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Realization problem of SISO nonlinear systems: a transfer function approach

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Abstract— **This paper studies the nonlinear realization problem, i.e. the problem of finding an observable state space representation of a SISO nonlinear system described by an input-output differential equation, within an alternative polynomial approach. To find the solution the so-called adjoint polynomials and adjoint transfer functions are introduced for nonlinear systems. Thereby the clear connection to the solution known from linear systems is established.**

I. INTRODUCTION

In control theory, systems are typically described either by higher order input-output diffrential equations or by sets of coupled first order differential equations, called the state equations. In the linear case, any control system, described by a higher order input-output differential equation, can be equivalently described by the state equations and vice versa. However, this is not valid anymore for nonlinear systems. Although for a given state space representation a corresponding input-output equation can be, at least locally, always found [2], the converse does not hold in general. The necessary and sufficient realizability conditions are rather restrictive and a generic nonlinear input-output equation cannot be realized in the state space form. Three different intrinsic realizability conditions were given in terms of integrability of the subspaces of differential one-forms [2], involutivity of the conditionally invariant distributions of the vector fields [19] and commutativity of iterative Lie brackets of vector fields [5]. Moreover, the algorithm based solutions [3], [9] are demonstrated for computing the integrable bases of the subspaces of one-forms in [16]. The algorithm to compute the sequence of subspaces of one-forms from the polynomial system description using polynomial methods is given in [20].

In the linear case, the state equations in the observer and controller canonical forms can be directly written out from the transfer function. However, the transfer function formalism was recently developed also for nonlinear systems [21], [10]. In doing so one associates with the control system two polynomials, as in [22], defined over the field of meromorphic functions. This can be understood also that way that the nonlinear system equations are linearized using Kähler diffrentials [14] and then the ideas similar to those applied for linear time-varying systems in [6] can be applied. See also [7] and references therein. That way it results in the linearized system description resembling time-varying linear system description except that now the time-varying coefficients of the polynomials are not necessarily independent [17].

The nonlinear transfer function formalism has been already employed in [18] to investigate some structural properties of nonlinear systems, in [13] to study the nonlinear model matching and in [12] to study the observer design. In this paper we continue in this direction and study the realization problem of nonlinear systems in terms of transfer functions. That is, we are looking for a more direct solution to the realization problem, that does not require to compute the sequence of subspaces of the one-forms [2] in order to find the differentials of the state coordinates, but rather allow to write down the diffrentials of the state equations directly from the transfer function. The state equations could then be found by integrating these equations. Our starting point is the observer canonical form in terms of differential one-forms. The results are similar to those in [15] addressing the time-varying linear case except that in the nonliner case the additional integrability conditions occur and the time-varying coefficients of the polynomials are not necessarily independent. From a technical point of view, the notion of the so-called adjoint polynomials plays a key role in both time-varying linear and nonlinear case.

II. LINEAR CASE

In the linear case, any SISO control system desribed by a higher order input-output differential equation of the form

$$y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_0y = b_{n-1}u^{(n-1)}+\cdots+b_0u \quad (1)$$

where $y,\ u \in \mathbf{R}$ and $a_i,\ b_i \in \mathbf{R}$, can be equivalently described by a set of coupled first order differential equation of the form

$$\begin{aligned} \dot{x} &= Ax+Bu \\ y &= Cx \end{aligned} \quad (2)$$

where $x \in \mathbf{R}^n$ and A, B, C are matrices over $\mathbf{R}$ with appropriate dimensions. Clearly, the converse holds too.

In switching between those two types of representations Laplace transforms and transfer functions play a key role

$$F(s) = C(sI-A)^{-1}B = \frac{b_{n-1}s^{n-1}+\cdots+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_0} \quad (3)$$

as generally known. In particular, one thus can easily find not only an input-otuput differential equation (1) from a state

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space representation (2) but also an observable state space representation (2) from an input-otuput differential equation (1), just by using the coefficients of polynomials in the tranfer function (3). For instance

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \quad B = \begin{pmatrix} b_0 \\ \vdots \\ b_{n-1} \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}$$

where the pair (A, C) is in the observer canonical form.
Note that also other types of canonical forms can be directly found from the transfer function (3), for instance the controller canonical form.

