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Kronecker's approximation theorem

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Dedicated to the memory of Johannes Gaultherus van der Corput on the 125th anniversary of his birth

Abstract

We review the various proofs of Kronecker's theorem concerning inhomogeneous Diophantine approximation, we discuss in detail the quantitative approaches of Turán (1960) and Chen (2000), and we derive strong localized versions of these theorems.

Keywords: Kronecker; Chebyshev; Fejér; Peak function

1. Historical review

Kronecker [31] addressed a general problem and mentioned that Jacobi [29] had already discussed a special case a half-century earlier. By elementary algebraic/arithmetic reasoning he reached a general result, which takes two forms.

Theorem A. *Let A be an $N \times M$ matrix with real entries, and let $\boldsymbol{\alpha} \in \mathbb{R}^N$. Then the following two assertions are equivalent:*

1. *For every $\varepsilon > 0$ there exists a point $\boldsymbol{x} \in \mathbb{R}^M$ such that each of the N coordinates of the vector $A\boldsymbol{x} - \boldsymbol{\alpha}$ are within ε of an integer.*
2. *If $\boldsymbol{u} \in \mathbb{Z}^N$ is a lattice point such that $\boldsymbol{u}A = \mathbf{0}$, then $\boldsymbol{u} \cdot \boldsymbol{\alpha} = 0$.*

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Theorem B. *Let A be an $N \times M$ matrix with real entries, and let $\boldsymbol{\alpha} \in \mathbb{R}^N$. Then the following two assertions are equivalent:*

1. *For every $\varepsilon > 0$ there exists a lattice point $\boldsymbol{x} \in \mathbb{Z}^M$ such that each of the N coordinates of the vector $A\boldsymbol{x} - \boldsymbol{\alpha}$ are within ε of an integer.*
2. *If $\boldsymbol{u} \in \mathbb{Z}^N$ is a lattice point such that $\boldsymbol{u}A \in \mathbb{Z}^N$, then $\boldsymbol{u} \cdot \boldsymbol{\alpha}$ is an integer.*

An exposition of these theorems is found in Cassels [16, pp. 53–59]. As to the distance from a real number x to the nearest integer, we shall find it convenient to write $\|x\| = \min_{n\in\mathbb{Z}} |x - n|$.

The special case $M = 1$ of the above is especially memorable, even more so when there are no linear dependencies. Thus we have

A(N) Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers that are linearly independent over $\mathbb{Q}$, and let $\alpha_1, \alpha_2, \ldots, \alpha_N$ be arbitrary real numbers. Then for any $\varepsilon > 0$, there exists a real number t such that $\|\lambda_n t - \alpha_n\| < \varepsilon$ for $n = 1, 2, \ldots, N$.

B(N) Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers such that $1, \lambda_1, \lambda_2, \ldots, \lambda_N$ are linearly independent over $\mathbb{Q}$, and let $\alpha_1, \alpha_2, \ldots, \alpha_N$ be arbitrary real numbers. Then for any $\varepsilon > 0$, there exists an integer t such that $\|\lambda_n t - \alpha_n\| < \varepsilon$ for $n = 1, 2, \ldots, N$.

For the purposes of our present discussion, we take our title to refer to these last two assertions (which are easily seen to be equivalent).

Hardy and Littlewood [26] gave several proofs for the case $N = 1$, and commented on the merits of the various approaches. Then they gave a simple inductive proof of A(N). Then, still using only elementary techniques, they showed that if K is a positive integer, and $\lambda_1, \lambda_2, \ldots, \lambda_N$ are real numbers such that $1, \lambda_1, \lambda_2, \ldots, \lambda_N$ are linearly independent over $\mathbb{Q}$, then as t runs over integers the fractional parts of the KN numbers $\lambda_n t^k$, $1 \le k \le K$, $1 \le n \le N$, considered as points in $\mathbb{T}^{KN}$, are dense. Here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle group of real numbers modulo 1.

Weyl [51] devised a criterion for uniform distribution, which asserts that a sequence of points $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots$ in $\mathbb{T}^N$ is uniformly distributed if and only if

$$\sum_{k=1}^{K} e(\boldsymbol{u} \cdot \boldsymbol{p}_k) = o(K) \tag{1}$$

as $K \to \infty$ for every nonzero lattice point $\boldsymbol{u} \in \mathbb{Z}^N$. We recall that if α is a real number, then

$$\Big|\sum_{k=1}^{K} e(k\alpha)\Big| \le \min\Big(K, \frac{1}{|\sin \pi\alpha|}\Big) \le \min\Big(K, \frac{1}{2\|\alpha\|}\Big). \tag{2}$$

Thus if in Weyl's criterion we take $\boldsymbol{p}_k = (k\lambda_1, k\lambda_2, \ldots, k\lambda_N)$, then $\boldsymbol{u}\cdot\boldsymbol{p}_k = k(u_1\lambda_1 + \cdots + u_N\lambda_N)$, and so the sum (1) that arises in Weyl's criterion is of the form of the sum in (2) with $\alpha = u_1\lambda_1 + \cdots + u_N\lambda_N$. Thus Weyl's criterion is satisfied if and only if this α is never an integer, which is to say if $1, \lambda_1, \lambda_2, \ldots, \lambda_N$ are linearly independent over $\mathbb{Q}$. Hence the condition under which the points are dense are precisely the conditions under which they are uniformly distributed.

Many further proofs of the approximation theorem have been given. Landau [33] gave a short trigonometric proof (in German and Hebrew). Lettenmeyer [34] gave a very elegant geometric proof that Hardy communicated to *PLMS*, and his argument is reproduced in Hardy and Wright [27]. Bohr [5–7] gave trigonometric proofs. Bohr and Jessen [10–12] gave trigonometric proofs of which the latter two are of considerable interest because they have contributed to

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subsequent developments. Skolem [22] gave a simple inductive proof, which is reproduced in Hardy and Wright [27]. Bohr [8,9] gave two additional trigonometric proofs. Skolem [43] gave a simple elementary proof, and Mahler [35] gave a simple proof based on the geometry of numbers, but Kueh [32] must hold the record for the simplest proof of all. He observes that A(1) is true because one can take $t = \alpha_1/\lambda_1$. Then he uses Dirichlet's theorem on simultaneous homogeneous approximation to show that A(N) implies B(N). Finally with a simple change of variables he shows that B(N) implies A($N + 1$), which completes the proof in less than a page.

