Symbolic Strategy Synthesis for Games on Pushdown Graphs

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Abstract. We consider infinite two-player games on pushdown graphs, the reachability game where the first player must reach a given set of vertices to win, and the Büchi game where he must reach this set infinitely often. We provide an automata theoretic approach to compute uniformly the winning region of a player and corresponding winning strategies, if the goal set is regular. Two kinds of strategies are computed: positional ones which however require linear execution time in each step, and strategies with pushdown memory where a step can be executed in constant time.

1 Introduction

Games are an important model of reactive computation and a versatile tool for the analysis of logics like the μ-calculus [5,6]. In recent years, games over infinite graphs have attracted attention as a framework for the verification and synthesis of infinite-state systems [8]. In the present paper we consider pushdown graphs (transition graphs of pushdown automata). It was shown in [12] that in two-player parity games played on pushdown graphs, winning strategies can be realized also by pushdown automata. The drawback of these results [8,12] are a dependency of the analysis on a given initial game position, and a lack of algorithmic description of the (computation of) winning strategies. Such an algorithmic (or "symbolic") solution must transform the finite presentation of the pushdown game into a finite description of the winning regions of the two players as well as of their strategies.

In this paper we develop such an algorithmic approach, also leading to uniform complexity bounds. The nodes of the game graph, *i.e.*, the game positions, are *unbounded* finite objects: stack contents of a given pushdown automaton. Our "symbolic approach" uses finite automata as defining devices for sets of game positions and for the description of strategies. This lifts the results of [1] and [2] from CTL and LTL model checking over pushdown systems to the level of program synthesis.

In this paper we only consider reachability games and Büchi games (where a winning play should reach a given regular set of nodes, respectively should pass infinitely often through this set). This restriction is justified by two aspects: First, reachability and Büchi games are the typical cases in the analysis of safety

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and liveness conditions. Secondly, as our construction shows, these cases can be handled with set-oriented computations (in determining fixed-points) which fits well to the symbolic approach. So far, it seems open whether in the more general case of parity games the treatment of individual game positions can be avoided in order to have a symbolic computation.

Our paper is structured as follows: we first exhibit a computation of the winning region of a reachability game. In third section, we derive from it two kinds of winning strategies. Finally in fourth section this material is used to solve Büchi games on pushdown graphs. The proofs can be found on the Web at `http://www-i7.informatik.rwth-aachen.de/~cachat/` .

2 Reachability Game, Computing the Attractor

2.1 Technical Preliminaries

A Pushdown Game System (PDS) $\mathcal{P}$ is a triple (P, Γ, Δ), where Γ is the finite stack alphabet, $P = P_0 \uplus P_1$ the partitioned finite set of control locations, where P_i indicates the game positions of Player i, and $\Delta \subseteq P \times \Gamma \times P \times \Gamma^*$ the finite set of (unlabelled) transition rules.

Each macro-state or *configuration* is a pair pw of a control location p and a stack content w. The set of nodes of the pushdown game graph $\mathcal{G} = (V, \hookrightarrow)$ is the set of *all* configurations: $V = P\Gamma^*$, and the arcs are exactly the pairs

$$p\gamma v \hookrightarrow qwv, \text{ for } (p, \gamma, q, w) \in \Delta, \text{ where } \gamma \in \Gamma \text{ and } v \in \Gamma^* \ .$$

Referring to the case $v = \epsilon$ we also write rules of Δ in the form $p\gamma \hookrightarrow qw$. In the following γ is always a single letter from Γ. If one needs a bottom stack symbol ($\bot$) one has to declare it explicitly in Γ and Δ. The set of nodes of Player 0 is $V_0 = P_0\Gamma^*$, that of Player 1 is $V_1 = P_1\Gamma^*$. Starting in a given configuration $\pi_0 \in V$, a play in $\mathcal{G}$ proceeds as follows: if $\pi_0 \in V_0$, Player 0 picks the first transition (move) to π_1, else Player 1 does, and so on from the new configuration π_1. A play is a (possibly infinite) maximal sequence $\pi_0\pi_1\cdots$

