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A bivariate generating function for zeta values and related supercongruences

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ABSTRACT
By using the Wilf–Zeilberger method, we prove a novel finite combinatorial identity related to a bivariate generating function for $\zeta(2+r+2s)$. Such identity, which is an extension of a Bailey–Borwein–Bradley Apéry-like formula for even zeta values, is then applied to show several supercongruences.

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1. Introduction

The bivariate formula

$$\sum_{k=1}^{\infty} \frac{k}{k^4 - a^2k^2 - b^4} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(5k^2-a^2)}{k\binom{2k}{k}} \cdot \frac{\prod_{j=1}^{k-1}((j^2-a^2)^2+4b^4)}{\prod_{j=1}^{k}(j^4-a^2j^2-b^4)} \tag{1}$$

has been first conjectured by H. Cohen and then proved independently by Rivoal [11, Theorem 1.1] and Bradley [3, Theorem 1] by reducing it to the finite combinatorial identity

$$\sum_{k=1}^{n} \binom{2k}{k} \frac{(5k^2-a^2)\prod_{j=1}^{k-1}((n^2-j^2)(n^2+j^2-a^2))}{\prod_{j=1}^{k}(n^2+(n-j)^2-a^2)(n^2+(n+j)^2-a^2)} = \frac{2}{n^2-a^2}, \tag{2}$$

and by Kh. Hessami Pilehrood and T. Hessami Pilehrood [6, Theorem 1] by applying the Wilf–Zeilberger theory. Since the left-hand side of (1) can be written as the generating function of $\zeta(3+2r+4s)$,

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{r+s}{r}\zeta(3+2r+4s)a^{2r}b^{4s},$$

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it follows that, by extracting the coefficients for $(r,s)=(0,0)$ and $(r,s)=(1,0)$, we obtain the Apéry-like identities

$$\zeta(3)=\frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}} \quad\text{and}\quad \zeta(5)=\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\binom{2k}{k}}\left(\frac{4}{k^5}-\frac{5H_{k-1}(2)}{k^3}\right), \tag{3}$$

where $H_n(s)=\sum_{j=1}^{n}\frac{1}{j^s}$ is the harmonic sum of weight s. For more details about Apéry-like series, see also [1, 4–6].

Here, we consider a similar bivariate formula

$$\sum_{k=1}^{\infty}\frac{1}{k^2-ak-b^2}=\sum_{k=1}^{\infty}\frac{(3k-a)}{k\binom{2k}{k}}\cdot\frac{\prod_{j=1}^{k-1}(j^2-a^2-4b^2)}{\prod_{j=1}^{k}(j^2-aj-b^2)}, \tag{4}$$

where the left-hand side is the generating function of $\zeta(2+r+2s)$,

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{r+s}{r}\zeta(2+r+2s)a^rb^{2s}.$$

For $a=0$, (4) yields a formula due to Bailey *et al.*

$$\sum_{s=0}^{\infty}\zeta(2+2s)b^{2s}=\sum_{k=1}^{\infty}\frac{1}{k^2-b^2}=3\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}}\cdot\frac{\prod_{j=1}^{k-1}(j^2-4b^2)}{\prod_{j=1}^{k}(j^2-b^2)},$$

which appeared in [2, Theorem 1.1]. Moreover, for $(r,s)=(1,0)$ and $(r,s)=(0,1)$, we get the Apéry-like identities

$$\zeta(3)=\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}}\left(\frac{2}{k^3}+\frac{3H_{k-1}(1)}{k^2}\right) \quad\text{and}\quad \zeta(4)=3\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}}\left(\frac{1}{k^4}-\frac{3H_{k-1}(2)}{k^2}\right). \tag{5}$$

Replacing a by $2a$ and then letting $x^2=a^2+b^2$ in (4), we find the equivalent identity

$$\sum_{k=1}^{\infty}\frac{1}{(k-a)^2-x^2}=\sum_{k=1}^{\infty}\frac{(3k-2a)}{k\binom{2k}{k}}\cdot\frac{\prod_{j=1}^{k-1}(j^2-4x^2)}{\prod_{j=1}^{k}((j-a)-x^2)}$$

which has been proved by Kh. Hessami Pilehrood and T. Hessami Pilehrood [7, (24)].

