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Demystifying Reachability in Vector Addition Systems

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***Abstract*—More than 30 years after their inception, the decidability proofs for reachability in vector addition systems (VAS) still retain much of their mystery. These proofs rely crucially on a decomposition of runs successively refined by Mayr, Kosaraju, and Lambert, which appears rather magical, and for which no complexity upper bound is known.**

We first offer a justification for this decomposition technique, by showing that it computes the ideal decomposition of the set of runs, using the natural embedding relation between runs as well quasi ordering. In a second part, we apply recent results on the complexity of termination thanks to well quasi orders and well orders to obtain a cubic Ackermann upper bound for the decomposition algorithms, thus providing the first known upper bounds for general VAS reachability.

***Key Words*—Vector addition system, reachability, well quasi order, ideal, fast-growing complexity**

I. Introduction

Vector addition systems (VAS), or equivalently Petri nets, find a wide range of applications in the modelling of concurrent, chemical, biological, or business processes. Their algorithmics, and in particular the decidability of their *reachability problem*, is a central component to many decidability results spanning from the verification of asynchronous programs [13] to the decidability of data logics [4, 9, 7]. Considered as one of the great achievements of theoretical computer science, the original 1981 decidability proof of Mayr [32] is the culmination of more than a decade of research into the topic, and builds notably on an incomplete proof by Sacerdote and Tenney [36]. The proof was simplified a year later by Kosaraju [22]; see also the account by Müller [33] and the self-contained and detailed monograph of Reutenauer [35] on this second proof. In spite of this success, as put by Lambert [24] "the complexity of the two proofs (especially in [32]) wrapped the result in mystery and no use of their original ideas" was made before he provided a further simplification ten years later in 1992, and employed it to prove results on VAS languages.

At the heart of the various proofs lies a *decomposition technique*, which we dub the Kosaraju-Lambert-Mayr-Sacerdote-Tenney (KLMST) decomposition in this article after its inventors. In a nutshell, the KLMST decomposition defines both a structure and a condition for this structure to represent in some way the set of all runs witnessing reachability. The algorithms advanced by Mayr, Kosaraju, and Lambert compute this decomposition by successive refinements of the structure until the condition is fulfilled. The KLMST decomposition is a powerful tool when reasoning about VAS runs, and it has notably been employed

- by Habermehl, Meyer, and Wimmel [15] to show that the downward-closure of a labelled VAS language is effectively computable—let us mention a new proof by Zetzsche [41], which does not explicitly rely on the KLMST decomposition—, and
- by Leroux [27] to derive a new algorithm for reachability based on Presburger inductive invariants—he would later re-prove the correction of this new algorithm *without* referring to the KLMST decomposition, yielding a compact self-contained decidability proof for VAS reachability [28].

Our feeling however is that the decidability of VAS reachability, and especially the KLMST decomposition, is still shrouded in mystery. The result is highly complex on two accounts:

a) On a conceptual level the various instances of the KLMST decomposition seem rather magical. How did Mayr come up with *regular constraint graphs* with a *consistent marking*? How did Kosaraju come up with *generalised VASS* and his θ *condition*? How did Lambert come up with his *perfect* condition on *marked graph-transition sequences*? Most importantly, which guidelines to follow in order to develop similar concepts for VAS extensions where the decidability of reachability is still open, e.g. for unordered data Petri nets [26], pushdown VASS [25], or branching VAS [37]? Arguably, the issue here is not to understand how these structures and conditions are used in the algorithms themselves, nor to check that they indeed yield the decidability of VAS reachability. Rather, the issue is to explain how these structures and conditions can be derived in a principled manner.

b) On a computational complexity level no complexity upper bound is known for the general VAS reachability problem, while the best known lower bound is ExpSpace-hardness [30]. The only known tight bounds pertain to the very specific case of 2-dimensional VAS with states, which were recently shown to have a PSpace-complete reachability problem [3]. As observed e.g. by Müller [33] the algorithms computing the KLMST decomposition are not primitive-recursive, but no one has been able to derive a complexity upper bound for these algorithms, while the new algorithm of Leroux [27, 28] using Presburger inductive invariants seems even harder to analyse from a complexity viewpoint.

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Our contributions in this paper are first to propose an explanation for the KLMST decomposition. Using a well quasi ordering of VAS runs defined by Jančar [18] and Leroux [28] and recalled in Sec. V, we show a *Decomposition Theorem* (Theorem VIII.1): the KLMST algorithm computes an *ideal decomposition* of the set of runs, i.e. a decomposition into irreducible downward-closed sets (see Sec. VIII). The effective representation of those ideals through finite structures turns out to match exactly the structures and conditions expressed by Lambert [24], see sections VI and VII. This provides a full formal framework in which the reachability problem in various VAS extensions might be cast, offering some hope to see progress on those open issues.

The second contribution in Sec. IX is the proof of a "cubic Ackermann" complexity upper bound on the complexity of the KLMST decomposition algorithm, i.e., an $\mathbf{F}_{\omega^3}$ upper bound in the fast-growing complexity hierarchy $(\mathbf{F}_\alpha)_\alpha$ defined in [38]. We apply to this end the recent results on bounding the length of controlled bad sequences over well quasi orders from [40, 39]. It yields the first known upper bound on VAS reachability. As a byproduct, it also yields the first complexity upper bound for numerous problems known decidable thanks to a reduction to VAS reachability, e.g. [4, 13, 9, 7] among many others.

We start in sections II, III, and IV by presenting the necessary background on VAS, well quasi orders, and ideals. Due to space constraints, some material is omitted but can be found in the full paper at the address http://arxiv.org/abs/1503.00745.

II. Vector Addition Systems

Vectors and sets of vectors in $\mathbb{Z}^d$ for some natural d are denoted in bold face. A *periodic set* is a subset $\boldsymbol{P}$ of $\mathbb{Z}^d$ that contains the zero vector $\mathbf{0} \stackrel{\text{def}}{=} (0, \ldots, 0)$ and such that $\boldsymbol{p} + \boldsymbol{q} \in \boldsymbol{P}$ for all $\boldsymbol{p}, \boldsymbol{q} \in \boldsymbol{P}$.

A *vector addition system* of dimension d in $\mathbb{N}$ is a finite set $\boldsymbol{A}$ of *actions* $\boldsymbol{a}$ in $\mathbb{Z}^d$ [21]. The operational semantics of VASs operates on *configurations*, which are vectors $\boldsymbol{c}$ in $\mathbb{N}^d$. A *transition* is then a triple $(\boldsymbol{u}, \boldsymbol{a}, \boldsymbol{v}) \in \mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d$ such that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}$, where addition operates componentwise; the set of transitions of $\boldsymbol{A}$ is denoted by $\mathrm{Trans}_{\boldsymbol{A}}$.

A *prerun* over $\boldsymbol{A}$ is a triple $\rho = (\boldsymbol{u}, w, \boldsymbol{v})$ where $\boldsymbol{u}$ and $\boldsymbol{v}$ are two configurations in $\mathbb{N}^d$ and w is a sequence of triples $(\boldsymbol{u}_1, \boldsymbol{a}_1, \boldsymbol{v}_1) \cdots (\boldsymbol{u}_k, \boldsymbol{a}_k, \boldsymbol{v}_k)$ in $(\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d)^*$. The configurations $\boldsymbol{u}$ and $\boldsymbol{v}$ are called respectively the *source* and *target* of ρ, and are denoted respectively by $\mathrm{src}(\rho)$ and $\mathrm{tgt}(\rho)$. The action sequence $\sigma = \boldsymbol{a}_1 \cdots \boldsymbol{a}_k$ is called the *label* of ρ. We write $\mathrm{PreRuns}_{\boldsymbol{A}}$ for the set of preruns over $\boldsymbol{A}$.

A prerun $(\boldsymbol{u}, w, \boldsymbol{v})$ is *connected* if w is a transition sequence $(\boldsymbol{u}_1, \boldsymbol{a}_1, \boldsymbol{v}_1) \cdots (\boldsymbol{u}_k, \boldsymbol{a}_k, \boldsymbol{v}_k)$ in $\mathrm{Trans}^*_{\boldsymbol{A}}$ such that

- either $w = \varepsilon$ is the empty sequence and then $\boldsymbol{u} = \boldsymbol{v}$,
- or $k > 0$ and $\boldsymbol{u} = \boldsymbol{u}_1$, $\boldsymbol{v} = \boldsymbol{v}_k$, and $\boldsymbol{u}_{j+1} = \boldsymbol{v}_j$ for all $0 \leq j < k$.

We call a connected prerun ρ a *run*. If there exists a run ρ from source $\boldsymbol{u}$ to target $\boldsymbol{v}$ labelled by σ, we denote by $\boldsymbol{u} \xrightarrow{\sigma} \boldsymbol{v}$ this unique run ρ. Notice that it implies $\boldsymbol{v} = \boldsymbol{u} + \sum_{j=1}^k \boldsymbol{a}_j$; note however that given $\boldsymbol{u}$, $\boldsymbol{v}$, and σ, $\boldsymbol{v} = \boldsymbol{u} + \sum_{j=1}^k \boldsymbol{a}_j$ does not necessarily imply that $\boldsymbol{u} \xrightarrow{\sigma} \boldsymbol{v}$.

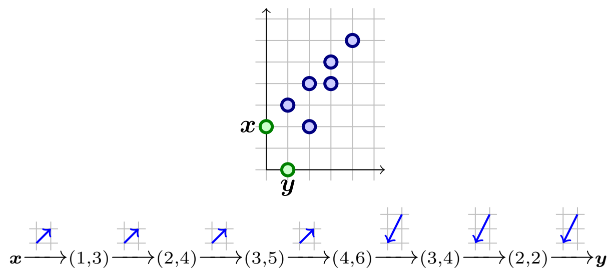

Figure 1. A run from $\boldsymbol{x} = (0, 2)$ to $\boldsymbol{y} = (1, 0)$ labelled by $(1, 1)^4(-1, -2)^3$.

We are interested in this paper in the following decision problem:

Problem: VAS Reachability.
input: A VAS $\boldsymbol{A}$, a source configuration $\boldsymbol{x}$, and a target configuration $\boldsymbol{y}$.
question: $\exists \sigma \in \boldsymbol{A}^*. \boldsymbol{x} \xrightarrow{\sigma} \boldsymbol{y}$?

Given two configurations $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{N}^d$, we define the *set of runs* of $\boldsymbol{A}$ from $\boldsymbol{x}$ to $\boldsymbol{y}$ as

$$\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) \stackrel{\text{def}}{=} \{\boldsymbol{x} \xrightarrow{\sigma} \boldsymbol{y} \mid \sigma \in \boldsymbol{A}^*\} . \tag{1}$$

The VAS reachability problem can then be recast as asking whether the set $\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ is non empty.

III. Well Quasi Orders

A *quasi-order* (qo) is a pair $(X, \leq)$ where X is a set and $\leq$ is a reflexive and transitive binary relation over X. We write $x < y$ if $x \leq y$ but $y \not\leq x$. Given a set $S \subseteq X$, we define its *upward-closure* $\uparrow S \stackrel{\text{def}}{=} \{x \in X \mid \exists s \in S \,.\, s \leq x\}$ and *downward-closure* $\downarrow S \stackrel{\text{def}}{=} \{x \in X \mid \exists s \in S \,.\, x \leq s\}$. When $S = \{s\}$ is a singleton, we write more succinctly $\uparrow s$ and $\downarrow s$. An *upward-closed* set $U \subseteq X$ is such that $U = \uparrow U$ and a *downward-closed* set $D \subseteq X$ such that $D = \downarrow D$. Observe that upward- and downward-closed sets are closed under arbitrary union and intersection, and that the complement over X of an upward-closed set is downward-closed and vice versa.

