When Locality Meets Preservation

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ABSTRACT

This paper investigates the expressiveness of a fragment of first-order sentences in Gaifman normal form, namely the positive Boolean combinations of basic local sentences. We show that they match exactly the first-order sentences preserved under local elementary embeddings, thus providing a new general preservation theorem and extending the Łós-Tarski Theorem.

This full preservation result fails as usual in the finite, and we show furthermore that the naturally related decision problems are undecidable. In the more restricted case of preservation under extensions, it nevertheless yields new well-behaved classes of finite structures: we show that preservation under extensions holds if and only if it holds locally.

CCS CONCEPTS

• **Theory of computation** → **Finite Model Theory**; • **Logic and verification**; • **Mathematics of computing** → **Discrete mathematics**;

KEYWORDS

Undecidability, Preservation theorem, Well quasi ordering, Tree depth, Locality, Gaifman normal form, Finite Model Theory.

1 INTRODUCTION

Preservation theorems. In classical model theory, preservation theorems characterise first-order definable sets enjoying some semantic property as those definable in a suitable syntactic fragment [e.g., 3, Section 5.2]. A well-known instance of a preservation theorem is the Łós-Tarski Theorem [15, 23]: a first-order sentence φ is preserved under extensions over all structures—i.e., $A \models \varphi$ and A is an induced substructure of B imply $B \models \varphi$—if and only if it is equivalent to an existential sentence. Similarly, the Lyndon Positivity Theorem applied to a unary predicate X [16] connects surjective homomorphisms that are *strong* in every predicate except X to sentences that are positive in X. These two preservation theorems can be seen as bridges between syntactic and semantic fragments in Figure 1.

As most of classical model theory, preservation theorems typically rely on compactness, which is known to fail in the finite case. Consequently, preservation theorems generally do not relativise to classes of structures, and in particular to the class $\text{Fin}(\sigma)$ of all finite structures [see the discussions in 21,Section 2 and 13,Section 3.4]. For example, Lyndon's Positivity Theorem fails even over the class of finite words [13]. Nevertheless, there are a few known instances of classes of finite structures where some preservation theorems hold [2, 12, 20, 21], and this type of question is still actively investigated [e.g. 4, 7, 13].

We say that a class C satisfies *preservation under extensions* whenever first order sentences are preserved under extensions (over C) if and only if they are equivalent (over C) to an existential sentence. In the particular case of the class $C = \text{Struct}(\sigma)$ of all structures, this is the Łós-Tarski Theorem. When investigating preservation under extensions that hinges on the interplay between the properties of the ordering and those of rist-order logic, one will typically focus on either of these two aspects.

Focusing on the ordering, one can ask for the class C of considered structures to be *well-quasi-ordered* with respect to the induced substructure ordering (hereafter written $\subseteq_i$). In order-theoretic terms, the set $[\![\varphi]\!]_C$ of models of a sentence φ in C is *upwards-closed* whenever φ is preserved under extensions. The assumption that C is well-quasi-ordered implies that $[\![\varphi]\!]_C$ has finitely many minimal elements for $\subseteq_i$ and this is well known to imply preservation under extensions. Practical instances where $\subseteq_i$ gives rise to a well-quasi-order are scarce; it is the case for graphs of bounded tree-depth [8], but to our knowledge the characterisation by Daligault, Rao, and Thomassé [5] of which classes of bounded clique-width are well-quasi-ordered yields the broadest known such class (see the left column in Figure 3).

Focusing on first-order logic, one can leverage finite model theory results over classes of structures provided they are somewhat locally well-behaved; this can be ensured for instance by some flavour of *sparsity*. Two instances of this are classes of bounded degree and the class of all graphs of treewidth less than k [1]. This is the direction we take in this paper.

Locality. Given a structure A over a finite relational signature σ, its *Gaifman graph* has the elements of A as vertices and an edge (a, b) whenever both a and b are in relation in A. The distance $d_A(a, b)$ between two elements of A is their distance in the Gaifman graph of A. For a tuple $\mathbf{a} \in A$, and a radius $r \in \mathbb{N}$ one can consider the r-neighborhood around $\mathbf{a}$ in A defined as $\mathcal{N}_A(\mathbf{a}, r) \triangleq \{a' \in A \mid \exists a \in \mathbf{a}, d_A(a, a') \leq r\} = \bigcup_{a \in \mathbf{a}} \mathcal{N}_A(a, r)$; we emphasise that this union is not required to be disjoint. Slightly
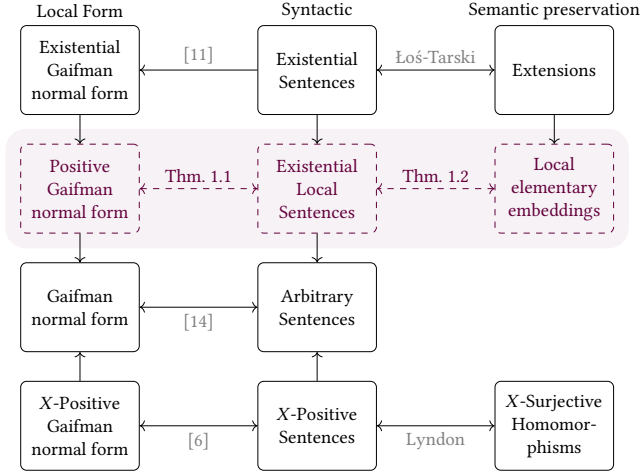
Figure 1: Comparison of the expressiveness of different fragments of FO[σ] **over** *general structures.*

abusing notations, we identify the set $\mathcal{N}_A(\mathbf{a}, r) \subseteq_i A$ with the corresponding induced substructure of A.

A first-order formula $\varphi(\mathbf{x})$ is said to be *r-local* if its evaluation over a structure A and a tuple $\mathbf{a}$ from A only depends on the r-neighborhood of $\mathbf{a}$ in A, i.e. $A, \mathbf{a} \models \varphi$ if and only if $\mathcal{N}_A(\mathbf{a}, r), \mathbf{a} \models \varphi$. In the particular case of $r = 0$, local sentences are equivalent to quantifier free sentences. Since σ is a finite relational signature, every formula φ can be *relativised* to a r-local formula, hereafter denoted by $\varphi_{\leq r}$, which coincides with φ over neighborhoods of size r, although $\varphi_{\leq r}$ might have a higher quantifier rank.

It is more delicate to craft a notion of locality for first order sentences since they have no free variables. The classical approach is to consider *basic local sentences* of the form $\exists \mathbf{x}. \bigwedge_{i \neq j} d(x_i, x_j) > 2r \wedge \bigwedge_i \psi_{\leq r}(x_i)$ where $\psi_{\leq r}$ is a formula relativised to the r neighborhood of its single free variable. Simply put, the evaluation of a local r-basic sentence is determined solely by the evaluation of $\psi_{\leq r}$ over *disjoint* neighborhoods of radius r. Note that the predicates $d(x, y) \leq r$ and $y \in \mathcal{N}(\mathbf{x}, r)$ are definable for each fixed $r \in \mathbb{N}$.

The Gaifman Locality Theorem [10] states that every first-order sentence is equivalent to a Boolean combination of basic local sentences. This can be thought of as FO[σ] being limited to describing the *local behaviour* of structures. In the study of preservation theorems such as the Łoś-Tarski Theorem over finite structures, a first step is often to use Gaifman's normal form and rely on the structural properties of models as a substitute for compactness [1, 2, 7, 12].

In Figure 1 several variants of the Gaifman normal form are listed along with their correspondence to syntactic fragments of first-order logic. For instance, in order to provide a faster model checking algorithm, Grohe and Wöhrle crafted an *existential Gaifman normal form* as follows: every existential first order sentence is equivalent to a *positive* Boolean combination of *existential* basic local sentences [11], where an existential basic local sentence is of the form $\exists \mathbf{x}. \bigwedge_{i \neq j} d(x_i, x_j) > 2r \wedge \bigwedge_i \psi_{\leq r}(x_i)$ and $\psi_{\leq r}$ is an *existential* formula relativised to the r-neighborhood of its one free variable. Note that this not an equivalence: $\exists x. \exists y. d(x, y) > 2$ is in existential Gaifman normal form but not preserved under extensions

nor equivalent to an existential sentence. Another variant of the Gaifman normal form, tailored to the study of Lyndon's Positivity Theorem over one unary predicate X is defined by Dawar, Grohe, Kreutzer, and Schweikardt [6, Theorem 2].It allows the authors to provide approximation schemes for evaluating sentences positive in one unary predicate.

1.1 Contributions

Our first main contribution is a new line of equivalent characterisations through local normal forms, syntactic restrictions, and preservation under some ordering, as embodied in Figure 1. This generalises the correspondences between the existential Gaifman normal forms defined by Grohe and Wöhrle, existential sentences, and preservation under extensions.

1.1.1 Existential Local Sentences, Positive Gaifman Normal Forms. This paper studies a positive variant of locality through the prism of existential closures of r-local formulas, abbreviated here as *existential local sentences*. Those are of the form $\exists \mathbf{x}. \tau(\mathbf{x})$ where τ is an r-local formula.

As opposed to basic local sentences, *existential local sentences* allow interaction between the existentially quantified variables, which increases their expressiveness. Existential local sentences also generalise existential sentences, as quantifier free formulas are 0-local (conversely, 0-local formulas can be rewritten as quantifier free formulas). Thus, by allowing formulas with a non-zero locality radius, we provide a middle ground between existential sentences and arbitrary sentences.

We prove that existential local sentences and *positive* Boolean combinations of basic local sentences are equally expressive, regardless of the class of structures considered. This theorem, proven in Section 3, is related to the existential Gaifman normal form of Grohe and Wöhrle [11] and relies, in part, on similar combinatorial arguments.

Theorem 1.1 (Positive Locality). *Let $C \subseteq \text{Struct}(\sigma)$ be a class of structures and $\varphi \in \text{FO}[\sigma]$ be a first-order sentence. The sentence φ is equivalent over C to an existential local sentence if and only if it is equivalent over C to a positive Boolean combination of basic local sentences.*

1.1.2 A Semantic Characterisation of Existential Local Sentences. Recall that a map $h \colon A \to B$ is an *elementary embedding* whenever for every first order sentence $\varphi(\mathbf{x})$ and every $\mathbf{a} \in A^k$ $A, \mathbf{a} \models \varphi$ if and only if $A, h(\mathbf{a}) \models \varphi$. A localised notion of elementary embeddings is obtained as follows: $h \colon A \to B$ is a *local elementary embedding* when for every $k \geq 1, r \geq 0, \mathbf{a} \in A^k$, and r-local formula φ, $A, \mathbf{a} \models \varphi$ if and only if $B, h(\mathbf{a}) \models \varphi$. This definition is a strengthening of the *induced substructure* ordering, defined by $A \subseteq_i B$ whenever there exists an injective morphism $h \colon A \to B$ such that for all relations $R \in \sigma$ and elements $\mathbf{a} \in A^k$, $A, \mathbf{a} \models R(\mathbf{x})$ if and only if $B, h(\mathbf{a}) \models R(\mathbf{x})$.

We define another natural ordering on structures by writing $A \Rightarrow_q^{r, k} B$ whenever, for all $\varphi = \exists x_1, \ldots x_k. \tau(\mathbf{x})$ where τ is an r-local formula of quantifier rank at most q, $A \models \varphi$ implies $B \models \varphi$. An existential local sentence is naturally preserved under $\Rightarrow_q^{r, k}$ for

some $r, q \geq 0$ and $k \geq 1$. We define the limit of those preorders as

$$\Rrightarrow_{\infty}^{\infty,\infty} \triangleq \bigcap_{r \geq 0} \bigcap_{q \geq 0} \bigcap_{k \geq 1} \Rrightarrow_q^{r,k} . \tag{1}$$

Theorem 1.2 (Local preservation). *Let φ be a sentence in* FO$[\sigma]$. *The following properties are equivalent over the class* Struct(σ) *of all structures.*

(a) *The sentence φ is equivalent to an existential local sentence.*

(b) *There exist $r, q, k \in \mathbb{N}$ such that φ is preserved under $\Rrightarrow_q^{r,k}$.*

(c) *The sentence φ is preserved under $\Rrightarrow_{\infty}^{\infty,\infty}$.*

(d) *The sentence φ is preserved under local elementary embeddings.*

1.1.3 Non-relativisation in the Finite. The proof of Theorem 1.2 does not relativise to classes of finite structures, except for the equivalence $(a) \Leftrightarrow (b)$. Our second main contribution in Section 5 is to show that Theorem 1.2 fails over Fin(σ) the class of finite structures, and to characterise for which parameters (r, k, q) preservation under $\Rrightarrow_q^{r,k}$ leads to an existential local form over Fin(σ).