III. Nonlinear case

However, the situation is different when systems are nonlinear.
For any state space representation of the form

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= g(x) \end{aligned} \tag{4}$$

where $x \in \mathbf{R}^n$, y, $u \in \mathbf{R}$ and f, g are meromorphic functions, a corresponding input-output differential equation of the form

$$y^{(n)} = \varphi(y, \dot{y}, \dots, y^{(n-1)}, u, \dot{u}, \dots, u^{(n-1)}) \tag{5}$$

where φ is a meromorphic function, can be (at least locally) always found [2]. We recall from [2] the state elimination algorithm (specified for SISO systems) since we use it later.

A. State elimination

Let k denote the minimum nonnegative integer such that

$$\operatorname{rank}\frac{\partial(g, \dots, g^{(k-1)})}{\partial x} = \operatorname{rank}\frac{\partial(g, \dots, g^{(k)})}{\partial x}$$

If $\partial g/\partial x = 0$ we define $k = 0$ and $y = g(\cdot)$ is the input-output equation. Note that $\frac{\partial(g,\dots,g^{(k-1)})}{\partial x}$ is, in fact, the observability matrix, see for instance [2], and if $k < n$ the system (4) is not observable.
Since

$$\operatorname{rank}\frac{\partial(g, \dots, g^{(k-1)})}{\partial x} = k$$

we can write (at least locally)

$$x = \phi(y, \dots, y^{(k-1)}, u, \dots, u^{(k-2)})$$

and the input-output equation we are looking for is

$$y^{(k)} = g^{(k)}(x, u, \dots, u^{(\nu)}) = g^{(k)}(\phi(\cdot), u, \dots, u^{(\nu)})$$

For more details see [2].

Example 1: Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 u \\ y &= x_1 \end{aligned}$$

Then $\dot{y} = x_2$ and thus $(x_1, x_2) = (y, \dot{y})$. So, the input-output equation

$$\ddot{y} = x_1 u = yu$$

B. Realization problem

However, although for any state space representation (4) a corresponding input-output differential equation (5) always exists, at least locally, the converse does not hold in general. There exists a class of input-output differential equations of the form (5) for which the state-space representation of the form (4) simply does not exist, see for instance [2]. A typical example is given by the system $\ddot{y} = \dot{u}^2$.

Remark 1: Note that it has been known for a long time that a necessary condition for realizability of an i/o equation is linearity in the highest derivative in control, see for example [4]. The above result also follows from Theorem 1 in [8].

Thus, the necessary and sufficient conditions, under which such a realization problem would be solvable in nonlinear case, had to be found. Since any transfer function formalism for nonlinear control systems was not developed until recently, the problem was solved within different approaches. Here, we recal the solution in terms of the algebraic formalism of one-forms introduced in [2].

Problem statement. For the nonlinear system (5) find, if possible, an observable state space representation of the form (4) such that it is its realization.

The system (5) defines a field of meromorphic functions $\mathcal{K}$ of variables $\{y, \dots, y^{(n-1)}, u^{(k)}; k \geq 0\}$. Let us define the formal vector space of differential one-forms

$$\mathcal{E} = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}\xi; \xi \in \mathcal{K}\}$$

and a filtration of $\mathcal{E}$ given by a sequence of subspaces $\{\mathcal{H}_k\}$ of $\mathcal{E}$ as follows:

$$\begin{aligned} \mathcal{H}_1 &= \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y, \dots, \mathrm{d}y^{(n-1)}, \mathrm{d}u, \dots, \mathrm{d}u^{(s)}\} \\ \mathcal{H}_{k+1} &= \operatorname{span}_{\mathcal{K}}\{\omega \in \mathcal{H}_k; \dot{\omega} \in \mathcal{H}_k\} \end{aligned}$$

where $s = \max\{i \geq 0; \partial\varphi/\partial u^{(s)} \neq 0\}$.

Proposition 1: For the system (5) there exists an observable state space realization of the form (4) if and only if

- $n > s$
- $\mathcal{H}_k$ is integrable for all $k = 1, \dots, s+2$.

Example 2: Consider the system $\ddot{y} = \dot{u}^2$ from [2] for which we get

$$\begin{aligned} \mathcal{H}_1 &= \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}u, \mathrm{d}\dot{u}\} \\ \mathcal{H}_2 &= \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}u\} \\ \mathcal{H}_3 &= \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y} - 2\dot{u}\mathrm{d}u\} \end{aligned}$$

Since $\mathcal{H}_3$ is not integrable there does not exist any system of the form (4) which, when applying the state elimination algorithm, would generate the input-output equation $\ddot{y} = \dot{u}^2$.

