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Over the reals these are precisely polynomial automata

No algorithmic results, i.e., zeroness / equivalence

Alternating weighted automata over commutative semirings ☆

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A B S T R A C T

We define and study alternating weighted automata over an arbitrary commutative semiring. Our model generalises the concept of alternation of existential and universal states in the classical Boolean setting to alternation of states performing addition and multiplication in the underlying commutative semiring. We completely characterise the class of commutative semirings S, for which alternating and non-alternating weighted automata over S are equally powerful, and we study some closure properties of the classes of formal power series realised by alternating weighted automata.

1. Introduction

Weighted automata, first introduced by Schützenberger [10], form a generalisation of nondeterministic finite automata, in which each transition carries a weight (usually) taken from some semiring. This weight represents some quantity related to the execution of a given transition, such as cost, reward, probability, reliability, etc. Instead of merely *recognising* languages, weighted automata *realise* formal power series – these generalise languages by assigning a weight from the underlying semiring to each word. The theory of weighted automata now forms a vast body of knowledge with various modifications and generalisations of the basic setting described above and with multiple practical applications as well. See the handbook [7] for the survey of most important topics.

The study of alternation in weighted automata has so far been limited to several particular settings directly motivated by problems arising in practice of quantitative formal verification. The branch of research going in this direction was initiated by Chatterjee, Doyen, and Henzinger [3], who have studied alternation in weighted automata over *infinite* words. Automata introduced in [3] compute the real weight of a single run using one of several operations considered in the article (such as limit superior or discounted sum). Branching is handled by two types of states, which correspond to the operations of minimum and maximum on reals. These two operations can be seen as alternating with each other – the setting described in [3] can thus be called "min–max"-alternation.

Alternating weighted automata over *finite* words have been studied by Almagor and Kupferman [1], who have focused on two different forms of alternation over the tropical (min-plus) semiring of reals. In the first variant, the "usual" states corresponding to the minimum operation alternate with states corresponding to the maximum operation – this setting of "min–max"-alternation is essentially the same as the one studied by Chatterjee, Doyen, and Henzinger [3], except that finite

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words are considered and the weight of a single run is always computed by taking the sum. In the second variant studied in [1], states performing the minimum alternate with summation states; this can be called "min-sum"-alternation. (Observe that addition in the tropical semiring alternates with multiplication in the tropical semiring in this setting.) It is argued by Almagor and Kupferman [1] that both variants are interesting for the purposes of formal verification.

The aim of this article is to initiate the study of alternating weighted automata over a general commutative semiring (and over finite words), while taking a more theoretical point of view. We shall introduce and study weighted automata over a commutative semiring S, in which states corresponding to addition over S alternate with states corresponding to multiplication in S.[1] This generalises both the classical Boolean setting of alternating finite automata [2] and "min-sum"-alternation of Almagor and Kupferman [1].

Our model will *not* incorporate "min–max"-alternation of Almagor and Kupferman [1] and the settings studied by Chatterjee, Doyen, and Henzinger [3]. However, any generalisation subsuming such models would clearly require structures with at least three operations – that is, structures beyond semirings. It thus seems to be reasonable to confine ourselves to the setting described above at this point. Nevertheless, we believe that a good enough understanding of weighted alternation over semirings can form a natural starting point for an even more general theory of alternating weighted automata. (We leave the task of developing such theory for future research.) Moreover, we shall limit ourselves to commutative semirings in this article, as commutativity appears to be a natural assumption when it comes to product states: non-commutative semirings would require some fixed ordering of transitions leading from such states.

The problems that we shall deal with in this article can roughly be summarised as follows.

Definitions One could imagine more than one natural and well motivated formalisation of the idea of alternating weighted automata as described above. We shall therefore give two different definitions in this article, representing two opposite sides of the spectrum. The definition of what we shall call simply *alternating weighted automata* mimics the customary definition of alternating finite automata without weights [2], in which existential and universal mode can be "combined" in a single state, leading to the representation of actions of symbols on states by means of Boolean formulae; in our weighted generalisation, a *polynomial* over the underlying semiring corresponds to each pair consisting of a state and a symbol. On the contrary, the definition of what we shall call *two-mode alternating weighted automata* follows strictly the intuitive idea of dividing the set of states into summation states and product states. Moreover, spontaneous transitions on the empty word ε will be allowed under some assumptions in two-mode alternating automata. We shall prove that alternating weighted automata and two-mode alternating weighted automata are equivalent in their expressive power and that ε-transitions can be removed from two-mode alternating automata. These results can be viewed as a sign of certain robustness of the concept of alternating weighted automata and hence also as an additional motivation for their study.

Systems of equations It is well known that (non-alternating) weighted automata are equivalent to systems of linear equations [8]. We shall prove a similar result for *alternating* weighted automata, linking them to what we shall call *systems of H-polynomial equations* (abbreviation for Hadamard-polynomial). These systems can be roughly described as linear systems, in which some additional Hadamard products of formal power series can take place.

Expressive power Weighted automata can be viewed just as a special case of alternating weighted automata and for this reason, alternating weighted automata are at least as powerful as weighted automata. Almagor and Kupferman showed that alternating weighted automata over the tropical semiring are strictly more powerful than non-alternating automata over the same semiring [1]. On the other hand, it is a well known fact that every alternating finite automaton (without weights) recognises a rational (regular) language [2]. We conclude that commutative semirings can be divided into two nonempty classes: the class of commutative semirings, for which alternation gains extra expressive power and the class of commutative semirings, for which alternating and non-alternating weighted automata are equally powerful. In our most significant result, we shall give a complete characterisation of these two classes of commutative semirings.

Closure properties We shall observe that the class of formal power series realised by alternating weighted automata over a commutative semiring S is closed under sum and Hadamard product for each S. On the other hand, we shall prove that there exists a commutative semiring S such that this class is closed neither under Cauchy product, nor under reversal.

2. Preliminaries

A *monoid* is a triple $(M, \cdot, 1)$, where M is a set, $\cdot$ is an associative binary operation on M, and 1 is an element of M such that $1 \cdot a = a \cdot 1 = a$ holds for all a in M. A monoid $(M, \cdot, 1)$ is *commutative* if $\cdot$ is commutative. A *semiring* is a quintuple $(S, +, \cdot, 0, 1)$, where S is a set, $+$ and $\cdot$ are binary operations (addition and multiplication) on S, and $0, 1$ are distinguished elements of S, such that $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, multiplication distributes over addition (both from left and from right), and $0 \cdot a = a \cdot 0 = 0$ holds for all a in S. A semiring $(S, +, \cdot, 0, 1)$ is *commutative* if the monoid $(S, \cdot, 1)$ is commutative. We shall often write S instead of $(S, +, \cdot, 0, 1)$ and denote multiplication by juxtaposition.

[1] To be more precise, we shall also allow "mixing" of both modes in a single state, which is the usual approach for alternating finite automata without weights [2].

Let a be an element of a semiring S and n a nonnegative integer. We shall write $na := \sum_{i=1}^{n} a$ and $a^n := \prod_{i=1}^{n} a$. In particular, $0a = 0$ and $a^0 = 1$. Moreover, to avoid possible confusion with the integers, we shall sometimes write 0_S instead of 0 and 1_S instead of 1 to denote the zero and the unity element of S. More generally, we shall use the notation n_S to denote the element $n1_S$ for a nonnegative integer n.

We say that a in S has *finite additive order* if the set $\{na \mid n \in \mathbb{N}\}$ is finite. Otherwise, we say that a has *infinite additive order*. Similarly, we say that a has *finite multiplicative order* if the set $\{a^n \mid n \in \mathbb{N}\}$ is finite, and we say that a has *infinite multiplicative order* otherwise. The reader can easily check that a in S has finite additive order (or finite multiplicative order) if and only if $na = ma$ (or $a^n = a^m$) for some distinct nonnegative integers n, m.

A *subsemiring* of a semiring $(S, +, \cdot, 0, 1)$ is a subset T of S that contains 0, 1 and that is closed under addition and multiplication. If T is a subsemiring of S, then T forms a semiring together with the operations $+$ and $\cdot$ restricted to T. One can easily show that if $\mathcal{U}$ is a collection of subsemirings of S, then $\bigcap_{T \in \mathcal{U}} T$ is a subsemiring of S as well. If X is a subset of S and $\mathcal{U}$ is the collection of all subsemirings of S that contain X, we say that $\bigcap_{T \in \mathcal{U}} T$ is the *subsemiring generated by* X. The subsemiring generated by X is the smallest subsemiring of S (with respect to inclusion) that contains X. A semiring S is *finitely generated* if it is generated by a finite subset of S and *locally finite* if every finitely generated subsemiring of S is finite.

For every commutative semiring S, we shall denote by $S[x_1, \ldots, x_n]$ the set of all *polynomials* in (commutative) indeterminates $x_1, \ldots, x_n$ with coefficients in S. This set, together with the operations of addition and multiplication of polynomials derived from the operations of the semiring S in the usual way, constitutes a commutative semiring. A *monomial* in indeterminates $x_1, \ldots, x_n$ is a polynomial in $S[x_1, \ldots, x_n]$ such that $m = c x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$ for some c in S and nonnegative integers $k_1, \ldots, k_n$. For each such monomial m, we shall denote the coefficient c of m by $\mathrm{coef}(m)$ and the exponent k_i of the indeterminate x_i by $\exp(m, i)$ for $i = 1, \ldots, n$. If $k_1 = k_2 = \ldots = k_n = 0$, we say that m is a *constant*. The whole semiring $S[x_1, \ldots, x_n]$ is generated by its monomials. If P in $S[x_1, \ldots, x_n]$ can be written as a sum of nonconstant monomials,[2] we say that P has *zero constant term*. We shall denote the subset of $S[x_1, \ldots, x_n]$ consisting of all polynomials with zero constant term by $S[x_1, x_2, \ldots, x_n]_{\mathrm{const}=0}$. Note that although this subset is closed under addition and multiplication, it is not a subsemiring of $S[x_1, \ldots, x_n]$, since it does not contain the unity of $S[x_1, \ldots, x_n]$.

Moreover, for each commutative semiring S we shall write $S(x_1, x_2, \ldots, x_n)$ to denote the set of all *polynomial functions* corresponding to polynomials from $S[x_1, \ldots, x_n]$. It is easy to see that the set $S(x_1, x_2, \ldots, x_n)$ constitutes a commutative semiring together with the pointwise addition and multiplication of functions.

A *formal power series* over a semiring S and over an alphabet Σ is a mapping $r \colon \Sigma^* \to S$. For every w in Σ^*, the value $r(w)$ is usually denoted by (r, w) and called the *coefficient of* w *in* r. The formal power series r itself is then written as

$$r = \sum_{w \in \Sigma^*} (r, w) w.$$

If there is a single w in Σ^* such that $(r, w) \neq 0$, then r can also be written as $r = (r, w)w$; a scalar s in S can be identified with the series $s\varepsilon$. If $(r, w) = 0$ for all w in Σ^*, we shall write $r = 0$. Moreover, coefficients equal to 1 can be omitted. The set of all formal power series over S and Σ is denoted by $S\langle\langle \Sigma^* \rangle\rangle$.

Let r_1 and r_2 be in $S\langle\langle \Sigma^* \rangle\rangle$. The *sum* of r_1 and r_2 is a formal power series $r_1 + r_2$ in $S\langle\langle \Sigma^* \rangle\rangle$ such that $(r_1 + r_2, w) = (r_1, w) + (r_2, w)$ for all w in Σ^*. The *Cauchy product* of r_1 and r_2 is a formal power series $r_1 \cdot r_2$ in $S\langle\langle \Sigma^* \rangle\rangle$ such that

$$(r_1 \cdot r_2, w) = \sum_{\substack{v_1, v_2 \in \Sigma^* \\ v_1 v_2 = w}} (r_1, v_1)(r_2, v_2)$$

for all w in Σ^*. The *Hadamard product* of r_1 and r_2 is a formal power series $r_1 \odot r_2$ in $S\langle\langle \Sigma^* \rangle\rangle$ defined for all w in Σ^* by $(r_1 \odot r_2, w) = (r_1, w)(r_2, w)$. The *$n$-th power* r^n of a series r is defined inductively by

$$r^0 = 1\varepsilon,$$
$$r^n = r^{n-1} \cdot r, \qquad n \geq 1.$$

Similarly, we define a formal power series $r^{\odot n}$ inductively by

$$r^{\odot 0} = \sum_{w \in \Sigma^*} 1w,$$
$$r^{\odot n} = r^{\odot n-1} \odot r, \qquad n \geq 1.$$

A *reversal* of a word $w = a_1 \ldots a_n$ in Σ^*, for $a_1, \ldots, a_n$ in Σ, is a word $w^R = a_n \ldots a_1$. A *reversal* of a formal power series r in $S\langle\langle \Sigma^* \rangle\rangle$ is a series r^R in $S\langle\langle \Sigma^* \rangle\rangle$ such that $(r^R, w) = (r, w^R)$ holds for each w in Σ^*.

[2] This is in particular true if $P = 0$, which is an empty sum of nonconstant monomials.

Let S be a semiring. A *weighted automaton* over S is a sextuple $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$, where: Q is a nonempty finite set of states; Σ is an alphabet; T is a finite set of transitions, with which we associate mappings $src, dst: T \to Q$ and $\sigma: T \to \Sigma \cup \{\varepsilon\}$; $\nu: T \to S$ is a transition weighting function; $\iota: Q \to S$ is an initial weighting function; $\tau: Q \to S$ is a terminal weighting function.

A weighted automaton can be viewed as a directed multigraph with labelled edges, where Q is the set of vertices and T is the set of edges. The mappings *src* and *dst* assign a source and a destination to each edge, while σ assigns labels.[3]

Let $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$ be a weighted automaton, p in Q, and z in $\Sigma \cup \{\varepsilon\}$. We shall write $T_{\mathcal{A}}(p)$ for the set of all transitions t in T such that $src(t) = p$ and $T_{\mathcal{A}}(p, z)$ for the set of all transitions t in T satisfying $src(t) = p$ and $\sigma(t) = z$. Moreover, we shall denote by $T^{\varepsilon}_{\mathcal{A}}$ the set of all ε-labelled transitions in $\mathcal{A}$. If clear from the context, we shall often omit the subscript denoting the automaton in consideration.

In the presence of spontaneous ε-labelled transitions, the behaviour of a weighted automaton cannot be at the same time defined for all semirings and for all automata. One of the usual solutions to this problem is to define the behaviour for so-called *cycle-free* automata [8] only. An automaton is always cycle-free if it does not contain a cycle of ε-labelled transitions. However, the definition is slightly more general – under some conditions, a cycle-free weighted automaton might still contain such cycles. It does not seem straightforward to make an analogous definition of cycle-free *alternating* weighted automata in such a manner that the definition does not become awkward. We shall therefore confine ourselves to the study of alternating weighted automata that contain no cycles of spontaneous transitions. We shall follow the same approach for "ordinary" weighted automata as well, since we want to view them as a special case of alternating weighted automata. We shall say that $\mathcal{A}$ is *without* ε*-cycles* if the directed graph with vertex set Q and edge set $T^{\varepsilon}_{\mathcal{A}}$ contains no cycle.

It can be shown that every cycle-free weighted automaton is equivalent to some weighted automaton that contains no ε-labelled transitions at all [8]. In a sense, this fact justifies our choice to consider weighted automata without ε-cycles only.

Let S be a semiring and $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$ a weighted automaton over S without ε-cycles. For each p in Q, let us define a formal power series $|\mathcal{A}|_p$ in $S\langle\langle \Sigma^* \rangle\rangle$ recursively as follows:

1. The coefficient of ε in $|\mathcal{A}|_p$ is defined by
$$(|\mathcal{A}|_p, \varepsilon) = \tau(p) + \sum_{t \in T(p,\varepsilon)} \nu(t) \cdot (|\mathcal{A}|_{dst(t)}, \varepsilon).$$
2. If c is in Σ and w is in Σ^*, then
$$(|\mathcal{A}|_p, cw) = \sum_{t \in T(p,c)} \nu(t) \cdot (|\mathcal{A}|_{dst(t)}, w) + \sum_{t \in T(p,\varepsilon)} \nu(t) \cdot (|\mathcal{A}|_{dst(t)}, cw).$$

The *behaviour* of $\mathcal{A}$ is a formal power series $|\mathcal{A}|$ in $S\langle\langle \Sigma^* \rangle\rangle$ defined for all w in Σ^* by
$$(|\mathcal{A}|, w) = \sum_{p \in Q} \iota(p) \cdot (|\mathcal{A}|_p, w).$$

A weighted automaton $\mathcal{A}$ *realises* a formal power series r if $|\mathcal{A}| = r$. We say that a formal power series r in $S\langle\langle \Sigma^* \rangle\rangle$ is *rational* over S if it is realised by some weighted automaton over S.

The theory of weighted automata and formal power series in non-commutative variables now forms a rich and active field of research. We have confined ourselves to stating several most important definitions in this section. For a more comprehensive source of information, we refer the reader to the handbook [7].

3. An introductory example

Before defining alternating weighted automata formally, let us first describe them on an intuitive level using a simple example. More precisely, the model that we shall intuitively describe in what follows corresponds to *two-mode* alternating weighted automata defined in Section 5. Either an additive or a multiplicative mode is assigned to each state of a two-mode alternating weighted automaton, which makes these automata easy to depict by diagrams. On the other hand, our basic definition of alternating weighted automata is given in Section 4 and allows "mixing" of both modes in a single state. Both models are nevertheless proved to be equivalent in their expressive power in Subsection 5.5.

Additional examples similar to the one below will be given in Subsection 5.2, following the formal definition of two-mode alternating weighted automata.

[3] Note that the definition allows parallel edges (transitions) with the same label in the graph. This makes our definition rather unusual in comparison with definitions of the majority of authors [7]. It is nevertheless obvious that the expressive power of weighted automata remains the same. Parallel transitions with the same label will play a significant role in our definition of *alternating* weighted automata – we have thus chosen to allow them in non-alternating weighted automata as well.

