European Journal of Combinatorics 41 (2014) 47–67

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Recurrence relations for graph polynomials on bi-iterative families of graphs

Tomer Kotek [a,1], Johann A. Makowsky [b]

[a] Institute for Information Systems, Vienna University of Technology, Vienna, Austria

[b] Faculty of Computer Science, Israel Institute of Technology, Haifa, Israel

ARTICLE INFO

Article history:
Received 16 September 2013
Accepted 19 March 2014
Available online 12 April 2014

ABSTRACT

We show that any graph polynomial from a wide class of graph polynomials yields a recurrence relation on an infinite class of families of graphs. The recurrence relations we obtain have coefficients which themselves satisfy linear recurrence relations. We give explicit applications to the Tutte polynomial and the independence polynomial. Furthermore, we get that for any sequence a_n satisfying a linear recurrence with constant coefficients, the sub-sequence corresponding to square indices a_{n^2} and related sub-sequences satisfy recurrences with recurrent coefficients.

1. Introduction

Recurrence relations are a major theme in the study of graph polynomials. As early as 1972, N.L. Biggs, R.M. Damerell and D.A. Sands [3] studied sequences of Tutte polynomials which are C-finite, i.e. satisfy a homogeneous linear recurrence relation with constant coefficients (or equivalently, sequences of coefficients of rational power series). More recently, M. Noy and A. Ribó [26] proved that over an infinite class of *recursively constructible families of graphs*, which includes e.g. paths, cycles, ladders and wheels, the Tutte polynomial is C-finite (see also [4]). The Tutte polynomials of many recursively constructible families of graphs received special treatment in the literature. Moreover, the Tutte polynomial can be defined through its famous deletion–contraction recurrence relation.

Similar recurrence relations have been studied for other graph polynomials, e.g. for the independence polynomial see e.g. [20,37]. E. Fischer and J.A. Makowsky [11] extended the result of Noy and

E-mail addresses: tomerkotek@gmail.com, kotek@forsyte.at, kotek@tuwien.ac.at (T. Kotek), janos@cs.technion.ac.il (J.A. Makowsky).

[1] Tel.: +43 158801184835.

http://dx.doi.org/10.1016/j.ejc.2014.03.012
0195-6698/

Ribó to an infinite class of graph polynomials definable in Monadic Second Order Logic (MSOL), which includes the matching polynomial, the independence polynomial, the interlace polynomial, the domination polynomial and many of the graph polynomials which occur in the literature. [11] applies to the wider class of *iteratively constructible graph families*. The class of MSOL-polynomials and variations of it were studied with respect to their combinatorial and computational properties, e.g. in [7,17,18,23]. L. Lovász treats MSOL-definable graph invariants in [21].

In this paper we consider recurrence relations of graph polynomials which go beyond C-finiteness. A sequence is C^2*-finite* if it satisfies a linear recurrence relation with C-finite coefficients. We start by investigating the set of C^2-finite sequences. The tools we develop apply to sparse sub-sequences of C-finite sequences. While C-finite sequences have received considerable attention in the literature, cf. e.g. [32, Chapter 4], and it is well-known that taking a linear sub-sequence a_{qn+r} of a C-finite sequence a_n yields again a C-finite sequence, it seems other types of sub-sequences have not been systematically studied. We show the following:

Theorem 1. *Let a_n be a C-finite sequence over $\mathbb{C}$. Let $c \in \mathbb{N}^+$ and $d, e \in \mathbb{Z}$. Then the sequence*

$$b_n = a_{c\binom{n}{2}+dn+e}$$

is C^2-finite.

In particular, a_{n^2} and $a_{\binom{n}{2}}$ are C^2-finite. The proof of Theorem 1 is given in Section 3. As an explicit example, we consider the Fibonacci numbers in Section 4.

Next, we show MSOL-polynomials satisfy C^2-recurrences on appropriate families of graphs. In Section 5 we introduce the notion of *bi-iteratively constructible graph families*, or *bi-iterative families* for short. This notion extends the notions of recursively constructible families of [26] and iteratively constructible families of [11]. In Section 6 we recall from the literature the definitions of two related classes of MSOL-polynomials and introduce a powerful theorem for them. The main theorem of the paper is:

Theorem 2 (*Informal*)**.** *MSOL-polynomials satisfy C^2-finite recurrences on bi-iterative families.*

Paths, cycles, cliques and stars are iterative families and therefore serve as simple examples of bi-iterative families. Some examples of bi-iterative families which are not iterative are depicted in Fig. 1.1:

1. G_n^1 is obtained from the disjoint union of the cycles $C_3, \ldots, C_{n+2}$ by selecting one vertex of each cycle and identifying the selected vertices in them.
2. G_n^2 is obtained from the disjoint union of the cliques $K_1, \ldots, K_{n+1}$ and the path of length P_{n+1} by identifying the ith vertex of the path with a vertex of the clique K_i.
3. G_n^3 is obtained from the disjoint union of the edgeless graphs $E_1, \ldots, E_{n+1}$ by adding all possible edges between E_i and E_{i+1}, $1 \le i \le n$.
4. G_n^4 is obtained from the disjoint union of the paths $P_1, \ldots, P_{n+1}$ as follows: denoting the leaves of the path P_i by a_i and b_i, the edges (a_i, a_{i+1}) and (b_i, b_{i+1}) are added for all $1 \le i \le n$.

More examples of bi-iterative families are given in Section 5 (see Fig. 5.2).

The exact statement of Theorem 2, namely Theorem 31, is given in Section 7 together with the proof. In Section 8 we compute explicit C^2-recurrences for the Tutte polynomial and the independence polynomial. Finally, in Section 9 we conclude and discuss future research.

2. C^2-finite sequences

In this section we define the recurrence relations we are interested in and give useful properties of sequences satisfying them.

Definition 3. Let $\mathbb{F}$ be a field. Let $a_n : n \in \mathbb{N}$ be a sequence over $\mathbb{F}$.

1. a_n is C-finite if there exist $s \in \mathbb{N}$ and $c^{(0)}, \ldots, c^{(s)} \in \mathbb{F}$, $c^{(s)} \neq 0$, such that for every $n \ge s$,

$$c^{(s)} a_{n+s} = c^{(s-1)} a_{n+s-1} + \cdots + c^{(0)} a_n.$$

We may assume w.l.o.g. that $c^{(s)} = 1$.

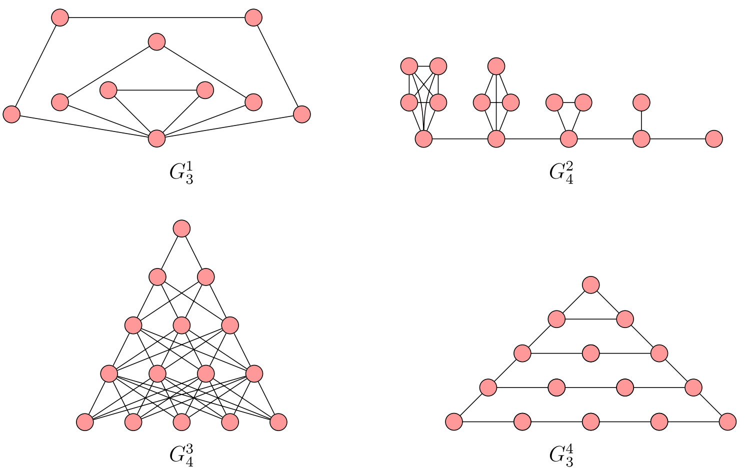

Fig. 1.1. Examples of graphs belonging to bi-iterative families.

2. a_n is P-recursive if there exist $s \in \mathbb{N}$ and $c_n^{(0)}, \ldots, c_n^{(s)}$ which are polynomials in n over $\mathbb{F}$, such that for every n we have $c_n^{(s)} \neq 0$, and for every $n \geq s$,

$$c_n^{(s)} a_{n+s} = c_n^{(s-1)} a_{n+s-1} + \cdots + c_n^{(0)} a_n. \quad (2.1)$$

3. a_n is C^2-finite if there exist $s \in \mathbb{N}$ and C-finite sequences $c_n^{(0)}, \ldots, c_n^{(s)}$, such that for every n we have $c_n^{(s)} \neq 0$, and for every $n \geq s$, Eq. (2.1) holds.

P-recursive (holonomic) sequences have been studied in their own right, but also as the coefficients of differentially finite generating functions [33], see also [29].

Example 4 (C^2-*Finite Sequences*)**.** Sequences with C^2-finite recurrences emerge in various areas of mathematics.

1. The q-derangement numbers $d_n(q)$ are polynomials in q related to the set of derangements of size n. A formula for computing them in analogy to the standard derangement numbers was found by I. Gessel [16] and M.L. Wachs [36]. This formula implies that the following C^2-recurrence holds:

$$d_n(q) = (q^n + [n])d_{n-1}(q) - q^n[n]d_{n-2}(q)\,,$$

see also [10]. We denote here $[n] = 1 + q + q^2 + \cdots + q^{n-1}$.

2. In knot theory, the colored Jones polynomial of a framed knot $\mathcal{K}$ in 3-space is a function from such knots to polynomials. The colored Jones function of the 0-framed right-hand trefoil satisfies the following C^2-recurrence [13]:

$$J_{\mathcal{K}}(n) = \frac{x^{2n-2} + x^8y^{4n} - y^n - x^2y^{2n}}{x(x^{2n}y - x^4y^n)} J_{\mathcal{K}}(n-1) + \frac{x^8y^{4n} - x^6y^{2n}}{x^4y^n - x^{2n}y} J_{\mathcal{K}}(n-2)$$

with $x = q^{1/2}$ and $y = x^{-2}$. See [14] and [15] for more examples.

