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The Early Development of the Algebraic Theory of Semigroups

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Abstract In the history of mathematics, the algebraic theory of semigroups is a relative new-comer, with the theory proper developing only in the second half of the twentieth century. Before this, however, much groundwork was laid by researchers arriving at the study of semigroups from the directions of both group and ring theory. In this paper, we will trace some major strands in the early development of the algebraic theory of semigroups. We will begin with the aspects of the theory which were directly inspired by, and were analogous to, existing results for both groups and rings, before moving on to consider the first independent theorems on semigroups: theorems with no group or ring analogues.

Contents

Dedicated to the memory of Professor W. Douglas Munn.

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1 Introduction

A *semigroup* is simply a set S which is closed under an associative binary operation; a *monoid* is a semigroup with a multiplicative identity. Obvious examples of semigroups are the natural numbers $\mathbb{N}$ under either addition or multiplication, and the collection of all mappings of a set into itself, under composition of functions (termed a *full transformation monoid*). As can be seen from these examples, the notion of a semigroup is an extremely natural one and it can be argued that semigroups, in the guise of $\mathbb{N}$, have been present in mathematics from its earliest origins. A true *theory* of semigroups, however, is a much more recent development. Broadly speaking, there are two overlapping theories of semigroups: the topological theory and the algebraic theory. The topological theory can be traced certainly as far back as Sophus Lie (Hofmann 2000), and perhaps even further to Niels Abel (Lawson 1996). The algebraic theory, on the other hand, is considerably younger, being firmly rooted in the twentieth century, and with most of the major developments taking place after the Second World War. There is already considerable literature on the history of topological semigroups; see, for example, Hofmann (1976, 1992, 2000) and Lawson (1996, 2002). The history of the algebraic theory, by comparison, has received little attention. The only two purely historical, and reasonably systematic, articles of which I am aware are Knauer (1980) and Gluskin (1968); the former gives a brief survey of the major aspects of the theory and draws connections with other areas, whilst the latter sketches out the development of the theory, but only in the Soviet Union. Other historical articles are Dubreil (1981) and Preston (1991), which focus on their authors’ experiences of the early theory of semigroups. Further sources which touch upon the history of algebraic semigroups are Bruck (1966), Clifford and Preston (1961, 1967) and Howie (2002). The present article is an attempt to give an overview of certain aspects of the early theory which had a profound effect on its subsequent development. From here on, whenever we use the term ‘semigroup theory’ without qualification, it can be taken to mean the algebraic theory.

It would be rather ambitious to try to trace the development of the theory up to the present day. We will therefore only take the story as far as 1941. There are four principal reasons why the early 1940s mark an epoch in the development of semigroup theory:

1. The early 1940s saw the publication of three highly influential papers (Rees 1940; Clifford 1941; Dubreil 1941) which have shaped the subsequent development of the theory.
2. After its initial coining in 1904, I have found the term ‘semigroup’ being used in four different senses (see Appendix A). But from 1940 onwards, the term seems to have become fixed with its modern definition (presumably due to the influence

of Rees (1940) and Clifford (1941), both of whom used ‘semigroup’ in its modern sense).
3. Up to 1941, most of the results obtained for semigroups had been analogues of results for groups or rings. However, as we shall see in Sect. 7, Clifford (1941) gave a result in a paper of 1941 which has been hailed by Howie (2002) as the first theorem of an independent semigroup theory.
4. Finally, there is a practical reason for taking the story as far as the early 1940s. During the 1940s, the number of semigroup papers being published exploded (presumably because of (1) above) and this therefore marks a good place to pause and take stock of the theory thus far.

We note that this historical survey is written with the non-specialist in mind; no semigroup theory will be assumed on the part of the reader. However, the rudiments of the theories of both groups and rings will be assumed. For a very brief introduction to semigroup theory, see Hollings (2007b); for a more comprehensive initiation, see Howie (1995). This latter textbook is also the place to look for any unexplained semigroup-theoretic terms which may be used in this article. Although it will not be an issue until the final section, we note that in this article we will follow a convention of Howie (1995) in writing functions on the right of their arguments and composing them from left to right; homomorphisms will be referred to simply as ‘morphisms’.

The structure of the article is as follows. In Sect. 2, we will give a brief overview of the development of the theory of algebraic semigroups, up to the publication of the first major textbooks in the 1960s. We will also discuss a number of aspects of the theory which it is necessary for the uninitiated reader to have a passing familiarity with. We then move to Sect. 3, in which we describe the group-theoretic circumstances which led to the initial definition of a ‘semigroup’.

In Sect. 4, we will describe some work of Heinrich Brandt of the 1920s, which foreshadowed some of the developments to be described later in the article. Although Brandt’s work was not truly semigroup-theoretic, he came very close; his work can be regarded as an early appearance of category theory.

