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Stochastic Games[1])

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Abstract: Stochastic Games have a value.

1. Introduction

A stochastic game is played in stages. At each stage, the game is in one of finitely many states and every player observes the current state z and chooses one of finitely many actions. The pair of actions at stage i, together with z determines the payoff x_i to be made by player II to player I at stage i, and the probability used by the referee to select the next state. All the referee's choices are made independently of the past. A player's (behavioural) strategy is a specification of a probability distribution over his actions at each stage conditional on the current state and the sequence of moves up to that stage. Any pair of strategies, σ of player I and τ of player II, induces together with the initial state z_1, a probability distribution on the stream $(x_1, x_2, \ldots)$ of payoffs. The definition of a value depends on how the players evaluate a distribution of streams of payoffs. *Shapley* [1953] proved that the λ-discounted game, i.e., the game with "evaluation" $E\left(\sum_{i=1}^{\infty} \lambda(1-\lambda)^{i-1} x_i\right)$ for $0 < \lambda \leqslant 1$, has a value and that both players have optimal stationary strategies. *Bewley/Kohlberg* [1976], proved that the value $v_\lambda(z)$ (respectively the optimal stationary strategy σ_λ) of the λ-discounted game with initial state z, has a convergent expansion in fractional powers of λ, and

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that the limit $v_\infty(z)$ of $v_\lambda(z)$ as $\lambda \to 0$ exists. The question as to whether or not the undiscounted stochastic games, i.e., the games with "evaluation" $E(\lim_{N\to\infty} \inf \bar{x}_N)$ where $\bar{x}_N = (1/N) \sum_{i=1}^{N} x_i$, always have a value, was open for many years. *Gilette* [1957] proved the existence of the value in two cases: first when all games have perfect information and also in the so called cyclic case. *Blackwell/Ferguson* [1968] found in a particular example ("The Big Match") two strategies that would prove to be basic for further generalizations. Extending their second strategy *Kohlberg* [1974] proved that all "games with absorbing states" have a value. Recently we answered in the affirmative the question as to whether or not all stochastic games have a value [*Mertens/Neyman*]. The proof however was long and involved and the present paper presents a simplified proof of the main theorem.

Theorem. For every stochastic game and for every $\epsilon > 0$ there exists a strategy σ of player I and $N > 0$ such that for every n, $n = N, N+1, \ldots, \infty$, and for every strategy τ of Player II,

$$E_{\sigma,\tau}(\bar{x}_n) \geqslant v_\infty - \epsilon$$

where by $\bar{x}_\infty$ we mean $\lim_{n\to\infty} \inf \bar{x}_n$.

This form of statement means that the strategy σ is ϵ-optimal both in the infinite game and in all sufficiently long finite games. The second implies in particular that the strategy σ is 2ϵ-optimal in all λ-discounted games with λ sufficiently small ($\lambda \leqslant \sqrt{\epsilon}/N$).

Independently of ourselves, *Monash* [1980] announced a weaker version of the present result: it is not claimed that the strategy σ is good, neither in the infinite game nor in sufficiently long finite games, but only that for every strategy τ of player II, there exists N such that for all finite $n \geqslant N (n \neq \infty)$, $E_{\sigma,\tau}(\bar{x}_n) \geqslant v_\infty - \epsilon$.

In section 2 we present the new short proof of the main theorem, and in section 3 a simplified proof of the ϵ-optimality of our original strategy is given. In section 4 we present sufficient conditions on stochastic games with infinite sets of states and actions to have a value. For instance if payoffs are uniformly bounded, the value v_λ of the λ-discounted games exists, and for every $\epsilon > 0$ there exists a sequence $0 < x_i \leqslant 1$ converging to zero and such that $x_{i+1} \geqslant (1-\epsilon)\, x_i$ and

$$\Sigma \, \| v_{x_{i+1}} - v_{x_i} \| < \infty$$

where $\| \;\; \|$ denotes the supremum norm (over the state space) then the undiscounted stochastic game has a value.

2. The Proof

In what follows we assume a fixed stochastic game. We will define a sequence $(\lambda_i)_{i=1}$ so that λ_i is a function of the past history, i.e., measurable with respect to the

σ-algebra F_i of all events preceding time i (including the choice of a new state z_i after time $i-1$). The $(\lambda_i)_{i=1}^{\infty}$ strategy of player I is to play on time i an optimal strategy in the λ_i-discounted game. For such a strategy, $v_{\lambda_i}(z_i) \leqslant E(\lambda_i x_i + (1-\lambda_i) v_{\lambda_i}(z_{i+1}) | F_i)$, i.e.,

$$E(v_{\lambda_i}(z_{i+1}) - v_{\lambda_i}(z_i) + \lambda_i (x_i - v_{\lambda_i}(z_{i+1})) \mid F_i) \geqslant 0. \tag{2.1}$$

