Irreducibility Conditions for Continuous-time Multi-input Multi-output Nonlinear Systems

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***Abstract*— The purpose of this paper is to present necessary and sufficient condition for irreducibility of continuous-time nonlinear multi-input multi-output system. The condition is presented in terms of the greatest common left divisor of two polynomial matrices related to the input-output equations of the system. The basic difference is that unlike the linear case the elements of the polynomial matrices belong to a *non-commutative* polynomial ring. This condition provides a bases for finding the equivalent minimal irreducible representation of the i/o equations which is a suitable starting point for constructing an observable and accessible state space realization.**

***Keywords*—nonlinear control systems, continuous-time systems, irreducibility conditions, accessibility, computer algebra system**

I. Introduction

Controllability is one of the fundamental concepts of modern mathematical control theory. In the nonlinear case, there are many different definitions of controllability. This is in contrast to linear systems, where many notions of controllability are equivalent to the Kalman rank condition. Among many different controllability notions, the accessibility property stands out as a computationally tractable and intuitively clear concept. In plain words, the special notion of controllability, (forward) accessibility refers to the case where every nonconstant function of the state is eventually influenced by the control variable of the system and hence cannot satisfy any autonomous differentiation.

Moreover, accessibility notion plays a crucial role in the minimal realization problem and is tightly related to the notion of irreducibility of a set of i/o differential equations. Namely, one ends up with minimal, i.e. accessible and observable state-space realization iff one starts with an irreducible i/o equation [1]. It is here where our motivation to study the irreducibility problem originates from.

The reduction problem of the *single-input single-output* equations has been studied in [2], where the irreducibility criterion was presented in terms of the greatest common left factor of two polynomials, related to the i/o equation. Note that in the discrete-time domain the polynomial irreducibility condition [3] has been extended to the multi-input multi-output case [4]. The purpose of this paper is to adapt the results of [4] to the continuous time multi-input multi-output case.

The paper is organized as follows. Section II describes the differential field and a polynomial matrix representation associated to the nonlinear systems. Using the polynomial matrix description, in Section III necessary and sufficient condition for irreducibility of nonlinear multi-input multi-output systems is given. In Section IV, implementation of the algorithm to check irreducibility in computer algebra system *Mathematica* is discussed and illustrative example is given.

II. Polynomial matrix description

Consider a continuous-time multi-input multi-output nonlinear system described by the set of higher order i/o differential equations relating the inputs u_j, $j = 1, \ldots, m$, the outputs y_i, $i = 1, \ldots, p$, and a finite number of their time derivatives:

$$\begin{array}{c} y_i^{(t+n_i)} = f_i(y_s, \ldots, y_s^{(n_{is}-1)}, u_k, \ldots, u_k^{(s_{ik})}), \\ s = 1, \ldots, p, \; k = 1, \ldots, m), \; i = 1, \ldots, p. \end{array} \tag{1}$$

We assume that the following assumptions hold for system (1).

Assumption 1: The maps f_i, $i = 1 \ldots, p$ are supposed to be meromorphic functions of their arguments.

Assumption 2: System (1) is strictly proper, i. e. that $s_{ik} < n_i$, for $i = 1 \ldots, p$, $k = 1, \ldots, m$.

Assumption 3: System (1) *is in a canonical form*, which means that $n_{is} < \min(n_i, n_s)$ and $n_1 + \ldots + n_p := n$ is the order of the system.

Assumption 3 implies that whenever (1) admits a Kalmanian realization, the indices n_i associated to each output y_i, $i = 1, \ldots, p$, are the observability indices of any observable state-space realization of order n. The form (1) is an extension of the echelon canonical matrix fraction description, introduced in [5] for linear systems. If the input-output equations are not in the form (1), one can apply the transformations in [6], [7] to bring the system into this form.

Assumption 4: $f_i(0, \ldots, 0) = 0$. We make this assumption because the mathematical tools we are going to employ require that instead of working with the equations themselves we work with their differentials and will not able to distinguish between the equations $f_i(0, \ldots, 0) = c$ for different c values.

