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# Graph Groups, Coherence, and Three-Manifolds

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## I. INTRODUCTION

We study groups given by presentations each of whose defining relations is of the form $xy = yx$ for some generators $x$ and $y$. To such a presentation we associate a graph $X$ whose vertices are the generators, two vertices $x$ and $y$ being adjacent in $X$ if and only if $xy = yx$ is a defining relation.

Given a graph $X$, we denote by $GX$ the group defined by the presentation associated to $X$ in this way. We call $GX$ a graph group. These groups have been studied by Kim and Roush [8], and by Dicks [3].

In this paper we prove the following:

THEOREM 1. *If $X$ is a finite graph, then the group $GX$ is coherent if and only if each circuit of $X$ of length greater than three has a chord.*

(Recall that a group is called coherent if each of its finitely generated subgroups is finitely presented.)

THEOREM 2. *If $X$ is a finite graph, then the group $GX$ is the fundamental group of a three-dimensional manifold if and only if each connected component of $X$ is either a tree or a triangle.*

## II. GRAPH-THEORETIC TERMINOLOGY

We refer the reader to [9] for terminology in graph theory not defined here. A full subgraph $U$ of a graph $X$ is a graph whose vertex set is a subset of the vertex set of $X$, two vertices being adjacent in $U$ if and only if they are adjacent in $X$. Since each full subgraph of $X$ is determined by its vertex set, we call $U$ the subgraph of $X$ induced by its vertex set. We denote by $\langle S \rangle$ the subgraph of $X$ induced by the subset $S$ of the vertices of $X$. Note



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that the subgroup of $GX$ generated by the elements of $S$ is isomorphic to $G\langle S\rangle$.

A vertex $x$ of the graph $X$ will be called central if $x$ is adjacent to all the other vertices of $X$.

## III. Group-Theoretic Preliminaries

Let $X$ be a finite graph. Given an element $g \in GX$, with $g = x_1^{e_1} x_2^{e_2} \cdots x_k^{e_k}$, where each $x_i$ is a vertex of $X$, we define

$$|g| = e_1 + e_2 + \cdots + e_k.$$

$|g|$ is independent of the expression of $g$ as a product of powers of generators, since each relator has exponent sum 0. Let $KX = \{g \in GX : |g| = 0\}$. Clearly $KX$ is a subgroup of $GX$.

If $U$ and $V$ are full subgraphs of $X$, with $X = U \cup V$ and $W = U \cap V$, then $GX = GU *_{GW} GV$, as follows easily by examining generators and relations. In particular, if $U \cap V$ is empty, then $GX = GU * GV$. Since free products of 3-manifold groups are 3-manifold groups [4, Lemma 3.2], and free products of coherent groups are coherent [7, Theorem 8], it will suffice to prove Theorems 1 and 2 for connected graphs.

Proposition. *Let $X$ be a finite connected graph, and let $U$ and $V$ be full subgraphs of $X$ with $X = U \cup V$ and $W = U \cap V$. Then*

$$KX = KU \underset{KW}{*} KV$$

*Proof.* Since $GX = GU *_{GW} GV$, [11, Theorem 13] implies that $GX$ acts on a directed tree $Y$, whose vertices are the left cosets of the subgroups $GU$ and $GV$ in $GX$, and whose edges are the left cosets of $GW$ in $GX$. Thus $KX$ acts on $Y$ also. In fact, $KX$ acts transitively on the edges of $Y$; to see this, let $w$ be any vertex of $W$, and let $gGW$ be any edge of $Y$. Then $w^{|g|}g^{-1} \in KX$, and $(w^{|g|}g^{-1})(gGW) = w^{|g|}GW = GW$, so there is only one orbit of edges under the action of $KX$. Since the vertices $GU$ and $GW$ lie in different orbits of the $KX$-action on $Y$, the quotient directed graph $Y/KX$ consists of two vertices joined by an edge, so again by [11, Theorem 13],

$$KX = KX \cap GU \underset{KX \cap GW}{*} KX \cap GV = KU \underset{KW}{*} KV.$$

Corollary. *Let $T$ be a finite tree with $n+1>0$ vertices. Then $KT$ is a free group of rank $n$. Further, $KT$ is freely generated by a set of elements*

*$k_1, k_2, \ldots, k_n$ in one-to-one correspondence with the $n$ edges of $T$; the generator corresponding to the edge joining the vertices $x$ and $y$ may be chosen equal to either $x^{-1}y$ or $y^{-1}x$.*

*Proof.* This is clear if $n=0$ or if $n=1$. If $n>1$, choose a pendent vertex $x$ of $T$, let $y$ be the unique vertex of $T$ adjacent to $x$, and let $T'$ denote the tree obtained from $T$ by deleting the vertex $x$ and the edge joining $x$ and $y$. Then $GT = GT' *_{G\langle y\rangle} G\langle x, y\rangle$, so by the above proposition, $KT = KT' *_{K\langle y\rangle} K\langle x, y\rangle$. By induction, $KT'$ is free of rank $n-1$. Clearly $K\langle x, y\rangle$ is infinite cyclic, and $K\langle y\rangle$ is trivial, so $KT$ is free of rank $n$. The assertion about generators follows from induction and the fact that each of $x^{-1}y$ and $y^{-1}x$ generates $K\langle x, y\rangle$.