The basic result of *Bewley/Kohlberg* [1976, Theorem 3.4] implies the existence of $0 < \lambda_0 < 1, 0 < r < 1, B > 0$, so that for any state z, $v_\lambda(z)$ is differentiable on $(0, \lambda_0]$ and $|(dv_\lambda(z))/(d\lambda)| \leqslant B\lambda^{-r}$ for $0 < \lambda \leqslant \lambda_0$. On $(\lambda_0, 1]$, v_λ is Lipschitz ($\| v_\lambda - v_\eta \| \leqslant (|\lambda - \eta| / \lambda) 2A$, e.g. by lemma 4.2), where $\| \ \|$ denotes the maximum norm on $\mathbf{R}^S$, S the state space, and A denotes the largest absolute value of payoffs appearing in the game matrices). Therefore, there exists a positive integrable function ψ: $(0, 1] \to \mathbf{R}_+$ such that for $0 < \lambda < \bar{\lambda} \leqslant 1$,

$$\| v_\lambda - v_{\bar{\lambda}} \| \leqslant \int_\lambda^{\bar{\lambda}} \psi(x)\, dx. \tag{2.2}$$

Let $\epsilon > 0$ be given. It suffices to consider $0 < \epsilon < A$. Define

$$s(y) = \frac{12A}{\epsilon} \int_y^1 \frac{\psi(x)}{x} dx + y^{-1/2}, 0 < y \leqslant 1.$$

Observe that $s(y)$ is a strictly decreasing continuous function from $(0, 1]$ onto $[1, \infty)$ so that an inverse continuous function λ: $[1, \infty) \to (0,1]$ exists. Our strategy will depend on the function λ and on two additional constants M, s_1 ($s_1 \geqslant M \geqslant 1$) sufficiently large to satisfy further requirements that will be specified later.

We now define inductively:

$$\lambda_i = \lambda(s_i)$$

$$s_{i+1} = \text{Max}\,[M, s_i + x_i - v_{\lambda_i}(z_{i+1}) + 4\epsilon].$$

We first observe the following:

$$| s_{i+1} - s_i | \leqslant 6A \tag{2.3}$$

$$x_i - v_{\lambda_i}(z_{i+1}) + 4\epsilon \leqslant s_{i+1} - s_i \leqslant x_i - v_{\lambda_i}(z_{i+1}) + 4\epsilon + 2A\, I(s_{i+1} = M) \tag{2.4}$$

where I denotes the indicator function.

Since $s((1-\epsilon)\lambda) - s(\lambda) \geqslant [(1-\epsilon)^{-1/2} - 1]\lambda^{-1/2} \to \infty$ as $\lambda \to 0$, (2.3) implies that there is $M_0 \geqslant 1$ such that for $M \geqslant M_0$

$$| \lambda_{i+1} - \lambda_i | \leqslant \epsilon \lambda_i / (6A). \tag{2.5}$$

Thus for $M \geqslant M_0$, by using (2.3), the definition of s, (2.5) and (2.2),

$$6A \geqslant | s_{i+1} - s_i | \geqslant \frac{12A}{\epsilon} \left| \int_{\lambda_i}^{\lambda_{i+1}} \frac{\psi(x)}{x} dx \right| \geqslant \frac{12A}{\epsilon} \frac{1}{2\lambda_i} \left| \int_{\lambda_i}^{\lambda_{i+1}} \psi(x)\, dx \right| \geqslant$$

$$\geqslant \frac{6A}{\epsilon\lambda_i} \| v_{\lambda_{i+1}} - v_{\lambda_i} \|.$$

Hence

$$\| v_{\lambda_i} - v_{\lambda_{i+1}} \| \leqslant \epsilon\, \lambda_i. \tag{2.6}$$

We now verify the integrability of λ on $(1, \infty)$.

$$1 + \int_1^\infty \lambda(s)\, ds = \int_0^1 s(y)\, dy = \int_0^1 y^{-1/2} dy + \frac{12A}{\epsilon} \int_0^1 \int_0^1 I(y \leqslant x \leqslant 1) \frac{\psi(x)}{x} dy\, dx =$$

$$= \int_0^1 y^{-1/2}\, dy + \frac{12A}{\epsilon} \int_0^1 \psi(x)\, dx < \infty.$$

Define t: $[1, \infty) \to \mathbf{R}_+$ by $t(s) = \int_s^\infty \lambda(y)\, dy$, $t_i = t(s_i)$. Observe that t is differentiable, $t'(s) = -\lambda(s)$ and that $t(s)$ decreases to 0 as s goes to ∞. Using the mean value theorem together with (2.3) and (2.5),

$$t_i - t_{i+1} \geqslant \lambda_i (s_{i+1} - s_i) - \epsilon\, \lambda_i. \tag{2.7}$$

Replacing in (2.1), $v_{\lambda_i}(z_{i+1})$ by $v_{\lambda_{i+1}}(z_{i+1})$, $\lambda_i (x_i - v_{\lambda_i}(z_{i+1}))$ by $\lambda_i (s_{i+1} - s_i) - 4\epsilon\lambda_i$, while using the left inequality of (2.4), and (2.6), we have for sufficiently large M,

$$E(v_{\lambda_{i+1}}(z_{i+1}) - v_{\lambda_i}(z_i) + \lambda_i (s_{i+1} - s_i) \mid F_i) \geqslant 3\epsilon\lambda_i. \tag{2.8}$$

