Information and Computation **178**, 499–514 (2002)
doi:10.1006/inco.2002.3157

Decidability for Left-Linear Growing Term Rewriting Systems[1]

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Published online August 20, 2002

A term rewriting system is called growing if each variable occurring on both the left-hand side and the right-hand side of a rewrite rule occurs at depth zero or one in the left-hand side. Jacquemard showed that the reachability and the sequentiality of linear (i.e., left-right-linear) growing term rewriting systems are decidable. In this paper we show that Jacquemard's result can be extended to left-linear growing rewriting systems that may have *right-nonlinear* rewrite rules. This implies that the reachability and the joinability of some class of right-linear term rewriting systems are decidable, which improves the results for right-ground term rewriting systems by Oyamaguchi. Our result extends the class of left-linear term rewriting systems having a decidable call-by-need normalizing strategy. Moreover, we prove that the termination property is decidable for almost orthogonal growing term rewriting systems.

Key Words: growing term rewriting system; reachability; sequentiality; joinability.

1. INTRODUCTION

The original idea of growing term rewriting systems (TRSs) was introduced by Jacquemard [15] for giving a better sufficient condition for sequential rewriting systems. A term rewriting system is called growing if each variable occurring on both the left-hand side and the right-hand side of a rewrite rule occurs at depth zero or one in the left-hand side. Jacquemard [15] proved the preservation of recognizability by linear growing term rewriting systems. By using this result, he showed that the reachability and the sequentiality of linear (i.e., left-right-linear) growing term rewriting systems are decidable. Jacquemard's result is a generalization of the decidable properties for linear shallow rewriting systems by Comon [2], in which each variable occurring on both the left-hand side and the right-hand side of a rewrite rule occurs at depth zero or one (this definition differs from the original one in Comon [2] but is essentially the same [7, 15]).

Similar decidable properties for monadic rewriting systems have been shown in [4, 10, 11, 16, 22]. Salomaa [22] showed that right-linear monadic rewriting systems preserve recognizability. A term rewriting system is called monadic if each left-hand side is a term of height at least one and each right-hand side is a term of height at most one. Coquidé *et al.* [4] proved the preservation of recognizability by linear semimonadic rewriting systems, in which each left-hand side is a term of height at least one and each variable in the right-hand side occurs at depth zero or one. Since a term rewriting system R is linear growing if the inverse system $\mathcal{R}^{-1}$ is linear semimonadic, the preservation of recognizability by Jacquemard [15] is a slight generalization of that by Coquidé *et al.* [4].

In this paper we extend Jacquemard's result to left-linear growing term rewriting systems that may have *right-nonlinear* rewrite rules. The key idea in our proof is to construct deterministic tree automata

[1] A preliminary version of this paper was presented at the 10th International Conference on Rewriting Techniques and Applications, Trento, July 1999. This work was partially supported by Grants 10139214 and 10680346 from Ministry of Education, Science and Culture of Japan.

0890-5401/02 $35.00

instead of the nondeterministic ones in Jacquemard [15]. The deterministic behavior of tree automata allows us to remove the right-linear restriction from growing term rewriting systems. This implies that the reachability and the joinability of a term rewriting system $\mathcal{R}$ are decidable if the inverse system $\mathcal{R}^{-1}$ is left-linear growing. This result extends the result by Oyamaguchi [20] that the reachability and the joinability of right-ground term rewriting systems are decidable.

Our result gives a better approximation of term rewriting systems, which extends the class of orthogonal term rewriting systems having a decidable call-by-need strategy [2, 7, 15]. Moreover, we prove that termination for almost orthogonal growing term rewriting systems is decidable. Our proof uses Gramlich's theorem [12] that a weakly innermost normalizing TRS $\mathcal{R}$ is terminating if every critical pair of $\mathcal{R}$ is a trivial overlay. Thus the decidability of termination is proven by showing that the set of all ground terms having normal forms by innermost reduction is recognized by a tree automaton for left-linear growing term rewriting systems.

This paper is organized as follows. Section 2 gives the definitions of term rewriting systems and tree automata. In Section 3, we show the recognizability concerning left-linear growing term rewriting systems. Using this result, Section 4 shows that the reachability and the joinability of right-linear term rewriting systems are decidable if their inverses are growing. In Section 5, we extend the class of orthogonal term rewriting systems having a decidable call-by-need strategy. Section 6 proves that termination for almost orthogonal growing term rewriting systems is decidable.

2. PRELIMINARIES

2.1. Term Rewriting Systems

We mainly follow the notation of [1, 6, 17]. Let $\mathcal{F}$ be a finite set of *function symbols* denoted by $f, g, h, \ldots$, and let $\mathcal{V}$ be a countably infinite set of *variables* denoted by $x, y, z, \ldots$, where $\mathcal{F} \cap \mathcal{V} = \phi$. The set of all *terms* built from $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{V})$. The set of variables occurring in a term t is denoted by $\mathcal{V}(t)$. Terms not containing variables are called *ground* terms. The set of all ground terms built from $\mathcal{F}$ is denoted by $\mathcal{T}(\mathcal{F})$. A term t is *linear* if every variable in t occurs only once in t. Identity of terms s and t is denoted by $s \equiv t$.

If p is a position in t then $t|_p$ denotes the *subterm* of t at p. A subterm s of t is *proper* if $s \not\equiv t$. We write $s \subset t$ to indicate that s is a proper subterm of t. $t[s]_p$ denotes the term obtained from t by replacing the subterm $t|_p$ with s. If t has an occurrence of some variable x then we write $x \in t$.

A *substitution* σ is a mapping from $\mathcal{V}$ into $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Substitutions are extended into homomorphisms from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ into $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We write $t\sigma$ instead of $\sigma(t)$. A term s is an *instance* of a term t if there exists a substitution σ such that $s \equiv t\sigma$.

A TRS $\mathcal{R}$ is a finite set of rewrite rules. A *rewrite rule* is a pair $\langle l, r\rangle$ of terms. (We do not assume the variable restriction that $l \notin \mathcal{V}$ and any variable in r also occurs in l.) We write $l \to r$ for $\langle l, r\rangle$. An instance of the left-hand side of a rewrite rule is a *redex*. The rewrite rules of a TRS $\mathcal{R}$ define a reduction relation $\to_{\mathcal{R}}$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ as follows: $t \to_{\mathcal{R}} s$ iff there exist a rewrite rule $l \to r \in \mathcal{R}$, a position p in t, and a substitution σ such that $t|_p \equiv l\sigma$ and $s \equiv t[r\sigma]_p$.

The transitive-reflexive closure of $\to_{\mathcal{R}}$ is denoted by $\xrightarrow{*}_{\mathcal{R}}$. The inverse relation of $\xrightarrow{*}_{\mathcal{R}}$ is denoted by $\xleftarrow{*}_{\mathcal{R}}$. A *normal form* is a term without redexes. We say that t *has a normal form* if $t \xrightarrow{*}_{\mathcal{R}} s$ for some normal form s. The set of all normal forms is denoted by $\mathrm{NF}_{\mathcal{R}}$. A TRS $\mathcal{R}$ is *terminating* (*strongly normalizing*) if there exists no infinite reduction sequence $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots$. A TRS $\mathcal{R}$ is *weakly normalizing* if every term has a normal form.

A rewrite rule $l \to r$ is *ground* (*linear*) if l and r are ground (linear). A rewrite rule $l \to r$ is *left-linear* (*right-linear*) if l (r) is linear. A TRS $\mathcal{R}$ is *ground* (*linear, left-linear, right-linear*) if every rewrite rule in $\mathcal{R}$ is ground (linear, left-linear, right-linear).

For a TRS $\mathcal{R}$, we define the inverse of $\mathcal{R}$ by $\mathcal{R}^{-1} = \{ r \to l \mid l \to r \in \mathcal{R} \}$. $\mathcal{R}^{-1}$ is also a TRS since we assume no restrictions on variables of rewrite rules.

Let $l \to r$ and $l' \to r'$ be two rules of $\mathcal{R}$. We assume that they are renamed to have no common variables. Suppose that p is a position of l such that $l|_p \notin \mathcal{V}$ and l' are unifiable with a most general unifier σ. Then the pair $\langle l[r']_p\sigma, r\sigma\rangle$ is called a *critical pair* of $\mathcal{R}$. If $l \to r$ and $l' \to r'$ are the same rule, then we do not consider the case $p = \varepsilon$. A critical pair $\langle l[r']_p\sigma, r\sigma\rangle$ with $p = \varepsilon$ is an *overlay*.

A critical pair $\langle t, s\rangle$ is *trivial* if $t \equiv s$. An *orthogonal* TRS is a left-liner TRS without critical pairs and whose rewrite rules satisfy the additional restriction that (i) the left-hand side is not a variable and (ii) variables occurring in the right-hand side occur also in the left-hand side. A left-linear TRS is *almost orthogonal* if all its critical pairs are trivial overlays and it satisfies the additional restriction on variables.

Note. In this paper, we regard pairs of terms as rewrite rules without the usual restrictions on variables, except for (almost) orthogonal TRSs. Hence the left-hand side of a rewrite rule may be a variable and the right-hand side of a rewrite rule may have a variable not occurring in the left-hand side. This is convenient for introducing the inverse of and approximations of TRSs later. Moreover, we consider rewriting on ground terms only. Replacing every variable in terms with a fresh constant, rewriting on nonground terms can be simulated by that on ground terms. Thus this restriction entails no loss of generality and would simplify matters.

2.2. Ω-Terms

Let $\mathcal{R}$ be a TRS. We add a new constant Ω to $\mathcal{F}$. Elements of $\mathcal{T}(\mathcal{F} \cup \{\Omega\}, \mathcal{V})$ are called *Ω-terms*. We say that an Ω-term t is a *normal form* if t contains neither redexes nor Ω's. Thus the set of all normal forms is denoted by $\mathrm{NF}_{\mathcal{R}} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$, which coincides with the set of normal forms of $\mathcal{R}$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. t_Ω denotes the Ω-term obtained from t by replacing all variables in t with Ω. The prefix ordering $\leq$ on $\mathcal{T}(\mathcal{F} \cup \{\Omega\}, \mathcal{V})$ is defined as follows:

- (i) $\Omega \leq t$ for all $t \in \mathcal{T}(\mathcal{F} \cup \{\Omega\}, \mathcal{V})$,
- (ii) $f(s_1, \ldots, s_n) \leq f(t_1, \ldots, t_n)$ if $s_i \leq t_i$ for any $1 \leq i \leq n$,
- (iii) $x \leq x$ for all $x \in \mathcal{V}$.

Two Ω-terms t and s are compatible, written $t \uparrow s$, if there exists an Ω-term r such that $t \leq r$ and $s \leq r$. In this case the least upper bound of t and s is denoted by $t \sqcup s$.

