Chapter 7: Algebraic Systems and Pushdown Automata

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1 Introduction

The theory of algebraic power series in non-commuting variables, as we understand it today, was initiated in [2] and developed in its early stages by the French school. The main motivation was the interconnection with context-free grammars: the defining equations were made to correspond to context-free productions. Then the coefficient of a word w in the series equals the degree of ambiguity of w according to the grammar.

We concentrate in this chapter on the core aspects of algebraic series, pushdown automata, and their relation to formal languages. We choose to follow

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here a presentation of their theory based on the concept of *properness*. Alternatively, one may present the theory in terms of complete semirings, as done for example in [11] and [3]. The main difference between the two presentations is in the handling of infinite sums, especially in connection with products and stars of infinite matrices. While such sums are well defined in the case of complete semirings, special care must be taken for arbitrary semirings. In the case of proper algebraic systems and proper pushdown automata, the infinite matrices have locally-finite stars. Consequently, if one considers the ambiguity of context-free grammars, one option is to assume that the ambiguity is in the semiring $\mathbb{N}^\infty$ and follow the results based on complete semirings, while the other option is to consider proper grammars and take the ambiguity in the semiring $\mathbb{N}$.

We introduce in Sect. 2 some auxiliary notions and results needed throughout the chapter, in particular the notions of discrete convergence in semirings and C-cycle free infinite matrices. In Sect. 3, we introduce the algebraic power series in terms of algebraic systems of equations. We focus on interconnections with context-free grammars and on normal forms. We then conclude the section with a presentation of the theorems of Shamir and Chomsky–Schützenberger. We discuss in Sect. 4 the algebraic and the regulated rational transductions, as well as some representation results related to them. Section 5 is dedicated to pushdown automata and focuses on the interconnections with classical (non-weighted) pushdown automata and on the interconnections with algebraic systems. We then conclude the chapter with a brief discussion of some of the other topics related to algebraic systems and pushdown automata.

2 Auxiliary Notions and Results

We introduce in this section the notion of discrete convergence in arbitrary semirings and in connection to it, a notion of convergence for column finite (infinite) matrices. This allows us then to define the notion of C-cycle free (infinite) matrices, needed in this chapter in connection with proper pushdown transition matrices. We give here only the elements that are essential for the purpose of this chapter, referring to [12] for more details, including other notions of convergence.

Definition 2.1. *A* sequence *in the semiring S is a mapping $\alpha : \mathbb{N} \to S$. We denote $\alpha_n = \alpha(n)$, for all $n \in \mathbb{N}$ and $\alpha = (\alpha_n)_{n\in\mathbb{N}}$. We denote the set of all sequences over S by $S^{\mathbb{N}}$. We say that α is* convergent *in S if there exist $a \in S$ and $n_0 \in \mathbb{N}$ such that $\alpha_n = a$, for all $n \geq n_0$. In this case, a is called the* limit *of α, denoted as $\mathsf{lim}_{n\to\infty} \alpha_n = a$.*

The notion of convergence defined above is often called the *discrete convergence* in the semiring S.

We note that the notion of cycle-free formal power series defined in Chap. 1 is equivalent with saying that, for $r \in S\langle\langle \Sigma^* \rangle\rangle$, the sequence $(r^n, \epsilon)_{n\in\mathbb{N}}$ is convergent in S and its limit is 0. It has been observed already in Chap. 1 that the star of any cycle-free series exists. We give here a short proof of this result that will be mirrored by a similar result for matrices.

Lemma 2.2. *For any cycle-free formal power series $r \in S\langle\langle \Sigma^* \rangle\rangle$ and any $w \in \Sigma^*$, there exists $n_w \in \mathbb{N}$ such that $(r^n, w) = 0$, for all $n \geq n_w$. Consequently, r^* exists and is locally finite.*

Proof. We prove the claim by induction on $|w|$. For $|w| = 0$, it follows from the definition of the limit in the discrete convergence that there exists $n_0 \in \mathbb{N}$ such that $(r^n, \epsilon) = 0$, for all $n \geq n_0$. Consider now an arbitrary $w \in \Sigma^+$ and assume that the claim holds for all words shorter than w. Then for any $n \geq n_0$,

$$\begin{aligned}
(r^n, w) &= \sum_{w=uv} (r^{n-n_0}, u)(r^{n_0}, v) \\
&= (r^{n-n_0}, w)(r^{n_0}, \epsilon) + \sum_{\substack{w=uv \\ |u|<|w|}} (r^{n-n_0}, u)(r^{n_0}, v) \\
&= \sum_{\substack{w=uv \\ |u|<|w|}} (r^{n-n_0}, u)(r^{n_0}, v).
\end{aligned}$$

If we choose $n_w \geq n_0 + n_u$, for all $|u| < |w|$, then the claim follows for $n \geq n_w$ based on the induction hypothesis. Indeed, in this case, $(r^{n-n_0}, u) = 0$. In particular, one may choose $n_w = n_0(|w| + 1)$, for all $w \in \Sigma^*$.

The second part of the lemma follows by observing that

$$(r^*, w) = \sum_{n\geq 0} (r^n, w) = \sum_{n=0}^{n_w-1} (r^n, w). \quad \square$$

Definition 2.3. *Let $(M_n)_{n\in\mathbb{N}} \in (S^{I\times I})^{\mathbb{N}}$ be a sequence of column finite (infinite) matrices. We say that $(M_n)_{n\geq 0}$ is convergent if the following two conditions are satisfied:*

(i) For all $j \in I$, there exists a finite set $I(j) \subseteq I$ such that $(M_n)_{i,j} = 0$, for all $n \in \mathbb{N}$ and all $i \in I \setminus I(j)$.
(ii) For all $i, j \in I$, the sequence $((M_n)_{i,j})_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ is convergent in S.

For $m_{i,j} = \mathsf{lim}_{n\to\infty}((M_n)_{i,j})$, we say that the matrix $M = (m_{i,j})_{i,j\in I}$ is the limit *of the sequence $(M_n)_{n\in\mathbb{N}}$, denoted $M = \mathsf{lim}_{n\to\infty} M_n$.*

Note that condition (i) in Definition 2.3 is not equivalent with the matrices M_n being column finite for all $n \in \mathbb{N}$. Note also that a different notion of convergence can be defined for row finite matrices; see [12].

Definition 2.4. *Let $M \in (S\langle\langle \Sigma^* \rangle\rangle)^{I\times I}$ be a column finite matrix. We say that M is* C-cycle free *if the sequence $(M^n, \epsilon)_{n\in\mathbb{N}} \in S^{I\times I}$ is convergent and its limit is the zero matrix.*

The notion of C-cycle free matrix is very similar with the notion of cycle-free power series. We indicate explicitly the letter C to stress that our notion is applied to column finite matrices only and also, to preserve the terminology in [12]. Note that a related notion of R-cycle free matrices may also be defined for row finite matrices. Note also that, based on the definition of discrete convergence and that of convergent matrices, a matrix $M \in (S\langle\langle \Sigma^* \rangle\rangle)^{I\times I}$ is C-cycle free if and only if the following two conditions are satisfied:

(i) For all $j \in I$, there exists a finite set $I(j) \subseteq I$ such that $(M^n, \epsilon)_{i,j} = 0$, for all $i \notin I(j)$ and all $n \geq 0$.
(ii) For all $j \in I$, there exists $n(j) \in \mathbb{N}$ such that $(M^n, \epsilon)_{i,j} = 0$, for all $n \geq n(j)$ and all $i \in I$.

The following result will be needed in connection with proper pushdown transition matrices and their stars.

Lemma 2.5. *The star of any C-cycle free matrix is locally finite.*

Proof. Let $M \in (S\langle\langle \Sigma^* \rangle\rangle)^{I\times I}$ be a C-cycle free matrix. We claim that for any $j \in I$ and any $w \in \Sigma^*$, there exists a non negative integer $n(j, w)$ such that $((M^n)_{i,j}, w) = 0$, for all $i \in I$ and all $n \geq n(j, w)$. Then we obtain that

$$\sum_{n\geq 0}\big((M^n)_{i,j}, w\big) = \sum_{n=0}^{n(j,w)}\big((M^n)_{i,j}, w\big),$$

showing that M^* exists and is locally finite.

We prove the claim by induction on $|w|$. For $|w| = 0$, the claim follows from the definition of C-cycle free matrices. Indeed, since $\mathsf{lim}_{n\to\infty}(M^n, \epsilon) = 0$, there exists $n(j) \in \mathbb{N}$ for all $j \in I$ such that $(M^n, \epsilon)_{i,j} = 0$, for all $i \in I$ and all $n \geq n(j)$.

Consider now $w \in \Sigma^*$ with $|w| \geq 1$ and assume inductively that the claim holds for all words shorter than w. Then for any $i, j \in I$, $n \geq n(j)$, $w \in \Sigma^*$, we have that

$$\begin{aligned}
\big((M^n)_{i,j}, w\big) &= \sum_{\substack{w=uv\\ k\in I}}\big((M^{n-n(j)})_{i,k}, u\big)\big((M^{n(j)})_{k,j}, v\big)\\
&= \sum_{\substack{w=uv\\ |u|<|w|\\ k\in I}}\big((M^{n-n(j)})_{i,k}, u\big)\big((M^{n(j)})_{k,j}, v\big)\\
&\quad + \sum_{k\in I}\big((M^{n-n(j)})_{i,k}, w\big)\big((M^{n(j)})_{k,j}, \epsilon\big).
\end{aligned}$$

Note now that $((M^{n(j)})_{k,j}, \epsilon) = 0$. Also, if $n - n(j) \geq n(k,u)$, then it follows by the induction hypothesis that $((M^{n-n(j)})_{i,k}, u) = 0$. Since there are only finitely many $k \in I$ with $(M^{n(j)})_{k,j} \neq 0$, to obtain that $((M^n)_{i,j}, w) = 0$, it is enough to define $n(j,w) \in \mathbb{N}$ as follows:

$$n(j,w) = \begin{cases} n(j), & \text{if } w = \epsilon, \\ \mathsf{max}\{n(j) + n(k,u) \mid (M^{n(j)})_{k,j} \neq 0, |u| < |w|\}, & \text{otherwise.} \end{cases} \qquad \square$$

3 Algebraic Power Series

This section introduces algebraic power series in terms of algebraic systems of equations and discusses various reduction, normal form, and characterization results. Special emphasis will be in the interconnection with context-free grammars and languages. The defining equations are algebraic, that is, polynomial equations. Moreover, they are of a somewhat special form. The form makes the interconnection with context-free grammars very direct.

The first comprehensive treatment about algebraic power series in noncommuting variables is in [15] where also references to earlier work, mainly by M.P. Schützenberger, are given.

3.1 Definition and Basic Reductions

Consider an alphabet $\Sigma = \{x_1, \ldots, x_k\}$, $k \geq 1$, and a commutative semiring S. Let $Y = \{y_1, \ldots, y_n\}$, $n \geq 1$, be another alphabet, the alphabet of *variables*.

