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FUNCTIONAL CHARACTERS OF SOLVABLE TERMS

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Introduction

One main motivation that induced Curry to introduce the theories of Illative Combinatory Logic (i.e. theories in which new objects and deduction rules are added to pure λ-calculus or Combinatory Logic) was to avoid certain paradoxes which demonstrate that some care must be taken in assigning "meanings" to terms. In particular, Curry shows that the interpretation of logical notions (such as implication or negation) inside Combinatory Logic leads to a kind of Russell's paradox.

The first and simplest of the illative systems proposed in [6] is the theory of basic functionality which is suggested by the very natural interpretation of terms as functions from terms to terms. In this system types or functional characters (as Curry calls them) are assigned to terms by means of a set of formal axioms and deduction rules.

From another point of view, when we see λ-calculus as a topic in the area of theoretical computer science, basic functionality theory represents an useful tool for studying the notion of type in programming languages [11].

Since types are in some sense related to the "meaning" of terms, we must expect that not all arbitrary terms can have a significant type in any consistent system of functionality, which in fact is so. An indication of the terms to which one can reasonably try to assign types (in some suitable system) could be found in the works in the area of λ-calculus models (see for example [15]) in which the problem of assigning "meanings" to terms is gone into very deeply (even though considered from another point of view). All these works agree on the fact that terms with a head normal form (h.n.f.) are all and only those to which a "meaning" different from "undefined" can be assigned. So it is reasonable to ask for a functionality theory in which all terms with a h.n.f. have some "meaningful" type. Types, moreover, should be preserved by convertibility. Another feature that is desirable in a functionality theory is the existence of some normal form theorem (similar to the one of [7, p. 325]) which characterizes all types that can be assigned only to terms having normal form (n.f.).

In basic functionality theory a normal form theorem holds (every typed term has a n.f.) but the terms which have a type are only a proper subset of the set of terms with n.f. ($\lambda x.xx$, for example, does not have any type). Moreover, types are preserved only by reduction but not, in general, by convertibility (to avoid this it is necessary to postulate, as Curry does, the (non-constructive) rule Eq′ [6, p. 279]).

Many recent theories (see the introduction of [14] for a review of them) have generalized functionality theory by modifying both type structures and type assignment

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rules, leading to more general systems. A feature of all these theories is that types are considered, more or less explicitly, functions from terms to types. In this case, however, the functional characters assigned to terms often become quite complex and this can be a problem when we are dealing with applications to computer science.

The aim of this paper is to introduce an extension of basic functionality theory (obtained by generalizing both type structure and type assignment rules) which satisfy all the previous requirements while preserving, in a great extent, the simplicity and clearness of CURRY's system. A modified form of the normal form theorem can also be proved in our system. Obviously, since we want to assign meaningful types to terms in h.n.f. (but not necessarly in n.f.), we cannot prove that a term which has an arbitrary type has a n.f.. Normal form theorem, indeed, will be shown to hold only for a particular subset of types that can be constructively defined, i.e. for the *proper types*. We prove also that the property of having a proper type characterizes all terms with a n.f., that is, a term possesses a n.f. iff it has a proper type.

The main tools used to obtain these results are:

i) the introduction of a universal type ω (similar to the object E of CURRY [6, p.240]) which allows the assignment of at least one type (i.e. ω) to every term (obviously we consider ω as a "non-meaningful" type),

ii) the introduction of a more powerful object (let us call it F′) for type formation in place of the object F of CURRY's system. In fact, while F builds a new type $F\sigma\tau$ from two types σ and τ, F′ builds a new type $F'[\sigma_1, \ldots, \sigma_n]\tau$ from $n + 1$ types $\sigma_1, \ldots, \sigma_n, \tau$. We call $[\sigma_1, \ldots, \sigma_n]$ a *sequence*. We will write shortly $[\sigma_1, \ldots, \sigma_n]\tau$ instead of $F'[\sigma_1, \ldots, \sigma_n]\tau$.

Sequences, as well as types, can be assigned to terms by a suitable deduction rule. All the deduction rules of basic functionality (Fe, Fi, Ap), obviously, are modified accordingly.

Sequences were introduced so as to allow different types to be assigned to different occurrences of the same variable in a term. This overcomes most of the difficulties which limit the power of basic functionality (in particular, the problem of the self-application). Consistently, a sequence $[\sigma_1, \ldots, \sigma_n]$ can be assigned to a term X iff σ_i can be assigned to X for all i ($1 \leqq i \leqq$ n). So if we consider, as suggested by CURRY, the naive interpretation of a type τ as the set of terms to which type τ can be assigned, we have that a sequence $[\sigma_1, \ldots, \sigma_n]$ can be interpreted as the intersection of the types $\sigma_1, \ldots, \sigma_n$.

In section 1 of this paper we shall introduce our system of type assignment formulated in a "natural deduction" style. In section 2 we shall prove the main results, i.e. convertibility theorem, h.n.f. theorem and normal form theorem. The last two, in particular, can be seen as proofs of the consistency of our system. In section 3 we shall prove that it is possible to give a characterization of the terms which possess a h.n.f. or a n.f. based on their types. In section 4 we shall introduce a restricted system, suggested by a functional interpretation of types and sequences, such that only types which are meaningful with respect to such interpretation can be deduced. All the results proved in sections 2 and 3 still hold for this restricted system.

1. General formulation of the system

The sets of types and sequences are defined (inductively) from a (denumerable) set of basic types $\{\psi_1, \psi_2, \ldots\}$ and a distinguished type ω by two operations of type formation which build, respectively, a sequence from a collection of types and a type from a sequence and another type.

Definition 1. The *sets of types and sequences* are so defined:

i) ω and each basic type is a type (*atomic types*),

ii) if $\sigma_1, \ldots, \sigma_n$ $(n > 0)$ are types then $[\sigma_1, \ldots, \sigma_n]$ is a sequence,

iii) if $[\sigma_1, \ldots, \sigma_n]$ is a sequence and τ is a type then $[\sigma_1, \ldots, \sigma_n]\,\tau$ is a type.

Let us notice that the set of types of basic functionality theory is isomorphic to the (proper) subset of the previous one obtained by eliminating ω from the atomic types and by restricting points ii) and iii) of Definition 1 to the case $n = 1$.

We shall indicate types with the greek letters $\sigma, \tau, \mu, \nu, \varrho$; φ will denote an atomic type; Δ will denote a sequence or a type.

