It Is Easy to Be Wise After the Event: Communicating Finite-State Machines Capture First-Order Logic with "Happened Before"

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Abstract -

Message sequence charts (MSCs) naturally arise as executions of communicating finite-state machines (CFMs), in which finite-state processes exchange messages through unbounded FIFO channels. We study the first-order logic of MSCs, featuring Lamport's happened-before relation. Our main result states that every first-order sentence can be transformed into an equivalent CFM. This answers an open question and settles the exact relation between CFMs and fragments of monadic second-order logic. As a byproduct, we obtain self-contained normal-form constructions for first-order logic over MSCs (and, therefore, over words). In particular, we show that first-order logic has the three-variable property.

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1 Introduction

First-order (FO) logic can be considered, in many ways, a reference specification language. It plays a key role in automated theorem proving and formal verification. In particular, FO logic over finite or infinite words is central in the verification of reactive systems. When a word is understood as a total order that reflects a chronological succession of events, it represents an execution of a sequential system. Apart from being a natural concept in itself, FO logic over words enjoys manifold characterizations. It defines exactly the star-free languages and coincides with recognizability by aperiodic monoids or natural subclasses of finite (Büchi, respectively) automata (cf. [7,25] for overviews). Moreover, linear-time temporal logics are usually measured against their expressive power with respect to FO logic. For example, LTL is considered the yardstick temporal logic not least due to Kamp's famous theorem, stating that LTL and FO logic are expressively equivalent [16].

While FO logic on words is well understood, a lot remains to be said once concurrency enters into the picture. When several processes communicate through, say, unbounded first-in first-out (FIFO) channels, actions are only partially ordered and the behavior, which is

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referred to as a message sequence chart (MSC), reflects Lamport's happened-before relation: an event e happens before an event f if, and only if, there is a "message flow" path from e to f [18]. Communicating finite-state machines (CFMs) [4] are to MSCs what finite automata are to words: a canonical model of finite-state processes that communicate through unbounded FIFO channels. Therefore, the FO logic of MSCs can be considered a canonical specification language for such systems. Unfortunately, its study turned out to be difficult, since algebraic and automata-theoretic approaches that work for words, trees, or Mazurkiewicz traces do not carry over. In particular, until now, the following central problem remained open:

Can every first-order formula be transformed into an equivalent communicating finite-state machine, without any channel bounds?

Partial answers were given for CFMs with bounded channel capacity [11, 15, 17] and for fragments of FO that restrict the logic to bounded-degree predicates [3] or to two variables [1].

In this paper, we answer the general question positively. To do so, we combine automata-theoretic techniques with new normal-form constructions for FO logic. Actually, we make a detour through a variant of propositional dynamic logic (PDL) [10] that we call star-free. In doing so, we provide a normal form of FO formulas that is of independent interest. It implies that every FO formula over message sequence charts, with arbitrarily many free variables, is equivalent to a disjunction of conjuncts that have at most two free variables. While this first reduction is based on purely logical considerations, the translation of star-free PDL into CFMs exploits a new gossiping technique, which allows us to cope with the intersection operator of PDL. As a byproduct, we establish that every FO sentence over MSCs can be rewritten into one that uses only three different variable names. Note that this result is already non-trivial in the special case of words, where it follows from Kamp's theorem [16]. As a further application, we obtain self-contained proofs of several results on channel-bounded CFMs whose original proofs refer to involved constructions for Mazurkiewicz traces.

Outline. In Section 2, we recall basic notions such as MSCs, FO logic, and CFMs. We also state known results as well as our main results. Section 3 presents our star-free version of PDL, which, as shown in Section 4, captures FO logic. In Section 5, we establish the translation of star-free PDL into CFMs. We conclude in Section 6 mentioning applications of our results. Some proof details can be found in the appendix.

2 Definitions and Results

We consider message-passing systems in which processes communicate through unbounded FIFO channels. We fix a nonempty finite set of processes P and a nonempty finite set of labels Σ . For all $p, q \in P$ such that $p \neq q$, there is a channel (p, q) that allows p to send messages to q. The set of channels is denoted Ch.

In the following, we define message sequence charts, which represent executions of a message-passing system, and logics to reason about them. Then, we recall the definition of communicating finite-state machines and state our main results.

2.1 Message Sequence Charts

A message sequence chart (MSC) (over P and Σ) is a graph $M=(E,\to,\lhd,loc,\lambda)$ with finite set of nodes E, edge relations $\to,\lhd\subseteq E\times E$, and node-labeling functions $loc:E\to P$

and $\lambda: E \to \Sigma$. An example MSC is depicted in Figure 1. A node $e \in E$ is an *event* that is executed by process $loc(e) \in P$. In particular, $E_p := \{e \in E \mid loc(e) = p\}$ is the set of events located on p. The label $\lambda(e) \in \Sigma$ may provide more information about e such as the message that is sent/received at e or "enter critical section" or "output some value".

Edges describe causal dependencies between events:

- The relation \to contains process edges. They connect successive events executed by the same process. That is, we actually have $\to \subseteq \bigcup_{p \in P} (E_p \times E_p)$. Every process p is sequential so that $\to \cap (E_p \times E_p)$ must be the direct-successor relation of some total order on E_p . We let $\leq_{\mathsf{proc}} := \to^*$ and $<_{\mathsf{proc}} := \to^+$.
- The relation \lhd contains message edges. If $e \lhd f$, then e is a send event and f is the corresponding receive event. Each event is part of at most one message edge. An event that is neither a send nor a receive event is called internal. Moreover, for all $(p,q) \in Ch$ and $(e,f),(e',f') \in \lhd \cap (E_p \times E_q)$, we have $e \leq_{\mathsf{proc}} e'$ iff $f \leq_{\mathsf{proc}} f'$ (which guarantees a FIFO behavior).

We require that $\to \cup \lhd$ be acyclic (intuitively, messages cannot travel backwards in time). The associated partial order is denoted $\leq := (\to \cup \lhd)^*$ with strict part $< = (\to \cup \lhd)^+$. We do not distinguish isomorphic MSCs.

Actually, MSCs are very similar to the space-time diagrams from Lamport's seminal paper [18], and \leq is commonly referred to as the *happened-before relation*.

▶ Example 1. Consider the MSC from Figure 1 over $P = \{p_1, p_2, p_3\}$ and $\Sigma = \{\Box, \Diamond, \diamondsuit\}$. We have, for instance, $E_{p_1} = \{e_0, \ldots, e_7\}$. The process relation is given by $e_i \to e_{i+1}$, $f_i \to f_{i+1}$, and $g_i \to g_{i+1}$ for all $i \in \{0, \ldots, 6\}$. Concerning the message relation, we have $e_1 \lhd f_0$, $e_4 \lhd g_5$, etc. Moreover, $e_2 \leq f_3$, but neither $e_2 \not\leq f_1$ nor $f_1 \not\leq e_2$.

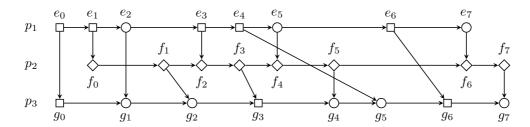


Figure 1 A message sequence chart (MSC)

It is worth noting that, when P is a singleton, an MSC with events $e_1 \to e_2 \to \ldots \to e_n$ can be identified with the word $\lambda(e_1)\lambda(e_2)\ldots\lambda(e_n)\in\Sigma^*$.

Bounded MSCs. The first logical characterizations of communicating finite-state machines were obtained for classes of *bounded* MSCs. Intuitively, this corresponds to restricting the channel capacity. Bounded MSCs are defined in terms of linearizations. A *linearization* of the MSC $M = (E, \rightarrow, \lhd, loc, \lambda)$ is a total order $\preceq \subseteq E \times E$ such that $\subseteq \subseteq \bot$. For $B \in \mathbb{N}$, we call $\preceq B$ -bounded if, for all $g \in E$ and $(p,q) \in Ch$, $|\{(e,f) \in \lhd \cap (E_p \times E_q) \mid e \preceq g \prec f\}| \subseteq B$. In other words, the number of pending messages in (p,q) never exceeds B. There are (at least) two natural definitions of bounded MSCs: We call $M \exists B$ -bounded if M has *some* bounded linearization. Accordingly, it is $\forall B$ -bounded if all its linearizations are B-bounded.

▶ Example 2. The MSC from Figure 1 is $\exists 1$ -bounded and $\forall 4$ -bounded. These bounds are tight: for all $B \in \{1, 2, 3\}$, the MSC is not $\forall B$ -bounded, because the four send events for, say, channel (p_1, p_3) can be scheduled before the first reception g_0 .

Let $\mathbb{MSC}(P,\Sigma)$ denote the set of MSCs over P and Σ . The sets of $\exists B$ -bounded and $\forall B$ -bounded MSCs are denoted by $\mathbb{MSC}_{\exists B}(P,\Sigma)$ and $\mathbb{MSC}_{\forall B}(P,\Sigma)$, respectively.

2.2 MSO Logic and Its Fragments

Next, we give an account of monadic second-order (MSO) logic and its fragments. We fix an infinite supply of first-order variables x, y, \ldots , which range over events of an MSC, as well as an infinite supply of second-order variables X, Y, \ldots , ranging over sets of events. The syntax of MSO (we consider that P and Σ are fixed) is given as follows:

$$\Phi ::= p(x) \mid a(x) \mid x = y \mid x \to y \mid x \lhd y \mid x \leq y \mid x \in X \mid \Phi \lor \Phi \mid \neg \Phi \mid \exists x. \Phi \mid$$

where $p \in P$, $a \in \Sigma$, x, y are first-order variables, and X is a second-order variable. We use the usual abbreviations to also include implication \Longrightarrow , conjunction \wedge , and universal quantification \forall . Moreover, the relation $x \leq_{\mathsf{proc}} y$ can be defined by $x \leq y \wedge \bigvee_{p \in P} p(x) \wedge p(y)$.

We write $\Phi(x_1,\ldots,x_m,X_1,\ldots,X_n)$ to emphasize that the free variables of Φ are among (but not necessarily equal to) $\{x_1,\ldots,x_m,X_1,\ldots,X_n\}$. Let $M=(E,\to,\lhd,loc,\lambda)$ be an MSC and $\nu:\{x_1,\ldots,x_m,X_1,\ldots,X_n\}\to E\cup 2^E$ be an interpretation mapping a first-order variable x_i to an event $\nu(x_i)\in E$, and a second-order variable X_i to a set of events $\nu(X_i)\subseteq E$. We write $M,\nu\models\Phi(x_1,\ldots,x_m,X_1,\ldots,X_n)$ if M satisfies Φ when $x_1,\ldots,x_m,X_1,\ldots,X_n$ are interpreted according to ν . Hereby, satisfaction is defined in the usual manner. For example, for $\Phi(x,y)=(x\lhd y)$, we have $M,[x\mapsto e,y\mapsto f]\models\Phi(x,y)$ iff $e\lhd f$.

We identify two important fragments of MSO logic: First-order (FO) formulas do not make use of second-order quantification (however, they may contain formulas $x \in X$). Moreover, existential MSO (EMSO) formulas are of the form $\exists X_1 \ldots \exists X_n . \Phi$ with $\Phi \in FO$.

Let $\mathcal{F} \subseteq MSO$ be a set of formulas. For $R \subseteq \{\rightarrow, \lhd, \leq\}$, we obtain the logic $\mathcal{F}[R]$ by restricting \mathcal{F} to formulas that do not make use of $\{\rightarrow, \lhd, \leq\} \setminus R$. Note that MSO equals $MSO[\rightarrow, \lhd, \leq]$. Moreover, for $k \in \mathbb{N}$, let $\mathcal{F}^k[R]$ be the set of formulas from $\mathcal{F}[R]$ that use at most k different first-order variables (however, a variable can occur several times in a formula).

