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Ambiguous classes in μ-calculi hierarchies[☆]

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Abstract

A classical result by Rabin states that if a set of trees and its complement are both Büchi definable in the monadic second order logic, then these sets are weakly definable. In the language of μ-calculi, this theorem asserts the equality between the complexity classes $\Sigma_2 \cap \Pi_2$ and $Comp(\Sigma_1, \Pi_1)$ of the fixed-point alternation-depth hierarchy of the μ-calculus of tree languages. It is natural to ask whether at higher levels of the hierarchy the ambiguous classes $\Sigma_{n+1} \cap \Pi_{n+1}$ and the composition classes $Comp(\Sigma_n, \Pi_n)$ are equal, and for which μ-calculi.

The first result of this paper is that the alternation-depth hierarchy of the games μ-calculus—whose canonical interpretation is the class of all complete lattices—enjoys this property. More explicitly, every parity game which is equivalent both to a game in Σ_{n+1} and to a game in Π_{n+1} is also equivalent to a game obtained by composing games in Σ_n and Π_n.

The second result is that the alternation-depth hierarchy of the μ-calculus of tree languages does not enjoy the property. Taking into account that any Büchi definable set is recognized by a nondeterministic Büchi automaton, we generalize Rabin's result in terms of the following separation theorem: if two disjoint languages are recognized by nondeterministic Π_{n+1} automata, then there exists a third language recognized by an alternating automaton in $Comp(\Sigma_n, \Pi_n)$ containing one and disjoint from the other.

Finally, we lift the results obtained for the μ-calculus of tree languages to the propositional modal μ-calculus: ambiguous classes do not coincide with composition classes, but a separation theorem is established for disjunctive formulas.

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0. Introduction

Roughly speaking, a μ-calculus [4] is a set of syntactical entities together with formal fixed-point operations μ and ν and a substitution operation. These entities come with a functional interpretation on a given class $\mathcal{K}$ of complete lattices: each entity t is interpreted as a monotonic mapping from $L^{ar(t)}$ to L, where L is a complete lattice in $\mathcal{K}$ and $ar(t)$ is the arity of t. The terms $\mu x.t$ and $\nu x.t$ of a μ-calculus are interpreted as the (parameterized) least and greatest fixed-points of the interpretation of t, while substitution is interpreted as functional composition; in particular $\theta x.t$ and $t[\theta x.t/x]$—for $\theta \in \{\mu, \nu\}$—denote the same object.

As an example, the collection of parity alternating automata (on infinite words, infinite complete trees, etc.) is such a μ-calculus. The only complete lattice in the class $\mathcal{K}$ is the powerset of the set of infinite words, infinite complete trees, etc. The interpretation of an automaton, as an entity of the μ-calculus with empty arity, coincides with the language of objects it accepts. The syntactical entities of a μ-calculus are often terms of a theory—in the usual sense from universal algebra—which happens to be an iteration theory [8] in two different ways. In this case the syntactical entities are called μ-terms.

The extremal fixed-point operations of μ-calculi are syntactic operators analogous to quantifiers. There have been a few proposals to classify μ-terms into classes according to the number of nested applications of fixed-point operations [23,25,34,26]; most of them turn out to be equivalent and give rise to the *alternation-depth hierarchy*. We recall here its algebraic definition—as found in [25]—which also gives a measure of the distance of μ-calculi from iteration theories. The class $\Sigma_0 = \Pi_0$ is the class of μ-terms with no application of the fixed-point operations; Σ_{n+1} (resp. Π_{n+1}) is the closure of Σ_n and Π_n under the composition (i.e., substitution) operation and the least fixed-point operation (resp. the greatest fixed-point operation). Also, the class $Comp(\Sigma_n, \Pi_n)$ is defined as the closure of Σ_n and Π_n under the composition operation. These classes are ordered by the inclusions as shown in Fig. 1. As far as we are dealing with the syntax these inclusions are obviously strict. However, if a μ-calculus comes with an intended interpretation, the relevant question is whether these inclusions are strict in the interpretation, i.e., whether for each class there exists a μ-term in it which is semantically equivalent to no term in a class of lower level. This problem—the strictness of the alternation-depth hierarchy—has no obvious answer. For the propositional modal μ-calculus [19] the strictness of the hierarchy was shown in [9,22].

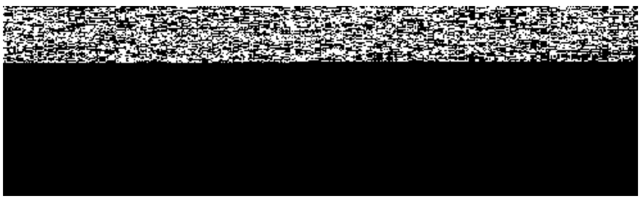

Fig. 1. The alternation-depth hierarchy.

For the μ-calculus of tree languages the hierarchy was shown to be strict in [1]. For the μ-calculus of parity games—which are interpreted in the class of all complete lattices—the hierarchy was proved strict in [30]. On the other hand, if parity games are interpreted only on distributive complete lattices, then every μ-term turns out to be equivalent to a term in Σ_0.

In this paper we investigate a different problem. It is easily seen that the relation

$$\Sigma_{n+1} \cap \Pi_{n+1} = Comp(\Sigma_n, \Pi_n) \tag{1}$$

holds in every μ-calculus at the syntactic level. It can be asked whether such equality still holds with respect to a given semantics: if a μ-term t is semantically equivalent both to a μ-term in Σ_{n+1} and to a μ-term in Π_{n+1}, is it equivalent to a term in $Comp(\Sigma_n, \Pi_n)$? The *ambiguous class* of level n is the collection of all terms semantically equivalent to a μ-term in Σ_{n+1} and to a μ-term in Π_{n+1}; deciding whether the above equality holds with respect to a given semantics is the ambiguous classes problem. The name is borrowed from the ambiguous classes in the Borel hierarchy [18, Section 22.E]. In this context the Hausdorff–Kuratowski theorem provides a constructive characterization of these classes. A positive answer to our ambiguous classes problem would provide an analogous constructive characterization.

The problem has its origins in a result of Rabin [29]: if a tree language is both Büchi and co-Büchi then it is definable in weak monadic second order logic. In [3] the authors succeeded to settle Rabin's result in the language of μ-calculi: a Büchi set is a language recognized by a nondeterministic automaton in Π_2, while a weakly definable set is a language recognizable by an alternating automaton—see [24]—in $Comp(\Sigma_1, \Pi_1)$. Since a language is recognized by an automaton in Π_2 if and only if its complement is recognized by an automaton in Σ_2 and every Π_2 language is Büchi definable [2,17,4], Rabin's result was transformed into: $\Sigma_2 \cap \Pi_2 = Comp(\Sigma_1, \Pi_1)$. A similar result $\Sigma_1 \cap \Pi_1 = Comp(\Sigma_0, \Pi_0) = \Sigma_0$ trivially holds.

Further evidences have motivated us to a deeper investigation of these relationships. For example, it is an easy exercise to show that if a language of infinite words is accepted both by a deterministic automaton in Σ_{n+1} and by a deterministic automaton in Π_{n+1}, then this language is accepted by a deterministic automaton in $Comp(\Sigma_n, \Pi_n)$. The reader should be aware that questions related to the alternation-depth hierarchy for deterministic automata on infinite words tend to be easy, in particular this hierarchy is decidable [37,28]. Recently, the results stating the identity between ambiguous classes and composition classes at the lower level of the hierarchy were lifted from the μ-calculus of tree languages to the propositional modal μ-calculus [20,21].

In this paper we investigate the ambiguous classes problem for the μ-calculus of parity games, the μ-calculus of tree languages, and for the propositional modal μ-calculus.

Parity games. Parity games are a fundamental tool in the theory of automata recognizing infinite objects and of the logics by which languages of these objects are defined [36]. Among these logics we list monadic second order logic, the propositional modal μ-calculus, and the collection of their fragments, i.e., logics of computation such as PDL, LTL, CTL, etc.

The spread use of parity games in these contexts should not be a surprise. In the monograph [4] it is shown that layered systems of positive Boolean equations are solved by finding winning strategies in parity games. The combinatorics of parity games is similar to the one of automata, thus a μ-calculus structure on the collection of parity games can be defined in analogy with the μ-calculus of automata. A major difficulty arises in defining the semantics of this μ-calculus. An obvious choice—analogous to the case of automata—is to say that the interpretation of a game is whether some distinguished position is winning or not. Since there are only two objects in the interpretation domain, such a μ-calculus does not look very interesting. Similarly, a semantics of parity games over the class of all distributive lattices would not be interesting, as a consequence of the fact that the alternation hierarchy is degenerate over this domain. In order to find an interesting semantics—with the goal of distinguishing behavioral aspects of games and strategies—one interprets parity games on the class of all complete lattices. [1] This is the interpretation of parity games studied in [32,30]. There a preorder $\leqslant$ on the collection $\mathcal{G}$ of parity games is constructively defined and the quotient $\mathcal{G}/\sim$ of this collection under the equivalence relation induced by $\leqslant$ is shown to be a lattice—although not a complete one—where the interpretation of $\theta x.G$ is indeed an extremal fixed-point. This is moreover the universal μ-lattice, that is the universal lattice in which every μ-term has an interpretation. This algebraic object is universal in that two μ-terms s, t satisfy $s \leqslant t$ in $\mathcal{G}/\sim$ if and only if this relation holds in every lattice where all μ-terms are interpretable. Moreover, it was proved that the relation $s \leqslant t$ holds in $\mathcal{G}/\sim$ if and only if it holds in every complete lattice.

In this paper we show that, for the μ-calculus of games with its interpretation in the class of all complete lattices, the equality (1) holds semantically, for every $n \geqslant 0$.

The proof of this heavily depends on the constructive characterization of the relation $\leqslant$. For two parity games G and H, the relation $G \leqslant H$ holds if and only if a chosen player has a winning strategy in a compound game $\langle\langle G, H\rangle\rangle$. A winning strategy for this player is treated as a formal proof, with the proviso that such a formal proof could have infinite branches or cycles. This is the reason for which in a previous work [31] we have called these winning strategies circular proofs. The tools used in the proof are of a proof-theoretic nature: the main technical proposition is an interpolation theorem [2] that we prove essentially with Maehara's method [35, Section 1.6.5].

Tree languages. Contrary to the case of games, we prove that identity (1) does not hold for tree languages, for $n \geqslant 2$. The proof of this result is quite similar to the proof of the strictness of the μ-calculus hierarchy [1,4] and uses the same diagonalization argument.

Considering the interpretation of Rabin's result [29] given in [3], a new question arises: why does this equality hold for $n = 1$ and what are the specific properties of Π_2 (or Σ_2) that imply the particular case? Our answer sounds as follows: Π_2 is the only class (with Π_1 and Σ_1) which has the property that an automaton is equivalent to a nondeterministic automaton in the same class [4]. This property suggests that, for tree automata, there is another possible generalization of relations such as (1) and $\Pi_2 \cap \Sigma_2 = Comp(\Sigma_1, \Pi_1)$. We prove

[1] Another possibility is to define a categorical semantics of parity games: in this case the algebraic interpretation of a parity game is a nontrivial set which turns out to coincide with the set of deterministic winning strategies for a given player. This idea was pursued in [33].

[2] This interpolation result concerns the hierarchy of fixed-points; it contrasts with the uniform interpolation property of the modal μ-calculus [10] which concerns the common language of two formulas and does not take into account the hierarchy of fixed-points, since the main tool to prove it are disjunctive normal forms.

the following generalization: if L and its complement are recognized by *nondeterministic* Π_{n+1} automata, then they are in $Comp(\Sigma_n, \Pi_n)$. We obtain this result as a corollary of a stronger separation result: if $\mathcal{A}$ and $\mathcal{B}$ are two nondeterministic automata in Π_{n+1} such that $L(\mathcal{A}) \cap L(\mathcal{B}) = \emptyset$, then there exists an alternating automaton $\mathcal{C}$ in $Comp(\Sigma_n, \Pi_n)$ such that $L(\mathcal{A}) \subseteq L(\mathcal{C}) \subseteq \overline{L(\mathcal{B})}$. The automaton $\mathcal{C}$ need not be nondeterministic. The proof combines a technique that we have introduced for parity games with the construction of an alternating automaton of [20].

The modal μ-calculus. Finally, we explicitly generalize the results obtained for tree languages to the propositional modal μ-calculus: with respect to the class of all Kripke frames, we show that equality (1) does not hold for $n \geqslant 2$. A separation theorem holds for the class of disjunctive formulas: [3] if $\mathscr{A}$ and $\mathscr{B}$ are two disjunctive formulas in Π_{n+1} such that $\mathscr{A} \wedge \mathscr{B}$ has no Kripke model, then there is a formula $\mathscr{C}$ in $Comp(\Sigma_n, \Pi_n)$ such that the formulas $\mathscr{A} \Rightarrow \mathscr{C}$ and $\mathscr{C} \Rightarrow \neg\mathscr{B}$ hold in every Kripke model.

1. The μ-calculus of parity games

Definition 1.1. A *parity game with draws* is a tuple

$$G = \langle Pos_E^G, Pos_A^G, Pos_D^G, M^G, \rho^G \rangle$$

where:

- $Pos_E^G, Pos_A^G, Pos_D^G$ are finite pairwise disjoint sets of positions (Eva's positions, Adam's positions, and draw positions),
- M^G, the set of moves, is a subset of $(Pos_E^G \cup Pos_A^G) \times (Pos_E^G \cup Pos_A^G \cup Pos_D^G)$,
- ρ^G is a mapping from $(Pos_E^G \cup Pos_A^G)$ to $\mathbb{N}$.