Again, in the same spirit of what has been done for (1), our proof of (4) is reduced to show the following novel finite identity:

$$\sum_{k=1}^{n}\binom{2k}{k}\frac{3k-2n+a}{k^2-a^2}\cdot\prod_{j=1}^{k-1}\frac{(j-n)(j-n+a)}{j^2-a^2}=\frac{2}{n-a}. \tag{6}$$

In [15, Theorem 4.2], the author established that for any prime $p > 5$, the next two congruences hold:

$$\sum_{k=1}^{p-1} \frac{1}{k}\binom{2k}{k} \equiv -\frac{8H_{p-1}(1)}{3} \pmod{p^4},$$

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2}\binom{2k}{k} \equiv \frac{4}{5}\left(\frac{H_{p-1}(1)}{p} + 2pH_{p-1}(3)\right) \pmod{p^4}.$$

Thanks to the finite identities (2) and (6), in Section 5, we improve the first one as follows:

$$\sum_{k=1}^{p-1} \frac{1}{k}\binom{2k}{k} \equiv -\frac{8H_{p-1}(1)}{3} - \frac{5p^2H_{p-1}(3)}{3} \pmod{p^5},$$

and we show that for any prime $p > 5$,

$$\sum_{k=1}^{p-1} \frac{1}{k^3}\binom{2k}{k} \equiv -\frac{2H_{p-1}(1)}{p^2} \pmod{p^2},$$

$$\sum_{k=1}^{p-1} \binom{2k}{k}\frac{H_k(2)}{k} \equiv \frac{2H_{p-1}(1)}{3p^2} \pmod{p^2}.$$

Notice that the last two congruences are known modulo p (see [8, Theorem 2] and [10, (38)]) and they confirm modulo p^2 the two conjectures by Z.-W. Sun: for each prime $p > 7$,

$$\sum_{k=1}^{p-1} \frac{1}{k^3}\binom{2k}{k} \equiv -\frac{2H_{p-1}(1)}{p^2} - \frac{13H_{p-1}(3)}{27} \pmod{p^4},$$

$$\sum_{k=1}^{p-1} \binom{2k}{k}\frac{H_k(2)}{k} \equiv \frac{2H_{p-1}(1)}{3p^2} - \frac{38H_{p-1}(3)}{81} \pmod{p^3}.$$

The first one appeared in [13, Conjecture 1.1] and the second one in [14, Conjecture 5.1].

In the last section, two more congruences related to the Apéry-like identities (3) and (5) are provided.

2. Preliminaries concerning multiple harmonic sums

We define the multiple harmonic sums as

$$H_n(s_1,\ldots,s_r) = \sum_{1\le k_1<k_2<\cdots<k_r\le n} \frac{1}{k_1^{s_1}k_2^{s_2}\cdots k_r^{s_r}},$$

where $n \ge r > 0$ and each s_i is a positive integer. The sum $s_1 + s_2 + \cdots + s_r$ is the weight of the multiple sum. Furthermore, by $\{s_1, s_2, \ldots, s_j\}^m$ we denote the sequence of length mj with m repetitions of $(s_1, s_2, \ldots, s_j)$.

By [12, Theorem 5.1]), for any prime $p > s+2$ we have

$$H_{p-1}(s) \equiv \begin{cases} -\dfrac{s(s+1)}{2(s+2)}\, p^2 B_{p-s-2} \pmod{p^3} & \text{if } s \text{ is odd,} \\ \dfrac{s}{s+1}\, p B_{p-s-1} \pmod{p^2} & \text{if } s \text{ is even,} \end{cases}$$

where B_n be the nth Bernoulli number.

