Solving Stochastic Büchi Games on Infinite Decisive Arenas

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Abstract

We consider games played on an infinite probabilistic arena where the first player aims at satisfying generalized Büchi objectives almost surely, i.e., with probability one. We provide a fixpoint characterization of the winning sets and associated winning strategies in the case where the arena is decisive. From this we directly deduce the decidability of these games on probabilistic lossy channel systems.

1 Introduction

2-player stochastic games are games where two players, Alice and Bob, interact in a probabilistic environment. Given an objective formalized, e.g., as an ω-regular condition, the goal for Alice is to maximize the probability to fulfill the condition, against any behaviour of her opponent. Qualitative questions ask whether Alice can win almost-surely (resp. positively) from a given initial configuration. Solving a stochastic game then amounts to deciding the latter question, as well as providing winning strategies for the players. In the case where the arena is finite, the literature offers several general results on the existence of optimal strategies, the determinacy of the games, and algorithmic methods for computing solutions, when the objectives range in complexity from simple reachability objectives to arbitrary Borel objectives [35, 22, 21].

For infinite arenas, general results are scarce and mostly concern purely mathematical, non-algorithmical, aspects, such as determinacy [28]. An obvious explanation for the lack of algorithmical results is that already solving a Büchi game with a single player and no stochastic aspects on the configurations of a Turing machine (a very regular arena) is a Σ_1^1-complete, hence "highly undecidable", problem.

Decidability can be regained for infinite arenas if it is known that they are generated in some specific way. In the field of algorithmic verification, the stochastic games with infinite arenas originate from classical infinite-state models. Prominent examples with positive results are stochastic games on systems with recursion [23, 18], on one-counter automata [16, 17], and on lossy channel systems [9, 2]. In all these examples, the description of winning sets and winning strategies is specific to the underlying infinite-state model, and rely on ad-hoc techniques.

In this paper we follow a more generic approach, and study stochastic games on infinite arenas for which we only assume decisiveness which roughly means that almost surely, if a set of configurations is always reachable, then it will be eventually visited.

Our contributions. Our first contribution is a simple fixpoint characterization of the winning sets and associated winning strategies for generalized Büchi objectives with probability one. The characterization is not concerned with computability and applies to any countable decisive arena. Using μ-calculus notation to define, and reason about, the winning sets and winning strategies makes the correctness proof rather direct: it is possible to give a fully detailed correctness proof in under three pages.

Our second contribution is an application of the above characterization to prove the computability of winning sets (for generalized Büchi objectives) in arenas generated by probabilistic lossy channel systems (PLCS). Rather than using ad-hoc reasoning, we just need to follow the approach advocated in [8] and use a generic finite-time convergence theorem for well-structured transition systems (more generally: for fixpoints over the powersets of WQO's). This allows us to infer the computability and the regularity of the winning sets directly from the fact that the fixpoint characterization uses "upward-guarded" fixpoint terms on regularity-preserving operators. The method easily accommodates arbitrary regular arena partition, PLCS extended with regular guards, and other kinds of unreliability.

Related work on lossy channel systems. An early positive result for stochastic games on probabilistic lossy channel systems is the decidability of single-player reachability or Büchi games with probability one (dually, safety or co-Büchi with positive probability) [9]. Then [2] proved the determinacy and decidability of two-player stochastic games on PLCS for (single) Büchi objectives with probability one. On PLCS, these positive results cannot be extended much —in particular to parity objectives— since Büchi games with positive probability are undecidable, already in the case of a single player [9]. Attempts to extend the decidability beyond (generalized) Büchi must thus abandon some generality in other dimensions, e.g., by restricting to finite-memory strategies, as in the one-player case [9].

Outline of the paper. Section 2 introduces the necessary concepts and notations on turn-based stochastic games. Section 3 provides the characterization of winning configurations in the general case of decisive arenas. Section 4 focuses on stochastic games on lossy channel systems and explains how decidability is obtained.

2 Stochastic games on decisive arenas

We consider general 2-player stochastic turn-based games on countable arenas. In such games, the two players choose moves in turns and the outcome of their choice is probabilistic.

Definition 2.1. *A* turn-based stochastic arena *is a tuple* $\mathcal{G} = (\mathit{Conf}, \mathsf{Moves}, \boldsymbol{P})$ *such that* Conf *is a countable set of configurations partitioned into* $\mathit{Conf}_A \sqcup \mathit{Conf}_B$*,* Moves *is a finite set of moves, and* $\boldsymbol{P} : \mathit{Conf} \times \mathsf{Moves} \to \mathit{Dist}(\mathit{Conf})$ *is a partial function whose values are probabilistic distribution of configurations. We say that move* m *is* enabled *in configuration* c *when* $\boldsymbol{P}(c, m)$ *is defined.* $\mathcal{G}$ *is* eternal *(also* deadlock-free*) if for all* c *there is some enabled* m*.*

The set of possible configurations *Conf* of the game is partitioned into configurations "owned" by each of the players: in some $c \in Conf_A$, player A, or "Alice", chooses the next move, while if $c \in Conf_B$, it is player B, "Bob", who chooses. It is useful to consider informally that, beyond Alice and Bob, there is a third party called "the environment" who is responsible for the probabilistic behaviors. This is why the game is stochastic: after each move m of one of the players, the environment chooses the next configuration probabilistically according to $\mathbf{P}(c, m)$. For a configuration c, when move $m \in \mathsf{Moves}$ is selected, we write $Post[m](c)$ for the set of possible configurations from c after m: $Post[m](c) \stackrel{\text{def}}{=} \{c' \in Conf \mid \mathbf{P}(c, m)(c') > 0\}$, and, symmetrically, $Pre[m](c)$ denotes the set of possible predecessors by m: $Pre[m](c) \stackrel{\text{def}}{=} \{c' \in Conf \mid \mathbf{P}(c', m)(c) > 0\}$.

Runs and strategies. For simplification purposes, and without any real loss of generality, we assume in the rest of this paper that all arenas are eternal, aka deadlock-free. A *run* of $\mathcal{G}$ is a (non-empty) sequence $\rho \in Conf^* \cup Conf^\omega$, finite or infinite, of configurations. A *strategy* for player A resolves all non-deterministic choices in $Conf_A$ by mapping every run ending in an A-configuration (i.e., a configuration $c \in Conf_A$) to a move enabled in c. Formally, a strategy σ for A is a mapping $\sigma : Conf^* Conf_A \to \mathsf{Moves}$ such that for every history run $\rho = c_0 c_1 \cdots c_n$ with $c_n \in Conf_A$, $\sigma(\rho) \in \mathsf{Moves}$ is enabled in c_n. Symmetrically, a strategy for player B is a mapping $\tau : Conf^* Conf_B \to \mathsf{Moves}$ which assigns an enabled move with each history run ending in $Conf_B$. The pair of strategies (σ, τ) is called a *strategy profile*. Note that in this paper we restrict to pure, also called deterministic, strategies. Allowing for randomization would not change the winning configurations [20].

Not all runs agree with a given strategy profile. We say that a finite or infinite run $\rho = c_0 c_1 \cdots c_n \cdots$ is *compatible* with (σ, τ) if for every prefix $\rho_i = c_0 \cdots c_i$ of ρ, $c_i \in Conf_A$ implies $\mathbf{P}(c_i, \sigma(\rho_i))(c_{i+1}) > 0$, and $c_i \in Conf_B$ implies $\mathbf{P}(c_i, \tau(\rho_i))(c_{i+1}) > 0$.

