

VC-dimension and Erdős–Pósa property

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ABSTRACT

Let $G = (V, E)$ be a graph. A k -neighborhood in G is a set of vertices consisting of all the vertices at distance at most k from some vertex of G . The hypergraph on vertex set V whose edge set consists of all the k -neighborhoods of G for all k is the *neighborhood hypergraph* of G . Our goal in this paper is to investigate the complexity of a graph in terms of its neighborhoods. Precisely, we define the *distance VC-dimension* of a graph G as the maximum taken over all induced subgraphs G' of G of the VC-dimension of the neighborhood hypergraph of G' . For a class of graphs, having bounded distance VC-dimension both generalizes minor closed classes and graphs with bounded clique-width.

Our motivation is a result of Chepoi et al. (2007) asserting that every planar graph of diameter 2ℓ can be covered by a bounded number of balls of radius ℓ . In fact, they obtained the existence of a function f such that every set \mathcal{F} of balls of radius ℓ in a planar graph admits a hitting set of size $f(v)$ where v is the maximum number of pairwise disjoint elements of \mathcal{F} .

Our goal is to generalize the proof of Chepoi et al. (2007) with the unique assumption of bounded distance VC-dimension of neighborhoods. In other words, the set of balls of fixed radius in a graph with bounded distance VC-dimension has the Erdős–Pósa property.

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1. Introduction

B -hypergraph and dominating sets. Let $G = (V, E)$ be a graph. A *dominating set* of G is a set X of vertices such that for every vertex v , there exists a vertex $x \in X$ satisfies either $x = v$ or v is a neighbor of x . In other words, all the vertices of V are at distance at most one from a vertex of X . In this paper we focus on a generalization of dominating sets called dominating sets at distance ℓ . A set X is a *dominating set at distance ℓ* if every vertex of the graph is at distance at most ℓ from a vertex of X .

A hypergraph is a pair (V, F) where V is a set of vertices and F is a set of subsets of V called hyperedges. For the study of dominating sets, a natural hypergraph arises: the B_1 -hypergraph. The B_1 -hypergraph of G has vertex set V and hyperedges are the closed neighborhoods of the vertices of the graph. Since we consider neighborhoods at distance ℓ in this paper, we naturally generalize the B_1 -hypergraph into the B_ℓ -hypergraph by replacing closed neighborhoods by balls of radius ℓ centered in every vertex of the graph. The B -hypergraph is the edge-union of the B_ℓ -hypergraphs for all ℓ .

A *hitting set* of a hypergraph $H = (V, F)$ is a subset of vertices intersecting every hyperedge. In other words, it is a subset X of vertices such that for every $e \in F$, $e \cap X \neq \emptyset$. One can note that a hitting set of the B_ℓ -hypergraph of a graph G is a dominating set at distance ℓ of the graph G (and the converse also holds). Indeed, let X be a hitting set of the B_ℓ -hypergraph

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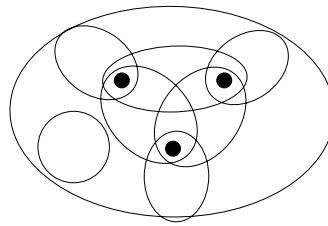


Fig. 1. A shattered set of size 3.

H of G . For every vertex $v \in V$, there exists $x \in X$ such that x is at distance at most ℓ from v . So the whole set of vertices is at distance at most ℓ from a vertex of X , i.e. X is a dominating set at distance ℓ of G . In the following we focus on hitting sets of the B_ℓ -hypergraphs. The minimum size of a hitting set, denoted by τ , is called the *transversality*. The *packing number*, denoted by ν , is the maximum number of pairwise disjoint hyperedges.

Complexity of graphs and VC-dimension. A set X of vertices is *shattered* (resp. *2-shattered*) if for every subset X' of X (resp. every subset of X of size 2) there exists a hyperedge e such that $e \cap X = X'$ (see Fig. 1). Introduced in [18,19], the *Vapnik–Chervonenkis dimension* (or *VC-dimension* for short) (resp. *2VC-dimension*) of a hypergraph H is the maximum size of a shattered set (resp. 2-shattered set). It is a good complexity measure of a hypergraph, for instance in the learnability sense. A bounded VC-dimension provides upper bounds on the number of hyperedges [18] but also on the transversality [6,13,16]. The VC-dimension has many applications, in learnability theory [13] and in computational geometry [4]. More recently, several applications were developed in graph theory, see [1,5,14] for instance.

One of our goals was to extend this notion on graphs to catch the complexity of a graph at large distance. The distance VC-dimension of a graph G could be defined as the VC-dimension of the B -hypergraph of the graph G . Since throughout this paper we only consider graphs closed under induced subgraphs, we define the *distance VC-dimension of a graph G* (resp. *distance 2VC-dimension of the graph G*) is the maximum over induced subgraphs of the distance VC-dimension (resp. 2VC-dimension) of the B -hypergraph. Since the VC-dimension “measures” the local randomness of hypergraphs, it is natural to think that classes with a lot of structure might have a bounded VC-dimension. In Section 3, we prove that two famous graph classes have bounded distance VC-dimension. First we show that the class of K_n -minor free graphs has distance VC-dimension at most $n - 1$. The proof is almost the proof of Chepoi, Estellon and Vaxès that the B_ℓ -hypergraph of planar graphs has distance VC-dimension at most 4.¹ Then we show that the class of bounded rankwidth graphs have bounded distance VC-dimension. Actually, we prove a slightly stronger statement for these two classes: their distance 2VC-dimension is bounded. We finally provide some graphs of bounded distance VC-dimension with an arbitrarily large distance 2VC-dimension.

Erdős–Pósa property. Chepoi, Estellon and Vaxès [5] proved that every planar graph of diameter 2ℓ can be covered by c balls of radius ℓ (where c does not depend on ℓ). It answered a conjecture of Gavaille, Peleg, Raspaud and Sopena [11]. Their proof uses the concept of VC-dimension but also planarity of the graph. One of our aims was to determine if the planarity arguments are necessary or if a purely combinatorial proof of this result exists.

Let G be a graph. We denote by respectively ν_ℓ and τ_ℓ the packing number and the transversality of the B_ℓ -hypergraph of G . Note that the B_ℓ -hypergraph of a planar graph of diameter 2ℓ satisfies $\nu_\ell = 1$. Indeed for every $u, v \in V$, since the diameter of the graph is at most 2ℓ , there exists a vertex x at distance at most ℓ from both u and v , so the hyperedges centered in u and in v intersect. Since τ_ℓ equals the minimum size of a dominating set at distance ℓ , we have $\tau_\ell \geq \nu_\ell$. A class of hypergraphs such that the transversality of every hypergraph is bounded by a function of its packing number is said to satisfy the *Erdős–Pósa property* (and the function is called the *gap function*). In their seminal paper [9], Erdős and Pósa proved that the minimum size of a feedback vertex set can be bounded by a function of the maximum number of vertex disjoint cycles: differently the cycle hypergraph of G has the Erdős–Pósa property (the vertices of the hypergraph are the vertices of the graph and the hyperedges are the cycles of the graph).

In Section 4, we first simplify and generalize the proof of Chepoi, Estellon and Vaxès. More precisely we prove that the B_ℓ -hypergraph of any graph G has a dominating set at distance ℓ of size $\mathcal{O}(\nu_\ell^{2d+1})$ where d denotes the distance 2VC-dimension of G . Note that the function depends on ν_ℓ but not directly on ℓ . Since planar graphs have distance 2VC-dimension at most 4, it ensures that the B_ℓ hypergraph of any planar graph of diameter 2ℓ satisfies $\tau_\ell \leq 35,200$. There is no doubt that this upper bound is still far away from the optimal one. For small diameters, better bounds exist. For instance every planar graph of radius 2 has a dominating set of size at most 3 [12,15].

Since some graphs of bounded distance VC-dimension have an arbitrarily large distance 2VC-dimension, it raises a natural question: is it possible to extend this result on graphs of bounded distance 2VC-dimension to graphs of bounded distance VC-dimension. Section 4 consists in proving that the answer to this question is positive. More formally, we prove that there exists a function f such that the B_ℓ -hypergraph of a graph of distance VC-dimension d has a hitting set of size at most $f(\nu_\ell, d)$. The original proof of Chepoi, Estellon and Vaxès for planar graphs is based on the same method but they conclude

¹ In their paper, Chepoi, Estellon and Vaxès noted that their proof for planar graphs can be extended to K_n -minor free graphs.

using topological properties of planar graphs. Since we only deal with combinatorial structures, our proof is more technically involved. Note nevertheless that the function f is exponential in the distance VC-dimension while the one provided by the distance 2VC-dimension is polynomial.

We will finally close this paper by some concluding remarks and open problems on distance VC-dimension and Erdős–Pósa property.

2. Preliminaries

It is sometimes convenient to see a hypergraph as its *incidence bipartite graph* B_H with vertex set $V \cup E$ in which there is an edge between $x \in V$ and $e \in E$ iff $x \in e$. Note that the pair (V, E) is oriented, and the hypergraph associated to the pair (E, V) is called the *dual hypergraph*. The vertices of the dual hypergraph are the hyperedges of the original one, and the hyperedges of the dual hypergraph are the subsets of E containing the vertex v , for every v . The *dual VC-dimension* of H is the VC-dimension of the dual hypergraph of H . The VC-dimension of H and the dual VC-dimension of H are equivalent up to an exponential function [3]. Similarly, the *dual 2VC-dimension* of H is the 2VC-dimension of the dual hypergraph of H . The 2VC-dimension is larger than or equal to the VC-dimension and the gap can be arbitrarily large. Indeed, consider the clique K_n . Its 2VC-dimension is equal to n whereas its VC-dimension is at most 2 (since no hyperedge contains 3 vertices). The same example ensures that no function links 2VC-dimension and dual 2VC-dimension.

A *transversal set* (or *hitting set*) of a hypergraph H is a set of vertices intersecting each hyperedge. The *transversality* τ of a hypergraph is the minimum size of a transversal set. The *packing number* ν of a hypergraph is the maximum number of vertex disjoint hyperedges. A class of hypergraphs \mathcal{H} has the *Erdős–Pósa property* if there exists a function f such that for all $H \in \mathcal{H}$, $\tau \leq f(\nu)$. We denote by ν_ℓ and τ_ℓ respectively the packing number and the transversality of the B_ℓ -hypergraph of G . Note that the B_ℓ -hypergraph of a graph G and its dual are the same since for every pair of vertices x, y , $x \in B(y, \ell)$ if and only if $y \in B(x, \ell)$. So:

Observation 1. *The B_ℓ -hypergraph is isomorphic to the dual of the B_ℓ -hypergraph.*

Let $G = (V, E)$ be a graph. Let $X \subseteq V$. The *graph induced by X* is the graph on vertex set X whose edges are edges of G with both endpoints in X . A *walk* of length k from $x \in V$ to $y \in V$ is a sequence of vertices $x = x_0, x_1, \dots, x_{k-1}, x_k = y$ where $x_i x_{i+1} \in E$ for each $0 \leq i \leq k-1$. A *path* is a walk with pairwise distinct vertices. The vertices x and y are the *endpoints* of the walk. The $x_i x_j$ -*subpath* is the path x_i, x_{i+1}, \dots, x_j . The *neighbors* of the vertex x_i on the path are the vertices x_{i-1} and x_{i+1} whenever they exist. A *minimum path* from x to y , also called *minimum xy -path*, is a path of minimum length from x to y . The *distance between x and y* , denoted by $d(x, y)$ is the length of a minimum xy -path when such a path exists and $+\infty$ otherwise. The *distance between a set X and a set Y* is the minimum for all $x, y \in X \times Y$ of the distance between x and y . The *ball of center x and radius k* , denoted by $B(x, k)$, is the set of vertices at distance at most k from x . The *neighbors* of x , denoted by $N(x)$ are the vertices of $B(x, 1)$ distinct from x .

