CARTESIAN CLOSED CATEGORIES AND TYPED λ-CALCULI

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This article consists of two parts. Sections 1 to 4, summing up my talk at Val d'Ajol, were composed on the instigation of Pierre-Louis Curien to give an exposition to computer scientists of one version of the relation between typed λ-calculi with surjective pairing and cartesian closed categories. (This is the version appearing in publications by Phil Scott and me; an alternative version by Curien will be explained by him in the present collection of articles.) There is an emphasis on the logical point of view; categories are introduced as deductive systems with equations between proofs and, in the spirit of Haskell Curry, functional completeness is viewed as a jazzed up version of the deduction theorem. Sections 5 and 6 were written in response to the discussion after my talk and contain material which I subsequently lectured on at the Autonomous National University of Mexico. They are concerned with a natural numbers object in a cartesian closed category and the question how such an object can be presented equationally in the sense of Albert Burroni.

While the material in the first part has been published before [L1974, 1980, LS1984], an attempt is made to look at some of it from a different point of view and to clarify some difficult points. At any rate, it is hoped that it will serve as an introduction to the forthcoming book "Introduction to higher order categorical logic", written in collaboration with Phil Scott.

The second part, on the other hand, contains some new material, however incomplete, although Section 5 recapitulates ideas that are surely known to many people and much of it is implicit in the excellent book by Goodstein [Go1957].

1. INTRODUCTION TO CATEGORY THEORY

I shall present a somewhat unorthodox definition of categories, by regarding them as deductive systems with equations between proofs. Deductive systems are defined by means of the notion of "graph", by which we here mean an oriented graph.

A *graph* consists of two classes, the class of *arrows* (= oriented edges) and the class of *nodes*, together with two mappings between them:

$$\{\text{arrows}\} \underset{\text{target}}{\overset{\text{source}}{\rightrightarrows}} \{\text{nodes}\}.$$

Thus each arrow f has a source, say A, and a target, say B. We usually represent this state of affairs by writing $f: A \to B$ or $A \xrightarrow{f} B$.

A *deductive system* is a graph with a specified arrow $1_A: A \to A$ for each node A, called the *identity* arrow and a rule for manufacturing a new arrow from two given arrows called *composition*:

$$\frac{f: A \to B \qquad g: B \to C}{gf: A \to C}.$$

It is customary to call the arrows of a deductive system *proofs* (= deductions) and the nodes *formulas*. Thus $f: A \to B$ may be viewed of a proof of the "entailment" $A \to B$, often written $A \vdash B$. In particular, $1_A: A \to A$ is the reflexivity axiom and composition is a rule of inference asserting transitivity of entailment.

A *category* is a deductive system in which the following equations are prescribed:

$$f1_A = f = 1_B f, \quad (hg)f = h(gf),$$

for all $f: A \to B$, $g: B \to C$ and $h: C \to D$.

It is customary to call the arrows of a category *morphisms* (or just "arrows", but not usually "proofs") and the nodes *objects* (rather than "formulas"). It is a pity that each branch of mathematics has its own terminology!

The categories familiar to most readers will be *concrete categories*, whose objects are sets endowed with some structure and whose morphisms are mappings which preserve this structure. Here are three examples:

(1) The category of sets has as objects all sets and as morphisms all mappings between sets.

(2) The category of monoids has as objects all monoids (semigroups with unity element) and as morphisms all homomorphisms between them.

(3) The category of preordered sets has as objects all preordered sets (sets with a reflexive and transitive relation $\leq$ on them) and as morphisms all monotone (order preserving) mappings between them.

Examples of (small) categories which are not concrete are the following.

(1') Sets: the objects of a set are its elements and there are no arrows except identity arrows.

(2') Monoids: each monoid has but one (anonymous) object and the arrows of a monoid are its elements.

(3') Preordered sets: the objects of a preordered set are its elements and the arrows are pairs (a,b) such that $a \le b$.

Sets are also called "discrete categories", monoids are categories with one object and preordered sets are categories with at most one arrow for any pair of objects.

Here is another example of a category, albeit in a bigger universe.

(4) The category of categories has as objects all categories and as arrows all functors between them. A *functor* $F:\mathcal{A} \to \mathcal{B}$ sends objects of $\mathcal{A}$ to objects of $\mathcal{B}$ and arrows of $\mathcal{A}$ to arrows of $\mathcal{B}$ such that

$$\frac{f:A \to B}{F(f):F(A) \to F(B)}$$

and

$$F(1_A) = 1_{F(A)}, \quad F(gf) = F(g)F(f).$$

The reader should convince himself that, if the objects of categories (1), (2) and (3) are viewed as small categories as in (1'), (2') and (3'), then the arrows in (1), (2) and (3) are precisely the functors between the small categories in question.

We add one final example of a category.

(5) Given two categories $\mathcal{A}$ and $\mathcal{B}$, the category $\mathcal{B}^{\mathcal{A}}$ has as objects all functors $F:\mathcal{A} \to \mathcal{B}$ and as morphisms $t:F \to G:\mathcal{A} \to \mathcal{B}$ what are called *natural transformations*, that is, for each object A of $\mathcal{A}$ an arrow $t(A):F(A) \to G(A)$ in $\mathcal{B}$ such that, for any arrow $f:A \to A'$ in $\mathcal{A}$, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{t(A)} & G(A) \\ F(f) \downarrow & & \downarrow G(f) \\ F(A') & \xrightarrow[t(A')]{} & G(A') \end{array} ,$$

that is, $G(f)t(A) = t(A')F(A)$.

Of special interest is the *functor category* $\mathrm{Sets}^{\mathcal{A}}$ when $\mathcal{A}$ is small. In particular, when $\mathcal{A}$ is (1') a set I, (2') a monoid M, (3') a preordered set P, $\mathrm{Sets}^{\mathcal{A}}$ will be the category of (1') I-indexed families of sets, (2') M-sets, (3') P-directed families of sets.

2. CARTESIAN CATEGORIES

A *conjunction calculus* is a deductive system $\mathcal{L}$ which has a specified formula T (= true) and a binary operation $\wedge$ (= and) between formulas producing from given formulas A and B their conjunction $A\wedge B$. Moreover, it has the following additional axioms:

$$\bigcirc_A : A \to T, \quad \pi_{A,B} : A\wedge B \to A, \quad \pi'_{A,B} : A\wedge B \to B,$$

and the additional rule of inference:

$$\frac{f:C \to A \qquad\qquad g:C \to B}{\langle f,g\rangle :C \to A\wedge B}\ .$$

Here is an example of a proof constructed from the axioms by the rules of inference:

$$\dfrac{\dfrac{(A\wedge B)\wedge C \to A\wedge B \qquad A\wedge B \to A}{(A\wedge B)\wedge C \to A} \qquad \dfrac{\dfrac{(A\wedge B)\wedge C \to A\wedge B \qquad A\wedge B \to B}{(A\wedge B)\wedge C \to B} \qquad (A\wedge B)\wedge C \to C}{(A\wedge B)\wedge C \to B\wedge C}}{(A\wedge B)\wedge C \to A\wedge(B\wedge C)}$$

As we shall have occasion to refer to this proof later, it is useful to give it a name:

$$\alpha_{A,B,C} \equiv \langle \pi_{A,B}\pi_{A\wedge B,C},\ \langle \pi'_{A,B}\pi_{A\wedge B,C},\ \pi'_{A,B,C}\rangle\rangle .$$

Although the conjunction calculus contains no symbol for implication, it admits the following form of the *deduction theorem*.

PROPOSITION 2.1. If $\varphi(x):B \to C$ is a proof from the assumption $x:T \to A$, there is a proof $\kappa_{x\in A}\varphi(x):A\wedge B \to C$ in $\mathscr{L}$ not depending on the assumption x.

Here proofs from the assumption $x:T \to A$ are generated inductively from old proofs in $\mathscr{L}$ and the assumption x by composition and pairing. They form a deductive system $\mathscr{L}(x)$. Technically speaking, one should distinguish between composition and pairing in $\mathscr{L}$ and in $\mathscr{L}(x)$, but we shall refrain from introducing distinct notations for these operations in $\mathscr{L}(x)$.

Proof. There are four cases in the proof of the deduction theorem:

(1) $\varphi(x) = k:B \to C$, a proof in $\mathscr{L}$;

(2) $\varphi(x) = x:T \to A$, where $B = T$ and $C = A$;

(3) $\varphi(x) = \chi(x)\psi(x)$, where $\psi(x):B \to D$ and $\chi(x):D \to C$;

(4) $\varphi(x) = \langle\psi(x),\chi(x)\rangle$, where $\psi(x):B \to D$, $\chi(x):B \to E$ and $C = D\wedge E$.

We define $\kappa_{x\in A}\varphi(x)$ by induction on the "length" of $\varphi(x)$:

(1) $\kappa_{x\in A}k = k\pi'_{A,B}$;

(2) $\kappa_{x\in A}x = \pi_{A,T}$;

(3) $\kappa_{x\in A}(\chi(x)\psi(x)) = \kappa_{x\in A}\chi(x)\langle\pi_{A,B},\ \kappa_{x\in A}\psi(x)\rangle$;

(4) $\kappa_{x\in A}\langle\psi(x),\ \chi(x)\rangle = \langle\kappa_{x\in A}\psi(x),\ \kappa_{x\in A}\chi(x)\rangle$.

