

# A Normal Form Characterization for Efficient Boolean Skolem Function Synthesis

Preety Shah, Aman Bansal, S. Akshay and Supratik Chakraborty

Department of Computer Science and Engineering

Indian Institute of Technology Bombay, Mumbai, India.

Email: {preeyshah, aman0456b}@gmail.com; {akshayss, supratik}@cse.iitb.ac.in

**Abstract**—Boolean Skolem function synthesis concerns synthesizing outputs as Boolean functions of inputs such that a relational specification between inputs and outputs is satisfied. This problem, also known as Boolean functional synthesis, has several applications, including design of safe controllers for autonomous systems, certified QBF solving, cryptanalysis etc. Recently, complexity theoretic hardness results have been shown for the problem, although several algorithms proposed in the literature are known to work well in practice. This dichotomy between theoretical hardness and practical efficacy has motivated the research into normal forms or representations of input specifications that permit efficient synthesis, thus explaining perhaps the efficacy of these algorithms.

In this paper we go one step beyond this and ask if there exists a normal form representation that can in fact precisely characterize “efficient” synthesis. We present a normal form called SAUNF that precisely characterizes tractable synthesis in the following sense: a specification is polynomial time synthesizable iff it can be compiled to SAUNF in polynomial time. Additionally, a specification admits a polynomial-sized functional solution iff there exists a semantically equivalent polynomial-sized SAUNF representation. SAUNF is exponentially more succinct than well-established normal forms like BDDs and DNNFs, used in the context of AI problems, and strictly subsumes other more recently proposed forms like SynNNE. It enjoys compositional properties that are similar to those of DNNF. Thus, SAUNF provides the right trade-off in knowledge representation for Boolean functional synthesis.

## I. INTRODUCTION

The history of Skolem functions can be traced all the way back to 1920’s to Thoralf Skolem and his simplified proof of the celebrated Löwenheim Skolem Theorem. A key step in the proof showed that any first order logic formula can be converted into Skolem normal form, that has no existential quantifiers, while preserving satisfiability. This process, called Skolemization, involves replacing existentially quantified variables by terms constructed out of new function symbols, called Skolem functions, and is now a standard technique with many applications, viz. automated theorem proving. While some applications only require the existence of Skolem functions, others depend upon being able to efficiently synthesize such Skolem functions.

Algorithmic synthesizability of Skolem functions has been studied extensively in the Boolean setting. Given disjoint sequences of Boolean variables  $\mathbf{I} = (i_1, \dots, i_n)$  and  $\mathbf{X} =$

$(x_1, \dots, x_m)$ , representing inputs and outputs respectively of a system, and given a Boolean formula  $\varphi(\mathbf{X}, \mathbf{I})$  specifying a desired relation between the system inputs and outputs, the *Boolean Skolem function synthesis* (BFnS) problem asks to synthesize a sequence of formulas  $\Psi(\mathbf{I}) = (\psi_1(\mathbf{I}), \dots, \psi_m(\mathbf{I}))$  that can be substituted for  $\mathbf{X}$  to satisfy the specification, i.e.,  $\forall \mathbf{I} (\varphi(\Psi(\mathbf{I}), \mathbf{I}) \Leftrightarrow \exists \mathbf{X} \varphi(\mathbf{X}, \mathbf{I}))$ . The formulas in  $\Psi$  indeed represent Boolean Skolem functions for  $\mathbf{X}$  in  $\exists \mathbf{X} \varphi(\mathbf{X}, \mathbf{I})$ <sup>1</sup>.

The above problem, also called *Boolean functional synthesis* in the literature, has several applications; we will just mention two here. Skolem functions (and their counterparts, called Herbrand functions) can be thought of as “certificates” that help us independently verify the results of satisfiability checking for Quantified Boolean Formulas, as done in [1]. QBF-satisfiability solving is used today in diverse applications [2], from planning to program repair to reactive synthesis and the like. Having certificates not only helps to verify correctness of QBF-satisfiability checks, but also has other benefits like providing a feasible plan in a planning problem. Yet another application of Skolem functions is motivated by cryptanalysis. Consider a system with a single  $2n$ -bit unsigned integer input  $\mathbf{I}$ , and two  $n$ -bit unsigned integer outputs  $\mathbf{X}_1$  and  $\mathbf{X}_2$ . Suppose the relational specification is given as  $F_{fact}(\mathbf{X}_1, \mathbf{X}_2, \mathbf{I}) \equiv ((\mathbf{I} = \mathbf{X}_1 \times_{[n]} \mathbf{X}_2) \wedge (\mathbf{X}_1 \neq \mathbf{1}) \wedge (\mathbf{X}_2 \neq \mathbf{1}))$ , where  $\times_{[n]}$  denotes  $n$ -bit unsigned integer multiplication and  $\mathbf{1}$  denotes an  $n$ -bit representation of the integer 1. This specification can be represented as a Boolean formula of size  $\mathcal{O}(n^2)$  over the variables in  $\mathbf{I}$ ,  $\mathbf{X}_1$  and  $\mathbf{X}_2$ . Finding Skolem functions for  $\mathbf{X}_1$  and  $\mathbf{X}_2$  in terms of  $\mathbf{I}$  effectively asks us to solve the ( $n$ -bit) factorization problem. Note that if  $\mathbf{I}$  represents a prime number, there are no values of  $\mathbf{X}_1, \mathbf{X}_2$  that satisfy the specification. Hence the specification is technically *unrealizable*; yet, it is of significant interest (e.g. in cryptanalysis) to synthesize Skolem functions for  $\mathbf{X}_1, \mathbf{X}_2$  that can be evaluated efficiently. It is worth noting that it is an open question whether there are polynomial time algorithms or polynomial sized circuits for integer factorization.

Given its significance, the Boolean Skolem function synthesis problem has received considerable interest over the last two decades, with a lot of work focussed towards design of practically efficient algorithms [3], [4], [5], [6], [7], [8],

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<sup>1</sup>We are conflating functions and formulas here for simplicity, the distinction will be made clear later.

[9], [10], [11], [12], [13]. These algorithms, using techniques ranging from CEGAR to decision tree learning, empirically work well on some large collections of benchmarks, but fail even for some small benchmarks. Further, and somewhat surprisingly, each of the tools seem to work well for a *different* set of benchmarks, often incomparable across tools. What is common about the approaches is that it is a priori unclear on which set of benchmarks, i.e., which precise class of formulas, a particular algorithm will be efficient (other than simple cases or heuristic guesses). In this background, in [12], a theoretical study was undertaken which showed that Boolean Skolem function synthesis requires super-polynomial space and time unless some well-regarded complexity-theoretic conjectures are falsified. In fact, they also showed that under some weaker assumptions, there cannot exist even sub-exponential algorithms for this problem.

This leads to a curious dichotomy, of theoretical worst-case hardness vs practical (sometimes unreasonable) efficiency. To resolve this dichotomy, researchers have searched for structure in the input specification that can result in provably efficient synthesis. It turns out that the representation used for input specification and Skolem functions also has a bearing on the efficiency of synthesis. For example, if the specification is given as a ROBDD [14] with *input-first variable ordering* (see [7] for details), there exists a polynomial-time algorithm that generates Skolem functions as ROBDDs [7]. In [15], a new normal form for specifications, called SynNNF, was proposed, which ensures polynomial-time synthesis, assuming both the specification and Skolem functions are represented as arbitrary Boolean circuits. However, these earlier studies only provide sufficient but not necessary conditions for efficient Skolem function synthesis. Significantly, it is not the case that every class of specifications that admit efficient Skolem function synthesis can be efficiently compiled to ROBDDs with input-first variable ordering, or even to SynNNF. Indeed, the authors of [15] give (counter-)examples of specifications that are not in SynNNF but admit efficient Skolem function synthesis.

In this paper, we address the above dichotomy, by presenting - to the best of our knowledge - for the first time, a normal form for Boolean circuits, called SAUNF (for *Subset-And-Unrealizable Normal Form*), that characterizes polynomial-time and polynomial-sized Boolean Skolem function synthesis. By a characterization, as formalized in Section III, we mean that for *every* class  $\mathcal{C}$  of circuits (i) Skolem functions can be synthesized in polynomial-time for specifications represented by circuits in  $\mathcal{C}$  iff these circuits can be compiled to semantically equivalent ones in SAUNF in polynomial-time, and (ii) specifications represented by circuits in  $\mathcal{C}$  admit polynomial-sized Skolem functions iff they can be compiled into polynomial-sized semantically equivalent circuits in SAUNF. We explore the proposed normal form in depth, and present several results about SAUNF :

- We show that SAUNF is (often exponentially) more succinct and strictly subsumes several other sub-classes (viz. DNNF, dDNNF [16], wDNNF [12], ROBDD, SynNNF).
- We present a polynomial-time algorithm to synthesize polynomial-sized Skolem functions from specifications in SAUNF.
- We study compositional properties of SAUNF including disjunction and conjunction operations.
- We show that checking membership in SAUNF is coNPhard and is in the second level of the polynomial hierarchy.
- We present a novel algorithm for compiling a Boolean relational specification in CNF to SAUNF.

Finally, we show an interesting application of SAUNF. Specifically, we show that in the context of the  $n$ -bit factorization problem mentioned above, there exist polynomial-sized SAUNF circuits relating specific bits of the input  $\mathbf{I}$  to the outputs  $\mathbf{X}_1$  and  $\mathbf{X}_2$ . While this does not solve the  $n$ -bit factorization problem yet, we point out that it is known that some of these bit relations require exponentially large ROBDDs [17], while sub-exponential circuits using normal forms like DNNF, dDNNF, SynNNF are not known.

Knowledge representations and normal forms for Boolean functions have been investigated extensively over the last decades [18], [19], [20], [21]. While a problem compiled to a normal form may allow the problem to be solved efficiently, compilation to the normal form may not always be easy. For instance, dDNNF allows polynomial-time model counting, but converting to dDNNF cannot always be done in polynomial time unless  $P = \#P$ . Despite the worst-case complexity of the compilation process, research in normal forms offers several benefits, such as better understanding of compositionality and other structural properties, explanations for practical performance of algorithms (e.g., on benchmarks in a normal form that permits efficient analysis) etc. Furthermore, the study of normal forms also feeds into research on normal form compilers, that have significant practical use. For example, multiple dDNNF compilers have been developed since the introduction of the normal form. We also point out that different types of normal forms have been studied earlier: syntactic or purely structural normal forms like CNF, DNF, DNNF allow efficient membership checking, while semantic normal forms like dDNNF require propositional satisfiability checks to determine membership. The proposed normal form (SAUNF) falls in the latter category, but like dDNNF, is relevant for the good properties it exhibits.

The paper is organized as follows. We start with preliminaries in Section II and problem statement in Section III. In Section IV, we introduce SAUNF with examples and compare it with other normal forms in Section V. We explain in Section VI how Skolem functions can be efficiently computed from SAUNF form and show compositionality properties in Section VII. Next, in Section VIII, we describe an algorithm to compile any circuit to SAUNF. Finally, we show applications to  $n$ -bit factorization in Section IX and conclude in Section X.

## II. PRELIMINARIES

Let  $\mathbf{V} = (v_1, \dots, v_r)$  be a finite sequence of Boolean variables. We use  $\text{set}(\mathbf{V})$  to denote the underlying set of the

sequence and  $|\mathbf{V}|$  to denote the length of the sequence. A *literal*  $\ell$  over  $\mathbf{V}$  is either  $v$  or  $\neg v$ , where  $v \in \text{set}(\mathbf{V})$ . The set of all literals over  $\mathbf{V}$  is denoted  $\text{lits}(\mathbf{V})$ . We use  $\top$  and  $\perp$  to represent the Boolean constants true and false respectively. A Boolean *formula*  $\varphi$  over  $\mathbf{V}$  is defined by the grammar:

$$\varphi ::= \neg\varphi \mid (\varphi) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \top \mid \perp \mid v_1 \mid \dots \mid v_r.$$

We write  $\varphi(\mathbf{V})$  to denote that the formula  $\varphi$  is defined over the sequence of variables  $\mathbf{V}$ . Special cases of formulas include *clauses*, disjunctions of literals and *cubes*, conjunction of literals. A formula is in *conjunctive normal form* (CNF) if it is a conjunction of clauses. Similarly, it is in *disjunctive normal form* (DNF) if it is a disjunction of cubes. A Boolean *function*  $F(\mathbf{V})$  is a mapping  $\{\perp, \top\}^{|\mathbf{V}|} \rightarrow \{\perp, \top\}$ . The semantics of the Boolean formula  $\varphi(\mathbf{V})$  is given by a Boolean function  $\llbracket \varphi \rrbracket(\mathbf{V}) : \{\perp, \top\}^{|\mathbf{V}|} \rightarrow \{\perp, \top\}$ . It is easy to see that every Boolean function  $F$  corresponds to at least one Boolean formula  $\varphi$  such that  $\llbracket \varphi \rrbracket = F$ .

Let  $\mathbf{U}$  be a sub-sequence of  $\mathbf{V}$  (this includes the possibility  $\mathbf{U} = \mathbf{V}$ ), and let  $\mathbf{V} \setminus \mathbf{U}$  denote the sequence obtained by removing from  $\mathbf{V}$  all variables present in  $\mathbf{U}$ . An *assignment* of  $\mathbf{U}$  is a mapping  $\sigma : \text{set}(\mathbf{U}) \rightarrow \{\perp, \top\}$ . We use  $\llbracket \varphi \rrbracket_\sigma$  to denote the Boolean function  $\{\perp, \top\}^{|\mathbf{V} \setminus \mathbf{U}|} \rightarrow \{\perp, \top\}$  obtained by substituting  $\sigma(v_j)$  for every variable  $v_j \in \text{set}(\mathbf{U})$  in  $\llbracket \varphi \rrbracket$ . We say that the formula  $\varphi(\mathbf{V})$  *reduces* to the formula  $\varphi'(\mathbf{V} \setminus \mathbf{U})$  under the assignment  $\sigma$  of  $\mathbf{U}$  iff  $\llbracket \varphi' \rrbracket = \llbracket \varphi \rrbracket_\sigma$ . We say that  $\sigma$  *satisfies*  $\varphi$  if  $\llbracket \varphi \rrbracket_\sigma$  always evaluates to  $\top$ .

