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On complexity of Ehrenfeucht–Fraïssé games

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ABSTRACT

In this paper, we initiate the study of Ehrenfeucht–Fraïssé games for some standard finite structures. Examples of such standard structures are equivalence relations, trees, unary relation structures, Boolean algebras, and some of their natural expansions. The paper concerns the following question that we call the Ehrenfeucht–Fraïssé problem. Given $n \in \omega$ as a parameter, and two relational structures $\mathcal{A}$ and $\mathcal{B}$ from one of the classes of structures mentioned above, how efficient is it to decide if Duplicator wins the n-round EF game $G_n(\mathcal{A}, \mathcal{B})$? We provide algorithms for solving the Ehrenfeucht–Fraïssé problem for the mentioned classes of structures. The running times of all the algorithms are bounded by constants. We obtain the values of these constants as functions of n.

1. Introduction

Ehrenfeucht–Fraïssé (EF) games [3,4] constitute an important tool in both finite and infinite model theory. For example, in infinite model theory these games can be used to prove the Scott Isomorphism Theorem showing that all countable structures are described (up to isomorphism) in $L_{\omega_1,\omega}$-logic [13]. In finite model theory, these games and their different versions are used for establishing expressibility results in first-order logic and its extensions [7]. These results can be found in standard books on finite and infinite model theory (e.g. [6,10]) and in relatively recent papers (e.g. [2,11]). In this paper all EF games are considered on finite structures.

Despite the significant use of EF games in finite and infinite model theory there has not been, with some exceptions, much work in addressing efficiency of these games. M. Grohe studied EF games with fixed number of pebbles and showed that the problem of deciding the winner is PTIME-complete [5]. E. Pezzoili showed that deciding the winner of EF games is PSPACE-complete [12]. In [8] P. Kolaitis and J. Panttaja prove that the following problem is EXPTIME-complete: given a natural number k and structures $\mathcal{A}$ and $\mathcal{B}$, does Duplicator win the k pebble existential EF game on $\mathcal{A}$ and $\mathcal{B}$? In [1] sufficient conditions are provided for Duplicator to win EF games. These conditions are then used to prove some inexpressibility results, e.g. reachability in undirected graphs is not in monadic NP. These results suggest that developing tools and algorithms for finding winners of EF games is of interest. We also point out that there has recently been an interest in EF games to collapse results in database theory [9,14]. In addition, we think that algorithms that solve EF games can be used in data matching and data transformation problems in databases.

In this paper, we initiate the study of EF games for some standard finite structures. Examples of such standard structures are equivalence relations, trees, unary relation structures, Boolean algebras, and some of their natural expansions. The paper concerns the following question that we call the *Ehrenfeucht–Fraïssé problem*.

> Given $n \in \omega$ as a parameter, and two relational structures $\mathcal{A}$ and $\mathcal{B}$ from a fixed (standard) class of structures, how efficient is it to decide if Duplicator wins the n-round EF game $G_n(\mathcal{A}, \mathcal{B})$?

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We provide algorithms for solving the Ehrenfeucht–Fraïssé problem for the structures mentioned above. The running times of all the algorithms are bounded by constants. We obtain the values of these constants as functions of n.

By a structure, we always mean a finite relational structure over a language without functional symbols. Let $\mathcal{A}$ and $\mathcal{B}$ be structures and $n \in \omega$. The n-round EF game on $\mathcal{A}$ and $\mathcal{B}$ is denoted by $G_n(\mathcal{A}, \mathcal{B})$. The game is played as follows. There are two players, Duplicator and Spoiler, both are provided with $\mathcal{A}$ and $\mathcal{B}$. The game consists of n rounds. Informally, Duplicator's goal is to show that these two structures are similar, while Spoiler needs to show the opposite. At round i, Spoiler selects structure $\mathcal{A}$ or $\mathcal{B}$, and then takes an element from the selected structure. Duplicator responds by selecting element from the other structure. Say the players have produced the following play consisting of pairs of elements $(a_1, b_1), \dots, (a_n, b_n)$, where $a_i \in A$ and $b_i \in B$ for $i = 1, \dots, n$. Note that if Spoiler selected a_i (or b_i) then Duplicator selected b_i (or a_i, respectively). Duplicator wins the play if the mapping $a_i \to b_i, i = 1, \dots, n$, extended by mapping the values of constant symbols $c^{\mathcal{A}}$ to $c^{\mathcal{B}}$, is a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$. It is clear that if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic then Duplicator wins game $G_n(\mathcal{A}, \mathcal{B})$ no matter what n is. The opposite is not always true. However, for large n, if Duplicator wins the game $G_n(\mathcal{A}, \mathcal{B})$ then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. Thus, solving the EF problem can be thought as an approximation to the isomorphism problem.

One can do the following rough estimates for finding the winner of game $\mathcal{G}_n(\mathcal{A}, \mathcal{B})$. There are finitely many, up to logical equivalence, formulas $\phi_1, \dots, \phi_k$ of quantifier rank n (see for example [10]). It is well known that Duplicator wins $G_n(\mathcal{A}, \mathcal{B})$ if and only if for all ϕ_i (with $i = 1, \dots, k$) the structure $\mathcal{A}$ satisfies ϕ_i if and only if $\mathcal{B}$ satisfies ϕ_i [10]. Thus, the question if Duplicator wins $G_n(\mathcal{A}, \mathcal{B})$ can be solved in polynomial time. However, there are two important issues here. The first issue concerns the number k that depends on n; k is approximately bounded by the n-repeated exponentiations of 2. The second issue concerns the degree of the polynomial for the running time that is bounded by n. Thus, questions arise as to for which standard structures the value of k is feasible as a function of n, and whether the degree of the polynomial for the running time can be pushed down. For example consider the class of linear orders. It is well known that Duplicator wins $G_n(\mathcal{A}, \mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are linear orders, if and only if either $|A| = |B|$ or both $|A| > 2^n$ and $|B| > 2^n$ (e.g. [10]). In this example, the number k roughly equals to 2^n. The degree of polynomial for the running time is 0. Thus, when n is fixed, the winner of the game can be found in constant time and the constant that bounds the time is 2^n.

A brief overview of this paper is as follows. The next section gives an elementary solution to EF games in the case when the language contains unary predicates only. The third, fourth and fifth sections are quite technical and devoted to solving EF games for equivalence structures and some of their extensions. Equivalence structures are natural models of university or large company databases. For example, in a university database there could be the *SameFaculty* and *SameDepartment* relations. The first relation stores all tuples (x, y) such that x and y belong to the same faculty; similarly, the second relation stores all tuples (u, v) such that u and v are in the same department. These relations are equivalence relations. Moreover, a natural connection between these two relations is as sets the relation *SameDepartment* is a subset of the relation *SameFaculty*. We call such structures *embedded equivalence relation structures*. Section 6 reduces the question of deciding EF games for trees of a given height to solving the EP games for embedded equivalence structures introduced in the previous sections. Finally, the main structures in the last section are Boolean algebras with distinguished ideals.

Each of these sections provides an algorithm that decides EF games $G_n(\mathcal{A}, \mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are from a given class of structures (considered in the section). In all of these results, it is proved that the winner can be found in a *constant time* with n being a parameter. We also bound the value of the constant as a function on n. Clearly, the constants obtained depend on the representations of the structures. In each case, it will be clear from the content how we represent our structures. As an example, we state two results of Sections 4 and 5. Section 4 is devoted to structures of the type $(A; E, P_1, \dots, P_s)$, where E is an equivalence relation on A and $P_1, \dots, P_s$ are unary predicates. We call these structures *equivalence structures with s colors*. The main result of Section 4 is the following:

Theorem 4.10. *Fix $n \in \omega$. There exists an algorithm that runs in constant time and decides whether Duplicator wins the game $G_n(\mathcal{A}, \mathcal{B})$ on equivalence structures with s colors. The constant that bounds the running time is n^{2^s+1}.*

Section 5 is devoted to the structures of type $(A; E_1, \dots, E_h)$, where each E_i is an equivalence relation on A and $E_1 \subseteq E_2 \subseteq \dots \subseteq E_h$. These structures are called embedded equivalence structures of height h. The main result of Section 5 is:

Theorem 5.6. *Fix $n \in \omega$. There exists an algorithm that runs in constant time and decides whether Duplicator wins the game $G_n(\mathcal{A}, \mathcal{B})$ on embedded equivalence structures of height h. The constant that bounds the running time is $< (n+1)^{\cdot^{\cdot^{\cdot^{(n+1)^n}}}}$ where the tower has height h.*

2. Simple example: Structures with unary predicates

This is an elementary section that gives a full solution for EF games in the case when the language contains unary predicates only. Here is the main result of this section.

Theorem 2.1. *Fix the language $L = (P_1, \dots, P_s)$, where each P_i is a unary predicate symbol. Let $n \in \omega$. There exists an algorithm that runs in constant time and decides whether Duplicator wins game $G_n(\mathcal{A}, \mathcal{B})$ on structures $\mathcal{A}$ and $\mathcal{B}$ of the language. The constant that bounds the running time is $2^s \cdot n$.*

Proof. Let $\mathcal{A} = (A; P_1, P_2, \dots, P_s)$ and $\mathcal{B} = (B; P_1, P_2, \dots, P_s)$ be structures of the language given. For structure $\mathcal{A} = (A; P_1, P_2, \dots, P_s)$, we set $P_{s+1} = \bigcap_i \neg P_i^A$.

