A Logical Approach to Asymptotic Combinatorics
I. First Order Properties

KEVIN J. COMPTON[1]

Wesleyan University, Middletown, Connecticut 06457

INTRODUCTION

We shall present a general framework for dealing with an extensive set of problems from asymptotic combinatorics; this framework provides methods for determining the probability that a large, finite structure, randomly chosen from a given class, will have a given property. Our main concern is the *asymptotic probability*: the limiting value as the size of the structure increases. For example, a common problem in elementary probability texts is to show that the asymptotic probabity that a permutation will have no fixed point is $1/e$. We shall say nothing about the closely related problem of determining rates of convergence, although the methods presented here may extend to such problems.

To develop a general approach we must fix a language for specifying properties of structures. Thus, our approach is logical; logic is the branch of mathematics that deals with problems of language. In this paper we consider properties expressible in the language of first order logic and speak of probabilities of first order sentences rather than properties. In the sequel to this paper we consider properties expressible in the more general language of monadic second order logic.

Clearly, we must restrict the classes of structures we consider in order for questions about asymptotic probabilities to be meaningful and significant. Therefore, we choose to consider only classes closed under disjoint unions and components (see Sect. 1 for definitions). This condition is general enough to include the classes of graphs, permutations, unary functions, and many other important examples. The main theorems of the paper are about classes satisfying this condition: Theorems 5.7 and 5.8 state that the existence of asymptotic probabilities for a special set of sentences called component-bounded sentences (defined in Sect. 5) is equivalent to a condition on the growth of such a class; Theorem 5.9 gives necessary and sufficient conditions for every first order sentence to have an asymptotic probability of 0 or 1 in nonfast growing classes. In Section 8 we present a long list of examples from the literature.

---

[1] Present address: University of Michigan, Ann Arbor, Michigan 48109.

Parts of this paper appeared in the author's Ph.D. thesis [9]. The author would like to thank H. J. Keisler, Richard Askey, Ward Henson, Doug Hoover, and Jim Lynch for advice and helpful discussions.

1. Preliminaries

We will assume a familiarity, but not an expertise, with basic concepts of model theory; the first chapter of Chang and Keisler [8] introduces the essential ideas. Throughout, L will denote a finite language without constant symbols, L-structures (models) will be denoted by upper case fraktur letters ($\mathfrak{A}$, $\mathfrak{A}^*$, $\mathfrak{B}$, $\mathfrak{K}$, etc.) and their universes by the corresponding upper case italic letters (A, A^*, B, K, etc.), and the set of first order sentences from L will be denoted $L_{\omega\omega}$.

There are two natural ways to choose a structure randomly when assigning a probability to a first order sentence. The first is to choose from a set of structures with a fixed universe; the second is to choose from a set of isomorphism types. We make those ideas precise in the following definitions.

DEFINITION. Let $\mathscr{C}$ be a class of L-structures closed under isomorphism, $\mathscr{A}_n$ the set of structures in $\mathscr{C}$ with universe $n = \{0, 1, ..., n-1\}$. For any sentence φ define $\mu_n(\varphi)$ to be the fraction of structures in $\mathscr{A}_n$ in which φ is true. Combinatorialists often refer to this kind of enumeration as *labeled* enumeration since every element in the universe of a structure in $\mathscr{A}_n$ is labeled with an ordinal number. Accordingly, we refer to the members of the $\mathscr{A}_n$'s as the labeled structures of $\mathscr{C}$ and define

$$\mu(\varphi) = \lim_{n \to \infty} \mu_n(\varphi)$$

to be the *labeled asymptotic probability* of φ whenever this limit exists.

$\mathscr{B}_n$ is a representative set from the isomorphism classes in $\mathscr{A}_n$; members of the $\mathscr{B}_n$'s are the *unlabeled structures* of $\mathscr{C}$ and the *unlabeled asymptotic probability* ν is defined in the same way as μ.

Asymptotic problems are the focus of much research in combinatorics. Erdös and Spencer [15] contains a wealth of examples, most having to do with classes of graphs and matrices. Many of the results in this area are due to Erdös and his collaborators (especcially Renyi, cf. Chap. 14 or Erdös [14]).

Other classes have been studied. Metropolis and Ulam [30] asked about the asymptotic probability of connectivity in random functions. Katz [25] answered the question (he derived an asymptotic expression for the number of connected unary functions, a computation requiring more sophisticated

analysis than a simple calculation of the asymptotic probability). Goncharov [20] showed that the asymptotic distribution of cycles in random permutations is Poisson. Shepp and Lloyd [37] extended his results; they calculated asymptotic probabilities of many statements about random permutations.

Fagin [16] proved that when $\mathscr{C}$ is the class of all L-structures, the asymptotic probability of any $L_{\omega\omega}$ sentence exists and, in fact, is either equal to 0 or 1. We will be especially interested in this kind of phenomenon.

DEFINITION. We say that $\mathscr{C}$ has an $L_{\omega\omega}$ *labeled* 0–1 *law* if $\mu(\varphi)$ exists and is equal to either 0 or 1 for all φ in $L_{\omega\omega}$. We define $L_{\omega\omega}$ *unlabeled* 0–1 *law* analogously.

Mycielski [32] and Lynch [28, 29] have looked at the question of the existence of asymptotic probabilities and the presence of 0–1 laws for a language with a relation symbol interpreted by total order and orientation relations. Grandjean [21] has determined the computational complexity required to decide, in the case of Fagin's 0–1 law, which sentences have probability 0 and which have probability 1. Our results relate 0–1 laws to growth conditions on the classes $\mathscr{A}_n$ and $\mathscr{B}_n$. The following types of generating series will figure prominently in the endeavor.

DEFINITION. Let $a_n = |\mathscr{A}_n|$, $b_n = |\mathscr{B}_n|$. In the labeled case the appropriate series to use is the *exponential generating series* for $\mathscr{C}$:

$$a(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

In the unlabeled case we use the *ordinary generating series* for $\mathscr{C}$:

$$b(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Asymptotic probabilities do not always exist for first order sentences. One reason is that there may be infinitely many positive integers that are not cardinalities of structures in $\mathscr{C}$, the class we have fixed. This is a minor technicality and in Section 6 we propose a slight change in the definition of asymptotic probability to overcome this difficulty. A more profound reason is that for some sentences φ, $\mu_n(\varphi)$ or $\nu_n(\varphi)$ may fail to converge. We need only consider a few familiar examples—the class of groups, for example—to see that this is not an uncommon occurrence. Hoover [24] has shown that $\mathscr{C}$ may be finitely axiomatizable by first order universal sentences and still have first order φ such that $\mu_n(\varphi)$ does not converge (this answers a question posed by Fagin [16]).

We want to restrict our attention to classes for which asymptotic probabilities are likely to exist, but not so narrowly that our examples are unnatural or uncommon. Our restrictions should not exclude structures, such as graphs, permutations, and unary functions, often considered in asymptotic combinatorics. These structures share the property that they may be uniquely represented as disjoint unions of *connected* structures (we will make this notion precise presently) from their respective classes: any graph is a disjoint union of connected graphs; beginning abstract algebra students learn that finite permutations can be decomposed into disjoint cycles; unary functions behave similarly. The classes we consider will have this property. We formalize this idea with the following:

DEFINITION. *Let $\mathfrak{A}$ be an L-structure. Define a binary relation $\sim$ as follows. Let $a, b \in A$. $a \sim b$ if for some relation symbol R in L and sequences* **x, y, z** *of element variables*

$$\mathfrak{A} \models (\exists \mathbf{x}, \mathbf{y}, \mathbf{z}) \, R(\mathbf{x}, a, \mathbf{y}, b, \mathbf{z}).$$

Let $\sim^*$ be the least equivalence relation extending $\sim$. The $\sim^*$ equivalence classes are called *components* of $\mathfrak{A}$. If $\mathfrak{A}$ is a graph, this corresponds to the graph theoretic notion of component.

Sometimes we will say that a structure $\mathfrak{K}$ is a component of $\mathfrak{A}$. By this, we mean that $\mathfrak{K}$ is a substructure of $\mathfrak{A}$ and K is a component of $\mathfrak{A}$. We will also say that $\mathfrak{A}$ is *connected* if it has just one component.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be *L*-structures. $\mathfrak{B}$ is a *closed substructure* of $\mathfrak{A}$ (equivalently, $\mathfrak{A}$ is a *closed extension* of $\mathfrak{B}$) if $\mathfrak{B}$ is a substructure of $\mathfrak{A}$ and is union of components of $\mathfrak{A}$.

The classes we will consider are those closed under disjoint unions and components; i.e., classes $\mathscr{C}$ which satisfy

    (i)   If $\mathfrak{A}, \mathfrak{B} \in \mathscr{C}$ then $\mathfrak{A} \amalg \mathfrak{B}$, the disjoint union of $\mathfrak{A}$ and $\mathfrak{B}$, is in $\mathscr{C}$;
and

    (ii) If $\mathfrak{A} \in \mathscr{C}$ and $\mathfrak{K}$ is a component of $\mathfrak{A}$ then $\mathfrak{K} \in \mathscr{C}$.

Combinatorialists have studied these classes extensively. In the next section, we will see that their generating series behave quite nicely and provide a useful means for enumeration.

2. PROPERTIES OF THE CLASSES

Cayley [7, Vol. 3, pp. 242–246] was first to use the properties of classes closed under disjoint unions and components for enumeration. In counting

the number of unlabeled oriented trees, he found it useful to consider unlabeled oriented forests (disjoint unions of trees). In a later work [7, Vol. 13, pp. 26–28] he enumerated the class of labeled free trees, again using forests (our terminology for the different kinds of trees, from Knuth [27], will be explained in Sect. 7). Cayley's work was the starting point for much fruitful investigation. Notable among his successors were Polya, whose paper [34] contained enumerations of many classes with the closure properties discussed in section 1, and Otter [33] who made the first enumeration of unlabeled free trees. A large part of Polya's paper is devoted to finding the relationship between the numbers of labeled and unlabeled structures. We shall not concern ourselves with this question, even though it is sometimes important in calculating asymptotic probabilities (e.g., Fagin's demonstration of an unlabeled 0–1 law for the class of all L-structures in [16] relies on an asymptotic relationship between the numbers of labeled and unlabeled L-structures; see [17]). Instead, we will present parallel developments of the theories of asymptotic probabilities for labeled and unlabeled structures.