We now turn to variants of the approximation theorem, which we shall refer to as *extended*, *quantitative*, and *localized*.

As might be inferred from the shape of Theorems A and B, one can relax the hypothesis that the λ_n are linearly independent, provided that any linear dependence among the λ_n is also satisfied by the α_n. That is, if $\boldsymbol{u} \in \mathbb{Z}^N$ and $\boldsymbol{u} \cdot \boldsymbol{\lambda} = 0$, then $\boldsymbol{u} \cdot \boldsymbol{\alpha} = 0$. This is called the *extended* Kronecker approximation theorem. Bohr [6] noted that his trigonometric proof applies in this extended setting, and this remark was renewed in Bohr and Jessen [12].

Let M be a positive integer, and suppose that instead of requiring that the λ_n are linearly independent, we only suppose that $\boldsymbol{u} \cdot \boldsymbol{\lambda} \neq 0$ for those nonzero lattice points $\boldsymbol{u}$ for which $|u_n| \le M$ for all n. Instead of asking for an approximation for all $\varepsilon > 0$, we ask only for an approximation to within $\varepsilon(M, N)$, where $\varepsilon(M, N) \to 0$ when N is fixed and $M \to \infty$. We call this a *quantitative* form of the Kronecker approximation theorem. The first results in this direction, due to Thomas [45] and Bacon [1] were of the form

$$\|t\lambda_n - \alpha_n\| \le \frac{c(N)}{M}.$$

Thomas did not specify the dependence of $c(N)$ on N, and Bacon had rather heavy dependence, worse than 2^{N^3}. Subsequent quantitative results have been given by Khintchine [30] and Cassels [16, pp. 97–99]. These authors employed tools from the geometry of numbers, but more recently Chen [19,20] adopted the trigonometric method of Bohr and Jessen [11], and has shown that there exists a real t for which

$$\sum_{n=1}^{N} \|t\lambda_n - \alpha_n\|^2 < c\frac{N}{M^2} \tag{3}$$

where c is an absolute constant. By means of examples he has shown that this is false if c is sufficiently small, so this result is best possible apart from determining the optimal c. Chen's approach depends on a trigonometric extremal problem that was solved by Fejér [23, pp. 77–79]. We give a brief account of Fejér's result in Section 3, and then discuss Chen's method in greater detail in Section 5.

Because linear independence is only a qualitative concept, there is no possibility of bounding the size of the t that gives a solution, unless one defines some quantitative measure of linear independence. However, if one is given some information as to the distribution of the linear forms $\boldsymbol{u} \cdot \boldsymbol{\lambda}$, with coefficients up to a given size, and in particular how close to 0 such a linear form may be, then one should be able to specify a number T such that there will be a solution in any interval of length T. We call such a form of Kronecker's theorem *localized*. The possibility of such a result was hinted at by Landau [33], and a weak result in this direction was given by Thomas [45]. The numbers $\log p$ are highly linearly independent and sums of the form $\sum_p a_p p^{-it}$ are very important in analytic number theory. Turán [47] adopted the method of Bohr and Jessen [12] to obtain a fairly sharp localized Kronecker theorem when the frequencies

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are the numbers $\frac{1}{2\pi}\log p$. His method addresses the sup norm $\max_n \|t\lambda_n - \alpha_n\|$ in contrast to the ℓ^2 norm of Chen (3), and involves a trigonometric extremal problem that was solved by Chebyshev [17]. In Section 2 we derive Chebyshev's result, and then in Section 4 we discuss Turán's method in more detail.

2. Chebyshev's extremal trigonometric polynomial

In this section, T_n denotes a Chebyshev polynomial of the first kind, which is to say that T_n is the unique polynomial such that $\cos n\theta = T_n(\cos\theta)$ for all θ. Suppose that $0 < \varepsilon < 1/2$, and that a trigonometric polynomial $U(x)$ with period 1 and degree not exceeding N has the property that $-1 \le U(x) \le 1$ for $\varepsilon \le x \le 1-\varepsilon$. Then how large can $U(0)$ be? This extremal problem is settled in the following theorem of Chebyshev [17, Sections 6–8].

Theorem 2.1. *Suppose that $0 < \varepsilon < 1/2$ and that N is a positive integer. Put*

$$U_0(x) = T_{2N}\Big(\frac{\cos\pi x}{\cos\pi\varepsilon}\Big).$$

If $U(x)$ is a trigonometric polynomial with period 1 and degree not exceeding N such that $-1 \le U(x) \le 1$ for $\varepsilon \le x \le 1-\varepsilon$, then $U(x) \le U_0(x)$ for $-\varepsilon \le x \le \varepsilon$.

This is an example of a family of such results, of which the most prominent is probably Marcel Riesz's Lemma; see M. Riesz [41] or Borwein and Erdélyi [14, p. 237].

Proof. Suppose that there is an $x_0 \in (-\varepsilon, \varepsilon)$ such that $U(x) > U_0(x)$. Choose η, $0 < \eta < 1$, so close to 1 that $\eta U(x_0) > U_0(x_0)$. Set $U_1(x) = \eta U(x)$. Thus $-1 < U_1(x) < 1$ for $\varepsilon \le x \le 1-\varepsilon$. Let numbers x_n be chosen so that $\varepsilon = x_0 < x_1 < \cdots < x_{2N} = 1-\varepsilon$ and $U_0(x_n) = (-1)^n$. Put $U_2(x) = U_0(x) - U_1(x)$. Then $U_2(x_n) > 0$ if n is even, and $U_2(x_n) < 0$ if n is odd, so $U_2(x)$ has at least $2N$ zeros in the interval $(\varepsilon, 1-\varepsilon)$. Since a trigonometric polynomial of degree $\le N$ has at most $2N$ zeros, it follows that U_2 can have no further zero. But $U_2(\varepsilon) > 0$ and $U_2(x_0) < 0$, so U_2 has a zero in $(-\varepsilon, \varepsilon)$, a contradiction. □

With a little more care one can show that $U(x) < U_0(x)$ for $-\varepsilon < x < \varepsilon$ unless $U = U_0$ identically (see Fig. 1).

