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Janet's approach to presentations and resolutions for polynomials and linear pdes.

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Abstract. Janet's algorithm to create normal forms for systems of linear pdes is outlined and used as a tool to construct resolutions for finitely generated modules over polynomial rings over fields as well as over rings of linear differential operators with coefficients in a differential field. The main result is that a Janet basis for a module allows to read off a Janet basis for the syzygy module. Two concepts are introduced: The generalized Hilbert series allowing to read off a basis (over the ground field) of the modules, once the Janet basis is constructed, and the Janet graph, containing all the relevant information connected to the Janet basis. In the context of pdes, the generalized Hilbert series enumerates the free Taylor coefficients for power series solutions. Rather than presenting Janet's algorithm as a powerful computational tool competing successfully with more commonly known Gröbner basis techniques, it is used here to prove theoretical results.

1. Introduction. One of the basic results in commutative algebra is Hilbert's Syzygy Theorem stating that each finitely generated module over the polynomial ring $K[x_1, \ldots, x_n]$ over a field K has a free resolution of length at most n. A similar result holds for finitely generated $K\langle\partial_1, \ldots, \partial_n\rangle$-modules, where K is a differential field with n commuting derivations $\partial_1, \ldots, \partial_n$ of K; e.g. K is of characteristic zero and consists of functions in $x_1, \ldots, x_n$, where $\partial_{x_1}, \ldots, \partial_{x_n}$ act as partial derivatives. The present paper will outline a procedure to turn any finite presentation of a module over one of these rings efficiently into a special form called Janet basis. It allows to read off a Janet basis for its relation module, which is better known as syzygy module. Iteration yields a constructive proof of Hilbert's Syzygy Theorem for both of these two rings. Moreover, the Janet basis of the starting module allows to read off the basic structure of the full resolution constructed in this way.

Here is a very short and incomplete outline of the history of the subject: The French mathematician Maurice Janet presented an algorithm to construct this kind of presentation in the case of linear partial differential equations (pdes) after a longer visit to Hilbert in Göttingen in the early twenties of the last century, cf. [14], [15], also for references for his predecessors. Independently, W. Gröbner introduced a device, nowadays called Gröbner basis, to compute in residue class rings of polynomial rings in the late thirties, cf. [11], [12],

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at that time restricted to the zero-dimensional case. In the 1960s, Gröbner basis techniques to compute with modules over the polynomial ring had an enormous boom as a consequence of both, B. Buchberger's thesis constructing Gröbner bases, cf. [5], and the general development of powerful computing devices. By 1980, F.-O. Schreyer, cf. [21], came up with a proof that Buchberger's so called S-polynomials come very close to a Gröbner basis of the syzygy module. After Janet's work had been ignored by the mathematical community for more than fifty years, J.-F. Pommaret, working on Spencer cohomology, became aware of Janet's work and pointed out that Janet's algorithm when applied to linear pdes with constant coefficients is a variant of Buchberger's algorithm and the Janet basis a special kind of Gröbner basis in this case ([unpublished work 1990] and [18]), though Janet's philosophy is completely different from Gröbner's philosophy. V. Gerdt and collaborators have shown that Janet's constructive ideas lead to very effective method cf. [8], [9], [23], [24]. They created an axiomatic framework for Janet's approach called involutive division and developed very efficient involutive division algorithms. For instance, the Singular package, cf. [10], recently started to use the Janet or involutive division algorithm to construct Gröbner bases. In [17], Pommaret uses Janet's ideas to construct compatibility conditions for the right hand sides of linear pdes in the form of new pdes. He proves that for certain kinds of coordinate systems, called δ-regular coordinate systems, a variant of Janet's algorithm yields not only a Janet basis for the original pdes but also for the compatibility conditions. Recently, W. Seiler has proved that the resolution obtained in this way for homogeneous equations yields even a resolution of minimal length, cf. [22]. Applications to control theory can be found in [20], [19], [6].

Janet's first basic idea is to construct explicitly a basis of the submodule of relations over the ground field by assigning so called multiplicative variables to each element of the Janet basis. This idea can already be explained in a combinatorial context. This, together with some useful bookkeeping devices, namely the Janet graph and the generalized Hilbert series, not mentioned explicitly in the classic literature on the subject, will be explained in Section 2. This part of the theory is completely different from Gröbner's approach. Section 3 discusses Janet's algorithm in view of its main theoretical consequences for the polynomial case. The main result of this paper, namely that Janet's algorithm does not only construct a Janet basis for the module but also for the syzygy module at the same time, follows naturally and easily. It should be noted that the generators of the syzygy module are in natural one-one correspondence with the non-multiplicative variables of the Janet basis elements (indexed by these elements). The next section gives consequences for the explicit determination of the affine Hilbert series of the factor module, in particular, not only the existence of the Hilbert polynomial but even its explicit form can be read off from the Janet basis. The final section deals with the case of linear pdes, which had been the original framework for Janet. In particular, the Hilbert series is shown to count the free Taylor coefficients for power series solutions of linear pde systems.

In this paper, algorithms are only used to prove theoretical results. Therefore they are only outlined and no effort is made towards efficiency. For illustrations of the results as well as for the possibility to work out examples with rather fast implementations, we refer to the Maple packages available working with involutive division described in [4].

This paper is self-contained in the sense that no previous knowledge on Gröbner basis techniques is required. It can serve as an introduction to Janet's theory and algorithm. All refined and axiomatized theory towards efficiency, like involutive division, has been avoided to make this paper accessible to non-specialists and to get more theoretically minded colleagues interested in the beautiful Janet approach. We should like to thank various colleagues, who have accompanied us on our walk into the Janet world; in chronological order: J. -F. Pommaret, V. Gerdt, A. Quadrat, and W. Seiler.

2. Underlying combinatorics. In this section let $R := K[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates over the arbitrary field K. Fix a natural number $q \in \mathbb{N}$ and let $(e_1, \ldots, e_q)$ be the standard R-basis of the free R-module R^q. The elements of

$$\mathrm{Mon}(R^q) := \{x^i e_s \mid 1 \leqq s \leqq q, i \in (\mathbb{Z}_{\geqq 0})^n\},$$

where $x^i := x_1^{i_1} \cdots x_n^{i_n}$ for any multi-index $i \in (\mathbb{Z}_{\geqq 0})^n$, are called monomials. One has an obvious multiplication $\mathrm{Mon}(R) \times \mathrm{Mon}(R^q) \to \mathrm{Mon}(R^q)$ of $\mathrm{Mon}(R^q)$ by the multiplicatively written monoid $\mathrm{Mon}(R)$ isomorphic to the monoid $((\mathbb{Z}_{\geqq 0})^n, +)$ of all multi-indices. A (non-empty) subset S of $\mathrm{Mon}(R^q)$ is called *multiple-closed*, if it is closed under multiplication by the elements of $\mathrm{Mon}(R)$. The aim is to develop a standard description for multiple-closed subsets S of $\mathrm{Mon}(R^q)$ as well as for their complements $\mathrm{Mon}(R^q) - S$. At the beginning of his monograph [15], Janet proves by an easy induction on the number of indeterminates n:

Lemma 1. *A sequence of monomials in* $\mathrm{Mon}(R^q)$ *in which none is a multiple of an earlier one is necessarily finite.*

There are two important consequences:

Corollary 2. *For a multiple-closed subset S of* $\mathrm{Mon}(R^q)$ *there are only finitely many elements of S which are not multiples of any other element of S.* (*Call these elements* canonical generators *of S*).