Lemma 5 (*Properties*)**.**

1. *Every C-finite sequence is P-recursive.*
2. *Every P-recursive sequence is* C^2*-finite.*

If $\mathbb{F} = \mathbb{R}$*, then:*

3. *For every C-finite sequence* a_n*, there exists* $\alpha \in \mathbb{N}$ *such that* $a_n \leq \alpha^n$ *for every large enough* n*.*
4. *For every P-recursive sequence* a_n*, there exists* $\alpha \in \mathbb{N}$ *such that* $a_n \leq n!^\alpha$ *for every large enough* n*.*
5. *For every* C^2*-finite sequence* a_n *for which* $c_n^{(s)}$ *from Eq. (2.1) is always integer, there exists* $\alpha \in \mathbb{N}$ *such that* $a_n \leq \alpha^{n^2}$ *for every large enough* n*.*

Proof. 1 and 2 follow directly from Definition 3. 3, 4 and 5 can be proven easily by induction on n. □

The following will be useful, see e.g. [32]:

Lemma 6 (*Closure Properties*)**.** *The C-finite sequences are closed under:*

1. *Finite addition;*
2. *Finite multiplication;*
3. *Given a C-finite sequence a_n, taking sub-sequences a_{tn+s}, $t \in \mathbb{N}^+$ and $s \in \mathbb{Z}$.*

The sets of C-finite sequences and P-recursive sequences form rings with respect to the usual addition and multiplication. However, they are not integral domains. For every $i \le p$ and every n let

$$\mathbb{I}_{n\equiv i\,(mod\,p)} = \begin{cases} 1 & n \equiv i\,(mod\,p) \\ 0 & n \not\equiv i\,(mod\,p). \end{cases}$$

For every $i \le p$, $\mathbb{I}_{n\equiv i\,(mod\,p)}$ is C-finite. While each of $\mathbb{I}_{n\equiv 0\,(mod\,2)}$ and $\mathbb{I}_{n\equiv 1\,(mod\,2)}$ is not identically zero, their product is. This obstacle complicates our proofs in the sequel, and is overcome using a classical theorem on the zeros of C-finite sequences:

Theorem 7 (*Skolem–Mahler–Lech Theorem*)**.** *If a_n is C-finite, then there exist a finite set $I \subseteq \mathbb{N}$, $n_1, p \in \mathbb{N}$, and $P \subseteq \{0, \dots, p-1\}$ such that*

$$\{n \mid a_n = 0\} = I \cup \bigcup_{i\in P} \{n \mid n > n_1,\ n \equiv i\,(mod\,p)\}.$$

Remark 8. Recently J.P. Bell, S.N. Burris and K. Yeats [1] extended the Skolem–Mahler–Lech theorem extends to a Simple P-recursive sequences, P-recursive sequences where the leading coefficient is a constant.

2.1. C-finite matrices

A notion of sequences of matrices whose entries are C-finite sequences will be useful. We define this exactly and prove some properties of these matrix sequences.

Definition 9. Let $r \in \mathbb{N}$ and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of $r \times r$ matrices over a field $\mathbb{F}$. We call $\{A_n\}_{n=1}^{\infty}$ a *C-finite matrix sequence* if for every $1 \le i, j \le r$, the sequence $A_n[i, j]$ is C-finite.

Lemma 10. *Let $r, n_0 \in \mathbb{N}$ and let $\{A_n\}_{n=n_0}^{\infty}$ be a C-finite matrix sequence of $r \times r$ matrices over $\mathbb{C}$. The following hold:*

1. *The sequence $\left\{A_n^T\right\}_{n=1}^{\infty}$ is a C-finite matrix sequence.*
2. *The sequence $\{|A_n|\}_{n=1}^{\infty}$ is in C-finite.*
3. *For any fixed i, j, the sequence of consisting of the (i, j)-th cofactor of A_n is C-finite, and the sequence $\{C_n\}_{n=1}^{\infty}$ of matrices of cofactors of A_n is a C-finite matrix sequence.*
4. *There exist n_1 and p such that, for every $0 \le i \le p-1$ and $n \in \mathbb{N}^+$, $\left|A_{i+n_1}\right| = 0$ iff $\left|A_{pn+i+n_1}\right| = 0$.*

Proof. 1. Immediate.
2. The determinant is a polynomial function of the entries of the matrix, so it is C-finite by the closure of the set of C-finite sequences under finite addition and multiplication.
3. The cofactor is a constant times a determinant, so again it is C-finite.
4. This follows from the Lech–Mahler–Skolem property of C-finite sequences and from the fact that the determinant is a C-finite sequence. □

Lemma 11. *Let M be an $r \times r$ matrix. Let $c, d \in \mathbb{Z}$ with $c > 0$. Let*

$$M_n = \begin{cases} M^{cn+d}, & cn + d \ge 0 \\ 0, & otherwise. \end{cases}$$

The sequence M_n is a C-finite matrix sequence.

Proof. Let

$$\chi(\lambda) = \sum_{t=0}^{r} e_t \lambda^t$$

be the characteristic polynomial of M^c, with $e_r \neq 0$. By the Cayley–Hamilton theorem, $\chi(M^c) = 0$, so

$$0 = \sum_{t=1}^{r} e_t M^{ct} \tag{2.2}$$

with $e_r \neq 0$. If $d \geq 0$, then by multiplying Eq. (2.2) by $M^{c(n-r)+d}$, we get that for every i, j, the entry (i, j) in the sequence of matrices $M_n : n \in \mathbb{N}$ satisfies the recurrence

$$M_n[i,j] = -\sum_{t=1}^{r-1} \frac{e_t}{e_r} M_{n-r+t}[i,j].$$

If $d < 0$, there exists $r > 0$ such that $cr > |d|$. We have $M^{cn+d} = M^{c(n-r)+cr-|d|}$. The claim follows similarly to the case of $d \geq 0$ by multiplying Eq. (2.2) by $M^{cr-|d|}$ and setting $t = n - r$. □

Lemma 12. *Let $r, m, \ell \in \mathbb{N}$ and let $\{A_n\}_{n=n_0}^{\infty}$, $\{B_n\}_{n=n_0}^{\infty}$ be C-finite matrix sequences of consisting of matrices of size $r \times m$ respectively $m \times \ell$ over $\mathbb{C}$. Then A_nB_n is a C-finite matrix sequence.*

Proof. Let $1 \leq i \leq r$ and $1 \leq j \leq \ell$. Then

$$(A_nB_n)_{ij} = \sum_{k=1}^{m} (A_n)_{ik} (B_n)_{kj}$$

is a polynomial in C-finite matrix sequences. Hence, by the closure of C-finite sequences under finite addition and multiplication, A_nB_n is a C-finite matrix sequence. □

3. Proof of Theorem 1

The proof of Theorem 1 relies on the notion of a *pseudo-inverse of a matrix*. This notion is a generalization of the inverse of square matrices to non-square matrices. For an introduction, see [2]. We need only the following theorem:

Theorem 13 (*Moore–Penrose Pseudo-Inverse*)**.** *Let $\mathbb{F}$ be a subfield of $\mathbb{C}$. Let $s, t \in \mathbb{N}^+$. Let M be a matrix over $\mathbb{F}$ of size $s \times t$ with $s \geq t$ whose columns are independent. Then there exists a unique matrix M^+ over $\mathbb{F}$ of size $t \times s$ which satisfies the following conditions:*

1. *M^*M is non-singular;*
2. *$M^+ = (M^*M)^{-1} M^*$;*
3. *Let $b \in \mathbb{C}^s$. If the equation $Mx = b$ is consistent, then M^+b is a solution to $Mx = b$.*

M^* is the Hermitian transpose of M, i.e. M^* is obtained by taking the transpose of M and replacing each entry with its complex conjugate.

M^+ is called the Moore–Penrose pseudo-inverse of M.

The following is the main lemma necessary for the proof of Theorem 1. It allows to extract C^2-recurrences for individual sequences of numbers from recursion schemes with C-finite coefficients for multiple sequences of numbers.

Lemma 14. *Let $\mathbb{F}$ be a subfield of $\mathbb{C}$ and let $r \in \mathbb{N}^+$. For every $n \in \mathbb{N}^+$, let $\overline{v_n}$ be a column vector of size $r \times 1$ over $\mathbb{F}$. Let w_n be a C-finite sequence which is always positive. Let M_n be a C-finite matrix sequence consisting of matrices of size $r \times r$ over $\mathbb{F}$ such that, for every n,*

$$\overline{v_{n+1}} = \frac{1}{w_n} M_n \overline{v_n}. \tag{3.1}$$

For each $j = 1, \ldots, r$, $\overline{v_n}[j]$ is C^2-finite. Moreover, all of the $\overline{v_n}[j]$ satisfy the same recurrence relation (possibly with different initial conditions).

Proof. For every $i = 0, \ldots, r^2$, let $M_n^{\{i\}} = \frac{1}{w_{n+i-1} \cdots w_{n-1}} M_{n+i-1} \cdots M_{n-1}$. By Eq. (3.1), for every n,

$$\overline{v_{n+i}} = M_n^{\{i\}} \overline{v_{n-1}}. \tag{3.2}$$

Let $N_n^{\{0\}}, \ldots, N_n^{\{r^2\}}$ be the column vectors of size $r^2 \times 1$ corresponding to $M_n^{\{0\}}, \ldots, M_n^{\{r^2\}}$ with $N_n^{\{i\}}[r(k-1)+\ell] = M_n^{\{i\}}[k, \ell]$. For every fixed n, $N_n^{\{0\}}, \ldots, N_n^{\{r^2\}}$ are members of the vector space of column vectors over $\mathbb{F}$ of size $r^2 \times 1$. Since this vector space is of dimension r^2, $N_n^{\{0\}}, \ldots, N_n^{\{r^2\}}$ are linearly dependent. Let $s_n \in \{1, \ldots, r^2\}$ be such that $N_n^{\{s_n\}}, \ldots, N_n^{\{r^2-1\}}$ are linearly independent, but $N_n^{\{s_n\}}, \ldots, N_n^{\{r^2\}}$ are linearly dependent.

For every $t = 0, \ldots, r^2 - 1$ let $N_{n,t}$ be the $r^2 \times (r^2 - t)$ matrix whose columns are $N_n^{\{t\}}, \ldots, N_n^{\{r^2-1\}}$. We have that $N_{n,s_n}^* N_{n,s_n}$ is non-singular. Let

$$\widetilde{N_{n,t}} = C\left(N_{n,t}^* N_{n,t}\right)^T N_{n,t}^*$$

where $C\left(N_{n,t}^* N_{n,t}\right)^T$ is the transpose of the cofactor matrix of $N_{n,t}^* N_{n,t}$. For every square matrix A, let $|A|$ denote the determinant of A. Then

$$N_{n,s_n}^+ = \frac{1}{|N_{n,s_n}^* N_{n,s_n}|} \widetilde{N_{n,s_n}} = \left(N_{n,s_n}^* N_{n,s_n}\right)^{-1} N_{n,s_n}^*$$

is the Moore–Penrose pseudo-inverse of N_{n,s_n}.

