ON A CONSTRUCTION OF THE UNIVERSAL FIELD OF FRACTIONS OF A FREE ALGEBRA

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§1. *Introduction.* In this note we present a simple way of obtaining the universal field of fractions of certain free rings as a subfield of an ultrapower of a (by no means unique) skew field. This method of embedding was discovered by Amitsur in [**1**]; our presentation uses Cohn's specialization lemma and the embedding is constructed in terms of full matrices over the rings in question (Theorem 3.2). In particular, if k is an infinite commutative field, the universal field of fractions of a free k-algebra can be realized as a subfield of an ultrapower of any skew extension of k, with centre k, which is infinite dimensional over k. Thus many problems concerning the universal field of fractions of a free k-algebra can be settled by studying skew extensions of k of relatively simple structure. More precisely and more generally, let E be a skew field with centre k and denote by R the free E-ring on X over k. Write U for the universal field of fractions of R and assume that U embeds in an ultrapower of a skew field D. Then by Łos' theorem, U inherits the first-order properties of D which can be expressed by universal sentences. In particular, we show that, if E is orderable, then so is U (Theorem 4.2) and further, that the algebraic elements of U over k are just the conjugates of the algebraic elements of E (Theorem 4.6). We also give an alternative proof of the following theorem of Cohn. The centralizer of every non-central element of the universal field of fractions of a free algebra is commutative.

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§2. *Notation and terminology.* Let R be a ring; the set of $m \times n$ matrices over R is denoted by ${}^mR^n$. We write R^n for ${}^1R^n$ and ${}^mR$ for ${}^mR^1$, further ${}^nR^n$ is abbreviated to R_n. Let $A \in R_n$ and $B \in R_m$, then the *diagonal sum* of A and B is defined as follows.

$$A \dotplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

By the term field we shall mean a, not necessarily commutative, division ring. Assume that α is a homomorphism of R into a field K. When K is generated as a field, by the image of R, we say that K is an *epic R-field* with respect to α. The set of epic R-fields can be made into a category whose morphisms are called specializations; a description of this category can be found in [**2**] or [**3**]. Here we only note that an epic R-field is characterized up to isomorphism by the set of matrices over R whose image is non-singular over K. Clearly, a necessary condition for a matrix A over R to be invertible over any epic R-field is that A should be square, say of order n, and the following condition should hold.

$$A = BC, \quad B \in {}^nR^s, \quad \text{and} \quad C \in {}^sR^n \quad \Longrightarrow \quad s \geqslant n.$$

Such a matrix A is said to be *full* over R.

All the rings we shall be concerned with are firs, that is, rings in which every left and every right ideal is free, of unique rank. We remark that over a fir, the diagonal sum of any number of matrices is full, if, and only if, each summand is full. Let R be a fir, then the category of epic R-fields has an initial element, called the *universal field of fractions* of R and denoted by $U(R)$, over which every full matrix becomes invertible (*cf.* [3, Theorem 4.C, p. 88]). Non-zero elements are full considered as 1×1 matrices, consequently we can view R as a subring of $U(R)$. It is shown in Chapter 7 of [2] how to construct $U(R)$ in terms of full matrices over R. For us it will suffice that, if p is any element of $U(R)$, then p can be obtained as the last entry of the unique solution of a system of linear equations over $U(R)$.

$$(A_0, A_*, A_n)\begin{pmatrix}1\\ u\\ p\end{pmatrix} = 0, \tag{1}$$

where $A_0, A_n \in {}^nR$, $A_* \in {}^nR^{n-1}$, $u \in {}^{n-1}U(R)$ and further (A_*, A_n) is full over R. We then say that (1) is an *admissible system* and (A_0, A_*, A_n) an *admissible matrix* for p. The above system can be rewritten as follows.

$$(A_*, A_n)\begin{pmatrix}I_{n-1} & u\\ 0 & p\end{pmatrix} = (A_*, -A_0).$$

It is clear now that p is non-zero, if, and only if, $(A_*, -A_0)$ is full.

