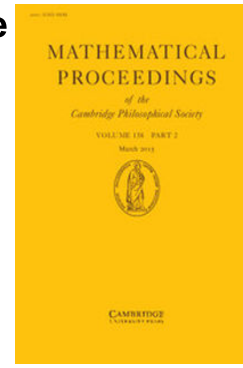

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Dedicated to P. M. Cohn on his 60th birthday

For a commutative field extension, $L \supset K$, it is clear that a left basis of L over K is also a right basis of L over K; however, for an extension of skew fields, this may easily fail, though it is hard to determine whether the right and left dimension may be different. Cohn ([**4**], ch. 5), however, was able to find extensions of skew fields such that the left and right dimensions were an arbitrary pair of cardinals subject only to the restrictions that neither were 1 and at least one of them was infinite. In this paper, I shall present a new approach that allows us to construct extensions of skew fields such that the left and right dimensions are arbitrary integers not equal to 1. In a subsequent paper, [**7**], I shall present related results and consequences; in particular, there is a construction of a hereditary artinian ring of finite representation type corresponding to the Coxeter diagram $I_2(5)$ answering the question raised by Dowbor, Ringel and Simson [**5**].

If R is a ring, an R-ring is a ring R' with a specified homomorphism from R to R'. Given a family of R-rings $\{R_i: i \in I\}$, we may form the ring coproduct of these in the category of R-rings; this is simply the universal R-ring, $\bar{R}$, with specified homomorphisms of R-rings from R_i to $\bar{R}$ for each $i \in I$. In this paper, R will usually be a skew field, and the ring coproduct in this case has received special attention in Bergman [**1**,**2**].

Given an extension of skew fields $F \supset G$, $[F:G]_l$ and $[F:G]_r$ are the left and the right dimension of F over G respectively.

Let x be an element of an R, S bimodule M for rings R and S; the left normalizer of x in R is the subring of R consisting of those elements t such that $tx = xt'$ for a suitable t' in S; the right normalizer is defined similarly. In the case where $R = S$ and $x \in R$, the left normalizer of x is usually called the left idealizer of x in R; we shall not make this distinction.

A fir is a domain such that all right and left ideals are isomorphic to free modules of uniquely defined rank. A full n by n matrix over a fir is one that cannot be factored as an n by $(n-1)$ times an $(n-1)$ by n matrix; Cohn has shown ([**3**], ch. 7) that there is a homomorphism from a fir to a skew field under which all full matrices become invertible. An atom in a fir is an element that cannot be written as the product of two elements that do not have inverses.

The basic construction

We shall construct our extension of skew fields of different but finite left and right dimension as the union of extensions $F_i \supset G_i$, $i \in \mathbb{N}$ such that

$$F_i \supset F_{i-1} \quad \text{and} \quad F_{i-1} \cap G_i = G_{i-1}.$$

The passage from $F_i \supset G_i$ to $F_{i+1} \supset G_{i+1}$ is a method for adjoining dependency relations on the left or the right between elements of F_i over G_{i+1}, whilst leaving everything alone on the other side; this will be clearer from the description of this construction in this section.

Let $F \supset G$ be an extension of skew fields; we should like to be able to construct an extension $F' \supset G'$ such that $F' \supset F$, $F \cap G' = G$, $[FG':G'] = m$, where m is some previously fixed integer, whilst $G'F$ should be isomorphic in the natural way to $G' \otimes_G F$. Of course, if such a construction is always possible for arbitrary m, there is an analogous construction that interchanges the roles of left and right; it is not hard to see how to use these two constructions to make the examples that we want; we refer the reader to the proof of Theorem 6 for the details. There are two problems that arise; firstly, if $F' \supset G'$ exists, then FG' is an F, G' bimodule such that $[FG':G'] = m$, and it has a generator that is G-centralizing, so we wish to know whether there exists such a skew field G' with an embedding of G in G' and an F, G' bimodule M having the properties we want; secondly, if we have such a skew field G' and a bimodule of the right sort, we must be able to construct a skew field F'. In each case, there is a particular universal construction that suggests itself.

