An Extension of the Knuth-Bendix Ordering with LPO-Like Properties

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Abstract. The Knuth-Bendix ordering is usually preferred over the lexicographic path ordering in successful implementations of resolution and superposition, but it is incompatible with certain requirements of hierarchic superposition calculi. Moreover, it does not allow non-linear definition equations to be oriented in a natural way. We present an extension of the Knuth-Bendix ordering that makes it possible to overcome these restrictions.

1 Introduction

In theorem proving calculi like Knuth-Bendix completion, resolution, or superposition, reduction orderings such as the Knuth-Bendix ordering (KBO) [11] or the lexicographic path ordering (LPO) by Kamin and Lévy [10] are crucial to reduce the search space. Among these orderings, the Knuth-Bendix ordering is usually preferred in state-of-the-art implementations of theorem provers. There are several reasons for this: it can be efficiently implemented – the most efficient known algorithm needs only linear time – and it correlates well with the sizes of terms; so, reductions w. r. t. a KBO usually lead to terms with fewer nodes. In comparison, computing term comparisons for the lexicographic path ordering requires at least quadratic time and reductions w. r. t. an LPO may result in arbitrarily larger terms.

On the other hand, it is exactly this correlation between the KBO and term sizes that renders the KBO incompatible with special requirements occurring in certain applications. One example is hierarchic theorem proving [3,6,15], where one considers two signatures $\Sigma \supseteq \Sigma_0$ and needs an ordering in which every ground term involving a symbol from $\Sigma \setminus \Sigma_0$ is larger than every ground term over Σ_0. With an LPO, this property is easy to establish, with a KBO it is usually impossible.

A second example is definitions of the form $f(t_1, \ldots, t_n) \approx t_0$ where f does not occur in t_0. Such definitions can easily be ordered from left to right using an LPO where f is larger in the precedence than every symbol occurring in t_0. With a KBO, however, we have the additional requirement that no variable occurs more often in t_0 than in $f(t_1, \ldots, t_n)$; non-linear definitions cannot be handled adequately using a KBO.

N. Dershowitz and A. Voronkov (Eds.): LPAR 2007, LNAI 4790, pp. 348–362, 2007.

In this paper, we present a variant of the Knuth-Bendix ordering that preserves as much as possible of the spirit of KBO, yet satisfies the requirements for hierarchic theorem proving and non-linear definitions. Like the original KBO, our ordering is a simplification ordering that can optionally be made total on ground terms. This is an essential property for theorem proving calculi such as superposition (Bachmair and Ganzinger [2]) and it goes beyond what can be shown using approaches for showing modular termination of rewrite systems (e. g., Fernández, Godoy, and Rubio [5]).

Due to lack of space we cannot give complete proofs in this paper, for which we refer the reader to (Ludwig [13]).

2 Preliminaries

We assume that the reader is familiar with standard concepts and notations in the area of rewriting (see Baader and Nipkow [1]). We use the notation $f/n \in \Sigma$ to denote that the signature Σ contains the n-ary function symbol f; if $n = 0$, f is also called a constant symbol. The set of terms over a signature Σ and a set X of variables is written $\mathrm{T}_\Sigma(X)$; $\mathrm{T}_\Sigma(\emptyset)$ is the set of ground terms over Σ. For a term $t \in \mathrm{T}_\Sigma(X)$, $|t|$ denotes the size, i. e., the number of nodes of t; if x is a variable in X, $|t|_x$ denotes the number of occurrences of x in t, and $\mathrm{P}(x, t)$ denotes the set of all positions of occurrences of x in t. Signatures are assumed to be finite.

Definition 1. Let X be a set of variables, let Σ be a signature, and let $\succ \subseteq \mathrm{T}_\Sigma(X) \times \mathrm{T}_\Sigma(X)$ be a binary relation on the terms over X and Σ. Then $\succ$ is said to be *compatible with Σ-operations*, if $s \succ s'$ implies $f(t_1, \ldots, t_{i-1}, s, t_{i+1}, \ldots, t_n) \succ f(t_1, \ldots, t_{i-1}, s', t_{i+1}, \ldots, t_n)$ for all symbols $f/n \in \Sigma$ with arity $n \in \mathbb{N}$, for all terms $s, s', t_1, \ldots, t_n \in \mathrm{T}_\Sigma(X)$ and for all coefficients $i \in \mathbb{N}, 1 \leq i \leq n$; $\succ$ is called *stable under substitutions* if $s \succ s'$ implies $s\sigma \succ s'\sigma$ for all terms $s, s' \in \mathrm{T}_\Sigma(X)$ and for all substitutions $\sigma\colon X \to \mathrm{T}_\Sigma(X)$. The relation $\succ$ has the *subterm property* if $s \succ s'$ whenever s' is a proper subterm of s; $\succ$ is a *rewrite relation* if $\succ$ is compatible with Σ-operations and stable under substitutions; it is a *rewrite ordering* if it is a strict partial ordering and a rewrite relation; it is a *simplification ordering* if $\succ$ is a rewrite ordering and has the subterm property.

The Knuth-Bendix ordering (KBO) is an example of a simplification ordering. It is parameterized by a "precedence" on signature symbols, and a weight function. The KBO was originally introduced by Knuth and Bendix [11] with a stricter variable condition; the version presented in this document can be found in (Dick, Kalmus, and Martin [4]) and also in (Baader and Nipkow [1]).[1]

First of all, in order to develop later a function that computes the weight of terms, we need to assign weights to signature symbols, which will be natural numbers in the case of the KBO.

[1] In fact, Dick, Kalmus, and Martin [4] and Baader and Nipkow [1] also permit positive real coefficients.

Let Σ be a signature, let X be a set of variables and let $\sqsupset$ be a strict partial ordering on Σ. A *(regular) symbol weight assignment* is a function $\lambda\colon \Sigma \cup X \to \mathbb{N}$. It is called *admissible for* $\sqsupset$ if the following two conditions are satisfied:

(i) There exists a $\lambda_0 \in \mathbb{N}^{>0}$ such that for all $x \in X$: $\lambda(x) = \lambda_0$ and for all $c/0 \in \Sigma$: $\lambda(c) \geq \lambda_0$.
(ii) If there exists an $f/1 \in \Sigma$ such that $\lambda(f) = 0$, then $f \sqsupseteq g$ for all $g \in \Sigma$.

A symbol weight assignment λ is extended recursively to a weight function $\mathrm{w}_\lambda\colon \mathrm{T}_\Sigma(X) \to \mathbb{N}$ on terms as follows:

- $\mathrm{w}_\lambda(x) = \lambda(x)$ for $x \in X$.
- $\mathrm{w}_\lambda\big(f(t_1, \ldots, t_n)\big) = \lambda(f) + \sum_{i=1}^{n} \mathrm{w}_\lambda(t_i)$ for $f/n \in \Sigma$, $n \in \mathbb{N}$.