§3. *The embedding.* Let E be a field with a subfield k, the free E-ring $E_k\langle X\rangle$ on X over k can be defined as the ring generated by E and X with defining relations $\alpha x = x\alpha$, for all $\alpha \in k$ and $x \in X$. For an alternative construction of $E_k\langle X\rangle$ the reader is referred to [3, pp. 111–115]; it is also indicated there that $E_k\langle X\rangle$ is a fir and hence possesses a universal field of fractions which is denoted by $E_k \langle\!\langle X \rangle\!\rangle$. When $E = k$ and k is commutative we obtain the free k-algebra $k\langle X\rangle$ whose universal field of fractions is denoted by $k\langle\!\langle X\rangle\!\rangle$. A homomorphism of $E_k\langle X\rangle$ into E which keeps E fixed is called an *evaluation*. Let f be an evaluation; it is clear that f is uniquely determined by its action on X. Notice also that E is an epic R-field with respect to f, with $R = E_k\langle X\rangle$. It follows that f extends to a specialization $E_k\langle\!\langle X\rangle\!\rangle \to E$ whose domain is the subring of $E_k\langle\!\langle X\rangle\!\rangle$ generated by the entries of the inverses of all the matrices over $E_k\langle X\rangle$ whose image under f becomes invertible (*cf.* [2, Theorem 7.2.3]). Suppose that E is a field with centre k such that (i) k is infinite and (ii) $[E:k] = \infty$. We then say that E satisfies *Amitsur's conditions.* The following is a basic result on evaluations.

Specialization Lemma ([3, Lemma 6.3.1]). *Let E be a field with centre k which satisfies Amitsur's conditions. Then for every full matrix A over $E_k\langle X\rangle$ there exists an evaluation, say f, such that A^f is non-singular over E.*

The specialization lemma can sometimes be applied, even if Amitsur's conditions are not assumed. We first need a definition. A homomorphism of rings is called *honest*, if it keeps full matrices full. Now let D be a field with centre C satisfying Amitsur's conditions. Let E be a subfield of D and put $k = E \cap C$. We then have a natural map

$$E_k\langle X\rangle \to D_C\langle X\rangle \tag{2}$$

induced by the inclusion $E \subseteq D$. If this map is honest, every full matrix over $E_k\langle X\rangle$ can be evaluated so that its image becomes non-singular over D. We note that Theorem 1 of [**6**] states that (2) is honest, if, and only if, E and C are *linearly disjoint* in D over k, that is, the natural map $E \bigotimes_k C \to D$ is injective.

We have seen that every evaluation induces a specialization. Suppose E satisfies Amitsur's conditions and let $p \in E_k \langle\!\langle X\rangle\!\rangle$. Further let (A_0, A_*, A_n) be a matrix over $E_k\langle X\rangle$, admissible for p. By the specialization lemma we can choose an evaluation $f: E_k\langle X\rangle \to E$ so that $(A_*, A_n)^f$ is invertible and p is then in the domain of the specialization, say ϕ, induced by f. If $p \neq 0$, $(A_*, -A_0)$ is full and hence so is $A = (A_*, A_n)\dotplus(A_*, -A_0)$. It follows that f can be chosen so that A^f is non-singular, and then $p^\phi \neq 0$. Moreover, let $\{p_i\}$ be a finite family of non-zero elements of $E_k \langle\!\langle X\rangle\!\rangle$ and, for each i, let $(A_0^{(i)}, A_*^{(i)}, A_{n_i}^{(i)})$ be admissible for p_i. We can choose f so that

$$\left\{\dotplus_i \left((A^{(i)}, A_{n_i}^{(i)})\dotplus(A^{(i)}, -A_0^{(i)})\right)\right\}^f$$

is non-singular. The domain of ϕ will then contain the family $\{p_i\}$, and $p_i^\phi \neq 0$ for all i. We have shown

