INFORMATION AND CONTROL 11, 3–29 (1967)

Algebraic Automata and Context-Free Sets

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The concepts of "equational" and "recognizable" are defined for sets of elements of an arbitrary abstract algebra. Context-free languages and finite-state languages become realizations of the same general concept (of equational sets) when the proper algebra is specified. A principal objective of the paper is to establish in the context of abstract algebra, relationships between such concepts as equational and recognizable sets.

I. INTRODUCTION

Algol-like or context-free languages over a finite alphabet Σ can be defined by systems of fixed point equations in which juxtaposition represents the operation of the semigroup $F(\Sigma)$ freely generated by Σ. A system of fixed-point equations determines an inductive process for generating elements of finitely many context-free sets simultaneously. Similarly, finite-state (regular) languages over the same alphabet are definable by fixed-point equations in which symbols of Σ denote operators of the generic monadic algebra M_Σ. The algebra M_Σ contains as elements Σ-strings, including the null string, and for each $\sigma \in \Sigma$ the operation $S \rightarrow S\sigma$ for all strings S.

The purpose of this investigation is to provide a basis for the theory of an arbitrary algebra, which will subsume the theory of $F(\Sigma)$ and M_Σ in such a way that context-free languages and finite-state languages become realizations of the same general concept when the *proper* algebra is specified.

The following concepts are defined for an arbitrary abstract algebra $\mathfrak{A}$, where α denotes a set of elements of $\mathfrak{A}$.

(1) α is equational iff it is definable by a system of fixed point equations in the syntax of $\mathfrak{A}$.

(2) α is recognizable iff it is a union of classes of a finite congruence on $\mathfrak{A}$.

A main result of the paper is a finite procedure for producing from a given system of equations, a second system of simpler form which is such that any set definable by the given system is a union of sets definable by the second system.

Another principal result is that if $\mathfrak{A}$ is a generic algebra with a finite number of operation symbols, then a set of elements of $\mathfrak{A}$ is equational if and only if it is recognizable. If finite algebras with distinguished terminal sets are taken to be generalized finite automata, then recognizability is a natural generalization of the concept of a set of strings being accepted by a finite automaton.

THEOREM. *A set of elements of an algebra $\mathfrak{A}$ with a finite number of operation symbols is equational if and only if it is the homomorphic image of a recognizable subset of the generic algebra with the same operation symbols as $\mathfrak{A}$.*

If $F(\Sigma)$ provides symbols for the elements of Σ, then equational sets of this algebra are exactly the context-free languages. The equational sets of M_Σ are exactly the finite-state languages. It is further possible, by introducing operators such as for example $S \rightarrow SS$, where S is a Σ-string, to obtain as equational sets, sets of Σ-strings that are not context-free.

We remark that the theorem above yields in particular that the context-free sets of $F(\Sigma)$ are homomorphic images of recognizable subsets of the free grouppoid generated by Σ.

The work presented in this paper grew out of reports by many authors on formal linguistics, automata and computability. The list of references gives an illustrative sample of such literature.

Consider the set L of the strings *ab*, *aabb*, *aaabbb*, $\cdots$ which is a context-free language [Chomsky and Schützenberger, 1963). L may be generated by

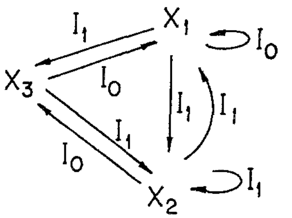
(1) placing in the set the string *ab*; and

(2) placing in the set the string *asb* when the string *s* is already present.

This process amounts to specifying L as the union of an increasing sequence of sets. With later developments in view, it should be noted that L may be defined as the smallest class of strings containing *ab* and "closed" with respect to the operation of "simultaneously flanking a string on the left by *a* and on the right by *b*." This definition amounts to specifying L as an intersection of closed sets.

The simple situation of defining the set L in isolation contrasts with the general case, illustrated by the problem of defining the set C of "proper parenthetical expressions." In the natural process for generating C, an auxiliary set C^* is employed, and since C, C^* are interdependent, the sets are defined simultaneously. (The process of simultaneous definition is carried out with the aid of the system $\mathcal{E}$ of equations of Definition 2.7, which under the appropriate interpretation will yield C^*, C as associated with the first and the second of its equations, respectively. A further discussion of this example is found in Section V.)

In the theory of automata, as well, simultaneous definition or generation of sets plays an important role (Rabin and Scott, 1959). A finite-state graph, such as the one in Fig. 1 may be interpreted as a simul-



X_3 is initial
FIG. 1.

taneous inductive definition of sets, one assigned to each state with (1) the null-string placed in each set assigned to a state that is distinguished as initial, and (2) in general strings placed in sets as indicated by the labeled arrows. Thus, in the example of Fig. 1, the null-string is placed in X_3, strings I_0, I_0I_0 placed in X_1, and I_0I_1 placed in both X_2, X_3.

(The system of equations of Example 3.4 will, under the appropriate interpretation, yield precisely the sets defined inductively by this graph).

A major objective of this investigation is to reformulate the several different types of inductive definitions, such as the ones illustrated above, in such a way that the common features can be abstracted to become basic components of a unified general theory.

Presupposing the set of all strings in the symbols a and b, and the operation of concatenation of strings, Ginsburg and Rice (1962) used equations of the fixed-point form to define "ALGOL-like languages," that is context-free sets such as L in the above example. Presupposing the same structure, Arden (1961) introduced a special form of equations with the property that a set of strings is definable by a system of Arden equations if and only if it is recognized by a finite automaton. In the Arden equations a variable could occur only at the beginning of a string.

If, however, one presupposes the set of all strings in the symbols a and b and the operations $s \to sa$, $s \to sb$, the Arden equations can be reinterpreted as fixed-point equations, with no special restrictions required on the form of such equations.

The point is, that the differences between context-free sets on one hand and finite-automaton recognizable sets on the other can be accounted for adequately by differences between the underlying structures and the methods of definition can be made analogous in the two cases. The underlying structures are a free semigroup and a generic monadic algebra [a structure determined by one-argument operators; see Büchi and Wright (1960)], respectively, both of which are abstract algebras. In our investigation, then, we wish to introduce the general concept of "an equational subset of an arbitrary algebra," as a set defined by a "system of simultaneous equations of the fixed-point form," appropriate for the given algebra. In particular, context-free sets are equational sets defined by semigroup equations and finite-automaton recognizable sets are equational sets defined by monadic algebra equations, both subsumed under the single concept of equational sets.

The theory in this paper is developed within the framework of "abstract" or "universal" algebra. By an "algebra," we will mean a set called the "carrier" together with a family of finitary operations on the carrier, indexed (or denoted) by given operator symbols.

In Section II, we define the concept of a term[1] (or well-formed algebraic expression), and given an algebra, we associate with a term a function over that algebra. The terms are viewed as being applicable to all the algebras for which the operator symbols that occur in a term have an interpretation. The use of capital letters in terms is meant to emphasize the fact that we are associating with terms "term functions" that have arguments, and take on values, which are *subsets*, rather than elements, of the carrier of the algebra under consideration.

$$1 - 2x_1X_3^{-1}, \qquad (1 - 2x_1^2)(\tfrac{1}{3} - x_3).$$

A system of equations of the fixed point form[2] is then defined as a

[1] The concept of term formalizes the intuitive notion of expression, as exemplified by the following two, written over the field of rationals:

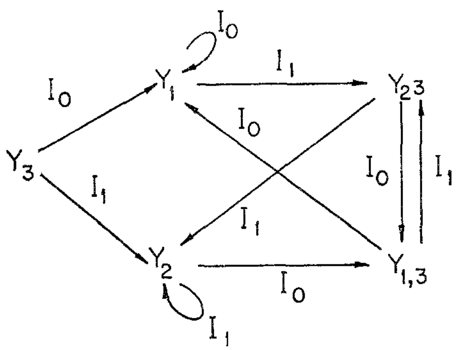
[2] For example,

$$X_1 = \{+1X_2\},$$
$$X_2 = \{+11, +X_2X_2, -X_2\}.$$

This system, with the usual interpretation of $+$, $-$, 1 over $\mathbf{Z}$, can be employed to obtain the set of odd and the set of even integers simultaneously.

finite sequence of equalities constrained in that each left-hand side consists of a (distinct) single variable, and in that each right-hand side is a finite collection of terms. Given an algebra, there is associated with a system of equations a "system function" over that algebra. If the system had m equations, the system function carries m-tuples of subsets to m-tuples of subsets of the carrier. An m-tuple of sets is a solution to the system of equations, i.e. causes the equalities to hold, if and only if it is a fixed-point of the associated system function. A set is "equational" if and only if it is a component of the *smallest* solution to such a system of equations of the fixed-point form.

The concept of finite-automaton recognizable set may also be generalized in the framework of abstract algebra. A finite automaton may be defined as a finite monadic algebra, a subset of whose carrier is distinguished as terminal (Büchi and Wrighl (1960)). In Fig. 2 appears the "graph" of such an algebra, with the elements of the carrier as the nodes and the one-argument functions represented by labeled and directed edges.



Y_3 is initial, $\{Y_{1,3}, Y_{2,3}\}$ is the terminal set.
FIG. 2.

A string is said to belong to the set recognized by a finite automaton if and only if 'it takes the automaton from its initial state to a terminal state'. This is expressed mathematically by defining a set of strings recognized by a finite automaton as the inverse image of the terminal set under the homomorphism that maps the generic monadic algebra referred to previously onto the finite automaton.

In the appropriate generalization, an arbitrary finite algebra with a distinguished terminal set will be referred to informally as a "generalized finite automaton', (Thatcher and Wright, 1966). For an arbitrary algebra, a recognizable set is defined as a union of classes of a finite congruence on the algebra. This definition is equivalent to the one based on generalized finite automaton because of the correspondence between homomorphisms and congruences. David Muller has proposed the definition of a "grammatical" set of an algebra as a homomorphic image of a set which is recognizable with respect to a finitely generated algebra. A principal objective of this paper is to establish, in the context of abstract algebra, relationships between such concepts as equational, recognizable and grammatical sets.

The principal results obtained in this paper may be summarized as follows.

Section III contains a number of finite procedures whose overall effect is the following. Given any system of equations, the procedure transforms it to a system in normal form, with the property that any component of a smallest solution to the original system is a union over components of the corresponding smallest solution to the normal system. A normal system is understood as one whose terms are of rank 1, with each term present in at most one equation and whose solutions have no empty components. (See Theorem 3.9 and Remark 4.7.)