Although we did not distinguish notationally between hg in $\mathcal{L}$ and hg in $\mathcal{L}(x)$, the reader should beware that $\kappa_{x\in A}$ of the former is $hg\pi'_{A,B}$, while $\kappa_{x\in A}$ of the latter is $h\pi'_{A,D}\langle\pi_{A,B},\ g\pi'_{A,B}\rangle$.

The reader should also note that the above argument remains valid for an assumption $x:T \to A$, even if there is a known proof $a:T \to A$ not involving this assumption or some other assumption $y:T \to A$.

A *cartesian category* is a conjunction calculus in which the following equations hold, in addition to those of a category:

$$f = \circ_A \quad \text{for all} \quad f:A \to T$$

$$\pi_{A,B}\langle f,g\rangle = f \quad \text{for all} \quad f:C \to A,\ g:C \to B;$$

$$\pi'_{A,B}\langle f,g\rangle = g \quad \text{for all} \quad f:C \to A,\ g:C \to B;$$

$$\langle\pi_{A,B}h,\ \pi'_{A,B}h\rangle = h \quad \text{for all} \quad h:C \to A\times B.$$

We note that these equations express the following isomorphisms:

$$\mathrm{Hom}(A,T) \cong \{*\}, \quad \mathrm{Hom}(C,A\wedge B) \cong \mathrm{Hom}(C,A)\times\mathrm{Hom}(C,B).$$

A cartesian category is thus a category with a specified terminal object and with specified binary products. To adopt the conventional notation, we shall replace T by 1 and $A \wedge B$ by $A \times B$.

We note the following distributive law for future reference:

$$\langle f,g\rangle h = \langle fh,gh\rangle \quad \text{for all} \quad f:C \to A, \quad g:C \to B, \quad h:D \to C.$$

This is proved by noting that $\pi_{A,B}\text{LHS} = \pi_{A,B}\text{RHS}$ and similarly for $\pi'_{A,B}$, and then applying the last of the above equations, which asserts that pairing is surjective.

One advantage of defining cartesian categories by means of equations is that we can adjoin indeterminate arrows, just as one adjoins an indeterminate element to a commutative ring R to form the polynomial ring R[x]. For the purpose of this exposition we confine attention to an indeterminate arrow of the form $x:1 \to A$, where A is a given object of a cartesian category $\mathscr{C}$. In view of the preliminary groundwork already done, we may regard x as an assumption $x:T \to A$ in the conjunction calculus $\mathscr{C}(x)$. *Polynomial expressions* are then proofs on the assumption x. The problem is now to define an appropriate congruence relation $\underset{x}{=}$ between polynomial expressions so that *polynomials* may be defined as equivalence classes modulo $\underset{x}{=}$. By a congruence relation we mean of course an equivalence relation which preserves the given operations, here composition and pairing, e.g., $\varphi(x) \underset{x}{=} \varphi'(x)$ and $\psi(x) \underset{x}{=} \psi'(x)$ should imply $\langle\varphi(x),\psi(x)\rangle \underset{x}{=} \langle\varphi'(x),\psi'(x)\rangle$.

What properties should this congruence relation $\underset{x}{=}$ have? Clearly, we want the polynomial category $\mathscr{C}[x] \equiv \mathscr{C}(x)/\underset{x}{=}$ to be a cartesian category, so we must stipulate that all the equations of a cartesian category hold, e.g.

$$\varphi(x) \underset{x}{=} \circ_B \quad \text{for all} \quad \varphi(x):B \to T,$$

$$\pi_{B,C}\langle\varphi(x),\psi(x)\rangle \underset{x}{=} \varphi(x) \quad \text{for all} \quad \varphi(x):D \to B \text{ and } \psi(x):D \to C,$$

and two other equations of the same nature. Moreover, we want the canonical embedding $H_x:\mathscr{C} \to \mathscr{C}[x]$ such that $H_x(A) = A$ and $H_x(f) = f$ to be a *cartesian functor*, that is, a functor which preserves the cartesian structure on the nose. Thus we require that

$$gf \underset{x}{=} h \quad \text{whenever} \quad gf = h \quad \text{in } \mathscr{C},$$

$$\langle f,g\rangle \underset{x}{=} h \text{ whenever } \langle f,g\rangle = h \quad \text{in } \mathscr{C}.$$

Finally, we stipulate that $\underset{x}{=}$ is the smallest congruence relation between polynomial expressions which satisfies these six conditions.

The polynomial category $\mathscr{C}[x] \equiv \mathscr{C}(x)/\underset{x}{=}$ has the following universal property.

PROPOSITION 2.2. Given a cartesian category $\mathscr{C}$, an object A of $\mathscr{C}$ and an indeterminate arrow $x:1 \to A$ over $\mathscr{C}$, let $F:\mathscr{C} \to \mathscr{D}$ be any cartesian functor into another cartesian category $\mathscr{D}$ and $a:1 \to F(A)$ any arrow from $1 = F(1)$ to $F(A)$ in $\mathscr{D}$. Then there exists a unique cartesian functor $F_a:\mathscr{C}[x] \to \mathscr{D}$ such that $F_aH_x = F$ and $F_a(x) = a$.

$$\begin{tikzcd}
\mathscr{C}[x] \arrow[dr, dashed, "F_a"] & & x \arrow[d, mapsto] \\
& \mathscr{D} & a \\
\mathscr{C} \arrow[uu, "H_x"] \arrow[ur, "F"'] & &
\end{tikzcd}$$

Proof. Clearly we must define $F_a(B) = B$ for every object B of $\mathscr{C}$. Before defining F_a on polynomials, we shall define it on polynomial expressions, that is, proofs from the assumption $x: T \to A$. Looking at the four cases in the proof of the deduction theorem, we define inductively:

$$F_a(k) = k,$$

$$F_a(x) = x,$$

$$F_a(\chi(x)\psi(x)) = F_a(\chi(x))F_a(\psi(x)),$$

$$F_a(\langle\psi(x),\chi(x)\rangle = \langle F_a(\psi(x)),F_a\chi(x)\rangle.$$

We now verify that F is in fact defined on polynomials, that is,

$$\varphi(x) \underset{x}{=} \psi(x) \Rightarrow F_a(\varphi(x)) = F_a(\psi(x)).$$

Indeed, let $\varphi(x) \sim \psi(x)$ mean that $F_a(\varphi(x)) = F_a(\psi(x))$. Then $\sim$ is seen to be a congruence relation between polynomial expressions which satisfies the six conditions required for equality between polynomials.

For example, to check the last condition, suppose $\langle f,g\rangle = h$ in $\mathscr{C}$, then

$$\begin{aligned} F_a(\langle f,g\rangle) &= \langle F_a(f),F_a(g)\rangle \\ &= \langle F(f),F(g)\rangle \\ &= F(\langle f,g\rangle) \\ &= F(h) = F_a(h). \end{aligned}$$

(We remind the reader that pairing in $\mathscr{C}(x)$ is not the same as pairing in $\mathscr{C}$, even if we have made no notational distinction.) Thus $\langle f,g\rangle = h$ in $\mathscr{C}$ implies $\langle f,g\rangle \sim h$. We leave the other five conditions as an exercise to the reader.

Once it has been shown that the congruence relation $\sim$ satisfies the required six conditions, since $\underset{x}{=}$ was the smallest congruence relation satisfying them, it follows that $\varphi(x) \underset{x}{=} \psi(x)$ implies $\varphi(x) \sim \psi(x)$, that is,

$F_a(\varphi(x)) = F_a(\psi(x))$. Thus F_a is defined on polynomials. It is easily seen that it is a cartesian functor such that $F_a H_x = f$ and $F_a(x) = a$ and that it is the only cartesian functor with this property.

COROLLARY 2.3. Given a cartesian category $\mathscr{C}$, an object A of $\mathscr{C}$ and an indeterminate arrow $x:1 \to A$ over $\mathscr{C}$. Then, for any arrow $a:1 \to A$ in $\mathscr{C}$ there is a unique cartesian functor $S_x^a:\mathscr{C} \to \mathscr{C}$ such that $S_x^a H_x = 1_{\mathscr{C}}$ and $S_x^a(x) = a$.

Proof. Take $F = 1_{\mathscr{C}}$ and $S_x^a = F_a$.

One calls S_x^a the *substitution functor* and writes $S_x^a(\varphi(x)) \equiv \varphi(a)$.

Just as every polynomial over a commutative ring has a normal form $\sum_{i=0}^{n} a_i x^i$, so every polynomial arrow $\varphi(x):B \to C$ over a cartesian category $\mathscr{C}$ has a simple normal form spelled out in the following result, which expresses what is usually called *functional completeness* of cartesian categories.

THEOREM 2.4. Given a cartesian category $\mathscr{C}$, an object A of $\mathscr{C}$, an indeterminate arrow $x:1 \to A$ over $\mathscr{C}$ and a polynomial arrow $\varphi(x):B \to C$ in $\mathscr{C}[x]$, there exists a unique arrow $f:A\times B \to C$ in $\mathscr{C}$ such that $\varphi(x) \underset{x}{=} f\langle x\bigcirc_B, 1_B\rangle$.

