Universal Horn Sentences and the Joint Embedding Property *

Manuel Bodirsky[1], Jakub Rydval[2], and André Schrottenloher[3]

[1] Institute of Algebra, TU Dresden, Germany
manuel.bodirsky@tu-dresden.de
[2] Institute of Theoretical Computer Science, TU Dresden, Germany
jakub.rydval@tu-dresden.de
[3] Cryptology Group, CWI, Amsterdam, The Netherlands
andre.schrottenloher@m4x.org

Abstract. The finite models of a universal sentence Φ are the age of a structure if and only if Φ has the *joint embedding property*. We prove that the computational problem whether a given universal sentence Φ has the joint embedding property is undecidable, even if Φ is additionally Horn and the signature is binary.

Keywords: joint embedding property · universal Horn sentence

1 Introduction

A first-order sentence Φ is called *universal* if it is of the form $\forall x_1, \ldots, x_n . \psi$ for a quantifier-free formula ψ. The preservation theorem of Łos and Tarski states that a first-order sentence Φ is equivalent to a universal sentence if and only if the class of models of Φ are preserved under taking substructures [Hod93]. Universal sentences can be used to describe the *ages* of structures: the *age* of a structure $\mathfrak{A}$ is the class of all finite structures that embed into $\mathfrak{A}$. A class $\mathcal{C}$ of finite relational structures is the age of a structure if and only if $\mathcal{C}$ is closed under isomorphism, substructures, and if it has the *joint embedding property (JEP)*: if $\mathfrak{A}$ embeds into $\mathcal{C}$ and $\mathfrak{B}$ embeds into $\mathcal{C}$, then $\mathcal{C}$ also contains a structure $\mathfrak{C}$ such that both $\mathfrak{A}$ and $\mathfrak{B}$ embed into $\mathfrak{C}$ (see [Hod93]). Braunfeld [Bra19] proved that the question whether the class of finite models of a given universal sentence Φ has the JEP is undecidable. His proof is based on a reduction from the tiling problem.

A universal sentence Φ is called *Horn* if its quantifier-free part is a conjunction of disjunctions each having at most one *positive* disjunct. Equivalently, every conjunct of Φ can be written as a *Horn implication*

$$(\phi_1 \wedge \cdots \wedge \phi_n) \Rightarrow \phi_0$$

* Supported by DFG GRK 1763 (QuantLA). Manuel Bodirsky has received funding from the European Research Council through the ERC Consolidator Grant 681988 (CSP-Infinity). André Schrottenloher is supported by ERC-ADG-ALSTRONGCRYPTO (project 740972).

where $\phi_0, \phi_1, \ldots, \phi_n$ are atomic formulas. We assume that equality $=$ as well as the symbol $\perp$ for falsity are always available when building formulas; thus, atomic formulas are of the form $\perp$, $x_i = x_j$ for $i, j \in \{1, \ldots, n\}$, and $R(y_{i_1}, \ldots, y_{i_k})$ for some relation symbol R of arity k and $i_1, \ldots, i_k \in \{1, \ldots, n\}$. The preservation theorem of McKinsey states that a universal sentence Φ is equivalent to a universal Horn sentence if and only if the class of models of Φ is closed under direct products [Hod93]. Universal Horn sentences are particularly relevant in theoretical computer science; we mention two contexts where they appear.

- Baader and Rydval [BR21] identified which relational structures whose age can be described by a universal sentence yield tractable extensions of the description logic $\mathcal{EL}$ when employed as a concrete domain if equalities are allowed in atomic formulas. This is the case if and only if the age of the structure can be described by a universal Horn sentence.
- The *constraint satisfaction problem* of a structure $\mathfrak{B}$ with a finite relational signature τ is the problem of deciding whether a given finite τ-structure $\mathfrak{A}$ has a homomorphism to $\mathfrak{B}$. This problem is in NP if the age of $\mathfrak{B}$ can be described by a universal sentence, and it is in P if the age of $\mathfrak{B}$ can be described by a universal Horn sentence.

We show that the question whether a given universal sentence is equivalent to a universal Horn sentence can be decided effectively. This motivates the question whether one can also decide whether the class of finite models of a given universal Horn sentence has the JEP. The theories used by Braunfeld to prove the undecidability of JEP for given universal sentences are not universal Horn. We prove that JEP remains undecidable even if the given universal sentence is additionally Horn, and if the signature is binary. Our proof is based on the undecidability of containment of context-free languages and shorter than the argument of Braunfeld.

