal{A}$ is it the case that there exists a positive rational number $c > 0$ such that $\mathcal{A}(w) \geqslant c$ for all $w \in \Sigma^*$. A sequence $w_1, w_2, \ldots$ of words with $\lim\limits_{i\to\infty} \mathcal{A}(w_i) = 0$ is a witness of nonisolated zero.

3 DETAILED STATE OF THE ART AND OUR RESULTS

We already remarked that for probabilistic automata both the $\leqslant$-threshold and $\geqslant$-threshold problems are well-known to be undecidable [32], even when the model is restricted to linearly ambiguous [16, Theorem 2]. The zero isolation problem is also undecidable for probabilistic automata [22]. Hence, these three problems are also undecidable for weighted automata over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$.

We remarked that the boundedness problem is not interesting for probabilistic automata, since all words have value bounded by 1. For weighted automata over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$ the problem is undecidable; we are not aware whether this fact is stated in the literature. Nevertheless, it can be proven within this paragraph (a similar argument appears e.g. in the proof of [6, Theorem 1]). Consider the undecidable $\leqslant$-threshold problem for probabilistic automata: given a probabilistic automaton $\mathcal{A}$ is it the case that $\mathcal{A}(w) \leqslant \frac{1}{2}$ for all $w \in \Sigma^*$. One can easily define $\mathcal{B}$ (which is no longer probabilistic, but over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$) such that $\mathcal{B}(w_1\# \ldots \# w_n) = 2\mathcal{A}(w_1)\cdot \ldots \cdot 2\mathcal{A}(w_n)$, where # is some fresh symbol, which intuitively restarts the automaton. Then $\mathcal{B}$ is bounded if and only if $\mathcal{A}(w) \leqslant \frac{1}{2}$ for all words w.

Corollary 3.1. *The $\leqslant$-threshold, $\geqslant$-threshold, zero isolation, and boundedness problems are undecidable for weighted automata over the semiring $\mathbb{Q}_{\geqslant 0}(+, \cdot)$. The first two problems are undecidable even for linearly ambiguous models.*

On the positive side, when ambiguity is restricted to be finitely ambiguous we can infer some decidability results for the $\leqslant$-threshold and $\geqslant$-threshold problems from [16].

Proposition 3.2. *For finitely ambiguous weighted automata over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$ the $\leqslant$-threshold problem and the boundedness problem are decidable, and the $\geqslant$-threshold problem and the zero isolation problem are decidable assuming Schanuel's conjecture is true.*

In this paper we are mostly interested in the boundedness and zero isolation problems over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$ for copyless linear CRA and Independent-CRA. Below we state our main results.

Theorem 3.3. *Boundedness for copyless linear CRA over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$ is decidable in polynomial time.*

Theorem 3.4. *Zero isolation for Independent-CRA in dimension 3 over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$ is decidable, subject to Schanuel's conjecture. For copyless CRA zero isolation is undecidable.*

As mentioned in the introduction, the main contribution of the results are: the techniques in the decidability result that we believe might generalise to arbitrary dimension; and the nontrivial connections with other problems. To prove Theorem 3.4 we will show that the zero isolation problem is essentially equivalent to the boundedness problem over $\mathbb{L}og\mathbb{Q}(\min, +)$. Thus the positive part of Theorem 3.4 will be a corollary of the following.

Theorem 3.5. *Zero isolation for Independent-CRA in dimension 3 over $\mathbb{L}og\mathbb{Q}(\min, +)$ is decidable, subject to Schanuel's conjecture. For Independent-CRA in dimension 3 over $\mathbb{Z}(\min, +)$ the boundedness problem is decidable in ExpTime (independent of Schanuel's conjecture).*

Proof of Proposition 3.2. *Threshold problems:* Consider the following containment problem: given two probabilistic automata $\mathcal{A}$ and $\mathcal{B}$ is it the case that $\mathcal{A}(w) \leqslant \mathcal{B}(w)$ for all words w. When $\mathcal{A}$ is finitely ambiguous and $\mathcal{B}$ is unambiguous then the problem is decidable [16, Proposition 16]. When $\mathcal{A}$ is unambiguous and $\mathcal{B}$ is finitely ambiguous then the problem is decidable, assuming Schanuel's conjecture is true [16, Theorem 17].

Consider an input for one of the threshold problems: a finitely ambiguous weighted automaton $\mathcal{A} = (\Sigma, I, F, (M_a)_{a\in\Sigma})$ over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$ and $c \in \mathbb{Q}_{\geqslant 0}$. Let N be the sum of all constants that appear in I, F, and M_a for all $a \in \Sigma$ and let $C = \max(c, N)$. We define the automaton $\mathcal{A}/C = (\Sigma, I', F', (M'_a)_{a\in\Sigma})$, where $I'(q) = I(q)/C$, $F(q) = F'(q)/C$, and $M'_a(p, q) = M_a(p, q)/C$. It is easy to see that $\mathcal{A}/C$ is a probabilistic automaton and that $\mathcal{A}/C(w) = \mathcal{A}(w)/C^{|w|+2}$ for all $w \in \Sigma^*$. It remains to observe that it is easy to define an unambiguous probabilistic automaton $\mathcal{B}$ such that $\mathcal{B}(w) = c/C^{|w|+2}$ for all $w \in \Sigma^*$. Thus the threshold problems can be reduced to the containment problems between $\mathcal{A}$ and $\mathcal{B}$. We conclude by the mentioned results from [16].

Boundedness: Since there are finitely many runs, check that at least one run is unbounded, which occurs if and only if some accessible cycle has weight greater than one.

Zero isolation: We reduce to the Big-O problem, which asks whether there exists $C > 0$ such that for all $w \in \Sigma^*$ $\mathcal{A}(w) \leqslant C\cdot\mathcal{B}(w)$. The problem is decidable for finitely-ambiguous $\mathcal{A}, \mathcal{B}$ assuming Schanuel's conjecture is true [11, Theorem 9.2]. Let $\mathcal{A}(w) = 1$ for all $w \in \Sigma^*$. Then there exists $C > 0$ such that $\mathcal{B}(w) \geqslant \frac{1}{C}$ for all $w \in \Sigma^*$ (*zero isolation*) if and only if $\mathcal{A}(w)$ is big-O of $\mathcal{B}(w)$. ◻

Organisation. In the following sections we will prove Theorems 3.3, 3.4 and 3.5. Section 4 proves decidability of the boundedness problem for copyless linear CRA over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$ (Theorem 3.3). Section 5 shows the chain of reductions from zero isolation for weighted automata over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$, through boundedness for weighted automata over $\mathbb{L}og\mathbb{Q}(\min, +)$, up to universal coverability in OVAS. Finally, Section 6 shows that universal coverability is decidable in dimension three, proving Theorem 3.4 and Theorem 3.5. Figure 2 presents the results also explaining how Independent-CRA and copyless linear CRA relate to other classes of weighted automata in terms of expressiveness. Omitted proofs (including the negative part of Theorem 3.4) can be found in the full version [14].

4 BOUNDEDNESS FOR COPYLESS LINEAR CRA OVER $\mathbb{Q}_{\geqslant 0}(+, \cdot)$

The goal of this section is to establish that the boundedness problem for copyless linear CRA over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$ is decidable in polynomial time, that is, Theorem 3.3.

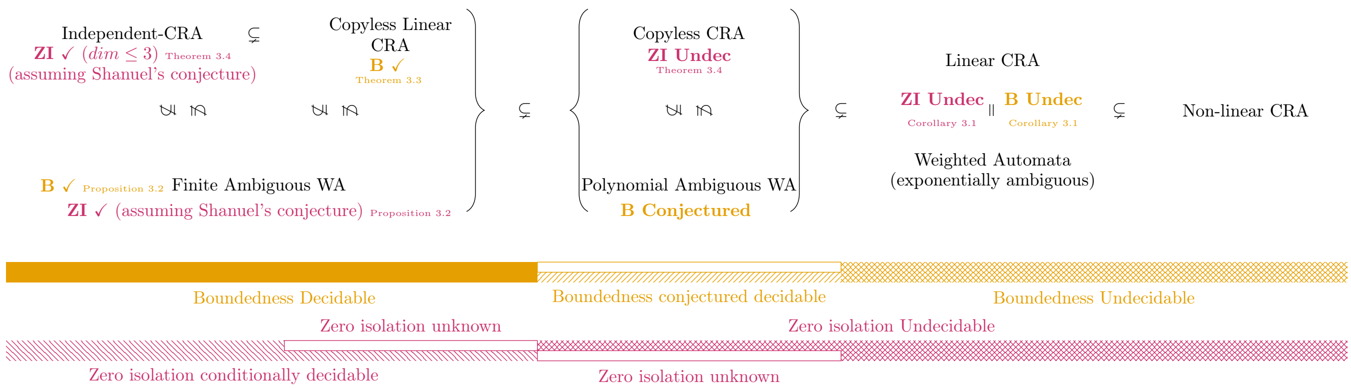


Figure 2: Decidability status of the boundedness (B) and zero isolation (ZI) problems in the semiring $\mathbb{Q}_{\geqslant 0}(+,\cdot)$. Notation $X \subsetneq Y$ indicates that Y recognises every function of X in every semiring, but there exists a semiring where Y recognises at least one other function not recognised by X. Notations $X \not\subseteq Y, Y \not\subseteq X$ mean that there exists some semiring where a function from X is not in Y and visa-versa. By Example 2.1 and Example 2.4 we see that the function $a^n \to n \mod 2$ is in the class of finitely-ambiguous weighted automata but not in Independent-CRA. Conversely, by Example 2.2 and Example 2.4 we see that the function $a^n \to n$ is in Independent-CRA but not in the class of finitely-ambiguous weighted automata. Other examples of non-inclusions go beyond copyless linear CRA and they are not always over $\mathbb{Q}_{\geqslant 0}(+,\cdot)$. See [1, 3, 29–31].

Our first step is to translate copyless linear CRA into WA with certain properties. More precisely, the WA will be nearly deterministic, except for a single state introducing ambiguity. The resulting automaton will be linearly ambiguous.