A. Characterisations

A finite or infinite sequence $x_0, x_1, x_2, \ldots$ of elements of a qo $(X, \leq)$ is *good* if there exist two indices $i < j$ such that $x_i \leq x_j$, and *bad* otherwise. A *well quasi order* (wqo) is a qo with the additional property that all its bad sequences are finite.

Example III.1 (Finite sets). As an example, a set X ordered by equality is a wqo if and only if it is finite: if finite, by the pigeonhole principle its bad sequences have length at most $|X|$; if infinite, any enumeration of infinitely many distinct elements yields an infinite bad sequence. □

There are many equivalent characterisations of wqos [23, 40]. For instance, $(X, \leq)$ is a wqo if and only if it is *well-founded*, i.e. there are no infinite descending sequences $x_0 > x_1 > \cdots$ of elements from X, and it has the *finite antichain* (FAC) property, i.e. any set of mutually incomparable elements from X is finite.

Example III.2 (Well orders). Any well-founded *linear* order, i.e. where $\leq$ is also antisymmetric and total, is a wqo: in that case, antichains have cardinal at most one. Examples include $(\mathbb{N}, \leq)$ the set of natural numbers, i.e. the ordinal ω. $\square$

We will also be interested in the following characterisation:

Fact III.3 (Descending Chain Property)**.** *A qo $(X, \leq)$ is a wqo if and only if any non-ascending chain $D_0 \supseteq D_1 \supseteq D_2 \supseteq \cdots$ of downward-closed subsets of X eventually stabilises, i.e. there exists a finite rank k such that $\bigcap_{i\in\mathbb{N}} D_i = D_k$.*

Another consequence of the definition of wqos is:

Fact III.4 (Finite Basis Property)**.** *Let $(X, \leq)$ be a wqo. If $U \subseteq X$ is upward-closed, then there exists a* finite basis *$B \subseteq U$ such that $\uparrow B = U$.*

B. Elementary Operations

Many constructions are known to yield new wqos from existing ones. In this paper we will employ the following elementary operations:

1) Cartesian Products: If $(X, \leq_X)$ and $(Y, \leq_Y)$ are wqos, then their Cartesian product $X \times Y$ is well quasi ordered by the *product (quasi-) ordering* defined by $(x, y) \leq (x', y')$ if and only if $x \leq_X x'$ and $y \leq_Y y'$. For instance, vectors in $\mathbb{N}^d$ along with the product ordering form a wqo. This result is also known as *Dickson's Lemma*.

2) Finite Sequences: If $(X, \leq_X)$ is a wqo, then the set X^* of finite sequences over X is well quasi ordered by the *sequence embedding* defined by $\sigma \leq_* \sigma'$ if and only if $\sigma = x_1 \cdots x_k$ and $\sigma' = \sigma'_0 x'_1 \sigma'_1 \cdots \sigma'_{k-1} x'_k \sigma'_k$ for some $x_j \leq_X x'_j$ in X for $1 \leq j \leq k$ and some σ'_j in X^* for $0 \leq j \leq k$. For instance, finite sequences in Σ^* for a finite alphabet $(\Sigma, =)$ form a wqo. This result is also known as *Higman's Lemma*.

In the following, we call *elementary* those wqos obtained from finite sets $(X, =)$ through finitely many applications of Dickson's and Higman's lemmas. Note that $(\mathbb{N}, \leq)$ is elementary since it is isomorphic with finite sequences over some unary alphabet with equality.

IV. WQO IDEALS

In this section, we recall a way of decomposing downward-closed sets, namely as finite unions of *ideals*. This is a classical notion—Fraïssé [12, Sec. 4.5] attributes finite ideal decompositions to Bonnet [6]—which has been rediscovered in the study of well structured transition systems [11]. Let us review the basic theory of ideals, as can be found in [6, 12, 20, 11]; see in particular [14] for a gentle introduction.

A. Ideals

A subset S of a qo $(X, \leq)$ is *directed* if for every $x_1, x_2 \in S$ there exists $x \in S$ such that both $x_1 \leq x$ and $x_2 \leq x$. An *ideal* I is a directed non-empty downward-closed set. The class of ideals of X is denoted by $\mathrm{Idl}(X)$.[1]

[1]The set of ideals equipped with the inclusion relation is also called the *completion* of the wqo $(X, \leq)$, see [11].

Example IV.1 (Well orders). In an ordinal α seen in set-theoretic terms as $\{\beta \mid \beta < \alpha\}$, any $\beta \leq \alpha$ is a downward-closed directed subset of α, and conversely any downward-closed directed subset of α is some $\beta \leq \alpha$. Hence the ideals of α are exactly the elements of $\alpha + 1$ except 0. $\square$

1) Ideals as Irreducible Downward-Closed Sets: An alternative characterisation of ideals shows that they are the *irreducible* downward-closed sets of a qo $(X, \leq)$:

Fact IV.2 (Ideals are Irreducible [20, 11, 14])**.** *Let I be a non-empty downward-closed set. The following are equivalent:*

1) *I is an ideal,*
2) *for every pair of downward-closed sets (D_1, D_2), if $I = D_1 \cup D_2$, then $I = D_1$ or $I = D_2$, and*
3) *for every pair of downward-closed sets (D_1, D_2), if $I \subseteq D_1 \cup D_2$, then $I \subseteq D_1$ or $I \subseteq D_2$.*

Example IV.3 (Finite sets). In a finite wqo $(X, =)$, any subset of X is downward-closed. The ideals are thus exactly the singletons over X: any other non-empty subset of X can be split into simpler sets. $\square$

Corollary IV.4. *An ideal I is included in a finite union $D_1 \cup \cdots \cup D_k$ of downward-closed sets $D_1, \ldots, D_k$ if and only if $I \subseteq D_j$ for some $1 \leq j \leq k$.*

2) Finite Decompositions: Observe that any downward-closed set of the form $\downarrow x$ is an ideal, hence any downward-closed set is a union of ideals. However, the main interest we find with ideals is that they provide *finite* decompositions for downward-closed subsets of wqos:

Fact IV.5 (Canonical Ideal Decompositions [20, 11, 14])**.** *Every downward-closed set over a wqo is the union of a unique finite family of incomparable (for the inclusion) ideals.*

B. Adherent Ideals

Consider some subset S of X. We call an ideal I of X an *adherent ideal* of S, and say that I is in the *adherence* of S, if there exists a directed subset $\Delta \subseteq S$ such that $\downarrow\Delta = I$.

By Fact IV.5, the downward-closure $\downarrow S$ has a canonical ideal decomposition. The following lemma shows that the ideals in this decomposition are in the adherence of S.

Lemma IV.6. *Let $S \subseteq X$. Then every maximal ideal of $\downarrow S$ is in the adherence of S.*

Later in Sec. V we will exploit Lemma IV.6 in a particular setting, where a downward-closed over-approximation D of S is known.

Lemma IV.7. *Let $S \subseteq D \subseteq X$ for D downward-closed and I be a maximal ideal of D. Then $I \subseteq \downarrow S$ if and only if I is in the adherence of S.*

C. Effective Ideal Representations

Thanks to Fact IV.5, any downward-closed set has a representation using finitely many ideals. Should we manage to find *effective* representations of wqo ideals, this will provide us with algorithmic means to manipulate downward-closed sets. This endeavour is the subject of [11, 14], and we merely provide pointers to their results here.

1) Natural Numbers: As seen in Example IV.1, the ideals of $(\mathbb{N}, \leq)$ are either $\downarrow n$ for some finite $n \in \mathbb{N}$, or the whole of $\mathbb{N}$ itself. As done classically in the VAS literature, we represent the latter using a new element noted "ω" with $n < \omega$ for all $n \in \mathbb{N}$, and denote the new set $\mathbb{N}_\omega \stackrel{\text{def}}{=} \mathbb{N} \uplus \{\omega\}$. For notational convenience, we write $\downarrow\omega$ for $\mathbb{N}$, so that an ideal of $(\mathbb{N}, \leq)$ can be written as $\downarrow x$ for x in $\mathbb{N}_\omega$.

2) Cartesian Products: Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be two wqos, and assume that we know how to represent the ideals in $\mathrm{Idl}(X)$ and $\mathrm{Idl}(Y)$. Then the ideals of $X \times Y$ equipped with the product ordering have a simple enough representation as pairs of ideals:

$$\mathrm{Idl}(X \times Y) = \{I \times J \mid I \in \mathrm{Idl}(X) \wedge J \in \mathrm{Idl}(Y)\} . \qquad (2)$$

a) Configurations: For example, configuration ideals can be represented as $\downarrow\boldsymbol{v}$ for a vector $\boldsymbol{v}$ in $\mathbb{N}_\omega^d$.

In this paper we often find it convenient to identify *partial* vectors $\boldsymbol{u}$ in $\mathbb{N}^F$ for some subset $F \subseteq \{1, \ldots, d\}$ with vectors $\boldsymbol{v}$ in $\mathbb{N}_\omega^d$ with finite values over F, such that $\boldsymbol{v}(i) = \omega$ if $i \notin F$ and $\boldsymbol{v}(i) = \boldsymbol{u}(i)$ otherwise. Then *projections* $\pi_F : \mathbb{N}_\omega^d \to \mathbb{N}_\omega^d$ on a set $F \subseteq \{1, \ldots, d\}$ can be defined for all $1 \leq i \leq d$ by

$$\pi_F(\boldsymbol{u})(i) \stackrel{\text{def}}{=} \begin{cases} \boldsymbol{u}(i) & \text{if } i \in F \\ \omega & \text{otherwise.} \end{cases} \qquad (3)$$

b) Transitions: By Dickson's Lemma, the product ordering over $\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d$ is a wqo.

A *transition ideal* is an ideal of $\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d$ that is the downward closure of a set of transitions of $\mathrm{Trans}_{\boldsymbol{A}}$. As seen in Example IV.3, the ideals of $\boldsymbol{A}$ are the singletons $\{\boldsymbol{a}\}$ for $\boldsymbol{a} \in \boldsymbol{A}$. By (2), the ideals of $\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d$ can thus be presented as downward-closures of triples $(\boldsymbol{u}, \boldsymbol{a}, \boldsymbol{v})$ in $\mathbb{N}_\omega^d \times \boldsymbol{A} \times \mathbb{N}_\omega^d$.

Transition ideals are going to form a particular class of such triples. Let us define addition over $\mathbb{Z} \uplus \{\omega\}$ by $k + \omega = \omega + k = \omega + \omega = \omega$. A *partial transition* is a triple $(\boldsymbol{u}, \boldsymbol{a}, \boldsymbol{v})$ in $\mathbb{N}_\omega^d \times \boldsymbol{A} \times \mathbb{N}_\omega^d$ such that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}$; we show in the full paper:

Lemma IV.8. *The transitions ideals of $\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d$ are exactly the sets $\downarrow t$ with t a partial transition.*

Partial transitions can also be viewed as projected transitions:

$$\pi_F((\boldsymbol{u}, \boldsymbol{a}, \boldsymbol{v})) \stackrel{\text{def}}{=} (\pi_F(\boldsymbol{u}), \boldsymbol{a}, \pi_F(\boldsymbol{v})) . \qquad (4)$$

3) Finite Sequences: In the case of sequences over a finite alphabet $(\Sigma, =)$, Jullien [19] first characterised the ideals using a simple form of regular expressions, which was later rediscovered by Abdulla et al. [1] for the verification of lossy channel systems. A representation of ideals for sequences over an arbitrary wqo $(X, \leq)$ was given by Kabil and Pouzet [20] and also rediscovered in the context of well-structured systems by Finkel and Goubault-Larrecq [11].