Example 3: Consider the system $\ddot{y} = \dot{y}u$ for which we get

$$\begin{aligned} \mathcal{H}_1 &= \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}u, \mathrm{d}\dot{u}\} \\ \mathcal{H}_2 &= \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}u\} \\ \mathcal{H}_3 &= \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}\} \end{aligned}$$

This time, $\mathcal{H}_3$ is integrable and choosing $x_1 = y$, $x_2 = \dot{y}$ yields the state space realization

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_2 u \\ y &= x_1 \end{aligned}$$

IV. Transfer function approach

Since recently the transfer function formalism has been introduced for nonlinear systems, it would be interesting to address the realization problem directly from the transfer function description and not from the higher order input-output diffrential equation. Here, we recall briefly such a concept and refer reader to [10] for more details.

The left skew polynomial ring $\mathcal{K}[s]$ of polynomials in s over $\mathcal{K}$ with the usual addition, and the (non-commutative) multiplication given by the commutation rule

$$sa = as + \dot{a} \tag{6}$$

where $a \in \mathcal{K}$, represents the ring of linear ordinary differential operators that act over the vector space of one-forms $\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi; \xi \in \mathcal{K}\}$ in the following way

$$\left(\sum_{i=0}^{k} a_i s^i\right) v = \sum_{i=0}^{k} a_i v^{(i)}$$

for any $v \in \mathcal{E}$. The commutation rule (6) actually represents the rule for differentiating.

Lemma 1 (Ore condition): For all non-zero $a, b \in \mathcal{K}[s]$, there exist non-zero $a_1, b_1 \in \mathcal{K}[s]$ such that $a_1 b = b_1 a$.

Thus, the ring $\mathcal{K}[s]$ can be embedded to the non-commutative quotient field $\mathcal{K}\langle s\rangle$ by defining quotients as

$$\frac{a}{b} = b^{-1} \cdot a$$

The addition and multiplication in $\mathcal{K}\langle s\rangle$ are defined as

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{\beta_2 a_1 + \beta_1 a_2}{\beta_2 b_1}$$

where $\beta_2 b_1 = \beta_1 b_2$ by Ore condition and

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{\alpha_1 a_2}{\beta_2 b_1}$$

where $\beta_2 a_1 = \alpha_1 b_2$ again by Ore condition.

Remark 2: Due to the non-commutative multiplication (6) they, of course, differ from the usual rules. In particular, in case of the multiplication we, in general, cannot simply multiply numerators and denominators, nor cancel them in a usual manner. We neither can commute them as the multiplication in $\mathcal{K}\langle s\rangle$ is non-commutative as well.

Once the fraction of two skew polynomials is defined we can introduce the transfer function of the nonlinear system (4) (respectively (5)) as an element $F(s) \in \mathcal{K}\langle s\rangle$ such that $\mathrm{d}y = F(s)\mathrm{d}u$.

When starting with the state space description (4) we, after differentiating, get

$$\begin{aligned} \mathrm{d}\dot{x} &= A\mathrm{d}x + B\mathrm{d}u \\ \mathrm{d}y &= C\mathrm{d}x \end{aligned} \tag{7}$$

where $A = (\partial f/\partial x)$, $B = (\partial f/\partial u)$, $C = (\partial g/\partial x)$. Or alternatively

$$\begin{aligned} s\mathrm{d}x &= A\mathrm{d}x + B\mathrm{d}u \\ \mathrm{d}y &= C\mathrm{d}x \end{aligned}$$

from which follows

$$F(s) = C(sI - A)^{-1}B$$

Remark 3: Note that in spite of the formal similarity to the transfer functions of linear time-invariant systems, this time we have to invert matrix $(sI - A)$ over the non-commutative quotient field $\mathcal{K}\langle s\rangle$, which is far from trivial. Clearly, the entries of $(sI - A)$ are skew polynomials from $\mathcal{K}[s]$. Thus, the inversion requires finding the solutions of a set of linear equations over the non-commutative field, see [10] and references therein for more details.

