J. RICHARD BÜCHI. *Weak second-order arithmetic and finite automata.* ***Zeitschrift für mathematische Logik und Grundlagen der Mathematik,*** vol. 6 (1960), pp. 66–92.

J. RICHARD BÜCHI. *On a decision method in restricted second order arithmetic.* ***Logic, methodology and philosophy of science, Proceedings of the 1960 International Congress,*** edited by Ernest Nagel, Patrick Suppes, and Alfred Tarski, Stanford University Press, Stanford, Calif., 1962, pp. 1–11.

Each of these papers is concerned with an interpreted system of logic, i.e., a specified set of well-formed formulas semantically interpreted but with no axioms or rules of inference specified. In each case, the decision question is answered in the affirmative and the problem of definability is solved conclusively, using the theory of automata as a method of attack. The second paper closes a question posed by Tarski that had been open for a number of years. As far as the reviewer knows, the author's research is unique in solving a problem of pure logic by means of its application to the theory of automata, or any such applied theory.

The two papers are alike in their specified sets of well-formed formulas. They both allow quantification over individual variables, interpreted as natural numbers, and over monadic predicate variables; the constants are zero, the successor functor, and the truth-functional operators. (Equality and inequalities are easily defined.) However, predicate variables as interpreted in the second paper range over all monadic predicates of natural numbers, while in the first paper they range over only special monadic predicates: a *special* monadic predicate is one that is, for some $k$, false for all numbers greater than $k$. The author calls the systems of the first and second papers weak second-order arithmetic (W2A), and the sequential calculus (SC), respectively.

The main theorem of the first paper (theorem 1) shows that, for every formula without free individual variables, one can effectively obtain a finite automaton whose $n$ binary inputs correspond to the $n$ free predicate variables of the formula, such that the output of the automaton at time $k + 1$ (and forever after, as it turns out) tells whether the formula is true or false when the input history up to time $k$ (as a set of $n$ predicates false for values greater than $k$) provides the interpretation of the $n$ predicate variables. The main theorem of the second paper (theorem 1) states that every formula of SC without free individual variables is effectively equivalent to a formula having a certain form (in which, e.g., all predicate variables are initially placed and existential); the proof requires a result of Ramsey's, namely theorem A of 295***12***.

Roughly, every formula of SC without free individual variables is equivalent to a statement about the behavior of a finite automaton (or, equivalently, about a finite set of finite automata) whose infinite input history is given by the free predicate variables of the formula. For example, it may be the statement that the automaton returns to a certain state an infinite number of times. The most enlightening general characterization of the expressive power of SC with regard to finite automata is given by lemma 10 of the second paper, which (together with theorem 1 and lemma 6) states (reminiscently of Ramsey's theorem A) that the set of infinite input histories satisfying an arbitrary formula $\phi$ of SC is a union of finitely many sets, $S_1, \ldots, S_n$, where, for each $i$, there are regular sets of words $\alpha$ and $\beta$ such that $S_i$ is the set of all infinite histories of the form $ab_1b_2 \ldots$, where $a$ is a word of $\alpha$ and each $b_j$ is a word of $\beta$.

Speaking loosely, W2A is adequate for describing the finite behavior of finite automata and no more, whereas SC is adequate for describing the finite behavior of finite automata and no more. But W2A and SC are exactly alike in the properties and relations on natural numbers they define. Thus a property is definable in either system if and only if it is ultimately periodic. The expressive power of SC is therefore not increased beyond W2A with regard to formulas without free predicate variables.

The main theorem of the first paper is a synthesis result. The corresponding analysis result states that, for every finite automaton with special output, there is a formula of W2A of a certain form corresponding to the automaton in the sense described above. This result involves a restriction, in that a *special* output must remain the same under that input transition in which all input predicates are false. It turns out that this restriction is relatively harmless for most purposes, as a long discussion in section 7 shows. However, the reviewer does not think that the author's method of handling the output problem is optimal for the purpose of using W2A for precise descriptions of the behavior of automata. The reviewer's choice would be to have, for each automaton, a formula with a free individual variable '$t$' stating the conditions under which the output has the value 1 at time $t$. It is the author's desire to deal with formulas without free individual variables that leads to the adoption of special outputs.

