Deciding Structural Liveness of Petri Nets

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Abstract. Place/transition Petri nets are a standard model for a class of distributed systems whose reachability spaces might be infinite. One of well-studied topics is the verification of safety and liveness properties in this model; despite the extensive research effort, some basic problems remain open, which is exemplified by the open complexity status of the reachability problem. The liveness problems are known to be closely related to the reachability problem, and many structural properties of nets that are related to liveness have been studied.

Somewhat surprisingly, the decidability status of the problem if a net is structurally live, i.e. if there is an initial marking for which it is live, has remained open, as also a recent paper (Best and Esparza, 2016) emphasizes. Here we show that the structural liveness problem for Petri nets is decidable.

A crucial ingredient of the proof is the result by Leroux (LiCS 2013) showing that we can compute a finite (Presburger) description of the reachability set for a marked Petri net if this set is semilinear.

1 Introduction

Petri nets are a standard tool for modeling and analysing a class of distributed systems; we can name [15] as a recent introductory monograph for this area. A natural part of the analysis of such systems is checking the safety and/or liveness properties, where the question of deadlock-freeness is just one example.

The classical version of place/transition Petri nets (exemplified by Fig. 1) is used to model systems with potentially infinite state spaces; here the decidability and/or complexity questions for respective analysis problems are often intricate. E.g., despite several decades of research the complexity status of the basic problem of *reachability* (can the system get from one given configuration to another?) remains unclear; we know that the problem is ExpSpace-hard due to a classical construction by Lipton (see, e.g., [4]) but the known upper complexity bounds are not primitive recursive (we can refer to [12] and the references therein for further information).

The *liveness* of a transition (modelling a system action) is a related problem; its complementary problem asks if for a given initial marking (modelling an initial system configuration) the net enables to reach a marking in which the transition is dead, in the sense that it can be never performed in the future.

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A marked net (N, M_0), i.e. a net N with an initial marking M_0, is live if all its transitions are live.

The close relationship of the problems of reachability and liveness has been clear since the early works by Hack [8,9]. Nevertheless, the situation is different for the problem of *structural* liveness that asks, given a net N, if there is a marking M_0 such that (N, M_0) is live. While semidecidability of structural liveness is clear due to the decidability of (reachability and) liveness, the decidability question has been open: see, e.g., the overview [16] and in particular the recent paper [3] where this problem (STLP) is discussed in Concluding Remarks.

Here we show the decidability of structural liveness, by showing the semidecidability of the complementary problem. The idea is to construct, for a given net N, a marked net (N', M'_0) (partly sketched in Fig. 2) that works in two phases (controlled by places added to N): in the first phase, an arbitrary marking M from the set $\mathcal{D}$ of markings with at least one dead transition is generated, and then N is simulated in the reverse mode from M. If N is not structurally live, then the projection of the reachability set of (N', M'_0) to the set P of places of N is the whole set $\mathbb{N}^P$; if N is structurally live, then there is $M \in \mathbb{N}^P$ such that the projection of any marking reachable from M'_0 differs from M.

In the first case (with the whole set $\mathbb{N}^P$) the reachability set of (N', M'_0) is semilinear, i.e. Presburger definable. Due to a result by Leroux [11], there is an algorithm that finishes with a Presburger description of the reachability set of (N', M'_0) when this set is semilinear (while it runs forever when not). This yields the announced semidecidability.

The construction of the above mentioned (downward closed) set $\mathcal{D}$ is standard; the crucial ingredient of our proof is thus the mentioned result by Leroux. Though we use the decidability of reachability (for semidecidability of the positive case), it is not clear if reachability reduces to structural liveness, and the complexity of the structural liveness problem is left open for future research.

Section 2 provides the formal background, and Sect. 3 shows the decidability result. In Sect. 4 a few comments are added, and in particular an example of a net is given where the set of live markings is not semilinear.

2 Basic Definitions

By $\mathbb{N}$ we denote the set $\{0, 1, 2, \dots\}$. For a set A, by A^* we denote the set of finite sequences of elements of A, and ε denotes the empty sequence.

Nets. A *Petri net*, or just a *net* for short, is a tuple $N = (P, T, W)$ where P and T are two disjoint finite sets of *places* and *transitions*, respectively, and $W : (P \times T) \cup (T \times P) \to \mathbb{N}$ is the *weighted flow function*. A *marking* M of N is an element of $\mathbb{N}^P$, a mapping from P to $\mathbb{N}$, often also viewed as a vector with $|P|$ components (i.e., an element of $\mathbb{N}^{|P|}$).

