JOURNAL OF COMPUTER AND SYSTEM SCIENCES 6, 151–167 (1972)

Locally Testable Languages*

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Received April 1, 1971

This paper studies the locally testable languages (or "events") introduced by McNaughton and Papert. We characterize these languages by means of their syntactic semigroups and obtain wreath product and direct product decompositions for these semigroups. As a by-product of our study, we find an algebraic characterization of A. Ginzburg's generalized definite languages.

1. INTRODUCTION

Chomsky and Schützerberger [16, p. 144] have introduced the notion of a standard regular event. A well-known theorem [16, p. 145] states that any regular language (or event) is a homomorphic image of a standard regular event. The theorem generalizes easily to tree automata, where the standard regular events have been termed "local sets" (Thatcher [19]).

The local sets can be shown to be sets of derivation trees of context-free grammars [19, pp. 343–345]. The standard regular events are related to context-free languages in yet another way: Another well-known theorem, due to Chomsky and Schützenberger, states that any context-free language is a homomorphic image of the intersection of a Dyck language and a standard regular event.

An obvious generalization of the notion of a standard regular event is that of a strictly locally testable event (McNaughton and Papert [10, Chapter 2]). An event E is strictly locally testable if and only if, for some $k > 0$, a word w of length at least k belongs to E if and only if w has prescribed prefix and suffix of length k, and all interior solid subwords of w of length k belong to a prescribed set.

From another point of view, the strictly locally testable events are a nontrivial generalization of the definite events.

* This research was sponsored by the Advanced Research Projects Agency of the Office of the Secretary of Defense (F44620-70-C-0107) and is monitored by the Air Force Office of Scientific Research.

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The strictly locally testable events are not closed under the boolean operations. Taking their closure under the boolean operations, one obtains the locally testable events (McNaughton and Papert [10, Chapter 2]).

Since the term "event" is archaic and misleading, we will replace it by the more modern term "language" (which is perhaps equally misleading). Thus we speak of the locally testable languages.

In this paper, we study the syntactic semigroups of the locally testable languages. Our work was motivated, in part, by Schützenberger's [15] early success in characterizing the semigroups of the star-free languages. The latter result had far-reaching implications and led to research culminating in the McNaughton–Papert monograph on noncounting automata [10]. We characterize the syntactic semigroups of the locally testable languages and, in turn, characterize these semigroups in terms of wreath product (Theorem 5) and direct product (Theorem 8) decompositions. The reader familiar with [10] will notice that our Theorem 1 may be considered, in some sense, an analog of $NC = GF$ while Theorem 5 may be considered an analog of $GF \cap NN = LF \cap NN$.

On the other hand, the author is not aware of any nontrivial direct-product decomposition results in the literature, analogous to Theorem 8.

As a by-product of our study, we obtain a characterization (Theorem 6) of the generalized definite languages, introduced by A. Ginzburg [5] and studied later by Steinby [17].

In order to make the paper accessible to a wider audience, we have kept the algebra to a minimum. Further results related to this paper, but requiring more algebraic sophistication, appear in [20] and [22].

Nearly all the semigroup theory needed is developed in Section 2. We only assume familiarity with finite automata, including the elementary theory of series-parallel decompositions (Ref. [7]). The Rees theorem is discussed in [4, chapter 3], [14].

2. Preliminaries

In this section we recall some basic definitions and establish notation. We abbreviate "if and only if" as "iff." For a finite set A, $|A|$ denotes the cardinality of A.

Let Σ be a finite, nonempty alphabet. Σ^* will denote the set of all words over Σ and Σ^+, the set of all nonnull words over Σ.

Elements of Σ^+ will be denoted by $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ $\sigma_i \in \Sigma$, or if there is no danger of confusion, simply by $\sigma_1\sigma_2 \cdots \sigma_n$. The latter notation is the customary one. However, on several occasions it will be necessary to use the former notation, to distinguish between a formal word in a semigroup S, i.e., an element of S^+ and the corresponding product in S. For $w \in \Sigma$ $|w|$ will denote the length of w.

A language (or event) is any subset of Σ^*. We will consider only nonnull languages, i.e., subsets of Σ^+. This involves no loss of generality since the null word is irrelevant for the questions studied in this paper and its exclusion will simplify the exposition. For $L \subseteq \Sigma^+$, let $A(L)$ be the reduced automation accepting L. L is regular iff $A(L)$ is finite. Throughout this paper, L will denote a nonnull regular language.

A machine or input–output map is a map $f : \Sigma^+ \to Y$ where Y is a finite, non-empty set. Σ^+ is a semigroup under concatenation. The semigroup f^S of the machine f is the quotient of Σ^+ modulo the equivalence relation $\equiv$ defined by $w_1 \equiv w_2$ iff for all v, $w \in \Sigma^*$, $f(vw_1w) = f(vw_2w)$. Let $L \subseteq \Sigma^+$, then the characteristic function χ_L of L is a machine $\chi_L : \Sigma^+ \to \{0, 1\}$. We call $\chi_L{}^S$, the syntactic semigroup (or simply the semigroup) of L and denote it by $S(L)$. If L is regular, then the abstract semigroup $S(L)$ is isomorphic to the semigroup of transformations on the states of $A(L)$ induced by the words of Σ^+. Given L, and $w \in \Sigma^+$, we will denote the equivalence class of w modulo $\equiv$ by $[w]$. McNaughton and Papert [9] is an excellent reference on regular languages and their semigroups.

The reader should note that we consider syntactic semigroups rather than syntactic monoids, as is done in [9, 10, 15]. The reason will become apparent later (see the remark following Theorem 1).

Let S be a semigroup. An element $e \in S$ is called idempotent iff $e^2 = e$. An element $z \in S$ is called a *right* (*left*) *zero* iff $xz = z$ (resp. $zx = z$) for all $x \in S$. S is *right* (*left*) *zero* iff every element of S is a right (resp. left) zero.

