Sequentiality and Strong Stability

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Abstract

We show that Kahn-Plotkin sequentiality can be expressed by a preservation property similar to stability and that this kind of generalized stability can be extended to higher order, giving rise to a new model of PCF.

Introduction

One of the main goals of denotational semantics is to find as adequate models as possible of purely functional programming languages like PCF [9], [3]. This consists in finding cartesian closed categories of domains where there are "as few points as possible", aiming at a situation where any finite point is definable by a term of PCF. In that situation, the model is fully abstract (see [8]). In this research, most people have followed the rule of not making any reference to the syntax of PCF in the constructions of domains. By the nature of the problem, that rule seems very sound, and we follow it.

The problem of full abstraction is still open, but several significative steps have been performed in the direction of its solution.

Plotkin [9] remarked that the presence of functions like "parallel or" in the Scott models prevents them from being fully abstract. Kahn, Plotkin ([6]) and Milner [8] tried to solve this problem by adding different "sequentiality" constraints on morphisms which turned out to be equivalent (at least for first order functions). However, they could not define any cartesian closed category of sequential functions.

These works gave rise to two main developments. The first one consisted in weakening the notion of sequentiality in a property extendable to higher orders, and this is what Berry [1] did introducing stable semantics. The second one, carried out by Berry and Curien (see [2]) was to stick firmly to the Kahn-Plotkin's notion of sequentiality, but the price to pay was the impossibility of keeping functions as morphisms; they were obliged to switch to sequential algorithms.

The model presented here is more in the spirit of the former development firstly because the morphisms of our category are functions and secondly because they are characterized by a preservation property similar to stability, this is why we call them "strongly stable". Indeed, in this model, the domains are endowed with a certain notion of "coherence" which is a predicate on the set of their finite subsets. A function has then to preserve this coherence and to commute with the glbs of coherent set. The interesting phenomenon is that Kahn-Plotkin sequentiality may be expressed in these terms. The main result is then the construction of a model where all morphisms are functions and, at ground types, these functions are sequential. The trouble with Kahn-Plotkin sequentiality is that it seems not to make sense at higher order in a model where morphisms are functions. This is why the question whether functionals of this model are sequential or not is probably unrelevant. However it would be interesting to find a connection with the category of sequential algorithms.

1 Qualitative domains and coherence spaces

We recall here some definitions and results about stable semantics as presented by Girard in [5].

Definition 1 *A qualitative domain (QD) is a pair $(|X|, X)$ where $|X|$ is a set and X is a subset of $\mathcal{P}(|X|)$ such that*

- $\emptyset \in X$.
- $\forall a \in |X| \quad \{a\} \in X$.
- $\forall x \in X\ \forall y \subseteq x \quad y \in X$.
- *For any $\subseteq$-directed subset D of X we have $\bigcup D \in X$.*

Furthermore, $|X|$ will always be countable.

We say that X is a coherence space if

$$x \in X \quad \textit{iff} \quad \forall a, b \in x\ \{a, b\} \in X$$

We call $|X|$ the web of the qualitative domain, atoms the elements of the web, and states the elements of X. Since $|X|$ is entirely determined by X, we shall confuse a qualitative domain and the set of its states. Following the usage of domain theory, we shall note $\perp$, $\leq$ and $\vee$ the set theoretic empty set, inclusion and union respectively. Last we note $\mathrm{K}X$ the set of (Scott)-compact states of X which are nothing but the finite states of X. If $a \in |X|$ we shall often identify a and the state $\{a\}$.

Definition 2 *Let X and Y be two QDs. A stable function f from X to Y is a continuous function $X \to Y$ such that for any bounded $x, y \in X$ we have $f(x \wedge y) = f(x) \wedge f(y)$.*
One says that f is linear if furthermore $f(\perp) = \perp$ and $f(x \vee y) = f(x) \vee f(y)$.

The natural ordering between stable (or linear) functions is the Berry's stable order (actually this choice is necessary if evaluation is intended to be stable):

Definition 3 *If $f, g : X \to Y$ are stable functions, we say that f is stably below g and we write $f \leq g$ iff for all $x, x' \in X$ such that $x \leq x'$ we have $f(x) = g(x) \wedge f(x')$.*

If $f : X \to Y$ is stable, its trace is defined by

$$\mathrm{tr}(f) = \{(c, b) \in \mathrm{K}X \times |X| \mid f(c) \geq b \text{ and } \forall c' < c\ f(c') \not\geq b\}$$

Proposition 1 *The trace enjoys the following properties:*

- *If $f : X \to Y$ is stable and $x \in X$ then*
$$f(x) = \bigvee \{b \mid \exists c \quad c \leq x \text{ and } (c, b) \in \mathrm{tr}(f)\}$$
- $f \leq g$ *iff* $\mathrm{tr}(f) \subseteq \mathrm{tr}(g)$
- *The set of all traces of stable functions from X to Y is a qualitative domain, the web of which is $\mathrm{K}X \times |Y|$, which is isomorphic to the set of stable functions from X to Y stably ordered.*

We note $\mathbf{O}$ the qualitative domain which has a singleton as web, say $\{*\}$, the state $\{*\}$ will be noted $\top$. The domain of booleans has a pair $\{T, F\}$ as web, with just $\{T\}$ and $\{F\}$ as non empty states.

If X is a QD, we note $X^\perp$ the set of linear functions from X to $\mathbf{O}$.