Corollary 3. *A properly ascending sequence $S_1 \subset S_2 \subset S_3 \subset \ldots$ of multiple-closed subsets of* $\mathrm{Mon}(R^q)$ *terminates after finitely many steps.*

The simplest multiple-closed subsets of $\mathrm{Mon}(R^q)$ are the ones with just one canonical generator v, i. e. they are of the form $\mathrm{Mon}(R)v$ for some $v \in \mathrm{Mon}(R^q)$ and hence consist of all multiples of v. Slightly more flexible building blocks for multiple-closed subsets of $\mathrm{Mon}(R^q)$ are cones defined as subsets C of $\mathrm{Mon}(R^q)$ for which there exists some $w \in \mathrm{Mon}(R^q)$ called the *vertex* of the cone C and a subset $M = M(C) \subseteqq \{x_1, \ldots, x_n\}$ of the set of variables of R called the *multiplicative variables* for the cone such that $C = \mathrm{Mon}(K[M])w$. Naturally, $|M|$ will denote the *dimension* of the cone C. Janet's idea was to decompose any multiple-closed subset S of $\mathrm{Mon}(R^q)$ as a disjoint union of finitely many cones by scanning S inductively as follows:

Lemma 4. 1) *Any multiple-closed subset S of* $\mathrm{Mon}(R^q)$ *is the disjoint union of its components $S_i := S \cap \mathrm{Mon}(R)e_i$ for $i = 1, \ldots, q$. The non-empty S_i are again multiple-closed subsets of* $\mathrm{Mon}(R^q)$*. It suffices to decompose such a non-empty component S_i into disjoint cones.*

2) *If $n = 1$, then each multiple-closed subset of* $\mathrm{Mon}(R)e_i$ *is a cone.*

3) *Let $S \subseteqq \mathrm{Mon}(R)e_i$ be a multiple-closed subset. For any $\gamma \in \mathbb{Z}_{\geqq 0}$ one has*

$$S^\gamma := \mathrm{Mon}(K[x_1, \ldots, x_{n-1}])x_n^\gamma e_i \cap S = S_{\mathrm{red}}^\gamma x_n^\gamma e_i$$

for a unique $S_{\mathrm{red}}^\gamma \subseteqq \mathrm{Mon}(K[x_1, \ldots, x_{n-1}])$, which is empty if and only if S^γ is empty and multiple-closed in $\mathrm{Mon}(K[x_1, \ldots, x_{n-1}])$ *otherwise. One has a smallest α with $S^\alpha \neq \emptyset$ and a smallest β such that $S_{\mathrm{red}}^\gamma = S_{\mathrm{red}}^\beta$ for all $\gamma \geqq \beta$. Because of $x_n S^\delta \subseteqq S^{\delta+1}$ for all $\delta \in \mathbb{Z}_{\geqq 0}$ one has*

$$\begin{aligned} \ldots \emptyset = S_{\mathrm{red}}^{\alpha-1} \subset S_{\mathrm{red}}^\alpha \subseteqq \ldots \subseteqq S_{\mathrm{red}}^{\beta-1} \subset S_{\mathrm{red}}^\beta = S_{\mathrm{red}}^{\beta+1} \\ = \ldots \subseteqq \mathrm{Mon}(K[x_1, \ldots, x_{n-1}]). \end{aligned}$$

Applying this lemma to a multiple-closed subset $S \subseteqq \mathrm{Mon}(R)e_i$ one arrives at Janet's partition of multiple-closed subsets of $\mathrm{Mon}(R^q)$ into cones, sometimes also referred to as Janet separation.

Corollary 5. *Let $S \subseteqq \mathrm{Mon}(R)e_i$ be multiple-closed. A partition of S into finitely many disjoint cones $C = \mathrm{Mon}(K[M(C)])v_C$ exists.*

Proof. The one-variable case is clear. In the terminology of Lemma 4. 3) one may assume that each S_{red}^δ with $\alpha \leqq \delta \leqq \beta$ is already decomposed into disjoint cones. In case $\delta < \beta$ we multiply these cones by $x_n^\delta e_i$ and obtain the cones for S^δ. In case $\delta = \beta$ each such cone $C' = \mathrm{Mon}(K[M(C')])v_{C'}$ yields the cone $C = \mathrm{Mon}(K[M(C)])v_C$ with $M(C) := M(C') \cup \{x_n\}$ and $v_C = x_n^\beta v_{C'} e_i$. (So x_n is non-multiplicative for each cone in the first case and multiplicative in the last case.) $\square$

It should be noted that there is an easy algorithm to work out the vertices of the cones C and their sets $M(C)$ of multiplicative variables for the Janet separation of multiple-closed sets. There are other decompositions possible; they are studied under the name of involutive division by V. Gerdt and others, cf. [8], [2], [22]. There are two combinatorial devices which make life easier.

Definition 6. The vertices v_C of the cones C in the decomposition of a multiple-closed subset S of $\mathrm{Mon}(R^q)$ form the *Janet basis* $J(S)$ of S.

The *Janet graph* of C is the labeled directed graph with vertex set $J(S)$ and the following edge set: For each $v_C \in J(S)$ and each $x_j \in \{x_1, \ldots, x_n\} - M(C)$ there is an edge $v_C \xrightarrow{x_j} v_D$ from v_C to v_D labeled by x_j, where D is the unique cone in the decomposition of S containing $x_j v_C$ (or equivalently $x_j C$).

For any subset $S \subseteqq \mathrm{Mon}(R^q)$ the *generalized Hilbert series* $H_S = H_S(x_1, \ldots, x_n)$ of S is the formal sum of all $m \in S$ considered as an element of the power series module $\oplus_i K[[x_1, \ldots, x_n]]e_i$.

So each multiple-closed subset S of $\mathrm{Mon}(R^q)$ can be described canonically in three ways: by its canonical generators, by its Janet graph, or by its generalized Hilbert series. The first description will rarely be used; note however, the second and third ones are equivalent because, using the geometric series, one has

$$H_S = \sum_C \left(\prod_{x_i \in M(C)} \frac{1}{1 - x_i} \right) v_C,$$

where the sum is taken over all cones in the Janet partition of S. (Of course this can be done for any noetherian involutive division.) Note, the connected components of the Janet graph are in natural bijection with the non-empty components of S. Each directed path, which cannot be extended, ends at the vertex representing the unique cone of full dimension n in the corresponding component. There are two kinds of vertices: the canonical generators and certain of their multiples, which must always have incoming edges. If a canonical generator v has an incoming edge $v_C \overset{x_j}{\to} v$, then $x_j v_C$ is a proper multiple of v. Finally, the Janet graph has no cycles, as an easy analysis of exponents shows.