Let S be a semigroup. S^1 will denote the monoid obtained from S by adjoining an identity element if S does not have an identity to begin with. The Green equivalence relations on S are defined as follows:

$$\begin{aligned}
s \mathscr{R} t \quad &\text{iff} \quad sS^1 = tS^1 \text{ i.e., there are elements } u, v \in S^1 \\
&\qquad\qquad\qquad \text{such that } su = t \text{ and } tv = s, \\
s \mathscr{L} t \quad &\text{iff} \quad S^1 s = S^1 t, \\
s \mathscr{J} t \quad &\text{iff} \quad S^1 s S^1 = S^1 t S^1, \\
s \mathscr{H} t \quad &\text{iff} \quad s \mathscr{R} t \quad \text{and} \quad s \mathscr{L} t.
\end{aligned}$$

The $\mathscr{R}$-relation has a more intuitive description in terms of the Graph G of S (see McNaughton and Papert [9, pp. 304–307]) $s \mathscr{R} t$ iff there are (directed) paths in G from s to t and from t to s.

A finite semigroup is *combinatorial* or *group-free* iff all its subgroups are trivial. As is well known, S is combinatorial iff there is a positive integer $p = p(S)$ such that $s^p = s^{p+1}$ for all $s \in S$ iff no two distinct elements of S are $\mathscr{H}$-related.

Let S be a semigroup. An *ideal* is a nonempty set $I \subseteq S$ such that for all $x \in I$, $s \in S$, xs, $sx \in I$. If S has a zero element 0 such that for all $s \in S$, $s0 = 0s = 0$, then $\{0\}$ is an ideal. S is *simple* iff it has no proper ideals. S is *0-simple* iff it has no proper ideals except $\{0\}$. A well-known theorem of Rees states that for any 0-simple combinatorial semigroup there are positive integers m, n and an $n \times m$ matrix C over $\{0, 1\}$ such that S is isomorphic to a set $(A \times B) \cup \{0\}$, $A = \{1, 2,..., m\}$, $B = \{1, 2,..., n\}$ with the multiplication

$$(i_1, j_1)(i_2, j_2) = \begin{cases} (i_1, j_2) & \text{if } C_{j_1 i_2} = 1 \\ 0 & \text{if } C_{j_1 i_2} = 0 \end{cases}.$$

Equivalently, the element with "coordinates" (i, j) can be represented by a $m \times n$ matrix M_{ij} having 1 at the ij-th entry and all other entries being zero. The product of $M_{i_1 j_1}$ and $M_{i_2 j_2}$ is the ordinary matrix product $M_{i_1 j_1} C M_{i_2 j_2}$.

If S is simple and combinatorial, then by the Rees theorem, S is isomorphic to $A \times B$ with the multiplication $(i_1, j_1)(i_2, j_2) = (i_1, j_2)$, i.e., to the direct product of a left-zero semigroup A with a right-zero semigroup B. Such a semigroup is called a *rectangular band*.

Let ρ be one of the Green relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{J}$, $\mathscr{H}$. The equivalence classes modulo ρ are called ρ-classes. We can define a partial ordering $\leqslant$ on the $\mathscr{J}$-classes of S by $J_1 \leqslant J_2$ iff $S^1 x S^1 \subseteq S^1 y S^1$ where $x \in J_1$, $y \in J_2$. A $\mathscr{J}$-class J is minimal (0-minimal, resp.) iff $S^1 x S^1$ is minimal (minimal, if the $\mathscr{J}$-class $\{0\}$ is excluded), for $x \in J$.

Let I be an ideal of S. S is a *nilpotent extension* of I iff there is a positive integer n (*the degree of nilpotence*) such that for all $s_1, s_2,..., s_n \in S$, the product $s_1 \cdots s_n$ belongs to I. S is *nilpotent* iff S has a zero element 0 and S is a nilpotent extension of $\{0\}$.

An element $s \in S$ is *regular* iff it has a "pseudo-inverse" $t \in S$ such that $sts = s$. A $\mathscr{J}$-class is *regular* iff all its elements are regular; otherwise it is called *null*. A semigroup is *regular* iff all its $\mathscr{J}$-classes (or equivalently all its elements) are regular.

The units U_1, U_2, U_3 of Krohn and Rhodes are defined as follows: U_1 is a two-element right-zero semigroup. U_2 is $\{0, 1\}$ under ordinary multiplication and U_3 is U_1 with an identity element adjoined.

Let S be a finite semigroup. $S^f : S^+ \to S$ is the machine defined by the multiplication in S. If $f : \Sigma^+ \to Y$ is a machine, $f^\sigma : \Sigma^+ \to Y^+$ is defined by

$$f^\sigma((\sigma_1,..., \sigma_n)) = (f(\sigma_1), f(\sigma_1, \sigma_2),..., f(\sigma_1,..., \sigma_n)).$$

If $h : \Sigma \to Y$, $h^\Gamma : \Sigma^+ \to Y^+$ is defined by

$$h^\Gamma((\sigma_1,..., \sigma_n)) = (h(\sigma_1), h(\sigma_2),..., h(\sigma_n)).$$

3. Locally Testable Languages and Semigroups

Definition. (McNaughton–Papert [10]). Let k be a positive integer. For $w \in \Sigma^+$ of length $\geqslant k$, let $L_k(w)$, $R_k(w)$ and $I_k(w)$ be, respectively, the prefix of w of length k, the suffix of w of length k, and the *set* of all interior solid subwords of w of length k. If $|w| = k$, then $L_k(w) = R_k(w) = w$. If $|w| = k$ or $k + 1$, then $I_k(w)$ is empty.

Let k be a positive integer. $L \subseteq \Sigma^*$ is *strictly k-testable* iff there are finite test-sets X, Y, Z of words over Σ such that for all $w \in \Sigma^+$, $| w | \geqslant k$, $w \in L$ iff $L_k(w) \in X$, $R_k(w) \in Y$ and $I_k(w) \subseteq Z$. L is *strictly locally* testable iff it is strictly k-testable, for some $k > 0$.

The languages mentioned in the first paragraph of Section 1 are strictly locally testable. On the other hand, the strictly locally testable languages are not closed under either union or complementation. As is well known, a language and its complement have the same semigroup. It follows in particular that the strictly locally testable languages do not admit a characterization in terms of their semigroups. For this and other reasons (see [10, Chapter 2] for further discussion) we introduce the following:

DEFINITION. [10]. A language $L \subseteq \Sigma^+$ is *k-testable* iff L is a boolean combination of strictly k-testable languages. L is *locally testable* iff it is k-testable for some $k > 0$.

