# Ellipsoidal Techniques for Reachability Analysis*

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**Abstract.** This report describes the calculation of the reach sets and tubes for linear control systems with time-varying coefficients and hard bounds on the controls through tight external and internal ellipsoidal approximations. These approximating tubes touch the reach tubes from outside and inside respectively at *every point* of their boundary so that the surface of the reach tube is totally covered by curves that belong to the approximating tubes. The proposed approximation scheme induces a very small computational burden compared with other methods of reach set calculation.
In particular such approximations may be expressed through ordinary differential equations with coefficients given in explicit analytical form. This yields exact parametric representation of reach tubes through families of external and internal ellipsoidal tubes. The proposed techniques, combined with calculation of external and internal approximations for intersections of ellipsoids, provide an approach to reachability problems for hybrid systems.

## Introduction

Recent activities to promote advanced automation of real-time processes have motivated new interest in the problem of reachability for controlled systems. This is also related to the problem of verification of hybrid systems [4]. Effective and implementable solutions to these problems must incorporate procedures for calculating reach sets and reach tubes for continuous-time systems [13]. Another demand for effectively performing such calculations comes from interval analysis in scientific computation. [11].
Among methods for reachability analysis are those based on ellipsoidal techniques, (see, for example [2], [3], [6]). Publications in this area were mostly concentrated on deriving a single equation that would produce a sub-optimal (with respect to volume) ellipsoidal approximation to the exact reach set.

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However, it turns that ellipsoidal methods allow *exact representations* of the reach sets and tubes for linear systems through *parametrized families of both external and internal ellipsoids* (see [6]). But to ensure effective calculation, an important open question is how to effectively single out such families of tightest ellipsoidal approximations to the reach tube that would touch its surface or the surface of its neighborhood at *every point*, (both from inside and outside !) and would thus *totally cover* this tube. A crucial point in organizing the calculation is to indicate such a parametrized variety of curves along which the procedure could be realized recurrently in time, without having to calculate the solution "afresh" for every new instant of time. A positive answer to the latter problem is given in this presentation for both external and internal approximations. It removes an unnecessary computational burden present in other methods and also opens new routes for deriving adequate numerical error estimates and new methods for systems other than those treated here [16], [12], [14]. The suggested approach is particularly relevant for hybrid systems since it allows further propagation to systems with resets. [1] An application of the proposed techniques to the verification of hybrid systems is given in paper [1].

In this paper we deal with reach tubes for control systems with linear dynamics and hard bounds on the control. We study the following question : given a reach tube (or its $\epsilon-$ neighborhood) and a smooth curve that runs along its surface, do there exist ellipsoid-valued external (internal) tubes that would contain (be contained in) the reach tube and touch the reach tube precisely along the given curve? The answer to this question is positive. However the properties of the respective ellipsoidal tubes do depend strongly on the given curve. The "good" situation is when the given curve may be realized *as a trajectory of the original control system.* [2] The required ellipsoidal tubes are then generated by ellipsoid-valued maps which satisfy the semigroup property and thus generate some generalized dynamical systems. Moreover, the approximating tubes are tight in the sense that there exists no other ellipsoidal tube that could be squeezed in between the approximation and the reach tube (for both external and internal ellipsoids). Lastly, the parameters of the ellipsoidal approximations are described by fairly simple ordinary differential equations. The paper also indicates the properties of the basic equations (18), (24) that allow them to be used correctly, without misunderstanding. Thus, it may be shown that when given is *any smooth curve* on the surface of the reach tube, which is not itself a system trajectory, there again exists ellipsoidal tubes that touch the reach sets along this curve. But now the respective ellipsoidal-valued maps may not satisfy the semigroup property and their evolution in time is not described by equations as simple as in the "good" case. The calculations then cannot be realized recur-

[1] These questions as well as the internal representations given here were not discussed in book [6].

[2] This happens when the given curve (a system trajectory) develops along the points of support for hyperplanes generated by vectors that are realized as the motions of the linear system adjoint to the homogeneous part of the control system under investigation.

sively. They require procedures that have to memorize additional items and are therefore computationally heavier than in the "good"case. A simplification of the computational procedure in this general case to the level of the "good" case results in *non-tight* approximations(!).