2 Preliminaries on Universal Sentences

Let τ be a finite relational signature. We always use capital Fraktur letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc to denote structures, and the corresponding capital Roman letters A, B, C, etc for their domains. If $\mathfrak{A}$ is a τ-structure and Φ is a τ-sentence, we write $\mathfrak{A} \models \Phi$ if $\mathfrak{A}$ satisfies Φ. If Φ and Ψ are τ-sentences, we write $\Phi \models \Psi$ if every τ-structure that satisfies Φ also satisfies Ψ. For universal sentences, this is equivalent to the statement that every *finite* τ-structure that satisfies Φ also satisfies Ψ, by a standard application of the compactness theorem of first-order logic. If Φ is a τ-sentence, we denote the class of all finite models of Φ by $[\![\Phi]\!]^{<\omega}$. We sometimes omit universal quantifiers in universal sentences (all first-order variables are then implicitly universally quantified).

3 Universal Horn Sentences

A universal τ-sentence Φ is called *convex* if for all atomic τ-formulas $\phi_1, \dots, \phi_n$, $\psi_1, \dots, \psi_k$, if

$$\Phi \models \bigwedge_{i \leq n} \phi_i \Rightarrow \bigvee_{j \leq k} \psi_j$$

then there exists $j \leq k$ such that

$$\Phi \models \bigwedge_{i \leq n} \phi_i \Rightarrow \psi_j.$$

We say that Φ is *preserved in products* if for every non-empty family $(\mathfrak{A}_i)_{i \in I}$ of models of Φ, the direct product $\prod_I \mathfrak{A}_i$ is also a model of Φ. The following is well known; e.g., the direction (3)⇒(1) is Corollary 9.1.7 in [Hod93]. Since we also need (4) we provide a proof for the convenience of the reader.

Theorem 1. *Let Φ be a universal sentence with relational signature τ. Then the following are equivalent.*

1. *Φ is convex;*
2. *Φ is equivalent to a universal Horn sentence;*
3. *Φ is preserved in products;*
4. *Φ is preserved in binary products of finite structures.*

Proof. (1)⇒(2): We may assume that Φ is in prenex normal form and that its quantifier-free part ϕ is in conjunctive normal form. Every conjunct in ϕ is equivalent to an implication of the form $\bigwedge_{i \leq n} \phi_i \Rightarrow \bigvee_{j \leq k} \psi_j$. Since Φ is convex, we can replace the conjunct by $\bigwedge_{i \leq n} \phi_i \Rightarrow \psi_j$ for some $j \leq k$. In this way, Φ can be rewritten into an equivalent universal Horn sentence.

(2)⇒(3): Corollary 9.1.6 in [Hod93].

(3)⇒(4): This direction is trivial.

(4)⇒(1): Suppose that Φ has m variables and is not convex, i.e., $\Phi \models \bigwedge_{i \leq n} \phi_i \Rightarrow \bigvee_{j \leq k} \psi_j$ but, for every $j \leq k$, there exists a model $\mathfrak{A}_j$ of Φ such that $\mathfrak{A}_j \models (\bigwedge_{i \leq n} \phi_i \wedge \neg\psi_j)(\bar{t}^j)$ for some tuple $\bar{t}^j \in A_j^m$. We may assume that each $\mathfrak{A}_j$ is finite; otherwise we replace it with its substructure on the coordinates of $\bar{t}^j$ while preserving the desired properties. For $\bar{s}^j \coloneqq ((t_1^1, \dots, t_1^j), \dots, (t_m^1, \dots, t_m^j))$ we have

$$\prod_{i \leq j} \mathfrak{A}_i \models (\bigwedge_{i \leq n} \phi_i \wedge \bigwedge_{i \leq j} \neg\psi_i)(\bar{s}^j).$$

It follows by induction on $j \leq k$ that if Φ is preserved in binary products of finite structures, then $\prod_{i \leq j} \mathfrak{A}_i \models \Phi$. We then obtain a contradiction for $j = k$. □

Proposition 1. *Deciding whether a given universal sentence Φ is equivalent to a universal Horn sentence is coNP-complete.*