We describe sets of configurations (and thus also winning conditions in pushdown games) by finite automata. We are thus interested in regular sets of configurations. We define them from alternating $\mathcal{P}$-*automata*. They are alternating word automata with a special convention about initial states. A $\mathcal{P}$-automaton $\mathcal{A}$ is a tuple $(\Gamma, Q, \longrightarrow, P, F)$, where Q is a finite set of states, $\longrightarrow \ \subseteq \ Q \times \Gamma \times 2^Q$ a set of transitions, $P \subseteq Q$ a set of initial states (which are taken here as the control locations of $\mathcal{P}$), and $F \subseteq Q$ a set of final states. For each $p \in P$ and $w \in \Gamma^*$, the automaton $\mathcal{A}$ accepts a configuration pw iff there exists a successful $\mathcal{A}$-run on w *from* p. Formally a transition has the form $r \xrightarrow{\gamma} \beta$, where β is a positive boolean formula over Q in Disjunctive Normal Form. To simplify the exposition we allow AND-transitions $r \xrightarrow{\gamma} r_1 \wedge \cdots \wedge r_n$, written as $r \xrightarrow{\gamma} \{r_1, \cdots, r_n\}$, and we capture disjunction by nondeterminism. So a transition like $r \xrightarrow{\gamma} (r_1 \wedge r_2) \vee (r_3 \wedge r_4)$ is represented here by *two* transitions $r \xrightarrow{\gamma} \{r_1, r_2\}$ and $r \xrightarrow{\gamma} \{r_3, r_4\}$.

We define the global transition relation of $\mathcal{A}$, the reflexive and transitive closure of $\longrightarrow$, denoted $\longrightarrow_* \ \subseteq \ Q \times \Gamma^* \times 2^Q$, as follows:

– $r \xrightarrow{\epsilon}_* \{r\}$, ($\epsilon$ is the empty word),
– $r \xrightarrow{\gamma} \{r_1, \cdots, r_n\} \ \wedge \ \forall i,\ r_i \xrightarrow{w}_* S_i \ \ \Rightarrow \ \ r \xrightarrow{\gamma w}_* \bigcup_i S_i$.

The automaton $\mathscr{A}$ accepts the word pw iff there *exists* a run $p \xrightarrow{w}_* S$ with $S \subseteq F$, *i.e.*, all finally reached states are final.

In section 3 we will need the description of a run. The run trees of an alternating automaton $\mathscr{A}$ (where the branching captures the AND-transitions) can be transformed to "run DAGs" (Directed Acyclic Graphs, see [9,10]). In such a run DAG, the states occurring on each level of the tree are collected in a set, and a transition $r \longrightarrow \{r_1, \cdots, r_k\}$ connects state r of level i with states $\{r_1, \cdots, r_k\}$ of level $i+1$. Note that every transition of level i is labelled by the same i-th letter of the input word. Let Φ be the set of partial functions from Q to the transition relation $\longrightarrow$ of $\mathscr{A}$. A run DAG from state p labelled by $w = \gamma_0 \cdots \gamma_n$ is described by a sequence $\sigma_0, \cdots, \sigma_n$ of elements of Φ and a sequence $Q_0, Q_1, \cdots, Q_n$ of subsets of Q, such that $Q_i = Dom(\sigma_i)$, and from each $q \in Q_i$ the transition $\sigma_i(q)$ is used from q:

$$Q_0 = \{p\} \xrightarrow[\sigma_0]{\gamma_0} Q_1 \xrightarrow[\sigma_1]{\gamma_1} \cdots \ Q_n \xrightarrow[\sigma_n]{\gamma_n} S \ .$$

So σ_i describes the step $Q_i \xrightarrow{\gamma_i} Q_{i+1}$ by the transitions used. We write shortly $\{p\} \xrightarrow[\sigma]{w}_* S$, assuming $\sigma = \sigma_0, \cdots, \sigma_n$, or just $\{p\} \xrightarrow{w}_* S$ to denote the run.

2.2 Reachability

We consider a regular *goal set* $R \subseteq P\Gamma^*$, defined by a $\mathscr{P}$-automaton $\mathscr{A}_R$. Player 0 wins a play iff it reaches a configuration of R. Our goal is to compute the winning region W_0 of this game: the set of nodes from which Player 0 can force the play to reach the set R or a deadlock for Player 1. The set W_0 is clearly the "0-attractor of R" (see [11]), denoted $Attr_0(R)$, and defined inductively by

$$\begin{aligned} Attr_0^0(R) &= R \ , \\ Attr_0^{i+1}(R) &= Attr_0^i(R) \cup \left\{u \in V_0 \mid \exists v, u \hookrightarrow v,\ v \in Attr_0^i(R)\right\} \\ &\qquad \cup \left\{u \in V_1 \mid \forall v, u \hookrightarrow v \Rightarrow v \in Attr_0^i(R)\right\} , \\ Attr_0(R) &= \textstyle\bigcup_{i \in \mathbb{N}} Attr_0^i(R) \ . \end{aligned}$$

As the degree of the game graph is finite, an induction on ω is sufficient. According to this definition, we adopt the convention that if the play is in a deadlock (before reaching R), the Player who should play has lost.

Our task is to transform a given automaton $\mathscr{A}_R$ recognizing R into an automaton $\mathscr{A}_{Att(R)}$ recognizing $Attr_0(R)$. Without loss of generality, we can assume that there is no transition in $\mathscr{A}_R$ leading to an initial state (a state of P).

