Game Semantics for Probabilistic μ-Calculi

Matteo Mio
CWI – Amsterdam

Game Semantics for Modal μ-Calculus

Modal μ-Calculus $\Leftrightarrow$ **2-Player Parity Games**

Model M + Formula ϕ $\rightsquigarrow$ Game $\mathtt{G}(M, \phi)$

ϕ holds at M iff Player Verifier ($\Diamond$) wins $\mathtt{G}(M, \phi)$

Determinacy: Either Verifier ($\Diamond$) wins or Refuter ($\Box$) wins.

Game Semantics for Probabilistic Modal μ-Calculus

Prob. Modal μ-Calculus $\Leftrightarrow$ $2\frac{1}{2}$**-Player Parity Games**

Probabilistic Model M
$+$
Formula ϕ $\rightsquigarrow$ Game $\mathtt{G}(M, \phi)$

ϕ holds at M with probability p iff Player Verifier ($\Diamond$) wins $\mathtt{G}(M, \phi)$ with probability p

Determinacy: $\mathtt{Val}(\Diamond) = 1 - \mathtt{Val}(\Box)$

- $\mathtt{Val}(\Diamond)$ = probability of winning that Player $\Diamond$ can ensure,
- $\mathtt{Val}(\Box)$ = probability of winning that Player $\Box$ can ensure.

Problem:

Probabilistic Modal μ-Calculus is not sufficiently expressive:

- e.g., cannot encode probabilistic CTL (**pCTL**).

My PhD thesis work

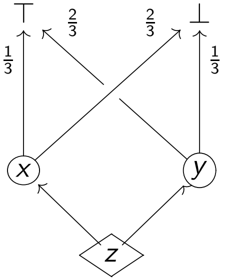
- Extensions of Prob. Modal μ-Calculus
  - Capable of encoding pCTL,
- with corresponding **Game Semantics**

New class of Games

- Generalizing $2\frac{1}{2}$-player Parity Games.

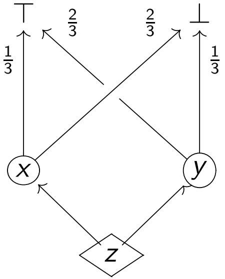
Ordinary $2\frac{1}{2}$-player games have three kinds of states:

- Verifier ($\Diamond$), Refuter ($\Box$) and Nature ($\bigcirc$)



Ordinary $2\frac{1}{2}$-player games have three kinds of states:

- Verifier ($\Diamond$), Refuter ($\Box$) and Nature ($\bigcirc$)



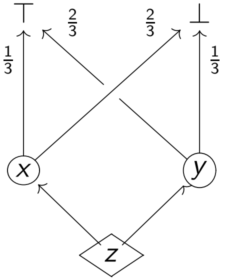
Possible outcomes:

$\frac{1}{3}$: $z \to x \to \top$ ✔

$\frac{2}{3}$: $z \to x \to \bot$ ✘

Ordinary $2\frac{1}{2}$-player games have three kinds of states:

- Verifier ($\Diamond$), Refuter ($\Box$) and Nature ($\bigcirc$)



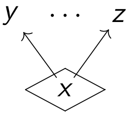
Possible outcomes:

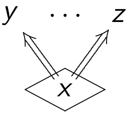
$\frac{1}{3}$: $z \to x \to \top$ ✔ $\frac{2}{3}$: $z \to x \to \bot$ ✘ OR: $\frac{2}{3}$: $z \to y \to \top$ ✔ $\frac{1}{3}$: $z \to y \to \bot$ ✘

Tree Games

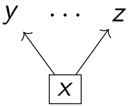
In Tree Games there are **FIVE** kinds of states:

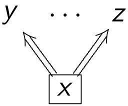
- Two types of Verifier ($\Diamond$) states,



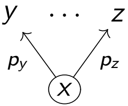


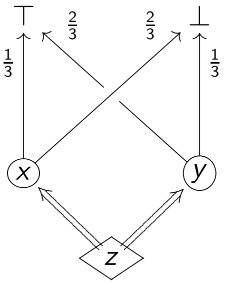
- Two types of Refuter ($\Box$) states,

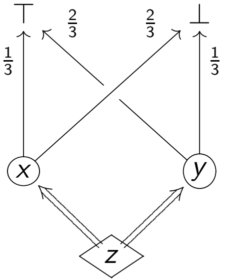




- Nature ($\bigcirc$)

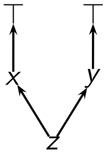


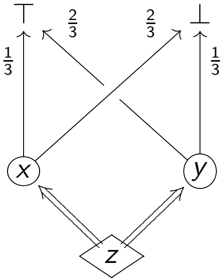

⊤
⊥
$\frac{1}{3}$
$\frac{2}{3}$
$\frac{2}{3}$
$\frac{1}{3}$
x
y
z



Possible outcomes:

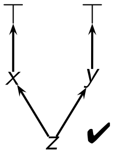
$\frac{2}{9}$:

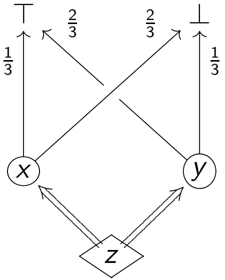




Possible outcomes:

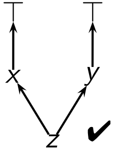
$\frac{2}{9}$:



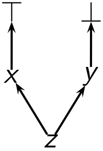


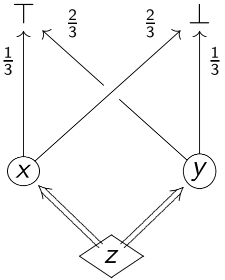
Possible outcomes:

$\frac{2}{9}$:



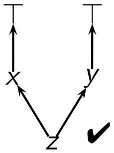
$\frac{1}{9}$:



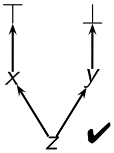


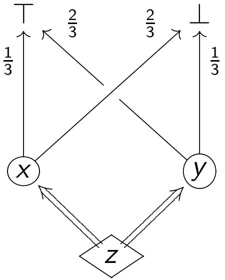
Possible outcomes:

$\frac{2}{9}$:

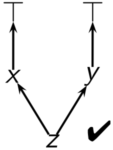


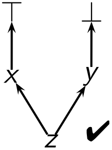
$\frac{1}{9}$:

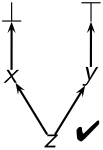


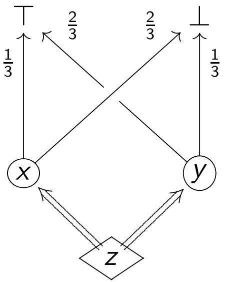


Possible outcomes:

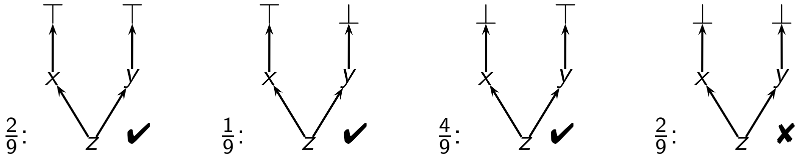
$\frac{2}{9}$: 

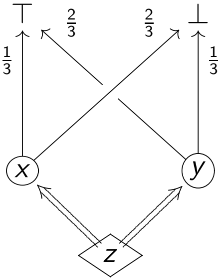
$\frac{1}{9}$: 

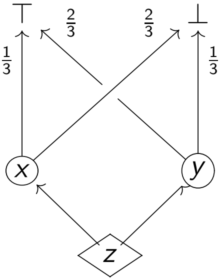
$\frac{4}{9}$: 



Possible outcomes:

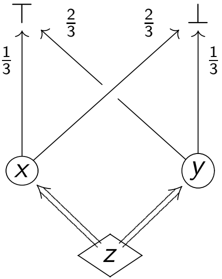


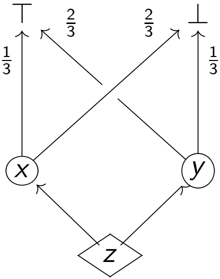




$$\mathtt{Val}(\Diamond) \quad = \quad \frac{1}{3} + \frac{2}{3} - \left(\frac{1}{3} \cdot \frac{2}{3}\right)$$
$$= \quad \frac{7}{9}$$

$$\mathtt{Val}(\Diamond) \quad = \quad \frac{1}{3} \sqcup \frac{2}{3}$$
$$\frac{6}{9}$$



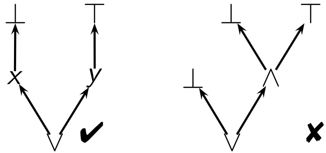


$$\texttt{Val}(\Diamond) = \frac{1}{3} + \frac{2}{3} - \left(\frac{1}{3} \cdot \frac{2}{3}\right) = \frac{7}{9}$$

$$\texttt{Val}(\Diamond) = \frac{1}{3} \sqcup \frac{2}{3} \quad \frac{6}{9}$$

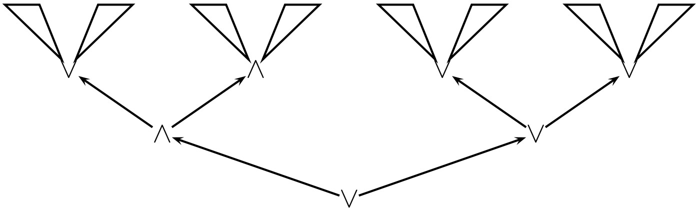
With probabilistic behavior: Parellel-OR $\neq$ Choice-OR

Outcomes of Tree Games are trees: **branching plays**.