At first, the Knuth-Bendix ordering compares two terms by using the weight function. If both terms have the same weight, the precedence is considered, and only ultimately, if the top symbol is equal as well, recursion is used to compare two terms.

Definition 2 (Knuth-Bendix Ordering). Let Σ be a signature and let X be a set of variables. Additionally, let $\sqsupset$ be a strict partial ordering, the precedence, on Σ and $\lambda\colon \Sigma \cup X \to \mathbb{N}$ be a regular symbol weight assignment that is admissible for $\sqsupset$. Finally, let $\mathrm{w} = \mathrm{w}_\lambda\colon \mathrm{T}_\Sigma(X) \to \mathbb{N}$ be the regular term weight function induced by λ.

We define the *Knuth-Bendix ordering* $\succ_{\mathrm{KBO}} \subseteq \mathrm{T}_\Sigma(X) \times \mathrm{T}_\Sigma(X)$ induced by $(\sqsupset, \lambda)$ on terms $s, t \in \mathrm{T}_\Sigma(X)$ in the following way: $s \succ_{\mathrm{KBO}} t$ if

(KBO1) $\forall x \in X\colon\ |s|_x \geq |t|_x$ and $\mathrm{w}(s) > \mathrm{w}(t)$,
or
(KBO2) $\forall x \in X\colon\ |s|_x \geq |t|_x$, $\mathrm{w}(s) = \mathrm{w}(t)$ and one of the following cases holds:
(KBO2a) $\exists\, f/1 \in \Sigma$, $\exists\, x \in X$, $\exists\, n \in \mathbb{N}^{>0}$ such that $s = f^n(x)$ and $t = x$,
(KBO2b) $\exists\, f/m, g/n \in \Sigma$ $(m, n \in \mathbb{N})$, $\exists\, s_1, \ldots, s_m, t_1, \ldots, t_n \in \mathrm{T}_\Sigma(X)$ such that $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$ with $f \sqsupset g$,
(KBO2c) $\exists\, f/m \in \Sigma$ $(m \in \mathbb{N}^{>0})$, $\exists\, s_1, \ldots, s_m, t_1, \ldots, t_m \in \mathrm{T}_\Sigma(X)$, $\exists\, i, 1 \leq i \leq m$ such that $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$ and such that $s_1 = t_1, \ldots, s_{i-1} = t_{i-1}$, $s_i \succ_{\mathrm{KBO}} t_i$.

The Knuth-Bendix ordering is a simplification ordering; moreover, if the precedence is a total ordering, it is total on ground terms. It can be computed in time $O(|s| + |t|)$, where s and t are the terms to be compared (Löchner [12]).[2]

3 Transfinite KBO

3.1 Motivation

The Knuth-Bendix ordering correlates well with the sizes of terms, which is often a desirable property, since it implies that reductions w. r. t. a KBO usually

[2] Using a machine model in which addition of numbers takes constant time.

lead to terms with fewer nodes. On the other hand, it is exactly this correlation between the KBO and term sizes that renders the KBO incompatible with some special requirements for certain applications.

One example is the problem of orienting definition equations in an intuitive direction. Suppose that we are given a sequence of signatures Σ_i $(0 \leq i \leq n)$ where $\Sigma_i = \{f_i\} \cup \Sigma_{i-1}$ for $i \geq 1$, and that we have a set of non-recursive definition equations of the form $f_i(s_{i1}, \ldots, s_{ik}) \approx t_i$, with $t_i, s_{ij} \in \mathrm{T}_{\Sigma_{i-1}}(X)$ and $\mathrm{Var}(t_i) \subseteq \mathrm{Var}(f_i(s_{i1}, \ldots, s_{ik}))$ (where the s_{ij} are often, but not necessarily, variables). If we use a lexicographic path ordering with a precedence $f_n \sqsupset \cdots \sqsupset f_2 \sqsupset f_1 \sqsupset \ldots$, then every term t with a top symbol f_i is larger than every term in $\mathrm{T}_{\Sigma_{i-1}}(\mathrm{Var}(t))$, i. e., all these equations can be oriented from left to right (and can hopefully be used to eliminate all occurrences of the f_i in the remainder of the specification completely). If we try to get a similar effect with a KBO, we face two problems: the KBO correlates with term sizes, so in general, a term cannot be larger than *every* term over some subsignature, and moreover a term cannot be larger than another term in which some variable occurs more often.

Another scenario where the Knuth-Bendix ordering does not work satisfactorily is hierarchic theorem proving. Standard first-order theorem provers are notoriously bad at dealing with integer or real arithmetic – encoding numbers in binary or unary is not really a viable solution in most application contexts. A hierarchic proof system adds theory knowledge to a saturation-based calculus by using a proof system for a base theory, say, a decision procedure for real arithmetic as a black box. The proof system is initially given a set of formulas over some extension of the base theory, e. g., over real arithmetic extended with data structures, free function symbols, etc. As usual, the deduction rules of the calculus are employed to generate formulas from premises and the conclusions are added to the set of formulas; in addition, all derived formulas belonging to the base domain are passed to the decision procedure. As soon as one of the two systems encounters a contradiction, the problem is solved. (Bachmair, Ganzinger, and Waldmann [3], Ganzinger, Sofronie-Stokkermans, and Waldmann [6], Prevosto and Waldmann [15]).

In hierarchic theorem proving calculi, one usually considers a signature Σ_0 of base symbols and a signature $\Sigma \supseteq \Sigma_0$ that extends Σ_0. Similarly to the ordinary superposition calculus, hierarchic superposition calculi are parameterized by a reduction ordering $\succ$ that is total on ground terms. In order to ensure refutational completeness, this ordering must have the property that every ground term in $\mathrm{T}_{\Sigma_0}(\emptyset)$ is strictly smaller than every ground term in $\mathrm{T}_{\Sigma}(\emptyset) \setminus \mathrm{T}_{\Sigma_0}(\emptyset)$. This requirement is easy to establish with an LPO – the precedence just needs to be defined in such a way that all the symbols from Σ_0 are smaller than all the symbols from Σ – but it is generally incompatible with the definition of the Knuth-Bendix ordering.[3]

Our goal is to find a computable (total) simplification ordering that generalises the KBO and satisfies the requirements of hierarchic theorem proving. We will

[3] Except if Σ_0 consists only of constant symbols and at most one unary function symbol.

show that such an ordering can be constructed using certain ordinal numbers as weights.

In the next sections, we start by presenting a very general version of the ordering, which is computable, but unfortunately not very efficiently. Restrictions that lead to a better runtime behaviour are discussed later.

3.2 Ordinal Numbers

A set α is an ordinal if α is totally ordered with respect to the subset relation and every element of α is also a subset of α. The class of all ordinals is denoted by **ON**. Ordinals are ordered by the element relation, or equivalently, by the subset relation, i. e., $\alpha < \beta$ if and only if $\alpha \in \beta$ if and only if $\alpha \subsetneq \beta$.