Let us conclude this section by an observation which ensures that we can restrict our study to connected subgraphs:

Observation 2. *The distance VC-dimension of a non connected graph is the distance VC-dimension of the maximum of its connected components.*

3. Graphs of bounded distance VC-dimension

In this section we prove that K_n minor-free graphs and bounded rank-width graphs have bounded distance 2VC-dimension. In addition we provide a class of graphs with arbitrarily large distance 2VC-dimension and distance VC-dimension at most 18.

3.1. K_d -minor-free graphs have bounded distance VC-dimension

A graph H is a *minor* of G if H can be obtained from G by contracting edges, deleting edges, and deleting vertices. Theorem 4 is roughly Proposition 1 of [5]. Since our definitions and statements are slightly different, we prove it for the sake of completeness. We first prove an easy lemma before stating the main theorem of this section.

Lemma 3. *If z is on a minimum xy -path, the ball $B(z, d(x, z))$ is included in $B(y, d(x, y))$.*

Proof. Since z is on a minimum xy -path, $d(x, y) = d(x, z) + d(z, y)$. Hence $B(y, d(y, z))$ contains z and then $B(y, d(y, z) + d(z, x))$ contains $B(z, d(x, z))$. \square

Theorem 4. *A K_d -minor-free graph has distance 2VC-dimension at most $d - 1$.*

Proof. Let G be a graph with distance 2VC-dimension d . Let $X = \{x_1, x_2, \dots, x_d\}$ be a set of vertices of G which is 2-shattered by the hyperedges of the B -hypergraph of G . Hence, for every pair (i, j) , there exists a vertex $c_{i,j}$ and an integer $r_{i,j}$ such that $B(c_{i,j}, r_{i,j}) \cap X = \{x_i, x_j\}$. We assume moreover that $r_{i,j}$ is minimum for all choices of $(c_{i,j}, r_{i,j})$. A *central path* $P_{i,j}$ is the concatenation of a minimum path from x_i to $c_{i,j}$ and a minimum path from $c_{i,j}$ to x_j .

Claim 1. A central path is indeed a path.

Proof. Assume by contradiction that x appears more than once in a central path $P_{i,j}$. Since $P_{i,j}$ is a concatenation of a shortest $x_i c_{i,j}$ -path and a shortest $c_{i,j} x_j$ -path, x appears once between x_i and $c_{i,j}$ and once between $c_{i,j}$ and x_j . Let us call Q_1 the subpath of $P_{i,j}$ from x to $c_{i,j}$ and Q_2 the subpath of $P_{i,j}$ from $c_{i,j}$ to x . Note that Q_1 and Q_2 are both shortest paths connecting $c_{i,j}$ and x , hence replacing Q_2 by the mirror of Q_1 gives another central path $P'_{i,j}$. The two neighbors of $c_{i,j}$ in $P'_{i,j}$ are the same vertex v , contradicting the minimality of $r_{i,j}$ since $B(v, r_{i,j} - 1) \cap X = \{x_i, x_j\}$. \square

Claim 2. If x belongs to two distinct central paths, then these paths are $P_{i,j}$ and $P_{i,l}$, and we both have $d(x, x_i) < d(x, x_j)$ and $d(x, x_i) < d(x, x_l)$.

Proof. Assume that x appears in $P_{i,j}$ and $P_{k,l}$, where $d(x, x_i) \leq d(x, x_j)$ and $d(x, x_k) \leq d(x, x_l)$. Free to exchange the roles of $P_{i,j}$ and $P_{k,l}$, we can also assume that $d(x, x_k) \leq d(x, x_i)$. By Lemma 3, $x_k \in B(c_{i,j}, r_{i,j})$, hence we have $x_k = x_i$ or $x_k = x_j$. Since $d(x, x_k) \leq d(x, x_i) \leq d(x, x_j)$ and x_k is either x_i or x_j , we have $d(x, x_k) = d(x, x_i)$. Hence $d(x, x_i) \leq d(x, x_k)$, and by the same argument, we have $x_i = x_k$ or $x_i = x_l$. Since the central paths are distinct, we necessarily have $x_i = x_k$. Observe that $d(x, x_i) = d(x, x_j)$, hence $d(x, x_j) \leq d(x, x_l)$, would give by the same argument $x_j = x_k$, hence a contradiction since we would have $x_i = x_j$. Therefore $d(x, x_i) < d(x, x_j)$, and for the same reason $d(x, x_i) < d(x, x_l)$. \square

Let us now construct some connected subsets X_i for all $1 \leq i \leq d$. For every path $P_{i,j}$, the vertices of $P_{i,j}$ closer to x_i than to x_j are added to X_i , the vertices of $P_{i,j}$ closer to x_j than to x_i are added to X_j , and the midvertex (if any) is arbitrarily added to X_i or to X_j .

The crucial fact is that the sets X_i are pairwise disjoint. Indeed, by Claims 1 and 2, if a vertex x appears in two distinct central paths, these are $P_{i,j}$ and $P_{i,l}$, where $d(x, x_i) < d(x, x_j)$ and $d(x, x_i) < d(x, x_l)$. In particular x belongs in both cases to X_i .

By construction, the sets X_i are connected and there is always an edge between X_i and X_j since their union contains $P_{i,j}$. Therefore if the distance 2VC-dimension is at least d , the graph contains K_d as a minor. \square

3.2. Bounded rankwidth graphs have bounded distance VC-dimension

Let us first recall the definition of rankwidth, introduced by Oum and Seymour in [17]. Let $G = (V, E)$ be a graph and (V_1, V_2) be a partition of V . Let M_{V_1, V_2} be the matrix of size $|V_1| \times |V_2|$ such that the entry $(x_1, x_2) \in V_1 \times V_2$ equals 1 if $x_1 x_2 \in E$ and 0 otherwise. The *cutrank* $cr(V_1, V_2)$ of (V_1, V_2) is the rank of the matrix M_{V_1, V_2} over the field \mathbb{F}_2 . A *ternary tree* is a tree with nodes of degree 3 or 1. The nodes of degree 3 are the *internal nodes*, the other nodes being the *leaves*. A *tree-representation* of G is a pair (T, f) where T is a ternary tree with $|V|$ leaves and f is a bijection from V to the set of leaves. Every edge e of T defines a partition of the leaves of T . Therefore it defines a partition of the vertex set V into (V_1^e, V_2^e) . The *rankwidth* rw of a graph G is defined by:

$$rw(G) = \min_{(T,f)} \max_{e \in E(T)} cr(V_1^e, V_2^e).$$

Before stating the main result, let us first state two lemmas concerning rankwidth and ternary trees.

Lemma 5. Let $G = (V, E)$ be a graph of rankwidth k and X, Y be the partition of V induced by an edge of a tree-representation of G of cutrank k . There exist partitions of X and Y into at most 2^k sets X_1, \dots, X_{2^k} and Y_1, \dots, Y_{2^k} such that for all i, j , $(X_i \times Y_j) \cap E = \emptyset$ or $(X_i \times Y_j) \cap E = X_i \times Y_j$.

Proof. Let T be a tree representation of G of cutrank at most k . Let e be an edge of the tree representation of G and (X, Y) be the partition of V induced by e . Since the cutrank is at most k , the matrix $M_{X,Y}$ has rank at most k . Hence there exists $j \leq k$ rows R_1, \dots, R_j which form a base of the rows of the matrix $M_{X,Y}$. By definition, every row corresponds to the neighborhood of a vertex of X into Y . Let us denote by x_i the vertex corresponding to R_i . We denote by \mathcal{B} the set $\{x_1, \dots, x_j\}$.

For every $\mathcal{B}' \subseteq \mathcal{B}$, $X(\mathcal{B}')$ denotes the subset of X which contains x if $N(x) \cap Y =_{\mathbb{F}_2} \sum_{x_i \in \mathcal{B}'} N(x_i)$. It induces a partition of X since $N(x_1), \dots, N(x_j)$ is a base of the neighborhoods of X in Y . Note that by definition all the vertices of $X(\mathcal{B}')$ have the same neighborhood in Y . Observe that a vertex $x \in X(\mathcal{B}')$ is connected to a vertex y iff an odd number of vertices of \mathcal{B}' are connected to y .

For every $\mathcal{B}' \subseteq \mathcal{B}$, $Y(\mathcal{B}')$ is the subset of Y containing y if $N(y) \cap \mathcal{B} = \mathcal{B}'$. It induces a partition of Y into at most 2^j sets with the same neighborhood in \mathcal{B} .

Let us finally prove that the partitions of $X(\mathcal{B}')_{\mathcal{B}' \subseteq \mathcal{B}}$ and $Y(\mathcal{B}')_{\mathcal{B}' \subseteq \mathcal{B}}$ satisfy the required properties. Let x, y be in $X(\mathcal{B}') \times Y(\mathcal{B}'')$ such that xy is an edge. Since xy is an edge, an odd number of vertices of \mathcal{B}' are connected to y . Since all the vertices of $Y(\mathcal{B}'')$ have the same neighborhood in \mathcal{B} , all the vertices of $Y(\mathcal{B}'')$ have an odd number of neighbors on \mathcal{B}' . Thus x is connected to all the vertices of $Y(\mathcal{B}'')$. Since all the vertices of $X(\mathcal{B}')$ have the same neighborhood in Y , $(X(\mathcal{B}'), Y(\mathcal{B}''))$ forms a complete bipartite graph. \square

Lemma 6. Every ternary tree T with $\alpha > 2$ labeled leaves has an edge e such that the partition induced by e has at least $\alpha/3$ labeled leaves in both of its two connected components.

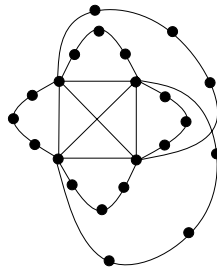


Fig. 2. The graph $G_{n,\ell}$ of Theorem 8 with $n = 4$ and $\ell = 2$. The vertices of the central clique are the vertices of X , the others are the vertices of Y .

Proof. Orient every edge of T from the component with less labeled leaves to the other one (when equality holds, orient arbitrarily). Observe that leaves are sources of this oriented tree. Let v be an internal node of T which is a sink. Consider a component C of $T \setminus v$ with at least $\alpha/3$ labeled leaves. Call $e = vw$ the edge of T inducing the partition $(T \setminus C, C)$. Since e is oriented from w to v , the component $T \setminus C$ has at least $\alpha/2$ labeled leaves, thus e is the edge we are looking for. \square

Theorem 7. The distance 2VC-dimension of a graph with rankwidth k is at most $3 \cdot 2^{k+1} + 2$.