We choose to represent both Boolean functions and Boolean formulas (modulo semantic equivalence) by Boolean circuits. For purposes of this paper, a Boolean circuit (or simply a circuit) is a rooted directed acyclic graph (DAG)  $G$  in which nodes with incoming edges, also called *internal nodes*, are labeled by  $\vee$ ,  $\wedge$  and  $\neg$  operators, and nodes with no incoming edges, also called *leaves*, are labeled either by variables in  $\mathbf{V}$  or by constants in  $\{\perp, \top\}$ . Every internal node labeled  $\wedge$  or  $\vee$  has incoming edges from exactly two children, while every internal node labeled  $\neg$  has an incoming edge from exactly one child. In order to ensure that a circuit doesn't have superfluous nodes, we require all nodes in a circuit to be descendants of the root. The size of a circuit  $G$ , denoted  $|G|$ , is the number of nodes in  $G$ . A circuit  $G$  represents a Boolean formula  $\varphi_G$  (alternatively, a Boolean function  $\llbracket \varphi_G \rrbracket$ , if  $G$  is used to represent a Boolean function) defined as follows: (i) if  $G$  consists of a single leaf labeled  $\lambda$ , then  $\varphi_G = \lambda$ ; (ii) if the root of  $G$  is labeled  $\text{op} \in \{\wedge, \vee\}$  and if the sub-circuits rooted at its children are  $G_1$  and  $G_2$ , then  $\varphi_G = \varphi_{G_1} \text{ op } \varphi_{G_2}$ ; (iii) if the root of  $G$  is labeled  $\neg$  and if the sub-circuit rooted at its (only) child is  $H$ , then  $\varphi_G = \neg\varphi_H$ .

A Boolean formula is said to be in *negation normal form* (or NNF) if the application of  $\neg$  is restricted to only the variables. Motivated by this, a circuit in which every  $\neg$  labeled node has a leaf labeled by a variable as its child is said to be an *NNF circuit*. It is well-known that every Boolean formula is semantically equivalent to a formula in NNF. Since we wish to reason about Boolean formulas/functions modulo semantic equivalence, it suffices to restrict our attention to NNF circuits. It is easy to see that an arbitrary Boolean circuit  $G$  can be

converted to an NNF circuit  $G'$  such that  $\llbracket \varphi_G \rrbracket = \llbracket \varphi_{G'} \rrbracket$ , and  $|\varphi_{G'}| \leq 2 \times |\varphi_G|$ . For notational convenience, we treat a  $\neg$  labeled node with a child labeled  $v$  in a NNF circuit, as a new leaf labeled  $\neg v$ . Thus, an NNF circuit can be viewed as a rooted DAG with  $\wedge$ - and  $\vee$ -labeled internal nodes and leaves labeled by literals over  $\mathbf{V}$ . Note that popular representations of Boolean functions, viz. lists of (implicitly conjoined) clauses, lists of (implicitly disjoined) cubes, and-inverter graphs [22], ROBDDs [14], DNNF/dDNNF circuits [19], [16] and the like, can all be translated to NNF circuits in linear time. Figure 1 shows an example of an NNF circuit. In this figure, the annotations  $G$ ,  $G_1$  and  $G_2$  represent the (sub-)circuits rooted at the nodes adjacent to the annotations. The leaves are designated  $L_0$  through  $L_{15}$  from left to right, for convenience of referring to them later. As is the case in this figure, multiple leaves of a circuit may have the same literal label. *Henceforth, all circuits are assumed to be NNF, unless stated otherwise.*

Let  $L$  be a subset of leaves of circuit  $G$ . We say  $L$  is *literal-consistent* in  $G$  if every leaf in  $L$  is labeled by the same literal. For a literal-consistent set of leaves  $L$  in  $G$  and for  $b \in \{\perp, \top\}$ , we use  $G|_{L:b}$  to denote the circuit obtained by re-labeling each leaf of  $G$  in the set  $L$  with  $b$ . For a literal  $\ell$  over  $\mathbf{V}$ , we use the term  $\ell$ -*leaves* of  $G$  to denote the set of all leaves of  $G$  labeled  $\ell$ . For a set of distinct literals  $\{\ell_1, \dots, \ell_i\}$  and (possibly same) labels  $b_1, \dots, b_i$ , we abuse notation and use  $G|_{\ell_1=b_1, \dots, \ell_r=b_r}$  to denote the circuit obtained by re-labeling all  $\ell_j$ -leaves of  $G$  by  $b_j$ , for all  $j \in \{1, \dots, r\}$ . Note that since  $\ell$  and  $\neg\ell$  are different literals, the notation  $G|_{\ell=b, \neg\ell=b}$  is meaningful (and useful), and represents the circuit obtained by re-labeling all  $\ell$ -leaves and  $\neg\ell$ -leaves of  $G$  by  $b$ .

Let  $\mathbf{I} = (i_1, \dots, i_m)$  and  $\mathbf{X} = (x_1, \dots, x_m)$  be disjoint sequences of Boolean variables representing inputs and outputs, respectively, of a hypothetical system. For clarity of exposition, we use "system inputs" to refer to  $\mathbf{I}$ , and "system outputs" to refer to  $\mathbf{X}$ . Consider a circuit  $G$  with leaves labeled by  $\text{lits}(\mathbf{X})$  and  $\text{lits}(\mathbf{I})$ . The formula  $\varphi_G(\mathbf{X}, \mathbf{I})$  represents a relational specification over the system inputs  $\mathbf{I}$  and system outputs  $\mathbf{X}$ . Given  $G$ , the *Boolean Skolem Function Synthesis* or BFnS problem requires us to find a sequence of Boolean formulas  $\Psi(\mathbf{I}) = (\psi_1(\mathbf{I}), \dots, \psi_m(\mathbf{I}))$  such that  $\forall \mathbf{I} (\varphi_G(\Psi(\mathbf{I}), \mathbf{I}) \Leftrightarrow \exists \mathbf{X} \varphi_G(\mathbf{X}, \mathbf{I}))$ . As seen earlier, this is an important problem with diverse applications. We call  $\psi_j(\mathbf{I})$  a *Skolem function*<sup>2</sup> for  $x_j$  in  $\varphi_G(\mathbf{X}, \mathbf{I})$ , and the sequence (or vector) of all such Skolem functions for  $x_1, \dots, x_m$  a *Skolem function vector* for  $\mathbf{X}$  in  $\varphi_G(\mathbf{X}, \mathbf{I})$ . Since we have chosen to represent all Boolean formulas and functions as circuits, we require each  $\psi_j(\mathbf{I})$  to be presented as a circuit.

**Example 1.** Let  $\mathbf{X} = (x_1, x_2)$  and  $\mathbf{I} = (i)$ . Let  $G$  be the circuit shown in Figure 1. Then  $\varphi_G(\mathbf{X}, \mathbf{I})$  is a relational specification over  $\mathbf{I}$  and  $\mathbf{X}$ , and one (of possibly many) Skolem function vectors for  $\mathbf{X}$  in  $\varphi_G$  is  $\Psi(\mathbf{I}) = (\psi_1(\mathbf{I}), \psi_2(\mathbf{I}))$ , where  $\psi_1(\mathbf{I}) =$

<sup>2</sup>Technically,  $\llbracket \psi_j \rrbracket$  is the Boolean Skolem function for  $x_j$  in  $\varphi_G$ . However, as we represent both Boolean functions and formulas as circuits, we use  $\psi_j$  and  $\llbracket \psi_j \rrbracket$  interchangeably for Boolean Skolem functions, to keep the notation simple.

$\neg i = \psi_2(\mathbf{I})$ . Indeed, it can be verified that  $\forall \mathbf{I} (\varphi_G(\Psi(\mathbf{I}), \mathbf{I}) \Leftrightarrow \exists \mathbf{X} \varphi_G(\mathbf{X}, \mathbf{I}))$ .

### III. MAIN PROBLEM STATEMENT

Earlier work [12] has established (conditional) time and space lower bounds for BFnS; therefore it is unlikely that efficient algorithms exist for solving the problem in general. Yet, several recent works [12], [7], [23], [24], [13], [25] have shown that BFnS indeed admits practically efficient solutions for several non-trivial benchmarks. This motivates us to ask the following question, where we are interested only in circuits representing relational specifications over system inputs and system outputs.

*Does there exist a class, say  $\mathcal{C}^*$ , of circuits such that the following hold?*

P0: For every circuit  $G$ , there is a semantically equivalent circuit  $G^* \in \mathcal{C}^*$ , i.e.  $\llbracket \varphi_G \rrbracket = \llbracket \varphi_{G^*} \rrbracket$ . In other words,  $\mathcal{C}^*$  is not semantically constraining.

P1: BFnS is solvable in polynomial-time for the class  $\mathcal{C}^*$ .

P2: For every class  $\mathcal{C}$  of circuits,

P2a: BFnS is solvable in polynomial-time for the class  $\mathcal{C}$  iff circuits in  $\mathcal{C}$  can be compiled to semantically equivalent ones in  $\mathcal{C}^*$  in polynomial-time.

P2b: Relational specifications represented by circuits in  $\mathcal{C}$  admit polynomial-sized Skolem function vectors iff circuits in  $\mathcal{C}$  admit polynomial-sized semantically equivalent circuits in  $\mathcal{C}^*$ .

We answer the above question positively in this paper, effectively providing a circuit normal form characterization of efficient Boolean Skolem Function Synthesis. In light of our characterization, the hardness results of [12] translate to hardness of computing  $G^* \in \mathcal{C}^*$  such that  $\llbracket \varphi_{G^*} \rrbracket = \llbracket \varphi_G \rrbracket$ .

### IV. A NORMAL FORM FOR SYNTHESIS

Let  $G$  be a circuit with leaves labeled by  $\text{lits}(\mathbf{I})$  and  $\text{lits}(\mathbf{X})$ . Let  $\ell$  be a literal labeling a leaf of  $G$ , and let  $v_\ell$  be the underlying variable of  $\ell$ . Throughout this section, we assume that  $w, w'$  are fresh variables not in  $\mathbf{I}$  or  $\mathbf{X}$ .

**Definition 1.** We say that  $\ell$  is  $\wedge$ -realizable in  $G$  iff there is an assignment  $\sigma : (\text{set}(\mathbf{I}) \cup \text{set}(\mathbf{X})) \setminus \{v_\ell\} \rightarrow \{\perp, \top\}$  such that  $\llbracket G \mid_{\ell=w, \neg \ell=w'} \rrbracket_\sigma = \llbracket (w \wedge w') \rrbracket$ . Furthermore, we say that  $\ell$  is  $\wedge$ -unrealizable in  $G$  iff it is not  $\wedge$ -realizable in  $G$ .

Intuitively,  $\ell$  is  $\wedge$ -realizable in  $G$  if  $\varphi_G$  reduces to  $w \wedge w'$  under some assignment of variables other than  $v_\ell$ , after  $\ell$  and  $\neg \ell$  are replaced by  $w$  and  $w'$  respectively in the leaves of  $G$ . Clearly, if  $\ell$  is  $\wedge$ -realizable (resp.  $\wedge$ -unrealizable) in  $G$ , then so is  $\neg \ell$ .

**Example 2.** Consider the circuit  $G$  in Figure 1, and let  $G_1$  and  $G_2$  denote the sub-circuits rooted at the left and right child, respectively of the root node. Then  $x_1$  is  $\wedge$ -realizable in  $G_2$  and also in  $G$  (use  $\sigma(i) = \sigma(x_2) = \perp$ ) but is  $\wedge$ -unrealizable in  $G_1$ .

We now extend the notion of  $\wedge$ -(un)realizability to that of sets of literal-consistent leaves. If  $S$  is the set of all  $\ell$ -leaves of

$G$ , the notion of  $S$  being  $\wedge$ -realizable (resp.  $\wedge$ -unrealizable) in  $G$  naturally coincides with that of literal  $\ell$  being  $\wedge$ -realizable (resp.  $\wedge$ -unrealizable) in  $G$ . However, if  $S$  does not contain all  $\ell$ -leaves of  $G$ , we must specify what to do with leaves labeled  $\ell$  but not in  $S$ . The following definition does exactly that.

**Definition 2.** Let  $S$  be a literal-consistent set of leaves of  $G$ , and let  $\ell$  be the literal labeling each leaf in  $S$ . Let  $S'$  be the set of all  $\ell$ -leaves of  $G$ . We say that  $S$  is  $\wedge$ -realizable (resp.  $\wedge$ -unrealizable) in  $G$  if  $\ell$  is  $\wedge$ -realizable (resp.  $\wedge$ -unrealizable) in  $G \mid_{S' \setminus S: \perp}$ .

Thus, all  $\ell$ -leaves that are not in  $S$  must be labeled  $\perp$  before we check whether  $\ell$  is  $\wedge$ -realizable in the resulting circuit.

**Example 3.** Referring back to Figure 1, we wish to check the  $\wedge$ -(un)realizability of  $S = \{L_3\}$  in  $G$ . The literal labeling  $L_3$  is  $x_1$  and the set of all  $x_1$ -leaves is  $S' = \{L_3, L_{10}\}$ . Hence  $S' \setminus S = \{L_{10}\}$ . To check the  $\wedge$ -(un)realizability of  $S$ , we re-label  $L_{10}$  with  $\perp$ , and  $L_3$  with a fresh variable  $w$ . Additionally, all leaves labeled  $\neg x_1$ , i.e.  $L_1$  and  $L_{14}$  are re-labeled with a fresh variable  $w'$ .

Let  $G'$  denote the resulting circuit. We now ask if there is an assignment  $\sigma : \{i, x_2\} \rightarrow \{\perp, \top\}$  such that  $\llbracket \varphi_{G'} \rrbracket_\sigma = \llbracket w \wedge w' \rrbracket$ . From the circuit structure of  $G'$ , we can see that there is only one leaf, viz.  $L_3$ , labeled  $w$ . Hence, in order to have  $\llbracket \varphi_{G'} \rrbracket_\sigma = \llbracket w \wedge w' \rrbracket$ , the assignment  $\sigma$  must not mask the value of  $L_3$  from "propagating" up to the root of  $G'$ . This implies that  $L_2$  must be labeled  $\top$ , i.e.  $\sigma(i) = \perp$ , and  $\sigma(x_2) = \neg \sigma(i) = \top$ . With this  $\sigma$ , it is now easy to verify that  $\llbracket \varphi_{G'} \rrbracket_\sigma = \llbracket w \rrbracket \neq \llbracket w \wedge w' \rrbracket$ . Hence, there is no assignment of  $x_2$  and  $i$  that renders the formula represented by  $G'$  semantically equivalent to  $w \wedge w'$ . It follows that  $S = \{L_3\}$  is  $\wedge$ -unrealizable in the circuit shown in Figure 1. A similar exercise shows that  $\tilde{S} = \{L_{10}\}$  is  $\wedge$ -realizable in the same circuit (use  $\sigma(i) = \sigma(x_2) = \perp$ ).

Finally, we use the above definitions to introduce a new normal form for circuits that precisely characterizes efficient Boolean Skolem Function Synthesis. We show in subsequent sections that this normal form defines a class  $\mathcal{C}^*$  of circuits that satisfies properties P0, P1 and P2 described in Section III.

