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On weak higher dimensional categories I: Part 1 ☆

Claudio Hermida[a,1], Michael Makkai[a,*,2], John Power[b,3]

[a] Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Street West, Montreal, Quebec, Canada H3A 2K6

[b] Department of Computer Science, University of Edinburgh, King's Building, Edinburgh EH9 3JZ, UK

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Dedicated to F. William Lawvere on his 60th birthday

Abstract

Inspired by the concept of opetopic set introduced in a recent paper by John C. Baez and James Dolan, we give a modified notion called multitopic set. The name reflects the fact that, whereas the Baez/Dolan concept is based on operads, the one in this paper is based on multicategories. The concept of multicategory used here is a mild generalization of the same-named notion introduced by Joachim Lambek in 1969. Opetopic sets and multitopic sets are both intended as vehicles for concepts of weak higher dimensional category. Baez and Dolan define weak n-categories as $(n+1)$-dimensional opetopic sets satisfying certain properties. The version intended here, multitopic n-category, is similarly related to multitopic sets. Multitopic n-categories are not described in the present paper; they are to follow in a sequel. The present paper gives complete details of the definitions and basic properties of the concepts involved with multitopic sets. The category of multitopes, analogs of opetopes of Baez and Dolan, is presented in full, and it is shown that the category of multitopic sets is equivalent to the category of set-valued functors on the category of multitopes. © 2000 Elsevier Science B.V. All rights reserved.

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☆ This is the first part of a three-part paper. Parts 2 and 3 are to appear in later issues of this Journal. Sections are numbered consecutively throughout the three parts.

* Corresponding author.

E-mail addresses: hermida@triples.math.mcgill.ca (C. Hermida), makkai@math.mcgill.ca (M. Makkai), ajp@dcs.ed.ac.uk (J. Power).

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Introduction

In [2] and [3], John C. Baez and James Dolan have introduced a concept of weak higher dimensional category. The present paper is inspired by the work of Baez and Dolan. It is the first of two papers in which a modification of the Baez/Dolan proposal is offered and described in detail.

There are other proposals for related concepts; see [4,22].

The problem of the identification of the weak higher-dimensional categories has been recognized for some time; see e.g. [6,18–21]. The motivations for the Baez/Dolan work were described in [1]. In [13,15], the second author of this paper describes another motivation, one that relates higher-dimensional categories to the foundations of mathematics. In [13], a program for a new type-theoretical foundation, termed *structuralist*, is described in which there is a hierarchy of totalities of higher and higher dimensions, starting with sets. In this framework, sets are taken to be totalities with an equality predicate. However, no equality is assumed between elements of different sets, and, essentially as a consequence, no equality of sets is contemplated. Because of this, sets do not form a set, or even a set-like totality like a class. Instead, sets form a *category*, the category of sets; and the role of equality as principle of identity is taken over by *isomorphism*, a concept derived from the structure of category. When we say that equality of objects is not part of the structure of the category, we have in mind a notion of category that is not the same as the one we deal with on the basis of the standard set-theoretical foundation. The negative statement of the denial of equality can be given objective content only by specifying a suitably constrained language to be adopted as the formal language of the structuralist foundation. The work [14] proposes first order logic with dependent sorts (FOLDS) as the basis for such a language.

Classically, categories form 2-categories; the latter concept can already be found in [11]. The structuralist foundation involves the program of revising category-theoretical concepts in which *equality of objects* of a category is replaced by *specified isomorphisms of objects*. As a matter of fact, it has been widely accepted among category theorists that equality of objects should be avoided; the tendency to replace equality of objects by isomorphism is a common one in category theory. Jean Benabou's notion [5] of bicategory is an instance of this tendency. In the case of a 2-category, the 1-arrows from a fixed 0-cell to another 0-cell form a(n ordinary) category. Applying the "isomorphisms-for-equality" treatment to the part of the definition of 2-category which explicitly refers to equality of 1-arrows (e.g., the associative law of composition of 1-arrows) results in the concept of bicategory. We do not simply require the existence of certain isomorphism-2-arrows, but introduce specified ones (*coherence isomorphisms*), and we attach them to the structure. Furthermore, certain natural *coherence conditions* are imposed on the coherence isomorphisms (the Mac Lane pentagon is an example; see [11, p. 158], formulated for monoidal categories, that is, bicategories with a single 0-cell). It should be emphasized that the concept of bicategory was motivated in the first place by more mathematical considerations than the ones connected to the structuralist foundation. Bicategories have turned out to be extremely useful, and a great deal more flexible than 2-categories.

The paper [12] deals with a more elementary instance of replacing equality of objects by isomorphism; the notion of (*saturated*) *anafunctor* is introduced, in which the value-object of a(n ana)functor at any given argument-object is determined (strictly) up to isomorphism. Anafunctors are "mathematically equivalent" to functors, but only at the cost of an application of the Axiom of Choice. The replacement of the composition-functors in the definition of a bicategory by anafunctors results in *anabicategories*, which are held, in [12,14], to be the right concept for totalities of categories, at least from the point of view of the structuralist foundation. Saturated anabicategories are equivalent to bicategories, again via Choice. Saturated anabicategories are equivalent in a canonical manner, without the use of Choice, to the Baez/Dolan weak 2-categories, and the *multitopic 2-categories* that the sequel to this paper will describe.

Besides being the first answer to a long-standing problem, the Baez/Dolan proposal has several remarkable features. The main one is a complete elimination of explicit lists of coherence structure and conditions. This feature is already fully apparent when one looks at the case $n=2$, a Baez/Dolan weak 2-category. It is related to a bicategory as a fibration is related to a pseudo-functor [7]. The coherence isomorphisms and conditions present in the definition of pseudo-functor are, in the corresponding fibration, eliminated in favor of a structure defined by a universal property, that of Cartesian arrows. Such an elimination of coherence takes place in a Baez/Dolan (B/D) weak n-category as well, for all n. For $n=2$, the composition of 1-cells is defined by a universal property, and accordingly, its result, the composite, is not a uniquely defined thing, but one which is determined up to isomorphism; recall that the last feature is present also in anabicategories. There are no coherence isomorphisms (such as the associativity isomorphism), no coherence conditions (such as Mac Lane's pentagon). The way this is achieved is similar to the case of fibrations inasmuch one adds *more entities* to the original (pseudo-functor, respectively, bicategory) to get the new structure (fibration, respectively, B/D weak 2-category). In the case of a fibration, the arrows between objects in *different fibers* of the total category are new with respect to the data of the pseudo-functor. In the case of the B/D weak 2-category, we have 2-cells whose domain is a composable string of 1-cells, of arbitrary finite lengths in fact, instead of just a single 1-cell. These "multi-arrows" are new entities with respect to the corresponding (ana)bicategory, and they are taken away when one passes from the B/D 2-category to the corresponding bicategory; of course, before being taken away, they are used to define the data for the bicategory.

Multitopic higher-dimensional categories, as we will call the objects that we intend to introduce, will share the above general aspects of the Baez/Dolan weak higher-dimensional categories.

Although the proposal to be explained here was directly inspired by the B/D proposal, its exposition will not make this fact clear. In fact, at the present time, we do not see the *precise* equivalence of the two proposals. A conspicuous difference is the absence here, and the presence in [3], of actions of permutation groups. It is possible to introduce an "up to isomorphism" variant of the basic notion of *multicategory* used in this paper (more on this will follow soon); this higher-dimensional variant of "multicategory" (in which, for instance, isomorphisms between arrows in a multicategory would appear) seems more directly related to [3] than what is found here.

On the order hand, even if there are close ties between the proposal of [3] and that of this paper, their mathematical forms are entirely different. The [3] concept is abstract and conceptual; ours here is concrete and geometric.

The above description concerning the two-dimensional case already indicates the starting point of the approach of the present paper. We define a concept of *k-dimensional cell*, or k-cell, for all $k = 0, 1, 2, 3, \ldots$, in an inductive way. For $k > 0$, a k-cell has a *domain* and a *codomain*; the codomain is an $(k-1)$-cell, but the domain is a *pasting diagram* of $(k-1)$-cells. The inductive character of the definition lies in the definition of pasting diagrams. These are related to what go under the same name in the literature (see e.g. [16,17]), but are greatly simplified by the fact that the codomains of cells are always single cells. Despite the fact that the Baez/Dolan concept is not explained in terms of cells whose domains are pasting diagrams of lower cells, the crucial restriction to single-cell codomains also originates in [3].

The present paper's approach is consciously geometrical. At the same time, great care is taken to express everything in algebraic terms. The main algebraic tool we use is the concept of *multicategory*, a modified form of the same-named notion introduced by Joachim Lambek in 1969; see [9,10]. It is worth remarking that one of the first uses Lambek made of multicategories was to proof-theory, for an algebraic formulation of Gentzen's proof-system for intuitionistic propositional logic.

Lambek's concept is closely related to monoidal categories. A multicategory may be said to be mathematically equivalent to a strictly associative monoidal category in which the monoid of the objects under the tensor-product is a free monoid (on the objects of the multicategory as generators). In a multicategory, we have objects and arrows; each arrow has a *source* which is a finite tuple of objects, and a *target*, a single object. The main distinguishing point about the notion of multicategory is that it is phrased in terms of a *composition*, a ternary operation, two of whose arguments are arrows, the third being the *place* where the target of one of the arrows is to fit into the source of the other; of course, the result of composition is an arrow. From the point of view of the arrows, we have a system of binary compositions. Two of the laws are an associative law and a commutative (or rather, *interchange*) law of composition suitably decorated with places.

We generalize Lambek's notion in two steps, one major and a minor. The major step is to make explicit and generalize the *amalgamation* that takes place in composition. When two arrows are composed, the source of the composite results by *amalgamating* the sources of the original arrows in a certain way. In the Lambek case, this amalgamation is the standard one of *inserting* the source of one of the arrows into the source of the other at the given place. In the generalized concept, the amalgamation is made arbitrary, subject to certain laws. It should be noted that for the precise statement of the laws of multicategory, one has to make an explicit reference to this amalgamation already in Lambek's case. Lambek does not make the amalgamation explicit, but there is an acknowledgement of the resulting incompleteness of the formulation in lines 12 and 11 from the bottom on p. 222 of [10].

