Adjointness in Foundations

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I. That pursuit of exact knowledge which we call mathematics seems to involve in an essential way two dual aspects, which we may call the Formal and the Conceptual. For example, we manipulate algebraically a polynomial equation and visualize geometrically the corresponding curve. Or we concentrate in one moment on the deduction of theorems from the axioms of group theory, and in the next consider the classes of actual groups to which the theorems refer. Thus the Conceptual is in a certain sense the subject matter of the Formal.

Foundations will mean here the study of what is universal in mathematics. Thus Foundations in this sense cannot be identified with any "starting-point" or "justification" for mathematics, though partial results in these directions may be among its fruits. But among the other fruits of Foundations so defined would presumably be guide-lines for passing from one branch of mathematics to another and for gauging to some extent which directions of research are likely to be relevant.

Being itself part of Mathematics, Foundations also partakes of the Formal-Conceptual duality. In its formal aspect, Foundations has often concentrated on the formal side of mathematics, giving rise to Logic. More recently, the search for universals has also taken a conceptual turn in the form of Category Theory, which began with viewing as a new mathematical object the totality of all morphisms of the mathematical objects of a given species A, and then recognizing that these new mathematical objects all belong to a common non-trivial species C which is independent of A. Naturally the formal tendency in Foundations can also deal with the conceptual aspect of mathematics, as when the semantics of a formalized theory T is viewed itself as another formalized theory T′, or in a somewhat different way, as in attempts to formalize the study of the category of categories. On the other hand, Foundations may conceptualize the formal aspect of mathematics, leading to BOOLEAN algebras, cylindric and polyadic algebras, and certain of the structures dicussed below.

This paper will have as one of its aims the giving of evidence for the universality of the concept of adjointness, which was first isolated and named in the conceptual sphere of category theory, but which also seems to pervade logic. Specifically, we describe in section III the notion of cartesian closed category, which appears to be the appropriate abstract structure for making explicit the known analogy between the theory of functionality and propositional logic which is sometimes exploited in proof theory. The structure of a cartesian closed category is entirely given by adjointness, as is the structure of a "hyperdoctrine", which includes quantification as well. Precisely analogous "quantifiers" occur in realms of mathematics normally considered far removed from the province of logic or proof theory. As we point out, recursion (at least on the natural numbers) is also characterized entirely by an appropriate adjoint; thus it is possible to give a theory, roughly proof theory of intuitionistic higher-order number theory, in which all important axioms (logical or mathematical) express instances of the notion of adjointness.

The above-discussed notions of Conceptual, Formal, and Foundations play no mathematical role in this paper; they were included in this introductory section only to provide one possible perspective from which to view the relationship of category theory in general and this paper in particular to other work of universal tendency. However, if one wished to take these notions seriously, it would seem to follow that an essential feature of any attempt to formalize Foundations would be a description of this claimed "duality" between the Formal and the Conceptual; indeed, both category theory and set theory succeed to some extent in providing such a description in certain cases. Concerning the latter point there is a remark at the end of section IV, which is otherwise devoted to a discussion of a class of adjoint situations differing from those of section III but likewise seeming to be of a universal significance. — these may be described briefly as a sort of globalized GALOIS connection.

II. The formalism of category theory is itself often presented in "geometric" terms. In fact, to give a category is to give a meaning to the word *morphism* and to the *commutativity* of diagrams like

$$A \xrightarrow{f} B \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd} \quad \text{etc.}$$

which involve morphisms, in such a way that the obvious associativity and identity conditions hold, as well as the condition that whenever

$$A \xrightarrow{f} B$$

and

$$B \xrightarrow{g} C$$ are commutative then there is just one h such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$ is commutative.

To save printing space, one also says that A is the *domain,* and B the *codomain* of f when

$$A \xrightarrow{f} B$$

is commutative, and in particular that h is the *composition* f.g if

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

is commutative. We regard *objects* as co-extensive with identity morphisms, or equivalently with those morphisms which appear as domains or codomains. As usual we call a morphism which has a two-sided inverse an *isomorphism.*

With any category $\mathbf{A}$ is associated another $\mathbf{A}^{op}$ obtained by maintaining the interpretation of "morphism" but reversing the direction of all arrows when re-interpreting diagrams and their commutativity. Thus $A \xrightarrow{f} B$ in $\mathbf{A}^{op}$ means $B \xrightarrow{f} A$ in $\mathbf{A}$, and h = f.g in $\mathbf{A}^{op}$ means h = g.f. in $\mathbf{A}$; clearly the interpretations of "domain", "codomain" and "composition" determine the interpretation of general "commutativity of diagrams". The category-theorist owes an apology to the philosophical reader for this unfortunately well-established use of the word "commutativity" in a context more general than any which could reasonably have a description in terms of "interchangeability".