Proof. To show the existence of f, take $f = \kappa_{x\in A}\varphi(x)$ as in the deduction theorem for the conjunction calculus and check that $\kappa_{x\in A}\varphi(x)\langle x\bigcirc_B, 1_B\rangle \underset{x}{=} \varphi(x)$, by looking at the four cases $\varphi(x) = k,\ x,\ \chi(x)\psi(x)$ and $\langle\psi(x),\chi(x)\rangle$ separately. For example, in the first case we have

$$\begin{aligned}\kappa_{x\in A}k\langle x\bigcirc_B, 1_B\rangle &\underset{x}{=} k\pi'_{A,B}\langle x\bigcirc_B, 1_B\rangle \\ &\underset{x}{=} k1_B \underset{x}{=} k.\end{aligned}$$

The other three cases are left to the reader. (The last case requires use of the distribution law mentioned earlier.)

At this point it might be worth noting the following special case of (3) in the definition of $\kappa_{x\in A}\varphi(x)$:

$$\begin{aligned}(3')\ \kappa_{x\in A}(h\varphi(x)) &= \kappa_{x\in A}h\langle\pi,\kappa_{x\in A}\varphi(x)\rangle && \text{by (3)}\\ &= h\pi'\langle\pi,\kappa_{x\in A}\varphi(x)\rangle && \text{by (1)}\\ &= h\kappa_{x\in A}\varphi(x).\end{aligned}$$

Before proving the uniqueness of f, let us show that $\kappa_{x\in A}\varphi(x)$ depends not just on the polynomial expression $\varphi(x)$, but on the polynomial, namely the equivalence class of $\varphi(x)$ modulo $\equiv_x$. In other words, we will show that

$$\varphi(x) \equiv_x \psi(x) \quad\Rightarrow\quad \kappa_{x\in A}\varphi(x) = \kappa_{x\in A}\psi(x).$$

Now $\equiv_x$ was the smallest congruence relation between polynomial expressions satisfying six conditions. Let us write $\varphi(x) \sim \psi(x)$ to mean that $\kappa_{x\in A}\varphi(x) = \kappa_{x\in A}\psi(x)$. The stated result will follow if we show that $\sim$ satisfies the same six conditions.

For example, one condition asserts that if $gf = h$ in $\mathcal{C}$ then $gf \sim h$. Indeed, we calculate

$$\begin{aligned}\kappa_x(gf) &= g\kappa_{x\in A}f && \text{by (3')}\\ &= gf\pi' && \text{by (1)}\\ &= h\pi' = \kappa_x h.\end{aligned}$$

The other five conditions are left to the reader.

Finally, to prove the uniqueness of f, suppose $f\langle x\circ_B, 1_B\rangle \equiv_x \varphi(x)$. Then, by the above, we have

$$\begin{aligned}\kappa_x\varphi(x) &= \kappa_x(f\langle x\circ_B, 1_B\rangle)\\ &= f\kappa_x\langle x\circ_B, 1_B\rangle\\ &= f\langle\kappa_x(x\circ_B), \kappa_x 1_B\rangle\\ &= f\langle\kappa_x x\langle\pi, \kappa_x\circ_B\rangle, \pi'\rangle\\ &= f\langle\pi\langle\pi, \circ_B\pi'\rangle, \pi'\rangle\\ &= f\langle\pi, \pi'\rangle = f1 = f.\end{aligned}$$

This completes the proof. The proof presented here is a bit simpler than its original version [L1974, 1980], the simplification having been suggested by Bill Hatcher.

COROLLARY 2.5. Given a cartesian category $\mathscr{C}$ and an object A of $\mathscr{C}$, if $\varphi(x):1 \to B$ is a polynomial in the indeterminate $x:1 \to A$, there is a unique arrow $g:A \to B$ in $\mathscr{C}$ such that $gx \underset{x}{=} \varphi(x)$.

Proof. Take $g = \kappa_{x\in A}\varphi(x)\langle 1_A, \circ_A\rangle$ and calculate $gx \underset{x}{=} \varphi(x)$ and $\kappa_{x\in A}(gx)\langle 1_A, \circ_A\rangle = g$.

3. CARTESIAN CLOSED CATEGORIES

A *positive intuitionistic propositional calculus* is a conjunction calculus with an additional binary operation $\Leftarrow$ (= if) between formulas and the following new axiom and rule of inference:

$$\epsilon_{A,B}:\ (A \Leftarrow B)\wedge B \to A$$

$$\frac{h:C\wedge B \to A}{\Lambda^{A}_{C,B}(h):C \to A \Leftarrow B}$$

We usually write $\Lambda^{A}_{C,B}(h) \equiv h^*$ where there is no question about the subscripts.

We can easily extend the deduction theorem to the positive intuitionistic propositional calculus by adding an extra case to the proof of Proposition 2.1, but more interesting is the following observation.

PROPOSITION 3.1. If $\mathcal{L}$ is a positive intuitionistic propositional calculus and $\mathcal{L}(x)$ is the conjunction calculus based on the assumption $x:T \to A$, then $\mathcal{L}(x)$ is also a positive intuitionistic propositional calculus.

Proof. For any proof $\chi(x):B\wedge C \to D$ in $\mathcal{L}(x)$, we shall define $\chi(x)^*:B \to D \Leftarrow C$ with the help of $h = \kappa_{x\in A}\chi(x):A\wedge(B\wedge C) \to D$. Indeed, $h\alpha_{A,B,C}:(A\wedge B)\wedge C \to D$, so $(h\alpha)^*:A\wedge B \to D \Leftarrow C$, and we define

$$\chi(x)^* \equiv (h\alpha)^*\langle x\circ_B, 1_B\rangle.$$

Note that $\kappa_{x\in A}(\chi(x))^*$ may now be calculated using the four clauses in the definition of $\kappa_{x\in A}$. There is a technical difference between the conjunction calculus $\mathcal{L}(x)$ in which $(-)^*$ is a composite operation and the positive intuitionistic propositional calculus generated by the assumption $x:T \to A$ in which $(-)^*$ is a primitive operation. In the latter we would have defined $\kappa_{x\in A}(\chi(x)^*) \equiv (h\alpha)^*$.

A *cartesian closed category* is a positive intuitionistic propositional calculus which satisfies the equations of a cartesian category and also the following:

$$\epsilon_{A,B}\langle h^*\pi_{C,B}, \pi'_{C,B}\rangle = h \qquad \text{for all } h:\ C\wedge B \to A;$$

$$(\epsilon_{A,B}\langle k\pi_{C,B}, \pi'_{C,B}\rangle)^* = k \qquad \text{for all } k:C \to A \Leftarrow B.$$

Inasmuch as we already replaced T by 1 and $A\wedge B$ by $A\times B$, we shall now replace $A \Leftarrow B$ by A^B. The new equations then express the isomorphism

$$\mathrm{Hom}(C\times B, A) \cong \mathrm{Hom}(C, A^B).$$

As a consequence of the new equations, we obtain a new "distributive law":

$$h^*k = (h\langle k\pi_{C,B},\ \pi'_{C,B}\rangle)^*,$$

where $k:C \to D$ and $h:D\times B \to A$.

This is proved thus:

$$\begin{aligned} h^*k &= (\epsilon\langle h^*k\pi,\ \pi'\rangle)^* \\ &= (\epsilon\langle h^*\pi,\ \pi'\rangle\ \langle k\pi,\ \pi'\rangle)^* \\ &= (h\langle k\pi,\ \pi'\rangle)^*. \end{aligned}$$

We may also derive the bijection:

$$\mathrm{Hom}(B,A) \cong \mathrm{Hom}(1\times B,\ A) \cong \mathrm{Hom}(1,\ A^B).$$

To put this more explicitly, we introduce the rules:

$$\frac{f:A \to B}{\ulcorner f \urcorner:1 \to B^A}\ , \qquad \frac{g:1 \to B^A}{g^{\ell}:A \to B}$$

where

$$\ulcorner f \urcorner \equiv (f\pi'_{1,A})^*$$

has been called "the name of f" by Lawvere and

$$g^{\ell} \equiv \epsilon_{B,A}\langle g\circ_A, 1_A\rangle$$

may be read as "g of". Thus we have the following rule:

$$\frac{g:1 \to B^A \qquad a:1 \to A}{g^{\ell}a:1 \to B}\ ,$$

producing from "elements" g of B^A and a of A an "element" $g^{\ell}a$ (= g of a) of B. (At the level of deductive systems, this is the rule of modus ponens and the "elements" are "proofs of formulas".) The reader will easily check the equations

$$\ulcorner f \urcorner^{\ell} = f, \qquad \ulcorner g^{\ell} \urcorner = g,$$

which establish the stated bijection.

Examples of cartesian closed categories are functor categories, Heyting algebras and the category of all small categories.

We could prove the functional completeness of cartesian closed categories as we did that of cartesian categories; but more interesting is the following sharper result.

PROPOSITION 3.2. If $\mathscr{C}$ is a cartesian closed category and $x: 1 \to A$ is an indeterminate arrow over $\mathscr{C}$ regarded as a cartesian category, then the cartesian category $\mathscr{C}[x]$ is also cartesian closed and the functor $H_x: \mathscr{C} \to \mathscr{C}[x]$ has the universal property with respect to cartesian closed categories.

Proof. To show that $\mathscr{C}[x]$ is cartesian closed, assume that $\chi(x): B\times C \to D$. We construct $\chi(x)^*: B \to D^C$ as in Proposition 3.1 and must show that it satisfies the required equations:

$$\epsilon_{B,C}\langle \chi(x)^* \pi_{D,C},\ \pi'_{D,C}\rangle \ \underset{x}{\equiv}\ \chi(x),$$

$$(\epsilon_{B,C}\langle \mu(x)\pi_{D,C},\ \pi'_{D,C}\rangle)^* \ \underset{x}{\equiv}\ \mu(x),$$

where $\mu(x): B \to D^C$.