It does not seem possible to relate the general concept of multicategory with that of monoidal category as closely as in the case of the Lambek multicategory. The new concept is "essentially geometric"; it has geometric instances (see below), but it does not seem to have "semantical" instances, apart from the standard Lambek case, which does have many "semantical" examples.

On the other hand, the generalized concept is a *mild* generalization. This is witnessed by the fact that the free multicategory in the Lambek sense on a set of objects and generating arrows is also the free multicategory on the same generating data in the generalized sense.

The main point of the new notion is that multicategories with non-standard amalgamation appear in nature. The *multicategory of function-replacement* derived from a free multicategory plays a central role in our work; it is needed for the definition of the domain, a $(k-1)$-dimensional pasting diagram, of a k-dimensional pasting diagram.

The first section of the paper is an extended informal introduction. After the next three sections on multicategories, on morphisms of multicategories, and free multicategories, respectively, Section 5 gives the construction of the multicategory of function-replacement.

Section 6 uses the preceding machinery to put together the definition of *multitopic set*, the main notion arrived at in this paper. A *multitopic n-category*, the main object we want, will, in the sequel to this paper, be defined as an $(n+1)$-dimensional multitopic set *with additional properties*; no new data are needed. Baez and Dolan used *opetopic sets* instead; the name of their notion is derived from *operads*, the abstract algebraic concept at the basis of their work. Let us note that by a multitopic set we mean what also could be called an ω-dimensional multitopic set; an n-dimensional one is in fact a truncated one.

Section 7 identifies a particular category, the category `Multitope` of *multitopes*, and identifies multitopic sets defined in the Section 6 as set-valued functors on the category of multitopes. More precisely, we prove that `MltSet`, the naturally defined category of multitopic sets, is equivalent to the category of functors from `Multitope` to `Set`. `Multitope` is related to the terminal object $\mathscr{T}$ of `MltSet`. The objects of `Multitope` are identical to the pasting diagrams of the multitopic set $\mathscr{T}$; on the other hand, the identification of the arrows of `Multitope` takes additional work. It should be emphasized that all the complexity involved in the definition of multitopic sets in general is already present in the definition of the terminal one, $\mathscr{T}$, despite the fact that this object is absolutely uniquely given.

The category `Multitope` and, for any $n=0,1,2,\ldots$, its truncation $\mathtt{Mlt}\restriction n$ to include k-pasting diagrams of $\mathscr{T}$ for $k=0,\ldots,n$, are fundamental from the point of view taken in this paper. In [14], a concept of L-equivalence, for variable signatures L for FOLDS, is introduced, and it is shown that, when used in conjunction with the ana-concepts of [12], L-equivalence becomes identified with categorical equivalence in many cases, for instance in the case of biequivalence for bicategories. $\mathtt{Mlt}\restriction n$ is the FOLDS-signature for *n-truncated multitopic sets*. In view of the fact that multitopic n-categories are $(n+1)$-truncated multitopic sets with additional properties *formulated in FOLDS*, $\mathtt{Mlt}\restriction(n+1)$ is the FOLDS-signature also for multitopic n-categories. Thus, even before we have given the further details of the definition of multitopic n-category, we have a notion of *n-equivalence* of multitopic n-categories.

The appendix contains some details of proofs for Sections 4 and 5.

It should be emphasized that this paper is only a part, in fact, just a beginning, of the work of establishing the concept of weak higher-dimensional category. Even when we have the full definition (which is given, in one form, by [3], and promised, in a modified form, to be given by the sequel to this paper), the accompanying structures are still to be provided.

1. An informal description

1.1. n-graphs and multitopic sets

In the classical, strict, concept of higher-dimensional category (HDC), an HDC $\boldsymbol{A}$ consists of *k-cells* in each of several *dimensions* k, where k ranges over a set $\{0,\ldots,n\}$ (*n-category*), or over all natural numbers (*ω-category*). Let us denote the class of all k-cells of $\boldsymbol{A}$ by C_k. For $k>0$, each k-cell a is "based on" two $(k-1)$-cells, the *domain* $\mathrm{d}a$ and *codomain* $\mathrm{c}a$ of a; when $b=\mathrm{d}a$, $c=\mathrm{c}a$, we write $a:b\to c$; we have the assignments $\mathrm{d}_k=\mathrm{d}:C_k\to C_{k-1}$, $\mathrm{c}_k=\mathrm{c}:C_k\to C_{k-1}$ as part of the structure of the HDC $\boldsymbol{A}$. The part of the structure of $\boldsymbol{A}$ so far described is an *n-graph* in the case of "n-category", *ω-graph* in the case of "*ω-category*"; the data for an n-graph can be summarized in the following diagram:

$$C_0 \underset{\mathrm{c}_1}{\overset{\mathrm{d}_1}{\leftleftarrows}} C_1 \underset{\mathrm{c}_2}{\overset{\mathrm{d}_2}{\leftleftarrows}} C_2 \cdots C_{n-1} \underset{\mathrm{c}_n}{\overset{\mathrm{d}_n}{\leftleftarrows}} C_n \tag{1}$$

A feature of n-graphs, is *globularity*: for any $a\in C_k$, $k\geq 1$, $b=\mathrm{d}a$ and $c=\mathrm{c}a$ must be *parallel*, that is, either $k-1=0$, or else $\mathrm{d}b=\mathrm{d}c$, $\mathrm{c}b=\mathrm{c}c$:

$$e \underset{c}{\overset{b}{\rightrightarrows}} f$$

where $e=\mathrm{d}b=\mathrm{d}c$, $f=\mathrm{c}b=\mathrm{c}c$. Put another way,

$$\mathrm{dd}=\mathrm{dc},\quad \mathrm{dc}=\mathrm{cc}, \tag{2}$$

where d and c ambiguously denote any of the domain, respectively codomain maps $\mathrm{d}_k: C_k\to C_{k-1}$, $\mathrm{c}_k: C_k\to C_{k-1}$, with the restriction that the composites intended should be meaningful. n-graphs are defined by having data as in (1), the domain/codomain assignments satisfying globularity (2). An n-category (in the usual sense) has several additional operations of *composition*; see, e.g., [19].

The notion of HDC of the present paper will retain the above general features, except for one thing; the *domain* of a cell is no longer a cell itself; rather, it is a *pasting* *diagram* (see below) of cells. Note the asymmetry: we only mentioned “domain”, not “codomain”; codomains will remain single cells.

The role of n-graphs is taken up by (n-dimensional) *multitopic sets*; below, there will be an explanation for the choice of the name of the concept. The data for a multitopic set are summarized in the diagram

$$
\begin{tikzcd}
P_0 & P_1 \arrow[l, "\mathrm{d}"'] \arrow[dl, "\mathrm{c}"] & P_2 \arrow[l, "\mathrm{d}"'] \arrow[dl, "\mathrm{c}"] \cdots & & P_k & P_{k+1} \arrow[l, "\mathrm{d}"'] \arrow[dl, "\mathrm{c}"] \cdots \\
C_0 \arrow[u, "\mathrm{i}"] & C_1 \arrow[l, "\mathrm{c}"] \arrow[u, "\mathrm{i}"] \arrow[ul, "\mathrm{d}"] & C_2 \arrow[l, "\mathrm{c}"] \arrow[u, "\mathrm{i}"] \arrow[ul, "\mathrm{d}"] \cdots & & C_k \arrow[u, "\mathrm{i}"] & C_{k+1} \arrow[l, "\mathrm{c}"] \arrow[u, "\mathrm{i}"] \arrow[ul, "\mathrm{d}"] \cdots
\end{tikzcd}
$$

where C_k is the set of k-cells, P_k the set of k-dimensional pasting diagrams (k-pd’s for short), each i is an inclusion map, and the d and c are domain and codomain maps. All meaningful instances of the globularity condition (2) will hold.

In the next subsection, we will explain the notion of pasting diagram; here, we note that they are not independent data governed by relations and properties; rather, they are defined explicitly in terms of cells. The most important point to keep in mind that there is an essential recursive character to the notion of multitopic set; this is because the notion of $(k+1)$-cell cannot be explained before we know what k-pd’s are, and k-pd’s, in turn, are defined in terms of k-cells.

The higher-dimensional categories, *multitopic n-categories*, whose definition is the eventual goal of the present paper, are based on multitopic sets, just as n-categories are based on n-graphs. As a compensation for the increased complexity in multitopic sets in comparison to higher-dimensional graphs, we have the fundamental fact that a multitopic n-category is an $(n+1)$-dimensional multitopic set *with additional properties* only; no additional data are required. (Note, however, the placing of the prefixes n and $n+1$ in this description.)

1.2. Pasting diagrams

The expression of “pasting diagram” refers to the idea of a *composable diagram*, one which, *if* a concept of composition of cells were available, would result in a single cell after all the meaningful compositions denoted in the diagram are performed. This is an approximate expression of an intuitive idea. It turns out that composability in higher dimensions is a difficult concept, and despite several contributions (e.g., [8,16–18]) it is not yet completely clarified. It is to be emphasized that the concept “composable diagram” is a geometric one in that it does not involve composition of cells in the algebraic sense. Composability is the *geometric precondition* of (iterated) composition.

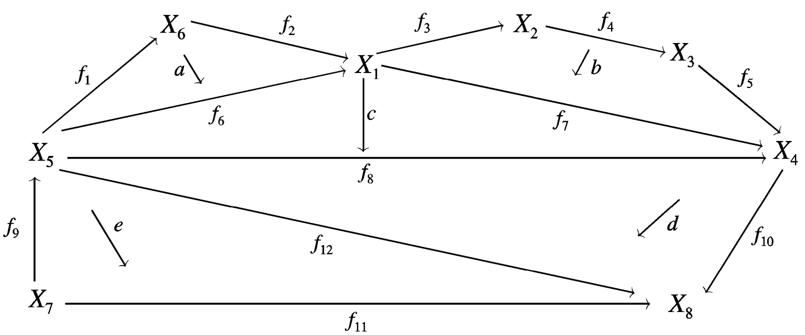
An important point for this paper, inspired by the Baez/Dolan work, is the restriction of cells to the form $a:\alpha \to b$, where α is a pasting diagram (pd), but b is a single cell. The first consequence is that the notion of pd itself becomes simple, and abstractly manageable, in comparison with the (potential) more comprehensive concept that would allow both the domain and the codomain to be arbitrary pd's. The "Baez/Dolan restriction" (as we may call the above-mentioned restriction) is not a *necessary* feature of the intended notion of HDC; it is, rather, a simplifying idea; the thus simplified notion of pd turns out to be *sufficient* for carrying the intended structure of an HDC.