Assume as before that we know how to represent the ideals in $\mathrm{Idl}(X)$. Define an *atom* A over X as a language $A \subseteq X^*$ of the form $A = D^*$ where D is a downward-closed set of X— i.e. a finite union of ideals from $\mathrm{Idl}(X)$—, or form $A = I \cup \{\varepsilon\}$ where I is an ideal from $\mathrm{Idl}(X)$ and ε denotes the empty sequence. A *product* $P \subseteq X^*$ over X is a finite concatenation $P = A_1 \cdots A_k$ of atoms $A_1, \ldots, A_k$ over X. We denote by $\mathrm{Prod}(X)$ the set of products over X.

Fact IV.9. *The ideals of X^* are the products over X.*

4) Effectiveness: In order to be usable in algorithms, wqo ideals need to be effectively represented. Following Goubault-Larrecq et al. [14], one can check that all the elementary wqos $(X, \leq)$ enjoy a number of effectiveness properties. Besides some basic desiderata, among which being able to decide whether (the representation of) two elements of X coincide or are related through $\leq$, and similarly for $\mathrm{Idl}(X)$ and the inclusion ordering, our elementary wqos are in particular equipped with (see [14] for details):

- II an algorithm taking any pair of (representations of) ideals I and J in $\mathrm{Idl}(X)$ and returning (a representation of) an ideal decomposition of $I \cap J$, and
- CU' an algorithm taking any (representation of an) element x in X and returning (a representation of) an ideal decomposition of $X \setminus \uparrow x$.

By combining those two algorithms, we get:

Corollary IV.10 ([14])**.** *Let $(X, \leq)$ be an elementary wqo. There is an algorithm taking any (representation of an) ideal I in $\mathrm{Idl}(X)$ and any (representation of an) element x in X and returning (a representation of) an ideal decomposition of $I \setminus \uparrow x$.*

V. A WQO ON RUNS

The key idea in our explanation of the KLMST decomposition is to see it as building the ideals of the downward-closure of $\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ for an appropriate well quasi ordering defined by Jančar [18] and Leroux [28]. The reachability problem can then be restated as asking whether $\downarrow \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ is non empty, i.e. whether the ideal decomposition of $\downarrow \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ is empty or not.

A. Ordering Preruns and Runs

There is a natural ordering $\trianglelefteq$ of preruns. The product ordering over $\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d$ can be lifted to an embedding between sequences of tuples in $(\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d)^*$. Finally, we denote by $\trianglelefteq$ the natural ordering over $\mathrm{PreRuns}_{\boldsymbol{A}}$ (see Fig. 2 for an illustration in the particular case of runs). For a set of runs Ω, we write $\downarrow\Omega$ for its downward-closure *inside* $\mathrm{PreRuns}_{\boldsymbol{A}}$, i.e.

$$\downarrow\Omega \stackrel{\text{def}}{=} \{\rho' \in \mathrm{PreRuns}_{\boldsymbol{A}} \mid \exists \rho \in \Omega . \rho' \trianglelefteq \rho\} . \qquad (5)$$

1) Transformer Relations: Embeddings between runs can also be understood in terms of *transformer relations* (aka production relations) à la Hauschildt [16] and Leroux [28, 29]: the relation $\stackrel{\boldsymbol{c}}{\curvearrowright}$ with *capacity* $\boldsymbol{c}$ in $\mathbb{N}^d$ is the relation included in $\mathbb{N}^d \times \mathbb{N}^d$ defined by $\boldsymbol{u} \stackrel{\boldsymbol{c}}{\curvearrowright} \boldsymbol{v}$ if there exists a run from $\boldsymbol{u} + \boldsymbol{c}$ to $\boldsymbol{v} + \boldsymbol{c}$.

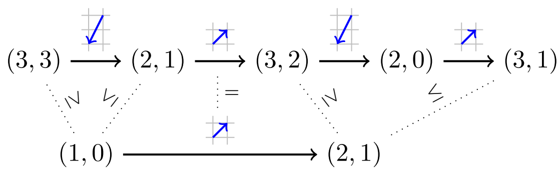

Figure 2. A run embedding for $\trianglelefteq$.

2) Run Amalgamation: Leroux [28] observed that, thanks to monotonicity, each $\overset{\boldsymbol{c}}{\curvearrowright}$ is a *periodic* relation (see Sec. II): $\mathbf{0} \overset{\boldsymbol{c}}{\curvearrowright} \mathbf{0}$, as witnessed by the empty run, and if $\boldsymbol{u} \overset{\boldsymbol{c}}{\curvearrowright} \boldsymbol{v}$ and $\boldsymbol{u}' \overset{\boldsymbol{c}}{\curvearrowright} \boldsymbol{v}'$, as witnessed by $\boldsymbol{u} + \boldsymbol{c} \xrightarrow{\sigma} \boldsymbol{v} + \boldsymbol{c}$ and $\boldsymbol{u}' + \boldsymbol{c} \xrightarrow{\sigma'} \boldsymbol{v}' + \boldsymbol{c}$ respectively, then $\boldsymbol{u} + \boldsymbol{u}' \overset{\boldsymbol{c}}{\curvearrowright} \boldsymbol{v} + \boldsymbol{v}'$ as witnessed by $\boldsymbol{u} + \boldsymbol{u}' + \boldsymbol{c} \xrightarrow{\sigma} \boldsymbol{v} + \boldsymbol{u}' + \boldsymbol{c} \xrightarrow{\sigma'} \boldsymbol{v} + \boldsymbol{v}' + \boldsymbol{c}$. Translated in terms of embeddings, the same reasoning shows:

Proposition V.1. *Let ρ_0, ρ_1, and ρ_2 be runs with $\rho_0 \trianglelefteq \rho_1, \rho_2$. Then there exists a run ρ_3 such that $\rho_1, \rho_2 \trianglelefteq \rho_3$.*

3) Prerun Ideals: By Fact IV.9 and Equation 2, the ideals of $\mathrm{PreRuns}_{\boldsymbol{A}}$ are of the form $\downarrow\boldsymbol{u} \times P \times \downarrow\boldsymbol{v}$ where $\boldsymbol{u}$ and $\boldsymbol{v}$ are in $\mathbb{N}^d_\omega$ and P is a product over $\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d$, i.e. can be represented as a regular expression over the alphabet $\mathbb{N}^d_\omega \times \boldsymbol{A} \times \mathbb{N}^d_\omega$.

B. Abstraction Refinement Procedure

Because runs are particular preruns, we can look at the downward-closure of $\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ *inside* $\mathrm{PreRuns}_{\boldsymbol{A}}$. By Fact IV.5, this set has a finite decomposition using prerun ideals from $\mathrm{Idl}(\mathrm{PreRuns}_{\boldsymbol{A}})$. This suggests an abstraction refinement procedure to compute the ideal decomposition of $\downarrow\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$.

1) A Procedure for Reachability: An idea that looks promising is to build a descending sequence of downward-closed sets $D_0 \supsetneq D_1 \supsetneq \cdots$ inside $\mathrm{PreRuns}_{\boldsymbol{A}}$ while maintaining $\downarrow\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) \subseteq D_n$ at all steps, until we find the ideal decomposition of $\downarrow\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$. By Fact IV.5 we can work with finite sets of incomparable ideals to represent the D_n's.

We start therefore with

$$D_0 \overset{\text{def}}{=} \mathrm{PreRuns}_{\boldsymbol{A}} \ . \tag{6}$$

Assume we are provided with an oracle to decide whether an ideal I from D_n is included in $\downarrow\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ and extract a counter-example otherwise. If $I \subseteq \downarrow\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ for all the (finitely many) maximal ideals I in D_n we stop; otherwise we find a maximal ideal I from the decomposition of D_n s.t.

$$\exists w \in I \setminus \downarrow\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) \tag{7}$$

and thanks to Corollary IV.10 we construct an ideal decomposition of

$$D' \overset{\text{def}}{=} I \setminus \uparrow w \tag{8}$$

and we can refine D_n and construct the downward-closed set for the next iteration—which involves removing redundant ideals—by

$$D_{n+1} \overset{\text{def}}{=} D' \cup (D_n \setminus I) \ . \tag{9}$$

The procedure terminates by Fact III.3 but depends on an oracle to perform (7).

2) Adherence Membership: Turning the previous abstraction refinement procedure into an algorithm hinges on the effective checking of $I \subseteq \downarrow\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ for a maximal prerun ideal I of D_n. By Lemma IV.7, and because $\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) \subseteq D_n$ for all n, we know that this containment check is equivalent to testing whether I is in the adherence of $\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$.

Problem: Adherence Membership of Prerun Ideals**.**

input: A d-dimensional VAS $\boldsymbol{A}$, two configurations $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{N}^d$, and an ideal I in $\mathrm{Idl}(\mathrm{PreRuns}_{\boldsymbol{A}})$.
question: Is I in the adherence of $\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$?

As we show in the full paper, this problem in its full generality is undecidable:

Theorem V.2. *The adherence membership of prerun ideals is already undecidable for ideals of the form $\downarrow\boldsymbol{x} \times D^* \times \downarrow\boldsymbol{x}$ for D a downward-closed subset of $\mathrm{Trans}_{\boldsymbol{A}}$ and $\boldsymbol{x}$ in $\mathbb{N}^d$.*

All is not lost however: we ask with the adherence membership problem for more than really needed. In the decomposition algorithm, I presents some further structure that can be exploited towards an algorithm. This motivates a deeper investigation of the properties of run ideals, which will be the object of the next sections.

VI. Locally Adherent Ideals

We start our investigation of the ideals of $\downarrow\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ by looking at rather restricted classes of runs. The treatment of this restricted case will turn out to contain most of the technical challenges of the next section on general run ideals, where we will assemble those local ideals into global ones.

More precisely, we focus on sets Ω_γ of runs of the form

$$\boldsymbol{c} + \boldsymbol{u} \xrightarrow{\sigma} \boldsymbol{c} + \boldsymbol{v} \tag{10}$$

where $\boldsymbol{c}$ is a configuration in $\mathbb{N}^d$, σ is a sequence in $\boldsymbol{A}^*$, and $(\boldsymbol{u}, \boldsymbol{v})$ is a pair of configurations in a periodic set (see Sec. II) $\boldsymbol{P}$ included in the transformer relation $\overset{\boldsymbol{c}}{\curvearrowright}$. We write γ for the pair $(\boldsymbol{c}, \boldsymbol{P})$. As we are going to see in Lemma VI.3, $\downarrow\Omega_\gamma$ is an ideal of a particular form, for which an effective representation can be found, see Sec. VI-B.

A. Periodic Transformer Subrelations

Formally, let γ denote a pair $(\boldsymbol{c}, \boldsymbol{P})$ where $\boldsymbol{c}$ is in $\mathbb{N}^d$ and $\boldsymbol{P} \subseteq \overset{\boldsymbol{c}}{\curvearrowright}$ is periodic. This is a familiar object, and we will reuse several statements from the literature. Following the notations from [29], let

- Ω_γ denote the set of runs of the form (10),
- $\boldsymbol{Q}_\gamma \subseteq \mathbb{N}^d$ denote the set of configurations $\boldsymbol{q}$ that appear along some run in Ω_γ—thus in particular $\boldsymbol{c}+\boldsymbol{u}$ and $\boldsymbol{c}+\boldsymbol{v}$ belong to $\boldsymbol{Q}_\gamma$ whenever $(\boldsymbol{u}, \boldsymbol{v})$ are in $\boldsymbol{P}$.