McNaughton and Papert show that an equivalent definition is the following:

Let us say that two words, w, $w' \in \Sigma^+$ of length $\geqslant k$ *have the same k-test vectors* iff $L_k(w) = L_k(w')$, $R_k(w) = R_k(w')$ and $I_k(w) = I_k(w')$. Then $L \subseteq \Sigma^*$ is k-testable iff for all w, $w' \in \Sigma$ of length $\geqslant k$, if w and w' have the same k-test vectors, then $w \in L$ iff $w' \in L$. Let LT denote the class of locally testable languages. It is easy to see that every locally testable language is regular [10, Chapter 2] and that a k-testable event is m-testable, for all $m \geqslant k$.

Our first observation is that the definition in the last paragraph suggests an analogous definition for semigroups. Thus we introduce the following:

DEFINITION. Let k be a positive integer. A semigroup S is *k-testable* iff for any two formal words $w = (s_1, s_2, \ldots, s_m)$, $w' = (t_1, t_2, \ldots, t_n) \in S^+$, $m, n \geqslant k$, if w and w' have the same k-test vectors, then $s_1 s_2 \cdots s_m = t_1 t_2 \cdots t_n$ (products in S). S is *locally testable* iff it is k-testable, for some $k > 0$. Let LTS denote the class of locally testable semigroups.

THEOREM 1. *Let L be a nonnull language. Then L is locally testable iff S(L) is locally testable.*

Proof. Assume $S(L)$ is k-testable and let $w = (\sigma_1, \ldots, \sigma_m)$ and $w' = (\sigma_1', \ldots, \sigma_n')$ be words in Σ^+, $m, n \geqslant k$. Suppose $L_k(w) = L_k(w')$, $R_k(w) = R_k(w')$ and $I_k(w) = I_k(w')$, then the words

$$([\sigma_1], \ldots, [\sigma_m]) \qquad \text{and} \qquad ([\sigma_1'], \ldots, [\sigma_n'])$$

of $S(L)^+$ will have the same k-test vectors. Thus

$$[\sigma_1 \cdots \sigma_m] = [\sigma_1' \cdots \sigma_n']$$

and so $w \in L$ iff $w' \in L$ and L is locally testable. Conversely, assume L is k-testable. Let $(s_1, \ldots, s_m)$, $(t_1, \ldots, t_n)$, $m, n \geqslant k$, be words in S^+ having the same k-test vectors. Pick an arbitrary but fixed representative in Σ^+ for each equivalence class in $S(L)$. Consider the two words $w, w' \in \Sigma^+$ obtained from $(s_1, \ldots, s_m)$ and $(t_1, \ldots, t_n)$, respectively, by replacing each s_i and t_j by their representatives. It is easy to see that xwy and $xw'y$ have the same k-test vectors, for any $x, y \in \Sigma^*$. Thus by k-testability $xwy \in L$ iff $xw'y \in L$, i.e., $[w] = [w']$. But w represents $s_1 \cdots s_m$ and w' represents $t_1 \cdots t_n$. Hence $s_1 \cdots s_m = t_1 \cdots t_n$. Q.E.D.

Remark. Theorem 1 is false if "syntactic semigroup" is replaced by "syntactic monoid." The latter is the standard construct in the literature (after Schützenberger). This is an important point, which can be seen as follows: Let M be a k-testable monoid with identity element 1. Let x, y be arbitrary elements of M. Let 1^k denote the word over M of length k consisting of all 1's. The words $1^k x 1^k y 1^k$ and $1^k y 1^k x 1^k$ (concatenation) have the same k-test vectors. Thus by k-testability, $xy = yx$ and M is commutative. Similarly, the words $1^k x 1^k x 1^k$ and $1^{k+1} x 1^k$ have the same k-test vectors, implying $x^2 = x$. So M must be a commutative semigroup in which every element is idempotent (a so-called "semilattice"). The class of languages whose syntactic monoids are semilattices is a very meager subclass of LT (for example, no definite language belongs to this class). Observe also that the restriction to nonnull languages in Theorem 1 is inessential since if L contains the null word λ, L is locally testable iff L-$\{\lambda\}$ is locally testable. Similar comments apply to Theorem 6 below.

The argument in the preceding paragraph yields a simple and easily verifiable necessary condition for local testability. This is shown in the next theorem.

THEOREM 2. *Let S be a locally testable semigroup. Then for every idempotent $e \in S$, eSe is a semilattice.*

Proof. eSe is a subsemigroup of S. Thus by Proposition 1 below, eSe is locally testable. Furthermore, eSe is a monoid with identity element e. Thus, by the argument in the preceding paragraph, eSe is a semilattice. Q.E.D.

Remark. In Ref. [20], we have proved the converse of Theorem 2, under mild restrictions on S, and conjectured that the full converse is true (for finite S). The result has some interesting algebraic consequences that are explored in Ref. [22]. Recently, McNaughton [11, 25], has proved the converse of Theorem 2. Theorem 2 and its converse have also been proved independently by Brzozowski and Simon [24].

Theorem 2 and its converse have the important corollary that there is an algorithm for checking languages L for local testability. The algorithm is simply checking whether $S(L)$ satisfies the condition of Theorem 2. This settles a problem posed by McNaughton and Papert in [10].

4. Wreath Product Decompositions

This section assumes familiarity with the elementary theory of series-parallel decomposition of machines. All undefined notation follows [7]. Before proceeding, we review a relatively unknown result about definite machines that will be needed in our proofs.

Definition. Let k be a positive integer. A machine $f : \Sigma^+ \to Y$ is *k-definite* iff for all $w \in \Sigma^+$, $| w | \geqslant k$ implies $f(w) = f(R_k(w))$. f is *definite* iff f is k-definite, for some $k > 0$.

Theorem 3 (Krohn, Mateosian and Rhodes [6], Stiffler [18]). *Let $f : \Sigma^+ \to Y$ be a machine. The following are equivalent:*

(1) *f is definite.*

(2) *f^S is a nilpotent extension of a finite right-zero kernel.*

(3) *f^S divides a wreath product of copies of U_1.*

Proof. The equivalence of (1) and (3) is the content of Proposition (7.1)(h) of [6]. The equivalence of (2) and (3) is proved in Stiffler [18, Theorem (3.4)(a)]. The equivalence of (1) and (2) is already implicit in [12] (the equivalence of definite events and definite automata). Q.E.D.