A zero-dimensional pd (or 0-pd) is just a 0-cell (object). A one-dimensional pd (or 1-pd) is a composable string of 1-cells:

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \rightarrow \cdots \xrightarrow{f_n} X_{n+1},$$

where the X_i's are 0-cells, the f_i's are 1-cells. $n = 0$ is allowed, in which case there are no arrows; but in this case, there is still an object, X_1, and we have the empty string of arrows starting and ending in X_1.

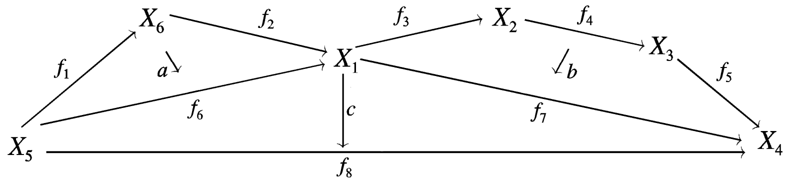
A 2-pd consists of 0-, 1- and 2-cells; each 2-cell in it is *from* a 1-pd, a string of one cells (possibly empty), *to* a single 1-cell; and the whole thing is *composable*. Here is an example of a 2-pd, which we denote by the single letter γ:



γ consists of the 0-cells X_i, $1 \le i \le 8$, 1-cells f_j, $1 \le j \le 11$ (numbered in no particular order), and the 2-cells a, b, c, d, e. The figure is supposed to make clear the domain/codomain relations among the cells and 1-pd's involved. Notice the constraint that each 2-cell targets a single 1-cell; in a 2-pd in a more general sense, both domains and codomains could be general 1-pd's. Perhaps it is superfluous to say that the 2-pd γ is the totality of the items listed; it is not the result of some kind of composition performed on those items. Of course, the relative position of its component 2-cells is part of the defining data of the 2-pd.

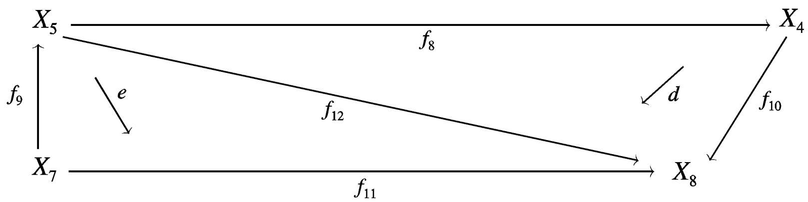
There are features of 2-pd's that become important elements of the general concept of a k-pd. The above 2-pd can be regarded as obtained by *composition*, in a new sense of "formal" composition, which applies to pd's rather than cells. This composition may also be called *grafting*. For instance, γ is obtained by grafting from the following two pd's α and β:

α:



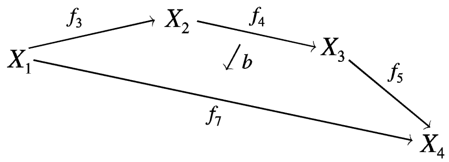
and

β:



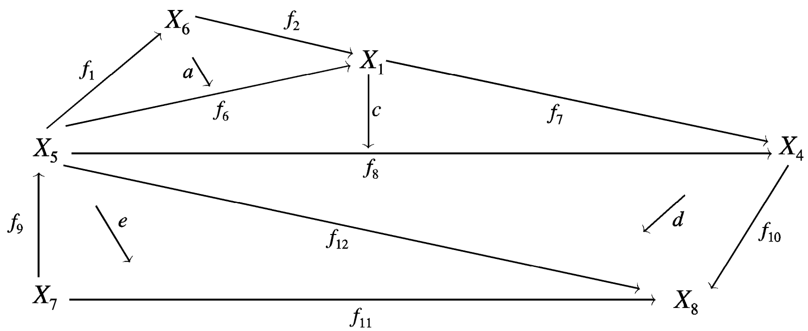
More precisely, we graft α into β at f_8, and obtain the original γ. Of course, the same pd γ can also be obtained in several other ways as the result of grafting, e.g. by grafting δ into ε, where

δ:



(which is a pd consisting of a single 2-cell), and

ε:



The grafting composition is a binary operation as far as the number of arguments that are pd's is concerned; but it also has a third argument, the *place* at which the grafting takes place. The two grafting compositions displayed are denoted as $\beta \circ_{f_8} \alpha = \gamma$, and $\varepsilon \circ_{f_7} \delta = \gamma$; read e.g. the first as α composed (grafted) into β at f_8 is γ.

Given β as above, and, say,

α':

$$\begin{tikzcd}
& X_6 \arrow[dr, "f_2"] \arrow[d, phantom, "a \Searrow"] & & X_2 \arrow[r, "f_4"] \arrow[d, phantom, "\Swarrow b"] & X_3 \arrow[ddr, "f_5"] & \\
& {} & X_1 \arrow[ur, "f_3"] \arrow[drrr, "f_7"'] \arrow[d, "c"] & {} & & \\
X_5' \arrow[uur, "f_1"] \arrow[urr, "f_6"'] \arrow[rrrrr, "f_8'"'] & & {} & & & X_4'
\end{tikzcd}$$

where the primed items may or may not be equal to the corresponding non-primed items in β, the composite $\beta \circ_{f_8} \alpha'$ is meaningful if and only if $f_8' = f_8$, and as a consequence, $X_5' = X_5$, $X_4' = X_4$. f_8' is distinguished as the *target-1-cell* of α; $\mathrm{t}(\alpha) \underset{\mathrm{def}}{=} f_8'$. For the given β, and an undetermined α', the condition for $\beta \circ_{f_8} \alpha'$ to be well-defined is that $\mathrm{t}(\alpha') = f_8$.

It is perfectly possible that several items in the above pd's that are now denoted by different symbols are actually the same. For instance, it is possible that all the 0-cells are the same, and all the 1-cells are the same. If so, the 2-cells a, c, d, e could all be the same, although b cannot be the same as those since its shape is different: its domain pd is a length-3 1-pd, whereas the domains of the others are of length 2. Assuming, e.g., that all the said coincidences actually take place, the subscript f_8 in $\beta \circ_{f_8} \alpha$ cannot refer to the f_8 simply as a 1-cell; it has to refer to *the place of* f_8; we have $f_8 = f_9 = f_{10}$, and we can just as well compose α into β at the *two other places*, now denoted f_9 and f_{10}, and the results of these compositions are all very different, distinguished already by their shapes. This tells us that in the concept of pd there has to be an essential element that we may call *place*; in a 2-pd, there are *places* for 1-cells, each of which carries the "occurrence" of a particular 1-cell.

Note that it does not make sense to compose anything into α at f_{11}, or into γ at places other than $f_9, f_1, f_2, f_3, f_4, f_5, f_{10}$; the result would not be a "composable diagram". The listed places of γ, the ones at which it is legitimate to compose something into γ, are the *source* places of γ; they are, together with the *target* place f_{11}, "outer places"; the "inner places" are the rest, f_6, f_7, f_8, f_{12}. $\mathrm{s}(\gamma)$ denotes the tuple $\langle f_{10}, f_5, f_4, f_3, f_2, f_1, f_9 \rangle$, and it is called the *source* of γ; the reason for the order will be explained below. In the example, $\mathrm{s}(\gamma)$ is a function on the set $[1,6] = \{1,2,3,4,5,6\}$, and its values are $\mathrm{s}(\gamma)\,(1) = f_{10}$, etc. The source places themselves of γ are identified with the natural numbers $1,2,3,4,5,6$; the place 1 carries an occurrence of f_{10}, the place 2 one of f_5, etc. Writing $|\mathrm{s}(\gamma)|$ for the domain of the function $\mathrm{s}(\gamma)$, the source-places of γ are the elements of $|\mathrm{s}(\gamma)|$.

Similarly, $\mathrm{s}(\beta) = \langle f_{10}, f_8, f_1 \rangle$. Since the place of f_8 in β is 2, we will write $\beta \circ_2 \alpha$ for $\beta \circ_{f_8} \alpha$; we have $\gamma = \beta \circ_2 \alpha$.

For general 2-pd's α and β,

$$\beta \circ_p \alpha \text{ makes sense if and only if } p \in |\mathrm{s}(\beta)| \text{ and } \mathrm{s}(\beta)(p) = \mathrm{t}(\alpha). \tag{3}$$

We have identified what we take to be the essential structure on pd's: the *placed composition* $\alpha \circ_p \beta$, a ternary operation as explained above.

1.3. Multicategories

The abstract concept of structure for the operation of placed composition is called *multicategory*. Multicategories were introduced by Lambek in 1969 [9]; one of the uses he made of them was to define a multicategory of proofs in the Gentzen formal system for intuitionistic logic, where the placed composition corresponds to the Cut-rule. A *Lambek multicategory* $\boldsymbol{C}$ has a set $O = \mathrm{O}(\boldsymbol{C})$ of objects, and a set $A = \mathrm{A}(\boldsymbol{C})$ of arrows; each arrow α has a source $\mathrm{s}(\alpha)$ which is a finite tuple of objects, and a target $\mathrm{t}(\alpha)$ which is a single object; when $\mathrm{s}(\alpha) = \vec{X}$, $\mathrm{t}(\alpha) = Y$, we write $\alpha : \vec{X} \to Y$; $\boldsymbol{C}$ has, for each object X, an identity arrow $1_X : \langle X \rangle \to X$; and $\boldsymbol{C}$ has a placed composition as in (3) above. These data are to satisfy certain laws, the first of which regulates the source and the target of a composite, with the remaining laws being two identity laws, an associativity law, and a commutativity law. The definition will be given in Section 2; the reader will notice that the definition in Section 2 is, initially, something more general and more complicated than the one indicated here; later in that section, however, it is pointed out what exactly the Lambek concept is as a special case. Later in this introduction we will turn to the reasons why we need the more general concept of multicategory.