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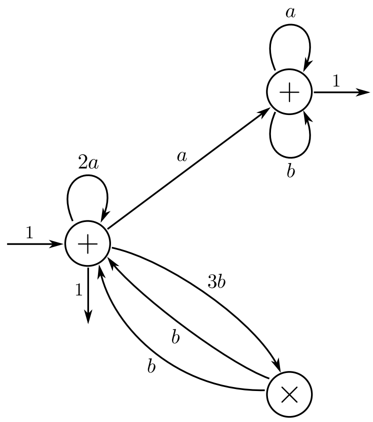

Fig. 1. The automaton $\mathcal{A}$.

Example 3.1. In Fig. 1, a two-mode alternating weighted automaton $\mathcal{A}$ over the alphabet $\Sigma = \{a, b\}$ and over the semiring $(\mathbb{N}, +, \cdot, 0, 1)$ of natural numbers with standard addition and multiplication is depicted. States of $\mathcal{A}$ are labelled either with "$+$" or with "$\times$". This label corresponds to the mode of the given state: states labelled by "$+$" are called *sum states* and states labelled by "$\times$" are called *product states*. Apart from this additional feature, Fig. 1 conforms to the usual conventions for diagrams of weighted automata.

For each state p of $\mathcal{A}$, we first define a formal power series $|\mathcal{A}|_p$, which may be described as the series realised by $\mathcal{A}$ if the run of the automaton was forced to start in p (having initial weight 1). If p is a sum state, then the definition of the power series $|\mathcal{A}|_p$ is the same as for (non-alternating) weighted automata; the coefficient of ε in $|\mathcal{A}|_p$ is equal to the terminal weight of p and for every c in Σ and w in Σ^*, the coefficient of cw in $|\mathcal{A}|_p$ is $\sum_{t \in T(p,c)} \nu(t)(|\mathcal{A}|_{dst(t)}, w)$, where $T(p, c)$ is the set of all transitions that start at p and are labelled with c, where $\nu(t)$ denotes the weight of the transition t, and where $dst(t)$ denotes the destination state of the transition t.[4] If p is a product state, the power series $|\mathcal{A}|_p$ is defined as follows: the coefficient of ε in $|\mathcal{A}|_p$ is equal to the terminal weight of p and for each c in Σ and w in Σ^*, the coefficient of cw in $|\mathcal{A}|_p$ is $\prod_{t \in T(p,c)} \nu(t)(|\mathcal{A}|_{dst(t)}, w)$, with the notation having the same meaning as above. In case $T(p, c)$ is empty, we define $(|\mathcal{A}|_p, cw) = 0$. The reason why the states of a two-mode alternating weighted automaton are called sum states and product states is now evident: a sum state performs addition to calculate the weight of a word w, while a product state performs multiplication. The behaviour of the two-mode alternating weighted automaton $\mathcal{A}$ is defined in the same way as it was defined for (non-alternating) weighted automata, i.e., $(|\mathcal{A}|, w) = \sum_{p \in Q} \iota(p)(|\mathcal{A}|_p, w)$ for all w in Σ^*, where Q is the set of states of $\mathcal{A}$ and $\iota(p)$ denotes the initial weight of p.

Let us finally examine the behaviour of the automaton $\mathcal{A}$. Clearly, we have $(|\mathcal{A}|, \varepsilon) = 1$ and $(|\mathcal{A}|, b) = 0$. The reader can easily check that for all w in Σ^*, we have $(|\mathcal{A}|, aw) = 2(|\mathcal{A}|, w) + 1$, $(|\mathcal{A}|, bbw) = 3(|\mathcal{A}|, w)^2$, and $(|\mathcal{A}|, baw) = 0$. These relations fully describe the formal power series $|\mathcal{A}|$.

4. Definition of alternating weighted automata

We shall now define *alternating weighted automata* formally. As we have already mentioned, our basic model of alternating weighted automata will be slightly more general than the one informally described in Section 3. We shall not require the set of states to be partitioned into summation states and product states; instead, we shall allow both modes to be "combined" in a single state. In Section 5, we shall give a definition of *two-mode alternating weighted automata*, which directly formalises the intuitive idea of dividing the states into two sets. We shall nevertheless observe that both models are equivalent in their expressive power.

The generalised approach that we shall follow in our definition of alternating weighted automata is usual in the Boolean setting when defining alternating finite automata without weights. In a typical definition [2], states of an alternating finite automaton need not be marked as existential or universal. Instead, a Boolean formula $\psi[q, c]$ is assigned to each state q and each symbol c (this formula is often required to be positive). For instance, if the state set is $\{q_1, \ldots, q_6\}$ and if $\psi[q_1, a] = (x_2 \wedge x_4) \vee (x_3 \wedge x_5 \wedge x_6)$, then there is a valid (accepting) run on a word aw starting in q_1 if and only if there are either valid runs on w starting both in q_2 and in q_4, or valid runs on w starting in q_3, q_5, and q_6.

We shall generalise this representation of transitions to the weighted setting by replacing positive Boolean formulae by polynomials without constants over the underlying semiring. A positive Boolean formula can obviously be viewed as a polynomial without constants over the Boolean semiring $(\mathbb{B}, \vee, \wedge, 0, 1)$.

[4] For simplicity, $\mathcal{A}$ has been chosen so that it contains no ε-labelled transitions. This restriction makes the definition of behaviour of two-mode alternating weighted automata simpler compared to the definition given in Section 5.

Definition 4.1. Let S be a commutative semiring. An *alternating weighted automaton* over the semiring S is a quintuple $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$, where: Q is a nonempty finite set of states with $|Q| = n$; Σ is an alphabet; $\psi : (Q \times \Sigma) \to S[x_1, \ldots, x_n]_{\text{const}=0}$ is a polynomial assigning function; P_0 in $S[x_1, \ldots, x_n]_{\text{const}=0}$ is an initial polynomial; $\tau : Q \to S$ is a terminal weighting function.

For each alternating weighted automaton $\mathcal{A}$, we shall always assume some linear ordering on its states to be given. Although this linear ordering is not part of the definition above, it is nevertheless important, as we shall see in the following definition of behaviour. Moreover, whenever we refer to "the i-th state", where i is a positive integer, we mean the i-th state with respect to the linear ordering of states of the automaton in consideration. For each alternating weighted automaton $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$, each p in Q, and each c in Σ, the polynomial $\psi[p, c]$ can also be denoted by $\psi[i, c]$ if p is the i-th state of $\mathcal{A}$. In some cases, we shall also write $\tau(i)$ instead of $\tau(p)$. We shall also shortly introduce some other notation, in which states are interchangeable with their numerical order.

We may now define the *behaviour* of an alternating weighted automaton $\mathcal{A}$ over S. Similarly as for non-alternating weighted automata, we shall first define a series $|\mathcal{A}|_p$ for each state p, which may be seen as the behaviour of $\mathcal{A}$ if the run of $\mathcal{A}$ was forced to start in state p (with initial weight 1_S). The behaviour $|\mathcal{A}|$ of $\mathcal{A}$ is then obtained from these auxiliary series by applying the initial polynomial.

Definition 4.2. Let $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ be an alternating weighted automaton over some commutative semiring S, let $n = |Q|$. For each p in Q, we define a formal power series $|\mathcal{A}|_p$ (also denoted by $|\mathcal{A}|_i$ if p is the i-th state) in $S\langle\langle\Sigma^*\rangle\rangle$ as follows:

1. The coefficient of ε in $|\mathcal{A}|_p$ is given by
$$(|\mathcal{A}|_p, \varepsilon) = \tau(p).$$
2. For each c in Σ and w in Σ^*, the coefficient of cw in $|\mathcal{A}|_p$ is given by
$$(|\mathcal{A}|_p, cw) = \psi[p, c]\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \ldots, (|\mathcal{A}|_n, w)\big).$$

The *behaviour* of $\mathcal{A}$ is a formal power series $|\mathcal{A}|$ in $S\langle\langle\Sigma^*\rangle\rangle$ defined for all w in Σ^* by
$$(|\mathcal{A}|, w) = P_0\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \ldots, (|\mathcal{A}|_n, w)\big).$$

Alternating weighted automata clearly form a generalisation of "ordinary" (non-alternating) weighted automata. Indeed, let S be a commutative semiring. Define $\mathcal{L}$ to be the set of all polynomials of the form $s_1x_1 + s_2x_2 + \ldots + s_nx_n$, where n is a nonnegative integer, $s_1, \ldots, s_n$ are in S, and $x_1, \ldots, x_n$ are indeterminates. An alternating weighted automaton $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ over S can be called "sum-only" if the polynomial P_0 is in $\mathcal{L}$ and so are the polynomials $\psi[p, c]$ for each p in Q and c in Σ. One can easily see that a (non-alternating) weighted automaton without ε-labelled transitions can be viewed as a sum-only alternating weighted automaton and vice versa.

In the rest of this section, we shall introduce an important notion of what we shall call *evaluation polynomials*. Let $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ be an alternating weighted automaton with $|Q| = n$. Assume that a word $w = a_1a_2 \ldots a_m$ is given, where $a_1, \ldots, a_m$ are in Σ, and that our task is to calculate the coefficient of w in $|\mathcal{A}|$. One obvious approach is the "bottom-up evaluation". We start with the values $\tau(p)$ for each p in Q. Substituting these values into the polynomial $\psi[p, a_m]$, we can evaluate $(|\mathcal{A}|_p, a_m)$ for each p in Q. If we substitute these values into the polynomial $\psi[p, a_{m-1}]$, we can evaluate $(|\mathcal{A}|_p, a_{m-1}a_m)$ for each p in Q. By repeating this process, we are eventually able to evaluate $(|\mathcal{A}|_p, a_1a_2 \ldots a_m)$. If we now substitute these values into the polynomial P_0, we obtain the weight of the word w.

Another approach to calculate the weight of the word w is the "top-down" method. We start with the polynomial P_0. Substituting $x_i = \psi[i, a_1]$ for $i = 1, \ldots, n$ into the polynomial P_0, we obtain a polynomial P_1. In the next step, we substitute $x_i = \psi[i, a_2]$ for $i = 1, \ldots, n$ into the polynomial P_1 and obtain a polynomial P_2. This process is repeated, successively constructing polynomials $P_3, P_4, \ldots$, until the polynomial P_m is constructed. If we now substitute $x_i = \tau(i)$ for $i = 1, \ldots, n$ into the polynomial P_m, we obtain the weight of the word w. The same approach can be used if one wishes to calculate the coefficient of the word w in $|\mathcal{A}|_i$ for some i in $\{1, \ldots, n\}$. The only difference is that we start with the polynomial x_i instead of P_0 in this case. We shall call the polynomials that can be constructed in one of these ways *evaluation polynomials*.

Definition 4.3. Let $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ be an alternating weighted automaton over a commutative semiring S, let $n = |Q|$. For each p in Q and w in Σ^*, we define the *evaluation polynomial* $P_{\mathcal{A}}[p, w]$ (also denoted by $P_{\mathcal{A}}[i, w]$ if p is the i-th state in $\mathcal{A}$) in $S[x_1, \ldots, x_n]$ as follows:

1. If p is the i-th state of $\mathcal{A}$, then
$$P_{\mathcal{A}}[p, \varepsilon] = x_i.$$

2. If c is in Σ and w is in Σ^*, then

$$P_{\mathcal{A}}[p, wc] = P_{\mathcal{A}}[p, w]\big(\psi[1, c], \psi[2, c], \ldots \psi[n, c]\big).$$

For all w in Σ^*, we define the *evaluation polynomial* $P_{\mathcal{A}}[w]$ in $S[x_1, \ldots, x_n]$ by

$$P_{\mathcal{A}}[w] = P_0\big(P_{\mathcal{A}}[1, w], P_{\mathcal{A}}[2, w], \ldots, P_{\mathcal{A}}[n, w]\big).$$

The straightforward proof of the following lemma is left to the reader. It should nevertheless be clear from our discussion above.

Lemma 4.4. *Let $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ be an alternating weighted automaton, let $n = |Q|$. If v, w are in Σ^* and p is in Q, then*

$$(|\mathcal{A}|_p, vw) = P_{\mathcal{A}}[p, v]\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \ldots, (|\mathcal{A}|_n, w)\big),$$
$$(|\mathcal{A}|, vw) = P_{\mathcal{A}}[v]\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \ldots, (|\mathcal{A}|_n, w)\big).$$

5. Two-mode alternating weighted automata

We shall now give an alternative definition of alternating weighted automata, introducing what we shall call *two-mode alternating weighted automata*. These can be viewed simply as weighted automata, states of which are partitioned into two subsets: summation states and product states. We shall allow spontaneous ε-labelled transitions in two-mode alternating weighted automata, provided there is no cycle of such transitions. The possibility of ε-labelled transitions can be useful in some constructions, which would otherwise often be less intuitive. Nevertheless, we shall prove in Subsection 5.4 that spontaneous transitions are not strictly necessary: for each two-mode alternating weighted automaton, there is an equivalent two-mode automaton that contains no ε-labelled transitions.

We shall prove in Subsection 5.5 that two-mode alternating weighted automata are equivalent in expressive power to alternating weighted automata in the sense of Definition 4.1. However, this formal justification is hopefully not necessary to convince the reader that both models are the same in their essence.

Two-mode alternating weighted automata can be particularly useful when gaining intuition about alternation in weighted automata, which largely stems from the fact that they can be naturally depicted by diagrams (see Section 3). Moreover, we shall see that two-mode automata admit a characterisation of their behaviour in terms of *run trees*, a fairly intuitive concept that we shall introduce in Subsection 5.3. On the other hand, this is at the expense of the definitions being a bit cumbersome compared to the definitions presented in Section 4.

5.1. Definitions

Definition 5.1. Let S be a commutative semiring. A *two-mode alternating weighted automaton* over the semiring S is a septuple $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$, where: $Q^{\oplus}$ and $Q^{\otimes}$ are finite sets of states such that $Q^{\oplus} \cup Q^{\otimes} \neq \emptyset$ and $Q^{\oplus} \cap Q^{\otimes} = \emptyset$; Σ is an alphabet; T is a finite set of transitions, with which we associate mappings $src, dst\colon T \to (Q^{\oplus} \cup Q^{\otimes})$ and $\sigma\colon T \to \Sigma \cup \{\varepsilon\}$; $\nu\colon T \to S$ is a transition weighting function; $\iota\colon (Q^{\oplus} \cup Q^{\otimes}) \to S$ is an initial weighting function; $\tau\colon (Q^{\oplus} \cup Q^{\otimes}) \to S$ is a terminal weighting function.

A two-mode alternating weighted automaton can be viewed as a directed multigraph with labelled and weighted edges, where $Q^{\oplus} \cup Q^{\otimes}$ is the set of vertices and T is the set of edges, while src, dst, σ, and ν are mappings that assign a source, a destination, a label, and a weight to each edge. The elements of $Q^{\oplus}$ and $Q^{\otimes}$ shall be called *sum states* and *product states*, respectively.

Let $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$ be a two-mode alternating weighted automaton, let p be in $Q^{\oplus} \cup Q^{\otimes}$, and z in $\Sigma \cup \{\varepsilon\}$. We shall write $T_{\mathcal{A}}(p)$ to denote the set of all transitions t in T such that $src(t) = p$ and $T_{\mathcal{A}}(p, z)$ to denote the set of all transitions t in T satisfying $src(t) = p$ and $\sigma(t) = z$. We shall denote the set of all ε-labelled transitions in $\mathcal{A}$ by $T^{\varepsilon}_{\mathcal{A}}$. If clear from the context, we shall often omit the subscript denoting the automaton in consideration.

The behaviour of a two-mode alternating weighted automaton will be defined only if the automaton contains no cycles of ε-labelled transitions.

Definition 5.2. Let $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$ be a two-mode alternating weighted automaton over some commutative semiring S. We say that $\mathcal{A}$ is *without ε-cycles* if the directed graph with vertex set $Q^{\oplus} \cup Q^{\otimes}$ and edge set T^{ε} contains no cycle.

From now on, all two-mode alternating weighted automata are assumed to be without ε-cycles (we shall usually not state this explicitly).

We are now prepared to define the behaviour $|\mathcal{A}|$ of a two-mode alternating weighted automaton $\mathcal{A}$. Once again, this is done via introducing an auxiliary series $|\mathcal{A}|_p$ for each state p of the automaton $\mathcal{A}$, which can be interpreted as the behaviour of $\mathcal{A}$ when all runs in $\mathcal{A}$ are forced to start in p (with initial weight 1_S). Apart from the intuitive meaning of sum states and product states explained already in Section 3, we have to pay some extra attention to boundary cases. For instance, if there is no c-labelled or ε-labelled transition leading from some state p, we shall define the coefficient of each word beginning by c in $|\mathcal{A}|_p$ to be 0_S regardless if p is a sum state or a product state. (Note that for product states, this is not consistent with the definition via empty product.) Similarly, terminal weights are added instead of multiplied to the rest of the expression in the definition of $(|\mathcal{A}|_p, \varepsilon)$ for a product state p. These choices are motivated mainly by the alternative characterisation of behaviour via *run trees*, introduced later in Subsection 5.3. However, it is not hard to see that similar details have no effect on the expressive power of the model.