Corollary 2.2. *Suppose that $0 < \varepsilon < 1/2$, that N is a positive integer, and put*

$$V_0(x) = \Big(T_N\Big(\frac{\cos\pi x}{\cos\pi\varepsilon}\Big)\Big)^2.$$

If $V(x)$ is a trigonometric polynomial of period 1 and degree not exceeding N such that $0 \le V(x) \le 1$ for $\varepsilon \le x \le 1-\varepsilon$, then $V(x) \le V_0(x)$ for $-\varepsilon \le x \le \varepsilon$.

Proof. Take $U(x) = 2V(x) - 1$. Then

$$2V(x) - 1 \le U_0(x) = 2V_0(x) - 1$$

since $T_{2N}(x) = 2T_N(x)^2 - 1$. □

To determine the size of $V_0(0)$ we recall that the Chebyshev polynomials are generated by the second order linear recurrence

$$T_{N+1}(x) = 2xT_N(x) - T_{N-1}(x)$$

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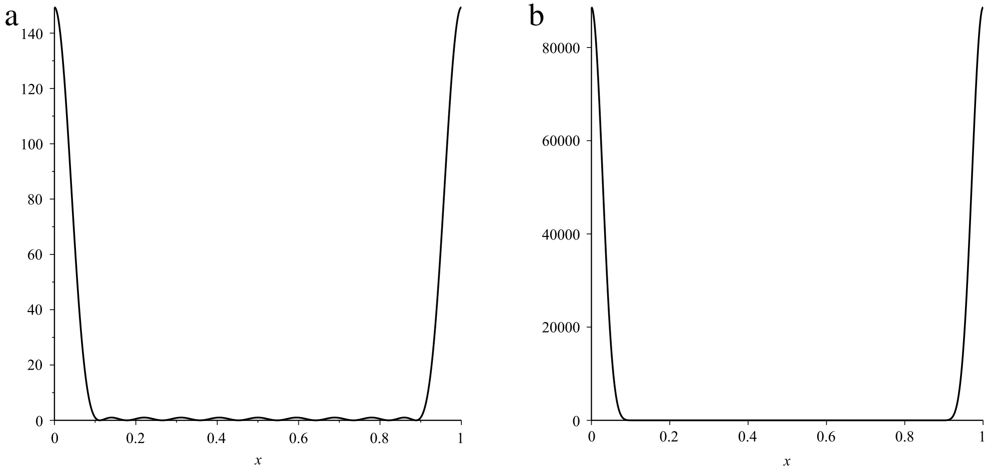

Fig. 1. Plots of $V_0(x)$ for $\varepsilon = 0.1$ and (a) $N = 10$; (b) $N = 20$.

with the initial conditions $T_0(x) = 1$ and $T_1(x) = x$. On solving this linear recurrence we discover that if $x \ge 1$, then

$$T_N(x) = \frac{1}{2}\big(x + \sqrt{x^2 - 1}\big)^N + \frac{1}{2}\big(x - \sqrt{x^2 - 1}\big)^N.$$

On setting $x = \sec \pi\varepsilon$ we find that

$$\begin{aligned} V_0(0) &= \frac{1}{4}(\sec \pi\varepsilon + \tan \pi\varepsilon)^{2N} + \frac{1}{2} + \frac{1}{4}(\sec \pi\varepsilon + \tan \pi\varepsilon)^{-2N} \\ &\ge \frac{1}{4}(\sec \pi\varepsilon + \tan \pi\varepsilon)^{2N}. \end{aligned}$$

Now if $0 \le x < \pi/2$, then

$$\log(\sec x + \tan x) = \int_0^x \sec u \, du \ge \int_0^x 1 \, du = x,$$

so $\sec \pi\varepsilon + \tan \pi\varepsilon \ge e^{\pi\varepsilon}$, and hence

$$V_0(0) \ge \frac{1}{4} e^{2\pi N\varepsilon}. \tag{4}$$

We are also interested in the question of how large $\int_0^1 V(x)\,dx$ can be, if V is a nonnegative trigonometric polynomial of degree not exceeding N that has the property that $V(x) \le 1$ for $\varepsilon \le x \le 1 - \varepsilon$. It may be that $V_0(x)$ is extremal for this problem also, but even if it is not, it is clear that

$$\max_V \int_0^1 V(x)\,dx \le \int_0^1 V_0(x)\,dx + 1, \tag{5}$$

so in any case V_0 is close to optimal. We note that if $T(x)$ is a nonnegative trigonometric polynomial with period 1 and of degree not exceeding N, then

$$T(0) \le \sum_{k=0}^{N} T(k/(N+1)) = (N+1) \int_0^1 T(x)\,dx.$$

Thus the mean value of V_0 satisfies the inequalities

$$\frac{V_0(0)}{N+1} \le \int_0^1 V_0(x)\,dx \le 2\varepsilon V_0(0) + 1. \tag{6}$$

For use in Section 4 we formulate the following further

Corollary 2.3. *Let M be a positive integer and suppose that $0 < \varepsilon < 1/2$. There is a trigonometric polynomial $P(u)$ of the form*

$$P(u) = \sum_{m=0}^{M} c(m)e(mu) \tag{7}$$

such that $\int_0^1 |P(u)|^2 = 1$ and

$$|P(u)|^2 \le \frac{4(M+1)}{\exp(2\pi M\varepsilon)}$$

when $\|u\| > \varepsilon$.

Proof. We take

$$P(u) = ce^{i\pi Mu} T_M\Big(\frac{\cos \pi u}{\cos \pi \varepsilon}\Big)$$

where c is chosen so that $\int_0^1 |P(u)|^2\,du = 1$. Write

$$T_M(x) = \sum_{k=0}^{M} a_k x^k.$$

We recall that $T_M(u)$ is an even or odd function of u according as M is even or odd. Thus if $a_k x^k$ is a monomial term of T_M with $a_k \ne 0$, then $k \equiv M \pmod 2$, and

$$\begin{aligned} e^{i\pi Mu}(\cos \pi u)^k &= e^{i\pi Mu} 2^{-k} \sum_{j=0}^{k} \binom{k}{j} e^{(2j-k)i\pi u} \\ &= 2^{-k} \sum_{j=0}^{k} \binom{k}{j} e\Big(\Big(j + \frac{M-k}{2}\Big)u\Big). \end{aligned}$$