The picture is even bleaker when applying the methodology of Chen and Flum [4] for the Łoś-Tarski Theorem or Kuperberg [13] for Lyndon's Positivity Theorem: we show that most decision problems ensuing this failure are undecidable. Namely, we show in Section 5.2 that it is not possible to decide whether a sentence is preserved under local elementary embedding in the finite setting, nor is it possible to decide whether a sentence preserved under elementary embedding is equivalent to an existential local one, and even under the promise that the sentence is equivalent to an existential local one such an equivalent sentence is not computable.

1.1.4 Application to Preservation Under Extensions. We leverage our understanding of existential local sentences to split the proof of preservation under extensions over a class C in two distinct steps: (⊶) C is *localisable*, i.e. sentences preserved under extensions over C are equivalent to existential local sentences C, (✳) C satisfies *existential local preservation under extensions*, i.e. existential local sentences preserved under extensions over C are equivalent to existential sentences over C. A class C that satisfies both (⊶) and (✳) satisfies preservation under extensions, hence this proof scheme is correct. Moreover, existential sentences are existential local therefore this proof scheme is complete. Section 6 is devoted to exploring the new classes of structures where preservation under extensions holds that are gained through this proof scheme. This is done by providing a finer understanding of the interaction between locality and combinatorial arguments in preservation under extensions.

(⊶) Let us say that a class $C \subseteq$ Fin(σ) is closed under disjoint unions when the disjoint union of one structure from C and a finite structure from Fin(σ) remains in C. When the class C is stable under induced substructures, we say that C is *hereditary*. We prove in Section 6.1 that hereditary classes of finite structures closed under disjoint unions are *localisable* (Theorem 6.3). This is the diagonal edge labelled (⊶) in Figure 2, that this bypasses the non-relativisation of Theorem 1.2 in the finite. As a consequence, the class Fin(σ) of finite structures satisfies (⊶). Moreover, we prove in Section 2 that closure under local elementary embeddings coincides with closure under disjoint unions in the case of finite structures. This highlights the crucial use of local elementary

embeddings in the literature [1, 2, 20], in the guise of disjoint unions.

(✳) We prove in Section 6.2 that a hereditary class C satisfies (✳) if and only if its "local neighbourhoods" Balls(C, r, k) satisfy preservation under extensions for $r, k \geq 0$ (Lemma 6.6). We construct the local neighbourhoods of a class C by collecting the neighbourhoods around k points in structures of C as follows:

$$\text{Balls}(C, r, k) \triangleq \left\{ \mathcal{N}_A(\mathbf{a}, r) \mid A \in C \wedge \mathbf{a} \in A^{\leq k} \right\} . \tag{2}$$

This allows to bridge the gap between existential sentences and existential local sentences preserved under extensions in Figure 2 via the edge labelled (✳).

The remaining of Section 6.3 is devoted to proving that the combination of (⊶) Theorem 6.3 and (✳) Lemma 6.6 strictly generalise previously known properties that imply preservation under extensions. To that end, we study classes C that are hereditary, closed under disjoint unions, and "locally well-behaved", i.e. such that Balls(C, r, k) is "well-behaved" for all $r, k \geq 0$. Instances of "well-behaved" are finite classes, classes of bounded tree-depth, or more generally classes that are well-quasi-ordered with respect to $\subseteq_i$, as depicted in the left column of Figure 3. By localising these properties in our Theorem 6.9, we obtain the right column of Figure 3, which still implies preservation under extensions, but strictly improves previously known results, with the exception of "locally finite classes" that coincides with the "wide classes" of Atserias et al. [1, Theorem 4.3]. This validates our proof scheme as we effectively decoupled the locality of first-order logic (⊶) from the combinatorial behaviour considered (✳) in our proofs of preservation under extensions.
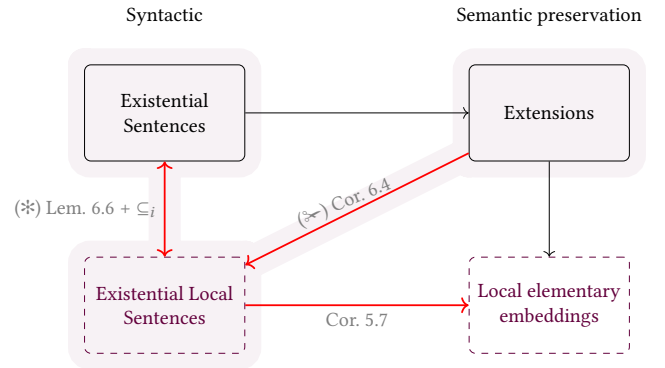
Figure 2: Comparison of the expressiveness of fragments of FO$[\sigma]$ over hereditary classes of finite structures stable under disjoint unions. Single headed arrows represent strict inclusions.

Due to page limitation, some proofs are omitted in the paper and can be found in its full version available on arxiv at `https://arxiv.org/abs/2204.02108`.

2 LOCALITY PREORDERS

As most of this paper is centered around preorders of the form $\Rrightarrow_q^{r,k}$, we start by illustrating them over several examples and by
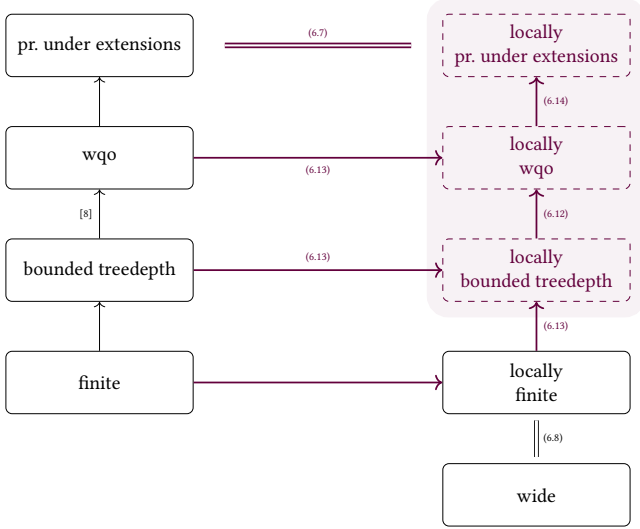
Figure 3: Implications of properties over *hereditary* classes of *finite* structures stable under disjoint unions.

relating them to well-known preorders for specific values of r, q and k.

Given a structure $A \in \text{Struct}(\sigma)$ and a tuple $\mathbf{a}$ of elements from A, let us write $\text{tp}_A^q(\mathbf{a}, r)$ for the (q, r)-local type of $\mathbf{a}$, i.e., the set of all formulas of quantifier rank at most q with $|\mathbf{a}|$ free variables that are r-local such that $(A, \mathbf{a}) \models \varphi(\mathbf{x})$. Note that there are only finitely many possible local types for a given pair $(q, r) \in \mathbb{N}^2$ and a given number of variables $|\mathbf{a}|$. Those local types can be collected to fully describe the local behaviour of A in

$$\text{Types}_r^{q,k}(A) \triangleq \left\{ \text{tp}_A^q(\mathbf{a}, r) \mid \mathbf{a} \in A^{\leq k} \right\} . \tag{3}$$

We shall often use this collection of types to reason with our preorders.

Fact 2.1 (Type collection). *Let $r, q, k \in \mathbb{N}$ with $k \geq 1$. For all $A, B \in$* Struct(σ), $A \Rightarrow_q^{r,k} B$ *if and only if* $\text{Types}_r^{q,k}(A) \subseteq \text{Types}_r^{q,k}(B)$.

Proof. Assume that $A \Rightarrow_q^{r,k} B$. Let us write T for the finite set (up to logical equivalence) of possible r-local formulas at radius r, quantifier rank q and k variables. Given a vector $\mathbf{a} \in A^k$, define $T_+ \triangleq \text{tp}_A^q(\mathbf{a}, r)$ and $T_- \triangleq T \setminus T_+$. Those are finite collections of r-local formulas of quantifier rank at most q with k free variables $x_1, \ldots x_k$. Let us write $\psi(\mathbf{x}) \triangleq \bigwedge_{t \in T_+} t(\mathbf{x}) \wedge \bigwedge_{t \in T_-} \neg t(\mathbf{x})$, which is r-local and of quantifier rank at most q. The structure A satisfies $\varphi \triangleq \exists \mathbf{x}.\psi(\mathbf{x})$ through the choice of the vector $\mathbf{a}$. Because $A \Rightarrow_q^{r,k} B$, $B \models \varphi$ and this provides a vector $\mathbf{b} \in B^k$ such that $B, \mathbf{b} \models \psi(\mathbf{x})$. In turn, this proves that the sentences of quantifier rank at most q and locality radius r that hold over $\mathbf{b}$ are exactly those in T_+. Finally, $\text{tp}_B^q(\mathbf{b}, r) = \text{tp}_A^q(\mathbf{a}, r)$.

Conversely, assume that $\text{Types}_r^{q,k}(A) \subseteq \text{Types}_r^{q,k}(B)$. Let φ be a sentence of the shape $\exists \mathbf{x}.\tau(\mathbf{x})$, where $|\mathbf{x}| = k$ and $\tau(\mathbf{x})$ is an r-local formula of quantifier rank at most q, that is true in A. There exists a vector $\mathbf{a} \in A^k$ such that $A, \mathbf{a} \models \tau(\mathbf{x})$, in particular $\tau(\mathbf{x}) \in \text{tp}_A^q(\mathbf{a}, r)$.

Since $\text{Types}_r^{q,k}(A) \subseteq \text{Types}_r^{q,k}(B)$, there exists $\mathbf{b} \in B^k$ such that $\text{tp}_A^q(\mathbf{a}, r) = \text{tp}_B^q(\mathbf{b}, r)$. Thus, $B, \mathbf{b} \models \tau(\mathbf{x})$ and therefore $B \models \varphi$. □

Fact 2.2 (Refinement). *If $(q, r, k) \leq (q', r', k')$ component-wise, then* $\Rightarrow_{q'}^{r',k'} \subseteq \Rightarrow_q^{r,k}$.

Example 2.3 (Inequalities between two structures). We consider undirected graphs as structures in Struct(σ) where $\sigma \triangleq \{(E, 2)\}$. Let P_n be a finite path of size n and C_m be a finite cycle of size m.

We can prove that $C_m \Rightarrow_q^{r,k} P_n$ whenever $n \geq m > k(2r + 1)$. Indeed, consider k points of C_m and their balls of radius r. As $m > k(2r + 1)$, the unions of these balls exclude at least one point of C_m, and are thus a finite union of paths. As $n \geq m$, it is possible to find points in P_n behaving similarly at radius r.

However, as soon as k, r and $q \geq 1$, P_n is not below C_m for $\Rightarrow_q^{r,k}$. Indeed, it suffices to select an endpoint of the path P_n, and to assert that it is of degree one using a sentence of quantifier rank 1 evaluated at radius 1. One cannot find a similar point in C_m as all nodes are of degree two.

The connection between disjoint unions, local elementary embeddings and induced substructures is provided in the following lemmas and examples that are detailed in the full paper.