Definition 5.3. Let $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$ be a two-mode alternating weighted automaton (without ε-cycles) over a commutative semiring S. For each p in $Q^{\oplus} \cup Q^{\otimes}$, we define a series $|\mathcal{A}|_p$ in $S\langle\langle\Sigma^*\rangle\rangle$ as follows:

1. If p is in $Q^{\oplus}$, then the coefficient of ε in $|\mathcal{A}|_p$ is given by

$$(|\mathcal{A}|_p, \varepsilon) = \tau(p) + \sum_{t \in T(p,\varepsilon)} \nu(t) \cdot (|\mathcal{A}|_{dst(t)}, \varepsilon).$$

   Moreover, for each c in Σ and w in Σ^*,

$$(|\mathcal{A}|_p, cw) = \sum_{t \in T(p,c)} \nu(t) \cdot (|\mathcal{A}|_{dst(t)}, w) + \sum_{t \in T(p,\varepsilon)} \nu(t) \cdot (|\mathcal{A}|_{dst(t)}, cw).$$

2. If p is in $Q^{\otimes}$, then the coefficient of ε in $|\mathcal{A}|_p$ is given by

$$(|\mathcal{A}|_p, \varepsilon) = \begin{cases} \tau(p) + \prod_{t \in T(p,\varepsilon)} \nu(t) \cdot (|\mathcal{A}|_{dst(t)}, \varepsilon) & \text{if } T(p,\varepsilon) \neq \emptyset, \\ \tau(p) & \text{otherwise.} \end{cases}$$

   Moreover, for each c in Σ and w in Σ^*,

$$(|\mathcal{A}|_p, cw) = \begin{cases} \left(\prod_{t \in T(p,c)} \nu(t) \cdot (|\mathcal{A}|_{dst(t)}, w)\right) \cdot \left(\prod_{t \in T(p,\varepsilon)} \nu(t) \cdot (|\mathcal{A}|_{dst(t)}, cw)\right) & \text{if } T(p,c) \cup T(p,\varepsilon) \neq \emptyset, \\ 0_S & \text{otherwise.} \end{cases}$$

The *behaviour* of $\mathcal{A}$ is a formal power series $|\mathcal{A}|$ in $S\langle\langle\Sigma^*\rangle\rangle$ defined for all $w \in \Sigma^*$ by

$$(|\mathcal{A}|, w) = \sum_{p \in Q^{\oplus} \cup Q^{\otimes}} \iota(p) \cdot (|\mathcal{A}|_p, w).$$

Evaluation polynomials, which we have defined for alternating weighted automata, can be defined for two-mode alternating weighted automata as well. Again, we shall always assume some linear ordering of states of the automaton in consideration to be given a priori. Whenever we refer to "the i-th state", we mean the i-th state with respect to this linear ordering; if p is the i-th state of $\mathcal{A}$, we shall also write $|\mathcal{A}|_i$, $\iota(i)$, and $\tau(i)$ instead of $|\mathcal{A}|_p$, $\iota(p)$, and $\tau(p)$, respectively.

Definition 5.4. Let $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$ be a two-mode alternating weighted automaton over some commutative semiring S, let $n = |Q^{\oplus} \cup Q^{\otimes}|$. For each w in Σ^* and p in $Q^{\oplus} \cup Q^{\otimes}$, we define the *evaluation polynomial* $P_{\mathcal{A}}[p, w]$ (also denoted by $P_{\mathcal{A}}[i, w]$ if p is the i-th state of $\mathcal{A}$) in $S[x_1, \ldots, x_n]$ as follows:

1. If p is the i-th state in $Q^{\oplus} \cup Q^{\otimes}$, then

$$P_{\mathcal{A}}[p, \varepsilon] = x_i.$$

2. If p is in $Q^{\oplus}$, c is in Σ, and w is in Σ^*, then

$$P_{\mathcal{A}}[p, cw] = \sum_{t \in T(p,c)} \nu(t) \cdot P_{\mathcal{A}}[dst(t), w] + \sum_{t \in T(p,\varepsilon)} \nu(t) \cdot P_{\mathcal{A}}[dst(t), cw].$$

3. If p is in $Q^{\otimes}$, c is in Σ, and w is in Σ^*, then

$$P_{\mathcal{A}}[p, cw] = \begin{cases} \left(\prod_{t \in T(p,c)} \nu(t) \cdot P_{\mathcal{A}}[dst(t), w]\right) \cdot \left(\prod_{t \in T(p,\varepsilon)} \nu(t) \cdot P_{\mathcal{A}}[dst(t), cw]\right) & \text{if } T(p,c) \cup T(p,\varepsilon) \neq \emptyset, \\ 0_S & \text{otherwise.} \end{cases}$$

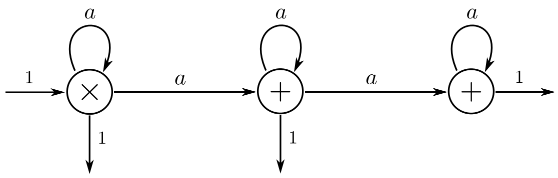

Fig. 2. The automaton $\mathcal{A}_1$.

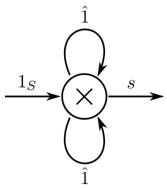

Fig. 3. The automaton $\mathcal{A}_2$.

For each w in Σ^*, we define the *evaluation polynomial* $P_{\mathcal{A}}[w]$ in $S[x_1, \ldots, x_n]$ by

$$P_{\mathcal{A}}[w] = \sum_{p \in Q^{\oplus} \cup Q^{\otimes}} \iota(p) \cdot P_{\mathcal{A}}[p, w].$$

The following lemma is an analogy to Lemma 4.4. We leave the proof of this fact for the reader.

Lemma 5.5. *Let $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$ be a two-mode alternating weighted automaton, and let us denote $n = |Q^{\oplus} \cup Q^{\otimes}|$. If v, w are in Σ^* and p is in $Q^{\oplus} \cup Q^{\otimes}$, then*

$$(|\mathcal{A}|_p, vw) = P_{\mathcal{A}}[p, v]\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \ldots, (|\mathcal{A}|_n, w)\big),$$
$$(|\mathcal{A}|, vw) = P_{\mathcal{A}}[v]\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \ldots, (|\mathcal{A}|_n, w)\big).$$

5.2. Examples

Let us now demonstrate the abilities of two-mode alternating weighted automata on several examples. Note that none of the automata constructed below contains transitions labelled by ε. It is thus obvious that they can be *directly interpreted* as alternating weighted automata in the sense of Definition 4.1 as well.[5]

Example 5.6. In Fig. 2, we give a diagram of a two-mode alternating weighted automaton $\mathcal{A}_1$ over the alphabet $\Sigma_1 = \{a\}$ and over the semiring of natural numbers with standard operations of addition and multiplication. It can be easily seen that the coefficient of a^n in $|\mathcal{A}_1|$ is $n!$ for all nonnegative integers n.

Example 5.7. In Fig. 3, a diagram of a two-mode alternating weighted automaton $\mathcal{A}_2$ over the alphabet $\Sigma_2 = \{\hat{1}\}$ and over some commutative semiring S is depicted. The semiring S is assumed to contain an element s. It is then easy to see that $(|\mathcal{A}_2|, (\hat{1})^n) = s^{2^n}$ for all n in $\mathbb{N}$.

Example 5.8. Let us now slightly generalise the previous example. In Fig. 4, we give a diagram of a two-mode alternating weighted automaton $\mathcal{A}_3$ over an alphabet $\Sigma_3 = \{\hat{0}, \hat{1}\}$ and over some commutative semiring containing an element s. From now on, we shall often label arrows in diagrams of two-mode alternating weighted automata by *sets* of symbols with or without coefficients. If an arrow leading from state p to state q is labelled with a set containing sc, where s is a coefficient taken from the semiring in consideration and c is a symbol, then there is a transition from p to q with label c and weight s. If some symbol in the set has no explicitly given coefficient, then this coefficient is understood to be 1_S. The label representing a set $\{d_1, d_2, \ldots, d_m\}$ is usually written simply as $d_1, d_2, \ldots, d_m$, omitting the braces. The diagram in Fig. 4 contains three arrows that should be interpreted in this way.

Each word w over the alphabet $\Sigma_3 = \{\hat{0}, \hat{1}\}$ can be viewed as a binary representation of some nonnegative integer; we shall write $\operatorname{int}(w)$ to denote this number. The reader can check that for each w in Σ_3^*, the coefficient of w in $|\mathcal{A}_3|$ is $s^{\operatorname{int}(w)}$.

[5] We shall see in Subsection 5.5 that two-mode alternating weighted automata are equivalent to alternating weighted automata in expressive power. This means that *any* two-mode alternating weighted automaton can be reworked to an alternating weighted automaton and vice versa.

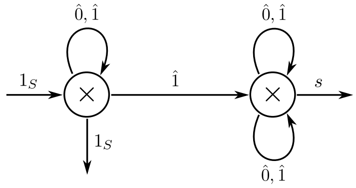

Fig. 4. The automaton $\mathcal{A}_3$.

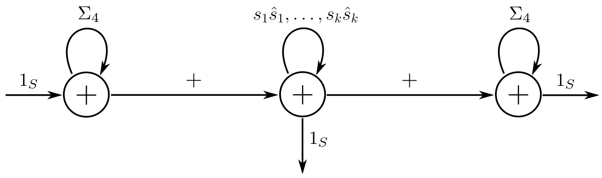

Fig. 5. The automaton $\mathcal{A}_4$.

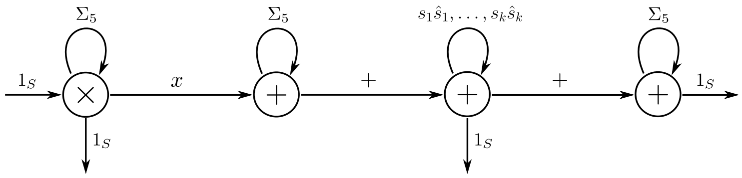

Fig. 6. The automaton $\mathcal{A}_5$.

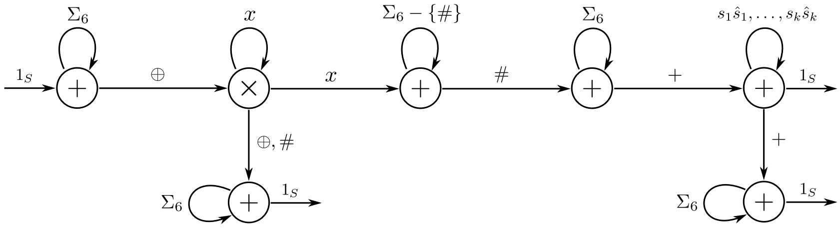

Fig. 7. The automaton $\mathcal{A}_6$.

Example 5.9. We shall now give an example of a (non-alternating) weighted automaton $\mathcal{A}_4$ over some commutative semiring S, which we shall later modify into a two-mode alternating weighted automaton. The automaton $\mathcal{A}_4$ is depicted in Fig. 5; the alphabet Σ_4 of the automaton $\mathcal{A}_4$ consists of the symbol "+" and of symbols $\hat{s}_1, \dots, \hat{s}_k$ for some particular elements $s_1, s_2, \dots, s_k$ of S.

Let L' be the language that consists of all words w over Σ_4 such that $w = \hat{s}_{i_1}\hat{s}_{i_2}\dots\hat{s}_{i_m}$, where m is a nonnegative integer and $i_1, \dots, i_m$ are in $\{1, \dots, k\}$. For each such w, we define $\mathrm{elem}(w)$ to be the semiring element $\prod_{j=1}^{m} s_{i_j}$. Moreover, let L_{elem} be the language consisting of all words w over Σ_4 such that $w = +u_1 + u_2 \dots + u_m$ for some nonnegative integer m and u_i in L' for $i = 1, \dots, m$. For each such w, we define $\mathrm{elem}(w) = \sum_{i=1}^{m} \mathrm{elem}(u_i)$. It is easy to see that $(|\mathcal{A}_4|, w) = \mathrm{elem}(w)$ for each w in L_{elem}. Coefficients of words from $\Sigma_4^* - L_{\mathrm{elem}}$ in $|\mathcal{A}_4|$ are not important for our purposes.

Example 5.10. We shall now modify the (non-alternating) weighted automaton $\mathcal{A}_4$ into a two-mode alternating weighted automaton $\mathcal{A}_5$ over S and over the alphabet $\Sigma_5 = \Sigma_4 \cup \{x\}$. This new automaton is depicted in Fig. 6. We can see that $(|\mathcal{A}_5|, x^m u) = (\mathrm{elem}(u))^m$ for each nonnegative integer m and each word u in L_{elem}. The automaton $\mathcal{A}_5$ has the ability to calculate exponents of semiring elements.

Example 5.11. We can go even further with the previous example. Let $\Sigma_6 = \Sigma_5 \cup \{\oplus, \#\}$. Let L'' consist of all words w over Σ_6 such that $w = \oplus x^{m_1} \oplus x^{m_2} \dots \oplus x^{m_l}$ for some nonnegative integers l and $m_1, \dots, m_l$. For each such w, let the polynomial $\sum_{i=1}^{l} x^{m_i}$ in $S[x]$ be denoted by $\mathrm{poly}[w]$. In Fig. 7, we give a diagram of a two-mode alternating weighted automaton $\mathcal{A}_6$ over S and Σ_6 such that $(|\mathcal{A}_6|, u\#v) = \mathrm{poly}[u](\mathrm{elem}(v))$ for each u in L'' and v in L_{elem}.

Example 5.12. Even more generally, let $\Sigma_7 = \Sigma_6 \cup \{\langle, \rangle\}$ (i.e., Σ_7 consists of symbols from Σ_6 and symbols for angle brackets). Let L_{poly} consist of all words w over Σ_7 such that $w = \oplus\langle u_1\rangle x^{m_1} \oplus \langle u_2\rangle x^{m_2} \dots \oplus \langle u_l\rangle x^{m_l}$ for some nonnega-

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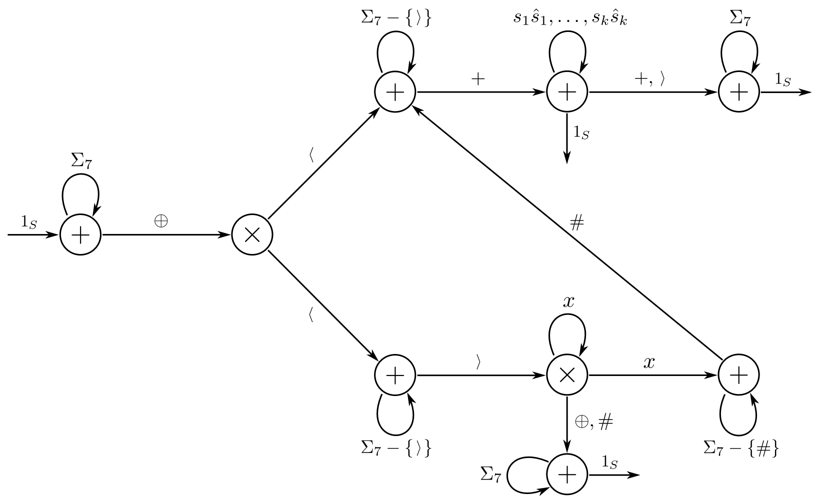

Fig. 8. The automaton $\mathcal{A}_7$.

tive integers l and $m_1, \ldots, m_l$ and words $u_1, \ldots, u_l$ in L_{elem}. For each such w, let the polynomial $\sum_{i=1}^{l} \mathrm{elem}(u_i)x^{m_i}$ in $S[x]$ be denoted by poly$[w]$. In Fig. 8, a two-mode alternating weighted automaton $\mathcal{A}_7$ over S and Σ_7 is depicted such that $(|\mathcal{A}_7|, u\#v) = \mathrm{poly}[u](\mathrm{elem}(v))$ for each u in L_{poly} and v in L_{elem}. The automaton $\mathcal{A}_7$ has the ability to substitute into polynomials.

5.3. Run trees

We shall now observe that the behaviour of a two-mode alternating weighted automaton admits a relatively natural characterisation in terms of *run trees*.

One can think of a run on a word $w = a_1 \ldots a_m$ (with m in $\mathbb{N}$ and $a_1, \ldots, a_m$ in Σ) in a two-mode alternating weighted automaton as follows: if a sum state is "visited" by the run before reading a_k for some k in $\{1, \ldots, m\}$, then the run chooses to follow *precisely one* transition leading from that state, which has to be labelled either by a_k, or by ε; if a product state is "visited", the run has to follow *all* a_k-labelled and ε-labelled transitions leading from the state. Moreover, if a sum state is "visited" after a_m is read, the run either chooses to follow *precisely one* ε-labelled transition, or halts; in a product state, the run either follows *all* ε-labelled transitions leading from the given state, or halts. The flow of a run in a two-mode alternating weighted automaton on a word w can thus be viewed as a rooted tree, which branches whenever the run passes a product state – we shall call this rooted tree a *run tree* on w.

The weight of a run tree will simply be given by the product of weights of all its constituent transitions, the initial weight of its root, and the terminal weights of its leaves. We shall prove that for each two-mode alternating weighted automaton $\mathcal{A}$ and each w over its input alphabet, the coefficient of w in the behaviour of $\mathcal{A}$ equals the sum of weights of all run trees on w in $\mathcal{A}$.

To give a formal definition of run trees, let us first recall that a *rooted tree* is a triple $\mathcal{T} = (X, E, r)$, where X is a nonempty finite set of nodes, $E \subseteq \binom{X}{2}$ is a set of edges, and r in X is a distinguished node called the *root*, with the usual conditions of connectedness and acyclicity. For each x in X, we shall denote by $d(x)$ the *depth* of x, i.e., the distance of x from the root (in particular, $d(r) = 0$). A *leaf* in $\mathcal{T}$ is a node x in X, for which there is no y in X such that $d(x) < d(y)$ and $\{x, y\}$ is in E. We shall denote the set of all leaves in $\mathcal{T}$ by $\ell(\mathcal{T})$. It is obvious that either $d(y) = d(x) + 1$ or $d(x) = d(y) + 1$ holds for each edge $\{x, y\}$ in E; we shall write $\{x, y\} = xy$ in the former case and $\{x, y\} = yx$ in the latter case.[6] It is also clear that precisely one node x in X exists for each y in $X - \{r\}$ such that xy is in E; we shall call this node x the *parent* of y and write $x = \mathrm{par}(y)$. Furthermore, we shall write $\omega(x) = \{xy \in E \mid y \in X\}$ for the set of all outgoing edges of a node x in X.

