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Myhill–Nerode type theory for fuzzy languages and automata☆

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Abstract

The Myhill–Nerode theory is a branch of the algebraic theory of languages and automata in which formal languages and deterministic automata are studied through right congruences and congruences on a free monoid. In this paper we develop a general Myhill–Nerode type theory for fuzzy languages with membership values in an arbitrary set with two distinguished elements 0 and 1, which are needed to take crisp languages in consideration. We establish connections between extensionality of fuzzy languages w.r.t. right congruences and congruences on a free monoid and recognition of fuzzy languages by deterministic automata and monoids, and we prove the Myhill–Nerode type theorem for fuzzy languages. We also prove that each fuzzy language possess a minimal deterministic automaton recognizing it, we give a construction of this automaton using the concept of a derivative automaton of a fuzzy language and we give a method for minimization of deterministic fuzzy recognizers. In the second part of the paper we introduce and study Nerode's and Myhill's automata assigned to a fuzzy automaton with membership values in a complete residuated lattice. The obtained results establish nice relationships between fuzzy languages, fuzzy automata and deterministic automata.

Keywords: Fuzzy automaton; Fuzzy language; Deterministic automaton; Minimal automaton; Derivative automaton; Syntactic monoid; Nerode's automaton; Myhill's automaton; Myhill–Nerode theorem

1. Introduction

The Myhill–Nerode theory is a branch of the algebraic theory of languages and automata in which formal languages and deterministic automata are studied through right congruences and congruences on a free monoid. The central place in this theory is held by the renowned Myhill–Nerode theorem, proved by Myhill in [49] and Nerode in [50]. It provides necessary and sufficient conditions for a language to be regular, which are in terms of right congruences and congruences of finite index on a free monoid. However, the Myhill–Nerode theory not only deal with this theorem, but it also considers many other important topics. In particular, right congruences on a free monoid have shown oneself to be very useful in the proof of existence and construction of the minimal deterministic automaton recognizing a given language, as well as in minimization of deterministic automata. Moreover, congruences on a free monoid have been studied in connection with recognition of languages by monoids, and they have been the starting point for the elegant

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Eilenberg classification theory of regular languages based on syntactic monoids and varieties of finite monoids. For more information about these topics we refer to [16,20,30,53–55,69,80].

The Myhill–Nerode theory has been generalized to different settings, including tree automata and languages (see the discussion in [29]), weighted automata and power series [21], weighted tree automata and tree series [6,43], etc. Various concepts of the Myhill–Nerode theory have been also generalized in the scope of the theory of fuzzy languages and automata. There are numerous papers dealing with syntactic right congruences and derivatives of fuzzy languages, syntactic congruences and syntactic monoids of fuzzy languages, recognition of fuzzy languages by monoids, transition monoids of fuzzy automata, and many other related questions. Myhill–Nerode type theorems for fuzzy languages with membership values in the Gödel structure have been given by Shen [67], Malik et al. [44], and Petković [51].

In the present paper we develop a general Myhill–Nerode type theory for fuzzy languages with membership values in an arbitrary set with two distinguished elements 0 and 1, which are needed to take crisp languages in consideration. These fuzzy languages will be studied from the aspect of their recognition by ordinary deterministic automata equipped with fuzzy sets of terminal states, which will be called deterministic fuzzy recognizers. It is well-known that each right congruence π on a free monoid defines a deterministic automaton $\mathscr{A}_\pi$, called the right congruence automaton associated with π. In Section 3 we study recognition of fuzzy languages by right congruence automata, and by Theorem 3.1 we show that a fuzzy language is recognized by $\mathscr{A}_\pi$ if and only if it is extensional w.r.t. π. It is worth noting that extensionality w.r.t. a crisp equivalence is an instance of a more general notion of extensionality w.r.t. a fuzzy equivalence, a very important notion of the theory of fuzzy sets (cf. [17,18,26–28]). Later, in Section 8, we will also study connections between extensionality w.r.t. fuzzy equivalences and recognition of fuzzy languages by fuzzy automata. Next, by Theorem 3.2 we show that for every fuzzy language f in a free monoid X^* there exists the greatest right congruence on X^* such that f is extensional w.r.t. it, which is called the syntactic right congruence of f. Syntactic right congruences have been already studied by Malik et al. [44] (see also the book [47]), and recently by Petković [51] and Bozapalidis and Louscou-Bozapalidou [9]. The main result of Section 3 is Theorem 3.4 asserting that the right congruence automaton associated with the syntactic right congruence of a given fuzzy language is a minimal deterministic automaton recognizing this fuzzy language.

A problem which one naturally imposes is how to construct the minimal deterministic automaton of a fuzzy language. The minimal deterministic automaton of a crisp language can be constructed by means of the concept of derivatives of this language, introduced by Rabin and Scott [61], Raney [62], and Elgot and Rutledge [22]. In the fuzzy framework, derivatives of fuzzy languages have been studied by Malik et al. [44], Mordeson and Malik [47], Petković [51] and Bozapalidis and Louscou-Bozapalidou [9]. Here, in Section 4, we introduce the concept of the derivative automaton of a fuzzy language, and by Theorem 4.1 we show that the derivative automaton $\mathscr{A}_f$ of a fuzzy language f is a minimal deterministic automaton of f. The main result of this section is Theorem 4.2 which asserts that the derivative automaton $\mathscr{A}_f$ is isomorphic to the accessible part of the direct product of the derivative automata of kernel languages of f. It is also a subdirect product of these automata. This result provides an effective method for construction of the derivative automaton of a fuzzy language, based on simultaneous construction of the derivative automata of its kernel languages. Another important result of Section 4 is Theorem 4.3, which says that a fuzzy language can be recognized by a deterministic finite automaton if and only if it has a finite rank and all its kernel languages (or cut languages) are recognizable. This theorem generalizes related theorems proved by Li and Pedrycz in [37,39] for fuzzy languages with membership values in lattice-ordered monoids and distributive lattices.

In Section 5 we consider another important problem, the minimization problem for deterministic fuzzy recognizers. The basic idea exploited in the papers by Basak and Gupta [2], Cheng and Mo [11], Lei and Li [32], Malik et al. [46] and Petković [52], and in the book by Mordeson and Malik [47], was to reduce the number of states of a fuzzy automaton by computing and merging indistinguishable states, what resembles the minimization algorithm for deterministic automata. However, the term minimization used in the mentioned papers does not mean the usual construction of the minimal one in the set of all fuzzy automata recognizing a given fuzzy language, but just the procedure of computing and merging indistinguishable states which do not necessarily results in a minimal fuzzy automaton. Actually, there are two major differences between minimization of fuzzy and deterministic automata. First, a minimal fuzzy automaton recognizing a given fuzzy language is not necessary unique up to an isomorphism (see Example 3.1). This is also true for non-deterministic automata. On the other hand, minimization of fuzzy automata, as well as of non-deterministic ones, is computationally hard. For that reason it is more interesting to look for state reduction methods which do not necessarily give a minimal fuzzy automaton, but they give a “reasonably” small fuzzy automaton which can be constructed efficiently. Such methods have been studied by Ćirić et al. in [13,14]. However, a minimal deterministic fuzzy recognizer recognizing a given fuzzy language is unique up to an isomorphism, and it can be constructed efficiently. An algorithm for minimization of deterministic fuzzy recognizers with membership values in a distributive lattice has been given in a recent paper by Li and Pedrycz [39]. Here we give an algorithm for minimization of deterministic fuzzy recognizers in a more general setting. For a given deterministic fuzzy recognizer $\mathscr{A}$ recognizing a fuzzy language f by a fuzzy set of terminal states τ, by Theorem 5.1 we show that the factor automaton $\mathscr{A}_\pi$ w.r.t. a congruence π on $\mathscr{A}$ recognizes the same language f if and only if τ is extensional w.r.t. π. Theorem 5.2 shows that there exists the greatest congruence π_τ on $\mathscr{A}$ such τ is extensional w.r.t. it, and if $\mathscr{A}$ is accessible, in Theorem 5.3 we prove that the factor automaton $\mathscr{A}_{\pi_\tau}$ w.r.t. a congruence π_τ is isomorphic to the derivative automaton $\mathscr{A}_f$, i.e., that it is a minimal deterministic automaton of f. This means that minimization of a deterministic fuzzy recognizer $\mathscr{A}$ amounts to construction of the greatest congruence contained in an equivalence $\ker \tau$, and hence, for minimization of deterministic fuzzy recognizers we can use the same algorithms as those used in minimization of crisp deterministic recognizers.

In Section 6 we study recognition of fuzzy languages by monoids, as well as syntactic congruences and syntactic monoids of fuzzy languages. As we have mentioned at the beginning of this Introduction, syntactic congruences and syntactic monoids of crisp languages play a very important role in the theory of crisp languages and, in particular, they have been the starting point for the elegant Eilenberg classification theory of regular languages based on syntactic monoids and varieties of finite monoids. Syntactic congruences of fuzzy languages have been studied by Shen [67], Malik et al. [44] Mordeson and Malik [47], Petković [51] and Bozapalidis and Louscou-Bozapalidou [8], and Petković [51] and Bozapalidis and Louscou-Bozapalidou [8,9] also studied recognition of fuzzy languages by monoids. Using a methodology similar to the one used in Section 3, we characterize recognizability of fuzzy languages in X^* by factor monoids of X^* in terms of their extensionality w.r.t. congruences on X^* (Theorem 6.1), and we show that for each fuzzy language f in X^* there exists the greatest congruence π_f on X^* such that f is extensional w.r.t. it (Theorem 6.2). The congruence π_f is called the syntactic congruence of f, and the corresponding factor monoid is called a syntactic monoid of f, and it is denoted by $\mathrm{Syn}(f)$. Then by Theorem 6.4 we prove that the syntactic monoid $\mathrm{Syn}(f)$ of a fuzzy language f is a minimal monoid of f, by Theorem 6.6 we prove that $\mathrm{Syn}(f)$ is isomorphic to the transition monoid of the derivative automaton $\mathscr{A}_f$, and by Theorem 6.9 we show that $\mathrm{Syn}(f)$ is a subdirect product of syntactic monoids of the kernel languages of f. By this subdirect representation it follows that the syntactic monoid of f can be constructed by simultaneous construction of the syntactic monoids of kernel languages of f. Finally, Theorem 6.10 asserts that a fuzzy language can be recognized by a finite monoid if and only if it has a finite rank and all its kernel languages can be recognized by finite monoids.

In Section 7 we collect some results from the previous sections and we state the Myhill–Nerode type theorem for fuzzy languages (Theorem 7.1). Among the other things, this theorem asserts that a fuzzy language can be recognized by a deterministic finite automaton if and only if it can be recognized by a finite monoid. It is worth noting that Bozapalidis and Louscou-Bozapalidou [8,9] studied recognizability of fuzzy languages by finite monoids in connection with their recognizability by certain types of fuzzy automata. If the structure of membership values is taken to be $\mathscr{L} = ([0,1], \wedge, \vee, \otimes, 0, 1)$, where $\otimes$ is the Gödel, Łukasiewicz or drastic t-norm, they proved that recognizability by finite monoids is equivalent to recognizability by fuzzy finite automata over $\mathscr{L}$. In all of these three cases the algebra $([0,1], \vee, \otimes, 0, 1)$ is a locally finite semiring, so this result can be viewed as a consequence of our result asserting that recognizability of a fuzzy language by a finite monoid is equivalent to recognizability by a deterministic finite automaton, and a result by Li and Pedrycz [37], asserting that recognizability of a fuzzy language by a fuzzy finite automaton over $\mathscr{L}$ is equivalent to recognizability by a deterministic finite automaton if and only if the semiring reduct of $\mathscr{L}$, w.r.t. join and multiplication, is a locally finite semiring.

In Sections 8–11 we consider fuzzy languages and automata in such a way that the structure $\mathscr{L}$ of membership values is required to be a complete residuated lattice. The concept of a right congruence automaton, considered in Section 3, has been generalized by Ignjatović et al. [25] to the fuzzy framework. It has been shown that each fuzzy right congruence E on a free monoid X^* determines a fuzzy automaton $\mathscr{A}_E$, called the fuzzy right congruence automaton associated with E. In [25] some basic properties of fuzzy right congruence automata have been described, and here, in Section 8, we give some new properties and new proofs of some results from [25]. The main result of this section is Theorem 8.1 asserting that a fuzzy right congruence automaton $\mathscr{A}_E$ recognizes a fuzzy language f if and only if f is extensional w.r.t. E.

Ignjatović et al. in [25] also proved that each initial fuzzy automaton $\mathscr{A}$ with a fuzzy set of initial states σ defines a fuzzy right congruence N_σ on the underlying free monoid X^*, called the Nerode's fuzzy right congruence of $\mathscr{A}$. Its crisp part $\widehat{N}_\sigma$ is a right congruence on X^*, and it is called the Nerode's right congruence of $\mathscr{A}$. In this paper, in Section 9, we show that every fuzzy automaton $\mathscr{A}$ determines a fuzzy congruence $M_{\mathscr{A}}$ on the free monoid X^*, which is called the Myhill's fuzzy congruence of $\mathscr{A}$, and its crisp part $\widehat{M}_{\mathscr{A}}$ is a congruence on X^*, called the Myhill's congruence of $\mathscr{A}$. By Theorem 9.4 we prove that the factor monoid of X^* w.r.t. the Myhill's congruence of $\mathscr{A}$ is isomorphic to the transition monoid of $\mathscr{A}$.

In Section 10 we study the right congruence automaton associated with the Nerode's right congruence of an initial fuzzy automaton $\mathscr{A}$ with a fuzzy set of initial states σ, which is called the Nerode's automaton of $\mathscr{A}$. The Nerode's automaton of $\mathscr{A}$ has been already studied in [25] and it was proved that it is isomorphic to a deterministic automaton $\mathscr{A}_\sigma$ obtained by determinization of $\mathscr{A}$ using the so-called accessible fuzzy subset construction. By Theorem 10.1 we restate a result from [25] which provides necessary and sufficient conditions for the Nerode's right congruence of an initial fuzzy automaton $\mathscr{A}$ to have a finite index, i.e., to provide finiteness of $\mathscr{A}_\sigma$. In particular, we prove that $\mathscr{A}_\sigma$ is finite if and only if every fuzzy language recognized by $\mathscr{A}$ can be recognized by a deterministic finite automaton. Theorem 10.2 shows that the automaton $\mathscr{A}_\sigma$ is isomorphic to the accessible part of the direct product of derivative automata of fuzzy languages $f^{(\sigma,a)}$, $a \in A$, where $f^{(\sigma,a)}$ is the fuzzy language which $\mathscr{A}$ recognizes by the crisp set $\{a\}$ of terminal states, and Theorem 10.3 says that $\mathscr{A}_\sigma$ is the minimal deterministic automaton which recognizes all fuzzy languages which can be recognized by $\mathscr{A}$. Similarly, in Section 11 we define the concept of a Myhill's automaton of a fuzzy automaton as the right congruence automaton associated with the Myhill's congruence of $\mathscr{A}$. By Theorem 11.3 we determine necessary and sufficient conditions for the Myhill's congruence $\widehat{M}_{\mathscr{A}}$ of a fuzzy automaton $\mathscr{A}$ to have a finite index. In particular, we show $\widehat{M}_{\mathscr{A}}$ has a finite index if and only if the transition monoid of $\mathscr{A}$ is finite, and also, if and only if every fuzzy language recognized by $\mathscr{A}$ can be recognized by a deterministic finite automaton. Then we give an algorithm for construction of the Myhill's automaton of a fuzzy automaton (Algorithm 11.1), by Theorem 11.4 we represent the Myhill's automaton of $\mathscr{A}$ as the accessible part of the direct product of derivative automata of fuzzy languages $f^{(a,b)}$, $(a,b) \in A^2$, where $f^{(a,b)}$ is the fuzzy language which $\mathscr{A}$ recognizes by the crisp set $\{a\}$ of initial states and the crisp set $\{b\}$ of terminal states, and by Theorem 11.5 we represent it as the accessible part of the direct product of Nerode's automata $\mathscr{A}_{\widehat{N}_a}$, $a \in A$. Finally, by Theorem 11.6 we show that the Myhill's automaton of $\mathscr{A}$ is the minimal deterministic automaton recognizing all fuzzy languages which can be recognized by $\mathscr{A}$.

2. Preliminaries

2.1. Fuzzy sets and relations

Fuzzy sets considered in this paper will take their membership values in two kinds of structures. In the first part of the paper, consisting of Sections 3–7, we will work with fuzzy sets taking membership values in a structure $\mathscr{L} = (L, 0, 1)$, where L is an arbitrary set, and 0 and 1 are two distinguished elements of L, which we need to take crisp languages into consideration. We do not impose any other requirement on $\mathscr{L}$.

In the second part, consisting of Sections 8–11, we will use complete residuated lattices as the structures of membership values. A *residuated lattice* is an algebra $\mathscr{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ such that

(L1) $(L, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,
(L2) $(L, \otimes, 1)$ is a commutative monoid with the unit 1,
(L3) $\otimes$ and $\to$ form an *adjoint pair*, i.e., they satisfy the *adjunction property*: for all $x, y, z \in L$,

$$x \otimes y \le z \iff x \le y \to z. \tag{1}$$

If, in addition, $(L, \wedge, \vee, 0, 1)$ is a complete lattice, then $\mathscr{L}$ is called a *complete residuated lattice*.

The operations $\otimes$ (called *multiplication*) and $\to$ (called *residuum*) are intended for modelling the conjunction and implication of the corresponding logical calculus, and supremum ($\vee$) and infimum ($\wedge$) are intended for modelling of the existential and general quantifier, respectively. An operation $\leftrightarrow$ defined by

$$x \leftrightarrow y = (x \to y) \wedge (y \to x), \tag{2}$$

called *biresiduum* (or *biimplication*), is used for modelling the equivalence of truth values. It can be easily verified that with respect to $\le$, $\otimes$ is isotonic in both arguments, and $\to$ is isotonic in the second and antitonic in the first argument.

Emphasizing their monoidal structure, in some sources residuated lattices are called integral, commutative, residuated ℓ-monoids [24].

It can be easily verified that with respect to $\leq$, $\otimes$ is isotonic in both arguments, $\to$ is isotonic in the second and antitonic in the first argument, and for all $x, y, z, x_1, x_2, y_1, y_2 \in L$ and $\{x_i\}_{i\in I}, \{y_i\}_{i\in I} \subseteq L$, the following hold:

$$x \leftrightarrow y \leq x \otimes z \leftrightarrow y \otimes z, \tag{3}$$

$$(x_1 \leftrightarrow y_1) \otimes (x_2 \leftrightarrow y_2) \leq (x_1 \otimes x_2) \leftrightarrow (y_1 \otimes y_2), \tag{4}$$

$$\left(\bigvee_{i\in I} x_i\right) \otimes x = \bigvee_{i\in I}(x_i \otimes x), \tag{5}$$

$$\bigwedge_{i\in I}(x_i \leftrightarrow y_i) \leq \left(\bigvee_{i\in I} x_i\right) \leftrightarrow \left(\bigvee_{i\in I} y_i\right). \tag{6}$$

For other properties of complete residuated lattices we refer to [3,5,24].

The most studied and applied structures of truth values, defined on the real unit interval $[0, 1]$ with $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$, are the *Łukasiewicz structure* ($x \otimes y = \max(x + y - 1, 0)$, $x \to y = \min(1 - x + y, 1)$), the *product structure* ($x \otimes y = x \cdot y$, $x \to y = 1$ if $x \leq y$ and $= y/x$ otherwise) and the *Gödel structure* ($x \otimes y = \min(x, y)$, $x \to y = 1$ if $x \leq y$ and $= y$ otherwise). More generally, an algebra $([0, 1], \wedge, \vee, \otimes, \to, 0, 1)$ is a complete residuated lattice if and only if $\otimes$ is a left-continuous t-norm and the residuum is defined by $x \to y = \vee\{u \in [0, 1] \mid u \otimes x \leq y\}$. Another important set of truth values is the set $\{a_0, a_1, \ldots, a_n\}$, $0 = a_0 < \cdots < a_n = 1$, with $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and $a_k \to a_l = a_{\min(n-k+l,n)}$. A special case of the latter algebras is the two-element Boolean algebra of classical logic with the support $\{0, 1\}$. The only adjoint pair on the two-element Boolean algebra consists of the classical conjunction and implication operations.