Lemma 2.1(i) expresses the relationship between the exponential generating series of a class closed under disjoint unions and components, and the exponential generating series of the subclass of connected structures. It is given as the defining equation for classes of structures in many papers; see, for example, Bender and Goldman [5], Gilbert [19], and Riordan [35]. To the best of our knowledge, we are the first to point out the closure properties shared by these classes. Foata [18] contains an extensive account of the exponential generating series of these classes. Roman and Rota [36], building on earlier work of Rota and his collaborators (see Mullin and Rota [31]), take a different approach to enumeration of these classes. They develop a rigorous version of Sylvester's umbral calculus for this purpose, rather than relying on generating series (the first attempt to formalize the umbral calculus was due to Bell [3]; his version was intended to apply to a wider range of classes than the one discussed here).

This is by no means a comprehensive history of the study of these classes. We hope only to give some indication of the interest in them and to demonstrate that our choice is a natural one. Harary and Palmer [22] contains a detailed bibliography and discussion of enumerative problems. See also the references cited above, particularly [27] and [36], for more complete bibliographies.

The following lemma summarizes the pertinent properties of classes closed under disjoint unions and components. For each finite connected structure $\Re$ and integer j let $\theta_{\Re,j}$ be a first order sentence that says "there are precisely j components isomorphic to $\Re$" (clearly this property is always first order expressible). $\sigma(\Re)$ is the number of symmetries of $\Re$.

LEMMA 2.1. *Let $\mathscr{C}$ be closed under disjoint unions and components with exponential generating series $a(x)$ and ordinary generating series $b(x)$.*

(i) *If $k(x)$ is the exponential generating series for the subclass of connected structures in $\mathscr{C}$ then*

$$a(x) = \exp(k(x)).$$

(ii) *If $\Re_0, \Re_1, ..., \Re_{q-1}$ are nonisomorphic finite connected structures, $|\Re_i| = m_i$, and $j_0, j_1, ..., j_{q-1} \in \omega$, then the exponential generating series for the subclass of connected structures in $\mathscr{C}$ wich satisfy $\bigwedge_{i<q} \theta_{\Re_i, j_i}$ is*

$$a(x) \prod_{i<q} \frac{1}{j_i!} \left(\frac{x^{m_i}}{\sigma(\Re_i)} \right)^{j_i} \exp\left(-\frac{x^{m_i}}{\sigma(\Re_i)} \right).$$

(iii) *If $c_j(x) = \sum_{n=0}^{\infty} c_{j,n}/n!$ is the exponential generating series for the subclass of structures in $\mathscr{C}$ that satisfy $\theta_{\Re,j}$, $|\Re| = m$, then*

$$\frac{c_{j,n-m}}{(n-m)!} = (j+1)\,\sigma(\Re)\,\frac{c_{j+1,n}}{n!}.$$

(iv) *If $l(x) = \sum_{n=0}^{\infty} l_n x^n$ is the ordinary generating series for the subclass of connected structures in $\mathscr{C}$ then*

$$b(x) = \prod_{n \geqslant 1} (1 - x^n)^{-l_n} = \exp\left(\sum_{m \geqslant 1} \frac{1}{m} l(x^m) \right).$$

(v) *With the same assumptions as in (ii), the ordinary generating series for the subclass of structures that satisfy $\bigwedge_{i<q} \theta_{\Re_i, j_i}$ is*

$$b(x) \prod_{i<q} x^{m_i j_i} (1 - x^{m_i}).$$

Proof. (i) and (iv) are well-known results from the theory of generating series (see the references mentioned above). The other results follow by similar arguments which we sketch here.

First, observe that if $a^*(x)$ and $a^{\#}(x)$ are the generating series for classes $\mathscr{C}^*$ and $\mathscr{C}^{\#}$, and no component of a structure in one class is isomorphic to a component of a structure in the other, then the exponential generating series for the class

$$\{ \mathfrak{A} \amalg \mathfrak{B} : \mathfrak{A} \in \mathscr{C}^*, \mathfrak{B} \in \mathscr{C}^{\#} \}$$

is $a^*(x)\, a^{\#}(x)$ (i.e., the result of formal multiplication). Also, if $k(x)$ is the exponential generating series for a class of connected structures, the

$k(x)^j/j!$ is the exponential generating series for the class of structures with precisely j components, each from this class. Thus we have (i),

$$a(x) = \sum_{j=0}^{\infty} k(x)^j/j! = \exp(k(x)).$$

If $\mathfrak{K}$ is connected, $|\mathfrak{K}| = m$, then there are $m!/\sigma(\mathfrak{K})$ labeled structures isomorphic to $\mathfrak{K}$, where $\sigma(\mathfrak{K})$ is the number of autormorphisms of $\mathfrak{K}$. Therefore, the exponential generating series for the class of structures isomorphic to $\mathfrak{K}$ is $x^m/\sigma(\mathfrak{K})$. The exponential generating series for the class of structures with precisely j components, each isomorphic to $\mathfrak{K}$, is $(x^m/\sigma(\mathfrak{K}))^j/j!$, and for the class of structures with all components isomorphic to $\mathfrak{K}$ is $\exp(x^m/\sigma(\mathfrak{K}))$. Thus

$$a(x) = \prod \exp\left(\frac{x^{|\mathfrak{K}|}}{\sigma(\mathfrak{K})}\right), \tag{2.1}$$

where the product is taken over a representative set from the isomorphism types of connected structures in $\mathscr{C}$, and

$$a(x) = \prod_{i<q} \frac{1}{j_i!}\left(\frac{x^{m_i}}{\sigma(\mathfrak{K}_i)}\right)^{j_i} \exp\left(-\frac{x^{m_i}}{\sigma(\mathfrak{K}_i)}\right) \tag{2.2}$$

is the exponential generating series for the subclass of structures satisfying $\bigwedge_{i<q} \theta_{\mathfrak{K}_i,j_i}$, establishing (ii). In the case $q=1$, the exponential generating series $c_j(x) = \sum_{n=0}^{\infty} (c_{j,n}/n!)\, x^n$ for the subclass of structures that satisfy $\theta_{\mathfrak{K},j}$ is

$$c_j(x) = a(x)\frac{1}{j!}\left(\frac{x^m}{\sigma(\mathfrak{K})}\right)^j \exp\left(-\frac{x^m}{\sigma(\mathfrak{K})}\right),$$

where $|\mathfrak{K}| = m$. Comparing this to

$$c_{j+1} = a(x)\frac{1}{(j+1)!}\left(\frac{x^m}{\sigma(\mathfrak{K})}\right)^{j+1} \exp\left(-\frac{x^m}{\sigma(\mathfrak{K})}\right),$$

we see that

$$c_j(x) = (j+1)\left(\frac{x^m}{\sigma(\mathfrak{K})}\right)^{-1} c_{j+1}(x),$$

so

$$\frac{c_{j,n-m}}{(n-m)!} = (j+1)\,\sigma(\mathfrak{K})\,\frac{c_{j+1,n}}{n!}.$$

This is (iii).

Now observe that if $b^*(x)$ and $b^{\#}(x)$ are the ordinary generating series for $\mathscr{C}^*$ and $\mathscr{C}^{\#}$, described above, then the ordinary generating series for the class

$$\{ \mathfrak{A} \amalg \mathfrak{B} \colon \mathfrak{A} \in \mathscr{C}^*, \mathfrak{B} \in \mathscr{C}^{\#} \}$$

is $b^*(x) b^{\#}(x)$. Also, if $\mathfrak{R}$ is connected, $|\mathfrak{R}| = m$, then the ordinary generating series for the class of structures with all components isomorphic to $\mathfrak{R}$ is

$$\sum_{i=0}^{\infty} x^{im} = (1 - x^m)^{-1}.$$

(iv) and (v) now follow by arguments similar to those used to establish (2.1) and (2.2), and by formal manipulation of generating series.

(We should remark that the formal manipulations used throughout the proof are legitimate and our methods are *combinatorially* valid (formal power series are discussed in several references already cited; cf. Bender and Goldman [5], Foata [18], Harary and Palmer [22], and Knuth [27]). The formal equivalence of these series shows that our methods are *analytically* valid when appropriate convergence conditions are satisfied.) ∎

3. LOGICAL RESULTS ABOUT THESE CLASSES

In this section we present, without proof, some logical results about classes closed under disjoint unions and components. The proofs use techniques which would take us far afield of our basic concerns here. The interested reader should consult [9] or [10]. The results are about first order sentences and theories holding in classes closed under disjoint unions and components. We need the following definition.

DEFINITION. A *connecting quantifier* is a quantifier of the form $(\forall \mathbf{x}.R(\mathbf{t}))$ or $(\exists \mathbf{x}.R(\mathbf{t}))$, where R is a relation symbol in L, $\mathbf{x}$ is a sequence of variables, $\mathbf{t}$ a sequence of variables and constant symbols, and the set of variables in $\mathbf{x}$ is properly contained in the set of variables and constant symbols in $\mathbf{t}$. The meanings of these quantifiers are given by

$$\models (\forall \mathbf{x}.R(\mathbf{t})) \, \varphi(\mathbf{t}, \mathbf{y}) \leftrightarrow \forall \mathbf{x}(R(\mathbf{t}) \to \varphi(\mathbf{t}, \mathbf{y})),$$

$$\models (\exists \mathbf{x}.R(\mathbf{t})) \, \varphi(\mathbf{t}, \mathbf{y}) \leftrightarrow \exists \mathbf{x}(R(\mathbf{t}) \wedge \varphi(\mathbf{t}, \mathbf{y})).$$

Connecting quantifiers connect the variables they bind to unbound variables via some relation symbol.

Using connecting quantifiers, we can define a set of first order formulas

closely associated with classes closed under disjoint unions and components.

DEFINITION. The set of *basic connected* formulas is the smallest set S of formulas satisfying the following.

(i) Atomic formulas and negated atomic formulas are in S.

(ii) S is closed under disjunctions and conjuctions of pairs of formulas.

