COMBINATORIAL RESOLUTION OF SYSTEMS OF DIFFERENTIAL EQUATIONS, I. ORDINARY DIFFERENTIAL EQUATIONS.

by

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§1. Introduction

There is a strong interplay between differential equations and enumerative combinatorics. A simple and classical example goes back to 1879 with the enumeration by André [AN] of **alternating permutations** $\sigma = \sigma(1) > \sigma(2) < \sigma(3) > \sigma(4) \dots$ of $[n] = \{1, 2, \dots, n\}$. The sequence of numbers E_n of such permutations (known as the **Euler numbers**) satisfies a certain recurrence relation which is equivalent to the following system of differential equations

$$\begin{aligned} y' &= 1 + y^2 , \quad & y(0) = 0 \\ z' &= yz , \quad & z(0) = 1 \end{aligned} \tag{1.1}$$

where $y = y(t) = \sum_{n\geq 0} E_{2n+1}\, t^{2n+1}/(2n+1)!$ and $z = z(t) = \sum_{n\geq 0} E_{2n}\, t^{2n}/(2n)!$, whose unique solution is given by the trigonometric (generating) functions $y = \tan t$ and $z = \sec t$.

One direction is the use of differential equations in order to obtain some information about the generating functions of finite combinatorial structures. For example, Collins et al. [CGJ] consider the enumeration of permutations of $[n]$ having a "periodic up-down sequence". They show that the corresponding generating functions satisfy a differential system of Riccati type. A similar enumeration problem involving the resolution of systems of differential equations can be found in [CA1], [CA2] where the so-called Ollivier functions appear. As another example, see [TU] where Tutte establishes a second order differential equation for the generating function of map colourings.

A second direction is to start from a (system of) differential equation(s) satisfied by a given function $f(t) = \sum_{n\geq 0} a_n\, t^n/n!$. Now the problem is to find finite combinatorial structures enumerated by the integers a_n. This gives a so-called "combinatorial interpretation" of the numbers a_n, or equivalently of the function $f(t)$. The combinatorialists

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want to derive the identities satisfied by these functions $f(t)$ from bijections and correspondences between the combinatorial objects themselves. These constructions constitute a "combinatorial" or "geometric" theory of the functions, giving a new insight into their analytic properties. Such a methodology can be applied to a wide class of functions, including, for example, elementary trigonometric functions or generating functions of orthogonal polynomials. A sample is given by the following trigonometric identities which can be proven directly from correspondences between finite combinatorial structures:

$$\sec^2 t = 1 + \tan^2 t \qquad (1.2) \qquad\qquad \tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} \qquad (1.3)$$

$$\int_0^t \tan x \, dx = \log \sec t \qquad (1.4) \qquad\qquad \int_0^t \sec x \, dx = \log(\tan t + \sec t) \qquad (1.5)$$

See [LO] in this volume for a combinatorial proof of (1.3) and §6 for (1.5). Another example is the combinatorial theory of Jacobi elliptic functions that has been initiated by Dumont [D1], [D2], Flajolet [FLA] and Viennot [V2] and where many questions remain open.

The first purpose of this paper is to give a systematic way to find combinatorial interpretations of the solutions of (systems of) differential equations of the form

$$y_i' = f_i(t, y_1, \ldots, y_k) \quad , \qquad y_i(0) = \alpha_i \quad , \qquad i = 1, \ldots, k \qquad (1.6)$$

where $f_1, \ldots, f_k$ are generating functions (in $k+1$ variables) of some combinatorial finite structures. The combinatorial objects may be weighted by some formal parameters. We will give a "**canonical**" **combinatorial interpretation** of the solution of such systems (see §3 and §6). In order to work at this level of generality, without expliciting the combinatorial structures having $f_1, \ldots, f_k$ as generating functions, we need an abstract formalization of what we mean by "combinatorial structures", together with some basic operations on these structures.

The fundamental notion used here is that of $\mathbb{L}$-species, a variant in the theory of combinatorial species of structures ([BL], [J1], [LA]). Intuitively, a $\mathbb{L}$-species corresponds to a certain kind of combinatorial objects constructed on totally ordered sets. An example of $\mathbb{L}$-species is the "alternating permutations" or, more generally, "permutations having a given up-down sequence". "Young tableaux" is another example. In comparison, the species of structures originally introduced by André Joyal (herein called $\mathbb{B}$-species, see §2) do not make use of a total order on the underlying sets. Examples of $\mathbb{B}$-species are "endofunctions", "permutations", "involutions", "linear lists", "graphs", "trees", "binary trees", etc.

In fact the differential equations themselves can be lifted to the combinatorial level and written, in the case of one equation, in the form

$$Y' = M(T,Y) \quad , \qquad Y'(0) = Z \quad , \tag{1.7}$$

where Y, Y' and M are certain 2-sorted $\mathbb{L}$-species and T and Z are variables corresponding to two distinct sorts. The standard operations on $\mathbb{L}$-species, such as derivation and substitution alluded to in (1.7), are defined in §2.

Our main result is that any differential equation (1.7) has a "canonical" solution which is given in terms of **M-enriched increasing arborescence**, as displayed in figure 3.2. , and that this solution is **unique**, up to a canonical isomorphism. A similar result also holds for combinatorial systems of the form (1.6). The unicity is in contrast with the case of $\mathbb{B}$-species: G. Labelle [L4] has shown that the same equation can have several non-isomorphic solutions (as $\mathbb{B}$-species).

We now illustrate the general method by giving the "canonical" combinatorial solution of the system (1.1). Here y and z will be denoted by Y and Z and considered as $\mathbb{L}$-species, that is, as constructions which can be performed on linearly ordered sets. Starting with the initial conditions, Y and Z are built up recursively using (1.1). The equations $Y' = 1 + Y^2$, $Y(0) = 0$, and $Z' = YZ$, $Z(0) = 1$, are respectively visualized in figures 1.1 and 1.2, where "min." designates the minimum element of the underlying sets.

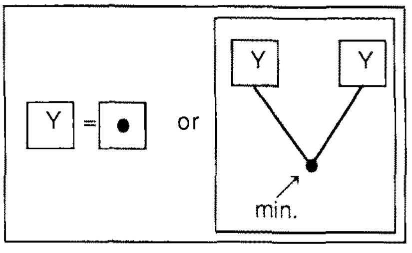


Figure 1.1

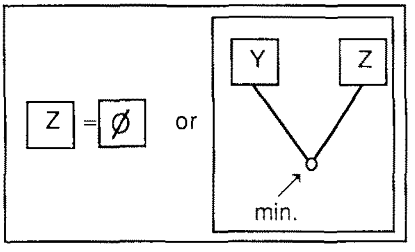


Figure 1.2

Iterating these procedures, one finds that Y can be canonically identified with the construction of so-called **complete increasing binary trees**. The same is true for Z, except that there remains an empty leaf at the extreme right end (see fig. 1.3 and 1.4 respectively). The sets of vertices of the trees are linearly ordered and "increasing" means that their elements are increasing when going away from the root.

This canonical solution, the $\mathbb{L}$-species of "increasing binary trees" should, by unicity, be isomorphic to that of "alternating permutations". In this case, the canonical isomorphism is the well known bijection due to Foata and Schützenberger (see [FS2, 2nd part] or [FST], [FR], [V1], [V3], [GJ, 5.2.14]) which associates to any increasing binary tree (not necessarily complete) a permutation, by "**projection**". For example the trees displayed on figure 1.3 (resp figure 1.4) give the permutation of an odd (resp. even) number of elements σ = 8 3 5 1 7 2 6 4 9 (respect. σ = 4 1 8 3 5 2 7 6). The alternating property comes from the fact that the binary tree is complete (resp. "almost" complete).

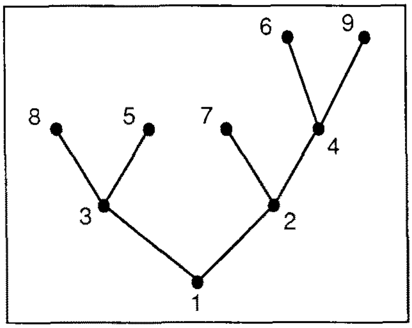


Figure 1.3

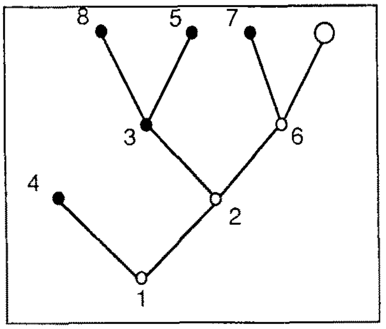


Figure 1.4

In fact, at least three different classes of permutations can be used to give a combinatorial interpretation of the Euler numbers: the above alternating permutations, the **André permutations** introduced by Foata and Schützenberger [FS1], [FST], and the **Jacobi permutations** defined by Viennot [V2]. These classes are naturally associated with the "canonical" interpretations of three different systems of differential equations (see §6 below).

Note that the canonical solutions of systems of differential equations given in terms of M-enriched increasing arborescences, remain at a certain "recursive level". This is the price to pay for a general method that works for any system of differential equations. Usually more combinatorial work needs to be done to go to other levels of combinatorics, as above in going back to the alternating, André or Jacobi permutations. This last step does not involve differential equations any more. It is taking place only at the "bijective combinatorics" level and is somewhat analoguous to the transformation of recursive programs into non-recursive ones in computer science. An advantage of our method is to bring some order inside the "zoo" of the many known different combinatorial interpretations of classical functions.

This paper is the first of a series devoted to a combinatorial theory of differential equations. Our objective is to lift as much as possible of the classical theory to the combinatorial or discrete level, thus obtaining a completely new insight and also new tools to effectively handle or solve differential equations.

In this paper we use the **M-enriched increasing arborescences** to give combinatorial ("bijective") proofs of some classical results such as, for example, the method of "variation of the constant" for linear differential equations. Using the beautiful idea of **eclosions** of G. Labelle [L2], we give in §5 a combinatorial proof of the classical **Lie-Gröbner formulas.** We recall basic concepts about $\mathbb{L}$-species in §2. For simplicity we work with only one equation in §3, §4 and §5. M-enriched increasing arborescences are formally introduced in §3 and the main theorem is proven there. In §6 we extend this theory to systems of differential equations and give some applications, in particular to the Jacobi elliptic functions and the **Duffing** equation.

The topics covered by the following papers will include a **combinatorial integral calculus** [LV1], the **method of separation of variables** [LV2], and **non linear systems with forced entries,** with particular combinatorial methods of resolution [LV3]. An interesting fact is that the combinatorial interpretation of systems with forced entries is contained in the interpretation of the corresponding ordinary system with no entries. Using Fliess theory of such systems (see for example [F1], [FL1], [FL2], [LL]) we will be able to find an explicit combinatorial interpretation of the coefficients of the so-called **Volterra kernels**. Then, using other combinatorial tools, such as the notion of "**histories**", we will give explicit new formulas for these coefficients in certain special cases, such as the one given in [FL2, p. 566]. The cases of the Duffing equation and of elliptic functions and integrals will also be treated in more detail in other papers.

ACKNOWLEDGMENTS

Much of this paper has been inspired by recent work of Gilbert Labelle on functional equations, differential equations, Lie Gröbner series and Newton-Raphson iteration, in the context of species of structures (see [L2], [L3] and [L4]), and also of Michel Fliess and his school on the resolution of non-linear differential equations with forced entries in the context of non-commuting formal power series (see [F1], [FL1], [FL2] and [LL]), together with some previous work of M.P. Schützenberger about the non-commutative solution of the differential equation related to André permutations [SC].