Proof. Assume by contradiction that the B -hypergraph of a graph G of rankwidth k admits a 2-shattered set S of size $3(2^{k+1} + 1)$. Let (T, f) be a tree decomposition of G achieving rankwidth k . By Lemma 6, there is an edge e of T such that the partition induced by e has at least $2^{k+1} + 1$ vertices of S in both connected components. Let V_1, V_2 (resp. X, Y) be the partition of V (resp. S) induced by e . Let $x_1, \dots, x_{2^{k+1}+1}$ and $y_1, \dots, y_{2^{k+1}+1}$ be distinct vertices of X and Y respectively.

Since S is 2-shattered, for each $(x_i, y_j) \in X \times Y$, there is a ball $B_{i,j}$ such that $B_{i,j} \cap S = \{x_i, y_j\}$ where $B_{i,j}$ is chosen with minimum radius.

Claim 3. One of the following holds:

- There is an i such that at least $2^k + 1$ balls $B_{i,j}$ have their centers in V_1 .
- There is a j such that at least $2^k + 1$ balls $B_{i,j}$ have their centers in V_2 .

Proof. Orient the edges of the complete bipartite graph with vertex set $X \cup Y$ such that $x_i \rightarrow y_j$ if $B_{i,j}$ has its center in V_1 and $x_i \leftarrow y_j$ otherwise. The average out-degree of the vertices of $X \cup Y$ is $2^k + \frac{1}{2}$. So a vertex has out-degree at least $2^k + 1$.

Assume that the vertex $x_i \in X$ has out-degree at least $2^k + 1$. There exist $2^k + 1$ vertices of Y , w.l.o.g. y_1, \dots, y_{2^k+1} , such that $x_i y_1, \dots, x_i y_{2^k+1}$ are arcs. So the balls $B_{i,j}$ have their centers in V_1 for all $j \in \{1, \dots, 2^k + 1\}$, and then the first point holds. If a vertex of Y has out-degree at least $2^k + 1$, a symmetric argument ensures that the second point holds, which achieves the proof. \square

By Claim 3, we can assume without loss of generality that $B(1, 1), B(1, 2), \dots, B(1, 2^k + 1)$ have their centers in V_1 . We denote by c_i and r_i respectively the center and the radius of $B(1, i)$ and by P_i a minimum $c_i y_i$ -path. By the pigeonhole principle, two P_i 's leave V_1 by the same set of vertices given by the partition of Lemma 5. Without loss of generality, we assume that these paths are P_1 and P_2 and we denote by z_1 and z_2 respectively their last vertices in V_1 . We finally assume that $d(z_1, y_1) \leq d(z_2, y_2)$. By Lemma 3, the ball $B(z_2, d(z_2, y_2))$ is included in $B(c_2, r_2)$ since z_2 is on a minimum path from c_2 to y_2 . Let $z_1 z'_1$ be the first edge of P_1 between z_1 and y_1 (hence z'_1 belongs to Y). By Lemma 5, z'_1 is also a neighbor of z_2 since z_1 and z_2 have the same neighborhood in Y . Thus $y_1 \in B(z_2, d(z_2, y_2))$. Thus $y_1 \in B(z_2, d(z_2, y_2))$ which contradicts the hypothesis. \square

Since the rankwidth is equivalent, up to an exponential function, to the cliquewidth [17], Theorem 7 implies that every class of graphs with bounded clique-width has bounded distance 2VC-dimension.

3.3. Unbounded distance 2VC-dimension with bounded distance VC-dimension

Theorem 8. Let n, ℓ be two integers. There exists a graph $G_{n,\ell}$ of distance VC-dimension at most 18 such that the 2VC-dimension of the B_ℓ -hypergraph of $G_{n,\ell}$ is at least n .

Proof. The following construction is illustrated in Fig. 2. The graph $G_{n,\ell}$ has vertex set $X \cup Y$. The set X contains n vertices denoted by $(x_i)_{1 \leq i \leq n}$ and Y is a set of $(2\ell - 1) \binom{n}{2}$ vertices denoted by $y_k^{i,j}$ where $1 \leq k \leq 2\ell - 1$ and $1 \leq i < j \leq n$. The graph restricted to X is a clique. The graph restricted to Y is a disjoint union of $\binom{n}{2}$ induced paths on $2\ell - 1$ vertices (whose endpoints will be connected to vertices of X). More formally, for every $1 \leq i < j \leq n$ and $k \leq 2\ell - 1$, the neighbors of the vertex $y_k^{i,j}$ are the vertices $y_{k-1}^{i,j}$ and $y_{k+1}^{i,j}$ where $y_0^{i,j}$ is x_i and $y_{2\ell}^{i,j}$ is x_j . For every $i < j$, the path $x_i, y_1^{i,j}, y_2^{i,j}, \dots, y_{2\ell-1}^{i,j}, x_j$ is called the long path between x_i and x_j .

The $2VC$ -dimension of the B_ℓ -hypergraph of $G_{n,\ell}$ is at least n . Indeed the set X is 2-shattered since for every $x_i, x_j \in X$, we have $B(y_\ell^{i,j}, \ell) \cap X = \{x_i, x_j\}$.

The remaining of the proof consists in showing that the distance VC -dimension of $G_{n,\ell}$ is at most 18. Consider an induced subgraph of $G_{n,\ell}$. The remaining vertices of X are in the same connected component since X is a clique. Connected components with no vertices of X form induced paths and then have distance VC -dimension at most two by [Theorem 4](#). Thus, by [Observation 2](#), [Theorem 8](#) holds if it holds for the connected component of $\{z_1, z_2, z_3, z_4\}$.

Claim 4. *A shattered set of size at least four has at most two vertices on each long path.*

Proof. Let z_1, z_2, z_3 be three vertices which appear in this order on the same long path P and z_4 be a vertex which is not between z_1 and z_3 on P . By construction, every path between z_2 and z_4 intersects either z_1 or z_3 . So no pair $z, p \in V \times \mathbb{N}$ satisfy $B(z, p) \cap X = \{z_2, z_4\}$, i.e. $\{z_1, z_2, z_3, z_4\}$ is not shattered. \square

Let Z' be a shattered set of size at least 19. By [Claim 4](#), we can extract from Z' a set Z of size 10 such that vertices of Z are in pairwise distinct long paths. For every vertex $z_i \in Z$, a *nearest neighbor* on X is a vertex x of X such that $d(x, z_i)$ is minimum. Each vertex has at most two nearest neighbors which are the endpoints of the long path containing z_i .

First assume z_1, z_2, z_3 in Z have a common nearest neighbor x , i.e. they are on long paths containing x as endpoint. Without loss of generality $d(z_3, X)$ is minimum. Let z, p be such that $\{z_1, z_2\} \subseteq B(z, p)$. Since z_1 and z_2 are not in the same long path, free to exchange z_1 and z_2 , a minimum zz_2 -path passes through a vertex y of X . If $y = x$, then $B(z, p)$ contains $B(x, d(x, z_2))$ by [Lemma 3](#), and then contains z_3 since $d(x, z_2) \geq d(x, z_3)$. Otherwise up to symmetry y is not an endpoint of the long path containing z_2 . Indeed the second endpoint of the long path containing z_1 and the second endpoint of the long path containing z_2 are distinct. Otherwise z_1, z_2 would be in the same long path since there is a unique long path between every pair of vertices of X . Hence a minimum path from y to z_2 is at least $d(z_2, X) + 1$. In addition a minimum path between y and z_3 has length at most $1 + d(z_3, X)$. So $d(y, z_2) \geq d(y, z_3)$. So z_3 is in $B(z, p)$ and $\{z_1, z_2, z_3\}$ cannot be shattered.

So each vertex of Z has at most two nearest neighbors in X and each vertex of X is the nearest neighbor of at most two vertices of Z . Thus every $z \in Z$ share a common nearest neighbor with at most two vertices of Z . Since $|Z| \geq 10$, at least four vertices z_1, z_2, z_3, z_4 of Z have distinct nearest neighbors. Assume w.l.o.g. that $d(z_4, X)$ is minimum.

Let $z, p \in V \times \mathbb{N}$ be such that $B(z, p)$ contains z_1, z_2, z_3 . Let x_1, x_2 be the endpoints of the long path containing z (if $z \in X$ we consider that $x_1 = x_2 = z$). Since nearest neighbors of z_1, z_2, z_3 are pairwise disjoint, we can assume w.l.o.g. that the nearest neighbors of z_3 are distinct from x_1 and from x_2 . So a minimum path from z to z_3 passes through x_1 or x_2 and we have $d(x_1, z_3) \geq d(x_1, z_4)$ and $d(x_2, z_3) \geq d(x_2, z_4)$. By [Lemma 3](#), $B(z, p)$ also contains z_4 , i.e. Z cannot be shattered. \square

Note that we did not make any attempt to exactly evaluate the distance VC -dimension of the graph $G_{n,\ell}$.

4. Erdős–Pósa property

Recall that ν_ℓ and τ_ℓ respectively denote the packing number and the transversality of the B_ℓ -hypergraph of G . Chepoi, Estellon and Vaxès proved in [\[5\]](#) that there is a constant c such that for all ℓ , every planar graph G of diameter 2ℓ can be covered by c balls of radius ℓ . It means that planar graphs of diameter 2ℓ satisfy $\tau_\ell \leq f(\nu_\ell)$ since two balls of radius ℓ necessarily intersect. They conjectured that there exists a linear function f such that for every ℓ and every planar graph we have $\tau_\ell \leq f(\nu_\ell)$. The following result due to Ding, Seymour and Winkler [\[6\]](#) ensures that a polynomial function f exists for any class of graphs of bounded distance $2VC$ -dimension.

Theorem 9 (Ding, Seymour, Winkler [\[6\]](#)). *Each hypergraph of dual $2VC$ -dimension d satisfies,*

$$\tau \leq 11 \cdot d^2 \cdot (d + \nu + 3) \cdot \binom{d + \nu}{d}^2.$$

Corollary 10. *Let d be an integer. For every graph $G \in \mathcal{G}$ and every integer ℓ , if the distance $2VC$ -dimension of G is at most d , then*

$$\tau_\ell \leq 11 \cdot d^2 \cdot (d + \nu_\ell + 3) \cdot \binom{d + \nu_\ell}{d}^2.$$

Proof. Let G be a graph. [Observation 1](#) ensures that the B_ℓ -hypergraph of G is isomorphic to its dual hypergraph. The B_ℓ -hypergraph of G is a sub-hypergraph (in the sense of hyperedges) of the B -hypergraph of G . Hence the dual $2VC$ -dimension of the B_ℓ -hypergraph of G is at most d and then [Theorem 9](#) can be applied. \square

[Theorems 4](#) and [7](#) and [Corollary 10](#) ensure that B_ℓ -hypergraphs of K_n -minor free graphs and of bounded rankwidth graphs have the Erdős–Pósa property. Note that the gap function is a polynomial function when the $2VC$ -dimension is fixed constant. In particular, [Corollary 10](#) implies that every planar graph of diameter 2ℓ has a dominating set at distance ℓ of size 35200 ($\nu_\ell = 1, d = 4$). Since [Theorem 8](#) ensures that there are some graphs with bounded distance VC -dimension and unbounded distance $2VC$ -dimension, [Corollary 10](#) raises a natural question. Does the same hold for graphs of bounded distance VC -dimension? The remaining of this section is devoted to answering this question.