A *functor* F involves a domain category $\mathbf{A}$, a codomain category $\mathbf{B}$, and a mapping assigning to every morphism x in $\mathbf{A}$ a morphism x F in $\mathbf{B}$ in a fashion which preserves the commutativity of diagrams. (Thus in particular a functor preserves objects, domains, codomains, compositions, and isomorphisms.) Utilizing the usual composition of mappings to define commutative diagrams of functors, one sees that functors are the morphisms of a super-category, usually called the *(meta-) category of categories.*

A category with exactly one morphism will be denoted by $\mathbf{1}$. It is determined uniquely up to isomorphism by the fact that for any category $\mathbf{A}$, there is exactly one functor $\mathbf{A} \longrightarrow \mathbf{1}$; the functors $\mathbf{1} \xrightarrow{A} \mathbf{A}$ corespond bijectively to the objects in $\mathbf{A}$.

A *natural transformation* φ involves a domain category $\mathbf{A}$, a codomain category $\mathbf{B}$, domain and codomain functors

$$\mathbf{A} \underset{F_1}{\overset{F_0}{\rightrightarrows}} \mathbf{B}$$

and a mapping assigning to each object A of $\mathbf{A}$ a morphism

$$AF_0 \xrightarrow{A\varphi} AF_1$$

in $\mathbf{B}$, in such a way that for every morphism

$$A \xrightarrow{a} A'$$

in $\mathbf{A}$, the diagram

$$\begin{tikzcd} AF_0 \arrow[r, "A\varphi"] \arrow[d, "aF_0"'] & AF_1 \arrow[d, "aF_1"] \\ A'F_0 \arrow[r, "A'\varphi"'] & A'F_1 \end{tikzcd}$$

is commutative in $\mathbf{B}$. For fixed $\mathbf{A}$ and $\mathbf{B}$, the natural transformations are the morphisms of a *functor category* $\mathbf{B}^{\mathbf{A}}$, where we write

$$F_0 \xrightarrow{\varphi} F_1 \quad \text{in } \mathbf{B}^{\mathbf{A}}$$

for the above-described situation, and define commutativity of triangles

$$\begin{tikzcd} F_0 \arrow[r, "\varphi"] \arrow[dr, "\varphi.\bar{\varphi}"'] & F_1 \arrow[d, "\bar{\varphi}"] \\ & F_2 \end{tikzcd} \quad \text{in } \mathbf{B}^{\mathbf{A}}$$

by the condition

$$A(\varphi.\bar{\varphi}) = (A\varphi).(A\bar{\varphi})$$

for all objects A in $\mathbf{A}$.

A second *Godement multiplication* is also defined for natural transformations, in a way which extends in a sense the composition of functors. Namely, if

$$F_0 \xrightarrow{\varphi} F_1 \text{ in } \mathbf{B}^{\mathbf{A}} \text{ and } G_0 \xrightarrow{\psi} G_1 \text{ in } \mathbf{C}^{\mathbf{B}}$$

then $F_0G_0 \xrightarrow{\varphi\psi} F_1G_1$ in $\mathbf{C}^{\mathbf{A}}$ is the natural transformation which assigns to each A of $\mathbf{A}$ the morphism of $\mathbf{C}$ which may be indifferently described as either composition in the commutative square

$$\begin{array}{ccc} AF_0G_0 & \xrightarrow{(A\varphi)\,G_0} & AF_1G_0 \\ {\scriptstyle (AF_0)\,\psi}\downarrow & & \downarrow{\scriptstyle (AF_1)\,\psi} \\ AF_0G_1 & \xrightarrow[(A\varphi)\,G_1]{} & AF_1G_1 \end{array}$$

The Godement multiplication is functional

$$\mathbf{B}^{\mathbf{A}} \times \mathbf{C}^{\mathbf{B}} \longrightarrow \mathbf{C}^{\mathbf{A}}$$

which means in particular that

$$(\varphi.\bar{\varphi})(\psi.\bar{\psi}) = (\varphi\psi).(\bar{\varphi}\bar{\psi}),$$

and associative, meaning that if also $H_0 \xrightarrow{\vartheta} H_1$ in $\mathbf{D}^{\mathbf{C}}$ then

$$(\varphi\psi)\,\vartheta = \varphi\,(\psi\vartheta) \text{ in } \mathbf{D}^{\mathbf{A}}$$

