THE JOURNAL OF SYMBOLIC LOGIC
Volume 37, Number 3, Sept. 1972

COUNTABLE HOMOGENEOUS RELATIONAL STRUCTURES AND $\aleph_0$-CATEGORICAL THEORIES

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A relational structure $\mathfrak{A}$ of cardinality $\aleph_0$ is called *homogeneous* by Fraissé [1] if each isomorphism between finite substructures of $\mathfrak{A}$ can be extended to an automorphism of $\mathfrak{A}$. In §1 of this paper it is shown that there are $2^{\aleph_0}$ isomorphism types of such structures for the first order language L_0 with a single (binary) relation symbol, answering a question raised by Fraissé. In fact, as is shown in §2, a family of $2^{\aleph_0}$ nonisomorphic homogeneous structures for L_0 can be constructed, each member $\mathfrak{A}$ of which satisfies the following conditions (where U is the homogeneous, $\aleph_0$-universal graph, the structure of which is considered in [4]):

(i) The relation R of $\mathfrak{A}$ is asymmetric ($R \cap R^{-1} = \varnothing$);

(ii) If A is the domain of $\mathfrak{A}$ and S is the symmetric relation $R \cup R^{-1}$, then (A, S) is isomorphic to U. That is, each $\mathfrak{A}$ may be regarded as the result of assigning a unique direction to each edge of the graph U.

Let T_0 be the first order theory of all homogeneous structures for L_0 which have cardinality $\aleph_0$. In §3 (which can be read independently of §2) it is shown that T_0 has $2^{\aleph_0}$ complete extensions (in L_0), each of which is $\aleph_0$-categorical. Moreover, among the complete extensions of T_0 are theories of arbitrary (preassigned) degree of unsolvability. In particular, there exists an undecidable, $\aleph_0$-categorical theory in L_0, which answers a question raised by Grzegorczyk [2], [3].

It follows from Theorem 6 of [3] that there are $\aleph_0$-categorical theories of partial orderings which have arbitrarily high degrees of unsolvability. This is in sharp contrast to the situation for linear orderings, which were the motivation for Fraissé's early work. Indeed, as is shown in [10], every $\aleph_0$-categorical theory of a linear ordering is finitely axiomatizable. (W. Glassmire [12] has independently shown the existence of $2^{\aleph_0}$ theories in L_0 which are all $\aleph_0$-categorical, and C. Ash [13] has independently shown that such theories exist with arbitrary degree of unsolvability.)

§1. Let L be a first order language with $=$ and finitely many relation symbols $P^1, \cdots, P^n$. In this section a method for obtaining countable homogeneous structures for L is given. (It should be noted that "homogeneous" is used, in model theory, in a weaker sense than Fraissé's. (Cf. [5], [6] and [7].) This should cause no confusion, since only Fraissé's concept will be used in this paper.)

The domain of a structure $\mathfrak{A}$ for L is denoted by $|\mathfrak{A}|$ and the relations of $\mathfrak{A}$ by $P^1_{\mathfrak{A}}, \cdots, P^n_{\mathfrak{A}}$. If $\mathfrak{A}$ and $\mathfrak{B}$ are structures for L, then $\mathfrak{A} \subset \mathfrak{B}$ means that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and $\mathfrak{A} \simeq \mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. The *union* of $\mathfrak{A}$ and

Received April 14, 1971, revised August 9, 1971.

$\mathfrak{B}$, denoted by $\mathfrak{A} \cup \mathfrak{B}$, is defined by letting $|\mathfrak{A} \cup \mathfrak{B}| = |\mathfrak{A}| \cup |\mathfrak{B}|$ and, for each $i = 1, \cdots, n$, $P^i_{\mathfrak{A}\cup\mathfrak{B}} = P^i_{\mathfrak{A}} \cup P^i_{\mathfrak{B}}$. The structures $\mathfrak{A}$ and $\mathfrak{B}$ are *compatible* if $|\mathfrak{A}| \cap |\mathfrak{B}|$ induces the same substructure in $\mathfrak{A}$ as in $\mathfrak{B}$. Equivalently, $\mathfrak{A}$ and $\mathfrak{B}$ are compatible if $\mathfrak{A} \subset \mathfrak{A} \cup \mathfrak{B}$ and $\mathfrak{B} \subset \mathfrak{A} \cup \mathfrak{B}$.

