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# The Decidability of the Equivalence Problem for DOL-Systems*

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The language and sequence equivalence problem for DOL-systems is shown to be decidable. In an algebraic formulation the sequence equivalence problem for DOL-systems can be stated as follows: Given homomorphisms $h_1$ and $h_2$ on a free monoid $\Sigma^*$ and a word $\sigma$ from $\Sigma^*$, is $h_1^n(\sigma) = h_2^n(\sigma)$ for all $n \geqslant 0$?

## INTRODUCTION

The DOL sequence equivalence problem can be stated algebraically as follows. Given two homomorphisms $h_1$, $h_2$ on a free monoid $\Sigma^*$ and a word $\sigma$ in $\Sigma^*$, is $h_1^n(\sigma) = h_2^n(\sigma)$ for all $n \geqslant 0$? This paper shows that this problem is decidable. The problem originated in Lindenmayer systems which are mathematical models of cellular development. In that context it can be restated as the problem of the developmental equivalence of two genetic encodings in filamental organisms developing deterministically without interaction. The Lindenmayer systems without interaction (OL-systems) were introduced in Lindenmayer (1971) and the equivalence problem for them was posed shortly afterwards (Problem Book, 1973). Its undecidability for nondeterministic OL-systems has been shown (e.g., Salomaa, 1973). The same question for deterministic OL-systems (DOL-systems) was conjectured to be decidable but remained open. Some partial results were obtained in Paz and Salomaa (1973), Johansen and Meiling (1974), Ehrenfeucht and Rozenberg (1974), Nielsen (1974), Culik (1975), Valiant (1975), and Karhumäki (1976). Our full solution is based on the results and methods shown in Culik (1975). A part of these results, namely, the decidability of the equivalence problem for smooth DOL-systems, appeared independently and using different terminology in Valiant (1975).

Now, we explain intuitively the basic ideas of our approach. The technical terms which are not fully explained in the introduction are enclosed in quotation marks on first use.

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We start by showing that, without loss of generality, the testing for equivalence may be restricted to "normal" systems. The essence of this paper is to show that every pair of equivalent normal systems has "bounded balance." It has been shown in Culik (1975) that the equivalence problem is decidable for each family of DOL-systems in which the equivalence implies bounded balance.

Neglecting many technical details, we will now informally describe the principal ideas of the proof that for normal systems the equivalence implies bounded balance. In Culik (1975) it has already been shown that "simple" systems have bounded balance. A normal system is simple iff it has no "subsystem" in the sense of general algebra. If a system has a subsystem, then the underlying set of the subsystem is called a "subalphabet."

For two equivalent systems which are not simple we find a common subalphabet and show that either all substrings of the language generated by the systems which are entirely in this subalphabet are "short" (such a subalphabet is called "limited") or the two systems "induced" by this subalphabet are equivalent. A second pair of normal systems is obtained by "removing" the subsystem (i.e., by omitting the symbols from the common subalphabet). As before, these "remainder" systems are equivalent because the original systems are equivalent. Since both the subsystem and the remainder system are systems over a smaller alphabet we can use the boundedness as an induction hypothesis. The base of the induction deals (essentially) with systems over one letter, so the claim is easy to verify. This allows us to assume that the remainder pair and (in the case of a subalphabet which is not limited) also the induced pair have bounded balance. As the case of limited subalphabets causes no problem, this allows us to construct a bound on the balance for the original pair.

Some of the more important technical details which were omitted above are as follows. In every step of the induction we have to consider the nonpropagating systems and another singular case separately. Since a propagating system may have a nonpropagating remainder system, we cannot include the propagating property in the requirements for normality.

Finally, and independently of the main result, we discuss in Section 6 an interesting property of pairs of equivalent DOL-systems which is equivalent to bounded balance. The property requires the existence of a regular set $R$ such that:

(i) $R$ contains the language generated by either of the systems.

(ii) The homomorphisms of the two systems are equal on every string in $R$.

An alternative algorithm for testing equivalence of DOL-systems can be based on this property. We conjecture that such a regular set exists for every pair of equivalent DOL-systems, i.e., every pair of equivalent systems has bounded balance. Note that although we solve the decision problem for all DOL-systems, the conjecture is shown correct for normal systems only.

## 1. Notation

Given an alphabet $\Sigma$, $\Sigma^*$ denotes the free monoid generated by $\Sigma$, with unit (empty string) $\epsilon$.

A *DOL-system* is a 3-tuple $G = (\Sigma, h, \sigma)$ consisting of *alphabet* $\Sigma$, *homomorphism* $h$, and a *starting string* $\sigma \in \Sigma^*$. $L(G)$, the language generated by $G$, is defined as $\{h^n(\sigma): n \geqslant 0\}$. $G$ is said to be *reduced*, if every symbol from $\Sigma$ occurs in at least one $h^n(\sigma)$, $n \geqslant 0$. To *reduce* $G$ means to omit from $\Sigma$ all symbols which do not have this property.

For $w \in \Sigma^*$ and $a \in \Sigma$, $\#_a w$ denotes the number of occurrences of $a$ in $w$. If $(a_1, \ldots, a_n)$ is an ordering of $\Sigma$, then $(\#_{a_1} w, \ldots, \#_{a_n} w)$ is called the *Parikh vector* of $w$ and is denoted by $[w]$. The matrix $M = (m_{ij})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n}$, where $m_{ij} = \#_{a_j} h(a_i)$ is called the *growth matrix* for $G$.

If $i$ is a number, $|i|$ denotes the absolute value of $i$; if $w$ is a string, $|w|$ denotes the length of $w$; later on $|A|$ is also used for length of a vector $A$ or maximum characteristic value of a matrix $A$.

For $w \in \Sigma^*$, let $\min(w) = \{a: a \text{ occurs in } w\}$.

Given $G = (\Sigma, h, \sigma)$, we say that $w$ is a $G$-prefix ($G$-substring, $G$-suffix) if $w$ is a prefix (substring, suffix) of $h^n(\sigma)$ for some $n \geqslant 0$.

Two DOL-systems $G_i = (\Sigma, h_i, \sigma_i)$, $i = 1, 2$ are called (*sequence*) *equivalent* if $h_1^n(\sigma_1) = h_2^n(\sigma_2)$ for all $n = 0, 1, \ldots$. Two DOL-systems $G_1$, $G_2$ are called *Parikh equivalent* if $[h_1^n(\sigma_1)] = [h_2^n(\sigma_2)]$ for all $n = 0, 1, \ldots$. The *balance* (with respect to $G_1$, $G_2$) of a string $w$ in $\Sigma^*$ is defined as in Culik (1975), $\beta(w) = ||h_1(w)| - |h_2(w)||$. If there exists $c \geqslant 0$ so that $\beta(x) \leqslant c$ for all $G_1$-prefixes, then the pair $(G_1, G_2)$ is said to have *bounded balance*. In this case the smallest such $c$ is called the balance of the pair $(G_1, G_2)$.

For two sets $A$, $B$, $A \cup B$ denotes their union. If $A$, $B$ are disjoint, we stress this by writing $A + B$ for the union. Finally, we will often write $a$ instead of $\{a\}$ for a one-element set.

## 2. The Normal Systems

Let $G = (\Sigma, h, \sigma)$ be a DOL-system. We define the function $m: \mathscr{P}(\Sigma) \to \mathscr{P}(\Sigma)$, where $\mathscr{P}(\Sigma)$ is the set of all subsets of $\Sigma$ by putting

$$\begin{aligned} m(\phi) &= \phi, \\ m(\{a\}) &= \min(h(a)) \qquad \text{for} \quad a \in \Sigma, \\ m(A \cup B) &= m(A) \cup m(B). \end{aligned}$$

It is easy to see that $m^i(a) = \min(h^i(a))$ for all $i \geqslant 1$. We will write $m(a)$ for $m(\{a\})$ and use $m_1$, $m_2$, $m_{12}$, etc. to denote similar functions based on $h_1$, $h_2$, $h_1 h_2$, etc.

DEFINITION 1. A DOL-system $G = (\Sigma, h, \sigma)$ is called an *lr*-system if $\Sigma = \Sigma_l + \Sigma_c + \Sigma_r$ is a decomposition of $\Sigma$ into three nonempty disjoint sets such that $h(a) \in \Sigma_l \Sigma_c^*$ for $a \in \Sigma_l$, $h(a) \in \Sigma_c^*$ for $a \in \Sigma_c$, $h(a) \in \Sigma_c^* \Sigma_r$ for $a \in \Sigma_r$, and $\sigma \in \Sigma_l \Sigma_c^* \Sigma_r$. We call $\Sigma_c$ the *core* of $\Sigma$, $\Sigma_l$ is called the *left side*, and $\Sigma_r$ the *right side* of $\Sigma$. The number of symbols in the core $\Sigma_c$ of $\Sigma$ is called the *order* of $G$.

DEFINITION 2. A DOL-system $G = (\Sigma, h, \sigma)$ is called *normal* if

$$G \text{ is an } lr\text{-system,} \tag{1}$$

$$G \text{ is reduced,} \tag{2}$$

$$\text{if } a \in m^j(b) \text{ for some } j > 0, \text{ then } a \in m(b) \text{ holds for every } a, b \in \Sigma_c. \tag{3}$$

The following lemma, which is used to prove that we may consider normal systems only, is given in somewhat more general form as needed for Lemma 7.