If there exists an F, G' bimodule M such that $[M:G'] = m$, there must be an embedding of F in $M_m(G')$ induced by the left action of F on M. Therefore, we look for a suitably universal ring T such that F embeds in $M_m(T)$. Because we wish T to be a G-ring, and we want our bimodule to have a G-centralizing generator, we consider the ring coproduct $M_m(G) \coprod_G F \cong M_m(T)$; it follows from Bergman [**2**] that T is a fir; so, by Cohn [**3**], it has a universal skew field, G'. Now, let $\{e_{ij}\}$ be the standard G-centralizing matrix units in the first factor of the ring coproduct; then $M_m(G')e_{11}$ is an F, G' bimodule, and e_{11} is a G-centralizing element; our first lemma shows that provided $[F:G]_r \geqslant m$, this is the bimodule we need.

LEMMA 1. *If $f_1, \ldots, f_m$ are right independent elements of F over G, then $f_1 e_{11}, \ldots, f_m e_{11}$ are right G' independent elements of $M_m(G')e_{11}$.*

Proof. We assume that $m \geqslant 2$ and if 1 is in the right F span of $f_1, \ldots, f_m$ then $f_1 = 1$.

If $f_1 e_{11}, \ldots, f_m e_{11}$ are G' dependent elements of $M_m(G')e_{11}$, the matrix over T whose columns are these elements is singular over $M_m(G')$, and so, it is a non-full matrix over the fir T, since all full matrices over T are invertible over F'; therefore there is a factorization over T:

$$\textstyle\sum_i f_i e_{1i} = yz \quad \text{with } ye_{mm} = 0, \quad \text{for } y, z \in M_m(T) = M_m(G) \coprod_G F. \tag{1}$$

We obtain a contradiction by arguing with normal forms for elements of the ring coproduct $M_m(G) \coprod_G F$ as in Bergman [**1**].

We write for convenience $A = M_m(G)$, $B = F$ and $C = A \coprod_G B$. We choose a right basis for A over G as $\{1\} \cup \{e_{ij}: (i,j) \neq (m,m)\} = \{1\} \cup \{a_i: i \in I\}$, and we extend the elements $f_1, \ldots, f_m$ to a basis of B over F of the form $\{1\} \cup \{b_j: j \in J\}$. It is easy to see that a basis for C over F may be chosen to be the set of all strings of elements alternately from $\{a_i\}$ and $\{b_j\}$ together with 1. We well-order the set $\{a_i\} \cup \{b_j\}$ such that $a_i > b_j$ for all i and j and this allows us to order our basis for C over F by ordering the strings length-lexicographically, and making 1 the initial element.

Given any element c of C, we express it with respect to this basis, and those strings that occur with non-zero coefficient are the support of c; the maximal element in the support of c is called the leading term of c and denoted by $\bar{c}$. Similarly, c can be expressed as a right A-linear combination of strings whose terminal letter comes from $\{b_j\}$; those strings that occur in this representation are called the A-support of c. For any such string, q, there is a co-efficient of q map $c_q: C \to A$ which is right A-linear.

We may assume that y in (1) has been chosen so that its A-support is minimal in the lexicographic ordering on descending arrangements. We choose a G basis $\{y_k\}$ for yA so that the leading terms are distinct, and their co-efficients in G are all 1. If an element of yA had leading term ending in some b_j, it would generate a free right A module, but $ye_{mm} = 0$, so this is not the case. We see by the well-ordering that any element of yC may be written in the form $yc = \sum y_k r_k$ where each $\overline{r_k}$ does not begin with a term a_i. Since the co-efficient of the leading term of y_k is 1, and this term ends in some a_i we see that $\overline{y_k r_k} = \overline{y_k}\,\overline{r_k}$; we claim that the $\overline{y_k r_k}$ are distinct. If not, say $\overline{y_1 r_1} = \overline{y_2 r_2}$ and $\overline{y_1}$ is at least as long as $\overline{y_2}$. Since $\overline{y_1}$ and $\overline{y_2}$ are distinct, this means that $\overline{y_2}$ is an initial segment of $\overline{y_1}$; so, $\overline{y_1} = \overline{y_2}s$ for some non-empty string, s, having no initial term from $\{a_i\}$. Let $q = \overline{y_1}$, $t = c_q(y)$, and write $y_2 = ya$; then $ta = c_q(y)\,a = c_q(y_2) = 0$, since y_2 is shorter than q. So, in (1), we may replace y by $y - yast$ and z by $z + astz$, which removes q from the A-support of y and replaces it by lower terms. Since this contradicts the minimality of the A-support of y, we must have that the $\overline{y_k r_k}$ are distinct.

