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Transverse Contraction Criteria for Existence, Stability, and Robustness of a Limit Cycle

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Abstract— **This paper derives a differential contraction condition for the existence of an orbitally-stable limit cycle in an autonomous system. This transverse contraction condition can be represented as a pointwise linear matrix inequality (LMI), thus allowing convex optimization tools such as sum-of-squares programming to be used to search for certificates of the existence of a stable limit cycle. Many desirable properties of contracting dynamics are extended to this context, including preservation of contraction under a broad class of interconnections. In addition, by introducing the concepts of differential dissipativity and transverse differential dissipativity, contraction and transverse contraction can be established for large scale systems via LMI conditions on component subsystems.**

I. Introduction

Dynamic systems with periodic solutions are important in many areas of engineering, including biologically-inspired robot locomotion, phase-locked loops, vortex shedding from aircraft wings, and combustion oscillations, to name just a few. In biology, oscillating systems seem to be the rule rather than the exception [1].

The basic question we address in this paper is the following: when does an autonomous system of the form

$$\dot{x} = f(x) \tag{1}$$

have the property that all solutions starting from a particular set K converge asymptotically to a unique limit cycle? It is well known that periodic solutions of an autonomous differential equation can never be asymptotically stable. This is clear from the fact that solutions which have initial conditions on the periodic orbit but offset in time will never converge.

There is a long and distinguished history of research into limit cycles for nonlinear systems. For example, the famous result of Poincaré-Bendixson gives a very simple condition for planar systems. An important generalization to monotone cyclic feedback systems was published in [2], however this depends on quite a special system structure and there are many application areas where it does not apply.

There are also interesting properties of the "global" structure of regions of attraction to periodic orbits. It is known that the region of attraction is a continuous deformation of a torus: the cartesian product of an open unit disc of dimension $n-1$, with a scalar circle coordinate [3]. These are often referred to as "transversal" and "phase" coordinates, respectively. In all cases except possibly with $n = 5$ it is guaranteed that the deformation is differentiable [4], due to the recent resolution of the Poincaré Hypothesis by Perelman. Birkhoff gave necessary conditions for periodic solutions in terms of the existence of particular "phase variables", or associated differential one-forms [5], [4].

However, all of these conditions imply the existence of *at least one* limit cycle, but give no insight into the number of limit cycles, or their stability. In recent years many efficient computational methods for proving stability of equilibria of nonlinear systems have been proposed, using optimization methods to search for "stability certificates" such as Lyapunov functions and barrier certificates [6], [7], [8]. In previous papers, the first author and others have extended this computational approach to limit cycles analysis using "transverse dynamics" and sum-of-squares programming [9], [10], [11], [12], however this method is not applicable when the system dynamics are uncertain, since uncertainty will generally change the location of the limit cycle in state space.

An alternative to Lyapunov methods is to search for a contraction metric [13], [14]. For the purposes of robust stability analysis of equilibria, an important difference is that a Lyapunov function must generally be constructed about a known equilibrium, whereas a contraction metric implies the existence of a stable equilibrium indirectly. This is particularly useful if the equilibrium point may change location depending on the unknown dynamics.

Historically, basic convergence results on contracting systems can be traced back to the 1949 results of Lewis in terms of Finsler metrics [15], and results of Hartman [16] and Demidovich [17]. To our knowledge, contraction to limit cycles was first investigated using an identity metric by Borg [18], and Hartman and Olech [19]. This line of analysis was extended to more complex attractors by Leonov and colleagues, see, e.g., [20].

In this paper, we introduce *transverse contraction*, extending the results of [18], [19] by exploiting generalized metrics and system combination properties as in [13]. Furthermore, we introduce a nonlinear change of variables that converts tranverse contraction to a linear matrix inequality (LMI) without conservatism. In Section IV we show that

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transverse contraction is preserved under several forms of interconnection with contracting systems. In Section V we introduce *differential dissipativity* and *transverse differential dissipativity*, as well as LMI conditions for each, giving a framework for optimization-based analysis of complex interconnections of nonlinear systems. Finally, in Section VI, we illustrate the applicability of the results on the Moore Greitzer jet engine model.