To operate this transformation, we need some additional notations for WA. Fix a WA $\mathcal{A} = (\Sigma, I, F, (M_a)_{a\in\Sigma})$ and let $\rho = q_0, v_1, q_1, \ldots, v_n, q_n$ be a run in $\mathcal{A}$. We say that ρ *starts* in q_0 and *ends* in q_n to indicate the first and the last state, respectively. We define $\mathrm{val}_{\leftrightarrow}(\rho) = v_1 \cdots v_n$ ($\mathrm{val}_{\leftrightarrow}(\rho) = 1$ if $n = 0$). Thus $\mathrm{val}(\rho) = I(q_0) \cdot \mathrm{val}_{\leftrightarrow}(\rho) \cdot F(q_n)$.

The run ρ is a q_0-*cycle* (or simply a cycle) if $q_0 = q_n$. A cycle is *simple* if $q_0, q_1 \ldots, q_{n-1}$ are all different (i.e. only the first and the last states are the same). We say that ρ' is a *sub-run* of ρ if $\rho' = q_i, v_{i+1}, q_{i+1}, \ldots, v_j, q_j$ for some $1 \leqslant i \leqslant j \leqslant n$. If ρ' is also a cycle then as $q_i = q_j$ the sequence $q_0, v_1, q_1, \ldots, q_{i-1}, v_i, q_j, v_{j+1}, \ldots v_n, q_n$ obtained by removing ρ' from ρ is a run of $\mathcal{A}$.

We translate copyless linear CRA into a new subclass of WA. Note that similar observations to the following definition and lemma were made in [1, Proposition 2].

Definition 4.1. A WA $\mathcal{B}$ is a *simple linearly-ambiguous weighted automaton* if its set of states can be written as $\{p, q_1, \ldots, q_n\}$ with the following properties:

(i) for all $a \in \Sigma$ there is a transition $p \xrightarrow{a|1} p$ and no other transition loops on, or enters, p; and
(ii) the automaton $\mathcal{B}$ restricted to the states $\{q_1, \ldots, q_n\}$ is deterministic.

We refer to the state p in Definition 4.1 as the *distinguished* state of the simple linearly-ambiguous WA. Notice that the only ambiguity in the automaton comes from the transitions that leave the distinguished state p to some other state.

Lemma 4.2. *Let $\mathcal{A}$ be a copyless linear CRA over $\mathbb{Q}_{\geqslant 0}(+,\cdot)$. One can build in polynomial time a simple linearly-ambiguous WA $\mathcal{B}$ over $\mathbb{Q}_{\geqslant 0}(+,\cdot)$ such that $\mathcal{A}$ is bounded if and only if $\mathcal{B}$ is bounded.*

Relying on Lemma 4.2 we may focus on simple linearly-ambiguous WA. We prove that unboundedness of such an automaton is characterised by certain patterns occurring in it. Lemma 4.3 shows what happens when such patterns are not present, and it is the key technical contribution in the proof. Then to prove Theorem 3.3 we only need to detect patterns violating the assumptions of Lemma 4.3.

We define specific sets of runs based on whether they exceed a given threshold: given $r \in \mathbb{Q}_{\geqslant 0}$ we set

$$Runs_{>r}(w) = \{\rho \mid \rho \text{ is a run over } w, \mathrm{val}_{\leftrightarrow}(\rho) > r\}\,.$$

Lemma 4.3. *Let $\mathcal{B} = (\Sigma, I, F, (M_a)_{a\in\Sigma})$ be a simple linearly-ambiguous WA with distinguished state p. Assume that for every word u and every q-cycle ρ over u, where $q \neq p$, both conditions hold:*

(i) $\mathrm{val}_{\leftrightarrow}(\rho) \leqslant 1$*;*
(ii) if $\mathrm{val}_{\leftrightarrow}(\rho) = 1$ *then* $M_u[p, q] = 0$.

Then $|Runs_{>\frac{1}{k}}(w)| \leqslant \mathrm{poly}\log(k)$ *for every* $k \geqslant 2$ *and every word* w.

The constants implied by the poly log in Lemma 4.3 depend on the rational numbers occurring in the transitions of $\mathcal{B}$. However, it is crucial that the bound on $|Runs_{>\frac{1}{k}}(w)|$ does not dependent on w. To get some intuition we show how the lemma concludes the proof of Theorem 3.3.

Sketch of Theorem 3.3. If the automaton violates the assumptions of Lemma 4.3, one can construct a witness for unboundedness. Conversely, divide all runs into $P_i = \{\rho \mid \frac{1}{c^i} < \mathrm{val}_{\leftrightarrow}(\rho) \leqslant \frac{1}{c^{i-1}}\}$ for $i \in \{1, \ldots, n\}$ and some constant c. By Lemma 4.3 we have

$|P_i| \leqslant \mathrm{poly}(\log c^i) = \mathrm{poly}(i \log c) = \mathrm{poly}(i)$. We obtain

$$\mathcal{B}(w) \leqslant \sum_{\rho \in Runs_{>0}(w)} \mathrm{val}(\rho) \leqslant \sum_{i=1}^{n} \frac{|P_i|}{c^{i-1}} \leqslant \sum_{i=1}^{\infty} \frac{\mathrm{poly}(i)}{c^{i-1}}.$$

Notice that the series converges, independent of w. □

Before establishing Lemma 4.3 we need to introduce some notation and intermediary results. Roughly, our goal is to obtain a finite representation of the set of matrices M_w. This will allows us to invoke Simon's Factorisation Forest Theorem that gives a tree representation on runs on w, such that nodes (corresponding to subwords of w) have height independent on w. Then, intuitively, the degree of poly log in Lemma 4.3 corresponds to the height of the node.

Let $\mathcal{B}$ be as in Lemma 4.3. As usual we will identify the dimensions of the vectors and matrices with the set of states $Q = \{p, q_1, \ldots, q_n\}$, where p is the distinguished state. Recall that: $\mathcal{B}$ is deterministic when restricted to $Q \setminus \{p\}$; $M_w[p,p] = 1$; and $M_w[q,p] = 0$ for every $q \in Q \setminus \{p\}$ and every word $w \in \Sigma^*$. Thus for every $q \neq p$ and every $w \in \Sigma^*$, there exists at most one q' such that $M_w[q,q'] > 0$. We further observe that for every pair of states $q, q' \in Q \setminus \{p\}$ and every word $w \in \Sigma^*$ there is at most one run ρ over w starting in q and ending in q' such that $\mathrm{val}_{\leftrightarrow}(\rho) > 0$. If there is such a run then we will denote $\rho = \overrightarrow{(q, w, q')}$ and $\mathrm{val}_{\leftrightarrow}(\rho) = M_w[q,q']$. We say that $r \in \mathbb{Q}_{\geqslant 0}$ is an admissible weight if there exists a word w and a run ρ over w such that $\mathrm{val}_{\leftrightarrow}(\rho) = r$.

For every $q \in Q \setminus \{p\}$ let

$$\begin{aligned} s_q &= \min_{w \in \Sigma^*} \left\{ \frac{1}{\mathrm{val}_{\leftrightarrow}(\rho)} \mid \rho \in Runs_{>0}(w), \rho \text{ starts in } q \right\}; \\ e_q &= \min_{w \in \Sigma^*} \left\{ \frac{1}{\mathrm{val}_{\leftrightarrow}(\rho)} \mid \rho \in Runs_{>0}(w), \rho \text{ ends in } q \right\}. \end{aligned} \tag{1}$$

Claim 4.4. *$0 < s_q, e_q \leqslant 1$ and both are computable rationals.*

Claim 4.5. *Let $x > 0$. There are finitely many admissible weights larger than x.*

Recall that $M_w \in (\mathbb{Q}_{\geqslant 0})^{Q \times Q}$ for every $w \in \Sigma^*$. Let $\varepsilon \notin \mathbb{Q}_{\geqslant 0}$ be a fresh symbol. We define the abstraction $\overline{M} \in (\mathbb{Q}_{\geqslant 0} \cup \{\varepsilon\})^{Q \times Q}$ as follows

$$\overline{M}[q,q'] = \begin{cases} 0 & \text{if } M[q,q'] = 0 \\ M[q,q'] & \text{if } q \neq p \text{ and } M[q,q'] \geqslant e_q \cdot s_{q'} \\ \varepsilon & \text{otherwise,} \end{cases} \tag{2}$$

for all $q, q' \in Q$, and e_q, $s_{q'}$ as defined in Equation (1). In words, $\overline{M}$ is the same as M, but some positive entries are replaced with ε. Notice that by Claim 4.5 this set of matrices is finite, as intended. The special symbol ε appears within $\overline{M}_w$ in two cases: it replaces $M_w[q,q']$ if $q \neq p$ and this value is small enough; and it is used to indicate whether there are any positive runs from p to q' (their exact values are not important). In particular, all non-zero weights of transitions from p are set to ε. An example translation is in Figure 3. The claim below states the purpose of ε formally.

Claim 4.6. *For every $w \in \Sigma^*$ and $q, q' \in Q \setminus \{p\}$:*

(1) if $\overline{M}_w[q,q'] \neq 0$, then $\overline{M}_w[q,q'] = \varepsilon$ if and only if, for every run ρ such that $\overrightarrow{(q, w, q')}$ is its subrun, $\mathrm{val}_{\leftrightarrow}(\rho) < 1$.
(2) $\overline{M}_w[p,q'] = \varepsilon$ if and only if $M_w[p,q'] > 0$.

We define the sum, product and order of ε with rationals. One can think that ε represents a number above zero but 'smaller' than the positive rationals. The only operation where this intuition breaks is addition, where ε is an absorbing element. This will be explained later. Formally: $0 < \varepsilon < r; \varepsilon \cdot 0 = 0 \cdot \varepsilon = 0$, $\varepsilon \cdot r = r \cdot \varepsilon = \varepsilon \cdot \varepsilon = \varepsilon; \varepsilon + x = x + \varepsilon = \varepsilon + \varepsilon = \varepsilon$ for every $x \in \mathbb{Q}_{\geqslant 0}$ and $r \in \mathbb{Q}_{>0}$.

We define the product of abstracted matrices. For every $M, N \in (\mathbb{Q}_{\geqslant 0} \cup \{\varepsilon\})^{Q \times Q}$ let MN be the usual product of matrices, i.e. $MN[q,q'] = \sum_{q'' \in Q} M[q,q''] \cdot N[q'',q']$. Then we define

$$M \otimes N[q,q'] = \begin{cases} \varepsilon & \text{if } 0 < MN[q,q'] < e_q \cdot s_{q'} \\ MN[q,q'] & \text{otherwise.} \end{cases}$$

For matrices $\overline{M}_w, \overline{M}_u$ the states in $Q \setminus \{p\}$ are deterministic, thus to define $\overline{M}_w \otimes \overline{M}_u[q,q']$ for $q \neq p$ we will need to sum elements at most one of which is nonzero. In case of $q = p$, we sum several positive elements. However, in this case we will only be interested in whether the transition is positive or zero; this explains our definition of addition with ε.