For a sentence $\Phi \in MSO$ (without free variables), we define $L(\Phi) := \{M \in MSC(P, \Sigma) \mid M \models \Phi\}$. Moreover, given a class $\mathcal{F} \subseteq MSO$ and $B \in \mathbb{N}$, we let $\mathcal{L}(\mathcal{F}) := \{L(\Phi) \mid \Phi \in \mathcal{F} \text{ is a sentence}\}$, $\mathcal{L}_{\exists B}(\mathcal{F}) := \{L(\Phi) \cap MSC_{\exists B}(P, \Sigma) \mid \Phi \in \mathcal{F} \text{ is a sentence}\}$, and $\mathcal{L}_{\forall B}(\mathcal{F}) := \{L(\Phi) \cap MSC_{\forall B}(P, \Sigma) \mid \Phi \in \mathcal{F} \text{ is a sentence}\}$.

Since the reflexive transitive closure of an MSO-definable binary relation is MSO-definable, we have $\mathcal{L}(MSO) = \mathcal{L}(MSO[\rightarrow, \lhd])$, i.e., MSO and MSO $[\rightarrow, \lhd]$ have the same expressive power. However, MSO $[\leq]$ (without the message relation) is strictly weaker than MSO [3].

Example 3. We give an FO formula that allows us to recover, at some event f, the most recent event e that happened in the past on, say, process p. More precisely, we define the predicate $latest_p(x,y)$ as $x \le y \land p(x) \land \forall z ((z \le y \land p(z)) \implies z \le x)$. The "gossip language" says that process q always maintains the latest information that it can have about p. Thus, it is defined by $\Phi_{p,q}^{\text{gossip}} = \forall x \forall y. ((latest_p(x,y) \land q(y)) \implies \bigvee_{a \in \Sigma} (a(x) \land a(y))) \in \text{FO}^3[\le]$. For example, for $P = \{p_1, p_2, p_3\}$ and $\Sigma = \{\Box, \bigcirc, \diamondsuit\}$, the MSC M from Figure 1 is contained in $L(\Phi_{p_1, p_3}^{\text{gossip}})$. In particular, M, $[x \mapsto e_5, y \mapsto g_5] \models latest_{p_1}(x, y)$ and $\lambda(e_5) = \lambda(g_5) = \bigcirc$.

2.3 Communicating Finite-State Machines

In a communicating finite-state machine, each process $p \in P$ can perform internal actions of the form $\langle a \rangle$, where $a \in \Sigma$, or send/receive messages from a finite set of messages Msg. A send action $\langle a, !_q m \rangle$ of process p writes message $m \in Msg$ to channel (p,q), and performs $a \in \Sigma$. A receive action $\langle a, ?_q m \rangle$ reads message m from channel (q,p). Accordingly, we let $Act_p(Msg) := \{\langle a \rangle \mid a \in \Sigma\} \cup \{\langle a, !_q m \rangle \mid a \in \Sigma, m \in Msg, q \in P \setminus \{p\}\} \cup \{\langle a, ?_q m \rangle \mid a \in \Sigma, m \in Msg, q \in P \setminus \{p\}\}$ denote the set of possible actions of process p.

A communicating finite-state machine (CFM) over P and Σ is a tuple $((\mathcal{A}_p)_{p\in P}, Msg, Acc)$ consisting of a finite set of messages Msg and a finite-state transition system $\mathcal{A}_p = (S_p, \iota_p, \Delta_p)$ for each process p, with finite set of states S_p , initial state $\iota_p \in S_p$, and transition relation $\Delta_p \subseteq S_p \times Act_p(Msg) \times S_p$. Moreover, we have an acceptance condition $Acc \subseteq \prod_{p\in P} S_p$.

Given a transition $t=(s,\alpha,s')\in\Delta_p$, we let source(t)=s and target(t)=s' denote the source and target states of t. In addition, if $\alpha=\langle a\rangle$, then t is an internal transition and we let label(t)=a. If $\alpha=\langle a,!_qm\rangle$, then t is a send transition and we let label(t)=a, msg(t)=m, and receiver(t)=q. Finally, if $\alpha=\langle a,?_qm\rangle$, then t is a receive transition with label(t)=a, msg(t)=m, and sender(t)=q.

A run ρ of \mathcal{A} over an MSC $M=(E,\to,\lhd,loc,\lambda)\in \mathbb{MSC}(P,\Sigma)$ is a mapping associating with each event $e\in E_p$ a transition $\rho(e)\in\Delta_p$, and satisfying the following conditions:

- **1.** for all events $e \in E$, we have $label(\rho(e)) = \lambda(e)$,
- **2.** for all \rightarrow -minimal events $e \in E$, we have $source(\rho(e)) = \iota_p$, where p = loc(e),
- **3.** for all process edges $(e, f) \in A$, we have $target(\rho(e)) = source(\rho(f))$,
- **4.** for all internal events $e \in E$, $\rho(e)$ is an internal transition, and
- **5.** for all message edges $(e, f) \in \triangleleft$, $\rho(e)$ and $\rho(f)$ are respectively send and receive transitions such that $msg(\rho(e)) = msg(\rho(f))$, $receiver(\rho(e)) = loc(f)$, and $sender(\rho(f)) = loc(e)$.

To determine whether ρ is accepting, we collect the last state s_p of every process p. If $E_p \neq \emptyset$, we let $s_p = target(\rho(e))$ where e is the last event of E_p . Otherwise, $s_p = \iota_p$. We say that ρ is accepting if $(s_p)_{p \in P} \in Acc$.

The language $L(\mathcal{A})$ of \mathcal{A} is the set of MSCs M such that there exists an accepting run of \mathcal{A} on M. Moreover, $\mathcal{L}(CFM) := \{L(\mathcal{A}) \mid \mathcal{A} \text{ is a CFM}\}$, and for $B \in \mathbb{N}$, we let $\mathcal{L}_{\exists B}(CFM) := \{L(\mathcal{A}) \cap \mathbb{MSC}_{\exists B}(P, \Sigma) \mid \mathcal{A} \text{ is a CFM}\}$ and $\mathcal{L}_{\forall B}(CFM) := \{L(\mathcal{A}) \cap \mathbb{MSC}_{\forall B}(P, \Sigma) \mid \mathcal{A} \text{ is a CFM}\}$. Recall that, for all these definitions, we have fixed P and Σ .

2.4 Known Facts and Main Results

Before presenting our main results, let us give a brief account of what is already known on the relation between logic and CFMs.

▶ Fact 4 ([5,9,26]). Suppose |P| = 1 (i.e., CMFs are essentially finite automata). We have $\mathcal{L}(MSO) = \mathcal{L}(CFM)$.

This classical result is known as the Büchi-Elgot-Trakhtenbrot theorem. It has been generalized to CFMs with bounded channels:

▶ Fact 5 ([11,12,15,17]). Let $B \in \mathbb{N}$. We have $\mathcal{L}_{\forall B}(MSO) = \mathcal{L}_{\forall B}(CFM)$ (and every formula is equivalent to a deterministic CFM) [15,17] and $\mathcal{L}_{\exists B}(MSO) = \mathcal{L}_{\exists B}(CFM)$ (and CFMs are inherently nondeterministic) [11,12].

The proofs of these characterizations reduce message-passing systems to finite-state shared-memory systems so that involved results from Mazurkiewicz trace theory [8] can be applied. This generic approach is no longer applicable when the restriction on the channel capacity is dropped. In fact, in general, CFMs do not capture MSO logic:

▶ Fact 6 ([1,3]).
$$\mathcal{L}(\text{EMSO}[\rightarrow, \triangleleft]) = \mathcal{L}(\text{CFM}) = \mathcal{L}(\text{EMSO}^2[\rightarrow, \triangleleft, \leq]) \subsetneq \mathcal{L}(\text{MSO}).$$

Thus, previous logical characterizations of CFMs restricted the set of binary relations or the number of first-order variables. Our main result states that CFMs and the *full* EMSO logic are expressively equivalent. This solves a problem that was stated as open in [12]:

▶ Theorem 7. $\mathcal{L}(\mathrm{EMSO}[\rightarrow, \lhd, \leq]) = \mathcal{L}(\mathrm{CFM}).$

It is standard to prove $\mathcal{L}(CFM) \subseteq \mathcal{L}(EMSO^2[\rightarrow, \lhd])$. As, moreover, the class $\mathcal{L}(CFM)$ is closed under projection, the proof of Theorem 7 comes down to the following inclusion:

▶ Proposition 8. $\mathcal{L}(FO[\rightarrow, \triangleleft, \leq]) \subseteq \mathcal{L}(CFM)$.

Note that the translation from FO[\rightarrow , \triangleleft , \leq] to CFMs is inherently nonelementary, already when |P|=1 [23].

The rest of the paper is dedicated to the proof of Proposition 8. As a byproduct, we will obtain the following result of own interest:

▶ Theorem 9. $\mathcal{L}(FO[\rightarrow, \triangleleft, \leq]) = \mathcal{L}(FO^3[\rightarrow, \triangleleft, \leq]).$

3 Star-Free Propositional Dynamic Logic

3.1 Syntax and Semantics

Originally, propositional dynamic logic (PDL) has been used to reason about program schema and transition systems [10]. Since then, PDL and its extension with intersection and converse have developed a rich theory with applications in artificial intelligence and verification [6, 13, 14, 19, 20]. It has also been applied in the context of MSCs [2, 22].

Here, we introduce a *star-free* version of PDL, denoted PDL_{sf}. It will serve as an "interface" between FO logic and CFMs. The syntax of PDL_{sf} and its fragment PDL_{sf}[Loop] is given by the following grammar:

$$\begin{split} \operatorname{PDL}_{\mathsf{sf}} &= \operatorname{PDL}_{\mathsf{sf}}[\mathsf{Loop}, \cup, \cap, \mathsf{c}] \\ \\ \operatorname{PDL}_{\mathsf{sf}}[\mathsf{Loop}] \quad \varphi &::= p \mid a \mid \varphi \vee \varphi \mid \neg \varphi \mid \langle \pi \rangle \, \varphi \mid \mathsf{Loop}(\pi) \\ \\ \pi &::= \rightarrow |\leftarrow| \vartriangleleft_{p,q} \mid \vartriangleleft_{p,q}^{-1} \mid \stackrel{\varphi}{\rightarrow} | \stackrel{\varphi}{\leftarrow} | \mathsf{jump}_{p,r} \mid \{\varphi\}? \mid \pi \cdot \pi \quad \pi \mid \pi \cap \pi \mid \pi^\mathsf{c} \end{split}$$

where $p, r \in P$, $q \in P \setminus \{p\}$, and $a \in \Sigma$. We refer to φ as an *event formula* and to π as a *path formula*. We name the logic star-free because we use the operators (\cup, \cap, c) of star-free regular expressions instead of the regular-expression operators $(\cup, *)$ of classical PDL. However, the formula $\xrightarrow{\varphi}$, whose semantics is explained below, can be seen as a restricted use of the construct π^* .

An event formula φ is evaluated wrt. a given MSC $M=(E,\to,\lhd,\log,\lambda)$ and an event $e\in E$. A path formula π , on the other hand, is interpreted over two events. In other words, it defines a binary relation $\llbracket\pi\rrbracket_M\subseteq E\times E$. We often write $M,e,f\models\pi$ to denote $(e,f)\in\llbracket\pi\rrbracket_M$. Moreover, for $e\in E$, we let $\llbracket\pi\rrbracket_M(e):=\{f\in E\mid (e,f)\in\llbracket\pi\rrbracket_M\}$. When M is clear from the context, we may write $\llbracket\pi\rrbracket$ instead of $\llbracket\pi\rrbracket_M$. The semantics of both event and path formulas is given in Table 1. As usual, logical equivalence of event/path formula is denoted by \equiv .