Let $\mathscr{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ be a complete residuated lattice. Omitting the infimum and residuum operations we obtain an algebra $\mathscr{L}^* = (L, \vee, \otimes, 0, 1)$ which is a commutative semiring, i.e., $(L, \vee, 0)$ and $(L, \otimes, 1)$ are commutative monoids, $\otimes$ distributes over $\vee$, and $0 \otimes x = 0$, for any $x \in L$. We call $\mathscr{L}^*$ the *semiring reduct* of $\mathscr{L}$. Recall that an algebra is called *locally finite* if any its finitely generated subalgebra is finite. We have that a semiring $\mathscr{L}^*$ is locally finite if and only if both monoids $(L, \vee, 0)$ and $(L, \otimes, 1)$ are locally finite. Since $(L, \vee, 0)$ is a semilattice, and every semilattice is locally finite, we also have that $\mathscr{L}^*$ is locally finite if and only if the monoid $(L, \otimes, 1)$ is locally finite (cf. [19,37]).

Let $\mathscr{L}$ be one of two structures mentioned at the beginning of this section. A *fuzzy subset* of a set A with membership values in $\mathscr{L}$, or a *fuzzy subset* of A *over* $\mathscr{L}$, or simply a *fuzzy subset* of A, is any mapping from A into L. The set of all fuzzy subsets of A will be denoted by $\mathscr{F}(A)$. Ordinary crisp subsets of A are considered as fuzzy subsets of A taking membership values in the set $\{0, 1\} \subseteq L$. Let f be a fuzzy subset of a set A. The *crisp part* of f, in notation $\widehat{f}$, is a crisp subset of A defined by $\widehat{f} = \{x \in A \mid f(x) = 1\}$. We also consider $\widehat{f}$ as a mapping $\widehat{f} : A \to L$ defined by $\widehat{f}(a) = 1$, if $f(a) = 1$, and $\widehat{f}(a) = 0$, if $f(a) < 1$. Note that many authors used the name "kernel" instead of "crisp part", but we use the name "kernel" in its usual meaning—for the crisp equivalence associated in a natural way with a mapping, i.e., the *kernel* of f, in notation $\ker f$, is a crisp equivalence on A defined by $\ker f = \{(x, y) \in A \times A \mid f(x) = f(y)\}$. The *image* of f, in notation $\operatorname{Im} f$, is a subset of L given by $\operatorname{Im} f = \{f(a) \mid a \in A\}$, and the cardinality of $\operatorname{Im} f$ is called the *rank* of f, and it is denoted by $\operatorname{ran} f$. For $f, g \in \mathscr{F}(A)$, the *equality* of f and g is defined as the usual equality of mappings, i.e., $f = g$ if and only if $f(x) = g(x)$, for every $x \in A$.

Further, let $\mathscr{L}$ be a complete residuated lattice. For $f, g \in \mathscr{F}(A)$, the *inclusion* $f \leq g$ is also defined pointwise: $f \leq g$ if and only if $f(x) \leq g(x)$, for every $x \in A$. Endowed with this partial order $\mathscr{F}(A)$ forms a complete residuated lattice, in which the meet (intersection) $\bigwedge_{i\in I} f_i$ and the join (union) $\bigvee_{i\in I} f_i$ of an arbitrary family $\{f_i\}_{i\in I}$ of fuzzy subsets of A are mappings from A into L defined by

$$\left(\bigwedge_{i\in I} f_i\right)(x) = \bigwedge_{i\in I} f_i(x), \quad \left(\bigvee_{i\in I} f_i\right)(x) = \bigvee_{i\in I} f_i(x),$$

and the product $f \otimes g$ is a fuzzy subset defined by $f \otimes g(x) = f(x) \otimes g(x)$, for every $x \in A$.

A *fuzzy relation* on A is any mapping from $A \times A$ into L, that is to say, any fuzzy subset of $A \times A$, and the equality, inclusion, joins, meets and ordering of fuzzy relations are defined as for fuzzy sets. For fuzzy relations R and S on A, their *composition* $R \circ S$ is a fuzzy relation on A defined by

$$(R \circ S)(a, b) = \bigvee_{c \in A} R(a, c) \otimes S(c, b), \tag{7}$$

for all $a, b \in A$, and for a fuzzy subset f of A and a fuzzy relation R on A, the *compositions* $f \circ R$ and $R \circ f$ are fuzzy subsets of A defined by

$$(f \circ R)(a) = \bigvee_{b \in A} f(b) \otimes R(b, a), \quad (R \circ f)(a) = \bigvee_{b \in A} R(a, b) \otimes f(b), \tag{8}$$

for any $a \in A$. Finally, for fuzzy subsets f and g of A we write

$$f \circ g = \bigvee_{a \in A} f(a) \otimes g(a). \tag{9}$$

The value $f \circ g$ can be interpreted as the "degree of overlapping" of f and g. We know that the composition of fuzzy relations is associative, and we can also easily verify that

$$(f \circ R) \circ S = f \circ (R \circ S), \quad (f \circ R) \circ g = f \circ (R \circ g), \tag{10}$$

for arbitrary fuzzy subsets f and g of A, and fuzzy relations R and S on A, and hence, the parentheses in (10) can be omitted. Note also that if A is a finite set with n elements, then R and S can be treated as $n \times n$ fuzzy matrices over $\mathscr{L}$ and $R \circ S$ is the matrix product, whereas $f \circ R$ can be treated as the product of a $1 \times n$ matrix f and an $n \times n$ matrix R, and $R \circ f$ as the product of an $n \times n$ matrix R and an $n \times 1$ matrix f^t (the transpose of f).

A fuzzy relation E on A is

(R) *reflexive* if $E(a, a) = 1$, for every $a \in A$;
(S) *symmetric* if $E(a, b) = E(b, a)$, for all $a, b \in A$;
(T) *transitive* if $E(a, b) \otimes E(b, c) \le E(a, c)$, for all $a, b, c \in A$.

A reflexive, symmetric and transitive fuzzy relation on A is called a *fuzzy equivalence relation*, or just a *fuzzy equivalence*, on A. For a fuzzy equivalence E on A and $a \in A$ we define a fuzzy subset E_a of A by

$$E_a(x) = E(a, x) \quad \text{for every } x \in A,$$

and we call E_a a *fuzzy equivalence class*, or just an *equivalence class*, of E determined by the element a. The set $A/E = \{E_a \mid a \in A\}$ is called the *factor set* of A with respect to E (cf. [3,12]). Cardinality of the factor set A/E, in notation $\mathrm{ind}(E)$, is called the *index* of E. The same notation we use for crisp equivalences, i.e., for an equivalence π on A, the related factor set is denoted by A/π, the equivalence class of an element $a \in A$ is denoted by π_a, and the index of π is denoted by $\mathrm{ind}(\pi)$.

The following properties of fuzzy equivalence relations will be useful in later work.

Lemma 2.1. *Let E be a fuzzy equivalence on a set A and let $\widehat{E}$ be its crisp part. Then $\widehat{E}$ is a crisp equivalence on A, and for any $a, b \in A$ the following conditions are equivalent*:

(i) $E(a, b) = 1$;
(ii) $E_a = E_b$;
(iii) $\widehat{E}_a = \widehat{E}_b$.

Consequently, $\mathrm{ind}(E) = \mathrm{ind}(\widehat{E})$.

Note that $\widehat{E}_a$ denotes the crisp equivalence class of $\widehat{E}$ determined by a.

2.2. Deterministic and fuzzy automata

A *deterministic automaton* is a triple $\mathscr{A} = (A, X, \delta)$, where A and X are sets, which are called, respectively, the *set of states* and the *input alphabet*, and $\delta : A \times X \to A$ is an ordinary crisp mapping, called the *transition function*. The input alphabet X will be always finite, but for methodological reasons we will allow the set of states A to be infinite. A deterministic automaton whose set of states is finite is called a *deterministic finite automaton*. Let X^* denote the free monoid over the alphabet X, and let $e \in X^*$ be the empty word. The mapping δ can be extended up to a mapping $\delta^* : A \times X^* \to A$ as follows: $\delta^*(a, e) = a$, for every $a \in A$, and $\delta^*(a, ux) = \delta(\delta^*(a, u), x)$, for all $a \in A$, $u \in X^*$ and $x \in X$. Without danger of confusion, δ^* is also called the *transition function* of $\mathscr{A}$. The transition function δ^* satisfies

$$\delta^*(a, uv) = \delta^*(\delta^*(a, u), v), \tag{11}$$

for all $a \in A$ and $u, v \in X^*$. Let $\mathscr{A} = (A, X, \delta)$ be a deterministic automaton, let $A' \subseteq A$ and let δ' be the restriction of δ to $A' \times X$. If for all $a \in A'$ and $x \in X$ we have that $\delta(a, x) \in A'$, then $\delta' : A' \times X \to A'$, and $\mathscr{A}' = (A', X, \delta')$ is a deterministic automaton, called a *subautomaton* of $\mathscr{A}$. Often we say simply that A' is a subautomaton of $\mathscr{A}$.

An *initial deterministic automaton* is a quadruple $\mathscr{A} = (A, a_0, X, \delta)$, where (A, X, δ) is a deterministic automaton and a_0 is a state in A, called the *initial state* of $\mathscr{A}$. Let $\mathscr{A} = (A, a_0, X, \delta)$ be an initial deterministic automaton. A state $a \in A$ is called *accessible* if there exists $u \in X^*$ such that $a = \delta^*(a_0, u)$. The set of all accessible states of $\mathscr{A}$ is called the *accessible part* of $\mathscr{A}$, and an initial deterministic automaton whose all states are accessible is called an *accessible deterministic automaton*.

Let $\mathscr{A} = (A, X, \delta_A)$ and $\mathscr{B} = (B, X, \delta_B)$ be two deterministic automata with the same input alphabet. A mapping $\phi : A \to B$ is a *homomorphism* of $\mathscr{A}$ to $\mathscr{B}$ if $\phi(\delta_A(a, x)) = \delta_B(\phi(a), x)$, for all $a \in A$ and $x \in X$, or equivalently, if $\phi(\delta^*_A(a, u)) = \delta^*_B(\phi(a), u)$, for all $a \in A$ and $u \in X^*$. If, in addition, ϕ is surjective, then it is called an *epimorphism*, and $\mathscr{B}$ is called a *homomorphic image* of $\mathscr{A}$, and if ϕ is bijective, then it is called an *isomorphism*, and $\mathscr{A}$ and $\mathscr{B}$ are said to be *isomorphic* deterministic automata. Furthermore, if $\mathscr{A} = (A, a_0, X, \delta_A)$ and $\mathscr{B} = (B, b_0, X, \delta_B)$ are initial deterministic automata, then a mapping $\phi : A \to B$ is a homomorphism if $\phi(\delta_A(a, x)) = \delta_B(\phi(a), x)$, for all $a \in A$ and $x \in X$, and $\phi(a_0) = b_0$.

Let $\mathscr{A} = (A, X, \delta)$ be a deterministic automaton. An equivalence relation π on A is a *congruence* on $\mathscr{A}$ if $(a, b) \in \pi$ implies $(\delta(a, x), \delta(b, x)) \in \pi$, for all $a, b \in A$ and $x \in X$, or equivalently, if $(a, b) \in \pi$ implies $(\delta^*(a, u), \delta^*(b, u)) \in \pi$, for all $a, b \in A$ and $u \in X^*$. If π is a congruence on $\mathscr{A}$ and $A_\pi = A/\pi$, then we can define a mapping $\delta_\pi : A_\pi \times X \to A_\pi$ by $\delta_\pi(\pi_a, x) = \pi_{\delta(a,x)}$, for any $a \in A$ and $x \in X$, what means that the triple $\mathscr{A}_\pi = (A_\pi, X, \delta_\pi)$ is a deterministic automaton, called the *factor automaton* of $\mathscr{A}$ w.r.t. π. In particular, if $\mathscr{A} = (A, a_0, X, \delta)$ is an initial deterministic automaton, then the factor automaton of $\mathscr{A}$ w.r.t. π is $\mathscr{A}_\pi = (A_\pi, \pi_{a_0}, X, \delta_\pi)$.

Let $\mathscr{A}_i = (A_i, a_i^0, X, \delta_i)$, $i \in I$, be a family of initial deterministic automata. The *direct product* of automata $\mathscr{A}_i$, $i \in I$, is an initial deterministic automaton $\mathscr{A} = (A, a_0, X, \delta)$, where

$$A = \prod_{i\in I} A_i = \left\{ a : I \to \bigcup_{i\in I} A_i \,\middle|\, (\forall i \in I)\, a(i) \in A_i \right\},$$

and $a_0 \in A$ and $\delta : A \times X \to A$ are defined by

$$a_0(i) = a_i^0, \quad \delta(a, x)(i) = \delta_i(a(i), x),$$

for any $i \in I$, $a \in A$ and $x \in X$. For any $i \in I$ let a mapping $\mathrm{pr}_i : A \to A_i$ be defined by $\mathrm{pr}_i(a) = a(i)$, for each $a \in A$. Then pr_i is an epimorphism, called the *projection homomorphism* of $\mathscr{A}$ onto $\mathscr{A}_i$. An initial deterministic automaton is a *subdirect product* of automata $\mathscr{A}_i$, $i \in I$, if it is isomorphic to a subautomaton $\mathscr{A}' = (A', a_0', X, \delta')$ of the direct product of automata $\mathscr{A}_i$, $i \in I$, having the property $\mathrm{pr}_i(A') = A_i$, for each $i \in I$. It is worth noting that direct and subdirect products of automata have a natural interpretation as parallel connections of automata.

In the sequel, let $\mathscr{L}$ be one of two structures mentioned at the beginning of the previous subsection. For a free monoid X^* over an alphabet X, by a *fuzzy language* in X^* we mean any fuzzy subset of X^*. A fuzzy recognizer $\mathscr{A} = (A, a_0, X, \delta, \tau)$, where (A, a_0, X, δ) is an ordinary deterministic automaton with a crisp initial state $a_0 \in A$ and τ is a fuzzy set of terminal states, will be called a *deterministic fuzzy recognizer*. A deterministic fuzzy recognizer $\mathscr{A}$ recognizes a fuzzy language $f \in \mathscr{F}(X^*)$ if for any $u \in X^*$ we have

$$f(u) = \tau(\delta^*(a_0, u)). \tag{12}$$

In the rest of this subsection the structure $\mathscr{L}$ of membership values is required to be a complete residuated lattice. By a *fuzzy automaton over* $\mathscr{L}$, or simply a *fuzzy automaton*, a triple $\mathscr{A} = (A, X, \delta)$ is meant, where A and X are sets, called the *set of states* and the *input alphabet*, and $\delta : A \times X \times A \to L$ is a fuzzy subset of $A \times X \times A$, called the *fuzzy transition function*. We can interpret $\delta(a, x, b)$ as the degree to which an input letter $x \in X$ causes a transition from a state $a \in A$ into a state $b \in A$. The input alphabet X will be always finite, but for methodological reasons we will allow the set of states A to be infinite. A fuzzy automaton whose set of states is finite is called a *fuzzy finite automaton*.

The mapping δ can be extended up to a mapping $\delta^* : A \times X^* \times A \to L$ as follows: If $a, b \in A$, then

$$\delta^*(a, e, b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise}, \end{cases} \tag{13}$$

and if $a, b \in A$, $u \in X^*$ and $x \in X$, then

$$\delta^*(a, ux, b) = \bigvee_{c \in A} \delta^*(a, u, c) \otimes \delta(c, x, b). \tag{14}$$

By (5) and Theorem 3.1 [37] (see also [56,57,59]), we have that

$$\delta^*(a, uv, b) = \bigvee_{c \in A} \delta^*(a, u, c) \otimes \delta^*(c, v, b), \tag{15}$$

for all $a, b \in A$ and $u, v \in X^*$, i.e., if $w = x_1 \cdots x_n$, for $x_1, \ldots, x_n \in X$, then

$$\delta^*(a, w, b) = \bigvee_{(c_1, \ldots, c_{n-1}) \in A^{n-1}} \delta(a, x_1, c_1) \otimes \delta(c_1, x_2, c_2) \otimes \cdots \otimes \delta(c_{n-1}, x_n, b). \tag{16}$$

Intuitively, the product $\delta(a, x_1, c_1) \otimes \delta(c_1, x_2, c_2) \otimes \cdots \otimes \delta(c_{n-1}, x_n, b)$ represents the degree to which the input word w causes a transition from a state a into a state b through the sequence of intermediate states $c_1, \ldots, c_{n-1} \in A$, and $\delta^*(a, w, b)$ represents the supremum of degrees of all possible transitions from a into b caused by w. Also, we can visualize a fuzzy finite automaton $\mathscr{A}$ representing it as a labelled directed graph whose nodes are states of $\mathscr{A}$, and an edge from a node a into a node b is labelled by pairs of the form $x/\delta(a, x, b)$, for any $x \in X$, as we will do in examples given in this paper.

If for any $u \in X^*$ we define a fuzzy relation δ_u on A by

$$\delta_u(a, b) = \delta^*(a, u, b), \tag{17}$$

for all $a, b \in A$, called the *transition relation* determined by u, then equality (15) can be written as

$$\delta_{uv} = \delta_u \circ \delta_v, \tag{18}$$

for all $u, v \in X^*$. If δ is a crisp subset of $A \times X \times A$, i.e., $\delta : A \times X \times A \to \{0, 1\}$, then $\mathscr{A}$ is an ordinary crisp non-deterministic automaton, and if δ is a mapping of $A \times X$ into A, then $\mathscr{A}$ is an ordinary deterministic automaton. Evidently, in these two cases we have that δ^* is also a crisp subset of $A \times X^* \times A$, and a mapping of $A \times X^*$ into A, respectively.

An *initial fuzzy automaton* is a quadruple $\mathscr{A} = (A, \sigma, X, \delta)$, where (A, X, δ) is a fuzzy automaton and σ is a fuzzy subset of A, called the fuzzy set of *initial states*, and a *fuzzy recognizer* is defined as a five-tuple $\mathscr{A} = (A, \sigma, X, \delta, \tau)$, where (A, σ, X, δ) is as above, and τ is a fuzzy subset of A, called the fuzzy set of *terminal states*. A fuzzy recognizer $\mathscr{A} = (A, \sigma, X, \delta, \tau)$ *recognizes* a fuzzy language $f \in \mathscr{F}(X^*)$ if for any $u \in X^*$ we have

$$f(u) = \bigvee_{a, b \in A} \sigma(a) \otimes \delta^*(a, u, b) \otimes \tau(b). \tag{19}$$

In other words, equality (19) means that the membership degree of the word u to the fuzzy language f is equal to the degree to which $\mathscr{A}$ recognizes or accepts the word u. Using notation from (8), and the second equality in (10), we can state (19) as

$$f(u) = \sigma \circ \delta_u \circ \tau. \tag{20}$$

The unique fuzzy language recognized by a fuzzy recognizer $\mathscr{A}$ is denoted by $L(\mathscr{A})$.

An initial fuzzy automaton $\mathscr{A} = (A, \sigma, X, \delta)$ is said to recognize a fuzzy language $f \in \mathscr{F}(X^*)$ if there exists a fuzzy set $\tau \in \mathscr{F}(A)$ such that the fuzzy recognizer $(A, \sigma, X, \delta, \tau)$ recognizes f. The class of all fuzzy languages in $\mathscr{F}(X^*)$ which can be recognized by this initial fuzzy automaton is denoted by $\mathrm{Rec}(\mathscr{A}, \sigma)$. Similarly, a fuzzy automaton $\mathscr{A} = (A, X, \delta)$ is said to recognize a fuzzy language $f \in \mathscr{F}(X^*)$ if there exist fuzzy sets $\sigma, \tau \in \mathscr{F}(A)$ such that the fuzzy recognizer $(A, \sigma, X, \delta, \tau)$ recognizes f. The class of all fuzzy languages in $\mathscr{F}(X^*)$ which can be recognized by this fuzzy automaton is denoted by $\mathrm{Rec}(\mathscr{A})$. A fuzzy language which can be recognized by a fuzzy finite automaton is called *FFA-recognizable*.

For undefined notions and notation we refer to [3,5,10,16,30,47,48,55,80].