Example VI.1. Let us consider the 3-dimensional VAS $\boldsymbol{A} = \{\boldsymbol{a}, \boldsymbol{b}\}$ where $\boldsymbol{a} = (1, 1, -1)$ and $\boldsymbol{b} = (-1, 0, 1)$, and the pair $\gamma = (\boldsymbol{c}, \boldsymbol{P})$ where $\boldsymbol{c} = (1, 0, 1)$ and $\boldsymbol{P} = \mathbb{N}(\mathbf{0}, \boldsymbol{y})$ with

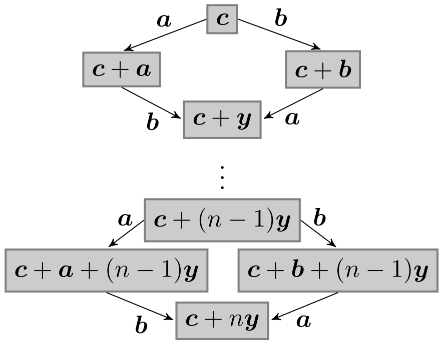

Figure 3. The set of runs Ω_γ in Example VI.1.

$\boldsymbol{y}=(0,1,0)$. Note that $\boldsymbol{P}$ is included in $\overset{\boldsymbol{c}}{\curvearrowright}$ since there exists a run $\boldsymbol{c}\xrightarrow{(\boldsymbol{ab})^n}\boldsymbol{c}+n\boldsymbol{y}$ for every n. We have

$$\Omega_\gamma=\{\boldsymbol{c}\xrightarrow{w_1\cdots w_n}\boldsymbol{c}+n\boldsymbol{y}\mid n\in\mathbb{N},w_j\in\{\boldsymbol{ab},\boldsymbol{ba}\}\}\ ,$$
$$\boldsymbol{Q}_\gamma=(\boldsymbol{c}+\boldsymbol{a}+\mathbb{N}\boldsymbol{y})\cup(\boldsymbol{c}+\mathbb{N}\boldsymbol{y})\cup(\boldsymbol{c}+\boldsymbol{b}+\mathbb{N}\boldsymbol{y})\ .$$

The set Ω_γ is depicted in Fig. 3. $\square$

1) Saturated Pairs: We denote by F^{in}_γ (resp. F^{out}_γ) the sets of indices i such that $\boldsymbol{u}(i)=0$ (resp. $\boldsymbol{v}(i)=0$) for every pair $(\boldsymbol{u},\boldsymbol{v})\in\boldsymbol{P}$. We say that a pair $(\boldsymbol{u},\boldsymbol{v})$ in $\boldsymbol{P}$ *saturates* $(F^{\text{in}}_\gamma,F^{\text{out}}_\gamma)$ if $\boldsymbol{u}(i)=0$ implies $i\in F^{\text{in}}_\gamma$ and $\boldsymbol{v}(i)=0$ implies $i\in F^{\text{out}}_\gamma$. Since $\boldsymbol{P}$ is periodic, by summing at most $2d$ pairs in $\boldsymbol{P}$, we see that there exist pairs in $\boldsymbol{P}$ that saturate $(F^{\text{in}}_\gamma,F^{\text{out}}_\gamma)$.

By projecting $\boldsymbol{c}$, we obtain two partial configurations $\boldsymbol{s}^{\text{in}}_\gamma$ and $\boldsymbol{s}^{\text{out}}_\gamma$:

$$\boldsymbol{s}^{\text{in}}_\gamma\overset{\text{def}}{=}\pi_{F^{\text{in}}_\gamma}(\boldsymbol{c})\ ,\qquad\boldsymbol{s}^{\text{out}}_\gamma\overset{\text{def}}{=}\pi_{F^{\text{out}}_\gamma}(\boldsymbol{c})\ .\tag{11}$$

Example VI.1 (continued). We have for our example:

$$F^{\text{in}}_\gamma=\{1,2,3\}\qquad F^{\text{out}}_\gamma=\{1,3\}\ ,$$
$$\boldsymbol{s}^{\text{in}}_\gamma=(1,0,1)\ ,\qquad\boldsymbol{s}^{\text{out}}_\gamma=(1,\omega,1)\ .$$

Note that $(\boldsymbol{0},\boldsymbol{y})$ saturates $(F^{\text{in}}_\gamma,F^{\text{out}}_\gamma)$. $\square$

B. Representation through Marked Witness Graphs

We investigate in this section how to effectively represent $\downarrow\Omega_\gamma$. In the sequel, we show that this ideal can be represented using the set of edges of a strongly connected graph called a *witness graph* (see Lemma VI.2) enjoying some *pumping* properties with respect to $\boldsymbol{s}^{\text{in}}_\gamma$ and $\boldsymbol{s}^{\text{out}}_\gamma$ (see Lemma VI.4). Such graphs will turn out to be exactly the ones employed by Lambert [24] in his variant of the KLMST decomposition (see also [27]).

1) Marked Witness Graphs: A *witness graph* is a strongly connected directed graph $G=(\boldsymbol{S},E,\boldsymbol{s})$ where $\boldsymbol{S}$ is a non-empty finite set of partial configurations in $\mathbb{N}^F$ for some $F\subseteq\{1,\ldots,d\}$, $E\subseteq\boldsymbol{S}\times\boldsymbol{A}\times\boldsymbol{S}$ is a finite set of partially defined transitions, and $\boldsymbol{s}$ is a distinguished state in $\boldsymbol{S}$.

A *marked witness graph* is a triple $M=(\boldsymbol{s}^{\text{in}},G,\boldsymbol{s}^{\text{out}})$ where G is a witness graph, and $\boldsymbol{s}^{\text{in}}$ and $\boldsymbol{s}^{\text{out}}$ are partial configurations in $\mathbb{N}^{F^{\text{in}}}$ and $\mathbb{N}^{F^{\text{out}}}$ for some $F^{\text{in}},F^{\text{out}}\supseteq F$

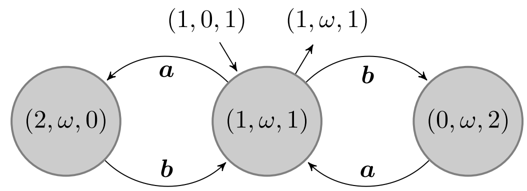

Figure 4. The graph G_γ with its input $\boldsymbol{s}^{\text{in}}_\gamma$ and output $\boldsymbol{s}^{\text{out}}_\gamma$ for Example VI.1.

such that $\pi_F(\boldsymbol{s}^{\text{in}})=\pi_F(\boldsymbol{s}^{\text{out}})=\boldsymbol{s}$. We associate with M the set Ω_M of runs ρ of the form $\boldsymbol{x}\xrightarrow{\sigma}\boldsymbol{y}$ where σ is the label of a cycle on $\boldsymbol{s}$ in G, and such that $\boldsymbol{s}^{\text{in}}=\pi_{F^{\text{in}}}(\boldsymbol{x})$ and $\boldsymbol{s}^{\text{out}}=\pi_{F^{\text{out}}}(\boldsymbol{y})$.

2) Projected Graphs: Let $F_\gamma\subseteq\{1,\ldots,d\}$ denote the set of indices i such that $\{\boldsymbol{q}(i)\mid\boldsymbol{q}\in\boldsymbol{Q}_\gamma\}$ is finite, i.e. the indices where $\boldsymbol{Q}_\gamma$ remains bounded. Note that this entails $F_\gamma\subseteq F^{\text{in}}_\gamma$ and $F_\gamma\subseteq F^{\text{out}}_\gamma$. We denote by π_γ the projection function π_{F_γ}.

Observe that the projection $\boldsymbol{S}_\gamma\overset{\text{def}}{=}\pi_\gamma(\boldsymbol{Q}_\gamma)$ of $\boldsymbol{Q}_\gamma$ is finite, and so is E_γ the set of partial transitions $(\pi_\gamma(\boldsymbol{q}),\boldsymbol{a},\pi_\gamma(\boldsymbol{q}'))$ where $(\boldsymbol{q},\boldsymbol{a},\boldsymbol{q}')$ appears in some run in Ω_γ. We distinguish $\boldsymbol{s}_\gamma\overset{\text{def}}{=}\pi_\gamma(\boldsymbol{c})$ as a particular state in $\boldsymbol{S}_\gamma$. We denote by $G_\gamma\overset{\text{def}}{=}(\boldsymbol{S}_\gamma,E_\gamma,\boldsymbol{s}_\gamma)$ the finite labelled directed graph defined by projecting the runs in Ω_γ, and $M_\gamma\overset{\text{def}}{=}(\boldsymbol{s}^{\text{in}}_\gamma,G_\gamma,\boldsymbol{s}^{\text{out}}_\gamma)$ the corresponding marked graph with input $\boldsymbol{s}^{\text{in}}_\gamma$ and output $\boldsymbol{s}^{\text{out}}_\gamma$.

Example VI.1 (continued). Projecting $\boldsymbol{Q}_\gamma$ on $F_\gamma=\{1,3\}$ yields $\pi_\gamma(\boldsymbol{c}+\boldsymbol{a}+n\boldsymbol{y})=(2,\omega,0)$, $\pi_\gamma(\boldsymbol{c}+n\boldsymbol{y})=(1,\omega,1)$, and $\pi_\gamma(\boldsymbol{c}+\boldsymbol{b}+n\boldsymbol{y})=(0,\omega,2)$:

$$\boldsymbol{s}_\gamma=(1,\omega,1)\ ,\quad\boldsymbol{S}_\gamma=\{(2,\omega,0),(1,\omega,1),(0,\omega,2)\}\ .$$

The graph G_γ is depicted on Fig. 4. $\square$

We associate to a prerun $\rho=(\boldsymbol{x},t_1\cdots t_k,\boldsymbol{y})$ and a set $F\subseteq\{1,\ldots,d\}$, the partial prerun:

$$\pi_F(\rho)\overset{\text{def}}{=}(\pi_F(\boldsymbol{x}),\pi_F(t_1)\cdots\pi_F(t_k),\pi_F(\boldsymbol{y}))$$

If ρ is a run in Ω_γ, then $\pi_\gamma(\rho)$ is a path inside G_γ, and by [29, Corollary VIII.5], $\pi_\gamma(\boldsymbol{x})=\pi_\gamma(\boldsymbol{y})=\boldsymbol{s}_\gamma$, which means that this path is actually a cycle in G_γ. This in turn shows that G_γ is strongly connected. This proves:

Lemma VI.2. *The marked graph M_γ is a marked witness graph such that $\Omega_\gamma\subseteq\Omega_{M_\gamma}$.*

3) Intraproductions: An *intraproduction* for γ is a vector $\boldsymbol{h}$ in $\mathbb{N}^d$ such that $\boldsymbol{c}+\boldsymbol{h}$ belongs to $\boldsymbol{Q}_\gamma$. We denote by $\boldsymbol{H}_\gamma$ the set of intraproductions for γ; note that it contains in particular $\boldsymbol{u}$ and $\boldsymbol{v}$ if $(\boldsymbol{u},\boldsymbol{v})\in\boldsymbol{P}$.

Leroux [29, Lemma VIII.3] shows that $\boldsymbol{H}_\gamma$ is periodic and $\boldsymbol{Q}_\gamma+\boldsymbol{H}_\gamma\subseteq\boldsymbol{Q}_\gamma$. Following the proof of that lemma, denoting by T_γ the set of transitions occurring along runs of Ω_γ, we deduce that if $t=(\boldsymbol{p},\boldsymbol{a},\boldsymbol{q})$ is in T_γ, and $\boldsymbol{h}$ in $\boldsymbol{H}_\gamma$ is an intraproduction, then the transition $t+\boldsymbol{h}\overset{\text{def}}{=}(\boldsymbol{p}+\boldsymbol{h},\boldsymbol{a},\boldsymbol{q}+\boldsymbol{h})$ also occurs in some run of Ω_γ, i.e. $t+\boldsymbol{h}\in T_\gamma$. It follows that, if $\boldsymbol{h}$ in $\boldsymbol{H}_\gamma$ is such that $\boldsymbol{h}(i)>0$ for some index i, then

i cannot belong to F_γ, since $\boldsymbol{c}+n\boldsymbol{h}$ is in $\boldsymbol{Q}_\gamma$ for all n. This entails in particular that $\boldsymbol{h}=\boldsymbol{0}$ if $F_\gamma=\{1,\ldots,d\}$.

