Infinite-state games with finitary conditions

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In a nutshell

This paper is about ωB-games, and the subclass of finitary games.

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- two-player turn-based games played over **infinite** graphs,
- the winning conditions involve counters,
- the first issue is to prove the existence of finite-memory strategies,
- the second issue is to construct algorithms to decide the winner.

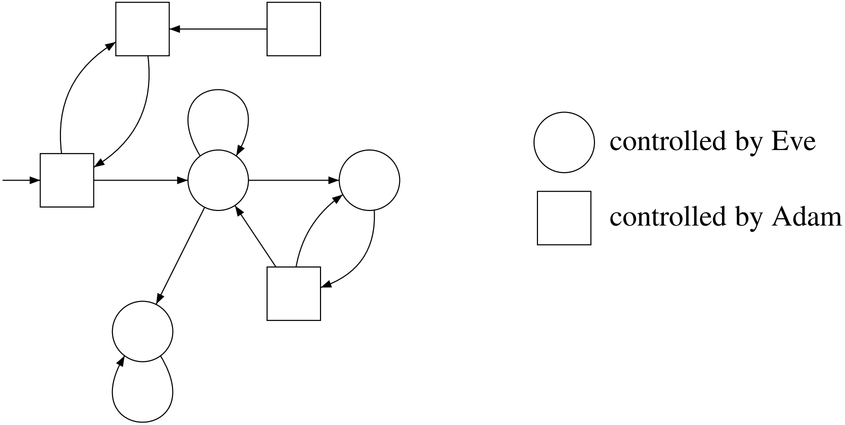
Motivation: expressing boundedness properties

$\text{MSO} + \mathbb{U}$

cost MSO

A lot is known, and even more is not known about those two logics!

Definition of ωB-games



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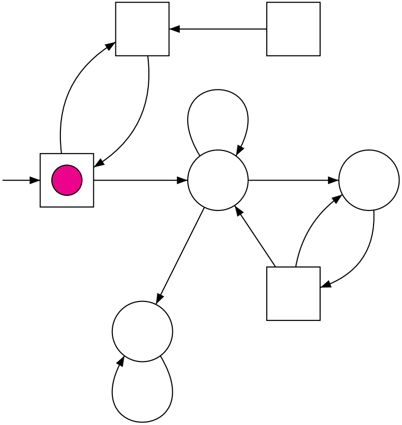

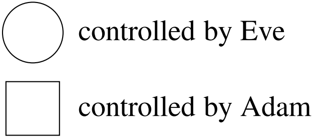

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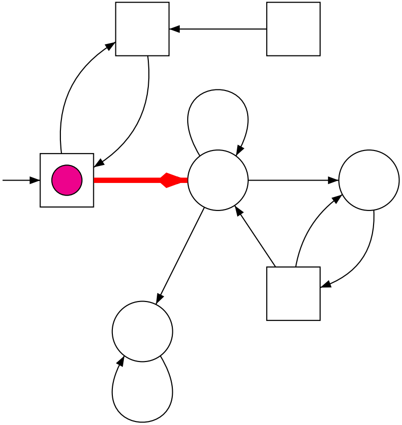

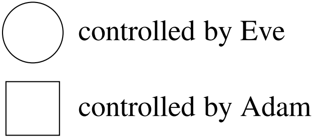

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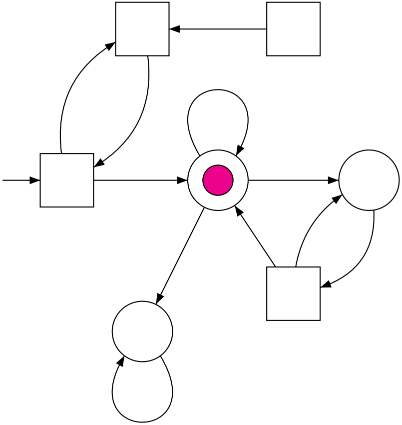