The Theorems 4.6, and 4.8 of Section IV can be jointly stated as saying that for any generic algebra, the recognizable sets are exactly the same as the equational sets. The sense of Theorem 5.5 of Section V is best understood in the light of extensions discussed as the end of that section. In those terms and with only broad limitations it may be said that for an arbitrary algebra, the equational sets are the same as the grammatical sets.

The last result in particular, yields the characterization of the context-free languages of the free semigroup on n generators as being the homomorphic images of the recognizable sets of the free grouppoid (algebra with a single binary operator) on n generators.

II. SYSTEMS OF EQUATIONS

DEFINITION 2.1. Let $\{\Omega_n : n \geqq 0\}$ be a collection of pair-wise disjoint sets and $\Omega = \bigcup \{\Omega_n : n \geqq 0\}$. Ω is referred to as the set of "operator symbols," an element of Ω_n is referred to as an "n-ary operator symbol." An "algebra" $\mathcal{A} = (S, h)$ consists of a non-empty set S, the "carrier" of $\mathcal{A}$, and of the mapping h that assigns to each operator symbol $J \in \Omega$ an operator h_J on S. If $J \in \Omega_n$ then $h_J : S^n \to S$ is an "n-ary operator." By convention, a zeroary operator h_J, $J \in \Omega_0$, is an element of S.

Given the algebra $\mathfrak{A} = (S, h)$ we wish to extend its operators to subsets of S, and, in fact, close the collection of operators on 2^S thus obtained under a generalized functional composition. To this end, we make the following definitions.

DEFINITION 2.2. Consider the algebra $\mathfrak{A} = (S, h)$. For each $J \in \Omega$, let the "complex operator" $\hat{h}_J$ be defined as follows:

1. if $J \in \Omega_0$, then $\hat{h}_J = \{h_J\}$
2. if $J \in \Omega_n$, $n > 0$, then for all $S_1, \cdots, S_n \subseteq S$,

$$\hat{h}_J(S_1, \cdots, S_n) = \{h_J(s_1, \cdots, s_n) : s_i \in S_i, 1 \leqq i \leqq n\}.$$

We note that we have thus defined an algebra $\hat{\mathfrak{A}} = (2^S, \hat{h})$ the "subset algebra" associated with $\mathfrak{A}$, and that $\mathfrak{A}$, $\hat{\mathfrak{A}}$ determine each other uniquely. Further, it may be verified that each operator $\hat{h}_J$ is "completely distributive" that is, given arbitrary families $\mathcal{S}_1, \cdots, \mathcal{S}_n \subseteq 2^S$,

$$\hat{h}_J(\bigcup_{\mathcal{S}_1} S_1, \cdots, \bigcup_{\mathcal{S}_n} S_n) = \bigcup_{\mathcal{S}_1} \cdots \bigcup_{\mathcal{S}_n} \{\hat{h}_J(S_1, \cdots, S_n)\}.$$

DEFINITION 2.3. Let $\mathcal{V} = \{X_1, X_2, \cdots\}$ be a set of "variables" ordered by an enumeration. The collection $\mathfrak{T}$ of "terms" is the smallest set such that

1. $\mathcal{V} \subseteq \mathfrak{T}$,
2. $\Omega_0 \subseteq \mathfrak{T}$,
3. for all $J \in \Omega_n$, $n > 0$ and all $T_1, \cdots, T_n \in \mathfrak{T}$:

$$JT_1 \cdots T_n \in \mathfrak{T}.$$

The "rank" of terms is defined as follows:

1. rank $(V) = 0$, for $V \in \mathcal{V}$
2. rank $(J) = 1$, for $J \in \Omega_0$
3. rank $(JT_1 \cdots T_n) = \max\{\text{rank}(T_i) : 1 \leqq i \leqq n\} + 1$.

For later use we call attention to the following two special classes of terms: 1) *linear*, in which each variable occurs no more than once, and 2) *linear distinguished* (L.d.) in which the leftmost variable to occur is X_1, the next leftmost is X_2, etc. Note that terms with no variables are L.d. All terms T may be associated with a unique L.d. term $\tilde{T}$, from which they may be obtained by a replacement of variables. For example, if $T = IX_3JX_5X_3$ with $I, J \in \Omega_2$, then $\tilde{T} = IX_1JX_2X_3$.

DEFINITION 2.4. Consider a term $T \in \mathfrak{T}$. Let Ω_T, $\mathcal{V}_T$, respectively, denote the collection of operator symbols, respectively, variables, that occur in T. Suppose an algebra $\mathfrak{A} = (S, h)$ is given. Since $\mathcal{V}_T$ inherits an order from $\mathcal{V}$, T determines in the natural way a "term function" $|\mathfrak{A}T|$ on 2^S, where $|\mathfrak{A}T| : (2^S)^n \to 2^S$, with $n = \text{card } \mathcal{V}_T$.

For example, given $J \in \Omega_0$, $|\mathfrak{A}J|$ is the constant function with the value $\hat{h}_J$, and given $V \in \mathcal{V}$, $|\mathfrak{A}V|$ is the identity mapping on 2^S. As a final example, let $T = IX_3JX_5X_3$ with $I, J \in \Omega_2$. Then, given S_1, $S_2 \subseteq S$ the value of $|\mathfrak{A}T|$ at (S_1, S_2) is

$$\hat{h}_I(S_1, \hat{h}_J(S_2, S_1)).$$

Note for later use that $|\mathfrak{A}T|$ may be obtained as an appropriate restriction of $|\mathfrak{A}\tilde{T}|$. Thus, for the above example, $\tilde{T} = IX_1JX_2X_3$ one has:

$$|\mathfrak{A}T|(S_1, S_2) = |\mathfrak{A}\tilde{T}|(S_1, S_2, S_1).$$

We have remarked (Definition 2.2) that the complex operators $\hat{h}_J$ are completely distributive, and this is true of all term functions associated with linear terms. In general, term functions are not completely distributive. They do, however, have the property of "distributivity over ω-chains." That is, if $\mathcal{V}_T = \{V_1, \cdots, V_k\}$ and $A_0^1 \subseteq A_1^1 \rightarrow \subseteq \cdots$, $A_0^2 \subseteq A_1^2 \subseteq \cdots, \cdots, A_0^k \subseteq A_1^k \subseteq \cdots$ and k nondecreasing sequences of subsets of S, i.e., "ω-chains over S," then:

$$|\mathfrak{A}T|(\bigcup \{A_i^1 : i = 0, 1, \cdots\}, \cdots, \bigcup \{A_j^k : j = 0, 1, \cdots\})$$
$$= \bigcup \{|\mathfrak{A}T|(A_i^1, \cdots, A_i^k) : i = 0, 1, \cdots\}$$

In particular, the term functions are "order-preserving" in the sense that, for $A_1, \cdots A_k, B_1, \cdots B_k \subseteq S$, if $A_1 \subseteq B_1, \cdots, A_k \subseteq B_k$, then

$$|\mathfrak{A}T|(A_1, \cdots A_k) \subseteq |\mathfrak{A}T|(B_1, \cdots, B_k).$$

Remark 2.5. Throughout the paper we have occasion to employ functions of arguments $\mathcal{V}_1$ as applied to some set $\mathcal{V}_2$ of arguments, $\mathcal{V}_1 \subseteq \mathcal{V}_2$. In all these cases, we intend, as is conventional, for the function to be vacuous in the arguments $\mathcal{V}_2 - \mathcal{V}_1$, while unchanged in arguments $\mathcal{V}_1$.

DEFINITION 2.6. Let E be a collection of terms, $E \subseteq \mathfrak{I}$. Let $\Omega_E = \bigcup \{\Omega_T : T \in E\}$, $\mathcal{V}_E = \bigcup \{\mathcal{V}_T : T \in E\}$. Again assume an algebra $\mathfrak{A} = (S, h)$ given. We associate with E a mapping $|\mathfrak{A}E|$, $|\mathfrak{A}E| : (2^S)^n \rightarrow S$, where $n = \text{card}(\mathcal{V}_E)$, as follows: For any $S_1, \cdots, S_n \subseteq S$,

$$|\mathfrak{A}E|(S_1, \cdots, S_n) = \bigcup \{|\mathfrak{A}T|(S_1, \cdots, S_n) : T \in E\}.$$

DEFINITION 2.7. A "system (of equations)" $\mathcal{E}$ is a finite sequence of expressions of the form

$$V_i = E_i$$

for $1 \leqq i \leqq m$, where the V are *distinct* variables and the E_i are sets of terms. Let $\mathcal{V}_\mathcal{E} = \{V_1, \cdots, V_m\}$ be the set of "unknowns of $\mathcal{E}$." $\mathcal{V}_\mathcal{E}$ is ordered and the equations are ordered accordingly. Unless otherwise specified, we assume:

1. E_i is finite for $1 \leqq i \leqq m$,
2. $\mathcal{V}_{E_i} \leqq \mathcal{V}_\mathcal{E}$ for $1 \leqq i \leqq m$.

We refer to the expression $V_i = E_i$ to be the set E_i as the "equation of V_i" and the "right-hand set of V_i," respectively. We define $E = \bigcup \{E_i : 1 \leqq i \leqq m\}$, $\Omega_\mathcal{E} = \bigcup \{\Omega_{E_1} : 1 \leqq i \leqq m\}$ to be the sets of terms of $\mathcal{E}$," and "operator symbols of $\mathcal{E}$."

For example, let $\Omega = \Omega_0 \cup \Omega_2$ with $\Omega_0 = \{A, B\}$, $\Omega_2 = \{\frown\}$. Let $\mathcal{E}$ be the system

$$X_3 = \{\frown AB, \frown\frown AX_5B\},$$

$$X_5 = \{\frown X_5X_3, X_3\}.$$

Then, for example: $V_1 = X_3$, $E_2 = \{\frown X_5X_3, X_3\}$, $\mathcal{V}_\mathcal{E} = \{X_3, X_5\}$.

It is convenient to present examples in terms of a more informal notation. In the informal notation, the unknowns are taken as distinct and unindexed symbols, the braces enclosing sets are dropped and a plus sign written to separate elements in a set. Hence the example of this definition would appear as

$$U = AB + (AW)B,$$

$$W = WU + U.$$

In this instance, the binary operator symbol has been replaced by the conventional device of concatenation and parentheses.