THEOREM 3.1. *Let E be a field with centre k satisfying Amitsur's conditions and let X be a set. Then, for each finite family $\{p_i\}$ of non-zero elements of $E_k \langle\!\langle X\rangle\!\rangle$, there is a subring S of $E_k \langle\!\langle X\rangle\!\rangle$ containing $E_k\langle X\rangle$ and $\{p_i\}$, and an E-ring homomorphism ϕ of S into E, such that $p_i^\phi \neq 0$ for all i.*

Let D be a field with centre C and let X be a set; put $R = D_C\langle X\rangle$ and $U = D_C \langle\!\langle X\rangle\!\rangle$. Since $\operatorname{ctr} D = C$, every map $X \to D$ induces an evaluation $R \to D$; thus the set of evaluations of R into D can be identified with D^X. Let A be a full matrix over R; by the *non-singularity support* of A we understand the set

$$s(A) = \{f \in D^X \mid A^f \text{ is non-singular}\}.$$

Suppose that D satisfies Amitsur's conditions. The specialization lemma ensures that $s(A)$ is non-empty. For any other full matrix B over R, $A \dotplus B$ is also full and

$$s(A \dotplus B) = s(A) \cap s(B),$$

as is easily checked. Hence the family

$$\{s(A) \subseteq D^X \mid A \text{ is full over } R\}$$

has the finite intersection property and therefore is contained in some ultrafilter of

$\mathscr{P}(D^X)$. Let $\mathscr{F}$ be such an ultrafilter; we construct

$$D^{(D^X)}/\mathscr{F},$$

an ultrapower of D. This is a field, say K. Let β be an element of $D^{(D^X)}$; its image in K will be denoted by $\bar{\beta}$. Now let $a \in R$, and write α for the element of $D^{(D^X)}$ defined by putting $\alpha(f) = a^f$ for all $f \in D^X$. The correspondence $a \mapsto \bar{\alpha}$ is clearly a homomorphism. We claim that the subfield of K, generated by the image of R, is U. Let $A = (a_{ij})$ be full over R; we construct its inverse over K, which will prove the claim. For each $f \in s(A)$, A^f is invertible over D; we put

$$(A^f)^{-1} = (b_{ij}^{(f)}), \qquad b_{ij}^{(f)} \in D.$$

Now define $\beta_{ij} \in D^{(D^X)}$ as follows:

$$\beta_{ij}(f) = \begin{cases} b_{ij}^{(f)}, & \text{if} \quad f \in s(A), \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$\sum_k \alpha_{ik}(f)\beta_{kj}(f) = \sum_k a_{ik}^f b_{kj}^{(f)} = \delta_{ij}$$

for all $f \in s(A)$. Thus $(\overline{\beta_{ij}})$ is the required inverse and we can now state

THEOREM 3.2. *Let D be a field with centre C satisfying Amitsur's conditions and let X be a set. Further let E be a subfield of D, put $k = E \cap C$ and assume that the natural map $E_k\langle X\rangle \to D_C\langle X\rangle$ is honest. Then $E_k\langle\!\langle X\rangle\!\rangle$ can be embedded in an ultrapower of D.*

Proof. We have seen that $D_C\langle\!\langle X\rangle\!\rangle$ embeds in an ultrapower of D. This proves the assertion since, by [3, Theorem 4.3.3], $E_k\langle\!\langle X\rangle\!\rangle$ embeds in $D_C\langle\!\langle X\rangle\!\rangle$.