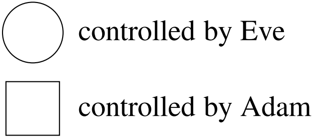

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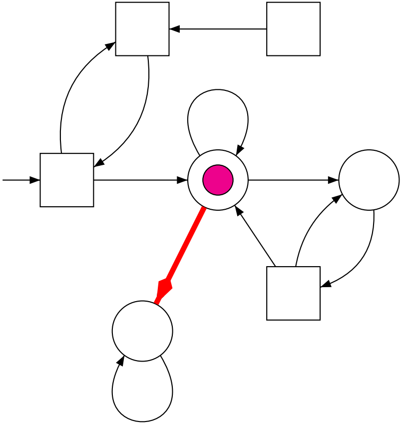

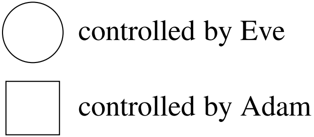

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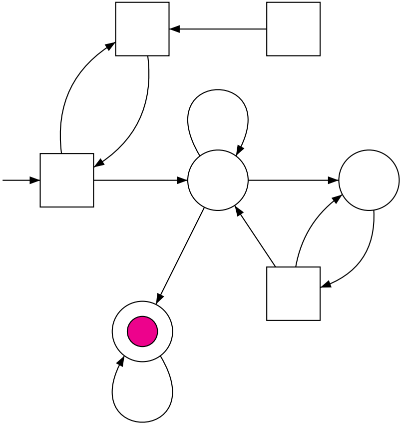

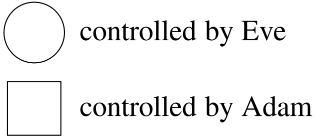

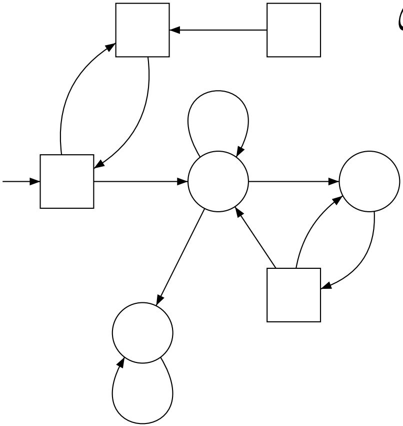

ωB winning condition:

parity
and
all counters
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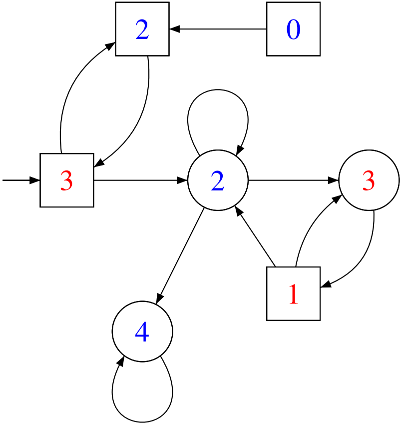

parity condition:

the minimal priority seen infinitely often is even

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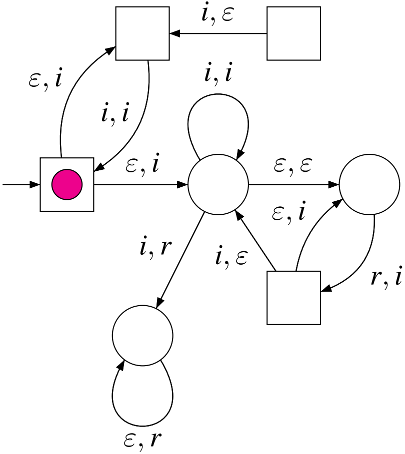

$$c_1 = 0$$
$$c_2 = 0$$

ε : nothing

i : increment

r : reset

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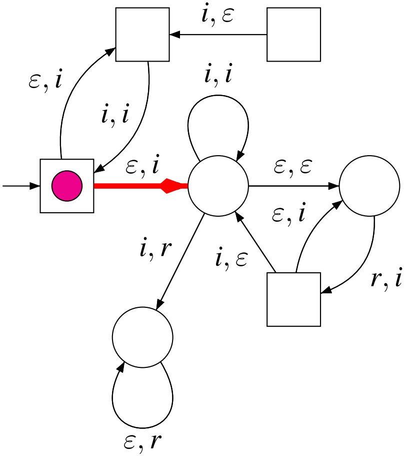