Example 4: Consider the system from Example 1. After differentiating we get

$$A = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ x_1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Note that

$$(sI - A)^{-1} = \begin{pmatrix} \frac{s}{s^2-u} & \frac{1}{s^2-u} \\ \frac{u}{s^2-\dot{u}/u-u} & \frac{s-\dot{u}/u}{s^2-\dot{u}/u-u} \end{pmatrix}$$

and thus

$$F(s) = C(sI - A)^{-1}B = \frac{x_1}{s^2 - u} = \frac{y}{s^2 - u}$$

When starting with the input-output description (5) we, after differentiating, get

$$\mathrm{d}y^{(n)} - \sum_{i=1}^{n-1} \frac{\partial \varphi}{\partial y^{(i)}} \mathrm{d}y^{(i)} = \sum_{i=0}^{m} \frac{\partial \varphi}{\partial u^{(i)}} \mathrm{d}u^{(i)}$$

or alternatively

$$a(s)\mathrm{d}y = b(s)\mathrm{d}u$$

where $a(s) = s^n - \sum_{i=1}^{n-1} \frac{\partial \varphi}{\partial y^{(i)}} s^i$, $b(s) = \sum_{i=0}^{m} \frac{\partial \varphi}{\partial u^{(i)}} s^i$ and $a(s), b(s) \in \mathcal{K}[s]$. Then the transfer function

$$F(s) = \frac{b(s)}{a(s)}$$

Note that this allows us to compute the transfer function even for nonrealizable systems.

Example 5: Consider the system from Example 2. After differentiating we get

$$\begin{aligned} \mathrm{d}\ddot{y} &= 2\dot{u}\mathrm{d}\dot{u} \\ s^2\mathrm{d}y &= 2\dot{u}s\mathrm{d}u \end{aligned}$$

and thus

$$F(s) = \frac{2\dot{u}s}{s^2}$$

A. State elimination

The possibility to find a corresponding input-output differential equation of a nonlinear system from its transfer function has been already mentioned in [10]. Note that $\mathrm{d}x = (sI - A)^{-1}B\mathrm{d}u$ and thus $\mathrm{d}y = C(sI - A)^{-1}B\mathrm{d}u + D\mathrm{d}u$. Hence, by computing the transfer function we directly treat the state elimination problem of the system (4).

Example 6: Consider the system from Example 4 with

$$F(s) = \frac{y}{s^2 - u}$$

Hence

$$\begin{aligned}(s^2 - u)\mathrm{d}y &= y\mathrm{d}u \\ \mathrm{d}\ddot{y} - u\mathrm{d}y &= y\mathrm{d}u \\ \ddot{y} &= yu\end{aligned}$$

However, the computation of an input-output differential equation from transfer function is more complicated than the state elimination procedure. Of course, the practical interest of the transfer functions will not lie in such an application. This serves mainly as a demonstration of the similarity to Laplace transforms and transfer functions of linear systems.

B. Realization problem

In the realization problem we are interested in finding a change of coordinates which transforms the input-output equation (5) into the state-space representation (4), as discussed in section III-B. In what follows, we present a transfer function approach to the realization problem having the contact points to that of [15].

The basic idea seems to be very easy; for a given transfer function $F(s) = \frac{b(s)}{a(s)}$ just find the matrices A, B and C, for instance in the observer canonical form, such that $F(s) = C(sI - A)^{-1}B$, like in the linear case. This is, of course, correct. However, the situation is not so straightforward for the coefficients of polynomials $b(s)$, $a(s)$ are not reals but, in general, meromorphic functions from $\mathcal{K}$. Stating matrices A, B and C in the observer canonical form, directly from transfer function like in linear case, does not yield, in general, required result, neither when working over differentials, as can be easily demonstrated.