Whenever an initial position is specified, these data define a game between player Eva and player Adam. The outcome of a finite play is determined according to the normal play condition: a player who cannot move loses. It can also be a draw, if a position in Pos_D^G is reached. [4] The outcome of an infinite play $\{ g_k \}_{k \geqslant 0}$—where $(g_k, g_{k+1}) \in M^G$ for all $k \geqslant 0$—is determined by means of the rank function ρ^G as follows: it is a win for Eva if and only if the number

$$\text{limsup}_k\, \rho^G(g_k) = \max \{ i \in \mathbb{N} \mid \exists \text{ infinitely many } k \text{ s.t. } \rho^G(g_k) = i \}$$

is even. To simplify the notation, we shall use $Pos_{E,A}^G$ for the set $Pos_E^G \cup Pos_A^G$ and use similar notations such as $Pos_{E,D}^G$, etc. We let $Max^G = \max \rho^G(Pos_{E,A}^G)$ if the set $Pos_{E,A}^G$ is not empty, and $Max^G = -1$ otherwise.

[3] Disjunctive formulas were introduced in [15] to generalize to the modal μ-calculus nondeterministic tree automata.

[4] Observe that there are no possible moves from a position in Pos_D^G.

1.1. Operations on parity games

We define now some operations on games as well as some constant games. When defining operations on games we shall always assume that the sets of positions of distinct games are pairwise disjoint.

Meets and joins. For any finite set I, $\bigwedge_I$ is the game defined by letting $Pos_E = \emptyset$, $Pos_A = \{p_0\}$, $Pos_D = I$, $M = \{ (p_0, i) \mid i \in I \}$ (where $p_0 \notin I$), $\rho(p_0) = 0$. The game $\bigvee_I$ is defined similarly, exchanging Pos_E and Pos_A.

Composition operation. Given two games G and H and a mapping $\psi : P_D^G \longrightarrow P_{E,A,D}^H$, the game $K = G \circ_\psi H$ is defined as follows:

- $Pos_E^K = Pos_E^G \cup Pos_E^H$,
- $Pos_A^K = Pos_A^G \cup Pos_A^H$,
- $Pos_D^K = Pos_D^H$,
- $$M^K = (M^G \cap (Pos_{E,A}^G \times Pos_{E,A}^G)) \cup M^H \cup \{ (p, \psi(p')) \mid (p, p') \in M^G \cap (Pos_{E,A}^G \times Pos_D^G) \}.$$
- ρ^K is such that its restrictions to the positions of G and H are respectively equal to ρ^G and ρ^H.

Sum operation. Given a finite collection of parity games G_i, $i \in I$, their sum $H = \sum_{i \in I} G_i$ is defined in the obvious way:

- $P_Z^H = \bigcup_{i \in I} P_Z^{G_i}$, for $Z \in \{E, A, D\}$,
- $M^H = \bigcup_{i \in I} M^{G_i}$,
- ρ^H is such that its restriction to the positions of each G_i is equal to ρ^{G_i}.

Fixed-point operations. If G is a game, a system on G is a tuple $S = \langle E, A, M \rangle$ where:

- E and A are pairwise disjoint subsets of Pos_D^G,
- $M \subseteq (E \cup A) \times Pos_{E,A,D}^G$.

Given a system S and $\theta \in \{\mu, \nu\}$, we define the parity game $\theta_S.G$:

- $Pos_E^{\theta_S.G} = Pos_E^G \cup E$,
- $Pos_A^{\theta_S.G} = Pos_A^G \cup A$,
- $Pos_D^{\theta_S.G} = Pos_D^G - (E \cup A)$,
- $M^{\theta_S.G} = M^G \cup M$,
- $\rho^{\theta_S.G}$ is the extension of ρ^G to $E \cup A$ such that:
  - if $\theta = \mu$, then $\rho^{\theta_S.G}$ takes on $E \cup A$ the constant value Max^G if this number is odd or $Max^G + 1$ if Max^G is even,
  - if $\theta = \nu$, then $\rho^{\theta_S.G}$ takes on $E \cup A$ the constant value Max^G if this number is even or $Max^G + 1$ if Max^G is odd.

Predecessor game. Let G be a game such that $Max^G \neq -1$, i.e., there is at least one position in $Pos_{E,A}^G$. Let $Top^G = \{ g \in Pos_{E,A}^G \mid \rho^G(g) = Max^G \}$, then the predecessor game G^- is defined as follows:

- $Pos_E^{G^-} = Pos_E^G - Top^G$,
- $Pos_A^{G^-} = Pos_A^G - Top^G$,
- $Pos_D^{G^-} = Pos_D^G \cup Top^G$,

- $M^{G^-} = M^G - (Top^G \times Pos^G_{E,A,D})$,
- ρ^{G^-} is the restriction of ρ^G to $Pos^{G^-}_{E,A}$.

1.2. Semantics of parity games

Given a complete lattice L, the interpretation of a parity game G is going to be a monotone mapping of the form

$$\|G\| : L^{P^G_D} \longrightarrow L^{P^G_{E,A}},$$

where, for an arbitrary set X, $L^X = \prod_{x\in X} L$ is the X-fold product lattice of L with itself. If $g \in Pos^G_{A,E}$ then $\|G_g\|$ will denote the projection of $\|G\|$ onto the g coordinate. Any parity game G can be reconstructed in a unique way from the predecessor game G^- by one application of some fixed-point operation θ_S; moreover the predecessor game is "simpler". Thus we define the interpretation of a parity game inductively. If $P^G_{E,A} = \emptyset$, then $L^{P^G_{E,A}} = L^\emptyset = 1$, the complete lattices with just one element, and there is just one possible definition of the mapping $\|G\|$. Otherwise, if Max^G is odd, then $\|G\|$ is the parameterized least fixed-point of the monotone mapping

$$L^{P^G_{E,A}} \times L^{P^G_D} \longrightarrow L^{P^G_{E,A}}$$

defined by the system of equations:

$$x_g = \begin{cases} \bigvee \{ x_{g'} \mid (g, g') \in M^G \} & \text{if } g \in Pos^G_E \cap Top^G, \\ \bigwedge \{ x_{g'} \mid (g, g') \in M^G \} & \text{if } g \in Pos^G_A \cap Top^G, \\ \|G^-_g\|(X_{Top^G}, X_{Pos^G_D}) & \text{otherwise.} \end{cases}$$

If Max^G is even, then $\|G\|$ is the parameterized greatest fixed-point of this mapping.

1.3. Parity games as a μ-calculus

Let X be a countable set of variables. A *pointed parity game* (with labeled draws) is a tuple $\langle G, p^G_\star, \lambda^G\rangle$ where G is a parity game, $p^G_\star \in Pos^G_{E,A,D}$ is a specified initial position, and $\lambda^G : Pos^G_D \longrightarrow X$ is a labeling of draw positions by variables. With $\mathcal{G}$ we shall denote the collection of all pointed parity games with labeled draws; as no confusion will arise, we will call a pointed parity game with labeled draws simply a "game". Similarly, we shall abuse the notation and write G to denote the entire tuple $\langle G, p^G_\star, \lambda^G\rangle$. With the notation G_g we shall denote the game that differs from G only in that the initial position is now g, i.e., $p^{G_g}_\star = g$.

We give the collection $\mathcal{G}$ the structure of a μ-calculus, as defined in [4, Section 2.1]. If x is a variable, the game $\hat{x}$ has just one draw position labeled by x. The arity of a game G, denoted by $ar(G)$, is the set of variables $\lambda^G(Pos^G_D)$.

A substitution is a mapping $\sigma : X \longrightarrow \mathcal{G}$; given a game G and a substitution σ, the composition of G and σ—for which we use the notation $G[\sigma]$—is defined as

$$G[\sigma] = \left(G \circ_{\psi} \sum_{x \in ar(G)} \sigma(x) \right),$$

where $\psi(g) = p_{\star}^{\sigma(\lambda^G(g))}$ for $g \in Pos_D^G$. Moreover, $p_{\star}^{G[\sigma]} = p_{\star}^G$ and $\lambda^{G[\sigma]}(p) = \lambda^{\sigma(x)}(p)$ whenever $p \in Pos_D^{\sigma(x)}$. Therefore $ar(G[\sigma]) = \bigcup_{x \in ar(G)} ar(\sigma(x))$.

Similarly, given G in $\mathcal{G}$ and $x \in X$, let $Pos_x = \{ g \in Pos_D^G \mid \lambda^G(g) = x \}$. Define the system S as $\langle \emptyset, Pos_x, Pos_x \times \{p_{\star}^G\} \rangle$. Then we define

$$\theta x.G = (\theta_S.G),$$

where moreover $\lambda^{\theta x.G}$ is the restriction of λ^G and $p_{\star}^{\theta x.G} = p_{\star}^G$. Remark that $\theta x.G = G$ if $x \notin ar(G)$.

The above constructions are analogous to those given in [4, Section7.2] for automata and therefore it is possible to mimic the proof presented there to show that $\mathcal{G}$, endowed with this structure, satisfies the axioms of a μ-calculus.

Observe that the operation of forming the predecessor game G^- can be extended to pointed parity games with labeled draws if we choose a variable $x_g \notin ar(G)$ for each $g \in Top^G$: we let in this case λ^{G^-} be the extension of λ^G such that $\lambda^{G^-}(g) = x_g$ for $g \in Top^G$.

1.4. The preorder on the class of parity games

In order to describe a preorder on the class $\mathcal{G}$, we shall define a new game $\langle\langle G, H \rangle\rangle$ for a pair of games G and H in $\mathcal{G}$. This is not a pointed parity game with draws as defined in the previous section; to emphasize this fact, the two players will be named Mediator and Opponent instead of Eva and Adam.

Before formally defining the game $\langle\langle G, H \rangle\rangle$, we give its informal description and explanation. Mediator's goal is to prove that the relation $\|G\| \leqslant \|H\|$ holds in any complete lattice; Opponent's goal is to show that this relation does not hold. For example, if $G = \bigvee_{i \in I} G_i$ has the shape of a join and $H = \bigwedge_{j \in J} H_j$ has the shape of a meet, then this is an Opponent's position: Mediator should be prepared to prove $\|G_i\| \leqslant \|H_j\|$ for any pair of indices i and j; Opponent should find a pair of indices (i, j) and show that $\|G_i\| \not\leqslant \|H_j\|$. If $G = \bigwedge_{i \in I} G_i$ is a meet and $H = \bigvee_{j \in J} H_j$ is a join, then this is a Mediator's position: Mediator should find either an i and show that $\|G_i\| \leqslant \|H\|$ or a j and show that $\|G\| \leqslant \|H_j\|$; Opponent should be prepared to disprove any such relation. [5]

Thus the game is played on the two boards, simultaneously. At a first approximation, a position of $\langle\langle G, H \rangle\rangle$ is a pair of positions from G and H. Since we code meets as Adam's positions and joins as Eva's positions, Mediator is playing with Adam on G and with

[5] These moves suffice for Mediator to reach his goal, as the relation $\leqslant$ which we shall define turns out to be transitive. This fact is analogous to a cut-elimination theorem and to Whitman's conditions characterizing free lattices [13].

Eva on H; Opponent is playing with Eva on G and with Adam on H. A pair (g, h) in $Pos_A^G \times Pos_E^H$ clearly belongs to Mediator and a pair (g, h) in $Pos_E^G \times Pos_A^H$ clearly belongs to Opponent. Pairs in $Pos_E^G \times Pos_E^H$ or $Pos_A^G \times Pos_A^H$ are ambiguous, as both players could play. The situation is not symmetric, however, as Opponent is obliged to play while Mediator is allowed to play, if he wants, but he can also decide to delay his move. In the formal definition, we code the fact that two players can play from the same pair by duplicating every pair into a Mediator's position and into an Opponent's position.

Definition 1.2. The game $\langle\!\langle G, H \rangle\!\rangle$ is defined as follows:

- The set of Mediator's positions is

$$Pos_{E,A,D}^G \times \{M\} \times Pos_{E,A,D}^H ,$$

  and the set of Opponent's positions is

$$Pos_{E,A,D}^G \times \{O\} \times Pos_{E,A,D}^H .$$

- We describe its set of moves[6] by cases:
  - If $(g, h) \in (Pos_E^G \times Pos_{A,D}^H) \cup (Pos_{E,D}^G \times Pos_A^H)$, then there is just one "silent" move

$$(g, M, h) \to (g, O, h)$$

    and moves of the form

$$(g, O, h) \to (g', M, h) \qquad (g, O, h) \to (g, M, h')$$

    for every move $(g, g') \in M^G$ and every move $(h, h') \in M^H$.
  - If $(g, h) \in (Pos_A^G \times Pos_{E,D}^H) \cup (Pos_{A,D}^G \times Pos_E^H)$, then there is just one silent move

$$(g, O, h) \to (g, M, h)$$

    and moves of the form

$$(g, M, h) \to (g', O, h) \qquad (g, M, h) \to (g, O, h')$$

    for every move $(g, g') \in M^G$ and every move $(h, h') \in M^H$.
  - If $(g, h) \in (Pos_E^G \times Pos_E^H)$ then there are moves of the form

$$(g, O, h) \to (g', M, h) \qquad (g, M, h) \to (g, O, h')$$

    for every move $(g, g') \in M^G$ and every move $(h, h') \in M^H$, and moreover a silent move

$$(g, M, h) \to (g, O, h) .$$

[6] As we wish to distinguish moves coming from G and moves coming from H, the underlying graph of this game can have distinct edges relating the same pair of vertices.

- ◦ Similarly, if $(g, h) \in (Pos_A^G \times Pos_A^H)$ then there are moves of the form

$$(g, M, h) \to (g', O, h) \qquad (g, O, h) \to (g, M, h')$$

for every move $(g, g') \in M^G$ and every move $(h, h') \in M^H$, and moreover a silent move

$$(g, M, h) \to (g, O, h)\,.$$

- ◦ Finally, if $(g, h) \in Pos_D^G \times Pos_D^H$, then: If $\lambda^G(g) = \lambda^H(h)$, then there is a move

$$(g, M, h) \to (g, O, h)$$

and no move from (g, O, h): that is, this is a winning position for Mediator. If $\lambda^G(g) \neq \lambda^H(h)$, then there is a move

$$(g, O, h) \to (g, M, h)$$

and no move from (g, M, h). The latter is a win for Opponent.

