

# Supported Sets – A New Foundation For Nominal Sets And Automata

Thorsten Wißmann 

Radboud University, Nijmegen, the Netherlands

---

## Abstract

The present work proposes and discusses the category of supported sets which provides a uniform foundation for nominal sets of various kinds, such as those for equality symmetry, for the order symmetry, and renaming sets. We show that all these differently flavoured categories of nominal sets are monadic over supported sets. Thus, supported sets provide a canonical finite way to represent nominal sets and the automata therein, e.g. register automata. Name binding in supported sets is modelled by a functor following the idea of de Bruijn indices. This functor lifts to the well-known abstraction functor in nominal sets. Together with the monadicity result, this gives rise to a transformation process that takes the finite representation of a register automaton in supported sets and transforms it into its configuration automaton in nominal sets.

**2012 ACM Subject Classification** Theory of computation

**Keywords and phrases** Nominal Sets, Monads, LFP-Category, Supported Sets, Coalgebra

**Related Version** *Full Version with Appendix:* <http://arxiv.org/abs/2201.09825>

**Funding** *Thorsten Wißmann:* Supported by the NWO TOP project 612.001.852

**Acknowledgements** The author thanks Jurriaan Rot, Joshua Moerman, and Frits Vaandrager for inspiring discussions. The definition of supported sets originated from discussions with Lutz Schröder, Dexter Kozen, and Stefan Milius.

## 1 Introduction

Nominal sets provide an elegant framework to reason about structures that involve names, the permutation of names, and name binding. Originally used 100 years ago by Fraenkel and Mostowski for the entirely different purpose of axiomatic set theory, Gabbay and Pitts [17] re-discovered them in 1999 to define  $\lambda$ -expressions modulo  $\alpha$ -equivalence as an inductive data type. Its associated structural recursion principle allowed it to define capture avoiding substitution as a total function. Since then, a plethora of results using nominal sets were established in many areas, including proof assistants [37, 5], calculi [16, 36], and automata models [7, 34, 25]. Nominal automata are capable of processing words over infinite alphabets (*data alphabets*), while having good computational properties. Their expressiveness is similar (and sometimes identical) to that of register automata [24, 12, 8], which are automata with a *finite* description processing infinite alphabets. This finiteness condition translates into the notion of *orbit-finiteness* in the nominal world, which requires extra work to obtain a finite description of nominal automata, because orbit-finite objects are infinite in general.

In recent years, more general concepts of nominal sets were considered that generalize from the permutation of names to other operations. One branch is called *renaming sets* [36, 19, 29] and allows that distinct names are mapped to the same identifier; in another branch, symmetries on other data alphabets [7, 41] were considered, for example monotone bijections on rational numbers  $\mathbb{Q}$ , called the *total order symmetry*.

Nominal sets are not the only categorical approach to name binding. Multiple presheaf-based approaches to name binding were developed over the years [14, 15] which are strongly related to nominal sets [21]. Also, the theory of *named sets* were introduced for the study of

history dependent automata [13, 31], they were shown to be equivalent (in the categorical sense) to nominal sets [21], and they also feature a name binding construction [9]. In contrast to nominal sets, the name binding in presheaf approaches and named sets follow the method of *de Bruijn indices* [11].

Despite their rich categorical structure, it is known that the category of nominal sets (in all above-mentioned flavours) is not *monadic over sets*, that is, their theory is not an algebraic theory that can be described by a monad on sets – unlike groups, rings, vector spaces, etc, which can be described by algebraic theories.

In order to still make algebraic methods applicable, we introduce the category of *supported sets* and show that nominal sets are monadic over those. Thus, we make nominal sets applicable to results and methods known from algebraic categories, such as the generalized powerset construction [35, 6] or results about the representation of algebras [1] that provide a canonical finite representation of orbit-finite nominal sets. Moreover, supported sets do not require symmetries on the data alphabet, so they can also serve as a categorical foundation for data alphabets that are not described by symmetries. In fact, we discovered supported sets while working on a learning algorithm for register automata for general data alphabets.

**Structure of the paper.** After recalling the basic definitions for nominal sets (Section 2), we introduce supported sets and discuss their basic categorical properties (Section 3). Having established the monadicity of nominal sets (Section 4), we show that supported sets have a functor for name binding that lifts to nominal sets (Section 5). This yields a construction from (register) automata in supported sets to nominal automata (Section 6). Full proofs and also additional explanations to definitions and examples can be found in the appendix.

## 2 Preliminaries on Nominal Sets

Before introducing supported sets in detail, we first recall some notations and basic concepts of nominal sets. We assume that the reader is familiar with basic category theory, but we restrict to set-theoretic definitions wherever possible.

► **Notation 2.1.** Given sets  $X, Y$ , the set of all maps  $X \rightarrow Y$  is written  $Y^X$ . Injective maps are denoted by  $\hookrightarrow$ , surjective maps by  $\twoheadrightarrow$ , and  $\cdot$  denotes map composition. We fix sets  $1 = \{0\}$ ,  $2 = \{0, 1\}$ . The cycle notation  $(a_0 a_1 \dots a_{n-1})$  for elements of a set  $A$  denotes the bijection  $A \rightarrow A$  sending  $a_0 \mapsto a_1$ ,  $a_1 \mapsto a_2$ ,  $\dots$ ,  $a_{n-1} \mapsto a_0$ , and fixing all other elements of  $A$ .

Nominal sets and various flavours thereof are built around the notion of monoid actions, which specifies how atoms nested in some structure can be permuted or renamed.

► **Definition 2.2.** Given a monoid  $(M, \cdot, e)$ , an  $M$ -set is a set  $X$  together with a map  $\cdot : M \times X \rightarrow X$ , called the action and written infix  $m \cdot x$  for  $x \in X$ ,  $m \in M$ , such that  $e \cdot x = x$  and  $(m \cdot m') \cdot x = m \cdot (m' \cdot x)$  for all  $m, m' \in M$ ,  $x \in X$ . Thus, we use  $\cdot$  both for the monoid multiplication and the action. A map  $f : X \rightarrow Y$  between  $M$ -sets  $(X, \cdot)$ ,  $(Y, \star)$  is equivariant if  $f(m \cdot x) = m \star f(x)$  for all  $m \in M$ ,  $x \in X$ .

Throughout the paper,  $M$  will be a submonoid of  $(A^A, \cdot, \text{id}_A)$  for a set  $A$ , written  $M \leq A^A$ , in which most of the results are parametric. The set  $A$  is called the set of *atoms*, which can intuitively be understood as names of registers or data-values that appear in the input of an automaton or as names used for binding operations. The submonoid  $M$  determines a subset of maps of interest, closed under composition. Thus, we use  $\cdot$  both for map composition and monoid-multiplication, and moreover, the unit of the monoid  $M$  is simply  $\text{id}_A$ .

► **Example 2.3.** The main instances of monoids  $M$  for nominal techniques are as follows:

1. For a set  $A$ , let  $\mathfrak{S}_f(A)$  be the bijections on  $A$  that modify only finitely many elements, i.e.

$$\mathfrak{S}_f(A) := \{\phi: A \rightarrow A \mid \phi \text{ bijective and } \{a \in A \mid \phi(a) \neq a\} \text{ is finite}\}$$

For nominal sets (over the equality symmetry), one fixes a countably infinite set  $\mathbb{A}$  (understood as *names*) and fixes  $M := \mathfrak{S}_f(\mathbb{A})$  [17, 33].

2. For the set  $\mathbb{Q}$  of rational numbers, let  $\text{Aut}(\mathbb{Q}, <)$  be the set of bijective and monotone maps  $\mathbb{Q} \rightarrow \mathbb{Q}$ . For nominal sets over the total order symmetry, one considers  $M := \text{Aut}(\mathbb{Q}, <)$  [7].
3. Let  $\text{Fin}(A) \subseteq A^A$  be the set of maps  $f: A \rightarrow A$  for which  $\{f(a) \neq a \mid a \in A\}$  is finite. For *nominal renaming sets*, one considers  $A := \mathbb{A}$  and  $M := \text{Fin}(\mathbb{A})$  [19].

An element  $x \in X$  of an  $M$ -set  $(X, \cdot)$  can be understood as a structure with elements from  $A$  embedded. We can alter these embedded elements according to  $m \in M$ , yielding  $m \cdot x \in X$ .

► **Example 2.4.** For every set  $A$  and  $M \leq A^A$ , the following sets are  $M$ -sets:

1. The set  $A$  itself with the action  $m \cdot a := m(a)$ , for  $m \in M, a \in A$ .
2. If  $X$  is an  $M$ -set, then so is  $\mathcal{P}_f(X)$ , the set of finite subsets of  $X$ , where the action is defined point-wise:  $m \cdot S := \{m \cdot x \mid x \in S\}$  for  $m \in M, S \in \mathcal{P}_f X$ .
3. Every set  $D$  can be equipped with the *discrete* action:  $m \cdot d = d$  for all  $m \in M, d \in D$ .
4. The set  $M$  itself is an  $M$ -set with monoid multiplication  $M \times M \rightarrow M$  as the action.

The outcome of  $m \cdot x$  only depends on what  $m$  does on the atoms  $S \subseteq A$  buried in  $x$ :

► **Definition 2.5.** We say that  $m, m' \in M$  are identical on  $S \subseteq A$ , written  $m \approx_S m'$ , if  $m(a) = m'(a)$  for all  $a \in S$ .

With the intuition that the action of an  $M$ -set  $X$  renames the atoms  $A$  in an element  $x \in X$ , we can derive from the action which atoms an element  $x \in X$  carries. Then, an  $M$ -set is *nominal*, if each  $x \in X$  carries only finitely many atoms.

► **Definition 2.6** [19, Def. 7]. A set  $S \subseteq A$  supports  $x \in X$  of an  $M$ -set  $X$ , if for all  $m \approx_S m'$  (in  $M$ ), we have  $m \cdot x = m' \cdot x$ . An  $M$ -set  $X$  is *nominal* if every element of  $X$  has some finite support, i.e. is supported by a finite set  $S \subseteq A$ .

If  $M$  happens to be a group, then the definition of support can be simplified slightly [17, 33].

► **Example 2.7.** Most of the  $M$ -sets from Example 2.4 are nominal:

1.  $A$  is nominal: every  $a \in A$  is supported by  $S := \{a\}$ , and also by any superset of  $S$ .
2. If  $X$  is a nominal  $M$ -set, then so is  $\mathcal{P}_f(X)$ . A set  $E \in \mathcal{P}_f(X)$  of elements is supported by the union of finite supports of the elements  $x \in E$ . This union is finite because  $E$  is so.
3. Every discrete  $M$ -set  $D$  is nominal because every  $x \in D$  is supported by the empty set.
4. For all monoids  $M$  of interest for nominal techniques,  $M$  considered as an  $M$ -set is not nominal: whenever  $M$  is a nominal  $M$ -set, then *every*  $M$ -set is nominal. For example,  $\mathfrak{S}_f(\mathbb{A})$  is not nominal because no  $\sigma \in \mathfrak{S}_f(\mathbb{A})$  has finite support.

► **Definition 2.8.** Let  $\text{Nom}(M)$  be the category of nominal  $M$ -sets and equivariant maps.

1. The category of *nominal sets* is denoted by  $\text{Nom} = \text{Nom}(\mathfrak{S}_f(\mathbb{A}))$  (slightly overloading notation), that is for  $A := \mathbb{A}$  and  $M := \mathfrak{S}_f(\mathbb{A})$ . This is called *the equality symmetry*.
2. *Ordered nominal sets* are  $\text{OrdNom} = \text{Nom}(\text{Aut}(\mathbb{Q}, <))$ , i.e. for  $A := \mathbb{Q}$ ,  $M := \text{Aut}(\mathbb{Q}, <)$ .
3. *Nominal renaming sets* are  $\text{RnNom} = \text{Nom}(\text{Fin}(\mathbb{A}))$ , i.e. for  $A := \mathbb{A}$ ,  $M := \text{Fin}(\mathbb{A})$ .

► **Definition 2.9** [7, Definition 4.11]. A monoid  $M \leq A^A$  admits least supports if each element of a nominal  $M$ -set has a least finite support. If so, we write  $\text{supp}_X: X \rightarrow \mathcal{P}_f(A)$  for the map that sends elements of the  $M$ -set  $X$  to their least finite support.

In all running examples of nominal set flavours, the monoid admits least supports, i.e.  $\text{Nom}$  [18, 33],  $\text{OrdNom}$  [7] and  $\text{RnNom}$  [19]. Intuitively,  $\text{supp}(x) \subseteq A$  can be understood as precisely the atoms that appear in  $x$ . For example,  $\text{supp}_A: A \rightarrow \mathcal{P}_f(A)$  sends  $a$  to  $\{a\}$ , and  $\text{supp}_{\mathcal{P}_f(X)}: \mathcal{P}_f(X) \rightarrow \mathcal{P}_f(A)$  is  $\text{supp}_{\mathcal{P}_f(X)}(E) = \bigcup \{\text{supp}_X(x) \mid x \in E\}$ . The opposite of an atom  $a \in A$  in the support is:

► **Definition 2.10.** An atom  $a \in A$  is fresh for  $x \in X$  in a nominal set  $X$  (notated  $a \# x$ ), if  $a \notin \text{supp}_X(x)$ . For multiple elements, we write  $a \# x, y$  to denote that  $a$  is fresh for  $x$  and  $y$ .

Both freshness and support are preserved by equivariant maps. Intuitively, equivariant maps can possibly forget about atoms, but can never introduce new atoms:

► **Lemma 2.11.** For an equivariant map, if  $S$  supports  $x$ , then  $S$  also supports  $f(x)$ . If  $X$  and  $Y$  have least finite supports, then  $\text{supp}_Y(f(x)) \subseteq \text{supp}_X(x)$ .

The set of elements in an  $M$ -set that can be reached from an element  $x$  is called the orbit:

► **Definition 2.12.** Given a group  $M$ , the orbit of an element  $x$  in an  $M$ -set is the subset  $\{m \cdot x \mid m \in M\} \subseteq X$ . A nominal  $M$ -set is orbit-finite if it consists of finitely many orbits.

In this definition, we assume  $M$  to be a group, because then, every nominal set  $X$  is a disjoint union of orbits. For  $M := \mathfrak{S}_f(A)$ , the orbit-finite nominal sets are precisely the finitely presentable objects in  $\text{Nom}$  [32, Proposition 2.3.7]. E.g.  $\mathbb{A}$  is orbit-finite,  $\mathcal{P}_f \mathbb{A}$  is not. In nominal automata, one requires the state set to be orbit-finite, as opposed to finiteness in classical automata theory.

### 3 Supported Sets

The central notion of the present paper are supported sets, which are parametric in the set  $A$  of atoms or data symbols of interest:

► **Definition 3.1.** For a set  $A$ , the category of supported sets  $\text{Supp}(A)$  contains the following:

1. a supported set  $X$  is a set  $X$  together with a map  $\mathbf{s}_X: X \rightarrow \mathcal{P}_f(A)$ .
2. a supported map  $f: (X, \mathbf{s}_X) \rightarrow (Y, \mathbf{s}_Y)$  is a map  $f: X \rightarrow Y$  with  $\mathbf{s}_Y(f(x)) \subseteq \mathbf{s}_X(x)$  for all  $x \in X$ . This means,  $f$  makes the right-hand triangle weakly commute.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \mathbf{s}_X & \swarrow \mathbf{s}_Y \\ & \mathcal{P}_f(A) & \end{array}$$

Whenever clear from the context, we simply speak of supported sets  $X, Y \in \text{Supp}(A)$  and supported maps  $f: X \rightarrow Y$ , leaving  $\mathbf{s}_X$  and  $\mathbf{s}_Y$  implicit.

The intuition for supported sets comes from the least finite support map  $\text{supp}$  of nominal sets. Hence, in a supported set  $X$ , the map  $\mathbf{s}_X(x)$  tells which atoms are carried by  $x \in X$ , but we can not permute or rename them in general.

The definition of the morphisms in supported sets comes from the property of equivariant maps: they possibly forget about atoms in the support, but they can never introduce new atoms (Lemma 2.11). This observation becomes the defining property of supported maps.

► **Example 3.2.** The set  $A$  itself is a support set with  $\mathbf{s}_A(a) = \{a\}$ . However, the singleton subset  $\{a\} \subseteq A$  is also a supported set with  $\mathbf{s}_{\{a\}}(a) = \{a\}$ . In general, every nominal  $M$ -set  $X$  (for  $M$  admitting least supports) yields a supported set by putting  $\mathbf{s}_X := \text{supp}_X$ .

The difference between nominal and supported sets is that **supp** in a nominal set  $X$  is a derived notion since it is implicit in the  $M$ -set action. On the other hand, for supported sets,  $\mathbf{s}_X$  is part of the syntactical structure. For the sake of clarity, we use different mathematical symbols for the semantical **supp** and the structural **s**.

► **Lemma 3.3.** *For every  $M \leq A^A$ , there is a faithful functor  $U: \mathbf{Nom}(M) \rightarrow \mathbf{Supp}(A)$  sending  $(X, \cdot)$  to a supported set  $(X, \mathbf{s}_X)$ , where  $\mathbf{s}_X(x)$  is the intersection of all finite supports of  $x$  in  $(X, \cdot)$ . If  $M \leq A^A$  admits least supports, then  $\mathbf{s}_{UX} = \mathbf{supp}_X$ . Equivariant maps in  $\mathbf{Nom}(M)$  are sent to their underlying map by  $U$ .*

Later, we show that this forgetful functor  $\mathbf{Nom}(M) \rightarrow \mathbf{Supp}(A)$  is right-adjoint and even monadic, so one can consider nominal sets as algebras on supported sets. Before, we establish some categorical properties of  $\mathbf{Supp}(A)$  (generic in the choice of  $A$ ) that will come in handy.

### 3.1 Universal Constructions and Finiteness

Unsurprisingly, supported sets have a very set-like nature. In the present Section 3.1, we let  $V$  denote the forgetful functor  $V: \mathbf{Supp}(A) \rightarrow \mathbf{Set}$  that forgets the support map and sends  $(X, \mathbf{s}_X)$  to the plain set  $X$ . Conversely, every set can be equipped with trivial support:

► **Lemma 3.4.** *The inclusion functor  $J: \mathbf{Set} \hookrightarrow \mathbf{Supp}(A)$  defined by  $JX = (X, \mathbf{s}_X)$  with  $\mathbf{s}_X(x) = \emptyset$  is right-adjoint to the forgetful  $V: \mathbf{Supp}(A) \rightarrow \mathbf{Set}$  ( $V \dashv J$ ) and  $VJ = \text{Id}_{\mathbf{Set}}$ .*

In other words,  $\mathbf{Set}$  is a reflective subcategory of  $\mathbf{Supp}(A)$ . Similarly to sets, universal constructions such as limits and colimits all exist in supported sets and are (almost) set-like. Given a diagram  $D: \mathcal{D} \rightarrow \mathbf{Set}$ , we write  $\text{pr}_X: \lim D \rightarrow DX$  for the limit projection map of  $X \in \mathcal{D}$  and  $\text{in}_X: DX \rightarrow \text{colim} D$  for the colimit injections.

► **Proposition 3.5.** *All colimits in  $\mathbf{Supp}(A)$  exist: Given a diagram  $D: \mathcal{D} \rightarrow \mathbf{Supp}(A)$ , the colimit is formed in  $\mathbf{Set}$  and then equipped with the support  $\mathbf{s}: \text{colim} VD \rightarrow \mathcal{P}_f(A)$ :*

$$\mathbf{s}(c) = \bigcap \{ \mathbf{s}_{DY}(y) \mid Y \in \mathcal{D}, y \in DY, \text{in}_Y(y) = c \}.$$

The intersection handles the case where elements of possibly different support are identified in the colimit. Thus, the intersection vanishes if there are no morphisms in the diagram:

► **Example 3.6.** The coproduct of supported sets  $X, Y$  is given by their disjoint union,  $X + Y$  equipped with support  $\mathbf{s}_{X+Y}(\text{in}_X(x)) = \mathbf{s}_X(x)$  and  $\mathbf{s}_{X+Y}(\text{in}_Y(y)) = \mathbf{s}_Y(y)$ .

