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Nonaffine Differential-Algebraic Curves Do Not Exist

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The text is both an algebraic lesson and a Cartesian
masterclass (a gift of Gods, one should say)
for metaphysicians, metaphysists and their followers.

Abstract—The paper outlines why the spectrum of maximal ideals $Spec_{\mathbb{C}}A$ of a countable-dimensional differential $\mathbb{C}$-algebra A of transcendence degree 1 without zero divisors is locally analytic, which means that for any $\mathbb{C}$-homomorphism $\psi_M : A \to \mathbb{C}$ ($M \in \mathrm{Spec}_{\mathbb{C}}A$) and any $a \in A$ the Taylor series $\widetilde{\psi}_M(a) \stackrel{\mathrm{def}}{=} \sum\limits_{m=0}^{\infty} \psi_M(a^{(m)})\frac{z^m}{m!}$ has nonzero radius of convergence depending on the element $a \in A$.

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1. Introduction. Let's start with an intuitionistic and purely algebraic trick allowing one to resolve certain types of ordinary differential equations relative to the highest derivative entering those equations.

Lemma on affine property of intermediate subalgebra. *Let a commutative associative integral domain A be an algebra (with a unit) over an arbitrary algebraically closed field k* (char$k \geq 0$)*, and its fraction field $Q(A)$ has over k the transcendence degree equal to* 1. *In this case, if in the chain of k-subalgebras $A \subseteq C \subseteq B \subset Q(A)$ the algebras A and B contain a finite number of generators, then the same is true for the subalgebra C.*

Proof. Since B has a finite number of generators, then C is a countable-dimensional k-algebra. Take elements $\{e_i | i = 1, 2, \dots\}$ in C so that they supplement the basis of the k-algebra A up to the basis of C. Assume $C_0 \stackrel{\mathrm{def}}{=} A$, $C_{i+1} \stackrel{\mathrm{def}}{=} C_i[e_{i+1}]$, $C_\infty \stackrel{\mathrm{def}}{=} B$. Integral closures ("normalizations") of all these finitely-generated subalgebras in $Q(A)$ are denoted by $\bar{C}_0, \bar{C}_1, \dots, \bar{C}_i, \dots, \bar{C}_\infty$, respectively. It is well known (see [1, Ch. 2]) that each such subalgebra have a finite number of generators over k. Moreover, it was established in the proof of this fact (see [1]) that the subalgebra $\bar{C}_i$ is finitely generated as a module over C_i and hence it is Noetherian. Show that increasing chains of k-subalgebras $\{\bar{C}_i\}, \{C_i\}, i = 1, 2, 3, \dots$, stabilize.

Proposition 1. *If in the chain of integral domains $F \subseteq G \subset Q(F)$ the k-subalgebras F and G have a finite number of generators, $F = \bar{F}$ (i.e., F is integral-closed), $\deg_k Q(F) = 1$, then the natural mapping $\nu : \mathrm{Spec}_k G \to \mathrm{Spec}_k F$ possesses the following properties:*

a) ν is an injective mapping;

b) if ν is surjective, then G coincides with F,

c) $(\mathrm{Spec}_k F) \setminus \nu(\mathrm{Spec}_k G)$ *is a finite set.*

This is an exact translation of the assertion of Corollary 2 of Theorem 2 of [1, Ch. 2, § 2, p. 136] to the language of commutative algebras.

Property b implies that if $\bar{C}_i \neq \bar{C}_{i+1}$, then some maximal ideal $M \in \mathrm{Spec}_k \bar{C}_i$ is not raised up to an ideal in $\mathrm{Spec}_k \bar{C}_{i+1}$, and hence property a implies that $M \cap \bar{C}_0 \in \mathrm{Spec}_k \bar{C}_0$ is not raised up to an ideal in $\mathrm{Spec}_k \bar{C}_\infty$. However, due to property c, there are a finite number of maximal ideals in $\bar{C}_0$ than are not raised up to an ideal in $\mathrm{Spec}_k \bar{C}_\infty$. Therefore, in the increasing chain of "normalizations" $\bar{C}_0 \subseteq \bar{C}_1 \subseteq \dots \bar{C}_m \subseteq \dots$ only a finite number of positions have strict inclusions, i.e., $\bar{C}_N = \bar{C}_{N+i}$ for sufficiently large $N \in \mathbb{N}$ ($i = 1, 2, \dots$) and $C_N \subseteq C = \bigcup\limits_m C_m \subseteq \bar{C}_N$. As was indicated above, the subalgebra $\bar{C}_N$ is Noetherian as a module over C_N. Therefore, its C_N-submodule C is finitely generated and must coincide with C_{N+q} for some $q \in \mathbb{N}$, which proves the lemma.