We would like to thank many colleagues for discussions and helpful suggestions, in particular François Bergeron, Michel Fliess, Dominique Foata, Gilbert Labelle, Françoise Lamnabhi-Lagarrigue, and Don Rawlings, and also Nantel Bergeron who MACDREW the figures of this paper.

§2. $\mathbb{L}$-species

The $\mathbb{L}$-species are introduced as a variant of the theory of species of structures to account for the combinatorial constructions that make use of a linear order on the underlying set, for example "alternating permutations" or "increasing binary trees". To give precise and concise definitions, we introduce the following categories:

$\mathbb{E}$ = the category of finite sets and functions
$\mathbb{B}$ = the category of finite sets and bijections
$\mathbb{L}$ = the category of finite linearly ordered sets and order preserving bijections.

Recall then (see [J1], [LA], or [L5] and [YE] in this volume) that a **species of structures** F (herein called $\mathbb{B}$-species to emphasize the distinction with $\mathbb{L}$-species) is a functor

$$F : \mathbb{B} \longrightarrow \mathbb{E}. \tag{2.1}$$

By contrast, an **$\mathbb{L}$-species** M is defined to be a functor

$$M : \mathbb{L} \longrightarrow \mathbb{E}. \tag{2.2}$$

This means that to each linearly ordered finite set ℓ, M associates a finite set, denoted by $M[\ell]$, whose elements are called **M-structures on** ℓ, and to each order preserving bijection $\varphi : \ell_1 \longrightarrow \ell_2$, a function

$$M[\varphi] : M[\ell_1] \longrightarrow M[\ell_2], \tag{2.3}$$

called the **transport of structures**, in a functorial way, that is such that

$$M[\varphi \circ \psi] = M[\varphi] \circ M[\psi] \quad \text{and} \quad M[1_\ell] = 1_{M[\ell]}. \tag{2.4}$$

A convenient and useful graphical representation of a generic or typical M-structure on a linearly ordered set is given by figure 2.1, where the curved arrow indicates the linear order on the set of points and the label M represents the M-structure.

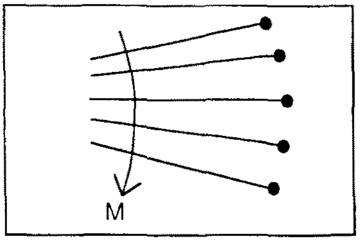


Figure 2.1. Generic M-structure

Two $\mathbb{L}$-species M and N are **isomorphic** if, by definition, there exists a natural isomorphism of functors

$$\alpha : M \xrightarrow{\sim} N. \tag{2.5}$$

In other words, there should exist a bijection $\alpha_\ell : M[\ell] \longrightarrow N[\ell]$, for each $\ell \in \mathbb{L}$, such that for any increasing bijection $\varphi : \ell \longrightarrow h$, the following diagram commutes:

$$\begin{array}{ccc} M[\ell] & \xrightarrow{\alpha_\ell} & N[\ell] \\ M[\varphi] \downarrow & & \downarrow N[\varphi] \\ M[h] & \xrightarrow{\alpha_h} & N[h] \end{array} \tag{2.6}$$

Given a $\mathbb{L}$-species M, it follows from the definition that the cardinality card $M[\ell]$ of $M[\ell]$ depends only on the cardinality of the linearly ordered set ℓ. The **cardinality** of M, or the (exponential) **generating function** of M, denoted by Card M or M(t) is defined to be the following formal power series of Hürwitz type

$$\mathrm{Card}\, M = M(t) = \textstyle\sum_{n\geq 0} \mathrm{Card}\, M[n]\, t^n/n!, \tag{2.7}$$

where, for any $n\geq 0$, we set $[n] = \{1,2,..,n\}$, including $[0] = \varnothing$, and we write $M[n]$ instead of $M[[n]]$. Of course, $[n]$ is equipped with its usual order $1 < 2 < ... < n$.

The following are examples of $\mathbb{L}$-species:

1. "Alternating permutations"; and also other classes of permutations determined by their up-down sequences.

2. "Increasing binary trees"; and other classes of rooted increasingly labelled trees. As we have seen, the $\mathbb{L}$-species of "complete increasing binary trees" and of "alternating descending odd permutations" are isomorphic.

3. If F is any $\mathbb{B}$-species, then F can be considered as an $\mathbb{L}$-species, after composition with the forgetful functor $\mathbb{L} \longrightarrow \mathbb{B}$; an F-structure on ℓ simply ignores its linear order. This class of species includes some important ones that we now point out:

- 0 : the "**empty**" species, with Card 0 = 0 ;
- 1 : the "**empty set**" species, with Card 1 = 1 ;
- T : the species of "**singletons**", with Card T = t ;
- T^2 : "**ordered pairs**", with Card $T^2 = t^2$; $T^2/2!$: "**doubletons**" ;
- T^3 : "**ordered triples**". And so on...
- L : the species of "**permutations**", considered as "**lists**" or "**linear orders**". We have $L(t) = 1/(1-t)$ and sometimes write $L = 1/(1 - T)$.
- E : the **uniform** species, with $|E[\ell]| = 1$ for any ℓ. There are several ways to describe the unique E-structure on ℓ; some of them, such as "the increasing (or decreasing) list of elements of ℓ", actualy use the linear order of ℓ. We have $E = \sum_{n\geq 0} T^n/n!$, $E(t) = \exp(t)$, and sometimes write $E = \exp(T)$.

Note that two non isomorphic $\mathbb{B}$-species can become isomorphic, when considered as $\mathbb{L}$-species. This is the case for the two species of permutations: L, for "linear orders", and P, for "bijective endofunctions", for which $L(t) = 1/(1-t) = P(t)$. The following proposition shows that $\mathbb{L}$-species constitute a more rigid combinatorial lifting of formal power series than $\mathbb{B}$-species.

Proposition 2.1. Two $\mathbb{L}$-species M and N are isomorphic if and only if $M(t) = N(t)$.

Proof. The condition certainly is necessary. To show that it is sufficient, assume $M(t) = N(t)$ and select for each $n \geq 0$ a bijection $\alpha_n : M[n] \longrightarrow N[n]$. Now for any linearly ordered set ℓ, say of cardinality n, there is a unique order preserving bijection $\varphi : [n] \longrightarrow \ell$. It is then easy to see that the family $\alpha = \{\alpha_\ell\}_{\ell \in \mathbb{L}}$, with α_ℓ defined as the composite

$$M[\ell] \xrightarrow{M[\varphi]^{-1}} M[n] \xrightarrow{\alpha_n} N[n] \xrightarrow{N[\varphi]} N[\ell], \tag{2.8}$$

is a natural isomorphism between M and N. □

Note. We often write $M = N$ to denote isomorphism of the $\mathbb{L}$-species M and N.

Operations are defined on $\mathbb{L}$-species, with the purpose of reflecting the usual operations on functions or formal power series. The definitions are similar to those of $\mathbb{B}$-species, with the additional following observations on linearly ordered sets ℓ.

We will denote by

$$\ell_1 + \ell_2 = \ell, \tag{2.9}$$

the situation where ℓ_1 and ℓ_2 are subsets of ℓ, with their linear order induced from that of ℓ, such that

$$\ell_1 \cap \ell_2 = \emptyset \quad \text{and} \quad \ell_1 \cup \ell_2 = \ell. \tag{2.10}$$

This supplies the category $\mathbb{L}$ with a decomposition law (see [J1] or [B1]) which determines the theory of $\mathbb{L}$-species.

Also, if $p \in R[\ell]$ = the set of all equivalence relations on ℓ, then each equivalence class $c \in \ell/p$ has an induced linear order and the factor set ℓ/p is itself linearly ordered according to the order of the smallest elements in each equivalence class. As usual, equivalence classes are required to be non empty. The empty set admits one equivalence relation with an empty set of equivalence classes.

In the following definitions, the operations + (and $\sum$) and × (and $\prod$) on sets are the disjoint union and cartesian product, respectively. We let $\min(\ell)$ denote the minimum element of ℓ and $1+\ell$ denote the ordered set obtained by adjunction of a new minimum element. Recall that $[0] = \emptyset$.

Let M and N be $\mathbb{L}$-species, and ℓ be a linearly ordered set. The following operations are defined:

The **addition**, $M + N$, by

$$(M + N)[\ell] = M[\ell] + N[\ell] . \tag{2.11}$$

The **product**, $M \cdot N$, (see fig. 2.2) by

$$(M \cdot N)[\ell] = \sum_{\ell_1 + \ell_2 = \ell} M[\ell_1] \times M[\ell_2] . \tag{2.12}$$

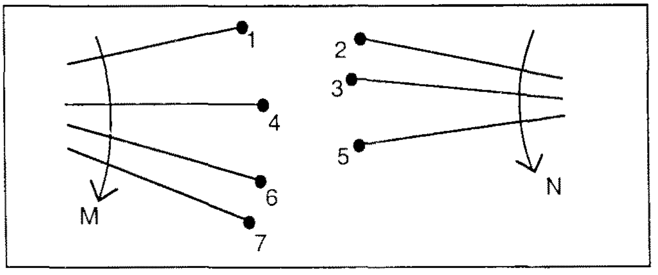


Figure 2.2. Generic M·N-structure

The **substitution**, $M(N)$, when $N[0] = \emptyset$, by

$$(M(N))[\ell] = \sum_{\rho \in R[\ell]} M[\ell/\rho] \times \prod_{c \in \ell/\rho} N[c] . \tag{2.13}$$

M(N)-structures are often called M-**assemblies of** N-**structures** (see fig. 2.3). If $M = E$, the uniform species, we simply speak of **assemblies** or **sets** of N-structures.

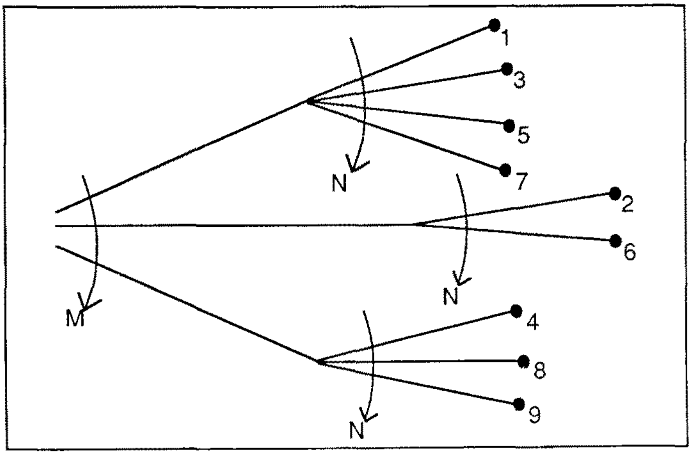


Figure 2.3. Generic M(N)-structure

The operation of substitution for $\mathbb{L}$-species is closely related to the concept of **composé partitionnel** of Foata and Schützenberger [FS1]. Note that substitution of the singleton species T is neutral so that we can write $M = M(T)$.