§4. *Applications.* Let L be a language for fields. Recall that a property P of fields is said to be a first-order property if there is an L-sentence σ, such that, for any field F,

$$F \text{ has property } P \Longleftrightarrow \sigma \text{ holds in } F.$$

An L-sentence σ is called universal if it is of form

$$\forall x_1, x_2, \dots, x_n \phi(x_1, x_2, \dots, x_n)$$

where ϕ is quantifier-free. It is clear that a first-order property expressed by a universal L-sentence is inherited by subfields. We begin by proving

THEOREM 4.1. *Retain the hypotheses of Theorem 3.2. Then $E_k\langle\!\langle X\rangle\!\rangle$ has every first-order property of D which can be expressed by a universal L-sentence.*

Proof. We know from Theorem 3.2 that $E_k\langle\!\langle X\rangle\!\rangle$ can be embedded in an ultrapower K of D. By Łos' theorem, the first-order properties of D and K coincide, and now the remark preceding this theorem completes the proof.

The first application is

THEOREM 4.2. *Let E be a field with centre k and let X be a set. Then $E_k \langle\!\langle X \rangle\!\rangle$ is orderable, if, and only if, E is so.*

Proof. The necessity of the condition is obvious. As is well known, orderability of a field can be expressed by a set of universal sentences; these sentences express that the sum of products of non-zero squares is non-zero. It follows from Theorem 4.1 that $E_k \langle\!\langle X \rangle\!\rangle$ can be ordered if E embeds in an ordered field D, such that the hypotheses of Theorem 3.2 are satisfied by D, $\operatorname{ctr} D$, E and k. When E itself can be ordered, the field of skew Laurent series $E(t)((x;\alpha))$, where α is the endomorphism of $k(t)$ induced by $t \mapsto t^2$, is such an extension of E. This completes the proof.

Consider now the universal L-sentence

$$\sigma = \forall x, y, t, u(\phi(x, y, t, u))$$

where

$$\phi(x, y, t, u) = \Big\{(xy \neq yx) \Longrightarrow \Big(((xt = tx) \wedge (xu = ux)) \Longrightarrow tu = ut\Big)\Big\}.$$

It is easy to see that σ expresses the first-order property, say CC, of fields: the centralizer of every non-central element is commutative. We shall show that the universal field of fractions of a free algebra has this property. First we establish

PROPOSITION 4.3. *Let D be a field with centre k satisfying Amitsur's conditions. Then* $\operatorname{ctr}(D_k \langle\!\langle X \rangle\!\rangle) = k$.

Proof. Set $U = D_k \langle\!\langle X \rangle\!\rangle$. Let $a \in \operatorname{ctr} U$; then in particular, a centralizes D, and hence every specialization $U \to D$, which is defined on a, maps a into k. Assume $a \notin k$ and consider the field $V = U_k \langle\!\langle z \rangle\!\rangle = D_k \langle\!\langle X \cup \{z\} \rangle\!\rangle$. Then clearly $az \neq za$ in V, and so by Theorem 3.1 we can find a specialization $s : V \to D$ which maps a, z and $az - za$ onto non-zero elements of D. This implies that $a^s \notin k$. Restricting s to the intersection of its domain with U, we obtain a specialization $U \to D$ which maps a outside k; a contradiction. Hence $\operatorname{ctr} U \subseteq k$; the reverse inclusion is obvious.

Let $T = \{t_i\}$ be a set of commuting indeterminates indexed by the integers and let α be the shift automorphism of $k(T)$, induced by $t_i \mapsto t_{i+1}$, for all $i \in \mathbb{Z}$. Write D for the field of skew Laurent series $k(T)((y;\alpha))$. Then $\operatorname{ctr} D = k$ and $[D:k] = \infty$, as is easily checked. Moreover, applying [2, Theorem 6.8.1] to $R = k(T)[[y;\alpha]]$ and then using the fact that $R \cup R^{-1} = D$, it is not hard to verify that D has the property CC. When k is infinite, from Theorem 4.1 and Proposition 4.3 we can deduce that elements of $D_k \langle\!\langle X \rangle\!\rangle$ outside k have commutative centralizers. Since $D_k \langle\!\langle X \rangle\!\rangle$ contains $k \langle\!\langle X \rangle\!\rangle$, the same is true of $k \langle\!\langle X \rangle\!\rangle$ and this proves part of