Figure 1 presents a net $N = (\{p_1, p_2, p_3\}, \{t_1, t_2, t_3\}, W)$ where $W(p_1, t_1) = 2$, $W(p_1, t_2) = 1$, $W(p_1, t_3) = 0$, etc.; we do not draw an arc from x to y when $W(x, y) = 0$, and we assume $W(x, y) = 1$ for the arcs (x, y) with no depicted

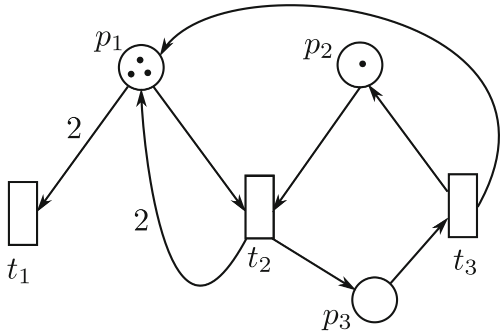

Fig. 1. Example of a net $N = (P, T, W)$, with marking $M = (3, 1, 0)$

numbers. Figure 1 also depicts a marking M by using black tokens, namely $M = (3, 1, 0)$, assuming the ordering (p_1, p_2, p_3) of places.

Reachability. Assuming a net $N = (P, T, W)$, for each $t \in T$ we define the following relation $\xrightarrow{t}$ on $\mathbb{N}^P$:

$$M \xrightarrow{t} M' \Leftrightarrow_{\mathrm{df}} \forall p \in P : M(p) \geq W(p,t) \wedge M'(p) = M(p) - W(p,t) + W(t,p).$$

By $M \xrightarrow{t}$ we denote that t is *enabled in* M, i.e., that there is M' such that $M \xrightarrow{t} M'$. The relations $\xrightarrow{t}$ are inductively extended to $\xrightarrow{u}$ for all $u \in T^*$: $M \xrightarrow{\varepsilon} M$; if $M \xrightarrow{t} M'$ and $M' \xrightarrow{u} M''$, then $M \xrightarrow{tu} M''$. The *reachability set for a marking* M is the set

$$[M\rangle = \{M' \mid M \xrightarrow{u} M' \text{ for some } u \in T^*\}.$$

For the net of Fig. 1 we have, e.g., $(3,1,0) \xrightarrow{t_2} (4,0,1) \xrightarrow{t_1} (2,0,1) \xrightarrow{t_1} (0,0,1) \xrightarrow{t_3} (1,1,0)$; we can check that the reachability set for $(3,1,0)$ is

$$\{\,(x,1,0) \mid x \text{ is odd}\,\} \cup \{\,(y,0,1) \mid y \text{ is even}\}. \qquad (1)$$

Liveness. For a net $N = (P, T, W)$, a *transition* t is *dead in* a *marking* M if there is no $M' \in [M\rangle$ such that $M' \xrightarrow{t}$. (Such t can be never performed in N when we start from M.)

A *transition* t is *live in* M_0 if there is no $M \in [M_0\rangle$ such that t is dead in M. (Hence for each $M \in [M_0\rangle$ there is $M' \in [M\rangle$ such that $M' \xrightarrow{t}$.) A *set* $T' \subseteq T$ *of transitions is live in* M_0 if each $t \in T'$ is live in M_0. (Another natural definition of liveness of a set T' is discussed in Sect. 4.)

A *marked net* is a pair (N, M_0) where $N = (P, T, W)$ is a net and M_0 is a marking, called *the initial marking.* A *marked net* (N, M_0) is *live* if each transition (in other words, the set T) is live in M_0 (in the net N). A *net* N is *structurally live* if there is M_0 such that (N, M_0) is live.

E.g., the net in Fig. 1 is structurally live since it is live for the marking $(3, 1, 0)$, as can be easily checked by inspecting the transitions enabled in the elements of the reachability set (1). We can also note that the net is not live for $(4, 1, 0)$, we even have that no transition is live in $(4, 1, 0)$, since $(4, 1, 0) \xrightarrow{t_1 t_1} (0, 1, 0)$ where all transitions are dead.

Liveness decision problems.

- The *partial liveness problem*, denoted PLP, asks, given a marked net (N, M_0) and a set T' of its transitions, if T' is live in M_0.
- The *liveness problem*, denoted LP, is a special case of PLP: it asks, given a marked net (N, M_0), if (N, M_0) is live (i.e., if all its transitions are live in M_0).
- The *partial structural liveness problem*, denoted PSLP, asks, given a net N and a set T' of its transitions, if there is M in which T' is live.
- The *structural liveness problem*, denoted SLP, is a special case of PSLP: it asks, given a net N, if there is M such that (N, M) is live.