(iii) S is closed under connecting quantifiers. The set of *universal connected* formulas is defined by conditions (i)–(iii) together with

(iv∀) S is closed under universal quantification. The set of *existential connected* formulas is defined by conditions (i)–(iii) together with

(iv∃) S is closed under existential quantification.

The first of our results syntactically characterizes theories and sentences preserved under closed substructures and closed extensions.

THEOREM 3.1. *For $\varphi \in L_{\omega\omega}$*

(i) *φ is preserved under closed extensions iff it is logically equivalent to an existential connected sentence,*

(ii) *φ is preserved under closed substructures iff it is logically equivalent to a universal connected sentence.*

This theorem will be useful in conjunction with

THEOREM 3.2. *Any $\varphi \in L_{\omega\omega}$ is logically equivalent to a Boolean combination of existential connected sentences, or, equivalently, to a Boolean combination of universal connected sentences.*

The next theorem is not needed for any later results, but should be included in any discussion of the logic of classes closed under disjoint unions and components.

THEOREM 3.3. *(Here we assume L does not contain constant symbols.)*

(i) *An $L_{\omega\omega}$ theory T is preserved under disjoint unions and components iff it has a set of axioms of the form*

$$\forall x\, \psi(x), \qquad \psi \text{ basic connected.} \tag{3.1}$$

(ii) *$\varphi \in L_{\omega\omega}$ is preserved under disjoint unions and components iff it is logically equivalent to a sentence of the form (3.1).*

The final theorem of this section says that every class of L-structures closed under disjoint unions and components and containing at least one finite structure is associated with a certain complete $L_{\omega\omega}$ theory. This theory, as we will see, is the set of $L_{\omega\omega}$ sentences true in almost all finite structures in the class whenever the class grows slowly enough. $\mathrm{Th}(\mathfrak{A})$ denotes the set of $L_{\omega\omega}$ sentences true in $\mathfrak{A}$.

THEOREM 3.4. (*We assume L contains no constant symbols.*) *Suppose $\mathscr{C}$ is a class of L-structures closed under disjoint unions and components, and that $\mathscr{C}$ contains at least one finite structure. The set of sentences*

$$T = \left\{ \neg \theta_{\mathfrak{R},j} : \mathfrak{R} \text{ connected}, j \in \omega \right\} \cup \bigcap_{\substack{\mathfrak{A} \in \mathscr{C} \\ \text{finite}}} \mathrm{Th}(\mathfrak{A})$$

is a complete, consistent theory.

4. DESCRIBING GROWTH RATES OF THE CLASSES

We will see in later sections that certain growth characteristics of these classes determine whether asymptotic probabilities exist. In this section, we introduce special notations to describe these characteristics.

DEFINITION. Let $\mathscr{C}$ be a class of structures with $a(x) = \sum_{n=0}^{\infty} (a_n/n!) x^n$, $b(x) = \sum_{n=0}^{\infty} b_n x^n$ the exponential and ordinary generating series for $\mathscr{C}$. We write $\mathscr{A} \to R$ for a nonnegative extended (i.e., ∞ is a permissible value) real number R to indicate that

$$\lim_{n \to \infty} \frac{a_{n-m}/(n-m)!}{a_n/n!} = R^m \tag{4.1}$$

for all m such that there is a connected structure of cardinality m in $\mathscr{C}$. We write $\mathscr{B} \to S$, S a nonnegative real number not greater than 1, to indicate that

$$\lim_{n \to \infty} \frac{b_{n-m}}{b_n} = S^m \tag{4.2}$$

for all m as above. These definitions may seem a little strange at first. They say that the radii of convergence of the generating series may be computed from ratios of certain coefficients. The following proposition shows that in most cases it suffices to consider ratios of successive coefficients.

PROPOSITION 4.1. *Suppose that $\mathscr{C}$ is closed under disjoint unions and components, and contains at least one finite structure.*

(i) *If $a(x)$ has radius of convergence R, $0 < R < \infty$, then $\mathscr{A} \to R$ iff*

$$\lim_{n \to \infty} \frac{a_{n-1}/(n-1)!}{a_n/n!} = R.$$

(ii) *If $b(x)$ has radius of convergence $S > 0$, then $\mathscr{B} \to S$ iff*

$$\lim_{n \to \infty} \frac{b_{n-1}}{b_n} = S.$$

Proof. We prove (ii), the proof of (i) being nearly identical. First, suppose b_{n-1}/b_n approaches S. Then

$$\lim_{n \to \infty} \frac{b_{n-m}}{b_n} = \lim_{n \to \infty} \frac{b_{n-m}}{b_{n-m+1}} \cdot \frac{b_{n-m+1}}{b_{n-m+2}} \cdots \frac{b_{n-1}}{b_n} = S^m$$

for any m, whether or not m is the cardinality of a connected structure.

Now suppose $\mathscr{B} \to S$. Let l_n be the number of connected structures in $\mathscr{C}$ of cardinality n. There must be an integer N such that $b_n \neq 0$ for all $n > N$, in order for the limit (4.2) to be defined. Hence, the greatest common divisor of integers in the set $\{m \in \omega : l_m \neq 0\}$ is 1. We may choose a finite number of elements $m_0, m_1, \dots, m_{p-1}$ from this set with greatest common divisor 1. Pick integers $\alpha_0, \alpha_1, \dots, \alpha_{p-1}$ such that

$$\sum_{i < p} \alpha_i m_i = 1.$$

Without loss of generality, we may assume for some $q \leqslant p$, $\alpha_i > 0$, when $i < q$ and $\alpha_i < 0$ when $q \leqslant i < p$. We may write

$$\sum_{i < q} \alpha_i m_i = 1 + \sum_{q \leqslant i < p} (-\alpha_i) m_i. \tag{4.3}$$

Now by definition of $\mathscr{B} \to S$ we know

$$\lim_{n \to \infty} \frac{b_{n-m_i}}{b_n} = S^{m_i}$$

for $i < p$. If $i, j < p$, then

$$\lim_{n \to \infty} \frac{b_{n-m_i-m_j}}{b_n} = \lim_{n \to \infty} \frac{b_{n-m_i-m_j}}{b_{n-m_i}} \frac{b_{n-m_i}}{b_n}$$

$$= S^{m_i + m_j}.$$

We may iterate this to show that

$$\lim_{n \to \infty} \frac{b_{n-u}}{b_n} = S^u$$

when u is a positive linear combination of the m_i's. Let $u = \sum_{i < q} \alpha_i m_i$, $u' = \sum_{q \leqslant i < p} (-\alpha_i) m_i$, so $u = 1 + u'$ by (4.3). Then

$$\lim_{n \to \infty} \frac{b_{n-u}}{b_n} = S^u, \qquad \lim_{n \to \infty} \frac{b_{n-1-u'}}{b_{n-1}} = S^{u'}.$$

Finally, since $S \to 0$, we conclude that

$$\lim_{n \to \infty} \frac{b_{n-1}}{b_n} = \lim_{n \to \infty} \frac{b_{n-u}/b_n}{b_{n-i-u'}/b_{n-i}}$$

$$= S^{u-u'}$$

$$= S. \quad \blacksquare$$

We now state several theorems which will help us determine which classes satisfy $\mathscr{A} \to R$ and $\mathscr{B} \to S$. The first theorems are due to Hayman [23]. To state those theorems, we must first give Hayman's definition of an admissible function.

DEFINITION. Suppose that $a(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$, where $a_n \in \mathbb{C}$, and for some $R_0 < R$, $a(r) > 0$ when $R_0 < r < R$. Let

$$f(z) = z \frac{a'(z)}{a(z)},$$

$$g(z) = z f'(z).$$

$a(z)$ is *admissible* if

$$\lim_{r \to R} g(r) = \infty$$

and there a function $\delta(r)$ defined for $R_0 < r < R$ with $0 < \delta(r) < \pi$ such that

$$a(re^{i\theta}) \sim a(r) \exp(i\theta f(r) - \tfrac{1}{2}\theta^2 g(r))$$

as $r \to R$, uniformly for $|\theta| \leqslant \delta(r)$, and

$$a(r e^{i\theta}) = o(a(r)/\sqrt{g(r)})$$

as $r \to R$, uniformly for $\delta(r) \leqslant |\theta| \leqslant \pi$ (here r ranges over the reals).

The following theorem (Corollary IV of Hayman [23]) enables us to establish the condition $\mathscr{A} \to R$ in many cases (see Sect. 7).

THEOREM 4.2. *If* $a(z) = \sum_{n=0}^{\infty} a_n z^n$ *is admissible then*

$$\lim_{n \to \infty} a_{n-1}/a_n = R.$$

The reader has no doubt observed that the definition of admissibility is unwieldy and difficult to apply. Hayman proves several theorems, summarized in Theorem 4.3, that allow us to easily show admissibility for a large class of functions.

THEOREM 4.3.   (i)   *If* $p(z)$ *is a polynomial with real coefficients and the coefficients of all large powers in the series* $\exp(p(z))$ *are positive, then* $\exp(p(z))$ *is admissible.*

(ii)   *If* $f(z)$ *and* $g(z)$ *are admissible, then* $f(z) g(z)$ *is admissible.*

(iii)   *If* $f(z)$ *is admissible, then* $\exp(f(z))$ *is admissible.*

(iv)   *If* $f(z)$ *is admissible with radius of convergence* R *and* $h(z)$ *is holomorphic in* $|z| < R$ *with real coefficients and* $\max_{|z| = r} |h(z)| = O(f(r)^{1-\delta})$ *for some* $\delta > 0$*, then* $f(z) + h(z)$ *is admissible.*

(v)   *If* $f(z)$ *is admissible and the leading coefficient of the polynomial* $p(z)$ *is positive, then*

$$p(z) \, f(z) \qquad \text{is admissible.}$$

The next theorem is a special case of a theorem due to Bateman and Erdös [2]. We will use it to show for some classes that $\mathscr{B} \to 1$.

THEOREM 4.4.   *If*

$$b(x) = \sum_{n=0}^{\infty} b_n x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-l_n},$$

where $l_n = 0$ *or* 1 *for all* n*, and the greatest common divisor of the elements of* $\{n \in \omega : l_n \neq 0\}$ *is* 1*, then* $\lim_{n \to 0} b_{n-1}/b_n = 1$.