Definition 3.1. *An S-*algebraic system *is a set of equations of the form*

$$y_i = p_i, \quad i = 1, \ldots, n,$$

where $p_i \in S\langle(\Sigma \cup Y)^*\rangle$. The system is termed proper *if, for all i and j, $(p_i, \epsilon) = 0$ and $(p_i, y_j) = 0$.*

A *solution* to the algebraic system consists of n power series $r_1, \ldots, r_n$ in $S\langle\langle\Sigma^*\rangle\rangle$ "satisfying" the system in the sense that if each variable y_i is replaced by the series r_i, then n valid equations result. This can be formalized as follows. Consider a column vector

$$R = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \in (S\langle\langle(\Sigma \cup Y)^*\rangle\rangle)^{n\times 1}$$

consisting of n power series, and define the morphism

$$h_R : (\Sigma \cup Y)^* \to S\langle\langle(\Sigma \cup Y)^*\rangle\rangle$$

by $h_R(y_i) = r_i$, $1 \le i \le n$, and $h_R(x) = x$, for $x \in \Sigma$. Defining

$$h_R(p) = \sum_{w \in (\Sigma \cup Y)^*)} (p, w) h_R(w),$$

where $p \in S\langle(\Sigma \cup Y)^*\rangle$, we now term R a *solution* of the original algebraic system if $r_i = h_R(p_i)$, for all i, $1 \le i \le n$.

So far, we have not used the assumption of the semiring S being commutative. Indeed, some parts of the theory such as the interconnection with pushdown automata remain valid without this assumption. We make this assumption because it is needed in important parts of the theory and, moreover, our main interest is in the semirings $\mathbb{Z}$ and $\mathbb{N}$ of integers and non-negative integers, as well as in the Boolean semiring $\mathbb{B}$.

An S-algebraic system does not always possess a solution because, for instance, an equation may be contradictory, such as the equation $y_1 = y_1 + x_1$ in the semiring $\mathbb{N}$. However, every proper S-algebraic system possesses a solution.

Theorem 3.2. *Every proper S-algebraic system possesses exactly one solution where each component is quasi-regular. In addition, it may have other solutions.*

Proof. The theorem is established by considering an *"approximation sequence"* R^i, $i = 0, 1, \ldots$, of n-tuples (or column vectors) of power series. By definition, R^0 consists of 0's, and R^{i+1} is obtained by applying h_{R^i} to each component of R^i, for $i \ge 0$. For $j \ge 0$, we consider also the truncation operator T_j defined for power series $r \in S\langle\langle\Sigma^*\rangle\rangle$ by

$$T_j(r) = \sum_{|w| \le j} (r, w) w.$$

The operator T_j is applied to n-tuples componentwise. An obvious induction on j shows that $T_j(R^j) = T_j(R^{j+t})$, for all j and t. This shows that the approximation sequence R^i converges (with respect to discrete convergence) to a specific

$$R = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \in (S\langle\langle\Sigma^*\rangle\rangle)^{n \times 1},$$

where each r_i is quasi-regular. Denoting

$$P = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix},$$

we see again inductively that $T_j(R) = T_j(h_R(P))$ holds for all $j \ge 0$, and consequently, R is a solution. Finally, if R' is another solution with quasi-regular components, we have $T_0(R) = T_0(R')$. Assuming inductively that $T_j(R) = T_j(R')$, we deduce

$$
\begin{aligned}
T_{j+1}(R) &= T_{j+1}\bigl(h_R(P)\bigr) = T_{j+1}\bigl(h_{T_j(R)}(P)\bigr) = T_{j+1}\bigl(h_{T_j(R')}(P)\bigr) \\
&= T_{j+1}\bigl(h_{R'}(P)\bigr) = T_{j+1}(R'),
\end{aligned}
$$

which completes the induction and shows that $R = R'$. □

A proper S-algebraic system may have other solutions where the components are not quasi-regular. For instance, the $\mathbb{N}$-algebraic system consisting of the single equation $y_1 = y_1 y_1$ has both of the power series $r_1 = 0$ and $r_2 = \epsilon$ as solutions. The solution constructed above as *the limit of the approximation sequence* is in the sequel referred to as the *strong solution*.

We are now ready for the basic definition.

Definition 3.3. *A formal power series $r \in S\langle\langle \Sigma^* \rangle\rangle$ is* S-algebraic, *in symbols $r \in S^{\mathrm{alg}}\langle\langle \Sigma^* \rangle\rangle$, if $r = (r,\epsilon)\epsilon + r'$, where r' is some component of the strong solution of a proper S-algebraic system.*

We have stated and established Theorem 3.2 for *proper* S-algebraic systems. Then the approximation sequence converges with respect to the discrete convergence, and the resulting solution was called strong. However, a more general result is valid for continuous semirings S. By the *fixpoint theorem*, the *least* solution of an S-algebraic system exists and is obtained by computing the least upper bound of the approximation sequence associated to it. The least solution is not necessarily strong. More details of this approach can be found in Chap. 2 and [11]. More specific results can be obtained if we are dealing with the Boolean semiring. The proofs of the following result can be found in [12].

Theorem 3.4. *Every $\mathbb{B}$-algebraic system possesses a strong solution. If r is a component of the strong solution of a $\mathbb{B}$-algebraic system, then the quasi-regular part of r is a component of the strong solution of a proper $\mathbb{B}$-algebraic system.*

We use in the next example and in several other places throughout the chapter the so-called *Dyck language* and *Dyck mapping*. The Dyck language L_D over the alphabet $\{x_1, x_2\}$ is the language of all correctly nested parentheses when x_1 and x_2 are viewed as the left and right parenthesis, respectively. More generally, for $X = X_1 \cup \overline{X}_1$, $\overline{X}_1 = \{\overline{x} \mid x \in X_1\}$, the Dyck language $L_D(X)$ consists of all words w such that $D(w) = \epsilon$, where D is the Dyck mapping,

$$D : X^* \to X^*$$

defined as follows. Intuitively, we view X_1 as a set of left parenthesis and $\overline{X}_1$ as the set of corresponding right parenthesis. Then D removes from a word over X all pairs of adjacent matching parenthesis, until no further removals are possible. Thus, $D(wx) = D(w)x$ for $x \in X_1$, and

$$
D(w\overline{x}) = \begin{cases} w_1 & \text{for } D(w) = w_1 x, \\ D(w)\overline{x} & \text{for } D(w) \notin X^* x, \end{cases}
$$

for $x \in X_1$.

Example 3.5. We consider supports of some $\mathbb{N}$-algebraic series. The considerations are also preparatory for the next subsection. For the proper $\mathbb{N}$-algebraic system

$$y = yy + x_1 y x_2 + x_1 x_2,$$

the approximation sequence R^i, $i = 0, 1, \ldots$, consists of singletons of power series because there is only one variable y. We obtain

$$\begin{aligned}
&R^0 = 0, \qquad R^1 = x_1x_2, \qquad R^2 = (x_1x_2)^2 + x_1^2x_2^2 + x_1x_2,\\
&R^3 = (x_1x_2)^4 + (x_1x_2)^2x_1^2x_2^2 + x_1^2x_2^2(x_1x_2)^2 + (x_1^2x_2^2)^2 + 2(x_1x_2)^3 + x_1^2x_2^2x_1x_2\\
&\qquad + x_1x_2x_1^2x_2^2 + x_1^2x_2x_1x_2^2 + x_1^3x_2^3 + x_1^2x_2^2 + (x_1x_2)^2 + x_1x_2.
\end{aligned}$$

The support of the resulting power series r equals the Dyck language L_D over the alphabet $\{x_1, x_2\}$ (without the empty word). However, r is not the characteristic series r_D of L_D because, as seen already from R^3, some coefficients in r are greater than 1. The characteristic series r_D is the first component of the solution of the $\mathbb{N}$-algebraic system

$$y = x_1yx_2 + x_1yx_2y + x_1x_2y + x_1x_2.$$

Also, the $\mathbb{N}$-algebraic system

$$\begin{aligned}
y_1 &= y_2 + y_1y_2,\\
y_2 &= x_1y_1x_2 + x_1x_2
\end{aligned}$$

(which is not proper) can be used for the same purpose. Then the first components of the vectors R^i, $0 \le i \le 4$, in the approximation sequence are

$$\begin{aligned}
&0, \quad 0, \quad x_1x_2, \quad (x_1x_2)^2 + x_1x_2,\\
&(x_1x_2)^2x_1^2x_2^2 + (x_1x_2)^3 + x_1x_2x_1^2x_2^2 + (x_1x_2)^2 + x_1^2x_2^2 + x_1x_2.
\end{aligned}$$

Also, now the approximation sequence converges, and r_D is the first component of the (strong) solution of the $\mathbb{N}$-algebraic system. Finally, also the $\mathbb{N}$-algebraic system

$$y = yx_1yx_2y + \epsilon$$

(which is also not proper) possesses a (strong) solution whose support equals L_D. The same result is obtained from the asymmetric systems, where either the first, or the last y on the right-hand side has been erased.

The next theorem deals with S-algebraic systems, such as the last one in our example, where the polynomials are not necessarily quasi-regular. The proof, similar to that of Theorem 3.2, is given in [12].

Theorem 3.6. *Assume that in an S-algebraic system $y_i = p_i$, $i = 1, \ldots, n$, the support of each polynomial p_i is contained in the language*

$$(\Sigma \cup Y)^* \Sigma (\Sigma \cup Y)^* \cup \{\epsilon\}.$$

Then the system possesses a unique solution, which moreover is strong.

3.2 Interconnections with Context-Free Grammars

Every context-free grammar and semiring S give rise to an S-algebraic system. Conversely, every S-algebraic system gives rise to a context-free grammar. Explicitly, this correspondence can be described as follows.

Given a context-free grammar G with Σ and $Y = \{y_1, \ldots, y_n\}$ as the terminal and non-terminal alphabets, y_1 as the initial symbol and R as the set of production rules, the *corresponding* S-algebraic system consists of the equations $y_i = p_i$, $i = 1, \ldots, n$, where $(p_i, w) = 1$ if $y_i \to w$ is a production in R, and $(p_i, w) = 0$, otherwise. (Since R is finite, each p_i is a polynomial.) Conversely, given an S-algebraic system $y_i = p_i$, $i = 1, \ldots, n$, the *corresponding* context-free grammar $\mathcal{G} = (\Sigma, Y, y_1, R)$ is defined by the condition: $y_i \to w$ is in R if and only if $(p_i, w) \neq 0$.

If we begin with an S-algebraic system, form the corresponding context-free grammar and then again the corresponding S-algebraic system, then the latter system does not necessarily coincide with the original one.

The most natural semiring for considerations dealing with formal languages is $\mathbb{N}$. A word generated by a context-free grammar appears in the support of the corresponding $\mathbb{N}$-algebraic power series. Moreover, its degree of ambiguity according to the grammar equals its coefficient in the series. If we want to deal with arbitrary context-free grammars, we should consider the semiring $\mathbb{N}^\infty$. We prefer dealing with $\mathbb{N}$ because every context-free grammar can be transformed to an equivalent one where no word has infinitely many leftmost derivations.

