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MONOID ORE EXTENSIONS

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ABSTRACT. Given a non-associative unital ring R , a commutative monoid G and a set of maps $\pi : R \rightarrow R$, we introduce a monoid Ore extension $R[\pi; G]$, and, in a special case, a differential monoid ring. We show that these structures generalize, in a natural way, the classical Ore extensions and differential polynomial rings, respectively. Moreover, we give necessary and sufficient conditions for differential monoid rings to be simple. We use this in a special case to obtain new and shorter proofs of classical simplicity results for differential polynomial rings previously obtained by Voskoglou and Malm by other means. We also give examples of new Ore-like structures defined by finite commutative monoids.

1. INTRODUCTION

In [15] Ore introduced a version of non-commutative polynomial rings, nowadays called Ore extensions, that have become a very important construction in ring theory. These extensions play an important role when investigating cyclic algebras, enveloping rings of solvable Lie algebras, and various types of graded rings such as group rings and crossed products, see e.g. [3], [7], [10] and [17]. They are also a natural source of examples and counter-examples in ring theory, see e.g. [1] and [2]. Furthermore, special cases of Ore

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extensions are used as tools in different analytical settings, such as differential-, pseudo-differential and fractional differential operator rings [4] and q -Heisenberg algebras [6].

Let us recall the definition of an Ore extension. Suppose that R is an associative unital ring with multiplicative identity 1. An Ore extension $S = R[x; \sigma, \delta]$ of R is defined to be the polynomial ring $R[x]$ as a left R -module equipped with a new multiplication induced by the relations $xr = \sigma(r)x + \delta(r)$, for $r \in R$, where $\sigma : R \rightarrow R$ is a ring endomorphism respecting 1 and $\delta : R \rightarrow R$ is a σ -derivation on R , i.e. δ is an additive map satisfying $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for all $r, s \in R$. In the special case when $\sigma = \text{id}_R$, then S is called a differential polynomial ring and δ is called a derivation. Let \mathbb{N} denote the set of non-negative integers. The product of two arbitrary monomials rx^a and sx^b in S , where $r, s \in R$ and $a, b \in \mathbb{N}$, is given by

$$(rx^a)(sx^b) = \sum_{c \in \mathbb{N}} r\pi_c^a(s)x^{c+b}, \quad (1)$$

where π_c^a denotes the sum of all the $\binom{a}{c}$ possible compositions of c copies of σ and $a - c$ copies of δ in arbitrary order (see [15, Equation (11)]). Here we use the convention that $\pi_c^a(s) = 0$, for every $s \in R$, whenever $c > a$. To motivate the approach taken in this article, let us, for a moment, dwell upon some properties of π , namely:

- (M1) $\pi_0^0 = \text{id}_R$;
- (M2) if $a \geq b$, then $\pi_b^a(1)$ equals the Kronecker delta function $\delta_{a,b}$;
- (M3) if $a + b \geq c$, then $\sum_{d+e=c} \pi_d^a \circ \pi_e^b = \pi_c^{a+b}$;
- (M4) if $a \geq b$ and $r, s \in R$, then $\sum_{b \leq c \leq a} \pi_c^a(r)\pi_b^c(s) = \pi_b^a(rs)$.
- (M4') if $a \geq b$, then π_b^a is left R^G -linear; (M4'') if $a \geq b$, then π_b^a is right R^G -linear.

Here R^G denotes the set of $r \in R$ satisfying $\pi_a^a(r) = r$ and $\pi_b^a(r) = 0$ if $a > b$. Property (M1) follows from the fact that S is a left R -module; (M2) holds since 1 is a multiplicative identity element of S ; (M3) and (M4) follow, respectively, from the equalities $(S, x^a, S) = \{0\}$, for $a \in \mathbb{N}$, and $(S, R, S) = \{0\}$. Recall that if A, B and C are additive subgroups of S , then (A, B, C) denotes the set of all finite sums of elements of the form $(a, b, c) := (ab)c - a(bc)$, for $a \in A, b \in B$ and $c \in C$. Properties (M4') and (M4'') follow from (M4). If S is a differential polynomial ring, then a sharpening of (M1) holds, namely:

- (M1') if $a \in \mathbb{N}$, then $\pi_a^a = \text{id}_R$.

Properties (M1)-(M4) parametrize all Ore extensions. Indeed, if $R[x]$ is equipped with a product (1) satisfying (M1)-(M4), then $R[x]$ is an Ore extension of R , if we put $\sigma = \pi_1^1$ and $\delta = \pi_0^1$.

In [12], the authors defined non-associative (strong) Ore extensions as $R[x]$, for a unital, possibly non-associative, ring R , equipped with a product (1) satisfying (M1)-(M3) (and either (M4') or (M4'')). If (M1') also held, then they called such an extension a (strong) non-associative differential polynomial ring. In loc. cit. a simplicity result for non-associative differential polynomial rings was obtained, thereby generalizing classical results by Amitsur and Jordan as well as more recent results by Öinert, Richter and Silvestrov [14].

The starting point of the present article is the observation that (M1)-(M4) make sense for a large class of commutative monoids. In fact, any commutative monoid G is endowed

with its *algebraic preordering* \leq defined by saying that if $a, b \in G$, then $a \leq b$ if there is $c \in G$ such that $a + c = b$ (see Proposition 1). To make the sums in (1) and (M3)-(M4) finite, we will make the assumption that all sublevel sets $G_a = \{b \in G \mid b \leq a\}$, for $a \in G$, are finite. Now we can define the objects of our study. Namely, let R be a unital, possibly non-associative, ring and suppose that for all $a, b \in G$ with $a \geq b$, there is an additive map $\pi_b^a : R \rightarrow R$. We will use the convention that $\pi_b^a(s) = 0$, for each $s \in R$, if $a \not\geq b$. We say that π is a (unital) G -derivation if (M1)-(M3) (and (M1')) hold. In that case, we say that the monoid ring $R[G]$ (see Definition 2), equipped with a product defined by (1), is a monoid Ore extension (differential monoid ring) and we will denote this structure by $R[G; \pi]$. We also say that π is a classical (unital) G -derivation if (M1)-(M4) (and (M1')) hold. In that case, we say that $R[G; \pi]$ is a classical monoid Ore extension (differential monoid ring). The usage of the term "classical" of course comes from the fact that if $G = \mathbb{N}$, then a classical monoid Ore extension is an ordinary Ore extension. For more details, see Definition 3.

The main objective of this article is to extend the simplicity results in [12] to monoid Ore extensions. The secondary objective is to apply this to particular cases of monoids to obtain new proofs of classical simplicity results for iterated Ore extensions as well as showing simplicity results for new Ore-like structures. Here is an outline of the article.

In Section 2, we state our conventions concerning relations, monoids and rings. After that, we define monoid Ore extensions and differential monoid rings (see Definition 3). Then we show some results concerning commutativity and associativity in monoid Ore extensions (see Propositions 7–13). At the end of the section, we obtain a characterization of the center of monoid Ore extensions (see Corollary 14).

In Section 3, we show the main result of the article (see Theorem 22) which gives us necessary and sufficient conditions for simplicity of strong non-associative G -differential polynomial rings $R[G; \pi]$ under the hypothesis that the algebraic preorder on G can be extended to a well-order.

In Section 4, we apply the main result of the previous section to the monoid $\mathbb{N}^{(I)}$ of functions f from a well-ordered set I to \mathbb{N} , satisfying $f(i) = 0$ for all but finitely many $i \in I$ (see Theorem 29). Using this result, we obtain generalizations of classical simplicity results by Voskoglou [18] and Malm [9] for differential polynomial rings in finitely many variables (see Theorem 38 and Theorem 40).

In Section 5, we study non-associative monoid Ore extensions for some finite monoids.

2. PRELIMINARIES AND DEFINITIONS

In this section, we state our conventions concerning relations, monoids and rings. After that, we define monoid Ore extensions and differential monoid rings (see Definition 3). Then we show some results concerning commutativity and associativity in monoid Ore extensions (see Proposition 7 – Lemma 13). These results are used at the end of the section to give a characterization of the center of monoid Ore extensions (see Corollary 14). This result will be used in the following sections, for particular cases of monoids.

Conventions on relations. Suppose that \sim is a relation on a set X . Recall that \sim is called *reflexive* if for every $a \in X$ we have that $a \sim a$; \sim is called *transitive* if all $a, b, c \in X$ that satisfy $a \sim b$ and $b \sim c$ must satisfy $a \sim c$; \sim is called *antisymmetric* if all $a, b \in X$ that satisfy $a \sim b$ and $b \sim a$ must satisfy $a = b$; \sim is called *total* if we, for all $a, b \in X$, must have $a \sim b$ or $b \sim a$. The relation \sim is called a *preorder* if it is reflexive and transitive. If \sim is a preorder which is antisymmetric and total, then \sim is called a *total order*. In that case, \sim is called a *well-order* if any non-empty subset of X has a least element.

Conventions on monoids. A commutative monoid is a non-empty set G equipped with a commutative and associative binary operation $G \times G \ni (a, b) \mapsto a + b \in G$ with a neutral element 0 . Take $a, b \in G$. We write $a \geq b$ if there is $c \in G$ such that $a = b + c$. We write $a > b$ if $a \geq b$ but $a \neq b$. We write $a \not\geq b$ if there is no $c \in G$ such that $a = b + c$. We write $b \leq a$ if $a \geq b$. Throughout this article, unless otherwise stated, G denotes a commutative monoid such that all sublevel sets $G_a = \{b \in G \mid b \leq a\}$, for $a \in G$, are finite.