We shall now use the notion of a rooted tree to define run trees of a given two-mode automaton $\mathcal{A}$. More precisely, we shall first define what we shall call *quasi run trees*, in which the initial weight of the root might not be the same as the initial weight of the corresponding state, but can be arbitrarily assigned. We shall define a *p-run tree* to be a quasi run tree such that the root corresponds to state p and has initial weight assigned to 1_S – such trees will be particularly useful in inductive proofs, as we shall see that they correspond to the series $|\mathcal{A}|_p$. Finally, we shall say that a quasi run tree is a *run tree* if the initial weight of the root is the same as the initial weight of the corresponding state.

Let us now define quasi run trees. We shall first introduce an auxiliary notion of a *structural run tree*, which can be seen as a rooted tree together with mappings assigning states to nodes and transitions to edges that are consistent with each

[6] That is, we shall write edges as directed from the root towards the leaves.

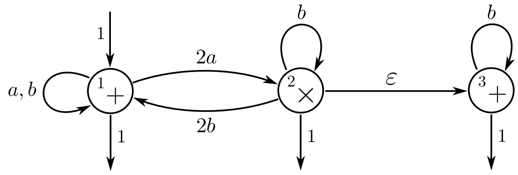

Fig. 9. The two-mode alternating weighted automaton $\mathcal{A}$.

other, and with a semiring element representing the initial weight of the root. By imposing several additional conditions, we shall be finally able to refine the notion of structural run trees in order to obtain a definition of quasi run trees, as described above.

Formally, let $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$ be a two-mode alternating weighted automaton over a semiring S. We may then define a *structural run tree* of $\mathcal{A}$ to be a quadruple $\mathcal{R} = (\mathcal{T}, \mathrm{st}, \mathrm{tr}, \varrho)$, where: $\mathcal{T} = (X, E, r)$ is a rooted tree; $\mathrm{st}\colon X \to Q^{\oplus} \cup Q^{\otimes}$ is a mapping assigning states of the automaton $\mathcal{A}$ to nodes; $\mathrm{tr}\colon E \to T$ is a mapping assigning transitions of $\mathcal{A}$ to edges such that $src(\mathrm{tr}(xy)) = \mathrm{st}(x)$ and $dst(\mathrm{tr}(xy)) = \mathrm{st}(y)$ holds for each xy in E; ϱ in S is the initial weight of a root. Moreover, let us define a mapping $word_{\mathcal{R}}\colon X \to \Sigma^*$ inductively by $word_{\mathcal{R}}(r) = \varepsilon$ and $word_{\mathcal{R}}(x) = word_{\mathcal{R}}(\mathrm{par}(x))\sigma(\mathrm{tr}(\mathrm{par}(x)x))$ for each x in $X - \{r\}$. We shall simply write $word$ instead of $word_{\mathcal{R}}$ when $\mathcal{R}$ is understood.

We shall say that a structural run tree $\mathcal{R} = (\mathcal{T}, \mathrm{st}, \mathrm{tr}, \varrho)$ of $\mathcal{A}$ with $\mathcal{T} = (X, E, r)$ is a *quasi run tree* of $\mathcal{A}$ on $w = a_1 \dots a_m$ in Σ^* (with m in $\mathbb{N}$ and $a_1, \dots, a_m$ in Σ) if the following conditions are satisfied:

1. The equality $word(x) = w$ holds for each leaf x in $\ell(\mathcal{T})$. (This also obviously implies that $word(x)$ is a prefix of w for each x in X.)
2. If x in $X - \ell(\mathcal{T})$ is a node such that $\mathrm{st}(x)$ is in $Q^{\oplus}$, then $\omega(x)$ is a singleton.
3. If x in $X - \ell(\mathcal{T})$ is a node such that $\mathrm{st}(x)$ is in $Q^{\otimes}$ and $word(x) = a_1 \dots a_k$ for some k in $\{0, \dots, m\}$, then $\mathrm{tr}(xy) \neq \mathrm{tr}(xz)$ holds for all xy, xz in E such that $xy \neq xz$, and

$$\mathrm{tr}(\omega(x)) = \begin{cases} T(\mathrm{st}(x), a_{k+1}) \cup T(\mathrm{st}(x), \varepsilon) & \text{if } k < m, \\ T(\mathrm{st}(x), \varepsilon) & \text{if } k = m. \end{cases}$$

We shall say that $\mathcal{R}$ is a *quasi run tree* of $\mathcal{A}$ if it is a quasi run tree of $\mathcal{A}$ on some word w in Σ^*.

For each quasi run tree $\mathcal{R} = (\mathcal{T}, \mathrm{st}, \mathrm{tr}, \varrho)$ of a two-mode alternating weighted automaton $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$, where $\mathcal{T} = (X, E, r)$, we shall define its *weight* $|\mathcal{R}|$ by

$$|\mathcal{R}| = \varrho \cdot \left(\prod_{e \in E} \nu(\mathrm{tr}(e)) \right) \cdot \left(\prod_{x \in \ell(\mathcal{T})} \tau(\mathrm{st}(x)) \right).$$

Finally, let $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$ be a two-mode alternating weighted automaton over S, p be in $Q^{\oplus} \cup Q^{\otimes}$, and $w = a_1 \dots a_m$ be in Σ^* (with m in $\mathbb{N}$ and $a_1, \dots, a_m$ in Σ). We shall define a *p-run tree* of $\mathcal{A}$ on w to be a quasi run tree $\mathcal{R} = ((X, E, r), \mathrm{st}, \mathrm{tr}, \varrho)$ of $\mathcal{A}$ on w such that $\mathrm{st}(r) = p$ and $\varrho = 1_S$. Moreover, we shall define a *run tree* of $\mathcal{A}$ on w to be a quasi run tree $\mathcal{R} = ((X, E, r), \mathrm{st}, \mathrm{tr}, \varrho)$ of $\mathcal{A}$ on w such that $\varrho = \iota(\mathrm{st}(r))$. We shall denote the set of all p-run trees of $\mathcal{A}$ on w by $\mathscr{R}_p(\mathcal{A}, w)$ and the set of all run trees of $\mathcal{A}$ on w by $\mathscr{R}(\mathcal{A}, w)$.

Example 5.13. Consider a two-mode alternating weighted automaton $\mathcal{A}$ over the alphabet $\Sigma = \{a, b\}$ and over the semiring $\mathbb{N}$ of natural numbers with standard addition and multiplication, depicted in Fig. 9. In addition to its mode, each state of the automaton $\mathcal{A}$ is labelled by its name in Fig. 9; the set of states of $\mathcal{A}$ is thus $\{1, 2, 3\}$.

In Fig. 10, all run trees of $\mathcal{A}$ on the word ab are shown. Nodes of the run trees are labelled by the corresponding state and by "+" or "×" depending on the mode of that state. Each edge is labelled according to the corresponding transition. Moreover, there is an ingoing arrow to the root in each of the depicted run trees. This is always labelled by the initial weight of the root (for run trees, this is the same as the initial weight of the state corresponding to the root). Similarly, there are outgoing arrows from the leaves of each of the run trees – these are labelled by terminal weights of the corresponding states.

We may now prove a characterisation of the behaviour $|\mathcal{A}|$ of a two-mode alternating weighted automaton $\mathcal{A}$ in terms of weights of run trees of $\mathcal{A}$. We shall do this by first proving an auxiliary lemma characterising the series $|\mathcal{A}|_p$, for each state p of $\mathcal{A}$, in terms of weights of p-run trees; the statement about $|\mathcal{A}|$ can be obtained as an easy consequence of this result.

Recall that all two-mode alternating weighted automata are understood to be without ε-cycles. This obviously implies that a partial ordering $\prec$ can be defined on states of each such two-mode alternating weighted automaton $\mathcal{A}$, such that $p \prec q$ if and only if p is reachable from q by a sequence of ε-labelled transitions. The minimal elements with respect to this

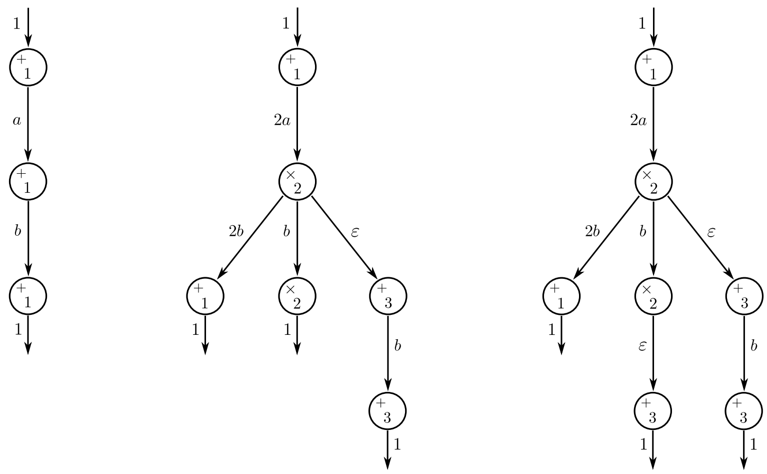

Fig. 10. All run trees of the automaton $\mathcal{A}$ from Fig. 9 on the word *ab*.

partial ordering are clearly the states of $\mathcal{A}$ with no outgoing ε-labelled transitions. We shall use these properties implicitly in the proof of the following lemma.

Lemma 5.14. *Let $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$ be a two-mode alternating weighted automaton over a semiring S, p in $Q^{\oplus} \cup Q^{\otimes}$, and w in Σ^*. Then*

$$(|\mathcal{A}|_p, w) = \sum_{\mathcal{R} \in \mathscr{R}_p(\mathcal{A}, w)} |\mathcal{R}|.$$

Proof. By induction on $|w|$ and on the partial ordering $\prec$ introduced above. First, let $w = \varepsilon$ and $T(p, \varepsilon) = \emptyset$. It follows by Definition 5.3 that $(|\mathcal{A}|_p, w) = \tau(p)$. On the other hand, it is clear that the only p-run tree of $\mathcal{A}$ on $w = \varepsilon$ consists of a single node – this is at the same time a root and a leaf – corresponding to the state p. The weight of such tree is $\tau(p)$ as well.

Next, let $w = \varepsilon$ and $T(p, \varepsilon) \neq \emptyset$, and suppose that the statement to be proved holds for $w = \varepsilon$ and for all states in $dst(T(p, \varepsilon))$. If p is in $Q^{\oplus}$, then it follows by Definition 5.3 that

$$(|\mathcal{A}|_p, \varepsilon) = \tau(p) + \sum_{t \in T(p,\varepsilon)} \nu(t) \cdot (|\mathcal{A}|_{dst(t)}, \varepsilon).$$

By the induction hypothesis, this rewrites to

$$(|\mathcal{A}|_p, \varepsilon) = \tau(p) + \sum_{t \in T(p,\varepsilon)} \nu(t) \cdot \sum_{\mathcal{R}' \in \mathscr{R}_{dst(t)}(\mathcal{A},\varepsilon)} |\mathcal{R}'| = \sum_{\mathcal{R} \in \mathscr{R}_p(\mathcal{A},\varepsilon)} |\mathcal{R}|,$$

where the second equality follows by the fact that $\mathscr{R}_p(\mathcal{A}, \varepsilon)$ consists of a tree with a single node corresponding to p (this has weight $\tau(p)$) and for each transition t in $T(p, \varepsilon)$, of a tree rooted in a node corresponding to p that is connected by an edge corresponding to t to the root of some subtree $\mathcal{R}'$ from $\mathscr{R}_{dst(t)}(\mathcal{A}, \varepsilon)$ (this has weight $\nu(t) \cdot |\mathcal{R}'|$). If p is in $Q^{\otimes}$, then Definition 5.3 gives us

$$(|\mathcal{A}|_p, \varepsilon) = \tau(p) + \prod_{t \in T(p,\varepsilon)} \nu(t) \cdot (|\mathcal{A}|_{dst(t)}, \varepsilon),$$

which rewrites by the induction hypothesis to

$$(|\mathcal{A}|_p, \varepsilon) = \tau(p) + \prod_{t \in T(p,\varepsilon)} \nu(t) \cdot \sum_{\mathcal{R}'_t \in \mathscr{R}_{dst(t)}(\mathcal{A},\varepsilon)} |\mathcal{R}'_t| = \sum_{\mathcal{R} \in \mathscr{R}_p(\mathcal{A},\varepsilon)} |\mathcal{R}|.$$

Here, the second equality follows by the fact that $\mathscr{R}_p(\mathcal{A}, \varepsilon)$ consists of a tree with a single node corresponding to p (this has weight $\tau(p)$) and of trees rooted in a node corresponding to p that is connected by edges in one-to-one correspondence with transitions t in $T(p, \varepsilon)$ to roots of some subtrees $\mathcal{R}'_t$ in $\mathscr{R}_{dst(t)}(\mathcal{A}, \varepsilon)$ (each such tree has weight $\prod_{t \in T(p,\varepsilon)} \nu(t) \cdot |\mathcal{R}'_t|$).

Now, suppose that the statement holds for some w in Σ^* and for all states of $\mathcal{A}$, and consider a word cw for some c in Σ. If p is a state in $Q^{\oplus}$ such that $T(p,\varepsilon)=\emptyset$, then it follows by Definition 5.3 that

$$(|\mathcal{A}|_p, cw) = \sum_{t\in T(p,c)} \nu(t)\cdot(|\mathcal{A}|_{dst(t)}, w).$$

By the induction hypothesis, this rewrites to

$$(|\mathcal{A}|_p, cw) = \sum_{t\in T(p,c)} \nu(t)\cdot \sum_{\mathcal{R}'\in\mathscr{R}_{dst(t)}(\mathcal{A},w)} |\mathcal{R}'| = \sum_{\mathcal{R}\in\mathscr{R}_p(\mathcal{A},cw)} |\mathcal{R}|,$$

where the second equality follows by the fact that $\mathscr{R}_p(\mathcal{A},cw)$ consists of all trees rooted in a node corresponding to p that is connected by an edge corresponding to a c-labelled transition t to the root of some subtree $\mathcal{R}'$ in $\mathscr{R}_{dst(t)}(\mathcal{A},w)$. If p is in $Q^{\otimes}$, $T(p,\varepsilon)=\emptyset$, and $T(p,c)\neq\emptyset$, then Definition 5.3 gives us

$$(|\mathcal{A}|_p, cw) = \prod_{t\in T(p,c)} \nu(t)\cdot(|\mathcal{A}|_{dst(t)}, w).$$

By the induction hypothesis, this rewrites to

$$(|\mathcal{A}|_p, cw) = \prod_{t\in T(p,c)} \nu(t)\cdot \sum_{\mathcal{R}'_t\in\mathscr{R}_{dst(t)}(\mathcal{A},w)} |\mathcal{R}'_t| = \sum_{\mathcal{R}\in\mathscr{R}_p(\mathcal{A},cw)} |\mathcal{R}|,$$

where the second equality follows by the fact that $\mathscr{R}_p(\mathcal{A},cw)$ consists of trees rooted in a node corresponding to p that is connected by edges in one-to-one correspondence with transitions t in $T(p,c)$ to roots of some subtrees $\mathcal{R}'_t$ in $\mathscr{R}_{dst(t)}(\mathcal{A},w)$. If $T(p,\varepsilon)=\emptyset$ and $T(p,c)=\emptyset$, then $(|\mathcal{A}|_p, cw)=0$, which is equal to the sum of $|\mathcal{R}|$ over $\mathcal{R}$ from the obviously empty set $\mathscr{R}_p(\mathcal{A},cw)$. Finally, if p is a state such that $T(p,\varepsilon)\neq\emptyset$, then the reasoning is similar as above; however, both c-labelled transitions and ε-labelled transitions leading to states preceding p in the partial ordering $\prec$ have to be taken into account. □

Theorem 5.15. *Let $\mathcal{A}=(Q^{\oplus},Q^{\otimes},\Sigma,T,\nu,\iota,\tau)$ be a two-mode alternating weighted automaton over a semiring S, and let w be in Σ^*. Then*

$$(|\mathcal{A}|, w) = \sum_{\mathcal{R}\in\mathscr{R}(\mathcal{A},w)} |\mathcal{R}|.$$

Proof. A run tree with root corresponding to a state p can be viewed as a p-run tree with the initial weight of the root changed from 1_S to $\iota(p)$. Moreover, this clearly introduces a bijection between run trees with root corresponding to p and p-run trees. Hence,

$$(|\mathcal{A}|, w) = \sum_{p\in Q^{\oplus}\cup Q^{\otimes}} \iota(p)|\mathcal{A}|_p = \sum_{p\in Q^{\oplus}\cup Q^{\otimes}} \sum_{\mathcal{R}\in\mathscr{R}_p(\mathcal{A},w)} \iota(p)|\mathcal{R}| = \sum_{\mathcal{R}\in\mathscr{R}(\mathcal{A},w)} |\mathcal{R}|,$$

and the theorem is proved. □

5.4. Elimination of spontaneous transitions

We have allowed spontaneous ε-labelled transitions in our definition of *two-mode* alternating weighted automata mainly in order to make constructions involving such automata easier and more intuitive. We shall now show that this feature of our definition has no effect on the expressive power of the model: we shall prove that ε-labelled transitions can be removed from two-mode alternating automata. We shall say that a two-mode alternating weighted automaton $\mathcal{A}$ is *ε-free* if it contains no ε-labelled transitions.

Theorem 5.16. *Let $\mathcal{A}_1$ be a two-mode alternating weighted automaton over a commutative semiring S. Then there exists an ε-free two-mode alternating weighted automaton $\mathcal{A}_2$ over S such that $|\mathcal{A}_2|=|\mathcal{A}_1|$.*

Proof. Let $\mathcal{A}_1=(Q_1^{\oplus},Q_1^{\otimes},\Sigma,T_1,\nu_1,\iota_1,\tau_1)$ and $|Q_1^{\oplus}\cup Q_1^{\otimes}|=n_1$. We shall construct an ε-free two-mode alternating weighted automaton $\mathcal{A}_2=(Q_2^{\oplus},Q_2^{\otimes},\Sigma,T_2,\nu_2,\iota_2,\tau_2)$ over S such that $|\mathcal{A}_2|=|\mathcal{A}_1|$.