The proofs of all theorems have been removed to meet page limits, but can be found online in an extended version of this paper [21].

II. Problem Setup and Preliminaries

We assume that $f : K \to \mathbb{R}^n$ in (1) is smooth and $x \in \mathbb{R}^n$, and that a unique solution of (1) exists. We refer to the Jacobian of f as $A(x) := \frac{\partial f}{\partial x}$. A set K is called *strictly forward invariant* under f if any solution of (1) starting with $x(0)$ in K is in the interior of K for all $t > 0$. A periodic solution $x^\star$ is one for which there exists some $T > 0$ such that $x^\star(t) = x^\star(t+T)$ for all t. Equilibria are trivially periodic for every T, but for oscillatory solutions – which are our main concern – there is some minimal time T such that the above holds and this is referred to as the *period.*

The *orbit* of a periodic solution is the set $\mathcal{X}^\star := \{x : x = x^\star(t) \text{ for some } t\}$. Note that while non-trivial periodic *solutions* cannot be asymptotically stable, their *orbits* can be, and in this case we say that the solution is *orbitally stable* (see, e.g., [22]). Define a *time reparametrization* $\tau(t)$ as a smooth function $\tau : [0,\infty) \to [0,\infty)$ such that $\tau(t)$ is monotonically increasing and $\tau(t) \to \infty$ as $t \to \infty$.

Two solutions $x_1(t), x_2(t)$ are said to converge exponentially with rate λ if $|x_1(t) - x_2(t)| \leq Ke^{\lambda t}|x_1(t) - x_2(t)|$ for some $K > 0$. A solution $x_1(t)$ is said to converge to the orbit of another solution $x_2(t)$ exponentially with rate λ if $\min_{\tau>0} |x_1(t) - x_2(\tau)| \leq Ke^{\lambda t}|x_1(t) - x_2(t)|$. This is clearly a weaker condition than exponential convergence of solutions.

A Finsler function [23] on a manifold M with tangent bundle TM is a smooth function $V : M \times TM \to \mathbb{R}$ satisfying *positive-definiteness*: For all $x \in M$, $V(x,0) = 0$ and $V(x,\delta) > 0$ for all $\delta \neq 0$, *homogeneity*: $V(x,\alpha\delta) = \alpha V(x,\delta)$ for $\alpha > 0$, and *convexity*: the Hessian matrix of V^2 with respect to δ is positive-definite for all x.

III. Contraction Conditions for Limit Cycles

In this section we introduce a *transverse contraction* condition for an autonomous dynamical system $\dot{x} = f(x),\ x \in M$, where M is a smooth, compact n-dimensional manifold. The condition is given in terms of a function $V(x,\delta_x)$, where $x \in M$ and $\delta_x \in \mathbb{R}^n$, which induces a distance function similar to a Riemannian or Finsler metric [23].

For most of this paper, we will assume a Riemannian-like contraction metric $V(x,\delta_x) := \sqrt{\delta_x' M(x)\delta_x}$ where $M(x)$ is positive-definite for all x, however the main results hold for more general structures such as Finsler metrics [23], [15], [24]. The following two theorems provide a generalization of the results of [18], [19], which considered the case $V(x,\delta_x) = |\delta_x|^2$.

Theorem 1: Let $K \subset \mathbb{R}^n$ be compact, smoothly path-connected, and strictly forward invariant. If there exists a Finsler function $V(x,\delta)$ satisfying

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial \delta_x} A(x)\delta_x \leq -\lambda V(x,\delta_x), \tag{2}$$

for all $\delta_x \neq 0$ such that $\frac{\partial V}{\partial \delta_x} f(x) = 0$, then for every two solutions x_1 and x_2 with initial conditions in K there exists time reparametrizations $\tau(t)$ such that $x_1(t) \to x_2(\tau(t))$ as $t \to \infty$. Furthermore, the convergence is exponential with rate λ.

