Exponentials with infinite multiplicities

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Abstract. Given a semi-ring with unit which satisfies some algebraic conditions, we define an exponential functor on the category of sets and relations which allows to define a denotational model of differential linear logic and of the lambda-calculus with resources. We show that, when the semi-ring has an element which is infinite in the sense that it is equal to its successor, this model does not validate the Taylor formula and that it is possible to build, in the associated Kleisli cartesian closed category, a model of the pure lambda-calculus which is not sensible. This is a quantitative analogue of the standard graph model construction in the category of Scott domains. We also provide examples of such semi-rings.
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Introduction

The category of sets and relations is a quite standard denotational model of linear logic which underlies most denotational models of this system (coherence spaces, hypercoherence spaces, totality spaces, finiteness spaces...). In this completely elementary setting, a formula is interpreted as a set, and a proof of that formula is interpreted as a subset of the set interpreting the formula.

Logical connectives are interpreted very simply: tensor product, par and linear implication are interpreted as cartesian products whereas direct product (with) and direct sums (plus) are interpreted as disjoint union. The linear negation of a set is the same set: it is a remarkable feature of linear logic that it admits such a "degenerate" semantics of types, which is nonetheless non trivial in the sense that proofs are not identified.

Exponentials are traditionally interpreted by the operation which maps a set X to the set of all finite multisets of elements of X (the origin of this idea can be found in [Gir88]). One might be tempted to use finite sets instead of finite multisets since, in the coherence space semantics, the exponential can be interpreted by an operation which maps a coherence space to the sets of its finite cliques (with a suitable coherence). In the relational model however, such an interpretation of the exponentials based on finite sets is not possible as it leads to a dereliction which is not natural (in the categorical sense).

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With this standard multiset-based interpretation of exponentials, the relational model interprets also the differential extensions of linear logic and of the lambda-calculus presented in [ER03,ER06b,EL10]. In the differential lambda-calculus, terms can be derived (differentiated): a term M of type $A \to B$ can be transformed into a term M' of type $A \to (A \to B)$ which is linear in its second parameter of type A (using a linear implication symbol " $\multimap$ ", the type of M' could be written $A \to (A \multimap B)$). The word "linear" can be taken here in its standard algebraic sense, or in its operational sense of "using its argument exactly once". This differentiation operation can be iterated, yielding a nth derivative $M^{(n)} : A \to (A^n \to B)$ which is n-linear in its n last arguments of type A, that is $M^{(n)} : A \to (A \otimes \cdots \otimes A \multimap B)$. The introduction of this new construction requires the possibility of freely adding terms of the same type: in the model **Rel**, this addition operation is interpreted as set union (remember that terms as interpreted as subsets of the interpretations of types). Also, each type has to contain a 0 element which, here, is the empty set.

This strongly suggests to consider the following "Taylor series", given a term M of type $A \to B$ and a term N of type A: $\sum_{n=0}^{\infty} \frac{1}{n!} M^{(n)}(0) \cdot (N, \ldots, N)$. In this formula, the map $N \mapsto \frac{1}{n!} M^{(n)}(0) \cdot (N, \ldots, N)$ is the approximation of degree n of the function M, that is the "part" of the function M which uses its argument exactly n times. For simplifying the setting and for dealing easily with untyped terms, it is suitable to consider a version of that formula where coefficients are all equal to one, and where addition of terms is an idempotent operation: terms form a complete lattice and the Taylor expansion of M can be written more simply $\bigvee_{n=0}^{\infty} M^{(n)}(0) \cdot (N, \ldots, N)$.

With Regnier, the second author studied this operation in [ER08,ER06a], introducing a *lambda-calculus with resources* which can be seen as the differential lambda-calculus where ordinary[3] application can be used only for applying a term to 0: this is the only ordinary application needed if we want to Taylor expand all the applications occurring in lambda-terms. In these two papers we proved in an untyped setting that, Taylor expanding completely a lambda-term M, one obtains a (generally infinite) linear combination of resource terms and that, if one normalizes each resource term occurring in that formal sum[4], one obtains the Taylor expansion of the Böhm tree of M.

This result implies that, in a denotational model which validates the Taylor expansion formula in the sense that the interpretation of a term M is equal to the interpretation of its Taylor expansion, the interpretation of an unsolvable lambda-term[5] is necessarily equal to 0. Since the multiset-based exponential of **Rel** validates the Taylor expansion formula, any model of the pure lambda-

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[3] In the differential lambda-calculus, there are two kinds of application: the ordinary application of a term to an argument, and the application of the nth derivative of a term to a n-tuple of terms. This latter application is n-linear in its arguments whereas the former is not linear.

[4] Resource terms are strongly normalizing, even if they are not typeable.

[5] We recall that a term is solvable iff its head reduction terminates, see [Kri93, Chapter 4].

calculus in the corresponding cartesian closed category, such as the model presented in [BEM07,BEM09], seems to be bound to be sensible (at least if differential operations are interpreted in the standard way). This seems to be a serious limitation in the equational expressive power of this kind of semantics.

This problem arose during a general investigation undertaken by the authors, whose scope is to develop an algebraic setting for differential extensions of the lambda-calculus, in the spirit of [PS98,MS09].

Content. The present paper proposes a solution to this problem, by introducing new exponential operations on **Rel**. The idea is quite simple: we replace the set $\mathbb{N}$ of natural numbers (which are used for counting multiplicities of elements in multisets) with more general semi-rings which typically contain "infinite elements" ω such that $\omega+1=\omega$. *Mutatis mutandis*, the various structures of the exponentials (functorial action, dereliction etc) are interpreted as with the ordinary multiset-based exponentials. For these structures to satisfy the required equations, some rather restrictive conditions have to be satisfied by the considered semi-ring: the semi-rings which satisfy these conditions are called "multiplicity semi-rings". We show that such a semi-ring must contain $\mathbb{N}$ and we exhibit multiplicity semi-rings with infinite elements.

In these models with infinite multiplicities, the differential constructions are available, but the Taylor formula does not hold. It is possible to find morphisms $f : A \to B$ (in the associated cartesian closed category) which are $\neq 0$ but are such that, for all n, the nth derivative $f^{(n)}(0) : A^n \to B$ is equal to 0. The Taylor expansion of such a function is the 0 map, and hence the function is different from its Taylor expansion. This is analogous to the well known smooth (C^∞) map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(0) = 0$ and $f(x) = e^{-1/|x|}$ for $x \neq 0$: all the derivatives of f at 0 are equal to 0 and hence there is no neighborhood of 0 where f coincides with its Taylor expansion at 0. In some sense, f is infinitely flat at 0, and we obtain a similar effect with our infinite multiplicities.

