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Guaranteed Control of Sampled Switched Systems using Semi-Lagrangian Schemes and One-Sided Lipschitz Constants

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***Abstract*— In this paper, we propose a new method for ensuring formally that a controlled trajectory stays inside a given safety set $\mathcal{S}$ for a given duration T. Using a finite gridding $\mathcal{X}$ of $\mathcal{S}$, we first synthesize, for a subset of initial nodes x of $\mathcal{X}$, an *admissible control* for which the Euler-based *approximate trajectories* lie in $\mathcal{S}$ at $t \in [0, T]$. We then give sufficient conditions which ensure that the *exact trajectories*, under the same control, also lie in $\mathcal{S}$ for $t \in [0, T]$, when starting at initial points "close" to nodes x. The statement of such conditions relies on results giving estimates of the deviation of Euler-based approximate trajectories, using *one-sided Lipschitz constants*. We illustrate the interest of the method on several examples, including a stochastic one.**

I. Introduction

Consider an ordinary differential equation (ODE) of the form $\dot{z} = f(z)$ on $\mathbb{R}^n$. Classically, one knows that, if the function f is Lipschitz continuous with Lipschitz constant L, the solution of the ODE starting at a given initial value exists and is unique. Besides, one has:

$$\|X_{t,z_1} - X_{t,z_2}\| \leq e^{Lt}\|z_1 - z_2\|, \tag{1}$$

where $\|\cdot\|$ denotes the Euclidean norm, and X_{t,z_i} denotes the value of the solution of the ODE at time t, starting at initial value z_i ($i = 1, 2$). This gives a rough *growth bound*, i.e. a function bounding the distance of neighboring trajectories as t evolves.

In the 90's, several researchers [11], [24] have obtained a more accurate growth bound, using the notion of "one-sided Lipschitz (OSL)" function. The function f is said to be OSL if there exists a constant $\lambda \in \mathbb{R}$ such that, for all $z_1, z_2 \in \mathbb{R}^n$:

$$\langle f(z_1) - f(z_2), z_1 - z_2\rangle \leq \lambda\|z_1 - z_2\|^2,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product of two vectors of $\mathbb{R}^n$. The real λ is called the OSL constant associated with f. In [11], it is proven that, if f is continuous and OSL with OSL constant λ, then the solution of the ODE starting at a given initial value exists and is unique, and, for all $z_1, z_2 \in \mathbb{R}^n$:

$$\|X_{t,z_1} - X_{t,z_2}\| \leq e^{\lambda t}\|z_1 - z_2\|. \tag{2}$$

This gives a more accurate growth bound because a Lipschitz function f is always OSL, and the associated OSL constant λ is always less than or equal to its Lipschitz counterpart L. Furthermore, in the case of "stiff" differential equations, we have $\lambda \ll L$ (see [11]). Note also that a function can be OSL but not Lipschitz (not even *locally* Lipschitz): inequation (2) then still applies while inequation (1) does not apply any longer. Using the OSL constant λ, it is also possible to bound the error $\|X_{t,z_1} - \tilde{X}_{t,z_2}\|$ in function of $\|z_1 - z_2\|$, where $\tilde{X}_{t,z_2}$ denotes the *Euler approximate* $z_2 + tf(z_2)$ of the solution X_{t,z_2}. In [6], we have derived some analytic forms of such error estimates when one focuses on a compact subdomain $\mathcal{S} \subset \mathbb{R}^n$ of solutions. We have also given an OSL-based error estimate for (a variant of) the Euler-Maruyama approximate solution in the case of *stochastic* ODEs [23]. These results have been used to synthesize controls that are "correct-by-construction", in the sense that they are guaranteed to satisfy given *safety constraints* [6], [23]. In this paper, we show how such error estimates can be integrated to *semi-Lagrangian (SL) schemes* in order to synthesize *optimal* controls for problems with safety constraints.

The plan of the paper is as follows: in Section II, we present the context of our work and the principle of the method; in Section III, we give formal sufficient conditions that guarantee the safety of the control; Section IV illustrates how the method can be extended for stochastic ODEs; we conclude in Section V.

II. Context and Principle of the Method

Let us present the context and the principle of our method.

A. Switched systems

A hybrid system is a system where the state evolves continuously according to several possible modes, and where the change of modes (switching) is done instantaneously.We consider here the special case of hybrid systems called "sampled switched systems" [17] where the change of modes occurs periodically with a period of τ seconds. We will suppose furthermore that the state keeps its value when the mode is changed (no jump). More formally, we denote the state of the system at time t by $z(t) \in \mathbb{R}^n$. The set of modes A is *finite*. With each mode $a \in A$ is associated a vector field f_a that governs the state $z(t)$, we have:

$$\dot{z}(t) = f_a(z(t)).$$

We make the following hypothesis:

$(H0)$ For all $a \in A$, f_a is a locally Lipschitz continuous map.

