

PROFILE AND HEREDITARY CLASSES OF ORDERED RELATIONAL STRUCTURES

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Specker sequence

ABSTRACT. Let \mathfrak{C} be a class of finite combinatorial structures. The *profile* of \mathfrak{C} is the function $\varphi_{\mathfrak{C}}$ which counts, for every integer n , the number $\varphi_{\mathfrak{C}}(n)$ of members of \mathfrak{C} defined on n elements, isomorphic structures being identified. The *generating function* of \mathfrak{C} is $\mathcal{H}_{\mathfrak{C}}(x) := \sum_{n \geq 0} \varphi_{\mathfrak{C}}(n)x^n$. Many results about the behavior of the function $\varphi_{\mathfrak{C}}$ have been obtained. Albert and Atkinson have shown that the generating series of several classes of permutations are algebraic. In this paper, we show how their results extend to classes of ordered binary relational structures; putting emphasis on the notion of hereditary well quasi order, we discuss some of their questions and answer one.

AMS Subject Classification: 05C30, 06F99, 05A05, 03C13.

Keywords: profile, well quasi-ordering, indecomposability, permutations.

1. INTRODUCTION

The context of this paper is the enumeration of finite relational structures. A relational structure \mathcal{R} is *embeddable* in a relational structure \mathcal{R}' , in notation $\mathcal{R} \leq \mathcal{R}'$, if \mathcal{R} is isomorphic to an induced substructure of \mathcal{R}' . The embeddability relation is a quasi order. Several significant properties of relational structures or classes of relational structures can be uniquely expressed in term of this quasi order. This is typically the case of hereditary classes: a class \mathfrak{C} of structures is *hereditary* if it contains every relational structure which can be embedded in some member of \mathfrak{C} . Interesting hereditary classes abound. In the late forties, Fraïssé, following the work of Cantor, Hausdorff and Sierpinski, pointed out the role of the quasi-ordering of embeddability and hereditary classes in the theory of relations (see his book [13] for an illustration). Recent years have seen a renewed interest for the study of these classes, particularly those made of finite structures. Many results have been obtained. Some are about obstructions allowing to define these classes, others on the behavior of the function $\varphi_{\mathfrak{C}}$, the *profile* of \mathfrak{C} which counts, for every integer n , the number $\varphi_{\mathfrak{C}}(n)$ of members of \mathfrak{C} defined on n elements, isomorphic structures being identified. General counting results have been obtained, as well as precise results, for graphs, tournaments and ordered graphs (see the survey [20]). Enumeration results on permutations, motivated by the *Stanley-Wilf conjecture*, solved by Marcus

Date: April 23, 2022.

2000 Mathematics Subject Classification. 05C30, 06F99, 05A05, 03C13.

Key words and phrases. ordered set, well quasi-ordering, relational structures, profile, indecomposability, graphs, tournaments, permutations.

*The author was supported by CMEP-Tassili grant.

and Tardös (2004), fall also under this frame, an important fact due to Cameron [10]. Indeed, to each permutation σ of $[n] := \{1, \dots, n\}$ we may associate the relational structure $C_\sigma := ([n], \leq, \leq_\sigma)$, that we call **bichain**, made of two linear orders on $[n]$ (\leq being the natural order on $[n]$ and \leq_σ the linear order defined by $i \leq_\sigma j$ if and only if $\sigma(i) \leq \sigma(j)$). As it turns out, **the order defined on permutations and the embeddability between bichains coincide** (see Subsection 3.2 for details and examples).

In this paper, we show how some results obtained by Albert and Atkinson [1] for classes of permutations extend to classes of ordered binary relational structures. We prove notably Theorem 5.7. For this purpose, we recall in Section 2 some basic definitions of the theory of relations, we survey in Section 3 some results concerning classes of permutations and show how permutations are related to relational structures. Then, we illustrate the role of indecomposable structures (see Section 4) and of well quasi order (see Section 5) in enumeration results. Finally, in Section 6, we present a conjecture and a partial solution, a special case answering a question of Albert and Atkinson [1].

Our results have been presented at the international conference on Discrete Mathematics and Computer Science (Dimacos'11) held in Mohammedia, Morocco, May-5-8, 2011, and at the International Symposium on Operational Research (Isor'11), held in Algiers, Algeria, May 30-June 2, 2011 [25]. We are pleased to thanks the organizers of these meetings for their help.

2. BASIC NOTIONS, EMBEDDABILITY, HEREDITARY CLASSES AND PROFILE

Our terminology agree with [13]. Let n be a positive integer. A n -ary relation with *domain* E is a subset ρ of the n -th power E^n of E ; for $n = 1$ and $n = 2$ we use the words *unary relation* and *binary relation*, in this later case we rather set $x\rho y$ instead of $(x, y) \in \rho$. A *relational structure* with *domain* E is a pair $\mathcal{R} := (E, (\rho_i)_{i \in I})$ made of a set E and a family $(\rho_i)_{i \in I}$ of n_i -ary relations ρ_i on E , each ρ_i being a subset of E^{n_i} . The family $\mu := (n_i)_{i \in I}$ is the *signature* of \mathcal{R} . We will denote by $V(\mathcal{R})$ the domain of \mathcal{R} . We denote by $\bar{\Omega}_\mu$ the class of these structures and by Ω_μ the subclass of the finite ones. A relational structure \mathcal{R} is *ordered* if it can be expressed as $\mathcal{R} := (E, \leq, (\rho_j)_{j \in J})$ where " \leq " is a linear order on E and the ρ_j 's are n_j -ary relations; the (truncated) signature in this case is $\mu = (n_j)_{j \in J}$. A relational structure $\mathcal{R} := (E, (\rho_i)_{i \in I})$ is a *binary relational structure*, *binary structure* for short, if each ρ_i is a binary relation; the class of those finite binary structures will be denoted by Ω_I instead of Ω_μ . Basic examples of ordered binary structures are chains ($J = \emptyset$), bichains ($J = \{1\}$ and ρ_1 is a linear order) and multichains (ρ_j is a linear order for all $j \in J$). We denote by Θ_d the collection of finite ordered binary structures made of a linear order and d binary relations. Let $\mathcal{R} := (E, (\rho_i)_{i \in I})$ be a relational structure; the *substructure induced by \mathcal{R} on a subset A of E* , simply called the *restriction of \mathcal{R} to A* , is the relational structure $\mathcal{R} \upharpoonright_A := (A, (\rho_i \upharpoonright_A)_{i \in I})$, where $\rho_i \upharpoonright_A := \rho_i \cap A^{n_i}$. Let $\mathcal{R} := (E, (\rho_i)_{i \in I})$ and $\mathcal{R}' := (E', (\rho'_i)_{i \in I})$ be two relational structures of the same signature $\mu := (n_i)_{i \in I}$. A map $f : E \rightarrow E'$ is an *isomorphism from \mathcal{R} onto \mathcal{R}'* if f is bijective and $(x_1, \dots, x_{n_i}) \in \rho_i$ if and only

if $(f(x_1), \dots, f(x_{n_i})) \in \rho'_i$ for every $(x_1, \dots, x_{n_i}) \in E^{n_i}$, $i \in I$. The relational structure \mathcal{R} is *isomorphic* to \mathcal{R}' if there is some isomorphism from \mathcal{R} onto \mathcal{R}' , it is *embeddable* into \mathcal{R}' , and we set $\mathcal{R} \leq \mathcal{R}'$, if \mathcal{R} is isomorphic to some restriction of \mathcal{R}' . The embeddability relation (called “abritement” by *Fraïssé* in french) is a quasi-order. A class \mathfrak{C} of relational structures is *hereditary* if $\mathcal{R} \in \mathfrak{C}$ and $\mathcal{S} \leq \mathcal{R}$ imply $\mathcal{S} \in \mathfrak{C}$; relational structures which are not in \mathfrak{C} are *obstructions* to \mathfrak{C} . The *age* of a relational structure \mathcal{R} is the class $\text{Age}(\mathcal{R})$ of finite \mathcal{S} which are embeddable into \mathcal{R} (equivalently, this is the set of finite restrictions of \mathcal{R} augmented of their isomorphic copies). An age is non-empty, hereditary and up-directed (that is for every $\mathcal{S}, \mathcal{S}' \in \text{Age}(\mathcal{R})$ there is some $\mathcal{T} \in \text{Age}(\mathcal{R})$ which embeds \mathcal{S} and \mathcal{S}'). In the terminology of posets, this is an *ideal* of Ω_μ . If the signature is finite, every ideal of Ω_μ is the age of some relational structure (*Fraïssé* 1954). If \mathfrak{B} is a subset of Ω_μ then $\text{Forb}(\mathfrak{B})$ denotes the subclass of members of Ω_μ which embed no member of \mathfrak{B} . Clearly, $\text{Forb}(\mathfrak{B})$ is an hereditary class. Moreover, every hereditary subclass \mathfrak{C} of Ω_μ has this form. This fact, due to *Fraïssé*, is based on the notion of bound: a *bound* of an hereditary subclass \mathfrak{C} of Ω_μ is every finite \mathcal{R} not in \mathfrak{C} such that every \mathcal{R}' which strictly embeds into \mathcal{R} belongs to \mathfrak{C} . Clearly, every finite obstruction to \mathfrak{C} contains a bound. Hence, if $\mathfrak{B}(\mathfrak{C})$ denotes the collection of bounds of \mathfrak{C} considered up to isomorphism then $\mathfrak{C} = \text{Forb}(\mathfrak{B}(\mathfrak{C}))$.

The *profile* of an hereditary class \mathfrak{C} is the function $\varphi_{\mathfrak{C}} : \mathbb{N} \longrightarrow \mathbb{N}$ which counts, for every n , the number of members of \mathfrak{C} defined on n elements, isomorphic structures been identified. The *generating function for \mathfrak{C}* is $\mathcal{H}_{\mathfrak{C}}(x) := \sum_{n \geq 0} \varphi_{\mathfrak{C}}(n) x^n$. These two notions are the specialization to hereditary classes of basic notions in enumeration. Many results on the enumeration of classes of permutations are about the enumeration of relational structures. Indeed, as mentioned in the introduction, permutations can be considered as special cases of binary structures, and more specifically ordered binary structures, in fact bichains. We introduce these notions below and point out the relationship between permutations and bichains in the next section.