We say that a sequence $[\sigma_1, \ldots, \sigma_n]$ *contains* a type τ iff for some i $(1 \leqq i \leqq n)$ $\sigma_i \equiv \tau$. We use $[\bar{\sigma}]$ to represent shortly a sequence $[\sigma_1, \ldots, \sigma_n]$ $(n > 0)$ (while $[\sigma]$ is the sequence which contains only σ).

Example 1. $\tau \equiv [\psi_1, \psi_2, \psi_3]\,[[\psi_1, \psi_2]\,\omega, \psi_1]\,\psi_3$ is a type.

Clearly for each type τ we have $\tau \equiv [\bar{\tau}_1] \ldots [\bar{\tau}_n]\,\varphi$ $(n \geqq 0)$ where φ is an atomic type. We say that τ is *tailproper* iff $\varphi \not\equiv \omega$.

One can define the *level* of a type occurrence τ in a type σ as follows:

i) if $\tau \equiv \sigma$ the level of τ in σ is 0,

ii) if $\sigma \equiv [\nu_1, \ldots, \nu_n]\,\mu$ the level of τ in σ is one plus the level of τ in ν_i if τ occurs in ν_i $(1 \leqq i \leqq n)$, and the level of τ in μ if τ occurs in μ.

We will say that a type τ is *proper* if either ω does not occur in τ or ω occurs in τ only at *odd* levels. Obviously a proper type is a particular tailproper type.

Example 2. τ as defined in Example 1 is a proper type since ω occurs only at level 1.

Types and sequences are assigned to terms by means of a system of inference rules which is based on the same idea as GENTZEN's "natural deduction" system [8]. Similarly as [6, p. 272] a *statement* is an expression ΔX where Δ is a type or a sequence and X is a term. X is referred as the *subject* of the statement and Δ as its *predicate*.

A *deduction* is a set of statements arranged as a tree according to a set of inference rules defined below. The statements in the leaves of this tree are the *premises* for the deduction of the statement which is in the root of the tree (*end statement*).

We force the *premises* of a deduction to be statements whose *subjects are variables*.

The inference rules are the following:

$(\omega\mathrm{i})$ $\qquad \dfrac{}{\omega X}$ for all terms X.

(Si) $\dfrac{\sigma_1 X \ \ldots \ \sigma_n X}{[\sigma_1, \ldots, \sigma_n]\, X}$ $(n > 0)$.

(F′e) $\dfrac{[\bar\sigma]\, \tau X \quad [\bar\sigma]\, Y}{\tau XY}$.

(F′i) If x is a variable and $\sigma_1 x, \ldots, \sigma_n x$ $(n \geqq 0)$ are the distinct premises, whose subject is x, which occur in the deduction of τX then:

$$\dfrac{\begin{matrix}[\sigma_1 x] \ \ldots \ [\sigma_n x] \\ \vdots \\ \tau X\end{matrix}}{[\bar\sigma]\, \tau \lambda x.X}$$

where $[\bar\sigma]$ is *any* sequence which contains $\sigma_1, \ldots, \sigma_n$. Moreover the premises $\sigma_1 x, \ldots, \sigma_n x$ must be cancelled when this rule is used (we indicate this by enclosing them between square brackets).

Rule ωi characterizes the universality of ω. Rule Si allows the assignment of sequences to terms. We could add an inverse rule Se to deduce $\sigma_i X$ from $[\sigma_1, \ldots, \sigma_n]\, X$ $(1 \leqq i \leqq n)$, but it can be proved as a metatheorem (see Property 3). Rules F′e and F′i generalize rules Fe and Fi of CURRY's system. Concerning rule F′i we must observe that it gives a certain degree of freedom in the choice of $[\bar\sigma]$ (this is not true for Fi). In particular:

i) $[\bar\sigma]$ can contain any (finite) number of occurrences of $\sigma_1, \ldots, \sigma_n$ as well as of any arbitrary type different from $\sigma_1, \ldots, \sigma_n$,

ii) the order of occurrences of all these types in $[\bar\sigma]$ is arbitrary.

In section 4 we will present a system in which rule F′i is modified in such a way that the statement deducible by it (in a given deduction) is unique (modulo a suitable equivalence relation).

It is easy to see that our system generalizes basic functionality theory. I.e. if we can deduce σX from some premises in basic functionality theory, then we can deduce σX from the same premises in our system, modulo the obvious correspondence between the types of the two systems.

We define a *basis* $\mathfrak{B}$ to be a set of statements σx where σ is a type and x is a variable and we write $\mathfrak{B} \vdash \Delta X$ if there is a deduction $\mathfrak{D}$ of ΔX in which the uncancelled premises are statements in $\mathfrak{B}$. Let us notice that statements which do not occur as premises in $\mathfrak{D}$ may also belong to $\mathfrak{B}$. This implies that, if $\mathfrak{B} \vdash \Delta X$, then for any basis $\mathfrak{B}'$ such that $\mathfrak{B} \subseteq \mathfrak{B}'$ we have $\mathfrak{B}' \vdash \Delta X$. If $\mathfrak{B}$ is the empty basis we write simply $\vdash \Delta X$. Let us notice that, if X is closed, $\mathfrak{B} \vdash \Delta X$ for some $\mathfrak{B}$ implies $\vdash \Delta X$. A basis $\mathfrak{B}$ is *proper* iff all types σ which are predicates of statements in $\mathfrak{B}$ are such that either ω does not occur in σ or ω occurs in σ only at *even* levels.

To simplify notations we shall write $\{\bar\sigma x\}$ instead of $\{\sigma_1 x, \ldots, \sigma_n x\}$ where $[\bar\sigma] \equiv [\sigma_1, \ldots, \sigma_n]$. Then we can write, for example, $\mathfrak{B} = \mathfrak{B}' \cup \{\bar\sigma x\}$.

