

Bounds for D-finite Substitution



Manuel Kauers · Institute for Algebra · JKU

Joint work with Gleb Pogudin

$f(x)$ is called **algebraic** if it satisfies a polynomial equation with polynomial coefficients:

$$p_0(x) + p_1(x)f(x) + \cdots + p_r(x)f(x)^r = 0.$$

Examples: $x^5 - 1$, $\sqrt{1-x}$, $\sqrt[3]{x^2 + 2x - 1} - \sqrt{1+x^9}$, ...

$f(x)$ is called **D-finite** if it satisfies a linear differential equation with polynomial coefficients:

$$p_1(x)f(x) + p_2(x)f'(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples: $\log(x)$, e^x , $\sqrt{1-x}$, $\log(1 - \sqrt{1-x})$, ...

Abel's theorem:

$$\text{algebraic} \Rightarrow \text{D-finite}$$

More generally:

$$f(x) \text{ D-finite} \wedge g(x) \text{ algebraic} \Rightarrow f(g(x)) \text{ D-finite}$$

Example:

$$f(x) = \log(1-x) \quad f'(x) + (x-1)f''(x) = 0 \quad \leftarrow \text{short}$$

$$g(x) = \sqrt{1-x} \quad (1-x) - g(x)^2 = 0 \quad \leftarrow \text{short}$$

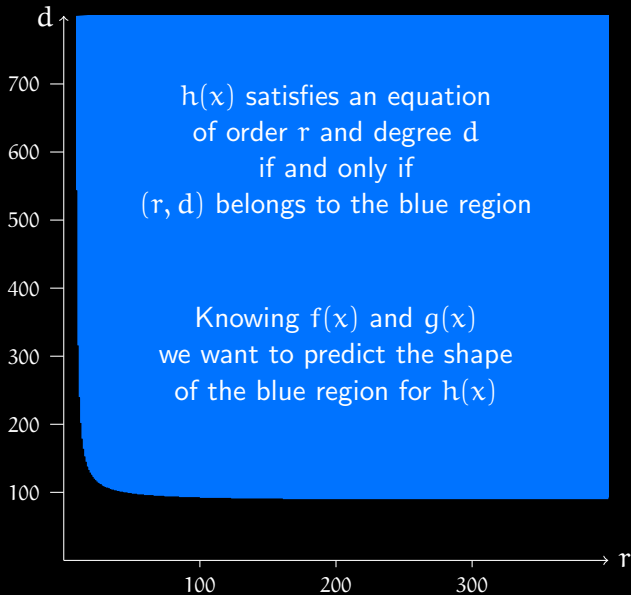
$$h(x) = f(g(x)) \quad \underbrace{3h'(x) + (7x-4)h''(x) + (2x^2-2x)h'''(x)}_{\text{longer}} = 0$$

Main Question:

How big is the equation for $h(x)$
in terms of the sizes of equations
for $f(x)$ and $g(x)$?

Subquestion A: how to measure the size of an equation?

Subquestion B: the equation of $h(x)$ is not unique; which equation is the smallest?



Similar questions have already been addressed for other operations:

- $f(x)$ algebraic $\Rightarrow f(x)$ D-finite
[Bostan, Chyzak, Salvy, Lecerf, Schost, 2007]
- $f(x, y)$ hyperexponential $\Rightarrow \int_x f(x, y)$ D-finite
[Chen, Kauers, 2012]
- $f(x), g(x)$ D-finite $\Rightarrow f(x) + g(x)$ and $f(x)g(x)$ D-finite
[Kauers, 2014]
- \vdots

In all these cases, it is not too hard to get a bound on the order.
Also for substitution, this is not too hard.

- If f satisfies a differential equation of order 4, then every higher order derivative of f can be rewritten as a $C(x)$ -linear combination of f, f', f'', f''' .
- If g satisfies a polynomial equation of degree 3, then every higher power of g can be rewritten as a $C(x)$ -linear combination of $1, g, g^2$.
- Moreover, also the derivative g' can be written in this form.

$$\begin{aligned}
h^{(8)}(x) &= \dots \\
&= (\text{ } + \text{ } g(x) + \text{ } g(x)^2) f(g(x)) \\
&+ (\text{ } + \text{ } g(x) + \text{ } g(x)^2) f'(g(x)) \\
&+ (\text{ } + \text{ } g(x) + \text{ } g(x)^2) f''(g(x)) \\
&+ (\text{ } + \text{ } g(x) + \text{ } g(x)^2) f'''(g(x))
\end{aligned}$$

h, h', h'', \dots all live in a $C(x)$ -vector space of dimension $4 \times 3 = 12$.

Therefore, $h, h', \dots, h^{(12)}$ are linearly dependent over $C(x)$.

Therefore, h satisfies a linear differential equation of order 12.

More generally:

$$\underbrace{f(x) \text{ D-finite}}_{\text{order } r_f} \wedge \underbrace{g(x) \text{ algebraic}}_{\text{degree } r_g} \Rightarrow \underbrace{f(g(x)) \text{ D-finite}}_{\text{order } \leq r_f r_g}$$

There can be equations of order $< r_f r_g$, but generically there aren't.

What about the degrees?

To bound the degrees, equate coefficients with respect to C rather than with respect to $C(x)$ and balance variables and equations.