Proof. We first prove containment in coNP. For a given universal sentence Φ, let $\phi(x_1, \dots, x_n)$ be the quantifier-free part of Φ. If Φ is not equivalent to a universal Horn sentence, then, by Theorem 1, Φ has two finite models $\mathfrak{A}, \mathfrak{B}$ such that $\mathfrak{A} \times \mathfrak{B}$ is not a model of Φ. This means that there exists $\bar{t} \in (A \times B)^n$ such that $\mathfrak{A} \times \mathfrak{B} \not\models \phi(\bar{t})$. But then, by the definition of product of structures, there exist substructures $\mathfrak{A}'$ of $\mathfrak{A}$ and $\mathfrak{B}'$ of $\mathfrak{B}$ of size n with $\bar{t} \in (A' \times B')^n$ and $\mathfrak{A}' \times \mathfrak{B}' \not\models \phi(\bar{t})$. Since models of Φ are preserved under taking substructures, we have $\mathfrak{A}' \models \Phi$ and $\mathfrak{B}' \models \Phi$. Conversely, if there exist two models of Φ of size n whose product is not a model of Φ, then clearly Φ is not equivalent to a universal Horn sentence by Theorem 1. This means that there is a sound and complete algorithm for the complement of the original problem which guesses two models of Φ of size n and then confirms in time polynomial in n that their product is not a model of Φ.

The coNP-hardness can be shown by a reduction from the propositional unsatisfiability problem. From a given propositional formula ϕ we obtain a universal sentence Φ in the unary signature $\tau \coloneqq \{C, L, R\}$ as follows. Let $X_1, \dots, X_n$ be the propositional variables which appear in ϕ. We represent them with fresh first-order variables $x_1, \dots, x_n$, and we also introduce one new first-order variable x_{n+1}. We obtain the formula $\phi'(x_1, \dots, x_n)$ from ϕ by replacing each occurrence of X_i with the literal $C(x_i)$. Now we define Φ as the τ-sentence

$$\forall x_1, \dots, x_{n+1} \big(\phi'(x_1, \dots, x_n) \Rightarrow (L(x_{n+1}) \vee R(x_{n+1}))\big).$$

If ϕ is unsatisfiable, then no τ-structure can satisfy the premise in Φ, which means that every τ-structure satisfies Φ, i.e., Φ is equivalent to the trivial Horn sentence $\bot \Rightarrow \bot$. Now suppose that ϕ is satisfiable. Let $f \colon \{X_1, \dots, X_n\} \to \{\mathit{true}, \mathit{false}\}$ be a satisfying assignment. We define τ-structures $\mathfrak{A}_1, \mathfrak{A}_2$ as follows:

- $A_1 = A_2 \coloneqq \{1, 2, \dots, n+1\}$;
- $C^{\mathfrak{A}_1} = C^{\mathfrak{A}_2} \coloneqq \{i \in [n] \mid f(X_i) = \mathit{true}\}$;
- $L^{\mathfrak{A}_1} = R^{\mathfrak{A}_2} \coloneqq \{1, 2, \dots, n+1\}$ and $R^{\mathfrak{A}_1} = L^{\mathfrak{A}_2} \coloneqq \emptyset$.

Then it is easy to see that $\mathfrak{A}_i \models \Phi$ for $i \in \{1, 2\}$ but $\mathfrak{A}_1 \times \mathfrak{A}_2 \not\models \Phi$. By Theorem 1, Φ is not equivalent to any universal Horn sentence. □

4 The Joint Embedding Property

This section contains some important observations about the JEP in the context of universal Horn theories which we later use in the proof of our main result. Let τ be a relational signature. We say that a Horn implication $\phi \Rightarrow \psi$ is *connected* if the graph with vertex set $\{x_1, \dots, x_n\}$ where x_i and x_j form an edge if they appear jointly in a conjunct of ϕ, is connected in the usual graph-theoretic sense. In the following, we fix a finite relational signature τ.

Proposition 2. *Let Φ be a conjunction of connected Horn implications. Then $[\![\Phi]\!]^{<\omega}$ has the JEP.*

Proof. The class $[\![\Phi]\!]^{<\omega}$ is even closed under the formation of disjoint unions, and therefore has the JEP. $\square$

Example 1. The class of all finite strict partial orders can be described by the conjunction of the following two connected Horn implications.

$$x < y \wedge y < z \Rightarrow x < z$$
$$x < x \Rightarrow \bot.$$

Example 2. The class of all finite layered directed graphs can be described by the conjunction of the following two connected Horn implications.

$$E(x, y) \wedge E(x, x') \wedge E(y, y') \Rightarrow E(x', y')$$
$$E(x, x) \Rightarrow \bot.$$

Example 3. The class of finite models of the Horn implication

$$A(x) \wedge B(y) \Rightarrow C(x, y)$$

is not closed under disjoint unions, but does have the JEP.