Applying (2.7), $E(v_{\lambda_{i+1}}(z_{i+1}) - v_{\lambda_i}(z_i) - t_{i+1} + t_i \mid F_i) \geqslant 2\epsilon\lambda_i$, or in other words, letting $Y_i = v_{\lambda_i}(z_i) - t_i$,

$$E(Y_{i+1} - Y_i \mid F_i) \geqslant 2\epsilon\lambda_i. \tag{2.9}$$

Since $\lambda_i \geqslant 0$, (2.9) means that Y_i is a submartingale. Obviously Y_i is bounded and thus it converges a.s., say to Y_∞, with $E(Y_\infty \mid F_1) > Y_1$. It follows also that for sufficiently large M,

$$4A \geqslant 2\,t(M) + 2A \geqslant E(Y_k - Y_1) \geqslant 2\epsilon E(\sum_{1 \leqslant i < k} \lambda_i), \tag{2.10}$$

so that by the monotone convergence theorem

$$E(\sum_{i<\infty} \lambda_i) \leqslant 2A/\epsilon, \quad \text{and} \tag{2.11}$$

$$E(\#\{i \mid \lambda_i \geqslant \eta\}) \leqslant \frac{2A}{\epsilon\eta}. \tag{2.12}$$

So that a.s., $\lambda_i \to 0$, $s_i \to \infty$, $t_i \to 0$, and therefore

$$v_{\lambda_i}(z_i) = Y_i + t_i \to Y_\infty \text{ a.s. and thus by (2.6) if } \int_0^{\lambda(M)} \psi(x)\,dx \leqslant \epsilon,$$

$$v_{\lambda_i}(z_{i+1}) \to Y_\infty \text{ with } E(Y_\infty \mid F_1) > Y_1 \geqslant v_\infty(z_1) - \epsilon - t_1. \tag{2.13}$$

Also by (2.10) and (2.6) for every i,

$$E(v_{\lambda_i}(z_{i+1})) \geqslant v_\infty(z_1) - 2\epsilon - t_1. \tag{2.14}$$

Now, summing the right hand inequalities of (2.4) for $1 \leqslant i < n$, we have

$$\sum_{i<n} x_i \geqslant \sum_{i<n} v_{\lambda_i}(z_{i+1}) + s_n - s_1 - 2A \sum_{i<n} I(s_{i+1} = M) - 4n\epsilon. \tag{2.15}$$

The result follows now by bounding the terms in (2.15) using (2.14), (2.13) and (2.12) and observing that for M sufficiently large t is sufficiently small.

3. Other ϵ-Optimal Strategies

Denote by A four times the largest absolute value of payoffs, and let $\epsilon > 0$. Assume without loss of generality $\epsilon \leqslant A$, and let $\delta = \epsilon/(12A)$. In what follows we denote $v(z, \lambda) = v_\lambda(z)$, $v(z, 0) = v_\infty(z)$.

Take two functions $L(s)$ and $\lambda(s)$ of a real variable s, with values in the positive integers for L and in $(0, 1]$ for λ. Choose them such that $\lambda(s)$ is decreasing and such that, for all s sufficiently large and all θ with $|\theta| \leqslant A$, and for every state z:

$$\frac{AL(s)}{s} \leqslant \delta \tag{i}$$

$$\left| \frac{\lambda(s + \theta L(s))}{\lambda(s)} - 1 \right| \leqslant \delta \tag{ii}$$

$$|v(z, \lambda(s+\theta L(s))) - v(z, \lambda(s))| \leqslant \delta A L(s) \lambda(s) \quad \text{(iii)}$$

$$\int_0^\infty \lambda(s)\, ds < +\infty. \quad \text{(iv)}$$

For instance, for a finite stochastic game where – using the *Bewley/Kohlberg* result [1976, Th. 3.4] – $|v(z, \lambda) - v(z, 0)| \leqslant B\lambda^{1-r}$, with $0 \leqslant r < 1$, one could take $\lambda(s) = s^{-\beta}$, $L(s)$ the minimal integer exceeding $2(A\delta)^{-1} B(\lambda(s))^{-r}$ – where $\beta > 1$ such that $\beta r < 1$ – (this yields the strategy used in *Mertens/Neyman* [1980]).

Or alternatively, as in the previous proof, $L(s) = 1$, $\lambda(s) = \eta s^{-1/r}$ for η sufficiently small – or still: $L(s) = 1$, $\lambda(s) = 1/(s \ln^2 s)$.

Our strategy will also depend on two constants M and s_0 ($s_0 \geqslant M$ – one can always choose $s_0 = M$) sufficiently large such as to satisfy further requirements. To begin with, M will be assumed such that

$$\forall s \geqslant M, \text{(i)} - \text{(iv) hold and } |v(z, \lambda(s)) - v_\infty(z)| \leqslant A\delta. \quad (3.0)$$

Define now inductively, using x_i (resp. z_i) for the payoff (resp. state) at stage i ($i = 1,2,\ldots$), starting with $s_0 \geqslant M$:

$$\lambda_k = \lambda(s_k), L_k = L(s_k);$$

$$B_0 = 1, B_{k+1} = B_k + L_k;$$

$$s_{k+1} = \text{Max}\,[M, s_k + \sum_{B_k \leqslant i < B_{k+1}} (x_i - v_\infty(z_{B_{k+1}}) + \epsilon/2)].$$

The strategy is to start playing from time B_k up to time B_{k+1} *a* ($\delta A L_k \lambda_k$-) optimal strategy in the λ_k-discounted game.

