Automaton and FO[$\mathbb{N}^r, <, \mathrm{mod}$]

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Problem

Strong logic	Weak Logic	Weaker Logic
$FO[+, V_b]$	$FO[+]$	$FO[<, \mathrm{mod}]$
$\|\|$		$FO[<]$
Automata		

Question

Decide if R definable in a strong logic is definable in a weak logic.

Automata reading r-tuple of integers

Example

Base $b = 3$, least digit first
$\overline{17}^{10} = \overline{2210}^{3}, \overline{18}^{10} = \overline{0020}^{3}$.

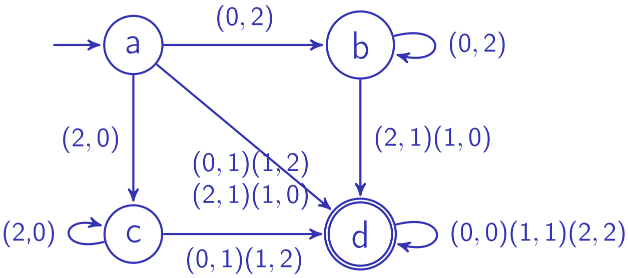
Example

Arity $r = 2$
$\binom{17}{18} = \binom{2}{0}\binom{2}{0}\binom{1}{2}\binom{0}{0}$

Definition

$A = (Q, [0, b-1]^r, \delta, q_0, F)$
A read an r-tuple of base b integer least digit first.

- $|A| = \{w \in [0, b-1]^{r*} \mid \delta(q_0, w) \in F\}$ its accepted set of tuples of words.
- $|\overline{A}| \subseteq \mathbb{N}^r$ its accepted set of integers.



$|\overline{A}| = \{(x, y) \mid x + 1 = y \vee y + 1 = x\}.$

Theorem (Büchi 60)

Let $R \subseteq \mathbb{N}^r$, R is accepted by an automaton in base b iff it is definable in $\mathrm{FO}[+, V_b]$.
$V_b(n) = b^k$ *when* $n = b^k c$ *and* b *does not divide* c.

First easy solutions, two 3-EXP algorithms

Theorem

Deciding if an automaton accepts a $\mathrm{FO}[<, \mathrm{mod}]$ *set is decidable in time 3-EXP.*

- Obtaining a polynomial-size $\mathrm{FO}[+]$-formula by Leroux 06
- Checking if the formula could be stated in $\mathrm{FO}[<, \mathrm{mod}]$ by Choffrut 08

- Stating ϕ in $\mathrm{FO}[+, R]$ "R is regular" by Milchior 13
- Rewriting ϕ as an automaton as in Muchnik 03 of size 3-EXP
- Checking if the formula is true on the automaton.

Our polynomial time algorithm

Theorem (Decidability)

Let $R \in \mathrm{FO}[+, V_b]$. There exists an algorithm in polynomial time that accepts iff R is in $\mathrm{FO}[<, \mathrm{mod}]$.

Its complexity is $O(2^r|Q|^2(r^2 b^r + 2^r \log(|Q|) + 8^r))$ when R is given as an automaton A.

Theorem (Computation of the formula)

There exists an algorithm that computes a $\mathrm{FO}[<, \mathrm{mod}, +C, = C]$-formula, if one exists, of polynomial size.

The size of the formula is $O(|Q|^2 r(b^r + |Q|^2 r \log(b) \log(|Q|)))$.

Open question:

- Is there a polynomial time algorithm for FO$[<]$?
- If the alphabet is $[0, b-1]^*$, is there a polynomial time algorithm ?

Remark

Everything still holds if $\mathbb{Z}$ replaces $\mathbb{N}$. The computation time is multiplied by 2^r.
The existence algorithm also works for FO[mod] *and when $C \subseteq \mathbb{N}$, $X \subseteq \{+C, =C, <\}$ it works for* $\Pi_0[X, \mathrm{mod}]$.

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Thank you