Removing Apparent Singularities of Systems of Linear Differential Equations with Rational Function Coefficients

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ABSTRACT

In this paper we present a new algorithm which, given a system of first order linear differential equations with rational function coefficients, constructs an equivalent system with rational function coefficients, whose finite singularities are exactly the non-apparent singularities of the original system. This algorithm is implemented in the computer algebra system Maple and is illustrated by examples.

Categories and Subject Descriptors

I.1.2 [**Symbolic And Algebraic Manipulation**]: Algorithms—*Algebraic algorithms*

General Terms

Algorithms, Theory

Keywords

Systems of linear ordinary differential equations, Apparent singularities, Desingularization, Computer algebra.

1. INTRODUCTION

Given a first-order differential system of size n with rational function coefficients in the complex variable z

$$[A] \qquad \partial X = A(z) X$$

where $\partial = \frac{d}{dz}$, $X = (x_1, \ldots, x_n)^t$ vector of lenght n and $A \in \mathbb{C}(z)^{n \times n}$. The finite singularities of system $[A]$ are the poles of $A(z)$ in $\mathbb{C}$. A singular point $z_0 \in \mathbb{C}$ is called an apparent singularity if there exists a fundamental matrix solution $\Phi(z)$ which is analytic at $z = z_0$. Consider, for example, the first-order differential system

$$\partial X = A(z) X, \quad A(z) = \begin{bmatrix} 0 & 1 \\ \frac{-2}{z} & 1 + \frac{2}{z}. \end{bmatrix} \tag{1}$$

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Clearly, this system is equivalent to the second-order scalar differential equation given by the monic operator L:

$$L := \partial^2 - \frac{z+2}{z} \partial + \frac{2}{z},$$

for which e^z and $1+z+\frac{z^2}{2}$ form a basis of solutions. It follows that the point $z = 0$ is an apparent singularity of system (1). *Desingularization*, i.e. the problem of constructing another operator $\tilde{L}$ of higher order such that the solution space of $\tilde{L}(x) = 0$ contains that of $L(x) = 0$, and for which the factor z is "removed" from the denominator, is an interesting problem of research. For instance, by ABH method [2], one can compute a desingularization of order 4 given by the operator

$$\begin{aligned} \tilde{L} &= \partial^4 + (-1 + 1/4\, z)\, \partial^3 + (-1/4 - 3/8\, z)\, \partial^2 \\ &+ (1/2 + 1/8\, z)\, \partial - 1/4 \end{aligned}$$

In the scalar case, several desingularization algorithms exist for differential, difference (e.g., [2]), and more generally, Ore operators (see, e.g. [11, 10] and references therein). However, the apparent singularity of system (1) (equivalently of L) at $z = 0$, can be also removed by acting directly on it. In fact, by setting [6]

$$X = T(z)\, Y, \quad T(z) = \begin{bmatrix} 1 & 0 \\ 1 & z^2 \end{bmatrix},$$

the new variable Y satisfies the equivalent first-order differential system of the same size as the order of L, given by $[B] \quad \partial Y = B\, Y$ where

$$B := T^{-1} A T - T^{-1} \partial T = \begin{bmatrix} 1 & z^2 \\ 0 & 0 \end{bmatrix}.$$

In this paper, we shall prove that, given any system $[A]$ with rational coefficients, it can be reduced to an *equivalent* system $[B]$ with rational coefficients, such that the finite singularities of $[B]$ coincide with the non-apparent singularities of $[A]$. Our method can, in particular, be applied to the companion system of any linear differential equation with arbitrary order n. We thus have an alternative method to the standard methods for removing apparent singularities of linear differential operators. However, it is also interesting by its own since first-order linear differential systems with apparent singularities arise naturally in applications (see, e.g., [15, 16, 9] and references therein, for applications within Feynman integrals and statistical physics). Moreover, such a desingularization can serve numerical methods.

This paper is organized as follows: In Section 2 we give the notations to be used in the sequel and some preliminary definitions. In Section 3, we give our main result: We show

how apparent singularities can be detected, prove the existence of desingularizations, and develop a desingularization algorithm over $\mathbb{C}(z)$. In Section 4, we show some examples of computations compared to two desingularization algorithms designed for scalar linear differential equations. In Section 5, we give a rational version of our algorithm. And finally, we give items for further investigation in Section 6.

2. PRELIMINARIES

Given system $[A] \quad \partial X = A(z)X$. In this section, we recall the useful notion of equivalent systems and the classification of singularities.

2.1 Equivalent systems

Let $T \in \mathrm{GL}(n, \mathbb{C}(z))$, the *gauge transformation* $X = TY$, leads to a new system $[B] \quad \partial Y = B(z)Y$, where

$$B = T[A] := T^{-1}AT - T^{-1}\partial T.$$

Systems $[A]$ and $[B]$ are called *equivalent over* $\mathbb{C}(z)$.

