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Extension of the transfer function approach to the realization problem of nonlinear systems to discrete-time case

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Abstract: In this paper the polynomial approach to the realization problem of nonlinear control systems, i.e. the problem of finding an observable state space representation of a SISO nonlinear system described by an input-output equation, is extended to the discrete-time case. To find the solution the so-called adjoint polynomials and adjoint transfer functions are employed establishing direct connection to the solution known from linear systems. In addition, in case a system is not realizable it is shown that, unlike in continuous-time case, there always exists a postcompensator which, when combined with the system in series connection, makes compensated system realizable. Due to the possibility to use the transfer function algebra when combining systems this problem can be solved in natural way.

Keywords: nonlinear discrete-time systems, realization problem, polynomial approach, transfer functions, algebraic approach

1. INTRODUCTION

Discrete-time control systems are usually described either by a higher order input-output difference equation or by a set of coupled first order difference equations called the state-space description. While the most results on parameter identification of real systems are available for systems described by an input-output difference equation, many techniques for system analysis and control are based on state-space description. In the linear case, any control system, described by a higher order input-output difference equation, can be equivalently described by the state-space equations and vice versa. In that respect the transfer function formalism introduced via the Z-transformation plays a key role. However, the situation is different for nonlinear systems. Although for a given state-space representation a corresponding input-output difference equation can be, at least locally, always found, the converse does not hold in general. There exists a class of nonlinear discrete-time input-output equations that do not admit usual state-space representation. Thus, a typical control problem here is to find conditions under which the state-space realization of a higher order input-output difference equation can be found. In case of continuous-time nonlinear systems different necessary and/or sufficient realizability conditions were found (Moog et al., 2002; Van der Schaft, 1987; Delaleau and Respondek, 1995; Conte et al., 2007). For discrete-time case the problem was studied for instance in Sadegh (2001) and Kotta et al. (2001). In the latter analogous conditions to those of Conte et al. (2007) were given.

In the linear case, different state-space representations of an input-output difference equations can be directly written out from the transfer function, for instance in the observer or controller canonical form. However, the transfer function formalism was recently developed also for nonlinear systems, see Zheng and Cao (1995); Halás (2008) for continuous-time case and Halás and Kotta (2007) for extension to discrete-time case. In doing so one associates with the control system two polynomials, as in Zhang and Zheng (2004), defined over the field of meromorphic functions. This can be understood also that way that the nonlinear system equations are linearized using Kähler diffrentials (Johnson, 1969) and then the ideas similar to those applied for linear time-varying systems in Fliess (1994) can be applied. That way it results in the linearized system description resembling time-varying linear system description except that now the time-varying coefficients of the polynomials are not necessarily independent (Li et al., 2008).

Such a transfer function formalism has been already employed in Perdon et al. (2007) to investigate some structural properties of nonlinear systems, in Halás et al. (2008) to study the nonlinear model matching and in Halás and Kotta (2008) to study the observer design. Recently, the transfer function formalism of nonlinear systems was applied in Halás and Kotta (2009) to study the realization problem of continuous-time systems. It was shown that analogical results to the linear case can be established. That is, a state-space realization of a nonlinear system can be find from its transfer function. In that respect the notion of adjoint polynomials (Abramov et al., 2005) and adjoint transfer function play a key role. In this paper we prove analogical results for discrete-time nonlinear sys-

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tems, providing the extension. However, besides the extension we study here also the possibility of overcoming the non-realizability of a system by adding a postcompensator. Different approaches were used to study this problem in Mullari et al. (2006) for continuous-time case and in Nõmm et al. (2005) for discrete-time case. It was shown that, unlike in continuous-time case, in discrete-time case any non-realizable system can always be combined with a postcompensator in series connection so that the compensated system becomes realizable. However, due to the availability of the transfer function algebra, this problem can be treated in very natural way. In particular, like in Nõmm et al. (2005) we, using the tranfer function formalism, show that for any non-realizable nonlinear discrete-time system there always exists a postcompensator such that the compensated system is realizable.