Recall that

$$\chi(x)^* \ \equiv\ (h\alpha)^*\langle x\circ_B, 1_B\rangle,$$

where $h = \kappa_{x\in A}\chi(x): A\times(B\times C) \to 0$ and $\alpha_{A,B,C}$ was defined in Section 2 by $\alpha \equiv \langle \pi\pi,\ \langle \pi'\pi,\ \pi'\rangle\rangle$. We shall check the first of the above equations, leaving the second to the reader.

$$\begin{aligned}
\epsilon\langle \chi(x)^*\pi, \pi'\rangle &\underset{x}{\equiv} \epsilon\langle (h\alpha)^*\langle x\circ_B, 1_B\rangle\pi,\ \pi'\rangle \\
&\underset{x}{\equiv} \epsilon\langle (h\alpha)^*\pi,\ \pi'\rangle\langle\langle x\circ_B, 1_B\rangle\pi,\ \pi'\rangle \\
&\underset{x}{\equiv} h\alpha\langle\langle x\circ_B\pi, \pi\rangle, \pi'\rangle \\
&\underset{x}{\equiv} h\langle x\circ_B\pi,\ \langle\pi, \pi'\rangle\rangle
\end{aligned}$$

$$\underset{x}{=} \; h\langle x\circ_{B\times C},\ 1_{B\times C}\rangle$$
$$\underset{x}{=} \; \chi(x).$$

Now consider the cartesian functor $H_x:\mathscr{C} \to \mathscr{C}[x]$. We first check that it is cartesian closed by taking $h:A\times B \to C$ in $\mathscr{C}$ and calculating $(H_x(h))^*$ in $\mathscr{C}[x]$ to show that it agrees with $H_x(h^*)$, where h^* is in $\mathscr{C}$. The calculation makes use of the new distributive law: $h^*k = (h\langle k\pi,\pi'\rangle)^*$:

$$\begin{aligned}((\kappa_{x\in A}h)\alpha)^*\langle x\circ_B,1_B\rangle &= (h\pi'\alpha)^*\langle x\circ_B,1_B\rangle \\ &= (h\pi'\alpha\langle\langle x\circ_B,1_B\rangle\pi,\pi'\rangle)^* \\ &= (h\pi'\langle x\circ_B,\ \langle\pi,\pi'\rangle\rangle)^* \\ &= (h\langle\pi,\pi'\rangle)^* \;=\; h^*.\end{aligned}$$

Now assume that $F:\mathscr{C} \to \mathscr{D}$ is a cartesian closed functor and that $a:1 \to F(A)$ in $\mathscr{D}$. We know already that there is a unique cartesian functor $F_a:\mathscr{C}[x] \to \mathscr{D}$ such that $F_aH_x = F$ and $F_a(x) = a$. It remains to show that F_a is cartesian closed.

Let $\chi(x):B\times C \to D$, we must show that $F_a(\chi(x)^*) = (F_a(\chi(x))^*$. Recalling the definition of $\chi(x)^*$ above, we have

$$\begin{aligned}F_a(\chi(x)^*) &= (F(h)\alpha)^*\langle a\circ_{F(B)},1_{F(B)}\rangle \\ &= (F(h)\alpha\langle\langle a\circ_{F(B)},\ 1_{F(B)}\rangle\pi,\ \pi'\rangle)^* \\ &= (F(h)\langle a\circ_{F(B)},\ 1_{F(B)}\rangle)^* \\ &= (F_a(\chi(x)))^*.\end{aligned}$$

COROLLARY 3.3. Given a cartesian closed category $\mathscr{C}$, if $\varphi(x):1 \to B$ is a polynomial arrow in the indeterminate $x:1 \to A$, there exists a unique arrow $h:1 \to B^A$ such that $h`x \underset{x}{=} \varphi(x)$.

Proof. This follows by taking $h = \ulcorner g \urcorner$, where $g = \kappa_{x \in A} \varphi(x) \langle 1_A, \circ_A \rangle$ was the unique arrow $A \to B$ of Corollary 2.5 such that $gx \underset{x}{=} \varphi(x)$.

We shall write $h = \lambda_{x \in A} \varphi(x)$, thus

$$\lambda_{x \in A} \varphi(x) \equiv \ulcorner \kappa_{x \in A} \varphi(x) \langle 1_A, \circ_A \rangle \urcorner$$

In view of Corollary 3.3, we have

$$\lambda_{x \in A} \varphi(x) {}^{\varsigma} x \underset{x}{=} \varphi(x), \qquad \lambda_{x \in A} (h {}^{\varsigma} x) = h,$$

where the first equation expresses the existence and the second equation the uniqueness of h. The second equation is usually called (η), while the first equation gives rise to the usual equation (β), namely

$$\lambda_{x \in A} \varphi(x) {}^{\varsigma} a = \varphi(a),$$

by applying the substitution functor $S^a_x : \mathscr{C}[x] \to \mathscr{C}$ of Corollary 2.3.

This suggests a connection between cartesian closed categories and a suitable formulation of the λ-calculus. What we need is a λ-calculus with types, including product types, and surjective pairing. This notion should be sufficiently general to include the "internal language" of any cartesian closed category.

4. TYPED λ-CALCULI

A *typed λ-calculus* consists of two classes, the class of types and the class of terms, and an assignment of types to terms (we write $a \in A$ to say that the term a has type A), together with certain equations between terms.

(a) The class of *types* contains the basic type 1 and is closed under two operations: if A and B are types so are $A \times B$ and A^B.

(b) The class of *terms* is, to begin with, freely generated from certain basic terms by certain term forming operations. Among the basic terms there are countably many variables of each type and certain constants, including the constant $* \in 1$. Among the term forming operations are the following:

$$\frac{a\in A \qquad b\in B}{\langle a,b\rangle \in A\times B}\,,\qquad \frac{c \in A\times B}{\pi_{A,B}(C) \in A}\,,\qquad \frac{c \in A\times B}{\pi'_{A,B}(C) \in B}\,,$$

$$\frac{f \in B^A \qquad a \in A}{f`a \equiv \epsilon_{B,A}(f,a) \in B}\,,\qquad \frac{x \in A \qquad \varphi(x) \in B}{\lambda_{x\in A}\varphi(x) \in B^A}\,,$$

where x is a variable.

(c) If X is any finite set of variables, $\underset{X}{=}$ is an equivalence relation on the subclass of all terms which contain no free occurrences of variables other than those in X (the words "free" and "bound" have the usual meaning) satisfying the following conditions:

$$\lambda_{x\in A}\varphi(x) \underset{X}{=} \lambda_{x'\in A}\varphi(x');$$

$$\frac{a \underset{X}{=} a'}{f`a \underset{X}{=} f`a'}$$

from which other substitution rules may be derived;

$$\frac{a \underset{X}{=} b}{a \underset{X\cup\{x\}}{=} b}\;;$$

$$\frac{\varphi(x) \underset{X\cup\{x\}}{=} \psi(x)}{\lambda_{x\in A}\varphi(x) \underset{X}{=} \lambda_{x\in A}\psi(x)}\;;$$

$$a \underset{X}{=} * \quad \text{if } a \in 1\,;$$

$$\pi(\langle a,b\rangle) \underset{X}{=} a \quad \text{if } a \in A,\ b \in B;$$

$$\pi'(\langle a,b\rangle) \underset{X}{=} b \quad \text{if } a \in A,\ b \in B;$$

$$\langle \pi(c),\ \pi'(c)\rangle \underset{X}{=} c \qquad \text{if} \quad c \in A\times B;$$

$$\lambda_{x\in A}\varphi(x)\text{`}a \underset{X}{=} \varphi(a) \qquad \text{if} \quad a \in A \quad \text{is substitutable for } x \text{ in } \varphi(x);$$

$$\lambda_{x\in A}(f\text{`}a) \underset{X}{=} f \qquad \text{if} \quad f \in B^A \quad \text{and } x \text{ is not in } X.$$

As a consequence of the above we also have:

$$\frac{\varphi(x) \underset{X\cup\{x\}}{=} \psi(x)}{\varphi(a) \underset{X}{=} \psi(a)} .$$

This definition leaves a lot of freedom. There are many typed λ-calculi. When the reader thinks of *the* typed λ-calculus, what he probably has in mind is the typed λ-calculus freely generated from a specified basic type N of so-called individuals. This λ-calculus has no types other than those freely generated from 1 and N by the two specified operations and there are no nontrivial equations between types. Moreover, there are no terms other than those freely generated from the constant $*$ and the countable number of variables of each type by the specified term forming operations. Finally, in this λ-calculus no equations hold unless they follow from the conditions laid down in (c) above.

Here we are interested in the typed λ-callculus associated with a given cartesian closed category $\mathscr{C}$. The internal language $L(\mathscr{C})$ of $\mathscr{C}$ is the typed λ-calculus described as follows.

(a) Its types are the objects of $\mathscr{C}$, 1 is the terminal object, $A\times B$ the cartesian product and A^B exponentiation in $\mathscr{C}$.