Consider the system of linear equations

$$N_{n,s_n} y_{n,s_n} = N_n^{\{r^2\}} \tag{3.3}$$

with y_{n,s_n} a column vector of size $\left(r^2 - s_n\right) \times 1$ of indeterminates $y_{n,s_n}[k]$. Since $N_n^{\{s_n\}}, \ldots, N_n^{\{r^2\}}$ are linearly dependent, Eq. (3.3) is consistent. Let

$$y'_{n,s_n} = \widetilde{N_{n,s_n}} N_n^{\{r^2\}}$$
$$y_{n,s_n} = \frac{1}{\left|N_{n,s_n}^* N_{n,s_n}\right|} y'_{n,s_n}.$$

By Theorem 13, $y_{n,s_n} = N_{n,s_n}^+ N_n^{\{r^2\}}$ is a solution of Eq. (3.3). This solution which can be rephrased as the matrix equation:

$$y'_{n,s_n}[1] M_n^{\{s_n\}} + \cdots y'_{n,s_n}[r^2 - s_n] M_n^{\{r^2-1\}} = \left|N_{n,s_n}^* N_{n,s_n}\right| M_n^{\{r^2\}}. \tag{3.4}$$

Moreover, by Lemmas 12 and 10, y'_{n,s_n} is a C-finite vector sequence. Multiplying Eq. (3.4) from the right by $\overline{v_{n-1}}$ and rearranging, we get

$$y'_{n,s_n}[1] \overline{v_{n+s_n}} + \cdots y'_{n,s_n}[r^2 - s_n] \overline{v_{n+r^2-1}} - \left|N_{n,s_n}^* N_{n,s_n}\right| \overline{v_{n+r^2}} = 0. \tag{3.5}$$

For every n and every $s \geq s_n$, $\left|N_{n,s}^* N_{n,s}\right| \neq 0$, and for every $s < s_n$, $\left|N_{n,s}^* N_{n,s}\right| = 0$ due to the choice of s_n. By Lemma 10, there exists n_1 such that for $n \geq n_1$, s_n is periodic and let p be the period. Using this periodicity we can remove the dependence of Eq. (3.5) on the infinite sequence s_n, and instead use for all $n \geq n_1$ a finite number of values, $s_{n_1+1}, \ldots, s_{n_1+p}$:

$$\sum_{i=1}^{p} \mathbb{I}_{n \equiv i \,(mod\, p)} \left(\left(\sum_{j=1}^{r^2 - s_{n_1+i}} y'_{n,s_{n_1+i}}[j] \overline{v_{n+s_{n_1+i}+j-1}} \right) - \left|N_{n,s_{n_1+i}}^* N_{n,s_{n_1+i}}\right| \overline{v_{n+r^2}} \right) = 0$$

which can be rewritten as

$$q_n^{\{0\}}\overline{v_n} + \cdots + q_n^{\{r^2-1\}}\overline{v_{n+r^2-1}} = q_n^{\{r^2\}}\overline{v_{n+r^2}}$$

with

$$q_n^{\{t\}} = \begin{cases} \sum_{i=1}^{p} \mathbb{I}_{n\equiv i\,(mod\,p)} y'_{n,s_{n_1+i}}[t+1-s_{n_1+i}], & 0 \le t \le r^2-1 \\ \sum_{i=1}^{p} \mathbb{I}_{n\equiv i\,(mod\,p)} \left| N^*_{n,s_{n_1+i}} N_{n,s_{n_1+i}} \right|, & t = r^2. \end{cases}$$

Note that, as the result of the closure of the C-finite sequences under finite addition and multiplication, $q_n^{\{t\}}$ is C-finite. Moreover, note $q_n^{\{r^2\}}$ is non-zero. □

We can now turn the proof of Theorem 1:

Proof of Theorem 1. Let $\zeta(n) = c\binom{n}{2} + dn + e$. Let $b'_n = a_{\zeta(n)}$. We have

$$\zeta(n) - \zeta(n-1) = cn + d.$$

Let a_n satisfy the C-recurrence

$$a_{n+s} = c^{(s-1)}a_{n+s-1} + \cdots + c^{(0)}a_n. \tag{3.6}$$

In order to write the latter equation in matrix form, let

$$M = \begin{pmatrix} c^{(s-1)} & \cdots & c^{(0)} \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

wherethe empty entries are taken to be 0. Let $\overline{u_n} = (a_n, \ldots, a_{n-s+1})^{tr}$. We have

$$\overline{u_n} = M\overline{u_{n-1}},$$

and consequently,

$$\overline{u_{\zeta(n)}} = M^{cn+d}\overline{u_{\zeta(n-1)}}.$$

For large enough values of n such that $cn + d \ge 0$, M^{cn+d} is C-finite by Lemma 11. Hence, the desired result follows from Lemma 14. □

As immediate consequences of Lemma 14, we get closure properties for C^2-finite sequences over $\mathbb{C}$.

Corollary 15. *Let a_n and b_n be* C^2*-finite sequences. The following hold:*

1. *$a_n + b_n$ is* C^2*-finite*
2. *a_nb_n is* C^2*-finite*

Proof. Let a_n and b_n satisfy the following recurrences

$$c_{n+1}^{(s+1)}a_{n+s+1} = c_{n+1}^{(s)}a_{n+s} + \cdots + c_{n+1}^{(0)}a_n$$
$$d_{n+1}^{(s'+1)}b_{n+s'+1} = d_{n+1}^{(s')}b_{n+s'} + \cdots + d_{n+1}^{(0)}b_n$$

where the sequences $c_n^{(i)}$ and $d_n^{(i)}$ are C-finite and $c_n^{(s+1)}$ and $d_n^{(s+1)}$ are non-zero. It is convenient to assume without loss of generality that $s = s'$. We apply Lemma 14 for both cases:

1. For addition, let $\overline{v_n} = (a_{n+s} + b_{n+s}, a_{n+s}, \ldots, a_n, b_{n+s}, \ldots, b_n)^{tr}$, $\gamma_n^p = c_n^{(p)}(c_n^{(s+1)})^{-1}$, $\delta_n^p = d_n^{(p)}(d_n^{(s+1)})^{-1}$, and M be the $(2s+3) \times (2s+3)$-matrix

$$M_n = \begin{pmatrix} 0 & \gamma_{n+1}^{(s)} & \cdots & \gamma_{n+1}^{(1)} & \gamma_{n+1}^{(0)} & \delta_{n+1}^{(s)} & \cdots & \delta_{n+1}^{(1)} & \delta_{n+1}^{(0)} \\ & \gamma_{n+1}^{(s)} & \cdots & \gamma_{n+1}^{(1)} & \gamma_{n+1}^{(0)} & & & & \\ & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 1 & & & & & \\ & & & & \delta_{n+1}^{(s)} & \cdots & \delta_{n+1}^{(1)} & \delta_{n+1}^{(0)} & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \end{pmatrix}$$

where the empty entries are taken to be 0. We have $\overline{v_{n+1}} = M_n \overline{v_n}$ and the claim follows from Lemma 14.
2. For multiplication, let $\overline{v_n} = \left(a_{n-p_1} b_{n-p_2} : 0 \le p_1, p_2 \le s\right)^{tr}$. Similarly to the case of addition, we can define M_n such that $\overline{v_{n+1}} = M_n \overline{v_n}$. M_n is using the following formulas which are obtained directly from the recurrences of a_n and b_n: for all $0 \le p \le s$,

$$c_{n+1}^{(s+1)} d_{n+1}^{(s+1)} a_{n+s+1} b_{n+s+1} = \sum_{p_1, p_2 = 0}^{s} c_{n+1}^{(p_1)} d_{n+1}^{(p_2)} a_{n+p_1} b_{n+p_2}$$

$$c_{n+1}^{(s+1)} a_{n+s+1} b_{n+p} = \sum_{p_1=0}^{s} c_{n+1}^{(p_1)} a_{n+p_1} b_{n+p}$$

$$d_{n+1}^{(s+1)} a_{n+p} b_{n+s+1} = \sum_{p_2=0}^{s} d_{n+1}^{(p_2)} a_{n+p} b_{n+p_2}.$$

4. Fibonacci numbers

The Fibonacci numbers of squares and of the triangular numbers are examples of sequences covered by Theorem 1.

The Fibonacci number F_n, given by the famous recurrence

$$F_{n+2} = F_{n+1} + F_n$$

with $F_1 = 1$, $F_2 = 1$, can also be described in terms of counting binary words. F_n counts the binary words of length $n-2$ which do not contain consecutive 1s. Similarly, F_{n-1} counts the binary words of length $n-2$ which begin with 0 (or, equivalently, end with 0), and F_{n-2} counts the binary words which begin (end) with 1. Let $W_n = F_{n+2}$.

Let $0 < k < m$, then

$$W_{m+k} = W_{k-1} W_m + W_{k-2} W_{m-1}$$

since $W_{k-1} W_m$ counts the binary words of length $m+k$ with no consecutive 1s which have 0 at index k, and $W_{k-2} W_{m-1}$ counts the binary words with no consecutive 1s which have 1 at index k and therefore 0 at index $k-1$. This translates back to the Fibonacci numbers as:

$$F_{m+k+2} = F_{k+1} F_{m+2} + F_k F_{m+1}.$$

So we have for the appropriate choices of m and k:

$$\begin{aligned} F_{(n+1)^2} &= F_{2n+1} F_{n^2+1} + F_{2n} F_{n^2} \qquad (4.1) \\ F_{n^2} &= F_{2n} F_{(n-1)^2} + F_{2n-1} F_{(n-1)^2-1} \\ F_{n^2+1} &= F_{2n+1} F_{(n-1)^2} + F_{2n} F_{(n-1)^2-1}. \end{aligned}$$

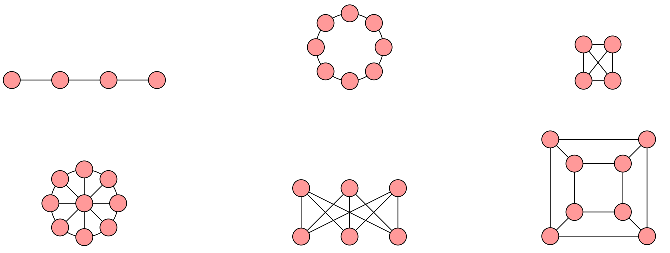

Fig. 5.1. Examples of graphs belonging to the iterative families: paths, cycles, cliques, wheels, complete bipartite graphs and prisms. They are also bi-iterative families.