Example 3. In fig. 1 we show a deduction of $[\tau]\, \tau(\lambda x.x(x(\lambda yt.t)\,(W_* W_*)))\,(\lambda v.v)$ (where $W_* \equiv \lambda u.uu$) from the empty basis. Let us notice that this term is not stratified in the sense of [6, p. 289].

$$\dfrac{[\tau t]}{[\tau]\,\tau\,\lambda t.t}\ \mathrm{F'i}$$

$$\dfrac{[\tau]\,\tau\,\lambda t.t}{[\omega]\,[\tau]\,\tau\,\lambda yt.t}\ \mathrm{F'i}$$

$$\dfrac{[\omega]\,[\tau]\,\tau\,\lambda yt.t}{[[\omega]\,[\tau]\,\tau]\,\lambda yt.t}\ \mathrm{Si}$$

$$\dfrac{[[[\omega]\,[\tau]\,\tau]\,[\omega]\,[\tau]\,\tau x] \qquad [[\omega]\,[\tau]\,\tau]\,\lambda yt.t}{[\omega]\,[\tau]\,\tau x(\lambda yt.t)}\ \mathrm{F'e}$$

$$\dfrac{}{\omega(\boldsymbol{W}_*\boldsymbol{W}_*)}\ \omega\mathrm{i}$$

$$\dfrac{\omega(\boldsymbol{W}_*\boldsymbol{W}_*)}{[\omega]\,(\boldsymbol{W}_*\boldsymbol{W}_*)}\ \mathrm{Si}$$

$$\dfrac{[\omega]\,[\tau]\,\tau x(\lambda yt.t) \qquad [\omega]\,(\boldsymbol{W}_*\boldsymbol{W}_*)}{[\tau]\,\tau x(\lambda yt.t)\,(\boldsymbol{W}_*\boldsymbol{W}_*)}\ \mathrm{F'e}$$

$$\dfrac{[\tau]\,\tau x(\lambda yt.t)\,(\boldsymbol{W}_*\boldsymbol{W}_*)}{[[\tau]\,\tau]\,x(\lambda yt.t)\,\boldsymbol{W}_*\boldsymbol{W}_*)}\ \mathrm{Si}$$

$$\dfrac{[[[\tau]\,\tau]\,[\tau]\,\tau x] \qquad [[\tau]\,\tau]\,x(\lambda yt.t)\,\boldsymbol{W}_*\boldsymbol{W}_*)}{[\tau]\,\tau x(x(\lambda yt.t)\,(\boldsymbol{W}_*\boldsymbol{W}_*))}\ \mathrm{F'e}$$

$$\dfrac{[\tau]\,\tau x(x(\lambda yt.t)\,(\boldsymbol{W}_*\boldsymbol{W}_*))}{[[[\omega]\,[\tau]\,\tau]\,[\omega]\,[\tau]\,\tau,\ [[\tau]\,\tau]\,[\tau]\,\tau]\,[\tau]\,\tau\lambda x.x(x(\lambda yt.t)\,(\boldsymbol{W}_*\boldsymbol{W}_*))}\ \mathrm{F'i}$$

$$\dfrac{[[\omega]\,[\tau]\,\tau v]}{[[\omega]\,[\tau]\,\tau]\,[\omega]\,[\tau]\,\tau\lambda v.v}\ \mathrm{F'i} \qquad \dfrac{[[\tau]\,\tau v]}{[[\tau]\,\tau]\,[\tau]\,\tau\lambda v.v}\ \mathrm{F'i}$$

$$\dfrac{[[\omega]\,[\tau]\,\tau]\,[\omega]\,[\tau]\,\tau\lambda v.v \qquad [[\tau]\,\tau]\,[\tau]\,\tau\lambda v.v}{[[[\omega]\,[\tau]\,\tau]\,[\omega]\,[\tau]\,\tau,\ [[\tau]\,\tau]\,[\tau]\,\tau]\,\lambda v.v}\ \mathrm{Si}$$

$$\dfrac{[[[\omega]\,[\tau]\,\tau]\,[\omega]\,[\tau]\,\tau,\ [[\tau]\,\tau]\,[\tau]\,\tau]\,[\tau]\,\tau\lambda x.x(x(\lambda yt.t)\,(\boldsymbol{W}_*\boldsymbol{W}_*)) \qquad [[[\omega]\,[\tau]\,\tau]\,[\omega]\,[\tau]\,\tau,\ [[\tau]\,\tau]\,[\tau]\,\tau]\,\lambda v.v}{[\tau]\,\tau(\lambda x.x(x(\lambda yt.t)\,(\boldsymbol{W}_*\boldsymbol{W}_*)))\,(\lambda v.v)}\ \mathrm{F'e}$$

Fig. 1. A deduction $\mathfrak{D}$ of $[\tau]\,\tau(\lambda x.x(x(\lambda yt.t)\,(W_*W_*)))\,(\lambda v.v)$ from the empty basis. The $\mathfrak{D}$-assignments for $\lambda v.v$ are underlined with a dashed line.

If $\mathfrak{D}$ is a deduction, any subtree of $\mathfrak{D}$ is, as usual, a *subdeduction* of $\mathfrak{D}$.

In CURRY's functionality theory, if we force the premises to be statements whose subjects are variables, there is a one-one correspondence between the structure of a term X and the structure of a deduction $\mathfrak{D}$ of a statement τX. Therefore, for each occurrence of a component Z of X there is exactly one statement σZ in $\mathfrak{D}$ (for some type σ). This is not true for our system since rule Si allows many different types to be assigned to Z so that many different statements σZ (for different types σ) may occur in $\mathfrak{D}$. Owing to rule ωi, moreover, Z can also occur in $\mathfrak{D}$ as a component of Y in statements ωY deduced by rule ωi (where obviously Y is a component of X). These observations motivate the introduction of the following definition.

Definition 2. Let $\mathfrak{D}$ be a deduction of ΔX and Z an occurrence of a component of X. A statement σZ which occurs in $\mathfrak{D}$ and which has not been obtained by an application of rule ωi is a *$\mathfrak{D}$-assignment* for Z.

Example 4. In the deduction $\mathfrak{D}$ shown in fig. 1 the $\mathfrak{D}$-assignments for $\lambda v.v$ are underlined with a dashed line.

If we recall that only statements whose subject is a variable are allowed in a basis, by the structure of types and deduction rules we can immediately prove the following properties.

Property 1. *If $\mathfrak{D}$ is a deduction of $\tau(XY)$ then either $\tau \equiv \omega$ and $\omega(XY)$ has been obtained by an application of rule ωi or the last step in $\mathfrak{D}$ has the form*

$$\frac{[\bar{\sigma}]\,\tau X \quad [\bar{\sigma}]\,Y}{\tau(XY)}\,\mathrm{F'e}$$

for some sequence $[\bar{\sigma}]$.