If a non-empty ordinal β has a largest element α, then it can be written as $\beta = \alpha \cup \{\alpha\}$. We say that β is the successor of α, denoted by $\beta = S(\alpha)$. A non-empty ordinal γ that is not a successor of another ordinal is called a limit ordinal. Every limit ordinal γ is the union (or least upper bound) of all ordinals that are smaller than γ.

The ordinals $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, and so on, are identified with the natural numbers $0, 1, 2, \ldots$. The smallest limit ordinal is denoted by ω, it corresponds to the set of all natural numbers.

The following operations on ordinal numbers can be seen as the standard extensions of the addition, multiplication and exponentiation on natural numbers, in particular they coincide with the latter if their arguments are natural numbers. For more information we refer to (Just and Weese [9]).

Definition 3 (Regular Ordinal Operations). Let $\alpha, \beta \in \mathbf{ON}$ be ordinals. The ordinal $\alpha + \beta$, $\alpha \cdot \beta$, and α^β are defined by recursion over β:

$$
\begin{aligned}
&\alpha + 0 = \alpha \\
&\alpha + \beta = S(\alpha + \gamma) \text{ if } \beta = S(\gamma) \\
&\alpha + \beta = \textstyle\bigcup_{\gamma<\beta} (\alpha + \gamma) \text{ if } \beta \text{ is a limit ordinal} > 0
\end{aligned}
$$

$$
\begin{aligned}
&\alpha \cdot 0 = 0 \\
&\alpha \cdot \beta = (\alpha \cdot \gamma) + \alpha \text{ if } \beta = S(\gamma) \\
&\alpha \cdot \beta = \textstyle\bigcup_{\gamma<\beta} (\alpha \cdot \gamma) \text{ if } \beta \text{ is a limit ordinal} > 0
\end{aligned}
$$

$$
\begin{aligned}
&\alpha^0 = 1 \\
&\alpha^\beta = \alpha^\gamma \cdot \alpha \text{ if } \beta = S(\gamma) \\
&\alpha^\beta = \begin{cases} 0 & \text{if } \beta \text{ is a limit ordinal} > 0 \text{ and } \alpha = 0 \\ \bigcup_{\gamma<\beta} \alpha^\gamma & \text{if } \beta \text{ is a limit ordinal} > 0 \text{ and } \alpha > 0 \end{cases}
\end{aligned}
$$

We cannot use the regular operations on ordinals to compute the weight of a term from the weights of its subterms, which is mainly due to the fact that ordinal addition and multiplication are only monotonic, but not strictly monotonic with respect to $>$. For instance, $1 > 0$, but $1 + \omega = \omega = 0 + \omega$, so $\alpha > \alpha'$ does in general not imply $\alpha + \beta > \alpha' + \beta$. There is an alternative set of operations on ordinals that is better suited for our purposes. Let us first define a subset of the ordinal numbers on which we will define those operations:

Definition 4 (Set O). We define the set $\mathbf{O} \subseteq \mathbf{ON}$ inductively as follows:

- $0 \in \mathbf{O}$.
- If there exists an $m \in \mathbb{N}^{>0}$, $n_1, \ldots, n_m \in \mathbb{N}^{>0}$, and $\delta_1, \ldots, \delta_m \in \mathbf{O}$ with $\delta_1 > \delta_2 > \cdots > \delta_m$, then

$$\sum_{i=1}^{m} (\omega^{\delta_i} \cdot n_i) \in \mathbf{O}.$$

The set $\mathbf{O}$ exactly contains those ordinals that are smaller than ε_0, where the limit ordinal ε_0 is the smallest ordinal such that $\varepsilon_0 = \omega^{\varepsilon_0}$. Elements of $\mathbf{O}$ are sums of finite sequences of ordinals $\omega^{\beta_i} \cdot n_i$, which we call the *basic building blocks*. The decomposition of an ordinal α into a sum $\sum_{i=1}^{m} (\omega^{\delta_i} \cdot n_i)$ with $\delta_1 > \delta_2 > \cdots > \delta_m$ is called the Cantor normal form of α; it is unique. We define

- $\deg(\alpha) = \delta_1$,
- $\text{Exponents}(\alpha) = \{\delta_1, \delta_2, \ldots, \delta_m\}$,
- $\text{coeff}(\alpha, \beta) = \begin{cases} n_i & \text{if } \beta = \delta_i \text{ for some } i \text{ with } 1 \leq i \leq m, \\ 0 & \text{otherwise.} \end{cases}$

For $\alpha = 0$, we define $\deg(\alpha) = -\infty$, $\text{Exponents}(\alpha) = \emptyset$, and $\text{coeff}(\alpha, \beta) = 0$.

The following operations on ordinals were introduced by Hessenberg [7]. Intuitively, they add and multiply ordinals in $\mathbf{O}$ as if they were polynomials in ω.

Definition 5 (Hessenberg Addition). The function $\oplus \colon \mathbf{O} \times \mathbf{O} \to \mathbf{O}$ is defined as follows:

- $0 \oplus \alpha = \alpha$ for $\alpha \in \mathbf{O}$.
- $\alpha \oplus 0 = \alpha$ for $\alpha \in \mathbf{O}$.
- Suppose that

$$\alpha = \sum_{i=1}^{m} (\omega^{\delta_i} \cdot n_i), \quad \beta = \sum_{i=1}^{m'} (\omega^{\delta'_i} \cdot n'_i) \in \mathbf{O}$$

  for natural numbers $m, m' \in \mathbb{N}^{>0}$, $n_1, \ldots, n_m$, $n'_1, \ldots, n'_{m'} \in \mathbb{N}^{>0}$, ordinals $\delta_1, \ldots, \delta_m$, $\delta'_1, \ldots, \delta'_{m'} \in \mathbf{O}$ such that $\delta_1 > \delta_2 > \cdots > \delta_m$ and $\delta'_1 > \delta'_2 > \cdots > \delta'_{m'}$. Then

$$\alpha \oplus \beta = \sum_{i=1}^{m''} \Big(\omega^{c_i} \cdot \big(\text{coeff}(\alpha, c_i) + \text{coeff}(\beta, c_i)\big)\Big)$$

  where we set $\text{Exponents}(\alpha) \cup \text{Exponents}(\beta) = \{c_1, c_2, \ldots, c_{m''}\}$ such that $m'' \in \mathbb{N}$ and $c_1 > c_2 > \cdots > c_{m''}$.

Definition 6 (Hessenberg Multiplication). The function $\odot \colon \mathbf{O} \times \mathbf{O} \to \mathbf{O}$ is defined as follows:

- $0 \odot \alpha = 0$ for $\alpha \in \mathbf{O}$.
- $\alpha \odot 0 = 0$ for $\alpha \in \mathbf{O}$.

