# The Reachability Problem for Petri Nets is Not Elementary (Extended Abstract)

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**Abstract**

Petri nets, also known as vector addition systems, are a long established and widely used model of concurrent processes. The complexity of their reachability problem is one of the most prominent open questions in the theory of verification. That the reachability problem is decidable was established by Mayr in his seminal STOC 1981 work, and the currently best upper bound is non-primitive recursive cubic-Ackermannian of Leroux and Schmitz from LICS 2015. We show that the reachability problem is not elementary. Until this work, the best lower bound has been exponential space, due to Lipton in 1976.

## 1 Introduction

Petri nets [16], also known as vector addition systems [6], [4, cf. Section 5.1], [5], are a long established model of concurrency with extensive applications in modelling and analysis of hardware, software, chemical, biological and business processes.

The central algorithmic problem for Petri nets is reachability: whether from the given initial configuration there exists a sequence of valid execution steps that reaches the given final configuration. It is one of the most well-known problems in theoretical computer science, and has had an interesting history over the past half a century.

The late 1970s and the early 1980s saw the initial burst of activity. After an incomplete proof by Sacerdote and Tenney [18], decidability was established by Mayr [14, 15], whose proof was then simplified by Kosaraju [7]. Building on the further progress made by Lambert in the 1990s [8], substantial progress over the past ten years has been made by Leroux, culminating in the joint work with Schmitz [11] that provides the first upper bound on the complexity, namely membership in fast-growing class $\mathbf{F}_{\omega^3}$ [20].

In contrast, since Lipton's landmark result over 40 years ago that the reachability problem requires exponential space [13], there has been virtually no progress on lower bounds. Moreover, it became common in the community to conjecture that the problem is ExpSpace-complete.

**Main Result and Its Significance.** We show that the reachability problem is not elementary, more precisely that it is hard for the class Tower of all decision problems that are solvable in time or space bounded by a tower of exponentials whose height is an elementary function of the input size [20, Section 2.3]. We see this result as important for several reasons:

- It shatters the conjecture of ExpSpace-completeness, establishing that the reachability problem is much harder than the coverability (i.e., state reachability) problem; the latter is also ubiquitous but has been known to be ExpSpace-complete since the late 1970s [13, 17].
- It narrows significantly the gap to the upper bound; in fact, there has been recent progress towards improving the latter to membership in $\mathbf{F}_\omega$ [21], the smallest class in the fast-growing hierarchy that contains non-primitive recursive problems.
- It implies that a plethora of problems from formal languages, logic, concurrent systems, process calculi and other areas, that are known to admit reductions from the Petri nets reachability problem, are also not elementary; for a comprehensive listing of such problems, we refer to Schmitz's recent survey [19].
- It makes obsolete the non-elementary lower bounds for the reachability problems for two key extensions of Petri nets: branching vector addition systems [9] and pushdown vector addition systems [10].

**Key Ideas and Their Presentation.** The proof of our main result involves a few ideas that we regard as worth singling out, and we have strived to organise the paper so that they are presented clearly.

- At the centre of our construction is an identity of the form $\prod_{i=0}^{k} (s_{k,i}/r_{k,i})^{2^{k-i}} = s_k/r_k$, where $s_{k,i}$, $r_{k,i}$, $s_k$ and $r_k$ are certain positive integers bounded exponentially in $k$, and such that $s_{k,i} > r_{k,i}$ and $s_k > r_k$. Thus for any $k$, we have a product of $k+1$ exponential powers of fractions greater than 1 which reduces to a single fraction, where the numerator and the denominator of each of those fractions are bounded exponentially. The details are in Section 4.
- We develop the idea of implementing the identity by Petri nets, where a counter is multiplied approximately by the left-hand side and the accuracy of the result is checked using

the right-hand side, to obtain gadgets that, assuming they can zero test counters bounded exponentially in $k$ (no zero tests are a priori permitted in Petri nets), guarantee to compute exactly a doubly exponential number $B$ as well as pairs of arbitrarily large numbers whose ratio is that $B$. This is presented in Section 5.

- We find a way to use $B$ together with a pair of sufficiently large numbers whose ratio is that $B$ to simulate zero tests of counters bounded by $B$. This allows us to link the gadgets into chains of arbitrary length $n$, so that we bootstrap from not having any zero tests to being able to simulate accurately zero tests of counters that are bounded by the tower of exponentials of height $n$. Some of the pieces for that are developed in Section 3, and everything is assembled in Section 6.

Having recalled that the coverability problem is in ExpSpace, it is not surprising that the underlying theme in all the key ideas here is that executions of the Petri nets we build are not constrained to be correct until the very end. In other words, bad nondeterministic choices may cause inaccuracies in computations, but our construction is such that as soon as that happens, a witness will remain and make reaching the final configuration impossible.

We also remark that the size of the Petri nets produced by our main construction is linear. A straightforward modification would show that there exists a small constant $D$ such that, for any $h$, the reachability problem for vector addition systems whose dimension (i.e., the number of counters) is $D \cdot h$ is $h$-ExpSpace-hard.

## 2 Counter Programs

Proving the main result of this paper, namely that solving the Petri nets reachability problem requires a tower of exponentials of time and space, involves some intricate programming. For ease of presentation, instead of working directly with Petri nets or vector addition systems, our primary language will be imperative programs that operate on variables (also called counters) that range over the naturals (i.e. the nonnegative integers). Such a program is a sequence of commands, each of which is of one of the following four types:

| | |
|---|---|
| $\mathsf{x} \mathrel{+}= 1$ | (increment variable $\mathsf{x}$) |
| $\mathsf{x} \mathrel{-}= 1$ | (decrement variable $\mathsf{x}$) |
| **goto** $L$ **or** $L'$ | (jump to either line $L$ or line $L'$) |
| **test** $\mathsf{x} = 0$ | (continue if variable $\mathsf{x}$ is zero), |

except that the last command is of the form:

**halt if** $\mathsf{x}_1, \ldots, \mathsf{x}_l = 0$ (terminate provided all the listed variables are zero).

We use a shorthand **halt** when no variable is required to be zero at termination.

To illustrate how the available commands can be used to express further constructs, addition $\mathsf{x} \mathrel{+}= m$ and substraction $\mathsf{x} \mathrel{-}= m$ of a natural constant $m$ can be written as $m$ consecutive increments $\mathsf{x} \mathrel{+}= 1$ and decrements $\mathsf{x} \mathrel{-}= 1$ (respectively). As another illustration, conditional jumps **if** $\mathsf{x} = 0$ **then goto** $L$ **else goto** $L'$ which feature in common definitions of Minsky machines can be written as:

1: **goto** 2 **or** 3
2: **test** $\mathsf{x} = 0$ **goto** $L$
3: $\mathsf{x} \mathrel{-}= 1$ $\mathsf{x} \mathrel{+}= 1$ **goto** $L'$,

where **goto** $L$ is a shorthand for the deterministic jump **goto** $L$ **or** $L$.

We emphasise that variables are not permitted to have negative values. In the example we have just seen, that is why the first two commands in line 3 work as a non-zero test.

Two more remarks may be useful. Firstly, our notion of counter programs only serves as a convenient medium for presenting both Petri nets and Minsky machines, and the exact syntax is not important; we were inspired here by Esparza's presentation [3, Section 7] of Lipton's lower bound [13]. Secondly, although the **halt if** $\mathsf{x}_1, \ldots, \mathsf{x}_l = 0$ commands could be expressed by zero tests followed by just **halt**, having them as atomic will facilitate clarity. Namely, as already indicated, much of the work in this paper can be seen as developing gadgets and mechanisms for eliminating zero tests of tower-bounded counters; however, zero tests that are at termination both do not need to be eliminated (since they can be enforced by the final configurations in the Petri nets reachability problem) and will play a key role in our construction.