In view of the theorem, we make the following definition:

Definition. A finite semigroup is *definite* iff it is a nilpotent extension of a right-zero kernel.

Theorem 4. *Let L be a strictly locally testable language. Then $S(L)$ divides $U_2 w D$, where D is definite.*

Proof. The proof will proceed by construction of a series-parallel decomposition of the characteristic function of L. The Krohn–Rhodes theorem cannot be applied since it uses flip-flops (whose semigroup is U_3 which is not locally testable).

For each positive integer m, define

$$f_m : \Sigma^+ \to \Sigma^{\leqslant m},$$

where $\Sigma^{\leqslant m}$ is the set of all words of length $\leqslant m$ over Σ, by

$$f_m(\sigma_1\sigma_2 \cdots \sigma_n) = \begin{cases} \sigma_{n-m+1} \cdots \sigma_n & \text{if } m < n \\ \sigma_1 \cdots \sigma_n & \text{if } n \leqslant m, \end{cases}$$

f_m is m-definite.

Let $L \subseteq \Sigma^+$ be strictly k-testable and let X, Y, Z be test sets for L. Let α, β be symbols not in Σ, let $\Sigma' = \Sigma \cup \{\alpha, \beta\}$ and let $F = \alpha L \beta \cap (\Sigma')^{\leqslant k+1}$. Then the characteristic function $\chi_{\alpha L \beta} : (\Sigma')^+ \to \{0, 1\}$ has the series-parallel decomposition

$$\chi_{\alpha L \beta} = \chi_{\Sigma^+ \beta} \times U_2{}^f h^\Gamma f^\sigma_{k+1} \qquad (*)$$

(notation as in Ref. [7, Section 2]), where $h : (\Sigma')^{\leqslant k+1} \to \{0, 1\}$ is defined by $h(w) = 1$ iff either

(1) $w = \alpha v$ and w is a prefix of a word in $F \cup \alpha X$, or

(2) $w = v\beta$ and $w \in Y\beta$, or

(3) $w = \sigma_1 \sigma_2 \cdots \sigma_n \sigma_{n+1}$ and both $\sigma_1 \cdots \sigma_n$ and $\sigma_2 \cdots \sigma_{n+1}$ belong to Z, or

(4) $w \in F$,

$h(w) = 0$ otherwise.

Clearly, for any $w \in \alpha L \beta$, the right-hand side of $(*)$ assigns 1 to w. Conversely, assume that the right-hand side of $(*)$ assigns 1 to w. Then w must end in β. w must also begin in α since otherwise the prefix of w of length 1 will be mapped by h into 0 ($k + 1 > 1$). It follows easily that $w \in L$. Since f_{k+1} is definite, it follows by Theorem 3 that $N = f^S_{k+1}$ is a definite semigroup. It follows from $(*)$, applying fact (2.9) of [7], that

$$S(\alpha L \beta) \mid (U_2{}^f h^\Gamma f^\sigma_{k+1})^S \times S(\Sigma^+ \beta).$$

$\Sigma^+ \beta$ is a definite language and so, by Theorem 3, its semigroup is definite. By Proposition (2.15) of [7], the semigroup of $U_2{}^f h^\Gamma f^\sigma_{k+1}$ divides $U_2 w N$ (recall that N is the semigroup of f_{k+1}). It follows by elementary properties of wreath products (see the proof of Theorem 5 below), that $S(\alpha L \beta)$ divides $U_2 w D$, where D is definite. It is easy to see that $S(L)$ is a subsemigroup of $S(\alpha L \beta)$. Thus $S(L)$ divides $U_2 w D$ and the theorem is proved.

THEOREM 5. *$L \subseteq \Sigma^+$ is locally testable if and only if $S(L)$ divides BwD, where D is definite and B is a semilattice.*

Proof.

Sufficiency. By Theorem 1, it will suffice to prove that $S = S(L)$ is locally testable.

We will need the following

PROPOSITION 1. *The class of locally testable semigroups is closed under subsemigroups, homomorphic images and finite direct products* (LTS *is thus a "pseudovariety" of semigroups*).

Proof. Closure under subsemigroups and finite direct products is immediate. Assume S is k-testable and let h be an onto homomorphism $S \to V$. Let $(x_1, \ldots, x_m)$, $(y_1, \ldots, y_n) \in V^+$, $m, n \geqslant k$ have the same k-test vectors. For each element $x_i \in V$ pick an arbitrary but fixed $s_i \in S$ such that $h(s_i) = x_i$ and similarly for y_j pick an arbitrary but fixed t_j such that $h(t_j) = y_j$. Then it is easy to see that $(s_1, \ldots, s_m)$ and $(t_1, \ldots, t_n)$ have the same k-test vectors. Thus $s = s_1 \cdots s_m = t_1 \cdots t_n = t$ and consequently $x_1 \cdots x_m = h(s) = h(t) = y_1 \cdots y_n$, so V is k-testable. Q.E.D.

By Proposition 1, LTS is closed under division. Thus it will suffice to prove that BwD is locally testable. Now, BwD is isomorphic to a semidirect product $F(D^1, B) \times_Y D$ (see [7, (1.4) and (2.1)]). Furthermore, $F(D^1, B)$ being a finite direct product of semilattices is again a semilattice. Thus it will suffice to prove that for D definite and B a semilattice, the semidirect product $B \times_Y D$, for an arbitrary connecting homomorphism Y is locally testable. We proceed to prove this.