Definition 3.7. *A language $L \subseteq \Sigma^*$ is S-algebraic if it equals the support of a power series in $S^{\mathrm{alg}}\langle\langle \Sigma^* \rangle\rangle$.*

Theorem 3.8. *A language is context-free if and only if it is $\mathbb{N}$-algebraic.*

Proof. It suffices to establish the result for ϵ-free languages and quasi-regular $\mathbb{N}$-algebraic series. Let $\mathcal{G}_r$ be the grammar corresponding to a given proper $\mathbb{N}$-algebraic system $y_j = p_j$, $1 \le j \le n$, where r is the first component of its strong solution. To show that $\mathrm{supp}(r) = L(\mathcal{G}_r)$, we first establish the inclusion $L(\mathcal{G}_r) \subseteq \mathrm{supp}(r)$ inductively. We consider the approximation sequence R^i as in the proof of Theorem 3.2, and denote by r_j^i, $1 \le j \le n$, the jth component of R^i. It is now straightforward to establish inductively on t the following claim. Whenever a word $w \in \Sigma^*$ possesses a derivation of length at most t from y_j, $1 \le j \le n$, then $w \in \mathrm{supp}(r_j^t)$. Indeed, the claim holds for $t = 1$ by the definition of the corresponding grammar. The inductive step is proven by dividing a $(t+1)$-step derivation into a 1-step and t-step derivation, and applying the induction hypothesis to the latter.

The inclusion $L(\mathcal{G}_r) \subseteq \mathrm{supp}(r)$ follows. To prove the reverse inclusion, it suffices to establish inductively on t the following claim. Whenever $w \in \mathrm{supp}(r_j^t)$, $1 \le j \le n$, then there is a derivation of w from y_j according to $\mathcal{G}_r$. (Observe that we do not specify the length of the derivation.) For $t = 0$, the claim holds vacuously. Assume the claim holds for a fixed value t and consider a word $w \in \mathrm{supp}(r_j^{t+1})$. Consequently, for some

$$w' = u_1 y_{j_1} u_2 \ldots u_m y_{j_m} u_{m+1} \in \mathrm{supp}(p_j), \quad u_k \in \Sigma^*,$$

we have

$$w = u_1 w_{j_1} u_2 \ldots u_m w_{j_m} u_{m+1}, \quad w_{j_k} \in \mathrm{supp}\big(r_{j_k}^t\big),\ 1 \le k \le m.$$

We now use the induction hypothesis and the fact that, according to the definition of $\mathcal{G}_r$, y_j directly derives w', and conclude that the claim holds for the value $t+1$. Thus, we have shown that $L(\mathcal{G}_r) = \mathrm{supp}(r)$.

The argument above shows that every $\mathbb{N}$-algebraic language is context-free. Given an ϵ-free context-free grammar, we first eliminate from it all chain productions, where a non-terminal goes to a non-terminal. Then the corresponding $\mathbb{N}$-algebraic system will be proper, and we can show the equality of the two languages exactly as above. □

The following generalization of Theorem 3.8 is easily obtained, [18]. Recall that a semiring S is *positive* if the mapping h of S into $\mathbb{B}$ defined by

$$h(0) = 0, \qquad h(s) = 1 \quad \text{for } s \neq 0,$$

is a morphism.

Theorem 3.9. *All of the following five statements are equivalent for a language L:*

(i) L is a context-free language.
(ii) L is $\mathbb{N}$-algebraic.
(iii) L is $\mathbb{B}$-algebraic.
(iv) L is S-algebraic for all positive semirings S.
(v) L is S-algebraic for some positive semiring S.

There are $\mathbb{Z}$-algebraic languages that are not context-free. An obvious way to obtain such languages is to consider the difference between the characteristic series of Σ^* and L, where $L \subseteq \Sigma^*$ is an unambiguous context-free language whose complement is not context-free.

There are several open language-theoretic problems in this area. For instance, no characterization is known for $\mathbb{Z}$-algebraic languages, in terms of some of the well-known language hierarchies. $\mathbb{N}$-algebraic languages over a one-letter alphabet coincide with regular languages, but it is not known whether this holds for $\mathbb{Z}$-algebraic languages as well.

Apart from the language generated by a context-free grammar $\mathcal{G}$, the corresponding $\mathbb{N}$-algebraic power series $r_{\mathcal{G}}$ tells the degree of ambiguity of each word in the language. In the following theorem, we assume that $\mathcal{G}$ is a context-free grammar without ϵ-rules and chain rules. Then the corresponding $\mathbb{N}$-algebraic system is proper.

Theorem 3.10. *The coefficient of each word w in the $\mathbb{N}$-algebraic power series $r_{\mathcal{G}}$ equals the degree of ambiguity of w according to $\mathcal{G}$. Consequently, $\mathcal{G}$ is unambiguous (resp. of bounded ambiguity) if and only if the coefficients in $r_{\mathcal{G}}$ are at most* 1 *(resp. bounded).*

Proof (outline). As before, we assume that the non-terminals of the given grammar are $y_1, \ldots, y_n$. We let each of them be the initial letter, obtaining the grammars $\mathcal{G}_i$, $1 \leq i \leq n$. We consider also the proper $\mathbb{N}$-algebraic system $g_i = p_i$, $1 \leq i \leq n$, corresponding to the grammar $\mathcal{G} = \mathcal{G}_1$. If the n-tuple $(r_1, \ldots, r_n)$ is the solution of the $\mathbb{N}$-algebraic system, it follows by Theorem 3.8 that $L(\mathcal{G}_i) = \mathrm{supp}(r_i)$, $1 \leq i \leq n$. For a word w, we denote by $\mathrm{amb}(\mathcal{G}_i, w)$ the ambiguity of w according to the grammar $\mathcal{G}_i$. It can now be shown by induction on $|w|$ that $\mathrm{amb}(\mathcal{G}_i, w) = (r_i, w)$, $1 \leq i \leq n$, whence the theorem follows. Indeed, it suffices to consider the approximation sequence for the solution, and separate the first step in a derivation according to $\mathcal{G}_i$. The details are given in [18]. □

The following generalization is again immediate. Observe that now the conditions corresponding to points (iii) and (v) in Theorem 3.9 are not applicable.

Theorem 3.11. *The following three statements are equivalent for a language L:*

(i) L is an unambiguous context-free language.
(ii) The characteristic series of L is $\mathbb{N}$-algebraic.
(iii) The characteristic series of L is S-algebraic for all positive semirings S.

We mention finally that Theorem 3.10 can be stated also without restrictions on the productions of $\mathcal{G}$. Then we have to deal with $\mathbb{N}^\infty$-algebraic series because the ambiguity of w may be ∞. The first component of the least solution of the corresponding $\mathbb{N}^\infty$-algebraic system indicates the ambiguity of each word in the language, [11]. However, the approximation sequence does not necessarily converge with respect to the discrete convergence.

3.3 Normal Forms

We already pointed out that the algebraic systems under consideration are of a special form, resembling the productions in a context-free grammar. (In fact, very little is known about more general algebraic systems.) We now take a step further by considering several "normal forms": we may assume that the polynomials p_i appearing on the right-hand sides of the equations satisfy certain additional conditions, without losing any power series as solutions. Such normal forms are customary in language theory, and indeed the ones considered below resemble those introduced for context-free grammars.

Definition 3.12. *An S-algebraic system*

$$y_i = p_i, \quad 1 \le i \le n, \qquad Y = \{y_1, \ldots, y_n\},$$

over the alphabet Σ is in the Chomsky *(resp.* operator*,* Greibach*) normal form if, for each i, $1 \le i \le n$,*

$$\mathrm{supp}(p_i) \subseteq \Sigma \cup Y^2$$

(resp. $\mathrm{supp}(p_i) \subseteq \{\epsilon\} \cup Y\Sigma \cup Y\Sigma Y$, $\mathrm{supp}(p_i) \subseteq \Sigma \cup \Sigma Y \cup \Sigma Y^2$*).*

Theorem 3.13. *The components of the strong solution of a proper S-algebraic system appear also as components of the strong solution of such a system in the Chomsky normal form. Moreover, the latter system can be effectively constructed from the former.*

Proof (outline). We first transform the given system into one, where the supports of the right-hand sides are contained in $\Sigma \cup YY^+$, by replacing letters $x \in \Sigma$ with new variables y and introducing the equations $y = x$. A similar introduction of new variables is then applied to catenations larger than 2. For instance, the equation $y_1 = y_1y_2y_3$ becomes

$$y_1 = y_1y_4, \qquad y_4 = y_2y_3. \quad \square$$

In the constructions in the next theorem, we need the operators w^{-1}, $w \in \Sigma^*$, customary in language theory. By definition, for $u \in \Sigma^*$, $w^{-1}u = v$ if $u = wv$, and $w^{-1}u = 0$, otherwise. The operator w^{-1} is defined similarly from the right, and extended additively to concern power series. The application of this operator explains the presence of ϵ in the supports defining the operator normal form.

Theorem 3.14. *The first component of the strong solution of a proper S-algebraic system appears also as the first component of the strong solution of such a system in the operator normal form, effectively obtainable from the given system.*

Proof (outline). By the preceding theorem, we may assume that the given S-algebraic system is in the Chomsky normal form. We separate in the equations the Σ-parts and Y^2-parts, obtaining the system

$$y_i = \sum_{x \in \Sigma} (p_i, x)x + \sum_{k,m=1}^{n} (p_i, y_ky_m)y_ky_m, \quad 1 \le i \le n.$$

We now construct a new S-algebraic system, with the set of variables

$$Y_1 = y_0 \cup \{y_{i,x} \mid 1 \le i \le n,\ x \in \Sigma\}.$$

The equations in the new system are

$$y_0 = \sum_{x \in \Sigma} y_{1,x} x,$$

$$y_{i,x} = (p_i, x)\epsilon + \sum_{x' \in \Sigma} \sum_{k,m=1}^{n} (p_i, y_k y_m) y_{k,x'} x' y_{m,x} = q_{i,x},$$

where x ranges over Σ and $1 \leq i \leq n$. Clearly, the new system is in the operator normal form. We now claim that the $y_{i,x}$-component of the strong solution of the new system is obtained by applying the operator x^{-1} from the right to the y_i-component of the strong solution of the original system. The theorem follows from this claim, by the equation for y_0.

Let R^j (resp. Q^j) be the approximation sequence associated to the system $y_i = p_i$ (resp. $y_{i,x} = q_{i,x}$), with the components r_i^j (resp. $r_{i,x}^j$). It can be shown, by an induction on j that $r_{i,x}^j = r_i^j x^{-1}$, whence the claim follows. This holds for $j = 0$, by the definition of the new system. The details of the inductive step are presented in [12]. Thereby the equations

$$\sum_{x \in \Sigma} (rx^{-1})x = r \quad \text{and} \quad r'(rx^{-1}) = (r'r)x^{-1}$$

are needed. The equations are valid only for quasi-regular power series r and r'. □

Theorem 3.15. *The first component of the strong solution of a proper S-algebraic system appears also as the first component of the strong solution of such a system in the Greibach normal form, effectively obtainable from the given system.*

Proof (outline). The proof consists of eliminating the left recursion from the equations. The elimination can be based either on the Chomsky normal form, [18], or on the operator normal form, [12]. Suppose we are dealing with the Chomsky normal form. If the given system is $y_i = p_i$, $1 \leq i \leq n$, we separate on the right-hand sides the Σ-parts and the Y^2-parts as in the preceding proof. The result can be written in the matrix form

$$Y = YM + P$$

where the ith entry of the row vector P equals $\sum_{x \in \Sigma}(p_i, x)x$ and M is an $n \times n$ matrix whose (j, k)th entry equals the polynomial

$$\sum_{i=1}^{n} (p_k, y_j y_i) y_i,$$

for $1 \leq j, k \leq n$. We now introduce a new variable y_{jk} for each entry of the matrix M. The resulting equations have the required form. For details, we refer to [18]. The theorem now follows by observing that the matrix M^+ exists. □

The arguments applied above show also many well-known facts about context-free languages. For instance, the argument in Theorem 3.14 shows that the family of context-free languages is closed under left and right derivatives. The following theorem summarizes some of the results obtainable in this fashion.