DEFINITION 2.8. Let $\mathcal{E}$ be a system of m equations, $\mathcal{A} = (S, h)$ an algebra. One associates with $\mathcal{E}$ the following "system function" $|\mathcal{A}\mathcal{E}|$, $|\mathcal{A}\mathcal{E}| : (2^S)^m \to (2^S)^m$. For all $S_1, \cdots, S_m, Z_1, \cdots, Z_m \subseteq S$,

$$|\mathcal{A}\mathcal{E}|(S_1, \cdots, S_m) = (Z_1, \cdots, Z_m) \text{ iff } Z_i = |\mathcal{A}E_i|(S_1, \cdots, S_m)$$

for each $1 \leqq i \leqq m$.

Let L be the lattice $(2^S)^m$ ordered by componentwise set inclusion, and 0 denote the m-tuple of empty sets. Since term functions are order-preserving, $|\mathcal{A}\mathcal{E}|$ is a monotone map of L into itself and the sequence

$\{|\mathcal{A}\mathcal{E}|^j(0)\}$ is a ω-chain. By virtue of the definitions, $|\mathcal{A}\mathcal{E}|$ inherits from the term functions the property of distributivity over ω-chains (see Definition 2.4) and one has

$$|\mathcal{A}\mathcal{E}|(\bigcup\{|\mathcal{A}\mathcal{E}|^j(0):j = 0, 1 \cdots\}) = \bigcup\{|\mathcal{A}\mathcal{E}|\,|\mathcal{A}\mathcal{E}|^j(0):$$
$$j = 0, 1 \cdots\} = \bigcup\{|\mathcal{A}\mathcal{E}|^j(0):j = 0, 1 \cdots\}.$$

We shall, in what follows, denote the union $\bigcup\{|\mathcal{A}\mathcal{E}|^j(0):j = 0, 1 \cdots\}$ by $[\mathcal{A}\mathcal{E}^\omega]$. The above equality says that $[\mathcal{A}\mathcal{E}^\omega]$ is a fixed point of $|\mathcal{A}\mathcal{E}|$. Since 0 is less than or equal to any fixed point of $|\mathcal{A}\mathcal{E}|$, and $|\mathcal{A}\mathcal{E}|$ is monotonic, $[\mathcal{A}\mathcal{E}^\omega]$ is the smallest fixed point of $|\mathcal{A}\mathcal{E}|$.

DEFINITION 2.9. A system $\mathcal{E}$ is of rank 1 iff all its terms are of rank 1.

DEFINITION 2.10. A system $\mathcal{E}$ of m equations is deterministic iff the collection $\{E_1, \cdots, E_m\}$ of nonempty right-hand sets of $\mathcal{E}$ is a partition of the set of all terms T such that rank $(T) = 1$ and $\mathcal{V}_T \subseteq \mathcal{V}_\mathcal{E}$. Note that deterministic systems exist iff Ω is finite. A deterministic system is necessarily of rank 1.

DEFINITION 2.11. A system $\mathcal{E}$ is reduced iff for all algebras $\mathcal{A}$, all components of $[\mathcal{A}\mathcal{E}^\omega]$ are nonempty.

III. REDUCED DETERMINISTIC SYSTEMS

LEMMA 3.1. *There exists a finite procedure, which when applied to a system $\mathcal{E}$ of m equations yields a system $\mathcal{G}$ with the following properties*:

1. *$\mathcal{G}$ is of rank* 1
2. *for any algebra $\mathcal{A}$, $[\mathcal{A}\mathcal{E}^\omega]$ is precisely the sequence of the first m components of $[\mathcal{A}\mathcal{G}^\omega]$.*

Proof. The procedure will consist of finitely iterated applications of the following two constructions.

Construction 1. For purposes of this construction, define the weights of terms and systems as follows. The weight $\omega(T)$ of term T is the excess of the number of its operator symbols over 1, i.e.,

$$\omega(T) = \begin{cases} \text{card }(\Omega_T)-1 & \text{if} \quad \text{card }(\Omega_T) \geqq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let the weight $\omega(\mathcal{E})$ of a system $\mathcal{E}$ be the sum of the weights of its terms. Clearly, $\omega(\mathcal{E}) = 0$ iff all terms of $\mathcal{E}$ are of ranks $\leqq 1$.

Let $\mathcal{H}$ be a system of m equations, T a term of $\mathcal{H}$ such that rank $(T) > 1$, hence $\omega(T) \geqq 1$. This construction produces a system $\mathcal{K}$ such that $\omega(\mathcal{K}) = \omega(\mathcal{H})-1$ and that the sequence of the first m components of $[\mathcal{A}\mathcal{K}^\omega]$ is precisely $[\mathcal{A}\mathcal{H}^\omega]$.

Let $T \in H_i$, for some $1 \leqq i \leqq m$. Since rank $(T) > 1$, there exist $J \in \Omega_n$, $n > 0$, terms $T_1, \cdots, T_n$ and a j, $1 \leqq j \leqq n$ such that $T = JT_1 \cdots T_{j-1}T_jT_{j+1} \cdots T_n$ and rank $(T_j) > 0$.

Let V_{m+1} be a variable such that $V_m < V_{m+1}$ within $\mathcal{V}$, hence in particular V_{m+1} is not an unknown of $\mathcal{H}$. Further denote by T' the term $JT_1 \cdots T_{j-1}V_{m+1}T_{j+1} \cdots T_n$.

The system $\mathcal{K}$ of $m + 1$ equations is given as follows. The first m equations of $\mathcal{K}$ are those of $\mathcal{H}$ *except* that $K_i = H_i - \{T\} \cup \{T'\}$, and the $(m + 1)$st equation of $\mathcal{K}$ is $V_{m+1} = \{T_j\}$.

We claim that the sequence of the first m components of $[\mathfrak{A}\mathcal{K}^\omega]$ is $[\mathfrak{A}\mathcal{H}^\omega]$. Let $\mathfrak{I}$ be the system consisting of $\mathcal{H}$ and the added equation $V_{m+1} = \{T_j\}$. Since $V_{m+1} \notin \mathcal{V}_{\mathcal{H}}$, "taking the initial m components" is a 1:1 association of the collection of fixed points of $| \mathfrak{A}\mathfrak{I} |$ onto those of $| \mathfrak{A}\mathcal{H} |$.

On the other hand, one obtains $\mathfrak{I}$ from $\mathcal{K}$ by replacement of V_{m+1} within T by T_j, where $V_{m+1} = \{T_j\}$ is an equation of $\mathcal{K}'$ and one obtains $\mathcal{K}$ from $\mathfrak{I}$ by "replacement of T_j within T" by V_{m+1}, where $V_{m+1} = \{T_j\}$ is an equation of $\mathfrak{I}'$. Hence, the collections of fixed points of $| \mathfrak{A}\mathfrak{I} |$, $| \mathfrak{A}\mathcal{K} |$ are identical, and the claim follows. It remains to observe that $\omega(T') + \omega(T_j) = \omega(T) - 1$; hence $\omega(\mathcal{K}) + \omega(\mathcal{H}) - 1$.

Let, now $\mathcal{E}_0, \mathcal{E}_1, \cdots$ be a sequence of systems for which $\mathcal{E}_0 = \mathcal{E}$ and $\mathcal{E}_{i+1}$ is obtained from $\mathcal{E}_i$ by way of Construction 1. It is clear that for $k = \omega(\mathcal{E})$, $\omega(\mathcal{E}_k) = 0$. Then $\mathfrak{F} = \mathcal{E}_k$ has the properties:

1. all terms of $\mathfrak{F}$ are of rank $\leqq 1$;
2. for any algebra $\mathfrak{A}$, the sequence of first m components of $[\mathfrak{A}\mathfrak{F}^\omega]$ is $[\mathfrak{A}\mathcal{E}^\omega]$.

To eliminate the occurrences of terms of rank 0—i.e., terms that are unknowns—we rely on the second construction.

Construction 2. Let $\mathcal{L}$ be a system of m equations and V_k, $1 \leqq k \leqq m$, an unknown of $\mathcal{L}$ that is a term (or rank 0) of $\mathcal{L}$. We construct a system $\mathfrak{N}$ such that it has one less term of rank 0 than $\mathcal{L}$ has, and such that $[\mathfrak{A}\mathcal{L}^\omega] = [\mathfrak{A}\mathfrak{N}^\omega]$.

Let $\mathfrak{M}$ be the system obtained from $\mathcal{L}$ by deleting V_k from L_k if $V_k \in L_k$, $\mathfrak{M} = \mathcal{L}$ otherwise. That is, $\mathfrak{M}$ is $\mathcal{L}$ with the change $M_k = L_k - \{V_k\}$. Then, let $\mathfrak{N}$ be obtained from $\mathfrak{M}$ by substituting for each occurrence of V_k (as a term) the set M_k; that is, $N_i = M_i - \{V_k\} \cup M_k$ if $V_k \in M_i$, $N_i = M_i$ otherwise, for $1 \leqq i \leqq m$. We wish to claim that $[\mathfrak{A}\mathcal{L}^\omega] = [\mathfrak{A}\mathfrak{N}^\omega]$. Since $V_k = M_k$ is an equation of $\mathfrak{M}$ it is clear that the collections of fixed points of $| \mathfrak{A}\mathfrak{M} |$, $| \mathfrak{A}\mathfrak{N} |$ respectively are the same.

Hence, it remains only to establish, for the case $\mathfrak{L} \neq \mathfrak{M}$, that is when $V_k \in L_k$, that $[\mathfrak{A}\mathfrak{L}^\omega] = [\mathfrak{A}\mathfrak{M}^\omega]$.

Consider the sets $|\mathfrak{A}\mathfrak{L}|^j(0)$, $|\mathfrak{A}\mathfrak{M}|^j(0)$. These are equal for $j = 0, 1$. Assume inductively that equality holds for some j, then:

$$\begin{aligned} |\mathfrak{A}\mathfrak{L}|^{j+1}(0) &= |\mathfrak{A}\mathfrak{M}|\,|\mathfrak{A}\mathfrak{L}|^j(0) \\ &\qquad \cup (\phi, \cdots, \phi_{k-1}, (|\mathfrak{A}\mathfrak{L}|^j(0))_k, \phi, \cdots, \phi) \\ &= |\mathfrak{A}\mathfrak{M}|^{j+1}(0) \cup (\phi, \cdots, \phi, (|\mathfrak{A}\mathfrak{M}|^j(0))_k, \phi, \cdots, \phi) \\ &= |\mathfrak{A}\mathfrak{M}|^{j+1}(0), \end{aligned}$$

where we have used the observation that $|\mathfrak{A}\mathfrak{M}|^{j+1}(0) \supseteq |\mathfrak{A}\mathfrak{M}|^j(0)$. Hence, $[\mathfrak{A}\mathfrak{L}^\omega] = [\mathfrak{A}\mathfrak{M}^\omega]$ and the claim is established.