Claim 4.7. *The set $\left\{\overline{M}_w \mid w \in \Sigma^*\right\}$ is finite and $\overline{M}_{wu} = \overline{M}_w \otimes \overline{M}_u$ for every $w, u \in \Sigma^*$. Thus $\left\{\overline{M}_w \mid w \in \Sigma^*\right\}$ is a finite monoid with the product $\otimes$.*

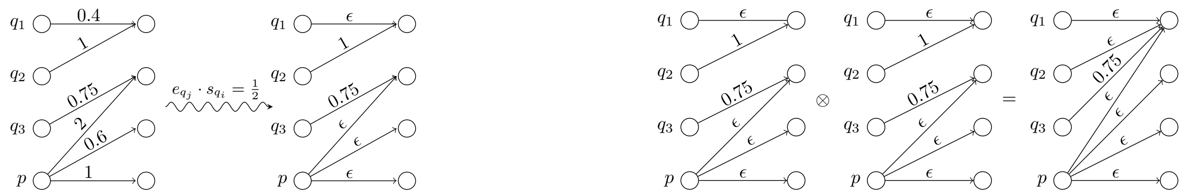
We let $\mathcal{M} = \left\{\overline{M}_w \mid w \in \Sigma^*\right\}$ denote the finite monoid of Claim 4.7. An element $M \in \mathcal{M}$ is *idempotent* if $M \otimes M = M$.

Consider a sequence of elements $e_1, e_2, \ldots e_n$ from $\mathcal{M}$. A factorisation of these elements is a labelled tree whose set of nodes is a subset of $\{(i,j) \mid 1 \leqslant i \leqslant j \leqslant n\}$. Intuitively, a node (i,j) corresponds to an infix $e_i, \ldots, e_j$. Formally: the leaves are $(1,1), \ldots, (n,n)$; the root is $(1,n)$; and for every (i,j) its children are $(i_0+1, i_1), (i_1+1, i_2)$, $\ldots, (i_{s-1}+1, i_s)$, where $i - 1 = i_0 < i_1 < i_2 \ldots < i_s = j$. The index $i_0 = i - 1$ is chosen so that even the first pair $(i_0+1, i_1) = (i, i_1)$ can be expressed as $(i_{x-1}+1, i_x)$. Every node (i,j) is labelled with $e_i \otimes e_{i+1} \otimes \ldots \otimes e_j \in \mathcal{M}$. Notice that the label of every parent is equal to the product of the labels of its children in the right order. We say that a node is idempotent if its label is idempotent. We will use the following result from [35].

Lemma 4.8 (Simon's Factorisation Forest Theorem). *Consider a sequence of elements $e_1, e_2, \ldots, e_n$ from a finite monoid S. There exists a factorisation into a tree of height at most $9|S|$ such that every inner node has either two children, or all its children are idempotents with the same label.*

We can now establish Lemma 4.3.

Proof of Lemma 4.3. Fix $k \geqslant 2$, $w = w_1 \ldots w_n \in \Sigma^*$ and let s_q, e_q be defined as in Equation (1) for all $q \in Q \setminus \{p\}$. Given $1 \leqslant s \leqslant t \leqslant n$ we will denote the infix $w_{s,t} = w_s \ldots w_t$. Let $a = \max\left\{\frac{1}{e_q \cdot s_{q'}} \mid q, q' \in Q \setminus \{p\}\right\}$. Notice that $a \geqslant 1$ and that all admissible weights are bounded by a. Let b be some rational number such that: $b < 1$; and for all $w \in \Sigma^*$ and $q, q' \in Q \setminus \{p\}$ if $\overline{M}_w[q,q'] = \varepsilon$ then $M_w[q,q'] < b$. Notice that by Claim 4.5 b is well-defined,



(a) Translation from M_w to $\overline{M}_w$. We assume that $e_{q_i} \cdot s_{q_j} = \frac{1}{2}$ for all i and j for simplicity.

(b) An example product of two elements.

Figure 3: Abstracted matrices $\mathcal{M} = \left\{\overline{M}_w \mid w \in \Sigma^*\right\}$.

unless ε does not occur in any matrix in $\mathcal{M}$ and then b's value will not be relevant (fix e.g. $b = \frac{1}{2}$ then). Consider a factorisation from Lemma 4.8 of $\overline{M}_{w_1}, \ldots, \overline{M}_{w_n}$ and let $H \leqslant 9|\mathcal{M}|$ be its height. We also fix a constant $\theta = \max(2, |Q|, 1 + H \log_{\frac{1}{b}} a)$ (the choice will become clear in the following). □

Claim 4.9. *Let $M \in \mathcal{M}$ be an idempotent and let $w^1, \ldots, w^m$ be words such that $\overline{M}_{w^i} = M$ for all $i \in \{1, \ldots, m\}$. Let $\rho \in Runs_{>0}(w^1 \ldots w^m)$ that starts in p and let $\rho_1, \ldots, \rho_m$ be subruns of ρ on the corresponding words $w^1, \ldots, w^m$. If $q_1, \ldots, q_m$ is the sequence of states where the $\rho_1, \ldots, \rho_m$ end, respectively, then there exists $i \in \{1, \ldots, m\}$ such that: $q_1 = q_2 \ldots = q_{i-1} = p$ and $q_{i+1} = q_{i+2} \ldots = q_m$. Moreover, either $i = m$ or $M[q_{i+1}, q_{i+1}] = \varepsilon$.*

Claim 4.10. *Let (s, t) be a node in the factorisation of height i, $0 \leqslant i \leqslant H$. Then*

$$\left|Runs_{>\frac{1}{k}a^{i-H}}(w_{s,t})\right| \leqslant \left(\theta(1 + \log_{\frac{1}{b}} k)\right)^{i+1} + |Q|.$$

Intuitively, either a node has not many children, then the number of runs cannot increase by a lot; or if there are many children then most runs will have a small value.

Proof of claim: For simplicity we will write log for $\log_{\frac{1}{b}}$. Since the automaton restricted to $Q \setminus \{p\}$ is deterministic, it suffices to prove that the number of runs starting in p is bounded by $(\theta(1+\log k))^{i+1}$. We proceed by induction on i. In the base case, when $i = 0$, (s, t) is a leaf and $w_{s,t}$ is a letter. Then there are at most $|Q|$ runs from p. We conclude since $(\theta(1 + \log k))^1 \geqslant |Q|$ for $k \geqslant 2$ by the choice of θ.

For the induction step assume that the claim holds for all $0, \ldots, i$ and we prove it true for $i + 1$. Since $i + 1 > 0$ (s, t) is an inner node. Let $(s_0 + 1, s_1)$, $(s_1 + 1, s_2), \ldots (s_{m-1} + 1, s_m)$ be the children of (s, t) such that $s - 1 = s_0 < s_1 < s_2 < \ldots < s_m = t$ and the height of every child is at most i. Consider a run $\rho \in Runs_{>\frac{1}{k}a^{i+1-H}}(w_{s,t})$ starting in p. Then ρ can be decomposed into m runs $\rho_1, \ldots, \rho_m$ over $w_{s_0+1,s_1}, \ldots, w_{s_{m-1}+1,s_m}$, respectively. Notice that $\mathrm{val}_{\leftrightarrow}(\rho) = \prod_{j=1}^{m} \mathrm{val}_{\leftrightarrow}(\rho_j)$. As $\mathrm{val}_{\leftrightarrow}(\rho) > \frac{1}{k}a^{i+1-H}$ this means that $\rho_x \in Runs_{>\frac{1}{k}a^{i-H}}(w_{s_{x-1}+1,s_x})$ for all $x \in \{1, \ldots, m\}$. Indeed, by the choice of a we know that

$$a \geqslant \prod_{j=1}^{x-1} \mathrm{val}_{\leftrightarrow}(\rho_j) \cdot \prod_{j=x+1}^{m} \mathrm{val}_{\leftrightarrow}(\rho_j),$$

hence

$$\mathrm{val}_{\leftrightarrow}(\rho_x)a \geqslant \mathrm{val}_{\leftrightarrow}(\rho) > \frac{1}{k}a^{i+1-H}.$$

We denote by q_j the ending state of ρ_j for $j \in \{1, \ldots, m\}$ (which is also the starting state of ρ_{j+1} for $j < m$). We consider two cases depending on the number of children m.

First, suppose there are two children, i.e. $m = 2$. Let us count the number of possible ρ, depending on whether $q_1 = p$ or $q_1 \neq p$. In the first case since there is exactly one run from p to p, the runs ρ differ only on ρ_2 and thus the number of such runs is bounded by $|Runs_{>\frac{1}{k}a^{i-H}}(w_{s_1+1,s_2})|$. In the second case since the transitions from $Q \setminus \{p\}$ are deterministic the number of runs is bounded by $|Runs_{>\frac{1}{k}a^{i-H}}(w_{s_0+1,s_1})|$. By the induction assumption altogether this is bounded $2(\theta(1 + \log k))^{i+1} \leqslant (\theta(1 + \log k))^{i+2}$ by the choice of θ.

Second, by Lemma 4.8 suppose that all children are idempotents with the same label, denote it M. By Claim 4.9, there is an index $x \in \{1, \ldots, m\}$ such that $q_1 = q_2 \ldots = q_{x-1} = p$, $q_{x+1} = q_{x+2} \ldots = q_m$, and if $x < m$ then $M(q_{x+1}, q_{x+1}) = \varepsilon$. By definition of b we get that $\mathrm{val}_{\leftrightarrow}(\rho_y) \leqslant b$ for $x < y \leqslant m$. Thus $\mathrm{val}_{\leftrightarrow}(\rho) \leqslant a \cdot b^{m-x}$ and since $\rho \in Runs_{>\frac{1}{k}a^{(i+1)-H}}(w_{s,t})$ we get $a \cdot b^{m-x} \geqslant \frac{a^{i+1-H}}{k}$, which implies $m - x \leqslant H \log a + \log k$. Thus there are at most $\theta + \log k \leqslant \theta(1 + \log k)$ valid indices for x. Let us count all possible ρ, depending on the value x. For a fixed x the number of possible ρ is bounded by $|Runs_{>\frac{1}{k}a^{i-H}}(w_{s_{x-1}+1,s_x})|$. This is because the automaton is deterministic on $Q \setminus \{p\}$. Thus by the induction assumption the number of all possible ρ is bounded by $\theta(1 + \log k) \cdot (\theta(1 + \log k))^{i+1} = (\theta(1 + \log k))^{i+2}$. ■

Lemma 4.3 follows by applying Claim 4.10 with $i = H$.