2.3. Tree Automata

A *tree automaton* is a tuple $\mathcal{A} = (\mathcal{F}, Q, Q^f, \Delta)$ where $\mathcal{F}$ is a finite set of function symbols, Q is a finite set of *states*, $Q^f \subseteq Q$ is a set of *final* states, and Δ is a set of ground rewrite rules of the form $f(q_1, \ldots, q_n) \to q$ or $q \to q'$ where $f \in \mathcal{F}, q_1, \ldots, q_n, q, q' \in Q$. The latter rules are called *ε-rules*. We use $\to_{\mathcal{A}}$ for the reduction relation $\to_\Delta$ on $\mathcal{T}(\mathcal{F} \cup Q)$. A term $t \in \mathcal{T}(\mathcal{F})$ is *accepted* by $\mathcal{A}$ if $t \stackrel{*}{\to}_{\mathcal{A}} q$ for some $q \in Q^f$. The tree language $L(\mathcal{A})$ recognized by $\mathcal{A}$ is the set of all terms accepted by $\mathcal{A}$. A set L is *recognizable* if there exists a tree automaton $\mathcal{A}$ such that $L = L(\mathcal{A})$. A tree automaton $\mathcal{A}$ is *deterministic* if there are neither ε-rules nor different rules with the same left-hand side. A tree automaton $\mathcal{A}$ is *complete* if there is at least one rule $f(q_1, \ldots, q_n) \to q$ in Δ for all $f \in \mathcal{F}$ and $q_1, \ldots, q_n \in Q$. The following properties of tree automata are well known [3, 8].

LEMMA 2.1. *The class of recognizable tree languages is closed under union, intersection, and complementation.*

LEMMA 2.2. *The emptiness problem for tree automata is decidable.*

3. LEFT-LINEAR GROWING TRSS

The definition of growing was given by Jacquemard in [15]. He showed that if $\mathcal{R}$ is a linear growing TRS then the set $\{t \in \mathcal{T}(\mathcal{F}) \mid \exists s \in L \;\; t \stackrel{*}{\to}_{\mathcal{R}} s\}$ is recognizable for every recognizable tree language L. In this section we improve this result by replacing *linear growing* (i.e., *left-right-linear)* with *left-linear growing*.

In the following definition, unlike Jacquemard, we do not assume the linearity for growing TRSs.

DEFINITION 3.1. A rewrite rule $l \to r$ is *growing* if all variables in $\mathcal{V}(l) \cap \mathcal{V}(r)$ occur at depth 0 or 1 in l. A TRS $\mathcal{R}$ is *growing* if every rewrite rule in $\mathcal{R}$ is growing.

EXAMPLE 3.1. Let

$$\mathcal{R} = \begin{cases} f(f(x, y), z) \to f(z, g(z)) \\ g(x) \to f(g(y), z). \end{cases}$$

Then $\mathcal{R}$ is growing. But the following $\mathcal{R}'$ is not growing.

$$\mathcal{R}' = \begin{cases} f(f(x, y), z) \to f(x, g(z)) \\ g(x) \to f(g(y), z). \end{cases}$$

Let R be a binary relation on a set A and let $B \subseteq A$. Then we define $R(B)$ as $\{y \in A \mid \exists x \in B\ (x, y) \in R\}$. Now, we are ready to prove our main result that if $\mathcal{R}$ is a left-linear growing TRS then the set $(\overset{*}{\leftarrow}_{\mathcal{R}})(L) = \{t \in \mathcal{T}(\mathcal{F}) \mid \exists s \in L\ t \overset{*}{\rightarrow}_{\mathcal{R}} s\}$ is recognizable for every recognizable tree language L.

Let $\mathcal{R}$ be a left-linear growing TRS and let L be a tree language recognized by $\mathcal{A}_L = (\mathcal{F}, Q_L, Q_L^f, \Delta_L)$. We now construct a tree automaton recognizing $(\overset{*}{\leftarrow}_{\mathcal{R}})(L)$ from $\mathcal{R}$ and $\mathcal{A}_L$. Let $\mathcal{L} = \{l \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \mid l \notin \mathcal{V},\ f(\ldots, l, \ldots) \to r \in \mathcal{R}\}$. Then every term in $\mathcal{L}$ is linear because of the left-linearity of $\mathcal{R}$. Since the set of all ground instances of a linear term is recognizable [3, 8], we have an automaton $\mathcal{A}_l = (\mathcal{F}, Q_l, Q_l^f, \Delta_l)$ with $L(\mathcal{A}_l) = \{l\sigma \mid \sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F})\}$ for each $l \in \mathcal{L}$. Without loss of generality, we assume $Q_a \cap Q_b = \phi$ for any $a, b \in \{L\} \cup \mathcal{L}$ with $a \neq b$. The tree automaton $\mathcal{A}_\cup = (\mathcal{F}, Q_\cup, Q_\cup^f, \Delta_\cup)$ is defined by $Q_\cup = \bigcup_{l \in \mathcal{L}} Q_l \cup Q_L$, $Q_\cup^f = Q_L^f$ and $\Delta_\cup = \bigcup_{l \in \mathcal{L}} \Delta_l \cup \Delta_L$.

Starting from $\mathcal{A}_0 = \mathcal{A}_\cup$, Jacquemard's method in [15] constructs *nondeterministic* tree automata $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$, which can define a *nondeterministic* tree automaton $\mathcal{A}_k$ as $\lim \mathcal{A}_i$ since the number of states is bounded. Then the obtained $\mathcal{A}_k$ accepts $(\overset{*}{\leftarrow}_{\mathcal{R}})(L)$ [15]. However, this method requires essentially not only the left-linearity but also the right-linearity and does not work for *left-linear* growing TRSs. Since the right-hand sides of rewrite rules of left-linear growing TRSs may have multiple occurrences of variables, a subterm in a redex can be duplicated through rewriting. However the *nondeterministic* tree automaton $\mathcal{A}_k$ does not guarantee to reduce the same duplicated subterm to the same state. Thus it cannot trace rewriting by non-right-linear rewrite rules.

The above observation naturally leads us to deterministic tree automata construction for tracing the behavior of left-linear growing TRSs. A naive construction method is to transform an induced nondeterministic automaton into the deterministic automaton at each step in Jacquemard's [15]. However, this method cannot guarantee $\lim \mathcal{A}_i$ because the transformation explodes the number of states; in fact, it requires exponentially many states at each step. To prevent this state of explosion we carefully construct a sequence of deterministic tree automata $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ as follows, using a fixed set $Q = 2^{Q_\cup}$ of states.

Let $\mathcal{A}_0 = (\mathcal{F}, Q, Q^f, \Delta_0)$ where $Q = 2^{Q_\cup}$, $Q^f = \{A \in Q \mid A \cap Q_\cup^f \neq \phi\}$, and Δ_0 contains the following rules:

$$f(A_1, \ldots, A_n) \to A$$
$$\text{if } A = \left\{q \in Q_\cup \,\middle|\, \exists q_1 \in A_1, \ldots, \exists q_n \in A_n\, f(q_1, \ldots, q_n) \overset{*}{\rightarrow}_{\mathcal{A}_\cup} q\right\}.$$

For $0 \leq i$, $\mathcal{A}_{i+1} = (\mathcal{F}, Q, Q^f, \Delta_{i+1})$ is obtained from $\mathcal{A}_i = (\mathcal{F}, Q, Q^f, \Delta_i)$ as follows:

If there exist $f(A_1, \ldots, A_n) \to A \in \Delta_i$, $l \to r \in \mathcal{R}$ and $A' \in Q$ satisfying the following Condition 1 or 2:

Condition 1.

1. $l \equiv f(l_1, \ldots, l_n)$,
2. for each $1 \leq j \leq n$, $l_j \notin \mathcal{V}$ implies $A_j \cap Q_{l_j}^f \neq \phi$,
3. there exists a substitution $\theta : \mathcal{V} \to Q$ such that
   (a) $r\theta \overset{*}{\rightarrow}_{\mathcal{A}_i} A'$,
   (b) for each $x \in r$, if $x \equiv l_j$ for some j then $x\theta = A_j$, otherwise $t \overset{*}{\rightarrow}_{\mathcal{A}_i} x\theta$ for some $t \in \mathcal{T}(\mathcal{F})$,

4. $A \subset A \cup A'$ (i.e., $A \cup A'$ properly includes A),

Condition 2.

1′. $l \in \mathcal{V}$,

2′. there exists a substitution $\theta : \mathcal{V} \to Q$ such that

  (a') $r\theta \xrightarrow{*}_{\mathcal{A}_i} A'$,

  (b') for each $x \in r$, if $x \equiv l$ then $x\theta = A$, otherwise $t \xrightarrow{*}_{\mathcal{A}_i} x\theta$ for some $t \in \mathcal{T}(\mathcal{F})$,

3′. $A \subset A \cup A'$ (i.e., $A \cup A'$ properly includes A),

then $\Delta_{i+1} = (\Delta_i \backslash \{f(A_1, \ldots, A_n) \to A\}) \cup \{f(A_1, \ldots, A_n) \to A \cup A'\}$.

From (4) of Condition 1 and (3′) of Condition 2, it is clear that the process of construction eventually stops with the resulting automaton $\mathcal{A}_k = (\mathcal{F}, Q, Q^f, \Delta_k)$ when no rule is modified by replacing the right-hand side A with $A \cup A'$ such that $A \subset A \cup A' \subseteq Q$. Note that $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_k$ are deterministic and complete.

EXAMPLE 3.2. Let $\mathcal{F} = \{a, b, f, g\}$ and consider the left-linear growing TRS

$$\mathcal{R} = \begin{cases} f(x) \to g(x, x) \\ a \to b. \end{cases}$$

Let $L = \{g(a, b)\}$ and $\mathcal{A}_L = (\mathcal{F}, Q_L, Q_L^f, \Delta_L)$ where $Q_L = \{q_a, q_b, q_f\}$, $Q_L^f = \{q_f\}$ and $\Delta_L = \{a \to q_a, b \to q_b, g(q_a, q_b) \to q_f\}$. Then $f(\{q_a, q_b\}) \to \phi \in \Delta_0$, $f(x) \to g(x, x) \in \mathcal{R}$ and $\{q_f\} \in Q$ satisfy Condition 1 because we have $g(\{q_a, q_b\}, \{q_a, q_b\}) \to_{\mathcal{A}_0} \{q_f\}$. Thus we can first replace the right-hand side of $f(\{q_a, q_b\}) \to \phi$ with $\{q_f\}$. Next the right-hand side of $a \to \{q_a\} \in \Delta_1$ can be replaced with $\{q_a, q_b\}$. Consequently, we obtain $\Delta_k = \Delta_2$. The term $f(a)$ in $(\xleftarrow{*}_{\mathcal{R}})(L)$ is accepted by $\mathcal{A}_k$ because $f(a) \to_{\mathcal{A}_k} f(\{q_a, q_b\}) \to_{\mathcal{A}_k} \{q_f\} \in Q^f$.

