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Fuzzy logic and arithmetical hierarchy

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Abstract

The aim of the paper is to show that fuzzy propositional logic (Pavelka's extension of real-valued Łukasiewicz's propositional logic) is a simple and elegant logical calculus but, on the other hand, it is badly undecidable (more undecidable than the classical Boolean propositional calculus).

Keywords: Logic; Completeness; Undecidability

0. Introduction

Following Zadeh let us distinguish fuzzy logic in broad sense (everything concerning fuzziness) from fuzzy logic in narrow sense (the formal logical calculus of fuzziness). This paper concerns the latter notion. We are going to investigate real-valued Łukasiewicz's logic (propositional calculus) with primitive connectives $\rightarrow, \neg$ (implication and negation) as well as the extension of Łukasiewicz's logic introduced and investigated by Pavelka [6]. Pavelka's papers are early, profound, important but poorly known. The two main ingredients are

1. investigation of fuzzy theories (fuzzy axiom systems) and
2. explicit work with propositional constants $\bar{r}$ for each $r \in [0,1]$ ($\bar{r}$ having constantly the truth value r) and graded proofs (proving the conclusion in some degree).

Pavelka's completeness theorem is formulated below and so is Rose–Rosser's completeness theorem for Łukasiewicz logic. Pavelka appears to have been ignorant of Rose–Rosser's theorem; the mere use of Rose–Rosser brings some simplification. We can also avoid the use of some rather tricky axioms of Pavelka. Moreover, we shall show that we can get rid of all constants $\bar{r}$ for irrational r and also of all irrationals in graded proofs. This makes proofs become finitary objects and, besides the achieved simplification, enables us to subject basic notions of fuzzy logic to a classical analysis by means of recursion theory. We show that for any recursive fuzzy theory T the corresponding notion of consequence is Π_2 and that for some recursive fuzzy theory T its notion of consequence is not simpler than Π_2 (is Π_2-complete). *Note that the reader interested only in the simple axiom system of fuzzy logic and the corresponding completeness proof may omit everything concerning recursion theory, Σ_1, Π_2 etc.*

Digression. For a reader not familiar with the arithmetical hierarchy we present a quick survey.

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A relation $R(n_1, \ldots, n_k)$ (on natural numbers or objects coded by natural numbers, like formulas, rational numbers, etc.) is recursive if there is an algorithm deciding for each input $(n_1, \ldots, n_k)$ whether $R(n_1, \ldots, n_k)$ holds or not. A relation $S(n_1, \ldots, n_{k-1})$ is Σ_1 (recursively enumerable) if there is a recursive relation $R(n_1, \ldots, n_k)$ such that, for each $n_1, \ldots, n_{k-1}$

$S(n_1, \ldots, n_{k-1})$ iff $(\exists n_k) R(n_1, \ldots, n_k)$.

A relation $T(n_1, \ldots, n_{k-2})$ is Π_2 if there is a recursive relation $R(n_1, \ldots, n_k)$ such that, for each $n_1, \ldots, n_{k-2}$

$T(n_1, \ldots, n_{k-2})$ iff $(\forall n_k)(\exists n_{k-1}) R(n_1, \ldots, n_k)$.

This shows how to define Σ_m and Π_m relations for each m; but we need only Σ_1 and Π_2. A relation T is Π_2-complete if it is Π_2 and each Π_2-relation is recursively reducible to T. (The exact definition can be derived from the proof below or found in [7].) If T is Π_2-complete then T is neither Σ_1 nor Π_1 (nor recursive). Note that sets are identified with unary relations.

In Boolean logic one shows that if T is a recursive theory (=set of axioms) then the set of all formulas provable in T is Σ_1 but for some such T the set of all formulas provable in T is not recursive (end of digression).

For the sake of completeness we mention the work of Biacino-Gerla [1]; they elaborate a generalization of Σ_1 sets that applies to subsets of the real interval $[0, 1]$ and analyze fuzzy logic using this notion. (cf. also, [4]; thanks are due to Novák for calling my attention to the work of Biacino–Gerla). Their work is of independent interest; but we stress the use of classical recursion theory and precise estimation of the complexity of fuzzy consequence by means of the usual arithmetical hierarchy.