We conjecture that the results can be generalised to polynomially ambiguous weighted automata.

5 FROM INDEPENDENT-CRA TO OVAS

Theorem 5.1. *For Independent-CRA the problems of zero-isolation over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$ and boundedness over $\mathbb{L}\mathrm{og}\mathbb{Q}(\min, +)$ are interreducible in polynomial time.*

The detailed proof can be found in the full version [14]. The rough intuition is that for a weighted automaton $\mathcal{A}$ over $\mathbb{Q}_{\geqslant 0}(+, \cdot)$ one can define $\mathcal{A}_{\log}$, where every weight c is replaced with $-\log c$. Notice that $c_i \to 0$ iff $-\log c_i \to +\infty$. If $\mathcal{A}$ would be considered over $\mathbb{Q}_{\geqslant 0}(\max, \cdot)$ (i.e. when accepting runs are aggregated with max instead of +) then this theorem is essentially a syntactic translation. Thus the crux of Theorem 5.1 is to show that it is equivalent to consider the maximum run, rather than the aggregation with +.

We recall some definitions to define a new VAS model. Given a positive integer $d \in \mathbb{N}$ an *orthant* in $\mathbb{R}^d$ is a subset of the form $\{\mathbf{x} = (x_1, \ldots, x_d) \colon \epsilon_1 x_1 \geqslant 0, \ldots, \epsilon_d x_d \geqslant 0\}$ for some $\epsilon_i \in \{-1, 1\}$. We write O_d for the set of all orthants. Notice that $|O_d| = 2^d$. For example when $d = 2$ there are four orthants also called quadrants. Let $A_\heartsuit, A_\diamondsuit \in O_d$, where $A_s = \left\{\mathbf{x} \mid \epsilon_1^s x_1 \geqslant 0, \ldots, \epsilon_d^s x_d \geqslant 0\right\}$ for $s \in \{\heartsuit, \diamondsuit\}$. We write $A_\heartsuit \preceq A_\diamondsuit$ if $\epsilon_i^\heartsuit \leqslant \epsilon_i^\diamondsuit$ for all $i \in \{1, \ldots, d\}$. This is a partial order on O_d, where the *negative orthant*, defined by $x_1 \leqslant 0, \ldots, x_d \leqslant 0$, is the smallest element; and the *positive orthant*, defined by $x_1 \geqslant 0, \ldots, x_d \geqslant 0$, is the largest element. Given an orthant $A = \{\mathbf{x} \mid \epsilon_1 x_1 \geqslant 0, \ldots, \epsilon_d x_d \geqslant 0\}$ we will be often interested in points $A^{\preceq} = \{\mathbf{x} \mid \exists A_\mathbf{x} \in O_d \text{ s.t. } A \preceq A_\mathbf{x} \text{ and } \mathbf{x} \in A_\mathbf{x}\}$. Let $i_1, \ldots, i_k$ be all indices such that $\epsilon_{i_j} = 1$. Notice that $A^{\preceq} = \left\{\mathbf{x} \mid \epsilon_{i_1} x_{i_1} \geqslant 0, \ldots, \epsilon_{i_k} x_{i_k} \geqslant 0\right\}$.

Notice that some vectors belong to more than one orthant, when some of their coordinates are zero. Given a vector $\mathbf{v}$ we denote by $A_\mathbf{v}^+$ and $A_\mathbf{v}^-$ the largest and the smallest orthants that contain $\mathbf{v}$, respectively. Notice that these are well defined since $\preceq$ induces a lattice on O_d.

We define a model related to vector addition systems over integers [25]. Consider a positive integer $d \in \mathbb{N}$. A *d-dimensional orthant vector addition system* (d-OVAS or OVAS if d is irrelevant) is $\mathcal{V} = (T_A)_{A \in O_d}$, where every T_A is a finite set of vectors $T_A \subseteq \mathbb{R}^d$ with the following property. If $A \preceq B$ then $T_A \subseteq T_B$. We will refer to this property as *monotonicity* of $\mathcal{V}$. It will be convenient to denote $T = \bigcup_{A \in O_d} T_A$. We define the norm of $\mathcal{V}$ as $\|\mathcal{V}\| = \max\{\|\mathbf{v}\| \mid \mathbf{v} \in T\}$. The transitions in $\mathcal{V}$ are encoded efficiently, i.e. for every $\mathbf{v} \in T$ it suffices to store the minimal orthants A such that $\mathbf{v} \in T_A$. Note that $\mathbf{v}$ may be minimal for multiple incomparable orthants.

A *run* from $\mathbf{v}_0$ over $\mathcal{V}$ is a sequence $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n$ such that $\mathbf{v}_{i+1} - \mathbf{v}_i \in T_{A^+_{v_i}}$ for all $i \in \{0, \ldots, n-1\}$. If such a run exists then we write $\mathbf{v}_0 \to^* \mathbf{v}_n$. We allow $n = 0$ and thus $\mathbf{v} \to^* \mathbf{v}$ for every vector $\mathbf{v}$.

The *universal coverability* problem is defined as follows. Given a d-OVAS $\mathcal{V}$ decide if for every vector $\mathbf{v} \in \mathbb{R}^d$, there exists $\mathbf{w}$ in the positive orthant (i.e. $\mathbf{w} \in \mathbb{R}^d_{\geqslant 0}$) such that there is a run $\mathbf{v} \to^* \mathbf{w}$. If there are such runs then we say that $\mathcal{V}$ is a positive instance of universal coverability.

The *coverability* problem is similar but the initial point is fixed. Formally, given a d-OVAS $\mathcal{V}$ and a vector $\mathbf{v} \in \mathbb{R}^d$ decide if there is a run $\mathbf{v} \to^* \mathbf{w}$ for some $\mathbf{w} \in \mathbb{R}^d_{\geqslant 0}$.

Theorem 5.2. *The boundedness problem for Independent-CRA over $\mathbb{L}\text{og}\mathbb{Q}(\min, +)$ and the universal coverability problem for OVAS are interreducible in polynomial time.*

Proof sketch. We only give an intuition. Given an Independent-CRA $\mathcal{A} = (\Sigma, I, F, \mathcal{X}, (\delta_a)_{a \in \Sigma})$ the dimension of the OVAS $\mathcal{V}$ is $|\mathcal{X}| = d$. For every letter $a \in \Sigma$ let $\delta_a(x) = \min(x + c_{a,x}, d_{a,x})$. The idea is that $c_{a,x}$ are the coordinates of a corresponding vector $\mathbf{c}_a$ in $\mathcal{V}$, while $d_{a,x}$ determine the orthants in which it is available. Intuitively, $\mathcal{A}$ consumes letters in the reversed order compared to applying the corresponding vectors in $\mathcal{V}$. Then $d_{a,x} = +\infty$ does not impose any restrictions, while $d_{a,x} < +\infty$, means that the value of register x needs to be big enough for the transition to be fired. □

5.1 OVAS with continuous semantics

Let $\mathcal{V} = (T_A)_{A \in O_d}$ be a d-OVAS. A *continuous run* from $\mathbf{v}$ over $\mathcal{V}$ is a sequence $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n$ such that for every $i \in \{0, \ldots, n-1\}$ there exists an orthant A_i, where: $\mathbf{v}_i, \mathbf{v}_{i+1} \in A_i$; and there exists $\delta_i \in \mathbb{R}_{>0}$ such that $\delta_i(\mathbf{v}_{i+1} - \mathbf{v}_i) \in T_{A_i}$. Notice that the former implies that orthants are crossed only by pausing on the boundaries. If such a run exists then we write $\mathbf{v}_0 \to_c^* \mathbf{v}_n$.

We say A_i is the orthant witnessing the transition if A_i is the maximal orthant such that $\mathbf{v}_i, \mathbf{v}_{i+1} \in A_i$, and then we write that $A_0, \ldots, A_{n-1}$ is the witnessing sequence of orthants.

We remark that it would be possible to drop the additional restriction of pausing at the boundaries. One would have to require $\delta_i \geqslant 1$ (otherwise, the vector $\delta_i(\mathbf{v}_{i+1} - \mathbf{v}_i)$ essentially becomes available in $T_{A_{i+1}}$). Moreover, with the restriction of pausing at the boundaries the behaviour within an orthant is similar to the standard continuous VAS model [7]. This will be convenient in Section 6, in particular to invoke Proposition 6.9.

Remark 5.3. A continuous run, where $\delta_i = 1$ for all i, is also a run.

The *universal continuous coverability* problem is defined as the universal coverability problem, where $\mathbf{v} \to^* \mathbf{w}$ is replaced with $\mathbf{v} \to_c^* \mathbf{w}$. Similarly, we will say e.g. that $\mathcal{V}$ is a positive instance of universal continuous coverability. In this subsection we will prove the following theorem.

Theorem 5.4. *Let $\mathcal{V} = (T_A)_{A \in O_d}$ be a d-OVAS. Then $\mathcal{V}$ is a positive instance of universal coverability if and only if it is a positive instance of universal continuous coverability.*

To prove the theorem we require several auxiliary lemmas about continuous runs over a d-OVAS $\mathcal{V} = (T_A)_{A \in O_d}$. Figure 4 shows geometric intuitions. In the following we assume that all vectors are over $\mathbb{R}^d$, unless specified otherwise.