Semantics. The behavior of $\mathcal{G}$ under strategy profile (σ, τ) is described by an infinite-state Markov chain $\mathcal{G}_{\sigma,\tau}$ where the states are finite runs compatible with (σ, τ), and where there is a transition from ρ_i to $\rho_{i+1} = \rho_i \cdot c_{i+1}$ with probability $\mathbf{P}(c_i, \sigma(\rho_i))(c_{i+1})$ if $c_i \in Conf_A$, and $\mathbf{P}(c_i, \tau(\rho_i))(c_{i+1})$ if $c_i \in Conf_B$. Standardly —see, e.g., [31] for details— with the Markov chain $\mathcal{G}_{\sigma,\tau}$ and a starting configuration c_0, is associated a probability measure on the set of runs of $\mathcal{G}$ starting with c_0 and where behaviors are ruled by (σ, τ).

It is well-known that given φ an LTL formula where atomic propositions are arbitrary sets of configurations, the set of runs that satisfy φ is measurable. Below we write $\mathbb{P}_{\sigma,\tau}(c_0 \models \varphi)$ for the measure of runs of $\mathcal{G}_{\sigma,\tau}$ that start with c_0 and satisfy φ, and use the standard "$\Box$", "$\Diamond$" and "$\bigcirc$" symbols for linear-time modalities "always", "eventually" and "next".

Objectives. Given a stochastic arena $\mathcal{G}$, the objective of the game describes the goal Alice aims at achieving. In this paper we consider generalized Büchi objectives. Let $R_1, \ldots, R_r \subseteq Conf$ be r sets of configurations, with an associated generalized Büchi property $\varphi = \bigwedge_{i=1}^{r} \Box\Diamond R_i$. We consider the game on $\mathcal{G}$ where Alice's objective is to satisfy φ with probability one.

We say that a strategy σ for Alice is *almost-surely winning* from c_0 for objective φ if for every strategy τ for Bob, $\mathbb{P}_{\sigma,\tau}(c_0 \models \varphi) = 1$. In this case, we say that configuration c_0 is *winning* (for Alice). The set of winning configurations is denoted: $\langle A\rangle^{=1}\varphi$.

Decisiveness. In this paper, we focus on a subclass of stochastic arenas, namely those that are decisive, following a terminology introduced for Markov chains in [3].

Definition 2.2. *A Markov chain is* decisive *if for every subset* U *of states and every initial state* s, $\mathbb{P}(s \models \Box\Diamond U \vee \Diamond\neg Pre^*(U)) = 1$ *—where* $Pre^*(U)$ *denotes the set of configurations from which there is a non-zero probability of reaching* U *in the chain—.*
An arena $\mathcal{G}$ *is* decisive *if for every strategy profile* (σ,τ)*, the Markov chain* $\mathcal{G}_{\sigma,\tau}$ *is decisive.*

Assuming $\mathcal{G}$ is decisive, the definition rewrites: for every $U \subseteq Conf$, for every initial configurations c_0 and for every strategy profile (σ,τ)

$$\mathbb{P}_{\sigma,\tau}(c_0 \models \Box\Diamond U \vee \Diamond\neg Pre^*_{\sigma,\tau}(U)) = 1\ , \tag{1}$$

where $\neg Pre^*_{\sigma,\tau}(U)$ denotes the set of configurations that can no longer reach U in $\mathcal{G}_{\sigma,\tau}$, formally $c \in Pre^*_{\sigma,\tau}(U) \stackrel{\text{def}}{\Leftrightarrow} \mathbb{P}_{\sigma,\tau}(c \models \Diamond U) > 0$. Examples of decisive arenas are coarse arenas (when there is some global lower-bound $p > 0$ such that $\mathbf{P}(c,m)(c') > 0$ implies $\mathbf{P}(c,m)(c') \geq p$) and arenas with a finite attractor (when there exists is a finite set $F \subseteq Conf$ which is visited infinitely often almost surely) [1, 7]. We want to stress that we do not require our infinite arenas to be *coarse* or even *finitely-branching* (when $Post[m](c)$ is finite for every c and m). We observe that the characterization of winning sets we provide in the next section is *not valid* for general countable arenas (a counterexample with only 1 player can be obtained, e.g., from the system built in [1]).

3 Solving generalized Büchi games

In this section we provide a simple fixpoint characterization of the set of winning configurations (and of the associated winning strategy) for games with a generalized Büchi objective that should be satisfied almost-surely. For this characterization and its proof of correctness, we use terms with fixpoints combining functions and constants over the complete lattice 2^{Conf} of all sets of configurations.

3.1 A μ-calculus for fixpoint terms

We assume familiarity with μ-calculus notation and only recall the basic concepts and notations we use below. The reader is referred to [6, 15] for more details.

The set of subsets of configurations with the inclusion, $(2^{Conf}, \subseteq)$, is a complete Boolean lattice. We consider *monotonic operators*, i.e., n-ary mappings $f : (2^{Conf})^n \to (2^{Conf})$ such that $f(U_1,\ldots,U_n) \subseteq f(V_1,\ldots,V_n)$ when $U_i \subseteq V_i$ for all $i = 1,\ldots,n$. (A *constant* $U \subseteq Conf$ is a 0-ary monotonic operator.) Formally, the language $L_\mu = \{\varphi,\psi,\ldots\}$ of *terms with fixpoints* is given by the following abstract grammar

$$\varphi ::= f(\varphi_1,\ldots,\varphi_n) \mid X \mid \mu X.\varphi \mid \nu X.\varphi$$

where f is any n-ary monotonic operator and X is any variable. Terms $\mu X.\varphi$ and $\nu X.\varphi$ are least and greatest fixpoint expressions.

The complementation operator $\neg$, defined with $\neg U = \mathit{Conf} \smallsetminus U$, may be used when writing down L_μ terms as long as any bound variable is under the scope of an even number of negations. Such terms can be rewritten in positive forms by using the *dual* $\widetilde{f}$ of any f, defined with $\widetilde{f}(U_1, \ldots, U_n) \stackrel{\text{def}}{=} \neg f(\neg U_1, \ldots, \neg U_n)$. Note that $\widetilde{f}$ is monotonic since f is.

The semantics of L_μ terms is as expected (see [8, 15]). Since we only use monotonic operators in our fixpoint terms, all the terms have a well-defined interpretation as a subset of Conf for closed terms, and more generally as a monotonic n-ary mapping over 2^{Conf} for terms with n-free variables. We sligthly abuse notation, letting e.g. $\varphi(X_1, \ldots, X_n)$ denote both a term in L_μ and its denotation as an n-ary monotonic operator. Similarly, $\varphi(\psi_1, \ldots, \psi_n)$ is the term obtained by substituting $\psi_1, \ldots, \psi_n \in L_\mu$ for the (free occurrences of) the X_i's in φ, but when $U_1, \ldots, U_n \subseteq \mathit{Conf}$ are constants, $\varphi(U_1, \ldots, U_n)$ also denotes the application of the operator defined by φ over the U_i's.