Lemma 2.4. *lemma Let A, B be two structures in* Struct(σ). *If $h: A \to B$ is a local elementary embedding then $A \Rightarrow_\infty^{\infty,\infty} B$.*

Proof. Fix $q, r, k \in \mathbb{N}$, and consider a tuple $\mathbf{a} \in A^k$. By construction, the tuple $h(\mathbf{a}) \in B^k$ satisfies the same local FO formulas, and in particular, $\text{tp}_A^q(\mathbf{a}, r) = \text{tp}_B^q(h(\mathbf{a}), r)$, thus $A \Rightarrow_q^{r,k} B$. □

Lemma 2.5 (Preorders in the finite). *Let A, B be two finite structures in* Fin(σ). *The following statements are equivalent.*

(1) *There exists C such that $A \uplus C = B$.*
(2) $A \Rightarrow_\infty^{\infty,\infty} B$.
(3) $A \Rightarrow_q^{r,\infty} B$ *for some $r, q \geq 1$.*
(4) *There exists a local elementary embedding from A to B.*

Proof of (1) $\Rightarrow$ (2). Given a vector $\mathbf{a} \in A^k$, a radius r, and quantifier rank q, one notices that $\mathcal{N}_A(\mathbf{a}, r) = \mathcal{N}_{A \uplus C}(\mathbf{a}, r)$, hence $\text{tp}_A^q(\mathbf{a}, r) = \text{tp}_B^q(\mathbf{a}, r)$. We have proven that $A \Rightarrow_\infty^{\infty,\infty} B$. □

Proof of (2) $\Rightarrow$ (3). By definition. □

Proof of (3) $\Rightarrow$ (4). Let us consider $\mathbf{a} \in A^{|A|}$ a vector containing all the points of A exactly once. There exists a vector $\mathbf{b} \in B^{|A|}$ such that $\text{tp}_A^1(\mathbf{a}, 1) = \text{tp}_B^1(\mathbf{b}, 1)$. As a consequence, $\mathcal{N}_B(\mathbf{b}, 1) = \mathcal{N}_B(\mathbf{b}, 0)$ since this equation holds for $\mathbf{a}$ and is expressible using one universal quantifier. The mapping $h: \mathbf{a} \mapsto \mathbf{b}$ is a local elementary embedding. Indeed, r-neighborhoods around $\mathbf{a}$ (resp. $\mathbf{b}$) are subsets of $\mathbf{a}$ (resp. $\mathbf{b}$). This proves that r-local formulas around $\mathbf{a}$ (resp. $\mathbf{b}$) can be rewritten as 0-local, and we conclude using the equality of their $(1, 1)$-local types. □

Proof of (4) $\Rightarrow$ (1). Let $h: A \to B$ be a local elementary embedding. Let D be the substructure induced by $h(A)$ in B; since h is a local elementary embedding, A and D are isomorphic. Let us consider a point $c \in B \setminus D$. Assume by contradiction that there exists a relation containing both c and an element of $d \in D$. Let $\varphi(\mathbf{x})$ be a 1-local formula of quantifier rank 1 with $|A|$ free variables stating that

there exists a point not in $\mathbf{x}$ connected to some element of $\mathbf{x}$. Since h is a local elementary embedding and $d \in h(A), B, h(A) \models \varphi(\mathbf{x})$ and $A, A \models \varphi(\mathbf{x})$. This is absurd, hence $B = D \uplus (B \setminus D) = A \uplus (B \setminus D)$. $\quad\square$

However, the existence of a local elementary embedding is in general not equivalent to $\Rrightarrow_{\infty}^{\infty,\infty}$, as shown next.

Example 2.6 (Preorder difference). Over the signature of graphs, let G be an infinite grid, and $G' \triangleq G \uplus G$. There exists no local elementary embedding from G' to G but $G' \Rrightarrow_{\infty}^{\infty,\infty} G$.

Over finite structures, when the parameter values of r, q or k are too small, one ends up with a preorder that is trivial, except for one specific combination where we obtain the *extension* preorder $\subseteq_i$.

Fact 2.7 (Trivial orders). Over finite structures, the following preorders are trivial, i.e. every pair of structures is related: $\Rrightarrow_q^{r,0}$ whenever $r, q \in \mathbb{N} \cup \{\infty\}$, and $\Rrightarrow_0^{\infty,1}$.

Lemma 2.8 (Extension preorder). *Over finite structures and for all* $q, r \in \mathbb{N} \cup \{\infty\}, \Rrightarrow_q^{0,\infty} = \Rrightarrow_0^{r,\infty} = \subseteq_i$.

3 POSITIVE GAIFMAN NORMAL FORM

The aim of this section is to provide a connection between the positive variant of Gaifman normal forms and existential local sentences. The theorem and its proof are heavily inspired by the combinatorics behind Grohe and Wöhrle's proof of the existential Gaifman normal form. In particular, the main issue arises from finding points with disjoint neighborhoods.

As basic local sentences are existential local, the only difficulty in Theorem 1.1 is converting an existential local sentence into a positive Boolean combination of basic local sentences. We split this transformation into intermediate syntactic steps

Existential local $\exists \mathbf{x}.\psi(\mathbf{x})$ where ψ is an r-local formula.
Almost basic local $\exists \mathbf{x}. \bigwedge_{i \neq j} d(x_i, x_j) > 2r \wedge \psi(\mathbf{x})$ where ψ is an r-local formula.
Asymmetric basic local $\exists \mathbf{x}. \bigwedge_{i \neq j} d(x_i, x_j) > 2r \wedge \bigwedge_i \psi_i(x_i)$ where $(\psi_i)_i$ is a family of r-local formulas with exactly one free variable.
Basic local $\exists \mathbf{x}. \bigwedge_{i \neq j} d(x_i, x_j) > 2r \wedge \bigwedge_i \psi(x_i)$ where ψ is an r-local formula with exactly one free variable.

Asymmetric basic local sentences already appear as an intermediate steps towards basic local sentences in the constructions of Grohe and Wöhrle [11] and Dawar et al. [6]. Most of the transformations will rely on the description of the 'spatial' repartition of elements in a given structure A. We handle this description through the following lemma

proven in the full paper.

Lemma 3.1. *For every* $k, r \geq 0$, *for every structure* $A \in \text{Struct}(\sigma)$ *and vector* $\mathbf{a} \in A^{\leq k}$ *there exists a vector* $\mathbf{b} \in A^{\leq k}$ *and a radius* $r \leq R \leq 4^k r$ *such that* $\mathcal{N}_A(\mathbf{a}, r) \subseteq \mathcal{N}_A(\mathbf{b}, R)$ *and* $\forall b \neq b' \in \mathbf{b}, \mathcal{N}_A(b, 3R) \cap \mathcal{N}_A(b', 3R) = \emptyset$.

3.1 From Existential Local Sentences to Asymmetric Basic Local Sentences

Using Lemma 3.1 it is already possible to transform an existential local sentence into a positive Boolean combination of *almost* basic local sentences.

Lemma 3.2 (From Existential local to Almost basic local). *Let* $\varphi(\mathbf{x})$ *be an* r-local formula. There exist $1 \leq n \leq 4^{|\mathbf{x}|}r$ and $\psi_1, \ldots, \psi_n$ *almost basic local sentences such that* $\exists \mathbf{x}.\varphi(\mathbf{x})$ *is equivalent to the disjunction* $\bigvee_{1 \leq i \leq k} \psi_i$ *over* $\text{Struct}(\sigma)$.

PROOF. Let us define $\Delta \triangleq \left\{ (k, R) \mid 0 \leq k \leq |\mathbf{x}| \wedge r \leq R \leq 4^{|\mathbf{x}|}r \right\}$ and

$$\begin{aligned} \psi_{(k,R)} \quad &\triangleq \quad \exists b_1, \ldots, b_k. \\ &\bigwedge_{1 \leq i < j \leq k} d(b_i, b_j) > 2R \\ &\wedge \exists \mathbf{x} \in \mathcal{N}(\mathbf{b}, R).\varphi(\mathbf{x}) \end{aligned} \tag{4}$$

To conclude it suffices to prove that $\exists \mathbf{x}.\varphi(\mathbf{x})$ is equivalent to $\bigvee_{(k,R) \in \Delta} \psi_{(k,R)} \triangleq \Psi$.

- Assume that $A \in \text{Struct}(\sigma)$ satisfies $\exists \mathbf{x}.\varphi(\mathbf{x})$. Then there exists $\mathbf{a} \in A^{|\mathbf{x}|}$ such that $\mathcal{N}_A(\mathbf{a}, r) \models \exists \mathbf{x}.\varphi(\mathbf{x})$ since φ is r-local around $\mathbf{x}$. Using Lemma 3.1, there exists a size $0 \leq k \leq |\mathbf{a}|$ a radius $r \leq R \leq 4^{|\mathbf{a}|}r$, and a vector $\mathbf{b} \in A^k$ such that $\mathcal{N}_A(\mathbf{a}, r) \subseteq \mathcal{N}_A(\mathbf{b}, R)$ and the balls of radius $3R$ around the points of $\mathbf{b}$ do not intersect.
  In particular, $\uplus_{b \in \mathbf{b}} \mathcal{N}_A(b, R) \models \exists \mathbf{x}.\varphi(\mathbf{x})$ since witnesses in $\mathcal{N}_A(\mathbf{a}, r)$ can still be found and φ is r-local. This proves that $A \models \psi_{(k,R)}$ hence that $A \models \Psi$.
- Assume that $A \in \text{Struct}(\sigma)$ satisfies Ψ. Then there exists (k, R) such that $A \models \psi_{k,R}$ thus proving that there exists $\mathbf{b} \in A^k$ such that $\uplus_{b \in \mathbf{b}} \mathcal{N}_A(b, R) \models \exists \mathbf{x}.\varphi(\mathbf{x})$. Since $r \leq R$ and φ is r-local this proves $A \models \exists \mathbf{x}.\varphi(\mathbf{x})$. $\quad\square$

An application of the Feferman-Vaught technique [9, 17] allows to transform *almost* basic local sentences into the *asymmetric* ones introduced by Grohe and Wöhrle [11], i.e. to sentences of the form $\exists \mathbf{x}. \bigwedge_{i \neq j} d(x_i, x_j) > 2r \wedge \bigwedge_i \psi_i(x_i)$ where each ψ_i is a r-local formula when i ranges from 1 to $|\mathbf{x}|$. The combination of this transformation

(detailed in the full paper)

with Lemma 3.2 generalises a similar statement over existential sentences [11, Theorem 6] to existential local sentences.

Lemma 3.3 (From Almost basic local to Asymmetric Basic local). *Every almost basic local sentence is equivalent to a disjunction of asymmetric basic local sentences.*

3.2 From Asymmetric Basic Local to Basic Local Sentences

We are now ready to build the final transformation between asymmetric basic local sentences and basic local sentences, reusing some of the combinatorics of Grohe and Wöhrle [11, Lemma 4].

As a convenience, let us write ϕ/i for the sentence φ where the variable x_i and local sentence ψ_i are 'removed.' For instance, if $\phi \triangleq \exists x_1, x_2.d(x_1, x_2) > 2r \wedge \psi_1(x_1) \wedge \psi_2(x_2)$ then $\phi/1 = \exists x_2.\psi_2(x_2)$ and $\phi/2 = \exists x_1.\psi_1(x_1)$.

Fact 3.4 (Removing variable weakens). If φ is an asymmetric basic local sentence of the form $\exists x_1, \ldots, x_k. \bigwedge_{i \neq j} d(x_i, x_j) > 2r \wedge \bigwedge_i \psi_i(x_i)$, $A \models \varphi$ and $1 \leq i \leq k$, then $A \models \varphi/i$.

The following lemma allows us to reduce the number of variables in an asymmetric basic local sentence under the assumption that some witness is *frequent*. To simplify notations, let us

write $\exists_r^{\geq n} x.\theta(x)$ as a shorthand for $\exists x_1, \ldots, x_n. \bigwedge_{i \neq j} d(x_i, x_j) > 2r \wedge \bigwedge_i \theta(x_i)$. When $\theta(x)$ is a r-local sentence, $\exists_r^{\geq n} x.\theta(x)$ is a basic local sentence.