Limits on the other hand are formed differently because of the finite support map:

► **Proposition 3.7.**  *$\mathbf{Supp}(A)$  is complete. The limit of a diagram  $D: \mathcal{D} \rightarrow \mathbf{Supp}(A)$  is the subset of finitely supported elements in the limit  $\lim VD$  in  $\mathbf{Set}$ :*

$$\lim D = \{ x \in \lim VD \mid \bigcup_{Y \in \mathcal{D}} \mathbf{s}_{DY}(\text{pr}_Y(x)) \text{ is finite} \} \quad (1)$$

The process of restricting to finitely supported elements also happens for limits in nominal sets. For finite limits, this side-condition disappears:

► **Corollary 3.8.**  *$V: \mathbf{Supp}(A) \rightarrow \mathbf{Set}$  preserves all finite limits.*

$\mathbf{Supp}(A)$  is cartesian closed, that is, we have an internal hom object. Similarly to nominal set, this internal hom object  $X^E$  in  $\mathbf{Supp}(A)$  contains more than just the hom set  $\text{hom}(E, X)$ :

► **Definition 3.9.** For supported sets  $X$  and  $E$ , let  $X^E$  contain those maps  $f: E \rightarrow X$  for which  $\mathbf{s}_{X^E}(f) = \bigcup_{e \in E} \mathbf{s}_X(f(e)) \setminus \mathbf{s}_E(e)$  is finite.

Note that ‘map’ really means ordinary maps between sets. Intuitively, a map  $f \in X^E$  may introduce finitely many atoms when mapping elements of  $E$  to  $X$ . For example, if  $E$  is a finite supported set, then  $X^E$  contains all maps  $E \rightarrow X$ . In contrast, a *supported* map may not introduce any new atoms at all, hence, a map  $f: E \rightarrow X$  is a supported map if and only if  $\mathbf{s}_{X^E}(f) = \emptyset$ . In particular,  $\mathbf{hom}(E, X) \subseteq X^E$ .

► **Proposition 3.10.**  $(-) \times E$  is left adjoint to  $(-)^E$ , for all supported sets  $E$ .

In contrast to nominal sets, categorical finiteness notions in  $\mathbf{Supp}(A)$  express actual finiteness. For the present paper, we do not need the precise definition of finitely presentable objects and locally finitely presentable (lfp) categories [3, 20]. For the present purposes, it is enough to mention that  $\mathbf{Supp}(A)$  is lfp and that the finitely presentable objects are the finite supported sets, i.e.  $\mathbf{Supp}(A)$  is locally finite. Detailed definitions can be found in the proof.

► **Proposition 3.11.**  $\mathbf{Supp}(A)$  is lfp and finite presentability is actual finiteness.

## 3.2 Injectivity, Surjectivity, Quasitopoi

Notions of surjective and injective maps generalize from  $\mathbf{Set}$ :

► **Lemma 3.12.** Monomorphisms in  $\mathbf{Supp}(A)$  are precisely the injective supported maps and epimorphisms are precisely the surjective supported maps.

However, not all bijective supported maps are isomorphisms in  $\mathbf{Supp}(A)$ , because they may drop atoms. That is, the difference between bijections and isomorphisms is the following:

► **Definition 3.13.** A map  $f: X \rightarrow Y$  is called support-reflecting if  $\mathbf{s}_Y \cdot f = \mathbf{s}_X$ .

► **Lemma 3.14.** The isomorphisms in  $\mathbf{Supp}(A)$  are the support-reflecting bijective maps.

► **Example 3.15.** The unit  $\eta_A: A \rightarrow JVA$  of the above adjunction  $V \dashv J$  to  $\mathbf{Set}$  is a bijective supported map but not support-reflecting, because  $\mathbf{s}_A(a) = \{a\}$  and  $\mathbf{s}_{JVA}(a) = \emptyset$ . Thus it is not an isomorphism in  $\mathbf{Supp}(A)$ .

This shows that  $\mathbf{Supp}(A)$  is not a topos – in contrast to  $\mathbf{Set}$  and  $\mathbf{Nom}$ . As we will see, it is a *quasitopos* [23, Def. 2.6.1], which entails that  $\mathbf{Supp}(A)$  is locally cartesian closed and that it has a subobject classifier for *regular* monomorphisms. The precise definition of regularity is not relevant here (but cf. [2, Rem 7.76(2)]), since it can be nicely characterized via support-reflection:

► **Lemma 3.16.** A monomorphism is regular iff it is support-reflecting. Moreover,  $t: 1 \rightarrow 2$ ,  $0 \mapsto 1$  is a regular-subobject classifier, i.e. for every supported set  $X$ , the support-reflecting monomorphisms  $m: S \rightarrow X$  are in correspondence to characteristic maps  $\chi_S: X \rightarrow 2$ .

In summary, for every set  $A$ ,  $\mathbf{Supp}(A)$  is (co)complete, cartesian closed, and locally finite, and we can express many of the above mentioned properties in one line:

► **Theorem 3.17.**  $\mathbf{Supp}(A)$  is a quasitopos (for both [23, Def. 2.6.1] and [2, Def. 28.7]).

Thus,  $\mathbf{Supp}(A)$  constitutes a rich basis to study algebraic theories on it, such as nominal sets.



## 4 Monadicity of Nominal $M$ -Sets

If a category is *monadic*, then intuitively, it is a well-behaved class of algebras. If so, it becomes applicable for many results and constructions, e.g. regarding representation and the generalized powerset construction. Algebraic theories have been studied extensively throughout the decades and are in correspondence to monads: given a monad  $T$  on a category  $\mathcal{C}$ , e.g. **Set**, its *Eilenberg-Moore category*  $\mathbf{EM}(T)$  contains the models of the algebraic theory defined by  $T$  (see e.g. [4, 10.3]). The forgetful functor  $U: \mathbf{EM}(T) \rightarrow \mathcal{C}$  is a right-adjoint functor and its left-adjoint  $F: \mathcal{C} \rightarrow \mathbf{EM}(T)$  sends an object  $X \in \mathcal{C}$  (e.g. a set of generators) to the *free algebra* on  $X$ :

$$F: \mathbf{Supp}(A) \rightarrow \mathbf{EM}(T) \dashv U: \mathbf{EM}(T) \rightarrow \mathbf{Supp}(A) \quad (\text{for } \mathcal{C} = \mathbf{Supp}(A)) \quad (2)$$

The category of  $M$ -sets is monadic over **Set**, but  $\mathbf{Nom}(M)$  fails to be monadic over **Set** in the instances of interest (Example 2.3). The reason for this is that infinite products in  $\mathbf{Nom}(M)$  are different than in **Set** [33], so the forgetful functor  $\mathbf{Nom}(M) \rightarrow \mathbf{Set}$  is not right-adjoint and therefore not monadic. As we will show,  $\mathbf{Nom}(M)$  is still monadic, but *over supported sets*.

► **Definition 4.1.** A functor  $U: \mathcal{D} \rightarrow \mathcal{C}$  is called *monadic* (over  $\mathcal{C}$ ) if there is a monad  $T$  on  $\mathcal{C}$  such that  $\mathcal{D}$  is  $\mathbf{EM}(T)$  and  $U$  is the forgetful functor.

We first show that  $U: \mathbf{Nom}(M) \rightarrow \mathbf{Supp}(A)$  is right-adjoint and then that it is monadic. Then, we can view nominal sets as algebras over  $\mathbf{Supp}(A)$ . There, the algebraic operations come from the following (supported) set:

► **Definition 4.2.** For  $M \leq A^A$  and a set  $S \subseteq A$ , put  $[m]_S$  for the  $\approx_S$ -equivalence class of  $m$  (cf. Definition 2.5) and  $M/S = \{[m]_S \mid m \in M\}$  for the supported set of equivalence classes with  $s_{M/S}([m]_S) := m[S] = \{m(a) \mid a \in S\}$ .

We can consider  $M/S$  either as equivalence classes of  $M$ -elements that are identical on  $S$  or alternatively as special maps  $S \rightarrow A$  that are obtained by restricting some  $m \in M \subseteq A^A$  to  $S \subseteq A$ . Intuitively,  $M/S$  is the free nominal set on one generator with support  $S$ . Hence, the free nominal  $M$ -set over a supported set  $X$  is the union of multiple  $M/S$ :

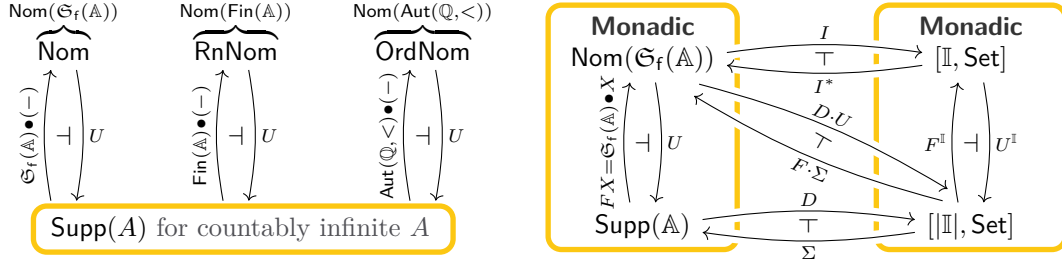
► **Definition 4.3.** Fix the functor  $M\bullet: \mathbf{Supp}(A) \rightarrow \mathbf{Supp}(A)$  by  $M\bullet X = \coprod_{x \in X} M/s(x)$ .

$M\bullet$  is the monad that will define nominal  $M$ -sets as an algebraic theory. Written with elements, we have  $M\bullet X = \{([m]_{s_X(x)}, x) \mid x \in X\}$  with  $s_{M\bullet X}([m]_{s_X(x)}, x) = m[s(x)]$ . For a supported set  $X$ , every such element  $([m]_{s_X(x)}, x)$  of the nominal set  $M\bullet X$  is equivalently an element  $x \in X$  together with an  $|s(x)|$ -tuple of atoms. Of course, such an equivalence class  $[m]_{s_X(x)}$  is not an arbitrary tuple of elements of  $A$  but only one that is obtained from restricting some  $m \in M \subseteq (A \rightarrow A)$  to  $s_X(x) \subseteq A$ . For example, for the equality symmetry  $M := \mathfrak{S}_f(\mathbb{A})$ , such a  $t \in M/s_X(x)$  is a tuple of *distinct* elements of  $A$ .

► **Proposition 4.4.**  $M\bullet X$  with the  $M$ -set action  $\ell \cdot ([m]_{s_X(x)}, x) = ([\ell \cdot m]_{s_X(x)}, x)$  is nominal and gives rise to a functor  $F: \mathbf{Supp}(A) \rightarrow \mathbf{Nom}(M)$ ,  $FX = M\bullet X$ .

Note that there is no statement about the least finite support of  $([m]_{s_X(x)}, x) \in M\bullet X$ . In the proof, we show that it is supported by  $m[s(x)]$ , but the least support might be smaller. Nevertheless, the existence of finite supports in  $M\bullet X$  suffices to show the adjunction  $F \dashv U$ :

► **Proposition 4.5.** If  $M \leq A^A$  admits least supports, then  $F: \mathbf{Supp}(A) \rightarrow \mathbf{Nom}(M)$  is left-adjoint to  $U: \mathbf{Nom}(M) \rightarrow \mathbf{Supp}(A)$  with unit  $\eta_X: X \rightarrow UFX$ ,  $\eta_X(x) = ([\text{id}_A]_{s_X(x)}, x)$ .



■ **Figure 1** Monadic adjunctions (Ex. 4.10)

■ **Figure 2** Relation to presheaves for  $M := \mathfrak{S}_f(\mathbb{A})$

This adjunction shows that every nominal  $M$ -set is an Eilenberg-Moore algebra and that every supported set generates a free nominal  $M$ -set satisfying a universal mapping property. The remaining direction to prove is that every Eilenberg-Moore algebra for  $M \bullet$  is indeed a nominal set, or concretely, that the adjunction is monadic. For doing so, we impose a property on the monoid  $M$  of interest, which will not only be used in the monadicity proof but also help us to characterize the least finite supports of the free nominal sets:

► **Definition 4.6** [7, Def. 9.6]. *A monoid  $M \leq A^A$  is called fungible if for all finite  $R \subseteq A$  and  $a \in A \setminus R$ , there is some  $\ell \in M$  with  $\ell \approx_R \text{id}_A$  and  $\ell(a) \neq a$ .*

Intuitively, the condition expresses that the atoms in the support can be renamed independently from each other: if an element is supported by  $R \cup \{a\}$ , we can always rename  $a$  to something fresh while keeping the rest of the support fixed.

► **Example 4.7.** All the leading examples (Ex. 2.3) have fungible monoids.

- For  $M := \mathfrak{S}_f(\mathbb{A})$  and  $M := \text{Fin}(\mathbb{A})$ , consider a finite  $R \subseteq \mathbb{A}$  and  $a \notin R$ . Then for  $b \notin R$ , the permutation  $\ell = (a b)$  fulfils the desired property  $(a b) \approx_R \text{id}_\mathbb{A}$ .
- For  $M := \text{Aut}(\mathbb{Q}, <)$ , the verification uses a notion called homogeneity [42, Lemma 5.2].

► **Lemma 4.8.** *If  $M$  is fungible, then  $M \bullet X$  is a nominal  $M$ -set with  $\text{supp}([m]_{s(x)}, x) = m[s(x)]$  for every  $m \in M$  and  $x \in X$ .*

We prove the adjunction to be monadic via Beck's theorem [28, Sec. VI.7], which will provide us with the monadicity of the leading examples of nominal sets by instantiating  $M$ .

► **Theorem 4.9.**  *$U : \text{Nom}(M) \rightarrow \text{Supp}(A)$  is monadic and  $\text{Nom}(M) = \text{EM}(M \bullet (-))$ , for every fungible  $M$  admitting least supports.*

► **Example 4.10.** The following categories are monadic over  $\text{Supp}(A)$ , for  $A$  being countably infinite. The monadic adjunctions are visualized in Figure 1, and the monads listed below.

1. The category  $\text{Nom}$  of nominal sets (with equality symmetry) is monadic over  $\text{Supp}(\mathbb{A})$ . The operations on a generator  $x$  in the corresponding theory are injective maps  $s(x) \rightarrow \mathbb{A}$ :

$$\mathfrak{S}_f(\mathbb{A}) \bullet X = \{(\pi, x) \mid x \in X, \pi : s_X(x) \rightarrow \mathbb{A}\}$$

2. The category  $\text{RnNom}$  of nominal renaming sets is monadic over  $\text{Supp}(\mathbb{A})$ . The operations on a generator  $x$  are arbitrary maps  $s(x) \rightarrow \mathbb{A}$  (not necessarily injective):

$$\text{Fin}(\mathbb{A}) \bullet X = \{(\pi, x) \mid x \in X, \pi : s_X(x) \rightarrow \mathbb{A}\}$$



3. The category **OrdNom** of nominal sets for the total order symmetry is monadic over **Supp**( $\mathbb{Q}$ ) (using  $\mathbb{Q} \cong A$ ). The operations on  $x$  are monotone injective maps  $\mathbf{s}(x) \rightarrow \mathbb{Q}$ :

$$\text{Aut}(\mathbb{Q}, <) \bullet X = \{(\pi, x) \mid x \in X, \pi: \mathbf{s}_X(x) \rightarrow \mathbb{Q} \forall q, p \in \mathbf{s}_X(x) : q < p \Rightarrow \pi(q) < \pi(p)\}$$

As a direct application of the monadicity, we can characterize orbit-finite nominal sets and finitely presentable objects in **Nom**( $M$ ) using a general result about algebraic categories [1, Thm. 3.7]:

► **Example 4.11.** A nominal  $M$ -set  $X$  is finitely presentable iff it can be described by a finite supported set  $G$  of *generators* and a finite subset  $E \subseteq (M \bullet G) \times (M \bullet G)$  of *equations*. This characterization means that given such finite  $G$  and  $E$ , we obtain projection maps

$$E \xrightarrow[r]{\ell} M \bullet G \quad \text{in } \text{Supp}(A) \quad \Longleftrightarrow \quad M \bullet E \xrightarrow[\bar{r}]{\bar{\ell}} M \bullet G \quad \text{in } \text{Nom}(M)$$

and their coequalizer in **Nom**( $M$ ) is  $X$ .

For  $M := \mathfrak{S}_f(\mathbb{A})$ , 1. every orbit-finite nominal set can be described by such finite supported sets  $G$  and  $E$ , and 2. any such finite system of equations  $E$  on  $G$  presents an orbit-finite nominal set. For example, the nominal set of unordered pairs of atoms can be described by one generator  $g$  encoded as a supported set  $G = \{g\}$ ,  $\mathbf{s}_G(g) := \{a, b\}$  (for fixed  $a, b \in \mathbb{A}$ ) and one equation  $E := \{(ab) \cdot g = \text{id} \cdot g\}$ . Here, we use intuitive notation to denote

$$E := \{([[(ab)]_{\{a,b\}}, g), ([\text{id}]_{\{a,b\}}, g)] \subseteq (M \bullet G) \times (M \bullet G).$$

With the projections  $\ell, r: E \rightarrow M \bullet G$  (in **Supp**( $A$ )) and their extensions  $\bar{\ell}, \bar{r}$  to **Nom**, we have the unordered pairs as a coequalizer diagram in **Nom**:

$$M \bullet E \xrightarrow[\bar{r}]{\bar{\ell}} M \bullet G \twoheadrightarrow \{\{c, d\} \mid c, d \in \mathbb{A}, c \neq d\}.$$

► **Remark 4.12 (Relation to Presheaves).** The category of supported sets **Supp**( $\mathbb{A}$ ) nicely fits into an existing diagram of Kurz, Petrisan, and Velebil [27] relating nominal sets **Nom** (for the equality symmetry  $\mathfrak{S}_f(\mathbb{A})$ ) with two presheaf categories:

1.  $[\mathbb{I}, \text{Set}]$  is the category of functors  $P: \mathbb{I} \rightarrow \text{Set}$  (*sets in context*), where the objects of  $\mathbb{I}$  are finite subsets of  $\mathbb{A}$  (i.e.  $|\mathbb{I}| = \mathcal{P}_f(\mathbb{A})$ ) and the morphisms are injective maps (i.e.  $\mathbb{I} \neq (\mathcal{P}_f(\mathbb{A}), \subseteq)$ ).
2.  $[|\mathbb{I}|, \text{Set}]$  is the category of functors  $P: |\mathbb{I}| \rightarrow \text{Set}$ , from the set  $\mathcal{P}_f(\mathbb{A})$  to **Set**, i.e.  $P$  is a  $\mathcal{P}_f(\mathbb{A})$ -indexed family of sets.

Figure 2 shows the result when extending the diagram of Kurz et al. [27] by **Supp**( $\mathbb{A}$ ). Let us go through the functors and adjunctions relating the categories:

1. The left adjoint  $\Sigma: [|\mathbb{I}|, \text{Set}] \rightarrow \text{Supp}(\mathbb{A})$  sends a family  $X: \mathcal{P}_f(\mathbb{A}) \rightarrow \text{Set}$  to the coproduct  $\Sigma(X) = \coprod_{S \in \mathcal{P}_f(\mathbb{A})} X(S)$ , where the component for  $S \in \mathcal{P}_f(\mathbb{A})$  has support  $S$ .
2. The right-adjoint  $D: \text{Supp}(\mathbb{A}) \rightarrow [|\mathbb{I}|, \text{Set}]$  forms down-sets: for a supported set  $X$ , the family  $DX: \mathcal{P}_f(\mathbb{A}) \rightarrow \text{Set}$  is given by  $DX(S) = \{x \in X \mid \mathbf{s}_X(x) \subseteq S\}$ .
3. Since adjunctions compose, we have the adjunction  $F \cdot \Sigma \dashv D \cdot U$ , which is *not monadic* [27], so the monad  $DUFR$  does not have **Nom** as its Eilenberg-Moore category.
4. Instead,  $[\mathbb{I}, \text{Set}] = \text{EM}(DUFR)$  [27], and  $F^\mathbb{I} \vdash U^\mathbb{I}$  is the corresponding adjunction.
5. The induced comparison functor from **Nom** is itself adjoint: **Nom** is (equivalent to) the full reflective subcategory of pullback preserving functors (cf. [21, 17]).

Note that **Supp**( $\mathbb{A}$ ) is the only category in Figure 2 that is not a topos, and therefore not a presheaf category.

► **Remark 4.13 (Relation to named sets).** Another notion to reason about names are *named sets* [13], which have been defined in slightly different ways over the years. For a fixed countably infinite set of names  $\mathbb{A} = \{v_1, v_2, \dots\}$  the definitions are as follows:

- In the definition by Ferrari, Montanari, Pistore [13, Def. 2], every element  $x$  of a named set  $X$  does not have an explicit support, but is only associated with a natural number  $n \in \mathbb{N}$ , and so its support is implicitly considered to be  $\{v_1, \dots, v_n\} \subseteq \mathbb{A}$ . Moreover, every element is equipped with a subgroup of  $\mathfrak{S}_f(n)$  that describes how the atoms  $v_1, \dots, v_n$  in the support may be permuted. In later works, this natural number  $n \in \mathbb{N}$  denoting only the size was replaced with a set of atoms:
- Montanari and Pistore [31] define a named set to be a set  $X$  together with a map  $n_X: X \rightarrow \mathcal{P}(\mathbb{A})$  (without any further information about permutations). A *named function*  $m: (X, n_X) \rightarrow (Y, n_Y)$  is a map on the sets  $m: X \rightarrow Y$  and for each  $x \in X$  an injective map  $m[x]: n_Y(f(x)) \hookrightarrow n_X(x)$ . A named set  $(X, n_X)$  is called *finitely named* if  $n_X(x)$  is finite for every  $x \in X$ .