Let $G_i = (\Sigma, h_i, \sigma)$, $i = 1, 2$, be two DOL-systems. Given $n \geqslant 1$ let $\mathbf{i} = (i_1, ..., i_n)$ be a sequence of length $n$ of integers $i_1, ..., i_n \in \{1, 2\}$. We denote $h^{(\mathbf{i})} = h_{i_1} \cdots h_{i_n}$, a composition of homomorphisms $h_1, h_2$, i.e., $h^{(\mathbf{i})}(x) = h_{i_1}(\cdots h_{i_n}(x) \cdots)$.

LEMMA 1. *Let* $G_i = (\Sigma, h_i, \sigma)$, $i = 1, 2$, $n \geqslant 1$, $\mathbf{i}_1 = (i_1, ..., i_n)$, $\mathbf{i}_2 = (j_1, ..., j_n)$ *be given. Denote* $\sigma_j = h_1^j(\sigma)$ *and let* $i_1 = 1$, $j_1 = 2$. *Under these assumptions* $G_1$, $G_2$ *are equivalent iff*

$$G_1^j = (\Sigma, h^{(\mathbf{i}_1)}, \sigma_j), \qquad G_2^j = (\Sigma, h^{(\mathbf{i}_2)}, \sigma_j) \tag{4}$$

*are equivalent for every* $j = 0, 1, ..., n - 1$ *and at the same time*

$$h_1^j(\sigma) = h_2^j(\sigma) \tag{5}$$

*also for every* $j = 0, 1, ..., n - 1$.

*Proof.* If $G_1$, $G_2$ are equivalent then Eq. (5) holds for every $j$ and thus $h^{(\mathbf{i}_1)}(\sigma) = h^{(\mathbf{i}_2)}(\sigma)$ for all possible sequences $\mathbf{i}_1$, $\mathbf{i}_2$. This means that Eq. (4) holds for all possible pairs.

Conversely, for each $l \geqslant 0$, $h_1^l(\sigma) = (h_1 h_{i_2} \cdots h_{i_n})^k h_1^m(\sigma) = (h^{(\mathbf{i}_1)})^k h_1^m(\sigma)$, $h_2^l(\sigma) = (h_2 h_{j_2} \cdots h_{j_n})^k h_2^m(\sigma) = (h^{(\mathbf{i}_2)})^k h_2^m(\sigma)$, where $l = kn + m$ and $0 \leqslant m < n$. Since $G_1^j$ and $G_2^j$ are equivalent and by Eq. (5) $h_1^m(\sigma) = h_2^m(\sigma)$ we have $h_1^l(\sigma) = h_2^l(\sigma)$, i.e., $G_1$, $G_2$ are equivalent. ∎

*Note.* It is sometimes more convenient to write $\bar{G}_i^j = (\Sigma, h^{(\mathbf{i}_1)}, h_i^j(\sigma))$ and instead Eqs. (4) and (5) require that $\bar{G}_1^j$, $\bar{G}_2^j$ be equivalent for $j = 0, 1, ..., n - 1$.

LEMMA 2. *Let* $G = (\Sigma, h, \sigma)$. *Then there is* $k \geqslant 1$ *such that in all the systems* $G^j = (\Sigma, h^k, h^j(\sigma))$, $j = 0, 1, ..., k - 1$, *Eq.* (3) *holds for all* $a, b \in \Sigma$.

*Proof.* As the validity of Eq. (3) does not depend on $j$ we may consider any single $j$. For every $a \in \Sigma$ consider the sets $m(a), m^2(a), \ldots$ where $m$ is based on the original $h$ of $G$. All the sets $m^j(a)$ are subsets of $\Sigma$, so we can find $r(a) > 0$, $d(a) > 0$ such that $m^{r(a)}(a) = m^{r(a)+d(a)}(a)$. From this $m^j(a) = m^l(a)$ for all $j$, $l \geqslant r(a)$ for which $j \equiv l$ (mod$(d(a))$. Consider the least common multiple $d = \text{l.c.m.}(d(a)\colon a \in \Sigma)$ and let $r$ be such that $r \geqslant r(a)$ for all $a \in \Sigma$ and $r \equiv 0$ (mod $d$).

Obviously $m^r(a) = m^{rj}(a)$ for all $a \in \Sigma$ and all $j = 1, 2, \ldots$. It is thus sufficient to take $k = r$. ∎

THEOREM 1. *The testing whether or not a pair $G_1$, $G_2$ is equivalent may be restricted to normal systems.*

*Proof.* Given any pair $G_i = (\Sigma, h_i, \sigma_i)$, $i = 1, 2$ of DOL-systems we can effectively construct a finite set $S$ of pairs of *normal* DOL-systems such that $G_1$, $G_2$ are equivalent iff each pair in $S$ is a pair of equivalent systems.

By Lemma 2 we can find $k_1$, $k_2$ for which $h_1^{k_1}$, $h_2^{k_2}$ meet Eq. (3). The systems constructed for $k = \text{l.c.m.}(k_1, k_2)$ meet Eq. (3) and $G_1$, $G_2$ are equivalent, by Lemma 1, iff all $G_1{}^j$, $G_2{}^j$ thus constructed are equivalent. Next, we reduce each $G_i{}^j$. Clearly $G_1{}^j$ and $G_2{}^j$ are equivalent iff the corresponding reduced systems are equivalent.

Finally, if $G_i{}^j$ is not yet an $lr$-system we may create the sides "artificially." Let $l$, $r$ be two distinct symbols $\notin \Sigma$. Put $\Sigma' = \{l\} + \Sigma + \{r\}$ and $h'(a) = h(a)$ for $a \in \Sigma$, while $h'(l) = l$, $h'(r) = r$ in each $G_i{}^j$. The new $G_i'^j$ is normal and again $G_1$, $G_2$ are equivalent iff all $G_1{}^j$, $G_2{}^j$ are equivalent. ∎

Note that systems obtained using the construction above meet Eq. (3) even for $a, b \in \Sigma$. We will, however, need the more general case subsequently.

The following definitions and facts from linear algebra are needed. A vector $x = (x_1, \ldots, x_p)$ and a matrix $M = (m_{ij})_{1 \leqslant i \leqslant p, 1 \leqslant j \leqslant p}$ will mean a vector and a matrix over real numbers. $|x| = \sum_{i=1}^{p} |x_i|$ is the length of $x$, $\|M\| = \sum_{j=1}^{p} \max_{1 \leqslant i \leqslant p} |m_{ij}|$ is the norm of $M$. $|M|$ will denote $\max_{1 \leqslant i \leqslant p} |r_i|$, where $r_i$ are the (generally complex) characteristic numbers. A vector $x$ and a matrix $M$ are called *positive* (*non-negative*) and denoted by $x > 0$, $M > 0$ ($x \geqslant 0, M \geqslant 0$) if $x_i > 0$, $m_{ij} > 0$ ($x_i \geqslant 0$, $m_{ij} \geqslant 0$) for all $1 \leqslant i, j \leqslant p$. Finally, $\langle x, y \rangle$ will denote the scalar product $\sum_{i=1}^{p} x_i y_i$, while $(x, y)$ will denote the direct sum of $x$ and $y$.

It is easy to establish the following facts.

PROPOSITION 1. *Let $M$ be a matrix and $q = |M|$, the absolute value of the largest characteristic value. Then for every vector $x$ $|xM^n| < q_0{}^n |x|$ for all sufficiently large $n$ and every $q_0 > q$.*

PROPOSITION 2. *Let $M = \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$ be a decomposition of a matrix $M$ where $A$*

*and $C$ are square matrices and $0$ a zero matrix. Assume that $C$ has a single characteristic vector $\bar{u}$ with respect to the maximal characteristic value $r = |C|$ which is real and positive. We will call such a vector the maximal characteristic vector. Let $\bar{v}$ be the characteristic vector of $C^T$ with respect to $r$. Denote by $u = (0, \bar{u})$ and $v = (0, \bar{v})$ the characteristic vectors of $M$ and $M^T$ respectively. Assume $|A| < r$ and $\bar{u} > 0$, $\bar{v} > 0$. From this $\langle u, v\rangle = \langle \bar{u}, \bar{v}\rangle > 0$, thus we may normalize them so that $\langle u, v\rangle = 1$. Finally, let $x = (y, z)$ be any vector also decomposed correspondingly to $M$. Now if $z \geqslant 0$, $z \neq 0$ then there exist constants $a$, $b$ and $r_0$ such that $a > 0$, $r_0 < r$ and*

$$|xM^n - ar^n u| < br_0^n \qquad \text{for all sufficiently large } n. \tag{6}$$

*Proof.* Let $x$, $\bar{u}$, $\bar{v}$, $u$, $v$ be as described. Writing $x = \langle x, v\rangle u + w_0$ we get $\langle w_0, v\rangle = 0$. Denote $a = \langle x, v\rangle = \langle z, \bar{v}\rangle > 0$. We have $xM^n = ar^n u + w_0 M^n$. Let $W = \{w \mid \langle w, v\rangle = 0\}$. By induction $w_0 M^n \in W$, thus $W$ is a subspace invariant with respect to $M$. Obviously, $u \notin W$. The characteristic value $r$ is simple, so all characteristic values of $M$ on $W$ are $< r$. Let $r_0 < r$ be any number larger than absolute values of all characteristic values of $M$ on $W$. From Proposition 1 above we get Eq. (6) immediately. ∎

PROPOSITION 3. *Let $M$, $u$, $x$ be as in Proposition 2. Consider the space $X = [x, xM, xM^2, \ldots]$, the space generated by the vectors $\{xM^i \mid i \geqslant 0\}$. It is closed (as any subspace in a finite-dimensional vector space) and there is a sequence of vectors from $X$, namely, the sequence $(1/r^i)\, xM^i$ which converges to $u$. Consequently, the maximal characteristic vector lies in every space $X$ generated by $\{xM^i\}$ starting with $x = (y, z)$ where $z \geqslant 0$, $z \neq 0$.*

The following definitions and facts about non-negative matrices can be found in Gantmacher (1960).