Let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of the variables $\{y_i, \dots, y_i^{(n_i-1)}, i = 1, \dots, p,\ u_k^{(l)},\ \ k = 0, \dots, m,\ l \geq 0\}$. Over the field $\mathcal{K}$ one can define a vector space, $\mathcal{E} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$ spanned by the differentials of the elements of $\mathcal{K}$. Consider a one-form $\omega \in \mathcal{E}$: $\omega = \sum_i \alpha_i \mathrm{d}\varphi_i$, $\alpha_i, \varphi_i \in \mathcal{K}$. Its derivative $\dot{\omega}$ is defined by $\dot{\omega} = \sum_i \dot{\alpha}_i \mathrm{d}\varphi_i + \alpha_i \mathrm{d}\dot{\varphi}_i$

A. Non-commutative ring of polynomials

The differential field $\mathcal{K}$ and the differentiation operator induce a ring of polynomials in the differentiation operator $s = \mathrm{d}/\mathrm{d}t$, denoted by $\mathcal{K}[s]$. A polynomial $p(s) \in \mathcal{K}[s]$ is written as

$$p(s) = a_m s^m + a_{m-1} s^{m-1} + \dots + a_1 s + a_0, \qquad (2)$$

where $a_i \in \mathcal{K}$ for $0 \leq i \leq m$. The degree of the polynomial $p(s)$ is m if $a_m \not\equiv 0$, and $p(s)$ is called monic if $a_m = 1$. Each polynomial $p(s) \in \mathcal{K}[s]$ is a mapping of $\mathcal{E}$ into itself. To evaluate $p(s)$ at any $\omega \in \mathcal{E}$, note that for nonlinear systems, the coefficients a_i in (2) are no more constants as in the linear case but meromorphic functions in a finite number of variables $y_s^{(l)}$, $s = 1, \dots, p$ and $u_k^{(l)}$, $k = 1, \dots, m$, $l \geq 0$. The latter implies that an element $a \in \mathcal{K}$ does not commute with the differentiation operator s, i. e. $a \cdot s \neq s \cdot a$ as $a \cdot s\mathrm{d}y_s = a\mathrm{d}\dot{y}_s$ is not equal to $s \cdot a\mathrm{d}y_s = a\mathrm{d}\dot{y}_s + \dot{a}\mathrm{d}y_s$. Since the multiplication between the differentiation operator s and an element $a \in \mathcal{K}$ is not commutative and can be defined by the following rule

$$s \cdot a = a \cdot s + \dot{a}, \qquad (3)$$

the ring $\mathcal{K}[s]$ thus defined is a non-commutative ring. If the multiplication is defined by (3), it is proved to satisfy both the left and right Øre condition, i. e. to be Øre ring [8]. If the non-commutative ring satisfies the Øre condition, one can construct the division ring of fractions, a process exactly like that of constructing the field of rational numbers from the ring of integers.

Note that all the other algebraic operations in the ring satisfy the operations in the field (of meromorphic functions) $\mathcal{K}$. The ring of polynomials $K[s]$ has all the properties of field except for the inverse, i. e. the inverse of a polynomial of degree one or more is not a polynomial.

In the differential field $\mathcal{K}$, there are no zero divisors in the sense that if $\varphi_1, \varphi_2 \in \mathcal{K}$ with $\varphi_1 \neq 0$, $\varphi_2 \neq 0$ then $\varphi_1\varphi_2 \neq 0$. Thus for three polynomials $p(s), p_1(s), p_2(s) \in \mathcal{K}[s]$, with $\deg p_1(s) = d_1 > 0$ and $\deg p_2(s) = d_2 > 0$, such that $p(s) = p_1(s)p_2(s)$, the degree of $p(s)$ satisfies $\deg p(s) = \deg p_1(s) + \deg p_2(s) = d_1 + d_2$.