Lemma 2.2. *Suppose $P_1, P_2, \ldots, P_s$ are pairwise disjoint. Then Duplicator wins $G_n(\mathcal{A}, \mathcal{B})$ if and only if for all $1 \le i \le s+1$ if $|P_i^A| < n$ or $|P_i^{\mathcal{B}}| < n$ then $|P_i^A| = |P_i^B|$. In particular, when Duplicator wins, it is the case that for all $1 \le i \le s+1$, $|P_i^A| > n$ if and only if $|P_i^B| > n$.*

To prove the lemma suppose that there is $1 \le i \le k+1$ such that $|P_i^A| < n$ but $|P_i^A| \ne |P_i^B|$. Assume $|P_i^B| < |P_i^A|$. Then Spoiler selects $|P_i^A|$ elements from P_i^A. This strategy is clearly winning for Spoiler. For the other direction, assume that the hypothesis of the lemma holds. Duplicator has a winning strategy as follows: At round k, assume that the players have produced the k-round play $(a_1, b_1), \ldots, (a_k, b_k)$ such that for each $1 \le i \le k$, $a_i \in A$, $b_i \in B$. If Spoiler selects $a_{k+1} \in A$, then Duplicator responds by selecting $b_{k+1} \in B$ as follows: If $a_{k+1} = a_i$ for some i then $b_{k+1} = b_i$. Otherwise if $a_{k+1} \in P_j^A$ for some $1 \le j \le k$, then $b_{k+1} \in P_j^B$ so that $b_{k+1} \notin \{b_1, \ldots, b_k\}$. The case when Spoiler selects an element from B is treated similarly. The strategy is clearly winning. □

Now assume that for a structure $\mathcal{A}$, the unary predicates $P_1, P_2, \ldots, P_s$ are not necessarily pairwise disjoint. For each element $x \in A$, define the *characteristic* of x, $ch(x)$, as a binary sequence $(t_1, t_2, \ldots, t_s)$ such that for each $1 \le i \le s$, $t_i \in \{0, 1\}$ if $x \in P_i$ and $t_i = 0$ otherwise. There are 2^s pairwise distinct characteristics, and we order them in lexicographic order: $ch_1, \ldots, ch_{2^s}$. Construct the structure $\mathcal{A}' = (A; Q_1, \ldots, Q_{2^s})$ such that for all $1 \le i \le 2^s$, $Q_i = \{x \in A \mid ch(x) = ch_i\}$. The following is now an easy lemma.

Lemma 2.3. *Duplicator wins $G_n(\mathcal{A}, \mathcal{B})$ if and only if Duplicator wins $G_n(\mathcal{A}', \mathcal{B}')$.* □

The above lemmas give us the following easy proof of the theorem. We represent each of $\mathcal{A}$ and $\mathcal{B}$ by 2^s lists, and the ith list lists all elements with characteristic ch_i. To solve the game $G_n(\mathcal{A}', \mathcal{B}')$, the algorithm checks the conditions in Lemma 2.2 by reading the lists. In each list it reads at most n elements. Hence the process takes time bounded by $2^s \cdot n$. □

3. Equivalence structures

In this section we study EF games played on *equivalence structure*. These are structures of the form $\mathcal{A} = (A; E)$ where E is an equivalence relation on A. We list all the equivalence classes of $\mathcal{A}$ as $A_1, \ldots, A_k$ such that $|A_i| \le |A_{i+1}|$ for all $1 \le i < k$. Let $q_{\mathcal{A}}$ be the number of equivalent classes in $\mathcal{A}$; for each $t < n$, let q_t^A be the number of equivalent classes in $\mathcal{A}$ with size t. Finally, let $q_{\ge r}^{\mathcal{A}}$ be the number of equivalence classes in $\mathcal{A}$ of size at least r. For an equivalence structure $\mathcal{B}$ we have similar notations as $B_1, B_2, \ldots$ to denote its equivalence classes, and the associated numbers $q_{\mathcal{B}}$, $q_t^{\mathcal{B}}$, and $q_{\ge r}^{\mathcal{B}}$.

Lemma 3.1. *If Duplicator wins game $G_n(\mathcal{A}, \mathcal{B})$ on equivalence structures $\mathcal{A}$ and $\mathcal{B}$, then the following must be true:*

1. *If $q_{\mathcal{A}} < n$ or $q_{\mathcal{B}} < n$ then $q_{\mathcal{A}} = q_{\mathcal{B}}$; and*
2. *$q_{\mathcal{A}} \ge n$ if and only if $q_{\mathcal{B}} \ge n$.*

To prove the lemma, we assumes that one of the two statements (1) or (2) is false. Then it is clear that Spoiler has a winning strategy in game $G_n(\mathcal{A}, B)$, contradicting the assumption of the lemma.

In our analysis below, by the above lemma, we always assume that $q_{\mathcal{A}} = q_{\mathcal{B}}$ or $q_{\mathcal{A}} \ge n$ if and only if $q_{\mathcal{B}} \ge n$. We need the following notation for the next lemma and definition. For $t \le n$, let $q^t = \min\{q_{\ge t}^{\mathcal{A}}, q_{\ge t}^{\mathcal{B}}\}$. Let $\mathcal{A}_t$ and $\mathcal{B}_t$ be equivalence structures obtained by taking out exactly q^t equivalence classes of size $\ge t$ from $\mathcal{A}$ and $\mathcal{B}$ respectively. We also set $n - q^t$ to be 0 in case $q^t \ge n$; and otherwise $n - q^t$ has its natural meaning.

Lemma 3.2. 1. *Assume that there is a $t < n$ such that $q_t^{\mathcal{A}} \ne q_t^{\mathcal{B}}$ and $n - \min\{q_t^{\mathcal{A}}, q_t^{\mathcal{B}}\} > t$. Then Spoiler wins game $G_n(\mathcal{A}, \mathcal{B})$.*
2. *Assume that there is a $t \le n$ such that $n - q^t > 0$ and one of the structures $\mathcal{A}_t$ or $\mathcal{B}_t$ has an equivalence class of size $\ge n - q^t$ and the other structure does not. Then Spoiler wins game $G_n(\mathcal{A}, \mathcal{B})$.*

Proof. To prove the first part of the lemma, assume that $q_t^{\mathcal{A}} > q_t^{\mathcal{B}}$ and $n - q_t^{\mathcal{B}} > t$. Spoiler's strategy is the following. First, select elements $a_1, \ldots, a_{q_t^{\mathcal{B}}}$ from distinct equivalence classes of size t in $\mathcal{A}$. Duplicator must select elements $b_1, \ldots, b_{q_t^{\mathcal{B}}}$ also from distinct equivalence classes of size t in $\mathcal{B}$ as otherwise, Duplicator will clearly lose. Next, Spoiler selects t distinct elements $x_1, \ldots, x_t$ in the equivalence class of size t in $\mathcal{A}$. If Duplicator responds by choosing elements $y_1, \ldots, y_t$ in an equivalence class of size $< t$ then Duplicator would clearly lose. Hence, Duplicator must select all $y_1, \ldots, y_t$ from an equivalence class Y of size $> t$. After t moves, Spoiler selects a new element in Y, thus winning the game. The case when $q_t^{\mathcal{B}} > q_t^{\mathcal{A}}$ and $n - q_t^{\mathcal{A}} > t$ is proved similarly.

For the second part, assume $\mathcal{A}_t$ has an equivalence class of size $\ge n - q^t$ and $\mathcal{B}_t$ does not, Spoiler has the following winning strategy. Spoiler selects q^t pairwise non-equivalent elements $a_1, \ldots, a_{q^t}$ in $\mathcal{A}$ from equivalence classes of size greater than or equal to t. Let $b_1, \ldots, b_{q^t}$ be elements selected by Duplicator. Note that for each i, the size of $[b_i]$ is greater than or equal to t. Otherwise, if the size of the equivalence class $[b_i]$ of b_i were smaller than t, then the size of $[b_i]$ would be smaller than $n - q^t$. Hence, in this case, Spoiler would win by selecting elements from $[a_i]$. Now let X be an equivalence class of size $\ge n - q^t$ as stipulated in the lemma. Spoiler wins the game by selecting $n - q^t$ distinct elements in X. □

We now single out the hypotheses of the lemma above and give the following definition.

Definition 3.3. 1. We say that $G_n(\mathcal{A}, B)$ *has small disparity* if there is a $t < n$ such that either $q_t^{\mathcal{A}} \ne q_t^{\mathcal{B}}$ and $n - \min\{q_t^{\mathcal{A}}, q_t^{\mathcal{B}}\}$ $> t$.

2. We say that $G_n(\mathcal{A}, B)$ *has large disparity* if there exists a $t \leq n$ such that $n - q^t > 0$ and one of the structures $\mathcal{A}_t$ or $\mathcal{B}_t$ has an equivalence class of size $\geq n - q^t$ and the other structure does not.

For the next lemma, recall that a *k-round play*, where $k \leq n$, in game $G_n(\mathcal{A}, \mathcal{B})$ is a sequence $(a_1, b_1), \ldots, (a_k, b_k)$ such that for each $1 \leq i \leq k$ we have $a_i \in A$ and $b_i \in B$; and if Spoiler has chosen a_i (or b_i), then Duplicator has responded by b_i (a_i).

Lemma 3.4. *Duplicator wins game $G_n(\mathcal{A}, \mathcal{B})$ if and only if $G_n(\mathcal{A}, \mathcal{B})$ has neither small nor large disparity.*

Proof. The previous lemma proves one direction. For the other, we assume that neither small nor large disparity occurs in the game. We describe a winning strategy for Duplicator.

Let us assume that the players have produced a k-round play $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$. In case $k = 0$, we are at the start of the game $G_n(\mathcal{A}, \mathcal{B})$. Our inductive assumptions on this k-round play are the following:

1. $E(a_i, a_j)$ is true in $\mathcal{A}$ if and only if $E(b_i, b_j)$ is true in $\mathcal{B}$, and the map $a_i \to b_i$ is one-to-one.
2. For all a_i, $|[a_i]| \geq n - i$ if and only if $|[b_i]| \geq n - i$, where $[x]$ denotes the equivalence class of x.
3. For a_i if $|[a_i]| < n - i$ then $|[a_i]| = |[b_i]|$.
4. Let $\mathcal{A}'$ and $\mathcal{B}'$ be the equivalence structures obtained by removing the equivalence classes $[a_1], \ldots, [a_k]$ from $\mathcal{A}$ and the equivalence classes $[b_1], \ldots, [b_k]$ from $\mathcal{B}$, respectively. We assume that $\mathcal{A}'$ and $\mathcal{B}'$ satisfy the following conditions:
   (a) In game $G_{n-k}(\mathcal{A}', \mathcal{B}')$ no small disparity occurs.
   (b) In game $G_{n-k}(\mathcal{A}', \mathcal{B}')$ no large disparity occurs.