EXAMPLE 3.3. Let $\mathcal{F} = \{f, g, a, b\}$ and consider the left-linear growing TRS

$$\mathcal{R} = \begin{cases} f(g(x), y) \to y \\ g(x) \to f(x, x) \\ a \to g(a). \end{cases}$$

Let $L = \{a\}$ and $\mathcal{A}_L = (\mathcal{F}, \{q_a\}, \{q_a\}, \{a \to q_a\})$. Then $\mathcal{L} = \{g(x)\}$ and we assume that the automaton $\mathcal{A}_{g(x)} = (\mathcal{F}, Q_{g(x)}, Q_{g(x)}^f, \Delta_{g(x)})$ is defined by $Q_{g(x)} = \{q_x, q_{g(x)}\}$, $Q_{g(x)}^f = \{q_{g(x)}\}$ and $\Delta_{g(x)} = \{a \to q_x,\ b \to q_x,\ f(q_x, q_x) \to q_x,\ g(q_x) \to q_x,\ g(q_x) \to q_{g(x)}\}$. We have the automaton $\mathcal{A}_0 = (\mathcal{F}, Q_0, Q_0^f, \Delta_0)$ where $Q_0 = 2^{\{q_a, q_x, q_{g(x)}\}}$, $Q_0^f = \{\{q_a\}, \{q_a, q_x\}, \{q_a, q_{g(x)}\}, \{q_a, q_x, q_{g(x)}\}\}$, and Δ_0 is the following set of rules:

$$\Delta_0 = \begin{cases} a \to \{q_a, q_x\} \\ b \to \{q_x\} \\ f(A_1, A_2) \to \{q_x\} & \text{if } q_x \in A_1 \text{ and } q_x \in A_2 \\ f(A_1, A_2) \to \phi & \text{if } q_x \notin A_1 \text{ or } q_x \notin A_2 \\ g(A) \to \{q_x, q_{g(x)}\} & \text{if } q_x \in A \\ g(A) \to \phi & \text{if } q_x \notin A. \end{cases}$$

We can see that $f(\{q_{g(x)}\}, \{q_{g(x)}\}) \to \phi \in \Delta_0$, $f(g(x), y) \to y \in \mathcal{R}$, and $\{q_{g(x)}\} \in Q_0$ satisfy Condition 1. Thus we first replace the right-hand side of the rule $f(\{q_{g(x)}\}, \{q_{g(x)}\}) \to \phi \in \Delta_0$ with $\{q_{g(x)}\}$. Then the right-hand side of the rule $g(\{q_{g(x)}\}) \to \phi \in \Delta_1$ can be replaced with $\{q_{g(x)}\}$ because we have

$f(\{q_{g(x)}\}, \{q_{g(x)}\}) \to_{\mathcal{A}_1} \{q_{g(x)}\}$. Consequently, Δ_k includes the following new rules:

$$\begin{cases} a \to \{q_a, q_x, q_{g(x)}\} \\ f(A_1, A_2) \to A_2 \quad \text{if } A_1 \in \big\{\{q_{g(x)}\},\ \{q_a, q_{g(x)}\}\big\} \quad \text{and} \\ \qquad\qquad\qquad\qquad A_2 \neq \phi \\ f(A_1, A_2) \to A_2 \quad \text{if } A_1 \in \big\{\{q_x, q_{g(x)}\},\ \{q_a, q_x, q_{g(x)}\}\big\} \quad \text{and} \\ \qquad\qquad\qquad\qquad A_2 \notin \{\phi,\ \{q_x\}\} \\ g(\{q_{g(x)}\}) \to \{q_{g(x)}\} \\ g(\{q_a, q_{g(x)}\}) \to \{q_a, q_{g(x)}\} \\ g(\{q_a, q_x, q_{g(x)}\}) \to \{q_a, q_x, q_{g(x)}\}. \end{cases}$$

Consider two terms $f(g(b), g(a)) \in (\overset{*}{\leftarrow}_{\mathcal{R}})(L)$ and $f(g(a), g(b)) \notin (\overset{*}{\leftarrow}_{\mathcal{R}})(L)$. We have

$$\begin{aligned} f(g(b), g(a)) &\overset{*}{\to}_{\mathcal{A}_k} f\big(g(\{q_x\}), g(\{q_a, q_x, q_{g(x)}\})\big) \\ &\overset{*}{\to}_{\mathcal{A}_k} f\big(\{q_x, q_{g(x)}\}, \{q_a, q_x, q_{g(x)}\}\big) \\ &\to_{\mathcal{A}_k} \{q_a, q_x, q_{g(x)}\} \in Q_k^f. \end{aligned}$$

Hence $f(g(b), g(a))$ is accepted by $\mathcal{A}_k$. The term $f(g(a), g(b))$ is not accepted by $\mathcal{A}_k$ because

$$\begin{aligned} f(g(a), g(b)) &\overset{*}{\to}_{\mathcal{A}_k} f\big(g(\{q_a, q_x, q_{g(x)}\}), g(\{q_x\})\big) \\ &\overset{*}{\to}_{\mathcal{A}_k} f\big(\{q_a, q_x, q_{g(x)}\}, \{q_x, q_{g(x)}\}\big) \\ &\to_{\mathcal{A}_k} \{q_x, q_{g(x)}\} \notin Q_k^f. \end{aligned}$$

Remark. Jacquemard's construction in [15] does not necessarily generate a tree automaton $\mathcal{A}$ such that $L(\mathcal{A}) = (\overset{*}{\leftarrow}_{\mathcal{R}})(L)$ for a *non-right-linear* TRS $\mathcal{R}$. Consider again the left-linear *non-right-linear* growing TRS $\mathcal{R}$ of Example 3.2:

$$\mathcal{R} = \begin{cases} f(x) \to g(x, x) \\ a \to b. \end{cases}$$

Let $L = \{g(a, b)\}$ and $\mathcal{A}_L = (\mathcal{F}, Q_L, Q_L^f, \Delta_L)$ where $Q_L = \{q_a,\ q_b,\ q_f\}$, $Q_L^f = \{q_f\}$, and $\Delta_L = \{a \to q_a,\ b \to q_b,\ g(q_a, q_b) \to q_f\}$. We add only the rule $a \to q_b$ to Δ_L at Jacquemard's construction process and hence we obtain the nondeterministic tree automaton $\mathcal{A} = (\mathcal{F}, Q_L, Q_L^f, \Delta_L \cup \{a \to q_b\})$. Note that the rule $f(q_a) \to q_f$ is not added to Δ_L because we do not have $g(q_a, q_a) \overset{*}{\to}_{\mathcal{A}} q_f$. Although we have $f(a) \overset{*}{\to}_{\mathcal{R}} g(a, b) \in L$, $f(a)$ is not accepted by $\mathcal{A}$. In order to accept $f(a)$ the automaton $\mathcal{A}$ needs to keep in a state the information that a can be reduced to both of q_a and q_b, but it is lost through nondeterministic behavior of $\mathcal{A}$.

In the following we prove that $L(\mathcal{A}_k) = (\overset{*}{\leftarrow}_{\mathcal{R}})(L)$. We write $t \overset{*}{\to}_{\mathcal{R}} \cdot \overset{*}{\to}_{\mathcal{A}} q$ if $t \overset{*}{\to}_{\mathcal{R}} s \overset{*}{\to}_{\mathcal{A}} q$ for some $s \in \mathcal{T}(\mathcal{F})$.

LEMMA 3.1. *Let $t \in \mathcal{T}(\mathcal{F}, \mathcal{V}), \theta : \mathcal{V} \to Q$ and $\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F})$ such that $x\sigma \overset{*}{\to}_{\mathcal{R}} \cdot \overset{*}{\to}_{\mathcal{A}_\cup} q'$ for any $x \in t$ and $q' \in x\theta$. For each $0 \le i \le k$, if $t\theta \overset{*}{\to}_{\mathcal{A}_i} A \in Q$ then $t\sigma \overset{*}{\to}_{\mathcal{R}} \cdot \overset{*}{\to}_{\mathcal{A}_\cup} q$ for any $q \in A$.*

Note. In the above claim the condition "$x\sigma \overset{*}{\to}_{\mathcal{R}} \cdot \overset{*}{\to}_{\mathcal{A}_\cup} q'$ for any $x \in t$ and $q' \in x\theta$" cannot be replaced with a simpler form "$x\sigma \overset{*}{\to}_{\mathcal{R}} \cdot \overset{*}{\to}_{\mathcal{A}_0} x\theta$ for any $x \in t$," because the first condition means

$\forall x \in t, \forall q' \in x\theta, \exists s \in \mathcal{T}(\mathcal{F})[x\sigma \xrightarrow{*}_{\mathcal{R}} s \xrightarrow{*}_{\mathcal{A}_\cup} q']$ but the second one means $\forall x \in t, \exists s \in \mathcal{T}(\mathcal{F}), \forall q' \in x\theta[x\sigma \xrightarrow{*}_{\mathcal{R}} s \xrightarrow{*}_{\mathcal{A}_\cup} q']$, which is different from the first one.

Proof. We prove the lemma by induction on i.

Base Step. We use induction on the structure of t. The case $t \equiv x$ is trivial. Let $t \equiv f(t_1, \dots, t_n)$. Assume $t\theta \equiv f(t_1, \dots, t_n)\theta \xrightarrow{*}_{\mathcal{A}_0} f(A_1, \dots, A_n) \rightarrow_{\mathcal{A}_0} A$. Let $q \in A$. Then by the definition of Δ_0 there exist $q_1 \in A_1, \dots, q_n \in A_n$ such that $f(q_1, \dots, q_n) \xrightarrow{*}_{\mathcal{A}_\cup} q$. By induction hypothesis, for each $1 \leq j \leq n$ there exists s_j such that $t_j\sigma \xrightarrow{*}_{\mathcal{R}} s_j \xrightarrow{*}_{\mathcal{A}_\cup} q_j$. Thus we have $t\sigma \equiv f(t_1\sigma, \dots, t_n\sigma) \xrightarrow{*}_{\mathcal{R}} f(s_1, \dots, s_n) \xrightarrow{*}_{\mathcal{A}_\cup} f(q_1, \dots, q_n) \xrightarrow{*}_{\mathcal{A}_\cup} q$.