A matrix $M \geqslant 0$ is called *irreducible* if $M$ cannot be written in the form $M = \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$, with $A$, $C$ square submatrices, $0$ a zero matrix, even after any permutation of rows and the same permutation of columns. If all $M^i$, $i = 1, 2, \ldots$, are irreducible, we call $M$ *primitive*.

PROPOSITION 4. *If $M$ is irreducible, but some power $M^d$ is reducible, then $M^d$ is fully reducible, i.e., it can be written (after a suitable permutation of rows and columns) as $M = \left(\begin{smallmatrix} A & 0 \\ 0 & C \end{smallmatrix}\right)$.*

Matrix $M$ is primitive iff some power $d$ of $M$ is positive: $M^d > 0$. Such a $d$, if it exists divides the order $m$ of $M$, i.e., in particular $d \leqslant m$.

A primitive matrix has a positive characteristic value $r$ which is simple, and $r > |r_i|$ for all other characteristic values $r_i$ of $M$. The characteristic vector belonging to $r$ is positive.

Finally, if $M = (m_{ik})$ is irreducible, then for the maximum characteristic value $r$ we have $r \geqslant \min_{1 \leqslant i \leqslant p} \sum_{k=1}^{p} m_{ik}$.

## 3. The *lr*-Simple Systems

Definition 3. Let $G = \langle \Sigma_l + \Sigma_c + \Sigma_r, h, \sigma \rangle$ be an *lr*-system. Homomorphism $h$ is called *lr-simple* if for every $a, b \in \Sigma_c$ and every $k > 0$ there is $j > 0$ such that $a \in m^{kj}(b)$. Equivalently, calling $h$ *lr-irreducible* if for every $a, b \in \Sigma_c$ there is $j > 0$ such that $a \in m^j(b)$, $h$ is *lr*-simple iff $h^k$ is *lr*-irreducible for all $k \geqslant 1$. We call $G$ *lr*-simple if $h$ is *lr*-simple.

If $G$ is *lr*-simple and normal, then from $a \in m^{kj}(b)$ we get $a \in m(b)$. Putting $a = b$ we get $a \in m(a)$, which implies in turn that $a \in m^k(b)$ for all $i \geqslant 1$. Thus if $G$ is normal, $G$ is *lr*-simple iff $m(b) = \Sigma_c$ for all $b \in \Sigma_c$. However, the following lemma is needed for systems not necessarily normal.

Lemma 3. *Let $G_i = (\Sigma, h_i, \sigma)$, $i = 1, 2$ be two DOL-systems, $G_1$ lr-simple, the order $m$ of $G_1$ at least two. If $G_1$, $G_2$ are Parikh equivalent then for every $\epsilon > 0$ there is $n_0 > 0$ such that for every $w \in \Sigma^*$, $w \notin (\Sigma_l + \Sigma_r)^*$*

$$\beta(h_1^n(w)) \leqslant \epsilon \mid h_1^n(w) \mid \qquad \textit{for all } n \geqslant n_0 . \tag{7}$$

*Proof.* Let $M_1$ be the growth matrix of $G_1$. If $\Sigma$ is suitable ordered we can write

$$M_1 = \begin{pmatrix} I_1 & 0 & A_1 \\ 0 & I_2 & A_2 \\ 0 & 0 & N \end{pmatrix},$$

where $I_1$, $I_2$ are matrices of the order $\mid \Sigma_l \mid$, $\mid \Sigma_r \mid$, respectively, with exactly one 1 in each row and all other elements zero. $A_1$, $A_2$ are rectangular matrices in general, and 0 denotes zero-matrices of appropriate orders. If the order of $G$ is $m$, then $N$ is $m \times m$ matrix which is primitive, in particular irreducible. Being primitive, $N^d$ is positive, for some $d \leqslant m$. The elements of $N$, and so of $N^d = (n_{i,j}^{(d)})$ are integers. Thus $\min_{1 \leqslant i \leqslant m} \sum_{j=1}^{m} n_{i,j}^{(d)} \geqslant m$. By Proposition 4, for the maximal characteristic value $r' = \mid N^d \mid$, we have $r' \geqslant m > 1$. Denoting $r = \mid N \mid$, we have $r' = r^d$, i.e., $r > 1$.

Let $u$ be the characteristic vector of $M_1$ with respect to $r$. Since all the characteristic values of matrices $I_1$, $I_2$ are in absolute value smaller than or equal to one, the assumptions of Proposition 2 are met for $A = \binom{I_1 \ 0}{0 \ I_2}$, $B = \binom{A_1}{A_2}$, $C = N$. Let $\bar{\sigma}$ be the Parikh vector of $\sigma$. From Proposition 3 we get $u \in [\bar{\sigma}, \bar{\sigma}M_1, \bar{\sigma}M_1^2, ...] = [\bar{\sigma}, \bar{\sigma}M_2, \bar{\sigma}M_2^2, ...] = M$, the first equality following from the Parikh equivalence of $G_1$ and $G_2$. For every vector $x \in M$ we have $xM_1 = xM_2$, thus, in particular, $u(M_1 - M_2) = 0$. Let $x$ be now the Parikh vector of $w$. As $w \notin (\Sigma_l + \Sigma_r)^*$, the conditions on $x$ in Proposition 2 are met and Eq. (6) holds. That is, for suitable $a, b > 0$, $r_0$ we have $\mid xM_1^n - ar^n u \mid \leqslant br_0^n$. From this $\mid xM_1^n(M_1 - M_2) \mid \leqslant b \| M_1 - M_2 \| r_0^n$. From Eq. (6) we further get $\mid xM_1^n \mid \geqslant \mid ar^n \mid u \mid - br_0^n \mid \geqslant$

$(a \mid u \mid/2)\, r^n$, again for sufficiently large $n$. These two inequalities combined give

$$\beta(h_1{}^n(w)) \leqslant \mid xM_1{}^n(M_1 - M_2)\mid \leqslant \frac{2b \parallel M_1 - M_2 \parallel}{a \mid u \mid} \left(\frac{r_0}{r}\right)^n \cdot \mid xM_1{}^n \mid.$$

As $r_0 < r$, Eq. (7) can be met if $n$ is large enough. ∎

LEMMA 4. *Under the assumptions of Lemma* 3

*for every* $\epsilon > 0$ *there is* $K > 0$ *such that for every* $G_1$*-prefix* $w$, $\mid w \mid > K$ *we have* $\beta(w) \leqslant \epsilon \mid w \mid$. (8)

*Proof.* Using Lemma 3, given $\epsilon/2$, we find $n_0$. Let $w$ be any $G_1$-prefix, i.e. $h_1{}^n(\sigma) = wx$ for suitable $n$, $x$. Assume $\mid w \mid > 1$, if $n \geqslant n_0$, then denote $u = h_1^{n-n_0}(\sigma)$. Let $u = u_1 a u_2$, where $a \in \Sigma$ be such that $h_1^{n_0}(u_1)$ is a prefix of $w$ but $w$ is a proper prefix of $h_1^{n_0}(u_1 a)$, i.e., $w = h_1^{n_0}(u_1)\, x_1$, $h_1^{n_0}(u_1 a) = wx_2$, $x_1, x_2 \in \Sigma^*$. Now

$$\beta(w) \leqslant \beta(h_1^{n_0}(u_1)) + \beta(x_1) \leqslant \frac{\epsilon}{2} \mid h_1^{n_0}(u_1)\mid + B \mid x_1 \mid \leqslant \frac{\epsilon}{2} \mid w \mid + BH^{n_0},$$

where $B = \max_{a \in \Sigma} \{\beta(a)\}$, and $H = \max_{a \in \Sigma} \mid h_1(a)\mid$. To prove Eq. (8) it is sufficient to take $H^{n_0}B/\mid w \mid \leqslant \epsilon/2$, i.e., to take $K > H^{n_0} \max(2B/\epsilon, 1)$. The second case in max-function guarantees that $n \geqslant n_0$. ∎

THEOREM 2. *Let* $G_i = (\Sigma, h_i, \sigma)$ *for* $i = 1, 2$ *be two lr-systems and let* $G_1$ *be lr-simple. Let* $G_1$ *and* $G_2$ *be sequence equivalent and let the order of* $G_1$ *be at least two. Then the pair* $(G_1, G_2)$ *has bounded balance.*

*Proof.* This result is shown in Culik (1975, Theorem 3.2) for pairs of equivalent simple DOL-systems. However, in the proof of this result only the following properties are essential:

(a) $h_1{}^n(a)$ is exponentially growing for each $a$ in $\Sigma$, except possibly for symbols which occur only as a first or last symbol in any $h_1{}^n(\sigma)$ for $n \geqslant 0$.