The left division algorithm is based on the fact that given two polynomials, $p_1(s)$ and $p_2(s)$ with $\deg p_1(s) > \deg p_2(s)$, there exists a unique (right) quotient polynomial $\gamma_1(s)$ and a unique remainder polynomial $p_3(s)$ such that

$$p_1(s) = p_2(s)\gamma_1(s) + p_3(s),$$

$$\deg p_3(s) < \deg p_2(s).$$

B. Polynomial system description

We now represent the nonlinear system (1) in terms of two polynomial matrices, with the polynomial entries in the polynomial ring, which is Øre ring. For that we apply the differentiation operation to (1) and use the notations $\mathrm{d}y_s^{(j)} = s^j \mathrm{d}y_s, \mathrm{d}u_k^{(r)} = s^r \mathrm{d}u_k$ to obtain

$$P(s)\mathrm{d}y = Q(s)\mathrm{d}u \qquad (4)$$

where $P(s)$ and $Q(s)$ are $p \times p$ and $p \times m$-dimensional matrices respectively, whose elements $p_{ij}, q_{ij} \in \mathcal{K}[s]$:

$$p_{is}(s) = s^{n_i} - \sum_{j=0}^{n_{is}} \frac{\partial f_i}{\partial y_s^{(j)}} s^j$$

$$q_{ik}(s) = \sum_{r=0}^{s_{ik}} \frac{\partial f_i}{\partial u_k^{(r)}} s^r$$

and $\mathrm{d}y = [\mathrm{d}y_1, \dots, \mathrm{d}y_p]^T$, $\mathrm{d}u = [\mathrm{d}u_1, \dots, \mathrm{d}u]^T$.

C. Polynomial matrices with elements in $\mathcal{K}[s]$

We now consider a class of matrices $P(s)$ whose elements are polynomials $p(s) \in \mathcal{K}[s]$ of finite, but unbounded degree. We write $\mathcal{K}^{p \times q}[s]$ for the set of $p \times q$ matrices with entries in $\mathcal{K}[s]$. The purpose of this subsection is to show that like in the linear case where the polynomials have real coefficients, the polynomial matrix with entries in $K[s]$ can be transformed by a sequence of elementary column operations to lower left triangular form. This result allows us to obtain the irreducibility criterion.

Definition 1: The following three elementary *column (row) operations* $E(s)$ on the polynomial matrix $P(s)$ are defined

1) Interchange of columns (rows) i and j.
2) Multiplication of column (row) i by nonzero scalar in $\mathcal{K}$.
3) Replacement of column (row) i by itself plus any polynomial multiplied by any other column (row) j.

Definition 2: A **unimodular matrix** $U(s)$ is defined as any square matrix which can be obtained from the identity matrix $\mathcal{I}$ by a finite number of elementary row and column operations on $\mathcal{I}$: $U(s) = E_k(s)E_{N-1}(s) \dots E_1(s)\mathcal{I}E_{k+1}(s) \dots\ E_{k+r}(s)$.

Any sequence of elementary row operations on $P(s)$ is equivalent to premultiplication (left multiplication) of $P(s)$ by an appropriate unimodular matrix $U_L(s)$. Similarly, any sequence of elementary column operations on $P(s)$ is equivalent to post multiplication (right multiplication) of $P(s)$ by an appropriate unimodular matrix $U_R(s)$.

Definition 3: Two polynomial matrices $P(s)$ and $\hat{P}(s)$ will be called **column equivalent** iff one of them can be obtained from the other by a sequence of elementary column operations.

$P(s)$ is thus column equivalent, to $\hat{P}(s)$ if and only if $P(s) = \hat{P}(s)U_R(s)$ where $U_R(s)$ is a unimodular matrix.