2. TRANSFER FUNCTIONS OF NONLINEAR DISCRETE-TIME SYSTEMS

We briefly recall the transfer function formalism of nonlinear discrete-time systems and refer reader to Halás and Kotta (2007) for additional details.

In what follows we use the notation ξ for any variable $\xi(t)$ and $\xi^{[k]}$ for its time shift $\xi(t+k)$.
Consider the SISO nonlinear discrete-time system defined by a state-space representation of the form

$$\begin{aligned} x^{[1]} &= f(x,u) \\ y &= g(x) \end{aligned} \tag{1}$$

with $x \in \mathbf{R}^n$, or by an input-output difference equation of the form

$$y^{[n]} = \varphi(y, \ldots, y^{[n-1]}, u, \ldots, u^{[s]}) \tag{2}$$

In both (1) and (2) the functions f, g and, respectively, φ are assumed to be elements of the field of meromorphic functions $\mathcal{K}$.
The system is assumed to be generically submersive, that is

$$\text{rank}_{\mathcal{K}} \frac{\partial f(\cdot)}{\partial(x,u)} = n$$

for (1) or, respectively

$$\text{rank}_{\mathcal{K}} \frac{\partial \varphi(\cdot)}{\partial(y,u)} \not\equiv 0 \tag{3}$$

for (2).

The left skew polynomial ring $\mathcal{K}[\delta]$ of polynomials in δ over $\mathcal{K}$ with the usual addition and the (non-commutative) multiplication given by the commutation rule

$$\delta a = a^{[1]}\delta \tag{4}$$

where $a \in \mathcal{K}$, represents the ring of linear shift operators that act over the vector space of one-forms

$$\mathcal{E} = \text{span}_{\mathcal{K}}\{\text{d}\xi; \xi \in \mathcal{K}\}$$

in the following way

$$\left(\sum_{i=0}^{k} a_i \delta^i\right) v = \sum_{i=0}^{k} a_i v^{[i]}$$

for any $v \in \mathcal{E}$.
The commutation rule (4) actually represents the rule for shifting.

Remark 1. Note that in case of rational or even algebraic functions the vector space $\mathcal{E}$ coincides with the vector space of Kähler differentials having been widely used to study nonlinear control systems, see for instance Fliess et al. (1995).

Lemma 2. (Ore condition). For all non-zero $a, b \in \mathcal{K}[\delta]$, there exist non-zero $a_1, b_1 \in \mathcal{K}[\delta]$ such that $a_1 b = b_1 a$.

Thus, the ring $\mathcal{K}[\delta]$ can be embedded to the non-commutative quotient field $\mathcal{K}\langle\delta\rangle$ by defining quotients as

$$\frac{a}{b} = b^{-1} \cdot a$$

The addition and multiplication in $\mathcal{K}\langle\delta\rangle$ are defined as

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{\beta_2 a_1 + \beta_1 a_2}{\beta_2 b_1}$$

where $\beta_2 b_1 = \beta_1 b_2$ by Ore condition and

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{\alpha_1 a_2}{\beta_2 b_1} \tag{5}$$

where $\beta_2 a_1 = \alpha_1 b_2$ again by Ore condition.
Due to the non-commutative multiplication (4) they, of course, differ from the usual rules. In particular, in case of the multiplication (5) we, in general, cannot simply multiply numerators and denominators, nor cancel them in a usual manner. We neither can commute them as the multiplication in $\mathcal{K}\langle\delta\rangle$ is non-commutative as well.

Once the fraction of two skew polynomials is defined we can introduce the transfer function of the nonlinear system (1) (respectively (2)) as an element $F(\delta) \in \mathcal{K}\langle\delta\rangle$ such that $\text{d}y = F(\delta)\text{d}u$.