Consider for instance a second-order nonlinear system with the transfer function

$$F(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

where all b_i, $a_i \in \mathcal{K}$. Then, if assuming a realization in terms of differentials (7) with

$$A = \begin{pmatrix} 0 & -a_0 \\ 1 & -a_1 \end{pmatrix} \quad B = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

we get

$$\begin{aligned}\mathrm{d}y &= \mathrm{d}x_2 \\ \mathrm{d}\dot{y} &= \mathrm{d}x_1 - a_1 \underbrace{\mathrm{d}x_2}_{\mathrm{d}y} + b_1\mathrm{d}u \\ \mathrm{d}\ddot{y} &= -a_0 \underbrace{\mathrm{d}x_2}_{\mathrm{d}y} + b_0\mathrm{d}u - a_1\mathrm{d}\dot{y} + b_1\mathrm{d}\dot{u} - \dot{a}_1\mathrm{d}y + \dot{b}_1\mathrm{d}u\end{aligned}$$

and clearly

$$F(s) = \frac{b_1 s + \dot{b}_1 + b_0}{s^2 + a_1 s + \dot{a}_1 + a_0}$$

which differs from that we started.

Obviously, the problem is caused by the fact that all b_i, a_i are, in general, functions from $\mathcal{K}$. Hence, in derivating $a\mathrm{d}\xi$, with $a \in \mathcal{K}$ and $\mathrm{d}\xi \in \mathcal{E}$, we have to consider that $\dot{\widehat{a\mathrm{d}\xi}} = a\mathrm{d}\dot{\xi} + \dot{a}\mathrm{d}\xi$, or in terms of polynomials $s\, a\mathrm{d}\xi = (as + \dot{a})\mathrm{d}\xi$. While in the linear case, $a \in \mathbf{R}$, it is only $\dot{\widehat{a\mathrm{d}\xi}} = a\mathrm{d}\dot{\xi}$, or in terms of polynomials $s\, a\mathrm{d}\xi = as\,\mathrm{d}\xi$.

However, this problem can be avoided elegantly by remaining the derivations unexpanded. This in terms of polynomials means that the indeterminate s has to be always from the *left*. Such an idea can be very transparently implemented by introducing the notion of adjoint polynomials.

1) Adjoint polynomials: Adjoint polynomials [1] represent in some sense dual objects to skew polynomials. We can get them by moving the indeterminate on the *left* of each summand. Formally, they are defined as follows.

Definition 1: The adjoint of a skew polynomial ring $\mathcal{K}[s]$ is defined as the skew polynomial ring $\mathcal{K}[s^*]$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}^* = -\frac{\mathrm{d}}{\mathrm{d}t}$$

That is, if $a(s) = a_n s^n + \ldots + a_1 s + a_0$ is a polynomial in $\mathcal{K}[s]$ then the adjoint polynomial $a^*(s^*)$ is defined by the formula

$$a^*(s^*) = {s^*}^n a_n + \ldots + s^* a_1 + a_0 \in \mathcal{K}[s^*] \tag{8}$$

Note that products ${s^*}^i a_i$ must be computed in the skew polynomial ring $\mathcal{K}[s^*]$, following the commutation rule

$$s^* a = a s^* - \dot{a}$$

Example 7: Consider the system from Example 3. After differentiating $\mathrm{d}\ddot{y} = u\mathrm{d}\dot{y} + \dot{y}\mathrm{d}u$ we get

$$(s^2 - us)\mathrm{d}y = \dot{y}\mathrm{d}u$$

where

$$\begin{aligned}a(s) &= s^2 - us \\ b(s) &= \dot{y}\end{aligned}$$

are polynomials in $\mathcal{K}[s]$. The adjoint of $\mathcal{K}[s]$ is the ring $\mathcal{K}[s^*]$ with $\frac{\mathrm{d}}{\mathrm{d}t}^* = -\frac{\mathrm{d}}{\mathrm{d}t}$ and the comutation rule $s^* a = a s^* - \dot{a}$. Thus, in according to the formula (8), the adjoint polynomials can be found by computing

$$\begin{aligned}a^*(s^*) &= {s^*}^2 - s^* u = \\ &= {s^*}^2 - u s^* + \dot{u} \\ b^*(s^*) &= \dot{y}\end{aligned}$$

This is, in fact, the formalization of the idea of moving the indeterminate on the *left* of each summand in original polynomials

$$\begin{aligned}a(s) &= s^2 - su + \dot{u} \\ b(s) &= \dot{y}\end{aligned}$$

Notice that we get the coefficients of adjoint polynomials.

Remark that in commutative case; that is, the case of linear systems when all coefficients are in $\mathbf{R}$, a polynomial and its adjoint are identical objects. Remark also that the adjoint is a bijective mapping and

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}^*\right)^* = \frac{\mathrm{d}}{\mathrm{d}t}$$

Moreover

$$(a^*)^* = a, \qquad (ab)^* = b^*a^*$$

for any $a, b \in \mathcal{K}[s]$. For more details see [1].

