A PARTIAL SOLUTION TO THE REACHABILITY-PROBLEM FOR VECTOR-ADDITION SYSTEMS*

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ABSTRACT. With geometrical techniques we hope to bring new insight into the reachability problem for vector-addition systems, which is pertaining in many areas in computer science theory but unsolved ever since it was proposed by Karp and Miller. We show that the reachability-problem is decidable in dim. ≤ 3 and give a partial solution for the general problem covering a wide instance of it. In both cases a semi-linear characterization is obtained for the complete set of solutions. Contrary to common belief the complete solution in dim. 3 does not immediately generalize, and our analysis gives evidence why. A few generalizations and application to long-standing classificational problems in language theory are discussed.

INTRODUCTION. Vector-addition systems were first introduced by Karp and Miller [12], and subsequently studied by Rabin (unpublished,[12]), Nash [15,16], Golan [19], Maurer [14], and others in varying context. Karp and Miller showed that it is decidable for any system and any given starting point whether or not an infinity of points can be reached. Rabin reportedly proved that the set of reachable points need not be semi-linear, and that it is undecidable to determine if two (equally dimensional) vector-addition systems have the same reachability-set.

The reachability problem is the question to decide for any system as described and any given starting point whether or not another, arbitrarily given configuration can be attained. For instance in grammatical derivations we usually want to know if we can arrive at a variable-free string, i.e., in the origin of the Parikh domain. Nash ([15],sect. 4.2, also [16]) gave some reductions but they do not imply a simplification. Here we attack the problem with combinatoric geometrical techniques, with an additional appeal to automata-theoretical arguments.

Effectiveness of the reachability-problem is equivalent to such problems as the emptiness-problem for context-free matrix grammars and termination-questions for program-schemata. There are further applications in studies of multi-counter machines and vector-games.

2. PRELIMINARIES.

Basic results from language and automata-theory are assumed as standard [2, 11, 23]. Various construction- and decision-problems in computer science theory have been reduced to and solved in discrete geometry, [5, 10, 3]. In Z^n let e_i ($i = 1\ldots n$) denote the unit coordinate vectors, π_i the projection on the i th coordinate axis, and write $v \geq w$ if $\pi_i(v) \geq \pi_i(w)$ for all i. Semi-linear varieties are finite unions of sets $v + F\lambda$, where v and F are a non-negative, integral vector and (n x m) matrix respectively, λ a variable over N^m. Semi-linear varieties form an effective Boolean algebra (the original proofs in [8] can be simplified considerably). Arcones are varieties $v + D\lambda$, with v and λ as before and D a diagonal 0-1 matrix. The number of 1's in D is the dimension of the arcone, n-dim. arcones $\subseteq N^n$ are called principal. The finite unions of arcones are an effective Boolean algebra as well.

Linear systems of equations $\sum_{j=1}^{m} a_{ij}\lambda_j = b_i$ ($i = 1\ldots n$) with $a_{ij}, b_j \in Z$ and solutions λ sought in N^m or Z^m are called (linear) diophantine. They were already investigated by e.g. [9] and [7].

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Theorem 2.1. The non-negative integer solutions to a linear diophantine system of equations form an effectively computable semi-linear variety.

Proof. When all sets $L_i = \{\lambda \in N^m | \sum_{j=1}^{m} a_{ij}\lambda_j = b_i\}$ are semi-linear, then by Boolean properties so is $\bigcap_{i=1}^{n} L_{i_\ell}$, the solution-set of the complete system. For i, let $\sum_{t=1}^{k} a_{ij_t}\lambda_{j_t} = |b_i| - \sum_{s=1}^{\ell} a_{ih_s}\lambda_{h_s}$ $(k + \ell = m)$ be the i th equation rearranged to non-negative coefficient form. If $k \cdot \ell = 0$ then the solution-set is empty or finite. For $k \cdot \ell \neq 0$ let $A = b_1^* \ldots b_k^* \,\mathrm{¢}\, b_{k+1} b_{k+2}^* \ldots b_{k+\ell+1}^*$ (regular), $B = \{a^n b^n | n \geq 0\}$ (context-free), and h the homomorphism induced by $h(b_t) = a^{a_{ij_t}}$ $(1 \leq t \leq k)$, $h(\mathrm{¢}) = \lambda$, $h(b_{k+1}) = b^{|b_i|}$, and $h(b_{k+s+1}) = b^{a_{ih_s}}$ $(1 \leq s \leq \ell)$. The language $h^{-1}(h(A) \cap B) \cap A \subseteq A$ is context-free, hence semi-linear by Parikh's theorem. Discarding ¢ and b_{k+1}, permuting coordinates back in order shows that L_i is semi-linear.