Proposition 1. \geq is a preorder which we will refer to as the algebraic preorder on G .

Proof. Take $a, b, c \in G$. Since $a = a + 0$, it follows that $a \geq a$. Thus \geq is reflexive. Now we show that \geq is transitive. To this end, suppose that $a \geq b$ and $b \geq c$. Then there are $d, e \in G$ such that $a = b + d$ and $b = c + e$. Then $a = b + d = c + d + e \geq c$. \square

Conventions on rings. Throughout this article, unless otherwise stated, R denotes a non-associative ring. By this we mean that R is an additive abelian group in which a multiplication is defined, satisfying left and right distributivity. We always assume that R is unital and that the multiplicative identity of R is denoted by 1 . The term "non-associative" should be interpreted as "not necessarily associative". Therefore all associative rings are non-associative. If a ring is not associative, we will use the term "not associative ring". Recall that the *commutator* $[\cdot, \cdot] : R \times R \rightarrow R$ and the *associator* $(\cdot, \cdot, \cdot) : R \times R \times R \rightarrow R$ are defined by $[r, s] = rs - sr$ and $(r, s, t) = (rs)t - r(st)$ for all $r, s, t \in R$, respectively. The *commuter* of R , denoted by $C(R)$, is the subset of R consisting of all elements $r \in R$ such that $[r, s] = 0$, for all $s \in R$. The *left, middle and right nucleus* of R , denoted by $N_l(R)$, $N_m(R)$ and $N_r(R)$, respectively, are defined by $N_l(R) = \{r \in R \mid (r, s, t) = 0, \text{ for } s, t \in R\}$, $N_m(R) = \{s \in R \mid (r, s, t) = 0, \text{ for } r, t \in R\}$, and $N_r(R) = \{t \in R \mid (r, s, t) = 0, \text{ for } r, s \in R\}$. The *nucleus* of R , denoted by $N(R)$, is defined to be equal to $N_l(R) \cap N_m(R) \cap N_r(R)$. From the so-called *associator identity* $u(r, s, t) + (u, r, s)t + (u, rs, t) = (ur, s, t) + (u, r, st)$, which holds for all $u, r, s, t \in R$, it follows that all of the subsets $N_l(R)$, $N_m(R)$, $N_r(R)$ and $N(R)$ are associative subrings of R . The *center* of R , denoted by $Z(R)$, is defined to be equal to the intersection $N(R) \cap C(R)$. It follows immediately that $Z(R)$ is an associative, unital and commutative subring of R .

Definition 2. The *monoid ring* $R[G]$, of G over R , is defined to be the set of formal sums $\sum_{a \in G} r_a x^a$, where $r_a \in R$ is zero for all but finitely many $a \in G$. The addition on $R[G]$ is defined point-wise and the multiplication on $R[G]$ is defined by the additive extension of the relations

$$(rx^a)(sx^b) = rsx^{a+b}$$

for $r, s \in R$ and $a, b \in G$.

Definition 3. Suppose that for all $a, b \in G$ there is an additive map $\pi_b^a : R \rightarrow R$. If $a \not\geq b$, then π_b^a is required to be the zero map. Let R^G denote the set of $r \in R$ satisfying $\pi_a^a(r) = r$ and $\pi_b^a(r) = 0$ if $a > b$. Consider the following axioms, for each pair of elements $a, b \in G$:

- (M1) $\pi_0^0 = \text{id}_R$;
- (M2) $\pi_b^a(1)$ equals the Kronecker delta function $\delta_{a,b}$;
- (M3) $\sum_{d+e=c} \pi_d^a \circ \pi_e^b = \pi_c^{a+b}$;
- (M4) if $r, s \in R$, then $\sum_{c \in G} \pi_c^a(r) \pi_b^c(s) = \pi_b^a(rs)$;
- (M1') $\pi_a^a = \text{id}_R$;
- (M4') π_b^a is left R^G -linear; (M4'') π_b^a is right R^G -linear.

We say that π is a *(unital) G -derivation* if (M1)–(M3) (and (M1')) hold. In that case, we say that the monoid ring $R[G]$ equipped with a product defined by (1), is a *monoid Ore extension (differential monoid ring)* and we will denote this structure by $R[G; \pi]$. Moreover, we say that π is a *strong (unital) G -derivation* if also (M4') or (M4'') holds. We say that π is a *classical (unital) G -derivation* if (M1)–(M4) (and (M1')) hold. In that case, we say that $R[G; \pi]$ is a *classical monoid Ore extension (differential monoid ring)*.

Remark 4. For the rest of this section, $S = R[G; \pi]$ denotes a monoid Ore extension.

Remark 5. (a) If T is a subring of R , then T^G denotes the set $T \cap R^G$.

(b) Let $a, b \in G$ with $a \geq b$. The additive map $\pi_b^a : R \rightarrow R$ may be extended to an additive map $\tilde{\pi}_b^a : S \rightarrow S$ by letting $\tilde{\pi}_b^a$ be the additive extension of the rule

$$\tilde{\pi}_b^a(r_c x^c) = \pi_b^a(r_c) x^c$$

for $c \in G$ and $r_c \in R$. By then defining S^G analogously to how R^G was defined, it turns out that S^G is equal to the set of all elements of S which are of the form $\sum_{a \in G} r_a x^a$, where $r_a \in R^G$ for each $a \in G$.

Proposition 6. *If π is strong, then the following assertions hold:*

- (a) R^G is a subring of R which we will refer to as the ring of constants of R ;
- (b) $Z(R)^G$ is a commutative subring of R .

Proof. (a) It is clear that R^G is an additive subgroup of R containing 1. Now we show that R^G is multiplicatively closed. Take $r, s \in R^G$ and $a \in G$. Suppose that (M4') holds. Then $\sum_{b \leq a} \pi_b^a(rs) x^b = \sum_{b \leq a} r \pi_b^a(s) x^b = r s x^a$ which shows that $rs \in R^G$. The case when (M4'') holds can be treated similarly and is therefore left to the reader.

(b) This follows from (a) since $Z(R)^G = Z(R) \cap R^G$. □

Proposition 7. *If $a \in G$, then $x^a \in N_m(S) \cap N_r(S)$.*

Proof. Take $r, t \in R$ and $b, c \in G$. First we show that $x^a \in N_m(S)$.

$$\begin{aligned}
(rx^b, x^a, tx^c) &= (rx^b \cdot x^a)tx^c - rx^b(x^a \cdot tx^c) = rx^{b+a} \cdot tx^c - \sum_{d \leq a} rx^b \cdot \pi_d^a(t)x^{d+c} \\
&= \sum_{e \leq b+a} r\pi_e^{b+a}(t)x^{e+c} - \sum_{d \leq a} \sum_{f \leq b} r(\pi_f^b \circ \pi_d^a)(t)x^{f+d+c} \\
&= \sum_{e \leq a+b} \sum_{f+d=e} r(\pi_f^b \circ \pi_d^a)(t)x^{e+c} - \sum_{d \leq a} \sum_{f \leq b} r(\pi_f^b \circ \pi_d^a)(t)x^{f+d+c} = 0.
\end{aligned}$$

Next we show that $x^a \in N_r(S)$.

$$\begin{aligned}
(rx^b, tx^c, x^a) &= (rx^b \cdot tx^c)x^a - rx^b(tx^c \cdot x^a) = \left(\sum_{d \leq b} r\pi_d^b(t)x^{d+c} \cdot x^a \right) - rx^b \cdot tx^{c+a} \\
&= \sum_{d \leq b} r\pi_d^b(t)x^{d+c+a} - \sum_{d \leq b} r\pi_d^b(t)x^{d+c+a} = 0.
\end{aligned}$$

□

Proposition 8. *If $s \in S^G$ and $a \in G$, then $[s, x^a] = 0$.*

Proof. Suppose that $s = \sum_{b \in G} r_b x^b \in S^G$. Then, for each $b \in G$, $r_b \in R^G$. Thus $[s, x^a] = sx^a - x^a s = \sum_{b \in G} r_b x^{b+a} - \sum_{b \in G} \sum_{c \leq a} \pi_c^a(r_b)x^{c+b} = \sum_{b \in G} r_b x^{b+a} - \sum_{b \in G} r_b x^{b+a} = 0$. □

Proposition 9. *Suppose that $s \in S^G$. Then $[s, S] \subseteq [s, R]S$. In particular, if $[s, R] = \{0\}$, then $s \in C(S)$.*

Proof. Suppose that $s = \sum_{a \in G} r_a x^a \in S^G$. Then, for each $a \in G$, $r_a \in R^G$. Take $t \in R$ and $b \in G$. From Proposition 7 and Proposition 8, it follows that $[s, tx^b] = s(tx^b) - (tx^b)s = (st)x^b - t(x^b s) = (st)x^b - t(sx^b) = (st)x^b - (ts)x^b = [s, t]x^b \in [s, R]S$. The last part of the statement follows immediately. □

Proposition 10. *The following assertions hold:*

- (a) (M_4') holds if and only if for every $a \in G$, $(x^a, S^G, R) = \{0\}$;
- (b) (M_4'') holds if and only if for every $a \in G$, $(x^a, R, S^G) = \{0\}$.