3. PERMUTATIONS, BICHAINS AND THEIR PROFILE

3.1. Permutations. Let n be a non negative integer. Let \mathfrak{S}_n be the set of permutations on $[n] := \{1, \dots, n\}$ and $\mathfrak{S} := \bigcup_{n \in \mathbb{N}} \mathfrak{S}_n$. An order relation on \mathfrak{S} is defined as follows: the permutation π of $[n]$ *contains* the permutation σ of $[k]$ and we write $\sigma \leq \pi$ if some subsequence of π of length k is order isomorphic to σ . More precisely, $\sigma \leq \pi$ if there exist integers $1 \leq x_1 < \dots < x_k \leq n$ such that for $1 \leq i, j \leq k$,

$$\sigma(i) < \sigma(j) \text{ if and only if } \pi(x_i) < \pi(x_j).$$

For example, $\pi := 391867452$ contains $\sigma := 51342$, as it can be seen by considering the subsequence $91672 (= \pi(2), \pi(3), \pi(5), \pi(6), \pi(9))$.

A subset \mathfrak{C} of \mathfrak{S} is *hereditary* if $\sigma < \pi \in \mathfrak{C}$ implies $\sigma \in \mathfrak{C}$. Its counting function, that we call the *profile* of \mathfrak{C} , is $\varphi_{\mathfrak{C}}(n) := |\mathfrak{C} \cap \mathfrak{S}_n|$. How much does $\varphi_{\mathfrak{C}}(n)$ drop

from $\varphi_{\mathfrak{S}}(n) = n!$ if $\mathfrak{C} \neq \mathfrak{S}$? The *Stanley-Wilf* conjecture asserted that it drops to exponential growth. The conjecture was proved in 2004 by *Marcus and Tardős* [23]:

Theorem 3.1. *If \mathfrak{C} is a proper hereditary set of permutations, then, for some constant c , $\varphi_{\mathfrak{C}}(n) < c^n$ for every n .*

Kaiser and Klazar [19] proved that if \mathfrak{C} is hereditary, then, either $\varphi_{\mathfrak{C}}$ is bounded by a polynomial and in this case is a polynomial, or is bounded below by an exponential, in fact the *generalized Fibonacci function* $F_{n,k}$.

We recall that the generalized Fibonacci number is given by the recurrence $F_{n,k} = 0$ for $n < 0$, $F_{0,k} = 1$ and $F_{n,k} = F_{n-1,k} + F_{n-2,k} + \dots + F_{n-k,k}$ for $n > 0$. $F_{n,k}$ is the coefficient of x^n in the power series expansion of the expression

$\frac{1}{1 - x - x^2 - \dots - x^k}$. The Kaiser and Klazar theorem reads as follow:

Theorem 3.2. *If \mathfrak{C} is an hereditary set of permutations, then exactly one of the four cases occurs.*

- (1) *For large n , $\varphi_{\mathfrak{C}}(n)$ is eventually constant.*
- (2) *There are integers a_0, \dots, a_k , $k \geq 1$ and $a_k > 0$, such that $\varphi_{\mathfrak{C}}(n) = a_0 \binom{n}{0} + \dots + a_k \binom{n}{k}$ for large n . Moreover, $\varphi_{\mathfrak{C}}(n) \geq n$ for every n .*
- (3) *There are constants c, k in \mathbb{N} , $k \geq 2$, such that $F_{n,k} \leq \varphi_{\mathfrak{C}}(n) \leq n^c F_{n,k}$ for every n .*
- (4) *One has $\varphi_{\mathfrak{C}}(n) \geq 2^{n-1}$ for every n .*

In the cases (1) to (3) the generating function is rational. *Albert and Atkinson* gave in 2005 examples of hereditary classes whose generating function is algebraic [1]. In order to state their result, we recall first that a power series $F(x) := \sum_{n \geq 0} a_n x^n$

with a_n in \mathbb{C} is *algebraic* if there exists a nonzero polynomial $Q(x, y)$ in $\mathbb{C}[x, y]$ such that $Q(x, F(x)) = 0$. The series F is *rational* if Q has degree 1 in y , that is, $F(x) = R(x)/S(x)$ for two polynomials in $\mathbb{C}[x]$ where $S \neq 0$. Recall next that a permutation $\pi = a_1 a_2 \dots a_n$ of $[n]$ is *simple* if no proper interval of $[n]$ ($\neq [n], \emptyset$ or $\{x\}$) is transformed into an interval. In other words, $\{a_i, a_{i+1}, \dots, a_j\}$ is not an interval in $[n]$ for every $1 \leq i < j \leq n$, and either $i \neq 1$ or $j \neq n$. If $n \leq 2$ all permutations are simple and called trivial. Albert and Atkinson's theorem is the following:

Theorem 3.3. *If \mathfrak{C} is an hereditary class of permutations containing only finitely many simple permutations, then the generating series of \mathfrak{C} , namely $\sum_{n \geq 0} \varphi_{\mathfrak{C}}(n) x^n$ is algebraic.*

As an illustration of this result, let us mention that the class of permutations not above 2413 and 3142 contain no non trivial simple permutation (these permutations are called *separable permutations*). The generating series of this class is $\frac{1 - x - \sqrt{1 - 6x + x^2}}{2}$ (see [3]).

The simple permutations of small degree are 1, 12, 21, 2413, 3142. Let S_n be the number of simple permutations of $[n]$. The values of S_n for $n = 1$ to 7 are:

1, 2, 0, 2, 6, 46, 338 [32]. Asymptotically, S_n goes to $\frac{n!}{e^2}$, a result obtained independently in [24, 2].

3.2. Permutations and bichains. Let σ be a permutation of $[n]$. To σ we associate the *bichain* $C_\sigma := ([n], \leq, \leq_\sigma)$ where \leq is the natural order on $[n]$ and \leq_σ the linear order defined by $i \leq_\sigma j$ if and only if $\sigma(i) \leq \sigma(j)$.

For example, let σ be the permutation of $\underline{10}$ given by the sequence of its values: 2468(10)13579. The sequence of elements of $\underline{10}$ ordered according to \leq_σ is: $6 <_\sigma 1 <_\sigma 7 <_\sigma 2 <_\sigma 8 <_\sigma 3 <_\sigma 9 <_\sigma 4 <_\sigma 10 <_\sigma 5$. Hence, this is the sequence of values of σ^{-1} , the inverse of σ . Let us represent σ by its graph in the product $\underline{n} \times \underline{n}$, that is the set $G(\sigma) := \{(i, \sigma(i)) : i \in \underline{n}\}$ and order this set componentwise, that is set $(i, \sigma(i)) \leq (j, \sigma(j))$ if $i \leq j$ and $\sigma(i) \leq \sigma(j)$. Since σ is bijective, the poset $G(f)$ is the intersection of two linear orders, given respectively by the natural order on the first and on the second coordinate. If we identify each i to $(i, \sigma(i))$, the order induced on \underline{n} is the intersection of \leq and \leq_σ . See Figure 1.

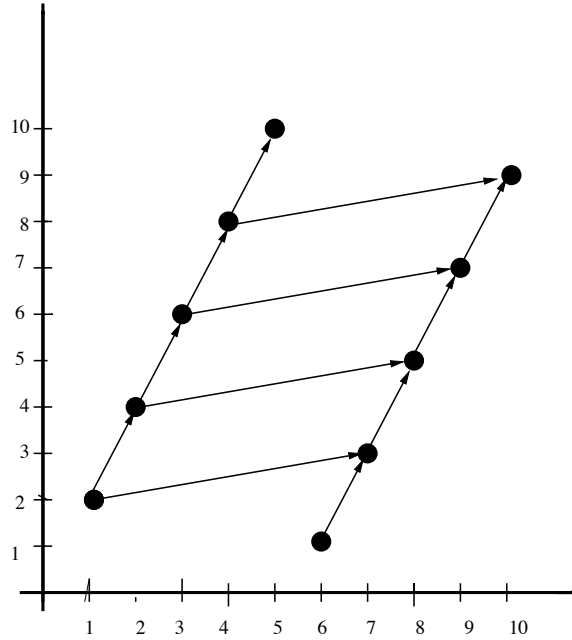


FIGURE 1. Representation of a permutation of ten elements.

Lemma 3.4. (1) If $B := (E, L_1, L_2)$ is a finite bichain then B is isomorphic to a bichain C_σ for a unique permutation σ on $[|E|]$.

(2) If σ and π are two permutations then $\sigma \leq \pi$ if and only if $C_\sigma \leq C_\pi$.

The correspondence between permutations and bichains was noted by Cameron [10] (who rather associated to σ the pair $(\leq, \leq_{\sigma^{-1}})$). It allows to study classes of permutations by means of the theory of relations. In particular, via this correspondence, hereditary classes of permutations correspond to hereditary classes of

bichains and, as we will see below, simple permutations correspond to indecomposable bichains.

4. INDECOMPOSABILITY AND LEXICOGRAPHIC SUM

Let $\mathcal{R} := (E, (\rho_i)_{i \in I})$ be a binary structure. A subset A of E is an *interval* of \mathcal{R} if for each $i \in I$:

$$(x\rho_i a \Leftrightarrow x\rho_i a') \text{ and } (a\rho_i x \Leftrightarrow a'\rho_i x) \text{ for all } a, a' \in A \text{ and } x \notin A.$$

The empty set, the singletons and the whole set E are intervals and said *trivial*. If \mathcal{R} has no non trivial interval it is *indecomposable*. For example, if $\mathcal{R} := (E, \leq)$ is a chain, its intervals are the ordinary intervals. If $\mathcal{R} := (E, \leq, \leq')$ is a bichain then A is an interval of \mathcal{R} if and only if A is an interval of (E, \leq) and (E, \leq') . Hence:

Fact 1. *A permutation σ is simple if and only if the bichain C_σ is indecomposable.*

The notion of indecomposability is rather old. The notion of interval goes back to Fraïssé [14], see also [15]. A fundamental decomposition result of a binary structures into intervals was obtained by Gallai [16] (see [12] for further extensions). Hence, it is not surprising that several results on simple permutations were already known (for example their asymptotic evaluation). Albert and Atkinson result recasted in terms of relational structures asserts that if \mathfrak{C} is an hereditary class of finite bichains containing only finitely many indecomposable bichains then the generating series of \mathfrak{C} is algebraic. The paper [21] contains several examples of infinite bichains \mathcal{B} whose infinitely many members of $\text{Age}(\mathcal{B})$ are indecomposable. We will establish an extension to ordered binary structures in Theorem 5.7.

In the sequel, we recall the facts we need on lexicographic sums and the links with the indecomposability notion. Some are old (the notion of lexicographic sum goes back to Cantor).