Assume that Spoiler selects an element $a_{k+1} \in A$. Duplicator responds to this move by choosing b_{k+1} as follows. If $a_{k+1} = a_i$ then $b_{k+1} = b_i$. Otherwise, if $E(a_i, a_{k+1})$ is true in $\mathcal{A}$ then Duplicator chooses a new b_{k+1} such that $E(b_i, b_{k+1})$ is true in $\mathcal{B}$. Assume a_{k+1} is not equivalent to any of the elements $a_1, \ldots, a_k$. If $|[a_{k+1}]| \geq n - k$ then Duplicator chooses b_{k+1} such that b_{k+1} is not equivalent to any of the elements $b_1, \ldots, b_k$ and $|[b_{k+1}]| \geq n - k$. Duplicator can select such an element as otherwise large disparity would occur in the game. If $|[a_{k+1}]| < n - k$ then Duplicator chooses b_{k+1} such that $|[b_{k+1}]| = |[a_{k+1}]|$ and b_{k+1} is not equivalent to any of the elements $b_1, \ldots, b_k$. Duplicator can select such an element as otherwise small disparity would occur in the game. The case when Spoiler selects an element from B is treated similarly.

We now show that the $(k+1)$-round play $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k), (a_{k+1}, b_{k+1})$ satisfies the inductive assumptions. Inductive assumptions (1), (2), and (3) can easily be checked to be preserved. To show that assumption (4) is preserved, consider the equivalence structures $\mathcal{A}''$ and $\mathcal{B}''$ obtained by removing the equivalence classes $[a_1], \ldots, [a_k], [a_{k+1}]$ from $\mathcal{A}$ and the equivalence classes $[b_1], \ldots, [b_k], [b_{k+1}]$ from $\mathcal{B}$, respectively. In game $G_{n-k-1}(\mathcal{A}'', \mathcal{B}'')$ small disparity does not occur as otherwise game $G_{n-k}(\mathcal{A}', \mathcal{B}')$ would have small disparity. Thus, assumption (4a) is also preserved. Similarly, if $G_{n-k-1}(\mathcal{A}'', \mathcal{B}'')$ had large disparity then game $G_{n-k}(\mathcal{A}', \mathcal{B}')$ would also have large disparity contradicting the inductive assumption. Hence the strategy described must be a winning strategy due to the fact that Duplicator preserves the inductive assumption (1) at each round. □

Theorem 3.5. *Fix $n \in \omega$. There exists an algorithm that runs in constant time and decides whether Duplicator wins the game $G_n(\mathcal{A}, \mathcal{B})$ on equivalence structures $\mathcal{A} = (A; E)$ and $\mathcal{B} = (B; E)$. The constant that bounds the running time is n.*

Proof. This result follows from the lemmas above. For the proof, we represent each equivalence structure $\mathcal{A}$ and $\mathcal{B}$ in two lists. For example, the first list for the structure $\mathcal{A}$ lists all equivalence classes of $\mathcal{A}$ in increasing order; the second list is $q^{\mathcal{A}}$, $q_1^{\mathcal{A}}, q_{\geq 1}^{\mathcal{A}}, q_2^{\mathcal{A}}, q_{\geq 2}^{\mathcal{A}}, \ldots$. The algorithm runs through the second lists for both $\mathcal{A}$ and $\mathcal{B}$, and for each $t \leq n$ checks whether or not small or large disparity occurs. If the algorithm detects disparity then Spoiler wins, otherwise, Duplicator wins. □

This theorem can be extended to equivalence structures expanded with unary predicates that act on equivalence classes uniformly as explained in the following definition.

Definition 3.6. A *homogeneously colored equivalence structure* is of the form $(A; E, P_1, \ldots, P_s)$ such that

- $(A; E)$ is an equivalence structure; and
- Each P_i is a homogeneous unary relation on A meaning that for all $x, y \in A$ if $E(x, y)$ then $x \in P_i$ if and only if $y \in P_i$.

Let $\mathcal{A} = (A; E, P_1, \ldots, P_s)$ be a homogeneously colored equivalence structure. As in the previous section, we define the characteristic $ch(x)$ of an element $x \in A$ as a binary sequence $(t_1, t_2, \ldots, t_s)$ such that for each $1 \leq i \leq s$, $t_i = 1$ if $x \in P_i$, and $t_i = 0$ otherwise. Since each predicate P_i is homogeneous, any pair of equivalent elements of $\mathcal{A}$ have the same characteristics. Therefore we can represent $\mathcal{A}$ as a disjoint union of equivalence structures $\mathcal{A}_1, \ldots, \mathcal{A}_{2^s}$, where A_ϵ is the subset of A consisting of elements with characteristic ϵ. The above theorem can thus be extended to the following result:

Theorem 3.7. *There exists an algorithm that runs in constant time and decides whether Duplicator wins game $G_n(\mathcal{A}, \mathcal{B})$ on homogeneously colored equivalence structures $\mathcal{A}$ and $\mathcal{B}$. The constant that bounds the running time is $2^s \cdot n$.* □

4. Equivalence structures with colors

In this section we study structures $\mathcal{A}$ of the form $(A; E, P_1, \ldots, P_s)$, where E is an equivalence relation on A and $P_1, \ldots, P_s$ are unary predicates on A. Note that $P_1, \ldots, P_s$ are not necessarily homogeneous unary predicates. We call these structures *equivalence structures with s colors*. Our goal is to give a full solution for EF games played on equivalence structures with s colors. We start with the case when $s = 1$. The case for $s > 2$ will be explained later.

Let $\mathcal{A} = (A; E, P)$ be an equivalence structure with one color. We say $x \in A$ is *colored* if $P(x)$ is true; otherwise we say x is *non-colored*. We say that an equivalence class X has *type* $tp(X) = (i, j)$, if the number of colored elements of X is i, non-colored elements is j; thus, $i + j = |X|$.

Definition 4.1. Given two types (i, j) and (i', j') respectively. We say that (i, j) is *colored n-equivalent* to (i', j'), denoted by $(i, j) \equiv_n^C (i', j')$, if the following holds.

1. If $i < n$ then $i' = i$, otherwise $i' \geq n$.
2. If $j < n - 1$ then $j' = j$, otherwise $j' \geq n - 1$.

We say that (i, j) is *non-colored n-equivalent* to (i', j'), denoted by $(i, j) \equiv_n^N (i', j')$, if the following holds.

1. If $j < n$ then $j' = j$, otherwise $j' \geq n$;
2. If $i < n - 1$ then $i' = i$, otherwise $i' \geq n - 1$.

For $X \subseteq A$, we use $(X; E \restriction X, P \restriction X)$ to denote the equivalence structure obtained by restricting E and P on X. Note that given two equivalence classes X and Y of types (i, j) and (i', j') respectively, if (i, j) is colored (non-colored) n-equivalent to (i', j'), then Duplicator wins the n-round game played on structures $(X; E \restriction X, P \restriction X)$ and $(Y, E \restriction Y, P \restriction Y)$, given the fact that Spoiler chooses a colored (non-colored) element in the first round. The following lemma follows from the definition above.

Lemma 4.2. *If* $(i', j') \equiv_n^C (i, j)$ *or* $(i', j') \equiv_n^N (i, j)$, *then* $(i', j') \equiv_{n-1}^C (i, j)$ *and* $(i', j') \equiv_{n-1}^N (i, j)$. $\square$

For an equivalence structure $\mathcal{A} = (A; E, P)$, we need the following notations:

- For type (i, j) and $k \geq 1$, Set $C^{\mathcal{A}}_{(i,j),k}$ be the set $\{X \mid X$ is an equivalence class of $\mathcal{A}$ and $tp(X) \equiv_k^C (i, j)\}$. Set $N^{\mathcal{A}}_{(i,j),k}$ as the set $\{X \mid X$ is an equivalence class of $\mathcal{A}$ and $tp(X) \equiv_k^N (i, j)\}$.
- For type (i, j) and $k \geq 1$, set

$$q^{\mathcal{A},C}_{(i,j),k} = |C^{\mathcal{A}}_{(i,j),k}| \quad \text{and} \quad q^{\mathcal{A},N}_{(i,j),k} = |N^{\mathcal{A}}_{(i,j),k}|.$$

- For $\mathcal{A}$ and $\mathcal{B}$, set

$$q^{C}_{(i,j),k} = \min\{q^{\mathcal{A},C}_{(i,j),k}, q^{\mathcal{B},C}_{(i,j),k}\} \quad \text{and} \quad q^{N}_{(i,j),k} = \min\{q^{\mathcal{A},N}_{(i,j),k}, q^{\mathcal{B},N}_{(i,j),k}\}.$$

- Set $\mathcal{A}^C((i, j), k)$ as the structure obtained from $\mathcal{A}$ by removing $q^C_{(i,j),k}$ equivalence classes in $C^{\mathcal{A}}_{(i,j),k}$.
- Set $\mathcal{A}^N((i, j), k)$ as the structure obtained from $\mathcal{A}$ by removing $q^N_{(i,j),k}$ equivalence classes in $N^{\mathcal{A}}_{(i,j),k}$.