(b) Equation (8) holds.

In our case for each $a$ in $\Sigma_c$, $h_1{}^n(a)$ grows because $G_1$ is *lr*-simple and of order at least two, therefore (a) is satisfied. By Lemma 4, (b) is satisfied. Therefore, the proof of Theorem 3.2 from Culik (1975) also proves our Theorem 2. The only modification required is that when comparing formulas (2) and (3) we may not say that without restriction of generality $\mid h_1(u')\mid \geqslant h_2(u')\mid$ since the assumptions of the theorem are not symmetric with respect to $G_1$ and $G_2$ here. However, the proof for the case $\mid h_1(u')\mid \leqslant \mid h_2(u')\mid$ is fully analogical since only the equivalence of $G_1$ and $G_2$ is used and this is a symmetric property. ∎

## 4. Subalphabets and Induced Systems

Given a DOL-system $G = \langle \Sigma, h, \sigma \rangle$, a set $\Pi$, $\phi \neq \Pi \subsetneq \Sigma_c$ is called a *subalphabet* if $h(a) \in \Pi^*$ for each $a \in \Pi$. Denote $\Omega = \Sigma - \Pi$. If $G$ is an *lr*-system we will also use $\Omega_c$ for $\Sigma_c - \Pi$. For every $z \in \Sigma^*$ we denote by $z^\Omega$ the string $z$ with all symbols from $\Pi$ omitted, thus $z^\Omega \in \Omega^*$. We define $G^\Omega$ as $\langle \Omega, h^\Omega, \sigma^\Omega \rangle$ where $h^\Omega(x) = (h(x))^\Omega$ for $x \in \Omega$. If for a sequence $s = s_1, s_2, ...$ we write $s^\Omega = s_1^\Omega, s_2^\Omega, ...$, then obviously

$$(s(G))^\Omega = s(G^\Omega), \tag{9}$$

where $s(G)$ is the sequence generated by $G$. Given two DOL-systems $G_1$, $G_2$, $\Pi$ is called their *common* subalphabet if $\Pi$ is a subalphabet of $G_i$ for $i = 1, 2$. From Eq. (9) we get immediately that if $G_1$, $G_2$ are equivalent and have a common subalphabet $\Pi$ then $G_1^\Omega$, $G_2^\Omega$ are equivalent. It is also obvious that if $G$ is normal, so is $G^\Omega$.

Lemma 5. *Let $G_i = \langle \Sigma, h_i, \sigma \rangle$, $i = 1, 2$ be two normal propagating equivalent DOL-systems. Then $G_1$ and $G_2$ have a common subalphabet $\Pi$, or the composite homomorphism $h_1 h_2$ is lr-simple.*

*Proof.* First, we will show that if there is no common subalphabet then $h_1 h_2$ is *lr*-irreducible. For $a, b \in \Sigma_c$ we say that *a immediately derives b*, written $a \Rightarrow b$, if $b \in m_1(a) \cup m_2(a)$. (See Section 2 for the definition of $m_1$, $m_2$.) Also, we say that $a$ derives $b$ *using* $m_1$ or $m_2$ if $b \in m_1(a)$ or $b \in m_2(a)$, respectively. Let $\Rightarrow^*$ be the reflexive and transitive closure of binary relation $\Rightarrow$. Finally, for $a \in \Sigma_c$, let $\tilde{m}(a) = \{b \in \Sigma_c : a \Rightarrow^* b\}$. Obviously, $m_i(\tilde{m}(a)) \subseteq \tilde{m}(a)$ for $i = 1, 2$; so either $\tilde{m}(a) = \Sigma_c$ or $\tilde{m}(a)$ is a common subalphabet of $G_1$ and $G_2$. This means that if there is no common subalphabet, then $a \Rightarrow^* b$ for any two $a, b \in \Sigma_c$.

Let $\Delta_i$ be the subset of $\Sigma_c$ of symbols which occur in $h_i^n(\sigma)$ for infinitely many $n \geqslant 0$, $i = 1, 2$. Since $G_1$ and $G_2$ are equivalent, $\Delta_1 = \Delta_2$. Assume that $\Delta_1 \subsetneq \Sigma_c$. Since $G_1$ is propagating $\Delta_1 \neq \phi$ and thus clearly $\Delta_1$ is a common subalphabet of $G_1$ and $G_2$. Therefore, if $G_1$ and $G_2$ have no common subalphabet $\Delta_1 = \Delta_2 = \Sigma_c$.

Consider arbitrary $a, b \in \Sigma_c$. Since $G_1$ is propagating, there exists $c \in \Sigma_c$ such that $c \in m_1(a)$. There exists $d \in \Sigma_c$ such that $b \in m_2(d)$, otherwise, i.e., if $G_2$ produces $b$ from a "side" only there obviously exists a common subalphabet. If there is no common subalphabet, then $c \Rightarrow^* d$. This means that $a$ can derive $b$ using $m_1$ in the first and $m_2$ in the last step of the derivation. From condition (3) of normality it follows that, if $x \Rightarrow^* y$ for $x, y \in \Sigma_c$ using only $m_1$ ($m_2$) in all steps, then $x \Rightarrow y$ using $m_1$ ($m_2$). Therefore, $a$ derives $b$ using $m_1$ and $m_2$ alternately starting with $m_1$ and ending with $m_2$. Thus we have shown that for every $a, b \in \Sigma_c$ there exist $n \geqslant 0$ and $c_1, ..., c_n \in \Sigma_c$ so that $c_1 \in m_{12}(a)$; $c_{j+1} \in m_{12}(c_j)$ for $j = 1, 2, ..., n - 1$; and $b \in m_{12}(c_n)$. We used the fact that the function $m_{12}$ as defined at the beginning of Section 2 is the composition of $m_1$ and $m_2$.

Thus we have shown that $h_1h_2$ is $lr$-irreducible and we proceed to show that $h_1h_2$ is $lr$-simple. A system is $lr$-simple iff its growth matrix restricted to $\Sigma_c$ is primitive. From results in Gantmacher (1960) it follows that, if the growth matrix is not primitive, then there exist $q > 1$ and a partition $\mathscr{P}$ of $\Sigma_c$ with $q$ classes such that for every $a, b \in \Sigma_c$, if $a \in m_{12}^2(b)$, then $a$ and $b$ belong to the same class of $\mathscr{P}$.

CLAIM 1. *Let $a, b \in \Sigma_c$. If $b \Rightarrow a$ then $a$ and $b$ belong to the same class of $\mathscr{P}$.*

*Proof.* Suppose that $a \in m_1(b)$. Since $G_1$ and $G_2$ are propagating there exists $c \in m_1(a)$, and similarly there exists $d \in m_2(c)$. Therefore $d \in m_{12}(a)$ and, since $G_1$ is normal $c \in m_1(b)$ (condition (3)), also $d \in m_{12}(b)$. This means that $m_{12}(a) \cap m_{12}(b) \neq 0$ and thus, since $G_1$ and $G_2$ are propagating, also $m_{12}^q(a) \cap m_{12}^q(b) \neq 0$. Therefore, $a$ and $b$ are in the same class of $\mathscr{P}$, namely, in the class including $m_{12}^{q-1}(d)$.

Similarly, suppose $a \in m_2(b)$. Since $\Delta_1 = \Delta_2 = \Sigma_c$ there exist $c, d \in \Sigma_c$ such that $b \in m_2(c)$ and $c \in m_1(d)$. Therefore, $b \in m_{12}(d)$ and using condition (3) of normality for $G_2$ we have $a \in m_2(c)$ and thus also $a \in m_{12}(d)$. Therefore, again $a$ and $b$ are in the same class of $\mathscr{P}$. ∎

Having proven the claim, let $a, b$ be again any two elements of $\Sigma_c$. We know that $a \Rightarrow^* b$. From the claim and the definition of $\Rightarrow^*$ through $\Rightarrow$, it follows that $a$ and $b$ belong to the same class of $\mathscr{P}$. Since this holds for arbitrary $a, b$ in $\Sigma_c$, partition $\mathscr{P}$ has a single class, i.e., $q = 1$, which shows that $h_1h_2$ is $lr$-simple. ∎

DEFINITION 4. Given $G = \langle \Sigma, h, \sigma \rangle$. A subalphabet $\Pi \subsetneq \Sigma$ is called *limited* if there is a constant $k$ such that for every substring $u \in \Pi^*$ of $L(G)$ we have $|u| < k$. Note that $\Pi$ is limited with respect to every DOL-system equivalent to $G$.