Let $p > 5$ be a prime, then by [15, Theorem 2.1],

$$H_{p-1}(2) \equiv -\frac{2H_{p-1}(1)}{p} - \frac{pH_{p-1}(3)}{3} \pmod{p^4}. \tag{7}$$

Moreover, by [8, Lemma 3],

$$H_{p-1}(1,2) \equiv -\frac{3H_{p-1}(1)}{p^2} - \frac{5H_{p-1}(3)}{12} \pmod{p^3}$$

and by [16, Proposition 3.7] and [9, Theorem 4.5],

$$H_{p-1}(1,1,2) \equiv -\frac{11H_{p-1}(3)}{12p} \pmod{p^2}, \quad H_{p-1}(1,1,1,2) \equiv -\frac{5H_{p-1}(3)}{6p^2} \pmod{p}.$$

Finally, by [16, Theorem 3.2],

$$H_{p-1}(2,2) \equiv \frac{H_{p-1}(3)}{3p}, \quad H_{p-1}(1,3) \equiv \frac{3H_{p-1}(3)}{4p} \pmod{p^2},$$

and by [16, Theorem 3.5],

$$H_{p-1}(2,1,2) \equiv 0, \quad H_{p-1}(1,2,2) \equiv \frac{5H_{p-1}(3)}{4p^2},$$
$$H_{p-1}(1,1,3) \equiv -\frac{5H_{p-1}(3)}{12p^2} \pmod{p}.$$

3. Proofs of the generating function (4) and the related combinatorial identity (6)

By partial fraction decomposition with respect to b^2, we get

$$\frac{\prod_{j=1}^{k-1}(j^2 - a^2 - 4b^2)}{\prod_{j=1}^{k}(j^2 - aj - b^2)} = \sum_{n=1}^{k} \frac{C_{n,k}(a)}{n^2 - an - b^2},$$

where

$$C_{n,k}(a) = \frac{\prod_{j=1}^{k-1}(j^2 - (a-2n)^2)}{\prod_{j=1, j\neq n}^{k}(j-n)(j+n-a)}.$$

Hence, by inverting the summations order, the identity (4) can be written as

$$\sum_{n=1}^{\infty}\frac{1}{n^2-an-b^2}=\sum_{k=1}^{\infty}\frac{(3k-a)}{k\binom{2k}{k}}\sum_{n=1}^{k}\frac{C_{n,k}(a)}{n^2-an-b^2}$$
$$=\sum_{n=1}^{\infty}\frac{1}{n^2-an-b^2}\sum_{k=n}^{\infty}\frac{(3k-a)C_{n,k}(a)}{k\binom{2k}{k}}.$$

It follows that (4) holds as soon as

$$1=\sum_{k=n}^{\infty}\frac{(3k-a)C_{n,k}(a)}{k\binom{2k}{k}}=\sum_{k=n}^{\infty}\frac{(3k-a)}{k\binom{2k}{k}}\cdot\frac{\prod_{j=1}^{k-1}(j^2-(a-2n)^2)}{\prod_{j=1,j\neq n}^{k}(j-n)(j+n-a)}. \tag{8}$$

Taking the same approach given in [11] for the proof of (1), the above formula is equivalent to this finite combinatorial identity

$$\sum_{k=1}^{n}\binom{2k}{k}(3k-a)\,\frac{\prod_{j=1}^{k-1}(j-n)(j+n-a)}{\prod_{j=1}^{k}(j^2-(a-2n)^2)}=\frac{2}{a-n}. \tag{9}$$

Both identities (8) and (9) are consequences of the next theorem after setting $z=2n-a$.