3 Structural Liveness of Nets Is Decidable

We aim to show the decidability of PSLP, and thus also of SLP:

Theorem 1. *The partial structural liveness problem (*PSLP*) is decidable.*

We prove the theorem in the rest of this section. We first recall the famous decidability result for reachability. The *reachability problem*, denoted RP, asks if $M \in [M_0\rangle$ when given N, M_0, M.

Lemma 2. *[13] The reachability problem (*RP*) is decidable.*

In Petri net theory this is a fundamental theorem; we call it a "lemma" here, since it is one ingredient used in proving the theorem of this paper (i.e. Theorem 1). The first proof of Lemma 2 was given by E.W. Mayr (see [13] for a journal publication), and there is a row of further papers dealing with this problem; we can refer to a recent paper [12] and the references therein for further information. As already mentioned, the complexity of the reachability problem remains far from clear.

There are long known, and straightforward, effective reductions among the reachability problem RP and the (partial) liveness problems (PLP and LP); we can find them already in Hack's works from 1970s [8,9]. This induces semidecidability of the partial structural liveness problem (PSLP): given N and T', we can systematically generate all markings of N, always deciding if T' is live in the currently generated M (and halt when the answer is positive). Hence the main issue is to establish the semidecidability of the complementary problem of PSLP; roughly speaking, we need to find a finite witness when (N, M) is non-live for all M.

We further assume a fixed net $N = (P, T, W)$ if not said otherwise.

Sets of "dead" markings are downward closed. A natural first step for studying (partial) liveness is to explore the sets

$$\mathcal{D}_{T'} = \{M \in \mathbb{N}^P \mid \text{ some } t \in T' \text{ is dead in } M\}$$

for $T' \subseteq T$. We note that the definition entails $\mathcal{D}_{T'} = \bigcup_{t \in T'} \mathcal{D}_{\{t\}}$. E.g., in the net of Fig. 1 we have $\mathcal{D}_{\{t_1\}} = \{(x, 0, 0) \mid x \leq 1\} \cup \{(0, x, 0) \mid x \in \mathbb{N}\}$, $\mathcal{D}_{\{t_2,t_3\}} = \{(x, 0, 0) \mid x \in \mathbb{N}\}$, and

$$\mathcal{D}_T = \{(0, x, 0) \mid x \in \mathbb{N}\} \cup \{(x, 0, 0) \mid x \in \mathbb{N}\}. \tag{2}$$

Due to the monotonicity of Petri nets (by which we mean that $M \xrightarrow{u} M'$ implies $M+\delta \xrightarrow{u} M'+\delta$ for all $\delta \in \mathbb{N}^P$), each $\mathcal{D}_{T'}$ is obviously downward closed. We say that $\mathcal{D} \subseteq \mathbb{N}^P$ is *downward closed* if $M \in \mathcal{D}$ implies $M' \in \mathcal{D}$ for all $M' \leq M$, where we refer to the component-wise order:

$$M' \leq M \Leftrightarrow_{\text{df}} \forall p \in P : M'(p) \leq M(p).$$

It is standard to characterize any downward closed subset $\mathcal{D}$ of $\mathbb{N}^P$ by the set of its maximal elements, using the extension $\mathbb{N}_\omega = \mathbb{N} \cup \{\omega\}$ where ω stands for an "arbitrarily large number" satisfying $\omega > n$ for all $n \in \mathbb{N}$. Formally we extend a downward closed set $\mathcal{D} \subseteq \mathbb{N}^P$ to the set

$$\widehat{\mathcal{D}} = \{M \in (\mathbb{N}_\omega)^P \mid \forall M' \in \mathbb{N}^P : M' \leq M \Rightarrow M' \in \mathcal{D}\}.$$

We thus have

$$\mathcal{D} = \{M' \in \mathbb{N}^P \mid M' \leq M \text{ for some } M \in \textsc{Max}(\widehat{\mathcal{D}})\}$$

where $\textsc{Max}(\widehat{\mathcal{D}})$ is the set of maximal elements of $\widehat{\mathcal{D}}$. By (the standard extension of) Dickson's Lemma, the set $\textsc{Max}(\widehat{D})$ is finite. (We can refer, e.g., to [5] where such completions by "adding the limits" are handled in a general framework.)

E.g., for the set $\mathcal{D}_T$ in (2) we have $\textsc{Max}(\widehat{\mathcal{D}_T}) = \{(0, \omega, 0), (\omega, 0, 0)\}$.