A class $\mathscr{K}$ of structures for L is *closed under unions* if $\mathfrak{A} \cup \mathfrak{B}$ is in $\mathscr{K}$ whenever $\mathfrak{A}, \mathfrak{B} \in \mathscr{K}$ and $\mathfrak{A}, \mathfrak{B}$ are compatible. $\mathscr{K}$ is a Γ-*class* (in the language of [1]) if there is a structure $\mathfrak{A}$ which satisfies:

(1) $\mathfrak{B}$ is in $\mathscr{K}$ if and only if every finite substructure of $\mathfrak{B}$ can be embedded in $\mathfrak{A}$.

Equivalently, $\mathscr{K}$ is a Γ-class if it is a universal class with the embedding property:

(2) if $\mathfrak{A}, \mathfrak{B} \in \mathscr{K}$, then there exists $\mathfrak{C} \in \mathscr{K}$ such that $\mathfrak{A}$ and $\mathfrak{B}$ can be embedded in $\mathfrak{C}$.

(See [1, Theorem III] and recall the Tarski–Łos characterization of universal classes [11].)

A structure $\mathfrak{A}$ for L is *decomposable* if it has proper substructures, $\mathfrak{A}_1$ and $\mathfrak{A}_2$, which satisfy $\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2$. Otherwise $\mathfrak{A}$ is *indecomposable*.

LEMMA 1.1. *Let $\mathfrak{A}$ be an indecomposable structure and $\mathfrak{B}_1$, $\mathfrak{B}_2$ be compatible structures for L. If $\mathfrak{A}$ can be embedded in $\mathfrak{B}_1 \cup \mathfrak{B}_2$, then $\mathfrak{A}$ can be embedded either in $\mathfrak{B}_1$ or in $\mathfrak{B}_2$.*

PROOF. Suppose that f as an embedding of $\mathfrak{A}$ into $\mathfrak{B}_1 \cup \mathfrak{B}_2$. For each $j = 1, 2$, let $\mathfrak{A}_j$ be the substructure of $\mathfrak{A}$ whose domain is $f^{-1}(|\mathfrak{B}_j|)$. If $(a_1, \cdots, a_k) \in P^i_{\mathfrak{A}}$, then $(fa_1, \cdots, fa_k)$ is in either $P^i_{\mathfrak{B}_1}$ or $P^i_{\mathfrak{B}_2}$. Thus $a_1, \cdots, a_k$ are all in $|\mathfrak{A}_1|$ or all in $|\mathfrak{A}_2|$. That is, $\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2$. Since $\mathfrak{A}$ is indecomposable, it follows that f maps $|\mathfrak{A}|$ into $|\mathfrak{B}_1|$ or into $|\mathfrak{B}_2|$. The fact that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are compatible implies that f is an embedding of $\mathfrak{A}$ into $\mathfrak{B}_1$ or into $\mathfrak{B}_2$.

For any class X of finite structures, let $\Gamma(X)$ denote the class of all structures $\mathfrak{A}$ for L which satisfy: no member of X can be embedded in $\mathfrak{A}$. X is a *base* if it is closed under isomorphism and satisfies (i) every member of X is indecomposable, and (ii) if $\mathfrak{A}, \mathfrak{B} \in X$ and $\mathfrak{A}$ can be embedded into $\mathfrak{B}$, then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.

THEOREM 1.2. (i) *Let X be any class of finite, indecomposable structures for L and suppose $\Gamma(X) \neq \varnothing$. Then $\Gamma(X)$ is a Γ-class which is closed under unions.*

(ii) *If $\mathscr{K}$ is any Γ-class which is closed under unions, then there is a unique base X which satisfies $\mathscr{K} = \Gamma(X)$.*

PROOF. (i) Evidently $\Gamma(X)$ is closed under isomorphism and under the formation of substructures. Since each member of X is finite, $\Gamma(X)$ is closed under unions of arbitrary chains. Each member of X is also indecomposable, so that by Lemma 1.1, $\Gamma(X)$ is closed under unions. By [1, Theorem III] it remains only to show that $\Gamma(X)$ satisfies (2). But if $\mathfrak{A}, \mathfrak{B} \in \Gamma(X)$, then the disjoint union of $\mathfrak{A}$ and $\mathfrak{B}$ is also in $\Gamma(X)$, and both $\mathfrak{A}$ and $\mathfrak{B}$ can be embedded in this structure.