Lemma 3.5 (Repetitions)**.** *If φ is an asymmetric basic local sentence of the form $\exists x_1, \ldots, x_k. \bigwedge_{i \neq j} d(x_i, x_j) > 2r \wedge \bigwedge_i \psi_i(x_i)$ and $A \models \varphi/i \wedge \exists_{2r}^{\geq k} x.\psi_i(x)$, then $A \models \varphi$.*

PROOF. Let A be a structure such that $A \models \varphi/i \wedge \exists_{2r}^{\geq k} x.\psi_i(x)$. By definition of φ/i there exists a vector $\mathbf{a} \in A^{k-1}$ of points at pairwise distance greater than $2r$ such that $A, a_j \models \psi_j(x)$ for $1 \leq j \neq i \leq k$. To prove that $A \models \varphi$, it suffices to find some witness a_i for ψ_i that is at distance greater than $2r$ of $\mathbf{a}$.

The fact that $A \models \exists_{2r}^{\geq k} x.\psi_i(x)$ guarantees that we can find k witnesses for ψ_i at pairwise distance greater than $4r$, let us write $\mathbf{b}$ this set of witnesses.

Assume by contradiction that $\forall b \in \mathbf{b}, \exists a \in \mathbf{a}. d(a, b) \leq 2r$. Since $|\mathbf{a}| = k - 1$ and $|\mathbf{b}| = k$, there exists a point $a \in \mathbf{a}$ such that two elements b_1 and b_2 of $\mathbf{b}$ are at distance less than $2r$ of a. The triangular inequality implies that $d(b_1, b_2) \leq 4r$ which is absurd.

We have proven that there exists a $b \in \mathbf{b}$ such that b is at distance greater than $2r$ of all elements in $\mathbf{a}$, therefore $A \models \varphi$. □

Let us temporarily fix a structure A. To transform an asymmetric basic local sentence into a positive Boolean combination of basic local sentences, one can proceed by induction on the number of outer existential quantifications. Whenever Fact 3.4 can be applied, it suffices to use the induction hypothesis. As a consequence, the base case of our induction is when Fact 3.4 cannot be applied at all. For such an asymmetric basic local sentence $\varphi \triangleq \exists x_1, \ldots, x_k. \bigwedge_{1 \leq i \neq j \leq k} d(x_i, x_j) > 2r \wedge \bigwedge_{1 \leq i \leq k} \psi_i(x_i)$, the set W of elements in A that satisfy at least one ψ_i enjoys some sparsity property. Namely, it is not possible to find more than $k(k-1)$ points in W whose neighborhoods of radius $2r$ do not intersect.

To handle this base case, we enumerate the possible behaviours of the set W through *template graphs*. Given a structure A, a non-empty finite set Q of r-local properties, a radius R and a vector $\mathbf{a}$ of elements in A such that every $a \in \mathbf{a}$ satisfies at least one property $p \in Q$, we build the template graph $G_{\mathbf{a}}^R$ as follows: its vertices are the elements of $\mathbf{a}$ and it has a labelled edge $(u, v, d_A(u, v))$ whenever $d_A(u, v) \leq R$. Moreover, a node $a \in V(G)$ is coloured by $p \in Q$ whenever $A, a \models p(x)$. Be careful that a vertex can have multiple colours; the set of colours of a vertex v is written $C(v)$.

Given a maximal size K, a radius R and a non-empty finite set Q of r-local properties, we define $\Delta(K, R, Q)$ to be the set of template graphs with at most K vertices, colours in Q and edges labelled with integers at most R. Graphs in $\Delta(K, Q, R)$ are ordered using $G \leq G'$ whenever there exists an isomorphism $h: G \to G'$ between the underlying graphs respecting edge labels such that $C(v) \subseteq C(h(v))$ for $v \in V(G)$. In a structure A, a graph G is *represented* by a vector $\mathbf{a}$ whenever $G_{\mathbf{a}}^R \geq G$.

From a graph $G \in \Delta(K, R, Q)$ where Q is a non-empty finite set of r-local properties, one can build the $(R + r)$-local formula $\theta_G^R(x)$ that finds a representative of G as follows

$$\theta_G^R(x) \triangleq \exists v_1, \ldots, v_{|V(G)|} \in \mathcal{N}(x, R).$$
$$\bigwedge_{(v_i, v_j, h) \in E(G)} d(v_i, v_j) = h$$
$$\wedge \bigwedge_{v_i \in V(G)} \bigwedge_{p \in C(i)} p(v_i)$$
$$\wedge \bigwedge_{v_i \in V(G)} \mathcal{N}(v_i, r) \subseteq \mathcal{N}(x, R).$$

Fact 3.6 (Graph representation)**.** Let A be a structure, $r, R \geq 1$, Q be a non-empty finite set of r-local properties, $G \in \Delta(K, R, Q)$ be a template graph, and $a \in A$ be an element of A. Then $A, a \models \theta_G^R(x)$ if and only if $\mathcal{N}_A(a, R)$ contains points $\mathbf{b}$ such that $G_{\mathbf{b}}^R \geq G$ and $\mathcal{N}_A(\mathbf{b}, r) \subseteq \mathcal{N}_A(a, R)$.

Template graphs miss one key compositional property, namely if A is a structure and $\mathbf{a}, \mathbf{b}$ are two vectors, it is not immediate to recover $G_{\mathbf{ab}}^R$ from $G_{\mathbf{a}}^R$ and $G_{\mathbf{b}}^R$. This is dealt with by restricting our attention to vectors that are far enough from one another.

Fact 3.7**.** Let A be a finite structure, Q a non-empty finite set of r-local properties, and $\mathbf{a}, \mathbf{b}$ be vectors of elements of A whose neighborhoods of radius $R \geq r$ do not intersect. Then the template graph $G_{\mathbf{ab}}^R$ is the disjoint union of $G_{\mathbf{a}}^R$ and $G_{\mathbf{b}}^R$.

To ensure that we can use Fact 3.7, we add *security cylinders* around points, as ensured by the following $\max(3R, R + r)$-local formula whose behaviour is detailed in the full paper

$$\pi_Q^R(x) \triangleq \forall y \in \mathcal{N}(x, 3R). \left(\bigvee_{p \in Q} p(u) \right) \implies \mathcal{N}(y, r) \subseteq \mathcal{N}(x, R).$$

Lemma 3.8 (Security Cylinders)**.** *For every radii $R, r \geq 1$, for every non-empty finite set Q of r-local properties, for every graph $G \in \Delta(K, R, Q)$, for every structure A, for every points $a, b \in A$ such that $d(a, b) \leq 2R$, $A, a \models \theta_G^R(x)$ and $A, b \models \pi_Q^R(x)$ implies $A, b \models \theta_G^R(x)$.*

Lemma 3.9 (From Asymmetric basic local to Basic Local)**.** *Every asymmetric basic local sentence is equivalent to a positive Boolean combination of basic local sentences.*

PROOF. Let φ be of the form $\exists x_1, \ldots, x_k. \bigwedge_{i \neq j} d(x_i, x_j) > 2r \wedge \bigwedge_i \psi_i(x_i)$. We prove by induction over k that φ is equivalent to a positive Boolean combination of basic local sentences. When $k \leq 1$, φ is already a basic local sentence hence we assume $k \geq 2$.

For $1 \leq i \leq k$, we apply the induction hypothesis on φ/i, which has strictly fewer existentially quantified variables, and call $\overline{\varphi/i}$ the obtained positive Boolean combination of basic local sentences.

Let $Q \triangleq \{\psi_1, \ldots, \psi_k\}$ and for convenience, let Δ be a shorthand for $\Delta(k, 4^{k^2} 6r, Q)$. We define $\mathbb{M}$ to be the multisets of $\wp(\Delta)$ with up to $k(k-1)$ repetitions per element of $\wp(\Delta)$. Given a multiset $M \in \mathbb{M}$ and a radius R we can write the following conjunction of basic local sentences

$$\Theta_M^R \triangleq \bigwedge_{S \in \wp(\Delta)} \exists_{3R}^{\geq M(S)} x.\pi_Q^R(x) \wedge \bigwedge_{G \in S} \theta_G^R(x) \wedge \bigwedge_{G \notin S} \neg \theta_G^R(x).$$

The set of graphs *obtainable* from a multiset $M \in \mathbb{M}$ is written $\mathrm{Obt}(M)$ and defined inductively by $\mathrm{Obt}(\emptyset) \triangleq \emptyset$ and $\mathrm{Obt}(\{S\} + M)$

defined as collection of graphs G, G' and $G \uplus G'$ for $G \in \text{Obt}(M)$ and $G' \in S$. We call a multiset M *valid* whenever there exists a graph G obtainable from M that contains vertices $v_1, \ldots, v_k$ at pairwise weighted distance greater than $2r$ and such that v_i is coloured by ψ_i. The finite set of valid multisets is written $\mathbb{M}_V$.

Let us prove that φ is equivalent to the positive Boolean combination of basic local sentences Ψ defined as

$$\Psi \triangleq \left(\bigvee_{1 \leq i \leq k} \overline{\varphi/i} \wedge \exists^{\geq k}_{2r} x. \psi_i(x) \right) \vee \left(\bigvee_{M \in \mathbb{M}_V} \bigvee_{6r \leq R \leq 4^{k^2} 6r} \Theta^R_M \right).$$

Assume first that $A \models \varphi$. Using Fact 3.4, $A \models \varphi/i$ for all $1 \leq i \leq k$ and by construction this proves that $A \models \overline{\varphi/i}$ for all $1 \leq i \leq k$. As a consequence, we only need to treat the case where $A \not\models \exists^{\geq k}_{2r} x. \psi_i(x)$ for all $1 \leq i \leq k$.

In such a structure A, let us call W the set of elements in A that satisfy at least one ψ_i. It is not possible to find more than $k(k-1)$ points in W whose neighborhoods of radius $2r$ do not intersect. This implies that there exists a vector $\mathbf{c}$ in A of size less or equal $k(k-1)$ such that $W \subseteq \mathcal{N}_A(\mathbf{c}, 6r)$.

Using Lemma 3.1 over $\mathbf{c}$ and $6r$ one obtains a radius $6r \leq R \leq 4^{k^2} 6r$ and a vector $\mathbf{b}$ such that $\mathcal{N}_A(\mathbf{c}, 6r) \subseteq \mathcal{N}_A(\mathbf{b}, R)$ and the neighborhoods of radius $3R$ around points in $\mathbf{b}$ do not intersect.

Given an element $b \in \mathbf{b}$, construct S_b the collection of the template graphs $G^R_{\mathbf{a}}$ when $\mathbf{a}$ ranges over the sets of k points of $W \cap \mathcal{N}_A(b, R)$. By construction, $S_b \in \wp(\Delta)$. Let us write $M_{\mathbf{b}}$ the multiset obtained by collecting the sets S_b for $b \in \mathbf{b}$.

We now prove that $M_{\mathbf{b}}$ is valid. As $A \models \varphi$, there exists a vector $\mathbf{a} \in A^k$ of points at pairwise distance greater than $2r$ such that $A, a_i \models \psi_i(x)$ hence it suffices to prove that $G^R_{\mathbf{a}} \in \text{Obt}(M_{\mathbf{b}})$ to conclude that $M_{\mathbf{b}}$ is valid.

Remark that $\mathbf{a}$ is included in the disjoint union of the $3R$ neighborhoods of the elements of $\mathbf{b}$. As a consequence of Fact 3.7 $G^R_{\mathbf{a}} = \biguplus_{b \in \mathbf{b}} G^R_{\mathbf{a}/b}$ where $\mathbf{a}/b$ is the vector of elements of $\mathbf{a}$ that are at distance less than $3R$ of b. This proves that $G^R_{\mathbf{a}} \in \text{Obt}(M_{\mathbf{b}})$.

As a consequence, $A \models \Theta^R_{M_{\mathbf{b}}}$ with M valid and $6r \leq R \leq 4^{k^2} 6r$ and therefore $A \models \Psi$.

Assume conversely that $A \models \Psi$. If $A \models \overline{\varphi/i} \wedge \exists^{\geq k}_{2r} x. \psi_i(x)$ for some i, then by definition $A \models \varphi/i \wedge \exists^{\geq k}_{2r} x. \psi_i(x)$ and using Lemma 3.5, $A \models \varphi$.