The following assertions are easily derived from the above result.

Theorem 1. *Any finitely generated differential k-algebra*[1] *without divisors of zero and having the transcendence degree equal to* 1 *has a finite number of generators as a commutative-associative algebra, in particular, these differential k-algebras are finitely determined*[2].

Corollary 1. *The spectrum of maximal ideals* $\mathrm{Spec}_{\mathbb{C}}A$ *of an arbitrary finitely generated differential commutative-associative $\mathbb{C}$-algebra A without divisors of zero and having the transcendence degree equal to* 1 *is analytic, i.e., for any $\mathbb{C}$-homomorphism* $\psi_M : A \to A/M \simeq \mathbb{C}$ $(M \in \mathrm{Spec}_{\mathbb{C}}A)$ *and under the Tailor homomorphism* $\widetilde{\psi}_M : A \to \mathbb{C}[[z]]$ $\left(\widetilde{\psi}_M(a) \overset{\mathrm{def}}{=} \sum\limits_{m=0}^{\infty} \psi_M(a^{(m)})\frac{z^m}{m!}\right)$ *all power series converge in a certain neighborhood of zero.*

Theorem 2. *Let X be an irreducible affine algebraic curve over an algebraically closed field k and $k[X]$ be its algebra of regular functions. In this case any k-subalgebra in $k[X]$ is generated by a finite number of its elements.*

Corollary 2. *Let the field K have the transcendence degree* 1 *over an algebraically closed field k and* $\mathrm{Der}_k K$ *be the Lie algebra of all k-differentiations $K \to K$. In this case for any $a_1, \ldots, a_n \in K$, $D_1, \ldots, D_l \in$* $\mathrm{Der}_k K$ *the least commutatively-associated k-subalgebra A in K such that $a_1, \ldots, a_m \in A$ and $D_i(A) \subset A (i = 1, \ldots, l)$ is finitely generated.*

Illustrate the above assertions by particular examples.

2. Picard's differential algebras (see [2]). Define the differential $\mathbb{C}$-algebra P by the generators $x_1, \ldots, x_n$ and n determining relations $x_i' = f_i(x_1, \ldots, x_n)$ $(i = 1, 2, \ldots, n)$, where all f_i are arbitrary fixed elements of the algebra of polynomials $\mathbb{C}[x_1, \ldots, x_n]$. Obviously, this algebra has no divisors of zero and it can be realized on $\mathbb{C}[x_1, \ldots, x_n]$ taking $D \overset{\mathrm{def}}{=} \sum\limits_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$ for the differentiation.

The spectrum of this differential algebra coincides with the affine space $\mathbb{C}^n$. For the coefficients of the Tailor series $\widetilde{\psi}_M(f) \overset{\mathrm{def}}{=} \sum\limits_{m=0}^{\infty} (D^m \times f)|_{x_i=\alpha_i,\ldots,x_n=\alpha_n} \cdot \frac{z^m}{m!}$ $(f \in \mathbb{C}[x_1, \ldots, x_n], M \overset{\mathrm{def}}{=} (\alpha_1, \ldots, \alpha_n))$, we immediately obtain the estimate $|((D^m \times f)|_{x=M})/m!| \le n^m a^{m+1}$, where a is the maximum of absolute values of the functions f, $f_1, \ldots, f_n$ and all their partial derivatives of arbitrary order at the point M. Therefore, all power series $\widetilde{\psi}_M(x_1), \ldots, \widetilde{\psi}_M(x_n)$ converge in a certain neighborhood of zero. The equality $\widetilde{\psi}_M(f) = f(\widetilde{\psi}_M(x_1), \ldots, \widetilde{\psi}_M(x_n))$ implies that for any $f \in \mathbb{C}[x_1, \ldots, x_n]$ the series $\widetilde{\psi}(f)$ converges in the same neighborhood. Since any finitely generated commutative associative $\mathbb{C}$-algebra A with a fixed differentiation $D \in \mathrm{Der}_{\mathbb{C}}A$ is a homomorphic image of the Picard algebra P for an appropriate choice of n and $f_1, \ldots, f_n$, then $\mathrm{Spec}_{\mathbb{C}}A$ is also analytic for any $D \in \mathrm{Der}_{\mathbb{C}}A$.

