# Modal Intuitionistic Logics as Dialgebraic Logics

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## Abstract

Duality is one of the key techniques in the categorical treatment of modal logics. From the duality between (modal) algebras and (descriptive) frames one derives e.g. completeness (via a syntactic characterisation of algebras) or definability (using a suitable version of the Goldblatt-Thomason theorem). This is by now well understood for classical modal logics and modal logics based on distributive lattices, via extensions of Stone and Priestley duality, respectively. What is conspicuously absent is a comprehensive treatment of modal intuitionistic logic. This is the gap we are closing in this paper. Our main conceptual insight is that modal intuitionistic logics do not appear as algebra/coalgebra dualities, but instead arise naturally as dialgebras. Our technical contribution is the development of dualities for dialgebras, together with their logics, that instantiate to large class of modal intuitionistic logics and their frames as special cases. We derive completeness and expressiveness results in this general case. For modal intuitionistic logic, this systematises the existing treatment in the literature.

***CCS Concepts:*** • **Theory of computation → Modal and temporal logics**.

***Keywords:*** modal logic, coalgebraic logic, dialgebras

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## 1 Introduction

Duality in logic has a long and venerated tradition, starting with the Stone representation theorem [82] in 1936 that establishes a categorical duality between Boolean algebras and certain topological spaces (now called Stone spaces). Subsequent notable results include the McKinsey-Tarski representation theorem for closure algebras [60], the Priestley duality theorem [68] linking distributive lattices and spaces now known as Priestley spaces, as well as Esakia duality for Heyting algebras [25, 27].

The mathematical appeal of duality lies in the fact that it offers conceptual understanding, as well as enables the transfer of results between the algebraic and the topological (or frame) setting.

In modal logic, the first representation theorem was established by Jónsson and Tarski in 1952 [47]. Despite the fact that *op. cit.* does not mention modal logic explicitly, it introduces relational semantics and uses the representation theorem to link relational and algebraic semantics. The full duality between Boolean algebras with operators and descriptive general frames was established by Goldblatt in 1976 [33, 34], and many other such dualities followed [28, 41, 63].

In the last 20 years, many dualities in modal logic have been recognised as an instance of a general algebra/coalgebra duality, where coalgebras play the rôle of frames. This has been instigated by Moss' coalgebraic logic [62]. It has lead to, e.g., recognising descriptive general frames as coalgebras for the Vietoris functor on Stone spaces [51] that are dual to modal algebras, understood as algebras over Boolean algebras as a base category. Similar results have been established for positive modal logic and coalgebras over Priestley spaces [63]. The coalgebraic view on modal logic has triggered a substantive body of research, see e.g. [52, 53, 65, 75], that remains active and relevant [4, 22, 42].

Interestingly, modal intuitionistic logic is conspicuously absent from the logical systems that have been subject to coalgebraic techniques, or general algebra/coalgebra duality, even though this is a seemingly obvious research direction.

A second noteworthy aspect of modal intuitionistic logic is that, in contrast to positive modal logic and "standard" modal logic over a classical base, there is no universally agreed semantics. Indeed, many variations have been put forward in the literature. This dates back to the 1960's [16, 17] and includes [15, 23, 49, 66, 76, 77, 88, 89] (see [81] for an overview). In addition to the modal intuitionistic logics that have a relational flavour, there is also interest in concurrent dynamic intuitionistic logic [86], epistemic intuitionistic logic [3, 44, 46, 69, 87] probabilistic intuitionistic logic [59], and intuitionistic public announcement logic [7, 58]. More recently, *conditional* intuitionistic logic has become prominent [21, 84, 85]. Many of these modal intuitionistic logics are also studied in philosophy, and conditional logics embody non-monotonic reasoning.

Indeed, it appears to be folklore that modal intuitionistic logic is not amenable to coalgebraic methods, as the frame semantics of modal intuitionistic logic does not "fit" with coalgebras nearly as neatly as the frame semantics for modal logic based on classical or positive logic, as we illustrate in Section 2.

In this paper, we show that *dialgebras* [39] allow us to give a general (co)algebraic analysis of many instances of modal intuitionistic logic, and so provide a solution to the research problem above. Specifically, we establish completeness, duality, and Hennessy-Milner type results for dialgebraic logics, that immediately specialise to various flavours of modal intuitionistic logic.

While our framework captures many existing modal intuitionistic logics (we give examples in Section 3), it does not encompass all. The most notable absentees are [66, 76, 77]. We refer to [88] for a discussion of how these fit in the framework of modal intuitionistic logic that we cover and we revisit this issue in the conclusion.

We begin our technical development by observing that the main impediment of a coalgebraic treatment of modal intuitionistic logic is a mismatch of *morphisms*, as coalgebraic modelling necessitates that both coalgebra structure maps and homomorphisms are drawn from the same category. We illustrate this in Section 2. Our main insight is that dialgebras present an elegant solution to resolve this mismatch. In Sections 3 and 4 we introduce dialgebras, give examples, and establish a general dual adjunction theorem. Section 5 introduces logics for dialgebras, and we proceed to establish a generic completeness result (Section 6) and a Hennessy-Milner type result (Section 7) for dialgebraic logic. Both results instantiate to a large class of modal intuitionistic logic and we show how it encompasses monotone modal intuitionistic logic [35, 36] and conditional intuitionistic logic [21, 84, 85] in Sections 8 and 9. Our principal technical tool is a dialgebraic notion of *general* and *descriptive frames*. A general frame is a frame together with a subalgebra of its complex algebra. General and descriptive frames (if they exist) provide the glue for duality between the algebraic semantics of a logic and certain (descriptive) frames.

***Related work.*** We are not aware of any previous work relating dialgebras and modal logic, but stand in the tradition of coalgebraic logic, logical duality and (intuitionistic) modal logic, as we have made explicit above.

## 2 Motivation

A conceptual and categorical treatment of duality for modal logics understands *modal algebras* as algebras for an endofunctor on a category of algebras **A** representing the underlying propositional logic, and *frames* as coalgebras on a category of spaces **C**. Typically, **C** is either dually equivalent to **A**, or to a sub-category of **A**. The duality between frames and algebras can then be understood as a duality between algebras and coalgebras that piggy-backs on a duality for **A**.

This explains, for example, the duality between descriptive general frames (coalgebras for the Vietoris functor on Stone spaces) and modal algebras [51] as well as the analogous correspondence for positive modal logics, and coalgebras over Priestly spaces [63].

Somewhat conspicuously, the same has *not* been carried out for modal intuitionistic logic, despite the fact that the basic setup seems obvious: modal intuitionistic algebras would be algebras over the category of Heyting algebras, and intuitionistic frames would be coalgebras for a suitable version of the Vietoris functor on Esakia spaces (or on intuitionistic Kripke frames).

The reason for this is simple: it does not work. More precisely, it fails to adequately describe the *morphisms* between standard and well-established examples of modal intuitionistic frames in the literature, and so stops any attempt at duality dead in its tracks. An argument for this has been made in [57, Remark 8] (but not in journal version [56] of the same paper). We showcase this phenomenon using $\Box$-frames from [88] and illustrate how a dialgebraic approach naturally emerges.

We consider the language $\mathbf{IntK}_\Box$ of intuitionistic logic with an additional unary operator $\Box$ which satisfies the axioms

$$\Box\varphi \wedge \Box\psi = \Box(\varphi \wedge \psi), \qquad \Box\top = \top.$$

This language can be interpreted in $\Box$-*frames*:

2.1. **Definition.** A $\Box$-frame is a tuple $(X, \leq, R_\Box)$ where $(X, \leq)$ is a poset and $R_\Box$ is a relation on $X$ satisfying

$$\leq \circ R_\Box \circ \leq \; = R_\Box. \tag{1}$$

The intuitionistic connectives are interpreted in the poset $(X, \leq)$ as usual. The $\Box$-operator is interpreted in a similar way as in normal modal logic: $x$ satisfies $\Box\varphi$ (written $x \Vdash \Box\varphi$) if all $R_\Box$-successors of $x$ satisfy $\varphi$.

Naively, our first goal is to model $\Box$-frames as coalgebras on the category **Pos** of posets and monotone functions.

2.2. **Definition.** Let $\mathsf{F}$ be an endofunctor on a category **C**. An $\mathsf{F}$-*coalgebra* is a pair $(C, \gamma)$ of an object $C \in \mathbf{C}$ (called the *state-space*) and a morphism $\gamma : C \to \mathsf{F}C$ in **C** (called the *structure map*). An $\mathsf{F}$-*coalgebra morphism* from $(C, \gamma)$ to $(C', \gamma')$ is a morphism $f : C \to C'$ in **C** that satisfies $\gamma' \circ f = \mathsf{F}f \circ \gamma$. We write $\mathbf{Coalg}(\mathsf{F})$ for the category of $\mathsf{F}$-coalgebras and $\mathsf{F}$-coalgebra morphisms.

To capture $\Box$-frames as coalgebras on **Pos**, the following endofunctor appears appropriate:

2.3. **Definition.** The *upper powerset functor* $\mathsf{P}^+_{\mathsf{up}}$ maps a poset $(X, \leq)$ to the collection of up-closed subsets of $(X, \leq)$ ordered by reverse inclusion. For a monotone function $f : (X, \leq) \to (X', \leq')$ define $\mathsf{P}^+_{\mathsf{up}} f$ by $\mathsf{P}^+_{\mathsf{up}} f(a) = {\uparrow_{\leq'}} f[a] = \{x' \in X' \mid \exists x \in a \text{ s.t. } f(x) \leq x'\}$.

With this definition, it is not hard to see that $\Box$-frames are precisely coalgebras for the upper powerset functor: simply identify the relation $R_\Box$ with the map $\gamma : (X, \leq) \to \mathsf{P}_{\mathsf{up}}(X, \leq)$ given by $\gamma(x) = \{y \in X \mid xR_\Box y\}$. Identity (1) holds iff $\gamma$ is monotone and $\gamma(x)$ is up-closed in $(X, \leq)$ for all $x \in X$.

While this looks like a success at first sight, the problem lies elsewhere: the category $\mathbf{Coalg}(\mathsf{P}^+_{\mathsf{up}})$ has "too many morphisms."

A morphism between $\Box$-frames $(X, \leq, R_\Box)$ and $(X', \leq', R'_\Box)$ needs to preserve truth of the formulas in $\mathbf{IntK}_\Box$. To guarantee preservation of all intuitionistic operators, $f$ should be a bounded morphism from $(X, \leq)$ to $(X', \leq')$. That is, a monotone morphism that additionally satisfies $\exists y \in X$ s.t. $x \leq y$ and $f(y) = y'$, whenever $f(x) \leq' y'$, for all $x \in X$ and $y' \in X'$. In a diagram:

$$\begin{tikzcd} y \arrow[r, dashed, "f"] & y' \\ x \arrow[u, dashed, "\leq"] \arrow[r, "f"] & f(x) \arrow[u, dash, "\leq'"'] \end{tikzcd}$$

To ensure that $x \Vdash \Box\varphi$ iff $f(x) \Vdash \Box\varphi$, $f$ also needs to be a bounded morphism $(X, R_\Box) \to (X', R_{\Box'})$. We call morphisms satisfying this $\Box$*-frame morphisms*. The collection of $\Box$-frames and $\Box$-frame morphisms constitutes the category $\mathbf{WZ}\Box$ (named after the authors of [88]).

On the other hand, a simple computation shows that the $\mathbf{Coalg}(\mathsf{P}^+_{\mathsf{up}})$-morphisms between (the coalgebraic rendering of) two $\Box$-frames $(X, \leq, R)$ and $(X', \leq', R')$ are precisely the bounded morphisms between $(X, R)$ and $(X', R')$, i.e. are not necessarily bounded with respect to the poset order. In other words, $\mathbf{WZ}\Box$ is isomorphic to the (non-full) subcategory of $\mathbf{Coalg}(\mathsf{P}^+_{\mathsf{up}})$ with the same objects, but whose morphisms satisfy the additional requirement that they are bounded morphisms between the underlying posets.

A seemingly self-evident solution here is a change of base category: instead of **Pos** one can consider the category **Krip** of posets with bounded morphisms. One fortuitous circumstance is that the functor $\mathsf{P}^+_{\mathsf{up}}$ restricts to **Krip**, but the price we have to pay is that the structure maps $\gamma : (X, \leq) \to \mathsf{P}^+_{\mathsf{up}}(X, \leq)$ of $\mathsf{P}^+_{\mathsf{up}}$-coalgebras are now required to be *bounded* morphisms (with respect to the poset structure). In other words, $\mathbf{Coalg}(\mathsf{P}^+_{\mathsf{up}})$ is the full subcategory of $\mathbf{WZ}\Box$ consisting of those $\Box$-frames where the coalgebraic rendering of the structure map is bounded, and so fails to contain *all* $\Box$-frames.