Example 7. Choosing the multiple-closed subset $S \subset \mathrm{Mon}(K[x_1, x_2, x_3])$ as generated by the canonical generators $x_2^2 x_3$, $x_1^2 x_3^3$ yields the following tree as Janet graph

$$x_1^2 x_3^3 \overset{x_2}{\to} x_1^2 x_2 x_3^3 \overset{x_2}{\to} x_2^2 x_3^3 \overset{x_3}{\leftarrow} x_2^2 x_3^2 \overset{x_3}{\leftarrow} x_2^2 x_3$$

with

$$H_S = \frac{x_1^2 x_3^3 + x_1^2 x_2 x_3^3}{(1 - x_1)(1 - x_3)} + \frac{x_2^2 x_3^3}{(1 - x_1)(1 - x_2)(1 - x_3)} + \frac{x_2^2 x_3^2 + x_2^2 x_3}{(1 - x_1)(1 - x_2)}.$$

Though we will not need it in the sequel, we are going to mention the following shelling property, which gives a good intuition on the structure of a multiple-closed subset of $\mathrm{Mon}(R^q)$. Without loss of generality, we take $q = 1$.

Proposition 8. *Let $S \subsetneqq \mathrm{Mon}(R)$ be a multiple-closed subset with Janet graph G. Construct an increasing sequence of induced subgraphs $G_0, G_1, \ldots$ with the following vertex sets: $V(G_0)$ consists of the unique vertex without outgoing edges, i.e. the vertex of the only n-dimensional cone. $V(G_{i+1})$ consists of all vertices in G_i and those having an edge ending in G_i. Then each G_i is the Janet graph of the multiple-closed subset of* $\mathrm{Mon}(R)$ *generated by its vertices.*

More important is a decomposition of the complement of a multiple-closed subset of $\mathrm{Mon}(R^q)$ into disjoint cones. We proceed analogously to Lemma 4.

Lemma 9. 1) *The complement $\overline{S}$ of any multiple-closed subset S of* $\mathrm{Mon}(R^q)$ *is the disjoint union of its components $\overline{S}_i := \overline{S} \cap \mathrm{Mon}(R)e_i$ for $i = 1, \ldots, q$. Each $\overline{S}_i \neq$* $\mathrm{Mon}(R)e_i$ *is again the complement of a multiple-closed subset S_i in* $\mathrm{Mon}(R)e_i$. *It suffices to decompose such a non-empty component $\overline{S}_i$ into disjoint cones.*

2) *If $n = 1$, then each multiple-closed subset of* $\mathrm{Mon}(R)e_i$ *has a finite set as its complement, i.e. a finite union of zero-dimensional cones.*
3) *Let* $S \subsetneqq \mathrm{Mon}(R)e_i$ *be a multiple-closed subset and define* S^γ, $S^\gamma_{\mathrm{red}}, \alpha, \beta$ *as in Lemma* 4. *Then for each* γ*:*

$$\begin{aligned}\overline{S}_i^\gamma &:= \mathrm{Mon}(K[x_1,\ldots,x_{n-1}])x_n^\gamma e_i \cap \overline{S}_i\\ &= \mathrm{Mon}(K[x_1,\ldots,x_{n-1}])x_n^\gamma e_i - S^\gamma\end{aligned}$$

and can be obtained as

$$\overline{S}_i^\gamma = (\mathrm{Mon}(K[x_1,\ldots,x_{n-1}]) - S^\gamma_{\mathrm{red}})x_n^\gamma e_i.$$

Corollary 10. *Let* $S \subsetneqq \mathrm{Mon}(R)e_i$ *be multiple-closed. A partition of* $\mathrm{Mon}(R)e_i - S$ *into finitely many disjoint cones* $C = \mathrm{Mon}(K[M(C)])v_C$ *exists.*

Proof. Analogous to the proof of Corollary 5 with Lemma 4 replaced by Lemma 9. □

More important than the generalized Hilbert series of multiple-closed subsets S is the generalized Hilbert series of their complements $\overline{S}$:

Corollary 11. *Let* $S \subsetneqq \mathrm{Mon}(R)e_i$ *be multiple-closed. The generalized Hilbert series of* $\overline{S}_i = \mathrm{Mon}(R)e_i - S$, *given by*

$$H_{\overline{S}_i}(x_1,\ldots,x_n) = \frac{1}{(1-x_1)\cdots(1-x_n)} - H_S(x_1,\ldots,x_n),$$

can be written as

$$H_{\overline{S}_i} = \sum_C \left(\prod_{x_i \in M(C)} \frac{1}{1-x_i}\right) v_C,$$

where the sum is taken over all cones C in the decomposition of $\overline{S}_i$ into disjoint cones. (*Note, some cones might be zero-dimensional, so the product is* 1 *in that case.*)

Example 12 (Example 7, continued). In the above example one gets the following generalized Hilbert series for the complement of S:

$$\frac{x_2x_3^3 + x_1x_2x_3^3 + x_3^3 + x_1x_3^3}{1-x_3} + \frac{x_3^2 + x_2x_3^2}{1-x_1} + \frac{x_3 + x_2x_3}{1-x_1} + \frac{1}{(1-x_1)(1-x_2)}.$$

3. Presentations. Keep the notation $R = K[x_1,\ldots,x_n], \mathrm{Mon}(R^q)$ etc. from the previous section. The bridge between sub- resp. factor modules of R^q and multiple-closed subsets of $\mathrm{Mon}(R^q)$ as treated in the previous section is provided by term orders on $\mathrm{Mon}(R^q)$. A term order is a total order $<$ on $\mathrm{Mon}(R^q)$ such that 1) $v < x_iv$ for all $i = 1,\ldots,n$ and $v \in \mathrm{Mon}(R^q)$ and 2) $v < w$ implies $x_iv < x_iw$ for all $i = 1,\ldots,n$ and

$v, w \in \mathrm{Mon}(R^q)$, cf. [1]. As a consequence, to any $0 \neq m \in R^q$ one can assign a *leading monomial* $\mathrm{Lt}(m) \in \mathrm{Mon}(R^q)$, i.e. the unique biggest monomial with respect to the term order $<$ occurring in the K-linear combination of m by elements of $\mathrm{Mon}(R^q)$. One has the following obvious lemma:

Lemma 13. *Let $\{0\} \neq X \leqq R^q$ be an R-submodule of R^q and $<$ a term order on R^q. Then $S = S(X, <) := \{\mathrm{Lt}(m) \mid 0 \neq m \in X\}$ is a multiple-closed subset of $\mathrm{Mon}(R^q)$ and the residue classes of the elements in $\overline{S} = \overline{S(X, <)} := \mathrm{Mon}(R^q) - S$ form a K-basis of the factor module R^q/X.*

Starting from a finite subset G of $R^q - \{0\}$, one will usually have $S(\langle \mathrm{Lt}(G)\rangle, <) \subsetneqq S(\langle G\rangle, <)$, where the computation of the left hand side is an easy combinatorial task discussed in the previous section and the computation of the right hand side will be discussed now, the strategy being to modify G without changing $\langle G\rangle$ until equality is achieved. For the subsequent discussion fix a term order $<$ on $\mathrm{Mon}(R^q)$. The first step to normalize a generating set for a submodule is a slight common generalization of both Gaussian elimination (for linear systems) and Euclid's algorithm (for univariate polynomials):

Algorithm 14 (Autoreduction). Given a finite subset $G \subset R^q - \{0\}$. By replacing an element of G by its difference with some multiple of some other element of G repeatedly, obtain a new finite generating set G' of $\langle G\rangle$ such that $\mathrm{Lt}(g)$ does not divide $\mathrm{Lt}(h)$ for any two different $g, h \in G'$.

The second step is dictated by the previous section:

Algorithm 15 (Completion). Given an autoreduced subset $G \subset R^q$. Complement G by multiples of its members to obtain G' (of minimal size) such that $\mathrm{Lt}(G')$ is the vertex set of the Janet partition of the multiple-closed set generated by $\mathrm{Lt}(G)$.

Once a completed set G of generators is obtained, any element of R^q can be normalized modulo G, controlled by the Janet partition of $S(\langle \mathrm{Lt}(G)\rangle, <)$ with $\mathrm{Lt}(G)$ as its set of cone vertices. If $\mathrm{Lt}(g)$ for some $g \in G$ is the vertex of such a cone C then write $M(g) := M(C)(=$ set of multiplicative variables of $g \in G)$ and $\overline{M}(g) := \{x_1, \ldots, x_n\} - M(C)$, i.e. $C = \mathrm{Mon}(K[M(g)])\mathrm{Lt}(g) = \mathrm{Lt}(K[M(g)]g - \{0\})$.

Algorithm 16 (Involutive reduction). Given a completed (finite) subset $G \subset R^q$ and some element $u \in R^q$. Set $u' := u$, $u'' := 0$. As long as there exist monomials m occurring in u' which lie in the cone of $\mathrm{Lt}(g)$ for some $g \in G$, select the $<$-largest m, subtract a suitable multiple $m'g$ with $m' \in K[M(g)]$ of g from u' to get rid of m in u' and add $m'g$ to u''. In the end one obtains the normal form $N_G(u) := u'$ of u with respect to G:

$$u = u' + u'' = N_G(u) + L_G(u)$$

with no monomial of $N_G(u)$ in $S(\langle \mathrm{Lt}(G)\rangle, <)$, the multiple-closed subset of $\mathrm{Mon}(R^q)$ spanned by $\mathrm{Lt}(G)$, and $L_G(u) = \sum_{g\in G} r_g g$ with $r_g \in K[M(g)]$.

It should be noted that not only the result of involutive reduction but also the succession of its steps is uniquely determined. Note also, if one adds a new generator to a completed set of generators, the combination of autoreduction and completion has obvious shortcuts.

Definition 17. A completed subset G of R^q is called *passive*, if $N_G(x_i g) = 0$ for all $g \in G$ and $x_i \in \overline{M}(g)$. A passive completed (finite) subset of R^q is called a *Janet basis*.

To force uniqueness for the Janet basis of a submodule, one can require that the coefficient of each leading monomial is equal to 1. It follows Janet's algorithm which terminates because of Corollary 3.

Algorithm 18 (Janet's Algorithm). Given a finite subset $G \subset R^q - \{0\}$. Construct a Janet basis $J = J(G)$ of $\langle G\rangle \leqq R^q$ as follows: Autoreduce and complete G to obtain G'. Let

$$T(G') := \{N_{G'}(x_i g) \mid g \in G', x_i \in \overline{M}(g)\} - \{0\}.$$

If $T(G') = \emptyset$, then $J(G) = G'$. Else replace G by $G' \cup T(G')$ and start from the beginning.

Having given a constructive proof of the existence of a Janet basis, we describe its benefits. The first three parts of the first main result are well known:

Theorem 19. *Let $J \subset R^q$ be a Janet basis with respect to a term order $<$ on* $\mathrm{Mon}(R^q)$ *and let $X := \langle J\rangle$ be the submodule of R^q generated by J. Then the following holds:*

1) *For $a, b \in R^q$ one has $a + X = b + X \in R^q/X$ if and only if $N_J(a) = N_J(b)$.*
2) *A K-basis B of X is*

$$B := \dot{\bigcup_{g\in J}} \mathrm{Mon}(K[M(g)])g.$$

   Lt *restricts to a bijection of B onto $S(X, <)$.*
3) *The cosets of the monomials in $\overline{S(X, <)} :=$* $\mathrm{Mon}(R^q) - S(X, <)$ *form a K-basis of R^q/X.*
4) *Let $\{\hat{g} \mid g \in J\}$ be standard generators of the free R-module $R^J \cong R^{|J|}$ and let $\pi : R^J \to X : \hat{g} \to g$ be the obvious epimorphism. Define*

$$*(J) := \{(g, x_i) \mid g \in J, x_i \in \overline{M}(g)\}.$$

   Then the kernel of π is generated by $J^{(1)} := \{g_{x_i} \mid (g, x_i) \in *(J)\}$, the set of standard syzygies

$$g_{x_i} := x_i\hat{g} - \sum_{h\in J} \alpha_{x_i,g,h}\hat{h},$$

where $\sum_{h \in J} \alpha_{x_i,g,h} h$ *with* $\alpha_{x_i,g,h} \in K[M(h)]$ *is the unique representation of* $x_i g$ *with respect to* J. *In fact, this* $J^{(1)}$ *forms a Janet basis of* $\ker \pi$ *in* R^J *with respect to a suitable term order.*

Proof. 2) We show that B generates X as a K-vector space. Let $u \in X$. Then there are $r_g \in R$ with $u = \sum_{g \in J} r_g g$. If all $r_g \in K[M(g)]$, we are done. In general, r_g is a K-linear combination of monomials $m_1 m_2$ with $m_1 \in \mathrm{Mon}(K[M(g)])$ and $m_2 \in \mathrm{Mon}(K[\overline{M}(g)])$. Call the degree of m_2 the defect $d(m_1 m_2)$ of $m_1 m_2$ in r_g and note that $d(m_1 m_2) = 0$ for all monomials of r_g means $r_g \in K[M(g)]$. If the defect of some monomial m in r_g is positive, say equal to d, then $m = m'x$ for some $x \in \overline{M}(g)$. By passivity, $mg = m'xg$ can be replaced by $m' L_G(xg)$ in the above representation of u, which has only terms of defect at most $d - 1$. Repeating this process until no monomials of positive defect are left, finally produces an expression $u = \sum_{g \in J} s_g g$ with $s_g \in K[M(g)]$, i.e. $u \in \langle B \rangle_K$, proving $\langle B \rangle_K = X$. The bijectivity of Lt from B to $S(X, <)$ is now clear and implies the linear independence of B.