## 1 The Reachability Problem

Consider the linear system

$$\dot{x} = A(t)x + B(t)u, \quad t_0 \leq t \leq t_1, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the control. The matrices $A(t), B(t)$ are continuous and the system is *completely controllable* (see [9]). The control $u = u(t)$ is any measurable function restricted by hard bounds $u(t) \in \mathcal{P}(t)$, for almost all $t$, where $\mathcal{P}(t)$ is a nondegenerate ellipsoid continuous in $t$, namely, $\mathcal{P}(t) = \mathcal{E}(q(t), Q(t))$, and

$$\mathcal{E}(q(t), Q(t)) = \{u : (u - q(t), Q^{-1}(t)(u - q(t)) \leq 1\}, \tag{2}$$

with $q(t) \in \mathbb{R}^m$ (the center of the ellipsoid) and positive definite matrix function $Q(t) \in \mathbb{R}^{m \times m}$ (the matrix of the ellipsoid) continuous in $t$. The *support function* of the ellipsoid is

$$\rho(l|\mathcal{E}(q(t), Q(t))) = \max\{(l, x)|x \in \mathcal{E}(q(t), Q(t)\} = (l, q(t)) \; + \; (l, Q(t)l)^{1/2}.$$

The continuity of $Q(t)$ means that its support function $\rho(l|Q(t))$ is continuous in $t$ uniformly in $l$ with $(l, l) \leq 1$.

**Definition 11** *Given position* $\{t_0, x^0\}$*, the* **reach set** *(or "attainability domain")* $\mathcal{X}(\tau, t_0, x^0)$ at time $\tau > t_0$ *from this position is the set*

$$\mathcal{X}[\tau] = \mathcal{X}(\tau, t_0, x^0) = \{x[\tau]\}$$

*of all states* $x[\tau] = x(\tau, t_0, x^0)$ *reachable at time* $\tau$ *by system (1), with* $x(t_0) = x^0$*, through all possible controls* $u$ *that satisfy the constraint (2). The set-valued function* $\tau \mapsto \mathcal{X}[\tau] = \mathcal{X}(\tau, t_0, x^0)$ *is known as the* **reach tube***.*
*The reach set* $\mathcal{X}(\tau, t_0, \mathcal{X}^0)$ *(at time* $\tau$*,* **from set** $\mathcal{X}^0 = \mathcal{X}(t_0)$*) is the union*

$$\mathcal{X}(\tau, t_0, X^0) = \cup\{\mathcal{X}(\tau, t_0, x^0)|x^0 \in \mathcal{X}^0\}.$$

*The set-valued function* $\tau \mapsto \mathcal{X}[\tau] = \mathcal{X}(\tau, t_0, \mathcal{X}_0)$ *is known as the* **reach tube from set** $\mathcal{X}^0$*.*

The following properties may be checked directly.

**Lemma 1.** *The set-valued map* $\mathcal{X}(t, t_0, \mathcal{X}^0)$ *satisfies the semigroup property*

$$\mathcal{X}(t, t_0, \mathcal{X}^0) = \mathcal{X}(t, \tau, \mathcal{X}(\tau, t_0, \mathcal{X}^0)). \tag{3}$$

In the sequel it is assumed that $\mathcal{X}^0 = \mathcal{E}(x^0, X^0)$ is an ellipsoid. It is worth noting that the set $\mathcal{X}[\tau]$ may also be treated as the cut $\mathcal{X}[\tau] = \mathcal{X}(\tau, t_0, \mathcal{E}(x^0, X^0))$ of the solution tube $\mathcal{X}(\cdot) = \{\mathcal{X}[t] : t \geq t_0\}$ to the differential inclusion

$$\dot{x} \in A(t)x + \mathcal{E}(B(t)q(t), B(t)Q(t)B'(t)), \quad t \geq t_0, \quad x^0 \in \mathcal{E}(x^0, X^0). \tag{4}$$

A standard calculation using convex analysis indicates the following (see, for example[6]).

**Lemma 2.** *The support function*

$$\rho(l|\mathcal{X}(t, t_0, \mathcal{E}(x^0, X^0)) = (l, x^\star(t)) + (l, X(t, t_0)X^0X'(t, t_0)l)^{1/2} + \tag{5}$$

$$+ \int_{t_0}^{t} (l, X(t, s)B(s)Q(s)B'(s)X'(t, s)l)^{1/2} ds.$$

Here $X(t, s)$ is the transition matrix for the homogeneous system (1),

$$\partial X(t, s)/\partial t = A(t)X(t, s), \;\; X(s, s) = I, \;\; \dot{x}^\star = A(t)x^\star + B(t)q(t), \; x^\star(t_0) = x^0,$$

where $I$ is the identity matrix. For a time-invariant system $A(t) = A = const$, and $X(t, s) = \exp(A(t - s))$. The last representation leads to the next result.