Example 3 shows that there are universal Horn sentences whose class of finite models has the JEP, but which are *not* equivalent to sets of connected Horn implications (since the latter are always closed under disjoint unions, while the class in Example 3 is not). However, there is a useful characterisation of the JEP for classes of finite structures defined by conjunctions of Horn implications.

Lemma 1. *Let Φ be a universal Horn sentence over the relational signature τ. Then the following are equivalent:*

1. *$[\![\Phi]\!]^{<\omega}$ has the JEP.*
2. *If ϕ_1, ϕ_2 are conjunctions of atomic formulas, χ is an atomic formula, and $\Phi \models \forall \bar{x}_1, \bar{x}_2 \big(\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2) \Rightarrow \chi(\bar{x}_1)\big)$, then*

$$\Phi \models \forall \bar{x}_1 \big(\phi_1(\bar{x}_1) \Rightarrow \chi(\bar{x}_1)\big)$$
$$\text{or} \quad \Phi \models \forall \bar{x}_2 \big(\phi_2(\bar{x}_2) \Rightarrow \bot\big).$$

Proof. (1)$\Rightarrow$(2): Suppose that Φ does not satisfy (2). Then for $i \in \{1, 2\}$ there are $\mathfrak{A}_i \in [\![\Phi]\!]^{<\omega}$ and tuples $\bar{a}_i$ over A_i such that $\mathfrak{A}_1 \models \phi_1(\bar{a}_1) \wedge \neg\chi(\bar{a}_1)$ and $\mathfrak{A}_2 \models \phi_2(\bar{a}_2)$. Suppose for contradiction that there exists a finite model $\mathfrak{C}$ of Φ together with embeddings $e_i \colon \mathfrak{A}_i \hookrightarrow \mathfrak{C}$ for $i \in \{1, 2\}$. By the assumption that $\Phi \models \forall \bar{x}_1, \bar{x}_2 \big(\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2) \Rightarrow \chi(\bar{x}_1)\big)$ we have that $\mathfrak{C} \models \chi(e_1(\bar{a}_1))$. But since e_1 is an embedding, we also have $\mathfrak{C} \models \neg\chi(e_1(\bar{a}_1))$, a contradiction. Thus, $[\![\Phi]\!]^{<\omega}$ does not have the JEP.

(2)$\Rightarrow$(1): Let $\mathfrak{A}_1, \mathfrak{A}_2 \in [\![\Phi]\!]^{<\omega}$. We construct a structure $\mathfrak{C} \in [\![\Phi]\!]^{<\omega}$ with $\mathfrak{A}_i \hookrightarrow \mathfrak{C}$ as follows. For $i \in \{1, 2\}$, let $\bar{x}_i$ be a tuple of variables representing the elements of A_i in some order, and let $\psi_i(\bar{x}_i)$ be the conjunction of all atomic τ-formulas which hold in $\mathfrak{A}_i$; suppose that the variables in $\bar{x}_1$ and in $\bar{x}_2$ are disjoint.

Let ψ be the conjunction of all atomic formulas implied by $\Phi \wedge \psi_1 \wedge \psi_2$. We claim that ψ does not contain $\bot$: otherwise, $\Phi \models \forall \bar{x}_1, \bar{x}_2(\psi_1(\bar{x}_1) \wedge \psi_2(\bar{x}_2) \Rightarrow \bot)$, and then (2) implies that $\Phi \models \forall \bar{x}_1(\psi_1(\bar{x}_1) \Rightarrow \bot)$ or $\Phi \models \forall \bar{x}_2(\psi_2(\bar{x}_2) \Rightarrow \bot)$, which is impossible since $\mathfrak{A}_1 \models \Phi$ and $\mathfrak{A}_2 \models \Phi$. Define $x \sim y$, for variables x, y from $\bar{x}_1, \bar{x}_2$, if ψ contains the conjunct $x = y$. Define $\mathfrak{C}$ as the structure whose domain are the equivalence classes of $\sim$, and where $([z_1]_\sim, \dots, [z_l]_\sim)$ is in a tuple of a relation of $\mathfrak{C}$ if ψ contains the conjunct $R(z_1, \dots, z_l)$. For $i \in \{1, 2\}$ and $x \in A_i$ define $e_i(x) \coloneqq [x]_\sim$. We claim that e_i is an embedding from $\mathfrak{A}_i$ to $\mathfrak{C}$. It is clear from the construction of $\mathfrak{C}$ that e_i is a homomorphism. Suppose for contradiction that there exist an atomic τ-formula χ and $z_1, \dots, z_l \in A_i$ such that $\mathfrak{A}_i \models \neg\chi(z_1, \dots, z_l)$ while $\mathfrak{C} \models \chi(e_i(z_1), \dots, e_i(z_l))$. For the sake of notation, we assume that $i = 1$; the case that $i = 2$ can be shown analogously. Note that the construction of $\mathfrak{C}$ implies that $\Phi \models \forall \bar{x}_1, \bar{x}_2(\psi_1(\bar{x}_1) \wedge \psi_2(\bar{x}_2) \Rightarrow \chi(\bar{x}_1))$. Then (2) implies that $\Phi \models \forall \bar{x}_1(\psi_1(\bar{x}_1) \Rightarrow \chi(\bar{x}_1))$ or $\Phi \models \forall \bar{x}_2(\psi_2(\bar{x}_2) \Rightarrow \bot)$. The first case is impossible by assumption, and the second case is impossible because $\mathfrak{A}_2 \models \Phi$. Thus, e_i is an embedding from $\mathfrak{A}_i$ to $\mathfrak{C}$, which concludes the proof of the JEP. $\square$

