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Bouziane's transformation of the Petri net reachability problem and incorrectness of the related algorithm

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ABSTRACT

The proceedings of FOCS'98 contain a paper by Zakariae Bouziane, who sketches a new representation of the Petri net reachability problem and claims to provide a new algorithm solving the problem. In this note, the essence of Bouziane's approach is explained, and a serious flaw of the algorithm is exposed.

1. Introduction

The Petri net reachability problem (PNRP), which asks if a given state is reachable from the initial state in a given place/transition Petri net, is well known also outside the Petri net community. It is surprisingly tricky from the complexity point of view. Its decidability had been a challenging open question for about a decade, starting in the early 1970s. Positive solutions for some subcases were given by several researchers, and then a "solution" for the general case was presented at STOC'77—but it turned out to be incomplete afterwards. The decidability was finally established by E.W. Mayr and presented at STOC'81 [1,2]. He used an involved combination of nice ideas and developed an algorithm with nonprimitive recursive complexity. But since it was a nontrivial task to verify Mayr's involved proof, some other researchers have further elaborated on it, aiming at improving its general understandability; the first one was Kosaraju at STOC'82 [3]. Nevertheless, nobody has improved the known upper bound on the complexity, which thus remains nonprimitive recursive. On the other hand, PNRP is known to be EXPSPACE-hard due to Lipton [4]; the proof can be also found, e.g., in [5]. The exact complexity of PNRP thus remains unclear and keeps to constitute an intellectual challenge.

In the proceedings of FOCS'98, Z. Bouziane [6] claims to provide a new algorithm for PNRP; it looks conceptually simpler than Mayr's algorithm and its complexity is claimed to be primitive recursive (double-exponential, in fact). Bouziane first transforms PNRP into an equivalent problem in a different setting. This new problem, let us denote it by BP, might resemble a special (and decidable) version of the Post correspondence problem, very roughly sketched as follows: for given pairs $(u_0,v_0),(u_1,v_1),(u_2,v_2),\ldots,(u_n,v_n)$, where each u_i (v_i) represents a tuple of sets of numbers, it asks if there is a finite sequence $i_1,i_2,\ldots,i_m$ of indices such that $u_0u_{i_1}u_{i_2},\ldots,u_{i_m}$ and $v_0v_{i_1}v_{i_2},\ldots,v_{i_m}$ represent (tuples of) sets of numbers which are linearly related in a certain sense.

Bouziane sketches an algorithm for BP, based on finite automata constructions, but in a technically complicated and unclear form.

Since Bouziane has not produced any elaborated version of the conference paper, the claimed result could not be really verified and had to be taken as unreliable (this was reflected, e.g., in [7]). The aim of this note is to make clear that the main

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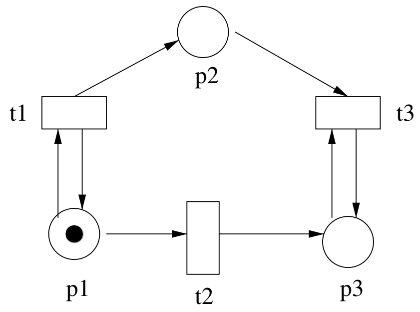

Fig. 1. Example of a Petrinet.

claim in [6] is, in fact, substantially wrong. Though a preliminary version of this note was made public on the Internet, it seems worth to make it an official reviewed publication since nonspecialists sometimes refer to the paper [6] as if it was valid; see, e.g., [8,9]. As a "side-effect", the note can also serve for attracting a new attention to an interesting and challenging problem.

Section 2 defines PNRP and provides a running example. Section 3 clarifies the transformation of PNRP to the equivalent BP (which is valid but hardly understandable from [6]). Section 4 exposes the flaw of Bouziane's algorithm, and Section 5 provides a short summary.

2. Petri net reachability problem

We now state PNRP and give a simple example on which Bouziane's algorithm will be later contradicted.