We will denote by X^a_{t,z_0} the solution at time t of $\dot{z}(t) = f_a(z(t))$ with $z(0) = z_0$. The existence of X^a_{t,z_0} is guaranteed by assumption $(H0)$. Let $\mathcal{S} \subset \mathbb{R}^n$ be a compact and convex set, typically a "rectangular set", i.e. a cartesian

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product on n closed intervals. We know by $(H0)$ that there exists a constant $L_a > 0$ such that:

$$\|f_a(z_1) - f_a(z_2)\| \leq L_a \|z_1 - z_2\| \quad \forall z_1, z_2 \in \mathcal{S}. \quad (3)$$

We also define, for all $a \in A$:

$$C_a = \sup_{z \in \mathcal{S}} L_a \|f_a(z)\|. \quad (4)$$

Let us denote by $\mathcal{T}$ a compact overapproximation of the set of trajectories starting in $\mathcal{S}$ for $0 \leq t \leq \tau$, i.e. $\mathcal{T}$ is such that

$$\mathcal{T} \supseteq \{X^a_{t,z_0} \mid a \in A, 0 \leq t \leq \tau, z_0 \in \mathcal{S}\}. \quad (5)$$

The existence of $\mathcal{T}$ is guaranteed by assumption $(H0)$. It follows from $(H0)$ that the vector fields f_a of the system are OSL on $\mathcal{T}$: for all $a \in A$, there exists a constant $\lambda_a \in \mathbb{R}$ such that

$$\langle f_a(z_1) - f_a(z_2), z_1 - z_2 \rangle \leq \lambda_a \|z_1 - z_2\|^2 \quad \forall z_1, z_2 \in \mathcal{T}. \quad (6)$$

We consider a *finite time horizon* problem: we suppose that time t belongs to interval $[0, k\tau]$, where k is a given integer number. Given a sequence of modes (or "pattern") $\pi := a_k \cdots a_1 \in A^k$, we denote by X^π_{t,z_0} the solution of the ODE of mode a_k for $t \in [0, \tau[$ with initial condition z_0, extended continuously with the solution of the ODE of mode a_{k-1} for $t \in [\tau, 2\tau[$, and so on iteratively until mode a_1 for $t \in [(k-1)\tau, k\tau]$.

B. Optimal problems

We consider the *cost function*: $J^\tau_k : \mathbb{R}^n \times A^k \to \mathbb{R}_{\geq 0}$ defined by:

$$J^\tau_k(z_0, \pi) = \|X^\pi_{k\tau,z_0} - z_{ref}\|,$$

and z_{ref} a given "target" state of $\mathbb{R}^n$.

We consider the *value function* $\mathbf{v}^\tau_k : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by:

$$\mathbf{v}^\tau_k(z_0) := \min_{\pi \in A^k} \{J^\tau_k(z_0, \pi)\} \equiv \min_{\pi \in A^k} \{\|X^\pi_{k\tau,z_0} - z_{ref}\|\}.$$

The function $\min$ is well-defined because the set A is finite.

We consider the following *finite time horizon optimal control problem*:

Given $k \in \mathbb{N}$ and $\tau \in \mathbb{R}_{>0}$, find for each $z \in \mathbb{R}^n$

- the *value* $\mathbf{v}^\tau_k(z_0)$, i.e.

$$\min_{\pi \in A^k} \{\|X^\pi_{k\tau,z_0} - z_{ref}\|\},$$

- and an *optimal pattern*:

$$\pi^\tau_k(z_0) := arg \min_{\pi \in A^k} \{\|X^\pi_{k\tau,z_0} - z_{ref}\|\}.$$

We are interested here in an optimal problem with *safety constraints*: we want that all the trajectories starting in $\mathcal{S}$ always stay in $\mathcal{S}$ for $t \in [0, k\tau]$. More precisely, we will focus on control patterns $\pi \in A^k$ that are "admissible for $z_0 \in \mathcal{S}$", i.e. such that: $X^\pi_{i\tau,z_0} \in \mathcal{S}$, for all $i \in \{1, \ldots, k\}$ (*discrete-time safety* constraint). We will also consider a stronger admissibility criterion requiring: $X^\pi_{t,z_0} \in \mathcal{S}$, for all $t \in [0, k\tau]$ (*continuous-time safety* constraint).