Induction Step. Let $f(A_1, \dots, A_n) \rightarrow A' \in \Delta_i \backslash \Delta_{i-1}$. We use induction on the number m of reduction steps using this rule in the reduction $t\theta \xrightarrow{*}_{\mathcal{A}_i} A$. If $m = 0$ then $t\theta \xrightarrow{*}_{\mathcal{A}_{i-1}} A$. Thus it follows from induction hypothesis on i that $t\sigma \xrightarrow{*}_{\mathcal{R}} \cdot \xrightarrow{*}_{\mathcal{A}_\cup} q$ for any $q \in A$. Let $m > 0$. Suppose

$$t\theta \equiv t\theta[f(t_1, \dots, t_n)\theta]_p \xrightarrow{*}_{\mathcal{A}_{i-1}} t\theta[f(A_1, \dots, A_n)]_p \rightarrow_{\mathcal{A}_i} t\theta[A']_p \xrightarrow{*}_{\mathcal{A}_i} A.$$

Let $\tilde{t} \equiv t[z]_p$ where $z \notin t$. We define $\tilde{\theta} : \mathcal{V} \rightarrow Q$ and $\tilde{\sigma} : \mathcal{V} \rightarrow \mathcal{T}(\mathcal{F})$ as follows: if $x \equiv z$ then $x\tilde{\theta} = A'$ and $x\tilde{\sigma} \equiv f(t_1, \dots, t_n)\sigma$, otherwise $x\tilde{\theta} = x\theta$ and $x\tilde{\sigma} \equiv x\sigma$. Clearly $\tilde{t}\tilde{\theta} \equiv t\theta[A']_p$ and $\tilde{t}\tilde{\sigma} \equiv t\sigma$. We will show the following claim:

$$x\tilde{\sigma} \xrightarrow{*}_{\mathcal{R}} \cdot \xrightarrow{*}_{\mathcal{A}_\cup} q \quad \text{for any } x \in \tilde{t} \text{ and } q \in x\tilde{\theta}.$$

Then by applying induction hypothesis on m to $\tilde{t}\tilde{\theta} \equiv t\theta[A']_p \xrightarrow{*}_{\mathcal{A}_i} A$, we can obtain $\tilde{t}\tilde{\sigma} \equiv t\sigma \xrightarrow{*}_{\mathcal{R}} \xrightarrow{*}_{\mathcal{A}_\cup} q$ for any $q \in A$. Thus the lemma holds.

Proof of the Claim. Let $x \in \tilde{t}$. If $x \not\equiv z$ then it follows from the assumption of the lemma that $x\tilde{\sigma} \xrightarrow{*}_{\mathcal{R}} \cdot \xrightarrow{*}_{\mathcal{A}_\cup} q$ for any $q \in x\tilde{\theta}$. We consider the case $x \equiv z$. Assume that $f(A_1, \dots, A_n) \rightarrow A'_1 \in \Delta_{i-1}$, $l \rightarrow r \in \mathcal{R}$ and $A'_2 \in Q$ satisfy Condition 1 or 2 and $A' = A'_1 \cup A'_2$. Since $f(t_1, \dots, t_n)\theta \xrightarrow{*}_{\mathcal{A}_{i-1}} f(A_1, \dots, A_n) \rightarrow_{\mathcal{A}_{i-1}} A'_1$, it follows from induction hypothesis on i that

$$f(t_1, \dots, t_n)\sigma \xrightarrow{*}_{\mathcal{R}} \cdot \xrightarrow{*}_{\mathcal{A}_\cup} q \quad \text{for any } q \in A'_1. \tag{1}$$

We distinguish two cases.

Case 1. Condition 1 is satisfied. Let $l \equiv f(l_1, \dots, l_n)$. By applying induction hypothesis on i to $t_j\theta \xrightarrow{*}_{\mathcal{A}_{i-1}} A_j$ for $1 \leq j \leq n$, we obtain $t_j\sigma \xrightarrow{*}_{\mathcal{R}} \cdot \xrightarrow{*}_{\mathcal{A}_\cup} q$ for any $q \in A_j$. For each $1 \leq j \leq n$, let s_j be a term such that if $l_j \in \mathcal{V}$ then $s_j \equiv t_j\sigma$; otherwise $t_j\sigma \xrightarrow{*}_{\mathcal{R}} s_j \xrightarrow{*}_{\mathcal{A}_\cup} q \in Q^f_{l_j}$. From the disjointness of the sets of states, $s_j \xrightarrow{*}_{\mathcal{A}_\cup} q \in Q^f_{l_j}$ implies $s_j \xrightarrow{*}_{\mathcal{A}_{l_j}} q \in Q^f_{l_j}$. Hence $f(s_1, \dots, s_n)$ is an instance of l by the linearity of l. Let $\theta' : \mathcal{V} \rightarrow Q$ be a substitution defined by 3 of Condition 1. Let $\sigma' : \mathcal{V} \rightarrow \mathcal{T}(\mathcal{F})$ be a substitution such that for any $y \in r$ if $y \equiv l_j$ for some j then $y\sigma' \equiv s_j$, otherwise $y\sigma' \xrightarrow{*}_{\mathcal{A}_{i-1}} y\theta'$. Then from the growingness of $\mathcal{R}$ we have the reduction $f(s_1, \dots, s_n) \rightarrow_{\mathcal{R}} r\sigma'$. Furthermore, we can see $y\sigma' \xrightarrow{*}_{\mathcal{A}_{i-1}} y\theta'$ for any $y \in r$. Therefore, by induction hypothesis on i, $y\sigma' \xrightarrow{*}_{\mathcal{R}} \cdot \xrightarrow{*}_{\mathcal{A}_\cup} q$ for any $y \in r$ and $q \in y\theta'$. Applying induction hypothesis on i to $r\theta' \xrightarrow{*}_{\mathcal{A}_{i-1}} A'_2$, it is obtained that $r\sigma' \xrightarrow{*}_{\mathcal{R}} \cdot \xrightarrow{*}_{\mathcal{A}_\cup} q$ for any $q \in A'_2$. Thus, since $f(t_1, \dots, t_n)\sigma \xrightarrow{*}_{\mathcal{R}} r\sigma'$, we have

$$f(t_1, \dots, t_n)\sigma \xrightarrow{*}_{\mathcal{R}} \cdot \xrightarrow{*}_{\mathcal{A}_\cup} q \quad \text{for any } q \in A'_2. \tag{2}$$

Because $z\tilde{\theta} = A' = A'_1 \cup A'_2$ and $z\tilde{\sigma} \equiv f(t_1, \dots, t_n)\sigma$, it follows from (1) and (2) that $z\tilde{\sigma} \xrightarrow{*}_{\mathcal{R}} \cdot \xrightarrow{*}_{\mathcal{A}_\cup} q$ for any $q \in z\tilde{\theta}$. Therefore the claim holds.

Case 2. Condition 2 is satisfied. Let $\theta' : \mathcal{V} \rightarrow Q$ be a substitution defined by $2'$ of Condition 2. Let $\sigma' : \mathcal{V} \rightarrow \mathcal{T}(\mathcal{F})$ be a substitution such that for any $y \in r$ if $y \equiv l$ then $y\sigma' \equiv f(t_1, \dots, t_n)\sigma$, otherwise

$y\sigma' \xrightarrow{*}_{\mathcal{A}_{i-1}} y\theta'$. Using (1) and induction hypothesis on i, we obtain $y\sigma' \xrightarrow{*}_{\mathcal{R}} \cdot \xrightarrow{*}_{\mathcal{A}_\cup} q$ for any $y \in r$ and $q \in y\theta'$. Applying induction hypothesis on i to $r\theta' \xrightarrow{*}_{\mathcal{A}_{i-1}} A_2'$, it is obtained that $r\sigma' \xrightarrow{*}_{\mathcal{R}} \cdot \xrightarrow{*}_{\mathcal{A}_\cup} q$ for any $q \in A_2'$. Since $f(t_1, \ldots, t_n)\sigma \rightarrow_{\mathcal{R}} r\sigma'$,

$$f(t_1, \ldots, t_n)\sigma \xrightarrow{*}_{\mathcal{R}} \cdot \xrightarrow{*}_{\mathcal{A}_\cup} q \quad \text{for any } q \in A_2'. \tag{3}$$

Therefore, it follows from (1) and (3) that $z\tilde{\sigma} \xrightarrow{*}_{\mathcal{R}} \cdot \xrightarrow{*}_{\mathcal{A}_\cup} q$ for any $q \in z\tilde{\theta}$. Hence the claim holds. ■

LEMMA 3.2. $L(\mathcal{A}_k) \subseteq (\xleftarrow{*}_{\mathcal{R}})(L)$.

Proof. Let $t \in L(\mathcal{A}_k)$, i.e., $t \xrightarrow{*}_{\mathcal{A}_k} A$ for some $A \in Q^f$. By the definition of Q^f, A has a final state q of $\mathcal{A}_L$. From Lemma 3.1, there exists $s \in \mathcal{T}(\mathcal{F})$ such that $t \xrightarrow{*}_{\mathcal{R}} s \xrightarrow{*}_{\mathcal{A}_\cup} q$. By the disjointness of the sets of states, we have $s \xrightarrow{*}_{\mathcal{A}_L} q \in Q_L^f$. Thus $t \in (\xleftarrow{*}_{\mathcal{R}})(L)$. ■

LEMMA 3.3. *Let $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Let $\theta, \theta' : \mathcal{V} \rightarrow Q$ with $x\theta \subseteq x\theta'$ for any $x \in t$. If $t\theta \xrightarrow{*}_{\mathcal{A}_i} A \in Q$ then $t\theta' \xrightarrow{*}_{\mathcal{A}_k} A'$ for some $A' \in Q$ with $A \subseteq A'$.*

Proof. We prove the lemma by induction on i.

Base Step. We use the induction on the structure of t. The case $t \equiv x$ is trivial. Let $t \equiv f(t_1, \ldots, t_n)$. Then we assume $f(t_1, \ldots, t_n)\theta \xrightarrow{*}_{\mathcal{A}_0} f(A_1, \ldots, A_n) \rightarrow_{\mathcal{A}_0} A \in Q$. By induction hypothesis, for each $1 \leq j \leq n$ there exists $A_j' \in Q$ such that $t_j\theta' \xrightarrow{*}_{\mathcal{A}_k} A_j'$ and $A_j \subseteq A_j'$. By the definition of $\mathcal{A}_0$, Δ_0 has a rule $f(A_1', \ldots, A_n') \rightarrow A'$ with $A \subseteq A'$. Then by the construction of $\mathcal{A}_k$, Δ_k has a rule $f(A_1', \ldots, A_n') \rightarrow A''$ with $A' \subseteq A''$. Thus we obtain $f(t_1, \ldots, t_n)\theta' \xrightarrow{*}_{\mathcal{A}_k} f(A_1', \ldots, A_n') \rightarrow_{\mathcal{A}_k} A''$ and $A \subseteq A''$.