3. "Rational" differential-algebraic parameterizations of plane affine irreducible algebraic curves. By X_H we denote the plane affine irreducible algebraic curve given by the equation $H(x, y) = 0$ $(H(x, y) \in k[x, y])$. Let $k[X_H]$ be its algebra of regular functions over an algebraically closed field k of arbitrary characteristic. In Sections 3.1–3.3, 4 we assume that differential k-algebras ("parameterizations") defined there by differential-algebraic relations naturally contain $k[X_H]$ as a subalgebra. Obviously, this poses certain restrictions on the irreducible polynomial $H(x, y)$. We directly specify necessary and sufficient conditions providing such inclusion for each of these cases:

a) $\frac{\partial H}{\partial y} \not\equiv 0$ in Sections 3.1, 3.3;

b) $x \cdot \frac{\partial H}{\partial x} + y \cdot \frac{\partial H}{\partial y} \notin k \cdot H(x, y)$ in Section 3.2;

c) $\left(\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2 \not\equiv 0$ in Section 4.

3.1. *One-generated differential-algebraic curves* (*proof of Theorem* 1). Define the differential k-algebra W_H with a unit by two generators ω, ω_1 and two determining relations $H(\omega, \omega_1) = 0$, $\omega' = \omega_1$, where $H(\omega, \omega_1)$ is an irreducible polynomial in $k[\omega, \omega_1]$ such that $\frac{\partial H}{\partial \omega_1} \not\equiv 0$. Unfortunately, it is not known now whether this commutative-associative algebra contains divisors of zero or not. To get rid of such virtual elements (for $\frac{\partial H}{\partial \omega_1} \not\equiv 0$), consider the differential ideal $I_H \overset{\mathrm{def}}{=} \{a \in W_H | \left(\frac{\partial H}{\partial \omega_1}\right)^m \cdot a = 0, m = m(a)\}$ in W_H and localize W_H with respect to the element $d \overset{\mathrm{def}}{=} \frac{\partial H}{\partial \omega_1} \in W_H$. In this case the kernel of the canonical homomorphism of differential algebras $\nu : W_H \to (W_H)_d \overset{\mathrm{def}}{=} \{\frac{b}{d^k} | b \in W_H\}$ coincides with the ideal I_H. Assume $\overline{W}_H \overset{\mathrm{def}}{=} \nu(W_H)$, $\bar{\omega} \overset{\mathrm{def}}{=} \nu(\omega)$, $\bar{d} \overset{\mathrm{def}}{=} \nu(d)$, $\bar{\omega}_1 \overset{\mathrm{def}}{=} \nu(\omega_1)$, $k[X_H] \overset{\mathrm{def}}{=} k[\bar{\omega}, \bar{\omega}_1]$. In this case the equality $\bar{\omega}' = \bar{\omega}_1$ implies that $\overline{W}_H$

[1] We say that a differential algebra is generated by elements $a_1, \ldots, a_m$ if it is generated as a commutative-associative algebra by all possible $a_i^{(j)}$, $i = 1, \ldots, m$, $j = 0, 1, 2, \ldots$, where $a_i^{(j)}$ is the result of j-fold application of the signature differentiation to the generator a_i.