It remains to note that $N = L - \{V_k\}$; hence the number of terms of zero rank has been diminished by one, and no new term introduced.

Let now $\mathfrak{F}_0, \mathfrak{F}_1, \cdots$ be a sequence of systems for which $\mathfrak{F}_0 = \mathfrak{F}$ and $\mathfrak{F}_{i+1}$ is obtained from $\mathfrak{F}_i$ by way of Construction 2. It is clear that for some $k = \text{card}\,(\mathfrak{V}_{\mathfrak{F}})$, $\mathfrak{F}_k$ is free of terms of rank 0. Then $\mathfrak{G} = \mathfrak{F}_k$ meets the conditions of the lemma.

EXAMPLE 3.2. Consider the system given as the example for Definition 2.7:

$$U = AB + (AW)B,$$
$$W = WU + U.$$

Replacing term AW by the additional unknown X yields

$$U = AB + XB,$$
$$W = WU + U,$$
$$X = AW.$$

Replacing term A by unknown Y yields

$$U = YB + XB, \qquad W = WU + U,$$
$$X = YW, \qquad Y = A.$$

Here A was replaced simultaneously in more than one term, which is clearly permissible. Final replacement of term B by unknown Z yields a system with term of rank ≤ 1:

$$U = YZ + XZ, \qquad W = WU + U,$$
$$X = YW, \qquad Y = A, \qquad Z = B.$$

Construction 2 is used to eliminate the unique term U of rank 0:

$$U = YZ + XZ,$$

$$W = WU + YZ + XZ,$$

$$X = YW, \qquad Y = A, \qquad Z = B,$$

This last system is rank 1, but not deterministic.

The right hand sets of a system are made disjoint by means of the following construction.

DEFINITION 3.3. Let $\mathcal{E}$ be a system of m equations. The "subset system" $\mathfrak{F}$ of $2^m - 1$ equations is constructed from $\mathcal{E}$ as follows.

1. For each nonempty subset M of the first m positive integers, let W_M name some variable not in $\mathcal{V}_{\mathcal{E}}$ and for distinct subsets M, M' of integers, let W_M, $W_{M'}$ be distinct also. Thus one obtains $2^m - 1$ distinct variables not in $\mathcal{V}_{\mathcal{E}}$. These are the variables taken for unknowns of $\mathfrak{F}$, and we denote the right hand set of W_M by F_M.

2. Informally the construction of the F_M may be described as follows. For each term $T \in E$, let a "derived term" T' be one obtained by replacing within T each $V_j \in \mathcal{V}_T$ by some W_M such that $i \in M$. Let the set E_i' be defined as the set of all terms "derived" from terms of E_i. Finally, F_M is the set of terms common precisely to those E_i' for which $i \in M$.

To facilitate the formal definition, let a relation $\hat{\in}$ be defined as follows. For variables V_i, W_M let $V_i \hat{\in} W_M$ iff $i \in M$, and for $J \in \Omega$, $J \hat{\in} J$. Extend $\hat{\in}$ to terms by:

$$T \hat{\in} T' \Leftrightarrow T = \alpha_1 \cdots \alpha_n \wedge T' = \beta_1 \cdots \beta_n \wedge \forall_1^n i[\alpha_i \hat{\in} \beta_i]$$

where $\alpha, \beta \in \mathcal{V} \cup \Omega$. This definition forces T, T' to be terms such that $\mathcal{V}_T \subseteq \mathcal{V}_{\mathcal{E}}$, $\mathcal{V}_{T^1} \subseteq \mathcal{V}_{\mathfrak{F}}$.

We note that $T \hat{\in} T' \Rightarrow \tilde{T} = \tilde{T}^1$, where $\tilde{T}$ is the L.d. term associated with T.

Finally, one defines $\mathfrak{F}$ as follows:

$$T' \in F_M \Leftrightarrow M = \{i : \exists T[T \in E_i \wedge T \hat{\in} T']\}$$

so that in particular:

$$\exists M[T' \in F_M \wedge i \in M] \Leftrightarrow \exists T[T \in E_i \wedge T \hat{\in} T'].$$

It is then immediate that the F_M are pairwise disjoint.

EXAMPLE 3.4. Let $\Omega = \Omega_0 \cup \Omega_1$ with $\Omega_0 = \{\Lambda\}$, $\Omega_1 = \{I_0, I_1\}$ and $\mathcal{E}$

be taken as the system

$$X_1 = I_0X_1 + I_0X_2 + I_0X_3\,,$$
$$X_2 = I_1X_1 + I_1X_2 + I_1X_3\,,$$
$$X_3 = \Lambda + I_1X_1 + I_0X_2$$

that is of rank 1. One obtains the associated subset system $\mathfrak{F}$ also of rank 1:

$$Y_1 = I_0Y_1 + I_0Y_3 + I_0Y_{1,3}\,,$$
$$Y_2 = I_1Y_2 + I_1Y_3 + I_1Y_{2,3}\,,$$
$$Y_3 = \Lambda, \qquad Y_{1,2} = \phi,$$
$$Y_{1,3} = I_0Y_2 + I_0Y_{1,2} + I_0Y_{2,3} + I_0Y_{1,2,3}\,,$$
$$Y_{2,3} = I_1Y_1 + I_1Y_{1,2} + I_1Y_{1,3} + I_1Y_{1,2,3}\,,$$
$$Y_{1,2,3} = \phi.$$

Here $\mathfrak{F}$ is deterministic.

LEMMA 3.5. Let $\mathcal{E}$ be a system of m equations, $\mathfrak{F}$ its associated subset system. For any algebra $\mathfrak{A}$, a component of $[\mathfrak{A}\mathcal{E}^\omega]$ is a union over the components of $[\mathfrak{A}\mathfrak{F}^\omega]$, and each component of $[\mathfrak{A}\mathfrak{F}^\omega]$ is contained in a component of $[\mathfrak{A}\mathcal{E}^\omega]$.

More precisely, let a mapping θ, $\theta:(2^S)^{2^m-1} \rightarrow (2^S)^m$ be defined such that for $D = (D_{M_1}, \cdots, D_{M_{2^m-1}})$ with $D_{M_j} \subseteq S$, $1 \leqq j \leqq 2^m-1$: $(\theta(D))_i = \bigcup\{D_M : i \in M\}$; then θ carries fixed points of $|\mathfrak{A}\mathfrak{F}|$ into fixed points of $|\mathfrak{A}\mathcal{E}|$ and, in particular, $\theta[\mathfrak{A}\mathfrak{F}^\omega] = [\mathfrak{A}\mathcal{E}^\omega]$.

Proof. We wish to show $\theta|\mathfrak{A}\mathfrak{F}| = |\mathfrak{A}\mathcal{E}|\theta$. As a preliminary, we show that given a term T such that $\mathcal{V}_T \subseteq \mathcal{V}\mathcal{E}$:

$$|\mathfrak{A}T|\theta(D) = \bigcup\{|\mathfrak{A}T'|(D) : T \mathrel{\widehat{\in}} T'\}.$$

Let $V_{i_1} \cdots V_{i_n}$ be the sequence of occurrences of variables in T, that is the sequence obtained by deleting the operator symbols from T. Then:

$$\begin{aligned}|\mathfrak{A}T|\theta(D) &= |\mathfrak{A}\tilde{T}|(\theta_{i_1}(D), \cdots, \theta_{i_n}(D))\\ &= |\mathfrak{A}\tilde{T}|(\bigcup\{D_{M_1} : i_1 \in M_1\}, \cdots, \bigcup\{D_{M_n} : i_n \in M_n\})\\ &= \bigcup\{|\mathfrak{A}\tilde{T}|(D_{M_i}, \cdots, D_{M_n}) : i_1 \in M_1, \cdots, i_n \in M_n\}.\end{aligned}$$

By the definition of θ and the distributivity of $|\mathfrak{A}\tilde{T}|$.

Now consider a term in the (finite) union above, that is a choice of $M_1, \cdots, M_n$. Let T' be the result of replacing in T, V_{i_1} by $W_{M_1}, \cdots, V_{i_n}$ by W_{M_n}. Then $T \mathrel{\widehat{\in}|} T'$ and $|\mathfrak{A}T'|(D) = |\mathfrak{A}\tilde{T}'|(D_{M_1}, \cdots, D_{M_n})$ since $\tilde{T} = \tilde{T}'$:

$$|\mathfrak{A}\tilde{T}|(D_{M_1}, \cdots, D_{M_n}) = |\mathfrak{A}\tilde{T}'|(DM_1, \cdots, D_{M_n}) = |\mathfrak{A}T'|(D).$$

The choices of $\mathrm{M}_1, \cdots, \mathrm{M}_n$ are in one to one correspondence with the terms T' such that $T \mathrel{\widehat{\in}} T'$; hence,

$$\bigcup\{|\mathfrak{A}\tilde{T}|(\mathrm{D}_{M_1}, \cdots, D_{M_n}) : i_1 \in \mathrm{M}_1, \cdots, i_n \in \mathrm{M}_n\}$$
$$= \bigcup\{|\mathfrak{A}T'|(D) : T \mathrel{\widehat{\in}} T'\}$$

as desired.

Returning to the proof of the lemma:

$$(\theta|\mathfrak{A}\mathfrak{F}\ (D))_i = \bigcup\{|\mathfrak{A}F_M|(D) : i \in M$$
$$= \bigcup\{|\mathfrak{A}T'|(D) : \exists M[T' \in F_M \wedge i \in M]$$

by virtue of the definition of θ and that of $|\mathfrak{A}\mathfrak{F}|$.

On the other hand:

$$(|\mathfrak{A}\mathcal{E}|\theta(D))_i = \bigcup\{|\mathfrak{A}T|\theta(D) : T \in E_i$$
$$= \bigcup\{|\mathfrak{A}T'|(D) : \exists T[T \in E_i \wedge T \mathrel{\widehat{\in}} \mathrm{T}']\}$$

by virtue of the definition of $|\mathfrak{A}\mathcal{E}|$ and of the preceeding preliminary result.