**Lemma 3.** *The reach set $\mathcal{X}[t] = \mathcal{X}(t, t_0, \mathcal{E}(x^0, X^0))$ is a convex compact set in $\mathbb{R}^n$ that evolves continuously in $t$.*

Points on the boundary of the reach set $\mathcal{X}[t]$ have an important characterization. Consider a point $x^*$ on the boundary $\partial\mathcal{X}[\tau]$ of the reach set $\mathcal{X}[\tau] = \mathcal{X}(\tau, t_0, \mathcal{E}(x^0, X^0))$.[3] Then there exists a related *support vector* $l^*$ such that

$$(l^*, x^*) = \rho(l^*|\mathcal{X}[\tau]). \tag{6}$$

The control $u = u^*(t)$ and the initial state $x(t_0) = x^{*0} \in \mathcal{E}(x^0, X^0)$ which transfer system (1) from state $x(t_0) = x^{*0}$ to $x(\tau) = x^*$ is specified by the well-known "maximum principle" (see details in [9]). However, the calculation of the reach sets directly from these relations, especially in large dimensions, is cumbersome. Among the effective methods for these problems are those that rely on ellipsoidal techniques, as given in [6].

Remark 1.1 Due to the controllability assumption we will further assume, without loss of generality, that $B(t) = I$. To return to the case $B(t) \neq I$ it suffices in the sequel to substitute everywhere $Q(t)$ by $B(t)Q(t)B'(t)$. However, in the last case, for computational purposes it may be useful to start the approximation process at time $t = t_0 + \delta, \delta > 0$, to have $W(t_0 + \delta, t_0) > 0$.

[3] The boundary $\partial\mathcal{X}[\tau]$ of set $\mathcal{X}[\tau]$ may be here defined as the set $\partial\mathcal{X}[\tau] = \mathcal{X}[\tau] \setminus int\mathcal{X}[\tau]$. Under the controllability assumption, set $\mathcal{X}[\tau]$ has a non-void interior $int\mathcal{X}[\tau] \neq \emptyset$ for $\tau > t_0$.

## 2 Ellipsoidal Approximation of Reach Sets

Although the initial set $\mathcal{E}(x^0, X^0))$ and the control set $\mathcal{E}(q(t), Q(t))$ are ellipsoids, the reach set $\mathcal{X}[t] = \mathcal{X}(t, t_0, \mathcal{E}(x^0, X^0))$ will *not* generally be an ellipsoid. As indicated in [6], the reachability set $\mathcal{X}[t]$ may be approximated both externally and internally by ellipsoids $\mathcal{E}_-$ and $\mathcal{E}_+$, with $\mathcal{E}_- \subseteq \mathcal{X}[t] \subseteq \mathcal{E}_+$. The approximations are said to be *tight* if for any ellipsoid $\mathcal{E}$ the inclusion $\mathcal{X}[t] \subseteq \mathcal{E} \subseteq \mathcal{E}_+$ implies $\mathcal{E} = \mathcal{E}_+$, while inclusion $\mathcal{E}_- \subseteq \mathcal{E} \subseteq \mathcal{X}[t]$ implies $\mathcal{E} = \mathcal{E}_-$. Here we shall deal with both tight *external and internal* approximations.

*Problem 2.1.* Given a vector function $l^*(t)$, $(l^*, l^*) = 1$, continuously differentiable in $t$, find external and internal ellipsoids $\mathcal{E}^*_-[t] \subseteq \mathcal{X}[t] \subseteq \mathcal{E}^*_+[t]$ such that **for all** $t \geq t_0$, the equalities

$$\rho(l^*(t)|\mathcal{X}[t]) = \rho(l^*(t)|\mathcal{E}_+[t]) = \rho(l^*(t)|\mathcal{E}_-[t]) = (l^*(t), x^*(t)), \tag{7}$$

hold, so that the supporting hyperplane for $\mathcal{X}[t]$ generated by $l^*(t)$, namely, the plane $(x - x^*(t), l^*(t)) = 0$ that touches $\mathcal{X}[t]$ at point $x^*(t)$, is also a supporting hyperplane for $\mathcal{E}^*_+[t], \mathcal{E}^*_-[t]$ and touch them at the same point.

The solutions to this problem are given within the following statements.