THEOREM 4.4. *Let k be a commutative field and let X be a set. Then the centralizer of every element of $k \langle\!\langle X \rangle\!\rangle$, outside k, is commutative.*

Proof. It remains to verify the assertion when k is finite. Set $V = k(t) \langle\!\langle X \rangle\!\rangle$; it is clear that V contains $k \langle\!\langle X \rangle\!\rangle$ and elements of $V \backslash k(t)$ have commutative centralizers. Let $a \in k(X) \backslash k$; then $a \notin k(t)$, and so the centralizer of a in V, hence also in $k \langle\!\langle X \rangle\!\rangle$, is commutative.

We easily obtain the

COROLLARY. *Let k be a commutative field and let X be a set. Then* $\operatorname{ctr}(k \prec X \succ) = k$.

We note that Proposition 3.3 and the above corollary are special cases of Theorem 4.7(iv) in [**5**]. Theorem 4.4 has been first proved in [**4**].

Before proving our final result we need a lemma.

LEMMA 4.5. *Let E be a field with centre k. Further let $T = \{t_i\}$ be a set of commuting indeterminates indexed by the integers, write α for the shift automorphism of $E(T)$ and set $D = E(T)((y;\alpha))$. Then* (i) $\operatorname{ctr} D = k$, (ii) $[D:k] = \infty$ *and* (iii) *every element of D, algebraic over k, is conjugate to an element of $E(T)$.*

The first two assertions are easily verified. Let $a \neq 0$ be an element of D, algebraic over k and suppose that in normal form

$$a = (a_0 + a'y)y^r,$$

where $0 \neq a_0 \in E(T)$, $a' \in E(T)[[y;\alpha]]$ and $r \in \mathbb{Z}$. Clearly if a^{-1} is conjugate to an element of D, so is a; hence without loss of generality we may assume that $r \geqslant 0$. Let p be a minimal polynomial for a; a straightforward normal form argument shows that $p(a_0) = 0$. Now by the Skolem–Noether theorem we can deduce that a is conjugate to a_0.

THEOREM 4.6. *Let E be a field with centre k and let X be a set. Then every element of $E_k \prec X \succ$, algebraic over k, is conjugate to an element of E.*

Proof. Put $U = E_k \prec X \succ$. Assume first that E satisfies Amitsur's conditions. Then $\operatorname{ctr} U = k$, by Proposition 4.3. Suppose $0 \neq u \in U$ is algebraic over k with minimal polynomial p. By Theorem 3.1 there is a specialization $s : U \to E$ whose domain includes a with $a^s \neq 0$. It is clear that a^s is a zero of p and so from the Skolem–Noether theorem we deduce that a and a^s are conjugates in U. When k is infinite but $[E:k]$ is finite, let D be as in the above lemma; then obviously E and k are linearly disjoint in D over k, so the natural map $E_k\langle X\rangle \to D_k\langle X\rangle$ is honest. It follows now from [3, Theorem 4.3.3] that we can consider U as a subfield of $D_k \prec X \succ$. Since D satisfies Amitsur's condition, every algebraic element of U is conjugate to an algebraic element of D and now the claim follows by the lemma. Finally, if k is finite, consider $E(t)$ whose centre is $k(t)$. Then E and $k(t)$ are linearly disjoint in $E(t)$ over k, so $E_k\langle X\rangle \to E(t)_{k(t)}\langle X\rangle$ is honest and we can view U as a subfield of $E(t)_{k(t)} \prec X \succ$. By the previous cases, every algebraic element of U is conjugate to an algebraic element of $E(t)$, which clearly must lie in E.

Putting $E = k$ in the theorem we obtain the

COROLLARY. *Let k be a commutative field and let X be a set. Then k is algebraically closed in $k \prec X \succ$.*

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