(b) its terms of type A are polynomial expressions $\mathscr{C}(x_1,\ldots,x_n):1 \to A$ over $\mathscr{C}$ which are freely generated from basic constants $a\in A$, namely arrows

$a: 1 \to A$ in $\mathscr{C}$, and variables $x \in A$, namely indeterminate arrows $x: 1 \to A$ over $\mathscr{C}$, by the term forming operations

$$\frac{a: 1 \to A \quad b: 1 \to B}{\langle a, b\rangle : 1 \to A \times B}, \quad \frac{a: 1 \to A}{fa \to B},$$

where $f: A \to B$ is any arrow in $\mathscr{C}$. Moreover we write $\pi_{A,B}(C)$ for $\pi_{A,B}C$ and similarly for π', $f^{\text{\textlangle}}a$ for $\epsilon_{B<A}\langle f, a\rangle$ and $\lambda_{x \in A} f(x)$ for $\ulcorner \kappa_{x \in A} \varphi(x) \langle 1_A, 0_A\rangle \urcorner$, where $\varphi(x): 1 \to B$.

A word of explanation: whereas previously we had only defined the deductive system and polynomial category $\mathscr{C}[x]$ in a single assumption or indeterminate $x: 1 \to A$, the definition is of course easily extended to any finite set $X = \{x_1, \ldots, x_n\}$ of assumptions or indeterminates. The reader will easily convince himself that $\mathscr{C}[x,y] \cong (\mathscr{C}[x])[y] \cong \mathscr{C}[z]$, where $x: 1 \to A$, $y: 1 \to B$ and $z: 1 \to A \times B$. In the same spirit, $\mathscr{C}[\phi] \cong \mathscr{C}[z]$ where $z: 1 \to 1$ is an indeterminate. Technically speaking, $\mathscr{C}[\phi]$ consists of expressions freely generated from arrows in $\mathscr{C}$ by the operations: composition, $\langle -,- \rangle$ and $(-)^*$. On $\mathscr{C}[X]$ one imposes an equality relation $\underset{X}{=}$ as in the special case $X = \{x\}$ already discussed. In particular, $\mathscr{C}[\phi] \equiv \mathscr{C}(\phi)$ modulo $\underset{\phi}{=}$ will be the same as $\mathscr{C}$.

(c) The equivalence relation $\underset{X}{=}$ between terms not containing any free variables other than those in X is now taken to be the equality relation $\underset{X}{=}$ in $\mathscr{C}[X]$. It is easily seen that it satisfies all the conditions it is supposed to satisfy.

This completes the construction of the internal language $L(\mathscr{C})$ of a cartesian closed category $\mathscr{C}$. In the converse direction, we shall show how from any typed λ-calculus $\mathscr{L}$ one may obtain a cartesian closed category $C(\mathscr{L})$ said to be *generated* by $\mathscr{L}$.

The objects of $C(\mathcal{L})$ are the types of $\mathcal{L}$.

The arrows $A \to B$ of $C(\mathcal{L})$ are pairs $(x \in A,\ \varphi(x) \in B)$, where $\varphi(x)$ contains no free variables other than x. We say that $(x \in A,\ \varphi(x) \in B) = (x' \in A,\ \psi(x') \in B)$ provided $\varphi(x) \underset{x}{=} \psi(x)$ in $\mathcal{C}[x]$.

$1_A : A \to A$ is $(x \in A,\ x \in A)$.

If $f = (x \in A, \varphi(x) \in B)$ and $g = (y \in B, \psi(y) \in C)$ then $gf = (x \in A,\ \psi(\varphi(x)) \in C)$.

$\circ_A \equiv (x \in A,\ * \in 1)$.

$\pi_{A,B} \equiv (z \in A{\times}B,\ \pi(z) \in A)$.

$\pi'_{A,B}$ similarly.

If $f = (z \in C,\ \varphi(z) \in A)$ and $g = (z \in C,\ \psi(z) \in B)$ then $\langle f,g\rangle = (z \in C,\ \langle\varphi(z),\psi(z)\rangle \in A{\times}B)$.

$\epsilon_{C,A} \equiv (y \in C^A{\times}A,\ \pi(y)`\pi'(y) \in C)$.

If $h = (z \in A{\times}B,\ \chi(z) \in C)$ then $h^* = (x \in A,\ \lambda_{y \in B}\chi(\langle x,y\rangle) \in C^B)$.

It is easily checked that $C(\mathcal{L})$ is indeed a cartesian closed category.

We shall refer the reader to [LS1985] for a proof of the following result.

PROPOSITION 4.1. For any cartesian closed category $\mathcal{C}$ and any typed λ-calculus $\mathcal{L}$,

$$CL(\mathcal{C}) \cong \mathcal{C},\quad LC(\mathcal{L}) \cong \mathcal{L}.$$

The first isomorphism presupposes the notion of morphism in the category of all cartesian closed categories: this is of course a cartesian closed functor. The second isomorphism presupposes the notion of morphism in the category of all typed λ-calculi; this is called a "translation" and is discussed in [LS1985]. Once we have ascertained the nature of the morphisms in the two categories in question, we can extend L and C to function between the categories and show that CL and LC are isormorphic to identity functors that is, L and C establish an equivalence of categories.

Let me end this section by making some controversial remarks about possible application of Occam's razor. One may argue that products in a cartesian closed category may be dispensed with because every arrow $C \to A\times B$ may be replaced by two arrows $C \to A$ and $C \to B$, while every arrow $A\times B \to C$ may be replaced by an arrow $A \to C^B$. We may also replace every arrow $A \to B$ by an arrow $1 \to B^A$ and then drop the terminal object 1 altogether, calling an arrow $1 \to C$ an "element" of C (or "proof" of C).

Although such an application of Occam's razor can do nothing but complicate our task, a search of the mathematical literature reveals that it has in fact been made three times.

In old books on logic, the propositional calculus is frequently presented in terms of implication and negation alone, other logical connectives such as conjunction being defined in terms of these, at the cost of working with an axiomatic system that deviates considerably from "natural deduction".

The early treatments of typed λ-calculus avoid product types altogether. Nonetheless, variables may be eliminated and one works with so-called "combinators". Thus Schönfinkel had shown essentially that every polynomial over the algebra of combinators has the form $f{}^{\prime}x$ where f is a constant combinator. (Actually he did this for the untyped λ-calculus, but the same argument works in the typed case.) Curry showed that the uniqueness of f could be assured provided the algebra of combinators satisfied five horrendous identities which were introduced for this purpose. It is now clear that these identities are essentially equivalent to the equations satisfied by a cartesian closed category.

More recently, Eilenberg and Kelly in their pioneering paper on cartesian closed categories [EK1966] again carried out the laborious exercise of eliminating products from cartesian closed categories, at the cost of producing commutative diagrams that cover whole pages.

5. NATURAL NUMBERS OBJECT

Lawvere has suggested the following definition of a *natural numbers object* (NNO) in a cartesian closed category: a diagram $1 \xrightarrow{0} N \xrightarrow{S} N$ such that for every diagram $1 \xrightarrow{a} A \xrightarrow{h} A$ there exists a unique arrow $f: N \to A$ making the following diagram commute:

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[d, equal] & N \arrow[r, "S"] \arrow[d, dotted, "f"] & N \arrow[d, dotted] \\ 1 \arrow[r, "a"'] & A \arrow[r, "h"'] & A \end{tikzcd}$$

that is, such that $f0 = a$ and $fS = hf$.

The existence of f can be expressed by saying that we have a mapping

$$J_A : \mathrm{Hom}(1,A) \times \mathrm{Hom}(A,A) \to \mathrm{Hom}(N,A)$$

such that, for all $a: 1 \to A$ and $h: A \to A$,

$$J_A(a,f)0 = a, \quad J_A(a,f)S = fJ_A(a,f).$$

The uniqueness of f is then expressed by saying that $fS = hf \Rightarrow f = J(f0,h)$, but there is no obvious way of rendering this implication by equations. If we have the existence but not necessarily the uniqueness of f we shall speak of a *weak* NNO.

The following result is very useful.

PROPOSITION 5.1. If $1 \xrightarrow{0} N \xrightarrow{S} N$ is a NNO (weak NNO) in a cartesian closed category $\mathcal{C}$ and if $x:1 \to A$ is an indeterminate arrow over $\mathcal{C}$, then $1 \xrightarrow{0} N \xrightarrow{S} N$ is also a NNO (weak NNO) in $\mathcal{C}[x]$.

Proof (sketched). Given a diagram $1 \xrightarrow{\beta(x)} B \xrightarrow{\eta(x)} B$ in $\mathcal{C}[x]$, we may use functional completeness to write $\beta(x) \equiv bx$ and $\eta(x) \equiv g\langle x\circ_B, 1_B\rangle$, where $b:A \to B$, $g:A\times B \to B$. Now consider the diagram $1 \xrightarrow{\ulcorner b \urcorner} B^A \xrightarrow{g^+} B^A$ in $\mathcal{C}$, where $g^+ \equiv (g\langle \pi'_{B^A,A}, \epsilon_{B,A}\rangle)^*$. We can then find $f:N \to B^A$ such that $f0 = \ulcorner b \urcorner$ and $fS = g^+f$. Writing $\varphi(x) \equiv \epsilon_{B,A}\langle f, x\circ_N\rangle : N \to B$, we may calculate $\varphi(x)0 \underset{x}{=} bx$ and $\varphi(x)S \underset{x}{=} \eta(x)\varphi(x)$. Moreover, if f is unique then $\varphi(x)$ is also unique, as is easily seen.