Property 2. *If $\mathfrak{D}$ is a deduction of $[\bar{\sigma}]\,\tau\,\lambda x.X$ then the last step in $\mathfrak{D}$ has the form*

$$\begin{array}{c} [\sigma_1 x] \;\; \ldots \;\; [\sigma_n x] \\ \vdots \\ \dfrac{\tau X}{[\bar{\sigma}]\,\tau\,\lambda x.X}\,\mathrm{F'i} \end{array}$$

where $\sigma_1 x, \ldots, \sigma_n x$ are the premises cancelled by the application of rule F'i.

Property 3. *If $\mathfrak{D}$ is a deduction of $[\sigma_1, \ldots, \sigma_n]\,X$ then the last step in $\mathfrak{D}$ has the form*

$$\frac{\sigma_1 X \;\; \ldots \;\; \sigma_n X}{[\sigma_1, \ldots, \sigma_n]\,X}\,\mathrm{Si}.$$

2. Convertibility and normal form theorems

One main feature of the present type assignment is that types are invariant by β-convertibility, i.e. β-convertible terms have the same set of types.

Lemma 1. *If R is a redex, R' its contractum, then $\mathfrak{B} \vdash \tau R$ iff $\mathfrak{B} \vdash \tau R'$.*

Proof. We construct, from any deduction $\mathfrak{D}$ of τR, a deduction $\mathfrak{D}'$ of $\tau R'$ which has the same premises (and viceversa). The construction is trivial if $\tau \equiv \omega$ and ωR or $\omega R'$ have been obtained by rule ωi.

($\Rightarrow$). Let $R \equiv (\lambda x.X)\,Y$. By Property 1 there exists a sequence $[\bar{\sigma}] \equiv [\sigma_1, \ldots, \sigma_n]$ such that in $\mathfrak{D}$ there is a subdeduction of $[\bar{\sigma}]\,\tau\lambda x.X$ and one of $[\bar{\sigma}]\,Y$ from $\mathfrak{B}$. By Property 2, moreover, there is a subdeduction $\mathfrak{D}_0$ of τX from $\mathfrak{B} \cup \{\sigma_1 x, \ldots, \sigma_n x\}$ and, by Property 3, there are subdeductions $\mathfrak{D}_i$ of $\sigma_i Y$ $(1 \leqq i \leqq n)$ from $\mathfrak{B}$. To obtain a deduction $\mathfrak{D}'$ of $\tau X[x/Y]$ from $\mathfrak{B}$ it is enough to replace in $\mathfrak{D}_0$ each occurrence of a premise $\sigma_i x$ by the deduction $\mathfrak{D}_i$ of $\sigma_i Y$ for $1 \leqq i \leqq n$ (and, obviously, to replace by Y all other occurrences of x in $\mathfrak{D}_0$).

($\Leftarrow$). Let R and R' be defined as in the proof of the "if part" (we can assume that x occurs neither in R' nor in Y). In what follows, by occurrences of Y in R' we mean only those occurrences of Y in R' which replace occurrences of x in X. Let $\mathfrak{D}'$ be a deduction of $\tau X[x/Y]$ from $\mathfrak{B}$ and $\{\sigma_1 Y, \ldots, \sigma_n Y\}$ be the (possibly empty) set of all *$\mathfrak{D}'$-assignments* for Y occurrences. We first obtain a deduction $\mathfrak{D}_0$ of τX from $\mathfrak{B} \cup \{\sigma_1 x, \ldots, \sigma_n x\}$ by replacing in $\mathfrak{D}'$ each subdeduction of $\sigma_i Y$ by a premise $\sigma_i x$ $(i \leqq n)$ (and by replacing all other occurrences of Y in $\mathfrak{D}'$ by x). Then we build a deduction $\mathfrak{D}_1$ of $[\bar{\sigma}]\,\tau\lambda x.X$ from $\mathfrak{B}$ by applying rule F'i to the end statement of $\mathfrak{D}_0$, where we choose $[\sigma] \equiv [\sigma_1, \ldots, \sigma_n]$ if $n > 0$, otherwise $[\bar{\sigma}] \equiv [\omega]$.[1]) If $[\bar{\sigma}] \not\equiv [\omega]$ we join together the subdeductions of $\sigma_i Y$ for $1 \leqq i \leqq n$ by an application of rule Si to obtain a deduction $\mathfrak{D}_2$ of $[\sigma_1, \ldots, \sigma_n]\,Y$ from $\mathfrak{B}$. If $[\bar{\sigma}] \equiv [\omega]$ $\mathfrak{D}_2$ is simply the deduction of $[\omega]Y$ by rules ωi and Si. Lastly, the deduction $\mathfrak{D}$ of τR from $\mathfrak{B}$ is the result of joining together $\mathfrak{D}_1$ and $\mathfrak{D}_2$ by means of rule F'e. □

Theorem 1. *If $X =_\beta X'$ and $\mathfrak{B} \vdash \sigma X$ then $\mathfrak{B} \vdash \sigma X'$.*

Proof. It is clearly sufficient to prove the theorem when X' differs from X for the contraction of only one occurrence of a redex. Let R be the occurrence of this redex and R' its contractum. If $\mathfrak{D}$ is a deduction of σX from $\mathfrak{B}$ and $\{\tau_1 R, \ldots, \tau_n R\}$ is the set of $\mathfrak{D}$-assignment for R, we can build a deduction of $\sigma X'$ from $\mathfrak{B}$ simply by replacing, in $\mathfrak{D}$, each subdeduction of $\tau_i R$ by a deduction of $\tau_i R'$ $(i \leqq n)$ obtained as in the proof of Lemma 1 and by replacing all other occurrences of R in $\mathfrak{D}$ by R'. □

In the rest of the present section we shall prove two results that generalize the normal form theorem of [7, p. 325]:

i) if $\mathfrak{B} \vdash \tau X$ and τ is a tailproper type, then X reduces to h.n.f.,

ii) if $\mathfrak{B} \vdash \tau X$ and $\mathfrak{B}$, τ are proper then X reduces to n.f.