Other rules follow from the facts that in any product category $\mathbf{L} \times \mathbf{M}$ (obvious definition) a sort of commutativity relation

$$\langle l, M\rangle . \langle L, m\rangle = \langle L', m\rangle . \langle l, M'\rangle$$

holds for

$$L' \xrightarrow{l} L \text{ in } \mathbf{L} \text{ and } M \xrightarrow{m} M' \text{ in } \mathbf{M},$$

and that if Godement multiplication is applied to identity natural transfomations, it reduces to composition of the corresponding functors. Two intermediate cases will be importance in the sequel : if

$$F_0 = \varphi = F_1 = F,$$

then
$$A(F\psi) = (AF)\,\psi$$

while if $G_0 = \psi = G_1 = G$, then $A(\varphi G) = (A\varphi)\,G$. Thus each functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$ induces via Godement multiplication by F a functor $\mathbf{C}^{\mathbf{B}} \xrightarrow{\mathbf{C}^F} \mathbf{C}^{\mathbf{A}}$ for any category $\mathbf{C}$, and similary each $\mathbf{B} \xrightarrow{G} \mathbf{C}$ induces $\mathbf{B}^{\mathbf{A}} \xrightarrow{G^{\mathbf{A}}} \mathbf{C}^{\mathbf{A}}$ for any category $\mathbf{A}$.

Now we come to the central concept, of which we will presently see many examples. An *adjoint situation* involves two categories $\mathbf{A}$ and $\mathbf{B}$, two functors

$$\mathbf{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{B}$$

and two natural transformations

$$\begin{array}{llll} \mathbf{A} & \xrightarrow{\eta} & \mathrm{FU} & \text{in } \mathbf{A}^{\mathbf{A}} \\ \mathrm{FU} & \xrightarrow{\epsilon} & \mathbf{B} & \text{in } \mathbf{B}^{\mathbf{B}} \end{array}$$

satisfying the two equations (commutative triangles)

$$\begin{array}{ll} \eta F \,.\, F\epsilon = F & \text{in } \mathbf{B}^{\mathbf{A}} \\ U\eta \,.\, \epsilon U = U & \text{in } \mathbf{A}^{\mathbf{B}}. \end{array}$$

We state immediately an equivalent form of the definition. Let $(F, \mathbf{B})$ denote the category whose morphisms are those quadruples of morphisms $A \xrightarrow{a} A'$ in $\mathbf{A}$, $B \xrightarrow{b} B'$, $AF \xrightarrow{h} B$, $A'F \xrightarrow{h'} B'$, in $\mathbf{B}$ for which

$$\begin{array}{ccc} AF & \xrightarrow{aF} & A'F \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ B & \xrightarrow[b]{} & B' \end{array}$$

is commutative in $\mathbf{B}$; commutative diagrams in $(F, \mathbf{B})$ are defined so that forgetting h, h′ in the above defines a functor

$$(F, \mathbf{B}) \longrightarrow \mathbf{A} \times \mathbf{B}.$$

The objects of $(F, \mathbf{B})$ thus correspond to pairs each consisting of a morphism $AF \xrightarrow{h} B$ in $\mathbf{B}$ with a given A in $\mathbf{A}$. Similary a category $(\mathbf{A}, U)$ is defined whose objects correspond to pairs each consisting of a morphism $A \xrightarrow{f} BU$ in $\mathbf{A}$ with a given B in $\mathbf{B}$, and which also has a forgetful functor

$$(\mathbf{A}, U) \longrightarrow \mathbf{A} \times \mathbf{B}$$

Now it can be shown that givineg η, ϵ satisfying the two conditions above in order to complete an adjoint situation is equivalent with giving a *functor*

$$(F, \mathbf{B}) \xrightarrow{Q} (\mathbf{A}, U)$$

which commutes with the forgetful functors to $\mathbf{A} \times \mathbf{B}$ and which a two sided inverse

$$(\mathbf{A}, U) \xrightarrow{Q^{-1}} (F, \mathbf{B}) .$$

In particular, Q assigns to the object $AF \xrightarrow{h} B$ of $(F, \mathbf{B})$ the object

$$A \xrightarrow{(A\eta)\,.\,(hU)} BU$$

of $\mathbf{A}$, U), while Q^{-1} assigns

$$AF \xrightarrow{(fF)\,.\,(B\epsilon)} B$$

to $A \xrightarrow{f} BU$. Sometimes Q is called an adjointness, η an adjunction, and ϵ a co-adjunction for F and U; if such exists, F is said to be *left-adjoint* to U and U is said to be *right adjoint* to F, and we write $F \dashv U$. Briefly, F and U may be placed in an adjoint situation iff bijections

$$(AF, B) \cong (A, BU)$$

can be given in a way which is "natural" or "functorial" when A and B vary; here the left-hand side denotes the set of all $AF \xrightarrow{h} B$ in $\mathbf{B}$ and the right-hand side denotes the set of all $A \xrightarrow{f} BU$ in $\mathbf{A}$.