In order to solve such optimal control problems, it is classical to *spatially discretize* the set $\mathcal{S} \subset \mathbb{R}^n$. Given a hyper-rectangle $\mathcal{S}$, we consider a *partition* of $\mathcal{S}$ into a finite number of hyper-rectangular cells. The *grid* $\mathcal{X}$ associated with $\mathcal{S}$ is the set of all the cell centers. We suppose furthermore that the radius of every cell is upper bounded by a given positive real ε: each cell C of center x is such that $\|z_0 - x\| \leq \varepsilon$, for all $z_0 \in C$. The center $x \in \mathcal{X}$ of a cell $C \subset \mathcal{S}$ is said to be the "ε-representative" of all points of C. Since the set of cells forms a partition of $\mathcal{S}$, each point $z_0 \in \mathcal{S}$ has a unique ε-representative $x \in \mathcal{S}$ with $\|z_0 - x\| \leq \varepsilon$.

In this context, the idea of a Semi-Lagrangian (SL) procedure is the following: we consider the points of $\mathcal{X}$ as the vertices of a finite oriented graph; there is a connection from $x \in \mathcal{X}$ to $x' \in \mathcal{X}$ if x' is the ε-representative of the Euler-based image $(x + \tau f_a(x))$ of x, for some $a \in A$. We then compute using dynamic programming the "path of length k with minimal cost" starting at x: such a path is a sequence of $n+1$ connected points $x\ x_k\ x_{k-1}\ \cdots\ x_1$ of $\mathcal{X}$ which minimises the distance $\|x_1 - z_{ref}\|$. The dynamic progamming procedure thus gives us a *spatially discrete value function* $\mathbf{v}^{\tau,\varepsilon}_k : \mathcal{X} \to \mathbb{R}_{\geq 0}$, and a *spatially discrete pattern function* $\pi^k_{\tau,\varepsilon} : \mathcal{X} \to A^k$ which "approximate" on $\mathcal{S}$ their counterparts $\mathbf{v}^\tau_k$ and π^k_τ respectively.

There is a vast literature on SL-schemes (see, e.g., [9], [14]) which gives numerous results to the following *convergence* problem P1:

"Under which conditions does the spatially-discrete value function $\mathbf{v}^{\tau,\varepsilon}_k$ converge to the value function $\mathbf{v}^\tau_k$ when $\varepsilon \to 0$?"

Actually, when ε decreases too much, the computations with SL-procedures become quickly impractical. We prefer to consider here that ε is *fixed* (as well as τ and k), and focus on the following (local) problem P2:

"Given $z_0 \in \mathcal{S}$, under which conditions does there exist a pattern $\pi \in A^k$ which guarantees:

1) the satisfaction of the safety constraint $X^\pi_{i\tau,z_0} \in S$ for all $i \in \{1, \ldots, k\}$ (or $X^\pi_{t,z_0} \in S$, for all $t \in [0, k\tau]$),
2) while minimizing $\|X^\pi_{k\tau,z_0} - z_{ref}\|$ as much as possible ?"

In order to solve problem P2, we use the SL-based procedure as sketched out above. Given a point $z \in \mathcal{S}$ of ε-representative $x \in \mathcal{X}$, we apply the SL procedure to x. The procedure generates a path of the form $x\ x_k\ x_{k-1}\ \cdots\ x_1$, where $x_k, \cdots, x_1$ are computed using an *Euler scheme*, and lie by construction in $\mathcal{S}$. The associated control pattern is of the form $a_k\ a_{k-1} \cdots a_1 \in A^k$. Let $\pi_{k,i} := a_k \cdots a_i$ for $1 \leq i \leq k$. In order, to ensure that the corresponding points $X^{\pi_1}_{\tau,z}, X^{\pi_2}_{2\tau,z}, \ldots X^{\pi_k}_{k\tau,z}$ of the *exact trajectory* lie also in $\mathcal{S}$, we need to establish a bound on the pairwise distances:

$\|x - z_0\|$, $\|x_k - X^{\pi_1}_{\tau,z_0}\|$, $\|x_{k-1} - X^{\pi_2}_{2\tau,z_0}\|$, ...,

$\|x_i - X^{\pi_{k,i}}_{(k-i+1)\tau,z_0}\|, \ldots, \|x_1 - X^{\pi_k}_{k\tau,z_0}\|$.
At time $t = 0$, the first distance $\|x - z_0\|$ is known to be bounded by ε. We will establish bounds $\Delta_1, \ldots, \Delta_k$ on the following distances using a recent result which gives an upper bound to the deviation of Euler-based trajectories with time (see [6]). More precisely, we will give an error function $\Delta(t)$ measuring the distance at time t between an approximate (Euler-based) trajectory starting at $x \in \mathcal{X}$ given by the SL-scheme, and an exact trajectory starting from the cell of x. In order to guarantee that the exact trajectory always lies in the hypercube $\mathcal{S}$ at times $t = \tau, 2\tau, \ldots, k\tau$, we merely perform two simple operations:

1) compute the "safety margin" of the Euler-based trajectory, i.e., its distance to the boundary of $\mathcal{S}$ at time $t = \tau, 2\tau, \ldots, k\tau$, and
2) check that this margin is always greater than the error $\Delta(t)$ at time $t = \tau, 2\tau, \ldots, k\tau$.