The idea behind the construction is to examine evaluation polynomials for words of length two in $\mathcal{A}_1$ – that is, polynomials $P_{\mathcal{A}_1}[p,ab]$ for p in $Q_1^{\oplus}\cup Q_1^{\otimes}$ and a,b in Σ. Each such polynomial can be written in the standard form as a sum $m_1+\ldots+m_k$ of some monomials $m_1,\ldots,m_k$. One can then proclaim p to be a sum state in $\mathcal{A}_2$ and lead k transitions labelled by a from p to some new product states corresponding to the monomials $m_1,\ldots,m_k$. If $m=cx_1^{i_1}\ldots x_{n_1}^{i_{n_1}}$ is any of these monomials, then one might weight the transition from p to the new state corresponding to m by c. Moreover, for $j = 1, \ldots, n_1$, one might add i_j transitions labelled by b and weighted by 1_S from the new state corresponding to m to the j-th state of the original automaton. The evaluation polynomial $P_{\mathcal{A}_2}[p, ab]$ will then clearly be the same as $P_{\mathcal{A}_1}[p, ab]$.

However, one has to address some additional technical details in order to make the construction work. First, in order to deal with runs ending by sequences of ε-labelled transitions, the terminal weight of each state p from $Q_1^{\oplus} \cup Q_1^{\otimes}$ in $\mathcal{A}_2$ has to be set to $(|\mathcal{A}_1|_p, \varepsilon)$ (the terminal weight of the new states corresponding to monomials will be 0_S). Moreover, the construction described up to now works for words of even length only. In order to deal with words of odd length, we shall add one new state q_d with terminal weight 1_S and no outgoing transitions, and lead an ε-labelled transition weighted by $(|\mathcal{A}_1|_p, c)$ from p to q_d for each p from $Q_1^{\oplus} \cup Q_1^{\otimes}$ and each c in Σ.

Let us now describe the construction formally. For each p in $Q_1^{\oplus} \cup Q_1^{\otimes}$ and w in Σ^*, the evaluation polynomial $P_{\mathcal{A}_1}[p, w]$ can be written as a sum of distinct nonzero monomials $m_1, m_2, \ldots, m_k$ in $S[x_1, \ldots, x_n]$. Let $M(p, w) = \{m_1, \ldots, m_k\}$.

First, let $Q_2^{\oplus} = Q_1^{\oplus} \cup Q_1^{\otimes} \cup \{q_d\}$, where q_d is a new state that is not in $Q_1^{\oplus} \cup Q_1^{\otimes}$, and let $Q_2^{\otimes}$ consist of all pairs (m, b), where b is in Σ and m is in $M(p, ab)$ for some p in $Q_1^{\oplus} \cup Q_1^{\otimes}$ and a in Σ. Let us assume some linear ordering of states of $\mathcal{A}_2$ to be given, in which states from $Q_1^{\oplus} \cup Q_1^{\otimes}$ keep their numerical order from the ordering of states of $\mathcal{A}_1$. Let $n_2 = |Q_2^{\oplus} \cup Q_2^{\otimes}|$.

The set of transitions T_2 will be constructed from the following sets[7]:

- T_α that consists of all tuples (p, c, s, q_d), where p is in $Q_1^{\oplus} \cup Q_1^{\otimes}$, c is in Σ, and $s = (|\mathcal{A}_1|_p, c)$.
- T_β that consists of all tuples $(p, a, s, (m, b))$, where p is in $Q_1^{\oplus} \cup Q_1^{\otimes}$, a, b are in Σ, m is a monomial in $M(p, ab)$, and $s = \mathrm{coef}(m)$.
- T_γ that consists of all tuples $((m, a), a, 1_S, p, i)$, where m is in $S[x_1, \ldots, x_n]$, a is in Σ, (m, a) is in $Q_2^{\otimes}$, p is the j-th state in $Q_1^{\oplus} \cup Q_1^{\otimes}$, and i is an integer such that $1 \le i \le \exp(m, j)$.

Now, let us take $T_2 = T_\alpha \cup T_\beta \cup T_\gamma$. For each t in T_2, let us define $src(t)$, $\sigma(t)$, $\nu(t)$, and $dst(t)$ to be the first, the second, the third, and the fourth entry of t, respectively.

Finally, we shall define the initial and the terminal weighting functions for all p in $Q_2^{\oplus} \cup Q_2^{\otimes}$ by

$$\iota_2(p) = \begin{cases} \iota_1(p) & \text{if } p \text{ is in } Q_1^{\oplus} \cup Q_1^{\otimes}, \\ 0_S & \text{otherwise,} \end{cases}$$

$$\tau_2(p) = \begin{cases} (|\mathcal{A}_1|_p, \varepsilon) & \text{if } p \text{ is in } Q_1^{\oplus} \cup Q_1^{\otimes}, \\ 1_S & \text{if } p = q_d, \\ 0_S & \text{otherwise.} \end{cases}$$

We shall now show that $|\mathcal{A}_2| = |\mathcal{A}_1|$. We shall first prove by induction on $|w|$ that $(|\mathcal{A}_2|_p, w) = (|\mathcal{A}_1|_p, w)$ for every p in $Q_1^{\oplus} \cup Q_1^{\otimes}$ and w in Σ^*.

First, let $w = \varepsilon$. Since $\mathcal{A}_2$ is ε-free, we have $(|\mathcal{A}_2|_p, \varepsilon) = \tau_2(p) = (|\mathcal{A}_1|_p, \varepsilon)$. Next, let $w = c$ for some c in Σ. Since $\mathcal{A}_2$ is ε-free, we have

$$(|\mathcal{A}_2|_p, c) = \sum_{t \in T_{\mathcal{A}_2}(p,c)} \nu_2(t) \cdot \tau_2(dst(t)).$$

There is a single transition t_d in $T_{\mathcal{A}_2}(p, c)$ that leads to q_d and this transition satisfies $\nu_2(t_d) = (|\mathcal{A}_1|_p, c)$ and $\tau_2(dst(t_d)) = 1$. Each other transition t in $T_{\mathcal{A}_2}(p, c)$ ends in $Q_2^{\otimes}$ and hence $\tau_2(dst(t)) = 0$ by the definition of τ_2. It follows that

$$(|\mathcal{A}_2|_p, c) = \nu_2(t_d) \cdot \tau_2(dst(t_d)) = (|\mathcal{A}_1|_p, c).$$

Finally, assume that $(|\mathcal{A}_2|_q, v) = (|\mathcal{A}_1|_q, v)$ for some v in Σ^* and each q in $Q_1^{\oplus} \cup Q_1^{\otimes}$, and let $w = abv$ for some a, b in Σ. The reader can easily check that $P_{\mathcal{A}_2}[p, ab] = P_{\mathcal{A}_1}[p, ab]$. By Lemma 5.5, we obtain

$$(|\mathcal{A}_2|_p, w) = P_{\mathcal{A}_2}[p, ab]((|\mathcal{A}_2|_1, v), \ldots, (|\mathcal{A}_2|_{n_2}, v)) = P_{\mathcal{A}_1}[p, ab]((|\mathcal{A}_1|_1, v), \ldots, (|\mathcal{A}_1|_{n_1}, v)) = (|\mathcal{A}_1|_p, w).$$

It is now easy to show that $|\mathcal{A}_2| = |\mathcal{A}_1|$. Indeed, for every w in Σ^*, we have

$$(|\mathcal{A}_2|, w) = \sum_{p \in Q_2^{\oplus} \cup Q_2^{\otimes}} \iota_2(p) \cdot (|\mathcal{A}_2|_p, w).$$

If p is in $Q_1^{\oplus} \cup Q_1^{\otimes}$, then $\iota_2(p) = \iota_1(p)$ and $(|\mathcal{A}_2|_p, w) = (|\mathcal{A}_1|_p, w)$, as we have already proved. For every other state p in $Q_2^{\oplus} \cup Q_2^{\otimes}$, we have $\iota_2(p) = 0$. These facts imply that

[7] See Section 2 for the definition of the notation $\mathrm{coef}(m)$ and $\exp(m, j)$ for a monomial m.

$$(|\mathcal{A}_2|, w) = \sum_{p \in Q_1^{\oplus} \cup Q_1^{\otimes}} \iota_1(p) \cdot (|\mathcal{A}_1|_p, w) = (|\mathcal{A}_1|, w).$$

The theorem is proved. □

5.5. Equivalence with alternating weighted automata

We have already mentioned that the definition of two-mode alternating weighted automata can be viewed just as an alternative definition of alternating weighted automata. The following theorem justifies this claim.

Theorem 5.17. *A formal power series r over a commutative semiring S and over an alphabet Σ is realised by an alternating weighted automaton over S if and only if it is realised by a two-mode alternating weighted automaton over S.*

Proof. Let $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ be an alternating weighted automaton over S. We shall construct a two-mode alternating weighted automaton $\mathcal{A}' = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau')$ over S such that $|\mathcal{A}'| = |\mathcal{A}|$.

For each p in Q and c in Σ, the polynomial $\psi[p,c]$ can be written as a sum of distinct nonzero monomials $m_1, m_2, \ldots, m_k$. Let $M(p,c) = \{m_1, \ldots, m_k\}$ and $M = \bigcup_{p \in Q, c \in \Sigma} M(p,c)$. Similarly, the polynomial P_0 can be written as a sum of distinct nonzero monomials $m'_1, m'_2, \ldots, m'_l$. Let $M_0 = \{m'_1, \ldots, m'_l\}$.

First, let us take $Q^{\oplus} = Q$ and $Q^{\otimes} = M \cup M_0$. The set T will be constructed from the following two components:

- The set T_α that consists of all tuples (p, c, s, m), where p is in Q, c is in Σ, m is a monomial in $M(p,c)$, and $s = \mathrm{coef}(m)$.
- The set T_β that consists of all tuples $(m, \varepsilon, 1_S, p, i)$, where m is in $M \cup M_0$, p is the j-th state in Q (with respect to the ordering of states of $\mathcal{A}$) and i is an integer such that $1 \le i \le \exp(m, j)$.

Let us now take $T = T_\alpha \cup T_\beta$. Moreover, for each t in T, let us define $src(t)$, $\sigma(t)$, $\nu(t)$, and $dst(t)$ to be the first, the second, the third, and the fourth entry of t, respectively.

If m is a monomial in M_0, let us define $\iota(m) = \mathrm{coef}(m)$. For every other state p in $Q^{\oplus} \cup Q^{\otimes}$, let $\iota(p) = 0_S$. Finally, let $\tau'(p) = \tau(p)$ for all p in Q and $\tau'(p) = 0_S$ for every other state p in $Q^{\oplus} \cup Q^{\otimes}$.

It is an easy exercise to show that $|\mathcal{A}'| = |\mathcal{A}|$. We shall thus leave the proof for the reader.

For the converse, let $\mathcal{A} = (Q^{\oplus}, Q^{\otimes}, \Sigma, T, \nu, \iota, \tau)$ be an ε-free two-mode alternating weighted automaton over S with $n = |Q^{\oplus} \cup Q^{\otimes}|$. We shall construct an alternating weighted automaton $\mathcal{A}' = (Q, \Sigma, \psi, P_0, \tau')$ over S such that $|\mathcal{A}'| = |\mathcal{A}|$.

Let $Q = Q^{\oplus} \cup Q^{\otimes}$ and let Q keep the linear ordering of $Q^{\oplus} \cup Q^{\otimes}$. Let us take $P_0 = \sum_{i=1}^{n} \iota(i) x_i$, and for each p in Q and c in Σ, let $\psi[p,c] = P_{\mathcal{A}}[p,c]$. The terminal weighting function τ' can then be defined by $\tau'(p) = \tau(p)$ for each p in Q.

It is straightforward to show that $|\mathcal{A}'| = |\mathcal{A}|$. We shall omit the proof of this fact as well. □

6. Systems of *H*-polynomial equations

The equivalence of (non-alternating) weighted automata and systems of linear equations is a well known result [8] that can be viewed as a weighted analogy of the classical result relating finite automata to right-linear (regular) grammars [9]. We shall now prove a similar characterisation for alternating weighted automata, relating them to what we shall call *systems of H-polynomial equations*. Here, "H-polynomial" is an abbreviation for "Hadamard-polynomial" – indeed, our H-polynomial systems can be viewed as linear systems that may contain some additional Hadamard products of formal power series. In yet another words, a H-polynomial system over a semiring S and over an alphabet Σ is simply an appropriately normalised system of polynomial equations over the semiring $\left(S\langle\langle\Sigma^*\rangle\rangle, +, \odot, 0, \sum_{w \in \Sigma^*} 1w\right)$.

Definition 6.1. A *system of H-polynomial equations* $\mathscr{P}$ over a commutative semiring S and over an alphabet Σ in indeterminates $X_1, X_2, \ldots, X_n$ is a system of n equations

$$X_i = \sum_{j=1}^{l_i} \left(s_{i,j} a_{i,j} \cdot \left(\bigodot_{k=1}^{n} X_k^{\odot m_{i,j,k}} \right) \right) + t_i \varepsilon, \qquad i = 1, \ldots, n, \tag{1}$$

where: l_i is a nonnegative integer; $s_{i,j}$ and t_i are in S for $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$; $a_{i,j}$ is in Σ for $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$; $m_{i,j,k}$ is a nonnegative integer for $i = 1, \ldots, n$, $j = 1, \ldots, l_i$, and $k = 1, \ldots, n$; the inequality $\sum_{k=1}^{n} m_{i,j,k} > 0$ holds for $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$.

An n-tuple $(r_1, r_2, \ldots, r_n)$ of power series in $S\langle\langle\Sigma^*\rangle\rangle$ is a *solution* to $\mathscr{P}$ if

$$r_i = \sum_{j=1}^{l_i} \left(s_{i,j} a_{i,j} \cdot \left(\bigodot_{k=1}^{n} r_k^{\odot m_{i,j,k}} \right) \right) + t_i \varepsilon$$

holds for $i = 1, \ldots, n$. We shall write $|\mathscr{P}|_i$ to denote r_i for $i = 1, \ldots, n$.

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Proposition 6.2. *Each system $\mathscr{P}$ of H-polynomial equations over a commutative semiring S and over an alphabet Σ has precisely one solution.*

Proof. Let $\mathscr{P}$ consist of n equations of the form (1). For $i = 1, \ldots, n$, we shall inductively define a formal power series r_i in $S\langle\langle\Sigma^*\rangle\rangle$. Let the coefficient of ε in r_i be t_i. For each c in Σ, let $J_{c,i}$ be the set of indices j, for which $a_{i,j} = c$, and let us define

$$(r_i, cw) = \sum_{j \in J_{c,i}} \left(s_{i,j} \cdot \prod_{k=1}^{n} (r_k, w)^{m_{i,j,k}} \right)$$

for each w in Σ^*. These relations define the formal power series r_i for $i = 1, \ldots, n$. Clearly, the n-tuple $(r_1, r_2, \ldots, r_n)$ is a solution to $\mathscr{P}$. Moreover, this n-tuple is evidently the only possible solution to $\mathscr{P}$. $\square$

We claim that a power series r over a commutative semiring S and over an alphabet Σ is realised by an alternating weighted automaton over S if and only if there is a system $\mathscr{P}$ of H-polynomial equations over S and Σ such that $|\mathscr{P}|_1 = r$. In order to prove this characterisation, we shall need the following lemma.

Lemma 6.3. *Let $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ be an alternating weighted automaton over a commutative semiring S. Then there exists an alternating weighted automaton $\mathcal{A}' = (Q', \Sigma, \psi', P_0', \tau')$ over S such that $P_0' = x_1$ and $|\mathcal{A}'| = |\mathcal{A}|$.*

Proof. Let $n = |Q|$. We shall construct an alternating weighted automaton $\mathcal{A}'' = (Q'', \Sigma, \psi'', P_0'', \tau'')$ with $n+1$ states such that $P_0'' = x_{n+1}$ and $|\mathcal{A}''| = |\mathcal{A}|$. Once this is done, the initial polynomial P_0'' can be changed to x_1 if the states of $\mathcal{A}''$ are suitably rearranged.

Let us define $Q'' = Q \cup \{q_0\}$, where q_0 is a new state that is not in Q. The numerical order of q_0 is $n+1$, while the rest of states in Q'' keeps its ordering from Q. For each c in Σ, let us define $\psi''[q_0, c] = P_0(\psi[1, c], \psi[2, c], \ldots, \psi[n, c])$, and let the terminal weight of the state q_0 be defined by $\tau''(q_0) = P_0(\tau(1), \tau(2), \ldots, \tau(n))$. For each p in $Q'' - \{q_0\}$ and c is in Σ, let $\psi''[p, c] = \psi[p, c]$ and $\tau''(p) = \tau(p)$. Finally, let $P_0'' = x_{n+1}$. It is easy to show that $|\mathcal{A}''| = |\mathcal{A}|$. $\square$

We are now ready to prove a theorem characterising formal power series realised by alternating weighted automata in terms of systems of H-polynomial equations.

Theorem 6.4. *Let r be a formal power series over a commutative semiring S and over an alphabet Σ. An alternating weighted automaton $\mathcal{A}$ over S such that $|\mathcal{A}| = r$ exists if and only if there exists a system $\mathscr{P}$ of H-polynomial equations over S and Σ such that $|\mathscr{P}|_1 = r$.*

Proof. Let $\mathscr{P}$ be a system of H-polynomial equations over S and Σ that consists of n equations of the form (1). We shall construct an alternating weighted automaton $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ such that $|\mathcal{A}| = |\mathscr{P}|_1$.