2.2 Classification of singularities

A singular point $z_0 \in \mathbb{C}$ is called a *regular singular point* for the system $[A]$ if in a neighborhood of z_0, there exists a fundamental matrix solution of the form $X(z) = \Phi(z)(z - z_0)^\Lambda$ where $\Lambda \in \mathbb{C}^{n\times n}$ is a constant matrix and $\Phi(z)$ is a matrix which is analytic at z_0; otherwise z_0 is called an *irregular singular point* (see, e.g. [13, Ch 4 Sec 2 pp 111]). Hence apparent singularities of $[A]$ are among regular singularities of $[A]$.
The change of variable $z \mapsto 1/z$ permits to classify the point $z = \infty$ as an ordinary, a regular singular or an irregular singular point of the system $[A]$.
This classification, based upon knowledge of a fundamental matrix, is not immediately apparent for a given differential system. It is well known (see [3]) that if $A(z)$ has a simple pole at $z = z_0$ then the point z_0 is a regular singularity for the system $[A]$. The converse is not true: even when $A(z)$ has a multiple pole at $z = z_0$ it is still possible for z_0 to be a regular singularity. However, it was proven by Moser [19] that in the case where z_0 is a regular singularity there exists a polynomial matrix function T of $z-z_0$ which is nonsingular for $z \neq z_0$ such that the transformation $X = TY$ transforms the system $[A]$ into an equivalent system $[B] \quad \partial Y = B(z)Y$, such that z_0 is a **simple** pole of $B(z)$.
Simple poles of $A(z)$ are called *first-kind singularities* of $[A]$, poles of higher order are singularities of *second kind*. Thus, a system has a regular singularity at a point z_0 if and only if it is equivalent to a system $[B]$ with a first-kind singularity at z_0. This latter system can be constructed using the so called *Rational Moser algorithm* developed in [5]. This algorithm, establishes *partial desingularization*, as it computes for a given system $[A]$ a polynomial transformation $T(z)$ with $\det T(z) \not\equiv 0$ that leads to a system $[B]$ such that (see [5, Sec. 6]):

(i) The finite singularities of $[B]$ are among the finite singularities of $[A]$.

(ii) **Every** finite singular point of $[B]$ has a **minimal** pole order among **all** equivalent systems.

3. DETECTING AND REMOVING APPARENT SINGULARITIES

Consider a system $[A] \ \partial X = A(z)X$ with $A(z) \in \mathbb{C}(z)^{n\times n}$.

Definition 1. *A system $[\tilde{A}] \ \partial\tilde{X} = \tilde{A}(z)\tilde{X}$ is called a desingularization of $[A]$ if:*

(i) There exists a **polynomial** *matrix $T(z)$ with $\det T(z) \not\equiv 0$ such that $\tilde{A} = T[A]$;*

(ii) The singularities of $[\tilde{A}]$ are the singularities of $[A]$ that are not apparent.

In the sequel we shall prove that desingularizations do exist and develop an algorithm that produces a desingularization of any system $[A]$ over $\mathbb{C}(z)$.
We first start by explaining how to remove one apparent singularity.

Proposition 1. *If $z = z_0$ is a finite apparent singularity of $[A]$ then one can construct a polynomial matrix $T(z)$ with $\det T(z) = c(z - z_0)^\alpha$, $c \in \mathbb{C}^*$ and $\alpha \in \mathbb{N}$ such that $T[A]$ has at worst a simple pole at $z = z_0$.*

Proof. (i) An apparent singularity is a regular singularity, hence $[A]$ can be reduced to an equivalent system $T[A]$ which has z_0 as a singularity of first kind. (ii) The transformation T can be constructed by the algorithm in [5] and hence it has the required property (see [5, Thm 2]).

Proposition 2. *Suppose that $A(z)$ has simple pole at $z = z_0$ and let*

$$A(z) = \frac{A_0}{(z - z_0)} + \sum_{i\geq 1} A_i(z - z_0)^{i-1}, \;\; A_i \in \mathbb{C}^{n\times n}.$$

If z_0 is an apparent singularity then the eigenvalues of the so-called residue matrix A_0, are nonnegative integers.

Proof. Suppose that A_0 possesses at least one eigenvalue which does not belong to $\mathbb{N}$ and let μ be an eigenvalue of A_0 such that $\mu \in \mathbb{C} \setminus \mathbb{N}$ and $\Re\mu$, its real part, is maximal. Then the system $[A]$ has a nonzero local vectorial solution of the form:
$X(z) = (z-z_0)^\mu \sum_{k=0}^{+\infty} X_k(z - z_0)^k$ with $X_k \in \mathbb{C}^n$ and $X_0 \neq 0$, the series being convergent in a disc centered at z_0 (see, e.g. [3, Ch 2 Thm 6 pp 32]). Such a solution is not analytic at $z = z_0$ because $\mu \notin \mathbb{N}$. This implies that z_0 is a singularity which is not apparent.