**Runs and Computed Relations.** A *run* of a program from an initial valuation of all its variables is *halted* if it has successfully executed its **halt** command (which is necessarily the program's last); otherwise, the run is either *partial* or *infinite*. Observe that, due to a decrement that would cause a counter to become negative, or due to an unsuccesful zero test, or due to an unsuccessful terminal check for zero, a partial run may be maximal because it is blocked from further execution. Moreover, due to nondeterministic jumps, the same program from the same initial valuation may have various runs in each of the three categories: halted runs, maximal partial runs, and infinite runs. We are mostly going to be interested in final variable valuations that are reached by halted runs.

We regard a run as *complete* if and only if it is halted and its initial valuation assigns zero to every variable. Let $\mathsf{x}_1, \ldots, \mathsf{x}_l$ be some (not necessarily all) of the variables in the program. We say that the *relation computed in* $\mathsf{x}_1, \ldots, \mathsf{x}_l$ by a program is the set of all tuples $\langle v_1, \ldots, v_l \rangle$ such that the program has a complete run whose final valuation assigns to every variable $\mathsf{x}_i$ the natural number $v_i$.

*Example* 1. Consider the following program, where $B$ is a natural constant:

1: $\mathsf{x}' \mathrel{+}= B$
2: **goto** 6 **or** 3
3: $\mathsf{x} \mathrel{+}= 1 \quad \mathsf{x}' \mathrel{-}= 1$
4: $\mathsf{y} \mathrel{+}= 2$
5: **goto** 2
6: **halt if** $\mathsf{x}' = 0$.

It repeats the block of three commands in lines 3–4 some number of times chosen nondeterministically (possibly zero, possibly infinite) and then halts provided counter $\mathsf{x}'$ is zero. Replacing the two jumps by more readable syntactic sugar, we may write this code as:

$\mathsf{x}' \mathrel{+}= B$
**loop**
  $\mathsf{x} \mathrel{+}= 1 \quad \mathsf{x}' \mathrel{-}= 1$
  $\mathsf{y} \mathrel{+}= 2$
**halt if** $\mathsf{x}' = 0$.

It is easy to see that the relation computed in $\mathsf{x}, \mathsf{y}$ is the set with the single tuple $\langle B, 2B \rangle$. $\square$

*Example* 2. We shall need to reason about properties of variable valuations at certain points in programs. As an example which will be useful later for eliminating zero tests of bounded counters, consider a fixed positive integer $B$ and assume that

$$\mathsf{x} + \mathsf{x}' \leq B \text{ and } \mathsf{d} \geq \mathsf{c} \cdot B \tag{1}$$

holds in a run at the entry to (i.e., just before executing) the program fragment

**loop**
  $\mathsf{x} \mathrel{+}= 1 \quad \mathsf{x}' \mathrel{-}= 1$

$\mathsf{d} \mathrel{-}= 1$
$\mathsf{c} \mathrel{-}= 1.$

The number of times the loop has been iterated by a run that also exits (i.e., completes executing) the program fragment is nondeterministic, so let us denote it by $K$. It is easy to see that property (1) necessarily also holds at the exit, since:

- the sum $\mathsf{x} + \mathsf{x}'$ is maintained by each iteration of the loop,
- we have that $K \leq B$, and
- counters $\mathsf{d}$ and $\mathsf{c}$ have been decreased by $K$ and 1 (respectively).

Continuing the example, if we additionally assume that the exit variable valuation satisfies $\mathsf{d} = \mathsf{c} \cdot B$, then we deduce that:

- necessarily $K = B$,
- $\mathsf{d} = \mathsf{c} \cdot B$ also held at the entry, and
- $\mathsf{x} = 0$ and $\mathsf{x}' = B$ at the entry, and their values at the exit are swapped.

We have thus seen two small arguments, one based on propagating properties of variable valuations forwards through executions of program fragments, and the other backwards. Both kinds will feature in the sequel. □

**Petri Nets Reachability Problem.** It is well known that Petri nets [16], vector addition systems [6], and vector addition systems with states [4, cf. Section 5.1], [5] are alternative presentations of the same model of concurrent processes, in the sense that between each pair there exist straightforward translations that run in polynomial time and preserve the reachability problem; for further details, see e.g. Schmitz's recent survey [19, Section 2.1].

Since counter programs with no zero tests can be seen as linear presentations of vector addition systems with states, where the latter are required to start with all vector components zero and to finish with vector components zero as specified by the **halt** command, the Petri nets reachability problem can be stated as:

**Input** A counter program with no zero test commands.

**Question** Does it have a complete run?

We remark that restricting further to programs where no counter is required to be zero finally (i.e., where the last command is just **halt**) turns this problem into the Petri nets *coverability* problem. In the terminology of vector addition systems with states, the latter problem is concerned with reachability of just a state, with no requirement on the final vector components. Lipton's ExpSpace lower bound [13] (cf. also Esparza's presentation [3, Section 7]) holds already for the coverability problem, which is in fact ExpSpace-complete [17].

**A Tower-Complete Problem.** We say that a counter $\mathsf{x}$ is *bounded by $B$* in a program if and only if $\mathsf{x}$ is always at most $B$ in every run with all counters initially zero.

To prove that the Petri nets reachability problem is not elementary, we shall provide a linear-time reduction from the following restriction of the halting problem, which is complete for the class Tower of all decision problems that are solvable in time or space bounded by a tower of exponentials whose height is an elementary function of the input size [20, Section 2.3].

**Input** A counter program of size $n$, all of whose counters are bounded by $\left.2^{\cdot^{\cdot^{\cdot^{2}}}}\right\} n \text{ times.}$

**Question** Does it have a complete run?

For confirming that this problem is TOWER-complete, we refer to [20, Section 4.1] and [20, Section 4.2] for the robustness of the class with respect to the choices of the fast-growing function hierarchy (here the Ackermann hierarchy) and of the computational model (here counter programs, which are essentially nondeterministic Minsky machines), respectively.

## 3 Eliminating Zero Tests of Bounded Counters

Programs that contain **test** $\mathsf{x} = 0$ commands only for counters $\mathsf{x}$ that are bounded by $B$ can be seen as being in between Petri nets (the extreme case where $B = 0$, i.e. there are no zero tests) and nondeterministic Minsky machines (the extreme case where $B = \infty$, i.e. unrestricted zero tests). In this section, assuming $B$ is such that a certain Petri net gadget exists, we define a special operator that transforms any program that zero tests only counters bounded by $B$ and composes it with the gadget, to obtain an equivalent program with no zero tests. The rest of the paper will then provide a construction of the required Petri net gadgets for $B$ that are towers of exponentials.

**Construction.** Suppose that:

- $B$ is a positive integer;
- $\mathcal{R}$ is a *gadget for ratio* $B$, i.e. a program with no zero tests and such that the relation it computes in counters $\mathsf{b}, \mathsf{c}, \mathsf{d}$ is

$$\{\langle b, c, d\rangle \ : \ b = B,\ c > 0,\ d = c \cdot b\};$$

- $\mathcal{P}$ is a program that zero tests only counters bounded by $B$.