Theorem 1: Any $p \times q$ ($p \leq q$) polynomial matrix $P(s)$ is column equivalent to the lower left triangular matrix shown below, i. e. one can always find a sequence of elementary column operations which reduces $P(s)$ to lower left triangular form where $\hat{P}(s) = [G_L(s) \;\; 0]$ and

$$G_L(s) = \begin{bmatrix} g_{11}(s) & & & \\ g_{21}(s) g_{22}(s) & & & \\ \vdots & & & \\ g_{p1}(s) g_{p2}(s) \ldots g_{pp}(s) & & & \end{bmatrix}. \tag{5}$$

Furthermore, in the above form, the polynomials $g_{k1}(s), \ldots, g_{k,k-1}(s)$ are of lower degree than $g_{kk}(s)$ for all $k = 1, \ldots, p$ if $\deg g_{kk}(s) > 0$, and are all zero, if $g_{kk}(s)$ is a nonzero scalar in $\mathcal{K}$.

Proof. If the first row of $P(s)$ is not identically zero we can choose a polynomial of least degree from its elements and, by permutation of the columns, make it the new $(1,1)$ entry $\tilde{p}_{11}(s)$. We then apply the Euclidean (left) **division algorithm** to every other nonzero entry in the first row i. e. we divide every other nonzero element $\tilde{p}_{1i}(s)$ in the first row by $\tilde{p}_{11}(s)$ obtaining the quotients $\tilde{q}_{1i}(s)$ and remainders $\tilde{r}_{1i}(s)$ according to the relationship

$$\tilde{p}_{1i}(s) = \tilde{p}_{11}(s)\tilde{q}_{1i}(s) + \tilde{r}_{1i}(s)$$

where either $\tilde{q}_{1i}(s) = 0$ or $\deg \tilde{r}_{1i}(s) < \deg \tilde{p}_{11}(s)$. We then subtract from each nonzero ith column, the first column multiplied by $\tilde{q}_{1i}(s)$. If not all of the remainders $\tilde{r}_{1i}(s)$ are zero, we choose one of the least degree and make it the new $(1,1)$ entry by another permutation of the columns. The purpose of repeating this process as many times as necessary is to continually reduce the degree of the polynomial element in the $(1,1)$ entry. Since the degree of the $(1,1)$ entry is finite, this repeated process must end at some stage, in particular, when all of the remaining elements of the first row are identically zero.

We next consider the second row of this matrix and, ignoring the first column for the moment, apply the above procedure to the elements beginning with the second column and second row. In this way we zero all the elements to the right of the $(2,2)$ entry.

If the $(2,1)$ element is of equal or higher degree than that of the $(2,2)$ element, the division algorithm can be employed to reduce the $(2,1)$ element to the remainder term associated with the division of the $(2,1)$ entry by the $(2,2)$ entry, or to zero if both elements are scalars. Continuing in this manner, with the elements beginning with the third column and third row next, we eventually reduce $P(s)$ to the appropriate form. ∎

Definition 4: If three polynomial matrices satisfy the relation $P(s) = C_L(s)Q(s)$, then $C_L(s)$ is called a left divisor of $P(s)$ and $P(s)$ is called a right multiple of $C_L(s)$. A greatest common left divisor (gcld) of two polynomial matrices $P(s)$ and $Q(s)$ is a common left divisor which is a right multiple of every common left divisor of $Q(s)$ and $P(s)$.

In view of this definition we can now state and establish

Theorem 2: Consider the pair $P(s)$, $Q(s)$ of polynomial matrices which have the same number of rows. If the composite matrix $[P(s) \vdots Q(s)]$ is reduced to lower left triangular form $[G_L(s) \vdots 0]$ as in Theorem 1, then $G_L(s)$ is a common greatest left divisor of $P(s)$ and $Q(s)$.

The greatest common left divisor is, in general, not unique, but it can be made unique by requiering that the polynomials in the diagonal of $G_L(s)$ are monic. The concept of a gcld now enables us to extend to the matrix case the notion of a pair of relatively left prime polynomials.

Definition 5: A pair $\{P(s), Q(s)\}$ of polynomial matrices which have the same number of rows is said to be relatively left prime if and only if their gcld are unimodular matrices.

The notion of a pair of relatively left prime polynomial matrices implies the inability to factor some non-unimodular matrix from the left side of both members of the pair.