The **derivative**, M', also denoted by (dM/dT), (see fig. 2.4), by

$$M'[\ell] = M[1+\ell] \quad . \tag{2.14}$$

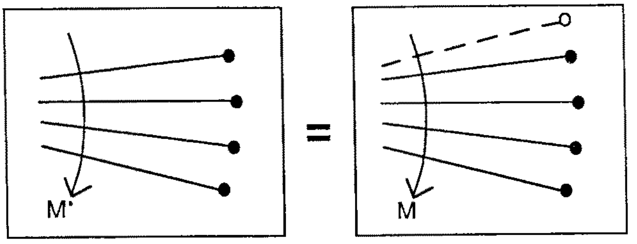


Figure 2.4. Generic M'-structure.

The **integral**, $F = F(T) = \int_0^T M(X)\,dX$, (see fig. 2.5), by

$$F[0] = \varnothing\,, \quad F[\ell] = M[\ell\backslash\{\min(\ell)\}], \quad \text{for } \ell \neq \varnothing. \tag{2.15}$$

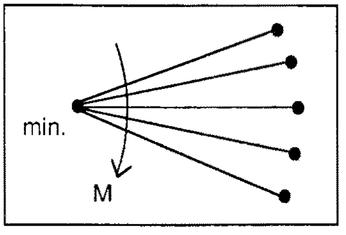


Figure 2.5. Generic $\int_0^T M(X)\,dX$-structure.

All the elementary properties, associativity, commutativity, distributivity, linearity, etc., of the operations are true at the combinatorial level, including the **Leibnitz rule** and the **chain rule** for the derivative:

$$(M \cdot N)' = M \cdot N' + M' \cdot N \quad , \tag{2.16}$$

$$M(N)' = M'(N) \cdot N' \quad , \tag{2.17}$$

with equality meaning isomorphism of $\mathbb{L}$-species (see also [J1] and [LA]). Properties particular to combinatorial integration (2.15) will be studied in a forthcoming paper [LV2]. The following proposition is now easily verified.

Proposition 2.2. For any $\mathbb{L}$-species M and N, we have

$$(M+N)(t) = M(t) + N(t) \quad , \tag{2.18}$$

$$(M \cdot N)(t) = M(t) \cdot N(t) \quad , \tag{2.19}$$

$$M(N)(t) = M(N(t)) \quad , \quad (N[0] = \varnothing) \tag{2.20}$$

$$M'(t) = dM(t)/dt \quad , \tag{2.21}$$

$$\mathrm{Card} \int_0^T M(X)dX = \int_0^t M(x)dx \quad . \tag{2.22}$$

□

A consequence of this is that any formula $\mathcal{F}(M_1, M_2, \ldots) = \mathcal{G}(M_1, M_2, \ldots)$, expressing an isomorphism of $\mathbb{L}$-species and involving the above operations, which is valid for any sequence of $\mathbb{L}$-species $M_1, M_2, \ldots$, will also be valid for their generating functions. By the principle of extension of algebraic identities, the formula will also be valid for any sequence of formal power series $m_1(t), m_2(t), \ldots$ and also for any sequence of analytic functions. This is the basic principle at the root of combinatorial proofs of analytic formulas such as the Leibnitz rule (2.16) or the chain rule (2.17) for derivation, the Lagrange inversion formulas [L1], etc...

We will also consider k-**sorted $\mathbb{L}$-species** ($k \geq 2$), that is functors of the type

$$\mathbb{L}^k = \mathbb{L} \times \mathbb{L} \times \ldots \times \mathbb{L} \longrightarrow \mathbb{E} \,, \tag{2.23}$$

and **mixed species**, for example of the type

$$\mathbb{L} \times \mathbb{B} \longrightarrow \mathbb{E} \,, \tag{2.24}$$

corresponding to multivariable functions and series, as well as **$\mathbb{K}$-weighted $\mathbb{L}$-species**, that is functors of the type

$$M : \mathbb{L} \longrightarrow \mathbb{E}_{\mathbb{K}} \,, \tag{2.25}$$

where $\mathbb{K}$ is a commutative ring with unity and $\mathbb{E}_{\mathbb{K}}$ denotes the category of **finite $\mathbb{K}$-weighted sets**, i.e. pairs (A, v) where A is a set and $v : A \longrightarrow \mathbb{K}$ is a **weight function**; a **morphism** between two $\mathbb{K}$-weighted sets (A, v_A) and (B, v_B) is a function $f : A \longrightarrow B$ such that $v_A = v_B \circ f$.

The reader is refered to [J1, §5, §6], [LA, §3] and [YE], for a discussion of k-sorted and $\mathbb{K}$-weigted $\mathbb{B}$-species, in particular for a definition of their operations and generating functions, which can easily be adapted to the case of $\mathbb{L}$-species. Perhaps is it worthwhile to give an explicit definition of partial derivatives: let $M = M(S,T)$ be a 2-sorted $\mathbb{L}$-species; then we set

$$(\partial M/\partial S)[h, \ell] = M[1+h, \ell] \quad , \tag{2.26}$$

$$(\partial M/\partial T)[h, \ell] = M[h, 1+\ell] \quad . \tag{2.27}$$

§3. Case of one differential equation

In this section we examine the case of one differential equation of the form

$$(dY/dT) = Y' = M(T,Y) \quad , \quad Y(0) = Z \tag{3.1}$$

where $M(T,Y)$ is a given 2-sorted $\mathbb{L}$-species, Z is an indeterminate which will correspond to an extra sort of points, and $Y(0)$ is to be interpreted as the $\mathbb{L}$-species obtained from Y by substitution of the empty species 0. Specific examples, with, for instance, $M(T,Y) = 1 + Y^2$, or $a_0 + a_1Y + a_2Y^2 + \ldots + a_nY^n$ (autonomous), and $M(T,Y) = G(T)Y + F(T)$ (linear), will be considered in §4.

Formally, a solution of (3.1) is defined to be a pair (A, ψ) where $A = A(T)$ $(=Y)$ is a $\mathbb{L}$-species such that $A(0) = Z$, and ψ is an isomorphism of $\mathbb{L}$-species

$$A'(T) \xrightarrow[\sim]{\psi} M(T, A(T)) . \tag{3.2}$$

Note that in fact A also depends on the initial condition Z and that we should write $A = A(T,Z)$ and

$$\partial A(T,Z)/\partial T \xrightarrow[\sim]{\psi} M(T, A(T,Z)) \quad , \quad A(0,Z) = Z . \tag{3.3}$$

Now keeping Z fixed, (3.3) is equivalent to the integral equation

$$A(T,Z) = Z + \int_0^T M(X, A(X,Z))dV . \tag{3.4}$$

By virtue of the definition (2.15) of the integral (see fig. 2.5) and of the definition of the substitution in a 2-sorted species (see [J1, Def. 19, p. 46] or [LA, p. 89]), this integral equation can be visualized as in figure 3.1, where:

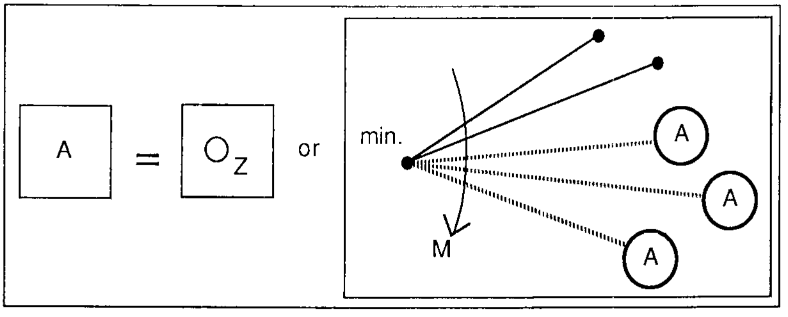


Figure 3.1

- the dots ■ and circles ○ represent singletons of the sorts T (and X) and Z, respectively, and will be called "points" and "buds", respectively,

- the circled A's represent A-structures on equivalence classes of the underlying set;
- the two sorts of elements on which the M-structure is constructed are symbolized by continuous and mashed lines respectively.

It now suffices to iterate this process to obtain a canonical combinatorial solution of (3.1), that is the $\mathbb{L}$-species $A = A_M(T,Z)$ of so-called M-**enriched increasing arborescences**, generically described by Figure 3.2. An $A_M(T,Z)$-structure lies over a couple (ℓ,s) of linearly ordered sets (in fig. 3.2, $\ell = \{1, 2, ..., 18\}$ and $s = \{a, b, ..., e\}$). Elements of ℓ and s will be called **points** (T-singletons) and **buds** (Z-singletons) respectively. We make the convention that "all points are smaller than all buds". In such an M-enriched increasing arborescence, a point is called **fertile** if it is the root of some A_M-substructure and **sterile** otherwise. Note that the buds, like the sterile points, do not have any sons.

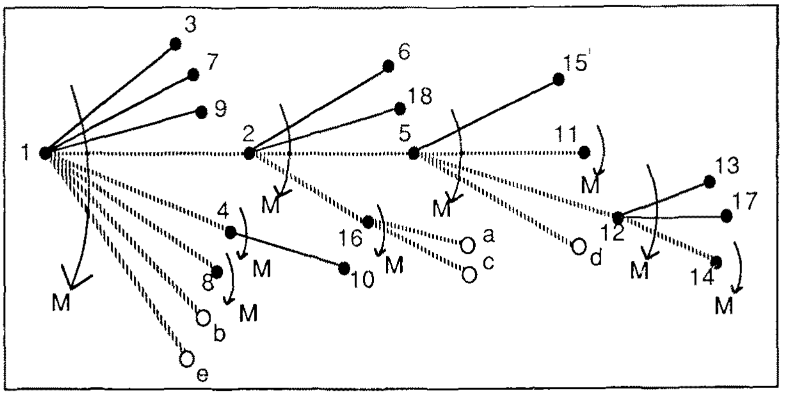


Figure 3.2. Generic $A_M(T,Z)$-structure

It should be clear to the reader that the $\mathbb{L}$-species $Y = A_M$ is indeed a solution of (3.1) since $A_M(0,Z) = Z$ and there exists an obvious isomorphism

$$\partial A_M(T,Z)/\partial T \xrightarrow[\sim]{\psi} M(T, A_M(T,Z)) \quad . \tag{3.5}$$

Now suppose that (B,φ) is another solution of 3.1, then there will be a unique **isomorphism of solutions**

$$\Phi : (A_M, \psi) \xrightarrow{\sim} (B, \varphi) , \tag{3.6}$$

that is an isomorphism of $\mathbb{L}$-species $\Phi : A_M \xrightarrow{\sim} B$ such that the following diagram commutes

$$\begin{array}{ccc} (\partial A_M/\partial T) & \xrightarrow[\sim]{\psi} & M(T,A_m) \\ {\scriptstyle (\partial\Phi/\partial T)}\downarrow\wr & & \wr\downarrow {\scriptstyle M(T,\Phi)} \\ (\partial B/\partial T) & \xrightarrow[\sim]{\varphi} & M(T,B) \end{array} \tag{3.7}$$

where the natural transformations $\partial\Phi/\partial T$ and $M(T,\Phi)$ are defined in an obvious fashion. This is shown by induction on the cardinality of ℓ, where (ℓ, s) is the couple of linearly ordered sets on which the A_M- and B-structures are taken: to start with, we have $A_M(0,Z) = Z = B(0,Z)$ and the unique choice for Φ_0 is the identity $Z \xrightarrow{\sim} Z$. Now, for $n \geq 0$, suppose that the natural bijection

$$\Phi_{(h,r)} : A[h,r] \longrightarrow B[h,r] \tag{3.8}$$

has been uniquely defined for all linearly ordered sets h of cardinality $\leq n$ and for all r, and let ℓ be of cardinality n. Then we have, for any s,

$$A_M[1+\ell,s] = (\partial A_M/\partial T)[\ell,s] \xrightarrow{\psi(\ell,s)} M(T,A_M)[\ell,s] \tag{3.9}$$

and also, by hypothesis,

$$B[1+\ell,s] = (\partial A_M/\partial T)[\ell,s] \xrightarrow{\varphi(\ell,s)} M(T,B)[\ell,s] \tag{3.10}$$