It follows that the arbitrary (integer) solutions if a linear dioph. equation form a set $\bigcup D_\delta(A_\delta)$, where D_δ is a Kronecker diagonal matrix, A_δ semi-linear. In particular it is decidable whether solutions exist or not (see also [4]).

Theorem 2.2 The non-negative integer solutions to a linear diophantine system of inequalities form an effectively computable semi-linear variety.

Proof. Add slack-variables, apply 2.1, and project to get rid of the slack-variables.

3. VECTOR-ADDITION SYSTEMS. We henceforth abbreviate the phrase "vector-addition" as "v.a.".

Definition. An (n-dim) v.a.-scheme is any finite set $W = \{w_1, \ldots, w_k\} \subseteq Z^n$. The v.a.-scheme $W^d = \{-w | w \in W\}$ is called the dual of W.

$X(w) = (\#_{w_1}(w), \ldots, \#_{w_k}(w))$ is called the folding of the path w, w an unfolding of $X(w)$.

Definition. An (n-dim) v.a.-system is a pair $V = \langle\sigma, W\rangle$ with $\sigma \in N^n$, W an n-dim. v.a.-scheme.

For $V = \langle\sigma, W\rangle$, $w \in W^*$ is said to encode a valid path if it runs entirely inside N^n when applied to σ. The (point-) reachability-problem $\langle V,\delta\rangle (\delta \in N^n)$ is to decide if there is a valid W-path from σ to δ [12].

Theorem 3.1. (Nash) Point-reachability is equivalent to zero-reachability.

Proof. Let [v, i] denote v augmented with coordinate i. $\langle V,\delta\rangle$ is equivalent to $\langle V', 0\rangle$ with $V' = \langle[\sigma, 1], \{[w, 0] | w \in W\} \cup \{[-\delta, -1]\}\rangle$.

Definition. The arcone-reachability problem for v.a.-systems $V = \langle\sigma, W\rangle$ is to decide for any given arcone A if there is a valid path from σ into A. The reachability language of A is the set $L(V,A) = \{w \in W^* | w$ is a valid path from σ to $A\}$.

Theorem 3.2. Let $V = \langle\sigma, W\rangle$, W a v.a.-scheme such that for each $w \in W$ there is a $1 \leq j \leq n$ with $w \geq -e_j$. The arcone-reachability problem for V is decidable.

Proof. Let $A = v + D\lambda$ be an arcone, $D = \mathrm{diag}(d_1, \ldots, d_n)$. Distinguish reaching A "directly" or through the origin. In the first case, construct a context-free grammar G_1 with variables $S, A_1, \ldots, A_n$ and productions $S \to A_1^{\pi_1(\sigma)} \ldots A_n^{\pi_n(\sigma)}$,

$A_i \to A_1^{\pi_1(w)} \ldots A_{i-1}^{\pi_{i-1}(w)} A_{i+1}^{\pi_{i+1}(w)} \ldots A_n^{\pi_n(w)}$ for all $w \in W$ with $\pi_i(w) = -1$

$(i \in 1, \ldots, n)$, and $A_i \to A_1^{\pi_1(w)} \ldots A_i^{\pi_i(w)+1} \ldots A_n^{\pi_n(w)}$ for all $w \geq 0$ in W and all i. Use language-theory to decide if G_1 derives a word of the (regular) language $\{z \in \{A_1, \ldots, A_n\}^* | \forall_{1<i<n}$ z has exactly (if $d_i = 0$) or at least (if $d_i = 1$) $\pi_i(v)$ occurrences of $A_i\}$. In the second case, use G_1 to decide if we can reach 0, then

G_2 with productions for S replaced by $S \to A_1^{\pi_1(w)} \ldots A_n^{\pi_n(w)}$ $(w \geq 0,\ w \in W)$ to decide similarly if we can reach A from 0.