This requires a more precise understanding of the clouds on the previous slide, which can be obtained by a lengthy calculation.

Theorem [Kauers, Pogudin, 2017]:

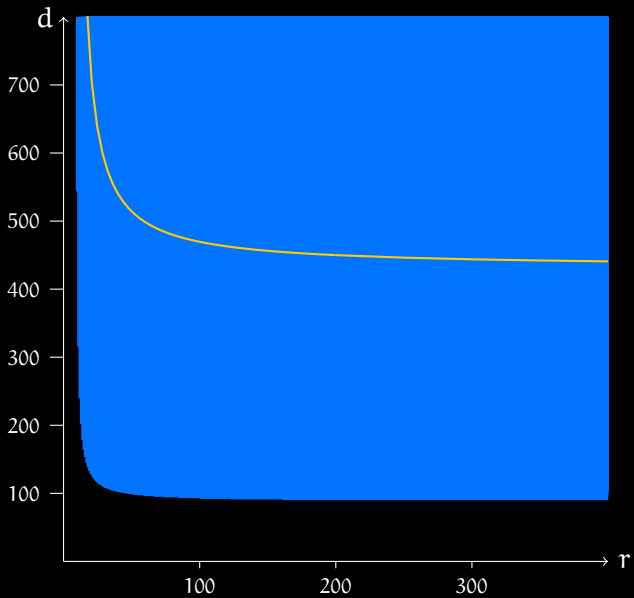
$$\underbrace{f(x) \text{ D-finite}}_{\substack{\text{order } r_f \\ x\text{-degree } d_f}} \wedge \underbrace{g(x) \text{ algebraic}}_{\substack{g\text{-degree } r_g \\ x\text{-degree } d_g}} \Rightarrow \underbrace{f(g(x)) \text{ D-finite}}_{\substack{\text{order } r \geq r_f r_g \\ x\text{-degree } d \text{ s.t.}}}$$

$d \leq \frac{r(3r_g + d_f - 1)d_g r_g r_f}{r + 1 - r_f r_g}$

The bound for the degree d depends rationally on the order r .

We get a hyperbolic curve.

How accurate is it?



Main Question:

Can we do better?

Subquestion A: Can we improve the left part of the curve?

Subquestion B: Can we improve the right part of the curve?

A Degree bounds for the operator of minimal order

- Setting $r = r_f r_g$ into our formula for the curve yields

$$d \leq (3r_g + d_f - 1) d_g r_g^2 r_f^2 = O((r_g + d_f) d_g r_g^2 r_f^2)$$

- Generalizing a theorem of [Bostan, Chyzak, Salvy, Lecerf, Schost, 2007], we can show that when $r \leq r_g r_f$ is the minimal order and d is the corresponding degree, then

$$\begin{aligned} d &\leq 2r^2 d_g - \frac{1}{2} r(r-1) + r d_g r_f (2r_g + d_f - 1) - \frac{1}{2} d_g r_f r_g (r_g - 1) \\ &= O((r_g + d_f) d_g r_g r_f^2). \end{aligned}$$

- We conjecture that generically the degree is

$$\begin{aligned} d &= r_f^2 (2r_g (r_g - 1) + 1) d_g + r_f r_g (d_g (d_f + 1) + 1) + d_f d_g - r_f^2 r_g^2 - r_f d_f d_g \\ &= O((r_g r_f + d_f) d_g r_g r_f). \end{aligned}$$

B Order-Degree Curve via Desingularization

- The order-degree curve is uniquely determined by the minimal order operator $L \in C[x][\partial]$, because all other operators are $C(x)[\partial]$ -left multiples of L .
- Left multiples of L may have lower degree than L , for example:

$$\left(\frac{1}{x}\partial^2\right) \underbrace{((x-1)x\partial + (2-x))}_{\substack{\text{order 1} \\ \text{degree 2}}} = \underbrace{(x-1)\partial^3 + 3\partial^2}_{\substack{\text{order 3} \\ \text{degree 1}}}$$

- Whether such a degree reduction is possible depends on the removable factors of L . A polynomial $p \in C[x]$ is called **removable at cost c** (from L) if

$$\exists P \in C(x)[\partial] : \deg_{\partial}(c) = P, \quad PL \in C[x][\partial], \quad \text{lc}(PL) = \text{lc}(L)/p.$$

B Order-Degree Curve via Desingularization

Lemma [Chen, Jaroschek, Kauers, Singer, 2013] Let $L \in \mathbb{C}[x][\partial]$, and let p be removable from L at cost c . Let $r \geq \deg_{\partial}(L)$ and

$$d \geq \deg_x(L) - \left(1 - \frac{c}{r - \deg_{\partial}(L) + 1}\right) \deg_x(p).$$

Then there is a $\mathbb{C}(x)[\partial]$ -left multiple of L of order r and degree d .

Theorem [Kauers, Pogudin, 2017]: Generically, $h(x) = f(g(x))$ satisfies a recurrence of order r and degree d if $r \geq r_f r_g$ and

$$d \geq (d_g(4r_f r_g - 2r_f + d_f) - \delta) \left(1 - \frac{1}{r - r_f r_g + 1}\right) + \delta.$$

Here, δ is a degree bound for the minimal order operator.