The only option in this situation is to investigate whether one can live with a smaller class of frames (with bounded structure maps): how much semantic richness is lost? Unfortunately, the answer here is everything, at least if we insist that our treatment should not only do justice do $\Box$-frames, but also cover $\Diamond$-frames.

To see this, we extend intuitionistic propositional logic with a $\Diamond$-modality that satisfies

$$\Diamond\varphi \vee \Diamond\psi = \Diamond(\varphi \vee \psi), \quad \Diamond\bot = \bot.$$

This language can be interpreted in $\Diamond$*-frames*: tuples $(X, \leq, R_\Diamond)$ where $(X, \leq)$ is a poset and $R_\Diamond$ satisfies $\geq \circ R_\Diamond \circ \geq = R_\Diamond$. These can be seen as $\mathsf{P}^+_{\mathsf{dn}}$-coalgebras, where $\mathsf{P}^+_{\mathsf{dn}} : \mathbf{Pos} \to \mathbf{Pos}$ is the functor which sends a poset $(X, \leq)$ to the collection of downwards closed subsets of $X$ ordered by inclusion, and a function $f : (X, \leq) \to (X', \leq')$ to $\mathsf{P}^+_{\mathsf{dn}} f = {\downarrow} f[-]$. A calculation similar to the above shows that $\Diamond$-frames are precisely equivalent to $\mathsf{P}^+_{\mathsf{dn}}$-coalgebras.

Contrary to $\mathsf{P}^+_{\mathsf{up}}$, the functor $\mathsf{P}^+_{\mathsf{dn}}$ does *not* restrict to **Krip**, as shown by the following example:

2.4. **Example.** Let $X = \{x\}$ be the singleton poset and $X' = \{x', y'\}$ the two-element poset ordered by equality. Then the map $f : X \to X' : x \mapsto x'$ trivially is a bounded morphism, but $\mathsf{P}_{\mathsf{dn}} f$ is not. To see this, note that we have $\mathsf{P}_{\mathsf{dn}} f(\{x\}) = \{x'\} \subseteq \{x', y'\}$, but there is not subset $a$ of $\{x\}$ such that $\mathsf{P}_{\mathsf{dn}} f(a) = \{x', y'\}$.

In other words, we can capture $\Diamond$-frames (but with the wrong morphisms) as $\mathsf{P}^+_{\mathsf{dn}}$-coalgebras, but any attempt at capturing a subclass of $\Diamond$-frames (with morphisms that are bounded with respect to the poset structure) is bound to fail.

It thus appears that we are in a double dead-end situation, as we need *structure maps* to be morphisms in **Pos**, and *coalgebra morphisms* to be morphisms in **Krip**. This is precisely where dialgebras come to the rescue, because they allow us to diligently discern morphisms (which should be bounded) and structure maps (which should not be). Indeed, dialgebras (formally introduced in Definition 3.1) allow us to describe $\mathbf{WZ}\Box$ (and also the analogously defined category $\mathbf{WZ}\Diamond$) as a category of *dialgebras*.

We write $\mathtt{i}$ for the obvious embedding $\mathbf{Krip} \to \mathbf{Pos}$ and let $\mathsf{P}_{\mathsf{up}}$ be the restriction of $\mathsf{P}^+_{\mathsf{up}}$ to **Krip** viewed as a functor $\mathbf{Krip} \to \mathbf{Pos}$. An $(\mathtt{i}, \mathsf{P}_{\mathsf{up}})$*-dialgebra* is a pair $(X, \gamma)$ of an object $X \in \mathbf{Krip}$ together with a morphism

$$\gamma : \mathtt{i}X \to \mathsf{P}_{\mathsf{up}} X$$

in **Pos**. A *dialgebra morphism* between $(X, \gamma)$ and $(X', \gamma')$ is a morphism $f : X \to X'$ (in **Krip**!) that makes

$$\begin{tikzcd} \mathtt{i}X \arrow[r, "\mathtt{i}f"] \arrow[d, "\gamma"] & \mathtt{i}X' \arrow[d, "\gamma'"] \\ \mathsf{P}_{\mathsf{up}} X \arrow[r, "\mathsf{P}_{\mathsf{up}} f"] & \mathsf{P}_{\mathsf{up}} X' \end{tikzcd}$$

commute. We write $\mathbf{Dialg}(\mathtt{i}, \mathsf{P}_{\mathsf{up}})$ for the category of $(\mathtt{i}, \mathsf{P}_{\mathsf{up}})$-dialgebras and morphisms. An easy verification reveals that

$$\mathbf{WZ}\Box \cong \mathbf{Dialg}(\mathtt{i}, \mathsf{P}_{\mathsf{up}}).$$

Thus, the categorical notion of dialgebras generalises that of coalgebras enough to describe $\Box$-frames.

We now have a look at the algebras corresponding to the language $\mathbf{IntK}_{\Box}$. These are *Heyting algebras with (unary) operators.*

2.5. **Definition.** A *Heyting algebra with operator* (HAO for short) is a pair $(H, \Box)$ where $H$ is a Heyting algebra and $\Box : H \to H$ is a function that satisfies for all $a, b \in H$,

$$\Box a \wedge \Box b = \Box(a \wedge b), \quad \Box\top = \top.$$

A morphism between HAOs $(H, \Box)$ and $(H', \Box')$ is a Heyting algebra morphism $f : H \to H'$ satisfying $\Box' \circ f = f \circ \Box$. We write **HAO** for the category HAOs and HAO morphisms.

The category **HAO** also arises as a category of dialgebras, see Example 3.3 below.

## 3 Dialgebras and Examples

We define dialgebras and give examples. Dialgebras were introduced in [39] to describe data types. They have also been used as a categorical semantics for inductive-inductive definitions [2] and furthermore occur in [20, 67]. The 2-categorically minded reader may appreciate the fact that dialgebras are precisely coinserters in the 2-category **CAT** of categories [43, Appendix A].

3.1. **Definition.** Let $\mathsf{F}, \mathsf{G} : \mathbf{C} \to \mathbf{D}$ be functors. An $(\mathsf{F}, \mathsf{G})$-dialgebra is a pair $(X, \gamma)$ of an object $X$ in $\mathbf{C}$ and a morphism $\gamma : \mathsf{F}X \to \mathsf{G}X$ in $\mathbf{D}$. An $(\mathsf{F}, \mathsf{G})$-dialgebra morphism $f : (X, \gamma) \to (X', \gamma')$ is a morphism $f : X \to X'$ in $\mathbf{C}$ satisfying $\mathsf{G}f \circ \gamma = \gamma' \circ \mathsf{F}f$. In diagrams,

$$\text{objects:} \quad \begin{tikzcd} \mathsf{F}X \arrow[d, "\gamma"] \\ \mathsf{G}X \end{tikzcd} \qquad \text{arrows:} \quad \begin{tikzcd} \mathsf{F}X \arrow[r, "\mathsf{F}f"] \arrow[d, "\gamma"'] & \mathsf{F}X' \arrow[d, "\gamma'"] \\ \mathsf{G}X \arrow[r, "\mathsf{G}f"] & \mathsf{G}X' \end{tikzcd}$$

We denote the category of $(\mathsf{F}, \mathsf{G})$-dialgebras and $(\mathsf{F}, \mathsf{G})$-dialgebra morphisms by $\mathbf{Dialg}(\mathsf{F}, \mathsf{G})$.

Evidently, both algebras and coalgebras are instances of dialgebras, where $\mathbf{C} = \mathbf{D}$ and either $\mathsf{F}$ or $\mathsf{G}$ is the identity. For basic constructions (like limits, colimits, subdialgebras, and quotients) in categories of dialgebras we refer to [13, Chapter 3]. Among other things, dialgebras describe some extraordinary biological phenomena.

3.2. **Example.** Unisexual salamanders [14] reproduce by stealing sperm from one or more donor species. The information relevant for a "family tree" for these salamanders is the mother of a child, and the different species which provided genetic material for its conception. Dialgebraically:

Let $S$ be the set of all (relevant) species and $\mathsf{P}_0$ and $\mathsf{P}_\omega$ the non-empty powerset functor and the finite powerset functor on **Set**, respectively. Define the functor $\mathsf{F} : \mathbf{Set} \to \mathbf{Set}$ by $\mathsf{F}X = X \times \mathsf{P}_0 S$. Then a family tree is a $(\mathsf{F}, \mathsf{P}_\omega)$-dialgebra. Morphisms in this category of dialgebras relate salamanders which are created via the same "genetic route."

One main motivating example are Heyting algebras with operators, i.e. the algebraic counterpart of $\Box$-frames [88, Section 2].

3.3. **Example.** Let $\mathtt{j} : \mathbf{HA} \to \mathbf{DL}$ be the embedding of Heyting algebras into distributive lattice. Let $\mathsf{N} : \mathbf{HA} \to \mathbf{DL}$ be the functor that sends a Heyting algebra $A$ to the free distributive lattice generated by $\Box a$, $a \in A$, subject to $\Box a \wedge \Box b = \Box(a \wedge b)$ and $\Box\top = \top$. For a morphism $h : A \to B$ let $\mathsf{N}h : \mathsf{N}A \to \mathsf{N}B$ be given by $\mathsf{N}h(\Box a) \mapsto \Box(ha)$. Then

$$\mathbf{HAO} \cong \mathbf{Dialg}(\mathsf{N}, \mathtt{j}).$$

We now restrict our attention to dialgebras that describe frames of modal (bi-)intuitionistic logic. Most of the references in these examples do not discuss morphisms, but in all cases there is an obvious notion of (bounded) morphism that preserves (bi-)intuitionistic connectives and modalities. We discuss monotone and conditional intuitionistic logic in Sections 8 and 9.

3.4. **Example.** In the previous section we have already seen the following isomorphisms of categories:

$$\Box\mathbf{WZ} \cong \mathbf{Dialg}(\mathtt{i}, \mathsf{P}_{\mathsf{up}}), \qquad \Diamond\mathbf{WZ} \cong \mathbf{Dialg}(\mathtt{i}, \mathsf{P}_{\mathsf{dn}}).$$

We note that the frames defined in [89, Definition 1.1.5] and [49, Definition 1] both coincide with $\Box$-frames.

A slight variation of $\Box$-frames and $\Diamond$-frames gives frame semantics for modal *bi*-intuitionistic logic:

3.5. **Example.** The category of *modal Kripke frames* as defined in [38, Definition 5.3] is isomorphic to the category $\mathbf{Dialg}(\mathtt{i}^*, \mathsf{P}^*_{\mathsf{up}} \times \mathsf{P}^*_{\mathsf{dn}})$ where $\mathtt{i}^*$, $\mathsf{P}^*_{\mathsf{up}}$ and $\mathsf{P}^*_{\mathsf{dn}}$ are restrictions of the analogously named functors to the category of posets and bi-bounded morphisms (Definition 3.1 of *op. cit.*).

We consider several more examples of frame semantics for modal intuitionistic logic. Most of these have a counterpart used for interpreting a diamond modality (like $\Diamond$-frames).

3.6. **Example.** A $H\Box$-frame [15, Definition 2] (found under a different name in [69, Definition 3.1]) is a tuple $(X, \leq, R)$ consisting of a pre-order $(X, \leq)$ and a binary relation $R$ on $X$ such that

$$\leq \circ R \subseteq R \circ \leq.$$

We can model these as dialgebras for the embedding $\mathtt{i}' : \mathbf{PreKrip} \to \mathbf{PreOrd}$, where **PreOrd** is the category of pre-orders and monotone maps, and **PreKrip** is the category of pre-orders and bounded morphisms. Define the functor

$\mathsf{P}_2^{\mathbf{pre}} : \mathbf{PreKrip} \to \mathbf{PreOrd}$ to send $(X, \leq)$ to the powerset $\mathsf{P}X$ of $X$ ordered by

$$a \sqsubseteq_2 b \quad \text{iff} \quad \forall y \in b \exists x \in a \text{ s.t. } x \leq y.$$

On morphisms $\mathsf{P}_2^{\mathbf{pre}}$ is defined to send $f$ to the direct image map. Then the category of $H\Box$-frames and truth-preserving morphisms is isomorphic to $\mathbf{Dialg}(\mathtt{i}', \mathsf{P}_2^{\mathbf{pre}})$.

3.7. **Example.** A condensed $H\Box$-frame [15, Definition 10] is a $H\Box$-frame which satisfies $\leq \circ R \subseteq R$. It can similarly be modelled as a dialgebra by using the functor which sends a pre-Kripke frame $(X, \leq)$ to $(\mathsf{P}X, \supseteq)$ and a morphism to the direct image map.