When starting with the state space description (1) we, after differentiating, get

$$\begin{aligned} \text{d}x^{[1]} &= A\text{d}x + B\text{d}u \\ \text{d}y &= C\text{d}x \end{aligned}$$

where $A = (\partial f/\partial x)$, $B = (\partial f/\partial u)$, $C = (\partial g/\partial x)$. Or alternatively

$$\begin{aligned} \delta\text{d}x &= A\text{d}x + B\text{d}u \\ \text{d}y &= C\text{d}x \end{aligned} \tag{6}$$

from which follows

$$F(\delta) = C(\delta I - A)^{-1}B$$

Remark 3. Note that in spite of the formal similarity to the transfer functions of linear time-invariant systems, this time we have to invert matrix $(\delta I - A)$ over the non-commutative quotient field $\mathcal{K}\langle\delta\rangle$, which is far from trivial. Clearly, the entries of $(\delta I - A)$ are skew polynomials from $\mathcal{K}[\delta]$. Thus, the inversion requires finding the solutions of a set of linear equations over the non-commutative field, see Halás (2008) and references therein for more details.

When starting with the input-output description (2) we, after differentiating, get

$$\text{d}y^{[n]} - \sum_{i=0}^{n-1} \frac{\partial \varphi}{\partial y^{[i]}} \text{d}y^{[i]} = \sum_{i=0}^{s} \frac{\partial \varphi}{\partial u^{[i]}} \text{d}u^{[i]}$$

or alternatively

$$a(\delta)\text{d}y = b(\delta)\text{d}u$$

where $a(\delta) = \delta^n - \sum_{i=0}^{n-1} \frac{\partial \varphi}{\partial y^{[i]}} \delta^i$, $b(\delta) = \sum_{i=0}^{s} \frac{\partial \varphi}{\partial u^{[i]}} \delta^i$ and $a(\delta), b(\delta) \in \mathcal{K}[\delta]$. Then the transfer function

$$F(\delta) = \frac{b(\delta)}{a(\delta)}$$

Note that this allows us to compute the transfer function even for nonrealizable systems.

Example 4. Consider the system

$$y^{[2]} = yu^{[1]} + u$$

which has according to Kotta et al. (2001) no state-space realization. However, the transfer function can be computed. After differentiating

$$\begin{aligned} \mathrm{d}y^{[2]} &= u^{[1]}\mathrm{d}y + y\mathrm{d}u^{[1]} + \mathrm{d}u \\ (\delta^2 - u^{[1]})\mathrm{d}y &= (y\delta + 1)\mathrm{d}u \end{aligned}$$

we get

$$F(\delta) = \frac{y\delta + 1}{\delta^2 - u^{[1]}}$$

3. REALIZATION PROBLEM

The realization problem of nonlinear discrete-time systems was solved using different approaches, see for instance Kotta et al. (2001); Sadegh (2001). Here, we recal that of Kotta et al. (2001) employing the algebraic formalism of one-forms, for later we couple solutions with the transfer function approach.

Problem statement. For the nonlinear system (2) find, if possible, an observable state space representation of the form (1) such that it is its realization.

Remark 5. Note that the system (1) is observable iff $\mathrm{rank}_{\mathcal{K}} \frac{\partial(y,\dots,y^{[n-1]})}{\partial x} = n$ and it is called a realization if it generates the input-output equation (2).

The system (2) defines a field of meromorphic functions $\mathcal{K}$ of variables $\{y, \dots, y^{[n-1]}, u^{[k]}; k \geq 0\}$. We define the formal vector space of differential one-forms $\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi; \xi \in \mathcal{K}\}$ and a filtration of $\mathcal{E}$ given by a sequence of subspaces $\{\mathcal{H}_k\}$ of $\mathcal{E}$ as follows:

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \dots, \mathrm{d}y^{[n-1]}, \mathrm{d}u, \dots, \mathrm{d}u^{[s]}\} \\ \mathcal{H}_{k+1} &= \mathrm{span}_{\mathcal{K}}\{\omega \in \mathcal{H}_k; \omega^{[1]} \in \mathcal{H}_k\} \end{aligned}$$

Proposition 6. For the system (2) there exists an observable state space realization of the form (1) if and only if

- $n > s$
- $\mathcal{H}_k$ is integrable for all $k = 1, \dots, s+2$.