Since $0 \le j + \frac{M-k}{2} \le M$, we see that P is of the form (7). The last property of $P(u)$ follows from (4) and (6), since $\left|T_M\big(\frac{\cos \pi u}{\cos \pi \varepsilon}\big)\right| \le 1$ when $\|u\| \ge \varepsilon$. □

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3. Fejér's extremal trigonometric polynomial

Suppose that $f \in L^1(\mathbb{T})$, that f is nonnegative, and that $\int_0^1 f(u)\,du = 1$. We set $r = |\widehat{f}(1)|$ and $2\pi\alpha = \arg \widehat{f}(1)$, so that $\widehat{f}(1) = r(\cos 2\pi\alpha + i \sin 2\pi\alpha)$ in polar coordinates. Then

$$0 < \int_0^1 \big(1 - \cos 2\pi(u+\alpha)\big) f(u)\,du = 1 - r,$$

so $|\widehat{f}(1)| = |\widehat{f}(-1)| < 1$. This inequality is best possible, as we see from the Fejér kernel,

$$\Delta_M(u) = \frac{1}{M}\Big(\frac{\sin \pi M u}{\sin \pi u}\Big)^2 = \sum_{m=-M+1}^{M-1} (1 - |m|/M) e(mu),$$

for which $\widehat{\Delta}_M(1) = 1 - 1/M$ is close to 1 if M is large. Nonnegative cosine polynomials $T(u)$ with large $\widehat{T}(1)$ were of great interest in the early twentieth century because of their use in establishing a zero-free region for the Riemann zeta function. Fejér [23, pp. 77–79] considered the problem of how close $\widehat{T}(1)$ can be to 1 as a function of M, where T is a cosine polynomial of degree $<M$. We give a short derivation of Fejér's result.

Perron [40] wrote a pioneering article about the eigenvalues and eigenvectors of matrices with positive coefficients, and Frobenius [24] extended this to the more complicated situation of matrices with nonnegative elements. We find it convenient to establish a small fragment in this area (see Corollary 1.12 in Chapter 2 of Berman and Plemmons [4]).

Lemma 3.1. *Let $A = [a_{mn}]$ be an $M \times M$ symmetric matrix with nonnegative coefficients. Suppose that an eigenvalue λ of A is known, with an associated eigenvector $\boldsymbol{v} = [v_n]$ all of whose coefficients are strictly positive. Then*

$$\Big| \sum_{1 \le m,n \le M} a_{mn} x_m x_n \Big| \le \lambda \sum_{m=1}^{M} |x_m|^2$$

for all $\boldsymbol{x} \in \mathbb{C}^M$.

Since equality is achieved by taking $\boldsymbol{x} = \boldsymbol{v}$, it follows that λ is the spectral radius of the matrix.

Proof. By the triangle inequality, the left hand side above is

$$\le \sum_{1 \le m,n \le M} a_{mn} |x_m|\,|x_n| = \sum_{1 \le m,n \le M} a_{mn} v_m v_n \frac{|x_m|}{v_m} \frac{|x_n|}{v_n}.$$

By applying the arithmetic–geometric mean inequality to the last two factors on the right, we see that the above is

$$\le \frac{1}{2} \sum_{1 \le m,n \le M} a_{mn} v_m v_n \Big(\frac{|x_m|^2}{v_m^2} + \frac{|x_n|^2}{v_n^2} \Big).$$

This gives rise to two terms. Since

$$\sum_{n=1}^{M} a_{mn} v_n = \lambda v_m$$

for all m, we see that

$$\sum_{1\le m,n\le M} a_{mn} v_m v_n \frac{|x_m|^2}{v_m^2} = \sum_{m=1}^{M} \frac{|x_m|^2}{v_m} \sum_{n=1}^{M} a_{mn} v_n = \lambda \sum_{m=1}^{M} |x_m|^2.$$

The second term is treated similarly, so we have the stated result. □

Lemma 3.2. *Suppose that $M > 1$. Then*

$$\Big|\sum_{m=1}^{M-1} x_m x_{m+1}\Big| \le \cos\frac{\pi}{M+1} \sum_{m=1}^{M} |x_m|^2$$

for all $\boldsymbol{x} \in \mathbb{C}^M$.

Proof. Let $A = [a_{mn}]$ be the $M \times M$ matrix for which $a_{mn} = 1$ if $m = n \pm 1$, and $a_{mn} = 0$ otherwise. Let $\boldsymbol{v} \in \mathbb{R}^M$ be the vector with coefficients

$$v_n = \sin\frac{\pi n}{M+1}$$

for $1 \le n \le M$. We note that $v_n > 0$ for all n. We show that $\boldsymbol{v}$ is an eigenvector of A with the eigenvalue $\lambda = 2\cos\frac{\pi}{M+1}$. For $m = 1$ we note that

$$\sum_{n=1}^{M} a_{1n} v_n = v_2 = \sin\frac{2\pi}{M+1} = 2\sin\frac{\pi}{M+1}\cos\frac{\pi}{M+1} = \lambda v_1.$$

For $1 < m < M$ we note that

$$\begin{aligned}\sum_{n=1}^{M} a_{mn} v_n &= v_{n+1} + v_{n-1} = \sin\frac{\pi(n+1)}{M+1} + \sin\frac{\pi(n-1)}{M+1}\\ &= 2\sin\frac{\pi n}{M+1}\cos\frac{\pi}{M+1} = \lambda v_m.\end{aligned}$$

Finally, for $m = M$ we note that

$$\sum_{n=1}^{M} a_{Mn} v_n = v_{M-1} = \sin\frac{\pi(M-1)}{M+1} = 2\sin\frac{\pi M}{M+1}\cos\frac{\pi}{M+1} = \lambda v_M.$$

Thus $\boldsymbol{v}$ is an eigenvector of A with eigenvalue λ. By the preceding lemma it follows that

$$\Big|\sum_{m=1}^{M-1} (x_m x_{m+1} + x_{m+1} x_m)\Big| \le \lambda \sum_{m=1}^{M} |x_m|^2,$$

so we have the stated result. □

What we have established suffices for our purposes, but Fejér proved more, namely that the eigenvalues of the matrix A considered above are precisely the numbers

$$2\cos\frac{\pi m}{M+1}$$

for $m = 1, 2, \ldots, M$.