Observe the following: If Spoiler selects a colored element from an equivalence class X in $\mathcal{A}$ and Duplicator responds by selecting another colored elements from an equivalence class Y in $\mathcal{B}$ such that $tp(Y) \equiv_n^C tp(X)$, there is no point for Spoiler to keep playing inside X because this will guarantee a win for Duplicator. Conversely, suppose Spoiler selects a colored element from an equivalence class X in $\mathcal{A}$, and there is no equivalence class in $\mathcal{B}$ whose type is colored n-equivalent to $tp(X)$. Then Spoiler has a winning strategy for the game by playing inside X and Y.

Definition 4.3. Consider game $G_n(\mathcal{A}, \mathcal{B})$ played on equivalence structures with one color. We say that a *colored disparity* occurs if there exists a type (i, j) and $n > k \geq 0$ such that the following holds:

1. $k = q^C_{(i,j),n-k}$;
2. In one of $\mathcal{A}^C((i, j), n - k)$ and $\mathcal{B}^C((i, j), n - k)$, there is an equivalence class whose type is colored $(n - k)$-equivalent to (i, j), and no such equivalence class exists in the other structure.

We say that a *non-colored disparity* occurs if there exists a type (i, j) and $n > k \geq 0$ such that the following holds:

1. $k = q^N_{(i,j),n-k}$;
2. In one of $\mathcal{A}^N((i, j), n - k)$ and $\mathcal{B}^N((i, j), n - k)$, there is an equivalence class whose type is non-colored $(n - k)$-equivalent to (i, j), and no such equivalence class exists in the other structure.

Lemma 4.4. *Suppose* $\mathcal{A}$ *and* $\mathcal{B}$ *are two equivalence structures with one color. Duplicator wins the game* $G_n(\mathcal{A}, \mathcal{B})$ *if and only if neither colored disparity nor non-colored disparity occurs in the game.*

Proof. Suppose a colored disparity occurs in game $G_n(\mathcal{A}, \mathcal{B})$ as witnessed by (i, j) and k. Suppose in $\mathcal{A}^C((i, j), n-k)$ there is an equivalence class whose type is colored $(n-k)$-equivalent to (i, j), and no such equivalence class exists in the other structure. We describe a winning strategy for Spoiler. The case when a non-colored disparity occurs is treated in a similar manner. To win the game, Spoiler selects $k = q^C_{(i,j),n-k}$ pairwise non-equivalent elements $a_1, \ldots, a_k$ in $\mathcal{A}$ from equivalence classes in $C^{\mathcal{A}}_{(i,j),n-k}$. Let $b_1, \ldots, b_k$ be elements selected by Duplicator in response. For each $1 \leq l \leq k$, let $[b_l]$ be the equivalence class of b_l. Note that $tp([b_l]) \equiv^C_{n-k} (i, j)$ as otherwise Spoiler would win. Now let X be an equivalence class in $\mathcal{A}$ such that $tp(X) \equiv^C_{n-k} (i, j)$ as stipulated in the lemma. Spoiler selects a colored element $x \in X$. Let y be the element selected by Duplicator in response to x and set $Y = [y]$. Note that $tp(Y)$ cannot be colored $(n-k)$-equivalent to $tp(X)$. By definition of $(n-k)$-equivalence, from now on, Spoiler uses a winning strategy inside X and Y and wins the game $G_n(\mathcal{A}, \mathcal{B})$.

Conversely, suppose neither colored disparity nor non-colored disparity occurs in game $G_n(\mathcal{A}, \mathcal{B})$, we then describe a strategy for Duplicator. Let us assume that the players have produced a k-round play $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$. Let (i_l, j_l) and (i'_l, j'_l) be the types of a_l and b_l,respectively with $1 \leq l \leq k$. Our inductive assumptions on this k-round play are the following:

1. For any $1 \leq l \leq k$, a_l is a colored element if and only if b_l is a colored element.
2. For any $1 \leq l, m \leq k$, $E(a_l, a_m)$ if and only if $E(b_l, b_m)$.
3. For any $1 \leq l \leq k$, $(i_l, j_l) \equiv^C_{n-l} (i'_l, j'_l)$ and $(i_l, j_l) \equiv^N_{n-l} (i'_l, j'_l)$.
4. Let $\mathcal{A}'$ and $\mathcal{B}'$ be the equivalence structures obtained by removing equivalence classes $[a_1], \ldots, [a_k]$ from $\mathcal{A}$ and $[b_1], \ldots, [b_k]$ from $\mathcal{B}$, respectively. We assume in game G_{n-k} that neither colored disparity nor non-colored disparity occurs.

Assume that Spoiler selects an element $a_{k+1} \in A$. Duplicator responds to this move by choosing b_{k+1} as follows. If $a_{k+1} = a_l$ then $b_{k+1} = b_l$. Otherwise, if $E(a_{k+1}, a_l)$ is true in $\mathcal{A}$, then Duplicator chooses a new b_{k+1} such that $E(b_{k+1}, b_l)$ and a_{k+1} is a colored element if and only if b_{k+1} is a colored element. By (3) of the inductive assumption, Duplicator can always select such an element b_{k+1}.

Assume a_{k+1} is not equivalent to any of the elements $a_1, \ldots, a_k$. Let X be the equivalence class of a_{k+1} in $\mathcal{A}$. If a_{k+1} is a colored element, then Duplicator chooses a colored element b_{k+1} from an equivalence class Y of $\mathcal{B}$ such that $tp(X) \equiv^C_{n-k} tp(Y)$. If a_{k+1} is a non-colored element, then Duplicator chooses a non-colored b_{k+1} from an equivalence class Y of $\mathcal{B}$ such that $tp(X) \equiv^N_{n-k} tp(Y)$. Note that such an equivalence class Y must exist in $\mathcal{B}$, as otherwise either colored or non-colored disparity would occur in $G_{n-k}(\mathcal{A}', \mathcal{B}')$ as witnessed by $tp(X)$ and 0. The case when Spoiler selects an element from B is treated in a similar manner.

On the play $(a_1, b_1), \ldots, (a_k, b_k), (a_{k+1}, b_{k+1})$, inductive assumption (1) and (2) can be easily checked to hold. To prove that inductive assumption (3) holds, let (i_{k+1}, j_{k+1}) and (i'_{k+1}, j'_{k+1}) be the type of $[a_{k+1}]$ and $[b_{k+1}]$, respectively. The strategy ensures one of $(i_{k+1}, j_{k+1}) \equiv^C_{n-k} (i'_{k+1}, j'_{k+1})$ and $(i_{k+1}, j_{k+1}) \equiv^N_{n-k} (i'_{k+1}, j'_{k+1})$ is true, and by Lemma 4.2, $(i_{k+1}, j_{k+1}) \equiv^C_{n-k-1} (i'_{k+1}, j'_{k+1})$ and $(i_{k+1}, j_{k+1}) \equiv^N_{n-k-1} (i'_{k+1}, j'_{k+1})$.

It remains for us to prove that inductive assumption (4) is preserved. Consider the structure $\mathcal{A}''$ and $\mathcal{B}''$ obtained by removing $[a_1], \ldots, [a_{k+1}]$ from $\mathcal{A}$ and $[b_1], \ldots, [b_{k+1}]$ from $\mathcal{B}$, respectively. Suppose a colored disparity occurs in $G_{n-k-1}(\mathcal{A}'', \mathcal{B}'')$ as witnessed by some type (i, j) and $0 \leq t < n-k-1$. There are two cases. If $(i, j) \equiv^C_{n-k-t-1} (i_{k+1}, j_{k+1})$, then by Lemma 4.2 $(i, j) \equiv^C_{n-k-t-1} (i'_{k+1}, j'_{k+1})$, and a colored disparity occurs in $G_{n-k}(\mathcal{A}', \mathcal{B}')$ as witnessed by (i, j) and $t+1$; If $(i, j) \not\equiv^C_{n-k-t-1} (i_{k+1}, j_{k+1})$, then by Lemma 4.2, $(i, j) \not\equiv^C_{n-k-t-1} (i'_{k+1}, j'_{k+1})$, and a colored disparity occurs in $G_{n-k}(\mathcal{A}', \mathcal{B}')$ as witnessed by (i, j) and t, contradicting our assumption. The case when a non-colored disparity occurs in $G_{n-k-1}(\mathcal{A}'', \mathcal{B}'')$ is treated in a similar way.

Hence the strategy is a winning strategy for Duplicator by inductive assumptions (1) and (2). The lemma is proved. □

Theorem 4.5. *Fix $n \in \omega$. There exists an algorithm that runs in constant time and decides whether Duplicator wins game $G_n(\mathcal{A}, \mathcal{B})$ on equivalence structures with one color $\mathcal{A}$ and $\mathcal{B}$. The constant that bounds the running time is n^3.*

Proof. This result follows from the lemmas above. We present colored equivalence structures $\mathcal{A}$ in three lists. The first one lists equivalence classes of $\mathcal{A}$ in increasing order of their types; the second and the third one list the sequence $\{q^{\mathcal{A},C}_{(i,j),k}\}_{0 \leq i,j,k \leq n}$ and $\{q^{\mathcal{A},N}_{(i,j),k}\}_{0 \leq i,j,k \leq n}$ respectively. The algorithm checks whether a colored or a non-colored disparity occurs by reading the second and third list. If the algorithm detects a disparity then Spoiler wins, otherwise, Duplicator wins. The running time for the process is bounded by n^3. □

Fix $s > 1$, let $\mathcal{A}$ be an equivalence structure with s colors. For each element x of $\mathcal{A}$, define the *characteristic* of x, $ch(x)$, as a binary sequence $(t_1, t_2, \ldots, t_s)$ such that for each $1 \leq i \leq s$, $t_i \in \{0, 1\}$, $t_i = 1$ if $x \in P_i$ and $t_i = 0$ otherwise. There are 2^s pairwise distinct characteristics, and we order them in lexicographic order: $ch_1, \ldots, ch_{2^s}$. Construct the structure $\mathcal{A}' = (A; E, Q_1, \ldots, Q_{2^s})$ such that for all $1 \leq i \leq 2^s$, $Q_i = \{x \in A \mid ch(x) = ch_i\}$.