Given two vectors $\mathbf{v}$ and $\mathbf{w}$, we define the set of maximal orthants on the path from $\mathbf{v}$ to $\mathbf{w}$:

$$O_{\mathbf{v},\mathbf{w}} = \left\{A^+_{\delta\mathbf{v} + (1-\delta)\mathbf{w}} \mid \delta \in [0,1]\right\}.$$

Lemma 5.5. *Fix some vectors $\mathbf{v}$ and $\mathbf{w}$. Suppose that, for all $C \in O_{\mathbf{v},\mathbf{w}}$, there exists $\delta_C \in \mathbb{R}_{>0}$ such that $\delta_C(\mathbf{w} - \mathbf{v}) \in T_C$. Then $\mathbf{v}' \to_c^* \mathbf{w}'$ for every $\mathbf{v}' = \lambda_1 \mathbf{v} + (1-\lambda_1)\mathbf{w}$, $\mathbf{w}' = \lambda_2 \mathbf{v} + (1-\lambda_2)\mathbf{w}$, where $0 \leqslant \lambda_2 \leqslant \lambda_1 \leqslant 1$.*

Lemma 5.6. *Let $\mathbf{v} \to_c^* \mathbf{w}$. Suppose, for all $C \in O_{\mathbf{v},\mathbf{w}}$, there exists $\delta_C \in \mathbb{R}_{>0}$ such that $\delta_C(\mathbf{w} - \mathbf{v}) \in T_C$. Let $\mathbf{v}' \geqslant \mathbf{v}$ and $\mathbf{w}' \geqslant \mathbf{w}$ be such that $\mathbf{w}' - \mathbf{v}' = \delta'(\mathbf{w} - \mathbf{v})$ for some $\delta' \in \mathbb{R}_{>0}$. Then $\mathbf{v}' \to_c^* \mathbf{w}'$.*

Lemma 5.7. *Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n$ be a continuous run such that $\mathbf{v}_0 \in \mathbb{R}^d_{<0}$ and let $\mathbf{a}_i = \mathbf{v}_i - \mathbf{v}_{i-1}$ for $i \in \{1, \ldots, n\}$. Let $m \geqslant 1$ and consider the sequence $\mathbf{v}'_0, \mathbf{v}'_1, \ldots, \mathbf{v}'_n$, defined by $\mathbf{v}'_0 = \mathbf{v}_0$, $\mathbf{v}'_i = \mathbf{v}'_{i-1} + m\mathbf{a}_i$. Then $\mathbf{v}'_i \to_c^* \mathbf{v}'_{i+1}$ for all $i \in \{0, \ldots, n-1\}$.*

We also need the following technical lemma, which is a direct consequence of the simultaneous version of the Dirichlet's approximation theorem. Intuitively, it says that given a finite set of reals we can multiply them all with the same natural number so that all resulting numbers are arbitrarily close to integers.

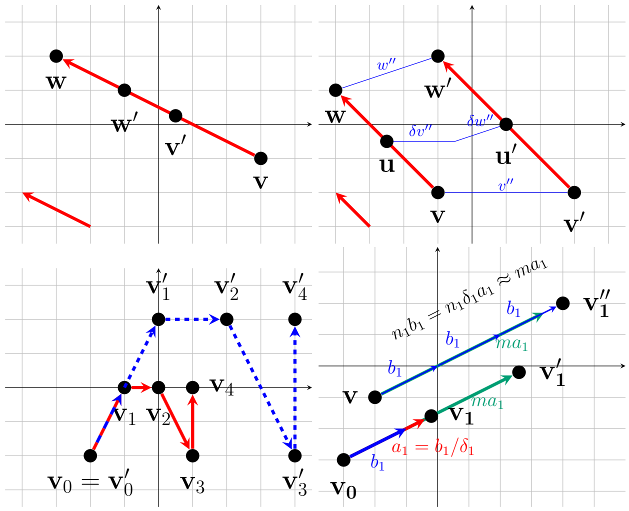


Figure 4: Examples are in dimension 2. Top left: a picture for Lemma 5.5. The transition $(-2,1)$ is available in some orthants, where v and w belong. Since $\mathbf{w}-\mathbf{v} = 3(-2,1)$ the lemma says that for all points $\mathbf{v}', \mathbf{w}'$ on the line from v to w there is a continuous run $\mathbf{v}' \to_c^* \mathbf{w}'$ (provided $\mathbf{v}'$ is closer than $\mathbf{w}'$ to v).
Top right: a picture for Lemma 5.6. The transition $(-1,1)$ is available in some orthants, where v and w belong. Since $\mathbf{w}-\mathbf{v} = 3(-1,1)$, $\mathbf{w}'-\mathbf{v}' = 4(-1,1)$, $\mathbf{v}' \geqslant \mathbf{v}$ and $\mathbf{w}' \geqslant \mathbf{w}$ there is a continuous run $\mathbf{v}' \to_c^* \mathbf{w}'$.
Bottom left: a picture for Lemma 5.7. The sequence $\mathbf{v}_0,\dots,\mathbf{v}_4$ is a continuous run and $\mathbf{v}_0$ has both coordinates negative. The sequence $\mathbf{v}_0',\dots,\mathbf{v}_4'$ mimics the first sequence but the distances between nodes are scaled by $m=2$. While $\mathbf{v}_0',\dots,\mathbf{v}_4'$ is not a continuous run (e.g. there is no orthant that contains both $\mathbf{v}_2'$ and $\mathbf{v}_3'$), there is a continuous run between each pair of consecutive nodes $\mathbf{v}_i' \to_c^* \mathbf{v}_{i+1}'$ (and thus $\mathbf{v}_0' \to_c^* \mathbf{v}_n'$).
Bottom right: a picture for proof of Theorem 5.4, the reduction from continuous semantics to discrete semantics, explaining $\mathbf{v}, \mathbf{v}_i, \mathbf{v}_i', \mathbf{v}_i''$ for first step.

Lemma 5.8. *[33, Theorem 1A] Let $r_1,\dots r_k \in \mathbb{R}$. For every $\epsilon > 0$ there exists $m \in \mathbb{N}\setminus\{0\}$ such that for every $i \in \{1,\dots,k\}$ there exists $z_i \in \mathbb{Z}$, where $|mr_i - z_i| < \epsilon$.*

Proof of Theorem 5.4. We show how to find a continuous witnessing run from a discrete run, and visa versa. A discrete run is not a continuous run, because a continuous run must pause at the boundary, and can only go further into the adjacent orthant if this direction remains available. We take a run from a sufficiently smaller starting point and shift it to our desired starting point. After the shift, if we cross orthants then directions will be always available due to monotonicity.

Conversely, when converting a continuous run to a discrete run, we face a continuous run using non-integer multiples of vectors. We use Lemma 5.7 to scale-up the run, using Lemma 5.8 to find a multiple so that the scaled real coefficients are sufficiently close to an integer.

Let $\mathcal{V} = (T_A)_{A\in O_d}$ be a d-OVAS. We start with the implication that if $\mathcal{V} = (T_A)_{A\in O_d}$ is a positive instance of universal coverability then it is also a positive instance of universal continuous coverability. Thus, fix any $\mathbf{v} \in \mathbb{R}^d$. We aim to prove that there is a continuous run $\mathbf{v} \to_c^* \mathbf{w}$ for some $\mathbf{w} \in \mathbb{R}^d_{\geqslant 0}$.

Let $\mathbf{u} = (\|\mathcal{V}\|,\dots,\|\mathcal{V}\|) \in \mathbb{R}^d$. By assumption there is a run $\mathbf{v}_0,\dots,\mathbf{v}_n$, where $\mathbf{v}_0 = (\mathbf{v}-\mathbf{u})$ and $\mathbf{v}_n \in \mathbb{R}_{\geqslant 0}$. Let $\mathbf{a}_{i+1} = \mathbf{v}_{i+1} - \mathbf{v}_i$ for $i \in \{0,\dots,n-1\}$, i.e. the differences between consecutive elements in the run. Since $\mathbf{a}_{i+1}$ is a transition, $\|\mathbf{a}_{i+1}\| \leqslant \|\mathbf{u}\|$. To conclude the proof we need to show that $\mathbf{v}_i + \mathbf{u} \to_c^* \mathbf{v}_{i+1} + \mathbf{u}$ for every $i \in \{0,\dots,n-1\}$; indeed, $\mathbf{v}_0 + \mathbf{u} = \mathbf{v}$ and $\mathbf{v}_n + \mathbf{u} \in \mathbb{R}^d_{\geqslant 0}$.

Since $\mathbf{v}_0,\dots,\mathbf{v}_n$ is a run, then $(\mathbf{v}_{i+1}-\mathbf{v}_i) \in T_{A^+_{\mathbf{v}_i}}$ for every $i \in \{0,\dots,n-1\}$. Let $\mathbf{v}_{i,\epsilon} = \epsilon(\mathbf{v}_{i+1}+\mathbf{u}) + (1-\epsilon)(\mathbf{v}_i+\mathbf{u})$, for $\epsilon \in [0,1)$. We argue that $\delta(\mathbf{v}_{i+1}-\mathbf{v}_i) = \delta((\mathbf{v}_{i+1}+\mathbf{u}) - (\mathbf{v}_i+\mathbf{u})) \in T_{A^+_{\mathbf{v}_{i,\epsilon}}}$ for some δ. Indeed

$$\begin{aligned}\mathbf{v}_{i,\epsilon} &= \epsilon(\mathbf{v}_{i+1}+\mathbf{u}) + (1-\epsilon)(\mathbf{v}_i+\mathbf{u})\\ &= \epsilon(\mathbf{v}_i + \mathbf{a}_{i+1}) + (1-\epsilon)(\mathbf{v}_i) + \mathbf{u} = \mathbf{v}_i + \mathbf{u} + \epsilon\mathbf{a}_{i+1}.\end{aligned}$$

Since $\|\mathbf{u}\| \geqslant \|\mathbf{a}_{i+1}\| \geqslant \|\epsilon\mathbf{a}_{i+1}\|$, we have $\mathbf{v}_{i,\epsilon} \geqslant \mathbf{v}_i$, thus, by monotonicity $\delta(\mathbf{v}_{i+1}-\mathbf{v}_i) \in T_{A^+_{\mathbf{v}_{i,\epsilon}}}$. Hence $\mathbf{v}_i + \mathbf{u} \to_c^* \mathbf{v}_{i+1} + \mathbf{u}$ by Lemma 5.5.