In other words, the equivalent of Figure 3.1 with A replaced by either A_M or B is valid. But each A_M- and B-substructures that appear in this decomposition (the circled A-structures in Fig. 3.1) lies over a couple (h,r) with $|h| \leq n$ and hence, by the induction hypothesis, correspond isomorphically to each other, using $\Phi_{(h,r)}$. Consequently the bijection $M(T,\Phi)_{(\ell,s)}$ can be constructed and afterwards, also $\Phi'_{(\ell,s)}$ by asking that the diagram (3.7), applied to (ℓ,s) commutes. This determines $\Phi_{(1+\ell,s)} = (\partial\Phi/\partial T)_{(\ell,s)}$ uniquely. We thus have proved the following:

Theorem 3.1. For any 2-sorted $\mathbb{L}$-species M, the 2-sorted $\mathbb{L}$-species $Y = A_M(T,Z)$ of M-enriched increasing arborescences with buds, described above, together with the natural isomorphism $\psi : (\partial A_M/\partial T)(T,Z) \xrightarrow{\sim} M(T,A_M(T,Z))$, is a (canonical) solution of the differential equation (3.1). Moreover, for any other solution (B,φ) of 3.1, there is a unique isomorphism of solutions $\Phi : (A_M,\psi) \xrightarrow{\sim} (B,\varphi)$.

□

We conclude this section by noting that, as shown by G. Labelle in [L4, theorem B], the combinatorial Newton-Raphson iteration scheme, first introduced in [DLL], can be applied in the resolution of a differential equation and gives a sequence of approximations with quadratic convergence. More precisely, for any $\mathbb{L}$-species F, we introduce the $\mathbb{L}$-species $F_{\leq n}$, the "truncation of F to sets of cardinality at most n", by

$$F_{\leq n}[\ell] = F[\ell], \text{ if } |\ell| \leq n, \text{ and } F_{\leq n}[\ell] = \emptyset, \text{ otherwise.}$$

We then have the following:

Theorem 3.2. Let $Y = A = A(T)$ be the solution of the equation $Y' = M(T,Y)$, $Y(0) = 0$ and set $\mathfrak{Q} = A_{\le n}$. Let $Y = \mathbb{B}$ be the solution of the 1^{rst} order linear differential equation

$$Y' = F(T)Y + G(T), \quad Y(0) = 0, \tag{3.11}$$

where

$$F(T) = \partial M(T,Y)/\partial Y \,\big|_{Y=\mathfrak{Q}(T)} \quad \text{and} \quad G(T) = M(T,\mathfrak{Q}(T)) - \mathfrak{Q}'(T).$$

Then the $\mathbb{L}$-species $\mathfrak{Q}^+ := \mathfrak{Q} + \mathbb{B}$ has a contact of order $2n + 2$ with A, i.e. there exists a canonical isomorphism

$$\mathfrak{Q}^+_{\le 2n+2} \xrightarrow{\ \sim\ } A_{\le 2n+2}. \tag{3.12}$$

Proof. See [LA, §3]. □

We will see in the next section how to deal combinatorialy with 1^{rst} order linear differential equations.

§4. Examples

In this section, we consider special cases of first order differential equations of the form (3.1), including autonomous and linear equations.

Note first that the initial condition $Y(0) = Z$, can take special forms, by substitution into Z, some of which actually make the solution independant of Z. In particular, $Y(0) = 0$, the **empty** species, is to be interpreted as "no buds are allowed", and $Y(0) = 1$, the **empty set** species, as "buds are unlabelled, indistinguishable and not accounted for". This last case however, the substitution of 1 for Z, is not always possible or legal. In particular, the generation of an infinite number of structures on any given ℓ should be avoided. More precisely, writing

$$Y(T,Z) = \textstyle\sum_{k\ge 0} Y_k(T)\, Z^k/k!\,, \tag{4.1}$$

then each $Y_k(T)$ should be combinatorialy divisible by $k!$ and the family $\{Y_k(T)/k!\}_{k\ge 0}$ of $\mathbb{L}$-species should be summable.

A first order differential equation is called **autonomous** if $M(T,Y) = G(Y)$ does not depend on T, i.e. if it is of the form

$$Y' = G(Y) \quad , \quad Y(0) = Z. \tag{4.2}$$

In this case, the M-enriched (or rather G-enriched) increasing arborescences will have no sterile points and, equivalently, only mashed edges will appear (these are then **unmashed** for simplicity of representation). See figure 4.1 for an illustration of this canonical solution. The following four examples are special cases of autonomous differential equations.

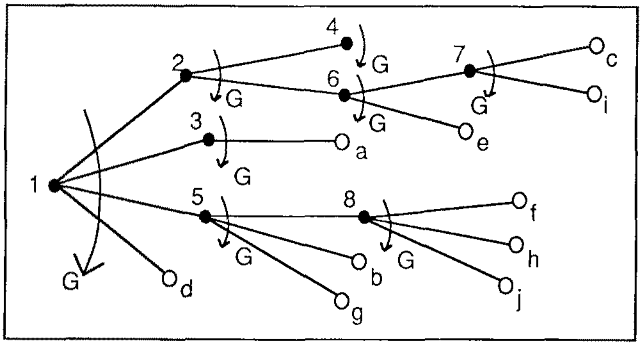


Figure 4.1

Example 4.1. Consider the autonomous differential equation

$$Y' = 1 + Y^2 \quad , \quad Y(0) = 0 \tag{4.3}$$

where $M(T,Y) = G(Y) = 1 + Y^2$ and $Z = 0$. Since there will be no buds, the canonical solution is that of "complete increasing binary trees", as we saw in §1 (see fig. 1.3). Moreover the unique isomorphism of solutions between this and the other solution of (4.1), that of "alternating descending odd permutations" was described in §1 as the "projection along the x-axis".

Example 4.2. The generating function $y = \tan t + \sec t$ of alternating permutations (without distinction between the odd and even case) is the solution of the following differential equation

$$y' = (1 + y^2)/2 \quad , \quad y(0) = 1 \ . \tag{4.4}$$

Defining $z = y - 1$, this is equivalent to

$$z' = 1 + z + z^2/2 \quad , \quad z(0) = 0 \ . \tag{4.5}$$

Similarly to example 4.1, we see that the canonical solution of (4.5) at the species level is that of the so-called **increasing 1-2 arborescences**, i.e. arborescences such that every vertex has at most two sons. Note that in the case of two sons, no distinction is made between left and right, contrarily to the case of binary trees. Adapting the bijection between increasing binary trees and permutations mentionned earlier, one can easily give a bijection between increasing 1-2 arborescences and "André permutations"; see [FS1], [FS2], [V3].

It is also possible to construct directly the canonical combinatorial solution of (4.4), where $(1 + Y^2)/2$ is to be interpreted as the $\mathbb{L}$-species $1 + Y^2$ weigted by $1/2$. The reader will easily show the equivalence between the corresponding weigted arborescences and the 1-2 arborescences.

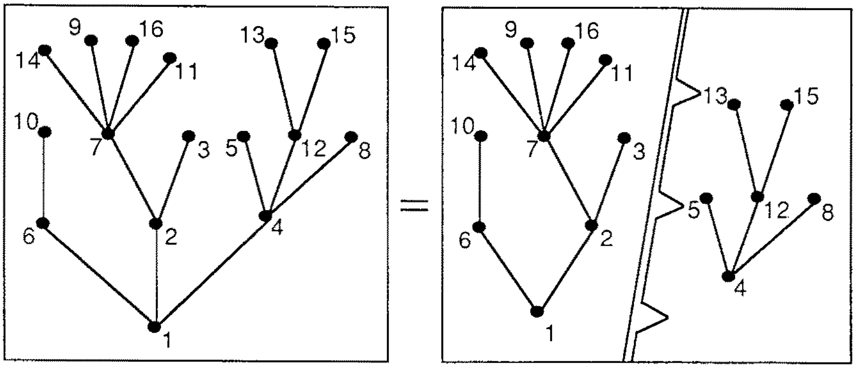
Example 4.3. **Planar trees** are, by definition, L-enriched arborescences (see [L1]), where $L(T) = 1/(1-T)$ is the $\mathbb{L}$-species of permutations, considered as lists. Now, from theorem 3.1, the $\mathbb{L}$-species $Y = Pla(T)$ of **increasing** planar trees (see Fig. 4.2,a)) is the solution of the differential equation

$$Y' = L(Y) \quad , \quad Y(0) = 0 \ . \tag{4.6}$$

This $\mathbb{L}$-species is also solution of the **functional** equation

$$Y = T + Y^2/2! \tag{4.7}$$

which says that an increasing planar tree is either a singleton or a set of two increasing planar trees. One way to realize this fact is to cut the right-most branch at the root of any increasing planar tree which is not a singleton (see fig. 4.2, b)).



a) **b)**

Figure 4.2

It follows from (4.7) that the generating function $y = Pla(t)$ satisfies the quadratic equation

$$y^2 - 2y + 2t = 0 \tag{4.8}$$

which can be solved to give

$$Pla(t) = 1 - \sqrt{1-2t} \tag{4.9}$$

Example 4.4. Let $a_0, a_1, \ldots, a_n$ be scalar parameters. From the previous example, the solution of the equation

$$Y' = a_0 1 + a_1 Y + \ldots + a_n Y^n \quad , \quad Y(0) = Z \tag{4.10}$$

is seen to be the $\mathbb{L}$-species of weighted increasing planar trees, such that each vertex has at most n sons, the weight of a vertex having i sons being a_i. We can also consider the infinite case

$$Y' = G(Y) = \sum_{i\geq 0} a_i Y^i \quad , \quad Y(0) = Z. \tag{4.11}$$

The solution is that of "weighted increasing planar trees". This point of view is different from the one adopted throughout this paper which considers $G = G(T)$ as an abstract $\mathbb{L}$-species rather than as the species of "weighted lists". It would be possible to develop the theory using these weighted increasing planar trees (see [BR1], [BR2]). Another option would be to start with $G(T) = \sum_{i\geq 0} a_i T^i/i!$ considered as the species of "weigted sets".

Example 4.5. The linear equation. The general first order linear differential equation can be expressed, at the combinatorial level, as follows:

$$Y' = F(T)Y + G(T) \quad , \quad Y(0) = Z \tag{4.12}$$

where F and G are given $\mathbb{L}$-species.

a) **The homogeneous case**. If $G(T) = 0$, we have the homogeneous equation

$$Y' = F(T)Y \quad , \quad Y(0) = Z \ . \tag{4.13}$$

Its canonical combinatorial solution, denoted by $Y = A_F(T)$, is given by enriched increasing arborescences of a special form, as illustrated in figure 4.3, with $m_1 < m_2 < m_3 < m_4$.