Theorem 3.1: *For any positive integer n,*

$$\sum_{k=1}^{n}\binom{2k}{k}(3k-2n+z)\,\frac{\prod_{j=1}^{k-1}(j-n)(j-n+z)}{\prod_{j=1}^{k}(j^2-z^2)}=\frac{2}{n-z} \tag{10}$$

and

$$\sum_{k=n}^{\infty}\frac{(3k-2n+z)}{k\binom{2k}{k}}\cdot\frac{\prod_{j=1}^{k-1}(j^2-z^2)}{\prod_{j=1,j\neq n}^{k}(j-n)(j-n+z)}=1. \tag{11}$$

Proof: Let

$$F(n,k)=\binom{2k}{k}(3k-2n+z)\frac{\prod_{j=0}^{k-1}(j-n)(j-n+z)}{\prod_{j=1}^{k}(j^2-z^2)}$$

and

$$G(n,k)=\frac{k(k^2-z^2)F(n,k)}{(2n-3k-z)(n+1-k)(n+1-k-z)}.$$

Then (F,G) is a Wilf–Zeilberger pair, or WZ pair, which means that its components satisfy the relation

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k).$$

In order to prove (10), it suffices to verify that $S_n:=\sum_{k=1}^{n}F(n,k)=2n$.

Now $S_1 = F(1,1) = 2$. Moreover,

$$S_{n+1} - S_n = \sum_{k=1}^{n+1} F(n+1,k) - \sum_{k=1}^{n+1} F(n,k) = \sum_{k=1}^{n+1} (G(n,k+1) - G(n,k))$$
$$= G(n,n+2) - G(n,1) = 2$$

because $F(n,n+1) = G(n,n+2) = 0$ and $G(n,1) = -2$.

In a similar way, we show (11) by considering the WZ pair given by

$$F(n,k) = \frac{(3k-2n+z)}{k\binom{2k}{k}} \cdot \frac{\prod_{j=1}^{k-1}(j^2-z^2)}{\prod_{j=1,j\neq n}^{k}(j-n)(j-n+z)}$$

and

$$G(n,k) = \frac{2(2k-1)(k-n)F(n,k)}{n(2n-3k-z)(n-z)}.$$

We have to prove that $S_n := \sum_{k=n}^{\infty} F(n,k) = 1$. It holds for $n = 1$, and for $n \geq 1$,

$$S_{n+1} - S_n = \sum_{k=n+1}^{\infty} F(n+1,k) - \sum_{k=n}^{\infty} F(n,k)$$
$$= -F(n,n) + \sum_{k=n+1}^{\infty} (G(n,k+1) - G(n,k))$$
$$= -F(n,n) - G(n,n+1) = 0. \qquad \blacksquare$$

4. More binomial identities

Here, we collect a few identities, apparently new, involving the binomial coefficients $\binom{2k}{k}$ and $\binom{n+k}{k}$ which will play a crucial role in the next sections.

Theorem 4.1: *For any positive integer n,*

$$\frac{3}{2}\sum_{k=1}^{n} \frac{1}{k}\binom{2k}{k} = \sum_{k=1}^{n} \frac{1}{k}\binom{n+k}{k} + H_n(1), \tag{12}$$

$$\sum_{k=1}^{n} \binom{2k}{k}\left(\frac{3H_k(1)}{2k} - \frac{1}{k^2}\right) = \sum_{k=1}^{n} \binom{n+k}{k}\frac{H_k(1)}{k} - H_n(2), \tag{13}$$

$$\sum_{k=1}^{n} \binom{2k}{k}\left(\frac{3H_k(2)}{k} - \frac{1}{2k^3}\right) = \sum_{k=1}^{n} \binom{n+k}{k}\frac{H_k(2)+H_n(2)}{k} + H_n(2)H_n(1) - H_n(1,2). \tag{14}$$