We denote by F_i ($i = 1, \ldots, \infty$) the σ-field of all events preceding stage i (including the choice of the state z_i) and let $G_k = F_{B_k}$.

Observe that

$$|s_{k+1} - s_k| \leqslant A L_k \quad (3.1)$$

and that, λ being decreasing and integrable,

$$\lim_{s \to \infty} s\lambda(s) = 0,$$

and therefore by (i)

$$\lim_{s \to \infty} \lambda(s) L(s) = 0.$$

In particular, assuming M sufficiently large,

$$\lambda(s) L(s) \leqslant \delta \text{ for } s \geqslant M. \tag{3.2}$$

Let $l_k = v(z_{B_k}, \lambda_k)$, and note that by (iii)

$$|v(z_{B_{k+1}}, \lambda_k) - l_{k+1}| \leqslant \delta A L_k \lambda_k. \tag{3.3}$$

Lemma 3.4.

$$E(l_{k+1} - l_k + \lambda_k(s_{k+1} - s_k) \mid G_k) \geqslant 2\delta A L_k \lambda_k.$$

Proof. Assume without loss of generality $k = 0$, and write λ, L, $v^1(\lambda)$ for λ_0, L_0, $v(Z_{B_1}, \lambda_0)$ respectively. Then, by the $\delta AL\lambda$-optimality,

$$l_0 \leqslant E(\lambda \sum_{i<L} (1-\lambda)^i x_{i+1} + (1-\lambda)^L v^1(\lambda)) + \delta A L \lambda$$

or

$$(\text{as } 1 - \lambda \sum_{i<L} (1-\lambda)^i = (1-\lambda)^L)$$

$$E(v^1(\lambda) - l_0 + \lambda \sum_{i<L} (1-\lambda)^i (x_{i+1} - v^1(\lambda)) \geqslant -\delta A L \lambda.$$

Using (3.3), $1 - \lambda L \leqslant (1-\lambda)^i \leqslant 1$ for $i < L$, (3.2) and (3.0) we get

$$E(l_1 - l_0 + \lambda \sum_{i<L} (x_{i+1} - v^1(0))) \geqslant -4\delta A L \lambda,$$

and therefore by the inequality $s_1 - s_0 \geqslant \sum_{i<L} (x_{i+1} - v^1(0) + 6\delta A)$ we have:

$$E(l_1 - l_0 + \lambda(s_1 - s_0)) \geqslant 2\delta A L \lambda.$$

Let now $t(s) = \int_s^\infty \lambda(x) dx$, $t_k = t(s_k)$. Observe that $t(s)$ decreases to zero (by iv), in particular we can assume $t(M) \leqslant \delta A$.

Lemma 3.5.

$$E[(l_{k+1} - t_{k+1}) - (l_k - t_k) \mid G_k] \geqslant \delta A L_k \lambda_k.$$

Proof. Using (ii) and (3.1):

$$t_{k+1} - t_k = \int_{s_{k+1}}^{s_k} \lambda(s) ds \leqslant \lambda_k(s_k - s_{k+1}) + \delta A L_k \lambda_k$$

and the result follows from lemma 3.4.

Let $k(i)$ be the $(G_k)_{k=0}^{\infty}$-stopping time $\inf\{k \mid B_k > i\}$, and let $\bar{\lambda}_i = \lambda_{k(i)-1}$, $\bar{l}_i = v_\infty(z_{B_{k(i)}})$. ($\bar{\lambda}_i$ is the discount rate "used at stage i".)

Proposition 3.6.

a) l_k converges a.s., say to l_∞; and s_k converges a.s. to $+\infty$.

b) For any $(G_k)_{k=0}^{\infty}$-stopping time T,

$$E(l_T \mid G_0) \geqslant l_0 - t_0 \ (\geqslant l_0 - \delta A)$$

c) $E(\sum_{i=1}^{\infty} \bar{\lambda}_i) \leqslant \delta^{-1}$.

Note in particular the following consequences (using (3.0) and the uniform convergence of v_λ to v_∞ (cfr. section 4)):

$$\bar{l}_i \text{ converges a.s. to } \bar{l}_\infty = l_\infty$$

$$\forall i = 0, 1, 2, \dots, \infty,\ E(\bar{l}_i \mid G_0) \geqslant \bar{l}_0 - 3\delta A \ (= v_\infty(z_1) - 3\delta A)$$

$$E \sum_k I(s_k = M) \leqslant 1/(\delta\lambda(M)).$$

Proof. By lemma 3.5, $Y_k = l_k - t_k$ is a bounded (by A) submartingale, and therefore converges a.s. to Y_∞.