Algorithm 1 (saturation procedure)
Input: *a PDS $\mathscr{P}$, a $\mathscr{P}$-automaton $\mathscr{A}_R$ that recognizes the goal set R, without transition to the initial states.*
Output: *a $\mathscr{P}$-automaton $\mathscr{A}_{Att(R)}$ that recognizes $Attr_0(R)$.*

Let $\mathscr{A}_{Att(R)} := \mathscr{A}_R$. *Transitions are added to $\mathscr{A}_{Att(R)}$ according to the following saturation procedure.*
repeat

(Player 0) *if* $p \in P_0$, $p\gamma \hookrightarrow qv$ *and* $q \xrightarrow{v}_* S$ *in* $\mathcal{A}_{Att(R)}$, *then add a new transition* $p \xrightarrow{\gamma} S$.

(Player 1) *if* $p \in P_1$, $\begin{cases} p\gamma \hookrightarrow q_1 v_1 \\ \vdots \qquad \vdots \\ p\gamma \hookrightarrow q_n v_n \end{cases}$ *are* all *the moves (rules) starting from* $p\gamma$ *and* $\begin{cases} q_1 \xrightarrow{v_1}_* S_1 \\ \vdots \qquad \vdots \\ q_n \xrightarrow{v_n}_* S_n \end{cases}$ *in* $\mathcal{A}_{Att(R)}$, *then add a new transition* $p \xrightarrow{\gamma} \bigcup_i S_i$.

until *no new transition can be added.*

Note that $\mathcal{A}_{Att(R)}$ has exactly the same state space as $\mathcal{A}_R$. The algorithm eventually stops because there are only finitely many possible new transitions, and the "saturation" consists in adding as many transitions as possible. The idea of adding a new transition $p \xrightarrow{\gamma} S$ for $p \in P_0$ is that, if $qv \in Attr_0(R)$, and $p\gamma \hookrightarrow qv$, then $p\gamma \in Attr_0(R)$ too, and then $p\gamma$ *should* have the same behavior as qv in the automaton. For $p \in P_1$, $Attr_0(R)$ is defined by a conjunction, expressed in $\mathcal{A}_{Att(R)}$ by the AND-transition. The algorithm and the proof is a generalization of [2,7] from nondeterministic automata (for simple reachability) to alternating automata (for game reachability). In [2], one deals with the case $P = P_0$, and the "winning region" is the set of "predecessors" of R, denoted $pre^*(R)$. In [1], alternating (pushdown) automata were already considered, but they were not used to solve a game, and winning strategies were not treated.

Theorem 2 *The automaton* $\mathcal{A}_{Att(R)}$ *constructed by Algorithm 1 recognizes the set* $Attr_0(R)$, *if* $\mathcal{A}_R$ *recognizes* R.

The algorithm runs in time $\mathcal{O}(|\Delta|\ 2^{c|Q|^2})$, where $|\Delta|$ is the sum of the lengths of the rules in Δ. An implementation of Algorithm 1 was developed in [4].

We have chosen a *regular* goal set R, and proved that $Attr_0(R)$ is also regular. If we consider a context free goal set R, the situation diverges for the cases of simple reachability and game reachability:

Proposition 3 *If the goal set* R *is a context free language, then* $pre^*(R)$ *is also context free, but* $Attr_0(R)$ *is not necessarily context free.*

The first part can be deduced from [3]. For the proof of the second part, we just remark that the intersection of two context free languages may not be context free. If R_1 and R_2 are two context free languages over $\{a, b, c\}$ and the first move of Player 1 goes from pu to $q_1 u$ or $q_2 u$, $u \in \{a, b, c\}^*$, and if the goal set is $q_1 R_1 \cup q_2 R_2$, then the winning region is $p(R_1 \cap R_2) \cup q_1 R_1 \cup q_2 R_2$.

2.3 Determining Membership in the Attractor

In [8] and [12], for a given initial position of the game, an EXPTIME procedure determines if it is in the winning region W_0 of Player 0. In contrast our solution

is uniform: after a single EXPTIME procedure, we can determine in linear time if any given configuration is in the winning region W_0.

To determine whether a given configuration belongs to $Attr_0(R)$, we can use a polynomial time algorithm, that searches backwards all the accepting runs of the automaton $\mathcal{A}_{Att(R)}$ (from now on, we skip corresponding claims for Player 1). We repeat here the classical algorithm because variants of it will be used in the next section. The correctness proof is easy and omitted here.