Otherwise $A \models \Theta^R_M$ where M is a valid multiset. Let us write $M = \{S_1, \ldots, S_n\}$ and each S_i is repeated $m_i \leq k(k-1)$ times. By construction of Θ^R_M, there exist points b^j_i for $1 \leq i \leq n$ and $1 \leq j \leq m_i$ in A such that $A, b^j_i \models \pi^R_Q(x) \wedge \bigwedge_{G \in S_i} \theta^R_G(x) \wedge \bigwedge_{G \notin S_i} \neg \theta^R_G(x)$. Moreover, for each i the points b^j_i are at distance greater than $6R$.

Our goal is to prove that the points b^j_i are at pairwise distance greater than $2R$ when i varies. Assume by contradiction that there exists i, j and i', j' such that $i \neq i'$ and $d\left(b^j_i, b^{j'}_{i'}\right) \leq 2R$. We are going to prove that $S_i = S_{i'}$. Assume by contradiction that there exists $G \in S_i \setminus S_{i'}$. We know that $A, b^j_i \models \theta^R_G(x)$ and $A, b^{j'}_{i'} \models \neg\theta^R_G(x)$. As $A, b^j_i \models \pi^R_Q(x)$, Lemma 3.8 implies that $A, b^{j'}_{i'} \models \theta^R_G(x)$ which is

absurd. Hence, $S_i = S_{i'}$, which is in contradiction with the definition of M, and finally, the points b^j_i must be at distance greater than $2R$.

As the multiset M is valid, there exists a graph $G \in \text{Obt}(M)$ with vertices $v_1, \ldots, v_k$ at pairwise distance greater than $2r$ such that v_i is coloured by ψ_i. By induction on the construction of G, and since the R-neighborhoods around the points b^j_i do not intersect, we can use Fact 3.7 to build a vector $\mathbf{c}$ in A such that $G^R_{\mathbf{c}} \geq G$. As a consequence, there exists k points $a_1, \ldots, a_k$ in $\mathbf{c}$ that are at pairwise distance greater than $2r$ and such that $A, a_i \models \psi_i(x)$. We have proven that $A \models \varphi$. $\square$

Proof of Theorem 1.1. This follows from lemmas 3.2 to 3.3. $\square$

Notice that we do not recover the existential locality of Grohe and Wöhrle using a radius 1 since we introduce negations in the construction of the basic local sentences. Conversely, we did not manage to apply directly their result to obtain Theorem 1.1.

4 A LOCAL PRESERVATION THEOREM

Positive Locality Theorem. In order to prove Theorem 1.2, we use a classical construction of model theory. Given a structure $A \in \text{Struct}(\sigma)$, one can build a new langage $\mathcal{L}_A$ consisting of the relations in σ extended with constants c_a for $a \in A$; the structure A is then canonically interpreted as a structure in $\mathcal{L}_A$ by interpreting c_a with the element a; this structure is written $\hat{A}$. Be careful that this construction makes us temporarily leave the realm of *finite relational signatures*. It is possible, given a fragment F of FO over the extended language to build a theory $T^+(\mathsf{F}, A)$ consisting of all the sentences of F true in $\hat{A}$.

Proof of Theorem 1.2. The implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ are consequences respectively of the definition of $\Rightarrow^{r,k}_q$, Fact 2.2 and Lemma 2.4. As a consequence, let us focus on the implication $(d) \Rightarrow (a)$.

Assume that A is a structure such that $A \models \varphi$. Let us build $\mathcal{L}_A$ the extended language of A and F the set of sentences that are r-local around the constants c_a.

Whenever $\hat{B}$ is a structure over the extended language such that $\hat{B} \models T^+(\mathsf{F}, A)$ we have an interpretation v of the constants c_a in B. Notice that the function $h: A \to B$ defined by $h(a) \triangleq v(c_a)$ is a local elementary embedding from A to B.

Because $A \models \varphi$, the theory $T^+(\mathsf{F}, A) \cup \{\neg\varphi\}$ is inconsistent in the language $\mathcal{L}_A$. Indeed, assume there exists a structure $\hat{B}$ such that $\hat{B} \models \neg\varphi$ and $\hat{B} \models T^+(\mathsf{F}, A)$. Then there exists an elementary embedding $h: A \to B$ and $A \models \varphi$ so $B \models \varphi$. As φ does not contain any constant of the form c_a this proves that $\hat{B} \models \varphi$ which is absurd.

This entails by the compactness theorem the existence of a sentence ψ_A, a finite conjunction of sentences in $T^+(\mathsf{F}, A)$ such that $\psi_A \implies \varphi$. Moreover, since finitely many constants appear in the sentence ψ_A, we can build a sentence θ_A by quantifying existentially over them. In particular, the sentence θ_A is a first-order sentence over the finite relational signature σ. Recall that $\hat{A} \models T^+(\mathsf{F}, A)$ by definition, hence $\hat{A} \models \psi_A$ and in particular $A \models \theta_A$.

Let us build $T_\varphi \triangleq \{\neg\theta_C \mid C \models \varphi\}$. We prove that $T_\varphi \cup \{\varphi\}$ is inconsistent as a model B of this theory is such that $\neg\theta_B \in T_\varphi$ hence $B \models \neg\theta_B$, but this contradicts the fact that $B \models \theta_B$.

A second use of the compactness theorem allows us to extract $C_1, \ldots, C_n$ such that $\varphi \implies \theta_{C_1} \vee \cdots \vee \theta_{C_n}$. Conversely, since $C_i \models \varphi$, whenever $B \models \theta_{C_i}$ the structure B models φ. As a consequence, $\varphi \iff \theta_{C_1} \vee \cdots \vee \theta_{C_n}$. Notice that this is a finite disjunction of finite conjunctions of sentences that are of the form $\exists \mathbf{x}.\tau(\mathbf{x})$ with τ local around $\mathbf{x}$, hence of the appropriate form. □

Adaptation to the Finite Case. As hinted in the introduction, the proof of Theorem 1.2 does not relativise to finite classes of structures due to its intensive use of the compactness theorem of first order logic. We know from Lemma 2.5 that (c) and (d) remain equivalent in the finite, and will prove in Section 5.1 that the equivalence between (a) and (c) fails in the finite. The following lemma salvages the equivalence between (a) and (b) in the finite.

Lemma 4.1 (Local preservation in the finite). *Let $C \subseteq \mathrm{Struct}(\sigma)$ be a class of structures. For a sentence $\varphi \in \mathrm{FO}[\sigma]$ the following properties are equivalent*

(a) *The sentence φ is equivalent (over C) to an existential local sentence.*
(b) *There exist $r, q, k \in \mathbb{N}$ such that φ is preserved under $\Rightarrow_q^{r,k}$ over C.*

PROOF. A sentence of the form $\exists \mathbf{x}.\tau$ with τ a r-local sentence of quantifier rank at most q is always preserved under $\Rightarrow_q^{r,|x|}$, which allows us to conclude $(a) \implies (b)$.

For the converse direction, assume that φ is preserved under $\Rightarrow_q^{r,k}$ for some $r, q, k \in \mathbb{N}$. Compute U as the image of $[\![\varphi]\!]_C$ through the map $\mathrm{Types}_r^{q,k}$. For a set T of local types in U, one can build the sentence $\psi_T \triangleq \bigwedge_{\tau(\mathbf{x}) \in T} \exists \mathbf{x}.\tau(\mathbf{x})$. We define $\psi \triangleq \bigvee_{T \in U} \psi_T$ and prove the following, where the second equality stems from our hypothesis (a)

$$[\![\psi]\!]_C = \left\{ B \in C \mid \exists A \in [\![\varphi]\!]_C, A \Rightarrow_q^{r,k} B \right\} = [\![\varphi]\!]_C . \tag{5}$$

Whenever $B \models \psi$, there exists $T \in U$ such that $B \models \psi_T$. Hence, there exists $A \in X$ such that $A \models \varphi$, and $T = \mathrm{Types}_r^{q,k}(A)$. Thanks to Fact 2.1, $A \Rightarrow_q^{r,k} B$.

Conversely, let $A \in C$ such that $A \models \varphi$ and $B \in C$ such that $A \Rightarrow_q^{r,k} B$. Let us write $T \triangleq \mathrm{Types}_r^{q,k}(A)$ and $T' \triangleq \mathrm{Types}_r^{q,k}(B)$. It is always the case that B satisfies its local types so $B \models \psi_{T'}$ and since $T \subseteq T'$ we can conclude that $B \models \psi_T$ which entails that $B \models \psi$.

We have proven that φ is equivalent to ψ which is a positive Boolean combination of existential local sentences, thus, equivalent to an existential local sentence. □

5 FAILURE IN THE FINITE CASE

The goal of this section is to prove that Theorem 1.2 does not relativise to finite structures, in particular we show in Section 5.1 that $(a) \Leftrightarrow (c)$ in the finite.

This naturally leads to a decision problem: given a sentence φ preserved under disjoint unions over $\mathrm{Fin}(\sigma)$, can it be rewritten as an existential local sentence? We prove in Section 5.2 that this problem, and two other associated decision problems, are undecidable.

We complete the picture in Section 5.3 by describing the parameters (r, q, k) such that first-order sentences preserved under $\Rightarrow_q^{r,k}$ over $\mathrm{Fin}(\sigma)$ are equivalent to an existential local sentence.

5.1 A Generic Counter Example

In the following, we will add whenever necessary a unary predicate B to the signatures in order to construct the following property: $\varphi_B \triangleq \forall x.\neg B(x) \vee \varphi_{CC}$, where φ_{CC} checks that the Gaifman graph of the structure has at least two connected components. Notice that φ_B may not be definable in $\mathrm{FO}[\sigma]$ since φ_{CC} may not be definable. Over a class C of finite structures where φ_{CC} is definable as an existential local sentence, the sentence φ_B is well-defined and preserved under disjoint unions. Whenever the class C contains large enough structures, φ_B cannot be expressed using an existential local sentence. Indeed, such an existential local sentence will not distinguish between a large connected component with one B node (not satisfying φ_B) and two connected components with one B node (satisfying φ_B). A prototypical example is the class of finite paths.

Example 5.1 (Finite paths). If $\sigma = \{(E, 2)\}$ and $\uplus P$ is the class of disjoint unions of finite paths, φ_B is definable, closed under disjoint unions, and is not expressible as an existential local sentence.

PROOF. One can detect the presence of two connected components using the fact that paths have at most two vertices of degree below 2 through φ_{CC} defined as the disjunction of the following sentences: $\exists x_1, x_2, x_3, x_4. \bigwedge_{1 \leq i \neq j \leq 4} x_i \neq x_j \wedge \bigwedge_{i=1}^4 \deg(x_i) = 1)$, $\exists x_1, x_2.x_1 \neq x_2 \wedge \deg(x_1) = \deg(x_2) = 0)$, and $\exists x_1, x_2, x_3.x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_3 \neq x_1 \wedge \deg(x_1) = \deg(x_2) = 1 \wedge \deg(x_3) = 0$.

Assume by contradiction that there exists a sentence $\psi = \exists \mathbf{x}.\theta(x)$ with $\theta(x)$ a r-local sentence of quantifier rank q such that φ_B is equivalent to ψ over $\uplus P$. Consider the family of structures $P_k^{\neg B}$ that are paths of length k labelled by $\neg B$ so that $P_k^{\neg B} \models \varphi_B$. Consider $k > |\mathbf{x}| \cdot (2r + 1)$. Since $P_k^{\neg B} \models \psi$, there exists a vector $\mathbf{a}$ with $|\mathbf{x}|$ elements of $P_k^{\neg B}$ such that $\mathcal{N}_{P_k^{\neg B}}(\mathbf{a}, r) \models \theta(x)$ and $\mathcal{N}_{P_k^{\neg B}}(\mathbf{a}, r) \subsetneq P_k^{\neg B}$. Consider a point $b \in P_k^{\neg B}$ that is not in $\mathcal{N}_{P_k^{\neg B}}(\mathbf{a}, r)$, and build P_k^B by colouring this point with B. The structure P_k^B is still a path but does not satisfy φ_B. However, $P_k^B \models \psi$ by construction, and this is absurd. □

Assume that C_{ord} is a class of finite structures defined over $\mathrm{Fin}(\sigma)$ through a finite axiomatisation $\mathcal{A}_{\mathrm{ord}}$ using *universal local sentences* (negations of existential local sentences). The following fact allows lifting arguments over C_{ord} to $\mathrm{Fin}(\sigma)$.