Proof. Take $a, b \in G$, $r \in R^G$ and $t \in R$.

(a) Suppose that (M_4') holds. Then, using Proposition 7, Proposition 8 we get that

$$\begin{aligned}
(x^a, rx^b, t) &= (x^a \cdot rx^b)t - x^a(rx^b \cdot t) = ((x^a r)x^b)t - x^a(r(x^b t)) \\
&= (rx^a)(x^b t) - x^a(r(x^b t)) = \sum_{c \leq b} ((rx^a)(\pi_c^b(t)x^c) - x^a(r\pi_c^b(t)x^c)) \\
&= \sum_{c \leq b} \sum_{d \leq a} (r\pi_d^a(\pi_c^b(t))x^{d+c} - \pi_d^a(r\pi_c^b(t))x^{d+c}) = 0.
\end{aligned}$$

Now suppose that the equality $(x^a, S^G, R) = \{0\}$ holds for any $a \in G$. Then, from Proposition 7 and Proposition 8, we get that $0 = (x^a, r, t) = (x^a r)t - x^a(rt) = (rx^a)t - x^a(rt) = r(x^a t) - x^a(rt) = \sum_{b \leq a} (r\pi_b^a(t)x^b - \pi_b^a(rt)x^b)$ from which (M_4') follows.

(b) Suppose that (M4'') holds. We wish to show that $(x^a, t, rx^b) = 0$. From Proposition 7, it follows that it is enough to show this equality for $b = 0$. Now, using Proposition 7 and Proposition 8, we get that

$$(x^a, t, r) = (x^a t)r - x^a(tr) = \sum_{c \leq a} ((\pi_c^a(t)x^c)r - \pi_c^a(tr)x^c) = \sum_{c \leq a} (\pi_c^a(t)r - \pi_c^a(tr))x^c = 0.$$

On the other hand, if the equality $(x^a, R, S^G) = \{0\}$ holds, then, in particular,

$$0 = (x^a, t, r) = (x^a t)r - x^a(tr) = \sum_{c \leq a} ((\pi_c^a(t)x^c)r - \pi_c^a(tr)x^c) = \sum_{c \leq a} (\pi_c^a(t)r - \pi_c^a(tr))x^c,$$

which shows that (M4'') holds. \square

Proposition 11. *Suppose that $s \in S$. Then $(s, S, S) \subseteq (s, R, R)S$. In particular, if $(s, R, R) = \{0\}$, then $s \in N_l(S)$.*

Proof. Suppose that $s = \sum_{a \in G} r_a x^a \in S$. Take $r, t \in R$ and $b, c \in G$. We wish to show that $(s, rx^b, tx^c) \subseteq (s, R, R)S$. From Proposition 7, it follows that it is enough to prove this inclusion for $c = 0$. Using Proposition 7 again, it follows that

$$\begin{aligned} (s, rx^b, t) &= (s \cdot rx^b)t - s(rx^b \cdot t) = ((sr)x^b)t - s(r(x^b t)) \\ &= (sr)(x^b t) - \sum_{d \leq b} s(r\pi_d^b(t)x^d) = \sum_{d \leq b} (sr)\pi_d^b(t)x^d - \sum_{d \leq b} (s(r\pi_d^b(t)))x^d \\ &= \sum_{d \leq b} ((sr)\pi_d^b(t) - s(r\pi_d^b(t)))x^d = \sum_{d \leq b} (s, r, \pi_d^b(t))x^d \subseteq (s, R, R)S. \end{aligned}$$

The last part of the statement follows immediately. \square

Proposition 12. *Suppose that $s \in S^G$.*

- (a) *If (M4') holds, then $(S, s, S) \subseteq (R, s, R)S$. In particular, if $(R, s, R) = 0$, then $s \in N_m(S)$.*
- (b) *If (M4'') holds, then $(S, S, s) \subseteq (R, R, s)S$. In particular, if $(R, R, s) = \{0\}$, then $s \in N_r(S)$.*

Proof. Take $r, t \in R$ and $b, c \in G$.

(a) Suppose that (M4') holds. We wish to show that $(rx^b, s, tx^c) \subseteq (R, s, R)S$. From Proposition 7, it follows that it is enough to show this inclusion for $c = 0$. Now, using Proposition 7, Proposition 8 and Proposition 10(a), we get that

$$\begin{aligned} (rx^b \cdot s)t - rx^b(s \cdot t) &= (r(x^b s))t - r(x^b(st)) = (r(sx^b))t - r((x^b s)t) = ((rs)x^b)t - r((sx^b)t) \\ &= (rs)(x^b t) - r(s(x^b t)) = \sum_{d \leq b} ((rs)\pi_d^b(t)x^d - r(s\pi_d^b(t))x^d) \\ &= \sum_{d \leq b} ((rs)\pi_d^b(t) - r(s\pi_d^b(t)))x^d = \sum_{d \leq b} (r, s, \pi_d^b(t))x^d \in (R, s, R)S. \end{aligned}$$

The second part follows immediately,

(b) Suppose that (M4'') holds. We wish to show that $(rx^b, tx^c, s) \subseteq (R, R, s)S$. From Proposition 7 and Proposition 8, we get that

$$\begin{aligned} (rx^b, tx^c, s) &= (rx^b \cdot tx^c)s - rx^b(tx^c \cdot s) = (rx^b \cdot t)(x^c s) - rx^b(t(x^c s)) \\ &= (rx^b \cdot t)(sx^c) - rx^b(t(sx^c)) = ((rx^b \cdot t)s)x^c - rx^b((ts)x^c) \\ &= ((rx^b \cdot t)s)x^c - ((rx^b)(ts))x^c = (rx^b, t, s)x^c. \end{aligned}$$

Thus, it is enough to prove the inclusion $(rx^b, tx^c, s) \subseteq (R, R, s)S$ for $c = 0$. Now, using Proposition 7, Proposition 8 and Proposition 10(b), we get that

$$\begin{aligned} (rx^b, t, s) &= (rx^b \cdot t)s - rx^b(t \cdot s) = (rx^b \cdot t)s - r(x^b(ts)) = (rx^b \cdot t)s - r((x^b t)s) \\ &= \sum_{d \leq b} ((r\pi_d^b(t)x^d)s - r(\pi_d^b(t)x^d \cdot s)) = \sum_{d \leq b} ((r\pi_d^b(t))(x^d s) - r(\pi_d^b(t)(x^d s))) \\ &= \sum_{d \leq b} ((r\pi_d^b(t))(sx^d) - r(\pi_d^b(t)(sx^d))) = \sum_{d \leq b} (((r\pi_d^b(t))s)x^d - (r(\pi_d^b(t)s))x^d) \\ &= \sum_{d \leq b} (((r\pi_d^b(t))s) - (r(\pi_d^b(t)s)))x^d = \sum_{d \leq b} (r, \pi_d^b(t), s)x^d \in (R, R, s)S. \end{aligned}$$

The second part follows immediately. \square

Lemma 13. *The following three equalities hold:*

$$Z(S) = C(S) \cap N_l(S) \cap N_m(S); \quad (2)$$

$$Z(S) = C(S) \cap N_l(S) \cap N_r(S); \quad (3)$$

$$Z(S) = C(S) \cap N_m(S) \cap N_r(S). \quad (4)$$

Proof. We only show (2). The equalities (3) and (4) are shown in a similar way and are therefore left to the reader. It is clear that $Z(S) \subseteq C(S) \cap N_l(S) \cap N_m(S)$. Now we show the reversed inclusion. Take $r \in C(S) \cap N_l(S) \cap N_m(S)$. We need to show that $r \in N_r(S)$. Take $s, t \in S$. We wish to show that $(s, t, r) = 0$, i.e. $(st)r = s(tr)$. Using that $r \in C(S) \cap N_l(S) \cap N_m(S)$ we get $(st)r = r(st) = (rs)t = (sr)t = s(rt) = s(tr)$. \square

Corollary 14. *If $s \in S^G$ and π is a strong G -derivation, then $s \in Z(S)$ if and only if s commutes and associates with all elements of R .*

Proof. The "only if" statement is immediate. Now we show the "if" statement.

Case 1: (M4') holds. Since $[s, R] = \{0\}$, we get, from Proposition 9, that $s \in C(S)$. Since $(s, R, R) = (R, s, R) = \{0\}$, we get, from Proposition 11 and Proposition 12(a), that $(s, S, S) = (S, s, S) = \{0\}$. Thus, from Lemma 13, we get that $s \in Z(S)$.

Case 2: (M4'') holds. Since $[s, R] = \{0\}$, we get, from Proposition 9, that $s \in C(S)$. Since $(s, R, R) = (R, R, s) = \{0\}$, we get, from Proposition 11 and Proposition 12(b), that $(s, S, S) = (S, S, s) = \{0\}$. Thus, from Lemma 13, we get that $s \in Z(S)$. \square

3. SIMPLICITY

Throughout this section, $S = R[\pi; G]$ denotes a monoid Ore extension. In this section, we introduce G -invariant ideals of R as well as G -simplicity of R (see Definition 15). Thereafter, we prove a number of propositions which lead up to the proof of the main result of this article (see Theorem 22).