Let l be the degree of nilpotence of D over its kernel and let $k > l$. We will show that $S = B \times_Y D$ is k-testable. Observe that D, being definite, is trivially k-testable. Furthermore, B is 1-testable. Let

$$((b_1, d_1), (b_2, d_2), \ldots, (b_m, d_m)), ((a_1, c_1), (a_2, c_2), \ldots, (a_n, c_n)) \in S^+, m, n \geqslant k,$$

have the same k-test vectors. We have to show that

$$(b_1\, Y(d_1)(b_2)\, Y(d_1 d_2)(b_3) \cdots Y(d_1 d_2 \cdots d_{m-1})(b_m), d_1 d_2 \cdots d_m)$$

$$= \prod_{i=1}^{m} (b_i, d_i) = \prod_{j=1}^{n} (a_j, c_j) = (a_1\, Y(c_1)(a_2) \cdots Y(c_1 c_2 \cdots c_{n-1})(a_n), c_1 \cdots c_n).$$

By the k-testability of D, $d_1 \cdots d_m = c_1 \cdots c_n$. By hypothesis, for $i \leqslant k$,

$$Y(d_1 \cdots d_{i-1})(b_i) = Y(c_1 \cdots c_{i-1})(a_i).$$

Furthermore, since any product of length $\geqslant l$ in D belongs to the kernel which is right-zero, it follows that for all $i > k$, $d_1 d_2 \cdots d_{i-1} = d_{i-k} \cdots d_{i-1}$. By hypothesis, there is $k < j < n$, such that $(b_{i-k}, d_{i-k}) \cdots (b_i, d_i) = (a_{j-k}, c_{j-k}) \cdots (a_j, c_j)$. Thus $d_i = c_j$ and $d_1 d_2 \cdots d_{i-1} = d_{i-k} \cdots d_{i-1} = c_{j-k} \cdots c_{j-1} = c_1 \cdots c_{j-1}$. Consequently, $Y(d_1 \cdots d_{i-1})(b_i) = Y(c_1 \cdots c_{j-1})(a_j)$. By a symmetric argument, for all $j > k$, there is $k < i < m$ such that $Y(c_1 \cdots c_{j-1})(a_j) = Y(d_1 \cdots d_{i-1})(b_i)$. Since B is commutative and idempotent, it follows easily that $\prod_{i=1}^{m} (b_i, d_i) = \prod_{j=1}^{n} (a_j, c_j)$. Thus S is k-testable.

Necessity. By Theorem 4, the semigroup of a strictly locally testable language divides a product $U_2 wD$, where D is definite. Furthermore, $(U_2 wD_1) \times (U_2 wD_2)$ divides $(U_2 x U_2)\, w(D_1 x D_2)$ (see [2, p. 134]). Since the class of definite semigroups is closed under direct products, it follows that the direct product of semigroups of the form U_2wD, D definite, divides BwD, D definite and B a direct product of U_2's. Since a finite semigroup S is a semilattice iff it is a subsemigroup of a direct product of U_2's, the proof will be completed if we can prove the following lemma.

LEMMA 1. *Let E_i, $1 < i < m$ be regular languages and let E be a boolean combination of the E_i's. Then $S(E)$ divides a direct product of copies of the $S(E_i)$'s.*

Proof. $\chi_{E_i \cup E_j} = h_1(\chi_{E_i} \times \chi_{E_j})$, where $h_1(0, 1) = h_1(1, 0) = h_1(1, 1) = 1$, $h_1(0, 0) = 0$, $\chi_{E_i \cap E_j} = h_2(\chi_{E_i} \times \chi_{E_j})$, where $h_2(1, 1) = 1$, $h_2(1, 0) = h_2(0, 1) = h_2(0, 0) = 0$. $\chi_{\bar{E}_i} = h_3 \chi_{E_i}$, where $h_3(0) = 1$, $h_3(1) = 0$. The lemma follows by fact (2.9) and Proposition (2.15) of [7].

This proves the theorem.

Remark. Theorem 5 can be easily reformulated as a result on the cascade decomposition of the minimal automaton accepting L.

5. PARALLEL DECOMPOSITIONS

While the characterization of Theorem 5 is quite satisfactory, it turns out that the locally testable semigroups have an even nicer characterization in terms of direct product (or in machine terminology, parallel) decompositions. Nontrivial parallel decompositions are quite rare so the result is somewhat surprising. Also, parallel decompositions[1] are right–left symmetric, showing clearly the closure of the locally testable semigroups under reversal, a fact which is not obvious from the decomposition of Theorem 5.

Let L be strictly k-testable with test sets X, Y, Z. Let Σ^k denote the set of all words of length k over Σ. Then, modulo a finite language, L equals

$$X\Sigma^* \cap \Sigma^* Y \cap \overline{\Sigma^*\Sigma(\Sigma^k - Z)\,\Sigma\Sigma^*}$$

(the bar denotes complement). Thus, by Lemma 1, $S(L)$ divides the direct product

$$S(X\Sigma^*) \times S(\Sigma^* Y) \times S(\Sigma^*\Sigma(\Sigma^k - Z)\,\Sigma\Sigma^*) \times T,$$

where T is the semigroup of a finite language. T is easily seen to be nilpotent. Thus our task is to find the semigroups of languages of the form $F\Sigma^*$, Σ^*F and $\Sigma^*F\Sigma^*$, where F is a finite language over Σ.

Languages of the form Σ^*F and $F\Sigma^*$ are well known. They are the definite and

[1] I am indebted to Maurice Nivat for suggesting the interest in, and the possibility of, parallel decompositions.

reverse-definite languages (events) (Refs. [12, 3]). The definite languages are a special case of the definite functions. Thus, by Theorem 3, L is definite iff $S(L)$ is definite.

For a semigroup S with multiplication "$\cdot$," $r(S)$ is the set S with multiplication $*$, where $s_1 * s_2 = s_2 \cdot s_1$. It is easy to see that for all languages L, $S(L^R) = r(S(L))$. Thus we immediately get the dual of Theorem 3.

THEOREM 3'. *Let $L \subseteq \Sigma^+$. The following are equivalent:*

(1) *L is reverse-definite.*

(2) *$S(L)$ is a nilpotent extension of a finite left-zero kernel.*

In view of Theorem 3', it is natural to call a finite semigroup S reverse-definite iff $r(S)$ is definite.

Observe that the nilpotent, the definite and the reverse-definite semigroups are special cases of a semigroup which is a nilpotent extension of a rectangular band. The class of semigroups that are a nilpotent extension of a rectangular band is easily seen to be closed under direct products and division. It follows by repeated application of Lemma 1, that all the terms T, $S(X\Sigma^*)$, $S(\Sigma^* Y)$ in the decomposition of $S(L)$ can be replaced by a single semigroup that is a nilpotent extension of a rectangular band.

It turns out that the languages whose semigroups are nilpotent extensions of a rectangular band can be characterized as the generalized-definite languages, introduced by A. Ginzburg [5]. First, we recall the definition.