Thus, the 2-pd's (in a given HDC $\boldsymbol{A}$) form a Lambek multicategory (the 1-pd's also do; in fact, they form an ordinary category). More is true: the 2-pd's form a *free* multicategory, with the 1-cells as objects, and the 2-cells as generating arrows. Hence, all 2-pd's are generated by the 2-cells by using the operation of placed composition. This should be seen as an intuitively natural fact about pasting (composable) diagrams. (Let us remind ourselves that here we are in the business of *defining* what pasting diagrams are; the definition is constrained by intuitive ideas, which we are trying to make explicit.) Freeness is meant here in the sense of a strict universal property; it will be crucial later that the free Lambek multicategory maintains its universal property in the larger context of all (generalized) multicategories in the sense of Section 2.

For precise definitions concerning morphisms of multicategories, and free multicategories, see Sections 3 and 4. Here we only give a brief idea.

Let O be a set of *objects*, L a set of *arrows*, with each $f \in L$ equipped with a source $\mathrm{s}(f) \in O^*$, and a target $\mathrm{t}(f) \in O$; data as described define a *language* $\mathscr{L}$. The terminology is natural, since $\mathscr{L}$ is exactly what is usually called a language (signature) for multi-sorted algebras; the elements of O are the sorts; the elements of L are the sorted operation symbols. The free multicategory, $\boldsymbol{C} = \mathscr{F}(\mathscr{L})$, on $\mathscr{L}$ is defined by the conditions that $\mathrm{O}(\boldsymbol{C}) = O$, $L \subset A(\boldsymbol{C})$, and any "interpretation" (a rather obvious notion) $\mathscr{L} \to \boldsymbol{D}$ to any multicategory $\boldsymbol{D}$ can be uniquely extended to a morphism $\boldsymbol{C} \to \boldsymbol{D}$.

It turns out that the concrete description of $\mathscr{F}(\mathscr{L})$ is very simple. Its arrows are the *terms*, in the sense used in describing the syntax of first-order logic, built up from sorted variables and the operation symbols of L, with the further simplification that we use only a single variable for each sort X, which variable therefore may just as well be identified with X itself.

Thus, we now have a *term representation* of 2-pd's. Turning to the examples above, we have the following:

$$\begin{aligned}
&\gamma : e(d(f_{10}, c(b(f_5, f_4, f_3), a(f_2, f_1))), f_9),\\
&\alpha : c(b(f_5, f_4, f_3), a(f_2, f_1)),\\
&\beta : e(d(f_{10}, f_8), f_9),\\
&\delta : b(f_5, f_4, f_3),\\
&\varepsilon : e(d(f_{10}, c(f_7, a(f_2, f_1))), f_9).
\end{aligned}$$

To understand these, consider the following. Any expression $x(y, z, \ldots)$ stands for a repeated composition; $x(y, z, \ldots) = \ldots(x \circ_1 y) \circ_{\bar{2}} z \ldots$; $\bar{2}$ is the place in $x \circ_1 y$ that "corresponds to" the place 2 in x. Each f_i stands for 1_{f_i}, the identity arrow

$$\langle f_i \rangle \xrightarrow{1_{f_i}} f_i.$$

Since $a(f_2, f_1)$ is a with identities composed into a, $a(f_2, f_1)$ equals a itself; we could write a in place of $a(f_2, f_1)$ above, except that in that case we would have not used the normal form which is intended by the term-representation. For $t_1 = b(f_5, f_4, f_3)$, $t_2 = a(f_2, f_1)$, the term $\alpha = c(t_1, t_2)$ is, really, the multicategory composite $(c \circ_1 t_1) \circ_{\bar{2}} t_2 = (c \circ_2 t_2) \circ_{\bar{1}} t_1$; the equality is the commutative law; $\bar{2} = 4$, $\bar{1} = 1$ (why?). We also see that placed composition corresponds to *substitution*: the fact that $\beta \circ_2 \alpha = \gamma$ is reflected in the fact that γ is the result of substituting α for f_8 in β.

The term-representation is a simple linear way of writing down 2-pd's; in fact, it will also be available for k-pd's for any k. However, note that in this notation, several elements that are clear in the geometric picture are suppressed. All 0-cells, and all but the input 1-cells are suppressed, although they can be recovered by the information concerning the *targets* of the 2-cells involved.

Let us note that the 1-pd's also admit a term representation, since they also form a multicategory, which in fact is an *ordinary category*, since only unary arrows appear. The 1-pd in (1) is represented by the term $f_n(\ldots(f_2(f_1(x_1)))\ldots)$. The source-assignment to 2-cells above follows the left-to-right order in the term-representation; this is the reason why we used the "reverse" order for those sources above.

1.4. 3-cells and three-dimensional pasting diagrams

Let us move from dimension 2 to dimension 3.

A 3-cell u is to have a 2-pd $\mathrm{d}u$ ($=\mathrm{d}_3 u$) as domain, and a 2-cell $\mathrm{c}u$ as codomain. Globularity requires that we should have $\mathrm{dd}u = \mathrm{dc}u$, $\mathrm{cd}u = \mathrm{cc}u$; however, we have not defined $\mathrm{d}\alpha, \mathrm{c}\alpha$ for 2-pd's α as yet, and we need them now for $\alpha = \mathrm{d}u$. The definition of the domain of a pd is a major issue in our enterprise; the codomain is easy. In the case of the example γ above, $\mathrm{d}\gamma$ is

$$\begin{tikzcd}
 & X_6 \arrow[dr, "f_2"] & & X_2 \arrow[dr, "f_4"] & & \\
 & & X_1 \arrow[ur, "f_3"] & & X_3 \arrow[dr, "f_5"] & \\
X_5 \arrow[uur, "f_1"] & & & & & X_4 \arrow[dl, "f_{10}"] \\
X_7 \arrow[u, "f_9"] & & & & X_8 &
\end{tikzcd}$$

that is,

$$X_7 \xrightarrow{f_9} X_5 \xrightarrow{f_1} X_6 \xrightarrow{f_2} X_1 \xrightarrow{f_3} X_2 \xrightarrow{f_4} X_3 \xrightarrow{f_5} X_4 \xrightarrow{f_{10}} X_8;$$

this is the "upper part of the contour (boundary) of γ". $\mathrm{c}\gamma$ is the 1-cell $X_7 \xrightarrow{f_{11}} X_8$, the "lower" part of the contour of γ, the cell that "closes off" $\mathrm{d}\gamma$. Thus, a 3-cell u for which $\mathrm{d}u = \gamma$, with γ as in the example, looks necessarily like $u : \gamma \to g$, where g is a 2-cell of the following "shape":

$$\begin{tikzcd}
 & X_6 \arrow[dr, "f_2"] & & X_2 \arrow[dr, "f_4"] & & \\
 & & X_1 \arrow[ur, "f_3"] \arrow[dd, "g"] & & X_3 \arrow[dr, "f_5"] & \\
X_5 \arrow[uur, "f_1"] & & & & & X_4 \arrow[dl, "f_{10}"] \\
X_7 \arrow[u, "f_9"] \arrow[rrrr, "f_{11}"'] & & {} & & X_8 &
\end{tikzcd}$$

which means that $\mathrm{d}g{=}\mathrm{d}\gamma$, $\mathrm{c}g{=}\mathrm{c}\gamma$. One cannot faithfully represent u in a two-dimensional drawing; but u has a good three-dimensional geometric representation; in this, the 2-pd γ is placed in the plane of the table, say; the 2-cell g is spanned out in a curved surface above the table, with its contour joining the contour of γ according to the identification inherent in the facts $\mathrm{d}g = \mathrm{d}\gamma$, $\mathrm{c}g = \mathrm{c}\gamma$; the 3-cell u "fills" the space between γ and g, "in the direction" from γ to g.

3-pd's will be construed as arrows in the free multicategory on the language whose objects (sorts) are the 2-cells, operation-symbols the 3-cells, and in which the sorting of the latter is given as follows. Every 3-cell u comes with $\mathrm{d}u$, a 2-pd; regard $\mathrm{d}u$ in the term-representation; look at all the operation-symbol occurrences in $\mathrm{d}u$, which are 2-cells; define $\mathrm{s}(u)$ to be left-to-right tuple $\langle \mathrm{d}u \rangle$ of those occurrences; $\mathrm{s}(u) \in C_2^*$ as it should be. $\mathrm{t}(u)$ is defined to be $\mathrm{c}(u)$.

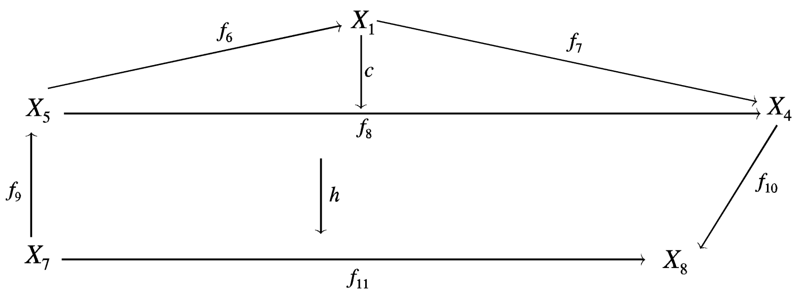
For instance, for $u : \gamma \to g$ considered above, $\mathrm{s}(u) = \langle e, d, c, b, a \rangle$.

It is important to realize that the arrow-notation is being heavily overcharged. We just wrote $u : \gamma \to g$ to indicate that for the 3-cell u, $\mathrm{d}u = \gamma$ and $\mathrm{c}u = g$. However, we also said that 3-cells are arrows in a particular free multicategory, in the *same* sense of "free multicategory" as used before. When we are in that multicategory, then u is considered an arrow $u\colon \mathrm{s}(u) \to \mathrm{t}(u)$, that is, $u\colon \langle e, d, c, b, a \rangle \to g$, which is different from $u\colon\gamma \to g$; in fact, $\mathrm{s}(u)$ is obtained by a "forgetful" process form $\mathrm{d}u$. When we were looking at a 2-cell, say a, before, $\mathrm{s}(a)$ and $\mathrm{d}(a)$ were still essentially the same. The role of the concept of multicategory in the notion of higher-dimensional pasting diagram is that of an ingredient which is used repeatedly, essentially as many times as the number of dimensions.