By using the formula for the sum of a geometric progression it is easy to show that

$$\sum_{m=1}^{M} v_m e(mu) = e\Big(\frac{M+1}{2}u\Big)\frac{\sin\frac{\pi}{M+1}\cos\pi(M+1)u}{2\sin\pi\Big(\frac{1}{2(M+1)}+u\Big)\sin\pi\Big(\frac{1}{2(M+1)}-u\Big)}. \tag{8}$$

Here the right hand side vanishes at the $M-1$ points $\frac{a}{2(M+1)}$ with $a = 3, 5, \dots, 2M-1$, but not when $a = 1$ or $a = 2M+1$ because the denominator is zero in these two cases.

To determine the norm of the eigenvector $\boldsymbol{v}$ employed in the above proof, it is convenient to observe that

$$\sum_{m=1}^{M+1} \cos 2\pi\Big(\frac{m}{M+1}+\alpha\Big) = 0 \tag{9}$$

for all α. Hence

$$\begin{aligned}\sum_{m=1}^{M} v_m^2 &= \sum_{m=1}^{M}\sin^2\frac{\pi m}{M+1} = \sum_{m=1}^{M+1}\sin^2\frac{\pi m}{M+1}\\ &= \frac{1}{2}\sum_{m=1}^{M+1}\Big(1-\cos\frac{2\pi m}{M+1}\Big) = \frac{M+1}{2}\end{aligned}$$

by (9) with $\alpha = 0$. Thus if we take

$$x_m = \sqrt{\frac{2}{M+1}}\sin\frac{\pi m}{M+1}, \tag{10}$$

then $\boldsymbol{x}$ is a unit vector and

$$\sum_{m=1}^{M-1} x_m x_{m+1} = \frac{2}{M+1}\sum_{m=1}^{M-1}\sin\frac{\pi m}{M+1}\sin\frac{\pi(m+1)}{M+1}.$$

The summands above vanish when $m = M$ or $m = M+1$, so the value of the sum is not changed by introducing these two further terms. Consequently the above is

$$\begin{aligned}&= \frac{2}{M+1}\sum_{m=1}^{M+1}\sin\frac{\pi m}{M+1}\sin\frac{\pi(m+1)}{M+1}\\ &= \frac{1}{M+1}\sum_{m=1}^{M+1}\Big(\cos\frac{\pi}{M+1}-\cos\frac{\pi(2m+1)}{M+1}\Big)\\ &= \cos\frac{\pi}{M+1}\end{aligned} \tag{11}$$

by (9) with $\alpha = 1/(2(M+1))$. This last calculation is a bit redundant, but it provides a self-contained proof of this important property of the x_m.

Theorem 3.1 (*Fejér*). *Suppose that $M > 1$. If $T(u)$ is a nonnegative trigonometric polynomial with period 1, degree $<M$, and $\widehat{T}(0) = 1$, then*

$$|\widehat{T}(\pm 1)| \le \cos\frac{\pi}{M+1}.$$

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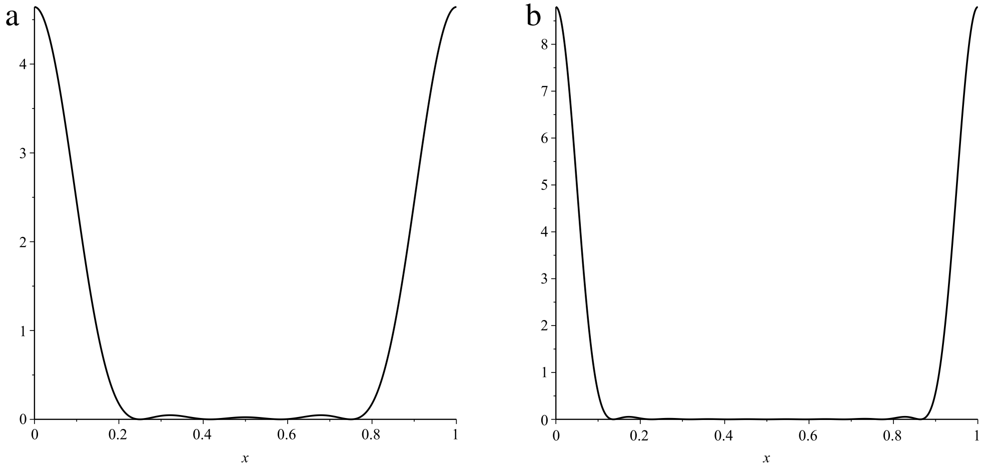

Fig. 2. Plots of Fejér's polynomial $T_M(u)$ for (a) $M = 5$ and (b) $M = 10$.

Moreover, there is a nonnegative cosine polynomial $T_M(u)$ with period 1*, degree $M - 1$, $\widehat{T}_M(0) = 1$, and $\widehat{T}_M(\pm 1) = \cos \frac{\pi}{M+1}$.*

It is significant that this value of $\widehat{T}(1)$ is much closer to 1 than the value $1 - 1/M$ that we obtained from the Fejér kernel $\Delta_M(u)$. Examples of Fejér's T_M are depicted in Fig. 2.

Proof. By the Fejér–Riesz theorem (as found, for example in Montgomery [38, Section 6.3]), if $T(u)$ is nonnegative with period 1 and degree $<M$, then there must exist numbers $x_1, x_2, \ldots, x_M$ such that

$$T(u) = \Big|\sum_{m=1}^{M} x_m e(mu)\Big|^2. \tag{12}$$

Then $1 = \widehat{T}(0) = \sum_{m=1}^{M} |x_m|^2$ and $\widehat{T}(1) = \sum_{m=1}^{M-1} x_{m+1}\overline{x_m}$, so the desired bound for $|\widehat{T}(1)|$ follows from Lemma 3.2. On the other hand, if we define $T_M(u)$ as in (12) with the x_m given as in (10), then $T_M(-u) = T_M(u)$ for all u because the x_m are real, and the values of $\widehat{T}_M(0)$ and $\widehat{T}_M(1)$ are given by (10) and (11). $\square$

Other proofs of Fejér's Theorem 3.1 have been given Szegö [44], Egerváry and Szász [49], and by Milovanovic et al. [36].