Since $Y_k - Y_0 \leqslant A$, the same lemma implies further

$$A \geqslant E(Y_k - Y_0) \geqslant \delta A\, E(\sum_{i<B_k} \bar{\lambda}_i).$$

c) Follows now by the monotone convergence theorem, and implies in particular that a.s. $\lambda_k \to 0$, thus $s_k \to \infty$, thus $t_k \to 0$ and therefore $l_k = Y_k + t_k$ converges a.s. to $l_\infty = Y_\infty$.

The stopping theorem for bounded submartingales implies now $E(l_T \mid G_0) \geqslant E(Y_T \mid G_0) \geqslant Y_0 = l_0 - t_0$, which completes the proof.

Lemma 3.7.

$$\sum_1^n x_i \geqslant \sum_1^n \bar{l}_i - 2s_0 - 8\delta A n - \delta M \sum_{k=1}^{\infty} I(s_k = M).$$

Proof. The definition of s_k implies that

$$s_{k+1} - s_k \leqslant \sum_{B_k \leqslant i < B_{k+1}} (x_i - \bar{l}_i) + 6\delta A L_k + I(s_{k+1} = M) A L_k / 2.$$

Since $AL(s)/s \leqslant \delta$, (3.1) implies that, when $s_{k+1} = M$, $AL_k \leqslant \delta M/(1-\delta)$;

$$s_{k+1} - s_k \leqslant \sum_{B_k \leqslant i < B_{k+1}} (x_i - \bar{l}_i) + 6\delta AL_k + \delta M I(s_{k+1} = M).$$

By summing

$$s_k - s_0 \leqslant \sum_{i < B_k} (x_i - \bar{l}_i) + 6\delta\, AB_k + \delta M \sum_{l=1}^{\infty} I(s_l = M),$$

and thus

$$\sum_1^n x_i \geqslant s_{k(n)} - s_0 - A(B_{k(n)} - n) + \sum_1^n \bar{l}_i - 6\delta\, AB_{k(n)} - \delta M \sum_{k=1}^{\infty} I(s_k = M)$$

$$\geqslant \sum_1^n \bar{l}_i - s_0 - 6\delta\, An - 2A(B_{k(n)} - n) - \delta M \sum_k I(s_k = M).$$

But $B_{k(n)} - n \leqslant L(s_{k(n)-1}) \leqslant A^{-1}\delta\, s_{k(n)-1} \leqslant \delta(A^{-1}s_0 + n)$, so the result follows.

The ϵ-optimality of the strategy follows now immediately by bounding the terms in lemma 3.7 using the consequences of proposition 3.6.

4. Infinite Stochastic Games

The finiteness hypothesis on the state space and the action sets we made in the definition of stochastic games are by no means necessary: the only thing required by our proof is

(1) that payoffs are uniformly bounded
(2) that the value v_λ of the λ-discounted games exists, and
(3) that for any $\delta > 0$ one can find functions $\lambda(s)$ and $L(s)$

satisfying the conditions of section 3, i.e. for s sufficiently large the following hold (all $|\theta| \leqslant A$);

a) $\lambda(s)$ has values in (0,1] and $L(s)$ in 1, 2, 3, . . .
b) $\lambda(s)$ is monotone and integrable
c) $\dfrac{AL(s)}{s} \leqslant \delta$
d) $\left|\dfrac{\lambda(s+\theta L(s))}{\lambda(s)} - 1\right| \leqslant \delta$
e) $|v_z[\lambda(s+\theta L(s))] - v_z[\lambda(s)]| \leqslant \delta\, AL(s)\lambda(s)$.

In this section we formulate (3) explicitly as a condition on v only.

Note that the monotone character of $\lambda(s)$ was introduced in section 3 as a matter of convenience, and that we used it only to guarantee such easy properties as $\lim_{s\to\infty} s\lambda(s) = 0$. For this reason we will show at the same time that this convencience was no restriction.

Theorem 4.1. If

(1) payoffs are uniformly bounded
(2) the values $v_\lambda(z)$ of the λ-discounted games exists
(3*) $\forall \alpha < 1$ there exists a sequence λ_i $(0 < \lambda_i \leqslant 1)$ such that $\lambda_{i+1} \geqslant \alpha\lambda_i$, $\lim_{i\to\infty} \lambda_i = 0$ and $\sum_i \sup_z | v_{\lambda_i}(z) - v_{\lambda_{i+1}}(z) | < \infty$

then $v_\infty(z) = \lim_{\lambda\to 0} v_\lambda(z)$ exists (uniformly in z) and the undiscounted game has v_∞ as value, in the sense that

$\forall \epsilon > 0\ \exists \sigma$ (a strategy of player 1) $\exists N$: $\forall \tau$ (strategy of player 2) $\forall z$ (initial state), $\forall n = N, N+1, N+2, \ldots, \infty$, $E^z_{\sigma,\tau}(\bar{x}_n) \geqslant v_\infty(z) - \epsilon$

where $\bar{x}_n$ denotes the average payoff up to stage n, $\bar{x}_\infty = \lim_{n\to\infty} \inf \bar{x}_n$.

A dual statement holds interchanging the two players.