For any multiplicity semi-ring which contains an infinite element, we build a model of the pure lambda-calculus, which is not sensible and, more precisely, where the term $\Omega = (\lambda x\,(x)\,x)\,\lambda x\,(x)\,x$ has a non-empty interpretation (we also exhibit a non solvable term whose interpretation is distinct from that of Ω). We use Krivine's notation for lambda-terms: the application of M to N is denoted as $(M)\,N$.

Warning. Most proofs are omitted and will be available in a longer version of this article.

1 The relational model of linear logic

Rel is the category whose objects are sets and with hom-sets $\mathbf{Rel}(X,Y) = \mathcal{P}(X \times Y)$. In this category, composition is the ordinary composition of relations: if $R \in \mathbf{Rel}(X,Y)$ and $S \in \mathbf{Rel}(Y,Z)$, then

$$S \cdot R = \{(a,c) \in X \times Z \mid \exists b \in Y \ (a,b) \in R \text{ and } (b,c) \in S\}\,.$$

and identities are the diagonal relations: $\mathsf{Id}_X = \{(a,a) \mid a \in X\}$.

This category has a well known symmetric monoidal structure (compact closed actually), with tensor product given on objects by $X_1 \otimes X_2 = X_1 \times X_2$ and on morphisms by

$$R_1 \otimes R_2 = \{((a_1,a_2),(b_1,b_2)) \mid (a_i,b_i) \in R_i \text{ for } i = 1,2\}$$

for any $R_i \in \mathbf{Rel}(X_i, Y_i)$ $(i = 1,2)$. The associativity and symmetry isomorphisms are the obvious bijections, the neutral object of the tensor product is the singleton set $1 = \{*\}$.

This monoidal category is closed: the object of morphisms from X to Y is $X \multimap Y = X \times Y$, with evaluation morphism $\mathsf{ev} \in \mathbf{Rel}((X \multimap Y) \otimes X, Y)$ given by $\mathsf{ev} = \{(((a,b),a),b) \mid a \in X,\, b \in Y\}$. Given $R \in \mathbf{Rel}(Z \otimes X, Y)$, the linear curryfication of R is $\mathsf{cur}(R) \in \mathbf{Rel}(Z, X \multimap Y)$. This category is star-autonomous, with dualizing object $\perp = 1$.

The category $\mathbf{Rel}$ is also cartesian: the cartesian product of a family of objects $(X_i)_{i\in I}$ is $\prod_{i\in I} X_i = \bigcup_{i\in I}(\{i\} \times X_i)$. The binary cartesian product of X and Y is denoted as $X \mathbin{\&} Y$ and the terminal object is $\top = \emptyset$. The projection $\pi_i \in \mathbf{Rel}(\prod_{i\in I} X_i, X_i)$ is $\pi_i = \{((i,a),a) \mid a \in X_i\}$ and, given a family $(R_i)_{i\in I}$ of morphisms $R_i \in \mathbf{Rel}(Y, X_i)$, the corresponding morphism $\langle R_i\rangle_{i\in I} \in \mathbf{Rel}(Y, \prod_{i\in I} X_i)$ is given by $\langle R_i\rangle_{i\in I} = \{(b,(i,a)) \mid i \in I \text{ and } (b,a) \in R_i\}$.

2 Exponentials

We present a way of building exponential functors, once a notion of multiplicity is given, as a semi-ring satisfying strong conditions.

2.1 Multiplicity semi-rings

Notational convention for indices. We shall use quite often multiple indices, written as subscript as in "a_{ijk}" which has three indices i, j and k. When there are no ambiguities, these indices will not be separated by commas. We insert commas when we use multiplication on these indices, as in "$a_{i,2j,k}$" for instance.

A semi-ring M is a *multiplicity semi-ring* if it is commutative, has a multiplicative unit and satisfies

(MS1) $\forall n_1, n_2 \in M \quad n_1 + n_2 = 0 \Rightarrow n_1 = n_2 = 0$ (we say that M is *positive*)
(MS2) $\forall n_1, n_2 \in M \quad n_1 + n_2 = 1 \Rightarrow n_1 = 0$ or $n_2 = 0$ (we say that M is *discrete*)
(MS3) $\forall n_1, n_2, p_1, p_2 \in M \quad n_1 + n_2 = p_1 + p_2 \Rightarrow \exists r_{11}, r_{12}, r_{21}, r_{22} \in M \quad n_1 = r_{11} + r_{12},\ n_2 = r_{21} + r_{22},\ p_1 = r_{11} + r_{21},\ p_2 = r_{12} + r_{22}$ (we say that M has the *additive splitting property*)
(MS4) $\forall m, p, n_1, n_2 \in M \quad pm = n_1 + n_2 \Rightarrow \exists p_1, p_2, m_{11}, m_{12}, m_{21}, m_{22} \in M \quad m_{11} + m_{21} = m_{12} + m_{22} = m,\ p_1 m_{11} + p_2 m_{12} = n_1,\ p_1 m_{21} + p_2 m_{22} = n_2$ and $p_1 + p_2 = p$ (we say that M has the *multiplicative splitting property*).

Remark 1. The motivation for Condition (MS4) is mainly technical: it is essential in the proof of Lemma 7. It has also an intuitive content, describing what happens when an element of M can be written both as a sum and as a product. The proof that this property holds in $\mathbb{N}$ is based on Euclidean division. We conjecture that this property is independent from Conditions (MS1), (MS2) and (MS3).

Generalized splitting properties. The splitting conditions are expressed in a binary way, we must generalize them to arbitrary arities. We first generalize Condition (MS3).

Lemma 1. *Let M be a semi-ring satisfying (MS1) and (MS3). Let $n_1, \ldots, n_l \in M$ and $p_1, \ldots, p_r \in M$ be such that $\sum_{i=1}^{l} n_i = \sum_{j=1}^{r} p_j$. Then there is a family $(s_{ij})_{i=1,j=1}^{l,r}$ of elements of M such that $\forall i \in \{1, \ldots, l\}$ $n_i = \sum_{j=1}^{r} s_{ij}$ and $\forall j \in \{1, \ldots, r\}$ $p_j = \sum_{i=1}^{l} s_{ij}$.*

Similarly, we generalize Condition (MS4).

Lemma 2. *Let M be a semi-ring satisfying (MS1), (MS3) and (MS4). Let $k \in \mathbb{N}$ with $k \neq 0$. Let $l = 2^{k-1}$. For all $n_1, \ldots, n_k, m, p \in M$, if $pm = n_1 + \cdots + n_k$, then there exist $(p_j)_{j=1}^{l} \in M$ and $(m_{ij})_{i=1,j=1}^{k,l}$ with*

- $p_1 + \cdots + p_l = p$
- $m_{1j} + \cdots + m_{kj} = m$ *for* $j = 1, \ldots, l$
- *and* $p_1 m_{i1} + \cdots + p_l m_{il} = n_i$ *for* $i = 1, \ldots, k$.