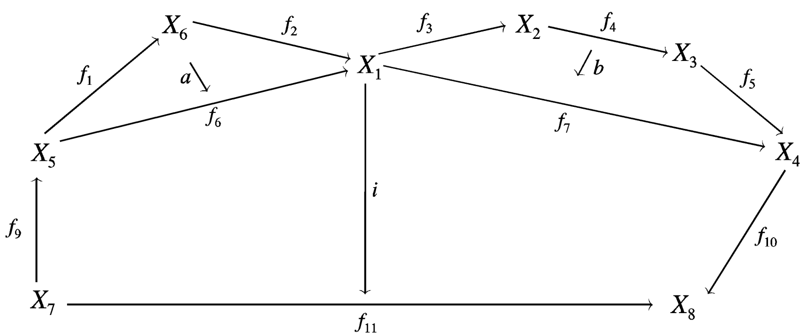
We will now describe a 3-pd φ which is *parallel* to the 3-cell u considered before. This involves the statement that $\mathrm{d}\varphi = \mathrm{d}u$, and therefore involves the determination of the domain $\mathrm{d}\varphi$ of a 3-pd φ. The systematic way of defining the domain of a pd is our main task.

Let us use the 2-pd's β and δ introduced above, as well as η and λ to follow below; we will use two new 2-cells, h and i:

$\eta = h(f_{10}, c(f_7, f_6), f_9)$:



$\lambda = i(f_{10}, b(f_5, f_6, f_3),\ a(f_2, f_1), f_9)$:



Now we introduce the 3-cells:

$$\beta \xrightarrow{v} h, \quad \eta \xrightarrow{w} i, \quad \delta \xrightarrow{x} b, \quad \lambda \xrightarrow{y} g.$$

The first thing to check is that these are well-formed, that is, in each case the assigned domain (a 2-pd) and codomain (a 2-cell) are parallel; this is true. Now, notice that these four 3-cells "line up" as follows:

$$\begin{tikzcd}
\delta \arrow[drr, "x"] & & & & \\
& & b \arrow[dr, phantom, "\in"] & & \\
& & & \lambda \arrow[r, "y"] & g \\
& & i \arrow[ur, phantom, "\in"] & & \\
& h \in \eta \arrow[ur, "w"] & & & \\
\beta \arrow[ur, "v"] & & & &
\end{tikzcd} \tag{3'}$$

In fact, we have

$$\mathrm{s}(v) = \langle e, d\rangle, \quad \mathrm{s}(w) = \langle h, c\rangle, \quad \mathrm{s}(x) = \langle b\rangle, \quad \mathrm{s}(y) = \langle i, b, a\rangle,$$

$$h = s(w)\ (1),\ \ i = \mathrm{s}(y)\ (1), \quad b = \mathrm{s}(y)\ (2),$$

and

$$\varphi \underset{\text{def}}{=} y(w(v(e,d),c),x(b),a)$$

is well-defined as a 3-pd. Note that, to an even larger extent than before, what φ really is cannot be directly seen on its defining expression; only by taking into account the descriptions of all the ingredients, which themselves were defined in similar ways, can we grasp what φ is. The faithful geometric representation of the 3-pd φ is a three-dimensional object, obtained by joining the three-dimensional cells v, w, x, y; the target 2-cell h of v is joined with the occurrence of h in η, similarly for i and b; we get a spherical (simply connected) three-dimensional object subdivided appropriately. The full entity φ involves four levels of ingredients: k-cells for all of $k = 0, 1, 2, 3$. The two-dimensional boundary of this object consists of the 2-pd γ as domain, and the 2-cell g as codomain; we have $\mathrm{d}\varphi = \gamma$, $\mathrm{c}\varphi = g$. The 2-cells h, i and one of the occurrences of b are "inner" 2-cells in φ, not denoted in the term representation. φ is indeed parallel to the 3-cell $u : \gamma \to g$; as a consequence, a 4-cell of the shape $\varphi \to g$ is possible.

1.5. The domain of a pasting diagram

We turn to explaining how $\mathrm{d}\varphi$, and in general, the domain of an arbitrary pd, is determined algebraically.

As explained before for the cases $k = 1, 2$ and 3, we construe the set P_k of k-pd's of the HDC $\boldsymbol{A}$ as the arrows of a free multicategory $\boldsymbol{C}_k^0$ whose objects are the elements of C_{k-1} ($(k-1)$-cells), and whose generating arrows are the elements of C_k. (We use the superscript 0 since there will be a modified ("twisted") variant $\boldsymbol{C}_k$ which will be the final version.) The k-cells $a \in C_k$ come with a domain $\mathrm{d}a \in P_{k-1}$ and a codomain

$\mathrm{c}a \in C_{k-1}$. For the determination of $\boldsymbol{C}_k^0$, we also need $\mathrm{s}a$ and $\mathrm{t}a$ for $a \in C_k$; as done above for low values of k, we put $\mathrm{s}a = \langle \mathrm{d}a \rangle$, and $\mathrm{t}a = \mathrm{c}a$.

Let $k \geq 1$ be arbitrary, and let $\alpha \in P_{k+1}$. For any $\gamma \in P_k$, we let $\langle \gamma \rangle$ denote the left-to-right list of function-symbol occurrences in γ. Thus, $\mathrm{s}\alpha$ is a tuple of elements of C_k, and $\mathrm{d}\alpha$ is to be defined in such a way that $\langle \mathrm{d}\alpha \rangle$ is also a tuple of elements of C_k.

The first fact on how $\mathrm{d}\alpha$ is defined is that $\mathrm{s}\alpha$ and $\langle \mathrm{d}\alpha \rangle$ are *almost* equal; one is obtained from the other by a permutation. That is, $|\mathrm{s}\alpha| = |\langle \mathrm{d}\alpha \rangle|$, and there is a permutation $\theta_\alpha = \theta : |\mathrm{s}\alpha| \xrightarrow{\cong} |\langle \mathrm{d}\alpha \rangle|$ such that

$$\begin{tikzcd}
{|\mathrm{s}\alpha|} \arrow[rr, "\theta"] \arrow[dr, "\mathrm{s}\alpha"'] & {} \arrow[d, phantom, "\mathrm{O}"] & {|\langle \mathrm{d}\alpha \rangle|} \arrow[dl, "\langle \mathrm{d}\alpha \rangle"] \\
& C_n &
\end{tikzcd}$$

Note that, by what was said above, for α a single cell, $\mathrm{d}\alpha$ is already defined, and $\mathrm{s}\alpha = \langle \mathrm{d}\alpha \rangle$; for such α, θ_α can be taken to be the identity.

The second, and main, fact about the way $\mathrm{d}\alpha$ is defined is that there is an operation assigning a new "composite" $\gamma \square_q \delta$ to any γ, $\delta \in P_k$ and $q \in |\langle \gamma \rangle|$ satisfying certain conditions of compatibility (that we will see below in detail) such that

$$\mathrm{d}(\alpha \circ_p \beta) = (\mathrm{d}\alpha) \square_{\theta(p)} (\mathrm{d}\beta), \tag{4}$$

that is, the domain of the grafting composite of two $(k+1)$-pd's is the $\square$-composite of the domains of the $(k+1)$-pd's. This, together with knowing what $\mathrm{d}\alpha$ is for single-cell pd's α determines the operation d.

Let us describe the operation $\square$. In fact, this can be done on an arbitrary-free multicategory.

Start with $\boldsymbol{C} = \mathscr{F}(\mathscr{L})$, the free Lambek multicategory on the arbitrary language $\mathscr{L}$; we use the notation we had before; $O = \mathrm{O}(\mathscr{L}) = \mathrm{O}(\boldsymbol{C})$ is the set of objects of $\boldsymbol{C}$; $A = \mathrm{A}(\boldsymbol{C})$ is the set of arrows of $\boldsymbol{C}$; we write $\mathrm{s}\alpha$ for $\mathrm{s}_{\boldsymbol{C}}(\alpha)$, $\mathrm{t}\alpha$ for $\mathrm{t}_{\boldsymbol{C}}(\alpha)$. For any $\alpha \in A$, we let $\langle \alpha \rangle$ denote the left-to-right list of function-symbol occurrences in α, as we did before. We let $\mathrm{T}(\alpha) \underset{\mathrm{def}}{=} (\mathrm{s}\alpha, \mathrm{t}\alpha)$. Note that $\mathrm{T}(\alpha) = \mathrm{T}(\beta)$ means that α and β are "parallel in the multicategory $\boldsymbol{C}$".

We are going to define a partial operation

$$(\alpha, \beta, p) \mapsto \alpha \square_p \beta \quad (\alpha, \beta \in A, p \in |\langle \alpha \rangle|; \alpha \square_p \beta \in A).$$

defined whenever

$$\mathrm{T}(\beta) = \mathrm{T}(\langle \alpha \rangle(p)). \tag{5}$$

The intuitive idea behind the operation $\square$, called *function-replacement*, is that $\alpha \square_p \beta$ is the function obtained by evaluating, at the place p and only at that place, the function-variable $\langle \alpha \rangle(p)$ as the composite function β. The condition (5) says that β is "*of the same type*" as $\langle \alpha \rangle(p)$, meaning that they have the same variables and the same value-type, and therefore, the said evaluation is possible.

Given $\alpha \in A$ and $p \in |\langle\alpha\rangle|$, let $f = \langle\alpha\rangle(p) \in L$. Then α can be written in the form

$$\alpha = \alpha' \circ_q f(\alpha_1, \dots, \alpha_n), \tag{6}$$

where $\alpha', \alpha_1, \dots, \alpha_n \in A$, and q is a suitable place $q \in \mathrm{s}(\alpha')$. Note that if f occurs in more than one place in α, then this *decomposition at* f of α is not unique; however, we have in mind the *decomposition of* α, *at the place* p, in which f "stands for the occurrence at p". What these obscure words mean is intuitively clear, and will be made precise in Section 5. The notation $f(\alpha_1, \dots, \alpha_n)$ follows the term-representation explained above; it is, structurally, a repeated (or *simultaneous*, because of the presence of an appropriate commutative law) composition, as it was also indicated above.