[2] A differential algebra is finitely determined if it is differentially isomorphic to a differential algebra determined by a finite number of differential generators and differential relations.

is differentially generated by one element $\bar{\omega}$, and the equality $0 = H' = \frac{\partial H}{\partial \omega}\omega' + \frac{\partial H}{\partial \omega_1}\omega''$ shows that all the elements $\nu(W_H) = \overline{W}_H$ lie in the commutative-associative algebra $(W_H)_d$ generated by the three elements ω, ω_1, $d^{-1} = (\omega_1 \cdot \frac{\partial H}{\partial \omega_1})^{-1}$. This allows us to realize $\overline{W}_H$ as a differential k-subalgebra in the field $k(X_H)$, where X_H is a plane irreducible affine algebraic curve given by the equation $H(\omega, \omega_1) = 0$ ($\frac{\partial H}{\partial \omega_1} \not\equiv 0$) when we take $D \overset{\text{def}}{=} \omega_1(\frac{\partial}{\partial \omega} - (\frac{\partial H}{\partial \omega}/\frac{\partial H}{\partial \omega_1})\frac{\partial}{\partial \omega_1})$ as the differentiation.

Therefore, we obtain a chain of k-algebras $k[X_H] \subseteq \overline{W}_H \subseteq (\overline{W}_H)_{\bar{d}} \subseteq k(X_H)$ satisfying all conditions of the lemma on affine property of the intermediate subalgebra, i.e., $\overline{W}_H$ is generated as a commutative-associative k-algebra by a finite number of elements. Obviously, *any one-generated differential subalgebra is a holomorphic image of* $\overline{W}_H$ *in an arbitrary differential integral domain* (*transcendence degree equal to one*) *under appropriate choice of* $H(\omega, \omega_1)$ $\left(\frac{\partial H}{\partial \omega_1} \not\equiv 0\right)$ *and hence is finitely generated as a commutative-associative k-algebra*. Since any m-generated differential k-algebra is a product of m its one-generated differential subalgebras, this proves Theorem 1 on affine property of differential-algebraic curves. (We especially point out that these arguments are valid for fields of positive characteristic p because the equalities $\frac{\partial H}{\partial w_1} \equiv 0$, $\frac{\partial H}{\partial \omega} \equiv 0$ imply $H(\omega, \omega_1) = (F(\omega, \omega_1))^p$, but this contradicts the irreducible nature of H.)

Complete this section with one simple (possibly useless, but memorable) version of Theorem 1.

Proposition 2. *If in a differential integral domain F over an algebraically closed field k the elements f and f' are connected by some nonzero polynomial relation $H(f, f') = 0$ ($H(x, y) \in k[x, y]$, $H \not\equiv 0$), then for some natural number N the "Nth derivative" $f^{(N)}$ can be polynomially expressed through previous f, f', f", $\ldots$, $f^{(N-1)}$.*

Corollary 3. *If an infinitely differentiable complex-valued function $f(t)$ is a solution to the differential equation $H(f, f') = 0$ on a real interval (a, b), where $H(x, y) \in \mathbb{C}[x, y]$ is an irreducible (nonzero) polynomial, then for some natural number N the function $f^{(N)}(t)$ can be polynomially expressed through $f(t)$, $f'(t)$, f"(t), $\ldots$, $f^{(N-1)}(t)$.*