Conversely, we fix $\mathbf{v} \in \mathbb{R}^d$ and aim to prove that there is a run $\mathbf{v} \to^* \mathbf{w}$ for some $\mathbf{w} \in \mathbb{R}^d_{\geqslant 0}$. Take any vector $\mathbf{v}_0 \in \mathbb{R}_{<0}$ such that $\mathbf{v}_0 + \mathbf{1} \leqslant \mathbf{v}$, where $\mathbf{1} = (1,\dots,1)$. By assumption there is a continuous run $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n$ for some $\mathbf{v}_n \in \mathbb{R}_{\geqslant 0}$. Let $A_0,\dots,A_{n-1}$ be the corresponding witnessing sequence of orthants; $\mathbf{a}_{i+1} = \mathbf{v}_{i+1} - \mathbf{v}_i$; $\delta_{i+1} \in \mathbb{R}_{>0}$ be such that $\delta_{i+1}\mathbf{a}_{i+1} \in T_{A_i}$ for all $i \in \{0,\dots,n-1\}$; and $\mathbf{b}_{i+1} = \delta_{i+1}\mathbf{a}_{i+1}$.

By Lemma 5.8 there exists $m \in \mathbb{N}\setminus\{0\}$ such that for every $i \in \{0,\dots,n-1\}$ there exist $n_{i+1} \in \mathbb{N}$ such that $|\frac{m}{\delta_{i+1}} - n_{i+1}| < \frac{1}{n\|\mathcal{V}\|}$. Consider the scaled run $\mathbf{v}_0',\dots,\mathbf{v}_n'$ defined by $\mathbf{v}_0' = \mathbf{v}_0$, $\mathbf{v}_i' = \mathbf{v}_{i-1}' + m\mathbf{a}_i$. By Lemma 5.7 $\mathbf{v}_i' \to_c^* \mathbf{v}_{i+1}'$ for all $i \in \{0,\dots,n-1\}$. Moreover, it is easy to see that $\mathbf{v}_n' \in \mathbb{R}_{\geqslant 0}$. We define the sequence of points $\mathbf{v}_0'',\dots,\mathbf{v}_n''$ as follows: $\mathbf{v}_0'' = \mathbf{v}$ and $\mathbf{v}_i'' = \mathbf{v}_{i-1}'' + n_i\mathbf{b}_i$. Intuitively, this is like the $\mathbf{v}_i'$ run approximated to integers, and shifted by $\mathbf{1}$ to compensate for errors. The rest of the proof is dedicated to show first that $\mathbf{v}_0'' \to_c^* \mathbf{v}_n''$ and then that $\mathbf{v}_0'' \to^* \mathbf{v}_n''$, which will conclude the proof. The first step is depicted in Figure 4.

We prove that $\mathbf{v}_i'' \to_c^* \mathbf{v}_{i+1}''$ for every $i \in \{0,\dots,n-1\}$. Notice that $\frac{1}{n_i}(\mathbf{v}_{i+1}'' - \mathbf{v}_i'') = \mathbf{b}_{i+1} \in T_{A_i}$. Moreover,

$$\begin{aligned}\mathbf{v}_i'' &= \mathbf{v} + \sum_{j=1}^{i} n_j\mathbf{b}_j \;\geqslant\; \mathbf{v}_0' + \mathbf{1} + \sum_{j=1}^{i} n_j\mathbf{b}_j\\ &\qquad\qquad\text{by the choice of } \mathbf{v}_0 = \mathbf{v}_0'\\ &\geqslant \mathbf{v}_0' + \mathbf{1} + \sum_{j=1}^{i}\left(\frac{m}{\delta_j}\mathbf{b}_j - \frac{1}{n\,\|\mathcal{V}\|}\,\|\mathbf{b}_j\|\right)\\ &\qquad\qquad\text{by the choice of } m\\ &\geqslant \mathbf{v}_0' + \mathbf{1} + \sum_{j=1}^{i}\left(m\mathbf{a}_j - \frac{1}{n}\mathbf{1}\right)\\ &\qquad\qquad \mathbf{b}_j = \delta_j\mathbf{a}_j \text{ and } \|\mathbf{b}_j\| \leqslant \|\mathcal{V}\|\\ &= \frac{n-i}{n}\mathbf{1} + \mathbf{v}_0' + \sum_{j=1}^{i} m\mathbf{a}_j \;\geqslant\; \mathbf{v}_i'.\end{aligned}$$

Since $\mathbf{v}'_i \to_c^* \mathbf{v}'_{i+1}$ by Lemma 5.6 we get $\mathbf{v}''_i \to_c^* \mathbf{v}''_{i+1}$.

To conclude we need to prove that $\mathbf{v}''_i \to^* \mathbf{v}''_{i+1}$ for every $i \in \{0, \dots, n-1\}$. Recall that $\mathbf{v}''_{i+1} - \mathbf{v}''_i = n_{i+1}\mathbf{b}_{i+1}$ and that n_{i+1} is a natural number. Let $\mathbf{u}_0, \dots, \mathbf{u}_{n_{i+1}}$ be defined as $\mathbf{u}_0 = \mathbf{v}''_i$ and $\mathbf{u}_{j+1} = \mathbf{u}_j + \mathbf{b}_{i+1}$. Notice that $\mathbf{u}_{n_{i+1}} = \mathbf{v}''_{i+1}$. By Lemma 5.5 and Remark 5.3 $\mathbf{u}_j \to^* \mathbf{u}_{j+1}$ for every $j \in \{0, \dots, n_{i+1} - 1\}$. Therefore, $\mathbf{v}''_i \to^* \mathbf{v}''_{i+1}$, which concludes the proof. □

6 COVERABILITY FOR OVAS

In this section we present our undecidability and decidability results for coverability and universal coverability, respectively. In the definition of d-OVAS for each orthant A the set of transitions T_A is a subset of $\mathbb{R}^d$. For a set $S \subseteq \mathbb{R}$ an OVAS *over* S is an OVAS using numbers only from S in its transitions, namely for each orthant A we have $T_A \subseteq S^d$.

Theorem 6.1. *The coverability problem for OVAS over $\mathbb{Z}$ is undecidable.*

Our two decidability results are the following.

Theorem 6.2. *The universal continuous coverability problem for 3-OVAS over $\mathbb{Q}$ is decidable in* ExpTime.

Theorem 6.3. *Assuming Schanuel's conjecture the universal continuous coverability problem for 3-OVAS over $\mathbb{L}\mathrm{og}\mathbb{Q}$ is decidable.*

Together with Theorem 5.1, Theorem 6.3 completes the proof of Theorem 3.5.

Remark 6.4. Computations with elements from $\mathbb{L}\mathrm{og}\mathbb{Q}$ require considerable care. Whilst they are easy to represent (e.g. by storing $2^x \in \mathbb{Q}$ in place of $x \in \mathbb{L}\mathrm{og}\mathbb{Q}$), note that $(\mathbb{L}\mathrm{og}\mathbb{Q}, +, \cdot)$ is not a semiring. In particular, the product of two elements, e.g. $\log_2(a)\log_2(b)$, is not necessarily an element of $\mathbb{L}\mathrm{og}\mathbb{Q}$. In general, such computations are curiously difficult; for example (unconditionally) deciding whether $\log_2(a)\log_2(b) \leqslant \log_2(c)\log_2(d)$ for $a, b, c, d \in \mathbb{Q}_{>0}$ is, to the best of our knowledge, open and related to the four exponential conjecture which asks if they can ever be equal (see e.g.,[38, Sec. 1.3 and 1.4]). However, Schanuel's conjecture implies decidability of the first order theory of the reals with exponential function $FO(+, \cdot, \exp, <)$ [27]. In particular, this allows arithmetic operations between elements of $\mathbb{L}\mathrm{og}\mathbb{Q}$.

The rest of this section is devoted to the proofs of Theorems 6.2 and 6.3. We slowly introduce required notions and at the end we show how the developed techniques allow to prove both theorems. Most of our steps will work for a d-OVAS in any dimension d.

We define a notion of a *separator* with a property that an OVAS V is a negative instance of the universal coverability problem if and only if there exist a separator for V. Finally we show that the existence of a separator in 3-OVAS over $\mathbb{Q}$ can be expressed in the first order logic $FO(+, \cdot, <)$ with bounded quantifier alternation which is decidable in ExpTime due to Tarski's theorem [23]. The existence of a separator in 3-OVAS over $\mathbb{L}\mathrm{og}\mathbb{Q}$ can be expressed in the first order logic $FO(+, \cdot, \exp, <)$ which is decidable, subject to Schanuel's conjecture [27].

Given a d-OVAS we define the *walls set* $W = \{\mathbf{v} \in \mathbb{R}^d \mid \exists i \in [1, d] : \mathbf{v}_i = 0\}$, i.e. vectors in $\mathbb{R}^d$ with some coordinate equal to zero. The set W contains all of the faces of d-dimensional orthants. Recall that the negative orthant is defined as $\mathbb{R}^d_{\leqslant 0}$. We define the *strictly negative orthant* as $\mathbb{R}^d_{<0} = \mathbb{R}_{\leqslant 0} \setminus W$. For a d-OVAS V let

$$\mathrm{reach}(V) = \left\{\mathbf{v} \mid \exists \mathbf{u} \in \mathbb{R}^d_{<0} : \mathbf{u} \to_c^* \mathbf{v}\right\}$$

be the set of all vectors reachable from the strictly negative orthant. We observe the following.

Claim 6.5. *A d-OVAS V is positive instance of universal continuous coverability problem if and only if* $\mathrm{reach}(V) \cap \mathbb{R}^d_{\geqslant 0} \neq \emptyset$.

By Claim 6.5 it is enough to focus on deciding whether $\mathrm{reach}(V)$ intersects the positive orthant. For $S \subseteq \mathbb{R}^d$ we define its *downward closure* as $S^{\downarrow} = \{\mathbf{v} \mid \exists \mathbf{s} \in S : \mathbf{v} \leqslant \mathbf{s}\}$. Suppose $S \subseteq T \subseteq \mathbb{R}^d$. We say that S is *downward closed inside* T if $S = S^{\downarrow} \cap T$. Notice that $\mathrm{reach}(V)$ intersects the positive orthant if and only if $\mathrm{reach}(V)^{\downarrow}$ intersects the positive orthant. Furthermore, let

$$\mathrm{reach}^{\downarrow}_W(V) = \mathrm{reach}(V)^{\downarrow} \cap W.$$

Claim 6.6. *For every d-OVAS V the following are equivalent:* $\mathrm{reach}^{\downarrow}_W(V) \cap \mathbb{R}^d_{\geqslant 0} \neq \emptyset$ *if and only if* $\mathrm{reach}(V) \cap \mathbb{R}^d_{\geqslant 0} \neq \emptyset$.