Several important properties hold for adjoint situations in general. One is that adjoints are unique (if they exist). That is, any two adjoint situations with a given $\mathbf{A} \xrightarrow{F} \mathbf{B}$ are uniquely isomorphic, and similarly for a given $\mathbf{B} \xrightarrow{U} \mathbf{A}$. More generally, given two adjoint situations with fixed $\mathbf{A}$ and $\mathbf{B}$ (with and without primes), then any natural transformation $F' \longrightarrow F$ induces a unique $U \longrightarrow U'$, and conversely. If $\mathbf{A} \xrightarrow{F} \mathbf{B}$ with $F \dashv U$, and $\mathbf{B} \xrightarrow{\widetilde{F}} \mathbf{C}$, with $\widetilde{F} \dashv \widetilde{U}$, then $F\widetilde{F} \dashv \widetilde{U}U$, and also $F^{\mathbf{D}} \dashv U^{\mathbf{D}}$ for any category $\mathbf{D}$. Thus a necessary condition for a functor U to have a left adjoint is that U commutes (up to isomorphism of functors) with all generalized limits which exist in $\mathbf{B}$ and $\mathbf{A}$, and dually a functor F can have a right adjoint only if F commutes up to isomorphism with all generalized colimits which exist in $\mathbf{A}$ and $\mathbf{B}$. Here, for any functor $\mathbf{D}' \xrightarrow{L} \mathbf{D}$, generalized limits in $\mathbf{A}$ along L and generalized colimits in $\mathbf{A}$ along L are, when they exist, respectively right and left adjoint to the induced functor $\mathbf{A}^{L}$. These generalized colimits include direct limits in the usual sense as well as free products and pushouts and evaluation adjoints; for many $\mathbf{A}$ of importance, they exist for all L with $\mathbf{D}'$ and $\mathbf{D}$ smaller than some fixed regular cardinal, but they will exist for all L iff $\mathbf{A}$ is a category in which the morphisms reduce to nothing more than a preordering on the objects, with respect to which the latter form a complete lattice.

The general adjoint functor theorem asserts that if a functor $\mathbf{B} \xrightarrow{U} \mathbf{A}$ satisfies the necessary continuity condition mentioned above, then it will have a left adjoint provided that $\mathbf{B}$ is suitably complete and U is suitably bounded. Freyd's Special Adjoint Functor Theorem shows that certain categories $\mathbf{B}$ (such as the category of sets or of abelian groups, but *not* the category of groups) are "compact" in the sens that every continuous U to a standard $\mathbf{A}$ (say the category of sets) is bounded. A schema formalizing these adjoint functor theorems for the case where $\mathbf{A}$ and $\mathbf{B}$ are replaced by metacategories comparable to the conceptual universe would, on the one hand, justify by itself a large portion of the existential richness of that universe, as follows at least to the extent of the higher types and of infinite objects from the following section, and on the other hand, provide a common rationale for the inevitability and basic properties of a large number of mathematical constructions, as the following examples indicate.

Letting Top denote the category of continuous mappings of topological spaces, the diagonal functor

$$\mathrm{Top} \longrightarrow \mathrm{Top} \times \mathrm{Top}$$

has a right adjoint, which forces the definition of product topology. If A denotes, say, the unit interval, then

$$\mathrm{Top} \xrightarrow{A \times (-)} \mathrm{Top}$$

also has a right adjoint, yielding the construction of the compact-open topology on path spaces. The inclusion functor Comp $\longrightarrow$ Top from the category of continuous mappings between compact spaces has a *left* adjoint, giving the Stone-Čech compactification construction.

Turning to algebra, the forgetful functor Gps $\longrightarrow$ Sets which views every group-homomorphism as a mapping of the carrying sets has a left adjoint, yielding the notion of free groups. The "commutator bracket" functor Assoc $\longrightarrow$ Lie, which interprets every homomorphism of associative linear algebra as a homomorphism of the same underlying vector spaces viewed only as Lie algebras, has also a left adjoint, yielding universal enveloping algebras. These two adjoint situations beong to a special large class, which also includes abelianization of groups, monoid rings, symmetric algebras, etc.