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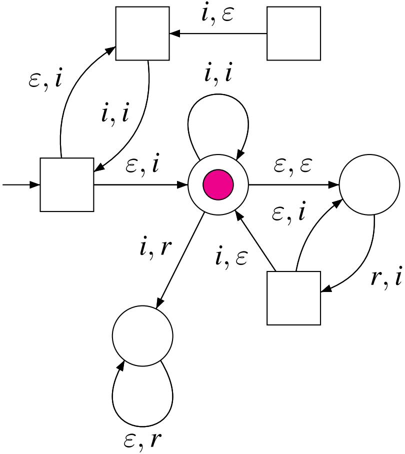

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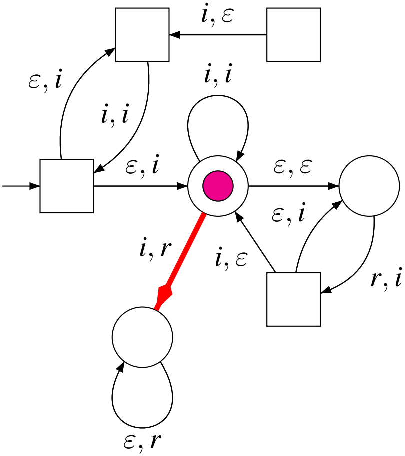

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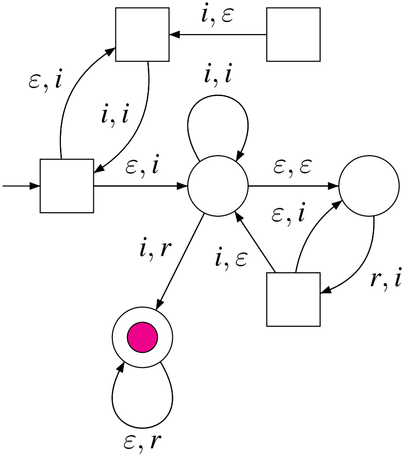

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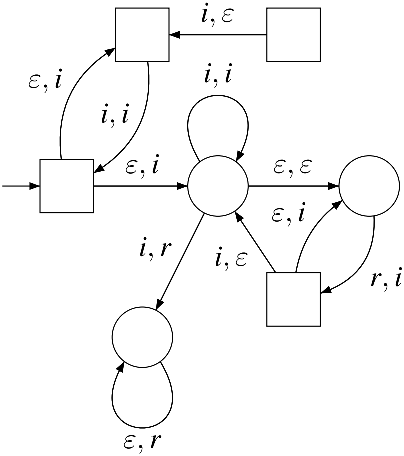

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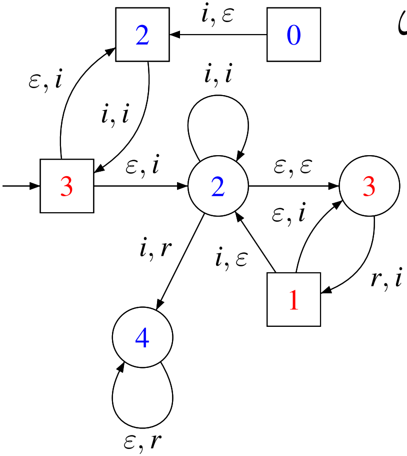

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General form

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What about ωB-games?

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What about ωB-games?

Finite-memory

$$\begin{cases} \sigma : V \times M \to V \\ \mu : M \times E \to M \end{cases}$$

Quantification

Eve wins means:

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non-uniform
($\mathrm{MSO} + \mathbb{U}$)

uniform
(cost MSO)

Why finite-memory strategies?