Remark: The third condition is for example obviously satisfied in each of the following cases:

- v_λ is of bounded variation – the variation being computed using the supremum norm $\| \cdot \|$ over the state space.
- For some sequence λ_i with $\lambda_i > 0$, $\inf \lambda_{i+1} / \lambda_i > 0 = \lim \lambda_i$ one has

  $$\Sigma \Delta v [\lambda_{i+1}, \lambda_i] < \infty$$

  where $\Delta v[x, y] = \sup \{ \| v_{\lambda_1}(\cdot) - v_{\lambda_2}(\cdot) \| \mid \lambda_i \in (0, 1], x \leqslant \lambda_i \leqslant y \}$ (and $\sup \phi = 0$).
- there exists a function $v_\infty(z)$ such that $\| v_\lambda - v_\infty \| / \lambda$ is integrable (indeed, this is equivalent to the integrability of $\| v_\lambda - v_\infty \|$ as a function of $\ln(\lambda)$, so that taking for λ_i the minimizer of $\| v_\lambda - v_\infty \|$ in the interval $i(1-\alpha) \leqslant -\ln\lambda \leqslant (i+1)(1-\alpha)$ yields the condition).

A property of v that is always valid is the following:

Lemma 4.2. If (1) and (2) hold then

$$| v_\lambda / \lambda - v_\eta / \eta | \leqslant A \mid \lambda^{-1} - \eta^{-1} \mid$$

where A is the upper bound of the absolute value of payoffs. In particular v_λ is Lipschitz in $\ln\lambda$ (and also $\| v_\lambda - v_\eta \| \leqslant 2A\lambda^{-1} \mid \lambda - \eta \mid$).

Proof. The difference of the payoff functions $| \Sigma (1-\lambda)^i x_i - \Sigma (1-\eta)^i x_i |$ is at most $A \mid \lambda^{-1} - \eta^{-1} \mid$ and thus the difference in the values $| v_\lambda / \lambda - v_\eta/\eta |$ is bounded by the same constant.

The theorem follows from the results of section 3 and the following (note that (4.7), together with lemma (4.2), implies the norm convergence of v_λ):

Proposition 4.3. The following conditions on v_λ are equivalent:

(4.4) For some $A > 0$, there exist for every $\delta > 0$ functions $L(s)$ and $\lambda(s)$ (not necessarily monotonic) satisfying conditions a) – e).

(4.5) There exist functions $L(s)$ and $\lambda(s)$ (strictly decreasing) that satisfy a) – e) whatever be $A > 0$ and $\delta > 0$.

(4.6) $\forall \alpha < 1$, there exists a sequence $0 < \lambda_i \leqslant 1$ converging to 0 and such that $\lambda_{i+1} \geqslant \alpha\lambda_i$ and

$$\sum_i \| v_{\lambda_{i+1}} - v_{\lambda_i} \| < \infty.$$

(4.7) There exists a strictly decreasing sequence λ_i such that $\lambda_0 = 1$, $\lim \lambda_i = 0$, $\lim \lambda_{i+1} / \lambda_i = 1$ and

$$\sum_i \| v_{\lambda_{i+1}} - v_{\lambda_i} \| < \infty.$$

Proof. We will prove (4.4) → (4.6) → (4.7) → (4.5) → (4.4). The last implication is obvious.

(4.4) → (4.6): Take $s_0 = 0$, $\lambda_i = \lambda(s_i)$, $s_{i+1} = s_i + L_i$ where $L_i = AL(s_i)$. Then $\| v_{\lambda_{i+1}} - v_{\lambda_i} \| \leqslant \delta\, \lambda_i L_i$ (by e)) and $\lambda_i L_i = \lambda_i (s_{i+1} - s_i) \leqslant (1-\delta)^{-1} \int_{s_i}^{s_{i+1}} \lambda(s)\, ds$ (by (d)). Thus $\Sigma \| v_{\lambda_{i+1}} - v_{\lambda_i} \| \leqslant \delta\, \Sigma \lambda_i L_i \leqslant \delta (1-\delta)^{-1}\, \Sigma \int_{s_i}^{s_{i+1}} \lambda(s)\, ds < \infty$. Since L_i is bounded away from 0, $\lambda_i \to 0$. Finally by d) $\lambda_{i+1} \geqslant \lambda_i (1-\delta)$.

(4.6) → (4.7): Remark first that there is no loss in assuming the sequence λ_i of (4.6) to be strictly decreasing: let $i_0 = 0$, $i_{k+1} = \inf\{i : \lambda_i < \lambda_{i_k}\}$ and $\tilde\lambda_k = \lambda_{i_k}$. Then $\tilde\lambda_k \to 0$, $\tilde\lambda_{k+1} \geqslant \epsilon\tilde\lambda_k$ and by the triangle inequality the variation on the subsequence $((\tilde\lambda_k)_{k=0}^{\infty})$ is at most the variation on the sequence. Observe also that there is no loss in assuming $\lambda_0 = 1$.