3.2. *Kepler's parameterizations of a plane curve.* Define a differential k-algebra G_H with a unit by generators x, y and two differential determining relations $H(x, y) = 0$, $xy' - x'y = \sigma$, where H is an irreducible polynomial in $k[x, y]$ such that $x \cdot \frac{\partial H}{\partial x} + y \cdot \frac{\partial H}{\partial y} \notin k \cdot H(x, y)$, and $0 \neq \sigma \in k$ (for example, $\sigma = \hbar/m_e$). Resolving the system of equations $0 = H' = \frac{\partial H}{\partial x} \cdot x' + \frac{\partial H}{\partial y} \cdot y'$, $-yx' + xy' = \sigma$ relative to x', y', we get $\mathcal{L}(x, y) \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} = \sigma \begin{pmatrix} -\partial H/\partial y \\ \partial H/\partial x \end{pmatrix}$, where $\mathcal{L} \overset{\text{def}}{=} \frac{\partial H}{\partial x} \cdot x + \frac{\partial H}{\partial y} \cdot y$. If the irreducible affine curve X_H (given by the equation $H(x, y) = 0$) is smooth, then the ideal generated by $\frac{\partial H}{\partial x}$, $\frac{\partial H}{\partial y}$ in $k[X_H]$ must coincide with the whole algebra. Therefore, $a\frac{\partial H}{\partial x} + b\frac{\partial H}{\partial y} = 1$ for some a, b from $k[X_H]$. Thus, $\mathcal{L} \cdot (-ax' + by') = \sigma$, i.e., the element $\mathcal{L}$ is invertible in G_H. This immediately implies that G_H as a commutative-associative k-algebra possesses the following properties: a) it is generated by the three its elements x, y, $\mathcal{L}^{-1}$; b) it is embedded into the field $k(X_H)$ and does not contain divisors of zero; c) it is realized as a differential subalgebra in $k(X_H)$ relative to the differentiation $D_H \overset{\text{def}}{=} \sigma \cdot \mathcal{L}^{-1}(-\frac{\partial H}{\partial y}\frac{\partial}{\partial x} + \frac{\partial H}{\partial x}\frac{\partial}{\partial y})$. In the general case we cannot exclude the situation when G_H contains divisors of zero. Let I be an arbitrary ideal in G_H such that G_H/I has no such elements. Suppose I is not intersected with the subalgebra $k[X_H]$ generated by x, y in G_H in nonzero way. In this case the quotient algebra $k[X_H]/(I \cap k[X_H])$ is zero-dimensional and the integral domain G_H/I must coincide with $k \cdot 1$, but this contradicts the equality $xy' - x'y = \sigma \neq 0$. If $I \cap k[X_H] = 0$, then the element $\mathcal{L} \in k[X_H]$ does not equal zero in the integral domain G_H/I and, localizing with respect to $\mathcal{L}$, we get $(G_H/I)_{\mathcal{L}} = (G_H)_{\mathcal{L}}/I_{\mathcal{L}}$. Therefore, the ideal I should coincide with the ideal $I(H) \overset{\text{def}}{=} \{a \in G_H | \mathcal{L}^m \cdot a = 0 \text{ in } G_H, m = m(a)\}$. Thus, there exists a unique integral domain $\bar{G}_H$ given by the generators x, y and the two differential relations $H(x, y) = 0$, $xy' - x'y = \sigma$ ($\sigma \in k, \sigma \neq 0$) and possessing the properties a) $\bar{G}_H$ is embedded into $k(X_H)$ relative to the differentiation $D_H \overset{\text{def}}{=} \mathcal{L}^{-1} \cdot \sigma(-\frac{\partial H}{\partial y}\frac{\partial}{\partial x} + \frac{\partial H}{\partial x}\frac{\partial}{\partial y})$; b) in this embedding we have $k[X_H] \subseteq \bar{G}_H \subseteq (\bar{G}_H)_{\mathcal{L}} \subset k(X_H)$ and the localization $(\bar{G}_H)_{\mathcal{L}}$ is generated as a commutative-associative k-algebra by three its elements $x, y, \mathcal{L}^{-1}$; c) $\bar{G}_H$ is a simple differential k-algebra and the signature differentiation does not vanish at any point of the spectrum $\mathrm{Spec}_k\bar{G}_H$. Thus, $X_{\bar{G}_H} = \mathrm{Spec}_k\bar{G}_H$ is a smooth affine irreducible algebraic curve and $\bar{G}_H$ contains $k[X_H^\nu]$, where X_H^ν is the normalization of the curve X_H. This indicates that Kepler's observer $\bar{G}_H$ excludes from consideration all nonruled branches X_H^ν, but moving slightly off the origin, may notice all ruled ones. Observing the curve X_H, such an observer is capable to act more radically, i.e., to run from the origin along the line $x = 0$ up on the "gallery."