In view of this proposition, the arrows $a:1 \to A$ and $h:A \to A$ in the equation satisfied by J_A may be replaced by indeterminates $x:1 \to A$ and $u':A \to A$, so that for $z:1 \to N$,

$$J_A(x,u')0 \underset{x,u}{=} x, \quad J_A(x,u')Sz \underset{x,u,z}{=} u'J_A(x,u')z.$$

Writing $I_A(x,u,z) \equiv J_A(x,u')z$, we obtain the equation satisfied by the so-called *iterator* in a typed λ-calculus.

By a typed λ-calculus *with iterator* we mean a typed λ-calculus with an additional basic type N, an additional basic term $0 \in N$ and additional term forming operations:

$$\frac{n \in N}{S(n) \in N}, \qquad \frac{a \in A \quad e \in A^A \quad n \in N}{I_A(a,e,n) \in N},$$

satisfying the additional equations:

$$I_A(a,e,0) \underset{X}{=} a, \quad I_A(a,e,S(x)) \underset{X\cup\{x\}}{=} e'I_A(a,e,x).$$

Proposition 4.1 may easily be extended to a correspondence between cartesian closed categories with weak NNO and typed λ-calculi with iterator. The former have been studied extensively by Marie-France Thibault [T 1982] and the latter by almost all those who have pursued arithmetic in typed λ-calculi. As far as I know, only Sanchis [S 1967] has looked at the analogue of a strong NNO in typed λ-calculus. In fact, he appears to have proved the λ-calculus analogue of our Proposition 5.1. However, most authors, including Gödel [Gö 1958] eschew the condition $fS = hf \Rightarrow f = J(f0,h)$, probably because it is not expressed in the form of equations. Whether it can be so expressed is a question to which we shall return presently.

For the moment, let us get some idea of the power of a strong natural numbers object by showing that much of ordinary arithmetic can be carried out inside any cartesian closed category with NNO.

PROPOSITION 5.2. (Primitive recursion). Let $\mathscr{C}$ be a cartesian closed category with NNO, A any object of $\mathscr{C}$ and $a:1 \to A$, $h:N\times A \to A$ given arrows in $\mathscr{C}$. Then there exists a unique $f:N \to A$ such that $f0 = a$ and $fS = h\langle 1_N, f\rangle$.

Proof. Consider the diagram

$$1 \xrightarrow{\langle 0,a\rangle} N\times A \xrightarrow{\langle S\pi_{N,A},h\rangle} N\times A,$$

let $\langle g,f\rangle \equiv J_{N\times A}(\langle 0,a\rangle, \langle S\pi_{N,A},h\rangle)$ and check that $g = J_N(0,1_N) = 1_N$.

We shall write $f \equiv R_A(a,h)$ so that the existence of f is expressed by:

$$R_A(a,h)0 = a, \quad R_A(a,h)S = h\langle 1_N, R_A(a,h)\rangle.$$

COROLLARY 5.3. Given arrows $a:1 \to A$ and $k:N \to A$, there exists a unique $f:N \to A$ such that $f0 = a$ and $fS = k$.

Proof. Put $h = k\pi_{N,A}$ in Proposition 5.2.

We shall write $f = R_A(a, k\pi_{N,A}) \equiv [a,k]$. Here both the existence and the uniqueness of f are easily expressed by equations:

$$[a,k]0 = a, \quad [a,k]S = k, \quad [f0,fS] = f.$$

They assert that $N \cong 1+N$ is a coproduct with canonical injections $a:1 \to N$ and $S:N \to N$.

Let us see, for example, how to do addition in a cartesian closed category $\mathscr{C}$ with NNO. Of course we must use Peano's recursive definition:

$$x+0 \underset{x}{=} x, \qquad x+Sy \underset{x,y}{=} S(x+y).$$

Let us write $f_x y \equiv x+y$, then $f_x:N \to N$ is an arrow in $\mathscr{C}[x]$ defined by

$$f_x 0 \underset{x}{=} x, \quad f_x S \underset{x}{=} Sf_x.$$

Thus $f_x \underset{x}{=} J_N(x,S)$. We may think of $+:N\times N \to N$ as the arrow defined by $+\langle x,y\rangle \underset{x,y}{=} f_x y \underset{x,y}{=} J(x,S)y$. However, for the the sake of readability, we shall continue to write x+y for $+\langle x,y\rangle$.

We claim that the usual laws of addition hold in any cartesian closed category with natural numbers object.

PROPOSITION 5.4. In a cartesian closed category with NNO,

$$(x+y)+z \underset{x,y,z}{=} x+(y+z), \qquad x+y \underset{x,y}{=} y+z.$$

Proof. Let us note that the usual proofs of these laws by mathematical induction are not valid as they stand. However, they can be modified as we shall show in case of the associative law.

Let $f_{x,y}, g_{x,y}:N \rightrightarrows N$ be the arrows in $\mathscr{C}[x,y]$ defined by $f_{x,y}z \equiv (x+y)+z$ and $g_{x,y}z \equiv x+(y+z)$.

We easily calculate that

$$f_{x,y}0 \underset{x,y}{=} x+y, \qquad f_{x,y}S \underset{x,y}{=} Sf_{x,y}$$

and similarly for $g_{x,y}$. Thus

$$f_{x,y} \underset{x,y}{=} J_N(x+y,S) \underset{x,y}{=} g_{x,y}.$$

This establishes the associative law. The commutative law may be proved similarly, with the help of two lemmas: $0+x \underset{x}{=} x$, $Sy+x \underset{x,y}{=} S(y+x)$, which themselves are proved in the same fashion.

The predecessor and naive difference are usually defined as follows:

$$P0 = 0, \quad PSz \underset{z}{=} z; \qquad x \dot{-} 0 \underset{x}{=} x, \quad x \dot{-} Sy \underset{x,y}{=} P(x \dot{-} y).$$

We may easily translate these into explicit definitions thus:

$$P \equiv [0,1_N], \qquad x \dot{-} y \equiv J_N(x,P)y.$$

Also useful are the "delta function" and the "minimum" defined by

$$\delta z \equiv 1 \dot{-} z; \qquad \min(x,y) \equiv y \dot{-} (y \dot{-} x).$$

The following properties of the naive difference will turn out to be useful.

PROPOSITION 5.5.

(a) $Sx \dot{-} Sy \underset{x,y}{=} x \dot{-} y$;

(b) $x \dot{-} x \underset{x}{=} 0$;

(c) $0 \dot{-} x \underset{x}{=} 0$;

(d) $(x+y) \dot{-} y \underset{x,y}{=} y$;

(e) $Sx \dot{-} y \underset{x,y}{=} (x \dot{-} Py) + \delta y$;

(f) $Sx \dot{-} y \underset{x,y}{=} (x \dot{-} y) + \delta(y \dot{-} x)$.

Proof. (a) We calculate

$$Sx \dot{-} S0 \underset{x}{=} P(Sx \dot{-} 0) \underset{x}{=} PSx \underset{x}{=} x,$$

$$Sx \dot{-} SSy \underset{x,y}{=} P(Sx \dot{-} Sy),$$

hence $Sx \dot{-} Sy \underset{x,y}{=} J_N(x,P)y = x \dot{-} y$.

(b) $0 \dot{-} 0 = 0$, $Sx \dot{-} Sx = x \dot{-} x$ by (a), hence $x \dot{-} x \underset{x}{=} J(0,1_N)x \underset{x}{=} 0o_Nx \underset{x}{=} 0$.

(c) $0 \dot{-} x \underset{x}{=} J_N(0,P)x \underset{x}{=} 0o_Nx \underset{x}{=} 0$.

(d) $(x+0) \mathbin{\dot-} 0 \underset{x}{=} x$, $(x+Sy) \mathbin{\dot-} Sy \underset{x,y}{=} S(x+y) \mathbin{\dot-} Sy \underset{x,y}{=} (x+y) \mathbin{\dot-} y$ by (a), hence $(x+y) \mathbin{\dot-} y \underset{x,y}{=} J_N(x,1_N)y \underset{x,y}{=} x\circ_N y \underset{x,y}{=} x$.

(e) We check that both sides are equal to $[Sx, J_N(x,P)]y$.

(f) Writing $\varphi(x,y)$ and $\psi(x,y)$ for the LHS and RHS respectively, we calculate:

$$\varphi(x,0) \underset{x}{=} Sx \underset{x}{=} \psi(x,0), \quad \varphi(0,y) \underset{y}{=} \delta y \underset{y}{=} \psi(0,y),$$
$$\varphi(Sx,Sy) \underset{x,y}{=} \varphi(x,y), \quad \psi(Sx,Sy) \underset{x,y}{=} \psi(x,y),$$

hence $\varphi(x,y) \underset{x,y}{=} \psi(x,y)$ by the following lemma.

LEMMA 5.6. Given $\alpha : N \to N$ and $\beta : N \to N$ such that $\alpha 0 = \beta 0$, there exists a unique $\varphi : N \times N \to N$ such that $\varphi\langle x,0\rangle \underset{x}{=} \alpha x$, $\varphi\langle 0,y\rangle \underset{y}{=} \beta y$, $\varphi\langle Sx,Sy\rangle \underset{x,y}{=} \varphi\langle x,y\rangle$.