Table 1 The semantics of PDL_{sf}

$$M, e \models p \text{ if } loc(e) = p \qquad \qquad M, e \models \langle \pi \rangle \varphi \text{ if } \exists f \in \llbracket \pi \rrbracket_M(e) : M, f \models \varphi$$

$$M, e \models a \text{ if } \lambda(e) = a \qquad \qquad M, e \models \text{Loop}(\pi) \text{ if } (e, e) \in \llbracket \pi \rrbracket_M$$

$$M, e \models \neg \varphi \text{ if } M, e \not\models \varphi \qquad \qquad M, e \models \varphi_1 \vee \varphi_2 \text{ if } M, e \models \varphi_1 \text{ or } M, e \models \varphi_2$$

$$\llbracket \rightarrow \rrbracket_M := \{(e, f) \in E \times E \mid e \rightarrow f\} \qquad \llbracket \triangleleft_{p,q} \rrbracket_M := \{(e, f) \in E_p \times E_q \mid e \triangleleft f\}$$

$$\llbracket \leftarrow \rrbracket_M := \{(f, e) \in E \times E \mid e \rightarrow f\} \qquad \llbracket \triangleleft_{p,q} \rrbracket_M := \{(f, e) \in E_q \times E_p \mid e \triangleleft f\}$$

$$\llbracket \downarrow \rightarrow \rrbracket_M := \{(e, f) \in E \times E \mid e \rightarrow f\} \qquad \llbracket \{\varphi\}? \rrbracket_M := \{(e, e) \mid e \in E : M, e \models \varphi\}$$

$$\llbracket \stackrel{\varphi}{\rightarrow} \rrbracket_M := \{(e, f) \in E \times E \mid e \rightarrow f\} \qquad \exists f \in E : (e, f) \in F \Rightarrow f \in F \Rightarrow$$

▶ **Example 10.** Consider again the MSC M from Figure 1 and the path formula $\pi = \lhd_{p_1,p_3}^{-1} \to \lhd_{p_1,p_2} \to \lhd_{p_2,p_3} \to$. We have $M,g_5 \models \mathsf{Loop}(\pi)$. Moreover, $(e_2,e_5) \in \llbracket \overset{\square}{\to} \rrbracket_M$ but $(e_2,e_6) \notin \llbracket \overset{\square}{\to} \rrbracket_M$.

We use the usual abbreviations for event formulas such as implication and conjunction. Moreover, $true := p \lor \neg p$ (for some arbitrary process $p \in P$) and $false := \neg true$. Finally, we define the path formulas $\langle \pi \rangle := \langle \pi \rangle \ true$, $\xrightarrow{+} := \xrightarrow{true}$, and $\xrightarrow{*} := \xrightarrow{+} \cup \{true\}$?.

Note that there are some redundancies in the logic. For example, $\to \equiv \xrightarrow{false}$, $\pi_1 \cap \pi_2 \equiv (\pi_1^c \cup \pi_2^c)^c$, and $\mathsf{Loop}(\pi) \equiv \{true\}? \cap \pi$. Some of them are necessary to define certain subclasses of $\mathsf{PDL}_{\mathsf{sf}}$. For every $R \subseteq \{\mathsf{Loop}, \cup, \cap, \mathsf{c}\}$, we let $\mathsf{PDL}_{\mathsf{sf}}[R]$ denote the fragment of $\mathsf{PDL}_{\mathsf{sf}}$ that does not make use of $\{\mathsf{Loop}, \cup, \cap, \mathsf{c}\} \setminus R$. In particular, $\mathsf{PDL}_{\mathsf{sf}} = \mathsf{PDL}_{\mathsf{sf}}[\mathsf{Loop}, \cup, \cap, \mathsf{c}]$. Note that, syntactically, $\stackrel{\star}{\to}$ is not contained in $\mathsf{PDL}_{\mathsf{sf}}[\mathsf{Loop}]$ since union is not permitted.

3.2 Basic Properties

We establish some fundamental properties of PDL_{sf} and its fragments. First notice that the converse of a PDL_{sf} formula is definable in PDL_{sf} (easy induction on π).

▶ Lemma 11. Let $R \subseteq \{\mathsf{Loop}, \cup, \cap, \mathsf{c}\}$ and $\pi \in \mathsf{PDL_{sf}}[R]$ be a path formula. There exists $\pi^{-1} \in \mathsf{PDL_{sf}}[R]$ such that, for all MSCs M, $[\![\pi^{-1}]\!]_M = [\![\pi]\!]_M^{-1} = \{(f, e) \mid (e, f) \in [\![\pi]\!]_M\}$.

Given a PDL_{sf}[Loop] path formula π , we denote by $\mathsf{Comp}(\pi)$ the set of pairs $(p,q) \in P \times P$ such that there may be a π -path from some event on process p to some event on process q. Formally, we let $\mathsf{Comp}(\to) = \mathsf{Comp}(\leftarrow) = \mathsf{Comp}(\xrightarrow{\varphi}) = \mathsf{Comp}(\xleftarrow{\varphi}) = \mathsf{Comp}(\{\varphi\}) = \mathsf{id}$ where $\mathsf{id} = \{(p,p) \mid p \in P\}$; $\mathsf{Comp}(\lhd_{p,q}) = \mathsf{Comp}(\lhd_{q,p}) = \{(p,q)\}$; $\mathsf{Comp}(\mathsf{jump}_{p,r}) = \{(p,r)\}$; and $\mathsf{Comp}(\pi_1 \cdot \pi_2) = \mathsf{Comp}(\pi_2) \circ \mathsf{Comp}(\pi_1) = \{(p,r) \mid \exists q : (p,q) \in \mathsf{Comp}(\pi_1), (q,r) \in \mathsf{Comp}(\pi_2)\}$.

Notice that, for all path formulas $\pi \in \mathrm{PDL}_{\mathsf{sf}}[\mathsf{Loop}]$, the relation $\mathsf{Comp}(\pi)$ is either empty or a singleton $\{(p,q)\}$ or the identity id. Moreover, $M,e,f \models \pi$ implies $(\mathsf{proc}(e),\mathsf{proc}(f)) \in$

 $\mathsf{Comp}(\pi)$. Therefore, all events in $\llbracket \pi \rrbracket(e)$ are on the same process and if this set is nonempty, i.e., if $M, e \models \langle \pi \rangle$, then $\min \llbracket \pi \rrbracket(e)$ and $\max \llbracket \pi \rrbracket(e)$ are well-defined.

▶ **Example 12.** Consider the MSC from Figure 1 and $\pi = \xrightarrow{+} \lhd_{p_1,p_2} \to \lhd_{p_2,p_3} \to$. We have $\mathsf{Comp}(\pi) = \{(p_1, p_3)\}$. Moreover, $\min[\![\pi]\!](e_2) = g_4$ and $\max[\![\pi]\!](e_2) = g_5$.

We say that a path formula π in PDL_{sf}[Loop] is *monotone* if, for all MSCs M and events e, f such that $M, e \models \langle \pi \rangle$, $M, f \models \langle \pi \rangle$, and $e \leq_{\mathsf{proc}} f$, we have $\min[\![\pi]\!](e) \leq_{\mathsf{proc}} \min[\![\pi]\!](f)$ and $\max[\![\pi]\!](e) \leq_{\mathsf{proc}} \max[\![\pi]\!](f)$. The next two lemmas are easy to prove by a simultaneous induction (see Appendix A).

▶ **Lemma 13.** Let $\pi_1, \pi_2 \in PDL_{sf}[\mathsf{Loop}]$ be path formulas, and $\pi = \pi_1 \cdot \pi_2$. For all MSCs M and events e such that $M, e \models \langle \pi \rangle$, we have

$$\min[\![\pi]\!](e) = \min[\![\pi_2]\!](\min[\![\pi_1 \cdot \{\langle \pi_2 \rangle\}?]\!](e)) \ and$$
$$\max[\![\pi]\!](e) = \max[\![\pi_2]\!](\max[\![\pi_1 \cdot \{\langle \pi_2 \rangle\}?]\!](e)).$$

▶ Lemma 14. All PDL_{sf}[Loop] path formulas are monotone.

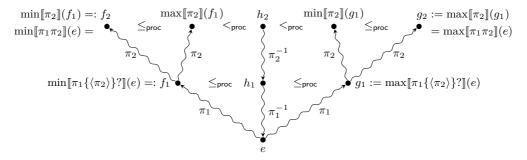
The following crucial lemma states that, for all path formulas $\pi \in PDL_{sf}[\mathsf{Loop}]$, the relation $\llbracket \pi \rrbracket$ can be characterized in terms of the interval delimited by $\min \llbracket \pi \rrbracket(e)$ and $\max \llbracket \pi \rrbracket(e)$.

▶ **Lemma 15.** Let π be a PDL_{sf}[Loop] path formula. For all MSCs M and events e such that $M, e \models \langle \pi \rangle$, we have

$$[\![\pi]\!](e) = \{f \in E \mid \min[\![\pi]\!](e) \leq_{\mathsf{proc}} f \leq_{\mathsf{proc}} \max[\![\pi]\!](e) \land M, f \models \langle \pi^{-1} \rangle \} \,.$$

Proof. The left-to-right inclusion is trivial. We prove the right-to-left inclusion by induction on π . The base cases are immediate.

Assume that $\pi = \pi_1 \cdot \pi_2$. For illustration, consider the figure below.



We let $f_1 = \min[\pi_1\{\langle \pi_2 \rangle\}?](e)$, $f_2 = \min[\pi_2](f_1)$, $g_1 = \max[\pi_1\{\langle \pi_2 \rangle\}?](e)$, and $g_2 = \max[\pi_2](g_1)$. By Lemma 13, we have $f_2 = \min[\pi_1\pi_2](e)$ and $g_2 = \max[\pi_1\pi_2](e)$. Let $h_2 \in E$ such that $f_2 \leq_{\mathsf{proc}} h_2 \leq_{\mathsf{proc}} g_2$ and $M, h_2 \models \langle (\pi_1\pi_2)^{-1} \rangle$. If $h_2 \leq_{\mathsf{proc}} \max[\pi_2](f_1)$, then by induction hypothesis, $M, f_1, h_2 \models \pi_2$, and we obtain $M, e, h_2 \models \pi_1\pi_2$. Similarly, if $\min[\pi_2](g_1) \leq_{\mathsf{proc}} h_2$, then $M, g_1, h_2 \models \pi_2$ and $M, e, h_2 \models \pi_1\pi_2$. So assume $\max[\pi_2](f_1) <_{\mathsf{proc}} h_2 <_{\mathsf{proc}} \min[\pi_2](g_1)$. Since $M, h_2 \models \langle \pi_2^{-1}\pi_1^{-1} \rangle$, there exists h_1 such that $M, h_1, h_2 \models \pi_2$ and $M, h_1 \models \langle \pi_1^{-1} \rangle$. Moreover, $\min[\pi_2](h_1) \leq_{\mathsf{proc}} h_2 <_{\mathsf{proc}} \min[\pi_2](g_1)$, hence $h_1 \leq_{\mathsf{proc}} g_1$ by Lemma 14 (notice that g_1 and h_1 must be on the same process). Similarly, $\max[\pi_2](f_1) <_{\mathsf{proc}} h_2 \leq_{\mathsf{proc}} \max[\pi_2](h_1)$, hence $f_1 \leq_{\mathsf{proc}} h_1$. We then have $f_1 \leq_{\mathsf{proc}} h_1 \leq_{\mathsf{proc}} g_1$, and $M, h_1 \models \langle \pi_1^{-1} \rangle$. By induction hypothesis, $M, e, h_1 \models \pi_1$. Hence, $M, e, h_2 \models \pi_1\pi_2$.