When reasoning on fixpoint terms, one often uses *unfoldings*, i.e., the following equalities that just state that a least or greatest fixpoint is indeed a fixpoint:

$$\mu X.\varphi(X, \ldots) = \varphi(\mu X.\varphi(X, \ldots), \ldots) , \quad \nu X.\varphi(X, \ldots) = \varphi(\nu X.\varphi(X, \ldots), \ldots) .$$

Recall that the least (or greatest) fixpoint is the least *pre-fixpoint* (greatest *post-fixpoint*):

$$\varphi(U) \subseteq U \text{ implies } \mu X.\varphi(X) \subseteq U , \qquad \varphi(U) \supseteq U \text{ implies } \nu X.\varphi(X) \supseteq U .$$

It is well-known (Kleene's fixpoint theorem) that when monotonic operators are $\bigcup$- and $\bigcap$-continuous, —i.e., satisfy $f(\bigcup_i U_i) = \bigcup_i f(U_i)$ and $f(\bigcap_i U_i) = \bigcap_i f(U_i)$—, their least and greatest fixpoints are obtained as the limits of ω-length sequences of approximants. We do not assume $\bigcap / \bigcup$-continuity in our setting (e.g., Pre is not $\bigcap$-continuous when finite-branching is not required) and fixpoints are obtained as the stationary limits of transfinite ordinal-indexed sequences of approximants (see [15]): for a set $U = \mu X.\varphi(X)$ defined as a least fixpoint, the approximants $(U_\alpha)_{\alpha \in \mathit{Ord}}$ are defined inductively with $U_0 \stackrel{\text{def}}{=} \emptyset$, $U_{\beta+1} \stackrel{\text{def}}{=} \varphi(U_\beta)$ for a successor ordinal, and $U_\lambda \stackrel{\text{def}}{=} \bigcup_{\beta<\lambda} U_\beta$ for a limit ordinal λ. For a greatest fixpoint $V = \nu X.\varphi(X)$, they are given by $V_0 \stackrel{\text{def}}{=} \mathit{Conf}$, $V_{\beta+1} \stackrel{\text{def}}{=} \varphi(V_\beta)$ and $V_\lambda \stackrel{\text{def}}{=} \bigcap_{\beta<\lambda} V_\beta$.

3.2 Characterization of winning configurations

We first introduce auxiliary operators that let us reason about strategies and characterize the winning sets. Let $\mathsf{Enabled}(c) \subseteq \mathsf{Moves}$ denote the set of moves enabled in configuration c and for $X, Y \subseteq \mathit{Conf}$ let

$$\mathit{Pre}^\exists(X, Y) \stackrel{\text{def}}{=} \{c \in \mathit{Conf} \mid \exists m \in \mathsf{Enabled}(c), \mathit{Post}[m](c) \subseteq X \text{ and } \mathit{Post}[m](c) \cap Y \neq \emptyset\} ,$$

$$\mathit{Pre}^\forall(X, Y) \stackrel{\text{def}}{=} \{c \in \mathit{Conf} \mid \forall m \in \mathsf{Enabled}(c), \mathit{Post}[m](c) \subseteq X \text{ and } \mathit{Post}[m](c) \cap Y \neq \emptyset\} .$$

It is clear that $\mathit{Pre}^\exists$ and $\mathit{Pre}^\forall$ are monotonic in both arguments if we reformulate them in terms of the more familiar Pre operator (recall that $c \in \widetilde{\mathit{Pre}}[m](\emptyset)$ iff

m is not enabled in c):

$$Pre^{\exists}(X,Y) = \bigcup_{m\in\mathsf{Moves}} \big[\widetilde{Pre}[m](X) \cap Pre[m](Y)\big] \ ,$$
$$Pre^{\forall}(X,Y) = \bigcap_{m\in\mathsf{Moves}} \big(\widetilde{Pre}[m](\emptyset) \cup \big[\widetilde{Pre}[m](X) \cap Pre[m](Y)\big]\big) \ .$$

We further define $Pre_A^{\otimes}(X,Y) \stackrel{\text{def}}{=} \big(Conf_A \cap Pre^{\exists}(X,Y)\big) \cup \big(Conf_B \cap Pre^{\forall}(X,Y)\big)$. In other words, $Pre_A^{\otimes}(X,Y)$ is exactly the set from where Alice can guarantee in one step to have X surely and Y with positive probability. This can be summarized as:

Fact 3.1. *Let $X,Y \subseteq Conf$.*

- *If $c \in Pre_A^{\otimes}(X,Y)$, then, A has a memoryless strategy σ such that, for every strategy τ for B: $\mathbb{P}_{\sigma,\tau}(c \models \bigcirc X) = 1$, and $\mathbb{P}_{\sigma,\tau}(c \models \bigcirc Y) > 0$.*
- *If $c \notin Pre_A^{\otimes}(X,Y)$, then, B has a memoryless strategy τ such that, for every strategy σ for A: $\mathbb{P}_{\sigma,\tau}(c \models \bigcirc X) < 1$, or $\mathbb{P}_{\sigma,\tau}(c \models \bigcirc Y) = 0$.*

We introduce auxiliary (unary) operators: for $i = 1,\ldots,r$, H_i is given by

$$H_i(X) \stackrel{\text{def}}{=} \mu Z.X \cap Pre_A^{\otimes}\big(X, R_i \cup Z\big) \ . \tag{2}$$

The intuition is that, from $H_i(X)$, Alice has a strategy ensuring a positive probability of reaching R_i later —which would be characterized by "$\mu Z.Pre_A^{\otimes}\big(Conf, R_i \cup Z\big)$"— *all the while staying surely in X*, hence the amendments. See Lemma 3.6 for a precise statement. Unfolding its definition, we see that $H_i(X) \subseteq X$, i.e., H_i is *contractive.*

Letting $H_{1,r}(X) \stackrel{\text{def}}{=} \bigcap_{i=1}^r H_i(X)$, we finally define the following fixpoint terms:

$$W \stackrel{\text{def}}{=} \nu X.H_{1,r}(X) = \nu X.\bigcap_{i=1}^{r} H_i(X) = \nu X.\bigcap_{i=1}^{r}\Big[\mu Z.X \cap Pre_A^{\otimes}\big(X, R_i \cup Z\big)\Big] \ ,$$

$$W' \stackrel{\text{def}}{=} \nu X.Pre_A^{\otimes}\Big(H_{1,r}(X), Conf\Big) = \nu X.Pre_A^{\otimes}\Big(\bigcap_{i=1}^{r}\Big[\mu Z.X \cap Pre_A^{\otimes}\big(X, R_i \cup Z\big)\Big], Conf\Big) \ ,$$

$$W_1 \stackrel{\text{def}}{=} \nu X.\mu Z.Pre_A^{\otimes}\big(X, R_1 \cup Z\big).$$

Theorem 3.2 (Fixpoint characterization of winning sets)**.** *For generalized Büchi objectives on decisive arenas, the winning set $\langle\!\langle A\rangle\!\rangle^{=1} \bigwedge_{i=1}^r \Box\Diamond R_i$ coincides with W. Moreover $W = W'$ and from W Alice has an almost-surely winning* finite memory *strategy σ_W.*
In the case $r = 1$ of simple Büchi objectives the winning set $\langle\!\langle A\rangle\!\rangle^{=1}\Box\Diamond R_1$ coincides with W_1 and the winning strategy σ_W is a memoryless *strategy.*

Let us first explain how, in the case where $r = 1$, one derives the correctness of W_1 from the correctness of W. Setting $r = 1$ in W yields $\langle\!\langle A\rangle\!\rangle^{=1}\Box\Diamond R_1 = \nu X.H_1(X) = \nu X.\mu Z.X \cap Pre_A^{\otimes}\big(X, R_1 \cup Z\big)$. In this situation, we can use Eq. (†), a purely algebraic and lattice-theoretical equality that holds for any monotonic binary f (see Appendix for a proof):

$$\nu X.\mu Z.X \cap f(X,Z) = \nu X.\mu Z.f(X,Z) \ . \tag{†}$$

Applying to $\nu X.H_1(X)$ with Eq. (2) yields $\nu X.H_1(X) = \nu X.\mu Z.\mathit{Pre}_A^{\otimes}\big(X, R_1 \cup Z\big)$, which is just W_1.