Algorithm 4 (Membership)
Input: *an alternating $\mathcal{P}$-automaton $\mathcal{B} = (\Gamma, Q, \longrightarrow, P, F)$ recognizing $Attr_0(R) = L(\mathcal{B})$, a configuration $pw \in P\Gamma^*$, $w = a_1 \dots a_n$.*
Output: *Answer whether $pw \in L(\mathcal{B})$ or $pw \notin L(\mathcal{B})$.*

Let $S := F$;
for $i := n$ down to 1 **do** $S := \{s \in Q \mid \exists\ (s \xrightarrow{a_i} X)$ in $\mathcal{B}, X \subseteq S\}$ **end for**
If $p \in S$, *answer "$pw \in L(\mathcal{B})$"* else *answer "$pw \notin L(\mathcal{B})$"*

The space complexity of Algorithm 4 is $\mathcal{O}(|Q|)$, the time complexity is $\mathcal{O}(nm|Q|)$ where m is the number of transitions of $\mathcal{B}$, and n is the length of the input configuration.

3 Winning Strategy for Player 0

A *strategy* for Player 0 is a function which associates to a prefix $\pi_0\pi_1 \cdots \pi_n \in V^*V_0$ of a play a "next move" π_{n+1} such that $\pi_n \hookrightarrow \pi_{n+1}$.

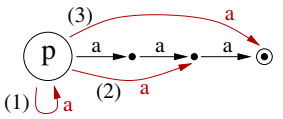
3.1 Preparation

A move of Player 0 consists in a choice of a PDS-Rule. Given a configuration $pw \in Attr_0(R)$, our aim is to extract such a choice from an accepting run of $\mathcal{A}_{Att(R)}$ on pw. In Algorithm 1, a new transition $p \xrightarrow{\gamma} S$ of $\mathcal{A}_{Att(R)}$ is generated by a (unique) rule $p\gamma \hookrightarrow qv$ of the PDS under consideration, if $p \in P_0$. We extend now the algorithm so that it computes the partial function *Rule* from $\longrightarrow$ to Δ. This function remembers the link between a new transition of the finite automaton for $Attr_0(R)$ and the rule of the Pushdown Graph that was used to construct it. We shall write in the algorithm $Rule(p \xrightarrow{\gamma} S) := p\gamma \hookrightarrow qv$. For transitions $p \xrightarrow{\gamma} S$ of the original automaton $\mathcal{A}$, $Rule(p \xrightarrow{\gamma} S)$ is undefined.

Now, given a configuration $p\gamma w \in V_0$ accepted by $\mathcal{A}_{Att(R)}$, with a run $\{p\} \xrightarrow{\gamma} S \xrightarrow{w}_* T$ (and if $p\gamma w \notin R$), a first idea would be to choose the move $Rule(p \xrightarrow{\gamma} S) = p\gamma \hookrightarrow qv$, *hoping* to get closer to R. Unfortunately this does not, in general, define a winning strategy. Still it ensures that we remain in the winning region. The following example illustrates this situation.

Example 5 *Let $\Gamma = \{a\}$, $P = P_0 = \{p\}$, $P_1 = \emptyset$ and $\Delta = \{pa \hookrightarrow p, pa \hookrightarrow paa\}$.*

It is clear that Player 0 can add and remove as many a's as he wants. Let $R = \{pa^3\}$ (R is regular), then the winning region is $Attr_0(R) = pa^+$, as shown



$$Rule((1)) = pa \hookrightarrow p$$
$$Rule((2)) = pa \hookrightarrow paa$$
$$Rule((3)) = pa \hookrightarrow paa \qquad p \hookleftarrow pa \leftrightarrows paa \leftrightarrows [paaa] \cdots$$

Fig. 1. Automaton from Algorithm 1 and Example 5, game graph

by the automaton in Figure 1, from Algorithm 1. There are two different runs of $\mathcal{A}_{Att(R)}$ that accept paa: through transitions (1)(3), or through (2). The strategy associated to (1)(3) plays to pa, and therefore is not successful. To define a winning strategy using finite automaton $\mathcal{A}_{Att(R)}$, we need to select the most suitable run on a given configuration, or to remember information about an accepting run, to play coherently the following moves. We give two solutions: the first one, a positional strategy, associates a cost to each transition added while constructing $\mathcal{A}_{Att(R)}$, in order to compute the distance to R. The second one, a pushdown strategy, uses a stack to remember how $\mathcal{A}_{Att(R)}$ accepts the current configuration.

3.2 Positional Min-Rank Strategy

The *rank* of a configuration pw is the smallest i such that $pw \in Attr_0^i(R)$ (it is ∞ if $pw \notin Attr_0(R) = W_0$). It is the "distance" of the configuration pw to R. In the following we consider only configurations in W_0. Then Player 0 will be able, from a configuration in $Attr_0^i(R)$, to move to $Attr_0^{i-1}(R)$, and Player 1 does this with each possible move. In order to implement this, during the construction of $\mathcal{A}_{Att(R)}$ we will attribute to each $\mathcal{A}_{Att(R)}$-transition τ a cost $Cost(\tau)$. Initially, each transition of $\mathcal{A}_R$ has the cost 0 (with these transitions $\mathcal{A}_{Att(R)}$ recognizes configurations that are already in R).