There are also adjoint situations in analysis, leading for example to l^1 spaces and almost-periodic functions. Needless to say, viewing all these constructions explicitly as adjoint situations seems to have certain formal and conceptual utility apart from any philosophical attempt to unify their necessity.

III. A *cartesian closed category* is a category $\mathbf{C}$ equipped with adjoint situations of the following three sorts:

(1) A *terminal object* $\mathbf{1} \xrightarrow{1} \mathbf{C}$, meaning a right adjoint to the unique $\mathbf{C} \longrightarrow \mathbf{1}$; if η denotes the adjunction, then $C \xrightarrow{C\eta} 1$ is the unique morphism in $\mathbf{C}$ with domain C and codomain 1, for each object C.

(2) A *product*, $\mathbf{C} \times \mathbf{C} \xrightarrow{\times} \mathbf{C}$, meaning a right adjoint to the diagonal functor $\mathbf{C} \longrightarrow \mathbf{C} \times \mathbf{C}$. Denoting the adjunction and co-adjunction by δ and π respectively, we have $X\delta$ as a *diagonal* morphism $X \longrightarrow X \times X$ and $\langle Y_0, Y_1\rangle \pi$ as a *projection* pair usually denoted for short by

$$Y_0 \times Y_1 \xrightarrow{\pi_0} Y_0, Y_0 \times Y_1 \xrightarrow{\pi_1} Y_1 .$$

Given morphisms $X \xrightarrow{f_0} Y_0$, $X \xrightarrow{f_1} Y_1$, then $\langle f_0, f_1\rangle = X\delta . (f_0 \times f_1)$ is the unique $X \longrightarrow Y_0 \times Y_1$ whose compositions with the π_i are the f_i. The projection $A \times 1 \longrightarrow A$ is an isomorphism for any object A.

(3) For each object A, *exponentiation by A*, $\mathbf{C} \xrightarrow{(\ )^A} \mathbf{C}$, meaning a right adjoint to the functor $\mathbf{C} \xrightarrow{A \times (\ )} \mathbf{C}$. Denote by λ_A and ϵ_A the adjunction and co-adjunction respectively. Then

$$X \xrightarrow{X\lambda_A} (A \times X)^A \text{ and } A \times Y^A \xrightarrow{Y\epsilon_A} Y,$$

and for any $A \times X \xrightarrow{h} Y$, one has that

$$X \xrightarrow{X\lambda_A . h^A} Y^A$$

is the unique morphism g for which $(A \times g) . Y\epsilon_A = h$. For any morphism $A \xrightarrow{f} Y$, one may consider in the above process the case $X = 1$, $h = (\langle A, 1\rangle \pi_1)^{-1} . f$, obtaining a morphism $1 \longrightarrow Y^A$ denoted for short by $\ulcorner f \urcorner$; every morphism $1 \longrightarrow Y^A$ is of form $\ulcorner f \urcorner$ for a unique $A \xrightarrow{f} Y$. Now for any $1 \xrightarrow{a} A$, $A \xrightarrow{f} Y$, one has that

$$\langle a, \ulcorner f \urcorner\rangle . Y\epsilon_A = a . f,$$

which justifies calling $Y\epsilon_A$ the *evaluation* morphism, though in most cartesian closed $\mathbf{C}$, a paucity of morphisms with domain 1 prevents the last equation from on its own strenght determining ϵ.

From the unqueness of adjoints it is clear that any two cartesian closed categories with a given category $\mathbf{C}$ are uniquely isomorphic. Most categories $\mathbf{C}$ cannot be made into cartesian closed categories at all. The category of all mappings between finite sets can be made cartesian closed, as can many larger categories of mappings between sets. Also, order preserving mappings between partially ordered sets can be made cartesian closed, Y^A then being necessarily isomorphic to the set of order— preserving mappings $A \longrightarrow Y$ equipped with the usual pointwise partial ordering. Another kind of example is provided by a Browerian semi-lattice, these essentially co-extensive with the "trivial" cartesian closed categories, that is those in which there is at most one morphism $X \longrightarrow Y$ for any given objects X and Y, or equivalently those in which all diagonal morphisms $X\delta$ are isomorphisms $X \longrightarrow X \times X$. More examples will appear presently.