- Now let us define the winning plays for Mediator in this game. As usual a maximal finite play is lost by the player who cannot move. For infinite plays, observe that any (maximal) play γ in $\langle\langle G, H\rangle\rangle$ defines two plays (not necessarily maximal) $\pi_G(\gamma)$ in G and $\pi_H(\gamma)$ in H. Generalizing what happens for finite plays we say that Mediator wins an infinite play γ if and only if either $\pi_G(\gamma)$ is a win for Adam on G, or $\pi_H(\gamma)$ is a win for Eva on H. An infinite play which is not a win for Mediator is a win for Opponent.

In the above definition we must explain the meaning of statements such as "$\pi_H(\gamma)$ is a win for Eva on H" whenever $\pi_H(\gamma)$ is a finite play which is not maximal. In this case, the last position of the play $\pi_H(\gamma)$ belongs either to Pos_E^H or to Pos_A^H: we say that this is a win for Adam in the first case and a win for Eva in the second case, with the intuition that the player who gives up playing loses.

This convention allows Mediator to play just on one board and to give up on the other if Adam has a winning strategy on G or Eva has a winning strategy on H. On the other hand, as soon as Opponent gives up on one board, he's going to lose. Notice that the game $\langle\langle G, H\rangle\rangle$ alternates between Opponent's positions and Mediator's positions, thus if a player among Mediator and Opponent gives up on one board, this is indeed his own responsibility.

Finally observe that the condition (1): "$\pi_G(\gamma)$ is a win for Adam on G, or $\pi_H(\gamma)$ is a win for Eva on H" implies but is not equivalent to (2): "if $\pi_G(\gamma)$ is a win for Eva on G, then $\pi_H(\gamma)$ is a win for Eva on H". The logic is complicated by the fact that $\pi_G(\gamma)$ could be a draw, but this is also the only obstacle to obtain the equivalence between (1) and (2).

Definition 1.3. If G and H belong to $\mathcal{G}$, then we declare that $G \leqslant H$ if and only if Mediator has a winning strategy in the game $\langle\langle G, H\rangle\rangle$ starting from position $(p_\star^G, O, p_\star^H)$.

In the following, we shall write $G \sim H$ to mean that $G \leqslant H$ and $H \leqslant G$. We continue by listing some useful facts about the game $\langle\langle G, H\rangle\rangle$ and the relation $\leqslant$.

Lemma 1.4. *In the game $\langle\langle G, H\rangle\rangle$ Mediator has a winning strategy from a position of the form (g, O, h) if and only if he has a winning strategy from (g, M, h).*

Proof. We assume first that Mediator has a winning strategy from (g, O, h).

If $g \in Pos_E^G$ or $h \in Pos_A^H$, then the move $(g, M, h) \to (g, O, h)$ is available to Mediator to reach a winning position. Similarly, if $(g, h) \in Pos_D^G \times Pos_D^H$, then relation $\lambda^G(g) = \lambda^H(h)$ holds since (g, O, h) is a winning position for Mediator; it follows that the move $(g, M, h) \to (g, O, h)$ is available to Mediator to reach a winning position.

If either $(g, h) \in Pos_A^G \times Pos_{E,D}^H$ or $(g, h) \in Pos_{A,D}^G \times Pos_E^H$, then the move $(g, O, h) \to (g, M, h)$ is the only one available to Opponent, and therefore Mediator has a winning strategy from (g, M, h).

We suppose now that Mediator has a winning strategy S from position (g, M, h) and construct a Mediator's winning strategy ∂S from position (g, O, h). We illustrate here its use: if in a position (g, M, h) with $(g, h) \in Pos_E^G \times Pos_E^H$ Mediator does not want to commit himself to a move on H, he can play the silent move $(g, M, h) \to (g, O, h)$ and continue with the "delayed" strategy ∂S.

To define the strategy ∂S, say that the position (g, O, h) is an exit position if either there exists a unique silent move $(g, O, h) \to (g, M, h)$, or if the strategy S suggests to Mediator the silent move $(g, M, h) \to (g, O, h)$. From an exit position Mediator can "catch up" and continue playing with S.

If (g, O, h) is not an exit position, then $(g, h) \in Pos_E^G \times Pos_E^H$ or $(g, h) \in Pos_A^G \times Pos_A^H$, and we must explain how Mediator plays from such a position.

We shall assume that $(g, h) \in Pos_E^G \times Pos_E^H$, and use an analogous argument if $(g, h) \in Pos_A^G \times Pos_A^H$. Suppose that Mediator's strategy S suggests the move $(g, M, h) \to (g, O, h')$, then, after Opponent's move $(g, O, h) \to (g', M, h)$, the strategy ∂S suggests the move $(g', M, h) \to (g', O, h')$. The following holds: from position (g', M, h') Mediator can play according to the given winning strategy S; as a consequence, Mediator can iterate this process from position (g', O, h'), and this defines a local winning strategy ∂S.

By using this strategy from a given position (g, O, h), either an exit position is eventually hit, thus Mediator eventually uses the strategy S and wins; or the play diverges to an infinite sequence of rounds of the form

$$(g_n, O, h_n) \to (g_{n+1}, M, h_n) \to (g_{n+1}, O, h_{n+1})$$

if $(g_n, h_n) \in Pos_E^G \times Pos_E^H$, or of the form

$$(g_n, O, h_n) \to (g_n, M, h_{n+1}) \to (g_{n+1}, O, h_{n+1})$$

if $(g_n, h_n) \in Pos_A^G \times Pos_A^H$. The two projections of this play are equal to the projections of the play

$$\begin{aligned} &\ldots (g_n, M, h_n) \to (g_n, O, h_{n+1}) \to (g_{n+1}, M, h_{n+1}) \ldots \\ &\quad \ldots (g_m, M, h_m) \to (g_{m+1}, O, h_m) \to (g_{m+1}, M, h_{m+1}) \ldots, \end{aligned}$$

where $(g_n, h_n) \in Pos_E^G \times Pos_E^H$ and $(g_m, h_m) \in Pos_A^G \times Pos_A^H$. This play is the outcome of playing according to the winning strategy S, hence it is a win for Mediator. $\square$

Definition 1.5. An *homomorphism* from a game G to a game H is a mapping f from the positions of G to the positions of H such that:

- $f(p_\star^G) = h_\star^H$,
- if g belongs to Pos_E^G (resp. Pos_A^G) then $f(g)$ belongs to Pos_E^H (resp. Pos_A^H) and $\rho^G(g) = \rho^H(f(g))$,
- if g belongs to Pos_D^G then $f(g)$ belongs to Pos_D^H and $\lambda^G(g) = \lambda^H(f(g))$,
- if $(g, g') \in M^G$ then $(f(g), f(g')) \in M^H$.

An homomorphism f from a game G to a game H is a *bisimulation* if moreover:

- for any position g of G, if $(f(g), h) \in M^H$ then there exists a position g' of G such that $(g, g') \in M^G$ and $h = f(g')$.

Lemma 1.6. *If there is a bisimulation from G to H, then $G \sim H$.*

Proof. We observe that both in the game $\langle\langle G, H\rangle\rangle$ and in the game $\langle\langle H, G\rangle\rangle$ Mediator can use a *copycat* strategy. We only show that $G \leqslant H$, the argument for $H \leqslant G$ being analogous.

Consider a position of the form $(g, O, f(g))$. If $g \in Pos_D^G$, then $f(g) \in Pos_D^H$ and this is a winning position for Mediator, since $\lambda^G(g) = \lambda^H(f(g))$. Otherwise, suppose that $g \in Pos_E^G$. If there are no moves on G, then this is a win for Mediator. If there is some move $g \to g'$ and Opponent plays $(g, O, f(g)) \to (g', M, f(g))$ then Mediator can replay $(g', M, f(g)) \to (g', O, f(g'))$. Similarly, if $g \in Pos_A^G$ and there are no moves on H from $f(g)$, then this is a win for Mediator. If there is some move $f(g) \to h'$ and Opponent plays $(g, O, f(g)) \to (g, M, h')$, then Mediator finds g' such that $f(g') = h'$ and then he plays $(g, M, h') \to (g', O, h')$.

Clearly if an infinite play γ is the outcome of playing with such a strategy, then $\pi_G(\gamma)$ and $\pi_H(\gamma)$ are both infinite plays. Therefore, if $\pi_G(\gamma)$ is not a win for Adam on G, then it is a win for Eva on G. This implies that $\pi_H(\gamma) = f(\pi_G(\gamma))$ is a win for Eva on H. $\square$

Lemma 1.6 is used to establish several equivalences. Let G be a game and $T \subseteq Pos_{E,A}^G$ be a collection of positions of G. Let $X_T \subseteq X$ be a subset of variables in bijection with T and such that $X_T \cap ar(G) = \emptyset$. The game $G^{T \hookleftarrow X_T}$ is obtained as follows: every position $t \in T$ is added to the set of draw positions and labeled by the variable x_t corresponding to t. Of course there are no more moves from a position $t \in T$ in the game $G^{T \hookleftarrow X_T}$. The relation $G_g \sim G_g^{T \hookleftarrow X_T}[G_t/x_t]_{t \in T}$ holds, as a consequence of the fact that there is a bisimulation from $G_g^{T \hookleftarrow X_T}[G_t/x_t]$ to G_g. Also, let G'_g be the game obtained from G_g by considering the reachable part from g. Again, we have $G_g \sim G'_g$ as the inclusion of the positions of G'_g into the positions of G_g is a bisimulation. Thus we are allowed to consider only games in $\mathcal{G}$ that are reachable from the initial position.

Proposition 1.7. *The relation $\leqslant$ has the following properties*:

(1) *It is reflexive and transitive*.
(2) *Composition is monotonic*: *If $G \leqslant H$ and if for all $x \in X$, $\sigma(x) \leqslant \sigma'(x)$ then $G[\sigma] \leqslant H[\sigma']$*.
(3) *For any game G and any substitution σ, $G \leqslant \bigwedge_I[\sigma]$ if and only if $G \leqslant \sigma(x_i)$ for all $i \in I$*.

(4) *For any game H and any substitution σ, $\bigvee_I[\sigma] \leqslant H$ if and only if $\sigma(x_i) \leqslant H$ for all $i \in I$.*
(5) *For $\theta \in \{\mu, \nu\}$, $\theta x.G \sim G[\theta x.G/x]$.*
(6) *If $G[H/x] \leqslant H$ then $\mu x.G \leqslant H$.*
(7) *If $G \leqslant H[G/x]$ then $G \leqslant \nu x.H$.*
(8) *It is the least relation on $\mathcal{G}$ having properties* 1 *to* 7.
(9) *It is sound and complete with respect to the class of all complete lattices*: $G \leqslant H$ *if and only if for any complete lattice L and any $v : X \longrightarrow L$*

$$\|G_{p^G_\star}\|(v \circ \lambda^G) \leqslant \|H_{p^H_\star}\|(v \circ \lambda^H) .$$

These properties were stated and proved in [32] for a restricted class of fair games and for a different relation $\preceq$ (similar to the one of [7,16]). However, we can prove the following: (a) the relation $\leqslant$ is indeed reflexive, transitive, and monotonic, (b) every game in $\mathcal{G}$ is $\leqslant$-equivalent to a fair game, (c) if G and H are fair games, then $G \leqslant H$ if and only if $G \preceq H$. From these properties, it follows that the quotient of the class of fair games under the equivalence relation induced by $\preceq$ is order isomorphic to the quotient of $\mathcal{G}$ under the equivalence relation $\sim$ and this quotient inherits all the properties proved in [32].

In particular the quotient $\mathcal{G}/\sim$ is a lattice where the greatest lower bound (resp. least upper bound) of the equivalence classes of $G_1, \ldots, G_k$ is the equivalence class of $\bigwedge_k(G_1, \ldots, G_k)$ (resp. $\bigvee_k(G_1, \ldots, G_k)$). It is a μ-*lattice* as well, meaning that all the μ-terms constructible from the signature $\langle \top, \wedge, \bot, \vee \rangle$ are interpretable as infima, suprema, least prefixed-points and greatest postfixed-points of previously defined operations. The μ-lattice $\mathcal{G}/\sim$ is freely generated by the set X, meaning that given any μ-lattice L and any mapping $\psi : X \longrightarrow L$, there exists a unique extension of ψ to a mapping $\psi' : \mathcal{G}/\sim \longrightarrow L$ that preserves the interpretation of μ-terms. From this property it readily follows that $\leqslant$ is the least preorder having the properties listed above.

1.5. Ambiguous classes in the games μ-calculus

1.5.1. A combinatorial characterization of the hierarchy

In the Introduction we have presented the alternation-depth hierarchy and its classes from an algebraic perspective. We present here an alternative definition of these classes that emphasizes the combinatorial aspects. The combinatorics will be more manageable in our proofs. The equivalence of the two perspectives is argued in [4, Section 8].

If G is a game then two mappings ρ and ρ' from $Pos^G_{E,A}$ to $\mathbb{N}$ are said to be equivalent with respect to G if any infinite path in G is a win for a player according to ρ if and only if it is a win for this player according to ρ', if and only if it is a win for this player according to ρ^G. Let G be a game and ρ be a mapping equivalent to ρ^G w.r.t. G. It is easily observed that the game G' obtained from G by substituting the rank function ρ with ρ^G is equivalent to G: $G \sim G'$.

Definition 1.8. We say that a game G belongs to $\Sigma_0 = \Pi_0$ if and only if it is acyclic. For $n \geqslant 1$, we say that a game G belongs to Σ_n (resp. Π_n) if there is a mapping ρ equivalent to ρ^G w.r.t. G, and an odd (resp. even) number $m \geqslant n-1$ such that $\rho(Pos^G_{E,A}) \subseteq \{m -$

$n+1, \ldots, m\}$. We say that a game belongs to $Comp(\Sigma_n, \Pi_n)$ if it can be obtained from games in Σ_n and Π_n by a sequence of applications of the composition operation of the μ-calculus.