Section 5 gives a brief biography of arguably the world’s first semigroup theorist: Anton Kazimirovich Suschkewitsch. This will lead us into Sect. 6, which describes the development of the so-called Rees Theorem, one of semigroup theory’s first major structure theorems. Section 6 will begin with the early work of Suschkewitsch on finite simple semigroups (Suschkewitsch 1928) and then move towards the infinite case, as studied by Rees (1940), taking in the work of Clifford (1933b) along the way. This latter paper saw the study of semigroups which are unions of groups. Clifford took up this study again a number of years later (Clifford 1941); this will be the subject of Sect. 7, which will feature semigroup theory’s first ‘independent’ theorem.

The paper concludes with two appendices: one (A) in which we discuss the various slightly different senses in which the term ‘semigroup’ was used before finally settling down with its modern definition, and another (B) where we comment briefly upon other semigroup-theoretic work which emerged in our chosen time-span.

Some comments should be made on the transliteration from Cyrillic letters of the Russian names which are used throughout. Take Anton Kazimirovich Suschkewitsch, for instance. Alternative transliterations which one sometimes sees are: Suškevič, Sushkevich or Suschkewitz. I have adopted the German spelling which appears not only in all of his papers written in German (see Suschkewitsch (1928), for example), but also in his single publication in English: Suschkewitsch (1929). Similarly, Anatoly Ivanovich Malcev has a number of different transliterations to his name (Maltsev, Mal'tsev, Mal'cev); I have simply opted for the shortest.

Biographies are available for many of the mathematicians appearing in this article. In particular, we have the following for Alfred Clifford: Preston (1974), Miller (1974, 1996) and Rhodes (1996). For Heinrich Brandt, see Fritzsche and Hoehnke (1986). Readily available references for Suschkewitsch are Gluskin and Schein (1972) and Hollings (to appear); other sources, slightly less accessible in the West, are Gluskin and Lyapin (1959) and Lyubich and Zhmud' (1989).

In a bid to avoid confusion when discussing earlier terminology throughout the paper, we will, where necessary, adopt the term *magma* (Bourbaki 1974) to mean a set which is closed under a given binary operation; a semigroup in the modern sense is therefore an associative magma.

2 An overview of the development of the theory

As we will see in the next section, the term 'semigroup' was first coined in 1904 to provide a name for certain systems which were not groups but which arose during attempts to extend results on finite groups to the infinite case; the definition of these early 'semigroups', however, differed slightly from the modern notion. With the benefit of hindsight, we can also see that semigroups were cropping up in other work in the early years of the twentieth century, for example, in the representation-theoretic work of Frobenius and Schur (1906), where it was observed that a group's inverses were unnecessary for the task at hand. Despite this, however, the work of Frobenius and Schur, as well as that of others who were unknowingly employing semigroups, cannot be regarded as semigroup *theory*; if ever semigroups did appear in the early twentieth century, it was by the bye.

The first 'proper' semigroup theory began to emerge in the 1920s with the work of the Russian mathematician Anton Kazimirovich Suschkewitsch. In fact, Suschkewitsch was doing (algebraic) semigroup theory before the rest of the world even knew there was such a thing! He was the first to prove a number of results which we now take for granted. For example, he proved that every semigroup may be embedded in a full transformation monoid (Suschkewitsch 1926)—the semigroup analogue of Cayley's Theorem for groups. However, despite the publication of a textbook *The Theory of Generalised Groups* (1937), Suschkewitsch's work remained relatively unknown for many years, particularly in the West, and his results were unwittingly rediscovered by later researchers. The embedding of a (finite) semigroup in a full transformation monoid was reproduced by Stoll (1944), for example. The availability of Suschkewitsch's work cannot have been helped by the fact that most of the copies of his book were kept in Kharkov, a Ukrainian city which changed hands several times during the Second World War, with consequent destruction (Schein 2008a).

During the 1930s, the study of semigroups began to take off, although at this early stage it was still heavily influenced by existing work on both groups and rings; semigroups were approached either by dropping selected group axioms, or by discarding an entire operation, namely addition, from a ring. As the decade progressed, the theory gradually gained momentum, culminating in the publication of three highly influential papers: Rees (1940), Clifford (1941) and Dubreil (1941)—see, for example, the relevant comments in Howie (2002). The papers of Rees and Clifford will be considered in Sects. 6 and 7, respectively. The paper of Dubreil, however, is rather different in character and, although it was no less influential than the other two, and was hailed by Clifford and Preston (1967, p. 174) as "ground-breaking", it would be slightly out of place in the present paper. For an indication of the content and influence of Dubreil (1941), see Lallement (1995).

David Rees' 1940 paper contains semigroup theory's first major structure theorem, now known, appropriately enough, as the *Rees Theorem*. This result completed a strand of research initiated by Suschkewitsch (1928), and is analogous to the Wedderburn–Artin Theorem for rings. The structure theorem given by Clifford in his 1941 paper, however, has no analogue in either group or ring theory, and can therefore be taken to mark the beginning of an independent theory of semigroups.