(ii) Let X be the class of all structures $\mathfrak{A}$ for L which satisfy: (i) $\mathfrak{A} \notin \mathscr{K}$ and (ii) each proper substructure of $\mathfrak{A}$ is in $\mathscr{K}$. Since $\mathscr{K}$ is a Γ-class the members of X are finite and $\mathscr{K} = \Gamma(X)$. Since $\mathscr{K}$ is closed under unions, the members of X are all indecomposable. Moreover, if $\mathfrak{A}, \mathfrak{B} \in X$, then since $\mathfrak{A}$ is not in $\mathscr{K}$, $\mathfrak{A}$ cannot be isomorphic to a proper substructure of $\mathfrak{B}$. This shows that X is a base. The uniqueness of X is clear from its definition.

If $\mathscr{K}$ is a Γ-class which is closed under unions, then it satisfies the amalgamation property ($\bar{D}$) of [1]. Also, $\mathscr{K}$ contains arbitrarily large finite structures (for example, disjoint unions of one-element structures). It follows [1, Theorem IV] that $\mathscr{K}$ contains a unique (up to isomorphism) homogeneous structure $\mathfrak{A}$ which has cardinality $\aleph_0$ and satisfies (1). Such a structure for $\mathscr{K}$ will be denoted by $\mathfrak{A}(X)$, where X is the unique base which satisfies $\mathscr{K} = \Gamma(X)$. Thus $\mathfrak{A}(X)$ is a homogeneous structure for L of cardinality $\aleph_0$ whenever X is a base and $\Gamma(X) \neq \varnothing$. (Note that for each base X, $\Gamma(X) = \varnothing$ if and only if X consists of the class of all structures for L with one element.)

THEOREM 1.3. *If X_1, X_2 and X are bases which satisfy $X_1 \cup X_2 \subset X$ and $\Gamma(X) \neq \varnothing$, then the following are equivalent:*

(i) $X_1 \supset X_2$;

(ii) $\Gamma(X_1) \subset \Gamma(X_2)$;

(iii) $\mathfrak{A}(X_1)$ *can be embedded in* $\mathfrak{A}(X_2)$.

PROOF. Obviously (i) implies (ii). If (ii) holds, then $\mathfrak{A}(X_1)$ is in $\Gamma(X_2)$. The $\aleph_0$-universality of $\mathfrak{A}(X_2)$ in $\Gamma(X_2)$ [1, Theorem V] shows that (iii) holds in that case.

Finally, suppose that (iii) holds. If $\mathfrak{A} \in X_2$, then $\mathfrak{A}$ cannot be embedded in $\mathfrak{A}(X_1)$, so that some member $\mathfrak{B}$ of X_1 can be embedded in $\mathfrak{A}$. Since $\mathfrak{A}$ and $\mathfrak{B}$ are in the base X, they must be isomorphic. That is, $X_2 \subset X_1$, which completes the proof.

COROLLARY 1.4. *If X_1, X_2 and X are bases which satisfy $X_1 \cup X_2 \subset X$ and $\Gamma(X) \neq \varnothing$, then*

$$\mathfrak{A}(X_1) \simeq \mathfrak{A}(X_2) \leftrightarrow X_1 = X_2.$$

PROOF. This follows immediately from Theorem 1.3 and the fact that countable homogeneous structures are determined up to isomorphism by their finite substructures.

Now let L_0 be the language with a single (binary) relation symbol P. To show that there are $2^{\aleph_0}$ isomorphism types of countable homogeneous structures for L_0, it suffices (using Corollary 1.4) to exhibit a base X which contains infinitely many isomorphism types.

For each $n \geq 1$, let π_n be the cyclic permutation of the set $\{0, 1, \cdots, n+1\}$: $\pi_n(0) = 1$, $\pi_n(1) = 2, \cdots, \pi_n(n) = n+1$, $\pi_n(n+1) = 0$. Let $\mathfrak{A}_n$ be the structure whose domain is $\{0, 1, \cdots, n+1\}$ and whose relation is defined by

$$P_{\mathfrak{A}_n} = \{(a, b) \mid 0 \leq a, b \leq n+1 \text{ and } \pi_n(a) \neq b\}.$$

Note that if a is in $|\mathfrak{A}_n|$, then $\pi_n(a)$ is the unique b in $|\mathfrak{A}_n|$ for which (a, b) is not in $P_{\mathfrak{A}_n}$. Since the domain of $\mathfrak{A}_n$ has at least 3 elements, it follows that $\mathfrak{A}_n$ is indecomposable.