Observe that, by construction, for every $n \geqslant 1$, if G belongs to Σ_n (resp. Π_n) then $\mu x.G$ belongs to Σ_n (resp. Π_{n+1}) and $\nu x.G$ belongs to Σ_{n+1} (resp. Π_n). Moreover, $Comp(\Sigma_0, \Pi_0) = \Sigma_0$ and in general $Comp(\Sigma_n, \Pi_n) \subseteq \Sigma_{n+1} \cap \Pi_{n+1}$. We shall show that the converse holds as well.

Lemma 1.9. *If a game G belongs to $\Sigma_1 \cap \Pi_1$ then it is acyclic.*

Proof. As G belongs to $\Sigma_1 \cap \Pi_1$ there are two mappings ρ and ρ' equivalent to ρ^G w.r.t. G whose images are respectively $\{m\}$ and $\{m'\}$, where m is odd and m' is even. If G is not acyclic, there exists a position p in G and a nonempty path γ from p to p. The infinite path γ^ω is a win for Adam, according to ρ, and a win for Eva, according to ρ'. This is a contradiction. □

Lemma 1.10. *If a game G is strongly connected and belongs to $\Sigma_{n+1} \cap \Pi_{n+1}$ then either it belongs to Σ_n, or it belongs to Π_n.*

Proof. If G belongs to $\Sigma_{n+1} \cap \Pi_{n+1}$ then there exist two mappings ρ and ρ', equivalent to ρ^G w.r.t. G, whose images are respectively included in $\{m-n, \ldots, m\}$ and $\{m'-n, \ldots, m'\}$ where m is odd and m' is even.

Suppose there are two positions $p, p' \in Pos^G_{E,A}$ such that $\rho(p) = m$ and $\rho'(p') = m'$. Since G is strongly connected, there exists a nonempty path γ from p to p' and a nonempty path γ' from p' to p. The maximal value of ρ (resp. ρ') which occurs infinitely often in the path $(\gamma\gamma')^\omega$ is m (resp. m'). Therefore this infinite path is a win for Adam according to ρ and a win for Eva according to ρ', a contradiction as ρ and ρ' are equivalent.

It follows that either ρ never takes the value m on $Pos^G_{E,A}$ or ρ' never takes the value m' on $Pos^G_{E,A}$. In the first case $\rho(Pos^G_{E,A}) \subseteq \{m-n, \ldots, m-1\}$ and $G \in \Pi_n$. In the second case $\rho'(Pos^G_{E,A}) \subseteq \{m'-n, \ldots, m'-1\}$ and $G \in \Sigma_n$. □

Corollary 1.11. *If a game G belongs to Σ_{n+1} and to Π_{n+1}, then it belongs to $Comp(\Sigma_n, \Pi_n)$.*

Proof. If $n = 0$ then this is Lemma 1.9. Otherwise we can construct G from its maximal strongly connected components G_i by means of a sequences of substitutions. According to Lemma 1.10, each of the G_i is either in Σ_n or in Π_n. Therefore $G \in Comp(\Sigma_n, \Pi_n)$. □

1.5.2. The semantical characterization of ambiguous classes

We have argued so far that equality (1) holds at the syntactic level. In the introduction we have stressed that the relevant question is whether such equality holds with respect to the given interpretation of all complete lattices. By the characterization in [32], this is the same as asking whether such equation holds up to the equivalence relation $\sim$ induced by the preorder $\leqslant$.

Definition 1.12. If $G \in \mathcal{G}$ then we say that $G \in \mathcal{S}_n$ if there exists a $G' \in \Sigma_n$ such that $G \sim G'$. Similarly, we say that $G \in \mathcal{P}_n$ if there exists a $G' \in \Pi_n$ such that $G \sim G'$, and that $G \in \mathcal{C}_n$ if there exists a $G' \in Comp(\Sigma_n, \Pi_n)$ such that $G \sim G'$.

The *ambiguous class* $\mathcal{D}_n$ is simply the intersection of $\mathcal{P}_n$ and $\mathcal{S}_n$. The main result of this section is the following theorem.

Theorem 1.13. *The ambiguous class* $\mathcal{D}_{n+1} = \mathcal{P}_{n+1} \cap \mathcal{S}_{n+1}$ *and the class* $\mathcal{C}_n$ *are equal, for every* $n \geqslant 0$.

The relation $\mathcal{C}_n \subseteq \mathcal{P}_{n+1} \cap \mathcal{S}_{n+1}$ immediately follows from the definition of the classes $\mathcal{C}_n, \mathcal{S}_{n+1}, \mathcal{P}_{n+1}$ and by relation (1). For the converse it is enough to prove the following proposition.

Proposition 1.14. *Let* G *and* H *be games in* Π_{n+1} *and* Σ_{n+1}, *respectively, and suppose that* $G \leqslant H$. *Then there exists a* $K \in Comp(\Sigma_n, \Pi_n)$ *such that* $G \leqslant K$ *and* $K \leqslant H$.

Indeed, if $G' \in \mathcal{S}_{n+1} \cap \mathcal{P}_{n+1}$, then let $G \in \Pi_{n+1}$ and $H \in \Sigma_{n+1}$, such that $G' \sim G \sim H$. If K is as in the statement of Proposition 1.14, then the relations

$$G' \leqslant G \leqslant K \leqslant H \leqslant G'$$

exhibit G' as a member of $\mathcal{C}_n$.

Proof of Proposition 1.14. Let us fix $G \in \Pi_{n+1}$ and $H \in \Sigma_{n+1}$, thus we shall assume that $\rho^G(Pos^G_{E,A}) \subseteq \{m-n, \ldots, m\}$ where m is even and that $\rho^H(Pos^H_{E,A}) \subseteq \{m'-n, \ldots, m'\}$ with m' odd. We also assume that $G \leqslant H$ and fix a winning strategy for Mediator in the game $\langle\langle G, H\rangle\rangle$ from position $(p^G_\star, O, p^H_\star)$. This game is almost[7] a game whose set of infinite winning plays is described by a Rabin acceptance condition. Thus, if Mediator has a winning strategy in this game, then he has a deterministic bounded memory winning strategy as well. Therefore we shall assume that the fixed winning strategy is deterministic and has a bounded memory. We shall represent it as the tuple $\langle S, U, s_\star, \psi\rangle$, where $\langle S, U, s_\star\rangle$ is a finite pointed graph, with set of memory states S, set of update transitions U, and an initial state $s_\star$; ψ is an homomorphism of graphs from $\langle S, U, s_\star\rangle$ to the graph of $\langle\langle G, H\rangle\rangle$ (mapping every memory state to a position and an update transition to a move) with the following properties:

- $\psi(s_\star) = (p^G_\star, O, p^H_\star)$,
- if $s \in S$ and $\psi(s) = (g, O, h)$ is an Opponent's position, then for every move $(g, O, h) \to (g', M, h')$ there exists a unique s' such that $s \to s'$ and $\psi(s') = (g', M, h')$,

[7] The winning condition can be described using Rabin pairs on the edges.

- if $s \in S$ and $\psi(s) = (g, M, h)$ is a Mediator's position, then there exists a unique transition $s \to s'$,
- if $s_0 \to s_1 \to \ldots$ is an infinite path in the graph $\langle S, U\rangle$, then the infinite play $\psi(s_0) \to \psi(s_1) \to \ldots$ is a win for Mediator.

Recall from 1.1 the definition of the predecessor game G^-. In particular, recall that $Top^G = \{ g \in Pos^G_{E,A} \,|\, \rho^G(g) = Max^G \}$ and that $Top^H = \{ h \in Pos^H_{E,A} \,|\, \rho^H(h) = Max^H \}$. Observe that, for the games G and H under consideration, G^- belongs to Σ_n and H^- belongs to Π_n. Intuitively, our next goal is to show that we can completely decompose the given winning strategy into a collection of local strategies that Mediator can play either in $\langle\langle G, H^-\rangle\rangle$, or in $\langle\langle G, H'\rangle\rangle$ for some game H' of the form $\bigwedge_I$, or in $\langle\langle G^-, H\rangle\rangle$ or in some $\langle\langle G', H\rangle\rangle$ for some game G' of the form $\bigvee_I$.

We shall denote by $[\, s\,]$ the maximal strongly connected component of the graph $\langle S, U\rangle$ to which s belongs. We observe that the following exhaustive and exclusive cases arise:

(Ac) The component $[\, s\,]$ is reduced to the singleton $\{s\}$. Observe that we cannot have a transition $s \to s$ as the graph of $\langle\langle G, H\rangle\rangle$ is bipartite. Therefore the component $[\, s\,]$ is acyclic.

(Cy) The component $[\, s\,]$ is cyclic (and contains at least two elements). We have the following subcases:

(CyA) The projection of $[\, s\,]$ onto H is stuck and belongs to Adam: let $s_1, s_2 \in [\, s\,]$ be such that $s_1 \to s_2$ and let $\psi(s_i) = (g_i, P_i, h_i)$ for $i = 1, 2$; then $h_1 = h_2 \in Pos^H_A$ and the move $(g_1, P_1, h_1) \to (g_2, P_2, h_1)$ is either a left move (i.e., $(g_1, g_2) \in M^G$) or it is a silent move.

(CyE) The projection of $[\, s\,]$ onto G is stuck and belongs to Eva: the formal definition is obtained by exchanging H with G and Adam with Eva in the definition of (CyA).

The previous conditions do not hold and:

(CyG) The projection of $[\, s\,]$ onto G contains a visit to Top^G: there exists an $s' \in [\, s\,]$ such that $\psi(s') = (g', P', h')$ and $\rho^G(g') = Max\ G$.

(CyH) The projection of $[\, s\,]$ onto H contains a visit to Top^H: there exists an $s' \in [\, s\,]$ such that $\psi(s') = (g', P', h')$ and $\rho^H(h') = Max^H$.

(CyN) None of the previous conditions hold. In particular, for all $s' \in [\, s\,]$, if $\psi(s') = (g', P', h')$, $g' \in Pos^G_{E,A}$ implies $\rho^G(g') < Max^G$ and $h' \in Pos^H_{E,A}$ implies $\rho^G(h') < Max^H$.

The reader should verify that the above cases are indeed disjoint. To see that (CyA) and (CyE) are disjoint, observe that a proper cycle in the graph of $\langle\langle G, H\rangle\rangle$ cannot be stuck both on G and on H. To see that (CyG) and (CyH) are disjoint consider a maximal strongly connected component $[\, s\,]$ that visits both Top^G and Top^H, and a path γ that visits all the states in $[\, s\,]$. The unique way the path $\psi(\gamma^\omega)$ can be a win in the game $\langle\langle G, H\rangle\rangle$ for Mediator is that the play $\psi(\gamma)$ is stuck on H on an Adam's position, in which case $[\, s\,]$ satisfies (CyA), or that this play is stuck on G on an Eva's position, in which case $[\, s\,]$ satisfies (CyE).

Definition 1.15. We say that $s \mapsto s'$ if and only if there exists a path $s = s_0 \to s_1 \to \ldots \to s_n = s'$, but s' does not belong to the strongly connected component of s.

Clearly, the relation $s \mapsto s'$ is irreflexive and acyclic, and therefore well founded.

Lemma 1.16. *Let $s \in S$ and $\psi(s) = (g, P, h)$, where $P \in \{O, M\}$. Suppose that the strongly connected component $[\, s\,]$ is of type (CyG) or of type (CyN). If for each $h' \in Top^H$ there exists $\kappa(h')$ such that $G_{g'} \leqslant \kappa(h')$ whenever $s \mapsto s'$ and $\psi(s') = (g', P', h')$, then*

$$G_g \leqslant H_h^-[\kappa(h')/y_{h'}]_{h' \in Top^H} .$$

Of course there is a dual lemma if the strongly connected component $[\, s\,]$ is of type (CyH); we leave to the reader to formulate it. Observe that in order to form a collection $\{\kappa(h')\}$ satisfying the hypothesis of the lemma, it is enough to let $\kappa(h') = \bigvee_\emptyset$ if there is no s' such that $s \mapsto s'$ and $\psi(s') = (g', P', h')$.

Proof of Lemma 1.16. The positions of the game $\langle\langle G, H_h^-[\kappa(h')/y_{h'}]_{h' \in Top^H} \rangle\rangle$ form a set which is the disjoint union of a component $Pos^G_{E,A,D} \times \{M, O\} \times (Pos^H_{E,A,D} - Top^H)$ and of components $Pos^G_{E,A,D} \times \{M, O\} \times Pos^{\kappa(h')}_{E,A,D}$ for $h' \in Top^H$. Moreover, in the latter components, the game is exactly as in $\langle\langle G, \kappa(h')\rangle\rangle$.

Mediator can use the strategy S from position $\psi(s)$ on the first component $Pos^G_{E,A,D} \times \{M, O\} \times (Pos^H_{E,A,D} - Top^H)$, as long as the strategy does not suggest a move $(g', P, h) \to (g', P', h')$ for some $h' \in Top^H$. If this is the case and if s' is the state of the strategy that lifts (g', P', h'), then $s \mapsto s'$, because $[\, s\,]$ cannot contain a visit to Top^H. Hence, by assumption, there is a winning strategy in the game $\langle\langle G_{g'}, \kappa(h')\rangle\rangle$ from both positions $(g', O, p_\star^{\kappa(h')})$ and $(g', M, p_\star^{\kappa(h')})$, by Lemma 1.4. The move $(g', P, h) \to (g', P', h')$ becomes a move to $(g', P', p_\star^{\kappa(h')})$ in $\langle\langle G, \kappa(h')\rangle\rangle$ and Mediator can continue with a winning strategy from the latter position. $\square$

We complete now the proof of Proposition 1.14 by proving the following stronger claim.