3.3. *Puiseux parameterization.* Consider a differential k-algebra P_H with a unit defined by the generators x, y and two differential relations $H(x, y) = 0$, $x' = c$ ($c \in k, c \neq 0$), where $H(x, y)$ is an irreducible polynomial such that $\frac{\partial H}{\partial y} \not\equiv 0$. The arguments presented in two previous sections guarantee that P_H contains a unique (probably zero) differential ideal $I \stackrel{\text{def}}{=} \{a \in P_H | \left(\frac{\partial H}{\partial y}\right)^m \cdot a = 0,\, m = m(a)\}$ such that the quotient algebra with respect to it does not contain divisors of zero. Denote it by $\bar{P}_H$. The equality $0 = H' = \frac{\partial H}{\partial x}c + \frac{\partial H}{\partial y}y'$ shows that he localization $\bar{P}_H$ with respect to the element $\frac{\partial H}{\partial y}$ is generated as a commutative-associative k-algebra by three its elements $x, y, \left(\frac{\partial H}{\partial y}\right)^{-1}$, and $\bar{P}_H$ is realized as a differential subalgebra in the field $k(X_H)$ relative to the differentiation $D = D(H) = c\left(\frac{\partial}{\partial x} - \left(\frac{\partial H}{\partial x}/\frac{\partial H}{\partial y}\right)\frac{\partial}{\partial y}\right)$. In this case, $k[X_H] \subset \bar{P}_H \subseteq (\bar{P}_H)_{\frac{\partial H}{\partial y}} \subset k(X_H)$, and due to the uniqueness of the ideal I, the differential k-algebra $\bar{P}_H$ is simple and its signature differentiation does not vanish at any point of $\mathrm{Spec}_k \bar{P}_H$ and, as well as in the previous example, $X_{\bar{P}_H} \stackrel{\text{def}}{=} \mathrm{Spec}_k \bar{P}_H$ is a smooth affine irreducible algebraic curve such that $k[X_{\bar{P}_H}] = \bar{P}_H$ contains $k[X_H^\nu]$, where X_H^ν is the normalization of the plane curve X_H^ν. In this case $\bar{P}_H$ does not exclude from consideration branches of the curve X_H^ν such that the projection of the tangent onto the plane Oxy is parallel to the line $x = 0$ (including nonruled branches).

4. Fermat's parameterization (natural parameter). Define a differential k-algebra F_H with a unit by the generators x, y and two determining relations $H(x, y) = 0$, $(x')^2 + (y')^2 = c^2$ (char $k \neq 2$), where $H(x, y)$ is an irreducible polynomial such that $\Delta \stackrel{\text{def}}{=} (\frac{\partial H}{\partial x})^2 + (\frac{\partial H}{\partial y})^2 \not\equiv 0$ and $0 \neq c \in k$. Evidently, the signature differentiation does not vanish at any point of the spectrum $\mathrm{Spec}_k F_H$ of the k-algebra F_H. Therefore, if we show that any homomorphic image $\bar{F}_H$ of the algebra F_H not containing divisors of zero has the transcendence degree equal to 1 over k, then $\bar{F}_H$ is a simple finitely-definite differential k-algebra with an analytic spectrum. Let $\phi : F_H \to \bar{F}_H$ be the corresponding epimorphism, $\bar{x} \stackrel{\text{def}}{=} \phi(x), \bar{y} \stackrel{\text{def}}{=} \phi(y)$, $k[X_H]$ be the algebra of regular functions of the plane affine curve X_H given by the equation $H(x, y) = 0$. It is clear that $k[X_H]$ is isomorphic to the k-subalgebra generated by x, y and F_H and $k[X_H] \cap \mathrm{Ker}\phi = 0$ (otherwise the zero-dimensional subalgebra $\phi(k[X_H])$ would generate $\bar{F}_H$ and $\bar{x}', \bar{y}'$ would be equal to zero, but this contradicts the equality $(x')^2 + (y')^2 = c^2 \neq 0$). The equalities $\frac{\partial H}{\partial x} \equiv 0, \frac{\partial H}{\partial y} \equiv 0$ are possible if char$k = p > 0$, but due to the fact that $H(x, y)$ is irreducible, we have either $\frac{\partial H}{\partial x} \not\equiv 0$, or $\frac{\partial H}{\partial y} \not\equiv 0$.