Theorem 3.16. *The following five statements are equivalent for a language L:*

(i) *L is context-free.*
(ii) *$L - \epsilon$ is generated by a context-free grammar without chain rules and ϵ-rules.*
(iii) *$L - \epsilon$ is generated by a context-free grammar where the right-hand side of every production is in $Y^2 \cup \Sigma$.*
(iv) *L is generated by a context-free grammar where the right-hand side of every production is in $Y \Sigma Y \cup Y \Sigma \cup \epsilon$.*
(v) *$L - \epsilon$ is generated by a context-free grammar where the right-hand side of every production is in $\Sigma Y^2 \cup \Sigma Y \cup \Sigma$.*

We now take a step further by considering "meta" normal forms, that is, classes of normal forms with parameters such that each of the (infinitely many) values of the parameter gives rise to a normal form. The approach has turned out to be very useful in language theory: in some cases a characterization of all possible normal forms has been obtained. The results below are stated in a form producing only quasi-regular series. The transition to general series is straightforward. If r is the first component of the strong solution of the system $y_i = p_i$, $1 \leq i \leq n$, then for any $s \in S$. $s\epsilon + r$ is the first component of the strong solution of the system

$$y_0 = s\epsilon + p_1, \qquad y_i = p_i, \quad 1 \leq i \leq n.$$

Theorem 3.17. *Assume that m_1, m_2, m_3 are non-negative integers. Then every power series $r \in S^{\mathrm{alg}}\langle\langle \Sigma^* \rangle\rangle$ can be effectively obtained from the strong solution of a proper S-algebraic system where the supports of the right-hand sides of the equations are included in the set*

$$\Sigma^+ \cup \Sigma^{m_1} Y \Sigma^{m_2} Y \Sigma^{m_3},$$

with Y being the alphabet of variables.

The proof of Theorem 3.17 is given in [12], the original ideas being due to [1, 13, 17]. Observe that the Chomsky, operator, and Greibach normal forms are essentially obtained from the triples $(0,0,0)$, $(0,1,0)$, and $(1,0,0)$. Theorem 3.17 can be generalized. Instead of the triple (m_1, m_2, m_3), one can consider an arbitrary t-tuple $(m_1, \ldots, m_t)$, $t \geq 3$, and show that the supports can be included in the set

$$\Sigma^+ \cup \Sigma^{m_1} Y \ldots Y \Sigma^{m_t}.$$

A further strengthening of Theorem 3.17 consists in restricting the powers of Σ in Σ^+ to lengths belonging to the length set of the support of the original series. If only such powers are used, the system is said to be *terminally balanced*. For instance, the system

$$y_1 = y_2 y_1 + x, \qquad y_2 = x^2$$

is in $(0,0,0)$-form, but not terminally balanced. The series

$$r_1 = \sum_{j=0}^{\infty} x^{2j+1}$$

is, however, obtained also from the following terminally balanced system in $(0,0,0)$-form:

$$y_1 = y_2 y_1 + x, \qquad y_2 = y_3 y_3, \qquad y_3 = x.$$

In general, such a simple construction does not work. The general construction of terminal balancing, due to [14], is presented in [12]. The construction works only for the Boolean semiring $\mathbb{B}$. It is an open problem to what extent the result can be extended to other semirings.

In conclusion for this subsection, some remarks about *closure properties* are in order. The general closure theory, the theory of abstract families of power series, [12, 7–9], is beyond the scope of this contribution. Some basic results are rather easily obtainable [18]. The family of S-algebraic power series generated by proper systems is closed under sum, product, and quasi-inverse. It is closed under semiring morphisms and non-erasing monoid morphisms, but not under arbitrary monoid morphisms. The Hadamard (or pointwise) product of an S-algebraic and S-rational series is S-algebraic [19]. This result corresponds to the well-known result about the intersection of context-free and regular languages. Every $\mathbb{Z}$-algebraic power series can be represented as the difference of two $\mathbb{N}$-algebraic series.

3.4 Theorems of Shamir and Chomsky–Schützenberger

We now discuss two famous theorems concerning algebraic power series. Both deal with the computation of the coefficients and consequently, also with degrees of ambiguity in derivations according to a context-free grammar. The methods for computing the coefficients, obtained by these theorems, are more direct than the iterative method of the approximation sequence, discussed above. The theorems of Shamir and Chomsky–Schützenberger were originally presented in [20] and [2], respectively. Our discussion uses also ideas from [15] and [18].

We consider first Shamir's theorem. An important auxiliary concept is that of an *involutive monoid*. Let X_1 be an alphabet, and denote $\overline{X}_1 = \{\overline{x} \mid x \in X_1\}$

and $X = X_1 \cup \overline{X}_1$. Then the *involutive monoid* $M(X)$ is the monoid generated by X, with the defining relations

$$x\overline{x} = \epsilon, \quad \text{for all } x \in X_1.$$

The monoid $M(X)$ can also be defined in terms of the Dyck mapping $D : X^* \to X^*$. The relation E_D defined by

$$wE_D w' \quad \Longleftrightarrow \quad D(w) = D(w')$$

is a congruence, and $M(X)$ can be defined as the factor monoid X^*/E_D. Observe that power series and polynomials can be defined for arbitrary monoids in the same way as for the free monoid Σ^*. Thus, $S\langle M(X)\rangle$ stands for polynomials over $M(X)$, with coefficients in S.

Theorem 3.18. *Let $r \in S\langle\langle \Sigma^* \rangle\rangle$ be a component in the strong solution of a proper S-algebraic system. Then there exist an alphabet $X = X_1 \cup \overline{X}_1$, $x_1 \in X_1$, and a morphism $h : \Sigma^+ \to S\langle M(X)\rangle$ such that the condition*

$$(r, w) = \big(h(w), \overline{x}_1\big)$$

is satisfied for all $w \in \Sigma^+$.

Proof. By Theorem 3.15, we assume that $r = r_1$, where $(r_1, \ldots, r_n)$ is the strong solution of the S-algebraic system $y_i = p_i$, $1 \le i \le n$, with

$$\operatorname{supp}(p_i) \subseteq \Sigma \cup \Sigma Y \cup \Sigma Y^2, \quad 1 \le i \le n, \qquad Y = \{y_1, \ldots, y_n\}.$$

Define the alphabets $X_1 = \{x_1, \ldots, x_n\}$ and $X = X_1 \cup \overline{X}_1$, as well as the morphism $h : \Sigma^+ \to S\langle M(X)\rangle$ by the condition

$$h(a) = \sum_{i,j,k} (p_i, ay_j y_k)\overline{x}_i x_k x_j + \sum_{i,j} (p_i, ay_j)\overline{x}_i x_j + \sum_i (p_i, a)\overline{x}_i,$$

for all $a \in \Sigma$. (Observe that the support of each $h(a)$ is contained in the set $\overline{X}_1 X_1^2 \cup \overline{X}_1 X_1 \cup \overline{X}_1$.) We have to prove that, for all $w \in \Sigma^+$,

$$(r_1, w) = \big(h(w), \overline{x}_1\big).$$

We do this by establishing the stronger *claim*

$$(r_i, w) = \big(h(w), \overline{x}_i\big), \quad \text{for all } i = 1, \ldots, n.$$

The proof is by induction on the length, $|w|$. The basis $|w| = 1$ is clear. Then $w = a \in \Sigma$, and we have

$$(r_i, w) = (p_i, a) = \big(h(w), \overline{x}_i\big).$$

Assume that the claim holds for all words of length at most t, and consider a word $w = aw_1$, where $a \in \Sigma$ and $|w_1| = t$. Considering the form of the supports $\mathrm{supp}(p_i)$, we obtain first

$$(r_i, w) = (r_i, aw_1) = \sum_{\substack{j,k \\ w_1 = u_1 u_2}} (p_i, ay_j y_k)(r_j, u_1)(r_k, u_2) + \sum_j (p_i, ay_j)(r_j, w_1).$$

This implies, by the inductive hypothesis and a slight modification of the first sum,

$$(r_i, w) = \sum_{j,k} (p_i, ay_j y_k) \sum_{w_1 = u_1 u_2} \big(h(u_1), \overline{x}_j\big)\big(h(u_2), \overline{x}_k\big) + \sum_j (p_i, ay_j)\big(h(w_1), \overline{x}_j\big).$$

Hence, because h is a morphism, we have to establish the equation

$$\begin{aligned}\big(h(a)h(w_1), \overline{x}_i\big) = & \sum_{j,k} (p_i, ay_j y_k) \sum_{w = u_1 u_2} \big(h(u_1), \overline{x}_j\big)\big(h(u_2), \overline{x}_k\big) \\ & + \sum_j (p_i, ay_j)\big(h(w_1), \overline{x}_j\big)\end{aligned}$$

to complete the induction. Denote the two sums on the right-hand side of the equation by A and B, respectively. We are interested in those terms of $h(w_1)$ only which together with $h(a)$ cancel in the Dyck mapping, to yield $\overline{x}_i$. This means that if $|w_1| = 1$ (resp. $|w_1| > 1$), we have to consider only B (resp. A) on the right side.

Assume that $w_1 = 1$, that is, w_1 is a letter. Then $(h(a)h(w_1), \overline{x}_i) = B$. This follows because the only terms in $h(a)h(w_1)$ canceling to $\overline{x}_i$ are obtained by multiplying a term with support $\overline{x}_i x_j$ in $h(a)$ and a term with support $\overline{x}_j$ in $h(w_1)$. The sum of such products equals B.

If $|w_1| > 1$, we have $(h(a)h(w_1), \overline{x}_i) = A$. Considering possible cancelations, the validity of this equation is first reduced to the validity of the equation

$$\big(h(w_1), \overline{x}_j \overline{x}_k\big) = \sum_{w_1 = u_1 u_2} \big(h(u_1), \overline{x}_j\big)\big(h(u_2), \overline{x}_k\big).$$

This equation holds because

$$\begin{aligned}\sum_{w_1 = u_1 u_2} \big(h(u_1), \overline{x}_j\big)\big(h(u_2), \overline{x}_k\big) &= \sum_{w_1 = u_1 u_2} (r_j, u_1)(r_k, u_2) \\ &= (r_j r_k, w_1) = \big(h(w_1), \overline{x}_j \overline{x}_k\big).\end{aligned}$$

This completes the induction, and we obtain Shamir's theorem. □

Example 3.19. Consider the $\mathbb{N}$-algebraic system over $\Sigma = \{a_0, a_1, a_2\}$, consisting of the single equation

$$y_1 = a_0 + a_1 y_1 + a_2 y_1^2.$$

The system is proper and in Greibach normal form. Following the notation in Shamir's theorem, we obtain

$$h(a_0) = \overline{x}_1, \qquad h(a_1) = \overline{x}_1 x_1, \qquad h(a_2) = \overline{x}_1 x_1^2.$$

Hence, all values $h(w)$ are monomials, with the coefficient 1. This implies that the resulting power series is the characteristic series of its language. The language is customarily referred to as the *Łukasiewicz language* and can be characterized as follows. Consider the morphism g of Σ^+ into the additive monoid of integers, defined by $g(a_i) = i - 1$, $0 \le i \le 2$. Then the Łukasiewicz language consists of all words w such that $g(w) = -1$ and $g(w') \ge 0$ for all proper prefixes w' of w.