III. Irreducibility of the I/O equations

Definition 6: [1] A function φ_r in $\mathcal{K}$ is said to be an autonomous variable for a system Σ of the form (1) if there exist an integer $\mu \geq 1$ and the non-zero meromorphic function F so that

$$F(\varphi_r, s\varphi_r, \ldots, s^{\mu}\varphi_r) = 0. \tag{6}$$

The notion of autonomous variable can be used to define (local) irreducibility of the nonlinear system (1) as follows.

Definition 7: The system (1) is said to be irreducible (forward accessible) if there does not exist any non-zero autonomous variable in $\mathcal{K}$.

Remark. φ_r denotes a variable as well as a function of y, u, and their time derivative. While φ_r is a function of $y^{(i)}$ and $u^{(j)}$, $0 \leq i \leq n-\mu$, $0 \leq j \leq n-\mu$, it is governed by the homogeneous differential equation (6). For any initial condition, the solution φ_r is uniquely determined by this homogeneous differential equation and is consequently independent of the external input u. In this sense φ_r is an autonomous variable which represents the lack of controllability of the nonlinear system. It follows that the nonlinear system (1) is irreducible (accessible) if and only if it contains no autonomous variables. Otherwise, it can be reduced.

Theorem 3: The nonlinear system (1) is reducible if and only if the polynomial matrices $P(s)$ and $Q(s)$ in (4) are not relatively left prime.

Proof. Sufficiency. Assume that the polynomial matrices $P(s)$ and $Q(s)$ are not relatively prime. The latter means that $P(s)$ and $Q(s)$ have a common left divisor $G_L(s)$ in the form (5), which is not unimodular, such that equation (4) can be written as

$$\begin{aligned} &P(s)\mathrm{d}y - Q(s)\mathrm{d}u = \\ &G_L(s)[\tilde{P}(s)\mathrm{d}y - \tilde{Q}(s)\mathrm{d}u] = 0, \end{aligned} \tag{7}$$

or equivalently,

$$G_L(s)\omega = 0$$

where $\omega = [\omega_1, \ldots, \omega_p]^T$ is a column-vector of differential one-forms. Using the results of [1], one can show that the one-forms, being the elements of $\tilde{P}(s)\mathrm{d}y - \tilde{Q}(s)\mathrm{d}u$ are exact one-forms (or can be made exact by multiplying the integrating factor), so we can define $\mathrm{d}\varphi_{ri} = \omega_i$ for $i = 1, \ldots, p$. From (7) we now obtain (for $\mathrm{d}\varphi_r = [\mathrm{d}\varphi_{r1}, \ldots, \mathrm{d}\varphi_{rp}]^T$)

$$G_L(s)\mathrm{d}\varphi_r = 0$$

which, because of non-unimodularity of $G_L(s)$, will imply the existence at least one F such that (6) holds. Hence the system is not irreducible.

Necessity. Suppose that the nonlinear system (1) is reducible. According to Definition 7, there exist functions $\varphi_r \in \mathcal{K}$ and $F \in \mathcal{K}$, such that (6) holds.

We can differentiate the functions φ_r and F in (6) and use $\mathrm{d}y_j^i = s^i \mathrm{d}y_j$, $\mathrm{d}u_k^i = s^i \mathrm{d}u_k$, $\mathrm{d}F^m = s^m \mathrm{d}F$ to obtain

$$\begin{array}{rcl} \mathrm{d}\varphi_r & = & \displaystyle\sum_{j=1}^{p}\sum_{i=0}^{n-\mu} \frac{\partial \varphi_r}{\partial y_j^{(i)}} \mathrm{d}y_j^{(i)} \\ & + & \displaystyle\sum_{k=1}^{m}\sum_{j=0}^{n-\mu} \frac{\partial \varphi_r}{\partial u_k^{(j)}} \mathrm{d}u_k^{(j)} \\ & = & \tilde{p}(s)\mathrm{d}y - \tilde{q}(s)\mathrm{d}u \\ \mathrm{d}F & = & \displaystyle\sum_{k=0}^{\mu} \frac{\partial F}{\partial \varphi_r^k} \mathrm{d}\varphi_r^k = g(s)\mathrm{d}\varphi_r = \\ & = & g(s)[\tilde{p}(s)\mathrm{d}y - \tilde{q}(s)\mathrm{d}u]. \end{array} \qquad (8)$$

The remaining part of the proof relies on the fact that the $p+1$ differential forms defined by p rows of $P(s)\mathrm{d}y - Q(s)\mathrm{d}u$ and $\mathrm{d}F$, are not independent.