The following example illustrates the use of Lemma 1 and is relevant for the proof of Theorem 2 below.

Example 4. Let Φ be the universal Horn sentence with the three Horn implications

$$\begin{aligned} R_{a_1}(x_1, x_2) \wedge R_{a_2}(x_2, x_3) \wedge R_{a_3}(x_3, x_4) \wedge T_1(z, z) &\Rightarrow R_b(x_1, x_4) \\ R_{a_1}(x_1, x_2) \wedge R_{a_2}(x_2, x_3) \wedge R_{a_3}(x_3, x_4) \wedge T_1(z, z) &\Rightarrow T_2(x_4, z) \\ R_b(x_1, x_2) \wedge T_2(x_2, z) \wedge R_{a_4}(x_2, x_3) &\Rightarrow \bot. \end{aligned}$$

Then $[\![\Phi]\!]^{<\omega}$ does not have JEP because Φ entails

$$R_{a_1}(x_1, x_2) \wedge R_{a_2}(x_2, x_3) \wedge R_{a_3}(x_3, x_4) \wedge R_{a_4}(x_4, x_5) \wedge T_1(z, z) \Rightarrow \bot$$

but neither $T_1(z, z) \Rightarrow \bot$ nor

$$R_{a_1}(x_1, x_2) \wedge R_{a_2}(x_2, x_3) \wedge R_{a_3}(x_3, x_4) \wedge R_{a_4}(x_4, x_5) \Rightarrow \bot.$$

5 Undecidability of JEP

In this section we prove that there is no algorithm that decides whether the class of all finite models of a given universal Horn sentence Φ has the joint embedding property. Our proof is based on a reduction from the problem of deciding containment for context-free languages [AN00]. A *context-free grammar* (CFG) is a 4-tuple $G = (S^n, S^t, P, s)$ where

- S^n is a finite set of *non-terminal symbols*,
- S^t is a finite set of *terminal symbols*,
- $P \subset S^n \times (S^n \cup S^t)^*$ is a finite set of *production rules*, and

– $s \in S^n$ is the *start symbol*.

For $u, v \in (S^n \cup S^t)^*$ we write $u \to_G v$ if there exist $x, y \in (S^n \cup S^t)^*$ and $(a, b) \in P$ such that $u = xay$ and $v = xby$. The transitive closure of $\to_G$ is denoted by $\to_G^*$. The *language of* G is $L(G) \coloneqq \{w \in (S^t)^* \mid s \to_G^* w\}$. We write ϵ for the *empty word*, i.e., the word of length 0.

Theorem 2. *For a given universal Horn sentence Φ the question whether $[\![\Phi]\!]^{<\omega}$ has the JEP is undecidable even if the signature is limited to binary relation symbols.*

Proof. Let $G = (S^n, S^t, P, s)$ be a CFG such that

– $(b, \epsilon) \notin P$ for every $b \in S^n$ (no *'empty productions'*), and
– s is not a subword of w for every pair $(b, w) \in P$.