Remark 1: Stability under time reparametrization is sometimes referred to as *Zhukovsky stability* and has been used in several recent papers on limit cycle stability, see e.g. [25], [26], [27], [11] and apparently goes back to Poincaré in its essential argument [22]. It is known that systems satisfying such a property have limit cycles [26], but we include a proof here since it is very straightforward using the constructions in the previous theorem.

Theorem 2: If the conditions of Theorem 1 are satisfied, then all solutions starting with $x(0) \in K$ converge to a unique limit cycle. Furthermore, convergence is exponential with rate λ and has the property of asymptotic phase: i.e. for any initial conditions $x_1(0), x_2(0)$, there exists a fixed τ such that $x_1(t) \to x_2(t+\tau)$ exponentially.

Remark 2: Note that transverse contraction is a strictly weaker condition than contraction, so every contracting system is also transverse contracting. Hence the periodic solution to which a transverse contracting system converges may be *trivially* periodic, i.e. an equilibrium.

Remark 3: In [28], [29] and [30], contraction transverse to a particular linear subspace was analyzed in the context synchronization. In this paper, contraction transverse to the system's vector field ensures asymptotically a form of "synchronization": in a periodic solution there is a single scalar variable (phase) that predicts all other states of the system. This concept may also be generalized to study higher-dimensional limit sets and non-autonomous systems.

A. Convex Formulation via Linear Matrix Inequalities

For the remainder of the paper we consider transverse contraction with a metric of the form $V(x,\delta_x) = \delta_x' M(x)\delta_x$. It will be shown in the next subsection that this class of metrics is sufficiently rich for testing orbital stability.

Theorem 3: A system $\dot{x} = f(x)$ is transverse contracting with rate λ on a set K if and only if there exists a function $\rho(x) \geq 0$ and a symmetric positive-definite matrix function $W(x)$ such that

$$W(x)A(x)' + A(x)W(x) - \dot{W}(x) + 2\lambda W(x) - \rho(x)Q(x) \leq 0 \tag{3}$$

for all $x \in K$, where $Q(x) := f(x)f(x)'$.

Note that this condition is linear in the unknown functions $W(x)$ and $\rho(x)$, i.e. it consists of a linear matrix inequality at each point x.

The above condition is convex and exact for each particular x. Such conditions can be verified over *regions* of the state space using sum-of-squares programming and positivstellensatz arguments [6], see [14] for an exposition of this approach for the case of strong contraction.

B. Generalized Jacobian and Transverse Linearization

The concept of a *generalized Jacobian* was introduced in [13] for analysing contracting systems. Consider a non-singular change of differential coordinates $\delta_z = \Theta(x)\delta_x$, then the dynamics in the new coordinates are given by $\dot{\delta}_z = F(x)\delta_z$ where the generalized Jacobian $F(x) := \Theta(x)A(x)\Theta(x)^{-1} + \dot{\Theta}(x)\Theta(x)^{-1}$. If such a change of coordinates exists such that $F(x)+F(x)' \leq -\lambda I$ then the system is contracting with rate λ. Furthermore, $M(x) = \Theta(x)'\Theta(x)$ is a valid contraction metric. Note that it is often easier to construct $\Theta(x)$ than a "global" change of coordinates $x \to z$.

A system is transverse contracting if there exists a differential change of coordinates such that $\delta_z(F+F')\delta_z < 0$ for all δ_z satisfying $\delta_z'\Theta(x)f(x) = 0$, where the latter condition follows from $\dot{z} = \Theta(x)\dot{x} = \Theta(x)f(x)$.