Proof: Let us consider the WZ pair

$$F(n,k) = \frac{1}{k}\binom{n+k}{k} \quad \text{and} \quad G(n,k) = \frac{k}{(n+1)^2}\binom{n+k}{k}$$

then

$$\begin{aligned} S_{n+1} - S_n &= F(n+1,n+1) + \sum_{k=1}^{n}(G(n,k+1) - G(n,k)) \\ &= F(n+1,n+1) + G(n,n+1) - G(n,1) \\ &= \frac{3/2}{n+1}\binom{2(n+1)}{n+1} - \frac{1}{n+1}, \end{aligned}$$

where $S_n := \sum_{k=1}^{n} F(n,k)$. Thus

$$S_n = \frac{3}{2}\sum_{k=1}^{n}\frac{1}{k}\binom{2k}{k} - H_n(1)$$

and we may conclude that (12) holds.

Now, let $S_n^{(1)} := \sum_{k=1}^{n} F(n,k)H_k(1)$. Then

$$\begin{aligned} S_{n+1}^{(1)} - S_n^{(1)} &= F(n+1,n+1)H_{n+1}(1) \\ &\quad + \sum_{k=1}^{n}\left(G(n,k+1)H_k(1) - G(n,k)\left(H_{k-1}(1) + \frac{1}{k}\right)\right) \\ &= F(n+1,n+1)H_{n+1}(1) + G(n,n+1)H_n(1) - \sum_{k=1}^{n}\frac{G(n,k)}{k} \\ &= \binom{2(n+1)}{n+1}\left(\frac{3H_{n+1}(1)}{2(n+1)} - \frac{1}{(n+1)^2}\right) + \frac{1}{(n+1)^2}, \end{aligned}$$

where we used $\sum_{k=1}^{n}\binom{n+k}{k} = \frac{1}{2}\binom{2(n+1)}{n+1} - 1$. Hence we find that

$$S_n^{(1)} = \sum_{k=1}^{n}\binom{2k}{k}\left(\frac{3H_k(1)}{2k} - \frac{1}{k^2}\right) + H_n(2)$$

which implies (13).

Let $S_n^{(2)} := \sum_{k=1}^{n} F(n,k)H_k(2)$ then

$$\begin{aligned} S_{n+1}^{(2)} - S_n^{(2)} &= F(n+1,n+1)H_{n+1}(2) \\ &\quad + \sum_{k=1}^{n}\left(G(n,k+1)H_k(2) - G(n,k)\left(H_{k-1}(2) + \frac{1}{k^2}\right)\right) \\ &= F(n+1,n+1)H_{n+1}(2) + G(n,n+1)H_n(2) - \sum_{k=1}^{n}\frac{G(n,k)}{k^2} \\ &= \binom{2(n+1)}{n+1}\left(\frac{3H_{n+1}(2)}{2(n+1)} - \frac{1}{2(n+1)^3}\right) - \frac{S_n}{(n+1)^2}, \end{aligned}$$

where we applied

$$\sum_{k=1}^{n} \frac{G(n,k)}{k^2} = \frac{1}{(n+1)^2} \sum_{k=1}^{n} F(n,k) = \frac{S_n}{(n+1)^2}.$$

Therefore,

$$\begin{aligned}
S_n^{(2)} &= \sum_{k=1}^{n} \binom{2k}{k} \left(\frac{3H_k(2)}{2k} - \frac{1}{2k^3} \right) - \sum_{k=1}^{n} \frac{S_{k-1}}{k^2} \\
&= \sum_{k=1}^{n} \binom{2k}{k} \left(\frac{3H_k(2)}{2k} - \frac{1}{2k^3} \right) - \frac{3}{2} \sum_{k=1}^{n} \frac{1}{k^2} \sum_{j=1}^{k-1} \frac{1}{j} \binom{2j}{j} + H_n(1,2) \\
&= \sum_{k=1}^{n} \binom{2k}{k} \left(\frac{3H_k(2)}{2k} - \frac{1}{2k^3} \right) - \frac{3}{2} \sum_{j=1}^{n} \frac{1}{j} \binom{2j}{j} (H_n(2) - H_j(2)) + H_n(1,2) \\
&= \sum_{k=1}^{n} \binom{2k}{k} \left(\frac{3H_k(2)}{k} - \frac{1}{2k^3} \right) - \frac{3H_n(2)}{2} \sum_{k=1}^{n} \frac{1}{k} \binom{2k}{k} + H_n(1,2)
\end{aligned}$$