DEFINITION. (Ginzburg [5]). A language $L \subseteq \Sigma^*$ is generalized-definite iff $L = F \cup (\bigcup_{i=1}^{n} A_i \Sigma^* B_i)$, where F, A_i, $B_i i = 1, 2,\ldots n$ are finite subsets of Σ^*.

It is easy to verify that a language is generalized-definite iff for some positive integer k, for all $w, w' \in \Sigma^*$ of length $\geqslant k$, $R_k(w) = R_k(w')$ and $L_k(w) = L_k(w')$ imply $w \in L \Leftrightarrow w' \in L$. Thus the generalized-definite languages form a subclass of the locally testable languages.

THEOREM 6 (see Footnote 2). *Let $L \subseteq \Sigma^+$. The following are equivalent:*

(1) *L is generalized definite.*

(2) *$S(L)$ is a nilpotent extension of a rectangular band.*

(3) *For every idempotent $e \in S(L)$, $eS(L)e = \{e\}$, and $S(L)$ is finite.*

Proof. (1) $\Rightarrow$ (2). Let L be generalized definite. Then L is a finite union of languages of the form $L_1 \cap L_2$, L_1 definite and L_2 reverse definite. By Lemma 1, the semigroup of a finite union of languages divides the direct product of the semigroups of the individual languages. Since, furthermore, the class of semigroups that are a

[2] Theorem 6 has been proved independently by Perrin [13].

nilpotent extension of a finite rectangular band is closed under direct products and division, it will suffice to prove that $S(L_1 \cap L_2)$ is of the required form. By another application of Lemma 1, $S(L_1 \cap L_2)$ divides $S(L_1) \times S(L_2)$.

Let $K_i = \ker(S(L_i))$, $i = 1, 2$. Then it follows easily that

$$K = \ker(S(L_1) \times S(L_2)) = K_1 \times K_2 ,$$

and furthermore, that $S(L_1) \times S(L_2)$ is a nilpotent extension of its kernel. Since K_1 and K_2 are right and left zero respectively, it follows that $K_1 \times K_2 = K$ is a rectangular band. Since K_1 and K_2 are finite, so is K.

(2) $\Rightarrow$ (1). Assume $S(L)$ is a nilpotent extension of degree p of a rectangular band $K = A \times B$.

Let $w = \sigma_1 \cdots \sigma_m$ and $w' = \sigma_1' \cdots \sigma_n'$, $m, n \geqslant p$. Then $[w] = [\sigma_1] \cdots [\sigma_m] \in K$ and $[w'] = [\sigma_1'] \cdots [\sigma_n'] \in K$ by hypothesis. Assume $L_p(w) = L_p(w')$ and $R_p(w) = R_p(w')$. Then

$$[\sigma_1] \cdots [\sigma_p] = [\sigma_1'] \cdots [\sigma_p'] = x \qquad \text{and} \qquad [\sigma_{m-p+1}] \cdots [\sigma_m] = [\sigma'_{n-p+1}] \cdots [\sigma_n'] = y.$$

It follows that $[w] = xs_1 = t_1 y$ and $[w'] = xs_2 = t_2 y$, where $x, y \in K$ and s_i, $t_j \in S(L)$. Let x have coordinates (a, b). Then for all $s \in S(L)$ $xs, sx \in K$ and, furthermore, the A-coordinate of xs equals a and the B-coordinate of sx equals b. This follows from the Schützenberger representation [4, p. 110] but can also be seen directly as follows. Let $xs = (a_1, b_1)$ and $sx = (a_2, b_2)$. Then $(a_1, b) = (xs)x = x(sx) = (a, b_2)$, thus $a_1 = a$ and $b_2 = b$ as asserted. It follows from the preceding that $[w]$ and $[w']$ have the same A-coordinate and the same B-coordinate and thus must be equal. In particular, $w \in L$ iff $w' \in L$ and so L is generalized definite.

(2) $\Rightarrow$ (3). All idempotents of $S(L)$ belong to the kernel K (if $e^2 = e$ and e is not in K, then for all n, $e^n = e$ is not in K, contradicting the hypothesis). Since K is a rectangular band, it follows from the argument in the preceding paragraph that for $s \in S(L)$, $ese = e$ and so $eS(L)e = \{e\}$. Finally, $S(L)$ is finitely generated, hence it follows easily from the hypothesis that it must be finite.

(3) $\Rightarrow$ (2). If $S(L)$ has a nontrivial subgroup G, then the identity element of G is an idempotent e and $eGe = G \subseteq eS(L)e = \{e\}$. Thus $S(L)$ is combinatorial. In particular, the kernel K of $S(L)$ is a rectangular band. Suppose there is an idempotent e not belonging to K. Let f be an arbitrary element of K. Then $efe \in K$, so $efe \neq e$. Consequently $\{e, efe\} \subseteq eS(L)e = \{e\}$, which is a contradiction. Thus all idempotents belong to K. It follows easily (as in [14, Fact 2.30]) that $S(L)$ is a nilpotent extension of K. Q.E.D.

As an immediate corollary of the preceding theorems, we obtain Ginzburg's results [5], that were obtained originally using fairly involved combinatorial arguments.

COROLLARY [5, 17]. *Let L be a generalized-definite (definite, reverse-definite, respectively) language. Let A be the reduced automaton accepting L. Let B be an automaton obtained from A by arbitrary changes of the initial and final states. Then L', the set of words accepted by B, is generalized-definite (definite, reverse-definite, respectively).*

Proof. The reduced automaton B' accepting L' is a homomorphic image of B, thus $S^{B'}$ is a homomorphic image of $S^B = S^A \simeq S(L)$. The corollary follows from Theorems 3, 3', and 6 since the classes of semigroups that are nilpotent extensions of right-zero, left-zero or combinatorial kernels are each closed under homomorphisms.

Remark. By analogy to Theorem 3, it can be further proved that L is generalized-definite iff $S(L)$ can be "built" from copies of U_1 via wreath products and reversals. The proof, together with other characterizations, appears in [21].

Our remaining task is characterizing $S(\Sigma^*F\Sigma^*)$. Since for any finite languages A, B over Σ,

$$\Sigma^*(A \cup B)\,\Sigma^* = \Sigma^*A\Sigma^* \cup \Sigma^*B\Sigma^*,$$

it will suffice, by Lemma 1, to find the structure of $S(\Sigma^*w\Sigma^*)$ for w in Σ^+.