**Theorem 21** *With $l(t) = l^*(t)$ given, the solution to Problem 2.1(external) is an ellipsoid $\mathcal{E}_+[t] = \mathcal{E}(x^*(t), X^*_+[t])$, where*

$$X^*_+[t] = \left(\int_{t_0}^{t} p^*_t(s)ds + p^*_0(t)\right)$$

$$\left(\int_{t_0}^{t} (p^*_t(s))^{-1} X(t,s)Q(s)X'(t,s)ds + p_0^{*-1}(t)X(t,t_0)X^0X'(t,t_0)\right), \tag{8}$$

*and*

$$p^*_t(s) = (l^*(t), X(t,s)Q(s)X'(t,s)l^*(t))^{1/2}, \tag{9}$$
$$p^*_0(t) = (l^*(t_0), X(t,t_0)X^0X'(t,t_0)l^*(t_0))^{1/2}.$$

This result follows from [6], [7]. Since the calculations have to be made for all $t$, the parametrizing functions $p_t(s), s \in [t_0, t], p_0(t)$ must depend on $t$. Note therefore that the result requires the evaluation of the integrals in (8) for each time $t$ and vector $l$. If the computation burden for each evaluation of (8) is $C_n t$, and we estimate the reach tube via (8) for $T$ values of time $t$ and $L$ values of $l$, the total computational burden would be $C_n TL$.

In other words, relations (8), (9) need to be solved "afresh" for each $t$. It may be more convenient for computational purposes to have them given in the form of recurrence relations. As indicated further, in the next Section, this could be done by selecting function $l^*(t)$ of Problem 2.1 in an appropriate way.

A similar result is available for internal approximations.

**Theorem 22** *With $l = l^*(t)$ given, the solution to Problem 2.1 (internal) is an ellipsoid $\mathcal{E}(x_-(t), X_-(t))$ , where*

$$X_-(t) = \tag{10}$$

$$= X(t,t_0)\Big(Q_0^{1/2}S'_{0t}(t_0) + \int_{t_0}^{t} X(t_0,\tau)Q^{1/2}(\tau)S'_t(\tau)d\tau\Big)'$$

$$\Big(S_{0t}(t_0)Q_0^{1/2} + \int_{t_0}^{t} S_t(\tau)Q^{1/2}(\tau)X'(t_0,\tau)\Big)X'(t,t_0). \tag{11}$$

*with* $S_0, S_t(\tau)$ *satisfying relations*

$$S_t(\tau)Q^{1/2}(\tau)X'(t,\tau)l^*(t) = \lambda_t(\tau)S_{0t}Q_0^{1/2}X'(t,t_0)l^*(t), \tag{12}$$

*and* $S'_{0t}S_{0t} = I; S'_t(\tau)S_t(\tau) \equiv I$ *for all* $t \geq t_0, \tau \in [t_0,t]$, *where*

$$\lambda_t(\tau) = (l^*(t), X(t,\tau)Q(\tau)X'(t,\tau)l^*(t))^{1/2}(l^*(t), X(t,t_0)Q_0X'(t,t_0)l^*(t))^{1/2}. \tag{13}$$

The parametrizing functions are *orthogonal matrix-valued functions* $S_t(\tau), S_{0t}$. They too are dependent on $t$, so that the calculations have to be done “afresh” for each $t$ as in the “external” case. Thus, the computaion in general is not recursive. To ease the computational burden we look for recurrence relations.

## 3 Recurrence Relations

There is a special selection of functions $l^*(t)$ that lead to recurrence relations.

**Assumption 31** *The function* $l^*(t)$ *is of the form,* $l^*(t) = X(t_0,t)l$ *, with* $l \in \mathbb{R}^n$ *given. For the time-invariant case* $l^*(t) = e^{-A'(t-t_0)}l$. [4]

Then $p_t^*(s), p_0^*(t), X_+^*[t]$ of (9), (8) transform into

$$p_t^*(s) = (l, X(t_0,s)Q(s)X'(t_0,s)l)^{1/2} = p^*(s);\ p_0^*(t) = (l, X^0 l)^{1/2} = p_0^*, \tag{14}$$

and

$$X_+^*[t] = X(t,t_0)X_+(t)X'(t,t_0),\ \ X_+[t] = \Big(\int_{t_0}^{t} p^*(s)ds + p_0^*\Big)\Psi(t), \tag{15}$$

where

$$\Psi(t) = \tag{16}$$

$$= \int_{t_0}^{t} (l, X(t_0,s)Q(s)X'(t_0,s)l)^{-1/2}X(t_0,s)Q(s)X'(t_0,s)ds + (l, X^0 l)^{-1/2}X^0.$$

In this particular case $p_t^*(s)$ does not depend on $t$ ($p_{t'}^*(s) = p_{t''}^*(s)$ for $t' \neq t$”) and the lower index $t$ may be dropped.

[4] Under this Assumption the vector $l^*(t)$ is the solution to equation

$$\dot{l^*} = -A'(t)l^*,\ l^*(t_0) = l,$$

which is the adjoint to the homogeneous part of equation (1).