Induction Step. We use the induction on the structure of t. The case $t \equiv x$ is trivial. Let $t \equiv f(t_1, \ldots, t_n)$. Assume $f(t_1, \ldots, t_n)\theta \xrightarrow{*}_{\mathcal{A}_i} f(A_1, \ldots, A_n) \rightarrow_{\mathcal{A}_i} A \in Q$. By induction hypothesis on the structure of t, for each $1 \leq j \leq n$ there exists $A_j' \in Q$ such that $t_j\theta' \xrightarrow{*}_{\mathcal{A}_k} A_j'$ and $A_j \subseteq A_j'$. Since $\mathcal{A}_k$ is deterministic and complete, there exists exactly one $A' \in Q$ such that $f(A_1', \ldots, A_n') \rightarrow A' \in \Delta_k$. We will show $A \subseteq A'$. If $f(A_1, \ldots, A_n) \rightarrow A \in \Delta_{i-1}$ then from induction hypothesis on i it follows that $A \subseteq A'$. Otherwise, we assume that $f(A_1, \ldots, A_n) \rightarrow B_1 \in \Delta_{i-1}$, $l \rightarrow r \in \mathcal{R}$ and $B_2 \in Q$ satisfy Condition 1 or 2 and $A = B_1 \cup B_2$. From induction hypothesis on i, we get $B_1 \subseteq A'$. We distinguish two cases.

Case 1. Condition 1 is satisfied. Let $l \equiv f(l_1, \ldots, l_n)$ and let $\theta_1 : \mathcal{V} \rightarrow Q$ be a substitution defined by (3) of Condition 1. Then let θ_2 be a substitution from $\mathcal{V}$ to Q such that for every $x \in r$ if $x \equiv l_j$ then $x\theta_2 = A_j'$, otherwise $t \xrightarrow{*}_{\mathcal{A}_k} x\theta_2$ for some $t \in \mathcal{T}(\mathcal{F})$ with $t \xrightarrow{*}_{\mathcal{A}_{i-1}} x\theta_1$. Using induction hypothesis on i, we can show that $x\theta_1 \subseteq x\theta_2$ for every $x \in r$. Applying induction hypothesis on i to $r\theta_1 \xrightarrow{*}_{\mathcal{A}_{i-1}} B_2$, we obtain $r\theta_2 \xrightarrow{*}_{\mathcal{A}_k} B_2'$ for some $B_2' \in Q$ with $B_2 \subseteq B_2'$. Therefore $f(A_1', \ldots, A_n') \rightarrow A' \in \Delta_k$, $l \rightarrow r \in \mathcal{R}$ and $B_2' \in Q$ satisfy (1), (2), and (3) of Condition 1. By the construction of $\mathcal{A}_k$, they must not satisfy (4) of Condition 1. Thus we have $A' = A' \cup B_2'$. Hence $A = B_1 \cup B_2 \subseteq A' \cup B_2' = A'$.

Case 2. Condition 2 is satisfied. Let $\theta_1 : \mathcal{V} \rightarrow Q$ be a substitution defined by (2′) of Condition 2. Then let $\theta_2 : \mathcal{V} \rightarrow Q$ be a substitution such that for every $x \in r$ if $x \equiv l$ then $x\theta_2 = A'$, otherwise $t \xrightarrow{*}_{\mathcal{A}_k} x\theta_2$ for some $t \in \mathcal{T}(\mathcal{F})$ with $t \xrightarrow{*}_{\mathcal{A}_{i-1}} x\theta_1$. Using induction hypothesis on i, we can show that $y\theta_1 \subseteq y\theta_2$ for every $y \in r$. Applying induction hypothesis on i to $r\theta_1 \xrightarrow{*}_{\mathcal{A}_{i-1}} B_2$, we obtain $r\theta_2 \xrightarrow{*}_{\mathcal{A}_k} B_2'$ for some $B_2' \in Q$ with $B_2 \subseteq B_2'$. Thus $f(A_1', \ldots, A_n') \rightarrow A' \in \Delta_k$, $l \rightarrow r \in \mathcal{R}$ and $B_2' \in Q$ satisfy (1′) and (2′) of Condition 2. By the construction of $\mathcal{A}_k$ they must not satisfy (3′) of Condition 2, i.e., $A' = A' \cup B_2'$. Hence $A = B_1 \cup B_2 \subseteq A' \cup B_2' = A'$. ■

LEMMA 3.4. *Let $t \in \mathcal{T}(\mathcal{F})$ and $t \xrightarrow{*}_{\mathcal{A}_k} A \in Q$. If $t \xrightarrow{*}_{\mathcal{A}_\cup} q \in Q_\cup$ then $q \in A$.*

Proof. Since $\mathcal{A}_0$ is complete, there exists $A' \in Q$ such that $t \xrightarrow{*}_{\mathcal{A}_0} A'$. By induction of the structure of t, we can show that $A' = \{q \in Q_\cup \mid t \xrightarrow{*}_{\mathcal{A}_\cup} q\}$. Thus, if $t \xrightarrow{*}_{\mathcal{A}_\cup} q \in Q_\cup$ then $q \in A'$. Because $\mathcal{A}_k$ is deterministic, we get $A' \subseteq A$ by Lemma 3.3. Hence $q \in A$. ■

LEMMA 3.5. $L(\mathcal{A}_k) \supseteq (\xleftarrow{*}_{\mathcal{R}})(L)$.

Proof. Assume that $t \xrightarrow{*}_{\mathcal{R}} s$ for some $s \in L$. We show that $t \in L(\mathcal{A}_k)$ by induction on the length m of this reduction. If $m = 0$ then $t \in L$. Thus $t \xrightarrow{*}_{\mathcal{A}_\cup} q$ for some $q \in Q^f_L$. Since $\mathcal{A}_k$ is complete, there exists $A \in Q$ such that $t \xrightarrow{*}_{\mathcal{A}_k} A$. According to Lemma 3.4, $q \in A$ and therefore $A \in Q^f$. Hence $t \in L(\mathcal{A}_k)$. Let $m > 0$. Then we assume that

$$t \equiv t[l\sigma]_p \to_{\mathcal{R}} t[r\sigma]_p \xrightarrow{*}_{\mathcal{R}} s \in L$$

with $l \to r \in \mathcal{R}$. By induction hypothesis, $t[r\sigma]_p$ is accepted by $\mathcal{A}_k$. Since $\mathcal{A}_k$ is deterministic, there exists $\theta : \mathcal{V} \to Q$ such that

$$t[r\sigma]_p \xrightarrow{*}_{\mathcal{A}_k} t[r\theta]_p \xrightarrow{*}_{\mathcal{A}_k} t[A]_p \xrightarrow{*}_{\mathcal{A}_k} B \in Q^f,$$

where $A \in Q$. By completeness of $\mathcal{A}_k$, we assume that

$$t \equiv t[f(t_1, \ldots, t_n)]_p \xrightarrow{*}_{\mathcal{A}_k} t[f(A_1, \ldots, A_n)]_p \to_{\mathcal{A}_k} t[A']_p \xrightarrow{*}_{\mathcal{A}_k} B' \in Q,$$

where $f(A_1, \ldots, A_n) \to A' \in \Delta_k$ and $n \geq 0$. We consider the following two cases.

Case 1. $l \equiv f(l_1, \ldots, l_n)$. If $l_j \notin \mathcal{V}$ then t_j is accepted by $\mathcal{A}_{l_j}$ and thus A_j has $q \in Q^f_{l_j}$ by Lemma 3.4. Because $\mathcal{A}_k$ is deterministic, for any $x \in r$, $x \equiv l_j$ implies $x\theta \equiv A_j$. Therefore $f(A_1, \ldots, A_n) \to A' \in \Delta_k$, $l \to r \in \mathcal{R}$, and $A \in Q$ fulfill (1), (2), and (3) of Condition 1. By the construction of $\mathcal{A}_k$, they must not satisfy (4) of Condition 1. Thus $A \subseteq A'$. Since Lemma 3.3 yields $B \subseteq B'$, we obtain $B' \in Q^f$. Therefore $t \in L(\mathcal{A}_k)$.

Case 2. $l \equiv x$ for some $x \in \mathcal{V}$. Because $\mathcal{A}_k$ is deterministic, if $x \in r$ then $x\theta \equiv A'$. Therefore $f(A_1, \ldots, A_n) \to A' \in \Delta_k, l \to r \in \mathcal{R}$ and $A \in Q$ fulfill (1′) and (2′) of Condition 2. By the construction of $\mathcal{A}_k$, they must not satisfy (3′) of Condition 2 and thus $A \subseteq A'$. According to Lemma 3.3, $B \subseteq B'$ and therefore $B' \in Q^f$. Hence $t \in L(\mathcal{A}_k)$. ■

Thus we obtain the following theorem.

THEOREM 3.1. *Let $\mathcal{R}$ be a left-linear growing TRS and let L be a recognizable tree language. Then the set $(\xleftarrow{*}_{\mathcal{R}})(L)$ is recognized by a tree automaton.*

Proof. From Lemmas 3.2 and 3.5, we have $L(\mathcal{A}_k) = (\xleftarrow{*}_{\mathcal{R}})(L)$. ■

Remark. The recognizability of $(\xrightarrow{*}_{\mathcal{R}})(L)$ was shown for right-linear monadic rewriting systems by Salomaa [22] and for linear semimonadic rewriting systems by Coquidé *et al.* [4]. If a TRS $\mathcal{R}$ is right-linear monadic or linear semimonadic, then $\mathcal{R}^{-1}$ is obviously left-linear growing and $(\xrightarrow{*}_{\mathcal{R}})(L) = (\xleftarrow{*}_{\mathcal{R}^{-1}})(L)$. Thus Theorem 3.1 extends both results. Gilleron and Tison [10] conjectured the recognizability of $(\xrightarrow{*}_{\mathcal{R}})(L)$ for a right-linear semimonadic rewriting system $\mathcal{R}$. Our result gives a positive answer for their conjecture as $\mathcal{R}^{-1}$ is again left-linear growing. Gyenizse and Vágvölgyi [11] proved the recognizability for linear generalized semimonadic rewriting systems, and Kitaoka *et al.* [16] extended this result to finite overlapping term rewriting systems. These results are incomparable to our result.[2]

If $\mathcal{R}$ is left-linear TRS then the set $\mathrm{NF}_{\mathcal{R}}$ of normal forms is a recognizable set [3, 8]. From Theorem 3.1 the set $(\xleftarrow{*}_{\mathcal{R}})(\mathrm{NF}_{\mathcal{R}})$ is recognizable for a left-linear growing $\mathcal{R}$. Thus the following corollary holds.

