ON THE TAYLOR COEFFICIENTS OF RATIONAL FUNCTIONS

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Dedicated to C. L. Siegel

Let $F(z)$ be a rational function of z which is regular at $z = 0$ and so possesses a convergent power series

$$F(z) = \sum_{h=0}^{\infty} f_h z^h.$$

The problem arises of characterizing those rational functions $F(z)$ that have *infinitely many vanishing Taylor coefficients* f_h. After earlier and more special results by Siegel (2) and Ward (4) I applied in 1934 (1) a p-adic method due to Skolem (3) to the problem and obtained the following partial solution.

Theorem 1. *Assume that all Taylor coefficients f_h of the rational function $F(z)$ are algebraic numbers, and that infinitely many of them vanish. Then two integers L and L_1 with $0 \leqslant L_1 < L$ exist such that f_h is zero for all sufficiently large $h \equiv L_1 \pmod{L}$.*

In the present paper, the restriction on the character of the coefficients f_h will be removed, by showing the

Theorem 2. *Theorem 1 remains valid when the coefficients f_h of $F(z)$ are arbitrary complex numbers.*

In the proof of this theorem, the assertion will be reduced to one relating to rational functions with algebraic Taylor coefficients, and it will be assumed that the truth of Theorem 1 has already been established.

1. If the difference of two functions is a polynomial, all but finitely many of their Taylor coefficients are the same. Also to a given rational function one can always add a unique polynomial such that the sum function vanishes at the point at infinity.

Hence, without loss of generality, we shall assume from now on that the rational function $F(z)$ is not only regular at $z = 0$, but also it vanishes at $z = \infty$. We then call $F(z)$ a *normed function*. The restriction to normed functions considerably shortens the discussion.

2. Let L and L_1 be two integers such that $0 \leqslant L_1 < L$. We say that $F(z)$ has the *zero sequence* $L_1 \pmod{L}$ if all but finitely many of the Taylor coefficients f_h with $h \equiv L_1 \pmod{L}$ are zero.

This property may also be expressed in another form. Put

$$\epsilon = e^{2\pi i/L} \quad \text{and} \quad E(z) = \sum_{j=0}^{L-1} \epsilon^{jL_1} F(\epsilon^{-j} z).$$

Evidently $$E(z) = \sum_{h=0}^{\infty} f_h z^h \sum_{j=0}^{L-1} \epsilon^{j(L_1-h)} = L \sum_{\substack{h=0 \\ h\equiv L_1 (\mathrm{mod}\, L)}}^{\infty} f_h z^h,$$

and so $E(z)$ reduces to a polynomial if $L_1 \pmod L$ is a zero sequence of $F(z)$. On the other hand, as $F(z)$ is normed, all terms $\epsilon^{jL_1}F(\epsilon^{-j}z)$ of $E(z)$ vanish at $z = \infty$. The same is then true for $E(z)$ itself, and so $E(z)$ *vanishes identically*. Hence the stronger property

$$f_h = 0 \text{ for all suffixes } h \equiv L_1 \pmod L$$

holds.

3. Assume again that $L_1 \pmod L$ is a zero sequence of $F(z)$. Let further $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the distinct poles of $F(z)$; by hypothesis, none of these poles lies at $z = 0$. Then $F(\epsilon^{-j}z)$ has the poles

$$\epsilon^j\alpha_1, \quad \epsilon^j\alpha_2, \quad \ldots, \quad \epsilon^j\alpha_n.$$

As was shown in § 2,

$$E(z) = \sum_{j=0}^{L-1} \epsilon^{jL_1}F(\epsilon^{-j}z) \equiv 0.$$

Hence the poles of $F(z)$ are cancelled by the poles of the $L-1$ other functions $\epsilon^{jL_1}F(\epsilon^{-j}z)$ where $j = 1, 2, \ldots, L-1$.