Example 7. Consider the system from Example 4 for which we compute

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}y^{[1]}, \mathrm{d}u, \mathrm{d}u^{[1]}\} \\ \mathcal{H}_2 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}y^{[1]}, \mathrm{d}u\} \\ \mathcal{H}_3 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}y^{[1]} - y^{[-1]}\mathrm{d}u\} \end{aligned}$$

Since $\mathcal{H}_3$ is not integrable the system is not realizable. However, following the lines of Nõmm et al. (2005) the problem can be overcome by adding a postcompensator,

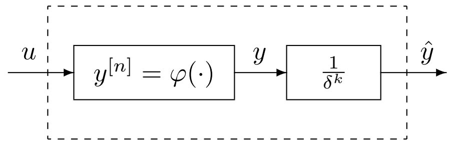

Fig. 1. Postcompensator

see Fig. 1. In this case the postcompensator $\hat{y}^{[1]} = y$ gives us the compensated system

$$\hat{y}^{[3]} = \hat{y}^{[1]}u^{[1]} + u$$

which is realizable in state-space form. The sequence of subspaces can be now computed as

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\hat{y}, \mathrm{d}\hat{y}^{[1]}, \mathrm{d}\hat{y}^{[2]}, \mathrm{d}u, \mathrm{d}u^{[1]}\} \\ \mathcal{H}_2 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\hat{y}, \mathrm{d}\hat{y}^{[1]}, \mathrm{d}\hat{y}^{[2]}, \mathrm{d}u\} \\ \mathcal{H}_3 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\hat{y}, \mathrm{d}\hat{y}^{[1]}, \mathrm{d}\hat{y}^{[2]} - \hat{y}\mathrm{d}u\} \end{aligned}$$

all of them integrable. Therefore we can find state variables $\hat{x}$ such that $\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\hat{x}\}$. If we choose $\hat{x}_1 = \hat{y}$, $\hat{x}_2 = \hat{y}^{[1]}$ and $\hat{x}_3 = \hat{y}^{[2]} - \hat{y}u$ the state-space realization becomes

$$\begin{aligned} \hat{x}_1^{[1]} &= \hat{x}_2 \\ \hat{x}_2^{[1]} &= \hat{x}_3 + \hat{x}_1 u \\ \hat{x}_3^{[1]} &= u \\ \hat{y} &= \hat{x}_1 \end{aligned}$$

Note that in continuous-time counterpart adding a post-compensator is of no help in overcoming non-realizability.

3.1 Transfer function approach

Since the transfer function of a nonlinear system can be computed from both a state-space representation (1) and an input-output difference equation (2) the basic idea how to solve the realization problem seems to be very easy. That is, for a nonlinear system (2) with the transfer function $F(\delta) = \frac{b(\delta)}{a(\delta)}$ find the matrices A, B and C such that $F(\delta) = C(\delta I - A)^{-1}B$, like in the linear case.
However, the situation is not so straightforward for the coefficients of polynomials $b(\delta)$, $a(\delta)$ are not reals but, in general, meromorphic functions from $\mathcal{K}$. Therefore, stating matrices A, B and C in the observer canonical form, directly from transfer function like in linear case, does not yield, in general, required result, as can be easily demonstrated.
Consider for instance a second-order nonlinear discrete-time system with the transfer function

$$F(\delta) = \frac{1}{\delta^2 + a_1\delta + a_0}$$

where $a_0, a_1 \in \mathcal{K}$. Then, if assuming a realization in terms of differentials (6) with

$$A = \begin{pmatrix} 0 & -a_0 \\ 1 & -a_1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

we get

$$\begin{aligned}\mathrm{d}y &= \mathrm{d}x_2\\ \mathrm{d}y^{[1]} &= \mathrm{d}x_1 - a_1\mathrm{d}y\\ \mathrm{d}y^{[2]} &= -a_0\mathrm{d}y + \mathrm{d}u - a_1^{[1]}\mathrm{d}y^{[1]}\end{aligned}$$

and clearly

$$F(\delta) = \frac{1}{\delta^2 + a_1^{[1]}\delta + a_0}$$

which differs from that we started.