Direct differentiation of $X_+[t]$ yields

$$\dot{X}_+[t] = \pi^*(t)X_+[t] + \pi^{*-1}(t)X(t_0,t)Q(t)X'(t_0,t),\ X_+[t_0] = X^0, \tag{17}$$

where

$$\pi^*(t) = p^*(t)\left(\int_{t_0}^t p^*(s)ds + p_0^*\right)^{-1}.$$

Calculating

$$(l, X_+[t]l) = \left(\int_{t_0}^t p^*(s)ds + p_0^*\right)(l, \Psi(t)l) = (\int_{t_0}^t p^*(s)ds + p_0^*)^2,$$

one may observe that

$$\pi^*(t) = (l, X(t_0,t)Q(t)X'(t_0,t)l)^{1/2}(l, X_+[t]l)^{-1/2}. \tag{18}$$

In order to pass to the matrix function $X_+^*[t]$ we note that

$$\begin{aligned}\dot{X}_+^*[t] = &A(t)X(t,t_0)X_+[t]X'(t,t_0) + X(t,t_0)X_+[t]X'(t,t_0)A(t)\\ &+X(t,t_0)\dot{X}_+[t]X'(t,t_0).\end{aligned}$$

After a substitution from (16) this gives

$$\dot{X}_+^* = A(t)X_+^* + X_+^*A'(t) + \pi^*(t)X_+^* + \pi^{*-1}(t)Q(t),\ X^*(t_0) = X^0. \tag{19}$$

We summarize these results as follows.

**Theorem 31** *Under Assumption 3.1 the solution to Problem 2.1(external) is given by the ellipsoid $\mathcal{E}_+^*[t] = \mathcal{E}(x_+(t), X_+^*[t])$, where $x_+(t) = x^\star(t)$ and $X_+^*[t]$ is a solution to equations (18), (16).*

Since the set $X_+^*[t]$ depends on vector $l$, we denote $X_+^*[t] = X_+^*[t]_l$.

**Theorem 32** *For any $t \geq t_0$ the reach set $\mathcal{X}[t]$ may be described as*

$$\mathcal{X}[t] = \cap\{\mathcal{E}(x_+(t), X_+^*[t]_l)\} |\ l : (l,l) = 1\}. \tag{20}$$

This is a direct consequence of Theorems 3.1.
Thus, if $l^*(t)$ satisfies Assumption 3.1, the complexity of computing a tight, external ellipsoidal approximation to the reach set for all $t$, is the same as computing the solution to the differential equation (18). If $L$ values of $l$ and $T$ values of $t$ are evaluated, the computational burden is $C_nTL$.
For the general(non-recursive) case, the relation corresponding to (18) is far more complicated and is actually a functional-differential equation which requires recalculations for each $t$. If however (18) is still used for the general case, the inclusion $\mathcal{X}[t] \subset \mathcal{E}_+[t]$ remains true but the tightness property is lost.
Throughout the previous discussion we have observed that under Assumption 3.1 the tight external ellipsoidal approximation $\mathcal{E}(x^\star, X_+^*(t))$ is governed by the

simple ordinary differential equations (18). Moreover, in this case the points $x^*(t)$ of support for the hyperplanes generated by vector $l(t)$ run along *a system trajectory* of (1) which is generated by a control that satisfies the maximum principle.

Similar facts are also true for internal approximations. We now again select function $l^*(t)$ to satisfy Assumption 3.1. Then substituting $l^*(t)$ in (12),(13), we observe that the relations for calculating $S_t(\tau), \lambda_t(\tau)$ transform into

$$S_t(\tau)Q^{1/2}(\tau)X'(t_0,\tau)l = \lambda_t(\tau)S_{0t}Q_0^{1/2}l; \;\; S_0'S_0 = I; S'(\tau)S(\tau) \equiv I \tag{21}$$

and

$$\lambda_t(\tau) = (l, X(t_0,\tau)Q(\tau)X'(t_0,\tau)l)^{1/2}/(l,Q_0l)^{1/2}. \tag{22}$$