Proposition 3. *Given $N = (P, T, W)$ and $T' \subseteq T$, the set $\mathcal{D}_{T'}$ is downward closed and the finite set $\textsc{Max}(\widehat{\mathcal{D}_{T'}})$ is effectively constructible.*

Proof. We consider a net $N = (P, T, W)$ and a set $T' \subseteq T$. As discussed above, the set $\mathcal{D}_{T'}$ is downward closed.

Instead of a direct construction of the finite set $\textsc{Max}(\widehat{\mathcal{D}_{T'}})$, we first show that the set $\mathcal{S}_{T'} = \textsc{Min}(\mathbb{N}^P \smallsetminus \mathcal{D}_{T'})$, i.e. the set of minimal elements of the (upward closed) complement of $\mathcal{D}_{T'}$, is effectively constructible.

For each $t \in T'$, we first compute $\mathcal{S}_t = \textsc{Min}(\mathbb{N}^P \smallsetminus \mathcal{D}_{\{t\}})$, i.e. the set of minimal markings in which t is not dead. One standard possibility for computing $\mathcal{S}_t$ is to use the following backward algorithm, where

$\textsc{MinPre}(t', M)$ is the unique marking in $\textsc{Min}(\{M' \mid \exists M'' \geq M : M' \xrightarrow{t'} M''\})$.

(For each $p \in P$, $\textsc{MinPre}(t', M)(p) = W(p, t') + \max\{M(p) - W(t', p), 0\}$.)

An algorithm for computing $\mathcal{S}_t$:

1. Initialize the variable $\mathcal{S}$, containing a finite set of markings, by
$$\mathcal{S} := \{\textsc{MinPre}(t, \mathbf{0})\}$$
where $\mathbf{0}$ is the zero marking ($\mathbf{0}(p) = 0$ for each $p \in P$).
2. Perform the following step repeatedly, as long as possible:
if for some $t' \in T$ and $M \in \mathcal{S}$ the marking $M' = \textsc{MinPre}(t', M)$ is not in the upward closure of $\mathcal{S}$ (hence $M' \not\geq M''$ for each $M'' \in \mathcal{S}$), then put
$$\mathcal{S} := \mathcal{S} \cup \{M'\} \smallsetminus \{M'' \in \mathcal{S} \mid M' \leq M''\}.$$

Termination is clear by Dickson's Lemma, and the final value of $\mathcal{S}$ is obviously the set $\mathcal{S}_t$ (of all minimal markings from which t can get enabled). We can remark that related studies in more general frameworks can be found, e.g., in [1,6].

Having computed the sets $\mathcal{S}_t = \textsc{Min}(\mathbb{N}^P \smallsetminus \mathcal{D}_{\{t\}})$ for all $t \in T'$, we can surely compute the set $\mathcal{S}_{T'} = \textsc{Min}(\mathbb{N}^P \smallsetminus \mathcal{D}_{T'})$ since
$$\mathcal{S}_{T'} = \textsc{Min}(\{M \in \mathbb{N}^P \mid (\forall t \in T')(\exists M' \in \mathcal{S}_t) : M \geq M'\}).$$
This also entails that the maximum $\mathrm{B} \in \mathbb{N}$ of values $M(p)$ where $M \in \mathcal{S}_{T'}$ (and $p \in P$) is bounded by the maximum value $M(p)$ where $M \in \mathcal{S}_t$ for some $t \in T'$. Since the finite (i.e., non-ω) numbers $M(p)$ in the elements M of $\textsc{Max}(\widehat{\mathcal{D}_{T'}})$ are obviously less than B, the set $\textsc{Max}(\widehat{\mathcal{D}_{T'}})$ can be constructed when given $\mathcal{S}_{T'}$. $\square$

Remark. Generally we must count with at least exponential-space algorithms for constructing $\textsc{Max}(\widehat{\mathcal{D}_{T'}})$ (or $\textsc{Min}(\mathbb{N}^P \smallsetminus \mathcal{D}_{T'})$), due to Lipton's ExpSpace-hardness construction that also applies to the coverability (besides the reachability). On the other hand, by Rackoff's results [14] the maximum B mentioned in the proof is at most doubly-exponential w.r.t. the input size, and thus fits in exponential space. Nevertheless, the precise complexity of computing $\textsc{Max}(\widehat{\mathcal{D}_{T'}})$ is not important in our context.