The category of all named sets and named functions is quite different from  $\text{Supp}(\mathbb{A})$ . For instance, the finitely named set  $(\mathbb{A}, n_{\mathbb{A}})$  with  $n_{\mathbb{A}}(a) = \{a\}$  has infinitely many automorphisms  $(\mathbb{A}, n_{\mathbb{A}}) \rightarrow (\mathbb{A}, n_{\mathbb{A}})$ , whereas both in  $\text{Nom}(\mathfrak{S}_f(\mathbb{A}))$  and in  $\text{Supp}(\mathbb{A})$ , there is only one morphism  $\mathbb{A} \rightarrow \mathbb{A}$ , the identity. Without the finiteness restriction, infinite limits in the category of named sets are **Set**-like and not **Nom**-like. More precisely, the infinite product  $\prod_{n \in \mathbb{N}} (\mathbb{A}, n_{\mathbb{A}})$  contains streams making use of infinitely many names.

- Gaducci, Miculan, and Montanari [21] restrict to *finitely named* sets and additionally equip every element  $x \in X$  of a named set  $(X, n_X)$  with a subgroup  $G_X(x)$  of the group  $\mathfrak{S}_f(n_X(x))$  of all permutations on the names  $n_X(x)$ . Here, a named function  $f: (X, n_X, G_X) \rightarrow (Y, n_Y, G_Y)$ , is a map  $f: X \rightarrow Y$  and additionally for each  $x \in X$  a non-empty set of injections  $f_x \subseteq (n_Y(f(x)) \hookrightarrow n_X(x))$  satisfying an additional coherence condition involving  $G_X$ . Details are not relevant here, because the resulting category **NSet** was shown to be equivalent to  $\text{Nom}(\mathfrak{S}_f(\mathbb{A}))$  in the same work [21, Prop. 29]. This has a couple of immediate implications regarding the relationship to supported sets:

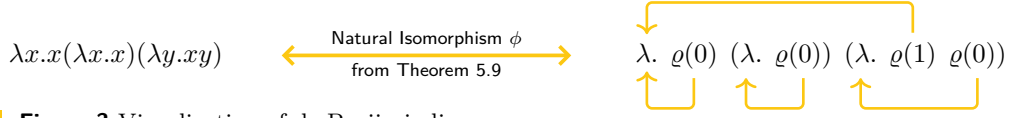
1. **NSet** is monadic over  $\text{Supp}(\mathbb{A})$ .
2. The right-adjoint of this monadic adjunction is a faithful functor  $\text{Supp}(\mathbb{A}) \rightarrow \text{NSet}$ , which is not full.

Having discussed categorical properties of supported sets and its relation to other name-aware notions, we now continue to see how name binding can be accomplished.

## 5 Name Abstraction and de Bruijn indices

In  $\lambda$ -calculus, the computational steps ( $\beta$ -reduction) essentially consist of a substitution rule  $(\lambda x. T) P \rightarrow_{\beta} T[x := P]$  which requires some bound variable names in  $T$  to be sufficiently fresh. Thus, it might be necessary to rename those bound variables ( $\rightarrow_{\alpha}$ ) in  $T$  before a  $\beta$  step can be performed. But even in such a renaming step  $\rightarrow_{\alpha}$ , a similar side-condition needs taken care of, because otherwise, the reduction would lead to wrong results.

Thus, there are several approaches to define  $\lambda$ -expressions directly modulo  $\alpha$ -equivalence, making substitution total and  $\beta$ -reduction always applicable to  $\lambda$ -expressions containing a reducible expression (redex). In 1972, de Bruijn [11] invented a technique of replacing the variable names with an index (the *de Bruijn index*) that counts the number binders between the variable and its corresponding binder. In 1999, Gabbay and Pitts [17] presented another way to define  $\lambda$ -expressions directly as  $\alpha$ -equivalence classes, in which  $\lambda$ -abstraction is a



■ **Figure 3** Visualization of de Bruijn indices

functor on *nominal sets* (called *FM-sets* back then). From now on, we stick to their setting of equality symmetry:

► **Assumption 5.1.** *For the rest of the paper, fix  $A := \mathbb{A}$  and  $M := \mathfrak{S}_f(\mathbb{A})$ . Hence, we may assume a bijection  $\varrho: \mathbb{N} \rightarrow \mathbb{A}$  and we simply write **Nom** for the  $M$ -nominal sets.*

For the name abstraction functor, the notion of  $\alpha$ -equivalence is first defined for arbitrary nominal sets, which is then used in the definition of the abstraction functor:

► **Definition 5.2** [17]. *For  $X \in \mathbf{Nom}$ , the equivalence relation  $\sim_\alpha$  on  $\mathbb{A} \times X$  is defined by*

$$(a, x) \sim_\alpha (b, y) \iff \exists c \# (a, b, x, y): (ca) \cdot x = (cb) \cdot y$$

The abstraction functor  $[\mathbb{A}]X: \mathbf{Nom} \rightarrow \mathbf{Nom}$  is given by  $[\mathbb{A}]X = (A \times X)/\sim_\alpha$ , where  $\langle a \rangle x$  denotes the equivalence class of  $(a, x) \in \mathbb{A} \times X$ .

In the equivalence class  $\langle a \rangle x$ ,  $a$  disappears from the support:  $\text{supp}_{[\mathbb{A}]X}(\langle a \rangle x) = \text{supp}_X(x) \setminus \{a\}$ .

► **Example 5.3** [17]. The initial algebra of the functor  $\Lambda X = \mathbb{A} + [\mathbb{A}]X + X \times X$  is carried by the nominal set of  $\lambda$ -expressions modulo  $\alpha$ -equivalence. The first summand  $\mathbb{A}$  describes variables  $a \in \mathbb{A}$ , the second summand  $[\mathbb{A}]X$  describes  $\lambda$ -abstractions  $\lambda x.T$ , where  $x \in \mathbb{A}$  and  $T$  is a  $\lambda$ -expression, and the third summand  $X \times X$  describes the application  $TS$  of one  $\lambda$ -expression to one other.

In the present paper, we define an abstraction functor on supported sets for which we require the set of atoms to be a countably infinite set. This functor in fact has a lifting to nominal sets and the lifting is (naturally isomorphic to) the above abstraction functor  $[\mathbb{A}]: \mathbf{Nom} \rightarrow \mathbf{Nom}$  on nominal sets.

The definition of  $[\mathbb{A}]$  makes use of the  $\mathfrak{S}_f(\mathbb{A})$ -action to capture  $\alpha$ -equivalence. When introducing abstraction as a functor directly on supported sets, we do not have renaming available, and so we use de Bruijn indices via the bijection  $\varrho: \mathbb{N} \rightarrow \mathbb{A}$  (Assumption 5.1).

► **Definition 5.4.** *The de Bruijn functor  $\mathcal{B}: \text{Supp}(\mathbb{A}) \rightarrow \text{Supp}(\mathbb{A})$  sends a supported set  $X$  to the same set  $\mathcal{B}X = X$  but with a different support function. To distinguish elements of  $X$  and  $\mathcal{B}X$ , we write  $\lambda.x \in \mathcal{B}X$  for  $x \in X$  (i.e.  $\lambda$  is a nameless binder). The support on  $\mathcal{B}X$  is*

$$\mathbf{s}_{\mathcal{B}X}: \mathcal{B}X \rightarrow \mathcal{P}_f(\mathbb{A}) \quad \mathbf{s}_{\mathcal{B}X}(\lambda.x) := \{\varrho(k) \mid \varrho(k+1) \in \mathbf{s}_X(x), k \in \mathbb{N}\}. \quad (3)$$

A supported map  $f: X \rightarrow Y$  is sent to the same map  $\mathcal{B}f: \mathcal{B}X \rightarrow \mathcal{B}Y$ ,  $\mathcal{B}f(\lambda.x) = \lambda.f(y)$ .

The definition of  $\mathbf{s}_{\mathcal{B}X}$  captures the idea of de Bruijn indices: we can think of the notation  $\lambda.x$  as a lambda abstraction of nameless binder  $\lambda.$  of a lambda term  $x$  (visualized in Figure 3):

- The variables referring to  $\lambda.$  have the de Bruijn index of 0 (at the level of  $x$ ), because there is no other binder between  $x$  and ' $\lambda.$ '. Since  $\varrho(0)$  is bound,  $\mathbf{s}_{\mathcal{B}X}(x)$  does not depend on whether  $\varrho(0) \in \mathbf{s}_X(x)$ .
- All other variables  $\varrho(k+1) \in \mathbf{s}_X(x)$  refer to variables that are free in  $\lambda.x$  and so refer to binders 'more above' than  $\lambda.x$ . A variable  $\varrho(k) \in \mathbf{s}_{\mathcal{B}X}(\lambda.x)$  refers to the same binder as the variable  $\varrho(k+1) \in \mathbf{s}_X(x)$  under ' $\lambda.$ ', because the latter is one level further down.

► **Lemma 5.5.**  $\mathcal{B}: \text{Supp}(\mathbb{A}) \rightarrow \text{Supp}(\mathbb{A})$  is a functor.

We use a slightly generalized notion of a *lifting* of the functor  $\mathcal{B}$  to nominal sets:

► **Definition 5.6.** For a monad  $T$  on a category  $\mathcal{C}$  and a functor  $H: \mathcal{C} \rightarrow \text{EM}(T)$ , a functor  $G: \text{EM}(T) \rightarrow \text{EM}(T)$  is called a *lifting* of  $H$  if  $HU$  and  $UG$  are naturally isomorphic functors. We say that a lifting is *strict* if  $HU = UG$ .

► **Remark 5.7.** Usually, not just a natural isomorphism but identity is required. Strict liftings  $G: \text{EM}(T) \rightarrow \text{EM}(T)$  are in one-to-one correspondence to distributive laws  $TH \rightarrow HT$  [22].

This generalization to natural isomorphisms is sound, because they induce strict liftings:

► **Lemma 5.8.** For every natural isomorphism  $\phi: HU \rightarrow UG$ , there is a unique strict lifting  $\bar{H}: \text{EM}(T) \rightarrow \text{EM}(T)$  such that  $\phi: H \rightarrow G$  is a natural isomorphism in  $\text{EM}(T)$ .

This means that for  $(C, \gamma) \in \text{EM}(T)$ ,  $\phi_{(C, \gamma)}$  is a  $T$ -algebra isomorphism  $\bar{H}(C, \gamma) \rightarrow G(C, \gamma)$ .

► **Theorem 5.9.** The abstraction functor  $[\mathbb{A}]: \text{Nom} \rightarrow \text{Nom}$  is a lifting of the de Bruijn functor  $\mathcal{B}: \text{Supp}(\mathbb{A}) \rightarrow \text{Supp}(\mathbb{A})$ . That is, there is a natural isomorphism  $\phi: \mathcal{B}U \rightarrow U[\mathbb{A}]$ .

Intuitively,  $\phi$  translates between the de Bruijn indices and the nominal abstraction functor: in the term  $\lambda.x \in \mathcal{B}X$ , we have  $\varrho(0)$  implicitly bound, like it is in  $\langle \varrho(0) \rangle(x) \in [\mathbb{A}]X$ . However, every  $\varrho(k+1)$  in  $x$  appears as  $\varrho(k)$  in the support of  $\lambda.x$ , so  $\phi$  essentially renames  $\varrho(\ell+1) \mapsto \varrho(\ell)$  in  $x$  for all  $\ell \geq 1$ .

► **Remark 5.10 (Abstraction in named sets).** A de Bruijn-style abstraction functor can also be defined directly on  $\text{Nom}$  [9, Def. 3.3]. The defining property (3) of abstraction on  $\text{Supp}(\mathbb{A})$  then reappears as a theorem of the support function **supp** in nominal sets [9, Thm. 4.1]. This adheres to the principle that in  $\text{Supp}(\mathbb{A})$ , the support is part of the data, whereas in nominal sets, **supp** is a derived notion. Since named sets are equivalent to nominal sets, we have name abstraction there, too, and an explicit definition is provided by Ciancia and Montanari [9, Sect. 7.1].

► **Remark 5.11 (Abstraction in presheaves).** Fiore et al. [14] also use de Bruijn indices in their abstraction functor, but treat support in a different way. They consider presheaves  $X \in \text{Set}^{\mathbb{F}}$  where  $\mathbb{F}$  is the full subcategory of  $\text{Set}$  containing only natural numbers as objects (considering natural numbers as finite cardinals). Thus for every natural number  $n \in \mathbb{N}$ , the set  $X(n)$  intuitively denotes the elements that make only use of the atoms  $\varrho(0), \dots, \varrho(n-1) \in \mathbb{A}$ . For each map  $\pi: n \rightarrow m$ , the map  $X(\pi): X(n) \rightarrow X(m)$  renames embedded variables. Their name abstraction functor  $\delta: \text{Set}^{\mathbb{F}} \rightarrow \text{Set}^{\mathbb{F}}$  sends a presheaf  $X$  to the presheaf  $\delta(X)(n) = X(n+1)$ , implicitly binding  $\varrho(0)$ , and ‘shifting’ the role of the other atoms. Despite this similarity, it is unclear whether there is a formal categorical connection between  $\text{Supp}(\mathbb{A})$  and  $\text{Set}^{\mathbb{F}}$  or between  $\mathcal{B}$  and  $\delta$ .

With name binding in supported sets, we can now study automata in supported sets and relate them to nominal automata.

## 6 Register Automata to Nominal Automata

The well-known powerset construction for automata is an instance of a more general principle called *generalized determinization* [35, 6] that *internalizes side-effects* modelled by a monad. The input of the standard powerset construction is a *non-deterministic* finite automaton (NFA), which can be understood as a deterministic automaton extended with

non-deterministic branching as a side-effect. With  $Q$  as the set of states of the NFA, the construction returns a deterministic automaton with states  $\mathcal{P}_f Q$ , which happens to be precisely the set of *configurations* of the original NFA. This construction generalizes from the (finite) powerset monad  $\mathcal{P}_f$  to arbitrary monads  $T: \mathcal{C} \rightarrow \mathcal{C}$  and from automata to state-based systems modelled via coalgebras for a functor  $H: \mathcal{C} \rightarrow \mathcal{C}$ . We apply this principle to the monad from the monadicity result in Section 4.

A *coalgebra* (for  $H: \mathcal{C} \rightarrow \mathcal{C}$ ) is an object  $Q \in \mathcal{C}$  together with a morphism  $Q \rightarrow HQ$ . E.g. an  $H$ -coalgebra for  $H: \mathbf{Set} \rightarrow \mathbf{Set}$ ,  $HX = 2 \times X^\Sigma$ , is a deterministic automaton, i.e. a set  $Q$  together with a map  $d: Q \rightarrow 2 \times Q^\Sigma$ . For a state  $q \in Q$ , the first component  $\text{pr}_1(d(q)) \in 2$  specifies the finality of  $q$ , and the second component  $\text{pr}_2(d(q)) \in Q^\Sigma$  sends input symbols  $a \in \Sigma$  to successor states in  $Q$  (the initial state  $q_0 \in Q$  is not important here).

On the other hand, a non-deterministic automaton is simply a coalgebra for the composed functor  $H\mathcal{P}_f: \mathbf{Set} \rightarrow \mathbf{Set}$ , i.e. a map  $c: Q \rightarrow 2 \times (\mathcal{P}_f Q)^\Sigma$ . The generalized determinization assumes that  $T: \mathcal{C} \rightarrow \mathcal{C}$  is a monad (with unit  $\eta$  and multiplication  $\mu$ ) and that  $H: \mathcal{C} \rightarrow \mathcal{C}$  is a functor that lifts to the Eilenberg-Moore category of  $T$ , i.e. that we have a functor  $G: \mathbf{EM}(T) \rightarrow \mathbf{EM}(T)$  with  $\phi: HU \xrightarrow{\cong} UG$  (Definition 5.6) – for example  $HX = 2 \times X^\Sigma$  and  $TX = \mathcal{P}_f X$ . Then the generalized determinization turns an  $HT$ -coalgebra on  $Q$  into a  $G$ -coalgebra on  $TQ$ , using the adjunction  $F \dashv U$  between  $\mathcal{C}$  and  $\mathbf{EM}(T)$  (cf (2)):

$$c: Q \rightarrow HTQ \text{ in } \mathcal{C} \xrightarrow{\text{Internalization}} d: FQ \rightarrow GFQ \text{ in } \mathbf{EM}(T)$$

In the instance  $TX = \mathcal{P}_f X$  of the powerset construction,  $\mathbf{EM}(T)$  is the category of join-semilattices, and every non-deterministic automaton on  $Q$  is turned into a deterministic automaton on  $\mathcal{P}_f Q$ . Since we internalized the side-effect of non-deterministic branching in the states, the resulting state space  $\mathcal{P}_f Q$  is the configuration space of the non-deterministic automaton and the induced transition structure  $d$  preserves joins. Instantiating the generalized determinization to supported sets and nominal sets yields a construction that internalizes the side-effect of rearranging atoms or registers. Here, we stick to the equality symmetry  $M := \mathfrak{S}_f(\mathbb{A})$ ,  $TX := \mathfrak{S}_f(\mathbb{A}) \bullet X$ , i.e. we stick to Assumption 5.1.

► **Construction 6.1.** Fix  $M := \mathfrak{S}_f(\mathbb{A})$ . For functors  $H: \mathbf{Supp}(\mathbb{A}) \rightarrow \mathbf{Supp}(\mathbb{A})$  and  $G: \mathbf{Nom} \rightarrow \mathbf{Nom}$  with  $HU \cong UG$ , we have the internalization process

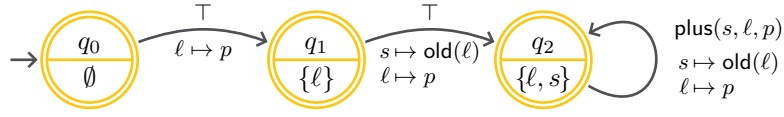
$$c: Q \rightarrow H(M \bullet Q) \text{ in } \mathbf{Supp}(\mathbb{A}) \xrightarrow{\text{Internalization}} d: M \bullet Q \rightarrow G(M \bullet Q) \text{ in } \mathbf{Nom}.$$

Here, the nominal set  $M \bullet Q$  is the configuration space (states + register assignments) of the original automaton  $(Q, c)$ . The configuration  $\text{id}_A \cdot q$  in  $M \bullet Q$  behaves like  $q \in Q$  in  $c$ , and the resulting transition structure  $d$  is equivariant. We can apply this construction to many different system types  $H$  and  $G$  that arise from the following functors. Here,  $\mathcal{P}_{\text{ufs}} X$  (for a nominal set  $X$ ) contains those subsets  $S \subseteq X$  for which  $\bigcup \{\text{supp}(x) \mid x \in S\}$  is finite [34].

► **Proposition 6.2.** The functors  $G$  on  $\mathbf{Nom}$  that are the lifting of a functor  $H$  on  $\mathbf{Supp}(\mathbb{A})$  contain  $\mathcal{P}_f$ ,  $\mathcal{P}_{\text{ufs}}$ ,  $[\mathbb{A}]$ , and all constant functors and are closed under all (possibly infinite) products and coproducts.

► **Example 6.3.** All  $\mathbf{Nom}$ -functors arising from binding signatures [26, 14] are such liftings. In particular,  $\Lambda X = \mathbb{A} + [\mathbb{A}]X + X \times X$  (Example 5.3) is the lifting of  $HX = \mathbb{A} + \mathcal{B}X + X \times X$ . The coalgebras of  $\Lambda$  are possibly infinite  $\lambda$ -trees modulo  $\alpha$ -equivalence [26]. Such a  $\lambda$ -tree can then be represented by a supported set  $X$  and a supported map

$$f: X \rightarrow \mathbb{A} + \mathcal{B}(\mathfrak{S}_f(\mathbb{A}) \bullet X) + (\mathfrak{S}_f(\mathbb{A}) \bullet X) \times (\mathfrak{S}_f(\mathbb{A}) \bullet X)$$



■ **Figure 4** Example (symbolic) register automaton

► **Example 6.4.** Nominal automata with name binding considered by Schröder et al. [34] are coalgebras for the **Nom**-functor  $KX = 2 \times \mathcal{P}_{\text{ufs}}([\mathbb{A}]X + \mathbb{A} \times X)$  which is a lifting of  $HX = 2 \times \mathcal{P}_{\text{ufs}}(\mathcal{B}X + \mathbb{A} \times X)$  on supported sets. Thus, these nominal automata can be represented by finite coalgebras in  $\text{Supp}(\mathbb{A})$ .