Proposition 1. *Any multiplicity semi-ring M contains an isomorphic copy of $\mathbb{N}$.*

We shall simply say that M contains $\mathbb{N}$, that is $\mathbb{N} \subseteq M$. In particular, a multiplicity semi-ring cannot be finite. An element m of a semi-ring will be said to be *infinite* if it satisfies $m = m + 1$.

Examples of multiplicity semi-rings. The elements of a multiplicity semi-ring should be considered as generalized natural numbers. We give here examples of such semi-rings.

Natural numbers. The most canonical example of multiplicity semi-ring is the set $\mathbb{N}$ of natural numbers, with the ordinary addition and multiplication. Of course, $\mathbb{N}$ has no infinite element.

Proposition 2. $\mathbb{N}$ *is a multiplicity semi-ring.*

Completed natural numbers. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\omega\}$ be the "completed set of natural numbers". We extend addition to this set by $n+\omega = \omega+n = \omega$, and multiplication by $0\omega = \omega 0 = 0$ and $n\omega = \omega n = \omega$ for $n \neq 0$, so that $\overline{\mathbb{N}}$ has exactly one infinite element, namely ω.

Proposition 3. $\overline{\mathbb{N}}$ *is a multiplicity semi-ring.*

A semi-ring with infinite and non-idempotent elements. A more interesting example is $\mathbb{N}_2 = (\mathbb{N}^+ \times \mathbb{N}) \cup \{0\}$. The element (n, d) of this set (with $n \neq 0$) will be denoted as $n\omega^d$. We extend this notation to the case where $n = 0$, identifying $0\omega^d$ with 0, which is quite natural with these notations. Addition is defined as follows (0 being of course neutral for this operation)

$$n\omega^d + n'\omega^{d'} = \begin{cases} (n+n')\omega^d & \text{if } d = d' \\ n\omega^d & \text{if } n \neq 0 \text{ and } d' < d \\ n'\omega^{d'} & \text{if } n' \neq 0 \text{ and } d < d' \end{cases}$$

and multiplication is defined by $n\omega^d n'\omega^{d'} = nn'\omega^{d+d'}$. This semi-ring has infinitely many infinite elements: all the elements $n\omega^d$ of $\mathbb{N}_2$ with $n \neq 0$ and $d \neq 0$ are infinite.

Proposition 4. *$\mathbb{N}_2$ is a multiplicity semi-ring.*

From now on, $\mathbb{M}$ denotes a multiplicity semi-ring.

2.2 The exponential functor

Given a set X, we define $!_{\mathbb{M}}X$ as the free $\mathbb{M}$-module $\mathbb{M}\langle X\rangle$ generated by X, that is, as the set of all functions $\mu : X \to \mathbb{M}$ such that $\mathsf{supp}(\mu) = \{a \in X \mid \mu(a) \neq 0\}$ (the *support* of μ) is finite. These functions will be called $\mathbb{M}$-*multisets* (of elements of X).

Given $a \in X$, we denote as $[a] \in !_{\mathbb{M}}X$ the function given by $[a](b) = \delta_{a,b}$ (the Kronecker symbol which takes value $0 \in \mathbb{M}$ if $a \neq b$ and $1 \in \mathbb{M}$ if $a = b$). We use the standard algebraic notations for denoting the operations in the $\mathbb{M}$-module $!_{\mathbb{M}}X$. If $\mu \in !_{\mathbb{M}}X$, we define the *cardinality of* μ by $\#\mu = \sum_{a\in\mathsf{supp}(\mu)} \mu(a) \in \mathbb{M}$.

Given $R \in \mathbf{Rel}(X, Y)$, we define $!_{\mathbb{M}}R \in \mathbf{Rel}(!_{\mathbb{M}}X, !_{\mathbb{M}}Y)$ as the set of all pairs (μ, ν) such that one can find $\sigma \in \mathbb{M}\langle X \times Y\rangle$ with $\mathsf{supp}(\sigma) \subseteq R$ and

$$\forall a \in X \quad \mu(a) = \sum_{b\in Y} \sigma(a, b) \quad \text{and} \quad \nu(b) = \sum_{a\in X} \sigma(a, b)\,.$$

We say then that σ *is a witness of* (μ, ν) *for* R. Observe that all these sums are finite because $\sigma \in \mathbb{M}\langle X \times Y\rangle$.

It is clear from this definition that $!_{\mathbb{M}} \mathsf{Id}_X = \mathsf{Id}_{!_{\mathbb{M}}X}$. Let $R \in \mathbf{Rel}(X, Y)$ and $S \in \mathbf{Rel}(Y, Z)$. We denote as $S \cdot R \in \mathbf{Rel}(X, Z)$ the relational composition of R and S.

Lemma 3. *$!_{\mathbb{M}}(S \cdot R) = !_{\mathbb{M}}S \cdot !_{\mathbb{M}}R$.*

Proof. This is essentially an application of Lemma 1. □

Lemma 4. *Let $R \subseteq X \times Y$ and let $(\mu_i, \nu_i) \in !_{\mathbb{M}}R$ and $p_i \in \mathbb{M}$ for $i = 1, \dots, n$. Then $(\sum_{i=1}^n p_i\mu_i, \sum_{i=1}^n p_i\nu_i) \in !_{\mathbb{M}}R$.*

Proof. For each i, choose a witness σ_i of (μ_i, ν_i) for R. Then $\sum_{i=1}^n p_i\sigma_i$ is a witness of $(\sum_{i=1}^n p_i\mu_i, \sum_{i=1}^n p_i\nu_i)$ for R. □

2.3 Comonad structure of the exponential

We introduce the fundamental comonadic structure of the exponential functor, which consists of two natural transformations usually called *dereliction* (the counit of the comonad) and *digging* (the comultiplication of the comonad).

Dereliction. We set $\mathsf{d}_X = \{([a], a) \mid a \in X\} \in \mathbf{Rel}(!_{\mathbb{M}}X, X)$.

Lemma 5. *d_X is a natural transformation from $!_{\mathbb{M}}$ to Id.*

Proof. One applies Conditions (MS1) and (MS2). □

Remark 2. One could consider taking $\mathbb{M} = \{0, 1\}$ with $1 + 1 = 1$, and then we would have $!_{\mathbb{M}}X = \mathcal{P}_{\text{fin}}(X)$, the set of all finite subsets of X. But this semiring does not satisfy Condition (MS2) and, indeed, dereliction is not natural as already mentioned in the Introduction.

Digging. This operation is more problematic and some preliminaries are required.