Theorem 3.2 provides two different characterizations of the winning set $\langle A\rangle^{=1}\big(\bigwedge_{i=1}^{r} \Box\Diamond R_i\big)$. Let us now prove its validity, in the general context of decisive stochastic arenas. The proof is divided in two parts: correctness of W' in Proposition 3.5, completeness of W in Proposition 3.7, and some purely algebraic reasoning closing the loop in Lemma 3.8.

3.3 Correctness for W'

We prove that W' only contains winning configurations for Alice by exhibiting a strategy with which she ensures almost surely $\bigwedge_i \Box\Diamond R_i$ when starting from some $c \in W'$. We first define r strategies $(\sigma_i)_{1\leq i\leq r}$, one for each goal set $R_1, \ldots, R_r$, and prove their relevant properties. It will then be easy to combine the σ_i's in order to produce the required strategy.

For $i = 1, \ldots, r$, observe that $H_i(W') = W' \cap \mathit{Pre}_A^{\otimes}(W', R_i \cup H_i(W'))$. We let σ_i be the memoryless A-strategy defined as follows: for $c \in \mathit{Conf}_A \cap H_i(W')$, Alice picks an enabled move m such that $\mathit{Post}(c)[m] \subseteq W'$ and $\mathit{Post}[m](c) \cap (R_i \cup H_i(W')) \neq \emptyset$, which is possible by definition of $\mathit{Pre}_A^{\otimes}$, while for $c \in \mathit{Conf}_A \cap W' \cap \neg H_i(W')$, Alice picks an enabled move m with $\mathit{Post}(c)[m] \subseteq H_{1,r}(W')$, which is possible since $W' = \mathit{Pre}_A^{\otimes}(H_{1,r}(W'), \mathit{Conf})$.

$H_{1,r}$ is contractive since the H_i's are, hence $H_{1,r}(W') \subseteq W'$ and we deduce

$$\forall c \in W' : \forall \tau : \mathbb{P}_{\sigma_i,\tau}(c \models \Box W') = 1 \, . \tag{3}$$

Lemma 3.3. *For all $c \in W'$ there exists some $\gamma_c > 0$ such that $\mathbb{P}_{\sigma_i,\tau}(c \models \Diamond R_i) \geq \gamma_c$ for all B-strategies τ.*

Proof. Consider first $c \in H_i(W')$. Writing $(Z_\alpha)_{\alpha\in Ord}$ for the approximants of $H_i(W')$, we prove, by induction on α, that $\gamma_c > 0$ exists when $c \in Z_\alpha$. The base case $\alpha = 0$ holds vacuously since $Z_0 = \emptyset$. For $\alpha = \lambda$ (a limit ordinal), $Z_\lambda = \bigcup_{\beta<\lambda} Z_\beta$ so each $c \in Z_\lambda$ is in some Z_β and the induction hypothesis applies.

Now to the successor case $\alpha = \beta + 1$. Here $Z_\alpha = W' \cap \mathit{Pre}_A^{\otimes}\big(W', R_i \cup Z_\beta\big)$ and, given σ_i and for any τ, from $c \in Z_\alpha$ Alice or Bob will pick a move m with $\mathit{Post}[m](c) \cap (R_i \cup Z_\beta) \neq \emptyset$. The probability that after probabilistic environment's move the play will be in R_i exactly at the next step is precisely $\gamma = \sum_{d\in R_i} \mathbf{P}(c, m)(d)$ and $\gamma > 0$ if $\mathit{Post}[m](c) \cap R_i \neq \emptyset$ (and only then). If $\gamma = 0$ then $\mathit{Post}[m](c) \cap R_i = \emptyset$ so that $\mathit{Post}[m](c) \cap Z_\beta \neq \emptyset$. Then there is a positive probability $\gamma' = \sum_{d\in Z_\beta} \mathbf{P}(c, m)(d)$ that after probabilistic decision the play will be in Z_β at the next step, hence (by induction hypothesis) a positive probability γ'' that it will be in R_i later, with $\gamma'' \geq \sum_{d\in Z_\beta} \gamma_d \cdot \mathbf{P}(c, m)(d)$. Here γ and the lower bound for γ'' depend on τ, or more precisely on what move m is chosen by Bob if $c \in \mathit{Conf}_B$. However, and since there are finitely many moves in Moves, we can pick a strictly positive value that is a lower bound for all the corresponding $\max(\gamma, \gamma'')$, proving the existence of $\gamma_c > 0$ for $c \in Z_\alpha$.

There remains the case where $c \in W' \cap \neg H_i(W')$: here σ_i ensures that the play will be in $H_i(W')$ in the next step, and we can take $\gamma_c \stackrel{\text{def}}{=} \min_{m\in\mathsf{Moves}}\{\mathbf{P}(c, m)(d)\cdot \gamma_d \mid \mathbf{P}(c, m)(d) > 0 \text{ and } d \in H_i(W')\}$. This definition ensures $\gamma_c > 0$, which concludes the proof. ∎

Lemma 3.4. $\mathbb{P}_{\sigma_i,\tau}(c \models \Box W' \wedge \Box\Diamond R_i) = 1$ *for all* $c \in W'$ *and all* B*-strategies* τ.

Proof. This is where we use that $\mathcal{G}$ is decisive. Let $c \in W'$ and τ be any strategy for Bob. Eq. (1) applied to set R_i, initial configuration c, and strategy profile (σ_i, τ) gives:

$$\mathbb{P}_{\sigma_i,\tau}(c \models \Box\Diamond R_i \vee \Diamond\neg \mathit{Pre}^*_{\sigma,\tau}(R_i)) = 1.$$

Using Eq. (3), we deduce $\mathbb{P}_{\sigma_i,\tau}\Big(c \models \Box W' \wedge \big(\Box\Diamond R_i \vee \Diamond\neg \mathit{Pre}^*_{\sigma,\tau}(R_i)\big)\Big) = 1$. Observe now that $W' \subseteq \mathit{Pre}^*_{\sigma,\tau}(R_i)$, thanks to Lemma 3.3. As a consequence, $\mathbb{P}_{\sigma_i,\tau}\big(c \models \Box W' \wedge \Diamond\neg \mathit{Pre}^*_{\sigma,\tau}(R_i)\big) = 0$, so that we conclude $\mathbb{P}_{\sigma_i,\tau}\big(c \models \Box W' \wedge \Box\Diamond R_i\big) = 1$. ☐

Proposition 3.5 (Correctness of W'). $W' \subseteq \langle A\rangle^{=1}\big(\bigwedge_{i=1}^{r} \Box\Diamond R_i\big)$.

Proof. By combining the strategies σ_i's, we define a finite-memory strategy σ_W that guarantees $\mathbb{P}_{\sigma_W,\tau}\big(c \models \bigwedge_i \Box\Diamond R_i\big) = 1$ for any $c \in W'$ and against any B-strategy τ.

More precisely, σ_W has r modes: $1, 2, \ldots, r$. In mode i, σ_W behaves like σ_i until R_i is reached, which is bound to eventually happen with probability 1 by Lemma 3.4. Note that the play remains constantly in W'. Once R_i has been reached, σ_W switches to mode $i + 1 (\bmod\ r)$, playing at least one move. This is repeated in a neverending cycle, ensuring $\Box\Diamond R_i$ with probability 1. ☐

3.4 Completeness of W

In order to prove that W contains the winning set for Alice, we show that $\langle A\rangle^{=1} \bigwedge_i \Box\Diamond R_i$ is a post-fixpoint of $H_{1,r}$, thus necessarily included in its greatest fixpoint W.