Suppose that f is an embedding of $\mathfrak{A}_n$ into $\mathfrak{A}_m$, so that $n \leq m$. By the observation above, for each a in $|\mathfrak{A}_n|$, $\pi_m(fa)$ must equal $f(\pi_n(a))$. Since π_m is an automorphism of $\mathfrak{A}_m$ it may be assumed that $f(0) = 0$. By induction it follows that $f(a) = a$ for each a in $|\mathfrak{A}_n|$. Then $0 = \pi_m(n+1)$ and hence $n = m$. That is, if $n \neq m$, then $\mathfrak{A}_n$ cannot be embedded into $\mathfrak{A}_m$.

For each subset S of $N = \{1, 2, \cdots\}$ the class defined by

$$X_S = \{\mathfrak{B} \mid \mathfrak{B} \simeq \mathfrak{A}_n \text{ for some } n \in N - S\}$$

is a base. (Note that the map taking S to X_S is order reversing.) For each subset

S of N let $\mathfrak{A}_S$ be the countable homogeneous structure $\mathfrak{A}(X_S)$. The following result is immediate from Theorem 1.3 and Corollary 1.4:

THEOREM 1.5. *There is a* 1-1 *mapping,* $S \mapsto \mathfrak{A}_S$, *from* $\mathscr{P}(N)$ *into the class of countable homogeneous structures for* L_0 *which satisfies*:

$$\mathfrak{A}_S \text{ can be embedded in } \mathfrak{A}_T \leftrightarrow S \subset T.$$

§2. A structure $\mathfrak{A}$ for L_0 is called a *graph* if the relation $P_\mathfrak{A}$ is symmetric and irreflexive. $\mathfrak{A}$ is called a *directed graph* if $P_\mathfrak{A}$ is asymmetric. Associated with each directed graph $\mathfrak{A}$ is a graph $\mathfrak{A}^*$ (the *underlying graph* of $\mathfrak{A}$) which is defined by letting $|\mathfrak{A}^*| = |\mathfrak{A}|$ and

$$P_{\mathfrak{A}^*} = \{(a, b) \mid (a, b) \text{ or } (b, a) \text{ is in } P_\mathfrak{A}\}.$$

Thus a directed graph $\mathfrak{A}$ may be regarded as the result of assigning a unique direction to each edge of the underlying graph $\mathfrak{A}^*$.

It is clear that the class $\mathscr{K}_0$ of all graphs and the class $\mathscr{K}_1$ of all directed graphs are Γ-classes and that both are closed under unions. Let X_0 and X_1 be the unique bases which satisfy $\mathscr{K}_0 = \Gamma(X_0)$ and $\mathscr{K}_1 = \Gamma(X_1)$.

Note that a structure $\mathfrak{A}$ for L_0 is decomposable if and only if there exist distinct elements a, b of $|\mathfrak{A}|$ such that neither (a, b) nor (b, a) is in $P_\mathfrak{A}$. In particular, a graph is indecomposable exactly when it is a *complete* graph. Since each complete graph can be embedded in any other complete graph of larger cardinality, it follows that any base X which properly contains X_0 must be of the form

$$X = X_0 \cup \{\mathfrak{A} \mid \mathfrak{A} \simeq K_p\},$$

where K_p denotes a complete graph of cardinality p. Thus the methods of this paper can yield only a countable number of homogeneous graphs (cf. [4]).

Let the homogeneous, $\aleph_0$-universal graph $\mathfrak{A}(X_0)$ be denoted by U (as in [4]). There does exist a base X which contains X_1 together with an infinite number of isomorphism types. Moreover, X can be chosen so that for each base Y which satisfies $X_1 \subset Y \subset X$ the underlying graph of $\mathfrak{A}(Y)$ is isomorphic to U.