Proposition 2 *The set $X^\perp$ endowed with stable order is isomorphic to the coherence space having $|X|$ as web and where a pair $\{a, b\}$ is a state iff $a = b$ or $\{a, b\}$ is not a state of X. This coherence space will also be noted $X^\perp$.*

The elements of $X^\perp$ will also be called linear open sets, and when we write $x \in \alpha$ for $x \in X$ and $\alpha \in X^\perp$ we mean that $\alpha(x) = \top$ or equivalently that $x \cap \alpha \neq \emptyset$. (The previous isomorphism is kept implicit.)

2 From sequentiality to strong stability

In this section, we shall try to give some mathematical motivations for our approach of "sequentiality" through a strong form of stability. Our basic intuition of sequentiality corresponds to the Kahn-Plotkin definition, but we shall simplify it a bit, getting rid of the CDS framework. Our framework captures however the essence of sequentiality.

The domains we consider are qualitative domains[1]. Each domain X is endowed with a certain subqualitative domain X^* of $X^\perp$ which separates X in the following sense: if $x, x' \in X$ are bounded, then they are equal iff they belong exactly to the same elements of X^*. This is equivalent to say that $|X^*| = |X|$. Such a pair (X, X^*) will be noted X and called sequential structure.

Remark that in a stable CDS S a cell c may be viewed as a linear map from S to $\mathbf{O}$, namely the one which maps a state $x \in S$ to $\top$ if c is filled in x and to $\perp$ otherwise. We replace here CDSs by qualitative domains, and then it seems natural to take as cells the elements of a subdomain of the orthogonal.

We can already give a simple definition of sequentiality which is essentially the one of Kahn and Plotkin [6].

Definition 4 *A Scott-continuous function $f : X \to Y$ is sequential iff for all $x \in X$, for all $\beta \in Y^*$ such that $f(x) \notin \beta$, there exists an $\alpha \in X^*$ such that $x \notin \alpha$ and for all $x' \geq x$, if $f(x') \in \beta$ then $x' \in \alpha$.*

According to the usual terminology of sequentiality, when x and β such that $f(x) \notin \beta$ are given, we call sequentiality index of f at x for β any α given by the previous definition.

In other words, this definition means that for any question not answered by $f(x)$, there is a question not answered by x which must be answered by any $x' \geq x$ such that $f(x')$ answeres β.

We shall relate this definition to a notion of linear stability that we define now.

Definition 5 *A finite part A of X will be said to be linearly coherent iff*

$$\forall \alpha \in X^* \quad A \subseteq \alpha \Rightarrow \bigwedge A \in \alpha\ .$$

We note $\mathcal{C}^{\mathrm{L}}(X)$ the set of these subsets of X.
A Scott-continuous function $f : X \to Y$ is linearly stable iff for any $A \in \mathcal{C}^{\mathrm{L}}(X)$ we have $f(A) \in \mathcal{C}^{\mathrm{L}}(Y)$ and $f(\bigwedge A) = \bigwedge f(A)$.

Remark that any linearly stable map is stable. The converse is false as we shall see.

The connection between these notions is very strong indeed.

[1]Indeed the theory may be carried out in the more general framework of DI-domains. For sake of shortness, we do not present here the general theory.

Proposition 3 *A function $f : X \to Y$ is sequential iff it is linearly stable.*

Proof: Let $f : X \to Y$ be sequential. Let $A \in \mathcal{C}^{\mathrm{L}}(X)$. If $f(A)$ was not linearly coherent, we could find a $\beta \in Y^*$ such that $f(A) \in \beta$ and $\bigwedge f(A) \notin \beta$. But $\bigwedge f(A) \geq f(\bigwedge A)$ and thus we would also have $f(\bigwedge A) \notin \beta$. Hence, since f is sequential, there would exist an $\alpha \in X^*$ such that $\bigwedge A \notin \alpha$ and for all $x \geq \bigwedge A$

$$f(x) \in \beta \Rightarrow x \in \alpha \ .$$

This implies $A \subseteq \alpha$ and this is a contradiction. Hence $f(A) \in \mathcal{C}^{\mathrm{L}}(Y)$. If we had $\bigwedge f(A) > f(\bigwedge A)$ we would get the same kind of contradiction since by separation we could find some $\beta \in Y^*$ such that $\bigwedge f(A) \in \beta$ and $f(\bigwedge A) \notin \beta$, henceforth f is linearly stable.

Conversely, let $f : X \to Y$ be any linearly stable map. Let $x \in X$ and let $\beta \in Y^*$ be such that $f(x) \notin \beta$. Let C be the set of all points $c \in X$ compatible with x and minimal such that $f(c) \in \beta$. (This set exists since f is stable, and its elements are pairwise unbounded.) If C is empty, there is nothing to say since this means that there is no $x' \geq x$ such that $f(x') \in \beta$. If C has a unique element c, then we know that $f(x \vee c) \in \beta$ and thus $x \vee c > x$, hence there is an $\alpha \in X^*$ such that $x \notin \alpha$ and $x \vee c \in \alpha$ and thus since α is linear $c \in \alpha$. If C has more than one element it cannot be linearly coherent otherwise $f(C)$ would be a linearly coherent subset of β and we would have $\bigwedge f(C) \in \beta$, that is $f(\bigwedge C) \in \beta$, but this is impossible since the elements of C are minimal. Hence there exists an $\alpha \in X^*$ such that $C \subseteq \alpha$ and $\bigwedge C \notin \alpha$. It remains to prove that $x \notin \alpha$. If it was the case, then x would be greater than one atom a in the trace of α and thus since x is compatible with all elements of C any atom a' of the trace of α such that there exists a $c \in C$ such that $c \geq a'$ would be equal to a and we would have $\bigwedge C \geq a$ and thus $\bigwedge C \in \alpha$ and this would be a contradiction.