Now, suppose, that, in addition, $\beta \in A$ such that (5). We put

$$\alpha \Box_p \beta \underset{\text{def}}{=} \alpha' \circ_q \beta(\alpha_1, \dots, \alpha_n). \tag{7}$$

Here, $\beta(\alpha_1, \dots, \alpha_n) = \beta(\alpha_1/1, \dots, \alpha_n/n)$ is simultaneous composition. $\mathrm{T}(\beta) = \mathrm{T}(f)$ implies that $\mathrm{s}(\beta) = \mathrm{s}(f)$, and so $\mathrm{t}(\alpha_i) = \mathrm{s}(f)\,(i) = \mathrm{s}(\beta)\,(i)$, which makes the term $\beta(\alpha_1, \dots, \alpha_n)$ well-defined; but also, $\mathrm{T}(\beta) = \mathrm{T}(f)$ implies that $\mathrm{t}(\beta) = \mathrm{t}(f)$ which ensures that $\mathrm{t}(\beta(\alpha_1, \dots, \alpha_n)) = \mathrm{t}(\beta) = \mathrm{t}(f) = \mathrm{s}(\alpha')(q)$, and thus, the composition at q is well-defined.

Let us see how this works for the examples of 3-pd's ($k = 2$) in the previous subsection. We are going to make the discussion easier to follow, by replacing the place-number p by the symbol which occurs at p in the given term; since the terms in the examples are *separated*, that is, have no repeated occurrences of symbols, this will not introduce ambiguity. Note that under this convention, with $f = (\mathrm{s}\alpha)\,(p) = \mathrm{t}\alpha$, (4) becomes

$$\mathrm{d}(\alpha \circ_f \beta) = (\mathrm{d}\alpha) \Box_f (\mathrm{d}\beta);$$

and the role of θ disappears (of course, for the general, non-separated case, the said simplification is not valid).

The 3-pd φ introduced in the previous subsection can be written in the following two ways:

$$\varphi = (y \circ_i (w \circ_h v)) \circ_b x = (y \circ_b x) \circ_i (w \circ_h v)$$

(compare $(3')$). Let us go through the definition of the domain of each of the constituent 3-pd's here.

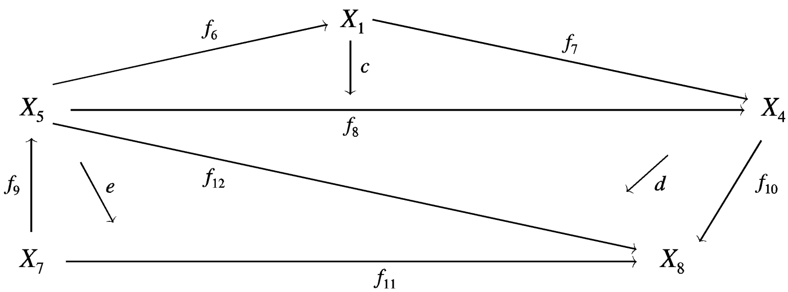
$$d(w \circ_h v) = (\mathrm{d}w) \Box_h (\mathrm{d}v) = \eta \Box_h \beta.$$

The decomposition of η at h has $\eta' = 1_{f_{11}}$ (we are writing η' for what was α' in the general case (6)); that is, now α' can be ignored in (6) and (7). (7) gives

$$\xi \underset{\text{def}}{=} \eta \Box_h \beta = \beta(f_{10}/f_{10},\ c(f_7, f_6)/f_8, f_9) = e(d(f_{10},\ c(f_7, f_6)), f_9),$$

that is,

ξ:



ξ is obtained by replacing h with β as it should.

Next,

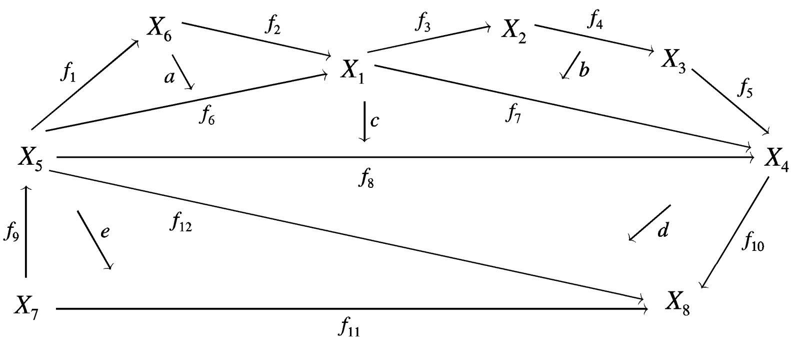
$$\mathrm{d}(y \circ_i (w \circ_h v)) = (\mathrm{d}y)\Box \circ_i \mathrm{d}(w \circ_h v)) = \lambda \Box_i \xi.$$

i is again the head-operation in λ, and so

$$\begin{aligned}\xi \underset{\text{def}}{=} \lambda \Box_i \xi = \xi(f_{10}/f_{10},\ b(f_5, f_6, f_3)/f_7, a(f_2, f_1)/f_6, f_9/f_9)\\ e(d(f_{10}, c(b(f_5, f_6, f_3),\ a(f_2, f_1)),\ f_9),\end{aligned}$$

that is,

ζ:



Note that ζ is the result of replacing i by ξ in λ.

Finally,

$$\mathrm{d}\varphi = \mathrm{d}(y \circ_i (w \circ_h v))\Box_b \mathrm{d}x = \zeta \Box_b \delta = \zeta \Box_b b = \zeta;$$

note that $\delta = b$ and when b is replaced by b, nothing happens. Of course, $\zeta = \gamma$, for our initial γ, so this calculation confirms what we said previously, in a "geometrical" language, about φ and γ.

Let us look at the other way of expressing φ. We have

$$\mathrm{d}(y \circ_b x) = \mathrm{d}y = \lambda,$$

for the same reason as in the preceding case. $\mathrm{d}(w \circ_h v) = \xi$ was calculated above. Then

$$\mathrm{d}\varphi = \mathrm{d}(y \circ_b x)\Box_i \mathrm{d}(w \circ_h v) = \lambda \Box_i \xi = \zeta = \gamma,$$

as it should be the case.

In this subsection, we described the way the domain-function $\mathrm{d} : P_{k+1} \to P_k$ is actually calculated, and saw that, in some examples at least, it agrees with the geometric intuition. However, thereby the problem of defining d is far from resolved. For instance, it is not clear that, in general, (4) is a compatible way of determining $\mathrm{d}\gamma$ for $\gamma \in P_{k+1}$; usually, γ can be written in more than one way as $\gamma = \alpha \circ_p \beta$, and we must see that the corresponding right-hand side expressions for $\mathrm{d}\gamma$ give the same result. There are other problems too. E.g., we have to see that if in (4), the left side is well-defined, so is the right side. Also note that we have not made any reference yet to the fact that d and c on C_{k+1} are determined so that the globularity condition (2) is satisfied. It is worth noting that that condition refers, besides d on P_{k+1}, also to d as defined on P_k. This suggests that d_k *must be defined recursively in* k.

1.6. Generalizing multicategories

The operation $\Box$ used in the last subsection looks like a multicategory operation. Let us start with $\boldsymbol{C}$, a free multicategory on $\mathscr{L}$ as we had in the last subsection for the purposes of defining the operation $\Box$ on the arrows of $\boldsymbol{C}$; let us use the same accompanying notation. We are going to define $\boldsymbol{D}$, a new multicategory, albeit in a somewhat generalized sense with respect to what we had above. $\boldsymbol{D}$ is called the *multicategory of function-replacement*. The arrows of $\boldsymbol{D}$ are the same as those of $\boldsymbol{C}$: $A(\boldsymbol{D}) = \mathrm{A}(\boldsymbol{C}) = \mathrm{A}$. The idea is to consider each $\alpha \in A$ to be a function not of its variable-occurrences, but of its function-symbol occurrences. The objects of $\boldsymbol{D}$ are pairs $(\vec{X}; Y)$ where $\vec{X} \in O^\star$ is a tuple of objects of $\boldsymbol{C}$, and Y is a single object: $\mathrm{O}(\boldsymbol{D}) = O^\star \times O$. If $\langle\alpha\rangle = \langle f_1, \dots, f_n\rangle$, then, by definition, $\mathrm{s}_{\boldsymbol{D}}(\alpha) = \mathrm{S}(\alpha) \underset{\mathrm{def}}{=} \langle \mathrm{T}f_1, \dots, \mathrm{T}f_n\rangle$ and $\mathrm{t}_{\boldsymbol{D}}(\alpha) \underset{\mathrm{def}}{=} \mathrm{T}(\alpha) = (\mathrm{s}\alpha, \mathrm{t}\alpha)$. The operation $\circ_{\boldsymbol{D}}$ is defined to be the operation $\Box$ explained in the previous subsection.

Before we say more on to what extent $\boldsymbol{D}$ is a multicategory, let us point out in what aspect it fails to be one.