A *Petri net* is a tuple $N=(P,T,F)$ where $P=\{p_1,p_2,\ldots,p_m\}$ and $T=\{t_1,t_2,\ldots,t_n\}$ are finite disjoint sets of *places* and *transitions*, respectively, and $F:(P\times T)\cup(T\times P)\to\{0,1\}$ is a flow function, i.e., the characteristic function of a set of *arcs*.
A *marking* M of the net N is a mapping $M:P\to\mathbb{N}$ associating a nonnegative number of *tokens* to each place; we implicitly assume an ordering of places and view M as a vector from $\mathbb{N}^m$. Addition, subtraction and ordering $\leq$ on $\mathbb{N}$ are extended to vectors component-wise.
For a transition t, we define ${}^\bullet t=\Big(F(p_1,t),F(p_2,t),\ldots,F(p_m,t)\Big)$ and $t^\bullet=\Big(F(t,p_1),F(t,p_2),\ldots,F(t,p_m)\Big)$.
We write $M\xrightarrow{t}M'$ (M changes to M' by performing transition t) iff $M\geq{}^\bullet t$ and $M'=M-{}^\bullet t+t^\bullet$; in the natural way, we extend the notation to $M\xrightarrow{w}M'$ where w is a finite sequence of transitions.

Fig. 1 shows an example of a net with three places (circles) and three transitions (rectangles); also the marking $(1,0,0)$ is depicted. We can check, e.g., that ${}^\bullet t_1=(1,0,0)$, $t_1^\bullet=(1,1,0)$, and that $(1,0,0)\xrightarrow{w}(0,2,1)$ for $w=t_1t_1t_1t_2t_3$.

Petri net reachability problem (PNRP) is defined as follows:

Instance: a Petri net N and two its markings M_0,M_f.

Question: is there a sequence w of transitions, called a *witness sequence*, such that $M_0\xrightarrow{w}M_f$?

If we put $M_0=(1,0,0)$ and $M_f=(0,0,1)$ in our example net on Fig. 1, we can observe that the witness sequences are precisely the sequences $w=(t_1)^q t_2(t_3)^q$, where $q\in\mathbb{N}$.

It is useful to observe that we can generally confine ourselves to the "cycle-free" witness sequences—these pass through any marking at most once (if $M_0\xrightarrow{u}M\xrightarrow{v}M\xrightarrow{z}M_f$ then also $M_0\xrightarrow{uz}M_f$).

3. Transformation of PNRP to BP

Bouziane transforms the problem of finding if there is a witness sequence to an equivalent problem of finding if there is a collection of finite subsets of $\mathbb{N}$ satisfying certain conditions. This is valid, though described unclearly in [6]. We now explain the transformation; it might be useful in the sense that it allows to view PNRP from a different perspective.

For illustration we use the example net from Fig. 1 where we put $M_0=(1,0,0)$ and $M_f=(0,0,1)$.

Definition 1. Given $m\in\mathbb{N}$, ϕ denotes the "Gödel coding" $\phi:\mathbb{N}^m\to\mathbb{N}$ defined as $\phi(x_1,x_2,\ldots,x_m)=prime_1^{x_1}prime_2^{x_2},\ldots,prime_m^{x_m}$ where $prime_i$ is the ith prime number.

Given a Petri net, with an ordering on its m places, the meaning of $\phi(M)$ for a marking M and of $\phi({}^\bullet t)$, $\phi(t^\bullet)$ for a transition t is induced.

We note that $M\xrightarrow{t}M'$ iff

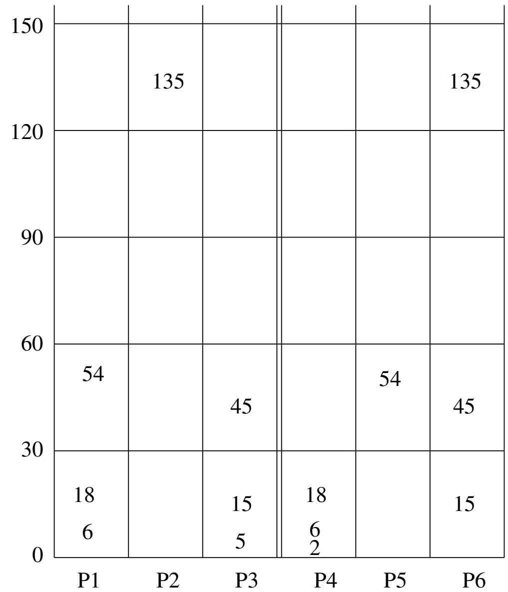

Fig. 2. Distribution of Markings on a path.