– Suppose that

$$\alpha = \sum_{i=1}^{m} (\omega^{\delta_i} \cdot n_i), \quad \beta = \sum_{j=1}^{m'} (\omega^{\delta'_j} \cdot n'_j)$$

for $m, m' \in \mathbb{N}^{>0}$, $n_1, \ldots, n_m$, $n'_1, \ldots, n'_{m'} \in \mathbb{N}^{>0}$, $\delta_1, \ldots, \delta_m$, $\delta'_1, \ldots, \delta'_{m'} \in \mathbf{O}$ such that $\delta_1 > \delta_2 > \cdots > \delta_m$ and $\delta'_1 > \delta'_2 > \cdots > \delta'_{m'}$. Then

$$\alpha \odot \beta = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m'} \Big(\omega^{\delta_i \oplus \delta'_j} \cdot \big(\mathrm{coeff}(\alpha,\, \delta_i) \cdot \mathrm{coeff}(\beta,\, \delta'_j)\big)\Big).$$

Lemma 7. *The following properties hold for all $\alpha, \beta, \gamma \in \mathbf{O}$:*

- $\alpha \oplus \beta = \beta \oplus \alpha$.
- $\alpha \odot \beta = \beta \odot \alpha$.
- $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$.
- $\alpha \odot (\beta \odot \gamma) = (\alpha \odot \beta) \odot \gamma$.
- $\alpha \odot (\beta \oplus \gamma) = \alpha \odot \beta \oplus \alpha \odot \gamma$.
- $\alpha < \beta$ *implies* $\alpha \oplus \gamma < \beta \oplus \gamma$.
- $\alpha < \beta$ *and* $\gamma > 0$ *imply* $\alpha \odot \gamma < \beta \odot \gamma$.

It is important to note that the Hessenberg addition $\oplus$ on the set $\mathbf{O}$ does not possess the continuity property, i. e., for two ordinals $\alpha, \beta \in \mathbf{O}$ such that $\alpha < \beta$ there does not necessarily exist an ordinal $\gamma \in \mathbf{O}$ such that $\alpha \oplus \gamma = \beta$. A simple example consists in the two ordinals 1 and ω: there is no ordinal $\alpha \in \mathbf{O}$ such that $1 \oplus \alpha = \omega$. This fact makes the proof of the following lemma (which is essential for proving that our ordering is stable under substitutions) rather tedious:

Lemma 8. *Let $\alpha, \beta, \gamma, \delta, \zeta \in \mathbf{O}$ be ordinals such that $\beta \leq \zeta$ and*

$$\alpha \oplus (\beta \odot \gamma) < \delta \oplus (\zeta \odot \gamma).$$

Furthermore, let $\eta \in \mathbf{O}$ be an ordinal such that $\deg(\eta) > \deg(\gamma)$. Then

$$\alpha \oplus (\beta \odot \eta) < \delta \oplus (\zeta \odot \eta).$$

3.3 Constructing the Ordering

We start by introducing two functions. Firstly, we assign an ordinal number, the so-called *symbol weight*, to every symbol in the signature and to every variable.

Definition 9 (Ordinal Symbol Weight Assignment). Let X be a set of variables and Σ be a signature. Then, an *ordinal symbol weight assignment* is a function $\Omega\colon \Sigma \cup X \to \mathbf{O}$.

Adding up ordinals as weights is sufficient for hierarchic theorem proving, but it is not sufficient for dealing with non-linear definitions. In addition, we need a factor for each signature symbol, called *subterm coefficient*, with which we multiply the weights of subterms before the weight of the top symbol is added:[4]

[4] The idea of multiplying weights of subterms by some factor can also be found in Otter's ad hoc ordering [14] (which is in general not a reduction ordering, though).

Definition 10 (Subterm Coefficient Function). Let Σ be a signature. Then, a *subterm coefficient function* is a mapping $\Psi\colon \Sigma \to \mathbf{O} \setminus \{0\}$.

Using the two previous definitions, we can construct a function that computes the (ordinal) weight of terms.

Definition 11 (Ordinal Term Weight). Let X be a set of variables and Σ be a signature. Furthermore, let $\Omega\colon \Sigma \cup X \to \mathbf{O}$ be an ordinal symbol weight assignment and $\Psi\colon \Sigma \to \mathbf{O} \setminus \{0\}$ be a subterm coefficient function. Then, we inductively define a function

$$\mathrm{W} = \mathrm{W}_{(\Omega,\,\Psi)}\colon \mathrm{T}_\Sigma(X) \to \mathbf{O}$$

which computes the *(ordinal) weight of a term* in the following way:

- For $x \in X$:
$$\mathrm{W}_{(\Omega,\,\Psi)}(x) = \Omega(x).$$
- For $n \in \mathbb{N}$ and terms $t_1, \ldots, t_n \in \mathrm{T}_\Sigma(X)$:
$$\mathrm{W}_{(\Omega,\,\Psi)}\big(f(t_1, \ldots, t_n)\big) = \Omega(f) \oplus \Big(\Psi(f) \odot \bigoplus_{i=1}^{n} \mathrm{W}_{(\Omega,\,\Psi)}(t_i)\Big).$$

We define now when an ordinal symbol weight assignment is *admissible* for a strict partial ordering on signature symbols.

Definition 12 (Admissible Symbol Weight Assignment). Let Σ be a signature, X be a set of variables and $\sqsupset$ be a strict partial ordering on Σ. We say then that the symbol weight assignment $\Omega\colon \Sigma \cup X \to \mathbf{O}$ is *admissible* for the ordering $\sqsupset$ if the following two conditions are satisfied:

(i) There exists an $\Omega_0 \in \mathbb{N}^{>0}$ such that for all $x \in X$: $\Omega(x) = \Omega_0$ and for all $c/0 \in \Sigma$: $\Omega(c) \geq \Omega_0$.
(ii) If there exists an $f/1 \in \Sigma$ such that $\Omega(f) = 0$, then $f \sqsupseteq g$ for all $g \in \Sigma$.

The coefficient of a position p in a term t is the product of all the coefficients of the function symbols on the path from the root of t to p:

Definition 13 (Coefficient of a Position). Let X be a set of variables and Σ be a signature. Furthermore, let $\Psi\colon \Sigma \to \mathbf{O} \setminus \{0\}$ be a subterm coefficient function, $t \in \mathrm{T}_\Sigma(X)$ be a term and $p \in \mathrm{pos}(t)$ be a position in t. We inductively define the coefficient $\mathcal{C}(p,\, t)$ of p in t as follows:

- $\mathcal{C}(\Lambda,\, t) = 1$, where Λ is the empty string.
- If $t = f(t_1, \ldots, t_n)$ for $n \in \mathbb{N}^{>0}$, terms $t_1, \ldots, t_n \in \mathrm{T}_\Sigma(X)$ and a position $p = ip'$ such that $1 \leq i \leq n$ and $p' \in \mathrm{pos}(t_i)$, then
$$\mathcal{C}(p,\, t) = \mathcal{C}\big(ip',\, f(t_1, \ldots, t_n)\big) = \Psi(f) \odot \mathcal{C}(p',\, t_i).$$

We can now define the *transfinite Knuth-Bendix ordering* (TKBO). Compared with the definition of the regular Knuth-Bendix ordering (KBO) (Def. 2) the variable occurrence condition is replaced by two separate conditions on term variables and coefficient sums. It is then possible for a smaller term (with respect to the TKBO) to contain a specific variable more often than the corresponding larger term, which allows to order non-linear term definitions. Note that we obtain the usual variable condition as a special case if $\Psi(f) = 1$ for every symbol f.