A *hyperdoctrine* shall consist of at least the following four data

1) A cartesian closed category $\mathbf{T}$. We will refer sometimes to the objects of $\mathbf{T}$ as types, to the morphisms in $\mathbf{T}$ as terms, and to the morphisms $1 \longrightarrow X$ in particular as constant terms of type X.
2) For each type X, a corresponding cartesian closed category P (X) called the category of attributes of type X. Morphisms of attributes will be called deductions over X, entailments, or inclusions as is appropriate. The basic functors giving P (X) its closed structure will be denoted by

$$1_X\,,\ \wedge_X\,,\ \alpha_X => (\ )$$

to distinguish them from the analogous

$$1\,,\quad \times\,,\qquad (\ )^A$$

in $\mathbf{T}$ (the subscripts X may be omitted when no confusion is likely). Thus the evaluation morphisms in P (X) for each pair of objects α, ψ in P (X) $\alpha \wedge_X (\alpha_X => \psi) \longrightarrow \psi$ may be sometimes more appropriately referred to as the modus ponens deductions.

3) For each term $X \xrightarrow{f} Y$ (morphism in $\mathbf{T}$) a corresponding functor $P(Y) \xrightarrow{f\cdot(\ )} P(X)$. Ten for any attribute ψ of type Y, $f\cdot\psi$ will be called the attribute of type X resulting from substituting f in ψ. We assume that for $Y \xrightarrow{g} Z$, $(f.g)\cdot\zeta = f\cdot(g\cdot\zeta)$ for all attributes ζ of type Z (at least up to coherent isomorphism) and similarly for deductions over Z.
4) For each $X \xrightarrow{f} Y$ in $\mathbf{T}$ two further functors $P(X) \xrightarrow{(\ )\Sigma f} P(Y)$ and $P(X) \xrightarrow{(\ )\Pi f} P(Y)$ which are (equipped with adjunctions

which make them) respectively left and right adjoint to substitution f. For any object φ in P(X) we call $\varphi\,\Sigma\,f$ (respectively $\varphi\,\Pi\,f$) the existential (respectively universal) quantification of φ along f. For ψ in P(Y), the adjointnesses reduce to two (natural) bijections, one between deductions $\varphi\,\Sigma\,f \longrightarrow \psi$ over Y and deductions $\varphi \longrightarrow f\cdot\psi$ over X, and the other betwenn deductions $\psi \longrightarrow \varphi\,\Pi\,f$ over Y and deductions $f\cdot\psi \longrightarrow \varphi$ over X. From the assumed functorality of substitution we have for a further term $Y \xrightarrow{g} Z$ that $\varphi\,\Sigma\,(f.g\cdot) = (\varphi\,\Sigma\,f)\,\Sigma\,g$ and $\varphi\,\Pi\,(f.g\cdot) = (\varphi\,\Pi\,f)\,\Pi\,g$, at least up to an invertible deduction in P(Z).

Now among general features of hyperdoctrines we note for example that the existence of existential quantification implies that substitution commutes with conjunction ; however in general substitution does not commute with implication, as one of our examples below shows. There is at least one interesting realm which has roughly all the features of a hyperdoctrine *except* existential quantification, namely sheaf theory, wherein $\mathbf{T}$ is the category of continuous mappings between Kelly spaces and P(X) is the category of (morphisms between) set-valued sheaves on X.

Also in a general hyperdoctrine we can construct for each term $X \xrightarrow{f} Y$ the attribute $1_X\,\Sigma\,\langle X, f\rangle$ of type $X \times Y$ which plays the role of the graph of f ; in particular the graph of the identity morphism on X is an object in $(P\,X \times X)$ having the rudimentary properties of equality. With help of equality, graphs, and the behaviour of quantification under composition of terms, one can in certain hyperdoctrines reduce the general quantification to quantification along very special terms, namely projections $Y \times Z \longrightarrow Y$.