- $\phi(M)$ is divisible by $\phi({}^\bullet t)$, and
- $\phi(M') = \phi(M) \cdot \delta(t)$ where $\delta(t) = \phi(t^\bullet)/\phi({}^\bullet t)$.

We also note that the conditions imply that $\phi(M')$ is divisible by $\phi(t^\bullet)$.

For brevity, we further say just "a marking" instead of "the (Gödel) code of an marking" etc. In the example, the (code of the) initial marking is 2, the final marking is 5. Let us consider one particular witness sequence and the corresponding path through markings:

$$2 \xrightarrow{t_1} 6 \xrightarrow{t_1} 18 \xrightarrow{t_1} 54 \xrightarrow{t_2} 135 \xrightarrow{t_3} 45 \xrightarrow{t_3} 15 \xrightarrow{t_3} 5.$$

When we distribute the markings on this path into classes $\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_n$, so that in $\mathcal{P}_i$ ($i = 1,2,\ldots,n$) are precisely those markings which are entered by transition t_i, we get ($n = 3$ and) $\mathcal{P}_1 = \{6,18,54\}$, $\mathcal{P}_2 = \{135\}$, $\mathcal{P}_3 = \{5,15,45\}$ (cf. Fig. 2, ignoring the horizontal "cut lines" at the moment). Similarly we can distribute the markings to classes $\mathcal{P}_{n+1},\mathcal{P}_{n+2},\ldots,\mathcal{P}_{2n}$, so that $\mathcal{P}_{n+i}$ ($i = 1,2,\ldots,n$) contains precisely those markings which are left by transition t_i; we get $\mathcal{P}_4 = \{2,6,18\}$, $\mathcal{P}_5 = \{54\}$, $\mathcal{P}_6 = \{15,45,135\}$.

The collection $\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_6$ is an example of a witness collection as defined by the following two definitions. Here $A \cdot b$, for $A \subseteq \mathbb{N}$ and a (rational) number b, denotes the set $\{\, x \cdot b \mid x \in A \,\}$.

Definition 2. Given a $(2n+2)$-tuple of natural numbers $(\alpha_1,\alpha_2,\ldots,\alpha_n,\beta_1,\beta_2,\ldots,\beta_n,\mathit{init},\mathit{fin})$, a *collection* $\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_{2n}$ of subsets of $\mathbb{N}$ is *successful* (with respect to the tuple) if the following two conditions hold:

Condition (1):

a/ $\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_n$ are pairwise disjoint and do not contain *init*, $\mathcal{P}_{n+1},\mathcal{P}_{n+2},\ldots,\mathcal{P}_{2n}$ are pairwise disjoint and do not contain *fin*,
b/ $\{\mathit{init}\} \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \ldots \cup \mathcal{P}_n = \{\mathit{fin}\} \cup \mathcal{P}_{n+1} \cup \mathcal{P}_{n+2} \cup \ldots \cup \mathcal{P}_{2n}$,
c/ each element of $\mathcal{P}_i$ ($i = 1,2,\ldots,n$) is divisible by α_i , and each element of $\mathcal{P}_{n+i}$ ($i = 1,2,\ldots,n$) is divisible by β_i .

Condition (2):
$\mathcal{P}_i = \mathcal{P}_{n+i} \cdot (\alpha_i/\beta_i)$ for each $i = 1,2,\ldots,n$.

Definition 3. Given a Petri net, with transitions $t_1,t_2,\ldots,t_n$, and its markings M_0, M_f, by a *witness collection* we mean any collection $\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_{2n}$ of finite subsets of $\mathbb{N}$ which is successful with respect to $\alpha_1 = \phi(t_1^\bullet),\alpha_2 = \phi(t_2^\bullet),\ldots,\alpha_n = \phi(t_n^\bullet)$, $\beta_1 = \phi({}^\bullet t_1),\beta_2 = \phi({}^\bullet t_2),\ldots,\beta_n = \phi({}^\bullet t_n)$, $\mathit{init} = \phi(M_0)$, $\mathit{fin} = \phi(M_f)$.

Generalizing our concrete example, we can easily verify that, given any Petri net and its markings M_0, M_f, any witness sequence translates into a witness collection. On the other hand, we can easily observe that any witness collection "contains" a witness sequence, which can be constructed by the following "program".