Example 3.20. Consider the alphabet $\Sigma = \{a, b\}$ and the proper $\mathbb{N}$-algebraic (actually right linear) system

$$\begin{aligned} y_1 &= ay_1 + 2by_2, \\ y_2 &= 3ay_1 + by_2 + b. \end{aligned}$$

We obtain now

$$h(a) = \overline{x}_1 x_1 + 3\overline{x}_2 x_1, \qquad h(b) = 2\overline{x}_1 x_2 + \overline{x}_2 x_2 + \overline{x}_2.$$

In this case, it is easy to analyze cancelations to $\overline{x}_1$. Corresponding to words of the form $a^i b^j$, we have the polynomial

$$(\overline{x}_1 x_1 + 3\overline{x}_2 x_1)^i (2\overline{x}_1 x_2 + \overline{x}_2 x_2 + \overline{x}_2)^j.$$

For any $i \ge 0$ and $j = 1$, we obtain the coefficient 2 and still have to cancel x_2. This requires arbitrarily many multiplications with $\overline{x}_2 x_2$ and the final multiplication with $\overline{x}_2$. Thus, every word in $a^* b b^*$ has the coefficient 2, whereas all other words in $a^* b^*$ have the coefficient 0. A similar analysis shows that every word in $b^+ a^+ b b^+$ has the coefficient 12. In general, the possibility of the cancelation to $\overline{x}_1$ shows that a change between the two letters in a word introduces a factor 2 or 3 to its coefficient. Observe that this example can be viewed also as a weighted grammar or a weighted finite automaton.

The converse of Shamir's theorem can be stated as follows. For a proof, see [18].

Theorem 3.21. *Assume that $r \in S\langle\langle \Sigma^* \rangle\rangle$ is quasi-regular, $h : \Sigma^+ \to S\langle M(X) \rangle$, $X = X_1 \cup \overline{X_1}$, is a morphism with the property that $h(a)$ is quasi-regular and non-zero for every $a \in \Sigma$, and $\gamma \in M(X)$, such that*

$$(r, w) = \big(h(w), \gamma\big)$$

holds for all $w \in \Sigma^+$. Then r is S-algebraic.

Instead of the involutive monoid $M(X)$, Shamir's theorem can be stated for the free group $\mathcal{G}(X_1)$ generated by X_1. (Thus, elements of $\overline{X}_1$ are inverses, not only right inverses as for $M(X)$.) Then the morphism h will be more complicated and more general normal forms for algebraic power series will be needed. The details are given in [18].

Finally, we present the *Chomsky–Schützenberger theorem*. It gives a method, similarly as Shamir's theorem, for computing the coefficients of an algebraic power series. While Shamir's theorem uses a morphism of the free monoid into a multiplicative monoid of polynomials, the Chomsky–Schützenberger theorem produces the coefficients by a morphism from the characteristic series of the intersection between a Dyck language and a regular language.

We omit the proof, [18], of the following Chomsky–Schützenberger theorem. The proof runs along the same lines as the corresponding result for context-free languages.

Theorem 3.22. *Let $r \in S\langle\langle \Sigma^* \rangle\rangle$ be a component in the strong solution of a proper S-algebraic system. Then there exist an alphabet $X = X_1 \cup \overline{X}_1$ and a regular language R over X such that r is a morphic image of the characteristic series of the intersection $L_D(X) \cap R$.*

4 Transductions

The theory of transductions originates from considerations about finite automata with outputs, generalized sequential machines, and pushdown transducers. Transductions can be viewed as mappings from $\mathbb{B}\langle\langle \Sigma_1^* \rangle\rangle$ into $\mathbb{B}\langle\langle \Sigma_2^* \rangle\rangle$ if only languages without multiplicities are considered. In general transductions between families of power series, the Boolean semiring is replaced by an arbitrary commutative semiring S. Direct generalizations of customary transductions between languages lead into difficulties because infinite sums over S may occur. Either one has to make strong summability assumptions about S, or else restrict the attention to cases not leading to infinite sums over S. The notion of a *regulated representation* is a convenient tool in the latter approach. We say that a morphism

$$h : \Sigma_1^* \to (S\langle\langle \Sigma_2^* \rangle\rangle)^{m \times m}$$

is a *regulated representation* if, for some positive integer t, all entries in all matrices $h(w)$ with $|w| \geq t$ are quasi-regular.

Let r be some component of the strong solution of a proper S-algebraic system. For brevity, we refer to such series r as *proper S-algebraic*. Hence, all proper S-algebraic series are quasi-regular.

Definition 4.1. *A mapping $\tau : S\langle\langle \Sigma_1^* \rangle\rangle \to S\langle\langle \Sigma_2^* \rangle\rangle$ is termed a* regulated semi-algebraic transduction *if, for $r \in S\langle\langle \Sigma_1^* \rangle\rangle$,*

$$\tau(r) = (r,\epsilon)r_0 + \sum_{w\in\Sigma_1^+} (r,w)\big(h(w)\big)_{1m},$$

where $r_0 \in S^{\mathrm{alg}}\langle\langle \Sigma_2^* \rangle\rangle$ *and*

$$h : \Sigma_1^* \to \big(S^{\mathrm{alg}}\langle\langle \Sigma_2^* \rangle\rangle\big)^{m\times m}$$

is a regulated representation. (As usual, M_{ij} *denotes the* (i,j)*th entry of a matrix* M*.) If in addition* r_0 *and all entries in every matrix* $h(a)$*,* $a \in \Sigma_1$*, are proper* S*-algebraic series, then* τ *is termed a* regulated algebraic transduction*.*

We are now ready for the fundamental result concerning regulated algebraic transductions.

Theorem 4.2. *A regulated semi-algebraic (resp. regulated algebraic) transduction maps every algebraic (resp. proper algebraic) series into an algebraic (resp. a proper algebraic) series.*

Proof. We use the notation in the definition above. We establish first the claim in parentheses, concerning regulated algebraic transductions. Consider a proper S-algebraic series r, and assume that all entries in the matrices $h(a)$, $a \in \Sigma_1$, are proper S-algebraic series. Hence, they are quasi-regular. Let $y_i = p_i$, $i = 1,\ldots,n$, be the proper S-algebraic system defining r. For each of the variables y_i, we associate the $m \times m$ matrix of variables

$$\begin{pmatrix} y_{11}^i & \cdots & y_{1m}^i \\ \vdots & & \vdots \\ y_{m1}^i & \cdots & y_{mm}^i \end{pmatrix}.$$

(Observe that each $a \in \Sigma_1$ is replaced by $h(a)$. No terms of S appear additively in any p_i, since the system is proper. Such terms would have to be multiplied by the identity matrix.) When the variables y_i in the original S-algebraic system are replaced by the associated matrices and the resulting equations are written out entry-wise, we obtain a proper S-algebraic system for the entries in the matrices, in particular, for the $(1,m)$th entry. However, in this new system, the coefficients on right-hand sides of the equations are power series in $S^{\mathrm{alg}}\langle\langle \Sigma^* \rangle\rangle$. It is shown in [11] that Theorem 3.2 holds for such systems as well. This establishes the claim concerning regulated algebraic transductions. (Observe that the commutativity of S is needed.)

Consider next the claim concerning regulated semi-algebraic transductions. It is no loss of generality to assume that the given series r is proper S-algebraic. For if $r = (r,\epsilon)\epsilon + r'$ and the claim holds for the proper S-algebraic r', then it clearly holds for r as well. Thus, we assume that the entries in all matrices $h(w)$, $|w| = t \geq 1$, are quasi-regular. The proof is now carried out by considering the words in Σ_1^t as new letters and reducing the argument to the (already established) case of regulated algebraic transductions.

Thus, consider the alphabet $\Sigma_3 = \{z_1, \ldots, z_l\}$ where $z_1, \ldots, z_l$ are all the words in Σ_1^t. Let

$$g : \Sigma_3^* \to \Sigma_1^*$$

be the natural morphism, mapping each z_i to the appropriate product of letters of Σ_1. Let $w \in \Sigma_1^+$ and define the series

$$r_w = \sum_{u \in \Sigma_3^*} \big(r, wg(u)\big) u.$$

It is easy to see that r_w is proper S-algebraic. (In fact, proper S-algebraic series are closed under inverse monoid morphisms.) By the first part of the proof, it follows that the entries of

$$\sum_{u \in \Sigma_3^*} \big(r, wg(u)\big) h\big(g(u)\big)$$

are proper S-algebraic. Because we can write

$$\tau(r) = \sum_{|w|<t} \tau(w)\tau(r_w),$$

we conclude that $\tau(r)$ is S-algebraic, which completes the proof. □

We now relax the requirement of the representation being regulated. In the following definition, we assume that our commutative semiring S is also *complete*.

Definition 4.3. *A mapping* $\tau : S\langle\langle \Sigma_1^* \rangle\rangle \to S\langle\langle \Sigma_2^* \rangle\rangle$ *is termed an* algebraic transduction *if, for* $r \in S\langle\langle \Sigma_1^* \rangle\rangle$,

$$\tau(r) = (r, \epsilon) r_0 + \sum_{w \in \Sigma_1^+} (r, w) \big(h(w)\big)_{1m},$$

where $r_0 \in S^{\mathrm{alg}}\langle\langle \Sigma_2^* \rangle\rangle$ *and*

$$h : \Sigma_1^* \to \big(S^{\mathrm{alg}}\langle\langle \Sigma_2^* \rangle\rangle\big)^{m \times m}$$

is a semiring morphism.

It is not known whether an algebraic transduction maps an algebraic series into an algebraic series. The problem goes essentially back to applying erasing morphisms to algebraic series. However, if S is continuous, then an algebraic transduction always maps an algebraic series to an algebraic series, [11]. The following result can be obtained in the general case.