Lemma 6. *Let X and Y be sets and let $R \subseteq X \times Y$. Let $\nu_1, \nu_2 \in !_{\mathbb{M}}Y$ and $\mu \in !_{\mathbb{M}}X$. If $(\mu, \nu_1 + \nu_2) \in !_{\mathbb{M}}R$, then one can find $\mu_1, \mu_2 \in !_{\mathbb{M}}X$ such that $\mu_1 + \mu_2 = \mu$ and $(\mu_i, \nu_i) \in !_{\mathbb{M}}R$ for $i = 1, 2$.*

Proof. We use Lemma 1. □

Given $M \in !_{\mathbb{M}}!_{\mathbb{M}}X$, we set

$$\Sigma(M) = \sum_{m \in !_{\mathbb{M}}X} M(m)m \in !_{\mathbb{M}}X\,.$$

Since M has a finite support, this sum is actually a finite sum (the linear combination, with coefficients $M(m) \in \mathbb{M}$, is taken in the module $!_{\mathbb{M}}X$).

We define $\mathsf{p}_X \in \mathbf{Rel}(!_{\mathbb{M}}X, !_{\mathbb{M}}!_{\mathbb{M}}X)$ by

$$\mathsf{p}_X = \{(\Sigma(M), M) \mid M \in !_{\mathbb{M}}!_{\mathbb{M}}X\}\,.$$

The next lemma is the main tool for proving the naturality of digging. It combines the two generalized splitting properties of $\mathbb{M}$.

Lemma 7. *Let X and Y be sets and let $R \subseteq X \times Y$ be finite. There exists $q(R) \in \mathbb{N}$ with the following property: for any $\mu \in !_{\mathbb{M}}X, \pi \in !_{\mathbb{M}}Y$ and $p \in \mathbb{M}$, if $(\mu, p\pi) \in !_{\mathbb{M}}R$, then one can find $p_1, \ldots, p_{q(R)} \in \mathbb{M}$ and $\mu_1, \ldots, \mu_{q(R)} \in !_{\mathbb{M}}X$ such that $\sum_{j=1}^{q(R)} p_j = p$, $\sum_{j=1}^{q(R)} p_j\mu_j = \mu$ and $(\mu_j, \pi) \in !_{\mathbb{M}}R$ for each $j = 1, \ldots, q(R)$.*

Proof. Let $I = \{a \in X \mid \exists b \in Y\ (a,b) \in R\}$ and $J = \{b \in Y \mid \exists a \in X\ (a,b) \in R\}$. Given $b \in J$, let $\mathsf{deg}_b(R) = \#\{a \in X \mid (a,b) \in R\} - 1 \in \mathbb{N}$ and let $\mathsf{deg}(R) = \sum_{b \in J} \mathsf{deg}_b(R)$. We prove the result by induction on $\mathsf{deg}(R)$.

Assume first that $\mathsf{deg}(R) = 0$, so that, for any $b \in J$, there is exactly one $a \in I$ such that $(a,b) \in R$, let us set $a = g(b)$: g is a surjective function from J to I whose graph coincides with R (in the sense that $R = \{(g(b), b) \mid b \in J\}$). Let σ be a witness of $(\mu, p\pi)$ for R. For all $b \in J$ we have $p\pi(b) = \sum_{a \in X} \sigma(a,b) = \sigma(g(b), b)$ and for all $a \in I$ we have $\mu(a) = \sum_{g(b)=a} \sigma(a,b) = p \sum_{g(b)=a} \pi(b)$. Let $\tau \in \mathbb{M}\langle X \times Y\rangle$ be defined by

$$\tau(a,b) = \begin{cases} \pi(b) & \text{if } g(b) = a \\ 0 & \text{otherwise.} \end{cases}$$

then clearly $\mathsf{supp}(\tau) \subseteq R$ and τ is a witness of (μ', π) for R, where $\mu' \in {!_{\mathbb{M}}}X$ is given by $\mu'(a) = \sum_{g(b)=a} \pi(a)$. Since $p\mu' = \mu$, we have obtained the required property (with $q(R) = 1$, $p_1 = p$ and $\mu_1 = \mu'$).

Assume now that $\mathsf{deg}(R) > 0$ and let us pick some $b \in J$ such that $k = \mathsf{deg}_b(R)+1 > 1$. Let σ be a witness of $(\mu, p\pi)$ for R. Let $a_1, \dots, a_k$ be a repetition-free enumeration of the elements a of I such that $(a,b) \in R$. We have

$$p\pi(b) = \sum_{i=1}^{k} \sigma(a_i, b)\,.$$

Let $l = 2^{k-1}$. By Lemma 2, there exist $p_1, \dots, p_l \in \mathbb{M}$ and $(m_{ij})_{i=1,j=1}^{k,l}$ elements of $\mathbb{M}$ with

- $p_1 + \cdots + p_l = p$
- $m_{1j} + \cdots + m_{kj} = \pi(b)$ for $j = 1, \dots, l$
- and $p_1 m_{i1} + \cdots + p_l m_{il} = \sigma(a_i, b)$ for $i = 1, \dots, k$.

Let $b_1, \dots, b_k$ be pairwise distinct new elements, which do not belong to X nor to Y, and let $Y' = (Y \setminus \{b\}) \cup \{b_1, \dots, b_k\}$. We define a new relation to which we shall be able to apply the inductive hypothesis as follows:

$$S = \{(a, b') \in R \mid b' \neq b\} \cup \{(a_i, b_i) \mid i = 1, \dots, k\}\,.$$

Then we have $\mathsf{deg}(S) = \mathsf{deg}(R) - k + 1 < \mathsf{deg}(R)$. Let $\tau \in \mathbb{M}\langle X \times Y'\rangle$ be given by

$$\tau(a,c) = \begin{cases} \sigma(a,c) & \text{if } c \notin \{b_1, \dots, b_k\} \\ \sigma(a_i, b) & \text{if } c = b_i \text{ and } a = a_i \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\mathsf{supp}(\tau) \subseteq S$. Moreover, τ is a witness of $(\mu, \sum_{j=1}^{l} p_j \pi_j)$ for S, where $\pi_j \in {!_{\mathbb{M}}}Y'$ is given by

$$\pi_j(c) = \begin{cases} \pi(c) & \text{if } c \notin \{b_1, \dots, b_k\} \\ m_{ij} & \text{if } c = b_i. \end{cases}$$

for each $j \in \{1, \ldots, l\}$. Indeed, for $a \in X$ we have

$$\begin{aligned}
\sum_{c \in Y'} \tau(a, c) &= \sum_{c \in Y' \setminus \{b_1, \ldots, b_k\}} \tau(a, c) + \sum_{i=1}^{k} \tau(a, b_i) \\
&= \sum_{c \in Y' \setminus \{b_1, \ldots, b_k\}} \sigma(a, c) + \sum_{i=1}^{k} \delta_{a, a_i} \sigma(a_i, b) \\
&= \sum_{c \in Y' \setminus \{b_1, \ldots, b_k\}} \sigma(a, c) + \sigma(a, b) = \sum_{b \in Y} \sigma(a, b) = \mu(a)
\end{aligned}$$

and for $c \in Y' \setminus \{b_1, \ldots, b_k\}$ we have

$$\sum_{a \in X} \tau(a, c) = \sum_{a \in X} \sigma(a, c) = p\pi(c) = \sum_{j=1}^{l} p_j \pi_j(c) \quad \text{since} \quad \begin{cases} \forall j \ \pi_j(c) = \pi(c) \\ \sum_{j=1}^{l} p_j = p \end{cases}$$

and last, for $c = b_i$ (with $i \in \{1, \ldots, k\}$), we have

$$\sum_{a \in X} \tau(a, c) = \sigma(a_i, b) = \sum_{j=1}^{l} p_j m_{ij} = \sum_{j=1}^{l} p_j \pi_j(c) \,.$$