We will focus on deciding whether $\mathrm{reach}^{\downarrow}_W(V)$ intersects the positive orthant. For each orthant A and $\mathbf{u}_0, \mathbf{u}_n \in A$ we write $\mathbf{u}_0 \to_{c,A}^* \mathbf{u}_n$ if there is a continuous run $\mathbf{u}_0, \dots, \mathbf{u}_n$ such that $\mathbf{u}_i \in A$ for all $0 \leqslant i \leqslant n$.

Definition 6.7. Given a d-OVAS V we say that $S \subseteq W$ is a *separator for* V if the following conditions are satisfied:

(1) S is closed under scaling, namely for every $\lambda > 0$ and $\mathbf{s} \in S$ we have $\lambda \cdot \mathbf{s} \in S$;
(2) S is downward closed inside W, namely $S = S^{\downarrow} \cap W$;
(3) for every orthant A if $u \in S$, $v \in W$ and $u \to_{c,A}^* v$ then $v \in S$;
(4) $S \cap \mathbb{R}^d_{\geqslant 0} = \emptyset$.

Lemma 6.8. *For every OVAS V:* $\mathrm{reach}^{\downarrow}_W(V) \cap \mathbb{R}^d_{\geqslant 0} = \emptyset$ *if and only if there exists a separator for V.*

We aim to show that the existence of a separator in 3-OVAS can be expressed in appropriate first order logics. It is helpful to use the following observation about continuous VASes from [7], which helps us to construct the needed first order sentences.

Proposition 6.9 (Reformulation of Proposition 3.2 in [7]). *Fix a d-OVAS V and an orthant $A \in O_d$. Consider two vectors of variables $\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{y} = (y_1, \dots, y_d)$. There is an existential formula $\varphi_A(\mathbf{x}, \mathbf{y})$ such that*

$$[\![\varphi_A]\!] = \left\{(\mathbf{v}, \mathbf{w}) \in A^2 \mid \mathbf{v} \to_{c,A}^* \mathbf{w}\right\}.$$

If V is over $\mathbb{Q}$ then $\varphi_A \in FO(+, \cdot, <)$, and if V is over $\mathbb{L}\mathrm{og}\mathbb{Q}$ then $\varphi_A \in FO(+, \cdot, \exp, <)$.

Notice that in our setting we work over reals, while in [7] they work over rationals. The results in [7] are stated for the logic $FO(+, <)$ over $\mathbb{Q}$, but it is easy to see that the same formulas work for our logics over $\mathbb{R}$. The reason why we need to consider irrational numbers is that whenever we deal with a number of the form $\log(\frac{p}{q})$ then we express this by $\exp(x) = \frac{p}{q}$.

The following lemma concludes the proofs in this section.

Lemma 6.10. *The existence of a separator in* 3*-OVAS:*

(1) over $\mathbb{Q}$ *is expressible in* $FO(+,\cdot,<)$ *over reals with fixed number of quantifier alternations;*
(2) over $\mathbb{L}\mathrm{og}\mathbb{Q}$ *is expressible in* $FO(+,\cdot,\exp,<)$ *over reals.*

Proof. The key observation is that a set $S \subseteq W_3$, which is downward closed and closed under scaling, can be described by at most 18 real numbers. Notice first that the set W_3 is a union of 12 quarters of a plane. Indeed, W_3 consists of three planes (defined by $x[1] = 0$, $x[2] = 0$ and $x[3] = 0$). Each of the three planes is divided into exactly four quarters. Thus a quarter Q is described by the choice of $i \in \{1, 2, 3\}$ such that $\mathbf{x} \in Q$ if $\mathbf{x}[i] = 0$ and two signs for the other coordinates. For example consider a quarter Q such that if $\mathbf{x} \in Q$ then $\mathbf{x}[3] = 0$. Then Q is determined by $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$, defining $Q = \{x \in \mathbb{R}^3 \mid x[1] \cdot \varepsilon_1 \geqslant 0, x[2] \cdot \varepsilon_2 \geqslant 0, x[3] = 0\}$. So every quarter is determined by a triple $(s_1, s_2, s_3) \in \{+, -, 0\}^3$ such that there is exactly one zero among s_1, s_2 and s_3. We will show that for every quarter Q the set $S \cap Q$ can be described using either one or two real numbers. To simplify the notation whenever Q is fixed we will think of quarters Q and S as subsets of $\mathbb{R}^2$ (projecting on the coordinates that are not fixed to 0 in Q).

Given a quarter Q consider two cases: 1) $\varepsilon_1 = \varepsilon_2$, 2) $\varepsilon_1 \neq \varepsilon_2$. We show that in the first case if $S \cap Q \neq \emptyset$ then $Q \subseteq S$. Indeed, assume $\mathbf{x} \in S \cap Q$ and let $\mathbf{y} \in Q$. We aim to show that $\mathbf{y} \in S$. Since $\varepsilon_1 = \varepsilon_2$, there are $\lambda \in \mathbb{R}_{\geqslant 0}$ and $\mathbf{v} \in \mathbb{R}^2_{\leqslant 0}$ such that $\mathbf{y} = \lambda\mathbf{x} + \mathbf{v}$. As $\mathbf{x} \in S$ and S is closed under scaling we have $\lambda\mathbf{x} \in S$. As S is downward closed and $\mathbf{v} \in \mathbb{R}^2_{\leqslant 0}$ we have $\mathbf{y} = \lambda\mathbf{x} + \mathbf{v} \in S$. To conclude either $S \cap Q$ is empty or it is the full quarter. Thus such quarters can be described by one variable $b \in \{0, 1\}$ (one bit of information: 0 for empty set, 1 for full set).

Consider the second case when $\varepsilon_1 \neq \varepsilon_2$ and assume without loss of generality that $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$. Thus $Q = \{(x_1, x_2) \mid x_1 \geqslant 0, x_2 \leqslant 0\}$. We observe that $Q \cap S$ is actually a part of Q which is below some line α. This will be the only step where we use the assumption that our OVAS is in 3-dimensions (which implies that Q is in 2 dimensions). Formally, a line $\{(x_1, x_2) \mid \alpha_1 x_1 = \alpha_2 x_2\}$ in Q is described by $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \in \mathbb{R}_{\geqslant 0}$, $\alpha_2, \in \mathbb{R}_{\leqslant 0}$, such that at least one $\alpha_i \neq 0$ (the sign of α_i comes from ε_i). □

Claim 6.11. $Q \cap S = \{(x_1, x_2) \in Q \mid \alpha_1 x_1 \lhd \alpha_2 x_2\}$ *, for some line* α *in* Q*, where* $\lhd$ *is either* $\leqslant$ *or* $<$*.*

We note that this claim is not true in higher dimensions, and it is the main obstacle for the proof to work in general. By Claim 6.11 the set $Q \cap S$ is described by two real numbers (recall that α_i also determines the value of ε_i). Summarising: 6 quarters need 1 bit of description, and 6 quarters need 2 real numbers. In total 18 numbers are needed to describe the downward closed set S, which is also closed under scaling.

Note that our description will satisfy conditions 1 and 2 in Definition 6.7. In order to check that given 18 real numbers describe a separator it remains to check that:

- the descriptions of quarters are consistent on the intersections of quarters;
- conditions 3 and 4 in Definition 6.7 are satisfied.

It is not hard to see that the first item can be described in $FO(+,\cdot,<)$, we just need to guarantee that quarters are consistent on the intersecting lines. In order to check condition 4 it is enough to guarantee that the quarters $(+, +, 0)$, $(+, 0, +)$ and $(0, +, +)$ are all described by the bit 0. The most involved part is to check condition 3. Here, we invoke Proposition 6.9 that defines the reachability formula $\varphi_A(\mathbf{x}, \mathbf{y})$.

We express condition 3 as follows: for every orthant A if $\mathbf{x} \in S$ and $\varphi_A(\mathbf{x}, \mathbf{y})$ then $\mathbf{y} \in S$. It is easy to transform the above description to sentences of first order logic. Moreover, observe that these sentences have quantifier alternation at most two: there exists a separator S, such that for all $\mathbf{x} \in S$ there exists $\mathbf{y} \in S$ fulfilling $\varphi_A(\mathbf{x}, \mathbf{y})$, where φ_A has only existential quantifiers. We have proved Lemma 6.10.

Remark 6.12. Notice that in d-OVAS for $d > 3$ it is not clear whether a separator can be described by a bounded number of real numbers as Claim 6.11 is no longer true. A natural generalisation of techniques used in the proof of Lemma 6.10 would result in expressing the existence of a separator in monadic second-order logic $MSO(+,\cdot,<)$ over the reals. However, the validation problem for $MSO(+,\cdot,<)$ is undecidable as it is easy to express natural numbers in $MSO(+,\cdot)$ as follows: $\mathbb{N}$ is the smallest set of numbers containing 1 and closed under adding 1. Thus decidability of $MSO(+,\cdot,<)$ over $\mathbb{R}$ would imply decidability of $MSO(+,\cdot,<)$ over $\mathbb{N}$ and in particular decidability of $FO(+,\cdot,<)$ over $\mathbb{N}$, which is well known to be undecidable [24]. Extending the techniques used in the proof of Lemma 6.10 would probably require showing that we can describe separators in higher dimensional spaces using a bounded number of real number.