Proposition 3. *Suppose that $z = z_0$ is a simple pole of $A(z)$ and that A_0 has only nonnegative integer eigenvalues. Then there exists a polynomial matrix $T(z)$ with $\det T(z) = c(z - z_0)^\alpha$ for some $c \in \mathbb{C}^*$ and $\alpha \in \mathbb{N}$ such that $B := T[A]$ has at worst a simple pole at $z = z_0$ and B_0 has a single eigenvalue: $B_0 = mI_n + N$ where $m \in \mathbb{N}$ and N nilpotent. Moreover, z_0 is an apparent singularity iff $N = 0$. In this case, the gauge transformation $Y = (z - z_0)^m\tilde{Y}$ leads to a system for which $z = z_0$ is an ordinary point.*

Proof. Let $m_1, \ldots, m_s \in \mathbb{N}$ be the eigenvalues of A_0. For $i = 1, \ldots, s$, denote by ν_i the multiplicity of m_i. Suppose that $m_1 > m_2 > \ldots > m_s$ and put $\ell_i = m_i - m_{i+1} \in \mathbb{N}^*$, $\;i = 1, \ldots, s-1$. By applying a constant gauge transformation we can assume that A_0 is in Jordan form:

$$A_0 = \begin{bmatrix} A_0^{11} & 0 \\ 0 & A_0^{22} \end{bmatrix}, \qquad (2)$$

where A_0^{11} is an ν_1 by ν_1 matrix having one single eigenvalue m_1:

$$A_0^{11} = m_1 I_{\nu_1} + N_1$$

N_1 being a nilpotent matrix. Applying the transformation $X = UY$, where

$$U = diag((z - z_0)I_{\nu_1}, I_{n-\nu_1}) \qquad (3)$$

yields the new system:

$$Y' = \tilde{A}(z)Y,$$

where

$$\tilde{A}(z) = U^{-1}A(z)U - U^{-1}U'.$$

Its residue matrix is given by:

$$\tilde{A}_0 = \left(A_0 + (z - z_0)U^{-1}A_1U - (z - z_0)U^{-1}U'\right)_{|z=z_0}.$$

Let A_1 be partitioned as A_0 :

$$A_1 = \begin{bmatrix} A_1^{11} & A_1^{12} \\ A_1^{21} & A_1^{22} \end{bmatrix}, \quad A_1^{11} \in \mathbb{C}^{\nu_1 \times \nu_1}$$

Then

$$\tilde{A}_0 = \begin{bmatrix} A_0^{11} - I_{\nu_1} & A_1^{12} \\ 0 & A_0^{22} \end{bmatrix}. \qquad (4)$$

Hence the eigenvalues of $\tilde{A}_0$ are: $m_1 - 1, m_2, \dots, m_s$, each with the same initial multiplicity ν_i.
By repeating this process ℓ_1 times where $m_1 - \ell_1 = m_2$, the eigenvalues become:

$$m_2, m_2, \dots, m_s.$$

Thus, after $\ell_1 + \dots + \ell_{s-1} = m_1 - m_s$ steps one gets an equivalent system $B := T[A]$ of the first kind with a residue matrix B_0 with a single eigenvalue m_s of multiplicity $\nu_1 + \dots + \nu_s = n$. Hence the matrix $N := B_0 - m_s I_n$ is nilpotent. Moreover the matrix T is the product of matrices that are either constant or of the form (3). Hence T is a polynomial matrix of degree at most $m_1 - m_s$ and its determinant is of the form $\det T(z) = c(z - z_0)^\alpha$ for some $c \in \mathbb{C}^*$ and $\alpha \in \mathbb{N}$. Due to the form of B_0 it follows that the system $[B]$ has, in the neighborhood of z_0, a fundamental matrix solution of the form

$$Y(z) = (z - z_0)^{m_s}\Phi(z)(z - z_0)^N$$

where

$$\Phi(z) = I_n + \sum_{k=1}^{+\infty} \Phi_k (z - z_0)^k \in \mathbb{C}\{(z - z_0)\}^{n\times n},$$

Hence z_0 is an apparent singularity of $[A]$ if and only if N is the zero matrix. Finally, if we put $Y = (z - z_0)^{m_s}\tilde{Y}$ the resulting system has the above matrix $\Phi(z)$ as a fundamental matrix solution around $z = z_0$. As $\Phi(z_0) = I_n$, the point z_0 is an ordinary point for the latter system.

Remark 1. *One can deduce from the above proof that a necessary (but not sufficient) condition for a first-kind singularity z_0 to be an apparent singularity for $[A]$ is that the residue matrix A_0 be* **diagonalizable** *with nonnegative integer eigenvalues. Indeed, if A_0 is not diagonalizable then in (2) at least one of the two blocks A_0^{11}, A_0^{22} has a nonzero nilpotent part. It follows from the form of (4) that transformation (3) cannot annihilate the nilpotent part of the residue matrix. Thus if the residue matrix A_0 of the input system (or one of the intermediate systems) is not diagonalizable then the output system $[B]$ has a residue matrix $B_0 = mI_n + N$ with $N \neq 0$.*

We thus proved the following theorem:

Theorem 1. *If $z = z_0$ is a finite apparent singularity of $[A]$ then one can construct a polynomial matrix $T(z)$ with $\det T(z) = c(z - z_0)^\alpha$, $c \in \mathbb{C}^*$ and $\alpha \in \mathbb{N}$ such that $B(z) := T[A]$ has no pole at $z = z_0$.*

Algorithm 1 Desingularization Algorithm

Input: $A(z)$;
Output: $T(z) \in GL(n, \mathbb{C}(z))$ and $T[A]$ such that $T[A]$ is a desingularization of the input system $[A]$. And, the two sets $\mathcal{A}pp$ and Σ of apparent singularities (which are "removed") and simple poles (which are not apparent singularities) respectively.