Claim 1.17. *For each $s \in S$ such that $\psi(s) = (g, P, h)$ there is a game K_s in the class $Comp(\Sigma_n, \Pi_n)$ such that $G_g \leqslant K_s \leqslant H_h$.*

The proof is by induction on the well-founded relation $\mapsto$ and it is subdivided into cases, according to the type of the strongly connected component $[\, s\,]$.

We suppose first that the type of $[\, s\,]$ is (**Ac**), so that if $s \to s'$ then $s \mapsto s'$. Observe that if $s \to s'$ is a transition lifting a silent move of the form

$$(g, O, h) \to (g, M, h) \quad (g, M, h) \to (g, O, h)$$

then there is essentially nothing to prove: we can let $K_s = K_{s'}$ since by the induction hypothesis $G_g \leqslant K_{s'} \leqslant H_h$.

If $g \in Pos^G_E$ and $P = O$, then for each move $(g, g') \in M^G$ there is a move $(g, O, h) \to (g', M, h)$ and a lifting $s \to s(g')$ of this move. By the induction hypothesis there are $K_{s(g')} \in Comp(\Sigma_n, \Pi_n)$ such that $G_{g'} \leqslant K_{s(g')} \leqslant H_h$. We can let $K_s = \bigvee_{(g,g') \in M^G} K_{s(g')} \in Comp(\Sigma_n, \Pi_n)$, it follows that

$$G_g \sim \bigvee_{(g,g') \in M^G} G_{g'} \leqslant \bigvee_{(g,g') \in M^G} K_{s(g')} \leqslant H_h .$$

Assume now that $g \in Pos_A^G$, $P = M$, and that the unique transition $s \to s'$ of the strategy is suggesting a move of the form $(g, M, h) \to (g', M, h)$ for some $(g, g') \in M^G$. We let $K_s = K_{s'} \in Comp(\Sigma_n, \Pi_n)$, and knowing that $G_{g'} \leqslant K_{s'} \leqslant H_h$ we derive

$$G_g \sim \bigwedge_{(g,g') \in M^G} G_{g'} \leqslant G_{g'} \leqslant K_s \leqslant H_h \,.$$

We can use a dual argument if $h \in Pos_A^H$ and $P = O$ or if $h \in Pos_E^H$ and $P = M$. If $g \in Pos_D^G$ and $h \in Pos_D^H$, then we let K_s be the game with only one position labeled by $\lambda^G(g)$.

We suppose now that the type of $[\, s\,]$ is **(CyA)**. Observe that if $s' \in [\, s\,]$ and $\psi(s') = (g', O, h)$ is an Opponent position, then for each move $(h, h') \in M^H$ there is a move $(g', O, h) \to (g', M, h')$ in $\langle\langle G, H\rangle\rangle$ and a lifting of this move $s' \to s'(h')$ in $\langle S, U\rangle$. By definition of the type (CyA), $s'(h') \notin [\, s\,]$, hence there exists a $K_{s'(h')}$ such that $G_{g'} \leqslant K_{s'(h')} \leqslant H_{h'}$. We can let $K_{s'} = \bigwedge_{(h,h') \in M^H} K_{s'(h')}$ since this game belongs to $Comp(\Sigma_n, \Pi_n)$ and

$$G_{g'} \leqslant \bigwedge_{(h,h') \in M^H} K_{s'(h')} \leqslant \bigwedge_{(h,h') \in M^H} H_{h'} \sim H_h \,.$$

If $s' \in [\, s\,]$ and $\psi(s') = (g', M, h)$, then there is a unique transition $s' \to s''$. If $s \mapsto s''$ then we can use the inductive hypothesis; otherwise, if $s'' \in [\, s\,]$, we observe that $\psi(s'') = (g'', O, h)$ is an Opponent position and that we have described in the previous paragraph how to construct $K_{s''}$ satisfying the claim. As the relation $G_{g'} \leqslant G_{g''}$ holds, we can let $K_{s'} = K_{s''}$, since

$$G_{g'} \leqslant G_{g''} \leqslant K_{s''} \leqslant H_h \,.$$

We suppose now that the type of $[\, s\,]$ is either **(CyG)** or **(CyN)**. For each $h' \in Top^H$ let

$$\kappa(h') = \bigvee_{\substack{s \;\mapsto\; s' \\ \psi(s') \;=\; (g', P', h')}} K_{s'},$$

where the $K_{s'} \in Comp(\Sigma_n, \Pi_n)$ have been previously constructed and satisfy the relation $G_{g'} \leqslant K_{s'} \leqslant H_{h'}$. Observe that $G_{g'} \leqslant \kappa(h')$ whenever $s \mapsto s'$ and $\psi(s') = (g', P', h')$, therefore by Lemma 1.16 the relation

$$G_g \leqslant H_h^-[\kappa(h')/y_{h'}]_{h' \in Top^H}$$

holds. Also, we have $\kappa(h') \leqslant H_{h'}$ for all $h' \in Top^H$ and therefore

$$H_h^-[\kappa(h')/y_{h'}]_{h' \in Top^H} \leqslant H_h^-[H_{h'}/y_{h'}]_{h' \in Top^H}\,,$$

where the last game is clearly equivalent to H_h. If we let K_s be the game $H_h^-\, [\kappa(h')/y_{h'}]_{h' \in Top^H}$, then K_s belongs to $Comp(\Sigma_n, \Pi_n)$, since $H_h^- \in \Pi_n$, $\Pi_n \subseteq Comp(\Sigma_n, \Pi_n)$, and for all $h' \in Top^H$ $\kappa(h') \in Comp(\Sigma_n, \Pi_n)$. Moreover we have shown that $G_g \leqslant K_s \leqslant H_h$.

We can use dual arguments if the strongly connected component is of type **(CyE)** or **(CyH)**; therefore the claim holds for every $s \in S$ and for $s_\star \in S$ in particular. As we have

$\psi(s_\star) = (p^G_\star, O, p^H_\star)$, the relations

$$G = G_{p^G_\star} \leqslant K_{s_\star} \leqslant H_{p^H_\star} = H\,,$$

prove Proposition 1.14. □

Finally we remark that if there exists a bounded memory winning strategy in the game $\langle\langle G, H\rangle\rangle$ for Mediator, then there exists a winning strategy for Mediator of size $|G| \times |H|$, where $|G| = \text{card}\, Pos^G_{E,A,D} + \text{card}\, M^G$ is the size of a game G. This follows from considerations developed in [11]. Thus effective bounds to construct K such that $G \leqslant K \leqslant H$ can be extracted out of this information.

2. The μ-calculus of tree automata

We shall consider here tree automata over binary trees. Given a set F of binary symbols, we recall that a binary F-tree is a mapping t from $\{l, r\}^*$ to F. The reader will convince himself that the tools introduced in this paper can be generalized to the case of G-trees where G is an arbitrary finite set of k-ary function symbols.

Definition 2.1. An *alternating tree automaton* is a tuple $\mathcal{A} = \langle X, \Delta, \rho\rangle$ where:
- X is a finite set of states (note that there are no initial states).
- For each $x \in X$ and each $f \in F$, $\Delta(x, f)$ is a positive Boolean combination of elements of the form $ⓓx$,[8] where d is a direction from the set $\{l, r\}$ and $x \in X$.
- ρ is a mapping from X to $\mathbb{N}$.

Using the distributive law and grouping together pairs with the same direction, we can assume that $\Delta(x, f)$ is normalized: that is, that it is written as

$$\Delta(x, f) = \bigvee_{j \in J} \left(ⓛ\bigwedge_{y \in L_j} y \wedge ⓡ\bigwedge_{z \in R_j} z \right).$$

In this case we think of $\Delta(x, f)$ as a set of *rules*, each rule j being a pair (L_j, R_j) of subsets of X. We shall often make the assumption that the sets of rules $\Delta(x, f)$ and each of the L_j, R_j are not empty. This assumption is harmless, since we shall be able to construct trivial automata with this property that recognize the empty and the total language respectively. If for each rule j the sets L_j and R_j are singletons, then we say that the automaton is *nondeterministic*.

Definition 2.1 can be generalized to endow the collection of tree automata with the structure of a μ-calculus, see [4]. Classes for the fixed-point alternation-depth hierarchy are defined as usual in these settings, see for example Section 1.5. For our goals it will be enough to recall the following combinatorial characterizations. An automaton is in Π_n (resp. Σ_n)

[8] An infinite binary tree is a Kripke model for a bimodal logic: states are words and the two functional transition relations l, r take a word w to wl and to wr, respectively. Functionality implies that the dual modal operators satisfy the equation $\langle d\rangle x = [d]x$, $d \in \{l, r\}$. For this reason we are using the next operator ⓓ of temporal logic whose standard interpretation is the successor relation on natural numbers.

if there is an even (resp. odd) integer $m \geqslant n-1$ such that $\rho(X) \subseteq \{m-n+1, \ldots, m\}$. An automaton is in *Comp*(Σ_n, Π_n) if there is a *preorder* $\succeq$ on X such that:

- for any x and f, if $(L, R) \in \Delta(x, f)$ and $y \in L \cup R$, then $x \succeq y$,
- for any equivalence class X' of X induced by $\succeq$ (x is equivalent to x' if $x \succeq x'$ and $x' \succeq x$), there exists $m \geqslant n-1$ such that $\rho(X') \subseteq \{m-n+1, \ldots, m\}$ or $\rho(X') \subseteq \{m-n, \ldots, m-1\}$.

If t is a tree and $\mathcal{A}$ is an automaton, the parity game $G(t, \mathcal{A})$ —see Definition 1.1—is defined as follows:

- Eva's positions are the pairs (w, x) with $w \in \{l, r\}^*$ and $x \in X$. The rank of (w, x) is $\rho(x)$.
- Adam's positions are all the pairs (w, j) where $j \in \Delta(x, t(w))$ for some $x \in X$. The rank of an Adam's position is always 0.
- There is an Eva's move from (w, x) to (w, j) if and only if $j \in \Delta(x, t(w))$.
- There is an Adam's move from (w, j) to (wl, y) for any $y \in L_j$ and to (wr, z) for any $z \in R_j$.

We say that t is *recognized* by $\mathcal{A}$ from a state x if Eva has a winning strategy in $G(t, \mathcal{A})$ from position (ε, x); we denote by $L_x(\mathcal{A})$ the set of trees recognized by $\mathcal{A}$ from x. We say that a tree language is in $\mathcal{P}_n$ (resp. $\mathcal{S}_n$, $\mathcal{C}_n$) if there is an automaton $\mathcal{A}$ in Π_n (resp. Σ_n, *Comp*(Σ_n, Π_n)) and a state x such that $L = L_x(\mathcal{A})$.

Clearly, if we give the automaton $\mathcal{A}$ its logical interpretation (its states are logical formulas of the propositional modal μ-calculus), Eva's goal from position (w, x) is to show that

$$w \vDash x \,.$$

A tree t is recognized by $\mathcal{A}$ if and only if $\varepsilon \vDash x$. The definition of the parity game $G(t, \mathcal{A})$ can be extended to the case of an automaton $\mathcal{A}$ where the Boolean expression $\Delta(x, f)$ are not normalized. Finally, we can assume that every automaton $\mathcal{A} = \langle X, \Delta, \rho \rangle$ in *Comp*(Σ_n, Π_n) is such that, for each $\succeq$-equivalence class X', $\rho(X')$ is included in $\{1, \ldots, n\}$ or in $\{2, \ldots, n+1\}$. If for some equivalence class X' this is not the case, then we can define a new automaton $\mathcal{A}' = \langle X, \Delta, \rho' \rangle$ letting $\rho'(x) = \rho(x) - 2$ if $x \in X'$ and $\rho'(x) = \rho(x)$ otherwise. Clearly $L_x(\mathcal{A}') = L_x(\mathcal{A})$ for each $x \in X$ and this transformation preserves the class *Comp*(Σ_n, Π_n). An automaton with the claimed property is obtained by iterating the transformation possibly on different classes.

We shall make use of the following facts:

- L is in $\mathcal{P}_n$ if and only if $\overline{L}$ is in $\mathcal{S}_n$.
- L is in $\mathcal{C}_n$ if and only if $\overline{L}$ is in $\mathcal{C}_n$.
- If $L \in \mathcal{C}_n$ then $L \in \mathcal{S}_{n+1} \cap \mathcal{P}_{n+1}$.

The first two statements are easily proved using the notion of a dual automaton, see [4], while the latter is a consequence of the syntactic equality (1).

2.1. The inequality theorem

The main result of this section is the following:

Theorem 2.2. *For any $n \geqslant 2$ there is a tree language in $\mathcal{S}_{n+1} \cap \mathcal{P}_{n+1}$ which is not in $\mathcal{C}_n$.*

$\star$	1	2	$\cdots$	i	$\cdots$	$n-1$	n
q_2	p_3	q_2	$\cdots$	q_i	$\cdots$	q_{n-1}	q_n
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
q_j	p_3	q_2	$\cdots$	q_i	$\cdots$	q_{n-1}	q_n
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
q_n	p_3	q_2	$\cdots$	q_i	$\cdots$	q_{n-1}	q_n
p_3	p_3	p_4	$\cdots$	p_{i+2}	$\cdots$	p_{n+1}	q_n
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
p_j	p_3	p_4	$\cdots$	p_{i+2}	$\cdots$	p_{n+1}	q_n
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
p_{n+1}	p_3	p_4	$\cdots$	p_{i+2}	$\cdots$	p_{n+1}	q_n

Fig. 2. Sketch of the transition relation of $\mathcal{M}_n$.

2.1.1. *Some tree languages*

With F_n we shall denote the set of binary symbols $\{a_i, e_i \mid 1 \leqslant i \leqslant n\}$. In the rest of this section we shall also assume that $n > 2$. A binary tree t over this alphabet can be interpreted as a parity game whose set of positions is infinite. A node of the tree $w \in \{l, r\}^*$ is an Eva's position if $t(w) \in \{e_i \mid 1 \leqslant i \leqslant n\}$, and otherwise it is an Adam's position. The rank of w is i provided that $t(w) \in \{a_i, e_i\}$.