Thomas Colcombet's habilitation:

> le fait 2.22 et en déduire que la domination entre formules de la logique monadique de coût est décidable sur les arbres infinis. Ainsi, la conjecture 9.2 implique la conjecture 9.1.
>
> En fait, il est possible de pointer avec encore plus de précision où se trouve la difficulté. Si l'on cherche à démontrer la conjecture 9.2, tout comme dans le cas des arbres finis, le point crucial est l'existence de stratégies gagnantes à mémoire finie. Il suffirait d'établir la conjecture suivante.
>
> **Conjecture 9.3.** *Les objectifs* hB $\wedge$ parité *et* $\neg$B $\wedge$ parité *sont à* $\approx$*-mémoire finie, sur toutes les arènes/sur les arènes chronologiques/sur les arènes «arborescentes».*

Existence of finite-memory strategies in (some) ωB-games
$\Longrightarrow$ Decidability of cost MSO over infinite trees
$\Longrightarrow$ Decidability of the index of the non-deterministic Mostowski's hierarchy (open for 40 years)!

Contributions of this paper (1/2)

Finitary conditions were introduced by Alur and Henzinger in 94, and are a subclass of ωB conditions where the counters and parity are not independent.

Theorem (not in this talk)

Over general graphs, Eve has finite-memory winning strategies in finitary games.

Theorem (not in this talk)

Solving pushdown finitary games with stack boundedness condition is EXPTIME-*complete.*

Contributions of this paper (2/2)

Theorem

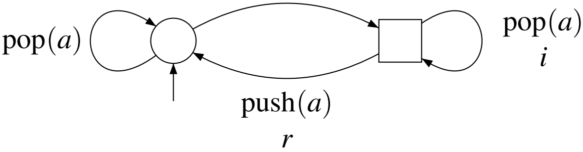
Over pushdown graphs, the uniform and non-uniform quantifications are ***almost*** *equivalent.*

Corollary

Solving pushdown ωB-games is decidable.

Outline

A first example



Eve should maintain a low stack.

Objective

Theorem

For all pushdown games, the following are equivalent:

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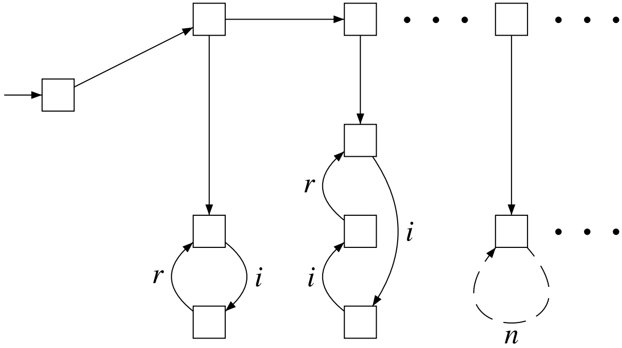
Counter-example for the general case



Eve wins but she does not know the bound!

Outline

A simple proof for the case of finite graphs

Condition: parity and all counters are bounded.

Define:

- $\mathcal{W}_E(N)$ the set of vertices where Eve wins for the bound N.
- $\mathcal{W}_E$ the set of vertices where Eve wins for some (non-uniform) bound.

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2. *There exists N such that $\mathcal{W}_E(N) = \mathcal{W}_E(N+1) = \cdots$.*
3. *For such N, Adam wins from $V \setminus \mathcal{W}_E(N)$, hence $\mathcal{W}_E = \mathcal{W}_E(N)$.*

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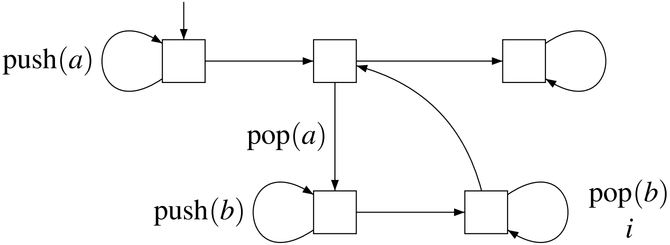

Adam may use the stack as "credit".

