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NOTE

RATIONAL ω-LANGUAGES ARE NON-AMBIGUOUS

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Abstract. We prove that every rational ω-language can be recognized by a *non-ambiguous* automaton, i.e., an automaton which accepts every infinite word in at most one way.

One knows (see [1] for example) that a rational ω-language cannot be recognized by a deterministic automaton. However, one can ask whether it can be recognized by a *non-ambiguous* automaton which, although nondeterministic, accepts a word in the ω-language in only one way. We answer this question by proving the following proposition.

Proposition. *Every rational ω-language is recognized by a non-ambiguous automaton.*

Notation. An *automaton* over a finite alphabet A is a 4-uple $\mathscr{A} = \langle Q, Q_0, Q_{\text{inf}}, \delta\rangle$ where

Q is a finite set of states,

$Q_0 \subset Q$ is the set of initial states,

$Q_{\text{inf}} \subset Q$ is a set of designated states,

$\delta : Q \times A \to \mathscr{P}(Q)$ is the transition mapping.

For every infinite word $u = u(1)u(2)\cdots u(n)\cdots \in A^\omega$ and for every state $q \in Q$, we define a *computation of u from q in $\mathscr{A}$* as being an infinite sequence $\{q_i\}_{i\geq 0}$ of states such that $q_0 = q$ and $q_i \in \delta(q_{i-1}, u(i))$ for $i \geq 1$. We say that a computation $\{q_i\}_i$ of u is *successful* iff $q_0 \in Q_0$ and $\{i \mid q_i \in Q_{\text{inf}}\}$ is infinite. The *ω-language recognized by $\mathscr{A}$* is the set $L(\mathscr{A})$ of all infinite words u which have a successful computation in $\mathscr{A}$. The automaton $\mathscr{A}$ is said to be *non-ambiguous* if for every u in $L(\mathscr{A})$ there exists only one successful computation of u in $\mathscr{A}$. Finally, an ω-language L is said to be *rational* if it is recognized by an automaton.

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The starting point of the proof of the proposition is the following version of the Büchi–MacNaughton Theorem.

Theorem. *Every rational ω-language can be recognized by a deterministic Muller automaton.*

Here a deterministic Muller automaton is a 4-uple $\mathcal{A} = \langle Q, Q_0, \mathscr{C}, \delta \rangle$ where Q, Q_0 and δ are as above but $\mathrm{Card}(\delta(q, a))$ is always less than 1, and $\mathscr{C} \subset \mathscr{P}(Q)$; the set of infinite words recognized by $\mathcal{A}$ is the set of words u such that the (unique if it exists) computation of u in $\mathcal{A}$ satisfies

$$q_0 \in Q_0 \quad \text{and} \quad \{q \in Q \mid \{i \mid q_i = q\} \text{ is infinite}\} \in \mathscr{C}.$$

Now we are ready for the proof. Let L be any rational ω-language and let $\mathcal{A} = \langle Q, Q_0, \mathscr{C}, \delta \rangle$ be a deterministic Muller automaton recognizing it.

Proof of the Proposition. First, let us define, for every T in $\mathscr{C}$, the deterministic Muller automaton $\mathcal{A}_T = \langle Q, Q_0, \{T\}, \delta \rangle$. Obviously, $L(\mathcal{A})$ is the disjoint union of the $L(\mathcal{A}_T)$, since if $u \in L(\mathcal{A}_T) \cap L(\mathcal{A}_{T'})$, the unique computation of u in $\mathcal{A}$ satisfies $T = \{q \in Q \mid \{i \mid q_i = q\} \text{ is infinite}\} = T'$. Now the disjoint union of ω-languages recognized by non-ambiguous automata is recognized by the disjoint union of these automata which is still non-ambiguous. Thus it remains to prove that $L(\mathcal{A}_T)$ is recognized by a non-ambiguous automaton.

Let us remark that any word u in $L(\mathcal{A}_T)$ can be written in a unique way in the form vaw (or w) such that

$$\begin{cases} a \in A, \qquad v \in A^*, \\ \delta(q_0, v) \notin T, \qquad \delta(q_0, va) \in T, \\ \forall q \in Q, \text{ the set } \{n \mid \delta(q_0, vaw(1)w(2) \cdots w(n)) = q\} \\ \text{is } \begin{cases} \text{empty} & \text{if } q \notin T, \\ \text{infinite} & \text{if } q \in T. \end{cases} \end{cases} \tag{$\star$}$$

Thus, assuming $T = \{s_0, s_1, \ldots, s_{n-1}\}$, we consider the automaton $\mathcal{A}' = \langle Q', Q'_0, Q'_{\mathrm{inf}}, \delta' \rangle$ where

$$Q' = Q \cup (T \times \{0, 1, \ldots, n\}), \qquad Q'_0 = Q_0 \cup ((T \cap Q_0) \times \{0\}),$$

$$Q'_{\mathrm{inf}} = \{\langle s_{n-1}, n \rangle\},$$

and δ' is defined by

if $q' = \delta(q, a)$, then

$q' \in \delta'(q, a)$,

$\langle q', 0 \rangle \in \delta'(q, a)$ iff $q \notin T$ and $q' \in T$,

$$\langle q', i\rangle \in \delta'(\langle q, i\rangle, a) \quad \text{iff} \quad q \in T \text{ and } q' \neq s_i,$$

$$\langle s_i, i+1\rangle \in \delta'(\langle q, i\rangle, a) \quad \text{iff} \quad q \in T, q' = s_i \text{ and } i < n,$$

$$\langle q', 0\rangle \in \delta'(\langle q, n\rangle, a).$$

It is just an exercise to prove that $L(\mathscr{A}') = L(\mathscr{A}_T)$. Moreover, to every successful computation $\{\bar{q}_i\}_i$ of u in $\mathscr{A}'$ we have either

(1) $\forall i \geqslant 0$: $\bar{q}_i = \langle q_i, n_i\rangle$ with $q_i \in T$, or

(2) $\exists k \geqslant 0$: $\bar{q}_k \in Q - T$ and $\forall i \geqslant k+1$, $\bar{q}_i = \langle q_i, n_i\rangle$ with $q_i \in T$.

In both cases we get a decomposition of u in w or vaw which satisfies (★). Since this decomposition is unique, u has only one successful computation in $\mathscr{A}'$ and $\mathscr{A}'$ is non-ambiguous. □

Some other properties of rational ω-languages can be derived from the previous construction of $\mathscr{A}'$.

(1) Like the automaton constructed by Karpinski in [2], $\mathscr{A}'$ is of 'nondeterministic rank' 2 and we get Theorem 2 of [2].

(2) More important is the following improvement of a part of the Büchi–MacNaughton theorem:

> *Every rational ω-language L has a non-ambiguous decomposition in the form* $\bigcup_{i=1,\dots,n} U_i V_i^{\omega}$

which means that every word u in L has a unique decomposition in the form $uv_1v_2\cdots v_n\cdots$ with $u \in U_i$ and $v_n \in V_i$.

References

[1] S. Eilenberg, *Automata, Languages and Machines*, Vol. A (Academic Press, New York, 1974).

[2] M. Karpinski, Almost deterministic ω-automata with output condition, *Proc. Amer. Math. Soc.* **53** (1975) 449–452.