Another related work is [5], where it is shown that one may get rid of all propositional constants $\bar{r}$ except one when a new unary connective $\surd$ is introduced. This is a nice result, but one may consider rational truth constants to be more natural than a "square-root" connective lacking natural interpretation; clearly, this is rather a matter of taste.

1. Formulas, truth-tables, tautologies

Formulas are built from propositional variables and connectives $\neg$, $\rightarrow$; other connectives ($\wedge$, $\vee$, &, $\underline{\vee}$) are understood as abbreviations. (Another choice of primitive connectives is $\neg$, $\underline{\vee}$). The "truth tables" are $\neg x = 1 - x$, $x \rightarrow y = \min(1, 1 - x + y)$ for $x, y \in [0, 1]$. (We use the same symbols for connectives and the corresponding truth functions; thus $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$, $x \& y = \max(0, x + y - 1)$, $x \underline{\vee} y = \min(x + y, 1)$. The notion of 1-tautologies is clear; basics can be found in [2]. In particular, as Rose–Rosser [8] proved, the following is a complete axiom system for 1-tautologies (with the only deduction rule being modus ponens):

$$\phi \rightarrow (\psi \rightarrow \phi),$$

$$(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi)),$$

$$(\neg \psi \rightarrow \neg \phi) \rightarrow (\phi \rightarrow \psi),$$

$$((\phi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \phi) \rightarrow \phi).$$

The 1-tautology problem (i.e. the provability problem for the Rose–Rosser system) is *decidable*, i.e. the set of all 1-tautologies is recursive; as Gottwald [2] shows, this is proved via Tarski's famous decidability result for real algebra.

Consider (crisp) axiomatic theories in Łukasiewicz logic. One apparent obstacle is the fact that the usual deduction theorem ($\phi \vdash \psi$ iff $\vdash \phi \rightarrow \psi$) fails in Łukasiewicz's logic; Novák [3] presents a deduction theorem of the following form: $\phi \vdash \psi$ iff for some n, $\vdash \phi^n \rightarrow \psi$ (where ϕ^n is $\phi \& \cdots \& \phi, n$ times, and $\phi \& \psi$ is $\neg(\phi \rightarrow \neg \psi)$). Recall *strong completeness* of theories in Boolean two-valued logic: if T is a set of special axioms then $T \vdash \phi$ iff ϕ is a semantic consequence of T, i.e. each evaluation of propositional atoms giving the value 1 to each element of T gives the value 1 to ϕ. But this is not true for Łukasiewicz's logic.

Lemma. *Strong completeness does not hold for arbitrary (possibly infinite) theories T in Łukasiewicz's logic.*

Proof. Let $n\phi$ be $\phi \veebar \cdots \veebar \phi$ (n times), let $T = \{n\phi \to \psi \mid n \in N\} \cup \{\neg\phi \to \psi\}$, then $T \models \psi$ but, for each finite theory $T_0 \subseteq T$, $T_0 \not\models \psi$. □

Remark. This shows that one possible formulation of the compactness theorem (if ψ is a consequence of T then it is a consequence of a finite subtheory of T) is false in fuzzy logic. But note that another formulation, equivalent to the former one in Boolean logic, is true for fuzzy logic, namely if for each finite $T_0 \subseteq T$, $(T_0 + \psi)$ has a model then $(T + \psi)$ has a model. This is easy to prove by the techniques of Section 2; see also [4].

Theorem. There is an infinite recursive theory T such that the set of all T-provable formulas is not recursive.

Proof (Easy, holds also for two-valued logic). Let $R(m,n)$ be recursive such that the set $\{m \mid (\exists n) R(m,n)\}$ is not recursive. Let

$$T = \{p_{m,n} \mid R(m,n)\} \cup \{p_{m,n} \to q_m \mid m, n \in \mathbb{N}\}$$

($\mathbb{N}$ is the set of all natural numbers); then $T \vdash q_m$ iff $(\exists n) R(m,n)$. Needless to say, if T is recursive then the set of all T-provable formulas is Σ_1 (recursively enumerable).