Finally, the right-hand unions are equal using the definition of $\widehat{\in}$. Thus, $\theta|\mathfrak{A}\mathfrak{F}| = |\mathfrak{A}\mathcal{E}|\theta$ and further for any $n > 0$, $\theta|\mathfrak{A}\mathfrak{F}|^n = |\mathfrak{A}\mathcal{E}|^n\theta$. If D is a fixed point of $|\mathfrak{A}\mathfrak{F}|$:

$$|\mathfrak{A}\mathcal{E}|\theta(D) = \theta|\mathfrak{A}\mathfrak{F}|(D) = \theta(D),$$

so that $\theta(D)$ is a fixed point of $|\mathfrak{A}\mathcal{E}|$.

Next, note that $\theta(0) = 0$ where the 0 denotes the appropriate tuple of empty sets, and that the mapping θ is completely distributive. Hence

$$\theta[\mathfrak{A}\mathfrak{F}^\omega] = \theta(\bigcup\{|\mathfrak{A}\mathfrak{F}|^j(0) : j = 0, 1, \cdots\})$$
$$= \bigcup\{\theta|\mathfrak{A}\mathfrak{F}|^j(0) : j = 0, 1, \cdots\}$$
$$= \bigcup\{|\mathfrak{A}\mathcal{E}|^j\theta(0) : j = 0, 1, \cdots\}$$
$$= \bigcup\{|\mathfrak{A}\mathcal{E}|^j(0) : j = 0, 1, \cdots\} = [\mathfrak{A}\mathcal{E}^\omega].$$

We wish next to eliminate from a system $\mathcal{E}$ those unknowns V_i for which $[\mathfrak{A}\mathcal{E}^\omega]_i = \phi$ (among these are those V_i for which $E_i = \phi$). It will turn out that this property is independent of $\mathfrak{A}$.

DEFINITION 3.6. Let $\mathcal{E}$ be a system of m equations. The "reduced system" $\mathcal{G}$ associated with $\mathcal{E}$ is constructed as follows:

1. Let the sequence of sets of variables U_0, U_1, $U_2 \cdots$ be defined by

$$U_0 = \mathcal{V}_\mathcal{E},$$

$$U_{j+1} = \{V_i \in \mathcal{V}_\mathcal{E}: \forall T[T \in E_i \Rightarrow \mathcal{V}_T \cap U_j \neq \phi]\}.$$

Then, U_0, $U_1 \cdots$ is a decreasing sequence of subsets of $\mathcal{V}_\mathcal{E}$. Since $U_j = U_{j+1} \Rightarrow U_{j+1} = U_{j+2}$, the sequence decreases properly until the first index k for which $U_k = U_{k+1}$ and thereafter all further terms of the sequence are identical. Note that $k < m$. Define $U \overset{\text{def}}{=} U_k$, $K \overset{\text{def}}{=} \{T \in E: \mathcal{V}_T \cap U \neq \phi\}$.

2. Take $\mathcal{V}_\mathcal{E} - U$ to be the variables of $\mathcal{G}$, $\mathcal{V}_\mathcal{E} - U = \{V_{i_1}, \cdots, V_{i_n}\}$ and let $G_{i_r} = E_{i_r} - K$. Informally speaking, $\mathcal{G}$ is obtained by deleting from $\mathcal{E}$ the equations of variables of U and then all terms of K.

EXAMPLE 3.7. Consider the subset system $\mathcal{F}$ of Example 3.4. In this case

$$U_1 = \{Y_1, Y_2, Y_{1,2}, Y_{1,3}, Y_{2,3}, Y_{1,2,3}\},$$

$$U_2 = \{Y_{1,2}, Y_{1,3}, Y_{2,3}, Y_{1,2,3}\},$$

$$U_3 = \{Y_{1,2}, Y_{1,2,3}\}.$$

Hence

$$K = \{I_0Y_{1,2}, I_0Y_{1,2,3}, I_1Y_{1,2}, I_1Y_{1,2,3}\}.$$

Finally, the reduced system is

$$\begin{aligned}
Y_1 &= I_0Y_1 + I_0Y_3 + I_0Y_{1,3}, \\
Y_2 &= I_1Y_2 + I_1Y_3 + I_1Y_{2,3}, \\
Y_3 &= \Lambda, \\
Y_{1,3} &= I_0Y_2 + I_0Y_{2,3}, \\
Y_{2,3} &= I_1Y_1 + I_1Y_{1,3}.
\end{aligned}$$

LEMMA 3.8. *Let $\mathcal{E}$ be a system and $\mathcal{G}$ the reduced system associated with $\mathcal{E}$. For any algebra $\mathfrak{A}$, the components of $[\mathfrak{A}\mathcal{G}^\omega]$ are precisely the nonempty components of $[\mathfrak{A}\mathcal{E}^\omega]$ in the same order.*

Proof. Let $\mathfrak{F}$ be the system with the terms of K removed, that is $F_i = E_i - K$. Then $\mathcal{G}$ may be obtained from $\mathfrak{F}$ by deleting the equations of variables that belong to U.

We claim for any $V_i \in \mathcal{V}_\mathcal{E}$ and natural number j

$$V_i \in U_j \Leftrightarrow (| \mathcal{A}\mathcal{E} |^j(0))_i = \phi. \tag{1}$$

For $j = 0$, the claim reduces to a tautology. Suppose the claim holds for j. Then, for any term $T \in E$,

$$\begin{aligned} | \mathcal{A}T | \, | \mathcal{A}\mathcal{E} |^j(0) = \phi &\Leftrightarrow \exists V_i \in \mathcal{V}_\mathcal{E}[V_i \in \mathcal{V}_T \wedge (| \mathcal{A}\mathcal{E} |^j(0))_i = \phi] \\ &\Leftrightarrow \exists V_i \in \mathcal{V}_\mathcal{E}[V_i \in \mathcal{V}_T \wedge V_i \in U_j] \\ &\Leftrightarrow \mathcal{V}_T \cap U_j \neq \phi. \end{aligned}$$

Hence

$$\begin{aligned} V_i \in U_{j+1} &\Leftrightarrow \forall T[T \in E_i \Rightarrow \mathcal{V}_T \cap U_j \neq \phi] \\ &\Leftrightarrow \forall T[T \in E_i \Rightarrow | \mathcal{A}T | \, | \mathcal{A}\mathcal{E} |^j(0) = \phi] \\ &\Leftrightarrow | \mathcal{A}E_i | \, | \mathcal{A}\mathcal{E} |^j(0) = (| \mathcal{A}\mathcal{E} |^{j+1}(0))_i = \phi. \end{aligned}$$

Further, we have

$$\begin{aligned} T \in K \Rightarrow \mathcal{V}_T \cap U_j \neq \phi \Rightarrow | \mathcal{A}T | \, | \mathcal{A}\mathcal{E} |^j(0) &= \phi, \\ | \mathcal{A}\mathfrak{F} | \, | \mathcal{A}\mathcal{E} |^j(0) &= | \mathcal{A}\mathcal{E} |^{j+1}(0). \end{aligned} \tag{2}$$

It is now a simple induction to show from (2) that $[\mathcal{A}\mathfrak{F}^\omega] = [\mathcal{A}\mathcal{E}^\omega]$. Since the terms of $\mathfrak{F}$ are free of the variables in U, $[\mathcal{A}\mathcal{G}^\omega]$ differs from $[\mathcal{A}\mathfrak{F}^\omega]$, hence $[\mathcal{A}\mathcal{E}^\omega]$, only in the absence of certain components. From (1), $V_i \in U \Leftrightarrow \forall_j[V_i \in U_j] \Leftrightarrow [\mathcal{A}\mathcal{E}^\omega]_i = \phi$; thus these components are precisely the empty ones.

COROLLARY 3.8. *Let $\mathcal{E}$ be any system. For algebras $\mathcal{A}$, $\mathcal{B}$:*

$$[\mathcal{A}\mathcal{E}^\omega]_i = \phi \Leftrightarrow [\mathcal{B}\mathcal{E}^\omega]_i = \phi.$$

THEOREM 3.9. *Let Ω be finite. There exists a finite procedure which, when applied to a system $\mathcal{E}$, yields a system $\mathcal{G}$ with the properties:*

(1) *$\mathcal{G}$ is deterministic*

(2) *$\mathcal{G}$ is reduced*

(3) *for any algebra $\mathcal{A}$, any component of $[\mathcal{A}\mathcal{E}^\omega]$ is a union over components of $[\mathcal{A}\mathcal{G}^\omega]$.*

Proof. Consider the sequence of constructions described in Lemma 3.1 and Definitions 3.3, 3.6, respectively. Applying to $\mathcal{E}$ the appropriate constructions, in that order, will yield a system $\mathfrak{F}$ that is of rank 1, that has properties (2) and (3) of the above theorem statement and such that, in addition, the F_i will be pairwise disjoint. (In general, however, contrary to the case of Example 3.7, the final system is not deterministic due to the fact that not all terms of rank 1 in variables $\mathfrak{V}_{\mathfrak{F}}$ may be present in the equations.) It should be noted in this connection that the constructions of Definitions 3.3 and 3.6 applied to a system of rank 1 yield systems of rank 1, and that the construction of Definition 3.6 does not add terms; hence, no construction undoes the effect of a preceding one. For some other order of carrying out the constructions, this may not be true.