Let $\mathcal{R} := (E, (\rho_i)_{i \in I})$ be a binary structure and $\mathfrak{F} := (\mathcal{S}_x)_{x \in E}$ be a family of binary structures $\mathcal{S}_x := (E_x, (\rho_i^x)_{i \in I})$, indexed by the elements of E . We suppose that E and the E_x are non-empty. The *lexicographic sum* of \mathfrak{F} over \mathcal{R} , denoted by $\bigoplus_{x \in \mathcal{R}} \mathcal{S}_x$, is the binary structure \mathcal{T} obtained by replacing each element $x \in E$ by the structure \mathcal{S}_x . More precisely, $\mathcal{T} = (Z, (\tau_i)_{i \in I})$ where $Z := \{(x, y) : x \in E, y \in E_x\}$ and for each $i \in I$, $(x, y)\tau_i(x', y')$ if either $x \neq x'$ and $x\rho_i x'$ or $x = x'$ and $y\rho_i^x y'$.

Trivially, if we replace each \mathcal{S}_x by an isomorphic binary structure \mathcal{S}'_x , then $\bigoplus_{x \in \mathcal{R}} \mathcal{S}'_x$ is isomorphic to $\bigoplus_{x \in \mathcal{R}} \mathcal{S}_x$. Hence, we may suppose that the domains of the \mathcal{S}_x 's are pairwise disjoint. In this case we may slightly modify the definition above, setting $Z := \bigcup_{x \in E} E_x$ and for two elements $z \in E_x$ and $z' \in E_{x'}$,

$$z\tau_i z' \text{ if either } x \neq x' \text{ and } x\rho_i x' \text{ or } x = x' \text{ and } z\rho_i^x z'.$$

With this definition, each set E_x is an interval of the sum \mathcal{T} . Furthermore, let Z/\equiv be the quotient of Z made of blocks of this partition into intervals, let $p : Z \rightarrow Z/\equiv$ be the natural projection and let \mathcal{R}' be the image of \mathcal{T} (that is $\mathcal{R}' = (Z/\equiv, (\rho'_i)_{i \in I})$

where $\rho'_i = \{(p(x_1), p(x_2)) : (x_1, x_2) \in \rho_i\}$. If we identify each block E_x to the element x , then \mathcal{R} and \mathcal{R}' coincide on pairs of distinct elements. They coincide if we consider only reflexive relations. Conversely, if $\mathcal{T} := (Z, (\tau_i)_{i \in I})$ is a binary structure and $(E_x)_{x \in E}$ is a partition of Z into non empty intervals of \mathcal{T} , then \mathcal{T} is the lexicographic sum of $(\mathcal{T} \upharpoonright_{E_x})_{x \in E}$ over the quotient Z/\equiv . In simpler words:

Fact 2. *The decompositions of a binary structure into lexicographic sums are in correspondence with the partitions of its domain into intervals.*

An important property of these decompositions is the following:

Fact 3. *The set of partitions of E into intervals of \mathcal{R} , once ordered by refinement, is a sublattice of the set of partitions of E .*

Let us illustrate. Let us say that a lexicographic sum $\bigoplus_{x \in \mathcal{R}} \mathcal{S}_x$ is *trivial* if $|E| = 1$ or $|E_x| = 1$ for all $x \in E$, otherwise it is *non trivial*; also a binary structure is *sum-indecomposable* if it can not be isomorphic to a non trivial lexicographic sum. We have immediately:

Fact 4. *A binary structure is sum-indecomposable if and only if it is indecomposable.*

Proposition 4.1. *Let \mathcal{R} be a finite binary structure with at least two elements. Then \mathcal{R} is isomorphic to a lexicographic sum $\bigoplus_{x \in \mathcal{S}} \mathcal{R}_x$ where \mathcal{S} is indecomposable with at least two elements. Moreover, when \mathcal{S} has at least three elements, the partition of \mathcal{R} into intervals is unique.*

If the set \mathcal{S} in Proposition 4.1 has two elements, then the decomposition is not necessary unique, a fact which leads to the notion of strong interval. We recall that an interval A of a binary structure \mathcal{R} is *strong* if it is non-empty and overlaps no other interval, meaning that if B is an interval such that $A \cap B \neq \emptyset$ then either $B \subseteq A$ or $A \subseteq B$. We say that A is *maximal* if it is maximal for inclusion among strong intervals distinct from the domain E of \mathcal{R} . The maximal strong interval form a partition of E , provided that some maximal exists; in this case E is *non-limit* [18] or, equivalently, *robust* [11]. Evidently, this partition exists whenever E is finite. The reader will easily check that when this partition exists and the quotient is indecomposable then every other non-trivial partition into intervals is finer. Hence, in Proposition 4.1 above, if \mathcal{S} has at least three elements, the intervals in the decomposition are strong and thus the decomposition is unique.

Let us say that $\mathcal{R} := (E, (\rho_i)_{i \in I})$ is *chainable* if there is a linear order, \leq , on E such that, for each i , $x \rho_i y \Leftrightarrow x' \rho_i y'$ for every x, y, x', y' such that $x \leq y \Leftrightarrow x' \leq y'$. If \mathcal{R} is reflexive, this amounts to the fact that each ρ_i is either the equality relation \triangle_E , the complete relation $E \times E$ or a linear order; moreover if ρ_i and ρ_j are two linear orders, they coincide or are opposite. Note that if furthermore \mathcal{R} is ordered then the ρ_i ' which are linearly ordered are equal or opposite to the given order.

We arrive to the fundamental decomposition theorem of Gallai [16] (see for example [18], [12], [11] for extensions to infinite structures)

Theorem 4.2. *Let \mathcal{R} be a finite binary structure with at least two elements, then \mathcal{R} is a lexicographic sum $\bigoplus_{x \in \mathcal{S}} \mathcal{R}_x$ where \mathcal{S} is either indecomposable with at least three elements or a chainable binary structure with at least two elements and the $V(\mathcal{R}_x)$'s are strong maximal intervals of \mathcal{R} .*

For our purpose, we need to introduce the following notion.

Let τ be an ordered structure with two elements. An ordered structure \mathcal{S} is said τ -indecomposable if it cannot be decomposed into a lexicographic sum indexed by τ . If \mathfrak{A} is a class of structures, we denote by $\mathfrak{A}(\tau)$ the set of members of \mathfrak{A} which are τ -indecomposable.

Lemma 4.3. *Let $\mathcal{S} := (\{0, 1\}, \leq, (\rho_i)_{i \in J})$, with $0 < 1$, be an ordered structure with two elements. If \mathcal{R} is a lexicographic sum $\bigoplus_{x \in \mathcal{S}} \mathcal{R}_x$ and \mathcal{R}_0 is \mathcal{S} -indecomposable, then the partition $\mathcal{R}_0, \mathcal{R}_1$ is unique.*

We extend the notion of lexicographic sum to collections of non-empty binary structures. Given a non-empty binary structure \mathcal{R} and classes \mathfrak{A}_x of non-empty binary structures for each $x \in \mathcal{R}$, let us denote by $\bigoplus_{x \in \mathcal{R}} \mathfrak{A}_x$ the class of all binary structures of the form $\bigoplus_{x \in \mathcal{R}} \mathcal{S}_x$ with $\mathcal{S}_x \in \mathfrak{A}_x$. If $\mathfrak{A}_x = \mathfrak{A}$ for every $x \in V(\mathcal{R})$ we denote this class by $\bigoplus_{\mathcal{R}} \mathfrak{A}$. If $\mathcal{R} := (\{0, 1\}, \leq, (\rho_i)_{i \in J})$, with $0 < 1$, $\mathfrak{A}_0 := \mathfrak{A}$ and $\mathfrak{A}_1 := \mathfrak{B}$, we set $\bigoplus_{x \in \mathcal{R}} \mathfrak{A}_x = \mathfrak{A} \oplus_{\mathcal{R}} \mathfrak{B}$. Also if \mathfrak{A} and \mathfrak{B} are two classes of binary structures, we set $\bigoplus_{\mathfrak{A}} \mathfrak{B} := \{ \bigoplus_{x \in \mathcal{R}} \mathcal{S}_x : \mathcal{R} \in \mathfrak{A}, \mathcal{S}_x \in \mathfrak{B} \text{ for each } x \in V(\mathcal{R}) \}$.

We say that a collection \mathfrak{C} of binary structures is *sum-closed* if $\bigoplus_{\mathfrak{C}} \mathfrak{C} \subseteq \mathfrak{C}$. The *sum-closure* $cl(\mathfrak{C})$ of \mathfrak{C} is the smallest sum-closed set that contains \mathfrak{C} . If we define $\mathfrak{C}_0 := \mathfrak{C}$ and $\mathfrak{C}_{n+1} := \bigoplus_{\mathfrak{C}} \mathfrak{C}_n$, then $cl(\mathfrak{C}) = \bigcup_{n=1}^{\infty} \mathfrak{C}_n$. If \mathfrak{C} is a class of bichains, $cl(\mathfrak{C})$ is also a class of bichains and the class of corresponding permutations is said *wreath-closed* [1]. If \mathfrak{C} is made of reflexive structures and contains a one element structure, say 1, then $cl(\mathfrak{C}) = \bigoplus_{cl(\mathfrak{C})} cl(\mathfrak{C})$. If in addition \mathfrak{C} contains the empty structure then $cl(\mathfrak{C})$ is hereditary.

We denote by $Ind(\Omega_I)$ the collection of finite indecomposable members of Ω_I . If \mathcal{R} is a binary structure, we denote by $Ind(\mathcal{R})$ the collection of its finite induced substructures which are indecomposable. For example, if \mathcal{R} is a cograph or a serie-parallel poset then the members of $Ind(\mathcal{R})$ have at most two elements (a graph (undirected) is a *cograph* if no induced subgraph is isomorphic to P_4 , the path on 4 vertices, and a poset is *serie-parallel* if its comparability graph is a cograph).

Let \mathfrak{D} be a hereditary class of $Ind(\Omega_I)$. Set $\sum \mathfrak{D} := \{ \mathcal{R} \in \Omega_I : Ind(\mathcal{R}) \subseteq \mathfrak{D} \}$.

Theorem 4.4. *If all members of \mathfrak{D} are reflexive, then $\sum \mathfrak{D} = cl(\mathfrak{D})$.*

Proof. Inclusion $\sum \mathfrak{D} \supseteq cl(\mathfrak{D})$ holds under assumption that all members of \mathfrak{D} are reflexive. Conversely, if $\mathcal{R} \notin cl(\mathfrak{D})$ then either \mathcal{R} is indecomposable in which case $\mathcal{R} \notin \mathfrak{D}$ or \mathcal{R} can not be expressed as a lexicographic sum of structures of \mathfrak{D} hence $\mathcal{R} \notin \sum \mathfrak{D}$. \square

In the sequel, we consider only ordered structures made of reflexive binary relations. Let Γ_d be the subclass of reflexive members of Θ_d .