The construction of X proceeds in two stages. First, for each $n \geq 1$, define $\mathfrak{A}_n$ by letting

$$|\mathfrak{A}_n| = \{0, 1, \cdots, n + 2\}$$

and

$$P_{\mathfrak{A}_n} = \{(a, b) \mid 0 \leq a, b \leq n + 2 \text{ and either } b = a + 1 \text{ or } b + 1 < a\}.$$

For $a < b$, either $a + 1 = b$ or $a + 1 < b$. That is, if $0 \leq a < b \leq n + 2$, then either (a, b) or (b, a) is in $P_{\mathfrak{A}_n}$. This shows that each $\mathfrak{A}_n$ is indecomposable. Note that

$$\text{if } a \leq b \text{ and } a, b \in |\mathfrak{A}_n|, \text{ then } (a, b) \in P_{\mathfrak{A}_n} \leftrightarrow a + 1 = b. \tag{3}$$

In particular each $\mathfrak{A}_n$ is a directed graph.

LEMMA 2.1. *If f is an embedding of $\mathfrak{A}_n$ into $\mathfrak{A}_m$, then $n \leq m$ and $f(a) = f(0) + a$ for each $a \in |\mathfrak{A}_n|$.*

PROOF. First, assume that f is an embedding of $\mathfrak{A}_1$ into $\mathfrak{A}_m$. Choose i_0 and j_0 so that $f(i_0)$ is the minimum and $f(j_0)$ is the maximum of the set $f(|\mathfrak{A}_1|)$. From (3) it follows that there is at most one i in $|\mathfrak{A}_1|$ which satisfies $(f(i_0), f(i)) \in P_{\mathfrak{A}_m}$. Similarly there is at most one j in $|\mathfrak{A}_1|$ which satisfies $(f(j), f(j_0)) \in P_{\mathfrak{A}_m}$. This implies i_0 is either 0 or 1 and j_0 is either 2 or 3. If $i_0 = 1$ and $j_0 = 2$, then $(f(i_0), f(j_0))$ is in $P_{\mathfrak{A}_m}$. But this implies that $f(j_0) = f(i_0) + 1$, by (3), which is impossible. Therefore $i_0 = 0$

or $j_0 = 3$. In either case, repeated use of (3) shows that $f(a) = f(0) + a$ for each $a \in |\mathfrak{A}_1|$.

In general, suppose f is an embedding of $\mathfrak{A}_n$ into $\mathfrak{A}_m$, so that $n \leq m$. For each subset A of $|\mathfrak{A}_n|$ which is of the form $\{a, a+1, a+2, a+3\}$, the substructure of $\mathfrak{A}_n$ with domain A is isomorphic to $\mathfrak{A}_1$ (via the function taking i to $a + i$). It follows that, for each $0 \leq i \leq 3$, $f(a+i) = f(a) + i$. By induction on $a \in |\mathfrak{A}_n|$ this shows that $f(a) = f(0) + a$.

The base X will be constructed in the form

$$X = X_1 \cup \{\mathfrak{B} \mid \mathfrak{B} \simeq \mathfrak{B}_n \text{ for some } n \geq 3\}$$

where each $\mathfrak{B}_n$ is a one-element extension of $\mathfrak{A}_n$, defined as follows. Let $|\mathfrak{B}_n| = |\mathfrak{A}_n| \cup \{\alpha\}$, where α is not a nonnegative integer. Let $P_{\mathfrak{B}_n}$ consist of $P_{\mathfrak{A}_n}$ together with the pairs $(\alpha, 0)$, $(\alpha, n+2)$ and, for each $1 \leq a \leq n+1$, (a, α). Evidently $\mathfrak{B}_n$ is an indecomposable directed graph, and has $\mathfrak{A}_n$ as a substructure. The next result shows that X is a base.

LEMMA 2.2. *If $n, m \geq 3$ and $n \neq m$, then $\mathfrak{B}_n$ cannot be embedded in $\mathfrak{B}_m$.*

PROOF. If otherwise, there exists an embedding f of $\mathfrak{B}_n$ into $\mathfrak{B}_m$, where $3 \leq n < m$. First, assume that, in addition, $f(|\mathfrak{A}_n|)$ is contained in $|\mathfrak{A}_m|$. Thus, for each $a \in |\mathfrak{A}_n|, f(a) = f(0) + a$, by Lemma 2.1. If $f(\alpha) = \alpha$, then $(\alpha, f(0))$ and $(\alpha, f(n+2))$ are in $P_{\mathfrak{B}_m}$. Therefore $f(0) = 0$ and $f(n+2) = n+2$, from which it follows that $n = m$, a contradiction. If $f(\alpha) \neq \alpha$, then $f(\alpha)$ must be in $|\mathfrak{A}_m|$. Thus $f(\alpha) < f(0)$ or $f(\alpha) > f(n+2)$. In the first case, $(f(\alpha), f(n+2))$ is not in $P_{\mathfrak{B}_m}$, which contradicts the fact that f is an embedding. In the second case, $(f(1), f(\alpha))$ is not in $P_{\mathfrak{B}_m}$, which is also a contradiction.