As a consequence we also have a proof of the consistency of our system. These results are obtained by associating to a deduction $\mathfrak{D}$ of τX from $\mathfrak{B}$ an element u (the "measure" of $\mathfrak{D}$) of a suitable well-ordered set and by showing that, if u is not the minimal element, there exists a term X' and a deduction $\mathfrak{D}'$ of $\tau X'$ such that $X \geqq_\beta X'$ and the measure of $\mathfrak{D}'$ is lower than that of $\mathfrak{D}$. To this aim we introduce some more definitions. The *length* $\|\tau\|$ of a type τ is defined as the number of occurrences of atomic types in τ. Let $\mathfrak{D}$ be a deduction of σX from $\mathfrak{B}$ and $R \equiv (\lambda x.Y)\,Z$ be the occurrence of a redex in X. The *characteristic set* of R in $\mathfrak{D}$ is the (possibly empty) set $\mathfrak{C}(R)$ of all types τ which are predicates of $\mathfrak{D}$-assignments for $\lambda x.Y$:

$$\mathfrak{C}(R) = \{\tau \mid \tau\lambda x.Y \text{ is a } \mathfrak{D}\text{-assignment for } \lambda x.Y\}.$$

[1]) We make this choice since we want the present proof to hold with merely trivial changes also for the restricted system which will be discussed in section 4.

The *height* $h(R)$ of R in $\mathfrak{D}$ is the maximum length of the types which belong to $\mathfrak{C}(R)$, i.e. $h(R) = \max\{\|\tau\| \mid \tau \in \mathfrak{C}(R)\}$. We notice that if $\tau \in \mathfrak{C}(R)$ then τ cannot be an atomic type, i.e. $\tau \equiv [\bar{\sigma}]\,\varrho$ for some sequence $[\bar{\sigma}]$ and type ϱ. This implies that, if $\mathfrak{C}(R)$ is not empty, then $h(R) \geqq 2$.

We can now associate to a deduction $\mathfrak{D}$ a pair of non-negative integers $\langle m(\mathfrak{D}), n(\mathfrak{D})\rangle$ (*measure* of $\mathfrak{D}$) defined as

$m(\mathfrak{D})$ = maximum height of redexes in $\mathfrak{D}$,

$n(\mathfrak{D})$ = number of redexes whose height is $m(\mathfrak{D})$ in $\mathfrak{D}$.

We assume $m(\mathfrak{D}) = 0$ if $\mathfrak{C}(R)$ is empty for all redexes R in $\mathfrak{D}$ and $n(\mathfrak{D}) = 0$ if $m(\mathfrak{D}) = 0$. Integer pairs can be ordered by the usual lexicographic order relation ($\lesssim$) defined as: $\langle h', k'\rangle \lesssim \langle h, k\rangle$ iff $h' < h$ or $h' = h$ and $k' \leqq k$.

Example 5. If $R \equiv (\lambda x.x(x(\lambda yt.t)\,(W_* W_*)))\,(\lambda v.v)$ and we consider the deduction $\mathfrak{D}$ of Example 3 then

$$\mathfrak{C}(R) = \{[[[\omega]\,[\tau]\,\tau]\,[\omega]\,[\tau]\,\tau,\ [[\tau]\,\tau]\,[\tau]\,\tau]\,[\tau]\,\tau\},$$

$$h(R) = 12, \qquad m(\mathfrak{D}) = 12, \qquad n(\mathfrak{D}) = 1.$$

Lemma 2. *Let $\mathfrak{D}$ be any deduction of σX from $\mathfrak{B}$ whose measure is not $\langle 0, 0\rangle$. Then there is a term X' and a deduction $\mathfrak{D}'$ of $\sigma X'$ from $\mathfrak{B}$ such that $X \geqq_\beta X'$ and the measure of $\mathfrak{D}'$ is less than that of $\mathfrak{D}$.*

Proof (constructive). Let us choose an occurrence of a redex $(\lambda x.Y)\,Z$ in X such that

i) its height is $m(\mathfrak{D})$,

ii) no redex with height $m(\mathfrak{D})$ occurs in it.

Since X is finite it is obvious that such a redex always exists. Let $\mathfrak{C}((\lambda x.Y)\,Z) = \{[\bar{\sigma}_i]\,\tau_i \mid 1 \leqq i \leqq n)\}$. Let X' be obtained from X by contracting $(\lambda x.Y)\,Z$. If we consider the deduction $\mathfrak{D}'$ of $\sigma X'$ from $\mathfrak{B}$ obtained as in the proof of Theorem 1, we observe that all redexes in X' which are residuals of some redex in X have in $\mathfrak{D}'$ the same characteristic sets as in $\mathfrak{D}$. Moreover the only redexes of X which have more than one residual in X' have heights lower than $m(\mathfrak{D})$ owing to the condition of point ii) above. The redexes in X' which are not residual of any redex in X have (possibly empty) characteristic sets which are either subsets of the set $\{\mu \mid \mu$ is contained in $[\bar{\sigma}_i]$ for some i $(1 \leqq i \leqq n)\}$ (if they are redexes of the form ZV for some component V of $Y[x/Z]$) or subsets of $\{\tau_1, \ldots, \tau_n\}$ (if there is a redex of the form $(Y[x/Z])\,W$, where W is a component of X). In both cases these redexes have heights lower than $m(\mathfrak{D})$. Therefore X contains exactly one occurrence of a redex of height $m(\mathfrak{D})$ more than X'. This means that either $m(\mathfrak{D}') < m(\mathfrak{D})$ (if $(\lambda x.Y)\,Z$ was the only redex of height $m(\mathfrak{D})$) or $m(\mathfrak{D}') = m(\mathfrak{D})$ and $n(\mathfrak{D}') = n(\mathfrak{D}) - 1$. □

Theorem 2. *If $\mathfrak{B} \vdash \tau X$ and τ is a tailproper type then X reduces to h.n.f.*

Proof. By induction on the measure of $\mathfrak{D}$, where $\mathfrak{D}$ is a deduction of τX from $\mathfrak{B}$.

First step. If the measure of $\mathfrak{D}$ is $\langle 0, 0\rangle$ we prove that X is in h.n.f. Let us suppose ad absurdum that X is not in h.n.f., i.e. $X \equiv \lambda x_1 \ldots x_n.RX_1 \ldots X_m$ where R is a redex. Since $\tau \not\equiv \omega$, by reiterated applications of Properties 1 and 2 we must have $\tau \equiv [\bar{\sigma}_1] \ldots [\bar{\sigma}_n]\,\varrho$ and there must be in $\mathfrak{D}$ a subdeduction $\mathfrak{D}_1$ of $[\bar{\mu}_1] \ldots [\bar{\mu}_m]\,\varrho R$ for some sequences $[\bar{\mu}_1], \ldots, [\bar{\mu}_m]$. Again by Property 1, if $R \equiv (\lambda x.Y)\, Z$, there must exist a sequence $[\bar{\nu}]$ such that the last step of $\mathfrak{D}_1$ is

$$\frac{[\bar{\nu}]\,[\bar{\mu}_1]\ldots[\bar{\mu}_m]\,\varrho\,\lambda x.Y \quad [\bar{\nu}]\,Z}{[\bar{\mu}_1]\ldots[\bar{\mu}_m]\,\varrho R}\,\mathrm{F'e}.$$

But, in this case, the statement $[\bar{\nu}]\,[\bar{\mu}_1]\ldots[\bar{\mu}_m]\,\varrho\,\lambda x.Y$ is a $\mathfrak{D}$-assignment for $\lambda x.Y$ and so $\mathfrak{C}(R)$ cannot be empty. This implies that the measure of $\mathfrak{D}$ cannot be $\langle 0, 0\rangle$ what is contrary to our hypothesis.