While the Fejér–Riesz theorem is very well-known, less familiar is the fact that Fejér also observed that among nonnegative trigonometric polynomials $T(u)$ of degree $<M$, the following two assertions are equivalent:

1. $T(-u) = T(u)$ for all u, which is to say that T is a cosine polynomial;
2. Among the representations of T in the form (12), there is at least one for which the x_m are all real.

That 2. implies 1. is trivial. What is more interesting is that 1. implies 2. To see why this is so, recall that in the course of proving the Fejér–Riesz theorem, one defines a rational function

$$R(z) = \sum_{m=-M+1}^{M-1} \widehat{T}(m) z^m;$$

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whence $T(u) = R(e(u))$. Since T is real-valued, we know that $\widehat{T}(-m) = \overline{\widehat{T}(m)}$. This allows one to prove that

$$R(1/\overline{z}) = \overline{R(z)} \tag{13}$$

for all nonzero z. This is the reflection with respect to the unit circle that is germane to the Fejér–Riesz theorem. To say that $T(-u) = T(u)$ for all u is equivalent to saying that $R(1/z) = R(z)$ when $|z| = 1$. But since this identity holds for infinitely many z, it follows that

$$R(1/z) = R(z) \tag{14}$$

for all nonzero z. When we apply this with z replaced by $\overline{z}$, and then combine the result with the first reflection (13), we find that $R(\overline{z}) = \overline{R(z)}$, so we have a reflection not just with respect to the unit circle, but also with respect to the real axis. Hence zeros of R form complex-conjugate pairs. One can then define a monic polynomial whose roots are the roots of R in the unit disc (with appropriate multiplicities in the case of zeros on the unit circle). The coefficients of this polynomial will be real, since its zeros are in complex-conjugate pairs. A second complementary polynomial can be formed similarly, and then one has only to follow the rest of the proof of the Fejér–Riesz theorem.

4. Turán's localized and quantitative Kronecker theorem

For the frequencies $\log p$, Turán [46] obtained an inferior result by following the method of Bohr and Landau [13]. Almost immediately, he switched to the method of Bohr and Jessen [12], which he developed further. Turán's frequencies were still $\log p$, but Weber [50] generalized the method to arbitrary frequencies with a quantitative measure of linear independence.

Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$ be given real numbers, and let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N$ be positive real numbers with $0 < \varepsilon_n < 1/2$ for all n. Our object is to show that if T is sufficiently large, then any interval I of length T will contain a real number t such that

$$\|t\lambda_n - \alpha_n\| \le \varepsilon_n \quad (n = 1, 2, \ldots, N). \tag{15}$$

To this end, for each n let $T_n(u)$ be a nonnegative trigonometric polynomial with $\int_0^1 T_n(u)\,du = 1$, and let η_n be chosen so that $T_n(u) \le \eta_n$ whenever $\|u\| > \varepsilon_n$. We put

$$F_0(\boldsymbol{x}) = \prod_{n=1}^{N} T_n(x_n) - \sum_{n=1}^{N} \eta_n \prod_{\substack{m=1\\ m\ne n}}^{N} T_m(x_m). \tag{16}$$

If $\|x_n\| > \varepsilon_n$, then

$$F_0(\boldsymbol{x}) \le \prod_{n=1}^{N} T_n(x_n) - \eta_n \prod_{\substack{m=1\\ m\ne n}}^{N} T_m(x_m) = \big(T_n(x_n) - \eta_n\big) \prod_{\substack{m=1\\ m\ne n}}^{N} T_m(x_m) \le 0.$$

Hence $F_0(\boldsymbol{x}) > 0$ only when $\|x_n\| \le \varepsilon_n$ for all n. We set $F(t) = F_0(t\lambda_1 - \alpha_1, t\lambda_2 - \alpha_2, \ldots, t\lambda_N - \alpha_N)$. Turán took the T_n to be of the form

$$T_n(u) = c_n\big(\cos \pi m_n\big)^{2k_n}.$$

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By taking $m_k \sim 1/\varepsilon_n$ and k_n so that $m_n k_n \le M_n$ we obtain a quite reasonable result, but we obtain better constants by using the Chebyshev polynomial as discussed in Section 2. Specifically, by an appeal to Corollary 2.3 we see that we may take T_n to be of the form

$$T_n(u) = \Big|\sum_{m=0}^{M_n} c_n(m)e(mu)\Big|^2$$

where $\int_0^1 T_n(u)\,du = \sum_{m=0}^{M_n} |c_n(m)|^2 = 1$, and then

$$\eta_n = \frac{M_n + 1}{\exp(2\pi M_n \varepsilon_n)} \tag{17}$$

is admissible. We note that

$$\int_{\mathbb{T}^N} F_0(\boldsymbol{x})\,d\boldsymbol{x} = 1 - \sum_{n=1}^{N} \eta_n.$$

Our intent is to show that there exist real numbers t for which $F(t) > 0$ by showing that F has positive mean value. Since this mean value is asymptotically the mean value of F_0, we need the above expression to be positive. To ensure that this is the case, we take

$$M_n = \Big\lceil \frac{1}{\varepsilon_n} \log \frac{N}{\varepsilon_n} \Big\rceil, \tag{18}$$

which implies that $\eta_n \le 1/(2N)$ for all n. We can now formulate our result.

Theorem 4.1. *For $N > 1$, let numbers λ_n, α_n, and ε_n be as described above, and let integers M_n be defined as in (18). Put*

$$\delta = \min |\boldsymbol{m} \cdot \boldsymbol{\lambda}|$$

where the minimum is extended over all lattice points $\boldsymbol{m} = (m_1, m_2, \ldots, m_N) \in \mathbb{Z}^N$ (except $\boldsymbol{m} = \boldsymbol{0}$) for which $|m_n| \le M_n$ for all n. We assume that $\delta > 0$. Then in any interval I of length $T \ge 4/\delta$ there is a real number t such that $\|t\lambda_n - \alpha\| \le \varepsilon_n$ for all n.