It follows therefore that *to every pole α_ν of $F(z)$ there is a second pole α_μ $(\mu \neq \nu)$ such that $\alpha_\nu/\alpha_\mu \neq 1$ is an L-th root of unity*, which, of course, need not be primitive. Furthermore, *$F(z)$ has at least two distinct poles.*

4. Let $\Sigma = \{\alpha_\nu/\alpha_\mu\}$ be the set of all those quotients $\alpha_\nu/\alpha_\mu \neq 1$ of distinct poles of $F(z)$ that are roots of unity. Unless Σ is the null set, there exists a smallest positive integer M such that Σ consists only of Mth roots of unity which, however, need not all be primitive.

Assume, in particular, that $L_1 \pmod L$ is a zero sequence of $F(z)$, and put

$$(L, M) = L^*, \quad L' = \frac{L}{L^*},$$

so that $$L^* = L\Lambda + M\mathrm{M}, \quad L = L^*L',$$

with certain integers Λ and M. By § 3, Σ is now certainly not the null set, because it contains elements that are Lth roots of unity. Denote by Σ^* the subset formed by all these elements of Σ that are Lth roots of unity. Thus the elements α_ν/α_μ of Σ^* satisfy both equations

$$\left(\frac{\alpha_\nu}{\alpha_\mu}\right)^L = 1 \quad \text{and} \quad \left(\frac{\alpha_\nu}{\alpha_\mu}\right)^M = 1,$$

and so also the equation

$$\left(\frac{\alpha_\nu}{\alpha_\mu}\right)^{L^*} = \left\{\left(\frac{\alpha_\nu}{\alpha_\mu}\right)^L\right\}^\Lambda \left\{\left(\frac{\alpha_\nu}{\alpha_\mu}\right)^M\right\}^{\mathrm{M}} = 1.$$

Therefore Σ^* *consists only of L^*th roots of unity.*

5. We introduce now the L' new functions

$$E_k(x) = \sum_{\substack{j=0 \\ j\equiv k (\mathrm{mod}\, L')}}^{L-1} \epsilon^{jL_1}F(\epsilon^{-j}z) \quad (k = 0, 1, 2, \ldots, L'-1).$$

As already shown, the sum of these functions

$$E(z) = \sum_{k=0}^{L'-1} E_k(z) = \sum_{j=0}^{L-1} \epsilon^{jL_1} F(\epsilon^{-j}z)$$

is *identically zero.*

It is obvious that $E_0(z)$ may have poles only at the points $\epsilon^j\alpha_1, \epsilon^j\alpha_2, \ldots, \epsilon^j\alpha_n$, where $j \equiv 0 \pmod{L'}$, $0 \leqslant j \leqslant L-1$, while, for $k = 1, 2, \ldots, L'-1$, poles of $E_k(z)$ can lie only at $\epsilon^\iota\alpha_1, \epsilon^\iota\alpha_2, \ldots, \epsilon^\iota\alpha_n$, where $\iota \equiv k \pmod{L'}$, $0 \leqslant \iota \leqslant L-1$. Let us suppose that $\epsilon^j\alpha_\nu$ is a pole of $E_0(z)$, and that $\epsilon^\iota\alpha_\mu$ is one of $E_k(z)$, where $1 \leqslant k \leqslant L'-1$. Then

$$\iota - j \equiv k \not\equiv 0 \pmod{L'}.$$

Hence $$L^*(\iota - j) \not\equiv 0 \pmod{L},$$

whence $$(\epsilon^{\iota-j})^{L^*} \neq 1, \quad \epsilon^{\iota-j} \neq 1.$$

Therefore, necessarily,

$$\epsilon^j\alpha_\nu \neq \epsilon^\iota\alpha_\mu,$$

because, $\epsilon^{\iota-j}$ being an Lth root of unity, the quotient

$$\frac{\alpha_\nu}{\alpha_\mu} = \epsilon^{\iota-j} \neq 1$$

would otherwise belong to Σ^* and be an L^*th root of unity; and this is not the case.