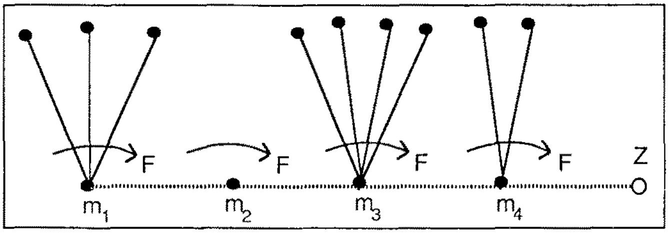


Figure 4.3. Generic A_F-structure

This is simply an increasing sequence of fertile points to which are attached F-structures on bigger elements, followed by a bud. Remembering how the integral $\int_0^T F(X)dX$ (or $\int F$ for short) was defined (see (2.15) and fig. 2.5), we see that we essentially have an assembly of $\int F$-structures, naturally written in the increasing order of the smallest elements (**increasing list**), multiplied by Z. In other words, the solution can be expressed as

$$A_F = \exp(\textstyle\int_0^T F(X)dX)\cdot Z \ . \tag{4.14}$$

b) **The general case**. If $Y = B_{F,G}(T)$ is the solution of

$$Y' = F(T)Y + G(T) \quad , \quad Y(0) = 0 \ , \tag{4.15}$$

that is with $Z = 0$, then it is easily checked that the solution of the general equation (4.12) is simply given by the sum

$$Y = A_F(T) + B_{F,G}(T) \ . \tag{4.16}$$

The solution $Y = B_{F,G}(T)$ of (4.15) also has a simple combinatorial description, given generically by figure 4.4. We see that a $B_{F,G}$-structure can be described as an increasing list of $\int F$-structures ended by an $\int G$-structure. Unfortunately the combinatorial operation "to end by" is not easily expressed in terms of the other operations.

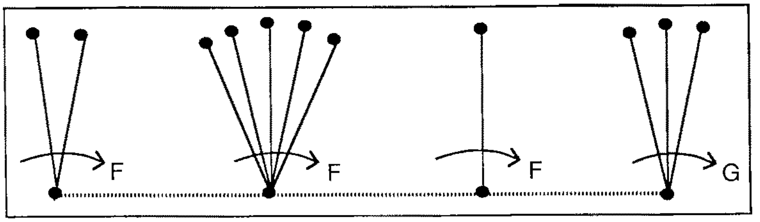


Figure 4.4 Generic $B_{F,G}$-structure.

Analytically, the classical method of "variation of the constant" gives the solution

$$Y = C_{F,G}(T) = \exp(\int_0^T F(X)dX) \cdot \int_0^T G(X) \cdot \exp(-\int_0^X F(U)dU)\,dX \qquad (4.17)$$

for equation (4.15). This expression can be interpreted combinatorially and shown to be equivalent to $B_{F,G}(T)$, in the context of **virtual $\mathbb{L}$-species**, that is formal differences $M - N$ of $\mathbb{L}$-species (with $M_1 - N_1 \cong M_2 - N_2 \iff M_1 + N_2 \cong M_2 + N_1$), as follows. Notice that $C_{F,G}(T)$ is a product. The left factor is simply an increasing list of $\int F$-structures while the right factor is an integral structure which is realized by setting aside the minimum element and structuring the rest with the integrand $G \cdot \exp(-\int F)$. This is itself the product of a G-structure and of an increasing list of $-\int F$-structures, that is of $\int F$-structures weighted by -1. The product of these -1 gives the global weight of the $C_{F,G}$-structure. See figure 4.5 for a generic $C_{F,G}$-structure, where the G-structure has been attached to the right-hand side minimum thus giving an $\int G$-structure denoted by g. Some of the labels are explicitely given and mashed edges are omitted in this figure.

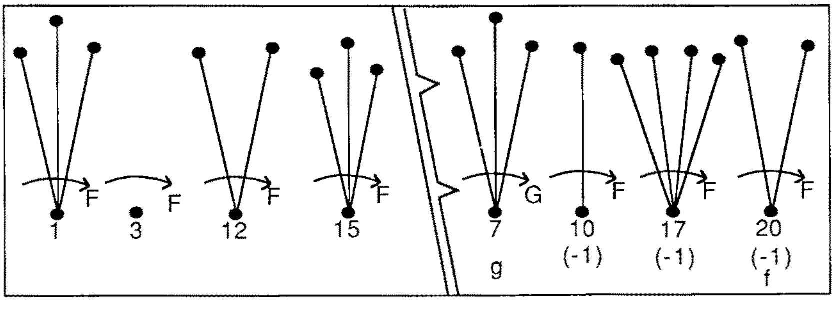


Figure 4.5. Generic $C_{F,G}$-structure.

There is a sign reversing involution Φ on $C_{F,G}$ defined as follows: let f denote the maximum $\int F$-structure (according to the roots, of course). If f is bigger than g then the involution Φ simply switches f from the right list to the left list or conversely and changes its sign accordingly. This cancels all $C_{F,G}$-structures except the fixed points of Φ which occur when g is actually the maximum of all integral structures of both sides. These are obviously isomorphic to $B_{F,G}$-structures. In other words, we have, as desired,

$$B_{F,G} \cong \text{Fix}\,\Phi \cong C_{F,G}\,. \tag{4.18}$$

§ 5. Lie-Gröbner formulas

In this section we adapt to $\mathbb{L}$-species the method of "combinatorial eclosions" first introduced by Gilbert Labelle in his work on multidimensional Lagrange inversion, the implicit function theorem and differential equations in the context of $\mathbb{B}$-species (see [L2], [L3] and [L4]. More precisely we will lift to the combinatorial level, the Lie-Gröbner type formulas (see [G1], [G2]) for the solution of the differential equation

$$Y' = M(T,Y) \quad , \quad Y(0) = Z. \tag{5.1}$$

The case of a system of equations will be considered in the following section.

Theorem 5.1 Let $M(T,Y)$ be a 2-sorted $\mathbb{L}$-species. The canonical solution of (5.1), the $\mathbb{L}$-species $Y = A_M(T)$ of M-enriched increasing arborescences, can be expressed as

$$A_M(T) = e^{\Gamma}Z\,\Big|_{X=0} \tag{5.2}$$

where $\Gamma = T\mathcal{D}$ and $\mathcal{D}$ is the differential operator defined by

$$\mathcal{D} = \partial/\partial X + M(X,Z)\,\partial/\partial Z\,. \tag{5.3}$$

Before proving the theorem, we have to understand the action of the operators $T\mathcal{D}$, $(T\mathcal{D})^n/n!$ $(n \geq 0)$, and $e^{T\mathcal{D}}$, on $\mathbb{L}$-species. Actually, these operators act on 3-sorted $\mathbb{L}$-species $\Psi = \Psi(T,X,Z)$. The 3 sorts of elements will be named as follows:

points, ●, minibuds, ○, maxibuds, ◯, corresponding to T, X, Z respectively.

Let $\Psi = \Psi(T,X,Z)$ be a 3-sorted $\mathbb{L}$-species and $\mathcal{D}$ be defined by (5.3) and set $\Gamma = T\mathcal{D}$. Then the 3-sorted $\mathbb{L}$-species $\Gamma\Psi(T,X,Z)$ is the sum

$$\Gamma\Psi(T,X,Z) = (T\partial/\partial X)\Psi(T,X,Z) + (TM(X,Z)\partial/\partial Z)\Psi(T,X,Z)\,. \tag{5.4}$$

Hence, using a proper combinatorial interpretation of the partial derivatives of $\mathbb{L}$-species (see 2.26, 2.27) and the idea of Gilbert Labelle, a $\Gamma\Psi$-structure can be described as a Ψ-structure which has undergone a **combinatorial eclosion** of one of the following two types:

Type I eclosion, $T\partial/\partial X$: "Replace the (phantomatic) minimum minibud by an arbitrary point". This point comes from the multiplication by T and is called the **eclosion point**. See figure 5.1 for a generic $(T\partial/\partial X\ \Psi)$-structure.

Type II eclosion, $TM(X,Z)\,\partial/\partial Z$: "Replace the (phantomatic) minimum maxibud by a $TM(X,Z)$-structure, that is an arbitrary (eclosion) point with an M-structure of minibuds and maxibuds attached". See figure 5.2 for a generic $(TM(X,Z)\,\partial/\partial Z\,\Psi)$-structure.

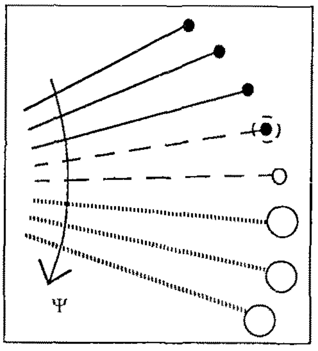


Figure 5.1.

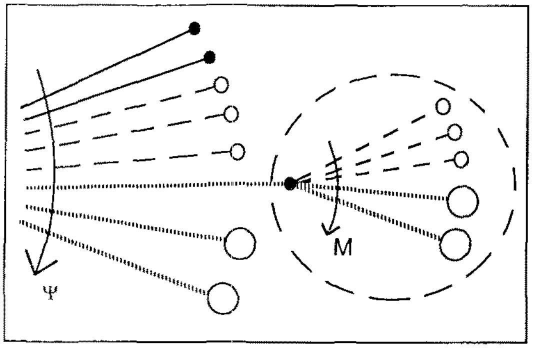


Figure 5.2.

For $n \geq 0$, $\Gamma^n\Psi$ is obviously obtained by n successive applications of $\Gamma = T\mathcal{D}$, that is by n successive eclosions of either type. The order of eclosion is important and can be recorded by noting it above the unique "eclosion point" that appears at each step (see fig. 5.3). This numbering is not to be confused with the labels of these points. Indeed, for a given set of n eclosion points there are n! different labellings of these in an otherwise fixed $\Gamma^n\Psi$-structure, since their occurence comes from n multiplications by T. In each case, this gives n! distinct structures; however there is only one of these whose labels are in the same order as the eclosions; we call this an **orderly labelled** $\Gamma^n\Psi$-structure. See for example the $\Gamma^n\Psi$-structure of figure 5.3 where $n = 13$ and the set of points is $\{a, b, ..., o\}$, in alphabetical order.

Now $\Gamma^n/n!$ can be interpreted as follows: a $(\Gamma^n/n!)\Psi$-structure is identified with an orderly labelled $\Gamma^n\Psi$-structure. Finally e^Γ is easily interpreted since it is defined as

$$e^\Gamma = \sum_{n\geq 0} \Gamma^n/n! \qquad (5.5)$$

so that a $e^\Gamma\Psi$-structure is an orderly labelled $\Gamma^n\Psi$-structure for some $n \geq 0$.

Proof of Theorem 5.1. If we take $\Psi(T,X,Z) = Z$, the starting structure is simply a maxibud. After one eclosion, we get $\Gamma(Z) = T\,M(X,Z)$ and after a few more eclosions we get an M-enriched arborescence on points or minibuds, and maxibuds. Moreover all the points are eclosion points so that in the properly labelled case, that is for $e^\Gamma Z$, the point labels can be identified with their order of apparition. Hence they should be increasing as we go away from the root since the eclosions occur in this way.