By Lemma 6, since $\left(\mu, \sum_{j=1}^{l} p_j \pi_j\right) \in {!_{\mathbb{M}}} S$, we can find $\mu_1, \ldots, \mu_l \in {!_{\mathbb{M}}} X$ such that $\sum_{j=1}^{l} \mu_j = \mu$ and $(\mu_j, p_j \pi_j) \in {!_{\mathbb{M}}} S$ for each $j \in \bar{l} = \{1, \ldots, l\}$. Since $\mathsf{deg}(S) < \mathsf{deg}(R)$, we can apply the inductive hypothesis for each $j \in \bar{l}$. So we can find a family $(p_{js})_{j=1,s=1}^{l,q(S)}$ of elements of $\mathbb{M}$ such that $p_j = \sum_{s=1}^{q(S)} p_{js}$ and we can find a family $(\mu_{js})_{j=1,h=1}^{l,q(S)}$ of elements of ${!_{\mathbb{M}}} X$ such that $\sum_{s=1}^{q(S)} p_{js} \mu_{js} = \mu_j$, and moreover $(\mu_{js}, \pi_j) \in {!_{\mathbb{M}}} S$ for each $j \in \bar{l}$ and $s \in \overline{q(S)}$. We conclude the proof by showing that $(\mu_{js}, \pi) \in {!_{\mathbb{M}}} R$. Let $\tau_{js} \in \mathbb{M}\langle X \times Y' \rangle$ be a witness of (μ_{js}, π_j) for S. Let $\sigma_{js} \in \mathbb{M}\langle X \times Y \rangle$ be given by

$$\sigma_{js}(a, b') = \begin{cases} \tau_{js}(a, b') & \text{if } b' \neq b \\ \sum_{i=1}^{k} \tau_{js}(a, b_i) & \text{if } b' = b. \end{cases}$$

For $b' \in Y \setminus \{b\}$, we have $\sum_{a \in X} \sigma_{js}(a, b') = \sum_{a \in X} \tau_{js}(a, b') = \pi_j(b') = \pi(b')$. Next we have

$$\begin{aligned}
\sum_{a \in X} \sigma_{js}(a, b) &= \sum_{a \in X} \sum_{i=1}^{k} \tau_{js}(a, b_i) \\
&= \sum_{i=1}^{k} \sum_{a \in X} \tau_{js}(a, b_i) \\
&= \sum_{i=1}^{k} \pi_j(b_i) = \sum_{i=1}^{k} m_{ij} = \pi(b) \,.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}\sum_{b'\in Y}\sigma_{js}(a,b') &= \sum_{b'\in Y\setminus\{b\}}\sigma_{js}(a,b')+\sigma_{js}(a,b)\\ &= \sum_{b'\in Y\setminus\{b\}}\tau_{js}(a,b')+\sum_{i=1}^{k}\tau_{js}(a,b_i)\\ &= \sum_{c\in Y'}\tau_{js}(a,c)=\mu_{js}(a)\,.\end{aligned}$$

It remains to prove that $\mathsf{supp}(\sigma_{js})\subseteq R$, but this results immediately from the definition of σ_{js} and from the fact that $\mathsf{supp}(\tau_{js})\subseteq S$.

Observe that we can take $q(R)=lq(S)$, so that in general $q(R)=2^{\mathsf{deg}(R)}$. □

Lemma 8. p_X *is a natural transformation from* $!_{\mathbb{M}}$ *to* $!_{\mathbb{M}}!_{\mathbb{M}}$.

Proof. This is essentially an application of Lemma 7. □

Comonad equations We prove that $\mathsf{d}_{!_{\mathbb{M}}X}\cdot\mathsf{p}_X=\mathsf{Id}_{!_{\mathbb{M}}X}$. Let $(\mu,\mu')\in\,!_{\mathbb{M}}X\times\,!_{\mathbb{M}}X$. Assume first that $(\mu,\mu')\in\mathsf{d}_{!_{\mathbb{M}}X}\cdot\mathsf{p}_X$. Then we can find $M\in\,!_{\mathbb{M}}!_{\mathbb{M}}X$ such that $(\mu,M)\in\mathsf{p}_X$ and $(M,\mu')\in\mathsf{d}_{!_{\mathbb{M}}X}$. This means that $M=[\mu']$ and hence $\Sigma(M)=\mu'$, hence $\mu=\mu'$. Conversely, for $\mu\in\,!_{\mathbb{M}}X$ we have $(\mu,[\mu])\in\mathsf{p}_X$, therefore $(\mu,\mu)\in\mathsf{d}_{!_{\mathbb{M}}X}\cdot\mathsf{p}_X$.

Next we prove that $!_{\mathbb{M}}\,\mathsf{d}_X\cdot\mathsf{p}_X=\mathsf{Id}_{!_{\mathbb{M}}X}$. Let $(\mu,\mu')\in\,!_{\mathbb{M}}\,\mathsf{d}_X\cdot\mathsf{p}_X$. Let $M\in\,!_{\mathbb{M}}!_{\mathbb{M}}X$ be such that $(\mu,M)\in\mathsf{p}_X$, that is $\Sigma(M)=\mu$, and $(M,\mu')\in\,!_{\mathbb{M}}\,\mathsf{d}_X$. Let $\sigma\in\mathbb{M}\langle!_{\mathbb{M}}X\times X\rangle$ be a witness of (M,μ') for d_X. This means that $\mu'(a)=\sum_{\nu\in!_{\mathbb{M}}X}\sigma(\nu,a)=\sigma([a],a)$ since $\mathsf{supp}(\sigma)\subseteq\mathsf{d}_X$, and that $M(\nu)=\sigma([a],a)$ if $\nu=[a]$, and $M(\nu)=0$ if $\#\nu\neq 1$. It follows that $\Sigma(M)=\sum_{\nu\in!_{\mathbb{M}}X}M(\nu)\nu=\sum_{a\in X}\sigma([a],a)[a]=\mu'$ and hence $\mu=\mu'$. Conversely, one has $(\mu,\mu)\in\,!_{\mathbb{M}}\,\mathsf{d}_X\cdot\mathsf{p}_X$, because $M\in\,!_{\mathbb{M}}!_{\mathbb{M}}X$ defined by $M(\nu)=\mu(a)$ if $\nu=[a]$ and $M(\nu)=0$ if $\#\nu\neq 0$ satisfies $(\mu,M)\in\mathsf{p}_X$ and $(M,\mu)\in\,!_{\mathbb{M}}\,\mathsf{d}_X$.