The function $E_0(z)$ has then no poles in common with the other terms $E_k(z)$ of $E(z)$, and all its poles are also poles of $E(z)$. Since $E(z)$ has no poles, $E_0(z)$ is thus a polynomial. But, from its definition in terms of $F(z)$, $E_0(z)$ is a *normed* rational function. Hence, finally,

$$E_0(z) \textit{ is identically zero.}$$

Put now $$\eta = \epsilon^{L'} = e^{2\pi i/L^*}.$$

Evidently $$E_0(z) = \sum_{j=0}^{L^*-1} \eta^{jL_1} F(\eta^{-j}z) = L^* \sum_{\substack{h=0 \\ h \equiv L_1 (\mathrm{mod}\, L^*)}}^{\infty} f_h z^h \equiv 0,$$

whence $$f_h = 0 \textit{ for all suffixes } h \equiv L_1 \pmod{L^*}.$$

The following result has thus been established.

LEMMA 1. *Let $L_1 \pmod{L}$ be a zero sequence of $F(z)$; let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the distinct poles of $F(z)$; and let M be the smallest integer such that all quotients $\frac{\alpha_\mu}{\alpha_\nu} \neq 1$, that are roots of unity, are M-th roots of unity. If $L^* = (L, M)$, then $F(z)$ admits the zero sequence $L_1 \pmod{L^*}$.*

This lemma is of importance for later, because L^* *is a divisor of* M; *and M depends only on the poles of $F(z)$.* We note that the lemma remains valid when $F(z)$ is not normed, but shall not use this fact.

6. We proceed now to the proof of Theorem 2.

The most general rational function $F(z) \not\equiv 0$ regular at $z = 0$ is of the form

$$F(z) = \frac{a_0 + a_1 z + \ldots + a_m z^m}{(z-\alpha_1)^{e_1}(z-\alpha_2)^{e_2}\ldots(z-\alpha_n)^{e_n}}.$$

Here $e_1, e_2, \ldots, e_n$ are arbitrary positive integers; $a_0, a_1, \ldots, a_m$ are arbitrary complex numbers with $a_m \neq 0$; and $\alpha_1, \alpha_2, \ldots, \alpha_n$, the poles of $F(z)$, are complex numbers that are all distinct and different from zero, but are otherwise arbitrary. The function $F(z)$ is assumed to be normed, and therefore the inequality

$$m < e_1 + e_2 + \ldots + e_n$$

holds.

If again

$$F(z) = \sum_{h=0}^{\infty} f_h z^h$$

is the power series of $F(z)$ in the neighbourhood of $z = 0$, let H be the set of all suffixes h for which

$$f_h = 0.$$

It is assumed that H is an infinite set; the problem is to prove that under this hypothesis $F(z)$ possesses at least one zero sequence.

7. From now on let X be the set of the $m+n+1$ parameters

$$X = \{a_0, a_1, \ldots, a_m, \alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_n^{-1}\}$$

that occur in $F(z)$; the use of α_ν^{-1} rather than α_ν will prove to be an advantage. Further put

$$e_0 = e_1 + e_2 + \ldots + e_n.$$

Then $F(z)$ may also be written in the form

$$F(z) = (-1)^{e_0} \prod_{\nu=1}^{n} (\alpha_\nu^{-1})^{e_\nu} \sum_{\mu=0}^{m} a_\mu z^\mu \prod_{\nu=1}^{n} (1 - a_\nu^{-1} z)^{-e_\nu}.$$

On developing here the last factor into a power series by means of the binomial theorem, we see immediately that, for $h = 0, 1, 2, \ldots$, *f_h is a polynomial with rational coefficients in the elements of X.*

Hence, if X consists only of algebraic numbers, the coefficients f_h are likewise algebraic. It is assumed that this is no longer the case; hence X includes at least one transcendental number.

Denote by R the Gaussian imaginary quadratic field. The elements of X generate a smallest extension field

$$\mathrm{P} = R(X) = R(a_0, a_1, \ldots, a_m, \alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_n^{-1})$$

over R. It is shown in algebra that this extension field may be constructed as follows.

We first adjoin to R a certain finite set of transcendental complex numbers

$$\sigma_1, \sigma_2, \ldots, \sigma_p$$

that are algebraically independent over R, so arriving at the purely transcendental extension

$$\mathrm{P}_0 = R(\sigma_1, \sigma_2, \ldots, \sigma_p).$$

The field P is now derived from P_0 by a simple algebraic extension

$$\mathrm{P} = \mathrm{P}_0(\tau) = R(\sigma_1, \sigma_2, \ldots, \sigma_p, \tau),$$

τ being a suitable complex number algebraic over P_0.