We close this section with the theorem which will allow us to infer the existence of asymptotic probabilities from the conditions $\mathscr{A} \to R$ or $\mathscr{B} \to S$.

THEOREM 4.5.   *Suppose that* $a(z) = \sum_{n=0}^{\infty} a_n z^n$*,* $a_n \in \mathbb{C}$*, and* $\lim_{n \to \infty} a_{n-k}/a_n = R^k$ *for some* $k > 0$ *and* $R \in \mathbb{R}$*,* $R \geqslant 0$*. Suppose also that* $b(z)/a(z) = c(z^k)$ *for some series* $b(z) = \sum_{n=0}^{\infty} b_n z^n$ *and* $c(z) = \sum_{n=0}^{\infty} c_n z^n$*. If* $c(z^k)$ *has radius of convergence* $S > R$ *and* $\lim_{z \to R} c(z^k) = P$ *then* $\lim_{n \to \infty} b_n/a_n = P$.

This is a slight extension of Theorem 2 in Bender [4]. The proof is a straightforward modification of the proof there (see also Compton [9]).

5. ASYMPTOTIC PROBABILITIES OF FIRST ORDER SENTENCES

This section contains the main results of the paper. Theorems 5.7 and 5.8 show that the existence of asymptotic probabilities for a set of sentences called *component-bounded* sentences is equivalent to the condition $\mathscr{A} \to R$ in the labeled case and $\mathscr{B} \to R$ in the unlabeled case. Theorem 5.9 uses these results to characterize the nonfast growing classes (i.e., those having generating series with positive radius of convergence) with $L_{\omega\omega}$ 0–1 laws.

Theorems 5.7 and 5.8 are effective: when $\mathscr{C}$ is recursive there is an algorithm for computing asymptotic probabilities of component-bounded sentences. Theorem 5.9, however, has no effective version. In Compton [11] it is shown that there is a finitely axiomatizable class, closed under disjoint unions and components, with an $L_{\omega\omega}$ unlabeled 0–1 law, but for which there is no algorithm to compute unlabeled asymptotic probabilities of $L_{\omega\omega}$ sentences. Thus, the general problem of computing asymptotic probabilities is undecidable.

First, we need several lemmas.

LEMMA 5.1. *Let Σ be a set of $L_{\omega\omega}$ sentences closed under conjunctions. For a consistent sentence φ the following are equivalent.*

(a) *φ is logically equivalent to some sentence in Σ.*

(b) *If $\mathfrak{A} \models \varphi$ and every sentence in Σ that holds in $\mathfrak{A}$ holds in $\mathfrak{B}$, then $\mathfrak{B} \models \varphi$.*

The proof is standard in model theory texts and will not be given here (see Chang and Keisler [8] for details).

This lemma is useful for computing asymptotic probabilities when used with

LEMMA 5.2. *Let Σ be a set of $L_{\omega\omega}$ sentences closed under conjunctions. Let Γ be the least set of $L_{\omega\omega}$ sentences containing Σ that is closed under logical equivalence and satisfies the following:*

(i) *Γ contains a valid sentence.*

(ii) *If $\varphi, \psi \in \Gamma$ and $\models \neg(\varphi \wedge \psi)$, then $\varphi \vee \psi \in \Gamma$*

(iii) *If $\varphi, \psi \in \Gamma$ and $\models \varphi \to \psi$ then $\neg\varphi \wedge \psi \in \Gamma$.*

Then Γ is the set of sentences equivalent to Boolean combination of sentences from Σ.

Proof. It is clear that every sentence in Γ is equivalent to a Boolean combination of sentences from Σ. To show the converse, assume that φ is a Boolean combination of sentences from Σ. We may assume that φ is in complete disjunctive normal form: $\varphi = \bigvee_{i < m} \psi_i$, where each ψ_i is a conjunction of sentences and negations of sentences from Σ, and $\models \neg(\psi_i \wedge \psi_j)$ when $i \neq j$. If we can show each ψ_i is in Γ then by (ii) ψ is in Γ. That is, we must show that any sentence ψ which is a conjunction of sentences and negations of sentences from Σ is in Γ. Since Σ is closed under conjunctions, we may assume that

$$\psi = \theta_0 \wedge \neg\theta_1 \wedge \neg\theta_2 \wedge \cdots \wedge \neg\theta_n, \tag{5.1}$$

where either θ_0 is a valid sentence or is in Σ, and $\theta_i \in \Sigma$, $1 \leqslant i \leqslant n$. We show by induction on n that any sentence of the form (5.1) is in Γ. It is certainly true when $n = 0$. Assume by the induction hypothesis that

$$\psi' = \theta_0 \wedge \neg\theta_1 \wedge \neg\theta_2 \wedge \cdots \wedge \neg\theta_{n-1},$$

$$\psi'' = (\theta_0 \wedge \theta_n) \wedge \neg\theta_1 \wedge \neg\theta_2 \wedge \cdots \wedge \neg\theta_{n-1}$$

are in Γ ($\theta_0 \wedge \theta_n \in \Sigma$ since Σ is closed under conjunctions.) But $\models \psi'' \rightarrow \psi'$ so by (iii) $\neg\psi'' \wedge \psi'$, which is equivalent to ψ, is in Γ and so $\psi \in \Gamma$. ∎

The next lemma is an effective version of the last one. We will use it to show the existence of algorithms for computing asymptotic probabilities. The proof is similar; we need only check effectiveness at each step.

LEMMA 5.3. *Let Σ and Γ be recursive sets which satisfy the conditions of the previous lemma. Let f be a function on Γ that is constant on logical equivalence classes. Moreover, suppose f is computable on Σ and satisfies the following.*

(i) *f is computable for some valid sentence.*

(ii) *If $\varphi, \psi \in \Gamma$ and $\models \neg(\varphi \wedge \psi)$ then $f(\varphi \wedge \psi)$ is computable from $f(\varphi)$ and $f(\psi)$.*

(iii) *If $\varphi, \psi \in \Gamma$ and $\models \varphi \rightarrow \psi$ then $f(\neg\varphi \wedge \psi)$ is computable from $f(\varphi)$ and $f(\psi)$.*

Then f is computable on Γ.

The next theorem illustrates how these lemmas are applied.

THEOREM 5.4. *Let $\mathscr{C}$ be any class of L-structures. If every existential connected sentence has an asymptotic probability then every $L_{\omega\omega}$ sentence has an asymptotic probability. If there is an algorithm to compute the asymptotic probability of each existential connected sentence then there is an*

algorithm to compute the asymptotic probability of each $L_{\omega\omega}$ sentence. (Asymptotic probability *may refer to either the labeled or unlabeled probability.*)

Proof. It is not difficult to see that the set Γ of sentences with asymptotic probabilities satisfies (i)–(iii) of Lemma 5.2, and the function f on Γ which maps a sentence to its asymptotic probability satisfies (i)–(iii) of Lemma 5.3. The result follows from Theorem 3.2. ∎

Note that by Theorem 3.1 this theorem is equivalent to the assertion that for any class $\mathscr{C}$, every $L_{\omega\omega}$ sentence has an asymptotic probability if every $L_{\omega\omega}$ sentence preserved under closed substructures has an asymptotic probability.

We now define component-bounded sentences.

DEFINITION. A *component-bounding quantifier* is a quantifier of the form $(\forall^{\leqslant m}x)$ and $(\exists^{\leqslant m}x)$ where m is a positive integer and x is an element variable. The quantifiers say "for all x in components of cardinality $\leqslant m$" and "there exists x in a component of cardinality $\leqslant m$." The intended semantics for these quantifiers should be clear. We will sometimes write $(\forall^{\leqslant m}x_0, x_1,..., x_{k-1})$ for the sequence of quantifiers $(\forall^{\leqslant m}x_0)(\forall^{\leqslant m}x_1)\cdots$ $(\forall^{\leqslant m}x_{k-1})$ and similarly $(\exists^{\leqslant m}x_0, x_1,..., x_{k-1})$ for $(\exists^{\leqslant m}x_0)(\exists^{\leqslant m}x_1)\cdots$ $(\exists^{\leqslant m}x_{k-1})$.

The set of *component-bounded formulas* is the smallest set that contains atomic and negated atomic formulas, and is closed under conjunctions, disjunctions, negations, and component-bounding quantifiers. Note that when L is finite, every component-bounded formula is logically equivalent to a first order formula. $(\forall^{\leqslant m}x)\,\varphi$ may be rewritten with first order quantifiers by saying "for all x if there exist $x_0,..., x_{m-1}$ which satisfy all the atomic and negated atomic sentences true in one of the (finitely many) components of cardinality $\leqslant m$, and no y other than $x_0,..., x_{m-1}$ is related to one of the $x_0,..., x_{m-1}$, and x is one of the $x_0,..., x_{m-1}$, then φ." $(\exists^{\leqslant m}x)\,\varphi$ may be rewritten similarly. The first order translations of component-bounded formulas are, in general, much larger than the original because the number of components of cardinality $\leqslant m$ grows quickly.

We observe that the sentences $\theta_{\mathfrak{A},j}$, which were introduced in Section 2, are logically equivalent to component-bounded sentences. The next lemma shows that component-bounded sentences cannot express much more than this.

LEMMA 5.5. *Every component-bounded sentence is equivalent to a Boolean combination of sentences of the form $\theta_{\mathfrak{A},j}$.*

Proof. We use Lemma 5.2. Let Σ be the set of Boolean combinations of sentences of the form $\theta_{\mathfrak{A},j}$. Since that $\mathfrak{A}\models\varphi$, δ is a component-bounded

sentence, and any sentence in Σ which holds in $\mathfrak{A}$ holds in $\mathfrak{B}$. We are done if we can show that $\mathfrak{B} \models \varphi$.