Inductive step. Immediate from Lemma 2. □

From this theorem it follows that an unsolvable term (in the sense of [15]) does not have tailproper types.

Theorem 3. *If $\mathfrak{B} \vdash \tau X$ and $\mathfrak{B}$, τ are proper then X reduces to n.f.*

Proof. By induction on the measure of $\mathfrak{D}$, where $\mathfrak{D}$ is a deduction of τX from $\mathfrak{B}$.

First step. The measure of $\mathfrak{D}$ is $\langle 0, 0\rangle$. We prove that X is in n.f. by structural induction on X. If X is a free variable the theorem is trivially true. Otherwise, by the argument given in the proof of the first step of Theorem 2, X is in h.n.f., i.e. $X \equiv \lambda x_1 \ldots x_n.zX_1 \ldots X_m$. From $\mathfrak{B} \vdash \tau X$, by reiterated applications of Properties 1 and 2, it follows that $\tau \equiv [\bar{\sigma}_1]\ldots[\bar{\sigma}_n]\,\varrho$ and, if $\mathfrak{B}' = \mathfrak{B} \cup \{\bar{\sigma}_1 x_1\} \cup \ldots \cup \{\bar{\sigma}_n x_n\}$, then there is in $\mathfrak{D}$ a subdeduction of $\varrho z X_1 \ldots X_m$ from $\mathfrak{B}'$. Moreover, there is a statement $[\bar{\mu}_1]\ldots[\bar{\mu}_m]\,\varrho\, z$ which belongs to $\mathfrak{B}'$ and there are in $\mathfrak{D}$ subdeductions of $[\bar{\mu}_i]\,X_i$ from $\mathfrak{B}'$ $(1 \leqq i \leqq m)$. By Property 3, moreover, we have that for all ν contained in $[\bar{\mu}_i]$ there are in $\mathfrak{D}$ subdeductions of νX_i from $\mathfrak{B}'$ $(1 \leqq i \leqq n)$. It is easy to verify that ν is a proper type and $\mathfrak{B}'$ is a proper basis. Therefore, by inductive hypothesis, each X_i $(1 \leqq i \leqq m)$ is in n.f. and this means that X is in n.f. too.

Inductive step. Immediate from Lemma 2. □

3. A characterization of some classes of typed terms

We will prove in this section that the property of possessing a tailproper (proper) type characterizes the set of terms which have a h.n.f. (n.f.). To do this it is now sufficient to prove that each term which is in h.n.f. (n.f.) possesses a tailproper (proper) type. By Theorem 1, in fact, this result immediately extends to terms which can be reduced to h.n.f. (or n.f.). Theorems 2 and 3, moreover, assure that the converse holds true.

Lemma 3. *If X is a term in h.n.f. then $\mathfrak{B} \vdash \tau X$ for some basis $\mathfrak{B}$ and tailproper type τ.*

Proof. Let $X \equiv \lambda x_1 \ldots x_n.xX_1 \ldots X_m$. It is easy to verify that $\underbrace{[\omega]\ldots[\omega]}_{m \text{ times}}\,\varphi x \vdash \varphi x X_1 \ldots X_m$ where φ is any basic type and the result follows immediately by n applications of rule F'i. □

Lemma 4. *If X is a term in n.f. then $\mathfrak{B} \vdash \tau X$ for some proper basis $\mathfrak{B}$ and proper type τ.*

Proof. By structural induction on X.

First step. If X is a free variable (say x), we have $\varphi x \vdash \varphi x$, where φ is any basic type.

Inductive step. By cases.

Case 1. $X \equiv xX_1 \dots X_m$ $(m > 0)$. By inductive hypothesis there are some proper bases $\mathfrak{B}_i$ and proper types σ_i such that $\mathfrak{B}_i \vdash \sigma_i X_i$ for $1 \leqq i \leqq m$. Then, if $\mathfrak{B} = \bigcup_{i=1}^{m} \mathfrak{B}_i \cup \{[\sigma_1] \dots [\sigma_m] \varphi x\}$, where φ is any basic type, we can deduce φX from $\mathfrak{B}$ by m applications of Si and F′e. It is immediate to verify that $\mathfrak{B}$ is a proper basis.

Case 2. $X \equiv \lambda x.\bar{X}$. By inductive hypothesis there is a deduction $\mathfrak{D}$ of $\tau\bar{X}$ from $\mathfrak{B}'$ for some proper $\mathfrak{B}', \tau$. By rule F″i, then, we can build a deduction of $[\bar{\sigma}]\tau X$ from $\mathfrak{B}$ where $\mathfrak{B} \subseteqq \mathfrak{B}'$ and $[\bar{\sigma}]$ contains only types which are predicates of statements belonging to $\mathfrak{B}'$ and whose subject is x, or $[\bar{\sigma}] \equiv [\omega]$ if no such statement exists. It is immediate to verify that both $\mathfrak{B}$ and $[\bar{\sigma}]\tau$ are proper.

Theorem 4. *Any term X possesses a h.n.f. iff there exist some basis $\mathfrak{B}$ and tailproper type τ such that $\mathfrak{B} \vdash \tau X$.*

Theorem 5. *Any term X possesses a n.f. iff there exist some proper basis $\mathfrak{B}$ and proper type τ such that $\mathfrak{B} \vdash \tau X$.*

4. A restricted system

As pointed out in the introduction, there is a very natural way of considering types as representations of the properties of terms when interpreted as functions from terms to terms. In this case it is obvious that ω represents the property of being a term uniquely and that a sequence $[\sigma_1, \dots, \sigma_n]$ represents the conjunction of the properties represented by $\sigma_1, \dots, \sigma_n$.