Something remains to be proved when C is infinite since we have required linearly coherent sets to be finite. We know however that no finite subset of C with more than one element is linearly coherent, by the previous argument. Let $(C_i)_{i \in \omega}$ be an increasing sequence of finite subsets of C having C as lub. (We know that X is ω-algebraic, since $|X|$ is counatble.) We also assume that $\bigwedge C_0 = \bigwedge C$ and this is possible because all elements of C are compact and thus have a finite number of lower bounds since we are in a QD. Of course we have also for any $i \in \omega$ that $\bigwedge C_i = \bigwedge C$. We define now a family $(L_i^j)_{i,j \in \omega}$ as follows:

$$L_i^0 = \{\alpha \in X^* \mid C_i \subseteq \alpha \ \ \bigwedge C \notin \alpha \text{ and } \alpha \text{ minimal}\}$$

and

$$L_i^{j+1} = \{\alpha \in L_i^j \mid \exists \alpha' \in L_{i+1}^j \ \alpha \leq \alpha'\} \ .$$

Remark that any L_i^j is a non empty finite set and that for any i the sequence $(L_i^j)_{j \in \omega}$ is decreasing and thus has a non empty finite glb L_i. Furthermore, this glb is reached in finite time and thus for any $\alpha \in L_i$ we may find a $\alpha' \in L_{i+1}$ such that $\alpha \leq \alpha'$. So we can find an increasing chain $(\alpha_i)_{i \in \omega}$ with $\alpha_i \in L_i$ for all i. Now let $\alpha = \bigvee_{i \in \omega} \alpha_i$. Since for all i we have $\bigwedge C \notin \alpha_i$ we have $\bigwedge C \notin \alpha$. Furthermore we know that $C_i \subseteq \alpha_i \subseteq \alpha$ for all i and thus $C \subseteq \alpha$. To show that α is a sequentiality index, we proced as above. ∎

Between two Scott domains, it is equivalent to say that a function preserves directed lubs or that it is continuous with respect to Scott topology (that is it preserves Scott open subsets under inverse image). Similarly, between two DI-domains, it is equivalent to say that a function is stable or that it preserves stable open subsets (those of which characteristic map to $\mathbf{O}$ is stable) under inverse image. Here, we have a similar property.

Definition 6 *A Scott-open subset U of X is linearly stable iff its characteristic map from X to $\mathbf{O}$ is linearly stable, or equivalently iff for any $A \in \mathcal{C}^{\mathrm{L}}(X)$, iff $A \subseteq U$, then $\bigwedge A \in U$. We write $\mathcal{O}_S(X)$ for the set of linearly stable open subsets of X.*

Then we have:

Proposition 4 *$f : X \to Y$ is linearly stable iff one of the two following equivalent conditions holds:*

i) For all $V \in \mathcal{O}_S(Y)$, we have $f^{-1}(V) \in \mathcal{O}_S(X)$.

ii) For all $\beta \in Y^*$, we have $f^{-1}(\beta) \in \mathcal{O}_S(X)$.

Proof: If f is linearly stable, then it statisfies (i) since the composite of two linearly stable maps is linearly stable. If it satisfies (i), it satisfies (ii) because any linear open subset of Y is linearly stable. We just have to prove that if f satisfies (ii), it is linearly stable. So assume that (ii) holds for f. Let $A \in \mathcal{C}^{\mathrm{L}}(X)$. Let $\beta \in Y^*$ be such that $f(A) \subseteq \beta$. We have $A \subseteq f^{-1}(\beta)$, and since $f^{-1}(\beta) \in \mathcal{O}_S(X)$, we have $\bigwedge A \in f^{-1}(\beta)$, that is $f(\bigwedge A) \in \beta$, and this implies $\bigwedge f(A) \in \beta$, so $f(A) \in \mathcal{C}^{\mathrm{L}}(Y)$. For the same reason (since Y^* separates Y) we have $\bigwedge f(A) = f(\bigwedge A)$. Let us prove that f is continuous. Let D be a directed subset of X. We know that $f(\bigvee D) \geq \bigvee f(D)$. Let $\beta \in Y^*$ be such that $f(\bigvee D) \in \beta$. That is $\bigvee D \in f^{-1}(\beta)$, so there exists an $x \in D$ such that $x \in f^{-1}(\beta)$, that is $f(x) \in \beta$, and hence $\bigvee f(D) \in \beta$, so $\bigvee f(D) = f(\bigvee D)$ since Y^* separates Y. ∎

To build a model of PCF, the most natural idea would be to take as morphisms linearly stable functions. Actually, this does not give rise to a cartesian closed category for evaluation is not linearly stable as shown by the following counterexample.

Let X and Y be two sequential structures. Then the canonical choice for $(X \times Y)^*$ is $X^* + Y^*$ (indeed, if we aim at a cartesian category, any acceptable choice must be a subspace of that one). Take $X = (\mathbf{B}^3 \to \mathbf{O})$ and $Y = \mathbf{B}^3$. Let $b_1 = (T, F, \bot)$, $b_2 = (F, \bot, T)$ and $b_3 = (\bot, T, F)$. Let f_i (for $i = 1,2,3$) be the elements of X the traces of which are $\{(b_i, \top)\}$. Then in $X \times Y$, the set $A = \{(f_i, b_i)\}_{i=1,2,3}$ is linearly coherent, since $\{b_i\}$ and $\{f_i\}$ are. Indeed, no linear open set contains all elements of the former, and the elements of the latter are pairwise bounded atoms. But evaluation maps all elements of A to $\top$, and its glb to $\bot$, so it is not linearly stable.