Like in continuous-time counterpart (Halás and Kotta, 2009), the problem is caused by the fact that coefficients of a tranfer function are, in general, functions from $\mathcal{K}$. Hence, in shifting $a\mathrm{d}\xi$, with $a \in \mathcal{K}$ and $\mathrm{d}\xi \in \mathcal{E}$, we have to consider that $(a\mathrm{d}\xi)^{[1]} = a^{[1]}\mathrm{d}\xi^{[1]}$, or in terms of polynomials $\delta a = a^{[1]}\delta$. While in the linear case, $a \in \mathbf{R}$, it is only $(a\mathrm{d}\xi)^{[1]} = a\mathrm{d}\xi^{[1]}$, or in terms of polynomials $\delta a = a\delta$.
However, this problem can be avoided elegantly by remaining all shifts unexpanded. This in terms of polynomials means that the indeterminate δ has to be always from the *left*. Such an idea can be very transparently implemented by introducing the notion of adjoint polynomials.

Adjoint polynomials. Adjoint polynomials (Abramov et al., 2005) represent in some sense dual objects to skew polynomials. We can get them by moving the indeterminate on the *left* of each summand. In general, they are defined for any skew polynomial ring. Here, we reduce the definition just for the case of polynomials representing shift operators.

Definition 8. The adjoint of a skew polynomial ring $\mathcal{K}[\delta]$ is defined as the skew polynomial ring $\mathcal{K}[\delta^*]$ with the commutation rule

$$\delta^* a = a^{[-1]}\delta^*$$

for any $a \in \mathcal{K}$.

That is, if $a(\delta) = a_n\delta^n + \ldots + a_1\delta + a_0$ is a polynomial in $\mathcal{K}[\delta]$ then the adjoint polynomial $a^*(\delta^*)$ is defined by the formula

$$a^*(\delta^*) = \delta^{*n}a_n + \ldots + \delta^* a_1 + a_0 \in \mathcal{K}[\delta^*] \tag{7}$$

Note that products $\delta^{*i}a_i$ must be computed in the skew polynomial ring $\mathcal{K}[\delta^*]$.
The adjoint operation is a bijective mapping and also have other useful properties, see Abramov et al. (2005) for more details.
Note that in commutative case; that is, the case of linear systems when all coefficients are in $\mathbf{R}$, a polynomial and its adjoint are identical objects. Note also that adjoint polynomials and other related tools are already implemented for instance in the computer algebra system Maple which significantly simplifies computations.

Example 9. Consider the system from Example 4 with the transfer function $F(\delta) = \frac{b(\delta)}{a(\delta)}$ where

$$\begin{aligned}a(\delta) &= \delta^2 - u^{[1]}\\ b(\delta) &= y\delta + 1\end{aligned}$$

Following the rule (7) the adjoint polynomials can be computed as

$$\begin{aligned}a^*(\delta^*) &= \delta^{*2} - u^{[1]}\\ b^*(\delta^*) &= \delta^*\, y + 1 = y^{[-1]}\delta^* + 1\end{aligned}$$

Note that this is exactly the idea of moving the indeterminate δ on the left of each summand in the original polynomial

$$b(\delta) = y\delta + 1 = \delta y^{[-1]} + 1$$

for following the comutation rule (4) in the original ring $\mathcal{K}[\delta]$ we have $\delta y^{[-1]} = y\delta$.
For that reason and for the sake of simplicity we will usualy use notation with the indeterminate δ on the left in original polynomials, for instance in the transfer function

$$F(\delta) = \frac{y\delta + 1}{\delta^2 - u^{[1]}} = \frac{\delta y^{[-1]} + 1}{\delta^2 - u^{[1]}}$$

while formally speaking of adjoint polynomials.