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REFERENCES

[1] Shaull Almagor, Michaël Cadilhac, Filip Mazowiecki, and Guillermo A. Pérez. 2020. Weak Cost Register Automata are Still Powerful. *Int. J. Found. Comput. Sci.* 31, 6 (2020), 689–709. https://doi.org/10.1142/S0129054120410026
[2] Rajeev Alur, Loris D'Antoni, Jyotirmoy V. Deshmukh, Mukund Raghothaman, and Yifei Yuan. 2013. Regular Functions and Cost Register Automata. In *28th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2013, New Orleans, LA, USA, June 25-28, 2013.* 13–22. https://doi.org/10.1109/LICS.2013.65
[3] Corentin Barloy, Nathanaël Fijalkow, Nathan Lhote, and Filip Mazowiecki. 2020. A Robust Class of Linear Recurrence Sequences. In *28th EACSL Annual Conference on Computer Science Logic, CSL 2020, January 13-16, 2020, Barcelona, Spain (LIPIcs, Vol. 152)*, Maribel Fernández and Anca Muscholl (Eds.). Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 9:1–9:16. https://doi.org/10.4230/LIPIcs.CSL.2020.9
[4] Amos Beimel, Francesco Bergadano, Nader H. Bshouty, Eyal Kushilevitz, and Stefano Varricchio. 2000. Learning functions represented as multiplicity automata. *J. ACM* 47, 3 (2000), 506–530. https://doi.org/10.1145/337244.337257
[5] Jean Berstel and Christophe Reutenauer. 1988. *Rational series and their languages.* EATCS monographs on theoretical computer science, Vol. 12. Springer. https://www.worldcat.org/oclc/17841475
[6] Vincent D. Blondel and John N. Tsitsiklis. 2000. The boundedness of all products of a pair of matrices is undecidable. *Systems & Control Letters* 41, 2 (2000), 135–140. https://doi.org/10.1016/S0167-6911(00)00049-9
[7] Michael Blondin, Alain Finkel, Christoph Haase, and Serge Haddad. 2017. The Logical View on Continuous Petri Nets. *ACM Trans. Comput. Log.* 18, 3 (2017), 24:1–24:28. https://doi.org/10.1145/3105908
[8] Mikolaj Bojanczyk. 2015. Star Height via Games. In *30th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2015, Kyoto, Japan, July 6-10, 2015.* IEEE Computer Society, 214–219. https://doi.org/10.1109/LICS.2015.29

[9] Georgina Bumpus, Christoph Haase, Stefan Kiefer, Paul-Ioan Stoienescu, and Jonathan Tanner. 2020. On the Size of Finite Rational Matrix Semigroups. In *47th International Colloquium on Automata, Languages, and Programming, ICALP 2020, July 8-11, 2020, Saarbrücken, Germany (Virtual Conference) (LIPIcs, Vol. 168)*, Artur Czumaj, Anuj Dawar, and Emanuela Merelli (Eds.). Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 133:1–133:19. https://doi.org/10.4230/LIPIcs.ICALP.2020.133

[10] Dmitry Chistikov, Stefan Kiefer, Andrzej S. Murawski, and David Purser. 2020. The Big-O Problem for Labelled Markov Chains and Weighted Automata. In *31st International Conference on Concurrency Theory, CONCUR 2020 (LIPIcs, Vol. 171)*. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 41:1–41:19. https://doi.org/10.4230/LIPIcs.CONCUR.2020.41

[11] Dmitry Chistikov, Stefan Kiefer, Andrzej S. Murawski, and David Purser. 2022. The Big-O Problem. *Log. Methods Comput. Sci.* 18, 1 (2022). https://doi.org/10.46298/lmcs-18(1:40)2022 Extended journal version of [10].

[12] Anne Condon and Richard J. Lipton. 1989. On the Complexity of Space Bounded Interactive Proofs (Extended Abstract). In *30th Annual Symposium on Foundations of Computer Science, Research Triangle Park, North Carolina, USA, 30 October - 1 November 1989*. IEEE Computer Society, 462–467. https://doi.org/10.1109/SFCS.1989.63519

[13] Wojciech Czerwinski, Slawomir Lasota, Ranko Lazic, Jérôme Leroux, and Filip Mazowiecki. 2021. The Reachability Problem for Petri Nets Is Not Elementary. *J. ACM* 68, 1 (2021), 7:1–7:28. https://doi.org/10.1145/3422822

[14] Wojciech Czerwiński, Engel Lefaucheux, Filip Mazowiecki, David Purser, and Markus A. Whiteland. 2022. The boundedness and zero isolation problems for weighted automata over nonnegative rationals. *CoRR* abs/2205.13516 (2022). arXiv:2205.13516 https://arxiv.org/abs/2205.13516 Full version of this paper.

[15] Laure Daviaud, Ismaël Jecker, Pierre-Alain Reynier, and Didier Villevalois. 2017. Degree of Sequentiality of Weighted Automata. In *Foundations of Software Science and Computation Structures - 20th International Conference, FOSSACS 2017, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2017, Uppsala, Sweden, April 22-29, 2017, Proceedings (Lecture Notes in Computer Science, Vol. 10203)*, Javier Esparza and Andrzej S. Murawski (Eds.). 215–230. https://doi.org/10.1007/978-3-662-54458-7_13

[16] Laure Daviaud, Marcin Jurdzinski, Ranko Lazic, Filip Mazowiecki, Guillermo A. Pérez, and James Worrell. 2021. When are emptiness and containment decidable for probabilistic automata? *J. Comput. Syst. Sci.* 119 (2021), 78–96. https://doi.org/10.1016/j.jcss.2021.01.006

[17] Manfred Droste, Werner Kuich, and Heiko Vogler. 2009. *Handbook of weighted automata*. Springer Science & Business Media.

[18] Nathanaël Fijalkow, Hugo Gimbert, Edon Kelmendi, and Youssouf Oualhadj. 2015. Deciding the value 1 problem for probabilistic leaktight automata. *Logical Methods in Computer Science* 11, 2 (2015). https://doi.org/10.2168/LMCS-11(2:12)2015

[19] Nathanaël Fijalkow, Cristian Riveros, and James Worrell. 2017. Probabilistic Automata of Bounded Ambiguity. In *28th International Conference on Concurrency Theory, CONCUR 2017, September 5-8, 2017, Berlin, Germany (LIPIcs, Vol. 85)*, Roland Meyer and Uwe Nestmann (Eds.). Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 19:1–19:14. https://doi.org/10.4230/LIPIcs.CONCUR.2017.19

[20] Emmanuel Filiot, Nicolas Mazzocchi, and Jean-François Raskin. 2019. Decidable weighted expressions with Presburger combinators. *J. Comput. Syst. Sci.* 106 (2019), 1–22. https://doi.org/10.1016/j.jcss.2019.05.005

[21] Michel Fliess. 1974. Matrices de hankel. *J. Math. Pures Appl* 53, 9 (1974), 197–222.

[22] Hugo Gimbert and Youssouf Oualhadj. 2010. Probabilistic Automata on Finite Words: Decidable and Undecidable Problems. In *Automata, Languages and Programming, 37th International Colloquium, ICALP 2010, Bordeaux, France, July 6-10, 2010, Proceedings, Part II (Lecture Notes in Computer Science, Vol. 6199)*, Samson Abramsky, Cyril Gavoille, Claude Kirchner, Friedhelm Meyer auf der Heide, and Paul G. Spirakis (Eds.). Springer, 527–538. https://doi.org/10.1007/978-3-642-14162-1_44

[23] Dima Grigoriev. 1988. Complexity of Deciding Tarski Algebra. *J. Symb. Comput.* 5, 1/2 (1988), 65–108.

[24] Kurt Gödel. 1931. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme. *Monatshefte für Mathematik und Physik* 38, 1 (1931), 173–198.

[25] Christoph Haase and Simon Halfon. 2014. Integer Vector Addition Systems with States. In *Reachability Problems - 8th International Workshop, RP 2014, Oxford, UK, September 22-24, 2014. Proceedings*. 112–124. https://doi.org/10.1007/978-3-319-11439-2_9

[26] Kosaburo Hashiguchi. 1988. Algorithms for Determining Relative Star Height and Star Height. *Inf. Comput.* 78, 2 (1988), 124–169. https://doi.org/10.1016/0890-5401(88)90033-8

[27] Angus Macintyre and Alex J. Wilkie. 1996. On the decidability of the real exponential field. *Kreiseliana. About and Around Georg Kreisel* (1996), 441–467.

[28] Arnaldo Mandel and Imre Simon. 1977. On Finite Semigroups of Matrices. *Theor. Comput. Sci.* 5, 2 (1977), 101–111. https://doi.org/10.1016/0304-3975(77)90001-9

[29] Filip Mazowiecki and Cristian Riveros. 2015. Maximal Partition Logic: Towards a Logical Characterization of Copyless Cost Register Automata. In *24th EACSL Annual Conference on Computer Science Logic, CSL 2015, September 7-10, 2015, Berlin, Germany (LIPIcs, Vol. 41)*, Stephan Kreutzer (Ed.). Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 144–159. https://doi.org/10.4230/LIPIcs.CSL.2015.144

[30] Filip Mazowiecki and Cristian Riveros. 2018. Pumping Lemmas for Weighted Automata. In *35th Symposium on Theoretical Aspects of Computer Science, STACS 2018, February 28 to March 3, 2018, Caen, France (LIPIcs, Vol. 96)*, Rolf Niedermeier and Brigitte Vallée (Eds.). Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 50:1–50:14. https://doi.org/10.4230/LIPIcs.STACS.2018.50

[31] Filip Mazowiecki and Cristian Riveros. 2019. Copyless cost-register automata: Structure, expressiveness, and closure properties. *J. Comput. Syst. Sci.* 100 (2019), 1–29. https://doi.org/10.1016/j.jcss.2018.07.002

[32] Azaria Paz. 1971. *Introduction to probabilistic automata*. Academic Press.

[33] Wolfgang M. Schmidt. 1980. *Simultaneous Approximation. In: Diophantine Approximation*. Springer Berlin Heidelberg, Berlin, Heidelberg, 27–47. https://doi.org/10.1007/978-3-540-38645-2_2

[34] Marcel Paul Schützenberger. 1961. On the Definition of a Family of Automata. *Information and Control* 4, 2-3 (1961), 245–270. https://doi.org/10.1016/S0019-9958(61)80020-X

[35] Imre Simon. 1990. Factorization Forests of Finite Height. *Theor. Comput. Sci.* 72, 1 (1990), 65–94. https://doi.org/10.1016/0304-3975(90)90047-L

[36] Imre Simon. 1994. On Semigroups of Matrices over the Tropical Semiring. *RAIRO Theor. Informatics Appl.* 28, 3-4 (1994), 277–294. https://doi.org/10.1051/ita/1994283-402771

[37] Paavo Turakainen. 1969. Generalized automata and stochastic languages. *Proc. Amer. Math. Soc.* 21, 2 (1969), 303–309.

[38] Michel Waldschmidt. 2000. *Diophantine Approximation on Linear Algebraic Groups*. Grundlehren der mathematischen Wissenschaften (A Series of Comprehensive Studies in Mathematics), Vol. 326. Springer, Berlin, Heidelberg.

[39] Andreas Weber and Helmut Seidl. 1991. On the Degree of Ambiguity of Finite Automata. *Theor. Comput. Sci.* 88, 2 (1991), 325–349. https://doi.org/10.1016/0304-3975(91)90381-B