This number τ may still be chosen in many different ways, and there is no loss of generality in assuming that τ is *integral over the polynomial ring* $R[\sigma_1, \sigma_2, \ldots, \sigma_p]$. The equation for τ has then the form

$$Q(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau) \equiv \tau^q + \sum_{\kappa=1}^{q} Q_\kappa(\sigma_1, \sigma_2, \ldots, \sigma_p) \tau^{q-\kappa} = 0,$$

where
$$Q_\kappa(\sigma_1, \sigma_2, \ldots, \sigma_p) \quad (\kappa = 1, 2, \ldots, q)$$

are polynomials in $R[\sigma_1, \sigma_2, \ldots, \sigma_p]$. It may also be assumed that $Q(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau)$, considered as a polynomial in $\sigma_1, \sigma_2, \ldots, \sigma_p, \tau$, is *irreducible over R*.

8. The elements a_μ and α_ν^{-1} of X are finite numbers in P. They can therefore be written as polynomials in τ, with coefficients that are rational functions of $\sigma_1, \sigma_2, \ldots, \sigma_p$ with numerical coefficients in R. Denote by $\Delta(\sigma_1, \sigma_2, \ldots, \sigma_p) \not\equiv 0$ the least common denominator of these rational functions; Δ is thus an element of $R[\sigma_1, \sigma_2, \ldots, \sigma_p]$. Then a_μ and α_ν^{-1} take the form

$$a_\mu = \frac{A_\mu(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau)}{\Delta(\sigma_1, \sigma_2, \ldots, \sigma_p)} \quad (\mu = 0, 1, \ldots, m)$$

and
$$\alpha_\nu^{-1} = \frac{\mathrm{A}_\nu(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau)}{\Delta(\sigma_1, \sigma_2, \ldots, \sigma_p)} \quad (\nu = 1, 2, \ldots, n).$$

Here the numerators

$$A_\mu(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau), \quad \mathrm{A}_\nu(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau)$$

belong to the polynomial ring $R[\sigma_1, \sigma_2, \ldots, \sigma_p, \tau]$.

On substituting these expressions for the elements of X, $F(z)$ becomes a rational function
$$F(z) = \Phi(z \mid \sigma_1, \sigma_2, \ldots, \sigma_p; \tau)$$

not only of z, but also of $\sigma_1, \sigma_2, \ldots, \sigma_p, \tau$, while its numerical coefficients lie in R. It follows further, from the representation of f_h as a polynomial in the elements of X with coefficients in R, that these Taylor coefficients may be written as

$$f_h = \frac{\phi_h(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau)}{\Delta(\sigma_1, \sigma_2, \ldots, \sigma_p)^{d_h}} \quad (h = 0, 1, 2, \ldots),$$

where the numerators
$$\phi_h(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau)$$

lie in the polynomial ring $R[\sigma_1, \sigma_2, \ldots, \sigma_p, \tau]$, while the exponents d_h are certain positive integers depending on h. One may, in fact, choose $d_h = e_0 + h + 1$; but we shall not need this. The hypothesis on f_h implies that

$$\phi_h(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau) = 0 \quad \text{if} \quad h \in H.$$

9. Let us now replace the algebraically independent complex numbers $\sigma_1, \sigma_2, \ldots, \sigma_p$ by independent complex variables

$$s_1, s_2, \ldots, s_p,$$

and the complex number τ for which

$$Q(\sigma_1, \sigma_2, \ldots, \sigma_p; \tau) = 0$$

by a dependent complex variable t satisfying

$$Q(s_1, s_2, \ldots, s_p;\, t) = 0.$$

We then obtain a new rational function

$$F^*(z) = \Phi(z \mid s_1, s_2, \ldots, s_p;\, t)$$

of z, as well as of $s_1, s_2, \ldots, s_p, t$, with numerical coefficients in R. This function has the explicit form