Proof. We note that

$$\prod_{n=1}^{N} T_n(t\lambda_n - \alpha_n) = \Big| \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^N \\ 0\le m_n\le M_n}} c(\boldsymbol{m}, \boldsymbol{\alpha}) e((\boldsymbol{m}\cdot\boldsymbol{\lambda})t)\Big|^2 \tag{19}$$

where

$$c(\boldsymbol{m}, \boldsymbol{\alpha}) = e(-(\boldsymbol{m}\cdot\boldsymbol{\alpha})) \prod_{n=1}^{N} c_n(m_n).$$

If $\boldsymbol{m}$ and $\boldsymbol{m}'$ are distinct lattice points occurring in the above sum, then $\boldsymbol{m} - \boldsymbol{m}'$ is a nonzero lattice point in the Cartesian product $\mathsf{X}_{n=1}^{N}[-M_n, M_n]$. Thus $|(\boldsymbol{m}\cdot\boldsymbol{\lambda}) - (\boldsymbol{m}'\cdot\boldsymbol{\lambda})| \ge \delta$. That is, the frequencies occurring in the almost periodic polynomial in (19) are spaced by at least δ. From a generalized form of Hilbert's inequality, Montgomery and Vaughan [39] established an approximate Parseval's identity, which states that if real numbers κ_n have the property that

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$|\kappa_m - \kappa_n| \ge \delta$ for $m \ne n$, then

$$\int_a^{a+T} \Big|\sum_{n=1}^N c_n e(\kappa_n t)\Big|^2 \, dt = (T + \theta/\delta) \sum_{n=1}^N |c_n|^2$$

for some $\theta \in [-1, 1]$. Thus if I is an interval of length T, then

$$\int_I F(t)\, dt \ge (T - 1/\delta) - \sum_{n=1}^N \eta_n (T + 1/\delta).$$

Now $\sum_{n=1}^N \eta_n \le 1/2$, so the above is

$$\ge \frac{1}{2}T - \frac{3}{2\delta} > 0$$

if $T \ge 4/\delta$. $\square$

One advantage of this approach is that Montgomery and Vaughan [39] also established a weighted Hilbert inequality that may sometimes lead to sharper estimates. Gonek and Montgomery [25] have already exploited this.

The task of constructing good trigonometric minorants in high dimensions is at the present not well-understood. The mechanism of (16) might appear clumsy, but it works pretty well. For other trigonometric minorants, see Baker [2], Cochrane [21], Harman [28, Lemma 6], Brüdern and Fouvry [15], and Barton et al. [3].

5. Chen's quantitative Kronecker theorem

Chen [18] used Bacon's quantitative form of Kronecker's theorem, but then turned to Bohr and Jessen [11] to obtain a sharper quantitative form. He achieved this in Chen [19], but the result was still somewhat imperfect because he followed Bohr and Jessen in using the Fejér kernel, which is not optimal in this context. By replacing the Fejér kernel by the Fejér polynomial as discussed in Section 3, Chen [20] obtained optimal results, which we now discuss.

Theorem 5.1. *For $1 \le n \le N$ let real numbers λ_n, α_n, positive integers M_n and positive real numbers w_n be given. Let*

$$\delta = \min \Big\| \sum_{n=1}^N m_n \lambda_n \Big\|$$

where the minimum is taken over those N-tuples $m_1, m_2, \ldots, m_N$ for which $|m_n| \le M_n$ for all n, and with not all m_n equal to 0. If $\delta > 0$, then for any $T \ge 1/\delta$, in any interval of length T there is a t such that

$$\sum_{n=1}^N w_n \|\lambda_n t - \alpha_n\|^2 \le \frac{\pi^2}{16} \sum_{n=1}^N \frac{w_n}{M_n^2} + \frac{1}{2T\delta} \sum_{n=1}^N w_n. \tag{20}$$

Chen obtained his result in the extended situation in which some linear dependencies are allowed, but with less attention to localization. We restrict ourselves to the case of linearly independent λ_n, in order that the main ideas should be easier to follow. If we take $w_n = 1$ and $M_n = M$ for all n, then we find that

$$\sum_{n=1}^{N} \|\lambda_n t - \alpha_n\|^2 \le \frac{\pi^2 N}{16M^2} + \frac{N}{2T\delta}. \tag{21}$$

By allowing T to tend to infinity we obtain the special case (3) already noted, with $c = \pi^2/16$. The most trivial bound for the left hand side above is $N/4$. Thus the right hand side becomes nontrivial as $T\delta$ becomes large, and continues to be more significant until T reaches the order of magnitude M^2/δ, at which point the estimate stabilizes. Indeed, Chen [20] has observed that if

$$\lambda_n = \frac{1}{(M+1)^{N-n+1}}, \qquad \alpha_n = \frac{1}{2M}\big((M+1)^n - 1\big)$$

for $n = 1, 2, \dots, N$, then

$$(M+1)(\lambda_n t - \alpha_n) - (\lambda_{n+1} t - \alpha_{n+1}) = \frac{1}{2}$$

for all real t, and $n = 1, 2, \dots, N-1$. Hence

$$(M+1)\|\lambda_n t - \alpha_n\| + \|\lambda_{n+1} t - \alpha_{n+1}\| \ge \frac{1}{2}.$$

By Cauchy's inequality it follows that

$$\sum_{n=1}^{N} \|\lambda_n t - \alpha_n\|^2 \ge \frac{2N}{21(M+1)^2}$$

for all real t, $N \ge 2$, and $M \ge 1$.

It would be interesting to know whether (20) might be replaced by

$$\sum_{n=1}^{N} w_n \|\lambda_n t - \alpha_n\|^2 \ll \left(1 + \frac{1}{T\delta}\right) \sum_{n=1}^{N} \frac{w_n}{M_n^2}. \tag{22}$$

Perhaps this is asking for too much; in any case it does not seem to follow by the current method.

In applications it is useful to be able to choose the w_n and M_n. This extra flexibility does not make the proof more complicated. If a weighted ℓ_2 bound is what is desired, then this theorem seems to be optimal. However, if one uses the above to derive a uniform bound, say $\|\lambda_n t - \alpha_n\| < \varepsilon$ for all n, then by Chen's method one must take $M \ge cN^{1/2}/\varepsilon$. This is in general much larger than the value of M indicated in (18). When M is larger, the associated value of δ is much smaller, which means that the localization will be inferior.