Fact 5.2 (Relativisation to C_{ord}). A sentence φ is equivalent over C_{ord} to an existential local sentence if and only if $\mathcal{A}_{\mathrm{ord}} \implies \varphi$ is equivalent to an existential local sentence over $\mathrm{Fin}(\sigma)$. Similarly, a sentence φ is preserved under disjoint union over C_{ord} if and only if $\mathcal{A}_{\mathrm{ord}} \implies \varphi$ is preserved under disjoint union over $\mathrm{Fin}(\sigma)$.

Note that the class $\uplus P$ of finite paths has no finite axiomatisation in $\mathrm{Fin}(\sigma)$. Thus, Fact 5.2 cannot be used to lift Example 5.1 to the class of all finite structures. As a workaround, we refine the counter example provided by Tait [22] in the case of the Łoś-Tarski Theorem and leverage the idea given by the class of finite paths. For this purpose, we let $\sigma \triangleq \{(\leq, 2), (S, 2), (E, 2)\}$. Define O_n to be the structure $\{1, \ldots, n\}$ with S interpreted as the successor relation, $\leq$ as the
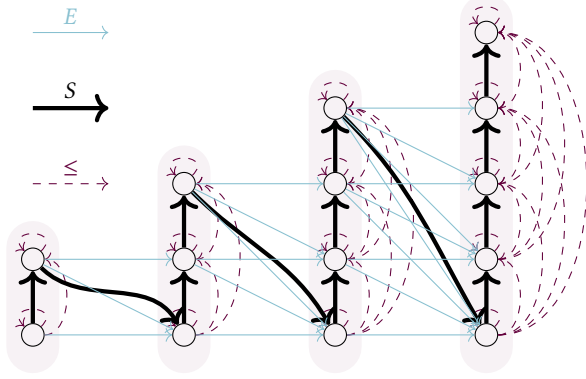
Figure 4: The structure $O_2 + O_3 + O_4 + O_5$

usual ordering of natural numbers and E the empty relation. Given natural numbers $2 \leq m \leq n$ one can build a structure denoted by $O_m + \cdots + O_n$ by extending the disjoint union $\biguplus_{m \leq i \leq n} O_i$ with new relations $S(a, b)$ whenever a is the last element of O_i and b the first of O_{i+1}, and $E(a, b)$ whenever $a \in O_i$, $b \in O_{i+1}$ and b is below a when interpreted as integers. An example of such a structure is given in Figure 4 with $m = 2$ and $n = 5$. We define then C_{ord} to be the class of finite disjoint unions of structures of the form $O_m + \cdots + O_n$ with $2 \leq m \leq n$.

Lemma 5.3 (Definition of φ_{CC} for C_{ord}). *There exists a sentence φ_{CC} that is existential local such that for all $A \in C_{\text{ord}}$, $A \models \varphi_{CC}$ if and only if A contains at least two connected components.*

PROOF. Define $\varphi_{CC} \triangleq \exists x_1, x_2.x_1 \neq x_2 \wedge \forall y. \neg S(y, x_1) \wedge \neg S(y, x_2)$. □

The translation of Example 5.1 to C_{ord} is a simple exercise detailed in the full paper.

Lemma 5.4 (Counter-example over C_{ord}). *The sentence $\varphi_B \triangleq \forall x.B(x) \vee \varphi_{CC}$ is preserved under disjoint unions over C_{ord} but is not equivalent to any existential local sentence over C_{ord}.*

We give the following axiomatic definition of C_{ord} as follows

(i) $\leq$ is transitive, reflexive and antisymmetric,
(ii) $\leq$ has connected components of size at least two: $\forall a, \exists b.a < b \vee b < a$,
(iii) S is an injective partial function without fixed points,
(iv) S and $\leq$ cannot conflict: $\forall a, b. \neg (S(a, b) \wedge b \leq a)$,
(v) There exists a proto-induction principle: $\forall a < b, \exists c, S(a, c) \wedge a \leq c \wedge c \leq b$,
(vi) Edges E can be factorised trough $(\leq)(S \backslash \leq)(\leq)$: $\forall a, b.E(a, b) \Rightarrow \exists c_1, c_2.a \leq c_1 \wedge c_1 S c_2 \wedge \neg(c_1 \leq c_2) \wedge c_2 \leq b$,
(vii) Pre-images through E form a suffix of the ordering: $\forall a, b, c.a \leq b \wedge E(a, c) \wedge S(a, b) \Rightarrow E(b, c)$,
(viii) Images through E form a prefix of the ordering: $\forall a, b, c.b \leq c \wedge E(a, c) \wedge S(b, c) \Rightarrow E(a, b)$,
(ix) Images through E are strictly increasing subsets: $\forall a, b.aSb \wedge a \leq b \Rightarrow \exists c.E(b, c) \wedge \neg E(a, b)$,
(x) Pre-images through E are strictly decreasing: $\forall a, b.aSb \wedge a \leq b \wedge (\exists c.E(c, a)) \Rightarrow \exists c.E(c, a) \wedge \neg E(c, b)$,

(xi) The last element of an order cannot be obtained through E: $\forall a, b.E(a, b) \Rightarrow \exists c.S(b, c) \wedge b \leq c$,
(xii) The relation $(\leq)(S \backslash \leq)$ is included in E: $\forall a, b.\exists c.a \leq c \wedge S(c, b) \wedge \neg(c \leq b) \Rightarrow E(a, b)$.

The goal of the rest of this section is to prove that $[\![\mathcal{A}_{\text{ord}}]\!]_{\text{Fin}(\sigma)} = C_{\text{ord}}$. To simplify notations, let us consider for a structure $A \in \text{Fin}(\sigma)$ the structure $(A, \leq)$ to be the structure A without the relations S and E. Using this convention, a $\leq$-component of a structure A is defined as a connected component of the Gaifman Graph of $(A, \leq)$.

The detailed proofs of lemmas 5.5 and 5.6 can be found in the full paper.

Lemma 5.5 (C_{ord} models $\mathcal{A}_{\text{ord}}$). *The class C_{ord} is contained in $[\![\mathcal{A}_{\text{ord}}]\!]_{\text{Fin}(\sigma)}$.*

Lemma 5.6 ($[\![\mathcal{A}_{\text{ord}}]\!]_{\text{Fin}(\sigma)} \subseteq C_{\text{ord}}$). *A structure A that models $\mathcal{A}_{\text{ord}}$ satisfies the following properties:*

(a) *If B is a $\leq$-component of A then the substructure induced by B in A is isomorphic to a total ordering of size greater than two with no E relations, i.e. $(\{1, \ldots, n\}, \leq, +1, \emptyset)$ with $n \geq 2$;*
(b) *If B_1 and B_2 are two $\leq$-components of A that are connected in A with the relation S, either the last element of B_1 is connected to the first one of B_2 or the last element of B_2 is connected to the first one of B_1;*
(c) *If B_1 and B_2 are two $\leq$-components of A connected through the relation E, then B_1 and B_2 are connected through the relation S;*
(d) *If B_1 and B_2 are two $\leq$-components of A connected through the relation S, the function $f : a \mapsto \max_{\leq} \{d \mid E(a, d)\}$ is a $\leq$-strictly increasing non-surjective function from B_1 to B_2, mapping the $\leq$-minimal element of B_1 to the $\leq$-minimal of B_2 satisfying $f(S(a)) = S(f(a))$.*
(e) *Connected components of A are in C_{ord}.*

Through Lemma 5.6 and Lemma 5.5 we learn that C_{ord} is definable using finitely many universal local sentences. We can lift the counter example provided in Lemma 5.4 using Fact 5.2.

Corollary 5.7 (Counter example over $\text{Fin}(\sigma)$). *There exists a sentence φ preserved under disjoint unions over $\text{Fin}(\sigma)$ but not equivalent to an existential local sentence over $\text{Fin}(\sigma)$.*

5.2 Undecidability

We have proven that deciding whether a sentence preserved under local elementary embeddings is equivalent to an existential local sentence is a non-trivial problem in $\text{Fin}(\sigma)$. We strengthen this by proving said problem is actually undecidable. This work was initially inspired by the work of Kuperberg [13] proving that Lyndon's Positivity Theorem fails for finite words and providing undecidability of the associated decision problem. Moreover, the statements and theorems can be interpreted as variations of those from Chen and Flum [4], who considers preservation under induced substructures.

Given a Universal Turing Machine U over an alphabet Σ and with control states Q, we extend the signature of C_{ord} with unary predicates $q/1$ for $q \in Q$, $P_a/1$ for $a \in \Sigma$, $P_{\$}/1$ and $P_{\square}/1$ to encode configurations of U in $\leq$-components. Without loss of generality, we assume that this Universal Turing Machine accepts only on a specific state $q_f^a \in Q$ and rejects only on a specific state $q_f^r \in Q$;

those two states being the only ones with no possible forward transitions. In a structure A, we call $C(a)$ the $\leq$-component of $a \in A$.

Fact 5.8. There exists a 1-local formula $\theta_C(x)$ such that for all structure $A \in C_{\mathrm{ord}}$, element $a \in A$, $A, a \models \theta_C(x)$ if and only if $C(a)$ represents a valid configuration of U.

The only difficulty in representing runs of the machine U is to map positions from one $\leq$-component to its successor. To that end, we exploit the first-order definability of the function $f : a \mapsto \max_{\leq}\{d \mid E(a, d)\}$ that links $\leq$-components (see the full paper).

Lemma 5.9 (Transitions are definable). *There exists a 1-local formula $\theta_T(x, y)$ such that for every structure $A \in C_{\mathrm{ord}}$ and points $a, b \in A$ the following two properties are equivalent:*

(i) The $\leq$-components of a and b are connected through S and represent valid configurations C and C' satisfying $C \rightarrow_U C'$,

(ii) $A, ab \models \theta_T(x, y)$.

Fact 5.10. Given a word $w \in \Sigma^*$ there exists a 1-local formula $\theta_I^w(x)$ such that for $A \in C_{\mathrm{ord}}$ and $a \in A$, $A, a \models \theta_I^w(x)$ if and only if $C(a)$ is the initial configuration of U on w and has no S-predecessor in A.

Fact 5.11. Given a word $w \in \Sigma^*$ there exists a 1-local formula $\theta_F(x)$ such that for $A \in C_{\mathrm{ord}}$ and $a \in A$, $A, a \models \theta_F(x)$ if and only if $C(a)$ is a final configuration of U and has no S-successor in A.

Fact 5.12. There exists a 1-local formula $\theta_N(x)$ such that for $A \in C_{\mathrm{ord}}$ and $a \in A$, $A, a \models \theta_N(x)$ if and only if $C(a)$ has no S-successor in A.

Theorem 5.13 (Undecidability). *It is in general not possible to decide whether a sentence φ closed under disjoint unions over $\mathrm{Fin}(\sigma)$ has an existential local equivalent form.*

PROOF. Without loss of generality thanks to Fact 5.2, we only work over C_{ord}, and we reduce from the halting problem.

Let M be a Turing Machine and $\langle M \rangle$ its code in the alphabet of the Universal Turing Machine U. Let φ_M be defined as the following existential local sentence: $\exists x.\theta_I^{\langle M \rangle}(x) \wedge \forall x, y.S(x, y) \wedge \neg(x \leq y) \implies \theta_T(x, y)$. We consider the sentence $\varphi \triangleq \varphi_M \vee \varphi_{CC}$ that is closed under disjoint unions over C_{ord}. This sentence is computable from the data $\langle M \rangle$.