**Definition 3.** Let  $G$  be a circuit with leaves labeled by  $\text{lits}(\mathbf{I})$  and  $\text{lits}(\mathbf{X})$ . Let  $S = (S_1, S_2, \dots, S_k)$  be a non-empty sequence of subsets of leaves of  $G$ . We say that  $G$  is in Subset And-Unrealizable Normal Form (SAUNF, for short) w.r.t.  $\text{set}(\mathbf{X})$  and  $S$  if the following hold:

- 1)  $S_j \cap S_l = \emptyset$  for all distinct  $j, l \in \{1, \dots, k\}$ .
- 2) For each  $j \in \{1, \dots, k\}$ , all leaves in  $S_j$  are labeled by the same literal over  $\mathbf{X}$ .
- 3)  $S_1$  is  $\wedge$ -unrealizable in  $G$ .
- 4) For each  $j \in \{2, \dots, k\}$ ,  $S_j$  is  $\wedge$ -unrealizable in  $G \mid_{S_1: \top, S_2: \top, \dots, S_{j-1}: \top}$ .
- 5)  $\llbracket \varphi_G \mid_{S_1: \top, S_2: \top, \dots, S_k: \top} \rrbracket$  is semantically independent of  $\mathbf{X}$ , i.e. its value doesn't depend on the assignment of  $\mathbf{X}$ .

A few points about Definition 3 are worth noting.

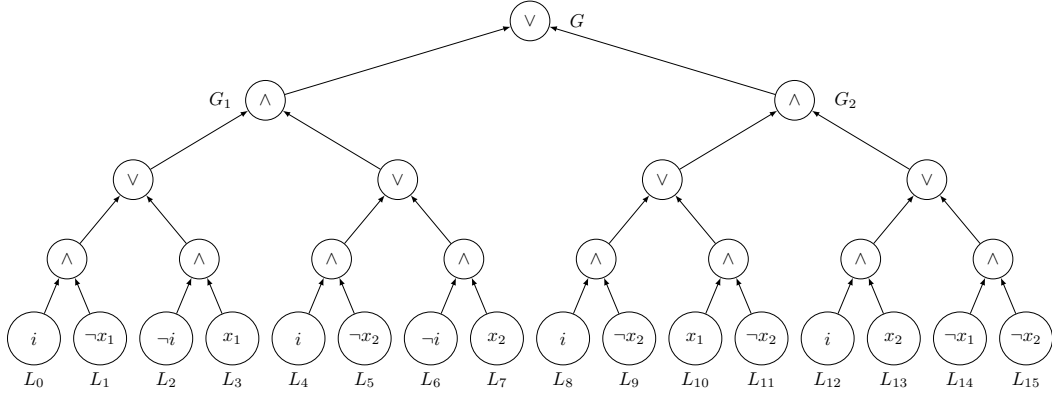


Fig. 1: Example of an NNF circuit.

- A circuit  $G$  may be in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and  $S$ , but not in SAUNF w.r.t. a different  $\text{set}(\mathbf{X}')$  and/or  $S'$ .
- Conditions 3, 4 and 5 are semantic in nature. Normal forms with such semantic conditions are not new. For example, the widely used disjoint decomposable negation normal form (dDNNF) uses a semantic condition in its definition (see [16]).
- $S_1 \cup \dots \cup S_k$  may not include all leaves of  $G$ , nor even all leaves labeled by a literal over  $\mathbf{X}$ .
- While the use of  $\top$  as labels for leaves in  $S_1, S_2, \dots$  in conditions 3 and 4 may seem arbitrary for now, we will soon see the significance of this in the synthesis of Boolean Skolem functions.

**Example 4.** Consider the circuit  $G$  in Figure 1 again, with  $\mathbf{I} = (i)$  and  $\mathbf{X} = (x_1, x_2)$ . Let  $S = (\{L_3\}, \{L_7\}, \{L_5\}, \{L_1\})$  be a sequence of (singleton) subsets of leaves. As seen above,  $\{L_3\}$  is  $\wedge$ -unrealizable in  $G$ . It can similarly be verified that  $\{L_7\}$  is  $\wedge$ -unrealizable in  $G \upharpoonright_{\{L_3\}:\top}$ ,  $\{L_5\}$  is  $\wedge$ -unrealizable in  $G \upharpoonright_{\{L_3\}:\top, \{L_7\}:\top}$  and  $\{L_1\}$  is  $\wedge$ -unrealizable in  $G \upharpoonright_{\{L_3\}:\top, \{L_7\}:\top, \{L_5\}:\top}$ . Finally, the function represented by  $G \upharpoonright_{\{L_3\}:\top, \{L_7\}:\top, \{L_5\}:\top, \{L_1\}:\top}$  is semantically equivalent to  $\top$ , and hence is independent of  $\mathbf{X} = (x_1, x_2)$ . Hence, the circuit  $G$  is in SAUNF w.r.t.  $\text{set}(\mathbf{X}) = \{x_1, x_2\}$  and  $S = (\{L_3\}, \{L_7\}, \{L_5\}, \{L_1\})$ . However,  $G$  is not in SAUNF w.r.t.  $\{x_1, x_2\}$  and  $S' = (\{L_{10}\}, \{L_7\}, \{L_5\}, \{L_1\})$ , since we have seen earlier that  $\{L_{10}\}$  is  $\wedge$ -realizable in  $G$ .

## V. RELATION WITH OTHER NORMAL FORMS

Several normal forms for Boolean circuits studied in the literature, viz. ROBDD [14], FDD [26], DNNF [19], dDNNF [16], wDNNF [12], SynNNF [15] and the like, admit efficient Boolean Skolem function synthesis, and satisfy properties P0 and P1 mentioned in Section III. However, none of these are known to satisfy property P2 described in the same Section, thereby failing to provide a characterization of efficient Boolean Skolem function synthesis. In contrast, as we show in this paper, the class of SAUNF circuits satisfies all the properties mentioned in Section III.

Among the various alternative normal forms, we discuss SynNNF [15] first. We say that a circuit normal form (or class)  $\mathcal{N}_1$  is exponentially more succinct than another normal form  $\mathcal{N}_2$  if (i) for every circuit  $G_2 \in \mathcal{N}_2$ , there exists a circuit  $G_1 \in \mathcal{N}_1$  such that  $|G_1| \leq |G_2|$  and  $\llbracket \varphi_{G_1} \rrbracket = \llbracket \varphi_{G_2} \rrbracket$ , and (ii) there is a circuit  $G_1 \in \mathcal{N}_1$  such that every circuit  $G_2 \in \mathcal{N}_2$  with  $\llbracket \varphi_{G_1} \rrbracket = \llbracket \varphi_{G_2} \rrbracket$  has  $|G_2| \in 2^{\mathcal{O}(|G_1|)}$ . The notion of super-polynomial succinctness is similarly defined. The authors of [15] showed a conditional succinctness result for SynNNF, namely SynNNF is super-polynomially more succinct than DNNF [19] and dDNNF [16], unless some long-standing complexity theoretic conjectures are falsified. We show the following stronger result for SAUNF.

**Lemma 1.** SAUNF is unconditionally exponentially more succinct than DNNF and dDNNF.

*Proof.* We use a result from [27] to prove the lemma. In Proposition 11 of [27], a family of Boolean functions  $\{JS_r \mid r \geq 2\}$  is defined. The formula  $JS_r$ , defined on  $\mathcal{O}(r^2)$  variables, asserts that for every triple  $(v_j, v_k, v_l)$  of variables in a carefully constructed set  $A_r$  of triples, at least one of  $v_j, v_k$  or  $v_l$  must be false. It is shown in [27] that  $|A_r| \in \mathcal{O}(r^2)$ . Therefore, a CNF formula representing  $JS_r$  has  $\mathcal{O}(r^2)$  clauses, with each clause having three negated variables as literals. Since no non-negated variables appear as literals in the formula, a circuit representation of the CNF formula cannot have any literal-consistent subset of leaves that is  $\wedge$ -realizable. This implies that  $JS_r$  can be represented in SAUNF in size  $\mathcal{O}(r^2)$ . It is also shown in [27] that any DNNF (and hence also dDNNF) representation of  $JS_r$  requires size  $2^{\Omega(r^2)}$ . Therefore, SAUNF is unconditionally exponentially more succinct compared to DNNF and dDNNF.  $\square$

Next, we show that SynNNF is, in fact, a special case of SAUNF. Towards this end, we recall the definition of SynNNF from [15], re-cast in our terminology.

**Definition 4.** A circuit  $G$  with leaves labeled by  $\text{lits}(\mathbf{I})$  and  $\text{lits}(\mathbf{X})$  is in SynNNF w.r.t.  $\mathbf{X}$  iff the following hold:

- $x_1$  is  $\wedge$ -unrealizable in  $G$ .

- For  $2 \leq i \leq |\mathbf{X}|$ ,  $x_i$  is  $\wedge$ -unrealizable in  $G|_{x_1=\top, \neg x_1=\top, \dots, x_{i-1}=\top, \neg x_{i-1}=\top}$ .

Note that  $\wedge$ -unrealizability of  $x_i$  also implies  $\wedge$ -unrealizability of  $\neg x_i$ . The following lemma shows that SAUNF strictly subsumes SynNNF.

**Lemma 2.** *Every SynNNF circuit  $G$  with leaves labeled by  $\text{lits}(\mathbf{I})$  and  $\text{lits}(\mathbf{X})$  is also a SAUNF circuit with a sequence of  $2 \cdot |\mathbf{X}|$   $\wedge$ -unrealizable subsets of leaves. However, there exist SAUNF circuits that are not in SynNNF.*

*Proof.* Suppose a circuit  $G$  is in SynNNF w.r.t.  $\mathbf{X}$ , and let  $|\mathbf{X}| = r$ . We define a sequence of  $2r$  literal-consistent subsets of leaves of  $G$  as follows. For each  $j \in \{1, \dots, r\}$ , we define  $S_{2j-1}$  to be the set of  $x_j$ -leaves of  $G$ , and  $S_{2j}$  to be the set of  $\neg x_j$ -leaves of  $G$ . It can now be seen from Definition 4 and Definition 3 that  $G$  is in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and the sequence  $S = (S_1, \dots, S_{2n})$  of subsets of literal-consistent leaves. This proves the first part of the lemma.

To show the second part, we must demonstrate a circuit that is in SAUNF but not in SynNNF for any permutation of the sequence of system outputs  $\mathbf{X}$ . We claim that the circuit  $G$  in Figure 1, already shown to be in SAUNF, suffices for this purpose. This is because Definition 4 entails that for  $G$  to be in SynNNF, at least one literal over  $\mathbf{X}$  must be  $\wedge$ -unrealizable in  $G$ . However, none of  $x_1, \neg x_1, x_2, \neg x_2$  are  $\wedge$ -unrealizable in the circuit  $G$  in Figure 1. Specifically,  $\llbracket \varphi_G|_{x_1=w, \neg x_1=w'} \rrbracket \sigma = \llbracket w \wedge w' \rrbracket$  when  $\sigma(i) = \sigma(x_2) = \perp$ , and  $\llbracket \varphi_G|_{x_2=w, \neg x_2=w'} \rrbracket \sigma' = \llbracket w \wedge w' \rrbracket$  when  $\sigma'(i) = \sigma'(x_1) = \top$ . Therefore, the circuit in Figure 1 is not in SynNNF w.r.t. any permutation of  $\mathbf{X}$ .  $\square$

It has been shown in [15] that every DNNF, dDNNF and wDNNF circuit is also in SynNNF, and SynNNF is super-polynomially more succinct than wDNNF unless  $P = NP$ . Furthermore, every ROBDD and FDD can be converted to a DNNF (hence SynNNF) circuit with at most linear blowup in size, although SynNNF can be exponentially more succinct than ROBDD or FDD [19], [15]. By virtue of Lemma 2, we now have the following result.

**Corollary 1.** *All subsumption and (conditional) succinctness results for SynNNF circuits hold for SAUNF circuits as well.*

Finally, since every Boolean specification can be represented as a SynNNF circuit [15], it follows from Lemma 2 that property P0 of Section III holds for the class of SAUNF circuits.

## VI. EFFICIENT SYNTHESIS OF SKOLEM FUNCTIONS FROM SAUNF SPECIFICATIONS

We now show how a Skolem function vector can be efficiently computed if the relational specification is given as a SAUNF circuit. Informally, the process involves transforming a given SAUNF circuit  $G$  with leaves labeled by  $\text{lits}(\mathbf{I})$  and  $\text{lits}(\mathbf{X})$  to a semantically different but related circuit  $H$  with leaves labeled by  $\text{lits}(\mathbf{I})$ ,  $\text{lits}(\mathbf{X})$  and  $\text{lits}(\mathbf{X}')$ , where  $\mathbf{X}'$  is a sequence of fresh system outputs, also called *auxiliary outputs*. The transformation is done in a way such that a Skolem

function vector for  $(\mathbf{X}, \mathbf{X}')$  in  $H$  can be found efficiently, and a projection of this Skolem function vector on the first  $|\mathbf{X}|$  components directly yields a Skolem function vector for  $\mathbf{X}$  in  $G$ . To formalize this notion, we begin with a few definitions.

**Definition 5. Equisynthesizable Under Projection.** *Let  $G$  be a circuit representing a relational specification over system inputs  $\mathbf{I}$  and systems outputs  $\mathbf{X}$ . Let  $H$  be another circuits representing a relational specification over  $\mathbf{I}$  and  $(\mathbf{X}, \mathbf{X}')$ , where  $\mathbf{X}'$  is a fresh sequence of system outputs or auxiliary outputs. We say that  $G$  is equisynthesizable to  $H$  under projection, denoted  $G \rightsquigarrow H$ , iff the following hold*

- $\forall \mathbf{I} \forall \mathbf{X} (\varphi_G(\mathbf{X}, \mathbf{I}) \Rightarrow \exists \mathbf{X}' \varphi_H(\mathbf{X}, \mathbf{X}', \mathbf{I}))$
- $\forall \mathbf{I} \forall \mathbf{X} \forall \mathbf{X}' (\varphi_H(\mathbf{X}, \mathbf{X}', \mathbf{I}) \Rightarrow \varphi_G(\mathbf{X}, \mathbf{I}))$

It follows from Definition 5 that  $\rightsquigarrow$  defines a transitive relation on circuits representing relational specifications. The following lemma is an easy consequence of Definition 5.