3) This follows now from Lemma 13.

1) In $a + X = b + X$ one may assume $a = N_J(a)$ and $b = N_J(b)$, i.e. none of the monomials in a or b lie in $S(J)$. Hence the same is true for $a - b$. But by 2) one can write $a - b = \sum_{g \in J} s_g g$ with $s_g \in K[M(g)]$. Hence $a - b = 0$.

4) The proof of 2) can be rephrased by saying that the standard syzygies are the only relations used to bring an arbitrary R-linear combination of the $g \in J$ to its normal form. Therefore it is clear that $J^{(1)}$ generates the kernel of π. Moreover, π induces a bijection

$$\hat{B} := \dot{\bigcup_{g \in J}} \mathrm{Mon}(K[M(g)])\hat{g} \to B : p\hat{g} \mapsto pg.$$

Note, $\hat{B} \subset \mathrm{Mon}(R^J) = \{x^i \hat{g} \mid g \in J, i \in (\mathbb{Z}_{\geqq 0})^n\}$ comes with a decomposition into disjoint cones, which are in natural bijection with the cones in the Janet partition of B. Whereas the cones of B have a topology described by the Janet graph, the cones of $\hat{B}$ are distributed into separate components. Because of L_G defined by involutive reduction in R^q, we have already a constructive tool to do computations in $R^J / \langle J^{(1)} \rangle$. We are going to show that $J^{(1)}$ is already a Janet basis with $\pi \circ N_{J^{(1)}} = L_J \circ \pi$. The main property for the term order $\prec$ to be defined on $\mathrm{Mon}(R^J)$ will be $\mathrm{Lt}_\prec(g_{x_i}) = x_i \hat{g}$ for all $(g, x_i) \in *(J)$. This is the definition of $\prec$: We first choose some total order $\ll$ of the J with the following property: Whenever there is a directed path in the Janet graph of $S(J)$ from $\mathrm{Lt}(g)$ to $\mathrm{Lt}(h)$ for $g, h \in J$, then $h \ll g$. Since the Janet graph has no closed cycles, $\ll$ exists. Now the term order on $\mathrm{Mon}(R^J)$ can be defined by

$$x^i \hat{g} \prec x^j \hat{h} \quad \text{iff} \quad \begin{cases} x^i \mathrm{Lt}(g) < x^j \mathrm{Lt}(h) \text{ or} \\ x^i \mathrm{Lt}(g) = x^j \mathrm{Lt}(h) \text{ and } g \ll h. \end{cases}$$

Clearly, working downwards this term order is the obvious way to proceed in the normalization of relations. Also $\mathrm{Lt}_\prec(g_{x_i}) = x_i \hat{g}$ for all $(g, x_i) \in *(J)$ is clear. To investigate the Janet partition of $S := S(\langle \mathrm{Lt}_{\prec}(g_{x_i}) \mid (g, x_i) \in *(J)\rangle, \prec)$ is now easy, since one can deal with each component $S_{\hat{g}} := S \cap \mathrm{Mon}(R)\hat{g}$ separately, the latter being generated by the $\mathrm{Lt}_{\prec}(g_{x_i})$ with $x_i \in \overline{M}(g)$. Obviously, these are the vertices of the cones in the Janet partition of $S_{\hat{g}}$ and $M(x_i\hat{g}) = \{x_1, x_2, \ldots, x_i\} \cup M(g)$. Now it is obvious that $\mathrm{Mon}(R^J)$ is the disjoint union of $\hat{B}$ and S, as it should be, and that $J^{(1)}$ is a completed subset of R^J. If $J^{(1)}$ were not passive, the test of passivity would produce relations $r \in R^J$ with $\mathrm{Lt}_{\prec}(r) \notin S$, i.e. $\mathrm{Lt}_{\prec}(r) \in \hat{B}$, contradicting the K-linear independence of B. $\square$

It might be useful to use a slightly more geometric language.

Definition 20. Let $X \leqq R^q$ be a submodule of R^q. The *Janet graph* $\Gamma(X, <)$ of X with respect to the term order $<$ on $\mathrm{Mon}(R^q)$ is given by the Janet graph of $\mathrm{Lt}(X - \{0\})$, where each vertex v is replaced by the unique element g of the Janet basis of X with $\mathrm{Lt}(g) = v$.

The proof of part 4 of Theorem 19 now yields the following.

Corollary 21. *In the situation of Theorem* 19 *the Janet graph of $J^{(1)}$ is the disjoint union of directed graphs σ_g with $g \in J$. The vertex set of σ_g is $V_g := \{g_{x_i} \mid x_i \in \overline{M}(g)\}$ and the directed edges* (*with label x_j*) *are given by* $(g_{x_i}, g_{x_j}) \in V_g \times V_g$ *with $i < j$.* (*For obvious reasons, σ_g will be referred to as directed simplex of dimension* $|\overline{M}(g)| - 1$.)

Iteration yields a free resolution of R^q/X.

Corollary 22. *In the situation of Theorem* 19 *let* $m := \max\{|\overline{M}(g)| \mid g \in J\}$. *Then there is a free resolution of R^q/X:*

$$0 \to R^{J^{(m-1)}} \to \ldots \to R^{J^{(1)}} \to R^J \xrightarrow{\pi} R^q \to R^q/X \to 0,$$

where each $J^{(k)}$ is defined as a set of symbols

$$J^{(k)} := \{g_{x_{i_1}\ldots x_{i_k}} \mid g \in J, x_{i_j} \in \overline{M}(g), i_1 < i_2 < \ldots < i_k\}.$$

Since

$$|J^{(k)}| = \sum_{g \in J} \binom{|\overline{M}(g)|}{k},$$

the Euler characteristic of R^q/X can easily be read off from the Janet graph $\Gamma(X, <)$. Obviously, it is equal to the dimension of $\mathrm{Quot}(R) \underset{R}{\otimes} R^q/X$ over the field of fractions $\mathrm{Quot}(R) = K(x_1, \ldots, x_n)$ of R. It is also equal to the number of n-dimensional cones for $S(X, <)$, also known as n-th Cartan character, cf. discussion before Corollary 31.