Let $Q = \{1, 2, \ldots, n\}$. For every i in Q and c in Σ, let $J_{c,i}$ be the set of indices j with $a_{i,j} = c$, and let

$$\psi[i, c] = \sum_{j \in J_{c,i}} s_{i,j} \prod_{k=1}^{n} x_i^{m_{i,j,k}}.$$

Let $P_0 = x_1$ and $\tau(i) = t_i$ for every i in Q. The proof of the fact $|\mathcal{A}| = |\mathscr{P}|_1$ is left for the reader.

For the converse, let $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ with $|Q| = n$ be an alternating weighted automaton over S. We shall construct a system $\mathscr{P}$ of H-polynomial equations over S and Σ such that $|\mathscr{P}|_1 = |\mathcal{A}|$.

By Lemma 6.3, we can assume that $P_0 = x_1$. The system of H-polynomial equations $\mathscr{P}$ shall consist of n equations. Let us construct the i-th equation of $\mathscr{P}$ for $i = 1, \ldots, n$. For each c in Σ, we can write

$$\psi[i, c] = \sum_{j=1}^{l_c} s_{c,j} \prod_{k=1}^{n} x_k^{m_{c,j,k}},$$

where l_c is a nonnegative integer, $s_{c,j}$ is in S for $j = 1, \ldots, l_c$, the exponent $m_{c,j,k}$ is a nonnegative integer for $j = 1, \ldots, l_c$ and $k = 1, \ldots, n$, and $\sum_{k=1}^{n} m_{c,j,k} > 0$ for $j = 1, \ldots, l_c$. Let the i-th equation of $\mathscr{P}$ be

$$X_i = \sum_{c \in \Sigma} \left(\sum_{j=1}^{l_c} s_{c,j} c \cdot \left(\bigodot_{k=1}^{n} X_k^{\odot m_{c,j,k}} \right) \right) + \tau(i)\varepsilon.$$

It is easy to show that $|\mathscr{P}|_1 = |\mathcal{A}|$. $\square$

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7. Expressive power

Since (non-alternating) weighted automata are just a special case of alternating weighted automata, the latter are at least as powerful as the former. A natural question is if alternating weighted automata are *strictly* more powerful. A positive answer to this question was already given by Almagor and Kupferman [1], who constructed a simple alternating weighted automaton over the tropical semiring $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$, and proved that no non-alternating weighted automaton over the same semiring is equivalent to this automaton. As a consequence, alternating weighted automata over the tropical semiring are strictly more powerful than non-alternating weighted automata over the tropical semiring. However, not the same can be said about the Boolean semiring, where it is well known that a language recognised by a Boolean alternating automaton is necessarily rational (regular) [2]. In other words, alternating and non-alternating weighted automata over the Boolean semiring are equally powerful.

We conclude that commutative semirings can be divided into two nonempty classes: the class of commutative semirings, for which alternating and non-alternating weighted automata are equally powerful and the class of commutative semirings, for which alternating weighted automata are strictly more powerful than their non-alternating counterparts. Let $\mathscr{S}$ denote the former class, i.e., let $\mathscr{S}$ be the class of all commutative semirings S such that the behaviour of every alternating weighted automaton over S is rational over S. For the rest of this section, we shall be proving the following characterisation of the class $\mathscr{S}$.

Theorem 7.1. *Let S be a commutative semiring. Then the following assertions are equivalent:*

1. *Alternating weighted automata over S are equally expressive as (non-alternating) weighted automata over S (i.e., S is in $\mathscr{S}$).*
2. *The semiring S is locally finite.*

As a first step, let us prove the following simple characterisation of locally finite commutative semirings, which we shall use to establish one of the implications of Theorem 7.1.

Lemma 7.2. *Let S be a commutative semiring. Then S is locally finite if and only if each element of S is of finite multiplicative order.*

Proof. Suppose that S is locally finite. In particular, this implies that the subsemiring T_s generated by s is finite for each s in S. The subsemiring T_s contains s^i for all nonnegative integers i and hence, there is only a finite number of such elements. This means that s has finite multiplicative order.

For the converse, suppose that every element of S has finite multiplicative order. First of all, let us note that then every element of S has finite *additive* order as well. Since 2_S has finite multiplicative order, there exist two distinct nonnegative integers k_1 and k_2 such that $(2_S)^{k_1} = (2_S)^{k_2}$. We have

$$2^{k_1} 1_S = (2^{k_1})_S = (2_S)^{k_1} = (2_S)^{k_2} = (2^{k_2})_S = 2^{k_2} 1_S,$$

which shows that there exist two distinct nonnegative integers $l_1 = 2^{k_1}$ and $l_2 = 2^{k_2}$ such that $l_1 1_S = l_2 1_S$. Therefore, if t is in S, then $l_1 t = (l_1 1_S)t = (l_2 1_S)t = l_2 t$. We have thus shown that $l_1 t = l_2 t$ for two distinct nonnegative integers l_1, l_2, which means that t has finite additive order.

Now, let T be a subsemiring of S generated by elements $s_1, s_2, \ldots, s_n$ in S. We shall show that T is finite. Let M be a subset of S that consists of all elements $\prod_{i=1}^{n} s_i^{k_i}$, where k_i is a nonnegative integer for $i = 1, \ldots, n$. Since s_i has finite multiplicative order for $i = 1, \ldots, n$, the set M is finite. Moreover, let K be a subset of S that consists of all elements $\sum_{t \in M} k_t t$, where k_t is a nonnegative integer for every t in M. Since every t in M has finite additive order, we can see that K is finite as well. Furthermore, it is easy to show that K is a subsemiring of S that contains $s_1, \ldots, s_n$ and is contained in every subsemiring of S that contains $s_1, \ldots, s_n$. This means that K is a subsemiring generated by $s_1, \ldots, s_n$ and hence, $K = T$. This proves that T is finite; the semiring S is thus locally finite. □

Our next aim is to prove the following direction of Theorem 7.1: if S is locally finite, then S is in $\mathscr{S}$. We shall first prove this claim when S is finitely generated, and hence finite.

The construction presented in the proof of the following lemma generalises the usual construction in the Boolean setting [2], in which a nondeterministic finite automaton equivalent to a given alternating finite automaton has states corresponding to finite conjunctions of variables corresponding to states of the original automaton, with no variable repeated. For instance, if the original alternating automaton has states $q_1, \ldots, q_6$, then $x_2 \wedge x_3 \wedge x_6$ can be a state of the equivalent nondeterministic finite automaton. Such conjunctions are clearly in bijection with the set of all Boolean *functions* "realised" by arbitrary conjunctions in the same variables, i.e., by monomials from $\mathbb{B}[x_1, \ldots, x_n]$ with coefficient 1.

The set of functions from a finite set to itself is always finite. As a result, the set of polynomial functions "realised" by monomials in $S[x_1, \ldots, x_n]$ with coefficient 1_S is finite whenever S is finite. In our construction, we shall simply replace the finite conjunctions described above by such "monomial functions".

Lemma 7.3. *Let S be a finite commutative semiring. Then S is in $\mathscr{S}$.*

Proof. Let $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ with $|Q| = n$ be an alternating weighted automaton over S. We shall construct a (non-alternating) weighted automaton $\mathcal{A}' = (Q', \Sigma, T, \nu, \iota, \tau')$ over S such that $|\mathcal{A}'| = |\mathcal{A}|$.

Since S is finite, there are finitely many functions $\mu_1, \mu_2, \ldots, \mu_l$ in $S(x_1, \ldots, x_n)$ corresponding to monomials from $S[x_1, \ldots, x_n]$ with coefficient 1_S. Each polynomial function η in $S(x_1, \ldots, x_n)$ is clearly a linear combination of $\mu_1, \ldots, \mu_l$, i.e., we can write $\eta = \sum_{i=1}^{l} s_i \mu_i$, where s_i is in S for $i = 1, \ldots, l$. For $i = 1, \ldots, l$, let m_i be a monomial from $S[x_1, \ldots, x_n]$ such that μ_i is its corresponding polynomial function.

Let us define $Q' = \{1, 2, \ldots, l\}$. For each k in Q' and each c in Σ, we shall construct the set of transitions $T(k, c)$ leading from k and labelled by c. Let η be the polynomial function that corresponds to the polynomial $m_k(\psi[1, c], \psi[2, c], \ldots, \psi[n, c])$. We can write $\eta = \sum_{i=1}^{l} s_i \mu_i$, where s_i is in S for $i = 1, \ldots, l$. For $i = 1, \ldots, l$, let $T(k, c)$ contain a transition $t_{k,i}$ with $\nu(t_{k,i}) = s_i$ leading to i if and only if s_i is nonzero. We may now define $T = \bigcup_{k \in Q', c \in \Sigma} T(k, c)$.

Next, let ϕ_0 be the polynomial function that corresponds to the polynomial P_0. Then we can write $\phi_0 = \sum_{i=1}^{l} s_i \mu_i$, where s_i is in S for $i = 1, \ldots, l$. For each k in Q', let us take $\iota(k) = s_k$. Finally, let us define $\tau'(k) = \mu_k(\tau(1), \tau(2), \ldots, \tau(n))$ for each k in Q'.

Let us now show that $|\mathcal{A}'| = |\mathcal{A}|$. First, we shall prove that

$$(|\mathcal{A}'|_k, w) = \mu_k\big((|\mathcal{A}|_1, w), (|\mathcal{A}|_2, w), \ldots, (|\mathcal{A}|_n, w)\big) \tag{2}$$

holds for each k in Q'. The proof will be done by mathematical induction on the length of the word w.

For $w = \varepsilon$, we have $(|\mathcal{A}|_i, \varepsilon) = \tau(i)$ for $i = 1, \ldots, n$. Since $\mathcal{A}'$ is ε-free, it follows that

$$(|\mathcal{A}'|_k, \varepsilon) = \tau'(k) = \mu_k(\tau(1), \ldots, \tau(n)) = \mu_k\big((|\mathcal{A}|_1, \varepsilon), \ldots, (|\mathcal{A}|_n, \varepsilon)\big).$$

We have thus proved (2) for $w = \varepsilon$.

Now, suppose that (2) holds for $w = v$ for some v in Σ^*. We shall prove that it holds also for $w = cv$, where c is in Σ. Let η be the polynomial function that corresponds to the polynomial $m_k(\psi[1, c], \psi[2, c], \ldots, \psi[n, c])$. For each i in Q', let s_i be the weight of the c-labelled transition of $\mathcal{A}'$ leading from k to i (if there is no such transition, let $s_i = 0_S$). By the definition of T, we have $\eta = \sum_{i=1}^{l} s_i \mu_i$. Since $\mathcal{A}'$ is ε-free, we also have $(|\mathcal{A}'|_k, cv) = \sum_{i=1}^{l} s_i (|\mathcal{A}'|_i, v)$. Moreover, by the induction hypothesis, $(|\mathcal{A}'|_i, v) = \mu_i\big((|\mathcal{A}|_1, v), \ldots, (|\mathcal{A}|_n, v)\big)$ for $i = 1, \ldots, l$. These facts imply

$$\begin{aligned}
(|\mathcal{A}'|_k, cv) &= \sum_{i=1}^{l} s_i (|\mathcal{A}'|_i, v) = \sum_{i=1}^{l} s_i \mu_i\big((|\mathcal{A}|_1, v), \ldots, (|\mathcal{A}|_n, v)\big) = \eta\big((|\mathcal{A}|_1, v), \ldots, (|\mathcal{A}|_n, v)\big) = \\
&= \mu_k\Big(\psi[1, c]\big((|\mathcal{A}|_1, v), \ldots, (|\mathcal{A}|_n, v)\big), \ldots, \psi[n, c]\big((|\mathcal{A}|_1, v), \ldots, (|\mathcal{A}|_n, v)\big)\Big) = \\
&= \mu_k\big((|\mathcal{A}|_1, cv), \ldots, (|\mathcal{A}|_n, cv)\big).
\end{aligned}$$

We have thus proved (2) for $w = cv$.

Finally, we shall prove that $(|\mathcal{A}'|, w) = (|\mathcal{A}|, w)$ for all w in Σ^*. Let ϕ_0 be the polynomial function that corresponds to the polynomial P_0. By the definition of the initial weighting function ι, we have $\phi_0 = \sum_{i=1}^{l} \iota(i) \mu_i$. This fact together with (2) implies

$$(|\mathcal{A}'|, w) = \sum_{i=1}^{l} \iota(i)(|\mathcal{A}'|_i, w) = \sum_{i=1}^{l} \iota(i) \mu_i\big((|\mathcal{A}|_1, w), \ldots, (|\mathcal{A}|_n, w)\big) = \phi_0\big((|\mathcal{A}|_1, w), \ldots, (|\mathcal{A}|_n, w)\big) = (|\mathcal{A}|, w).$$

The lemma is proved. □

We now wish to prove the more general claim: if S is locally finite, then S is in $\mathscr{S}$, regardless of whether S is finite or not. This claim follows easily from the special case proved above. The reason is that even if S is not finite, every alternating weighted automaton over S still "makes use" just of some finite part of S. Let us suppose that $\mathcal{A}$ is some two-mode alternating weighted automaton over S. If X is the set of all elements of S that are either carried by some transition of $\mathcal{A}$ or are assigned to some state as an initial or a terminal weight, then $\mathcal{A}$ can be viewed as a two-mode alternating weighted automaton over the semiring T generated by X. The set X is clearly finite, so the semiring T is finitely generated. The same reasoning stands also if $\mathcal{A}$ is a (general) alternating weighted automaton. That is, for every alternating weighted automaton $\mathcal{A}$ over a commutative semiring S, there exists a finitely generated subsemiring T of S and an alternating weighted automaton $\mathcal{A}'$ over T such that $|\mathcal{A}'| = |\mathcal{A}|$.

Lemma 7.4. *Let S be a locally finite commutative semiring. Then S is in $\mathscr{S}$.*

Proof. Let $\mathcal{A}$ be an alternating weighted automaton over S. Then there exists a finitely generated subsemiring T of S and an alternating weighted automaton $\mathcal{A}'$ over T such that $|\mathcal{A}'| = |\mathcal{A}|$. The semiring T is finite and so it is in $\mathscr{S}$ by Lemma 7.3. Therefore, there exists a (non-alternating) weighted automaton $\mathcal{B}$ over T such that $|\mathcal{B}| = |\mathcal{A}'| = |\mathcal{A}|$. Clearly, $\mathcal{B}$ can be viewed as a (non-alternating) weighted automaton over S. □

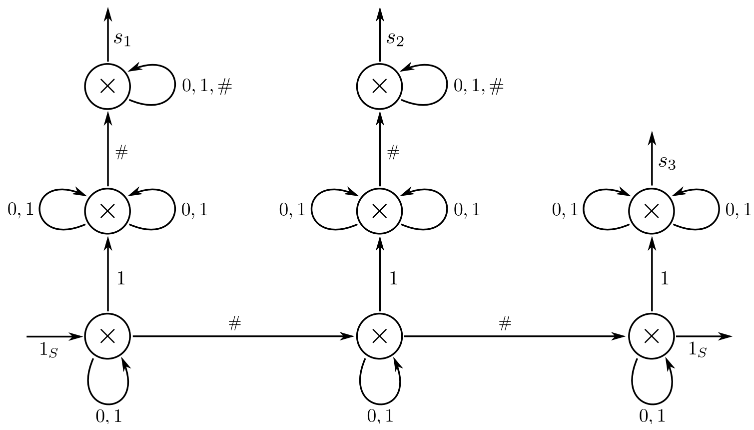

Fig. 11. A two-mode alternating weighted automaton $\mathcal{A}_X$ over the alphabet $\{0, 1, \#\}$ and over a commutative semiring S, where $X = \{s_1, s_2, s_3\}$ for some particular elements s_1, s_2, s_3 of S. The formal power series $|\mathcal{A}_X|$ satisfies $(|\mathcal{A}_X|, \langle k_1\rangle\#\langle k_2\rangle\#\langle k_3\rangle) = s_1^{k_1} s_2^{k_2} s_3^{k_3}$ for every triplet of nonnegative integers k_1, k_2, k_3.

To finish the proof of Theorem 7.1, it remains to prove the following claim: if a commutative semiring S is in $\mathscr{S}$, then S is locally finite. We shall apply Lemma 7.2 and prove an equivalent statement: if S is in $\mathscr{S}$, then each element of S is of finite multiplicative order. This is the most intricate part of our proof of Theorem 7.1, which requires some preparations.

For every nonnegative integer n, let $\langle n\rangle$ denote the binary representation of n ($\langle n\rangle$ is a word over the alphabet $\{0, 1\}$). Let $X = \{s_1, s_2, \ldots, s_z\}$ be a finite subset of a commutative semiring S. It is easy to construct an alternating weighted automaton $\mathcal{A}_X$ over S and over the alphabet $\{0, 1, \#\}$ such that

$$(|\mathcal{A}_X|, \langle k_1\rangle\#\langle k_2\rangle\#\ldots\#\langle k_z\rangle) = \prod_{i=1}^{z} s_i^{k_i} \tag{3}$$

for every z-tuple of nonnegative integers $k_1, k_2, \ldots, k_z$.[8] A diagram of one such automaton for $z = 3$ is depicted in Fig. 11. For every finite subset $X = \{s_1, \ldots, s_z\}$, let us denote the series $|\mathcal{A}_X|$ by r_X. We claim that if the formal power series r_X is rational over S for every finite subset X of S, then every element of S has finite multiplicative order. We shall now give some definitions that will help us to prove this.

For every finite subset $X = \{s_1, s_2, \ldots, s_z\}$ of S and every nonnegative real number C, let $G_X(C)$ be the set that consists of all elements s in S such that

$$s = \prod_{i=1}^{z} s_i^{k_i}$$

for some nonnegative integers $k_1, k_2, \ldots, k_z$ satisfying $\sum_{i=1}^{z} k_i \le C$.[9] Furthermore, for every pair of nonnegative real numbers C, D, let $H_X(C, D)$ be the set that consists of all elements s in S such that

$$s = \sum_{i=1}^{k} g_i,$$

where k is a nonnegative integer satisfying $k \le D$ and g_i is in $G_X(C)$ for $i = 1, \ldots, k$. If X contains only one element s, we shall usually write $G_s(C)$ and $H_s(C, D)$ instead of $G_{\{s\}}(C)$ and $H_{\{s\}}(C, D)$.