2. Fuzzy theories, Pavelka's theorem

Pavelka [6] deals with fuzzy theories, i.e. fuzzy sets of formulas. His formulas are built from propositional connectives and propositional constants $\bar{a}$ for each $a \in [0,1]$ using connectives. If F is a fuzzy theory and e is an evaluation of propositional variables then e is said to *respect* F iff for each formula ϕ, $e(\phi) \geqslant F(\phi)$. (Clearly, $F(\phi)$ is the degree of membership of the formula ϕ into the fuzzy axiom set F.) Following Novák's notation we write $F \models_a \phi$ if $a = \inf\{e(\phi) \mid e$ respects $F\}$. Pavelka presents a fuzzy set of logical axioms; combining his approach with that of Rose and Rosser we may define the axioms as follows:

Rose–Rosser's axioms in degree 1
$\bar{a}$ in degree a for each $a \in [0,1]$
$\neg\bar{a} \leftrightarrow \overline{\neg a}$, $(\bar{a} \to \bar{b}) \leftrightarrow \overline{a \to b}$ in degree 1
(Pavelka has also some other axioms but they are superfluous by our results)
every other formula in degree 0.

A *graded formula* is a pair (ϕ, r) where ϕ is a formula and $r \in [0,1]$; a *graded proof* from F is a finite sequence of graded formulas such that each element either is a logical axiom or a special axiom (i.e. a pair (ϕ, r) where $F(\phi) = r$) or follows from some previous elements using one of the following graded deduction rules:

- from (ϕ, r) and $(\phi \to \psi, s)$ infer $(\psi, r \& s)$,
- from (ϕ, r) deduce $(\bar{s} \to \phi, s \to r)$.

We say that ϕ has a proof from F of value r (notation: $F \vdash^{\circ}_{r} \phi$, F is a fuzzy set of axioms) if there is a proof from F whose last element is (ϕ, r). $F \vdash_r \phi$ means that $r = \sup_s (F \vdash^{\circ}_{s} \phi)$.

Pavelka's completeness theorem. $F \vdash_r \phi$ iff $F \models_r \phi$.

If we want to analyze the arithmetical complexity of the notions involved it is preferable to deal with finitary objects. A graded proof with irrational truth-values is difficult to handle as a finitary object; but if all values are rational then there is no problem. Thus let us change our syntax: *eliminate* constants $\bar{r}$ for irrational r and consider only graded formulas, fuzzy axiom systems and graded proofs with rational truth values. The resulting logic may be called *Rational-valued Pavelka's Logic* RPL. (Caution: We continue to consider general real-valued evaluations of atoms, semantics is not changed!) From now on, $F \vdash^{\circ}_{r} \phi$ means that F is a rational-valued fuzzy theory and ϕ has a RPL-proof from F of value r.

Theorem (Pavelka's completeness revisited). *If F is a rational-valued fuzzy theory, ϕ is a formula and $F \models_r \phi$ then $r = \sup\{s$ rational $\mid F \vdash^{\circ}_{s} \phi\}$, i.e. r is the supremum of values of rational proofs of ϕ from F.*

To prove this one shows that for r such that $F \models_r \phi$:
(a) if $F \vdash^{\circ}_{s} \phi$ then $s \leqslant r$ (soundness),
(b) if not $F \vdash^{\circ}_{s} \phi$ then $r \leqslant s$.

We sketch a proof of (b) in the rest of this section ((a) being elementary).

Definition (cf. Novák [3]). A theory F is *contradictory* (or inconsistent) if for some rational $a > 0$, $F \vdash^{\circ}_{a} \bar{0}$; otherwise it is *consistent*.

Lemma (cf. [3]). (1) *F is contradictory iff for each formula ϕ, $F \vdash_1 \phi$.*

(2) *If $F \nvdash^{\circ}_{a} \phi$ then $F \cup \{(\phi \to \bar{a}, 1)\}$ is consistent.*

Definition. F is *complete* if for each rational a and each formula ϕ, $F \vdash^{\circ}_{1} \bar{a} \to \phi$ or $F \vdash^{\circ}_{1} \phi \to \bar{a}$.