Before we state the theorem, we need a definition and some notation from automata theory. Let A be an automaton with input set Σ and state set Q. For q in Q and w in Σ^+, qw is the state reached by a path starting at q and spelling out w, while $q\lambda = q$, where λ is the null word. An automaton A with state set Q is *almost strongly connected* (or, in algebraic parlance, 0-transitive) iff A has a dead state d and for any two states q_1, q_2 in $Q - \{d\}$ there is $w \in \Sigma^*$ such that $q_1w = q_2$.

The locally testable languages over a one-symbol alphabet are trivial. They are finite or cofinite and their semigroups are nilpotent. Thus we may restrict ourselves to languages over alphabets containing at least two distinct symbols.

THEOREM 7. *Let $L = \Sigma^*w\Sigma^*$. The following are equivalent:*

(1) *Either Σ contains at least three distinct symbols or $\Sigma = \{\sigma_1, \sigma_2\}$ and w is not of the form $\sigma_1{}^n\sigma_2$ or $\sigma_1\sigma_2{}^n$, $n \geqslant 1$.*

(2) *$A(L)$, the reduced automaton accepting L, is almost strongly connected.*

(3) *$S(L)$ is a nilpotent extension of a finite 0-simple combinatorial ideal.*

Proof. Let $w = \sigma_1\sigma_2 \cdots \sigma_m$. The reduced automaton $A(L)$ accepting L has $m + 1$ states $\{q_0, q_1, ..., q_m\} = Q$ and for $1 \leqslant i \leqslant m$, $q_i = q_0\sigma_1\sigma_2 \cdots \sigma_i$. Furthermore, $q_m = d$ is a dead state and is the unique accepting state of $A(L)$. We will need the following lemma.

LEMMA 2. *Let L be as in the theorem and $A(L)$ as above. For every state q_i, $i < m$ in $A(L)$ and every word $v \in \Sigma^+$, $q_iv = d$ iff w is a subword of $\sigma_1 \cdots \sigma_iv$ and otherwise $q_iv = q_j$, where $\sigma_1 \cdots \sigma_j$ is the longest suffix of $\sigma_1 \cdots \sigma_iv$ which is a prefix of w.*

Proof. Let $x = \sigma_1 \cdots \sigma_i v = y\sigma_1 \cdots \sigma_j$. Then no suffix of y is a prefix of w. Hence it is easy to see that the states q_0 and $q_0 y$ are indistinguishable. Since $A(L)$ is reduced, it follows that $q_0 y = q_0$. Thus $q_i v = q_j$. Q.E.D.

(1) $\Rightarrow$ (2). Since all states are reachable from q_0, it will suffice to prove that q_0 is reachable from q_{m-1}.

The proof proceeds by an examination of cases and repeated application of Lemma 2. Let $\sigma \neq \sigma_m$ and let $v = \sigma^k$, $k > m$. If $\sigma_1 \neq \sigma$, then no suffix of v is a prefix of w and so, by Lemma 2, $q_{m-1}v = q_0$. Otherwise, again by Lemma 2, $q_{m-1}v = q_{i-1}$, where i is the first index for which $\sigma_i \neq \sigma$. Thus it remains to be shown that q_0 is reachable from q_i (by hypothesis, $i < m - 1$). Assume first that $w = \sigma^{i-1}(\sigma')^{m-i+1}$. By hypothesis $2 < i$ and $2 \leqslant m - i + 1$. By the lemma, $q_i\sigma = q_1$ and $q_1\sigma' = q_0$ and we are done. Otherwise, let j be the first index greater than i such that $\sigma_j \neq \sigma'$. By hypothesis, $w = \sigma^{i-1}(\sigma')^{j-i}t$, where t is not null. By another application of Lemma 2, $q_i(\sigma')^{j-i+1} = q_0$. Finally, if $|\Sigma| \geqslant 3$, and $w = \sigma^p(\sigma')^k$, then $q_{m-1}\sigma'' = q_0$, where σ'' is any symbol in Σ different from σ and σ'. Thus $A(L)$ is almost strongly connected.

(2) $\Rightarrow$ (3). For this part of the proof we have to review some elementary concepts related to transformation semigroups. A transformation semigroup (X, S) is a semigroup S that is isomorphic to a semigroup of transformations on X (acting on the right). For $s \in S$, Xs is called the *range* of s and the cardinality of Xs is called the *rank* of s. Furthermore, s defines an equivalence relation and a corresponding partition on X by $x_1 \equiv x_2$ iff $x_1 s = x_2 s$.

Let the notation be as in the first paragraph of the proof. $S(L)$ is isomorphic to the semigroup of transformations induced on the states of $A(L)$ by Σ^+ and so we may identify $S(L)$ with the latter transformation semigroup. $S(L)$ has a zero element 0, namely, the transformation whose range is $\{d\}$. The proof will proceed in two stages. First, we will show that all elements of rank 2 belong to a single regular $\mathscr{J}$-class J. Then we will show that every product in $S(L)$ of length at least $|Q|$ has rank at most 2. Thus $S(L)$ is a nilpotent extension of $J \cup \{0\}$.

Since $S(L)$ is combinatorial, it follows easily that two elements having the same range and the same partition must be equal. Also, since d is a dead state, it is in the range of all elements of $S(L)$. In particular, the range of an element of rank 2 is $\{q, d\}$ and is thus determined by the single parameter q. Let s, t be arbitrary elements of rank 2. Let range$(s) = \{q_i, d\}$ and range$(t) = \{q_j, d\}$. If $q_i \neq q_j$, then, since $A(L)$ is almost strongly connected, there is u in Σ^+ such that $q_i u = q_j$. Thus range$(s[u]) = \{q_j, d\} =$ range(t). Let $x = [u]$. Now, for any two elements a, b of rank 2, $ab \neq 0$ iff range$(a) = \{q, d\}$ and q is not mapped into d by b. Furthermore, if $ab \neq 0$ then ab will have the same partition as a and the same range as b. Let q be any state that is not mapped into q_j by sx and let y be any element of $S(L)$ that maps q_j into q. Then $tysx \neq 0$ and thus t and $(ty)\,sx$ have the same range and the same partition and consequently are equal. Since the argument is symmetric in s and t, it follows that all elements of rank two are $\mathscr{J}$-related and thus belong to the same $\mathscr{J}$-class J. Furthermore, it is trivial that $\mathscr{J}$ equivalent elements have the same rank. It follows that J consists precisely of the elements of rank 2. To show that J is regular, it will suffice, by [4, Theorem 2.11(a)] (and since for a finite semigroup, $\mathscr{J} = \mathscr{D}$, see [14, Fact 1.15]) to show that J contains an idempotent. Let $\sigma \neq \sigma_1$ and let $v = \sigma^p$, $p \geqslant m$. Then by Lemma 2, $[v]$ has range $\{q_0, d\}$ and since $q_0 v = q_0$ and $dv = d$, $[v]$ is idempotent.