4. WEBS and THE PRINCIPAL ARCONE REACHABILITY-PROBLEM.

Fix n and a v.a.-scheme W.

Definition. Let $\sigma \in N^n$, $A \subseteq N^n$. A web of σ with respect to A is a collection L of W-paths such that

(i) all paths in L are valid paths from σ "up" into A
(ii) for each valid path w from σ into L there is a $y \in L$ with $X(y) \leq X(w)$.
(iii) foldings of paths in L are pairwise incomparable.

Dual webs are defined analogously, for paths running "down" from A to σ. Thus, dual webs are webs in the dual scheme.

Theorem 4.1. (Dual) webs are always finite.

Proof. König's theorem.

In our analysis of the reachability problem it is crucial to determine webs effectively. It can be done for principal arcones, and in that case an even stronger result can be obtained which, as will turn out, holds the major clue to further results.

Theorem 4.2. Let A be a principal arcone in N^n. There is a finite partition of N^n such that for each block there is a single web which applies to all its points uniformly. The partition and the corresponding webs can all be determined effectively.

Proof. Induct on n. For $n = 1$ it follows from 2.2 and the fact that in this case $N - A$ is finite.

Assume the theorem holds for dimensions $n = 1, \ldots, \ell$. Let A be a principal arcone in $N^{\ell+1}$, squeezing finitely many ℓ-dimensional arcones (half-hyperplanes) between its faces and those of $N^{\ell+1}$. A is the first block (with trivial webs) and the construction proceeds into each of the squeezed arcones. Say that "going down" inductively we arrived at a j-dimensional arcone B (which is a half-space) which is to be partitioned further $(1 \leq j \leq \ell)$. Project A and B on the 0-centered arcone spanned by the $\ell + 1 - j$ unit-vectors orthogonal to B, to be identified with $N^{\ell+1-j}$ and subjected to the correspondingly projected scheme (retaining, however, the distinguished names of vectors even when their projections are the same). Denoting the projections by $\bar{A}$ (a principal arcone) and $\bar{B}$ (a point), it follows from the induction-hypothesis that (after locating $\bar{B}$ in the appropriate partitioning) the "projected" web of $\bar{B}$ with respect to $\bar{A}$ can be effectively computed. Unfolding it in $N^{\ell+1}$ at the corner of B, the web is valid in $\ell + 1 - j$ directions but may be "sticking out" in the remaining directions. However by 4.1 the web is finite and we can push it into B to be a valid web for all points in a j-dimensional sub-arcone of B, thus locating a new block of the partition and its corresponding web. The construction then proceeds with the finitely many $(j - 1)$-dimensional arcones (which again are 0-centered) squeezed between the sides of B and the faces of the newly found block.

As the dimension steadily decreases from step to step in the algorithm (ran in parallel on all further squeezed arcones), termination of the construction is guaranteed. In the last step we are stuck with initial parts of all the 0-centered lines the algorithm came to consider, the points of which form a finite set γ. The webs for points in γ are constructed in an alternative way. Note that W defines a transition-function on γ and the one-step reachable points in its neighborhood. Points of A are final, for remaining neighborhood points we continue along the web that was determined for them into A. With finite-state techniques the web for each point of γ can be determined effectively, and the individual points of γ are the last blocks to complete the partition.

Theorem 4.3. For each $\sigma \in N^n$ and principal arcone A the (dual) web of σ with respect to A can be effectively determined.

A similar result with non-principal arcones is still open except for dimensions ≤ 3 (but we need the more sophisticated results of the next sections for that). In view of reachability-problems we are able to conclude what is already implicit in Karp and Miller from their tree-approach.