The following is an easy lemma:

Lemma 4.6. *Let $\mathcal{A} = (A; E, P_1, \ldots, P_s)$ be an equivalence structure with s unary predicates.*

1. *For any two distinct characteristics ch_i and ch_j, we have $Q_i \cap Q_j = \emptyset$.*
2. *$\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if $\mathcal{A}'$ and $\mathcal{B}'$ are isomorphic.* □

For an equivalence class X, we define the *type* of X, $tp(X)$, as a sequence $(i_1, i_2, \ldots, i_{2^s})$ such that in X, the number of elements with characteristic ch_j is i_j for all $1 \le j \le 2^s$; thus $\sum_{j=1}^{2^s} i_j = |X|$.

Definition 4.7. Let $\kappa = (i_1, \ldots, i_{2^s})$ and $\lambda = (i'_1, \ldots, i'_{2^s})$ be two types of equivalence classes. For $1 \le j \le 2^s$, we say that κ is (j, n)-*equivalent* to λ, denoted by $\kappa \equiv_n^j \lambda$, if the following holds.

1. If $i_j < n$ then $i'_j = i_j$, otherwise $i'_j \ge n$; and
2. For all $1 \le l \le 2^s$ where $l \ne j$, if $i_l < n - 1$ then $i'_l = i_l$, otherwise $i'_l \ge n - 1$.

Let X and Y be equivalence classes of types κ and λ respectively. If $\kappa \equiv_n^j \lambda$, then Duplicator wins the n-round EF game played on structures $(X; E \upharpoonright X, P_1 \upharpoonright X, \ldots, P_s \upharpoonright X)$ and $(Y; E \upharpoonright Y, P_1 \upharpoonright Y, \ldots, P_s \upharpoonright Y)$, given that Spoiler selects an element $x \in X$ with characteristic ch_j.

For type λ, $1 \le j \le 2^s$ and $k \ge 1$, we set $C_{\lambda,k}^{\mathcal{A},j}$ be the set $\{X \mid X$ is an equivalence class of $\mathcal{A}$ and $tp(X) \equiv_k^j \lambda\}$. Similar to the case of equivalence structures with one color, we introduce the notions $q_{\lambda,k}^{A,j}$, $q_{\lambda,k}^{j}$ and $\mathcal{A}^j(\lambda, k)$.

Definition 4.8. Consider game $G_n(\mathcal{A}, \mathcal{B})$ played on equivalence structures with s colors. For $1 \le j \le 2^s$, we say that a *disparity occurs with respect to* ch_j if there exists a type $\lambda = (i_1, \ldots, i_{2^s})$ and $n > k \ge 0$ such that the following holds:

1. $k = q_{\lambda,n-k}^{j}$;
2. In one of $\mathcal{A}^j(\lambda, n-k)$, there is an equivalence class whose type is $(j, n-k)$-equivalent to λ, and no such equivalence class exists in the other structure.

The following lemma can thus be proved in a similar manner as Lemma 4.4.

Lemma 4.9. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are two equivalence structures with s colors. Duplicator wins the game $G_n(\mathcal{A}, \mathcal{B})$ if and only if disparity does not occur with respect to ch_j for any $1 \le j \le 2^s$.* $\square$

By the lemma above, we can extend Theorem 4.5 to the following results.

Theorem 4.10. *Fix $n \in \omega$. There exists an algorithm that runs in constant time and decides whether Duplicator wins the game $G_n(\mathcal{A}, \mathcal{B})$ on equivalence structures with s colors. The constant that bounds the running time is n^{2^s+1}.* $\square$

5. Embedded equivalence structures

In this section we study *embedded equivalence structure of height h*; these are structures of the form $\mathcal{A} = (A; E_1, E_2, \ldots, E_h)$ such that each E_i where $1 \le i \le h$ is an equivalence relation and $E_i \subseteq E_j$ for $i < j$. In this section, we give a full solution for EF games played on embedded equivalence structures of height h. We start with the case where $h = 2$. The case for $h > 2$ will be explained later.

Let $\mathcal{A} = (A; E_1, E_2)$ be an embedded equivalence structure of height 2. We say that an E_2-equivalence class X has type $tp(X) = (q_1, \ldots, q_t)$ if the largest E_1-equivalence class contained in X has size t and for all $1 \le i \le t$, q_i is the number of E_1-equivalence classes of size i contained in X. Thus, $\sum_{i=1}^{t}(q_i \times i) = |X|$. For two types $\sigma = (q_1, \ldots, q_{t_1})$ and $\tau = (q'_1, \ldots, q'_{t_2})$, we say $\sigma = \tau$ if $t_1 = t_2$ and $q_i = q'_i$ for all $1 \le i \le t_1$.

For $X \subseteq A$, we use $(X; E_1 \upharpoonright X)$ to denote the equivalence structure obtained by restricting E_1 on X. Given two E_2-equivalence classes X and Y of types σ and τ respectively, we say that σ is n-equivalent to τ, denoted by $\sigma \equiv_n \tau$, if Duplicator wins the n-round game played on structures $(X; E_1 \upharpoonright X)$ and $(Y; E_1 \upharpoonright Y)$. Note that if $\sigma \equiv_n \tau$, then $\sigma \equiv_i \tau$ for all $i \le n$.

We need the following notations:

- For type σ and $i \ge 1$, set

  $$C_{\sigma,i}^{\mathcal{A}} = \{X \mid X \text{ is an } E_2\text{-equivalence class of } \mathcal{A} \ \wedge \ tp(X) \equiv_i \sigma\}.$$

- Set $q_{\sigma,i}^{\mathcal{A}} = |C_{\sigma,i}^{\mathcal{A}}|$.
- For embedded equivalence structure $\mathcal{A}$ and $\mathcal{B}$, set $q^{\sigma,i} = \min\{q_{\sigma,i}^{\mathcal{A}}, q_{\sigma,i}^{\mathcal{B}}\}$.
- Set $\mathcal{A}(\sigma, i)$ be the embedded equivalence structure of height 2 obtained from $\mathcal{A}$ by removing $q^{\sigma,i}$ equivalence classes whose types are i-equivalent to σ.

Observe that in round k of game $G_n(\mathcal{A}, \mathcal{B})$, if Spoiler selects an element from an E_2-equivalence class X in $\mathcal{A}$, and Duplicator responds by selecting another element from an E_2-equivalence class Y in $\mathcal{B}$ such that $tp(Y) \equiv_{n-k} tp(X)$, there is no reason for Spoiler to keep playing inside X because this will guarantee a win for Duplicator. Intuitively, $\mathcal{A}(\sigma, n-k)$ contains all the E_2-equivalence classes for Spoiler to choose elements from after $q^{\sigma,n-k}$ many E_2-equivalence classes whose types are $(n-k)$-equivalent to σ have been chosen.

Definition 5.1. Consider a game $G_n(\mathcal{A}, \mathcal{B})$ played on embedded equivalence structures of height 2. We say that a *disparity* occurs if there exists a type σ and $n > k \ge 0$ such that the following holds.

1. $k = q^{\sigma,n-k}$.
2. In one of $\mathcal{A}(\sigma, n-k)$ and $\mathcal{B}(\sigma, n-k)$, there is an E_2-equivalence class whose type is $(n-k)$-equivalent to σ, and no such E_2-equivalence class exists in the other structure.

Lemma 5.2. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are two embedded equivalence structures of height 2. Duplicator wins the game $G_n(\mathcal{A}, \mathcal{B})$ if and only if no disparity occurs.*

Proof. Suppose disparity occurs in game $G_n(\mathcal{A}, \mathcal{B})$ as witnessed by σ and k, in $\mathcal{A}(\sigma, n-k)$ there is an E_2-equivalence class whose type is $(n-k)$-equivalent to σ, and no such E_2-equivalence class exists in $\mathcal{B}(\sigma, n-k)$. We describe a winning strategy for Spoiler as follows. Spoiler selects $k = q^{\sigma,n-k}$ pairwise non-E_2-equivalent elements $a_1, \ldots, a_k$ in $\mathcal{A}$ from E_2-equivalence classes whose types are $(n-k)$-equivalent to σ. Let $b_1, \ldots, b_k$ be elements selected by Duplicator in response. For each i, let $[b_i]_{E_2}$ be E_2-equivalence class that b_i is in. Note that $tp([b_i]) \equiv_{n-k} \sigma$ as otherwise Spoiler would win. Now let X be an equivalence class in $\mathcal{A}$ such that $tp(X) \equiv_{n-k} \sigma$ as stipulated in the lemma. Spoiler selects an element $x \in X$. Let y be the element selected by Duplicator in response to x and set $Y = [y]_{E_2}$. Note that $tp(Y)$ cannot be $(n-k)$-equivalent to $tp(X)$. By definition of $(n-k)$-equivalence, henceforth Spoiler uses a winning strategy inside X and Y and wins game $G_n(\mathcal{A}, \mathcal{B})$.

Conversely, suppose no disparity occurs in the game $G_n(\mathcal{A}, \mathcal{B})$, we then describe a strategy for Duplicator. Let us assume that the players have produced a k-round play $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$. Let σ_i and τ_i be the types of a_i and b_i, respectively with $1 \le i \le k$. Our inductive assumptions on this k-round play are the following:

1. The map $a_i \to b_i$ is partial isomorphism.
2. For all $1 \le i \le k$, $\sigma_i \equiv_{n-i} \tau_i$.
3. Let $\mathcal{A}'$ and $\mathcal{B}'$ be the equivalence structures obtained by removing the E_2-equivalence classes $[a_1]_{E_2}, \ldots, [a_k]_{E_2}$ from $\mathcal{A}$ and the equivalence classes $[b_1]_{E_2}, \ldots, [b_k]_{E_2}$ from $\mathcal{B}$, respectively. We assume in game $G_{n-k}(\mathcal{A}', \mathcal{B}')$ that no disparity occurs.