We may assume $\mathfrak{A}$ and $\mathfrak{B}$ are countable by the Downward Löwenheim-Skolem Theorem (see Chang and Keisler [8]). Since the sets of sentences of the form $\theta_{\mathfrak{R},j}$ true in $\mathfrak{A}$ and $\mathfrak{B}$ are the same, $\mathfrak{A}$ and $\mathfrak{B}$ will have the same number of components isomorphic to each finite $\mathfrak{R}$. Therefore, there is an isomorphism f from the structure composed of finite components in $\mathfrak{A}$ to the structure composed of finite components in $\mathfrak{B}$. A trivial induction on formulas shows that for any component-bounded formula ψ and $a_0,..., a_{n-1} \in A$,

$$\mathfrak{A} \models \psi(a_0,..., a_{n-1}) \text{ iff } \mathfrak{B} \models \psi(fa_0,..., fa_{n-1}).$$

Thus $\mathfrak{B} \models \varphi$. ∎

Recall that for a class $\mathscr{C}$ of L-structures, $a(x)$ denotes the exponential generating series for the class and $b(x)$ denotes the ordinary generating series. The next theorem tells us how to compute the asymptotic probabilities of conjunctions of the $\theta_{\mathfrak{R},j}$'s.

THEOREM 5.6. *Let $\mathscr{C}$ be closed under disjoint unions and components, and $\mathfrak{R}_0,..., \mathfrak{R}_{q-1} \in \mathscr{C}$ be finite and connected with $|\mathfrak{R}_i| = m_i$, $i < q$. Also, let $j_0,..., j_{q-1}$ be nonnegative integers.*

(i) *If $\mathscr{A} \to R$ then*

$$\mu\left(\bigwedge_{i<q} \theta_{\mathfrak{R}_i, j_i}\right) = \prod_{i<q} \frac{\lambda_i^{j_i}}{j_i!} \exp(-\lambda_i),$$

where $\lambda_i = R^{m_i}/\sigma(\mathfrak{R}_i)$. In case $R = \infty$, we take this to mean

$$\mu\left(\bigwedge_{i<q} \theta_{\mathfrak{R}_i, j_i}\right) = 0.$$

(ii) *If $\mathscr{B} \to S$ then*

$$\nu\left(\bigwedge_{i<q} \theta_{\mathfrak{R}_i, j_i}\right) = \prod_{i<q} S^{m_i j_i}(1 - S^{m_i}).$$

Remark. (i) says that the labeled structures have asymptotic component distributions which are Poisson and independent of each other. That is, for each finite connected $\mathfrak{R}$ we let $X_{\mathfrak{R}}$ be a random variable on the space of structures in $\mathscr{C}$ with universe $\subseteq \omega$ and define a probability measure on the space so that

$$P(X_{\mathfrak{R}_0} = j_0,..., X_{\mathfrak{R}_{q-1}} = j_{q-1}) = \mu\left(\bigwedge_{i<q} \theta_{\mathfrak{R}_i, j_i}\right)$$

then the $X_{\mathfrak{R}}$'s have Poisson distribution, and $X_{\mathfrak{R}}$ and $X_{\mathfrak{R}'}$ are independent when $\mathfrak{R}$ is not isomorphic to $\mathfrak{R}'$. The same statement holds for (ii) except that the distribution is geometric rather than Poisson.

Proof of Theorem. First we prove (i) for the case $R < \infty$. Let

$$c_n = \left| \left\{ \mathfrak{A} \in \mathscr{A}_n : \mathfrak{A} \models \bigwedge_{i < q} \theta_{\mathfrak{R}i, q_i} \right\} \right|,$$

$$c(x) = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n.$$

Recall that in Lemma 2.1(ii) we showed that

$$c(x) = a(x) \prod_{i < q} \frac{1}{j_i!} \left(\frac{x^{m_i}}{(\sigma(\mathfrak{R}_i))} \right)^{j_i} \exp\left(-\frac{x^{m_i}}{\sigma(\mathfrak{R}_i)} \right). \tag{5.2}$$

If $R > 0$ then by Proposition 4.1(i)

$$\lim_{n \to \infty} \frac{a_{n-1}/(n-1)!}{a_n/n!} = R$$

and so by Lemma 4.5(i)

$$\lim_{n \to \infty} \frac{c_n}{a_n} = \lim_{n \to \infty} \frac{c_n/n!}{a_n/n!}$$

$$= \lim_{x \to R} \frac{c(x)}{a(x)}$$

$$= \prod_{i < q} \frac{\lambda_i^{j_i}}{j_i!} \exp(-\lambda_i)$$

since the radius of convergence of $c(x)/a(x)$ is ∞. If $R = 0$ we compute $\mu(\theta_{\mathfrak{R},j})$ from Lemma 4.5(i) by setting $q = 1$ in (5.2). Then

$$\frac{c(x)}{a(x)} = \frac{1}{j!} \left(\frac{x^m}{\sigma(\mathfrak{R})} \right)^j \exp\left(-\frac{x^m}{\sigma(\mathfrak{R})} \right)$$

is a power series in x^m. Since $|\mathfrak{R}| = m$ and $\mathscr{A} \to 0$ we know that $\lim_{n \to \infty} (a_{n-m}/(n-m)!)/(a_n/n!) = 0$. Hence

$$\mu(\theta_{\mathfrak{R},j}) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The result follows easily from this.

To prove the case $R = \infty$ we find an upper bound for $\mu_n(\theta_{\Re,j})$. Let

$$c_{j,n} = |\{\mathfrak{A} \in \mathscr{A}_n : \mathfrak{A} \models \theta_{\Re,j}\}|$$

and suppose that $|\Re| = m$. Then by Lemma 2.1(iii)

$$\frac{c_{j,n-m}}{(n-m)!} = (j+1)\,\sigma(\Re)\,\frac{c_{j+1,n}}{n!}$$

and hence

$$\frac{c_{j,n-m}}{a_{n-m}} = (j+1)\,\sigma(\Re)\,\frac{c_{j+1,n/n!}}{a_{n-m}/(n-m)!}$$

$$\leqslant (j+1)\,\sigma(\Re)\left(\frac{a_{n-m}/(n-m)!}{a_n/n!}\right)^{-1}.$$

Since $\mathscr{A} \to \infty$, the right side of this inequality approaches 0 and $n \to \infty$. We conclude that $\mu(\theta_{\Re,j}) = 0$.

Part (ii) follows from Lemma 4.5(i) applied to Lemma 2.1(v). Note that $S \leqslant 1$ so we do not have to consider the case where the radius of convergence is infinite as we did for (i). ∎

Theorem 5.6 tells us that the asymptotic probabilities of the conjunctions of $\theta_{\Re,j}$'s exist when $\mathscr{A} \to R$ or $\mathscr{B} \to S$. It is not difficult to extend this to component-bounded sentences. The following theorems show the hypotheses cannot be weakened; for $\mathscr{C}$ closed under disjoint unions and components, the existence of asymptotic probabilities for component-bounded sentences is equivalent to $\mathscr{A} \to R$ or $\mathscr{B} \to S$.

THEOREM 5.7. *The following are equivalent for $\mathscr{C}$ closed under disjoint unions and components.*

  (i) $\mathscr{A} \to R$ *for some extended real* R.

  (ii) $\mu(\varphi)$ *exists for every sentence* φ *of the form* $\bigwedge_{i<q} \theta_{\Re_i,j_i}$.

  (iii) $\mu(\varphi)$ *exists for every component-bounded sentence* φ.

When these conditions hold, we may calculate $\mu(\varphi)$ *for any component-bounded sentence* φ *by setting*

$$c_n = |\{\mathfrak{A} \in \mathscr{A}_n : \mathfrak{A} \models \varphi\}|,$$

$$c(x) = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n.$$

Then $\mu(\varphi) = \lim_{x \to R} c(x)/a(x)$. *If $\mathscr{C}$ is recursive, there is an algorithm for*

computing $\mu(\varphi)$ *for component-bounded* φ *in terms of* R, *exp and arithmetic operations.*

Proof. (i) $\Rightarrow$ (ii) is Theorem 5.6(i).
(ii) $\Rightarrow$ (iii) follows immediately from Lemmas 5.2 and 5.5.
(iii) $\Rightarrow$ (i) We must show that

$$\lim_{n \to \infty} \frac{a_{n-m}/(n-m)!}{a_n/n!} = R^m$$

for each m such that there is a connected structure $\mathfrak{R} \in \mathscr{C}$, $|\mathfrak{R}| = m$. We know that $\mu(\theta_{\mathfrak{R},j})$ exists for each j since $\theta_{\mathfrak{R},j}$ is a component-bounded sentence. First, suppose that $\mu(\theta_{\mathfrak{R},j}) = 0$ for all $j \in \omega$. Recall that

$$c_j(x) = \sum_{n=0}^{\infty} \frac{c_{j,n}}{n!} x^n$$

is the exponential generating series for the subclass of structures that satisfy $\theta_{\mathfrak{R},j}$, $|\mathfrak{R}| = m$, and

$$\frac{c_{j,n-m}}{(n-m)!} = (j+1)\, \sigma(\mathfrak{R}) \frac{c_{j+1,n}}{n!}$$

by Lemma 2.1(iii). But

$$\frac{a_n}{n!} = \sum_{j=0}^{\infty} \frac{c_{j,n}}{n!}$$

and hence

$$\frac{a_n}{n!} = \frac{c_{0,n}}{n!} + \frac{1}{\sigma(\mathfrak{R})} \sum_{j=0}^{\infty} \frac{1}{j+1} \frac{c_{j,n-m}}{(n-m)!}.$$

Subtracting $c_{0,n}/n!$ from both sides and multiplying by $(a_{n-m}/(n-m)!)^{-1}(1 - c_{j,n}/a_n)^{-1}$ we have

$$\left(\frac{a_{n-m}/(n-m)!}{a_n/n!} \right)^{-1} = \left(1 - \frac{c_{j,n}}{a_n} \right)^{-1} \frac{1}{\sigma(\mathfrak{R})} \sum_{j=0}^{\infty} \frac{c_{j,n-m}}{a_{n-m}}. \tag{5.3}$$

For a given $\varepsilon > 0$ choose N large enough that $1/(N+1) < \varepsilon/2$. Since $\sum_{j=0}^{\infty} c_{j,n-m}/a_{n-m} = 1$,

$$\sum_{j=0}^{\infty} \frac{1}{j+1} \frac{c_{j,n-m}}{a_{n-m}} \leqslant \sum_{j=0}^{N} \frac{1}{j+1} \frac{c_{j,n-m}}{a_{n-m}} + \frac{\varepsilon}{2}.$$

By hypothesis, $\lim_{n \to \infty} c_{j,n-m}/a_{n-m} = 0$ so we may make this sum less than

ε by choosing n large enough. Thus the right side of (5.3) can be made arbitrarily small so

$$\lim_{n \to \infty} \frac{a_{n-m}/(n-m)!}{a_n/n!} = \infty.$$

Now suppose there is a j such that $\mu(\theta_{\Re,j}) \neq 0$. We again use

$$\frac{c_{j,n-m}}{(n-m)!} = (j+1)\, \sigma(\Re)\, \frac{c_{j+1,n}}{n!}$$

to deduce that

$$\frac{a_{n-m}/(n-m)!}{a_n/n!} = (j+1)\, \sigma(\Re)\, \frac{c_{j+1,n}/a_n}{c_{j,n-m}/a_{n-m}}.$$

Since the denominator of the fraction on the right side of this equality does not approach 0, the right side approaches a limit.