Definition 14 (Transfinite Knuth-Bendix Ordering). Let Σ be a signature and X be a set of variables. Additionally, let $\sqsupset$ be a strict ordering on Σ, let $\Omega\colon \Sigma \cup X \to \mathbf{O}$ be an ordinal symbol weight assignment that is admissible for $\sqsupset$ and let $\Psi\colon \Sigma \to \mathbf{O} \setminus \{0\}$ be a subterm coefficient function. Finally, let $\mathrm{W} = \mathrm{W}_{(\Omega,\Psi)}\colon \mathrm{T}_\Sigma(X) \to \mathbf{O}$ be the ordinal term weight function induced by Ω and Ψ.

We define the *transfinite Knuth-Bendix ordering* (TKBO) $\succ_\mathrm{T} \subseteq \mathrm{T}_\Sigma(X) \times \mathrm{T}_\Sigma(X)$ induced by $(\sqsupset, \Omega, \Psi)$ on terms $s, t \in \mathrm{T}_\Sigma(X)$ in the following way:
$s \succ_\mathrm{T} t$ if

(TKBO1) $\mathrm{Var}(s) \supseteq \mathrm{Var}(t)$, $\mathrm{W}(s) > \mathrm{W}(t)$ and

$$\forall\, x \in \mathrm{Var}(t)\colon \bigoplus_{p \in \mathrm{P}(x,\, s)} \mathcal{C}(p,\, s) \geq \bigoplus_{p \in \mathrm{P}(x,\, t)} \mathcal{C}(p,\, t)$$

or
(TKBO2) $\mathrm{Var}(s) \supseteq \mathrm{Var}(t)$, $\mathrm{W}(s) = \mathrm{W}(t)$,

$$\forall\, x \in \mathrm{Var}(t)\colon \bigoplus_{p \in \mathrm{P}(x,\, s)} \mathcal{C}(p,\, s) \geq \bigoplus_{p \in \mathrm{P}(x,\, t)} \mathcal{C}(p,\, t)$$

and one of the following cases occurs:
(TKBO2a) $\exists\, f/1 \in \Sigma$, $\exists\, x \in X$, $\exists\, n \in \mathbb{N}^{>0}$ such that $s = f^n(x)$ and $t = x$,
(TKBO2b) $\exists\, f/m, g/n \in \Sigma$ $(m, n \in \mathbb{N})$, $\exists\, s_1, \dots, s_m, t_1, \dots, t_n \in \mathrm{T}_\Sigma(X)$ such that $s = f(s_1, \dots, s_m)$, $t = g(t_1, \dots, t_n)$ with $f \sqsupset g$,
(TKBO2c) $\exists\, f/m \in \Sigma$ $(m \in \mathbb{N}^{>0})$, $\exists\, s_1, \dots, s_m, t_1, \dots, t_m \in \mathrm{T}_\Sigma(X)$, $\exists\, i \in \mathbb{N}, 1 \leq i \leq m$ such that $s = f(s_1, \dots, s_m)$, $t = f(t_1, \dots, t_m)$ with $s_1 = t_1, \dots, s_{i-1} = t_{i-1}$, $s_i \succ_\mathrm{T} t_i$.

Example 15. Let $\Omega(x) = 1$, $\Omega(h) = \Psi(h) = 1$, $\Omega(g) = \Psi(g) = \omega$, $\Omega(f) = \Psi(f) = \omega^\omega$. Let $s = f(h(x))$ and $t = g(g(x,x), g(x,x))$. Then $\mathrm{W}(s) = \omega^\omega \cdot 3 > \mathrm{W}(t) = \omega^2 \cdot 6 + \omega$. Furthermore $\bigoplus_{p \in \mathrm{P}(x,\, s)} \mathcal{C}(p,\, s) = \omega^\omega > \bigoplus_{p \in \mathrm{P}(x,\, t)} \mathcal{C}(p,\, t) = \omega^2 \cdot 4$. Hence $f(h(x)) \succ_\mathrm{T} g(g(x,x), g(x,x))$ by (TKBO1).

The following two theorems are proved analogously to the corresponding theorems for KBO (e.g. in Baader and Nipkow [1]):

Theorem 16. *The transfinite Knuth-Bendix ordering $\succ_T$ is a simplification ordering.*

Theorem 17. *If the precedence $\sqsupset$ is a total ordering, then the transfinite Knuth-Bendix ordering $\succ_T$ is total on ground terms.*

For terms built from symbols with subterm coefficient 1 and natural numbers as weights, the transfinite Knuth-Bendix ordering $\succ_{\mathrm{T}}$ agrees with $\succ_{\mathrm{KBO}}$:

Theorem 18. *Let Σ_0 be a subsignature of Σ such that $\Psi(f) = 1$ and $\Omega(f) \in \mathbb{N}$ for all $f \in \Sigma_0$. Let $\succ_{KBO}$ be the regular Knuth-Bendix ordering on $\mathrm{T}_{\Sigma_0}(X)$ with $\lambda(f) = \Omega(f)$ for $f \in \Sigma_0$. Then, for all terms $s, t \in \mathrm{T}_{\Sigma_0}(X)$, $s \succ_T t$ if and only if $s \succ_{KBO} t$.*

4 Ordering Definition Equations

The TKBO is able to orient every set of non-recursive, but possibly non-linear definition equations from left to right, i.e. in an intuitive way. Moreover, if the set of definition equations is finite and given a priori, this is possible even with natural numbers as weights and subterm coefficients:

Suppose that we have a sequence of signatures Σ_i $(0 \leq i \leq n)$ where $\Sigma_i = \{f_i\} \cup \Sigma_{i-1}$ for $i \geq 1$, and that we have a set of non-recursive definition equations of the form $f_i(s_1, \ldots, s_k) \approx t$, with $t, s_j \in \mathrm{T}_{\Sigma_{i-1}}(X)$ and $\mathrm{Var}(t) \subseteq \mathrm{Var}$ $(f_i(s_1, \ldots, s_k))$ (where the s_j are not necessarily variables). We start with arbitrary natural numbers as weights and subterm coefficients for the symbols in Σ_0. Then, for $i = 1, \ldots, n$, we recursively choose $\Omega(f_i)$ and $\Psi(f_i)$ in such a way that $\Omega(f_i) > \mathrm{W}(t)$ and $\Psi(f_i) \geq \max_{x \in \mathrm{Var}(t)} \bigl(\sum_{p \in \mathrm{P}(x,t)} \mathcal{C}(p,t)\bigr)$ for every definition equation $f_i(s_1, \ldots, s_k) \approx t$ for f_i. It is clear that this construction implies $f_i(s_1, \ldots, s_k) \succ_{\mathrm{T}} t$ by condition (TKBO1).