Theorem 4: If a system $\dot{x} = f(x)$ has a unique limit cycle to which all solutions starting in K converge exponentially, then there exists a transverse contraction metric of the form $V(x,\delta_x) = \sqrt{\delta_x M(x)\delta_x}$ satisfying

$$\frac{\partial V}{\partial x}f(x) + \frac{\partial V}{\partial \delta_x}A(x)\delta_x < 0$$

for all δ_x with strict inequality for δ_x satisfying $\frac{\partial V}{\partial \delta_x}f(x) = 0$. The generalized Jacobian is of the form

$$F = \begin{bmatrix} 0 & \star \\ 0 & F_\perp \end{bmatrix}$$

where $F_\perp + F_\perp' < 0$ and $F+F'$ has eigenvalues $0 = \lambda_{max} > \lambda_2 \geq \lambda_3 ... \geq \lambda_n$.

In the above, $A_\perp(x)$ is the transverse linearization that was used to construct Lyapunov functions for limit cycles in [31] and [11]. Note that in those works, it was necessary for the limit cycle to be known and fixed to prove convergence, whereas transverse contraction decouples the question of convergence from knowledge of a particular solution.

IV. Properties of Transverse Contracting Systems

In many applications in which exact models are unavailable or very complex, it is desirable to characterize parameter ranges or interconnection structures over which the qualitative behaviour of the system remains the same. Engineering motivations are well known, but robustness analysis has also become of interest recently in biology, including as a measure of model validity [32]. E.g., in [32] robustness of limit cycles is assessed by gridding over parameter ranges and simulating the nonlinear system until convergence can be ascertained. Gridding and simulation becomes very expensive computationally for systems with large state dimension or many parameters, so alternative methods are desirable.

A. Hierarchical Compositions of Systems

A relatively simple application of the above theorem is to consider the composition of a contracting system and a transverse-contracting system.

$$\dot{x}_1 = f_1(x_1), \quad \dot{x}_2 = f_2(x_1, x_2)$$

Theorem 5: Suppose for each fixed x_1, f_2 is transverse contracting with metric $M_2(x_2, x_1)$, i.e.

$$\delta_2'(F_2'M_{22} + M_2F_{22} + \frac{\partial}{\partial x_2}M_2 f(x_2) + 2\lambda_2 M_2)\delta_2 \leq 0$$

for all δ_2 satisfying $\delta_2' M_2 f_2 = 0$ and f_1 is strongly contracting in the sense of [13], i.e. there exists $M_1(x_1)$ such that

$$F_1'M_1 + M_1F_1 + \frac{\partial}{\partial x_1}M_1 f(x_1) + 2\lambda_1 M_1 \leq 0$$

then the composed system is transverse contracting, and hence has a unique stable limit cycle.

The opposite composition, a transverse-contracting system driving a contracting system clearly converges to a periodic solution due to natural input-to-state stability properties of contracting systems. In a sense, the second system can be considered as being driven by a periodic input [13]. This line of reasoning can be extended to show that any system interconnection represented by a directed acyclic graph, with a single transverse contracting system and the remaining systems contracting, is transverse constracting.

B. Robustness to Parametric Variation

Suppose the system dynamics depend on some *parameter vector* θ, i.e. $\dot{x} = f(x,\theta)$. When studying robustness of equilibria of such systems, a widely-used method is to search for a parameter-dependent Lyapunov function (see, e.g., [7]).

In the context of the present paper, we assume that a particular set K is robustly forward invariant – which can be verified using the methods of [7] – then robust existence of a single globally stable (within K) limit cycle is ensured if one can find a parameter-dependent contraction metric $M(x,\theta)$ which satisfies

$$\delta'(\dot{M}(x,\theta) + 2F(x,\theta)'M(x,\theta) + 2\lambda M(x,\theta))\delta \leq 0$$

for all δ such that $\delta' M(x,\theta)f(x,\theta) = 0$, and for all $x \in K$ and θ in some set Θ, where K is a forward invariant set. Note that this condition can be expressed as a parameter-dependent LMI as in (3), and verified via either sum-of-squares [6] or sample-based methods [33].