Sets of "live" markings are more complicated. Assuming $N = (P, T, W)$, for $T' \subseteq T$ we define
$$\mathcal{L}_{T'} = \{M \in \mathbb{N}^P \mid T' \text{ is live in } M\}.$$
The set $\mathcal{L}_{T'}$ is not the complement of $\mathcal{D}_{T'}$ in general, but our definitions readily yield the following equivalence:

Proposition 4. *$M \in \mathcal{L}_{T'}$ iff $[M\rangle \cap \mathcal{D}_{T'} = \emptyset$.*

We note that $\mathcal{L}_{T'}$ is not upward closed in general. We have already observed this on the net in Fig. 1, where $\mathcal{D}_T = \{(0, x, 0) \mid x \in \mathbb{N}\} \cup \{(x, 0, 0) \mid x \in \mathbb{N}\}$ (i.e., $\textsc{Max}(\widehat{\mathcal{D}_T}) = \{(0, \omega, 0), (\omega, 0, 0)\}$). It is not difficult to verify that in this net we have
$$\mathcal{L}_T = \{\, M \in \mathbb{N}^{\{p_1,p_2,p_3\}} \mid M(p_2) + M(p_3) \geq 1 \text{ and } M(p_1) + M(p_3) \text{ is odd}\,\}. \quad (3)$$
Proposition 4 has the following simple corollary:

Proposition 5. *The answer to an instance $N = (P, T, W)$, T' of* PSLP *(the partial structural liveness problem) is*

1. *YES if $\mathcal{L}_{T'} \neq \emptyset$, i.e., if $\{M \in \mathbb{N}^P; [M\rangle \cap \mathcal{D}_{T'} \neq \emptyset\} \neq \mathbb{N}^P$.*
2. *NO if $\mathcal{L}_{T'} = \emptyset$, i.e., if $\{M \in \mathbb{N}^P; [M\rangle \cap \mathcal{D}_{T'} \neq \emptyset\} = \mathbb{N}^P$.*

It turns out important for us that in the case 2 (NO) the set $\{M \in \mathbb{N}^P; [M\rangle \cap \mathcal{D}_{T'} \neq \emptyset\}$ is semilinear. We now recall the relevant notions and facts, and then we give a proof of Theorem 1.

Semilinear sets. For a fixed (dimension) $d \in \mathbb{N}$, a *set* $\mathscr{L} \subseteq \mathbb{N}^d$ is *linear* if there is a (basic) vector $\rho \in \mathbb{N}^d$ and (period) vectors $\pi_1, \pi_2, \ldots, \pi_k \in \mathbb{N}^d$ (for some $k \in \mathbb{N}$) such that

$$\mathscr{L} = \{\, \rho + x_1\pi_1 + x_2\pi_2 + \cdots + x_k\pi_k \mid x_1, x_2, \ldots, x_k \in \mathbb{N}\,\}.$$

Such vectors $\rho, \pi_1, \pi_2, \ldots, \pi_k$ constitute a *description* of the set $\mathscr{L}$.

A *set* $\mathscr{S} \subseteq \mathbb{N}^d$ is *semilinear* if it is the union of finitely many linear sets; a *description* of $\mathscr{S}$ is a collection of descriptions of $\mathscr{L}_i$, $i = 1, 2, \ldots, m$ (for some $m \in \mathbb{N}$), where $\mathscr{S} = \mathscr{L}_1 \cup \mathscr{L}_2 \cup \cdots \cup \mathscr{L}_m$ and $\mathscr{L}_i$ are linear.

It is well known that an equivalent formalism for describing semilinear sets are Presburger formulas [7], the arithmetic formulas that can use addition but no multiplication (of variables); we also recall that the truth of (closed) Presburger formulas is decidable. E.g., all downward (or upward) closed sets $\mathcal{D} \subseteq \mathbb{N}^P$ are semilinear, and also the above sets (1) and (3) are examples of semilinear sets. Moreover, given the set $\textsc{Max}(\widehat{\mathcal{D}})$ for a downward closed set $\mathcal{D}$, constructing a description of $\mathcal{D}$ as of a semilinear set is straightforward.

It is also well known that the reachability sets $[M\rangle$ are not semilinear in general; similarly the sets $\mathcal{L}_{T'}$ (of live markings) are not semilinear in general. (We give an example in Sect. 4.) But we have the following result by Leroux [11]; it is again an important theorem in Petri net theory that we call a "lemma" in our context (since it is an ingredient for proving Theorem 1).

Lemma 6. *[11] There is an algorithm that, given a marked net (N, M_0), halts iff the reachability set $[M_0\rangle$ is semilinear, in which case it produces a description of this set.*

Roughly speaking, the algorithm guaranteed by Lemma 6 generates the reachability graph for M_0 while performing certain "accelerations" when possible (which captures repeatings of some transition sequences by simple formulas); this process is creating a sequence of descriptions of increasing semilinear subsets of the reachability set $[M_0\rangle$ until the subset is closed under all steps $\xrightarrow{t}$ (which can be effectively checked); in this case the subset (called an inductive invariant in [11]) is equal to $[M_0\rangle$, and the process is guaranteed to reach such a case when $[M_0\rangle$ is semilinear. (A consequence highlighted in [11] is that in such a case all reachable markings can be reached by sequences of transitions from a bounded language.)