COROLLARY 3.1. *The weakly normalizing property of left-linear growing TRSs is decidable.*

Proof. A left-linear growing TRS $\mathcal{R}$ is weakly normalizing iff the complement of $(\xleftarrow{*}_{\mathcal{R}})(\mathrm{NF}_{\mathcal{R}})$ is empty. From Lemmas 2.1 and 2.2, the claim follows. ■

[2] Takai *et al.* recently extended the result by Kitaoka *et al.* to the class of right-linear finite path overlapping term rewriting systems, which includes the class of left-linear growing term rewriting systems: See [25].

4. REACHABILITY AND JOINABILITY

The reachability problem for $\mathcal{R}$ is the problem of deciding whether $t \stackrel{*}{\rightarrow}_{\mathcal{R}} s$ for given two terms t and s. It is well known that this problem is undecidable for general TRSs. Oyamaguchi [20] has shown that this problem is decidable for right-ground TRSs. Decidability for linear growing TRSs was shown by Jacquemard [15]. Since a singleton set of a term is recognizable, we can extend these results by using Theorem 3.1.

THEOREM 4.1. *The reachability problem for left-linear growing TRSs is decidable.*

Proof. Let t and s be two terms. Then $t \stackrel{*}{\rightarrow}_{\mathcal{R}} s$ iff $(\stackrel{*}{\leftarrow}_{\mathcal{R}})(\{s\}) \cap \{t\} \neq \phi$. By Theorem 3.1, $(\stackrel{*}{\leftarrow}_{\mathcal{R}})(\{s\})$ is recognizable as $\{s\}$ is recognizable. Thus from Lemmas 2.1 and 2.2 the theorem follows. ∎

It is clear that $t \stackrel{*}{\rightarrow}_{\mathcal{R}} s$ iff $s \stackrel{*}{\rightarrow}_{\mathcal{R}^{-1}} t$. By Theorem 4.1, we obtain the following theorem.

THEOREM 4.2. *Let $\mathcal{R}$ be a TRS such that $\mathcal{R}^{-1}$ is left-linear growing. The reachability problem for $\mathcal{R}$ is decidable.*

If $\mathcal{R}$ is right-ground TRS then $\mathcal{R}^{-1}$ is left-linear growing. Thus, the above theorem is a generalization of Oyamaguchi's result.

Gyenizse and Vágvölgyi [11] showed that the joinability and the local confluence property are decidable for term rewriting systems preserving recognizable. Following Gyenizse and Vágvölgyi, we next prove that the joinability and the local confluence property are decidable for term rewriting systems the inverse of which is left-linear growing.

The joinability problem for a TRS $\mathcal{R}$ is the problem of deciding given finite number of terms $t_1, \ldots, t_n$, whether there exists a term s such that $t_i \stackrel{*}{\rightarrow} s$ for any $1 \leq i \leq n$. Oyamaguchi [20] has shown that this problem is decidable for right-ground TRSs. This result is extended as follows.

THEOREM 4.3. *Let $\mathcal{R}$ be a TRS such that $\mathcal{R}^{-1}$ is left-linear and growing. The joinability problem for $\mathcal{R}$ is decidable.*

Proof. Let $t_1, \ldots, t_n$ be terms. Then $t_1, \ldots, t_n$ are joinable iff

$$(\stackrel{*}{\rightarrow}_{\mathcal{R}})(\{t_1\}) \cap \cdots \cap (\stackrel{*}{\rightarrow}_{\mathcal{R}})(\{t_n\}) \neq \phi.$$

By Theorem 3.1, $(\stackrel{*}{\rightarrow}_{\mathcal{R}})(\{t_i\}) = (\stackrel{*}{\leftarrow}_{\mathcal{R}^{-1}})(\{t_i\})$ is recognizable for any $1 \leq i \leq n$. Thus from Lemmas 2.1 and 2.2 the theorem follows. ∎

A TRS $\mathcal{R}$ is locally confluent if $t \rightarrow_{\mathcal{R}} t'$ and $t \rightarrow_{\mathcal{R}} t''$ imply $t' \stackrel{*}{\rightarrow}_{\mathcal{R}} s$ and $t'' \stackrel{*}{\rightarrow}_{\mathcal{R}} s$ for some s. It is well known that $\mathcal{R}$ is locally confluent iff every critical pair of $\mathcal{R}$ is joinable [1]. Applying Theorem 4.3, we have the following corollary.

COROLLARY 4.1. *Let $\mathcal{R}$ be a TRS such that $\mathcal{R}^{-1}$ is left-linear and growing. Then it is decidable whether $\mathcal{R}$ is locally confluent.*

5. DECIDABLE APPROXIMATIONS

Huet and Lévy [14] investigated normalizing one-step reduction strategies for orthogonal TRSs. A redex position p in a term t is *needed* if in every reduction sequence from t to a normal form a redex at some descendant of p is contracted. We also say that the redex at position p is *needed*. A reduction $t \rightarrow_{\mathcal{R}} s$ by applying a rule at position p is *needed* (or *call-by-need*) if p is needed. Huet and Lévy [14] showed that the needed reduction is a normalizing reduction strategy for orthogonal TRSs; i.e., repeated contraction of needed redexes eventually results in a normal form if it exists. Unfortunately, needed redexes are undecidable in general. Thus, in order to obtain a decidable class of orthogonal (or left-linear) TRSs having the decidable needed reduction strategy, several decidable approximations of TRSs were introduced in the literature [2, 7, 9, 14, 15, 18, 19, 21, 24].

The first idea of decidable approximations was proposed by Huet and Lévy [14] as the *strongly sequential* approximation of orthogonal TRSs, which is obtained by replacing the right-hand side of every rewrite rule with a fresh variable not occurring in the left-hand side. Oyamaguchi [21] gave a better approximation, the *NV-sequential* approximation, which is obtained by replacing all variables in the right-hand side of every rewrite rule with distinct fresh variables. Comon [2] showed that the *linear shallow* approximation is decidable, and Jacquemard [15] introduced the *linear growing* approximation which is finer than other ones. Here the linear shallow approximation (resp. the linear growing approximation) is obtained by replacing the variables in the right-hand side which do not satisfy the condition of linear shallowness (resp. linear growingness) with distinct fresh variables. We now give a better decidable approximation of TRSs than all of them, based on the recognizability result of Section 3.

A TRS $\mathcal{R}'$ is an *approximation* of a TRS $\mathcal{R}$ if $\xrightarrow{*}_{\mathcal{R}} \subseteq \xrightarrow{*}_{\mathcal{R}'}$. An *approximation mapping* τ is a mapping from TRSs to TRSs such that $\tau(\mathcal{R})$ is an approximation of $\mathcal{R}$ for every TRS $\mathcal{R}$.

DEFINITION 5.1. Let $\mathcal{R} = \{l_i \to r_i \mid 1 \le i \le n\}$ be a left-linear TRS. The *left-linear growing approximation* of $\mathcal{R}$ is a left-linear growing TRS $\{l_i' \to r_i \mid 1 \le i \le n\}$ where for any $1 \le i \le n$, l_i' is obtained from l_i by replacing the variables which do not satisfy the condition of left-linear growingness with distinct fresh variables.

DEFINITION 5.2. An approximation mapping τ is *left-linear growing* (resp. *strongly sequential*, *NV-sequential*, *linear shallow*, *linear growing*) if $\tau(\mathcal{R})$ is a left-linear growing (resp. strongly sequential, NV-sequential, linear shallow, linear growing) approximation of $\mathcal{R}$ for every TRS $\mathcal{R}$.

If $\mathcal{R}$ is a left-linear growing TRS then the left-linear growing approximation of $\mathcal{R}$ is $\mathcal{R}$ itself. If τ is a left-linear growing mapping then $\mathrm{NF}_{\mathcal{R}} = \mathrm{NF}_{\tau(\mathcal{R})}$ for every left-linear TRS $\mathcal{R}$.

Note. In the *left-linear growing approximation* we replace the variables in the left-hand side instead of those in the right-hand side not satisfying the left-linear growingness. This modification can give a slight better approximation because of keeping nonlinear variables in the right-hand side. For example, the left-linear growing approximation of the rewrite rule $f(g(x), y) \to f(x, f(y, x))$ is $f(g(z), y) \to f(x, f(y, x))$ or $f(g(x), y) \to f(z, f(y, z))$ after a variable renaming), but if the variables in the right-hand side are replaced instead then we have a worse approximation $f(g(x), y) \to f(z, f(y, z'))$.

EXAMPLE 5.1. Let

$$\mathcal{R} = \begin{cases} f(g(x), y) \to f(x, f(y, x)) \\ g(x) \to f(x, x). \end{cases}$$

Then, the strongly sequential approximation of $\mathcal{R}$ is

$$\mathcal{R}_{st} = \begin{cases} f(g(x), y) \to z \\ g(x) \to z, \end{cases}$$

the NV-sequential approximation of $\mathcal{R}$ is

$$\mathcal{R}_{nv} = \begin{cases} f(g(x), y) \to f(z, f(z', z'')) \\ g(x) \to f(z, z'), \end{cases}$$

the linear shallow approximation of $\mathcal{R}$ is

$$\mathcal{R}_{sh} = \begin{cases} f(g(x), y) \to f(z, f(z', z'')) \\ g(x) \to f(x, z), \end{cases}$$

the linear growing approximation of $\mathcal{R}$ is

$$\mathcal{R}_{lg} = \begin{cases} f(g(x), y) \to f(z, f(y, z')) \\ g(x) \to f(x, z), \end{cases}$$

and the left-linear growing approximation of $\mathcal{R}$ (after a variable renaming) is

$$\mathcal{R}_{llg} = \begin{cases} f(g(x), y) \to f(z, f(y, z)) \\ g(x) \to f(x, x). \end{cases}$$

It is clear that $\overset{*}{\to}_{\mathcal{R}} \subset \overset{*}{\to}_{\mathcal{R}_{llg}} \subset \overset{*}{\to}_{\mathcal{R}_{lg}} \subset \overset{*}{\to}_{\mathcal{R}_{sh}} \subset \overset{*}{\to}_{\mathcal{R}_{nv}} \subset \overset{*}{\to}_{\mathcal{R}_{st}}$. Hence the left-linear growing approximation is better than others.

Durand and Middeldorp [7] presented a simpler framework for decidable approximations of TRSs without notions of index and sequentiality. The following notions and results originate from [7]. The redex at a position p in $t \in \mathcal{T}(\mathcal{F})$ is $\mathcal{R}$-*needed* if there exists no $s \in \mathrm{NF}_{\mathcal{R}}$ such that $t[\Omega]_p \overset{*}{\to}_{\mathcal{R}} s$. Note that a normal form s does not contain Ω's. Then the following proposition gives an easy alternative definition of neededness without the notion of descendent.