Definition 15. Let I be an ideal of R . We say that I is G -invariant if for all $a, b \in G$, the inclusion $\pi_b^a(I) \subseteq I$ holds. We say that R is G -simple if $\{0\}$ and R are the only G -invariant ideals of R . By following Remark 5 we may extend the maps $\pi_b^a : R \rightarrow R$, for $a, b \in G$, to additive maps $S \rightarrow S$. Using these extensions we can speak of G -simplicity of S . It is clear that $S^G = \sum_{a \in G} R^G x^a$.

Proposition 16. *If S is G -simple, then R is G -simple.*

Proof. Suppose that I is a non-zero G -invariant ideal of R . Let J denote the non-zero additive group $\sum_{g \in G} Ix^g$. It is clear that J is G -invariant. We claim that J is an ideal of S . If we assume that the claim holds, then, by G -simplicity of S , we get that $J = S$, and, hence, that $I = R$. Now we show the claim. Take $r \in R$, $i \in I$ and $a, b \in G$. Since I is a G -invariant ideal of R , we get that $(rx^a)(ix^b) = \sum_{c \leq a} r\pi_c^a(i)x^{c+b} \in \sum_{c \leq a} RIx^{c+b} \subseteq J$. On the other hand, we also get that $(ix^b)(rx^a) = \sum_{c \leq a} i\pi_c^b(r)x^{c+a} \in \sum_{c \leq a} IRx^{c+b} \subseteq J$. \square

Proposition 17. *If S is G -simple and π is right (left) $Z(S)^G$ -linear, then $Z(S)^G$ is a field.*

Proof. Suppose that, for $a \geq b$, $\tilde{\pi}_b^a$ is right S^G -linear. The left-handed case is shown in an analogous fashion. From Proposition 6 we know that $Z(S)^G$ is a commutative ring. Take a non-zero $s \in Z(S)^G$. Let $I = Ss$. Then I is a non-zero ideal of S . Take $a, b \in G$. Then $\tilde{\pi}_b^a(I) = \tilde{\pi}_b^a(Ss) = \tilde{\pi}_b^a(S)s \subseteq Ss = I$. Therefore I is G -invariant. From the G -simplicity of S , we get that there is $t \in S$ such that $st = ts = 1$. We know that $t \in Z(S)$. What is left to show is that $t \in S^G$. First of all $\tilde{\pi}_a^a(t) = \tilde{\pi}_a^a(t)st = \tilde{\pi}_a^a(ts)t = \tilde{\pi}_a^a(1)t = 1t = t$. If $b < a$, then we get that $\tilde{\pi}_b^a(t) = \tilde{\pi}_b^a(t)st = \tilde{\pi}_b^a(ts)t = \tilde{\pi}_b^a(1)t = 0t = 0$. \square

Proposition 18. *Take $a, b \in G$ with $a \geq b$. If $\pi_b^a : R \rightarrow R$ is right R^G -linear, then the extension $\tilde{\pi}_b^a : S \rightarrow S$ is right S^G -linear.*

Proof. Suppose that $a, b, d, e \in G$ satisfy $a \geq b$. Take $r \in R$ and $s \in R^G$. Suppose that π_b^a is right R^G -linear. Then $\tilde{\pi}_b^a(rx^d \cdot sx^e) = \tilde{\pi}_b^a(\sum_{c \leq d} r\pi_c^d(s)x^{c+e}) = \tilde{\pi}_b^a(rsx^{d+e}) = \pi_b^a(rs)x^{d+e} = \pi_b^a(r)sx^{d+e} = \sum_{c \leq d} \pi_b^a(r)\pi_c^d(s)x^{c+e} = \pi_b^a(r)x^d \cdot sx^e = \tilde{\pi}_b^a(rx^d)sx^e$. \square

Definition 19. We say that π is commutative if for any $a, b, c, d \in G$, with $a \geq b$ and $c \geq d$, the relation $\pi_b^a \circ \pi_d^c = \pi_d^c \circ \pi_b^a$ holds for the maps $\pi_b^a : R \rightarrow R$ and $\pi_d^c : R \rightarrow R$.

Proposition 20. *Take $a, b \in G$ with $a \geq b$. If $\pi_b^a : R \rightarrow R$ is left R^G -linear and π is commutative, then the extension $\tilde{\pi}_b^a : S \rightarrow S$ is left S^G -linear.*

Proof. Suppose that $a, b, d, e \in G$ satisfy $a \geq b$. Take $r \in R$ and $s \in R^G$. Then $\tilde{\pi}_b^a(sx^e \cdot rx^d) = \tilde{\pi}_b^a(\sum_{c \leq e} s\pi_c^e(r)x^{c+d}) = \sum_{c \leq e} \pi_b^a(s\pi_c^e(r))x^{c+d} = \sum_{c \leq e} s\pi_b^a(\pi_c^e(r))x^{c+d} = \sum_{c \leq e} s\pi_c^e(\pi_b^a(r))x^{d+c} = sx^e \cdot \pi_b^a(r)x^d = sx^e \cdot \tilde{\pi}_b^a(rx^d)$. \square

Definition 21. Suppose that \preceq is a total order on G . Suppose also that \preceq extends \leq , i.e. $\leq \subseteq \preceq$. Define the *degree map* $\deg : S \setminus \{0\} \rightarrow G$ in the following way. Take a non-zero element $s = \sum_{g \in G} r_g x^g \in S$. Since $\text{supp}(s) := \{g \in G \mid r_g \neq 0\}$ is finite, $\text{supp}(s)$ has a greatest element with respect to \preceq . Let us call this element $\deg(s)$.

Theorem 22. Suppose that π is strong and that $S = R[G; \pi]$ is a differential monoid ring such that \leq can be extended to a well-order \preceq on G .

- (a) If (M_4'') holds, then S is G -simple if and only if R is G -simple and $Z(S)^G$ is a field.
- (b) If (M_4') holds and π is commutative, then S is G -simple if and only if R is G -simple and $Z(S)^G$ is a field.

Proof. The "only if" parts in (a) and (b) follow from Proposition 16, Proposition 17, Proposition 18 and Proposition 20. Now we show the "if" parts of (a) and (b) simultaneously. Suppose that R is G -simple and that $Z(S)^G$ is a field. Take a non-zero G -invariant ideal I of S . Let m be the least degree of non-zero elements of I . Define the non-empty subset J of R by saying that $r \in J$ if there are $r_g \in R$, for $g \in G$, with $g \prec m$, such that $rx^m + \sum_{g \prec m} r_g x^g \in I$. It is clear that J is a non-zero left ideal of R . Since S is a differential monoid ring over R it follows that J is also a right ideal of R . Since I is G -invariant it follows that J is G -invariant. From G -simplicity of R we get that $J = R$. In particular, we get that there are $r_g \in R$, for $g \in G$ with $g \prec m$, such that $y := x^m + \sum_{g \prec m} r_g x^g \in I$. Since I is G -invariant, we get, from minimality of m , that $y \in S^G$. Take $r \in R$. Put $z = ry - yr$. Then $\deg(z) < m$ which implies that $z = 0$, since $z \in I$. Thus, y commutes with all elements of R . Next we show that $y \in N(S)$. By Proposition 14, it is enough to show that y associates with all elements of R . Take $r, s \in R$. It is easy to see that the degrees of all the elements (y, r, s) , (r, y, s) and (r, s, y) are less than m . Hence, by minimality of m , we get that they are all zero. Therefore, y is a non-zero element in the field $Z(S)^G$. This implies that $I = S$. Thus, S is G -simple. \square

4. THE MONOID $\mathbb{N}^{(I)}$

In this section, we apply the main result of the previous section to the monoid $\mathbb{N}^{(I)}$ (see Definition 23 and Theorem 29). Using this result, we obtain generalizations of classical simplicity results by Voskoglou [18] and Malm [9] for differential polynomial rings in finitely many variables (see Theorem 38 and Theorem 40).

Definition 23. Let \mathbb{N} denote the commutative monoid having the non-negative integers as its elements and addition as its operation. Let I be a set which is well-ordered with respect to a relation \ll . For a function $f : I \rightarrow \mathbb{N}$ we let $\text{supp}(f)$ denote the support of f , i.e. the set $\{i \in I \mid f(i) \neq 0\}$. Let $\mathbb{N}^{(I)}$ denote the set of functions $I \rightarrow \mathbb{N}$ with finite support. Take $f, g \in \mathbb{N}^{(I)}$. Define $f + g \in \mathbb{N}^{(I)}$ from the relations $(f + g)(i) = f(i) + g(i)$, for $i \in I$. With this operation $\mathbb{N}^{(I)}$ is a commutative monoid. Let \leq denote the algebraic preordering on $\mathbb{N}^{(I)}$ induced from the monoid structure on $\mathbb{N}^{(I)}$ (cf. Proposition 1 and the conventions preceding it). Given $f \in \mathbb{N}^{(I)}$, with cardinality of $\text{supp}(f)$ equal to $m \in \mathbb{N}$, we will often, for simplicity of notation, assume that $\text{supp}(f) = \{0, \dots, m-1\} \subseteq I$.