Using the above defined notion of adjoint polynomials we introduce the notion of the adjoint transfer function.

Definition 2: Let

$$F(s) = \frac{b(s)}{a(s)}$$

where $b(s)$, $a(s) \in \mathcal{K}[s]$, be the transfer function of a nonlinear system (4) (respectively (5)). Then

$$F^*(s^*) = \frac{b^*(s^*)}{a^*(s^*)}$$

where $b^*(s^*)$, $a^*(s^*) \in \mathcal{K}[s^*]$, is said to be the adjoint transfer function.

Example 8: Consider the system from the previous example. Transfer function and the adjoint transfer function can be stated as

$$F(s) = \frac{\dot{y}}{s^2 - us}$$

and, respectively

$$F^*(s^*) = \frac{\dot{y}}{{s^*}^2 - us^* + \dot{u}}$$

Now, we will discuss main results of this paper. Firstly, we show how the realization over differentials (7) looks like. This is in agreement with [15] for linear time-varying case.

Lemma 2: Let

$$F(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

and

$$F^*(s^*) = \frac{b^*_{n-1}{s^*}^{n-1} + \cdots + b^*_0}{{s^*}^n + a^*_{n-1}{s^*}^{n-1} + \cdots + a^*_0}$$

be the tranfer function and, respectively, the adjoint transfer function of the nonlinear system (5). Then

$$\begin{array}{rcl} \mathrm{d}\dot{\xi}_1 &=& -a^*_0\mathrm{d}\xi_n + b^*_0\mathrm{d}u \\ \mathrm{d}\dot{\xi}_2 &=& \mathrm{d}\xi_1 - a^*_1\mathrm{d}\xi_n + b^*_1\mathrm{d}u \\ &\vdots& \\ \mathrm{d}\dot{\xi}_n &=& \mathrm{d}\xi_{n-1} - a^*_{n-1}\mathrm{d}\xi_n + b^*_{n-1}\mathrm{d}u \\ \mathrm{d}y &=& \mathrm{d}\xi_n \end{array} \tag{9}$$

is the realization in terms of differentials.

Sketch of the proof: It follows the same line as in [12] or [15]. □

Clearly, for the system (5) to be realizable the equations (9) have to result in certain cequences of exact/integrable one-forms.

Proposition 2: Let

$$F(S) = \frac{b_{n-1}s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

and

$$F^*(s^*) = \frac{b^*_{n-1}{s^*}^{n-1} + \cdots + b^*_0}{{s^*}^n + a^*_{n-1}{s^*}^{n-1} + \cdots + a^*_0}$$

be the tranfer function and, respectively, the adjoint transfer function of the nonlinear system (5). Let $\omega_i = b^*_{i-1}\mathrm{d}u - a^*_{i-1}\mathrm{d}y$ for $i = 1, \ldots, n$. Then:

1) there exists an observable state space realization in the observer cannonical form if and only if all ω_i's are exact; that is,
$$\mathrm{d}\omega_i = 0$$
for $i = 1, \ldots, n$.
2) there exists an observable state space realization of the form (4) if and only if
$$\mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y} - \omega_n, \mathrm{d}\ddot{y} - \dot{\omega}_n - \omega_{n-1}, \ldots, \\ \mathrm{d}y^{(n-1)} - \omega_n^{(n-2)} - \omega_{n-1}^{(n-3)} - \ldots - \omega_2\}$$
is integrable.

Sketch of the proof: For part 1 it follows the same line as in [12].