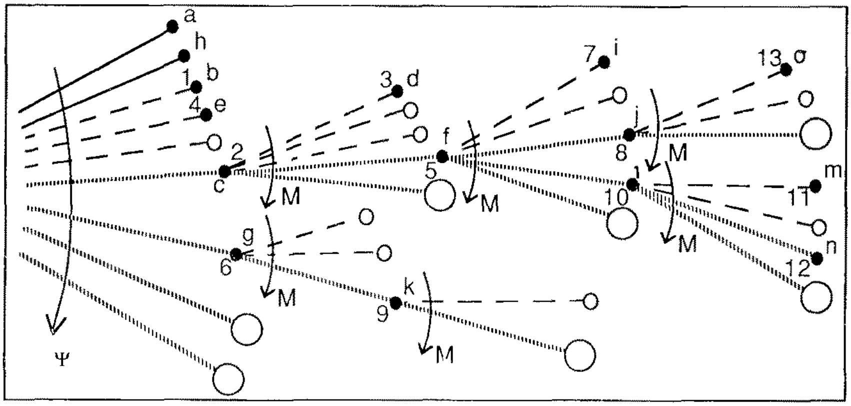


Figure 5.3. Generic orderly labelled $\Gamma^n\Psi$-structure, $n = 13$.

Finally, setting $X = 0$ in $e^{\Gamma}Z$ simply imposes the restriction that there should be no minibuds remaining in the structures. These can obviously be identified with the increasing M-enriched arborescences defined in § 3 (see fig. 3.2). In other words we have

$$A_M \cong e^{T\mathcal{D}} Z \Big|_{X=0} . \qquad \square$$

If we start with $\Psi(T,X,Z) = F(Z)$, that is with an F-assembly of maxibuds, where F is any given $\mathbb{L}$-species, and if we apply e^{Γ} and set $X = 0$, we obviously obtain F-assemblies of A_M-structures; in other words we have the slightly more general result:

Proposition 5.2. For any $\mathbb{L}$-species F, we have

$$F(A_M) = e^{\Gamma}F(Z) \,\big|_{X=0} \tag{5.6}$$

where A_M and Γ are defined as in theorem 5.1.

□

In the case of the autonomous equation

$$Y' = G(Y) , \qquad Y(0) = Z , \tag{5.7}$$

where G is a given $\mathbb{L}$-species, we have $M(T,Y) = G(Y)$ and, as seen in § 4, the canonical solution $Y = A(T,Z)$ is given by "G-enriched increasing rooted trees, with buds" (see fig. 4.1). We then have the following:

Corollary 5.3. The solution $Y = A(T,Z)$ of the autonomous equation (5.7) can be expressed as

$$A(T,Z) = \exp(TG(Z)\partial/\partial Z)\,Z \tag{5.8}$$

and

$$A(T,Z) = \textstyle\sum_{n\geq 0} (G(Z)\partial/\partial Z)^n Z\, T^n/n!\ . \tag{5.9}$$

Moreover, for any $\mathbb{L}$-species F, we have

$$F(A(T,Z)) = \textstyle\sum_{n\geq 0} (G(Z)\partial/\partial Z)^n F(Z)\, T^n/n!\,. \tag{5.10}$$

Proof. Since $M(X,Z) = G(Z)$, minibuds will never appear in the combinatorial eclosions applied to any $\Psi(T,X,Z) = \Psi(T,Z)$. In other words, we have $\partial/\partial X = 0$ and $\Gamma = T\mathcal{D} = TG(Z)\partial/\partial Z$ so that theorem 5.1 immediately gives (5.8).

Furthermore we also have (5.9) since

$$\begin{aligned}\exp(TG(Z)\partial/\partial Z)\,Z &= \textstyle\sum_{n\geq 0} ((TG(Z)\partial/\partial Z)^n/n!)\,Z\\ &= \textstyle\sum_{n\geq 0} (G(Z)\partial/\partial Z)^n Z\; T^n/n!\,.\end{aligned}$$

Formula (5.10) is obtained similarly from proposition 5.2. □

The expression (5.9) is interpreted as follows: First perform all eclosions with buds only, keeping track of the order of eclosions (see fig. 5.4), and, when this is completed, put the points into their unique position (compare with fig. 4.1).

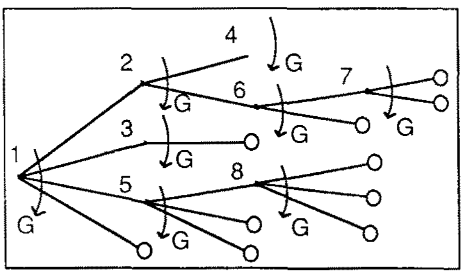


Figure 5.4. $(G(Z)\,\partial/\partial Z)^n Z$ $(n = 8)$

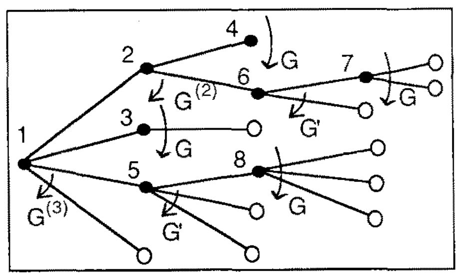


Figure 5.5

Moreover each $(G(Z)\partial/\partial Z)^n Z$-structure such as that of figure 5.4 can be transformed isomorphically into a structure given generically by figure 5.5. From this observation results the following proposition.

Proposition 5.4. [BR1] The $\mathbb{L}$-species $H_n(Z) = (G(Z)\,\partial/\partial Z)^n Z$, for $n \geq 1$, can be written in the form

$$H_n(Z) = \textstyle\sum_{\boldsymbol{\alpha}} C_n(\boldsymbol{\alpha})\, G^{\alpha_0} (G')^{\alpha_1} (G'')^{\alpha_2} \ldots (G^{(p)})^{\alpha_p} \tag{5.11}$$

where $\boldsymbol{\alpha}$ ranges over all sequences $\boldsymbol{\alpha} = (\alpha_0, \alpha_1 \ldots)$ of non negative integers and $C_n(\boldsymbol{\alpha})$ is the number of increasing arborescences on $[n]$ having, for $p = 0, 1, \ldots,$ α_p vertices with p sons. □

Note that the conditions

$$n = \sum_{p\geq 0} \alpha_p \qquad \text{and} \qquad \alpha_0 = 1 + \sum_{p\geq 1} (p-1)\alpha_p \tag{5.12}$$

are necessary for having $C_n(\boldsymbol{\alpha}) \neq 0$, so that the expression (5.11) is a finite sum. The first values of $H_n(Z)$ are

$$\begin{aligned} H_1(Z) &= G(Z) \\ H_2(Z) &= G(Z)\,G'(Z) \\ H_3(Z) &= G^2(Z)\,G''(Z) + G(Z)\,(G'(Z))^2 \\ H_4(Z) &= G^3(Z)\,G^{(3)}(Z) + 4\,G^2(Z)\,G'(Z)\,G''(Z) + G(Z)\,(G'(Z))^3. \end{aligned} \tag{5.13}$$

Another observation which follows from (5.9), or from figure 5.4, is that, for $n \geq 0$,

$$(G(Z)\,\partial/\partial Z)^n\, Z = (\partial/\partial T)^n\, A(T,Z)\big|_{T=0}. \tag{5.14}$$

Similar interpretations can be given for the general expansion of the differential operator $(G(Z)\,\partial/\partial Z)^n$ applied to any $F(Z)$, using equation (5.10). See [CO] and [BR1].

Example 5.5. We now consider a simple example to illustrate the use of combinatorial eclosions. Let $Y = \mathrm{Ter}\,(T,Z)$ be the solution of the autonomous differential equation

$$Y' = Y^3 \quad , \quad Y(0) = Z. \tag{5.15}$$

It can be viewed as the $\mathbb{L}$-species of increasing ternary trees, with buds as leaves (see fig. 5.6). Setting $Z = 1$, we get

$$\mathrm{Ter}(T,Z)\big|_{Z=1} = \mathrm{Ter}(T), \tag{5.16}$$

the $\mathbb{L}$-species of **ternary trees**.

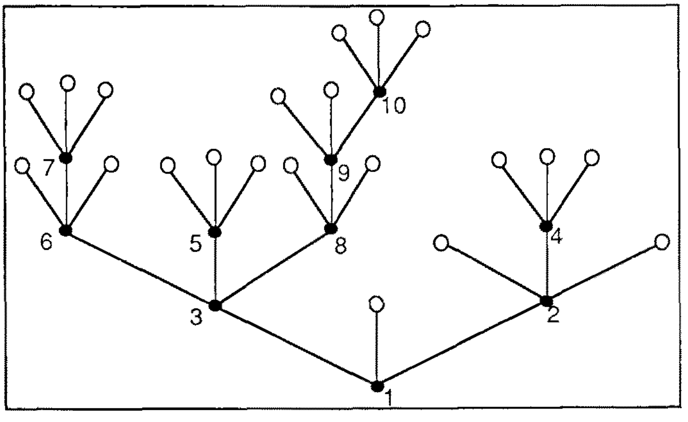


Figure 5.6

Now corollary 5.3 can be applied, with $G(Z) = Z^3$, to get

$$\mathrm{Ter}(T,Z) = e^{TZ^3\,\partial/\partial Z}\, Z. \tag{5.17}$$

Hence, to get an increasing ternary tree with buds, we can do the following:

1. Start with a single bud: O

2. Apply a certain number of times the combinatorial eclosion $TZ^3\,\partial/\partial Z$ which can be interpreted as "replace a bud by a TZ^3-structure", as in Figure 5.7.

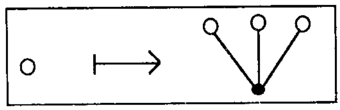


Figure 5.7. $TZ^3\,\partial/\partial Z$

3. Label the eclosion points according to their order of apparition.

However formula (5.17) can be given a different interpretation by changing slightly the first two rules given above, as follows:

1 bis. Start with a single bud, with a stem planted in a flower box labelled by zero (Fig. 5.8)

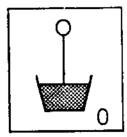


Figure 5.8

2 bis. The combinatorial eclosion $T\,Z^3\partial/\partial Z$ is to be interpreted as "replace a stemmed bud by a TZ^3-structure as in Figure 5.9".