3.8. **Example.** An $H\Box$-frame $(X, \leq, R)$ is *strictly condensed* [15, Definition 10] if $\leq \circ R \circ \leq \subseteq R$. These are dialgebras for the functor which sends $(X, \leq)$ to the collection of upsets of $(X, \leq)$ ordered by reverse inclusion, $(\mathsf{Up}X, \supseteq)$. Note that in this case $\supseteq$ coincides with $\sqsubseteq_2$.

The same frames are used in [80], to interpret tense intuitionistic logic. To guarantee preservation of tense operators by the morphisms, we have to restrict the functor above to the category of pre-orders and *bi-bounded* morphisms.

3.9. **Example.** EK-structures [46, Definition 1] interpret intuitionistic epistemic logic. Their underlying frames are dialgebras for the embedding $\mathtt{i}' : \mathbf{PreKrip} \to \mathbf{PreOrd}$ and the functor $\mathsf{E}$, where $\mathsf{E}$ sends a preorder $(X, \leq)$ to the $n$-fold product of $(\mathsf{P}X, \subseteq)$ in $\mathbf{PreOrd}$, and a morphism $f : (X, \leq) \to (X', \leq')$ in $\mathbf{PreKrip}$ to the $n$-fold product of the direct image map $f[-] : (\mathsf{P}X, \subseteq) \to (\mathsf{P}X', \subseteq)$. The morphisms in $\mathbf{Dialg}(\mathtt{i}', \mathsf{E})$ preserve truth of all formulas defined in *op. cit.* (including the common knowledge operator!).

Lastly, we show how to view *descriptive* $\Box$-frames [88, Section 2] as dialgebras.

3.10. **Definition.** A general $\Box$-frame is a tuple $(X, \leq, R_\Box, A)$ where $(X, \leq, R_\Box)$ is a $\Box$-frame and $A \subseteq \mathsf{Up}(X, \leq)$ is a collection of upsets such that $(X, \leq, A)$ is a general intuitionistic frame (see e.g. [18, §8.1]) and $A$ is closed under the map $m_\Box : \mathsf{Up}(X, \leq) \to \mathsf{Up}(X, \leq)$ given by

$$m_\Box(a) = \{x \in X \mid xR_\Box y \text{ implies } y \in a\}. \tag{2}$$

A *general $\Box$-frame morphism* $(X, \leq, R_\Box, A) \to (X', \leq', R'_\Box, A')$ is a $\Box$-frame morphism $f$ which satisfies $f^{-1}(a') \in A$ whenever $a' \in A'$.

A general $\Box$-frame $(X, \leq, R_\Box, A)$ is *descriptive* if $(X, \leq, A)$ is a descriptive intuitionistic frame and $R_\Box$ satisfies

$$xR_\Box y \quad \text{iff} \quad \forall a \in A (x \in \Box a \to y \in a).$$

We write $\mathbf{D\text{-}WZ\Box}$ for the category of descriptive (general) $\Box$-frames and general $\Box$-frame morphisms.

3.11. **Example.** Let $\mathtt{I} : \mathbf{ES} \to \mathbf{Pries}$ be the embedding of Esakia spaces into Priestley spaces and $\mathsf{V}_{\mathsf{up}} : \mathbf{ES} \to \mathbf{Pries}$ the *upper Vietoris functor* defined below. Then

$$\mathbf{D\text{-}WZ\Box} \cong \mathbf{Dialg}(\mathtt{I}, \mathsf{V}_{\mathsf{up}}).$$

The upper Vietoris functor is a variation of the Vietoris functors on Stone spaces [51] and Priestley spaces [63, Definition 29][11, Definition 2.12]. It resembles the Smyth powerdomain [78], although upward closure need not be with respect to the specialisation order.

We write $\mathbb{X}$ for ordered topological spaces (like Priestley spaces and Esakia spaces) and $X$ for their underlying sets. In order to avoid clutter, we often suppress the order $\leq$.

3.12. **Definition.** For an Esakia space $\mathbb{X}$, let $\mathsf{V}_{\mathsf{up}}\mathbb{X}$ be the collection of closed upsets ordered by reverse inclusion and topologised by the clopen subbase

$$\boxdot a = \{c \in \mathsf{V}_{\mathsf{up}}\mathbb{X} \mid c \subseteq a\}, \quad \diamonddot b = \{c \in \mathsf{V}_{\mathsf{up}}\mathbb{X} \mid c \cap b \neq \emptyset\},$$

where $a$ ranges over the clopen upsets and $b$ over the clopen downsets of $\mathbb{X}$. For an Esakia morphism $f : \mathbb{X} \to \mathbb{X}'$, define

$$\mathsf{V}_{\mathsf{up}}f : \mathsf{V}_{\mathsf{up}}\mathbb{X} \to \mathsf{V}_{\mathsf{up}}\mathbb{Y} : c \mapsto {\uparrow_{\leq_{\mathbb{Y}}}} f[c].$$

3.13. **Proposition.** $\mathsf{V}_{\mathsf{up}} : \mathbf{ES} \to \mathbf{Pries}$ *is a functor.*

One easily sees that $\mathsf{V}_{\mathsf{up}}$ extends to an endofunctor on $\mathbf{Pries}$. For future reference, we also note:

3.14. **Proposition.** $\mathsf{V}_{\mathsf{up}}$ *defines an endofunctor on* $\mathbf{ES}$.

We can now substantiate the claim of Example 3.11.

*Proof sketch.* Recall that the category of Esakia spaces is isomorphic to the category of descriptive intuitionistic frames. Knowing this, on objects, the proof is given by identifying $\gamma(x)$ with the successor set of $x$ under $R_\gamma$. The claim for morphisms follows from a straightforward computation. □

Analogously, we can define the downward Vietoris functor: $\mathsf{V}_{\mathsf{dn}}\mathbb{X}$ is the collection of closed downwards closed sets, ordered by inclusion and topologised by the clopen subbase

$$\boxdot b = \{c \in \mathsf{V}_{\mathsf{dn}}\mathbb{X} \mid c \subseteq b\}, \quad \diamonddot a = \{c \in \mathsf{V}_{\mathsf{dn}}\mathbb{X} \mid c \cap a \neq \emptyset\},$$

where $b$ ranges over the clopen downsets and $a$ ranges over the clopen upsets of $\mathbb{X}$. For an Esakia morphism $f$ we define $\mathsf{V}_{\mathsf{dn}}f$ to be ${\downarrow} f[-]$. Again, we obtain a functor $\mathbf{ES} \to \mathbf{Pries}$.

3.15. **Example.** The category of descriptive $\Diamond$-frames is isomorphic to $\mathbf{Dialg}(\mathtt{I}, \mathsf{V}_{\mathsf{dn}})$. We leave the details to the reader.

## 4 A Dual Adjunction Theorem

We give a sufficient condition for the existence of a dual adjunction between categories of dialgebras. We then apply this to obtain a dual adjunction (in fact, a dual equivalence) between $\mathbf{D\text{-}WZ\Box}$ and $\mathbf{HAO}$.

The following theorem is a modest generalisation of [43, Theorem 2.14]. In view of the application of dialgebras in the (dual) rôles of frame and algebraic semantics, we formulate the results in the form of *dual* adjunctions. As in [48], we use crossed arrows to indicate dual adjunctions.

4.1. **Theorem.** *Let* $\mathsf{P} : \mathbf{C} \to \mathbf{D}$ *and* $\mathsf{P}' : \mathbf{C}' \to \mathbf{D}'$ *be contravariant functors. Suppose we have two natural transformations* $\alpha$ *and* $\beta$ *in the situation*

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\mathsf{P}"] \arrow[dr, phantom, "\searrow\alpha"] & \mathbf{D} \\ \mathbf{C}' \arrow[u, "\mathsf{F}"] \arrow[r, "\mathsf{P}'"'] & \mathbf{D}' \arrow[u, "\mathsf{F}'"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{C} \arrow[r, "\mathsf{P}"] \arrow[dr, phantom, "\nwarrow\beta"] & \mathbf{D} \\ \mathbf{C}' \arrow[u, "\mathsf{G}"] \arrow[r, "\mathsf{P}'"'] & \mathbf{D}' \arrow[u, "\mathsf{G}'"'] \end{tikzcd}$$

*This induces a functor* $\bar{\mathsf{P}} : \mathbf{Dialg}(\mathsf{F}, \mathsf{G}) \to \mathbf{Dialg}(\mathsf{G}', \mathsf{F}')$ *which sends a* $(\mathsf{F}, \mathsf{G})$*-dialgebra* $\gamma : \mathsf{F}c \to \mathsf{G}c$ *to the composition*

$$\mathsf{G}'\mathsf{P}'c \xrightarrow{\beta_c} \mathsf{PG}c \xrightarrow{\mathsf{P}\gamma} \mathsf{PF}c \xrightarrow{\alpha_c} \mathsf{F}'\mathsf{P}'c.$$

*Suppose furthermore that* $\mathbf{C} \underset{\mathsf{S}}{\overset{\mathsf{P}}{\rightleftarrows}} \mathbf{D}$ *and* $\mathbf{C}' \underset{\mathsf{S}'}{\overset{\mathsf{P}'}{\rightleftarrows}} \mathbf{D}'$ *are dual adjunctions. Let* $\eta : \mathrm{id}_{\mathbf{C}} \to \mathsf{SP}$ *and* $\theta : \mathrm{id}_{\mathbf{D}} \to \mathsf{PS}$ *be the units of the left dual adjunction, and similarly define* $\eta'$ *and* $\theta'$*. Let* $\beta''$ *be the composition*

$$\mathsf{GS}' \xrightarrow{\eta_{\mathsf{GS}'}} \mathsf{SPGS}' \xrightarrow{\mathsf{S}\beta_{\mathsf{S}'}} \mathsf{SG}'\mathsf{P}'\mathsf{S}' \xrightarrow{\mathsf{SG}'\theta'} \mathsf{SG}'.$$

*We call* $\beta''$ *the* adjoint buddy *of* $\beta$.[1] *If both* $\alpha$ *and* $\beta''$ *are natural isos, then we can define a functor* $\bar{\mathsf{S}} : \mathbf{Dialg}(\mathsf{G}', \mathsf{F}') \to \mathbf{Dialg}(\mathsf{F}, \mathsf{G})$ *which sends* $\delta : \mathsf{G}'d \to \mathsf{F}'d$ *to*

$$\mathsf{FS}'d \xrightarrow{\alpha'_d} \mathsf{SF}'d \xrightarrow{\mathsf{S}\delta} \mathsf{SG}'d \xrightarrow{\beta'_d} \mathsf{GS}'d,$$

*where* $\beta' = (\beta'')^{-1}$ *and* $\alpha'$ *is defined by the composition*

$$\mathsf{FS}' \xrightarrow{\eta_{\mathsf{FS}'}} \mathsf{SPFS}' \xrightarrow{\mathsf{S}\alpha^{-1}_{\mathsf{S}'}} \mathsf{SF}'\mathsf{P}'\mathsf{S}' \xrightarrow{\mathsf{SF}\theta'} \mathsf{SF}.$$

*Moreover, the functors* $\bar{\mathsf{P}}$ *and* $\bar{\mathsf{S}}$ *constitute a dual adjunction between* $\mathbf{Dialg}(\mathsf{F}, \mathsf{G})$ *and* $\mathbf{Dialg}(\mathsf{G}', \mathsf{F}')$.

4.2. **Remark.** Note that $\beta$ and $\beta''$ are interdefinable: taking the adjoint buddy twice gives identity. If $\mathsf{P}'$ and $\mathsf{S}'$ form a dual equivalence, then the same holds for the dual adjunction obtained via Theorem 4.1.

In our applications, the left square above will often consist of two embeddings and $\mathsf{P}'$ and $\mathsf{S}'$ will be restriction of $\mathsf{P}$ and $\mathsf{S}$, respectively. In this case, however, the pair $(\mathsf{P}', \mathsf{S}')$ is not necessarily a dual adjunction itself (see Example 4.4 below). If they *do* form a dual adjunction, we get a trivial natural isomorphism $\alpha$. The following lemma states two easy conditions that guarantee that the restriction of the dual adjunction is again a dual adjunction.

4.3. **Lemma.** *Let* $\mathsf{P} : \mathbf{C} \to \mathbf{D}$ *and* $\mathsf{S} : \mathbf{D} \to \mathbf{C}$ *form a dual adjunction. Let* $\mathtt{i} : \mathbf{C}' \to \mathbf{C}$, $\mathtt{j} : \mathbf{D}' \to \mathbf{D}$ *be embeddings such that the restriction* $\mathsf{P}'$ *of* $\mathsf{P}$ *to* $\mathbf{C}'$ *lands in* $\mathbf{D}'$ *and similar for* $\mathsf{S}'$.