Definition 24. Suppose that for each $i \in I$, $\delta_i : R \rightarrow R$ is an additive map satisfying $\delta_i(1) = 0$. Put $\Delta = \{\delta_i\}_{i \in I}$ and $R_\Delta = \bigcap_{i \in I} \ker(\delta_i)$. Let J be an ideal of R . We say that J is Δ -invariant if for each $i \in I$, $\delta_i(J) \subseteq J$. If $\{0\}$ and R are the only Δ -invariant ideals of R , then R is said to be Δ -simple. We say that Δ is commutative if for all $i, j \in I$, the relation $\delta_i \circ \delta_j = \delta_j \circ \delta_i$ holds. Furthermore, Δ is said to be a set of left (right) kernel derivations on R if for each $i \in I$, δ_i is left (right) R_Δ -linear. Take $f, g \in \mathbb{N}^{(I)}$ and $m \in \mathbb{N}$ such that the support of f is contained in $\{1, \dots, m\}$. Define $\delta^f : R \rightarrow R$ by $\delta^f(r) = (\delta_1^{f(1)} \circ \dots \circ \delta_m^{f(m)})(r)$, for $r \in R$. If $f \geq g$, then put $\binom{f}{g} = \prod_{i \in I} \binom{f(i)}{g(i)}$. In that case, define $\pi_g^f : R \rightarrow R$ by $\pi_g^f(r) = \binom{f}{g} \delta^{f-g}(r)$, for $r \in R$.

Remark 25. We will use the convention that $\binom{n}{m} = 0$ if $m > n$.

Proposition 26. *With the above notation the following assertions hold:*

- (a) π satisfies (M1), (M1') and (M2);
- (b) π satisfies (M3) $\Leftrightarrow \pi$ is commutative $\Leftrightarrow \Delta$ is commutative;
- (c) Suppose that Δ is commutative. Then π satisfies (M4) \Leftrightarrow each δ_i , $i \in I$, is a derivation on R ;
- (d) $R^{\mathbb{N}^{(I)}} = R_\Delta$;
- (e) π satisfies (M4') (or (M4'')) $\Leftrightarrow \Delta$ is a set of left (or right) kernel derivations on R ;
- (f) Suppose that Δ is commutative. Then R is Δ -simple $\Leftrightarrow R$ is $\mathbb{N}^{(I)}$ -simple.

Proof. (a) and (b) follow immediately from the definition of π .

(c) Suppose that (M4) holds. Take $i \in I$ and define $f_i \in \mathbb{N}^{(I)}$ by the relations $f_i(j) = 1$, if $j = i$, and $f_i(j) = 0$, otherwise. For any $r, s \in R$ we get

$$\delta_i(rs) = \binom{f_i}{0} \delta^{f_i}(rs) = \pi_0^{f_i}(rs) = \pi_0^{f_i}(r) \pi_0^0(s) + \pi_{f_i}^{f_i}(r) \pi_0^{f_i}(s) = \delta_i(r)s + r\delta_i(s),$$

which implies that δ_i is a derivation on R .

Now suppose that for each $i \in I$, δ_i is a derivation on R . We first prove that, for any $i \in I$, $r, s \in R$ and $n \in \mathbb{N}$, the equation

$$\delta_i^n(rs) = \sum_{k=0}^n \binom{n}{k} \delta_i^{n-k}(r) \delta_i^k(s) \quad (5)$$

holds.

Take $i \in I$ and $r, s \in R$. Equation (5) clearly holds if $n = 0$ or $n = 1$. We will prove the general case by induction. To this end, suppose that Equation (5) holds for n . For clarity,

we will write δ instead of δ_i . We now get

$$\begin{aligned}
\delta^{n+1}(rs) &= \delta(\delta^n(rs)) = \delta\left(\sum_{k=0}^n \binom{n}{k} \delta^{n-k}(r) \delta^k(s)\right) \\
&= \left(\sum_{k=0}^n \binom{n}{k} (\delta^{n+1-k}(r) \delta^k(s) + \delta^{n-k}(r) \delta^{k+1}(s))\right) \\
&= \sum_{k=0}^n \binom{n}{k} \delta^{n+1-k}(r) \delta^k(s) + \sum_{k=0}^n \binom{n}{k} \delta^{n-k}(r) \delta^{k+1}(s) \\
&= \sum_{k=0}^n \binom{n}{k} \delta^{n+1-k}(r) \delta^k(s) + \sum_{k=1}^{n+1} \binom{n}{k-1} \delta^{n+1-k}(r) \delta^k(s) \\
&= \sum_{k=0}^{n+1} \left(\binom{n}{k} + \binom{n}{k-1}\right) \delta^{n+1-k}(r) \delta^k(s) = \sum_{k=0}^{n+1} \binom{n+1}{k} \delta^{n+1-k}(r) \delta^k(s)
\end{aligned}$$

and by induction we conclude that Equation (5) holds for any $n \in \mathbb{N}$.

We will first show (M4) in a special case. Suppose that $f(i) = n$, $g(i) = m \leq n$ and $f(j) = g(j) = 0$ if $i \neq j$. Then we get

$$\begin{aligned}
\pi_g^f(rs) &= \binom{n}{m} \delta^{n-m}(rs) = \binom{n}{m} \sum_{k=0}^{n-m} \binom{n-m}{k} \delta^{n-m-k}(r) \delta^k(s) \\
&= \sum_{k=0}^{n-m} \binom{n}{m} \binom{n-m}{k} \delta^{n-m-k}(r) \delta^k(s) = \sum_{k=m}^n \binom{n}{m} \binom{n-m}{k-m} \delta^{n-k}(r) \delta^{k-m}(s) \\
&= \sum_{k=m}^n \binom{n}{m} \binom{n-m}{n-k} \delta^{n-k}(r) \delta^{k-m}(s) = \sum_{k=m}^n \binom{n}{n-k} \binom{k}{m} \delta^{n-k}(r) \delta^{k-m}(s) \\
&= \sum_{k=m}^n \binom{n}{k} \binom{k}{m} \delta^{n-k}(r) \delta^{k-m}(s) = \sum_{k=m}^n \binom{n}{k} \delta^{n-k}(r) \binom{k}{m} \delta^{k-m}(s) = \sum_{h \in \mathbb{N}^{(I)}} \pi_h^f(r) \pi_g^h(s).
\end{aligned}$$

In the above calculation we have used an identity for binomial coefficients that will be generalized in Proposition 34.

We have proven that (M4) holds for π_g^f if the support of f only contains one element. The general case can be proven by induction on the size of the support. To this end, suppose that we have proven (M4) if the functions involved have support of size at most n . Let f be a function with a support of size $n+1$ and let $g \leq f$. We can write $f = f' + \bar{f}$ and $g = g' + \bar{g}$, where the support of f' and \bar{f} are disjoint, the support of \bar{f} has size n , the support of f' has size 1 and $g' \leq f'$ and $\bar{g} \leq \bar{f}$. Then $\pi_g^f = \pi_{g'}^{f'} \circ \pi_{\bar{g}}^{\bar{f}}$. By the induction

hypothesis we have

$$\begin{aligned} \pi_g^f(rs) &= \pi_{g'}^{f'} \circ \pi_{\bar{g}}^{\bar{f}}(rs) = \pi_{g'}^{f'} \left(\sum_{\bar{h}} \pi_{\bar{h}}^{\bar{f}}(r) \pi_{\bar{g}}^{\bar{h}}(s) \right) = \sum_{\bar{h}} \sum_{h'} \pi_{h'}^{f'}(\pi_{\bar{h}}^{\bar{f}}(r)) \pi_{g'}^{h'}(\pi_{\bar{g}}^{\bar{h}}(s)) \\ &= \sum_{g \leq h' + \bar{h} \leq f} \pi_{h'}^{f'}(\pi_{\bar{h}}^{\bar{f}}(r)) \pi_{g'}^{h'}(\pi_{\bar{g}}^{\bar{h}}(s)) = \sum_{g \leq h \leq f} \pi_h^f(r) \pi_g^h(s) = \sum_h \pi_h^f(r) \pi_g^h(s). \end{aligned}$$

(d) First we show the inclusion $R^{\mathbb{N}^{(I)}} \supseteq R_\Delta$. Take $r \in R_\Delta$. Then, for each $i \in I$, the equality $\delta_i(r) = 0$ holds. Take $f, g \in \mathbb{N}^{(I)}$ such that $f \geq g$. Then, from the definition of π_g^f , it follows that $\pi_g^f(r) = 0$. Thus, $r \in R^{\mathbb{N}^{(I)}}$. Now we show the reversed inclusion $R^{\mathbb{N}^{(I)}} \subseteq R_\Delta$. Take $r \in R^{\mathbb{N}^{(I)}}$ and $i \in I$. Define $f_i \in \mathbb{N}^{(I)}$ by the relations $f_i(j) = 1$, if $j = i$, and $f_i(j) = 0$, otherwise. Since $r \in R^{\mathbb{N}^{(I)}}$, we get, in particular, that $0 = \pi_0^{f_i}(r) = \binom{f_i}{0} \delta^{f_i}(r) = \delta_i(r)$. Thus $r \in \ker(\delta_i)$. Hence $r \in R_\Delta$.

(e) This follows immediately from (d).