Theorem 11. *There exists a function f such that, for every ℓ , every graph of distance VC-dimension d can be covered by $f(v_\ell, d)$ balls of radius ℓ , i.e. $\tau_\ell \leq f(v_\ell, d)$.*

Our proof is based on a result of Matoušek linking (p, q) -property and Erdős–Pósa property [16] (Chepoi, Estellon and Vaxès use this method in their paper). Nevertheless our proof is more technically involved since we cannot use topological properties as for planar graphs in [5]. A hypergraph has the (p, q) -property if for every set of p hyperedges, q of them have a non-empty intersection, i.e. there is a vertex v in at least q of the p hyperedges. The following result, due to Matoušek [16], generalizes a result of Alon and Kleitman [2].

Theorem 12 (Matoušek [16]). *There exists a function f such that every hypergraph H of dual VC-dimension d satisfying the $(p, d+1)$ -property satisfies*

$$\tau(H) \leq f(p, d).$$

Let d be an integer. Let G be a graph of distance VC-dimension d . By Observation 1, the dual VC-dimension of the B_ℓ -hypergraph is at most d . Hence if there exists a function p such that, for every ℓ and every graph G of distance VC-dimension d , the B_ℓ -hypergraph of G satisfies the $(p(v_\ell, d), d+1)$ -property, then Theorem 12 will ensure that Theorem 11 holds. So for proving Theorem 11, it suffices to show that the size of a set of balls of radius ℓ which does not contain $(d+1)$ balls intersecting on a same vertex is bounded by a function of v_ℓ and d . The remaining of this section is devoted to proving this result.

4.1. A lower bound for the distance VC-dimension of a graph

Let A and B be two disjoint sets. An *interference matrix* $M = (A, B)$ is a matrix with $|A|$ rows and $|B|$ columns such that for every $(a, b) \in A \times B$, the entry $m(a, b)$ is a subset of $(A \cup B) \setminus \{a, b\}$. The size of an entry is its number of elements. A k -interference matrix M is an interference matrix whose entries have size at most k . If $A' \subseteq A$ and $B' \subseteq B$, the submatrix M' of M induced by $A' \times B'$ is the matrix restricted to the set of rows A' and the set of columns B' whose entries are $m'(a', b') = m(a', b') \cap (A' \cup B')$. A 0-interference matrix is called a *proper matrix* (in the sense that no element interferes with any entry of the matrix). A matrix is *square* if $|A| = |B|$. The size of a square matrix is its number of rows.

Lemma 13. *Let $k > 0$. A k -interference square matrix with no proper submatrix of size n has size less than kn^3 .*

Proof. Let us show that if $M = (A, B)$ is a k -interference matrix with size $m = kn^3 + 1$, then it contains a proper submatrix of size n . A triple $(i, j, l) \in A \times B \times (A \cup B)$ is a *bad triple* if $l \in m(i, j)$ (and then $l \neq i$ and $l \neq j$). A bad triple (i, j, l) is *bad for* (X, Y) with $X \subseteq A, Y \subseteq B, |X| = |Y| = n$ if $i \in A, j \in B$ and l is in A or B .

For a given bad triple (i, j, l) , let us count the number of pairs (X, Y) where $X \subseteq A, Y \subseteq B$, and $|X| = |Y| = n$ containing (i, j, l) as a bad triple. Let us consider the case $l \in A$ (the case $l \in B$ is obtained similarly). The number of X 's containing both i and l is $\binom{m-2}{n-2}$ since $i \neq l$. The number of Y 's containing j is $\binom{m-1}{n-1}$. Since M is a k -interference matrix, the total number of bad triples is at most $k \cdot m^2$. Thus the total number of pairs X, Y with $X \subseteq A, Y \subseteq B, |X| = |Y| = n$ is $\binom{n}{m}^2$. So if the number of such pairs is larger than the number of pairs containing a bad triple, the conclusion holds. In other words, if

$$\binom{m-2}{n-2} \cdot \binom{m-1}{n-1} \cdot km^2 < \binom{n}{m}^2$$

there is a pair (X, Y) with $X \subseteq A, Y \subseteq B, |X| = |Y| = n$ which does not contain a bad triple. This latter inequality is equivalent with $kn^2 \cdot (n-1) < m-1$. \square

Given a path P from x to y and a path Q from y to z , the *concatenation* of P and Q denoted by PQ is the walk consisting on the edges of P followed by the edges of Q . The length of a path P is denoted by $|P|$. Let $G = (V, E)$ be a graph and $<_l$ be a total order on E . We extend $<_l$ on paths, for any paths P_1 and P_2 as follows:

- If P_1 has no edges, then $P_1 <_l P_2$.
- If $P_1 = P'_1.e_1$ and $P_2 = P'_2.e_1$, where e_1 is the last edge of P_1 and P_2 , then $P_1 <_l P_2$ if and only if $P'_1 <_l P'_2$.
- If $P_1 = P'_1.e_1$ and $P_2 = P'_2.e_2$, where $e_1 \neq e_2$, then $P_1 <_l P_2$ if and only if $e_1 <_l e_2$.

The order $<_l$ is called the *lexicographic order* (note nevertheless that paths are compared from their end to their beginning). The *minimum path* from x to z , also called the *xz-path* and denoted by P_{xz} , is the path of minimum length with minimum lexicographic order from x to z . Observe that two minimum paths going to the same vertex z and passing through the same vertex u coincide between u and z . We note $u \preceq_{xz} v$ if u appears before v on the xz -path. Given a path from a to b passing through c , the *suffix path on c* (resp. *prefix path on c*) is the cb -subpath (resp. ac -subpath) of the ab -path. Note that every suffix of a minimum path is a minimum path. Given two sets X and Z , the *XZ-paths* are the xz -paths for all $x, z \in X \times Z$.

Let x_1, x_2 and z be three vertices. Two distinct edges v_1u_2 and u_1v_2 form a *cross* between the x_1z -path and the x_2z -path if for $i \in \{1, 2\}$, $u_i \preceq_{x_iz} v_i$ (see Fig. 3).

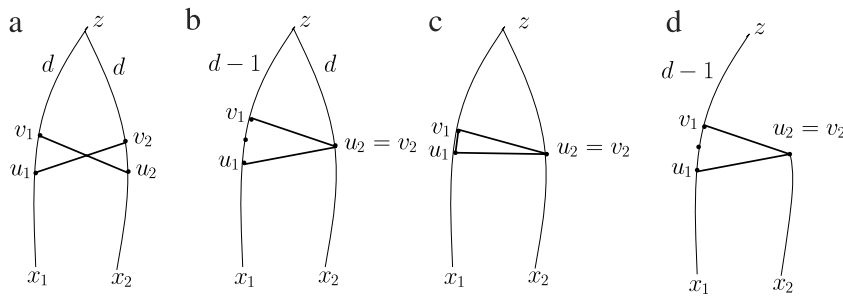


Fig. 3. 4 types of crosses. Lemma 14 ensures that, up to symmetry, only (c) and (d) are authorized. The thick chords are edges of the graph. Thin chords represent paths. Distances are denoted by d or $d - 1$. In the case of Fig. 3(d), the path Q_{v_2} is $v_2v_1Q_{v_1}$.

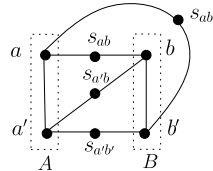


Fig. 4. The sets $S_{ab} = \{s_{ab}\}$ are 2-disconnecting for A, B .

Lemma 14. Let x_1, x_2, z be three vertices. If the edges u_1v_2 and v_1u_2 form a cross between the x_1z -path and the x_2z -path, then free to exchange x_1 and x_2 we have:

- either $u_2 = v_2$ and u_1v_1 is an edge.
- Or $u_2 = v_2$ and the v_2z -path is the edge v_2v_1 concatenated with the v_1z -path.

In other words, only cases (c) and (d) of Fig. 3 can occur.

Proof. For $i \in \{1, 2\}$, we denote by Q_{u_i} (resp. Q_{v_i}) the suffix of the $x_i z$ -path on u_i (resp. v_i). Since suffixes of minimum paths are minimum paths, these four paths are minimum paths. We prove that if a cross does not satisfy the condition of Lemma 14, then one of these paths is not minimum.

A *real cross* is a cross for which $u_1 \neq v_1$ and $u_2 \neq v_2$ (Fig. 3(a)). A *degenerated cross* is a cross for which, up to symmetry, $u_2 = v_2$ and $Q_{v_2} \neq v_2v_1.Q_{v_1}$ (Fig. 3(b) and (c)).

A real cross satisfies $|Q_{v_1}| = |Q_{v_2}|$. Indeed if $|Q_{v_1}| < |Q_{v_2}|$ then $u_2v_1.Q_{v_1}$ has length at most $|Q_{v_2}|$. This path is strictly shorter than Q_{u_2} (since $u_2 \neq v_2$, indeed the cross is a real cross), contradicting the minimality of Q_{u_2} . So $|Q_{v_1}| = |Q_{v_2}|$. Free to exchange x_1 and x_2 , we have $Q_{v_1} \prec_l Q_{v_2}$. So $u_2v_1.Q_{v_1} \prec_l Q_{u_2}$ (recall that we first compare the last edge) and $|u_2v_1.Q_{v_1}| \leq |Q_{u_2}|$. So Q_{u_2} is not minimum, a contradiction. Hence there is no real cross.

Consider a degenerated cross such that $u_1v_1 \notin E$. In particular u_1 and v_1 are at distance 2. So we have $|Q_{v_1}| < |Q_{v_2}|$ otherwise $u_1v_2.Q_{v_2}$ would be strictly shorter than Q_{u_1} , a contradiction. In addition, $|Q_{v_2}|$ and $|Q_{v_1}|$ differ by at most one since v_1v_2 is an edge. So $|Q_{v_1}| + 1 = |Q_{v_2}|$. Assume now that we are not in the case of Fig. 3(d), in other words, $Q_{v_2} \neq v_2v_1.Q_{v_1}$. If $Q_{v_2} \prec_l Q_{v_1}$ then $u_1v_2.Q_{v_2}$ is not longer than Q_{u_1} (since u_1 and v_1 are at distance 2) and has a smaller lexicographic order, a contradiction with the minimality of Q_{u_1} . If $Q_{v_1} \prec_l Q_{v_2}$ then $v_2v_1.Q_{v_1}$ is not longer and has a smaller lexicographic order, a contradiction with the minimality of Q_{v_2} . So either the degenerated cross satisfies $u_1v_1 \in E$ or $Q_{v_2} = v_2v_1.Q_{v_1}$. \square

Let ℓ be an integer and A, B be two disjoint subsets of vertices. To every pair $(a, b) \in A \times B$, we associate a set of vertices $S_{a,b}$ which is disjoint from $A \cup B$. We say that the set of subsets $\mathcal{S} = \{(S_{a,b})_{(a,b) \in A \times B}\}$ is ℓ -*disconnecting* if for every subset C of \mathcal{S} and every pair (a, b) , we have $d(a, b) > \ell$ in $G \setminus \bigcup C$ if and only if $S_{a,b} \in C$. If such a family of sets exists, then A, B are said to be ℓ -*disconnectable*. Another way of defining ℓ -disconnecting families would be to say that $d(a, b) > \ell$ in $G \setminus S_{a,b}$ and $d(a, b) \leq \ell$ in $G \setminus (\mathcal{S} \setminus S_{a,b})$, or roughly speaking that $S_{a,b}$ is the only set whose deletion can increase $d(a, b)$ above ℓ . In Fig. 4, the sets A, B are 2-disconnectable. Indeed the deletion of any vertex s_{ab} eliminates all the paths of length at most 2 from a to b . Note nevertheless that a and b are still in the same connected component after this operation.