**Lemma 3.** *If  $G \rightsquigarrow H$  holds and  $(\Psi(\mathbf{I}), \Psi'(\mathbf{I}))$  is a Skolem function vector for  $(\mathbf{X}, \mathbf{X}')$  in  $\varphi_H(\mathbf{X}, \mathbf{X}', \mathbf{I})$ , then  $\Psi(\mathbf{I})$  is a Skolem function vector for  $\mathbf{X}$  in  $\varphi_G(\mathbf{X}, \mathbf{I})$ .*

*Proof.* Since  $\forall \mathbf{I} (\varphi_G(\Psi(\mathbf{I}), \mathbf{I}) \Rightarrow \exists \mathbf{X} \varphi_G(\mathbf{X}, \mathbf{I}))$  holds trivially, we show below that  $\forall \mathbf{I} (\exists \mathbf{X} \varphi_G(\mathbf{X}, \mathbf{I}) \Rightarrow \varphi_G(\Psi(\mathbf{I}), \mathbf{I}))$ .

From Definition 5 and from the definition of Skolem functions, we have  $\forall \mathbf{I} (\exists \mathbf{X} \varphi_G(\mathbf{X}, \mathbf{I}) \Rightarrow \exists \mathbf{X} \exists \mathbf{X}' \varphi_H(\mathbf{X}, \mathbf{X}', \mathbf{I}) \Rightarrow \varphi_H(\Psi(\mathbf{I}), \Psi'(\mathbf{I}), \mathbf{I}) \Rightarrow \varphi_G(\Psi(\mathbf{I}), \mathbf{I}))$ .  $\square$

### A. Role of auxiliary outputs

We now investigate how auxiliary outputs can be introduced in a principled manner, so that they help in generating Skolem functions. We start with a circuit  $G$  with leaves labeled by  $\text{lits}(\mathbf{I})$  and  $\text{lits}(\mathbf{X})$ . Let  $G_1$  and  $G_2$  be two sub-circuits of  $G$  such that  $G_1$  is not a sub-circuit of  $G_2$  and vice versa. For a fresh auxiliary variable  $p \notin \text{set}(\mathbf{X}) \cup \text{set}(\mathbf{I})$  and for  $j \in \{1, 2\}$ , define a circuit transformation  $\tau_j^p$  that replaces the sub-circuit  $G_j$  in  $G$  with the circuit representing  $\varphi_{G_j} \wedge p$ .

**Lemma 4.** *If  $\varphi_{G_1} \wedge \varphi_{G_2}$  is unsatisfiable, then  $G \rightsquigarrow \tau_1^p(\tau_2^{-p}(G))$ .*

*Proof.* Let  $H$  denote the circuit  $\tau_1^p(\tau_2^{-p}(G))$ , and let  $H_1$  and  $H_2$  denote the newly introduced sub-circuits representing  $p \wedge \varphi_{G_1}$  and  $\neg p \wedge \varphi_{G_2}$  respectively in  $H$ . We show below that the conditions for  $G \rightsquigarrow H$  (see Definition 5) are satisfied.

Let  $\sigma : \text{set}(\mathbf{X}) \cup \text{set}(\mathbf{I}) \rightarrow \{\perp, \top\}$  be an assignment for which  $\llbracket \varphi_G \rrbracket$  evaluates to  $\top$ . We have four cases to analyze depending on what  $\llbracket \varphi_{G_1} \rrbracket$  and  $\llbracket \varphi_{G_2} \rrbracket$  evaluate to under  $\sigma$ .

- $\llbracket \varphi_{G_1} \rrbracket = \perp = \llbracket \varphi_{G_2} \rrbracket$ : Then for any assignment to  $p$ ,  $\llbracket \varphi_{H_1} \rrbracket$  and  $\llbracket \varphi_{H_2} \rrbracket$  also evaluate to  $\perp$ , and hence  $\llbracket \varphi_H \rrbracket$  evaluates to the same value, viz.  $\top$ , as  $\llbracket \varphi_G \rrbracket$ .
- If  $\llbracket \varphi_{G_1} \rrbracket = \top$ ,  $\llbracket \varphi_{G_2} \rrbracket = \perp$ , then with  $p$  assigned  $\top$ ,  $\llbracket \varphi_{H_1} \rrbracket = \top$  and  $\llbracket \varphi_{H_2} \rrbracket = \perp$ , and hence  $\llbracket \varphi_H \rrbracket$  evaluates to the same value, viz.  $\top$ , as  $\llbracket \varphi_G \rrbracket$ .
- By a similar argument, if  $\llbracket \varphi_{G_1} \rrbracket = \perp$ ,  $\llbracket \varphi_{G_2} \rrbracket = \top$ , assigning  $p$  to  $\perp$  causes  $\llbracket \varphi_H \rrbracket$  to evaluate to  $\top$ .
- The case of  $\llbracket \varphi_{G_1} \rrbracket = \llbracket \varphi_{G_2} \rrbracket = \top$  doesn't arise since  $\varphi_{G_1} \wedge \varphi_{G_2}$  is unsatisfiable.

This shows that  $\forall \mathbf{I} \forall \mathbf{X} (\varphi_G(\mathbf{X}, \mathbf{I}) \Rightarrow \exists p \varphi_H(\mathbf{X}, p, \mathbf{I}))$ .

Consider any assignment  $\sigma' : \text{set}(\mathbf{X}) \cup \{p\} \cup \text{set}(\mathbf{I}) \rightarrow \{\perp, \top\}$  that renders  $\llbracket \varphi_H \rrbracket = \top$ . Let  $\sigma : \text{set}(\mathbf{X}) \cup \text{set}(\mathbf{I}) \rightarrow \{\perp, \top\}$  be the projection of  $\sigma'$  on  $\text{set}(\mathbf{X}) \cup \text{set}(\mathbf{I})$ . By definition,  $\llbracket \varphi_H \rrbracket_{p=\top, \neg p=\top} = \llbracket \varphi_G \rrbracket$ . Since all internal gates in  $H$ , viz.  $\wedge$  and  $\vee$  gates, are monotone,  $\sigma$  must cause  $\llbracket \varphi_H \rrbracket_{p=\top, \neg p=\top}$  to evaluate to  $\top$  as well. Hence,  $\sigma$  causes  $\llbracket \varphi_G \rrbracket(\mathbf{X}, \mathbf{I})$  to evaluate to  $\top$ . This shows that  $\forall \mathbf{I} \forall \mathbf{X} \forall p (\varphi_H(\mathbf{X}, p, \mathbf{I}) \Rightarrow \varphi_G(\mathbf{X}, \mathbf{I}))$ .  $\square$

The argument in the above proof can be easily generalized to prove the following.

**Lemma 5.** Let  $\mathcal{G}_1 = \{G_{1,1}, \dots, G_{1,s}\}$  and  $\mathcal{G}_2 = \{G_{2,1}, \dots, G_{2,t}\}$  be two sets of sub-circuits of  $G$  such that (a) there are no distinct  $G_{k,i}$  and  $G_{l,j}$  where one is a sub-circuit of the other, and (b)  $\bigvee_{i=1}^s \varphi_{G_{1,i}} \Rightarrow \bigwedge_{j=1}^t \neg \varphi_{G_{2,j}}$ . Let  $\tau_{\mathcal{G}_k}^p$  denote the circuit transformation that replaces every sub-circuit  $G_{k,i} \in \mathcal{G}_k$  with a subcircuit representing  $\varphi_{G_{k,i}} \wedge p$ , where  $p$  is a fresh variable. Then  $G \rightsquigarrow \tau_{\mathcal{G}_1}^p(\tau_{\mathcal{G}_2}^p(G))$ .

A particularly easy application of Lemma 5 is obtained by choosing any literal  $\ell$  that labels leaves of  $G$ , and by choosing  $\mathcal{G}_1$  and  $\mathcal{G}_2$  to be subsets of  $\ell$ -leaves and  $\neg \ell$ -leaves, respectively. Note that if  $L$  is a subset of  $\ell$ -leaves of  $G$ , then  $\tau_L^p$  gives the same circuit as  $G|_{L:p \wedge \ell}$ .

Given a relational specification over  $\mathbf{I}$  and  $\mathbf{X}$  specified by a circuit  $G$ , and a literal  $\ell$  over  $\mathbf{X}$ , let  $v_\ell$  be the underlying variable in  $\mathbf{X}$ , and  $S_\ell$  be the set of all  $\ell$ -leaves of  $G$ .

**Theorem 1.** Suppose  $S \subseteq S_\ell$  is  $\wedge$ -unrealizable in  $G$ . For a fresh auxiliary variable  $p \notin \text{set}(\mathbf{X}) \cup \text{set}(\mathbf{I})$ , let  $E$  denote the circuit  $G|_{S_\ell \setminus S: (p \wedge \ell), S_\ell \setminus S: (\neg p \wedge \neg \ell)}$  and let  $H$  denote the circuit  $E|_{\ell=\top, \neg \ell=\top}$ . Note that the literals labeling leaves of  $G$  are from  $\text{lits}(\mathbf{X})$ ,  $\text{lits}(\mathbf{I})$ , those labeling leaves of  $E$  are from  $\text{lits}(\mathbf{X})$ ,  $\text{lits}(\mathbf{I})$  and  $\{p, \neg p\}$ , while the literals labeling leaves of  $H$  are from  $\text{lits}(\mathbf{X} \setminus (v_\ell))$ ,  $\text{lits}(\mathbf{I})$  and  $\{p, \neg p\}$ . Then the following statements hold.

- 1)  $\exists v_\ell \varphi_G \iff \exists p \varphi_H \iff \exists v_\ell \varphi_{G|_{S:\top}}$
- 2) If  $\Psi_H(\mathbf{I})$  is a Skolem function vector for  $(\mathbf{X} \setminus (v_\ell), p)$  in  $H$ , then the projection of  $\Psi_H$  on  $\mathbf{X} \setminus (v_\ell)$  augmented with the Skolem function  $\varphi_{E|_{\ell=\top, \neg \ell=\top}}(\Psi_H(\mathbf{I}), \mathbf{I})$  for  $\ell$  gives a Skolem function vector  $\Psi_G(\mathbf{I})$  for  $\mathbf{X}$  in  $G$ .

*Proof.* By Lemma 5, we have  $G \rightsquigarrow E$ . It then follows from Definition 5 that  $\exists v_\ell \varphi_G \iff \exists v_\ell \exists p \varphi_E$ . Furthermore, since  $S$  is  $\wedge$ -unrealizable in  $G$ , it follows from the definition of  $E$  and from Definition 2 that  $\ell$  is  $\wedge$ -unrealizable in  $E$ . From this, we will now show that  $\exists v_\ell \varphi_E$  is equivalent to  $\varphi_{E|_{\ell=\top, \neg \ell=\top}}$ . To see this, in one direction, we observe that  $\exists v_\ell \varphi$  is always equivalent to  $\varphi_{E|_{\ell=\top, \neg \ell=\top}} \vee \varphi_{E|_{\ell=\perp, \neg \ell=\top}}$ . This logically implies  $\varphi_{E|_{\ell=\top, \neg \ell=\top}}$  as all internal gates in an NNF circuit are monotone. In the other direction,  $\varphi_{E|_{\ell=\top, \neg \ell=\top}} \wedge \neg(\varphi_{E|_{\ell=\top, \neg \ell=\top}} \vee \varphi_{E|_{\ell=\perp, \neg \ell=\top}})$  is unsatisfiable if  $\ell$  is  $\wedge$ -unrealizable in  $E$  (follows from definition of  $\wedge$ -unrealizability). Therefore, we have  $\exists v_\ell \varphi_E \iff \varphi_{E|_{\ell=\top, \neg \ell=\top}} \iff \varphi_H$ . Hence,  $\exists v_\ell \exists p \varphi_E \iff \exists p \varphi_H$ . Finally, from the definitions of circuits  $E$  and  $H$ , we have

$$\exists p \varphi_H \iff \exists p \varphi_{E|_{\ell=\top, \neg \ell=\top}} \iff \exists p \varphi_{G|_{S:\top, S_\ell \setminus S: p, S_\ell \setminus S: \neg p}}.$$

By renaming  $p$  to  $v_\ell$  in the last formula, we get  $\exists v_\ell \varphi_{G|_{S:\top}}$ .

Since  $\exists v_\ell \varphi_E \iff \varphi_H$ , a Skolem function vector for  $(\mathbf{X} \setminus (v_\ell), p)$  in  $E$  is obtained from  $\Psi_H(\mathbf{I})$ . The Skolem function for  $\ell$  in  $E$  is then given by  $\varphi_{E|_{\ell=\top, \neg \ell=\top}}(\Psi_H(\mathbf{I}), \mathbf{I})$ . To see why, note that  $\varphi_E$  with  $\Psi_H(\mathbf{I})$  substituted for  $(\mathbf{X} \setminus (v_\ell), p)$  represents a specification with a single system output  $\ell$  and system inputs  $\mathbf{I}$ . Let us call this  $\phi(\ell, \mathbf{I})$ . Then,  $\phi(\top, \mathbf{I})$  serves as a Skolem function, say  $\psi^\ell(\mathbf{I})$ , for  $\ell$  in  $\phi$ , i.e.,  $\exists \ell \phi(\ell, \mathbf{I}) \iff \phi(\psi^\ell(\mathbf{I}), \mathbf{I})$ . Indeed, suppose for some  $\mathbf{I}$ ,  $\psi^\ell(\mathbf{I}) = \phi(\top, \mathbf{I}) = \top$ . Then  $\phi(\psi^\ell(\mathbf{I}), \mathbf{I}) = \phi(\top, \mathbf{I}) = \top$ . Conversely, if  $\psi^\ell(\mathbf{I}) = \phi(\top, \mathbf{I}) = \perp$ , we consider two cases: (a) if  $\phi(\perp, \mathbf{I}) = \top$ , then  $\phi(\psi^\ell(\mathbf{I}), \mathbf{I}) = \top$ ; (b) if  $\phi(\perp, \mathbf{I}) = \perp$ , then we have  $\forall \ell \phi(\ell, \mathbf{I}) = \perp$ . Therefore, in all cases, we have  $\exists \ell \phi(\ell, \mathbf{I}) \iff \phi(\psi^\ell(\mathbf{I}), \mathbf{I})$ . The above method of obtaining a Skolem function for a single system output is also called self-substitution [12], [1], [7]. The second part of the theorem now follows from the observation that  $G \rightsquigarrow E$ .  $\square$

### B. Generating Skolem functions from SAUNF circuits

Theorem 1 suggests an efficient algorithm for generating a Skolem function vector from a specification given as a SAUNF circuit. Algorithm 1 presents the pseudo-code of algorithm *SkGen*. The purpose of sub-routines used in *SkGen* is explained in the comments.