4. Increasing filtrations. All gradings $R = \bigoplus_{i \geqq 0} R_i$ and $R^q = \bigoplus_{i \geqq i_0} (R^q)_i$ for some $i_0 \in \mathbb{Z}$ of R and of R^q into finite dimensional K-subspaces with $R_i R_j \subseteqq R_{i+j}$, and $R_i (R^q)_j \subseteqq (R^q)_{i+j}$, and the additional property that each R_i and $(R^q)_j$ has a K-basis of monomials, interact remarkably well with Janet's algorithm. Note, the leading monomial $\mathrm{Lt}(u)$ of some homogeneous element $u \in (R^q)_j$ is also homogeneous of the same degree j. Therefore one has the following obvious remark:

Remark 23. Let $G \subset R^q$ consist of homogeneous elements only, where R and R^q are graded as described above. Then the Janet basis of $X := \langle G \rangle$ also consists of homogeneous elements only. The Janet algorithm never produces new elements by involutive reduction, i.e. it suffices to apply autoreduction and completion.

The Hilbert series $H_{R^q/X}(t) := \sum_i \dim_K(R^q/X)_i \, t^i$ of the graded factor module R^q/X can be obtained by the following substitution process from the generalized Hilbert series $H_{\overline{S}}(x_1, \ldots, x_n)$ of the complement $\overline{S} = \mathrm{Mon}(R^q) - S$ of $S := S(X, <)$ in $\mathrm{Mon}(R^q)$ as follows:

($*$) For each vertex v of a cone in the Janet partition of $\overline{S}$ substitute v by $t^{\theta(v)}$, where $v \in (R^q)_{\theta(v)}$, and replace each x_i by $t^{d(x_i)}$, where $x_i \in R_{d(x_i)}$.

The corresponding procedure yields $H_X(t)$ from $H_S(x_1, \ldots, x_n)$. Note, from Corollary 11 one gets almost directly Hilbert's basic result:

Corollary 24. $\dim_K(R^q/X)_i$ *is a polynomial in* i *for* $i \geqq r$ *for some* r. *In fact, this polynomial can be read off from the Janet partition of* $\overline{S}$, *and* r *equals the maximum of the degrees of the vertices* v *of the cones in the Janet partition of* $\overline{S(X, <)}$.

In general, G might have non-homogeneous elements as well. In this case, the above grading on R^q yields via the increasing filtration of R^q by the

$$(R^q)_{\leqq i} := \bigoplus_{j \leqq i} (R^q)_j, \quad i \geqq i_0,$$

the filtration of X:

$$\begin{aligned} \{0\} =: X_{i_0 - 1} \subset X_{i_0} \subseteqq X_{i_0+1} \subseteqq \ldots \subseteqq X_i \\ := X \cap (R^q)_{\leqq i} \subseteqq \ldots \subseteqq X = \bigcup_{i \geqq i_0} X_i \end{aligned}$$

and the filtration of $Y := R^q/X$:

$$\begin{aligned} \{0\} =: Y_{i_0 - 1} \subseteqq Y_{i_0} \subseteqq Y_{i_0+1} \subseteqq \ldots \subseteqq Y_i \\ := ((R^q)_{\leqq i} + X)/X \subseteqq \ldots \subseteqq Y = \bigcup_{i \geqq i_0} Y_i. \end{aligned}$$

As usual $\mathrm{gr}(R) := \bigoplus_{i \geqq 0} (R_{\leqq i}/R_{\leqq i-1}) \cong \bigoplus_{i \geqq 0} R_i = R$ is a graded ring for which $\mathrm{gr}(Y) := \bigoplus_{i \geqq i_0} (Y_i/Y_{i-1})$ (and also $\mathrm{gr}(X)$) is a graded module. Janet's algorithm provides a natural basis for Y_i/Y_{i-1} (as well as for X_i/X_{i-1}) provided the term order is compatible with the grading of R^q:

Definition 25. The term order $<$ on $\mathrm{Mon}(R^q)$ is called (*filtration-*)*compatible with the grading* $R^q = \bigoplus_{i \geqq i_0} (R^q)_i$ of R^q if $\mathrm{Lt}((R^q)_{\leqq i} - (R^q)_{\leqq i-1}) \subseteqq (R^q)_i$ for all $i \geqq i_0$.

Obviously, we can now complement part 3) of Theorem 19 as follows:

Theorem 26. *Let R^q be graded as above and assume that the term order $<$ on $\mathrm{Mon}(R^q)$ is compatible with this grading.*

1) *For any submodule $X \leqq R^q$ the cosets of the elements of $\overline{S(X,<)} \cap (R^q)_i$ modulo X form a K-basis of a K-complement of Y_{i-1} in Y_i, where $Y := R^q/X$ and Y_i is defined as above, $i \geqq i_0$.*
2) *A Janet basis of a graded R-module isomorphic to $\mathrm{gr}(Y)$ is obtained from the Janet basis of X by replacing each of its elements $g = g_{\theta(g)} + g_{\theta(g)-1} + \cdots + g_{\theta(g)-\varepsilon(g)}$ with $g_i \in (R^q)_i$ and $g_{\theta(g)} \neq 0$ by its top component $g_{\theta(g)}$.*

Note, the passage from g to $g_{\theta(g)}$, if performed on an arbitrary generating set of X (in the non-homogeneous case), might lead to a graded module having $\mathrm{gr}(Y)$ as a proper epimorphic image. In case the term order $<$ is not compatible with the grading, one still can construct a graded module from Y, by replacing each element of the Janet basis by its leading term. One obtains a Hilbert series again by applying the comments on the homogeneous case at the beginning of this section. This, however, needs some care for its interpretation.

Corollary 27. *In the situation of Theorem 26 the specialization procedure $(*)$ above turns the generalized Hilbert series of $\overline{S(X,<)}$ into the series $H_{\mathrm{gr}(Y)}(t)$, whose coefficient of t^i also counts the elements of $\overline{S(X,<)}$ lying in $(R^q)_i$.*

Analogously for X instead of Y, B from Theorem 19 is a basis adapted to the filtration, i.e. the elements g of B with $\mathrm{Lt}(g) \in (R^q)_{\leqq i}$ form a K-basis of X_i. The submodule of R^q generated by the $g_{\theta(g)}$ is isomorphic to $\mathrm{gr}(X)$ and its Hilbert series can be obtained by the procedure $(*)$ applied to $S(X,<)$ instead of $\overline{S(X,<)}$.

We note that a special case of Theorem 26 and Corollary 27 together with applications to linear pdes with constant coefficients was announced in [16]. However, the decreasing filtration given there has to be replaced by the increasing filtration given here.