A kind of converse property sometimes holds: we say that an intraproduction $\boldsymbol{h}$ in $\boldsymbol{H}_\gamma$ *saturates* F_γ if whenever $\boldsymbol{h}(i)=0$, then i belongs to F_γ, and therefore $F_\gamma=\{i \mid \boldsymbol{h}(i)=0\}$. Leroux [29, Lemma VIII.3] shows there exist intraproductions $\boldsymbol{h}$ in $\boldsymbol{H}_\gamma$ that saturate F_γ.

Example VI.1 (continued). To continue with our example, the set of intraproductions is $\boldsymbol{H}_\gamma=\mathbb{N}\boldsymbol{y}$. The only non-saturated intraproduction is $\boldsymbol{0}$, as any $n\boldsymbol{y}$ with $n>0$ saturates F_γ. $\square$

By similarly shifting every word $w=t_1\ldots t_k$ of transitions in T_γ^* to the word $w+\boldsymbol{h}\stackrel{\text{def}}{=}(t_1+\boldsymbol{h})\cdots(t_k+\boldsymbol{h})$ where $\boldsymbol{h}$ is an intraproduction that saturates F_γ, we can show the following characterisation of $\downarrow\Omega_\gamma$:

Lemma VI.3. *The following equality holds:*

$$\downarrow\Omega_\gamma=\downarrow\boldsymbol{s}_\gamma^{\text{in}}\times(\downarrow E_\gamma)^*\times\downarrow\boldsymbol{s}_\gamma^{\text{out}}\ .$$

Leroux [29, Lemma VIII.11] shows that $\boldsymbol{S}_\gamma$ is a set of incomparable partial configurations. Therefore the partial transitions in E_γ are incomparable. The previous lemma then shows that E_γ is the unique finite set of incomparable elements in $\mathbb{N}_\omega^d\times\boldsymbol{A}\times\mathbb{N}_\omega^d$ satisfying Lemma VI.3.

4) Pumpable Configurations: A partial configuration $\boldsymbol{x}$ in $\mathbb{N}_\omega^d$ is said to be *forward pumpable* by a witness graph $G=(\boldsymbol{S},E,\boldsymbol{s})$ if there exists a cycle on $\boldsymbol{s}$ labelled by a word σ_+, and a run using this label $\boldsymbol{x}\xrightarrow{\sigma_+}\boldsymbol{x}'$ with $\boldsymbol{x}\le\boldsymbol{x}'$ such that $\downarrow\boldsymbol{s}=\bigcup_n\downarrow\boldsymbol{x}_n$, where $\boldsymbol{x}_n$ is the configuration defined by $\boldsymbol{x}\xrightarrow{\sigma_+^n}\boldsymbol{x}_n$ (such a configuration exists by monotonicity). Symmetrically, a partial configuration $\boldsymbol{y}$ in $\mathbb{N}_\omega^d$ is said to be *backward pumpable* by a witness graph $G=(\boldsymbol{S},E,\boldsymbol{s})$ if there exists a cycle on $\boldsymbol{s}$ labelled by a word σ_-, and a run $\boldsymbol{y}'\xrightarrow{\sigma_-}\boldsymbol{y}$ with $\boldsymbol{y}\le\boldsymbol{y}'$ such that $\downarrow\boldsymbol{s}=\bigcup_n\downarrow\boldsymbol{y}_n$ where $\boldsymbol{y}_n$ is the configuration defined by $\boldsymbol{y}_n\xrightarrow{\sigma_-^n}\boldsymbol{y}$.

Saturated intraproductions also provide a way to prove that the graph input $\boldsymbol{s}_\gamma^{\text{in}}$ and output $\boldsymbol{s}_\gamma^{\text{out}}$ are pumpable.

Lemma VI.4. *The input $\boldsymbol{s}_\gamma^{\text{in}}$ is forward pumpable by G_γ, and the output $\boldsymbol{s}_\gamma^{\text{out}}$ is backward pumpable by G_γ.*

Proof. Let $\boldsymbol{h}$ be an intraproduction that saturates F_γ. There exists a run $\rho\stackrel{\text{def}}{=}\boldsymbol{c}+\boldsymbol{u_h}\xrightarrow{\sigma_+}\boldsymbol{c}+\boldsymbol{h}\xrightarrow{\sigma_-}\boldsymbol{c}+\boldsymbol{v_h}$ in Ω_γ. The projection $\pi_\gamma(\rho)$ shows that σ_+,σ_- are cycles on $\boldsymbol{s}_\gamma$. Moreover, by projecting over F_γ^{in} the run $\boldsymbol{c}+\boldsymbol{u_h}\xrightarrow{\sigma_+}\boldsymbol{c}+\boldsymbol{h}$ we see that $\boldsymbol{s}_\gamma^{\text{in}}\xrightarrow{\sigma_+}\boldsymbol{s}_\gamma^{\text{in}}+\boldsymbol{h}$. Hence $\boldsymbol{s}_\gamma^{\text{in}}$ is forward pumpable by G_γ. Symmetrically $\boldsymbol{s}_\gamma^{\text{out}}$ is backward pumpable by G_γ. $\square$

VII. Globally Adherent Ideals

Our understanding of the KLMST decomposition is that it builds an ideal decomposition of $\downarrow\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})$ inside $\mathrm{PreRuns}_{\boldsymbol{A}}$. We have seen in Sec. V-A how to represent prerun ideals. However we should expect the maximal ideals of $\downarrow\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})$ to have additional properties besides adherence, and indeed we shall see they can be represented using the structures employed in the KLMST decomposition.

The starting point for our characterisation of run ideals is to consider some finite basis B of $(\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}),\trianglelefteq)$: if we consider the upward closure $\uparrow\rho\cap\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})$ of each run ρ in B *inside* $\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})$, we obtain again

$$\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})=\bigcup_{\rho\in B}\uparrow\rho\cap\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})\ . \tag{12}$$

Taking the downward-closure inside $\mathrm{PreRuns}_{\boldsymbol{A}}$ then yields

$$\downarrow\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})=\bigcup_{\rho\in B}\downarrow(\uparrow\rho\cap\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}))\ , \tag{13}$$

prompting the study of $\downarrow(\uparrow\rho\cap\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}))$. By Proposition V.1, each set $\downarrow(\uparrow\rho\cap\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}))$ is an ideal, for which we want to find a representation.

A. Perfect Runs

Let us accordingly fix a run $\rho=\boldsymbol{c}_0\xrightarrow{\boldsymbol{a}_1}\boldsymbol{c}_1\cdots\boldsymbol{c}_{k-1}\xrightarrow{\boldsymbol{a}_k}\boldsymbol{c}_k$ with $\boldsymbol{x}=\boldsymbol{c}_0$ and $\boldsymbol{y}=\boldsymbol{c}_k$ throughout this subsection.

1) Transformer Relations Along a Run: Consider the relation $\boldsymbol{R}$ of tuples $((\boldsymbol{u}_0,\boldsymbol{v}_0),\ldots,(\boldsymbol{u}_k,\boldsymbol{v}_k))$ of pairs in $\mathbb{N}^d\times\mathbb{N}^d$ such that:

$$\boldsymbol{0}=\boldsymbol{u}_0\overset{\boldsymbol{c}_0}{\curvearrowright}\boldsymbol{v}_0=\boldsymbol{u}_1\overset{\boldsymbol{c}_1}{\curvearrowright}\boldsymbol{v}_1\cdots=\boldsymbol{u}_k\overset{\boldsymbol{c}_k}{\curvearrowright}\boldsymbol{v}_k=\boldsymbol{0} \tag{14}$$

and let us introduce the relation $\boldsymbol{P}_j$ defined for $0\le j\le k$ by:

$$\boldsymbol{P}_j\stackrel{\text{def}}{=}\{(\boldsymbol{u}_j,\boldsymbol{v}_j)\mid((\boldsymbol{u}_0,\boldsymbol{v}_0),\ldots,(\boldsymbol{u}_k,\boldsymbol{v}_k))\in\boldsymbol{R}\}\ . \tag{15}$$

Informally, each $\boldsymbol{P}_j$ is the subset of $\overset{\boldsymbol{c}_j}{\curvearrowright}$ that can be completed into some run in $\uparrow\rho\cap\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})$. We can check that $\boldsymbol{R}$ and each $\boldsymbol{P}_j$ is a periodic relation since each transformer relation is periodic.

2) Global Ideal Representation: Denoting by γ_j the pair $(\boldsymbol{c}_j,\boldsymbol{P}_j)$, we derive from Lemma VI.3 the following equality:

$$\downarrow\Omega_{\gamma_j}\ =\ \downarrow\boldsymbol{s}_{\gamma_j}^{\text{in}}\times(\downarrow E_{\gamma_j})^*\times\downarrow\boldsymbol{s}_{\gamma_j}^{\text{out}}\ . \tag{16}$$

Notice that $\boldsymbol{s}_{\gamma_0}^{\text{in}}=\boldsymbol{x}$ and $\boldsymbol{s}_{\gamma_k}^{\text{out}}=\boldsymbol{y}$. Moreover, the triple $e_j\stackrel{\text{def}}{=}(\boldsymbol{s}_{\gamma_{j-1}}^{\text{out}},\boldsymbol{a}_j,\boldsymbol{s}_{\gamma_j}^{\text{in}})$ is a partial transition for every $1\le j\le k$. Observe that $\downarrow(\uparrow\rho\cap\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}))$ is included in

$$\downarrow\boldsymbol{x}\times(\downarrow E_{\gamma_0})^*\cdot A_0\cdot(\downarrow E_{\gamma_1})^*\cdots A_k\cdot(\downarrow E_{\gamma_k})^*\times\downarrow\boldsymbol{y} \tag{17}$$

where A_j is the atom $\downarrow e_j\cup\{\varepsilon\}$. The converse inclusion will be a consequence of Lemma VII.2 and Lemma VII.4.

In the upcoming subsection, we derive a condition satisfied by the following sequence ξ_ρ of interspersed marked witness graphs and actions, which allows to represent the ideal (17):

$$\xi_\rho\stackrel{\text{def}}{=}M_{\gamma_0},\boldsymbol{a}_1,M_{\gamma_1},\ldots,\boldsymbol{a}_k,M_{\gamma_k}\ . \tag{18}$$

B. Perfect Marked Witness Graph Sequences

A *marked witness graph sequence* ξ is a sequence

$$\xi\ =\ M_0,\boldsymbol{a}_1,M_1,\ldots\boldsymbol{a}_k,M_k\ , \tag{19}$$

where $M_0,\ldots,M_k$ are marked witness graphs and $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_k$ are actions in $\boldsymbol{A}$. In the sequel, M_j denotes the marked witness graph $(\boldsymbol{s}_j^{\text{in}},G_j,\boldsymbol{s}_j^{\text{out}})$ where G_j is the witness graph

$(\boldsymbol{S}_j, E_j, \boldsymbol{s}_j)$. The sets $F_j^{\text{in}}, F_j, F_j^{\text{out}}$ denote the finite coordinates of $\boldsymbol{s}_j^{\text{in}}, \boldsymbol{s}_j, \boldsymbol{s}_j^{\text{out}}$. The two partial configurations $\boldsymbol{s}_0^{\text{in}}$ and $\boldsymbol{s}_k^{\text{out}}$ are assumed to be respectively $\boldsymbol{x}$ and $\boldsymbol{y}$. Such sequences ξ are also called *marked graph-transition sequences* in [24], and are the structures maintained throughout the KLMST decomposition algorithm.