$c_1 := \phi(M_0)$; $i := 1$
while $c_i \neq \phi(M_f)$ **do**
(find $\mathcal{P}_{n+j}$ s.t. $c_i \in \mathcal{P}_{n+j}$; $c_{i+1} := c_i \cdot \phi(t_j^\bullet)/\phi({}^\bullet t_j)$; $i := i+1$)

We note that a witness collection can also contain "separated cycles" (e.g., by adding 10 to $\mathcal{P}_3$ and $\mathcal{P}_4$ and 30 to $\mathcal{P}_1$ and $\mathcal{P}_6$); these "cycles" can even not correspond to (the codes of) markings (when, e.g., adding 70 to $\mathcal{P}_3$ and $\mathcal{P}_4$ and 210 to $\mathcal{P}_1$ and $\mathcal{P}_6$). Nevertheless, we have shown the following lemma.

Lemma 4. *For a given Petri net, M_f is reachable from M_0 iff there is a witness collection.*

Remark. An idea of replacing the problem of finding a witness sequence by the problem of finding a collection of finite sets of markings was (indirectly) present, e.g., in [10] (Section 3); in fact, Mayr and Meyer deal with polynomials (with m variables) which can naturally represent sets of markings (for m places). Bouziane also uses polynomials, namely one-variable polynomials since he codes markings by integers; but this serves just as a notation and the approach can be presented without using polynomials—as we do in this note.

Let BP denote the problem of deciding whether there is a successful collection with respect to a given $(2n+2)$-tuple; we assume the numbers given in binary. We have thus proved the following theorem.

Theorem 5. *PNRP is polynomially reducible to BP.*

Remark. BP can be reduced to PNRP, using the prime decomposition of the given numbers. Polynomiality of such a reduction is not clear but this is not our concern here.

4. Exposing the flaw

After establishing (an equivalent of) Lemma 4, Bouziane shows that the collections which satisfy Condition (1) of Definition 2 and need not satisfy Condition (2) have a regular structure. To demonstrate this, we use some technical definitions, including condition (1^{hom}) which is the "homogeneous" version of (1) (where *init* and *fin* are omitted).

Definition 6. Given a $(2n+2)$-tuple $(\alpha_1,\alpha_2,\ldots,\alpha_n,\beta_1,\beta_2,\ldots,\beta_n,init,fin)$, a *collection* $\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_{2n}$ of sub sets of $\mathbb{N}$ is called a *(1)-collection* if it satisfies condition (1) of Definition 2.

It is called a *(1^{hom})-collection* if the following condition (1^{hom}) holds:

a/ $\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_n$ are pairwise disjoint, and $\mathcal{P}_{n+1},\mathcal{P}_{n+2},\ldots,\mathcal{P}_{2n}$ are pairwise disjoint,
b/ $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_n = \mathcal{P}_{n+1} \cup \mathcal{P}_{n+2} \cup \cdots \cup \mathcal{P}_{2n}$,
c/ each element of $\mathcal{P}_i$ ($i = 1,2,\ldots,n$) is divisible by α_i, each element of $\mathcal{P}_{n+i}$ ($i = 1,2,\ldots,n$) is divisible by β_i .

Definition 7. Given a $(2n+2)$-tuple $(\alpha_1,\alpha_2,\ldots,\alpha_n,\beta_1,\beta_2,\ldots,\beta_n,init,fin)$, a number $\ell_0 \in \mathbb{N}$ is a *cut number* iff ℓ_0 is divisible by all α_i and by all β_i ($i = 1,2,\ldots,n$) and is greater than both *init* and *fin*. By a *basic segment* (with respect to a cut number ℓ_0) we mean any (1)-collection whose union is a subset of $\{0,1,\ldots,\ell_0-1\}$. By a *basic hom-segment* we mean any (1^{hom})-collection whose union is a subset of $\{0,1,\ldots,\ell_0-1\}$. By an *rth segment* ($r = 1,2,\ldots$) we mean any (1^{hom})-collection whose union is a subset of $\{r\ell_0,r\ell_0+1,\ldots,r\ell_0+\ell_0-1\}$.

Definition 8. The *mod-ℓ_0-image* of a set $A \subseteq \mathbb{N}$ is the set $\{\, x \bmod \ell_0 \mid x \in A \,\}$. The *mod-$\ell_0$-image* of a collection of sets arises by replacing each set in the collection by its mod-ℓ_0-image.