*Example* 3. As an example to be used later, when $B$ is sufficiently small to write $B$ consecutive increments explicitly, it is very easy to code a gadget for ratio $B$:

$\mathsf{b} \mathrel{+}= B$ $\quad\to$ set $\mathsf{b}$ to constant $B$
$\mathsf{c} \mathrel{+}= 1 \quad \mathsf{d} \mathrel{+}= B$
**loop**
  $\mathsf{c} \mathrel{+}= 1 \quad \mathsf{d} \mathrel{+}= B$
**halt**. □

Under the stated assumptions, we now define a construction of a program $\mathcal{R} \rhd \mathcal{P}$ without zero tests. The idea is to supplement each zero tested counter $\mathsf{x}$ of $\mathcal{P}$ by a new counter $\mathsf{x}'$ and to ensure that the invariant $\mathsf{x} + \mathsf{x}' = B$ is maintained, so that zero tests of $\mathsf{x}$ can be replaced by loops that $B$ times increment $\mathsf{x}$ and then $B$ times decrement $\mathsf{x}$. Counter $\mathsf{b}$ provided by $\mathcal{R}$ is employed to initialise each complement counter $\mathsf{x}'$, whereas counters $\mathsf{c}$ and $\mathsf{d}$ are used to ensure that if $\mathsf{d}$ is zero at the end of the run then all the loops in the simulations of the zero tests iterated $B$ times as required. Concretely, the program $\mathcal{R} \rhd \mathcal{P}$ is constructed as follows:

(i) variables are renamed if necessary so that no variable occurs in both $\mathcal{R}$ and $\mathcal{P}$;

(ii) letting $\mathsf{x}_1, \ldots, \mathsf{x}_l$ be the counters that are zero tested in $\mathcal{P}$, new counters $\mathsf{x}'_1, \ldots, \mathsf{x}'_l$ are introduced and the following code is inserted at the beginning of $\mathcal{P}$:

**loop**
  $\mathsf{x}'_1 \mathrel{+}= 1 \quad \cdots \quad \mathsf{x}'_l \mathrel{+}= 1$
  $\mathsf{b} \mathrel{-}= 1 \quad \mathsf{d} \mathrel{-}= 1$
$\mathsf{c} \mathrel{-}= 1$

(we shall show that complete runs necessarily iterate this loop $B$ times, i.e. until counter $\mathsf{b}$ becomes zero);

(iii) every $\mathsf{x}_i \mathrel{+}= 1$ command in $\mathcal{P}$ is replaced by two commands

$$\mathsf{x}_i \mathrel{+}= 1 \quad \mathsf{x}'_i \mathrel{-}= 1;$$

(iv) every $\mathsf{x}_i \mathrel{-}= 1$ command in $\mathcal{P}$ is replaced by two commands

$$\mathsf{x}_i \mathrel{-}= 1 \quad \mathsf{x}'_i \mathrel{+}= 1;$$

(v) every **test** $\mathsf{x}_i = 0$ command in $\mathcal{P}$ is replaced by the following code:

**loop**
    $\mathsf{x}_i \mathrel{+}= 1 \quad \mathsf{x}'_i \mathrel{-}= 1$
    $\mathsf{d} \mathrel{-}= 1$
$\mathsf{c} \mathrel{-}= 1$
**loop**
    $\mathsf{x}_i \mathrel{-}= 1 \quad \mathsf{x}'_i \mathrel{+}= 1$
    $\mathsf{d} \mathrel{-}= 1$
$\mathsf{c} \mathrel{-}= 1$

(we shall show that complete runs necessarily iterate each of the two loops $B$ times, i.e. they check that $\mathsf{x}_i$ equals 0 through checking that $\mathsf{x}'_i$ equals $B$ by transferring $B$ from $\mathsf{x}'_i$ to $\mathsf{x}_i$ and then back);

(vi) letting $\mathsf{y}_1, \ldots, \mathsf{y}_m$ (respectively, $\mathsf{z}_1, \ldots, \mathsf{z}_h$) be the counters that are required to be zero at termination of $\mathcal{R}$ (respectively, $\mathcal{P}$), the code of $\mathcal{R} \triangleright \mathcal{P}$ consists of the code of $\mathcal{R}$ concatenated with the code of $\mathcal{P}$ modified as stated, both without their **halt** commands, and ending with the command

$$\textbf{halt if } \mathsf{d}, \mathsf{y}_1, \ldots, \mathsf{y}_m, \mathsf{z}_1, \ldots, \mathsf{z}_h = 0.$$

We remark that simulating zero tests of counters bounded by some $B$ using transfers from and to their complements is a well-known technique that can be found already in Lipton [13]; the novelty here is the cumulative verification of such simulations, through decreasing appropriately the two counters $\mathsf{d}$ and $\mathsf{c}$ whose ratio is $B$, and checking that $\mathsf{d}$ is zero finally.

**Correctness.** The next proposition states that the construction of $\mathcal{R} \triangleright \mathcal{P}$ is correct in the sense that its computed relations in counters of $\mathcal{P}$ are the same as those of $\mathcal{P}$. (We shall treat any renamings of variables in step (i) of the construction as implicit.) In one direction, the proof proceeds by observing that $\mathcal{R} \triangleright \mathcal{P}$ can simulate faithfully any complete run of $\mathcal{P}$. In the other direction we argue that although some of the loops introduced in step (v) may iterate fewer than $B$ times and hence erroneously validate a zero test, the ways in which counters $\mathsf{c}$ and $\mathsf{d}$ are set up by $\mathcal{R}$ and used in the construction ensure that no such run can continue to a complete one. Informally, as soon as a loop in a simulation of a zero test iterates fewer than $B$ times, the equality $\mathsf{d} = \mathsf{c} \cdot B$ turns into the strict inequality $\mathsf{d} > \mathsf{c} \cdot B$ which remains for the rest of the run, preventing counter $\mathsf{d}$ from reaching zero.

**Proposition 1.** *For every valuation of counters of $\mathcal{P}$, it occurs after a complete run of $\mathcal{R} \triangleright \mathcal{P}$ if and only if it occurs after a complete run of $\mathcal{P}$.*

*Proof.* The 'if' direction is straightforward: from a complete run of $\mathcal{P}$ with a total of $q$ zero tests, obtain a complete run of $\mathcal{R} \triangleright \mathcal{P}$ with the same final valuation of counters of $\mathcal{P}$ by

- running $\mathcal{R}$ to termination with $\mathsf{b} = B$, $\mathsf{c} = 2q + 1$, $\mathsf{d} = \mathsf{c} \cdot B$ and all of $\mathsf{y}_1, \ldots, \mathsf{y}_m$ equal to 0, where the latter variables will remain untouched for the rest of the run and hence satisfy the requirement to be zero finally (cf. step (vi) of the construction),
- iterating the loop in step (ii) $B$ times to initialise each complement counter $\mathsf{x}'_i$ to $B$, which also subtracts $B$ and 1 from $\mathsf{d}$ and $\mathsf{c}$ (respectively), decreases $\mathsf{b}$ to 0, and
- in place of every zero test in $\mathcal{P}$, iterating both loops in step (v) $B$ times, which subtracts $2B$ and 2 from $\mathsf{d}$ and $\mathsf{c}$ (respectively), eventually decreasing them both to 0.

For the 'only if' direction, consider a complete run of $\mathcal{R} \triangleright \mathcal{P}$. Extracting from it a complete run of $\mathcal{P}$ with the same final valuation of counters of $\mathcal{P}$ is easy once we show that, for each simulation of a **test** $\mathsf{x}_i = 0$ command by the code in step (v) of the construction, the values of $\mathsf{x}_i$ at the start and at the finish of the code are 0.

Firstly, by step (vi) and the fact that counters $\mathsf{y}_1, \ldots, \mathsf{y}_m$ are not used after its termination, we have that the values of $\mathsf{b}$, $\mathsf{c}$ and $\mathsf{d}$ that have been provided by $\mathcal{R}$ satisfy $\mathsf{b} = B$ and $\mathsf{d} = \mathsf{c} \cdot B$. After the code in step (ii) we therefore have that $\mathsf{x}_i + \mathsf{x}'_i \leq B$ for all $i$. Recalling the reasoning in Example 2 in Section 2 and arguing forwards through the run, we infer that

$$\mathsf{x}_i + \mathsf{x}'_i \leq B \text{ for all } i, \text{ and } \mathsf{d} \geq \mathsf{c} \cdot B$$

is an invariant that is maintained by the rest of the run.