Theorem 4.4. The principal arcone reachability problem for vector-addition systems is effectively solvable.

5. TRANSFORMATION-AREAS. Results are going to deviate more and more from what seems feasible in tree-based arguments. Again fix n and a v.a.-scheme W.

Lemma 5.1. Let σ and $\delta \in N^n$, w a path connecting σ to δ, and $1 \leq j \leq n$. There is a path w' with $X(w') = X(w)$ connecting σ to δ such that the projection on the j^{th} coordinate axis runs entirely on the non-negative part of it.

Proof. Rearrange the steps in w to have all those with non-negative j^{th} coordinate first.

Definition. A principal arcone $A \subseteq N^n$ is called a W-transformation area if for any two points σ and $\delta \in A$ and for each path w connecting σ to δ the steps of w can be permuted to form a path connecting σ to δ running entirely inside N^n.

For points in transformation-areas reachability is easily decided: we only have to test for connectivity and that is decidable with 2.1. Note that when A is a transformation-area, B a principal arcone, then $B \cap A$ is a transformation-area.

It is not evident a priori that transformation-areas always exist. However they do, and may even be required to have a slightly more special form. For the proof we need some auxiliaries.

Definition. Let $\sigma \in N^n$, $A \subseteq N^n$. A free (dual) web of σ with respect to A is a (dual) web with the condition that only valid paths are considered omitted.

Free webs are again finite, and this time they can be effectively determined for each point and arbitrary arcone (extend 2.2. for mixed systems).

Lemma 5.2. Let A be a principal arcone in N^n. There is a finite partition of N^n such that for each block there is a single free web which applies to all its points uniformly. The partition and the corresponding free webs can all be determined effectively.

Proof. Induct on n. For $n = 1$ it is trivial. The induction step proceeds as in 4.2. The unfolding of "projected" free webs needs to be pushed "inwards" this time for the sole reason that the unfolded end-points need not immediately lie in A. Only the end of the construction is slightly different. Let γ be finite set of points left over at the end (see 4.2). Let $A = v + I\lambda$. For each $\sigma \in \gamma$ solve $\sigma + W\mu \geq v$, yielding a semi-linear solution-set $\bigcup_i (\tau_i + F_i\mu_i)$ for μ. Then determine a maximal incomparable subset of $\{\tau_i\}_i$, and unfold it to a desired free web for σ. The individual points of γ again are the last blocks completing the partition.

Theorem 5.3. For each $1 \leq j \leq n$ there is a W-transformation area of the form $v + I\lambda$ with $\pi_j(v) = 0$.

Proof. Let $d = \max\{|\pi_i(w)| \mid w \in W, 1 \leq i \leq n\}$, and $A' = v' + I\lambda$ (a principal arcone), where $\pi_i(v') = d(1 - \delta_{ij})$. Determine a free web partition with respect to A' as in 5.2, in all (finitely many) free webs permuting steps to have projections on the j^{th} coordinate axis all on the non-negative part of it (5.1). In the $n - 1$ remaining dimensions however webs may be sticking out in negative areas, but all together no more than finitely far since the partition is finite and blocks that are infinite are along main axis-directions. Thus we can shift A' and the entire partition into the orthoplement of e_j sufficiently far to pull all webs entirely inside N^n. Let A be the shifted arcone, surrounded on $(n - 1)$ sides by a layer of d $(n - 1)$-dimensional, partitioned half-hyperplanes. Note that what originally was a free web for each block has now become an ordinary web with valid paths! We will show that A (a principal arcone of the required form) is a transformation-area. Induct on the length m of connecting paths. For $m = 0, 1$ there is nothing to prove. Let w be a path of length $m + 1$ connecting σ and $\delta \in A$. By 5.1 we may assume that w has a non-negative projection along the j^{th} coordinate axis, but it may stick out of N^n through any of the bordering half-hyperplanes distinct from e_j's orthoplement. If so, let δ' be the first point reached by w outside A. Since in one step we cannot go more than d in any dimension, δ' belongs to one of the partitioned hyperplanes surrounding A and we have its web. Let w' be the remaining part of w connecting δ' to δ.