Using Lemma 13, we now show that, if π is a PDL_{sf}[Loop] formula, then the relation $\{(e, \min[\![\pi]\!](e))\}$ can also be expressed in PDL_{sf}[Loop] (and similarly for max).

▶ Lemma 16. Let $R = \emptyset$ or $R = \{\text{Loop}\}$. For every path formula $\pi \in \text{PDL}_{\mathsf{sf}}[R]$, there exist $\text{PDL}_{\mathsf{sf}}[R]$ path formulas $\min \pi$ and $\max \pi$ such that $M, e, f \models \min \pi$ iff $f = \min[\![\pi]\!](e)$, and $M, e, f \models \max \pi$ iff $f = \max[\![\pi]\!](e)$.

Proof. We construct by induction on π formulas $\min (\pi \cdot \{\psi\}?)$ for all $PDL_{sf}[R]$ event formulas ψ . For $\pi \in \{\rightarrow, \leftarrow, \lhd_{p,q}, \lhd_{p,q}^{-1}, \{\varphi\}?\}$, we let $\min (\pi \cdot \{\psi\}?) = \pi \cdot \{\psi\}?$. Then,

$$\begin{split} \min \ &(\xrightarrow{\varphi} \cdot \{\psi\}?) = \xrightarrow{\varphi \wedge \neg \psi} \cdot \{\psi\}? \\ \min \ &(\xleftarrow{\varphi} \cdot \{\psi\}?) = \xleftarrow{\varphi} \cdot \{\psi \wedge (\neg \varphi \vee \neg \langle \xleftarrow{\varphi} \rangle \psi)\}? \\ \min \ &(\mathrm{jump}_{p,q} \cdot \{\psi\}?) = \mathrm{jump}_{p,q} \cdot \{\psi \wedge \neg \langle \xleftarrow{+} \rangle \psi\}? \\ \min \ &(\pi_1 \cdot \pi_2 \cdot \{\psi\}?) = \min \ (\pi_1 \cdot \{\langle \pi_2 \rangle \psi\}?) \cdot \min \ (\pi_2 \cdot \{\psi\}?) \,. \end{split}$$

The construction of $\max \pi$ is similar.

For all $PDL_{sf}[Loop]$ path formulas π, π' , we define the $PDL_{sf}[Loop]$ event formulas

$$\min \ \pi \leq_{\mathsf{proc}} \min \ \pi' := \mathsf{Loop}(\min \ \pi \cdot (\min \ \pi')^{-1}) \vee \mathsf{Loop}(\min \ \pi \cdot \xrightarrow{+} \cdot (\min \ \pi')^{-1}) \ \mathrm{and}$$
$$\max \ \pi \leq_{\mathsf{proc}} \max \ \pi' := \mathsf{Loop}(\max \ \pi \cdot (\max \ \pi')^{-1}) \vee \mathsf{Loop}(\max \ \pi \cdot \xrightarrow{+} \cdot (\max \ \pi')^{-1}) \,.$$

▶ **Lemma 17.** For all PDL_{sf}[Loop] path formulas π, π' , MSCs M, and events e, we have

$$M, e \models \min \pi \leq_{\mathsf{proc}} \min \pi' \quad iff \quad M, e \models \langle \pi \rangle \land \langle \pi' \rangle \quad and \quad \min[\![\pi]\!](e) \leq_{\mathsf{proc}} \min[\![\pi']\!](e)$$

 $M, e \models \max \pi \leq_{\mathsf{proc}} \max \pi' \quad iff \quad M, e \models \langle \pi \rangle \land \langle \pi' \rangle \quad and \quad \max[\![\pi]\!](e) \leq_{\mathsf{proc}} \max[\![\pi']\!](e)$

Finally, we prove that any boolean combination of $PDL_{sf}[Loop]$ formulas is equivalent to a positive one, i.e., one that does not use c.

▶ Lemma 18. For all path formulas $\pi \in \mathrm{PDL}_{\mathsf{sf}}[\mathsf{Loop}]$, there exist $\mathrm{PDL}_{\mathsf{sf}}[\mathsf{Loop}]$ path formulas $(\pi_i)_{1 \leq i \leq |P|^2 + 3}$ such that $\pi^{\mathsf{c}} \equiv \bigcup_{1 \leq i \leq |P|^2 + 3} \pi_i$.

Proof. We show $\pi^{c} \equiv \sigma$ where

$$\sigma = (\min \, \pi \cdot \stackrel{+}{\longleftrightarrow}) \cup (\max \, \pi \cdot \stackrel{+}{\longleftrightarrow}) \cup (\pi \cdot \stackrel{+}{\longleftrightarrow} \cdot \{ \neg \, \langle \pi^{-1} \rangle \}?) \cup \bigcup_{(p,q) \in P^2} \{ \neg \, \langle \pi \rangle \, q \}? \cdot \mathsf{jump}_{p,q} \, .$$

Let $M = (E, \to, \lhd, loc, \lambda)$ be an MSC and $e, f \in E$. We write p = loc(e), q = loc(f). Let us show that $M, e, f \models \pi^c$ iff $M, e, f \models \sigma$. If $M, e \models \neg \langle \pi \rangle q$, then both $M, e, f \models \pi^c$ and $M, e, f \models \sigma$ hold. In the following, we assume that $M, e \models \langle \pi \rangle q$, and thus that $\min[\![\pi]\!](e)$ and $\max[\![\pi]\!](e)$ are well-defined and on process q. Again, if $f <_{\mathsf{proc}} \min[\![\pi]\!](e)$ or $\max[\![\pi]\!](e) <_{\mathsf{proc}} f$, then both $M, e, f \models \pi^c$ and $M, e, f \models \sigma$ hold. And if $\min[\![\pi]\!](e) \le_{\mathsf{proc}} f \le_{\mathsf{proc}} \max[\![\pi]\!](e)$, then, by Lemma 15, we have $M, e, f \models \pi^c$ iff $M, f \models \neg \langle \pi^{-1} \rangle$, iff $M, e, f \models \sigma$.

4 From FO to PDL_{sf}[Loop]

In this section, we show that, for formulas with one free variable, the logics $FO[\rightarrow, \lhd, \leq]$, $FO^3[\rightarrow, \lhd, \leq]$, PDL_{sf} , and $PDL_{sf}[Loop]$ are expressively equivalent.

We start by a simple observation, which is shown along a straightforward induction:

- ▶ **Proposition 19.** Every PDL_{sf} formula is equivalent to some FO³[\rightarrow , \triangleleft , \leq] formula:
- 1. For every PDL_{sf} path formula π , there exists an FO³[\rightarrow , \triangleleft , \leq] formula $\widetilde{\pi}(x,y)$ with two free variables such that $M, e, f \models \pi$ iff $M, [x \mapsto e, y \mapsto f] \models \widetilde{\pi}(x,y)$.
- **2.** For every PDL_{sf} event formula φ , there exists an FO³[\rightarrow , \triangleleft , \leq] formula $\widetilde{\varphi}(x)$ with a single free variable such that $M, e \models \varphi$ iff $M, [x \mapsto e] \models \widetilde{\varphi}(x)$.

In the remainder of the paper, we will simply write $\pi(x,y)$ for $\widetilde{\pi}(x,y)$, and $\varphi(x)$ for $\widetilde{\varphi}(x)$.

We are now ready to state the main result of this section, that is, the translation from $FO[\rightarrow, \lhd, \leq]$ to $PDL_{sf}[Loop]$:

- ▶ Theorem 20. Let $\Phi(x_1, ..., x_n) \in FO[\rightarrow, \lhd, \leq]$ with $n \geq 1$. There exists a formula $\Phi'(x_1, ..., x_n) \in FO[\rightarrow, \lhd, \leq]$ such that
- for all MSCs $M = (E, \rightarrow, \triangleleft, loc, \lambda)$ and interpretations $\nu : \{x_1, \ldots, x_n\} \rightarrow E$, we have $M, \nu \models \Phi(x_1, \ldots, x_n)$ iff $M, \nu \models \Phi'(x_1, \ldots, x_n)$, and
- $\Phi'(x_1,...,x_n)$ is a finite disjunction of formulas of the form $\bigwedge_j \pi_j(y_j,z_j)$ where $y_j,z_j \in \{x_1,...,x_n\}$ and all π_j are PDLsf[Loop] path formulas.

Note that, if the free variables of $\Phi(x_1, \ldots, x_n)$ are *strictly* included in $\{x_1, \ldots, x_n\}$ (in particular, if $\Phi(x_1, \ldots, x_n)$ is a sentence), then $\Phi'(x_1, \ldots, x_n)$ may have more free variables than $\Phi(x_1, \ldots, x_n)$.

Since $(\mathsf{Loop}(\pi))(x) \equiv \pi(x,x)$, Theorem 20 implies that every $FO[\to, \lhd, \leq]$ formula with one free variable is equivalent to some $PDL_{\mathsf{sf}}[\mathsf{Loop}]$ event formula, and every $FO[\to, \lhd, \leq]$ formula with two free variables is equivalent to a finite union of intersections of $PDL_{\mathsf{sf}}[\mathsf{Loop}]$ path formulas. Combined with Proposition 19, this also implies Theorem 9.

Proof of Theorem 20. We proceed by induction.

Base cases. For $x, y \in \{x_1, \dots, x_n\}$, we have

$$p(x) \equiv \{p\}?(x,x) \qquad x \to y \equiv \to (x,y) \qquad x = y \equiv \{true\}?(x,y)$$

$$a(x) \equiv \{a\}?(x,x) \qquad x \lhd y \equiv \bigvee_{(p,q) \in Ch} \lhd_{p,q}(x,y)$$

Moreover, $x \leq y$ is equivalent to the disjunction of the formulas $(\pi \cdot \triangleleft_{p_1,p_2} \cdot \xrightarrow{+} \cdot \triangleleft_{p_2,p_3} \cdot \cdot \xrightarrow{+} \cdot \triangleleft_{p_2,p_3} \cdot \cdot \xrightarrow{+} \cdot \triangleleft_{p_m-1,p_m} \cdot \pi')(x,y)$ where $1 \leq m \leq |P|, p_1,\ldots,p_m \in P$ are such that $p_i \neq p_{i+1}$ for all $i \in \{1,\ldots,m-1\}$, and $\pi,\pi' \in \{\xrightarrow{+},\{True\}?\}$.

Disjunction. The case $\Phi = \Phi_1 \vee \Phi_2$ is trivial.

Negation. Suppose $\Phi = \neg \Psi$. By induction hypothesis, $\neg \Psi(x_1, \ldots, x_n)$ is logically equivalent to a finite *conjunction* of formulas of the form $\bigvee_j \pi_j^{\mathsf{c}}(y_j, z_j)$ where all π_j are $\mathrm{PDL}_{\mathsf{sf}}[\mathsf{Loop}]$ path formulas. By Lemma 18, each π_j^{c} is equivalent to a disjunction of $\mathrm{PDL}_{\mathsf{sf}}[\mathsf{Loop}]$ path formulas. We conclude by rewriting the formula into disjunctive normal form.