Consider the case $\frac{\partial H}{\partial y} \not\equiv 0$. We have $d \stackrel{\text{def}}{=} \phi\left(\frac{\partial H}{\partial y}\right)$ is the zero element in the integral domain $\bar{F}_H$ and the equalities $0 = \phi(H') = \phi(\frac{\partial H}{\partial x})\bar{x}' + \phi(\frac{\partial H}{\partial y})\bar{y}'$ and $(\bar{x}')^2 + (\bar{y}')^2 = c^2$ imply $\bar{y}' = -(\phi(\frac{\partial H}{\partial x})/d)\bar{x}'$, $(\bar{x}')^2(1 + (\phi(\frac{\partial H}{\partial x})/d)^2) = c^2$ in the field of fractions $Q(\bar{F}_H)$ of the algebra $\bar{F}_H$. The latter relation implies that a) the k-subalgebra E generated by $\bar{x}$, $\bar{y}$, $\bar{x}'$, $\bar{y}'$ is contained in the "quadratic extension" of the filed $\phi(k(X_H))$; b) $\bar{x}^{(i)}, \bar{y}^{(i)} \in Q(E)$, $i = 2, 3, \ldots$. This proves that the integral domain $\bar{F}_H$ is contained in $Q(E)$ and $\deg_k \bar{F}_H = \deg_k Q(E) = 1$. According to Theorem 1, the commutative-associative k-algebra $\bar{F}_H$ is generated by a finite number of its elements and is finitely definite as a differential k-algebra. Since the signature differentiation $'$ does not vanish at any point $X_{\bar{F}_H} \stackrel{\text{def}}{=} \mathrm{Spec}_k \bar{F}_H$, then $\bar{F}_H$ is an integrally closed k-algebra and $\bar{F}_H$ contains $k[X_H^\nu]$, where X_H^ν is the normalization of the curve X_H.

5. Nonaffine differential-algebraic surfaces do exist. Define a differential $\mathbb{C}$-algebra E (with a unit) by the generators x, y and determining relations $x' = 1$, $x^2 \cdot y' + y - x = 0$. Assume $\bar{x}(z) \stackrel{\text{def}}{=} z$, $\bar{y}(z) \stackrel{\text{def}}{=} \sum\limits_{m=0}^{\infty} (-1)^m m! \cdot z^{(m+1)}$ and generate the differential $\mathbb{C}$-subalgebra $\bar{E}$ relative to the differentiation $\frac{d}{dz}$ by the elements $\bar{x}$, $\bar{y}$ in power series $\mathbb{C}[[z]]$. Direct checking shows that $\frac{d\bar{x}}{dz} = 1$, $\bar{x}^2 \cdot \frac{d\bar{y}}{dz} + \bar{y} - \bar{x} = 0$ in $\mathbb{C}[[z]]$. Therefore, the integral domain $\bar{E}$ is a homeomorphic image of E for $\phi : E \to \bar{E}$ ($\phi(x) = \bar{x}$, $\phi(y) = \bar{y}$) and we successively obtain the following assertions:

a) $\mathrm{Ker}\phi = \{a \in E | x^{2m} \cdot a = 0 \ \ (m = m(a))\}$;

b) for the maximal ideal $M \in \mathrm{Spec}_{\mathbb{C}} \bar{E}$ being the intersection of $\bar{E}$ with the unique maximal ideal in $\mathbb{C}[[z]]$ under the Tailor homomorphism $\widetilde{\psi}_M : \bar{E} \to \mathbb{C}[[z]]$ we have $\widetilde{\psi}_M(\bar{x}) = \bar{x}$, $\widetilde{\psi}_M(\bar{y}) = \bar{y}$, i.e., $\mathrm{Spec}_{\mathbb{C}} \bar{E}$ is not analytic at the point M;

c) $\bar{x}$, $\bar{y}$ are algebraically independent over $\mathbb{C}$ (otherwise $\bar{E}$ would coincide with some Puiseux parameterization $\bar{P}_H$ ($H(x, y) \in \mathbb{C}[x, y]$) and $\mathrm{Spec}_{\mathbb{C}} \bar{E}$ would be analytic);

d) the algebra $\bar{E}$ can be realized in the field of rational functions $\mathbb{C}(x, y)$ as a differential $\mathbb{C}$-subalgebra relative to the differentiation $D \stackrel{\text{def}}{=} \frac{\partial}{\partial x} + \frac{x-y}{x^2}\frac{\partial}{\partial y}$.