5. Linear partial differential equations. This is the environment where Janet's algorithm was originally developed: The Hilbert series counts power series solutions and the generalized Hilbert series marks free Taylor coefficients. In this section let K be a differential field of characteristic zero with n commuting derivations $\partial_1, \ldots, \partial_n$. Because of $\partial_i f = \partial_i(f) + f\partial_i$ for all $f \in K$ and $i = 1, \ldots, n$, where functions are interpreted as multiplication operators for K, one can use these relations to form the iterated skew polynomial ring $R := K\langle\partial_1, \ldots, \partial_n\rangle$, which will remain fixed throughout this section. (Actually, Janet's algorithm works for the slightly more general class of skew polynomial rings considered in [7].) The elements of $p \in R$ can be written in the normal form

$$p = \sum_{i \in I} a_i \partial^i, \quad I \subset (\mathbb{Z}_{\geqq 0})^n \text{ finite}, \quad a_i \in K \quad \text{for all} \quad i \in I,$$

where $\partial^i := \partial_1^{i_1} \cdots \partial_n^{i_n}$. The ring R is non-commutative unless the field K happens to be the field of constants $K_0 := \{f \in K \mid \partial_i(f) = 0, i = 1, \ldots, n\}$, in which case

$R = K_0[\partial_1, \ldots, \partial_n]$ is a commutative polynomial ring as treated in the previous sections. Since R is non-commutative in general, we view R^q as left R-module and want to study its (left) submodules $X \leqq R^q$ and factor modules $Y := R^q/X$. Clearly, X is necessarily finitely generated and both, the definitions of $\mathrm{Mon}(R)$ and of $\mathrm{Mon}(R^q)$ carry over from the beginning of Section 2 where the monomial x^i is replaced by ∂^i. The results of Section 2 carry over trivially to the present case. From Section 3 also the definition of term order on $\mathrm{Mon}(R^q)$ can be kept together with the definition of leading monomials which are now called *leading derivatives* $\mathrm{Lt}(p) \in \mathrm{Mon}(R^q)$ for all $p \in R^q - \{0\}$. The basic property $\mathrm{Lt}(mp) = m\mathrm{Lt}(p)$ for all $p \in R^q - \{0\}$, $m \in \mathrm{Mon}(R)$, is retained. All the results including proofs of Section 3 remain valid with the obvious modifications: replacing x_i by ∂_i, multiple by left multiple, submodule by left submodule etc..

Coming to Section 4, the changes are of a more serious nature because in the non-commutative case R is not graded but only filtered. In particular, Remark 23 makes no sense and has no counterpart.

Lemma 28. 1) *Choose* $d : \{\partial_1, \ldots, \partial_n\} \to \mathbb{N}$ *and extend* d *to a monoid homomorphism* $d : \mathrm{Mon}(R) \to \mathbb{Z}_{\geqq 0}$ *and define for* $i \geqq 0$*:*

$$R_i := \bigoplus_{\substack{m \in \mathrm{Mon}(R) \\ d(m)=i}} Km, \qquad R_{\leqq i} := \bigoplus_{j \leqq i} R_j$$

to obtain an increasing filtration $\{0\} =: R_{\leqq -1} \subset R_{\leqq 0} \subseteqq R_{\leqq 1} \subseteqq \ldots \subseteqq R = \cup_i R_{\leqq i}$ *of* R, *i.e.* $R_{\leqq i} R_{\leqq j} \subseteqq R_{\leqq i+j}$. *The associated graded ring* $\mathrm{gr}(R) := \bigoplus_{i \geqq 0} (R_{\leqq i}/R_{\leqq i-1})$ *is isomorphic to the polynomial ring* $K[\partial_1, \ldots, \partial_n]$ *with the grading also defined by* d.

2) *In addition, assign values to the standard basis of* R^q, *i. e. define* $\theta : \{e_1, \ldots, e_q\} \to \mathbb{Z}$ *and extend* θ *to* $\mathrm{Mon}(R^q)$ *by* $\theta(me_i) = d(m) + \theta(e_i)$ *for all* $m \in \mathrm{Mon}(R)$, $i = 1, \ldots, q$. *Then*

$$(R^q)_{\leqq i} := \bigoplus_{j \leqq i} (R^q)_j \quad \text{with} \quad (R^q)_j := \bigoplus_{\substack{m \in \mathrm{Mon}(R^q) \\ \theta(m)=j}} Km$$

turns R^q *into a filtered* R*-module:* $\{0\} =: (R^q)_{\leqq i_0 - 1} \subset (R^q)_{\leqq i_0} \subseteqq (R^q)_{\leqq i_0+1} \subseteqq \ldots \subseteqq R^q = \cup_i (R^q)_{\leqq i}$ *with* $R_{\leqq i}(R^q)_{\leqq j} \subseteqq (R^q)_{\leqq i+j}$ *for all* $i \geqq 0$, $j \geqq i_0$. *Moreover,* $\mathrm{gr}(R^q)$ *is isomorphic as graded module to the free* $\mathrm{gr}(R)$*-module* $\mathrm{gr}(R)^q$.

The proof is straightforward. A (θ-)compatible term order $<$ on $\mathrm{Mon}(R^q)$ is defined by exactly the same conditions as in Definition 25. Now Theorem 26 remains valid for θ-compatible term orders rather than term orders compatible with the grading given there. Finally, Corollary 27 also remains true and the proofs need just minor adjustments.

Now follows the application to solution theory of linear pdes. Since we want to count power series solutions, the differential field K needs further specification: Assume that K is the field of fractions of some field of (real or complex) analytic functions in the independent variables $x_1, \ldots, x_n$ and let ∂_i be the partial derivatives with respect to x_i for $i = 1, \ldots, n$.

Let $\mathcal{S}$ be a left R-module, whose elements we usually think of as functions in $x_1, \ldots, x_n$, on which ∂_i acts as partial derivative with respect to x_i, though the module is arbitrary at this stage. Then each $p = p(\partial) \in R^q$ can be viewed as a row in $R^{1\times q}$ acting on the columns in $\mathcal{S}^{q\times 1}$ by matrix multiplication. Since columns in $\mathcal{S}^{q\times 1}$ can also be considered as R-module homomorphisms in $\operatorname{Hom}_R(R^{1\times q}, \mathcal{S})$, the well known algebraic interpretation of a solution $u \in \mathcal{S}^{q\times 1}$ of the linear pde-system $p(\partial)u = 0$ with $p(\partial) \in G$ for some finite subset $G \subset R^q$ is simply that of a homomorphism in $\operatorname{Hom}_R(R^q/X, \mathcal{S})$ with $X := \langle G\rangle_R$. Unfortunately, this algebraic way of viewing solutions is rather restricted. To get to an interpretation of the (generalized) Hilbert series, we introduce the following notion for points with coordinates in the field of constants K_0 of K:

Definition 29. A point $(a_1, \ldots, a_n) \in K_0^n$ is called *regular* for a (finite) subset $G \subset R^q$, if there is a K_0-subalgebra A of K closed under the partial derivatives ∂_i satisfying

1) $G \subset A\langle\partial_1, \ldots, \partial_n\rangle^q$,
2) the Janet basis of $\langle G\rangle_R$ can be obtained by computation over $A\langle\partial_1, \ldots, \partial_n\rangle$ only,
3) $(a_1, \ldots, a_n)$ is not a pole for any of the elements of A.

