Quine's Fluted Fragment is Non-elementary

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Outline

Overview of results

Lower bound proof

Conclusion

- Fragment identified by W.V.Quine in 1968:
  - homogeneous *m*-adic formulas
    (generalization of monadic fragment)
  - later generalized to fluted fragment
- Examples of fluted formulas:

  No student admires every professor
  $\forall x_1(\text{student}(x_1) \rightarrow \neg\forall x_2(\text{prof}(x_2) \rightarrow \text{admires}(x_1, x_2)))$

  No lecturer introduces any professor to every student
  $$\forall x_1(\text{lecturer}(x_1) \rightarrow \neg\exists x_2(\text{prof}(x_2) \wedge \forall x_3(\text{student}(x_3) \rightarrow \text{intro}(x_1, x_2, x_3)))).$$

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- Order of quantification of variables matches order of appearance in predicates.

- Let $x_1, x_2, \ldots$ be a fixed sequence of variables.
- The fluted fragment with k free variables, $\mathcal{FL}^{[k]}$, is defined by simultaneous induction for all k:
  - any atom $p(x_\ell, \ldots, x_k)$ is in $\mathcal{FL}^{[k]}$;
  - $\mathcal{FL}^{[k]}$ is closed under Boolean operations;
  - $\mathcal{FL}^{[k]}$ contains $\exists x_{k+1}\varphi$ and $\forall x_{k+1}\varphi$ for any $\varphi \in \mathcal{FL}^{[k+1]}$.
- The fluted fragment, $\mathcal{FL}^{[k]}$ is the union:

$$\mathcal{FL} = \bigcup_{k \geq 0} \mathcal{FL}^{[k]}.$$

- For all $m > 0$, we define $\mathcal{FL}^m$, to be the set of fluted formulas containing at most the variables $x_1, \ldots, x_m$, free or bound.

- History:
  - Noah (1980): the generalization of the homogeneous *m*-adic fluted formulas to the fluted fragment makes decidability of satisfiability non-obvious.
  - Purdy (1996): $\mathcal{FL}$ has the finite model property; hence its satisfiability problem is decidable.
  - Purdy (2002): $\mathcal{FL}$ has the exponential-sized model property; hence its satisfiability problem is in $\mathrm{NE{\scriptsize XP}T{\scriptsize IME}}$.

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  - Purdy (2002): $\mathcal{FL}$ has the exponential-sized model property; hence its satisfiability problem is in NExpTime.
- The claims in Purdy 2002 are false:
  - satisfiable formulas of $\mathcal{FL}^{2m}$ force *m*-tuply exponential models;
  - the satisfiability problem for $\mathcal{FL}^{2m}$ is *m*-NExpTime-hard.

Outline

- Let int_1 and $p_0, \ldots, p_{n-1}$ be unary predicates. We refer to any object satisfying int_1 (in some structure) as a 1-integer.
- For any 1-integer b, define $\mathrm{val}_1(b)$ to be the integer in the range $[0, 2^n]$ determined by b's satisfaction of $p_0, \ldots, p_{n-1}$.
- It is routine to define (fluted) formulas fixing the predicates

$$\mathrm{zero}_1 \qquad \mathrm{eq}_1 \qquad \mathrm{pred}_1$$

to have the expected meaning, and enforcing the property:

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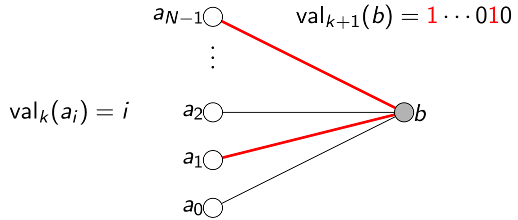
Here:

$$\mathfrak{t}(m, n) = 2^{\left.2^{\cdot^{\cdot^{\cdot^{2^n}}}}\right\} m \text{ 2's}}$$

- For all k $(2 \leq k \leq m)$ we introduce a unary predicate int_k. Any object satisfying int_k will be called a k-integer.
- For all k $(1 \leq k < m)$ we introduce a binary predicate in_k, and for any $(k+1)$-integer b, we define a function $\mathrm{val}_{k+1}(b)$



$$s_i = \begin{cases} 1 & \text{if } \mathfrak{A} \models \mathrm{in}_k[a,b] \text{ for some 1-integer } a \text{ s.t. } \mathrm{val}_k(a) = i; \\ 0 & \text{otherwise.} \end{cases}$$

where $0 \leq i < N = \mathfrak{t}(k,n)$.

- Let zero_k be a unary predicate and eq_k, pred_k binary predicates.
- By insisting that $\mathfrak{A}$ satisfies certain fluted formulas, we can ensure that, for all k-integers b and b':

$$\mathfrak{A} \models \text{eq}_k[b, b'] \Leftrightarrow \text{val}_k(b) = \text{val}_k(b')$$
$$\mathfrak{A} \models \text{pred}_k[b, b'] \Leftrightarrow \text{val}_k(b') = \text{val}_k(b) - 1 \quad \text{mod } \mathfrak{t}(k, n)$$
$$\mathfrak{A} \models \text{zero}_k[b] \Leftrightarrow \text{val}_k(b') = 0.$$

- Suppose $\mathfrak{A}$ also makes the following true:

$$\exists x_1(\text{int}_k(x_1) \wedge \text{zero}_k(x_1))$$
$$\forall x_1(\text{int}_k(x_1) \rightarrow \exists x_2(\text{int}_k(x_2) \wedge \text{pred}_k(x_1, x_2))).$$

Then $\mathfrak{A}$ satisfies the property

k-covering: $\text{val}_k : \text{int}_k^{\mathfrak{A}} \rightarrow [0, \mathfrak{t}(k, n) - 1]$ is surjective.

Outline

- The claims of (Purdy 2002) are false. For all $m \geq 1$, the complexity of the satisfiability problem for $\mathcal{FL}^m$ is

$$\lfloor m/2 \rfloor\text{-}\textsc{NExpTime}\text{-hard}.$$

- Using a corrected version of the argument in that paper, we can also show that, for $m \geq 3$, any satisfiable formula of $\mathcal{FL}^m$ has a model of $(m-2)$-tuply exponential size; hence, the satisfiability problem for $\mathcal{FL}^m$ is

$$\text{in } (m-2)\text{-}\textsc{NExpTime}.$$

- Furthermore, the satisfiability problems for $\mathcal{FL}^1$ and $\mathcal{FL}^2$ are $\textsc{NPTime}$- and $\textsc{NExpTime}$-complete, respectively.
- These bounds leave a gap when $m \geq 5$.