Lemma 10. Let

$$\begin{aligned}F(\delta) &= \frac{b_{n-1}\delta^{n-1} + \cdots + b_0}{\delta^n + a_{n-1}\delta^{n-1} + \cdots + a_0}\\ &= \frac{\delta^{n-1}b_{n-1}^* + \cdots + b_0^*}{\delta^n + \delta^{n-1}a_{n-1}^* + \cdots + a_0^*}\end{aligned}$$

be the tranfer function and, respectively, the adjoint transfer function of the nonlinear system (2). Then

$$\begin{array}{rcll}\mathrm{d}\xi_1^{[1]} &=& & -a_0^*\mathrm{d}\xi_n + b_0^*\mathrm{d}u\\ \mathrm{d}\xi_2^{[1]} &=& \mathrm{d}\xi_1 & -a_1^*\mathrm{d}\xi_n + b_1^*\mathrm{d}u\\ &\vdots& & \\ \mathrm{d}\xi_n^{[1]} &=& \mathrm{d}\xi_{n-1} & -a_{n-1}^*\mathrm{d}\xi_n + b_{n-1}^*\mathrm{d}u\\ \mathrm{d}y &=& \mathrm{d}\xi_n & \end{array} \tag{8}$$

is the realization in terms of differentials.

Proof. Note that (8) is in the observer canonical form and therefore the proof follows the same line as in Halás and Kotta (2008).

Clearly, for the system (2) to be realizable the equations (8) have to result in certain sequences of exact/integrable one-forms.

Theorem 11. Let

$$\begin{aligned}F(\delta) &= \frac{b_{n-1}\delta^{n-1} + \cdots + b_0}{\delta^n + a_{n-1}\delta^{n-1} + \cdots + a_0}\\ &= \frac{\delta^{n-1}b_{n-1}^* + \cdots + b_0^*}{\delta^n + \delta^{n-1}a_{n-1}^* + \cdots + a_0^*}\end{aligned}$$

be the tranfer function and, respectively, the adjoint transfer function of the nonlinear system (2). Let $\omega_i = b_{i-1}^*\mathrm{d}u - a_{i-1}^*\mathrm{d}y$ for $i = 1, \ldots, n$. Then:

(1) there exists an observable state space realization in the observer cannonical form if and only if all ω_i's are exact; that is,

$$\mathrm{d}\omega_i = 0$$

for $i = 1, \ldots, n$.

(2) there exists an observable state space realization of the form (1) if and only if

$$\mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}y^{[1]} - \omega_n, \mathrm{d}y^{[2]} - \omega_n^{[1]} - \omega_{n-1}, \ldots,\\ \mathrm{d}y^{[n-1]} - \omega_n^{[n-2]} - \omega_{n-1}^{[n-3]} - \ldots - \omega_2\}$$

is integrable.

Proof. For part 1 it follows the same line as in Halás and Kotta (2008).
Part 2: Note that from (8) we can write

$$\begin{aligned}
\mathrm{d}y &= \mathrm{d}\xi_n \\
\mathrm{d}y^{[1]} &= \mathrm{d}\xi_{n-1} + \omega_n \\
\mathrm{d}y^{[2]} &= \mathrm{d}\xi_{n-2} + \omega_{n-1} + \omega_n^{[1]} \\
&\vdots \\
\mathrm{d}y^{[n-1]} &= \mathrm{d}\xi_1 + \omega_2 + \ldots + \omega_{n-1}^{[n-3]} + \omega_n^{[n-2]}
\end{aligned}$$

from which one can derive $\mathrm{d}\xi_n = \mathrm{d}y$, $\mathrm{d}\xi_{n-1} = \mathrm{d}y^{[1]} - \omega_n$, $\ldots, \mathrm{d}\xi_1 = \mathrm{d}y^{[n-1]} - \omega_n^{[n-2]} - \omega_{n-1}^{[n-3]} - \ldots - \omega_2$, i.e. $\mathcal{H}_{s+2} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi_1, \ldots, \mathrm{d}\xi_n\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}y^{[1]} - \omega_n, \mathrm{d}y^{[2]} - \omega_n^{[1]} - \omega_{n-1}, \ldots, \mathrm{d}y^{[n-1]} - \omega_n^{[n-2]} - \omega_{n-1}^{[n-3]} - \ldots - \omega_2\}$.