Theorem 15. Let $G = (V, E)$ be a graph and ℓ be an integer. If there exist two subsets A, B of V with $|B| = 2^{|A|}$ which are ℓ -disconnectable, then the distance VC-dimension of G is at least $|A|$.

Proof. Let us prove that the set A can be shattered in the B_ℓ -hypergraph of an induced subgraph of G . Associate in a one to one way every vertex b of B to a subset A_b of A . Since A, B are ℓ -disconnectable, there exists a family \mathcal{S} of subsets which is ℓ -disconnecting for A, B . Let C be the collection of \mathcal{S} consisting of all the sets $S_{a,b}$ such that $a \in A_b$. Since \mathcal{S} is ℓ -disconnecting, $B(b, \ell) \cap A = A \setminus A_b$ in $G \setminus C$, for all $b \in B$ (the deletion of $S_{a,b}$ eliminates the paths of length at most ℓ between a and b). Hence the set A is shattered by balls of radius ℓ in $G \setminus C$. Therefore the distance VC-dimension of G is at least $|A|$. \square

4.2. Sparse sets

Let G be a graph of distance VC-dimension d and q, ℓ be two integers. Most of the following definitions depend on ℓ . Nevertheless, in order to avoid heavy notations, this dependence will be implicit in the terminology. A set of balls of radius ℓ is q -sparse if no vertex of the graph is in more than q balls of the set. Note that a subset of a q -sparse set is still q -sparse. By abuse of notation, a set X of vertices is called q -sparse if the set of balls of radius ℓ centered in X is q -sparse.

Assume that the B_ℓ -hypergraph of a graph G does not satisfy the $(p, d + 1)$ -property. Then there exist p balls of radius ℓ such that no vertex is in at least $(d + 1)$ of these p balls, i.e. there is a d -sparse set of size p . In other words, a d -sparse set of size p is a certificate that the $(p, d + 1)$ -property does not hold. In order to prove [Theorem 11](#), we just have to show that p can be bounded by a function of d and v_ℓ . The remaining of this section is devoted to show that there exists a function f such that the size of a d -sparse set is at most $f(d, v_\ell)$.

A set X of vertices is d -localized if the vertices of X are pairwise at distance at least $\ell + 1$ and at most $2\ell - 2^{d+2} - 3$. A d -localized set is defined only if this value is positive. A pair A, B of disjoint sets of vertices is q -sparse if $A \cup B$ is. A disjoint pair A, B of vertices is d -localized if the vertices of $A \cup B$ are pairwise at distance at least $\ell + 1$, and if for every $a, b \in A \times B$, $d(a, b) \leq 2\ell - 2^{d+2} - 3$. A subpair of a d -localized pair is d -localized. The size of a pair A, B is $\min(|A|, |B|)$.

Theorem 16. (Ramsey) *There exists a function r_k such that every complete edge-colored graph G with k colors with no monochromatic clique of size n has at most $r_k(n)$ vertices.*

All along the paper, logarithms are in base 2.

Theorem 17. *Let G be a graph and X be a subset of vertices pairwise at distance exactly r . Assume also that no vertex of G belongs to q balls of radius $\lceil r/2 \rceil$ with centers in X . Then the distance VC-dimension of G is at least $(\log |X| - \log 2q)/3$.*

Proof. Let r' be equal to $\lceil r/2 \rceil$. Free to remove one vertex from X , we can assume that X is even, and we consider a partition A, B of X with $|A| = |B|$. For every pair $(a, b) \in A \times B$, we denote the minimum ab -path by P_{ab} . By abuse of notation, we still denote by G the restriction of G to the vertices of the union of the paths P_{ab} for all $a \in A$ and $b \in B$. Observe that we preserve the hypothesis of [Theorem 17](#) apart from the fact that the distance between vertices inside A (resp. inside B) may have increased above r . Let $a \in A$ and $b \in B$. Let y be a vertex of X distinct from a and b . If $B(y, r') \cap P_{ab} \neq \emptyset$, then denote by x a vertex in this set. We have $d(a, x) \geq \lfloor r/2 \rfloor$ since $d(a, y) \geq r$ and $d(y, x) \leq \lceil r/2 \rceil$. By symmetry, we also have $d(b, x) \geq \lfloor r/2 \rfloor$. Hence x is a *midvertex* of P_{ab} , i.e. a vertex of P_{ab} at distance $\lfloor r/2 \rfloor$ or $\lceil r/2 \rceil$ from a (and thus also from b). Recall that a midvertex x of P_{ab} belongs to at most $q - 1$ balls of radius r' (including $B(a, r')$ and $B(b, r')$).

Consider the interference matrix $M = (A, B)$ where $m(a, b) = \{y \in (A \cup B) \setminus \{a, b\} \mid B(y, r') \cap P_{ab} \neq \emptyset\}$. Since P_{ab} has at most two midvertices and each of these belongs to at most $q - 3$ balls $B(y, r')$ with y different from a and b , the matrix M is a $(2q - 6)$ -interference matrix. To avoid tedious calculations and free to increase the interference value, we only assume that M is a $2q$ -interference matrix (with $2q \geq 1$). By [Lemma 13](#), there is a proper submatrix M' of size $N = (|X|/2q)^{1/3}$. Let us denote by A' the set of rows and B' the set of columns of the extracted matrix. Let us still denote by G the restriction of the graph to the vertices of the paths $(P_{ab})_{(a,b) \in A' \times B'}$.

Let $a, a' \in A'$ and $b' \in B'$. The key-observation is that if $B(a, r')$ intersects $P_{a'b'}$, then $a = a'$. Indeed, by definition of M , we have $a \in m(a', b)$, contradicting the fact that M' is a proper submatrix.

Let M_{ab} be the set of midvertices of P_{ab} , where $a, b \in A' \times B'$. We claim that M_{ab} is disjoint from $P_{a'b'}$, whenever $P_{a'b'} \neq P_{ab}$. Indeed if $x \in M_{ab} \cap P_{a'b'}$, we have in particular both $d(a, x) \leq r'$ and $d(b, x) \leq r'$, and thus by the key-observation $a = a'$ and $b = b'$. In other words, deleting M_{ab} never affects $P_{a'b'}$, whenever $P_{a'b'} \neq P_{ab}$.

Another crucial remark is that every path P of length r from a to b intersects M_{ab} . Indeed, let x be a vertex of P with both $d(a, x) \leq r'$ and $d(b, x) \leq r'$. Since x is in G , it belongs to some path $P_{a'b'}$. By the key-observation, we both have $a' = a$ and $b' = b$, hence $x \in M_{ab}$.

To conclude, observe that the deletion of M_{ab} ensures that the distance $d(a, b)$ is more than r whereas deleting the union of all $M_{a'b'}$ different from M_{ab} does not affect $d(a, b)$ which is still equal to r . Consequently, the sets $(M_{ab})_{(a,b) \in A' \times B'}$ are r -disconnecting for A', B' . Hence, by [Theorem 15](#), the distance VC-dimension of G is at least $\log(N) = (\log |X| - \log 2q)/3$. \square

Lemma 18. *Let G be a graph of distance VC-dimension at most d . There exists a function f such that:*

- Either G contains a d -localized set of size p which is d -sparse,*
- Or the $(f(v_\ell, d, p), d + 1)$ -property holds.*

Proof. Let $D = 2^{d+2} + 2$ and $N = \max(p, v_\ell + 1, 2^{3d+3+\log(4d+2)})$. Let f be a function such that $f(v_\ell, d, p) \geq r_{D+4}(N) + 1$. Let us show that function f satisfies [Lemma 18](#). Assume that point (b) does not hold, i.e. the $(f(v_\ell, d, p), d + 1)$ -property does not hold. So there is a subset X of vertices of size $r_{D+4}(N) + 1$ such that the set X is d -sparse. Let us show that point (a) holds.

Consider the complete $(D + 4)$ -edge-colored graph G' with vertex set X such that, for every $x, y \in X$, xy has color:

- c with $0 \leq c \leq D$ if $d(x, y) = 2\ell - c$,
- $D + 1$ if $d(x, y) \leq \ell$,
- $D + 2$ if $d(x, y) > 2\ell$,
- $D + 3$ otherwise.

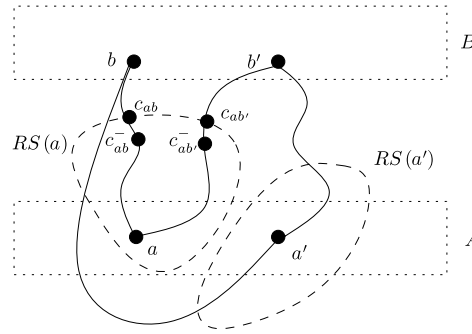


Fig. 5. Minimum paths with root sections (dashed parts), critical vertices and pre-critical vertices.

Theorem 16 ensures that there is a monochromatic clique K of size N . Let K' be a clique of color $D + 1$ and $x \in K'$. Then $K' \subseteq B(x, \ell) \cap X$. Thus the size of K' is at most d since X is d -sparse. At most v_ℓ balls of radius ℓ centered in X are vertex disjoint by definition of the packing number. Thus the size of a clique of color $D + 2$ is at most $v_\ell < N$. Since X is d -sparse, then K also is. Then, for every $0 \leq c \leq D$, no vertex of G belongs to $(d + 1)$ balls of radius $\lceil (2\ell - c)/2 \rceil \leq \ell$ centered in X . Therefore the color of K cannot be in $0 \leq r \leq D$. Otherwise **Theorem 17** would ensure that the distance VC-dimension of G is at least $\log(N)/3 - \log(4d + 2)/3 \geq d + 1$. So the clique K of size $N \geq p$ has color $D + 3$. A clique of color $D + 3$ defines a d -localized set. Moreover K is d -sparse since X is. Thus K satisfies (a). \square

The vertices of a d -localized set have to be pairwise at distance at least $\ell + 1$ and at most $2\ell - 2^{d+2} - 3$. The edge-colored graph of **Lemma 18** was constructed in order to ensure this property.

4.3. Localized and independent pairs

In this section we introduce a notion of independence for every pair of vertices. We first give some properties of independent pairs and we will finally show that any large enough d -sparse and d -localized pair contains a large enough independent subpair.