Existential quantification. Suppose $\Phi(x_1,\ldots,x_{n-1})=\exists x_n.\Psi(x_1,\ldots,x_n)$ with n>1. Without loss of generality, we assume that $\Psi(x_1,\ldots,x_n)$ is a finite disjunction of formulas of the form $\bigwedge_j \pi_j(y_j,z_j)$ where $y_j=x_{i_1}$ and $z_j=x_{i_2}$ such that $i_1\leq i_2$. The latter condition

can be guaranteed by replacing π_j with π_j^{-1} whenever needed. Then, $\exists x_n.\Psi(x_1,\ldots,x_n)$ is logically equivalent to a finite disjunction of formulas of the form

$$\bigwedge_{j \in I} \pi_j(y_j, z_j) \wedge \underbrace{\exists x_n . \left(\bigwedge_{j \in J} \pi_j(y_j, x_n) \wedge \bigwedge_{j \in J'} \pi_j(x_n, x_n) \right)}_{=: \Upsilon(x_1, \dots, x_{n-1})}$$

for three finite, pairwise disjoint index sets I, J, J' such that $y_j \in \{x_1, \dots, x_{n-1}\}$ for all $j \in I \cup J$, and $z_j \in \{x_1, \dots, x_{n-1}\}$ for all $j \in I$. If $J = \emptyset$, then

$$\Upsilon(x_1,\dots,x_{n-1}) \equiv \bigvee_{p,q \in P} \Big(\mathsf{jump}_{p,q} \cdot \{ \bigwedge_{j \in J'} \mathsf{Loop}(\pi_j) \} ? \cdot \mathsf{jump}_{q,p} \Big) (x_1,x_1) \,.$$

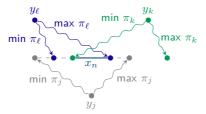
So assume $J \neq \emptyset$. Set

$$\Upsilon'(x_1,\ldots,x_{n-1}) := \bigvee_{k,\ell \in J} \left(\begin{array}{c} \bigwedge_{j \in J} ((\min \, \pi_j) \cdot \stackrel{*}{\rightarrow} \cdot (\min \, \pi_k)^{-1})(y_j,y_k) \\ \wedge \ \bigwedge_{j \in J} ((\max \, \pi_\ell) \cdot \stackrel{*}{\rightarrow} \cdot (\max \, \pi_j)^{-1})(y_\ell,y_j) \\ \wedge \ (\pi_k \cdot \{\psi\}? \cdot \pi_\ell^{-1})(y_k,y_\ell) \end{array} \right)$$

where $\psi = \bigwedge_{i \in J} \langle \pi_i^{-1} \rangle \wedge \bigwedge_{i \in J'} \mathsf{Loop}(\pi_i)$.

▶ Claim 21. We have
$$\Upsilon(x_1, \ldots, x_{n-1}) \equiv \Upsilon'(x_1, \ldots, x_{n-1})$$
.

The proof of Claim 21 can be found in Appendix B. Intuitively, by Lemma 15, we know that $\Upsilon(x_1,\ldots,x_{n-1})$ holds iff the intersection of the intervals $[\min[\pi_j](y_j),\max[\pi_j](y_j)]$ contains some event satisfying ψ . The formula $\Upsilon'(x_1,\ldots,x_{n-1})$ identifies some π_k such that $\min[\pi_k](y_k)$ is maximal (first line), some π_ℓ such that $\max[\pi_\ell](y_\ell)$ is minimal (second line), and tests that there exists an event x_n satisfying ψ between the two (third line). This is illustrated in the figure below.



Thus, $\Upsilon(x_1,\ldots,x_{n-1})$ is equivalent to some positive combination of formulas $\pi(x,y)$ with $\pi \in \mathrm{PDL}_{\mathsf{sf}}[\mathsf{Loop}]$ and, therefore, so is $\Phi(x_1,\ldots,x_{n-1})$. Note that the two formulas $((\min \pi_j) \cdot \overset{*}{\to} \cdot (\min \pi_k)^{-1})(y_j,y_k)$ and $((\max \pi_\ell) \cdot \overset{*}{\to} \cdot (\max \pi_j)^{-1})(y_\ell,y_j)$ are not $\mathrm{PDL}_{\mathsf{sf}}[\mathsf{Loop}]$ formulas (since $\overset{*}{\to}$ is not). However, they are disjunctions of $\mathrm{PDL}_{\mathsf{sf}}[\mathsf{Loop}]$ formulas, for instance, $((\min \pi_j) \cdot \overset{*}{\to} \cdot (\min \pi_k)^{-1})(y_j,y_k) \equiv ((\min \pi_j) \cdot (\min \pi_k)^{-1})(y_j,y_k) \vee ((\min \pi_j) \cdot \overset{*}{\to} \cdot (\min \pi_k)^{-1})(y_j,y_k)$.

5 From PDL_{sf}[Loop] to CFMs

Letter-to-letter MSC transducers. For the translation of $FO[\rightarrow, \lhd, \leq]$ sentences into CFMs, we will need to introduce MSC transducers to handle subformulas with one free variable, or, equivalently, $PDL_{sf}[\mathsf{Loop}]$ event formulas. More precisely, we will associate with an event

formula φ a transducer that evaluates φ at all events, and outputs 1 when the formula holds, and 0 otherwise.

Let Γ be a nonempty finite output alphabet. A *(nondeterministic) letter-to-letter MSC transducer* (or simply, transducer) \mathcal{A} over P and from Σ to Γ is a CFM over P and $\Sigma \times \Gamma$. The transducer \mathcal{A} accepts the relation $[\![\mathcal{A}]\!] = \{((E, \to, \lhd, loc, \lambda), (E, \to, \lhd, loc, \gamma)) \mid (E, \to, \lhd, loc, \lambda \times \gamma) \in L(\mathcal{A})\}$. Transducers are closed under product and composition, using standard constructions:

▶ **Lemma 22.** Let \mathcal{A} be a transducer from Σ to Γ , and \mathcal{A}' a transducer from Σ to Γ' . There exists a transducer $\mathcal{A} \times \mathcal{A}'$ from Σ to $\Gamma \times \Gamma'$ such that

$$\begin{split} \llbracket \mathcal{A} \times \mathcal{A}' \rrbracket &= \left\{ \left((E, \rightarrow, \lhd, loc, \lambda), (E, \rightarrow, \lhd, loc, \gamma \times \gamma') \right) \mid \\ & \left((E, \rightarrow, \lhd, loc, \lambda), (E, \rightarrow, \lhd, loc, \gamma) \right) \in \llbracket \mathcal{A} \rrbracket, \\ & \left((E, \rightarrow, \lhd, loc, \lambda), (E, \rightarrow, \lhd, loc, \gamma') \right) \in \llbracket \mathcal{A}' \rrbracket \right\}. \end{split}$$

▶ **Lemma 23.** Let \mathcal{A} be a transducer from Σ to Γ , and \mathcal{A}' a transducer from Γ to Γ' . There exists a transducer $\mathcal{A}' \circ \mathcal{A}$ from Σ to Γ' such that

$$\llbracket \mathcal{A}' \circ \mathcal{A} \rrbracket = \llbracket \mathcal{A}' \rrbracket \circ \llbracket \mathcal{A} \rrbracket = \{ (M, M'') \mid \exists M' \in \mathbb{MSC}(P, \Gamma) : (M, M') \in \llbracket \mathcal{A} \rrbracket, (M', M'') \in \llbracket \mathcal{A}' \rrbracket \} .$$

Translation of PDL_{sf}[Loop] **Event Formulas into CFMs.** For a PDL_{sf}[Loop] event formula φ and an MSC $M=(E,\to,\lhd,loc,\lambda)$ over P and Σ , we define an MSC $M_{\varphi}=(E,\to,\lhd,loc,\gamma)$ over P and $\{0,1\}$, by setting $\gamma(e)=1$ if $M,e\models\varphi$, and $\gamma(e)=0$ otherwise. Our goal is to construct a transducer \mathcal{A}_{φ} such that $[\![\mathcal{A}_{\varphi}]\!]=\{(M,M_{\varphi})\mid M\in\mathbb{MSC}(P,\Sigma)\}$.

We start with the case of formulas from $PDL_{sf}[\emptyset]$, i.e., without Loop. A straightforward induction (see Appendix C) shows:

▶ Lemma 24. Let φ be a PDL_{sf}[\emptyset] event formula. There exists a transducer \mathcal{A}_{φ} such that $[\![\mathcal{A}_{\varphi}]\!] = \{(M, M_{\varphi}) \mid M \in \mathbb{MSC}(P, \Sigma)\}.$

Next, we look at a single loop where the path is of the form $\min \pi$ or $\max \pi$ (and therefore, deterministic).

▶ Lemma 25. Let π be a PDL_{sf}[\emptyset] path formula of the form $\pi = \min \pi'$ or $\pi = \max \pi'$, and let $\varphi = \text{Loop}(\pi)$. There exists a transducer \mathcal{A}_{φ} such that $\llbracket \mathcal{A}_{\varphi} \rrbracket = \{(M, M_{\varphi}) \mid M \in \mathbb{MSC}(P, \Sigma)\}$.

Proof. We can assume that $\mathsf{Comp}(\pi) \subseteq \mathsf{id}$. We define \mathcal{A}_{φ} as the composition of three transducers that will guess and check the evaluation of φ . More precisely, \mathcal{A}_{φ} will be obtain as an inverse projection α^{-1} , followed by the intersection with an MSC language K, followed by a projection β .

We first enrich the labeling of the MSC with a color from $\Theta = \{\Box, \blacksquare, \bigcirc, \bigcirc\}$. Intuitively, colors \Box and \blacksquare will correspond to a guess that the formula φ is satisfied, and colors \bigcirc and \blacksquare to a guess that the formula is not satisfied. Consider the projection $\alpha \colon \mathbb{MSC}(P, \Sigma \times \Theta) \to \mathbb{MSC}(P, \Sigma)$ which erases the color from the labeling. The inverse projection α^{-1} can be realized with a transducer \mathcal{A} , i.e., $\llbracket \mathcal{A} \rrbracket = \{(\alpha(M'), M') \mid M' \in \mathbb{MSC}(P, \Sigma \times \Theta)\}$.

Define the projection $\beta \colon \mathbb{MSC}(P, \Sigma \times \Theta) \to \mathbb{MSC}(P, \{0, 1\})$ by $\beta((E, \to, \lhd, loc, \lambda \times \theta)) = (E, \to, \lhd, loc, \gamma)$ where $\gamma(e) = 1$ if $\theta(e) \in \{\Box, \blacksquare\}$, and $\gamma(e) = 0$ otherwise. The projection β can be realized with a transducer \mathcal{A}'' : we have $[\![\mathcal{A}'']\!] = \{(M', \beta(M')) \mid M' \in \mathbb{MSC}(P, \Sigma \times \Theta)\}$.

Finally, consider the language $K \subseteq \mathbb{MSC}(P, \Sigma \times \Theta)$ of MSCs $M' = (E, \rightarrow, \lhd, loc, \lambda \times \theta)$ satisfying the following two conditions:

- **1.** Colors \square and \blacksquare alternate on each process $p \in P$: if $e_1 < \dots < e_n$ are the events in $E_p \cap \theta^{-1}(\{\square, \blacksquare\})$, then $\theta(e_i) = \square$ if i is odd, and $\theta(e_i) = \blacksquare$ if i is even.
- 2. For all $e \in E$, $\theta(e) \in \{\Box, \blacksquare\}$ iff there exists $f \in E$ such that $M, e, f \models \pi$ and $\theta(e) = \theta(f)$. The first property is trivial to check with a CFM. Using Lemma 24, we can easily show that the second property can also be checked with a CFM. We deduce that there is a transducer \mathcal{A}' such that $[\![\mathcal{A}']\!] = \{(M', M') \mid M' \in K\}$. We let $\mathcal{A}_{\varphi} = \mathcal{A}'' \circ \mathcal{A}' \circ \mathcal{A}$. Notice that $[\![\mathcal{A}_{\varphi}]\!] = \{(\alpha(M'), \beta(M')) \mid M' \in K\}$. From the following two claims, we deduce immediately that $[\![\mathcal{A}_{\varphi}]\!] = \{(M, M_{\varphi}) \mid M \in \mathbb{MSC}(P, \Sigma)\}$.
- ▶ Claim 26. For all $M \in MSC(P, \Sigma)$ there exists $M' \in K$ with $\alpha(M') = M$.