Lemma. *For each consistent F there is a stronger theory $\hat{F}$ which is consistent and complete.*

Proof. Arrange all pairs (ϕ, a) (ϕ a formula, a rational) into a sequence $p_0, p_1, p_2, \ldots$ of pairs. Put $F_0 = F$; if F_n is defined and p_n is (ϕ, a), either $F_n \vdash^{\circ}_{1} (\bar{a} \to \phi)$ and $F_{n+1} = F_n$ or else $F_{n+1} = F_n \cup \{(\phi \to \bar{a}, 1)\}$ (which means that $F_{n+1}(\psi) = F_n(\psi)$ for ψ different from $\phi \to \bar{a}$, $F_{n+1}(\phi \to \bar{a}) = 1$). We show that F_{n+1} is consistent if F_n is (cf. the preceding lemma). Assume F_n consistent and $F_n \cup \{(\phi \to \bar{a}, 1)\}$ inconsistent, thus $F_n \cup \{(\phi \to \bar{a}, 1)\} \vdash^{\circ}_{1} \bar{0}$, hence for some k, $F_n \vdash^{\circ}_{1} (\phi \to \bar{a})^k \to \bar{0}$, thus $F_n \vdash^{\circ}_{1} \neg(\phi \to \bar{a})^k$. On the other hand, $F_n \vdash^{\circ}_{1} (\phi \to \bar{a})^k \veebar (\bar{a} \to \phi)^k$ and $F_n \vdash^{\circ}_{1} (\bar{a} \to \phi)^k$ (due to obvious 1-tautologies of Łukasiewicz's logic) and hence $F_n \vdash^{\circ}_{1} (\bar{a} \to \phi)$. Thus each F_n is consistent and so is $\hat{F}$ defined thus: $\hat{F}(\psi) = \max\{F_n(\psi) \mid n \in N\}$ (observe that if $\hat{F} \vdash^{\circ}_{1} \phi$ then for some $n \in \mathbb{N}$, $F_n \vdash^{\circ}_{1} \phi$). Moreover, $\hat{F}$ is complete. □

Definition. If F is consistent and complete, for each ψ, let $e(\psi) = \alpha$ iff $F \vdash_{\alpha} \psi$. Note that

$$e(\psi) = \sup\{a \mid F \vdash^{\circ}_{1} (\bar{a} \to \psi)\}$$
$$= \inf\{a \mid F \vdash^{\circ}_{1} (\psi \to \bar{a})\}.$$

Lemma. *If F is consistent and complete and e is as above then e is a valuation,* i.e. $e(\neg\psi) = 1 - e(\psi)$, $e(\psi \to \chi) = (e(\psi) \to e(\chi))$.

Proof. (1) $e(\neg\psi) = \sup\{t \mid F \vdash^{\circ}_{1} \bar{t} \to \neg\psi\} = \sup\{t \mid F \vdash^{\circ}_{1} \psi \to \overline{(1-t)}\} = \sup\{(1-s) \mid F \vdash^{\circ}_{1} \psi \to \bar{s}\} = 1 - \inf\{s \mid F \vdash^{\circ}_{1} \psi \to \bar{s}\} = 1 - e(\psi)$.

(2) $e(\psi) \to e(\chi) = \inf\{r \mid F \vdash^{\circ}_{1} \psi \to \bar{r}\} \to \sup\{s \mid F \vdash^{\circ}_{1} \bar{s} \to \chi\} = \sup\{r \to \sup\{s \mid F \vdash^{\circ}_{1} \bar{s} \to \chi\} \mid F \vdash^{\circ}_{1} \psi \to \bar{r}\} = \sup\{r \to s \mid F \vdash^{\circ}_{1} \psi \to \bar{r}, F \vdash^{\circ}_{1} \bar{s} \to \chi\} \leqslant e(\psi \to \chi)$, since $F \vdash^{\circ}_{1} \psi \to \bar{r}$ and $F \vdash^{\circ}_{1} \bar{s} \to \chi$ implies $F \vdash^{\circ}_{r \to s} \psi \to \chi$. Conversely, assume $(e(\psi) \to e(\chi)) < t < t_1 < e(\psi \to \chi)$ for some rational t, t_1. Take $r < e(\psi)$, $s > e(\chi)$ such that $t = (r \to s)$. Then $F \vdash^{\circ}_{1} \bar{r} \to \psi$ and $F \vdash^{\circ}_{1} \chi \to \bar{s}$, which implies $F \vdash^{\circ}_{1} (\psi \to \chi) \to \overline{(r \to s)}$; but $F \vdash^{\circ}_{1} \bar{t}_1 \to (\psi \to \chi)$, thus $F \vdash^{\circ}_{1} \bar{t} \leftrightarrow \bar{t}_1$ and hence F is contradictory. (If $a < 1$ and $F \vdash^{\circ}_{1} \bar{a}$ then $F \vdash^{\circ}_{1} \bar{a}^n$ $(a \& \cdots \& a)$ for each n, hence $F \vdash^{\circ}_{1} \bar{0}$. Take $a = 1 - t_1 + t$.)