3. Minimal deterministic automaton of a fuzzy language

As we have noted in the previous section, fuzzy sets considered in this section, as well as in Sections 4–7, will take their membership values in a structure $\mathscr{L} = (L, 0, 1)$, where L is an arbitrary set, and 0 and 1 are two distinguished elements of L, which we need to take crisp languages in consideration. We do not impose any other requirement on $\mathscr{L}$. Under these conditions it is not possible to define fuzzy automata, because we cannot define the composition of transition relations, but in Sections 3–7 we will study fuzzy languages from the aspect of their recognition by ordinary deterministic automata equipped with fuzzy sets of terminal states, i.e., by deterministic fuzzy recognizers.

Let π be an equivalence on a set A. A fuzzy subset $f \in \mathscr{F}(A)$ is said to be *extensional* w.r.t. π if $\pi \subseteq \ker f$. Hence, $\ker f$ is the greatest equivalence on A such that f is extensional with respect to it. The class of all fuzzy subsets of A extensional w.r.t. π is denoted by $\mathscr{H}_\pi$.

Lemma 3.1. *Let π be an equivalence on a set A, and for any $f \in \mathscr{H}_\pi$, let $\tau_f \in \mathscr{F}(A/\pi)$ be a fuzzy set defined by $\tau_f(\pi_a) = f(a)$, for each $a \in A$.*

Then τ_f is well-defined and $f \mapsto \tau_f$ is a bijective mapping of $\mathscr{H}_\pi$ onto $\mathscr{F}(A/\pi)$.

Proof. By extensionality of f w.r.t. π it follows that τ_f is a well-defined fuzzy set, i.e., if $\pi_a = \pi_b$, for some $a, b \in A$, then $f(a) = f(b)$. Further, if $\tau_f = \tau_g$, for some $f, g \in \mathscr{H}_\pi$, then for each $a \in A$ we have that $f(a) = \tau_f(\pi_a) = \tau_g(\pi_a) = g(a)$, so $f = g$. Therefore, the mapping $f \mapsto \tau_f$ is injective. Finally, to any $\tau \in \mathscr{F}(A/\pi)$ we can assign $f \in \mathscr{F}(A)$ defined by $f(a) = \tau(\pi_a)$, and we can easily check that $f \in \mathscr{H}_\pi$ and $\tau = \tau_f$. Hence, we have proved that $f \mapsto \tau_f$ is a bijective mapping of $\mathscr{H}_\pi$ onto $\mathscr{F}(A/\pi)$. $\square$

Let S be semigroup and π a crisp equivalence on S. If $(a, b) \in \pi$ implies $(ax, bx) \in \pi$, for all $a, b, x \in S$, then π is called a *right congruence*, if $(a, b) \in \pi$ implies $(xa, xb) \in \pi$, for all $a, b, x \in S$, then it is called a *left congruence*, and if π is both a left and right congruence, then it is called a *congruence* on S.

It is well-known that to any right congruence π on the free monoid X^* we can associate a deterministic automaton $\mathscr{A}_\pi = (A_\pi, X, \delta_\pi)$, where $A_\pi = X^*/\pi$, the factor set of X^* with respect to π, and the transition function $\delta_\pi : A_\pi \times X \to A_\pi$ is defined by

$$\delta_\pi(\pi_u, x) = \pi_{ux}, \tag{21}$$

for all $u \in X^*$ and $x \in X$ (cf. [16,80]), where π_u is the equivalence class of π determined by u. We also know that δ_π can be extended up to a mapping $\delta^*_\pi : A_\pi \times X^* \to A_\pi$ so that

$$\delta^*_\pi(\pi_u, v) = \pi_{uv}, \tag{22}$$

for all $u, v \in X^*$. We usually take π_e, the equivalence class of π determined by the empty word e, as an initial state of $\mathscr{A}_\pi$. The automaton $\mathscr{A}_\pi$ is known as the *right congruence automaton* associated with π.

The following theorem establishes a relationship between extensionality of fuzzy languages and their recognition by right congruence automata.

Theorem 3.1. *Let π be a right congruence on a free monoid X^*.*

A fuzzy language $f \in \mathscr{F}(X^*)$ is recognized by $\mathscr{A}_\pi$ if and only if f is extensional with respect to π.

Proof. Let $\mathscr{A}_\pi$ recognize the fuzzy language *f* by a fuzzy set of terminal states $\tau \in \mathscr{F}(A_\pi)$, i.e., let

$$f(u) = \tau(\delta_\pi^*(\pi_e, u)) = \tau(\pi_u),$$

for each $u \in X^*$. If $(u, v) \in \pi$, then $\pi_u = \pi_v$, and $f(u) = \tau(\pi_u) = \tau(\pi_v) = f(v)$. Therefore, $\pi \subseteq \ker f$, and hence, *f* is extensional w.r.t. π.

Conversely, let *f* be extensional w.r.t. π. Let us define a fuzzy subset $\tau \in \mathscr{F}(A_\pi)$ by $\tau(\pi_u) = f(u)$, for any $\pi_u \in A_\pi$, $u \in X^*$. If $\pi_u = \pi_v$, for some $u, v \in X^*$, then we have that $(u, v) \in \pi \subseteq \ker f$, so $f(u) = f(v)$. Thus, τ is a well-defined fuzzy subset of A_π.

Now, $f(u) = \tau(\pi_u) = \tau(\delta_\pi^*(\pi_e, u))$, for every $u \in X^*$, and we conclude that $\mathscr{A}_\pi$ recognizes *f* by a fuzzy set of terminal states τ. $\square$

Now we show that every fuzzy language *f* possess the greatest right congruence such that *f* is extensional w.r.t. it.

Theorem 3.2. *For any fuzzy language $f \in \mathscr{F}(X^*)$, the relation ϱ_f on X^* defined by*

$$(u, v) \in \varrho_f \;\Leftrightarrow\; (\forall p \in X^*)\, f(up) = f(vp), \tag{23}$$

is the greatest right congruence on X^* such that f is extensional w.r.t. it.

Proof. It is well-known that for any equivalence λ on X^*, a relation λ_r^0 on X^* defined by

$$(u, v) \in \lambda_r^0 \;\Leftrightarrow\; (\forall p \in X^*)\,(up, vp) \in \lambda,$$

for all $u, v \in S$, is the greatest right congruence on X^* contained in λ. By this it follows that ϱ_f is the greatest right congruence on X^* contained in $\ker f$, i.e., the greatest right congruence on X^* such that *f* is extensional w.r.t. it. $\square$

The relation ϱ_f will be called the *syntactic right congruence* of a fuzzy language *f*. In some sources dealing with crisp languages, syntactic right congruences were called *Nerode's right congruences* of languages.

In the sequel we will see that the right congruence automaton associated with the syntactic right congruence of a fuzzy language plays an outstanding role in recognition of this fuzzy language.

Theorem 3.3. *For any fuzzy language $f \in \mathscr{F}(X^*)$, the right congruence automaton $\mathscr{A}_{\varrho_f}$ is a homomorphic image of any accessible deterministic automaton recognizing f.*

Proof. For the sake of simplicity set $\varrho_f = \varrho$.

Let $\mathscr{A} = (A, a_0, X, \delta)$ be any accessible deterministic automaton which recognizes *f* by a fuzzy set of terminal states $\tau \in \mathscr{F}(A)$. Then for any $a \in A$ there exists at least one $u \in X^*$ such that $\delta^*(a_0, u) = a$, and we define a mapping $\varphi : A \to A_\varrho$ as follows:

$$\varphi(a) = \varrho_u \;\Leftrightarrow\; \delta^*(a_0, u) = a, \tag{24}$$

for $a \in A$ and $u \in X^*$. Let $u, v \in X^*$ such that $\delta^*(a_0, u) = \delta^*(a_0, v) = a$. Then for any $p \in X^*$ we have

$$f(up) = \tau(\delta^*(a_0, up)) = \tau(\delta^*(a, p)) = \tau(\delta^*(a_0, vp)) = f(vp),$$

and we conclude that $(u, v) \in \varrho$, i.e., $\varrho_u = \varrho_v$. Therefore, φ is a well-defined mapping. For any $u \in X^*$ we have that $\varrho_u = \varphi(\delta^*(a_0, u))$, so φ is also surjective.

Next, consider arbitrary $a \in A$ and $w \in X^*$. If $\varphi(a) = \varrho_u$, for some $u \in X^*$, i.e., $\delta^*(a_0, u) = a$, then we have that

$$\varphi(\delta^*(a, w)) = \varphi(\delta^*(a_0, uw)) = \varrho_{uw} = \delta_\varrho^*(\varrho_u, w) = \delta_\varrho^*(\varphi(a), w),$$

and therefore, φ is a homomorphism of $\mathscr{A}$ onto $\mathscr{A}_\varrho$. $\square$

A deterministic automaton $\mathscr{A}$ is called a *minimal deterministic automaton* of a fuzzy language $f \in \mathscr{F}(X^*)$ if it recognizes *f* and $|\mathscr{A}| \leq |\mathscr{A}'|$, for any deterministic automaton $\mathscr{A}'$ recognizing *f*.

The next two theorems are the main results of this section.

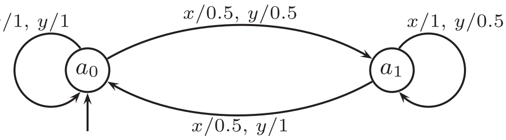


Fig. 1. The transition graph of $\mathscr{A}$.

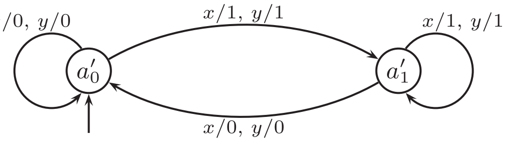


Fig. 2. The transition graph of $\mathscr{A}'$.

Theorem 3.4. *For any fuzzy language $f \in \mathscr{F}(X^*)$, the right congruence automaton $\mathscr{A}_{\varrho_f}$ is a minimal deterministic automaton of f.*

Proof. For the sake of simplicity set $\varrho_f = \varrho$.

By Theorems 3.1 and 3.2 we have that $\mathscr{A}_\varrho$ recognizes f. Let $\mathscr{A} = (A, a_0, X, \delta)$ be any deterministic automaton which recognizes f by a fuzzy set of terminal states $\tau \in \mathscr{F}(A)$, and consider its accessible part, i.e., a deterministic automaton $\mathscr{A}' = (A', a_0, X, \delta')$, where A' is the set of all accessible states of $\mathscr{A}$ and δ' is the restriction of δ to $A' \times X$. We have that $\mathscr{A}'$ is an accessible automaton which recognizes f by a fuzzy set of terminal states $\tau' \in \mathscr{F}(A')$, where τ' is the restriction of τ to A', and according to Theorem 3.3, $\mathscr{A}_\varrho$ is a homomorphic image of $\mathscr{A}'$. Therefore, $|\mathscr{A}_\varrho| \le |\mathscr{A}'| \le |\mathscr{A}|$, and we have proved that $\mathscr{A}_\varrho$ is a minimal deterministic automaton of a fuzzy language f. □

Theorem 3.5. *If a fuzzy language $f \in \mathscr{F}(X^*)$ has a finite minimal deterministic automaton, then all minimal deterministic automata of f are mutually isomorphic.*

Proof. By the hypothesis, all minimal automata of f, including the automaton $\mathscr{A}_{\varrho_f}$, have the same number of elements, and evidently, they are accessible. Now, according to Theorem 3.3, $\mathscr{A}_{\varrho_f}$ is a homomorphic image of any minimal automaton of f, and since any surjective mapping between two finite sets with the same number of elements must be bijective, we conclude that $\mathscr{A}_{\varrho_f}$ is isomorphic to any minimal automaton of f. □

The previous theorem says that if a fuzzy language is DFA-recognizable, then it possess a minimal deterministic automaton which is unique up to an isomorphism. The next example shows that this is not true in the case of fuzzy automata, i.e., a minimal fuzzy automaton of an FFA-recognizable fuzzy language is not necessary unique up to an isomorphism.

Example 3.1. Let $\mathscr{L}$ be the Gödel structure, and consider initial fuzzy finite automata $\mathscr{A} = (A, a_0, X, \delta)$ and $\mathscr{A}' = (A', a_0', X, \delta')$ over $\mathscr{L}$, where $A = \{a_0, a_1\}$, $A' = \{a_0', a_1'\}$ and $X = \{x, y\}$, which are given by the transition graphs in Figs. 1 and 2.

Also, consider a fuzzy language $f \in \mathscr{F}(X^*)$ defined by

$$f(u) = \begin{cases} 0.5 & \text{if } u \in X^+, \\ 0 & \text{if } u = e. \end{cases}$$

Then $\mathscr{A}$ recognizes f by a crisp set $\{a_1\}$, and $\mathscr{A}'$ recognizes f by a fuzzy set $\tau \in \mathscr{F}(A')$ given by $\tau(a_0') = 0$ and $\tau(a_1') = 0.5$. It can be easily proved that f cannot be recognized by any fuzzy automaton with a single state, so both $\mathscr{A}$ and $\mathscr{A}'$ are minimal fuzzy automata recognizing f. It is clear that they are not isomorphic.

Let us also note that $\mathscr{A}'$ is a deterministic automaton, obtained by determinization of $\mathscr{A}$ by means of the accessible fuzzy subset construction developed in [25], and therefore, $\mathscr{A}'$ is the minimal deterministic automaton of f.

4. The derivative automaton of a fuzzy language

A problem which one naturally imposes is how to construct the minimal deterministic automaton of a fuzzy language. The minimal deterministic automaton of a crisp language can be constructed by means of the concept of derivatives of this language, introduced by Rabin and Scott [61], Raney [62], and Elgot and Rutledge [22]. In the fuzzy framework, derivatives of fuzzy languages have been already studied by Malik et al. [44], Mordeson and Malik [47], Petković [51] and Bozapalidis and Louscou-Bozapalidou [9]. Here we will use derivatives of a fuzzy language to construct its minimal deterministic automaton.

For a fuzzy language $f \in \mathscr{F}(X^*)$ and $u \in X^*$, a fuzzy language $f_u \in \mathscr{F}(X^*)$ defined by

$$f_u(v) = f(uv), \tag{25}$$

for each $v \in X^*$, is called a *derivative* of *f* with respect to u.

In addition, if *f* is a crisp language, then its derivative f_u w.r.t. u is a crisp language given by

$$f_u = \{v \in X^* \mid uv \in f\}. \tag{26}$$

Derivatives of crisp languages are also known as *right quotients*, *quotients* or *residuals* of languages.

Let $A_f = \{f_u \mid u \in X^*\}$ be the set of all derivatives of *f*, and define a mapping $\delta_f : A_f \times X \to A_f$ by

$$\delta_f(g, x) = g_x, \tag{27}$$

for all $g \in A_f$ and $x \in X$. We have the following:

Theorem 4.1. *For any fuzzy language $f \in \mathscr{F}(X^*)$, the mapping δ_f is well-defined and $\mathscr{A}_f = (A_f, f, X, \delta_f)$ is a minimal deterministic automaton of f.*

Furthermore, $\mathscr{A}_f$ recognizes f with a fuzzy set of terminal states $\tau \in \mathscr{F}(A_f)$ defined by

$$\tau(g) = g(e), \tag{28}$$

for any $g \in A_f$.

Proof. Consider arbitrary $g \in A_f$ and $x \in X$. Then $g = f_u$, for some $u \in X^*$, and for any $w \in X^*$ we have that $g_x(w) = g(xw) = f_u(xw) = f(uxw) = f_{ux}(w)$, so $g_x = f_{ux} \in A_f$. Therefore, δ_f is well-defined, and $\mathscr{A}_f$ is a deterministic automaton.

Next, to prove that $\mathscr{A}_f$ is a minimal deterministic automaton of *f*, it is enough to prove that it is isomorphic to $\mathscr{A}_{\varrho_f}$. For the sake of simplicity set $\varrho_f = \varrho$, and define a mapping $\phi : A_f \to A_\varrho$ by $\phi(f_u) = \varrho_u$, for any $u \in X^*$. According to (23), for any $u, v \in X^*$ we have that $f_u = f_v$ if and only if $(u, v) \in \varrho$, i.e., if $\varrho_u = \varrho_v$, and hence, ϕ is a well-defined bijective mapping. Also, $\phi(\delta_f(f_u, x)) = \delta_\varrho(\phi(f_u), x)$, for all $u \in X^*$ and $x \in X$, so ϕ is an isomorphism of deterministic automata.

Finally, for any $u \in X^*$ we have that

$$\tau(\delta_f(f, u)) = \tau(\delta_f(f_e, u)) = \tau(f_u) = f_u(e) = f(u),$$

so $\mathscr{A}_f$ recognizes *f* with τ. $\square$

The above constructed automaton $\mathscr{A}_f$ will be called the *derivative automaton* of a fuzzy language *f*.

In the sequel we will show that construction of the derivative automaton of a fuzzy language amounts to construction of the derivative automata of particular crisp languages.

Let $f \in \mathscr{F}(X^*)$ be an arbitrary fuzzy language. For any $i \in \operatorname{Im} f$, define a crisp language $f^{[i]}$ by

$$f^{[i]} = \{u \in X^* \mid f(u) = i\}. \tag{29}$$

Clearly, the languages $f^{[i]}$, $i \in \operatorname{Im} f$, form the set of all equivalence classes of the equivalence relation $\ker f$, and they will be called the *kernel languages* of *f*.

The operations of forming derivatives and kernel languages can be combined, i.e., for a fuzzy language $f \in \mathscr{F}(X^*)$, $i \in \operatorname{Im} f$ and $u \in X^*$, we can first form the derivative f_u, and then the kernel language $(f_u)^{[i]}$ of f_u, or to form the kernel language $f^{[i]}$, and then the derivative $(f^{[i]})_u$ of $f^{[i]}$. The next lemma shows that in both cases we obtain the same result.

Lemma 4.1. *For any $f \in \mathscr{F}(X^*)$, $i \in \operatorname{Im} f$ and $u \in X^*$, we have that $(f_u)^{[i]} = (f^{[i]})_u$.*

Proof. For any $v \in X^*$ we have the following sequence of equivalences

$$v \in (f_u)^{[i]} \;\Leftrightarrow\; f_u(v) = i \;\Leftrightarrow\; f(uv) = i \;\Leftrightarrow\; uv \in f^{[i]} \;\Leftrightarrow\; v \in (f^{[i]})_u.$$

Therefore, $(f_u)^{[i]} = (f^{[i]})_u$. □

Now we are ready to state and prove the main result of this section.

Theorem 4.2. *The derivative automaton of any fuzzy language is a subdirect product of derivative automata of its kernel languages.*

In addition, the derivative automaton of a fuzzy language is isomorphic to the accessible part of the direct product of derivative automata of its kernel languages.

Proof. Consider an arbitrary fuzzy language $f \in \mathscr{F}(X^*)$. For the sake of simplicity, I will stand for $\operatorname{Im} f$, and $\mathscr{A}_i = (A_i, f^{[i]}, X, \delta_i)$ will stand for $\mathscr{A}_{f^{[i]}} = (A_{f^{[i]}}, f^{[i]}, X, \delta_{f^{[i]}})$, for any $i \in I$. Let $\mathscr{A} = (A, a_0, X, \delta)$ be the direct product of automata $\mathscr{A}_i$, $i \in I$, i.e.,

$$A = \prod_{i \in I} A_i,$$

and an initial state $a_0 \in A$ and a transition function $\delta : A \times X \to A$ are defined by

$$a_0(i) = f^{[i]}, \quad \delta(a, x)(i) = \delta_i(a(i), x),$$

for any $a \in A$, $x \in X$ and $i \in I$. Now, define a mapping $\phi : A_f \to A$ by

$$\phi(g)(i) = g^{[i]},$$

for any $g \in A_f$. According to Lemma 4.1, for any $g \in A_f$ we have that $g = f_u$, for some $u \in X^*$, whence $g^{[i]} = (f_u)^{[i]} = (f^{[i]})_u \in A_{f^{[i]}}$, for each $i \in I$, so ϕ is well-defined.