Because the web for δ' was constructed with respect to arbitrary paths beforehand, it contains a path y (which is a valid path now, due to shifting) with $X(y) \leq X(w') < X(w)$ already leading from δ' back into A, say to σ'. By induction-hypothesis the remaining path from σ' to δ, which has a folding $X(w') - X(y) < X(w) = m + 1$, can be rearranged to run entirely inside N^n as well. The final path from σ to δ' to σ' to δ runs entirely inside N^n then, and is a rearrangement of w.

6. PARTIAL SOLUTIONS OF THE REACHABILITY-PROBLEM. All theorems in this section make essential use of webs and transformation-areas. Let $n \geq 1$, and fix an arbitrary v.a.-scheme W.

Theorem 6.1. There is an effectively computable constant C_W (depending only on W) such that for all $V = \langle\sigma, W\rangle$ and $\delta \in N^n$ with σ or δ having at least $n - 1$ coordinates $\geq C_W$ it is decidable whether δ is reachable in V or not.

Proof. For $1 \leq j \leq n$ let $v^{(j)} + I\lambda$ $(w^{(j)} + I\lambda)$ be a $W(W^d)$-transformation area with $\pi_j(v^{(j)}) = \pi_j(w^{(j)}) = 0$ as guaranteed by 5.3. Let $C_W = \max\{\pi_i(v^{(j)}), \pi_i(w^{(j)}) \mid 1 \leq i \leq n,\ 1 \leq j \leq n\}$. By duality it is no restriction to assume that δ has $n - 1$ coordinates $\geq C_W$, and it follows that δ must be in one of the W-transformation areas determined above, say in A. If $\sigma \in A$ we are through, otherwise determine a web for σ with respect to A (4.3). Since A is a transformation-area it follows that δ is reachable from σ iff δ is reachable from an endpoint of a web-path. Since there are finitely many of them and all are $\in A$, this case is reduced to connectivity again, which is decidable by 2.1.

Thus 6.1 shows that when at least one of the points lies far enough inside the space (except for, possibly, in one dimensional direction) then reachability can be decided. A more precise analysis of the procedure even shows

Theorem 6.2. In the situation of 6.1 $X(L(V, \delta))$ is an effectively computable semi-linear variety.

Proof. Continuing the proof of 6.1, if $\sigma \in A$ then $X(L(V, \delta))$ is the solution-set of a linear diophantine system of equations (apply 2.1). If $\sigma \notin A$ then $X(L(V, \delta))$ equals $\bigcup (X(z) + L_z)$, the (finite) union taken over all z in the web of σ and L_z denoting the semi-linear solution-variety of the diophantine equation system describing connectivity of the endpoint of z and δ.

It would be desirable to have similar characterizations for all v.a.-systems, but no more general results have been obtained so far. We can however do much better for dimensions ≤ 3.

Theorem 6.3. The reachability problem for vector-addition systems is decidable for $n \leq 3$.

Proof. Assume without restriction that $n = 3$. Let W be an arbitrary 3-dim v.a.-scheme, and let C_W be determined as in 6.1. Then $(C_W, C_W, 0) + I\lambda$, $(C_W, 0, C_W) + I\lambda$, and $(0, C_W, C_W) + I\lambda$ are both W-and W^d-transformation area, together spanning most of N^3 but leaving (at most) three "tubes" of finitely many lines along each of the coordinate axes. Let $V = \langle\sigma, W\rangle$ and $\delta \in N^3$. If σ or δ is in one of the three transformation areas, apply the method of 6.1. If not then there are two possibilities:

(i) there is a connecting, valid path through one of the transformation-areas. Then δ is reachable from σ iff there is a transformation-area in which an end-point of the dual web of δ is reachable from an endpoint of the ordinary web of σ (all with respect to the particular area), in this way again reducing it to finitely many connectivity-problems which are decidable.