Now, due to step (vi) again, $\mathsf{d}$ is zero finally, and so the inequality $\mathsf{d} \geq \mathsf{c} \cdot B$ is finally an equality. Therefore, $\mathsf{c}$ is zero finally as well. Recalling again the reasoning in Example 2 in Section 2 and arguing backwards through the run, we conclude that in fact $\mathsf{d} = \mathsf{c} \cdot B$ has been maintained and that, for each simulation of a **test** $\mathsf{x}_i = 0$ command, each of the two loops has been iterated exactly $B$ times, and hence the values of $\mathsf{x}_i$ at its start and at its finish have been 0. Also, the loop introduced in step (ii) has been iterated $B$ times, and $\mathsf{b}$ is zero finally. ☐

# 4 Small Products of Exponential Powers of Fractions

To construct gadgets for ratios that are towers of exponentials (cf. Section 6) and thereby obtain our main result, we shall show how to produce repeatedly from a gadget for ratio $B$ a gadget for ratio exponential in $B$ (cf. Section 5). The latter is the technical core of the paper, and is based on an identity involving exponential powers of fractions that we now present together with establishing some of its remarkable properties.

**Key Definitions and Their Properties.** For any natural $k$, let us define numbers $e_k$, $s_k$ and $r_k$, and for any $i = 0, \ldots, k$, numbers $e_{k,i}$, $s_{k,i}$, $r_{k,i}$, as follows:

$$\begin{array}{rclrclrcl} e_{k,i} & = & e_k + 2^i & s_{k,i} & = & (e_k)^i e_{k,i} & r_{k,i} & = & e_k \prod_{j=0}^{i-1} e_{k,j} \\ e_k & = & 2^{2k} & s_k & = & \prod_{j=0}^{k} e_{k,j} & r_k & = & (e_k)^{k+1}. \end{array} \tag{2}$$

Observe that all the numbers defined are bounded by an exponential in $k$. Nevertheless, we prove that all of the fractions $s_{k,i}/r_{k,i}$ and $s_k/r_k$ are greater than 1, and that a certain product of the former fractions raised to powers exponential in $k$ equals the latter fraction:

**Proposition 2.** *For all natural $k$, we have:*

$$\frac{s_{k,i}}{r_{k,i}} > 1 \textit{ for all } i = 0, 1, \ldots, k, \textit{ and } \frac{s_k}{r_k} > 1$$

$$\prod_{i=0}^{k} \left( \frac{s_{k,i}}{r_{k,i}} \right)^{2^{k-i}} = \frac{s_k}{r_k}.$$

*Proof.* The claimed equality is presentable as:

$$\left(\frac{s_{k,0}}{r_{k,0}}\right)^{2^k} \cdot \left(\frac{s_{k,1}}{r_{k,1}}\right)^{2^{k-1}} \cdot \quad \cdots \quad \cdot \left(\frac{s_{k,k-1}}{r_{k,k-1}}\right)^{2^1} \cdot \left(\frac{s_{k,k}}{r_{k,k}}\right)^{2^0} = \frac{s_k}{r_k}. \tag{3}$$

For convenience, decompose the fractions into simpler ones:

$$\frac{s_{k,i}}{r_{k,i}} = \frac{p_i}{p_0 \cdot \cdots \cdot p_{i-1}}, \qquad \frac{s_k}{r_k} = p_0 \cdot \cdots \cdot p_k, \qquad \text{where} \quad p_i = \frac{e_{k,i}}{e_k} = 1 + \frac{1}{2^{2k-i}}.$$

In particular, $s_{k,0}/r_{k,0} = p_0$. Using the decomposition we easily prove the claimed equality (3), as it is the multiplication of $k+1$ equalities, for $i = 0, \ldots, k$, of the following form:

$$\left(\frac{p_i}{1}\right)^{2^{k-i}} \cdot \left(\frac{1}{p_i}\right)^{2^{k-(i+1)}} \cdot \quad \cdots \quad \cdot \left(\frac{1}{p_i}\right)^{2^1} \cdot \left(\frac{1}{p_i}\right)^{2^0} = \frac{p_i}{1}.$$

Now we concentrate on the claimed inequalities. The inequalities $s_k/r_k > 1$ and $s_{k,0}/r_{k,0} > 1$ are immediate, as $p_i > 1$ for $i = 0 \ldots k$. To show the remaining inequalities $s_{k,i}/r_{k,i} > 1$ we need to prove:

$$p_0 \cdot p_1 \cdot \cdots \cdot p_{i-1} < p_i, \text{ for } i = 1, \ldots, k. \tag{4}$$

Fix $i$. As $p_j < {p_{j-1}}^2$ for $j = 1, \ldots, k$, by induction on $j$ one shows $p_j < {p_0}^{2^j}$; therefore the left-hand side is dominated by:

$${p_0}^{2^0+2^1+\cdots+2^{i-1}} = {p_0}^{2^i-1}.$$

It is thus sufficient to prove

$${p_0}^{2^i-1} < p_i. \tag{5}$$

For better readability we put $a := 2^i - 1$ and $b := e_k = 2^{2k}$; with this shorthands, the inequality (5) is rephrased as

$$\left(1 + \frac{1}{b}\right)^a < 1 + \frac{a+1}{b}. \tag{6}$$

By inspecting the expansion of the left-hand side we observe that the left-hand side is bounded by the sum of the first $a+1$ elements of a geometric progression, which, in turn, is bounded by the sum of the whole infinite one:

$$\left(1 + \frac{1}{b}\right)^a \leq 1 + \frac{a}{b} + \frac{a^2}{b^2} + \cdots + \frac{a^a}{b^a} < \frac{1}{1 - \frac{a}{b}}. \tag{7}$$

Thus, recalling the right-hand side of the inequality (6), it is sufficient to prove:

$$1 < \left(1 - \frac{a}{b}\right)\left(1 + \frac{a+1}{b}\right).$$

One readily verifies that the inequality holds as long as $a(a+1) < b$, which holds due to $i \leq k$. The inequalities (4) are thus proved, and hence so is Proposition 2. $\square$

The products of all the numerator powers and all the denominator powers in the identity of Proposition 2, which we now name, will be important in the sequel:

$$B_k = \textstyle\prod_{i=0}^{k} (s_{k,i})^{2^{k-i}} \qquad A_k = \prod_{i=0}^{k} (r_{k,i})^{2^{k-i}}. \tag{8}$$

A property of the parameterised quantity $B_k$ which is going to be central for exponentiating the ratios of gadgets (cf. Section 5) is that it is exponentially larger than all the numbers defined in (2), as witnessed by the following bounds:

**Lemma 3.** *For all natural $k$ and $i = 0, \ldots, k$, we have:*

$$e_k, r_k, e_{k,i}, s_{k,i}, r_{k,i} \leq s_k \leq 2^{2(k+1)^2} \text{ and } B_k \geq 2^{2k \cdot 2^k}.$$

*Proof.* We first show the inequalities $e_k, r_k, e_{k,i}, s_{k,i}, r_{k,i} \leq s_k$. Proposition 2 gives us that $s_k > r_k$, and that $s_{k,i} > r_{k,i}$ for all $i$. It is easy to check that $e_{k,k} \geq e_k$, and that $e_{k,k} \geq e_{k,i}$, $s_{k,k} \geq s_{k,i}$ and $s_{k,k} \geq e_{k,k}$ for all $i$. We are therefore left with needing to show $s_{k,k} \leq s_k$. Observe that $s_k = \left(\prod_{j=0}^{k-1} e_{k,j}\right) e_{k,k} \geq \left(\prod_{j=0}^{k-1} e_k\right) e_{k,k} = (e_k)^k e_{k,k} = s_{k,k}$.

To obtain the explicit bounds, we estimate that $e_{k,k} = 2^{2k} + 2^k \leq 2^{2(k+1)}$, and so $s_k \leq (e_{k,k})^{k+1} \leq 2^{2(k+1)^2}$. Finally, $B_k \geq (s_{k,0})^{2^k} \geq 2^{2k \cdot 2^k}$. □

We remark that the properties we have established in Proposition 2 and Lemma 3 are more important than the exact definition of the numbers $e_k$ and $e_{k,i}$, and that our choice of the latter is not the only workable one.