Interpreting  $\mathbb{A}$  as the names of registers, register automata in the style of Cassel et al. [8] straightforwardly adapt to *HT*-coalgebras in supported sets:

► **Example 6.5.** Let  $\mathcal{G}$  be a nominal set of *guards*, where we understand  $g \in \mathcal{G}$  as a predicate involving register names  $\text{supp}_G(g) \in \mathcal{P}_f(\mathbb{A})$ , e.g.  $\text{iszero}(a)$ ,  $\text{divides}(a, b)$ ,  $\text{plus}(a, b, c)$ . It is important that the atoms  $a \in \mathbb{A}$  are the register names, and not the data itself. So e.g.  $\text{iszero}(a_3)$  is satisfied if the data value stored in register  $a_3$  is zero – this is the *symbolic* semantics of register automata [39, 40]. Using the forgetful  $U: \text{Nom} \rightarrow \text{Supp}(\mathbb{A})$ , we can also use  $\mathcal{G}$  as a supported set where the support of  $g \in \mathcal{G}$  is the set of register names  $a \in \mathbb{A}$  appearing in  $g$ . Now, a register automaton is a *HT*-coalgebra in  $\text{Supp}(\mathbb{A})$

$$c: Q \rightarrow 2 \times \mathcal{BP}_f(U\mathcal{G} \times \mathfrak{S}_f(\mathbb{A}) \bullet Q) \quad \text{for } HX = 2 \times \mathcal{BP}_f(\mathcal{G} \times X) \text{ and } TX = \mathfrak{S}_f(\mathbb{A}) \bullet X.$$

As before, the first component of  $c(q)$  defines the finality of a state  $q \in Q$ . While in state  $q$ , the registers  $\mathbf{s}(q) \subseteq \mathbb{A}$  are filled with data and the second component of  $c(q)$  binds the next input data symbol into a fresh register ( $\mathcal{B}$ ) and then provides a finite number of transitions of the form  $q \xrightarrow{g, \pi} q'$ . In such a transition, the guard  $g \in \mathcal{G}$ , specifies when this transition can be taken, depending on the register contents of  $q$  and the freshly read input symbol. When a transition is taken, the registers are rearranged via the permutation  $\pi$  before state  $q'$  is entered. The map  $\pi: \mathbf{s}(q') \rightarrow \mathbb{A}$  specifies for each register  $r \in \mathbf{s}(q')$  of the target set, where the data symbol is drawn from. The map  $c$  being a supported map ensures that whenever we transfer to state  $q'$ , then all registers  $\mathbf{s}(q')$  can be filled with data, coming for the support of the previous state  $q$  or from the ‘input register’ in  $\mathcal{B}$  (a concrete register automaton is discussed in Example 6.6). The internalization process turns  $c$  into an equivariant map  $d: \bar{Q} \rightarrow 2 \times [\mathbb{A}]\mathcal{P}_f(\mathcal{G} \times \bar{Q})$ , which is the nominal automaton of configurations  $\bar{Q} = \mathfrak{S}_f(\mathbb{A}) \bullet Q$ .

The construction also nicely interacts with *initial* states. An initial state of a register automaton is simply a supported map  $i: 1 \rightarrow Q$ , that is, an element of  $Q$  with empty support. Applying the functor  $\mathfrak{S}_f(\mathbb{A}) \bullet$  to  $i$  yields an equivariant map,  $\mathfrak{S}_f(\mathbb{A}) \bullet i: \mathfrak{S}_f(\mathbb{A}) \bullet 1 \rightarrow \bar{Q}$ . Since the only element of  $1 = \{0\}$  has empty support, we have  $\mathfrak{S}_f(\mathbb{A}) \bullet 1 \cong 1$ , so  $\mathfrak{S}_f(\mathbb{A}) \bullet i$  is equivalent to an equivariant map  $i': 1 \rightarrow \bar{Q}$ .

► **Example 6.6.** An example register automaton is visualized in Figure 4: the upper half of the nodes provide the state names  $Q = \{q_0, q_1, q_2\}$ , the lower half specifies their support  $\mathbf{s}_Q(q_0) = \emptyset$ ,  $\mathbf{s}_Q(q_1) = \{\ell\}$ ,  $\mathbf{s}_Q(q_2) = \{s\}$ , where  $\ell, s \in \mathbb{A}, \ell \neq s$  are arbitrary (standing for *last* and *second last*). The initial state is  $q_0$ , i.e. the supported map  $i: 1 \rightarrow Q$  is  $i(0) = q_0$ , and all states are final. We mimic existing register automata notation [8, 39] by defining  $p := \varrho(0) \in \mathbb{A}$  (note that we do *not* require  $\ell, s$  to be distinct from  $p$ ). Also, we use  $\text{old}: \mathbb{A} \rightarrow \mathbb{A}$  defined by  $\text{old}(\varrho(k)) = \varrho(k+1)$ , which satisfies  $a \in \mathbf{s}_{\mathcal{B}X}(\lambda.x)$  iff  $\text{old}(a) \in \mathbf{s}_X(x)$  for all supported sets  $X$  and  $x \in X$ . The nominal set of guards is defined by

$$\mathcal{G} := \{\top\} + \{\text{plus}\} \times \mathbb{A}^3,$$



so the constant  $\top$  represents ‘true’ and the ternary relation symbol  $\text{plus}(a, b, c)$  represents that the sum of the register contents of  $a$  and  $b$  equals the register content of  $c$ . Since we use symbolic semantics, the concrete data domain does not need to be specified here; but of course one can interpret **plus** over rational numbers for example. The supported map  $c: Q \rightarrow 2 \times \mathcal{BP}_f(UG \times \mathfrak{S}_f[\mathbb{A}]Q)$  for transitions in the automaton is sincerely visualized in Figure 4:

- The transition  $q_0 \xrightarrow{g, \pi} q_1$  has the guard  $g = \top$  and the register reassignment  $\pi: \mathfrak{s}(q_1) \rightarrow \mathbb{A}$  is defined by  $\pi(\ell) = p$ , meaning that when entering state  $q_1$ , the register  $\ell$  will be filled with what was in the input  $p$  before:  $c(q_0) = (1, \lambda. \{(\top, (\pi, q_1))\})$ . This satisfies support preservation because  $\mathfrak{s}(\top, (\pi, q_1)) = \pi[\mathfrak{s}(q_1)] = \{p\}$  and so  $\mathfrak{s}(\lambda. (\top, (\pi, q_1))) = \emptyset \subseteq \mathfrak{s}(q_0)$ .
- The transition  $q_1 \xrightarrow{g, \sigma} q_2$  has the same guard  $g = \top$  and  $\sigma: \mathfrak{s}(q_2) \rightarrow \mathbb{A}$  is defined by  $\sigma(s) = \text{old}(\ell)$  and  $\sigma(\ell) = p$ . Again,  $c(q_1) = (1, \lambda. \{(\top, (\sigma, q_2))\})$  preserves support because
 
$$\mathfrak{s}(\top, (\sigma, q_2)) = \sigma[\mathfrak{s}(q_2)] = \sigma[\{\ell, s\}] = \{p, \text{old}(\ell)\} \quad \text{and} \quad \mathfrak{s}(\lambda. \{(\top, (\sigma, q_2))\}) = \{\ell\} \subseteq \mathfrak{s}(q_1).$$
- The loop  $q_2 \xrightarrow{g', \sigma} q_2$  has the guard  $g' = \text{plus}(s, \ell, p)$  and literally the same  $\sigma: \mathfrak{s}(q_2) \rightarrow \mathbb{A}$  as in the previous transition. Support preservation holds for the mapping  $c(q_1) = (1, \lambda. \{(\top, (\sigma, q_2))\})$ , because
 
$$\mathfrak{s}(\text{plus}(s, \ell, p), (\sigma, q_2)) = \{\text{old}(s), \text{old}(\ell), p\} \quad \text{and} \quad \mathfrak{s}(\lambda. \{(\text{plus}(s, \ell, p), (\sigma, q_2))\}) = \{s, \ell\} \subseteq \mathfrak{s}(q_2).$$

The coherence axioms of register automata naturally translate into  $c$  being a supported map. Construction 6.1 transforms this finite coalgebra in  $\mathbf{Supp}(\mathbb{A})$  into a nominal automaton, in the style of symbolic semantics [39] of register automata.

## 7 Conclusions and Future Work

We have seen that by going from the base category of sets to supported sets, nominal sets for various symmetries surprisingly turn out to be monadic. Supported sets have a functor for name binding, which even lifts to the abstraction functor in nominal sets. It remains for future work whether a similar name binding functor can be found for other data alphabets, most notably for the total order symmetry on  $\mathbb{Q}$ , and whether multiple atoms can be bound simultaneously, as it is possible in nominal sets [10]. It can be conjectured that such generalizations are not possible on the level of supported sets.

On the positive side, due to the little structure of supported sets, it provides a common foundation for the nominal sets for different symmetries, in the sense of being described by monads on supported sets. The monadicity can be used to relate nominal automata with register automata, which have a natural definition in supported sets. It remains for future investigation how the *data semantics* of register automata can be phrased in supported sets. We are optimistic that it helps to develop a categorical semantics for register automata for data alphabets and signatures beyond symmetries (e.g. those for priority queues [8]). When developing algorithms, in particular learning algorithms and minimization algorithms, for register or nominal automata [8, 38, 30], supported sets directly yield a finite representation that can help in the implementation and complexity analysis.

---

## References

- 1 Jiří Adámek, Stefan Milius, Lurdes Sousa, and Thorsten Wißmann. Finitely presentable algebras for finitary monads. *Theory and Applications of Categories*, 34(37):1179–1195, 11 2019. URL: <http://www.tac.mta.ca/tac/volumes/34/37/34-37abs.html>.

- 2 Jiří Adámek, Horst Herrlich, and George E. Strecker. Abstract and concrete categories. the joy of cats, 2004.
- 3 Jiří Adámek and Jiří Rosický. *Locally Presentable and Accessible Categories*. Cambridge University Press, 1994.
- 4 Steve Awodey. *Category Theory*. Oxford Logic Guides. OUP Oxford, 2010. URL: <http://books.google.de/books?id=-MCJ6x21C7oC>.
- 5 Brian E. Aydemir, Aaron Bohannon, and Stephanie Weirich. Nominal reasoning techniques in coq: (extended abstract). *Electron. Notes Theor. Comput. Sci.*, 174(5):69–77, 2007. doi:10.1016/j.entcs.2007.01.028.
- 6 Falk Bartels. *On generalized coinduction and probabilistic specification formats: Distributive laws in coalgebraic modelling*. PhD thesis, Vrije Universiteit Amsterdam, 2004.
- 7 Mikolaj Bojanczyk, Bartek Klin, and Slawomir Lasota. Automata theory in nominal sets. *Log. Methods Comput. Sci.*, 10, 2014. doi:10.2168/LMCS-10(3:4)2014.
- 8 Sofia Cassel, Falk Howar, Bengt Jonsson, and Bernhard Steffen. Extending automata learning to extended finite state machines. In Amel Bennaceur, Reiner Hähnle, and Karl Meinke, editors, *Machine Learning for Dynamic Software Analysis: Potentials and Limits - International Dagstuhl Seminar 16172*, volume 11026 of *LNCS*, pages 149–177. Springer, 2018. doi:10.1007/978-3-319-96562-8\_6.
- 9 Vincenzo Ciancia and Ugo Montanari. A name abstraction functor for named sets. In Jiri Adámek and Clemens Kupke, editors, *Proceedings of the Ninth Workshop on Coalgebraic Methods in Computer Science, CMCS 2008*, volume 203 of *Electronic Notes in Theoretical Computer Science*, pages 49–70. Elsevier, 2008. doi:10.1016/j.entcs.2008.05.019.
- 10 Ranald Clouston. Generalised name abstraction for nominal sets. In Frank Pfenning, editor, *Foundations of Software Science and Computation Structures - 16th International Conference, FOSSACS 2013, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2013, Rome, Italy, March 16-24, 2013. Proceedings*, volume 7794 of *Lecture Notes in Computer Science*, pages 434–449. Springer, 2013. doi:10.1007/978-3-642-37075-5\_28.
- 11 N.G de Bruijn. Lambda calculus notation with nameless dummies, a tool for automatic formula manipulation, with application to the Church-Rosser theorem. *Indagationes Mathematicae (Proceedings)*, 75(5):381–392, 1972. doi:10.1016/1385-7258(72)90034-0.
- 12 Stéphane Demri and Ranko Lazic. LTL with the freeze quantifier and register automata. *ACM Trans. Comput. Log.*, 10(3):16:1–16:30, 2009. doi:10.1145/1507244.1507246.
- 13 Gian Luigi Ferrari, Ugo Montanari, and Marco Pistore. Minimizing transition systems for name passing calculi: A co-algebraic formulation. In Mogens Nielsen and Uffe Engberg, editors, *Foundations of Software Science and Computation Structures, 5th International Conference, FOSSACS 2002. Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2002 Grenoble, France, April 8-12, 2002, Proceedings*, volume 2303 of *Lecture Notes in Computer Science*, pages 129–158. Springer, 2002. doi:10.1007/3-540-45931-6\_10.
- 14 Marcelo Fiore, Gordon Plotkin, and Daniele Turi. Abstract syntax and variable binding (extended abstract). In *Proc. 14<sup>th</sup> LICS Conf.*, pages 193–202. IEEE, Computer Society Press, 1999.
- 15 Marcelo P. Fiore and Sam Staton. Comparing operational models of name-passing process calculi. *Inf. Comput.*, 204(4):524–560, 2006. doi:10.1016/j.ic.2005.08.004.
- 16 Murdoch Gabbay and James Cheney. A sequent calculus for nominal logic. In *19th IEEE Symposium on Logic in Computer Science (LICS 2004), 14-17 July 2004, Turku, Finland, Proceedings*, pages 139–148. IEEE Computer Society, 2004. doi:10.1109/LICS.2004.1319608.
- 17 Murdoch Gabbay and Andrew M. Pitts. A new approach to abstract syntax involving binders. In *14th Annual IEEE Symposium on Logic in Computer Science, Trento, Italy, July 2-5, 1999*, pages 214–224. IEEE Computer Society, 1999. doi:10.1109/LICS.1999.782617.
- 18 Murdoch Gabbay and Andrew M. Pitts. A new approach to abstract syntax with variable binding. *Formal Aspects Comput.*, 13(3-5):341–363, 2002. doi:10.1007/s001650200016.

- 19 Murdoch James Gabbay and Martin Hofmann. Nominal renaming sets. In Iliano Cervesato, Helmut Veith, and Andrei Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning, 15th International Conference, LPAR 2008*, volume 5330 of *LNCS*, pages 158–173. Springer, 2008. doi:10.1007/978-3-540-89439-1\_11.
- 20 Peter Gabriel and Friedrich Ulmer. *Lokal präsentierbare Kategorien*, volume 221 of *Lecture Notes Math.* Springer-Verlag, 1971.
- 21 Fabio Gadducci, Marino Miculan, and Ugo Montanari. About permutation algebras, (pre)sheaves and named sets. *Higher Order Symbol. Comput.*, 19(2-3):283–304, September 2006. doi:10.1007/s10990-006-8749-3.
- 22 Peter T. Johnstone. Adjoint Lifting Theorems for Categories of Algebras. *Bull. London Math. Soc.*, 7(3):294–297, nov 1975.
- 23 Peter T Johnstone. *Sketches of an elephant: a Topos theory compendium*. Oxford logic guides. Oxford Univ. Press, New York, NY, 2002.
- 24 Michael Kaminski and Nissim Francez. Finite-memory automata. *Theor. Comput. Sci.*, 134(2):329–363, 1994. doi:10.1016/0304-3975(94)90242-9.
- 25 Dexter Kozen, Konstantinos Mamouras, Daniela Petrisan, and Alexandra Silva. Nominal kleene coalgebra. In Magnús M. Halldórsson, Kazuo Iwama, Naoki Kobayashi, and Bettina Speckmann, editors, *ICALP 2015, Proceedings*, volume 9135 of *LNCS*, pages 286–298. Springer, 2015. doi:10.1007/978-3-662-47666-6\_23.
- 26 Alexander Kurz, Daniela Petrisan, Paula Severi, and Fer-Jan de Vries. Nominal coalgebraic data types with applications to lambda calculus. *Logical Methods in Computer Science*, 9(4), 2013.
- 27 Alexander Kurz, Daniela Petrisan, and Jiri Velebil. Algebraic theories over nominal sets. *CoRR*, abs/1006.3027, 2010. URL: <http://arxiv.org/abs/1006.3027>.
- 28 Saunders Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer New York, 1998. URL: <http://books.google.de/books?id=eBvhyc4z8HQC>.
- 29 Joshua Moerman and Jurriaan Rot. Separation and Renaming in Nominal Sets. In Maribel Fernández and Anca Muscholl, editors, *CSL 2020*, volume 152 of *LIPIcs*, pages 31:1–31:17, Dagstuhl, Germany, 2020. LIPIcs. doi:10.4230/LIPIcs.CSL.2020.31.
- 30 Joshua Moerman, Matteo Sammartino, Alexandra Silva, Bartek Klin, and Michal Szynwelski. Learning nominal automata. In Giuseppe Castagna and Andrew D. Gordon, editors, *POPL 2017*, pages 613–625. ACM, 2017. doi:10.1145/3009837.3009879.
- 31 Ugo Montanari and Marco Pistore. History-dependent automata: An introduction. In Marco Bernardo and Alessandro Bogliolo, editors, *Formal Methods for Mobile Computing, 5th International School on Formal Methods for the Design of Computer, Communication, and Software Systems, SFM-Moby 2005, Bertinoro, Italy, April 26-30, 2005, Advanced Lectures*, volume 3465 of *Lecture Notes in Computer Science*, pages 1–28. Springer, 2005. doi:10.1007/11419822\_1.
- 32 Daniela Petrişan. *Investigations into Algebra and Topology over Nominal Sets*. dissertation, University of Leicester, 2011. URL: <http://www.cs.le.ac.uk/people/dlp10/publications/thesis-final.pdf>.
- 33 Andrew M. Pitts. *Nominal Sets: Names and Symmetry in Computer Science*, volume 57 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2013.
- 34 Lutz Schröder, Dexter Kozen, Stefan Milius, and Thorsten Wißmann. Nominal automata with name binding. In Javier Esparza and Andrzej Murawski, editors, *FoSSaCS 2017*, volume 10203 of *LNCS*, pages 124–142. Springer, 2017. doi:10.1007/978-3-662-54458-7\_8.
- 35 Alexandra Silva, Filippo Bonchi, Marcello M. Bonsangue, and Jan J. M. M. Rutten. Generalizing determinization from automata to coalgebras. *Log. Methods Comput. Sci.*, 9(1), 2013. doi:10.2168/LMCS-9(1:9)2013.
- 36 Sam Staton. Name-passing process calculi: operational models and structural operational semantics. Technical Report UCAM-CL-TR-688, University of Cambridge, Computer Laboratory, June 2007. doi:10.48456/tr-688.

- 37 Christian Urban and Christine Tasson. Nominal techniques in isabelle/hol. In Robert Nieuwenhuis, editor, *CADE-20*, volume 3632 of *LNCs*, pages 38–53. Springer, 2005. doi:10.1007/11532231\_4.
- 38 Henning Urbat and Lutz Schröder. Automata learning: An algebraic approach. In Holger Hermanns, Lijun Zhang, Naoki Kobayashi, and Dale Miller, editors, *LICS '20: 35th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 900–914. ACM, 2020. doi:10.1145/3373718.3394775.
- 39 Frits W. Vaandrager and Abhisek Midya. A myhill-nerode theorem for register automata and symbolic trace languages. In Violet Ka I Pun, Volker Stolz, and Adenilso Simão, editors, *ICTAC 2020*, volume 12545 of *LNCs*, pages 43–63. Springer, 2020. doi:10.1007/978-3-030-64276-1\_3.
- 40 Frits W. Vaandrager and Abhisek Midya. A myhill-nerode theorem for register automata and symbolic trace languages. *Theor. Comput. Sci.*, 912:37–55, 2022. doi:10.1016/j.tcs.2022.01.015.
- 41 David Venhoek, Joshua Moerman, and Jurriaan Rot. Fast computations on ordered nominal sets. In Bernd Fischer and Tarmo Uustalu, editors, *ICTAC 2018*, volume 11187 of *LNCs*, pages 493–512. Springer, 2018. doi:10.1007/978-3-030-02508-3\_26.
- 42 David Venhoek, Joshua Moerman, and Jurriaan Rot. Fast computations on ordered nominal sets. *CoRR*, abs/1902.08414, 2019. URL: <http://arxiv.org/abs/1902.08414>, arXiv:1902.08414.