To show that $S(L)$ is a nilpotent extension of $J \cup \{0\}$, we have to prove that for all sufficiently long words v, the state transformation induced by v has rank at most 2. Let $|v| \geqslant m$. By Lemma 2, for all $1 \leqslant i \leqslant m$,

$$q_i v = q_0 \sigma_1 \cdots \sigma_i v = \begin{cases} d & \text{if } w \text{ is a solid subword of } \sigma_1 \cdots \sigma_i v \\ q_j & \text{otherwise,} \end{cases}$$

where $\sigma_1 \cdots \sigma_j$ is the longest suffix of $\sigma_1 \cdots \sigma_i v$ which is a prefix of w. Since $|v| \geqslant m$, if $q_i v \neq d$, then $q_i v$ is *independent of* i. Thus the transformation induced by v has rank at most 2 and so (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). The following proposition determines the semigroups of L for $w = \sigma_1^{m-1}\sigma_2$ and $w = \sigma_1 \sigma_2^{m-1}$, if $|\Sigma| = 2$. None of these semigroups is a nilpotent extension of a 0-simple ideal. This completes the proof of the theorem.

PROPOSITION 2. *Let* $\Sigma = \{0, 1\}$. *For* $n \geqslant 1$, *let* $R_n = S(\Sigma^* 0 1^n \Sigma^*)$ *and* $L_n = S(\Sigma^* 0^n 1 \Sigma^*)$. *Then* L_n *is isomorphic to* $r(R_n)$ *and* R_n *has, besides the* $\mathscr{J}$-*class consisting of the zero element alone, two other regular* $\mathscr{J}$-*classes*

$$J_1 = \{[1^n]\}, \qquad J_2 = \{[1^i 0 1^j] : 0 \leqslant i \leqslant n-1, 0 \leqslant j \leqslant n-1\}.$$

Furthermore, $J_3 = \{[1^n 0 1^i] : 0 \leqslant i \leqslant n-1\}$ *is a* 0-*minimal null* $\mathscr{J}$-*class. In particular,* R_n *and* L_n *are not nilpotent extensions of a* 0-*simple ideal.*

Proof. The proof is by direct verification and is left to the reader.

THEOREM 8. *Let* L *be a regular language. The following are equivalent:*

(1) L *is locally testable.*

(2) $S(L)$ *divides a direct product*

$T_1 \times T_2 \times \cdots \times T_n$, *where each* T_i *is either a nilpotent extension of a finite rectangular band, a nilpotent extension of a finite combinatorial* 0-*simple ideal or one of the semigroups* R_n, L_n.

Proof. (1) $\Rightarrow$ (2). By the preceding discussion, $S(L)$ divides a direct product $T_1 \times \cdots \times T_n$, where each T_i is either a nilpotent extension of a finite rectangular band or else is $S(\Sigma^* w \Sigma^*)$. By Theorem 7 and Proposition 2, $S(\Sigma^* w \Sigma^*)$ is either a nilpotent extension of a finite combinatorial 0-simple ideal or else is one of R_n or L_n.

(2) $\Rightarrow$ (1). By Proposition 1, LTS is closed under direct products and division. Thus it will suffice to prove that each of the following is locally testable: a nilpotent extension of a rectangular band, a nilpotent extension of a combinatorial 0-simple ideal, R_n, L_n. R_n and L_n are locally testable by Theorem 1 and Proposition 2. The local testability of the remaining semigroups will follow from the next two lemmas.

LEMMA 3. *A rectangular band is* 1*-testable. A finite* 0*-simple combinatorial semigroup is* 2*-testable.*

Proof. Let S be a rectangular band. Then $S = A \times B$. If $s = s_1 s_2 \cdots s_m$, the A-coordinate of s equals the A-coordinate of s_1 and the B-coordinate of s equals the B-coordinate of s_m. Thus $s = s_1 s_m$ and S is 1-testable.

Let S be combinatorial 0-simple. If $s = s_1 \cdots s_m \neq 0$, then, as before, $s = s_1 s_m$. Furthermore, $s = 0$ iff for some i, $s_i s_{i+1} = 0$ (by the Rees theorem). Thus S is 2-testable.

LEMMA 4. *A nilpotent extension S of degree n of a k-testable ideal I is nk-testable.*

Proof. Let $m \geqslant nk$. Pick p such that $k \leqslant p \leqslant m/n$. Then any product $s_1 \cdots s_m$ in S equals a product $x_1 \cdots x_p$, where

$$x_1 = s_1 \cdots s_n, \qquad x_2 = s_{n+1} \cdots s_{2n}, \ldots, x_i = s_{(i-1)n+1} \cdots s_{in}, \ldots, x_p = s_{(p-1)n+1} \cdots s_m .$$

Since S is a nilpotent extension of degree n of I, $x_i \in I$, $1 \leqslant i \leqslant p$. The result follows easily by the k-testability of I. Q.E.D.

This completes the proof of Theorem 8.

Refinements and extensions of the results of this paper appear in [22].

ACKNOWLEDGMENT

I am indebted to Bob McNaughton and Bret Tilson for several interesting discussions on the subject of this paper. Maurice Nivat has suggested the possibility of parallel decompositions. Imre Simon, Peter Johansen and Richard Lipton have detected inaccuracies in the manuscript. Finally, thanks are due to Jean-Francois Perrot for his savage criticism of the manuscript and several helpful suggestions on exposition (which were not always followed).

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