LEMMA 6. *Let $G_1$, $G_2$ be two equivalent systems, with a common subalphabet $\Pi$. If $\Pi$ is limited and if the pair $(G_1^{\Omega}, G_2^{\Omega})$ has a bounded balance, then the pair $(G_1, G_2)$ has bounded balance.*

*Proof.* Let the balance of $(G_1^{\Omega}, G_2^{\Omega})$ be $c$ and let $k$ be such that $|u| \leqslant k$ for all $G_1$-substrings $u$ from $\Pi^*$. Then the balance of the pair $(G_1, G_2)$ is clearly smaller or equal to $(c + 1)k + c$. ∎

DEFINITION 5. Let $G_1$, $G_2$ be a pair of DOL-systems, $G_i = (\Sigma, h_i, \sigma)$. Given $\mathbf{i} = i_1 i_2 \cdots i_n$ with $n \geqslant 1$ and $i_1, \ldots, i_n \in \{1, 2\}$, the set $S = \{G_1{}^j, G_2{}^j: 0 \leqslant j \leqslant n\}$ of pairs of DOL-systems is called $\mathbf{i}$-*combination* of $(G_1, G_2)$ where $G_i{}^j = (\Sigma, \bar{h}_i, \sigma_{ij})$, for $i = 1, 2; j = 0, \ldots, n$, $\bar{h}_1 = h_1 h_{i_1} h_{i_2} \cdots h_{i_n}$, $\bar{h}_2 = h_2 h_{i_1} h_{i_2} \cdots h_{i_n}$, $\sigma_{i,0} = \sigma$ and $\sigma_{i,j} = h_{i_j} \cdots h_{i_n}(\sigma)$ for $i = 1, 2$ and $j = 1, \ldots, n$. Finally, we reduce each system $G_i{}^j$, if necessary.

Instead of 1-combination we will say just combination. If $\mathbf{i} = (21)^k$ for the minimal $k > 0$ such that each $G_i^j$ is normal we call the $\mathbf{i}$-combination the *normal combination* of $(G_1, G_2)$. We show that for normal systems $G_1, G_2$ such $k$ always exists. We find $k$ according the proof of Lemma 2 for $G = (\Sigma, h_2h_1, \sigma)$. So, we have $m_{21}^k(a) = m_{21}^{ks}(a)$ for all $a \in \Sigma$ and $s = 1, 2,\ldots$. Therefore also $m_i(m_{21}^k(a)) = m_i(m_{21}^{ks}(a))$ for $i = 1, 2$ and $s = 1, 2,\ldots$.

Now, to show that the homomorphisms of the normal combination satisfy condition (3) of normality we note that

$$m_2^{k_1}m_1^{k_2} \cdots m_2^{k_{2n-1}}m_1^{k_{2n}}(a) = m_{21}^n(a)$$

for each $a \in \Sigma$, $n \geqslant 1$ and arbitrary $k_1, \ldots, k_{2n} \geqslant 1$; since, because of normality of $G_1$ and $G_2$, the repetitions of the same homomorphisms are irrelevant. Specifically,

$$[m_2m_{21}^k]^s(a) = m_{21}^{ks}(a) = m_{21}^k(a) = m_2m_{21}^k(a)$$

and

$$[m_1m_{21}^k]^s(a) = m_1m_{21}^{ks}(a) = m_1m_{21}^k(a)$$

for each $a \in \Sigma$ and $s \geqslant 1$, which shows that the systems of a normal combination satisfy condition (3) of normality.

*Note.* The normal combinations have been introduced in the revised version of this paper to close a gap pointed out to the authors by K. Ruohonen.

We will say that the set $S$ has bounded blance if each pair $(G_1{}^j, G_2{}^j) \in S$ has bounded balance.

LEMMA 7. *Let $(G_1, G_2)$ be a pair of DOL-systems. Let $S$ be their* $\mathbf{i}$*-combination for some $\mathbf{i} \in \{1, 2\}^+$. Then*

(i) *$G_1, G_2$ are equivalent iff for all $(G_1{}^j, G_2{}^j) \in S$, $G_1{}^j$, $G_2{}^j$ are equivalent.*

(ii) *Let $G_1$ and $G_2$ be equivalent. Then $(G_1, G_2)$ has bounded balance iff their* $\mathbf{i}$*-combination $S$ has bounded balance.*

*Proof.* Part (i) has already been proven in Lemma 1. Now, let $k = |\mathbf{i}|$ and assume that $(G_1{}^j, G_2{}^j)$ has bounded balance and let $w$ be a $G_1$-prefix, say, $ww' = h_1{}^n(\sigma)$ for some $n \geqslant 0$ and some $w' \in \Sigma^*$. When proving that the balance is bounded on a set of strings we may neglect finitely many strings, so let $n \geqslant k$. Let $\mathbf{i} = i_1i_2 \cdots i_k$ and $h = h_{i_1}h_{i_2} \cdots h_{i_k}$. Let $ua$ with $u \in \Sigma^*$, $a \in \Sigma$ be a prefix of $h_1^{n-k}(\sigma)$ such that $h(u)$ is a prefix of $w$, but $w$ is a proper prefix of $h(ua)$ (such $ua$ exists if $w$ is a proper prefix, but if $w$ is the whole string $h_1{}^n(\sigma)$ then $\beta(w) = 0$, so again we may ignore this), i.e., $h(u)x = w$ for some $x \in \Sigma^*$, and $x$ is a prefix of $h(a)$, from which $|x| \leqslant H^k$ and $\beta(x) \leqslant BH^k$, i.e., $\beta(w) \leqslant \beta(h(u)) + BH^k$, where $H = \max_{a\in\Sigma} |h_1(a), h_2(a)|$ and $B = \max_{a\in\Sigma} \beta(a)$. The boundedness of $\beta(w)$ follows from the fact that $\beta(h(u)) = | \, |\bar{h}_1(u)| - |\bar{h}_2(u)| \, | = \beta_j(u)$, where we denoted by $\beta_j$ the balance in $(G_1{}^j, G_2{}^j)$ which is bounded, and $j$ is chosen so that $w$ is a $G_1{}^j$-prefix.

The converse, namely, that if $(G_1, G_2)$ has bounded balance so has each $(G_1{}^j, G_2{}^j)$ is obvious and is not in fact needed in our proofs. ∎

DEFINITION 6. Let $G = (\Sigma, h, \sigma)$ be a DOL-system and let $\Pi \subset \Sigma$ be a subalphabet, and assume that $h^\Omega$ is propagating. For every $avb \in \Omega\Pi^*\Omega$ we define an *induced system* $G^{avb} = (\Sigma^a + \Pi' + {}^b\Sigma, \hat{h}, \bar{a}v\bar{\bar{b}})$ as follows.

For $a \in \Omega$, we write $h(a) = xcv$, where $c \in \Omega$, $v \in \Pi^*$. (Note that such decomposition is possible because $h^\Omega$ is propagating and is obviously unique.) We denote $l(a) = c$, $l'(a) = v$. Similarly, writing $h(a) = v'c'y$, where $c' \in \Omega$, $v' \in \Pi^*$, we define $r(a) = c'$, $r'(a) = v'$.

We define $\Sigma^a = \{\bar{c}$: there is $n \geqslant 0$ and a sequence $c_0 = a, c_1, \ldots, c_{n-1}, c_n = c$, $c_j \in \Omega$ such that $c_j = l(c_{j-1})$, $j = 1, 2, \ldots, n\}$, where $\bar{c}$ is one new symbol for each $c \in \Omega$. Similarly, we define ${}^b\Sigma$ starting with $c_0 = b$ and using $r$ instead of $l$: ${}^b\Sigma = \{\bar{\bar{c}}$: there is $m \geqslant 0$ and a sequence $c_0 = b, c_1, \ldots, c_m = c$, $c_j \in \Omega$ and $c_j = r(c_{j-1})$ for $j = 1, 2, \ldots, m\}$, and where $\bar{\bar{c}}$ is another new symbol, one for each $c \in \Omega$. Let

$$\hat{h}(\bar{a}) = \overline{l(a)}\, l'(a) \qquad \text{for} \quad a \in \Omega,$$

$$\hat{h}(\bar{\bar{a}}) = r'(a)\, \overline{\overline{r(a)}} \qquad \text{for} \quad a \in \Omega,$$

$$\hat{h}(d) = h(d) \qquad \text{for} \quad d \in \Pi,$$

Finally, $\Pi'$ is the subset of $\Pi$ of symbols actually used when the homomorphism $\hat{h}$ is repeatedly applied to $v$. That completes the definition of $G^{avb}$. When starting with $G_1$ or $G_2$ we will, as usual, talk about $\hat{h}_1$, $\hat{h}_2$, $G_1^{avb}$, and $G_2^{avb}$.