IV. Symbolic implementation using *Mathematica*

The procedure for checking irreducibility, described in the paper, has been implemented in *Mathematica* in the form of the function **Reduction**. By calling the function **Reduction** one can check if the system is in the irreducible form and find the reduced i/o equations. Next example illustrates implementation of the function **Reduction** step by step .

Example 1: Consider the system, described by the input/output differential equations

$$\begin{array}{rcl} \ddot{y}_1 & = & y_1\dot{u}_1 + u_1\dot{y}_1 - 3\dot{y}_1 + 3y_1u_1 \\ \ddot{y}_2 & = & u_2\dot{y}_1 + y_1\dot{u}_2 - \dot{y}_1 - 2\dot{y}_2 + 2u_2y_1 - 2y_1 \end{array} \qquad (9)$$

Next we show which functions and in which sequence **Reduction** calls, together with intermediate results. To make the computations with the system of equations (9) more comfortable, one first assigns the name **ioeq** to it. Next function differentiates the system equations and presents the result in the form $P(s)\mathrm{d}y = Q(s)\mathrm{d}u$:

MakeFMF[ioeq]; PQ = MapThread[Join, {P, Q} = FromEquationsToMatrix[ioeq]];

MatrixForm[P]

$$\begin{pmatrix} s^2 + (3 - u_1)s - \dot{u}_1 + 3u_1 & 0 \\ (1 - u_2)s + 2 - 2u_2 - \dot{u}_2 & s^2 + 2s \end{pmatrix}$$

MatrixForm[Q]

$$\begin{pmatrix} y_1 s + (\dot{y}_1 + 3y_1) & 0 \\ 0 & \dot{y}_1 + 2y_1 + y_1 s \end{pmatrix}$$

According to Theorem 2, one can find a common greatest left divisor $G_L(s)$ of $P(s)$ and $Q(s)$ by reducing the composite matrix $[P(s) \vdots Q(s)]$ into the lower left triangular form $[G_L(s) \vdots 0]$. For that purpose the function **MatrixForm[G = LowerLeftTriangularMatrix[PQ]]** has been developed, which, for our system, gives the following result

$$\begin{pmatrix} y_1 s + (\dot{y}_1 + 3y_1) & 0 \\ 0 & y_1 s + (\dot{y}_1 + 2y_1) \end{pmatrix}.$$

Since $G_L(s)$ is not a unimodular matrix (elements of main diagonal depend on the differentiation operator s), the system (9) can be reduced; that is we can find the polynomial matrices $P_1(s)$ and $Q_1(s)$ defining the reduced system $P_1(s)\mathrm{d}y = Q_1(s)\mathrm{d}u$ by solving the equations $P(s) = G_L(s)P_1(s)$ and $Q(s) = G_L(s)Q_1(s)$ for $P_1(s)$ and $Q_1(s)$, respectively:

MatrixForm[P1 = PolynomialRightQuotient[P, Q]]

$$\begin{pmatrix} \dfrac{1}{y_1}s - \dfrac{u_1}{y_1} & 0 \\ \dfrac{1 - u_2}{y_1} & \dfrac{1}{y_1}s \end{pmatrix} \qquad (10)$$

MatrixForm[Q1 = PolynomialRightQuotient[P, Q]]

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (11)$$

On the basis of the matrices (10) and (11), next function finds two one-forms, the integration of which, will lead to the reduced i/o equations.