Let $M = (E, \to, \lhd, loc, \lambda) \in \mathbb{MSC}(P, \Sigma)$. Let $E_1 = \{e \in E \mid M, e \models \varphi\}$ and $E_0 = E \setminus E_1$. Consider the graph $G = (E, \{(e, f) \mid M, e, f \models \pi\})$. Since $\pi = \min \pi'$ or $\pi = \max \pi'$, every vertex has outdegree at most 1, and, by Lemma 14, there are no cycles except for self-loops. So the restriction of G to E_0 is a forest, and there exists a 2-coloring $\chi \colon E_0 \to \{\circlearrowleft, \bullet\}$ such that, for all $e, f \in E_0$ with $M, e, f \models \pi$, we have $\chi(e) \neq \chi(f)$. There exists $\theta \colon E \to \Theta$ such that $\theta(e) = \chi(e)$ for $e \in E_0$, and $\theta(e) \in \{\Box, \blacksquare\}$ for $e \in E_1$ is such that Condition 1 of the definition of K is satisfied. It is easy to see that Condition 2 is also satisfied. Indeed, if $\theta(e) \in \{\Box, \blacksquare\}$ then $e \in E_1$ and $M, e, e \models \pi$. Now, if $\theta(e) \notin \{\Box, \blacksquare\}$ then $e \in E_0$ and either $M, e \not\models \langle \pi \rangle$ or, by definition of χ , we have $\theta(e) \neq \theta(f)$ for the unique f such that $M, e, f \models \pi$.

▶ Claim 27. For all $M' \in K$, we have $\beta(M') = M_{\varphi}$ where $M = \alpha(M')$.

Let $M' = (E, \rightarrow, \lhd, loc, \lambda \times \theta) \in K$ and $M = \alpha(M')$. Suppose towards a contradiction that $M_{\varphi} \neq \beta(M) = (E, \rightarrow, \lhd, loc, \gamma)$. By Condition 2, for all $e \in E$ such that $\gamma(e) = 0$, we have $M, e \not\models \varphi$. So there exists $f_0 \in E$ such that $\gamma(f_0) = 1$ and $M, f_0 \not\models \varphi$. Notice that $\theta(f_0) \in \{\Box, \blacksquare\}$. For all $i \in \mathbb{N}$, let f_{i+1} be the unique event such that $M, f_i, f_{i+1} \models \pi$. Such an event exists by Condition 2, and is unique since $\pi = \min \pi'$ or $\pi = \max \pi'$. Note that, for all $i, \theta(f_{i+1}) = \theta(f_i) \in \{\Box, \blacksquare\}$. Suppose $f_0 <_{\mathsf{proc}} f_1$ (the case $f_1 <_{\mathsf{proc}} f_0$ is similar). By Condition 1, there exists g_0 such that $f_0 <_{\mathsf{proc}} g_0 <_{\mathsf{proc}} f_1$ and $\{\theta(f_0), \theta(g_0)\} = \{\Box, \blacksquare\}$. Again, for all $i \in \mathbb{N}$, let g_{i+1} be the unique event such that $M, g_i, g_{i+1} \models \pi$. Note that all f_0, f_1, \ldots have the same color, in $\{\Box, \blacksquare\}$, and all g_0, g_1, \ldots carry the complementary color. Thus, $f_i \neq g_j$ for all $i, j \in \mathbb{N}$. But, by Lemma 14, this implies $f_0 <_{\mathsf{proc}} g_0 <_{\mathsf{proc}} f_1 <_{\mathsf{proc}} g_1 <_{\mathsf{proc}} \cdots$, which contradicts the fact that we deal with finite MSCs.

The general case is more complicated. We first show how to rewrite an arbitrary loop formula using loops on paths of the form $\max \pi$ or $(\max \pi) \cdot \stackrel{+}{\leftarrow}$ or $(\min \pi) \cdot \stackrel{+}{\leftarrow}$.

▶ **Lemma 28.** For all PDL_{sf}[Loop] path formulas π ,

$$\mathsf{Loop}(\pi) \equiv \mathsf{Loop}(\mathsf{max}\ \pi) \lor \left(\langle \pi^{-1} \rangle \land \mathsf{Loop}((\mathsf{max}\ \pi) \cdot \xleftarrow{+}) \land \neg \mathsf{Loop}((\mathsf{min}\ \pi) \cdot \xleftarrow{+}) \right) \, .$$

Proof. The result follows from Lemma 15. Indeed, if we have $M, e \models \mathsf{Loop}(\pi)$ and $M, e \not\models \mathsf{Loop}(\mathsf{max}\ \pi)$, then $\mathsf{min}[\![\pi]\!](e) \leq_{\mathsf{proc}} e <_{\mathsf{proc}} \mathsf{max}[\![\pi]\!](e)$ and $M, e \models \langle \pi^{-1} \rangle$, hence $M, e \models \langle \pi^{-1} \rangle \wedge \mathsf{Loop}((\mathsf{max}\ \pi) \cdot \overset{+}{\leftarrow}) \wedge \neg \mathsf{Loop}((\mathsf{min}\ \pi) \cdot \overset{+}{\leftarrow})$. Conversely, if $M, e \models \mathsf{Loop}(\mathsf{max}\ \pi)$, then $M, e \models \mathsf{Loop}(\pi)$, and if $M, e \models (\langle \pi^{-1} \rangle \wedge \mathsf{Loop}((\mathsf{max}\ \pi) \cdot \overset{+}{\leftarrow}) \wedge \neg \mathsf{Loop}((\mathsf{min}\ \pi) \cdot \overset{+}{\leftarrow}))$, then $M, e \models \langle \pi^{-1} \rangle$ and $\mathsf{min}[\![\pi]\!](e) \leq_{\mathsf{proc}} e <_{\mathsf{proc}} \mathsf{max}[\![\pi]\!](e)$, hence $M, e, e \models \pi$, i.e., $M, e \models \mathsf{Loop}(\pi)$.

▶ Theorem 29. For all PDL_{sf}[Loop] event formulas φ , there exists a transducer \mathcal{A}_{φ} such that $\llbracket \mathcal{A}_{\varphi} \rrbracket = \{(M, M_{\varphi}) \mid M \in \mathbb{MSC}(P, \Sigma)\}.$

Proof. By Lemma 28, we can assume that all loop subformulas in φ are of the form $\mathsf{Loop}((\max \pi) \cdot \stackrel{+}{\leftarrow})$ or $\mathsf{Loop}(\max \pi)$ (notice that $\min \pi = \max \min \pi$). We prove Theorem 29 by induction on the number of loop subformulas in φ . The base case is stated in Lemma 24.

Let $\psi = \mathsf{Loop}(\pi')$ be a subformula of φ such that π' contains no loop subformulas. Let us show that there exists \mathcal{A}_{ψ} such that $[\![\mathcal{A}_{\psi}]\!] = \{(M, M_{\psi}) \mid M \in \mathbb{MSC}(P, \Sigma)\}$. If $\pi' = \max \pi$, then we apply Lemma 25. Otherwise, $\pi' = (\max \pi) \cdot \stackrel{+}{\leftarrow}$ for some $\mathrm{PDL}_{\mathsf{sf}}[\emptyset]$ path formula π . So we assume from now on that $\psi = \mathsf{Loop}((\max \pi) \cdot \stackrel{+}{\leftarrow})$.

We start with some easy remarks. Let $p \in P$ be some process and $e \in E_p$. A necessary condition for $M, e \models \psi$ is that $M, e \models \langle \pi \rangle \land \neg \mathsf{Loop}(\mathsf{max}\ \pi)$. Also, it is easy to see that $M, e \models \mathsf{Loop}(\mathsf{min}\ (\stackrel{+}{\rightarrow} \cdot \pi^{-1}))$ is a sufficient condition for $M, e \models \psi$.

We let E_p^{π} be the set of events $e \in E_p$ satisfying $\langle \pi \rangle$. For all $e \in E_p^{\pi}$ we let $e' = [\![\max \pi]\!](e) \in E_p$. The transducer \mathcal{A}_{ψ} will compute for each $e \in E_p^{\pi}$ whether $e' <_{\mathsf{proc}} e$, e' = e or $e <_{\mathsf{proc}} e'$ and it will output 1 if $e <_{\mathsf{proc}} e'$ and 0 otherwise. The case e' = e means $M, e \models \mathsf{Loop}(\max \pi)$ and can be checked with the help of Lemma 25. So the difficulty is to distinguish between $e' <_{\mathsf{proc}} e$ and $e <_{\mathsf{proc}} e'$ when $M, e \models \langle \pi \rangle \land \neg \mathsf{Loop}(\max \pi)$.

The following two claims are proved in Appendix C.

- ▶ Claim 30. Let f be the minimal event in E_p^{π} (assuming this set is nonempty). Then, $M, f \models \psi$ iff $M, f \models \mathsf{Loop}(\min(\xrightarrow{+} \cdot \pi^{-1}))$.
- ▶ Claim 31. Let e, f be consecutive events in E_p^{π} , i.e., $e, f \in E_p^{\pi}$ and $M, e, f \models \frac{\neg \langle \pi \rangle}{}$.
- 1. If $M, e \not\models \psi$ then $[M, f \models \psi \text{ iff } M, f \models \mathsf{Loop}(\mathsf{min}\ (\xrightarrow{+} \cdot \pi^{-1}))].$
- **2.** If $M, e \models \psi$ then $[M, f \not\models \psi \text{ iff } M, f \models \mathsf{Loop}(\mathsf{max}\ \pi) \lor \mathsf{Loop}(\mathsf{max}\ ((\mathsf{max}\ \pi) \cdot \xrightarrow{\neg \langle \pi \rangle}))].$

To conclude the proof, let $\varphi_1 = \langle \pi \rangle$, $\varphi_2 = \mathsf{Loop}(\mathsf{max}\ \pi)$, $\varphi_3 = \mathsf{Loop}(\mathsf{min}\ (\xrightarrow{+} \cdot \pi^{-1}))$ and $\varphi_4 = \mathsf{Loop}(\mathsf{max}\ ((\mathsf{max}\ \pi) \cdot \xrightarrow{\neg \langle \pi \rangle}))$. By Lemmas 24 and 25, we already have transducers \mathcal{A}_{φ_i} for $i \in \{1, 2, 3, 4\}$. We let $\mathcal{A}_{\psi} = \mathcal{A} \circ (\mathcal{A}_{\varphi_1} \times \mathcal{A}_{\varphi_2} \times \mathcal{A}_{\varphi_3} \times \mathcal{A}_{\varphi_4})$, where, at an event f labeled (b_1, b_2, b_3, b_4) , the transducer \mathcal{A} outputs 1 if $b_3 = 1$ or if $(b_1, b_2, b_3, b_4) = (1, 0, 0, 0)$ and the output was 1 at the last event e on the same process satisfying φ_1 (to do so, each process keeps in its state the output at the last event where b_1 was 1), and 0 otherwise.

Consider the formula φ' over $\Sigma \times \{0,1\}$ obtained from φ by replacing ψ by $\bigvee_{a \in \Sigma} (a,1)$, and all event formulas a, with $a \in \Sigma$, by $(a,0) \vee (a,1)$. It contains fewer Loop operators than φ , so by induction hypothesis, we have a transducer $\mathcal{A}_{\varphi'}$ for φ' . We then let $\mathcal{A}_{\varphi} = \mathcal{A}_{\varphi'} \circ (\mathcal{A}_{Id} \times \mathcal{A}_{\psi})$ where \mathcal{A}_{Id} is the transducer for the identity relation.