$$F^*(z) = \frac{a_0^* + a_1^* z + \ldots + a_m^* z^m}{(z-\alpha_1^*)^{e_1}(z-\alpha_2^*)^{e_2}\ldots(z-\alpha_n^*)^{e_n}},$$

where
$$a_\mu^* = \frac{A_\mu(s_1, s_2, \ldots, s_p;\, t)}{\Delta(s_1, s_2, \ldots, s_p)} \quad (\mu = 0, 1, \ldots, m)$$

and
$$\alpha_\nu^{*-1} = \frac{\mathrm{A}_\nu(s_1, s_2, \ldots, s_p;\, t)}{\Delta(s_1, s_2, \ldots, s_p)} \quad (\nu = 1, 2, \ldots, n).$$

Further it possesses the power series

$$F^*(z) = \sum_{h=0}^{\infty} f_h^* z^h,$$

where
$$f_h^* = \frac{\phi_h(s_1, s_2, \ldots, s_p;\, t)}{\Delta(s_1, s_2, \ldots, s_p)^{d_h}} \quad (h = 0, 1, 2, \ldots).$$

Since Δ does not vanish identically, and since the change from $\sigma_1, \sigma_2, \ldots, \sigma_p, \tau$ to $s_1, s_2, \ldots, s_p, t$ maps $\mathrm{P} = R(\sigma_1, \sigma_2, \ldots, \sigma_p, \tau)$ isomorphically onto $R(s_1, s_2, \ldots, s_p, t)$, it is clear that also

$$\phi_h(s_1, s_2, \ldots, s_p, t) = 0 \text{ and } f_h^* = 0 \quad \text{if} \quad h \in H.$$

It is further obvious from the construction that for

$$s_1 = \sigma_1, \quad s_2 = \sigma_2, \quad \ldots, \quad s_p = \sigma_p, \quad t = \tau,$$

the equations
$$F^*(z) = F(z), \quad a_\mu^* = a_\mu, \quad \alpha_\nu^* = \alpha_\nu, \quad f_h^* = f_h$$
hold.

10. To simplify the notation, we introduce the p-dimensional space C^p of all points

$$\mathbf{s} = (s_1, s_2, \ldots, s_p), \quad \mathbf{s}' = (s_1', s_2', \ldots, s_p'), \quad \boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_p), \quad \ldots,$$

with arbitrary real or complex coordinates, and we make C^p a metric space by defining the distance $\rho(\mathbf{s}, \mathbf{s}')$ of two points $\mathbf{s}, \mathbf{s}'$ by

$$\rho(\mathbf{s}, \mathbf{s}') = \{|s_1 - s_1'|^2 + |s_2 - s_2'|^2 + \ldots + |s_p - s_p'|^2\}^{\frac{1}{2}}.$$

Let R^p similarly be the set of all points in C^p the coordinates of which lie in the Gaussian field R; *thus R^p is dense in C^p*. We can then select in many ways an infinite sequence of points

$$S = \{\mathbf{s}^{(1)}, \mathbf{s}^{(2)}, \mathbf{s}^{(3)}, \ldots\}, \quad \text{where} \quad \mathbf{s}^{(k)} = (s_1^{(k)}, s_2^{(k)}, \ldots, s_p^{(k)}),$$

in R^p such that
$$\lim_{k\to\infty} \mathbf{s}^{(k)} = \boldsymbol{\sigma}, \quad \text{i.e.} \quad \lim_{k\to\infty} \rho(\mathbf{s}^{(k)}, \boldsymbol{\sigma}) = 0.$$