## IV. Proof of Theorem 1

Suppose every circuit of $X$ of length greater than three has a chord. If $X$ is complete, then $GX$ is finitely generated free abelian, and so coherent. Otherwise, $X$ has a separating set $A$ of vertices which induces a complete subgraph of $X$ [9, Solution to Problem 9.29b]. That is, there are proper full subgraphs $X_1$ and $X_2$ of $X$ such that $X = X_1 \cup X_2$, $\langle A\rangle = X_1 \cap X_2$, and $\langle A\rangle$ is complete. Thus,

$$GX = GX_1 \underset{G\langle A\rangle}{*} GX_2$$

Every circuit of either $X_1$ or $X_2$ of length greater than three has a chord, so by induction, $GX_1$ and $GX_2$ are coherent. $G\langle A\rangle$ is finitely generated free abelian, so by [7, Theorem 8], $GX$ is also coherent.

Now suppose that the graph $X$ is a circuit of length greater than three and let $x$ and $y$ be two nonadjacent vertices of $X$. Then there are proper full subgraphs $X_1$ and $X_2$ of $X$ such that $X = X_1 \cup X_2$, $X_1 \cap X_2 = \langle x, y\rangle$, and $X_1$ and $X_2$ are trees. Thus,

$$KX = KX_1 \underset{K\langle x, y\rangle}{*} KX_2.$$

Each of $KX_1$ and $KX_2$ is a finitely generated free group, so $KX$ is finitely generated. $K\langle x, y\rangle$ is the normal closure in the free group $G\langle x, y\rangle$ of $x^{-1}y$, so $K\langle x, y\rangle$ is not finitely generated. By [1], $KX$ is not finitely presented, so $GX$ is not coherent. It follows that if some circuit of $X$ of length greater than 3 has no chord, then $GX$ has a noncoherent subgroup, and is thus itself not coherent.

## V. An Ordering of the Vertices of a Tree

Let $T$ be a finite tree, and let $x_0$ be a pendent vertex of $T$. We will describe a linear ordering of the vertices of $T$ which we will use in the proof of Theorem 2. Given a vertex $y$, denote by $\text{star}^+(y)$ the set $\{z: y \text{ lies on the path joining } x_0 \text{ to } z\}$. This is well defined since $T$ is a tree. Define $\text{star}(y)$ to be the set of vertices in $\text{star}^+(y)$ which are adjacent to $y$. We order the vertices of $T$ as follows: first, for each vertex $y$, arbitrarily order the set $\text{star}(y)$. Then, given 2 vertices $y$ and $z$ of $T$, set $y < z$ if either of the two conditions below is satisfied:

(i) $z \in \text{star}^+(y)$,

(ii) there are vertices $v$, $y_0$, and $z_0$ with $y_0, z_0 \in \text{star}(v)$, $y \in \text{star}^+(y_0)$, $z \in \text{star}^+(z_0)$, and $y_0 < z_0$ in the ordering chosen on $\text{star}(v)$.

$<$ is a linear ordering of the vertices of $T$, since $T$ is a tree.

## VI. Proof of Theorem 2

By [10], any 3-manifold group is coherent, so we need only consider connected graphs in which every circuit of length greater than three has a chord.

If the graph $X$ is a triangle, then $GX = \pi_1(S^1 \times S^1 \times S^1)$, so $GX$ is a 3-manifold group. Let $T$ be a finite tree and let $x_0$ be a pendent vertex of $T$. We will show that $GT$ is a three-manifold group. Let $s: GT \to gp\langle x_0 \rangle$ be the homomorphism determined by setting $s(y) = x_0$ for each vertex $y$ of $T$. Then $\ker(s) = KT$, so there is a split exact sequence

$$1 \longrightarrow KT \longrightarrow GT \xrightarrow{s} gp\langle x_0 \rangle \longrightarrow 1.$$

Thus, $GT$ is isomorphic to the semidirect product $KT \ltimes gp\langle x_0 \rangle$.