Proof of Proposition 8. By Theorem 20, every $FO[\rightarrow, \lhd, \leq]$ formula $\Phi(x)$ with a single free variable is equivalent to some $PDL_{sf}[\mathsf{Loop}]$ state formula, for which we obtain a transducer \mathcal{A}_{Φ} using Theorem 29. It is easy to build from \mathcal{A}_{Φ} CFMs for the sentences $\forall x.\Phi(x)$ and $\exists x.\Phi(x)$. Closure of $\mathcal{L}(CFM)$ under union and intersection takes care of disjunction and conjunction.

6 Discussion

Though the translation of EMSO/FO formulas into CFMs is interesting on its own, it allows us to obtain some difficult results for bounded CFMs as corollaries. We will briefly sketch some of them. Details can be found in Appendix D.

First, note that the set $\mathbb{MSC}_{\exists B}(P,\Sigma)$ of $\exists B$ -bounded MSCs is FO-definable (essentially due to [21]). By Theorem 7, we obtain [11, Proposition 5.14] stating that $\mathbb{MSC}_{\exists B}(P,\Sigma)$ is recognized by some CFM. Second, we obtain [11, Proposition 5.3], a Kleene theorem for

 $\exists B$ -bounded MSCs, as a corollary of Theorem 7 in combination with a linearization normal form from [24].

Since (bounded) MSCs can be seen as a special case of Mazurkiewicz traces [8], we also get Zielonka's theorem [27] (though a weaker, nondeterministic version, and without guarantee on the size of the constructed automaton).

We leave open whether there is a temporal logic (with a finite set of modalities) over MSCs that is expressively complete for FO logic.

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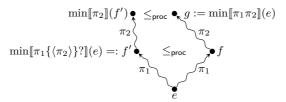
A Missing Proofs of Section 3

We first show that Lemma 13 holds when π_2 is monotone. We then use this to prove by induction that all PDL_{sf} formulas are monotone (Lemma 14), and thus that Lemma 13 is always true.

▶ Lemma 32. Let $\pi_1, \pi_2 \in PDL_{sf}[Loop]$ be path formulas, and $\pi = \pi_1 \cdot \pi_2$. If π_2 is monotone, then for all MSCs M and events e such that $M, e \models \langle \pi \rangle$, we have

$$\min[\![\pi]\!](e) = \min[\![\pi_2]\!](\min[\![\pi_1 \cdot \{\langle \pi_2 \rangle\}]\!](e)) \text{ and } \\ \max[\![\pi]\!](e) = \max[\![\pi_2]\!](\max[\![\pi_1 \cdot \{\langle \pi_2 \rangle\}]\!](e)).$$

Proof. We let $g = \min[\pi_1 \pi_2](e)$. Since $M, e, g \models \pi_1 \pi_2$, there exists f such that $M, e, f \models \pi_1$, and $M, f, g \models \pi_2$. In particular, $M, e, f \models \pi_1 \{\langle \pi_2 \rangle\}$?, so $f' = \min[\pi_1 \{\langle \pi_2 \rangle\}]$? (e) is well-defined and $f' \leq_{\mathsf{proc}} f$. Since π_2 is monotone, $\min[\pi_2](f') \leq_{\mathsf{proc}} \min[\pi_2](f) \leq_{\mathsf{proc}} g$. Also, $M, e, \min[\pi_2](f') \models \pi_1 \pi_2$. Hence $g \leq_{\mathsf{proc}} \min[\pi_2](f')$. Therefore, $g = \min[\pi_2](f')$. The proof is illustrated by the figure below.



The proof that $\max[\pi_1 \pi_2](e) = \max[\pi_2](\max[\pi_1 \{\langle \pi_2 \rangle\}?](e))$ is similar.

Proof of Lemma 14. We show that for all PDL_{sf}[Loop] event formulas ψ , $\pi \cdot \{\psi\}$? is monotone, by induction on π . Let e, f be events such that $e \leq_{\mathsf{proc}} f$, $M, e \models \pi \cdot \{\psi\}$? and $M, f \models \pi \cdot \{\psi\}$?. Let $e' = \min[\pi \cdot \{\psi\}]$? e and $f' = \min[\pi \cdot \{\psi\}]$? f. We show that $e' \leq_{\mathsf{proc}} f'$. The proof that $\max[\pi \cdot \{\psi\}]$? e $\exp_{\mathsf{proc}} \max[\pi \cdot \{\psi\}]$? e is similar. We start with the base cases.

If $\pi=\{\varphi\}$?, we have $e'=e\leq_{\mathsf{proc}} f=f'$. The result is also trivial for $\pi=\to$ or $\pi=\leftarrow$. It follows from the fact that channels are FIFO for $\pi=\lhd_{p,q}$ or $\pi=\lhd_{p,q}^{-1}$. When $\pi=\mathsf{jump}_{p,q}$ we have e'=f'. Suppose that $\pi=\overset{\varphi}{\to}$. It is easy to see that either $e'\leq_{\mathsf{proc}} f<_{\mathsf{proc}} f'$ or e'=f'. Similarly, when $\pi=\overset{\varphi}{\leftarrow}$ we have either $e'<_{\mathsf{proc}} e\leq_{\mathsf{proc}} f'$ or e'=f'.

The proof for $\pi = \pi_1 \cdot \pi_2$ is illustrated in the figure below.

$$\min[\pi_2 \cdot \{\psi\}?](e'') = e' \bullet \qquad \leq_{\mathsf{proc}} \qquad \bullet f' = \min[\pi_2 \cdot \{\psi\}?](f'')$$

$$\min[\pi_1 \cdot \{\langle \pi_2 \rangle \psi\}?](e) =: e'' \bullet \qquad \qquad \bullet f'' := \min[\pi_1 \cdot \{\langle \pi_2 \rangle \psi\}?](f)$$

$$\pi_1 \qquad \qquad \bullet f'' := \min[\pi_1 \cdot \{\langle \pi_2 \rangle \psi\}?](f)$$

$$\pi_1 \qquad \qquad \bullet f'' := \min[\pi_1 \cdot \{\langle \pi_2 \rangle \psi\}?](f)$$

By induction, the path formulas $\pi_1 \cdot \{\langle \pi_2 \rangle \psi\}$? and $\pi_2 \cdot \{\psi\}$? are monotone. So we can apply Lemma 32 to $\pi_1 \cdot (\pi_2 \cdot \{\psi\}?)$. Let $e'' = \min[\pi_1 \cdot \{\langle \pi_2 \rangle \psi\}?]](e)$ and $f'' = \min[\pi_1 \cdot \{\langle \pi_2 \rangle \psi\}?]](f)$. We have $e'' \leq_{\mathsf{proc}} f''$ since $\pi_1 \cdot \{\langle \pi_2 \rangle \psi\}$? is monotone. By Lemma 32, we have $e' = \min[\pi_2 \cdot \{\psi\}?]](e'')$ and $f' = \min[\pi_2 \cdot \{\psi\}?]](f'')$. We get $e' \leq_{\mathsf{proc}} f'$ since $\pi_2 \cdot \{\psi\}$? is monotone.

B Proof Details of Claim 21 (Theorem 20)

Assume $M, \nu \models \Upsilon(x_1, \ldots, x_{n-1})$. There exists an extension ν' of ν to $\{x_1, \ldots, x_n\}$ such that for all $j \in J$, $M, \nu'(y_j), \nu'(x_n) \models \pi_j$, and for all $j \in J'$, $M, \nu'(x_n) \models \text{Loop}(\pi_j)$. In particular, all $\min[\![\pi_j]\!](\nu'(y_j))$ and $\max[\![\pi_j]\!](\nu'(y_j))$ for $j \in J$ are well-defined and on a same process. Let $k \in J$ such that $\min[\![\pi_k]\!](\nu'(y_k))$ is maximal, i.e., $\min[\![\pi_j]\!](\nu'(y_j)) \leq_{\text{proc}} \min[\![\pi_k]\!](\nu'(y_k))$ for all $j \in J$. Then, for all $j \in J$, we have $M, \nu'(y_j), \nu'(y_k) \models (\min \pi_j) \xrightarrow{*} \cdot (\min \pi_k)^{-1}$. Similarly, let $\ell \in J$ such that $\max[\![\pi_\ell]\!](\nu'(y_\ell))$ is minimal. Then, for all $j \in J$, $M, \nu'(y_\ell), \nu'(y_j) \models (\max \pi_\ell) \cdot \xrightarrow{*} \cdot (\max \pi_j)^{-1}$. In addition, we have $M, \nu'(x_n) \models \psi$, $M, \nu'(y_k), \nu'(y_k), \nu'(y_k) \models \pi_k$, and $M, \nu'(x_n), \nu'(y_\ell), \models \pi_\ell^{-1}$, hence $M, \nu'(y_k), \nu'(y_\ell) \models \pi_k \cdot \{\psi\}$? $\cdot \pi_\ell^{-1}$. So we have $M, \nu' \models \Upsilon'(x_1, \ldots, x_{n-1})$, i.e., $M, \nu \models \Upsilon'(x_1, \ldots, x_{n-1})$.

Conversely, assume $M, \nu \models \Upsilon(x_1, \ldots, x_{n-1})$. Let $k, \ell \in J$ such that the corresponding sub-formula is satisfied. There exists e such that $M, \nu(y_k), e \models \pi_k, M, e \models \psi$, and $M, e, \nu(y_\ell) \models \pi_\ell^{-1}$. Note that we have $\min[\![\pi_k]\!](\nu(y_k)) \leq_{\mathsf{proc}} e \leq_{\mathsf{proc}} \max[\![\pi_\ell]\!](\nu(y_\ell))$. We extend ν to $\nu' : \{x_1, \ldots, x_n\} \to E$ by $\nu'(x_n) = e$. For all $j \in J'$, we have $M, \nu'(x_n) \models \mathsf{Loop}(\pi_j)$, i.e., $M, \nu' \models \pi_j(x_n, x_n)$. Now, let $j \in J$. We have $M, \nu'(y_j), \nu'(y_k) \models \min \pi_j \cdot \overset{*}{\to} \cdot (\min \pi_k)^{-1}$, hence $\min[\![\pi_j]\!](\nu'(y_j)) \leq_{\mathsf{proc}} \min[\![\pi_k]\!](\nu'(y_k)) \leq_{\mathsf{proc}} \nu'(x_n)$. Similarly, $M, \nu'(y_\ell), \nu'(y_j) \models (\max \pi_\ell) \cdot \overset{*}{\to} \cdot (\max \pi_j)^{-1}$, hence $\nu'(x_n) \leq_{\mathsf{proc}} \max[\![\pi_\ell]\!](\nu'(y_\ell)) \leq_{\mathsf{proc}} \max[\![\pi_j]\!](\nu'(y_j))$. In addition, since $M, \nu'(x_n) \models \psi$, we have $M, \nu'(x_n) \models \langle \pi_j^{-1} \rangle$. Applying Lemma 15, we get $M, \nu'(y_j), \nu'(x_n) \models \pi_j$. Hence $M, \nu \models \Upsilon(x_1, \ldots, x_{n-1})$.

C Missing Proofs of Section 5

Proof of Lemma 24. Any $PDL_{sf}[\emptyset]$ event formula is equivalent to some formula φ over the syntax

$$\varphi ::= p \mid a \mid \varphi \vee \varphi \mid \neg \varphi \mid \left< \lhd_{p,q} \right> \varphi \mid \left< \lhd_{p,q}^{-1} \right> \varphi \mid \left< \frac{\varphi}{\rightarrow} \right> \varphi \mid \left< \frac{\varphi}{\rightarrow} \right> \varphi \mid \left< \text{jump}_{p,q} \right> \varphi \mid \left< \frac{\varphi}{\rightarrow} \varphi \mid \left< \frac{\varphi}{\rightarrow} \right> \varphi \mid \left< \frac{\varphi}{\rightarrow} \Rightarrow \varphi \mid \left< \frac{\varphi}{\rightarrow$$

Indeed, we have $\langle \pi_1 \cdot \pi_2 \rangle \varphi \equiv \langle \pi_1 \rangle (\langle \pi_2 \rangle \varphi)$, and $\langle \{\varphi\}? \rangle \psi \equiv \varphi \wedge \psi$. Notice that $\to \equiv \xrightarrow{false}$ and $\leftarrow \equiv \xleftarrow{false}$.