Here the known functions used for calculating $S_t(\tau), \lambda_t(\tau)$ *do not depend on* $t$. Therefore, the unknown functions $S_t(\tau), \lambda_t(\tau)$ *do not depend on* $t$ either, no matter what is the interval $[t_0, t]$. The lower indices $t$ in $S_{0t}, S_t, \lambda_t$ may be dropped. Differentiating (10) in view of the last remark, we come to

$$\dot{X}_- = A(t)X_- + X_-A'(t) + \dot{Q}_*'Q_* + Q_*'\dot{Q}_*, \tag{23}$$

where

$$Q_*(t) = S_0Q_0^{1/2}X'(t,t_0) + \int_{t_0}^{t} S(\tau)Q^{1/2}(\tau)X'(t,\tau)d\tau,$$

$$\dot{Q}_*(t) = S(t)Q^{1/2}(t), Q_*(t_0) = S_0Q_0.$$

Using the notation

$$H(t) = Q_*^{-1}(t)S(t)Q^{1/2}(t) = Q_*^{-1}(t)\dot{Q}_*(t), \tag{24}$$

we further come to equation

$$\dot{X}_- = A(t)X_- + X_-A'(t) + H'(t)X_-(t) + X_-(t)H(t), \;\; X_(t_0) = Q_0. \tag{25}$$

and also observe that the center $x_-(t) = x_+(t) = x^\star(t)$. This leads to the following theorem.

**Theorem 33** *Under Assumption 3.1 the solution to Problem 3.1 (internal) is given by ellipsoid $\mathcal{E}(x_-(t), X_-(t))$ where $X_-(t)$ is given by equations (24), (23), and the functions $S(t), \lambda(t)$ involved in the calculation of $H(t)$ satisfy together with $S_0$ the relations (20), (21), where the lower indices $t$ in $S_{0t}, S_t, \lambda_t$ are to be dropped.*

Function $H(t) = Q_*^{-1}(t)S(t)Q^{1/2}(t)$ in (23) may be also expressed through equation

$$\dot{Q}_* = Q_*A'(t) + S(t)Q^{1/2}(t), \; Q_*(t_0) = S_0Q_0^{1/2}. \tag{26}$$

This gives the result

**Lemma 4.** *The ellipsoid $\mathcal{E}(x_-(t), X_-(t))$ of Theorem 3.3 given by equations (23)-(25) depends on the selection of the orthogonal matrix function $S(t)$ and for any such $S(t)$ the inclusion*

$$\mathcal{E}(x_-(t), X_-(t)) \subseteq \mathcal{X}[t], \quad t \geq t_0, \tag{27}$$

*is true with equalities (7)(internal) attained under conditions (20), (21).The following relation is true*

$$\mathcal{X}[t] = cl\{\cup\{\mathcal{E}(x_-(t), X_-[t]_l)\}| \, l : (l, l) = 1\}\}.$$

*where $clY$ stands for the closure of set $Y$.*

The boundary of $\mathcal{X}[t]$ is thus described as a function of a *finite-dimensional parameter* $l \in \mathbb{R}^n$.
Let us now suppose that function $l(t)$ of Problem 2.1 (internal) is *any* continuous curve on the surface of $\mathcal{X}[t]$. Then one has to use formula (10), keeping in mind that $S_{0t}, S_t(\tau)$ do depend on $t$. After a differentiation of (10) in $t$, one may observe that (25) transforms into

$$\dot{X}_- = A(t)X_- + X_-A'(t) + H'(t)X_-(t) + X_-(t)H(t) + \Phi(t, \cdot), \quad X_(t_0) = Q_0. \tag{28}$$

where $\Phi(t, \cdot)$ is a functional of $S_t(\tau), S_{0t}$. The calculations are then far more cumbersome than under Assumption 3.1. If in this general case we still use the simpler equation (24), then the inclusion (26) will still be true, but the property of tightness will be lost. Note that under Assumption 3.1 the term $\Phi(t, \cdot)$ disappears.

## 4 The Reach Tube

The results of the previous Sections may be thus summarized as follows. Suppose Assumption 3.1 is fulfilled. then the points $x^*(t)$ of support for vector $l^*(t) = X'(t, t_0)l, \; l \in \mathbb{R}^n$, namely, those for which the equalities

$$(l^*(t), x^*(t)) = \rho(l^*(t)|\mathcal{X}[t]) = \rho(l^*(t)|\mathcal{E}(x^\star(t), X_+^*[t])) \tag{29}$$

are true **for all** $t \geq t_0$, may be reached from initial state

$$x^{*0} = x^*(t_0) = \frac{X^0 l}{(l, X^0 l)^{1/2}} + x^0. \tag{30}$$

and from a *trajectory* $x^*(t)$ that satisfies the following "maximum relation":

$$(l^*(t), x^*(t)) = \max\{(l^*(t), x)|x \in \mathcal{E}(x^\star, X_+^*[t])\}, \tag{31}$$