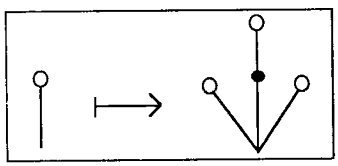


Figure 5.9

For example, if we apply a sequence of 10 eclosions, as in figure 5.6, but with this new interpretation, we get a planar tree like that of figure 5.10. Setting $Z = 1$ gives figure 5.11 which is nothing but an increasing planar tree (see example 4.3) on the linearly ordered set $1+[10]$ (zero is the new minimum element), that is a $(d/dT)\mathrm{Pla}(T)$-structure on $[10]$. Thus, in general, we have

$$\mathrm{Ter}(T) = e^{(TZ\partial/\partial Z)}\, Z\Big|_{Z=1} = (d/dT)\,\mathrm{Pla}\,(T) \tag{5.18}$$

and, for the generating function, using (4.9),

$$\mathrm{Ter}(t) = (d/dt)\,\mathrm{Pla}(t) = 1/\sqrt{1-2t}. \tag{5.19}$$

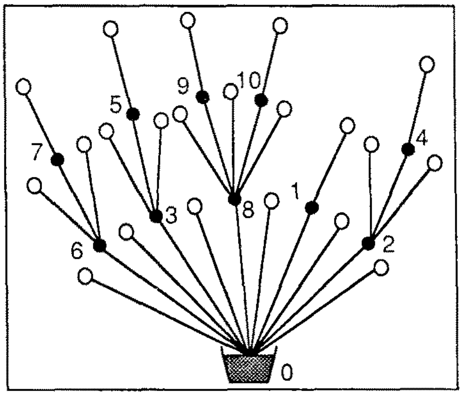


Figure 5.10

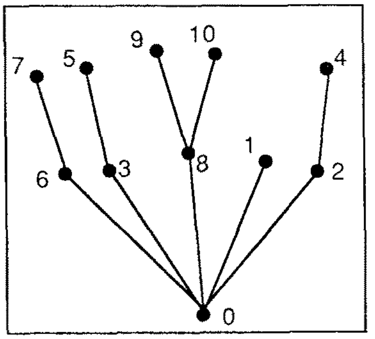


Figure 5.11

§ 6. Systems of differential equations

Most of the theory developed so far can be easily extended to systems of differential equations. We now sketch briefly the main results in this case. Let $M_1,\ldots,M_p$ be given p-sorted $\mathbb{L}$-species. We will consider the following system of first order autonomous differential equations:

$$dY_i/dT = Y_i' = M_i(Y_1,\ldots,Y_p) \ , \qquad Y_i(0) = Z_i \quad , \quad i = 1, 2, \ldots, p \ . \tag{6.1}$$

There is no loss of generality in assuming, as we do, that the equations are autonomous since we can always reduce ourselves to this case at the cost of the supplementary equation (6.2), if necessary.

$$Y_0' = 1 \ , \qquad Y_0(0) = 0 \tag{6.2}$$

The initial conditions $Z_1, \ldots, Z_p$ are considered as p extra sorts of linearly ordered sets (of buds) and the solutions will in fact depend also on them, altough it will not be explicitely written, to lighten the notations.

A solution of (6.1) is a family (A_i, ψ_i) where for $i = 1,\ldots,p$, A_i is a $\mathbb{L}$-species such that $A_i(0) = Z_i$ and ψ_i is an isomorphism

$$dA_i/dT \xrightarrow[\sim]{\psi_i} M_i(A_i(T), \ldots, A_p(T)) \ . \tag{6.3}$$

As in the case of one equation, the system (6.1) is equivalent to the integral equations

$$A_i(T) = Z_i + \int_0^T M_i(A_1(X),\ldots,A_p(X))\, dX \ , \qquad i = 1, \ldots, p \ . \tag{6.4}$$

For each i , this equation can be represented combinatorially as in Figure 6.1, where in the non-empty case, the minimum element is marked by the number i and the M_i-structure lies over a p-tuple of sets of lines of p sorts: a line of sort j is used to attach an A_j-structure to the minimum element.

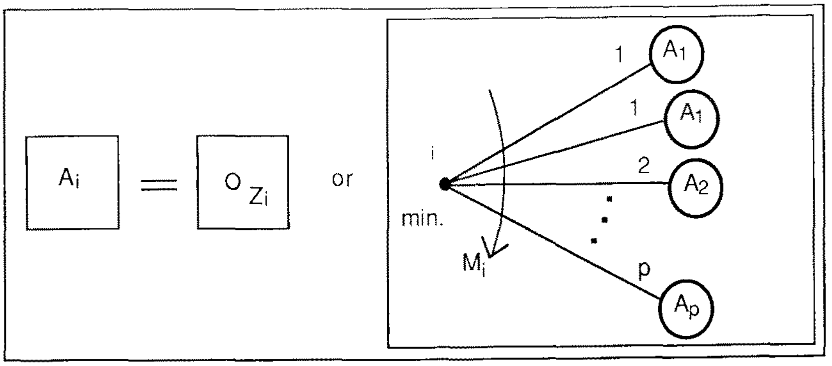


Figure 6.1

Since this process can be iterated,we obtain a canonical solution $(A_{\mathbf{M},1},\ldots,A_{\mathbf{M},p})$ where $A_{\mathbf{M},i}(T)$ is the $\mathbb{L}$-species of p-colored $\mathbf{M}$-enriched increasing arborescences as represented in figure 6.2 (where $p = 2$), where each point is marked with some color j , $1 \leq j \leq p$, and is the root of an $A_{\mathbf{M},j}$-structure; the buds of sort i are empty $A_{\mathbf{M},i}$-structures and are marked by Z_i.

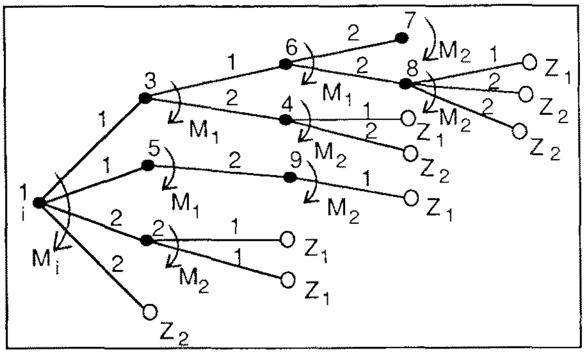


Figure 6.2 $A_{\mathbf{M},i}$-structure

Theorem 6.1. The $\mathbb{L}$-species of p-colored $\mathbf{M}$-enriched increasing arborescences is, up to isomorphism, the unique solution $A_{\mathbf{M},1},\ldots,A_{\mathbf{M},p}$ of the system (6.1) Moreover this solution can be expressed as, for $i = 1,\ldots,p$,

$$A_{\mathbf{M},i}(T) \;=\; e^{T\mathcal{D}}Z_i \tag{6.5}$$

where $\mathcal{D}$ is the combinatorial differential operator

$$\mathcal{D} = \sum_{j=1}^{p} M_j(Z_1,\dots,Z_p)\,\partial/\partial Z_j \tag{6.6}$$

Proof. The expression (6.5) is very similar to (5.8), in the case of one autonomous equation (see cor. 5.3). The only difference here is that the eclosions can be of p different sorts: for $j = 1,..,p$, the operator $TM_j(Z_1,\dots,Z_p)\partial/\partial Z_j$ is interpreted as "replace the (phantom) minimum bud of sort j by a point (marked j) with an M_j-assembly of buds attached". □

Note also that (6.5) can be written as

$$A_{\mathbf{M},i}(T) = \sum_{n\geq 0} \mathcal{D}^n Z_i\, T^n/n! \tag{6.7}$$

which says that pointless eclosions can be performed first and then the points placed in position according to the order of eclosions as in corollary 5.3.

The system of differential equations that we considered in the introduction (see (1.1)) can be written as

$$\begin{aligned} Y_1' &= 1 + Y_1^2 \quad , \quad & Y_1(0) = 0 \\ Y_2' &= Y_1Y_2 \quad , \quad & Y_2(0) = 1 \end{aligned} \tag{6.8}$$

Its canonical solution, as given then, is that of complete increasing binary trees for Y_1 and complete (except for an empty bud at the far right) increasing binary trees for Y_2. See figures 1.3 and 1.4 where colors 1 and 2 are modestly realized as black and white respectively.

The tangent and secant functions are also solutions of the following slightly different system of equations:

Example 6.2. Consider the system of equations

$$\begin{aligned} Y_1' &= Y_2^2, \quad & Y_1(0) = 0 \\ Y_2' &= Y_1Y_2, \quad & Y_2(0) = 1. \end{aligned} \tag{6.9}$$

Here the two sorts of eclosions are as follows:

1. $TZ_2^2\,\partial/\partial Z_1$ (see fig. 6.3). **2.** $TZ_1Z_2\,\partial/\partial Z_2$ (see fig. 6.4).

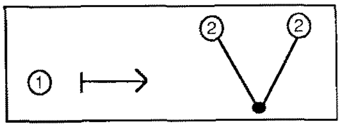


Figure 6.3

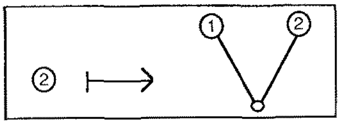


Figure 6.4

In order to get the canonical solution $Y_1 = A_1(T)$, $Y_2 = A_2(T)$, we can then start with a bud Z_1 or Z_2 and iterate the eclosion operators a certain number of times. If we set $Z_1 = 0$ and $Z_2 = 1$, we get, for $A_1(T)$, increasing binary trees of the form illustrated by Figure 6.8 and for $A_2(T)$, increasing binary trees like those rooted at white points in Figure 6.8 (the labels a, c and d on points, and C_0 and D_0 on buds should be disregarded in Figure 6.8 for the time being. Define a **left branch** to be a maximal subtree having only left edges. These increasing binary trees, called **Jacobi arborescences**, are characterized by the following properties:

A_1: All the left branches are even (number of points, not counting buds), except the leftmost one which is odd.

A_2: All the left branches are even.

This combinatorial model of the tangent and secant functions is appropriate for establishing the integral (1.5) which can be equivalently written as

$$\tan T + \sec T = \exp\left(\int_0^T \sec X \, dX\right). \tag{6.10}$$

Indeed, if you cut the edges of the leftmost left branch of an A_1- or A_2-structure, what you get is an assembly (in the form of a decreasing list), of $\int A_2$-structures.

Note also that this model can take other interesting forms. For example, when the bijection between increasing binary trees and permutations (projection on the x-axis) is applied to A_1 and A_2, it gives the class of **Jacobi permutations** introduced by one of the authors in [V2] in order to obtain a combinatorial interpretation for the Jacobi elliptic functions (see below).

Another possibility is to apply the bijection between permutations, or increasing binary trees, and forests of increasing arborescences (see [BU] , [V1] or [V3]). This gives forests, that is assemblies (odd or even), of increasing arborescences where all points have an even number of sons. For example, the forest of figure 6.5 corresponds to the binary tree of figure 6.8 under this bijection.

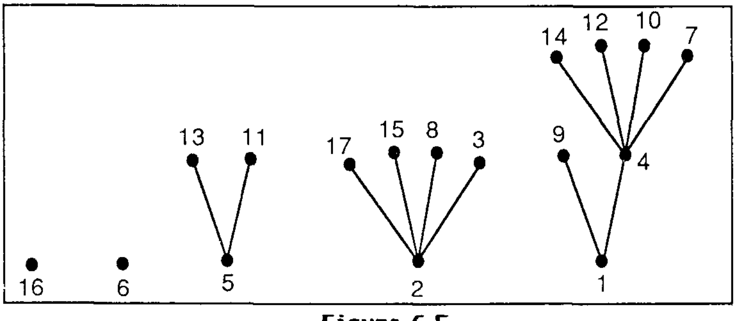


Figure 6.5

Example 6.3. Elliptic functions. We consider the following system

$$
\begin{aligned}
S' &= aCD\ , & S(0) &= S_0\ ,\\
C' &= cDS\ , & C(0) &= C_0\ ,\\
D' &= dSC\ , & D(0) &= D_0\ ,
\end{aligned}
\tag{6.11}
$$

where a, c, d are some scalar parameters. The classical Jacobi elliptic functions correspond to this system (sn, cn, dn, respectively, for S, C, D) with $a = 1$, $c = -1$, $d = -k^2$, $S_0 = 0$, $C_0 = D_0 = 1$. They can be expanded in the form

$$
\begin{aligned}
sn(t, k) &= \textstyle\sum_{n\geq 0} (-1)^n J_{2n+1}(k)\, t^{2n+1}/(2n+1)!\ ,\\
cn(t, k) &= 1 + \textstyle\sum_{n\geq 1} (-1)^n J_{2n}(k)\, t^{2n}/(2n)!,\\
dn(t, k) &= 1 + \textstyle\sum_{n\geq 1} (-1)^n k^{2n} J_{2n}(1/k)\, t^{2n}/(2n)!.
\end{aligned}
\tag{6.12}
$$

Here $J_{2n+1}(k)$ and $J_{2n}(k)$ are even polynomials with non negative integer coefficients, of degree respectively $2n$ and $2n-2$.