Corollary. *If $F \nvdash^{\circ}_{a} \phi$ then there is an evaluation e respecting F and such that $e(\phi) \leqslant a$. This completes the completeness proof.*

3. Relation to arithmetical hierarchy

Remark. Obviously, the relation $Pr^F_o = \{(\phi, r) \mid F \vdash^{\circ}_{r} \phi\}$ is $\Sigma_1(F)$ (recursively enumerable in F); thus if F is recursive then Pr^F_o is recursively enumerable.

Lemma. *If F is a recursive rational theory then the relation $Pr^F = \{(\phi, r) \mid F \vdash_r \phi$, r rational$\}$ is Π_2.*

Proof. $F \vdash_r \phi$ iff $(\forall s < r)(F \vdash^{\circ}_{s} \phi)$ and $(\forall s > r)(F \nvdash^{\circ}_{s} \phi)$ (r, s ranging over rationals from $[0,1]$). Now the condition $F \vdash^{\circ}_{s} \phi$ is Σ_1, thus $(\forall s < r)(F \vdash^{\circ}_{s} \phi)$ is Π_2, whereas $(\forall s > r)(F \nvdash^{\circ}_{s} \phi)$ is Π_1; thus $F \vdash_r \phi$ is Π_2. □

Theorem. *There is an infinite recursive rational theory F such that the relation*

$Pr^F = \{(\phi, r) \mid F \vdash_r \phi, r$ *rational*$\}$

is not recursively enumerable (is Π_2-complete).

Proof. We shall exhibit an F such that the set $\{\phi \mid F \vdash_1 \phi\}$ is Π_2-complete. Let R be a recursive relation such that the set $A = \{k \mid (\forall z)(\exists y > z) \times R(k, y)\}$ is Π_2-complete (see [7]). Let $c_m = (m-1)/m$ and take the following (crisp) axiom system for F (k, l, m are natural numbers):

$\overline{c_m} \to p_{km}$ for $R(k, m)$,

$p_{km} \to q_k$.

Then $F \vdash_1 q_k$ iff $k \in A$. □

Remark. We have seen that for a recursive F the set Pr^F may be very ineffective, even not recursively enumerable. (In the language of Turing degrees of undecidability, we have exhibited a recursive fuzzy theory F such that for any recursive classical Boolean theory T, the Turing degree of Pr^F is strictly greater than the Turing degree of the set of formulas provable in T.) Call the r such that $F \vdash_r \phi$ the *provability limit* of ϕ (from F). Observe that for each positive rational r we can find a recursive F such that the set of all ϕ with provability limit in the closed interval $[r, 1]$ is Π_2-complete. On the other hand, the provability limit of ϕ is strictly bigger than a given $r < 1$ iff there is an $r' > r$ such that ϕ has a rational proof of value r' (i.e. $F \vdash^{\circ}_{r'} \phi$). Thus for rational $r < 1$ the set of formulas with provability limit bigger than r is recursively enumerable. This should be compared with the fact that for each rational $r < 1$, the set of all $(r, 1]$-tautologies of Łukasiewicz's predicate logic is recursively enumerable whereas the set of all $[r, 1]$-tautologies of that logic is not recursively enumerable (see [2]).

A continuation of this paper, generalizing the present results to the fuzzy predicate calculus, is in final stages of preparation.

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