Thus, the differential integral domain $\bar{E}$ has the transcendence degree equal to 2 over $\mathbb{C}$, its spectrum of maximal ideals is not analytic and hence $\bar{E}$ cannot be generated by a finite number of its elements as a commutative-associative $\mathbb{C}$-algebra.

As exercises, we leave to the reader the verification of two more properties of the $\mathbb{C}$-algebra $\bar{E}$:

e) $\mathbb{C}[x, y] \subset \bar{E} \subset \mathbb{C}[x, y, x^{-1}] \subset \mathbb{C}(x, y)$;

f) $\bar{E}$ is a simple differential $\mathbb{C}$-algebra.

6. Proof of Theorem 2. Since the algebra $k[X]$ is finitely generated, then any its k-subalgebra C is countable dimensional. If C contains the unit element of the algebra $k[X]$, than take a basis $\{e_i | i = 0, 1, \dots\}$ in C so that $e_0 \stackrel{\text{def}}{=} 1$. Assume $C_0 \stackrel{\text{def}}{=} k \cdot e_0$, $C_{i+1} \stackrel{\text{def}}{=} C_i[e_{i+1}]$, $i = 0, 1, 2, \dots$. Since the field k is algebraically closed, then the k-algebra C_1 is isomorphic to the algebra of polynomials $k[e_1]$. Consider the increasing chain of fraction fields $Q(C_i)$ of k-algebras C_i. Since $k[X]$ is finitely generated and $\deg_k k(X) = 1$, then the field $k(X)$ is a finite extension of the subfield $Q(C_1)$ and $\dim_{Q(C_1)} Q(C_i) \leq \dim_{Q(C_1)} Q(C_{i+1}) \leq \dim_{Q(C_1)} k(X)$. Therefore, the increasing chain of fields $Q(C_i)$, $i = 0, 1, 2, \dots$, stabilizes beginning with some number N, i.e., $Q(C_N) = Q(C_{N+i})$, $i = 1, 2, \dots$. Assume $A \stackrel{\text{def}}{=} C_N$. In this case $Q(A) = Q(C) \subseteq k(X)$ and the embedding $A \subset k[X]$ defines a regular mapping $\nu : X \to X_A \stackrel{\text{def}}{=} \operatorname{Spec}_k A$. Since $\deg_k k(X) = 1$, then the set $X_A \backslash \nu(X)$ is finite and X_A contains a finite number of singular points. Therefore, we can take an element d in the k-algebra A so that

a) the localization $A_d \stackrel{\text{def}}{=} A[d^{-1}] \subset Q(A)$ of the algebra A with respect to the element d is an integrally closed k-algebra;

b) the localization $(k[X])_d$ consists of algebraic elements over A_d and any ideal from $\operatorname{Spec}_k A_d$ is raised up to an ideal from $\operatorname{Spec}_k (k[X])_d$, in particular, up to an ideal from $\operatorname{Spec}_k (C_{N+i})_d$.

Applying Proposition 1 for $F = A_d$, $G = (C_{N+i})_d$, we conclude that $A_d = (C_{N+i})_d = C_d$, and we get the chain of subalgebras $A \stackrel{\text{def}}{=} C_N \subseteq C \subseteq B \stackrel{\text{def}}{=} A_d = (C_{N+i})_d \subseteq Q(A)$ satisfying all the conditions of the lemma on affine property of the intermediate subalgebra. This completes the proof of the theorem in the case when the k-subalgebra C contains a unit.

If $1 \notin C$, consider the k-subalgebra $C_{\text{id}} \stackrel{\text{def}}{=} k \cdot 1 \oplus C$, which, according to above proof, is generated by some its elements $e_i = \lambda_i \cdot 1 \oplus c_i$ $i = 1, \dots, m$, $m = m(C)$, $c_i \in C$, $\lambda_i \in k$. But in this case $c_1, \dots, c_m \in C$ generate C. Theorem 2 is completely proved.

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