Suppose that $\phi(g) = \phi(h)$, for some $g, h \in A_f$, i.e., $g^{[i]} = h^{[i]}$, for every $i \in I$. Then for any $u \in X^*$ we have that $g(u) = i$, for some $i \in L$, and by $g \in A_f$ it follows that $i \in I$, what yields $u \in g^{[i]} = h^{[i]}$, i.e., $h(u) = i = g(u)$. Thus, we have obtained that $g = h$, and we conclude that ϕ is an injective mapping.

Further, for any $g \in A_f$ and $x \in X$, by Lemma 4.1 it follows that

$$\phi(\delta_f(g, x))(i) = \phi(g_x)(i) = (g_x)^{[i]} = (g^{[i]})_x = \delta_i(g^{[i]}, x) = \delta_i(\phi(g)(i), x) = \delta(\phi(g), x)(i),$$

for every $i \in I$, whence $\phi(\delta_f(g, x)) = \delta(\phi(g), x)$. Therefore, ϕ is a homomorphism, i.e., it is an isomorphism of $\mathscr{A}_f$ onto a subautomaton $\operatorname{Im}\phi$ of $\mathscr{A}$.

Finally, consider arbitrary $i \in I$ and $K \in A_{f^{[i]}}$. Then there exists $u \in X^*$ such that

$$K = (f^{[i]})_u = (f_u)^{[i]} = \operatorname{pr}_i(\phi(f_u)) = \phi \circ \operatorname{pr}_i(f_u),$$

where pr_i denotes the projection homomorphism of $\mathscr{A}$ onto $\mathscr{A}_i$. Hence, $\mathscr{A}_f$ is a subdirect product of automata $\mathscr{A}_i$, $i \in I$.

Consider an arbitrary $g \in A_f$, and assume that $g = f_u$, for some $u \in X^*$. In the above notation,

$$\phi(g)(i) = g^{[i]} = (f_u)^{[i]} = (f^{[i]})_u = \delta_i(f^{[i]}, u) = \delta_i(\phi(f)(i), u) = \delta(\phi(f), u)(i),$$

for every $i \in I$, so $\phi(g) = \delta(\phi(f), u) = \delta(a_0, u)$. On the other hand, for any $u \in X^*$ we have that $\delta(a_0, u) = \delta(\phi(f), u) = \phi(f_u)$. Hence, $\operatorname{Im}\phi$ is the accessible part of $\mathscr{A}$, what completes the proof of the theorem. □

Let us note that the previous theorem provides an effective method for construction of the derivative automaton of a fuzzy language, which is based on simultaneous construction of the derivative automata of its kernel languages. Also, note again that direct and subdirect products of automata can be interpreted as their parallel connections.

Another important result of this section is the following theorem which was proved by Li and Pedrycz [37,39] for fuzzy languages with membership values in lattice-ordered monoids and distributive lattices.

Theorem 4.3. *A fuzzy language f is DFA-recognizable if and only if it has a finite rank and all its kernel languages are recognizable.*

Proof. Let f be DFA-recognizable, i.e., let the automaton $\mathscr{A}_f$ be finite. Let $\psi : A_f \to \operatorname{Im} f$ be a mapping defined by $\psi(g) = g(e)$, for any $g \in A_f$. If we assume that $g = f_u$, for some $u \in X^*$, then we have that $g(e) = f_u(e) = f(u) \in \operatorname{Im} f$, so ψ is well-defined. Furthermore, for every $i \in \operatorname{Im} f$ there exists $u \in X^*$ such that $i = f(u) = f_u(e) = \psi(f_u)$, so ψ is surjective, and since A_f is finite, we conclude that $\operatorname{Im} f$ is also finite. Next, according to Theorem 4.2, for any $i \in I$ the automaton $\mathscr{A}_{f^{[i]}}$ is a homomorphic image of the automaton $\mathscr{A}_f$, so it is finite. Hence, any kernel language $f^{[i]}$ is recognizable.

The converse follows immediately by Theorem 4.2. □

We also have the following.

Theorem 4.4. *Let $\mathscr{A} = (A, a_0, X, \delta)$ be an accessible deterministic automaton.*
Then $\mathscr{A}$ recognizes a fuzzy language $f \in \mathscr{F}(X^*)$ if and only if it recognizes every kernel language of f.

Proof. Let $\mathscr{A}$ recognize f by a fuzzy set of terminal states $\tau \in \mathscr{F}(A)$. Then for any $i \in \operatorname{Im} f$ and any $u \in X^*$ we have that

$$u \in f^{[i]} \iff f(u) = i \iff \tau(\delta^*(a_0, u)) = i \iff \delta^*(a_0, u) \in \tau^{[i]},$$

and therefore, $\mathscr{A}$ recognizes the kernel language $f^{[i]}$ by a set of terminal states $\tau^{[i]}$.

Conversely, for each $i \in \operatorname{Im} f$, let $\mathscr{A}$ recognize the kernel language $f^{[i]}$ by a set of terminal states $T_i \subseteq A$, i.e., let

$$u \in f^{[i]} \iff \delta^*(a_0, u) \in T_i.$$

Since $\{f^{[i]}\}_{i\in \operatorname{Im} f}$ is a partition of X^* and $\mathscr{A}$ is accessible, then $\{T_i\}_{i\in \operatorname{Im} f}$ is a partition of A, and if for any $a \in A$ we set

$$\tau(a) = i \iff a \in T_i,$$

then τ is a well-defined fuzzy subset of A and the automaton $\mathscr{A}$ recognizes the fuzzy language f by the fuzzy set of terminal states τ. □

Theorem 4.5. *For any fuzzy language $f \in \mathscr{F}(X^*)$, the derivative automaton $\mathscr{A}_f$ is a minimal deterministic automaton recognizing every kernel language of f.*

Proof. This assertion is an immediate consequence of Theorem 4.4. □

Let us notice that if the set L of truth values is a partially ordered set with the least element 0 and the greatest element 1, then for any fuzzy language $f \in \mathscr{F}(X^*)$ and any $i \in \operatorname{Im} f$ we can define a crisp language $f^{(i)}$ by

$$f^{(i)} = \{u \in X^* \mid f(u) \geq i\}, \tag{30}$$

and the languages $f^{(i)}$, $i \in \operatorname{Im} f$, are called the *cut languages* of f. Under these conditions on L, Theorems 4.2–4.5 remain valid if kernel languages are replaced by cut languages. Let us note that cut languages have been studied by Bozapalidis and Louscou-Bozapalidou [8], in the context of recognition of fuzzy languages by finite monoids, and by Li and Pedrycz [37], in the context of recognition of fuzzy languages by fuzzy finite automata. Li and Pedrycz [37] proved that cut languages of a fuzzy language recognized by a fuzzy finite automaton are not necessary recognizable languages. Recently, collections of cuts of recognizable tree series over a partially ordered semiring have been studied in [7,63].

If, in addition, L is a lattice, then the following is true

Theorem 4.6. *A fuzzy language $f \in \mathscr{F}(X^*)$ is DFA-recognizable if and only if it can be represented as*

$$f = i_1K_1 \vee i_2K_2 \vee \cdots \vee i_nK_n, \tag{31}$$

where $i_1, i_2, \ldots, i_n \in L$ and $K_1, K_2, \ldots, K_n$ are crisp recognizable languages.

Proof. Let a fuzzy language f be DFA-recognizable. According to Theorem 4.3, we can assume that $\operatorname{Im} f = \{i_1, i_2, \ldots, i_n\}$, for some $n \in \mathbb{N}$. Since the kernel languages of f are mutually disjoint, f can be represented as

$$f = i_1 f^{[i_1]} \vee i_2 f^{[i_2]} \vee \cdots \vee i_n f^{[i_n]}, \tag{32}$$

and by Theorem 4.3, the kernel languages $f^{[i_k]}$, $1 \le k \le n$, are recognizable.

Conversely, let f be represented by (31). For each $k \in \{1, 2, \ldots, n\}$, the language K_k is recognizable, and assume that $\mathscr{A}_k = (A_k, a_0^{(k)}, X, \delta_k)$ is a deterministic finite automaton recognizing K_k by a crisp set of terminal states $T_k \subseteq A_k$. Moreover, let $\tau_k \in \mathscr{F}(A_k)$ be a fuzzy set defined by

$$\tau_k(a_k) = \begin{cases} i_k & \text{if } a_k \in T_k, \\ 0 & \text{if } a_k \in A_k \setminus T_k, \end{cases}$$

and let $\mathscr{A} = (A, a_0, X, \delta)$ be the direct product of automata $\mathscr{A}_k$. Then the automaton $\mathscr{A}$ recognizes f by a fuzzy set $\tau \in \mathscr{F}(A)$ defined by

$$\tau(a_1, a_2, \ldots, a_n) = \bigvee_{k=1}^{n} \tau_k(a_k),$$

for each $(a_1, a_2, \ldots, a_n) \in A$. Consequently, f is DFA-recognizable. $\square$

Let us note that the previous theorem was proved by Li and Pedrycz [37] for fuzzy languages with membership values in a lattice-ordered monoid.

5. Minimization of deterministic fuzzy recognizers

In the previous section we have considered the problem of construction of a minimal deterministic automaton of a given fuzzy language. However, a fuzzy language is often represented as a fuzzy language recognized by some (deterministic) fuzzy recognizer. Therefore, it is interesting to consider a problem how to construct a minimal deterministic automaton of a fuzzy language starting from any deterministic fuzzy recognizer which recognizes this fuzzy language, i.e., the problem of minimization of deterministic fuzzy recognizers.

As in the crisp case, the central place in minimization of a deterministic fuzzy recognizer $\mathscr{A}$ is held by congruences on $\mathscr{A}$. Making factor automata w.r.t. congruences on $\mathscr{A}$ we reduce the number of states, but the factor automaton do not necessarily recognize the same fuzzy language. The next theorem determines necessary and sufficient conditions for a congruence to the corresponding factor automaton recognize the same fuzzy language as the original one.

Theorem 5.1. *Let $\mathscr{A} = (A, a_0, X, \delta, \tau)$ be an accessible deterministic fuzzy recognizer recognizing a fuzzy language $f \in \mathscr{F}(X^*)$, and let π be a congruence on $\mathscr{A}$.*

The factor automaton $\mathscr{A}_\pi = (A_\pi, \pi_{a_0}, X, \delta_\pi)$ recognizes the fuzzy language f if and only if τ is extensional w.r.t. π.

Proof. Let $\mathscr{A}_\pi$ recognize f by a fuzzy set of terminal states $\theta \in \mathscr{F}(A_\pi)$. Since for any $a \in A$ there exists $u \in X^*$ such that $\delta^*(a_0, u) = a$, we have that

$$\tau(a) = \tau(\delta^*(a_0, u)) = f(u) = \theta(\delta_\pi^*(\pi_{a_0}, u)) = \theta(\pi_{\delta^*(a_0,u)}) = \theta(\pi_a).$$

Now, if $a, b \in A$ such that $(a, b) \in \pi$, then $\pi_a = \pi_b$ and $\tau(a) = \theta(\pi_a) = \theta(\pi_b) = \tau(b)$, so we have that $(a, b) \in \ker \tau$. Thus, $\pi \subseteq \ker \tau$, and hence, τ is extensional w.r.t. π.

Conversely, let τ be extensional w.r.t. π, and define $\theta \in \mathscr{F}(A_\pi)$ by $\theta(\pi_a) = \tau(a)$, for any $a \in A$. By extensionality of τ w.r.t. π it follows that θ is a well-defined fuzzy subset of A_π, i.e., if $a, b \in A$ such that $\pi_a = \pi_b$, then $(a, b) \in \pi \subseteq \ker \tau$, so $\tau(a) = \tau(b)$.

Next, for any $u \in X^*$ we have that

$$f(u) = \tau(\delta^*(a_0, u)) = \theta(\pi_{\delta^*(a_0,u)}) = \theta(\delta^*_\pi(\pi_{a_0}, u)),$$

and therefore, $\mathscr{A}_\pi$ recognizes f by a fuzzy set of terminal states θ. $\square$

Next we prove that each deterministic fuzzy recognizer possess the greatest congruence such that its fuzzy set of terminal states is extensional w.r.t. this congruence.

Theorem 5.2. *Let $\mathscr{A} = (A, a_0, X, \delta, \tau)$ be a deterministic fuzzy recognizer. The relation π_τ on A defined by*

$$(a, b) \in \pi_\tau \iff (\forall u \in X^*)\, \tau(\delta^*(a, u)) = \tau(\delta^*(b, u)), \tag{33}$$

for all $a, b \in A$, is the greatest congruence on $\mathscr{A}$ such that τ is extensional w.r.t. it.

Proof. It is well-known that for any equivalence λ on A, a relation λ^0 on A defined by

$$(a, b) \in \lambda^0 \iff (\forall u \in X^*)\, (\delta^*(a, u), \delta^*(b, u)) \in \lambda,$$

for all $a, b \in A$, is the greatest congruence on $\mathscr{A}$ contained in λ. Consequently, π_τ is the greatest congruence on $\mathscr{A}$ contained in $\ker \tau$, i.e., the greatest congruence on $\mathscr{A}$ such that τ is extensional w.r.t. it. $\square$

Now we are ready to state and prove the main result of this section.

Theorem 5.3. *Let $\mathscr{A} = (A, a_0, X, \delta, \tau)$ be an accessible deterministic fuzzy recognizer recognizing a fuzzy language $f \in \mathscr{F}(X^*)$.*

Then the factor automaton $\mathscr{A}_{\pi_\tau}$ is isomorphic to the derivative automaton $\mathscr{A}_f$ of f.

Proof. For the sake of simplicity set $\pi_\tau = \pi$. According to Theorems 5.1 and 5.2, the factor automaton $\mathscr{A}_\pi = (A_\pi, \pi_{a_0}, X, \delta_\pi)$ recognizes f. In order to prove that $\mathscr{A}_\pi$ is isomorphic to the derivative automaton $\mathscr{A}_f$, define a mapping $\phi : A_\pi \to A_f$ by

$$\phi(\pi_a) = f_u \iff \delta^*(a_0, u) = a, \tag{34}$$

for $a \in A$ and $u \in X^*$. By accessibility of $\mathscr{A}$, for every $a \in A$ there exists at least one $u \in X^*$ such that the right-hand side of (34) is satisfied. To prove that ϕ is single-valued and injective, let us consider $a, b \in A$ and $u, v \in X^*$ such that $\pi_a = \pi_b$, $\delta^*(a_0, u) = a$ and $\delta^*(a_0, v) = b$. Then by (33) and the fact that $\mathscr{A}$ recognizes f by τ we obtain that

$$\begin{aligned}
\pi_a = \pi_b &\iff (\forall w \in X^*)\, \tau(\delta^*(a, w)) = \tau(\delta^*(b, w))\\
&\iff (\forall w \in X^*)\, \tau(\delta^*(a_0, uw)) = \tau(\delta^*(a_0, vw))\\
&\iff (\forall w \in X^*)\, f(uw) = f(vw)\\
&\iff (\forall w \in X^*)\, f_u(w) = f_v(w)\\
&\iff f_u = f_v,
\end{aligned}$$

and hence, ϕ is a well-defined and injective mapping. Moreover, for every $u \in X^*$ we have that $f_u = \phi(\pi_{\delta^*(a_0,u)})$, so ϕ is also surjective.

Finally, to prove that ϕ is a homomorphism, consider arbitrary $a \in A$ and $v \in X^*$. Assume that $\phi(\pi_a) = f_u$, for some $u \in X^*$, i.e., $\delta^*(a_0, u) = a$. Then,

$$\phi(\delta^*_\pi(\pi_a, v)) = \phi(\pi_{\delta^*(a,v)}) = \phi(\pi_{\delta^*(a_0,uv)}) = f_{uv} = \delta^*_f(f_u, v) = \delta^*_f(\phi(\pi_a), v),$$

and we conclude that ϕ is a homomorphism.

Summarizing the proof we establish that ϕ is an isomorphism of $\mathscr{A}_\pi$ onto $\mathscr{A}_f$. $\square$

According to Theorem 5.3, minimization of a deterministic fuzzy recognizer $\mathscr{A} = (A, a_0, X, \delta, \tau)$ can be carried out constructing the greatest congruence on $\mathscr{A}$ contained in the crisp equivalence $\ker \tau$. By this it follows that minimization of deterministic fuzzy recognizers can be carried out using the same algorithms which are used in minimization of deterministic crisp recognizers. Construction of the greatest congruence contained in $\ker \tau$ is described by the next theorem. The proof is the same as in the crisp case so it will be omitted.

Theorem 5.4. *Let $\mathscr{A} = (A, a_0, X, \delta, \tau)$ be an accessible deterministic fuzzy recognizer, and let a sequence $\{\varrho_k\}_{k\in\mathbb{N}}$ of equivalences on A be defined inductively, as follows*:

$$\varrho_1 = \ker \tau, \quad \varrho_{k+1} = \{(a, b) \in \varrho_k \mid (\forall x \in X)\,(\delta(a, x), \delta(b, x)) \in \varrho_k\} \quad \textit{for each } k \in \mathbb{N}.$$

Then

(a) $\pi_\tau \subseteq \cdots \subseteq \varrho_{k+1} \subseteq \varrho_k \subseteq \cdots \subseteq \varrho_1 = \ker \tau$.
(b) *If* $\varrho_k = \varrho_{k+m}$, *for some* $k, m \in \mathbb{N}$, *then* $\varrho_k = \varrho_{k+1} = \pi_\tau$.
(c) *If* $\mathscr{A}$ *is finite*, *then* $\varrho_k = \pi_\tau$ *for some* $k \in \mathbb{N}$.

It is worth noting that the basic idea exploited in the papers by Basak and Gupta [2], Cheng and Mo [11], Lei and Li [32], Maliket al. [46] and Petković [52], and in the book by Mordeson and Malik [47], was to reduce the number of states of a fuzzy automaton by computing and merging indistinguishable states, what resembles the minimization algorithm for deterministic automata. However, the term minimization used in the mentioned papers does not mean the usual construction of the minimal one in the set of all fuzzy automata recognizing a given fuzzy language, but just this procedure of computing and merging indistinguishable states which do not necessarily results in a minimal fuzzy automaton. Actually, there are two key differences between minimization of fuzzy and deterministic automata. First, a minimal fuzzy automaton recognizing a given fuzzy language is not necessary unique up to an isomorphism (see Example 3.1). This is also true for non-deterministic automata. On the other hand, minimization of fuzzy automata, as well as of non-deterministic ones, is computationally hard. For that reason it is more interesting to look for state reduction methods which do not necessarily give a minimal fuzzy automaton, but they give a "reasonably" small fuzzy automaton which can be constructed efficiently. Such methods have been studied by Ćirić et al. in [13,14]. However, we have proved that a minimal deterministic fuzzy recognizer recognizing a given fuzzy language is unique up to an isomorphism, and it can be constructed efficiently. An algorithm for minimization of deterministic fuzzy recognizers with membership values in a distributive lattice was given in a recent paper by Li and Pedrycz [39].

6. Recognition of fuzzy languages by monoids

As we have mentioned in the Introduction, recognition of crisp languages by monoids, as well as syntactic congruences and syntactic monoids of crisp languages, plays a very important role in the theory of crisp languages and, in particular, they have been the starting point for the elegant Eilenberg classification theory of regular languages based on syntactic monoids and varieties of finite monoids. Syntactic congruences of fuzzy languages have been studied by Shen [67], Malik et al. [44] Mordeson and Malik [47], Petković [51] and Bozapalidis and Louscou-Bozapalidou [8], and Petković [51] and Bozapalidis and Louscou-Bozapalidou [8,9] also studied recognition of fuzzy languages by monoids. Here we develop a general theory of recognition of fuzzy languages by monoids.

A fuzzy language $f \in \mathscr{F}(X^*)$ is said to be *recognized by a monoid S* if there exist a homomorphism $\phi : X^* \to S$ and a fuzzy subset $\tau \in \mathscr{F}(S)$ such that $f = \phi \circ \tau$, that is, $f(u) = \tau(\phi(u))$, for each $u \in X^*$. In this case we also say that the monoid S recognizes f by a homomorphism ϕ and a fuzzy subset τ. Let us note that f is a crisp language if and only τ is a crisp subset of S. In other words, a monoid S recognizes a crisp language $K \subseteq X^*$ if and only if there are a homomorphism $\phi : X^* \to S$ and a crisp subset T of S such that $u \in K$ if and only if $\phi(u) \in T$ holds for each $u \in X^*$. A fuzzy language is said to be *FM-recognizable* if it can be recognized by a finite monoid.