We will briefly show how the general theory gives back the interpretations of Viennot [V2] and of Schett's polynomials [SCH]. There are three types of eclosions, as displayed on figure 6.6.

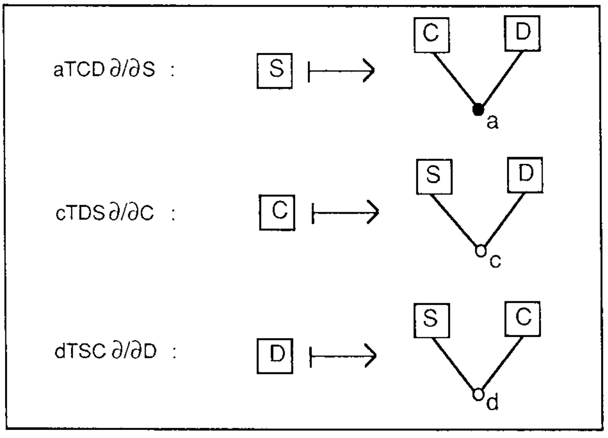


Figure 6.6

Let us assume that $S_0 = 0$. The underlying binary trees of the solutions $S(T)$, $C(T)$, $D(T)$ of (6.11) are of the same type as in the previous example (see fig. 6.8)). Moreover, we have to consider the weight of these trees in terms of the parameters a, c and d. In fact, once the underlying binary tree is known, the weight $a^i c^j d^h$ of an S- (or C- or D-) structure can easily be determined recursively, starting at the root.

As pointed out in the introduction a little more work has to be done in order to define this weight in a "global" form, without recursivity. Define the **right-height** of a vertex in a binary tree as the number of right edges of the path going from the root to that vertex. We redefine the eclosions of the first type as follows:

If the eclosion appears at an even right-height vertex, use fig 6.7, a):

If the eclosion appears at an odd right-height vertex, use fig 6.7, b):

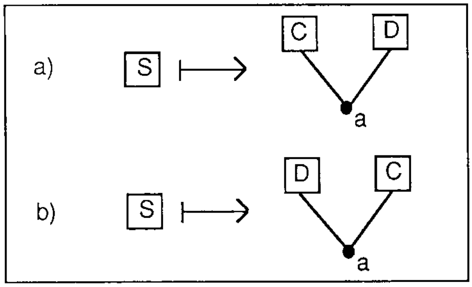


Figure 6.7

The binary trees produced are the same as before. In particular all the left branches are even, except the leftmost in the case of S(T). See the figure 6.8, where the weights a, c and d as well as the bud labels S_0, C_0, D_0, have been displayed according to these new eclosion rules.

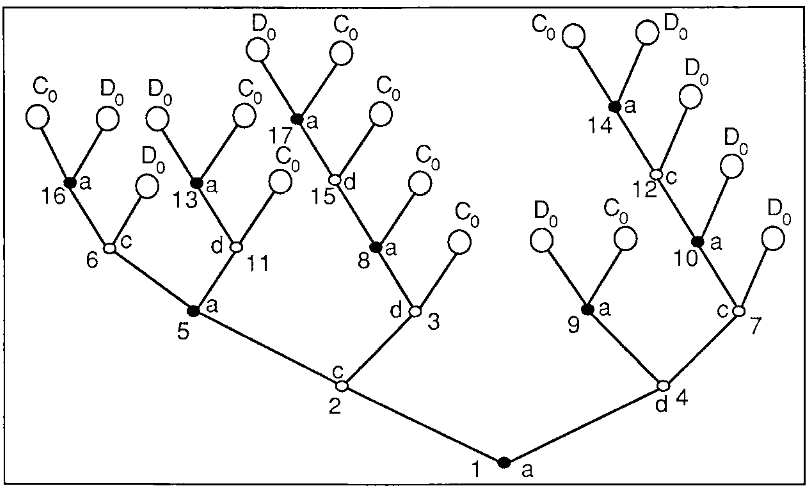


Figure 6.8

It is easy to see that all the eclosions of the third type appear at an odd right-height in the binary tree (vertices labelled d). Also, on each left branch containing such vertices, exactly half of them are labelled d (the others being labelled a). Thus the weight of a S-binary tree of size $2n+1$ (buds are not counted) is $a^{n+1}c^{n-j}d^{j}$, where j is equal to half the number of nodes having an odd right-height. Setting $a = 1$, $c = -1$ and

$d=-k^2$ and taking generating functions, we obtain from (6.12) that, for example,

$$J_{2n+1}(k) = \sum_{j=0}^{n} J_{2n+1,2j}\, k^{2j} , \qquad (6.13)$$

where $J_{2n+1,2j}$ is the number of Jacobi arborescences with $2n+1$ vertices among which exactly $2j$ have an odd right height. By projection of these binary trees, we get the Jacobi permutations and the interpretation of Viennot [V2]. Also, as in the previous example, using the bijection between increasing binary trees and assemblies of increasing arborescences (see fig. 6.5), one gets another result of [V2]: $J_{2n+1,2j}$ is equal to the number of forests of increasing arborescences of size $2n+1$, with all vertices of even degree and having $2j$ vertices at an odd height.

The functions cn and dn can be deduced similarly.

Dumont's interpretation of the Jacobi elliptic functions is based on Schett's method [SCH]. Schett introduced (in a slightly different form) the following polynomials:

$$S_m(x,y,z) = \mathcal{D}^m x \quad , \qquad m \geq 1 , \qquad (6.14)$$

where

$$\mathcal{D} = yz\partial/\partial x + zx\partial/\partial y + xy\partial/\partial z \ . \qquad (6.15)$$

These polynomials can be written explicitly as

$$\begin{aligned} S_{2n+1}(x,y,z) &= \textstyle\sum_{i,j\geq 0} a_{2n+1,i,j}\, x^{2i} y^{2j+1} z^{2n-2i-2j+1} , \\ S_{2n}(x,y,z) &= \textstyle\sum_{i,j\geq 0} a_{2n,i,j}\, x^{2i+1} y^{2j} z^{2n-2i-2j} . \end{aligned} \qquad (6.16)$$

Schett's result is that the coefficients $a_{2n+1,0,j}$ (resp. $a_{2n,i,0}$) are precisely the coefficients $J_{2n+1,2j}$ of the polynomials $J_{2n+1}(k)$ (resp. $J_{2n,2i}$ of $J_{2n}(k)$).

This can be shown as follows. The operator $\mathcal{D}$ is the operator of theorem 6.1. Thus $S_m(x,y,z)$ is the polynomial enumerating the $m!$ increasing binary trees according to the number of buds labelled S_0, C_0 and D_0. Let $a^e c^g d^j$ be the weight of such a tree; i.e. there have been e eclosions of first type, g of second type, and j of third type and $e+g+j=m$. Let α (resp. γ, resp. δ) be the number of buds labelled S_0 (resp. C_0, resp D_0). We have the equations

$$\alpha = 1-e+g+j , \qquad \gamma = e-g+j , \qquad \delta = e+g-j . \qquad (6.17)$$

The case of sn is given by putting $\alpha = 0$ (no buds labelled S_0) and $m = 2n+1$. Eliminating e gives $\gamma = 2j+1$ and $\delta = 2n-2j+1$ and we get Schett's result. The functions cn and dn can be treated similarly. We will give more results, in particular the relationship between the present general theory and Dumont's work [D1], [D2] in another paper.

Example 6.4 Duffing's equation

Higher order differential equation are generally reduced to systems of first order differential equations. We give here the classical example of the cubic anharmonic oscillator, commonly known as Duffing's equation

$$y'' = ay' + by + cy^3 , \qquad y(0) = \alpha . \tag{6.18}$$

Denoting $\beta = y'(0)$, this is equivalent to the system

$$\begin{aligned} y' &= u , & y(0) &= \alpha , \\ u' &= au + by + cy^3 , & u(0) &= \beta . \end{aligned} \tag{6.19}$$

There are four types of eclosions giving birth to the weighted increasing rooted planar trees as shown in figure 6.9 :

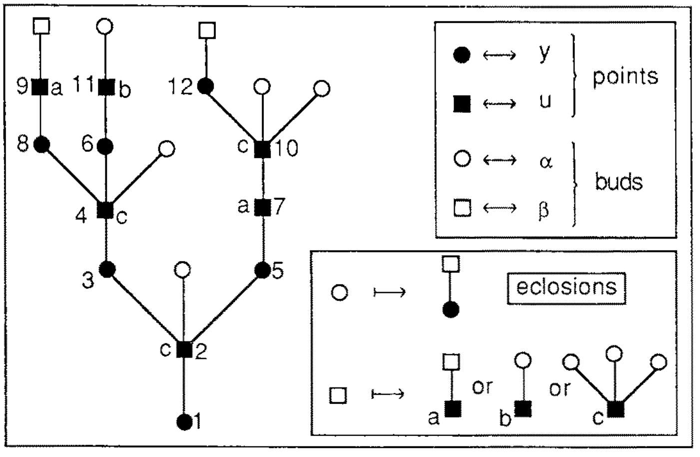


Figure 6.9.

The tree of figure 6.9 gives the contribution $a^2bc^3\alpha^4\beta^2$ to the coefficient of $t^{12}/12!$ of the Taylor expansion of the solution. This kind of planar trees can be drawn in a more compact form, as shown in figure 6.10.

There are three types of nodes:

1) some nodes contain two numbers (unordered),

2) some contain one number and a "□",

3) some contain a single number.

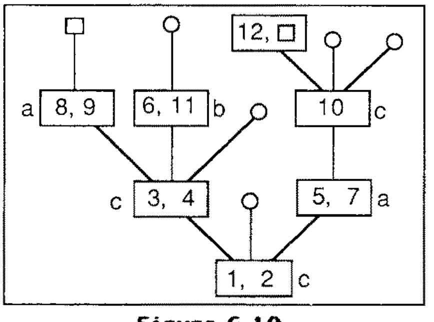


Figure 6.10

In case 1 or 2, the father is weighted b or c (depending if there is one or three sons). In case 3, the father is weighted a. These planar rooted trees are 1-3 (one or three sons) and still have the increasing property.

The case $a=0$ is particularly important since it contains that of Jacobi elliptic functions sn, cn and dn. The equation $y'' = 2y^3$, $y(0) = 1$, $y'(0) = 1$, gives birth to increasing ternary trees with nodes formed by pairs (unordered) of integers or, possibly for external nodes, by a single integer. Such trees can be put in bijection with permutations. Increasing binary trees correspond to binary search trees in Computer Science (see Françon [FR]). The above ternary trees also have a corresponding concept in Computer Science, as shown by Jonassen & Knuth [JK]. The analysis of the average cost of comparisons involves elliptic functions.

We will give more details in another paper, combining the general concepts of this paper with the concept of "histories". This will be particularily important when the Duffing equation has forcing terms. Starting from Fliess & Lamnabhi-Lagarrigue's paper [FL1] we will give more explicit computations of the coefficients of the functional expansion of the solution, in terms of certain weighted paths.

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