The complexity of these operations is very low.

C. Comparison with related work

We distinguish between works dealing with problem P1 and those dealing with P2.

- Problem P1: In many papers of the literature on SL methods with state constraints (see, e.g., [12]), the authors enforce the trajectory system to stay in $\mathcal{S}$ by introducing a (somehow artificial) "penalization" term in the cost function J, making the cost of crossing the boundary of $\mathcal{S}$ prohibitive (cf. [13]). In order to guarantee the result of convergence of $v^{\tau,\varepsilon}$ to v^τ, they also often make a restrictive assumption of "controllability". Note however that, in works like [1], [4], [5], no controllability assumption is made.
  In [25] (cf. [26]), the authors constructs a sequence of abstractions which are more and more precise. The sequence of value function associated with each abstraction *converges* to the optimal value function associated the original problem. The abstract transition function computes an over-approximation of the set of trajectories starting at neighbouring points. This over-approximation is computed using a growth bound (bounding the distance of neighboring trajectories) based on the Jacobian matrix of f_a. More precisely, the growth bound is a function mapping any $\mathbf{r} \in \mathbb{R}^n_{\geq 0}$ to $e^{Mt}\mathbf{r}$, where M is a $n \times n$-matrix whose (i,j)-entry is $D_j f^i_a(z)$, if $i = j$ and $|D_j f^i_a(z)|$ otherwise, and $f^i_a(z)$ denotes the i-th component of vector $f_a(z)$. Here, $D_j f^i_a$ denotes the partial derivative with respect to the jth component of the argument of f^i_a. By comparison, our work here can be seen as a particular case of [25] where one uses, for each of the n components, a uniform growth bound, mapping $r \in \mathbb{R}_{\geq 0}$ to $e^{\lambda_a t} r$. The counterpart of the convergence result of [25] for the value function, would state in our context that the synthesized control converge towards the optimal control as ε tends to 0. However, this does not seem true (unless adding very restrictive assumptions), which leads us to focus on problem P2 instead of P1.
- Problem P2: In the work of [27], [28], the authors pursue an objective similar to ours: providing a (finite time-horizon) optimal control procedure with a formal guarantee of constraint satisfaction (safety). However they do not use SL-schemes, but perform a reachability analysis based on over-approximative state set representations (*zonotopes*, cf. [16], [2]).
  In [10], the authors also provide a formal guarantee of safety property. Contrarily to [27], [28], they do use SL-schemes. They also focus to (periodically) sampled systems as we do. However, they still perform a form of reachability analysis similar to [27], [28], using convex polytopes as state set representations. Their growth bound are not based on OSL constants as here, but rather on overapproximations of Lagrange remainders in Taylor series.

III. Sufficient Conditions for Reachability with Safety

Given a starting point $z_0 \in \mathcal{S}$ and a mode $a \in A$, we denote by $\tilde{X}^a_{\tau,y}$ the Euler-based image of z_0 at time $t = \tau$ via a. We have:

$$\tilde{X}^a_{\tau,z_0} := z_0 + \tau\ f_a(z_0).$$

The *set of admissible modes for* $x \in \mathcal{X}$ is defined by:

$$A_\tau(x) := \{a \in A \mid \tilde{X}^a_{\tau,x} \in \mathcal{S}\}.$$

The function $next^a : \mathcal{X} \to \mathcal{X} \cup \{\bot\}$ is defined by:

- if $a \in A_\tau(x)$, then: $next^a(x) = x'$, where x' is the ε-representative of $\tilde{X}^a_{\tau,x}$,
- otherwise (i.e., $\tilde{X}^a_{\tau,x} \notin S$): $next^a(x) = \bot$.

For a pattern $\pi \in A^k$, the function $next^\pi : \mathcal{X} \to \mathcal{X} \cup \{\bot\}$ is defined as follows:

- if $\pi = a$ for some $a \in A$, then $next^\pi(x) = next^a(x)$,
- if π is of the form $a \cdot \pi'$;
  - if $next^a(x) \neq \bot$, then $next^\pi(x) = next^{\pi'}(next^a(x))$,
  - otherwise, $next^\pi(x) = \bot$.

It is easy to show, using the definition of $next$:

Proposition 1: Let $x \in \mathcal{X}$, and $\pi_k \in A^k$ a pattern of the form $a_k\ a_{k-1} \cdots\ a_1$. Let us write $\pi_{k,i} := a_k \cdots a_i$ for $1 \leq i \leq k$, and $x_{k+1} := x$.

