p is a subset of the truth-set of q. The main notion of logical consequence considered is that according to which p follows from S when every model of S is a model of p.

A deductive system in the style of Gentzen sequents is presented for which a modified Henkin-style completeness proof is given.

Call two interpretations of L (elementarily) equivalent when they have the same truth-valuation. It is natural to think of a theory as "complete" when all of its models are equivalent. Relative to the above language and semantics, it is possible to define three unary sentence operators T, F, and U such that under any interpretation: Ts is t if s is t and f otherwise, Fs is t if s is f and f otherwise, and Us is t if s is u and f otherwise. The above notion of "complete" holds of a theory R if and only if for all s in L, R implies Ts, R implies Fs, or R implies Us. By completeness (and soundness) of the underlying logic, the latter is equivalent to the condition: For all s in L, $R \vdash Ts$, $R \vdash Fs$, or $R \vdash Us$. The author presents a definition of completeness of theories which is equivalent to the last-stated condition.

Concerning the relationships between a given theory relative to a standard underlying logic and "the same" theory relative to the above free logic, the author defines for each set C of non-logical constants a set of sentences PC which holds exactly in the interpretations in which the constants in C are total, and he shows that the standard deductive closure of a set R of sentences is complete if and only if the free deductive closure of $R + PC$ is complete. In addition, he indicates the correct forms for conditional definitions so that definitional extensions of complete theories are complete.

Except for the inherent limitations of the Gentzen-sequent approach, the author's system and his defined concepts seem remarkably faithful to standard practice. This may well be a definitive work on analysis of actual deductive practice in mathematics.

JOHN CORCORAN and JOHN HERRING

ROBERT MCNAUGHTON. *Testing and generating infinite sequences by a finite automaton.* ***Information and control,*** vol. 9 (1966), pp. 521–530.

McNaughton proves that exactly the regular ω-events are defined by deterministic (finite) automata working on ω-sequences. As for the well-known corresponding theorem on finite sequences and regular events, the proof amounts essentially to showing that, for any given non-deterministic automaton A, there exists an equivalent deterministic one, B. To construct B, McNaughton runs several copies of A, starting one after the other on the same input. The crucial observation is: Since A has only finitely many states, and since one can identify two copies which have reached the same state, one can get along with finitely many copies. Thus, the behavior of all the copies can be described by a single (deterministic) automaton B. B is "by far the most intricate finite automaton this writer has seen in action" (Büchi, ***Bulletin of the American Mathematical Society***, vol. 71 (1965), pp. 767–770). Its construction is described informally, leaving hard but not essential work to the reader.

McNaughton's theorem implies that the regular ω-events are closed under complementation, and thus yields, as he points out, another proof of the decidability of the monadic second-order theory (MST) of ω (Büchi). Indeed, McNaughton's result has proved to be fundamental in decision procedures for MSTs. Büchi's proof of the decidability of the MST of countable ordinals (cited above) uses McNaughton automata. Also Rabin (see following review) uses the method described above, translated to tree automata.

DIRK SIEFKES

MICHAEL O. RABIN. *Decidability of second-order theories and automata on infinite trees.* ***Bulletin of the American Mathematical Society,*** vol. 74 (1968), pp. 1025–1029.

MICHAEL O. RABIN. *Decidability of second-order theories and automata on infinite trees.* ***Transactions of the American Mathematical Society,*** vol. 141 (1969), pp. 1–35.

Solving a problem of Büchi, the author gives a decision procedure for the monadic second-order theory (MST) of two successor functions (S2S), i.e., the theory of the full binary tree which allows quantification over both elements and subsets of the tree. This result generalizes and uses not only the decidability proof for the MST of one successor by Büchi (XXVIII 100(2)) but also that for the weak MST of two successors by Doner (see the review following).