Let A, B be a d -localized pair. In the following we consider the restriction of the graph to $\cup_{a \in A, b \in B} P_{ab}$. Recall that P_{ab} is the minimum path with minimum lexicographic order from a to b , also called the ab -path. Note that the sets A and B are not treated symmetrically since we only consider the minimum paths from A to B . Let $a, a' \in A$ and $b \in B$. Note that, since $d(a, a') > \ell$, $d(a', b) > \ell$ and $d(a, b) < 2\ell$, the vertex a' does not belong to P_{ab} .

For every pair $a, b \in A \times B$, the critical vertex c_{ab} (resp. c_{ba}) is the vertex of P_{ab} at distance $\ell - 3$ from a (resp. b) and the pre-critical vertex c_{ab}^- is the vertex of P_{ab} at distance $\ell - 4$ from a (see Fig. 5). Such vertices exist since $d(a, b) > \ell$. Moreover c_{ab} and c_{ab}^- are adjacent. Note that both c_{ab} and c_{ba} are vertices of P_{ab} . In the following, we mostly need the vertex c_{ba} in order to ensure some distance properties (and then we do not use the minimality of the lexicographic order for these vertices). On the contrary, the vertex c_{ab} will be used for both distance and lexicographic arguments. The root section of $a \in A$ (resp. $b \in B$), denoted by $RS(a)$ (resp. $RS(b)$), is the set of vertices of the ac_{ab} -subpaths (resp. $c_{ba}b$ -subpaths) of P_{ab} for all $b \in B$ (resp. $a \in A$). We denote by $RS(A)$ the set $\cup_{a \in A} RS(a)$.

Since $d(a, b) \leq 2\ell - 7$, the vertex c_{ba} precedes the vertex c_{ab} on the path P_{ab} . In particular we have $P_{ab} \subseteq RS(a) \cup RS(b)$, hence every vertex of G belongs to some root section. In fact, we have the following slightly stronger observation:

Observation 19. For every a, b in $A \times B$, the critical vertex c_{ab} and the pre-critical vertex c_{ab}^- are in $RS(b)$.

A d -localized pair A, B is independent, if for every $a, b \in A \times B$, the ball $B(c_{ab}, \ell)$ intersects $A \cup B$ on $\{a, b\}$ and $B(c_{ba}, \ell) \cap (A \cup B) = \{a, b\}$. A subpair of an independent pair is still independent. In addition, A, B is still independent in the graph induced by the vertices of the AB -paths.

Lemma 20. The size of a d -sparse and d -localized pair with no independent subpair of size p is at most $2d \cdot p^3$.

Proof. Let A, B be a d -sparse and d -localized pair of size $2d \cdot p^3 + 1$. For every vertex u , $I(u)$ denotes $B(u, \ell) \cap (A \cup B)$. Since $A \cap B = \emptyset$, the matrix $M = (A, B)$ where $m(a, b) = (I(c_{ab}) \cup I(c_{ba})) \setminus \{a, b\}$, is a well-defined interference matrix. The pair A, B is d -sparse, then $|I(u)| \leq d$ for every vertex u . Thus M is a $2d$ -interference matrix.

By **Lemma 13**, M has a proper submatrix (A', B') of size p . Thus for every $a', b' \in A' \times B'$, $B(c_{a'b'}, \ell) \cap (A' \cup B') = \{a', b'\}$ and the same holds for $c_{b'a'}$, i.e. A', B' is independent. \square

Lemma 21. Let A, B be an independent pair.

- (a) Every pair of vertices of AB -paths whose endpoints are disjoint are at distance at least 4.
- (b) For every pair a, a' in A (resp. b, b' in B), $d(RS(a), RS(a')) \geq 4$ (resp. $d(RS(b), RS(b')) \geq 4$).

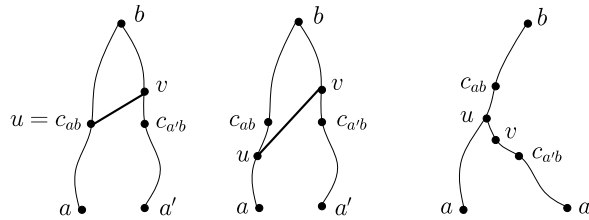


Fig. 6. Examples of escapes. The right one is an edge of the $a'b$ -path.

Proof. Let us first prove (b). We prove it for vertices of A , the case of vertices of B will handle symmetrically (indeed the proof rely on distance arguments and not lexicographic ones). Let $a \neq a'$ with $u \in RS(a)$ and $u' \in RS(a')$. There exist b and b' in B such that u is in the prefix path on c_{ab} of the ab -path and u' is in the prefix path on $c_{a'b'}$ of the $a'b'$ -path. Free to exchange a and a' , $d(a, u) \leq d(a', u')$. Since $d(a', c_{a'b'}) = \ell - 3$, we have $d(a, u) + d(u', c_{a'b'}) \leq d(a', u') + d(u', c_{a'b'}) = \ell - 3$. Since A, B is independent, $d(a, c_{a'b'}) > \ell$, we have $\ell < d(a, u) + d(u, u') + d(u', c_{a'b'}) \leq \ell - 3 + d(u, u')$, and then $d(u, u') \geq 4$. So (b) holds.

Let u be a vertex of the ab -path, and u' be a vertex of the $a'b'$ -path such that $a \neq a'$ and $b \neq b'$. By part (b) of Lemma 21 we may assume without loss of generality that $u \in RS(a)$ and $u' \in RS(b')$. In addition, we can assume that $d(a, u) \leq d(b', u')$. So $d(a, u) + d(u', c_{b'a'}) \leq d(b', u') + d(u', c_{b'a'}) = \ell - 3$. So $\ell < d(a, c_{a'b'}) \leq d(a, u) + d(u, u') + d(u', c_{b'a'}) \leq \ell - 3 + d(u, u')$. Hence $d(u, u') \geq 4$. \square

An edge leaves a set S if exactly one of its endpoints is in S .

Observation 22. Let A, B be an independent pair and $a \in A$. For all $b \neq b'$, we have $c_{ab}^- \neq c_{ab'}^-$ (and then $c_{ab} \neq c_{ab'}$). Moreover the edges of the aB -paths leaving $RS(a)$ form an induced matching.

Recall that, by lexicographic minimality, when two aB -paths separate, they never meet again, so if $c_{ab}^- \neq c_{ab'}^-$, we immediately have $c_{ab} \neq c_{ab'}$.

Proof. Observation 19 ensures that $c_{ab}^- \in RS(b)$ and $c_{ab'}^- \in RS(b')$. So Lemma 21(b) ensures that $c_{ab}^- \neq c_{ab'}^-$. The lexicographic minimality ensures that edges of aB -paths leaving $RS(a)$ are vertex disjoint, i.e. they form a (non necessarily induced) matching. By Observation 19, the edge of P_{ab} leaving $RS(a)$ is an edge with both endpoints in $RS(b)$. Thus Lemma 21(b) ensures that the matching is induced. \square

4.4. Escape property

Let A, B be an independent pair. In the following we consider the restriction of the graph to the vertices of the AB -paths. Let a in A . An escape uv from a is an edge leaving $RS(a)$ such that uv is not an edge of any P_{ab} for $b \in B$. By convention, when uv is an escape from a , we still denote by u the vertex in $RS(a)$ and by v the vertex which is not in $RS(a)$. The vertex u is called the *beginning* of the escape and v the *end* of the escape.

Let uv be an escape from a . Since $u \in RS(a)$, there exists $b \in B$ such that the vertex u is in P_{ab} . Since we have considered the restriction of the graph to the vertices of the AB -paths, the vertex v is in the path $P_{a'b'}$ for $a' \in A$ and $b' \in B$. Lemma 21(a) ensures that either $a = a'$ or $b = b'$. If $a = a'$ then $d(a, v) > \ell - 3$ (otherwise v would be in $RS(a)$). So we have $d(a, u) = \ell - 3$ since $u \in RS(a)$ and uv is an edge. Though the induced matching property of Observation 22 ensures that there is no edge between c_{ab} and v (otherwise the edges of P_{ab} and $P_{ab'}$ leaving $RS(a)$ do not form an induced matching). So $a \neq a'$, i.e. $b = b'$. Thus every escape uv is an escape from a to a' for b . In Fig. 6, the edges uv are escapes from a to a' for b . An escape can be an edge of a minimum path (see the rightmost example of Fig. 6).

A *deep escape* is an escape such that u is neither a critical vertex nor a pre-critical vertex. Let us define two graphs: the *escape graph* of b (resp. *deep escape graph* of b) is a directed graph with vertex set A where aa' is an arc if there is an escape (resp. a deep escape) from a to a' for b . In Fig. 6, the leftmost escape is not a deep escape since $u = c_{ab}$.

If a vertex x which is not in $RS(A)$ has a neighbor in $RS(a)$, a is called an *origin root section* on x . Lemma 21(b) ensures that every vertex has at most one origin root section (otherwise two root sections would be at distance 2). Note that if uv is an escape from a , then a is the origin root section of v .

Let us informally explain why we introduce escapes. As long as a path from a to B follow edges of aB -paths, then we can understand the structure of the path. In particular, if such a path passes through a critical (or pre-critical) vertex we can “evaluate” its length using the fact that $d(a, c_{ab}) = \ell - 3$. If a path uses an escape, it can “escape” $RS(a)$ without passing through such a vertex, which implies that the length of the path is somehow harder to evaluate. Let us first show that the structure of the (deep) escape graph can be constrained.

Lemma 23. Let A, B be an independent pair. For every $b \in B$, the escape graph of b has no circuit.

Proof. Assume that there is a circuit $a_0, a_1, \dots, a_k, a_0$. In the following indices have to be understood modulo $k+1$. For every i , let $u_i v_i$ be an escape from a_i to a_{i+1} for b . Since $u_i \in RS(a_i)$ and $u_{i+1} \in RS(a_{i+1})$, Lemma 21(b) ensures that $d(u_i, u_{i+1}) \geq 4$, then $d(v_i, u_{i+1}) \geq 3$. Hence $d(b, u_i) \leq d(b, v_i) + 1 < d(b, v_i) + d(v_i, u_{i+1}) = d(b, u_{i+1})$. The first inequality comes from the fact that $u_i v_i$ is an edge and the last equality comes from the fact that the path is a minimum path. A propagation of these inequalities along the arcs of the circuit leads to $d(b, u_0) < d(b, u_0)$, a contradiction. \square

The deep escape graph of b is a subgraph, in the sense of arcs, of the escape graph of b . Thus the deep escape graph of b has no circuit. For every b , the order inherited from b is a partial order on A such that $a < a'$ if and only if there is an escape from a to a' for b . An independent pair A, B has the escape property if for every $b \in B$, the deep escape graph of b is a transitive tournament.

Lemma 24. The size of an independent pair with no subpair of size 2^{d+1} satisfying the escape property is at most $r_{2^{d+2}}(2^{d+1})$.

Proof. Let (A, B) be an independent pair of size $r_{2^{d+2}}(2^{d+1}) + 1$.

Claim 5. A, B has a subpair X, Z of size 2^{d+1} such that:

- (1) either the pair X, Z does not contain a deep escape,
- (2) or the pair X, Z satisfies the escape property.