1. *If* $\mathtt{i}$ *and* $\mathtt{j}$ *are full embeddings, then* $\mathsf{P}'$ *and* $\mathsf{S}'$ *form a dual adjunction.*
2. *If* $\mathsf{P}$ *and* $\mathsf{S}$ *form a dual equivalence and both embeddings reflect isos, then* $\mathsf{P}'$ *and* $\mathsf{S}'$ *form a dual equivalence.*

[1] This looks like the adjoint mate of $\beta$ but is slightly different because the units come from potentially different (dual) adjunctions, hence the name "adjoint buddy".

In fact, it suffices that the units are still morphisms in the subcategories. Both conditions in Lemma 4.3 guarantee that this is the case, but it is not automatic:

4.4. **Example.** The dual adjunction between **BA** and **Set**, given by ultrafilters and (contravariant) powerset, does not restrict to an adjunction between (the non-full subcategories of) Boolean algebras with only monomorphic homomorphisms, and sets with surjective functions.

We use the remainder of this section to prove that the category of descriptive $\Box$-frames (Example 3.11) is dually equivalent to that of HAOs (Example 3.3). For objects this was proved in [88].

4.5. **Theorem.** *We have a dual equivalence*

$$\mathbf{D\text{-}WZ}\Box \equiv^{\mathrm{op}} \mathbf{HAO}.$$

We work with the following inclusion functors and dual adjunctions:

$$\begin{tikzcd} \mathbf{Pries} \arrow[r, bend left, "\mathtt{ClpUp}"] \arrow[r, phantom, "\equiv^{\mathrm{op}}"] & \mathbf{DL} \arrow[l, bend left, "\mathtt{pt}"] \\ \mathbf{ES} \arrow[u, "\mathtt{I}"] \arrow[r, bend left, "\mathtt{ClpUp}'"] \arrow[r, phantom, "\equiv^{\mathrm{op}}"] & \mathbf{HA} \arrow[u, "\mathtt{j}"'] \arrow[l, bend left, "\mathtt{pt}'"] \end{tikzcd} \tag{3}$$

Here both $\mathtt{I}$ and $\mathtt{j}$ are embeddings, and the horizontal arrows are Priestley duality [68] and Esakia duality [26, 27].

By Theorem 4.1 and Remark 4.2 we need to find natural isomorphisms $\alpha : \mathtt{ClpUp} \cdot \mathtt{I} \to \mathtt{j} \cdot \mathtt{ClpUp}'$ and $\beta'' : \mathsf{V}_{\mathsf{up}} \cdot \mathtt{pt}' \to \mathtt{pt} \cdot \mathsf{N}$. Since Esakia duality is a restriction of Priestley duality, the diagram in (3) naturally commutes and we trivially get $\alpha$. So we focus on finding $\beta''$.

Let $\beta$ be defined by $\beta_X : \mathsf{N} \cdot \mathtt{ClpUp}X \to \mathtt{ClpUp} \cdot \mathsf{V}_{\mathsf{up}}X : \Box a \mapsto \boxdot a$. This is easily seen to be a natural transformation. The adjoint buddy $\beta''$ of $\beta$ is a natural transformation $\mathsf{V}_{\mathsf{up}} \cdot \mathtt{pt} \to \mathtt{pt} \cdot \mathsf{N}$. We aim to prove that it is a natural isomorphism.

For $U \in \mathsf{V}_{\mathsf{up}} \cdot \mathtt{pt}A$ and $a \in A$ we have

$$\begin{aligned} \Box a \in \beta''_A(U) \quad &\text{iff} \quad \beta(\mathsf{N}\theta(\Box(a))) \in \widetilde{U} \\ &\text{iff} \quad U \in \beta(\mathsf{N}\theta(\Box a)) = \beta(\Box(\theta(a))) = \beta(\Box\hat{a}) \\ &\text{iff} \quad U \subseteq \hat{a} \end{aligned}$$

where $\hat{a} = \{q \in \mathtt{pt}A \mid a \in q\}$. (Note that $\hat{a}$ is a clopen upset of $\mathtt{pt}A$, hence an element of $\mathsf{V}_{\mathsf{up}} \cdot \mathtt{pt}A$.) Guided by this, to prove that $\beta''$ is an isomorphism on objects, we define a map in the converse direction by

$$\xi_A : \mathtt{pt} \cdot \mathsf{N}A \to \mathsf{V}_{\mathsf{up}} \cdot \mathtt{pt}A : Q \mapsto \bigcap\{\hat{b} \mid \Box b \in Q\}.$$

This is well defined because the arbitrary intersection of clopen upsets is a closed upset. It is monotone (i.e. a morphism in **Pos**) because $Q \subseteq Q'$ implies $\xi_A(Q) \supseteq \xi_A(Q')$ and $\mathsf{V}_{\mathsf{up}}$ is ordered by reverse inclusion. Furthermore we have:

4.6. **Lemma.** *The assignment* $\xi_A : \mathtt{pt} \cdot \mathsf{N}A \to \mathsf{V}_{\mathsf{up}} \cdot \mathtt{pt}A$ *given by* $Q \mapsto \bigcap\{\hat{b} \mid \Box b \in Q\}$ *satisfies for all* $a \in A$,

$$\Box a \in Q \quad \textit{iff} \quad \xi_A(Q) \subseteq \hat{a}.$$

*Proof.* Left to right is immediate from the definition. Suppose that $\Box a \notin Q$. Then the filter $B_a = \{b \in A \mid \Box b \in Q\}$ is disjoint from the ideal $\downarrow a$ (because we must have $b \nleq a$ for all $b \in B$, for otherwise $\Box a \in Q$). By the prime filter lemma (see e.g. [61, Lemma 1.4]) $B_a$ extends to a prime filter $u$ disjoint from $\downarrow a$. We then have $u \in \xi_A(Q)$ while $u \notin \hat{a}$, hence $\xi_A(Q) \nsubseteq \hat{a}$. □

It follows from the lemma that $\xi_A^{-1}(\boxdot \hat{a}) = \widehat{\Box a}$, so that:

4.7. **Corollary.** *For every $A \in \mathbf{HA}$, $\xi_A$ is a Priestley morphism.*

We now have all ingredients to prove Theorem 4.5.

*Proof of Theorem 4.5.* By Theorem 4.1 it suffices to show that the $\beta''$ is a natural isomorphism. Note that a closed upset $c \in \mathsf{V}_{\mathsf{up}} \cdot \mathtt{pt}A$ is determined by the clopen upsets in which it is contained, and a point in $\mathtt{pt} \cdot \mathsf{N}A$ is uniquely determined by the sets of the form $\Box a$ it contains. It then follows from Lemma 4.6 that $\xi_A = (\beta''_A)^{-1}$, hence $\beta''$ is a natural isomorphism. □

## 5 Logic for Dialgebras

We turn to a more restricted setup of dialgebras, where one of the two functors determining the category of dialgebras is an embedding. We define logic in a similar way as usual in abstract coalgebraic logic, namely via a natural transformation $\rho$ (see e.g. [48, Definition 3.1]) that we call a *logical connection*. We then explain how to characterise a logic by predicate liftings and axioms, and show how these give rise to such a logical connection. We illustrate this using modal intuitionistic logic based on the frames from Sections 2 and 3. We fix the following setup for the rest of the paper.

5.1. **Setup.** Let $\mathsf{P} : \mathbf{C} \to \mathbf{A}$ be a contravariant functor and $\mathtt{i} : \mathbf{C}' \to \mathbf{C}$ and $\mathtt{j} : \mathbf{A}' \to \mathbf{A}$ embeddings such that $\mathsf{P}[\mathbf{C}'] \subseteq \mathbf{A}'$. We denote the restriction of $\mathsf{P}$ to $\mathbf{C}'$ by $\mathsf{P}'$.

We think of $\mathbf{C}$ as the category of carriers of frames, and $\mathsf{P}(C)$ as the algebra of predicates over $C \in \mathbf{C}$, and a guiding example is to take $\mathbf{A}' = \mathbf{HA} \subseteq \mathbf{A} = \mathbf{DL}$. For frames, one can think of $\mathbf{C}' = \mathbf{Krip} \subseteq \mathbf{C} = \mathbf{Pos}$, or alternatively $\mathbf{C}' = \mathbf{ES} \subseteq \mathbf{C} = \mathbf{Pries}$. The prime examples of functors $\mathsf{P}$ defining the predicates are (clopen) upsets.

5.2. **Definition.** We call such a setup *structured* if $\mathsf{P}$ is the dual adjoint of some contravariant $\mathsf{S} : \mathbf{A} \to \mathbf{C}$ and $\mathsf{S}[\mathbf{A}'] \subseteq \mathbf{C}'$. We denote the units of the dual adjunction by $\eta : \mathrm{id}_{\mathbf{C}} \to \mathsf{SP}$ and $\theta : \mathrm{id}_{\mathbf{A}} \to \mathsf{PS}$.

If moreover $\theta_{A'} \in \mathbf{A}'$ for all $A' \in \mathbf{A}'$ then we call the setup *well-structured*. In this case we write $\theta'$ for the restriction of $\theta$ to $\mathbf{A}'$. In particular this implies $\theta_{\mathtt{j}A'} = \mathtt{j}\theta'_{A'}$ for all $A' \in \mathbf{A}'$.

For the running examples, we note that the prime filter functor $\mathsf{S} = \mathtt{pf}$ witnesses that both $\mathtt{ClpUp} : \mathbf{Pries} \to \mathbf{DL}$ and $\mathtt{Up} : \mathbf{Pos} \to \mathbf{DL}$ are well-structured.

A functor $\mathsf{T} : \mathbf{C}' \to \mathbf{C}$ gives rise to a category $\mathbf{Dialg}(\mathtt{i}, \mathsf{T})$ of dialgebras. We introduce logic for dialgebras generalising the abstract view of coalgebraic logic [52, Section 3.3].

5.3. **Definition.** A *modal logic* for $\mathbf{Dialg}(\mathtt{i}, \mathsf{T})$ is a pair $(\mathsf{L}, \rho)$ where $\mathsf{L}$ is a functor $\mathbf{A}' \to \mathbf{A}$ and $\rho$ is a natural transformation

$$\rho : \mathsf{LP}' \to \mathsf{PT}.$$

The *complex (di)algebra* of a $(\mathtt{i}, \mathsf{T})$-dialgebra $(X, \gamma)$ is the object $(\mathsf{P}'X, \gamma^*) \in \mathbf{Dialg}(\mathsf{L}, \mathtt{j})$ given by the concatenation

$$\mathsf{LP}'X \xrightarrow{\rho_X} \mathsf{PT}X \xrightarrow{\mathsf{P}\gamma} \mathsf{P}\mathtt{i}X \xrightarrow{\cong} \mathtt{j}\mathsf{P}'X$$

where we will often elide the embedding $\mathtt{j}$.

5.4. **Proposition.** *The assignment $(X, \gamma) \mapsto (\mathsf{P}'X, \gamma^*)$ extends to a functor $(\cdot)^* : \mathbf{Dialg}(\mathtt{i}, \mathsf{T}) \to \mathbf{Dialg}(\mathsf{L}, \mathtt{j})$ by putting $f^* = \mathsf{P}'f$ for morphisms.*

Specifically, the initial $(\mathsf{L}, \mathtt{j})$-dialgebra gives rise to the interpretation of modal formulae, and plays the rôle of the Lindenbaum-Tarski algebra of the logic.

5.5. **Definition** (Interpretation)**.** Assume $\mathbf{Dialg}(\mathsf{L}, \mathtt{j})$ has initial object $\psi : \mathsf{L}\Psi \to \mathtt{j}\Psi$. Then the semantics of a $(\mathtt{i}, \mathsf{T})$-dialgebra $\gamma : \mathtt{i}X \to \mathsf{T}X$ is the unique map $[\![\cdot]\!]_\gamma$ that makes the following diagram commute:

$$\begin{tikzcd} \mathsf{L}\Psi \arrow[r, "\mathsf{L}[\![\cdot]\!]_\gamma"] \arrow[d, "\psi"'] & \mathsf{LP}'X \arrow[d, "\gamma^*"] \\ \mathtt{j}\Psi \arrow[r, "\mathtt{j}[\![\cdot]\!]_\gamma"] & \mathtt{j}\mathsf{P}'X \end{tikzcd}$$

For structured setups, we define the *theory map* $\mathrm{th}_\gamma : \mathtt{i}X \to \mathtt{i}\mathsf{S}'\Psi$ for $(X, \gamma)$ as the transpose of $\mathtt{j}[\![\cdot]\!]_\gamma$. Concretely, $\mathrm{th}_\gamma$ is given by the composition

$$\mathtt{i}X \xrightarrow{\eta^{\mathsf{SP}}_X} \mathsf{SP}\mathtt{i}X = \mathtt{i}\mathsf{S}'\mathsf{P}'X \xrightarrow{\mathtt{i}\mathsf{S}'[\![\cdot]\!]_\gamma} \mathtt{i}\mathsf{S}'\Psi.$$

The map $\mathrm{th}_\gamma$ need not be in $\mathbf{C}'$ because $\eta_{\mathtt{i}X}$ need not be.