3 Strong stability

In this section we generalize the previous situation to an abstract notion of domains endowed with coherence, and we get a cartesian closed category.

Remark that the usual notion of coherence (that is being upper-bounded) is preserved by subsets, this means that if A is coherent and if $B \subseteq A$ then B is coherent too. But linear coherence does not enjoy this property; consider in $\mathbf{B}^3$ the three points $(\bot, T, F)$, $(F, \bot, T)$ and $(T, F, \bot)$. As we have seen they are lineraly coherent. Nevertheless $(\bot, T, F)$ and $(F, \bot, T)$ are not linearly coherent since the linear open set the trace of which is $\{(\bot, \bot, T), (\bot, \bot, F)\}$ contains both but not their lub.

Anyway, it is clear that any singleton is linearly coherent. Furthermore, linear coherence is preserved for the Egli-Milner preorder between subsets.

Definition 7 *If $(D, \leq)$ is a poset and if $A, B \subseteq D$, we say that A is Egli-Milner smaller than B (we write $A \sqsubseteq B$) if*

$$\forall a \in A\ \exists b \in B\ a \leq b \quad \textit{and} \quad \forall b \in B\ \exists a \in A\ a \leq b\ .$$

Proposition 5 *If $A \in \mathcal{C}^{\mathrm{L}}(X)$, $B \sqsubseteq A$ and B is finite then $B \in \mathcal{C}^{\mathrm{L}}(X)$.*

Proof: Let $\alpha \in X^*$ be such that $B \subseteq \alpha$. Since every element of A has a lower bound in B, we have $A \subseteq \alpha$ and hence $\bigwedge A \in \alpha$. So there exists an $a \in \mathrm{tr}(\alpha)$ such that $a \leq \bigwedge A$. Let $x \in B$; since $x \in \alpha$ there exists an $a' \in \mathrm{tr}(\alpha)$ such that $a' \leq x$. But x has an upper bound in A, which is consequently an upper bound of a and a'. Thus $a' = a$ and hence $\bigwedge B \in \alpha$. ∎

In the sequel the previous property and the preservation by directed lub's will be our only requirements about coherence.

If P is a set we note $\mathcal{P}_{\mathrm{fin}}(P)$ the set of its finite subsets.

Definition 8 *A qualitative domain with coherence (QDC) is a qualitative domain X endowed with a certain subset $\mathcal{C}(X)$ of $\mathcal{P}(X)$ which satisfies the following conditions:*

- *Each element of $\mathcal{C}(X)$ is finite.*
- $\forall x \in X \quad \{x\} \in \mathcal{C}(X)$.
- $\forall A \in \mathcal{C}(X)\ \forall B \in \mathcal{P}_{\mathrm{fin}}(X) \quad B \sqsubseteq A \Rightarrow B \in \mathcal{C}(X)$.
- *If $D_1, \ldots, D_n$ are directed subsets of X such that for any family $x_1 \in D_1, \ldots, x_n \in D_n$ we have $\{x_1, \ldots x_n\} \in \mathcal{C}(X)$, then $\{\bigvee D_1, \ldots, \bigvee D_n\} \in \mathcal{C}(X)$.*

Such a subset of $\mathcal{P}(X)$ will be called an acceptable coherence for X.

A strongly stable function from X to Y is a continuous function such that for any $A \in \mathcal{C}(X)$ we have $f(A) \in \mathcal{C}(Y)$ and $\bigwedge f(A) = f(\bigwedge A)$.

Remark that a bounded part of X is always in $\mathcal{C}(X)$ and thus any strongly stable function is stable. We shall often use this fact.

The following is useful:

Proposition 6 *If $f, g : X \to Y$ are continuous and f is strongly stable, and if $g \leq f$, then g is strongly stable.*

Proof: Let $A \in \mathcal{C}(X)$. We have $g(A) \sqsubseteq f(A) \in \mathcal{C}(Y)$ and thus $g(A) \in \mathcal{C}(Y)$. Furthermore, if $x \in A$, we have $g(\bigwedge A) = g(x) \wedge f(\bigwedge A)$. Thus $g(\bigwedge A) = \bigwedge g(A) \wedge \bigwedge f(A) = \bigwedge g(A)$. ∎

If P and Q are two sets and if E is a subset of $P \times Q$, we note E_P (resp. E_Q) the projection of E on P (resp. the projection of E on Q). If $A \subseteq P$ and $B \subseteq Q$, we call pairing of A and B any subset E of $P \times Q$ such that $E_P = A$ and $E_Q = B$.

Proposition 7 *If X and Y are two QDCs, the usual cartesian product $X \times Y$ endowed with the coherence*

$$\mathcal{C}(X \times Y) = \{C \subseteq X \times Y \mid C_X \in \mathcal{C}(X) \text{ and } C_Y \in \mathcal{C}(Y)\}$$

is the cartesian product of X and Y in the category of QDCs and strongly stable maps.

The proof is straightforward. Projections and pairing are defined as usual.

