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Membership Problem for Differential Ideals Generated by a Composition of Polynomials

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Abstract—The question of whether a polynomial belongs to a finitely generated differential ideal remains open. This problem is solved only in some particular cases. In the paper, we propose a method, which reduces the test of membership for fractional ideals generated by a composition of differential polynomials to another, simpler, membership problem.

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1. INTRODUCTION

The problem of whether a differential polynomial belongs to a finitely generated differential ideal is solved at present only in some particular cases. Among them are the following: the case of radical ideals, isobaric (i.e., homogeneous with respect to a weight) ideals, and ideals with a finite or parametrical standard basis. In the first part of the paper, we propose an algorithm that constructs the minimal standard basis (if it is finite) for some monomial orderings. Note that, at present, no universal method is known that determines whether a given polynomial belongs to an ideal, even in the case where the ideal is generated by one differential polynomial.

We consider fractional ideals generated by a composition of two differential polynomials. To solve the membership problem for such an ideal, we use the so-called preparation polynomial. This polynomial was introduced by J. Ritt [1], who was the actual founder of modern differential algebra.

In the paper, we also discuss some theorems on finiteness of standard bases. We propose a refined process of Ollivier, which can be used for constructing finite standard bases under δ-lexicographical orderings.

2. BASIC CONCEPTS

The basic concepts of differential algebra can be found in monographs by Ritt [1] and Kolchin [2], as well as in survey [3].

A differential ring is a unitary commutative ring, on which pairwise commuting derivation operators $\delta_1, \ldots, \delta_m$ act. These operators must be linear and satisfy the Leibniz rule. We denote the set of all derivation operators by Δ and set $\Theta = \{\delta_1^{i_1}, \ldots, \delta_m^{i_m} | i_1, \ldots, i_m \geq 0\}$. The case of $\Delta = \{\delta\}$ is referred to as *ordinary.* If $\mathcal{R}$ is an ordinary differential ring, then the *ring of differential polynomials* over $\mathcal{R}$ (denoted by $\mathcal{R}\{x\}$) is the ring of ordinary polynomials in a countable set of variables $\mathcal{R}[x, \delta(x), \delta^2(x), \ldots]$ considered as a differential ring. The differential polynomials are analogs of algebraic differential equations, where the role of the unknown function is played by the independent variable x. For the sake of convenience, we denote $\delta^n(x)$ by x_n. We consider only differential polynomials over a field of constants $\mathcal{F}$ of characteristic zero.

Let $f \in \mathcal{F}\{x\}$ be a differential polynomial. The *differential order* of the polynomial f (denoted by ord$f$) is defined as the maximal n such that x_n is present in f; in this case, the variable x_n is the *leader* of f. We denote the leader by u_f. Let us consider f as a polynomial in one variable u_f:

$$f = I_f u_f^d + a_1 u_f^{d-1} + \ldots + a_d.$$

The polynomial I_f is referred to as the *initial* of f.

Applying $\delta \in \Delta$ to f, we obtain

$$\delta f = \frac{\partial f}{\partial u_f}\delta u_f + \delta I_f u_f^d + \delta a_1 u_f^{d-1} + \ldots + \delta a_d.$$

The leader of δf is δu_f (it is present in δf in degree one). The initial of the differential polynomial δf is the *separant* of f. We denote it by S_f and the product $S_f I_f$ by H_f. It should be noted that, for any $\theta \in \Theta$, $\theta \neq 1$, the initial of θf is equal to S_f.

A polynomial $f \in \mathcal{F}\{x\}$ is said to be *partially pseudoreduced* with respect to g if f is free of variables of the form $\delta^k u_g$, $k > 0$. A partially pseudoreduced polynomial f is *pseudoreduced* if $\deg_{u_g} f < \deg_{u_g} g$.

Let $F \subset R\{x\}$. By $[F]$, we denote the minimal differential ideal containing F.

3. MEMBERSHIP PROBLEM FOR A POLYNOMIAL AND A DIFFERENTIAL IDEAL

Let a differential ideal I of the ring $\mathcal{F}\{x\}$ be specified by a finite set of generators:[1] $I = [f_1, \ldots, f_n]$. For a given polynomial $g \in \mathcal{F}\{x\}$, it is required to determine whether g belongs to I. If, for ideals of a certain class, this problem can be algorithmically solved for any g, then these ideals are referred to as *solvable*. Let us list important classes of solvable ideals.