**An Exercise.** To prepare for the developments and arguments in the next section that are based on the definitions and observations in this one, we suggest an exercise that links the identity in the statement of Proposition 2 with counter programs.

Recall that complete runs of programs have initial valuations that assign zero to all variables. To initialise a variable to another value in a program $\mathcal{P}$, let us write $\mathcal{P}[\mathsf{x} := m]$ for the program

$\mathsf{x} \mathrel{+}= m$
$<\mathcal{P}>$

that assigns the natural $m$ to the variable $\mathsf{x}$ and then proceeds like $\mathcal{P}$.

Consider program $\mathcal{E}$ defined in Algorithm I, where the **for** and **exactly** loops, as well as the computations of the quantities enclosed in boxes, are macros that have the expected effects and that will be defined properly in Section 5. For any natural $k$, the program $\mathcal{E}[\mathsf{k} := k]$ does the following:

- initialises $\mathsf{x}$ and $\mathsf{y}$ to some positive integer $a$ chosen nondeterministically, which will be kept unchanged in counter $\mathsf{y}$ until the final loop;
- in each iteration of the outer two main loops, uses counter $\mathsf{x}'$ to attempt to multiply counter $\mathsf{x}$ by the fraction $s_{k,\mathsf{i}}/r_{k,\mathsf{i}}$;
- by the final loop and the terminal check that counter $\mathsf{y}$ is zero, halts provided the value of $\mathsf{x}$ is at least $a \cdot s_k/r_k$ (in which case it will be exactly $a \cdot s_k/r_k$).

The first and easier part of the exercise is to show that, for any positive $a$ divisible by $(e_k)^{2^k+1} = 2^{2k\cdot(2^k+1)}$, there exists a complete run of $\mathcal{E}[\mathsf{k} := k]$ that initialises $\mathsf{x}$ and $\mathsf{y}$ to $a$, then multiplies $\mathsf{x}$ exactly by all the powers of fractions on the left-hand side of the identity of Proposition 2, and finally checks that $\mathsf{x}$ equals $a$ times the fraction on the right-hand side.

The second part is to show the converse, i.e. that any complete run of the program $\mathcal{E}[\mathsf{k} := k]$ is of that form, in particular that the value to which counters $\mathsf{x}$ and $\mathsf{y}$ are set by the initial loop is necessarily divisible by $2^{2k\cdot(2^k+1)}$. As a hint, we remark that this is the case because as soon as a multiplication of $\mathsf{x}$ by a fraction $s_{k,\mathsf{i}}/r_{k,\mathsf{i}}$ does not complete accurately (i.e. either the first inner loop does not decrease $\mathsf{x}$ to exactly zero, or the second inner loop does not decrease $\mathsf{x}'$ to exactly zero), it will not be possible to repair that error in the rest of the run, in the sense that the value of $\mathsf{x}$ at the beginning of the final loop will necessarily be strictly smaller than $a \cdot s_k/r_k$ and thus it will be impossible to complete the run. We also remark that this vitally depends on the fact that all the fractions $s_{k,\mathsf{i}}/r_{k,\mathsf{i}}$ are strictly greater than 1.

**Algorithm I** Counter program $\mathcal{E}$.

$\mathsf{x} \mathrel{+}= 1 \quad \mathsf{y} \mathrel{+}= 1$
**loop**
  $\mathsf{x} \mathrel{+}= 1 \quad \mathsf{y} \mathrel{+}= 1$
**for** $\mathsf{i} = 0, \ldots, \mathsf{k}$ **do**
  **loop exactly** $\boxed{2^{\mathsf{k}-\mathsf{i}}}$ **times**
    **loop**
      $\mathsf{x} \mathrel{-}= \boxed{r_{\mathsf{k},\mathsf{i}}} \quad \mathsf{x}' \mathrel{+}= \boxed{s_{\mathsf{k},\mathsf{i}}}$
    **loop**
      $\mathsf{x}' \mathrel{-}= 1 \quad \mathsf{x} \mathrel{+}= 1$
**loop**
  $\mathsf{x} \mathrel{-}= \boxed{s_{\mathsf{k}}} \quad \mathsf{y} \mathrel{-}= \boxed{r_{\mathsf{k}}}$
**halt if** $\mathsf{y} = 0$

For readers interested in flat vector addition systems with states (for example, it is well known that two-dimensional VASS are flattable [12]), we additionally remark that, by inlining the two outer main loops in the program $\mathcal{E}[\mathsf{k} := k]$, we obtain a three-counter unary linear path scheme that has complete runs but of length at least exponential in the scheme's size. It is then not difficult, employing some of the other ideas from this paper, to extend the construction to a reduction from the subset-sum problem and conclude that the reachability problem for unary flat VASS is NP-hard already for a small fixed dimension. Those observations resolve some of the main open questions left by the recent works of Blondin et al. [1] and Englert et al. [2].

## 5 Lifting Ratio Gadgets Exponentially

Building on the previous section which is the algebraic centre of the paper, we now develop the technical core by providing a way to exponentiate ratios of gadgets. More precisely, we liberalise the notion of gadget for a ratio from Section 3 by allowing them to zero test counters whose values do not exceed some bound. We then define and prove correct programs that, for any natural $k$, zero test only counters bounded by $s_k$ and are gadgets for ratio $B_k$ (cf. (2) and (8) for the definitions of those quantities). Composing these programs using the operator from Section 3 with gadgets for ratio at least $s_k$ will hence produce gadgets for ratio $B_k$, both without zero tests (cf. Proposition 1). Recalling Lemma 3, the exponentiation will be achieved since $B_k$ is indeed exponentially larger than $s_k$.

As auxiliary building blocks for the main program, we are going to make use of programs that compute several exponentially bounded parameterised quantities. They may involve zero tests, but all their counters are also exponentially bounded.

**Proposition 4.** *There exist counter programs $\mathcal{A}_1, \ldots, \mathcal{A}_5$ such that:*

- *they compute the functions $s_k$, $r_k$, $2^{k-i}$, $s_{k,i}$ and $r_{k,i}$, placing their outputs in counter* $\mathsf{a}$*, and preserving their inputs in counters* $\mathsf{k}$ *and* $\mathsf{i}$*:*

| *for any natural $k$ and $i = 0, \ldots, k$* | *computes in counters* $\mathsf{k}, \mathsf{i}, \mathsf{a}$ |
|---|---|
| $\mathcal{A}_1[\mathsf{k} := k][\mathsf{i} := i]$ | $\{\langle k, i, 2^{k-i} \rangle\}$ |
| $\mathcal{A}_2[\mathsf{k} := k][\mathsf{i} := i]$ | $\{\langle k, i, s_k \rangle\}$ |
| $\mathcal{A}_3[\mathsf{k} := k][\mathsf{i} := i]$ | $\{\langle k, i, r_k \rangle\}$ |
| $\mathcal{A}_4[\mathsf{k} := k][\mathsf{i} := i]$ | $\{\langle k, i, s_{k,i} \rangle\}$ |
| $\mathcal{A}_5[\mathsf{k} := k][\mathsf{i} := i]$ | $\{\langle k, i, r_{k,i} \rangle\}$; |

- *the programs $\mathcal{A}_q[\mathsf{k} := k][\mathsf{i} := i]$ have all counters bounded by $s_k$;*

- *the* ***halt*** *command of each program $\mathcal{A}_q$ does not require any counters to be zero.*

*Proof.* By routine programming of Minsky machines whose counters are bounded by $s_k$ and may be zero tested. ☐

The definition of our main program $\mathcal{R}$ in Algorithm II is presented at a high level for readability. It features a macro **loop** introduced earlier (cf. Section 2), and several new macros that we shall expand shortly. The latter include the boxes that contain values to be computed using the programs from Proposition 4.