The proof combines in an ingenious way a great many ideas, some of which go back to work of Büchi (XXVIII 100(1, 2)) and McNaughton (see preceding review). The heart of all the proofs is that formulas of these MST's can be brought into normal forms which one can read (by interpreting a k-tuple of predicates as a function into the set $\{0, 1\}^k$) as acceptance conditions of appropriate finite automata. In this way the theory of finite automata enters the decidability proofs for MST's. Conversely, the problem of MST's and their use as condition languages has stimulated automata theory; in particular it has created the generalization to infinite domains.

Despite these similarities Rabin's proof surpasses the other quoted proofs in its extreme difficulty and complexity. In this connection, three points should be mentioned: (i) Tree automata appear with both a McNaughton-type and a combinatorial output condition. There might be another output condition which could replace both. (ii) The use of automata makes the proof steps elementary but the whole proof involved. One might hope for underlying combinatorial theorems by which the structure of the proof would be better exhibited. (iii) The proof makes use of the axiom of choice. It is not clear whether this is necessary (but of course avoiding the axiom of choice would contradict the tendency of remark (ii)).

The author proves the decidability of many other theories by constructing conservative interpretations into S2S: MST of k successors, $k \leqq \omega$; MST of countable linear orderings; MST of a unary function with a countable domain (this extends to the MST of a unary function by a result of LeTourneau); elementary theory of countable Boolean algebras plus quantification over ideals; elementary theory of the lattice of closed subsets of the real numbers (this solves a problem of Grzegorczyk XVIII 73). This impressive list best shows the importance of the result.

Errata. On page 23, last line, read $\leq$ instead of $<$. On page 33, last line, and on page 34 line 11, read 3.10 instead of 3.9.

DIRK SIEFKES

JOHN DONER. *Tree acceptors and some of their applications.* ***Journal of computer and system sciences***, vol. 4 (1970), pp. 406–451.

Tree acceptors are finite automata which work on finite trees as input, instead of the usual strings. Like other kinds of generalized finite automata (e.g. finite automata working on infinite strings, or on infinite trees), tree acceptors are invented to solve the decision problem of a monadic second-order theory (MST). In this paper, the decidability of the weak monadic second-order theory (WST) of p successors, $p < \omega$, is shown, thus settling a problem of Büchi. To this end the theory of tree acceptors is developed to some extent, mainly generalizing facts from ordinary finite automata theory. (Note, however, that deterministic tree acceptors work down the trees, from the treetop to the base, since upward deterministic tree acceptors are too weak.)

Tree acceptors are also used to give new characterizations of the class of regular sets and of context-free languages. Finally, the decidability of the WST of p successors is used to prove the decidability of various other theories by giving conservative interpretations of the latter into the former: for example, the WST of any order type which is built up from the finite types, ω, and η, by finitely many applications of the operations of order type addition, multiplication, and converse; and the WST of locally free algebras with only unary operations. It is pointed out that the latter result can be extended to the MST of locally free algebras with unary operations, if one uses Rabin's result of the decidability of the MST of p successors (see preceding review).

The paper of Doner is historically earlier than Rabin's, as the abstract of it appeared in ***Notices of the American Mathematical Society***, vol. 12 (1965), p. 819, two years before corresponding abstracts by Rabin (ibid., vol. 14 (1967), p. 826 and p. 941), but it was not published in final form until 1970.

Erratum. On p. 409, line 21, read $\tau[w/\tau']$ instead of $\tau[w/\tau]$.

DIRK SIEFKES

J. W. THATCHER and J. B. WRIGHT. *Generalized finite automata theory with an application to a decision problem of second-order logic.* ***Mathematical systems theory***, vol. 2 (1968), pp. 57–81.

Most of the results of this paper are similar to some of the contents of the paper of Doner (see preceding review), but were obtained independently and only slightly later. The theory of generalized finite automata (i.e., tree acceptors) is presented in a very nice algebraic formulation, developing ideas of Büchi and Wright. Of special interest is the generalization of Kleene's