• *Radical differential ideals*, i.e., differential ideals I such that $f^n \in I \Rightarrow f \in I$. An algorithm for testing whether a polynomial belongs to a radical differential ideal has been proposed by Ritt [1] and Kolchin [2]. In the early 1990s, effective implementations of this algorithm appeared. They represent the ideal as a finite intersection of some simpler ideals. The MAPLE computer algebra system contains a diffalg package with such an implementation (the Rosenfeld–Gröbner algorithm).[2] At present, the diffalg package is supported by E. Hubert [5, 6].

• *Ideals generated by homogeneous polynomials.* We mean the generators that are homogeneous with respect to a weight function [7]. For a given polynomial g, the membership problem for such ideals reduces to a solvable algebraic problem.

• *Ideals with finite differential standard bases.* Such bases are generalizations of the Gröbner bases to the case of rings of differential polynomials. This case is considered in detail below.

We also propose a method for solving the membership problem for another class of ideals, namely, for ideals generated by a composition of two differential polynomials (under some additional assumptions).

4. DIFFERENTIAL STANDARD BASES

Let $\mathbb{M}$ denote the set of all differential monomials of the ring $\mathcal{F}\{x\}$. Let $m = x_0^{\alpha_0} \ldots x_k^{\alpha_k} \in \mathbb{M}$. Recall that the degree of the monomial is defined as $\deg m = \sum_{i=0}^{k} \alpha_i$, and its weight is the number $\mathrm{wt}\, m = \sum_{i=1}^{k} i\alpha_i$. An *admissible ordering* is a linear order $<$ on $\mathbb{M}$ such that

(1) $M < N \Rightarrow MP < NP \quad \forall M, N, P \in \mathbb{M}$;

(2) $1 \leqslant P \quad \forall P \in \mathbb{M}$;

(3) $x_i < x_j \Leftrightarrow i < j$.

These properties guarantee that the set $\mathbb{M}$ is well ordered [8]. An example of an admissible ordering is the *purely lexicographic* ordering (lex).

Let us fix an admissible ordering $<$. By $\mathrm{lm}_< f$, we denote the *leading monomial* of polynomial $f \notin \mathcal{F}$. Consider a differential ideal I in $\mathcal{F}\{x\}$. A set $G \subset I$ is a *differential standard basis* of the ideal I if ΘG is an (infinite) algebraic Gröbner basis of the ideal I considered in the ring $\mathcal{F}[x_0, x_1, \ldots]$. A differential standard basis is called *reduced* if any polynomial $g \in G$ is reduced with respect to $\Theta(G\backslash\{g\})$.[3]

Definition 1 [9]. An admissible ordering $<$ is called δ-*lexicographical* if the following equivalent conditions are satisfied:

• $\mathrm{lm}_<\delta M = \mathrm{lm}_{\mathrm{lex}}\delta M$ for any monomial $M \neq 1$;

• for any monomial M and any $n > 0$, all monomials present in the polynomial $\delta^n M$ are compared with respect to $<$ in the same way as for the lexicographical ordering;

• $x_i x_j < x_{i-1}x_{j+1}$ for any $i \leq j$; i.e., homogeneous monomials of degree 2 are compared with respect to $<$ lexicographically.

Examples of δ-lexicographical orderings are the following: purely lexicographical ordering (lex); first, by degree, then lexicographical ordering (deglex); and, first by weight, then lexicographical ordering (wt-lex).

Definition 2. A polynomial f is *quasilinear* with respect to an admissible ordering $<$ if $\deg \mathrm{lm}_< f = 1$.

The δ-lexicographical orderings possess remarkable properties that allow us to prove a finiteness criterion for differential standard bases [9]. First, this is the *concordance with quasi-linearity*: the derivative of any quasilinear polynomial is quasilinear. Second, the δ-*fixity* of the ordering: for any polynomial $f \notin \mathcal{F}$, there are a monomial M and an index r such that, starting from a certain order of differentiation, the equality $\mathrm{lm}_<\delta^n f = Mx_{r+n}$ holds.

Theorem 1 (finiteness criterion for differential standard bases (see [9, 10])). *Let $<$ be a δ-lexicographical ordering and I a proper differential ideal of $\mathcal{F}\{x\}$. The ideal I has a finite differential standard basis G with respect to $<$ if and only if I contains a $<$-quasilinear polynomial.*

Example. The ideal $[x_1^3 + x]$ has the finite standard lex-basis

$$\{x_1^3 + x_0;\ 3x_0x_2 - x_1^2;\ 3x_1x_2^2 + x_2;\ x_3 - 3x_2^3\}.$$

We see that it contains the quasilinear polynomial $x_3 - 3x_2^3$.