We are going to be interested in complete runs of the programs $\mathcal{R}[\mathsf{k} := k]$, i.e. of program $\mathcal{R}$ with counter $\mathsf{k}$ initialised to number $k$. In such a run, first counter $\mathsf{b}$ is initialised to 1, and counters $\mathsf{c}$, $\mathsf{d}$, $\mathsf{x}$ and $\mathsf{y}$ are set to some positive integer chosen nondeterministically. In the **for** loop, the program attempts to divide $\mathsf{c}$ by $A_k$, multiply $\mathsf{d}$ and $\mathsf{x}$ by $B_k/A_k$, and set $\mathsf{b}$ to $B_k$, in which case we also have $\mathsf{d} = \mathsf{c} \cdot \mathsf{b}$. (Cf. (8) for the definitions of quantities $B_k$ and $A_k$.)

In order to confirm that indeed $\mathsf{d} = \mathsf{c} \cdot \mathsf{b}$ and $\mathsf{b} = B_k$ (which is difficult since $\mathsf{d}$ and $\mathsf{c}$ are unbounded, and $B_k$ is exponentially larger than the bound $s_k$ on zero-testable counters), the last loop is used together with the requirement that $\mathsf{y}$ is zero finally to verify that all the attempted multiplications and divisions have been accurate. It checks that $\mathsf{x}$ is at least $\mathsf{y}$ times $s_k/r_k$, where the latter fraction equals $B_k/A_k$ by Proposition 2.

**Algorithm II** Counter program $\mathcal{R}$.

$\mathsf{b} \mathrel{+}= 1 \quad \mathsf{c} \mathrel{+}= 1 \quad \mathsf{d} \mathrel{+}= 1 \quad \mathsf{x} \mathrel{+}= 1 \quad \mathsf{y} \mathrel{+}= 1$
**loop**
    $\mathsf{c} \mathrel{+}= 1 \quad \mathsf{d} \mathrel{+}= 1 \quad \mathsf{x} \mathrel{+}= 1 \quad \mathsf{y} \mathrel{+}= 1$
**for** $\mathsf{i} = 0, \ldots, \mathsf{k}$ **do**
    **loop exactly** $\boxed{2^{\mathsf{k}-\mathsf{i}}}$ **times**
        **loop**
            $\mathsf{c} \mathrel{-}= \boxed{r_{\mathsf{k},\mathsf{i}}} \quad \mathsf{c}' \mathrel{+}= 1$
            **loop at most** $\mathsf{b}$ **times**
                $\mathsf{d} \mathrel{-}= \boxed{r_{\mathsf{k},\mathsf{i}}} \quad \mathsf{d}' \mathrel{+}= \boxed{s_{\mathsf{k},\mathsf{i}}} \quad \mathsf{x} \mathrel{-}= \boxed{r_{\mathsf{k},\mathsf{i}}} \quad \mathsf{x}' \mathrel{+}= \boxed{s_{\mathsf{k},\mathsf{i}}}$
        **loop**
            $\mathsf{b} \mathrel{-}= 1 \quad \mathsf{b}' \mathrel{+}= \boxed{s_{\mathsf{k},\mathsf{i}}}$
        **loop**
            $\mathsf{b}' \mathrel{-}= 1 \quad \mathsf{b} \mathrel{+}= 1$
        **loop**
            $\mathsf{c}' \mathrel{-}= 1 \quad \mathsf{c} \mathrel{+}= 1$
            **loop at most** $\mathsf{b}$ **times**
                $\mathsf{d}' \mathrel{-}= 1 \quad \mathsf{d} \mathrel{+}= 1 \quad \mathsf{x}' \mathrel{-}= 1 \quad \mathsf{x} \mathrel{+}= 1$
**loop**
    $\mathsf{x} \mathrel{-}= \boxed{s_\mathsf{k}} \quad \mathsf{y} \mathrel{-}= \boxed{r_\mathsf{k}}$
**halt if** $\mathsf{y} = 0$

We now proceed to expand the new macros that feature in Algorithm II. That is going to introduce a fixed number of further counters that may be zero tested but are bounded by $s_\mathsf{k}$. Whenever we use a program $\mathcal{A}_q$ from Proposition 4, we assume that the names of its counters do not clash with the rest of $\mathcal{R}$, except for counters $\mathsf{k}$ and $\mathsf{i}$.

**loop down** $\mathsf{z}$ <*body*>**:** This macro is used intensively in other ones, for different counters $\mathsf{z}$, possibly with the empty <*body*>. Note that counter $\mathsf{z}$ is reset to zero by this loop.

```
loop
    <body>
    z −= 1
test z = 0
```

**for** $\mathsf{i} = 0, \ldots, \mathsf{k}$ **do** <*body*>**:** The **for** loop is easy to write by means of the **down** loop.

```
loop down k
    k′ += 1    k″ += 1        → save k in k′ and k″
loop down k′
    k += 1        → restore k from k′
loop down k″
    <body>
    i += 1
<body>
```

**loop exactly** $\boxed{2^{\mathsf{k}-\mathsf{i}}}$ **times** <*body*>**:** We use the program $\mathcal{A}_1$ from Proposition 4.

```
⋮
loop down w        → reset every counter w of A_1 except k and i
⋮
<A_1 with halt removed>        → number 2^(k−i) is computed in counter a
loop down a
    <body>
```

$\mathsf{z} \mathrel{+}= \boxed{a_{\mathsf{k},\mathsf{i}}}$**:** To add to counter z one of the quantities $s_{\mathsf{k}}$, $r_{\mathsf{k}}$, $s_{\mathsf{k},\mathsf{i}}$ or $r_{\mathsf{k},\mathsf{i}}$, we use the program $\mathcal{A}_q$ from Proposition 4 for $q = 2, 3, 4$ or 5 (respectively).

```
⋮
loop down w        → reset every counter w of A_q except k and i
⋮
<A_q with halt removed>        → number a_{k,i} is computed in counter a
loop down a
    z += 1
```

$\mathsf{z} \mathrel{-}= \boxed{a_{\mathsf{k},\mathsf{i}}}$**:** Subtracting is analogous, with the increment of z replaced by decrement.

**loop at most** b **times** <*body*>**:** To express this construct, we employ the auxiliary counter b′ to which the value of b is transferred and then transferred back. Provided b′ is zero at the start, the body is indeed performed at most b times.

```
loop
    b −= 1    b′ += 1
loop
    b′ −= 1    b += 1
    <body>
```

At last, we turn to arguing the correctness. While reading the next lemma and its proof, we suggest comparing and contrasting with the exercise in Section 4, since program $\mathcal{E}$ (cf. Algorithm I) was obtained from program $\mathcal{R}$ essentially by removing counters b, c, d and their primed counterparts.

Table 1: Equalities for counter values in complete runs of programs $\mathcal{R}[\mathsf{k} := k]$, for all $i = 0, \ldots, k$ and $j = 1, \ldots, 2^{k-i}$.

$$\begin{array}{lll lll lll}
\mathsf{b}_{0,0} & = & 1 & \mathsf{c}_{0,0} & = & a & \mathsf{d}_{0,0} & = & a \\
\mathsf{b}'_{0,0} & = & 0 & \mathsf{c}'_{0,0} & = & 0 & \mathsf{d}'_{0,0} & = & 0 \\
\bar{\mathsf{b}}_{i,j} & = & 0 & \bar{\mathsf{c}}_{i,j} & = & 0 & \bar{\mathsf{d}}_{i,j} & = & 0 \\
\bar{\mathsf{b}}'_{i,j} & = & \mathsf{b}_{i,j-1} \cdot s_{k,i} & \bar{\mathsf{c}}'_{i,j} & = & \mathsf{c}_{i,j-1}/r_{k,i} & \bar{\mathsf{d}}'_{i,j} & = & \mathsf{d}_{i,j-1} \cdot s_{k,i}/r_{k,i} \\
\mathsf{b}_{i,j} & = & \bar{\mathsf{b}}'_{i,j} & \mathsf{c}_{i,j} & = & \bar{\mathsf{c}}'_{i,j} & \mathsf{d}_{i,j} & = & \bar{\mathsf{d}}'_{i,j} \\
\mathsf{b}'_{i,j} & = & 0 & \mathsf{c}'_{i,j} & = & 0 & \mathsf{d}'_{i,j} & = & 0
\end{array}$$

**Lemma 5.** *For any natural $k$, the program $\mathcal{R}[\mathsf{k} := k]$ computes in counters $\mathsf{b}, \mathsf{c}, \mathsf{d}$ the relation*

$$\{\langle b, c, d\rangle \ : \ c, d > 0, \ b = B_k, \ d = c \cdot b\}$$

*and zero tests only counters bounded by $s_k$.*

*Proof.* The bound on the zero-tested counters is clear from the definition of $\mathcal{R}$ and Lemma 3.