Thus, for every $n \geqslant 1$, there are strictly decreasing sequences $(x_i^n)_{i=0}^{\infty}$ with $x_0^n = 1$, $\ln(x_i^n / x_{i+1}^n) \leqslant 2^{-n}$, $x_i^n \to 0$ as $i \to \infty$, so that

$$\sum_i \| v_{x_{i+1}^n} - v_{x_i^n} \| < \infty.$$

Therefore, there is a sequence $(a_n)_{n=1}^{\infty}$ with $a_1 = 1$, $\ln(a_n/a_{n+1}) > 1$ so that

$$\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}\|v_{x_{i+1}^n}-v_{x_i^n}\|I(x_i^n\leqslant a_n)<\infty.$$

The decreasing sequence λ_i generated by the union of the nonempty sets $\Lambda_n=\{x_i^n\mid a_{n+1}<x_i^n\leqslant a_n\}$ does the job. Obviously $\lambda_0=1$, and $\lambda_i\to 0$. To verify the other conditions on (λ_i), let $\bar{a}_n=\min\{x_i^{n-1}\mid x_i^{n-1}>a_n\}$, $\underline{a}_n=\max\{x_i^n\mid x_i^n\leqslant a_n\}$. As $\ln\bar{a}_n/\underline{a}_n\leqslant\ln\bar{a}_n/a_n+\ln a_n/\underline{a}_n\leqslant 2^{-n+1}+2^{-n}$, $\lambda_{i+1}/\lambda_i\to 1$, and also $\|v_{\bar{a}_n}-v_{\underline{a}_n}\|\leqslant K(2^{-n+1}+2^{-n})$ (by lemma 4.2), which is summable, and thus

$$\Sigma\|v_{\lambda_{i+1}}-v_{\lambda_i}\|\leqslant\sum_{n=1}^{\infty}\sum_{x_i^n\leqslant a_n}\|v_{x_{i+1}^n}-v_{x_i^n}\|+\sum_{n=1}^{\infty}\|v_{\bar{a}_n}-v_{\underline{a}_n}\|<\infty.$$

(4.7) → (4.5): Let $(\lambda_i)_{i=1}^{\infty}$ be the sequence of (4.7), and define $\bar{v}(\lambda)$ by linear interpolation from the values $\bar{v}(\lambda_i)=v_{\lambda_i}$. Let $n\geqslant 1$, $y=n/(n+1)$, $l_i=\Delta\bar{v}[y^{i+1},y^i]$ (thus $l_i=0$ for $i<0$). Observe that $\Sigma l_i\leqslant\Sigma\|v_{\lambda_{k+1}}-v_{\lambda_k}\|<\infty$. Let also $\bar{l}_i=\sum_{|j|\leqslant 2}l_{i+j}$, $g(x)=2n[I(x\leqslant 1)+\bar{l}_i y^{-i}]$ for $y^{i+1}<x\leqslant y^i$, and let h be defined by linear interpolation from the values $h(y^i)=n\sum_{j<i}[g(y^{j+1})+g(y^j)+g(y^{j-1})]$. Then $g\geqslant n$ on $(0,1]$ and h is continuous, decreasing, $\geqslant n\,g$ and integrable (e.g.
$\int h\,dx\leqslant\Sigma h(y^i)(y^{i-1}-y^i)=n(y^{-1}-1)\sum_{j<i}y^i[g(y^{j+1})+g(y^j)+g(y^{j-1})]$
$=n\sum_j y^j[g(y^{j+1})+g(y^j)+g(y^{j-1})]\leqslant\frac{3n}{y}\sum_j y^j g(y^j)=\frac{6n^2}{y}[1/(1-y)+\Sigma\bar{l}_i]<\infty.$)
Further, for $y^{i+1}<x\leqslant y^i$, $\Delta\bar{v}[xy^2,xy^{-2}]\leqslant\Delta\bar{v}[y^{i+3},y^{i-2}]\leqslant\bar{l}_i\leqslant(1/2n)y^i g(y^i)\leqslant$ $\leqslant(1/n)x\;g(x)$ (recall that $y\geqslant 1/2$).
Since on the interval $(y^{i+1},y^i]$, $h(xy^k)$ is linear (in x), and $g(x)=g(y^i)$, the equality $h(y^{i+1})=h(y^i)+n[g(y^{i+1})+g(y^i)+g(y^{i-1})]$ implies that

$$h(x)+n\,g(x)\leqslant h(xy)\quad\text{and}\quad h(x)-n\,g(x)\geqslant h(xy^{-1})$$

hold in every interval $(y^{i+1},y^i]$. Altogether we found for each $n\geqslant 1$ a pair of functions on $(0,\infty)$, h_n and g_n (the integer part of g), such that g_n is integer valued, and $\geqslant n$ on $(0,1]$, while h_n is continuous, decreasing, $\geqslant n\,g_n$, integrable, and such that letting $y_n=n/(n+1)$

$$\Delta\bar{v}[x\,y_n^2,x\,y_n^{-2}]\leqslant\frac{1}{n}x\,g_n(x)$$

and

$$h_n(x)+n\,g_n(x)\leqslant h_n(x\,y_n),\;\;h_n(x)-n\,g_n(x)\geqslant h_n(x\,y_n^{-1}).$$

Take a sequence $(a_n)_{n=1}^{\infty}$ satisfying $1/2 \geqslant a_n > a_{n+1} > 0$,

$$\inf\{\lambda_i/\lambda_{i-1} \mid \lambda_i \leqslant 2a_n\} \geqslant y_n,$$

and

$$\sum_{1\leqslant n<\infty} \int_0^{a_n} h_n(x)\,dx < \infty; \text{ and set } \bar{h}_n(x) = h_n(x)\,(2 - x/a_n)^+.$$

Then $\bar{h}(x) = \sum_1^{\infty} \bar{h}_n(x)$ is continuous, decreasing (strictly on $(0,2a_1]$) and integrable.