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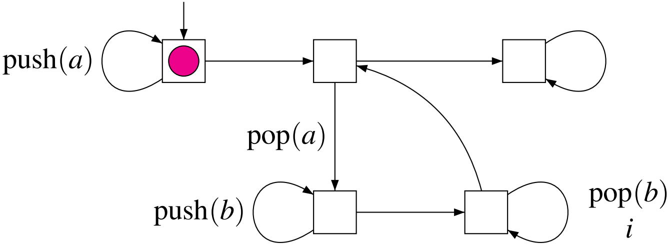

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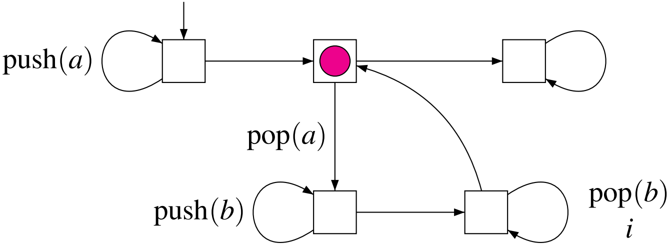

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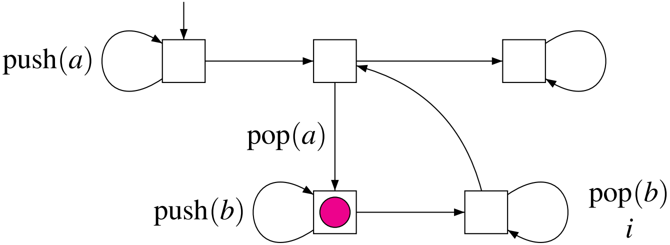

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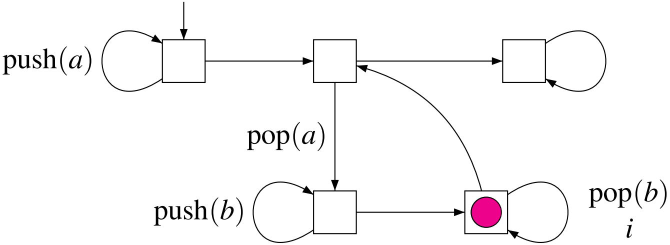

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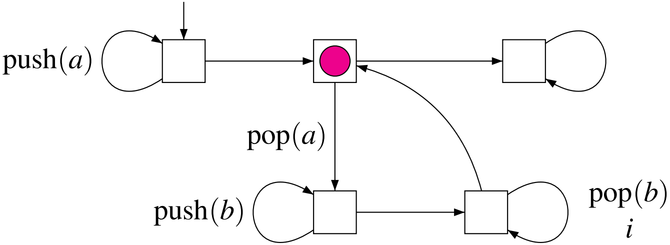

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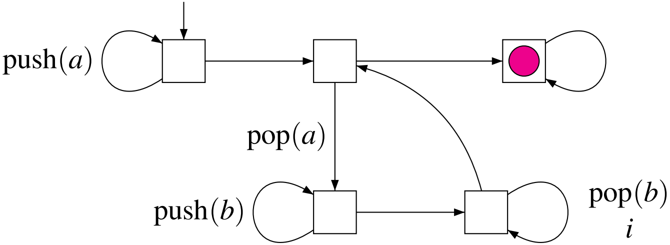

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Proof sketch

Condition: parity and all counters are bounded.

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3. *For such N, Adam wins from $V \setminus \mathcal{W}_E(N)$, hence $\mathcal{W}_E = \mathcal{W}_E(N)$.*

Why is 2. true?

Regularity of the winning regions

Theorem (derived from Serre)

*For all N, $\mathcal{W}_E(N)$ is a regular set of configurations, recognized by an alternating automaton of size $|Q|$ (**independent of** N).*

Decidability

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Remark: one can show that the collapse bound is doubly-exponential!