Let $\mathcal{W}_n$ be the nondeterministic automaton (in Π_n if n is even and in Σ_n otherwise) whose set of states is $\{q_i \mid 1 \leqslant i \leqslant n\} \cup \{q_\top\}$, where the rank of q_i is i and the rank of $q_\top$ is 2, and whose transition function Δ is defined as follows:

- for any i, $\Delta(q_\top, a_i) = \Delta(q_\top, e_i) = \{(q_\top, q_\top)\}$,
- for any i and j, $\Delta(q_j, a_i) = \{(q_i, q_i)\}$ and $\Delta(q_j, e_i) = \{(q_i, q_\top), (q_\top, q_i)\}$.

Let $\mathcal{M}_n$ be the nondeterministic automaton (in Σ_n if n is even and in Π_n otherwise) whose set of states is $\{q_i \mid 2 \leqslant i \leqslant n\} \cup \{p_i \mid 3 \leqslant i \leqslant n+1\} \cup \{q_\top\}$, where the rank of q_i and of p_i is i and the rank of $q_\top$ is 2, and whose transition function Δ is defined as follows:

- for any i, $\Delta(q_\top, a_i) = \Delta(q_\top, e_i) = \{(q_\top, q_\top)\}$,
- for any i and any $q \neq q_\top$, $\Delta(q, a_i) = \{(q \star i, q \star i)\}$,
  and $\Delta(q, e_i) = \{(q \star i, q_\top), (q_\top, q \star i)\}$.

Here $q \star i$ is defined as a function of q and i by means of the table in Fig. 2.

Let $W_n = L_{q_1}(\mathcal{W}_n)$ and $M_n = L_{q_2}(\mathcal{M}_n)$. One of them is in $\mathcal{S}_n$ and the other one in $\mathcal{P}_n$. Observe that W_n is the language of game-trees where Eva has a winning strategy from the root.

Finally, let K_n be the set of all trees over F_n such that on each branch the set of symbols which occur infinitely often is included in $\{a_i, e_i \mid 1 \leqslant i \leqslant n-1\}$ or in $\{a_i, e_i \mid 2 \leqslant i \leqslant n\}$. Since the condition that a tree has at least one branch belonging to a given regular ω-language is a Büchi condition, the complement $\overline{K_n}$ is in $\mathcal{P}_2$; therefore the language K_n is in $\mathcal{S}_2$.

Proposition 2.3. $W_n \cap K_n = M_n \cap K_n$.

Proof. A strategy S in the games $G(t, \mathcal{W}_n)$ and $G(t, \mathcal{M}_n)$ consists in selecting one successor (left or right) at each node labeled by some e_i. Let t_S be the (partial) tree obtained by cutting out the non-selected successors. With each branch b of t_S we associate the infinite word $\tilde{b} \in \{1, \ldots, n\}^\omega$ by substituting i for e_i or a_i. The strategy S is winning if for each branch b of t_S:

$\mathcal{W}_n$ —the largest number that occurs infinitely often in $\tilde{b}$ is even,

$\mathcal{M}_n$ —$\tilde{b}$ is recognized by the parity word automaton whose transitions are given in the previous table (with q_2 as initial state).

It is easy to check that if $\tilde{b}$ is in $\{1, \ldots, n\}^*(\{1, \ldots, n-1\}^\omega \cup \{2, \ldots, n\}^\omega)$ then these two conditions are equivalent. $\Box$

An immediate consequence of this proposition is that $W_n \cup \overline{K}_n = M_n \cup \overline{K}_n$. Since $\overline{K}_n \in \mathcal{P}_2 \subseteq \mathcal{S}_n \cap \mathcal{P}_n$, we obtain the following proposition.

Proposition 2.4. *$W_n \cup \overline{K}_n = M_n \cup \overline{K}_n \in \mathcal{S}_n \cap \mathcal{P}_n$ and $\overline{W}_n \cap K_n \in \mathcal{S}_n \cap \mathcal{P}_n$.*

2.1.2. The diagonal argument

Let us assume that $\overline{W}_n \cap K_n$ is in $\mathcal{C}_{n-1} \subseteq \mathcal{S}_n \cap \mathcal{P}_n$: there is an automaton $\mathcal{A} \in Comp(\Sigma_{n-1}, \Pi_{n-1})$ and a state $x_\star$ such that $\overline{W}_n \cap K_n = L_{x_\star}(\mathcal{A})$, and for each $\succeq$-equivalence class X', $\rho(X')$ is included in $\{1, \ldots, n-1\}$ or in $\{2, \ldots, n\}$.

On the algebra of infinite binary trees over F_n, define the operations

$$
\begin{aligned}
A_i^1(y_1) &= a_i(y_1, y_1) \\
A_i^{k+1}(y_1, \ldots, y_{k+1}) &= a_i(y_1, A_i^k(y_2, \ldots, y_{k+1})) \\
E_i^1(y_1) &= e_i(y_1, y_1) \\
E_i^{k+1}(y_1, \ldots, y_{k+1}) &= e_i(y_1, E_i^k(y_2, \ldots, y_{k+1}))\,.
\end{aligned}
$$

Under the game theoretic interpretation, an operation A_i^k encodes an Adam's choice at rank i among k possibilities; similarly for E_i^k and Eva. With each tree t over F_n and each state x of $\mathcal{A}$ we associate the tree $G_x(t)$ over F_n defined as follows. Let i be the rank of x, let $t = f(t_l, t_r)$ with $f \in F_n$, and let $\Delta(x, f) = \{(L_1, R_1), \ldots, (L_k, R_k)\}$. We define

$$G_x(t) = E_i^k(t_1, \ldots, t_k),$$

where, if $L_j = \{y_1, \ldots, y_{k_j}\}$ and $R_j = \{z_1, \ldots, z_{l_j}\}$, then

$$t_j = a_i\left(A_i^{k_j}(G_{y_1}(t_l), \ldots, G_{y_{k_j}}(t_l)),\ A_i^{l_j}(G_{z_1}(t_r), \ldots, G_{z_{l_j}}(t_r))\right)\,.$$

It is proved in [1,4] that this mapping has the following properties:

Proposition 2.5. *A tree t belongs to $L_x(\mathcal{A})$ if and only if the tree $G_x(t)$ belongs to W_n. Moreover, each mapping G_x has a unique fixed point t_x.*

Since $\mathcal{A}$ is in $Comp(\Sigma_{n-1}, \Pi_{n-1})$, we have the additional property:

Proposition 2.6. *For any t and any x, $G_x(t)$ is in K_n.*

We can now complete our goal.

Proof of Theorem 2.2. We are assuming that $\mathcal{A}$ is an automaton in the class $Comp(\Sigma_{n-1}, \Pi_{n-1})$ such that for some $x_\star$ the language $L_{x_\star}(\mathcal{A})$ is equal to $\overline{W_n} \cap K_n$. According to the considerations we have developed, the tree $t_{x_\star}$ is in $\overline{W_n} \cap K_n$ if and only $G_{x_\star}(t_{x_\star}) \in W_n$, that is, if and only if $t_{x_\star} \in W_n$. Since $t_{x_\star} = G_{x_\star}(t_{x_\star}) \in K_n$, $t_{x_\star} \in \overline{W_n} \cap K_n$ if and only $t_{x_\star} \in \overline{W_n}$. Therefore $t_{x_\star} \in W_n$ if and only if $t_{x_\star} \in \overline{W_n}$, i.e., we have reached a contradiction. $\square$

2.2. The separation theorem

Say that a language L is in $\mathcal{P}_n^{\mathrm{nd}}$ if there is a nondeterministic automaton $\mathcal{A}$ in Π_n and a state x such that $L = L_x(\mathcal{A})$. Although $\mathcal{P}_n = \mathcal{P}_n^{\mathrm{nd}}$ for $n = 2$, this equality does not hold for $n > 2$ [2,4]. We are going to prove the following theorem.

Theorem 2.7. *Let L and L′ be two disjoint tree languages over an alphabet F. If both are in* $\mathcal{P}_{n+1}^{\mathrm{nd}}$ *(with* $n \geqslant 2$*) then there exists* $K \in \mathcal{C}_n$ *such that* $L \subseteq K \subseteq \overline{L'}$.

Note that the language K need not be recognized by a nondeterministic automaton. We give the proof when the alphabet F has only binary symbols. The generalization to any alphabet is straightforward.

2.2.1. A game for deciding nonemptiness

Let $\mathcal{A} = \langle X, \Delta^{\mathcal{A}}, \rho^{\mathcal{A}} \rangle$ and $\mathcal{B} = \langle Y, \Delta^{\mathcal{B}}, \rho^{\mathcal{B}} \rangle$ be two nondeterministic automata over an alphabet F of binary symbols. Let us consider the (biparity) game $G(\mathcal{A}, \mathcal{B})$ defined as follows:

- Eva's positions are all the pairs $(x, y) \in X \times Y$.
- Adam's positions are all the pairs (j, k) where $j \in \Delta^{\mathcal{A}}(x, f)$ and $k \in \Delta^{\mathcal{B}}(y, f)$ for some pair $(x, y) \in X \times Y$ and some $f \in F$.[9]
- Eva can move from (x, y) to (j, k) if and only if $j \in \Delta^{\mathcal{A}}(x, f)$ and $k \in \Delta^{\mathcal{B}}(y, f)$ for some $f \in F$.
- Adam can move from (j, k) either to (x_l, y_l) or to (x_r, y_r), provided that $j = (x_l, x_r)$ and $k = (y_l, y_r)$.

W.l.o.g. we shall assume that $\Delta^{\mathcal{A}}(x, f)$ and $\Delta^{\mathcal{B}}(y, f)$ are never empty, hence all the maximal plays in this game are infinite. A play is winning for Eva if the sequence $\{(x_n, y_n)\}_{n \geqslant 0}$ of Eva's positions along this play is such that both $\limsup \rho^{\mathcal{A}}(x_n)$ and $\limsup \rho^{\mathcal{B}}(y_n)$ are even. The proof of the following result is easy and appears in [5].

Proposition 2.8. $L_x(\mathcal{A}) \cap L_y(\mathcal{B})$ *is not empty if and only if the position* (x, y) *is winning for Eva.*

[9] We assume here that all the sets involved are disjoint. Hence given an Adam's position (j, k) we can compute the unique triple (x, y, f) such that $j \in \Delta^{\mathcal{A}}(x, f)$ and $k \in \Delta^{\mathcal{B}}(y, f)$.

By the previous proposition, for any two states x and y, $L_x(\mathcal{A}) \cap L_y(\mathcal{B}) = \emptyset$ if and only if the position (x, y) in $G(\mathcal{A}, \mathcal{B})$ is not winning for Eva. In this case Adam has a winning strategy, that is in any position (j, k) he chooses either the left or the right direction. It is a standard fact that Adam's winning strategy can be chosen to be with finite-memory, see [36].

We represent such a strategy as a finite graph $\langle S, U \rangle$ whose set of nodes is the disjoint sum of two sets S_E and S_A. Each position $s \in S_E$ is a tuple (x, y, h) where h is a memory from a finite set H. Consequently, we are given projection functions $\pi_X(s) \in X$ and $\pi_Y(s) \in Y$. Transitions of $\langle S, U \rangle$ are as follows: for each $s = (x, y, h) \in S_E$, $f \in F$, and $(j, k) \in \Delta^{\mathcal{A}}(x, f) \times \Delta^{\mathcal{B}}(y, f)$ there exists a unique vertex $s'_{j,k} \in S_A$ and a transition from s to $s'_{j,k}$; each vertex $s_{j,k} \in S_A$ has a unique successor in S_E—i.e., Adam's choice—which we denote by $\chi(s_{j,k})$; moreover if $\chi(s_{j,k}) = (x', y', h')$ then $(j, k) \to (x', y')$ is a move of $G(\mathcal{A}, \mathcal{B})$. If Adam's choice from $s_{j,k}$ is a left choice then we let $\delta(s_{j,k})$ be the modal operator $\textcircled{l}$, and otherwise we let $\delta(s_{j,k})$ be $\textcircled{r}$.

The proof of the following lemma is easy and is left to the reader.

Lemma 2.9. *For any infinite path p in $\langle S, U \rangle$, the restriction $\{s_i\}_{i \geqslant 0}$ of p to the nodes in S_E is such that* $\limsup_i \rho^{\mathcal{A}}(\pi_X(s_i))$ *is odd or* $\limsup_i \rho^{\mathcal{B}}(\pi_Y(s_i))$ *is odd.*

2.2.2. The separation property

Let $\mathcal{A}$ and $\mathcal{B}$ be two nondeterministic automata in Π_{n+1} such that for some pair of states (x, y) we have $L_x(\mathcal{A}) \cap L_y(\mathcal{B}) = \emptyset$. Without loss of generality, we may assume that there is an even m such that $\rho^{\mathcal{A}}(X)$ and $\rho^{\mathcal{B}}(Y)$ are both included in $\{m - n, \ldots, m\}$. Let us consider the graph $\langle S, U \rangle$ induced by an Adam's winning strategy in $G(\mathcal{A}, \mathcal{B})$ and the set S_E of vertices of the form $s = (x, y, h)$; we are going to define two automata $\mathcal{C}^1, \mathcal{C}^2$ on S_E. To this goal, let the preorder $\succeq$ on S_E be defined saying that by $s \succeq s'$ if and only if there is a path from s to s'. Two states s and s' are equivalent (with respect to the equivalence induced by the preorder $\succeq$) if and only if they belong to the same strongly connected component of $\langle S, U \rangle$.