Assume that Spoiler selects an element $a_{k+1} \in A$. Duplicator responds to this move by choosing b_{k+1} as follows. If $a_{k+1} = a_i$ then $b_{k+1} = b_i$. Otherwise, if $E_1(a_i, a_{k+1})$ is true in $\mathcal{A}$, then Duplicator chooses a new b_{k+1} such that $E_1(b_i, b_{k+1})$. If $E_2(a_i, a_{k+1})$ is true in $\mathcal{A}$ and there is no j such that $E_1(a_j, a_{k+1})$, then Duplicator chooses a new b_{k+1} such that $E_2(b_i, b_{k+1})$ and there is no j such that $E_1(b_j, b_{k+1})$. By (2) of the inductive assumption Duplicator can always select such an element b_{k+1} by following its winning strategies.

Assume a_{k+1} is not equivalent to any of the elements $a_1, \ldots, a_k$. Let X be the E_2-equivalence class in $\mathcal{A}$ that contains a_{k+1}. Duplicator selects b_{k+1} from an E_2-equivalence class Y in $\mathcal{B}$ such that $tp(X) \equiv_{n-k} tp(Y)$. Duplicator is able to select such an element as otherwise disparity would occur as witnessed by the type of X and 0.

The case when Spoiler selects an element from B is treated similarly.

Inductive assumption (1) and (2) can be easily checked to hold on the play $(a_1, b_1), \ldots, (a_k, b_k), (a_{k+1}, b_{k+1})$. To show that assumption (3) is preserved, consider the structures $\mathcal{A}''$ and $\mathcal{B}''$ obtained by removing $[a_1]_{E_2}, \ldots, [a_k]_{E_2}, [a_{k+1}]_{E_2}$ and $[b_1]_{E_2}, \ldots, [b_k]_{E_2}, [b_{k+1}]_{E_2}$ from $\mathcal{A}$ and $\mathcal{B}$, respectively. Suppose disparity occurs in $G_{n-k-1}(\mathcal{A}'', \mathcal{B}'')$ as witnessed by some type τ and $t < n-k-1$. There are two cases. If $tp([a_{k+1}]) \equiv_{n-k-t-1} \tau$, then $tp([b_{k+1}]) \equiv_{n-k-t-1} \tau$, and disparity must occur in $G_{n-k}(\mathcal{A}', \mathcal{B}')$ as witnessed by τ and $t+1$. If $tp([a_{k+1}]) \not\equiv_{n-k-t-1} \tau$, then $tp([b_{k+1}]) \not\equiv_{n-k-t-1} \tau$, and disparity must occur in $G_{n-k}(\mathcal{A}', \mathcal{B}')$ as witnessed by τ and t, contradicting our assumption. Hence the strategy is a winning strategy for Duplicator by inductive assumption (1). □

Theorem 5.3. *Fix $n \in \omega$. There exists an algorithm that runs in constant time and decides whether Duplicator wins game $G_n(\mathcal{A}, \mathcal{B})$ on embedded equivalence structures of height 2. The constant that bounds the running time is $(n+1)^n$.*

Proof. This result follows from the lemma above. We represent structure $\mathcal{A} = (A; E_1, E_2)$ by a tree and a list. The tree has height 3. The leaves of the tree are all elements in A. Two leaves x, y have the same parent if $E_1(x, y)$, and x, y have the same ancestor at level 1 if $E_2(x, y)$. Intuitively, we can view the root of tree as A, the internal nodes at level 1 represent all E_2-equivalence classes on A, and the children of each E_2-equivalence class X at level 2 are all E_1-equivalence classes contained in X. We further require that representations of E_2- and E_1-equivalence classes are put in left-to-right order according to their cardinalities.

The list is $q^{\mathcal{A}}_{\sigma_1,1}, \ldots, q^{\mathcal{A}}_{\sigma_t,1}, q^{\mathcal{A}}_{\sigma_1,2}, \ldots, q^{\mathcal{A}}_{\sigma_1,2}, \ldots, q^{\mathcal{A}}_{\sigma_1,n}, \ldots, q^{\mathcal{A}}_{\sigma_t,n}$ where each σ_i is a type of E_2-equivalence class, and $q^{\mathcal{A}}_{\sigma_i,j}$ is as defined above. Each $q^{\mathcal{A}}_{\sigma_i,j}$ has a value between 0 and n and if it is greater than n, we set it to n.

The algorithm checks whether disparity occurs in $G_n(\mathcal{A}, \mathcal{B})$ by examining the list. There can be at most $(n+1)^n$ pairwise non-n-equivalent types. Therefore, checking disparity requires a time bounded by $(n+1)^{n+1}$. □

For the case when $\mathcal{A}$ and $\mathcal{B}$ are two embedded equivalence structures of height h, where $h > 2$, we give a similar definition of the type of an E_h-equivalence class. We can then describe the winning conditions for Spoiler and Duplicator in a similar way.

Let $\mathcal{A}$ be an embedded equivalence structure of height h where $h > 2$. For an E_h-equivalence class X, we recursively define $tp(X)$, the type of X. Set $tp(X)$ be $(q_{\sigma_1}, \ldots, q_{\sigma_t})$ that satisfies the following properties.

1. Each σ_i is the type of an E_{h-1}-equivalence class.
2. σ_t is the maximum type in lexicographic order among all types of E_{h-1}-equivalence classes contained in X.

3. The list $\sigma_1, \ldots, \sigma_t$ contains all possible types of E_{h-1}-equivalence classes less or equal to σ_t ordered lexicographically.
4. For all $1 \le i \le t$, q_{σ_i} is the number of all E_{h-1}-equivalence classes contained in X whose type are σ_i.

Let $\kappa = (q_{\sigma_1}, \ldots, q_{\sigma_s})$ and $\lambda = (q'_{\sigma_1}, \ldots, q'_{\sigma_t})$ be types of two E_h-equivalence classes X and Y, respectively. We say $\kappa = \lambda$ if $s = t$ and $q_{\sigma_i} = q'_{\sigma_i}$ for all $1 \le i \le s$. We say $\kappa \equiv_n \lambda$ if the structures $(X; E_1 \upharpoonright X, \ldots, E_{h-1} \upharpoonright X)$ and $(Y; E_1 \upharpoonright Y, \ldots, E_{h-1} \upharpoonright Y)$ are n-equivalent.

The following proposition shows that $tp(X)$ are isomorphism invariants of the E_h-equivalence classes.

Proposition 5.4. *Let X and Y be two E_h-equivalence classes in an embedded equivalence structure $\mathcal{A} = (A; E_1, \ldots, E_h)$. Then $tp(X) = tp(Y)$ if and only if the structures $(X; E_1 \upharpoonright X, \ldots, E_{h-1} \upharpoonright X)$ and $(Y; E_1 \upharpoonright Y, \ldots, E_{h-1} \upharpoonright Y)$ are isomorphic. In particular, the isomorphism problem for embedded equivalence structure of height h is linear on the size of the structure.*

Proof. The first part of the proposition easily follows from the definition. To prove the second part of the proposition, suppose $\mathcal{A}$ and $\mathcal{B}$ are two embedded equivalence structures of height h. We represent them by listing E_h-equivalence classes in a manner that their types are lexicographically ordered. Suppose $\mathcal{A}$ and $\mathcal{B}$ are represented by listing E_h-equivalence classes $X_1, X_2, \ldots, X_{k_1}$ and $Y_1, Y_2, \ldots, Y_{k_2}$, respectively. Then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if $k_1 = k_2$ and for all $1 \le i \le k_1$, $(X_i; E_1 \upharpoonright X_i) \cong (Y_i; E_1 \upharpoonright Y_i)$, which is same as $tp(X_i) = tp(Y_i)$. $\square$

Similarly to the case of embedded equivalence structures of height 2, we re-introduce the notions $C^{\mathcal{A}}_{\sigma,i}$, $q^{\mathcal{A}}_{\sigma,i}$, $q^{\sigma,i}$, $\mathcal{A}(\sigma, i)$ and disparity in game $G_n(\mathcal{A}, \mathcal{B})$. The only difference would be that in the new definition, we refer to the E_h-equivalence classes wherever we refer to E_2-equivalence classes in the original definition. The following lemma can thus be proved in a similar manner as Lemma 5.2.

Lemma 5.5. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are two embedded equivalence structures of height h where $h \ge 2$. Duplicator wins game $G_n(\mathcal{A}, \mathcal{B})$ if and only if no disparity occurs.* $\square$

A simple calculation reveals that the number of pairwise non-n-equivalent types of E_h-equivalence classes is at most $(n+1)^{\cdots^{(n+1)^n}}$ where the tower of $(n+1)$ has height h. Therefore, by the lemma above, we can extend Theorem 5.3 to the following result.

Theorem 5.6. *Fix $n \in \omega$. There exists an algorithm that runs in constant time and decides whether Duplicator wins game $G_n(\mathcal{A}, \mathcal{B})$ on embedded equivalence structures of height h $\mathcal{A} = (A; E_1, \ldots, E_h)$ and $\mathcal{B} = (B; E_1, \ldots, E_h)$. The constant that bounds the running time is $< (n+1)^{\cdots^{(n+1)^{(n+1)}}}$ where the tower of $(n+1)$ has height h.* $\square$

6. Trees with level predicates

In this section we study EF games played on *trees with level predicates*. Trees are finite structures of the type $\mathcal{T} = (T, \le)$, where the relation $\le$ is a partial order on T such that $\mathcal{T}$ has the greatest element (the root), and the set $\{y \mid x \le y\}$ for any given $x \in T$ is a linearly ordered set under $\le$. We call an element a *leaf* of the tree $\mathcal{T}$ if it is a minimal element; otherwise we call it an *internal node*. A *path* in $\mathcal{T}$ is a maximal linearly order subset of T. The length of a given path is the number of elements in the path. The *height* of $\mathcal{T}$ is the length of the largest path in $\mathcal{T}$. We say that the *level* of a node $x \in T$ is j if the distance from x to the root is j. A *tree with level predicates* is $(T, \le, L_0, \ldots, L_h)$ where $(T, \le)$ is a tree of height h, and for $0 \le i \le h$, L_i is a unary predicate such that an element $x \in T$ belongs to L_i if and only if x has level i. We fix number $h \ge 2$ and restrict ourselves to the class $\mathcal{K}_h$ of all trees with level predicates of height at most h. Deciding Ehrenfeucht–Fraïssé games on trees from $\mathcal{K}_h$ can be done directly by using the techniques from the previous section. Instead, we reduce the problem of deciding Ehrenfeucht–Fraïssé games on trees with level predicates in K_h to one for embedded equivalence structures of height $h+1$.