Lemma 9. *Let* $\mathcal{M}\in\,!_{\mathbb{M}}!_{\mathbb{M}}!_{\mathbb{M}}X$. *Then* $\Sigma(\Sigma(\mathcal{M}))=\sum_{N\in!_{\mathbb{M}}!_{\mathbb{M}}X}\mathcal{M}(N)\Sigma(N)$.

Proof. We have

$$\begin{aligned}\Sigma(\Sigma(\mathcal{M})) &= \sum_{\nu\in!_{\mathbb{M}}X}\Sigma(\mathcal{M})(\nu)\nu\\ &= \sum_{\nu\in!_{\mathbb{M}}X}\left(\sum_{N\in!_{\mathbb{M}}!_{\mathbb{M}}X}\mathcal{M}(N)N(\nu)\right)\nu\\ &= \sum_{N\in!_{\mathbb{M}}!_{\mathbb{M}}X}\mathcal{M}(N)\left(\sum_{\nu\in!_{\mathbb{M}}X}N(\nu)\nu\right)\end{aligned}$$

and we are done. $\square$

Using Lemma 9, one proves the last comonad equation, namely $\mathsf{p}_{!_{\mathbb{M}}X} \cdot \mathsf{p}_X = !_{\mathbb{M}}\, \mathsf{p}_X \cdot \mathsf{p}_X$.

Fundamental isomorphism. One of the most important properties of the exponential is that it maps cartesian products to tensor products. Combined with the monoidal closure of **Rel**, this property leads to the cartesian closeness of the Kleisli category $\mathbf{Rel}_!$.

Proposition 5. *Given two sets X_1 and X_2, there is an natural bijection $\mathsf{n}_{X_1,X_2} : !_{\mathbb{M}}X_1 \otimes !_{\mathbb{M}}X_2 \to !_{\mathbb{M}}(X_1 \ \&\ X_2)$ and a bijection $\mathsf{n}_0 : 1 \to !_{\mathbb{M}}\top$.*

Given $(\mu_1, \mu_2) \in !_{\mathbb{M}}X_1 \otimes !_{\mathbb{M}}X_2$, we define $\nu = \mathsf{n}(\mu_1, \mu_2) \in !_{\mathbb{M}}(X_1 \ \&\ X_2)$ by $\nu(i, a) = \mu_i(a)$ for $i = 1, 2$, and $\mathsf{n}_0(*)$ is unique element of $!_{\mathbb{M}}\top$ (the empty multiset).

Structural morphisms. They are used for interpreting the *structural rules* of linear logic, associated with the exponentials. The weakening morphism is $\mathsf{weak}_X : !_{\mathbb{M}}X \to 1$ is $\mathsf{weak}_X = \{([], *)\}$. The contraction morphism is $\mathsf{contr}_X : !_{\mathbb{M}}X \to !_{\mathbb{M}}X \otimes !_{\mathbb{M}}X$ is obtained by applying the $!_{\mathbb{M}}$ functor to the diagonal map $X \to X \ \&\ X$, so that $\mathsf{contr}_X = \{(\lambda + \rho, (\lambda, \rho)) \mid \lambda, \rho \in !_{\mathbb{M}}X\}$.

There are other equations to check for proving that we have defined a model of linear logic (see [Bie95]), the corresponding verifications are straightforward.

2.4 The Kleisli cartesian closed category

The objects of the Kleisli category $\mathbf{Rel}_!$ of the comonad "$!_{\mathbb{M}}$" are the sets, and $\mathbf{Rel}_!(X, Y) = \mathbf{Rel}(!_{\mathbb{M}}X, Y)$. Identity in this category is dereliction $\mathsf{d}_X \in \mathbf{Rel}_!(X, X)$ and composition is defined as follows: let $R \in \mathbf{Rel}_!(X, Y)$ and $S \in \mathbf{Rel}_!(Y, Z)$, then $S \circ R = S \cdot !_{\mathbb{M}}R \cdot \mathsf{p}_X$. We give a direct characterization of this composition law.

Proposition 6. *Let $(\mu, c) \in !_{\mathbb{M}}X \times Z$, we have $(\mu, c) \in S \circ R$ iff there exist $b_1, \ldots, b_n \in Y$ (not necessarily distinct), $p_1, \ldots, p_n \in \mathbb{M}$ and $\mu_1, \ldots, \mu_n \in !_{\mathbb{M}}X$ such that*

$$\forall i \in \{1, \ldots, n\}\ (\mu_i, b_i) \in R\,, \quad \left(\sum_{i=1}^{n} p_i[b_i], c\right) \in S \quad \textit{and} \quad \mu = \sum_{i=1}^{n} p_i \mu_i\,.$$

Proof. Assume first that $(\mu, c) \in S \circ R$. Let $M \in !_{\mathbb{M}}!_{\mathbb{M}}X$ such that $(\mu, M) \in \mathsf{p}_X$ and let $\nu \in !_{\mathbb{M}}Y$ be such that $(\nu, c) \in S$ and $(M, \nu) \in !_{\mathbb{M}}R$. We have $\Sigma(M) = \mu$. Let $\sigma \in \mathbb{M}\langle !_{\mathbb{M}}X \times Y\rangle$ be a witness of (M, ν) for R, and let $(\mu_1, b_1), \ldots, (\mu_n, b_n)$ be a repetition-free enumeration of the set $\mathsf{supp}(\sigma) \subseteq R$. Taking $p_i = \sigma(\mu_i, b_i)$, we have $\sum_{i=1}^{n} p_i[b_i] = \nu$ and $\sum_{i=1}^{n} p_i[\mu_i] = M$, and therefore $\mu = \sum_{i=1}^{n} p_i \mu_i$.

Assume conversely that (μ, c) satisfies the conditions stated in the proposition. Then we take $\nu = \sum_{i=1}^{n} p_i[b_i]$ and $M = \sum_{i=1}^{n} p_i[\mu_i]$. We have $(\nu, c) \in S$

and $(\mu, M) \in \mathsf{p}_X$ and we have just to check that $(M, \nu) \in !_{\mathbb{M}}R$. We define $\sigma = \sum_{i=1}^{n} p_i[(\mu_i, b_i)]$; this is a witness of (M, ν) for R, as easily checked. $\square$

We recall that the cartesian product of X and Y in this category is $X \,\&\, Y$, with projections obtained by composing π_1 and π_2 with $\mathsf{d}_{X\&Y}$ in **Rel**. The function space of X and Y is $!_{\mathbb{M}}X \multimap Y$. Evaluation $\mathsf{Ev} \in \mathbf{Rel}_!(X \,\&\, (!_{\mathbb{M}}X \multimap Y), Y) \simeq \mathbf{Rel}(!_{\mathbb{M}}X \otimes !_{\mathbb{M}}(!_{\mathbb{M}}X \multimap Y), Y)$ is

$$\mathsf{Ev} = \{((\mu, [(\mu, b)]), b) \mid \mu \in !_{\mathbb{M}}X \text{ and } b \in Y\}\,.$$

Curryfication is defined as follows: let $R \in \mathbf{Rel}_!(Z \,\&\, X, Y) \simeq \mathbf{Rel}(!_{\mathbb{M}}Z \otimes !_{\mathbb{M}}X, Y)$, then $\mathsf{Cur}(R) = \{(\pi, (\mu, b)) \mid ((\pi, \mu), b) \in R\} \in \mathbf{Rel}_!(Z, !_{\mathbb{M}}X \multimap Y)$.