which is attained at

$$x^*(t) = x^\star(t) + X_+^*[t]l^*(t)(l^*(t), X_+^*[t]l^*(t))^{-1/2}, \tag{32}$$

where $X_+^*[t] = X[t]$ is the solution to equations (18), (17).
For $B(t) \equiv I$ and $Q(t)$ nondegenerate the same trajectory (31) may be attained through *internal ellipsoids* with

$$x^*(t) = x^\star(t) + X_-[t]l^*(t)(l^*(t), X_-[t]l^*(t))^{-1/2}, \tag{33}$$

where $X_-[t]$ is a solution to (24), (23). The same property holds if $Q(t)$ is nondegenerate and the system (1) is controllable.
Denoting $x^*(t) = x[t,l]$, we thus come to a two-parameter surface $x[t,l]$ that defines the boundary $\partial\mathcal{X}$ of the reachability tube $\mathcal{X} = \cup\{X[t],\ t \geq t_0\}$. With $t = t'$ fixed and $l \in \mathcal{S}$ varying,($\mathcal{S}$ is a unit sphere), the vector $x[t',l]$ runs along the boundary $\partial\mathcal{X}[t']$. On the other hand, with $l = l'$ fixed and with $t$ varying, the vector $x[t,l']$ moves along one of the trajectories $x^*(t)$ that touch the reachability set $\mathcal{X}[t]$ according to (7). Then

$$\cup\{x[t,l] | l \in \mathcal{S}\} = \partial\mathcal{X}[t],\ \cup\{x[t,l] | l \in \mathcal{S},\ t \geq t_0\} = \partial\mathcal{X}$$

*Remark* 4.1. The possibility of using both external and internal representations is important for treating hybrid dynamics for systems that allow resets. Thus, if for example set $\mathcal{X}^0 = \cap\mathcal{E}_i^0$, one may introduce approximations of type $\mathcal{E}_-^0 \subseteq \mathcal{X}^0 \subseteq \mathcal{E}_+^0$ to start the calculations of the reach set $\mathcal{X}[t]$. On the other hand, if for some $t' > t_0$ we have $\mathcal{X}[t'] \cap \mathcal{E}_M$, where $\mathcal{E}_M$ stands for a given guard, we may introduce approximations of type

$$\begin{aligned}\mathcal{E}_-^*[t'] &\subseteq \cup\{\mathcal{E}(x_-(t'), X_-(t')) | (l,l) \leq 1\} \cap \mathcal{E}_M \\ &\subseteq \cap\{\mathcal{E}(x_+(t'), X_+(t')) | (l,l) \leq 1\} \cap \mathcal{E}_M \subseteq \mathcal{E}_+^*[t']\end{aligned}$$

for the resets and proceed for $t \geq t'$ with the procedures of Sections 2-4.

## 5 An Example

Taking system

$$\dot{x}_1 = x_2,\ \ \dot{x}_2 = u,$$
$$x_1(0) = x_1^0, x_2(0) = x_2^0,\ \ |u| \leq \mu,\ \mu > 0,\ X^0 = \{x : (x,x) \leq \epsilon^2\}.$$

and omitting the calculations, we indicate the external and internal ellipsoidal approximations of the respective reach set $\mathcal{X}[t] = \mathcal{X}(t, 0, X^0)$.
Here the “good” curves of Assumption 3.1 have the form of straight lines: $l^*(t) = \exp(-A't)l$ or $l_1^* = l_1, l_2^* = l_2 - tl_1$.They are shown in fig.1 for $\epsilon > 0$. The external and internal approximations that touch the reach set $\mathcal{X}[t]$ along these lines are shown in fig.2 and fig.3 for $\epsilon = 0$ and in fig.4 for $\epsilon > 0$.

## 6 Conclusion

This paper specifies and studies the behavior of the tight external and internal ellipsoidal approximations of reach sets and reach tubes for linear time-variant

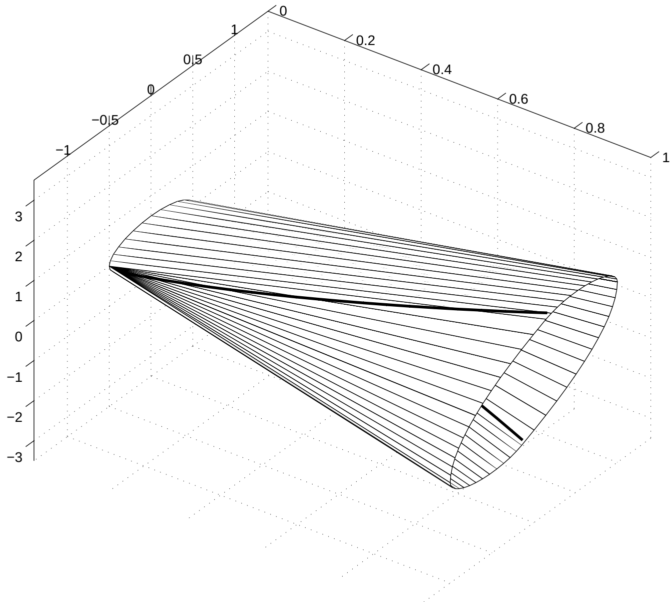
0
0.2
0.4
0.6
0.8
1
1
0.5
0
−0.5
−1
3
2
1
0
−1
−2
−3