If we want to have the property that then *every* term t with a top symbol f_i is larger than *every* term in $\mathrm{T}_{\Sigma_{i-1}}(\mathrm{Var}(t))$, this is still possible with the transfinite Knuth-Bendix ordering, but now we have to use ordinal numbers beyond ω:

Theorem 19. *Let Σ_0 be a subsignature of Σ and let $i \in \mathbb{N}$ such that $\Psi(f) < \omega^{\omega^i}$ and $\Omega(f) < \omega^{\omega^i}$ for all $f \in \Sigma_0$ and $\Psi(f) \geq \omega^{\omega^i}$ and $\Omega(f) \geq \omega^{\omega^i}$ for all $f \in \Sigma \setminus \Sigma_0$. Let s be a term with top symbol in $\Sigma \setminus \Sigma_0$, let $t \in \mathrm{T}_{\Sigma_0}(\mathrm{Var}(s))$ be a term over Σ_0 and the variables of s. Then $s \succ_T t$ holds.*

Corollary 20. *If we have a a sequence of signatures Σ_i, $i = 0, \ldots, n$, where $\Sigma_i = \{f_i\} \cup \Sigma_{i-1}$ for $i \geq 1$, and an arbitrary KBO $\succ_{KBO}$ on $\mathrm{T}_{\Sigma_0}(X)$ with weights in $\mathbb{N}$, then defining $\Psi(f_i) = \Omega(f_i) = \omega^{\omega^i}$ yields a transfinite KBO that agrees with $\succ_{KBO}$ on $\mathrm{T}_{\Sigma_0}(X)$ and in which moreover every term s with top symbol f_i is larger than every term in $\mathrm{T}_{\Sigma_{i-1}}(\mathrm{Var}(s))$.*

It is clear that the transfinite Knuth-Bendix ordering is computable: Ordinals from **O** can easily be encoded as nested list structures, on which the Hessenberg operations can be performed. Neither addition nor multiplication can be performed in constant time, though. Consequently, the efficiency advantage of the KBO over the LPO is essentially lost, and Cor. 20 is mostly a theoretical result. On the other hand, both the criteria from Thm. 18 and from Thm. 19 can be efficiently checked, and together, they are often sufficient in practice. Cor. 20 then ensures that completeness proofs, etc., which require the existence of a reduction ordering total on ground terms still hold.

5 Hierarchic KBO

5.1 Simple Simplification Orderings

As mentioned earlier, for refutational completeness a hierarchic proof calculus that operates on a base signature Σ_0 and an extension $\Sigma \supseteq \Sigma_0$ of Σ_0 needs a reduction ordering $\succ$ that is total on ground terms and has the property that every ground term in $\mathrm{T}_{\Sigma_0}(\emptyset)$ is strictly smaller than every ground term in $\mathrm{T}_{\Sigma}(\emptyset) \setminus \mathrm{T}_{\Sigma_0}(\emptyset)$. It is easy to see that the transfinite Knuth-Bendix ordering satisfies this property, for instance, if the weight symbol assignment Ω maps every symbol in Σ_0 to a natural number and every symbol in $\Sigma \setminus \Sigma_0$ to an ordinal number $\omega \cdot m + n$ with $m > 0$, and if $\Psi(f) \in \mathbb{N}$ for all $f \in \Sigma$. Note that ordinals of the form $\omega \cdot m + n$ with $m \geq 0$, $n \geq 0$, can be written as tuples (m, n); the Hessenberg addition then corresponds to the componentwise addition of tuples, the Hessenberg multiplication with positive integers to scalar multiplication, and the ordering on ordinals is equivalent to the lexicographic ordering over $\mathbb{N} \times \mathbb{N}$.[5]

Moreover, a small refinement of the transfinite Knuth-Bendix ordering for the hierarchic case is possible: In hierarchic superposition calculi there may be variables for which we only have to consider instantiations with terms from $\mathrm{T}_{\Sigma_0}(X)$ and other variables for which we only have to consider instantiations with terms from $\mathrm{T}_{\Sigma}(X) \setminus \mathrm{T}_{\Sigma_0}(X)$.[6] This motivates a relaxation of the definitions of reduction and simplification orderings.

Definition 21 (Simple Substitution). Let Σ_0, Σ be two signatures such that $\Sigma_0 \subseteq \Sigma$ and let X_l, X_s, X_u be disjoint sets of variables with $X = X_\mathrm{l} \cup X_\mathrm{s} \cup X_\mathrm{u}$. We say that a substitution $\sigma\colon X \to \mathrm{T}_{\Sigma}(X)$ is a *simple substitution* for $(X_\mathrm{l}, X_\mathrm{s}, X_\mathrm{u})$ and $\Sigma_0 \subseteq \Sigma$ if $\sigma(x) \in \mathrm{T}_{\Sigma_0}(X_\mathrm{s})$ for all $x \in X_\mathrm{s}$ and $\sigma(x) \in \mathrm{T}_{\Sigma}(X) \setminus \mathrm{T}_{\Sigma_0}(X_\mathrm{s} \cup X_\mathrm{u})$ for all $x \in X_\mathrm{l}$.

In other words, variables in X_s ("small variables") may only be mapped to terms over base symbols and small variables ("small terms"); variables in X_l ("large variables") may only be mapped to terms containing at least one proper extension symbol or large variable ("large terms"); for variables in X_u ("unspecified variables") there is no restriction.

The next definition is analogous to Def. 1 and introduces the concept of simple simplification orderings.

[5] A very restricted case of such a behaviour can also be found in the DomPred mechanism implemented in Vampire [16] and SPASS [17].

[6] A similar requirement appears in superposition for finite domains (Hillenbrand and Weidenbach [8]), where it is sufficient to consider substitutions that map variables to a given set of constant symbols (which are smaller than complex terms).

Definition 22 (Simple Simplification Ordering). Let Σ_0, Σ be two signatures such that $\Sigma_0 \subseteq \Sigma$ and let X_l, X_s, X_u be disjoint sets of variables with $X = X_\mathrm{l} \cup X_\mathrm{s} \cup X_\mathrm{u}$. Furthermore, let $\succ \subseteq \mathrm{T}_\Sigma(X) \times \mathrm{T}_\Sigma(X)$ be a binary relation on terms. Then we say that

- $\succ$ is *stable under simple substitutions* if for all terms $s, s' \in \mathrm{T}_\Sigma(X)$ and for all simple substitutions $\sigma \in \mathrm{Subst}_{X,\Sigma}$ for $(X_\mathrm{l}, X_\mathrm{s}, X_\mathrm{u})$ and $\Sigma_0 \subseteq \Sigma$ it holds that $s \succ s' \implies s\sigma \succ s'\sigma$,
- $\succ$ is a *simple rewrite relation* if $\succ$ is compatible with Σ-operations and stable under simple substitutions,
- $\succ$ is a *simple rewrite ordering* if $\succ$ is a strict partial ordering and a simple rewrite relation,
- $\succ$ is a *simple simplification ordering* if $\succ$ is a simple rewrite ordering and has the subterm property.