Let \mathfrak{A} be a subclass of Γ_d ; for $i \in \mathbb{N}$ let $\mathfrak{A}_{(i)}$, resp. $\mathfrak{A}_{(\geq i)}$, be the subclass made of its members which have i elements, resp. at least i elements.

Lemma 4.5. *Let \mathfrak{D} be a class made of non-empty indecomposable members of Γ_d such that $\mathfrak{D}_{(1)}$ is reduced to the one-element structure **1**. Let \mathfrak{A} be the sum-closure of \mathfrak{D} and for each $\mathcal{S} \in \mathfrak{D}_{(2)}$, let $\mathfrak{A}(\mathcal{S})$ be the subclass of \mathcal{S} -indecomposable members of \mathfrak{A} . Set $\mathfrak{A}_{\mathcal{S}} := \bigoplus_{\mathcal{S}} \mathfrak{A}$ if $\mathcal{S} \in \mathfrak{D}_{(\geq 3)}$ and otherwise set $\mathfrak{A}_{\mathcal{S}} := \mathfrak{A}(\mathcal{S}) \bigoplus_{\mathcal{S}} \mathfrak{A}$ if $\mathcal{S} \in \mathfrak{D}_{(2)}$ and $\mathcal{S} := (\{0, 1\}, \leq, (\rho_i)_{i \in J})$ with $0 < 1$. Then:*

$$(4.1) \quad \mathfrak{A} = \{1\} \cup \bigcup_{\mathcal{S} \in \mathfrak{D}_{(\geq 2)}} \mathfrak{A}_{\mathcal{S}}$$

and

$$(4.2) \quad \mathfrak{A}(\mathcal{S}) = \mathfrak{A} \setminus \mathfrak{A}_{\mathcal{S}}$$

for every $\mathcal{S} \in \mathfrak{D}_{(2)}$.

Furthermore, all sets in equation (4.1) are pairwise disjoint.

Proof. Let's denote by (1) (respectively by (2)) the left-hand side (respectively the right-hand side) of Equation 4.1. Inclusion (2) \subseteq (1) is obvious because \mathfrak{A} is sum-closed according to Theorem 4.4. To prove inclusion (1) \subseteq (2), let \mathcal{R} be in (1), if \mathcal{R} has one element then it is in (2), otherwise, according to Theorem 4.2, \mathcal{R} is a lexicographic sum $\bigoplus_{x \in \mathcal{S}} \mathcal{R}_x$ where \mathcal{S} is either indecomposable with at least three elements or a chainable binary structure with at least two elements and each $\mathcal{R}_x \in \mathfrak{A}$ is a strong interval of \mathcal{R} for each $x \in \mathcal{S}$. In the first case, \mathcal{R} is in $\bigcup_{\mathcal{S} \in \mathfrak{D}_{(\geq 3)}} \bigoplus_{x \in \mathcal{S}} \mathfrak{A}$,

hence in (2). In the second case, \mathcal{S} is chainable with n elements, $n \geq 2$, and we may set $\mathcal{S} := (\{0, 1, \dots, n-1\}, \leq, (\rho_i)_{i \in I})$ with $0 < 1 < \dots < n-1$. Set $\mathcal{S}' := \mathcal{S} \upharpoonright_{\{0,1\}}$, $\mathcal{S}'' := \mathcal{S} \upharpoonright_{\{1, \dots, n-1\}}$, $\mathcal{R}'_0 := \mathcal{R}_0$ and $\mathcal{R}'_1 := \bigoplus_{x \in \mathcal{S}''} \mathcal{R}_x$. We have obviously $\mathcal{R} = \mathcal{R}'_0 \bigoplus_{\mathcal{S}'} \mathcal{R}'_1$. Since \mathcal{R}_0 is a strong interval of \mathcal{R} , \mathcal{R}'_0 is \mathcal{S}' -indecomposable, hence \mathcal{R} belongs to $\bigcup_{\mathcal{S}' \in \mathfrak{D}_{(2)}} (\mathfrak{A}(\mathcal{S}') \bigoplus_{\mathcal{S}'} \mathfrak{A})$ which is a subset of (2). The fact that these sets are pairwise disjoint follows from Proposition 4.1 and Lemma 4.3.

Equality 4.2 is obvious: the \mathcal{S} -indecomposable members of \mathfrak{A} are those which cannot be written as \mathcal{S} -sums. \square

In the sequel, we count. Our structures being ordered we may choose a unique representative of an n -element structure on the set $\{0, \dots, n-1\}$, the ordering being the natural order. Let \mathcal{H} and \mathcal{K} be the generating series of \mathfrak{A} and $\mathfrak{D}_{(\geq 3)}$ and let $\mathcal{K}(\mathcal{H})$ be the series obtained by substituting the indeterminate x by \mathcal{H} . Let $\mathcal{H}_{\mathfrak{A}(\mathcal{S})}$ and $\mathfrak{A}_{\mathcal{S}}$ be the generating series of $\mathfrak{A}(\mathcal{S})$ and $\mathfrak{A}_{\mathcal{S}}$ for $\mathcal{S} \in \mathfrak{D}_{(\geq 2)}$. And let p be the cardinality of $\mathfrak{D}_{(2)}$.

Lemma 4.6.

$$(4.3) \quad (p-1)\mathcal{H}^2 + (x-1+\mathcal{K}(\mathcal{H}))\mathcal{H} + x + \mathcal{K}(\mathcal{H}) = 0.$$

$$(4.4) \quad \mathcal{H}_{\mathfrak{A}(\mathcal{S})} = \frac{\mathcal{H}}{1+\mathcal{H}} \text{ for every } \mathcal{S} \in \mathfrak{D}_{(2)}.$$

Proof.

Let us prove that Equation 4.4 holds. Let $\mathcal{S} \in \mathfrak{D}_{(2)}$. Since by definition in Lemma 4.5, $\mathfrak{A}_{\mathcal{S}} = \mathfrak{A}(\mathcal{S}) \oplus_{\mathcal{S}} \mathfrak{A}$, we have $\mathcal{H}_{\mathfrak{A}_{\mathcal{S}}} = \mathcal{H}_{\mathfrak{A}(\mathcal{S})} \cdot \mathcal{H}$. From Equation (4.2), we deduce $\mathcal{H}_{\mathfrak{A}_{\mathcal{S}}} = \mathcal{H} - \mathcal{H}_{\mathfrak{A}_{\mathcal{S}}} \cdot \mathcal{H}$. Since the coefficients of \mathcal{H} are non-negative, the series $1 + \mathcal{H}$ is invertible, hence $\mathcal{H}_{\mathfrak{A}_{\mathcal{S}}} = \frac{\mathcal{H}}{1+\mathcal{H}}$ as claimed in Equation (4.4).

Let us prove that Equation 4.3 holds. Let $\mathcal{S} \in \mathfrak{D}_{(n)}$, with $n \geq 3$. Since by definition in Lemma 4.5, $\mathfrak{A}_{\mathcal{S}} = \oplus_{\mathcal{S}} \mathfrak{A}$, we have $\mathcal{H}_{\mathfrak{A}_{\mathcal{S}}} = \mathcal{H}^n$. From this, we deduce that the generating series of $\bigcup_{\mathcal{S} \in \mathfrak{D}_{(\geq 3)}} \mathfrak{A}_{\mathcal{S}}$ is equal to $\mathcal{K}(\mathcal{H})$. From Equation (4.4), we deduce

that the generating series of $\mathcal{H}_{\mathfrak{A}_{\mathcal{S}}}$ is $\frac{\mathcal{H}^2}{1+\mathcal{H}}$. Hence the generating series of $\bigcup_{\mathcal{S} \in \mathfrak{D}_{(2)}} \mathfrak{A}_{\mathcal{S}}$ is

equal to $p \frac{\mathcal{H}^2}{1+\mathcal{H}}$.

Substituting these values in Equation (4.1), we obtain

$$(4.5) \quad \mathcal{H} = x + p \frac{\mathcal{H}^2}{1+\mathcal{H}} + \mathcal{K}(\mathcal{H}).$$

A straightforward computation yields Equation (4.3). □

Let us say that a class of finite structures is *algebraic* if its generating series is algebraic.

Corollary 4.7. *Let \mathfrak{D} be a class made of non-empty indecomposable members of Γ_d such that $\mathfrak{D}_{(1)}$ is reduced to the one-element structure $\mathbf{1}$. If \mathfrak{D} is algebraic then its sum-closure and the subclass $\mathfrak{A}_{\mathcal{S}}$ consisting of the \mathcal{S} -indecomposable members of the sum-closure \mathfrak{A} are algebraic for each $\mathcal{S} \in \mathfrak{D}_{(2)}$.*

5. WELL-QUASI-ORDERED HEREDITARY CLASSES

Let \mathfrak{C} be a subclass of $\overline{\Omega}_{\mu}$ and \mathcal{A} be a poset. Set $\mathfrak{C}.\mathcal{A} := \{(\mathcal{R}, f) : \mathcal{R} \in \mathfrak{C}, f : V(\mathcal{R}) \rightarrow \mathcal{A}\}$ and $(\mathcal{R}, f) \leq (\mathcal{R}', f')$ if there is an embedding $h : \mathcal{R} \rightarrow \mathcal{R}'$ such that $f(x) \leq f'(h(x))$ for all $x \in V(\mathcal{R})$.

We recall that \mathcal{A} is *well-quasi ordered* (wqo) if \mathcal{A} contains no infinite antichain and no infinite descending chain. We say that \mathfrak{C} is *hereditary wqo* if $\mathfrak{C}.\mathcal{A}$ is wqo for every wqo \mathcal{A} . It is clear that every class which is hereditary wqo is wqo. If \mathfrak{C} is reduced to a single structure \mathcal{R} , it is hereditary wqo provided that \mathcal{R} is finite (this follows from the fact that if \mathcal{A} is wqo then its power \mathcal{A}^n ordered coordinatewise is wqo for each integer n). If \mathcal{R} is infinite, this does not hold. Also, a finite union of hereditary wqo classes is hereditary wqo; hence every finite subclass \mathfrak{C} of Ω_{μ} is hereditary wqo.

A longstanding open question ask whether \mathfrak{C} is hereditary wqo whenever the class \mathfrak{C}_2 of the elements of \mathfrak{C} labelled by $\underline{2}$, the 2-element antichain, is wqo.