5.6. **Definition.** If the setup is well-structured we can define the *adjoint buddy* $\rho^\flat : \mathsf{TS}' \to \mathsf{SL}$ of $\rho$ as the composition

$$\mathsf{TS}' \xrightarrow{\eta_{\mathsf{TS}'}} \mathsf{SPTS}' \xrightarrow{\mathsf{S}\rho_{\mathsf{S}'}} \mathsf{SLP}'\mathsf{S}' \xrightarrow{\mathsf{SL}\theta'} \mathsf{SL}.$$

To capture our running examples, we demonstrate how to define concrete logics using predicate liftings [64, 74], originally introduced for set-based coalgebras. They have been used over different base categories in [24, Definition 2.5] and [10, Definition 8]. We assume that $\mathbf{A}$ is a variety of single-sorted algebras over $\mathbf{Set}$ and $\mathsf{U} : \mathbf{A} \to \mathbf{Set}$ is the forgetful functor.

5.7. **Definition.** An $n$-ary *predicate lifting* is a natural transformation $\lambda : \mathsf{U}(\mathtt{j}\mathsf{P}')^n \to \mathsf{UPT}$, where $(\mathtt{j}\mathsf{P}')^n X$ is the $n$-fold product of $\mathsf{P}X$ in $\mathbf{A}$.

For a collection $\Lambda$ of predicate liftings for $\mathsf{T}$ and a set $X$, let $\mathsf{L}'X$ be the free $\mathsf{A}$-algebra generated by the set

$$\{\heartsuit^\lambda(x_1, \ldots, x_n) \mid \lambda \in \Lambda, x_i \in X\}.$$

A $\Lambda$-*axiom* is a pair $(\varphi, \psi)$ of two elements in $\mathsf{L}'X$, where $X$ is some set of variables.

For a set $\Lambda$ of predicate liftings for $\mathsf{T}$ and a collection Ax of $\Lambda$-axioms, define a functor $\mathsf{L}_{(\Lambda,\mathrm{Ax})} = \mathsf{L} : \mathbf{A}' \to \mathbf{A}$ as follows: For $A \in \mathbf{A}'$ let $\mathsf{L}A$ be the free $\mathbf{A}$-algebra generated by

$$\left\{\heartsuit^{\lambda}(a_1, \dots, a_n) \mid \lambda \in \Lambda, a_i \in \mathtt{j}A\right\}$$

subject to the relations $R$ from Ax, where the variables are substituted by elements from $A$. For a morphism $f : A \to B$ define $\mathsf{L}f$ on objects by

$$\mathsf{L}f(\heartsuit^{\lambda}(a_1, \dots, a_n)) = \heartsuit^{\lambda}(f(a_1), \dots, f(a_n)).$$

We call Ax *sound* if it induces a logical connection, i.e. a well-defined natural transformation $\rho_{(\Lambda,\mathrm{Ax})} : \mathsf{LP}' \to \mathsf{PT}$ via

$$\rho_{(\Lambda,\mathrm{Ax}),X}([\heartsuit^{\lambda}(a_1, \dots, a_n)]_R) = \lambda_X(a_1, \dots, a_n)$$

that is independent of the choice of representative relative to $R$-equivalence classes $[\cdot]_R$. In this case, naturality is an immediate consequence of naturality of predicate liftings.

5.8. **Definition.** The logic given by collections $\Lambda$ of predicate liftings and Ax of sound $\Lambda$-axioms is $(\mathsf{L}_{(\Lambda,\mathrm{Ax})}, \rho_{(\Lambda,\mathrm{Ax})})$.

The predicate liftings and axioms give rise to a new class of algebras. We define the signature $\Sigma^+$ as the signature $\Sigma'$ of $\mathbf{A}'$ plus an operator $\spadesuit^{\lambda}$ for each $\lambda \in \Lambda$. We let $E^+$ to be the collection of equations for $\mathbf{A}'$ plus Ax. Let $\mathbf{A}^+ = \mathbf{Var}(\Sigma^+, E^+)$. Then $\mathbf{A}^+$ is a subvariety of $\mathbf{A}'$. Write $\Psi$ for the free $\mathbf{A}^+$-algebra on zero generators. Then $\Psi$ is also a $\mathbf{A}'$-algebra. We claim that $\Psi$ is initial in $\mathbf{Dialg}(\mathsf{L}, \mathtt{j})$.

5.9. **Lemma.** *View $\Psi$ as an object of $\mathbf{A}'$ and define $\psi : \mathsf{L}\Psi \to \mathtt{j}\Psi$ by $\heartsuit^{\lambda}(a_1, \dots, a_n) \mapsto \spadesuit^{\lambda}(a_1, \dots, a_n)$. Then $(\Psi, \psi)$ is initial in $\mathbf{Dialg}(\mathsf{L}, \mathtt{j})$.*

5.10. **Remark.** Propositional variables can be expressed as nullary operators in algebraic signature. The logical connection $\rho$ then also defines their valuation.

We return to the examples of Sections 2 and 3.

5.11. **Example.** Recall the dialgebraic rendering of $\Box$-frames from Example 3.11. Define $\lambda^{\Box} : \mathsf{Up} \to \mathsf{Up} \cdot \mathsf{P}_{\mathsf{up}}$ by

$$\lambda^{\Box}_X : \mathsf{Up}X \to \mathsf{Up} \cdot \mathsf{P}_{\mathsf{up}}X : a \mapsto \{b \in \mathsf{P}_{\mathsf{up}}X \mid b \subseteq a\}.$$

It is easy to see that this is a natural transformation. The interpretation of the box modality in a $(\mathtt{i}, \mathsf{P}_{\mathsf{up}})$-dialgebra $(X, \gamma)$ now is as desired: $x \Vdash \varphi$ iff $\gamma(x) \subseteq [\![\varphi]\!]$, which intuitively translates to every $R_{\Box}$-successor of $x$ satisfies $\varphi$. Imposing the axioms

$$\Box\varphi \wedge \Box\psi = \Box(\varphi \wedge \psi) \quad \text{and} \quad \Box\top = \top$$

yields the functor $\mathsf{N}$ from Example 3.3 via the procedure from Definition 5.7.

One can define an interpretation of the same logic in $\mathsf{V}_{\mathsf{up}}$ dialgebras in the same manner. Furthermore, the $\Box$ and $\Diamond$ modalities in [15, 69, 89] can be defined likewise.

## 6 General Frames and Completeness

We introduce categorical notions of completeness and general frames, and define descriptive semantics and use this for a general completeness theorem.

Consider a setting as in the previous section, where the semantics of a language $(\mathsf{L}, \rho)$ is given via an initial dialgebra. Intuitively, the logic is complete if every two expressions (elements) in the initial $(\mathsf{L}, \mathtt{j})$-dialgebra $(\Psi, \psi)$ can be distinguished in some $(\mathtt{i}, \mathsf{T})$-dialgebra. That is, for every two elements $a, b \in \Psi$ there exists $(X, \gamma) \in \mathbf{Dialg}(\mathtt{i}, \mathsf{T})$ such that the initial map $[\![\cdot]\!]_{\gamma} : (\Psi, \psi) \to (\mathsf{P}'X, \gamma^*)$ satisfies $[\![a]\!]_{\gamma} \neq [\![b]\!]_{\gamma}$. This idea underlies the following definition.

6.1. **Definition.** We say that the logic $(\mathsf{L}, \rho)$ is *complete* if the source (or cosink)

$$\left\{ [\![\cdot]\!]_{\gamma} : \Psi \to \mathsf{P}'X \right\}_{(X,\gamma)\in\mathbf{Dialg}(\mathtt{i},\mathsf{T})}$$

is jointly monic in $\mathbf{A}'$.

For standard (e.g. normal) modal logics, $\Psi$ is the algebra of formulae, quotiented by logical equivalence, and the $\mathsf{P}'X$ correspond to the complex algebras of the class of frames under consideration.

It is easy to see that soundness of Ax (Definition 5.7) immediately implies soundness of the logic, as formulae identified by axioms are equivalent in the codomain of $\rho$, and therefore have the same denotation.

In practice one often finds a single monomorphism from the initial algebra to some complex algebra of a frame, e.g. the canonical model, which then makes the cosink jointly monic.

We now give a categorical definition of general frames. Intuitively, a general frame is a $(\mathtt{i}, \mathsf{T})$-dialgebra $(X, \gamma)$ together with a "collection of admissible subsets" (a subobject of $\mathsf{P}'X$) which is closed under certain operations. General frames provide the glue between algebraic and frame semantics that we need to prove completeness via duality, because they arise as the duals of Lindenbaum-Tarski algebras.

6.2. **Definition.** A *general frame* for the logic $(\mathsf{L}, \rho)$ is a triple $(X, \gamma, m)$ where $(X, \gamma)$ is a $(\mathtt{i}, \mathsf{T})$-dialgebra and $m : A \to \mathsf{P}'X$ is a monomorphism in $\mathbf{A}'$ (i.e. $m$ is a subobject of $\mathsf{P}'X$) such that the composition $\gamma^* \circ \mathsf{L}m$ factors through $\mathtt{j}m : \mathtt{j}A \to \mathtt{j}\mathsf{P}'X$.

A morphism from $(X, \gamma, m)$ to $(X', \gamma', m')$ is an $(\mathtt{i}, \mathsf{T})$-dialgebra morphism $h : (X, \gamma) \to (X', \gamma')$ such that $\mathsf{P}'h \circ m'$ factors through $m$. In diagrams:

objects: $\begin{array}{ccc} \mathsf{L}A & \xrightarrow{\mathsf{L}m} & \mathsf{LP}'X \\ \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle \gamma^*} \\ \mathtt{j}A & \xrightarrow{\mathtt{j}m} & \mathtt{jP}'X \end{array}$ morphisms: $\begin{array}{ccc} A' & \xrightarrow{m'} & \mathsf{P}'X' \\ \downarrow{\scriptstyle \hat{h}} & & \downarrow{\scriptstyle \mathsf{P}'h} \\ A & \xrightarrow{m} & \mathsf{P}'X \end{array}$

We write $\mathbf{Gen}(\rho)$ for the category of general frames and general frame morphisms, and identify the subobject $m : A \to \mathsf{P}'X$ with $A$ if there is no danger of confusion.

6.3. **Remark.** If $\mathtt{j}$ preserves monos then both fill-ins $\mu_A$ and $\hat{h}$ in the diagrams above are unique as $m$ and $\mathtt{j}m$ are monic.

6.4. **Example.** Let $(X, \leq, R_\Box, A)$ be a general $\Box$-frame (Definition 3.10), and let $((X, \leq), \gamma)$ be the dialgebra representing $(X, \leq, R_\Box)$, carried by the poset $(X, \leq)$. Then $((X, \leq), \gamma, A)$ is a general frame in the sense of Definition 6.2 above.

6.5. **Examples.** Since coalgebras are special cases of dialgebras, Definition 6.2 gives a notion of general frame for coalgebraic logic. This specialises to the usual notion of general frame in well-known settings like normal modal logic (see [18, §8.1] or [12]) and monotone modal logic (see [41]). Specialising this further by viewing $\mathbf{C} \cong \mathbf{Coalg}(\mathsf{T})$ where $\mathsf{T}c = 1$ and $\mathbf{A} \cong \mathbf{Alg}(\mathrm{id}_0)$ for final/initial objects 0 and 1, respectively, we obtain that e.g. general frames for the adjunction between sets and boolean algebras are fields of sets in the standard way, and analogous instances for distributive lattices (which yields rings of sets) and Heyting algebras (general intuitionistic Kripke frames).