Summary of the first paper beyond the main theorem: The author seeks to find a first-order arithmetic that is exactly intertranslatable with W2A. He falsely states (theorem 4) that such a system is the one containing $=$, $+$, and the constant monadic predicate, "is a power of 2." It is true that this system is translatable into W2A, but the proof the other way rests on an incorrect definition (p. 81, top) in this first-order system of the binary predicate "is a power of 2 occurring in the representation (as a sum of powers of 2) of." The reviewer conjectures that this error is incorrigible, and that theorem 4 is false. This error affects theorem 9 and most of the interesting discussion following it in section 7 (which are not reviewed here). As a first-order arithmetic exactly intertranslatable with W2A, the reviewer makes the obvious suggestion of using as primitive the binary predicate mentioned, instead of the weaker monadic predicate. (Theorem 9 and the discussion in section 7 could then be modified following the revised theorem 4.)

Theorem 3 gives a normal form for W2A, and corollary 3 of this theorem states that a well-ordering is definable in W2A if and only if it is of type less than $\omega^2$. The author's flat assertion in the proof of corollary 3 that "there is no strictly increasing $\omega^2$-sequence of ultimately periodic sets of natural numbers" is false. (Let $A_i$ be the set of all positive integers $\equiv 2^i \bmod 2^{i+1}$, and let $B_{\omega i+j}$ consist of all numbers in $A_0, \ldots, A_i$ together with the first $j$ numbers of $A_{i+1}$. Then the sequence of $B$'s is a counterexample.) What is true is that there is no such sequence where the period is bounded. Using this fact and the fact that there is an upper bound on the period of any ultimately periodic set definable from the formula $\mathfrak{C}(x, y)$, one can then prove that any well-ordering definable in W2A is of type less than $\omega^2$, thereby saving corollary 3.

Theorem 5 states that the first-order theory with $=$, $+$, and a constant monadic predicate $P$ is undecidable in case $P(x)$ is satisfied ultimately by values of a polynomial of degree greater than 1, and is decidable in case $P(x)$ is satisfied ultimately by values of a geometric progression. The second half is a generalization (without proof) of a valid corollary of theorem 4. The first half is a generalization (again without proof) of a result by Putnam (XXIII 446).

In a rather brief passage the author apologizes for presenting a two-valued language for automata in which only binary conditions can be represented directly. He states (p. 84) that a language adequate for representing all $k$-ary conditions directly would require a kind of infinite-valued logic that has not yet been developed. He does not explain further, but the reviewer would guess that this logic would have to be based on an infinite-valued propositional calculus having, for every $k$, the functionally complete $k$-valued logic as a part.

Summary of the second paper beyond the main theorem: It is shown that a relation on natural numbers is definable in SC if and only if it is definable in W2A. As R. M. Robinson reports (*Restricted set-theoretical definitions in arithmetic*, ***Proceedings of the American Mathematical Society***, vol. 9 (1958), pp. 238–242), Tarski posed two questions, whether addition is definable in SC, and whether SC is decidable. The author does not discuss this matter in detail, but from his results (including a detailed description of the decision method for SC) and from what is said in the Robinson paper, one can infer that the answers to the questions are, respectively, "no" and "yes'.

Further results are: (1) that the restriction of predicate variables to ultimately periodic predicates does not alter the truth-value of a formula without free variables; and (2) that the first-order theory is decidable whose domain of discourse is the set of non-negative real (or, alternatively, rational) numbers with addition and the two constant predicates, "is an integral power of 2" and "is an integer."