Lemma 3.6. $H_i(X) \supseteq \big\{c \;\big|\; \exists\sigma\ \forall\tau,\ \mathbb{P}_{\sigma,\tau}(c \models \Box X) = 1$ *and* $\mathbb{P}_{\sigma,\tau}(c \models \bigcirc\Diamond R_i) > 0\big\}$.

Proof. We actually prove a stronger claim: we show that there exists a memoryless strategy τ for Bob such that for every $c \notin H_i(X)$ and every strategy σ for Alice, either $\mathbb{P}_{\sigma,\tau}(c \models \Diamond\neg X) > 0$, or $\mathbb{P}_{\sigma,\tau}(c \models \bigcirc\Box\neg R_i) = 1$.

Let $c \notin H_i(x)$. By definition $\neg H_i(X) = \neg X \cup \neg \mathit{Pre}^{\otimes}_A(X, R_i \cup H_i(X))$. If $c \notin X$, then trivially $\mathbb{P}_{\sigma,\tau}(c \models \Diamond\neg X) > 0$ for any (σ, τ) so we do not care how τ is defined here. Consider now $c \notin \mathit{Pre}^{\otimes}_A(X, R_i \cup H_i(X))$. By Fact 3.1, Bob has a (memoryless) strategy τ_c such that against any strategy σ for Alice, $\mathbb{P}_{\sigma,\tau_c}(c \models \bigcirc X) < 1$ or $\mathbb{P}_{\sigma,\tau_c}\big(c \models \bigcirc(R_i \cup H_i(X))\big) = 0$, which can be reformulated as $\mathbb{P}_{\sigma,\tau_c}(c \models \bigcirc\neg X) > 0$ or $\mathbb{P}_{\sigma,\tau_c}\big(c \models \bigcirc(\neg R_i \cap \neg H_i(X))\big) = 1$. For $c \in \mathit{Conf}_B$, we define $\tau(c)$ as the move given by $\tau_c(c)$. The resulting strategy τ guarantees, starting from $\neg H_i(X)$, that the game will either always stay in $\neg R_i \cap \neg H_i(X)$ (after the 1st step) or has a positive probability of visiting $\neg X$ eventually. ☐

Proposition 3.7 (Completeness of W). $\langle A\rangle^{=1}\big(\bigwedge_{i=1}^{r} \Box\Diamond R_i\big) \subseteq W$.

Proof. Let $c \in \langle A\rangle^{=1} \bigwedge_i \Box\Diamond R_i$, and σ be a strategy ensuring $\bigwedge_i \Box\Diamond R_i$ with probability 1 from c. Consider $E = \big\{d \in \mathit{Conf} \mid \exists\tau : \mathbb{P}_{\sigma,\tau}(c \models \Diamond d) > 0\big\}$, i.e., the set of configurations that can be visited under strategy σ. Obviously

$c \in E$. Furthermore, for any $d \in E$ and any B-strategy τ, $\mathbb{P}_{\sigma',\tau}(d \models \Box E) = 1$ holds, where σ' is a suffix strategy of σ after d is visited, that is, σ' behaves from d like σ would after some prefix ending in d. Since furthermore $\mathbb{P}_{\sigma',\tau}\big(d \models \bigwedge_i \Box\Diamond R_i\big) = 1$ by assumption, we deduce in particular $\mathbb{P}_{\sigma',\tau}\big(d \models \bigcirc\Diamond R_i\big) = 1$ for any $i = 1, \ldots, r$. Hence $E \subseteq H_i(E)$ for any i by Lemma 3.6, and thus $E \subseteq H_{1,r}(E)$. Finally E is a post-fixpoint of $H_{1,r}$, and is thus included in its greatest fixpoint. We conclude that $c \in \nu X.H_{1,r}(X) = W$. $\square$

The loop is closed, and Theorem 3.2 proven, with the following lattice-theoretical reasoning:

Lemma 3.8. *$W \subseteq W'$.*

Proof. Recall that $W = \nu X.H_{1,r}(X)$, so that $W = H_{1,r}(W)$. Similarly, using Eq. (2), we deduce $H_{1,r}(W) = \bigcap_i H_i(W) = \bigcap_i \big(W \cap \mathit{Pre}_A^{\otimes}(W, R_i \cup W)\big)$, hence $H_{1,r}(W) \subseteq \mathit{Pre}_A^{\otimes}(W, \mathit{Conf})$ by monotonicity of $\mathit{Pre}_A^{\otimes}$ in its second argument. Combining these two points gives $W = H_{1,r}(W) \subseteq \mathit{Pre}_A^{\otimes}(H_{1,r}(W), \mathit{Conf})$, hence W is a post-fixpoint of $X \mapsto \mathit{Pre}_A^{\otimes}(H_{1,r}(X), \mathit{Conf})$ and is included in its greatest fixpoint W'. $\square$

4 Stochastic games on lossy channel systems

Theorem 3.2 entails the decidability of generalized Büchi games on channel systems with probabilistic message losses, or PLCS's. This is obtained by applying a generic and powerful "finite-time convergence theorem" for fixpoints defined on WQO's.

4.1 Channel systems with guards

A *channel system* is a tuple $S = (Q, \mathsf{C}, \mathsf{M}, \Delta)$ consisting of a finite set $Q = \{q, q', \ldots\}$ of *locations*, a finite set $\mathsf{C} = \{\mathtt{ch}_1, \ldots, \mathtt{ch}_d\}$ of *channels*, a finite *message alphabet* $\mathsf{M} = \{a, b, \ldots\}$ and a finite set $\Delta = \{\delta, \ldots\}$ of *transition rules*. Each transition rule has the form (q, g, op, q'), written $q \xrightarrow{g,op} q'$, where g is a *guard* (see below), and op is an *operation* of one of the following three forms: $\mathtt{ch}!a$ (sending message $a \in \mathsf{M}$ along channel $\mathtt{ch} \in \mathsf{C}$), $\mathtt{ch}?a$ (receiving message a from channel $\mathtt{ch}$), or $\surd$ (an internal action to some process, no I/O-operation).

Let S be a channel system as above. A *configuration* of S is a pair $c = (q, w)$ where q is a location of S and $w : \mathsf{C} \to \mathsf{M}^*$ is a mapping, that describes the current channel contents, and we write $\mathit{Conf}_S \stackrel{\text{def}}{=} Q \times \mathsf{M}^{*\mathsf{C}}$.

A *guard* is a predicate on channel contents used to constrain the firability of rules. In this paper, a guard is a tuple $g = (L_1, \ldots, L_d) \in \mathbf{Reg}(\mathsf{M})^{|\mathsf{C}|}$ of regular languages, one for each channel. For a configuration $c = (q, w_1, \ldots, w_d)$, we write $c \models g$, and say that c *respects* g, when $w_i \in L_i$ for all $i = 1, \ldots, d$.

Rules give rise to transitions in the operational semantics. Let $\delta = (q_1, g, op, q_2)$ be a rule in Δ and let $c = (q, w)$, $c' = (q', w')$ be two configurations of S. We write $c \xrightarrow{\delta} c'$, and say that δ is enabled in c, if $q = q_1$, $q' = q_2$, $c \models g$, and w' is the valuation obtained from w by applying op. Formally $w' = w$ if $op = \surd$, and otherwise if $op = \mathtt{ch}_i!a$ (resp. if $op = \mathtt{ch}_i?a$) then $w'_i = w_i.a$ (resp. $a.w'_i = w_i$) and $w'_j = w_j$ for all $j \neq i$.

For simplicity, we assume in the rest of the paper that S denotes a fixed channel system $S = (Q, \mathsf{C}, \mathsf{M}, \Delta)$ that has no deadlock configurations, i.e., every $c \in Conf_S$ has an enabled rule: this is no loss of generality since it is easy (when guards are allowed) to add rules going to a new sink location exactly in configurations where none of the original rules is enabled.