$T \leftarrow$ Use the *Rational Moser Algorithm* of [5] to compute a polynomial matrix $T(z)$ with $\det T(z) \not\equiv 0$ such that

- The roots of $\det T(z) = 0$ belong to $\mathcal{P}(A)$ (*this implies that the poles of $T^{-1}(z)$ are among the poles of A and hence $\mathcal{P}(T[A]) \subset \mathcal{P}(A)$*)
- The orders of the poles of $T[A]$ are minimal among all equivalent systems

$A \leftarrow T[A]$;
$\mathcal{A}pp \leftarrow$ {set of simple poles of A, z_i : $1 \le i \le \mu$};

$\Sigma \leftarrow \{\}$;

$i \leftarrow 1$;
while $i \neq \mu + 1$ **do**

- Compute A_{z_i} the residue matrix of A at $z = z_i$
- Compute the SN decomposition of A_{z_i}, namely $A_i = S_{z_i} + N_{z_i}$ where S_{z_i} semi-simple, N_{z_i} nilpotent, and S_{z_i} and N_{z_i} commute. We remark that a rational decomposition exists (see, e.g. [17])

if $N_{z_i} \neq 0_n$ or S_{z_i} has at least one eigenvalue in $\mathbb{C} \setminus \mathbb{N}$ **then** $\mathcal{A}pp \leftarrow \mathcal{A}pp \setminus \{z_i\}$; $\Sigma \leftarrow \Sigma \cup \{z_i\}$;
else Use the method presented in the proof of Proposition 3 to compute a polynomial matrix T_{z_i} such that $T_{z_i}[A]$ has at worst a simple pole at $z = z_i$ with residue matrix of the form $A_{z_i} = m_{z_i} I_n + N_{z_i}$ where $m_{z_i} \in \mathbb{N}$ and N_{z_i} is nilpotent.
if $N_{z_i} \neq 0_n$ **then** $\mathcal{A}pp \leftarrow \mathcal{A}pp \setminus \{z_i\}$; $\Sigma \leftarrow \Sigma \cup \{z_i\}$;
else $A \leftarrow T_{z_i}[A]$; $T \leftarrow T \star (z - z_i)^{m_{z_i}} T_{z_i}$;
end if
end if

- $i \leftarrow i + 1$;

end while
return (T, A, Σ, $\mathcal{A}pp$).

Due to the form of its determinant, the gauge transformation $T(z)$ in the previous theorem does not affect the other finite singularities of $[A]$. This means that the apparent singularity at z_0 is removed without introducing new finite singularities or changing the pole order of the other finite singularties of $[A]$, as illustrated by the following simple example.

EXAMPLE 1. *Given the system* $[A]\ \partial X = A(z)X$ *where*

$$A = \begin{bmatrix} 0 & 1 \\ 2\,\frac{-1+2\,z^2}{z^2+2} & -\frac{3\,z^2-4}{z\left(z^2+2\right)} \end{bmatrix}$$

It has a simple pole at $z=0$ *with a residue matrix*

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Our algorithm computes the gauge transformation given by

$$T = \begin{bmatrix} 1 & 0 \\ z & -z^2 \end{bmatrix}$$

The matrix of the new equivalent system is

$$B = T^{-1}(AT - \partial T) = \begin{bmatrix} z & -z^2 \\ 1 & -\frac{z\left(z^2+7\right)}{z^2+2} \end{bmatrix}$$

As expected it has $z=0$ *as an ordinary point. Moreover, neither new finite singularities are introduced, nor the pole order of the other non-apparent finite singularties, i.e. the roots of* (z^2+2)*, are changed.*

Thus by successively applying Theorem 1 to each finite apparent singularity of $[A]$, we get the following:

THEOREM 2. *Given a system* $[A]$*, one can construct a polynomial matrix* $T(z)$ *which is invertible in* $\mathbb{C}(z)$ *such that the finite poles of* $B := T[A]$ *are exactly the poles of* A *that are not apparent singularities for* $[A]$*.*

REMARK 2. *If the point at infinity of the original system is singular regular then it will be also singular regular of the computed desingularization. However, the order of the pole at infinity may increase. This follows immediately from the fact that the two systems are gauge equivalent.*

Consider a system $[A]\ \partial X = A(z)X$ and let $\mathcal{P}(A)$ denote the set of finite poles of $A(z)$. We thus have the desingularization Algorithm 1.

4. APPLICATION: DESINGULARIZATION OF SCALAR DIFFERENTIAL EQUATION

The interest in desingularization of scalar differential equations dates back to the 19^{th} century. Since then, several algorithms have been developed for such and more general scalar equations (see, e.g., the introductions of [2, 11] and the references therein). The desingularization algorithms developed specifically for scalar equations are based on computing a least common left multiple of the operator in question and an appropriately chosen operator. This outputs in general an equation whose solution space contains strictly the solution space of the input equation. As we mentioned in the introduction, the algorithm developed in this paper can be used as well for the desingularization of a companion system of any scalar differential equation. This desingularization is based on an adequate choice of a gauge transformation. Thus the desingularized output system is always equivalent to the input system and the dimension of the solution space is preserved. However, a scalar differential equation equivalent to the desingularized system (see, e.g., [4, 12]) would generally feature apparent singularities. Thus, when dealing with scalar differential equations, our algorithm is well-suited to situations where adhering to a scalar representation is insignificant, e.g. reduction prior to computing solutions near singularities via numerical methods.
In this section, we use Algorithm 1 to desingularize companion systems of two equations which are already treated by existing algorithms. But first we recall the definition of desingularization in the scalar case. Let $L \in \mathbb{C}(z)[\partial]$ be a monic differential operator of order n,

$$L = \partial^n + c_{n-1}(z)\partial^{n-1} + \cdots + c_0(z).$$

We denote by $S(L)$ the set of finite singularities of L, i.e. the set of the poles of the c_i's, $0 \le i \le n-1$.