Part 2: Following the state elimination algorithm discussed in section III-A applied to (9) we get

$$\begin{array}{rcl} \mathrm{d}y &=& \mathrm{d}\xi_n \\ \mathrm{d}\dot{y} &=& \mathrm{d}\xi_{n-1} + \omega_n \\ \mathrm{d}\ddot{y} &=& \mathrm{d}\xi_{n-2} + \omega_{n-1} + \dot{\omega}_n \\ &\vdots& \\ \mathrm{d}y^{(n-1)} &=& \mathrm{d}\xi_1 + \omega_2 + \omega_n^{(n-2)} \end{array}$$

from which one can derive $\mathcal{H}_{s+2} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi_1, \ldots, \mathrm{d}\xi_n\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y} - \omega_n, \mathrm{d}\ddot{y} - \dot{\omega}_n - \omega_{n-1}, \ldots, \mathrm{d}y^{(n-1)} - \omega_n^{(n-2)} - \omega_{n-1}^{(n-3)} - \ldots - \omega_2\}$. □

V. Examples

Example 9: Consider the system from Example 2 with the transfer function

$$F(s) = \frac{2\dot{u}s}{s^2}$$

We showed that the system is not realizable in the state space representation of form (4). The same result can be found employing adjoints. The adjoint transfer function is

$$F^*(s^*) = \frac{2\dot{u}s^* - 2\ddot{u}}{{s^*}^2}$$

and the realization in terms of differentials

$$\begin{array}{rcl} \mathrm{d}\dot{\xi}_1 &=& -2\ddot{u}\mathrm{d}u \\ \mathrm{d}\dot{\xi}_2 &=& \mathrm{d}\xi_1 + 2\dot{u}\mathrm{d}u \\ \mathrm{d}y &=& \mathrm{d}\xi_2 \end{array}$$

Neither $\omega_1 = -2\ddot{u}\mathrm{d}u$ nor $\omega_2 = 2\dot{u}\mathrm{d}u$ are exact, therefore the realization in the ovserver cannonical form does not exist. In addition, neither any observable realization of the form (4) exists. Note that employing the state elimination algorithm

$$\begin{aligned} \mathrm{d}y &= \mathrm{d}\xi_2 \\ \mathrm{d}\dot{y} &= \mathrm{d}\xi_1 + 2\dot{u}\mathrm{d}u \end{aligned}$$

from which $\mathrm{d}\xi_2 = \mathrm{d}y$, $\mathrm{d}\xi_1 = \mathrm{d}\dot{y} - 2\dot{u}\mathrm{d}u$ and thus $\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\dot{y} - 2\dot{u}\mathrm{d}u, \mathrm{d}y\}$ which is not integrable.

Example 10: Consider the system from Example 3 with the transfer function

$$F(s) = \frac{\dot{y}}{s^2 - us}$$

We showed that the system is realizable. The adjoint transfer function is

$$F^*(s^*) = \frac{\dot{y}}{s^{*2} - us^* + \dot{u}}$$

and the realization in terms of differentials

$$\begin{aligned} \mathrm{d}\dot{\xi}_1 &= -\dot{u}\mathrm{d}\xi_2 + \dot{y}\mathrm{d}u \\ \mathrm{d}\dot{\xi}_2 &= \mathrm{d}\xi_1 + u\mathrm{d}\xi_2 \\ \mathrm{d}y &= \mathrm{d}\xi_2 \end{aligned}$$

Also here neither $\omega_1 = -\dot{u}\mathrm{d}y + \dot{y}\mathrm{d}u$ nor $\omega_2 = u\mathrm{d}y$ are exact, therefore the realization in the ovserver cannonical form does not exist. However, an observable realization of the form (4) exists. Employing the state elimination algorithm

$$\begin{aligned} \mathrm{d}y &= \mathrm{d}\xi_2 \\ \mathrm{d}\dot{y} &= \mathrm{d}\xi_1 + u\mathrm{d}\xi_2 \end{aligned}$$

from which $\mathrm{d}\xi_2 = \mathrm{d}y$, $\mathrm{d}\xi_1 = \mathrm{d}\dot{y} - u\mathrm{d}y$ and thus $\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y} - u\mathrm{d}y\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}\}$ which is, this time, integrable. Choosing $x_1 = y$, $x_2 = \dot{y}$ yields the same realization as in Example 3.

VI. Conclusions

In this work an alternative polynomial approach to the realization problem of nonlinear control systems was given. The notion of adjoint polynomials and adjoint transfer functions played a key role and established the direct connection to the results known from linear systems. Results of this paper can be easily extended to the case of nonlinear discrete-time systems, for the corresponding algebraic setting see [11] and [12] and, in addition, following the ideas of [15] one can try to extend such a concept even for nonlinear time-delay systems.

References

[1] S.A. Abramov, H.Q. Le and Z. Li, Univariate ore polynomial rings in computer algebra, *Journal of Mathematical Sciences*, vol. 131, 2005, pp 5885-5903.