1) Ideals and Runs: A marked witness graph sequence ξ defines a prerun ideal

$$I_\xi \stackrel{\text{def}}{=} \downarrow\boldsymbol{x} \times (\downarrow E_0)^* \cdot A_1 \cdot (\downarrow E_1)^* \cdots A_k \cdot (\downarrow E_k)^* \times \downarrow\boldsymbol{y} \quad (20)$$

where $A_j \stackrel{\text{def}}{=} \downarrow(\boldsymbol{s}_{j-1}^{\text{out}}, \boldsymbol{a}_j, \boldsymbol{s}_j^{\text{in}}) \cup \{\varepsilon\}$ for all $1 \leq j \leq k$. It is also associated with a set of runs Ω_ξ of the form

$$\boldsymbol{x}_0 \xrightarrow{\sigma_0} \boldsymbol{y}_0 \xrightarrow{\boldsymbol{a}_1} \boldsymbol{x}_1 \xrightarrow{\sigma_1} \boldsymbol{y}_1 \cdots \xrightarrow{\boldsymbol{a}_k} \boldsymbol{x}_k \xrightarrow{\sigma_k} \boldsymbol{y}_k \quad (21)$$

where each $\boldsymbol{x}_j \xrightarrow{\sigma_j} \boldsymbol{y}_j$ is a run in Ω_{M_j}. Note that $\downarrow\Omega_\xi \subseteq I_\xi$.

We show next in Lemma VII.2 that for marked witness graph sequences ξ which satisfy the *perfectness* condition of Lambert [24]—which is mostly equivalent to Kosaraju's θ *condition*—, the prerun ideal I_ξ associated with ξ is adherent. This condition is not arbitrary, but stems from the properties of the sequences ξ_ρ we derived in sections VI and VII.

2) Perfectness Condition: Perfectness is defined by introducing a linear system over the natural numbers that denotes a set L_ξ of solutions. This linear system relies on a binary relation $\overset{\psi}{\dashrightarrow}$ over configurations in $\mathbb{N}^d$, where $\psi\colon E \to \mathbb{N}$ denotes some function defined on a finite set E of partial transitions. The relation is defined by $\boldsymbol{x} \overset{\psi}{\dashrightarrow} \boldsymbol{y}$ if $\boldsymbol{y} = \boldsymbol{x} + \sum_{e \in E} \psi(e)\Delta(e)$, where $\Delta(e) \stackrel{\text{def}}{=} \boldsymbol{a}$ for a partial transition e labelled by $\boldsymbol{a}$.

Let L_ξ be the set of tuples $(\boldsymbol{x}_0, \psi_0, \boldsymbol{y}_0, \ldots, \boldsymbol{x}_k, \psi_k, \boldsymbol{y}_k)$ where $\psi_j\colon E_j \to \mathbb{N}$ is a function satisfying for every $\boldsymbol{s} \in \boldsymbol{S}_j$:

$$\sum_{e \in E_j | \text{tgt}(e) = \boldsymbol{s}} \psi_j(e) = \sum_{e \in E_j | \text{src}(e) = \boldsymbol{s}} \psi_j(e)$$

and $\boldsymbol{x}_0, \boldsymbol{y}_0, \ldots, \boldsymbol{x}_k, \boldsymbol{y}_k$ are configurations in $\mathbb{N}^d$ such that

$$\boldsymbol{x}_0 \overset{\psi_0}{\dashrightarrow} \boldsymbol{y}_0 \xrightarrow{\boldsymbol{a}_1} \boldsymbol{x}_1 \overset{\psi_1}{\dashrightarrow} \boldsymbol{y}_1 \cdots \boldsymbol{x}_k \overset{\psi_k}{\dashrightarrow} \boldsymbol{y}_k$$

and such that for every $0 \leq j \leq k$

$$\pi_{F_j^{\text{in}}}(\boldsymbol{x}_j) = \boldsymbol{s}_j^{\text{in}} \ \wedge \ \pi_{F_j^{\text{out}}}(\boldsymbol{y}_j) = \boldsymbol{s}_j^{\text{out}} \ .$$

Notice that L_ξ is defined as solutions of a linear system. Moreover, for every run in Ω_ξ of the form (21), by introducing the Parikh image $\psi_j\colon E_j \to \mathbb{N}$ of the cycle on $\boldsymbol{s}_j$ labelled by σ_j, we get a sequence $((\boldsymbol{x}_0, \psi_1, \boldsymbol{x}_1), \ldots, (\boldsymbol{x}_k, \psi_k, \boldsymbol{y}_k))$ in L_ξ.

Definition VII.1. *A marked witness graph sequence is said to be* perfect *if it satisfies the following conditions for all j:*

- *$\boldsymbol{s}_j^{\text{in}}$ and $\boldsymbol{s}_j^{\text{out}}$ are respectively forward and backward pumpable by G_j,*
- *$\sup \boldsymbol{X}_j = \boldsymbol{s}_j^{\text{in}}$ and $\sup \boldsymbol{Y}_j = \boldsymbol{s}_j^{\text{out}}$,*
- *$\sup \Psi_j(e) = \omega$ for every $e \in E_j$, and*

where $\boldsymbol{X}_j$, Ψ_j, and $\boldsymbol{Y}_j$ are resp. the sets of elements $\boldsymbol{x}_j$, ψ_j, and $\boldsymbol{y}_j$ such that $((\boldsymbol{x}_0, \psi_0, \boldsymbol{y}_0), \ldots, (\boldsymbol{x}_k, \psi_k, \boldsymbol{y}_k)) \in L_\xi$.

Perfect witness graph sequences denote adherent ideals:

Lemma VII.2. *If ξ is a perfect marked witness graph sequence, then I_ξ is in the adherence of* $\text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ *and* $I_\xi = \downarrow\Omega_\xi$.

Proof. The proof comes from [24, Lemma 4.1] and shows that a directed family of runs of the following form can always be extracted from a perfect marked witness graph sequence:

$$\boldsymbol{x}_{0,n} \xrightarrow{\sigma_{+,0}^n \sigma_0^n w_0 \sigma_{-,0}^n} \boldsymbol{y}_{0,n} \xrightarrow{\boldsymbol{a}_1} \boldsymbol{x}_{1,n} \cdots \boldsymbol{x}_{k,n} \xrightarrow{\sigma_{+,k}^n \sigma_k^n w_k \sigma_{-,k}^n} \boldsymbol{y}_{k,n} \quad (22)$$

such that each run family $\boldsymbol{x}_{j,n} \xrightarrow{\sigma_{+,j}^n \sigma_j^n w_j \sigma_{-,j}^n} \boldsymbol{y}_{j,n}$ is directed with $\downarrow\Omega_{M_j}$ as downward-closure. Intuitively, $\sigma_{+,j}$ pumps up the components in $F_j^{\text{in}} \setminus F_j$, $\sigma_{-,j}$ pumps down those in $F_j^{\text{out}} \setminus F_j$, and σ_j is the label of a cycle on $\boldsymbol{s}_j$ such that every transition in E_j occurs at least once along the cycle. The sequence w_j comes from a solution of the linear system L_ξ. $\square$

3) Deciding Perfectness: We can decide if a marked witness graph sequence is perfect as follows. First of all, observe that checking if a partial configuration $\boldsymbol{x} \in \mathbb{N}_\omega^d$ is pumpable (either backward or forward) by a witness graph $G = (\boldsymbol{S}, E, \boldsymbol{s})$ can be performed in exponential space since this problem reduces to the place boundedness problem for vector addition systems [2, 8]. Moreover, since we can compute the unbounded components of the set of solutions of a linear system on $\mathbb{N}$ in nondeterministic polynomial time, we can effectively do this computation on sets L_ξ of solutions for marked witness graph sequences ξ. Hence:

Lemma VII.3. *The perfectness of a marked witness graph sequence is decidable in exponential space.*

C. Run Ideals

We have seen that the downward closed set $\downarrow \text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ can be decomposed as a finite union of ideals I_{ξ_ρ} where ξ_ρ is the marked witness graph sequence associated to ρ. By the following lemma, this implies that $\downarrow \text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ can be represented using a finite set of perfect marked witness graph sequences.

Lemma VII.4. *The marked witness graph sequence ξ_ρ is perfect for every run ρ.*

Proof. By Lemma VI.4, for all j, $\boldsymbol{s}_{\gamma_j}^{\text{in}}$ and $\boldsymbol{s}_{\gamma_j}^{\text{in}}$ are resp. forward and backward pumpable by G_{γ_j}.

Regarding the conditions on L_{ξ_ρ}, for every tuple $((\boldsymbol{u}_0, \boldsymbol{v}_0), \ldots, (\boldsymbol{u}_k, \boldsymbol{v}_k))$ in $\boldsymbol{R}$, every sequence family $(\sigma_j)_{1 \leq j \leq k}$ in $\boldsymbol{A}^*$ such that $\rho_j \stackrel{\text{def}}{=} (\boldsymbol{c}_j + \boldsymbol{u}_j \xrightarrow{\sigma_j} \boldsymbol{c}_j + \boldsymbol{v}_j)$, and every $n \in \mathbb{N}$, we observe that

$$((\boldsymbol{c}_0 + n\boldsymbol{u}_0, n\psi_0, \boldsymbol{c}_0 + n\boldsymbol{v}_0), \ldots, (\boldsymbol{c}_k + n\boldsymbol{u}_k, n\psi_k, \boldsymbol{c}_k + n\boldsymbol{v}_k))$$

is in L_{ξ_ρ} where $\psi_j\colon E_j \to \mathbb{N}$ is the Parikh image of the cycle $\pi_{\gamma_j}(\rho_j)$ on $\boldsymbol{s}_j$ in G_j. In particular, if $\boldsymbol{s}_j^{\text{in}}(i) = \omega$ for some $i \in F_{\gamma_j}^{\text{in}}$ and some $0 \leq j \leq k$, then there exists $(\boldsymbol{u}_j, \boldsymbol{v}_j) \in \boldsymbol{P}_j$ such that $\boldsymbol{u}_j(i) > 0$. By completing this pair as a tuple $((\boldsymbol{u}_0, \boldsymbol{v}_0), \ldots, (\boldsymbol{u}_k, \boldsymbol{v}_k))$ in $\boldsymbol{R}$, we deduce that $\sup \boldsymbol{X}_j(i) = \omega$.

Thus $\sup \boldsymbol{X}_j = \boldsymbol{s}^{\mathrm{in}}_{\gamma_j}$, and we get similarly $\sup \boldsymbol{Y}_j = \boldsymbol{s}^{\mathrm{out}}_{\gamma_j}$ and $\sup \Psi_j(e) = \omega$ for every $e \in E_j$. Thus ξ_ρ is perfect. $\square$

Theorem VII.5. *For any perfect marked witness graph sequence ξ, $I_\xi \subseteq \downarrow \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})$. Moreover, there exists a* finite *set Ξ of perfect marked witness graph sequences such that*

$$\downarrow \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}) = \bigcup_{\xi\in\Xi} I_\xi \;.$$

VIII. The Decomposition Algorithm

We explain succinctly in this section how the classical KLMST algorithm of Mayr, Kosaraju, and Lambert computes the decomposition of $\downarrow \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})$ into ideals. By Theorem VII.5 these ideals can be presented as finite families of perfect marked witness graph sequences.

The KLMST algorithm operates along the same general lines as the abstraction refinement procedure of Sec. V-B. It refines successively a finite family Ξ_n of marked witness graph sequences from $\boldsymbol{x}$ to $\boldsymbol{y}$ while maintaining as an invariant

$$\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}) = \bigcup_{\xi\in\Xi_n} \Omega_\xi \tag{23}$$

for all n. Because $\downarrow\Omega_\xi \subseteq I_\xi$ for all ξ, this implies

$$\downarrow \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}) \subseteq D_n \stackrel{\text{def}}{=} \bigcup_{\xi\in\Xi_n} I_\xi \tag{24}$$

as in the abstraction refinement procedure.