Proposition 8 *Let X and Y be QDCs. The domain of strongly stable functions from X to Y, endowed with the stable ordering, is a qualitative domain. It will be noted $[X \to Y]$.*

Proof: We already know that the space of stable functions from X to Y is a QD and that any subset of the trace of a strongly stable map is the trace of a strongly stable map (this results from propositions 1 and 6). So all we have to prove is that the lub of a directed family $\mathcal{D}$ of strongly stable functions is strongly stable. We already know that this lub is a stable function g defined by

$$g(x) = \bigvee_{f \in \mathcal{D}} f(x)\ .$$

Let $A \in \mathcal{C}(X)$. We prove first that $g(A) \in \mathcal{C}(Y)$. We know that for all $x \in A$ the set $\mathcal{D}(x) = \{f(x) \mid f \in \mathcal{D}\}$ is directed. Let $B = (y_x)_{x \in A}$ be a family of points of Y such that $\forall x \in A\ y_x \in \mathcal{D}(x)$. By finiteness of A, there exists a function $f \in \mathcal{D}$ such that $B \sqsubseteq f(A)$ and hence since f is strongly stable we have $B \in \mathcal{C}(Y)$. So $g(A) \in \mathcal{C}(Y)$.

$$\begin{aligned} g(\bigwedge A) &= (\bigvee \mathcal{D})(\bigwedge A) \\ &= \bigvee_{f \in \mathcal{D}} f(\bigwedge A) \\ &= \bigvee_{f \in \mathcal{D}} \bigwedge_{x \in A} f(x) \\ &\leq \bigwedge_{x \in A} \bigvee_{f \in \mathcal{D}} f(x) \end{aligned}$$

Take any atom $b \leq \bigwedge_{x \in A} \bigvee_{f \in \mathcal{D}} f(x)$. For all $x \in A$ we choose an $f_x \in \mathcal{D}$ such that $b \leq f_x(x)$. Since A is finite and $\mathcal{D}$ is directed, we can find a $f \in \mathcal{D}$ such that $f \geq f_x$ for any $x \in A$. So $b \leq f(x)$ for all $x \in A$ and thus $b \leq \bigvee_{f \in \mathcal{D}} \bigwedge_{x \in A} f(x)$.

Remark that all the axioms of coherence are needed in this proof. ∎

We note Ev the evaluation map $[X \to Y] \times X \to Y$.

Definition 9 *We say that $\mathcal{F} \subseteq [X \to Y]$ is coherent iff it is finite and for all $A \in \mathcal{C}(X)$ and for all pairing $\mathcal{E}$ of $\mathcal{F}$ and A the set $\mathrm{Ev}(\mathcal{E}) = \{f(x) \mid (f, x) \in \mathcal{E}\}$ is in $\mathcal{C}(Y)$ and furthermore*

$$(\bigwedge \mathcal{F})(\bigwedge A) = \bigwedge \mathrm{Ev}(\mathcal{E}) \ .$$

Proposition 9 *The set of all these subsets of $\mathcal{P}([X \to Y])$ is an acceptable coherence for $[X \to Y]$.*

Proof: First, if $f \in [X \to Y]$, then $\{f\} \in \mathcal{C}([X \to Y])$ since f is strongly stable. Next, let $\mathcal{F}$ and $\mathcal{F}'$ be finite subsets of $[X \to Y]$ such that $\mathcal{F} \in \mathcal{C}([X \to Y])$ and $\mathcal{F}' \sqsubseteq \mathcal{F}$. Let $A \in \mathcal{C}(X)$ and $\mathcal{E}'$ be any pairing of $\mathcal{F}'$ and A. Let $\mathcal{E}$ be given by

$$\begin{aligned} \mathcal{E} = \ & \{(f, x) \in \mathcal{F} \times A \mid \\ & \quad \exists f' \in \mathcal{F}'\ f' \leq f \text{ and } (f', x) \in \mathcal{E}'\} \ . \end{aligned}$$

Since $\mathcal{F}' \sqsubseteq \mathcal{F}$, this is a pairing of $\mathcal{F}$ and A. We have $\mathrm{Ev}(\mathcal{E}) \in \mathcal{C}(Y)$ and $\mathrm{Ev}(\mathcal{E}') \sqsubseteq \mathrm{Ev}(\mathcal{E})$ and thus $\mathrm{Ev}(\mathcal{E}') \in \mathcal{C}(Y)$.

Let us prove now that, for any $x \in X$ we have $(\bigwedge \mathcal{F}')(x) = \bigwedge_{f' \in \mathcal{F}'} f'(x)$. The direction $\leq$ is clear. Consider now an atom b of Y such that $\bigwedge_{f' \in \mathcal{F}'} f'(x) \geq b$, that is, for any $f' \in \mathcal{F}'$, $f'(x) \geq b$. For $f' \in \mathcal{F}'$, let $c(f')$ be the unique compact of X such that $c(f') \leq x$ and $(c(f'), b) \in \mathrm{tr}(f')$. We have also

$$(\bigwedge \mathcal{F})(x) = \bigwedge_{f \in \mathcal{F}} f(x) \geq \bigwedge_{f' \in \mathcal{F}'} f'(x) \geq b$$

and thus there is a compact c in X such that $c \leq x$ and $(c, b) \in \mathrm{tr}(\bigwedge \mathcal{F})$, that is $(c, b) \in \mathrm{tr}(f)$ for all $f \in \mathcal{F}$. Now, for any $f' \in \mathcal{F}'$, there exists an $f \in \mathcal{F}$ such that $\mathrm{tr}(f') \subseteq \mathrm{tr}(f)$. For such an f, we have $(c(f'), b) \in \mathrm{tr}(f)$ and $(c, b) \in \mathrm{tr}(f)$. But $c(f')$ and c are both bounded by x and hence they must be equal. So, for any $f' \in \mathcal{F}'$ we have $(c, b) \in \mathrm{tr}(f')$ and we conclude.