Proof of Theorem 1 (decidability of PSLP).

Given $N = (P, T, W)$ and $T' \subseteq T$, we will construct a marked net (N', M'_0) where $N' = (P \cup P_{new}, T \cup T_{new}, W')$ so that we will have:

(a) if $\mathcal{L}_{T'} = \emptyset$ in N (i.e., T' is non-live in each marking of N) then $[M'_0\rangle$ is semilinear and the projection of $[M'_0\rangle$ to P is equal to $\mathbb{N}^P$;

(b) if $\mathcal{L}_{T'} \neq \emptyset$, then the projection of $[M'_0\rangle$ to P is not equal to $\mathbb{N}^P$ (and might be non-semilinear).

This construction of (N', M'_0) yields the required decidability proof, since we can consider two algorithms running in parallel:

– One is the algorithm of Lemma 6 applied to (N', M'_0); if it finishes with a semilinear description of $[M'_0\rangle$, which surely happens in the case (a), then we can effectively check if the projection of $[M'_0\rangle$ to P is $\mathbb{N}^P$, i.e. if $\mathcal{L}_{T'} = \emptyset$. (A projection of a semilinear set is effectively semilinear, the set-difference of two semilinear set is also effectively semilinear [7], and checking emptiness of a semilinear set is trivial.)
– The other algorithm generates all $M \in \mathbb{N}^P$ and for each of them checks if there is $M' \in [M'_0\rangle$ such that $M'{\restriction_P}$ (i.e., M' projected to P) is equal to M. It thus finds some M with the negative answer if, and only if, $\mathcal{L}_{T'} \neq \emptyset$ (the case (b)). The existence of the algorithm checking the mentioned property for M follows from a standard extension of the decidability of reachability (Lemma 2); for our concrete construction below this extension is not needed, and just the claim of Lemma 2 will suffice.

The construction of (N', M'_0) is illustrated in Fig. 2; we create a marked net that first generates an element of $\mathcal{D}_{T'}$ on the places P, and then simulates N in the reverse mode. More concretely, we assume the ordering $(p_1, p_2, \ldots, p_n)$ of the set P of places in N, and compute a description of the semilinear set $\mathcal{D}_{T'} \subseteq \mathbb{N}^{|P|}$ (by first constructing the set $\textsc{Max}(\widehat{\mathcal{D}_{T'}})$; recall Proposition 3). We thus get

$$\mathcal{D}_{T'} = \mathscr{L}_1 \cup \mathscr{L}_2 \cup \cdots \cup \mathscr{L}_m,$$

given by descriptions $\rho_i, \pi_{i1}, \pi_{i2}, \ldots, \pi_{ik_i}$ of the linear sets $\mathscr{L}_i$, for $i = 1, 2, \ldots, m$.

Remark. We choose this description of $\mathcal{D}_{T'}$ to make clear that the construction can be applied to any semilinear set, not only to a downward closed one.

The construction of (N', M'_0), where $N' = (P \cup P_{new}, T \cup T_{new}, W')$, is now described in detail:

1. Given $N = (P, T, W)$, create the "reversed" net $N_{rev} = (P, T, W_{rev})$, where $W_{rev}(p, t) = W(t, p)$ and $W_{rev}(t, p) = W(p, t)$ for all $p \in P$ and $t \in T$. (By induction on the length of u it is easy to verify that $M \xrightarrow{u} M'$ in N iff $M' \xrightarrow{u_{rev}} M$ in N_{rev}, where u_{rev} is defined inductively as follows: $\varepsilon_{rev} = \varepsilon$ and $(tu)_{rev} = u_{rev}t$.)
2. To get N', extend N_{rev} as described below; we will have $W'(p, t) = W_{rev}(p, t)$ and $W'(t, p) = W_{rev}(t, p)$ for all $p \in P$ and $t \in T$.