The sets $G_X(C)$ and $H_X(C, D)$ are clearly finite for every finite subset X of S and for every pair of nonnegative real numbers C, D. Moreover, if X consists of z elements, then the sizes of these sets can be estimated, using the standard formula for combinations with repetition,[10] by

[8] Note that (3) does not specify value of $|\mathcal{A}_X|$ on each word in $\{0, 1, \#\}^*$. However, this is not important for our purposes.

[9] This is really interesting only if C is a natural number, as obviously $G_X(C) = G_X(\lfloor C\rfloor)$ for all X and C. The reason why $G_X(C)$ is defined with nonnegative *real* C is that we shall often substitute estimated values for C, which would otherwise have to be rounded. The same remark applies to the notation $H_X(C, D)$ introduced below.

[10] The size of $G_X(C)$ can obviously be estimated from above by the number of nonnegative integer solutions to the inequality $k_1 + \ldots + k_z \le \lfloor C\rfloor$, which is the same as the number of nonnegative integer solutions to the equation $k_1 + \ldots + k_z + k_{z+1} = \lfloor C\rfloor$. Thus, the estimate is given by the number of $\lfloor C\rfloor$-combinations with repetition of a multiset containing $z + 1$ distinct elements. The estimate for the size of $H_X(C, D)$ is obtained similarly.

$$|G_X(C)| \leq \binom{\lfloor C \rfloor + z}{z} \leq (C+z)^z,$$
$$|H_X(C,D)| \leq \binom{\lfloor D \rfloor + |G_X(C)|}{|G_X(C)|} \leq (D + |G_X(C)|)^{|G_X(C)|} \leq (D + (C+z)^z)^{(C+z)^z}.$$

For our purposes, we shall manage with the weaker estimates

$$|G_X(C)| \leq (C+z)^z, \tag{4}$$
$$|H_X(C,D)| \leq (D + (C+z)^z)^{(C+z)^z}. \tag{5}$$

Recall that our goal is to prove that if S is a commutative semiring in $\mathscr{S}$, then each element of S is of finite multiplicative order. In a nutshell, we shall do this by proving that if S is in $\mathscr{S}$, then $G_s(2^n)$ contains less than 2^n elements for each s in S and all sufficiently large n (it is clear that s is of finite multiplicative order in that case). To get this result, we shall prove that $G_s(2^n)$ is contained, if n is large enough, in a set $H_Y(C(n), D(n))$ for some suitable Y, $C(n)$, and $D(n)$, so that the estimate (5) can be used to obtain the upper bound 2^n for its size.

We shall prove this containment of the sets $G_s(2^n)$ in some suitable $H_Y(C(n), D(n))$ gradually in the following three lemmata. The first of them, Lemma 7.5, is a variation on a well-known property of weighted automata [4,5]. We shall nevertheless give a proof for convenience.

Lemma 7.5. *Let S be a commutative semiring and Σ be an alphabet. If r in $S\langle\langle\Sigma^*\rangle\rangle$ is rational over S, then there exists a finite subset Y of S and a nonnegative real number C so that (r, w) is in $H_Y(|w|+1, C^{|w|})$ for every w in Σ^*.*

Proof. Let r be realised by a weighted automaton $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$ over S. We can assume that $\mathcal{A}$ is ε-free and that $\iota(p) = 0$ for every p in Q except for some q_0 in Q for which $\iota(q_0) = 1$. (It is a well known fact that every rational power series is realised by some weighted automaton that satisfies these conditions [8].) Clearly, we can also assume that there are no parallel transitions with the same label in $\mathcal{A}$.

Let Y be the set that consists of all elements of S that are carried by some transition in T or are assigned to some state in Q as a terminal weight, i.e.,

$$Y = \{\nu(t) \mid t \in T\} \cup \{\tau(p) \mid p \in Q\}.$$

Let $C = |Q|$. We claim that for each p in Q and w in Σ^*, the coefficient of w in $|\mathcal{A}|_p$ is in $H_Y(|w|+1, C^{|w|})$. Once we prove this claim, the proof of Lemma 7.5 is finished, as $|\mathcal{A}| = |\mathcal{A}|_{q_0}$. We shall prove the claim by induction on the length of w.

If $w = \varepsilon$, then $(|\mathcal{A}|_p, w) = \tau(p)$. As $\tau(p)$ is in Y, the coefficient of w in $|\mathcal{A}|_p$ belongs to $H_Y(1,1) = H_Y(|w|+1, C^{|w|})$. Let us now assume that $w = cv$ for some c in Σ and v in Σ^*. We have

$$(|\mathcal{A}|_p, cv) = \sum_{t \in T(p,c)} \nu(t) \cdot (|\mathcal{A}|_{dst(t)}, v). \tag{6}$$

By the induction hypothesis, $(|\mathcal{A}|_{dst(t)}, v)$ is in $H_Y(|v|+1, C^{|v|})$ for every t in $T(p,c)$ and so $\nu(t) \cdot (|\mathcal{A}|_{dst(t)}, v)$ is in $H_Y(|v|+2, C^{|v|})$ for every t in $T(p,c)$. Moreover, there are no more than $C = |Q|$ transitions in $T(p,c)$, since $\mathcal{A}$ has no parallel transitions with the same label. So the sum in (6) is taken over at most C elements from $H_Y(|v|+2, C^{|v|})$ and hence, $(|\mathcal{A}|_p, w)$ is in $H_Y(|v|+2, C^{|v|+1}) = H_Y(|w|+1, C^{|w|})$. □

Recall that r_X is a series satisfying $(r_X, \langle k_1\rangle\#\langle k_2\rangle\#\ldots\#\langle k_z\rangle) = s_1^{k_1} \cdot s_2^{k_2} \cdot \ldots \cdot s_z^{k_z}$ for each finite subset $X = \{s_1, \ldots, s_z\}$ of S and all nonnegative integers $k_1, k_2, \ldots, k_z$ with their binary representations denoted by $\langle k_1\rangle, \langle k_2\rangle, \ldots, \langle k_z\rangle$. Recall also that this series is always realised by an alternating weighted automaton $\mathcal{A}_X$. All logarithms are understood to be binary in what follows.

Lemma 7.6. *Let S be a commutative semiring in $\mathscr{S}$. For every finite subset X of S, there exists a nonnegative integer n_1, a nonnegative real number D_1, and a finite subset Y of S such that the inclusion $G_X(n) \subseteq H_Y(D_1 \log n, n^{D_1})$ holds for all $n \geq n_1$.*

Proof. Let $X = \{s_1, s_2, \ldots, s_z\}$. Since S is in $\mathscr{S}$, the series r_X is rational over S. Let Y be a finite subset of S and C be a real number such that $C \geq 1$ and

$$(r_X, w) \in H_Y(|w|+1, C^{|w|}) \tag{7}$$

for all w over the alphabet $\{0, 1, \#\}$. Existence of such Y and C is guaranteed by Lemma 7.5. Let n_1 be an integer and D_1 be a real number such that

$$z(\log n + 2) + 1 \le D_1 \log n \quad \text{and} \quad C^{2z} n^{z \log C} \le n^{D_1} \tag{8}$$

holds for every $n \ge n_1$.

Let n be an integer satisfying $n \ge n_1$, let s be in $G_X(n)$. Our goal is to prove that s is in $H_Y(D_1 \log n, n^{D_1})$. We can write

$$s = \prod_{i=1}^{z} s_i^{k_i},$$

where $\sum_{i=1}^{z} k_i \le n$. The element s is also the coefficient of the word $w := \langle k_1 \rangle \# \langle k_2 \rangle \# \ldots \# \langle k_z \rangle$ in r_X. The reader can easily check that $|w| \le z(\log n + 2)$. By (7), the element s is in

$$H_Y(z(\log n + 2) + 1, C^{z(\log n + 2)}) = H_Y(z(\log n + 2) + 1, C^{2z} n^{z \log C}).$$

By (8), this set is included in $H_Y(D_1 \log n, n^{D_1})$. So s is in $H_Y(D_1 \log n, n^{D_1})$ and the lemma is proved. □

Lemma 7.7. *Let S be a commutative semiring in $\mathscr{S}$. For every s in S, there exists a nonnegative integer n_0, nonnegative real numbers C_1, C_2, and a finite subset Y of S such that $G_s(2^n)$ is included in $H_Y(C_1 \log n, n^{C_2})$ for all $n \ge n_0$.*

Proof. Let n_1 be an integer, D_1 be a real number, and X be a finite subset of S such that

$$G_s(n) \subseteq H_X(D_1 \log n, n^{D_1}) \tag{9}$$

holds for every $n \ge n_1$. Existence of such n_1, D_1, and X is guaranteed by Lemma 7.6. Assume that $D_1 \ge 1$. If X does not contain 2_S, let us add this semiring element to X (this has no effect on the inclusion (9)). Similarly, let n_2 be an integer, D_2 be a real number, and Y be a finite subset of S such that

$$G_X(n) \subseteq H_Y(D_2 \log n, n^{D_2}) \tag{10}$$

holds for every $n \ge n_2$. Let us denote $z = |Y|$. Let n_3 be an integer and C_1 be a real number such that

$$D_2 \log(2D_1 n) \le C_1 \log n, \tag{11}$$

holds for every $n \ge n_3$. Let n_4 be an integer and C_2 be a real number such that

$$\tfrac{1}{2}(2D_1 n)^{D_2+1}(D_1 n + z)^z \le n^{C_2} \tag{12}$$

holds for every $n \ge n_4$. Finally, let $n_0 = \max\{n_1, n_2, n_3, n_4\}$.

Let n be an integer satisfying $n \ge n_0$, let t be in $G_s(2^n)$. Our goal is to prove that t is in $H_Y(C_1 \log n, n^{C_2})$. By (9), t is in $H_X(D_1 \log 2^n, (2^n)^{D_1}) = H_X(D_1 n, 2^{D_1 n})$. This means that we can write

$$t = \sum_{g \in G_X(D_1 n)} k_g g, \tag{13}$$

where k_g is a nonnegative integer satisfying

$$k_g \le 2^{D_1 n} \tag{14}$$

for every g in $G_X(D_1 n)$. Let us pick some g from $G_X(D_1 n)$ and examine the semiring element $h := k_g g$. If we take into account the binary representation of the integer k_g, we can see that

$$k_g = \sum_{i=0}^{l} 2^{m_i}$$

for some $l \le \log(k_g)$ and $m_i \le \log(k_g)$ for $i = 0, 1, \ldots, l$. We thus have

$$h = \sum_{i=0}^{l} h_i, \tag{15}$$

where $h_i := (2_S)^{m_i} g$ for $i = 0, 1, \ldots, l$. Since g is in $G_X(D_1 n)$, the element h_i is in $G_X(m_i + D_1 n)$ for $i = 0, 1, \ldots, l$. The set $G_X(m_i + D_1 n)$ is included in $G_X(2D_1 n)$, since $m_i \le \log k_g \le \log 2^{D_1 n} = D_1 n$ (the second inequality follows from (14)). Furthermore, the inclusion (10) implies that $G_X(2D_1 n)$ is included in $H_Y(D_2 \log(2D_1 n), (2D_1 n)^{D_2})$. We conclude that h_i is in $H_Y(D_2 \log(2D_1 n), (2D_1 n)^{D_2})$ for $i = 0, 1, \ldots, l$. Together with (15), this implies that h is in $H_Y(D_2 \log(2D_1 n), (2D_1 n)^{D_2} l)$. As $l \le \log(k_g) \le \log 2^{D_1 n} = D_1 n$ (the second inequality follows from (14)), this set is a subset of

$$H_Y(D_2\log(2D_1n), (2D_1n)^{D_2}D_1n) = H_Y(D_2\log(2D_1n), \tfrac{1}{2}(2D_1n)^{D_2+1}).$$

Therefore, h is in $H_Y(D_2\log(2D_1n), \frac{1}{2}(2D_1n)^{D_2+1})$. The element h was chosen as $k_g g$ for arbitrary g in $G_X(D_1n)$, so we conclude that

$$k_g g \in H_Y(D_2\log(2D_1n), \tfrac{1}{2}(2D_1n)^{D_2+1}) \tag{16}$$

for each g in $G_X(d_1n)$.

Finally, let us return to equality (13). By (4), $G_X(D_1n)$ contains no more than $(D_1n+z)^z$ elements. Together with (16), this upper bound implies that t is in $H_Y(D_2\log(2D_1n), \frac{1}{2}(2D_1n)^{D_2+1}(D_1n+z)^z)$. By inequalities (11) and (12), this set is included in $H_Y(C_1\log n, n^{C_2})$. We have thus shown that t is in $H_Y(C_1\log n, n^{C_2})$ and the lemma is proved. □

We are now ready to prove the last ingredient needed to establish Theorem 7.1.

Lemma 7.8. *If a commutative semiring S is in $\mathscr{S}$, then every element of S is of finite multiplicative order.*

Proof. Let s be in S. We shall prove that s has finite multiplicative order. Let n_0 be an integer, C_1, C_2 be real numbers and Y be a finite subset of S such that $G_s(2^n)$ is included in $H_Y(C_1\log n, n^{C_2})$ for all $n \ge n_0$. We have just proven in Lemma 7.7 that this assumption is valid. Let $z = |Y|$. By (5), the size of $H_Y(C_1\log n, n^{C_2})$ is at most

$$((C_1\log n + z)^z + n^{C_2})^{(C_1\log n+z)^z} = 2^{\log((C_1\log n+z)^z+n^{C_2})(C_1\log n+z)^z}.$$

Clearly, there exists an integer n_1 such that $|H_Y(C_1\log n, n^{C_2})| < 2^n$ for all $n \ge n_1$. If $n = \max\{n_0, n_1\}$, then $G_s(2^n)$ is a subset of $H_Y(C_1\log n, n^{C_2})$ and thus also $|G_s(2^n)| < 2^n$. This implies that $s^{m_1} = s^{m_2}$ for two distinct nonnegative integers m_1, m_2, which means that s has finite multiplicative order. □

Now we may finally collect the results obtained in this section to prove Theorem 7.1.

Proof of Theorem 7.1. By Lemma 7.4, the second assertion of Theorem 7.1 implies the first. On the other hand, it follows by Lemma 7.8 coupled with Lemma 7.2 that the first assertion implies the second. □

Now that we have proved Theorem 7.1, we can use it to examine the expressive power of alternating weighted automata over some particular commutative semirings. The class of commutative semirings, for which alternating and non-alternating weighted automata are equally powerful, includes the following semirings:

- All finite commutative semirings, e.g., the Boolean semiring $(\mathbb{B}, \vee, \wedge, 0, 1)$, the semiring $(\mathbb{Z}_k, +, \cdot, 0, 1)$ of integers modulo k (for some $k \ge 2$) with standard operations of addition and multiplication, etc.
- The semiring $(\mathscr{P}(U), \cup, \cap, \emptyset, U)$ on the powerset $\mathscr{P}(U)$ of an arbitrary set U with union as addition and intersection as multiplication (or more generally, any bounded distributive lattice [6]).

On the contrary, we shall now list some commutative semirings, for which alternating weighted automata are strictly more powerful than (non-alternating) weighted automata:

- The semiring $(\mathbb{R}, +, \cdot, 0, 1)$ of real numbers with the standard operations of addition and multiplication.
- The tropical semiring of reals, i.e., the semiring $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$ on the set of real numbers with positive infinity, together with minimum as addition and the standard addition of real numbers as multiplication.
- The arctic semiring of reals, i.e., the semiring $(\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$ on the set of real numbers with negative infinity, together with maximum as addition and the standard addition of real numbers as multiplication.
- The semiring of polynomials $S[x_1, \ldots, x_n]$ for an arbitrary commutative semiring S and any positive integer n.
- The semiring $(2^{\{a\}^*}, \cup, \cdot, \emptyset, \{\varepsilon\})$ of formal languages over a singleton alphabet $\{a\}$, together with union as addition and concatenation as multiplication.

8. Closure properties

For any individual commutative semiring S, one can examine the closure properties of the class of formal power series realised by alternating weighted automata over S. Naturally, these closure properties might vary for different semirings. We shall examine several standard operations on formal power series in this section and for each one of them, we shall determine if the class of formal power series realised by alternating weighted automata over S is closed under the operation in consideration *for all* commutative semirings S.

Let us start with the sum and Hadamard product of formal power series. We shall prove that the class of series realised by alternating weighted automata *is closed* under these two operations. The constructions that we shall use to establish this fact are straightforward: for each pair of alternating weighted automata $\mathcal{A}_1$ and $\mathcal{A}_2$, we shall construct an automaton $\mathcal{A}$, which essentially contains $\mathcal{A}_1$ and $\mathcal{A}_2$ as "two independent subautomata". We shall then specify the initial polynomial of $\mathcal{A}$ so that the automaton realises the sum or the Hadamard product of these two "subautomata" – i.e., $|\mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2|$ or $|\mathcal{A}| = |\mathcal{A}_1| \odot |\mathcal{A}_2|$.

Theorem 8.1. *Let S be a commutative semiring. The class of formal power series realised by alternating weighted automata over S is closed under sum.*

Proof. Let $\mathcal{A}_1 = (Q_1, \Sigma_1, \psi_1, P_{0,1}, \tau_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma_2, \psi_2, P_{0,2}, \tau_2)$ be alternating weighted automata over the semiring S with $|Q_1| = n_1$ and $|Q_2| = n_2$. Let us assume that $Q_1 \cap Q_2 = \emptyset$. We shall construct an alternating weighted automaton $\mathcal{A} = (Q, \Sigma_1 \cup \Sigma_2, \psi, P_0, \tau)$ over S such that $|\mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2|$.