To complete the proof, then, define for a given set U of variables, $\mathfrak{I}U$ to be the collection of those terms T for which rank $(T) = 1$ and $\mathfrak{V}_T \subseteq U$. We consider the case in which $\mathfrak{F}$ is not deterministic, that is, $F \subset \mathfrak{I}\mathfrak{V}_{\mathfrak{F}}$. If $\mathfrak{F}$ has m equations, choose V_{m+1} to be some variable, $V_{m+1} > V_m$; hence, V_{m+1} is not in $\mathfrak{V}_{\mathfrak{F}}$. Denote $\mathfrak{V}_{\mathfrak{F}} \cup \{V_{m+1}\}$ by $\mathfrak{V}_{\mathfrak{G}}$, a notation justified in what follows. Since Ω is finite, so is $\mathfrak{I}\mathfrak{V}_{\mathfrak{G}}$, and we define the system $\mathfrak{G}$ of $m + 1$ equations as follows:

$\mathfrak{G}$ is taken to be $\mathfrak{F}$ together with the additional equation $V_{m+1} = \mathfrak{I}\mathfrak{V}_{\mathfrak{G}} - F$. Then, $\mathfrak{I}$ is of rank 1 and the G_i are pairwise disjoint. But, $\mathfrak{I}\mathfrak{V}_{\mathfrak{G}} = G$; hence $\mathfrak{G}$ is deterministic. Further, since $\mathfrak{F}$ was reduced and $\mathfrak{I}\mathfrak{V}_{\mathfrak{F}} \cap G_{m+1} \neq \phi$, $\mathfrak{G}$ is reduced. Since $V_{m+1} \notin \mathfrak{V}_{\mathfrak{F}}$, $[\mathfrak{A}\mathfrak{F}^{\omega}]$ is precisely the sequence of first n components of $[\mathfrak{A}\mathfrak{G}^{\omega}]$; hence the theorem.

One can draw a parallel between the results of this section and some results reported previously in the literature concerning context-free languages and finite-state (fs) automata.

Thus, for example, Lemma 3.1 is related to the following result: Given a context-free language, it has a grammar with the property that the right-hand side of any production is either (1) a terminal letter or (2) a string of two nonterminal letters. Similarly, the construction of the reduced system (Definition 3.6) is related to the result: There exists a finite procedure for determining, given a context-free grammar, whether the language it generates is empty or not.

The construction of the subset system (Definition 3.3) is a generalization of a procedure employed in deriving from a nondeterministic fs automaton a behaviorally equivalent deterministic fs automaton. This claim may be argued easier in connection with a modified special case of of the construction actually given in Definition 3.3. Namely, let Ω be finite and $\mathcal{E}$ of rank 1. In this case, modify the definition by (1) including among the variables W_M the variable W_ϕ that corresponds to $M = \phi$ and (2) admitting to F_M only terms of rank 1. The subset system $\mathfrak{F}$ that results under these conditions is then deterministic.

IV. GENERIC ALGEBRAS

In this section we characterize the minimal fixed point $[\mathfrak{D}\mathcal{E}^\omega]$ of a system $\mathcal{E}$ applied to the generic algebra $\mathfrak{D}$.

DEFINITION 4.1. The generic algebra $\mathfrak{D} = (\mathfrak{J}_\Omega, e)$ is defined as follows. The carrier $\mathfrak{J}_\Omega$ is the collection of those terms $T \in \mathfrak{J}$ that are free of variables, i.e., $\mathcal{V}_T = \phi$. The mapping e is given by

(1) for $J \in \Omega_0$, $e_J = J$,

(2) for $J \in \Omega_n$, $n > 0$, and $T_1, \cdots, T_n \in \mathfrak{J}_\Omega$, $e_J(T_1, \cdots, T_n) = JT_1 \cdots T_n$.

With respect to $\mathfrak{D}$, the following situation arises. Given any term T', the term function $| \mathfrak{D}T' |$ has sets of terms (contained in $\mathfrak{J}_\Omega$) as both its arguments and values. If $T \in \mathfrak{J}_\Omega$, then $| \mathfrak{D}T |$ is zeroary, with value $\{T\}$.

Remark 4.2. In the literature, there appears a definition of generic or "anarchic" algebra which employs the concept of a set of generators. In the preceding definition, Ω_0 plays the role of such a set of generators, which consequently is fixed for a given Ω. If $\Omega_0 = \phi$, then the generic algebra is not defined. As a further consequence of the Definition 4.1, a generic algebra has no proper subalgebras.

DEFINITION 4.3. Given a congruence relation on the carrier of an algebra $\mathfrak{A}$, we refer to its collection of congruence classes as a "congruence on $\mathfrak{A}$."

LEMMA 4.4. *Let $\mathcal{G}$ be a system that is both deterministic and reduced, $\mathfrak{D}$ the generic algebra. The collection of the components of $[\mathfrak{D}\mathcal{G}^\omega]$ is a finite congruence on $\mathfrak{D}$.*

Proof. Let $[\mathfrak{D}\mathcal{G}^\omega] = \mathcal{B} = (\mathcal{B}_1, \cdots, \mathcal{B}_m)$. We use throughout the observation that since $\mathcal{B}$ is a fixed point of $| \mathfrak{D}\mathcal{G} |$, $T \in \mathcal{B}_i$ iff there exists $T' \in G_i$ such that $T \in | \mathfrak{D}T' | (\mathcal{B})$.

(1) Since $\mathcal{G}$ is reduced, the $\mathcal{B}_i$ are non-empty.

(2) $\bigcup \{\mathcal{B}_i : 1 \leqq i \leqq m\} \supseteq \mathfrak{J}_\Omega$. The argument is by induction on the rank ($\geqq 1$) of terms T in $\mathfrak{J}_\Omega$.

If rank $(T) = 1$, then for some $1 \leqq j \leqq n$, $T \in G_j$, since $\mathcal{G}$ is deterministic. Since rank $(T) = 1$, $T \in \Omega_0$ and $T \in | \mathfrak{D}T | \subseteq \mathcal{B}_j$. Assume inductively that if rank $(T) \leqq k$, then there exists $1 \leqq j \leqq m$ such that $T \in \mathcal{B}_j$. Consider a term T, rank $(T) = k + 1$. Let $T = JT_1 \cdots$

T_n, with rank $(T_1), \cdots$ rank $(T_n) \leqq k$. Using the inductive assumption there exist $j_1, \cdots, j_n$ such that $T_1 \in \mathcal{B}_{j_1}, \cdots, T_n \in \mathcal{B}_{j_n}$. Consider the term $T' = JV_{j_1} \cdots V_{j_n}$, since $\mathcal{G}$ is deterministic, there exists $1 \leqq j \leqq n$ such that $T' \in G_j$. Hence, $T \in |\mathfrak{D}T'|(\mathcal{B}) \subseteq \mathcal{B}_j$.

(3) The $\mathcal{B}_i$ are pairwise disjoint. The argument is once more by induction on the rank of terms in $\mathfrak{I}_\Omega$.

Let rank $(T) = 1$, $T \in \mathcal{B}_j \cap \mathcal{B}_k$ for some $1 \leqq j, k \leqq m$. There exist terms $T' \in G_j$, $T'' \in G_k$ such that $T \in |\mathfrak{D}T'|(\mathcal{B}) \cap |\mathfrak{D}T''|(\mathcal{B})$. Since rank $(T) = 1$, $T \in |\mathfrak{D}T'|(\mathcal{B})$ implies $T = T'$. Analogously, $T = T''$ and $T \in G_j \cap G_k$. But $\mathcal{G}$ is deterministic; hence $j = k$.

Assume inductively that for all $1 \leqq j, k \leqq m$ if there exists a term $T \in \mathcal{B}_j \cap \mathcal{B}_k$ with rank $(T) \leqq n$, then $j = k$. Now let $T \in \mathcal{B}_j \cap \mathcal{B}_k$ with rank $(T) = n + 1$. There exist $T' \in G_j$, $T'' \in G_k$ such that $T \in |\mathfrak{D}T'|(\mathcal{B}) \cap |\mathfrak{D}T''|(\mathcal{B})$. Since $\mathcal{G}$ is deterministic rank $(T') =$ rank $(T'') = 1$ and since rank $(T) > 1$, T', T'' are of the forms $T' = J'V_{i_1} \cdots V_{i_s}$, $T'' = J''V_{r_1} \cdots V_{r_t}$. Since $T \in (\mathfrak{D}T')(\mathcal{B})$, $T = J'T_{i_1} \cdots T_{i_s}$ with $T_{i_1} \in \mathcal{B}_{i_1}, \cdots, T_{i_s} \in \mathcal{B}_{i_s}$. Analogously, $T = J''T_{r_1} \cdots T_{r_t}$ with $T_{r_1} \in \mathcal{B}_{r_1}, \cdots, T_{r_t} \in \mathcal{B}_{r_t}$. By the unique factorization property of T, $J' = J''$, $s = t$, $T_{i_1} = T_{r_1}, \cdots, T_{i_s} = T_{r_s}$. Now $T_{i_1}, \cdots, T_{i_s}$ are of ranks $\leqq n$, and $T_{i_1} \in \mathcal{B}_{i_1} \cap \mathcal{B}_{r_1}, \cdots, T_{i_s} \in \mathcal{B}_{i_s} \cap \mathcal{B}_{r_s}$. Using the inductive assumption, $i_1 = r_1, \cdots, i_s = r_s$ and $T' = T'' \in G_j \cap G_k$. Since $\mathcal{G}$ is deterministic, $j = k$.

(4) To show $\{\mathcal{B}_1, \cdots, \mathcal{B}_m\}$ is a congruence on $\mathfrak{D}$, it now suffices to establish that for any $J \in \Omega_n$, $n > 0$ and any $\mathcal{B}_{i_1}, \cdots, \mathcal{B}_{i_n}$ there exists a $\mathcal{B}_j$ such that $\hat{e}_J(\mathcal{B}_{i_1}, \cdots, \mathcal{B}_{i_n}) \subseteq \mathcal{B}_j$. But $\hat{e}_J(\mathcal{B}_{i_1}, \cdots, \mathcal{B}_{i_n}) = |\mathfrak{D}T'|(\mathcal{B})$ where $T' = JV_{i_1} \cdots V_{i_n}$. Since $\mathcal{G}$ is deterministic, there exists a $1 \leqq j \leqq n$ such that $T' \in G_j$, and $\hat{e}_J(\mathcal{B}_{i_1}, \cdots, \mathcal{B}_{i_n}) \subseteq \mathcal{B}_j$.

DEFINITION 4.5. Let $\mathfrak{A} = (S, h)$ be an algebra. A subset C of S is recognizable with respect to $\mathfrak{A}$ iff there exists a finite congruence Σ over $\mathfrak{A}$ such that C is a union over Σ.

THEOREM 4.6. *Let Ω be finite, $\mathcal{E}$ a system of equations and $\mathfrak{D}$ the generic algebra. Each component of $[\mathfrak{D}\mathcal{E}^\omega]$ is recognizable with respect to $\mathfrak{D}$.*

Proof. According to Theorem 3.9 there exists a deterministic and reduced system $\mathcal{G}$ such that a component of $[\mathfrak{D}\mathcal{E}^\omega]$ is a union over components of $[\mathfrak{D}\mathcal{G}^\omega]$. From Lemma 4.4 the collection of components of $[\mathfrak{D}\mathcal{G}^\omega]$ is a finite congruence on $\mathfrak{D}$. Hence, the theorem.