From the form of the equation

$$Q(s_1, s_2, \ldots, s_p, t) = 0$$

for t, it is further possible to associate with each point $\mathbf{s}^{(k)}$ of S a complex root, $t^{(k)}$ say, of the equation

$$Q(s_1^{(k)}, s_2^{(k)}, \ldots, s_p^{(k)}, t^{(k)}) = 0,$$

such that also

$$\lim_{k\to\infty} t^{(k)} = \tau.$$

11. Denote, for $k = 1, 2, 3, \ldots$, by

$$F^{(k)}(z), \quad a_\mu^{(k)}, \quad \alpha_\nu^{(k)}, \quad f_h^{(k)},$$

the expressions into which $\quad F^*(z), \quad a_\mu^*, \quad \alpha_\nu^*, \quad f_h^*,$

respectively, are changed on putting

$$s_1 = s_1^{(k)}, \quad s_2 = s_2^{(k)}, \quad \ldots, \quad s_p = s_p^{(k)}, \quad t = t^{(k)}.$$

Then $F^{(k)}(z)$ is the rational function

$$F^{(k)}(z) = \frac{a_0^{(k)} + a_1^{(k)} z + \ldots + a_m^{(k)} z^m}{(z-\alpha_1^{(k)})^{e_1}(z-\alpha_2^{(k)})^{e_2}\ldots(z-\alpha_n^{(k)})^{e_n}} = \Phi(z \mid s_1^{(k)}, s_2^{(k)}, \ldots, s_p^{(k)}, t^{(k)})$$

of z with the Taylor series

$$F^{(k)}(z) = \sum_{h=0}^{\infty} f_h^{(k)} z^h,$$

and here

$$f_h^{(k)} = 0 \quad \text{if} \quad h \in H.$$

We must, however, assume that k is already sufficiently large, i.e. that $\mathbf{s}^{(k)}$ is sufficiently near to $\boldsymbol{\sigma}$, so as to exclude the possibility that one of the expressions $a_\mu^{(k)}$, $\alpha_\nu^{(k)}$, $f_h^{(k)}$ becomes infinite, or that one of the poles $\alpha_\nu^{(k)}$ vanishes, or that two of these poles coincide. Assume, say, that these cases are excluded when $k \geqslant k_0$.

It follows now, from the continuity properties of a rational function, that

$$\lim_{k\to\infty} a_\mu^{(k)} = a_\mu, \quad \lim_{k\to\infty} \alpha_\nu^{(k)} = \alpha_\nu, \quad \lim_{k\to\infty} f_h^{(k)} = f_h$$

for all values of the suffixes μ, ν and h.

12. The equation

$$Q(s_1^{(k)}, s_2^{(k)}, \ldots, s_p^{(k)}; t^{(k)}) = 0$$

for $t^{(k)}$ is of degree q, and its coefficients lie in R; for both the numerical coefficients of Q, and the coördinates of $\mathbf{s}^{(k)}$, belong to R. Therefore $t^{(k)}$ is an algebraic number at most of degree q over the Gaussian field, hence at most of degree $2q$ over the rational field. Denote by $K^{(k)} = R(t^{(k)})$ the algebraic extension of R generated by $t^{(k)}$; this field has likewise a degree not greater than $2q$ over the rational field. From their definitions, it is clear that the numbers

$$a_\mu^{(k)}, \quad \alpha_\nu^{(k)}, \quad f_h^{(k)}$$

all are elements of $K^{(k)}$, as soon as $k \geqslant k_0$.

In particular, the Taylor coefficients f_h^* of

$$F^{(k)}(z) = \sum_{h=0}^{\infty} f_h^{(k)} z^h$$

are algebraic numbers, and furthermore infinitely many of these coefficients vanish,

$$f_h^{(k)} = 0 \quad \text{if} \quad h \in H.$$

The hypothesis of Theorem 1 is thus satisfied. Hence, for every $k \geqslant k_0$, $F^{(k)}(z)$ possesses at least one zero sequence $L_1 \pmod{L}$. Here we may assume that $0 \leqslant L_1 < L$. Both $L = L^{(k)}$ and $L_1 = L_1^{(k)}$ may still depend on k. We note that, by hypothesis,

$$m < e_1 + e_2 + \ldots + e_n.$$

Hence also $F^{(k)}(z)$ is normed, so that *all* its Taylor coefficients $f_h^{(k)}$ satisfying $h \equiv L_1 \pmod{L}$ are zero.

13. Lemma 1 enables us to construct a zero sequence $L_1 \pmod{L}$ of $F^{(k)}(z)$ with *bounded* L, hence also with *bounded* L_1.