We have now shown that for any $\Re$ with $|\Re| = m$, $(a_{n-m}/(n-m)!)/(a_n/n!)$ approaches a limit. The limit must be R^m, where R is the radius of convergence of $a(x)$. Thus $\mathscr{A} \to R$.

To show the latter part of the theorem, we note that the method described to calculate $\mu(\varphi)$ works for $\varphi = \theta_{\Re,j}$ by Theorem 5.6(i). Next observe that the set of sentences for which this method gives labeled asymptotic probabilities satisfies conditions (i)–(iii) of Lemma 5.2 and therefore by Lemma 5.5 the method works for component-bounded sentences. The existence of an algorithm to compute labeled asymptotic probabilities follows from Lemma 5.3. ∎

Of course, there is an analogue to this theorem in the unlabeled case.

THEOREM 5.8. *The following are equivalent for $\mathscr{C}$ closed under disjoint unions and components.*

   (i) *$\mathscr{B} \to S$ for some real S, $0 \leqslant S \leqslant 1$.*

   (ii) *$\nu(\varphi)$ exists for every sentence φ of the form $\bigwedge_{i < q} \theta_{\Re_i, j_i}$.*

   (iii) *$\nu(\varphi)$ exists for every component-bounded sentence φ.*

When these conditions hold for $\mathscr{C}$, we may calculate $\nu(\varphi)$ for any component-bounded sentences φ by setting

$$d_n = |\{\mathfrak{A} \in \mathscr{B}_n : \mathfrak{A} \models \varphi\}|,$$

$$d(x) = \sum_{n=0}^{\infty} d_n x^n.$$

Then $v(\varphi) = \lim_{x \to S} d(x)/b(x)$. *If* $\mathscr{C}$ *is recursive, there is an algorithm for computing* $v(\varphi)$ *for component-bounded* φ *in terms of* S *and arithmetic operations.*

The proof is identical to the proof of Theorem 5.7.

The next theorem tells when nonfast growing classes will have $L_{\omega\omega}$ 0–1 laws.

THEOREM 5.9.   *Let* $\mathscr{C}$ *be closed under disjoint unions and components.*

   (i)   *Suppose that the radius of convergence of* $a(x)$, *the exponential generating series for* $\mathscr{C}$, *is greater than* 0. *Then* $\mathscr{C}$ *has an* $L_{\omega\omega}$ *labeled* 0–1 *law iff* $\mathscr{A} \to \infty$.

   (ii)   *Suppose that the radius of convergence of* $b(x)$, *the ordinary generating series for* $\mathscr{C}$, *is greater than* 0. *Then* $\mathscr{C}$ *has an* $L_{\omega\omega}$ *unlabeled* 0–1 *law iff* $\mathscr{B} \to 1$ (*i.e., iff* $\lim_{n \to \infty} b_{n-1}/b_n = 1$).

Proof. Again, the proofs for the labeled and unlabeled cases are nearly identical, so we prove only (i).

Suppose that $\mathscr{C}$ has a labeled 0–1 law. Then by Theorem 5.7 $\mathscr{A} \to R$ for some R, since every component-bounded sentence has an asymptotic probability. Theorem 5.6 shows that $R = \infty$, since there will be labeled asymptotic probabilities not equal to 0 or 1 when $0 < R < \infty$.

Now suppose $\mathscr{A} \to \infty$. To show that $\mathscr{C}$ has a labeled 0–1 law, we show that there is a complete $L_{\omega\omega}$ theory with axioms each having labeled asymptotic probability 1. By Gödel's Compactness Theorem (see Chang and Keisler [8]) any consequence of the theory is a consequence of finitely many of the axioms and so will also have probability 1, its negation then having probability 0. By Theorem 3.4.

$$\{\neg \theta_{\mathfrak{K},j} : \mathfrak{K} \in \mathscr{C} \text{ is connected}, j \in \omega\} \cup \bigcap_{\substack{\mathfrak{A} \in \mathscr{C} \\ \text{finite}}} \text{Th}(\mathfrak{A})$$

is complete. Theorem 5.6(i) says that $\mu(\neg \theta_{\mathfrak{K},j}) = 1$ when $\mathscr{A} \to \infty$. Any sentence that is true in all finite structures will also have labeled asymptotic probability 1. We conclude that $\mathscr{C}$ has an $L_{\omega\omega}$ labeled 0–1 law.   ∎

6. ASYMPTOTIC PROBABILITIES FOR CLASSES WITH COINFINITE SPECTRA

In this section, we show how to extend the theorems in Section 5 to classes for which asymptotic probabilities fail to exist for the trivial reason that there are infinitely many positive integers that are not cardinalities of structures in the class.

DEFINITION. Let $\mathscr{C}$ be a class of L-structures. Denote by $\mathrm{sp}(\mathscr{C})$ the *finite spectrum* of $\mathscr{C}$; this is the set of integers that are cardinalities of structures in $\mathscr{C}$.

Let $k_0 < k_1 < \cdots$ be the integers in $\mathrm{sp}(\mathscr{C})$ and φ be a sentence. Set

$$a_n = |\mathscr{A}_n|,$$
$$c_n = |\{\mathfrak{A} \in \mathscr{A}_n : \mathfrak{A} \models \varphi\}|.$$

We define

$$\mu^*(\varphi) = \lim_{n \to \infty} \frac{c_{k_n}}{a_{k_n}}$$

to be the *generalized labeled asymptotic probability* of φ whenever this limit exists. We define $v^*(\varphi)$, the *generalized unlabeled asymptotic probability* of φ, analogously. We also define the notions of *generalized 0–1 laws* in the obvious manner.

Classes closed under disjoint unions and components have nice spectra. Let d be the greatest common divisor of the elements of $\mathrm{sp}(\mathscr{C})$. From elementary number theory we know $\mathrm{sp}(\mathscr{C})$ will contain all large multiples of d. We call d the *period* of $\mathscr{C}$.

We now extend the notions $\mathscr{A} \to R$ and $\mathscr{B} \to S$. Let $\mathscr{C}$ be closed under disjoint unions and components, and d be the period of $\mathscr{C} \cdot \mathscr{A} \xrightarrow{*} R$ means that

$$\lim_{n \to \infty} \frac{a_{nd-m}/(nd-m)!}{a_{nd}/(nd)!} = R^m$$

for each m such that there is a connected structure of cardinality m in $\mathscr{C}$. $B \xrightarrow{*} S$ is defined similarly. We state a simple proposition which shows that in most cases $\mathscr{A} \xrightarrow{*} R$ ($\mathscr{B} \xrightarrow{*} S$) is equivalent to the condition that the ratio of the successive non-zero coefficients of the exponential (ordinary) generating series approaches a limit. The proof is a straightforward modification of the proof of Proposition 4.1.

PROPOSITION 6.1. *Suppose that $\mathscr{C}$ is closed under disjoint unions and components and has period d.*

(i) *If $a(x)$ has radius of convergence strictly between 0 and ∞ then $\mathscr{A} \xrightarrow{*} R$ iff*

$$\lim_{n \to \infty} \frac{a_{(n-1)d}/((n-1)d)!}{a_{nd}/(nd)!} = R^d.$$

(ii) *If $b(x)$ has radius of convergence strictly greater than 0 then*
$\mathscr{B} \xrightarrow{*} S$ *iff*

$$\lim_{n \to \infty} \frac{b_{(n-1)d}}{b_{nd}} = S^d.$$

We can now prove generalizations of Theorems 5.6–5.9. In these theorems replace every occurrence of $\to$ with $\xrightarrow{*}$, of μ with μ^*, of ν with ν^*, and of 0–1 *law* with *generalized* 0–1 *law*. The proofs are nearly the same; Theorem 4.5 was stated in a sufficiently general form to apply here.

7. EXAMPLES

In this section, we shall present examples of classes closed under disjoint unions and components. For each it is possible to give a set of axioms of the form (5.1) but we leave that to the interested reader. We describe the growth rates of the class using the notation of section 4.

EXAMPLE 7.1. The class of all L-structures. If L contains only unary relation symbols then $\mathscr{A} \to \infty$ and $\mathscr{B} \to 1$; otherwise $\mathscr{A} \to 0$ and since $b_n \sim a_n/n!$ (see Fagin [16]) $\mathscr{B} \to 0$. This is the class for which Fagin [16] demonstrated labeled and unlabeled 0–1 laws.

EXAMPLE 7.2. The class of directed graphs. L contains just one relation symbol R. $R(x, y)$ will mean "there is a directed edge from x to y." The class may be described by just one axiom which says "no element is connected to itself by an edge." This class is virtually the same as the previous example. $\mathscr{A} \to 0$, $\mathscr{B} \to 0$, and a minor modification of Fagin's argument shows that it has both a labeled and unlabeled 0–1 law.

EXAMPLE 7.3. The class of graphs. To the previous example add an axiom saying R is symmetric. The remarks for the previous example apply here.

EXAMPLE 7.4. The class of oriented graphs. These are directed graphs in which there is at most one edge between any two vertices. Again, $\mathscr{A} \to 0$, $\mathscr{B} \to 0$, and labeled and unlabeled 0–1 laws pertain.