If a monoid S recognizes a fuzzy language $f \in \mathscr{F}(X^*)$ by a homomorphism $\phi : X^* \to S$ and a fuzzy subset $\tau \in \mathscr{F}(S)$, then the submonoid $S' = \operatorname{Im} \phi$ of S recognizes f by the same homomorphism ϕ and a fuzzy subset $\tau' \in \mathscr{F}(S')$ given by $\tau'(a) = \tau(a)$, for every $a \in S'$. We can consider S' as the factor monoid of X^* with respect to the congruence $\pi = \ker \phi$, and the homomorphism ϕ as the natural homomorphism of π, so it is especially interesting to consider recognition of fuzzy languages from $\mathscr{F}(X^*)$ by factor monoids of X^* and corresponding natural homomorphisms.

For a congruence π on the free monoid X^*, the corresponding factor monoid of X^* will be denoted by S_π, and we say that S_π recognizes a fuzzy language $f \in \mathscr{F}(X^*)$ if it recognizes f by the natural homomorphism $\pi^\natural$ of X^* onto S_π and some fuzzy subset of S_π.

We have the following:

Theorem 6.1. *Let π be a congruence on the free monoid X^*.*

A fuzzy language $f \in \mathscr{F}(X^*)$ is recognized by the factor monoid S_π if and only if f is extensional with respect to π.

Proof. Let $\phi : X^* \to S_\pi$ be the natural homomorphism of the congruence π.

Let S_π recognizes f, i.e., let $f = \phi \circ h$, for some fuzzy set $h : S_\pi \to L$. Then we have that $\pi = \ker \phi \subseteq \ker f$, and hence, f is extensional w.r.t. π.

Conversely, let f be extensional w.r.t. π. Define a mapping $h : S_\pi \to L$ by $h(\pi_u) = f(u)$, for any $u \in X^*$. Since f is extensional w.r.t. π, we have that h is well-defined, and clearly, $f = \phi \circ h$. Therefore, S_π recognizes f. □

Theorem 6.2. *For any fuzzy language $f \in \mathscr{F}(X^*)$, the relation π_f on X^* defined by*

$$(u, v) \in \pi_f \;\Leftrightarrow\; (\forall p, q \in X^*)\, f(puq) = f(pvq), \tag{35}$$

for all $u, v \in X^*$, is the greatest congruence on X^* such that f is extensional w.r.t. it.

Proof. It is well-known that for any equivalence λ on X^*, a relation λ^0 on X^* defined by

$$(u, v) \in \lambda^0 \;\Leftrightarrow\; (\forall p, q \in X^*)\,(puq, pvq) \in \lambda,$$

for all $u, v \in S$, is the greatest congruence on X^* contained in λ. Consequently, π_f is the greatest congruence on X^* contained in $\ker f$, i.e., the greatest congruence on X^* such that f is extensional w.r.t. it. □

The relation π_f is called the *syntactic congruence* of a fuzzy language f. In some sources dealing with crisp languages, syntactic congruences are called *Myhill's congruences* of languages. The factor monoid S_{π_f} is denoted by $\mathrm{Syn}(f)$, and it is called the *syntactic monoid* of a fuzzy language f. The natural homomorphism of π_f is called the *syntactic homomorphism* of f. A monoid S is said to *divide* a monoid T if S is a homomorphic image of a submonoid of T.

Theorem 6.3. *A monoid S recognizes a fuzzy language $f \in \mathscr{F}(X^*)$ if and only if the syntactic monoid* $\mathrm{Syn}(f)$ *divides S.*

Proof. For the sake of simplicity set $\pi = \pi_f$.

Let S recognize f, i.e., let $f = \phi \circ h$, for some homomorphism $\phi : X^* \to S$ and some fuzzy set $h : S \to L$. For the congruence $\pi = \ker \phi$ we have that f is extensional w.r.t. π, and by Theorem 6.2 we obtain that $\pi \subseteq \pi_f$. By this it follows that $\mathrm{Syn}(f) = X^*/\pi_f$ is a homomorphic image of $S_\pi = X^*/\pi$, and since $S_\pi \cong \mathrm{Im}\phi$, then $\mathrm{Syn}(f)$ is a homomorphic image of a submonoid of S, i.e., $\mathrm{Syn}(f)$ divides S.

Conversely, let $\mathrm{Syn}(f)$ divides S, i.e., let there exists a submonoid T of S and a surjective homomorphism $\psi : T \to \mathrm{Syn}(f)$. For any $x \in X^*$, the set $\psi^{-1}(\pi_x)$ is non-empty, and we can choose an arbitrary element $\varphi(x)$ from $\psi^{-1}(\pi_x)$. In this way we have defined a mapping $\varphi : X \to T$, which can be extended up to a homomorphism $\phi : X^* \to T$.

Now we have that $\psi(\phi(x)) = \psi(\varphi(x)) = \pi_x$, for each $x \in X$, and since ψ and ϕ are homomorphisms, then $\psi(\phi(u)) = \pi_u$, for each $u \in X^*$, so we have that $\phi \circ \psi = \pi^\natural$, where $\pi^\natural$ is the natural homomorphism of π. We also have that $\mathrm{Syn}(f)$ recognizes f by a fuzzy set $h : \mathrm{Syn}(f) \to L$, i.e., $f = \pi^\natural \circ h$, whence it follows that $f = (\phi \circ \psi) \circ h = \phi \circ (\psi \circ h)$, and hence, T recognizes f by a fuzzy set $\psi \circ h : T \to L$.

Finally, we can consider ϕ as a homomorphism of X^* into S, and we can easily extend the mapping $\psi \circ h : T \to L$ up to a mapping $g : S \to L$ such that $f = \phi \circ g$, so we conclude that S recognizes f. □

A monoid S is called a *minimal monoid* of a fuzzy language $f \in \mathscr{F}(X^*)$ if it recognizes f and $|S| \leq |S'|$, for any monoid S' recognizing f. By the next two theorems we prove existence of a minimal monoid of a fuzzy language, and its uniqueness, if this fuzzy language is FM-recognizable.

Theorem 6.4. *For any fuzzy language $f \in \mathscr{F}(X^*)$, the syntactic monoid* $\mathrm{Syn}(f)$ *is a minimal monoid of f.*

Proof. This is an immediate consequence of Theorem 6.3. □

Theorem 6.5. *If a fuzzy language $f \in \mathscr{F}(X^*)$ has a finite minimal monoid, then all minimal monoids of f are mutually isomorphic.*

Proof. Let S be an arbitrary minimal monoid of f. By Theorem 6.3 and minimality of S we obtain that there exists a surjective homomorphism $\phi : S \to \mathrm{Syn}(f)$. Moreover, by the hypothesis, all minimal monoids of f, including S and $\mathrm{Syn}(f)$, have the same number of elements, and since any surjective mapping between two finite sets with the same number of elements must be bijective, we conclude that ϕ is an isomorphism. Hence, any minimal monoid of f is isomorphic to $\mathrm{Syn}(f)$. □

The next theorem establishes a relationship between the syntactic monoid of a fuzzy language and its derivative automaton.

Theorem 6.6. *For any fuzzy language $f \in \mathscr{F}(X^*)$, the syntactic monoid* $\mathrm{Syn}(f)$ *is isomorphic to the transition monoid of the derivative automaton of f.*

Proof. Let $\mathscr{A}_f = (A_f, f, X, \delta_f)$ be the derivative automaton of f, and for each $u \in X^*$, let the transition function of $\mathscr{A}_f$ determined by u be denoted by δ_u. We know that a mapping $\phi : X^* \to T(\mathscr{A}_f)$ given by $\phi(u) = \delta_u$, is a homomorphism of X^* onto $T(\mathscr{A}_f)$, and hence, $T(\mathscr{A}_f) \cong X^*/\ker\phi$. On the other hand, for any $u, v \in X^*$ we have that

$$\begin{aligned}
\delta_u = \delta_v \;&\Leftrightarrow\; (\forall g \in A_f)\, \delta_u(g) = \delta_v(g) \\
&\Leftrightarrow\; (\forall g \in A_f)\, g_u = g_v \\
&\Leftrightarrow\; (\forall p \in X^*)\, f_{pu} = f_{pv} \\
&\Leftrightarrow\; (\forall p, q \in X^*)\, f_{pu}(q) = f_{pv}(q) \\
&\Leftrightarrow\; (\forall p, q \in X^*)\, f(puq) = f(pvq) \\
&\Leftrightarrow\; (u, v) \in \pi_f,
\end{aligned}$$

and consequently, $\ker\phi = \pi_f$. Therefore, $T(\mathscr{A}_f) \cong X^*/\pi_f = \mathrm{Syn}(f)$. □

According to (35), to any fuzzy subset f of a free monoid X^* we assign a congruence π_f on X^*. Similarly, if S is an arbitrary monoid and h is a fuzzy subset of S, then we can define a congruence π_h on S by

$$(a, b) \in \pi_h \;\Leftrightarrow\; (\forall s, t \in S)\, h(sat) = h(sbt), \tag{36}$$

for all $a, b \in S$. If π_h is the equality relation on S, i.e., if $(a, b) \in \pi_h$ implies $a = b$, then h is called a *disjunctive fuzzy subset* of S.

Theorem 6.7. *For a given set L of truth values, a monoid S is the syntactic monoid of some fuzzy language over L if and only if S contains a disjunctive fuzzy subset over L.*

Proof. Let S be the syntactic monoid of a fuzzy language $f : X^* \to L$, and let $\phi : X^* \to S$ be the syntactic homomorphism of f. By Theorem 6.4, S recognizes f, i.e., there exists a fuzzy subset $h : S \to L$ of S such that $f = \phi \circ h$. Let $(a, b) \in \pi_h$, i.e., let $h(sat) = h(sbt)$, for all $s, t \in S$. By surjectivity of ϕ, there exist $u, v \in X^*$ such that $a = \phi(u)$ and $b = \phi(v)$, and for arbitrary $p, q \in X^*$ we have that

$$f(puq) = h(\phi(puq)) = h(\phi(p)a\phi(q)) = h(\phi(p)b\phi(q)) = h(\phi(pvq)) = f(pvq),$$

so $(u, v) \in \pi_f = \ker\phi$, and hence, $a = \phi(u) = \phi(v) = b$. Thus, h is a disjunctive fuzzy subset of S.

Conversely, let S possess a disjunctive fuzzy subset $h : S \to L$. It is well-known that any monoid is a homomorphic image of some free monoid, i.e., there exists a free monoid X^* and a surjective homomorphism $\phi : X^* \to S$.

Set $f = \phi \circ h : X^* \to L$. Then for any $u, v \in X^*$, by surjectivity of ϕ we obtain that

$$\begin{aligned}
(u, v) \in \pi_f &\Leftrightarrow (\forall p, q \in X^*)\, f(puq) = f(pvq)\\
&\Leftrightarrow (\forall p, q \in X^*)\, h(\phi(puq)) = h(\phi(pvq))\\
&\Leftrightarrow (\forall p, q \in X^*)\, h(\phi(p)\phi(u)\phi(q)) = h(\phi(p)\phi(v)\phi(q))\\
&\Leftrightarrow (\forall s, t \in S)\, h(s\phi(u)t) = h(s\phi(v)t)\\
&\Leftrightarrow (\phi(u), \phi(v)) \in \pi_h\\
&\Leftrightarrow \phi(u) = \phi(v)\\
&\Leftrightarrow (u, v) \in \ker \phi,
\end{aligned}$$

so $\pi_f = \ker \phi$. Therefore, $S \cong X^*/\ker \phi = X^*/\pi_f = \mathrm{Syn}(f)$. □

Theorem 6.8. *For every monoid S there exists a fuzzy language f with truth values in some set L such that S is the syntactic monoid of f.*

Proof. Let L be an arbitrary set of truth values such that $|S| \leq |L|$, i.e., such that there exists an injective mapping $h : S \to L$. It is routine matter to show that h is a disjunctive fuzzy subset of S, and by Theorem 6.7 we obtain that S is the syntactic monoid of some fuzzy language f with truth values in L. □

Let us observe that the set L in Theorem 6.8 can be chosen to be a complete residuated lattice, lattice-ordered monoid, or any other particular ordered algebraic structure used in fuzzy set theory. Namely, if L' is an arbitrary ordered algebraic structure of a given type, then we can define L to be the direct power $(L')^S$ of L', so all algebraic properties of L' are preserved by L (cf. [70]), and $|S| \leq |L|$.

Theorem 6.9. *For any fuzzy language* $f \in \mathscr{F}(X^*)$, *the syntactic monoid* $\mathrm{Syn}(f)$ *is a subdirect product of syntactic monoids of the kernel languages of f.*

Proof. For the sake of simplicity, for each $i \in \mathrm{Im}\, f$ let the syntactic congruence of the kernel language $f^{[i]}$ be denoted by π_i.

Then for every $u, v \in X^*$ we have that

$$\begin{aligned}
(u, v) \in \pi_f &\Leftrightarrow (\forall p, q \in X^*)\, f(puq) = f(pvq)\\
&\Leftrightarrow (\forall p, q \in X^*)(\forall i \in \mathrm{Im}\, f)(f(puq) = i \Leftrightarrow f(pvq) = i)\\
&\Leftrightarrow (\forall i \in \mathrm{Im}\, f)(\forall p, q \in X^*)(puq \in f^{[i]} \Leftrightarrow pvq \in f^{[i]})\\
&\Leftrightarrow (u, v) \in \bigcap_{i \in \mathrm{Im}\, f} \pi_i,
\end{aligned}$$

and therefore,

$$\pi_f = \bigcap_{i \in \mathrm{Im}\, f} \pi_i .$$

Now, according to a well-known result of universal algebra, the monoid $X^*/\pi_f = \mathrm{Syn}(f)$ is a subdirect product of monoids $X^*/\pi_i = \mathrm{Syn}(f^{[i]})$, $i \in \mathrm{Im}\, f$. □

Now we state and prove one of the main results of this section.

Theorem 6.10. *A fuzzy language* $f \in \mathscr{F}(X^*)$ *is FM-recognizable if and only if it has a finite rank and all its kernel languages are FM-recognizable.*

Proof. Let f be FM-recognizable, i.e., let there exist a finite monoid S, a homomorphism $\phi : X^* \to S$ and a fuzzy subset $h \in \mathscr{F}(S)$ such that $f = \phi \circ h$. Then $\mathrm{Im}\, f \subseteq \mathrm{Im}\, h$, so we have that $|\mathrm{Im}\, f| \leq |\mathrm{Im} h| \leq |S|$, and hence, f has a finite rank. According to Theorem 6.4, the syntactic monoid $\mathrm{Syn}(f)$ is finite, and by Theorem 6.9, the syntactic monoid of each kernel language of f is finite, so each kernel language of f is *FM*-recognizable.

The converse follows immediately by Theorem 6.9. □

Here we also have that if L is a partially ordered set, then Theorems 6.9 and 6.10 remain valid if kernel languages are replaced by cut languages.

7. The Myhill–Nerode type theorem

Here we collect some results from the previous sections and we state and prove the Myhill–Nerode type theorem for fuzzy languages.

Theorem 7.1 (*The Myhill–Nerode type theorem*). *The following conditions on a fuzzy language $f \in \mathscr{F}(X^*)$ are equivalent*:

(i) *f is DFA-recognizable*;
(ii) *f is extensional w.r.t. a right congruence of finite index*;
(iii) *The syntactic right congruence of f has a finite index*;
(iv) *f is FM-recognizable*;
(v) *f is extensional w.r.t. a congruence of finite index*;
(vi) *The syntactic congruence of f has a finite index.*

Proof. The implication (i)⇒(iii) follows immediately by Theorem 3.4, (iii)⇒(ii) follows by Theorem 3.2, and (ii)⇒(i) is an immediate consequence of Theorem 3.1.

On the other hand, (iv)⇒(vi) follows by Theorem 6.4, (vi)⇒(v) follows by Theorem 6.2, and (v)⇒(iv) by Theorem 6.1.

Finally, the equivalence (i)⇔(iv) follows by Theorems 4.3 and 6.10, and the well-known fact that a crisp language can be recognized by a finite deterministic automaton if and only if it can be recognized by a finite monoid. □

According to the previous theorem, a fuzzy language can be recognized by a deterministic finite automaton if and only if it can be recognized by a finite monoid. It is worth noting that Bozapalidis and Louscou-Bozapalidou [8,9] studied recognizability of fuzzy languages by finite monoids in connection with their recognizability by certain types of fuzzy automata. If the structure of membership values is taken to be $\mathscr{L} = ([0,1], \wedge, \vee, \otimes, 0, 1)$, where $\otimes$ is the Gödel, Łukasiewicz or drastic t-norm, they proved that recognizability by finite monoids is equivalent to recognizability by fuzzy finite automata over $\mathscr{L}$. In all of these three cases the algebra $([0,1], \vee, \otimes, 0, 1)$ is a locally finite semiring, so this result can be viewed as a consequence of our result asserting that recognizability of a fuzzy language by a finite monoid is equivalent to recognizability by a deterministic finite automaton, and a result by Li and Pedrycz [37], asserting that recognizability of a fuzzy language by a fuzzy finite automaton over $\mathscr{L}$ is equivalent to recognizability by a deterministic finite automaton if and only if the semiring reduct of $\mathscr{L}$, w.r.t. join and multiplication, is a locally finite semiring. Bozapalidis and Louscou-Bozapalidou [8,9] also studied a more general type of fuzzy automata defined by means of t-norms and t-conorms. In the general case a t-norm does not distribute over a t-conorm, and Bozapalidis and Louscou-Bozapalidou started study of fuzzy (and weighted) automata defined by means of non-distributive pairs of operations, what is a very interesting and important problem.

8. Fuzzy right congruence automata

In the rest of the paper we will consider fuzzy sets, relations, languages and automata with membership values in a complete residuated lattice $\mathscr{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$.

The concept of a right congruence automaton, considered in Section 3, has been generalized by Ignjatović et al. in [25] to the fuzzy framework. Namely, let E be a fuzzy right congruence on the free monoid X^*, and let $A_E = X^*/E$. We define a mapping $\delta_E : A_E \times X \times A_E \to L$ by

$$\delta_E(E_u, x, E_v) = E_{ux}(v), \tag{37}$$

for all $u, v \in X^*$ and $x \in X$. Let us note that

$$E_{ux}(v) = E(ux, v) = \bigwedge_{w \in X^*} E_{ux}(w) \leftrightarrow E_v(w), \tag{38}$$

(cf. [12]), so $\delta_E(E_u, x, E_v)$ can be interpreted as the degree of relationship of the words ux and v w.r.t. E, or the degree of equality of the fuzzy equivalence classes E_{ux} and E_v.

It was proved in [25] that δ_E is a well-defined mapping, and $\mathscr{A}_E = (A_E, X, \delta_E)$ is a fuzzy automaton, called the *fuzzy right congruence automaton* associated with E. It was also proved that

$$\delta^*_E(E_u, p, E_v) = E_{up}(v) = E(up, v), \tag{39}$$

for all $u, v \in X^*$ and $p \in X^* \setminus \{e\}$, and $\delta^*_E(E_u, p, E_v) = 1$ if and only if $E_{up} = E_v$. Hence, if E is a crisp right congruence on X^*, then this construction yields the construction of a right congruence automaton given in Section 3.