A mapping $\rho^{\mathcal{C}} : S_E \to \mathbb{N}$ is defined as follows. Let C be a strongly connected component in $\langle S, U \rangle$, which contains at least one node of S_E. If C is trivial (it contains only one s) then we set $\rho^{\mathcal{C}}(s) = m - 1$. If C is nontrivial there cannot be in C an s and an s' with $\rho^{\mathcal{A}}(\pi_X(s)) = \rho^{\mathcal{B}}(\pi_Y(s')) = m$. Therefore either $\rho^{\mathcal{A}}$ never takes the value m on $\pi_X(C)$ or $\rho^{\mathcal{B}}$ never takes the value m on $\pi_Y(C)$. In the first case we set $\rho^{\mathcal{C}}(s) = \rho^{\mathcal{A}}(\pi_X(s))$. In the second case we set $\rho^{\mathcal{C}}(s) = \rho^{\mathcal{B}}(\pi_Y(s)) + 1$. The definition of two alternating automata $\mathcal{C}^1 = \langle S_E, \Delta^1, \rho^{\mathcal{C}} \rangle$ and $\mathcal{C}^2 = \langle S_E, \Delta^2, \rho^{\mathcal{C}} \rangle$ is completed by letting:

$$\Delta^1(s, f) = \bigvee_{j \in \Delta^{\mathcal{A}}(\pi_X(s), f)} \; \bigwedge_{k \in \Delta^{\mathcal{B}}(\pi_Y(s), f)} \delta(s'_{j,k})\chi(s'_{j,k}),$$
$$\Delta^2(s, f) = \bigwedge_{k \in \Delta^{\mathcal{B}}(\pi_Y(s), f)} \; \bigvee_{j \in \Delta^{\mathcal{A}}(\pi_X(s), f)} \delta(s'_{j,k})\chi(s'_{j,k}) \,.$$

Proposition 2.10. *Both $\mathcal{C}^1$ and $\mathcal{C}^2$ are in $Comp(\Sigma_n, \Pi_n)$ and moreover, for any $s \in S_E$, $L_s(\mathcal{C}^1) \subseteq L_s(\mathcal{C}^2)$.*

Proof. Using the preorder $\succeq$ on S_E and the definition of $\rho^{\mathcal{C}}$, it is easy to see that $\mathcal{C}^1$ and $\mathcal{C}^2$ are in $Comp(\Sigma_n, \Pi_n)$. Since the Boolean formula

$$\bigvee_{j \in \Delta^{\mathcal{A}}(\pi_X(s), f)} \bigwedge_{k \in \Delta^{\mathcal{B}}(\pi_Y(s), f)} \delta(s'_{j,k})\chi(s'_{j,k})$$

logically implies

$$\bigwedge_{k \in \Delta^{\mathcal{B}}(\pi_Y(s), f)} \bigvee_{j \in \Delta^{\mathcal{A}}(\pi_X(s), f)} \delta(s'_{j,k})\chi(s'_{j,k}),$$

we obviously have $L_s(\mathcal{C}^1) \subseteq L_s(\mathcal{C}^2)$. $\square$

Note that if we exchange the roles of $\mathcal{A}$ and $\mathcal{B}$ in the above construction we obtain two automata $\mathcal{D}^1$ and $\mathcal{D}^2$. The dual automaton $\widetilde{\mathcal{D}^1}$ of $\mathcal{D}^1$, which satisfies $L_s(\widetilde{\mathcal{D}^1}) = \overline{L_s(\mathcal{D}^1)}$, is the automaton $\langle S_E, \widetilde{\Delta}^1, \rho^{\mathcal{C}}_{+1} \rangle$ where $\rho^{\mathcal{C}}_{+1}(s) = \rho^{\mathcal{C}}(s) + 1$ and

$$\widetilde{\Delta}^1(s, f) = \bigwedge_{k \in \Delta^{\mathcal{B}}(\pi_Y(s), f)} \bigvee_{j \in \Delta^{\mathcal{A}}(\pi_X(s), f)} \delta(s_{j,k})\chi(s_{j,k})\,.$$

It is easy to see that $\widetilde{\mathcal{D}^1}$ is equivalent to $\mathcal{C}^2$. We shall prove that $L_{\pi_X(s)}(\mathcal{A}) \subseteq L_s(\mathcal{C}^1)$ since then

$$L_{\pi_X(s)}(\mathcal{A}) \subseteq L_s(\mathcal{C}^1) \subseteq L_s(\mathcal{C}^2) = L_s(\widetilde{\mathcal{D}^1}) = \overline{L_s(\mathcal{D}^1)} \subseteq \overline{L_{\pi_Y(s)}(\mathcal{B})}$$

achieves the proof of the Separation Theorem.

Proposition 2.11. *For any $s \in S_E$, $L_{\pi_X(s)}(\mathcal{A}) \subseteq L_s(\mathcal{C}^1)$.*

Proof. We shall construct an Eva's winning strategy T^+ in $G(t, \mathcal{C}^1)$ given an Eva's winning strategy T in $G(t, \mathcal{A})$.

From a position (w, s) such that $(w, \pi_X(s))$ has been reached playing with T, Eva chooses the same rule $j \in \Delta(\pi_X(s), t(w))$ she would choose according to T. In the game $G(t, \mathcal{C}^1)$ [10] from the position (w, j) Adam must choose a rule $k \in \Delta(\pi_Y(s), t(w))$ and then the play continues from position $(wd, \tilde{s})$, where $\tilde{s} = \chi(s'_{j,k})$ and the direction d is determined by ⓓ $= \delta(s'_{i,j})$. Since the move $(w, j) \to (wd, \pi_X(\tilde{s}))$ is an Adam's move, the position $(wd, \pi_X(\tilde{s}))$ is necessarily reachable by playing with T: thus Eva can iterate this process.

Consider now the restriction $\{(w_i, s_i)\}_{i \geqslant 0}$ to Eva's position of an infinite play that is the outcome of playing according to this strategy. Since the (restriction of the) infinite play $\{(w_i, \pi_X(s_i))\}_{i \geqslant 0}$ has been played according to the winning strategy T, we have that $\limsup_i \rho^{\mathcal{A}}(\pi_X(s_i))$ is even. It is therefore enough to show that if $\limsup_i \rho^{\mathcal{A}}(\pi_X(s_i))$ is even then $\limsup_i \rho^{\mathcal{C}}(s_i)$ is even as well. Let $k = \limsup_i \rho^{\mathcal{A}}(\pi_X(s_i))$, $k' = \limsup_i \rho^{\mathcal{B}}(\pi_Y(s_i))$, and let $k'' = \limsup_i \rho^{\mathcal{C}}(s_i)$. Since from some n, the set $\{s_i \mid i \geqslant n\}$ is included in a nontrivial

[10] The game $G(t, \mathcal{C}^1)$ has not been explicitly defined, since the transition Δ^1 is not normalized. This game is defined in the expected way, according to the syntax tree of the expression $\Delta(s)$. Explicit definitions of this kind of games appear in [12, Section 3], [6, Section 2], and [4, Sections 4,6].

strongly connected component of $\langle S, U\rangle$, either $k'' = k$, or $k'' = k' + 1$. If k is even, then by Lemma 2.9, k' is odd, thus k'' is certainly even. $\square$

2.2.3. The case of nondeterministic Σ_n

To complete the picture, we show that the separation property does not hold for nondeterministic $\mathcal{S}_n$ languages (for $n \geqslant 3$). We give the proof for n even; the proof for n odd is quite similar if we consider trees over the alphabet $\{a_i, e_i \mid 0 \leqslant i \leqslant n-1\}$ instead of $\{a_i, e_i \mid 1 \leqslant i \leqslant n\}$.

We already know from Section 2.1 that $W_n \cup \overline{K}_n = M_n \cup \overline{K}_n$. If n is even and greater than 3, then $W_n \in \mathcal{P}_n$ and $M_n \in \mathcal{S}_n$. Since both M_n and $\overline{K}_n$ are recognized by nondeterministic automata in Σ_n then so is $W_n \cup \overline{K}_n$. We will prove below that $\overline{W}_n \cap K_n$ is nondeterministic $\mathcal{S}_n$. If the separation property holds also for languages that are nondeterministic $\mathcal{S}_n$, $\overline{W}_n \cap K_n$ has to be in $\mathcal{C}_{n-1}$, but we already know that this is not the case.

Proposition 2.12. *If $n > 2$ is even, $\overline{W}_n \cap K_n$ is nondeterministic $\mathcal{S}_n$.*

Proof. The language $\overline{W}_n$ is the set of game-trees where Adam has a winning strategy. Therefore this set is recognized by the nondeterministic automaton $\widetilde{\mathcal{W}}_n$ in Σ_n. Also, the language of infinite words $\{1, \ldots, n\}^*(\{1, \ldots, n-1\}^\omega \cup \{2, \ldots, n\}^\omega)$ is recognized by a deterministic Σ_2 word automaton $\mathcal{K}_n$.

Let us consider the direct product $\widetilde{\mathcal{W}}_n \times \mathcal{K}_n$. It is a nondeterministic biparity automaton: each state (q, s) with q a state of $\widetilde{\mathcal{W}}_n$ and s a state of $\mathcal{K}_n$ has a rank $\rho^{\widetilde{\mathcal{W}}}(q) \in \{0, \ldots, n-1\}$ (with $n-1$ odd) and a rank $\rho^{\mathcal{K}}(s) \in \{0, 1\}$. An infinite play of the game $G(t, \widetilde{\mathcal{W}}_n \times \mathcal{K}_n)$—whose restriction to Eva's positions is the sequence $\{(w_i, (q_i, s_i))\}_{i \geqslant 0}$—is a win (or it is accepted) if and only if $\limsup_i \rho^{\widetilde{\mathcal{W}}}(q_i)$ is even and $\limsup_i \rho^{\mathcal{K}}(s_i) = 0$.

We define a new rank function $\rho^\times$ by

$$\rho^\times(q, s) = \begin{cases} \rho^{\widetilde{\mathcal{W}}}(q) & \text{if } \rho^{\mathcal{K}}(s) = 0, \\ n-1 & \text{if } \rho^{\mathcal{K}}(s) = 1\,. \end{cases}$$

If $\limsup_i \rho^{\mathcal{K}}(s_i) = 0$ then for any i large enough $\rho^{\mathcal{K}}(s_i) = 0$, hence $\rho^\times(q_i, s_i) = \rho^{\widetilde{\mathcal{W}}}(q_i)$ and $\limsup_i \rho^\times(q_i, s_i) = \limsup_i \rho^{\mathcal{A}}(q_i)$. If $\limsup_i \rho^{\mathcal{K}}(s_i) = 1$ then $\limsup_i \rho^\times(q_i, s_i) = n-1$, which is odd. It follows that $\limsup_i \rho^\times(q_i, s_i)$ is even if and only both $\limsup_i \rho^{\widetilde{\mathcal{W}}}(q_i)$ and $\limsup_i \rho^{\mathcal{K}}(s_i)$ are even. Therefore, using the rank function $\rho^\times$, we can transform the biparity automaton $\widetilde{\mathcal{W}}_n \times \mathcal{K}_n$ into an equivalent parity automaton in Σ_n. $\square$

3. The propositional modal μ-calculus

In this section we extend the results obtained in the previous section for tree automata to the propositional (uni)modal μ-calculus [19,4].

A *modal automaton* $\mathscr{A}$ is a tuple $\langle X, \Delta, \rho\rangle$. X is a finite set of states or variables, $\rho : X \to \mathbb{N}$ is a rank function. For each $q \in X$, $\Delta(q)$ is a term constructed from the literals

a, $\neg a$,—where *a* ranges over a set of propositional constants *Prop*—the variables in *X*, using the positive Boolean operations $\bigvee$, $\bigwedge$, the unary modal operators $\langle\ \rangle$, $[\ ]$, and the modal operator $\rightarrow$. The arity of the operators $\bigvee$, $\bigwedge$, $\rightarrow$ is an arbitrary set of finite cardinality. The relation

$$\rightarrow Y = \bigwedge_{y\in Y} \langle\ \rangle y \wedge [\ ] \bigvee_{y \in Y} y$$

defines the modal operator $\rightarrow$ [11] from $\langle\ \rangle$, $[\ ]$; these are standard from modal logic K [14]. The relations

$$\langle\ \rangle y = \rightarrow\{y, \top\} \quad [\ ] y = \rightarrow\{y\} \vee \rightarrow\emptyset\,,$$

show that the two sets of modal operations $\{\langle\ \rangle, [\ ]\}$ and $\{\rightarrow\}$ are interdefinable; consequently, we shall often assume that each term $\Delta(q)$ is built using only one set of modal operators. Taking into account that every formula of the μ-calculus is equivalent to a guarded one [19] and the possibility of introducing new states in an automaton, we can also assume that each occurrence of a variable in $\Delta(q)$ is guarded by a single layer of modal operators. For a *formula* we shall mean either a pair $\mathscr{A}_q = (\mathscr{A}, q)$ where $q \in X^{\mathscr{A}}$, or a formula of the propositional modal μ-calculus as this is usually presented. The standard equivalence between scalar and vectorial μ-calculi [4, Section 2.7] ensures that we can confuse these notions. We shall also feel free to use the sloppy notation $\mathscr{A}$ for a formula $\mathscr{A}_q$ when the designated state q is understood.

A *Kripke model* over the set of propositional constants *Prop* is a tuple $\langle V, E, \lambda\rangle$ where V is a set of states, $E \subseteq V \times V$ is an accessibility relation between states, and λ is a labeling of states by subsets of *Prop*. By setting $I_\lambda(p) = \{v \in V \mid p \in \lambda(v)\}$, we recognize in λ an interpretation of the basic propositional constants. The semantics of the operator $\rightarrow$ is as follows: a vertex v of a Kripke model $\langle V, E, \lambda\rangle$ satisfies $\rightarrow Y$ if and only if a bisimulation step can be established between the vertex and the formula; that is, for any successor v' of v there is a $y \in Y$ such that v' satisfies *y*, and for any $y \in Y$ there is a successor v' of v such that v' satisfies *y*.