LEMMA 8. *Let $G_1$, $G_2$ be two equivalent DOL-systems with a common subalphabet $\Pi$. Assume both $h_1$ and $h_1{}^\Omega$ are propagating and there exists a constant $k$ such that for every $G_1$-prefix of the form $xav$, where $a \in \Omega$, $x \in \Sigma^*$, and $v \in \Pi^*$ we have*

$$\text{if } |v| > k, \text{ then } h_1{}^\Omega(xa) = h_2{}^\Omega(xa). \tag{10}$$

*Then for every $avb \in \Omega\Pi^*\Omega$, $|v| > k$, $avb$ a substring of $L(G_1)$ the systems $G_1^{avb}$, $G_2^{avb}$ are equivalent.*

*Proof.* As $avb$ is a $G_1$-substring, we can write $xavby = h_1{}^j(\sigma)$ for some $x, y \in \Sigma^*$ and some $j \geqslant 0$. From Eq. (10) we have $h_1(xa) = x'l_1(a)\, l_1'(a)$, $h_2(xa) = x'l_2(a)\, l_2'(a)$, where $l_1$, $l_1'$ and $l_2$, $l_2'$ are the functions from Definition 6 based here on $h_1$ and $h_2$. Similarly, $h_i(xavb) = x'l_i(a)\, l_i'(a)\, h_i(v)\, r_i'(b)\, r_i(b)\, x_i''$, for some $x'$, $x_i'' \in \Sigma^*$, $i = 1, 2$. Strings $h_1(xa)$, $h_2(xa)$ and $h_1(xavb)$, $h_2(xavb)$ are prefixes of the same string $h_1^{j+1}(\sigma) = h_2^{j+1}(\sigma)$, so $l_1(a) = l_2(a) \in \Omega$; $l_1'(a)\, h_1(v)\, r_1'(b)$ and $l_2'(a)\, h_2(v)\, r_2'(b) \in \Pi^*$, but they are equal as the next symbol $r_1(b) = r_2(b) \in \Omega$.

That is, $|\, v \,| > k$ implies (through $h_1{}^{\Omega}(xa) = h_2{}^{\Omega}(xa)$ that $\hat{h}_1(\bar{a} v \bar{\bar{b}}) = \hat{h}_2(\bar{a} v \bar{\bar{b}})$. As $h_1$, and thus $\hat{h}_1$, are propagating also $|\, h_1(v)| \geqslant |\, v \,| > k$. This proves that $G_1^{avb}$, $G_2^{avb}$ are equivalent. ∎

Lemma 9. *Let* $G = \langle \Sigma, h, \sigma \rangle$ *be a normal DOL-system. Denote* $H = \max(|\, h(a)|\colon a \in \Sigma)$. *Let* $\Pi \subset \Sigma$ *be a subalphabet and* $v_0 a_1 \cdots a_n v_n$ *a decomposition of a substring of* $h_1{}^m(\sigma)$, *where* $n \geqslant 1$; $a_1, \ldots, a_n \in \Omega$; $v_0, \ldots, v_n \in \Pi^*$. *Assume that* $h^{\Omega}$ *is propagating. Assume further that* $m \geqslant n$, *and* $|\, v_0 \,|, |\, v_n \,| > H^n$. *Then*

$$h^{\Omega}(a_i) = a_i \qquad \textit{for all } i = 1, 2, \ldots, n.$$

*Proof.* Suppose that for some $a \in \{a_1, \ldots, a_n\}$, $h^{\Omega}(a) \neq a$. Let $c_0$ be "the father of degree $n$ of our $a$", i.e., assume that the following picture is a part of the derivation tree in $G$

$$\begin{array}{lc}
m - n & c_0 \\
 & \diagup \mid \diagdown \\
m - n + 1 & c_1 \\
 & \diagup \mid \diagdown \\
\cdots & \cdots \\
 & \diagup \mid \diagdown \\
m - 1 & c_{n-1} \\
 & \mid \\
m & v_0 a_1 \cdots a \cdots a_n v_n
\end{array}$$

There are two possibilities:

(i) There exists $b \in \Omega$, $b \in m^{\Omega}(a)$, and $b \neq a$. As $a \in (m^{\Omega})^j(c_{n-j})$ and $G$ is normal, we have $\{a, b\} \subseteq m^{\Omega}(c_j)$ for all $0 \leqslant j \leqslant n - 1$. From this we get $|(h^{\Omega})^n(c_0)| \geqslant n + 1$.

(ii) $h^{\Omega}(a) = a^r$ for some $r \geqslant 2$. As before, from the normality and from $a \in (m^{\Omega})^n(c_0)$ we get $a \in m^{\Omega}(c_0)$. From this

$$|(h^{\Omega})^n(c_0)| \geqslant r^n \geqslant n + 1 \qquad \text{if } n \geqslant 1.$$

Thus in both cases $h_1{}^n(c_0)$ has at least $n + 1$ occurrences of symbols from $\Omega$. In other words, either $v_0$ or $v_n$ must be a substring of $h_1{}^n(c_0)$, but from this $|\, v_0 \,|$ or $|\, v_n \,| \leqslant H^n$. ∎

## 5. The Main Theorem

Theorem 3. *Every pair of normal equivalent DOL-systems has bounded balance.*

*Proof.* Let $G_i = (\Sigma, h_i, \sigma)$ for $i = 1, 2$. Denote by $r$ the order of $G_1$ (same as $G_2$). The proof will be by induction on $r$.

Base of induction, $r = 1$. Let $\Sigma_c = \{a\}$. For $i = 1, 2$ we have:

(i) For each $b \in \Sigma_l$, $h_i(b) = ca^{\alpha_{i,b}}$ for some $c \in \Sigma_l$ and $\alpha_{i,b} \geqslant 0$.

(ii) $h_i(a) = a^{\beta_i}$ for some $\beta_i \geqslant 0$.

(iii) For each $b \in \Sigma_r$, $h_i(b) = a^{\alpha_{i,b}}c$ for some $c \in \Sigma_r$ and $\alpha_{i,b} \geqslant 0$.

Since $G_1$ and $G_2$ are equivalent, obviously, $\beta_1 = \beta_2$ and the balance of the pair $(G_1, G_2)$ is at most $\max_{i=1,2;b\in\Sigma_l} \alpha_{i,b}$, i.e., the pair $(G_1, G_2)$ has bounded balance.

We now make the induction hypothesis that the assertion holds for systems of order smaller than a fixed $r > 1$, and consider a pair of systems of order $r$, i.e., $|\Sigma_c| = r \geqslant 2$.

*Case I.* Assume that $h_1(a) = h_2(a) = \epsilon$ for some $a \in \Sigma_c$. Then $\Pi = \{a\}$ is a common subalphabet. Let $\Omega = \Sigma - \Pi$. Since $G_1$ and $G_2$ are equivalent $G_1{}^\Omega$ and $G_2{}^\Omega$ are also equivalent and since $|\Omega_c| < |\Sigma_c|$ the pair $(G_1{}^\Omega, G_2{}^\Omega)$ has bounded balance by induction hypothesis. Subalphabet $\Pi$ is clearly limited and therefore the pair $(G_1, G_2)$ has bounded balance by Lemma 6.

*Case* II. Assume that $h_1(a) = \epsilon$ for some $a \in \Sigma_c$ but not necessarily $h_2(a) = \epsilon$. Consider the normal combination of $(G_1, G_2)$. Clearly, we have $\bar{h}_1(a) = \epsilon$, $\bar{h}_2(a) = \epsilon$, so by Case I, $(G_1{}^i, G_2{}^i)$ has bounded balance for $i = 1, 2$ and so has $(G_1, G_2)$ by Lemma 7.

*Case* III. We may now assume that both $G_1$ and $G_2$ are propagating. By Lemma 5 either the combination of $(G_1, G_2)$ is simple, this implying using Theorem 2 and Lemma 7, that $(G_1, G_2)$ has bounded balance, or there is a common subalphabet $\Pi$. Denote $\Omega = \Sigma - \Pi$ and $\Omega_c = \Sigma_c - \Pi$. We may assume that $\Pi$ is maximal, i.e., there is no subalphabet $\Pi'$ so that $\Pi \subsetneq \Pi' \subsetneq \Sigma_c$. We may further assume without loss of generality that either $\Omega_c$ has exactly one element or $h_1{}^\Omega$ and $h_2{}^\Omega$ are propagating. This is so for the following reasons. In view of Lemma 7, in order to prove that the pair $(G_1, G_2)$ has bounded balance we may show this for the normal combination of $(G_1, G_2)$ instead. Note also that every common subalphabet with respect to $G_1$, $G_2$ is also a common subalphabet with respect to each combination of $(G_1, G_2)$, i.e., with respect to each pair of systems from the combination. Suppose now that the assumption above is not valid, i.e., for some $a$ in $\Omega_c$ either $h_1(a) = \epsilon$ or $h_2(a) = \epsilon$ and $\Omega_c - \{a\} \neq \varnothing$. Then for the homomorphisms $\bar{h}_1$, $\bar{h}_2$ from the normal combination of $(G_1, G_2)$ (or $(G_2, G_1)$) we have $\bar{h}_1{}^\Omega(a) = \bar{h}_2{}^\Omega(a) = \epsilon$. Therefore, $\Pi \cup \{a\}$ is also a common subalphabet with respect to the combination of $(G_1, G_2)$. It might not be a maximal one but can be enlarged to such. If this new subalphabet does not satisfy our assumption we repeat the above construction. After a finite number of steps we get a maximal subalphabet, which meets the assumption.

Since $G_1$ and $G_2$ are equivalent, $G_1{}^\Omega$ and $G_2{}^\Omega$ are also equivalent, and, since they are of order smaller than $r$ and normal, the pair $(G_1{}^\Omega, G_2{}^\Omega)$ has bounded balance by the induction hypothesis. For the rest of the proof we will use the following notation. The balance of $(G_1{}^\Omega, G_2{}^\Omega)$ is denoted by $c$ and $H = \max_{i=1,2} (\max_{a\in\Sigma} | h_i(a)|)$.

Now, as a part of Class III we formulate and prove the following.