Extracting $F_{(n-1)^2-1}$ from the second equation, we get:

$$F_{(n-1)^2-1} = \frac{F_{n^2} - F_{2n}F_{(n-1)^2}}{F_{2n-1}}$$

and substituting $F_{(n-1)^2-1}$ in the third equation, we have:

$$F_{n^2+1} = \frac{F_{2n}F_{n^2} + (F_{2n-1}F_{2n+1} - F_{2n}^2)F_{(n-1)^2}}{F_{2n-1}}$$

and substituting into Eq. (4.1), we have

$$F_{2n-1}F_{(n+1)^2} = F_{2n}\left(F_{2n+1} + F_{2n-1}\right)F_{n^2} + F_{2n+1}\left(F_{2n-1}F_{2n+1} - F_{2n}^2\right)F_{(n-1)^2}$$

where F_{2n-1}, $F_{2n}\left(F_{2n+1} + F_{2n-1}\right)$ and $F_{2n+1}\left(F_{2n-1}F_{2n+1} - F_{2n}^2\right)$ are C-finite by the closure properties of C-finite sequences in Lemma 6. Similarly, we can derive the following C^2-recurrence for $F_{\binom{n+1}{2}}$:

$$F_{n-1}F_{\binom{n+1}{2}} = \left(F_{n-1}F_{n+1} + F_nF_{n-2}\right)F_{\binom{n}{2}} + \left(F_nF_{n-1}^2 - F_{n-2}F_n^2\right)F_{\binom{n-1}{2}}.$$

The sequences F_{n^2} and $F_{\binom{n}{2}}$ are cataloged in the On-Line Encyclopedia of Integer Sequences [27] as (A054783) and (A081667). $\square$

5. Bi-iterative graph families

In this section we define the notion of a bi-iterative graph family, give examples for some simple families which are bi-iterative and provide some simple lemmas for them. The graph families we are interested in are built recursively by applying *basic operations* on k-graphs. A *k-graph* is of the form

$$G = (V, E; R_1, \ldots, R_k)$$

where (V, E) is a simple graph and $R_1, \ldots, R_k \subseteq V$ partition V. The sets $R_1, \ldots, R_k$ are called *labels*. The labels are used technically to aid in the description of the graph families, but we are really only interested in the underlying graphs. Before we give precise definitions and auxiliary lemmas for constructing bi-iterative graph families, we give some examples of bi-iterative graph families.

Example 16 (*Bi-Iterative Graph Families*)**.** See Figs. 1.1, 5.1 and 5.2 for illustrations of the following graph families.

1. Iteratively families, such as paths, cycles, and cliques, serve as simple examples of bi-iteratively constructible families.
2. G_0^1 is a single vertex labeled 2. For each n, G_n^1 has one vertex labeled 2 and all others are labeled 1. G_n^1 is obtained from G_{n-1}^1 by adding a cycle of size $n+2$ and identifying one vertex of the cycle with the vertex labeled 2 is in G_{n-1}^1. All other vertices in the cycle are labeled 1.
3. G_n^2 is obtained from a path of length $n+1$ by adding, for each vertex $1 \le i \le n+1$, a new clique of size i, and identifying one vertex of the clique with i.

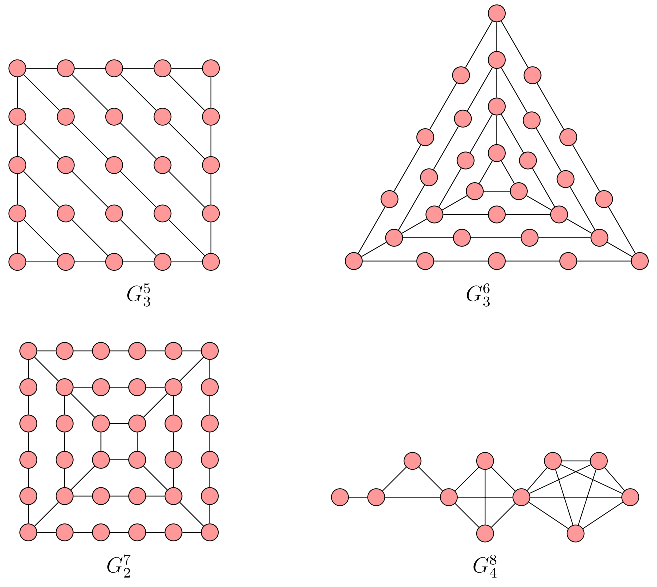

Fig. 5.2. Examples of graphs belonging to some bi-iterative families from Example 16.

4. G_0^3 is a single vertex labeled 2. G_n^3 is obtained from G_{n-1}^3 by adding $n+1$ isolated vertices labeled 3, adding all possible edges between vertices labeled 2 and vertices labeled 3, and relabeling 2 to 1, and then from 3 to 2.
5. G_0^4 consists of a triangle in which the vertices are labeled 1, 2, 3. G_n^4 is obtained from G_{n-1}^4 by adding a path P_{n+2} whose end-points are labeled 4 and 5. Then, the edges $\{2, 4\}$ and $\{3, 5\}$ are added, and the labels are changed so that the endpoints of the P_{n+2} path are now labeled 2 and 3, and all other vertices in G_n^4 are labeled 1.
6. G_0^5 is obtained by taking two disjoint copies of G_0^4 and respectively identifying the vertices labeled 2 and 3. G_n^5 is obtained from two disjoint copies of G_{n-1}^4 by adding a path P_{n+2} and connecting each of its endpoints to the corresponding end-points labeled 2 and 3 of the two copies of G_n^5.
7. G_0^6 consists of a triangle in which the vertices are labeled 2, 3, 4. G_n^6 is obtained by adding to G_{n-1}^6 a cycle of size $3n+3$ in which three vertices are labeled 5, 6, 7. Between each of the pairs $(5, 6)$, $(6, 7)$ and $(5, 7)$ there are n vertices labeled 1. Then, 2, 3, 4 are connected to 5, 6, 7 respectively, and the labels are changed so that only the vertices labeled 5, 6, 7 remain labeled, and their new labels are 2, 3, 4.
8. The family G_n^7 is similar to G_n^6, except we add a cycle of size $8n+4$, we have four distinguished vertices separated by n vertices labeled 1, etc.
9. G_0^3 is a single vertex labeled 1. G_n^8 is obtained from G_{n-1}^3 by adding $n-1$ isolated vertices labeled 2 and one isolated vertex labeled 3, adding all possible edges between vertices labeled 1, 2 and 3, and relabeling 1 and 2 to 4, and then from 3 to 1.

Now we proceed to define the precise definitions which allow us to build such families.

Definition 17 (*Basic and Elementary Operations*)**.** The following are the *basic operations* on k-graphs:

1. $Add_i(G)$: A new vertex is added to G, where the new vertex belongs to R_i;
2. $\rho_{i\to j}(G)$: All the vertices in R_i are moved to R_j, leaving R_j empty;
3. $\eta_{i,j}(G)$: All possible edges between vertices labeled i and vertices labeled j are added;
4. $\eta_{i,j}^b(G)$: If $R_i \cup R_j \leq b$, then $\eta_{i,j}^b(G) = \eta_{i,j}(G)$; otherwise $\eta_{i,j}^b(G) = G$;
5. $\delta_{i,j}(G)$: All edges between vertices labeled i and vertices labeled j are removed.

An operation F on k-graphs is *elementary* if F is a finite composition of any of the basic operations on k-graphs. We denote by *id* the elementary operation which leaves the k-graph unchanged.

Definition 18 (*Bi-Iterative Graph Families*)**.** Let $k \in \mathbb{N}$, G_0 be a k-graph and F, H, L be elementary operations on k-graphs,

1. The sequence $F(G_n) : n \in \mathbb{N}$ is called an F-iteration family and is said to be an *iteratively constructible family*.
2. The sequence $G_{n+1} = H(F^n(L(G_n))) : n \in \mathbb{N}$ is called an (H, F, L)-bi-iteration family and is said to be a *bi-iteratively constructible family*. By $F^n(G)$ we mean the result of performing n consecutive applications of F on G.

Let $G_n : n \in \mathbb{N}$ be a family of graphs. This family is (bi-)iteratively constructible if there exists $k \in \mathbb{N}$ and a family $G'_n : n \in \mathbb{N}$ of k-graphs which is (bi-)iteratively constructible, such that G_n is obtained from G'_n by ignoring the labels.

It is sometimes convenient to describe G_0 using basic operations on the empty graph $\emptyset$. Note that it is always possible to construct any fixed graph G_0 with n vertices using basic operations provided that $k \geq n$.

We can now prove the observation from Example 16(1):

Lemma 19. *Every iteratively constructible family is bi-iteratively constructible.*

Proof. If F is an elementary operation such that $G_n : n \in \mathbb{N}$ is an F-iteration family, then $G_n : n \in \mathbb{N}$ is also an (F, id, id)-bi-iteration family. □

All of the families in Example 16 are bi-iteratively constructible families which are not iteratively constructible. They all grow too quickly to be iteratively constructible. Now consider for instance G_n^3. Let[2] $F = Add_3$, $H = Add_3 \circ \eta_{2,3} \circ \rho_{2\to 1} \circ \rho_{3\to 2}$ and $L = \emptyset$. We have $G_{n+1}^3 = H(F^n(L(G_n)))$.

In the sequel we will want to distinguish a particular type of bi-iterative families, in which every application of $\eta_{i,j}$ only adds at most a fixed amount of edges.

Definition 20 (*Bounded Bi-Iterative Families*)**.** A basic operation is *bounded* if it not of the type $\eta_{i,j}$. A bi-iteratively constructible graph family $G_n : n \in \mathbb{N}$ is *bounded* if its construction uses only bounded basic operations.

Example 21. Considering the families of Example 16, it is not hard to see that G_n^1, G_n^4, G_n^5,G_n^6 and G_n^7 are bounded bi-iterative families, while G_n^2, G_n^3 and G_n^8 are bi-iterative families which are not bounded.

5.1. Lemmas for building bi-iterative graph families

Here we give some lemmas which are useful to make the construction of bi-iterative families easier. Their aim is to help the reader understand which families of graph are bi-iterative.

Lemma 22. *Let $G_n^A, G_n^B : n \in \mathbb{N}$ be two bi-iteratively constructible families. The family $G_n^A \sqcup G_n^B : n \in \mathbb{N}$ obtained by taking the disjoint union of the two families is bi-iteratively constructible. In particular, if both families $G_n^A, G_n^B : n \in \mathbb{N}$ are iteratively constructible, then so is $G_n^A \sqcup G_n^B : n \in \mathbb{N}$.*

Proof. Let H_O, F_O, L_O be elementary operations such that $G_n^O : n \in \mathbb{N}$ is an (H_O, F_O, L_O)-bi-iteration family for $O = A, B$. We can assume w.l.o.g. that the labels of the two families are disjoint; if they are not, we can simply rename the labels used by one of the families. The family $G_n^A \sqcup G_n^B : n \in \mathbb{N}$ is an $(H_A \circ H_B, F_A \circ F_B, L_A \circ L_B)$-bi-iteration family, where $\circ$ denotes the composition of operations. The case in which $G_n^A, G_n^B : n \in \mathbb{N}$ are iteratively constructible is similar. □

[2] We use $\circ$ to denote the composition of functions as follows: $g(f(x)) = (f \circ g)(x)$.