and (14) is established. ■

5. Proofs of the main results

Theorem 5.1: *For any prime $p > 3$,*

$$\sum_{k=1}^{p-1} \frac{1}{k} \binom{2k}{k} \equiv -\frac{8H_{p-1}(1)}{3} - \frac{5p^2 H_{p-1}(3)}{3} \pmod{p^5}. \tag{15}$$

Moreover, for any prime $p > 5$,

$$\sum_{k=1}^{p-1} \frac{1}{k^3} \binom{2k}{k} \equiv -\frac{2H_{p-1}(1)}{p^2} \pmod{p^2}, \tag{16}$$

$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{H_k(2)}{k} \equiv \frac{2H_{p-1}(1)}{3p^2} \pmod{p^2}. \tag{17}$$

Proof: We first note that

$$\binom{p-1+k}{k} = \frac{p}{k} \binom{p+k-1}{k-1} = \frac{p}{k} \prod_{j=1}^{k-1} \left(1 + \frac{p}{j}\right) = \frac{1}{k} \sum_{j=0}^{k-1} p^{j+1} H_{k-1}(\{1\}^j). \tag{18}$$

Therefore, by (12) with $n = p-1$, we obtain the desired congruence (15),

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{1}{k}\binom{2k}{k} &= \frac{2}{3}\left(H_{p-1}(1)+\sum_{j=0}^{p-2}p^{j+1}H_{p-1}(\{1\}^j,2)\right)\\ &\equiv \frac{2}{3}\big(H_{p-1}(1)+pH_{p-1}(2)+p^2H_{p-1}(1,2)\\ &\quad +p^3H_{p-1}(1,1,2)+p^4H_{p-1}(1,1,1,2)\big)\\ &\equiv -\frac{8H_{p-1}(1)}{3}-\frac{5p^2H_{p-1}(3)}{3} \pmod{p^5}.\end{aligned}$$

By letting $z = 2n$ in (10), we have

$$\sum_{k=1}^{n}\binom{2k}{k}\frac{k}{k^2-4n^2}\prod_{j=1}^{k-1}\frac{j^2-n^2}{j^2-4n^2} = -\frac{2}{3n}.$$

Let $n = p > 5$ be a prime and move the pth term of the sum to the right-hand side

$$\sum_{k=1}^{p-1}\frac{1}{k}\binom{2k}{k}\frac{1}{1-\frac{4p^2}{k^2}}\prod_{j=1}^{k-1}\frac{1-\frac{p^2}{j^2}}{1-\frac{4p^2}{j^2}} = \frac{2}{3p}\left(\frac{1}{2}\binom{2p}{p}\prod_{j=1}^{p-1}\frac{1-\frac{p^2}{j^2}}{1-\frac{4p^2}{j^2}}-1\right).$$

The left-hand side modulo p^4 is congruent to

$$\sum_{k=1}^{p-1}\frac{1}{k}\binom{2k}{k}\left(1+\frac{4p^2}{k^2}\right)\prod_{j=1}^{k-1}\left(1+\frac{3p^2}{j^2}\right) \equiv \sum_{k=1}^{p-1}\frac{1}{k}\binom{2k}{k}+p^2\sum_{k=1}^{p-1}\binom{2k}{k}\left(\frac{1}{k^3}+\frac{3H_k(2)}{k}\right).$$