From (1), we see that $f_i e_{1j} \in yC$ for all i, j; so $f_i e_{1j} = \bar{y}_k q$ for some string q with no initial term from $\{a_i\}$. Clearly, then $\bar{y}_k = f_i e_{1j}$, which shows that yA contains m^2 right F independent elements, contradicting the fact that $ye_{mm} = 0$.

In particular, we see that if $ae_{11} = e_{11}\, b$ for $a \in F$, $b \in G'$, then $a \in G$ and $a = b$. We set $M = M_m(G')\,e_{11}$ and $x = e_{11}$. From the skew fields F and G', and the F, G' bimodule M with G-centralizing generator x, we wish to construct a skew field F' containing F and G' such that FG' is isomorphic to M as F, G' bimodule to M by a map that sends 1 to x. Our hope is to find a suitably universal construction of such a skew field so that $G'F$ has a chance to be naturally isomorphic to $G' \otimes_G F$.

For the purposes of the next part of our construction, we assume that we have skew fields F and G' and a cyclic F, G' bimodule M generated by x such that the left (and hence right) normalizer of x is G; by suitably identifying the embeddings of G in F and G', we may assume that G centralizes x. Lemma 1 shows us that the situation we are interested in satisfies these conditions. There is a standard way to construct a skew field from F, G' and M; first, we form the ring coproduct $F \coprod_G G'$ which is a fir, by the coproduct theorems, [**2**], and has a universal skew field $F \circ_G G'$ (Cohn, [**3**], ch. 7); we form the $F \circ_G G'$ bimodule $(F \circ_G G') \otimes_F M \otimes_{G'} (F \circ_G G') = M'$, and we consider the tensor ring on this bimodule, $F \circ_G G'\langle M'\rangle$. We identify M with the image of $1 \otimes M \otimes 1$ in this ring, which is a fir since it is a ring satisfying the weak algorithm (Cohn [**3**], ch. 3) with respect to the standard grading. Let F' be the universal skew field of this fir. We have two skew subfields F and $xG'x^{-1}$; $FxG'x^{-1}$ as F, $xG'x^{-1}$ bimodule is isomorphic to M as F, G' bimodule as we see by right multiplication by x; we are left with checking the bimodule $xG'x^{-1}F$ as $xG'x^{-1}$, F bimodule. In order to work this out, we shall need some information on the element x in the ring $F \circ_G G'\langle M'\rangle$.

LEMMA 2. *The element x is an atom in $F \circ_G G'\langle M'\rangle$ and its left normalizer in $F \circ_G G'\langle M'\rangle$ is $G + x(F\circ_G G'\langle M'\rangle)$.*

Proof. x has degree 1 with respect to the natural grading on $F\circ_G G'\langle M'\rangle$ which satisfies the weak algorithm (Cohn, [3], ch. 3) with respect to this grading; so, x is an atom, and the left normalizer of x lies in $F\circ_G G' + x(F \circ_G G'\langle M'\rangle)$.

$$F \circ_G G'xG' \cong F\circ_G G' \otimes_F M,$$

from which we see that those elements of $F \circ_G G'$ normalizing x must lie in F, and hence, they must lie in G, since G is the left normalizer of x in F.

Our next result allows us to calculate the bimodule $xG'x^{-1}F$.

LEMMA 3. *Let R be a fir with universal skew field U; let y be an atom in R with left normalizer N and $K = N/yR$; then the map given by multiplication from $Ry^{-1} \otimes_N R$ to $Ry^{-1}R$ is an isomorphism. Therefore, the map given by multiplication from*

$$Ry^{-1}/R \otimes_K R/yR$$

to $Ry^{-1}R/R$ is an isomorphism.

Proof. We note first that in $Ry^{-1} \otimes_N R$, the two natural embeddings of R are the same since $ryy^{-1} \otimes 1 = r \otimes 1 = y^{-1}yr \otimes 1 = y^{-1} \otimes yr = 1 \otimes r$.