CLAIM 2. *Suppose that for every $G_1$-prefix of the form $wav$, where $w \in \Sigma^*$, $a \in \Omega$, and $v \in \Pi^*$ with $| v | > H^c$*

$$\beta^\Omega(wa) = 0. \tag{11}$$

*Then the pair $(G_1, G_2)$ has bounded balance.*

*Proof.* Let $Q = H^c$ and let $S = \{w \in \Omega\Pi^*\Omega: Q < | w | \leqslant HQ\}$. Now, consider the pairs of induced systems (cf. Definition 6) $(G_1{}^w, G_2{}^w)$ for each $w \in S$. By Eq. (11) and Lemma 8 the systems $G_1{}^w$ and $G_2{}^w$ are equivalent for each $w \in S$. Clearly, $G_i{}^w$ is normal for each $w \in S$ and $i = 1, 2$.

Hence, by the induction hypothesis the pair $(G_1{}^w, G_2{}^w)$ has bounded balance for every $w \in S$. Let the balance of $(G_1{}^w, G_2{}^w)$ be $c_w$, and let $c_\Pi = \max_{w\in S} c_w$, which is well defined since $S$ is finite.

We now proceed in the proof of Claim 2 by considering all $G_1$-prefixes, and show that their balances are bounded. Every $G_1$-prefix $x$ can be written uniquely in the form $x = a_d v_d a_{d-1} v_{d-1} \cdots a_1 v_1$ for some $d \geqslant 1$, and $a_i \in \Omega$, $v_i \in \Pi^*$ for $i = 1, 2,..., d$. We will consider four cases. In the first three we assume that $x$ is a prefix of $h_1{}^t(\sigma)$ for some $t \geqslant c$.

*Case* A. Let $d \leqslant c$ and $| v_i | \leqslant Q$ for $i = 1, 2,..., d$. In this case we have $\beta(x) \leqslant dQ + dH \leqslant c(Q + H)$.

*Case* B. Let $d > c$ and $| v_i | \leqslant Q$ for $i = 1, 2,..., c + 1$. Without loss of generality we may assume that $h_1(x)$ is a prefix of $h_2(x)$, i.e., $h_2(x) = h_1(x)z$ for some $z \in \Sigma^*$. Since $\beta^\Omega(x) \leqslant c$, $z$ contains at most $c$ occurrences of symbols from $\Omega$; at the same time $G_2$ is propagating and therefore $z$ is a suffix of $h_2(v_{c+1}a_c v_c \cdots a_1 v_1)$ (see Fig. 1), thus $\beta(x) = | z | \leqslant H | v_{c+1} a_c v_c \cdots a_1 v_1 | \leqslant (c + 1)(Q + 1)H$.

*Case* C. Let there exist an $m$ such that $1 \leqslant m \leqslant \min(d, c + 1)$ and $| v_m | > Q$; assume that $m$ is the smallest such index, i.e., $| v_j | \leqslant Q$ for $1 \leqslant$

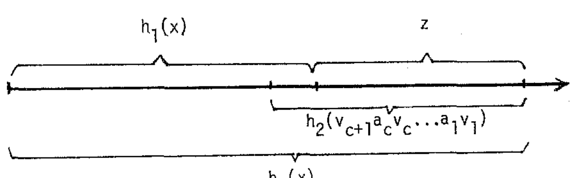


FIGURE 1

$j < m$. By Eq. (11) we have $\beta^{\Omega}(a_d v_d \cdots v_{m+1} a_m) = 0$, this implies that $h_i(a_d v_d \cdots v_{m+1} a_m) = z u_i$ for some $z \in \Sigma^* \Omega$ and $u_i \in \Pi^*$ where $u_i$ is a suffix of $h_i(a_m)$, for $i = 1, 2$. Therefore $\beta(a_d v_d \cdots v_{m+1} a_m) \leqslant H$. Also $\beta(v_m) \leqslant \beta'(a_m v_m) + 2H \leqslant c_{\Pi} + 2H$, where $\beta'$ is the balance with respect to the pair $(G_1{}^w, G_2{}^w)$ for a suitable $w \in S$. Such a $w$ exists since every $G_1$-substring $y$ such that $y \in \Omega\Pi^*$ and $|y| \geqslant Q + 1$ is a $G_1{}^w$-prefix for some $w \in S$. Finally, $\beta(a_{m-1} v_{m-1} \cdots a_1 v_1) \leqslant (m-1) H(Q+1) \leqslant c(Q+1)H$. Since $\beta(x) \leqslant \beta(a_d v_d \cdots v_{m+1} a_m) + \beta(v_m) + \beta(a_{m-1} v_{m-1} \cdots a_1 v_1)$, $\beta(x)$ is bounded for all $G_1$-prefixes belonging to Case C.

*Case* D. There are only finitely many $G_1$-prefixes not considered in the previous cases, thus we may conclude that the balance is bounded on all $G_1$-prefixes ▌

We have completed the proof of Claim 2 and will continue with Case III of the proof of Theorem 3. We will consider four subcases.

*Subcase* IIIA. Let $\Sigma_c = \Pi \cup \{a\}$, i.e., $\Omega_c = \Sigma_c - \Pi = \{a\}$, and $h_1{}^{\Omega}(a) = h_2{}^{\Omega}(a) = a$. Let $p \geqslant 1$ be the smallest integer such that if $\sigma = bud$, then $h_1{}^p(bud) = bvd$, for some $v$ in $\Sigma^*$. Then for all $n \geqslant 0$ the first (last) symbol of $h_1{}^n(\sigma)$ and of $h_1^{n+p}(\sigma)$ are the same.

Consider any pair of $lr$-systems from the i-combination of $(G_1, G_2)$, say $(G_1{}^m, G_2{}^m)$ where $G_i{}^m = (\Sigma, \bar{h}_i, \sigma_m)$ for $i = 1, 2$. We proceed to show that $(G_1{}^m, G_2{}^m)$ has bounded balance. Let $\sigma_m = bud$ for some $b, d \in \Omega$, clearly $\bar{h}_1{}^n(\sigma_m) \in b\Pi^* d$ for all $n \geqslant 0$.

Denote by $l_i$, $r_i$ the number of occurrences of $a$ in $\bar{h}_i(b)$ and $\bar{h}_i(d)$, respectively $(i = 1, 2)$. As $l_i + r_i$ is the number by which the number of occurrences of $a$ is increased when $\bar{h}_i$ is applied to any string $bwd$ with $w \in \Sigma_c{}^*$, we have $l_1 + r_1 = l_2 + r_2$. Without loss of generality we may assume that $l_1 \geqslant l_2$.

If $l_1 = l_2$, then also $r_1 = r_2$ and clearly $\beta^{\Omega}(x) = 0$ for every $G_1{}^m$-prefix. Therefore, by Claim 2 the pair $(G_1{}^m, G_2{}^m)$ has bounded balance. Since this is true for every pair from the $p$-combination of $(G_1, G_2)$ the pair $(G_1, G_2)$ has also bounded balance by Lemma 7.

It remains to consider the case $l_1 > l_2$. For each $n \geqslant 0$ we can write $\bar{h}_1{}^n(\sigma) = bv_1^{(n)} a v_2^{(n)} \cdots a v_{s_n}^{(n)} d$, where $v_j^{(n)} \in \Pi^*$ for $j = 1, \ldots, s_n$. The number of occurrences of $a$ in $\bar{h}_1{}^n(b)$ is $nl_1$, thus $bv_1^{(n)} a \cdots a v_{nl_1} a$ is a prefix of $h_1^{n'}(b)$ for each $n' \geqslant n$. Therefore $v_j^{(n')} = v_j^{(n)}$ for all $n$, $n'$ and $j = 1, 2, \ldots, \min(n, n')\, l_1$. Symmetrically we get $v_{s_{n'}-j}^{(n')} = v_{s_n-j}^{(n)}$ for $j = 1, 2, \ldots, \min(n, n')\, r_2$.

Let $q > (l_1 + r_1 + s_0)/(l_1 - l_2)$. Consider any $v_j^{(n)}$ for $n > q$. If $j \leqslant (n-1)l_1$, then

$$v_j^{(n)} = v_j^{(n-1)}; \tag{12}$$

if $j \geqslant s_{n-1} - (n-1)r_2$, then

$$v_j^{(n)} = v_{s_{n-1}-j}^{(n-1)}. \tag{13}$$

Since $s_n = s_0 + n(l_1 + r_1)$ we have $s_{n-1} - (n-1)\, r_2 - (n-1)\, l_1 =$

$s_0 + (n-1)(l_1 + r_1) - (n-1)\, r_2 - (n-1)\, l_1 = s_0 - (n-1)(r_2 - r_1) = s_0 - (n-1)(l_1 - l_2) < s_0 - (l_1 + r_1 + s_0) < 0$. The inequality follows from the choice of $q$ and $n$ above. Hence, all $j = 1, 2,..., s_n$ are considered in either Eq. (12) or (13). Since this is so for all $n > q$ we conclude by induction that, for each $n > q$, all the substrings of $\bar{h}_1{}^n(\sigma)$ occurring between two consecutive $a$'s have already occurred in $\bar{h}_1{}^q(\sigma)$. Therefore, there is only a finite number of distinct substrings from $\Pi$, thus $\Pi$ is limited and the pair $(G_1{}^m, G_2{}^m)$ has bounded balance by Lemma 6. Since this is true for each pair in the **i**-combination of $(G_1, G_2)$ the pair $(G_1, G_2)$ also has bounded balance by Lemma 7. This concludes Subcase IIIA.