We have an obvious forgetful functor $\mathtt{f} : \mathbf{Gen}(\rho) \to \mathbf{Dialg}(\mathtt{i}, \mathsf{T})$. Conversely, we can view every $(\mathtt{i}, \mathsf{T})$-dialgebra $(X, \gamma)$ as a general $\rho$-frame via $(X, \gamma, \mathsf{P}'X)$ with the inclusion $\mathrm{id}_{\mathsf{P}'X}$. This assignment extends to a functor $\mathtt{g} : \mathbf{Dialg}(\mathtt{i}, \mathsf{T}) \to \mathbf{Gen}(\rho)$ by setting $\mathtt{g}h = h$, for every morphism $h$ in $\mathbf{Dialg}(\mathtt{i}, \mathsf{T})$. We immediately see that $\mathtt{f} \circ \mathtt{g} = \mathrm{id}_{\mathbf{Dialg}(\mathtt{i},\mathsf{T})}$. Furthermore, for every general frame $(X, \gamma, A)$ the identity on $X$ is a general frame morphism $\mathtt{g} \cdot \mathtt{f}(X, \gamma, A) = (X, \gamma, \mathsf{P}'X) \to (X, \gamma, A)$. Now it is easy to verify that we have:

6.6. **Proposition.** *The functors* $\mathtt{f}$ *and* $\mathtt{g}$ *constitute an adjunction* $\mathtt{g} \dashv \mathtt{f}$ *and exhibit that* $\mathbf{Dialg}(\mathtt{i}, \mathsf{T})$ *is a coreflective subcategory of* $\mathbf{Gen}(\rho)$ *(because the unit is a natural isomorphism).*

Crucially for completeness, we can mediate between general frames and the associated algebras.

6.7. **Proposition.** *If* $\mathtt{j}$ *preserves monos, we have a contravariant functor* $(\cdot)^+ : \mathbf{Gen}(\rho) \to \mathbf{Dialg}(\mathsf{L}, \mathtt{j})$ *given by* $(X, \gamma, A)^+ = (A, \mu_A)$ *and* $h^+ = \hat{h}$*, with* $\mu_A$ *and* $\hat{h}$ *as in Definition 6.2.*

Keeping Definition 6.1 in mind, we are interested in whether an $(\mathsf{L}, \mathtt{j})$-dialgebra $(A, \alpha)$ maps to the complex algebra of some $(\mathtt{i}, \mathsf{T})$-dialgebra and, if such a morphism $f : (A, \alpha) \to (X, \gamma)^*$ exists, whether the underlying map $f : A \to \mathsf{P}'X$ is mono. Put differently, we want to know whether $(\cdot)^+$ is surjective on objects. If this is the case we view it as a special property:

6.8. **Definition.** A logic $(\mathsf{L}, \rho)$ has *weak descriptive semantics* if we have a map $(\cdot)_+$ from the objects of $\mathbf{Dialg}(\mathsf{L}, \mathtt{j})$ to the objects of $\mathbf{Gen}(\rho)$ such that for every $(\mathsf{L}, \mathtt{j})$-dialgebra $\mathfrak{B}$ we have $(\mathfrak{B}_+)^+ \cong \mathfrak{B}$. We say that the logic has *descriptive semantics* if $(\cdot)_+$ extends to a section of $(\cdot)^+$ (i.e, a functor such that $(\cdot)^+ \circ (\cdot)_+ = \mathrm{id}_{\mathbf{Dialg}(\mathsf{L},\mathtt{j})}$).

If the logic has descriptive semantics, then by definition we have $\mathfrak{B} \cong (\mathfrak{B}_+)^+$ for all $\mathfrak{B} \in \mathbf{Dialg}(\mathsf{L}, \mathtt{j})$. We call the general frames in the image of $(\cdot)_+$ *descriptive*, and we call a $(\mathtt{i}, \mathsf{T})$-dialgebra *descriptive* if it lies in the image of $\mathtt{f} \circ (\cdot)_+$. For every descriptive general frame $\mathfrak{F} = \mathfrak{B}_+$ we have $(\mathfrak{F}^+)_+ \cong \mathfrak{F}$ because $(\mathfrak{F}^+)_+ = ((\mathfrak{B}_+)^+)_+ \cong \mathfrak{B}_+ = \mathfrak{F}$. In fact, the image of $(\cdot)_+$ is dually equivalent to $\mathbf{Dialg}(\mathsf{L}, \mathtt{j})$. Thus, a logic has descriptive semantics if we can choose a subcategory of $\mathbf{Gen}(\rho)$ which is dually equivalent to $\mathbf{Dialg}(\mathsf{L}, \mathtt{j})$. We have:

6.9. **Proposition.** *Suppose* $(\mathsf{L}, \rho)$ *has weak descriptive semantics. Then the logic is complete for* $(\mathtt{i}, \mathsf{T})$*-dialgebras.*

*Proof.* Let $(\Psi, \psi)_+ = (X, \gamma, A)$. Then we have a monomorphism $m_A : A \to \mathsf{P}'X$ which is also a morphism from $(X, \gamma, A)^+$ to $(X, \gamma)^* = (\mathsf{P}'X, \gamma^*)$. By assumption $(X, \gamma, A)^+ \cong (\Psi, \psi)$ and $m_A$ is monic. □

The following Proposition gives a sufficient condition for the existence of (weak) descriptive semantics. It is a modification of results in [55].

6.10. **Proposition.** *Suppose we work in a well-structured setup and* $\theta'$ *is pointwise monic.*

1. *If there exists a (not necessarily natural) transformation* $\tau : \mathsf{SL} \to \mathsf{TS}'$ *such that* $\rho^\flat \circ \tau = \mathrm{id}$ *then* $(\mathsf{L}, \rho)$ *has weak descriptive semantics.*
2. *If moreover* $\tau$ *is natural, then* $(\mathsf{L}, \rho)$ *has descriptive semantics.*

*Proof.* We prove only item 1. Given a $(\mathsf{L}, \mathtt{j})$-dialgebra $(A, \alpha)$, define $\alpha_*$ to be the composition

$$\mathtt{i}\mathsf{S}'A \xrightarrow{\cong} \mathsf{S}\mathtt{j}A \xrightarrow{\mathsf{S}\alpha} \mathsf{SL}A \xrightarrow{\tau_A} \mathsf{TS}'A$$

and let $(A, \alpha)_+ = (\mathsf{S}'A, \alpha_*, A)$. By assumption we have a monomorphism $\theta'_A : A \to \mathsf{P}'\mathsf{S}'A$. We claim that

$$\begin{tikzcd} \mathsf{L}A \arrow[r, "\mathsf{L}\theta'_A"] \arrow[d, dashed, "\alpha"] & \mathsf{LP}'\mathsf{S}'A \arrow[d, "(\alpha_*)^*"] \\ \mathtt{j}A \arrow[r, "\mathtt{j}\theta'_A"] & \mathtt{j}\mathsf{P}'\mathsf{S}'A \end{tikzcd} \tag{4}$$

commutes. This can be proven in a way similar to Theorem 6.4 in [55]. □

As an example, we re-establish the completeness for $\Box$-frames from [88, Theorem 1].

6.11. **Example.** There is a forgetful functor $\mathbf{Dialg}(\mathtt{I}, \mathsf{V}_{\mathsf{up}}) \to \mathbf{Gen}(\rho)$, where $\rho$ corresponds to the logic given in Example 5.11. Since $\mathbf{Dialg}(\mathtt{I}, \mathsf{V}_{\mathsf{up}})$ is dually equivalent to $\mathbf{Dialg}(\mathsf{L}, \mathtt{j})$ (by Theorem 4.5), this proves that the functor $(\cdot)^+ : \mathbf{Gen}(\rho) \to \mathbf{Dialg}(\mathsf{L}, \mathtt{j})$ has a section, which in turn proves completeness of $\mathbf{IntK}_\Box$ with respect to $\Box$-frames.

Alternatively, this can be obtained via Proposition 6.10.

6.12. **Example.** Minor modification of Example 6.11 yields the same results for *bi*-intuitionistic counterpart of normal modal intuitionistic logic, see [38, Section 5].

## 7 Prime Filter Extensions and Expressivity

We investigate when a logic $(\mathsf{L}, \rho)$ is expressive for (a class of) $(\mathtt{i}, \mathsf{T})$-dialgebras, a notion similar to the Hennessy-Milner property. We generalise this to *expressivity-somewhere-else*, to capture notions of bisimilarity-somewhere-else as found in [12, 50]. To do this, we introduce *prime filter extensions*, which build on our theory of general and descriptive frames from the previous section. (We choose the term "prime filter extension" by analogy to ultrafilter extensions for modal logic over a classical base, although concrete instances are not necessarily based on prime filters.) The abstract notion of the "collection of prime filters" of $X \in \mathbf{C}'$ is the object $\mathsf{S}'\mathsf{P}'X \in \mathbf{C}'$. In Proposition 6.10 we have seen that, under certain conditions, the spaces underlying descriptive frames corresponding to complex algebras are of the form $\mathsf{S}'\mathsf{P}'X$. Therefore we define the prime filter extension of $(X, \gamma)$ as a truth-preserving map in $\mathbf{C}$ to a certain descriptive frame.

7.1. **Definition.** Write $\mathtt{pf}$ for the map on objects given by the composition

$$\mathbf{Dialg}(\mathtt{i}, \mathsf{T}) \xrightarrow{\mathsf{g}} \mathbf{Gen}(\rho) \xrightarrow{(\cdot)^+} \mathbf{Dialg}(\mathsf{L}, \mathtt{i}) \xrightarrow{(\cdot)_+} \mathbf{Gen}(\rho) \xrightarrow{\mathsf{f}} \mathbf{Dialg}(\mathtt{i}, \mathsf{T}).$$

Let $(X, \gamma)$ be a $(\mathtt{i}, \mathsf{T})$-dialgebra and let $\mathtt{pf}(X, \gamma) = (\hat{X}, \hat{\gamma})$. A *prime filter extension* of a $(\mathtt{i}, \mathsf{T})$-dialgebra is a morphism $u$ in $\mathbf{C}$ from $\mathtt{i}X$ to $\mathtt{i}\hat{X}$ such that the following commutes:

$$\begin{tikzcd} \mathtt{i}X \arrow[r, "u"] \arrow[rr, bend right, "\mathrm{th}_\gamma"'] & \mathtt{i}\hat{X} \arrow[r, "\mathrm{th}_{\hat{\gamma}}"] & \mathtt{i}\mathsf{S}'\Psi \end{tikzcd} \tag{5}$$

In the following special case, prime filter extensions exist.

7.2. **Proposition.** *If weak descriptive semantics are given via Proposition 6.10, then every dialgebra has a prime filter extension.*

Prime filter extensions play a rôle in the expressivity-somewhere-else result below. We define what we mean by expressivity and expressivity-somewhere-else. The former is a minor adaptation of [48, Definition 4.1], and intuitively expresses that non-equivalent formulae can be semantically distinguished. A dialgebra is expressive-somewhere-else if it can be embedded into an expressive one.

7.3. **Definition.** A $(\mathtt{i}, \mathsf{T})$-dialgebra $(X, \gamma)$ is said to be *expressive* if the theory map $\mathrm{th}_\gamma$ factors through a dialgebra morphism followed by a monomorphism in $\mathbf{C}'$.

A $(\mathtt{i}, \mathsf{T})$-dialgebra $(X, \gamma)$ is said to be *expressive somewhere else* if the theory map $\mathrm{th}_\gamma$ factors through morphism in $\mathbf{C}$ followed by a theory map of an expressive dialgebra.

General expressivity results can be proven in a way similar to [48]. We proceed differently and use the theory of general and descriptive frames, developed in the previous section, to prove expressivity and expressivity-somewhere-else.

7.4. **Proposition** (Expressivity-somewhere-else)**.** *If every descriptive* $(\mathtt{i}, \mathsf{T})$*-dialgebra is expressive and the* $(\mathtt{i}, \mathsf{T})$*-dialgebra* $(X, \gamma)$ *has a prime filter extension. Then* $(X, \gamma)$ *is expressive somewhere else.*

*Proof.* The diagram in (5) commutes and we assumed $\mathrm{th}_{\hat{\gamma}}$ to correspond to an expressive dialgebra. □

The following theorem is the main theorem of this section.