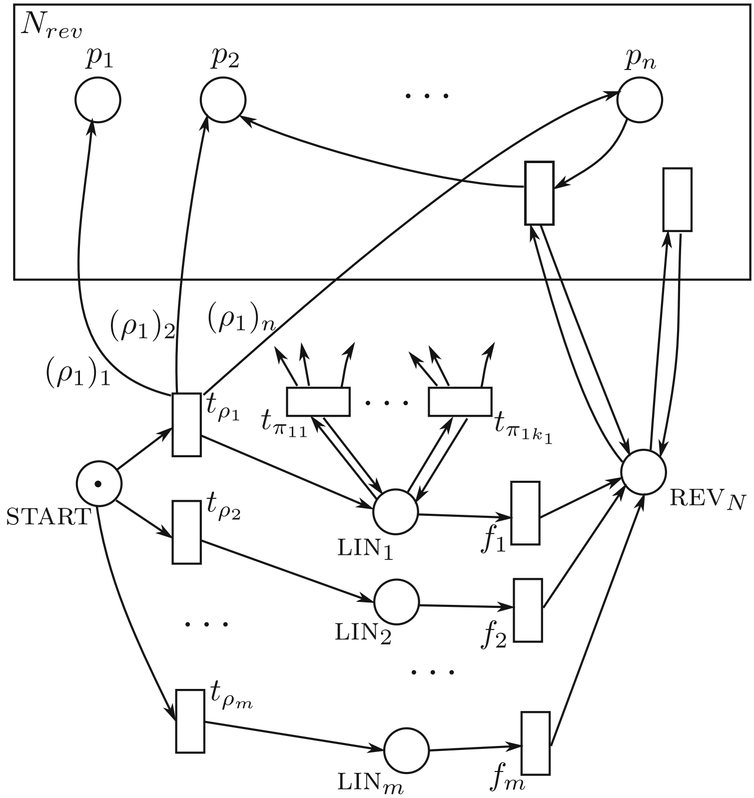

Fig. 2. Construction of (N', M'_0) for deciding the (partial) structural liveness (PSLP)

3. Create the set P_{new} of additional places

$$P_{new} = \{\text{START}, \text{LIN}_1, \text{LIN}_2, \ldots, \text{LIN}_m, \text{REV}_N\}$$

   and the set T_{new} of additional transitions

$$T_{new} = \bigcup_{i \in \{1,2,\ldots,m\}} \{t_{\rho_i}, f_i, t_{\pi_{i1}}, t_{\pi_{i2}}, \ldots, t_{\pi_{ik_i}}\}$$

   (as partly depicted in Fig. 2.)
4. Put $M'_0(\text{START}) = 1$ and $M'_0(p) = 0$ for all other places $p \in P \cup P_{new}$.
5. For each $i \in \{1, 2, \ldots, m\}$, put $W'(\text{START}, t_{\rho_i}) = W'(t_{\rho_i}, \text{LIN}_i) = 1$, and $W'(t_{\rho_i}, p_j) = (\rho_i)_j$ for all $j \in \{1, 2, \ldots, n\}$, where $(\rho_i)_j$ is the j-th component of the vector $\rho_i \in \mathbb{N}^n$. (We tacitly assume that the value of W' is 0 for the pairs (p, t) and (t, p) that are not mentioned.)
6. For each $t_{\pi_{i\ell}}$ ($i \in \{1, 2, \ldots, m\}$, $\ell \in \{1, 2, \ldots, k_i\}$) put $W'(\text{LIN}_i, t_{\pi_{i\ell}}) = W'(t_{\pi_{i\ell}}, \text{LIN}_i) = 1$, and $W'(t_{\pi_{i\ell}}, p_j) = (\pi_{i\ell})_j$ for all $j \in \{1, 2, \ldots, n\}$.
7. For each f_i put $W'(\text{LIN}_i, f_i) = W'(f_i, \text{REV}_N) = 1$.
8. For each transition $t \in T$ in N_{rev} put $W'(\text{REV}_N, t) = W'(t, \text{REV}_N) = 1$.

In the resulting (N', M'_0) we have only one token moving on P_{new}; more precisely, the set $[M'_0\rangle$ can be expressed as the union

$$[M'_0\rangle = \mathcal{S}_{\text{START}} \cup \mathcal{S}_{\text{LIN}_1} \cup \cdots \cup \mathcal{S}_{\text{LIN}_m} \cup \mathcal{S}_{\text{REV}_N}$$

of the disjoint sets $\mathcal{S}_p = \{M \mid M \in [M'_0\rangle \text{ and } M(p) = 1\}$, for $p \in \{\text{START}, \text{LIN}_1, \ldots, \text{LIN}_m, \text{REV}_N\}$. It is clear that each of the sets $\mathcal{S}_{\text{START}}$, $\mathcal{S}_{\text{LIN}_1}$, $\ldots$, $\mathcal{S}_{\text{LIN}_m}$ is linear, and that the projection of $\mathcal{S}_{\text{REV}_N}$ to $P = \{p_1, p_2, \ldots, p_n\}$ is the set $\{M \in \mathbb{N}^P; [M\rangle \cap \mathcal{D}_{T'} \neq \emptyset\}$ where $[M\rangle$ refers to the net N.