## A Omitted Proofs and Further Details

### Details for Notation 2.1

The precise definition of the cycle notation is as follows: Given elements  $a_0, \dots, a_{n-1} \in A$ , the notation  $(a_0 \ a_1 \ \dots \ a_{n-1})$  denotes the bijective map  $f: A \rightarrow A$  given by

$$f(x) = \begin{cases} a_{k+1 \bmod n} & \text{if there is some } k < n \text{ with } x = a_k \\ x & \text{otherwise.} \end{cases} \quad \blacktriangleleft$$

### Details for Example 2.4

All examples listed here satisfy the  $M$ -set axioms:

1.  $A$  is an  $M$ -set because indeed  $\text{id}_A \cdot a = \text{id}_A(a) = a$  and  $(\ell \cdot m) \cdot a = \ell(m(a)) = \ell \cdot (m \cdot a)$ .
2. & 3. The verification of the  $M$ -set axioms for  $\mathcal{P}_f(X)$  and discrete  $M$ -sets are straightforward.
4. For disambiguation, let us use  $\circ$  for monoid multiplication  $(M, \circ, e)$  and  $\star$  for the monoid action, i.e. for  $m, x \in M$ , we define

$$m \star x := m \circ x$$

Then, the monoid laws directly imply that  $\star$  satisfies axioms of the  $M$ -set action:

$$e \star x = e \circ x = x$$

$$m' \star (m \star x) = m' \circ (m \circ x) = (m' \circ m) \circ x = (m' \circ m) \star x$$

### Details for Definition 2.6

Given that  $M$  is a group,  $S \subseteq A$  supports  $x \in X$  iff for all  $m \in M$  with  $\forall a \in S: m(a) = a$ , we have  $m \cdot x = x$ .

This is indeed equivalent to Definition 2.6: For ‘only if’, put  $m' := \text{id}_A$ . For ‘if’, take  $m, m' \in M$  with  $m(a) = m'(a)$  for all  $a \in S$ . Hence,  $(m'^{-1} \cdot m)(a) = a$  for all  $a \in S$ , hence  $m \cdot x = m' \cdot m'^{-1} \cdot m \cdot x = m' \cdot x$ .

The definition can be simplified in the same way for nominal renaming sets [19, Lem. 13].  $\blacktriangleleft$

### Details for Example 2.7

The examples of nominal  $M$ -sets Example 2.7 are all standard. It only remains to verify the non-example in item 4, i.e. let us verify that whenever  $M$  is a nominal  $M$ -set, then all  $M$ -sets are nominal.

If  $M$  is a nominal  $M$ -set, let  $S \subseteq A$  be a finite set that supports  $\text{id}_A \in M$ . Then,  $S$  is the support of all elements  $x \in X$  in all  $M$ -sets  $X$ : consider  $m, m' \in M$  with  $m \approx_S m'$ . Then  $m = m \cdot \text{id}_A = m' \cdot \text{id}_A = m'$  because  $S$  supports  $\text{id}_A$ . Hence,  $m \cdot x = m' \cdot x$ , showing that  $S$  also supports  $x \in X$ .  $\blacktriangleleft$

### Proof of Lemma 2.11

If  $S$  supports  $x$ , consider  $m, m' \in M$  with  $m \approx_S m'$ . Then,  $m \cdot f(x) = f(m \cdot x) = f(m' \cdot x) = m' \cdot f(x)$ .

For least finite supports, we have that  $\text{supp}_X(x)$  supports  $f(x)$  by the previous statement. Hence, the least finite support  $\text{supp}_Y(f(x))$  is smaller or equal to  $\text{supp}_X(x)$  (w.r.t. subset inclusion). ◀

### Proof of Lemma 3.3

For arbitrary  $M \leq A^A$ , define the functor  $U: \text{Nom}(M) \rightarrow \text{Supp}(A)$  sending  $(X, \cdot)$  to the set  $X$  with the support

$$\mathbf{s}_X(x) = \bigcap \{S \subseteq A \mid S \text{ finite and supports } x\}$$

Since every element  $x \in X$  has some finite support (Definition 2.6), the intersection in the definition of  $\mathbf{s}_X$  is not empty, and hence  $\mathbf{s}_X(x)$  is a finite set and so  $U$  sends nominal  $M$ -sets to supported sets.

For an equivariant map  $f: X \rightarrow Y$  and  $x \in X$ , we have by Lemma 2.11:

$$\begin{aligned} & \{S \subseteq A \mid S \text{ finite and supports } f(x)\} \\ & \supseteq \{S \subseteq A \mid S \text{ finite and supports } x\}. \end{aligned}$$

When taking the intersection of these families of subsets of  $A$ , the inclusion reverses and we obtain

$$\begin{aligned} \mathbf{s}_Y(f(x)) &= \bigcap \{S \subseteq A \mid S \text{ finite and supports } f(x)\} \\ &\subseteq \bigcap \{S \subseteq A \mid S \text{ finite and supports } x\} = \mathbf{s}_X(x) \end{aligned}$$

Thus,  $Uf$  is a supported map. Clearly,  $U$  preserves identities and composition. Note that we did not require here that *least* finite supports exist in nominal  $M$ -sets.

If they exist however, we have that the least finite support  $\text{supp}_X(x)$  of  $x \in X$  is the intersection of all finite supports of  $x$ , i.e.  $\mathbf{s}_X(x) = \text{supp}_X(x)$ , as claimed. ◀

### Proof of Lemma 3.4

Define  $J: \text{Set} \hookrightarrow \text{Supp}(A)$  by  $JX = (X, \mathbf{s}_X)$ ,  $\mathbf{s}_X(x) = \emptyset$ . Every map  $f: X \rightarrow Y$  then is a supported map  $f: JX \rightarrow JY$ . Clearly,  $VJX = X$  and  $VJ(f: X \rightarrow Y) = f$ , hence  $VJ = \text{Id}_{\text{Set}}$ .

For the adjunction, we verify that there are supported maps  $\eta_X: X \rightarrow JVX$  such that for every supported map  $h: X \rightarrow JY$  ( $Y \in \text{Set}$ ), there is a unique  $h': VJX \rightarrow Y$  with  $h = Jh' \cdot \eta_X$ :

$$\begin{array}{ccc} JVX & \xrightarrow{Jh'} & JY \\ \eta_X \uparrow & \nearrow h & \\ X & & \end{array}$$

Note that the identity  $\text{id}_X: X \rightarrow X$  gives rise to the supported map  $\eta_X: X \rightarrow JVX$ . Hence,  $Jh' \cdot \eta_X = h$  and that  $h' := Vh$ . Since  $\eta_X$  is surjective and  $J$  is faithful,  $h'$  is the unique map fulfilling this property. ◀



### Proof of Proposition 3.5

Consider a diagram  $D: \mathcal{D} \rightarrow \mathbf{Supp}(A)$  and its colimiting cocone in **Set**:

$$\mathrm{in}_Y: DY \rightarrow C \quad \text{in } \mathbf{Set}.$$

In order to make  $C$  a supported set, define  $\mathbf{s}: C \rightarrow \mathcal{P}_f(A)$  by

$$\mathbf{s}(c) = \bigcap \{ \mathbf{s}_{DY}(y) \mid Y \in \mathcal{D}, y \in DY, \mathrm{in}_Y(y) = c \}$$

Since colimit injections in **Set** are jointly surjective, the above intersection is non-empty, and thus yields a finite subset of  $A$ .

This lets the cocone lift to  $\mathbf{Supp}(A)$ : every  $\mathrm{in}_Z: DZ \rightarrow C$  is a supported map, because for every  $z \in DZ$  we have

$$\begin{aligned} \mathbf{s}_C(\mathrm{in}_Z(z)) &= \bigcap \{ \mathbf{s}_{DY}(y) \mid Y \in \mathcal{D}, y \in DY, \mathrm{in}_Y(y) = \mathrm{in}_Z(z) \} \\ &\subseteq \mathbf{s}_{DZ}(z). \end{aligned}$$

In order to see that  $(\mathrm{in}_Y)_{Y \in \mathcal{D}}$  is a colimiting cocone, consider a cocone  $(e_Y: DY \rightarrow E)_{Y \in \mathcal{D}}$  in  $\mathbf{Supp}(A)$ . Since this is also a competing cocone in **Set**, we obtain a cocone morphism  $u: C \rightarrow E$ . For the verification that  $u$  is a supported map, note that for every  $c \in C$  and every  $Y \in \mathcal{D}$ ,  $y \in DY$  with  $\mathrm{in}_Y(y) = c$ , we have  $\mathbf{s}_E(u(c)) \subseteq \mathbf{s}_Y(y)$ . By the universal property of intersection, this implies that

$$\mathbf{s}_E(u(c)) \subseteq \bigcap \{ \mathbf{s}_Y(y) \mid Y \in \mathcal{D}, y \in DY, \mathrm{in}_Y(y) = c \} = \mathbf{s}_C(c),$$

so  $u: C \rightarrow E$  is a supported map. Since  $V: \mathbf{Supp}(A) \rightarrow \mathbf{Set}$  is faithful,  $u$  is a cocone morphism in  $\mathbf{Supp}(A)$  and moreover unique.  $\blacktriangleleft$

### Verification of Example 3.6

We show the statement about coproducts directly for  $I$ -indexed families  $(X_i)_{i \in I}$ : by Proposition 3.5, the coproduct  $\coprod_{i \in I} X_i$  in  $\mathbf{Supp}(A)$  is given by the coproduct in **Set**. For every  $c \in \coprod_{i \in I} X_i$  there is precisely one  $i \in I$ ,  $x \in X_i$  with  $\mathrm{in}_i(x) = c$ . Hence,  $\mathbf{s}_{\coprod_{i \in I} X_i}(c) = \mathbf{s}_{X_i}(x)$ .  $\blacktriangleleft$

### Proof of Proposition 3.7

Consider a diagram  $D: \mathcal{D} \rightarrow \mathbf{Supp}(A)$  and its limiting cone in **Set**:

$$\mathrm{pr}_Y: P \rightarrow DY \quad \text{in } \mathbf{Set}.$$

Define the map to all – i.e. possibly infinite – subsets of  $A$ :

$$p: P \rightarrow \mathcal{P}(A) \quad p(x) = \bigcup \{ \mathbf{s}_{DY}(\mathrm{pr}_Y(x)) \mid Y \in \mathcal{D} \}$$

The limit in  $\mathbf{Supp}(A)$  is the restriction of  $P$  to finitely supported elements:

$$L := \{ x \in P \mid p(x) \text{ is finite} \} \quad \mathbf{s}_L(x) := p(x).$$

By definition,  $L$  is a finitely supported map. The projections are the restrictions of  $\mathrm{pr}_Y$  to  $L \subseteq P$ :  $\ell_Y: L \rightarrow DY$  and  $\ell(x) = \mathrm{pr}_Y(x)$  for all  $Y \in \mathcal{D}$ . Hence,

$$\mathbf{s}_L(x) = \bigcup \{ \mathbf{s}_{DY}(\ell_Y(x)) \mid Y \in \mathcal{D} \}.$$

It is immediate that every  $\ell_Y$  is a supported map:

$$\mathbf{s}_{DY}(\ell_Y(x)) \subseteq \bigcup \{\mathbf{s}_Z(\ell_Z(x)) \mid Z \in \mathcal{D}\} = \mathbf{s}_L(x)$$

The family  $(\ell_Y: L \rightarrow DY)_{Y \in \mathcal{D}}$  is a cone because it was defined as a restriction of the cone  $P$  in **Set**. For the verification of the universal property, consider another cone

$$(e_Y: E \rightarrow DY)_{Y \in \mathcal{D}} \quad \text{in } \mathbf{Supp}(A).$$

This is also a cone for the diagram  $VD: \mathcal{D} \rightarrow \mathbf{Set}$ , so we obtain a unique cone morphism  $u: E \rightarrow P$  (i.e.  $\mathbf{pr}_Y \cdot u = e_Y$  for all  $Y \in \mathcal{D}$ ). For every  $x \in E$  and  $Y \in \mathcal{D}$ , we have

$$\mathbf{s}_{DY}(\mathbf{pr}_Y(u(x))) = \mathbf{s}_{DY}(e_Y(x)) \subseteq \mathbf{s}_E(x)$$

since  $e_Y$  is a supported map. So by the universal property of union, we obtain

$$p(u(x)) = \bigcup \{\mathbf{s}_{DY}(\mathbf{pr}_Y(u(x))) \mid Y \in \mathcal{D}\} \subseteq \mathbf{s}_E(x),$$

which proves the finiteness of  $p(u(x)) \subseteq A$ . Hence,  $u: E \rightarrow P$  restricts to  $u: E \rightarrow L$  which is thus also a supported map ( $\mathbf{s}_L(u(x)) = p(u(x)) \subseteq \mathbf{s}_E(x)$ ). Uniqueness of  $u: U \rightarrow L$  follows directly from the faithfulness of  $V: \mathbf{Supp}(A) \rightarrow \mathbf{Set}$ . Hence,  $(\ell_Y: L \rightarrow DY)_{Y \in \mathcal{D}}$  is a limiting cone.  $\blacktriangleleft$

### Proof of Corollary 3.8

Consider a diagram  $D: \mathcal{D} \rightarrow \mathbf{Supp}(A)$  and its limit

$$(\mathbf{pr}_Y: \lim VD \rightarrow VDY)_{Y \in \mathcal{D}} \quad \text{in } \mathbf{Set}.$$

If  $\mathcal{D}$  has only finitely many objects, then

$$\bigcup \{\mathbf{s}_{DY}(\mathbf{pr}_Y(x)) \mid Y \in \mathcal{D}\}$$

is finite for all  $x \in \lim VD$ . Hence, the limit in  $\mathbf{Supp}(A)$  given by Proposition 3.7 is identical to the limit  $\lim VD$  in **Set**.  $\blacktriangleleft$

### Details for Definition 3.9

For a supported set  $E$ , the full definition of the functor  $(-)^E: \mathbf{Supp}(A) \rightarrow \mathbf{Supp}(A)$  is

$$X^E := \{f: E \rightarrow X \mid \bigcup_{e \in E} \mathbf{s}_X(f(e)) \setminus \mathbf{s}_E(e) \text{ is finite}\} \quad \mathbf{s}_{X^E}(f) = \bigcup_{e \in E} \mathbf{s}_X(f(e)) \setminus \mathbf{s}_E(e)$$

on supported sets  $X$ . For supported maps  $g: X \rightarrow Y$  we put  $g^E: X^E \rightarrow Y^E$ ,  $g^E(f) = g \cdot f$ .  $\blacktriangleleft$

### Proof of Proposition 3.10

Define  $\eta_X: X \rightarrow (X \times E)^E$  by

$$\eta_X(x)(e) = (x, e).$$

For every  $x \in X$ , clearly  $\eta_X(x)$  is in  $(X \times E)^E$  since

$$\mathbf{s}_{(X \times E)^E}(\eta_X(x)) = \bigcup_{e \in E} \mathbf{s}_{X \times E}(x, e) \setminus \mathbf{s}_E(e) = \bigcup_{e \in E} \mathbf{s}_X(x) = \mathbf{s}_X(x)$$

is finite; this also shows that  $\eta_X$  is a supported map. Moreover,  $\eta_X$  is natural in  $X$  since

$$\begin{aligned} (g \times E)^E(\eta_X(x)) &= (g \times E) \cdot (e \mapsto (x, e)) \\ &= (e \mapsto (g(x), e)) = \eta_Y(g(x)). \end{aligned}$$

For the verification of the universal mapping property, consider  $g: X \rightarrow Y^E$  and define

$$h: X \times E \rightarrow Y \quad \text{by} \quad h(x, e) = g(x)(e).$$

By the definition of  $\mathbf{s}_{Y^E}$ , we have

$$\mathbf{s}_Y(g(x)(e)) \setminus \mathbf{s}_E(e) \subseteq \mathbf{s}_{Y^E}(g(x))$$

and so

$$\mathbf{s}_Y(g(x)(e)) \subseteq \mathbf{s}_{Y^E}(g(x)) \cup \mathbf{s}_E(e) \subseteq \mathbf{s}_X(x) \cup \mathbf{s}_E(e) = \mathbf{s}_{X \times E}(x, e),$$

hence,  $h$  is a supported map. For all  $x \in X$ , we have

$$\begin{aligned} h^E(\eta_X(x)) &= h \cdot (\eta(x)) = h \cdot (e \mapsto (x, e)) \\ &= (e \mapsto h(x, e)) = (e \mapsto g(x)(e)) = g(x), \end{aligned}$$

so  $h^E \cdot \eta_X = g$ . For uniqueness, consider a supported map  $u: X \times E \rightarrow Y$  with  $u^E \cdot \eta_X = g$ . Necessarily,

$$g(x) = u^E(\eta_X(x)) = u \cdot (e \mapsto (x, e)) = (e \mapsto u(x, e))$$

i.e.  $u(x, e) = g(x)(e) = h(x, e)$  and  $u = h$ . ◀

### Proof of Proposition 3.11

Let us first recall the definition of fp objects and lfp categories:

- **Definition A.1** [3, 20]. 1. A directed colimit is the colimit for a diagram  $D: \mathcal{D} \rightarrow \mathcal{C}$  where  $\mathcal{D}$  is a poset in which any two objects have an upper bound.
2. An object  $X$  in a category  $\mathcal{C}$  is called finitely presented (fp) if the hom-functor  $\mathcal{C}(X, -): \mathcal{C} \rightarrow \mathbf{Set}$  preserves colimits of directed diagrams.
3. A category  $\mathcal{C}$  is called locally finitely presentable (lfp) if it is cocomplete and every object is the directed colimit of finitely presentable objects.

Given  $X \in \mathbf{Supp}(A)$  and a directed diagram  $D: \mathcal{D} \rightarrow \mathbf{Supp}(A)$ , note that the colimit preservation

$$\mathbf{Supp}(A)(X, \text{colim} D) = \text{colim} \mathbf{Supp}(A)(X, D(-))$$

boils down to the following condition [3, Def. 1.1]: for every supported map  $f: X \rightarrow \text{colim} D$ , there exists  $Y \in \mathcal{D}$  such that

1. there are  $Y \in \mathcal{D}$  and  $g: X \rightarrow DY$  with  $\text{in}_Y \cdot f' = f$ :

$$\begin{array}{ccc} X & \xrightarrow{f} & \text{colim} D \\ & \searrow g & \uparrow \text{in}_Y \\ & & DY \end{array} \quad \text{in } \mathbf{Supp}(A)$$

2.  $Y$  and  $g$  are essentially unique in the sense that if  $f = \text{in}_{Y'} \cdot g'$  for  $g': X \rightarrow DY'$ , then  $D(Y \rightarrow Z) \cdot g = D(Y' \rightarrow Z) \cdot g'$  for some  $Z \geq Y$  in  $\mathcal{D}$  (recall that  $\mathcal{D}$  is a poset, so we can denote connecting morphisms simply by  $Y \rightarrow Z$  and  $Y' \rightarrow Z$ )

We now prove the following statements:

1. Every supported set  $X$  is the directed colimit of its finite subsets:  
 Let  $\mathcal{D} = \{Y \mid Y \subseteq X\}$ ,  $D: \mathcal{D} \rightarrow \mathbf{Supp}(A)$ ,  $DY = (Y, \mathbf{s}_{DY})$  where  $\mathbf{s}_{DY}$  is the restriction of  $\mathbf{s}_X: X \rightarrow \mathcal{P}_f(A)$  to  $Y \subseteq X$ . In  $\mathcal{D}$ , any two elements  $Y, Y' \in \mathcal{D}$  have an upper bound:  $Y \cup Y' \in \mathcal{D}$ . By Proposition 3.5, we have that  $\text{colim} D$  is carried by  $X$ , and

$$\begin{aligned} \mathbf{s}_{\text{colim} D}(x) &= \bigcap \{\mathbf{s}_{DY}(x) \mid Y \in \mathcal{D}, x \in Y\} \\ &= \bigcap \{\mathbf{s}_X(x) \mid Y \in \mathcal{D}, x \in Y\} = \mathbf{s}_X(x). \end{aligned}$$

So  $\text{colim} D \cong X$ , as desired.

2. Every finitely presentable supported set  $X$  is finite:

Consider the directed colimit of subsets of  $X$ . By item 1, this directed colimit yields  $X$ , so we have an isomorphism  $\phi: X \rightarrow \text{colim} D$ . Since  $X$  is finitely presentable, this factors through some finite subset  $Y \subseteq X$ :

$$\begin{array}{ccc} X & \xrightarrow[\cong]{\phi} & \text{colim} D \\ & \searrow g & \uparrow \text{in}_Y \\ & & DY \end{array} \quad \text{in } \mathbf{Supp}(A)$$

Note that  $\text{in}_Y$  is support-preserving by the definition of  $\mathcal{D}$  and injective since it's the same colimit injection as in  $\mathbf{Set}$ . Also,  $\text{in}_Y$  is surjective, since  $\text{in}_Y \cdot g = \phi$ . Hence, it is a regular monomorphism, an epimorphism and thus an isomorphism, showing finiteness of  $X$ .