C. Skew-Symmetric Feedback Interconnection

In this section and the next one we consider feedback interconnections of two systems of the form:

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2). \tag{4}$$

Theorem 6: Suppose System 1 is partially contracting with respect to x_1, i.e. there exists a differential change of coordinates $\Theta_1(x_1)$ such that $F_1 := \Theta_1 \frac{\partial f_1}{\partial x_1}\Theta_1^{-1} + \dot{\Theta}_1\Theta_1^{-1}$ satisfies $F_1 + F_1' < 0$.

Suppose also that System 2 is partially transverse contracting with respect to x_2, i.e. by Theorem 4 there exists a differential change of coordinates $\Theta_2(x_2)$ such that $F_2 := \Theta_2 \frac{\partial f_2}{\partial x_2}\Theta_2^{-1} + \dot{\Theta}_2\Theta_2^{-1}$ satisfies $F_2 + F_2' \leq 0$ and $\delta_2(F_2 + F_2')\delta_2 < 0$ when $\delta \neq 0$ satisfies $\delta_2\Theta_2 f_2 = 0$.

Define $G_{12} := \Theta_1 \frac{\partial f_1}{\partial x_2}\Theta_2^{-1}$ and $G_{21} := \Theta_2 \frac{\partial f_2}{\partial x_1}\Theta_1^{-1}$ and suppose $G_{12} = -kG_{21}'$ for some $k > 0$, then the interconnection (4) is transverse contracting.

D. Bounded Feedback Interconnections

A more general theorem was presented in [28] for contracting systems. Here we discuss how it extends to transverse contraction. Suppose we have a general feedback interconnection, and construct F as above. Define

$$F_s := F + F' = \begin{bmatrix} F_{1s} & G_s \\ G_s' & F_{2s} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} =: F\delta$$

where $F_{1s} := F_1 + F_1'$ and $F_{2s} := F_2 + F_2'$ and $G_s := \Theta_1 \frac{\partial f_1}{\partial x_2}\Theta_2^{-1} + \left(\Theta_2 \frac{\partial f_2}{\partial x_1}\Theta_1^{-1}\right)'$.

Suppose system 1 is transverse contracting, so $F_{1s} \leq 0$ and $z'F_{1s}z < 0$ for all $z'\Theta_1 f_1 = 0$. In [28] the Schur complement was used to derive conditions for contraction:

$$F_{1s} \leq G_s F_{2s}^{-1} G_s' \Leftrightarrow F_s \leq 0.$$

Note that in the case of transverse contracting systems, $z'F_{1s}f_1(x)z$ when $z'\Theta f = 0$ and $z'F_{2s}z < 0$ otherwise. Since F_{2s} is nonsingular, for the inequality on the left hand side to hold, it must be the case that $G_s'\Theta f = 0$. A very simple condition for $G_s = 0$, which is equivalent to the skew-symmetric condition in the previous section, i.e. $\Theta_1 \frac{\partial f_1}{\partial x_2}\Theta_2^{-1} = -\left(\Theta_2 \frac{\partial f_2}{\partial x_1}\Theta_1^{-1}\right)'$

Another sufficient condition for $Gf_1(x) = 0$ would be for both of these terms to be zero. For the first term, this implies that perturbations in System 2 only affect the *transversal* states of System 1, not the phase. For the second term, this means that perturbations in the phase of system 1 do not affect system 2. This would correspond to a decomposition of System 1 into a phase and transversal system, only the latter of which interacts with System 2.

Suppose that $G'\Theta f_1 = 0$ then a sufficient condition for transverse contraction of the interconnection is

$$\lambda_2(F_{1s})\lambda_{max}(F_{2s}) < \sigma^2(G)$$

by a similar argument to [28]. Note that $\lambda_2(F_{1s})$ is the rate of transverse contraction of System 1 and $\lambda_{max}(F_{2s})$ is the exponential rate of contraction of System 2.