DEFINITION 2. *An operator* $\tilde{L} \in \mathbb{C}[z][\partial]$ *is called a desingularization of* L *if :*

(i) $\tilde{L} = RL$ *for some* $R \in \mathbb{C}(z)[\partial]$ *,*

(ii) $S(\tilde{L}) = \{z_0 \in S(L) \mid z_0 \text{ not apparent}\}$

An algorithm developed in [2] constructs, for a given a monic operator $L \in \mathbb{C}(z)[\partial]$ of order n, a monic operator $\tilde{L} \in \mathbb{C}(z)[\partial]$ with minimal order $m+1 \ge n$ satisfying (i) and (ii), m being the maximum of the of the set of all local exponent at the different finite apparent singularities of L. This algorithm has been implemented in Maple and is referred to in this paper by *ABH* method. The system of Example 1 is in fact the companion system of the following differential equation which we treat below by the ABH method.

EXAMPLE 2. *Consider the operator*

$$L = \partial^2 + \frac{(3z^2-4)}{z(z^2+2)}\partial + 2\frac{1-2z^2}{z^2+2}.$$

It has an apparent singularity at $z=0$ *with local exponents* 0 *and* 3*. The desingularization computed by ABH method is the following operator of order* 4*:*

$$\begin{aligned} \tilde{L} \;&=\; \partial^4 + \frac{z(24+7z^2)}{2(z^2+2)}\partial^3 + \frac{(58z^2+88+27z^4)}{2(z^2+2)^2}\partial^2 \\ &-\; \frac{z(-4z^2+4+93z^4+28z^6)}{2(z^2+2)^3}\partial \\ &-\; \frac{4(44z^2+16+42z^4+7z^6)}{(z^2+2)^3} \end{aligned}$$

The classical algorithm for differential equations takes the least common left multiple of the given differential operator and a well-chosen auxiliary one (see, e.g., [14]). A “three-fold generalization” of this algorithm to more general operators is given recently in [11]. The following example has been treated therein. However, the removal of one apparent singularity, namely at $z_0 = 0$, introduces new singularities. The latter can then be removed by using a trick introduced in ABH algorithm (see [2, Thm 2 and Step 6 in Algo “t-desing”] and [11, Sec 3]). As illustrated below, Algorithm 1 removes all apparent singularities at one stroke without introducing any new ones.

EXAMPLE 3. *Let*

$$L = \partial^2 - \frac{(z^2-3)(z^2-2z+2)}{(z-1)(z^2-3z+3)z}\partial + \frac{(z-2)(2z^2-3z+3)}{(z-1)(z^2-3z+3)z}.$$

The apparent singularities of L are $z=0$ and the roots of $z^2-3z+3=0$. In what follows, we seek their removal using different algorithms:
(i) A desingularization computed by the classical algorithm [11, Example 1]:

$$\begin{aligned}\tilde{L}_{Classical} &= (z-1)(z^4-z^3+3z^2-6z+6)\partial^4\\ &- (z^5-2z^4+z^3-12z^2+24z-24)\partial^3\\ &- (3z^3+9z^2)\partial^2+(6z^2+18z)\partial-(6z+18).\end{aligned}$$

$\tilde{L}_{Classical}$ is a desingularizaiton of L at $z=0$ and $z^2-3z+3=0$. However, new apparent singularities, i.e. the roots of $z^4-z^3+3*z^2-6*z+6=0$, are introduced.

(ii) A desingularization computed by the probabilistic of [11], which we refer to as CKS algorithm, (see Example 7(1) therein):

$$\begin{aligned}\tilde{L}_{CKS} &= (z-1)(z^6-3z^5+3z^4-z^3+6)\partial^4\\ &- (2z^6-9z^5+15z^4-11z^3+3z^2-24)\partial^3\\ &- (z^7-4z^6+6z^5-4z^4+z^3+6z-6)\partial\\ &+ (2z^6-9z^5+15z^4-11z^3+3z^2-24).\end{aligned}$$

$\tilde{L}_{CKS}$ is a desingularizaiton of L at $z=0$ and $z^2-3z+3=0$. However, new apparent singularities, i.e. the roots of $z^6-3z^5+3z^4-z^3+6=0$, are introduced.