Proof. Let $B' = \{b_1, \dots, b_{2^{d+2}}\}$ be a subset of B of size 2^{d+2} . Consider the complete edge-colored graph G' on vertex set A . The colors are binary integers of 2^{d+2} digits. The i th digit of the color of aa' is 1 if there is a deep escape from a to a' (or from a' to a) for b_i and 0 otherwise. Theorem 16 ensures that G' contains a monochromatic clique X of size 2^{d+1} . Let us denote by c the color of the edges of $G'[X]$. At least 2^{d+1} digits of c are equal. Denote by Z the subset of B' corresponding to these digits. If the digits equal 0 then (1) holds, otherwise (2) holds. \square

Let us prove by contradiction that Claim 5(1) cannot hold. Let X, Z be an independent pair with no deep escape. Consider the restriction of the graph to $\bigcup_{x,z} P_{xz}$. For every x, z , the private part of xz , denoted by $PP(x, z)$, is the set of vertices which belong to P_{xz} and which do not belong to any other path in P_{xz} .

Claim 6. $PP(x, z)$ separates x from c_{xz} and from c_{xz}^- in the graph induced by $RS(x)$.

Proof. Let P be a path from x to c_{xz} in $RS(x)$ and let u be the last vertex of P which is on $P_{xz'}$ for some $z' \neq z$. The vertex u exists since $c_{xz} \neq c_{xz'}$ and $x \in P_{xz'}$ for every $z' \neq z$. Let v be the vertex after u in P . By maximality of u , the vertex v is in P_{xz} (since $v \in RS(x)$). So if $v \notin PP(x, z)$ then $v \in P_{x'z'}$ for some $x' \neq x$. By Lemma 21(a), we have $z = z'$. Thus a vertex of $P_{xz'}$ and a vertex of $P_{x'z}$ are adjacent, contradicting Lemma 21(a).

Let P be a path from x to c_{xz}^- which does not pass through $PP(x, z)$. Since $P_{c_{xz}^-} c_{xz}$ is a path from x to c_{xz} , the first part of the proof ensures that $c_{xz} \in PP(x, z)$. Since $c_{xz}^- \notin PP(x, z)$, the lexicographic minimality ensures that c_{xz}^- is in $P_{x'z'}$ for $z \neq z'$. Lemma 21(a) ensures that $x = x'$. By Observation 19, we have $c_{xz}^- \in RS(z')$ and $c_{xz} \in RS(z)$, contradicting Lemma 21(b). \square

Let us finally prove that X, Z is $(2\ell - 5)$ -disconnectable with the sets $PP(x, z)$. Let $x, z \in X, Z$. Since X, Z is d -localized, P_{xz} has length at most $2\ell - 7$. In addition $PP(x, z)$ does not intersect $P_{x'z'}$ if $x \neq x'$ or $z \neq z'$; so the deletion of $PP(x, z)$ does not delete all the paths from $x'z'$ of length at most $2\ell - 7$. Let us finally show that all the paths of length at most $2\ell - 5$ from x to z pass through $PP(x, z)$.

Since there is no deep escape, any edge leaving $RS(x)$ intersects a critical or a pre-critical vertex. By independence, if $z \neq z'$ then we have $d(c_{xz'}, z) \geq \ell + 1$ and $d(c_{xz}^-, z) \geq \ell$. Moreover, we have $d(x, c_{xz'}) = \ell - 3$ and $d(x, c_{xz}^-) = \ell - 4$. Thus the length of a path from x to z passing through $c_{xz'}$ or c_{xz}^- is at least $2\ell - 4$. Therefore every path of length at most $2\ell - 5$ from x to z passes through c_{xz} or c_{xz}^- . By Claim 6, there is no path of length at most $2\ell - 5$ from x to z in $G[V \setminus PP(x, z)]$. By Theorem 15, the distance VC-dimension is at least $(d + 1)$, a contradiction. So case (1) of Claim 5 cannot hold, i.e. case (2) holds. \square

4.5. Escape property implies large distance VC-dimension

The outline of the proof of Lemma 24 consisted in finding a $(2\ell - 5)$ -disconnecting pair. The approach is the same when the escape property holds even if the proof is more involved.

Definitions of this paragraph are illustrated in Fig. 7. Let $a', b \in A \times B$. The incoming vertex $v_{a'b}$ of the $a'b$ -path is the first vertex in $P_{a'b}$ (from a' to b) for which there exists an escape $u_{a'b} v_{a'b}$ from a to a' for b for some $a \in A$. In other words, it is the first vertex of $P_{a'b}$ at distance one from $RS(A) \setminus RS(a')$. The edge $u_{a'b} v_{a'b}$ is a first-in escape to a' for b . Note that several first-in escapes to a' can exist, but the incoming vertex is unique. The free section of the $a'b$ -path, denoted by $FS(a', b)$, is the $c_{a'b} v_{a'b}$ -subpath of the $a'b$ -path where $c_{a'b}$ is not included but $v_{a'b}$ is included. Lemma 21(b) ensures that the free section exists and has length at least 3.

Lemma 25. Let A, B be a pair satisfying the escape property. Then there is no edge between two free sections of AB -paths.

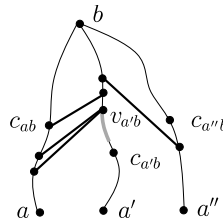


Fig. 7. The vertex v is the incoming vertex of the $a'b$ -path. The gray part (where $v_{a'b}$ is included but not $c_{a'b}$) is the free section of the $a'b$ -path.

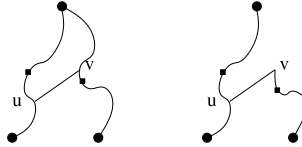


Fig. 8. The minimum path (at the left) is transformed into the jump path (at the right).

Proof. Consider an edge xy where $x \in FS(a', b')$ and $y \in FS(a, b)$. Let us prove that there is a forbidden cross (see Lemma 14). Notice that $x \in RS(b')$ since $x \in P_{a'b'}$ ($FS(a', b')$ is a subpath of $P_{a'b'}$) and $x \notin RS(a')$ (it is after $c_{a'b'}$). Similarly, $y \in RS(b)$. So Lemma 21(b) ensures that $b = b'$. Assume w.l.o.g. that $a < a'$ in the order inherited from b . Hence there is a deep escape uv from a to a' for b . By definition of deep escape, y is strictly after c_{ab} in P_{ab} and u is strictly before c_{ab}^- in P_{ab} . So we have $d(u, y) \geq 3$ (since P_{ab} is a minimum path). Moreover x is before v on $P_{a'b}$ by definition of the free section of $P_{a'b}$. Finally edges xy and uv contradict Lemma 14. \square

Lemma 26. The size of a pair with the escape property is at most $2^{d+1} - 1$.

Proof. Assume by contradiction that a pair A, B of size 2^{d+1} satisfies the escape property. Let $b \in B$. Let us denote by $a_1, \dots, a_{2^{d+1}}$ the vertices of A ordered along the order inherited from b . For every $i \geq 2$, we denote by v_i the incoming vertex and by $u_i v_i$ a first-in escape to a_i for b . By convention we put $v_1 = b$ and $FS(a_1, b)$ is the subpath of $P_{a_1 b}$ from $c_{a_1 b}$ to b . Recall that there exists $j < i$ such that $u_i \in RS(a_j)$. Note that v_j is after u_i on $P_{a_j b}$. Indeed u_i appears before $c_{a_j b}$ since $u_i \in RS(a_j)$ and v_j appears after $c_{a_j b}$. Therefore the following collection of Ab -paths, called jump paths (for b), is well-defined:

- The jump path of $a_1 b$ is the $a_1 b$ -path.
- The jump path of $a_i b$ is the $a_i v_i$ -subpath of $P_{a_i b}$, the edge $v_i u_i$ of origin root section a_j and the suffix path on u_i of the jump path of $a_j b$ (see Fig. 8).

Note that jump paths can be equal to minimum paths (see the rightmost part of Fig. 6).

Let us analyze a bit the structure of jump paths. The jump path of $a_i b$ starts with the $a_i v_i$ -subpath of $P_{a_i b}$. In particular both paths coincide in $RS(a_i)$. Then the jump path of $a_i b$ contains the first in-escape to a_i , namely the edge $v_i u_i$. By definition of the order, the vertex u_i is in $RS(a_j)$ for $j < i$ and then $u_i v_i$ is an escape from a_j to a_i for b . Thus u_i is in $P_{a_j b} \cap RS(a_j)$. So it is on the jump path of $a_j b$. After this “rerouting” the two jump paths are the same and do not quit each other before the end of the path.

Jump paths follow minimum AB -paths except on incoming vertices in which they are “rerouted”. A rerouting edge is an edge e such that there exists i satisfying $e = u_i v_i$. Since after a rerouting edge, the jump path of $a_i b$ coincides with the jump path $a_j b$ for $j < i$, every jump path has at most 2^{d+1} rerouting edges. Moreover each rerouting edge increases the length of the path by at most two since $|d(u_i, b) - d(v_i, b)| \leq 1$ ($u_i v_i$ is an edge and $P_{a_j b}$ is minimum). Since a pair with the escape property is d -localized (each path P_{ab} has length at most $2\ell - 2^{d+2} - 3$), the length of the jump path of ab is at most

$$(2\ell - 2^{d+2} - 3) + 2^{d+1} \cdot 2 = 2\ell - 3 \quad (1)$$

for every pair a, b . Let us now state a claim on the structure of the paths.

Claim 7. Any vertex of a jump path is either in $RS(A)$ or in a free section $FS(a, b)$. Moreover any vertex of a jump path for b is in $\bigcup_{a \in A} P_{ab}$.

Proof. By induction on the order inherited from b . It holds for the jump path of $a_1 b$. The jump path of $a_i b$ coincides with the $a_i b$ -path from a to the incoming vertex, i.e. on $RS(a_i)$ and on $FS(a_i, b)$. By induction, it holds for the remaining vertices since the remaining of the jump path of $a_i b$ is included in the jump path of $a_j b$ for $j < i$. \square

In the remaining of the proof we consider the restriction of the graph to the vertices of the jump paths of ab for every $a, b \in A \times B$. Let $a_i \in A, b \in B$. Remind that the first vertex of $FS(a_i, b)$ is the vertex after $c_{a_i b}$ in $P_{a_i b}$ and the last one is v_i , the incoming vertex of $P_{a_i b}$.

Claim 8. Let $i \geq 2$. The vertices of $FS(a_i, b)$ induce a subpath $w_1, \dots, w_k = v_i$ of $P_{a_i b}$. The only neighbors of these vertices are the following:

- For every $1 \leq q \leq k$, the vertex w_q is incident to w_{q-1} and w_{q+1} (if they exist).
- The vertex $w_k = v_i$ has neighbors in $RS(a_j)$ where a_j is the origin root section of v_i (in particular $j < i$ in the order inherited from b).
- The vertex w_1 is incident to $c_{a_i b}$.

Proof. Claim 7 ensures that every vertex is either in $RS(A)$ or in $FS(A, B)$. By Lemma 25, there is no edge between two free sections. So an edge leaving $FS(a_i, b)$ has an endpoint in $RS(A)$. By definition of incoming vertex, no vertex of $FS(a_i, b)$ distinct from v_i is incident to a vertex of $RS(a_j)$ with $j \neq i$. Moreover, since $P_{a_i b}$ is a minimum path, w_1 is the unique vertex of $FS(a_i, b)$ which can be incident to $RS(a_j)$.