Let n be minimal such that there is a relation $\sum_{i=1}^{n} r_i y^{-1} s_i = r \in R$, that is not a consequence of relations in $Ry^{-1} \otimes_N R$. One checks by an elementary calculation that the matrix

$$\left(\begin{array}{cccc|c} y & & & & s_1 \\ & y & 0 & & \cdot \\ & & \cdot & & \cdot \\ & 0 & & \cdot & \cdot \\ & & & y & s_n \\ \hline r_1 & \cdot & \cdot & r_n & r \end{array}\right)$$

must be singular over U. Since R is a fir, this matrix cannot be full, since all full matrices are invertible over U (Cohn [3], ch. 7); therefore, we have an equation over R:

$$\left(\begin{array}{cccc|c} y & & & & s_1 \\ & y & 0 & & \cdot \\ & & \cdot\ \cdot & & \cdot \\ & 0 & & & \cdot \\ & & & y & s_n \\ \hline r_1 & \cdot & \cdot & r_n & r \end{array}\right) = \left(\begin{array}{c} \\ B \\ \\ \hline b_1 \ \cdots\ b_n \end{array}\right) \left(\begin{array}{c|c} & c_1 \\ C & \vdots \\ & c_n \end{array}\right)$$

where we may ensure by internal multiplication by an invertible matrix and its inverse on the right-hand side of this equation that B and C are lower triangular, since yI_n certainly is, and R is a fir (Cohn [3]). So, $b_{ii}c_{ii} = y$; since y is an atom, one of b_{ii} and c_{ii} is a unit, and by suitably adjusting, we may ensure that one of b_{ii} and c_{ii} is 1 whilst the other is y. However, $b_{11}c_1 = s_1$; if $b_{11} = y$, this would allow us to shorten the relation given to $\Sigma r_i y^{-1} s_i = r + r_1 c_1$. Therefore, we may assume that $b_{11} = 1$. Let j be the first index such that $b_{jj} = y$; then the matrix $(b_{kl})_{k,l<j}$ is invertible, which implies that we can ensure that it is the identity matrix, whilst $(c_{kl})_{k,l<j}$ must be yI_{j-1} and $c_k = s_k$ for $k < j$. We consider the jth row of B; by multiplying with the kth column of C for $k < j$, we find that $b_{kj}y + yc_{kj} = 0$; that is, b_{kj} lies in the normalizer of x in R. We also have the equation $\sum_{k<j} b_{kj} s_k + xc_j = s_j$; substituting this equation into our original relation allows us to shorten it, and so, we need only consider the case where $b_{ii} = 1$ for all i. In this case, $c_{nn} = y$, so $b_n y = r_n$ and once again we can shorten our relation by one that must hold in $Ry^{-1} \otimes_N R$.

The last sentence follows from the right exactness of the tensor product; it is also precisely what our proof has shown.

We return to the notation used before and during lemma 2.

COROLLARY 4. *$xG'x^{-1}F$ is isomorphic as $xG'x^{-1}$, F bimodule to $xG'x^{-1} \otimes_G F$ by a map that sends 1 to $1 \otimes 1$.*

Proof. By lemma 2, x is an atom in $F \circ_G G'\langle M' \rangle$, which we call R, and its left normalizer is $G + xR$. Since $xG'x^{-1}$ embeds in Rx^{-1}/R and F embeds in R/xR, we may apply the last sentence of lemma 3 to show that the image of $xG'x^{-1}F$ in $Rx^{-1}R/R$ is isomorphic to $xG'x^{-1} \otimes_G F$ in the natural way; since $xG'x^{-1}F$ is a quotient of $xG'x^{-1} \otimes_G F$ in the natural way, this completes the proof of the corollary.

We use these lemmas to complete the first part of the construction.