(f) If R is $\mathbb{N}^{(I)}$ -simple, then clearly R is also Δ -simple. Now suppose that R is not $\mathbb{N}^{(I)}$ -simple. We want to show that R is not Δ -simple. Let J be a non-zero proper $\mathbb{N}^{(I)}$ -invariant ideal of R . Take $i \in I$. Define $f_i \in \mathbb{N}^{(I)}$ by the relations $f_i(j) = 1$, if $j = i$, and $f_i(j) = 0$, otherwise. For any $r \in J$ we get $\delta_i(r) = \binom{f_i}{0} \delta^{f_i}(r) \in J$. This shows that J is Δ -invariant and hence R is not Δ -simple. \square

Definition 27. Now we will define a well-order \preceq on $\mathbb{N}^{(I)}$ which extends \leq . To this end, take $f, g \in \mathbb{N}^{(I)}$ with $f \neq g$ and put $|f| = \sum_{i \in I} f(i)$. Case 1: if $|f| > |g|$, then put $f \succ g$. Case 2: if $|f| < |g|$, then put $f \prec g$. Case 3: Suppose that $|f| = |g|$. Then there is $j \in I$ such that $f(i) = g(i)$, for $i \gg j$, but $f(j) \neq g(j)$. If $f(j) > g(j)$, then put $f \succ g$. If $f(j) < g(j)$, then put $f \prec g$. We will refer to \preceq as the graded lexicographical ordering on $\mathbb{N}^{(I)}$.

Remark 28. For each $i \in I$, the map $\delta_i : R \rightarrow R$ may be extended to an additive map $\tilde{\delta}_i : S \rightarrow S$ by defining $\tilde{\delta}_i(\sum_{f \in \mathbb{N}^{(I)}} r_f x^f) = \sum_{f \in \mathbb{N}^{(I)}} \delta_i(r_f) x^f$.

Theorem 29. If Δ is a commutative set of left (right) kernel derivations on R , then the differential monoid ring $S = R[\mathbb{N}^{(I)}; \pi]$ is simple if and only if R is Δ -simple and $Z(S)$ is a field.

Proof. Put $G = \mathbb{N}^{(I)}$ and equip G with the graded lexicographical ordering. All ideals J of S are G -invariant. Indeed, take $s = \sum_{f \in G} r_f x^f \in J$ and $i \in I$. Then $J \ni x^{f_i} s - s x^{f_i} = \sum_{f \in G} \delta_i(r_f) x^f = \tilde{\delta}_i(s)$. By induction, $\pi_g^f(s) \in J$ for all $f, g \in G$ with $f \geq g$. This implies that G -simplicity of S is equivalent to simplicity of S . Also, $Z(S)^G = Z(S)$. In fact, given $s = \sum_{f \in G} r_f x^f \in Z(S)$ and $i \in I$, we have $0 = x^{f_i} s - s x^{f_i} = \sum_{f \in G} \delta_i(r_f) x^f = \tilde{\delta}_i(s)$. The claim now follows immediately from Theorem 22. \square

Remark 30. Theorem 29 generalizes [14, Theorem 4.15] both to the case of several variables and to the non-associative situation. In fact, if R is associative, I is the finite set $\{1, \dots, n\}$ and each δ_i , for $i \in I$, is a derivation on R , then $R[\mathbb{N}^{(I)}; \pi]$ coincides with the

differential polynomial ring $R[x_1, \dots, x_n; \delta_1, \dots, \delta_n]$. Recall that the latter ring is defined to be the ordinary ring of polynomials with variables x_1, \dots, x_n with ordinary addition and a multiplication defined by the distributive and associative laws subject to the relations $x_i r = \delta_i(r) + r x_i$ and $x_i x_j = x_j x_i$, for all $i, j \in I$ (for more details concerning this construction, see e.g. [9] or [18]).

Now we wish to proceed to prove generalizations (see Theorem 38 and Theorem 40) of results of Voskoglou [18] and Malm [9] for non-associative differential polynomial rings in, possibly, infinitely many variables. Therefore, for the rest of this section, we assume that Δ is a commutative set of derivations on a (possibly non-associative) ring R and we put $S = R[\mathbb{N}^{(I)}; \pi]$. To this end, we first prove a few useful propositions.

Proposition 31. *If $f, g, h \in \mathbb{N}^{(I)}$ are chosen so that $f = g + h$, then, for every $l \in \mathbb{N}^{(I)}$, we get that $\sum_{p+q=l} \binom{g}{p} \binom{h}{q} = \binom{f}{l}$.*

Proof. Suppose that the cardinality of the support of f is n . We may suppose that the support of f is $\{1, \dots, n\}$. Since $f = g + h$, we may suppose that the supports of g and h also are contained in $\{1, \dots, n\}$. The claimed equality coincides with Vandermonde's identity in the case $n = 1$. The general case now follows directly:

$$\sum_{p+q=l} \binom{g}{p} \binom{h}{q} = \sum_{p+q=l} \prod_{i=1}^n \binom{g(i)}{p(i)} \binom{h(i)}{q(i)} = \prod_{i=1}^n \sum_{p(i)+q(i)=l(i)} \binom{g(i)}{p(i)} \binom{h(i)}{q(i)} = \prod_{i=1}^n \binom{f(i)}{l(i)} = \binom{f}{l}.$$

□

Definition 32. If $f \in \mathbb{N}^{(I)}$, then we put $|f| = \sum_{i \in \mathbb{N}} f(i)$ and $(-1)^f = (-1)^{|f|}$.

Proposition 33. *If $r \in R$ and $f \in \mathbb{N}^{(I)}$, then $rx^f = \sum_{g \leq f} (-1)^g \binom{f}{g} x^{f-g} \delta^g(r)$.*

Proof. Suppose that the cardinality of the support of f is n . We will show the claim by induction over n .

Base case: $n = 1$. This has already been proven in [12, Proposition 19].

Induction step: Suppose that $n > 1$ and that the claim holds for all elements in $\mathbb{N}^{(I)}$ such that the cardinality of the support of the element is less than n . Suppose that $f = g + h$ for some $g, h \in \mathbb{N}^{(I)}$ chosen so that the cardinalities of the supports of g and h are less than n . From the induction hypothesis and Proposition 31, we now get that

$$\begin{aligned} rx^f &= rx^{g+h} = (rx^g)x^h = \sum_{p \leq g} (-1)^p \binom{g}{p} x^{g-p} \delta^p(r) x^h \\ &= \sum_{p \leq g, q \leq h} (-1)^p (-1)^q \binom{g}{p} \binom{h}{q} x^{g-p} x^{h-q} (\delta^p \circ \delta^q)(r) \\ &= \sum_{p \leq g, q \leq h} (-1)^{p+q} \binom{g}{p} \binom{h}{q} x^{f-(p+q)} \delta^{p+q}(r) \\ &= \sum_{l \leq f} \sum_{p+q=l} (-1)^l \binom{g}{p} \binom{h}{q} x^{f-l} \delta^l(r) = \sum_{l \leq f} (-1)^l \binom{f}{l} x^{f-l} \delta^l(r). \end{aligned}$$

□

Proposition 34. *If $f, g, h \in \mathbb{N}^{(I)}$, then $\binom{f}{g} \binom{f-g}{h} = \binom{f}{h} \binom{f-h}{g}$.*

Proof. Suppose first that f, g and h have support in the same one element set. We can thus suppose that $f, g, h \in \mathbb{N}$. Suppose that $\binom{f}{g} \binom{f-g}{h} \neq 0$. Then there are $l, m \in \mathbb{N}$ such that $f = g + l$ and $f - g = h + m$. Therefore $f - h = g + m$ and hence $\binom{f}{h} \binom{f-h}{g} \neq 0$. On the other hand, suppose that $\binom{f}{h} \binom{f-h}{g} \neq 0$. Then there are $p, q \in \mathbb{N}$ such that $f = h + p$ and $f - h = g + q$. Therefore $f - g = h + q$ and hence $\binom{f}{g} \binom{f-g}{h} \neq 0$. To show the equality in this case it thus suffices to consider the case when $\binom{f}{g} \binom{f-g}{h} \neq 0 \neq \binom{f}{h} \binom{f-h}{g}$. Now, in that case, it is easy to see that $\binom{f}{g} \binom{f-g}{h} = \frac{f!}{g!h!(f-g-h)!} = \binom{f}{h} \binom{f-h}{g}$. In the general case we can suppose that f, g and h have support in $\{1, \dots, n\}$. From the previous case, we now get that

$$\binom{f}{g} \binom{f-g}{h} = \prod_{i=1}^n \binom{f(i)}{g(i)} \binom{f(i)-g(i)}{h(i)} = \prod_{i=1}^n \binom{f(i)}{h(i)} \binom{f(i)-h(i)}{g(i)} = \binom{f}{h} \binom{f-h}{g}.$$

□

Proposition 35. *If $a \in Z(S)$, then, for each $g \in \mathbb{N}^{(I)}$, we get that $\sum_{f \geq g} x^{f-g} \binom{f}{g} a_f \in Z(S)$.*