Now, let us prove that $(\bigwedge \mathcal{F}')(\bigwedge A) = \bigwedge_{(f', x) \in \mathcal{E}'} f'(x)$.

We already know that

$$(\bigwedge \mathcal{F}')(\bigwedge A) = \bigwedge_{f' \in \mathcal{F}'} f'(\bigwedge A)$$

But, for any $(f', x) \in \mathcal{E}'$ and any $f \in \mathcal{F}$ such that $f' \leq f$ we have $f'(\bigwedge A) = f(\bigwedge A) \wedge f'(x)$. Thus

$$\begin{aligned} & \bigwedge_{f' \in \mathcal{F}'} f'(\bigwedge A) \\ &= \bigwedge_{f' \in \mathcal{F}'} \{f'(x) \wedge f(\bigwedge A) \mid \\ & \qquad f \in \mathcal{F},\ x \in A,\ f \geq f',\ (f', x) \in \mathcal{E}'\} \\ &= \bigwedge_{f' \in \mathcal{F}'} (\bigwedge \{f'(x) \mid x \in A\ (f', x) \in \mathcal{E}'\} \\ & \qquad \wedge \bigwedge \{f(\bigwedge A) \mid f \in \mathcal{F},\ f' \leq f\}) \\ &= \bigwedge_{f' \in \mathcal{F}'} \bigwedge \{f'(x) \mid x \in A\ (f', x) \in \mathcal{E}'\} \\ & \qquad \wedge \bigwedge_{f' \in \mathcal{F}'} \bigwedge \{f(\bigwedge A) \mid f \in \mathcal{F},\ f' \leq f\} \\ &= \bigwedge_{(f', x) \in \mathcal{E}'} f'(x) \wedge \bigwedge_{f \in \mathcal{F}} f(\bigwedge A) \end{aligned}$$

since $\mathcal{F}' \sqsubseteq \mathcal{F}$. Now, since $\mathcal{F}$ is coherent

$$\begin{aligned} \bigwedge_{f' \in \mathcal{F}'} f'(\bigwedge A) &= \bigwedge_{(f', x) \in \mathcal{E}'} f'(x) \wedge \bigwedge_{(f, x) \in \mathcal{E}} f(x) \\ &= \bigwedge_{(f', x) \in \mathcal{E}'} f'(x) \end{aligned}$$

by definition of $\mathcal{E}$.

Last, let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be directed parts of $[X \to Y]$ such that, for any $f_1 \in \mathcal{D}_1, \ldots, f_n \in \mathcal{D}_n$ we have $\{f_1, \ldots, f_n\} \in \mathcal{C}([X \to Y])$. For $1 \leq i \leq n$, let $g_i = \bigvee \mathcal{D}_i$. We prove that $\mathcal{G} = \{g_1, \ldots, g_n\} \in \mathcal{C}([X \to Y])$. Let $A \in \mathcal{C}(X)$, we take a pairing of $\mathcal{G}$ and A that we define as a subset E of $\{1, \ldots, n\} \times A$. First, $\{g_i(x)\}_{(i,x) \in E} \in \mathcal{C}(Y)$. Actually, for any $(i, x) \in E$, let $D_{i,x} = \mathcal{D}_i(x)$. Take any family $(f_{i,x}(x))_{(i,x)\in E}$ in these sets. Since A is finite, we may find functions $f_1 \in \mathcal{D}_1, \ldots, f_n \in \mathcal{D}_n$ such that, for all $x \in A$, $f_i \geq f_{i,x}$, and thus $\{f_{i,x}(x)\}_{(i,x)\in E} \sqsubseteq \{f_i(x)\}_{(i,x)\in E}$. But the latter set is in $\mathcal{C}(Y)$ since $\{f_i\}_{i=1,\ldots,n} \in \mathcal{C}([X \to Y])$. Next, we prove that $(\bigwedge \mathcal{G})(\bigwedge A) = \bigwedge_{(i,x)\in E} g_i(x)$. On one hand

$$\bigwedge_{(i,x)\in E} g_i(x) = \bigwedge_{(i,x)\in E} \left(\bigvee_{f\in\mathcal{D}_i} f(x) \right)$$

call this point u. On the other hand,

$$(\bigwedge \mathcal{G})(\bigwedge A) = \bigvee_{\vec{f}\in\mathcal{D}} \left(\bigwedge_{i=1}^{n} f_i(\bigwedge A) \right)$$

by distributivity, where $\mathcal{D} = \mathcal{D}_1 \times \ldots \times \mathcal{D}_n$. So

$$(\bigwedge \mathcal{G})(\bigwedge A) = \bigvee_{\vec{f}\in\mathcal{D}} \left(\bigwedge_{(i,x)\in E} f_i(x) \right)$$

and call v this point. That $v \leq u$ is clear. Take any atom b of Y such that $b \leq u$, that is

$$\forall (i,x) \in E\ \exists f_{i,x} \in \mathcal{D}_i \quad f_{i,x}(x) \geq b$$

Take such a family $f_{i,x}$. Since A is finite, we may find some family $\vec{f} \in \mathcal{D}$ such that $\forall (i,x) \in E\ f_i \geq f_{i,x}$, and so

$$\exists \vec{f} \in \mathcal{D}. \forall (i,x) \in E \quad f_i(x) \geq b$$

that is $v \geq b$. ∎

From this fact, it results that Ev is a strongly stable map. In fact we have taken the greatest possible coherence on $[X \to Y]$ making Ev strongly stable. The fact that this drastic choice give rise to a cartesian closed category is quite surprising. In some sense, this semantics is the most constrained "stable" (this means that the only condition [2] on morphisms is preservation of some glbs) semantics for PCF.