If $next^{\pi_k}(x) \in \mathcal{X}$, then there exists a sequence of points of the form $x_{k+1}x_k \cdots x_1 \in \mathcal{X}^{n+1}$ with, for all $1 \leq i \leq k$:

- $\tilde{X}^{a_i}_{\tau,x_{i+1}} \in \mathcal{S}$,
- $x_i = next^{a_i}(x_{i+1}) = next^{\pi_{k,i}}(x)$, and
- $\|x_i - \tilde{X}^{a_i}_{\tau,x_{i+1}}\| \leq \varepsilon$.

Definition 1: For all points $x \in \mathcal{X}$, the *spatially discrete value function* $\mathbf{v}^{\tau,\varepsilon}_k : \mathcal{X} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is defined by:

- for $k = 0$, $\mathbf{v}^{\tau,\varepsilon}_k(x) = \|x\|$,
- for $k \geq 1$,
  - if $A_\tau(x) = \emptyset$: $\mathbf{v}^{\tau,\varepsilon}_k(x) = \infty$,

– if $A_\tau(x) \neq \emptyset$:
$\mathbf{v}_k^{\tau,\varepsilon}(x) = \min_{a\in A_\tau(x)}\{\mathbf{v}_{k-1}^{\tau,\varepsilon}(next^a(x))\}$.

If $\mathbf{v}_k^{\tau,\varepsilon}(x) \neq \infty$, one defines the *approximate optimal pattern of length* k associated to x, denoted by $\pi_k^{\tau,\varepsilon}(x) \in A^k$, recursively by:

- if $k = 0$, $\pi_k^{\tau,\varepsilon}(x) =$ nil,
- if $k \geq 1$, $\pi_k^{\tau,\varepsilon}(x) = \mathbf{a}_k(x) \cdot \pi'$ where

$$\mathbf{a}_k(x) = arg \min_{a\in A_\tau(x)} \{\mathbf{v}_{k-1}^{\tau,\varepsilon}(next^a(x))\}$$

and $\pi' = \pi_{k-1}^{\tau,\varepsilon}(x')$ with $x' = next^{\mathbf{a}_k(x)}(x)$.

The initialization with nil means that the pattern is initialized empty, and the notation $\mathbf{a}_k(x) \cdot \pi'$ is used to add $\mathbf{a}_k(x)$ as a prefix to π', i.e. $a \cdot \pi' = aa_k a_{k-1} \dots a_1$ if $a \in A$ and $\pi' = a_k a_{k-1} \dots a_1 \in A^k$.

Using the value function $\mathbf{v}_k^{\tau,\varepsilon}$ it is thus easy to construct an SL procedure $PROC_k^{\tau,\varepsilon}$ which takes a point $x \in \mathcal{X}$ as input, and returns, in case of success (i.e., when $\mathbf{v}_k^{\tau,\varepsilon}(x) \geq 0$), a pattern $\pi_k^{\tau,\varepsilon} \in A^k$ with $next^{\pi_k^{\tau,\varepsilon}}(x) \in \mathcal{X}$. We now define, for such a pattern $\pi_k^{\tau,\varepsilon}$ output by $PROC_k^{\tau,\varepsilon}(x)$, a value $\Delta(\pi_k^{\tau,\varepsilon})$ which gives us an *upperbound* to $\|X_{k\tau,z_0}^{\pi_k^{\tau,\varepsilon}} - next^{\pi_k^{\tau,\varepsilon}}(x)\|$, for any $z_0 \in B(x,\varepsilon)$ (i.e., any z_0 such that : $\|z_0 - x\| \leq \varepsilon$).

Definition 2: Let μ be a given positive constant. Let us define, for all $a \in A$ and $t \in [0,\tau]$, $\delta_{t,\mu}^a$ as follows:

- if $\lambda_a < 0$:

$$\delta_{t,\mu}^a = \left(\mu^2 e^{\lambda_a t} + \frac{C_a^2}{\lambda_a^2}\left(t^2 + \frac{2t}{\lambda_a} + \frac{2}{\lambda_a^2}\left(1 - e^{\lambda_a t}\right)\right)\right)^{\frac{1}{2}}$$

- if $\lambda_a = 0$:

$$\delta_{t,\mu}^a = \left(\mu^2 e^t + C_a^2(-t^2 - 2t + 2(e^t - 1))\right)^{\frac{1}{2}}$$

- if $\lambda_a > 0$:

$$\delta_{t,\mu}^a = \Big(\mu^2 e^{3\lambda_a t} + \frac{C_a^2}{3\lambda_a^2}\left(-t^2 - \frac{2t}{3\lambda_a} + \frac{2}{9\lambda_a^2}\left(e^{3\lambda_a t} - 1\right)\right)\Big)^{\frac{1}{2}}$$

where C_a and λ_a are constants defined in Section II-A.