Theorem 4.4. *Every algebraic transduction can be represented as the composition of a projection and a regulated semi-algebraic transduction.*

Proof (outline). The argument is commonly used in language theory: introduce a new letter x' to the alphabet Σ_2. Multiply then in each entry of the matrices of the algebraic transduction the coefficient of ϵ by x'. A regulated semi-algebraic transduction results (with $\Sigma_2 \cup \{x'\}$ instead of Σ_2 as the target alphabet.) After applying this regulated transduction, apply the projection erasing x' and keeping the letters of Σ_2 fixed. □

A *regulated rational transduction* is defined exactly as a regulated semi-algebraic transduction (Definition 4.1) except that now $r_0 \in S^{\mathrm{rat}}\langle\langle \Sigma_2^* \rangle\rangle$ and the target semiring of h is $(S^{\mathrm{rat}}\langle\langle \Sigma_2^* \rangle\rangle)^{m\times m}$. A regulated rational transduction maps a series in $S^{\mathrm{rat}}\langle\langle \Sigma_1^* \rangle\rangle$ into a series in $S^{\mathrm{rat}}\langle\langle \Sigma_2^* \rangle\rangle$. The reader is referred to [18] for further details, as well as for the proof of the following result which can be viewed as another formulation of the Chomsky–Schützenberger Theorem.

Theorem 4.5. *For every proper S-algebraic series r, there is an alphabet $X = X_1 \cup \overline{X}_1$ and a regulated rational transduction τ such that*

$$r = \tau\big(\mathrm{char}\big(L_D(X)\big)\big).$$

An alternative way of presenting the theory of transductions is to consider power series in the product monoid $\Sigma_1^* \times \Sigma_2^*$. We now define (general) rational transductions using this approach.

Definition 4.6. *Assume that S is complete and $\tau : S\langle\langle \Sigma_1^* \rangle\rangle \to S\langle\langle \Sigma_2^* \rangle\rangle$ is a mapping such that $\tau(r) = \sum (r,w)\tau(w)$. If*

$$\sum w \times \tau(w) \in S^{\mathrm{rat}}\langle\langle \Sigma_1^* \times \Sigma_2^* \rangle\rangle,$$

then τ is said to be a rational transduction.

Both rational transductions and regulated rational transductions are closed under composition. The following result is a restatement of relations concerning rational power series in product monoids. (See [18].)

Theorem 4.7. *A mapping $\tau : S\langle\langle \Sigma_1^* \rangle\rangle \to S\langle\langle \Sigma_2^* \rangle\rangle$ is a rational transduction if and only if there are a series $r_0 \in S^{\mathrm{rat}}\langle\langle \Sigma_2^* \rangle\rangle$ and a representation $h : \Sigma_1^* \to (S^{\mathrm{rat}}\langle\langle \Sigma_2^* \rangle\rangle)^{m\times m}$ such that*

$$\tau(r) = (r,\epsilon)r_0 + \sum_{w\neq\epsilon} (r,w)\big(h(w)\big)_{1m}.$$

Observe that this theorem shows that every regulated rational transduction is a rational transduction. The following theorem tells explicitly the interconnection between rational transductions and regulated rational transductions. The result may be proved with arguments similar as those used for Theorem 4.4.

Theorem 4.8. *Every rational transduction can be expressed as the composition of a projection and a regulated rational transduction.*

We conclude this section with the following *Nivat's theorem*. The Hadamard product is denoted by $\odot$.

Theorem 4.9. *A mapping $\tau : S\langle\langle \Sigma_1^* \rangle\rangle \to S\langle\langle \Sigma_2^* \rangle\rangle$ is a rational transduction if and only if, for some alphabet Σ_3, projections $g : \Sigma_3^* \to \Sigma_1^*$ and $h : \Sigma_3^* \to \Sigma_2^*$, and for some series $r_0 \in S^{\mathrm{rat}}\langle\langle \Sigma_3^* \rangle\rangle$, we have*

$$\tau(r) = h\big(g^{-1}(r) \odot r_0\big).$$

Much of the fundamental work concerning algebraic transductions is due to [15] and [6]. We have also used the above arguments from [18].

5 Pushdown Automata

This section introduces $S\langle\langle \Sigma^* \rangle\rangle$-pushdown automata and discusses on one hand the interconnection with classical pushdown automata over finite alphabets (without weights) and on the other hand, the interconnection with algebraic systems.

5.1 Pushdown Transition Matrices

Throughout this section, $\Sigma = \{x_1, \ldots, x_k\}$ will denote a finite alphabet and S a commutative semiring. Also, Q will denote a finite non-empty set (of states) and Γ a finite alphabet (of pushdown symbols), not necessarily distinct from Σ.

Definition 5.1. *A matrix $M \in ((S\langle\langle \Sigma^* \rangle\rangle)^{Q\times Q})^{\Gamma^* \times \Gamma^*}$ is called an $S\langle\langle \Sigma^* \rangle\rangle$-* pushdown transition matrix *if the following two conditions are satisfied:*

(i) For any $p \in \Gamma$, there exist only finitely many $\pi \in \Gamma^*$ such that $M_{p,\pi} \neq 0$.
(ii) For any $\pi_1, \pi_2 \in \Gamma^*$,

$$M_{\pi_1,\pi_2} = \begin{cases} M_{p,\pi}, & \text{if } \pi_1 = p\pi', \pi_2 = \pi\pi', \text{ for some } \pi' \in \Gamma^*, \\ 0, & \text{otherwise.} \end{cases}$$

If all entries of M are in $S\langle \Sigma \cup \{\epsilon\} \rangle$, then we call M an $S\langle \Sigma \cup \{\epsilon\} \rangle$-pushdown transition matrix.

It follows directly from the definition that any pushdown transition matrix is finitely specified by the blocks $M_{p,\pi}$, with $p \in \Gamma$ and $\pi \in \Gamma^*$. In particular, any such matrix is both row and column finite. Consequently, the product of pushdown transition matrices and their arbitrary powers are well defined. However, without special assumptions about either the semiring or the matrix itself, the star may not exist because infinite sums may arise. For instance, if the semiring is complete, the infinite sums are well defined and the star always exists. For a presentation of pushdown transition matrices and pushdown automata in the case of complete semirings, we refer to [11] and [3]. We give here a different presentation where the semiring is not assumed to be complete, but rather the matrices are assumed to satisfy such properties as to obtain a locally finite star matrix, thus avoiding infinite sums. This allows us, e.g., to consider pushdown automata with multiplicities in $\mathbb{N}$ rather than $\mathbb{N}^\infty$, which is a desirable feature from the point of view of weighted automata and formal languages. In many essentials, we follow here the presentation in [12].

It is important to note that based on the semiring isomorphisms described in Chap. 1 of this handbook a pushdown transition matrix may be considered in $((S\langle\langle\Sigma^*\rangle\rangle)^{Q\times Q})^{\Gamma^*\times\Gamma^*}$, but also in $((S^{Q\times Q})\langle\langle\Sigma^*\rangle\rangle)^{\Gamma^*\times\Gamma^*}$, or in $(S^{Q\times Q})^{\Gamma^*\times\Gamma^*}\langle\langle\Sigma^*\rangle\rangle$. We will use also both of the latter semirings in our considerations without risk of confusion. For example, when discussing the star of a pushdown matrix, we will base our discussion on the semiring $(S^{Q\times Q})^{\Gamma^*\times\Gamma^*}\langle\langle\Sigma^*\rangle\rangle$, but the definition below of a proper pushdown transition matrix is based on $((S^{Q\times Q})\langle\langle\Sigma^*\rangle\rangle)^{\Gamma^*\times\Gamma^*}$.

Definition 5.2. *An $S\langle\langle\Sigma^*\rangle\rangle$-pushdown transition matrix is called proper if for all $p\in\Gamma$ and $\pi\in\Gamma^*$, $(M_{p,\pi},\epsilon)\neq 0$ implies that $|\pi|\geq 2$.*

The next result shows that a proper pushdown transition matrix is C-cycle free, and so based on Lemma 2.5, its star exists.

Theorem 5.3. *Let M be an $S\langle\langle\Sigma^*\rangle\rangle$-pushdown transition matrix. If M is proper, then it is C-cycle free. Moreover, $(M^*)_{p,\epsilon}$ is quasi-regular for all $p\in\Gamma$.*

Proof. We prove first that $(M^n,\epsilon)_{\pi_1,\pi_2}=0$, for all $\pi_1,\pi_2\in\Gamma^*$ and all $n\geq 0$, with $|\pi_2|\leq|\pi_1|+n-1$. We prove the claim by induction on n.

For $n=0$, the claim holds vacuously. Also, in case $|\pi_1|=0$, the claim follows directly from the definition of a pushdown transition matrix. Let $n\geq 1$ and $\pi_1,\pi_2\in\Gamma^*$ such that $|\pi_2|\leq|\pi_1|+n-1$. We may assume without loss of generality that $|\pi_1|\geq 1$, i.e., $\pi_1=p\pi_1'$, for some $p\in\Gamma$, $\pi_1'\in\Gamma^*$. Then

$$\begin{aligned}\left(M^n,\epsilon\right)_{p\pi_1',\pi_2} &= \sum_{\pi\in\Gamma^*,|\pi|\geq 2}(M,\epsilon)_{p\pi_1',\pi\pi_1'}\left(M^{n-1},\epsilon\right)_{\pi\pi_1',\pi_2}\\ &= \sum_{\pi\in\Gamma^*,|\pi|\geq 2}(M,\epsilon)_{p,\pi}\left(M^{n-1},\epsilon\right)_{\pi\pi_1',\pi_2}.\end{aligned}$$

Note now that $|\pi\pi_1'|+(n-1)-1\geq|\pi_1'|+n=|\pi_1|+n-1\geq|\pi_2|$, and so by the induction hypothesis, $(M^{n-1},\epsilon)_{\pi\pi_1',\pi_2}=0$, proving our claim.

To prove that M is C-cycle free, we have to show, by definition that $\mathsf{lim}^C_{n\to\infty}(M^n,\epsilon)=0$. This is equivalent with the following two conditions:

(i) For all $\pi_2\in\Gamma^*$, there exists a finite set $I(\pi_2)\subseteq\Gamma^*$ such that $(M^n,\epsilon)_{\pi_1,\pi_2}=0$, for all $\pi_1\notin I(\pi_2)$ and all $n\geq 0$.

(ii) For all $\pi_2 \in \Gamma^*$, there exists a non-negative integer $n(\pi_2)$ such that $(M^n, \epsilon)_{\pi_1,\pi_2} = 0$, for all $n \geq n(\pi_2)$ and all $\pi_1 \in \Gamma^*$.

Part (i) follows from our claim for $I(\pi_2) = \{\pi_1 \in \Gamma^* \mid |\pi_1| \leq |\pi_2|\}$. Part (ii) follows from our claim for $n(\pi_2) = |\pi_2| + 1$.

Applying again our claim, this time for $\pi_1 = p \in \Gamma$, $\pi_2 = \epsilon$, it follows that $(M^n, \epsilon)_{p,\epsilon} = 0$, for all $n \geq 0$, i.e., $(M^*)_{p,\epsilon}$ is quasi-regular. □

The following two results will be useful in the next section when proving the equivalence of algebraic systems and pushdown automata. For proofs, we refer to [12], where the results are stated also for C-cycle free (and other types of) pushdown transition matrices.