The constructed (N', M'_0) clearly satisfies the above conditions (a) and (b). In the algorithm verifying b), it suffices to generate the markings M of N' that satisfy $M(\text{REV}_N) = 1$, $M(\text{START}) = M(\text{LIN}_1) = \cdots = M(\text{LIN}_m) = 0$, and to check the (non)reachability from M'_0 for each of them (recall Lemma 2).

Remark. We also have another option (than Lemma 2) for establishing the non-reachability of M from M'_0, due to another result by Leroux (see, e.g., [10]): namely to find a description of a semilinear set that contains M'_0, does not contain M, and is closed w.r.t. all steps $\xrightarrow{t}$ (being thus an inductive invariant in the terminology of [10]).

4 Additional Remarks

Sets of live markings can be non-semilinear. In Petri net theory, there are many results that relate liveness to specific structural properties of nets. We can name [2] as an example of a cited paper from this area. Nevertheless, the general structural liveness problem is still not fully understood; one reason might be the fact that *the set of live markings of a given net is not semilinear in general.*

We give an example. If the set $\mathcal{L}_T$ of live markings for the net $N = (P, T, W)$ in Fig. 3 was semilinear, then also its intersection with the set $\{(x_1, 0, 1, 0, 1, x_6) \mid x_1, x_6 \in \mathbb{N}\}$ would be semilinear (i.e., definable by a Presburger formula). But is is straightforward to verify that the markings in this set are live if, and only if, $x_6 > 2^{x_1}$, which makes the set clearly non-semilinear. Indeed, any marking M where p_4 is marked (forever), i.e. $M(p_4) \geq 1$, is clearly live, and we can get at most 2^{x_1} tokens in p_5 as long as p_4 is unmarked; if $x_6 \leq 2^{x_1}$, then there is a reachable marking where all transitions are dead, but otherwise p_4 gets necessarily marked.

Another version of liveness of a set of transitions. Given $N = (P, T, W)$, we defined that a set T' of transitions is live in a marking M if each $t \in T'$ is live in M. Another option is to view T' as live in M if in each $M' \in [M\rangle$ at least one $t \in T'$ is not dead. But the problem if T' is live in M in this sense can be easily reduced to the problem if a specific transition is live. (We can add a place $\bar{p}$ and a transition $\bar{t}$, putting $W(\bar{p}, \bar{t}) = 1$. For each $t \in T'$ we then add t' and put $W(t', \bar{p}) = 1$ and $W(p, t') = W(t', p) = W(p, t)$ for each $p \in P$. Then T' is live in M in the new sense iff $\bar{t}$ is live in M.) The above nuances in definitions thus make no substantial difference.

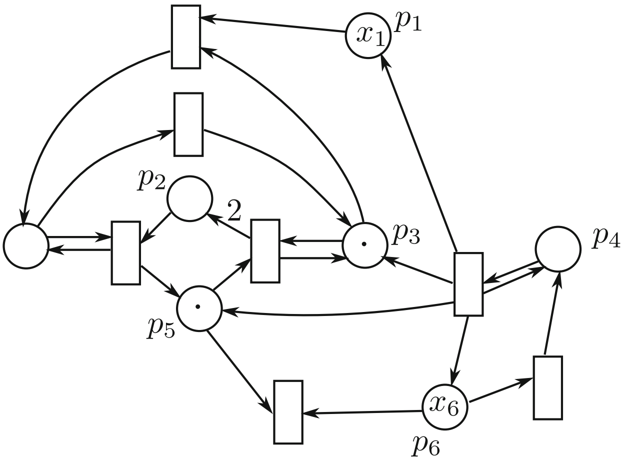

Fig. 3. Sets of live markings can be non-semilinear

Open complexity status. We have not clarified the complexity of the (partial) structural liveness problem (PSLP, SLP). The complexity of the (partial) liveness problem (PLP, LP) is "close" to the complexity of the reachability problem RP (as follows already by the constructions in [8]), but it seems natural to expect that the *structural* liveness problem might be easier. (E.g., the boundedness problem, asking if $[M_0\rangle$ is finite when given (N, M_0), is ExpSpace-complete, by the results of Lipton and Rackoff, but the structural boundedness problem is polynomial; here we ask, given N, if (N, M_0) is bounded for all M_0, or in the complementary way, if (N, M_0) is unbounded for some M_0.)

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