Example 12. Consider the system from Example 4 with the transfer function and the adjoint transfer function

$$F(\delta) = \frac{y\delta + 1}{\delta^2 - u^{[1]}} = \frac{\delta y^{[-1]} + 1}{\delta^2 - u^{[1]}}$$

The realization in terms of differentials is

$$\begin{aligned}
\mathrm{d}\xi_1^{[1]} &= u^{[1]}\mathrm{d}\xi_2 + \mathrm{d}u \\
\mathrm{d}\xi_2^{[1]} &= \mathrm{d}\xi_1 + y^{[-1]}\mathrm{d}u \\
\mathrm{d}y &= \mathrm{d}\xi_2
\end{aligned}$$

Neither $\omega_1 = u^{[1]}\mathrm{d}\xi_2 + \mathrm{d}u$ nor $\omega_2 = y^{[-1]}\mathrm{d}u$ are exact, therefore the realization in the ovserver cannonical form does not exist.
In addition, neither any observable realization of the form (1) exists. For $\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}y^{[1]} - \omega_2\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}y^{[1]} - y^{[-1]}\mathrm{d}u\}$ is not integrable.

Postcompensator. However, in case of nonlinear discrete-time systems it is always possible to overcome non-realizability of a system by adding a postcompensator of the form $\frac{1}{\delta^k}$, see Fig. 1, as was proved in Nõmm et al. (2005). This is in big contrast to continuous-time case where the use of such a postcompensator is of no help. Here we study the problem in terms of the transfer function formalism which is, due to the possibility to use the transfer function algebra when combining systems in series connection, a very natural way.
Consider the nonlinear discrete-time system (2) and let

$$F(\delta) = \frac{b_{n-1}\delta^{n-1} + \cdots + b_0}{\delta^n + a_{n-1}\delta^{n-1} + \cdots + a_0}$$

be its transfer function. If we consider a postcompensator of the form

$$\hat{y}^{[k]} = y$$

with the transfer function

$$\frac{1}{\delta^k}$$

then clearly the compensated system will have the transfer function

$$G(\delta) = \frac{1}{\delta^k} \cdot F(\delta) = \frac{b_{n-1}\delta^{n-1} + \cdots + b_0}{\delta^{n+k} + a_{n-1}\delta^{n+k-1} + \cdots + a_0\delta^k}$$

Hence, the question of the existence of a suitable post-compensator reslults simply in checking the conditions of Theorem 11 for $k = 1, 2, \ldots$ until they are satisfied.

Example 13. We will consider the system from Example 4 for which we show in Examples 7 and 12 that it is not realizable. However, as shown in Example 7 a postcompensator making the compensated system realizable can be found. Here, the solution is found in terms of transfer functions.
The transfer function of the system was

$$F(\delta) = \frac{y\delta + 1}{\delta^2 - u^{[1]}}$$

We start with adding a postcompensator $\hat{y}^{[1]} = y$ with the transfer function

$$\frac{1}{\delta}$$

The transfer function and the adjoint transfer function of the compensated system can be expressed as

$$G(\delta) = \frac{1}{\delta} \cdot F(\delta) = \frac{y\delta + 1}{\delta^3 - u^{[1]}\delta} = \frac{\delta y^{[-1]} + 1}{\delta^3 - \delta u} = \frac{\delta\hat{y} + 1}{\delta^3 - \delta u}$$

Now, we have $\omega_1 = \mathrm{d}u$, $\omega_2 = \hat{y}\mathrm{d}u + u\mathrm{d}\hat{y}$ and $\omega_3 = 0$, all of them exact, note that $\omega_2 = \mathrm{d}(\hat{y}u)$. Hence, this time the realization exists directly in the observer cannonical form

$$\begin{aligned}
\hat{x}_1^{[1]} &= u \\
\hat{x}_2^{[1]} &= \hat{x}_1 + \hat{x}_3 u \\
\hat{x}_3^{[1]} &= \hat{x}_2 \\
\hat{y} &= \hat{x}_3
\end{aligned}$$

yielding the same result as in Example 7.

Finally, we show that for any system (2) there exists a postcompensator making the compensated system realizable.

Theorem 14. Given a system (2), there always exists a postcompensator $\hat{y}^{[k]} = y$ with $0 \leq k \leq s$ such that the compensated system $\hat{y}^{[k+n]} = \varphi(\cdot)$ is realizable.