By definition of v_i , there exist edges between v_i and the origin root section of v_i , namely $RS(a_j)$. Since every vertex has at most one origin root section, the second point holds.

The vertex w_1 is incident to $c_{a_i b}$ since they are consecutive in $P_{a_i b}$. Others neighbors of w_1 in $RS(a_i)$ must be critical vertices since $d(a_i, w_1) = \ell - 2$ (indeed $d(c_{a_i b}, a_i) = \ell - 3$ and $P_{a_i b}$ is minimum). Thus the matching property of Observation 22 ensures that w_1 has no other neighbor in $RS(a_i)$, which concludes the proof of Claim 8. \square

In particular, Claim 8 ensures that any path P leaving $FS(a_i, b)$ has to enter in $RS(a_i)$ or in $RS(a_j)$. Conversely, you can notice that any neighbor of a vertex in $RS(a_i)$ is either in $RS(a_i)$ or is in some $FS(a_j, b')$ for $j > i$. These two observations are the most important pieces of the proofs of the remaining statements.

Remind that any path of length at most $2\ell - 3$ from a to b does not pass through $c_{a'b'}$ with $b \neq b'$. Indeed by independence, $d(a, c_{a'b'}) \geq \ell - 3$ and $d(b, c_{a'b'}) > \ell$.

Claim 9. Any path of length at most $2\ell - 3$ from a_i to b does not contain any vertex in $RS(a_j)$ for $j > i$ (in the order inherited from b).

Proof. Assume by contradiction that such a path P exists and denote by j the maximum index such that P passes through $RS(a_j)$. Note that $j \geq 2$. Let u be the first vertex of P in $RS(a_j)$ and let v be the vertex before u in P . The path P cannot enter in $RS(a_j)$ through $c_{a_j b'}$ with $b' \neq b$ since P has length at most $2\ell - 3$. Lemma 21(a) ensures that $v \notin RS(A)$. So Claim 7 ensures that $v \in FS(a_k, b')$.

Assume first that $a_k \neq a_j$. Since $u \in RS(a_j)$, uv is an escape from a_j to a_k for b' . In particular, it means that $k > j$. Since u is the first vertex of P in $RS(a_j)$, the path P cannot enter in $FS(a_k, b')$ through $RS(a_j)$. So Claim 8 ensures that P enters in $FS(a_k, b')$ through $RS(a_k)$, contradicting the maximality of j .

Assume now that $a_k = a_j$. Claim 8 ensures that the unique vertex of $FS(a_j, b')$ with a neighbor in $RS(a_j)$ is the first vertex of $FS(a_j, b')$, so v is this vertex. Moreover, the unique neighbor of v in $RS(a_j)$ is the vertex $c_{a_j b'}$ by Claim 8. Since P cannot pass through $c_{a' b'}$ with $a' \neq a_i$ and $b' \neq b$, we have $b' = b$. So $u = c_{a_j b}$ and v is the first vertex of $FS(a_j, b)$. Let us now denote by w the last vertex of P in $RS(a_j)$. Note that $w \neq c_{a_j b'}$ for $b' \neq b$. Moreover the vertex after w in P cannot be in $FS(a_j, b)$ since otherwise this vertex would be v , and then P would not be a path (v would appear twice in P). So the edge used to leave $RS(a_j)$ is an escape to a_ℓ for b'' . In particular, $\ell > j$. By Claim 8, vertices of $FS(a_\ell, b'')$ only have neighbors in $RS(a_j)$ and in $RS(a_\ell)$. Since w is the last vertex in $RS(a_j)$, when P leaves $FS(a_\ell, b'')$ it enters in $RS(a_\ell)$, contradicting the maximality of j . \square

Claim 10. The vertex c_{ab} is in every path P from a to b of length at most $2\ell - 3$. Moreover if a vertex of the ac_{ab} -subpath of P is not in $RS(a)$, then the next one is.

Proof. Let P be a path from a to b of length at most $2\ell - 3$. Let u be the last vertex of u in $RS(a)$. Let v be the vertex after u in P . For distance reasons, the vertex u is not $c_{ab'}$ for $b' \neq b$. Let us show that P does not leave $RS(a)$ using an escape. Assume by contradiction that v is in $FS(a_j, b')$ for $a_j > a$ (in the order inherited from b'). Let us denote by w the first vertex of P after v which is not in $FS(a_j, b')$. By Claim 8, w is either in $RS(a)$ or is $c_{a_j b'}$. Since w is after u in P , $w \notin RS(a)$, so $w = c_{a_j b'}$. If $b \neq b'$, we have a distance contradiction since both a and b are at distance more than ℓ from $c_{a_j b'}$. If $b = b'$, then $a_j > a$ in the order inherited from b , contradicting Claim 9.

So the vertex u is the vertex c_{ab} . In addition, in the ac_{ab} -subpath of P , if a vertex is not in $RS(a)$, then it is in $FS(a_j, b)$ where $a_j > a$. Claims 8 and 9 ensure that the next vertex is in $RS(a)$. \square

The jump private part of a and b , denoted by $JPP(a, b)$, is the set of vertices which are in the jump path of ab and in no other jump path.

Claim 11. All the paths of length at most $2\ell - 3$ from a to b pass through $JPP(a, b)$.

Proof. Let P be a path from a to b of length at most $2\ell - 3$. Claim 10 ensures that P passes through c_{ab} . Assume by contradiction that the subpath of P between a and c_{ab} does not pass through $JPP(a, b)$. Let u be the last vertex of the ac_{ab} -subpath of P which is in the path $P_{ab'}$ for $b' \neq b$. Such a vertex exists since c_{ab} is not in $P_{ab'}$ for $b' \neq b$ by Observation 22. Let v be the vertex after u on P . And, for every $b' \in B$, a is in the path $P_{ab'}$.

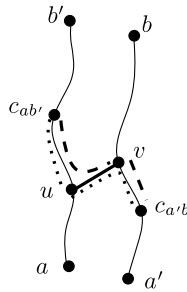


Fig. 9. Illustration of Claim 11. The two dotted paths are the two sides of an inequality. And the two dashed paths are the two sides of the other one.

If $v \notin RS(a)$ then Claim 10 ensures that the vertex after v is in $RS(a)$. So v is in $P_{a'b''}$ with $a' \neq a$. The vertex after v is in $P_{ab''}$ since it is in $RS(a)$. By maximality of u , we have $b = b''$. Thus $u \in P_{ab'}$ for $b' \neq b$ (by definition of u) and $v \in P_{a'b}$ for $a' \neq a$, a contradiction with Lemma 21(a).

So $v \in RS(a)$ and then $v \in P_{ab}$. Assume by contradiction that $v \notin JPP(a, b)$. So the vertex v is in the jump path of $a'b$ for some $a' \neq a$. Free to modify a' , we may assume that the jump path of $a'b$ has been rerouted only once before v . The vertex v is on the $c_{a'b}$ -subpath of the jump path of $a'b$ and u is on the $ac_{ab'}$ -subpath of $P_{ab'}$. The following two inequalities, illustrated in Fig. 9, provide a contradiction.

First $d(u, c_{ab'}) + 3 < d(v, c_{a'b}) + 1$ since $d(a, c_{ab'}) \leq \ell - 3$ and $d(a, c_{a'b}) > \ell$. Indeed, by definition of critical vertex, $d(a, c_{ab'}) = \ell - 3$ (even in this induced subgraph) and $d(a, c_{a'b}) > \ell$ is a consequence of the independence. Since u is on a minimum $ac_{ab'}$ -path, the inequality holds.

Second $d(v, c_{a'b}) < d(u, c_{ab'}) + 1$ since $d(b, c_{a'b}) \leq \ell$ and $d(b, c_{ab'}) > \ell$ and uv is an edge. The first inequality is due to the fact that jump paths have length at most $2\ell - 3$ and that the length of the $a'c_{a'b}$ -subpath of the jump path of $a'b$ is exactly $\ell - 3$. The second inequality is a consequence of the independence of A, B .

The sum of these two inequalities gives $3 < 2$, a contradiction. \square

To conclude the proof of Lemma 26, we apply Theorem 15 with the sets $JPP(a, b)$ for paths of length at most $2\ell - 3$. Eq. (1) ensures that the jump path of ab has length at most $2\ell - 3$, so if $JPP(a, b)$ is not selected, there remain paths of length at most $2\ell - 3$. The sets $JPP(a, b)$ are pairwise disjoint and are only on the jump path of ab . Claims 10 and 11 ensure that the sets $JPP(a, b)$ are $(2\ell - 3)$ -disconnecting for A, B . So the graph G has distance VC-dimension at least $d + 1$, a contradiction. \square

By combining Theorem 12 and Lemmas 18, 20, 24 and 26, we obtain Theorem 11.

5. Concluding remarks

In Section 4, we did not make any attempt to improve the gap function. We made exponential extractions at several steps as Ramsey's extractions and the function of Theorem 12 is not expressed in the original paper of Matoušek. Finding a polynomial gap instead of an exponential one is an interesting problem, though probably a hard one. We can also study this problem for particular classes of graphs. Chepoi, Estellon and Vaxès conjectured that the gap function between v_ℓ and τ_ℓ for planar graphs is linear. More formally they conjectured the following.

Conjecture 27 (Chepoi, Estellon, Vaxès [5]). *There exists a constant c such that $\tau_\ell(G) \leq c \cdot v_\ell(G)$ for every ℓ and every planar graph G .*

Dvůřák proved in [7] that $\tau_\ell \leq c(\ell)v_\ell$ for bounded expansion classes. Moreover the function c is a polynomial function.

In graph coloring, we need some structure to bound the chromatic number. The *chromatic number* $\chi(G)$ of the graph G is the minimum number of colors needed to color properly the vertices of G , i.e. such that two adjacent vertices of G receive distinct colors. The size of the maximum clique of G , denoted by $\omega(G)$, is a lower bound on the chromatic number $\chi(G)$. The gap between χ and ω can be arbitrarily large since there exist triangle-free graphs with an arbitrarily large chromatic number (Erdős was the first to construct some of them in [10]). A class of graphs \mathcal{C} is χ -bounded if there exists a function f such that for every graph $G \in \mathcal{C}$, every induced subgraph G' of G satisfies $\chi(G') \leq f(\omega(G'))$. Dvůřák and Král' proved in [8] that graphs of bounded rankwidth are χ -bounded. Actually they proved it for classes of graphs with cuts of small rank. Since the distance VC-dimension catches the complexity of the intersection of neighborhoods at large distance, the same might be extended for graphs of bounded distance VC-dimension.

Conjecture 28. *Let \mathcal{G} be a class of graphs. If there exists a function f such that the distance VC-dimension of $G \in \mathcal{G}$ is at most $f(\omega(G))$ then \mathcal{G} is χ -bounded.*

We also conjecture that the following graph classes, known to be χ -bounded, have a bounded distance VC-dimension.

Conjecture 29. *The distance VC-dimension of every P_ℓ -free graph G is bounded by a function of ℓ and $\omega(G)$. Similarly the distance VC-dimension of every circle graph G is bounded by a function of $\omega(G)$.*

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