We illustrate the running of *SkGen* by considering its execution on the circuit  $G$  shown in Fig. 1. Here,  $\mathbf{X} = (x_1, x_2)$  and  $\mathbf{I} = (i)$ . As discussed earlier, we use  $L_0$  through  $L_{15}$  to denote the leaves of the circuit  $G$  in left-to-right order. We have also seen earlier that  $G$  is in SAUNF for the sequence of subsets of leaves  $(S_1, S_2, S_3, S_4)$ , where  $S_1 = \{L_3\}$ ,  $S_2 = \{L_7\}$ ,  $S_3 = \{L_5\}$  and  $S_4 = \{L_1\}$ .

As algorithm *SkGen* proceeds, labels of different leaves of  $G$  need to be updated. For notational convenience, we use  $G^{(r)}$ ,  $H^{(r)}$  and  $E^{(r)}$  to refer to the circuits  $G$ ,  $H$  and  $E$  in the  $r^{\text{th}}$  level of recursion of *SkGen*. Table I shows how  $G^{(r)}$ ,  $H^{(r)}$  and  $E^{(r)}$  are obtained by replacing the labels of suitable leaves of  $G$ . Each entry in this table lists which leaf labels of  $G$  must be updated, where  $L_{\{i,j,k\}} : f$  denotes updation of the label of each leaf in  $\{L_i, L_j, L_k\}$  by  $f$ . All leaves whose label updates are not specified are assumed to have the same labels as in  $G$ . It can be verified that  $G$  with leaf labels updated as in the table entry corresponding to  $H^{(4)}$  simplifies to  $\top$  by constant propagation. Hence  $H^{(4)}$  is semantically independent of output variables. This is not a coincidence, but is guaranteed by the definition of SAUNF. Hence, at recursion level 5 of *SkGen*, any vector of functions  $(f_3(i), f_4(i))$  can be returned in line 2 of Algorithm 1 as a Skolem function vector for  $(p_3, p_4)$  in  $H^{(2)}$ .

As the recursive calls return, we obtain  $E^{(4)}(p_3 = f_3(i), p_4 = f_4(i), p_1 = \top, i)$  as Skolem function for  $p_1$  in  $E^{(4)}$ . Call this function  $f_1(i)$ . Next, we get  $f_2(i) = E^{(3)}(p_1 = f_1(i), p_3 = f_3(i), p_2 = \top, i)$  as Skolem function for  $p_2$  in  $E^{(3)}$ . Continuing further, we obtain  $f_{x_2(i)} = E^{(2)}(p_1 = f_1(i), p_2 = f_2(i), x_2 = \top, i)$  as Skolem function for  $b$  in  $E^{(2)}$ , and  $f_{x_1(i)} = E^{(1)}(x_2 = f_{x_2(i)}, p_1 = f_1(i), x_1 = \top, i)$

**Algorithm 1:** SkGen( $G, S, r$ )

---

**Input:**  $G$ : Relational spec in SAUNF;  
 $S = (S_1, S_2 \dots S_k)$ : Sequence of  $\wedge$ -unrealizable subsets of lits( $\mathbf{X}$ )-labeled leaves of  $C$ ;  $r$ : Recursion level  
**Output:**  $\Psi_G(\mathbf{I})$ : Skolem function vector for  $C$

---

```

1 if  $r = k+1$  then
2    $\Psi_G(\mathbf{I}) := \text{GetAnyFuncVec}(|\mathbf{X}|, \mathbf{I})$ ;
   // Returns an  $|\mathbf{X}|$ -dim vector of
   // (arbitrary) functions of  $\mathbf{I}$ 
3 else
4    $\ell := \text{Literal label of leaves in } S_r$ ;
5    $p_r := \text{newOutputVar}()$ ; //  $p_r$  is auxiliary
   // output variable added at recursion
   // level  $r$ 
6    $E := \text{GetCkt}(G, S_r, \ell, p_r)$ ; // Replace all
   //  $\ell$ -labeled leaves of  $G$  other than
   // those in  $S_r$  by  $\ell \wedge p_r$ , and replace
   // all  $\neg\ell$ -labeled leaves by  $\neg\ell \wedge \neg p_r$ 
7    $S := \text{GetNewSeq}(S, r, \ell, p_r)$ ; // Replace  $\ell$  by
   //  $p_r$  and  $\neg\ell$  by  $\neg p_r$  in all elements
   // (leaves) of  $S_j$  for  $j > r$ 
8    $H := \text{CPropSimp}(E |_{\ell=\top, \neg\ell=\top})$ ;
   // CPropSimp propagates constants
   // and eliminates gates with constant
   // outputs
9    $\Psi_H(\mathbf{I}) = \text{SkGen}(H, S, r+1)$ ;
10   $\psi_E^\ell(\mathbf{I}) := \varphi_{E|_{\ell=\top, \neg\ell=\perp}}(\Psi_H(\mathbf{I}), \mathbf{I})$ ; //  $\psi_E^\ell$ 
   // gives Skolem function for  $\ell$  in  $\varphi_E$ 
11   $\Psi_G(\mathbf{I}) = (\Psi_H(\mathbf{I}) \setminus (\psi_H^{p_r}, \psi_E^\ell(\mathbf{I})))$ ; //  $\psi_H^{p_r}$  is
   // Skolem function for  $p_r$  in  $\psi_H$ 
12 return  $\Psi_G(\mathbf{I})$ ;
```

---

	$G^{(r)}$	$E^{(r)}$	$H^{(r)}$
$r=1$ :	None	$L_{\{1,14\}}: (\neg x_1 \wedge \neg p_1),$ $L_{\{10\}}: (x_1 \wedge p_1)$	$L_{\{1,14\}}: \neg p_1, L_{\{3\}}: \top,$ $L_{\{10\}}: p_1$
$r=2$ :	Same as in $H^{(1)}$	$L_{\{1,14\}}: \neg p_1,$ $L_{\{3\}}: \top, L_{\{10\}}: p_1,$ $L_{\{5,9,11,15\}}: \neg x_2 \wedge \neg p_2,$ $L_{\{13\}}: x_2 \wedge p_2$	$L_{\{1,14\}}: \neg p_1,$ $L_{\{3,7\}}: \top, L_{\{10\}}: p_1,$ $L_{\{5,9,11,15\}}: \neg p_2,$ $L_{\{13\}}: p_2$
$r=3$ :	Same as in $H^{(2)}$	$L_{\{1,14\}}: \neg p_1,$ $L_{\{3,7\}}: \top, L_{\{10\}}: p_1,$ $L_{\{9,11,15\}}: \neg p_2 \wedge p_3,$ $L_{\{5\}}: \neg p_2,$ $L_{\{13\}}: p_2 \wedge \neg p_3$	$L_{\{1,14\}}: \neg p_1,$ $L_{\{3,5,7\}}: \top, L_{\{10\}}: p_1,$ $L_{\{9,11,15\}}: p_3,$ $L_{\{13\}}: \neg p_3$
$r=4$ :	Same as in $H^{(3)}$	$L_{\{1\}}: \neg p_1,$ $L_{\{14\}}: \neg p_1 \wedge p_4,$ $L_{\{3,5,7\}}: \top,$ $L_{\{10\}}: p_1 \wedge \neg p_4,$ $L_{\{9,11,15\}}: p_3,$ $L_{\{13\}}: \neg p_3$	$L_{\{14\}}: p_4,$ $L_{\{1,3,5,7\}}: \top,$ $L_{\{10\}}: \neg p_4,$ $L_{\{9,11,15\}}: p_3,$ $L_{\{13\}}: \neg p_3$

TABLE I: Run of Algorithm 1 on Fig. 1

as Skolem function for  $x_1$  in  $E^{(1)}$ . The final return gives  $(f_{x_1(i)}, f_{x_2(i)})$  as a Skolem function vector for  $(x_1, x_2)$  in  $G$ . Note that different choices of  $f_3(i), f_4(i)$  yield different Skolem function vectors of  $G$ , all of which are correct.

**Theorem 2.** Suppose Algorithm *SkGen* is invoked with a circuit  $G$  as input, that is in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and a sequence  $S$  of subsets of leaves. Assuming the vector returned in line

2 of the algorithm can be constructed in time  $\mathcal{O}(|S|^2 \cdot |G|)$ , the algorithm returns a Skolem function vector of size in  $\mathcal{O}(|S|^2 \cdot |G|)$  in time  $\mathcal{O}(|S|^2 \cdot |G|)$ .

*Proof.* We show that Algorithm *SkGen* generates a Skolem function vector from an inductive application of Theorem 1. Specifically, for each level  $i$  of recursion, if the Skolem function  $\Psi_H$  returned in line 9 by the  $i+1^{\text{th}}$  recursive call of *SkGen* is correct for  $H$ , Theorem 1 ensures that the Skolem function  $\Psi_G$  computed in lines 10 and 11 of the  $i^{\text{th}}$  recursive call is correct for  $G$ .

To see why the terminating case of this recursion yields correct Skolem functions, note that when the recursion level is  $k+1$  (lines 1-2 of Algorithm 1), by Definition 3, the function represented by  $G$  is semantically independent of  $\mathbf{X}$ . Hence any Skolem function vector for  $\mathbf{X}$  suffices in line 2 of Algorithm 1.

Algorithm *SkGen* has exactly  $k+1$  recursive calls, and in each of the first  $k$  calls, the steps in lines 4, 5, 6, 7 and 8 take time linear in  $|G|$  and generate circuits  $E$  and  $H$  that are of size in  $\mathcal{O}(|G|)$ . Indeed, the circuit  $H$  in each recursion level  $\leq k$  is simply  $G$  with the literal labels of some of its leaves replaced by other literals or by Boolean constants (possibly followed by simplification by constant propagation). Therefore,  $|H| \leq |G|$  in each recursion level  $\leq k$ . The circuit  $E$  is similarly obtained by replacing some leaves of  $G$  with Boolean constants, other literals or conjunctions of two literals. Therefore,  $|E| \leq 2 \times |G|$  in each recursion level  $\leq k$ . In the  $k+1^{\text{th}}$  recursive call, line 2 is executed, and as discussed above, we restrict it to take time in  $\mathcal{O}(|S|^2 \cdot |G|)$ . This also ensures that the size of the Skolem function vector returned in line 2 is in  $\mathcal{O}(|S|^2 \cdot |G|)$ .

Once the recursive calls start returning, lines 10 and 11 of Algorithm *SkGen* are executed. Note that in line 10, the Skolem function for  $\ell$  is obtained by feeding into the inputs of circuit  $E$  (as obtained in the current level of recursion) the outputs of Skolem functions computed in later (or higher) levels of the recursion. We have already seen above that  $|E|$  is at most  $2 \times |G|$  in each recursion level  $\leq k$ . A Skolem function computed at recursion level  $j$  can potentially feed into  $E$  at all recursion levels in  $\{1, \dots, j-1\}$ . Therefore, a maximum of  $\sum_{j=2}^{|S|+1} (j-1) \in \mathcal{O}(|S|^2)$  connections may need to be created between the output of a Skolem function generated at some recursion level and the input of  $E$  at a lower level of recursion. Therefore, constructing the entire Skolem function vector at recursion level 1 requires time (and, hence space) in  $\mathcal{O}(|S|^2 \cdot |G|)$ .  $\square$

In the above analysis, we assumed that the vector of functions used in line 2 of Algorithm 1 has size in  $\mathcal{O}(|S|^2 \cdot |G|)$ . Since an arbitrary vector of functions of  $\mathbf{I}$  suffices in line 2, we can choose a  $|\mathbf{X}|$ -dimensional constant function vector, say  $(\perp, \dots, \perp)$  as the output of *GetAnyFuncVec*. Hence, the above assumption can always be satisfied. Note that Theorem 2 guarantees that SAUNF enjoys property P1 of Section III.



### C. Generating SAUNF circuits from Skolem functions

Next, we show that if we already know one (out of possibly many) Skolem function vector of a relational specification  $G$  given as a circuit, we can easily derive a semantically equivalent circuit in SAUNF.

**Theorem 3.** *Let  $\Psi(\mathbf{I})$  be a Skolem function vector for  $\mathbf{X}$  in  $\varphi_G(\mathbf{X}, \mathbf{I})$ . Define  $G'$  to be  $G$  with all labels  $x_i$  (resp.  $\neg x_i$ ) in  $\text{lits}(\mathbf{X})$  replaced by  $\psi_i(\mathbf{I})$  (resp.  $\neg\psi_i(\mathbf{I})$ ), i.e.  $\varphi_{G'}(\mathbf{I}) = \varphi_G(\Psi(\mathbf{I}), \mathbf{I})$ . Define  $F$  to be the circuit representing the formula  $\bigwedge_{i=1}^m ((x_i \wedge \psi_i(\mathbf{I})) \vee (\neg x_i \wedge \neg\psi_i(\mathbf{I})))$ , and  $H$  to be the circuit representing  $(\varphi_F \vee \varphi_G) \wedge \varphi_{G'}$ . Then  $H$  is in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and a sequence of literal-consistent subsets of leaves, and  $\varphi_G(\mathbf{X}, \mathbf{I}) \iff \varphi_H(\mathbf{X}, \mathbf{I})$ .*

*Proof.* We first show that  $\varphi_G(\mathbf{X}, \mathbf{I}) \iff \varphi_H(\mathbf{X}, \mathbf{I})$ . This involves showing two implications.

- $\varphi_H(\mathbf{X}, \mathbf{I}) \Rightarrow \varphi_G(\mathbf{X}, \mathbf{I})$  : We know from the definitions that  $\varphi_{G'}(\mathbf{I}) \iff \varphi_G(\Psi(\mathbf{I}), \mathbf{I}) \Rightarrow \forall \mathbf{X} (\varphi_F(\mathbf{X}, \mathbf{I}) \Rightarrow \varphi_G(\mathbf{X}, \mathbf{I}))$ . The last implication follows from the observation that  $\varphi_F(\mathbf{X}, \mathbf{I})$  simply asserts that  $\bigwedge_{i=1}^m (x_i \iff \psi_i(\mathbf{I}))$  holds, and hence  $\varphi_G(\Psi(\mathbf{I}), \mathbf{I}) \Rightarrow \forall \mathbf{X} (\varphi_F(\mathbf{X}, \mathbf{I}) \Rightarrow \varphi_G(\mathbf{X}, \mathbf{I}))$ . We also know from the definition of  $H$  that  $\varphi_H(\mathbf{X}, \mathbf{I}) \Rightarrow \varphi_{G'}(\mathbf{I})$  and  $\varphi_H(\mathbf{X}, \mathbf{I}) \Rightarrow (\varphi_F(\mathbf{X}, \mathbf{I}) \vee \varphi_G(\mathbf{X}, \mathbf{I}))$ . However, since  $\varphi_{G'}(\mathbf{I}) \Rightarrow \forall \mathbf{X} (\varphi_F(\mathbf{X}, \mathbf{I}) \Rightarrow \varphi_G(\mathbf{X}, \mathbf{I}))$ , it follows that  $\varphi_H(\mathbf{X}, \mathbf{I}) \Rightarrow (\varphi_{G'}(\mathbf{I}) \wedge \varphi_G(\mathbf{X}, \mathbf{I}))$ . Hence  $\varphi_H(\mathbf{X}, \mathbf{I}) \Rightarrow \varphi_G(\mathbf{X}, \mathbf{I})$ .
- $\varphi_G(\mathbf{X}, \mathbf{I}) \Rightarrow \varphi_H(\mathbf{X}, \mathbf{I})$  : We know that  $\varphi_G(\mathbf{X}, \mathbf{I}) \Rightarrow \exists \mathbf{X} \varphi_G(\mathbf{X}, \mathbf{I}) \iff \varphi_G(\Psi(\mathbf{I}), \mathbf{I}) \iff \varphi_{G'}(\mathbf{I})$  by definition. It follows that  $\varphi_G(\mathbf{X}, \mathbf{I}) \Rightarrow (\varphi_G(\mathbf{X}, \mathbf{I}) \wedge \varphi_{G'}(\mathbf{I}))$ . However, from the definition of  $H$ , we know that  $(\varphi_G(\mathbf{X}, \mathbf{I}) \wedge \varphi_{G'}(\mathbf{I})) \Rightarrow \varphi_H(\mathbf{X}, \mathbf{I})$ . Hence,  $\varphi_G(\mathbf{X}, \mathbf{I}) \Rightarrow \varphi_H(\mathbf{X}, \mathbf{I})$ .