Note that if $X_\mathrm{s} = X_\mathrm{l} = \emptyset$, the notion of simple simplification ordering coincides with the notion of (regular) simplification ordering.

5.2 Constructing the Ordering

In order to turn the transfinite Knuth-Bendix ordering into a simple simplification ordering, we use ordinals of the form $\omega \cdot m + n$ with $m \geq 0$, $n \geq 0$ as weights and positive integers as subterm coefficients.

Definition 23 (Admissible Hierarchic Symbol Weight Assignment). Let Σ_0, Σ be signatures such that $\Sigma_0 \subseteq \Sigma$ and let X_l, X_s, X_u be pairwise disjoint sets of variables with $X = X_\mathrm{l} \cup X_\mathrm{s} \cup X_\mathrm{u}$. Additionally, let $\sqsupset$ be a strict partial ordering on Σ and let $\Omega\colon \Sigma \cup X \to \mathbf{O}$ be a symbol weight assignment. We say that Ω *is admissible for* $\sqsupset$ *and* Σ_0 if the following conditions are satisfied:

(i) There exists an $\Omega_0 \in \mathbb{N}^{>0}$ such that for all $x \in X_\mathrm{s} \cup X_\mathrm{u}$: $\Omega(x) = \Omega_0$ and such that for all $c/0 \in \Sigma$: $\Omega(c) \geq \Omega_0$;
(ii) For all $f \in \Sigma_0$: $\Omega(f) \in \mathbb{N}$.
(iii) There exists an $\Omega_1 = \omega \cdot m + n$ with $m \in \mathbb{N}^{>0}$, $n \in \mathbb{N}$, such that for all $x \in X_\mathrm{l}$: $\Omega(x) = \Omega_1$ and such that for all $f \in \Sigma \setminus \Sigma_0$: $\Omega(f) = \omega \cdot m' + n' \geq \Omega_1$;
(iv) If there is a symbol $f/1 \in \Sigma$ such that $\Omega(f) = 0$, then $f \sqsupseteq g$ for all $g \in \Sigma$.

The extension from symbol weights to term weights is defined as before. We can now introduce the *hierarchic Knuth-Bendix ordering* (HKBO). Compared with the transfinite Knuth-Bendix ordering (Def. 14) there are two major differences: the new case (HKBO1$'$) implies that small variables can essentially be ignored if the weight difference of the two terms is large enough, and a change in the definition of admissible symbol weight assignments enforces large variables to get assigned the weight Ω_1, which is greater than the weight of every symbol from Σ_0.

Definition 24 (Hierarchic Knuth-Bendix Ordering). Let Σ_0, Σ be signatures such that $\Sigma_0 \subseteq \Sigma$, let X_l, X_s, X_u be pairwise disjoint sets of variables with

$X = X_\mathrm{l} \cup X_\mathrm{s} \cup X_\mathrm{u}$. In addition, let $\sqsupset$ be a strict partial ordering, the precedence, on Σ, let $\Omega\colon \Sigma \cup X \to \{\, \omega \cdot m + n \mid m, n \in \mathbb{N}\,\}$ be a hierarchic symbol weight assignment that is admissible for $\sqsupset$ and Σ_0, and let $\Psi\colon \Sigma \to \mathbb{N}^{>0}$ be a subterm coefficient function. Finally, let $\mathrm{W} = \mathrm{W}_{(\Omega,\,\Psi)}\colon \mathrm{T}_\Sigma(X) \to \mathbf{O}$ be the ordinal term weight function induced by Ω and Ψ.

We define the *hierarchic Knuth-Bendix ordering* (HKBO) $\succ_\mathrm{H} \subseteq \mathrm{T}_\Sigma(X) \times \mathrm{T}_\Sigma(X)$ induced by $(\sqsupset, \Omega)$ on terms $s, t \in \mathrm{T}_\Sigma(X)$ in the following way:
$s \succ_\mathrm{H} t$ if

(HKBO1) $\mathrm{Var}(s) \supseteq \mathrm{Var}(t)$, $\mathrm{W}(s) > \mathrm{W}(t)$ and

$$\forall\, x \in \mathrm{Var}(t)\colon \bigoplus_{p \in \mathrm{P}(x,\,s)} \mathcal{C}(p,\,s) \geq \bigoplus_{p \in \mathrm{P}(x,\,t)} \mathcal{C}(p,\,t)$$

or
(HKBO1′) $\mathrm{Var}(s) \supseteq \mathrm{Var}(t) \cap (X_\mathrm{l} \cup X_\mathrm{u})$, $\mathrm{W}(s) = \omega \cdot m + n$, $\mathrm{W}(t) = \omega \cdot m' + n'$, $m > m'$ and

$$\forall\, x \in \mathrm{Var}(t) \cap (X_\mathrm{l} \cup X_\mathrm{u})\colon \bigoplus_{p \in \mathrm{P}(x,\,s)} \mathcal{C}(p,\,s) \geq \bigoplus_{p \in \mathrm{P}(x,\,t)} \mathcal{C}(p,\,t)$$

or
(HKBO2) $\mathrm{Var}(s) \supseteq \mathrm{Var}(t)$, $\mathrm{W}(s) = \mathrm{W}(t)$,

$$\forall\, x \in \mathrm{Var}(t)\colon \bigoplus_{p \in \mathrm{P}(x,\,s)} \mathcal{C}(p,\,s) \geq \bigoplus_{p \in \mathrm{P}(x,\,t)} \mathcal{C}(p,\,t)$$

and one of the following cases occurs:
(HKBO2a) $\exists\, f/1 \in \Sigma$, $\exists\, x \in X$, $\exists\, n \in \mathbb{N}^{>0}$ such that $s = f^n(x)$ and $t = x$,
(HKBO2b) $\exists\, f/m, g/n \in \Sigma$ $(m, n \in \mathbb{N})$, $\exists\, s_1, \ldots, s_m, t_1, \ldots, t_n \in \mathrm{T}_\Sigma(X)$ such that $s = f(s_1, \ldots, s_m)$, $t = g(t_1, \ldots, t_n)$ with $f \sqsupset g$,
(HKBO2c) $\exists\, f/m \in \Sigma$ $(m \in \mathbb{N}^{>0})$, $\exists\, s_1, \ldots, s_m, t_1, \ldots, t_m \in \mathrm{T}_\Sigma(X)$, $\exists\, i \in \mathbb{N}, 1 \leq i \leq m$ such that $s = f(s_1, \ldots, s_m)$, $t = f(t_1, \ldots, t_m)$ with $s_1 = t_1, \ldots, s_{i-1} = t_{i-1}$, $s_i \succ_\mathrm{T} t_i$.