THEOREM 5. *Let $F \supset G$ be an extension of skew fields, and let m be an integer such that $[F:G]_r \geqslant m$; then there exists an extension of skew fields $F' \supset G'$ such that $F' \supset F$, $G' \cap F = G$, and $[FG':G'] = m$, whilst $G'F \cong G' \otimes_G F$ in the natural way. Moreover, if $f_1, \ldots, f_m$ are right independent elements of F over G, then $\sum_i f_i G' = FG'$.*

Proof. First, we construct G' using the technique of Lemma 1; that is, we define G' by the equation $M_m(G') \cong M_m(G) \circ_G F$, the universal localization of the ring $M_m(G) \coprod_G F$ at the full matrices over it.

Since $[F:G]_r \geqslant m$, the F, G' bimodule $M_m(G')e_{11} = M$ has as generator $e_{11} = x$ by Lemma 1, and $\sum_i f_i xG' = M$; therefore, we can use the method of Lemma 2 to Corollary 4 to construct a skew field $F' \cong F_G \circ G'\langle M' \rangle$, which contains skew subfields F and $xG'x^{-1}$ which satisfy the properties required in the Theorem as we showed in Corollary 4.

Of course, there is a corresponding Theorem that interchanges the role of left and right throughout. We shall refer to this version as Theorem 5′ during the rest of this paper.

We have all the results we need to finish our constructions.

THEOREM 6. *Let $F \supset G$ be an extension of skew fields, and let m, n be integers greater than 1. Then there exists an extension of skew fields $\bar{F} \supset \bar{G}$ such that $\bar{F} \supset F$, $F \cap \bar{G} = G$, and $[F:G]_r = m$, whilst $[F:G]_l = n$.*

Proof. We ensure that $[F:G]_r$ and $[F:G]_l$ are greater than m and n respectively by replacing F with $F(y)$ where y is a commuting indeterminate if necessary. Let $t_1, \ldots, t_m$ be elements of F right independent over G, and let $s_1, \ldots, s_n$ be elements of F left independent over G. For the purposes of our induction, we label F as F_0 and G as G_0.

At an odd stage of our construction, we assume that we have constructed $F_{2k} \supset G_{2k}$ such that $F_{2k} \supset F$, $F \cap G_{2k} = G$, $t_1, \ldots, t_m$ are right independent and $s_1, \ldots, s_n$ are left independent over G_{2k}. We construct by applying Theorem 5 an extension of skew fields $F_{2k+1} \supset G_{2k+1}$ such that $F_{2k+1} \supset F_{2k}$, $F_{2k} \cap G_{2k+1} = G_{2k}$, $[F_{2k}G_{2k+1}:G_{2k+1}] = m$, $\sum_i t_i G_{2k+1} = F_{2k}G_{2k+1}$, and $G_{2k+1}F_{2k} \cong G_{2k+1} \otimes_{G_{2k}} F_{2k}$ in the natural way; in particular, $s_1, \ldots, s_n$ remain left independent and $t_1, \ldots, t_m$ remain right independent over G_{2k+1} inside F_{2k+1}.

At an even stage, we assume that we have an extension of skew fields $F_{2k-1} \supset G_{2k-1}$ such that $F_{2k-1} \supset F$, $F \cap G_{2k-1} = G$, $t_1, \ldots, t_m$ are right independent and $s_1, \ldots, s_n$ are left independent over G_{2k-1}; we construct using Theorem 5′ an extension of skew fields $F_{2k} \supset G_{2k}$ such that $F_{2k} \supset F_{2k-1}$, $F_{2k-1} \cap G_{2k} = G_{2k-1}$, $[G_{2k}F_{2k-1}:G_{2k}] = n$,

$$\sum G_{2k}s_i = G_{2k}F_{2k-1} \quad \text{and} \quad F_{2k-1}G_{2k} \cong F_{2k-1} \otimes G_{2k-1}G_{2k}$$

in the natural way; in particular, $s_1, \ldots, s_n$ remain left independent and $t_1, \ldots, t_m$ remain right independent over G_{2k}.

Let $\bar{F} = \cup_i F_i$ and let $\bar{G} = \cup_i G_i$; by construction, $\bar{F} = \Sigma t_i \bar{G} = \Sigma \bar{G} s_j$ and the elements $\{t_i\}$ are right independent over $\bar{G}$, whilst the elements $\{s_j\}$ are left independent; therefore, $[\bar{F}:\bar{G}]_r = m$ and $[\bar{F}:\bar{G}]_l = n$.

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