Usually we consider $\mathscr{A}_E$ as an initial fuzzy automaton with the crisp initial state E_e, the fuzzy equivalence class of E determined by the empty word e, and in this case we write $\mathscr{A}_E = (A_E, E_e, X, \delta_E)$. In particular, when we talk about recognition of fuzzy languages by this fuzzy automaton, we always assume that $\mathscr{A}_E$ starts from the crisp initial state E_e. In other words, the automaton $\mathscr{A}_E$ recognizes a fuzzy language $f \in \mathscr{F}(X^*)$ by a fuzzy set of terminal states $\tau \in \mathscr{F}(A_E)$ if for every $u \in X^*$ we have

$$\begin{aligned} f(u) &= \bigvee_{\xi \in A_E} \delta^*_E(E_e, u, \xi) \otimes \tau(\xi) = \bigvee_{w \in X^*} \delta^*_E(E_e, u, E_w) \otimes \tau(E_w) \\ &= \bigvee_{w \in X^*} E(u, w) \otimes \tau(E_w). \end{aligned} \tag{40}$$

Let f be a fuzzy subset of a set A and let E be a fuzzy equivalence on A. Then f is said to be *extensional* with respect to E if for all $x, y \in A$ we have

$$f(x) \otimes E(x, y) \le f(y). \tag{41}$$

According to (41), symmetry of E and the adjunction property, f is extensional w.r.t. E if and only if

$$E(x, y) \le f(x) \leftrightarrow f(y), \tag{42}$$

for all $x, y \in A$ (cf. [17,18,26–28]). The set of all fuzzy subsets of A extensional w.r.t. E will be denoted by $\mathscr{H}_E$. If we define a fuzzy equivalence E_f on A setting $E_f(x, y) = f(x) \leftrightarrow f(y)$, for all $x, y \in A$, then f is extensional w.r.t. E if and only if $E \le E_f$, and hence, E_f is the greatest fuzzy equivalence on A such that f is extensional with respect to it. In particular, if E is a crisp equivalence on A, then this definition turns into the definition of a fuzzy set extensional w.r.t. a crisp equivalence, introduced in Section 3.

Theorem 8.1. *Let E be a fuzzy right congruence on a free monoid X^*.*
A fuzzy language $f \in \mathscr{F}(X^*)$ is recognized by $\mathscr{A}_E$ if and only if f is extensional with respect to E.

Proof. Let $\mathscr{A}_E$ recognize the fuzzy language f by a fuzzy set of terminal states $\tau \in \mathscr{F}(A_E)$. According to (5), for any $u, v \in X^*$ we have that

$$\begin{aligned} f(u) \otimes E(u, v) &= \left[\bigvee_{w \in X^*} \delta^*_E(E_e, u, E_w) \otimes \tau(E_w) \right] \otimes E(u, v) \\ &= \bigvee_{w \in X^*} [\delta^*_E(E_e, u, E_w) \otimes E(u, v) \otimes \tau(E_w)] \\ &= \bigvee_{w \in X^*} [E(u, w) \otimes E(u, v) \otimes \tau(E_w)] \\ &\le \bigvee_{w \in X^*} [E(v, w) \otimes \tau(E_w)] \\ &= \bigvee_{w \in X^*} [\delta^*_E(E_e, v, E_w) \otimes \tau(E_w)] = f(v). \end{aligned}$$

Therefore, f is extensional with respect to E.

Conversely, let f be extensional with respect to E. For any $u \in X^*$ let $\tau(E_u) = f(u)$. If $u, v \in X^*$ such that $E_u = E_v$, then $E(u, v) = 1$, whence $f(u) = f(u) \otimes E(u, v) \le f(v)$, and similarly, $f(v) \le f(u)$, so $f(u) = f(v)$. Hence, we have proved that τ is a well-defined fuzzy subset of A_E.

Now, for any $u \in X^*$, by reflexivity of E it follows that

$$f(u) = E(u, u) \otimes f(u) \le \bigvee_{w \in X^*} E(u, w) \otimes f(w) = \bigvee_{w \in X^*} \delta_E^*(E_e, u, E_w) \otimes \tau(E_w),$$

and by extensionality of f with respect to E we have that

$$\bigvee_{w \in X^*} \delta_E^*(E_e, u, E_w) \otimes \tau(E_w) = \bigvee_{w \in X^*} E(u, w) \otimes f(w) \le f(u).$$

Therefore,

$$f(u) = \bigvee_{w \in X^*} \delta_E^*(E_e, u, E_w) \otimes \tau(E_w),$$

what means that $\mathscr{A}_E$ recognizes f by τ. $\square$

Theorem 8.2. *Let E be a fuzzy right congruence on X^* and let $\widehat{E}$ be its crisp part. Then*:

(a) *The crisp part of $\mathscr{A}_E$ is a deterministic automaton isomorphic to $\mathscr{A}_{\widehat{E}}$.*
(b) *Any fuzzy language $f \in \mathscr{F}(X^*)$ recognized by $\mathscr{A}_E$ is also recognized by $\mathscr{A}_{\widehat{E}}$.*

Proof. This theorem was proved in [25], but for the sake of completeness, we will repeat the proof of (a), and we will give a simpler proof of (b).

(a) According to Lemma 2.1, for arbitrary $u, v, p \in X^*$ we have that

$$\delta_E^*(E_u, p, E_v) = 1 \Leftrightarrow E(up, v) = 1 \Leftrightarrow E_{up} = E_v \Leftrightarrow \widehat{E}_{up} = \widehat{E}_v \Leftrightarrow \delta_{\widehat{E}}^*(\widehat{E}_u, p) = \widehat{E}_v.$$

This means that $\widehat{\delta_E^*}$ is a mapping of $A_E \times X^*$ into A_E, and the mapping $E_u \mapsto \widehat{E}_u$ is an isomorphism of the deterministic automaton $\widehat{\mathscr{A}_E} = (A_E, X, \widehat{\delta}_E)$ onto the deterministic automaton $\mathscr{A}_{\widehat{E}} = (A_{\widehat{E}}, X, \delta_{\widehat{E}})$.

(b) According to Theorem 8.1, if f is recognized by $\mathscr{A}_E$, then it is extensional w.r.t. E, i.e. $E \le E_f$, what implies that $\widehat{E} \subseteq \widehat{E_f} = \ker f$. Therefore, f is extensional w.r.t. $\widehat{E}$, and by Theorem 3.1 we conclude that f is recognized by $\mathscr{A}_{\widehat{E}}$. $\square$

The converse of the assertion (b) of Theorem 8.2 does not hold, i.e., there is a fuzzy right congruence E on X^* and a fuzzy language $f \in \mathscr{F}(X^*)$ which can be recognized by $\mathscr{A}_{\widehat{E}}$, but it cannot be recognized by $\mathscr{A}_E$. This is demonstrated by the following example.

Example 8.1. Let $\mathscr{L}$ be the Gödel structure and let $X = \{x, y\}$. Consider a right congruence π on X^* having three equivalence classes

$$\pi_e = \{e\}, \quad \pi_y = \{yx^n \mid n \ge 0\}, \quad \pi_x = X^+ \setminus \pi_y,$$

a fuzzy right congruence E on X^* defined by

$$E(u, v) = \begin{cases} 1 & \text{if } (u, v) \in \pi_x \times \pi_x \cup \pi_y \times \pi_y \cup \pi_e \times \pi_e, \\ 0.5 & \text{if } (u, v) \in \pi_x \times \pi_y \cup \pi_y \times \pi_x, \\ 0 & \text{otherwise}, \end{cases}$$

for any $u, v \in X^*$, and a fuzzy language $f \in \mathscr{F}(X^*)$ defined by

$$f(u) = \begin{cases} 1 & \text{if } u \in \pi_x, \\ 0 & \text{otherwise}, \end{cases}$$

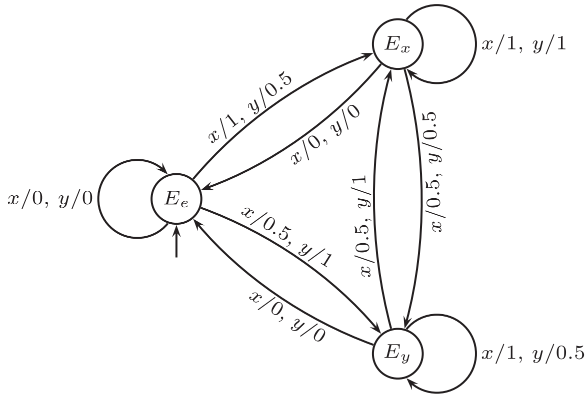


Fig. 3. The fuzzy automaton $\mathscr{A}_E$.

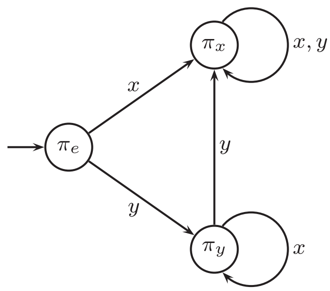


Fig. 4. The deterministic automaton $\mathscr{A}_\pi$.

for any $u \in X^*$ (i.e., f is a crisp language equal to the equivalence class π_x). We have that π is the crisp part of E, and f is extensional w.r.t. π, since $\pi \subset \ker f$, but f is not extensional w.r.t. E, since for any $u \in \pi_x$ and $v \in \pi_y$ we have that

$$f(u) \otimes E(u, v) = 1 \otimes 0.5 = 0.5 > 0 = f(v).$$

By this it also follows that f can be recognized by $\mathscr{A}_{\widehat{E}}$, but it cannot be recognized by $\mathscr{A}_E$, so we have that $\mathrm{Rec}(\mathscr{A}_E, E_e) \subsetneq \mathrm{Rec}(\mathscr{A}_{\widehat{E}}, \widehat{E}_e)$.

The fuzzy automaton $\mathscr{A}_E$ and the deterministic automaton $\mathscr{A}_\pi$ ($\pi = \widehat{E}$) are represented by Figs. 3 and 4.

If E is a fuzzy equivalence on a set A, then the class $\mathscr{H}_E$ of all fuzzy subsets of A extensional w.r.t. E is a complete sublattice of the lattice $\mathscr{F}(A)$ (cf. [27,17]). In particular, if π is a crisp equivalence on A, then by Lemma 2.1, the mapping $f \mapsto \tau_f$ is an order-isomorphism of $\mathscr{H}_\pi$ onto $\mathscr{F}(A/\pi)$, and hence, $\mathscr{H}_\pi$ and $\mathscr{F}(A/\pi)$ are isomorphic complete lattices (cf. [15]). Therefore, if π is a right congruence on X^*, then for any fuzzy language $f \in \mathscr{H}_\pi$ there is a unique fuzzy set $\tau_f \in \mathscr{F}(A/\pi)$ of such that $\mathscr{A}_\pi$ recognizes f by τ_f. This does not necessary hold for all fuzzy right congruence automata.

Example 8.2. The fuzzy right congruence automaton $\mathscr{A}_E$ given in Fig. 3 recognizes a fuzzy language $f \in \mathscr{F}(X^*)$ defined by

$$f(u) = \begin{cases} 1 & \text{if } u \in \pi_x, \\ 0.5 & \text{if } u \in \pi_y, \\ 0 & \text{if } u = e, \end{cases}$$

by any fuzzy set $\tau \in \mathscr{F}(A_E)$ given by

$$\tau = \begin{pmatrix} E_e & E_x & E_y \\ 0 & 1 & r \end{pmatrix},$$

where $r \in [0, 0.5]$ is an arbitrary number.

9. Myhill's and Nerode's fuzzy and crisp relations of fuzzy automata

In the previous section we have demonstrated how to construct a fuzzy automaton starting from a given fuzzy right congruence on the free monoid X^*. The opposite problem can be also considered: How to assign, in a natural way, a fuzzy right congruence on X^* to a given fuzzy automaton $\mathscr{A}$? It is well-known that a crisp relation η on X^* can be assigned to a crisp initial deterministic automaton $\mathscr{A} = (A, a_0, X, \delta)$ in the following way: for any $u, v \in X^*$,

$$(u, v) \in \eta \quad \Leftrightarrow \quad \delta^*(a_0, u) = \delta^*(a_0, v). \tag{43}$$

The relation η is a right congruence on X^*, and we call it the *Nerode's right congruence* associated with $\mathscr{A}$ (cf. [30, Section 9.6]). This concept has been generalized to the fuzzy framework by Ignjatović et al. in [25], where the concept of a Nerode's fuzzy right congruence associated with an initial fuzzy automaton has been introduced.

Let $\mathscr{A} = (A, \sigma, X, \delta)$ be an initial fuzzy automaton. For any word $u \in X^*$, a fuzzy set $\sigma_u \in \mathscr{F}(A)$ is defined by

$$\sigma_u(a) = \bigvee_{b \in A} \sigma(b) \otimes \delta^*(b, u, a) = (\sigma \circ \delta_u)(a),$$

for any $a \in A$. The same formula is used to define a fuzzy language $f^{(\sigma,a)} \in \mathscr{F}(X^*)$, for each $a \in A$, i.e., for any $u \in X^*$ we set $f^{(\sigma,a)}(u) = \sigma_u(a)$. Now, a fuzzy relation N_σ on the free monoid X^* is defined by

$$N_\sigma(u, v) = \bigwedge_{a \in A} \sigma_u(a) \leftrightarrow \sigma_v(a) = \bigwedge_{a \in A} f^{(\sigma,a)}(u) \leftrightarrow f^{(\sigma,a)}(v), \tag{44}$$

for $u, v \in X^*$, and it is called the *Nerode's fuzzy relation* of $\mathscr{A}$. In particular, for any $a \in A$ we define $\sigma^a \in \mathscr{F}(A)$ by $\sigma^a(b) = 1$, if $b = a$, and $\sigma^a(b) = 0$, if $b \neq a$, and we denote the corresponding Nerode's fuzzy relation by N_a. In this case we have that $\sigma^a_u(b) = \delta^*(a, u, b)$, for each $u \in X^*$ and $b \in A$, and N_a can be represented by

$$N_a(u, v) = \bigwedge_{b \in A} \delta^*(a, u, b) \leftrightarrow \delta^*(a, v, b). \tag{45}$$

Comparing (43) and (45), we can realize that the Nerode's fuzzy relation of a fuzzy automaton is an immediate generalization of the concept of a Nerode's right congruence of a crisp deterministic automaton.

It was proved in [25] that the following is true:

Theorem 9.1. *For any initial fuzzy automaton $\mathscr{A} = (A, \sigma, X, \delta)$, the Nerode's fuzzy relation N_σ is a fuzzy right congruence on X^*.*

In view of the previous theorem, the relation N_σ will be further called the *Nerode's fuzzy right congruence* of $\mathscr{A}$. Its crisp part $\widehat{N}_\sigma$ is a right congruence on X^*, and we will call it the *Nerode's right congruence* of $\mathscr{A}$. It is worth noting that for any $u, v \in X^*$ the following is true:

$$(u, v) \in \widehat{N}_\sigma \quad \Leftrightarrow \quad \sigma_u = \sigma_v. \tag{46}$$

Next, let $\mathscr{A} = (A, \sigma, X, \delta, \tau)$ be a fuzzy recognizer, and let N be the Nerode's fuzzy right congruence of $\mathscr{A}$. In the way shown in the previous section, we can form the fuzzy right congruence automaton (A_N, N_e, X, δ_N) associated with N, with a crisp initial state N_e, the fuzzy equivalence class determined by the empty word e. Moreover, we can define a fuzzy set $\tau_N : A_N \to L$ of terminal states as follows:

$$\tau_N(N_u) = \sigma_u \circ \tau = L(\mathscr{A})(u), \tag{47}$$

for any fuzzy equivalence class $N_u \in A_N$, $u \in X^*$. If $N_u = N_v$, for some $u, v \in X^*$, then $(u, v) \in \widehat{N}$, and by (46) we have that $\sigma_u = \sigma_v$, whence $\tau_N(N_u) = \sigma_u \circ \tau = \sigma_v \circ \tau = \tau_N(N_v)$. Thus, τ_N is a well-defined mapping and $\mathscr{A}_N = (A_N, N_e, X, \delta_N, \tau_N)$ is a fuzzy recognizer.

We have the following:

Theorem 9.2. *Let $\mathscr{A} = (A, \sigma, X, \delta, \tau)$ be a fuzzy recognizer, let N be the Nerode's fuzzy right congruence of $\mathscr{A}$ and $\mathscr{A}_N = (A_N, N_e, X, \delta_N, \tau_N)$. Then $L(\mathscr{A}_N) = L(\mathscr{A})$.*

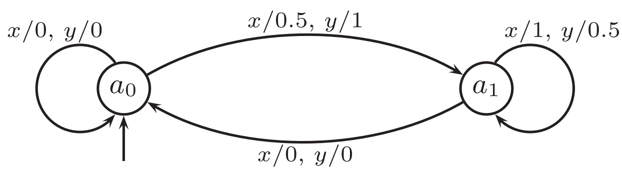
Proof. This theorem was proved in [25], but here we give a simpler proof. For the sake of simplicity set $L(\mathscr{A}) = f$. Then for any $u, v \in X^*$, by (3) and (6) it follows that

$$\begin{aligned} N(u,v) &= \bigwedge_{a\in A} \sigma_u(a) \leftrightarrow \sigma_v(a) \le \bigwedge_{a\in A} [\sigma_u(a) \otimes \tau(a) \leftrightarrow \sigma_v(a) \otimes \tau(a)] \\ &\le \left[\bigvee_{a\in A} \sigma_u(a) \otimes \tau(a)\right] \leftrightarrow \left[\bigvee_{a\in A} \sigma_v(a) \otimes \tau(a)\right] = f(u) \leftrightarrow f(v), \end{aligned}$$

and hence, f is extensional w.r.t. N. Now, according to (47) and the proof of Theorem 8.1 we conclude that $f = L(\mathscr{A}_N)$. □

By the above theorem it follows that any fuzzy language recognized by an initial fuzzy automaton $\mathscr{A}$ can be recognized by the fuzzy automaton $\mathscr{A}_N$, where N is the Nerode's fuzzy right congruence of $\mathscr{A}$. The converse does not hold, as the following example shows.

Example 9.1. Let $\mathscr{L}$ be the Gödel structure, and let an initial fuzzy finite automaton $\mathscr{A} = (A, a_0, X, \delta)$ over $\mathscr{L}$, where $A = \{a_0, a_1\}$ and $X = \{x, y\}$, be given by the following transition graph:



This fuzzy automaton was considered in Example 4.2 of [25], and according to this example, we can easily verify that the Nerode's fuzzy right congruence N of $\mathscr{A}$ is exactly the fuzzy right congruence E from Example 8.1, so $\mathscr{A}_N$ is exactly the fuzzy automaton represented by Fig. 3. Now, consider a fuzzy language $f \in \mathscr{F}(X^*)$ defined by

$$f(u) = \begin{cases} 0 & \text{if } u = e, \\ 0.6 & \text{if } u \in \pi_x, \\ 0.8 & \text{if } u \in \pi_y, \end{cases}$$

for any $u \in X^*$, where π is a right congruence on X^* defined in Example 8.1. It can be easily shown that f is extensional w.r.t. N, so $\mathscr{A}_N$ recognizes f.

On the other hand, suppose that $\mathscr{A}$ recognizes f by a fuzzy set of terminal states $\tau \in \mathscr{F}(A)$. Then

$$\begin{aligned} f(e) &= (\delta^*(a_0, e, a_0) \wedge \tau(a_0)) \vee (\delta^*(a_0, e, a_1) \wedge \tau(a_1)) \\ &= (1 \wedge \tau(a_0)) \vee (0 \wedge \tau(a_1)) = \tau(a_0), \end{aligned}$$

whence $\tau(a_0) = 0$, and

$$\begin{aligned} f(x) &= (\delta^*(a_0, x, a_0) \wedge \tau(a_0)) \vee (\delta^*(a_0, x, a_1) \wedge \tau(a_1)) \\ &= (0 \wedge 0) \vee (0.5 \wedge \tau(a_1)) = 0.5 \wedge \tau(a_1) \le 0.5, \end{aligned}$$

what is not possible, because $f(x) = 0.6$. Therefore, f cannot be recognized by $\mathscr{A}$, and by this fact and Example 8.1 we obtain that $\mathrm{Rec}(\mathscr{A}, a_0) \subsetneq \mathrm{Rec}(\mathscr{A}_N, N_e) \subsetneq \mathrm{Rec}(\mathscr{A}_{\widehat{N}}, \widehat{N}_e)$.

To a fuzzy automaton $\mathscr{A} = (A, X, \delta)$, we can also assign a fuzzy relation $M_{\mathscr{A}}$ on the free monoid X^* defined by

$$M_{\mathscr{A}}(u, v) = \bigwedge_{a\in A} N_a(u, v) = \bigwedge_{a,b\in A} \delta^*(a, u, b) \leftrightarrow \delta^*(a, v, b) = \bigwedge_{a,b\in A} \delta_u(a, b) \leftrightarrow \delta_v(a, b), \tag{48}$$

for $u, v \in X^*$, which is called the *Myhill's fuzzy relation* of the fuzzy automaton $\mathscr{A}$. By this definition, for any $u, v \in X^*$, $M_{\mathscr{A}}(u, v)$ can be interpreted as the degree of equality of transition relations δ_u and δ_v.