If Ch is the class of finite chains, $Ch.\mathcal{A}$ identifies to the set \mathcal{A}^* of finite words over the alphabet \mathcal{A} equipped with the Higman ordering. The fact that Ch is hereditary wqo is a famous result due to Higman [17]. We also note the following fact:

Fact 5. *If a subclass \mathfrak{C} of Ω_μ (with I finite) is hereditary wqo, then $\downarrow \mathfrak{C}$, the least hereditary subclass of Ω_μ containing \mathfrak{C} , is hereditary wqo.*

We recall the following result of [26].

Theorem 5.1. *If the signature is finite, a subclass of Ω_μ which is hereditary and hereditary wqo has finitely many bounds.*

Behavior of the profile of special hereditary classes, the *ages* of Fraïssé, and the link with wqo classes were considered by the second author in the early seventies (see [27] and [28] for a survey). The case of graphs, tournaments and other combinatorial structures was elucidated more recently (see the survey of [20]).

Proposition 5.2. *If a hereditary class \mathfrak{D} of $Ind(\Omega_I)$ is hereditary wqo then $\sum \mathfrak{D}$ is hereditary wqo and $\sum \mathfrak{D}$ has finitely many bounds.*

Proof. The second part of the proposition follows from Theorem 5.1 above. In our case of binary structures, we may note that the proof is straightforward. The first part uses properties of wqo posets, and follows from Higman's theorem on algebras preordered by divisibility (1952) [17]. Instead of recalling the result we give a direct proof. Let \mathcal{A} a poset which is wqo and consider $(\sum \mathfrak{D}).\mathcal{A}$. If $(\sum \mathfrak{D}).\mathcal{A}$ is not wqo, then according to one of preliminary result of Higman, it contains some non finitely generated final segment (\mathcal{F} is a final segment if $x \in \mathcal{F}$ and $x \leq y$ imply $y \in \mathcal{F}$). According to Zorn lemma, there is a maximal one, say \mathcal{F} , with respect to inclusion among final segments having this property. Let $\mathcal{I} := (\sum \mathfrak{D}).\mathcal{A} \setminus \mathcal{F}$ be the complement of \mathcal{F} in $(\sum \mathfrak{D}).\mathcal{A}$. The set \mathcal{I} is then wqo. Let $\mathfrak{R} := (\mathcal{R}_0, f_0), \dots, (\mathcal{R}_n, f_n), \dots$ be an infinite antichain of minimal elements of \mathcal{F} . As $\mathfrak{D}.\mathcal{A}$ is wqo because \mathfrak{D} is hereditary wqo, we can suppose that no element of this antichain is in $\mathfrak{D}.\mathcal{A}$. Then, according to Proposition 4.1 and Theorem 4.2, for every $i \geq 0$ there exists an indecomposable structure \mathcal{S}_i and non-empty structures $(\mathcal{R}_{ix})_{x \in V(\mathcal{S}_i)}$ such that $\mathcal{R}_i = \bigoplus_{x \in \mathcal{S}_i} \mathcal{R}_{ix}$. Since $(\mathcal{R}_{ix}, f_i \upharpoonright_{V(\mathcal{R}_{ix})})$ strictly embeds into (\mathcal{R}_i, f_i) we have $(\mathcal{R}_{ix}, f_i \upharpoonright_{V(\mathcal{R}_{ix})}) \in \mathcal{I}$ for every $i \geq 0$ and $x \in \mathcal{S}_i$. Since \mathcal{I} is wqo, and \mathfrak{D} is hereditary wqo, $\mathfrak{D}.\mathcal{I}$ is wqo, thus the infinite sequence $(\mathcal{S}_0, g_0), \dots, (\mathcal{S}_i, g_i), \dots$ of $\mathfrak{D}.\mathcal{I}$, where $g_i(x) := (\mathcal{R}_{ix}, f_i \upharpoonright_{V(\mathcal{R}_{ix})})$, contains an increasing pair $(\mathcal{S}_p, g_p) \leq (\mathcal{S}_q, g_q)$ for some $p < q$. Which means that there is an embedding $h : \mathcal{S}_p \rightarrow \mathcal{S}_q$ such that $g_p(x) \leq g_q(h(x))$ for all $x \in V(\mathcal{S}_p)$, that is $(\mathcal{R}_{px}, f_p \upharpoonright_{V(\mathcal{R}_{px})}) \leq (\mathcal{R}_{qh(x)}, f_q \upharpoonright_{V(\mathcal{R}_{qh(x)})})$ for all $x \in \mathcal{S}_p$. It follows that $(\mathcal{R}_p, f_p) = \bigoplus_{x \in \mathcal{S}_p} (\mathcal{R}_{px}, f_p \upharpoonright_{V(\mathcal{R}_{px})}) \leq \bigoplus_{x \in \mathcal{S}_q} (\mathcal{R}_{qx}, f_q \upharpoonright_{V(\mathcal{R}_{qx})}) = (\mathcal{R}_q, f_q)$ which contradicts that \mathfrak{R} is an antichain. Thus $(\sum \mathfrak{D}).\mathcal{A}$ is wqo and hence $(\sum \mathfrak{D})$ is hereditary wqo. \square

Proposition 5.2 particular holds if \mathfrak{D} is finite. If \mathfrak{D} is the class $Ind_k(\Omega_I)$ of indecomposable structures of size at most k then according to a result of Schmerl and Trotter, 1993 ([31]), the bounds of $\sum Ind_k(\Omega_I)$ have size at most $k + 2$. When \mathfrak{D} is made of bichains, Proposition 5.2 was obtained by Albert and Atkinson [1].

An immediate corollary is:

Corollary 5.3. *If a hereditary class of Ω_I contains only finitely many indecomposable members then it is wqo and has finitely many bounds.*

We say that a class \mathfrak{C} of relational structures is *hereditary rational*, resp. *hereditary algebraic* if the generating function of every hereditary subclass of \mathfrak{C} is rational, resp. algebraic. Albert, Atkinson and Vatter [3] proved that hereditary rational classes of permutations are wqo. This fact can be extended to hereditary algebraic classes.

Lemma 5.4. *A hereditary class \mathfrak{C} which is hereditary algebraic is wqo.*

Proof. If \mathfrak{C} contains an infinite antichain, there are uncountably many hereditary subclasses of \mathfrak{C} and in fact an uncountable chain of subclasses; these classes provides uncountably many generating series. Some of these series cannot be algebraic. Indeed, according to C. Retenauer [29], a generating series with rational coefficients which is algebraic over \mathbb{C} is algebraic over \mathbb{Q} . Since the generating series we consider have integer coefficients, there are algebraic over \mathbb{Q} , hence there are only countably many such series. \square

If \mathfrak{C} and \mathfrak{D} are two hereditary classes, then the generating series satisfy the identity $\mathcal{H}_{\mathfrak{C} \cup \mathfrak{D}} = \mathcal{H}_{\mathfrak{C}} + \mathcal{H}_{\mathfrak{D}} - \mathcal{H}_{\mathfrak{C} \cap \mathfrak{D}}$. From this simple equality we have:

Lemma 5.5. *The union of two hereditary rational (resp. algebraic) classes is hereditary rational (resp. algebraic).*

Corollary 5.6. *A minimal non-hereditary rational or a minimal non-hereditary algebraic class \mathfrak{C} is the age of some relational structure.*

Proof. According to Lemma 5.5, \mathfrak{C} cannot be the union of two proper hereditary subclasses, hence this is an ideal, thus an age. \square

Theorem 5.7. *Let d be an integer. If an hereditary class \mathfrak{C} of Γ_d contains only finitely many indecomposable members then it is algebraic.*

We follows essentially the lines of Albert-Atkinson proof. We do an inductive proof over the hereditary subclasses of \mathfrak{C} . But for that, we need to prove more, namely that \mathfrak{C} and each $\mathfrak{C}(\mathcal{S})$ for $\mathcal{S} \in Ind(\mathfrak{C})_{(2)}$, are algebraic (this is the only difference with Albert-Atkinson proof). To avoid unessential complications, we take out the empty relational structure of Γ_d , that is we suppose that \mathfrak{C} is made of non-empty structures. Let $\mathfrak{A} := \sum Ind(\mathfrak{C})$. If $\mathfrak{C} = \mathfrak{A}$ then by Corollary 4.7, \mathfrak{C} and each $\mathfrak{C}(\mathcal{S})$ for $\mathcal{S} \in Ind(\mathfrak{C})_{(2)}$, are algebraic. Thus the result is proved. If $\mathfrak{C} \neq \mathfrak{A}$, we may suppose that for each proper hereditary subclass \mathfrak{C}' of \mathfrak{C} , both \mathfrak{C}' and $\mathfrak{C}'(\mathcal{S})$ for each $\mathcal{S} \in Ind(\mathfrak{C}')_{(2)}$, are algebraic. Indeed, otherwise, since by Corollary 5.3, \mathfrak{C} is wqo, it contains a minimal hereditary subclass not satisfying this property and we may replace \mathfrak{C} by this subclass. Let $\mathcal{S} \in Ind(\mathfrak{C})_{(2)}$. Let $\mathfrak{C}(\mathcal{S})$ be the subclass of \mathfrak{C} made

of \mathcal{S} -indecomposable members of \mathfrak{C} . Let 0 and 1, with $0 < 1$, be the two elements of $V(\mathcal{S})$, we set $\mathfrak{C}_{\mathcal{S}} := (\mathfrak{C}(\mathcal{S}) \oplus_{\mathcal{S}} \mathfrak{C}) \cap \mathfrak{C}$. Let $\mathcal{S} \in \text{Ind}(\mathfrak{C})_{(\geq 3)}$ we set $\mathfrak{C}_{\mathcal{S}} := (\oplus_{\mathcal{S}} \mathfrak{C}) \cap \mathfrak{C}$.

As in Lemma 4.5 we have

$$(5.1) \quad \mathfrak{C} = \{1\} \cup \bigcup_{\mathcal{S} \in \text{Ind}(\mathfrak{C})_{(\geq 2)}} \mathfrak{C}_{\mathcal{S}}$$

and

$$(5.2) \quad \mathfrak{C}(\mathcal{S}) = \mathfrak{C} \setminus \mathfrak{C}_{\mathcal{S}} \text{ for every } \mathcal{S} \in \text{Ind}(\mathfrak{C})_{(2)}.$$

Let \mathcal{H} and \mathcal{K} be the generating series of \mathfrak{C} and $\text{Ind}(\mathfrak{C})_{(\geq 3)}$ respectively.