Fig.1

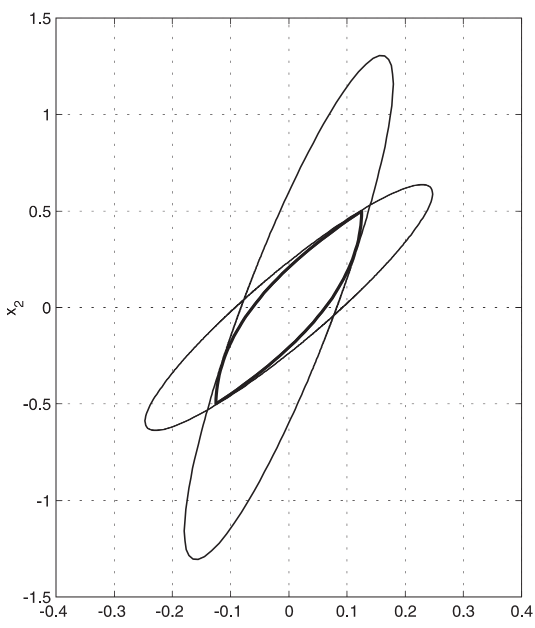
1.5
1
0.5
0
-0.5
-1
-1.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
$x_1$
$x_2$

Fig.2

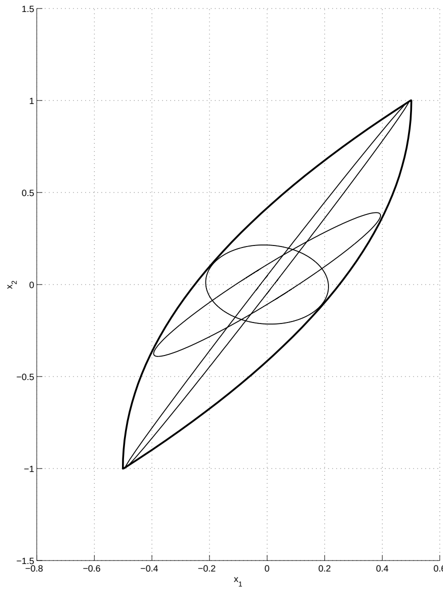


Fig.3

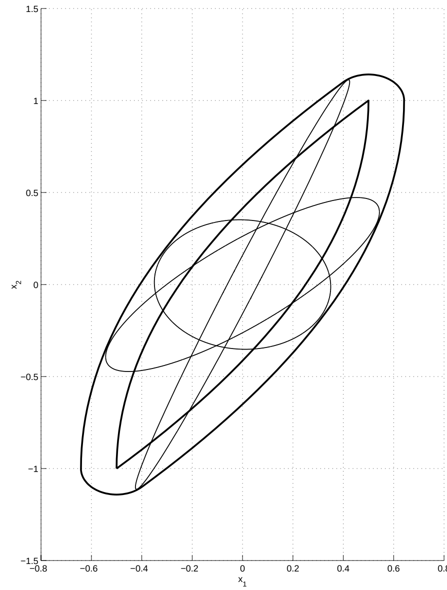


Fig.4

control systems. It shows that equations (18), (24) with appropriately chosen parametrizing functions $\pi(t), S(t)$ generate two family of tight (external and internal) ellipsoidal aproximations to the reach tube $\mathcal{X}[t]$ which touch it along a certain family of "good" curves that *cover the whole tube*. It gives analytical representations that allow to achieve a substantial reduction of the computation burden for calculating these sets as compared to direct methods and thus gives effective techniques for calculating the reach tubes in a compact recursive form. The analytical relations developed in this paper open routes to the investigation of precise error estimates in ellipsoidal approximations for problems of evolution, estimation and control as well as to the development of new computational tools for classes of systems more complicated than those treated in this paper. In particular, they indicate convenient tools for the treatment of hybrid dynamics (see [1]).

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