3.1. A negative result on ambiguous classes

Our first goal is to generalize Theorem 2.2 to the modal μ-calculus. To this end, let $Prop = F_n = \{a_i, e_i \mid i = 0, \ldots, n\}$; we define first a formula $\mathscr{T}_n$ whose Kripke models look like the F_n-trees considered in the previous section. Consider the operator

$$[\ ]^* y = \nu x.(\ y\ \wedge [\ ] x\ )$$

and define the formula

$$\mathscr{T}_n = [\ ]^* \left(\ \langle\ \rangle \top\ \wedge \bigvee_{f \in F_n} f\ \wedge \bigwedge_{f,g \in F_n, f \neq g} f \Rightarrow \neg g\ \right).$$

Clearly, a rooted tree $(\mathcal{T}, r)$ satisfies $\mathscr{T}_n$ if and only if it is a complete tree and every node is labeled by exactly one symbol among those in F_n. Let $\mathscr{K}_n$ be a modal formula such that,

[11] This operator was introduced in [15].

for every rooted tree $(\mathcal{T}, r)$ which satisfies $\mathscr{T}_n$, $\mathcal{T}, r \vDash \mathscr{K}_n$ if and only if on every infinite branch the set of indices visited infinitely often is either included in $\{1, \ldots, n-1\}$ or in $\{2, \ldots, n\}$; such a formula is in the class Σ_2. Finally, let

$$\mathscr{Z}_n = \mathscr{K}_n \wedge \mathscr{T}_n .$$

The modal automaton $\mathscr{W}_n$ has the same states and ranking function as the tree automaton $\mathcal{W}_n$ presented in Section 2.1; its rules are

$$\Delta^{\mathscr{W}_n}(q_j) = \bigwedge_{i=1}^{n} (a_i \Rightarrow [\ ]q_i \wedge e_i \Rightarrow \langle\ \rangle q_i) .$$

In a similar way we can construct a modal automaton $\mathscr{M}_n$ analogous to the tree automaton $\mathcal{M}_n$ of Section 2.1.

Proposition 3.1. *For every rooted tree $(\mathcal{T}, r)$ the relation $\mathcal{T}, r \vDash \mathscr{W}_n \wedge \mathscr{Z}_n$ holds if and only if the relation $\mathcal{T}, r \vDash \mathscr{M}_n \wedge \mathscr{Z}_n$ holds. Hence the two formulas $\mathscr{W}_n \wedge \mathscr{Z}_n$ and $\mathscr{M}_n \wedge \mathscr{Z}_n$ are equivalent over the class of all Kripke models.*

The proof is analogous to the proof of Proposition 2.3. As before, Boolean transformations lead to the equivalence $\mathscr{W}_n \vee \neg\mathscr{Z}_n = \mathscr{M}_n \vee \neg\mathscr{Z}_n$ and therefore to following statement.

Corollary 3.2. *The formula $\neg\mathscr{W}_n \wedge \mathscr{Z}_n$ is in the semantic class $\mathcal{S}_n \cap \mathcal{P}_n$.*

We can now derive the following proposition.

Proposition 3.3. *The formula $\neg\mathscr{W}_n \wedge \mathscr{Z}_n$ is not equivalent to any formula with propositional constants only from F_n belonging to the syntactic class $Comp(\Sigma_{n-1}, \Pi_{n-1})$.*

Proof. Suppose that the formula $\neg\mathscr{W}_n \wedge \mathscr{Z}_n$ is equivalent to a formula $\mathscr{C}_q$ where $\mathscr{C} = \langle Q, \Delta, \rho \rangle$ is a modal automaton in the class $Comp(\Sigma_{n-1}, \Pi_{n-1})$. Construct a tree automaton $\mathcal{C}$ by replacing in each term $\Delta(q)$ modal operators according to the following rule:

$$\langle\ \rangle x \rightsquigarrow ⓛ\, x \vee ⓡ x \qquad [\ ] x \rightsquigarrow ⓛ\, x \wedge ⓡ x .$$

Such a transformation produces an alternating tree automaton $\mathcal{C}$ in the class $Comp(\Sigma_{n-1}, \Pi_{n-1})$ recognizing the language $\neg W_n \wedge K_n$. According to the proof of Theorem 2.2 this cannot be the case. $\square$

3.2. Separation for the modal μ-calculus

We shall assume—without loss of generality—that each node v of a Kripke structure K has a unique label $\lambda(v)$ taken from the powerset F of local properties. The transition function $\Delta^{\mathscr{A}}$ of a *disjunctive modal automaton* associates with each state $x \in X$ and each symbol $f \in F$ a set $\Delta^{\mathscr{A}}(x, f)$ of rules, possibly empty, where each rule $j \in \Delta^{\mathscr{A}}(x, f)$ has the form $\to R_j$ for some subset R_j of X. [12] In the informal development, we shall confuse the rule j with the subset R_j. We shall prove the following theorem.

Theorem 3.4. *Let $\mathscr{A}$, $\mathscr{B}$ be two disjunctive modal formulas in the class Π_n with no common model. Then there exists a modal automaton $\mathscr{C}$ in the class $Comp(\Sigma_{n-1}, \Pi_{n-1})$ such that both $\mathscr{A} \Rightarrow \mathscr{C}$ and $\mathscr{C} \Rightarrow \neg\mathscr{B}$ are valid formulas.*

If $\mathscr{A} = \langle X, \Delta^{\mathscr{A}}, \rho^{\mathscr{A}} \rangle$ and $\mathscr{B} = \langle Y, \Delta^{\mathscr{B}}, \rho^{\mathscr{B}} \rangle$ are two disjunctive modal automata, the game $G(\mathscr{A}, \mathscr{B})$ for deciding emptiness is defined as follows:

- In a position $(x, y) \in X \times Y$ Eva chooses a symbol f, a rule $j \in \Delta^{\mathscr{A}}(x, f)$, and a rule $k \in \Delta^{\mathscr{B}}(y, f)$, and moves to (j, k).
- In a position $(j, k) \in \Delta^{\mathscr{A}}(x, f) \times \Delta^{\mathscr{B}}(y, f)$ Adam chooses either $x' \in j$ and moves to (x', j), or chooses $y' \in k$ and moves to (j, y').
- In position (x', k) Eva chooses $y' \in k$ and moves to (x', y'). In a position (j, y') Eva chooses $x' \in j$ and moves to (x', y').

As usual, this is a biparity game where, for an infinite path $\{p_i\}_{i \geqslant 0}$, Eva wins if and only if both $\limsup_i \rho^{\mathscr{A}}(\pi_X(p_i))$ and $\limsup_i \rho^{\mathscr{B}}(\pi_Y(p_i))$ are even. Moreover, a proof of the following result can be found in [27].

Proposition 3.5. *There exists a Kripke model $\mathcal{M}$ and a state v such that*

$$\mathcal{M}, v \vDash \mathscr{A}_x \wedge \mathscr{B}_y$$

if and only if Eva has a winning strategy from position (x, y) in the game $G(\mathscr{A}, \mathscr{B})$.

Observe that the positions of this game have four different shapes: they are pairs of states (x, y), pairs of rules (j, k), left positions (x, k), and right positions (j, y). If $\langle S, U \rangle$ is the graph of an Adam's winning strategy in $G(\mathscr{A}, \mathscr{B})$, then we distinguish accordingly its vertexes by indexing them by the positions of the game they are related to: thus we have vertexes $s_{x,y}$, $s_{j,k}$, $s_{x,k}$, and $s_{j,y}$. The structure of the graph $\langle S, U \rangle$ is then described as follows: for any $s_{x,y}$, $f \in F$, and any $(j, k) \in \Delta^{\mathscr{A}}(x, f) \times \Delta^{\mathscr{B}}(y, f)$, there exists a unique successor of $s_{x,y}$ having the form $s'_{j,k}$. For each $s_{j,k}$ there exists a unique successor—Adam's deterministic choice—$\chi(s_{j,k})$ of $s_{j,k}$ and this is a left choice $\chi(s_{j,k}) = s'_{x,k}$, or a right choice $\chi(s_{j,k}) = s'_{j,y}$. For each vertex $s_{x,k}$ and each $y \in k$ there is a unique successor $s'_{x,y}$. For each vertex $s_{j,y}$ and each $x \in j$ there is a unique successor $s'_{x,y}$.

Our next goal is to define separating modal automata $\mathscr{C}^1, \mathscr{C}^2$ on the subset $S_P \subseteq S$ of nodes the form $s_{x,y}$. For each node of the form $s_{j,k}$, we define the modal expression $M(s_{j,k})$ as follows: if $\chi(s_{j,k}) = s'_{x,k}$ is a left choice then we let

$$M(s_{j,k}) = \langle\ \rangle \bigwedge_{y \in k} s''_{x,y}\,.$$

[12] To be precise, the transition function of such an automaton is defined by $\Delta^{\mathscr{A}}(x) = \bigwedge_{f \in F} f \Rightarrow \Delta^{\mathscr{A}}(x, f)$, where $\Delta^{\mathscr{A}}(x, f)$ has the form $\bigvee_j \to R_j$.

Otherwise, if $\chi(s_{j,k}) = s'_{j,y}$ is a right choice, we let

$$M(s_{j,k}) = [\ ] \bigvee_{x \in j} s''_{x,y} .$$

We define $\rho^{\mathscr{C}} : S_P \to \mathbb{N}$ as in the case of trees. The modal automata $\mathscr{C}^1 = \langle S_P, \Delta^1, \rho^{\mathscr{C}} \rangle$ and $\mathscr{C}^2 = \langle S_P, \Delta^2, \rho^{\mathscr{C}} \rangle$ in $Comp(\Sigma_n, \Pi_n)$ are defined as follows:

$$\Delta^1(s_{x,y}, f) = \bigvee_{j \in \Delta^{\mathscr{A}}(x,f)} \bigwedge_{k \in \Delta^{\mathscr{B}}(y,f)} M(s'_{j,k}),$$

$$\Delta^2(s_{x,y}, f) = \bigwedge_{k \in \Delta^{\mathscr{B}}(y,f)} \bigvee_{j \in \Delta^{\mathscr{A}}(x,f)} M(s'_{j,k}) .$$

Note that in the above definition, a union over an empty set is equal to $\perp$ and an intersection over an empty set is equal to $\top$.

Observe that (in every interpretation) $\mathscr{C}^1_s \leqslant \mathscr{C}^2_s$, and that the dual of $\mathscr{C}^2$ is the automaton $\mathscr{D}^1$ obtained as $\mathscr{C}^1$ by exchanging the role of $\mathscr{A}$ and $\mathscr{B}$. Therefore, in order to establish the chain of inequalities

$$\mathscr{A}_x \leqslant \mathscr{C}^1_{s_{x,y}} \leqslant \mathscr{C}^2_{s_{x,y}} = \neg \mathscr{D}^1_{s_{y,x}} \leqslant \neg \mathscr{B}_y ,$$

and proving Theorem 3.4, it will be enough to prove the following proposition.

Proposition 3.6. *For any Kripke model $\mathcal{M}$ and any state v of $\mathcal{M}$, if $\mathcal{M}, v \vDash \mathscr{A}_x$, then $\mathcal{M}, v \vDash \mathscr{C}^1_{s_{x,y}}$ too.*

Proof. To prove the proposition, we define a local strategy for Eva in the game $G(\mathcal{M}, \mathscr{C}^1)$ from each position $(v, s_{x,y})$ given an Eva's winning strategy *T* in the game $G(\mathcal{M}, \mathscr{A})$ from position (v, x).

Suppose that (v, x) has been reached by playing with *T*; from a position $(v, s_{x,y})$ Eva chooses the same rule $j \in \Delta(\lambda(v), x)$ she would choose according to *T*. The game $G(\mathcal{M}, \mathscr{C}^1)$ continues with Adam choosing a rule $k \in \Delta(\lambda(v), y)$ and ends in the position $(v, M(s'_{j,k}))$. We must describe how Eva can fulfill the local condition $v \vDash M(s'_{j,k})$ and iterate the process. There are two cases.

If $\chi(s'_{j,k}) = t_{x',k}$ is a left choice, then $M(s'_{j,k}) = \langle\ \rangle \bigwedge_{y' \in k} t'_{x',y'}$. Since $(v, \to j)$ is a position reached with *T* and $x' \in j$, $v \vDash \langle\ \rangle x'$. Hence there exists a transition $v \to v'$ such that the position (v', x') is reached by *T*. It follows that the same transition $v \to v'$ in $\mathcal{M}$ is such that for every $y' \in k$ (v', x') is reachable with *T*—the last statement implying that Eva can iterate this process from $(v', t'_{x',y'})$.

If $\chi(s'_{j,k}) = t_{j,y'}$ is a right choice, then $M(s'_{j,k}) = [\ ] \bigvee_{x' \in j} t'_{x',y'}$. Since $(v, \to j)$ is a position reached with *T*, $v \vDash [\ ] \bigvee_{x' \in j} x'$. Therefore if $v \to v'$ is a transition of $\mathcal{M}$, then there exists $x' \in j$ such that (v', x') is reachable with *T*. It follows that for every transition $v \to v'$ of $\mathcal{M}$ there exists $x' \in j$ such that Eva can iterate the local strategy from $(v', t'_{x',y'})$.

Eva's local strategy consists in iterating this process. We show that this strategy is winning: let $\{(v_i, s_{x_i,y_i})\}_{i \geqslant 0}$ be an infinite play played according to this strategy. Since the infinite play $\{(v_i, x_i)\}_{i \geqslant 0}$ has been played according to the winning strategy *T*, $\limsup_i \rho^{\mathscr{A}}(x_i)$ is even. It is therefore enough to argue that is $\limsup_i \rho^{\mathscr{A}}(x_i)$ is even, then $\limsup_i \rho^{\mathscr{C}^1}(s_{x_i,y_i})$ is also even. This is done as in the proof of Proposition 2.11. □

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