Let $\mathcal{H}_{\mathfrak{C}_{(2)}}$ and $\mathcal{H}_{\mathfrak{C}_{(\geq 3)}}$ be the generating series of $\mathfrak{C}_{(2)} := \bigcup_{\mathcal{S} \in \text{Ind}(\mathfrak{C})_{(2)}} \mathfrak{C}_{\mathcal{S}}$ and of

$$\mathfrak{C}_{(\geq 3)} := \bigcup_{\mathcal{S} \in \text{Ind}(\mathfrak{C})_{(\geq 3)}} \mathfrak{C}_{\mathcal{S}}.$$

We have:

$$(5.3) \quad \mathcal{H}_{\mathfrak{C}} = x + \mathcal{H}_{\mathfrak{C}_{(2)}} + \mathcal{H}_{\mathfrak{C}_{(\geq 3)}}$$

and

$$(5.4) \quad \mathcal{H}_{\mathfrak{C}(\mathcal{S})} = \mathcal{H}_{\mathfrak{C}} - \mathcal{H}_{\mathfrak{C}_{\mathcal{S}}} \text{ for every } \mathcal{S} \in \mathfrak{D}_{(2)}.$$

We deduce that $\mathcal{H}_{\mathfrak{C}}$ and $\mathcal{H}_{\mathfrak{C}(\mathcal{S})}$ are algebraic for every $\mathcal{S} \in \text{Ind}(\mathfrak{C})_{(2)}$, from the following claims that we will prove afterwards

Claim 1. *The generating series of $\mathcal{H}_{\mathfrak{C}_{(\geq 3)}}$ is a polynomial in the generating series $\mathcal{H}_{\mathfrak{C}}$ whose coefficients are algebraic series.*

Claim 2. *For each $\mathcal{S} \in \text{Ind}(\mathfrak{C})_{(2)}$, the generating series $\mathcal{H}_{\mathfrak{C}(\mathcal{S})}$ of $\mathfrak{C}(\mathcal{S})$ is either a linear polynomial in the generating series $\mathcal{H}_{\mathfrak{C}}$ of the form*

$$(5.5) \quad \mathcal{H}_{\mathfrak{C}(\mathcal{S})} = \frac{(1 - \alpha)\mathcal{H}_{\mathfrak{C}} - \delta}{1 + \beta};$$

whose coefficients are algebraic series or is a rational fraction of the form

$$(5.6) \quad \mathcal{H}_{\mathfrak{C}(\mathcal{S})} = \frac{\mathcal{H}_{\mathfrak{C}}}{1 + \mathcal{H}_{\mathfrak{C}}}.$$

Substituting in formula 5.3 the values of $\mathcal{H}_{\mathfrak{C}_{(\geq 3)}}$ and $\mathcal{H}_{\mathfrak{C}_{(2)}}$ given by Claim 1 and Claim 2 we obtain a polynomial in $\mathcal{H}_{\mathfrak{C}}$ whose coefficients are algebraic series. This polynomial is not identical to zero. Indeed, it is the sum of a polynomial $A = a_0 + a_1\mathcal{H}_{\mathfrak{C}} + a_2\mathcal{H}_{\mathfrak{C}}^2$ and $B = b_0 + b_1\mathcal{H}_{\mathfrak{C}} + \dots + a_k\mathcal{H}_{\mathfrak{C}}^k$ whose coefficients are algebraic series (in fact, $B = \mathcal{H}_{\mathfrak{C}_{(\geq 3)}}(1 + \mathcal{H}_{\mathfrak{C}})$). The valuation of A and B as series in x are distinct. Indeed, the valuation of A is 1 (notice that $a_0 = x + \delta$ where δ is either zero or an algebraic series of valuation at least 2). Hence, if $B \neq 0$ (when $\text{Ind}(\mathfrak{C})_{(\geq 3)}$ is non empty) its valuation is at least 3. Since A and B don't have the same valuation, then $A + B$ is not identical to zero. Being a solution of a non zero polynomial, $\mathcal{H}_{\mathfrak{C}}$ is algebraic. With this result and claim 2, $\mathcal{H}_{\mathfrak{C}(\mathcal{S})}$ is algebraic. With this, the proof of Theorem 5.7 is complete.

In order to prove our claims, we need the following lemmas (respectively Lemma 15 and Lemma 18 in [1]).

Lemma 5.8. *Let \mathcal{S} be an indecomposable ordered structure and $\mathfrak{A} := (\mathfrak{A}_x)_{x \in \mathcal{S}}$, $\mathfrak{B} := (\mathfrak{B}_x)_{x \in \mathcal{S}}$ be two sequences of subclasses of ordered binary structures indexed by the elements of \mathcal{S} . If \mathcal{S} has at least three elements then*

$$\left(\bigoplus_{x \in \mathcal{S}} \mathfrak{A}_x \right) \cap \left(\bigoplus_{x \in \mathcal{S}} \mathfrak{B}_x \right) = \bigoplus_{x \in \mathcal{S}} (\mathfrak{A}_x \cap \mathfrak{B}_x).$$

If $\mathcal{S} := (\{0, 1\}, \leq, (\rho_i)_{i \in J})$ with $0 < 1$ then

$$(\mathfrak{A}_0(\mathcal{S}) \oplus_{\mathcal{S}} \mathfrak{A}_1) \cap (\mathfrak{B}_0(\mathcal{S}) \oplus_{\mathcal{S}} \mathfrak{B}_1) = (\mathfrak{A}_0(\mathcal{S}) \cap \mathfrak{B}_0(\mathcal{S})) \oplus_{\mathcal{S}} (\mathfrak{A}_1 \cap \mathfrak{B}_1).$$

Proof. The first equality follows from Proposition 4.1 and the second one follows from Lemma 4.3. \square

Let \mathfrak{C} be a class of finite structures and $\overline{\mathcal{B}} := \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_l$ be a sequence of finite structures, we will set $\mathfrak{C} < \overline{\mathcal{B}} > := \mathfrak{C} < \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_l > := \text{Forb}(\{\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_l\}) \cap \mathfrak{C}$.

If \mathfrak{C} is hereditary, a proper hereditary subclass \mathfrak{C}' of \mathfrak{C} is *strong* if every bound of \mathfrak{C}' in \mathfrak{C} is embeddable in some bound of \mathfrak{C} . Note that the intersection of strong subclasses is strong.

Let $\mathfrak{A} := (\mathfrak{A}_x)_{x \in \mathcal{S}}$, where \mathcal{S} is indecomposable with at least three elements. A *decomposition of a binary structure \mathcal{B} over \mathfrak{A}* is a map $h : \mathcal{B} \rightarrow \mathcal{S}$ such that $\mathcal{B} = \bigoplus_{x \in \mathcal{S} \upharpoonright_{\text{range}(h)}} \mathcal{B} \upharpoonright_{h^{-1}(x)}$ and $\mathcal{B} \upharpoonright_{h^{-1}(x)} \in \mathfrak{A}_x$ for all $x \in \text{range}(h)$. Hence, each $\mathcal{B} \upharpoonright_{h^{-1}(x)}$ is an interval of \mathcal{B} . Let $H_{\mathcal{B}}$ be the set of all such decompositions of \mathcal{B} .

Lemma 5.9. *Let \mathcal{S} be an indecomposable ordered structure, $\mathfrak{A} := (\mathfrak{A}_x)_{x \in \mathcal{S}}$ be a sequences of subclasses of ordered binary structures indexed by the elements of \mathcal{S} , $\overline{\mathcal{B}} := \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_l$ be a sequence of finite structures, and $\mathfrak{C} := \left(\bigoplus_{x \in \mathcal{S}} \mathfrak{A}_x \right) < \overline{\mathcal{B}} >$. If \mathcal{S} has at least three elements then \mathfrak{C} is a union of sets of the form $\bigoplus_{x \in \mathcal{S}} \mathfrak{D}_x$ where each \mathfrak{D}_x is either $\mathfrak{A}_x < \overline{\mathcal{B}} >$ or one of its strong subclasses.*

Proof. We prove the result for $l = 1$ and we set $\mathcal{B} := \mathcal{B}_1$. For that we prove that:

$$(5.7) \quad \mathfrak{C} = \bigcap_{h \in H_{\mathcal{B}}} \bigcup_{x \in \text{range}(h)} \bigoplus_{y \in \mathcal{S}} \mathfrak{A}_y^{(x)}$$

where $\mathfrak{A}_x^{(x)} := \mathfrak{A}_x < \mathcal{B} \upharpoonright_{h^{-1}(x)} >$ and $\mathfrak{A}_y^{(x)} := \mathfrak{A}_y$ for $y \neq x$.

Let's call by (1) (respectively by (2)) the left-hand side (respectively the right-hand side) of Equation 5.7. Inclusion (1) \subseteq (2) holds without any assumption. Indeed, let \mathcal{T} in (1). We prove that \mathcal{T} is in (2). If h is a decomposition of \mathcal{B} , we want to find $x \in \text{range}(h)$ such that $\mathcal{T} \in \bigoplus_{y \in \mathcal{S}} \mathfrak{A}_y^{(x)}$. Since \mathcal{T} is in (1), it has a decomposition over \mathcal{S} . Let $h \in H_{\mathcal{B}}$, since $\mathcal{B} \not\leq \mathcal{S}$, there exist $x \in \mathcal{S} \upharpoonright_{\text{range}(h)}$ such that $\mathcal{B} \upharpoonright_{h^{-1}(x)} \not\leq \mathcal{T}$, hence $\mathcal{T} \in \bigoplus_{y \in \mathcal{S}} \mathfrak{A}_y^{(x)}$.

Inclusion $(2) \subseteq (1)$ holds under the assumption that a structure in (2) has a unique decomposition over \mathcal{S} and that it is ordered, (what means that \mathcal{S} is rigid, that is \mathcal{S} has no automorphism distinct from the identity). Let \mathcal{T} in (2) , then for every $h \in H_{\mathcal{B}}$ there exist $x_h \in \text{range}(h)$ such that $\mathcal{T} \in \bigoplus_{y \in \mathcal{S}} \mathfrak{A}_y^{(x_h)}$, thus, $\mathcal{T} \in \bigcap_{h \in H_{\mathcal{B}}} \bigoplus_{y \in \mathcal{S}} \mathfrak{A}_y^{(x_h)}$. We have $\mathcal{T} \in \bigoplus_{y \in \mathcal{S}} \mathfrak{A}_y$ because, $\mathfrak{A}_y^{(x_h)} \subseteq \mathfrak{A}_y$ for every h . Hence, $\mathcal{T} = \bigoplus_{y \in \mathcal{S}} \mathcal{T}_y$. We claim that $\mathcal{B} \not\leq \mathcal{T}$. Suppose $\mathcal{B} \leq \mathcal{T}$ let f be an embedding of \mathcal{B} into \mathcal{T} and $h := p \circ f$, where p is the projection map from \mathcal{T} into \mathcal{S} , we must have $\mathcal{B} \upharpoonright_{h^{-1}(x)} \leq \mathcal{T}_x$ for $x \in \text{rang}(h)$ which is a contradiction with the fact that $\mathcal{T} \in \bigcap_{h \in H} \bigoplus_{y \in \mathcal{S}} \mathfrak{A}_y^{(x_h)}$.