EXAMPLE 7.5. The class of unary functions. Since we do not allow function symbols in L, we use a binary relation symbol R instead and stipulate that $R(x, y)$ means "x is mapped onto y." This is the class considered by Metropolis and Ulam [30] and Katz [25]. Lynch [29] has

shown that $\mu(\varphi)$ exists for every first order sentence about this class. $a_n = n^n$ so

$$\lim_{n \to \infty} \frac{a_{n-1}/(n-1)!}{a_n/n!} = \frac{1}{e}.$$

That is, $\mathscr{A} \to 1/e$. Harary and Palmer [22] derive an expression for the ordinary generating series of this class from which it can be shown that $\mathscr{B} \to S$ for some $S > 0$. Thus, component bounded sentences have unlabeled asymptotic probabilities; it is not known if this is true for all $L_{\omega\omega}$ sentences.

EXAMPLE 7.6. The class of permutations. Take the axioms in the previous example together with an axiom that says that R is one-to-one, (or onto—for finite structures they are equivalent). Since $a_n = n!$, we know $a(x) = (1-x)^{-1}$ and $\mathscr{A} \to 1$. There is exactly one unlabeled connected structure (a cycle) of each positive finite cardinality so by Lemma 2.1(iv),

$$b(x) = \prod_{n \geqslant 1} (1 - x^n)^{-1}.$$

Theorem 4.2 implies $\mathscr{B} \to 1$. Thus, by Theorems 5.7 and 5.8, the class has an unlabeled 0–1 law but not a labeled 0–1 law. Asymptotic probabilities for labeled permutations are investigated in Goncharov [20], and Shepp and Lloyd [37].

EXAMPLE 7.7. The class of partial orders. They satisfy the reflexive, antisymmetric and transitive axioms. The asymptotics for labeled partial orders are worked out in Kleitman and Rothschild [26]. Their results show immediately that $\mathscr{A} \to 0$. It is possible to show that $\mathscr{B} \to 0$, and that labeled and unlabeled 0–1 laws hold (see Compton [13]).

EXAMPLE 7.8. The class of oriented forests. An *oriented tree* (this term is from Knuth [27]) is a tree with a distinguished node called the root. These trees are often called *rooted trees* in the literature. A forest is just a set of trees (only mathematicians and a few visionary poets would refer to a single tree as a forest; the null forest is a refuge for those who find inner peace contemplating nothingness). We identify oriented forests with partial orders that have the property that all the elements below any given element are linearly ordered.

This is the class first considered by Cayley. The exponential generating series for the class is among the most well known in enumerative combinatorics:

$$a(x) = \sum_{n=0}^{\infty} \frac{n^{n-1}}{n!} x^n$$

(see, e.g., Harary and Palmer [22]). Thus, $\mathscr{A} \to 1/e$. It is not difficult to show, using the same techniques, that the ordinary generating series $l(x)$ for the subclass of connected structures (trees) is implicitly given by

$$l(x) = x \exp\left(\sum_{m \geqslant 1} \frac{1}{m} l(x^m) \right).$$

From this and Lemma 2.1(iv) one can show that $\mathscr{B} \to S$ for some S, $0 < S < 1$. It is not known whether $\mu(\varphi)$ and $\nu(\varphi)$ exist for all first order φ.

EXAMPLE 7.9. Oriented binary forests. To the previous example add an axiom which says that every element has either 0 or 2 immediate successors. Letting $k(x)$ be the exponential generating series for the class of oriented binary trees, we can show that

$$x\left(1 + \frac{k(x)^2}{2} \right) = k(x).$$

We solve to get

$$k(x) = \frac{1 - \sqrt{1 - 2x^2}}{x}$$

so

$$a(x) = \exp\left(\frac{1 - \sqrt{1 - 2x^2}}{x} \right). \tag{7.1}$$

It is *not* the case that $\mathscr{A} \to R$ for any R. To find the ordinary generating series $l(x)$ for the class of oriented binary trees, we have to be a little more careful. We can show that

$$x\left(1 + \frac{l(x)^2 + l(x^2)}{2} \right) = l(x),$$

$$b(x) = \exp\left(\sum_{m \geqslant 1} \frac{1}{m} l(x^m) \right).$$

It is possible, using methods described Otter [33], to get an asymptotic expression for the number of unlabeled oriented binary forests. We shall not carry out these computations, but we note that it is false that $\mathscr{B} \to S$. Therefore, there is a component-bounded sentence without a labeled asymptotic probability and a component-bounded sentence without an unlabeled asymptotic probability.

EXAMPLE 7.10. The class of oriented unary-binary forests. This example shows how small changes in axioms may drastically change asymptotic properties. Rather than the axiom added in the previous example we add an axiom which says that every element has 0, 1, or 2 immediate successors. The methods of the previous example carry over. We find that

$$x\left(1 + k(x) + \frac{k(x)^2}{2}\right) = k(x),$$

$$k(x) = (1 - x - \sqrt{1 - 2x - x^2})/x,$$

$$a(x) = \exp((1 - x - \sqrt{1 - 2x - x^2})/x). \qquad (7.2)$$

From this one can show that $\mathscr{A} \to -1 + \sqrt{2}$. Also,

$$x\left(1 + l(x) + \frac{l(x)^2 + l(x^2)}{2}\right) = l(x),$$

$$b(x) = \exp\left(\sum_{m \geqslant 1} \frac{1}{m} l(x^m)\right).$$

From this one can show that $\mathscr{B} \to S$, $0 < S < 1$, in contrast to the previous example. It is not known whether $\mu(\varphi)$ and $\nu(\varphi)$ exist for all first order φ.

EXAMPLE 7.11. Classes of ordered forests. When mathematicians use the term *tree*, they often implicitly assume a linear order on the nodes of the tree. If we take R to be any of the forest relations described above (Examples 7.8–7.10), we can specify that a binary relation S is a linear order on each tree and require that the order S is inherited by successors. Now to specify an order uniquely, we must decide the ordering of each node with respect to its successors. The most obvious choices are to have a node come before its successors (this is called a preorder) or after its successor (this is called a postorder). We define another order for binary or unary–binary forest by putting a node between its two immediate successors to form an *inorder* (sometimes called *symmetric order*). See Knuth [27] for details on the properties of these ordered trees.

Ordered forests are easier to work with than unordered forests because their ordinary generating series are a little simpler. Since each finite ordered tree has only trivial automorphisms, the exponential and ordinary generating series for the subclass of connected structures are the same. We summarize the pertinent facts for different classes of ordered forests.

Oriented forests with pre- or postorder:

$$k(x) = l(x) = (1 - \sqrt{1 - 4x})/2;$$

$$a(x) = \exp(k(x)), \qquad \mathscr{A} \to \tfrac{1}{4};$$

$$b(x) = \exp\left(\sum_{m \geq 1} \frac{1}{m} l(x^m)\right), \qquad \mathscr{B} \to \tfrac{1}{4}.$$

Binary forests with pre-, post-, or inorder:

$$k(x) = l(x) = (1 - \sqrt{1 - 4x^2})/2x;$$

$$a(x) = \exp(k(x)) \qquad \text{false that } \mathscr{A} \to R;$$

$$b(x) = \exp\left(\sum_{m \geq 1} \frac{1}{m} l(x^m)\right) \qquad \text{false that } \mathscr{B} \to S.$$

Unary–binary forests with pre- or postorder:

$$k(x) = l(x) = (1 - x - \sqrt{1 - 2x - 3x^2})/2x;$$

$$a(x) = \exp(k(x)), \qquad \mathscr{A} \to \tfrac{1}{3};$$

$$b(x) = \exp\left(\sum_{m \geq 1} \frac{1}{m} l(x^m)\right), \qquad \mathscr{B} \to \tfrac{1}{3}.$$

Unary–binary forests with inorder:

$$k(x) = l(x) = (1 - 2x - \sqrt{1 - 4x})/2x;$$

$$a(x) = \exp(k(x)), \qquad \mathscr{A} \to \tfrac{1}{4};$$

$$b(x) = \exp\left(\sum_{m \geq 1} \frac{1}{m} l(x^m)\right), \qquad \mathscr{B} \to \tfrac{1}{4}.$$

It is clear that in the case of binary forest with orders that there is a component-bounded sentence without a labeled asymptotic probability and a component-bounded sentence without an unlabeled asymptotic probability. In the other cases it is not known whether $\mu(\varphi)$ and $\nu(\varphi)$ exist for all first order φ.

EXAMPLE 7.12. Oriented forests of height 1. This example illustrates the power of Theorem 4.3. To the axioms for oriented forests (Example 7.8) add an axiom that says all nodes are distance at most 1 from a root. There are n labeled connected structures of cardinality n so $k(x) = \sum_{n=1}^{\infty} x^n/(n-1)! = xe^x$ and thus $a(x) = \exp(xe^x)$. It is easy to see by Theorem 4.3 that this function is admissible so $\mathscr{A} \to \infty$ by Theorem 4.2. Also, there is exactly one connected structure of each finite cardinality so

$b(x) = \prod_{n \geqslant 1} (1 - x^n)^{-1}$ and $\mathcal{B} \to 1$ by Theorem 4.4. Thus, this class has both a labeled and unlabeled $L_{\omega\omega}$ 0–1 law.

THEOREM 7.13.   The class of free forests. Graph theorists usually reserve the term *forest* for graphs without cycles. Knuth [27] uses the term *free tree* for connected graphs without cycles and so by extension we use *free forest* for the graph theorists' forest. This is our ony example which is not finitely axiomatizable. One possible set of axioms for this class consists of the axioms for graphs in Example 7.3 and a sentence for every n saying "there is no cycle of length n." We saw in Example 7.8 that there are n^{n-1} labeled oriented trees of cardinality n. Clearly, the number of labeled free trees must be n^{n-2} because of the obvious correspondence between oriented trees and free trees with a distinguished point. Hence

$$a(x) = \exp\left(\sum_{n=0}^{\infty} \frac{n^{n-2}}{n!} x^n \right).$$

It can be shown that $\mathscr{A} \to 1/e$. Otter [33] enumerated unlabeled free trees and found an asymptotic expression for the growth of b_n. His results imply that $\mathscr{B} \to S$ for some S, $0 < S < 1$. It is not known whether $\mu(\varphi)$ and $v(\varphi)$ exist for all first order φ.