7.5. **Theorem.** *Suppose the logic* $(\mathsf{L}, \rho)$ *for* $(\mathtt{i}, \mathsf{T})$*-dialgebras has descriptive semantics given as in Proposition 6.10. Then:*

1. *Every descriptive* $(\mathtt{i}, \mathsf{T})$*-dialgebra is expressive.*
2. *Every* $(\mathtt{i}, \mathsf{T})$*-dialgebra is expressive somewhere else.*

*Proof.* We show that for each $(X, \gamma)$ underlying a descriptive frame, the theory map is a dialgebra morphism. (Clearly this suffices.) So suppose $(X, \gamma, A) = (A, \alpha)_+$ is a descriptive general frame. Then we have an initial morphism $t : (\Psi, \psi) \to (A, \alpha)$ and this induces a morphism $t_+ = \mathsf{S}'t : (\Psi, \psi)_+ \to (A, \alpha)_+ = (X, \gamma, A)$. By construction this is such that the left diagram below commutes, and taking complex algebras (see Proposition 5.4) proves that the right diagram commutes:

$$\begin{tikzcd} \mathtt{i}X \arrow[r, "\mathtt{i}\mathsf{S}'t"] \arrow[d, "\gamma"'] & \mathtt{i}\mathsf{S}'\Psi \arrow[d, "\psi_+"] \\ \mathsf{T}X \arrow[r, "\mathsf{T}\mathsf{S}'t"] & \mathsf{T}\mathsf{S}'\Psi \end{tikzcd} \qquad \begin{tikzcd} \mathtt{j}\mathsf{P}'\mathsf{S}'\Psi \arrow[r, "\mathtt{j}\mathsf{P}'\mathsf{S}'t"] \arrow[d, "(\psi_+)^*"'] & \mathtt{j}\mathsf{P}'X \arrow[d, "\gamma^*"] \\ \mathsf{L}\mathsf{P}'\mathsf{S}'\Psi \arrow[r, "\mathsf{L}\mathsf{P}'\mathsf{S}'t"] & \mathsf{L}\mathsf{P}'X \end{tikzcd}$$

Besides, we have seen in the proof of Proposition 6.10 that $\theta'_\Psi$ is an $(\mathsf{L}, \mathtt{j})$-dialgebra morphism from $(\Psi, \psi)$ to $(\mathsf{P}'\mathsf{S}'\Psi, (\psi_+)^*)$. Therefore the following diagram commutes:

$$\begin{tikzcd} \mathtt{j}\Psi \arrow[r, "\mathtt{j}\theta'_\Psi"] \arrow[d, "\psi"'] & \mathtt{j}\mathsf{P}'\mathsf{S}'\Psi \arrow[r, "\mathtt{j}\mathsf{P}'\mathsf{S}'t"] \arrow[d, "(\psi_+)^*"] & \mathtt{j}\mathsf{P}'X \arrow[d, "\gamma^*"] \\ \mathsf{L}\Psi \arrow[r, "1\theta'_\Psi"] & \mathsf{L}\mathsf{P}'\mathsf{S}'\Psi \arrow[r, "\mathsf{L}\mathsf{P}'\mathsf{S}'t"] & \mathsf{L}\mathsf{P}'X \end{tikzcd}$$

Since $(\Psi, \psi)$ is the initial $(\mathsf{L}, \mathtt{j})$-dialgebra, the interpretation $[\![\cdot]\!] : \Psi \to \mathsf{P}'X$ satisfies

$$\mathtt{j}[\![\cdot]\!] = \mathtt{j}\mathsf{P}'\mathsf{S}'t \circ \mathtt{j}\theta'_\Psi = \mathsf{P}\mathtt{i}\mathsf{S}'t \circ \theta_{\mathtt{j}\Psi}.$$

Here the second equality holds by the remark in Definition 5.2. But this shows that $[\![\cdot]\!]$ is the adjoint buddy of $\mathtt{i}\mathsf{S}'t : \mathtt{i}X \to \mathtt{i}\mathsf{S}'\Psi$ and therefore $\mathtt{i}\mathsf{S}'t = \mathrm{th}_\gamma$ is the theory map from Definition 5.5.

But $\mathsf{S}'t$ is a dialgebra morphism $(X, \gamma) \to (\mathsf{S}'\Psi, \psi_+)$, so this proves that the theory map factors as a dialgebra morphism $(\mathsf{S}'t)$ followed by a monomorphism (the identity on $\mathtt{i}\mathsf{S}'\Psi$), hence $(X, \gamma)$ is expressive.

Item 2 follows from the first and Proposition 7.2. □

7.6. **Example.** Descriptive □-frames are expressive because the map $\xi$ defined in Lemma 4.6 is a natural isomorphism (as inverse of a natural isomorphism). Moreover, □-frames have prime filter extensions by Proposition 7.2 because □-frames have weak descriptive semantics via Proposition 6.10 (Example 6.11, note that the section in this example need not be natural). It then follows from Proposition 7.4 that every $\Box$-frame is expressive-somewhere-else.

## 8 Monotone Modal Intuitionistic Logic

We showcase the versatility of our approach by showing how it encompasses *monotone* modal intuitionistic logic [36, Section 6]. This logic is closely related to its classical counterpart [19, 40, 41], except that the underlying propositional logic is intuitionistic. We first define the algebras corresponding to this logic.

8.1. **Definition.** A *Heyting algebra with monotone operator* (HAM) is a pair $(H, \triangle)$ of a Heyting algebra $H$ and a map $\triangle : H \to H$ satisfying $\triangle(a \wedge b) \leq \triangle a$ for all $a, b \in H$. A morphism between HAMs $(H, \triangle)$ and $(H', \triangle')$ is a Heyting algebra homomorphism $f : H \to H'$ satisfying $\triangle' \circ f = f \circ \triangle$. Write **HAM** for the category of HAMs and HAM morphisms.

HAMs are $(\mathtt{M}, \mathtt{j})$-dialgebras in a way similar to Example 3.3, where $\mathtt{j}$ is the embedding $\mathbf{HA} \to \mathbf{DL}$ and $\mathtt{M}A$ is the free distributive lattice generated by $\triangle a$, where $a \in A$, subject to $\triangle(a \wedge b) \leq \triangle a$ (or equivalently $\triangle(a \wedge b) \wedge \triangle a = \triangle(a \wedge b)$).

The frame semantics of this logic are given by a form of (monotone) neighbourhood semantics, called *neighbourhood spaces* in [36, Subsection 6.4.1].

8.2. **Definition.** Let $(X, \leq)$ be a poset. A *monotone neighbourhood* for $x \in X$ is a collection $W$ of up-closed sets in $(X, \leq)$ such that $a \in W$ and $a \subseteq b$ implies $b \in W$. A *monotone frame* is a tuple $(X, \leq, N)$ where $(X, \leq)$ is a poset and $N$ assigns to each $x \in X$ a monotone neighbourhood, such that $a \in N(x)$ and $x \leq y$ implies $a \in N(y)$.

A *monotone frame morphism* $f : (X, \leq, N) \to (X', \leq', N')$ is a bounded morphism $f : (X, \leq) \to (X', \leq')$ such that for all up-closed $a' \subseteq X'$ and $x \in X$:

$$a' \in N'(f(x)) \quad \text{iff} \quad f^{-1}(a') \in N(x).$$

Monotone frames and morphism form the category **Mon**.

We can view **Mon** as a category of dialgebras. Interestingly, the crucial functor to achieve this is the composition of two functors we have already encountered: $\mathsf{P}_{\mathsf{up}}^{-}$ and $\mathsf{P}_{\mathsf{dn}}$, where $\mathsf{P}_{\mathsf{up}}^{-}$ is the restriction of $\mathsf{P}_{\mathsf{up}}$ to **Krip** (see Section 2).

8.3. **Theorem.** *We have* $\mathbf{Mon} \cong \mathbf{Dialg}(\mathtt{i}, \mathsf{P}_{\mathsf{dn}}\mathsf{P}_{\mathsf{up}}^{-})$.

Define the predicate lifting $\lambda^{\triangle} : \mathtt{j} \cdot \mathsf{Up} \to \mathsf{P}_{\mathsf{dn}}\mathsf{P}_{\mathsf{up}}^{-} \cdot \mathsf{Up}$ by

$$\lambda^{\triangle}_{(X, \leq)}(a) = \{W \in \mathsf{P}_{\mathsf{dn}}\mathsf{P}_{\mathsf{up}}^{-}X \mid a \in W\}.$$

One readily sees that this yields the modalilty $\triangle$. We now prove completeness. Using Propositions 6.9 and 6.10, completeness follows from a simple application of the prime filter lemma (in the proof of Lemma 8.4). Monotone modal intuitionistic logic, as a logic in the sense of Definition 5.3, is given by $\rho : \mathtt{M} \cdot \mathsf{Up}' \to \mathsf{Up} \cdot \mathsf{P}_{\mathsf{dn}}\mathsf{P}_{\mathsf{up}}^{-}$, which is defined by

$$\rho_{(X, \leq)}(\triangle a) = \{W \in \mathsf{P}_{\mathsf{dn}}\mathsf{P}_{\mathsf{up}}^{-}(X, \leq) \mid a \in W\}$$

and $\mathtt{M}$ is as introduced after Definition 8.1. Let $\rho^{\flat} : \mathsf{P}_{\mathsf{dn}}\mathsf{P}_{\mathsf{up}}^{-} \cdot \mathtt{pt}' \to \mathtt{pt} \cdot \mathtt{M}$ be the adjoint buddy of $\rho$. We have

$$\triangle a \in \rho^{\flat}_{A}(W) \quad \text{iff} \quad \widetilde{a} \in W, \tag{6}$$

where $\widetilde{a} = \{u \in \mathtt{pt}A \mid a \in u\}$. We define a potential inverse $\xi : \mathtt{pt} \cdot \mathtt{M} \to \mathsf{P}_{\mathsf{dn}}\mathsf{P}_{\mathsf{up}}^{-} \cdot \mathtt{pt}'$ by

$$\xi_A(U) = \{b \subseteq \mathtt{pt}'A \mid \exists a \in A \text{ with } \widetilde{a} \subseteq b \text{ and } \triangle a \in U\}.$$

8.4. **Lemma.** *Let* $A \in \mathbf{HA}$ *we have* $\rho^{\flat}_{A} \circ \xi_A = \mathrm{id}_{\mathtt{pt}\cdot\mathtt{N}A}$.

Combining Propositions 6.9 and 6.10 and Lemma 8.4 gives:

8.5. **Theorem.** *Monotone modal intuitionistic logic is complete with respect to monotone frames.*

In concrete terms, an axiomatisation of monotone modal intuitionistic logic is given by an axiomatisation of intuitionistic logic, together with the axiom $\triangle(\varphi \wedge \psi) \to \triangle\varphi$, and the theorem above gives completeness of this axiomatisation with respect to monotone frames.

8.6. **Definition.** A *general monotone frame* is a tuple $(X, \leq, N, A)$ where $(X, \leq, N)$ is a monotone frame, $(X, \leq, A)$ is a general intuitionistic frame, and $A$ is closed under $m_{\triangle} : \mathsf{Up}(X, \leq) \to \mathsf{Up}(X, \leq)$ given by

$$m_{\triangle}(a) = \{x \in X \mid a \in N(x)\}.$$

We call $A$ the collection of *admissible subsets*. A morphism between such frames is a monotone frame morphism whose inverse image preserves admissibles.

An upset in a general monotone frame is called *closed* if it is the intersection of admissible upsets. A general monotone frame is called *descriptive* (compare [41, Subsection 2.4.2]) if

- $(D_1)$ For every closed upset $c$, $c \in N(x)$ iff $[c \subseteq a \to a \in N(x)]$, where $a$ ranges over $A$;
- $(D_2)$ For any upset $b$, $b \in N(x)$ iff there is a closed $c \subseteq b$ with $c \in N(x)$.

We denote the category of general monotone frames and morphisms and its full subcategory of descriptive frames by **G-Mon** and **D-Mon** respectively.

Every monotone frame can be viewed as a general monotone frame where $A$ is the collection of all upsets. Also, a general monotone frame $(X, \leq, N, A)$ gives rise to the HAM $(A, m_{\triangle})$, where $A$ is the Heyting algebra given by the general intuitionistic frame $(X, \leq, A)$. This assignment extends to a functor **G-Mon** $\to$ **HAM**. In the converse direction we have:

8.7. **Definition.** Let $(H, \triangle)$ be a HAM and let $(\mathtt{pf}H, \subseteq, \widetilde{H})$ be the dual intuitionistic Kripke frame ($\mathtt{pf}$ are prime filters). For each $u \in \mathtt{pf}H$ define a neighbourhood $N(u)$ by:

- If $k$ is the intersection of sets in $\widetilde{H}$, then $k \in N(u)$ iff $k \subseteq \widetilde{a}$ implies $\widetilde{a} \in N(u)$;
- For any up-closed $q$ in $(\mathtt{pf}H, \subseteq)$, $q \in N(u)$ iff there is an intersection $k$ of elements in $\widetilde{H}$ such that $k \subseteq q$ and $k \in N(u)$.

It is straightforward to verify that $(\mathsf{pf}H, \subseteq, N, \widetilde{H})$ is a descriptive monotone frame. Moreover, although we omit the details, note that this gives rise to a dual equivalence

$$\mathbf{D\text{-}Mon} \equiv^{\mathrm{op}} \mathbf{HAM}. \tag{7}$$

Just like descriptive $\Box$-frames, descriptive monotone frames can also be viewed as dialgebras for functors $\mathtt{I}, \mathtt{W} : \mathbf{ES} \to \mathbf{Pries}$, where $\mathtt{I}$ is the embedding of Esakia spaces into Priestley spaces and $\mathtt{W}$ is defined as follows.