[2] G. Conte, C.H. Moog and A.M. Perdon, *Algebraic Methods for Nonlinear Control Systems. Theory and Applications*, Springer-Verlag, London, 2007.

[3] P. Crouch and F. Lamnabhi-Lagarrigue, State space realizations of nonlinear systems defined by input-output differential equations. In: A. Bensouan and J. Lion. *Analysis and optimization systems*, Lecture Notes in Control Information Science, Springer, vol. 111, 1988, pp 138-149.

[4] P. Crouch, F. Lamnabhi-Lagarrigue and D. Pinchon, A realization algorithm for input-output systems, *International Journal of Control*, vol. 62, 1995, pp 941-960.

[5] E. Delaleau and W. Respondek, Lowering the orders of derivatives of controls in generalized state space systems, *J. Math. Systems Estim. Control* vol. 5, 1995, pp 1-27.

[6] M. Fliess, Une interprétation algébrique de la transformation de Laplace et des matrices de transfert, *Linear Algebra and its Applications*, vol. 203, 1994, pp 429-442.

[7] M. Fliess, C. Join and H. Sira-Ramírez, Non-linear estimation is easy, *Int. J. Modelling, Identification and Control*, vol. 4, 2008, pp 12-27.

[8] M. Freedman and J. Willems, Smooth representation of systems with diffrentiated inputs, *IEEE Transactions on Automatic Control*, vol. 23, 1978, pp 16-21.

[9] S.T. Glad, Nonlinear state space and input-output descriptions using differential polynomials. In: J. Descusse, M. Fliess, A. Isidori and P. Leborne. *New trends in nonlinear control theory*, Lecture Notes in Control and Information Science, Springer, vol. 122, 1989, pp 182-189.

[10] M. Halás, An algebraic framework generalizing the concept of transfer functions to nonlinear systems, *Automatica*, vol. 44, 2008, pp 1181-1190.

[11] M. Halás and Ü. Kotta, "Pseudo-linear algebra: a powerful tool in unification of the study of nonlinear control systems", *in 7th IFAC Symposium NOLCOS*, Pretoria, South Africa, 2007.

[12] M. Halás and Ü. Kotta, "Synthesis of Observers for Nonlinear Systems: A Polynomial Approach", *47th Conference on Decision and Control*, Cancun, Mexico, 2008.

[13] M. Halás, Ü. Kotta and C.H. Moog, "Transfer function approach to the model matching problem of nonlinear systems", *17th IFAC World Congress*, Seoul, Korea, 2008.

[14] J. Johnson, Kähler differentials and differential algebra, *Annals of Mathematics*, vol. 89, 1969, pp 92-98.

[15] E. W. Kamen, Representation and realization of operational differential equations with time-varying coefficients, *Journal of the Franklin Institute*, vol. 301, 1976, pp 559-571.

[16] Ü. Kotta and T. Mullari, Equivalence of different ralization methods for higher order nonlinear input-output differential equations, *European Journal of Control*, vol. 11, 2005, pp 185-193.

[17] Z. Li, M. Ondera and H. Wang, "Simplifying skew fractions modulo differential and difference relations", *International Symposium on Symbolic and Algebraic Computation ISSAC*, Linz, Austria, 2008.

[18] A.M. Perdon, C.H. Moog and G. Conte, "The pole-zero structure of nonlinear control systems", *7th IFAC Symposium NOLCOS*, Pretoria, South Africa, 2007.

[19] A.J. Van der Schaft, On realization of nonlinear systems described by higher-order differential equations, *Math Syst Theory* vol. 19, 1987, 239-275.

[20] M. Tõnso and Ü. Kotta, "Realization of continuous-time nonlinear input-output equations: polynomial approach", *12th International Conference on Computer Aided Systems Theory*, Las Palmas de Gran Canaria, Spain, 2009.

[21] Y. Zheng and L. Cao, Transfer function description for nonlinear systems, *Journal of East China Normal University (Natural Science)*, vol. 2, 1995, pp 15-26.

[22] Y. Zheng, J. Willems and C. Zhang, A polynomial approach to nonlinear system controllability, *IEEE Transactions on Automatic Control*, vol. 46, 2001, pp 1782-1788.