On Theorem 1, the completion process proposed by the authors in paper [9] is based.[4] The **DiffComplete**(G, r) algorithm used in this process returns the set consisting of all elements of G and their derivatives which lie in the ring $\mathcal{F}[x_0, x_1, \ldots, x_r]$. The **DiffAutoreduce**$(G, <)$ algorithm reduces each element g of the set G with respect to the elements of $G\backslash\{g\}$ and their derivatives. **GröbnerBasis**$(H, <)$ returns the reduced

[1] We restrict our consideration to only finitely generated ideals, because the membership problem for infinitely generated differential ideals is, in the general case, undecidable [4].

[2] See http://www-sop.inria.fr/cafe/Evelyne.Hubert/diffalg.

[3] Here, we mean the ordinary algebraic reduction of polynomials.

[4] This process is referred to as the *improved Ollivier process.* In contrast to the original process of Ollivier, this process surely stops if the ideal has a finite basis.

Gröbner basis of the ideal (H) with respect to $<$, and **ContainsQuasiLinearPolynomial**$(G, <)$ checks whether G contains a $<$-quasilinear polynomial.

Improved Ollivier process

Input: $F \subset \mathcal{F}\{x\}$ is a finite set of polynomials; $<$ is a δ-fixed ordering concordant with quasi-linearity.

Output: Differential Gröbner basis of the ideal $[F]$ if it is finite (otherwise, the process does not stop).

$G := F$; $H := \varnothing$;
$s := \max_{f \in F} \operatorname{ord} f$; $k := 0$;
repeat
  $G_{old} := \varnothing$
  while $G \neq G_{old}$ **do**
    $G_{old} := G$;
    $G :=$ **GröbnerBasis** $(H, <)$;
  end do;
  $k := k + 1$;
until $G \subset \mathcal{F}$ or
ContainsQuasiLinearPolynomial $(G, <)$;
return DiffAutoreduce $(G, <)$;

Theorem 2 [9]. *Let $<$ be a δ-fixed ordering which is concordant with quasi-linearity.*[5] *The improved Ollivier process stops if and only if* $[F]$ *has a finite differential standard basis with respect to $<$. In this case, it returns the reduced differential standard basis of the ideal* $[F]$.

The following fact should be noted [9]: if an ideal has a finite differential standard basis for a δ-lexicographical ordering, then it has such a basis for the purely lexicographical ordering too.

Now, consider orderings that are not δ-lexicographical.

Example. By wt-revlex, we denote the ordering such that monomials are compared, first, by their weights and, then, by the reverse lexicographical ordering. In other words, if $p = x^{\alpha_0} \dots x_k^{\alpha_k}$ and $q = x^{\beta_0} \dots x_k^{\beta_k}$, then $p < q$ means that either $\mathrm{wt} p < \mathrm{wt} q$ or $\mathrm{wt} p = \mathrm{wt} q$ and there exists an index i such that $\alpha_0 = \beta_0, \dots, \alpha_i = \beta_i$ and $\alpha_{i+1} > \beta_{i+1}$. The degrevlex ordering differs from this ordering in that, first, the degrees of monomials are compared.

Such orderings are of interest, because, contrary to the δ-lexicographical orderings, the set of ideals with finite differential standard bases with respect to these orderings is not limited by ideals containing quasilinear polynomials. For instance, the differential standard bases of the ideal x^p with respect to degrevlex and wt-revlex consist of a single element (see [8]).

Conjecture. *An ideal I has a finite standard basis with respect to the orderings* $<$ = degrevlex *or* $<$ = wt-degrevlex *if and only if either I contains a $<$-quasilinear polynomial or $I = [f^p]$, where f is a $<$-quasilinear polynomial.*

The sufficiency of the conjecture assumption is not difficult to prove.[6] Whether it is necessary has not been proved yet. Note also that it is not clear whether the sufficiency holds in the case of partial derivatives. As for the purely lexicographical ordering, Carrà Ferro has proved [10] that the presence of a quasilinear polynomial is necessary for ideals with a finite standard basis regardless of the number of derivation operators.

Of course, there are much more examples where the standard basis is infinite than those with a finite basis. However, in some cases, a parametrical differential standard basis can be constructed. Moreover, sometimes, the membership problem for a particular ideal can be solved by representing this ideal as an intersection of ideals with finite standard bases.

Example. Consider $I = [xx_1]$. Standard bases of the ideal I with respect to lex and degrevlex are infinite; however, we can represent I as an intersection of ideals with finite standard bases, namely, $I = [x_2] \cap [x_1]$. It is also interesting that the ideal I has a finite *normalizing system* [7] consisting of a single element. This means that any element belonging to the ideal I is reducible to 0 in two stages; i.e., its remainder with respect to the system $\Theta(xx_1)$ for the degrevlex-ordering is reducible to 0 with respect to the lex-ordering.