The function $Cost$ from the transition relation $\longrightarrow$ of $\mathcal{A}_{Att(R)}$ to $\mathbb{N}$ is extended to a function $Cost^*$ from the run DAGs to $\mathbb{N}$. Given a fixed run $\{q\} \xrightarrow{w}_* S$ of the automaton $\mathcal{A}_i$ (obtained at step i in the construction of $\mathcal{A}_{Att(R)}$), its cost $Cost^*(\{q\} \xrightarrow{w}_* S)$ is the maximal sum of the costs of the transitions along a single path (branch) of the run DAG $\{q\} \xrightarrow{w}_* S$. Inductively $Cost^*$ is defined by the following clauses:
$Cost^*(\{q\} \xrightarrow{\epsilon}_* \{q\}) = 0$
$Cost^*(\{q\} \xrightarrow{\gamma} \{q_1, \cdots, q_n\} \xrightarrow{u}_* \bigcup_i S_i) =$

$$Cost(q \xrightarrow{\gamma} \{q_1, \cdots, q_n\}) + \max_{1 \leqslant i \leqslant n} (Cost^*(\{q_i\} \xrightarrow{u}_* S_i)) \ .$$

When adding a new transition $p \xrightarrow{\gamma} S$ to $\mathcal{A}_i$, to obtain $\mathcal{A}_{i+1}$, its cost is computed by an extension of Algorithm 1, using the costs of the existing transitions. In the main loop of Algorithm 1, we add the following assignments:
- if $p \in P_0 \ \ldots$, let $Cost(p \xrightarrow{\gamma} S) := 1 + Cost^*(\{q\} \xrightarrow{v}_* S)$,
- if $p \in P_1 \ \ldots$, let $Cost(p \xrightarrow{\gamma} \bigcup_i S_i) := 1 + \max_j(Cost^*(\{q_j\} \xrightarrow{v_j}_* S_j))$.

The significance of $Cost^*$ follows clearly from next proposition:

Proposition 6 *For any configuration $pw \in Attr_0(R)$,*

$$rank(pw) = \min\{Cost^*(\{p\} \xrightarrow{w}_* S) \mid \{p\} \xrightarrow{w}_* S \subseteq F \text{ in } \mathscr{A}_{Att(R)}\} .$$

In the Example 5, one gets $Cost((1)) = 1$, $Cost((2)) = 1$, $Cost((3)) = 2$. So using the transitions (1) and (3) is not the best way to accept *paa*, and transition (2) is taken. We are now able to define the desired strategy.

Min-rank Strategy for Player 0
Input: alternating automaton $\mathscr{A}_{Att(R)}$ for $Attr_0(R)$, functions $Rule$ and $Cost$ (as computed from Algorithm 1 from PDS $\mathcal{P}$), configuration $pw \in Attr_0(R)$, $p \in P_0$.
Output: "next move" from configuration pw.

Find an accepting run $\{p\} \xrightarrow{w}_* S \subseteq F$ of $\mathscr{A}_{Att(R)}$ with minimal cost $Cost(\{p\} \xrightarrow{w}_* S)$.
If the cost is 0, $pw \in R$ and the play is won, else decompose this run: $w = \gamma w'$, $\{p\} \xrightarrow{\gamma} T \xrightarrow{w'}_* S$, and choose the rule $Rule(p \xrightarrow{\gamma} T)$.

Theorem 7 *The min-rank strategy is positional, winning from all configurations of the winning region W_0 of Player 0. It can be computed in time $\mathcal{O}(n)$ in the length n of the input configuration.*

Algorithm 4 can be easily extended to compute the distance to R and the strategy. By Proposition 6, the min-rank strategy is optimal in the sense that it finds a shortest path to R. It reevaluates its choices at each step of the game (particularly if Player 1 goes much "closer" to R than needed). We will present in the next subsection a strategy that is not necessarily optimal but easier to compute.