We transform trees from the class $\mathcal{K}_h$ into the class of embedded equivalence structures of height h in the following manner. Let $\mathcal{T}$ be a tree in $\mathcal{K}_h$. We now define an embedded equivalence structure $\mathcal{A}(\mathcal{T})$ as follows. The domain D of $\mathcal{A}(T)$ is now $T \cup \{a_x \mid x \text{ is a leaf of } \mathcal{T}\}$. We define the equivalence relation E_i, $1 \le i \le h$, on the domain as follows. The relation E_1 is the minimal equivalence relation that contains $\{(x, a_x) \mid x \text{ is a leaf of } \mathcal{T}\}$. Let $x_1, \ldots, x_s$ be all elements of $\mathcal{T}$ at level $h-i+1$ where $1 \le i < h$. Let $\mathcal{T}_1, \ldots, \mathcal{T}_s$ be the subtrees of $\mathcal{T}$ whose roots are $x_1, \ldots, x_s$, respectively. Set E_i be the minimal equivalence relation that contains $E_{i-1} \cup T_1^2 \cup \cdots \cup T_s^2$. It is clear that $E_i \subseteq E_{i+1}$ for all $1 \le i \le h$. Thus we have the embedded equivalence structure $\mathcal{A}(T) = (D; E_1, \ldots, E_h)$.

Lemma 6.1. *For trees with level predicates $\mathcal{T}_1$ and $\mathcal{T}_2$, $\mathcal{T}_1 \cong \mathcal{T}_2$ if and only if $\mathcal{A}(\mathcal{T}_1) \cong \mathcal{A}(\mathcal{T}_2)$. In particular, Duplicator wins game $G_n(\mathcal{T}_1, \mathcal{T}_2)$ if and only if Duplicator wins $G_n(\mathcal{A}(\mathcal{T}_1), \mathcal{A}(T_2))$.*

Proof. Suppose $\mathcal{T}$ is a tree in the class $\mathcal{K}_h$. Take an element $x \in T$. By construction of $\mathcal{A}(\mathcal{T})$, the following statements are true.

- x is a leaf in $\mathcal{T}$ if and only if $|\{y \mid E_1(x, y)\}| = 2$ in $\mathcal{A}(\mathcal{T})$.
- x is the root of $\mathcal{T}$ if and only if $|\{y \mid E_h(x, y)\}| = 1$ in $\mathcal{A}(\mathcal{T})$.

We define the **level** of x in $\mathcal{A}(\mathcal{T})$ as follows. If x is the root of $\mathcal{T}$, the level of x is 0. Otherwise, if x is an internal node, the level of x in $\mathcal{A}(\mathcal{T})$ is the largest l such that $|\{y \mid E_{h-l+1}(x,y)\}| > 1$. If x is a leaf, we define the level of x in $\mathcal{A}(\mathcal{T})$ to be the largest $l+1$ such that there is an internal node y such that $E_{h-l+1}(x,y)$.

By definition, for all $x \in T$, the level of x in $\mathcal{T}$ coincides with the level of x in $\mathcal{A}(\mathcal{T})$. For $x, y \in T$, $x \leq y$ in $\mathcal{T}$ if and only in $\mathcal{A}(\mathcal{T})$ x has level s and y has level t such that $s \geq t$ and $E_{h-t+1}(x,y)$. Therefore given two trees from $\mathcal{K}_h$, $\mathcal{T}_1$ and $\mathcal{T}_2$, and a mapping $f : T_1 \to T_2, f$ is an isomorphism between $\mathcal{T}_1$ and $\mathcal{T}_2$ if and only if f is an isomorphism between $\mathcal{A}(\mathcal{T}_1)$ and $\mathcal{A}(\mathcal{T}_2)$.

To prove the second part of the lemma, assume that there is a winning strategy for Spoiler on game $G_n(\mathcal{T}_1, \mathcal{T}_2)$. It is easy to see that this strategy is also a winning strategy for Spoiler on game $G_n(\mathcal{A}(\mathcal{T}_1), \mathcal{A}(\mathcal{T}_2))$, as otherwise Duplicator would win the game $G_n(\mathcal{T}_1, \mathcal{T}_2)$.

Conversely, assume that there is a winning strategy for Duplicator on the n-round game $G_n(\mathcal{T}_1, \mathcal{T}_2)$. We describe a strategy for Duplicator on the game $G_n(\mathcal{A}(\mathcal{T}_1), \mathcal{A}(\mathcal{T}_2))$ where $\mathcal{A}(\mathcal{T}_1) = (D_1; E_1, \ldots, E_h)$ and $\mathcal{A}(\mathcal{T}_2) = (D_2; E_1, \ldots, E_h)$. Let us assume that the players have produced a k-round play $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$. Assume on this k-round play that the map $x_i \to y_i$ is a partial isomorphism between $\mathcal{A}(\mathcal{T}_1)$ and $\mathcal{A}(\mathcal{T}_2)$.

Assume that Spoiler selects an element $x_{k+1} \in D_1$. Duplicator responds to this move by choosing x_{k+1} as follows. If $x_{k+1} = x_i$ then $y_{k+1} = y_i$. Otherwise, if $x_{k+1} \in T_1$, then Duplicator selects an element $y_{k+1} \in T_2$ according to its winning strategy on $G_n(\mathcal{T}_1, \mathcal{T}_2)$. If $x_{k+1} = a_x$ for some leaf $x \in \mathcal{T}_1$, then Duplicator responds by selecting $y_{k+1} = a_y$ where y is the leaf in T_2 that corresponds to x in Duplicator's winning strategy in $G_n(\mathcal{T}_1, \mathcal{T}_2)$. It is clear that $x_i \to y_i$ where $1 \leq i \leq k+1$ is also a partial isomorphism between $\mathcal{A}(\mathcal{T}_1)$ and $\mathcal{A}(\mathcal{T}_2)$. Therefore the strategy described is a winning strategy for Duplicator on game $G_n(\mathcal{A}(\mathcal{T}_1), \mathcal{A}(\mathcal{T}_2))$. □

Theorem 6.2. *Fix $n \in \omega$. There exists an algorithm that runs in constant time and decides whether Duplicator wins game $G_n(\mathcal{T}_1, \mathcal{T}_2)$ on trees with level predicates $\mathcal{T}_1 = (T_1; \leq)$ and $\mathcal{T}_2 = (T_2; \leq)$ from the class $\mathcal{K}_h$. The constant that bounds the running time is $< (n+1)^{\cdots^{(n+1)^{(n+1)}}}$ where the tower has height h.*

Proof. To prove the theorem, the trees with level predicates $\mathcal{T}_1$ and $\mathcal{T}_2$ are represented by the two embedded equivalence structures of height h $\mathcal{A}(\mathcal{T}_1)$ and $\mathcal{A}(\mathcal{T}_2)$, respectively. By the lemma above, Duplicator wins game $G_n(\mathcal{A}(\mathcal{T}_1), \mathcal{A}(\mathcal{T}_2))$ if and only if Duplicator wins game $G_n(\mathcal{T}_1, \mathcal{T}_2)$. By Theorem 5.6, we have a constant time algorithm that decides who wins EF game $G_n(\mathcal{A}(\mathcal{T}_1), \mathcal{A}(\mathcal{T}_2))$ in which the constant is bounded by $(n+1)^{\cdots^{(n+1)^{(n+1)}}}$ where the tower has height h. □

7. Boolean algebras with distinguished ideals

In this section we study EF games played on *Boolean algebras with distinguished ideals*; these are structures of the form $\mathcal{A} = (A; \leq, 0, 1, I_1, \ldots, I_s)$, where $(A; \leq, 0, 1)$ forms a Boolean algebra and each I_j is an ideal of the algebra $(A; \leq, 0, 1)$. The set of **atoms** of $\mathcal{A}$, denoted $At(\mathcal{A})$, is the set $\{a \mid \forall y(0 \leq y \leq a \to y = 0 \vee y = a)\}$. Since we restrict ourselves to finite structures, the Boolean algebra $(A; \leq, 0, 1)$ can be identified with the structure $(2^{X_A}; \subseteq, \emptyset, X_A)$, where $X_A = At(\mathcal{A})$ and 2^{X_A} is the collection of all subsets of X_A. Moreover, for each ideal I_j there exists a set $A_j \subset At(\mathcal{A})$ such that $I_j = 2^{A_j}$. Hence the original structure $\mathcal{A}$ can be identified with the following structure:

$$(2^{X_A}; \subseteq, \emptyset, X_A, 2^{A_1}, \ldots, 2^{A_s}).$$

For each element $x \in At(\mathcal{A})$, define the *characteristic* of x, $ch(x)$, as a binary sequence $(t_1, t_2, \ldots, t_s)$ such that for each $1 \leq i \leq s$, $t_i \in \{0,1\}$, $t_i = 1$ if $x \in A_i$ and $t_i = 0$ otherwise. For each characteristic $\epsilon \in \{0,1\}^s$ consider the set $A_\epsilon = \{x \in At(\mathcal{A}) \mid ch(x) = \epsilon\}$. This defines the ideal I_ϵ in the Boolean algebra $(2^{X_A}; \subseteq, \emptyset, X_A)$. Moreover, we can also identify this ideal with the Boolean algebra $(2^{A_\epsilon}; \subseteq, \emptyset, A_\epsilon)$. There are 2^s pairwise distinct characteristics. Let $\epsilon_1, \ldots, \epsilon_{2^s}$ be the list of all characters. We denote by $\mathcal{A}'$ the following structure:

$$(2^X; \subseteq, \emptyset, X, 2^{A_{\epsilon_1}}, \ldots, 2^{A_{\epsilon_{2^s}}}).$$