E. Robustness to Bounded Disturbance

Consider the global coordinates $x_\perp, \tau$ – either implicitly or explicitly defined. Since $\tau \in S^1$ the dynamics of $x_\perp$ can be considered a periodic differential equation with a transformation of time. This makes it clear that any internal perturbation in f which keeps $\dot{\tau} > 0$ and $F_\perp(x)$ contracting still results in a limit cycle (c.f. above).

Bounded external perturbations will also have bounded effect on behavior. Denote $x^\star$ the periodic orbit of a transverse contracting system $\dot{x} = f(x)$. Letting $R(x) = \min_\tau \int_{x^\star(\tau)}^{x} V(\gamma(s), \frac{\partial \gamma}{\partial s})ds$ we have $\dot{R} + \lambda R \leq 0$. Consider a bounded external disturbance, i.e. $\dot{x} = f(x) + d(t)$, where $|d| \leq d_{\max}$, then we have $\dot{R} + \lambda R \leq |\Theta d(t)|$, so after exponentially-forgotten transients, the perturbed system is within a ball of radius R around the original limit cycle. For further details on such analysis, see [13].

V. Differential Dissipativity and Transverse Differential Dissipativity

Methods related to dissipation inequalities are central to quantitative results in systems analysis, including input-output methods such as small-gain and passivity [34], robust control design [35], and integral quadratic constraints [36], [37]. In this section, we introduce concepts of differential dissipativity, closely related to incremental small gain and passivity [34].

Roughly speaking, a system is differentially dissipative if the *linearization* along every solution is dissipative, however the results are exact and global, not local. The concept has been used several times before – though not under that name – in constructing small gain theorems for contracting systems [38] and in bounding the simulation error of identified models [39], [40], [41]. To the authors knowledge, the first general presentation of this concept was in [21], although the potential usefulness of such a construction was suggested in the introductory remarks of [24] and recently pursued in [42].

For this section we consider systems with external inputs and outputs:

$$\dot{x} = f(x, w), y = g(x, w) \tag{5}$$

which has the differential system:

$$\dot{\delta}_x = A(x)\delta_x + B(x)\delta_w, \;\; \delta_y = C(x)\delta_x + D(x)\delta_w, \tag{6}$$

where $A(x) := \frac{\partial f}{\partial x}, B(x) := \frac{\partial f}{\partial w}, C(x) := \frac{\partial g}{\partial x}, D(x) := \frac{\partial g}{\partial w}$.

A statement about *differential dissipativity* relates the system (5), (6) to a particular form $\sigma(x, w, \delta_x, \delta_w)$ which in applications is usually quadratic in δ_x, δ_w. In particular, along all solutions of (5), the differential system (6) satisfies

$$\int_0^T \sigma(x, w, \delta_x, \delta_w)dt \geq -\kappa(x(0), \delta_x(0)) \tag{7}$$

for all $T > 0$ and for some $\kappa : TM \to \mathbb{R}$. A shorthand notation for this is $\sigma(x, w, \delta_x, \delta_w) \succ 0$, c.f. the notion of a "complete IQC" in [37].

For example, one can define differential versions of the classical small-gain condition with $\sigma_\gamma = \gamma|\delta_w|^2 - |\delta_y|^2$ and passivity with $\sigma_p = \delta_w'\delta_y$, where the latter assumes the input and output have matching dimensions.

Inspired by IQC analysis [37], if a number of system properties are encoded in dissipativity relations of the form $\sigma_i \succ 0, i = 1, 2, ...p$, then a desired property (e.g. stability or bounded gain) encoded as $\sigma^\star \succ 0$, and then one searches for constants $\tau_i \geq 0, i = 1, 2, ...p$ satisfying $\sigma^\star - \sum_{i=1}^p \tau_i\sigma_i \succ 0$.