Therefore, f does not map $|\mathfrak{A}_n|$ into $|\mathfrak{A}_m|$ and there must exist $a \in |\mathfrak{A}_n|$ which satisfies $f(a) = \alpha$. Since f is an embedding, there can be at most two elements b of $|\mathfrak{B}_n|$ which satisfy $(a, b) \in P_{\mathfrak{B}_n}$. This implies that a is either 0 or 1. Let $\mathfrak{A}$ be the substructure of $\mathfrak{A}_n$ whose domain is $\{x \mid a+1 \leq x \leq n+2\}$. Then $\mathfrak{A}$ has at least 4 elements and is isomorphic in the obvious way to $\mathfrak{A}_{n-a-1}$. Applying Lemma 2.1, it follows that, for each $b \in |\mathfrak{A}|$, $f(b) = f(a+1) + (b - a - 1)$. Now $(a, a+1) \in P_{\mathfrak{B}_n}$, so that $(\alpha, f(a+1)) \in P_{\mathfrak{B}_m}$. This implies that $f(a+1)$ must be 0, since $f(a+1) < m+2$. It follows that $f(a+2) < f(n+2) < f(\alpha)$, and hence that $(f(\alpha), f(a+2)) \in P_{\mathfrak{B}_m}$. But this implies that $(\alpha, a+2)$ is in $P_{\mathfrak{B}_n}$, which contradicts the fact that $0 < a+2 < n+2$ (recall that $a = 0$ or 1 and $n \geq 3$). This completes the proof that the embedding f cannot exist.

LEMMA 2.3. *Let Y be a base which satisfies $X_1 \subset Y \subset X$ and let $\mathfrak{A}$ be a finite substructure of $\mathfrak{A}(Y)$. For each finite graph $\mathfrak{B}$ which satisfies $\mathfrak{A}^* \subset \mathfrak{B}$, there is an embedding of $\mathfrak{B}$ into $\mathfrak{A}(Y)^*$ which is the identity on $|\mathfrak{A}|$.*

PROOF. Suppose $|\mathfrak{A}| = \{a_1, \cdots, a_n\}$ and $|\mathfrak{B}| = \{a_1, \cdots, a_n, b_1, \cdots, b_m\}$, where it may be assumed that $n, m \geq 1$. Define a directed graph $\mathfrak{C}$ by letting $|\mathfrak{C}| = |\mathfrak{B}|$ and letting $P_{\mathfrak{C}}$ consist of $P_{\mathfrak{A}}$ together with (i) all pairs (a_i, b_j) which satisfy $(a_i, b_j) \in P_{\mathfrak{B}}$ and (ii) all pairs (b_j, b_k) which satisfy $j < k$ and $(b_j, b_k) \in P_{\mathfrak{B}}$. Then $\mathfrak{A}$ is a substructure of $\mathfrak{C}$, while the underlying graph of $\mathfrak{C}$ is $\mathfrak{B}$.

Thus it suffices to show that there is an embedding of $\mathfrak{C}$ into $\mathfrak{A}(Y)$ which is the identity on $|\mathfrak{A}|$. Since $\mathfrak{A}(Y)$ is $\aleph_0$-universal in $\Gamma(Y)$, it must only be shown that no member of Y may be embedded in $\mathfrak{C}$. Suppose otherwise that g is an embedding of

$\mathfrak{B}_n$ into $\mathfrak{C}$, where $n \geq 3$. Since $\mathfrak{A}$ is in $\Gamma(Y)$, the range of g is not contained in $|\mathfrak{A}|$. Let j_0 be the largest j which satisfies $1 \leq j \leq m$ and $b_j \in g(|\mathfrak{B}_n|)$. Choose $b \in |\mathfrak{B}_n|$ such that $g(b) = b_{j_0}$. There is an element a of $|\mathfrak{B}_n|$ such that $(b, a) \in P_{\mathfrak{B}_n}$ and therefore $(g(b), g(a)) \in P_{\mathfrak{C}}$. But this implies that $g(a) = b_j$ for some $j > j_0$, which is a contradiction. This completes the proof.