On the other hand, by [15, Theorem 2.4],

$$\frac{1}{2}\binom{2p}{p} \equiv 1+2pH_{p-1}(1)+\frac{2p^3H_{p-1}(3)}{3} \equiv 1-p^2H_{p-1}(2)-\frac{p^4H_{p-1}(4)}{2} \pmod{p^6}, \tag{19}$$

and the right-hand side is

$$\begin{aligned}\frac{1}{2}\binom{2p}{p}\prod_{j=1}^{p-1}\frac{1-\frac{p^2}{j^2}}{1-\frac{4p^2}{j^2}} &\equiv \frac{1}{2}\binom{2p}{p}\prod_{j=1}^{p-1}\left(1+\frac{3p^2}{j^2}+\frac{12p^4}{j^4}\right)\\ &\equiv \left(1-p^2H_{p-1}(2)-\frac{p^4H_{p-1}(4)}{2}\right)\\ &\quad\cdot\big(1+3p^2H_{p-1}(2)+12p^4H_{p-1}(4)+9p^4H_{p-1}(2,2)\big)\\ &\equiv 1+2p^2H_{p-1}(2)+p^4\left(\frac{17H_{p-1}(4)}{2}+3H_{p-1}(2,2)\right)\\ &\equiv 1+2p^2H_{p-1}(2) \pmod{p^5},\end{aligned}$$

where $2H_{p-1}(2,2) = (H_{p-1}(2))^2 - H_{p-1}(4) \equiv 0 \pmod{p}$. Finally, by (15),

$$\sum_{k=1}^{p-1} \binom{2k}{k} \left(\frac{1}{k^3} + \frac{3H_k(2)}{k} \right) \equiv \frac{8H_{p-1}(1)}{3p^2} + \frac{5H_{p-1}(3)}{3} + \frac{4pH_{p-1}(2)}{3} \equiv 0 \pmod{p^2}, \tag{20}$$

where we used (7).

By (14), with $n = p-1$, we have that

$$\begin{aligned}\sum_{k=1}^{p-1} \binom{2k}{k} \left(\frac{3H_k(2)}{k} - \frac{1}{2k^3} \right) &= p \sum_{k=1}^{p-1} \prod_{j=1}^{k-1} \left(1 + \frac{p}{j} \right) \frac{H_k(2) + H_{p-1}(2)}{k^2} \\ &\quad + H_{p-1}(2)H_{p-1}(1) - H_{p-1}(1,2) \\ &\equiv p \sum_{k=1}^{p-1} \frac{H_k(2)}{k^2} - H_{p-1}(1,2) \\ &= pH_{p-1}(2,2) + pH_{p-1}(4) - H_{p-1}(1,2) \\ &\equiv -H_{p-1}(1,2) = \frac{3H_{p-1}(1)}{p^2} \pmod{p^2}. \end{aligned} \tag{21}$$

The proofs of (16) and (17) are complete as soon as we properly combine congruences (20) and (21). ■

6. Finale : two Appéry-like congruences

We conclude with two more congruences which are related, respectively, to the first series in (5), and to the second series in (3).

Theorem 6.1: *For any prime $p > 3$,*

$$\sum_{k=1}^{p-1} \binom{2k}{k} \left(\frac{2}{k^2} - \frac{3H_k(1)}{k} \right) \equiv \frac{2H_{p-1}(1)}{p} + 3pH_{p-1}(3) \pmod{p^4}, \tag{22}$$

$$\sum_{k=1}^{p-1} (-1)^k \binom{2k}{k} \left(\frac{4}{k^4} + \frac{5H_k(2)}{k^2} \right) \equiv -H_{p-1}(4) \pmod{p^2}. \tag{23}$$