(iii) The desingularization computed by ABH method:

$$\begin{aligned}\tilde{L}_{ABH} &= \partial^4+\frac{(16z^4-55z^3+63z^2-42z+36)}{9(z-1)}\partial^3\\ &- \frac{(64z^5-316z^4+591z^3-468z^2+123z+42)}{9(z-1)^2}\partial^2\\ &+ \frac{\beta}{9(z-1)^3}\partial\\ &- \frac{96z^5-570z^4+1333z^3-1597z^2+993z-219}{9(z-1)^3},\end{aligned}$$

where

$$\beta = (48z^6-197z^5+148z^4+488z^3-1162z^2+999z-288).$$

(iv) The desingularization computed by algorithm 1: The companion system of L is given by:

$$[A] \quad \partial X = \begin{bmatrix} 0 & 1 \\ \frac{(z-2)(2z^2-3z+3)}{(z-1)(z^2-3z+3)z} & \frac{(z^2-3)(z^2-2z+2)}{(z-1)(z^2-3z+3)z}\end{bmatrix} X. \qquad (5)$$

The gauge transformation $X=TY$ where

$$T = \begin{bmatrix} 1 & 0 \\ 1 & (-z^2+3z-3)z^2\end{bmatrix} \qquad (6)$$

results in the following equivalent system

$$[B] \qquad \partial Y = \begin{bmatrix} 1 & -z^2(z^2-3z+3) \\ 0 & \frac{2}{1-z}\end{bmatrix} Y. \qquad (7)$$

Observe that in (iii) and (iv) no new apparent singularities are introduced while old ones are removed.

Note that, as system (5) has rational function coefficients, the transformation (6) and the equivalent system (7), computed by our algorithm, have rational function coefficients as well. In the following section, we describe how such a rationality is preserved by our algorithm.

5. RATIONAL VERSION OF THE ALGORITHM

So far, we have presented our algorithm over $\mathbb{C}$ for the sake of clarity. However, in practice, the base field can be taken as any commutative field k of characteristic zero ($\mathbb{Q} \subseteq k \subset \bar{k} \subset \mathbb{C}$). Consider now a system

$$[A] \;\; \partial X = A(z)X, \quad \text{with} \quad A(z) \in k(z)^{n\times n}. \qquad (8)$$

Let $\Omega = \{\alpha_1, \dots \alpha_d\} \subset \bar{k}$ be a set of conjugate simple poles of $A(z)$ over k. We aim to find an equivalent system which is a desingularization of (8) at each of the points of Ω. One possible method is the successive application of Algorithm 1 to each singularity individually. That is, we first compute a transformation T_1 such that the equivalent system $T_1[A]$ is a desingularization in α_1. We then compute a transformation T_2 such that the equivalent system $T_2[T_1[A]] = (T_2T_1)[A]$ is a desingularization in α_2. Eventually, this yields an equivalent system $(\prod_{i=1}^d T_i)[A]$ which is a desingularization of $[A]$ at all points of Ω. However, the entries of T_j and $(\prod_{i=1}^j T_i)[A]$, $1 \leq j \leq d$, belong to $k(\alpha_1, \dots, \alpha_j)[z]$. Thus, this individual treatment of singularities in d steps, requires an algebraic field extension $k(\alpha_1, \dots \alpha_d)$.
This section describes our “rational” algorithm, i.e. the algorithm which avoids computations with individual singularities by representing them by the irreducible polynomial $p(z)$. Consequently, it replaces d steps by only one step and the computations of intermediate steps are limited to $k(z)/(p)$. We remark however that neither the former nor the latter method require a field extension for the final output, i.e. the equivalent system and the gauge transformation.
For this purpose, we work, similar to [5], with the irreducible polynomial $p(z) = \prod_{i=1}^d (z-\alpha_i) \in k[z]$, and consider the p-adic expansions rather than Laurent series expansions at the α_i's.
Let p be an irreducible polynomial in $k[z]$, i.e. a finite “point”. If f is a non-zero element of $k(z)$, we define $ord_p(f)$ (read order of f at p) to be the unique integer n such that :

$$f = \frac{a}{b}p^n, \text{ with } a, b \in k[z]\setminus\{0\}, \quad p \nmid a \text{ and } p \nmid b.$$

By convention, $ord_p(0) = +\infty$. The *local ring* at p is $\mathcal{O}_p = \{f \in k(z) \ : \ \mathrm{ord}_p(f) \geq 0\}$. If $f \in \mathcal{O}_p$ then $f = a/b \in k(z)$, where $\gcd(a,b) = 1$ and $p \nmid b$. The residue field of $k(z)$ at p is $\mathcal{O}_p/p\mathcal{O}_p$, which is isomorphic to the field $k[z]/(p)$.
Let $f \in k(z)$ then it has a unique $p-$adic expansion given by:

$$f = p^{ord_p f}(f_{0,p} + pf_{1,p} + \cdots)$$

where the $f_{i,p}$'s are polynomials of degree $< \deg p$, and $f_{0,p} \neq 0$ is called the *leading coefficient*.
In analogy, let $A = (a_{i,j})$ be a matrix in $k(z)^{n\times n}$. We define the *order* at p of A, notation $\mathrm{ord}_p(A)$, by

$$\underset{p}{\mathrm{ord}}(A) = \min_{i,j}(\underset{p}{\mathrm{ord}}(a_{i,j})).$$

We say that A has a pole at p if $\mathrm{ord}_p(A) < 0$. Similarly, the leading coefficient is $A_{0,p} \neq 0_n$ in the p-adic expansion of A given by:

$$A = p^{\mathrm{ord}_p(A)}(A_{0,p} + pA_{1,p} + \cdots).$$

5.1 The residue matrix at p

The following lemma leads to a definition of the *residue matrix at p*.