Proof. Define $\eta : N \times N^N \to N^N$ by

$$\eta\langle x,u\rangle \equiv \ulcorner [\alpha Sx, u^{\backprime}] \urcorner$$

where $x : 1 \to N$, $u : 1 \to N^N$ are indeterminate arrows. Then define $\psi : N \to N^N$ by recursion thus:

$$\psi 0 = \ulcorner \beta \urcorner, \quad \psi Sx \underset{x}{=} \eta\langle x, \psi x\rangle,$$

that is, $\psi \equiv R_{N^N}(\ulcorner\beta\urcorner, \eta)$. Putting $\varphi\langle x,y\rangle \underset{x,y}{=} (\psi x)^{\backprime} y$, we calculate

$$\varphi\langle 0,0\rangle = (\psi 0)^{\backprime} 0 = \ulcorner\beta\urcorner^{\backprime} 0 = \beta 0 = \alpha 0,$$

$$\varphi\langle Sx,0\rangle \underset{x}{=} (\psi Sx)^{\backprime} 0 \underset{x}{=} \eta\langle x,\psi x\rangle^{\backprime}\, 0 \underset{x}{=} [\alpha Sx, (\psi x)^{\backprime}]0 \underset{x}{=} \alpha Sx,$$

$$\varphi\langle 0,y\rangle \underset{y}{=} (\psi 0)^{\backprime} y \underset{y}{=} \ulcorner\beta\urcorner^{\backprime} y \underset{y}{=} \beta y,$$

$$\varphi\langle Sx,Sy\rangle \underset{x,y}{=} (\psi Sx)^{\backprime} Sy \underset{x,y}{=} \eta\langle x,\psi x\rangle^{\backprime} Sy \underset{x,y}{=} [\alpha Sx, (\psi x)^{\backprime}]Sy$$
$$\underset{x,y}{=} (\psi x)^{\backprime} y \underset{x,y}{=} \varphi\langle x,y\rangle.$$

Note that $\varphi\langle x,0\rangle \underset{x}{=} [\alpha 0, \alpha S]x \underset{x}{=} \alpha x$.

We shall end this section by obtaining a property of the minimum which will be used in the next section.

PROPOSITION 5.7.

(a) $\min(x,Sy) \underset{x,y}{=} \min(x,y)+\delta(Sy \dot{-} x)$,

(b) $(x \dot{-} y) + \min(x,y) \underset{x,y}{=} x$.

Proof. (a) We easily check that both sides of (a) are $[Sy, J_N(y,P)]z$.

(b) $x \dot{-} 0 + \min(x,0) \underset{x}{=} x$,

$$\begin{aligned}(x \dot{-} Sy) + \min(x,Sy) &\underset{x,y}{=} (x \dot{-} Sy) + \min(x,y) + \delta(Sy \dot{-} x) \quad \text{by (a)}\\ &\underset{x,y}{=} (Sx \dot{-} Sy) + \min(x,y) \quad \text{by Proposition 5.5(f)}\\ &\underset{x,y}{=} (x \dot{-} y) + \min(x,y) \quad \text{by Proposition 5.5(a),}\end{aligned}$$

hence

$$x \dot{-} y + \min(x,y) \underset{x,y}{=} J_N(x,1_N)y \underset{x,y}{=} x o_N y \underset{x,y}{=} x.$$

6. CAN THE NNO BE PRESENTED EQUATIONALLY?

Given a weak NNO in a cartesian closed category. We shall call it a NNO *with respect to* the object A if, for all $h:A \to A$, $fS = hf$ implies $f = J_A(f0,h)$. To say that N is a NNO then means that it is one with respect to every object A.

PROPOSITION 6.1. Let $(N,0,S)$ be a weak natural numbers object in a cartesian closed category. A sufficient condition for this to be a NNO with respect to A is the statement $\Gamma(A)$: there exist

$$\omega_A \in \mathrm{Hom}(N,A),$$

$$\varphi_A: \mathrm{Hom}(N,A)^2 \to \mathrm{Hom}(N,A),$$

$$\psi_A: \mathrm{Hom}(1,A)\times\mathrm{Hom}(A\times A)\times\mathrm{Hom}(N,A)^2 \to \mathrm{Hom}(N,A)$$

such that, for all $f:N \to A$,

$$\varphi_A(f,f) = \omega_A,$$

$$\psi_A(f0,h,\varphi_A(fS,hf),\varphi_A(hf,fS)) = f.$$

If the object A is isomorphic to a finite product of internal powers N^B of N, this condition is also necessary.

Note that the last assumption is satisfied by all objects A of $\mathscr{C}$ when $\mathscr{C}$ is the *free* cartesian closed category with natural numbers object generated by the empty graph. This corresponds to the *pure* typed λ-calculus with iterator in which there are no types, terms or equations except those which it has to have by virtue of its definition as typed λ-calculus with iterator.

Proof. To prove the sufficiency of the condition, suppose $f0 = a$, $fS = hf$, $g0 = a$, $gS = hg$, then $f = \psi_A(a,h,\omega_A,\omega_A) = g$.

To prove the necessity, first consider the case $A = N$. Define

$$\omega_A \equiv 0o_N : N \to 1 \to N,$$

$$\varphi_A(f,g)x \equiv fx \dot{-} gx.$$

$$\psi_A(a,h,g,g') \equiv R_A(a,\eta_A(h,g,g')),$$

where

$$\eta_A(h,g,g')\langle x,y\rangle \equiv gx + (hy \dot{-} g'x).$$

Then clearly

$$\varphi_A(f,f)x \underset{x}{=} fx \dot{-} fx \underset{x}{=} 0,$$

so $\varphi_A(f,g) = 0o_N = \omega_A$. To prove the equation involving ψ_A, it suffices to show that

$$f = R_A(a,\eta_A(h,g,g'))$$

in case $a = f0$, $g = \varphi_A(fS,hf)$, $g' = \varphi_A(hf,fS)$. Indeed, we have

$$f0 = a,$$

and we show that

$$fSx = \eta_A(h,g,g')\langle x,fx\rangle,$$

because the

$$\begin{aligned} \text{RHS} &= gx + (hfx \dot{-} g'x) \\ &= (fSx \dot{-} hfx) + (hfx \dot{-} (hfx \dot{-} fSx)) \\ &= (fSx \dot{-} hfx) + \min(fSx,hfx) \\ &= fSx, \end{aligned}$$

by Proposition 5.7.

Next, consider the case $A = N^B$. The same argument will work, because we can define $+$ and $\dot{-}$ as arrows $A\times A \to A$ to satisfy the same identities as before, provided we take $\omega_A = (\lambda_{y\in B}0)\circ_A$. For example, if u and v are indeterminate arrows $1 \to N^B$ and $y:1 \to B$ is another indeterminate, we define

$$(u \dot{-} v)^{\epsilon}y \equiv u^{\epsilon}y \dot{-} v^{\epsilon}y,$$

hence, e.g.,

$$(u \dot{-} u)^{\epsilon}y \underset{x,y}{=} 0.$$

Replacing u by fx, where $f:N \to A$, we get

$$\varphi_A(f,f)x \underset{x}{=} fx \dot{-} fx \underset{x}{=} \lambda_{y\in B}0 \underset{x}{=} \omega_A x,$$

and so

$$\varphi_A(f,f) = \omega_A.$$

Finally, suppose $A = A_1\times A_1$, where we have already defined $+_1:A_1\times A_1 \to A_1$ and $+_2:A_2\times A_2 \to A_2$, then we define $+:A\times A \to A$ elementwise by

$$\langle x_1,x_2\rangle + \langle x_1',x_2'\rangle \equiv \langle x_1+x_1', x_2+x_2'\rangle$$

and similarly for $\dot{-}$ and 0. Clearly all identities are preserved and the same argument will work again.

We can get a little more from the above argument. Consider the canonical arrow $\eta_A:A \to N^{(N^A)}$ defined by $\eta_A x \equiv \hat{x}$, where $\hat{x}$ is defined by $\hat{x}^{\epsilon}u \equiv u^{\epsilon}x$.

Here $x:1 \to A$ and $u:1 \to N^A$ are indeterminate arrows. It is easily seen that η is a natural transformation from the identity functor on $\mathscr{C}$ to the functor $N^{N^{(-)}}$.

Let us call two arrows $f,g:C \rightrightarrows A$ *quasi-equal* if $\eta_A f = \eta_A g$. By a *quasi-NNO* we shall understand a weak natural numbers object in which the arrow $f:N \to A$ such that $f0 = a$ and $fS = hf$ is unique up to quasi-equality.

PROPOSITION 6.2. Given a cartesian closed category with a weak natural numbers object $(N,0,S)$. A necessary and sufficient condition for this to be a quasi-NNO is that for each object A we have $\Gamma(N^{N^A})$ (or, equivalently, $\Gamma(N^B)$ for each object B).

Proof. First, let us show the sufficiency of the condition. Suppose $f0 = a$, $fS = hf$, $g0 = a$, $gS = hg$. Consider the following diagram:

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[d, equal] & N \arrow[r, "S"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & N \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\
1 \arrow[r, "a"'] \arrow[d, equal] & A \arrow[r, "h"'] \arrow[d, "\eta_A"'] & A \arrow[d, "\eta_A"] \\
1 \arrow[r, "\hat{a}"'] & N^{N^A} \arrow[r, "N^{N^h}"'] & N^{N^A}
\end{tikzcd}$$

By naturality of η, the lower right square commutes, and it is easily checked that $\eta_A a = \hat{a}$.