Let us again consider our example, and let us take $\ell_0 = 2 \cdot 3 \cdot 5 = 30$ as a cut number. Any collection $\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_{2n}$ can be thought as cut into segments of height ℓ_0; we illustrate this on Fig. 2 for our above given collection. We note that the mod-ℓ_0-images of the 1st, 2nd, 3rd and 4th segment are basic hom-segments—where the mod-ℓ_0-images of the 2nd and 3rd segment are the same (the collection of empty sets in this case). Generally, we observe:

Observation 9. For a cut number ℓ_0, any collection whose union is a subset of $\{r\ell_0,r\ell_0+1,\ldots,r\ell_0+\ell_0-1\}$, $r \geq 1$, is an rth segment iff its mod-ℓ_0-image is a basic hom-segment.

For a $(2n+2)$-tuple of numbers and a cut number, we can easily construct the set of all (finitely many) basic segments—and index the set, e.g., by numbers. Similarly we can construct (and index) the set of all (finitely many) basic-hom segments. Using Observation 9, we note that the (1)-collections correspond precisely to the following (1)-words:

Definition 10. Given a $(2n+2)$-tuple of numbers and a cut number, a *(1)-word* is a sequence bu where b is (the index of) a basic segment and u is a finite sequence of (the indices of) basic hom-segments.

Observation 11. (Given a $(2n+2)$-tuple of numbers and a cut number), there is a bijection between the set of (1)-collections and the set of (1)-words.

Bouziane tries to show that a finite automaton can recognize precisely such (1)-words which correspond to the successful collections (i.e., those satisfying both (1) and (2)). In principle, he tries to prove the following claim; but we will lead this to a contradiction.

Claim 12 (*false !*). *Given a Petri net and M_0, M_f, we can (effectively) construct a finite automaton which accepts precisely the witness words, i.e., the (1)-words corresponding to witness collections.*

If the claim were true, the reachability question would be equivalent to the question if the automaton accepts at least one word, and the size of the automaton could serve for deriving some (elementary) complexity upper bound for the reachability problem; this is what Bouziane did.

But we show that the language of witness words can be nonregular; so there exists no such finite automaton then. It is sufficient to consider our simple example in Fig. 1. We note that the sets $\mathcal{P}_2$ and $\mathcal{P}_5$ in the witness collection corresponding to a witness sequence $(t_1)^q t_2 (t_3)^q$ are singletons and they contain two greatest markings. Let us write $\mathcal{P}_2 = \{m_2\}$, $\mathcal{P}_5 = \{m_5\}$; we note that $m_2 = 5 \cdot 3^q$, $m_5 = 2 \cdot 3^q$, and $m_2 = \frac{5}{2} m_5$—as required by (2). Hence the difference between the two greatest markings increases with increasing q. Let us assume that a respective finite automaton A exists (accepting precisely the witness words for our example net and markings $(1,0,0)$, $(0,0,1)$), and denote the number of its states by r. We can now take a witness sequence $(t_1)^q t_2 (t_3)^q$ for sufficiently large q so that the difference between the two greatest markings is greater than $\ell_0 \cdot (r+2)$; the respective witness collection thus corresponds to a witness word of the form $v0^r a$—where we suppose 0 to be the index of the empty segment (i.e., of the collection of empty sets) while a is the index of a nonempty (basic hom-) segment. A accepts the word $v0^r a$ according to our assumption. Due to the well-known pumping lemma, A would also accept a word $v0^s a$ where $s > r$; in the respective collection, $\mathcal{P}_2$ and $\mathcal{P}_5$ remain singletons but the condition $m_2 = \frac{5}{2} m_5$ is obviously violated—thus we have got a contradiction.

5. Conclusion

We have explained the essence of the approach from [6] and shown that Bouziane's construction of the automaton (which uses further coding and other technical results) contains a serious mistake. It is not necessary to find the exact point of that irreparable mistake; nevertheless the sentence "If we put m_1 to be ... and we put m_2 to be … then $m_1 = m_2$" in the proof of Lemma 3.4. in [6] seems to be that point.

In the whole, the paper [6] brings no real contribution to the study of reachability, and surely should not be cited in the literature as if it did.

The complexity of the reachability problem for Petri nets remains a challenging question.

6. Acknowledgments

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