The containment problem for the languages of such CFGs is known to be undecidable [AN00]. Let τ_G be the signature that contains for every element $a \in S^n \cup S^t$ a binary relation symbol R_a. Let Φ_G be the universal Horn sentence that contains for every $(b, a_1 \dots a_n) \in P$ the Horn implication

$$\begin{array}{lll} & R_{a_1}(x_1, x_2) \wedge \dots \wedge R_{a_n}(x_n, x_{n+1}) \Rightarrow R_b(x_1, x_{n+1}) & \text{if } b \neq s \\ \text{and} & R_{a_1}(x_1, x_2) \wedge \dots \wedge R_{a_n}(x_n, x_{n+1}) \Rightarrow \bot & \text{if } b = s. \end{array}$$

Note that all conjuncts of Φ_G are connected, which means that $[\![\Phi_G]\!]^{<\omega}$ has the JEP by Proposition 2. The following can be shown via a straightforward induction over the length of paths in $\to_G$.

Claim 1 *For every $w = a_1 \dots a_n \in (S^n \cup S^t)^*$ and every $b \in N \setminus \{s\}$, we have*

$$\begin{aligned} & b \to_G^* w \quad \textit{if and only if} \\ & \Phi_G \models \forall x_1, \dots, x_{n+1}\big(R_{a_1}(x_1, x_2) \wedge \dots \wedge R_{a_n}(x_n, x_{n+1}) \Rightarrow R_b(x_1, x_{n+1})\big). \end{aligned}$$

For $b = s$, the same statement holds with $\bot$ instead of $R_b(x_1, x_{n+1})$.

Given an instance of the containment problem consisting of two context-free grammars $G_1 = (S_1^n, S_1^t, P_1, s_1)$ and $G_2 = (S_2^n, S_2^t, P_1, s_1)$ as above, we may assume that $S_1^n \cap S_2^n = \emptyset$ and $S_1^t = S_2^t =: S_*^t$. Let Φ'_{G_1} be the universal Horn sentence in a new signature τ'_{G_1} obtained from Φ_{G_1} as follows. The signature τ'_{G_1} differs from τ_{G_1} in that it contains two new binary relation symbols T_1 and T_2. We extend every Horn implication in Φ_{G_1} by a new universally quantified variable z and replace every literal $R_b(x_i, x_j)$ for $b \in S_1^n$ with the conjunction $R_b(x_i, x_j) \wedge T_2(x_j, z)$. Note that the resulting formula can be rewritten into an equivalent conjunction of Horn implications. Finally, for every Horn implication whose premise only contains binary literals coming from terminal symbols, we expand its premise by the literal $T_1(z, z)$. See Example 4 for an illustration of this construction.

Claim 2 *$[\![\Phi'_{G_1} \wedge \Phi_{G_2}]\!]^{<\omega}$ has the JEP iff $L(G_1) \subseteq L(G_2)$.*

"$\Rightarrow$": Suppose that $[\![\Phi'_{G_1} \wedge \Phi_{G_2}]\!]^{<\omega}$ has the JEP. Let $a_1 \dots a_n \in L(G_1)$. Claim 1 implies that $\Phi_{G_1} \models \forall x_1, \dots, x_{n+1} \big(R_{a_1}(x_1, x_2) \wedge \cdots \wedge R_{a_n}(x_n, x_{n+1}) \Rightarrow \bot \big)$. It follows from the construction of Φ'_{G_1} that

$$\Phi'_{G_1} \models \forall x_1, \dots, x_{n+1}, z \big(R_{a_1}(x_1, x_2) \wedge \cdots \wedge R_{a_n}(x_n, x_{n+1}) \wedge T_1(z, z) \Rightarrow \bot \big).$$

Since $[\![\Phi'_{G_1} \wedge \Phi_{G_2}]\!]^{<\omega}$ has the JEP, by Lemma 1 we must have

$$\begin{aligned} & \Phi'_{G_1} \wedge \Phi_{G_2} \models \forall x_1, \dots, x_{n+1} \big(R_{a_1}(x_1, x_2) \wedge \cdots \wedge R_{a_n}(x_n, x_{n+1}) \Rightarrow \bot \big) \quad (\dagger) \\ \text{or} \quad & \Phi'_{G_1} \wedge \Phi_{G_2} \models \forall z \big(T_1(z, z) \Rightarrow \bot \big). \end{aligned}$$