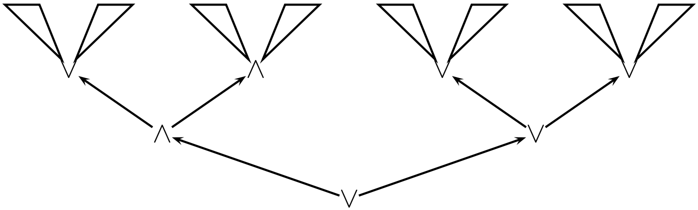
Outcomes of Tree Games are trees: **branching plays**.

How do we make sense of infinite branching plays?



Outcomes of Tree Games are trees: **branching plays**.

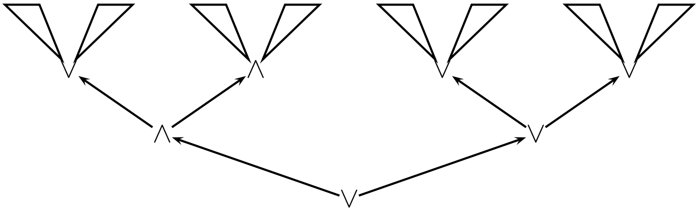
How do we make sense of infinite branching plays?



They can be regarded as infinitary Boolean formulas.

Outcomes of Tree Games are trees: **branching plays**.

How do we make sense of <u>infinite</u> branching plays?

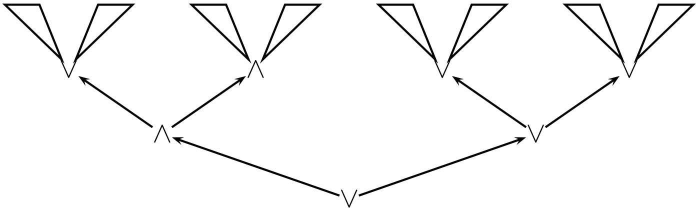


They can be regarded as infinitary Boolean formulas.

- Infinite branches are either wining for $\Diamond$ or for $\Box$.
  - Parity Condition.

Outcomes of Tree Games are trees: **branching plays**.

How do we make sense of infinite branching plays?



They can be regarded as infinitary Boolean formulas.

- Infinite branches are either wining for $\Diamond$ or for $\Box$.
    - Parity Condition.
- "Value" of the branching play is given by a Parity Game.

We call these games

Two-Player Stochastic meta- Parity Games

- The moves of $\Diamond$, $\Box$ and $\bigcirc$ determine a play
- The play is a tree (Branching play),
- itself interpreted as a Parity Game to determine winner.

Two-Player Stochastic meta-Parity games

- are games of **imperfect information**.
  - Each thread executes **ignoring** other parallel threads.
- Do not seem to be straightforwardly reducible to other known classes of games.

Outcomes are not finite or infinite **paths**,

- they are finite or infinite trees: **branching plays**.
- As usual, space $\mathtt{BP}$ of possible branching plays is **uncountable**
- Naturally a Polish space.

Outcomes are not finite or infinite **paths**,

- they are finite or infinite trees: **branching plays**.
- As usual, space $\mathtt{BP}$ of possible branching plays is **uncountable**
- Naturally a Polish space.

The set $\mathcal{W} \subseteq \mathtt{BP}$ of branching plays winning for $\Diamond$ is:

$$\mathbf{\Delta}^1_2 \text{ in the Projective Hierarchy.}$$

Outcomes are not finite or infinite **paths**,

- they are finite or infinite trees: **branching plays**.
- As usual, space $\mathtt{BP}$ of possible branching plays is **uncountable**
- Naturally a Polish space.

The set $\mathcal{W} \subseteq \mathtt{BP}$ of branching plays winning for $\Diamond$ is:

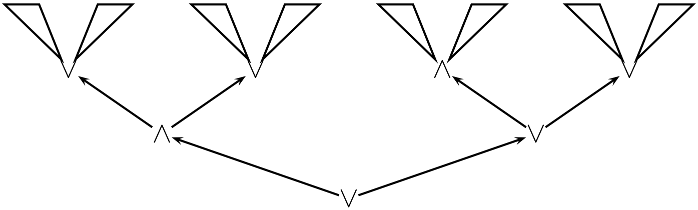
$$\mathbf{\Delta}^1_2 \text{ in the Projective Hierarchy.}$$

DANGER : It is consistent with ZFC that some $\mathbf{\Delta}^1_2$ set is not measurable!!!