Remark 4.7. This theorem holds with the restriction to finite Ω removed. In particular, one may use a system such as $\mathfrak{F}$ of Theorem 3.9, which is not deterministic but has the three properties of being of rank 1, of having pairwise disjoint right-hand sets, and of being reduced. For a system with these properties, and without the requirement that Ω be finite, a Lemma analogous to Lemma 4.4 can be proved.

THEOREM 4.8. *Let Ω be finite, $\mathfrak{D}$ the generic algebra and $\Sigma = \{\sigma_1, \cdots, \sigma_m\}$ a finite congruence on $\mathfrak{D}$. There exists a system $\mathcal{G}$, $\mathcal{G}$ both deterministic and reduced, such that $[\mathfrak{D}\mathcal{G}^\omega] = (\sigma_1, \cdots \sigma_m)$.*

Proof. Choose for $\mathcal{V}_\mathcal{G}$ a collection of variables in a 1:1 correspondence with the classes $\sigma_1, \cdots, \sigma_m$. The set $\mathfrak{I}\mathcal{V}_\mathcal{G}$ of terms (see Theorem 3.9 for definition) is finite because Ω is. Let the right-hand sides G_i of $\mathcal{G}$ be determined by the requirement that for $1 \leqq i \leqq m$:

$$G_i = \{T' \in \mathfrak{I}\mathcal{V}_\mathcal{G}: | \mathfrak{D}T' |(\sigma_1, \cdots, \sigma_m) \subseteq \sigma_i\}.$$

Clearly, $\mathcal{G}$ is deterministic.

Let $[\mathfrak{D}\mathcal{G}^\omega] = (\mathfrak{B}_i, \cdots, \mathfrak{B}_m)$. We show that $\sigma_i \subseteq \mathfrak{B}_i$, for all $1 \leqq i \leqq m$, by induction on the rank of terms.

Let T be a term of rank 1. If $T \in \sigma_i$; hence $T \in \Omega_0$, $\{T\} = | \mathfrak{D}T | \subseteq \sigma_i$. Thus $T \in G_i$ and $T \in | \mathfrak{D}T | \subseteq \mathfrak{B}_i$. Assume inductively that for all terms T of rank $\leqq k$ and all $1 \leqq i \leqq m$, if $T \in \sigma_i$ then $T \in \mathfrak{B}_i$.

Consider a term T, rank $(T) = k + 1$. Let $T = JT_1 \cdots T_n$, let $\sigma_{i_1}, \cdots, \sigma_{i_n}$ be the uniquely determined congruence classes such that $T_1 \in \sigma_{i_1}, \cdots, T_n \in \sigma_{i_n}$ and consider the term $T' = JV_{i_1} \cdots V_{i_n}$. Then $T \in | \mathfrak{D}T' | (\sigma_1, \cdots, \sigma_m)$.

Assume that $T \in \sigma_i$; hence $T \in | \mathfrak{D}T' | (\sigma_1, \cdots, \sigma_m) \cap \sigma_i$. Since Σ is a congruence, $| \mathfrak{D}T' | (\sigma_1, \cdots, \sigma_n) = \hat{e}_J(\sigma_{i_1}, \cdots, \sigma_i) \subseteq \sigma_i$. Since $T' \in T\mathcal{V}_\mathcal{G}$, $T' \in G_i$.

Now rank $(T_1), \cdots$, rank $(T_n) \leqq k$ and using the inductive assumption, $T_1 \in \mathfrak{B}_{i_1}, \cdots, T_n \in \mathfrak{B}_{i_n}$. Hence, $T \in | \mathfrak{D}T' | (\mathfrak{B}_1, \cdots, \mathfrak{B}_m)$. But $(\mathfrak{B}_1, \cdots, \mathfrak{B}_m)$ is a fixed point of $| \mathfrak{D}\mathcal{G} |$ and $| \mathfrak{D}T' | (\mathfrak{B}_1, \cdots, \mathfrak{B}_m) \subseteq \mathfrak{B}_i$. Finally, $T \in \mathfrak{B}_i$.

Since Σ is a congruence, $\phi \neq \sigma_i \subseteq \mathfrak{B}_i$ for all $1 \leqq i \leqq m$, and $\mathcal{G}$ is reduced. By Lemma 4.4, $\{\mathfrak{B}_1, \cdots, \mathfrak{B}_m\}$ is a congruence on $\mathfrak{D}$. But Σ is a congruence on $\mathfrak{D}$ and $\sigma_i \subseteq \mathfrak{B}_i$ for all $1 \leqq i \leqq m$; hence $\sigma_i = \mathfrak{B}_i$ for all $1 \leqq i \leqq m$ and $(\sigma_1, \cdots, \sigma_m) = [\mathfrak{D}\mathcal{G}^\omega]$.

Remark 4.9. Suppose the definition of a system $\mathcal{E}$ (Definition 2.7) were changed as follows:

(1) The restriction to finite E_i removed, permitting arbitrary right-hand sets;

(2) The restriction imposed that $\mathcal{E}$ must be of rank 1. (In particular, the number of equations of $\mathcal{E}$ would remain finite.)

With this definition of a system, a system could be deterministic without Ω being finite. Hence, Theorem 3.9, consequently Theorem 4.6, and Theorem 4.8 would all hold without the requirement that Ω be finite.

V. EQUATIONAL SUBSETS IN ALGEBRAS

We begin by extending the results of the last section to algebras other than generic with the aid of the concept of homomorphism.

DEFINITION 5.1. Let $\mathfrak{A} = (S, h)$, $\mathfrak{B} = (Z, k)$ be algebras. A mapping $\eta: S \to Z$ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ iff:

(1) for all $J \in \Omega_0$, $\eta h_J = k_J$;

(2) for all $J \in \Omega_n$, $n > 0$ and all $s_1, \cdots, s_n \in S$, $\eta h_J(s_1, \cdots, s_n) = k_J(\eta(s_1), \cdots, \eta(s_n))$.

Remark 5.2. If the mapping η is extended to $\hat{\eta}: 2^S \to 2^Z$ in the natural way; that is, for any $C \subseteq S$,

$$\hat{\eta}(C) = \{\eta(s): s \in C\}, \qquad \hat{\eta}(\phi) = \phi,$$

it may be verified that

(1) for all $J \in \Omega_0$, $\hat{\eta}\hat{h}_J = \hat{k}_J$,

(2) for all $J \in \Omega_n$, $n > 0$ and all $S_1, \cdots, S_n \subseteq S$,

$$\hat{\eta}\hat{h}_J(S_1, \cdots, S_n) = \hat{k}_J(\hat{\eta}(S_1), \cdots, \eta(S_n)).$$

That is, $\hat{\eta}$ is a homomorphism of $\hat{\mathfrak{A}}$ onto $\hat{\mathfrak{B}}$. As a further consequence, if $\hat{\eta}$ is extended to n-tuples of sets componentwise, for any term T,

$$\hat{\eta}| \mathfrak{A}T | = | \mathfrak{B}T | \hat{\eta}.$$

LEMMA 5.3. *Let $\mathcal{E}$ be a system, $\mathfrak{A}$, $\mathfrak{B}$ two algebras, and η a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Then, $\hat{\eta}$ carries fixed points of $| \mathfrak{A}\mathcal{E} |$ into fixed points of $| \mathfrak{B}\mathcal{E} |$ and in particular, $\hat{\eta}[\mathfrak{A}\mathcal{E}^\omega] = [\mathfrak{B}\mathcal{E}^\omega]$.*

Proof. The mapping $\hat{\eta}$ is completely distributive; hence distributive over finite unions. As a consequence of this and of the preceding remark, $\hat{\eta} | \mathfrak{A}\mathcal{E} | = | \mathfrak{B}\mathcal{E} | \hat{\eta}$. Hence $\hat{\eta}$ carries fixed points of $| \mathfrak{A}\mathcal{E} |$ into fixed points of $| \mathfrak{B}\mathcal{E} |$. From the property of complete distributivity of $\hat{\eta}$, and addition to the fact that $\hat{\eta}(0) = 0$, it follows that $\hat{\eta}[\mathfrak{A}\mathcal{E}^\omega] = \hat{\eta}(\bigcup \{| \mathfrak{A}\mathcal{E} |^j(0): j = 0, 1, \cdots\}) = [\mathfrak{B}\mathcal{E}^\omega]$.

DEFINITION 5.4. Let $\mathfrak{A} = (S, h)$ be an algebra, C a subset of S. C is equational with respect to $\mathfrak{A}$ iff there exists a system $\mathcal{E}$ such that C is a component of $[\mathfrak{A}\mathcal{E}^\omega]$.

THEOREM 5.5. *Let Ω be finite and $\Omega_0 \neq \phi$. Let $\mathfrak{A} = (S, h)$ be an algebra,*

C a subset of S. C is equational with respect to $\mathcal{A}$ if *C is the homomorphic image of a set recognizable with respect to the generic algebra.*

Proof. Let C be equational; hence there exists a system $\mathcal{E}$ such that C is a component of $[\mathcal{A}\mathcal{E}^\omega]$. By the assumption $\Omega_0 \neq \phi$ the generic algebra $\mathfrak{D}$ exists and by Theorem 4.6 each component of $[\mathfrak{D}\mathcal{E}^\omega]$ is recognizable with respect to $\mathfrak{D}$. By Lemma 5.3, taking η as the natural (unique) homomorphism of $\mathfrak{D}$ into $\mathcal{A}$, $\hat{\eta}[\mathfrak{D}\mathcal{E}^\omega] = [\mathcal{A}\mathcal{E}^\omega]$. Hence $C = \hat{\eta}(D)$, D recognizable with respect to the generic algebra. In this direction, the proof would carry through with the restriction to finite Ω removed (see Remark 4.7).