Proposition 2: [6] Given $x \in \mathbb{R}^n$, we have, for all $a \in A$ and all $z_0 \in B(x,\varepsilon)$ (i.e., z_0 such that $\|z_0 - x\| \leq \varepsilon$):

$$\|X_{\tau,z_0}^a - \tilde{X}_{\tau,x}^a\| \leq \delta_{\tau,\varepsilon}^a.$$

Definition 3: Let us define $\Delta(a_k \cdots a_1)$ recursively by:

- $\Delta(a_i) = \delta_{\tau,\varepsilon}^{a_i}$ for $i = 1$, and
- $\Delta(a_i \cdots a_1) = \delta_{\tau,\mu}^{a_i}$ with $\mu = \varepsilon + \Delta(a_{i-1} \cdots a_1)$, for $i \geq 2$.

In the rest of the paper, we suppose that $k \in \mathbb{N}$ and $\tau, \varepsilon \in \mathbb{R}_{>0}$ are given and fixed. So, for the sake of notation simplicity, we will abbreviate $\mathbf{v}_k^{\tau,\varepsilon}$ as $\mathbf{v}_k$. We abbreviate similarly $\pi_k^{\tau,\varepsilon}$ and $PROC_k^{\tau,\varepsilon}$ as π_k and $PROC_k$ respectively. We will suppose also that we are given a compact set $\mathcal{S} \subset \mathbb{R}^n$ as well as a "target" set $R \subset \mathcal{S}$.[1] We have:

Lemma 1: Let $x \in \mathcal{X}$ and $\pi_k \equiv a_k \cdots a_1 \in A^k$ the pattern generated by $PROC_k(x)$ with $next^{\pi_k}(x) \in \mathcal{X}$. We have, for all $z_0 \in B(x,\varepsilon)$:

1) $\|X_{k\tau,z_0}^{\pi_k} - \tilde{X}_{\tau,x_2}^{a_1}\| \leq \Delta(\pi_k)$,
with $x_2 := next^{a_k \cdots a_2}(x)$ for $k \geq 2$, and $x_2 := x$ for $k = 1$;
2) $\|X_{k\tau,z_0}^{\pi_k} - next^{\pi_k}(x)\| \leq \Delta(\pi_k) + \varepsilon$.

The proof is given in the extended version of this paper [7].

Using item 2 of Lemma 1, it is easy to show:

Theorem 1: (sufficient conditions of safety and k-reachability) Let $x \in \mathcal{X}$, and $\pi_k \equiv a_k \cdots a_1 \in A^k$ the pattern generated by $PROC_k(x)$ with $next^{\pi_k}(x) \in \mathcal{X}$. Suppose, for all $1 \leq i \leq k$:

- (H_1^i): $B(next^{\pi_{k,i}}(x), \Delta(\pi_{k,i}) + \varepsilon) \subseteq \mathcal{S}$, and
- (H_2^k): $B(next^{\pi_k}(x), \Delta(\pi_k) + \varepsilon) \subseteq R$,

where $\pi_{k,i} := a_k \cdots a_i$. Then we have, for all $z_0 \in B(x,\varepsilon)$[2]

- $X_{(k-i+1)\tau,z_0}^{\pi_{k,i}} \in S$ for all $1 \leq i \leq k$ (*discrete-time safety*),

and

- $X_{k\tau,z_0}^{\pi_k} \in R$ (k-*reachability*).

Furthermore, assuming that, for all $a \in A$, $\delta_{t,\varepsilon}^a$ is a convex function for $t \in [0,\tau]$ (i.e., $\frac{d^2(\delta_t^a)}{dt^2} > 0$ for all $t \in [0,\tau]$ [3]), we have:

$X_{t,z_0}^{\pi_k} \in \mathcal{S}$ for all $t \in [0, k\tau]$ (*dense-time safety*).

Suppose in particular that conditions (H_1^i)-(H_2^k) hold for a set of points $\mathcal{Y} \subseteq \mathcal{X}$ which ε-*covers* R, i.e., such that: $R \subseteq \bigcup_{x\in\mathcal{Y}} B(x,\varepsilon)$. In this case, the procedure $PROC_k$ gives us a guarantee of "(R,S)-stability" as defined in [15]. By Theorem 1, we know indeed that, for all $z_0 \in R$ of representative $x \in \mathcal{X}$, the pattern π_k generated by $PROC_k(x)$ applied to z_0 yields a trajectory that reaches at $t = k\tau$ a point z' of R (while always staying in $\mathcal{S}$ for $0 \leq t \leq k\tau$); the process can then be iterated to z', and so on repeatedly. This means that, via the set of patterns π_k associated to elements of $\mathcal{Y}$, one can control any trajectory starting at R in order to make it return to R periodically every $k\tau$ seconds, and stay in $\mathcal{S}$ for all $t \geq 0$ ("(R,S)-stability").[4]

The SL-based procedure $PROC_k$ can thus replace advantageously the brute-force enumeration strategy implemented

[1] We suppose implicitly that R contains the target point z_{ref}, so R can be seen as a neighborhood of z_{ref}.