Remark (About guards)**.** *Allowing guards in transition rules is useful (e.g., for expressing priorities) but departs from the standard models of channel systems [29]. Indeed, testing the whole contents of a fifo channel is not a realistic feature when modeling distributed asynchronous systems. However, (unreliable) channel systems are now seen more broadly as a fundamental computational model with algorithmic applications beyond distributed protocols: see, e.g., [26, 5, 11, 14]. In such settings, simple guards have been considered and proved useful: see, e.g., [14, 13, 24].*

Using additional control states and messages, it is sometimes possible to simulate guards in (lossy) channel systems. We note that the known simulations only preserve nondeterministic reachability, not game-theoretical properties in stochastic environments. $\square$

4.2 Probabilistic message losses

PLCS's are channel systems where messages can be lost (following some probabilistic model) while they are in the channels [30, 10, 4, 1, 32]. In this paper, we shall consider two kinds of unreliability caused by a stochastic environment: message losses on one hand, and combinations of message losses and duplications on the other hand.

Message losses are traditionally modeled via the subword relation: given two words $u, v \in \mathsf{M}^*$, we write $u \sqsubseteq v$ when u is a *subword*, i.e., a scattered subsequence, of v. For two configurations $c = (q, w)$ and $c' = (q', w')$, we let $c \sqsubseteq c' \overset{\text{def}}{\Leftrightarrow} [q = q' \text{ and } w_i \sqsubseteq w'_i \text{ for all } i = 1, \ldots, d]$. In other words, $c \sqsubseteq c'$ when c is the result of removing some messages (possible none) at arbitrary places in the channel contents for c'.

Message duplications are modeled by a rational transduction $\mathcal{T}_{\text{dup}} \subseteq \mathsf{M}^* \times \mathsf{M}^*$ over sequences of messages, where every single message $a \in \mathsf{M}$ is replaced by either a or aa. We write $u \preceq_{\text{dup}} v$ when $(u, v) \in \mathcal{T}_{\text{dup}}$ and we extend to configurations with $(q, w) \preceq_{\text{dup}} (q', w') \overset{\text{def}}{\Leftrightarrow} [q = q' \text{ and } w_i \preceq_{\text{dup}} w'_i \text{ for all } i = 1, \ldots, d]$.

For PLCS's with only message losses, we write $c \rightsquigarrow c'$ when $c \sqsupseteq c'$ ($\overset{\text{def}}{\Leftrightarrow} c' \sqsubseteq c$). For PLCS's with losses and duplications, $c \rightsquigarrow c'$ means that $c \preceq_{\text{dup}} c'' \sqsupseteq c'$ for some c''.

In PLCS's, message perturbations are probabilistic events. Formally, we associate a distribution $D_{env}(c) \in Dist(Conf_S)$ with every configuration $c \in Conf_S$ and we say that "$D_{env}(c, c')$ is the probability that c becomes c' by message losses and duplications (in one step)". Given D_{env} and a partition $Conf_S = Conf_A \sqcup Conf_B$, the channel system S with probabilistic losses defines a stochastic arena $\mathcal{G}_S = (Conf_S, \Delta, \mathbf{P})$ where the moves available to the players are exactly the rules of S, and the probabilistic transition function $\mathbf{P}$ is formalized by: for every $c \in Conf_S$ and δ enabled in c, $\mathbf{P}(c, \delta) \overset{\text{def}}{=} D_{env}(c')$ where $c \xrightarrow{\delta} c'$.

The qualitative properties that we are interested in do not depend on the exact choices made for D_{env}. In this paper, we only require that D_{env} is *well-behaved*, i.e., satisfies the following two properties:

Compatibility with nondeterministic semantics: $D_{env}(c)(c') > 0$ iff $c \rightsquigarrow c'$.

Decisiveness: The arena $\mathcal{G}_S$ is decisive.

A now standard choice for D_{env} in PLCS's models message losses (and duplications) as independent events. One assumes that at every step, each individual message can be lost with a fixed probability $\lambda \in (0, 1)$, duplicated with a fixed probability $\lambda' \in [0, 1)$ (and remains unperturbed with probability $1 - \lambda - \lambda'$). This is the so-called *local-fault* model from [4, 32, 34], and it gives rise to a well-behaved D_{env} when only message losses are considered, i.e., when $\lambda' = 0$, or when losses are more probable than duplications, i.e., when $0 < \lambda' < \lambda$. In particular, the set $F_0 \stackrel{\text{def}}{=} \{(q, \varepsilon, \ldots, \varepsilon) \mid q \in Q\}$ of configurations with empty channels is a finite attractor in $\mathcal{G}_S$, which entails decisiveness [3]. The interested reader can find in [4, sections 5&6] some detailed computations of $D_{env}(c)(c')$ in the local-fault model, but s/he must be warned that the qualitative outcomes on PLCS's do not depend on these values as long as D_{env} is well-behaved.

4.3 Regular model-checking of channel systems

Regular model-checking [12, 25] is a symbolic verification technique where one computes infinite but regular sets of configurations using representations from automata theory or from constraint solving.

Definition 4.1. *A* (regular) region *of S is a set $R \subseteq Conf_S$ of configurations that can be written under the form $R = \bigcup_{i \in I} \{q_i\} \times L_i^1 \times \cdots \times L_i^d$ with a* finite *index set I, and where, for $i \in I$, q_i is some location $\in Q$, and each L_i^j for $j = 1, \ldots, d$ is a regular language $\in \mathbf{Reg}(\mathsf{M})$.*

Let $\mathcal{R} \subseteq 2^{Conf_S}$ denote the set of all regions of S. A monotonic operator f is *regularity-preserving*, if $f(R_1, \ldots, R_n) \in \mathcal{R}$ when $R_1, \ldots, R_n \in \mathcal{R}$. A regularity-preserving f is *effective* if a representation for $f(R_1, \ldots, R_n)$ can be computed uniformly from representations for the R_i's (and from S). For example, the set-theoretical $\cap$, $\cup$ are regularity-preserving and effective. While not a monotonic operator, complementation is regularity-preserving and effective. Hence the dual $\widetilde{f}$ of any f is regularity-preserving and effective when f is.

For the verification of (lossy) channel systems in general, and the resolution of games in particular, some useful operators are the unary pre-images $Pre_S[\delta]$ for $\delta \in \Delta$, and the upward- and downward-closures $C_\uparrow$ and $C_\downarrow$, defined with

$$Pre_S[\delta](U) \stackrel{\text{def}}{=} \{c \in Conf_S \mid \exists c' \in U : c \xrightarrow{\delta} c'\}\,, \quad C_\uparrow(U) \stackrel{\text{def}}{=} \{c \in Conf_S \mid \exists c' \in U : c' \sqsubseteq c\}\,,$$
$$Pre_S(U) \stackrel{\text{def}}{=} \textstyle\bigcup_{\delta \in \Delta} Pre_S[\delta](U)\,, \qquad C_\downarrow(U) \stackrel{\text{def}}{=} \{c \in Conf_S \mid \exists c' \in U : c \sqsubseteq c'\}\,.$$

Observe that $Pre_S[\delta]$ and Pre_S are pre-images for steps of channel systems without/before message perturbations, while $C_\uparrow$ and $C_\downarrow$ are pre- and post-images for the message-losing relation. $C_\uparrow$ and $C_\downarrow$ are closure operators. Their duals are *interior operators*: $K_\downarrow(U) \stackrel{\text{def}}{=} \widetilde{C_\uparrow}(U)$ and $K_\uparrow(U) \stackrel{\text{def}}{=} \widetilde{C_\downarrow}(U)$ are the largest downward-closed and, resp., upward-closed, subsets of U. Finally, we are also interested in pre-images for $\preceq_{\rm dup}$: we write $\mathcal{T}_{\rm dup}^{-1}(U)$ for $\{c \mid \exists c' \in U : c \preceq_{\rm dup} c'\}$. We remark that $\mathcal{T}_{\rm dup}^{-1}(\mathit{Conf}_S) = \mathit{Conf}_S$, and that $\mathcal{T}_{\rm dup}^{-1}(C_\uparrow U) = C_\uparrow(\mathcal{T}_{\rm dup}^{-1}(C_\uparrow U)) = C_\uparrow \mathcal{T}_{\rm dup}^{-1}(U)$, i.e., the definition of $c \rightsquigarrow c'$ is not sensitive to the order of perturbations.