Lemma 23. *Let $G_n : n \in \mathbb{N}$ be an iteratively constructible family of k-graphs and let H and L be two elementary operations over k-graphs. Let D_0 be a k-graph, and $D_{n+1} = H(L(D_n) \sqcup G_n)$. The family $D_n : n \in \mathbb{N}$ is bi-iteratively constructible.*

Proof. Let F be an elementary operation such that $G_n : n \in \mathbb{N}$ is an F-iteration family. Let F' and G'_0 be the same as F and G_0, except that the labels they use are changed as follows. If a basic operation in F uses label i, then the corresponding operation in F' uses label $i+k$. For every $i = 1, \dots, k$, let $\rho_i = \rho_{i+k\to i}$. Let ρ be the composition $\rho_1 \circ \cdots \circ \rho_k$. If a vertex in G_0 has label i, then the corresponding vertex in G'_0 has label $i+k$. For every vertex v of G_0 with label i, let $a_v = Add_{i+k}$. Let a be the composition of a_v, $v \in V(G_0)$. We have $D_{n+1} = H(\rho(F'^n(a(L(D_n)))))$, and therefore $D_n : n \in \mathbb{N}$ is a bi-iteratively constructible family of $2k$-graphs. □

Using Lemma 23, it is easy to show that some families from Example 16 are indeed bi-iterative.

Example 24. Consider G_n^4 from Example 16. From Lemma 19 we get that $\tilde{P}_n = P_{n+3}^{4,5}$ is an iterative family. We define $E_{n+1} = H(L(E_n) \sqcup \tilde{P}_n)$ with $L = \rho_{1\to5} \circ \rho_{2\to6}$ and $H = \eta_{2,4} \circ \eta_{3,5} \circ \rho_{2\to1} \circ \rho_{3\to1} \circ \rho_{4\to2} \circ \rho_{3\to5}$. We get $G_n^4 = E_n$.

Subfamilies of iteratively constructible families give rise to many other related bi-iteratively constructible families:

Lemma 25. *Let $G_n : n \in \mathbb{N}$ be iteratively constructible.*

1. *$G_{\binom{n}{2}} : n \in \mathbb{N}$ and $G_{n^2} : n \in \mathbb{N}$ are bi-iteratively constructible.*
2. *Let $c \in \mathbb{N}^+$ and $d, e \in \mathbb{Z}$. There exists $r \in \mathbb{N}$ such that $H_n = G_{cm^2+dm+e} : m \in \mathbb{N}, m = n + r$ is bi-iteratively constructible.*

Proof. Let F be an elementary operation such that $G_n : n \in \mathbb{N}$ is an F-iteration family.

1. $G_{\binom{n}{2}} : n \in \mathbb{N}$ is an (id, F, id)-bi-iteration family. The proof is by induction on n with $G_{\binom{0}{2}} = G_0$ and
$$id\left(F^n\left(id\left(G_{\binom{n}{2}}\right)\right)\right) = F^{n+\binom{n}{2}}(G_0) = F^{\binom{n+1}{2}}(G_0) = G_{\binom{n+1}{2}}.$$
$G_{n^2} : n \in \mathbb{N}$ is an (id, F^2, F)-bi-iteration family. Again by induction with $G_{0^2} = G_0$ and
$$id\left(F^{2n}\left(F(G_{n^2})\right)\right) = F^{2n+1+n^2}(G_0) = F^{(n+1)^2}(G_0) = G_{(n+1)^2}.$$
2. Since $c > 0$, there exists $r \in \mathbb{N}$ such that $c(n+r)^2 + d(n+r) + e = cn^2 + d'n + e'$ and $d', e' \geq 0$. Let $H_0 = G_{e'}$, then $H_n : n \in \mathbb{N}$ is an $(id, F^{2c'}, F^{d'+1})$-bi-iteration family. Here again the proof is by induction on n. □

5.2. Families which are not bi-iterative

Clique-width is a graph parameter which generalizes tree-width, and is very useful for designing efficient algorithms for NP-hard problems, see e.g. [8,28].

Definition 26. The clique-width $cwd(G)$ of a graph G is the minimal $k \in \mathbb{N}$ such that there exists a k-graph H whose underlying graph is isomorphic to G and which can be obtained from $\emptyset$ by applying the basic operations Add_i, $\rho_{i\to j}$, $\eta_{i,j}$ and $\delta_{i,j}$ from Definition 17.

Bi-iterative families have bounded clique-width. Using this fact we easily get examples of families which are not bi-iterative.

Lemma 27. *If $G_n : n \in \mathbb{N}$ is a bi-iterative family of k-graphs, then for every n, G_n has clique-width at most k.*

Proof. Let G_n be a (H, F, L)-bi-iteration family of k-graphs. Since G_0 is a k-graph, it can be expressed by the basic operations Add_i, $\rho_{i\to j}$, and $\eta_{i,j}$ on $\emptyset$. For every $n > 0$, G_n is a composition of the operations H,

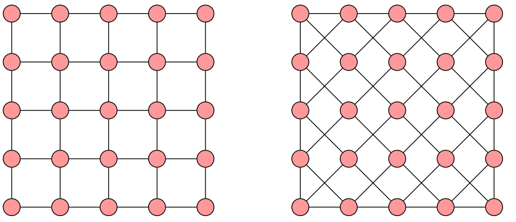

Fig. 5.3. Examples of graphs belonging to two families which are not bi-iterative, because they have unbounded clique-width.

F and L, which are in turn compositions of basic operations. Therefore, for every n, G_n can be obtained from $\emptyset$ by applying operations of the form Add_i, $\rho_{i\to j}$, $\eta_{i,j}$, $\delta_{i,j}$, and $\eta^b_{i,j}$. It remains to notice that whenever an operation $\eta^b_{i,j}$ is applied to a k-graph G', it can be either replaced by $\eta_{i,j}$ or omitted, depending on whether the number of vertices in G' labeled i or j is smaller or equal to b or not. Therefore, for every n, G_n can be obtained from $\emptyset$ by applying operations of the form Add_i, $\rho_{i\to j}$, $\eta_{i,j}$ and $\delta_{i,j}$ (but no operations of the form $\eta^b_{i,j}$). Therefore, each G_n is of clique-width as most k. □

Graph families which have unbounded clique-width, like square grids and other lattice graphs, are not bi-iterative. It is instructive to compare the graphs in Fig. 5.3 with the graphs of Figs. 1.1 and 5.2.

6. Graph polynomials and MSOL

We consider in this paper two related rich families of graph polynomials with useful decomposition properties. These graph polynomials are defined using a simple logical language on graphs.

6.1. Monadic second order logic of graphs, MSOL

We define the logic MSOL of graphs inductively. We have three types of variables: $x_i : i \in \mathbb{N}$ which range over vertices, $U_i : i \in \mathbb{N}$ which range over sets of vertices and $B_i : i \in \mathbb{N}$ which range over sets of edges. We assume our graphs are *ordered*, i.e. that there exists an order relation $\leq$ on the vertices. Atomic formulas are of the form $x_i = x_j$, $(x_i, x_j) \in E$, $x_i \leq x_j$, $x_i \in U_j$ and $(x_i, x_j) \in B_\ell$. The logical formulas of MSOL are built inductively from the atomic formulas by using the connectives $\vee$ (or), $\wedge$ (and), $\neg$ (negation) and $\to$ (implication), and the quantifiers $\forall x_i$, $\exists x_i$, $\forall U_i$, $\exists U_i$, $\forall B_i$, $\exists B_i$ with their natural interpretation.

If no variable B_i occurs in the formula, then the formula is said to be in MSOL_G, MSOL on graphs. Otherwise, the formula is said to be on hypergraphs.[3] Sometimes additional modular quantifiers are allowed, giving rise to the extended logic CMSOL. The counting quantifiers are of the form $C_q x\, \varphi(x)$, whose semantics is that the number of elements from the universe satisfying φ is zero modulo q. On structures containing an order relation, as is the case here, CMSOL and MSOL are equivalent, cf. [6].

Example 28. 1. We can express in MSOL that a set of edges B_1 is a matching:

$$\varphi_{match}(B_1) = \forall x_1 \forall x_2 \forall x_3\, ((x_1, x_2) \in B_1 \wedge (x_2, x_3) \in B_1 \to x_1 = x_3)\,.$$

2. We can express in MSOL that a set of vertices U_1 is an independent set:

$$\varphi_{ind}(U_1) = \forall x_1 \forall x_2\, ((x_1, x_2) \in E \to (x_1 \notin U_1 \vee x_2 \notin U_1))$$

where write e.g. $x_1 \notin U_1$ as shorthand for $\neg(x_1 \in U_1)$. Note $\varphi_{ind}(U_1)$ is a MSOL_G formula.

3. A graphs is 3-colorable iff it satisfies the following MSOL_G formula:

$$\exists U_1 \exists U_2 \exists U_3 \left(\varphi_{partition}(U_1, U_2, U_3) \wedge \varphi_{ind}(U_1) \wedge \varphi_{ind}(U_2) \wedge \varphi_{ind}(U_3)\right)$$

3 MSOL_G is referred to as node-MSOL in [21], as MS_1 in [6], and as $\mathrm{MSOL}(\tau_{graph})$ in [18]. Full MSOL is sometimes referred to as MS_2 or as $\mathrm{MSOL}(\tau_{hypergraph})$. τ_{graph} and $\tau_{hypergraph}$ are vocabularies whose structures represent graphs in different ways, the later of which can also be used to represent hypergraphs.

where $\varphi_{partition}$ expresses that U_1, U_2, U_3 form a partition of the vertices:

$$\varphi_{partition}(U_1, U_2, U_3) = \forall x_1 (x_1 \in U_1 \lor x_1 \in U_2 \lor x_1 \in U_3) \land \forall x_1 \neg (x_1 \in U_1 \land x_1 \in U_2) \land \forall x_1 \neg (x_1 \in U_2 \land x_1 \in U_3) \land \forall x_1 \neg (x_1 \in U_1 \land x_1 \in U_3)$$

4. We can express in MSOL that a vertex x_1 is the first element is its connected component in the graph spanned by B_1 with respect to the ordering of the vertices:

$$\varphi_{fconn}(x_1, B_1) = \forall x_2 (\varphi_{sc}(x_1, x_2) \to x_1 \le x_2)$$

where $\varphi_{sc}(x_1, x_2)$ says that x_1 and x_2 belong to the same connected component in the graph spanned by B_1:

$$\varphi_{sc}(x_1, x_2, B_1) = \forall U_1 ((x_1 \in U_1 \land x_2 \notin U_1) \to \exists x_3 \exists x_4 (B_1(x_3, x_4) \land x_3 \in U_1 \land x_4 \notin U_1)).$$

The formula $\varphi_{fconn}(x_1, B_1)$ will be useful when we discuss the definability of the Tutte polynomial.