Any theory formalized in higher-order logic yields a hyperdoctrine in which types are just all expressions $1, V, V^V, V \times V, (V \times V)^{(V\times V\times V)^V}$, etc. and in which terms are identified if they are in the theory provably equal (we assume that "higher-order logic" does involve at least terms corresponding to the $\delta, \pi, \lambda, \in$ necessary to yield a cartesian closed category by this procedure). In this hyperdoctrine, we let the objects of P(X) be those formulas of the theory whose free variables are of type X (for example if $X = V \times V^V$, P(X) consists of the formulas with two free variables, one an "element" variable and the other a "function" variable, but with bound variables of arbitrary type) and the morphisms of P(X) be simply entailments deducible in the theory Thus the diagonal $\varphi \longrightarrow \varphi_X \wedge \varphi$ is an isomorphism in this example. Also each P (X) has a *coterminal* object O_X (falsity) with a deduction $\varphi \longrightarrow (\varphi_X \Longrightarrow O_X) \Longrightarrow_X O_X$ which is an isomorphism if the logic is classical. If the theory is number theory, then $\mathbf{T}$ participates in another adjoint situation (setting $\omega = V$)

$$\mathbf{T}^{\circlearrowleft} \underset{\omega \times (\ )}{\overset{}{\rightleftarrows}} \mathbf{T}$$

where the unlabeled right adjoint functor is the forgetful functor from the category whose objects correspond to endomorphisms $X \xrightarrow{t} X$ in $\mathbf{T}$ and whose morphisms are diagrams

$$\begin{tikzcd} X \arrow[loop above, "t"] \arrow[r, "h"] & Y \arrow[loop above, "s"] \end{tikzcd}$$

in $\mathbf{T}$ with t.h. = h.s. Conversely any cartesian closed category in which this forgetful functor has a left adjoint will contain a definite morphism corresponding to each higher-type primitive recursive function, and if the category contains somewhere at least one non-identity endomorphism, then there will be an infinite number of morphisms $1 \rightarrow \omega$ the category.

We also obtain a hyperdoctrine if we consider as types arbitrary sets, as terms arbitrary mappings, and as attributes arbitrary subsets, (i.e. $P(X) = 2^X$) and take substitution as the inverse image operator, which forces existential quantification to be the direct image operator. With the appropriate definition of morphism between hyperdoctrines, such a morphism into this one from one of the examples of the previous paragraph involves nothing more nor less than a *model* of the higher-order theory in question.

A different hyperdoctrine with the same category of types and terms (namely arbitrary sets and mappings) is obtained by defining $P(X) = (\mathbf{T}, X) =$ the category of "sets over X" whose morphisms are arbitrary commutative triangles

$$\begin{tikzcd} A \arrow[rr, "d"] \arrow[dr, "\varphi"'] & & A' \arrow[dl, "\varphi'"] \\ & X & \end{tikzcd}$$

of mappings of sets, and taking for $f\cdot\psi$ (when $X \xrightarrow{f} Y$ and $B \xrightarrow{\psi} Y$) the projection to X of the set of pairs $\langle x, b\rangle$ with $xf = b\psi$. Then existential quantification is simply composition $\varphi\Sigma f = \varphi.f$. Since $P(1) = \mathbf{T}$, for any element $1 \xrightarrow{x} X$ and "property" φ in $P(X)$, $x\cdot\varphi$ is simply a set, the "fiber" of φ over x. Calling a deduction $1_X \longrightarrow \varphi$ over X a "proof" of φ, and noting that in the present example 1_X is just the identity mapping, we see that a proof of φ is a mapping assigning to each x a proof of $x\cdot\varphi$. Then a proof of $\varphi\Sigma f$ is a mapping assigning to each y a specific x such that $xf = y$, together with a proof of $x\cdot\varphi$. Since the fiber over y of a universal quantification $y.(\varphi\Pi f) = \Pi_{xf=y}\, x\cdot\varphi$, a proof of $\varphi\Pi f$ involves a mapping assigning to each y a mapping assigning to each x with $xf = y$ a proof that $x\cdot\varphi$, and more generally deductions with premises other than 1_X have similar descriptions. The validity of the axiom of choice in $\mathbf{T}$ then implies that

$$(\mathbf{T}, X) \xrightarrow{\text{direct image}} 2^X$$

commutes with all the "propositional" and quantificational operations and hence defines the strictest sort of morphism from the present hyperdoctrine to the previous one. Since in this case every ψ in $P(Y)$ is of the form $1_X\,\Sigma\, f = \psi$ for a unique X and f, it is possible to view conjunction and implication as particular cases of substitution and universal quantification. In particular a deduction over X of a conjunction $\varphi_1 \wedge_X \varphi_2$ involves an ordered pair of mappings, each of which may be an arbitrary deduction of φ_1, or φ_2 respectively; this language makes clear the reasonableness of the non-idempotence of conjunction.