First, let us take $Q = Q_1 \cup Q_2$. Let us fix an ordering of states in Q so that the i-th state of $\mathcal{A}_1$ is the i-th state of $\mathcal{A}$ and the i-th state of $\mathcal{A}_2$ is the $(i + n_1)$-th state of $\mathcal{A}$.

For each P in $S[x_1, \ldots, x_{n_2}]$, we shall write shift(P) for the polynomial in $S[x_{n_1+1}, x_{n_1+2}, \ldots, x_{n_1+n_2}]$ that is obtained from P after replacing each occurrence of the indeterminate x_i by x_{i+n_1} for $i = 1, \ldots, n_2$. Now, if p is in Q_1, let us define $\psi[p, c] = \psi_1[p, c]$ for each c in Σ_1 and $\psi[p, c] = 0$ for each c in $\Sigma_2 - \Sigma_1$. If p is in Q_2, we shall define $\psi[p, c] = \text{shift}(\psi_2[p, c])$ for each c in Σ_2 and $\psi[p, c] = 0$ for each c in $\Sigma_1 - \Sigma_2$.

Finally, let $P_0 = P_{0,1} + \text{shift}(P_{0,2})$, and let each state in Q keep its terminal weight from $\mathcal{A}_1$ or $\mathcal{A}_2$. It is easy to show that $|\mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2|$. $\square$

Theorem 8.2. *Let S be a commutative semiring. The class of formal power series realised by alternating weighted automata over S is closed under Hadamard product.*

Proof. Let $\mathcal{A}_1 = (Q_1, \Sigma_1, \psi_1, P_{0,1}, \tau_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma_2, \psi_2, P_{0,2}, \tau_2)$ be alternating weighted automata over S with $|Q_1| = n_1$ and $|Q_2| = n_2$. An alternating weighted automaton $\mathcal{A} = (Q, \Sigma_1 \cup \Sigma_2, \psi, P_0, \tau)$ over S such that $|\mathcal{A}| = |\mathcal{A}_1| \odot |\mathcal{A}_2|$ can be constructed similarly to the automaton realising the sum from the proof of Theorem 8.1. The only difference is in the initial polynomial P_0, where one takes $P_0 = P_{0,1} \cdot \text{shift}(P_{0,2})$. It is easy to show that if $\mathcal{A}$ is constructed in this way, then $|\mathcal{A}| = |\mathcal{A}_1| \odot |\mathcal{A}_2|$ holds. $\square$

We shall now show that there exists a commutative semiring S such that the class of formal power series realised by alternating weighted automata over S is closed *neither* under reversal, *nor* under Cauchy product. More specifically, we shall show this for the semiring $\mathbb{B}[y]$ of polynomials in indeterminate y with coefficients in the Boolean semiring $\mathbb{B}$ (isomorphic to the semiring of finite languages over a unary alphabet).

Let $\Sigma = \{a, b, \#\}$ and let r_B be a series in $\mathbb{B}[y]\langle\langle\Sigma^*\rangle\rangle$ such that $(r_B, a^i\#b^j) = (1 + y^i)^j$ for each pair of nonnegative integers i, j and $(r_B, w) = 0$ for all other w in Σ^*. For each i and j, we can equivalently write $(r_B, a^i\#b^j) = \sum_{k=0}^{j} y^{ki}$. We shall first prove that r_B is not realised by an alternating weighted automaton over $\mathbb{B}[y]$. Later, we shall show that the reversal of r_B can be realised by an alternating weighted automaton and that r_B can be expressed as a Cauchy product of two series realised by alternating automata.

To prove that r_B cannot be realised by an alternating weighted automaton, some preparations need to be made. Let S be a commutative semiring and X an arbitrary set. Let S^X be the semiring of all functions from X to S with the usual addition and multiplication of functions: that is, the sum of functions f_1, f_2 in S^X is a function $f_1 + f_2$ in S^X such that $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ for all x in X and the product of functions f_1, f_2 in S^X is a function $f_1 f_2$ in S^X such that $(f_1 f_2)(x) = f_1(x) f_2(x)$ for all x in X. In the proof of Lemma 8.3, we shall work with the semiring $(\mathbb{B}[y])^{\mathbb{N}}$.

Let S be a commutative semiring, X be an arbitrary set, and F be a set of functions from X to S. We shall say that functions in F *have the same support* if $f_1(x) = 0_S$ if and only if $f_2(x) = 0_S$ for each f_1, f_2 in F and all x in X. Moreover, let P be a polynomial in $S[x_1, \ldots, x_n]$. We shall denote by $\#(P)$ the *number of terms* in P – that is, the smallest number k such that P can be obtained as a sum of k monomials. If m is a monomial in $S[x_1, \ldots, x_n]$, we shall denote by J_m the set of all indices i in $\{1, \ldots, n\}$ such that x_i occurs in m.

Lemma 8.3. *The formal power series r_B is not realised by an alternating weighted automaton over $\mathbb{B}[y]$.*

Proof. Suppose for the purpose of contradiction that the series r_B is realised by an alternating weighted automaton $\mathcal{A} = (Q, \Sigma, \psi, P_0, \tau)$ over $\mathbb{B}[y]$. Let $n = |Q|$. For $i = 1, \ldots, n$, let f_i be a mapping from $\mathbb{N}$ to $\mathbb{B}[y]$ such that $f_i(k) = (|\mathcal{A}|_i, b^k)$ for all nonnegative integers k. For each nonnegative integer i, let g_i be a mapping from $\mathbb{N}$ to $\mathbb{B}[y]$ such that $g_i(k) = (1 + y^i)^k$ for all nonnegative integers k, and let $P_i = P_{\mathcal{A}}[a^i\#]$. We have

$$g_i(k) = (|\mathcal{A}|, a^i\#b^k) = P_{\mathcal{A}}[a^i\#]\big((|\mathcal{A}|_1, b^k), \ldots, (|\mathcal{A}|_n, b^k)\big) = P_i(f_1(k), \ldots, f_n(k)) \tag{17}$$

for all nonnegative integers k. Let us now interpret each coefficient c in P_i as a constant function h_c from $\mathbb{N}$ to $\mathbb{B}[y]$ with $h_c(k) = c$ for all k. Thus, P_i can be viewed as a polynomial in $(\mathbb{B}[y])^{\mathbb{N}}[x_1, \ldots, x_n]$. Under this interpretation, the equation (17) says that $P_i(f_1, \ldots, f_n) = g_i$ for every nonnegative integer i.

Let i be a nonnegative integer. The polynomial P_i can be written as a sum of some monomials with nonzero coefficients. For each such monomial m, we would like the functions in $\{f_j\}_{j\in J_m}$ to have the same support. Of course, this might not be the case. To make this true, we shall modify the polynomials $P_1, P_2, P_3, \ldots$ to polynomials $P'_1, P'_2, P'_3, \ldots$ in $(\mathbb{B}[y])^{\mathbb{N}}[x_1, \ldots, x_{n'}]$ (where n' is some nonnegative integer) and replace the set of functions $F = \{f_1, \ldots, f_n\}$ by some other set of functions $F' = \{f'_1, \ldots, f'_{n'}\}$ in $(\mathbb{B}[y])^{\mathbb{N}}$. This needs to be done in such a way that we still have $P'_i(f'_1, \ldots, f'_{n'}) = g_i$ for every nonnegative integer i.

Let us first construct F'. For $i = 1, \ldots, n$, let χ_i be a function from $\mathbb{N}$ to $\mathbb{B}[y]$ such that $\chi_i(k) = 0$ if $f_i(k) = 0$ and $\chi_i(k) = 1$ otherwise. Let us define F' to be the set of all functions $f\prod_{j\in J}\chi_j$, where f is in F and J is a subset of $\{1, \ldots, n\}$. The set F' is clearly finite; let n' be its size and let $F' = \{f'_1, \ldots, f'_{n'}\}$.

We shall now show how the polynomial P'_i can be constructed for a given nonnegative integer i. We can write $P_i = \sum_{m\in M} c_m m$, where M is a finite set of monomials with coefficient 1 and c_m is a constant function in $(\mathbb{B}[y])^{\mathbb{N}}$ for each m in M. Then $g_i = \sum_{m\in M} c_m m(f_1, \ldots, f_n)$. Let m be in M and let us look at the function $m(f_1, \ldots, f_n)$. By its definition, this function can be obtained as a finite product of functions from F. For each j in J_m, let us replace each occurrence of f_j in this product by $f_j\prod_{k\in J_m}\chi_k$. We obtain a product of functions in F' that evaluates to $m(f_1, \ldots, f_n)$ and all functions in this product have the same support. Now it is clear that we can find a monomial m' with coefficient 1 such that $m'(f'_1, \ldots, f'_{n'}) = m(f_1, \ldots, f_n)$ and functions in $\{f'_j\}_{j\in J_{m'}}$ have the same support. We can do this for each m in M, and so we conclude that we can construct a polynomial P'_i such that $P'_i(f'_1, \ldots, f'_{n'}) = P_i(f_1, \ldots, f_n) = g_i$ and functions in $\{f'_j\}_{j\in J_{m'}}$ have the same support for each monomial m' that occurs in P'_i.

Now, let J^∞ be the set of indices j in $\{1, \ldots, n'\}$ such that x_j occurs in P'_i for infinitely many indices i. We shall show that for every j in J^∞ and every nonnegative integer k, the number of terms in the polynomial $f'_j(k)$ in $\mathbb{B}[y]$ is at most 1 (i.e., either $f'_j(k) = y^i$ for some positive integer i, or $f'_j(k) = 1$, or $f'_j(k) = 0$). In order to obtain a contradiction, let us suppose that $f'_j(k) = y^{l_1} + y^{l_2} + B_1$ for some j in J^∞, nonnegative integers k, l_1, and l_2 satisfying $l_1 < l_2$, and B_1 in $\mathbb{B}[y]$. Let i be a nonnegative integer such that $i > l_2 - l_1$ and P'_i contains x_j (such i exists, since x_j occurs in infinitely many polynomials in $\{P'_1, P'_2, P'_3, \ldots\}$). Let m be a monomial in P'_i that contains x_j. We can write $m = x_j m'$, where m' is a nonzero monomial. Thus,

$$m(f'_1, \ldots, f'_{n'})(k) = f'_j(k)m'(f'_1, \ldots, f'_{n'})(k) = (y^{l_1} + y^{l_2} + B_1)m'(f'_1, \ldots, f'_{n'})(k).$$

As $f'_j(k)$ is nonzero and functions in $\{f'_j\}_{j\in J_m}$ have the same support, $m'(f'_1, \ldots, f'_{n'})(k)$ is nonzero as well. We can write $m'(f'_1, \ldots, f'_{n'})(k) = y^{l_3} + B_2$, where l_3 is a nonnegative integer and B_2 is in $\mathbb{B}[y]$. (Note that y^{l_3} might be 1 and B_2 might be 0.) We have

$$m(f'_1, \ldots, f'_{n'})(k) = (y^{l_1} + y^{l_2} + B_1)m'(f'_1, \ldots, f'_{n'})(k) = (y^{l_1} + y^{l_2} + B_1)(y^{l_3} + B_2) = y^{l_1+l_3} + y^{l_2+l_3} + B_3$$

for some B_3 in $\mathbb{B}[y]$. We conclude that $P'_i(f'_1, \ldots, f'_{n'})(k) = y^{l_1+l_3} + y^{l_2+l_3} + B_4$ for some B_4 in $\mathbb{B}[y]$ and

$$\sum_{j'=0}^{k} (y^i)^{j'} = g_i(k) = P'_i(f'_1, \ldots, f'_{n'})(k) = y^{l_1+l_3} + y^{l_2+l_3} + B_4.$$

This is clearly a contradiction, since $i > l_2 - l_1$.

We have thus proved that if j is in J^∞ and k is a nonnegative integer, then the number of terms in $f'_j(k)$ is at most 1. Let i be a nonnegative integer such that every indeterminate that occurs in P'_i is in $\{x_j\}_{j\in J^\infty}$. We can write $P'_i = \sum_{m\in M} c_m m$, where M is some finite set of monomials with coefficient 1 and c_m is a constant function in $(\mathbb{B}[y])^{\mathbb{N}}$. Let k be a nonnegative integer. We have $g_i(k) = \sum_{m\in M} c_m(k)m(f'_1, \ldots, f'_{n'})(k)$. Since the number of terms in $f'_j(k)$ is at most 1 for every j in J_m, the number of terms in $m(f'_1, \ldots, f'_{n'})(k)$ is also at most 1 for every monomial m in M. This implies that the number of terms in $c_m(k)m(f'_1, \ldots, f'_{n'})(k)$ is at most the number of terms in $c_m(k) = c_m(0)$ for each monomial m in M. We conclude that the number of terms in $g_i(k) = \sum_{m\in M} c_m(k)m(f'_1, \ldots, f'_{n'})(k)$ is at most $\sum_{m\in M} \#(c_m(0))$ for all nonnegative integers k. This is clearly a contradiction. □

We are now finally prepared to prove that the class of formal power series realised by alternating weighted automata over $\mathbb{B}[y]$ is closed neither under reversal, nor under Cauchy product.

Theorem 8.4. *The class of formal power series realised by alternating weighted automata over $\mathbb{B}[y]$ is not closed under reversal.*

Proof. In Fig. 12, a two-mode alternating weighted automaton $\mathcal{A}$ over $\mathbb{B}[y]$ is depicted such that $|\mathcal{A}|^R = r_B$. By Lemma 8.3, the series r_B is not realised by an alternating weighted automaton over $\mathbb{B}[y]$. This means that the class of formal power series realised by alternating weighted automata over the semiring $\mathbb{B}[y]$ is not closed under reversal. □

Theorem 8.5. *The class of formal power series realised by alternating weighted automata over $\mathbb{B}[y]$ is not closed under Cauchy product.*

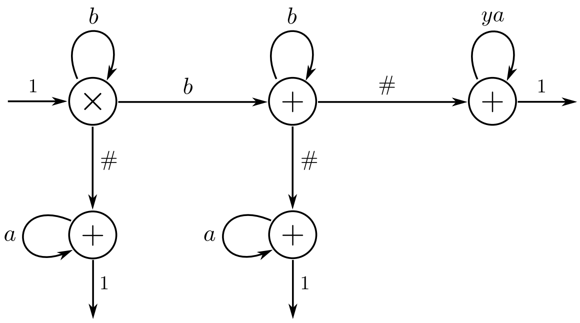

Fig. 12. A two-mode alternating weighted automaton $\mathcal{A}$, which realises the reversal of r_B.

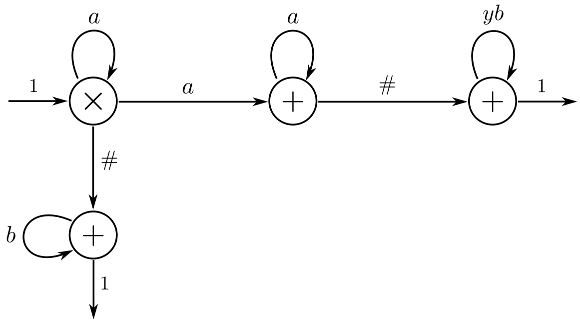

Fig. 13. A two-mode alternating weighted automaton $\mathcal{A}_1$ such that $(|\mathcal{A}_1|, a^i\#b^j) = y^{ij}$ for each pair of nonnegative integers i, j and $(|\mathcal{A}_1|, w) = 0$ for all other words w in $\{a, b, \#\}^*$.

Proof. In Fig. 13, a two-mode alternating weighted automaton $\mathcal{A}_1$ over $\mathbb{B}[y]$ and $\Sigma = \{a, b, \#\}$ is depicted such that $(|\mathcal{A}_1|, a^i\#b^j) = y^{ij}$ holds for each pair of nonnegative integers i, j and such that $(r_1, w) = 0$ for all other w in Σ^*. Moreover, it is easy to construct an alternating weighted automaton $\mathcal{A}_2$ over $\mathbb{B}[y]$ and Σ such that $(|\mathcal{A}_2|, w) = 1$ for each w in $\{b\}^*$ and $(r_2, w) = 0$ for all other w in Σ^*. The reader can easily check that $r_1 r_2 = r_B$. By Lemma 8.3, the series r_B is not realised by an alternating weighted automaton over $\mathbb{B}[y]$. This means that the class of formal power series realised by alternating weighted automata over $\mathbb{B}[y]$ is not closed under Cauchy product. $\square$

9. Conclusion

We have defined and initiated the study of alternating weighted automata over an arbitrary commutative semiring S. The model introduced in this article generalises both classical alternating finite automata (without weights) [2] and "min-sum"-alternating automata over the tropical semiring of reals introduced by Almagor and Kupferman [1]. We have shown that the usual equivalent approaches to the formal definition of alternating finite automata (without weights) remain equivalent in our weighted setting as well and proved an equational characterisation of formal power series realised by alternating weighted automata.

In our main result, we have proved a complete characterisation of the class of commutative semirings, for which alternating and non-alternating weighted automata are equally powerful: a commutative semiring belongs to this class if and only if it is locally finite. Alternating weighted automata are strictly more powerful than their non-alternating counterparts for commutative semirings that are not locally finite.

We have also observed that the class of formal power series realised by alternating weighted automata over S is closed under sum and Hadamard product for every commutative semiring S. On the other hand, we have proved that this class is not closed under reversal and Cauchy product for at least one commutative semiring S.

One might still think of some reasonable and well motivated settings, which can be described as a form of alternation in weighted automata, but cannot be incorporated into the framework of alternating weighted automata over semirings. Most notably, this is the case of "min–max"-alternation [1,3], for which a suitable generalisation seems to require structures with at least three operations. A systematic research of weighted alternation at this higher level of generality is the main task that we leave open for future research. Moreover, an interesting question that we have not touched upon in this article is the power of *finite* alternation in two-mode alternating weighted automata. Finally, we believe that an extension to infinite words can be beneficial mainly due to possible applications in quantitative formal verification [3].

Does not add expressiveness over the reals

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