Differential structure and the Taylor expansion. Without giving precise definitions, let us mention that the *differential structure* of this model, which consists of natural linear morphisms $\partial_X \in \mathbf{Rel}\,(X, !_{\mathbb{M}}X)$ (*codereliction*), $\mathsf{coweak}_X \in \mathbf{Rel}\,(1, !_{\mathbb{M}}X)$ (*coweakening*) and $\mathsf{cocontr}_X \in \mathbf{Rel}\,(!_{\mathbb{M}}X \otimes !_{\mathbb{M}}X, !_{\mathbb{M}}X)$ (*cocontraction*) allows to associate, with any morphism $R \in \mathbf{Rel}_!(X, Y)$, its Taylor expansion $R^* \in \mathbf{Rel}_!(X, Y)$. When $\mathbb{M} = \mathbb{N}$, one has $M^* = M$ but this equation does not hold anymore when $\mathbb{M}$ has an infinite element ω. In that case, if $R = \{(\omega[*], *)\} \in \mathbf{Rel}_!(1, 1)$, one has $R^* = \emptyset \neq R$.

3 Graph models in Rel

Graph models [Bar84] have been isolated by Scott and Engeler in the continuous semantics. We develop here a similar construction, in the relational semantics. Let A be a non-empty set whose elements will be called atoms, and are not pairs. Let $\iota : A \to (!_{\mathbb{M}}A \multimap A)$ be a partial injective map.

We define a sequence $(D^\iota_n)_{n\in\mathbb{N}}$ of sets as follows: $D^\iota_0 = A$ and $D^\iota_{n+1} = D^\iota_n \cup ((!_{\mathbb{M}}D^\iota_n \multimap D^\iota_n) \setminus \iota(A))$. This sequence is monotone, and we set $D^\iota = \bigcup_{n\in\mathbb{N}} D^\iota_n$. We have $!_{\mathbb{M}}D^\iota \multimap D^\iota = \bigcup_{n\in\mathbb{N}}(!_{\mathbb{M}}D^\iota_n \multimap D^\iota_n)$.

We define a function $\varphi : D^\iota \to (!_{\mathbb{M}}D^\iota \multimap D^\iota)$ by

$$\varphi(\alpha) = \begin{cases} \iota(a) & \text{if } \alpha = a \in A \\ \alpha & \text{if } \alpha \notin A \end{cases}$$

and a function $\psi : (!_{\mathbb{M}}D^\iota \multimap D^\iota) \to D^\iota$ by

$$\psi(\mu, \alpha) = \begin{cases} a & \text{if } (\mu, \alpha) = \iota(a) \text{ where } a \in A \\ (\mu, \alpha) & \text{if } (\mu, \alpha) \notin \iota(A)\,. \end{cases}$$

This definition makes sense because ι is injective, and because, if $(\mu, \alpha) \in (!_{\mathbb{M}}D^\iota_n \multimap D^\iota_n) \setminus \iota(A)$, then $(\mu, \alpha) \in D^\iota_{n+1} \subseteq D^\iota$. Let $(\mu, \alpha) \in !_{\mathbb{M}}D^\iota \multimap D^\iota$. If $(\mu, \alpha) \in \iota(A)$, let a be the unique element of A such that $\iota(a) = (\mu, \alpha)$. We have $\varphi(\psi(\mu, \alpha)) = \varphi(a) = \iota(a) = (\mu, \alpha)$. If $(\mu, \alpha) \notin \iota(A)$, we have $\varphi(\psi(\mu, \alpha)) = \varphi(\mu, \alpha) = (\mu, \alpha)$ because $(\mu, \alpha) \notin A$, since no element of A is a pair.

So we have $\varphi \circ \psi = \mathsf{Id}$. We define two morphisms $\mathsf{App} = \{([\alpha], \varphi(\alpha)) \mid \alpha \in D^\iota\} \in \mathbf{Rel}_!(D^\iota, !_{\mathbb{M}} D^\iota \multimap D^\iota)$ and $\mathsf{Lam} = \{([(\mu, \alpha)], \psi(\mu, \alpha)) \mid (\mu, \alpha) \in !_{\mathbb{M}} D^\iota \multimap D^\iota\} \in \mathbf{Rel}_!(!_{\mathbb{M}} D^\iota \multimap D^\iota, D^\iota)$. Then we have $\mathsf{App} \circ \mathsf{Lam} = \mathsf{Id}_{!_{\mathbb{M}} D^\iota \multimap D^\iota}$, so that D^ι is a reflexive object in $\mathbf{Rel}_!$, whatever be the choice of the multiplicity semi-ring $\mathbb{M}$.

3.1 Interpreting terms

Given a lambda-term M and a repetition-free list of variables $\boldsymbol{x} = (x_1, \dots, x_n)$ which contains all free variables of M, the interpretation $[M]_{\boldsymbol{x}} \in \mathbf{Rel}_!(D^{\iota n}, D^\iota)$ (where $D^{\iota n}$ is the cartesian product of D^ι with itself, n times) is defined by induction on M as follows

- $[x_i]_{\boldsymbol{x}} = \pi_i$ (the ith projection from $(D^\iota)^n$ to D^ι)
- $[\lambda x\, N]_{\boldsymbol{x}} = \mathsf{Lam} \circ \mathsf{Cur}([M]_{\boldsymbol{x},x})$, assuming that x does not occur in $\boldsymbol{x}$
- $[(N)\, P]_{\boldsymbol{x}} = \mathsf{Ev} \circ \langle \mathsf{App} \circ [N]_{\boldsymbol{x}}, [P]_{\boldsymbol{x}} \rangle$

Using the cartesian closeness of $\mathbf{Rel}_!$ and the fact that $\mathsf{App} \circ \mathsf{Lam} = \mathsf{Id}_{!_{\mathbb{M}} D^\iota \multimap D^\iota}$, one proves that if M and M' are beta-equivalent, and $\boldsymbol{x}$ is a repetition-free list of variables which contain all the free variables of M and M', one has $[M]_{\boldsymbol{x}} = [M']_{\boldsymbol{x}}$. This requires to prove first a substitution lemma, see [AC98].