In the rest of the proof, we shall be considering runs of $\mathcal{R}[\mathsf{k} := k]$ whose initial valuation assigns zero to every variable, and which are either halted or blocked at the **halt** command because $\mathsf{y}$ is not zero. In particular, any such run will have completed the outer two main loops (the **for** and **exactly** loops in Algorithm II). Hence, we can introduce the following notations for counter values during the $i^{\text{th}}$ iteration of the **for** loop ($i = 0, \ldots, k$) and the $j^{\text{th}}$ iteration of the **exactly** loop ($j = 1, \ldots, 2^{k-i}$), where $\mathsf{v}$ is any of the counters $\mathsf{b}$, $\mathsf{b}'$, $\mathsf{c}$, $\mathsf{c}'$, $\mathsf{d}$, $\mathsf{d}'$:

$\bar{\mathsf{v}}_{i,j}$**:** the final value of $\mathsf{v}$ after lines 6–11;

$\mathsf{v}_{i,j}$**:** the final value of $\mathsf{v}$ after lines 12–17.

It will also be convenient to write $\mathsf{v}_{i,0}$ for the value of $\mathsf{v}$ during the $i^{\text{th}}$ iteration of the **for** loop and at the start of the **exactly** loop. Observe that, for $i > 0$, we have $\mathsf{v}_{i,0} = \mathsf{v}_{i-1,2^{k-(i-1)}}$. We emphasise that these notations are relative to the run under consideration, which for readability is not written explicitly.

The rest of the proof works for any natural $k$ and consists of two parts, that establish the two inclusions between the relation computed by $\mathcal{R}[\mathsf{k} := k]$ in counters $\mathsf{b}, \mathsf{c}, \mathsf{d}$ and the relation in the statement of the lemma.

The first part, where we assume $b = B_k$, $c > 0$ and $d = c \cdot b$, and argue that the program $\mathcal{R}[\mathsf{k} := k]$ has a complete run whose final values of counters $\mathsf{b}, \mathsf{c}, \mathsf{d}$ are exactly $b, c, d$, is the easier part.

*Claim* 1. For any $a$ divisible by $A_k$, the program $\mathcal{R}[\mathsf{k} := k]$ has a complete run which satisfies the equalities in Table 1.

*Proof of claim.* Such a run can be built by iterating each nondeterministic loop the maximum number of times. Namely, during iteration $i, j$ of the two outer main loops:

- the loop at line 6 is iterated $\mathsf{c}_{i,j-1}/r_{k,i}$ times and in each pass the loop at line 8 is iterated $\mathsf{b}_{i,j-1}$ times;
- the loop at line 10 is iterated $\mathsf{b}_{i,j-1}$ times;
- the loop at line 12 is iterated $\bar{\mathsf{b}}_{i,j}$ times;
- the loop at line 14 is iterated $\bar{\mathsf{c}}'_{i,j}$ times and in each pass the loop at line 16 is iterated $\mathsf{b}_{i,j}$ times.

The divisibility of $a$ by $A_k$ ensures that all division in the statement of the claim yield integers.

To see that the run thus obtained can be completed, observe that from the equalities in Table 1 it follows that

$$\begin{aligned} \mathsf{b}_{k,1} &= \textstyle\prod_{i=0}^{k}(s_{k,i})^{2^{k-i}} = B_k \\ \mathsf{c}_{k,1} &= a \cdot \textstyle\prod_{i=0}^{k}(1/r_{k,i})^{2^{k-i}} = a/A_k \\ \mathsf{d}_{k,1} &= a \cdot \textstyle\prod_{i=0}^{k}(s_{k,i}/r_{k,i})^{2^{k-i}} = a \cdot B_k/A_k. \end{aligned}$$

In particular, at the start of final loop (at line 18), counter $\mathsf{x}$ equals counter $\mathsf{d}$ and hence has value $a \cdot B_k/A_k = a \cdot s_k/r_k$ (cf. Proposition 2), and counter $\mathsf{y}$ has value $a$. Iterating the final loop $a/r_k$ times therefore reduces $\mathsf{y}$ (and $\mathsf{x}$) to zero as required. □

To obtain $b, c, d$ as the final values of counters $\mathsf{b}, \mathsf{c}, \mathsf{d}$, we apply Claim 1 with $a = c \cdot A_k$.

We now turn to the remaining second part of the proof of the lemma, where we consider any complete run and need to show that the final values $b, c, d$ of counters $\mathsf{b}, \mathsf{c}, \mathsf{d}$ satisfy $b = B_k$, $c > 0$ and $d = c \cdot b$.

*Claim* 2. For all $i = 0, \ldots, k$ and $j = 1, \ldots, 2^{k-i}$, we have:

- $\bar{\mathsf{d}}_{i,j} + \bar{\mathsf{d}}'_{i,j} \leq (\mathsf{d}_{i,j-1} + \mathsf{d}'_{i,j-1}) \cdot s_{k,i}/r_{k,i}$;
- $\bar{\mathsf{d}}_{i,j} + \bar{\mathsf{d}}'_{i,j} = (\mathsf{d}_{i,j-1} + \mathsf{d}'_{i,j-1}) \cdot s_{k,i}/r_{k,i}$ if and only if $\bar{\mathsf{d}}_{i,j} = \mathsf{d}'_{i,j-1} = 0$;
- $\mathsf{d}_{i,j} + \mathsf{d}'_{i,j} = \bar{\mathsf{d}}_{i,j} + \bar{\mathsf{d}}'_{i,j}$.

*Proof of claim.* Straightforward calculation based on $s_{k,i}/r_{k,i} > 1$ (cf. Proposition 2). □

Let $a$ denote the value of counters $\mathsf{c}$, $\mathsf{d}$, $\mathsf{x}$ and $\mathsf{y}$ at the start of the **for** loop.

*Claim* 3. The equalities in Table 1 for the values of counters $\mathsf{d}$ and $\mathsf{d}'$ are satisfied.

*Proof of claim.* From Claim 2, we first deduce that at the start of the final loop (at line 18), counter $\mathsf{x}$ has value at most $a \cdot s_k/r_k$. Since counter $\mathsf{y}$ has value $a$ at that point and the run is complete, it must actually be the case that the value of $\mathsf{x}$ here equals $a \cdot s_k/r_k$. By Claim 2 again, we infer that for all $i = 0, \ldots, k$ and $j = 1, \ldots, 2^{k-i}$, we indeed have:

$$\bar{\mathsf{d}}_{i,j} = 0 \qquad \bar{\mathsf{d}}'_{i,j} = \mathsf{d}_{i,j-1} \cdot s_{k,i}/r_{k,i} \qquad \mathsf{d}_{i,j} = \bar{\mathsf{d}}'_{i,j} \qquad \mathsf{d}'_{i,j} = 0.$$ □

*Claim* 4. We have that $a$ is divisible by $A_k$ and that the equalities in Table 1 for the values of counters $\mathsf{b}$, $\mathsf{b}'$, $\mathsf{c}$ and $\mathsf{c}'$ are satisfied.