Proposition 5.1 [7]. *Let $\mathcal{R}$ be an orthogonal TRS. Then a redex is needed iff it is $\mathcal{R}$-needed.*

Let τ be an approximation mapping. The redex at a position p in $t \in \mathcal{T}(\mathcal{F})$ is $\tau(\mathcal{R})$-*needed* if there exists no $s \in \mathrm{NF}_{\mathcal{R}}$ such that $t[\Omega]_p \overset{*}{\to}_{\tau(\mathcal{R})} s$. From the definitions and the above proposition it immediately follows that every $\tau(R)$-needed redex is *needed* if $\mathcal{R}$ is an orthogonal TRS. Thus $\tau(\mathcal{R})$-needed reduction strategy gives a needed reduction strategy, i.e., a normalizing reduction strategy for $\mathcal{R}$ [7].

The class $\mathcal{C}_\tau$ of TRSs is defined as follows: $\mathcal{R} \in \mathcal{C}_\tau$ iff every term not in normal form has a $\tau(\mathcal{R})$-needed redex. Let st, nv, sh, lg, llg be a strongly sequential approximation map, a NV-sequential approximation map, a linear shallow approximation map, a linear growing approximation map, and a left-linear growing approximation map, respectively. Then it was shown [7, 15] that $\mathcal{C}_{st} \subset \mathcal{C}_{nv} \subset \mathcal{C}_{sh} \subset \mathcal{C}_{lg}$.

The following sufficient condition was given by Durand and Middeldorp [7] for proving uniformly the decidability of $\tau(\mathcal{R})$-neededness and membership of $\mathcal{C}_\tau$ for various approximation maps τ.

Theorem 5.1 [7]. *Let $\mathcal{R}$ be a left-linear TRS. Let τ be an approximation mapping. If the set $\{t \in \mathcal{T}(\mathcal{F} \cup \{\Omega\}) \mid \exists s \in \mathrm{NF}_{\mathcal{R}}\ t \overset{*}{\to}_{\tau(\mathcal{R})} s\}$ is recognizable then*

(1) *it is decidable whether a redex in a term is $\tau(\mathcal{R})$-needed,*

(2) *it is decidable whether $\mathcal{R} \in \mathcal{C}_\tau$.*

Let $\mathcal{R}$ be an orthogonal TRS and $\mathcal{R} \in \mathcal{C}_\tau$. Then since every $\tau(\mathcal{R})$-needed redex is needed for orthogonal TRSs, the above theorem guarantees that $\tau(\mathcal{R})$-needed reduction strategy works as a decidable normalizing reduction strategy for $\mathcal{R}$.

Corollary 5.1 [2, 7, 14, 15, 21]. *Let $\mathcal{R}$ be a left-linear TRS and τ in $\{st, nv, sh, lg\}$.*

(1) *It is decidable whether a redex in a term is $\tau(R)$-needed.*

(2) *It is decidable whether $\mathcal{R} \in \mathcal{C}_\tau$.*

The set $\mathrm{NF}_{\mathcal{R}}$ is recognizable if $\mathcal{R}$ is left-linear. Hence we have the following decidability result from Theorems 3.1 and 5.1.

Theorem 5.2. *Let $\mathcal{R}$ be a left-linear TRS. Let llg be a left-linear growing approximation mapping.*

(1) *It is decidable whether a redex in a term is $llg(\mathcal{R})$-needed.*

(2) *It is decidable whether $\mathcal{R} \in \mathcal{C}_{llg}$.*

Let $\mathcal{R}$ be an orthogonal TRS. From Proposition 5.1 it follows that if $\tau(\mathcal{R}) = \mathcal{R}$ then $\tau(R)$-neededness coincides with neededness [7]. It was also shown by Huet and Lévy [14] that every term not in normal from has a needed redex. Thus we have the following corollary.

Corollary 5.2. *Let $\mathcal{R}$ be an orthogonal growing TRS. Then the neededness is decidable and we have $\mathcal{R} \in \mathcal{C}_{llg}$ for every left-linear growing approximation mapping llg.*

The following theorem shows that left-linear growing approximations extend the class of orthogonal TRSs having a decidable needed reduction strategy.

THEOREM 5.3. *Let llg be a left-liner growing approximation mapping and let lg be a linear growing approximation mapping. Then $\mathcal{C}_{lg} \subset \mathcal{C}_{llg}$ even if these classes are restricted to orthogonal TRSs.*

Proof. For every TRS $\mathcal{R}$, $lg(\mathcal{R})$-neededness implies $llg(\mathcal{R})$-neededness because we have $\xrightarrow{*}_{llg(\mathcal{R})} \subseteq \xrightarrow{*}_{lg(\mathcal{R})}$. Thus $\mathcal{C}_{lg} \subseteq \mathcal{C}_{llg}$. Let $\mathcal{R} = \{g(x) \to f(x,x,x)\} \cup \mathcal{R}'$ where $\mathcal{R}' = \{f(a,b,x) \to a,\ f(b,x,a) \to a,\ f(x,a,b) \to b\}$. From Corollary 5.2 we have $\mathcal{R} \in \mathcal{C}_{llg}$. We will show that $\mathcal{R} \notin \mathcal{C}_{lg}$. If $lg(\mathcal{R}) = \{g(x) \to f(y,z,x)\} \cup \mathcal{R}'$ then $g(b) \xrightarrow{*}_{lg(\mathcal{R})} a$ and $g(b) \xrightarrow{*}_{lg(\mathcal{R})} b$. Therefore, the term $f(g(b), g(b), g(b))$ does not have $lg(\mathcal{R})$-needed redexes. Similarly, we can show that $f(g(a), g(a), g(a))$ does not have $lg(\mathcal{R})$-needed redexes for other linear growing approximations of $\mathcal{R}$. Hence $\mathcal{R} \notin \mathcal{C}_{lg}$. ∎

6. TERMINATION OF ALMOST ORTHOGONAL GROWING TRSS

Termination is decidable for ground TRSs [13], right-ground TRSs [5], and right-linear monadic TRSs [23]. In this section, we show that termination of almost orthogonal growing TRSs is decidable. If a TRS $\mathcal{R}$ contains a rewrite rule which does not satisfy the variable restriction then $\mathcal{R}$ is not terminating. Thus we may assume that $\mathcal{R}$ satisfies the variable restriction. We first explain the theorem of Gramlich [12], which is used in our proof.

A reduction $t \to_{\mathcal{R}} s$ by applying a rule at position p is *innermost* if every proper subterm of $t|_p$ is a normal form. The innermost reduction is denoted by $\to_{\mathcal{I}}$. We say that a term t is *weakly innermost normalizing* if $t \xrightarrow{*}_{\mathcal{I}} s$ for some normal form s. A TRS $\mathcal{R}$ is *weakly innermost normalizing* if every term t is weakly innermost normalizing.

THEOREM 6.1 [12]. *Let $\mathcal{R}$ be a TRS such that every critical pair of $\mathcal{R}$ is a trivial overlay.*

(*a*) *$\mathcal{R}$ is terminating iff $\mathcal{R}$ is weakly innermost normalizing.*

(*b*) *For any term t, t is terminating iff t is weakly innermost normalizing.*

According to Theorem 6.1, if we can prove the decidability of weakly innermost normalizing then termination is decidable. We show that the set of all ground terms being weakly innermost normalizing is recognizable. From here on we assume that $\mathcal{R}$ is a left-linear growing TRS.

We must construct a tree automaton which recognizes the set of all ground terms being weakly innermost normalizing. We start with the deterministic and complete tree automaton $\mathcal{A}_{NF}$ by Comon [2] which accepts ground normal forms. The set $\mathcal{S}_{\mathcal{R}}$ is defined as follows: $\mathcal{S}_{\mathcal{R}} = \{t \in \mathcal{T}_{\Omega} \mid t \subset l_{\Omega},\ l \to r \in \mathcal{R}\}$. $\mathcal{S}^*_{\mathcal{R}}$ is the smallest set such that $\mathcal{S}_{\mathcal{R}} \subseteq \mathcal{S}^*_{\mathcal{R}}$ and if $t, s \in \mathcal{S}^*_{\mathcal{R}}$ and $t \uparrow s$ then $t \sqcup s \in \mathcal{S}^*_{\mathcal{R}}$. $\mathcal{A}_{NF} = (\mathcal{F}, Q_{NF}, Q^f_{NF}, \Delta_{NF})$ is defined by $Q_{NF} = \{q_t \mid t \in \mathcal{S}^*_{\mathcal{R}}$ and t does not contain redexes$\} \cup \{q_{\Omega}, q_{\mathrm{red}}\}$, $Q^f_{NF} = Q_{NF} \setminus \{q_{\mathrm{red}}\}$, and Δ_{NF} consists of the following rules:

- $f(q_{t_1}, \ldots, q_{t_n}) \to q_t$
  if $f(t_1, \ldots, t_n)$ is not a redex and
  t is maximal Ω-term w.r.t. $\leq$ such that $t \leq f(t_1, \ldots, t_n)$ and $q_t \in Q^f_{NF}$,
- $f(q_{t_1}, \ldots, q_{t_n}) \to q_{\mathrm{red}}$ if $f(t_1, \ldots, t_n)$ is a redex,
- $f(q_1, \ldots, q_n) \to q_{\mathrm{red}}$ if $q_{\mathrm{red}} \in \{q_1, \ldots, q_n\}$.

The following lemma shows that $\mathcal{A}_{NF}$ recognizes the set of ground normal forms.

LEMMA 6.1 [2]. *Let $t \in \mathcal{T}(\mathcal{F})$.*

(i) *$\mathcal{A}_{NF}$ is deterministic and complete.*

(ii) *If $t \xrightarrow{*}_{\mathcal{A}_{NF}} q_s \in Q^f_{NF}$ then t is a normal form, $s \leq t$ and $u \leq s$ for any $q_u \in Q^f_{NF}$ with $u \leq t$.*

(iii) *If $t \xrightarrow{*}_{\mathcal{A}_{NF}} q_{\mathrm{red}}$ then t is not a normal form.*

We inductively construct tree automata $\mathcal{A}_0, \mathcal{A}_1, \ldots$ as follows. Let $\mathcal{A}_0 = (\mathcal{F}, Q, Q^f, \Delta_0) = (\mathcal{F}, Q_{NF}, Q^f_{NF}, \Delta_{NF}) = \mathcal{A}_{NF}$. For $0 \leq i$, $\mathcal{A}_{i+1} = (\mathcal{F}, Q, Q^f, \Delta_{i+1})$ is obtained from $\mathcal{A}_i = (\mathcal{F}, Q, Q^f, \Delta_i)$ as follows:

If there exist $q_{t_1} \in Q^f, \ldots, q_{t_n} \in Q^f$, $f(l_1, \ldots, l_n) \to r \in \mathcal{R}$ and $q \in Q$ such that

(1) $f(l_1, \ldots, l_n)_\Omega \le f(t_1, \ldots, t_n)$,

(2) there exists a substitution $\theta : \mathcal{V} \to Q$ such that $r\theta \xrightarrow{*}_{\mathcal{A}_i} q$ and $x \equiv l_j$ implies $x\theta = q_{t_j}$ for every $x \in r$ and $1 \le j \le n$,

(3) $f(q_{t_1}, \ldots, q_{t_n}) \to q \notin \Delta_i$,

then $\Delta_{i+1} = \Delta_i \cup \{f(q_{t_1}, \ldots, q_{t_n}) \to q\}$.