Assume that M halts in at most k' steps, there exists a bound k for the run of the universal Turing Machine U. Given a size $n \in \mathbb{N}$, we define φ_M^n to be the following existential local sentence: $\exists x_1, \ldots, x_n.\theta_I^{\langle M \rangle}(x_1) \wedge \theta_T(x_1, x_2) \wedge \cdots \wedge \theta_T(x_{n-1}, x_n) \wedge \theta_N(x_n)$. It is a routine check that φ is equivalent to $\varphi_{CC} \vee \bigvee_{1 \leq k \leq n} \varphi_M^n$ over C_{ord}.

Assume that M does not halt. The universal Turing Machine U does not halt on the word $\langle M \rangle$. Assume by contradiction that φ is equivalent to a sentence $\psi \triangleq \exists x_1, \ldots, x_k.\theta$ where θ is r-local. Find a run of size greater than $k \cdot (2r + 1)$ and evaluate ψ. It cannot look at all the configurations simultaneously; change the state of the one not seen: it still satisfies ψ but this is no longer a run of U, which is absurd. □

An analogous proof allows us to conclude that the semantic property of closure under disjoint unions is also undecidable (see the full

paper). Theorem 5.13 is then strengthened to prove that equivalent existential sentences are uncomputable in general.

Theorem 5.14 (Undecidable semantic property). *It not possible, given a sentence φ, to decide whether or not it is closed under disjoint unions over $\mathrm{Fin}(\sigma)$.*

Theorem 5.15 (Uncomputable equivalence). *There is no algorithm, which given a sentence φ that is equivalent to an existential local sentence over $\mathrm{Fin}(\sigma)$ computes such a sentence.*

PROOF. Without loss of generality thanks to Fact 5.2, we only work over C_{ord}, and we reduce from the halting problem. Let M be a Turing Machine, $\varphi_M \triangleq \exists x.\theta_I^{\langle M \rangle}(x) \wedge \exists x.\theta_F(x) \wedge \forall x, y.S(x, y) \wedge \neg(x \leq y) \implies \theta_T(x, y)$, and $\varphi \triangleq \varphi_M \vee \varphi_{CC}$.

Assume that M halts. Then φ is equivalent to an existential local sentence $\varphi' \vee \varphi_{CC}$, as noticed in the proof of Theorem 5.13. Assume that M does not halt. Then φ is equivalent to φ_{CC}, which is existential local. We have proven that φ is equivalent to an existential local sentence in all cases.

Assume by contradiction that there exists an algorithm computing an existential local sentence μ that is equivalent to $\varphi_M \vee \varphi_{CC}$ over C_{ord}. One can use μ to decide whether M halts. Let us write k for the number of existential quantifiers of μ and r for the locality radius of its inner formula. If the sentence μ accepts the coding of a run of size greater than $2k \cdot (2r + 1)$, then it accepts structures that are not coding runs of U, and μ cannot be equivalent to φ. As a consequence, if M terminates, it must terminate in at most $2k \cdot (2r + 1)$ steps, which is decidable. □

5.3 Generalisation to Weaker Preorders

We characterise in this section the set of parameters (r, q, k) for which a sentence preserved under $\Rightarrow_q^{r,k}$ over $\mathrm{Fin}(\sigma)$ is equivalent to an existential local sentence through a finer analysis of $\mathcal{A}_{\mathrm{ord}}$.

Lemma 5.16 (Failure at arbitrary radius). *There exists a sentence φ_B preserved under $\Rightarrow_q^{\infty,k}$ over $\mathrm{Fin}(\sigma)$ for $k \geq 2$ and $q \geq 1$ that is not equivalent to an existential local sentence over $\mathrm{Fin}(\sigma)$.*

PROOF. Using Fact 2.2 it suffices to consider the case where $k = 2$ and $q = 1$ to conclude.

A first check is that C_{ord} can alternatively be defined by sentences that have at most 2 universal quantifiers over variables x_1 and x_2 and then one variable y at distance at most 1 from the tuple $x_1 x_2$. This proves that C_{ord} is definable and downwards closed for $\Rightarrow_1^{1,2}$. In particular, sentences preserved under $\Rightarrow_1^{\infty,2}$ over C_{ord} are preserved under $\Rightarrow_1^{\infty,2}$ over $\mathrm{Fin}(\sigma)$, which is a strengthening of Fact 5.2.

Using the same syntactical analysis, φ_{CC} is preserved under $\Rightarrow_1^{\infty,2}$ over C_{ord}. As a consequence it is an easy check that $\varphi_B \triangleq \forall x.\neg B(x) \vee \varphi_{CC}$ is preserved under $\Rightarrow_1^{\infty,2}$ over C_{ord}. Moreover, it was stated in Lemma 5.4 that φ_B cannot be defined as an existential local sentence over C_{ord}. □

Using similar techniques one can tackle the case $q = \infty$, $k \geq 2$ and $r \geq 1$. When $k = 1$ such methods will not apply because the preorders $\Rightarrow_q^{r,1}$ cannot distinguish a structure A from $A \uplus A$. We use this fact in the full paper to provide a positive answer in this case.

Lemma 5.17 (Failure at arbitrary quantifier rank). *For every $r \geq 1, k \geq 2$, there exists a sentence φ preserved under $\Rightarrow_{\infty}^{r,k}$ over $\mathrm{Fin}(\sigma)$ but not equivalent to an existential local sentence over $\mathrm{Fin}(\sigma)$.*

Lemma 5.18 (Success at $k = 1$). *For every $0 \leq r, q \leq \infty$, for every sentence φ preserved under $\Rightarrow_q^{r,1}$ over $\mathrm{Fin}(\sigma)$, there exists an existential local sentence ψ that is equivalent to φ over $\mathrm{Fin}(\sigma)$.*

We provide in Figure 5 a panel of the existence of an existential local form for different values for r, q and k over the class of finite structures $\mathrm{Fin}(\sigma)$, collecting the results from lemmas 5.16 to 5.18, Lemma 4.1, and the forthcoming Corollary 6.4 in the case of induced substructures.
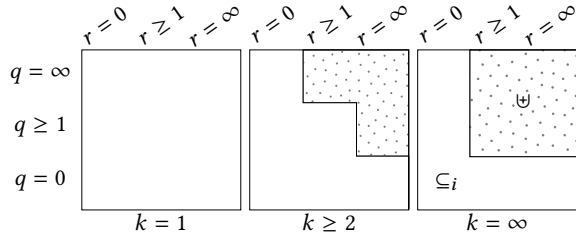

Figure 5: Parameters (r, q, k) leading to an existential normal form (white), those with a counter example (dots).

6 LOCALISING THE ŁOŚ-TARSKI THEOREM

Let us recall our proof scheme to handle preservation under extensions over subclasses of $\mathrm{Fin}(\sigma)$ in two distinct steps: (⇝) C is *localisable*, i.e. sentences preserved under extensions over C are equivalent to existential local sentences C, (✳) C satisfies *existential local preservation under extensions*, i.e. existential local sentences preserved under extensions over C are equivalent to existential sentences over C. We prove in Section 6.1 that step (⇝) can be done on every hereditary class of finite structures stable under disjoint unions, generalising the proof of preservation under extensions over structures of bounded degree of Atserias et al. [1, Theorem 4.3]. Moreover, we show in Section 6.2 that (✳) can be done under very mild assumptions on the class. These two results are represented as a path from sentences preserved under extensions to existential sentences in Figure 2, which allows us to significantly improve known classes of finite structures enjoying preservation under extensions in Section 6.3, as depicted in Figure 3. Let us recall hereafter the definition of $\mathrm{Balls}(C, r, k)$

$$\mathrm{Balls}(C, r, k) \triangleq \left\{ \mathcal{N}_A(\mathbf{a}, k) \mid A \in C, \mathbf{a} \in A^{\leq k} \right\} \quad (6)$$

6.1 Localisable Classes (⇝)

Given a sentence φ preserved under extensions, our goal is to prove that it has an existential local normal form. As a typical first step in preservation theorems, we will focus on the $\subseteq_i$-minimal models of φ. For this study to make sense we restrict our attention to classes C that are preserved under induced substructures, also known as hereditary classes.

Lemma 6.1 (Minimal models). *Let C be a hereditary class of finite structures. A sentence φ preserved under extensions over C has an*

equivalent existential local normal form if and only if there exists $k \geq 1, r \geq 0$ such that its $\subseteq_i$-minimal models are all found in $\mathrm{Balls}(C, r, k)$.

PROOF OF ⇒. If $\varphi \equiv \exists \mathbf{x}.\tau(\mathbf{x})$ over C where $\tau(x)$ is a r-local formula, then a minimal model $A \in C$ of φ necessarily contains a vector $\mathbf{a}$ such that $\mathcal{N}_A(\mathbf{a}, r), \mathbf{a} \models \tau$. This shows that $\mathcal{N}_A(\mathbf{a}, r) \models \varphi$ where $\mathcal{N}_A(\mathbf{a}, r) \subseteq_i A$ by definition. Since C is hereditary, $\mathcal{N}_A(\mathbf{a}, r) \in C$, thus $A = \mathcal{N}_A(\mathbf{a}, r)$ by minimality and $A \in \mathrm{Balls}(C, r, |\mathbf{x}|)$. □

PROOF OF ⇐. Assume that the minimal models of φ are all found in $\mathrm{Balls}(C, r, k)$. Let q be the quantifier rank of φ. We are going to show that φ is preserved under $\Rightarrow_{qr}^{r,k}$ over C and deduce by Lemma 4.1 that φ has an equivalent existential local sentence over C.

Let $A \models \varphi$ and $A \Rightarrow_{qr}^{r,k} B$. Since $A \models \varphi$, there exists a minimal model $A_0 \in \mathrm{Balls}(C, r, k)$ with $A_0 \subseteq_i A$. Let $\mathbf{a} \in A^k$ be the centers of the balls of radius r in A that contain A_0. Since $A \Rightarrow_{qr}^{r,k} B$ there exists a vector $\mathbf{b} \in B^k$ such that $\mathrm{tp}_A^{qr}(\mathbf{a}, r) = \mathrm{tp}_B^{qr}(\mathbf{b}, r)$.

Notice that $A_0 \subseteq_i \mathcal{N}_A(\mathbf{a}, r)$, hence $\mathcal{N}_A(\mathbf{a}, r) \models \varphi$ since φ is preserved under extensions. Thus, $\mathcal{N}_A(\mathbf{a}, r), \mathbf{a} \models \varphi_{\leq r}(\mathbf{x})$ where $\varphi_{\leq r}(\mathbf{x})$ is an r-local formula around its k variables $\mathbf{x}$ with quantifier rank qr. This shows that $\mathcal{N}_B(\mathbf{b}, r), \mathbf{b} \models \varphi_{\leq r}(\mathbf{x})$ using the equivalence of types up to quantifier rank qr. To conclude, observe that this entails $\mathcal{N}_B(\mathbf{b}, r) \models \varphi$, hence $B \models \varphi$ since $\mathcal{N}_B(\mathbf{b}, r) \subseteq_i B$ and φ is preserved under extensions. □

The proof of preservation under extensions over some specific classes provided by Atserias et al. [1, Theorem 4.3] is done by contradiction, using the fact that minimal structures that are large enough must contain large scattered sets of points. Forgetting about the size of the structure, this actually proves that minimal models do not have large scattered sets, hence are in some $\mathrm{Balls}(C, r, k)$ for well-chosen parameters.

See the full paper for a proof of the following variant of Atserias et al.'s result.

Lemma 6.2 (Minimal models). *Let $C \subseteq \mathrm{Fin}(\sigma)$ be a hereditary class of finite structures closed under disjoint unions and $\varphi \in \mathrm{FO}[\sigma]$ be a sentence preserved under $\subseteq_i$ over C. There exist R, K such that the minimal models of φ are in $\mathrm{Balls}(C, R, K)$.*

Theorem 6.3 (Localisable Classes ⇝). *If $C \subseteq \mathrm{Fin}(\sigma)$ is hereditary and closed under disjoint unions then it is localisable.*

PROOF. Consider a sentence φ preserved under local elementary embeddings over C. Using Lemma 6.2 its minimal models are in some $\mathrm{Balls}(C, r, k)$, and using Lemma 6.1 this provides an equivalent existential local sentence over C. □

Corollary 6.4. *A sentence φ preserved under extensions over $\mathrm{Fin}(\sigma)$ is equivalent over $\mathrm{Fin}(\sigma)$ to an existential local sentence.*

The use of disjoint unions was crucial in the construction, and removing the assumption that C is closed under this operation provides counter-examples to Theorem 6.3 (see the full paper).