The poles $\alpha_1^{(k)}, \alpha_2^{(k)}, \ldots, \alpha_n^{(k)}$ of $F^{(k)}(z)$ lie in $K^{(k)}$, and the same is therefore true for the quotients of two such poles. Denote by $\Sigma = \Sigma^{(k)}$ the set of all those quotients

$$\frac{\alpha_\mu^{(k)}}{\alpha_\nu^{(k)}} \neq 1$$

that are roots of unity; we know, from § 3, that Σ is not the null set. Hence a smallest positive integer $M = M^{(k)}$ exists such that all elements of Σ are Mth roots of unity.

By Lemma 1, $F^{(k)}(z)$ admits also the larger zero sequence

$$L_1 \pmod{L^*}, \quad \text{where} \quad L^* = (L, M).$$

This zero sequence is identical with $L_1^* \pmod{L^*}$, where L_1^* is the integer for which

$$L_1^* \equiv L_1 \pmod{L^*}, \quad 0 \leqslant L_1^* < L^*.$$

The roots of unity which are the elements of Σ lie in the algebraic field $K^{(k)}$, and this field is at most of degree $2q$. On the other hand, there are only finitely many roots of unity that are algebraic numbers at most of degree $2q$. Denote by M_0 the least common multiple of the orders of all these roots of unity. Then evidently

$$M^{(k)} \leqslant M_0 \quad \text{for} \quad k \geqslant k_0.$$

Since L^* is a divisor of $M^{(k)}$, this implies that also

$$0 \leqslant L_1^* < L^* \leqslant M_0 \quad \text{for} \quad k \geqslant k_0.$$

14. On dropping now again the asterisk, the last result may be formulated as follows:

If $k \geqslant k_0$, then $F^{(k)}(z)$ possesses at least one zero sequence

$$L_1 \pmod{L}, \quad \textit{where} \quad 0 \leqslant L_1 < L \leqslant M_0,$$

and where M_0 is independent of k. Moreover, all Taylor coefficients $f_h^{(k)}$ of $F^{(k)}(z)$ with $h \equiv L_1 \pmod{L}$ are zero.

There exist only *finitely many* zero sequences $L_1 \pmod{L}$ for which $0 \leqslant L_1 < L \leqslant M_0$, the zero sequences

$$Z_1, \quad Z_2, \quad \ldots, \quad Z_v,$$

say. For each $k \geqslant k_0$ denote by $u = u^{(k)}$ the smallest suffix such that Z_u is a zero sequence of $F^{(k)}(z)$. This function $u = u^{(k)}$ has only v possible values. Hence there is an infinite sequence of indices

$$k = k_1, k_2, k_3, \ldots, \quad \text{where} \quad k_0 \leqslant k_1 \leqslant k_2 \leqslant k_3 \leqslant \ldots,$$

for which $u = u^{(k)}$ assumes one and the same fixed value u_0. For all these indices, $F^{(k)}(z)$ possesses the same zero sequence Z_{u_0}, or say $L_1^0 \pmod{L^0}$, and all Taylor coefficients $f_h^{(k)}$ with $h \equiv L_1^0 \pmod{L^0}$ are zero. However, as was proved in § 11,

$$\lim_{k\to\infty} f_h^{(k)} = f_h \quad \text{for all } h.$$

Therefore, on allowing k to run over the sequence $k_1, k_2, k_3, \ldots$ to infinity, it follows at once that also

$$f_h = 0 \quad \text{if} \quad h \equiv L_1^0 \pmod{L^0}.$$

Hence the original function $F(z)$ likewise admits the zero sequence $L_1^0 \pmod{L^0}$. This proves the assertion.

15. Theorem 2 implies a slightly stronger result.

THEOREM 3. *Let*

$$F(z) = \sum_{h=0}^{\infty} f_h z^h$$

be a rational function of z which is regular at $z = 0$ and has infinitely many vanishing Taylor coefficients f_h. Then a positive integer L and at most L non-negative integers $L_1, L_2, \ldots, L_l$ with

$$L_j \not\equiv L_k \pmod L \quad \textit{for} \quad j \neq k$$

exist such that f_h vanishes exactly when

$$h \equiv L_j \pmod L, \quad h \geqslant L_j \quad (j = 1, 2, \ldots, l)$$

and for at most finitely many other values of h.