6.2. MSOL-polynomials

MSOL-polynomials are a class of inductively defined graph polynomials given e.g. in [17]. It is convenient to refer to them in the following normal form:

$$p = \sum_{U_1,\ldots,U_\ell,B_1,\ldots,B_m : \Phi(\bar{U},\bar{B})} X_1^{|U_1|} \cdots X_{\ell'}^{|U_{\ell'}|} X_{\ell'+1}^{|B_1|} \cdots X_{\ell'+m'}^{|B_{m'}|}$$

where Φ is an MSOL formula with the iteration variables indicated and $\ell' \le \ell$, $m' \le m$. $\bar{U}, \bar{B}$ is short for $U_1, \ldots, U_\ell, B_1, \ldots, B_m$. If $m = 0$ and all the formulas are MSOL_G formulas, then we say p is a MSOL_G-polynomial. It is often convenient to think of the indeterminates X_i as *multiplicative weights* of vertices and edges.

While every MSOL_G-polynomial is a MSOL-polynomial, the converse is not true. The independence polynomial, the interlace polynomial [5], the domination polynomial and the vertex cover polynomial are MSOL_G-polynomials. The Tutte polynomial, the matching polynomial, the characteristic polynomial and the edge cover polynomial are MSOL-polynomials. We illustrate this for the independence polynomial and the Tutte polynomial.

6.3. The independence polynomial

The independence polynomial is the generating function of independent sets,

$$I(G) = \sum_{j=0}^{n} ind_G(j) X^j,$$

where $ind_G(j)$ is the number of independent sets of size j and n is the number of vertices in G. It is a MSOL_G-polynomial, given by

$$I(G) = \sum_{U_1 : \Phi_{ind}(U_1)} X^{|U_1|}$$

where $\Phi_{ind} = \varphi_{ind}$ from Example 28 says U_1 is an independent set.

6.4. The Tutte polynomial and the chromatic polynomial

The chromatic polynomial is defined in terms of counting proper colorings, but it can be written as a subset expansion which resembles an MSOL-polynomial as follows:

$$\chi(G) = \sum_{A \subseteq E} (-1)^{|A|} X^{k(A)} \qquad (6.1)$$

where $k(A)$ is the number of connected components in the spanning subgraph of G with edge set A.

Therefore, $\chi(G)$ is an evaluation of the dichromatic polynomial given by

$$Z(G) = \sum_{A \subseteq E} Y^{|A|} X^{k(A)}$$

which is an MSOL-polynomial:

$$Z(G) = \sum_{U_1, B_1 : \Phi_1} Y^{|B_1|} X^{|U_1|}$$

with Φ_1 says that U_1 is the set of vertices which are minimal in their connected component in the graph (V, B_1) with respect to the ordering on the vertices

$$\Phi_1 = \forall x \left(x \in U_1 \leftrightarrow \varphi_{fconn}(x_1, B_1) \right),$$

where φ_{fconn} is from Example 28. The dichromatic polynomial is related to the Tutte polynomial via the following relation:

$$T(G, X, Y) = \frac{Z(G, (X-1)(Y-1), Y-1)}{(X-1)^{k(E)}(Y-1)^{|V|}}.$$

The Tutte polynomial can also be shown to be an MSOL-polynomial via its definition in terms of spanning trees.

6.5. *A decomposition theorem for MSOL-polynomials*

The main technical tool from model theory that we use in this paper is a decomposition property for MSOL-polynomials, which resembles decomposition theorems for formulas of First Order Logic, FOL, and MSOL. For an extensive survey of the history and uses of such decomposition theorems see [24].

In Theorem 29 we rephrase Theorem 6.4 of [24]. For simplicity, we do not introduce the general machinery that is used there, e.g. instead of the notion of MSOL-*smoothness of binary operations* we limit ourselves to our elementary operations (see Section 4 of [24] for more details). Some other small differences follow from the proof of Theorem 6.4.

Theorem 29 ([24], *See Also* [11]). *Let k be a natural number. Let P be a finite set of MSOL-polynomials. Then there exists a finite set of MSOL-polynomials $P' = \{p_0, \dots, p_\alpha\}$ such that $P \subseteq P'$ and for every elementary operation σ on k-graphs, the following holds. If either all members of P are* MSOL$_G$*-polynomials, or σ consists only of bounded basic operations, then there exists a matrix M_σ such that for every graph G,*

$$\left(p_0(\sigma(G), \bar{X}), \dots, p_\alpha(\sigma(G), \bar{X})\right)^{tr} = M_\sigma \left(p_0(G, \bar{X}), \dots, p_\alpha(G, \bar{X})\right)^{tr}.$$

M_σ is a matrix of size $\alpha \times \alpha$ of polynomials with indeterminates $\bar{X}$ with integer coefficients. Additionally, if all members of P are MSOL $_G$-polynomials, then the same is true for P'.

For bi-iterative families of graphs we prove the following result, which we will use in the proof of our main theorem.

Lemma 30. *Let k be a natural number. Let p be an MSOL-polynomial and let $G_n : n \in \mathbb{N}$ be a bi-iterative graph family. If p is an MSOL $_G$-polynomial, or $G_n : n \in \mathbb{N}$ is bounded, then there exist a finite set of MSOL-polynomials $P' = \{p_0, \dots, p_\alpha\}$ and a C-finite sequence $M_n : n \in \mathbb{N}$ such that $p \in P'$ and such that*

$$\left(p_0(G_{n+1}, \bar{X}), \dots, p_\alpha(G_{n+1}, \bar{X})\right)^{tr} = M_n \left(p_0(G_n, \bar{X}), \dots, p_\alpha(G_n, \bar{X})\right)^{tr}.$$

Additionally, if p is an MSOL $_G$-polynomial, then the same is true for all members of P'. The entries of M_n are polynomials with integer coefficients.

Proof. Let F, H and L be elementary operations such that $G_{n+1} = H\left(F^n(L(G_n))\right)$. Let $P' = \{p_0, \dots, p_\alpha\}$ be the set of MSOL-polynomials guaranteed in Theorem 29 for $P = \{p\}$. We have

$$\left(p_0(\sigma(G), \bar{X}), \dots, p_\alpha(\sigma(G), \bar{X})\right)^{tr} = M_\sigma \left(p_0(G, \bar{X}), \dots, p_\alpha(G, \bar{X})\right)^{tr}$$

for $\sigma \in \{L, F, H\}$. Therefore,

$$\left(p_0(G_{n+1}, \bar{X}), \ldots, p_\alpha(G_{n+1}, \bar{X})\right)^{tr} = M_H M_F^n M_L \left(p_0(G, \bar{X}), \ldots, p_\alpha(G, \bar{X})\right)^{tr}.$$

By Lemmas 11 and 12, $A_n = M_H M_F^n M_L$ is a C-finite sequence of matrices. □

7. Statement and proof of Theorem 2

We are now ready to state Theorem 2 exactly and prove it.

Theorem 31. *Let k be a natural number. Let p be an MSOL-polynomial and let $G_n : n \in \mathbb{N}$ be a bi-iterative graph family. If p is an MSOL $_G$-polynomial, or $G_n : n \in \mathbb{N}$ is bounded, then the sequence $p(G_n) : n \in \mathbb{N}$ is C^2-finite.*

To transfer Theorem 31 to C-finite sequences over a polynomial ring, we will use the following lemma:

Lemma 32. *Let $\mathbb{F}$ be a countable subfield of $\mathbb{C}$. For every $\xi \in \mathbb{N}$, there exists a set $D_\xi = \{d_1, \ldots, d_\xi\} \subseteq \mathbb{R}$ such that the partial function $sub_\xi : \mathbb{F}[x_1, \ldots, x_\xi] \to \mathbb{C}$ given by*

$$sub_\xi(p) = p(d_1, \ldots, d_\xi)$$

is injective.

Proof. We prove the claim by induction on ξ. For the case $\xi = 0$ we have $D_\xi = \emptyset$ and $sub_\xi(p) = p$, which is injective.

Now assume there exists $D_{\xi-1}$ such that $sub_{\xi-1}$ is injective. sub_ξ and $sub_{\xi-1}$ are linear maps. Therefore, $ker\, sub_{\xi-1} = \{0\}$ and it is enough to show that $ker\, sub_\xi = \{0\}$.

Let $B_{\xi-1}$ be the set of real numbers which are roots of non-zero polynomials in the polynomial ring $\mathbb{F}[d_1, \ldots, d_{\xi-1}][x_\xi]$ of polynomials in the indeterminate x_ξ whose coefficients are polynomials in $d_1, \ldots, d_{\xi-1}$ with coefficients in $\mathbb{F}$. The cardinality of $B_{\xi-1}$ is $\aleph_0$, implying that there exists $d_\xi \in \mathbb{R} \backslash B_{\xi-1}$. Let $D_\xi = D_{\xi-1} \cup \{d_\xi\}$. For every $r \in ker\, sub_\xi$, since $d_\xi \notin B_{\xi-1}$ and $r(d_1, \ldots, d_\xi) = 0$, x_ξ must have degree 0 in r, so r belongs to $ker\, sub_{\xi-1}$, implying that r is the zero polynomial. Hence, $ker\, sub_\xi = \{0\}$. □

Lemma 33. *Let $\mathbb{F}$ be a subfield of $\mathbb{C}$ and let $r \in \mathbb{N}^+$ and let $r \in \mathbb{N}^+$. For every $n \in \mathbb{N}^+$, let $\overline{v_n}$ be a column vector of size $r \times 1$ of polynomials in $\mathbb{F}[x_1, \ldots, x_k]$. Let M_n be a C-finite sequence of matrices of size $r \times r$ over $\mathbb{F}[x_1, \ldots, x_k]$ such that, for every n,*

$$\overline{v_{n+1}} = M_n \overline{v_n}. \tag{7.1}$$

For each $j = 1, \ldots, r$, $\overline{v_n}[j]$ is C^2-finite. Moreover, all of the $\overline{v_n}[j]$ satisfy the same recurrence relation (possibly with different initial conditions).