The following is an easy lemma:

Lemma 7.1. *Let $\mathcal{A} = (2^{X_A}; \subseteq, \emptyset, X_A, 2^{A_1}, \ldots, 2^{A_s})$ be a Boolean algebra with distinguished ideals*

1. *For any two distinct characteristics ϵ and δ we have $I_\epsilon \cap I_\delta = \{\emptyset\}$.*
2. *For any element $a \in 2^X$ there are elements $a_\epsilon \in I_\epsilon$ such that $a = \cup_\epsilon a_\epsilon$.*
3. *The Boolean algebra $(2^{X_A}; \subseteq, \emptyset, X_A)$ is isomorphic to the Cartesian product of the Boolean algebras I_ϵ.*
4. *$\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if $\mathcal{A}'$ and $\mathcal{B}'$ are isomorphic.* □

The next lemma connects the structure $\mathcal{A}'$ and $\mathcal{A}$ in terms of characterizing the winner of the game $G_n(\mathcal{A}, \mathcal{B})$.

Lemma 7.2. *Duplicator wins the game $G_{n+1}(\mathcal{A}, \mathcal{B})$ if and only if each of the following two conditions are true:*

1. *For each characteristic ϵ, $|A_\epsilon| \geq 2^n$ if and only if $|B_\epsilon| \geq 2^n$.*
2. *For each characteristic ϵ, if $|A_\epsilon| < 2^n$ then $|A_\epsilon| = |B_\epsilon|$.*

Proof. Assume that for some ϵ, we have $|A_\epsilon| \neq |B_\epsilon|$ and $|B_\epsilon| < 2^n$. Let us assume that $|A_\epsilon| \geq 2^n$. The case when $|A_\epsilon| < 2^n$ is treated in a similar manner. We describe a winning strategy for Spoiler. Spoiler starts by taking elements $a_1, a_2, \ldots$ in A_ϵ. For each $i \leq n$ the element a_i is such that $|At(a_i)| \geq 2^{n-i}$ where $At(a)$ denotes the set of atoms below a. The elements $a_1, a_2, \ldots$ are such that for each i, either $a_i \subset a_{i-1}$ or $a_i \cap a_{i-1} = \emptyset$. Consider the k-round play $(a_1, b_1), \ldots, (a_k, b_k)$ where $k < n$. Let $e < k$ be the last round for which $a_k \subset a_e$. If no such e exists, let $a_e = 2^{A_\epsilon}$ and $b_e = 2^{B_\epsilon}$. We have the following inductive assumptions.

- $|At(a_k)| \geq 2^{n-k}$ and $|At(a_e \setminus (a_{e+1} \cup \ldots \cup a_k))| \geq 2^{n-k}$.
- Either $|At(b_k)| < 2^{n-k}$ or $|At(b_e \setminus (b_{e+1} \cup \ldots \cup b_k))| < 2^{n-k}$.

There are two cases.

Case 1. Assume that $|At(b_k)| < 2^{n-k}$ and $|At(a_k)| \geq 2^{n-k}$. In this case, Spoiler selects a_{k+1} such that $a_{k+1} \subset a_k$, $a_{k+1} \neq \emptyset$, $|At(a_{k+1})| \geq 2^{n-k-1}$, and $|At(a_k \setminus a_{k+1})| \geq 2^{n-k-1}$. Note that Duplicator must choose b_{k+1} strictly below b_k. Then either $|At(b_{k+1})| < 2^{n-k-1}$ or $|At(b_k \setminus b_{k+1})| < 2^{n-k-1}$.

Case 2. Assume that $|At(b_k)| \geq 2^{n-k}$ and $|At(a_k)| \geq 2^{n-k}$. In this case, Spoiler selects a_{k+1} such that $a_{k+1} \subset a_e$, $a_{k+1} \neq \emptyset$, $a_{k+1} \cap (a_{e+1} \cup \ldots \cup a_k) = \emptyset$, $|At(a_{k+1})| \geq 2^{n-k-1}$, and $|At(a_e \setminus (a_{e+1} \cup \ldots \cup a_{k+1}))| \geq 2^{n-k-1}$. Note that by definition of e, $|At(b_e)| < 2^{n-k}$ and for each $e + 1 \leq i \leq k - 1$, $|At(b_i)| \geq 2^{n-i}$ as otherwise b_k would be below b_i. Hence $|At(b_k \setminus (b_{e+1} \cup \ldots \cup b_k))| < 2^{n-k}$. Duplicator must choose b_{k+1} strictly below b_e and disjoint with $b_{e+1}, \ldots, b_k$. Therefore, either $|At(b_{k+1})| < 2^{n-k-1}$ or $|At(b_e) \setminus At(b_{e+1} \cup \ldots \cup b_{k+1})| < 2^{n-k-1}$.

After n rounds, by the inductive assumption, it is either $|At(b_n)| = 0$ or $|At(b_e \setminus (b_{e+1} \cup \ldots \cup b_n))| = 0$. If the former, then Spoiler wins by selecting $a_{n+1} \subset At(a_n)$; otherwise, Spoiler wins by selecting $a_{n+1} \subset a_e \setminus (a_{e+1} \cup \ldots \cup a_n)$.

We now prove that the conditions stated in the lemma suffice Duplicator to win the $(n + 1)$-round game $G_{n+1}(\mathcal{A}, \mathcal{B})$. Let us assume that the players have produced a k-round play $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$. Our inductive assumptions on this k-round play are the following:

1. The map $a_i \to b_i$ is a partial isomorphism.
2. For each a_i, $1 \leq i \leq k$, let $a_i = \cup_\epsilon a_\epsilon$ be as stipulated in Lemma 7.1(2). For each a_ϵ, let e be the last round such that $a_\epsilon \subseteq a_e$; if there is no such round, then assume $a_e = At(I_\epsilon)$. Let d be the last round such that $a_d \subseteq a_\epsilon$; if there is no such round, then assume $a_d = \emptyset$. Let $b_i = \cup_\epsilon b_\epsilon$. The conditions for b_ϵ are the following:
   - $|At(a_\epsilon \setminus a_d)| \geq 2^{n-i}$ if and only if $|At(b_\epsilon \setminus a_d)| \geq 2^{n-i}$; $|At(a_e \setminus a_\epsilon)| \geq 2^{n-i}$ if and only if $|At(b_e \setminus b_\epsilon)| \geq 2^{n-i}$.
   - If $|At(a_\epsilon \setminus a_d)| < 2^{n-i}$ then $|At(b_\epsilon \setminus a_d)| = |At(a_\epsilon \setminus a_d)|$; If $|At(a_e \setminus a_\epsilon)| < 2^{n-i}$ then $|At(b_e \setminus b_\epsilon)| = |At(a_e \setminus a_\epsilon)|$.

Assume that Spoiler selects an element $a_{k+1} \in A$. Duplicator responds to this move by choosing b_{k+1} as follows. If $a_{k+1} = a_i$ then $b_{k+1} = b_i$. Otherwise, suppose $a_{k+1} = \cup a_\epsilon$ as stipulated in Lemma 7.1(2). For each a_ϵ, let d, e be as described in the inductive assumptions. We select each b_ϵ by the following rules.

- If $|At(a_\epsilon \setminus a_d)| \geq 2^{n-k-1}$ then select b_ϵ such that $|At(b_\epsilon \setminus a_d)| \geq 2^{n-k-1}$; If $|At(a_e \setminus a_\epsilon)| \geq 2^{n-k-1}$ then $|At(b_e \setminus b_\epsilon)| \geq 2^{n-k-1}$.
- If $|At(a_\epsilon \setminus a_d)| < 2^{n-k-1}$ then select b_ϵ such that $|At(b_\epsilon \setminus a_d)| = |At(a_\epsilon \setminus a_d)|$; If $|At(a_e \setminus a_\epsilon)| < 2^{n-k-1}$ then $|At(b_e \setminus b_\epsilon)| = |At(a_e \setminus a_\epsilon)|$.

Finally, Duplicator selects $b_{k+1} \in B$ such that $b_{k+1} = \cup_\epsilon b_\epsilon$.

Note the inductive assumptions guarantee that Duplicator is able to make such a move. It is clear that the inductive assumptions also hold on the $(k+1)$-round play $(a_1, b_1), \ldots, (a_{k+1}, b_{k+1})$. Hence the strategy described must be a winning strategy due to the fact that Duplicator preserves inductive assumption (1) at each round. The lemma is proved. □

Theorem 7.3. *Fix $n \in \omega$. There exists an algorithm that runs in constant time and decides whether Duplicator wins the game $G_{n+1}(\mathcal{A}, \mathcal{B})$ on Boolean algebras $\mathcal{A} = (2^{X_A}; \subseteq, \emptyset, X_A, 2^{A_1}, \ldots, 2^{A_s})$ and $\mathcal{B} = (2^{X_B}; \subseteq, \emptyset, X_B, 2^{B_1}, \ldots, 2^{B_s})$. The constant that bounds the running time is $2^s \cdot 2^n$.*

Proof. In order to prove the theorem, we represent the Boolean algebras by listing their atoms in 2^s lists. The ith list lists all atoms with characteristic ϵ_i. To solve game $G_{n+1}(\mathcal{A}, \mathcal{B})$, the algorithm checks the condition in the lemma above by reading the lists. In each list, it reads at most 2^n elements. Therefore the process requires time bounded by $2^s \cdot 2^n$. □

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