To show that  $H$  is in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and a suitably defined sequence  $S$  of subsets of leaves, we first observe that the circuit  $F$  that naturally represents  $\bigwedge_{i=1}^n ((x_i \wedge \psi_i(\mathbf{I})) \vee (\neg x_i \wedge \neg\psi_i(\mathbf{I})))$  is a 3-level circuit with leaves labeled by  $\text{lits}(\mathbf{X})$  and  $\text{lits}(\mathbf{I})$ . It is easy to see from the structure of this circuit that for  $i \in \{1, \dots, m\}$ , literals  $x_i$  (and  $\neg x_i$ ) are  $\wedge$ -unrealizable in  $F$ . Let  $S = (S_1, S_2, \dots, S_{2m})$  be a sequence of subsets of leaves of  $F$ , where  $S_{2i-1}$  is the set of all  $x_i$ -labeled leaves of  $F$ , and  $S_{2i}$  is the set of all  $\neg x_i$ -labeled leaves labeled of  $F$ . Since the literals  $x_i$  and  $\neg x_i$  are  $\wedge$ -unrealizable in  $F$  for  $i \in \{1, \dots, m\}$ , we also have that  $S_j$  is  $\wedge$ -unrealizable in  $F$  for every  $j \in \{1, \dots, 2m\}$ . Finally,  $F|_{S_1:\top, \dots, S_{2m}:\top}$  has no literal in  $\text{lits}(\mathbf{X})$  labeling any leaf. Hence,  $F$  satisfies all the conditions of Definition 3, and is in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and  $S$ .

We now claim that the circuit, say  $R$ , representing  $(\varphi_F \vee \varphi_G) \wedge \varphi_{G'}$  is in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and the sequence  $S$  of subsets of leaves of  $F$  described above. To see why this is so, observe that all subsets  $S_j \in S$  are mutually disjoint and contain leaves labeled by  $\text{lits}(\mathbf{X})$ . Hence the first 2 conditions of Definition 3 are satisfied. To see why conditions 3 and 4 of Definition 3 are satisfied, recall from Definition 2 that when checking  $\wedge$ -unrealizability of any  $S_i \in S$  in  $R$ , all leaves of the sub-circuit  $G$  (of the circuit  $R$ ) that are labeled by the same

literal as leaves in  $S_i$ , must be re-labeled  $\perp$ . This, coupled with the fact that  $S_i$  is  $\wedge$ -unrealizable in  $F$ , ensures that  $S_i$  is  $\wedge$ -unrealizable in  $R$  as well. Finally, as we will see in Section VII (see Lemma 7),  $\varphi_F|_{x_1=\top, \neg x_1=\top, \dots, x_n=\top, \neg x_n=\top} \iff \exists \mathbf{X} \varphi_F(\mathbf{X}, \mathbf{I}) \iff \top$ . The last equivalence follows from the definition of  $F$ ; specifically  $\varphi_F(\Psi(\mathbf{I}), \mathbf{I}) = \top$  for all assignments of  $\mathbf{I}$ . Therefore  $\varphi_R|_{S_1:\top, \dots, S_{2m}:\top} \iff (\top \vee \varphi_G) \wedge \varphi_{G'} \iff \varphi_{G'}$ . Since  $\varphi_{G'}$  is semantically independent of  $\mathbf{X}$  by definition, condition 5 of Definition 3 is satisfied for  $R$ . Hence,  $R$  is in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and the sequence  $S = (S_1, S_2, \dots, S_{2m})$ .  $\square$

The proof of Theorem 3 gives the following corollary.

**Corollary 2.** *Given  $G$ , and a Skolem function vector  $\Psi(\mathbf{I})$  for  $\mathbf{X}$  in  $\varphi_G(\mathbf{X}, \mathbf{I})$ , a SAUNF circuit  $H$  semantically equivalent to  $G$  can be constructed in  $\mathcal{O}(|G| + |\Psi|)$  time. Furthermore,  $|H| \in \mathcal{O}(|G| + |\Psi|)$ .*

Finally, Corollary 2 and Theorem 2 immediately yield the following theorem.

**Theorem 4.** *For every class  $\mathcal{C}$  of circuits representing relational specifications,*

- 1) *Boolean Skolem function synthesis can be solved in polynomial-time for  $\mathcal{C}$  iff every circuit in  $\mathcal{C}$  can be compiled to a semantically equivalent SAUNF circuit in polynomial-time.*
- 2) *A Skolem function vector of polynomial size exists for every specification in  $\mathcal{C}$  iff every circuit in  $\mathcal{C}$  can be compiled to a polynomial-sized semantically equivalent SAUNF circuit.*

Thus, SAUNF satisfies properties P2a and P2b of Section III. In other words, SAUNF truly characterizes efficient Boolean functional synthesis. Note that Theorem 4 is significantly stronger than sufficient conditions for efficient synthesis given in [12], [15].

## VII. OPERATIONS ON SAUNF

In this section, we discuss the application of basic operations like conjunction, disjunction and existential quantification of variables on formulas represented by SAUNF circuits, and also examine the complexity of checking membership in SAUNF. Throughout the section, we assume that all specification circuits are over system inputs  $\text{set}(\mathbf{I})$  and system outputs  $\text{set}(\mathbf{X})$  unless otherwise specified. To reduce notational clutter, given circuits  $G$  and  $H$ , we abuse notation and use  $G \vee H$  (resp.  $G \wedge H$ ) to denote the circuit consisting of an  $\vee$ - (resp.  $\wedge$ -)labeled root node with the roots of  $G$  and  $H$  as its children. We first look at disjunction.

**Lemma 6.** *Suppose  $G$  is in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and a sequence  $S^G$ , and  $H$  is in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and  $S^H$ . Then the circuit  $G \vee H$  is in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and  $(S^G, S^H)$ .*

*Proof.* The proof follows from the claim that  $(S^G, S^H)$  is a sequence of subsets of leaves of  $G \vee H$  that satisfies the conditions of Definition 3. To see why this is so, note that by Definition 2, when considering a subset, say  $S_i^G$ , of  $\ell$ -labeled

leaves in  $S^G$ , all  $\ell$ -labeled leaves of  $H$  must be re-labeled  $\perp$ . Hence,  $H$  can only contribute  $\neg\ell$ , and can, at worst, combine with  $\ell$  contributed by  $G$  at the  $\vee$ -labeled root of  $G \vee H$ . Since the set of  $\ell$ -labeled leaves in  $S_i^G$  was already  $\wedge$ -unrealizable in  $G$ , we find that  $S_i^G$  is  $\wedge$ -unrealizable in  $G \vee H$  as well. By repeating this argument, we find that conditions 1, 2, 3 and 4 of Definition 3 are satisfied by the circuit  $G \vee H$ . To see why condition 5 is also satisfied, observe that since  $G$  (resp.  $H$ ) is SAUNF w.r.t  $\text{set}(\mathbf{X})$  and  $S^G$  (resp.  $S^H$ ), when all subsets of leaves in  $(S^G, S^H)$  are re-labeled to  $\top$ , the circuit  $G \vee H$  represents the disjunction of two formulas, each of which is semantically independent of  $\mathbf{X}$ . Hence,  $G \vee H$  is in SAUNF w.r.t  $\text{set}(\mathbf{X})$  and  $(S^G, S^H)$ .  $\square$

Significantly, Lemma 6 does not require any assumptions on the relation between ordering of subsets in  $S^G$  and  $S^H$ . Other popular normal forms do not enjoy this property. For example, disjoining two ROBDDs constructed with different ordering of variables does not always yield an ROBDD in polynomial-time. Similarly, combining two SynNNF circuits with an  $\vee$  gate may not result in a SynNNF circuit unless the ordering of output variables in both circuits are the same. This shows that disjunction is more efficiently computable in SAUNF than in ROBDDs and in SynNNF.

Next, we observe that existential quantification of *all* system outputs is easy for SAUNF representations.

**Lemma 7.** *Suppose  $G$  in SAUNF w.r.t  $\text{set}(\mathbf{X})$  and a sequence  $S$ . Let  $L$  be the set of all leaves of  $G$  that are labeled by a literal over  $\mathbf{X}$ . Then  $\exists \mathbf{X} G \Leftrightarrow G|_{L:\top}$ .*

*Proof.* Follows from Theorem 1(1) and Definition 3.  $\square$

Next, we move to the more difficult case of conjunction.

**Lemma 8.** *Suppose  $G$  is in SAUNF w.r.t  $\text{set}(\mathbf{X})$  and a sequence  $S^G$ , and  $H$  is in SAUNF w.r.t  $\text{set}(\mathbf{X})$  and sequence  $S^H$ . If there is no literal  $\ell$  over  $\mathbf{X}$  such that  $G$  has an  $\ell$ -labeled leaf and  $H$  has a  $\neg\ell$ -labeled leaf, the circuit  $G \wedge H$  is in SAUNF w.r.t  $\text{set}(\mathbf{X})$  and  $(S^G, S^H)$ . Otherwise, a SAUNF circuit semantically equivalent to  $G \wedge H$  cannot be constructed in time polynomial in  $|G|, |H|$  unless  $P = NP$ . Further, such a circuit cannot have size polynomial in  $|G|, |H|$  unless  $\Pi_2^P = \Sigma_2^P$  (i.e. unless the polynomial hierarchy collapses to the second level).*

*Proof.* If there is no literal  $\ell$  over  $\mathbf{X}$  such that  $G$  has an  $\ell$ -labeled leaf and  $H$  has a  $\neg\ell$ -labeled leaf, it is easy to see that a leaf of  $G$  and a leaf of  $H$  cannot participate together to make any literal in  $\text{lits}(\mathbf{X})$   $\wedge$ -realizable in the circuit  $G \wedge H$ . Since  $G$  is in SAUNF w.r.t  $\text{set}(\mathbf{X})$  and  $S^G$ , and  $H$  is in SAUNF w.r.t  $\text{set}(\mathbf{X})$  and  $S^H$ , it then follows that  $G \wedge H$  is also in SAUNF w.r.t  $\text{set}(\mathbf{X})$  and the sequence  $(S^G, S^H)$  (or alternatively,  $(S^H, S^G)$ ).

The proof for the remainder of the lemma is more intricate. For this part of the discussion, we consider circuits  $G$  in which the labels of all leaves are considered to be from  $\text{lits}(\mathbf{X})$ . In other words  $\mathbf{I}$  is assumed to be empty. Let  $G^-$  denote the circuit  $G|_{\neg x_1=x'_1, \dots, \neg x_m=x'_m}$  where  $\mathbf{X}' = (x'_1, \dots, x'_m)$  is

a sequence of fresh variables. This is sometimes called the *positive form* of the circuit.

**Claim 1.** *For every circuit  $G$ , the circuit  $G^-$  is in SAUNF w.r.t  $\text{set}(\mathbf{X}) \cup \text{set}(\mathbf{X}')$  for any sequence of literal-consistent subsets of leaves.*

*Proof.* There is no label of a leaf of  $G^-$  whose negation is also the label of some other leaf of  $G^-$  (since there are no negated output literals at all in the labels of leaves). This eliminates the possibility of a subset of literal-consistent leaves being  $\wedge$ -realizable in  $G^-$ .  $\square$

Further, given  $G$ , let  $G^+$  be the circuit representing the formula  $\bigwedge_{i=1}^n ((x'_i \wedge \neg x_i) \vee (\neg x'_i \wedge x_i))$  where  $x'_i$  are the variables introduced in  $G^-$ . Note that  $\varphi_{G^+} \Leftrightarrow \bigwedge_{i=1}^n (x'_i \Leftrightarrow \neg x_i)$

**Claim 2.** *For every circuit  $G$ , the circuit  $G^+$  is in SAUNF w.r.t  $\text{set}(\mathbf{X}) \cup \text{set}(\mathbf{X}')$  and the sequence of literal-consistent leaves  $S = (S_1, \dots, S_{2n}, S'_1, \dots, S'_{2n})$ , where  $S_{2i-1}$  (resp.  $S'_{2i-1}$ ) is the set of all leaves labeled  $x_i$  (resp.  $x'_i$ ), and  $S_{2i}$  (resp.  $S'_{2i}$ ) is the set of all leaves labeled  $\neg x_i$  (resp.  $\neg x'_i$ ) in  $G^+$ .*

*Proof.* It is easy to see that for the sequence  $(S_1, S_2, \dots, S_{2n})$ , the first 4 conditions of Definition 3 are satisfied. Moreover  $\varphi_{G^+|_{S_1:\top, S_2:\top, \dots, S_{2n}:\top}} \Leftrightarrow \top$ . Therefore, condition 5 of Definition 3 is also satisfied, and  $G^+$  is in SAUNF w.r.t  $\text{set}(\mathbf{X}) \cup \text{set}(\mathbf{X}')$  and  $(S_1, \dots, S_{2n})$ , and hence also w.r.t. the sequence  $S = (S_1, \dots, S_{2n}, S'_1, \dots, S'_{2n})$ .  $\square$

It is easy to see that  $\varphi_{G^-} \wedge \varphi_{G^+}$  is equisatisfiable to  $\varphi_G$ . Now, consider an arbitrary instance of the classical Boolean satisfiability (SAT) problem, i.e., given a Boolean circuit  $G$  over  $\mathbf{X} = (x_1, \dots, x_m)$ , we must determine if  $\varphi_G$  is satisfiable. We interpret  $\varphi_G$  as a relational specification over system outputs  $\mathbf{X}$ , with the system inputs  $\mathbf{I}$  being absent.