3. Every finite supported set  $X$  is finitely presentable:

Since finitely presentable objects are closed under finite coproducts [3, Prop. 1.3] and every finite set is the finite coproduct of singleton sets (Example 3.6), we can assume wlog that  $X$  is singleton  $X = \{x\}$ . For the verification that  $X$  is fp, consider a supported map  $f: X \rightarrow \text{colim} D$  for a directed diagram  $D: \mathcal{D} \rightarrow \mathbf{Supp}(A)$ . Since colimit injections are jointly surjective in  $\mathbf{Set}$  and thus also in  $\mathbf{Supp}(A)$ , there are  $Z \in \mathcal{D}$ ,  $z \in DZ$  with  $\text{in}_Z(z) = f(x)$ . However, the support of  $z$  could possibly contain more than  $\mathbf{s}(f(x))$ . Denote this difference by the finite set

$$R := \mathbf{s}_{DZ}(z) \setminus \mathbf{s}_{\text{colim} D}(f(x)) \subseteq A.$$

- If  $R := \emptyset$ , then  $\mathbf{s}_{DZ}(z) = \mathbf{s}_{\text{colim} D}(f(x)) \subseteq \mathbf{s}_X(x)$  and we have the desired supported map  $g: X \rightarrow DZ$ ,  $g(x) = z$ .
- By the characterization of colimits in  $\mathbf{Supp}(A)$  (Proposition 3.5), we know that

$$\mathbf{s}_{\text{colim} D}(f(x)) = \bigcap \{\mathbf{s}_Y(y) \mid Y \in \mathcal{D}, \text{in}_Y(y) = f(x)\}$$

So for every  $a \in R$ , there must be some  $Y_a \in \mathcal{D}$  and  $y_a \in DY_a$  with

$$\text{in}_{Y_a}(y_a) = f(x) \text{ and } a \notin \mathbf{s}_{DY_a}(y_a).$$

Let  $U \in \mathcal{D}$  be the upper bound of  $\{Y_a \mid a \in R\}$  in  $\mathcal{D}$ , which exists by assumption. Then  $u := D(Y_a \rightarrow U)(y_a)$  fulfils

$$\mathbf{s}_U(u) = \mathbf{s}_{\text{colim} D}(f(x)) \subseteq \mathbf{s}_X(x) \text{ and } \text{in}_U(u) = f(x)$$

so we have the desired supported map  $g: X \rightarrow U$ ,  $g(x) = u$ .

The essential uniqueness of the factorization follows on the level of **Set**: consider two factorizations:

$$g: X \rightarrow DY \quad g': X \rightarrow DY' \quad \text{in}_Y \cdot g = f = \text{in}_{Y'} \cdot g'$$

Since  $g(x)$  and  $g'(x)$  are identified in the colimit ( $\text{in}_Y(g(x)) = \text{in}_{Y'}(g'(x))$ ) and since  $\mathcal{D}$  is directed, by [3, Ex. 1a.2.ii] there is some  $Z \in \mathcal{D}$  with  $Z \geq Y$ ,  $Z \geq Y'$  and

$$D(Y \rightarrow Z)(g(x)) = D(Y' \rightarrow Z)(g'(x)).$$

4. Since  $\text{Supp}(A)$  is cocomplete, and since we have characterized the finitely presentable objects as the finite supported sets, item 1 shows that  $\text{Supp}(A)$  is lfp.

### Proof of Lemma 3.12

- Epi = surjective: Every surjective supported map is an epimorphism because the forgetful  $V: \text{Supp}(A) \rightarrow \text{Set}$  is faithful. Conversely, every epimorphism is surjective, because  $V$  is left-adjoint (Lemma 3.4).
- Mono = injective: Every injective supported map is a monomorphism because the forgetful  $V: \text{Supp}(A) \rightarrow \text{Set}$  is faithful. Conversely, for a monomorphism  $m: X \rightarrow Y$  in  $\text{Supp}(A)$ , consider  $a, b \in X$  with  $m(a) = m(b)$ . Define

$$Z = 1 = \{0\} \text{ with } s_Z(0) = s_X(a) \cup s_X(b),$$

hence we have supported maps  $f_a, f_b: Z \rightarrow X$  with  $f_a(0) = a$ ,  $f_b(0) = b$ . Thus,  $m \cdot f_a = m \cdot f_b$ . Since  $m$  is monic, we obtain  $f_a = f_b$  and so  $a = b$  as desired.

### Proof of Lemma 3.14

If a bijective map  $f: X \rightarrow Y$  is support reflecting, then  $f^{-1}: Y \rightarrow X$  fulfils  $s_Y = s_X \cdot f^{-1}$  and hence  $f^{-1}$  is a supported map; thus  $f$  is an isomorphism.

Conversely, if  $f: X \rightarrow Y$  is an isomorphism, then  $f$  is bijective and we have a supported map  $f^{-1}: Y \rightarrow X$  with

$$s_Y(f(x)) \supseteq s_X(f^{-1}(f(x))) = s_X(x).$$

Thus,  $s_Y \cdot f = s_X$  in total, i.e.  $f$  is support-reflecting. ◀

### Proof of Lemma 3.16

We prove both statements about regular monomorphisms via the following steps:

1. There is a subobject classifier for support-preserving monomorphisms.
2. Every regular monomorphism is support-preserving.
3. Every support-preserving monomorphism is regular (using item 1).

For the verification:

1. For a support-preserving monomorphism  $m: S \rightarrow X$ , We need to define  $\chi_S: X \rightarrow 2$  such that

$$\begin{array}{ccc} S & \xrightarrow{m} & X \\ \downarrow ! & & \downarrow \chi_S \\ 1 & \xrightarrow{t} & 2 \end{array}$$

is a pullback square. Define  $\chi_S: X \rightarrow 2$  to be the usual characteristic function, i.e. the subobject classifier in **Set**:

$$\chi_S(x) = \begin{cases} 1 & \text{if there is some } s \in S \text{ with } m(s) = x \\ 0 & \text{otherwise.} \end{cases}$$

Hence,  $\chi_S \cdot m = t \cdot !$  since the faithful  $V: \mathbf{Supp}(A) \rightarrow \mathbf{Set}$  sends this to the commutative square of the subobject classifier in **Set**.

For the verification of the universal property of the pullback, consider a cone  $(C, c, d)$ , i.e. supported maps  $!_C: C \rightarrow 1$ ,  $d: C \rightarrow X$  with  $\chi_S \cdot d = t \cdot !_C$ .

$$\begin{array}{ccccc} C & & & & \\ & \searrow^d & & & \\ & & S & \xrightarrow{m} & X \\ & \searrow^u & \downarrow ! & & \downarrow \chi_S \\ & & 1 & \xrightarrow{t} & 2 \\ & \searrow^{!_C} & & & \end{array}$$

The universal property of the subobject classifier in **Set** induces a map  $u: C \rightarrow S$  with  $m \cdot u = d$  and  $! \cdot u = !_C$ . All we need to verify is that  $u$  is a supported map. Since  $m$  is support-preserving, we have  $s_X \cdot m = s_S$  and thus for all  $y \in C$

$$s_S(u(y)) = s_X(m(u(y))) = s_X(d(y)) \subseteq s_C(y)$$

as required.

The uniqueness of  $u$  is clear because  $V: \mathbf{Supp}(A) \rightarrow \mathbf{Set}$  is faithful.

2. Consider a regular monomorphism  $m: S \rightarrow X$ , that is, the equalizer of a parallel pair of morphisms  $f, g: X \rightarrow Y$ :

$$S \xrightarrow{m} X \rightrightarrows Y$$

By Proposition 3.7, the support on the limit  $S$  is

$$s_S(z) \stackrel{(1)}{=} s_X(m(z)) \cup s_Y(f(m(z))) = s_X(m(z))$$

where we use that  $f$  is a supported map in the second equation.

3. Given a support-preserving monomorphism  $m: S \rightarrow X$ , we have a pullback square

$$\begin{array}{ccc} S & \xrightarrow{m} & X \\ \downarrow ! & \swarrow ! & \downarrow \chi_S \\ 1 & \xrightarrow{t} & 2 \end{array}$$

Since 1 is the terminal object, we have that  $m$  is the equalizer of  $\chi_S$  and  $t \cdot !$ :

$$S \xrightarrow{m} X \rightrightarrows 2$$

(Note that this is just the argument that if a monomorphism  $m$  has a subobject classifier, then it is a regular monomorphism).  $\blacktriangleleft$



### Proof of Theorem 3.17

There exist slightly different definitions of *quasitopos* in the literature, which vary in the class of monomorphisms considered.

There is an entire hierarchy of monomorphisms classes [2, Rem 7.76(2)], containing extremal, strong, and regular monomorphisms, among others. Details are not important for the present work, because the entire hierarchy collapses into a single notion whenever regular monomorphisms are closed under composition [2, Prop. 14.14]. This is the case in  $\mathbf{Supp}(A)$ , since regular monos are the support-reflecting monos (Lemma 3.16), a notion clearly closed under composition. Hence, we only need to distinguish monomorphisms and support-reflecting monomorphisms in  $\mathbf{Supp}(A)$ .

For the verification that  $\mathbf{Supp}(A)$  is a quasitopos, we consider the textbooks definitions by Adámek et al. and Johnstone:

**Adámek, Herrlich, and Strecker** [2, Def. 28.7] require that a quasitopos

- is finitely cocomplete,
- is cartesian closed, and
- has representable extremal partial morphisms.

We do not recall the last condition [2, Def. 28.1] in full generality here. The condition simplifies in  $\mathbf{Supp}(A)$  because extremal monomorphisms are the support-reflecting ones and because  $\mathbf{Supp}(A)$  has all pullbacks. Instantiated to our setting, it suffices to verify that for all support-reflecting monomorphisms  $m: X \rightarrow Y$  and supported maps  $f: X \rightarrow B$ , there exists a unique map  $g: Y \rightarrow B + 1$  such that

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ f \downarrow & & \downarrow g \\ B & \xrightarrow{\text{in}_1} & B + 1 \end{array}$$

is a pullback square. We can simply define

$$g: Y \rightarrow B + 1 \quad g(y) = \begin{cases} \text{in}_1(f(x)) & \text{if there exists } x \in X \text{ with } m(x) = y \\ \text{in}_2(0) & \text{otherwise.} \end{cases}$$

Since  $m$  is injective,  $g$  is well-defined and since  $m$  is support-reflecting,  $g$  is a supported map. Pullbacks are finite limits, so formed as in **Set**. Hence, the uniqueness of  $g$  and the verification of the pullback's universal property is just as in **Set** [2, Ex. 28.2(1)].

**Johnstone** [23, Def. 2.6.1] requires a quasitopos to be

- finitely cocomplete (called ‘cocartesian’ in [23]),
- locally cartesian closed (a property slightly stronger than cartesian closed),
- equipped with a subobject classifier for strong monomorphisms (called ‘cocothers’ in [23]).

For the verification, we have already seen that  $\mathbf{Supp}(A)$  is finitely cocomplete and that  $\mathbf{Supp}(A)$  has a regular-subobject classifier 2 (Lemma 3.16), and that strong and regular monomorphisms coincide in  $\mathbf{Supp}(A)$ .

The remaining property to verify is locally cartesian closedness, which means that every slice category  $\mathbf{Supp}(A)/I$  (for arbitrary  $I \in \mathbf{Supp}(A)$ ) is cartesian closed. Recall that an object of  $\mathbf{Supp}(A)/I$  is a pair  $(B, b)$  where  $B \in \mathbf{Supp}(A)$  and  $b: B \rightarrow I$ . The

morphisms in  $\mathbf{Supp}(A)/I$  are commutative triangles  $(f: (B, b) \rightarrow (C, c))$  is  $f: B \rightarrow C$  with  $c \cdot f = b$ . For example, the slice category  $\mathbf{Supp}(A)/1 = \mathbf{Supp}(A)$  is indeed cartesian closed (Proposition 3.10). For a general supported set  $I$ , an object  $(B, b) \in \mathbf{Supp}(A)/I$  consists of an  $I$ -indexed supported set  $B$ . By writing  $B_i := \{x \in B \mid b(x) = i\}$  for component  $i \in I$ , we have that  $(B, b)$  is equivalently a family of supported sets

$$(B_i)_{i \in I} \text{ with } s_{B_i}(x) \supseteq s_I(i) \text{ for all } x \in B_i.$$

Hence,  $\mathbf{Supp}(A)/I$  is equivalent to a product of slice categories:

$$\mathbf{Supp}(A)/I \cong \prod_{i \in I} \mathbf{Supp}(A)/\{i\}$$

In a slice category  $\mathbf{Supp}(A)/\{i\}$  for a singleton  $\{i\}$  (for  $i \in I$ ), all elements of all supported sets have at least support  $s_I(i)$ , and the morphisms  $f: B \rightarrow C$  in  $\mathbf{Supp}(A)/\{i\}$  are just ordinary supported maps  $f: B \rightarrow C$  in  $\mathbf{Supp}(A)$  since  $\{i\}$  is singleton. Thus,  $\mathbf{Supp}(A)/\{i\} \cong \mathbf{Supp}(A \setminus s_I(i))$ , which is cartesian closed by Proposition 3.10. Thus, the category

$$\mathbf{Supp}(A)/I \cong \prod_{i \in I} \mathbf{Supp}(A)/\{i\} \cong \prod_{i \in I} \mathbf{Supp}(A \setminus \{i\}).$$

is the (possibly infinite) product of cartesian closed categories and therefore cartesian closed itself.  $\blacktriangleleft$

## Details for Definition 4.2

Note that  $M/S$  is the image of the composition  $M \rightarrowtail A^A \twoheadrightarrow A^S$  of the submonoid inclusion  $M \rightarrowtail A^A$  and the restriction of maps  $A^A \twoheadrightarrow A^S$  to  $S \subseteq A$ :

$$\begin{array}{ccc} M & \rightarrowtail & A^A \\ \downarrow & & \downarrow \\ M/S & \rightarrowtail & A^S \end{array} \quad (\text{in sets}) \tag{4}$$

## Verification of Definition 4.3

The functor  $M\bullet: \mathbf{Supp}(A) \rightarrow \mathbf{Supp}(A)$  sends  $X$  to

$$M\bullet X = \coprod_{x \in X} M/s(x) = \{([m]_{s_X(x)}, x) \mid x \in X\}$$

and a supported map  $f: X \rightarrow Y$  to the supported map

$$\begin{aligned} M\bullet f: M\bullet X &\rightarrow M\bullet Y \\ (M\bullet f)([m]_{s_X(x)}, x) &= ([m]_{s_Y(f(x))}, f(x)). \end{aligned}$$

For the verification that  $M\bullet$  is a functor, consider a supported map  $f: X \rightarrow Y$  and  $([m]_{s(x)}, x) \in M\bullet X$ . The map  $M\bullet f$  is well-defined, because

$$s_Y(f(x)) \subseteq s_X(x) \quad \text{implies} \quad [m]_{s_X(x)} \subseteq [m]_{s_Y(f(x))},$$

so if  $[m]_{s(x)} = [m']_{s(x)}$ , then also  $[m]_{s_Y(f(x))} = [m']_{s_Y(f(x))}$ . It is straightforward to see that  $M\bullet(-)$  preserves identities and composition.  $\blacktriangleleft$

### Proof of Proposition 4.4

We first show a couple of auxiliary results:

► **Lemma A.2.** *For every  $S \subseteq A$ ,  $\approx_S$  is an equivalence relation and a congruence w.r.t. multiplication from the left. That is,  $m \approx_S m'$  implies  $\ell \cdot m \approx_S \ell \cdot m'$  for all  $m, m', \ell \in M$ .*

**Proof.** By definition,  $\approx_S$  is reflexive, symmetric, and transitive.

We now verify that  $m \approx_S m'$  implies  $\ell \cdot m \approx_S \ell \cdot m'$  for all  $\ell, m, m' \in M$ . If  $m \approx_S m'$ , then  $m(a) = m'(a)$  for all  $a \in S$ . Hence,  $\ell(m(a)) = \ell(m'(a))$  for all  $a \in S$ , i.e.  $\ell \cdot m \approx_S \ell \cdot m'$ . ◀

► **Lemma A.3.**  *$M/S$  is a nominal  $M$ -set via  $\ell \cdot [m]_S := [\ell \cdot m]_S$  and  $m[S]$  supports  $[m]_S$ .*

**Proof.** Since  $\approx_S$  is a congruence for left-multiplication (Lemma A.2), the  $M$ -set structure on  $M/S$  by  $\ell \cdot [m]_S := [\ell \cdot m]_S$  is well-defined. For the verification that  $m[S]$  supports  $[m]_S$ , take  $\ell, \ell' \in M$  with  $\ell \approx_{m[S]} \ell'$ . For all  $a \in S$  we have  $m(a) \in m[S]$  and so  $\ell(m(a)) = \ell'(m(a))$ . Thus,  $[\ell \cdot m]_S = [\ell' \cdot m]_S$  and

$$\ell \cdot [m]_S = [\ell \cdot m]_S = [\ell' \cdot m]_S = \ell' \cdot [m]_S. \quad \blacktriangleleft$$

By Lemma A.3,  $M/\mathbf{s}(x)$  is a nominal  $M$ -set, and hence the disjoint union

$$M \bullet X = \coprod_{x \in X} M/\mathbf{s}(x)$$

is also a nominal  $M$ -set.

For supported map  $f: X \rightarrow Y$ ,  $M \bullet f: M \bullet X \rightarrow M \bullet Y$  is equivariant because

$$\begin{aligned} \ell \cdot (M \bullet f)([m]_{\mathbf{s}_X(x)}, x) &= \ell \cdot ([m]_{\mathbf{s}_Y(f(x))}, f(x)) \\ &= ([\ell \cdot m]_{\mathbf{s}_Y(f(x))}, f(x)) \\ &= (M \bullet f)([\ell \cdot m]_{\mathbf{s}_X(x)}, x) \end{aligned}$$

### Proof of Proposition 4.5

The proposed unit  $\eta_X$  is a supported map  $\eta_X: X \rightarrow UFX$  because

$$\begin{aligned} \mathbf{s}_{UF X}(\eta_X(x)) &= \mathbf{s}_{UF X}([\text{id}_A]_{\mathbf{s}(x)}, x) \\ &= \text{supp}([\text{id}_A]_{\mathbf{s}(x)}, x) && \text{(Lemma 3.3)} \\ &\subseteq \text{id}_A[\mathbf{s}_X(x)] = \mathbf{s}_X(x). && \text{(Lemma A.3)} \end{aligned}$$

For the universal property, consider a supported map  $f: X \rightarrow UY$  (in  $\text{Supp}(A)$ ) for some nominal  $M$ -set  $Y$ . We need to show that there is a unique  $g: M \bullet X \rightarrow Y$  in  $\text{Nom}(M)$  with  $g([\text{id}_A]_{\mathbf{s}(x)}, x) = g(\eta_X(x)) = f(x)$ :

$$\begin{array}{ccc} U(M \bullet X) & \xrightarrow{Ug} & UY \\ \eta_X \uparrow & \nearrow f & \\ X & & \end{array}$$

Define  $g$  by

$$g([m]_{\mathbf{s}(x)}, x) := m \cdot f(x)$$

which clearly fulfils  $g(\eta_X(x)) = f(x)$ . For well-definedness, consider  $m \approx_{s(x)} m'$ , and thus  $m(a) = m'(a)$  for all  $a \in s_X(x)$ . Since  $M$  admits least supports, we have

$$\text{supp}_Y(f(x)) = s_{UY}(f(x)) \subseteq s_X(x),$$

and so  $m(a) = m'(a)$  for all  $a \in \text{supp}_Y(f(x))$ . Thus,  $m \cdot f(x) = m' \cdot f(x)$  (Definition 2.6). Clearly,  $g$  is equivariant:

$$\begin{aligned} g(\ell \cdot ([m]_{s(x)}, x)) &= g([\ell \cdot m]_{s(x)}, x) = \ell \cdot m \cdot f(x) \\ &= \ell \cdot g([m]_{s(x)}, x) \quad \text{for all } \ell \in M. \end{aligned}$$

For uniqueness, consider another equivariant  $g' : M \bullet X \rightarrow Y$  with  $g'([\text{id}_A]_{s(x)}, x) = g'(\eta_X(x)) = f(x)$ . By equivariance, we obtain

$$\begin{aligned} g'([m]_{s(x)}, x) &= g'([m \cdot \text{id}_A]_{s(x)}, x) = g'(m \cdot ([\text{id}_A]_{s(x)}, x)) \\ &= m \cdot g'([\text{id}_A]_{s(x)}, x) \\ &= m \cdot f(x) = g([m]_{s(x)}, x), \end{aligned}$$

i.e.  $g' = g$ . ◀

### Equivalent formulation of Definition 4.6

In proofs, we will use the contrapositive of Definition 4.6:

► **Lemma A.4.**  $M \leq A^A$  is fungible iff for all  $a \in A$  and finite  $R \subseteq A$  we have that

$$(\forall \ell \in M, \ell \approx_R \text{id}_A : \ell(a) = a) \quad \text{implies} \quad a \in R.$$

**Proof.** To see that this is indeed equivalent, we have the following chain of equivalences for all  $a \in A$  and finite  $R \subseteq A$ :

$$\begin{aligned} a \notin R &\text{ implies there exists } \ell \in M \text{ with } \ell \approx_R \text{id}_A \text{ and } \ell(a) \neq a. \\ \Leftrightarrow a \notin R &\implies \exists \ell \in M : \ell \approx_R \text{id}_A \wedge \ell(a) \neq a \\ \Leftrightarrow (\neg \exists \ell \in M : \ell \approx_R \text{id}_A \wedge \ell(a) \neq a) &\implies a \in R \\ \Leftrightarrow (\forall \ell \in M : \neg(\ell \approx_R \text{id}_A) \vee \ell(a) = a) &\implies a \in R \\ \Leftrightarrow (\forall \ell \in M, \ell \approx_R \text{id}_A : \ell(a) = a) &\implies a \in R \end{aligned} \quad \blacktriangleleft$$

### Verification of Example 4.7

For plain Nom and renaming nominal sets, one simply picks some  $b \in A$  fresh for  $R$  and  $\{a\}$  and puts  $\ell = (ab)$ , i.e.  $\ell(a) = b$ ,  $\ell(b) = a$ , and  $\ell(x) = x$  for  $x \notin \{a, b\}$ .