3.3 Pushdown Strategy

A *pushdown strategy*, as defined in [12] is a deterministic pushdown automaton with input and output. It "reads" the moves of Player 1 and outputs the moves (choices) of Player 0, like a pushdown transducer. For simplicity, we will restrict our presentation to the following form of pushdown strategy:

Definition 8 *Given a PDS (P, Γ, Δ), $P = P_0 \uplus P_1$, where Δ_i is the set of transition rules in Δ departing from Player i configurations, a* pushdown strategy *for Player 0 in this game is a* deterministic *pushdown automaton $\mathcal{S} = (P, A, \Pi)$, where $A = \Gamma \times \Sigma$, Σ is any alphabet, $\Pi \subseteq ((P_1 \times A \times \Delta_1) \times (P \times A^*)) \cup ((P_0 \times A) \times (P \times A^* \times \Delta_0))$ is a finite set of transition rules.*

A transition of $\mathcal{S}$ either reads a move of Player 1 or outputs a move for Player 0, in both cases updating its stack. We will now define a pushdown strategy, starting

from the automaton $\mathcal{A}_{Att(R)}$. Given a configuration $pw \in Attr_0(R)$, there is an accepting run $\{p\} \xrightarrow{w}_* S$ of $\mathcal{A}_{Att(R)}$:

$$Q_0 = \{p\} \xrightarrow[\sigma_0]{\gamma_0} Q_1 \xrightarrow[\sigma_1]{\gamma_1} \cdots \; Q_n \xrightarrow[\sigma_n]{\gamma_n} S, \qquad w = \gamma_0\gamma_1\cdots\gamma_n \; .$$

Our aim is to store in the stack of the strategy the description of this run. The corresponding configuration of $\mathcal{S}$ is $p(\gamma_0, \sigma_0)\cdots(\gamma_n, \sigma_n)$. We fix for the alphabet Σ the set Φ (see Section 2.1).

At the beginning of the play, if the initial configuration pw is in $Attr_0(R)$, we have to initialize the stack of $\mathcal{S}$ with the description of an accepting run of $\mathcal{A}_{Att(R)}$ (not necessarily the cheapest according to the costs defined above). Algorithm 4 can initialize the stack at the same time when searching an accepting run (in linear time). We define now the unique transition rule of Π from $(p, (\gamma_0, \sigma_0))$ or $(p, (\gamma_0, \sigma_0), \delta_1)$ (with $\delta_1 \in \Delta_1$). By construction $\sigma_0(p)$ is the "good" transition $\tau = p \xrightarrow{\gamma_0} Q_1$ used in the run of $\mathcal{A}_{Att(R)}$.

- If $p \in P_0$ then output the move $Rule(\sigma_0(p)) = p\gamma_0 \hookrightarrow qv$ that corresponds to τ. Remove the first letter of the stack. Push on the stack the description of the run $\{q\} \xrightarrow{v}_* Q_1$ used in Algorithm 1) to generate τ. Go to control state q.
- If $p \in P_1$ and Player 1 chooses the transition $\delta_1 = p\gamma_0 \hookrightarrow qv$ in Δ_1, by construction of the automaton $\mathcal{A}_{Att(R)}$, $q \xrightarrow{v}_* S$, and S is a subset Q_1. Go to control state q, remove the first letter of the stack, push the description of the run $\{q\} \xrightarrow{v}_* S$ used in Algorithm 1) to generate τ.

For Example 5 (Figure 1), we can see that the pushdown strategy is winning even if the initialization is not optimal. The configuration paa is in the winning region and an accepting run is coded on the stack of the strategy:

$$p(a, \{(p, 1)\})(a, \{(p, 3)\}) \; .$$

According to the strategy, the following play is generated (the symbol "$-$" denotes a value that is not relevant):

$$\begin{array}{rll} (\text{by } Rule((1)) = pa \hookrightarrow p) & \text{proceed to} & p(a, \{(p, 3)\}) \\ (\text{by } Rule((3)) = pa \hookrightarrow paa) & \text{proceed to} & p(a, \{(p, 2)\})(a, -) \\ (\text{by } Rule((2)) = pa \hookrightarrow paa) & \text{proceed to} & p(a, -)(a, -)(a, -) \; . \end{array}$$

Theorem 9 *One can construct effectively a pushdown strategy that is winning from each node of the winning region of a pushdown reachability game. Its transition function is defined uniformly for the whole winning region. The initialization of the stack is possible in linear time in the length of the initial game position, and the computation of the "next move" is in constant time (for fixed Δ).*

Although there is no need to compute costs to define this strategy, it is useful to refer to the costs of the previous subsection for the correctness proof. The strategy for Player 1 in this "safety game" is much easier to define and compute: he just has to stay in $V \backslash Attr_0(R)$.

3.4 Discussion

The stack of the pushdown strategy needs to be initialized at the beginning of a play (in linear time in the length of the configuration); then the computation of the "next move" is done in constant time (execution of one transition of the strategy). In contrast, the min-rank strategy needs for each move a computation in linear time in the length of the configuration. So we can say that in the case of pushdown graphs a positional strategy can be more expensive than a strategy with memory. This effect does not appear over finite-state game graphs.

4 Büchi Condition

Given $\mathcal{P}$ and R as in the preceding sections, the (Büchi) winning condition is now the following:

Player 0 wins a play iff it meets infinitely often the goal set R, or ends in a deadlock for Player 1.