8.8. **Definition.** Define $\mathtt{W} : \mathbf{ES} \to \mathbf{Pries}$ by $\mathtt{W} = \mathsf{V}_{\mathsf{dn}} \cdot \mathsf{V}^{-}_{\mathsf{up}}$, where $\mathsf{V}^{-}_{\mathsf{up}}$ is the restriction of $\mathsf{V}_{\mathsf{up}}$ to $\mathbf{ES}$, see Proposition 3.14.

An element of $\mathtt{W}\mathbb{X}$ is a closed collection $A$ of closed upsets in $\mathbb{X}$ satisfying $c \in A$ and $c \subseteq c'$ implies $c' \in A$ (i.e. $A$ itself, as well as the elements of $A$ need to be closed).

8.9. **Theorem.** *We have* $\mathbf{D\text{-}Mon} \cong \mathbf{Dialg}(\mathtt{I}, \mathtt{W})$.

The categorical duality in (7) can also be achieved using Theorem 4.1 and the characterisation of descriptive monotone frames as dialgebras.

8.10. **Remark.** Similarly, one can define a "dual" operator $\triangledown$, interpreted by $x \Vdash \triangledown\varphi$ iff $X \setminus [\![\varphi]\!] \notin N(x)$. This means that the collection of neighbourhoods of $x$ consists of downsets. The category of frames corresponding to this setting is $\mathbf{Dialg}(\mathtt{i}, \mathsf{P}^{+}_{\mathsf{up}}\mathsf{P}_{\mathsf{dn}})$ and the category of descriptive frames is $\mathbf{Dialg}(\mathtt{I}, \mathsf{V}^{+}_{\mathsf{up}}\mathsf{V}_{\mathsf{dn}})$, where $\mathsf{P}^{+}_{\mathsf{up}}$ and $\mathsf{V}^{+}_{\mathsf{up}}$ are the extensions of $\mathsf{P}_{\mathsf{up}}$ and $\mathsf{V}_{\mathsf{up}}$ to endofunctors on $\mathbf{Pos}$ and $\mathbf{Pries}$, respectively.

## 9 Conditional Intuitionistic Logic

Conditional intuitionistic logics [21, 84, 85] combine non-monotonic reasoning with an intuitionistic base logic. From a philosophical viewpoint, this allows us to separate logical principles that are conditional from those that are induced by the base logic being classical. We show how this fits in our framework.

The language of conditional intuitionistic logic is intuitionistic logic with an additional binary modality $\rhd$ which is non-monotonic in its first argument and normal in its second argument. That is, $\rhd$ satisfies

$$\varphi \rhd (\psi \wedge \theta) = (\varphi \rhd \psi) \wedge (\varphi \rhd \theta), \quad \varphi \rhd \top = \top.$$

The algebraic semantics are defined as follows.

9.1. **Definition.** A *Heyting algebra with conditional operator* (HAC) is a pair $(H, \rhd)$ of a Heyting algebra $H$ together with a binary map $\rhd : H \times H \to H$ which satisfies

$$a \rhd (b \wedge c) = (a \rhd b) \wedge (a \rhd c), \quad a \rhd \top = \top.$$

A HAC morphism $(H, \rhd) \to (H', \rhd')$ is a Heyting algebra morphism $h : H \to H'$ satisfying $h(a \rhd b) = h(a) \rhd' h(b)$. Write **HAC** for the category of HACs and HAC morphisms.

The object part of this definition corresponds to [84, Definition 3]. Definition 1 of *op. cit.* defines frame semantics for conditional intuitionistic logic as follows.

9.2. **Definition.** A *conditional Kripke frame* is a tuple $(X, \leq, \{R_a \mid a \in \mathsf{P}X\})$, where $(X, \leq)$ is a preorder and for each subset $a \subseteq X$ we have a relation $R_a \subseteq X \times X$, subject to

$$\leq \circ R_a \subseteq R_a \circ \leq.$$

These can in fact be seen an (intuitionistic) adaptation of selection function frames (see e.g. [19, Section 10], [52, Section 2]), that is, as tuples $(X, \leq, s)$ where $(X, \leq)$ is a preorder and

$$s : X \times \mathsf{P}X \to \mathsf{P}X$$

is a function satisfying $x \leq y$ implies $s(y, a) \subseteq {\uparrow_{\leq}} s(x, a)$. The correspondence with conditional Kripke frames arises by identifying $s(x, a)$ with the set of $R_a$-successors of $x$.

Since truth sets of intuitionistic formulas are upsets in $(X, \leq)$, it makes sense to restrict the domain of $s$ to $X \times \mathsf{Up}(X, \leq)$. Since furthermore we are interested in whether or not $s(x, a)$ is a subset of some truth set $[\![\varphi]\!]$, and $s(x, a) \subseteq [\![\varphi]\!]$ if and only if ${\uparrow_{\leq}} s(x, a) \subseteq [\![\varphi]\!]$, we stipulate the codomain of $s$ to consist of upsets only.

9.3. **Definition.** An *intuitionistic selection function frame* is a tuple $(X, \leq, s)$ consisting of a preorder $(X, \leq)$ and a function

$$s : X \times \mathsf{Up}(X, \leq) \to \mathsf{Up}(X, \leq)$$

such that $x \leq y$ implies $s(y, a) \subseteq s(x, a)$. A morphism $f : (X, \leq, s) \to (X', \leq', s')$ is a bounded morphism $f : (X, \leq) \to (X', \leq')$ that satisfies for all $x \in X$ and $a' \in \mathsf{Up}(X, \leq)$:

$$f[s(x, f^{-1}(a'))] = s'(f(x), a'). \tag{8}$$

(Unravelling the definitions shows that this is precisely required to preserve truth of $\rhd$.) Write **SFF** for the category of intuitionistic selection function frames and morphisms.

We show how to view such frames as dialgebras.

9.4. **Definition.** For a preorder $(X, \leq)$ define $\mathsf{D}(X, \leq)$ as the collection of functions $\mathsf{Up}(X, \leq) \to \mathsf{Up}(X, \leq)$ ordered by $g \leq h$ iff $h(a) \subseteq g(a)$ for all $a \in \mathsf{Up}(X, \leq)$, viewed as a preorder. For a bounded morphism $f : (X, \leq) \to (X', \leq')$ in **PreKrip** define $\mathsf{D}f$ by $\mathsf{D}f(h)(a') = f[h(f^{-1}(a'))]$.

Recall that $\mathtt{i}'$ denotes the embedding **PreKrip** $\to$ **PreOrd**. We leave it to the reader to verify:

9.5. **Theorem.** *We have* $\mathbf{SFF} \cong \mathbf{Dialg}(\mathtt{i}', \mathsf{D})$.

We now define general and descriptive frames, connect these to the frames used in [21], and prove completeness. In what follows, we write "frame" instead of "intuitionistic selection function frame."

9.6. **Definition.** A *general frame* is a tuple $(X, \leq, s, A)$ where $(X, \leq, s)$ is a frame, $(X, \leq, A)$ is a general intuitionistic frame and $A$ is closed under $m_{\rhd} : \mathsf{Up}(X, \leq) \times \mathsf{Up}(X, \leq) \to \mathsf{Up}(X, \leq)$ given by $m_{\rhd}(a, b) = \{x \in X \mid s(x, a) \subseteq b\}$.

A general frame is called *descriptive* if moreover:

- For all $a \in A$ we have $s(x, a) = \bigcap\{b \in A \mid s(x, a) \subseteq b\}$;
- For all $a \notin A$ we have $s(x, a) = \emptyset$.

A morphism between general frames is a morphism between the underlying frames (as in Definition 9.3) which is simultaneously a general intuitionistic frame morphism. Write **G-SFF** and **D-SFF** for the category of general frames and morphism, and its full subcategory of descriptive frames.

The second clause of descriptive frames is rather arbitrary. We could have chosen to set $s(x, a) = a$ or $s(x, a) = X$ for all $a \notin A$. The crux is that we want to determine the action of $s$ on $a \notin A$ uniquely. Additional axioms of the logic, for example inspired by [29–32], may require a more involved definition.

9.7. **Remark.** Interestingly, general frames with $s(x, a) = \emptyset$ for all $a \notin A$ correspond bijectively with the frames underlying the models in [21, Definition 3].

We complete this section by proving completeness (which again follows from an easy application of the prime filter lemma). As usual, we have a functor $(\cdot)^+ : \mathbf{G\text{-}SFF} \to \mathbf{HAC}$, sending a general frame $(X, \leq, s, A)$ to $(X, m_{\triangleright})$ and a morphism $f$ to $f^{-1}$. Conversely:

9.8. **Definition.** Suppose given a HAC $(H, \triangleright)$. Let $(X, \leq, A)$ be the descriptive intuitionistic frame dual to $H$ and recall that $A = \{\widetilde{a} \mid a \in H\}$. Let $(H, \triangleright)_+ = (X, \leq, s, A)$ where $s : X \times \mathsf{Up}(X, \leq) \to \mathsf{Up}(X, \leq)$ is defined by:

- If $\widetilde{a} \in A$, then $s(u, \widetilde{a}) = \bigcap\{\widetilde{b} \in A \mid a \triangleright b \in u\}$;
- If $d \in \mathsf{Up}(X, \leq)$ and $d \notin A$, let $s(u, d) = \emptyset$.

It is obvious that the construction in Definition 9.8 does indeed give a descriptive frame. The next lemma provides the crucial argument for proving completeness. Its proof resembles the proof of Lemma 4.6.

9.9. **Lemma.** *In the setting of Definition 9.8, for all $\widetilde{a}, \widetilde{b} \in A$ we have $s(u, \widetilde{a}) \subseteq \widetilde{b}$ iff $a \triangleright b \in u$.*

Lemma 9.9 entails $((H, \triangleright)_+)^+ = (A, m_{\triangleright}) \cong (H, \triangleright)$. So the logic has weak descriptive semantics and by Proposition 6.9:

9.10. **Theorem.** *Conditional intuitionistic logic is (sound and) complete with respect to the intuitionistic selection function frames from Definition 9.3.*

## 10 Conclusion

We have established duality, completeness, and Hennessy-Milner type results for dialgebraic logics. Crucially, this allows us to give a general, categorical description of various flavours of modal intuitionistic logics based on algebra/coalgebra duality, addressing an open research question in the field.

Some technical questions remain open. For example, under which conditions $\theta'$ in Proposition 6.10 is pointwise monic? A similar question has been treated in [48, Section 4], although the setup in *op. cit.* is different. Another question is: under which conditions can the natural transformation $\tau$ in Proposition 6.10 be found? A similar question, but in a more restricted setting, has been investigated in [55].

Conceptually, our research can be extended along several independent and mutually orthogonal directions.

*Additional Axioms.* An important direction for future research is the investigation of intuitionistic modal logics not yet covered by our paper, for example Simpson's **IK** which adds additional interaction axioms between implication and modal operators. More generally, it would be intriguing to develop a theory of definability along the lines of Goldblatt Thomason [37, 54], and investigate Sahlqvist type theorems [72, 73].

*Different Modalities.* Other modal intuitionistic logics that potentially fit the framework include probabilistic intuitionistic logic [59] and intuitionistic public announcement logic [7, 58]. More generally, it would be fascinating to investigate *intuitionistic fragments of (set-based) coalgebraic logics*, in analogy with positive fragments of coalgebraic logics [6, 22]. Such a line of research may benefit from existing results about the *posetification* of a set-functor, i.e. the lifting of an endofunctor on **Set** to one on **Pos** [5, 22].

*Different Base Logics.* Although this paper focussed on modal intuitionistic logics, the theory is developed in more generality. Evidently, it also captures modal extensions of dual-intuitionistic and bi-intuitionistic logic [38, 70, 83], but it would be interesting to see whether more (logical) paradigms fall within the presented framework. For example, it may give rise to a dialgebraic treatment of modal logics based on the fragment of intuitionistic logic with connectives $\wedge$ and $\to$ [9, 71]. Algebraic semantics for this logic are given by *implicative meet-semilattices*, for which an Esakia-style duality result is already in place [8].

*Extending the Framework.* Apart from further investigating the framework defined in this paper, it would be fascinating to see if it can be generalised to the setting of arbitrary dialgebras. That is, can we avoid the assumption that one of the functors defining a category of dialgebras is an embedding? In particular, this would cover several flavours of automata, modelled as dialgebras for two functors on **Set** [13].

*Equivalence Notions.* Finally, the theory would benefit from a high-level comparison of dialgebraic equivalence notions in the style of Staton [79]. It would furthermore be interesting to see how these interact with the notions of bisimulations for modal (bi-)intuitionistic logics given in [1, 38, 45].

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