As a final example of a hyperdoctrine, we mention the one in which types are finite categories and terms arbitrary functors between them, while $P(\mathbf{A}) = \mathbf{S}^{\mathbf{A}}$ where $\mathbf{S}$ is the category of finite sets and mappings, with substitution as the special Godement multiplication. Quantification must then consist of generalized limits and colimits, while implication works like this: $\alpha \Rightarrow \psi$ is a functor whose value at an object A in $\mathbf{A}$ is the set of natural transformations $h_A \times \alpha \longrightarrow \psi$, where $h_A : \mathbf{A} \longrightarrow \mathbf{S}$ is the "representable" functor assigning to A′ the set of morphisms $\mathbf{A}(A, A')$. By focusing on those $\mathbf{A}$ with one object and all morphisms isomorphisms, one sees that this hyperdoctrine includes the theory of permutation groups; in fact, such $\mathbf{A}$ *are* groups and a "property" of $\mathbf{A}$ is nothing but a depresentation of $\mathbf{A}$ by permutations. Quantification yields "induced representations" and implication gives a kind of "intertwining representation". Deductions are of course equivariant maps.

IV. If $\mathbf{O}_1$ and $\mathbf{O}_2$ are partially ordered sets the traditional notion of a *Galois connection* between them is easily seen to be equivalent to an adjoint situation

$$\mathbf{O}_1\ \mathrm{op} \rightleftarrows \mathbf{O}_2$$

between the corresponding trivial categories (one of the turned opposite). Recently it has begun to appear that the basic examples of Galois connections are really just fragments of more global adjoints which involve non-trivial categories and which carry more information.

For example, Galois theory itself has profitably been treated as the study of adjoint situations

$$\mathrm{Alg}_k^{\mathrm{op}} \rightleftarrows \text{sets } \mathrm{Gal}\,(\bar{k}, k)$$

where the left hand category has as objects all commutative and associative linear algebras over the field k and the right hand all continuous permutation representations of the compact Galois group of the separable closure $\bar{k}$ of k, while the left-to-right right adjoint functor assigns to every k-algebra A the representation by permutations of the set $\mathrm{Alg}_k\,(A, \bar{k})$ of all algebra homomorphisms $A \longrightarrow \bar{k}$. Restricting to subalgebras of $\bar{k}$ on the one hand and to quotients of the regular representation on the other retrieves in effect the usual Galois connection. Similarly, in algebraic geometry the Galois connection between ideals in a polynomial ring $k\,[x_1, \ldots, x_n]$ and subsets of k^n has been globalized to an adjoint situation

$$\mathrm{Alg}_k^{\mathrm{op}} \underset{T}{\overset{\mathrm{spec}}{\rightleftarrows}} \mathrm{Schema}_k$$

with considerable profit.

Finally, in Foundations there is the familiar Galois connection between sets of axioms and classes of models, for a fixed set of relation variables R_i. Globalizing to an adjoint pair allows making precise the semantical effects, not only of increasing the axioms, but also of omitting some relation symbols or reinterpreting them, in a unified way, and if we deal with *categories* of models, allows the latter to determine their own full sets of natural relation variables, thus giving definability theory a new significance outside the realm of axiomatic classes. To do this for a given species — equational, elementary, higher-order, etc — of, say, I-sorted theories, one defines an adjoint situation

$$\text{Theories}^{\text{op}} \underset{\text{structure}}{\overset{\text{semantics}}{\rightleftarrows}} (\text{Cat}, [\text{Sets}^{\text{I}}])$$

in which the right hand side denotes a category whose morphisms are commutative triangles

$$\begin{tikzcd} \mathbf{C} \arrow[rr] \arrow[dr] & & \mathbf{C}' \arrow[dl] \\ & \text{Sets}^{\text{I}} & \end{tikzcd}$$

of functors with $\mathbf{C}$ and $\mathbf{C}'$ more or less arbitrary categories. The invariant notion of theory here appropriate has, in all cases considered by the author, been expressed most naturally by identifying a theory T itself with a category of a certain sort, in which case the semantics (category of models) of T is a certain subcategory of the category of functors $\text{T} \longrightarrow \text{Sets}^{\text{I}}$. There is then a further adjoint situation

$$\text{Formal} \rightleftarrows \text{Theories}$$

describing the presentation of the invariant theories by means of the formalized languages appropriate to the species. Composing this with the above, and tentatively indentifying the Conceptual with categories of the general sort $(\text{Cat}, [\text{Sets}^{\text{I}}])$, we arrive at a family of adjoint situations

$$\text{Formal}^{\text{op}} \rightleftarrows \text{Conceptual}$$

(one for each species of theory) which one may reasonably hope constitute the fragments of a precise description of the duality with which we began our discussion.

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