By definition,  $|G^-|, |G^+| \in \mathcal{O}(|G|)$ . Using Claims 1,2, each of these circuits is also in SAUNF w.r.t  $\text{set}(\mathbf{X}) \cup \text{set}(\mathbf{X}')$  and an appropriate sequence of subsets of leaves. Suppose there exists a polynomial-time algorithm  $A$  that takes two SAUNF circuits as inputs and produces a SAUNF circuit semantically equivalent to the conjunction of the formulas represented by the input circuits. We use algorithm  $A$  to obtain a circuit  $\hat{G}$  that is semantically equivalent to  $\llbracket \varphi_{G^-} \wedge \varphi_{G^+} \rrbracket$ . Clearly,  $|\hat{G}|$  has size polynomial in  $|G^-|$  and  $|G^+|$ , and therefore polynomial in  $|G|$ . Since every SAUNF circuit yields Skolem functions for all outputs in time polynomial in the size of the circuit (see Theorem 2), we can compute a Skolem function for every output of  $\hat{G}$  in time polynomial in  $|G|$ . Since there are no inputs, each of these Skolem functions must simplify to a Boolean constant. From the definition of Skolem functions, we also know that  $\hat{G}$  is satisfiable iff the Skolem functions obtained above (Boolean constants for variables in  $\text{set}(\mathbf{X}) \cup \text{set}(\mathbf{X}')$ ) cause  $\llbracket \hat{G} \rrbracket$  to evaluate to  $\top$ . In other words, we can determine if  $\varphi_{\hat{G}}$ , and hence  $\varphi_{G^-} \wedge \varphi_{G^+}$ , is satisfiable in time polynomial in  $|G|$ . Since  $\varphi_{G^-} \wedge \varphi_{G^+}$  is equisatisfiable to  $\varphi_G$ , this effectively solves the Boolean satisfiability problem in polynomial time. Therefore, algorithm  $A$  cannot run in polynomial time unless  $P = NP$ .

Suppose for every two SAUNF circuits, there exists a polynomial sized SAUNF circuit that is semantically equivalent to the conjunction of the formulas represented by the two input circuits. Let  $\tilde{G}$  be the circuit obtained in this manner for  $\llbracket \varphi_{G-} \wedge \varphi_{G+} \rrbracket$ . By Theorem 2, Skolem functions synthesized from  $\tilde{G}$  must have size polynomial in  $|G|$ . By the same argument as above, it now follows that Boolean satisfiability must be in P/Poly. This implies that  $\text{NP} \subseteq \text{P/Poly}$ . By Karp-Lipton Theorem, however, we know that this can happen only if the polynomial hierarchy collapses to the second level, i.e. if  $\Pi_2^P = \Sigma_2^P$ .  $\square$

Finally, we ask how difficult it is to check if a given circuit  $G$  is in SAUNF.

**Theorem 5.** 1) *Given  $G$  and a sequence  $S$  of literal-consistent disjoint subsets of leaves, checking if  $G$  is in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and  $S$  is coNP-complete.*  
 2) *Given  $G$ , checking if  $G$  is in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and some (unspecified) sequence of subsets of leaves is coNP-hard and in  $\Sigma_2^P$ .*

*Proof.* First, we show that identifying whether a given circuit is in SAUNF for a given sequence of subsets of leaves is in coNP. This is equivalent to asking whether the complement problem, i.e. if the given circuit is *not* in SAUNF for the given sequence of subsets of leaves, is in NP. We will define a non-deterministic polynomial-time Turing machine  $M$  that solves this complement problem. The machine  $M$  first checks whether the input circuit (say  $G$ ) is in NNF. This can be done by checking if each internal node of  $G$  is labeled either  $\wedge$  or  $\vee$ , and if all negations (if any) are on the labels of leaves. Clearly, this check can be done in time polynomial in the size of  $G$ . If the circuit is found to be not in NNF, the machine  $M$  accepts, since  $G$  cannot be in SAUNF in this case. Otherwise (i.e. if  $G$  is in NNF), the machine  $M$  non-deterministically chooses a subset  $S_i$  of literal-consistent leaves in the given sequence  $S$  and executes the following operations.

Suppose the literal labeling leaves in the subset  $S_i$  is  $\ell_i$ . The machine  $M$  (a) constructs  $G^+ = G \upharpoonright_{S_1:\top \dots S_{i-1}:\top}$ , (b) sets all leaves of  $G^+$  that are not in  $S_i$  but are labeled  $\ell_i$  to  $\perp$ , (c) replaces all remaining labels  $\ell$  on leaves by  $w$  and all labels  $\neg \ell$  on leaves by  $w'$ , (d) guesses an assignment  $\sigma$  to all variables other than  $w$  and  $w'$  labeling leaves in the resulting circuit, and (e) checks if the resulting circuit represents the Boolean function  $w \wedge w'$  for the assignment  $\sigma$  to other variables. Note that after step (d), the resulting circuit represents a function of only  $w$  and  $w'$  (at most). Hence the check in step (e) can be performed by setting  $(w, w')$  to each of  $(\top, \top)$ ,  $(\top, \perp)$ ,  $(\perp, \top)$  and  $(\perp, \perp)$  and checking if the resulting circuit evaluates to  $\top$ ,  $\perp$ ,  $\perp$  and  $\perp$  respectively. Clearly, all the steps from (a) through (e) above can be done in time polynomial in the size of  $G$ . If after step (e), the resulting circuit is found to represent  $w \wedge w'$ , then machine  $M$  accepts. In this case,  $G$  is not in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and the given sequence  $S$  of subsets of leaves. Conversely, if the circuit is not in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and the given sequence  $S$ , then there is a subset  $S_i$  of leaves

in  $S$  that is  $\wedge$ -realizable in  $G$  for the assignment described above. Hence the problem of identifying whether a circuit is *not* in SAUNF for a given sequence of subsets of leaves is in NP. Thus, the problem of identifying whether  $G$  is in SAUNF for a given sequence of subsets of leaves is in co-NP.

Next, we show that the problem is co-NP hard. We reduce the problem of identifying whether a propositional formula represented by a CNF circuit is unsatisfiable to identifying whether an appropriately constructed circuit is in SAUNF for a specific sequence of subsets of literal-consistent leaves. For this reduction, we consider the specification  $G \wedge x \wedge \neg x$ , where  $x$  is the sole output of the specification, and the inputs are the variables labeling leaves of  $G$ . Since there is only one output variable, there are only two (equivalent) ordering of subsets of leaves labeled by output literals. It is easy to see that  $x$  (equivalently,  $\neg x$ ) is  $\wedge$ -realizable if and only if  $G$  is satisfiable. Hence, identifying whether a problem is in SAUNF for a given sequence of subsets of leaves is coNP-hard.

To prove the second part of the theorem, we use the following result from the polynomial hierarchy:  $\Sigma_2^P = \text{NP}^{\text{NP}}$ . Specifically, we show that checking whether a given  $G$  is in SAUNF w.r.t. some (unspecified) sequence of subsets of output literal-consistent leaves can be solved by an non-deterministic polynomial-time Turing machine  $M$  that invokes an NP oracle, i.e.  $\in \text{NP}^{\text{NP}}$ . Given specification  $G$  with system inputs  $\mathbf{I}$  and system outputs  $\mathbf{X}$  the machine  $M$  does the following: (i) Guesses an sequence  $S$  of disjoint subsets of literal-consistent leaves. (ii) Reduces the problem of deciding whether  $G$  is not in SAUNF w.r.t  $\text{set}(\mathbf{X})$  and the sequence  $S$  to checking the satisfiability of an appropriately constructed propositional formula  $\varphi$ . This reduction is similar to what we discussed above in the proof of part (1). (iii) Feeds  $\varphi$  to the NP oracle. (iv) Accepts if and only if the NP oracle rejects. It follows that  $M$  accepts if and only if there is a sequence  $S$  of subsets of output literal-consistent leaves for which  $G$  is in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and  $S$ . Hence, we have proved that our problem is contained in  $\text{NP}^{\text{NP}}$ . The proof that the problem is coNP-hard follows similar lines as the earlier coNP hardness proof.  $\square$

## VIII. CONVERSION TO SAUNF

We present an algorithm for compiling a circuit  $G$  representing a CNF formula over  $\mathbf{X}$  and  $\mathbf{I}$  to a semantically equivalent circuit in SAUNF. Algorithm `GetSaunf` (see Algorithm 2) takes  $G$  as input and produces a circuit  $F$  and sequence  $S$  of subset of leaves such that  $\llbracket \varphi_G \rrbracket = \llbracket \varphi_F \rrbracket$  and  $F$  is in SAUNF w.r.t.  $\text{set}(\mathbf{X})$  and  $S$ . `GetSAUNF` uses a routine named `GetSubset` (which will be detailed in Algorithm 3) to obtain an  $\wedge$ -unrealizable subset  $U$  of leaves labeled by a chosen literal  $\ell$ . The set  $U$  thus obtained is then used to decompose the problem into two sub-problems: (i) a circuit  $G'$  representing conjunction of all clauses that have  $\ell$ -labeled leaves in  $U$  feeding them, and (ii) a circuit representing conjunction of all other clauses. While  $G'$  does not require any recursive application of `GetSAUNF`, the second sub-problem is recursively solved using Shannon-style decomposition. Finally, the sequence  $S$  of

---

**Algorithm 2:** GetSAUNF( $G$ )

---

**Input:**  $G$ : Circuit representing a CNF formula  
**Output:**  $F$ : SAUNF ckt sem. equiv. to  $G$   
 $S$ : Sequence of literal-consistent subsets of leaves of  $F$

```

1 if  $\llbracket G \rrbracket$  is semantically independent of  $\mathbf{X}$  then
2   return  $(G, \emptyset)$ ;
3  $\ell := \text{ChooseLiteral}(G)$ ; // Gives a literal
   over  $\mathbf{X}$  labeling a leaf in  $G$ 
4  $U := \text{GetSubset}(G, \ell)$ ; //  $U$  is an
    $\wedge$ -unrealizable subset of  $\ell$ -labeled
   leaves in  $G$ 
5  $G' := \text{CktFromClausesWithLeaves}(U)$ ; //  $G'$ 
   represents conjunction of clauses (in
   the CNF formula represented by  $G$ )
   containing leaves in  $U$ 
6  $D := G|_{U:\top}$ ;
7 if  $\llbracket D \rrbracket$  is semantically independent of  $\ell$  then
8    $(G_1, U_1) := \text{GetSAUNF}(D)$ ;
9   return  $(G' \wedge G_1, (U, U_1))$ ;
10  $(G_1, U_1) := \text{GetSAUNF}(D|_{\ell=\top})$ ;
11  $(G_2, U_2) := \text{GetSAUNF}(D|_{\ell=\perp})$ ;
12  $F := G' \wedge ((\ell \wedge G_1) \vee (\neg \ell \wedge G_2))$ ;
13  $U_\ell := \{\ell\text{-leaf of } F \text{ in sub-circuit for } (\ell \wedge G_1)\}$ ;
14  $U_{\neg\ell} := \{\neg\ell\text{-leaf of } F \text{ in sub-circuit for } (\neg\ell \wedge G_2)\}$ ;
15 return  $(F, (U_\ell, U, U_{\neg\ell}, U_1, U_2))$ ;
```

---

subsets of leaves is obtained from the subset  $U$ , and from the sequences obtained from recursive applications of GetSAUNF.

**Lemma 9.** *Algorithm GetSAUNF returns a SAUNF circuit  $F$  semantically equivalent to the input circuit  $G$ , with a worst-case running time exponential in  $|G|$ , and the worst-case size of  $F$  also exponential in  $|G|$ .*

*Proof.* Since the circuit  $G'$  represents a conjunction of a subset of clauses, each of which contains the literal  $\ell$ , the circuit  $D = G|_{U:\top}$  obtained in line 6 of Algorithm GetSAUNF simply represents the conjunction of all remaining clauses. In the circuit  $F$  constructed in line 11 of Algorithm 2, the set  $U_\ell$  containing only the leaf  $\ell$  is  $\wedge$ -unrealizable as it meets up  $\neg\ell$  at an  $\vee$  gate. In  $F|_{U_\ell:\top}$ , we can show that the set  $U$  of  $\ell$ -leaves of  $G'$  is  $\wedge$ -unrealizable as it was already  $\wedge$ -unrealizable in the circuit  $G$ .

For  $U$  to be  $\wedge$ -realizable in  $F|_{U_\ell:\top}$ , there must be an assignment  $\sigma : \text{set}(\mathbf{X}) \setminus \{v_\ell\} \cup \text{set}(\mathbf{I}) \rightarrow \{\perp, \top\}$  such that  $\llbracket \varphi_{G'} \rrbracket$  evaluates to  $\ell$ ,  $\llbracket \varphi_{G_1} \rrbracket$  to  $\perp$  and  $\llbracket \varphi_{G_2} \rrbracket$  to  $\top$ . However, if this were possible, then  $U$  would have  $\wedge$ -realizable in  $G$  (using the same assignment  $\sigma$ ). However, this is a contradiction, since  $U$  is  $\wedge$ -unrealizable in  $G$ . In  $F|_{U_\ell:\top, U:\top}$ , all  $\ell$ -leaves of  $F$  have been re-labeled to  $\top$ , and therefore we can set the only  $\neg\ell$ -leaf (in set  $U_{\neg\ell}$ ) to  $\top$ . Finally,  $F|_{U_\ell:\top, U:\top, U_{\neg\ell}:\top} = G_1 \vee G_2$ , which is in SAUNF assuming the recursive calls return correct representations and using Lemma 6. The base condition is when  $G$  is independent of  $X$ , for which it already returns the correct value.