Conversely, let $\mathfrak{D}$ be the generic algebra, $\Sigma = \{\sigma_1, \cdots \sigma_m)$ a finite congruence on $\mathfrak{D}$ and $D = \sigma_{i_1} \cup \cdots \cup \sigma_{i_k}$. Let η be the homomorphism of $\mathfrak{D}$ into $\mathcal{A}$ and $\hat{\eta}(D) = C$. Since Ω is finite, by Theorem 4.8, there exists a system $\mathcal{G}$ such that $[\mathfrak{D}\mathcal{G}^\omega] = (\sigma_1, \cdots, \sigma_m)$. With $V_1, \cdots V_m$ the unknowns of $\mathcal{G}$, choose a variable $V_{m+1} > V_m$ and let $\mathcal{E}\ \hat{e}\ \mathcal{G}$ with the added equation:

$$V_{m+1} = \{V_{i_1}, \cdots, V_{i_k}\}$$

Then, $[\mathfrak{D}\mathcal{E}^\omega] = (\sigma_1, \cdots, \sigma_m, D)$. From Lemma 5.3, $\hat{\eta}[\mathfrak{D}\mathcal{E}^\omega] = [\mathcal{A}\mathcal{E}^\omega] = (\hat{\eta}(\sigma_1), \cdots, \hat{\eta}(\sigma_m), C)$. Hence C is equational with respect to $\mathcal{A}$.

Consider an algebra $\mathcal{A} = (S, h)$ for which S is the collection of all strings of right and left parentheses, $\Omega = \Omega_0 \cup \Omega_2$, $\Omega_0 = \{A, B\}$, $\Omega_2 = \{\frown\}$ and $h_A = ($ [redacted], $h_B =)$, $h_\frown$ is the juxtaposition operator. Let $\mathcal{E}$ be the system given in Definition 2.7 as illustration. Then, the second component of $[\mathcal{A}\mathcal{E}^\omega]$ is the set $C = \{(\ ), ((\ )), (\ )(\ ), \cdots\}$ of proper parenthetical expressions. C is equational with respect to $\mathcal{A}$, and Theorem 5.5 applies; hence C is the homomorphic image of a set recognizable with respect to the appropriate generic algebra $\mathfrak{D}$. Note $\mathfrak{D}$ has the carrier $\{A, B, \frown AA, \frown AB, \cdots\}$.

Except for the distinction of (,) as zeroary operators, $\mathcal{A}$ is identical to the free semigroup $\mathcal{S}$ generated by $\{(,)\}$. That is to say, the algebras $\mathcal{A}$, $\mathcal{S}$ have the same carrier and their non-zeroary operators are identical. Analogously, except for the distinction of A, B as zeroary operators, $\mathfrak{D}$ is identical to the free grouppoid $\mathcal{G}$ generated by $\{\mathcal{A}, \mathcal{B}\}$. Conventionally, the free algebras $\mathcal{S}$, $\mathcal{G}$ are regarded as having only a binary operator.

Now C is a subset of the carrier of $\mathcal{S}$, but under our definitions, C is not equational with respect to $\mathcal{S}$. This situation is due to the fact that for expository reasons, we have chosen to provide in the definitions only the zeroary operator symbols as a means for naming elements of the carrier of an algebra. While our choice allows simple definitions such as Definition 2.7 and a simple deduction for results such as Lemma 5.3, it has the following consequence: if the set of zeroary operator symbols is empty, the unique equational set is the empty set. Hence, in order to extend the results to their proper range of application, and, in particular, reach conclusions concerning finitely generated algebras such as $\mathcal{S}$, $\mathcal{G}$ we are obliged to add some further device.

The device that is useful in properly generalizing definitions and in obtaining results analogous to Theorems 4.6, 4.8. 5.5 is the addition and deletion of zeroary operator symbols. Such a device is successful due to the following reasons.

Let $\mathcal{A}$, $\mathcal{A}'$ be algebras that differ merely in that some elements of the carrier S common to both are distinguished as zeroary operators within $\mathcal{A}$, but not within $\mathcal{A}'$. Then, the collections of subsets of S recognizable with respect to $\mathcal{A}$, $\mathcal{A}'$ respectively are identical. Further, regardless of whether S is considered the carrier of $\mathcal{A}$ or of $\mathcal{A}'$, one obtains the same collection of subsets of S that are homomorphic images of recognizable sets.

In the illustration, take C as a subset of S considered as the carrier of $\mathcal{S}$. C may be taken equational with respect to $\mathcal{S}$ in an extended sense, because C is equational with respect to $\mathcal{A}$ where $\mathcal{A}$ is obtained by adding zeroary operator symbols and taking for the zeroary operators the generators of $\mathcal{S}$. We had concluded that C is the homomorphic image o set recognizable with respect to $\mathcal{D}$. We now delete the zeroary operato symbols and may claim that C is also a homomorphic image of a set recognizable with respect to $\mathcal{G}$.

Let, in general, $\mathcal{S}$ be the free semigroup with n generators. The sets equational with respect to $\mathcal{S}$ (in an extended sense) are precisely those called "definable" by Ginsburg and Rice (1962), who further showed that these are exactly the context-free languages of $\mathcal{S}$. With our discussion in mind, we conclude that: "A set is a 'context-free language' of the free semigroup with n generators iff it is the homomorphic image of a set that is recognizable with respect to the free grouppoid with n generators." This result is due to D. Muller[3] (personal communication).

VI. CONCLUSION

In this section, applications of the foregoing theory to special kinds of algebras, such as monadic algebras and semigroups, will be considered.

[3] The following terminology has been introduced by Muller: A subset C of the carrier of an algebra is grammatical iff C is the homomorphic image of a set recognizable with respect to a finitely generated algebra.

A monadic algebra is an algebra in which, except for zeroary operations, every primitive operation of the algebra is a mapping of the carrier into itself. Monadic algebras arise in a very natural way in the theory of finite automata.

Let S be a set of states, $S_0 \in S$ the initial state, and Σ a set of input symbols. Assume that every element of Σ induces a transition function, mapping S into S. This part of the description can be given a precise mathematical formulation as a monadic algebra $\mathfrak{A} = (S, h)$ with a set Ω of operation symbols consisting of Σ together with λ. In the algebra $\mathfrak{A}$, the set of states S is the carrier, the zeroary operation λ denotes S_0, and each symbol in Σ denotes the transition function which it induces on S. Note that in order to constitute a finite automaton, this description must be supplemented by some provision for the production of output information, such as a distinguished terminal subset of the set of states Let M_Σ represent the generic monadic algebra based on the operation symbols under consideration and let H stand for the (unique) homomorphism from M_Σ into $\mathfrak{A}$. Let T be a term belonging to the carrier of M_Σ. T is interpreted as a sequence of input symbols to $\mathfrak{A}$, and as such is read from right to left, except for its rightmost symbol (λ). One says that T is "accepted by the finite automaton" iff $H(T)$ is terminal. Thus, the set of input sequences accepted by the finite automaton is the inverse image, under H, of the terminal set of the automaton, and is therefore, a recognizable subset of M_Σ. It follows from familiar algebraic considerations that a set of input terms (i.e. a subset of the carrier of M_Σ) is accepted by some finite automaton iff it is a recognizable subset of M_Σ. Furthermore, Theorems 4.6 and 4.8 imply that the recognizable subsets of M_Σ are exactly its equational sets.

As a further application of the theory to monadic algebras, consider the 'finite state transductions' defined by Elgot and Mezei (1965), as follows: A (binary) finite state transduction over an alphabet Σ is a set of pairs, of strings, determined by paths between prescribed beginning and end points in a finite directed graph, a graph whose edges are labelled by pairs of Σ-strings. Let P_Σ represent the algebra defined as follows. The carrier of P_Σ is the set of pairs of Σ-strings. The set Ω of operation symbols for P_Σ consists of: a zeroary λ denoting the pair of empty strings, and for each $\sigma \in \Sigma$, the unary operation symbols f_σ, g_σ. For $\sigma \in \Sigma$, $f_\sigma(x, y) = (\sigma x, y)$ and $g_\sigma(x, y) = (x, \sigma y)$. Note that this monadic algebra P_Σ, although free, is not generic. It may be verified that the equational sets of P_Σ are exactly the binary finite-state transductions over Σ.

In the literature of automata theory, there exist numerous versions of the structure that underlies the concept of a finite automaton, versions that differ in superficial ways. One of these versions is the above presented finite monadic algebra $\mathfrak{A} = (S, h)$. Another is the state graph. A state graph G can be constructed from $\mathfrak{A}$, with the carrier S of $\mathfrak{A}$ as the set of "vertices" of states in G, and the state S_i connected to S_j by an arrow labelled σ, for $\sigma \in \Sigma$, iff $\sigma(S_i) = S_j$. In G, the distinguished element $H(\lambda)$ is labelled as the initial state. A system $\mathcal{E}$ of equations can also be associated with $\mathfrak{A}$, in the following manner. Establish a 1:1 correspondence between S and a set of variables for $\mathcal{E}$, so that S_i corresponds to the variable V_i. For each V_i there is an equation in $\mathcal{E}$ with V_i on the left, and the term σV_j belongs to the right-hand set of that equation iff $\sigma(S_j) = S_i$. The symbol λ is inserted in the equation corresponding to the initial state $H(\lambda)$.

Clearly, with these constructions in mind, finite monadic algebras correspond exactly to the deterministic state graphs and also to the deterministic monadic algebra equations. Were the operators of monadic algebras permitted to take on values that are subsets of the carrier, the correspondence could be extended to include these structures, all the finite state graphs, and all systems of monadic algebra equations of rank one. Even further, the use of "boxes" with several inputs and one output instead of "arrows" in finite state graphs, makes it possible to extend these remarks to algebras which are not monadic.

The question as to what sets of Σ-strings are equational depends, naturally, on the structure associated with the set Σ^* of all Σ-strings. Consider, for example, the function $\gamma: S \to SS$, which maps any string S into its double SS. Let $\mathfrak{A}$ now be the algebra consisting of the carrier $\{A, B\}^*$, the null string λ, the unit strings A and B, the concatenation operation, and the additional operation γ.

Consider the following system of equations (written in the informal notation).

$$X = XA + XB + \lambda, \qquad Y = \gamma(X).$$

The second component of the minimum fixed point of this system over $\mathfrak{A}$ is a set Q of all strings of the form SS. It is known that Q is not context-free [Chomsky and Schützenberger (1963)] although it is equational relative to $\mathfrak{A}$.

ACKNOWLEDGMENT

The direction in which this work developed was influenced considerably by a personal communication from Prof. D. Muller. We wish to express our gratitude to Dr. C. C. Elgot for his critical reading of the initial version and for his helpful suggestions.

RECEIVED: March 9, 1966; revised August 25, 1966

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