Consider a language $\mathscr{L}$ in which we have sorts U, V, W, X, Y and function-symbols

$$f : \langle U, V\rangle \to W, \quad g : \langle X\rangle \to U, \quad h : \langle U, Y\rangle \to W, \quad i : \langle V\rangle \to Y. \tag{8}$$

Let $\beta = f(g(X), V)$, $\alpha = h(U, i(V))$, terms in $\mathrm{A}(\mathscr{L})$. We have $\alpha : \langle U, V\rangle \to W$, thus $\mathrm{T}(\alpha) = \mathrm{T}(f)$, and so $\beta \Box_1 \alpha = \beta \Box_f \alpha$ is well-defined. Now, we have $\beta = 1_W \circ_1 f(g(X), V)$ as the decomposition of β at 1 (at f), so

$$\beta \Box_1 \alpha = 1_W \circ_1 \alpha(g(X)/U, V/V) = \alpha(g(X)/U,\ V/V) = h(g(X),\ i(V)).$$

Also,

$$\langle\beta\rangle = \langle f, g\rangle, \quad S(\beta) = \langle \mathrm{T}f, \mathrm{T}g\rangle = \langle (\langle U, V\rangle; W),\ (\langle X\rangle; U)\rangle,$$

$$\langle\alpha\rangle = \langle h, i\rangle, \quad S(\alpha) = \langle \mathrm{T}h, \mathrm{T}i\rangle = \langle((\langle U, Y\rangle; W),\ (\langle V\rangle; Y)\rangle,$$

$$\langle\beta \square_1 \alpha\rangle = \langle h, g, i\rangle,$$

$$\mathrm{s}\langle\beta \square_1 \alpha\rangle = \langle \mathrm{T}h, \mathrm{T}g, \mathrm{T}i\rangle = \langle((\langle U, Y\rangle; W),\ (\langle X\rangle; U),\ (\langle V\rangle; Y)\rangle.$$

In a Lambek multicategory $\boldsymbol{E}$, if $\mathrm{s}_{\boldsymbol{E}}(\beta) = \langle b_1, \ldots, b_n\rangle$, $\mathrm{s}_{\boldsymbol{E}}(\alpha) = \langle a_1, \ldots, a_m\rangle$, then for $\beta \circ_p \alpha = \beta \circ_p^{(\boldsymbol{E})} \alpha$, we have

$$\mathrm{s}_{\boldsymbol{E}}(\beta \circ_p \alpha) = \langle b_1, \ldots, b_{p-1},\ a_1, \ldots, a_m,\ b_{p+1}, \ldots, b_n\rangle;$$

$\mathrm{s}_{\boldsymbol{E}}(\alpha)$ is inserted in the place of a_p; this is what we mean by *standard amalgamation* of the sources. The operation $\square$ is much like a multicategory composition, except for the standard amalgamation. If $\boldsymbol{D}$ had standard amalgamation, $\mathrm{S}(\beta \square_1 \alpha)$ would have to be the result of inserting $\langle \mathrm{T}h, \mathrm{T}i\rangle$ into $\langle \mathrm{T}f, \mathrm{T}g\rangle$ in the place of $\mathrm{T}f$, resulting in $\langle \mathrm{T}h, \mathrm{T}i, \mathrm{T}g\rangle$; but $\mathrm{S}(\beta \square_1 \alpha)$ is, rather, $\langle \mathrm{T}h, \mathrm{T}g, \mathrm{T}i\rangle \neq \langle \mathrm{T}h, \mathrm{T}i, \mathrm{T}g\rangle$.

We cannot hope that another simple "rule of amalgamation" applies, either. Suppose that in the above, $U = V$, but all other objects listed are distinct; so we have the previous example, still with non-standard amalgamation. But also, for $\beta' = f(U, g(X))$, $\beta' \square_f \alpha = h(U, i(g(X))$, and $\mathrm{S}(\beta' \square_f \alpha) = \langle h, i, g\rangle \neq \mathrm{S}(\beta \square_f \alpha)$, despite the fact that $\mathrm{S}(\beta') = \mathrm{S}(\beta)$. That is, the source of a composite does not depend just on the sources (and targets) of the composed arrows, unlike in the ordinary, Lambek, multicategory.

There is a generalized notion of "multicategory" which allows for "non-standard" amalgamation. In this we have, as part of the structure, so-called *amalgamating maps* $\psi = \psi[\beta, \alpha, p]$, $\varphi = \varphi[\beta, \alpha, p]$:

$$\mathrm{s}(\beta) \setminus p \xrightarrow{\psi} \mathrm{s}(\beta \circ_p \alpha) \xleftarrow{\varphi} \mathrm{s}(\alpha)$$

associated with any meaningful composition $(\beta, \alpha, p) \mapsto \beta \circ_p \alpha$, which puts together the source of $\beta \circ_p \alpha$ in a specific, but a priori undetermined, way from the source of β (take away the symbol at place p) and the source of α. The notation abbreviates the following: ψ is a map from the set $|\mathrm{s}(\beta)| - \{p\}$ to the set $|\mathrm{s}(\beta \circ_p \alpha)|$ (where $|s| = \mathrm{dom}(s)$, and $s \setminus p = s \restriction (|s| - \{p\})$) such that

$$\begin{tikzcd}
{|\mathrm{s}(\beta)| - \{p\}} \arrow[rr, "\psi"] \arrow[dr, "\mathrm{s}(\beta)\setminus p"'] \arrow[rr, phantom, shift right=4ex, "\mathrm{O}"] & & {|\mathrm{s}(\beta \circ_p \alpha)|} \arrow[dl, "\mathrm{s}(\beta\circ_p\alpha)"] \\
& \mathrm{O} &
\end{tikzcd}$$

and similarly for φ. In the standard case, the amalgamating maps correspond to the fact that in $\mathrm{s}(\beta \circ_p \alpha)$, "$\mathrm{s}\alpha$ is inserted in $\mathrm{s}\beta$ in the place p". In the generalized concept, there are coherence conditions on the amalgamating maps, one for each of the four laws: the unit laws, the associative law, and the commutative law. The above-described structure $\boldsymbol{D}$ is a multicategory in the generalized sense (in comparing this part with the official definition of Section 2, and the definition of $\boldsymbol{D}$ in Section 5, note that the concept being described here is a 1-*level* multicategory as opposed to the more general

2-level version given in those sections; we will comment on the reason for the 2-level version later in this introductory section).

The reason for the general concept of multicategory and for the particular multicategory $\boldsymbol{D}$ is to provide a concept under which $\mathrm{d}: P_{k+1} \to P_k$ becomes a morphism of multicategories. A *morphism* $F:\boldsymbol{C} \to \boldsymbol{D}$ *of multicategories* maps objects to objects, arrows to arrows, but, instead of being compatible strictly with the source-assignments, it has a system of transition isomorphisms $\theta_\alpha : |\mathrm{s}_{\boldsymbol{C}}(\alpha)| \xrightarrow{\cong} |\mathrm{s}_{\boldsymbol{D}}(F\alpha)| \quad (\alpha \in \mathrm{A}(\boldsymbol{C}))$ such that

$$\begin{array}{ccc} |\mathrm{s}_{\boldsymbol{C}}(\alpha)| & \xrightarrow{\theta_\alpha} & |\mathrm{s}_{\boldsymbol{D}}(F\alpha)| \\ {\scriptstyle \mathrm{s}_{\boldsymbol{C}}(\alpha)}\downarrow & \mathrm{O} & \downarrow{\scriptstyle \mathrm{s}_{\boldsymbol{D}}(F\alpha)} \\ \mathrm{O}(\boldsymbol{C}) & \xrightarrow[F]{} & \mathrm{O}(\boldsymbol{D}) \end{array}.$$

F is to preserve placed composition; in formulating this, the transition isomorphisms play a role: given that $\beta\circ_p\alpha$ is well-formed in $\boldsymbol{C}$, $F\beta\circ_q F\alpha$ for $q=\theta_\beta(p)$ is well-formed in $\boldsymbol{D}$; we require that $F(\beta\circ_p\alpha)=F\beta\circ_q F\alpha$. It is also required that the θ_α be compatible with the amalgamating maps.

There is a trade-off between amalgamating maps and transition isomorphisms. Given any morphism F: $\boldsymbol{C} \to \boldsymbol{D}$ of multicategories, there is a factorization of F,

$$\begin{tikzcd} \boldsymbol{C} \arrow[rr, "F"] \arrow[dr, "\Phi"', "\cong"] & & \boldsymbol{D} \\ & \boldsymbol{C}' \arrow[ur, "F'"'] & \end{tikzcd},$$

in which Φ is an isomorphism, and in fact, it is an identity on both objects and arrows; and F' is *strict*, its transition isomorphisms are all identities. In other words, by changing the domain to an isomorphic copy, albeit with "twisted" amalgamating maps, it is possible to turn a morphism into a strict one.

1.7. Constructing higher-dimensional cells

We are ready to summarize the construction of higher-dimensional cells. Assuming that we have a set C_k of k-cells for $k=0,1,\ldots,n$, and we have defined k-pd's for the same k's, with domain and codomain maps $\mathrm{d}: P_{k+1} \to P_k$, $\mathrm{c}: P_{k+1} \to C_k$, we introduce $(n+1)$-cells $a \in C_{n+1}$ by declaring each $\mathrm{d}a = \mathrm{d}_{n+1}(a)$ and $\mathrm{c}a = \mathrm{c}_{n+1}(a)$ to be a specific n-pd $\alpha = \mathrm{d}a$, resp. n-cell $b = \mathrm{c}a$ *such that* $\mathrm{d}\alpha = \mathrm{d}b$, $\mathrm{c}\alpha = \mathrm{c}b$, that is,

$$\mathrm{dd}a = \mathrm{dc}a, \quad \mathrm{cd}a = \mathrm{cc}a. \tag{9}$$

We let $\boldsymbol{D}_n$ be the multicategory of function-replacement based on $\boldsymbol{C}_n$, the free multicategory with arrows the n-pd's, and $\boldsymbol{C}^0_{n+1}$ be the free multicategory with standard amalgamation, and with objects the n-cells, and generating arrows the $(n+1)$-cells just declared; in other words, $\boldsymbol{C}^0_{n+1} = \mathscr{F}(\mathscr{L})$ where $\mathrm{O}(\mathscr{L}) = C_n$, $\mathrm{L}(\mathscr{L}) = C_{n+1}$, and in which $\mathrm{s}_{\mathscr{L}}(a) = \langle \mathrm{d}a \rangle$, $\mathrm{t}_{\mathscr{L}}(a) = \mathrm{c}a (a \in C_{n+1})$. P_{n+1} is the set of arrows in $\boldsymbol{C}^0_{n+1}$. The main step in the definition is to define $\mathrm{d}^0 = \mathrm{d}^0_{n+1} : \boldsymbol{C}^0_{n+1} \to \boldsymbol{D}_n$ by the freeness of

$\boldsymbol{C}^0_{n+1}$ so as to extend the determination of d on C_{n+1}. For this, it is crucial that $\boldsymbol{C}^0_{n+1}$, although it is defined as a Lambek multicategory, it remains free on $\mathscr{L}$ in the larger category of all multicategories with possibly non-standard amalgamation. Finally, we alter $\boldsymbol{C}^0_{n+1}$ to the *isomorphic copy* $\boldsymbol{C}_{n+1}$ by "twisting" the amalgamation maps to ensure that $\mathrm{d} : \boldsymbol{C}_{n+1} \to \boldsymbol{D}_n$ is strict. As a result, we get the *main formula* saying that

$$\mathrm{d}(\beta \circ_p \alpha) = (\mathrm{d}\beta) \Box_p (\mathrm{d}\alpha) \tag{10}$$

every time $\beta \circ_p \alpha$ is a meaningful composition in $\boldsymbol{C}_{n+1}$.