---

**Algorithm 3:** GetSubset( $G, \ell$ )

---

**Input:**  $G$ : circuit representing CNF formula,  $\ell$ : Literal  
**Output:**  $T$ :  $\wedge$ -unrealizable subset of  $\ell$ -leaves in  $G$

```

1  $AllS := CurrS := \emptyset$ ;
2  $D := G$ ;
3 repeat
4    $\sigma := \text{GetAssignment}(D, \ell)$ ; // Assignment
   of vars except  $v_\ell$  for which  $\ell$  is
    $\wedge$ -realizable in  $D$ 
5    $CurrS := \text{GetClausesEvaluatingToL}(G, \sigma, \ell)$ ;
   // Set of all clauses (of CNF
   formula represented by  $G$ )
   containing  $\ell$  that do not become  $\top$ 
   under  $\sigma$ 
6    $AllS := AllS \cup \{CurrS\}$ ;
7    $D := D \wedge \text{DisjoinWithoutLit}(CurrS, \ell)$ ;
   // DisjoinWithoutLit( $CurrS, \ell$ ) gives
   disjunction of clauses in  $CurrS$ 
   after dropping  $\ell$ 
8 until  $\ell$  is  $\wedge$ -unrealizable in  $D$ ;
9  $HitS := \text{SatisfiableHittingSet}(AllS)$ ;
10  $T := \text{Set of all } \ell\text{-leaves of } G \text{ that don't feed into any}$ 
   clause in  $HitS$ ;
11 return  $T$ ;
```

---

Note that the sequence of subsets returned in line 15 of Algorithm GetSAUNF is  $(U_\ell, U, U_{\neg\ell}, U_1, U_2)$ , which corresponds to the sequence of setting leaves to  $\top$  as discussed above. The algorithm would be correct if it had returned  $(U_\ell, U, U_{\neg\ell}, U_2, U_1)$  as the sequence of subsets of leaves as well.

In the worst-case, Algorithm GetSAUNF can reduce to brute-force Shannon expansion if  $U$  computed in line 4 of the algorithm always returns the empty set of leaves. In this case, both the running time and the size of  $F$  can grow exponentially with  $|G|$ .  $\square$

It remains to discuss the sub-routine GetSubset. The pseudo-code for this sub-routine is shown in Algorithm 3. Let  $v_\ell$  denote the underlying variable of the literal  $\ell$ .  $CurrS$  is a subset of clauses containing  $\ell$  such that there exists an assignment  $\sigma : \text{set}(\mathbf{X}) \setminus \{v_\ell\} \cup \text{set}(\mathbf{I}) \rightarrow \{\perp, \top\}$  for which all (and only) these clauses of the underlying CNF formula evaluate to  $\ell$ , and  $\ell$  is  $\wedge$ -realizable in  $G$  under  $\sigma$  (see Definition 1). Such an assignment  $\sigma$  can be obtained by effectively finding a satisfying assignment of  $\forall w \forall w' (\varphi_{D|_{\ell=w, \neg\ell=w'}} \Leftrightarrow (w \wedge w'))$ , and therefore  $\varphi_{D|_{\ell=\top, \neg\ell=\top}} \wedge \neg\varphi_{D|_{\ell=\top, \neg\ell=\perp}} \wedge \neg\varphi_{D|_{\ell=\perp, \neg\ell=\top}} \wedge \neg\varphi_{D|_{\ell=\perp, \neg\ell=\perp}}$ . To ensure that  $CurrS$  in the current iteration does not include any such set obtained in previous iterations of the loop, we conjoin  $D$  with the clause returned by DisjoinWithoutLit( $CurrS, \ell$ ). All sets  $CurrS$  obtained as the repeat-until loop iterates are collected in  $AllS$ . Finally when  $\ell$  becomes  $\wedge$ -unrealizable in circuit  $D$ , we obtain a satisfiable minimal hitting set (set cover)  $HitS$  of  $AllS$ , i.e. a subset of clauses that is jointly satisfiable with  $\ell$  set to  $\perp$  and that

includes a clause from every set in  $AllS$ . Given  $AllS$ , finding a minimal  $HitS$  can be reduced to a MaxSAT problem. Once  $HitS$  is as obtained, we exclude all  $\ell$ -leaves that appear in the clauses in  $HitS$  to obtain a (maximal)  $\wedge$ -unrealizable subset of  $\ell$ -leaves. With this, we can state the correctness and complexity of Algorithm 3.

**Lemma 10.** *Algorithm `GetSubset` returns a  $\wedge$ -unrealizable subset of  $\ell$ -leaves of  $G$ , and takes worst-case time exponential in  $|G|$ .*

*Proof.* Suppose Algorithm `GetSubset` returned an  $\wedge$ -realizable subset of  $\ell$ -leaves of  $G$ . Let  $\sigma$  be the corresponding assignment of  $\text{set}(\mathbf{X}) \setminus \{v_\ell\} \cup \text{set}(\mathbf{I})$ . The set  $S$  of clauses of  $\varphi_G$  that do not become  $\top$  under  $\sigma$  must then be either equal to or a superset of  $CurrS$  in some iteration of the repeat-until loop of lines 3-7. Therefore,  $HitS$  must include some clause from  $S$ . This implies that  $T$  contains at least one  $\ell$ -labeled leaf that feeds into a clause in  $HitS$  – a contradiction.

The worst-case running time is dominated by the product of the number of times the repeat-until loop of lines 3–8 iterates and the time required to obtain  $\sigma$  (line 4) and check the loop termination condition (line 8). The count of loop iterations can be as high as the count of all subsets of  $\ell$ -labeled leaves. This is exponential in  $|G|$  in the worst-case. Computing  $\sigma$  and checking the loop termination condition also require time exponential in  $|G|$  in the worst-case. Hence, the worst-case running time of Algorithm `GetSubset` is exponential in  $|G|$ .  $\square$

Note that we can modify the loop termination condition in Algorithm `GetSubset` by incorporating a timeout. In case a timeout happens, we conservatively return  $\emptyset$  as  $T$ . This reduces the worst-case running time, providing a tradeoff between running time and precision of computation.

## IX. SOME INTERESTING APPLICATIONS

In the introduction, Section I, we described the  $n$ -bit factorization problem – a problem of immense interest in cryptanalysis. We now show some interesting partial results using SAUNF circuits. We start with a relational specification  $R$  over inputs  $\mathbf{I}$  and outputs  $(\mathbf{X}, \mathbf{Y})$  defined by  $(\mathbf{X} \times_{[n]} \mathbf{Y} = \mathbf{I})$  where  $\times_{[n]}$  denotes  $n$ -bit unsigned integer multiplication. This specification evaluates to true iff the given product relation holds, where  $\mathbf{X}, \mathbf{Y}$  are  $n$ -bit output vectors and  $\mathbf{I}$  is a  $2n$  bit input vector. For  $1 \leq l \leq j \leq 2n$ , we define a parametrized specification  $R[l, j]$  over  $\mathbf{X}, \mathbf{Y}$  and  $\mathbf{I}$  that evaluates to true if and only if the bits from position  $l$  to  $j$  of  $\mathbf{X} \times_{[n]} \mathbf{Y}$  match the corresponding bits of  $\mathbf{I}$ .

Let  $1$  denotes an  $n$ -bit representation of the integer 1. If  $R[1, 2n] \wedge (\mathbf{X} \neq 1) \wedge (\mathbf{Y} \neq 1)$  can be represented as a polynomial (in  $n$ ) sized SAUNF circuit, our results show that Skolem functions of size polynomial in  $n$  can be obtained for  $n$ -bit factorization with non-trivial (i.e.  $\neq 1$ ) factors. This would have serious ramifications for cryptanalysis. While we are not close to achieving such a result, our initial studies show some interesting results in trying to represent  $R[l, j]$

in SAUNF. Note that it is already known from [17] that representing  $R[n, n]$  requires exponentially large ROBDDs, and sub-exponential representations using DNNF, dDNNF, wDNNF or SynNNF are not known. With SAUNF circuits however, we obtain a significant improvement.

**Theorem 6.** *For  $l \leq j \leq 2n, j - l < n, R[l, j]$  is representable by a polynomial (in  $n$ ) sized SAUNF circuit.*

*Proof.* By Theorem 4, a polynomial-sized SAUNF circuit can be generated from a polynomial-sized Skolem function vector. Therefore, we focus on obtaining a polynomial-sized Skolem function vector for  $R[l, j]$ ,  $l \leq j \leq 2n, j - l < n$ . We use the notation  $\mathbf{X}_l$  to denote the  $l^{th}$  least significant bit of  $\mathbf{X}$  and  $\mathbf{X}_{l,j}$  to denote the bit-slice of  $\mathbf{X}$  from  $l$  to  $j$  (both included). Using similar notations for  $\mathbf{Y}$  and  $\mathbf{I}$ , we consider two cases:

- $l \leq n$ : A Skolem function vector for  $(\mathbf{X}, \mathbf{Y})$  is given by  $\psi^{x_k} = \perp$  (resp.  $\top$ ), if  $k \neq l$  (resp.  $k = l$ ), and  $\psi^{y_k} = \mathbf{I}_{k+l-1}$  (resp.  $\perp$ ) if  $k \leq j+1-l$  (resp.  $k > j+1-l$ ). Intuitively,  $\mathbf{X}$  represents  $2^{l-1}$  and  $\mathbf{Y} = \mathbf{I}_{l,j}$ .
- $l > n$ : Following similar logic, the Skolem function vector is given by  $\psi^{x_k} = \perp$  (resp.  $\top$ ) if  $k \neq n$  (resp.  $k = n$ ), and  $\psi^{y_k} = \mathbf{I}_{k+n-1}$  (resp.  $\perp$ ) if  $l-n < k \leq j+1-n$  (resp. otherwise). Intuitively,  $\mathbf{X}$  represents  $2^{n-1}$  and  $\mathbf{Y}$  represents  $2^{l-n} \times_{[n]} \mathbf{I}_{l,j}$ .

Having generated a polynomial-sized Skolem function vector for  $R[l, j]$ ,  $j - l < n$ , we can generate a corresponding polynomial-sized SAUNF circuit using Theorem 4.  $\square$

Surprisingly, we can use Theorem 6 to also show that a restricted version of division has a polynomial sized SAUNF representation. Consider the same relational specification  $\mathbf{X} \times_{[n]} \mathbf{Y} = \mathbf{I}$  considered earlier. For the division problem, we treat  $(\mathbf{I}, \mathbf{Y})$  as system inputs and  $\mathbf{X}$  as system outputs, and write the relation as  $\mathbf{X} = \mathbf{I}/\mathbf{Y}$ . Then, we have,

**Theorem 7.** *The relation  $\mathbf{X} = \mathbf{I}/\mathbf{Y}$ , with inputs  $\mathbf{I}, \mathbf{Y}$  restricted to odd numbers (i.e. the relation evaluates to  $\perp$  if  $\mathbf{I}$  or  $\mathbf{Y}$  is even), is representable as a polynomial (in  $n$ ) sized SAUNF circuit.*

*Proof.* For notational convenience, we use  $\mathbf{X}, \mathbf{I}$  and  $\mathbf{Y}$  to denote both sequences of Boolean variables, and also the unsigned integers represented by the corresponding bit-vectors. We use  $\times$  instead of  $\times_{[n]}$  to denote  $n$ -bit unsigned integer multiplication, and  $+$  (resp.  $-$ ) to denote  $n$ -bit unsigned integer addition (resp. subtraction).

We give below a polynomial-sized Skolem function vector for division, which can be used to obtain a SAUNF form by Theorem 4.

Suppose we have inputs  $\mathbf{I}$  and  $\mathbf{Y}$  and we have to find  $\mathbf{X}$  such that  $\mathbf{X} \times \mathbf{Y} = \mathbf{I}$ . We first show that for odd valued inputs, if  $(\mathbf{X} \times \mathbf{Y}) \bmod 2^n = \mathbf{I} \bmod 2^n$  and if  $\mathbf{I}$  is divisible by  $\mathbf{Y}$  then  $\mathbf{X} \times \mathbf{Y} = \mathbf{I}$ . Suppose there are two values  $\mathbf{X}^1, \mathbf{X}^2$  such that  $(\mathbf{X}^1 \times \mathbf{Y}) \bmod 2^n = \mathbf{I} \bmod 2^n$  and  $(\mathbf{X}^2 \times \mathbf{Y}) \bmod 2^n = \mathbf{I} \bmod 2^n$ . Then  $(\mathbf{X}^1 - \mathbf{X}^2) \times \mathbf{Y} \equiv 0 \bmod 2^n$ . However,  $\mathbf{Y}$  is odd and therefore co-prime to  $2^n$ ; hence  $(\mathbf{X}^1 - \mathbf{X}^2) \equiv 0 \bmod 2^n$ . Since,  $\mathbf{X}^1, \mathbf{X}^2 < 2^n$ , we must

have  $\mathbf{X}^1 = \mathbf{X}^2$ . Therefore, the generated  $\mathbf{X}$  from the Skolem function vector is correct if there exists a solution that matches the least significant  $n$  bits of  $\mathbf{I}$ . Using the notation defined above, denote  $\mathbf{X}_l$  to be the  $l^{\text{th}}$  least significant bit of  $\mathbf{X}$  and  $\mathbf{X}_{l,j}$  to denote the bit vector from  $\mathbf{X}_l$  to  $\mathbf{X}_j$  (both included). Now, since the inputs  $\mathbf{I}$  and  $\mathbf{Y}$  are odd,  $\mathbf{Y}_1 = \top, \mathbf{I}_1 = \top$ . Therefore  $\psi^{\mathbf{X}_1} = \top$ .

Now note that  $(\mathbf{X} \times \mathbf{Y})_i = (\mathbf{X}_{1,i} \times \mathbf{Y}_{1,i})_i = (\mathbf{X}_{1,i-1} \times \mathbf{Y}_{1,i} + \mathbf{X}_i \cdot \mathbf{Y}_1 \times (2^{i-1}))_i = \mathbf{X}_i \cdot \mathbf{Y}_1 \oplus (\mathbf{X}_{1,i-1} \times \mathbf{Y}_{1,i})_i$  (using the structure of multiplication), where “ $\cdot$ ” denotes 1-bit multiplication and  $\oplus$  denotes 1-bit addition modulo 2. Therefore,  $\mathbf{I}_i = \mathbf{X}_i \oplus (\mathbf{X}_{1,i-1} \times \mathbf{Y}_{1,i})_i$ , or equivalently,  $\mathbf{X}_i = \mathbf{I}_i \oplus (\mathbf{X}_{1,i-1} \times \mathbf{Y}_{1,i})_i$ .

It is now easy to see that once we obtain a Skolem function for  $\psi^{\mathbf{X}_1}$  to  $\psi^{\mathbf{X}_{i-1}}$  in this manner, we can recursively generate the skolem function for  $\mathbf{X}_i$ , giving the entire Skolem function vector for  $\mathbf{X}$ .  $\square$

Note that while we have used a specific Skolem function vector above, once the SAUNF form is obtained, it can be used to generate other Skolem function vectors as well (from Algorithm 1).

## X. CONCLUSION

In this paper, we presented a normal form for Boolean relational specifications that characterizes efficient Skolem function synthesis. This is a significantly stronger characterization than those used in earlier works. SAUNF is exponentially more succinct than DNNF, dDNNF while enjoying similar composability properties. It also strictly subsumes the recently proposed SynNNF. As future work, we plan to improve the compilation algorithm and apply it to challenging benchmarks.

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