For the order symmetry, a more involved argument is necessary. Venhoek et al. [42, Lemma 5.2] show a property called *homogeneity*: for any two finite  $C \subseteq \mathbb{Q}$ ,  $C' \subseteq \mathbb{Q}'$ , if  $|C| = |C'|$  then there is a  $\pi \in \text{Aut}(\mathbb{Q}, <)$  with  $\pi[C] = C'$ .

Apply this for

$$b := a + \frac{1}{2} \cdot \min_{x \in R} |a - x| \quad C := R \cup \{a\} \quad C' := R \cup \{b\}$$

Since  $a \notin R$ , we have  $b \neq a$  and also  $b \notin R$ . Hence,  $|C| = |C'| = |R| + 1$  and homogeneity provides us with  $\pi \in \text{Aut}(\mathbb{Q}, <)$  with  $\pi[C] = C'$ . Since  $\pi$  is monotone and  $b$  is closer to  $a$  than any element in  $R$ , we necessarily have  $\pi(x) = x$  for all  $x \in R$  and  $\pi(a) = b$ .

Note that for general monoids  $M$  (i.e.  $A$  instead of  $\mathbb{Q}$  and  $M$  instead of  $\text{Aut}(\mathbb{Q}, <)$ ), homogeneity is a different notion to fungibility. Take for instance  $A := \Sigma \times \mathbb{A}$  for a finite set  $\Sigma$ , and let  $M$  be the finite bijections on  $A$  that fix  $\Sigma$ :

$$M := \{f \in \mathfrak{S}_f(\Sigma \times \mathbb{A}) \mid \pi_1(f(\sigma, a)) = \sigma \text{ for all } (\sigma, a) \in A\}$$

Then,  $M$  is clearly fungible: given  $(\sigma, a) \in A \setminus R$ , just consider the transposition  $\ell = ((\sigma, a) (\sigma, b))$  for some fresh  $b$ . However,  $M$  does not fulfil homogeneity. Simply put  $C = \{(\sigma, a)\}$ ,  $C' = \{(\sigma, b)\}$  for distinct  $a, b \in \mathbb{A}$  and some  $\sigma \in \Sigma$ ; then, there is no  $\pi \in M$  with  $\pi[C] = C'$ .  $\blacktriangleleft$

### Proof of Lemma 4.8

After Lemma A.3 it remains to show that  $m[S] \subseteq R$  for every finite  $R \subseteq A$  that supports  $[m]_S$ .

Consider an arbitrary  $\ell \in M$  with  $\ell \approx_R \text{id}_A$ . Since  $R$  supports  $[m]_S$  (Definition 2.6), this implies  $\ell \cdot [m]_S = \text{id}_A \cdot [m]_S$ . Thus,  $[\ell \cdot m]_S = [m]_S$  and so  $\ell(m(a)) = m(a)$  for all  $a \in S$ . So  $M$  and  $R$  fulfil  $\forall \ell \in M, \ell \approx_R \text{id}_A: \ell(m(a)) = m(a)$  for all  $a \in S$ . Hence by Lemma A.4, we obtain  $m(a) \in R$  for all  $a \in S$ . In other words,  $m[S] \subseteq R$ .  $\blacktriangleleft$

### Proof of Theorem 4.9

We have seen in Proposition 4.5 that  $U: \text{Nom}(M) \rightarrow \text{Supp}(A)$  is right-adjoint if  $M$  admits least supports. For the monadicity, we use Beck's theorem (see e.g. [28, Section VI.7]):

► **Theorem A.5** (Beck's theorem). *For every right-adjoint functor  $U: \mathcal{D} \rightarrow \mathcal{C}$  the following are equivalent:*

1.  $U: \mathcal{D} \rightarrow \mathcal{C}$  is monadic.
2. The functor  $U: \mathcal{D} \rightarrow \mathcal{C}$  creates split coequalizers: concretely, if  $f, g: X \rightarrow Y$  in  $\mathcal{D}$  and morphisms  $e, r, t$  in  $\mathcal{C}$  make

$$\begin{array}{ccccc} & & \text{id}_{UY} & & \\ & \xrightarrow{\quad} & \downarrow & \xrightarrow{\quad} & \\ UY & \xrightarrow{t} & UX & \xrightarrow{Uf} & UY \\ e \downarrow & & \downarrow Ug & & \downarrow e \\ E & \xrightarrow{r} & UY & \xrightarrow{e} & E \\ & & \text{id}_E & & \end{array} \quad \text{in } \mathcal{C}$$

commute, then  $f, g$  have a coequalizer  $e': Y \rightarrow E'$  in  $\mathcal{D}$  with  $Ue' = e$ .

Also, we will use the following result on the **supp** maps:

► **Lemma A.6** [19, Lem. 11]. *If  $(X, \cdot)$  has least supports, then we have for all  $m \in M$  and  $x \in X$ :*

1.  $\text{supp}_X(m \cdot x) \subseteq m \cdot \text{supp}_X(x)$ .
2. If  $m$  is invertible, then  $\text{supp}_X(m \cdot x) = m \cdot \text{supp}_X(x)$ .

**Proof of Lemma A.6.** The proof for  $M := \text{Fin}(A)$  can be found in [19, Lem. 11] and straightforwardly adapts to arbitrary monoids  $M$ :

1. We show that  $m[\text{supp}_X(x)]$  is a support of  $m \cdot x$ . Hence, consider  $k, k' \in M$  with  $k \approx_{m[\text{supp}_X(x)]} k'$ . Thus,  $k \cdot m \approx_{\text{supp}_X(x)} k' \cdot m$  and so  $(k \cdot m) \cdot x = (k' \cdot m) \cdot x$  (since  $\text{supp}_X(x)$  supports  $x$ ). So  $k \cdot (m \cdot x) = k' \cdot (m \cdot x)$ , showing that  $m[\text{supp}_X(x)]$  supports  $m \cdot x$ .

2. Let  $m^{-1} \in M$  be the (left and right) inverse of  $m$ . Then we verify the other inclusion direction:

$$\begin{aligned} m \cdot \text{supp}_X(x) &= m \cdot \text{supp}_X(m^{-1} \cdot m \cdot x) \subseteq m \cdot m^{-1} \cdot \text{supp}_X(m \cdot x) && \text{(by item 1)} \\ &= \text{supp}_X(m \cdot x) && \blacktriangleleft \end{aligned}$$

**Main proof of Theorem 4.9.** Let us verify that  $U: \text{Nom}(M) \rightarrow \text{Supp}(A)$  reflects split coequalizers: Consider a parallel pair of equivariant maps  $f, g: X \rightarrow Y$  in  $\text{Nom}(M)$  and consider supported maps  $e, r, t$  such that

$$\begin{array}{ccccc} & & \text{id}_{UY} & & \\ & \swarrow & & \searrow & \\ UY & \xrightarrow{t} & UX & \xrightarrow{Uf} & UY \\ e \downarrow & & \downarrow Ug & & \downarrow e \\ E & \xrightarrow{r} & UY & \xrightarrow{e} & E \\ & \swarrow & \text{id}_E & \searrow & \end{array} \quad \text{in } \text{Supp}(A) \quad (5)$$

commutes.

**Monoid action**  $(E, \cdot)$ . First, define an  $M$ -set structure on  $E$  using the nominal structure on  $Y$ :

$$m \cdot x := e(\underbrace{m \cdot r(x)}_{\text{in } Y}) \quad \text{for all } m \in M, x \in E$$

This definition implies that

$$m \cdot e(y) = e(m \cdot y) \quad \text{for all } m \in M, y \in Y, \quad (6)$$

because we have

$$\begin{aligned} m \cdot e(y) &= e(m \cdot r(e(y))) && \text{(Def.)} \\ &= e(m \cdot Ug(t(y))) && (r \cdot e = Ug \cdot t \text{ (5)}) \\ &= e(Ug(\underbrace{m \cdot t(y)}_{\text{in } X})) && (g \text{ equivariant}) \\ &= e(Uf(m \cdot t(y))) && (e \cdot Ug = e \cdot Uf \text{ (5)}) \\ &= e(m \cdot Uf(t(y))) && (f \text{ equivariant}) \\ &= e(m \cdot y). && (Uf \cdot t = \text{id}_{UY} \text{ (5)}) \end{aligned}$$

Since  $e$  is surjective, the defined  $M$ -set action on  $E$  fulfils the required axioms:

$$\begin{aligned} \text{id}_A \cdot e(y) &= e(\text{id}_A \cdot y) = e(y) \\ (k \cdot m) \cdot e(y) &= e((k \cdot m) \cdot y) = e(k \cdot (m \cdot y)) \\ &= k \cdot e(m \cdot y) = k \cdot (m \cdot e(y)). \end{aligned}$$

Also, equation (6) implies that  $e$  is an equivariant map. By Lemma 2.11, every element of  $E$  is finitely supported. Hence,  $(E, \cdot)$  is a nominal  $M$ -set, and  $e: (Y, \cdot) \rightarrow (E, \cdot)$  is an equivariant map (i.e. it is in  $\text{Nom}(M)$ ) with  $e \cdot f = e \cdot g$ . For clarity, we explicitly write  $(E, \mathbf{s}_E)$  for the originally given supported set and  $(E, \cdot)$  to denote the constructed nominal set.

**Verification of  $U(E, \cdot) = E$ .** We need to verify that  $U(E, \cdot) = E$  and in particular  $\mathbf{s}_{U(E, \cdot)} = \mathbf{s}_E$ . Since  $M$ , admits least supports, we have  $\mathbf{s}_{U(E, \cdot)} = \text{supp}_{(E, \cdot)}$  (Lemma 3.3). We show the two inclusions of  $\mathbf{s}_E(x) = \text{supp}_{(E, \cdot)}(x)$  for all  $x \in E$  separately:



- For the proof of  $\mathbf{s}_E(x) \supseteq \mathbf{supp}_{(E,\cdot)}(x)$ , we verify

$$\begin{aligned}
 \mathbf{supp}_{(E,\cdot)}(x) &= \mathbf{supp}_{(E,\cdot)}(e(r(x))) \\
 &\subseteq \mathbf{supp}_Y(r(x)) && (e \text{ equivariant, Lemma 2.11}) \\
 &= \mathbf{s}_{UY}(r(x)) && (\text{Def. } U, \text{ Lemma 3.3}) \\
 &\subseteq \mathbf{s}_E(x) && (r \text{ supported map})
 \end{aligned}$$

- For the proof of  $\mathbf{s}_E(x) \subseteq \mathbf{supp}_{(E,\cdot)}(x)$ , consider  $a \in \mathbf{s}_E(x)$ . We use the contrapositive formulation of  $M$  being fungible (Lemma A.4) for

$$R := \mathbf{supp}_{(E,\cdot)}(x) \cup (\mathbf{s}_E(x) \setminus \{a\}).$$

Hence, we first show that the condition of Lemma A.4 applies, namely that

$$\forall m \in M: \text{ if } m \approx_R \text{id}_A \text{ then } m(a) = a. \quad (*)$$

Consider  $m \in M$  with  $m \approx_R \text{id}_A$ . By definition,  $R \supseteq \mathbf{supp}_{(E,\cdot)}(x)$  supports  $x$ , hence  $x = m \cdot x$  and

$$\begin{aligned}
 a \in \mathbf{s}_E(x) &= \mathbf{s}_E(m \cdot x) = \mathbf{s}_E(e(m \cdot r(x))) && (\text{Def. } m \cdot x) \\
 &\subseteq \mathbf{s}_{UY}(m \cdot r(x)) && (e \text{ supported map}) \\
 &= \mathbf{supp}_Y(m \cdot r(x)) && (\mathbf{s}_{UY} = \mathbf{supp}_Y \text{ by Lemma 3.3}) \\
 &\subseteq m[\mathbf{supp}_Y(r(x))] && (\text{Lemma A.6}) \\
 &= m[\mathbf{s}_{UY}(r(x))] \subseteq m[\mathbf{s}_E(x)] && (r \text{ supported map})
 \end{aligned}$$

Hence,  $a \in m[\mathbf{s}_E(x)]$ , so there is some  $b \in \mathbf{s}_E(x)$  with  $m(b) = a$ .

- If  $b \in R$ , then  $m(b) = \text{id}_A(b)$  since  $m \approx_R \text{id}_A$ , and so  $a = b$ .
- If  $b \notin R$ , then in particular  $b \notin \mathbf{s}_E(x) \setminus \{a\}$ . Together with  $b \in \mathbf{s}_E(x)$ , this implies  $b = a$ .

In any case, we have deducted  $b = a$  and  $m(a) = m(b) = a$  the conclusion of  $(*)$ . Having proven  $(*)$ , fungibility of  $M$  provides us with

$$a \in R = \mathbf{supp}_{(E,\cdot)}(x) \cup (\mathbf{s}_E(x) \setminus \{a\}).$$

Since  $a$  is clearly not in the right-hand disjunct, it must be in the left disjunct  $a \in \mathbf{supp}_{(E,\cdot)}(x)$ .

Both inclusions together show that  $\mathbf{s}_E = \mathbf{supp}_{(E,\cdot)}$ , and so  $U(E, \cdot) = E$  and automatically  $Ue = e$ .

**Universal Property of  $(E, \cdot)$ .** It remains to show that  $e: Y \rightarrow (E, \cdot)$  is indeed the coequalizer of  $f$  and  $g$  in  $\mathbf{Nom}(M)$ . Since  $e \cdot Uf = e \cdot Ug$  in  $\mathbf{Supp}(A)$ , we also have  $e \cdot f = e \cdot g$  in  $\mathbf{Nom}(M)$ . Given another cocone, i.e. an equivariant map  $h: Y \rightarrow H$  with  $h \cdot f = h \cdot g$ , let  $u: E \rightarrow UH$  be induced by the coequalizer in  $\mathbf{Supp}(A)$ , i.e.  $u$  is the unique supported map with  $u \cdot e = h$ . This supported map is equivariant, because for all  $m \in M$  and  $x \in E$ , we straightforwardly have

$$\begin{aligned}
 m \cdot u(x) &= m \cdot u(e(r(x))) = m \cdot h(r(x)) = h(m \cdot r(x)) \\
 &= u(e(m \cdot r(x))) \stackrel{(6)}{=} u(m \cdot e(r(x))) = u(m \cdot x).
 \end{aligned}$$

Hence,  $u: (E, \cdot) \rightarrow H$  is the desired cocone morphism in  $\mathbf{Nom}(M)$ . Uniqueness of  $u$  is clear because  $U: \mathbf{Nom}(M) \rightarrow \mathbf{Supp}(A)$  is faithful.  $\blacktriangleleft$

### Details for Remark 4.12

The full definitions on the functors mentioned in Remark 4.12 are as follows:

- The left adjoint  $\Sigma: [\mathbb{I}, \mathbf{Set}] \rightarrow \mathbf{Supp}(\mathbb{A})$  sends  $X: \mathcal{P}_f(\mathbb{A}) \rightarrow \mathbf{Set}$  to the supported set

$$\Sigma(X) = \coprod_{S \in \mathcal{P}_f(\mathbb{A})} X(S) \quad \mathbf{s}_{\Sigma(X)}(\mathbf{in}_S(x)) = S$$

and a natural transformation  $f: X \rightarrow Y$  to the supported map

$$\Sigma(f): \Sigma X \rightarrow \Sigma Y \quad \Sigma(f)(\mathbf{in}_S(x)) = \mathbf{in}_S(f(x)).$$

- The right-adjoint  $D: \mathbf{Supp}(\mathbb{A}) \rightarrow [\mathbb{I}, \mathbf{Set}]$  forms down-sets: for a supported set  $X$ , the family  $DX: \mathcal{P}_f \mathbb{A} \rightarrow \mathbf{Set}$  is given by

$$DX(S) = \{x \in X \mid \mathbf{s}_X(x) \subseteq S\}.$$

For a supported map  $f: X \rightarrow Y$  the natural transformation  $Df: DX \rightarrow DY$  is given by

$$(Df)_S: DX(S) \rightarrow DY(S) \quad (Df)_S(x) = f(x).$$

- The composition  $D \cdot U: \mathbf{Nom} \rightarrow [\mathbb{I}, \mathbf{Set}]$  sends a nominal set  $X$  to a family  $DUX$  where for  $S \in \mathcal{P}_f(X)$ ,  $DUX(S) \subseteq X$  contains all elements supported by  $S$ ; this is precisely the functor mentioned by Kurz et al. [27]. ◀

### Proof of Lemma 5.5

For every supported set  $X$  (with  $\mathbf{s}_X$ ),  $\mathcal{B}X$  (with  $\mathbf{s}_{\mathcal{B}X}$ ) is a supported set by definition. A supported map  $f: X \rightarrow Y$  is sent by  $\mathcal{B}$  to the map

$$\mathcal{B}f: \mathcal{B}X \rightarrow \mathcal{B}Y \quad \mathcal{B}f(\lambda.x) = \lambda.f(x).$$

This is again a supported map because

$$\begin{aligned} \mathbf{s}_{\mathcal{B}Y}(\mathcal{B}f(\lambda.x)) &= \mathbf{s}_{\mathcal{B}Y}(\lambda.f(x)) \\ &= \{\varrho(k) \mid \varrho(k+1) \in \mathbf{s}_Y(f(x)), k \in \mathbb{N}\} \\ &\subseteq \{\varrho(k) \mid \varrho(k+1) \in \mathbf{s}_X(x), k \in \mathbb{N}\} \\ &= \mathbf{s}_{\mathcal{B}X}(\lambda.x) \end{aligned} \tag{3}$$

Since  $\mathcal{B}f$  just has  $f$  as its underlying map,  $\mathcal{B}$  preserves identities and compositions of supported maps. ◀

### Proof of Lemma 5.8

In the following, we denote the algebra structure of an Eilenberg-Moore-algebra  $(C, \alpha)$  by

$$\mathbf{struct}(C, \alpha) := \alpha.$$

Since the carrier of an Eilenberg-Moore-algebra is given by  $U(C, \alpha) = C$ , we can write every  $T$ -algebra  $X := (C, \alpha)$  as a morphism

$$TUX \xrightarrow{\mathbf{struct}(X)} UX$$

without needing to ‘unpack’ the tuple  $(C, \alpha)$ . In order to be a strict lifting, the functor  $\bar{H}: \mathbf{EM}(T) \rightarrow \mathbf{EM}(T)$  needs to send a  $T$ -algebra  $X := (C, \alpha)$  to a  $T$ -algebra on  $HC$ , so it only remains to define  $\mathbf{struct}(\bar{H}X)$  as the composition:

$$\begin{array}{ccc} THUX & \xrightarrow{T\phi_X} & TUGX \\ \mathbf{struct}(\bar{H}X) \downarrow \scriptstyle{:=} & & \downarrow \mathbf{struct}(GX) \\ HUX & \xleftarrow{\phi_X^{-1}} & UGX \end{array}$$

Hence, we have that  $\phi_X$  is a homomorphism of  $T$ -algebras by definition and that  $\mathbf{struct}(\bar{H}UX)$  is the unique morphism with this property. The unit  $\eta$  of the monad  $T$  is preserved, because  $\mathbf{struct}(\bar{H}X) \cdot \eta_{HUX} = \mathbf{id}_{HUX}$ .