The latter case is impossible: otherwise, Claim 1 implies that $G \to_G^* \epsilon$. But recall that we disallow empty productions, and hence if $b \to_G^* w$ then $|w| > 0$. In the former case, $(\dagger)$ implies that $\bot$ can be derived from $R_{a_1}(x_1, x_2) \wedge \cdots \wedge R_{a_n}(x_n, x_{n+1}) \wedge \Phi'_{G_1} \wedge \Phi_{G_2}$ by SLD resolution, since SLD resolution is a sound and complete proof system for entailment of Horn sentences by Horn sentences (see, e.g., Theorem 7.10 in [NdW97]). Note that there is no Horn implication in Φ'_{G_1} whose premise only contains S^t_*-literals, which means that $R_{a_1}(x_1, x_2) \wedge \cdots \wedge R_{a_n}(x_n, x_{n+1}) \wedge \Phi'_{G_1}$ is satisfiable. Moreover, the conclusion of each Horn implication in Φ_{G_2} is either $\bot$ or an S^n_2-literal where $S^n_2 \cap \tau'_{G_1} = \emptyset$, which means that $\bot$ must have a syntactic derivation from $R_{a_1}(x_1, x_2) \wedge \cdots \wedge R_{a_n}(x_n, x_{n+1}) \wedge \Phi_{G_2}$. By Claim 1, we have $a_1 \dots a_n \in L(G_2)$.

"$\Leftarrow$": Suppose that $L(G_1) \subseteq L(G_2)$. Let $\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2) \Rightarrow \chi(\bar{x}_1)$ be an arbitrary Horn implication entailed by $\Phi'_{G_1} \wedge \Phi_{G_2}$. We show that $\Phi'_{G_1} \wedge \Phi_{G_2}$ already entails $\phi_1(\bar{x}_1) \Rightarrow \chi(\bar{x}_1)$ or $\phi_2(\bar{x}_2) \Rightarrow \bot$. Again, since SLD resolution is a sound and complete proof system for entailment of Horn sentences by Horn sentences, $\chi(\bar{x}_1)$ can be syntactically derived from $\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2) \wedge \Phi'_{G_1} \wedge \Phi_{G_2}$. Note that the conclusion of each Horn implication in Φ'_{G_1} or Φ_{G_2} has one of the following forms:

(1) $\bot$,
(2) $R_b(x_i, x_j)$ with $b \in S^n_1$,
(3) $T_2(x_j, z)$,
(4) $R_b(x_i, x_j)$ with $b \in S^n_2$.

Thus, since $S^n_2 \cap \tau'_{G_1} = \emptyset$ and $S^n_1 \cap \tau_{G_2} = \emptyset$, the atom $\chi(\bar{x}_1)$ can already be syntactically derived from $\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2) \wedge \Phi'_{G_1}$ or from $\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2) \wedge \Phi_{G_2}$. In the latter case, by Lemma 1, we must have $\Phi_{G_2} \models \forall \bar{x}_1 \big(\phi_1(\bar{x}_1) \Rightarrow \chi(\bar{x}_1) \big)$ or $\Phi_{G_2} \models \forall \bar{x}_2 \big(\phi_2(\bar{x}_2) \Rightarrow \bot \big)$, because $[\![\Phi_{G_2}]\!]^{<\omega}$ has the JEP (Proposition 2).

Suppose that $\chi(\bar{x}_1)$ can be syntactically derived from $\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2) \wedge \Phi'_{G_1}$. Note that the conclusion of each Horn implication in Φ'_{G_1} is of the form (1), (2), or (3). Thus, if $\chi(\bar{x}_1)$ is not of the form (1), (2), or (3), then it has to already be present in $\phi_1(\bar{x}_1)$ as a subformula and we are done. Thus we may assume that $\chi(\bar{x}_1)$ is of the form (1), (2), or (3). Suppose that $\chi(\bar{x}_1)$ is of the form (2) or (3). By our assumption, there exists a sequence $\phi^1, \dots, \phi^\ell$ of conjunctions of atomic τ'_{G_1}-formulas such that

- $\phi^1 = \phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2)$,
- $\phi^{i+1} = \phi^i \wedge \chi^{i+1}$ for a single atomic τ'_{G_1}-formula χ^{i+1} that results from the application of a single Horn implication from Φ'_{G_1} to ϕ^i,
- ϕ^ℓ contains $\chi(\bar{x}_1)$ as a subformula whereas $\phi^{\ell-1}$ does not.