For system evolution on an invariant compact set, taking $\sigma^\star := -|\delta_x|^2 \succ 0$ implies contraction, since it implies that

δ_x converges to zero via Barbalat's lemma [43]. Differential contraction versions of the small-gain theorem and the passivity theorem are special cases of this formulation.

For a system of the form (5), a sufficient condition for (7) is the existence of a metric function $V(x, \delta_x) = \delta' M(x)\delta > 0$ such that

$$\frac{d}{dt}V(x, \delta_x) \leq \sigma(x, w, \delta_x, \delta_w) \quad (8)$$

where the path integral of V plays the role of an incremental storage function between solutions.

We define a system as *transverse differentially dissipative* (TDD) with a supply rate $\sigma(x, w, \delta_x, \delta_w)$ if (8) holds for all δ_x such that $\frac{\partial V}{\partial \delta_x} f(x, w) = 0$.

We give the following theorem, which can easily be extended to more than two system.

Theorem 7: Given two systems $\dot{x}_1 = f_1(x_1, w_1)$ and $\dot{x}_2 = f_2(x_2, w_2)$, and consider the interconnection $w_1 = g_2(x_2, w_2), w_2 = g_1(x_1, w_1)$. Suppose System 1 is transverse differentially dissipative with respect to supply rate $\sigma_1(x_1, w_1, \delta_{x1}, \delta_{w1})$ and System 2 satisfies $\sigma_2(x_2(t), w_2(t), \delta_{x2}(t), \delta_{w2}(t)) \geq 0$ for all t. Then if there exists nonnegative constants τ_1, τ_2 such that $0 < \tau_1\sigma_1(x_1, w_1, \delta_{x1}, \delta_{w1}) + \tau_2\sigma_2(x_2, w_2, \delta_{x2}, \delta_{w2})$ on a forward-invariant set of the interconnected system, then the interconnection is transverse contracting and has a unique stable periodic solution.

The proof of this theorem follows standard S-Procedure arguments in robust control theory [35], [37].

For example, for a dynamic system in feedback with a time-varying but non-dynamic mapping $w = \Delta(y, t)$ where Δ is slope-restricted with respect to y, one can choose $\sigma_1(x, w, \delta_x, \delta_w)\sigma_c(\delta_y, \delta_w) := (\delta_y - \alpha\delta_w)(\beta\delta_w - \delta_y)$ and $\sigma_2 = -\sigma_c(\delta_w, \delta_y)$. In doing so, we recover a differential form of the circle criterion that proves existence of a limit cycle in feedback with sector-bounded and slope-restricted nonlinearities.

A. Linear Matrix Inequalities for DD and TDD

The convex formulation from Section III-A can be extended to differential dissipativity conditions where the supply rate has the form

$$\sigma = \delta_x' H(x, u)\delta_x + 2\delta_x' N(x, u)\delta_u + \delta_u' R(x, u)\delta_u$$

as long as $H(x, u)$ is negative semidefinite, which is the case for common supply rates such as passivity and small gain. Note that it is necessary that $R(x, u)$ to be positive semidefinite for a lower bound to exist in (7).

Using the S-Procedure, the following condition is equivalent to transverse differential dissipativity:

$$\begin{aligned}\eta'(-WA' - AW + \dot{W} + \rho Q + WHW)\eta& \\ +2\eta'(-B + WN)\delta_u + \delta_u R\delta_u \leq 0& \quad (9)\end{aligned}$$

where we have dropped dependence of matrices on x and u for the sake of space and clarity. Note that although this inequality is quadratic in W it is still convex (when $H \leq 0$) and it can be transformed via the Schur complement to give a linear matrix inequality, which can be solved via convex optimization, see [21].