Since the set of states is fixed, the number of new rules is bounded. Thus, the process of construction eventually stops with the resulting automaton $\mathcal{A}_k = (\mathcal{F}, Q, Q^f, \Delta_k)$ when there is no new rule to add. Note that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are nondeterministic. In the following we prove that

$$L(\mathcal{A}_k) = \{t \in \mathcal{T}(\mathcal{F}) \mid t \text{ is weakly innermost normalizing}\}.$$

LEMMA 6.2. *Let $t \in \mathcal{T}(\mathcal{F})$. For any $0 \le i \le k$, if $t \xrightarrow{*}_{\mathcal{A}_i} q \in Q$ then $t \xrightarrow{*}_{\mathcal{I}} s \xrightarrow{*}_{\mathcal{A}_{NF}} q$ for some $s \in \mathcal{T}(\mathcal{F})$.*

Proof. We prove the lemma by induction on i. *Base step.* Trivial. *Induction step.* Assume that $q_{s_1} \in Q^f, \ldots, q_{s_n} \in Q^f$, $f(l_1, \ldots, l_n) \to r \in \mathcal{R}$ and $q_1 \in Q$ satisfy the conditions of construction and Δ_i is obtained by adding the rule $f(q_{s_1}, \ldots, q_{s_n}) \to q_1$ to Δ_{i-1}. We use induction on the number m of applications of the rule $f(q_{s_1}, \ldots, q_{s_n}) \to q_1$ in the reduction $t \xrightarrow{*}_{\mathcal{A}_i} q$. If $m = 0$ then $t \xrightarrow{*}_{\mathcal{A}_{i-1}} q$. Thus it follows from induction hypothesis on i that $t \xrightarrow{*}_{\mathcal{I}} s \xrightarrow{*}_{\mathcal{A}_{NF}} q$ for some $s \in \mathcal{T}(\mathcal{F})$. Let $m > 0$. Suppose that

$$t \equiv t[f(t_1, \ldots, t_n)]_p \xrightarrow{*}_{\mathcal{A}_{i-1}} t\left[f\left(q_{s_1}, \ldots, q_{s_n}\right)\right]_p \to_{\mathcal{A}_i} t[q_1]_p \xrightarrow{*}_{\mathcal{A}_i} q.$$

For every $1 \le j \le n$, we obtain $u_j \in \mathcal{T}(\mathcal{F})$ such that $t_j \xrightarrow{*}_{\mathcal{I}} u_j \xrightarrow{*}_{\mathcal{A}_{NF}} q_{s_j}$ by applying induction hypothesis on i to $t_j \xrightarrow{*}_{\mathcal{A}_{i-1}} q_{s_j}$. According to Lemma 6.1 (ii), $f(s_1, \ldots, s_n) \le f(u_1, \ldots, u_n)$ and $u_1, \ldots, u_n$ are normal forms. Because we have $f(l_1, \ldots, l_n)_\Omega \le f(s_1, \ldots, s_n)$ by the condition (1), we obtain the following reduction sequence:

$$f(t_1, \ldots, t_n) \xrightarrow{*}_{\mathcal{I}} f(u_1, \ldots, u_n) \equiv f(l_1, \ldots, l_n)\sigma \to_{\mathcal{I}} r\sigma.$$

Let θ be a substitution which is satisfied in the condition (2) of construction. Then from the growingness of $\mathcal{R}$ we have $r\sigma \xrightarrow{*}_{\mathcal{A}_{NF}} r\theta$ and hence $r\sigma \xrightarrow{*}_{\mathcal{A}_{i-1}} q_1$. Applying induction hypothesis on m to $t[r\sigma]_p \xrightarrow{*}_{\mathcal{A}_{i-1}} t[q_1]_p \xrightarrow{*}_{\mathcal{A}_i} q$, we obtain $s \in \mathcal{T}(\mathcal{F})$ such that $t[r\sigma]_p \xrightarrow{*}_{\mathcal{I}} s \xrightarrow{*}_{\mathcal{A}_{NF}} q$. Thus we have $t \xrightarrow{*}_{\mathcal{I}} s \xrightarrow{*}_{\mathcal{A}_{NF}} q$ since $t \to_{\mathcal{I}} t[r\sigma]_p$. ■

LEMMA 6.3. *$L(\mathcal{A}_k) \subseteq \{t \in \mathcal{T}(\mathcal{F}) \mid t$ is weakly innermost normalizing$\}$.*

Proof. From Lemmas 6.1 and 6.2. ■

LEMMA 6.4. *Let $t \in \mathcal{T}(\mathcal{F})$ be a normal form. Then there exists exactly one q in Q such that $t \xrightarrow{*}_{\mathcal{A}_k} q$. Furthermore, q is the state q_s in Q^f such that $s \le t$ and $u \le s$ for any $q_u \in Q^f$ with $u \le t$.*

Proof. By Lemma 6.2, $t \xrightarrow{*}_{\mathcal{A}_k} q$ iff $t \xrightarrow{*}_{\mathcal{A}_{NF}} q$. Thus, from Lemma 6.1 the claim follows. ■

LEMMA 6.5. *$L(\mathcal{A}_k) \supseteq \{t \in \mathcal{T}(\mathcal{F}) \mid t$ is weakly innermost normalizing$\}$.*

Proof. Assume that $t \xrightarrow{*}_{\mathcal{I}} s$ for some normal form s. We show that $t \in L(\mathcal{A}_k)$ by induction on the length m of this reduction. Let $m = 0$. Then t is a normal form and hence $t \in L(\mathcal{A}_{NF}) \subseteq L(\mathcal{A}_k)$. Let $m > 0$. We assume that

$$t \equiv t[f(l_1, \ldots, l_n)\sigma]_p \to_{\mathcal{I}} t[r\sigma]_p \xrightarrow{*}_{\mathcal{I}} s$$

with $f(l_1, \ldots, l_n) \to r \in \mathcal{R}$. By induction hypothesis, $t[r\sigma]_p$ is accepted by $\mathcal{A}_k$, i.e., $t[r\sigma]_p \xrightarrow{*}_{\mathcal{A}_k} q$ for some $q \in Q^f$. Because $x\sigma$ is a normal form for every $x \in r$, Lemma 6.4 yields $\theta : \mathcal{V} \to Q$ such that

$$t[r\sigma]_p \xrightarrow{*}_{\mathcal{A}_k} t[r\theta]_p \xrightarrow{*}_{\mathcal{A}_k} t[q_1]_p \xrightarrow{*}_{\mathcal{A}_k} q,$$

where $q_1 \in Q$. For any $1 \le j \le n$, by Lemma 6.4 we have exactly one $q_{s_j} \in Q$ with $l_j\sigma \xrightarrow{*}_{\mathcal{A}_k} q_{s_j}$ because $l_j\sigma$ is a normal form. Note that if $l_j \equiv x$ and $x \in r$ then $x\theta = q_{s_j}$. For any $1 \le j \le n$, $q_{l_{j\Omega}} \in Q^f$ since $l_{j\Omega} \in \mathcal{S}^*_{\mathcal{R}}$ and $l_{j\Omega}$ does not contain redexes. According to Lemma 6.4 $f(l_1, \ldots, l_n)_\Omega \le f(s_1, \ldots, s_n)$. Therefore $q_{s_1} \in Q^f, \ldots, q_{s_n} \in Q^f$, $f(l_1, \ldots, l_n) \to r \in \mathcal{R}$ and $q_1 \in Q$ satisfy the conditions (1) and (2) of construction. By the construction of $\mathcal{A}_k$, Δ_k has the rule $f(q_{s_1}, \ldots, q_{s_n}) \to q_1$. Thus, since

$$t \equiv t[f(l_1, \ldots, l_n)\sigma]_p \xrightarrow{*}_{\mathcal{A}_k} t\big[f\big(q_{s_1}, \ldots, q_{s_n}\big)\big]_p \to_{\mathcal{A}_k} t[q_1]_p \xrightarrow{*}_{\mathcal{A}_k} q \in Q^f,$$

t is accepted by $\mathcal{A}_k$. ■

Thus we obtain the following result.

LEMMA 6.6. *Let $\mathcal{R}$ be a left-linear growing TRS. The set of ground terms being weakly innermost normalizing is recognized by a tree automaton.*

THEOREM 6.2. *Termination is decidable for almost orthogonal growing TRSs.*

Proof. Let $\mathcal{R}$ be an almost orthogonal growing TRS. According to Lemma 6.1, $\mathcal{R}$ is strongly normalizing iff every ground term is weakly innermost normalizing. From Lemmas 2.1, 2.2, and 6.6, it is decidable whether every ground term is weakly innermost normalizing. ■

7. CONCLUSION

We have introduced the notion of left-linear growing term rewrite systems, which (or the inverse of which) is a generalization of well-known term rewriting systems: ground term rewriting systems, linear shallow term rewriting systems, linear growing term rewriting systems, right-linear monadic rewriting systems, linear semimonadic rewriting systems, and right-linear semimonadic rewriting system. We have shown that left-linear growing term rewriting systems preserve the recognizability. Several applications of this result have been presented:

1. The decidability for the reachability and the joinability of a term rewriting system the inverse of which is left-linear growing,
2. A better decidable approximation of term rewriting systems, which extends the class of orthogonal term rewriting systems having a decidable call-by-need strategy,
3. The decidability for termination of almost orthogonal growing term rewriting systems.

We now raise some open problems.

1. Considering complexity issues: Comon [2] showed that deciding strong sequentiality of any left-linear term rewriting system is in EXPTIME. Huet and Lév [14] showed that finding strongly sequential needed redex is in linear time of the size of term. Oyamaguchi [21] showed that finding NV-sequential needed redex is in polynomial time of the size of the system and of the term. It is still open whether finding left-linear growing needed redex is in polynomial time.
2. The decidability for termination of arbitrary left-linear growing term rewriting systems without almost orthogonality: We believe that this conjecture is positive, though we have never proven it. In the proof presented in Section 6, almost orthogonality is essential because it guarantees the equivalence of *termination* and *weakly innermost normalizing*, which is the point for applying tree automaton techniques. Thus different proof techniques seem necessary.

ACKNOWLEDGMENT

We thank the anonymous referees for their useful comments, which improved the presentation of our results.

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