5. COMPOSITION OF DIFFERENTIAL POLYNOMIALS

Let $f \in \mathcal{F}\{y\}$ and $g \in \mathcal{F}\{x\}$. The *composition* $f \circ g$ is defined to be the image of the polynomial f under the differential homomorphism of substitution ϕ: $\mathcal{F}\{y\} \longrightarrow \mathcal{F}\{x\}$ such that $y \mapsto g$. It is clear that any such composition can be extended to the composition $\mathcal{F}\{x\}\{y\} \longrightarrow \mathcal{F}\{x\}$.

Definition 3 ([1, p. 63]). The (partial) *preparation polynomial* $p_{g,n} \in \mathcal{F}\{x\}\{y\}$ of a polynomial $p \in \mathcal{F}\{x\}$ with respect to a polynomial $g \in \mathcal{F}\{x\}$ can be determined by the equality $S_g^n p = p_{g,n} \circ g$ and the condition that all coefficients (which lie in $\mathcal{F}\{x\}$) of the polynomial $p_{g,n}$ are partially pseudoreduced with respect to g and are not divisible by g.

Ritt proved (see [1, p. 64]) that, for any p and g, there is an exponent n such that $p_{g,n}$ exists. The proof is constructive; therefore, we may claim that there exists an algorithm that finds a number n and the preparation polynomial $p_{g,n}$ for given polynomials p and g. This algorithm is based on $\operatorname{ord} p - \operatorname{ord} g$ differentiations of the polynomial g and algebraic calculations in the ring $\mathcal{F}[x_0, \dots, x_n]$, where $n = \operatorname{ord} p$. In the same monograph, it is proved that, for any n such that $p_{g,n}$ exists, the partial preparation polynomial is unique. The partial preparation polynomial differs from the preparation polyno-

[5] For instance, the δ-lexicographical ordering.

[6] See http://shade.msu.ru/difalg/DSB.

mial in the Kolchin sense by the fact that p is not multiplied by I_g.

Obviously, if $p_{g,n}$ exists, then $p_{g,k}$ also exists for all $k \geq n$. Indeed, being multiplied by the separant S_g, coefficients of $p_{g,n}$ remain partially pseudoreduced with respect to g (though, in the general case, they may become divisible by g; hence, the general form of $p_{g,n+1}$ may be changed). Therefore, when determining the partial preparation polynomial for $S_g p_{g,n}$, no additional multiplication by S_g is needed, and one obtains $p_{g,n+1}$.

The number $n \geq 0$ is the *index* of the partial preparation polynomial $p_{g,n}$. The partial preparation polynomial with the minimal possible index will also be denoted simply by p_g.

It is clear that, if g is irreducible and n is the index of p_g, then $p_{g,n+k} = S_g^k p_g$.

Suppose that an ideal $[f]$ is solvable. Our goal is to investigate the solvability of the ideal $[f \circ g] : S_g^\infty$.

Theorem 3. *Let $f \in \mathcal{F}\{y\}$. Then,*

$$p \in [f \circ g]: S_g^\infty \Leftrightarrow p_{g,n} \in [f]$$

for a certain n, where the ideal $[f]$ is considered in the ring $\mathcal{F}\{x\}\{y\}$.

Proof. One can easily see that

$$[f \circ g] = ((\delta^i f) \circ g, i \geq 0)$$

as an algebraic ideal in $\mathcal{F}\{x\}$.

Let $p_{g,n} \in [f] \subset \mathcal{F}\{x\}\{y\}$. Then, $p_{g,n} = \sum_{i=0}^{k} q_i \delta^i f$, where $q_i \in \mathcal{F}\{x\}\{y\}$. Hence,

$$S_g^n p = p_{g,n} \circ g = \sum_{i=0}^{k} (q_i \circ g)(\delta^i f) \circ g \in [f \circ g].$$

Therefore, $p \in [f \circ g]: S_g^\infty$.

Conversely, let $p \in [f \circ g]: S_g^\infty$. Then, for a certain s, we have $S_g^s p = \sum_{i=0}^{l} r_i ((\delta^i f) \circ g)$, where $r_i \in \mathcal{F}\{x\}$. Let us set

$$\tilde{p} = \sum_{i=0}^{l} (r_i)_{g,t} \delta^i f \in [f] \subset \mathcal{F}\{x\}\{y\},$$

where t is the common index for all preparation polynomials constructed for coefficients r_i. Then, all coefficients from $\mathcal{F}\{x\}$ of the polynomial $\tilde{p}$ are coefficients from $\mathcal{F}\{x\}$ of the polynomials $(r_i)_{g,t}$; hence, they are partially pseudoreduced with respect to g and indivisible by g. Moreover,

$$\tilde{p} \circ g = \sum_{i=0}^{l} S_g^t r_i ((\delta^i f) \circ g) = S_g^{s+t} p.$$

By virtue of the uniqueness of the partial preparation polynomial, this implies that $p_{g,s+1} = \tilde{p} \in [f]$.