VI. Application Example: Moore Greitzer Model of Combustion Oscillation

The Moore-Greitzer model, a simplified model of surge-stall dynamics of a jet engine [44], has motivated substantial development in nonlinear control design (see, e.g., [45] and references therein). In [14], sum-of-squares programming was applied for automated construction of verification of contraction metrics. The following form of the Moore Greitzer model was examined, with δ considered an uncertain parameter:

$$\begin{bmatrix}\dot{\phi} \\ \dot{\psi}\end{bmatrix} = \begin{bmatrix}-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 + \delta \\ 3\phi - \psi\end{bmatrix}.$$

Contraction, and hence existence of a stable equilibrium, was established that values of δ with $|\delta| < 1.023$ using a contraction metric with each element a degree-six polynomial. In fact the system is also contracting for values of $\delta > 1.023$, but at $\delta \approx -1.023$ a Hopf bifurcation occurs.

Using the S-procedure formulation for transverse contraction from Section III-A of the present paper, we have established that for values of $\delta < -1.023$ the Moore Greitzer model exhibits stable oscillations.

Let $H(x) = A(x)W(x) + W(x)A(x)' - \dot{W}(x) + \lambda W(x)$, and let $\Sigma[x]$ denote the set of sum-of-squares polynomials in x, and $\Sigma_n[x]$ denote the set of $n \times n$ matrices verified positive semidefinite via sum-of-squares i.e. matrices $R(x)$ satisfying $y' R(x) y \in \Sigma[x, y]$.

Using a positivstellensatz construction [6] we derive the following conditions for transverse contraction, restricted to a set K which is a disc of radius ρ with a small region around the unstable equilibrium deleted.

$$\begin{aligned}W(x) - (f(x)'f(x) - 0.1)L_1(x)& & \\ -(\rho - x'x)L_2(x) &\in &\Sigma_n[x], \\ -H(x) - \alpha(x)f(x)f(x)'& & \\ -(f(x)'f(x) - \epsilon)L_3(x) - (\rho^2 - x'x)L_4(x) &\in &\Sigma_n[x], \\ L_1(x), L_2(x), L_3(x), L_4(x), &\in &\Sigma_n[x], \\ \alpha(x) &\in &\Sigma[x].\end{aligned}$$

We found that these conditions could be verified with $\rho = 10, \epsilon = 0.1$, and $W(x)$ a matrix of degree-four polynomials, and $L_i(x), \alpha(x)$ degree-two. The MATLAB code used to verify these conditions has been made available online [46].

A further example of the applicability is given by system identification of live neurons in culture, which naturally exhibit stable limit cycles of varying frequency. See [40], [21], [41].

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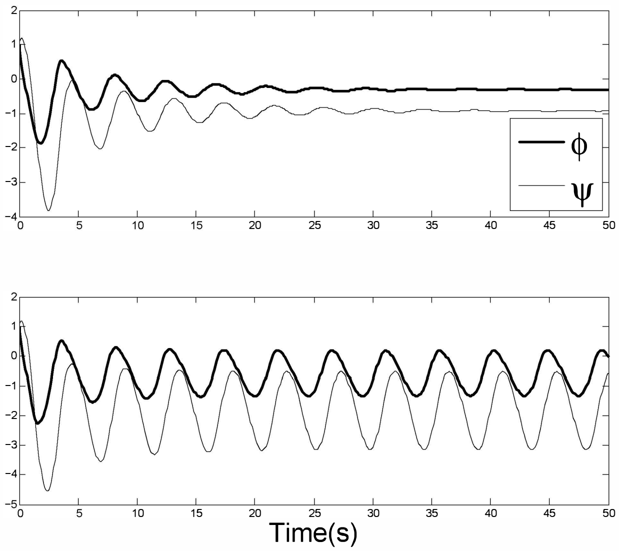

Fig. 1. Moore-Greitzer jet engine model response with $\delta = -0.8$ (top) and $\delta = -1.2$ (bottom).

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