LEMMA 1. *Consider the system*

$$[A]\ \partial X = A(z)X, \quad with \quad A(z) \in k(z)^{n\times n}.$$

Let $\Omega = \{\alpha_1, \dots \alpha_d\} \subset \mathbb{C}$ *be a set of conjugate apparent singularities and* $p(z) = \prod_{i=1}^{d}(z-\alpha_i) \in k[z]$ *be the irreducible polynomial representing them. Consider the p-adic and* α_i*-Laurent expansions of* $A(z)$ *given respectively by*

$$\begin{aligned} A(z) &= \frac{1}{p}(A_{0,p} + pA_{1,p} + \cdots) \\ A(z) &= \frac{1}{(z-\alpha_i)}(A_{0,\alpha_i} + (z-\alpha_i)A_{1,\alpha_i} + \cdots),\ 1 \le i \le d. \end{aligned}$$

Then we have,

$$\frac{1}{\partial p(\alpha_i)}\ A_{0,p}(\alpha_i) = A_{0,\alpha_i},\ 1 \le i \le d.$$

PROOF. From the above expansions, it follows that

$$\begin{aligned} \frac{A_{0,p}}{p}(z) &= \sum_{i=1}^{d} \frac{A_{0,\alpha_i}}{(z-\alpha_i)} \\ &= \sum_{i=1}^{d} \frac{A_{0,\alpha_i}}{p} \prod_{1\le j\ne i\le d} (z-\alpha_j). \end{aligned}$$

But $\partial p(\alpha_i) = \prod_{1\le j\ne i\le d}(z-\alpha_j)$, which completes the proof. □

REMARK 3. *In the following each equivalent* g *of* $k[x]/(p)$ *is represented by the unique polynomial of degree* $< \deg p$ *belonging to* g*. The operations of addition and multiplication in* $k[x]/(p)$ *are performed on the representatives considered as polynomials and the results are reduced modulo* p*. For inverting a nonzero element of* $k[x]/(p)$ *we use the extended Euclidean algorithm.*

Thus, the folllowing definition is well-justified.

DEFINITION 3. *The matrix given by*

$$\frac{A_{0,p}(z)}{\partial p(z)} \in (k[z]/(p))^{n\times n}$$

is called the residue matrix of $A(z)$ *at* p*. We shall denote by* $R_{0,p}(z)$ *its representative in* $k[z]^{n\times n}$*. The latter is of degree strictly less than* d *and can be computed as:* $(uA_{0,p} \bmod p)$*, where* u *denotes the inverse of* $(\partial p \bmod p)$*.*

EXAMPLE 4. *Given*

$$[A] \quad \partial X = A(z)X = \frac{1}{1+z^2}\begin{bmatrix} 1-z & z \\ -z & 1+z \end{bmatrix} X.$$

Let $p := 1+z^2$*, then* p *is an irreducible polynomial over* $\mathbb{Q}[z]$ *and its roots are given by* $\pm i$ *over* $\mathbb{Q}(i)$*. Then* $u = -\frac{z}{2}$ *is the inverse of* ∂p *modp. Thus,* $R_{0,p}(z)$ *is given by* $uA_{0,p} \bmod p$ *and so we have*

$$R_{0,p}(z) = \frac{1}{2}\begin{bmatrix} 1-z & -1 \\ -1 & -1-z \end{bmatrix}.$$

Indeed, one can verify that the residue matrices at $\pm i$*, are given by*

$$\begin{bmatrix} \frac{1-i}{2i} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1+i}{2i} \end{bmatrix} \quad and \quad \begin{bmatrix} \frac{-1-i}{2i} & \frac{1}{2} \\ -\frac{1}{2} & \frac{-1+i}{2i} \end{bmatrix}.$$

We now proceed to giving a rational algorithm for testing whether the eigenvalues of $R_{0,p}(z)$ are nonnegative integers or not.

5.2 Computing the integer eigenvalues of the residue matrix

Given $R_{0,p}(z) \in k[z]^{n\times n}$, we wish to compute its integer eigenvalues, in the course of reduction, to identify the nature of the singularity. Let the characteristic polynomial of $R_{0,p}(z)$ be given by

$$\chi_R(z,\lambda) = \lambda^n + a_{n-1}(z)\lambda^{n-1} + \cdots + a_0(z),$$

where $a_i(z) \in k[z]$ s.t. $deg_z(a_i) < deg_z(p) = d$. Then, $\chi_R(z,\lambda)$ can be rewritten equivalently as

$$\chi_R(z,\lambda) = \sum_{i=0}^{d-1} b_i(\lambda) z^i,$$

where $b_i(\lambda) \in k[\lambda]$, $0 \le i \le d-1$, are of maximal degree n. Let $h(\lambda) =\ gcd\ \{b_i(\lambda)\ ,\ 0 \le i \le d-1\}$. It follows that the set of integer roots of $h(\lambda)$ coincides with the set of integer roots of $\chi_R(z,\lambda)$.

Additionally, we remark that in order to compute the integer roots of $\chi_R(z,\lambda)$, it suffices to compute those of $det(\partial p\lambda - A_{0,p})\ mod\ p$. Similar arguments hold true for operations of Proposition 3.

5.3 Examples

In this subsection, we illustrate the rational version of our algorithm with two examples. We first treat the introductory example of [10], and then an example from [9] arising in statistical physics. We show how the corresponding systems can be desingularized by "rational" transformations at irreducible polynomials of degrees 3 and 4 respectively. A third example with an irreducible polynomial of degree 37 is available at [18].