Thus, if we let $g_0 = 1, y_0 = 1/2$, and $n(x) = \#\{n \mid a_n \geqslant x\}, \bar{g}(x) = g_{n(x)}(x)$, we have $\bar{g}(x) \geqslant n(x) \to \infty$ as $x \to 0$.

$$\Delta\bar{v}[xy^2_{n(x)}, xy^{-2}_{n(x)}] \leqslant \frac{1}{n(x)}\, x\,\bar{g}(x),$$

and

$$\bar{h}(x) - \bar{h}(xy^{-1}_{n(x)}) \geqslant \bar{h}_{n(x)}(x) - \bar{h}_{n(x)}(xy^{-1}_{n(x)}) \geqslant h_{n(x)}(x) - h_{n(x)}(xy^{-1}_{n(x)}) \geqslant$$

$$\geqslant n(x) g_{n(x)}(x) = n(x)\,\bar{g}(x) \text{ (thus } \bar{h}(x)/\bar{g}(x) \geqslant h(x) \to \infty \text{ as } x \to 0)$$

and similarly $\bar{h}(xy_{n(x)}) - \bar{h}(x) \geqslant n(x)\,\bar{g}(x)$.

Therefore for $x \leqslant a_N$, $\bar{h}(x) - N\bar{g}(x) \leqslant \bar{h}(z) \leqslant \bar{h}(x) + N\bar{g}(x)$ implies $z \in [xy_{n(x)}, xy^{-1}_{n(x)}]$,

$$\text{so if further } \lambda_{i+1} \leqslant z \leqslant \lambda_i, \text{ then } \lambda_i, \lambda_{i+1} \in [xy^2_{n(x)}, xy^{-2}_{n(x)}] \tag{*}$$

$$\lambda_{i+1} \leqslant xy^{-1}_{n(x)} \leqslant a_{n(x)}\, y^{-1}_{n(x)} \leqslant 2a_{n(x)}, \text{ and thus } \frac{\lambda_{i+1}}{\lambda_i} \geqslant y_{n(x)} \quad .$$

By our above results, $\bar{h}^{-1}(s)$ is well defined on R_+, and is continuous, strictly decreasing, and integrable.

Let $\lambda(s)$ be the closest λ_i to $\bar{h}^{-1}(s)$ (selected in a monótonic way), and $L(s) = \bar{g}(\bar{h}^{-1}(s))$. By (*), if we let $\bar{h}(x) = s$, then for all $|\theta| \leqslant n(x)$, $\lambda(s + \theta L(s)) \in [x\,y^2_{n(x)}, x\,y^{-2}_{n(x)}]$.

It follows now immediately that, whatever be $A > 0$ and $\delta > 0$, conditions a) to e) are satisfied for s sufficiently large.

Finally, make $\lambda(s)$ strictly decreasing – without changing $L(s)$ – using the continuity of v_λ at each λ_i (lemma 4.2). This finishes the proof.

One might wonder whether the discontinuities in the function $\lambda(s)$ are really necessary. Note however that our proof proved at the same time the following:

Proposition 4.8. The following conditions on v_λ are equivalent:

(4.9) For some $A > 0$ and $1 > \delta > 0$ there exist a function $L(s)$ and a continuous function $\lambda(s)$ (not necessarily monotonic) satisfying conditions a) to e);

(4.10) There exist functions $L(s)$ and $\lambda(s)$ (continuous and strictly decreasing) that satisfy conditions a) – e) whatever be $A > 0$, $\delta > 0$;

(4.11) There exists a sequence such that λ_i, $\liminf \lambda_{i+1}/\lambda_i > 0 = \lim \lambda_i$ and that $\sum_i \Delta\nu[\lambda_{i+1}, \lambda_i] < \infty$;

(4.12) For any sequence statisfying $\limsup \lambda_{i+1}/\lambda_i < 1$, $\lambda_i > 0$ one has $\sum_i \Delta\nu[\lambda_{i+1}, \lambda_i] < \infty$.

The proof is 4.9 ⇒ 4.11 ⇒ 4.12 ⇒ 4.10 ⇒ 4.9. Each of those is essentially the same as the corresponding step in the proof of proposition 4.3 – except for the implication 4.11 ⇒ 4.12, which is trivial: just note that there is a bounded number of terms in the sequence of 4.12 that can fall between any two successive terms of the sequence of 4.11 (first made decreasing). Also in the implication 4.12 ⇒ 4.10, the sequence λ_i is not used; one wokrs directly with the $\Delta\nu$'s, and uses 4.12 to guarantee that $\Sigma\, \Delta\nu\, [y^{i+1}, y^i] < \infty$. $\lambda(s)$ is defined simply as $\bar{h}^{-1}(s)$.

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