To conclude that the category is cartesian closed, it remains to prove

Proposition 10 *If $f : Z \times X \to Y$ is a strongly stable function, then for all $z \in Z$ the function $f^z : X \to Y$ defined by $f^z(x) = f(z,x)$ is strongly stable and the function $g : Z \to [X \to Y]$ defined by $g(z) = f^z$ is strongly stable.*

Proof: Actually, we already know the corresponding result for stable functions, and in that case, remember that simply

$$\mathrm{tr}(g) = \{(d,(c,b)) \mid ((d,c),b) \in \mathrm{tr}(f)\}$$

and that, for all $z \in Z$,

$$\mathrm{tr}(f^z) = \{(c,b) \mid \exists d \leq z \quad ((d,c),b) \in \mathrm{tr}(f)\}\ .$$

[2] apart from Scott continuity

The fact that, for any $z \in Z$, the function f^z is strongly stable is clear, since $\{z\} \in \mathcal{C}(Z)$.

We prove now that, for any $C \in \mathcal{C}(Z)$, the following holds:

$$f^{\bigwedge C} = \bigwedge_{z\in C} f^z$$

The direction $\leq$ is clear. Let us prove the converse. Take any $(c,b) \in \mathrm{tr}(\bigwedge_{z\in C} f^z) = \bigcap_{z\in C}\mathrm{tr}(f^z)$. Let $C' = \{d \mid \exists z \in C d \leq z \text{ and } ((d,c),b) \in \mathrm{tr}(f)\}$. We have $C' \sqsubseteq C$ and we know that C' is finite, indeed its cardinality is bounded by the one of C. Actually, if $d, d' \in C'$ are bounded by the same element of C, they must be equal. Thus $C' \in \mathcal{C}(Z)$ and hence we must have $f(\bigwedge C', c) \geq b$. But this implies that C' is a singleton $\{d_0\}$ because the elements of C' are pairwise incompatible and are the minimal points $d \in Z$ such that $f(d,c) \geq b$. Hence we conclude

$$\exists d \forall z \in C\ d \leq z \text{ and } ((d,c),b) \in \mathrm{tr}(f)$$

and thus $(c,b) \in \mathrm{tr}(f^{\bigwedge C})$.

Take any $C \in \mathcal{C}(Z)$. We prove that $g(C) \in \mathcal{C}([X \to Y])$. We take a pairing of $g(C)$ with a $A \in \mathcal{C}(X)$, that is a pairing E of C with A. The fact that $\{f^z(x) \mid (z,x) \in E\} \in \mathcal{C}(Y)$ results from the fact that f is strongly stable. Next, we compute

$$\begin{aligned}
\bigwedge_{(z,x)\in C} f^z(x) &= \bigwedge_{(z,x)\in C} f(z,x) \\
&= f(\bigwedge C, \bigwedge A) \\
&= f^{\bigwedge C}(\bigwedge A) \\
&= (\bigwedge_{z\in C} f^z)(\bigwedge A)
\end{aligned}$$

and this is what we wanted. And we have proven above precisely that $g(\bigwedge C) = \bigwedge g(C)$ so g is a strongly stable function. ∎

To summarize:

Proposition 11 *The category of QDC's with strongly stable functions is cartesian closed.*

Using previous propositions, it is routine to prove this fact. See [7] for categorical details.

In order to get a model of PCF, cartesian closedness is not sufficient, we need furthermore fixpoint operators. They are defined as usual. Precisely, let X be a QDC. By cartesian closedness, each functional $Y_n : [X \to X] \to X$ defined by $Y_n(f) = f^n(\bot)$ is strongly stable. Furthermore the sequence $(Y_n)_{n\in\omega}$ is increasing with respect to stable order (it is already true in the stable case). Its limit is the least fixpoint operator.

The notion of coherence previously defined for function spaces makes the property expressed in proposition 4 false in general. Precisely, in a QDC X we may define in the usual way the set $\mathcal{O}_S(X)$ of strongly stable open sets, and it is false in general that if a function $f : X \to Y$ preserves these open sets under inverse image it is strongly stable. Actually, take $X = \mathbf{B}^3$ and $Y = (X \to \mathbf{O})$, and consider the function $f : X \to Y$ the trace of which is $\{(b_i, (b_i, \top))\}$. This function is not strongly stable. Actually the image of the coherent set $\{b_i\}_{i=1,2,3}$ is $\{(b_i, \top)\}$ which is not coherent, because of the pairing $\{((b_i, \top), b_i)\}_{i=1,2,3}$. But any strongly stable open subset of Y which contains two different elements of the image of f is Y itself, because the $(b_i, \top)$'s are pairwise bounded in Y, and of course $f^{-1}(Y) \in \mathcal{O}_S(X)$. So a strongly stable open subset of Y different from Y contains either none of the (b_i, T)'s and then its inverse image under f is empty, or it contains just one of them, say (b_1, T), and its inverse image under f is the set of upper bounds of b_1. Hence f preserves strongly stable open sets under inverse image. However there is a weakening of proposition 4 which remains true.