Let us see the effect of the above general procedure for some particular 3-cells and 3-pd's. In what follows, $U, V, W, \ldots$ denote 1-cells, $f, g, h, \ldots$ 2-cells, u, v, 3-cells; Greek letters are used to denote pd's of various dimensions.

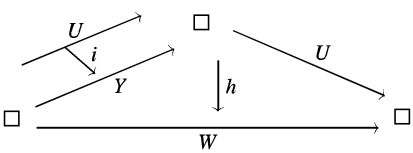
We adopt a single 0-cell that we indicate by $\Box$; the 1-cells U, W, X, Y are all like $\Box \to \Box$. The 2-cells f, g, h, i are as in (8), but we stipulate that $V = U$. We add

$$k : \langle X, U \rangle \to W, \quad l : \langle U, X \rangle \to W.$$

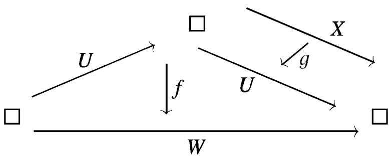
We are assuming that $U = V$, but other 1-cells denoted differently are distinct.

Consider the 2-pd's $\alpha = h(U, i(U))$ and $\beta = f(g(X), U)$:

α :

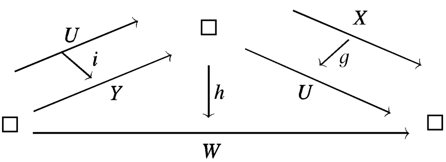


β :



We have $\beta \Box_1 \alpha = h(g(X), i(U))$:

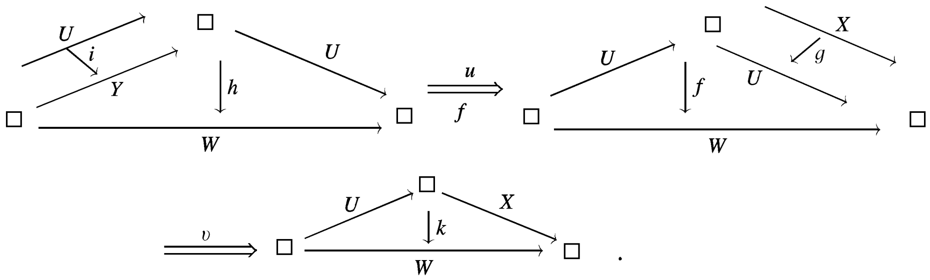
$\beta \Box_1 \alpha$:



. (11)

We introduce the 3-cells u and v by declaring $\mathrm{d}u = \alpha$, $\mathrm{c}u = f$ and $\mathrm{d}v = \beta$, $\mathrm{c}v = k$; the globularity conditions (9) are satisfied. We let

$$\psi = v \circ_1 u = v(u(h, i), g):$$



We have

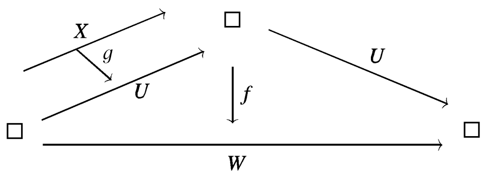
$$\mathrm{d}\psi = \mathrm{d}(v \circ_1 u) = (\mathrm{d}v) \square_1 (\mathrm{d}u) = \beta \square_1 \alpha \quad (\text{see } (11)),$$

so

$$\mathrm{s}_{C_3}(\psi) = \langle \mathrm{d}\psi \rangle = \langle h, g, i \rangle.$$

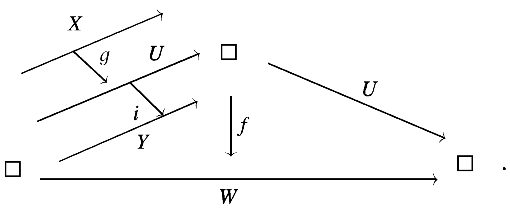
Now, look at

$$\beta' = f(U, g(X)):$$



and

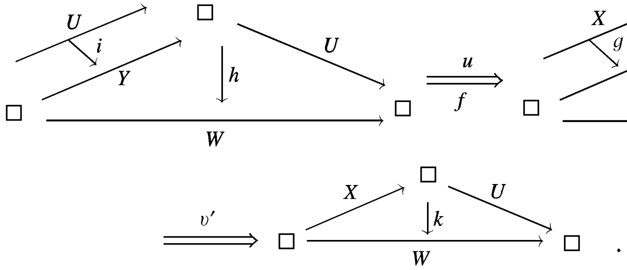
$$\beta' \square_1 \alpha = h(U, i(g(X))):$$



(12)

We let $v' \in C_3$ with $\mathrm{d}v' = \beta'$, $\mathrm{c}v' = k$ ($\mathrm{d}\beta' = \mathrm{d}k$, $\mathrm{c}\beta' = \mathrm{c}k$ hold), and

$$\psi' = v' \circ_1 u = v'(u(h, i), g):$$



We have

$$\mathrm{d}\psi' = \beta' \Box_1 \alpha \quad \text{(see (12))}$$

and

$$\mathrm{s}_{\boldsymbol{C}_3}(\psi') = \langle \mathrm{d}\psi' \rangle = \langle h, i, g \rangle.$$

We just have to get used to the fact that

$$\mathrm{s}_{\boldsymbol{C}_3}(v(u(h,i),g)) = \langle h, g, i \rangle$$

and

$$\mathrm{s}_{\boldsymbol{C}_3}(v'(u(h,i),g)) = \langle h, i, g \rangle$$

at the same time. Of course, this does not look so surprising if we look at the *full* representations of the two 3-pd's $\psi = v(u(h,i),g)$ and $\psi' = v'(u(h,i),g)$, which are different "geometrically".

1.8. Introducing two levels of objects

Some remarks concerning the "2-leveled" version for the notion of multicategory, for whose definition we refer to Section 2. This is introduced purely for technical convenience. The 2-leveled notion packs more structure into the multicategory $\boldsymbol{D}$ of function-replacement, structure that is already there "naturally". For instance, instead of having the source of α as $\mathrm{s}_{\boldsymbol{D}}(\alpha) = \langle \mathrm{T}(f_1), \ldots, \mathrm{T}(f_n) \rangle$, we have it, in the 2-leveled version of $\boldsymbol{D}$, as $\mathrm{s}_{\boldsymbol{D}}(\alpha) = \langle f_1, \ldots, f_n \rangle = \langle \alpha \rangle$. The effect is to *restrict the scope* of the composition operation $\Box$; composition in the 2-leveled version remains the same as in the 1-leveled version, but it is defined for a subset of the domain of the 1-leveled composition. For $\gamma, \delta \in \boldsymbol{D}$, the composite $\delta \Box_p \gamma$ is meaningful, in the 2-leveled version, if and only if $p \in |\langle \delta \rangle|$, and for $f = \langle \delta \rangle(p)$, we have $\mathrm{d}f = \mathrm{d}\gamma$ and $\mathrm{c}f = \mathrm{c}\gamma$. This is in fact the case exactly when the function-replacement composite is *meaningful geometrically*. Under the 1-leveled version, the multicategory $\boldsymbol{D}$ has composites that cannot be realized geometrically in Euclidean space.

The 2-leveled concept helps technically. An example is the equality $\mathrm{d}\alpha = \langle \alpha \rangle$ holding for all $\alpha \in \boldsymbol{C}_{n+1}$. This is immediate if d is defined by the freeness of $\boldsymbol{C}_{n+1}$ with respect to the 2-leveled version of "multicategory"; it would require additional arguments if we used the 1-leveled version.

1.9. *Final remarks*

Obviously, for any fixed n, n-graphs are the objects of a category of the form $\mathrm{Set}^{\mathrm{g}_n}$; here, g_n is the category whose shape is given in (1). It turns out that n-dimensional multitopic sets, with a natural notion of morphism, also form a category of the form Set^E with a suitable exponent category $E = \mathrm{Mlt} \restriction n$. In this case, the exponent category $E = \mathrm{Mlt} \restriction n$, the category of n-dimensional *multitopes*, is less easy to describe. In fact, there is, apparently, no other way of describing $\mathrm{Mlt} \restriction n$ than by the same recursive process that serves defining multitopic sets in general. The objects of $\mathrm{Mlt} \restriction n$ are the same as the pasting diagrams (elements of the P_n-component) in the *terminal* n-dimensional multitopic set, the one that has exactly one cell in each possible type (where "type" means domain/codomain pair; in fact, here "domain" suffices). The arrows of $\mathrm{Mlt} \restriction n$ are more difficult to explain. The definition of the $\mathrm{Mlt} \restriction n$ and the proof of their connection to multitopic sets in general are given in Section 7.

The fact just stated is the justification for the name "multitopic set". It is similar construction to "simplicial set", with "simplices" in the background, and also to "opetopic set" of [3], based on "opetopes", in which *operads*, the basic abstract concept for [3], are referred to. We copied and modified "opetope" and "opetopic set" of [3], bearing in mind multicategories as the basic abstract concept, replacing operads.

We note that "higher dimensional (or: n-dimensional) multicategory", a term that may seem at first to be the appropriate one for our concept of multitopic set, is in fact incorrect and misleading. "Higher-dimensional multicategory" would rightly be expected to generalize "multicategory"; however, in our multitopic sets only *special* multicategories figure, namely, the free ones, and another particular kind, the multicategories of function replacement, closely tied to the free multicategories. For multitopic sets, *particular* multicategories are used as a tool to describe a specific geometric arrangement, that of cells of various dimensions fitting together in pasting diagrams. Of course, this is similar to the use of operads in [3].

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