Proof. First note that due to the C-finiteness of M_n and Eq. (7.1), we may assume w.l.o.g. that $\mathbb{F}$ is a finite extension field of $\mathbb{Q}$. In particular, we need that $\mathbb{F}$ is countable.

Let $D_k = \{d_1, \ldots, d_k\}$ be the set guaranteed in Lemma 32. For every n, let $\overline{u_n}$ and L_n be the real vector respectively real matrix obtained from $\overline{v_n}$ respectively M_n by substituting $x_1, \ldots, x_k$ with $d_1, \ldots, d_k$. L_n is a C-finite sequence of matrices over $\mathbb{F}(d_1, \ldots, d_k)$ the extension field of $\mathbb{F}$ with D_k. We have for every n,

$$\overline{u_{n+1}} = L_n \overline{u_n}.$$

By Lemma 14, there exists n_0 and C-finite sequences over $\mathbb{F}(d_1, \ldots, d_k)$, $c_n^{\{0\}}, \ldots, c_n^{\{r^2\}}$, such that for every $n > n_0$,

$$c_n^{\{0\}} \overline{u_n} + \cdots + c_n^{\{r^2-1\}} \overline{u_{n+r^2-1}} = c_n^{\{r^2\}} \overline{u_{n+r^2}}$$

and $c_n^{\{r^2\}}$ is non-zero. Using Lemma 32, there exist unique polynomials

$$q_n^{\{0\}}(x_1, \ldots, x_\xi), \ldots, q_n^{\{r^2\}}(x_1, \ldots, x_\xi)$$

such that for every $i = 0, \ldots, r^2$,

$$q_n^{\{i\}}(d_1, \ldots, d_\xi) = c_n^{\{i\}}(d_1, \ldots, d_\xi).$$

Let $t(x_1, \ldots, x_\xi)$ be the polynomial given by

$$t(x_1, \ldots, x_\xi) = q_n^{\{0\}}\overline{v_n} + \cdots + q_n^{\{r^2-1\}}\overline{v_{n+r^2-1}} - q_n^{\{r^2\}}\overline{v_{n+r^2}}$$

substituting $d_1, \ldots, d_\xi$ on both sides of the latter equation, we get $sub_\xi(t) = 0$, but this implies that $t(x_1, \ldots, x_\xi)$ is identically zero, since $sub_\xi(0) = 0$ and sub_ξ is injective. □

Proof of Theorem 31. Let $P' = \{p_0, \ldots, p_\alpha\}$ and $M_n : n \in \mathbb{N}$ be as guaranteed by Lemma 30. We have

$$\left(p_0(G_{n+1}, \bar{X}), \ldots, p_\alpha(G_{n+1}, \bar{X})\right)^{tr} = M_n \left(p_0(G_n, \bar{X}), \ldots, p_\alpha(G_n, \bar{X})\right)^{tr}.$$

By Lemma 33, $p(G_n) : n \in \mathbb{N}$ is C^2-finite. □

Remark 34. The C^2-finite recurrences obtained in Theorem 31 have coefficients which are polynomials with integer coefficients, since the matrices M_n guaranteed by Lemma 30 contain polynomials with integer coefficients.

Remark 35. Some natural counting functions are known not to be MSOL-definable, for instance the counting functions of harmonious colorings, non-repetitive colorings and convex colorings [17,18]. The methods of this paper do not seem to extend to these counting functions. It remains open whether these counting functions satisfy recurrence relations on iterative and bi-iterative families.

The extension of Theorem 31 with recursive graph families which do not have bounded clique-width such as the grids may also seem appealing. However, since it is possible to express computation of Turing machines on grids in MSOL, it seems hard to believe that interesting recurrence relations could be found for grids.

Remark 36. In order to compute the recurrence relations of Theorem 31 explicitly, it is necessary to compute the matrix M from the decomposition theorem of MSOL, Theorem [24], and M can be large. For practical purposes, it would be useful to develop and use decomposition theorems for classes of MSOL-polynomials, e.g. those which count homomorphisms, where the matrix M is not so large.

8. Examples of recurrences for bi-iterative families

Here we give explicit applications of Theorem 2. The applications follow the basic ideas underlying the proof, but can be significantly simplified given specific choices of a graph polynomial and a bi-iterative family.

8.1. *The independence polynomial on G_n^2*

Let G_n^2 be as described in Example 16. We denote by $v_0, \ldots, v_n$ the vertices of the underlying path of G_n^2. Let $I_A(G_n^2, x)$ ($I_B(G_n^2, x)$) be the generating functions counting independent sets U_1 in G_n^2 such that v_n belongs (resp. does not belong) to U_1. Then,

$$I(G_n^2, x) = I_A(G_n^2, x) + I_B(G_n^2, x). \tag{8.1}$$

Now we give a matrix equation for computing $I_A(G_{n+1}^2, x)$, $I_B(G_{n+1}^2, x)$ and $I(G_{n+1}^2, x)$ from $I_A(G_n^2, x)$, $I_B(G_n^2, x)$ and $I(G_n^2, x)$: for all m,

$$\begin{pmatrix} I_A(G_{m+1}^2, x) \\ I_B(G_{m+1}^2, x) \end{pmatrix} = M \begin{pmatrix} I_A(G_m^2, x) \\ I_B(G_m^2, x) \end{pmatrix} \tag{8.2}$$

where

$$M = \begin{pmatrix} 0 & x \\ 1+nx & 1+nx \end{pmatrix}.$$

The first row reflects the facts that if v_{n+1} belongs to the sets U_1 counted by $I_A(G^2_{n+1}, x)$, v_{n+1} and v_n may not belong to the same U_1, and v_{n+1} contributes a multiplicative factor of x. The second row reflects that v_{n+1} does not belong to the sets U_1 counted in $I_B(G^2_{n+1}, x)$, so independent of whether v_n is in U_1, there are two options: either exactly one of the clique vertices adjacent to v_{n+1} belong to U_1 and contributes a factor of x, or no vertex of that clique belongs to U_1, contributing a factor of 1.

Eq. (8.2) holds both for n and $n+1$, leading to the recurrence relation

$$I(G^2_{n+1}, x) = (1+nx)I(G^2_n, x) + x(1+(n-1)x)I(G^2_{n-1}, x)$$
$$I(G^2_0, x) = 1+x$$
$$I(G^2_1, x) = 1+3x+x^2$$

using Eq. (8.1). This is a C^2-finite recurrence, which is also a P-recurrence.

The number of independent sets of G^2_{n+1} is $I(G^2_{n+1}, 1)$. Interestingly, the sequence $I(G^2_{n+1}, 1) : n \in \mathbb{N}$ is in fact equal to the seemingly unrelated sequence (A052169) of [27]. This implies $I(G^2_{n+1}, 1)$ has an alternative combinatorial interpretation as the number of non-derangements of $1, \ldots, n+3$ divided by $n+2$. See [30] for a treatment of the related (A002467).

8.2. *The dichromatic polynomial on G^4_n*

Let $Z_t(P_{n+2})$ denote the dichromatic polynomial of P_{n+2} such that the end-points of P_{n+2} belong to the same connected component iff $t = 1$, for $t = 0, 1$. $Z_t(G^4_n)$ is defined similarly with respect to the most recently added path.

We have

$$Z_0(G^4_n) = \left(\frac{v}{q}+1\right)^2 Z_0(P_{n+2}) \cdot Z_0(G^4_{n-1}) + \left(2\frac{v}{q}+1\right) Z_0(P_{n+2}) Z_1(G^4_{n-1})$$

$$Z_1(G^4_n) = \frac{v^2}{q^2} Z_0(P_{n+2}) Z_1(G^4_{n-1}) + \left(\frac{v^2}{q}+2\frac{v}{q}+1\right) Z_1(P_{n+2}) Z_1(G^4_{n-1}) + \left(\frac{v}{q}+1\right)^2 Z_1(P_{n+2}) Z_0(G^4_{n-1})$$

by dividing into cases by considering the end-points u, v of P_{n+2} and the end-points u', v' of the P_{n+1} in G^4_{n-1} and the edges $\{u, v\}$ and $\{u', v'\}$ with respect to the iteration variable of $Z_t(G^4_n)$. For example, the coefficient of $Z_1(P_n)Z_1(G^4_{n-1})$ corresponds exactly to the case that u, v are in the same connected components in the graph spanned by A (A is the iteration variable in the definition of Z in Eq. (6.1)). If at least one of the edges $\{u, v\}$ and $\{u', v'\}$ belongs to A, then G^4_{n-1} and G^4_n have the same number of connected components, but in $Z_1(P_{n+2})Z_1(G^4_{n-1})$ we have that $Z_1(P_{n+2})$ contributes an additional factor of q which should be cancelled, so the weight in the case is $\frac{v^2+2v}{q}$. If none of the two edges belongs to A, then u, v are in a different connected component from u', v', so no correction is needed and the weight is 1.