$$\begin{array}{ccccc} HUX & \xrightarrow{\phi_X} & UGX & & \\ \eta_{HUX} \downarrow \text{Naturality} & & \downarrow \eta_{UGX} & & \\ THUX & \xrightarrow{T\phi_X} & TUGX & & \\ \mathbf{struct}(\bar{H}X) \downarrow \text{Def.} & & \downarrow \mathbf{struct}(GX) & & \\ HUX & \xleftarrow{\phi_X^{-1}} & UGX & \xleftarrow{\mathbf{id}_{UGX}} & \end{array}$$

With the same successive application of the definition of  $\mathbf{struct}(\bar{H}X)$ , it preserves the multiplication  $\mu$  of  $T$ :

$$\begin{array}{ccccc} TTHUX & \xrightarrow{T\mathbf{struct}(\bar{H}X)} & THUX & & \\ \downarrow \mu_{THUX} & \searrow T\phi_X & \downarrow T\phi_X^{-1} & \nearrow T\phi_X & \downarrow \\ & TTUGX & \xrightarrow{T\mathbf{struct}(GX)} & TUGX & \\ \text{Naturality} \downarrow & \mu_{TUGX} \downarrow & \text{Axiom} \downarrow & \downarrow \mathbf{struct}(GX) & \\ THUX & \xrightarrow{T\phi_X} & TUGX & \xrightarrow{\mathbf{struct}(GX)} & UGX \\ & \nearrow \mathbf{struct}(\bar{H}X) & \downarrow \phi_X^{-1} & & \downarrow \\ THUX & \xrightarrow{\mathbf{struct}(\bar{H}X)} & HUX & & \end{array}$$

For the functoriality of  $\bar{H}$ , it is easy to see that every  $T$ -algebra homomorphism  $f: X \rightarrow Y$  is sent to a homomorphism again since  $G$  is functorial:

$$\begin{array}{ccccccc} & & \mathbf{struct}(\bar{H}X) & & & & \\ & \xrightarrow{\quad} & & \xrightarrow{\quad} & & & \\ THUX & \xrightarrow{T\phi_X} & TUGX & \xrightarrow{\mathbf{struct}(GX)} & UGX & \xrightarrow{\phi_X^{-1}} & HUX \\ THUf \downarrow & \text{Naturality} & \downarrow TUGf & & UGf \downarrow & \text{Naturality} & \downarrow HUf \\ THUY & \xrightarrow{T\phi_Y} & TUGY & \xrightarrow{\mathbf{struct}(GY)} & UGY & \xrightarrow{\phi_Y^{-1}} & HUY \\ & \xleftarrow{\quad} & & \xleftarrow{\quad} & & & \end{array}$$

Finally, preservation of identities and composition is immediate. ◀

### Proof of Theorem 5.9

The natural isomorphism  $\phi: \mathcal{BU} \rightarrow U[\mathbb{A}]$  is defined by

$$\begin{aligned} \phi_X: \mathcal{BU}X &\rightarrow U[\mathbb{A}]X \\ \phi_X(\lambda.x) &= \sigma_{\max \text{idx}(x)}^{-1} \cdot \langle \varrho(0) \rangle x \end{aligned}$$

where  $\text{maxidx}(x) \in \mathbb{N}$  and  $\sigma_m \in \mathfrak{S}_f(\mathbb{A})$  ( $m \in \mathbb{N}$ ) are given by:

$$\begin{aligned} \text{maxidx}(x) &= 1 + \max \{n \in \mathbb{N} \mid \varrho(n) \in \mathfrak{s}(x)\} \\ \sigma_m(\varrho(\ell)) &= (\varrho(0) \cdots \varrho(m)) = \begin{cases} \varrho(\ell + 1) & \text{if } \ell < m \\ \varrho(0) & \text{if } \ell = m \\ \varrho(\ell) & \text{else.} \end{cases} \end{aligned}$$

We verify that  $\phi$  is indeed a natural isomorphism between  $\mathcal{B}U$  and  $U[\mathbb{A}]$  by showing that every  $\phi_X$  is a support-reflecting, bijective map (Lemma 3.14), and that it is natural in  $X$ :

**Support-reflecting:** For  $\lambda.x \in \mathcal{B}UX$ , let  $m := \text{maxidx}(x)$ :

$$\begin{aligned} & \mathfrak{s}_{U[\mathbb{A}]X}(\phi_X(\lambda.x)) \\ &= \mathfrak{s}_{U[\mathbb{A}]X}(\sigma_m^{-1} \cdot \langle \varrho(0) \rangle x) \\ &= \sigma_m^{-1} \cdot \text{supp}_{[\mathbb{A}]X}(\langle \varrho(0) \rangle x) && \text{(Lemma A.6)} \\ &= \sigma_m^{-1} \cdot (\text{supp}_X(x) \setminus \varrho(0)) \\ &= \sigma_m^{-1} \cdot \{\varrho(k) \mid \varrho(k) \in \text{supp}_X(x), k \neq 0\} \\ &= \{\sigma_m^{-1}(\varrho(k)) \mid \varrho(k) \in \text{supp}_X(x), k \neq 0\} \\ &= \{\varrho(k-1) \mid \varrho(k) \in \text{supp}_X(x), k \neq 0\} && \text{(Def. } \sigma_m) \\ &= \{\varrho(k) \mid \varrho(k+1) \in \text{supp}_X(x), k \in \mathbb{N}\} \\ &= \{\varrho(k) \mid \varrho(k+1) \in \mathfrak{s}_{UX}(x), k \in \mathbb{N}\} \\ &= \mathfrak{s}_{\mathcal{B}UX}(\lambda.x) && \text{(by (3))} \end{aligned}$$

So  $\phi_X$  is indeed support-reflecting.

**Surjective:** For the surjectivity of  $\phi_X: \mathcal{B}UX \rightarrow U[\mathbb{A}]X$ , consider  $\langle \varrho(k) \rangle y \in U[\mathbb{A}]X$ . With

$$m := \max\{\text{maxidx}(y), k\} + 1$$

and

$$x := ((\varrho(0) \varrho(k+1)) \cdot \sigma_m \cdot y),$$

we will that  $\phi_X(\lambda.x) = \langle \varrho(k) \rangle y$ .

For the definition of  $\phi_X$ , let us first investigate  $\text{maxidx}(x)$ :

$$\text{supp}(\sigma_m \cdot y) = \{\varrho(\ell + 1) \mid \varrho(\ell) \in \text{supp}(y)\}$$

and moreover

$$\text{supp}(x) \subseteq \{\varrho(0)\} \cup \{\varrho(\ell + 1) \mid \varrho(\ell) \in \text{supp}(y)\}$$

Hence,  $\text{maxidx}(x) \leq \text{maxidx}(y) < m$ . Every  $\varrho(\ell)$  in the support of  $\langle \varrho(0) \rangle x$  is in the range  $1 \leq \ell < m$ , so

$$\sigma_{\text{maxidx}(x)}^{-1}(\varrho(\ell)) = \varrho(\ell) - 1 = \sigma_m^{-1}(\varrho(\ell))$$

and  $\sigma_m^{-1} \approx_S \sigma_{\text{maxidx}(x)}^{-1}$  for  $S := \text{supp}(\langle \varrho(0) \rangle x)$  (Definition 2.5). Since  $S$  supports  $\langle \varrho(0) \rangle x$  (Definition 2.6), these permutations act identically on it, i.e.

$$\sigma_{\text{maxidx}(x)}^{-1} \cdot \langle \varrho(0) \rangle x = \sigma_m^{-1} \cdot \langle \varrho(0) \rangle x$$

and we verify:

$$\begin{aligned}\phi_X(\lambda.x) &= \sigma_{\max \text{idx}(x)}^{-1} \cdot \langle \varrho(0) \rangle x \\ &= \sigma_m^{-1} \cdot \langle \varrho(0) \rangle x\end{aligned}\tag{Def.  $\phi_X$ }$$

Note that  $\varrho(0) \notin \text{supp}(\sigma_m \cdot y)$ , so  $\varrho(k+1) \notin \text{supp}(x)$ , i.e.  $\varrho(k+1)$  is fresh for  $x$ . Hence,

$$\langle \varrho(0), x \rangle \sim_\alpha \langle \varrho(k+1), (\varrho(0) \varrho(k+1)) \cdot x \rangle$$

and we conclude:

$$\begin{aligned}\phi_X(\lambda.x) &= \sigma_m^{-1} \cdot \langle \varrho(0) \rangle x \\ &= \sigma_m^{-1} \cdot \langle \varrho(k+1) \rangle ((\varrho(0) \varrho(k+1)) \cdot x) \\ &= \sigma_m^{-1} \cdot \langle \varrho(k+1) \rangle (\sigma_m \cdot y) && \text{(Def. } x) \\ &= \langle \sigma_m^{-1}(\varrho(k+1)) \rangle (\sigma_m^{-1} \cdot \sigma_m \cdot y) && \text{(Def. } [\mathbb{A}]) \\ &= \langle \sigma_m^{-1}(\varrho(k+1)) \rangle (y) \\ &= \langle \varrho(k) \rangle (y) && (k < m)\end{aligned}$$

**Injective:** Consider  $x, y$  with

$$\begin{aligned}\phi_X(\lambda.x) &= \sigma_{\max \text{idx}(x)}^{-1} \cdot \langle \varrho(0) \rangle x \\ &= \sigma_{\max \text{idx}(y)}^{-1} \cdot \langle \varrho(0) \rangle y = \phi_X(\lambda.y).\end{aligned}$$

So by assumption already,  $\langle \varrho(0) \rangle x, \langle \varrho(0) \rangle y \in [\mathbb{A}]X$  are in the same orbit. Since we have proven the support-reflectivity of  $\phi_X$  already, we have

$$\begin{aligned}\mathbf{s}_{\mathcal{B}UX}(\lambda.x) &= \text{supp}_{[\mathbb{A}]X}(\phi_X(\lambda.x)) \\ &= \text{supp}_{[\mathbb{A}]X}(\phi_X(\lambda.y)) = \mathbf{s}_{\mathcal{B}UX}(\lambda.y).\end{aligned}$$

By the definition of  $\mathbf{s}_{\mathcal{B}UX}$ , we have

$$\mathbf{s}_{UX}(x) \cup \varrho(0) = \mathbf{s}_{UX}(y) \cup \varrho(0).$$

Since  $\langle \varrho(0) \rangle x, \langle \varrho(0) \rangle y$  are in the same orbit,  $\varrho(0) \in \text{supp}(x)$  iff  $\varrho(0) \in \text{supp}(y)$ . Hence,

$$\mathbf{s}_{UX}(x) = \mathbf{s}_{UX}(y)$$

and  $\max \text{idx}(x) = \max \text{idx}(y)$ . So in fact  $\langle \varrho(0) \rangle x = \langle \varrho(0) \rangle y$  and by [33, Lemma 4.2] also  $x = y$  and  $\lambda.x = \lambda.y$  as desired.

**Natural:** We need to verify that for every equivariant map  $f: X \rightarrow Y$ , the diagram

$$\begin{array}{ccc}\mathcal{B}UX & \xrightarrow{\phi_X} & U[\mathbb{A}]X \\ \mathcal{B}Uf \downarrow & & \downarrow U[\mathbb{A}]f \\ \mathcal{B}UY & \xrightarrow{\phi_Y} & U[\mathbb{A}]Y\end{array}$$

commutes. To this end, we verify:

$$\begin{aligned}U[\mathbb{A}]f(\phi_X(\lambda.x)) \\ = U[\mathbb{A}]f(\sigma_{\max \text{idx}(x)}^{-1} \cdot \langle \varrho(0) \rangle x)\end{aligned}$$

$$\begin{aligned}
&= \sigma_{\text{maxidx}(x)}^{-1} \cdot U[\mathbb{A}]f(\langle \varrho(0) \rangle x) && ([\mathbb{A}]f \text{ equivariant}) \\
&= \sigma_{\text{maxidx}(x)}^{-1} \cdot \langle \varrho(0) \rangle (f(x))
\end{aligned}$$

Since every  $\varrho(\ell) \in \text{supp}(\langle \varrho(0) \rangle (f(x)))$  is in the range

$$1 \leq \ell < \text{maxidx}(f(x)) \leq \text{maxidx}(x)$$

we have

$$\sigma_{\text{maxidx}(x)}^{-1}(\varrho(\ell)) = \varrho(\ell - 1) = \sigma_{\text{maxidx}(f(x))}^{-1}(\varrho(\ell)).$$

Thus, we can conclude:

$$\begin{aligned}
&U[\mathbb{A}]f(\phi_X(\lambda.x)) \\
&= \sigma_{\text{maxidx}(x)}^{-1} \cdot \langle \varrho(0) \rangle (f(x)) \\
&= \sigma_{\text{maxidx}(f(x))}^{-1} \cdot \langle \varrho(0) \rangle (f(x)) \\
&= \phi_Y(\lambda.f(x)) && (\text{Def. } \phi_Y) \\
&= \phi_Y(BUf(\lambda.x))
\end{aligned}$$

## Details for Construction 6.1

Apply the adjunction of the Eilenberg-Moore category

$$\text{EM}(M\bullet(-)) = \text{Nom}$$

to

$$Q \xrightarrow{c} H(M\bullet Q) \xrightarrow{\phi_Q} UG(M\bullet Q).$$

This yielding the desired equivariant map

$$M\bullet Q \xrightarrow{d} G(M\bullet Q).$$

If  $Q$  is finitely presentable (cf. Proposition 3.11), then by [3, Corollary 2.75],  $M\bullet Q$  is finitely presentable in  $\text{Nom}$ , i.e. orbit-finite.  $\blacktriangleleft$

## Proof of Proposition 6.2

► **Definition A.7.** *The uniformly supported powerset functor  $\mathcal{P}_{\text{ufs}}: \text{Nom} \rightarrow \text{Nom}$  is given by*

$$\mathcal{P}_{\text{ufs}}(X) = \{S \subseteq X \mid \bigcup_{x \in S} \text{supp}_X(x) \text{ is finite}\}$$

*On equivariant maps  $f$ ,  $\mathcal{P}_{\text{ufs}}f$  performs the direct image.*

For the verification of Proposition 6.2:

- Finite powerset  $\mathcal{P}_f: \text{Nom} \rightarrow \text{Nom}$  has precisely the same definition in  $\text{Nom}$  and  $\text{Supp}(\mathbb{A})$ . In fact, they both lift from  $\mathcal{P}_f$  on  $\text{Set}$ .

- The uniformly supported powerset functor on  $\mathbf{Nom}$  is a lifting of the obvious functor  $\mathcal{P}'_{\text{ufs}}$  on  $\mathbf{Supp}(\mathbb{A})$ :

$$\begin{aligned}\mathcal{P}'_{\text{ufs}}: \mathbf{Supp}(\mathbb{A}) &\rightarrow \mathbf{Supp}(\mathbb{A}) \\ \mathcal{P}'_{\text{ufs}}(X) &= \{S \subseteq X \mid \bigcup_{x \in S} s_X(x) \text{ is finite}\} \\ s_{\mathcal{P}'_{\text{ufs}}(X)}(S) &= \bigcup_{x \in S} s_X(x)\end{aligned}$$

Since  $s_{UX}(x) = \text{supp}_X(x)$  for all nominal sets  $X$ , we have that

$$\mathcal{P}'_{\text{ufs}}UX = U\mathcal{P}_{\text{ufs}}X$$

- Every constant functor  $C: \mathbf{Nom} \rightarrow \mathbf{Nom}$ ,  $CX = N$  is the lifting of the constant functor  $C': \mathbf{Supp}(\mathbb{A}) \rightarrow \mathbf{Supp}(\mathbb{A})$ ,  $C'X = UN$ .
- If the family  $G_i: \mathbf{Nom} \rightarrow \mathbf{Nom}$ ,  $i \in I$  are liftings of the functors  $H_i: \mathbf{Supp}(\mathbb{A}) \rightarrow \mathbf{Supp}(\mathbb{A})$ , then
  - $\prod_{i \in I} G_i$  is the lifting of  $\prod_{i \in I} H_i$ , because the right-adjoint  $U: \mathbf{Nom} \rightarrow \mathbf{Supp}(\mathbb{A})$  preserves products:

$$U \prod_{i \in I} G_i \cong \prod_{i \in I} UG_i \cong \prod_{i \in I} H_i U$$

- Coproducts are given by disjoint union, both in  $\mathbf{Nom}$  and in  $\mathbf{Supp}(\mathbb{A})$ . Hence,  $U: \mathbf{Nom} \rightarrow \mathbf{Supp}(\mathbb{A})$  preserves coproducts and so  $\prod_{i \in I} G_i$  is the lifting of  $\prod_{i \in I} H_i$ :

$$U \prod_{i \in I} G_i \cong \prod_{i \in I} UG_i \cong \prod_{i \in I} H_i U$$

## Details for Example 6.5

In Example 6.5, we simplified the presentation a lot, especially the definition of guards; the full definitions are as follows.

A *(relational) signature* is a set  $\mathcal{R}$  with an arity map  $\text{ar}: \mathcal{R} \rightarrow \mathbb{N}$ ; The nominal set of *guards*  $\mathcal{G}$  over this relational signature  $\mathcal{R}$  is

$$\mathcal{G} = \left(2 \times \prod_{r \in \mathcal{R}} \mathbb{A}^{\text{ar}(r)}\right)^*$$

Intuitively, the words  $(-)^*$  represent conjunction, 2 encodes a possible negation, the coproduct over  $\mathcal{R}$  is the set of relational terms over  $\mathbb{A}$ . Hence, a guard  $g \in \mathcal{G}$  is a finite conjunction of possibly negated terms of the form  $r(a_1, \dots, a_n)$ ,  $n = \text{ar}(r)$ ,  $a_1, \dots, a_n \in \mathbb{A}$ . Hence, the support of a guard  $g \in \mathcal{G}$  is the set of register names  $a \in \mathbb{A}$  that appear in it. Then, a register automaton (in the style of [8]) for  $\mathcal{R}$  is a tuple  $(Q, q_0, f, \Gamma)$  where

1.  $Q$  is a finite supported set of *locations*
2.  $q_0: 1 \rightarrow Q$  is a supported map, the *initial location*,
3.  $f: Q \rightarrow 2$  is a supported map, indicating *finality*,
4.  $\Gamma: Q \rightarrow \mathcal{BP}_f(\mathcal{G} \times \mathfrak{S}_f(\mathbb{A}) \bullet Q)$  is a supported map, the *transitions*.

Here, the properties of supported sets and maps encode precisely the coherence conditions of register automata:



1. Every location has access to finitely many registers.
2. The initial location has all registers uninitialized.
3. There is no side condition on  $f$  since 2 is a set.
4. Every transition  $q \xrightarrow{g, \pi} q'$  is in such a way that:
  - the guard  $g$  may only use registers from  $\mathbf{s}(q)$  and the input data value.
  - $\pi$  tells for each register of  $q'$ ,  $\pi$  its value (from the previous register contents or the input data value).

If the register contents and the input value satisfy the guard  $g$ , then  $q$  makes a transition to location  $q'$  with the registers rearranged with respect to the injective map

$$\pi: \mathbf{s}_Q(q') \mapsto \{\varrho(0)\} \cup \{\text{old}(a) \mid a \in \mathbf{s}_Q(q)\}$$

where  $\varrho(0)$  is the input value and  $\text{old}(\varrho(k)) = \varrho(k+1)$  refers to the old registers contents, i.e.  $\text{old}(\varrho(k))$  has the role that  $\varrho(k)$  had above the binder  $\mathcal{B}$ .

In one line, we have an  $HT$ -coalgebra on  $Q$  for

$$HX = 2 \times \mathcal{BP}_f(\mathcal{G} \times X) \quad \text{and} \quad TX = \mathfrak{S}_f(\mathbb{A}) \bullet X.$$

Since  $\mathcal{G}$  is a nominal set, we can consider it as a constant functor  $C_{\mathcal{G}}: \mathbf{Nom} \rightarrow \mathbf{Nom}$ ,  $C_{\mathcal{G}}(X) = \mathcal{G}$ . This functor is the lifting of the constant  $C_{U\mathcal{G}}: \mathbf{Supp}(\mathbb{A}) \rightarrow \mathbf{Supp}(\mathbb{A})$ ,  $C_{U\mathcal{G}}(X) = U\mathcal{G}$  because  $U \circ C_{\mathcal{G}} = C_{U\mathcal{G}} \circ U$ . By Proposition 6.2,  $H$  lifts to a functor

$$G: \mathbf{Nom} \rightarrow \mathbf{Nom} \quad G(X) = 2 \times [\mathbb{A}] \mathcal{P}_f(\mathcal{G} \times X)$$

and so Construction 6.1 transforms a register automaton into a nominal set  $\bar{Q}$  and an equivariant map

$$d: \bar{Q} \longrightarrow 2 \times [\mathbb{A}] \mathcal{P}_f(\mathcal{G} \times \bar{Q})$$

that is, a nominal automaton. ◀