To show that $GT$ is a 3-manifold group, we will use a different generating set for $KT$ than that described above. Let $<$ denote an ordering of the vertices of $T$ as defined in section III. Given a vertex $x$ other than $x_0$, set $\hat{x} = y^{-1}x$, where $y$ is the unique vertex of $T$ for which $x \in \text{star}(y)$. By the above corollary the set $\{\hat{x} \mid x \neq x_0\}$ freely generates $KT$. For $x \neq x_0$, let $x^* = \hat{x}\hat{x}_k^{-1}\hat{x}_{k-1}^{-1} \cdots \hat{x}_1^{-1}$, where $\text{star}(x) = \{x_1, x_2, \ldots, x_k\}$ and $x_1 < x_2 < \cdots < x_k$. (If $\text{star}(x)$ is empty, we define $x^* = \hat{x}$.) A routine computation shows that if $\text{star}^+(x) = \{x_1, x_2, \ldots, x_m\}$, with $x_1 < x_2 < \cdots < x_m$, then $x^* x_1^* x_2^* \cdots x_m^* = \hat{x}$. Thus, the set $\{x^* \mid x \neq x_0\}$ also freely generates $KT$. Let $a: KT \to KT$ be the automorphism defined by $a(k) = x_0^{-1} k x_0$ for each $k \in KT$. If $x$ is a vertex of $T$ other than $x_0$, then the elements $x^*$ and $x$

of $GT$ commute, so that $a(x^*) = (x^{-1}x_0)^{-1}x^*(x^{-1}x_0)$. Therefore, since $x^{-1}x_0 \in KT$, $a(x^*)$ is conjugate in $KT$ to $x^*$. Furthermore, if the vertex set of $T$ is the set $\{x_0, x_1, \dots, x_n\}$, with $x_0 < x_1 < \cdots < x_n$, then $x_1^* x_2^* \cdots x_n^* = \hat{x}_1 = x_0^{-1}x_1$. Because $T$ is connected, the vertices $x_0$ and $x_1$ must be adjacent, so

$$a(x_1^* x_2^* \cdots x_n^*) = x_0^{-1}(x_0^{-1}x_1)\, x_0 = x_0^{-1}x_1 = x_1^* x_2^* \cdots x_n^* .$$

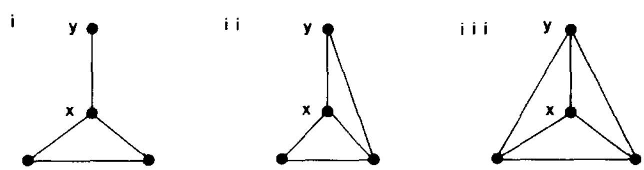
Let $D_n^2$ denote the space obtained by removing $n$ interior points from the disk $D^2$. Then $\pi_1(D_n^2) = KT$ and by [2, Theorem 1.10], there is a homeomorphism $h$ of $D_n^2$ which fixes the boundary of $D^2$ pointwise and for which $a = h_*$, the automorphism of $\pi_1(D_n^2)$ induced by $h$. Let $\sim$ be the least equivalence relation on the space $D_n^2 \times [0, 1]$ for which $[p, 0] \sim [h(p), 1]$, and let $M = D_n^2 \times [0, 1]/\sim$. Clearly, $M$ is a 3-manifold. The fundamental group of $M$ is isomorphic to the semidirect product $\pi_1(D_n^2) \ltimes Z$, where $Z$ is an infinite cyclic group with generator $t$, and, for each $g \in \pi_1(D_n^2)$, $t^{-1}gt = h_*(g)$ [2, proof of Theorem 2.2]. Since $h_* = a$, this group is isomorphic to $GT$.

To complete the proof of Theorem 2, we shall need the following:

LEMMA. *Let $X$ be a finite graph with central vertex $x$, and suppose that $GX$ is a 3-manifold group. If $y$ is any vertex of $X$ other than $x$, then the graph $Y$ obtained from $X$ by deleting the vertices $x$ and $y$ is totally disconnected.*

*Proof.* Since $X$ is finite, $GX$ is finitely generated, so by [6], $GX$ is the fundamental group of a compact 3-manifold. Let $X'$ be the graph obtained from $X$ by deleting the vertex $x$. Then $GX'$ is a normal subgroup of $GX$ with infinite cyclic quotient, so by [12], $GX'$ is the fundamental group of a surface. $GY$ is a subgroup of infinite index in $GX'$, so by [5], $GY$ is free. Thus, $Y$ must be totally disconnected, since otherwise, $GY$ would have a free abelian subgroup of rank two.

Now, suppose that the graph $X$ is neither a tree nor a triangle, and that every circuit of $X$ of length greater than three has a chord. Then $X$ must have an induced subgraph of one of the following forms:




It follows from the lemma that none of the graph groups associated with these graphs is a 3-manifold group. Because every subgroup of the fundamental group of a 3-manifold is itself a 3-manifold group, $GX$ is not a 3-manifold group.

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