Using that $Z(G^4_n) = Z_0(G^4_n) + Z_1(G^4_n)$, we get:

$$Z_0(G^4_n) = \frac{v^2}{q^2} Z_0(P_{n+2}) \cdot Z_0(G^4_{n-1}) + \left(2\frac{v}{q}+1\right) Z_0(P_{n+2}) Z(G^4_{n-1}) \tag{8.3}$$

$$Z(G^4_n) = \left(\left(\frac{v}{q}+1\right)^2 Z(P_{n+2}) + \frac{v^2}{q}\left(1-\frac{1}{q}\right) Z_1(P_{n+2})\right) Z(G^4_{n-1}) - \frac{v^2}{q}\left(1-\frac{1}{q}\right) Z_1(P_{n+2}) Z_0(G^4_{n-1}). \tag{8.4}$$

Let $m \in \mathbb{N}$. Eqs. (8.3) and (8.4) hold for every n, in particular for m and $m+1$, and from these equations we can extract a recurrence relation for $Z(G^4_{m+1})$ using $Z(G^4_m)$ and $Z(G^4_{m-1})$ by canceling out $Z_0(G^4_m)$ and $Z_0(G^4_{m-1})$:

$$\begin{aligned}
Z(G^4_{m+1}) &= Z(G^4_m)\left(\left(\frac{v}{q}+1\right)^2 Z(P_{m+3}) + \frac{v^2}{q}\left(1-\frac{1}{q}\right)Z_1(P_{m+3})\right)\\
&\quad - Z(G^4_{m-1})\left[\frac{v^2}{q}\left(1-\frac{1}{q}\right)Z_1(P_{m+3})\cdot\left(Z_0(P_{m+2})\frac{v^2}{q^2}\right.\right.\\
&\quad \left.\left. + \frac{\left(\frac{v}{q}+1\right)^2 Z_0(P_{m+2})Z(P_{m+2})}{(q-1)\,Z_1(P_{m+2})} + \left(2\frac{v}{q}+1\right)Z_0(P_{m+2})\right)\right].
\end{aligned}$$

Using this recurrence relation, it is easy to compute the dichromatic and Tutte polynomials. E.g., $Z(G^4_m, 3, -1)$, the number of 3-proper colorings of G^4_m, and $|Z(G^4_m, -1, -1)|$, the number of acyclic orientations of G^4_m, are given, for $m = 0, \ldots, 6$, by

$$\begin{aligned}
Z(G^4_m, 3, -1) &: 6\ 30\ 318\ 6762\ 288354\ 24601830\ 4198550862\\
|Z(G^4_m, -1, -1)| &: 6\ 90\ 2826\ 179874\ 22988394\ 5882561010\ 3011536790874.
\end{aligned}$$

9. Conclusion and further research

We introduced a natural type of recurrence relations, C^2-recurrences, and proved a general theorem stating that a wide class of graph polynomials have recurrences of this type on some families of graphs. We gave explicit applications to the Tutte polynomial and the independence set polynomial. We further showed that quadratic sub-sequence of C-finite sequences are C^2-finite.

A natural generalization of the notion of C^2-recurrences could be to allow even sparser sub-sequences. We say a sequence a_n is C^1*-finite* if it is C-finite. We say a sequence is C^r*-finite* if it has a linear recurrence relation of the form

$$c_n^{(s)} a_{n+s} = c_n^{(s-1)} a_{n+s-1} + \cdots + c_n^{(0)} a_n$$

where $c_n^{(0)}, \ldots, c_n^{(s)}$ are C^{r-1}-finite. This definition coincides with the definition of C^2-finite.

Problem 37. Can we find families of graphs for which the Tutte polynomial and other MSOL-polynomials have C^r-recurrences?

Recently the development of an algebraic theory of graph homomorphisms and an analytic theory of convergence of graph sequences and their limits has emerged as a stimulating and rapidly evolving research area, see [21]. Homomorphism counting functions are used to define convergent graph limits. Many classic counting functions in combinatorics can be represented in terms of weighted counting homomorphisms from input graphs to fixed graphs, e.g. the independent set polynomial and the number of k-colorings, $k \in \mathbb{N}$. Weighted homomorphism counting functions are MSOL-polynomials, and hence the results of this paper apply to them. Weighted homomorphism counting functions were characterized with respect to the algebraic properties of infinite matrices related to them called *connection matrices* [22,12]. In particular, the connection matrices of homomorphism counting functions have finite rank. The algebraic properties of the related edge-connection matrices were studied e.g. in [34,31,9]. In [17,18] it was shown more generally that MSOL-polynomials have connection matrices of finite rank. The results of this paper could be expressed in terms of graph invariants with finite connection matrices ranks rather than MSOL-polynomials. The connection matrices of homomorphisms counting functions satisfy further algebraic properties.

Problem 38. Can further properties of the C^2-recurrences of homomorphism counting functions, deriving from the algebraic properties of their connection matrices, be found?

In order to deal with graph invariants over tropical rings (max-plus algebras), and more generally over arbitrary rings, the notion of connection matrices was extended in [19] to join matrices, and the implication of finite row-rank was studied. Hence, the ideas of this paper may apply to such graph invariants.

Polynomial link invariants have received considerable attention in the literature of knot theory. The Jones polynomial and Kauffman polynomial, which are amongst the most prominent invariants of knot theory, they are easily computable for alternating links from the Tutte polynomials of the signed graph representing the link by a classic result and therefore are MSOL-polynomials [35,25]. For general links the colored Tutte polynomial has to be used instead. Knots and links can be presented as labeled planar graphs. The colored Jones polynomial of a framed knot $\mathcal{K}$ in 3-space is a function from such knots to polynomials. The colored Jones polynomial from Example 4 is a sequence of Jones polynomials of a sequence of knots, obtained from an initial knot by an operation called cabling. [13] showed that the colored Jones polynomial is C^2-recursive.

Problem 39. Can the approach of this paper be used to prove the C^2-recursiveness of the Jones polynomial and other knot invariants? What knot operations fit our framework?

Acknowledgments

The authors are grateful to the reviewers for their useful and constructive comments and suggestions. The first author would like to acknowledge support by the Austrian National Research Network S11403-N23 (RiSE) of the Austrian Science Fund (FWF) and by the Vienna Science and Technology Fund (WWTF) through grants PROSEED, ICT12-059, and VRG11-005.

References

[1] J.P. Bell, S.N. Burris, K. Yeats, On the set of zero coefficients of a function satisfying a linear differential equation, Math. Proc. Camb. Phil. Soc. 153 (2012) 235–247. 9.
[2] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, CMS Books in Mathematics. Springer, 2003.
[3] N.L. Biggs, R.M Damerell, D.A Sands, Recursive families of graphs, J. Combin. Theory Ser. 12 (2) (1972) 123–131.
[4] C. Bouey, C. Graves, A. Ostrander, G. Palma, Non-recursively constructible recursive families of graphs, Electron. J. Combin. 19 (2) (2012) P9.
[5] B. Courcelle, A multivariate interlace polynomial and its computation for graphs of bounded clique-width, Electron. J. Combin. 15 (1) (2008).
[6] B. Courcelle, J. Engelfriet, Graph Structure and Monadic Second-Order Logic: A Language-Theoretic Approach, Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2012.
[7] B. Courcelle, J.A. Makowsky, U. Rotics, Linear time solvable optimization problems on graphs of bounded clique-width, Theory Comput. Syst. 33 (2) (2000) 125–150.
[8] B. Courcelle, S. Olariu, Upper bounds to the clique width of graphs, Discrete Appl. Math. 101 (1–3) (2000) 77–114.
[9] J. Draisma, G. Regts, Tensor invariants for certain subgroups of the orthogonal group, J. Algebraic Combin. 38 (2) (2013) 393–405.
[10] J. Désarménien, M.L. Wachs, Descent classes of permutations with a given number of fixed points, J. Combin. Theory Ser. A 64 (2) (1993) 311–328.
[11] E. Fischer, J.A. Makowsky, Linear recurrence relations for graph polynomials, in: Pillars of Computer Science, in: Lecture Notes in Computer Science, vol. 4800, Springer, 2008, pp. 266–279.
[12] M. Freedman, L. Lovász, A. Schrijver, Reflection positivity, rank connectivity, and homomorphism of graphs, J. Amer. Math. Soc. 20 (1) (2007) 37–51.
[13] S. Garoufalidis, T.T.Q. Lê, The colored Jones function is q-holonomic, Geom. Topol. 9 (2005) 1253–1293.
[14] R. Gelca, Non-commutative trigonometry and the a-polynomial of the trefoil knot, Math. Proc. Camb. Phil. Soc. 133 (2002) 311–323. 9.
[15] R. Gelca, J. Sain, The noncommutative a-ideal of $a(2, 2p+1)$-torus knot determines its Jones polynomial, J. Knot Theory Ramifications 12 (02) (2003) 187–201.
[16] I.M. Gessel, C. Reutenauer, Counting permutations with given cycle structure and descent set, J. Combin. Theory Ser. A 64 (2) (1993) 189–215.
[17] B. Godlin, T. Kotek, J.A. Makowsky, Evaluations of graph polynomials, in: Workshop on Graph Theoretic Concepts in Computer Science (WG) (2008) pp. 183–194.
[18] T. Kotek, J.A. Makowsky, Connection matrices and the definability of graph parameters. in: Computer Science Logic (CSL), (2012).
[19] N. Labai, J.A. Makowsky, Finiteness conditions for graph algebras over tropical semirings. preprint (2013).

[20] V.E. Levit, E. Mandrescu, The independence polynomial of a graph—a survey, in: Proceedings of the 1st International Conference on Algebraic Informatics, (2005) pp. 223–254,.
[21] L. Lovász, Large Networks and Graph Limits, American Mathematical Society Colloquium Publications. American Mathematical Society, 2012.
[22] L. Lovász, The rank of connection matrices and the dimension of graph algebras, European J. Combin. 27 (6) (2006) 962–970.
[23] J.A. Makowsky, Coloured Tutte polynomials and Kauffman brackets for graphs of bounded tree width, Discrete Appl. Math. 145 (2) (2005) 276–290.
[24] J.A. Makowsky, Algorithmic uses of the Feferman-Vaught theorem, Annals of Pure and Applied Logic 126 (1-3) (2004) 159–213.
[25] J.A. Makowsky, Colored Tutte polynomials and Kaufman brackets for graphs of bounded tree width, in: Proceedings of the Twelfth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA '01, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2001, pp. 487–495.
[26] M. Noy, A. Ribó, Recursively constructible families of graphs, Adv. Appl. Math. 32 (2004) 350–363.
[27] The On-Line Encyclopedia of Integer Sequences, 2013. published electronically at http://oeis.org.
[28] S. Oum, P. Seymour, Approximating clique-width and branch-width, Journal of Combinatorial Theory, Series B 96 (4) (2006) 514–528.
[29] M. Petkovšek, H.S. Wilf, D. Zeilberger, $A = B$, A.K. Peters, Wellesley, MA, 1996.
[30] A. Petojevic, The function $vm_m(s; a; z)$ and some well-known sequences, J. Integer Seq. 5 (2002) Article 02.1.7.
[31] G. Regts, The rank of edge connection matrices and the dimension of algebras of invariant tensors, European J. Combin. 33 (6) (2012) 1167–1173.
[32] R.P. Stanley, Enumerative Combinatorics, vol. 1, Cambridge University Press, 1986.
[33] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, 1999.
[34] B. Szegedy, Edge coloring models and reflection positivity, J. Amer. Math. Soc 20 (4) (2007) 969–988.
[35] M.B. Thistlethwaite, A spanning tree expansion of the Jones polynomial, Topology 26 (3) (1987) 297–309.
[36] M.L. Wachs, On q-derangement numbers, Proc. Amer. Math. Soc. 106 (1) (1989) 273–278.
[37] Y. Wang, B. Zhu, On the unimodality of independence polynomials of some graphs, European J. Combin. 32 (1) (2011) 10–20.