Theorem 5.4. *Let M be a proper $S\langle\langle\Sigma^*\rangle\rangle$-pushdown transition matrix. Then $(M^*)_{\pi_1\pi_2,\epsilon} = (M^*)_{\pi_1,\epsilon}(M^*)_{\pi_2,\epsilon}$, for all $\pi_1, \pi_2 \in \Gamma^*$.*

Theorem 5.5. *Let M be a proper $S\langle\langle\Sigma^*\rangle\rangle$-pushdown transition matrix. For any $p \in \Gamma$, let $S_p \in (S\langle\langle\Sigma^*\rangle\rangle)^{Q\times Q}$ be quasi-regular. Also, let S_ϵ be the $Q \times Q$ unity matrix and $S_{p\pi} = S_p S_\pi$, for all $p \in \Gamma$, $\pi \in \Gamma^*$. If*

$$S_p = \sum_{\pi\in\Gamma^*} M_{p,\pi} S_\pi,$$

then $S_\pi = (M^*)_{\pi,\epsilon}$, for all $\pi \in \Gamma^*$.

5.2 $S\langle\langle\Sigma^*\rangle\rangle$-Pushdown Automata

We define in this section the notion of $S\langle\langle\Sigma^*\rangle\rangle$-pushdown automata and their behavior.

Definition 5.6. *An* $S\langle\langle\Sigma^*\rangle\rangle$-pushdown automaton $\mathcal{P}$ *is a structure*

$$\mathcal{P} = (Q, \Gamma, M, q_0, p_0, P),$$

where:

(i) Q *is a finite set of* states.
(ii) Γ *is a finite alphabet of* pushdown symbols.
(iii) M is an $S\langle\langle\Sigma^*\rangle\rangle$-pushdown transition matrix.
(iv) $q_0 \in Q$ *is an* initial state.
(v) $p_0 \in \Gamma$ *is an* initial pushdown symbol.
(vi) $P \in (S\langle\{\epsilon\}\rangle)^{Q\times 1}$ *is a* final state vector.

We say that $\mathcal{P}$ is an $S\langle\Sigma \cup \{\epsilon\}\rangle$-pushdown automaton if M is an $S\langle\Sigma \cup \{\epsilon\}\rangle$-pushdown transition matrix. We also say that $\mathcal{P}$ is proper *if M is a proper pushdown transition matrix.*

The behavior $\|\mathcal{P}\| \in S\langle\langle\Sigma^*\rangle\rangle$ of $\mathcal{P}$ is defined by

$$\|\mathcal{P}\| = e_{q_0}(M^*)_{p_0,\epsilon}P = \big((M^*)_{p_0,\epsilon}P\big)_{q_0},$$

provided that M^* *exists, where* $e_{q_0}\in(S\langle\{\epsilon\}\rangle)^Q$*, with* $(e_{q_0})_{q_0}=\epsilon$ *and* $(e_{q_0})_q=0$*, for all* $q \in Q \setminus \{q_0\}$*. We say that two pushdown automata* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are* equivalent *if* $\|\mathcal{P}_1\| = \|\mathcal{P}_2\|$.

Note that the behavior of a pushdown automaton is well defined if its pushdown transition matrix is proper. Note also that, using standard terminology of formal language theory, the mode of acceptance of a pushdown automaton is defined here through reaching a final state, while emptying the pushdown stack. Two other (equivalent) modes of acceptance are often considered: through emptying the pushdown stack (regardless of the state), or by reaching a final state (regardless of the pushdown stack).

Example 5.7. Let $\Sigma = \{a, b\}$ and consider the $\mathbb{N}\langle \Sigma \cup \{\epsilon\}\rangle$-pushdown automaton $\mathcal{P} = (Q, \Gamma, M, q_1, p_0, P)$, where $Q = \{q_1, q_2\}$, $\Gamma = \{p_0, a, b\}$, $P_{q_1} = 0$, $P_{q_2} = \epsilon$ and $M \in ((\mathbb{N}\langle\langle \Sigma^*\rangle\rangle)^{Q\times Q})^{\Gamma^*\times\Gamma^*}$ is defined as follows:

$$M_{p_0,a} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \qquad M_{a,aa} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \qquad M_{a,\epsilon} = \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix}.$$

Clearly, M and by consequence $\mathcal{P}$, are proper. Then M^* exists and $\|\mathcal{P}\| = ((M^*)_{p_0,\epsilon})_{q_1,q_2}$.

Based on the definition of M, we obtain that

$$(M^*)_{p_0,\epsilon} = M_{p_0,a}(M^*)_{a,\epsilon} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} (M^*)_{a,\epsilon},$$
$$(M^*)_{a,\epsilon} = M_{a,\epsilon} + M_{a,aa}(M^*)_{aa,\epsilon} = \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \left((M^*)_{a,\epsilon}\right)^2.$$

If

$$(M^*)_{a,\epsilon} = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix},$$

with $p_i \in \mathbb{N}\langle\langle \Sigma^*\rangle\rangle$, $1 \le i \le 4$, then it follows that $p_3 = 0$, $p_4 = b$, $p_1 = ap_1^2$, and $p_2 = ap_1p_2 + ap_2b + b$. Consequently, $p_1 = 0$ and $p_2 = ap_2b + b$. Then $p_2 = \sum_{n\ge 0} a^n b^{n+1}$, and so,

$$\|\mathcal{P}\| = \sum_{n\ge 0} a^{n+1} b^{n+1}.$$

Definition 5.8. *An* $S\langle\langle \Sigma^*\rangle\rangle$*-pushdown automaton* $\mathcal{P} = (Q, \Gamma, M, q_0, p_0, P)$ *is called* normalized *if:*

(i) $(M_{\pi_1,\pi_2})_{q,q_0} = 0$*, for all* $\pi_1, \pi_2 \in \Gamma^*$*,* $q \in Q$*.*
(ii) There is $t \in Q \setminus \{q_0\}$ *such that* $P_t = \epsilon$*,* $P_q = 0$*, for all* $q \in Q \setminus \{t\}$*, and* $(M_{\pi_1,\pi_2})_{t,q} = 0$*, for all* $\pi_1, \pi_2 \in \Gamma^*$*,* $q \in Q$*.*

It is not difficult to see that for any $S\langle \Sigma \cup \{\epsilon\}\rangle$-pushdown automaton, an equivalent normalized one can be constructed. The construction is the one often encountered in automata theory: one adds a new initial state and a new final state, and extends the pushdown matrix in a suitable way so that no transitions into the initial state and no transitions from the final state exist. Moreover, the new pushdown automaton remains proper if the initial one was so. We state the following result without proof, referring to [12] for a detailed construction and proof.

Theorem 5.9. *For any proper $S\langle \Sigma \cup \{\epsilon\}\rangle$-pushdown automaton, an equivalent normalized proper one exists.*

We recall that a (classical) pushdown automaton over an alphabet is a structure $\mathcal{A} = (Q, \Gamma, M, q_0, p_0, F)$, where $F \subseteq Q$ is a set of final states, $M \in (\mathcal{P}(\Sigma \cup \{\epsilon\})^{Q\times Q})^{\Gamma^* \times \Gamma^*}$ is a row and column finite pushdown transition matrix and the significance of the other components is the same as in the case of an $S\langle\langle \Sigma^* \rangle\rangle$-pushdown automaton. In particular, note that the matrix M may be seen as an $\mathbb{B}\langle \Sigma \cup \{\epsilon\}\rangle$-pushdown transition matrix, and F may be seen as a final state vector in $(\mathbb{B}\langle \{\epsilon\}\rangle)^{Q\times 1}$. Consequently, we may consider any (classical) pushdown automaton $\mathcal{A}$ as an $\mathbb{B}\langle \Sigma \cup \{\epsilon\}\rangle$-pushdown automaton. In this case M^* always exists over $\mathbb{B}$ and $\|\mathcal{A}\| \in \mathbb{B}\langle\langle \Sigma^* \rangle\rangle$ is the characteristic series of the language accepted by $\mathcal{A}$. Similarly, if M is proper, $\mathcal{A}$ may also be seen as an $\mathbb{N}\langle \Sigma \cup \{\epsilon\}\rangle$-pushdown automaton. The correspondence between the language $\mathcal{L}(\mathcal{A})$ accepted by $\mathcal{A}$ and its behavior $\|\mathcal{A}\| \in \mathbb{N}\langle\langle \Sigma^* \rangle\rangle$ is given in the next result.

Theorem 5.10. *For any (classical) proper pushdown automaton $\mathcal{A}$ over Σ and any $w \in \Sigma^*$, $(\|\mathcal{A}\|, w)$ is the number of distinct successful computations of $\mathcal{A}$ on input w, where the acceptance mode of $\mathcal{A}$ is with empty pushdown stack and final state.*

Proof. We prove the more general claim that for any $\pi_1, \pi_2 \in \Gamma^*$, $q_1, q_2 \in Q$, $w \in \Sigma^*$, $n \geq 0$, $(((M^n)_{\pi_1,\pi_2})_{q_1,q_2}, w)$ is equal to the number of distinct n-step computations in $\mathcal{A}$ changing the state from q_1 to q_2 and the stack content from π_1 to π_2 while reading the input w. Then the theorem follows with $\pi_1 = p_0$, $\pi_2 = \epsilon$, $q_1 = q_0$, and $q_2 \in F$.

For $n = 0$ and $n = 1$ the claim is trivial. For $n > 0$, assume the claim holds up to n. Then any $n+1$-step computation with input w, changing the state from q_1 to q_2 and the stack from π_1 to π_2, can be decomposed into:

(i) An n-step computation with input u, changing the state from q_1 to q and the stack from π_1 to π, and
(ii) A 1-step computation with input v, changing the state from q to q_2 and the stack from π to π_2

where $w = uv$, $q \in Q$, and $\pi \in \Gamma^*$. Thus, based on the induction hypothesis, the number of such distinct $(n+1)$-step computations is the following:

$$\sum_{\pi\in\Gamma^*}\sum_{q\in Q}\sum_{\substack{u,v\in\Sigma^*\\ w=uv}}\big(\big((M^n)_{\pi_1,\pi_2}\big)_{q_1,q},u\big)\big((M_{\pi,\pi_2})_{q,q_2},v\big)$$
$$=\big(\big((M^{n+1})_{\pi_1,\pi_2}\big)_{q_1,q_2},w\big).\quad\square$$

5.3 Equivalence with Algebraic Systems

We prove in this section that the algebraic systems and the pushdown automata are equivalent, in the sense that the set of behaviors of proper $S\langle\Sigma\cup\{\epsilon\}\rangle$-pushdown automata is exactly $S^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$. We prove first that any algebraic series is the behavior of a pushdown automaton.