To determine the winning region of this game one defines $Attr_0^+$ inductively, similarly to $Attr_0$: for $T \subseteq V$, let

$$\begin{aligned} X_0(T) &= \emptyset \ , \\ X_{i+1}(T) &= X_i(T) \cup \{u \in V_0 \mid \exists v, u \hookrightarrow v,\ v \in T \cup X_i(T)\} \\ &\quad \cup \{u \in V_1 \mid \forall v, u \hookrightarrow v \Rightarrow v \in T \cup X_i(T)\} \ , \\ Attr_0^+(T) &= \textstyle\bigcup_{i \geqslant 0} X_i(T) \qquad \text{(the degree of the game graph is bounded).} \end{aligned}$$

Following the approach as in [11] (where the definition of the X_i's needs adjustment) the following claims are easy:

Proposition 10 *$Attr_0^+(T)$ is the set of nodes from which Player 0 can join T in* at least *one move, whatever Player 1 does.*

Proposition 11 *Let $Y_0 = V$, and $\forall i \geqslant 0, Y_{i+1} = Attr_0^+(Y_i \cap R)$, then the fixed point $Y^\infty = \bigcap_{i \geqslant 0} Y_i$ is the winning region of the Büchi game with goal set R.*

In the case of finite graphs, this induction can be effectively carried out: the sequence $(Y_i)_{i>0}$ is strictly decreasing until it reaches a fixed point. For pushdown graphs, the regular languages Y_i are also smaller and smaller, but the question of convergence is nontrivial. Following the symbolic approach, we will construct finite automata that are strictly "decreasing" until they reach a fixed point.

In the following we will proceed in two steps: firstly Algorithm 12 computes an automaton that recognizes Y_i for a given i, secondly Algorithm 14 computes directly an automaton for Y^∞. The first algorithm helps to understand the second, but is not a pre-computation.

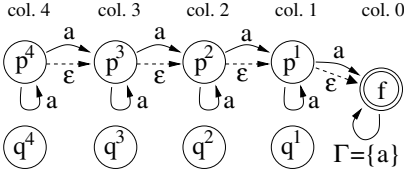
Remark 1. In this section we will consider a simple goal set of the form $R = F\Gamma^*$ for some $F \subseteq P$ (it only depends on the control state). This is not an essential restriction: given a PDS $\mathcal{P} = (P, \Gamma, \Delta)$ that defines the original game and a regular set R of configurations, we can reduce to the case of simple goal sets by proceeding to a new PDS $\mathcal{P} \times \mathcal{D}$, where $\mathcal{D}$ is a finite automaton recognizing R.

So from now on we consider a simple goal set $R = F\Gamma^*$ for some $F \subseteq P$. Of course it is regular. Algorithm 1 is modified (by new steps involving ϵ-transitions) to compute $Attr_0^+(R)$. The intersection with R is easy to compute, and Algorithm 12 determines successively $Y_1, Y_2, Y_3 \ldots$

Algorithm 12 (computation of Y_j)
Input: *PDS $\mathcal{P} = (P, \Gamma, \Delta)$, $j \geqslant 1$ and $F \subseteq P$ that defines the goal set $R = F\Gamma^*$.*
Output: *a $\mathcal{P}$-automaton $\mathcal{B}_j$ that recognizes Y_j.*

Initialization: *the state space of $\mathcal{B}_j$ is $\{f\} \cup (P \times [1, j])$, and f is the unique final state (we write (p, i) as p^i). For all $\gamma \in \Gamma$, $f \xrightarrow{\gamma} f$. For all $p \in P$, p^0 is set to be f.*

for *$i := 1$ to j* ***do***
(compute generation i, consider now the p^i's as initial states.)
Add an ϵ-transition from p^i to p^{i-1} for each $p \in F$ (only). (If $i = 1$, p^0 is f.)
Add new transitions to $\mathcal{B}_j$ according to Algorithm 1:
repeat
(Player 0) if $p \in P_0$, $p\gamma \hookrightarrow qw$ *and* $q^i \xrightarrow{w}_* S$ *in the current automaton,* then *add a new transition* $p^i \xrightarrow{\gamma} S$.
(Player 1) if $p \in P_1$, $\{p\gamma \hookrightarrow q_1 w_1, \cdots, p\gamma \hookrightarrow q_n w_n\}$ *are* all *the moves (rules) starting from $p\gamma$ and $\forall k$, $q_k^i \xrightarrow{w_k}_* S_k$ in the current automaton,* then *add a new transition* $p^i \xrightarrow{\gamma} \bigcup_k S_k$.
until *no new transition can be added*
remove the ϵ-transitions. (generation number i is done)
end for



Example of execution where $j = 4$. We consider $\Delta = \{pa \hookrightarrow p\}$, $R = pI^*$ ($F = \{p\}$), $P = P_0 = \{p, q\}$, $P_1 = \emptyset$. The i-th generation is given by the states of column i. The a-edges from p^i are added in the construction of generation i, since $pa \hookrightarrow p$ and $p^i \xrightarrow{\epsilon}_* p^i$, $p^i \xrightarrow{\epsilon}_* p^{i-1}$.