Using distributivity of intersection over union, we may write (2) as a union of terms, each of which is an intersection of terms like $\bigoplus_{y \in \mathcal{S}} \mathfrak{A}_y < \mathcal{B}_x >$, where \mathcal{B}_x is an interval of \mathcal{B} such that, there exist a decomposition h of \mathcal{B} and $\mathcal{B}_x = \mathcal{B} \upharpoonright_{h^{-1}(x)}$. These intersections, by lemma 5.8 and the fact that among all decompositions of \mathcal{B} are all ones which send \mathcal{B} into a single element x of \mathcal{S} have the form $\bigoplus_{x \in \mathcal{S}} \mathfrak{D}_x$ where each \mathfrak{D}_x is of the form $\mathfrak{A}_x < \mathcal{B}, \dots >$ where the structures occurring after \mathcal{B} (if any) are intervals of \mathcal{B} . Hence, \mathfrak{D}_x is either $\mathfrak{A}_x < \mathcal{B} >$ or one of its strong subclasses. The case $l > 1$ follows by induction. \square

Proof of Claim 1. Since \mathfrak{A} is wqo and \mathfrak{C} is a proper hereditary subclass, we have $\mathfrak{C} = \mathfrak{A} < \overline{\mathcal{B}} >$ for some finite family $\overline{\mathcal{B}} := \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_l$ of elements of \mathfrak{A} . Let $\mathcal{S} \in \text{Ind}(\mathfrak{C})_{(\geq 3)}$, Lemma 5.9 asserts that $\mathfrak{C}_{\mathcal{S}}$ is an union of classes, not necessarily disjoint, of the form $\bigoplus_{x \in \mathcal{S}} \mathfrak{C}_x$ where each \mathfrak{C}_x is either \mathfrak{C} or one of its strong subclasses. The generating series of $\bigoplus_{x \in \mathcal{S}} \mathfrak{C}_x$ is a monomial in the generating series $\mathcal{H}_{\mathfrak{C}}$ of \mathfrak{C} whose coefficient is a product of generating series of proper strong subclasses of \mathfrak{C} . From the induction hypothesis, the generating series of these strong subclasses are algebraic series, hence this coefficient is an algebraic series. Using the principle of inclusion-exclusion, we get that the generating series $\mathcal{H}_{\mathfrak{C}_{\mathcal{S}}}$ of $\mathfrak{C}_{\mathcal{S}}$ is a polynomial in the generating series $\mathcal{H}_{\mathfrak{C}}$ whose coefficients are algebraic series. Since the $\mathfrak{C}_{\mathcal{S}}$'s are pairwise disjoint, the generating series $\mathcal{H}_{\mathfrak{C}_{(\geq 3)}}$ is also a polynomial in the generating series $\mathcal{H}_{\mathfrak{C}}$ whose coefficients are algebraic series. \square

Lemma 5.10. *If \mathcal{S} has two elements 0 and 1, $\mathcal{S} := (\{0, 1\}, \leq, (\rho_i)_{i \in J})$ with $0 < 1$, then $(\mathfrak{A}(\mathcal{S}) \oplus \mathfrak{A}) < \overline{\mathcal{B}} >$ is an union of classes of the form $(\mathfrak{A}'(\mathcal{S}) < \overline{\mathcal{B}} >) \oplus (\mathfrak{A}'' < \overline{\mathcal{B}} >)$, where $\mathfrak{A}' < \overline{\mathcal{B}} >$ and $\mathfrak{A}'' < \overline{\mathcal{B}} >$ are either equal to $\mathfrak{A} < \overline{\mathcal{B}} >$ or to some strong subclasses of $\mathfrak{A} < \overline{\mathcal{B}} >$.*

Proof. As above we suppose first $l = 1$. Equation 5.7 yields

$$(\mathfrak{A}(\mathcal{S}) \oplus \mathfrak{A}) < \mathcal{B} > = \bigcap_{h \in H_{\mathcal{B}}} \left[(\mathfrak{A}(\mathcal{S}) < \mathcal{B} \upharpoonright_{h^{-1}(0)} > \oplus \mathfrak{A}) \bigcup (\mathfrak{A}(\mathcal{S}) \oplus (\mathfrak{A} < \mathcal{B} \upharpoonright_{h^{-1}(1)} >)) \right]$$

An induction take care of the case $l > 1$. \square

Proof of Claim 2. Let $\mathcal{S} \in \text{Ind}(\mathfrak{C})_{(\geq 3)}$. Lemma 5.10 asserts that $\mathfrak{C}_{\mathcal{S}}$ is an union of classes, not necessarily disjoint, of the form $\mathfrak{C}'(\mathcal{S}) \oplus_{\mathcal{S}} \mathfrak{C}''$, where \mathfrak{C}' and \mathfrak{C}'' are either equal to \mathfrak{C} or to some strong subclasses of \mathfrak{C} . The generating series of these classes are of the form $\mathcal{H}_{\mathfrak{C}(\mathcal{S})}\mathcal{H}_{\mathfrak{C}}$ or $\alpha\mathcal{H}_{\mathfrak{C}}$ or $\beta\mathcal{H}_{\mathfrak{C}(\mathcal{S})}$, where α and β are algebraic series. Using the principle of inclusion-exclusion, we get that the generating series $\mathcal{H}_{\mathfrak{C}_{\mathcal{S}}}$ is either of the form $\mathcal{H}_{\mathfrak{C}(\mathcal{S})}\mathcal{H}_{\mathfrak{C}}$ or of the form $\alpha\mathcal{H}_{\mathfrak{C}} + \beta\mathcal{H}_{\mathfrak{C}(\mathcal{S})} + \delta$, where α, β and δ are algebraic series. In particular $\mathcal{H}_{\mathfrak{C}_{\mathcal{S}}}$ is of the form $\alpha_{\mathcal{S}}\mathcal{H}_{\mathfrak{C}(\mathcal{S})} + \beta_{\mathcal{S}}$ where $\alpha_{\mathcal{S}}$ and $\beta_{\mathcal{S}}$ are polynomials in $\mathcal{H}_{\mathfrak{C}}$ of degree at most 1 with algebraic series as coefficients. Using Equation 5.2 we obtain

$$(5.9) \quad \mathcal{H}_{\mathfrak{C}(\mathcal{S})} = \frac{\mathcal{H}_{\mathfrak{C}}}{1 + \mathcal{H}_{\mathfrak{C}}};$$

when all bounds \mathcal{B}_i of \mathfrak{C} in \mathfrak{A} are \mathcal{S} -indecomposable or

$$(5.10) \quad \mathcal{H}_{\mathfrak{C}(\mathcal{S})} = \frac{(1 - \alpha)\mathcal{H}_{\mathfrak{C}} - \delta}{1 + \beta};$$

if at least one bound \mathcal{B}_i is not \mathcal{S} -indecomposable. □

The conclusion of Theorem 5.7 above does not hold with structures which are not necessarily ordered.

Example 5.11. Let $K_{\infty, \infty}$ be the direct sum of infinitely many copies of the complete graph on an infinite set. As it is easy to see the generating function of $\text{Age}(K_{\infty, \infty})$ is the integer partition function. This generating series is not algebraic. However, $\text{Age}(K_{\infty, \infty})$ contains no indecomposable member with more than two elements. More generally, note that the class $\text{Forb}(P_4)$ of finite cographs contains no indecomposable cograph with more than two vertices and that this class is not hereditary algebraic. Finite cographs are comparability graphs of serie-parallel posets which in turn are intersection orders of separable bichains. By Albert-Atkinson's theorem, the class of these bichains is hereditary algebraic. This tells us that algebraicity is not necessarily preserved by the transformation of a class into another via a process as above (processes of this type are the free-operators of Fraïssé [13]).

6. A CONJECTURE AND SOME QUESTIONS

In their paper [1], Albert and Atkinson indicate that there are infinite sets of simple permutations whose sum closure is algebraic but, as it turns out, some hereditary subclasses are not necessarily algebraic. An example is the collection of decreasing oscillations (see the end of the section). In order to extend their proof to some other classes, they ask whether there exists an infinite set of simple permutations whose sum-closure is well quasi ordered. As we indicate in Proposition 6.1 below, the set of exceptional permutations has this property. In fact, it is hereditary wqo. We guess that this notion of hereditary wqo is the right concept for extending Albert-Atkinson theorem.

Exceptional permutations correspond to bichains which are *critical* in the sense of Schmerl and Trotter. Let us recall that a binary structure \mathcal{R} with domain E is *critical* if \mathcal{R} is indecomposable but $\mathcal{R} \upharpoonright_{E \setminus \{x\}}$ is not indecomposable for every $x \in E$. Schmerl and Trotter [31] gave a description of critical posets. They fall into two infinite classes: $\mathfrak{P} := \{\mathcal{P}_n : n \in \mathbb{N}\}$ and $\mathfrak{P}' := \{\mathcal{P}'_n : n \in \mathbb{N}\}$ where $\mathcal{P}_n := (V_n, \leq_n)$, $V_n := \{0, \dots, n-1\} \times \{0, 1\}$, $(x, i) <_n (y, j)$ if $i < j$ and $x \leq y$; $\mathcal{P}'_n := (V_n, \leq'_n)$ and $(x, i) <'_n (y, j)$ if $j \leq i$ and $x < y$.

These posets are two-dimensional. That is, they are intersection of two linear orders which are respectively $L_{n,1} := (0, 0) < (0, 1) < \dots < (i, 0) < (i, 1) \dots < (n-1, 0) < (n-1, 1)$ and $L_{n,2} := (n-1, 0) < \dots < (n-i, 0) < \dots < (0, 0) < (n-1, 1) < \dots < (n-i, 1) \dots < (0, 1)$ for \mathcal{P}_n and $L'_{n,1} := L_{n,1}$ and $L'_{n,2} := (L_{n,2})^*$ for \mathcal{P}'_n .