Proof. Put $b = \sum_{f \geq g} x^{f-g} \binom{f}{g} a_f$. We will check the conditions in Corollary 14. Since a commutes with every x^h , for $h \in \mathbb{N}^{(I)}$, we get that $a_f \in R_\Delta$, for $f \in \mathbb{N}^{(I)}$. Next we show that b commutes with every $r \in R$. Since $a_f \in R_\Delta$, for $f \in \mathbb{N}^{(I)}$, we can write $a = \sum_{f \in \mathbb{N}^{(I)}} x^f a_f$. Since $ar = ra$, we may use Proposition 33 to conclude that

$$\begin{aligned} \sum_f x^f a_f r &= ar = ra = \sum_f r x^f a_f = \sum_{f,g} (-1)^g \binom{f}{g} x^{f-g} \delta^g(r) a_f \\ &= \sum_{f,h} (-1)^{f-h} \binom{f}{f-h} x^h \delta^{f-h}(r) a_f, \end{aligned}$$

where we in the last sum have put $f - g = h$. Hence, for each $h \in \mathbb{N}^{(I)}$, we get that

$$a_h r = \sum_f (-1)^{f-h} \binom{f}{f-h} \delta^{f-h}(r) a_f. \quad (6)$$

Thus

$$rb = \sum_f r x^{f-g} \binom{f}{g} a_f = \sum_{f,h} x^{f-g-h} (-1)^h \binom{f-g}{h} \delta^h(r) \binom{f}{g} a_f.$$

If we now put $v = f - h$ in the last sum and use Proposition 34, then we get

$$\sum_{f,v} x^{v-g} \binom{f}{g} \binom{f-g}{f-v} (-1)^{f-v} \delta^{f-v}(r) a_f = \sum_{f,v} x^{v-g} \binom{f}{f-v} \binom{v}{g} (-1)^{f-v} \delta^{f-v}(r) a_f.$$

Finally, using (6), the last sum equals $\sum_v x^{v-g} \binom{v}{g} a_v r = br$.

Finally, we show that b associates with all elements in R . From the relations $(R, R, a) = \{0\}$ and $(a, R, R) = \{0\}$ it follows that for each $f \in \mathbb{N}^{(I)}$, $(R, R, a_f) = (a_f, R, R) = \{0\}$. Hence we get that $(R, R, b_f) = (b_f, R, R) = \{0\}$, for $f \in \mathbb{N}^{(I)}$. Thus, $(b, R, R) = (R, R, b) = \{0\}$. Since $[b, R] = \{0\}$, we automatically get that $(R, b, R) = \{0\}$, using an argument similar to the proof of (3) in Lemma 13. \square

Definition 36. Take $a = \sum_{f \in \mathbb{N}^{(I)}} a_f x^f \in S$. Recall from Definition 21 that $\deg(a)$ is the largest $f \in \mathbb{N}^{(I)}$, with respect to \preceq , such that $a_f \neq 0$. If a is non-zero and $a_{\deg(a)} = 1$, then we say that a is monic. We say that a is constant if $\deg(a) = 0$. We say that a is linear if a is non-constant, $a_0 = 0$ and the set $\text{supp}(\deg(a))$ contains exactly one element. Note that if $f \in \mathbb{N}^{(I)}$ has $\text{supp}(f) = \{i\}$ for some $i \in I$, then, for every $r \in R$, the relation $x^f r = \delta_i(r) + r x^f$ holds.

Proposition 37. Suppose that R is Δ -simple and that $\text{char}(R) = 0$. Put $F = Z(R)_\Delta$.

- (a) If $Z(S)$ only contains constants, then $Z(S) = F$.
- (b) If $Z(S)$ contains non-constants, then $Z(S)$ contains a unique, up to addition by elements from F , non-constant monic a of least graded lexicographical degree. In that case, $Z(S)$ is not a field and there is $c \in R_\Delta$, $m \in \mathbb{N}$ and $c_i \in F$, for $i \in \{1, \dots, m-1\}$, such that $a = x_m + \sum_{i=1}^{m-1} c_i x_i - c$ and, hence, $\delta_c = \delta_m + \sum_{i=0}^{m-1} c_i \delta_i$.

Proof. (a) This is clear.

(b) Suppose that $Z(S)$ is not contained in R . Let a be non-constant in $Z(S)$ of least graded lexicographical degree $h \in \mathbb{N}^{(I)}$. For each $g \in \mathbb{N}^{(I)}$, let $b_g = \sum_{f \geq g} x^{f-g} \binom{f}{g} a_f$. From Proposition 35 and the definition of a it follows that if $g > 0$, then $a_g = b_g \in Z(S)$. Thus, if $g > 0$, then $a_g \in F$. In particular, $a_h \in F \setminus \{0\}$. We can thus, from now on, assume that $a_h = 1$ so that a is monic. We claim that $a - a_0$ is linear. If we assume that the claim holds, then, by the definition of the graded lexicographical ordering, there is $c \in R_\Delta$, $m \in \mathbb{N}$ and $c_i \in F$, for $i \in \{1, \dots, m-1\}$, such that $a = x_m + \sum_{i=1}^{m-1} c_i x_i - c$. From the relations $ar = ra$, for $r \in R$, it follows that $\delta_c = \delta_m + \sum_{i=0}^{m-1} c_i \delta_i$. Now we show the claim. Seeking a contradiction, suppose that $\text{supp}(\deg(a))$ contains more than one element. Then there is $g \in \mathbb{N}^{(I)}$ such that $h > g > 0$. Since $\text{char}(R) = 0$, we get that $\binom{h}{g} \in F \setminus \{0\}$ and thus $0 < \deg(b_g) < \deg(a)$ which contradicts the choice of a . \square

Theorem 38. If $\text{char}(R) = 0$ and we put $F = Z(R)_\Delta$, then S is simple if and only if R is Δ -simple and no non-trivial finite F -linear combination of elements from Δ is an inner derivation on R defined by an element from R_Δ .

Proof. The "if" statement follows from Theorem 29 and Proposition 37. Now we show the "only if" statement. Suppose that S is simple. By [12, Proposition 9] we know that $Z(S)$ is a field. Suppose that there are $c_1, \dots, c_n \in F$, not all of them equal to zero, such that $c_1 \delta_1 + \dots + c_n \delta_n = \delta_c$. Take $c \in R_\Delta$ and consider the element $p = c_1 x_1 + \dots + c_n x_n - c$. Then it is clear that $p x_i = x_i p$ for all $i \in I$. Now take $r \in R$. Then $p r = r p + r c + c_1 \delta_1(r) + \dots + c_n \delta_n(r) - c r = r p + c_1 \delta_1(r) + \dots + c_n \delta_n(r) - \delta_c(r) = r p$. Therefore $p \in Z(S)$. But this is a contradiction since $Z(S)$ is a field and p is non-constant. \square

In the proof of the next proposition, we will use the following notation. Let $N = \{-\infty\} \cup \mathbb{N}$. Let p be a prime number. We will formally write $p^{-\infty} = 0$. Let $N^{(I)}$ denote the set of functions $f : I \rightarrow N$ with the property that $f(i) = -\infty$ for all but finitely many $i \in I$. Given $f \in N^{(I)}$, let $p^f \in \mathbb{N}^{(I)}$ be defined by $(p^f)(i) = p^{f(i)}$, for $i \in I$.

Proposition 39. *Suppose that R is Δ -simple and that $\text{char}(R) = p$. Put $F = Z(R)_\Delta$.*

- (a) *If $Z(S)$ only contains constants, then $Z(S) = F$.*
- (b) *If $Z(S)$ contains non-constants, then $Z(S)$ contains a unique, up to addition by elements from F , non-constant monic a of least graded lexicographical degree. In that case, if for every $i \in \{1, \dots, n\}$, the maps $\delta_i^{p^j}$, for $j \in \{0, 1, 2, \dots\}$, are F -linearly independent, then there is $c \in R_\Delta$, $m, n \in \mathbb{N}$ and $c_{ij} \in F$, for $1 \leq i \leq m$ and $0 \leq j \leq n$, such that $c_{mn} = 1$ and $a = \sum_{i=1}^m \sum_{j=0}^n c_{ij} x_i^{p^j} - c$. Thus, $\delta_c = \sum_{i=1}^m \sum_{j=0}^n c_{ij} \delta_i^{p^j}$.*

Proof. (a) This is clear.