EXAMPLE 5 (INTRODUCTION, [10]). *Consider the differential operator*

$$\begin{aligned} L &= (1+z)(23-20z-z^2+2z^3)\partial^2 \\ &+ \ 2(33-9z-3z^2-z^3)\partial - (45+25z-35z^2-z^3+2z^4). \end{aligned}$$

whose companion system is given by:

$$[A] \quad \partial X = A(z)X$$

where

$$A(z) = \begin{bmatrix} 0 & 1 \\ \frac{(45+25z-35z^2-z^3+2z^4)}{(1+z)(23-20z-z^2+2z^3)} & -\frac{2(33-9z-3z^2-z^3)}{(1+z)(23-20z-z^2+2z^3)} \end{bmatrix}.$$

The gauge transformation $X = TY$ *where*

$$T = \begin{bmatrix} 1 & \frac{-14}{143}z^2 + \frac{3}{11}z + \frac{153}{143} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2z^3 - z^2 - 20z + 23 & 0 \\ 0 & 1 \end{bmatrix},$$

results in the following system desingularized at $p = (23-20z-z^2+2z^3)$*:*

$$[B] \quad \partial Y = \begin{bmatrix} \frac{14z^3-39z^2-258z-175}{143(z+1)} & \frac{98z^2-497z-1385}{-20449(z+1)} \\ \frac{2z^4-z^3-35z^2+25z+45}{z+1} & \frac{14z^3-39z^2-258z+111}{-143(z+1)} \end{bmatrix} Y.$$

The algorithm gives as well a negative response for whether $[A]$ *can be desingularized at* $(1+z)$*. In fact, the eigenvalues of the residue matrix are* 0 *and* -2*, which is a negative integer.*

EXAMPLE 6 (THE ISING MODEL,[9]). *Given*

$$[A] \quad X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-4\alpha_1}{z^2(-1+16z)^3 q} & \frac{-4\alpha_2}{z^2(-1+16z)^2 q} & \frac{-2\alpha_3}{z(-1+16z)q} \end{bmatrix} X,$$

where

$$\begin{aligned} q &= (4\,z-1)\left(4352\,z^4+3607\,z^3-1678\,z^2+252\,z-8\right) \\ \alpha_1 &= 89128960\,z^7+74981376\,z^6-97687536\,z^5 \\ &+ 33948640\,z^4-4652220\,z^3+84480\,z^2+9469\,z \\ &- 294 \\ \alpha_2 &= 17825792\,z^7+13139200\,z^6-16119599\,z^5 \\ &+ 5128290\,z^4-689440\,z^3+28373\,z^2-185\,z-6 \\ \alpha_3 &= 1183744\,z^6+770128\,z^5-872579\,z^4+252146\,z^3 \\ &- 30499\,z^2+1172\,z-12. \end{aligned}$$

We are interested in desingularization at

$$p = \left(4352\,z^4+3607\,z^3-1678\,z^2+252\,z-8\right).$$

Our algorithm computes the gauge transformation $X = T_1 T_2 Y$ *where*

$$T_1 = \begin{bmatrix} 1 & \gamma_1\gamma_2 & \gamma_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \textit{and}$$

$$\begin{aligned} \gamma_1 &= -6128505692416\,z^3-5454831630087\,z^2 \\ &+ 2041133482952\,z-215817804724 \\ \gamma_2 &= \frac{10598786\,z^3}{22324211786901375}-\frac{145270123\,z^2}{714374777180844000} \\ &- \frac{70951\,z}{71437477718084400}+\frac{79111}{44648423573802750} \\ \gamma_3 &= \frac{169580576\,z^3}{525308635}-\frac{145270123\,z^2}{1050617270}-\frac{70951\,z}{105061727} \\ &+ \frac{632888}{525308635}. \end{aligned}$$

The system is not desingularizable at any of the other polynomials. In fact, the algorithm gives as well the following information:

- $z = 0$ *is a simple pole, and a partial desingularization can be computed. However, only two of the eigenvalues of the residue matrix are nonnegative integers.*
- *The root of* $16z-1=0$ *is a simple pole, and a partial desingularization can be computed. However, none of the eigenvalues of the residue matrix are nonnegative integers.*
- *The root of* $4z-1=0$ *is a simple pole. However, only two of the eigenvalues of the residue matrix are nonnegative integers.*

The resulting desingularizaton at p *and the partial desingularizations at* z *and* $16z-1$ *are available at [18].*

6. CONCLUSION

In this paper, we give a method for detecting and removing the apparent singularities of linear differential systems via a rational algorithm, i.e. an algorithm which avoids the computations with individual conjugate singularities. The Maple package is available for download at [18] with examples. Our method can be used, in particular, for the desingularization of differential operators in the scalar case.
One field of investigation is the generalization of our algorithm to treat more general systems, e.g. systems with parameters as well as investigating the case of difference systems. First steps in this direction, namely reductions in the parameter and the partial desingularization, are established in [8, 1] respectively.
Another field of investigation is the complexity study of the various algorithms existing for the scalar case, as well as this new algorithm which can be applied to the companion system, so that their efficiency can be compared. Partial results in this direction are already obtained in [7].

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