The aim of this section is to present a system obtained from that of section 1 by restricting both the syntax of types and the assignment rules, in such a way that all types which are meaningless from this point of view are ruled-out.

First of all, let us notice that a type $[\bar{\sigma}]\omega$ represents a property of terms which, applied to a particular class of terms (i.e. terms in $[\bar{\sigma}]$), give a term. But this property characterizes all terms and, so, it is natural to identify with ω all types of the shape $[\bar{\sigma}]\omega$ and, in general, all non-tailproper types. By a similar argument we are led to identify a sequence $[\bar{\sigma}]$ with a sequence $[\bar{\sigma}']$ obtained by eliminating from $[\bar{\sigma}]$ all occurrences of non-tailproper types (unless, obviously, $[\bar{\sigma}] \equiv [\omega]$). These arguments justify the elimination of all non-tailproper types (except ω) and of all sequences which contain non-tailproper types (except $[\omega]$).

Definition 3. The set of *normalized* types and sequences is so defined:

i) each atomic type is a normalized type,
ii) $[\omega]$ is a normalized sequence. If $\sigma_1, \dots, \sigma_n$ are normalized types and $\sigma_i \not\equiv \omega$ for $1 \leqq i \leqq n$ then $[\sigma_1, \dots, \sigma_n]$ is a normalized sequence,
iii) if $[\sigma]$ is a normalized sequence, τ is a normalized type and $\tau \not\equiv \omega$ then $[\bar{\sigma}]\tau$ is a normalized type.

Let us observe that the only normalized non-tailproper type is ω.

Moreover we notice that, if we interpret a sequence as a conjunction of properties, it is meaningless to distinguish between two sequences which differ only in the order of types they contain. From another point of view, moreover, it is obvious that, if we can deduce $[\sigma]\,\tau\lambda x.X$ by rule F'i or $[\sigma]\,Y$ by rule Si then we can also deduce $[\bar{\sigma}']\,\tau\lambda x.X$ or $[\sigma']\,Y$ where $[\sigma']$ is any permutation of $[\sigma]$. This justifies the following definition of the equivalence relation $\sim$ between types and sequences:

Definition 4. The relation $\sim$ is so defined:

i) $\varphi \sim \varphi$ for any atomic type φ,

ii) $[\sigma_1, \ldots, \sigma_i, \sigma_{i+1}, \ldots, \sigma_n] \sim [\sigma'_i, \ldots, \sigma'_{i+1}, \sigma'_i, \ldots, \sigma'_n]$ iff $\sigma_j \sim \sigma'_j$ for $1 \leqq j \leqq n$,

iii) $[\bar{\sigma}]\,\tau \sim [\sigma']\,\tau'$ iff $[\sigma] \sim [\sigma']$ and $\tau \sim \tau'$.

Duplication of types in sequences is also meaningless. To take this into account we could extend the equivalence relation $\sim$ in an obvious way.

With regard to the deduction rules, we observe that, when rule F'i is applied to deduce $[\bar{\sigma}]\,\tau\,\lambda x.X$ from τX, only the types ϱ which occur as predicates of premises of the kind ϱx really need to be contained in $[\bar{\sigma}]$. For, if we introduce in $[\bar{\sigma}]$ a type μ which is not such, we require a possible argument Y of $\lambda x.X$ to possess a property (i.e. that of having type μ) which is not needed to assure that $(\lambda x.X)\,Y$ has type τ. Moreover, if no premise whose subject is x has been used to deduce τX, this obviously means that $(\lambda x.X)\,Y$ has type τ for *every* term Y and this property is better represented by saying that $\lambda x.X$ has type $[\omega]\,\tau$. Lastly, since we do not want to deduce non-normalized types we must forbid rule F'i to be applied to statements of the shape ωX. These arguments suggest a restricted form of rule F'i (we will call it F*i). Rule Si, moreover, also needs a slight modification to avoid the introduction of non-normalized sequences.

We will now give the deduction rules of the restricted system. In the following we always consider types and sequences modulo $\sim$ (also the identity relation $\equiv$ between types and sequences is considered modulo $\sim$). Moreover we force the *premises* of a deduction to be statements whose subjects are variables and whose *predicates are normalized types different from* ω.

(ωi) $\quad \dfrac{}{\omega X}$ for all terms X.

(S*i) $\quad$ If $\sigma_i \not\equiv \omega$ for $1 \leqq i \leqq n$ then $\dfrac{\sigma_1 X \;\ldots\; \sigma_n X}{[\sigma_1, \ldots, \sigma_n]\,X}$ else $\dfrac{\omega X}{[\omega]\,X}$.

(F'e) $\quad \dfrac{[\sigma]\,\tau X \quad [\sigma]\,Y}{\tau XY}$.

(F*i) $\quad$ Let X be a term, τ a type different from ω and x a variable. If $\sigma_1 x, \ldots, \sigma_n x$ ($n \geqq 1$) are all and only the premises, whose subject is x, which occur in the deduction of τX then

$$\dfrac{\begin{array}{c}[\sigma_1 x] \;\ldots\; [\sigma_n x]\\ \vdots\\ \tau X\end{array}}{[\sigma_1, \ldots, \sigma_n]\,\tau\,\lambda x.X}$$

and the premises $\sigma_1 x, \ldots, \sigma_n x$ must be cancelled. Else, if in the deduction of τX there is no premise whose subject is x then

$$\dfrac{\tau X}{[\omega]\,\tau\,\lambda x.X}.$$

Let us notice that, since we consider sequences modulo $\sim$, by an application of rule F*i to a statement τX in a given deduction we can deduce *only one type* for $\lambda x.X$. We write $\mathfrak{B} \vdash^* \Delta X$ iff there is a deduction of ΔX, using the rules of the restricted system, in which the uncancelled premises are statements in $\mathfrak{B}$.

Let us notice that this new system is not, in a strict sense, a generalization of CURRY's functionality theory. In fact, for example, we have $[\tau]\,[\omega]\,\tau\;\lambda xy.x$ but, because of the structure of rule F*i, we cannot deduce $[\tau]\,[\sigma]\,\tau\lambda xy.x$ for $\sigma \not\equiv \omega$ (which actually is a deducible statement in basic functionality).