The set $\mathcal{W} \subseteq \mathtt{BP}$ is defined as

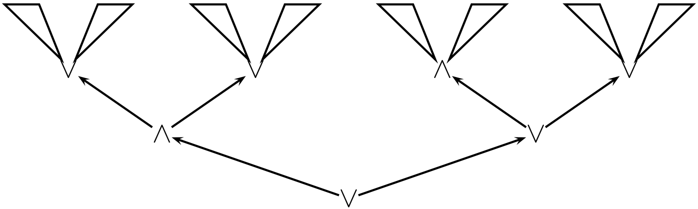
The set $\mathcal{W} \subseteq \mathtt{BP}$ is defined as

- those branching play T which are winnable by $\Diamond$.



The set $\mathcal{W} \subseteq \mathtt{BP}$ is defined as

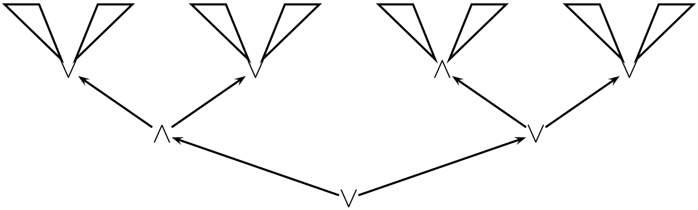
- those branching play T which are winnable by $\Diamond$.



- But hard to work with with this definition.

The set $\mathcal{W} \subseteq \mathtt{BP}$ is defined as

- those branching play T which are winnable by $\Diamond$.



- But hard to work with with this definition.

Inductive Characterization: $\mathcal{W} = \bigcup_{\alpha < \omega_1} \mathcal{W}^\alpha$

- $\mathcal{W}$ is the limit of a transfinite increasing chain.
- ω_1 is the smallest uncountable ordinal. $|\omega_1| = \aleph_1$.

Measure theory works well with **countable** chains:
$X_0 \subseteq X_1 \subseteq \dots X_n \dots$

$$\text{if } X = \bigcup_{\alpha<\omega} X^{\alpha} \qquad \text{then} \quad \mu(X) = \bigsqcup_{\alpha<\omega} \mu(X^{\alpha})$$

But not much can be said about transfinite chains:

$$\text{if } X = \bigcup_{\alpha<\omega_1} X^{\alpha} \qquad \text{then} \quad \mu(X) \geq \bigsqcup_{\alpha<\omega_1} \mu(X^{\alpha})$$

Set Theory: Martin's Axiom at the cardinal $\aleph_1$ ($\mathrm{MA}_{\aleph_1}$)

Set Theory: Martin's Axiom at the cardinal $\aleph_1$ ($\mathsf{MA}_{\aleph_1}$)

- $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1}$ is relatively consistent with ZFC.
- $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1} \vdash$ Every $\mathbf{\Delta}^1_2$ set is measurable.
- $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1} \vdash$ if $X = \bigcup_{\alpha<\omega_1} X^\alpha$ then $\mu(X) = \bigsqcup_{\alpha<\omega_1} \mu(X^\alpha)$,

Set Theory: Martin's Axiom at the cardinal $\aleph_1$ ($\mathsf{MA}_{\aleph_1}$)

- $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1}$ is relatively consistent with ZFC.
- $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1} \vdash$ Every $\boldsymbol{\Delta}^1_2$ set is measurable.
- $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1} \vdash$ if $X = \bigcup_{\alpha < \omega_1} X^\alpha$ then $\mu(X) = \bigsqcup_{\alpha < \omega_1} \mu(X^\alpha)$,

Good consequences for us:

- $\mathcal{W} \subseteq \mathtt{BP}$ is measurable, and
- Inductive characterization can be exploited in proofs.

Set Theory: Martin's Axiom at the cardinal $\aleph_1$ ($\mathsf{MA}_{\aleph_1}$)

- $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1}$ is relatively consistent with ZFC.
- $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1} \vdash$ Every $\mathbf{\Delta}^1_2$ set is measurable.
- $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1} \vdash$ if $X = \bigcup_{\alpha<\omega_1} X^\alpha$ then $\mu(X) = \bigsqcup_{\alpha<\omega_1} \mu(X^\alpha)$,

Good consequences for us:

- $\mathcal{W} \subseteq \mathtt{BP}$ is measurable, and
- Inductive characterization can be exploited in proofs.

Remark: $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1} \vdash \aleph_1 < 2^{\aleph_0}$,

Theorem ($\mathsf{ZFC} + \mathsf{MA}_{\aleph_1}$) [Thesis]: Determinacy.

$$\mathtt{Val}(\Diamond) = 1 - \mathtt{Val}(\Box)$$

Theorem $(\mathsf{ZFC} + \mathsf{MA}_{\aleph_1})$ **[Thesis]**: Determinacy.

$$\mathtt{Val}(\Diamond) = 1 - \mathtt{Val}(\Box)$$

Problem left open: is ZFC sufficient?

- Our theorems can not be refuted in ZFC, if ZFC is consistent.

THANK YOU