It is easy to show that terms built over Σ_0 and "small" variables are smaller w. r. t. the HKBO than terms which contain at least one large variable or one symbol from $\Sigma \setminus \Sigma_0$, as required for hierarchic superposition:

Lemma 25. *For every term $s \in \mathrm{T}_\Sigma(X) \setminus \mathrm{T}_{\Sigma_0}(X_\mathrm{s} \cup X_\mathrm{u})$ and for every term $t \in \mathrm{T}_{\Sigma_0}(X_\mathrm{s})$ we have $s \succ_H t$.*

Proof. By part (iii) of Def. 23, we have $\mathrm{W}(s) \geq \omega$; by part (i) and (ii) we have $\omega > \mathrm{W}(t)$; hence $s \succ_\mathrm{H} t$ by (HKBO1′).

The following theorems are proved analogously to the corresponding propositions for TKBO:

Theorem 26. *The hierarchic Knuth-Bendix ordering $\succ_H$ is a simple simplification ordering.*

Theorem 27. *If the precedence $\sqsupset$ is a total ordering, then the hierarchic Knuth-Bendix ordering $\succ_H$ is total on ground terms.*

Furthermore, if we restrict to subterm coefficient functions that map every symbol to 1, then Löchner's proof [12] that KBO can be computed in linear time can easily be extended to HKBO:

Theorem 28. *If $\Psi(f) = 1$ for all $f \in \Sigma$, then there exists an algorithm with worst-case time complexity $\mathrm{O}(|s| + |t|)$ that tests for two terms s and t whether $s = t$, $s \succ_H t$, $t \succ_H s$, or s and t are incomparable.*

6 Conclusions

We have described a generalisation of the Knuth-Bendix ordering that possesses certain properties that are typical for LPO, such as the usability for hierarchic theorem proving or the ability to handle non-linear definition equations adequately.

As long as we restrict ourselves to subterm coefficient functions that map every signature symbol to 1, the transfinite and the hierarchic KBO not only inherit the general computation scheme of KBO but also its runtime behaviour, which in particular turns the HKBO into a useful tool for actual implementations of hierarchic theorem proving. In SPASS+T [15], we have implemented a three-level version of the HKBO, with numeric constants on the lowest level, numeric operators and predicates on the middle level, and other operators and predicates on the top level. This ordering ensures that (a) terms and literals are primarily compared using their non-numeric parts; (b) terms that differ only by their numeric constants are essentially compared by the sum of the absolute values of these constants, e. g., $g(20, 4) \succ g(5, 5)$ and $g(4, 20) \succ g(5, 5)$; (c) complex numeric expressions are always larger than the numbers to which they evaluate, e. g., $4 \cdot 5 \succ 20$.

On the other hand, choosing subterm coefficients that are larger than 1 is clearly detrimental to the runtime behaviour of the TKBO. This holds already when positive integers greater than 1 are used as subterm coefficients (in this case one needs arbitrary precision integer arithmetic), and even more so when one uses ordinal numbers beyond ω as subterm coefficients. In its full generality, the transfinite KBO is mostly a theoretical device which ensures that using the regular KBO on "small terms" and applying definition equation on "large terms" are both compatible with a single reduction ordering over the whole signature that is total on ground terms and whose existence may be required for refutational completeness of a calculus. Actual computation of the TKBO is possible, but it is essentially a last resort.

Acknowledgements. We are grateful to Andrei Voronkov and the LPAR reviewers for providing useful comments on previous versions of this paper.

References

1. Baader, F., Nipkow, T.: Term rewriting and all that. Cambridge University Press, New York (1998)
2. Bachmair, L., Ganzinger, H.: Rewrite-based equational theorem proving with selection and simplification. Journal of Logic and Computation 4(3), 217–247 (1994)
3. Bachmair, L., Ganzinger, H., Waldmann, U.: Refutational theorem proving for hierarchic first-order theories. Applicable Algebra in Engineering, Communication and Computing (AAECC) 5(3/4), 193–212 (1994)
4. Dick, J., Kalmus, J., Martin, U.: Automating the Knuth-Bendix ordering. Acta Informatica 28(2), 95–119 (1990)
5. Fernández, M.-L., Godoy, G., Rubio, A.: Recursive path orderings can also be incremental. In: Sutcliffe, G., Voronkov, A. (eds.) LPAR 2005. LNCS (LNAI), vol. 3835, pp. 230–245. Springer, Heidelberg (2005)
6. Ganzinger, H., Sofronie-Stokkermans, V., Waldmann, U.: Modular proof systems for partial functions with Evans equality. Information and Computation 204, 1453–1492 (2006)
7. Hessenberg, G.: Grundbegriffe der Mengenlehre. Vandenhoeck & Ruprecht, Göttingen (1906)
8. Hillenbrand, T., Weidenbach, C.: Superposition for finite domains. Research Report MPI-I-2007-RG1-002, Max-Planck-Institut für Informatik, Stuhlsatzenhausweg 85, 66123 Saarbrücken, Germany (April 2007)
9. Just, W., Weese, M.: Discovering modern set theory. I: The Basics, Graduate Studies in Mathematics, vol. 8. American Mathematical Society (1996)
10. Kamin, S., Lévy, J.-J.: Attempts for generalising the recursive path orderings. Manuscript Department of Computer Science, University of Illinois, Urbana-Champaign (1980), available at `http://perso.ens-lyon.fr/pierre.lescanne/not_accessible.html`
11. Knuth, D.E., Bendix, P.B.: Simple word problems in universal algebras. In: Leech, J. (ed.) Computational Problems in Abstract Algebra, pp. 263–297. Pergamon Press, Oxford (1970)
12. Löchner, B.: Things to know when implementing KBO. Journal of Automated Reasoning 36, 289–310 (2006)
13. Ludwig, M.: Extensions of the Knuth-Bendix ordering with LPO-like properties. Diploma thesis, Universität des Saarlandes, Saarbrücken, Germany (July 2006)
14. McCune, W.: Otter 3.3 Reference Manual. Argonne National Laboratory, Argonne, IL, USA, Technical Memorandum No. 263 (August 2003)
15. Prevosto, V., Waldmann, U.: SPASS+T. In: Sutcliffe, G., Schmidt, R., Schulz, S. (eds.) ESCoR: FLoC 2006 Workshop on Empirically Successful Computerized Reasoning, Seattle, WA, USA. CEUR Workshop Proceedings, vol. 192, pp. 18–33 (August 2006)
16. Riazanov, A., Voronkov, A.: The design and implementation of Vampire. AI Communications 15, 91–110 (2002)
17. Weidenbach, C., Brahm, U., Hillenbrand, T., Keen, E., Theobalt, C., Topić, D.: SPASS version 2.0. In: Voronkov, A. (ed.) CADE-18. LNCS (LNAI), vol. 2392, pp. 275–279. Springer, Heidelberg (2002)