Clearly, the set of regular points is Zariski dense in K_0^n; in fact, for reasonably well behaved K, it can be easily obtained by keeping track of the denominators occurring in G and during the run of Janet's algorithm. For these regular points, the Hilbert series counts the free Taylor coefficients for power series solutions. To set up notation, let $(a_1, \ldots, a_n) \in K_0^n$ be a regular point for $G \subset R^q$, and let $\mathcal{S} := K_0[[x_1 - a_1, \ldots, x_n - a_n]]$ be the ring of formal power series over the field K_0 in the $x_i - a_i$. Note, $\mathcal{S}$ is an $A\langle\partial_1, \ldots, \partial_n\rangle$-module for a suitable A, as just defined. The elements of $K_0^{\operatorname{Mon}(R^q)}$, i.e. the mappings $f : \operatorname{Mon}(R^q) \to K_0$, are in bijection to $\mathcal{S}^{q\times 1}$:

$$\begin{aligned} K_0^{\operatorname{Mon}(R^q)} &\to \mathcal{S}^{q\times 1} : f \mapsto s \\ &= \left(\sum_{i \in (\mathbb{Z}_{\geqq 0})^n} f(\partial^i e_1) \frac{(x-a)^i}{i!}, \ldots, \sum_{i \in (\mathbb{Z}_{\geqq 0})^n} f(\partial^i e_q) \frac{(x-a)^i}{i!} \right)^{tr}, \end{aligned}$$

where $(x-a)^i := (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}$ and $i! := i_1! \cdots i_n!$. Let $X := \langle G\rangle_R$. The associated multiple-closed set of leading monomials of its non-zero elements gives rise to the partition $\operatorname{Mon}(R^q) = \overline{S(X,<)} \,\dot\cup\, S(X,<)$ and hence, via the above bijection, to a direct sum decomposition $\mathcal{S}^{q\times 1} = \mathcal{S}^{q\times 1}|_{\overline{S(X,<)}} \oplus \mathcal{S}^{q\times 1}|_{S(X,<)}$. The elements of $\overline{S(X,<)}$ are referred to as *parametric derivatives* and the ones of $S(X,<)$ as *principal derivatives* of the system of equations. The values of the former can be freely chosen, the values of the latter are then uniquely determined, cf. [15]:

Proposition 30. *Let $a := (a_1, \ldots, a_n) \in K_0^n$ be a regular point for $G \subset R^q$, $\mathcal{S} := K_0[[x_1 - a_1, \ldots, x_n - a_n]]$ and denote the K_0-space of solutions of G in $\mathcal{S}^{q\times 1}$ by $\mathcal{L}$. Then the projection $\pi_{\mathrm{par}} : \mathcal{S}^{q\times 1} \to \mathcal{S}^{q\times 1}|_{\overline{S(X,<)}}$ restricts to a K_0-isomorphism of $\mathcal{L}$ onto $\mathcal{S}^{q\times 1}|_{\overline{S(X,<)}}$.*

Proof. Let $J \subset R^q$ be the Janet basis of X with involutive reduction N_J. For any principal derivative $\partial^i e_k \in S(X,<)$, by definition $\partial^i e_k$ and $N_J(\partial^i e_k)$ are in the same coset modulo the equations, i.e. modulo X. Hence any solution $s \in \mathcal{S}^{q\times 1}$, represented by some $f \in K_0^{\mathrm{Mon}(R^q)}$ as above, satisfies $(\partial^i e_k)s = N_J(\partial^i e_k)s$. Substituting a for x gives $f(\partial^i e_k)$ in terms of the $f(v)$, where v are monomials occurring in $N_J(\partial^i e_k)$, i.e. in terms of parametric derivatives. This proves that π_{par} becomes injective upon restriction to $\mathcal{L}$. On the other hand, it also gives a construction of some $s \in \mathcal{S}^{q\times 1}$ from some $s' \in \mathcal{S}^{q\times 1}|_{\overline{S(X,<)}}$. We have to prove $gs = 0$ for all $g \in J$. Note first, $gs \in \mathcal{S}$; moreover $t \in \mathcal{S}$ is zero, if and only if $(mt)_{|x=a} = 0$ for all $m \in \mathrm{Mon}(R)$, because this procedure extracts the Taylor coefficients of t. But $m(gs) = (mg)s$ and each mg can be written (via L_J) in the form $\sum_{h\in J} p_h h$ with $p_h \in K\langle M(h)\rangle$. Therefore it suffices to show $(mgs)_{|x=a} = 0$ for all $m \in \mathrm{Mon}(K\langle M(g)\rangle)$ and all $g \in J$. But $mg = \mathrm{Lt}(mg) - N_J(\mathrm{Lt}(mg))$ for all $m \in \mathrm{Mon}(K\langle M(g)\rangle)$, $g \in J$, and these $\mathrm{Lt}(mg)$ lie in $S(X,<)$, i.e. $((\mathrm{Lt}(mg) - N_J(\mathrm{Lt}(mg)))s)_{|x=a} = 0$ are exactly the equations used above to define s. Hence $gs = 0$ for all $g \in J$. $\square$

Note, selecting an $s' \in \mathcal{S}^{q\times 1}|_{\overline{S(X,<)}}$ amounts to choosing certain power series over K_0 in certain $x_i - a_i$ according to the multiplicative variables of the cones in the Janet partition of $\overline{S(X,<)}$ as preassigned data. Janet applies the Cauchy-Kovalevskaya Theorem to prove convergence of the components of s, if these preassigned series converge, cf. also [17]. The numbers of cones of fixed dimension in the Janet partition of $\overline{S(X,<)}$, or certain refined partitions, are called (Cartan) characters in [15] resp. [17], where, however, δ-regular coordinates are assumed. Clearly, only the Cartan character for the top dimension is a structural invariant. But all this information can be read off from the partial fraction expansion of the Hilbert series as obtained in Corollary 11 and even much more precisely from the generalized Hilbert series of $\overline{S(X,<)}$, which enumerates the parametric derivatives.

Corollary 31. *In the situation of Proposition* 30 *assign standard degrees $d(\partial_i) = 1$ for $i = 1,\dots,n$ and $\theta(e_j) = 0$ for $j = 1,\dots,q$. Then the substitution process* $(*)$ *of Section 4 turns the generalized Hilbert series of $\overline{S(X,<)}$ into the Hilbert series which is the generating function for the number of parametric derivatives according to* (*differential*) *order.*

The key to an interpretation of free resolutions as outlined in Corollary 22 in the present situation is given by the passage from homogeneous linear pdes to inhomogeneous pdes: each homomorphism in the free resolution can be seen as a homogeneous linear pde system for the possible right hand sides of the previous pde system. Details on this iterated compatibility conditions can be found in [18].

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