*Subcase* IIIB. Let $\Omega_c = \{a\}$ and $h_1{}^\Omega(a) = h_2{}^\Omega(a) = \epsilon$. Since here the symbol $a$ can occur only in $h_i(b)$ for $b \in \Sigma_l \cup \Sigma_r$, we can write the string $h_1{}^n(\sigma)$ for each $n \geqslant 1$ in the form $lu_1a_1u_2 \cdots u_k a_k w b_m v_m \cdots b_1 v_1 r$ where $l \in \Sigma_l$, $r \in \Sigma_r$, $a_j \in \Omega_c$, $u_j \in \Pi^*$, $|u_j| < H$, for $j = 1,..., k$, $b_j \in \Omega_c$, $v_j \in \Pi^*$, $|v_j| < H$, for $j \in 1, 2,..., m$ and $w \in \Pi^*$.

Since $G_1$ and $G_2$ are equivalent we have $h_1{}^\Omega(l') = la_1 \cdots a_k = h_2{}^\Omega(l')$ where $l'$ is the first symbol in $h_1^{n-1}(\sigma)$. Since $\beta^\Omega(u_1a_1 \cdots u_h a_h) = 0$, we have $\beta^\Omega(lu_1a_1 \cdots u_h a_h) = |la_1 \cdots a_h| - |la_1 \cdots a_h| = 0$. As $w$ is the only maximal (i.e., with neighbors from $\Omega$) substring over $\Pi$ which can be longer than $H^c$ we can apply Claim 2 and conclude that the pair $(G_1, G_2)$ has bounded balance.

*Subcase* IIIC. Let $\Omega_c = \{a\}$, $h_1{}^\Omega(a) = \epsilon$ and $h_2{}^\Omega(a) \neq \epsilon$. We consider the combination of $(G_1, G_2)$. For the homomorphisms $\bar{h}_1$, $\bar{h}_2$ from the combination we have $\bar{h}_1(a) = \bar{h}_2(a) = \epsilon$, which is the Subcase IIIB. Finally, the pair $(G_1, G_2)$ has bounded balance by Lemma 7. Similarly for $h_1{}^\Omega(a) \neq \epsilon$ and $h_2{}^\Omega(a) = \epsilon$.

*Subcase* IIID. Let $h_1{}^\Omega$ and $h_2{}^\Omega$ be propagating and either $\Omega_c$ contains more than one symbol, or if $\Omega_c = \{a\}$, then $h_1{}^\Omega(a) \neq a$.

We show that the assumption of Claim 2 is satisfied. Let $wav$ be a $G_1$-prefix, where $w \in \Sigma^*$, $a \in \Omega_c$ and $v \in \Pi^*$ with $|v| > H^c$. Denote $\beta^\Omega(wa)$ by $p$ and assume that $p > 0$, i.e., one of the strings $h_1(wa)$ and $h_2(wa)$ is a proper prefix of the other, say $h_2(wa) = h_1(wa)z$, where $z$ contains $p$ occurrences of symbols from $\Omega_c$. We may write (see Fig. 2)

$$h_2(wav) = h_1(wa)\, z h_2(v) = h_1(wa)\, u_0 b_1 \cdots b_p u_p \tag{14}$$

where $b_1, b_2,..., b_p \in \Omega_c$ and $u_0, u_1,..., u_p \in \Pi^*$. Note that $h_2(v)$ is a suffix of $u_p$ and since $G_2$ is propagating we have $|u_p| > H^c$.

Now, we will show that

$$|u_j| \leqslant H^c, \qquad \text{for} \quad j = 0,..., p-1. \tag{15}$$

If Eq. (15) does not hold, there is $s$, $0 \leqslant s \leqslant p-1$, such that $|u_s| > H^c$ and, by Lemma 9, $h_i{}^\Omega(b_j) = b_j$ for all $j = s+1,..., p$ and $i = 1, 2$. This is in contra-

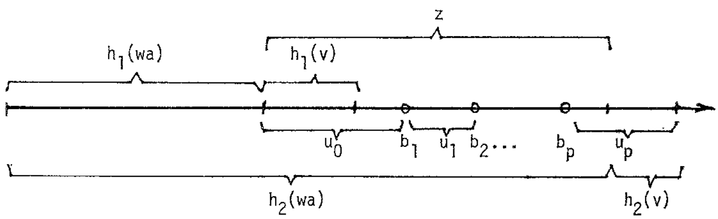


FIGURE 2

diction with the assumption that $\Pi$ is a maximal subalphabet as we can add any one of the $b_j$ ($j = s + 1,..., p$) to $\Pi$ to obtain a larger subalphabet. Note that since $\Omega_c$ does not consist of a single symbol $a$ such that $h_1{}^{\Omega}(a) = h_2{}^{\Omega}(a) = a$ the enlargement of $\Pi$ is properly contained in $\Sigma_c$, and therefore it is in fact a subalphabet. Hence Eq. (15) is established.

However, using Eq. (14) we see that $h_1(v)$ is a prefix of $u_0$ and since $G_1$ is propagating we have $| u_0 | \geqslant | h_1(v)| \geqslant | v | > H^c$, which is in contradiction to Eq. (15). Thus the assumption $p > 0$ is false, and we have $\beta^{\Omega}(wa) = 0$. Finally, we conclude using Claim 2 that the pair $(G_1, G_2)$ has bounded balance also in this last subcase. That completes the proof of Theorem 3. ▮

COROLLARY 1. *The sequence equivalence problem for DOL-system is decidable.*

*Proof.* Theorem 3 shows that the family of normal systems is smooth in the terminology of Culik (1975); therefore, the sequence equivalence problem is decidable for this family by Theorem 2.1 from Culik (1975). Thus, by Theorem 1, the problem is decidable for all DOL-systems. ▮

COROLLARY 2. *Given two DOL-systems $G_1$, $G_2$, it is decidable whether $L(G_1) = L(G_2)$.*

*Proof.* By Corollary 1 and Nielsen (1974). ▮

## 6. REGULAR ENVELOPES

We have shown that every pair of equivalent normal DOL-systems has bounded balance. This bounded balance was then used to construct a decision algorithm to test the equivalence. There is another property which is equivalent to bounded balance and which is quite interesting, but as the following facts are not needed for the main result we will state them without a proof.

DEFINITION 7. Let $G_i = (\Sigma, h_i, \sigma)$, $i = 1, 2$ be two DOL-systems. We say that a set $R$ is a *true envelope* for the pair $(G_1, G_2)$ if

(i) $L(G_1) \subseteq R$ and $L(G_2) \subseteq R$,

(ii) $h_1(x) = h_2(x)$ for all $x \in R$.

Obviously, if a pair $(G_1, G_2)$ has a true envelope then $G_1, G_2$ are equivalent.

THEOREM 4. *Let $G_i = (\Sigma, h_i, \sigma)$, $i = 1, 2$ be two equivalent DOL-systems. Then the pair $(G_1, G_2)$ has bounded balance iff there exists a regular set $R$ which is a true envelope of $(G_1, G_2)$.*

The proof is independent of Theorem 3 and the main idea is in the fact that the bound on the balance is also a bound on the number of states of an automaton which compares prefixes of $L(G_1)$ and $L(G_2)$. In more details, if $x$ is an $G_1$ prefix then either

$$h_1(x) = h_2(x)z \tag{16}$$

or

$$h_2(x) = h_1(x)z \tag{17}$$

for some $z \in \Sigma^*$. The relations (16) and (17) enable us to introduce a congruence relation $x \equiv x'$ if (16) or (17) holds with the same $z$. If the congruence is finite, we have a finite automaton, but this also gives the bound on the balance as the maximum length of $z$. ∎

The existence of a regular true envelope also gives an alternative, but essentially the same construction for the algorithm which decides a possible equivalence.

THEOREM 5. *If every pair of equivalent DOL-systems has a regular true envelope, then the sequence equivalence problem for DOL-systems is recursively decidable.*

*Proof.* Let $R_1, R_2, \ldots, R_k, \ldots$ be any effective enumeration of regular sets (more precisely their representatives, say finite automata), which of course exists. For each $k = 1, 2, \ldots$, check whether $R_k$ is a true envelope of $(G_1, G_2)$. Condition (i) is equivalent to $L(G_1) \cap \bar{R} = 0$, $\bar{R}$ is again regular, and for a DOL-system and a regular set we can effectively find EOL-system $G'$ so that $L(G') = L(G_1) \cap \bar{R}$. Finally, emptiness problem is decidable for EOL-systems. Condition (ii) can clearly be checked since it is enough to check it for finitely many strings, e.g., only for simple paths and loops of a finite automaton representing $R$. From our assumption we know that if $G_1, G_2$ are equivalent then there exists a true envelope for $(G_1, G_2)$ and we will find this true envelope in our enumeration, therefore our procedure will always halt in that case and gives a semi-decision procedure for equivalence. Since a semi-decision procedure for nonequivalence obviously exists we have completed the proof. ∎

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