Let Y be any base which satisfies $X_1 \subset Y \subset X$. It follows from Lemma 2.3 and Theorem 5.5.1 of [1] that $\mathfrak{A}(Y)^*$ is homogeneous and that every finite graph can be embedded into $\mathfrak{A}(Y)^*$. Therefore $\mathfrak{A}(Y)^*$ is isomorphic to U. For each subset S of N let Y_S be the base $X_1 \cup \{\mathfrak{B} \mid \mathfrak{B} \simeq \mathfrak{B}_n$ for some $n \in N - S\}$. Let $\mathfrak{A}_S$ be the structure $\mathfrak{A}(Y_S)$. The argument above yields the following improvement of Theorem 1.5.

THEOREM 2.4. *There is a* 1-1 *mapping,* $S \mapsto \mathfrak{A}_S$, *from* $\mathscr{P}(N)$ *into the class of countable homogeneous structures for* L_0 *which satisfies*

(i) $\mathfrak{A}_S$ *is a directed graph whose underlying graph is isomorphic to* U, *and*

(ii) $\mathfrak{A}_S$ *can be embedded in* $\mathfrak{A}_T \leftrightarrow S \subset T$, *for each* S, T *in* $\mathscr{P}(N)$.

In particular, there are $2^{\aleph_0}$ nonisomorphic homogeneous directed graphs of cardinality $\aleph_0$. It would be interesting to know how many isomorphism types of countable homogeneous structures there are in other restricted classes; for example in the class of graphs or the class of partial orderings. It seems likely that for each of these classes the answer is $\aleph_0$.

§3. Let T_0 be the theory of the class of all homogeneous structures for L_0 which have cardinality $\aleph_0$.

THEOREM 3.1. (i) T_0 *is axiomatizable.*

(ii) *Every countable model of* T_0 *is homogeneous.*

(iii) T_0 *has* $2^{\aleph_0}$ *complete extensions in* L_0, *each of which is* $\aleph_0$-*categorical.*

PROOF. Let $\mathfrak{A}$ be a finite structure for L_0 with $|\mathfrak{A}| = \{a_1, \cdots, a_n\}$. Let $F(\mathfrak{A})$ denote an open formula (with variables $v_1, \cdots, v_n$) with the property: given any structure $\mathfrak{B}$ for L_0 and a function f mapping $|\mathfrak{A}|$ into $|\mathfrak{B}|$, f is an embedding if and only if

$$\mathfrak{B} \models F(\mathfrak{A})[fa_1, \cdots, fa_n].$$

For any pair $\mathfrak{A}_1, \mathfrak{A}_2$ of finite structures for L_0 which satisfy $\mathfrak{A}_1 \subset \mathfrak{A}_2$, let $G(\mathfrak{A}_1, \mathfrak{A}_2)$ denote the sentence

$$(\exists v_1) \cdots (\exists v_{n+m}) F(\mathfrak{A}_2) \rightarrow (\forall v_1) \cdots (\forall v_n)[F(\mathfrak{A}_1) \rightarrow (\exists v_{n+1}) \cdots (\exists v_{n+m}) F(\mathfrak{A}_2)]$$

where $|\mathfrak{A}_1| = \{a_1, \cdots, a_n\}$ and $|\mathfrak{A}_2| = \{a_1, \cdots, a_{n+m}\}$. Then $G(\mathfrak{A}_1, \mathfrak{A}_2)$ is valid in $\mathfrak{B}$ if and only if either (i) $\mathfrak{A}_2$ cannot be embedded in $\mathfrak{B}$, or (ii) every embedding of $\mathfrak{A}_1$ into $\mathfrak{B}$ can be extended to an embedding of $\mathfrak{A}_2$ into $\mathfrak{B}$.

Let S_0 be the set of all sentences $G(\mathfrak{A}_1, \mathfrak{A}_2)$ where $\mathfrak{A}_1 \subset \mathfrak{A}_2$ are finite structures for L_0. It follows from Theorem 5.5.1 of [1] that the countable models of S_0 are exactly the countable homogeneous structures for L_0. This proves (i) and (ii).