Theorem 9.3. *For any fuzzy automaton $\mathscr{A} = (A, X, \delta)$, the Myhill's fuzzy relation $M_{\mathscr{A}}$ is a fuzzy congruence on X^*.*

Proof. Seeing that $M_{\mathscr{A}}$ is the intersection of fuzzy right congruences N_a, $a \in A$, we have that it is also a fuzzy right congruence. To prove that $M_{\mathscr{A}}$ is a fuzzy left congruence, consider arbitrary $u, v, w \in X^*$. Then for any $a, b, c \in A$, by (3) it follows that

$$M_{\mathscr{A}}(u, v) \leq \delta^*(c, u, b) \leftrightarrow \delta^*(c, v, b) \leq \delta^*(a, w, c) \otimes \delta^*(c, u, b) \leftrightarrow \delta^*(a, w, c) \otimes \delta^*(c, v, b),$$

and by this and (6) we have that

$$\begin{aligned} M_{\mathscr{A}}(u, v) &\leq \bigwedge_{c\in A} [\delta^*(a, w, c) \otimes \delta^*(c, u, b) \leftrightarrow \delta^*(a, w, c) \otimes \delta^*(c, v, b)] \\ &\leq \left[\bigvee_{c\in A} \delta^*(a, w, c) \otimes \delta^*(c, u, b)\right] \leftrightarrow \left[\bigvee_{c\in A} \delta^*(a, w, c) \otimes \delta^*(c, v, b)\right] \\ &= \delta^*(a, wu, b) \leftrightarrow \delta^*(a, wv, b), \end{aligned}$$

for every $a, b \in A$, whence

$$M_{\mathscr{A}}(u, v) \leq \bigwedge_{a,b\in A} [\delta^*(a, wu, b) \leftrightarrow \delta^*(a, wv, b)] = M_{\mathscr{A}}(wu, wv).$$

Therefore, $M_{\mathscr{A}}$ is a fuzzy left congruence on X^*. $\square$

In view of Theorem 9.3, the relation $M_{\mathscr{A}}$ will be further called the *Myhill's fuzzy congruence* of $\mathscr{A}$. Its crisp part $\widehat{M}_{\mathscr{A}}$ is a congruence on X^*, and it will be called it the *Myhill's congruence* of $\mathscr{A}$. It is worth noting that for any $u, v \in X^*$ the following is true:

$$\begin{aligned} (u, v) \in \widehat{M}_{\mathscr{A}} &\Leftrightarrow (\forall a, b \in A)\, \delta^*(a, u, b) = \delta^*(a, v, b) \\ &\Leftrightarrow \delta_u = \delta_v. \end{aligned} \tag{49}$$

Myhill's congruences of fuzzy automata have been already studied by Malik et al. in [45] (see also [47, Section 6.2]), and the following has been shown:

Theorem 9.4. *Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton and let $\widehat{M}$ be the Myhill's congruence of $\mathscr{A}$. Then the factor monoid $X^*/\widehat{M}$ is isomorphic to the transition monoid $T(\mathscr{A})$ of $\mathscr{A}$.*

Proof. For the sake of completeness we will give a brief proof of this assertion.

According to (15), the mapping $\phi : X^* \to T(\mathscr{A})$ defined by $\phi(u) = \delta_u$, for any $u \in X^*$, is a homomorphism of X^* onto $T(\mathscr{A})$, and by (49) it follows that $\ker\phi = \widehat{M}$, whence it follows that $T(\mathscr{A})$ is isomorphic to $X^*/\ker\phi = X^*/\widehat{M}$. $\square$

Theorem 9.5. *Let E be a fuzzy right congruence on X^*. Then*

(a) *Nerode's fuzzy right congruence of $\mathscr{A}_E$ coincides with E.*
(b) *Myhill's fuzzy congruence of $\mathscr{A}_E$ is the fuzzy (left) congruence opening of E.*

Proof. Let the Nerode's fuzzy right congruence of $\mathscr{A}_E$ be denoted by N. By Eq. (18) of [12], and its consequence given also in [12], for arbitrary $u, v \in X^*$ we have that

$$N(u,v) = \bigwedge_{w\in X^*} \delta_E^*(E_e,u,E_w) \leftrightarrow \delta_E^*(E_e,v,E_w) = \bigwedge_{w\in X^*} E(u,w) \leftrightarrow E(v,w)$$
$$= \bigwedge_{w\in X^*} E_u(w) \leftrightarrow E_v(w) = E(u,v),$$

and hence, $N = E$. On the other hand, for any $p \in X^*$, Eq. (18) of [12] yields

$$N_{E_p}(u,v) = \bigwedge_{w\in X^*} \delta_E^*(E_p,u,E_w) \leftrightarrow \delta_E^*(E_p,v,E_w) = \bigwedge_{w\in X^*} E(pu,w) \leftrightarrow E(pv,w)$$
$$= \bigwedge_{w\in X^*} E_w(pu) \leftrightarrow E_w(pv) = E(pu,pv),$$

by which it follows

$$M_{\mathscr{A}_E}(u,v) = \bigwedge_{p\in X^*} \varrho_{E_p}(u,v) = \bigwedge_{p\in X^*} E(pu,pv) = E^0(u,v).$$

Therefore, $M_{\mathscr{A}_E} = E^0$. □

10. Nerode's automaton associated with a fuzzy automaton

Let $\mathscr{A} = (A, \sigma, X, \delta)$ be an initial fuzzy automaton. The right congruence automaton associated with the Nerode's right congruence $\widehat{N}_\sigma$ will be called the *Nerode's automaton* of $\mathscr{A}$. The Nerode's automaton of $\mathscr{A}$ has been already studied by Ignjatović et al. in [25] and it was proved that it is isomorphic to the deterministic automaton $\mathscr{A}_\sigma$ obtained by determinization of $\mathscr{A}$ using the so-called accessible fuzzy subset construction.

By the next theorem we restate another result from [25], which provides necessary and sufficient conditions for an initial fuzzy automaton $\mathscr{A}$ to the Nerode's right congruence of $\mathscr{A}$ have a finite index, i.e., to the automaton $\mathscr{A}_\sigma$ be finite.

Theorem 10.1. *Let $\mathscr{A} = (A, \sigma, X, \delta)$ be an initial fuzzy finite automaton. Then the following conditions are equivalent*:

(i) *The Nerode's right congruence $\widehat{N}_\sigma$ of $\mathscr{A}$ has a finite index.*
(ii) *The automaton $\mathscr{A}_\sigma$ is finite.*
(iii) *The fuzzy language $f^{(\sigma,a)}$ has a finite rank, for each $a \in A$.*
(iv) *Any fuzzy language from* $\mathrm{Rec}(\mathscr{A}, \sigma)$ *is DFA-recognizable.*

For the proof of the above theorem we refer to [25].

Next, we give the following subdirect representation of the Nerode's automaton $\mathscr{A}_\sigma$.

Theorem 10.2. *For any initial fuzzy automaton $\mathscr{A} = (A, \sigma, X, \delta)$, the automaton $\mathscr{A}_\sigma$ is a subdirect product of derivative automata of fuzzy languages $f^{(\sigma,a)}$, $a \in A$.*

In addition, $\mathscr{A}_\sigma$ is isomorphic to the accessible part of the direct product of derivative automata of fuzzy languages $f^{(\sigma,a)}$, $a \in A$.

Proof. Let the direct product of derivative automata $\mathscr{A}_{f^{(\sigma,a)}} = (A_{f^{(\sigma,a)}}, f^{(\sigma,a)}, X, \delta_{f^{(\sigma,a)}})$, $a \in A$, be denoted by $\mathscr{A}_\Pi = (A_\Pi, \alpha_0, X, \delta_\Pi)$, i.e., let

$$A_\Pi = \prod_{a\in A} A_{f^{(\sigma,a)}},$$

and an initial state $\alpha_0 \in A_\Pi$ and a transition function $\delta_\Pi : A_\Pi \times X \to A_\Pi$ are defined by

$$\alpha_0(a) = f^{(\sigma,a)}, \quad \delta_\Pi(\alpha, x)(a) = \delta_{f^{(\sigma,a)}}(\alpha(a), x),$$

for any $a \in A$, $\alpha \in A_\Pi$ and $x \in X$. Now, define a mapping $\phi : A_\sigma \to A_\Pi$ by

$$\phi(\sigma_u)(a) = f_u^{(\sigma,a)},$$

for any $u \in X^*$ and $a \in A$ (where $f_u^{(\sigma,a)}$ denotes the derivative of $f^{(\sigma,a)}$ w.r.t. u).

For arbitrary $u, v \in X^*$ we have that

$$\begin{aligned}\sigma_u = \sigma_v &\Leftrightarrow (u, v) \in \widehat{N}_\sigma \\ &\Leftrightarrow (\forall w \in X^*)\,(uw, vw) \in \widehat{N}_\sigma \\ &\Leftrightarrow (\forall w \in X^*)(\forall a \in A)\, f^{(\sigma,a)}(uw) = f^{(\sigma,a)}(vw) \\ &\Leftrightarrow (\forall a \in A)(\forall w \in X^*)\, f_u^{(\sigma,a)}(w) = f_v^{(\sigma,a)}(w) \\ &\Leftrightarrow (\forall a \in A)\, f_u^{(\sigma,a)} = f_v^{(\sigma,a)},\end{aligned}$$

and hence, ϕ is single-valued and injective.

Further, for any $u, v \in X^*$ and $a \in A$ we have that

$$\begin{aligned}\phi(\delta_\sigma^*(\sigma_u, v))(a) &= \phi(\sigma_{uv})(a) = f_{uv}^{(\sigma,a)} = \delta_{f^{(\sigma,a)}}^*(f_u^{(\sigma,a)}, v) = \delta_{f^{(\sigma,a)}}^*(\phi(\sigma_u)(a), v) \\ &= \delta_\Pi^*(\phi(\sigma_u), v)(a),\end{aligned}$$

whence $\phi(\delta_\sigma^*(\sigma_u, v)) = \delta_\Pi^*(\phi(\sigma_u), v)$, and we have obtained that ϕ is a homomorphism, i.e., it is an isomorphism of $\mathscr{A}_\sigma$ onto a subautomaton $\mathrm{Im}\phi$ of $\mathscr{A}_\Pi$.

Next, consider arbitrary $a \in A$ and $f \in A_{f^{(\sigma,a)}}$. Then there exists $u \in X^*$ such that

$$f = f_u^{(\sigma,a)} = \mathrm{pr}_a(\phi(\sigma_u)) = \phi \circ \mathrm{pr}_a(\sigma_u),$$

where pr_a denotes the projection homomorphism of $\mathscr{A}_\Pi$ onto $\mathscr{A}_{f^{(\sigma,a)}}$, and therefore, $\mathscr{A}_\sigma$ is a subdirect product of automata $\mathscr{A}_{f^{(\sigma,a)}}$, $a \in A$.

Finally, for any $u \in X^*$ we have that

$$\phi(\sigma_u)(a) = f_u^{(\sigma,a)} = \delta_{f^{(\sigma,a)}}^*(f^{(\sigma,a)}, u) = \delta_{f^{(\sigma,a)}}^*(\alpha_0(a), u) = \delta_\Pi^*(\alpha_0, u)(a),$$

for each $a \in A$, whence $\phi(\sigma_u) = \delta_\Pi^*(\alpha_0, u)$. Hence, $\mathrm{Im}\,\phi$ is the accessible part of $\mathscr{A}_\Pi$, what completes the proof of the theorem. □

Now we prove that the automaton $\mathscr{A}_\sigma$ possess the following minimal properties.

Theorem 10.3. *Let $\mathscr{A} = (A, \sigma, X, \delta)$ be an initial fuzzy automaton. Then*

(a) *$\mathscr{A}_\sigma$ is a minimal deterministic automaton which recognizes all fuzzy languages $f^{(\sigma,a)}$, $a \in A$.*
(b) *$\mathscr{A}_\sigma$ is a minimal deterministic automaton which recognizes all fuzzy languages from* $\mathrm{Rec}(\mathscr{A}, \sigma)$.

Proof. (a) By Theorem 4.1 of [25], the automaton $\mathscr{A}_\sigma$ recognizes every fuzzy language from $\mathrm{Rec}(\mathscr{A}, \sigma)$, and hence, it recognizes all fuzzy languages $f^{(\sigma,a)}$, $a \in A$. Consider an arbitrary deterministic automaton $\mathscr{B} = (B, b_0, X, \lambda)$ which recognizes all fuzzy languages $f^{(\sigma,a)}$, $a \in A$. Without loss of generality we can assume that $\mathscr{B}$ is accessible.

According to Theorem 5.3, for each $a \in A$ there exists a factor automaton of $\mathscr{B}$ isomorphic to the derivative automaton $\mathscr{A}_{f^{(\sigma,a)}}$ of $f^{(\sigma,a)}$, what means that there exists a homomorphism ψ^a of $\mathscr{B}$ onto $\mathscr{A}_{f^{(\sigma,a)}}$ such that $\psi^a(b_0) = f^{(\sigma,a)}$. Using notation from the proof of Theorem 10.2, let us define a mapping $\psi : B \to A_\Pi$ by

$$(\psi(b))(a) = \psi^a(b), \tag{50}$$

for every $b \in B$ and $a \in A$. Then for every $b \in B$, $u \in X^*$ and $a \in A$ we have that

$$\begin{aligned}[\psi(\lambda^*(b, u))](a) &= \psi^a(\lambda^*(b, u)) = \delta_{f^{(\sigma,a)}}^*(\psi^a(b), u) = \delta_{f^{(\sigma,a)}}^*([\psi(b)](a), u) \\ &= [\delta_\Pi^*(\psi(b), u)](a)\end{aligned}$$

so $\psi(\lambda^*(b,u)) = \delta^*_{\Pi}(\psi(b),u)$, what means that ψ is a homomorphism. We also have that $\psi(b_0) = \alpha_0$, since $(\psi(b_0))(a) = \psi^a(b_0) = f^{(\sigma,a)} = \alpha_0(a)$, for each $a \in A$. Therefore, the factor automaton $\mathscr{B}/\ker\psi$ is isomorphic to a subautomaton $\mathscr{A}'$ of $\mathscr{A}_{\Pi}$ containing the initial state α_0 of $\mathscr{A}_{\Pi}$. On the other hand, by Theorem 10.2, the automaton $\mathscr{A}_\sigma$ is isomorphic to the accessible part $\mathscr{A}^*_{\Pi}$ of $\mathscr{A}_{\Pi}$, and since $\mathscr{A}^*_{\Pi}$ is the least subautomaton of $\mathscr{A}_{\Pi}$ containing the initial state α_0, we conclude that $\mathscr{A}^*_{\Pi}$ is contained in $\mathscr{A}'$. Now,

$$|\mathscr{A}_\sigma| = |\mathscr{A}^*_{\Pi}| \leq |\mathscr{A}'| = |\mathscr{B}/\ker\psi| \leq |\mathscr{B}|.$$

Hence, $\mathscr{A}_\sigma$ is a minimal deterministic automaton recognizing all fuzzy languages $f^{(\sigma,a)}$, $a \in A$.

(b) As we have mentioned earlier, the automaton $\mathscr{A}_\sigma$ recognizes every fuzzy language from $\mathrm{Rec}(\mathscr{A},\sigma)$. If $\mathscr{B}$ is an arbitrary deterministic automaton which recognizes all fuzzy languages from $\mathrm{Rec}(\mathscr{A},\sigma)$, then $\mathscr{B}$ recognizes all fuzzy languages $f^{(\sigma,a)}$, $a \in A$, and by (a) we obtain that $|\mathscr{A}_\sigma| \leq |\mathscr{B}|$. Therefore, we have proved that $\mathscr{A}_\sigma$ is a minimal deterministic automaton which recognizes all fuzzy languages from $\mathrm{Rec}(\mathscr{A},\sigma)$. □

11. Myhill's automaton associated with a fuzzy automaton

In this section we define and study the concept of a Myhill's automaton of a fuzzy automaton.

Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton and let $\widehat{M}$ be the Myhill's congruence of $\mathscr{A}$. The right congruence automaton $\mathscr{A}_{\widehat{M}} = (A_{\widehat{M}}, \widehat{M}_e, X, \delta_{\widehat{M}})$ associated with $\widehat{M}$ will be called the *Myhill's automaton* of $\mathscr{A}$.

We will show that the Myhill's automaton of $\mathscr{A}$ can be also constructed in another way. Namely, let T denote the set of all transition relations of $\mathscr{A}$, i.e., $T = \{\delta_u \mid u \in X^*\}$, and let the mapping $\delta_T : T \times X \to T$ be defined by

$$\delta_T(\delta_u, x) = \delta_{ux}, \tag{51}$$

for any $\delta_u \in T$, $u \in X^*$, and any $x \in X$. If $\delta_u = \delta_v$, for some $u, v \in X^*$, then by (49) it follows that $(u,v) \in \widehat{M}$, what implies $(ux, vx) \in \widehat{M}$, and again by (49) we obtain $\delta_{ux} = \delta_{vx}$. Therefore, δ_T is a well-defined mapping and $\mathscr{A}_T = (T, \delta_e, X, \delta_T)$ is an initial deterministic automaton.

Moreover, we have the following:

Theorem 11.1. *Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton, let $\widehat{M}$ be the Myhill's congruence of $\mathscr{A}$, and let T be the set of all transition relations of $\mathscr{A}$.*

Then the automaton $\mathscr{A}_T = (T, \delta_e, X, \delta_T)$ is isomorphic to the Myhill's automaton $\mathscr{A}_{\widehat{M}}$.

Proof. According to (49), the mapping $\phi : T \to A_{\widehat{M}}$ defined by $\phi(\delta_u) = \widehat{M}_u$, for every $u \in X^*$, is well-defined and injective, and it is evident that it is surjective. Moreover, it can be easily verified that ϕ is a homomorphism. Therefore, ϕ is an isomorphism of $\mathscr{A}_T$ onto $\mathscr{A}_{\widehat{M}}$. □

We can also prove the following:

Theorem 11.2. *Any fuzzy language recognized by a fuzzy automaton $\mathscr{A} = (A, X, \delta)$ is also recognized by the Myhill's automaton of $\mathscr{A}$.*

Proof. Assume that $\mathscr{A}$ recognizes a fuzzy language $f \in \mathscr{F}(X^*)$ by a fuzzy set of initial states $\sigma \in \mathscr{F}(A)$ and a fuzzy set of terminal states $\tau \in \mathscr{F}(A)$. Let the Myhill's congruence of $\mathscr{A}$ be denoted by $\widehat{M}$. If $(u,v) \in \widehat{M}$, for some $u, v \in X^*$, then by (49) it follows $\delta_u = \delta_v$, so $f(u) = \sigma \circ \delta_u \circ \tau = \sigma \circ \delta_v \circ \tau = f(v)$. Therefore, $\widehat{M} \subseteq \ker f$, i.e., f is extensional w.r.t. $\widehat{M}$, and according to Theorem 3.1, the Myhill's automaton $\mathscr{A}_{\widehat{M}}$ recognizes f. □

For any pair $(a,b) \in A^2$ let us define a fuzzy language $f^{(a,b)} \in \mathscr{F}(X^*)$ by

$$f^{(a,b)}(u) = \delta^*(a,u,b), \tag{52}$$

for any $u \in X^*$. These fuzzy languages will play an important role in the further work.

The next theorem is one of the main results of this section. It provides necessary and sufficient conditions for a fuzzy automaton $\mathscr{A}$ to the Myhill's congruence $\widehat{M}_{\mathscr{A}}$ have a finite index, i.e., to Myhill's automaton of $\mathscr{A}$ be finite.

Theorem 11.3. *Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy finite automaton. Then the following conditions are equivalent*:

(i) *The Myhill's congruence $\widehat{M}_{\mathscr{A}}$ of $\mathscr{A}$ has a finite index.*
(ii) *The Nerode's right congruence $\widehat{N}_a$ of $\mathscr{A}$ has a finite index, for each $a \in A$.*
(iii) *The transition monoid $T(\mathscr{A})$ of $\mathscr{A}$ is finite.*
(iv) *The fuzzy language $f^{(a,b)}$ has a finite rank, for every $(a, b) \in A^2$.*
(v) *Any fuzzy language from* Rec($\mathscr{A}$) *is DFA-recognizable.*

Proof. (i)⇔(iii). This is an immediate consequence of Theorem 9.4.