[2] In particular, for any $z_0 \in S$ of ε-representative x.

[3] The sign of $\frac{d^2(\delta_t^a)}{dt^2}$ on $[0,\tau]$ depends on the value of the constants C_a and λ_a occurring in Definition 2; knowing these constant values, the sign is easy to determine (see [6]).

[4] R can be seen as a special case of *viability kernel* for $\mathcal{S}$ (see, e.g., [3]) since any trajectory starting from R can be controlled in order to stay inside $\mathcal{S}$ forever.

in tool MINIMATOR [20]: the time complexity of MINIMATOR procedure is indeed $O(m^k \times N)$ where m is the number of modes, N the number of cells and k the time-horizon length, while the complexity of $PROC_k$ is $O(m \times k \times N)$.

Description of the implementation

The procedure is implemented in a program called "OSLator" [21], implemented in Octave. It is composed of 9 functions and a main script totalling 500 lines of code. For comparison, the tool MINIMATOR uses 28 functions for a total of 2000 lines of code.

The computations are realised in a virtual machine running Ubuntu 18.06 LTS, having access to one core of a 2.3GHz Intel Core i5, associated to 3.5GB of RAM memory.

Note that the accuracy of the Euler approximation can be optionally increased by using a smaller time step. The time-step h used for Euler approximation is not necessarily equal to the control sampling period, but is in general a *submultiple* of τ ($\tau = p \times h$ where p is a natural number greater than 1).

Example 1: (2-tanks)

In this example, we illustrate the approach given above for (R, S)-stability on a two tank example. The two-tank system is a linear example taken from [19]. The system consists of two tanks and two valves. The first valve adds to the inflow of tank 1 and the second valve is a drain valve for tank 2. There is also a constant outflow from tank 2 caused by a pump. The system is linearized at a desired operating point. The objective is to keep the water level in both tanks within limits using a discrete open/close switching strategy for the valves. Let the water level of tanks 1 and 2 be given by x_1 and x_2 respectively. The behavior of x_1 is given by $\dot{x}_1 = -x_1 - 2$ when the tank 1 valve is closed, and $\dot{x}_1 = -x_1 + 3$ when it is open. Likewise, x_2 is driven by $\dot{x}_2 = x_1$ when the tank 2 valve is closed and $\dot{x}_2 = x_1 - x_2 - 5$ when it is open.

Let $S = [-2, 3] \times [-1, 2]$, $R = [-1.5, 2.5] \times [-0.5, 1.5]$, $N = 10 \times 10$ the number of cells, $\varepsilon = 0.33$, $\tau = 0.1$. The proof of (R, S)-stability is obtained for $k = 5$, it takes 7.34 seconds. By comparison, MINIMATOR takes 25.53 seconds to obtain a controller without any optimality result. A simulation of the (R, S)-stability controller is given in Figure 1.

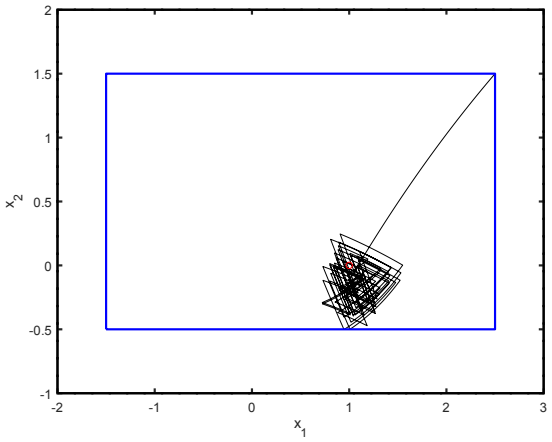

Fig. 1. Simulation of the (R, S)-stability controller on the two tank example. The safety set is $S = [-2, 3] \times [-1, 2]$, the recurrence set $R = [-1.5, 2.5] \times [-0.5, 1.5]$, the blue box is the set R, the red circle is the objective here chosen as $(1.0, 0.0)$. The trajectory of the system is in black for the initial condition $(2.5, 1.5)$.