We present now this interpretation as a typing system (a variation of de Carvalho's system R [DC08]). A type is an element of D^ι. Given $\mu \in !_{\mathbb{M}} D^\iota$ and $\alpha \in D^\iota$, we set $\mu \to \alpha = \psi(\mu, \alpha)$. A typing context is a finite partial function from variables to $!_{\mathbb{M}} D^\iota$. If $\Gamma_1, \dots, \Gamma_k$ are contexts with the same domain and $p_1, \dots, p_k \in \mathbb{M}$, the sum $\sum_{i=1}^k p_i \Gamma_i$ is defined pointwise (using the addition of $!_{\mathbb{M}} D^\iota$). The typing rules are

$$\frac{}{x_1 : [], \dots, x_n : [], x : [\alpha] \vdash x : \alpha} \qquad \frac{\Gamma, x : \mu \vdash M : \alpha}{\Gamma \vdash \lambda x\, M : \mu \to \alpha}$$

$$\frac{\Gamma \vdash M : (\sum_{i=1}^n p_i [\beta_i]) \to \alpha \qquad \forall i \in \overline{n} \quad \Gamma_i \vdash N : \beta_i}{\Gamma + \sum_{i=1}^n p_i \Gamma_i \vdash (M)\, N : \alpha}$$

In the last rule, all contexts involved must have same domain, and the β_i's need not be distinct.

Proposition 7. *The judgment $\Gamma \vdash M : \alpha$ is derivable iff $(\Gamma(x_1), \dots, \Gamma(x_n), \alpha) \in [M]_{\boldsymbol{x}}$ where $\boldsymbol{x} = (x_1, \dots, x_n)$ is a repetition-free enumeration of the domain of Γ, which is assumed to contain all the free variables of M.*

Proof. Straightforward induction on the judgment, using Proposition 6. □

We take for $\mathbb{M}$ a multiplicity semi-ring which contains an infinite element ω (remember that this means that $\omega + 1 = \omega$). Let $A = \{a\}$, $\iota : A \to (!_{\mathbb{M}} A \multimap A)$ be defined by $\iota(a) = (\omega[a], a)$, so that $(\omega[a] \to a) = a$. Let $\Omega = (\delta)\, \delta$ where $\delta = \lambda x\, (x)\, x$.

Proposition 8. *In the model D^ι, we have $[\Omega] = \{a\}$.*

Proof. We have the following deduction tree (we have inserted in this tree the equations between types or $\mathbb{M}$-multisets of types that we use)

$$
\dfrac{\dfrac{\dfrac{x:[a]\vdash x:a=\omega[a]\to a \qquad x:[a]\vdash x:a}{x:[a]+\omega[a]=\omega[a]\vdash (x)\,x:a}}{\vdash \lambda x\,(x)\,x:\omega[a]\to a} \qquad \dfrac{\text{(same derivation)}}{\vdash \lambda x\,(x)\,x:\omega[a]\to a=a}}{\vdash (\lambda x\,(x)\,x)\,\lambda x\,(x)\,x:a}
$$

Therefore $a\in[\Omega]$.

Conversely, let $\alpha\in D^\iota$ and assume that $\vdash\Omega:\alpha$. There must exist $\mu\in{!_{\mathbb{M}}}D^\iota$ such that $\vdash\delta:\mu\to\alpha$ and $\forall\beta\in\mathsf{supp}(\mu)\ \ \vdash\delta:\beta$. Form the first of these two judgments we get $x:\mu\vdash(x)\,x:\alpha$ and hence there must exist $\nu\in{!_{\mathbb{M}}}D^\iota$ such that $\mu=\nu+[\nu\to\alpha]$. From the second judgment we get $\vdash\delta:\nu\to\alpha$ and $\forall\beta\in\mathsf{supp}(\nu)\ \ \vdash\delta:\beta$. Iterating this process, we build a sequence $(\mu_i)_{i=1}^{\infty}$ of elements of ${!_{\mathbb{M}}}D^\iota$ such that $\vdash\delta:\mu_i\to\alpha$, $\forall\beta\in\mathsf{supp}(\mu_i)\ \ \vdash\delta:\beta$ and $\mu_i=\mu_{i+1}+[\mu_{i+1}\to\alpha]$ for all i. Let $\beta_i=\mu_i\to\alpha$, it follows that $\forall i\ \beta_i\in\mathsf{supp}(\mu_1)$ and since $\mathsf{supp}(\mu_1)$ is finite, we can find i and $n>0$ such that $\beta_{i+n}=\beta_i$. We have $\beta_i=(\mu_i\to\alpha)=((\mu_{i+1}+[\beta_{i+1}])\to\alpha)=\cdots=((\mu_{i+n}+[\beta_{i+1}]+\cdots+[\beta_{i+n}])\to\alpha)$ and since $\beta_i=\beta_{i+n}=(\mu_{i+n}\to\alpha)$, we get $\mu_{i+n}=\mu_{i+n}+[\beta_{i+1}]+\cdots+[\beta_{i+n}]$ (because ψ is injective) and hence $\beta_{i+n}\in\mathsf{supp}(\mu_{i+n})$. But $\beta_{i+n}=(\mu_{i+n}\to\alpha)$ and hence we must have $\beta_{i+n}=a$. Indeed, if $\beta_{i+n}\notin A$ then by definition of ψ we have $\beta_{i+n}=(\mu_{i+n},\alpha)$ and, if k is the least integer such that $\beta_{i+n}\in D^\iota_k$, we have $k>0$ and $\beta\in D^\iota_{k-1}$ for all $\beta\in\mathsf{supp}(\mu_{i+n})$. This is impossible since $\beta_{i+n}\in\mathsf{supp}(\mu_{i+n})$. Since $(\mu_{i+n}\to\alpha)=a$, we have $\alpha=a$ and we are done. □

Since $([]\to a)\in[\lambda y\,\Omega]$ and $a\neq([]\to a)$, we have found two unsolvable terms (namely Ω and $\lambda y\,\Omega$) with distinct interpretations in D^ι and hence this model is not sensible.

Conclusion

We have introduced the algebraic concept of multiplicity semi-ring, which can be used for generalizing the standard exponential construction of the relational model of linear logic. Such a semi-ring must contain $\mathbb{N}$ as a sub-semi-ring but can also have infinite elements ω such that $\omega+1=\omega$. In that case, the corresponding model of linear logic is a model of the differential lambda-calculus which does not satisfy the Taylor formula, and it is possible to build non sensible models of the lambda-calculus in the corresponding Kleisli cartesian closed category. This shows that models of the pure differential lambda-calculus can have non sensible theories and provides a new way of building models of the pure lambda-calculus where non termination is taking into account in a quantitative way by means of these infinite multiplicities.

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