The second paper closes with three unsolved problems: First, if $\lambda x(2x + 1)$ and $\lambda x(2x + 2)$ are used instead of successor, is SC still decidable? Second, if the domain of individuals of SC is enlarged to the ordinals less than $\omega^2$ and the ordering on these ordinals is added as a primitive, is the system still decidable? And third, is there an effective manner of determining whether or not there exists a finite automaton satisfying a given condition expressed in SC?

Exposition and technical matters: The author's style in both papers, together with an inherent subtlety in the subject matter, makes for difficult reading. Too often is part of a proof omitted which the author expects the reader to fill in for himself. The rule should be that such an omission must be something that the reader can fill in by a mere computation; this rule is violated frequently in both papers, considerable ingenuity being required on the part of the reader. Another fault is the tendency to disguise as a remark or corollary an important theorem whose difficult proof is omitted.

There is a minor technical defect in the proof of theorem 2 of the first paper (p. 76). The clause $U[\boldsymbol{r}(t)]$ in the line marked (1) does not work for special predicates $\boldsymbol{r}$. $U$ is a special output and will therefore be stable under the null input transition; but there is no guarantee that the $\boldsymbol{r}$-states are null after time $x$, and so special $\boldsymbol{r}$-predicates can be guaranteed to represent the states only for a finite amount of time. The reviewer suggests replacing this clause by $U[\boldsymbol{r}(x)]$. Another more complicated change would then have to be made in the line marked (2).

In the second paper, the last paragraph of the proof of the crucial and difficult lemma 9 is obscure to the reviewer, who had to finish the proof to convince himself of the validity of the lemma. The reviewer also finds the connection between the proof and the statement of lemma 4 to be obscure; it would appear that the second and third occurrences of $R(\mathbf{i}, \mathrm{x}, \mathrm{y})$ in the statement should be changed to $\mathrm{R}(\mathbf{i}, \mathrm{z}, \mathrm{y})$, with a corresponding change in the second line of the proof. The proof of lemma 8 has an error in that s as recursively defined does not always satisfy the condition stated. (When $\mathrm{p}_1(0) \equiv \mathrm{T}$, $\mathrm{p}_2(0) \equiv \mathrm{F}$ and, for all t, $\mathrm{p}_1(\mathrm{t}') \equiv \mathrm{p}_2(\mathrm{t}') \equiv \mathrm{F}$, this condition, which is a biconditional, has its right side true and left side false.) However, if $\mathrm{s}_1$ is defined by $\mathrm{s}_1(\mathrm{t}) \equiv \mathrm{s}(\mathrm{t})\, \mathrm{p}_2(\mathrm{t})$ then the condition is satisfied with $\mathrm{s}_1$ in place of s. (The reviewer acknowledges the aid of his student, Miss Amy Eliasoff, in this matter.)

Errata, first paper: p. 72, line 15, '$\boldsymbol{i}^n$' for '$\boldsymbol{i}^m$'; p. 73, line 11, insert '$\wedge W(\boldsymbol{Y})$' immediately to the left of '$\cdot\supset\cdot$'; p. 74, line 5, '$\boldsymbol{E}$' for '$\boldsymbol{G}$'; p. 74, line 9, 'lemma 5' for 'lemma 4'; p. 75, last line, '$U_1$' for '$U$'; p. 76, line 7, insert '$\sim$' after '$\equiv$'; p. 76, line 15, '$\bar{U}_5$' for '$U_5$'; same correction on line 16; p. 90, line 8, '$\boldsymbol{u}$' for '$\boldsymbol{y}$'; p. 91, something has been dropped from the sentence ending on line 16.

Second paper: p. 6, line 11, all occurrences of 'i' should be bold-face; p. 4, lines 22 through 28, all occurrences of 'r' should be bold-face.

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A. GRZEGORCZYK. *A theory without recursive models.* ***Bulletin de l'Académie Polonaise des Sciences,*** Série des sciences mathématiques, astronomiques et physiques, vol. 10 (1962), pp. 63–69.