*Proof of claim.* That $a$ is divisible by $A_k$ will follow once we establish the equalities for the values of $\mathsf{c}$ and $\mathsf{c}'$, since they involve dividing $a$ by $\prod_{i=0}^{k}(r_{k,i})^{2^{k-i}} = A_k$.

For the rest of the claim, we argue inductively, where the hypothesis is that the equalities for the values of $\mathsf{b}$, $\mathsf{b}'$, $\mathsf{c}$ and $\mathsf{c}'$ are satisfied for all indices $i'$ and $j'$ such that either $i' < i$, or $i' = i$ and $j' < j$. Consequently, recalling Claim 3, we have that

$$\mathsf{d}_{i,j-1} = \mathsf{c}_{i,j-1} \cdot \mathsf{b}_{i,j-1}. \tag{9}$$

Consider the iteration $i, j$ of the outer two main loops. We infer from Claim 3 that the commands in line 9 must have been performed $\mathsf{d}_{i,j-1}/r_{k,i}$ times. Hence, as the values of counters $\mathsf{b}$ and $\mathsf{b}'$ remain unchanged until line 10, using (9) we deduce that the commands in line 7 must have been performed $\mathsf{c}_{i,j-1}/r_{k,i}$ times, and we have:

$$\bar{\mathsf{c}}_{i,j} = 0 \qquad \bar{\mathsf{c}}'_{i,j} = \mathsf{c}_{i,j-1}/r_{k,i}.$$

Also by Claim 3, the commands in line 17 must have been performed $\bar{\mathsf{d}}'_{i,j} = \mathsf{d}_{i,j-1} \cdot s_{k,i}/r_{k,i}$ times. From what we have just shown, that number equals $\bar{\mathsf{c}}'_{i,j} \cdot \mathsf{b}_{i,j-1} \cdot s_{k,i}$, and so we conclude that indeed:

$$\begin{aligned} \bar{\mathsf{b}}_{i,j} &= 0 & \mathsf{b}_{i,j} &= \bar{\mathsf{b}}'_{i,j} & \mathsf{c}_{i,j} &= \bar{\mathsf{c}}'_{i,j} \\ \bar{\mathsf{b}}'_{i,j} &= \mathsf{b}_{i,j-1} \cdot s_{k,i} & \mathsf{b}'_{i,j} &= 0 & \mathsf{c}'_{i,j} &= 0. \end{aligned}$$ □

As in the first part, we now conclude that the final values $b, c, d$ of counters $\mathsf{b}, \mathsf{c}, \mathsf{d}$ are $B_k, a/A_k, a \cdot B_k/A_k$, and in particular $c \cdot b = a \cdot B_k/A_k = d$. □

# 6 Main Result

As already indicated, the bulk of the work for our headline result, namely TOWER-hardness of the reachability problem for Petri nets, is showing how to construct a zero-test free gadget for a ratio which is a tower of exponentials. Most of the pieces have already been developed in the previous sections, and here we put them together to obtain a linear-time construction (although any elementary complexity would suffice for the TOWER-hardness).

Let $t(m) = 2^{m/2}$, and let $t^n$ denote the $n^{\text{th}}$ iterate of the function $t$, i.e. $\overbrace{t \circ \cdots \circ t}^{n \text{ times}}$, where $t^0$ is the identity function.

Writing $k_n$ for the number $t^n(6)$ (the first few values are $6, 8, 16, 256, 2^{128}, \ldots$), the following lemma confirms that we are going to be able to iterate the operation of producing from a gadget for ratio $s_k$ a gadget for ratio $B_k$.

**Lemma 6.** *For all natural $n$, we have $s_{k_{n+1}} \leq B_{k_n}$.*

*Proof.* By Lemma 3, we have $s_{k_{n+1}} \leq 2^{2(k_{n+1}+1)^2}$ and $B_{k_n} \geq 2^{2k_n \cdot 2^{k_n}}$. It remains to observe:

$$2(k_{n+1}+1)^2 \leq 3(k_{n+1})^2 = 3 \cdot 2^{k_n} \leq 2k_n \cdot 2^{k_n}.$$ □

We also need another auxiliary program akin to those in the statement of Proposition 4.

**Proposition 7.** *There exists a counter program $\mathcal{K}$ such that:*

- *for any natural $n$, the program $\mathcal{K}[\mathsf{n} := n]$ computes in counters $\mathsf{n}, \mathsf{k}$ the pair $\langle n, k_n \rangle$;*
- *the program $\mathcal{K}[\mathsf{n} := n]$ has all counters bounded by $s_{k_n}$;*
- *the **halt** command of program $\mathcal{K}$ does not require any counters to be zero.*

*Proof.* Again, by routine programming of a bounded Minsky machine. □

Here is the main lemma.

**Lemma 8.** *A zero-test free gadget for ratio $B_{k_n}$ is computable in time $O(n)$.*

*Proof.* Letting $\mathcal{B}_0$ be a trivial zero-test free gadget for ratio $B_{k_0} = B_6$ (cf. Example 3 in Section 3), from Proposition 1, Lemma 5, Lemma 6 and Proposition 7, it is straightforward to deduce that the program

$$\Big(\big(\mathcal{B}_0 \rhd (\mathcal{K}[\mathsf{n} := 1] \fatsemi \mathcal{R})\big) \rhd (\mathcal{K}[\mathsf{n} := 2] \fatsemi \mathcal{R}) \cdots \Big) \rhd (\mathcal{K}[\mathsf{n} := n] \fatsemi \mathcal{R}),$$

where the concatenation operation $\fatsemi$ removes the **halt** command of the program on its left, is a zero-test free gadget for ratio $B_{k_n}$.

To achieve the computability in linear time, we have to compact the initialisations of counter $\mathsf{n}$ in the programs $\mathcal{K}[\mathsf{n} := m]$ because they are causing the composite program to be of quadratic size. Since counter $\mathsf{n}$ is not used by program $\mathcal{R}$ and its value is preserved by program $\mathcal{K}$ (cf. Proposition 7), that is not difficult. It suffices to modify the composition operator so that it does not rename counter $\mathsf{n}$ but keeps it common to the two programs being composed, and to increment $\mathsf{n}$ at the start of each lifting. The resulting program

$$\Big(\big(\mathcal{B}_0 \rhd_{\mathsf{n}} (\mathsf{n} \mathrel{+}= 1\ \mathcal{K} \fatsemi \mathcal{R})\big) \rhd_{\mathsf{n}} (\mathsf{n} \mathrel{+}= 1\ \mathcal{K} \fatsemi \mathcal{R}) \cdots \Big) \rhd_{\mathsf{n}} (\mathsf{n} \mathrel{+}= 1\ \mathcal{K} \fatsemi \mathcal{R})$$

with $n$ composition operators has the required properties and is computable in time $O(n)$. □

**Theorem 9.** *The Petri nets reachability problem is* TOWER*-hard.*

*Proof.* We reduce in linear time from the TOWER-complete halting problem for counter programs of size $n$ with all counters bounded by $\mathit{tower}(n) = 2^{\cdot^{\cdot^{\cdot^{2}}}}\Big\}{\scriptstyle n \text{ times}}$ (cf. Section 2).

Let $\mathcal{M}$ be such a program. Since $k_n \geq 4 \cdot \mathit{tower}(n)$ by an easy induction, we have $B_{k_n} > \mathit{tower}(n)$ and so the counters of $\mathcal{M}$ are also bounded by $B_{k_n}$.

By Lemma 8, a gadget $\mathcal{B}_n$ for ratio $B_{k_n}$ is computable in time $O(n)$, and from the construction in Section 3, the same is hence true for the composite program $\mathcal{B}_n \triangleright \mathcal{M}$.

It remains to recall that $\mathcal{B}_n \triangleright \mathcal{M}$ has no zero test commands, and that by Proposition 1, it has a complete run if and only if $\mathcal{M}$ does. $\square$

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