Theorem 5.11. *Let r be the first component of the strong solution of a proper algebraic system. Then there exists a proper $S\langle\Sigma\cup\{\epsilon\}\rangle$-pushdown automaton $\mathcal{P}$ such that $\|\mathcal{P}\|=r$.*

Proof. Let

$$y_i=p_i,\quad i=1,\ldots,n,\tag{1}$$

be a proper algebraic system with r as the first component of its strong solution, where $p_i\in S\langle(\Sigma\cup Y)^*\rangle$, for all $1\le i\le n$. We consider the $S\langle\Sigma\cup\{\epsilon\}\rangle$-pushdown automaton

$$\mathcal{P}=\big(\{q\},\Sigma\cup Y,M,q,y_1,(\epsilon)\big),$$

where M is defined as follows:

$$M_{y_i,y_j\gamma}=(p_i,y_j\gamma)\epsilon+\sum_{x\in\Sigma}(p_i,xy_j\gamma)x,\quad\text{for }\gamma\in(\Sigma\cup Y)^*,\ 1\le i,j,\le n,$$
$$M_{y_i,x\gamma}=\sum_{x'\in\Sigma}(p_i,x'x\gamma)x',\quad\text{for }\gamma\in(\Sigma\cup Y)^*,\ x\in\Sigma,\ 1\le i\le n,$$
$$M_{y_i,\epsilon}=\sum_{x\in\Sigma}(p_i,x)x,\quad\text{for }1\le i\le n,$$
$$M_{x,\epsilon}=x,\ \text{for }x\in\Sigma,$$
$$M_{\pi_1,\pi_2}=0,\quad\text{in all other cases.}$$

Note that M is a proper pushdown matrix. Indeed, if $(M_{q,\pi},\epsilon)\neq 0$, for some $q\in\Gamma$, $\pi\in\Gamma^*$, it implies that $q=p_i$ and $(p_i,\pi)\neq 0$, for some $1\le i\le n$. However, since the algebraic system is proper, it follows that $|\pi|\ge 2$. Thus, it follows by Theorem 5.3 that M^* exists and $(M^*)_{y_i,\epsilon}$ is quasi-regular for all $1\le i\le n$.

We write the algebraic system (1) as follows:

$$
\begin{aligned}
y_i =& \sum_{j=1}^{n} \sum_{\gamma \in (\Sigma \cup Y)^*} (p_i, y_j\gamma) y_j\gamma + \sum_{j=1}^{n} \sum_{\gamma \in (\Sigma \cup Y)^*} \sum_{x \in \Sigma} (p_i, x y_j \gamma) x y_j \gamma \\
&+ \sum_{\gamma \in (\Sigma \cup Y)^*} \sum_{x \in \Sigma} \sum_{x' \in \Sigma} (p_i, x' x \gamma) x' x \gamma + \sum_{x \in \Sigma} (p_i, x) x \qquad (2) \\
=& \sum_{j=1}^{n} \sum_{\gamma \in (\Sigma \cup Y)^*} M_{y_i, y_j\gamma} y_j \gamma + \sum_{x \in \Sigma} \sum_{\gamma \in (\Sigma \cup Y)^*} M_{y_i, x\gamma} x\gamma + M_{y_i, \epsilon}.
\end{aligned}
$$

We claim now that the system (1) is satisfied when substituting $(M^*)_{y_i,\epsilon}$ for y_i, for all $1 \leq i \leq n$. Based on Theorem 5.4, that means that when checking the equalities in (1), we will substitute $(M^*)_{\pi,\epsilon}$ for all $\pi \in (\Sigma \cup Y)^*$.

It is easy to see that $M^n_{x,\epsilon} = 0$, for all $x \in \Sigma$ and $n \geq 2$ and so, $(M^*)_{x,\epsilon} = x$. For $(M^*)_{y_i,\epsilon}$, based on the definition of M, we obtain that

$$
\begin{aligned}
(M^*)_{y_i,\epsilon} =& \sum_{\pi \in (\Sigma \cup Y)^*} M_{y_i,\pi} (M^*)_{\pi,\epsilon} \\
=& \sum_{j=1}^{n} \sum_{\gamma \in (\Sigma \cup Y)^*} M_{y_i, y_j\gamma} (M^*)_{y_j\gamma,\epsilon} \\
&+ \sum_{x \in \Sigma} \sum_{\gamma \in (\Sigma \cup Y)^*} M_{y_i, x\gamma} (M^*)_{x\gamma,\epsilon} + M_{y_i,\epsilon},
\end{aligned}
$$

for all $1 \leq i \leq n$, i.e., the refined version (2) of system (1) is verified, proving the claim.

Note now that based on Theorem 5.3, $(M^*)_{y_i,\epsilon}$ is a quasi-regular series. Since a proper algebraic system has only one solution with all components quasi-regular, see Theorem 3.2. It follows now that $r = (M^*)_{y_1,\epsilon} = \|\mathcal{A}\|$, concluding our proof. □

Example 5.12. Consider the proper $\mathbb{N}$-algebraic system

$$y = yy + x_1 y x_2 + x_1 x_2$$

of Example 3.5. Based on Theorem 5.11, we construct a pushdown automaton $\mathcal{P}$ such that $\|\mathcal{P}\|$ is the strong solution of the system. We consider $\mathcal{P} = (\{q\}, \{x_1, x_2, y\}, M, q, y, (\epsilon))$, where the pushdown transition matrix M is defined as follows: $M_{x_1,\epsilon} = x_1$, $M_{x_2,\epsilon} = x_2$, $M_{y,x_2} = x_1$, $M_{y,x_2y} = x_1$, $M_{y,yx_2} = x_1$, $M_{y,yx_2y} = x_1$. It follows then by Theorem 5.11 that $\|\mathcal{P}\| = (M^*)_{y_1,\epsilon}$ is the strong solution of the algebraic system above.

We prove now the reverse transition, from a pushdown automaton to an algebraic system.

Theorem 5.13. *Let $\mathcal{P}$ be a proper $S\langle \Sigma \cup \{\epsilon\}\rangle$-pushdown automaton. Then there exists a proper $S\langle\langle \Sigma^* \rangle\rangle$-algebraic system with $\|\mathcal{P}\|$ as the first component of its strong solution.*

Proof. By Theorem 5.9, we may assume without loss of generality that $\mathcal{P} = (Q, \Gamma, M, q_0, p_0, P)$ is a normalized proper $S\langle \Sigma \cup \{\epsilon\}\rangle$-pushdown automaton. Thus, $\|\mathcal{P}\| = ((M^*)_{p_0,\epsilon})_{q_0,t}$, where $t \in Q$, $P_t = \epsilon$ (and it is the only non-zero component of P).

Consider the alphabet

$$Y = \{y^p_{q_1,q_2} \mid p \in \Gamma,\ q_1, q_2 \in Q\}.$$

We consider the matrices $Y_p \in (S\langle Y\rangle)^{Q\times Q}$, defined by $(Y_p)_{q_1,q_2} = y^p_{q_1,q_2}$, for all $q_1, q_2 \in Q$. We then extend our definition to $Y_\pi \in (S\langle Y\rangle)^{Q\times Q}$, for all $\pi \in \Gamma^*$ in the following way:

$$Y_\epsilon = E, \qquad Y_{p\pi} = Y_p Y_\pi,$$

for all $\pi \in \Gamma^*$, where we denote by E the unity matrix (in this case a $Q \times Q$ matrix).

Consider now the algebraic system written in the following matrix notation:

$$Y_p = \sum_{\pi \in \Gamma^*} M_{p,\pi} Y_\pi, \quad \text{for all } p \in \Gamma. \tag{3}$$

Clearly, since M is proper, so is our algebraic system. Consequently, it follows by Theorem 5.5 that the strong solution of (3) is given by $(M^*)_{p,\epsilon}$, $p \in \Gamma$ (substituted for Y_p in the system (3)). It follows in particular that the component of the strong solution of (3) corresponding to $y^{p_0}_{q_0,t}$ is $((M^*)_{p_0,\epsilon})_{q_0,t} = \|\mathcal{P}\|$. □

One should observe that the variables $y^p_{q_1,q_2}$ in the proof of Theorem 5.13 correspond to the well-known triple construction $[q_1, p, q_2]$, used in the transition from (classical) pushdown automata to context-free grammars. The construction and the transition are originally due to Evey [4].

Example 5.14. Let $\Sigma = \{a, b\}$ and consider the proper $\mathbb{N}\langle\langle \Sigma \cup \{\epsilon\}\rangle\rangle$-pushdown automaton in Example 5.7. We construct a proper algebraic system with $\|\mathcal{P}\|$ as a component of its strong solution as follows. Let

$$Y = \{y^p_{q,q'} \mid p \in \{p_0, a, b\},\ q, q' \in \{q_1, q_2\}\}.$$

Let also

$$Y_{p_0} = \begin{pmatrix} y^{p_0}_{q_1,q_1} & y^{p_0}_{q_1,q_2} \\ y^{p_0}_{q_2,q_1} & y^{p_0}_{q_2,q_2} \end{pmatrix}, \qquad Y_a = \begin{pmatrix} y^a_{q_1,q_1} & y^a_{q_1,q_2} \\ y^a_{q_2,q_1} & y^a_{q_2,q_2} \end{pmatrix},$$

and consider the following algebraic system:

$$\begin{cases} Y_{p_0} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} Y_a, \\ Y_a = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} Y_a^2 + \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix}. \end{cases} \tag{4}$$

A simple calculation shows that (4) implies that $y^a_{q_2,q_1} = 0$, $y^a_{q_2,q_2} = b$, $y^a_{q_1,q_1} = a(y^a_{q_1,q_1})^2$, $y^a_{q_1,q_2} = ay^a_{q_1,q_1}y^a_{q_1,q_2} + ay^a_{q_1,q_2}b + b$. However, this implies that any solution of (4) will have 0 on the component corresponding to $y^a_{q_1,q_1}$ and so, we obtain that in (4), we may replace the equation corresponding to $y^a_{q_1,q_2}$ with the equation $y^a_{q_1,q_2} = ay^a_{q_1,q_2}b + b$. A suitable change of notation leads to the following proper algebraic system:

$$\begin{cases} z_1 = az_2, \\ z_2 = az_2b + b, \end{cases}$$

where $\|\mathcal{P}\|$ is the first component of its strong solution.

As noted already earlier in this chapter, any (classical) proper pushdown automaton may be seen as an $\mathbb{B}\langle\Sigma \cup \{\epsilon\}\rangle$- and as an $\mathbb{N}\langle\Sigma \cup \{\epsilon\}\rangle$-pushdown automaton. Based on this analogy and on Theorems 5.11 and 5.13, the following result may be proved.

Theorem 5.15.

(i) An ϵ-free language is context-free if and only if it is the behavior of a proper pushdown automaton.
(ii) For any epsilon-free context-free grammar G without chain rules, there exists a proper pushdown automaton $\mathcal{A}_G$ such that the ambiguity of any word $w \in \Sigma^*$ in $L(G)$ is $(\|\mathcal{A}_G\|, w)$.
(iii) For any proper pushdown automaton $\mathcal{A}$, there exists a context-free grammar $G_{\mathcal{A}}$ such that the ambiguity of any word $w \in \Sigma^*$ in $L(G_{\mathcal{A}})$ is $(\|\mathcal{A}\|, w)$.

6 Other Topics

Several other topics may be considered in connection with algebraic systems and pushdown automata. We mention here briefly two such topics. A result of Gruska [5] on a characterization of context-free languages may be generalized to a Kleene theorem for algebraic power series. One may prove (see [10] for a presentation in terms of complete semirings) that the algebraic power series coincide with the least equationally closed semiring containing all monomials. One may also consider the algebraic power series over the free commutative monoid $\Sigma^{\oplus}$ rather than Σ^*: this corresponds to the case where all variables are commuting. As it is well known from the theory of formal languages, the commuting case yields very different behavior; one example in this respect is the theorem of Parikh [16]. In the case of formal power series, several interesting decidability results may be given in the commutative case, based on tools from mathematical analysis and algebraic geometry. We refer to [12] for more details on the topic.

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