EXAMPLE 7.14.   The class of linear forests. Each tree is a linear order. To the axioms for oriented forests (Example 7.8) add an axiom saying that two nodes with a common ancestor are comparable. There are $n!$ labeled linear orders of each cardinality $n > 0$ so $a(x) = \exp(x/(1-x))$. It is not difficult to show that $a(x)$ is admissible (taking $\delta(r) = (1-r)^{7/5}$ in the definition of admissibility) so $\mathscr{A} \to 1$. Since there is exactly one unlabeled linear order of each finite cardinality,

$$b(x) = \prod_{n \geqslant 1} (1 - x^n)^{-1}.$$

$\mathscr{B} \to 1$ by Theorem 6.5. Thus, this class has both a labeled and unlabeled $L_{\omega\omega}$ 0–1 law.

EXAMPLE 7.15.   Classes with finitely many connected structures. Given finite connected L-structures $\mathfrak{R}_0, \mathfrak{R}_1,..., \mathfrak{R}_{q-1}$, we can say "each component is isomorphic to some $\mathfrak{R}_i$, $i < q$" with a first order sentence. Suppose $|\mathfrak{R}_i| = m_i$, $i < q$. If the m_i's are relatively prime, then by Theorems 6.3 and 6.4, $\mathscr{A} \to \infty$. Also, under this hypothesis we can easily show that $\mathscr{B} \to 1$ by partial fraction decomposition of the ordinary generating series

$$b(x) = \prod_{i < q} (1 - x^{m_i})^{-1}$$

and explicit evaluation of the series (this is done in Bateman and Erdös [2]). Thus, this class has both a labeled and unlabeled 0–1 law.

EXAMPLE 7.16.   The class of equivalence relations or partitions. The familiar axiomatization says the relation is reflexive, symmetric, and transitive. The exponential generating series is

$$a(x) = \exp(e^x - 1)$$

since there is one connected partition of each finite cardinality. The a_n's are the Bell numbers. De Bruijn [6] and Bender [4] find an asymptotic expression for the sequence of Bell numbers to illustrate asymptotic methods. We invoke Theorems 4.2 and 4.3 to show $\mathscr{A} \to \infty$. It is also clear from Theorem 4.4 that $\mathscr{B} \to 1$. Thus, this class has both a labeled and unlabeled $L_{\omega\omega}$ 0–1 law.

EXAMPLE 7.17.   Partitions with selected subsets. This example appears in Roman and Rota [36]. The language has two relation symbols, R, which interprets an equivalence relation, and S, a unary relation symbol. Add an axiom which says that each R-equivalence class contains an element which is in S. Now any set of cardinality n has $2^n - 1$ nonempty subsets so the exponential generating series for the connected structures in this class is

$$k(x) = \sum_{n=1}^{\infty} \frac{2^n - 1}{n!} x^n = e^x(e^x - 1).$$

We have

$$a(x) = \exp(e^x(e^x - 1)).$$

This is admissible by Theorem 4.3 so $\mathscr{A} \to \infty$. There are n connected unlabeled structures of cardinality n so

$$b(x) = \prod_{n \geqslant 1} (1 - x^n)^{-n}.$$

It follows from a theorem of Meinardus that $\mathscr{B} \to 1$ (see Andrews [1, Chap. 11, Ex. 6). Thus, this class has both a labeled and unlabeled 0–1 law.

8. EPILOGUE: OPEN PROBLEMS

In the following $\mathscr{C}$ is closed under disjoint unions and components.

QUESTION 8.1.   Does $\mathscr{A} \to R$ imply $\lim_{n \to \infty} (a_{n-1}/(n-1)!)/(a_n/n!) = R$ when $R = 0$ or ∞? Does $\mathscr{B} \to 0$ imply $\lim_{n \to \infty} b_{n-1}/b_n = 0$? An affirmative

answer to these questions would simplify our definitions of $\mathscr{A} \to R$ and $\mathscr{B} \to S$ in Section 4.

QUESTION 8.2. Is it true that $\mathscr{A} \xrightarrow{*} \infty$ whenever the exponential generating series for $\mathscr{C}$ has radius of convergence ∞? Is it true that $\mathscr{B} \to 1$ whenever the ordinary generating series has radius of convergence 1? An affirmative answer would imply, by Theorem 5.9, that all slow growing classes closed under disjoint unions and components have $L_{\omega\omega}$ 0–1 laws. We conjecture that the answer is negative.

QUESTION 8.3. Find easily verifiable sufficient conditions for $\mathscr{A} \xrightarrow{*} \infty$ and $\mathscr{B} \xrightarrow{*} 1$. For example, if l_n is the number of unlabeled connected structures in $\mathscr{C}$ and $l_n = O(n^k)$ for some $k \geqslant 0$ then is is true that $\mathscr{B} \xrightarrow{*} 1$? A result like this would be a useful extension of Theorem 4.4. Is there a similar theorem for the labeled case that could be used in place of admissibility.

QUESTION 8.4. We saw in Example 7.6 that a class may have an unlabeled 0–1 law but not a labeled one. Are there any natural examples of classes with a labeled 0–1 law but not an unlabeled one? One approach to this problem is to find a class for which $\mathscr{A} \to \infty$, $\mathscr{B} \to S < 1$. A priori considerations suggest that the finite connected structures for such a class would have many symmetries.

QUESTION 8.5. Note that S, the radius of convergence of the ordinary generating series for $\mathscr{C}$, cannot exceed R, the radius of convergence of the exponential generating series, since $a_n/n! \leqslant b_n$. Thus, if $R = 0$ then $S = 0$ and there is a possibility that the class has both a labeled and unlabeled 0–1 law. Is it true that when $R = 0$ the class has a labeled 0–1 iff it has an unlabeled 0–1 ?

REFERENCES

1. G. E. ANDREWS, "Theory of Partitions," Addison–Wesley, Reading, Mass., 1976.
2. P. BATEMAN AND P. ERDÖS, Monotonicity of partition problems, *Mathematika* 3 (1956), 1–14.
3. E. T. BELL, Postulational bases for the umbral calculus, *Amer. J. Math.* 62 (1940), 717–724.
4. E. A. BENDER, Asymptotic methods in enumeration, *SIAM Rev.* 16 (1974), 485–515.
5. E. A. BENDER AND J. R. GOLDMAN, Enumerative uses of generating functions, *Indiana Math. J.* 20 (1971), 753–765.
6. N. G. DE BRUIJN, "Asymptotic Methods in Analysis," North-Holland, Amsterdam, 1958.
7. A. CAYLEY, "Collected Papers," Cambridge Univ. Press, Cambridge, 1897.

8. C. C. CHANG AND H. J. KEISLER, "Model Theory," North-Holland, Amsterdam, 1973.
9. K. COMPTON, "Applications of Logic to Finite Combinatorics," Ph. D. thesis, University of Wisconsin, 1980.
10. K. COMPTON, Some useful preservation theorems, *J. Symbolic Logic* **48** (1983), 427–440.
11. K. COMPTON, An undecidable problem in finite combinatorics, *J. Symbolic Logic*, in press.
12. K. COMPTON, A logical approach to asymptotic combinatorics. II. Monadic second order properties, submitted.
13. K. COMPTON, the computational complexity of asymptotic problems. I. Partial orders, in preparation.
14. P. ERDŐS, "The Art of Counting" (J. Spencer, Ed.), M.I.T. Press, Cambridge, Mass., 1973.
15. P. ERDŐS AND J. SPENCER, "Probabilistic Methods in Combinatorics," Academic Press, New York, 1974.
16. R. FAGIN, Probabilities on finite models, *J. Symbolic Logic* **41** (1976), 50–58.
17. R. FAGIN, The number of finite relational structures, *Discrete Math.* **19** (1977), 17–21.
18. D. FOATA, "La Série Génératice Exponentielle dans les Problèmes d'Enumération," Les Presses de l'Université de Montréal, Montréal, 1974.
19. E. N. GILBERT, Enumeration of labelled graphs, *Can. J. Math.* **8** (1956), 405–411.
20. V. GONCHAROV, Sur la distribution des cycles dans les permutations, *C. Acad. Sci. USSR* **35** (1942), 267–269.
21. E. GRANDJEAN, Complexity of the first-order theory of almost all finite structures, *Inform. and Control* **57** (1983), 180–204.
22. F. HARARY AND E. PALMER, "Graphical Enumeration," Academic Press, New York, 1973.
23. W. K. HAYMAN, A generalisation of Stirling's formula, *J. Reine Angew. Math.* **196** (1956), 67–95.
24. D. N. HOOVER, Probabilities on models of universal sentences, preprint.
25. L. KATZ, The problem of in decomposability of random matching functions, *Ann. Math. Stat.* **26** (1955), 512–517.
26. D. J. KLEITMAN AND B. L. ROTSCHILD, Asymptotic enumeration and partial orders on a finite set, *Trans. Amer. Math. Soc.* **205** (1975), 205–220.
27. D. E. KNUTH, "The Art of Computer Programming," Vol. 1, Addison–Wesley, Reading, Mass., 1968.
28. J. F. LYNCH, Almost sure theories, *Ann. Math. Logic* **18** (1980), 91–135.
29. J. F. LYNCH, Probabilities of first-order sentences about unary functions, preprint.
30. N. METROPOLIS AND S. ULAM, A property of randomness in an arithmetical function, *Amer. Math. Monthly* **61** (1954), 392–397.
31. R. MULLIN AND G.-C. ROTA, On the foundations of combinatorial theory III, "Graph Theory and its Applications" (Harris, Ed.), Academic Press, New York, 1970.
32. J. MYCIELSKI, A 0–1 law in finite space, *Notices Amer. Math. Soc.* **23** (1976), A–288.
33. R. OTTER, The number of trees, *Ann. Math.* **49** (1948), 583–599.
34. G. POLYA, Kombinatorische Anzahlbestimmen für Gruppen, and chemische Verbindungen, *Acta Math.* **27** (1978), 146–254.
35. J. RIORDAN, Forests of labelled trees, *J. Combin. Theory* **5** (1968), 90–103.
36. S. ROMAN AND G.-C. ROTA, The umbral calculus, *Advan. in Math.* **27** (1978), 95–188.
37. L. A. SHEPP AND S. P. LLOYD, Ordered cycle lengths in random permutations, *Trans. Amer. Math. Soc.* **121** (1966), 340–357.