In view of Proposition 6.1, $\Gamma(N^{N^A})$ assures that $\eta_A f = \eta_A g$, since both have the form $J_{N^{N^A}}(\hat{a}, N^{N^h})$. Thus f and g are quasi-equal.

To prove the necessity of the condition, let us assume that N is a quasi-NNO. Thus, given $f, g : N \rightrightarrows A$ such that $f0 = a$, $fS = hf$, $g0 = a$, $gS = hg$, we have $\eta_A f = \eta_A g$. Now assume $A = N^B$, then we have $\eta_B : B \to N^{N^B}$ and therefore $N^{\eta_B} : N^{N^A} \to A$. It is easily checked that $N^{\eta_B}\eta_A = 1_A$, so it follows that $f = g$. Thus N is a strong NNO with respect to N^B and so Proposition 6.1 shows that $\Gamma(N^B)$ holds for each object B, in particular, when B itself is an internal power of N.

For which objects A does a quasi-NNO act like a NNO, that is, when is η_A a monomorphism?

PROPOSITION 6.3. The class of objects for which η_A is a monomorphism contains N and is closed under subobjects, finite products and internal powers (i.e. if A is in the class and B is any object then A^B is in the class).

Proof. That $N \cong N^1$ is in the class follows from the proof of Proposition 6.2, where it was shown that N^B is in the class. The other assertions follow from inspection of the following three diagrams.

(1)
$$\begin{tikzcd}
C \arrow[dr, shift left, "f"] \arrow[dr, shift right, "g"'] & & \\
& B \arrow[r, "\eta_B"] \arrow[d, "m"'] & N^{N^B} \arrow[d, "N^{N^m}"] \\
& A \arrow[r, "\eta_A"'] & N^{N^A}
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
C \arrow[dr, shift left, "f"] \arrow[dr, shift right, "g"'] & & \\
& A \times B \arrow[r, "\eta_{A\times B}"] \arrow[d, "\pi_{A,B}"'] & N^{N^{A\times B}} \arrow[d, "N^{N^{\pi_{A,B}}}"] \\
& A \arrow[r, "\eta_A"'] & N^{N^A}
\end{tikzcd}$$

(and similarly for π')

(3)
$$\begin{tikzcd}
C \arrow[dr, shift left, "f"] \arrow[dr, shift right, "g"'] & & \\
& A^B \arrow[r, "\eta_{A^B}"] \arrow[d, "p_y"'] & N^{N^{A^B}} \arrow[d, "N^{N^{p_y}}"] \\
& A \arrow[r, "\eta_A"'] & N^{N^A}
\end{tikzcd}$$

Here p_y is defined for each indeterminate arrow $y: 1 \to B$ by $p_y{}^{\text{‘}}u \equiv u^{\text{‘}}y$, where $u: 1 \to A^B$ is an indeterminate arrow.

(1) We are given that η_A and m are monomorphisms, hence so is $N^{N^m}\eta_B$, hence so is η_B.

(2) We are given that η_A and η_B are monomorphisms. Suppose $\eta_{A\times B}f = \eta_{A\times B}g$, then $\pi_{A,B}f = \pi_{A,B}g$ and similarly $\pi'_{A,B}f = \pi'_{A,B}g$. Therefore $f = g$.

(3) We are given that η_A is a monomorphism. Suppose $\eta_{A^B} f = \eta_{A^B} g$, then $p_y f \underset{y}{=} p_y g$. Let $z: 1 \to C$ be an indeterminate arrow, then $p_y fz \underset{y,z}{=} p_y gz$, that is, $fz^{\text{‘}}y \underset{y,z}{=} gz^{\text{‘}}y$. By functional completeness, $f = g$.

REMARK 6.4. The class of objects A for which η_A is a monomorphism contains the subclass for which η_A is a retract. This also contains N and is closed under finite products, internal powers and retracts.

The class of objects for which η_A is a monomorphism (retract) may also be described as the class of subobjects (retracts) of internal powers of N.

It may be useful to have a criterion for two arrows $f,g: C \rightrightarrows A$ in $\mathscr{C}$ to be quasi-equal. The statement $\eta_A f = \eta_A g$ easily translates into $u^{\text{‘}}f \underset{u}{=} u^{\text{‘}}g$ where $u: 1 \to N^A$ is an indeterminate. A necessary condition for this is that $hf = hg$ for any $h: A \to N$; just replace u by $\ulcorner h \urcorner$. Unfortunately, this condition is not sufficient. For example, in the free topos $\mathscr{F}$ one has $h_\top = h_\bot$ for any $h: \Omega \to N$ (see [LS 1985]), yet $\eta_\Omega \top = \eta_\Omega \bot$ in $\mathscr{F}$ would imply the same statement in any topos, in particular in the category of sets, where it is surely false. However, the following condition is both necessary and sufficient.

PROPOSITION 6.5. Two arrows $f,g: C \rightrightarrows A$ in a cartesian closed category $\mathscr{C}$ are quasi-equal if and only if, for every cartesian closed functor $F: \mathscr{C} \to \mathscr{D}$ and every arrow $h: F(A) \to F(N)$ in $\mathscr{D}$, $hF(f) = hG(g)$.

Proof. The reader should consult propositions 2.2 and 3.2 for the universal property of $\mathscr{C}[u]$.

To prove the sufficiency of the above criterion, take the special case $\mathscr{D} = \mathscr{C}[u]$, $F = H_u: \mathscr{C} \to \mathscr{C}[u]$ and $h = u^{\text{‘}}: N \to A$.

To prove the necessity of the criterion, apply the cartesian closed functor $F_{\ulcorner h \urcorner} : \mathscr{C}[u] \to \mathscr{D}$ to the equation $u^{\text{‘}} f \underset{u}{=} u^{\text{‘}} g$ and obtain $hf = hg$.

How could this criterion be useful in showing that η_A is a monomorphism? For example, suppose 1 is a generator in $\mathscr{C}$, F is faithful and $\mathscr{D}$ is a Boolean topos. Suppose $f \neq g : C \rightrightarrows A$ in $\mathscr{C}$, then we can find $c : 1 \to C$ such that $fc \neq gc$, hence $F(fc) \neq F(gc)$. Define $h : F(A) \to F(N)$ by stipulating that, for $x \in F(A)$, $hx = F(0)$ if $x = F(fc)$ and $hx = F(1)$ if $x \neq F(fc)$. (This can be done in the internal language of the topos $\mathscr{D}$.) Then surely $hF(fc) \neq hF(gc)$, hence $hF(f) \neq hF(g)$, so $\eta_A f \neq \eta_A g$.

The proof of the following presupposes some knowledge of toposes and their internal language (see e.g. [LS1985]).

PROPOSITION 6.6. In a topos $\mathscr{T}$, η_A is a monomorphism for all objects A of $\mathscr{T}$ if and only if $\mathscr{T}$ is Boolean.

Proof. Suppose η_A is a monomorphism. This is expressed in the internal language of the topos by the statement

$$\forall_{x \in A} \forall_{x' \in A} \, (\eta_A x = \eta_A x' \Rightarrow x = x'),$$

which holds in $\mathscr{T}$. We argue in the internal language as follows. Suppose $\neg\neg(x = x')$, then also $\neg\neg(u^{\text{‘}}x = u^{\text{‘}}x')$ for any $u \in N^A$. But the NNO is decidable, so $u^{\text{‘}}x = u^{\text{‘}}x'$. Thus $\forall_{u \in N^A} (u^{\text{‘}}x = u^{\text{‘}}x')$, that is, $\eta_A x = \eta_A x'$, hence $x = x'$. We have thus shown that

$$\forall_{x \in A} \forall_{x' \in A} \, (\neg\neg(x = x') \Rightarrow x = x').$$

In particular, take $A = \Omega$ and $x' = T$, then we obtain $\forall_{x \in \Omega}(\neg\neg x \Rightarrow x)$, that is, $\mathscr{T}$ is Boolean.

Conversely, suppose $\mathcal{T}$ is Boolean. We again argue informally in the internal language of $\mathcal{T}$ and assume that $\eta_A x = \eta_A x'$, that is, $\forall_{u \in N^A} (u`x = u`x')$. Let us replace u by $\ulcorner \delta_x \urcorner$, where $\delta_x : A \to N$ is defined thus:

$$\delta_x x' = \begin{cases} 1 & \text{if } x' = x \\ 0 & \text{if } \neg(x' = x) \end{cases} .$$

Then $1 = \delta_x `x = \delta_x `x'$, so $x' = x$. Therefore η_A is a monomorphism.

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NOTES ADDED IN PROOF

The editors have suggested that I add the following clarifying comments.

(1) In saying that a functor preserves the Cartesian structure "on the nose", as we did just prior to Proposition 2.2, we mean to indicate that the structure is preserved exactly, not just up to isomorphism, e.g.

$H_x(\langle f,g\rangle) = \langle H_x(f),H_x(g)\rangle$

(2) On second thought, only (a) and (f) in Proposition 5.5 seem to be used later. The other properties listed have been used in a superceded argument.

(3) The necessary and sufficient condition in Proposition 6.2 may be expressed more clearly by saying: (N,O,S) is a NNO with respect to N^B for each object B.

(4) The reference [C1983] is now available as

[C1986] P-L. Curien, Categorical Combinators, Sequential Algorithms and Functional Programming (Pitman, 1985).