DifferentialForm[sp = FromMatrixToSpan [P1, Q1, Join[Yt, Ut],t]];

$$\{-u_1\mathrm{d}y_1 + \mathrm{d}\dot{y}_1 - y_1\mathrm{d}u_1, (1 - u_2)\mathrm{d}y_1 + \mathrm{d}\dot{y}_2 - y_1\mathrm{d}u_2\}$$

For this example those one-forms can be easily integrated to provide the reduced input-output differential equations: **IntegrateOneForms[sp]**

$$\{-y_1u_1 + \dot{y}_1, -u_2y_1 + \dot{y}_2\}.$$

So, the reduced i/o equations are

$$\begin{array}{rcl} \dot{y}_1 & = & y_1u_1 \\ \dot{y}_2 & = & u_2y_1 - y_1 \end{array}$$

Example 2: Consider the system, described by the input/output differential equations

$$\begin{array}{rcl} \ddot{y}_1 & = & \dot{y}_2u_1 - y_1u_1u_2 + \dot{u}_1u_2 + u_1\dot{u}_2 \\ \ddot{y}_2 & = & \dot{u}_1u_2 - 2\dot{y}_2 + y_1\dot{u}_2 + 2y_1u_2 \end{array}$$

By assigning the name **ioeq2** to this system and calling the function **Reduction[ioeq2]** one can get the following reduced i/o equations

$$\begin{array}{rcl} \ddot{y}_1 & = & u_2\dot{u}_1 + u_1\dot{u}_2 + u_1\dot{y}_2 - u_1u_2y_1 \\ \dot{y}_2 & = & u_2y_1 \end{array}$$

Unlike in the previous example lower left triangular matrix $G_L(s)$ has a very complicated form and can not be shown here due to the limitations imposed on the layout of the page.

Example 3: Consider the system, described by the input/output differential equations

$$\begin{array}{rcl}\ddot{y}_1 & = & \dot{y}_2u_1 - y_1u_1u_2 + \dot{u}_1u_2 + u_1\dot{u}_2 \\ \ddot{y}_2 & = & u_1u_2y_1 - y_1\dot{u}_2 + u_2\dot{y}_1 - y_1\dot{y}_1\end{array}$$

and assign the name **ioeq3** to it. The function **Reduction[ioeq3]** returns: **The system has irreducible form**. Like in the Example 2, lower left triangular matrix $G_L(s)$ has a very complicated form which makes it impossible to present here.

A. Comparison with a discrete-time case

While the general structure of the function **Reduction** for continuous-time case is pretty much similar to that of written for the discrete-time systems, there are few important differences which we describe below.

First, construction of the field $\mathcal{K}$ differs from the discrete-time case due to the fact that instead of forward shift one has to use the standard time-derivative. However, the most important difference is in the multiplication rule between the differentiation operator s (or forward-shift operator δ, in the discrete-time case) and an element $a \in \mathcal{K}$. Whereas in the continuous-time case this rule is defined by (3), in the discrete-time case it is as follows

$$\delta \cdot a = a^+ \cdot \delta, \tag{12}$$

where a^+ is a forward shift of the function a. As our experience shows, the rule (3) yields more complex expressions and longer computation time.

V. Conclusions

This paper presents a necessary and sufficient condition for irreducibility of nonlinear multi-input multi-output differential equations which is formulated in terms of the greatest common left divisor of the polynomial matrices associated with the system equations. Note that the polynomials are defined over the differential field, and unlike the linear case, belong to a non-commutative polynomial ring. On the bases of the above condition a constructive procedure using the Euclidean left division algorithm is suggested to examine the system irreducibility. The proposed condition and procedure are consistent with those for the linear system and SISO nonlinear system. Note that though the irreducibility property can be checked within the polynomial approach, this is not so with system reduction. The algorithm described in this paper results in the one-forms related to the irreducible system description. To find the equations themselves, one has to integrate the one-forms. Though the one-forms are, in principle, integrable, this sometimes needs to find the integrating factors to make the one-forms found by the algorithm, exact.

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