Fact 4.2. *$\mathit{Pre}_S[\delta]$, Pre_S, $C_\uparrow$, $C_\downarrow$, $\mathcal{T}_{\rm dup}^{-1}$ and their duals are regularity-preserving and effective (monotonic) operators.*

When using effective regularity-preserving operators, one can evaluate any closed L_μ term that does not include fixpoints. For a closed $U = \mu X.\varphi(X)$, or $V = \nu X.\varphi(X)$, term with a single fixpoint, any approximant U_k and V_k for a finite $k \in \mathbb{N}$ can be evaluated but there is no guarantee that the fixpoint is reached in finite time, or that the fixpoint is a regular region. However, for fixpoints over a WQO like Conf_S, there exists a generic finite-time convergence theorem.

Definition 4.3 (Guarded L_μ terms)**.** *1. A variable Z is* upward-guarded *in an L_μ term φ if every occurrence of Z in φ is under the scope of an upward-closure $C_\uparrow$ or upward-interior $K_\uparrow$ operator.*
2. It is downward-guarded *in φ if all its occurrences in φ are under the scope of a downward-closure $C_\downarrow$ or downward-interior $K_\downarrow$ operator.*
3. A term φ is guarded *if every least fixpoint subterm $\mu Z.\psi$ of φ has Z upward-guarded in ψ, and every greatest fixpoint subterm $\nu Z.\psi$ has Z downward-guarded in ψ.*

Theorem 4.4 (Effective & regularity-preserving fixpoints)**.** *Any guarded L_μ term $\varphi(X_1, \ldots, X_n)$ built with regularity-preserving and effective operators denotes a regularity-preserving and effective n-ary operator. Furthermore the denotation of a closed term can be evaluated by computing its approximants which are guaranteed to converge after finitely many steps.*

Theorem 4.4 is a special case of the main result of [8] (see also [27]) where it is stated for arbitrary well-quasi-ordered sets (WQO's) and a generic notion of "effective regions". We recall that, by Higman's lemma, $(\mathit{Conf}_S, \sqsubseteq)$ is a *well-quasi-ordered set*, i.e., a quasi-ordered set —$\sqsubseteq$ is reflexive and transitive— such that every infinite sequence $c_0, c_1, c_2, \ldots$ contains an increasing subsequence $c_i \sqsubseteq c_j$ (with $i < j$).

4.4 Stochastic games on lossy channel systems

In the context of section 4.2 and the decisive stochastic arena $\mathcal{G}_S$, we can reformulate the *Pre* operator used in Section 3 as a regularity-preserving and effective operator.

When we only consider message losses, $\mathit{Pre}[\delta](X) = \mathit{Pre}_S[\delta](C_\uparrow X)$ (thanks to the assumption that D_{env} is *compatible* with the nondeterministic semantics). If also duplications are considered, then $\mathit{Pre}[\delta](X) = \mathit{Pre}_S[\delta]\big(\mathcal{T}_{\rm dup}^{-1}(C_\uparrow X)\big)$. In order to deal uniformly with the two cases we shall let $\mathcal{T}_{\rm dup}$ be the identity relation when duplications are not considered. By duality $\widetilde{\mathit{Pre}}[\delta](X) = \widetilde{\mathit{Pre}_S}[\delta](K_\downarrow \widetilde{\mathcal{T}_{\rm dup}^{-1}}(X))$ and the derived operators satisfy $\mathit{Pre}^\exists(X, Y) = \mathit{Pre}^\exists(K_\downarrow X, C_\uparrow Y)$, $\mathit{Pre}^\forall(X, Y) = \mathit{Pre}^\forall(K_\downarrow X, C_\uparrow Y)$ and $\mathit{Pre}_A^\otimes(X, Y) = \mathit{Pre}_A^\otimes(K_\downarrow X, C_\uparrow Y)$. Thus

Theorem 3.2 rewrites:

$$\langle A\rangle^{=1} \bigwedge_{i=1}^{r} \Box\Diamond R_i \;=\; \nu X. \bigcap_{i=1}^{r} H_i(X) \;=\; \nu X. Pre_A^{\otimes}\Big(K_{\downarrow} \bigcap_{i=1}^{r} H_i(X), Conf_S\Big), \quad (4)$$

with $H_i(X) \stackrel{\text{def}}{=} \mu Z. X \cap Pre_A^{\otimes}\big(K_{\downarrow}X, C_{\uparrow}(R_i \cup Z)\big)$. Observe how the closure properties of $Pre_A^{\otimes}$ let us easily rewrite W' into a guarded term. The same technique does not apply to the simpler term W and this explains why we developed two characterizations of the winning set in Section 3. However, in the case where $r = 1$, the characterization with W can be simplified in W_1 and Theorem 3.2 yields the following *guarded term* for stochastic Büchi games on lossy channel systems: $\langle A\rangle^{=1}\Box\Diamond R_1 = \nu X.\mu Z. Pre_A^{\otimes}\big(K_{\downarrow}X, C_{\uparrow}(R_1 \cup Z)\big)$.

Since $H_i(X)$ and $\langle A\rangle^{=1} \bigwedge_{i=1}^{r} \Box\Diamond R_i$ have guarded L_μ expressions, the following decidability result is an immediate application of Theorem 4.4 to Eq. (4).

Theorem 4.5 (Decidability of Generalized Büchi games with probability 1)**.** *In stochastic games on lossy channel system S with regular arena partition $Conf_S = Conf_A \sqcup Conf_B$ and for regular goal regions $R_1, \ldots, R_r$, the winning set $\langle A\rangle^{=1} \bigwedge_{i=1}^{r} \Box\Diamond R_i$ is a regular region that can be computed uniformly from S and $R_1, \ldots, R_r$.*

Furthermore, the winning strategies have simple finite representations. One first computes the regular region W $(= W')$. Then for each rule $\delta \in \Delta$, and each $i = 1, \ldots, r$, one computes $V_i^{\delta} \stackrel{\text{def}}{=} Conf_A \cap Pre_S[\delta]\big(K_{\downarrow}W \cap C_{\uparrow}\big(R_i \cup H_i(W)\big)\big)$, these are again regular regions. The strategy σ_i for Alice is then "when in V_i^{δ}, choose δ" and the strategy σ_W is just a combination of the σ_i's using finite memory and testing when we are in the R_i's.