Proof: As regards (22), by (13) with $n = p-1$, and (18), we have that

$$\begin{aligned}&\sum_{k=1}^{p-1} \binom{2k}{k} \left(\frac{3H_k(1)}{2k} - \frac{1}{k^2} \right) \\ &= \sum_{k=1}^{p-1} \frac{H_k(1)}{k^2} \sum_{j=0}^{k-1} p^{j+1} H_{k-1}(\{1\}^j) - H_{p-1}(2) \\ &\equiv p \sum_{k=1}^{p-1} \frac{H_{k-1}(1) + \frac{1}{k}}{k^2} \left(1 + pH_{k-1}(1) + p^2 H_{k-1}(1,1) \right) - H_{p-1}(2)\end{aligned}$$

$$
\begin{aligned}
&\equiv -H_{p-1}(2) + pH_{p-1}(1,2) + pH_{p-1}(3) \\
&\quad + 2p^2H_{p-1}(1,1,2) + p^2H_{p-1}(2,2) + p^2H_{p-1}(1,3) \\
&\quad + 3p^3H_{p-1}(1,1,1,2) + p^3H_{p-1}(2,1,2) + p^3H_{p-1}(1,2,2) + p^3H_{p-1}(1,1,3) \\
&\equiv -\frac{H_{p-1}(1)}{p} - \frac{3pH_{p-1}(3)}{2} \pmod{p^4},
\end{aligned}
$$

where at the last step we applied the results mentioned in the preliminaries.

By comparing the coefficient of a^2 in the expansion of both sides of (2) at $a = 0$, we have

$$
\sum_{k=1}^{n} \binom{2k}{k} \frac{5k^2}{4n^4 + k^4} \prod_{j=1}^{k-1} \frac{n^4 - j^4}{4n^4 + j^4} \left(\frac{1}{5k^2} + \sum_{j=1}^{k-1} \frac{1}{n^2 + j^2} - 2\sum_{j=1}^{k} \frac{2n^2 + j^2}{4n^4 + j^4} \right) = -\frac{2}{n^4}.
$$

Let $n = p > 3$ be a prime. Then we move to the right-hand side the pth term of the sum on the left. It follows that the left-hand side is congruent modulo p^2 to the left-hand side of the (23)

$$
\sum_{k=1}^{p-1} (-1)^{k-1} \binom{2k}{k} \frac{5}{k^2} \left(\frac{1}{5k^2} + H_{k-1}(2) - 2H_k(2) \right) = \sum_{k=1}^{p-1} (-1)^k \binom{2k}{k} \left(\frac{4}{k^4} + \frac{5H_k(2)}{k^2} \right).
$$

The right-hand side multiplied by p^4 is

$$
-2 - \binom{2p}{p} \prod_{j=1}^{p-1} \frac{p^4 - j^4}{4p^4 + j^4} \left(\frac{1}{5} + p^2 \sum_{j=1}^{p-1} \frac{1}{p^2 + j^2} - 2p^2 \sum_{j=1}^{p} \frac{2p^2 + j^2}{4p^4 + j^4} \right), \tag{24}
$$

and therefore it remains to verify that it is congruent to $-p^4H_{p-1}(4)$ modulo p^6.

We note that

$$
\begin{aligned}
&\prod_{j=1}^{p-1} \frac{p^4 - j^4}{4p^4 + j^4} \equiv 1 - 5p^4H_{p-1}(4) \pmod{p^6}, \\
&p^2 \sum_{j=1}^{p-1} \frac{1}{p^2 + j^2} \equiv p^2H_{p-1}(2) - p^4H_{p-1}(4) \pmod{p^6}, \\
&2p^2 \sum_{j=1}^{p} \frac{2p^2 + j^2}{4p^4 + j^4} \equiv \frac{6}{5} + 2p^2H_{p-1}(2) + 4p^4H_{p-1}(4) \pmod{p^6}.
\end{aligned}
$$

Hence, by (19), (24) simplifies to

$$
-2 + 2\left(1 - p^2H_{p-1}(2) - \frac{p^4H_{p-1}(4)}{2}\right)\left(1 + p^2H_{p-1}(2)\right) \equiv -p^4H_{p-1}(4) \pmod{p^6}
$$

and the proof is finished. ■

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