Fig. 2. Automaton from Algorithm 12

Proposition 13 *Algorithm 12 constructs a $\mathcal{P}$-automaton $\mathcal{B}_j$ that recognizes exactly Y_j (using the states p^j as initial states).*

The sequence (Y_i) might be strictly decreasing, so that the previous algorithm can not reach a fixed point: for the example above one gets $Y_i = pa^i\Gamma^*$, $\forall i > 0$. But on the symbolic level we can modify Algorithm 12 to obtain an automaton for Y^∞ directly. As a preparation one defines a function ϕ which is the translation

one column to the right of the elements of S in the figure above, with the convention that f stays f. For all finite sets $S \subseteq P \times \mathbb{N}$ let

$$\phi(S) = \begin{cases} \{q^i \mid q^{i+1} \in S\} \cup \{f\} & \text{if } f \in S \text{ or } \exists q^1 \in S \\ \{q^i \mid q^{i+1} \in S\} & \text{else} \end{cases}$$

We also use the projection $\pi^i(S)$ of the states in $P \times [1,i]$ to the i-th column (except for f). For all $i > 0$ and sets $S \subseteq P \times [1,i] \cup \{f\}$, let

$$\pi^i(S) = \{q^i \mid \exists i \geqslant k > 0, q^k \in S\} \cup \{f \mid f \in S\} \ .$$

Algorithm 14 (computation of Y^∞)
Input: *PDS $\mathcal{P}$, $F \subseteq P$ that defines the goal set $R = F\Gamma^*$*
Output: *a $\mathcal{P}$-automaton $\mathcal{C}$ that recognizes Y^∞*

Initialization: *the state space of $\mathcal{C}$ is a subset of $(P \times \mathbb{N}) \cup \{f\}$, where (p,i) is denoted by p^i and f is the unique final state. For all $\gamma \in \Gamma, f \xrightarrow{\gamma} f$ in $\mathcal{C}$. By convention p^0 is f for all $p \in F$.*
$i := 0$.

repeat
$i := i + 1$ *(consider now the p^i's as initial states.)*
Add an ϵ-transition from p^i to p^{i-1} for each $p \in F$ (only)
Add new transitions to $\mathcal{C}$ according to Algorithm 1 (see inner loop of Algorithm 12). Remove the ϵ-transitions.
Replace each transition $p^i \xrightarrow{\gamma} S$ by $p^i \xrightarrow{\gamma} \pi^i(S)$
until $i > 1$ *and the outgoing transitions from the p^i's are "the same" as from the p^{i-1}'s:*

$$p^i \xrightarrow{\gamma} S \Leftrightarrow p^{i-1} \xrightarrow{\gamma} \phi(S) \ .$$

Applying the algorithm to the simple example above, we see that the ϵ-transitions and the a-transitions from p^2 to p^1 and from p^3 to p^2 are deleted (by the projection), and that only three generations are created.

Theorem 15 *The automaton constructed by Algorithm 14 recognizes Y^∞, the winning region of the Büchi game given by the PDS $\mathcal{P}$ and goal set R.*

The theorem implies that Y^∞ is regular. Similarly to Section 2 each execution of the inner loop (saturation procedure) is done in time $|\Delta|\ 2^{\mathcal{O}(|Q|^2)}$, where $|Q| = 2|P| + 1$. At each step of the outer loop, we "lose" at least one transition. Since there are at most $|\Gamma|\ |P|\ 2^{|Q|}$ of them, the global time complexity of the algorithm is $\mathcal{O}(|\Gamma|\ |\Delta|\ 2^{c|P|^2})$. With an extension of the arguments of the previous section we obtain corresponding winning strategies:

Theorem 16 *For a Büchi game given by a PDS $\mathcal{P}$ and goal set R, one can compute from the automaton constructed by Algorithm 14 a min-rank (positional) winning strategy and a pushdown winning strategy, uniformly for the winning region of Player 0.*

5 Conclusion

We developed a symbolic approach to solve pushdown games with reachability and Büchi winning conditions. It allows to handle uniformly the whole winning region (of a given player) and to define uniformly two types of winning strategies, a positional one and a pushdown strategy. As an extension to the results presented here, one could consider parity games on pushdown graphs, or more general winning conditions. In this context it would be interesting to connect the present symbolic approach in a tighter way with the work of Walukiewicz [12]. Another generalization is to study games on prefix recognizable graphs.

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