If every marked witness graph sequence in Ξ_n is perfect (which is decidable by Lemma VII.3), the algorithm stops, since by Lemma VII.2

$$\downarrow \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}) = \bigcup_{\xi\in\Xi_n} I_\xi \;. \tag{25}$$

Otherwise, the family Ξ_n is decomposed into a new family Ξ_{n+1} as follows: we pick a marked witness graph sequence $\xi \in \Xi_n$ that is not perfect. The imperfectness of ξ provides a way of computing a new finite family $\mathrm{dec}(\xi)$ of marked witness graph sequences from $\boldsymbol{x}$ to $\boldsymbol{y}$ (see Sec. VIII-B) with

$$\Omega_\xi = \bigcup_{\xi'\in\mathrm{dec}(\xi)} \Omega_{\xi'} \;. \tag{26}$$

The family Ξ_{n+1} is then defined as

$$\Xi_{n+1} \stackrel{\text{def}}{=} (\Xi_n \setminus \{\xi\}) \cup \mathrm{dec}(\xi) \;. \tag{27}$$

Termination is ensured through a ranking function relating ξ with each sequence in $\mathrm{dec}(\xi)$, see Sec. VIII-C. This shows:

Theorem VIII.1 (Decomposition Theorem)**.** *The ideal decomposition of $\downarrow \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})$ inside $\mathrm{PreRuns}_{\boldsymbol{A}}$ is effectively computable.*

Because $\downarrow \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}) = \emptyset$ if and only if $\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}) = \emptyset$, this yields:

Theorem VIII.2 (Mayr [32], Kosaraju [22], Lambert [24])**.** *VAS reachability is decidable.*

A. Initial Family

The KLMST algorithm starts with an initial family Ξ_0 containing a single marked witness graph sequence ξ_0, itself reduced to a single marked witness graph $M \stackrel{\text{def}}{=} (\boldsymbol{x}, G, \boldsymbol{y})$ where $G \stackrel{\text{def}}{=} (\boldsymbol{S}, E, \boldsymbol{s})$ is defined by $\boldsymbol{s} = (\omega,\ldots,\omega)$, $\boldsymbol{S} = \{\boldsymbol{s}\}$, and $E = \boldsymbol{S}\times\boldsymbol{A}\times\boldsymbol{S}$. Note that $\Omega_{\xi_0} = \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})$ and

$$\downarrow \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}) \subseteq D_0 = {\downarrow}\boldsymbol{x} \times (\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d)^* \times {\downarrow}\boldsymbol{y} \;. \tag{28}$$

B. Decomposition

Let us fix a marked witness graph sequence ξ that is not perfect, and let us recall how the finite family $\mathrm{dec}(\xi)$ is obtained in the KLMST algorithm. We assume that

$$\xi \;=\; M_0, \boldsymbol{a}_1, M_1, \ldots \boldsymbol{a}_k, M_k \;,$$

where $M_0,\ldots,M_k$ are marked witness graphs, and $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_k$ are actions in $\boldsymbol{A}$. In the sequel, M_j denotes the marked witness graph $(\boldsymbol{s}^{\mathrm{in}}_j, G_j, \boldsymbol{s}^{\mathrm{out}}_j)$ and G_j is the witness graph $(\boldsymbol{S}_j, E_j, \boldsymbol{s}_j)$. We let F^{in}_j, F_j, F^{out}_j be respectively the finite components of $\boldsymbol{s}^{\mathrm{in}}_j$, $\boldsymbol{s}_j$ and $\boldsymbol{s}^{\mathrm{out}}_j$.

Remark VIII.3. The main difference between the KLMST algorithm and the abstraction refinement procedure from Sec. V-B lies in the decomposition step. Because some of the ideals I_ξ denoted by the various sequences ξ in Ξ_n might be comparable, a decomposition step (27) might leave $D_n = D_{n+1}$ unchanged. $\square$

1) Unpumpable Case: If $\boldsymbol{s}^{\mathrm{in}}_j$ is not forward pumpable by G_j, the algorithm of Karp and Miller [21] provides an effective way of computing an index $i \notin F_j$ and a constant c such that configurations occurring in any run ρ in Ω_{M_j} are bounded by c on component i. The same property holds if symmetrically $\boldsymbol{s}^{\mathrm{out}}_j$ is not backward pumpable by G_j.

In such cases the graph G_j can be synchronised with a finite state automaton $\mathcal{A}$ with states in $S = \{0,\ldots,c\}$ and transitions of form $(n, \boldsymbol{a}, m) \in S\times\boldsymbol{A}\times S$ satisfying $m = \boldsymbol{a}(i)+n$. This synchronisation might produce a graph that is no longer strongly connected, but it can be decomposed into strongly connected components. This way we obtain a finite family $\mathrm{dec}(\xi)$ of marked witness graph sequences where the graph G_j in ξ is replaced by sequences of subgraphs of $G_j\times\mathcal{A}$ where the finite components F_j of G_j are replaced by a larger set $F_j\cup\{i\}$.

2) Input/Output Bounded Solutions: Now, let us assume that ξ is not perfect due to the conditions on the set of solutions L_ξ. Following the notations introduced in Definition VII.1, recall that we can check in nondeterministic polynomial time whether $\sup \boldsymbol{X}_j(i) < \omega$ for a component i such that $\boldsymbol{s}^{\mathrm{in}}_j(i) = \omega$. If it is not the case, we obtain a component $i \notin F^{\mathrm{in}}$ such that $\sup \boldsymbol{X}_j(i) = c$ is finite. Such a bound is computable in deterministic polynomial time. Now, just observe that component i of $\boldsymbol{s}^{\mathrm{in}}_j$ can be replaced by all the possible values up to c. We obtain in this way a finite family $\mathrm{dec}(\xi)$ where the set F^{in}_j is replaced by $F^{\mathrm{in}}_j\cup\{i\}$. The same construction can be applied symmetrically when $\sup \boldsymbol{Y}_j$ does not match $\boldsymbol{s}^{\mathrm{out}}_j$.

3) Edge Bounded Solutions: Finally, assume that $\{\psi_j(e) \mid \psi_j \in \Psi_j\}$ is bounded. Once again, we can effectively compute in deterministic polynomial time an upper bound c of this set. Notice that in this case, every run $\rho_j \in \Omega_{M_j}$ labelled by a word σ provides a cycle on $\boldsymbol{s}_j$ in G_j in such a way that e occurs at most c times. By removing from G_j the edge e we obtain a graph that may not be strongly connected any more. However, by computing strongly connected components, we obtain in this way a finite family $\mathrm{dec}(\xi)$ such that the graph G_j has been replaced by sequences of up to c graphs, each with a set of edges included in $E_j \setminus \{e\}$.

C. Ranking Function

We present the usual termination argument for the KLMST algorithm by explicitly giving a ranking function r from marked witness graph sequences into an ordinal, such that $r(\xi) > r(\xi')$ for all ξ' in $\mathrm{dec}(\xi)$.

1) Ordinals: Rather than the usual multiset ordering over triples in $\mathbb{N}^3$ ordered lexicographically used in the KLMST algorithm, we use an equivalent formulation using ordinals. Recall that an ordinal $\alpha < \varepsilon_0$ can be written in Cantor normal form (CNF) as $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ where $\alpha > \alpha_1 \geq \cdots \geq \alpha_n$, or equivalently as $\alpha = \omega^{\alpha_1} \cdot c_1 + \cdots + \omega^{\alpha_n} \cdot c_n$ with $\alpha > \alpha_1 > \cdots > \alpha_n$ and finite c_i's. One can compare two ordinals $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ using their CNFs: $\alpha < \beta$ if and only if there exists $k \leq m$ such that $\alpha_j = \beta_j$ for all $1 \leq j < k$ with $j \leq n$, and $n < k$ or $\alpha_k < \beta_k$. The natural sum of two ordinals $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ is defined as $\alpha \oplus \beta \stackrel{\text{def}}{=} \omega^{\gamma_1} + \cdots + \omega^{\gamma_{n+m}}$ such that $\gamma_1 \geq \cdots \geq \gamma_{n+m}$ is a reordering of the α_i's and β_j's.

2) Rank of a Marked Witness Graph: We associate with a marked witness graph $M = (\boldsymbol{s}^{\text{in}}, G, \boldsymbol{s}^{\text{out}})$ an ordinal β_M in ω^3 defined as

$$\beta_M \stackrel{\text{def}}{=} \omega^2 \cdot (d - |F|) + \omega \cdot |E| + (2d - |F^{\text{in}}| - |F^{\text{out}}|) \quad (29)$$

where $G = (\boldsymbol{S}, E, \boldsymbol{s})$, and F^{in}, F, F^{out} are respectively the defined components of $\boldsymbol{s}^{\text{in}}$, $\boldsymbol{s}$, $\boldsymbol{s}^{\text{out}}$. Note that this is equivalent to a lexicographic ordering over triples in $\mathbb{N}^3$.

3) Rank of a Sequence: We associate with a marked witness graph sequence $\xi = M_0, \boldsymbol{a}_1, M_1, \ldots, \boldsymbol{a}_k, M_k$ the ordinal $r(\xi)$ in ω^{ω^3} defined by

$$r(\xi) \stackrel{\text{def}}{=} \bigoplus_{1 \leq j \leq k} \omega^{\beta_{M_j}} \ . \quad (30)$$

Note that this is equivalent to a multiset ordering over the β_{M_j}.

4) Termination Argument: By seeing the KLMST algorithm as constructing a tree with ξ labelling the parent node of ξ' if ξ is imperfect and $\xi' \in \mathrm{dec}(\xi)$, this ranking function shows that the tree has finite height. Since the families Ξ_0 and $\mathrm{dec}(\xi)$ are finite, this tree is also of finite degree, and is therefore finite by Kőnig's Lemma.

IX. Fast-Growing Upper Bounds

We establish in this section an $\mathbf{F}_{\omega^3}$ upper bound on the complexity of the KLMST decomposition algorithm, which yields the first upper bound on the complexity of VAS reachability. Without loss of generality, we can assume that the actions in $\boldsymbol{A}$ are in $\{-1, 0, 1\}^d$.

A. Subrecursive Hierarchies

As noted early on e.g. by Müller [33], the complexity of the decomposition algorithm of Mayr, Kosaraju, and Lambert is not primitive-recursive. As a consequence, we have to employ some lesser known complexity classes in order to express upper bounds on the running time and space of this algorithm.

1) The Hardy Hierarchy: A convenient tool to this end is found in the *Hardy hierarchy* of functions. Given some monotone expansive function $h: \mathbb{N} \to \mathbb{N}$, this is an ordinal-indexed hierarchy of functions $(h^\alpha: \mathbb{N} \to \mathbb{N})_\alpha$ defined by transfinite induction by

$$h^0(x) \stackrel{\text{def}}{=} x, \quad h^{\alpha+1}(x) \stackrel{\text{def}}{=} h^\alpha(h(x)), \quad h^\lambda(x) \stackrel{\text{def}}{=} h^{\lambda(x)}(x),$$

where λ denotes a limit ordinal and $\lambda(x)$ the xth element of its *fundamental sequence*. The latter is usually defined for limit ordinals below ε_0 by

$$(\gamma + \omega^{\beta+1})(x) \stackrel{\text{def}}{=} \gamma + \omega^\beta \cdot (x+1) \ ,$$
$$(\gamma + \omega^\lambda)(x) \stackrel{\text{def}}{=} \gamma + \omega^{\lambda(x)} \ .$$

Observe that h^k for some finite k is the kth iterate of h. At index ω, $\omega(x) = x + 1$ and thus $h^\omega(x) = h^{x+1}(x)$; more generally, h^α is a transfinite iteration of the function h, using a kind of diagonalisation to handle limit ordinals.