Sketch of the proof. One can show that if the compensated system with $k = 0$, i.e. an original system (2), is not realizable then it becomes realizable at least for $k = s$. Let

$$\begin{aligned}
F(\delta) &= \frac{b_s\delta^s + \cdots + b_0}{\delta^n + a_{n-1}\delta^{n-1} + \cdots + a_0} \\
&= \frac{\delta^s b_s^* + \cdots + b_0^*}{\delta^n + \delta^{n-1}a_{n-1}^* + \cdots + a_0^*}
\end{aligned}$$

be the (adjoint) transfer function of a system (2).
Then the (adjoint) transfer function of the compensated system, $G(\delta) = \frac{1}{\delta^s} \cdot F(\delta)$, reads

$$\begin{aligned}
G(\delta) &= \frac{b_s\delta^s + \cdots + b_0}{\delta^{n+s} + a_{n-1}\delta^{n+s-1} + \cdots + a_0\delta^s} \\
&= \frac{\delta^s b_s^* + \cdots + b_0^*}{\delta^{n+s} + \delta^{n+s-1}a_{n-1}^{*[-s]} + \cdots + \delta^s a_0^{*[-s]}}
\end{aligned}$$

from which by Theorem 11 one has

$$\begin{aligned}
\omega_1 &= b_0^*\mathrm{d}u & \omega_{s+2} &= -a_1^{*[-s]}\mathrm{d}\hat{y} \\
&\vdots & &\vdots \\
\omega_s &= b_{s-1}^*\mathrm{d}u & \omega_{n+s} &= -a_{n-1}^{*[-s]}\mathrm{d}\hat{y} \\
\omega_{s+1} &= b_s^*\mathrm{d}u - a_0^{*[-s]}\mathrm{d}\hat{y} & &
\end{aligned}$$

Thus, the subspace

$$\mathcal{H}_{s+2} = \text{span}_{\mathcal{K}}\,\{\mathrm{d}\hat{y}, \mathrm{d}\hat{y}^{[1]} - \omega_{n+s}, \ldots, \\ \mathrm{d}\hat{y}^{[n-1]} - \omega_{n+s}^{[n-2]} - \ldots - \omega_{s+2}, \\ \mathrm{d}\hat{y}^{[n]} - \omega_{n+s}^{[n-1]} - \ldots - \omega_{s+1}, \ldots, \\ \mathrm{d}\hat{y}^{[n+s-1]} - \omega_{n+s}^{[n+s-2]} - \ldots - \omega_2\}$$

Due to the special form of the ω_i's the first n elements of the basis can be transformed (each of them is a linear combination of the previous ones) and then the subspace

$$\mathcal{H}_{s+2} = \text{span}_{\mathcal{K}}\,\{\mathrm{d}\hat{y}, \mathrm{d}\hat{y}^{[1]}, \ldots, \mathrm{d}\hat{y}^{[n-1]}, \\ \mathrm{d}\hat{y}^{[n]} - \omega_{n+s}^{[n-1]} - \ldots - \omega_{s+1}, \ldots, \\ \mathrm{d}\hat{y}^{[n+s-1]} - \omega_{n+s}^{[n+s-2]} - \ldots - \omega_2\}$$

is, by Frobenius theorem, always integrable.

Note that the non-integrability could be caused only by the fact that some coefficients of the one-forms contain the system variables y and u at negative time shifts, which can be a result of the adjoint operation. However, this cannot happen now, for all possible negative time shifts of the original output y are compensated by adding the postcompensator $\hat{y}^{[s]} = y$ and remaining possible negative time shifts of the input u can be, due to the submersivity assumption (3), restated in terms of non-negative time shifts of $\hat{y}$ and u. See Mullari et al. (submitted).

4. CONCLUSIONS

In this paper the transfer function approach to the realization problem of nonlinear control systems was extended to the discrete-time case. The notion of adjoint polynomials and adjoint transfer functions played a key role and established direct connection to the results known from linear systems. Using the transfer function formalism it was show that if a system is not realizable then, unlike in continuous-time case, there always exists a postcompensator such that the compensated system becomes realizable.

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