As it is well known, an indecomposable two-dimensional poset $\mathcal{P} := (V, L)$ has a unique realizer (that is there is a unique pair $\{L_1, L_2\}$ of linear orders whose intersection is the order L of \mathcal{P}). Hence, there are at most two bichains, namely (V, L_1, L_2) and (V, L_2, L_1) such that $L_1 \cap L_2 = L$. The critical posets described above yield four kind of bichains, namely $(V_n, L_{n,1}, L_{n,2})$, $(V_n, L_{n,2}, L_{n,1})$, $(V_n, L_{n,1}, (L_{n,2})^*)$ and $(V_n, (L_{n,2})^*, L_{n,1})$. These bichains are critical. Indeed, a bichain is indecomposable if and only if the intersection order is indecomposable ([30] for finite bichains and [33] for infinite bichains). The isomorphic types of these bichains are described in Albert and Atkinson's paper in terms of permutations of $1, \dots, 2m$ for $m \geq 2$:

- (i) $2.4.6 \dots 2m.1.3.5 \dots 2m-1$.
- (ii) $2m-1.2m-3 \dots 1.2m.2m-2 \dots 2$.
- (iii) $m+1.1.m+2.2 \dots 2m.m$.
- (iv) $m.2m.m-1.2m-1 \dots 1.m+1$.

For example, the type of the bichain $(V_m, L_{m,1}, L_{m,2})$ is the permutation given in (iv), whereas the type of $(V_m, L_{m,2}, L_{m,1})$ is its inverse, given in (ii) (enumerate the elements of V_m into the sequence $1, \dots, 2m$, this according to the order $L_{m,1}$, then reorder this sequence according to the order $L_{m,2}$; this yields the sequence $\sigma^{-1} := \sigma^{-1}(1), \dots, \sigma^{-1}(2m)$; according to our definition the type of $(V_m, L_{m,1}, L_{m,2})$ is the permutation σ , this is the one given in (iv)). For $m = 2$, the permutations given in (i) and (iv) coincide with 2413 whereas those given in (ii) and (iii) coincide with 3142; for larger values of m , they are all different.

The four classes of indecomposable bichains are obtained from $\mathfrak{B} := \{(V_n, L_{n,1}, L_{n,2}) : n \in \mathbb{N}\}$ by exchanging the two orders in each bichain or by reversing the order of the first one, or by reversing the second one. Hence the order structure w.r.t. embeddability of these classes is the same, and it remains the same if we label the elements of these bichains.

Proposition 6.1. *The class of critical bichains is hereditary wqo.*

Proof. This class is the union of four classes hence, in order to prove that it is hereditary wqo, it suffices to prove that each one of these classes is hereditary wqo. According to the observation above, it suffices to prove that one, for example \mathfrak{B} , is

hereditary wqo. Let \mathcal{A} be a wqo poset. We have to prove that $\mathfrak{B}.\mathcal{A}$ is wqo. For that, set $\mathcal{B} := \mathcal{A}^2$, where $\mathcal{A}^2 := \{e : \{0, 1\} \rightarrow \mathcal{A}\}$, and order \mathcal{B} componentwise. Let \mathcal{B}^* be the set of all words over the ordered alphabet \mathcal{B} . We define an order preserving map F from \mathcal{B}^* onto $\mathfrak{B}.\mathcal{A}$. This will suffice. Indeed, \mathcal{B} is wqo as a product of two wqo sets; hence, according to Higman theorem on words over ordered alphabets [17], \mathcal{B}^* is wqo. Since $\mathfrak{B}.\mathcal{A}$ is the image of a wqo by an order preserving map, it is wqo. We define the map F as follows. Let $w := w(0)w(1) \cdots w(n-1) \in \mathcal{B}^*$. Set $F(w) := (\mathcal{R}, f_w) \in \mathfrak{B}.\mathcal{A}$ where $\mathcal{R} := (V_n, L_{n,1}, L_{n,2})$ and $f_w(i, j) := w(i)(j)$ for $j \in \{0, 1\}$. We observe first that $w \leq w'$ in \mathcal{B}^* implies $F(w) \leq F(w')$ in $\mathfrak{B}.\mathcal{A}$. Indeed, if $w \leq w'$ there is an embedding h of the chain $0 < \cdots < n-1$ into the chain $0 < \cdots < n'-1$ such that $w(i) \leq w'(h(i))$ for all $i < n$. Let $\bar{h} : \{0, \dots, n-1\} \times \{0, 1\} \rightarrow \{0, \dots, n'-1\} \times \{0, 1\}$ defined by setting $\bar{h}(i, j) := (h(i), j)$. As it is easy to check, \bar{h} is an embedding of $F(w)$ into $F(w')$. Next, we note that F is surjective. Indeed, if $(\mathcal{R}, f) \in \mathfrak{B}.\mathcal{A}$ with $\mathcal{R} := (V_n, L_{n,1}, L_{n,2})$, then the word $w := w(0)w(1) \cdots w(n-1)$ with $w(i)(j) := f(i, j)$ yields $F(w) = (\mathcal{R}, f)$. \square

With Proposition 5.2, we have:

Corollary 6.2. *The sum-closure of the class of critical bichains is wqo.*

In [1] it is mentioned that this class has finitely many bounds. The generating series of the class of critical bichains is rational (the class is covered by four chains). According to Corollary 13 of [1] their sum-closure is algebraic.

Question 1. *Is the sum-closure of the class of critical bichains hereditary algebraic?*

We conjecture that the answer is positive. This will be a consequence of a conjecture for hereditary classes of ordered binary structures that we formulate below.

Conjecture 6.3. *If \mathfrak{D} is a hereditary class of indecomposable ordered binary structures which is hereditary wqo and hereditary algebraic, then its sum-closure is hereditary algebraic.*

The requirement that \mathfrak{D} is wqo will not suffice in Conjecture 6.3.

Indeed, let $\mathcal{P}_{\mathbb{Z}}$ be the doubly infinite path whose vertex set is \mathbb{Z} and edge set $E := \{(n, m) \in \mathbb{Z} \times \mathbb{Z} : |n - m| = 1\}$. The edge set E has two transitive orientations, e.g. $P := \{(n, m) \in \mathbb{Z} \times \mathbb{Z} : |n - m| = 1 \text{ and } n \text{ is even}\}$ and its dual P^* . As an order, P is the intersection of the linear orders $L_1 := \cdots < 2n < 2n - 1 < 2(n + 1) < 2n + 1 < \cdots$ and $L_2 := \cdots < 2(n + 1) < 2n + 3 < 2n < 2n + 1 < \cdots$. Let $\mathcal{C} := (\mathbb{Z}, L_1, L_2)$ and $\mathfrak{D} := \text{Ind}(\mathcal{C})$.

Lemma 6.4. *\mathfrak{D} is wqo but not hereditary wqo.*

Proof. Members of \mathfrak{D} of size n are obtained by restricting \mathcal{C} to intervals of size n , $n \neq 3$, of the chain (\mathbb{Z}, \leq) (observe first that the graph $\mathcal{P}_{\mathbb{Z}}$ is indecomposable as all its restrictions to intervals of size different from 3 of the chain (\mathbb{Z}, \leq) and furthermore there are no others indecomposable restrictions; next, use the fact that the indecomposability of a comparability graph amounts to the indecomposability of its orientations [18], and that the indecomposability of a two-dimensional poset

amounts to the indecomposability of the bichains associated with the order [33]). Up to isomorphism, there are two indecomposable bichains of size n , $n \neq 3$, namely $\mathcal{C}_n := \mathcal{C}_{\uparrow\{0,\dots,n-1\}}$ and $\mathcal{C}_n^* := \mathcal{C}_{\uparrow\{0,\dots,n-1\}}^*$ where $\mathcal{C}^* := (\mathbb{Z}, L^*_1, L^*_2)$. These two bichains embed all members of \mathfrak{D} having size less than n . Being covered by two chains, \mathfrak{D} is wqo. To see that \mathfrak{D} is not hereditary wqo, we may associate to each indecomposable member of \mathfrak{D} the comparability graph of the intersection of the two orders and observe that this association preserves the embeddability relation, even though labels are added. The class of graphs obtained from this association consists of paths of size distinct from 3. It is not hereditary wqo. In fact, as it is immediate to see, if a class \mathfrak{G} of graphs contains infinitely many paths of distinct sizes, then $\mathfrak{G}.2$ is not wqo. Indeed, if we label the end vertices of each path by 1 and label the other vertices by 0, we obtain an infinite antichain. Thus $\mathfrak{D}.2$ is not wqo. \square

The generating series of \mathfrak{D} is rational (its generating function is $\frac{x+x^2}{1-x}$). In fact, \mathfrak{D} is hereditary algebraic (every hereditary subclass of \mathfrak{D} is finite). By Corollary 13 of [1], the sum-closure $\sum \mathfrak{D}$ is algebraic. (in fact, if D is the generating function of $\sum \mathfrak{D}$, then $2D^5 + 2D^4 - D^3 + (2-x)D^2 - D + x = 0$.) But $\sum \mathfrak{D}$ is not hereditary algebraic. For that, it suffices to observe that it is not wqo and to apply Lemma 5.4. The fact that $\sum \mathfrak{D}$ is not wqo is because we may embed the poset $\mathfrak{D}.2$ into $\sum \mathfrak{D}$ via an order preserving map. A simpler argument consist to observe first that the family $(G_n)_{n \in \mathbb{N}}$, where G_n is the graph obtained from the n -vertex path P_n by replacing its end-vertices by a two-vertex independent set, is an antichain, next that these graphs are comparability graphs associated to members of \mathfrak{D} .

The permutations corresponding to the members of \mathfrak{D} are called *decreasing oscillations*. They have been the object of several studies:

The downward closure $\downarrow \mathfrak{D}$ is $\text{Age}(\mathcal{C})$, the age of \mathcal{C} ; this age has four obstructions, it is rational: the generating series is $\frac{1-x}{1-2x-x^3}$, the generating function being the sequence A05298 of [32], starting by 1, 1, 2, 5, 11, 24. For all of this see [9].

6.1. Questions. Is it true that:

- (1) a hereditary class of indecomposable ordered binary structures \mathfrak{D} is hereditary wqo whenever its sum closure is hereditary algebraic?
- (2) the generating series of a hereditary class of relational structures is rational whenever the profile of this class is bounded by a polynomial? This is true for graphs [4] and tournaments [7].
- (3) the profile of a wqo hereditary class of relational structures is bounded above by some exponential?

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