It is easy to see that all properties proved in sections 2 and 3 for the system of section 1 remain true in the present restricted system too (the proofs have been given in such a way that they are valid by making only trivial changes). We can prove, indeed, a convertibility result stronger than that of Theorem 1. I.e., if there is a deduction $\mathfrak{D}$ of τX from $\mathfrak{B}$ such that $\mathfrak{B}$ is the set of *all and only* the premises which occur in $\mathfrak{D}$ and $X =_\beta X'$, then there is a deduction $\mathfrak{D}'$ of $\tau X'$ (obtained as in the proof of Theorem 1) from $\mathfrak{B}$ such that $\mathfrak{B}$ is the set of all and only the premises which occur in $\mathfrak{D}'$. This is not true, in general, for the first system since, for example, we have a deduction of $\omega(\lambda x.a)\,b$ from the premise φb but we cannot use this premise to deduce ωa (let us notice that, in the restricted system, we can deduce $\omega(\lambda x.a)\;b$ only by rule ωi).

In the restricted system we can deduce only normalized types and sequences:

Theorem 6. *If $\mathfrak{B} \vdash^* \Delta X$ then Δ is normalized.*

Proof. Δ is normalized since we force the premises to have normalized predicates and it is easy to verify, by a straightforward case analysis, that we cannot deduce, by any of the deduction rules, a non-normalized type or sequence. □

It is obvious that $\mathfrak{B} \vdash^* \Delta X$ implies $\mathfrak{B} \vdash \Delta X$, while the opposite is trivially not true. Moreover, even if $\mathfrak{B}$ and Δ are normalized, $\mathfrak{B} \vdash \Delta X$ does not imply $\mathfrak{B} \vdash^* \Delta X$. This is due to the fact that rule F'i allows the construction of sequences which contain "redundant" types and which cannot be constructed by rule F*i. For example, $\vdash[\varphi_1, \varphi_2]\,\varphi_1\;\lambda x.x$ but not $\vdash^*[\varphi_1, \varphi_2]\,\varphi_1\;\lambda x.x$. In fact, in this case, rule F*i only allows $[\varphi_1]\,\varphi_1\;\lambda x.x$ (or $[\varphi_2]\,\varphi_2\;\lambda x.x$) to be deduced. However, from a functional point of view, $[\varphi_1]\,\varphi_1$ is a "better" type for $\lambda x.x$ than $[\varphi_1, \varphi_2]\,\varphi_1$.

Lastly, we examine briefly the behaviour of types with respect to η-conversion. In both systems, types are not invariant by η-expansion, since for example $\vdash^*[\varphi]\,\varphi\;\lambda x.x$ but not $\vdash[\varphi]\,\varphi\;\lambda xy.xy$ (when φ is any basic type). In the first system types are also not invariant by η-reduction again for the freedom of introducing arbitrary types in sequences given by rule F'i (for example, $\vdash[[\sigma]\,\tau]\,[\sigma, \varrho]\,\tau\;\lambda xy.xy$ but not $\vdash[[\sigma]\,\tau]\,[\sigma, \varrho]\,\tau\;\lambda x.x$). This property, on the contrary, holds in the restricted system.

Theorem 7. *If $\mathfrak{B} \vdash^* \tau X$ and $X \geqq_\eta Y$ then $\mathfrak{B} \vdash^* \tau Y$.*

Proof. It is enough to prove the property for a single η-redex $\lambda x.Yx$, since the general result can be obtained from the previous one as in the proof of Theorem 1. If $\tau \equiv \omega$ then $\mathfrak{B}$ is empty and the property is trivially true. Otherwise, if $\mathfrak{D}$ is a deduction of $\tau\lambda x.Yx$ from $\mathfrak{B}$ in the restricted system it is easy to verify that $\tau \equiv [\sigma]\,\varrho$ and the last two steps of $\mathfrak{D}$ are

$$\cfrac{\cfrac{[\sigma]\,\varrho Y \quad [\sigma x]}{\varrho Y x}\,\mathrm{F'e}}{[\sigma]\,\varrho\,\lambda x.Yx}\,\mathrm{F^{*}i}$$

if $\sigma \not\equiv \omega$, and otherwise

$$\cfrac{\cfrac{[\omega]\,\varrho Y \quad \cfrac{}{\omega x}\,\omega\mathrm{i}}{\varrho Y x}\,\mathrm{F'e}}{[\omega]\,\varrho\,\lambda x.Yx}\,\mathrm{F^{*}i}$$

Let us notice that in $\mathfrak{D}$ there are no other premises whose subject is x since x does not occur free in Y. □

Conclusion

As pointed out it the introduction, some features of the present type system can be related to properties of terms provable in λ-calculus models. Some recent results have been obtained by the authors in this direction. In [2] it is shown that the finite nature of types which are assigned to terms in these systems can be explained by means of the notion of approximant (in the usual sense of [15]), i.e. if τ is a type for X then there is an approximant A of X which has type τ and viceversa. As a consequence of this we have that, if $\sim_G$ is the following equivalence relation between terms

$$X \sim_G Y \text{ iff } \mathfrak{B} \vdash \tau X \text{ implies } \mathfrak{B} \vdash \tau Y \text{ and viceversa}$$

then $\sim_G$ coincides with the equality in the model P_ω [13]. In [4], moreover, the authors prove that it is possible to define a λ-calculus model (according to [10]) whose domain is the power set of the set of normalized types. In the same paper they investigate the possibility of extending the notion of principal type scheme of basic functionality theory (see [5], [9]) to the restricted system presented in section 4 of the present paper. As proved in [4], this extension is possible only if we introduce, besides substitution, a new (effective) operation of "expansion" (to generate, from the principal type scheme of a term X, all the types which can be assigned to X) and if we extend the set of types to contain "infinite" ones as well.

The idea of allowing different types to be assigned to the same variable was first developed in [1] where, nevertheless, all the types assignable to a variable were bound to be "lower" (according to a suitable partial order relation) than a unique type. We did not need, in this case, the introduction of sequences but the power of the system was reduced. In [1] all terms in normal form have a type but types are not preserved, in general, by convertibility. A normal form theorem holds. In [3] a system is presented which extends basic functionality theory by introducing the notion of sequence but not that of universal type; this system is fully satisfactory (in the sense explained in the introduction) only within λ-I-calculus. In [12] a generalization of the system of [1] is obtained by introducing sequences and the "universal type" ω. In [1] and [12], however, types are "interpreted" objects and, so, they are meaningful only with respect to a particular property of terms (i.e. termination property).

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