Corollary 1. *Let g be an irreducible polynomial. Then,*

$$p \in [f \circ g]: S_g^\infty \Leftrightarrow p_g \in [f];$$

i.e., the theorem holds true even for p_g.

Proof. Let n be the index of p_g. In the last equality of the theorem, we have

$$p_{g,s+t} = S_g^{s+t-n} p_g = \tilde{p} \in [f].$$

Hence, $p_g \in [f]: S_g^\infty$. However, $f \in \mathcal{F}\{y\}$, while $S_g \in \mathcal{F}\{x\}$. Therefore, $[f]: S_g^\infty = [f]$ in $\mathcal{F}\{x\}\{y\}$.

Remark 1. The condition that g is irreducible is essential. Indeed, let $f = y^2$ and $g = x^2$. Then, $f \circ g = x^4$ and $[f \circ g]: S_g^\infty = [1]$. However, for $p = x$, we have $p_g = p_{g,0} = x \notin [f]$. The minimal admissible index is $n = 3$: $p_{g,3} = y^2 \in [f]$.

Corollary 2. *Let g be an irreducible polynomial and p is such that p_g has index n. Then,*

$$p \in [f \circ g]: S_g^n \Leftrightarrow p_g \in [f].$$

Corollary 3. *Let g be a lexicographically quasilinear polynomial, i.e., $S_g = 1$. Then,*

$$p \in [f \circ g] \Leftrightarrow p_g \in [f].$$

Proof. Indeed, in this case, g is irreducible and p_g has index 0 for any p.

Remark 2. In the case where $S_g = 1$ and $f = y^p$, we can construct an admissible ordering such that the standard basis of the ideal $[g^p]$ consists of a single element.

Remark 3. A similar theory may be developed for the preparation polynomials in the Kolchin sense, i.e., for the polynomials $p_{g,m,n}$ satisfying the property

$$I_g^m S_g^n p = p_{g,m,n} \circ g.$$

In this case, the main theorem asserts that

$$p \in [f \circ g]: H_g^\infty \Leftrightarrow p_{g,m,n} \in [f]$$

for some m and n. Here, the key role is played by the uniqueness condition for $p_{g,m,n}$ under fixed admissible m and n and the possibility of construction of the preparation polynomials with common indices for a certain finite set of polynomials.

Example. Consider the polynomials $f = y^2$ and $g = xx_1 + 1$. Note that the ideal $[xx_1 + 1]$ is solvable since it contains the quasilinear polynomial $x_2 - x_1^3$ and, hence, has a finite standard basis. By Corollary 1 of Theorem 3, the ideal $I = [(xx_1 + 1)^2]: x^\infty$ is also solvable. Consider, for instance, the polynomial $p = (x_2 - x_1^3)^2$. First, it lies in the radical of the ideal I. The preparation polynomial of p with respect to g is the polynomial

$$p_{g,2} = (y_1 - x_1^2 y)^2 = y_1^2 - 2x_1^2 y y_1 + x_1^4 y^2.$$

Since $y^2, yy_1 \in [y^2]$ and $y_1^2 \notin [y^2]$ (this can easily be verified, because $[y^2]$ has a finite standard degrevlex-basis), it follows that $p_{g,2} \notin [y^2]$ and, by Theorem 3, we have $p \notin I$. Now, we determine the preparation polynomial for $q = (x_2 - x_1^3)^3$:

$$x^3 q = (y_1 - x_1^2 y)^3 = y_1^3 - 3x_1^2 y y_1^2 + 3x_1^4 y^2 y_1 - x_1^6 y^3.$$

Here, we see that the preparation polynomial (considered as a differential polynomial in y with coefficients in the ring $\mathcal{F}\{x\}$) lies in the ideal $[y^2]$. By virtue of Theorem 3, this means that $(x_2 - x_1^3)^3 \in I$.

Note that the ideal I falls in none of the classes mentioned in the beginning of the paper: it is not radical, inhomogeneous, and has no finite standard basis (at least, for δ-lexicographical orderings). Thus, the method presented in this paper actually extends the totality of known classes of solvable ideals.

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