Proposition 12 *Let X and Y be two QDC, the latter being endowed with a linear coherence. Then $f : X \to Y$ is strongly stable iff one of the two following equivalent conditions holds:*

i) For all $V \in \mathcal{O}_S(Y)$, we have $f^{-1}(V) \in \mathcal{O}_S(X)$.

ii) For all $\beta \in Y^*$, we have $f^{-1}(\beta) \in \mathcal{O}_S(X)$.

The proof is essentially the same as the one of proposition 4.

This proposition is interesting because, up to uncurryfication, the codomain of any term of PCF is a ground type, and so may be endowed with a linear coherence in a strongly stable semantics of the language.

If X is a qualitative domain, call $\mathcal{C}^s(X)$ the set of its finite bounded subsets. If X and Y are endowed with this notion of coherence (the usual one in stability theory of Berry and Girard), it is plain that the induced coherence on $X \times Y$ is $\mathcal{C}^s(X \times Y)$. A natural question to ask is whether the same thing happens for exponential. The answer is no. This means that even in that case, the functionals of our category are more constrained that in the category of stable functions.

However, the case of two stable functions is interesting:

Proposition 13 *Assume that $\mathcal{C}(X) = \mathcal{C}^s(X)$ and $\mathcal{C}(Y) = \mathcal{C}^s(Y)$. If $\{f, g\} \in \mathcal{C}([X \to Y])$ then f and g are stably bounded.*

Proof: We may consider, for any $x \in X$ the set $\{f(x), g(x)\}$ and we know that it is always in $\mathcal{C}^s(Y)$, and thus $h(x) = f(x) \vee g(x)$ does exist and depends continuously on x (and this is true even for more than two functions). Of course h is an extensional upper bound for f and g. We prove that h is stably greater than g and f, and that it is a stable function. Let $x \leq x'$ be elements of X. We have $h(x) \wedge f(x') = f(x) \vee (g(x) \wedge f(x'))$, but $\{x, x'\} \in \mathcal{C}^s(X)$ and $\{f, g\} \in \mathcal{C}([X \to Y])$, thus $h(x) \wedge f(x') = f(x)$ and so $f \leq h$ and the same holds for g. Next let $\{x, x'\} \in \mathcal{C}^s(X)$. We know that $f(x) = h(x) \wedge f(x \vee x')$ and similarly for x', and the same for g. So

$$\begin{aligned} h(x \wedge x') &= (f(x) \wedge f(x')) \vee (g(x) \wedge g(x')) \\ &= (h(x) \wedge h(x') \wedge f(x \vee x')) \\ &\quad \vee (h(x) \wedge h(x') \wedge g(x \vee x')) \\ &= h(x) \wedge h(x') \wedge h(x \vee x') \end{aligned}$$

and we conclude. ∎

The interesting fact is that this phenomenon is specific to the case of two functions. Actually, consider the three functions from $\mathbf{O}^2$ to $\mathbf{O}$, the trace of which are $\{((T, \bot), T)\}$, $\{((\bot, T), T)\}$ and $\emptyset$. They form a coherent set of functions (in our sense), but are not stably bounded.

As far as coherence is restricted to be pairwise, if it is compatility at ground types, it remains compatibility at all types.

Conclusion

The goal of settling down a new framework for dealing with full abstraction has been achieved, but much work remains to be done. Actually the model of QDC and strongly stable maps is far from beeing fully abstract, because all the counterexamples of [4] page 269 are still present. The first one of these counterexamples can be rejected by a model construction where all domains are QDCs which can be embedded in a corresponding domain of the Scott semantics, and where morphisms are strongly stable functions extendable to morphisms of the Scott model. This is the subject of another paper. However this method does not eliminate the two last Curien's counterexamples and this means that they exist both in the strongly stable and continuous semantics.

In spite of this problem, strongly stable semantics seems to have an interest for itself, since it appears as more canonical than the stable one in the sense that we take at higher order the coherence imposed by evaluation whereas in stable semantics coherence it is always choosen a priori as compatibility.

From a logical point of view the complexity of coherence seems to increase as functionality of types grows. A deeper study of the combinatorial meaning of strong stability could be interesting.

References

[1] G. Berry. "Modèles complètement adéquats et stables des lambda-calculs typés". Thèse d'Etat.

[2] G. Berry and P.L. Curien. "Sequential algorithms on concrete data structures". TCS 20, 1982.

[3] G. Berry, P.L. Curien, J.J. Levy. "Full abstraction for sequential lamguages: the state of the art". Algebraic mehods in Semantics, Cambridge University Press 1985.

[4] P.L. Curien. "Categorical Combinators, Sequential Algorithms and Functional Programming". Research Notes in Theoretical computer Science, Pitman (1986).

[5] J.Y. Girard. "The system F of variable types, fifteen years later". TCS.

[6] G. Kahn and G. Plotkin. "Domaines Concrets". Rapport IRIA-LABORIA 336, 1978.

[7] S. MacLane. "Categories for the Working Mathematician". Springer-Verlag (1971).

[8] R. Milner. "Fully abstract models of typed lambda-calculi". TCS 4, 1977.

[9] G. Plotkin. "LCF considered as a programming language". TCS 5, december 1977, p. 223-256.