Example IX.1. For instance, starting with the successor function $H(x) \stackrel{\text{def}}{=} x+1$, we see that $H^\omega(x) = H^x(x+1) = 2x+1$. The next limit ordinal occurs at $H^{\omega\cdot 2}(x) = H^{\omega+x}(x+1) = H^\omega(2x+1) = 4x+3$. Fast-forwarding a bit, we get for instance a function of exponential growth $H^{\omega^2}(x) = 2^{x+1}(x+1) - 1$, and later a non-elementary function H^{ω^3}, an "Ackermannian" non primitive-recursive function H^{ω^ω}, and a "hyper-Ackermannian" non multiply recursive-function $H^{\omega^{\omega^\omega}}$. □

2) Complexity Classes: Although we could derive upper bounds in terms of Hardy functions, it is more convenient to work with coarser-grained complexity classes. For $\alpha > 2$, we define respectively the *fast-growing function* classes $(\mathscr{F}_\alpha)_\alpha$ of Löb and Wainer [31] and the associated *fast-growing complexity* classes $(\mathbf{F}_\alpha)_\alpha$ of [38] by

$$\mathscr{F}_{<\alpha} \stackrel{\text{def}}{=} \bigcup_{\beta < \omega^\alpha} \mathrm{FS{\scriptsize PACE}}\big(H^\beta(n)\big) \ , \quad (31)$$

$$\mathbf{F}_{h,\alpha} \stackrel{\text{def}}{=} \bigcup_{p \in \mathscr{F}_{<\alpha}} \mathrm{S{\scriptsize PACE}}\big(h^{\omega^\alpha}(p(n))\big) \ , \quad \mathbf{F}_\alpha \stackrel{\text{def}}{=} \mathbf{F}_{H,\alpha} \ , \quad (32)$$

where $\mathrm{FS{\scriptsize PACE}}(s(n))$ (resp. $\mathrm{S{\scriptsize PACE}}(s(n))$) denotes the set of functions computable (resp. problems decidable) in space $O(s(n))$ and H is the successor function $H(x) \stackrel{\text{def}}{=} x + 1$. This defines for instance $\mathscr{F}_{<\omega}$ as the set of primitive-recursive functions, and $\mathbf{F}_\omega$ as the class of problems that can be solved in Ackermann time of some primitive-recursive function of their input size. Here $\mathbf{F}_{\omega^3}$ is not primitive-recursive, but among the lowest multiply-recursive classes.

B. Length Function Theorems

Given some wqo $(X, \leq)$, let us posit a norm $|.|_X : X \to \mathbb{N}$ over X such that $X_{\leq n} \stackrel{\text{def}}{=} \{x \in X \mid |x|_X \leq n\}$ is finite for every n. Given a *control function* $g : \mathbb{N} \to \mathbb{N}$ which is monotone expansive and some *initial norm* $n \in \mathbb{N}$, we say that a sequence $x_0, x_1, \dots$ over X is (g, n)*-controlled* if for all i, $|x_i|_X \leq g^i(n)$ the ith iterate of g. Then there exists maximal (g, n)-controlled bad sequences over $(X, \leq)$, and we write $L_{g,X}(n)$ for their length.

Length function theorems provide upper bounds on this maximal length $L_{g,X}(n)$. The upper bounds we use from [40, 39] are expressed in terms of another hierarchy of functions called the *Cichoń hierarchy* $(h_\alpha : \mathbb{N} \to \mathbb{N})_\alpha$. The relation with the Hardy hierarchy is that, if a controlled sequence is of length bounded by some $h_\alpha(x)$ from the Cichoń hierarchy, then the norm of all its elements is bounded by

$$h^{h_\alpha(x)}(x) = h^\alpha(x) \tag{33}$$

in the Hardy hierarchy.

For instance, upper bounds for $(\mathbb{N}^d \times Q, \leq)$ for some finite set Q, along with the product ordering, can be found in [40, Theorem 2.34], where the norm of a pair $(\boldsymbol{x}, q)$ from $\mathbb{N}^d \times Q$ is $\max_{1 \leq i \leq d} \boldsymbol{x}(i)$:

Fact IX.2 ([40])**.** *Let $H(x) \stackrel{\text{def}}{=} x + 1$ and $n, d > 0$. Then $L_{H,\mathbb{N}^d \times Q}(n) \leq H_{\omega^d \cdot |Q| d}(dn) \leq H_{\omega^{d+1}}(|Q|dn)$.*

Another example from [39, Theorem 3.3] is a length function theorem for ordinals below ε_0, where the norm $N(\alpha)$ of an ordinal $\alpha = \omega^{\alpha_1} \cdot c_1 + \cdots + \omega^{\alpha_n} \cdot c_n$ with $\alpha > \alpha_1 > \cdots > \alpha_n \geq 0$ and $\omega > c_1, \dots, c_n \geq 0$ is the largest constant that appears in it: $N(\alpha) \stackrel{\text{def}}{=} \max_{1 \leq i \leq n}\{c_i, N(\alpha_i)\}$:

Fact IX.3 ([39])**.** *Let $\alpha < \varepsilon_0$ be of norm $N(\alpha) \leq n$. Then $L_{g,\alpha}(n) = g_\alpha(n)$.*

C. Controlling the KLMST Decomposition

Recall from Sec. VIII-C that the KLMST algorithm terminates because any descending sequence of ordinals in ω^{ω^3} is finite. As remarked in Example III.2, descending sequences over an ordinal are bad sequences. From the previous discussion of length function theorems, in order to apply the bounds from [39] on the norms in bad sequences over ω^{ω^3}, we need to find a control function for any sequence

$$r(\xi_0) > r(\xi_1) > \cdots \tag{34}$$

of ordinals in ω^{ω^3} found along a branch of the tree described in Sec. VIII-C4.

1) A Measure on Marked Witness Graph Sequences: Let us write $\|\boldsymbol{v}\| \stackrel{\text{def}}{=} \max_{i \in F} \boldsymbol{v}(i)$ for the infinite norm of partial vectors in $\mathbb{N}^d_\omega$ and $\|\boldsymbol{V}\| \stackrel{\text{def}}{=} \max_{\boldsymbol{v} \in \boldsymbol{V}}(|\boldsymbol{V}|, \|\boldsymbol{v}\|)$ for a set $\boldsymbol{V}$ of partial vectors. Using the norm function N over ε_0 defined above on the ordinals in (29) and (30), we see that $N(r(\xi))$ is bounded by

$$\|\xi\| \stackrel{\text{def}}{=} \max_{0 \leq j \leq k}(2d, k, |E_j|, \|\boldsymbol{s}_j^{\text{in}}\|, \|\boldsymbol{s}_j^{\text{out}}\|, \|\boldsymbol{S}_j\|) \tag{35}$$

for $\xi = M_0, \boldsymbol{a}_1, \dots, \boldsymbol{a}_k, M_k$ where M_j is the marked graph $(\boldsymbol{s}_j^{\text{in}}, G_j, \boldsymbol{s}_j^{\text{out}})$ and $G_j = (\boldsymbol{S}_j, E_j, \boldsymbol{s}_j)$. Note that $\|\xi_0\| = \max(2d, 1, |\boldsymbol{A}|)$ initially.

2) Controlling Decompositions: We are going to exhibit a control function g such that $\|\xi_i\| \leq g^i(\|\xi_0\|)$ for all descending sequences (34) and index i, which will therefore also be a control function on (34) for the ordinal norm. It suffices to show that $\|\xi'\| \leq g(\|\xi\|)$ whenever $\xi' \in \operatorname{dec}(\xi)$. Let us analyse how this measure evolves in the different decomposition cases:

1) In the unpumpable case, the constant c can be bounded using Fact IX.2 by $H^{\omega^{d+1}}(d^2 \cdot |\boldsymbol{S}_j| \cdot \max(\|\boldsymbol{s}_j^{\text{in}}\|, \|\boldsymbol{s}_j^{\text{out}}\|))$ (see also [17, Theorem 2.10] or [10, Sec. VII-C] for similar enough bounds in terms of the *fast-growing function* $F_{d+1} = H^{\omega^{d+1}}$). The resulting sequences ξ' in $\operatorname{dec}(\xi)$ satisfy therefore $\|\xi'\| \leq H^{\omega^{d+1}}(\|\xi\|^4)$.
2) In the input/output bounded case, the constant c is at most exponential in the size of the linear system L_ξ, which is of polynomial size in $\|\xi\|$. Thus $\|\xi'\| \leq 2^{p(\|\xi\|)}$ for some fixed polynomial p.
3) In the edge bounded case, the constant c is similarly at most exponential in the size of L_ξ and again $\|\xi'\| \leq 2^{p(\|\xi\|)}$ for some fixed polynomial p.

Assuming $d \geq 1$, $H^{\omega^{d+1}}(x) > 2^x$, hence we can choose $g(x) \stackrel{\text{def}}{=} H^{\omega^{d+1}}(p(x))$ for some fixed polynomial p as our control function. This is a primitive-recursive function in $\mathscr{F}_{<\omega}$ for any fixed d, and is in $\mathscr{F}_{<\omega+1}$ when d is part of the input.

D. Complexity Bounds

Assuming $\|\xi_0\| \geq 3$, by Fact IX.3 the norm of the elements in any sequence (34) controlled by g is at most $g^{\omega^{\omega^3}}(\|\xi_0\|)$. This function can be computed in space $g^{\omega^{\omega^3}}(e(\|\xi_0\|))$ for some elementary function e by [38, Theorem 5.1]. This yields the same bound on the space used by a nondeterministic version of the KLMST decomposition algorithm, which guesses a branch like (34) that leads to a perfect marked witness graph sequence if there is one. Finally, because our function g yields $\mathbf{F}_{g,\omega^3} = \mathbf{F}_{\omega^3}$ by [38, Theorem 4.4], we obtain:

Theorem IX.4. *VAS reachability is in $\mathbf{F}_{\omega^3}$.*

E. A Combinatorial Algorithm

The bounds in Sec. IX-D allow to propose a conceptually simple algorithm for VAS Reachability, based on a *small run property*. If there is a run in $\operatorname{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$, it must belong to some Ω_ξ for a perfect ξ constructed by the KLMST decomposition. Thus this ξ is of measure $\|\xi\|$ bounded by $g^{\omega^{\omega^3}}(\|\xi_0\|)$. Using Lemma VII.2 we can extract a run of commensurate length ℓ. The combinatorial algorithm is a nondeterministic algorithm that first computes ℓ and then guesses a run ρ in $\operatorname{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ of length at most ℓ. Its complexity is similar to that of the KLMST decomposition algorithm, in $\mathbf{F}_{\omega^3}$.

X. Conclusion

The KLMST decomposition algorithm of Mayr, Kosaraju, and Lambert is most certainly a stroke of genius, allowing to prove the decidability of reachability in VAS. What was however sorely lacking until now was an explanation for this decomposition that could be adapted and extended in various directions. Far from closing the subject, we expect this demystification to span a whole research programme.

The first natural question is how easily one can use the framework of ideals on runs for various VAS extensions. A good test is the case of VAS with hierarchical zero tests, which were proven to enjoy a decidable reachability problem by Reinhardt [34]. A wqo on runs using nested applications of Higman's Lemma for this extension is defined by Bonnet [5] in his alternative decidability proof using Presburger inductive invariants. Using the algebraic framework of Sec. IV-C, we see that prerun ideals for this new ordering are essentially nested products, and thus bear at least a superficial resemblance to the structures manipulated by Reinhardt [34]. The framework could also shed new light on reachability in other VAS extensions [26, 37, 25].

A second question is whether we can significantly improve the $\mathbf{F}_{\omega^3}$ upper bound provided in Sec. IX. The best known lower bound on the running time of the algorithm is Ackermannian, i.e. $\mathbf{F}_{\omega}$, leaving a huge gap on the complexity of the KLMST algorithm, and a gigantic gap on the complexity of VAS reachability, which is only known to be EXPSPACE-hard.

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