That the complete extensions of T_0 are all $\aleph_0$-categorical follows from (ii) and the fact that countable homogeneous structures for L_0 are determined up to isomorphism by their finite substructures. Theorem 1.5 then implies that T_0 has $2^{\aleph_0}$ complete extensions.

THEOREM 3.2. *For each degree of unsolvability* d *there is a complete* (*and thus* $\aleph_0$-*categorical*) *extension of* T_0 *which has degree* d.

PROOF. For each finite structure $\mathfrak{A}$ for L_0 let $H(\mathfrak{A})$ be the sentence

$$(\exists v_1)\cdots(\exists v_n)F(\mathfrak{A})$$

where $|\mathfrak{A}| = \{a_1, \cdots, a_n\}$.

Let X be the base constructed in §1 and let S be a subset of N which has degree d. Let Y be the base

$$Y = \{\mathfrak{A} \mid \mathfrak{A} \simeq \mathfrak{A}_n \text{ for some } n \in S\}$$

so that $Y \subset X$. Let T be the theory of $\mathfrak{A}(Y)$, so that T is a complete extension of T_0. Evidently a set of axioms for T may be constructed as the union of a set of axioms for T_0, the set $S_0 = \{\neg H(\mathfrak{A}_n) \mid n \in S\}$ and the set

$$S_1 = \{H(\mathfrak{A}) \mid \mathfrak{A} \text{ is finite and can be embedded in } \mathfrak{A}(Y)\}.$$

By the definition of $\mathfrak{A}(Y)$, S_1 consists of exactly those $H(\mathfrak{A})$ for which $\mathfrak{A}$ is finite and, for each $n \in S$, $\mathfrak{A}_n$ cannot be embedded in $\mathfrak{A}$. The construction of X insures that the set S_1 has the same degree as S. Therefore T has a set of axioms of degree d. Since T is a complete theory, this implies that T has degree d, and completes the proof.

REFERENCES

[1] ROLAND FRAISSÉ, *Sur l'extension aux relations de quelques proprietés des ordres*, ***Annales Scientifiques l'École Normale Supérieure***, vol. 71 (1954), pp. 361–388.

[2] ANDRZEJ GRZEGORCZYK, *Logical uniformity by decomposition and categoricity in* $\aleph_0$, ***Bulletin de l'Académie Polonaise des Sciences, Série Sciences, Mathematiques, Astronomiques et Physiques***, vol. 16 (1968), pp. 687–692.

[3] ———, *Decision procedures for theories categorical in* $\aleph_0$, ***Symposium on automatic demonstration***, Lecture Notes in Mathematics, No. 125, Springer-Verlag, Berlin, 1970, pp. 87–100.

[4] C. WARD HENSON, *A family of countable homogeneous graphs*, ***Pacific Journal of Mathematics***, vol. 38 (1971), pp. 69–83.

[5] BJARNI JÓNSSON, *Universal relational systems*, ***Mathematica Scandinavica***, vol. 4 (1956), pp. 193–208.

[6] ———, *Homogeneous universal relational systems*, ***Mathematica Scandinavica***, vol. 8 (1960), pp. 137–142.

[7] MICHAEL MORLEY and ROBERT VAUGHT, *Homogeneous universal models*, ***Mathematica Scandinavica***, vol. 11 (1962), pp. 37–57.

[8] RICHARD RADO, *Universal graphs and universal functions*, ***Acta Arithmetica***, vol. 9 (1964), pp. 331–340.

[9] ———, *Universal graphs*, ***A seminar in graph theory*** (F. Harary, editor) Holt, Rinehart and Winston, New York, 1967, pp. 83–85.

[10] JOSEPH G. ROSENSTEIN, $\aleph_0$*-categoricity of linear orderings*, ***Fundamenta Mathematicae***, vol. 64 (1969), pp. 1–5.

[11] JOSEPH SHOENFIELD, ***Mathematical logic***, Addison-Wesley, Reading, Mass., 1969.

[12] WILLIAM GLASSMIRE, *A problem in categoricity*, ***Notices of the American Mathematical Society***, vol. 17 (1970), p. 295, Abstract #70T-E7.

[13] CHRISTOPHER ASH, *Undecidable* $\aleph_0$*-categorical theories*, ***Notices of the American Mathematical Society***, vol. 18 (1971), p. 423, Abstract #71T-E10.

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