(ii) there is a connecting, valid path entirely inside the tubes. But since the tubes are "one-dimensional" line-bundles, such walks can be modeled on a (deterministic) counter-machine which in its states remembers in what tube and on what line it is and in its counter keeps the current height in the tube. It can switch from one tube into the other only when the counter is below $\backsim C_W + d$, also to be stored in finite control. The reachability-problem reduces to the halting-problem for counter-machines.

With more sophisticated constructions one can even show that in dim ≤ 3 the half-hyperplane reachability problem is effectively solvable (the principal arcone reachability is decidable in all cases, 4.4). From 6.3 it becomes apparent why the methods do not immediately generalize to a solution in dimension 4. The tubes then consist of finitely many half-planes, and a simulation along current lines would require 2-counter machines, which in general have a non-solvable halting-problem.

Carefully analyzing all instances of the algorithm in 6.3 and using Parikh's theorem for context-free languages it follows.

Theorem 6.4. For all $\leq$ 3-dim. vector-addition systems V and points δ $X(L(V, \delta))$ is an effectively computable semi-linear variety.

7. SOME APPLICATIONS. Vector-addition systems appear in the word-problem for commutative semigroups, termination-questions in parallel program-schemata, recursiveness-problems for context-free like rewriting systems, in vector and number games, and in studies of multi-counter machines.

Context-free matrix grammars [1] have productions of the form $[A_1 \to w_1, \ldots, A_k \to w_k]$ (not necessarily $A_i \neq A_j$), upon application of which all rules have to be carried out in the order listed in the matrix in one step. They appear in a variety of equivalent formulations: context-free grammars with regular control [21], success-only context-free programmed grammars ([20], also [25]), and context-free parallel leveled grammars of height 1 [15]. The emptiness-, and hence the recursiveness-problem for these classes is still open, and so far little is known. Extending the diophantine techniques of [25], it can be shown that the emptiness-problem is unsolvable when $\{a^n b^{n^2} \mid n \geq 1\}$ is a context-free matrix language [18].

Definition. A context-free matrix grammar is in normal form if in no (scattered) substring entirely replaced by a matrix in one step there is a symbol which also occurs in the newly created (correspondingly scattered) substring.

Theorem 7.1. All context-free matrix grammars have a normal form.

Proof. Each matrix can be replaced by an equivalent, finite set of matrices the productions in which have to be carried out in parallel (instead of, possibly, nested). Rename, and add matrices to turn new names back into old names.

With the same technique it easily follows.

Theorem 7.2. [15, 19]. The reachability-problem for vector-addition systems is equivalent to the emptiness-, and hence the recursiveness problem for context-free matrix grammars.

The sentential forms of a context-free matrix grammar are a context-free matrix language as well, and the general arcone reachability problem can be reformulated as whether the intersection of a context-free matrix language and a particular regular set is empty or not.

Theorem 7.3. The general arcone reachability-problem is equivalent to the point-reachability problem.

It should be noted, however, that in reducing the arcone reachability problem dimensionality is usually not preserved.

Definition. A language L is called k-uniform with respect to a context-free matrix grammar G (with, say, n variables) in normal form if there is a context-free language M and a finite set of strings $\{\sigma_1, \ldots, \sigma_\ell\}$ with each σ_j such that $X_{\mathrm{var.}}(\sigma_j) \geq (k, \ldots, k)$ such that $L = M \cup \{w \in \Sigma^* \mid \exists_j \; \sigma_j \overset{*}{\underset{G}{\Rightarrow}} w\}$.

With Parikh's theorem and the results of section 6 we have

Theorem 7.4. Languages which are k-uniform with respect to a context-free matrix grammar in normal form for sufficiently large k have the semi-linear property (and therefore a solvable emptiness-problem).

This result may indicate what types of matrix languages remain to be investigated regarding the general emptiness-problem for this class. It also shows that unary context-free matrix languages which are k-uniform for sufficiently large k (with respect to the number of variables needed) are regular, thus further narrowing down the search for a non-regular one [22].

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