Example 6.5 (Counter example without disjoint unions). Let C be the downwards closure of the class of finite cycles for $\subseteq_i$. The sentence $\varphi \triangleq \forall x. \deg(x) = 2$ is preserved under extensions over C but not equivalent to an existential local sentence over C.

6.2 Existential Local Preservation Under Extensions (✳)

Given an existential local sentence φ preserved under extensions, we want to prove that φ is equivalent to an existential sentence. As existential local sentences focus on neighbourhoods of the structures, we decompose our class C of finite structures into *local neighbourhoods*, that is Balls(C, r, k) with r, k ranging over the natural numbers. As we assume C to be hereditary, this is a subset Balls$(C, r, k) \subseteq C$ of the structures in C. Quite surprisingly, we do not need closure under disjoint unions to carry on step (✳).

Lemma 6.6 (Locally well-behaved classes). *Let C be a hereditary class of finite structures, the following properties are equivalent:*

(i) Balls(C, r, k) *satisfies preservation under extensions for* $r, k \geq 0$.
(ii) *Existential local sentences preserved under extensions over C are equivalent to existential sentences.*

PROOF OF $(i) \implies (ii)$. Let us consider a sentence φ preserved under extensions over C. As C is localisable, φ is equivalent to an existential local sentence and Lemma 6.1 provides k and r such that the minimal models of φ are all found in Balls(C, r, k).

As Balls$(C, r, k) \subseteq C$, the sentence φ is also preserved under extensions over Balls(C, r, k). Hence, there exists an existential sentence θ equivalent to φ over Balls(C, r, k). Moreover, C is hereditary, which entails that θ has finitely many $\subseteq_i$-minimal models $A_1, \ldots, A_m$ in C.

Let us define $M \triangleq \max_{1 \leq i \leq m} |A_i|$. Consider a minimal model B of φ in C and assume by contradiction that $|B| > M + k + r \cdot M$. Since $B \models \varphi$ and $B \in$ Balls(C, r, k), $B \models \theta$ and there exists a $\subseteq_i$-minimal model A of θ such that $A \subseteq_i B$. There exists a vector $\mathbf{b} \in B^k$ such that $B = \mathcal{N}_B(\mathbf{b}, r)$. It is then possible to define an induced substructure $A' \subseteq_i B$ that contains both A and $\mathbf{b}$ and belongs to Balls(C, r, k): this is ensured by adding paths taken from B of length at most r from each element of A to an element of $\mathbf{b}$. Hence, $A' \models \theta$ because $A \subseteq_i A'$, and therefore $A' \models \varphi$ because $A' \in$ Balls(C, r, k). But $|A'| \leq |A| + k + r \cdot |A| \leq M + k + r \cdot M < |B|$, which is absurd.

See the full paper for a graphical representation of this construction.

Therefore, $\subseteq_i$-minimal models of φ have bounded size. It is a usual consequence that φ is definable as an existential sentence. $\quad\square$

PROOF OF $(ii) \implies (i)$. Let φ be a sentence preserved under extensions over Balls(C, r, k). Let us write $\varphi_{\leq r}(\mathbf{x})$ the relativisation of φ to the r neighbourhood of its k fresh free variables $\mathbf{x}$.

The sentence $\psi \triangleq \exists \mathbf{x}. \varphi_{\leq r}(\mathbf{x})$ is preserved under extensions over C. Let $A \subseteq_i B$ through a map $h \colon A \to B$ and $A \models \psi$. There exists $\mathbf{a} \in A$ such that $\mathcal{N}_A(\mathbf{a}, r) \models \varphi$. Notice that $\mathcal{N}_A(a, r) \subseteq_i \mathcal{N}_B(h(\mathbf{b}), r)$ through h, hence that $\mathcal{N}_B(h(\mathbf{b}), r) \models \varphi$, since φ is preserved under extensions over Balls(C, r, k). As a consequence, $B \models \psi$.

We conclude that ψ is equivalent to an existential sentence θ over C. Remark that φ is equivalent to ψ over Balls(C, r, k). As a consequence, φ is equivalent to an existential sentence θ over Balls(C, r, k). $\quad\square$

6.3 Preservation under Extensions on Locally Well-Behaved Classes

We can now combine our study of step (✴) in Section 6.1 and step (✳) in Section 6.2 to harvest new classes enjoying preservation under extensions, by characterising classes satisfying preservation under extensions as those locally satisfying preservation under extensions. Quite surprisingly given the non-relativisation properties of preservation theorems, we are able to state an equivalence between preservation under extensions over a set C and preservation under extensions on its local neighbourhoods.

Theorem 6.7 (Local preservation under extensions). *Let C be a hereditary class of finite structures stable under disjoint unions. Preservation under extensions holds over C if and only if preservation under extension holds over* Balls(C, r, k) *for all $r \geq 0$ and $k \geq 1$.*

PROOF. Assume that the preservation under extensions holds over Balls(C, r, k) for $r, k \geq 0$. Let φ be a sentence preserved under extensions over C. Because C is hereditary and closed under disjoint unions, we can apply Theorem 6.3, and C satisfies (✴). Hence, φ is equivalent to an existential local sentence ψ over C. Since C is hereditary and locally satisfies preservation under extensions, we can apply Lemma 6.6, and C satisfies (✳). Therefore, ψ is equivalent over C to an existential sentence. We have proven that C satisfies preservation under extensions.

Conversely, assume that the preservation under extensions holds over C. In particular, existential local sentences preserved under extensions are equivalent over C to existential sentences. Thanks to Lemma 6.1, this proves that Balls(C, r, k) satisfies preservation under extensions for $r, k \geq 0$. $\quad\square$

The spaces Balls(C, r, k) appear naturally in the study of *sparse* structures [19], through the notions of *wideness* and *quasi-wideness*, which were already at play in recent proofs of preservation theorems [1, 2]. Indeed, they rely crucially on the existence of (r, m)-scattered sets, that is, sets of m points with disjoint neighborhoods of size r. Recall that a class $C \subseteq$ Fin(σ) is *wide* when there exists $\rho \colon \mathbb{N}^2 \to \mathbb{N}$, such that for all $n, m \in \mathbb{N}$, for all $A \in C$ of size greater than $\rho(r, m)$, there exists a (r, m)-scattered set in A. In particular Atserias, Dawar, and Grohe prove that for a *wide, hereditary, closed under disjoint unions* class of structures C, preservation under extensions holds [1, Theorem 4.3]. Hereditary wide classes are exactly those that are locally finite.

Fact 6.8 (Nešetřil and Ossona de Mendez [18, Theorem 5.1]). For a hereditary class C it is equivalent to ask for C to be *wide* or for Balls(C, r, k) to be *finite* for all $r, k \geq 1$.

Over a finite set of finite models, every sentence φ is equivalent to an existential sentence, hence preservation under extensions holds trivially. We can recover Atserias et al.'s result: hereditary wide classes closed ender disjoint unions enjoy preservation under extensions. Indeed, hereditary wide classes are locally finite (Fact 6.8), and hereditary locally finite classes stable under disjoint unions satisfy preservation under extensions (Theorem 6.7). This proof scheme is generalised in the following theorem.

Theorem 6.9 (New Classes Satisfying Preservation Under Extensions). *The following properties imply preservation under extensions over C assuming that C is hereditary and closed under disjoint unions.*

(i) C is locally finite (i.e., wide).
(ii) C has locally bounded tree-depth.
(iii) C is locally wqo.
(iv) C locally satisfies preservation under extensions.

Moreover, $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ and these implications are strict.

The remaining of this section is devoted to proving Theorem 6.9, together with their relation with previously known properties as illustrated in Figure 3. Recall that in this figure arrows represent strict inclusions of properties and dashed boxes are the new properties introduced in this paper. In particular, the examples prove that we strictly generalise previously known properties implying preservation under extensions. As no logic is involved in the generation of these new classes, we effectively *decoupled* our proofs of preservation theorems in a *locality argument* followed by a *combinatorial argument*.

Recall here that the *tree-depth* td(G) of a graph G is the minimum height of the comparability graphs F of partial orders such that G is a subgraph of F [19, Chapter 6]. This extends as usual to structures by considering the tree-depth of the Gaifman graphs of said structures. We shall say that C has locally bounded tree-depth, if for all $r, k \geq 1$, there is a bound on the tree-depth of the structures in Balls(C, r, k).

Note that working with Balls(C, r, k) rather than Balls($C, r, 1$) is a somewhat uncommon way to localise properties. Thankfully, for properties that are well-behaved with respect to disjoint unions, the localisation using a single ball or several ones will coincide; examples of such properties are wideness, exclusion of a minor, or bounded clique-width. The following proposition illustrates this point in the case of bounded tree-depth.

Lemma 6.10 (Locally bounded tree-depth). *A class $C \subseteq$ Struct(σ) has locally bounded tree-depth if and only if $\exists \rho \colon \mathbb{N} \rightarrow \mathbb{N}, \forall A \in C, \forall a \in A, \forall r \geq 1, \text{td}(\mathcal{N}_A(a, r)) \leq \rho(r)$.*

Example 6.11 (Stars). The class Star of stars is of tree-depth bounded by 1, hence of locally bounded tree-depth, but is not locally finite.

Example 6.12 (Cliques). Consider the class $\uplus$K of finite disjoint unions of cliques. This class is not of locally bounded tree-depth but locally well-quasi-ordered.

Example 6.13 (Diamonds). Let us call D_n the cycle of length n extended with two new points a and b such that both are connected to every node of the cycle using a path of length n. Consider the class $\uplus$D of induced subgraphs of finite disjoint unions of some D_n. This class is not well-quasi-ordered nor locally finite, but is of locally bounded tree-depth.

Example 6.14 (Pointed graphs). Consider the class Δ_2 of graphs of degree bounded by 2. The class $P\Delta_2$ is obtained by adding one point connected to every other point in a structure of Δ_2. The class is not locally well-quasi-ordered but locally satisfies preservation under extensions.

PROOF OF THEOREM 6.9. Properties (i) to (iv) all imply that the class C locally satisfies preservation under extensions. Thanks to Theorem 6.7, this proves that C satisfies preservation under extensions. The implications are strict thanks to Examples 6.11, 6.12 and 6.14.   □

Moreover, notice that, as explained in Figure 3, "locally wqo" strictly generalises "wqo" (see Example 6.13), which strictly generalises locally bounded treedepth (see Example 6.12). Notice that locally bounded treedepth already strictly generalises "wqo" via Example 6.13.

7   CONCLUDING REMARKS

We investigate the notion of positive locality through three fragments of first-order logic: existential local sentences, a positive variant of the Gaifman normal form, and sentences preserved under local elementary embeddings. We prove that those three fragments are equally expressive in the case of arbitrary structures, but that this fails in the finite. Following the line of undecidability results for preservation theorems [4, 13], we prove that most of the associated decision problems are undecidable in the case of finite structures.

Maybe surprisingly, the study of this seemingly arbitrary notion of positive locality has a direct application in the study of preservation under extensions over classes of finite structures. In the case of finite structures, our notion of local elementary embeddings describes exactly disjoint unions, and might explain why they featured so prominently in the study of preservation under extensions [1, 2, 12, 20].

We prove that under mild assumptions on the class C of structures considered, sentences preserved under extensions can be rewritten as existential local sentences. We leverage this to craft a locality principle relating preservation under extensions over the neighborhoods of a class and preservation under extensions over the whole class.

This allows us to build new classes of structures where preservation under extensions holds by localising known properties. This proof scheme does not always yield new classes: for instance, *nowhere dense* classes [see 20, Chapter 5] are locally nowhere dense and vice-versa.

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