(b) Suppose that $Z(S)$ is not contained in R . Let a be non-constant polynomial in $Z(S)$ of least graded lexicographical degree $h \in \mathbb{N}^{(I)}$. For each $g \in \mathbb{N}^{(I)}$, let $b_g = \sum_{f \geq g} X^{f-g} \binom{f}{g} a_f$. From Proposition 35 and the definition of a it follows that if $g > 0$, then $a_g = b_g \in Z(S)$. Thus, if $g > 0$, then $a_g \in F$. In particular, $a_h \in F \setminus \{0\}$. We can thus, from now on, assume that $a_h = 1$ so that a is monic. Take $f \in \mathbb{N}^{(I)}$ such that $a_f \neq 0$. From Proposition 35 and Lucas' theorem it follows that $f = p^t$ for some $t \in N^{(I)}$. We say that t is *more than singly supported* if there are $i, j \in I$, with $i < j$, such that $t(i) \neq -\infty \neq t(j)$. Seeking a contradiction, suppose that the set $Z = \{t \in N^{(I)} \mid t \text{ is more than singly supported and } a_{p^t} \neq 0\}$ is non-empty. To this end, let s denote the unique element from Z of least graded lexicographical degree. Thus, there are $i, j \in I$, with $i < j$, such that $s(i) \neq -\infty \neq s(j)$ and $s(k) = -\infty$, for $k > j$. Given $n \in \mathbb{N}$, define $s_n \in N^{(I)}$ by the relations $s_n(l) = s(l)$, for $l < j$, and $s_n(j) = n$, otherwise. In the relation $ar - ra = 0$, considering the coefficient of $x^{p^{s_n(j)}}$ yields the relation $\sum_{n=0}^j a_{p^{s_n}} \delta_n^{p^{s_n(j)}} = 0$ which is a contradiction since $a_{p^{s_j}} \neq 0$. Hence $Z = \emptyset$ and the desired result follows. \square

Theorem 40. *If $\text{char}(R) = p$ and we put $F = Z(R)_\Delta$, then S is simple if and only if R is Δ -simple and no non-trivial finite F -linear combination of $\delta_i^{p^j}$, for $i \in I$ and $j \in \mathbb{N}$, is an inner derivation on R defined by an element from R_Δ .*

Proof. The "if" statement follows from Theorem 29 and Proposition 39. Now we show the "only if" statement. Suppose that S is simple. By [12, Proposition 9], we know that $Z(S)$ is a field. Seeking a contradiction, suppose that there are $c_{ij} \in F$, not all of them equal to zero, and $c \in R_0$ such that $\sum_{ij} c_{ij} \delta_i^{p^j} = \delta_c$. Consider now the element $a = \sum_{ij} c_{ij} x_i^{p^j} - c$. Then it is clear that $ax_i = x_i a$ for all $i \in I$. Take $r \in R$. Then $ar - ra = rc + c_1 \delta_1(r) + \dots + c_n \delta_n(r) - cr = c_1 \delta_1(r) + \dots + c_n \delta_n(r) + rc - cr = c_1 \delta_1(r) + \dots + c_n \delta_n(r) - \delta_c(r) = 0$. Therefore $a \in Z(S)$. But this is a contradiction since $Z(S)$ is a field and a is non-constant. \square

5. FINITE MONOIDS

In this section, we study non-associative monoid Ore extensions for some finite commutative monoids G . Throughout this section, R denotes a unital non-associative algebra over a field F such that $F \subseteq R$ and $1 = 1_R \in F$. We will also assume that π is a unital G -derivation on R . Recall that this means that the following axioms from Definition 3 hold:

- (M2) if $a \geq b$, then $\pi_b^a(1)$ equals the Kronecker delta function $\delta_{a,b}$;
- (M3) if $a + b \geq c$, then $\sum_{d+e=c} \pi_d^a \circ \pi_e^b = \pi_c^{a+b}$;
- (M1') if $a = b$, then $\pi_b^a = \text{id}_R$.

Throughout this section, we let $S = R[G; \pi]$ denote the corresponding differential monoid ring.

Example 41. Suppose that $G = \{0\}$ is the unique monoid with one element. Then, since $\pi_0^0 = \text{id}_R$, we get that $S = R$.

Example 42. Suppose that $G = \{0, g\}$ is the unique monoid with two elements forming a group. Then $g + g = 0$. This implies that $0 \leq g \leq 0$. Thus, \leq can not be extended to a well-order on G . From (M2), (M3) and (M1') it follows that $\pi_0^0 = \pi_g^g = \text{id}_R$, $\pi_g^0 = 0$, $(\pi_0^g)^2 = 0$ and $2\pi_0^g = 0$. We now consider two cases.

Case 1: $\text{char}(F) \neq 2$. Then $\pi_0^g = 0$ and $S = R + Rx^g$. This ring is isomorphic to the group ring $R[G]$, which in turn, is isomorphic to $R \times R$. Indeed, an explicit isomorphism is given by $R[G] \ni r + sg \mapsto (r - s, r + s) \in R \times R$.

Case 2: $\text{char}(F) = 2$. Put $N = \pi_0^g$. Then $N^2 = 0$, i.e. $\text{im}(N) \subseteq \ker(N)$. Then S equals the generalized group ring $R_N[G] = \{r + sg \mid r, s \in R\}$ where multiplication is given by $(r + sg)(r' + s'g) = rr' + ss' + sN(r') + (rs' + sr' + sN(s'))g$.

Example 43. Suppose that $G = \{0, g\}$ is the unique cyclic monoid with two elements not forming a group. Furthermore, suppose that $\pi_0^g \neq 0$. Then $g + g = g$. Then \leq is a well-order on G . From (M2), (M3) and (M1') it follows that $\pi_0^0 = \pi_g^g = \text{id}_R$, $\pi_0^g(1) = 0$, $\pi_0^g \circ \pi_0^g = \pi_0^g$, and $\text{char}(F) = 2$. The last conclusion follows from the equality

$$\text{id}_R = \pi_g^g = \pi_g^{g+g} = \pi_g^g \pi_0^g + \pi_0^g \pi_g^g + \pi_g^g \pi_g^g$$

which simplifies to

$$\text{id}_R = 2\pi_0^g + \text{id}_R.$$

Put $P = \pi_0^g$. We can write $R = \ker(P) \oplus V$ for some F -vector subspace V of R and notice that $R^G = \ker(P) = F$. Thus, it follows that π_0^g is both right and left R^G -linear. Hence π is strong. It is easy to check that $Z(S)^G = F$. By Theorem 22 we get that S is G -simple if and only if R is G -simple. From this it is easy to construct examples of simple non-associative differential monoid rings. Namely, if we let $Q : R \rightarrow F$ denote the projection, then we say that R has the Q -property if for any non-zero $r \in R$, there is $r' \in R$ such that $Q(rr') \neq 0$ or $Q(r'r) \neq 0$. It follows that if R has the Q -property, then S is simple. Notice that if R is graded by a group H with identity element e , and $R_e = F$, then it is easy to see that R has the Q -property if the grading is non-degenerate in the sense of [13, Definition 2].

Example 44. Suppose that G is the cyclic commutative monoid with three elements $\{0, g, h\}$ satisfying the relations $g + g = h$ and $g + h = h$. From (M3), with $a = b = g$ and $c = 0$, it follows that

$$\pi_0^h = \pi_0^g \circ \pi_0^g. \quad (7)$$

From (M1') and (M3), with $a = b = c = g$, we get that

$$\pi_g^h = 2\pi_0^g. \quad (8)$$

From (M1') and (M3), with $a = c = h$ and $b = g$, it follows that

$$\text{id}_R = \pi_g^h + \pi_0^g + \text{id}_R$$

which, in combination with (8), shows that

$$0 = 3\pi_0^g. \quad (9)$$

Finally, from (M3), with $a = b = c = h$, we get that

$$\text{id}_R = 2\pi_0^h + \pi_g^h \circ \pi_g^h + 2\pi_g^h + \text{id}_R$$

which, in combination with (7), (8) and (9), shows that $\pi_0^g = 0$. It now follows from (7) and (8) that $\pi_0^h = 0$ and $\pi_g^h = 0$. Thus, there are no non-trivial unital G -derivations on any ring R , for this monoid.

Example 45. Suppose that G is the commutative monoid with four elements $\{0, g, h, p\}$ with the relations $g + g = g$, $h + h = h$ and $g + h = g + p = h + p = p + p = p$. We notice that g and h are not comparable and thus $\pi_g^h = \pi_h^g = 0$. From (M2), (M3) and (M1'), we get that π satisfies the following relations:

$$\pi_0^a \circ \pi_0^b = \pi_0^{a+b}, \quad (10)$$

for $a, b \in G$,

$$2\pi_0^g = 2\pi_0^h = 0, \quad (11)$$

and

$$\pi_g^p = \pi_0^h, \quad \pi_h^p = \pi_0^g. \quad (12)$$

For example, the last two equalities follow from the calculations

$$\pi_g^p = \pi_g^{g+h} = \pi_g^g \circ \pi_0^h + \pi_0^g \circ \pi_g^h + \pi_g^g \circ \pi_g^h = \pi_0^h$$

and

$$\pi_h^p = \pi_h^{h+g} = \pi_h^h \circ \pi_0^g + \pi_0^h \circ \pi_h^g + \pi_h^h \circ \pi_h^g = \pi_0^g.$$

Now we consider two cases.

Case 1: $\text{char}(F) \neq 2$. Then, from (10), (11) and (12), it follows that $\pi_b^a = 0$ whenever $a > b$.

Case 2: $\text{char}(F) = 2$. Consider the two projections $\pi_0^g : R \rightarrow R$ and $\pi_0^h : R \rightarrow R$. Put $V' = \text{im}(\pi_0^g)$ and $W' = \text{im}(\pi_0^h)$. Let V and W be two vector subspaces of R such that $V \cap W = F$ and $R = V \oplus V' = W \oplus W'$. Then $R^G = \ker(\pi_0^g) \cap \ker(\pi_0^h) = V \cap W = F$ and it thus follows that π_b^a is both right and left R^G -linear, for all $a \geq b$. Hence π is strong. If $V \not\subseteq W$ and $W \not\subseteq V$, then it is easy to check that $Z(S)^G = F$, in which case Theorem 22 shows that S is G -simple if and only if R is (π_0^g, π_0^h) -simple.

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