Proof. For $1 \le n \le N$ let T_n be a cosine polynomial of degree $M_n - 1$ chosen as in Section 3, so that $T_n(x) \ge 0$ for all x, $\int_0^1 T_n(x)\,dx = \widehat{T}_n(0) = 1$, and $\widehat{T}_n(\pm 1) = \cos \pi/(M_n + 1)$. We define a weighting function

$$W(t) = \prod_{n=1}^{N} T_n(\lambda_n t - \alpha_n) = \prod_{n=1}^{N} \Bigg(\sum_{m_n = -M_n+1}^{M_n - 1} \widehat{T}_n(m_n) e\big(m_n(\lambda_n t - \alpha_n)\big) \Bigg). \tag{23}$$

In preparation for expanding this, we set $\boldsymbol{m} = (m_1, m_2, \ldots, m_N)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, and

$$c(\boldsymbol{m}, \boldsymbol{\alpha}) = e(-\boldsymbol{m}\cdot\boldsymbol{\alpha}) \prod_{n=1}^{N} \widehat{T_n}(m_n)$$

for $|m_n| < M_n$, and $c(\boldsymbol{m}, \boldsymbol{\alpha}) = 0$ otherwise. Thus we find that

$$W(t) = \sum_{\boldsymbol{m}} c(\boldsymbol{m}, \boldsymbol{\alpha}) e\big((\boldsymbol{m}\cdot\boldsymbol{\lambda})t\big). \tag{24}$$

Let I be an interval of length $T \ge 1/\delta$. In order to derive an upper bound for $\int_I W(t)\,dt$, we recall that Selberg [42, pp. 213–219] defined functions $S_\pm \in L^1(\mathbb{R})$ such that

$$S_-(t) \le \chi_I(t) \le S_+(t),$$

for all t,

$$\int_{-\infty}^{\infty} S_\pm(t)\,dt = T \pm 1/\delta,$$

and $\widehat{S_\pm}(x) = 0$ when $|x| \ge \delta$. Expositions of this work of Selberg are found for example in Montgomery [37] and Vaaler [48]. If $\boldsymbol{m} \ne \boldsymbol{0}$ and $c(\boldsymbol{m}, \boldsymbol{\alpha}) \ne 0$, then $\|\boldsymbol{m}\cdot\boldsymbol{\lambda}\| \ge \delta$, so

$$\int_I W(t)\,dt \le \int_{-\infty}^{\infty} W(t)S_+(t)\,dt = T + 1/\delta. \tag{25}$$

For given positive real numbers $w_1, w_2, \ldots, w_N$, we define

$$F(t) = \sum_{k=1}^{N} w_k\big(1 + \cos 2\pi(\lambda_k t - \alpha_k)\big) = 2\sum_{k=1}^{N} w_k \cos^2 \pi(\lambda_k t - \alpha_k), \tag{26}$$

and consider how F correlates with W. We note that

$$F(t)W(t) = \sum_{k=1}^{N} w_k\big(W(t) + R_+(k,t) + R_-(k,t)\big) \tag{27}$$

where

$$R_\pm(k,t) = \frac{1}{2}\sum_{\boldsymbol{m}} c(\boldsymbol{m}, \boldsymbol{\alpha}) e(\mp\alpha_k) e\big((\boldsymbol{m}\cdot\boldsymbol{\lambda} \pm \lambda_k)t\big). \tag{28}$$

As concerns R_+, we note that $\|\boldsymbol{m}\cdot\boldsymbol{\lambda} + \lambda_k\| \ge \delta$ unless $\boldsymbol{m}$ is of the form

$$m_n = \begin{cases} -1 & n = k, \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

for which

$$c(\boldsymbol{m}, \boldsymbol{\alpha}) e(-\alpha_k) = \widehat{T_k}(-1) = \cos\frac{\pi}{M_k + 1}.$$

Similarly for R_-, we find that $\|\boldsymbol{m}\cdot\boldsymbol{\lambda} - \lambda_k\| \ge \delta$ unless

$$m_n = \begin{cases} 1 & n = k, \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

in which case

$$c(\boldsymbol{m}, \boldsymbol{\alpha})e(\alpha_k) = \widehat{T_k}(1) = \cos \frac{\pi}{M_k + 1}.$$

We note that here we are assuming not just that $\boldsymbol{m} \cdot \boldsymbol{\lambda} \geq \delta$ when $|m_n| < M_n$, but also in the slightly larger set of $\boldsymbol{m}$ for which $|m_n| \leq M_n$ (of course with the exception of $\boldsymbol{m} = \mathbf{0}$). Hence

$$\int_I F(t)W(t)\,dt \geq \int_{-\infty}^{\infty} F(t)W(t)S_-(t)\,dt$$
$$= (T - 1/\delta) \sum_{n=1}^{N} w_n \Big(1 + \cos \frac{\pi}{M_n + 1}\Big).$$

Since $\cos x \geq 1 - x^2/2$ for all x, the above is

$$\geq (T - 1/\delta) \sum_{n=1}^{N} w_n \Big(2 - \frac{\pi^2}{2(M_n + 1)^2}\Big). \tag{31}$$

On comparing (25) with (31), we deduce that there is a $t \in I$ such that

$$F(t) \geq \frac{T - 1/\delta}{T + 1/\delta} \sum_{n=1}^{N} w_n \Big(2 - \frac{\pi^2}{2(M_n + 1)^2}\Big).$$

Now $\cos 2\pi x = 1 - 2\sin^2 \pi x$, and $|\sin \pi x| \geq 2\|x\|$, so $\cos 2\pi x \leq 1 - 8\|x\|^2$. Hence from the definition (26) of $F(t)$ we see that

$$F(t) \leq 2 \sum_{n=1}^{N} w_n - 8 \sum_{n=1}^{N} w_n \|\lambda_n t - \alpha_n\|^2.$$

On rearranging these inequalities we deduce that

$$8 \sum_{n=1}^{N} w_n \|\lambda_n t - \alpha_n\|^2 \leq 2\Big(1 - \frac{T - 1/\delta}{T + 1/\delta}\Big) \sum_{n=1}^{N} w_n + \frac{T - 1/\delta}{T + 1/\delta} \sum_{n=1}^{N} w_n \frac{\pi^2}{2(M_n + 1)^2}.$$

Now $1 - \frac{1-x}{1+x} = \frac{2x}{1+x} \leq 2x$ for $x \geq 0$, so the above is

$$\leq \frac{\pi^2}{2} \sum_{n=1}^{N} \frac{w_n}{(M_n + 1)^2} + \frac{4}{T\delta} \sum_{n=1}^{N} w_n,$$

and the proof is complete. □

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