It is easy to define \mathcal{A}_{φ} for formulas $\varphi = p$ with $p \in P$, or $\varphi = a$ with $a \in \Sigma$, as well as for the simple formulas over P and $\{0,1\}^2$ or P and $\{0,1\}$ needed below. We then let

$$\begin{split} \mathcal{A}_{\varphi_{1}\vee\varphi_{2}} &= \mathcal{A}_{(1,1)\vee(0,1)\vee(1,1)}\circ\left(\mathcal{A}_{\varphi_{1}}\times\mathcal{A}_{\varphi_{2}}\right) & \mathcal{A}_{\neg\varphi} &= \mathcal{A}_{(0)}\circ\mathcal{A}_{\varphi} \\ \mathcal{A}_{\langle \lhd_{p,q}\rangle\,\varphi} &= \mathcal{A}_{\langle \lhd_{p,q}\rangle\,(1)}\circ\mathcal{A}_{\varphi} & \mathcal{A}_{\langle \lhd_{p,q}^{-1}\rangle\,\varphi} &= \mathcal{A}_{\langle \lhd_{p,q}^{-1}\rangle\,(1)}\circ\mathcal{A}_{\varphi} \\ \mathcal{A}_{\langle \xrightarrow{\varphi_{1}}\rangle\,\varphi_{2}} &= \mathcal{A}_{\langle \xrightarrow{(1,0)\vee(1,1)}\rangle\,(0,1)\vee(1,1)}\circ\left(\mathcal{A}_{\varphi_{1}}\times\mathcal{A}_{\varphi_{2}}\right) & \mathcal{A}_{\langle \mathsf{jump}_{p,q}\rangle\,\varphi} &= \mathcal{A}_{\langle \mathsf{jump}_{p,q}\rangle\,(1)}\circ\mathcal{A}_{\varphi} \\ \mathcal{A}_{\langle \xleftarrow{\varphi_{1}}\rangle\,\varphi_{2}} &= \mathcal{A}_{\langle \xleftarrow{(1,0)\vee(1,1)}\rangle\,(0,1)\vee(1,1)}\circ\left(\mathcal{A}_{\varphi_{1}}\times\mathcal{A}_{\varphi_{2}}\right). \end{split}$$

Proof of Claim 30. The right to left implication holds without any hypothesis. Conversely, assume $f \stackrel{+}{\to} f' = [\![\max \pi]\!](f)$. Then, $M, f, f \models \stackrel{+}{\to} \cdot \pi^{-1}$, and $g = [\![\min (\stackrel{+}{\to} \cdot \pi^{-1})]\!](f) \leq_{\mathsf{proc}} f$. Moreover, $M, g \models \langle \pi \rangle$ and by minimality of f in E_p^{π} , we conclude that g = f.

Proof of Claim 31. 1. Again, the right to left implication holds without any hypothesis. Conversely, assume that $M, e \not\models \psi$ and $M, f \models \psi$, i.e., $e' \leq_{\mathsf{proc}} e$ and $f <_{\mathsf{proc}} f'$. As in the proof of Claim 30, we obtain $M, f \models \langle \stackrel{+}{\to} \cdot \pi^{-1} \rangle$ and $g = [\min(\stackrel{+}{\to} \cdot \pi^{-1})](f) \leq_{\mathsf{proc}} f$. Moreover, by Lemma 13, $g = [\min \pi^{-1}](g')$ for some g' with $f <_{\mathsf{proc}} g' \leq_{\mathsf{proc}} f'$. Notice that $g \in E_p^{\pi}$. If $g <_{\mathsf{proc}} f$ then we get $g \leq_{\mathsf{proc}} e$ and using Lemma 14 we obtain $e <_{\mathsf{proc}} f <_{\mathsf{proc}} g' \leq_{\mathsf{proc}} [\max \pi](g) \leq_{\mathsf{proc}} [\max \pi](e) = e' \leq_{\mathsf{proc}} e$, a contradiction. Therefore, g = f and $M, f \models \mathsf{Loop}(\min(\stackrel{+}{\to} \cdot \pi^{-1}))$.

2. The right to left implication holds trivially. Conversely, assume that $M, e \models \psi$, $M, f \not\models \psi$ and $M, f \not\models \mathsf{Loop}(\mathsf{max}\ \pi)$, i.e., $e <_{\mathsf{proc}}\ e'$ and $f' <_{\mathsf{proc}}\ f$. From Lemma 14 we get $e' \leq_{\mathsf{proc}}\ f'$ and since e, f are consecutive in E_p^π we obtain $M, f', f \models \xrightarrow{\neg \langle \pi \rangle}$. Therefore, $M, f \models \mathsf{Loop}((\mathsf{max}\ \pi) \cdot \xrightarrow{\neg \langle \pi \rangle}))$.

D Applications

We show that Theorem 7 can be used to give alternative proofs to two results on existentially-bounded MSCs from [11].

Let $M=(E,\to,\lhd,loc,\lambda)$ be some MSC, and $e_1 \prec e_2 \cdots \prec e_n$ a linearization of M. Given $e \in E$, we write $\operatorname{type}(e)=p$ if e is an internal event on process p, $\operatorname{type}(e)=p!q$ if e is a write on channel (p,q), and $\operatorname{type}(e)=p?q$ if e is a read from channel (q,p). We associate with the linearization \preceq a word w_{\preceq} over the alphabet $\Sigma_{lin}=\Sigma\times(P\cup\{p?q,p!q\mid (p,q)\in Ch\})$. More precisely, we let $w_{\preceq}=a_1\ldots a_n$ where $a_i=(\lambda\times\operatorname{type})(e_i)$. Note that M can be retrieved from w_{\prec} . We let $Lin^B(M)=\{w_{\prec}\mid \preceq \text{ is a }B\text{-bounded linearization of }M\}$.

- ▶ Fact 33 ([11, Theorem 4.1]). Let $B \in \mathbb{N}$ and L be a set of $\exists B$ -bounded MSCs. The following are equivalent:
- 1. L = L(A) for some CFM A.
- **2.** $L = L(\Phi)$ for some MSO formula Φ .
- **3.** $Lin^B(L)$ is a regular language.

The proof given in [11] relies on the theory of Mazurkiewicz traces. Another major part of the proof was the construction of a CFM recognizing the set $\mathbb{MSC}_{\exists B}(P,\Sigma)$ of $\exists B$ -bounded MSCs [11, Proposition 5.14]. We show that this CFM can in fact be obtained as a simple application of Theorem 7. Moreover, we give an alternative proof of (3) \Longrightarrow (1) (Section 5 in [11]).

A CFM for existentially bounded MSCs. The set $\mathbb{MSC}_{\exists B}(P,\Sigma)$ of $\exists B$ -bounded MSCs is in fact $FO[\lhd, \to, \leq]$ -definable, and thus, we can apply Theorem 7 to construct a CFM $\mathcal{A}_{\exists B}$ recognizing $\mathbb{MSC}_{\exists B}(P,\Sigma)$. We describe below a formula $\Phi_{\exists B}$ such that $L(\Phi_{\exists B}) = \mathbb{MSC}_{\exists B}(P,\Sigma)$.

We first recall a characterization of $\exists B$ -bounded MSCs. Let $M=(E,\to,\lhd,loc,\lambda)$ be an MSC. We define a relation $rev\subseteq E\times E$ which associates a receive event f from channel (p,q) with the send event g that is the B-th send on channel (p,q) after the event e such that $e\lhd f$. This can be defined in FO $[\to,\lhd,\leq]$ as follows:

$$rev(x,y) := \exists z_0, z_1, \dots, z_B. \ z_0 \lhd x \land z_B = y \land \bigwedge_{0 \leq i \leq B} \exists x_i. \ z_i \lhd x_i \land x \leq_{\mathsf{proc}} x_i \land \\ \bigwedge_{0 \leq i < B-1} z_i <_{\mathsf{proc}} z_{i+1} \land \neg (\exists z', x'. \ z_i <_{\mathsf{proc}} z' <_{\mathsf{proc}} z_{i+1} \land z' \lhd x' \land x \leq_{\mathsf{proc}} x').$$

▶ Fact 34 ([21]). M is $\exists B$ -bounded iff the relation ($< \cup rev$) is acyclic.

Note that if $(\leq \cup rev)$ contains a cycle, then it contains one of size at most |P|. So M is $\exists B$ -bounded iff it satisfies the formula

$$\Phi_{\exists B} = \bigwedge_{1 \leq n \leq |P|} \neg \Big(\exists x_0, \dots, x_n. \ x_0 = x_n \land \bigwedge_{0 \leq i < n} x_i < x_{i+1} \lor rev(x_i, x_{i+1})\Big).$$

Proof of (3) \Longrightarrow **(1).** We give a canonical B-bounded linearization of $\exists B$ -bounded MSCs, adapted from [24, Definition 14] where the definition was given for traces. We fix some total order \sqsubseteq on P. Let $M = (E, \to, \lhd, loc, \lambda)$ be an $\exists B$ -bounded MSC, and $\leq_B = (\leq \cup rev)^*$. Note that a linearization of M is B-bounded iff it contains \leq_B . Given $e, f \in E$, we let $P_{\uparrow e - \uparrow f} = \{loc(g) \mid e \leq_B g \land f \not\leq_B g\}$. We then define a relation \prec_M by

 $e \prec_M f$ iff $e <_B f$ or $(e \not\leq_B f, f \not\leq_B e$, and $\min P_{\uparrow e - \uparrow f} \sqsubset \min P_{\uparrow f - \uparrow e})$. Notice that for all $e \neq f$, we have either $e \prec_M f$ or $f \prec_M e$, but not both. Indeed, for all $e \neq f$, $P_{\uparrow e - \uparrow f}$ and $P_{\uparrow f - \uparrow e}$ are disjoint, and if $e \parallel f$, then they are non-empty as $loc(e) \in P_{\uparrow e - \uparrow f}$ and $loc(f) \in P_{\uparrow f - \uparrow e}$. Finally, notice that \prec_M is $FO[\rightarrow, \lhd, \leq]$ -definable.

A case analysis identical to [24, Lemma 15] shows that \prec_M is transitive, hence is a strict linear order on E. Moreover, \preceq_M contains \leq_B , hence it is a B-bounded linearization of M. Let L be a set of $\exists B$ -bounded MSCs such that $Lin^B(L)$ is regular. There exists an EMSO sentence Φ_{lin} such that $Lin^B(L) = L(\Phi_{lin})$. Since \prec_M is $FO[\to, \lhd, \leq]$ -definable, it is easy to translate Φ_{lin} into an EMSO $[\to, \lhd, \leq]$ formula Φ such that for all $\exists B$ -bounded MSC M, we have $M \models \Phi$ iff $w_{\prec_M} \models \Phi_{lin}$. Let $\mathcal A$ be a CFM such that $L(\mathcal A) = L(\Phi \land \Phi_{\exists B})$. Then for all $M \in L$, M is $\exists B$ -bounded and $w_{\preceq_M} \models \Phi_{lin}$, hence $M \in L(\mathcal A)$. Conversely, if $M \in L(\mathcal A)$, then \preceq_M is a linearization of M and $w_{\preceq_M} \in Lin^B(L)$, hence $M \in L$.