Proof. It may again be assumed that $F(z)$ is normed. Denote by M the same positive integer as in Lemma 1. By this lemma, it suffices to consider those zero sequences $L_j \pmod L$ of $F(z)$ for which L is a divisor of M. As such sequences can be subdivided into sequences $\pmod M$, it further suffices to prove the theorem with L replaced by M.

Denote by $L_1, L_2, \ldots, L_l \pmod M$ all distinct residue classes $h \equiv L_j \pmod M$ that contain infinitely many suffixes h for which $f_h = 0$. The assertion is proved if it can be shown that each $L_j \pmod M$ is a zero sequence of $F(z)$. It will be enough to consider $L_1 \pmod M$.

We assume thus that

$$f_h = 0 \text{ for infinitely many } h \equiv L_1 \pmod M.$$

Similarly as before, put

$$\epsilon = e^{2\pi i/M}, \quad E(z) = \sum_{j=0}^{M-1} \epsilon^{L_1 j} F(\epsilon^{-j} z);$$

further write

$$\zeta = z^M.$$

Then

$$z^{-L_1} E(z) = M z^{-L_1} \sum_{\substack{h=0 \\ h \equiv L_1 \pmod M}}^{\infty} f_h z^h = M \sum_{k=0}^{\infty} f_{L_1+kM} z^{kM} = \mathrm{E}(\zeta),$$

where

$$\mathrm{E}(\zeta) = M \sum_{k=0}^{\infty} f_{L_1+kM} \zeta^k$$

evidently is a rational function of ζ. This new function $\mathrm{E}(\zeta)$ is regular at $\zeta = 0$, vanishes at $\zeta = \infty$, and has infinitely many vanishing Taylor coefficients f_{L_1+kM}.

Hence it follows from Theorem 2 that $E(\zeta)$ possesses at least one zero sequence

$$k \equiv \kappa_1 \pmod{\kappa}.$$

As the function is normed, this implies that

$$f_{L_1+kM} = 0 \quad \text{if} \quad k \equiv \kappa_1 \pmod{\kappa},$$

or, what is the same,

$$f_h = 0 \quad \text{if} \quad h \equiv L_1 + \kappa_1 M \pmod{\kappa M}.$$

This relation means that the original function $F(z)$ has the zero sequence $L_1 + \kappa_1 M$ $(\mathrm{mod}\, \kappa M)$. But then, by Lemma 1, it also admits the larger zero sequence $L_1 + \kappa_1 M$ $(\mathrm{mod}\, M)$, hence also the zero sequence L_1 $(\mathrm{mod}\, M)$. This concludes the proof.

16. It is well known that, for sufficiently large h, the Taylor coefficient f_h of the rational function $F(z)$ has the explicit representation

$$f_h = \sum_{\nu=1}^{n} p_\nu(h)\,\beta_\nu^h$$

where $p_1(h), p_2(h), \ldots, p_n(h)$ are polynomials in the variable h not identically zero, while $\beta_1, \beta_2, \ldots, \beta_n$ are distinct constants different from zero, viz. the reciprocals of the poles of $F(z)$. Conversely, every expression of this kind defines the Taylor coefficients of a rational function regular at $z = 0$, and the same is true if h is replaced by $-h$. The following result is then implicit in Theorem 2.

THEOREM 4. *Let $\beta_1, \beta_2, \ldots, \beta_n$ be finitely many complex numbers that are distinct, different from zero, and such that no quotient of two of them is a root of unity. Also let $p_1(h), p_2(h), \ldots, p_n(h)$ be an equal number of polynomials not identically zero with arbitrary complex coefficients. Then the equation*

$$\sum_{\nu=1}^{n} p_\nu(h)\,\beta_\nu^h = 0$$

has at most finitely many solutions in rational integers.

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