IV. Extension to stochastic switched systems

We now explain how to extend the method to stochastic ODEs. Consider a stochastic switched system defined by

$$\mathrm{d}X_t = f_a(X_t)\mathrm{d}t + g_a(X_t)\mathrm{d}W_t, \qquad X_0 = x_0, \qquad (7)$$

where W_t is a standard $m-dimensional$ Brownian motion, and suppose that for all $a \in A$:

(H1) $f_a : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function whose derivative grows at most polynomially,

(H2) $g_a = (g_{a_{i,j}})_{i \in \{1,\ldots,n\}, j \in \{1,\ldots,m\}} : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ is a globally Lipschitz continuous function,

(H3) f_a is globally one-sided Lipschitz.

Under the above-mentioned hypotheses, we can establish bounds $\delta^a_{t,\varepsilon}$ similar to Definition 2 for stochastic switched systems using the *tamed Euler scheme* [18]. See [22] for more details, and [23] for the proofs of the error bounding for stochastic switched systems. The result is stated as follows for a single switching step integration:

Proposition 3 ([23]): Consider two points x_0 and z in $\mathbb{R}^n$, and a positive real number ε. Suppose that $x_0 \in B(z, \varepsilon)$. Let us denote by $\tilde{X}_{t,z}$ the tamed Euler approximation of X_t starting from initial point z in (7). Then $\mathbb{E}X_{t,x_0} \in B(\tilde{X}_{t,z}, \delta^a_{t,\varepsilon})$ for all $t \in [0, \tau]$, where $\mathbb{E}$ is the symbol of expected value.

Example 2: (Stochastic system) Consider the system (see ([30], [29])):

$dx_1 = (-0.25x_1 + ux_2 + (-1)^u 0.25)dt + 0.01x_1 dW^1_t$

$dx_2 = ((u-3)x_1 - 0.25x_2 + (-1)^u(3-u))dt + 0.01x_2 dW^2_t$

where $u = 1, 2$.

We can apply the above procedure $PROC^{\tau,\varepsilon}_k$ in order to minimize the average distance of the state to the origin after a given number of steps. We consider a switching period $\tau = 0.5$ subdivided in time steps of size $\Delta_t = 10^{-4}$. Consider the interest set $R = B((0, 0), \rho)$ with $\rho = 7$, discretized with an accuracy $\varepsilon = 0.57$. We compute (sub)optimal patterns for the entire set, using different lengths of patterns, and simulate the induced controller for 200 initial conditions randomly selected in R. Simulations are given in Figure 2. The procedure took 11.8 seconds of computation for patterns of length 1, and 47.4 seconds for patterns of length 3.

V. Final Remarks

We have presented a new SL-based method for synthesizing a provably safe finite-time horizon control. We have illustrated the interest of the method on a classical example (2-tanks) and shown how to extend it to stochastic ODEs. The potential application of such methods to Model Predictive Control has been pointed in [27]. The approach

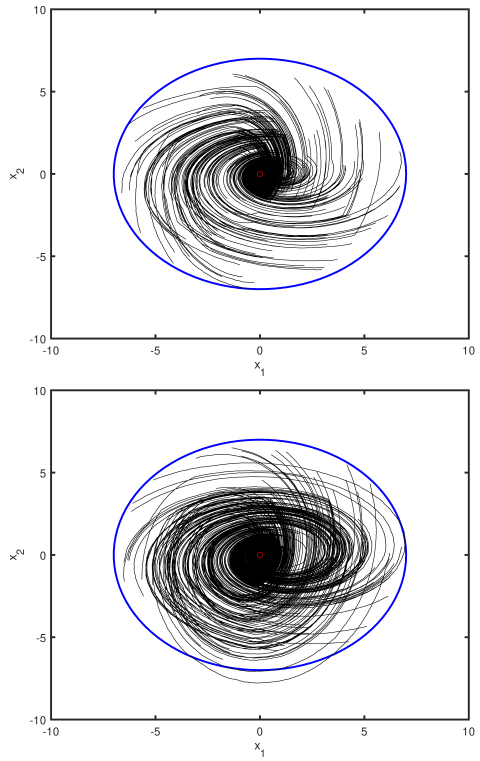

Fig. 2. Simulations of Example 2 with the controller induced by $PROC_k$, for patterns of length 1 (top), length 3 (bottom). The blue circle is the set $R = B((0,0),7)$, the red marker is the target state (the origin), the black lines are the controlled trajectories.

has also been extended to reaction-diffusion models using model reduction in [8].

A defect of our method is that, in order to satisfy the sufficient conditions of Theorem 1, one may have to decrease the cell size ε too much, thus making the number of cells explode, as often in SL methods. In this case, methods using symbolic reachability analysis, such as in [10], [27], [28], may be more efficient. A comparative experimental work between the two kinds of method is planned for future work.

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