On complexity. Theorem 4.4 does not only show that $W = \langle A\rangle^{=1} \bigwedge_i \Box\Diamond R_i$ is computable from S and $R_1, \ldots, R_r$. It also shows that W is obtained by computing the sequence of approximants $(W_k)_{k\in\mathbb{N}}$ —given by $W_0 = Conf_S$ and $W_{k+1} = Pre_A^{\otimes}(K_{\downarrow}H_{1,r}(W_k), Conf_S)$— until the sequence stabilizes, which is guaranteed to eventually occur. Furthermore, computing $H_{1,r}(W_k)$, i.e., $\bigcap_{i=1}^{r} H_i(W_k)$, involves r fixpoint computations that can use the same technique: sequences of approximants guaranteed to converge in finite time by Theorem 4.4.

There now exist generic upper bounds on the convergence time of such sequences, see [33]. In our case, they entail that the above symbolic algorithm computing the regular region $\langle A\rangle^{=1}(\bigwedge_{i=1}^{r} \Box\Diamond R_i)$ runs in time $F_{\omega^\omega}(O(n))$, where n is the size of a description of $S, R_1, \ldots, R_r$, and where F_{ω^ω} is the first function in the extended Grzegorczyck hierarchy that is not multiply-recursive (a kind of "Hyper-Ackermann" function).

This bound is optimal: deciding whether $c \in \langle A\rangle^{=1}(\bigwedge_{i=1}^{r} \Box\Diamond R_i)$ is F_{ω^ω}-hard since this generalizes reachability questions (in lossy channel systems) that are F_{ω^ω}-hard [19].

Corollary 4.6. *Deciding whether $c \in \langle A\rangle^{=1}(\bigwedge_{i=1}^{r} \Box\Diamond R_i)$ for given S, c, R_1, $\ldots$, R_r is F_{ω^ω}-complete.*

5 Concluding remarks

We gave a simple fixpoint characterization of winning sets and winning strategies for 2-player stochastic games where a generalized Büchi objective should be satisfied almost-surely. The characterization is correct for any countable decisive arena.

Such fixpoint characterizations lead to symbolic model-checking and symbolic strategy-synthesizing algorithms for infinite-state systems and programs. The main issue here is the finite-time convergence of the fixpoint computations. For well-quasi-ordered sets, one can use generic results showing the finite-time convergence of so-called "guarded" fixpoint expressions as we demonstrated by showing the decidability of generalized Büchi games on probabilistic lossy channel systems, a well-quasi-ordered model that induces decisive arenas.

We believe Theorem 4.4 has more general applications for games, stochastic or not, on well-quasi-ordered infinite-state systems. We would like to mention quantitative objectives as an interesting direction for future works (see [32, 36]).

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A Proof of Equation (†) page 6

Section 3 relies on the following Lemma for simplifying the characterization of winning sets for simple Büchi objectives:

Lemma A.1 (Contractive $\boldsymbol{\nu}$-$\boldsymbol{\mu}$ fixpoint)**.** *For any binary (monotonic) operator f, $\nu X.\mu Y.X \cap f(X,Y) = \nu X.\mu Y.f(X,Y)$.*

This is a purely algebraic and lattice-theoretical result that is not specific to stochastic games or channel systems. We include its proof here for the sake of completeness.

We start with a simpler lemma: let h be a unary (monotonic) operator.

Lemma A.2. *Assume $U = \mu Y.h(Y)$ and $V \supseteq U$. Then $\mu Y.V \cap h(Y) = U$.*

Proof. Write W for $\mu Y.V \cap h(Y)$. Now $V \cap h(Y) \subseteq h(Y)$ entails $\mu Y.V \cap h(Y) \subseteq \mu Y.h(Y)$, i.e., $W \subseteq U$, by monotonicity.

For the other inclusion, we consider the approximants $(U_\alpha)_{\alpha \in Ord}$ of U and show, by induction over α, that $U_\alpha \subseteq W$ for all α, which is sufficient since $U = \bigcup_\alpha U_\alpha$.

The base case $\alpha = 0$ is clear since $U_0 = \emptyset$. For the inductive case $\alpha = \beta + 1$, one has $U_\alpha \stackrel{\text{def}}{=} h(U_\beta)$. From $U_\beta \subseteq W$ (the ind. hyp.) we deduce $h(U_\beta) \subseteq h(W)$. From $U_\beta \subseteq U$ and $h(U) = U$, we deduce $h(U_\beta) \subseteq h(U) = U \subseteq V$. Thus $U_\alpha \subseteq V \cap h(W) = W$. Now for a limit U_λ, we obtain $U_\lambda \subseteq W$ from $U_\lambda = \bigcup_{\beta<\lambda} U_\beta$ and the ind. hyp. ☐

We may now prove Lemma A.1. Write $g(X,Y)$ for $X \cap f(X,Y)$ and let $U \stackrel{\text{def}}{=} \nu X.\mu Y.f(X,Y)$ and $V \stackrel{\text{def}}{=} \nu X.\mu Y.g(X,Y)$. From $g(X,Y) \subseteq f(X,Y)$ we derive $V \subseteq U$ by monotonicity.

For the reverse inclusion, let $(V_\alpha)_{\alpha \in Ord}$ be the approximants of V. We claim that they satisfy the following inclusions and equalities:

$$\mu Y.f(V_\alpha, Y) \subseteq V_\alpha\,, \qquad \mu Y.f(V_\alpha, Y) = V_{\alpha+1}\,, \qquad U \subseteq V_\alpha\,, \qquad (\mathrm{P}_\alpha,\ \mathrm{P}'_\alpha,\ \mathrm{P}''_\alpha)$$

Note that (P'_α) entails (P_α) since $V_{\alpha_1} \subseteq V_{\alpha_2}$ when $\alpha_1 \geq \alpha_2$. Reciprocally (P_α) entails (P'_α) since assuming (P_α) and applying Lemma A.2 on $h(Y) \stackrel{\text{def}}{=} f(V_\alpha, Y)$ gives $\mu Y.f(V_\alpha, Y) = \mu Y.V_\alpha \cap f(V_\alpha, Y) = \mu Y.g(V_\alpha, Y)$, which is the definition of $V_{\alpha+1}$. Therefore it is sufficient to prove (P_α) and (P''_α), which we do by induction over α.

For the base case, (P_0) and (P''_0) are clear since $V_0 \stackrel{\text{def}}{=} \mathit{Conf}$.

For the successor case $\alpha = \beta + 1$, we start with $\mu Y.f(V_\alpha, Y) \subseteq \mu Y.f(V_\beta, Y)$ —by monotonicity, since $V_\alpha \subseteq V_\beta$— and combine with the ind. hyp. (P'_β), i.e., $\mu Y.f(V_\beta, Y) = V_\alpha$, to obtain (P_α). For (P''_α), we use the ind. hyp. $U \subseteq V_\beta$ from which we deduce $\mu Y.f(U, Y) \subseteq \mu Y.f(V_\beta, Y)$, i.e., $U \subseteq V_\alpha$, since $U = \mu Y.f(U, Y)$ by definition of U, and $V_\alpha = \mu Y.f(V_\beta, Y)$ is the ind. hyp. (P'_β).

For the limit case $\alpha = \lambda$, one obtains (P''_λ) directly from the ind. hyp. and the definition $V_\lambda = \bigcap_{\beta<\lambda} V_\beta$. For (P_λ), we know $\mu Y.f(V_\lambda, Y) \subseteq \mu Y.f(V_\beta, Y)$ for all $\beta < \lambda$ since $V_\lambda \subseteq V_\beta$. Hence $\mu Y.f(V_\lambda, Y) \subseteq \bigcap_{\beta<\lambda} \mu Y.f(V_\beta, Y) \subseteq \bigcap_{\beta<\lambda} V_\beta$ (by ind. hyp.) $= V_\lambda$.

Finally, since (P''_α) holds for all α and since $V = \bigcap_\alpha V_\alpha$, we deduce $U \subseteq V$.