(i)⇒(v). This follows immediately by Theorem 11.2.

(v)⇒(iv). Consider arbitrary $a, b \in A$. It is easy to verify that $\mathscr{A}$ recognizes the fuzzy language $f^{(a,b)}$ by the crisp initial state $\{a\}$ and the crisp terminal state $\{b\}$, and according to the hypothesis, $f^{(a,b)}$ is DFA-recognizable. Now, by Theorem 4.3 we obtain that $f^{(a,b)}$ has a finite rank.

(iv)⇒(i). According to (49) and (52) we have that

$$\widehat{M}_{\mathscr{A}} = \bigcap_{(a,b)\in A^2} \ker f^{(a,b)}, \tag{53}$$

and since A^2 is finite and fuzzy languages $f^{(a,b)}$ have finite ranks, then

$$\mathrm{ind}(\widehat{M}_{\mathscr{A}}) \leq \prod_{(a,b)\in A^2} \mathrm{ind}(\ker f^{(a,b)}) = \prod_{(a,b)\in A^2} \mathrm{ran}\, f^{(a,b)}.$$

Therefore, $\widehat{M}_{\mathscr{A}}$ has a finite index.

(i)⇒(ii). By (48) it follows that

$$\widehat{M}_{\mathscr{A}} = \bigcap_{a\in A} \widehat{N}_a, \tag{54}$$

and for an arbitrary $a \in A$ we have that $\widehat{M}_{\mathscr{A}} \subseteq \widehat{N}_a$, what implies $\mathrm{ind}(\widehat{N}_a) \leq \mathrm{ind}(\widehat{M}_{\mathscr{A}})$. By the hypothesis, $\mathrm{ind}(\widehat{M}_{\mathscr{A}})$ is finite, so we conclude that $\mathrm{ind}(\widehat{N}_a)$ is finite, for each $a \in A$.

(ii)⇒(i). Since A is finite and any $\widehat{N}_a$ has a finite index, by (54) it follows that

$$\mathrm{ind}(\widehat{M}_{\mathscr{A}}) \leq \prod_{a\in A} \mathrm{ind}(\widehat{N}_a),$$

what implies that $\widehat{M}_{\mathscr{A}}$ has a finite index. □

Now we are ready to give an algorithm for construction of the Myhill's automaton of a fuzzy automaton.

Algorithm 11.1 (*Construction of the Myhill's automaton*). The input of this algorithm is a fuzzy finite automaton $\mathscr{A} = (A, X, \delta)$ over a complete residuated lattice $\mathscr{L}$, and the output is the Myhill's automaton $\mathscr{A}_{\widehat{M}} = (A_{\widehat{M}}, \widehat{M}_e, X, \delta_{\widehat{M}})$, that is, the isomorphic automaton $\mathscr{A}_T = (T, \delta_e, X, \delta_T)$. The procedure is to construct the *transition tree* of $\mathscr{A}_T$ directly from $\mathscr{A}$. It is constructed inductively in the following way:

(1) The root of the tree is the fuzzy relation δ_e, and we put $T_0 = \{\delta_e\}$.
(2) After the i-th step let a tree T_i has been constructed, and vertices in T_i have been labelled either 'closed' or 'non-closed'. The meaning of these two terms will be made clear in the sequel.
(3) In the next step we make a tree T_{i+1} constructing, for each non-closed leaf $\eta \in T_i$ and each $x \in X$, a vertex $\eta \circ \delta_x$ and an edge from η to $\eta \circ \delta_x$ labelled by x. If, in addition, $\eta \circ \delta_x$ is a repeat of any state that has already been constructed, then we say that it is *closed* and mark it with a ×. The procedure terminates when all leaves are marked closed.
(4) When the transition tree of $\mathscr{A}_T$ is constructed, we erase all × marks and glue leaves to interior vertices with the same label. The diagram that results is the transition graph of $\mathscr{A}_T$.

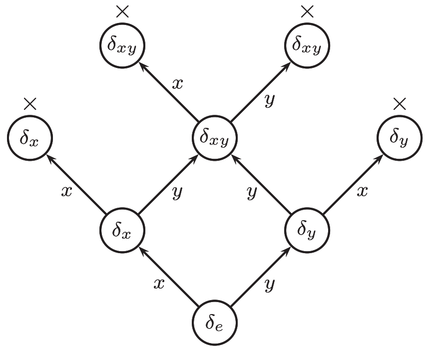


Fig. 5. The transition tree of $\mathscr{A}_T$.

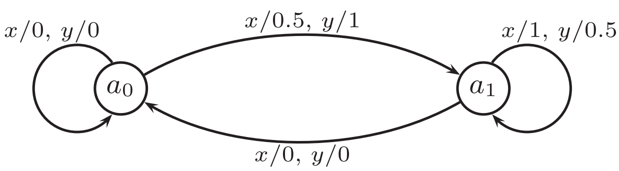
If the reduct $\mathscr{L}^*$ of $\mathscr{L}$ is locally finite, then the algorithm terminates in a finite number of steps, for any fuzzy finite automaton over $\mathscr{L}$, and the result is a finite deterministic initial automaton.

On the other hand, if $\mathscr{L}^*$ is not locally finite, then the algorithm terminates in a finite number of steps under conditions determined by Theorem 11.3.

Let us note that we can use the same algorithm to compute the transition monoid of $\mathscr{A}$.

Work of the above algorithm will be demonstrated by the following example.

Example 11.1. Let $\mathscr{L}$ be the Gödel structure, and let a fuzzy automaton $\mathscr{A} = (A, X, \delta)$ over $\mathscr{L}$, where $A = \{a_0, a_1\}$ and $X = \{x, y\}$, be given by the following transition graph:



If we represent δ_e, δ_x and δ_y as fuzzy matrices over $\mathscr{L}$, i.e.,

$$\delta_e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \delta_x = \begin{bmatrix} 0 & 0.5 \\ 0 & 1 \end{bmatrix}, \quad \delta_y = \begin{bmatrix} 0 & 1 \\ 0 & 0.5 \end{bmatrix},$$

then we have that

$$\delta_{x^2} = \delta_x \circ \delta_x = \begin{bmatrix} 0 & 0.5 \\ 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & 0.5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 \\ 0 & 1 \end{bmatrix} = \delta_x,$$

$$\delta_{xy} = \delta_x \circ \delta_y = \begin{bmatrix} 0 & 0.5 \\ 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 \\ 0 & 0.5 \end{bmatrix},$$

$$\delta_{yx} = \delta_y \circ \delta_x = \begin{bmatrix} 0 & 1 \\ 0 & 0.5 \end{bmatrix} \circ \begin{bmatrix} 0 & 0.5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0.5 \end{bmatrix} = \delta_y,$$

$$\delta_{y^2} = \delta_y \circ \delta_y = \begin{bmatrix} 0 & 1 \\ 0 & 0.5 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 \\ 0 & 0.5 \end{bmatrix} = \delta_{xy},$$

$$\delta_{xy} \circ \delta_x = \delta_{xy}, \quad \delta_{xy} \circ \delta_y = \delta_{xy},$$

and we obtain the transition tree of $\mathscr{A}_T$ represented by Fig. 5. Erasing $\times$ marks and merging leaves to interior vertices with the same label we obtain the transition graph of $\mathscr{A}_T$ represented by Fig. 6.

By this it follows that $\mathrm{ind}(\widehat{M}_{\mathscr{A}}) = 4$. On the other hand, Example 4.2 from [25] shows that $\mathrm{ind}(\widehat{N}_{a_0}) = 3$, and we can easily verify that $\mathrm{ind}(\widehat{N}_{a_1}) = 2$.

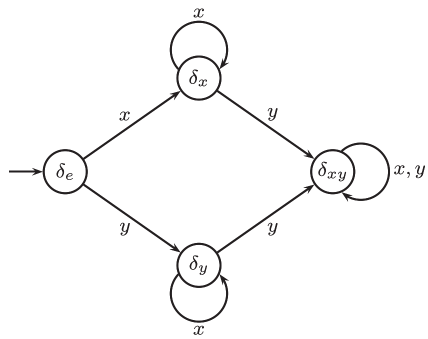
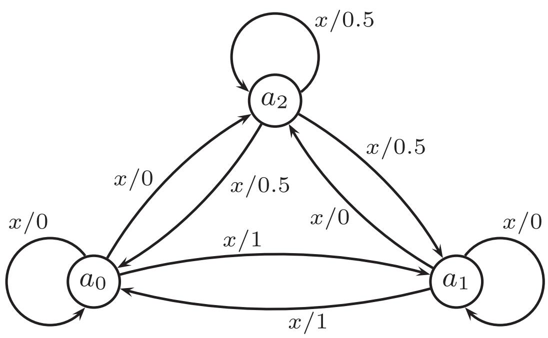


Fig. 6. The transition graph of $\mathscr{A}_T$.

Another example shows that the Myhill's congruence of a fuzzy automaton can have an infinite index, although some of the Nerode's right congruences appearing in (ii) have finite indices.

Example 11.2. Let $\mathscr{L}$ be the product structure, and let a fuzzy automaton $\mathscr{A} = (A, X, \delta)$ over $\mathscr{L}$, where $A = \{a_0, a_1, a_2\}$ and $X = \{x\}$, be given by the following transition graph:



Then σ_e, and δ_x, for any $x \in X$, can be represented as fuzzy matrices

$$\delta_e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \delta_x = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0.5 & 0.5 & 0.5 \end{bmatrix},$$

and for any $n \in \mathbb{N}$ we have that

$$\delta_{x^{2n}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.5 & 0.5 & (0.5)^{2n} \end{bmatrix}, \quad \delta_{x^{2n+1}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0.5 & 0.5 & (0.5)^{2n+1} \end{bmatrix}.$$

Therefore, the Myhill's congruence $\widehat{M}_{\mathscr{A}}$ has an infinite index. Moreover, it can be easily checked that $\mathrm{ind}(\widehat{N}_{a_0}) = \mathrm{ind}(\widehat{N}_{a_1}) = 2$, but $\widehat{N}_{a_2}$ has an infinite index.

Next we give two subdirect representations of Myhill's automata.

Theorem 11.4. *For any fuzzy automaton $\mathscr{A} = (A, X, \delta)$, the Myhill's automaton $\mathscr{A}_{\widehat{M}}$ of $\mathscr{A}$ is a subdirect product of derivative automata of fuzzy languages $f^{(a,b)}$, $(a, b) \in A^2$.*

In addition, $\mathscr{A}_{\widehat{M}}$ is isomorphic to the accessible part of the direct product of derivative automata of fuzzy languages $f^{(a,b)}$, $(a, b) \in A^2$.

Proof. Let $\widehat{M}$ denote the Myhill's congruence of $\mathscr{A}$, and let $\mathscr{A}_\Pi = (A_\Pi, \alpha_0, X, \delta_\Pi)$ be the direct product of derivative automata $\mathscr{A}_{f^{(a,b)}} = (A_{f^{(a,b)}}, f^{(a,b)}, X, \delta_{f^{(a,b)}})$, $(a, b) \in A^2$.

Define a mapping $\phi : A_{\widehat{M}} \to A_{\Pi}$ by

$$\phi(\widehat{M}_u)(a,b) = f_u^{(a,b)},$$

for any $u \in X^*$ and $(a,b) \in A^2$ (where $f_u^{(a,b)}$ is the derivative of $f^{(a,b)}$ w.r.t. u). In a similar way as in the proof of Theorem 10.2 we show that ϕ is an isomorphism of $\mathscr{A}_{\widehat{M}}$ onto a subautomaton $\operatorname{Im}\phi$ of $\mathscr{A}_{\Pi}$, and $\phi \circ \mathrm{pr}_{(a,b)}$ is a surjective homomorphism of $\mathscr{A}_{\widehat{M}}$ onto $\mathscr{A}_{f^{(a,b)}}$, for each $(a,b) \in A^2$. Therefore, $\mathscr{A}_{\widehat{M}}$ is a subdirect product of automata $\mathscr{A}_{f^{(a,b)}}$, $(a,b) \in A^2$. It can be also easily verified that $\operatorname{Im}\phi$ is the accessible part of $\mathscr{A}_{\Pi}$. This completes the proof of the theorem. □

Theorem 11.5. *For any fuzzy automaton $\mathscr{A} = (A, X, \delta)$, the Myhill's automaton $\mathscr{A}_{\widehat{M}}$ of $\mathscr{A}$ is a subdirect product of Nerode's automata $\mathscr{A}_{\widehat{N}_a}$, $a \in A$.*

In addition, $\mathscr{A}_{\widehat{M}}$ is isomorphic to the accessible part of the direct product of Nerode's automata $\mathscr{A}_{\widehat{N}_a}$, $a \in A$.

Proof. Let $\widehat{M}$ denote the Myhill's congruence of $\mathscr{A}$, and let $\mathscr{A}_{\Pi} = (A_{\Pi}, \alpha_0, X, \delta_{\Pi})$ be the direct product of Nerode's automata $\mathscr{A}_{\widehat{N}_a}$, $a \in A$.

Define a mapping $\phi : A_{\widehat{M}} \to A_{\Pi}$ by

$$\phi(\widehat{M}_u)(a) = \sigma_u^a,$$

for any $u \in X^*$ and $a \in A$ (where σ_u^a is the characteristic function of the set $\{a\}$). It is not hard to check that ϕ is an isomorphism of $\mathscr{A}_{\widehat{M}}$ onto a subautomaton $\operatorname{Im}\phi$ of $\mathscr{A}_{\Pi}$, and $\phi \circ \mathrm{pr}_a$ is a surjective homomorphism of $\mathscr{A}_{\widehat{M}}$ onto $\mathscr{A}_{\widehat{N}_a}$, for each $a \in A$. Therefore, $\mathscr{A}_{\widehat{M}}$ is a subdirect product of automata $\mathscr{A}_{\widehat{N}_a}$, $a \in A$. It can be also shown that $\operatorname{Im}\phi$ is the accessible part of $\mathscr{A}_{\Pi}$. This completes the proof of the theorem. □

Finally, we show that the Myhill's automaton also possess certain minimal properties.

Theorem 11.6. *Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton. Then*

(a) *The Myhill's automaton $\mathscr{A}_{\widehat{M}}$* of *$\mathscr{A}$ is a minimal deterministic automaton which recognizes all fuzzy languages $f^{(a,b)}$, $(a,b) \in A^2$.*
(b) *The Myhill's automaton $\mathscr{A}_{\widehat{M}}$ of $\mathscr{A}$ is a minimal deterministic automaton which recognizes all fuzzy languages from* $\mathrm{Rec}(\mathscr{A})$.

Proof. This theorem can be proved in a similar way as Theorem 10.3, using Theorems 11.2 and 11.4. □

12. Related work on fuzzy automata and languages

Fuzzy automata and languages have been studied first by Santos [64–66], Wee [71], Wee and Fu [72], Lee and Zadeh [31], and later in numerous papers. From late 1960s until early 2000s mainly fuzzy automata and languages with membership values in the Gödel structure have been considered. Malik and Mordeson's book [47] gives the most complete overview of the results obtained in this period. In recent years researcher's attention has been aimed mostly to fuzzy automata and languages with membership values in more general structures, such as complete residuated lattices, lattice ordered monoids, and other kinds of lattices.

Fuzzy automata and languages taking membership values in a complete residuated lattice have been first investigated by Qiu, who has considered in [56,57] some basic concepts, and later, he and his coworkers have carried out extensive research on these fuzzy automata and languages. They have proved Pumping lemmata for fuzzy regular languages and fuzzy context-free languages [59,73], they have proved equivalence between fuzzy context-free grammars and fuzzy pushdown automata [75], they have studied equivalence between fuzzy automata and their reduction [76], and in [74] fuzzy automata have been studied from the aspect of category theory. Authors of this paper have studied fuzzy automata from different point of view. They have given a new determinization algorithm [25], and powerful state reduction methods [13,14].

Another contemporary mainstream is study of fuzzy automata and languages with membership values in a lattice-ordered monoid. It started by Li and Pedrycz [37], who have considered regular operations on fuzzy languages, fuzzy regular expressions and fuzzy regular languages, and have given the Kleene type theorem for fuzzy automata and languages over a lattice-ordered monoid. They have also studied DFA-recognizability of fuzzy languages, relationship between FFA- and DFA-recognizability, and they have determined necessary and sufficient conditions on the underlying lattice-ordered monoid under which FFA- and DFA-recognizability are equivalent. Equivalence between fuzzy regular grammars and fuzzy finite automata has been proved in [68,23], and in [33] fuzzy regular languages have been characterized in terms of closeness under regular operations on fuzzy languages. From various aspects fuzzy automata and languages over a lattice-ordered monoid have been also studied in [32,34,38,40,41].

Except lattice-ordered monoids and complete residuated lattices, different structures of truth values for fuzzy automata and languages have been also used, such as complete lattices, distributive lattices, and other types of lattices [1,4,39], and the real unit interval with various kinds of t-norms and t-conorms [8,9,36].

The structure of membership values considered in the first part of this paper is very general, and includes as its special cases not only all structures used for modelling membership values in the fuzzy set theory, but it also includes semirings, used for modelling weights in the theory of weighted automata. Therefore, results from the first part of this paper are applicable to fuzzy languages over any structure of truth values used in the fuzzy set theory, as well as to formal power series over semirings. It is worth noting that there is a lot of similarities, but also a lot of differences between fuzzy and weighted automata, and these similarities and differences are a very interesting topic for research.

As we have already mentioned, Bozapalidis and Louscou-Bozapalidou [8,9] have proposed study of fuzzy automata defined by means of a pair consisting of a t-norm and a t-conorm on the real unit interval, and they have found especially interesting to study fuzzy automata defined by means of nondistributive pairs. Recently, Droste, Stüber and Vogler (private communication) have started study of weighted automata over a strong bimonoid, a structure which is not necessary distributive and generalizes both lattices and semirings. They have shown that lack of distributivity makes it possible to define behavior of such weighted automata in several different ways. It is worth noting that there are many natural examples of such structures and weighted automata over them. Among the most important ones are complete orthomodular lattices, which serve as a basis of quantum logic. Automata based on quantum logic, i.e., automata over a complete orthomodular lattice, may be viewed as a logical approach of quantum computation. They have been first studied by Ying [77,78], and then in papers of the same author [79], Qiu [58,60], and Li [35].

13. Concluding remarks

The Myhill–Nerode theory is a branch of the algebraic theory of languages and automata in which formal languages and deterministic automata are studied through right congruences and congruences on a free monoid. In this paper we develop a general Myhill–Nerode type theory for fuzzy languages with membership values in an arbitrary set with two distinguished elements 0 and 1, which are needed to take crisp languages in consideration. We establish connections between extensionality of fuzzy languages w.r.t. right congruences and congruences on a free monoid and recognition of fuzzy languages by deterministic automata and monoids, and we prove the Myhill–Nerode type theorem for fuzzy languages. Related results have been given by Malik et al. [44], Shen [67], Mordeson and Malik [47], Petković [51] and Bozapalidis and Louscou-Bozapalidou [8,9].

We also prove that each fuzzy language possess a minimal deterministic automaton recognizing it, we give a construction of this automaton using the concept of a derivative automaton of a fuzzy language and we give a method for minimization of deterministic fuzzy recognizers. For some related results we refer to Li and Pedrycz in [37,39], Petković [51] and Bozapalidis and Louscou-Bozapalidou [8,9].

In the second part of the paper we introduce and study Nerode's and Myhill's automata assigned to a fuzzy automaton with membership values in a complete residuated lattice. It is worth noting that the Nerode's automaton of an initial fuzzy automaton $\mathscr{A}$ is isomorphic to the determinization of $\mathscr{A}$ by means of the accessible fuzzy subset construction (cf. [25]), and it is a minimal deterministic automaton which recognizes every fuzzy language which can be recognized by the initial fuzzy automaton $\mathscr{A}$. Similarly, the Myhill's automaton of a fuzzy automaton $\mathscr{A}$ is a minimal deterministic automaton which recognizes every fuzzy language which can be recognized by this fuzzy automaton $\mathscr{A}$.

The results obtained in this paper give nice relationships between fuzzy languages, fuzzy automata and deterministic automata.

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