Consider any such sequence $\phi^1, \ldots, \phi^\ell$ of the smallest possible length, i.e., without any unnecessary derivations. Note that formulas of the form (2) or (3) only occur in Φ'_{G_1} simultaneously as a conjunction where the variable x_j is identical in both conjuncts. In particular, a formula of the form (2) can be syntactically derived from $\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2) \wedge \Phi'_{G_1}$ if and only if a formula of the form (3) can be syntactically derived from $\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2) \wedge \Phi'_{G_1}$. Consequently, the application of a Horn implication from Φ'_{G_1} with a conclusion of the form (2) or (3) either does not trigger any new Horn implications from Φ'_{G_1} with different premises, or it results in connectedness of all variables that appear in its premise. Since the variables of χ are properly contained in the variables of ϕ_1, no χ^{i+1} can contain any variables from $\bar{x}_2$. Thus, $\Phi'_{G_1} \models \forall \bar{x}_1 \big(\phi_1(\bar{x}_1) \Rightarrow \chi(\bar{x}_1)\big)$ and we are done. Finally, suppose that $\chi(\bar{x}_1)$ is of the form (1). If it is not the case that

- $\phi_1(\bar{x}_1)$ contains a subformula of the form $R_{a_1}(x_1, x_2) \wedge \cdots \wedge R_{a_n}(x_n, x_{n+1})$ for some $a_1 \ldots a_n \in L(G_1)$ and
- $\phi_2(\bar{x}_2)$ contains a subformula of the form $T_1(z)$

or vice versa, then we can use a similar *'connectivity'* argument as before to show that $\Phi'_{G_1} \models \forall \bar{x}_1 \big(\phi_1(\bar{x}_1) \Rightarrow \bot\big)$ or $\Phi'_{G_1} \models \forall \bar{x}_2 \big(\phi_2(\bar{x}_2) \Rightarrow \bot\big)$ and we are done. Otherwise, we obtain $\Phi_{G_2} \models \forall \bar{x}_1 \big(\phi_1(\bar{x}_1) \Rightarrow \bot\big)$ or $\Phi_{G_2} \models \forall \bar{x}_2 \big(\phi_2(\bar{x}_2) \Rightarrow \bot\big)$ using Claim 1 because $a_1 \ldots a_n \in L(G_1) \subseteq L(G_2)$. Thus, $[\![\Phi'_{G_1} \wedge \Phi_{G_2}]\!]^{<\omega}$ has the JEP. This concludes the proof of the claim.

We have thus found a reduction from the undecidable containment problem $L(G_1) \subseteq L(G_2)$ to the decidability problem of the JEP for $\Phi'_{G_1} \wedge \Phi_{G_2}$; note that the latter sentence is universal Horn and can be computed from G_1 and from G_2 in polynomial time. □

6 Open Problems

An important open question about universal sentences over relational signatures is the decidability of the *amalgamation property (AP)*, which is a strong form of the joint embedding property, and which is of fundamental importance in constraint satisfaction [Bod21]. Unlike the JEP, the AP is decidable if all relation symbols in the signature are at most binary (see, e.g., [BKS20]). For general relational signatures, we ask the following question.

Question 1. Is it decidable whether the class of finite models of a given universal Horn sentence Φ has the amalgamation property?

Note that if the class of finite models of Φ does have amalgamation, then we can compute an amalgam syntactically by computing implied atomic formulas just as in the proof of Lemma 1.

References

AN00. Peter RJ Asveld and Anton Nijholt. The inclusion problem for some subclasses of context-free languages. *Theoretical computer science*, 230(1-2):247–256, 2000.

BKS20. Manuel Bodirsky, Simon Knäuer, and Florian Starke. ASNP: A tame fragment of existential second-order logic. In Marcella Anselmo, Gianluca Della Vedova, Florin Manea, and Arno Pauly, editors, *Beyond the Horizon of Computability - 16th Conference on Computability in Europe, CiE 2020, Fisciano, Italy, June 29 - July 3, 2020, Proceedings*, volume 12098 of *Lecture Notes in Computer Science*, pages 149–162. Springer, 2020.

Bod21. Manuel Bodirsky. *Complexity of Infinite-Domain Constraint Satisfaction*. Cambridge University Press, 2021. To appear in the LNL Series.

BR21. Franz Baader and Jakub Rydval. An algebraic view on p-admissible concrete domains for lightweight description logics, 2021. To appear in JELIA 2021, available at https://tu-dresden.de/inf/lat/reports#BaRy-LTCS-20-10.

Bra19. Samuel Braunfeld. The undecidability of joint embedding and joint homomorphism for hereditary graph classes. *Discrete Mathematics & Theoretical Computer Science*, 21, 2019. Preprint available at arXiv:1903.11932.

Hod93. Wilfrid Hodges. *Model theory*. Cambridge University Press, 1993.

NdW97. Shan-Hwei Nienhuys-Cheng and Ronald de Wolf. *Foundations of Inductive Logic Programming*, volume 1228 of *Lecture Notes in Computer Science*. Springer, 1997.