Thanks to the influence of the above-mentioned papers, the theory of semigroups went from strength to strength in the 1940s, with an increasing number of papers appearing. Of course, the theory did not immediately emerge fully-formed, and so we find the following in the preface to Jacobson's *Lectures in Abstract Algebra*, in reference to semigroups:

> Though this notion appears to be useful in many connections, the theory of semi-groups is comparatively new and it certainly cannot be regarded as having reached a definitive stage. (Jacobson 1951, p. 15)

The 1950s saw the introduction of three broad concepts which continue to be of use and interest in the modern theory of semigroups: *Green's relations*, *regular semigroups* and *inverse semigroups* (see below). It was also around this time that semigroup theory really began to take off in the USSR and, thanks to the relative lack of mathematical communication across the Iron Curtain, certain results in the growing theory of semigroups were duplicated in East and West. Inverse semigroups were a prime example of this, being introduced independently by Wagner (1952, 1953) in the Soviet Union, and by Preston (1954a,b,c) in Great Britain. As the 1950s progressed, however, each side was becoming more aware of the work of the other, and, at the very least, Russian papers were becoming more easily accessible in the West (Munn 2008).

By the end of the decade, the subject had expanded to the extent that the first textbooks since Suschkewitsch's were beginning to appear. First came Lyapin's *Semigroups* (1960), with the English translation appearing in 1963. In 1961, however, the first volume of Clifford and Preston's classic *The Algebraic Theory of Semigroups* was published. This book, together with its eventual second volume (1967), is arguably the most influential semigroup textbook to date; as well as collating many of the existing results on semigroups, it added new ones and, perhaps most importantly, standardised the notation and terminology of the subject. Indeed, this came not a moment too soon; Preston comments on the semigroup theory of the 1950s that

> … virtually no two authors agreed on their definitions and terminology. It was becoming most difficult to keep track of what were often minor, but essential,

> differences from paper to paper. You would try to apply a theorem you had learnt to a new situation, failing to note that a slight difference in definitions made the theorem inapplicable. (Preston 1991, p. 28)

This standardisation was extremely effective: the modern semigroup theorist still tends to use the notation set down by Clifford and Preston; although over 40 years have passed since its publication, the terminology of Clifford and Preston needs no deciphering. The semigroup textbooks of the 1960s provided a firm foundation for the subsequent development of the theory, and, indeed, the algebraic theory of semigroups has thrived ever since. The foundation in 1970 of a journal, *Semigroup Forum*, devoted exclusively to semigroups gave the theory an extra boost along the way—see Hofmann (1995).

Although we are only taking the story of semigroup theory as far as 1941, it will still be useful for us to bear in mind some of the later aspects of the theory—we will see these developments foreshadowed in some of the earlier work. At this point, we therefore take a few paragraphs to précis some major features of semigroup theory which appeared after 1941.

The first of the notions we wish to introduce, *Green's relations*, first appeared in a paper by Green (1951); simply put, these are a collection of equivalence relations which may be defined within a given semigroup (in terms of its principal ideals) to enable us to study its 'large-scale' structure. To define an ideal in a semigroup, we simply take the definition of an ideal in a ring and delete all reference to addition. Thus a subsemigroup I of a semigroup S forms a *right ideal* if, whenever $i \in I$ and $s \in S$, we have $is \in I$. Similarly, I is a *left ideal* if $si \in I$. If I is both a left and a right ideal, then it is called a *two-sided ideal* or, simply, an *ideal*. By analogy with the situation in a ring, we may also define so-called *principal ideals*, for which we need another new concept: that of a *semigroup with identity adjoined*. It is often an advantage to work with a monoid rather than a semigroup, so if the semigroup S in question has no identity, then we simply adjoin an extra symbol 1 to S and define 1 to behave as an identity. The *semigroup with identity adjoined*, denoted S^1, is then defined as follows:

$$S^1 = \begin{cases} S & \text{if } S \text{ is already a monoid;} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

Principal ideals of S are defined in terms of S^1: given an element $a \in S$, the *principal right ideal generated by* a is simply the ideal $aS^1 = \{as : s \in S^1\}$. Similarly, $S^1a = \{sa : a \in S^1\}$ is the *principal left ideal generated by* a and $S^1aS^1 = \{sat : s, t \in S^1\}$ is the *principal (two-sided) ideal generated by* a. Principal ideals are defined in terms of S^1 rather than S in order to ensure that a belongs to the principal left/right/two-sided ideal which it generates. The principal left/right/two-sided ideal generated by an element a is the smallest left/right/two-sided ideal to contain a.

Two elements a, b of a semigroup S are related by Green's relation $\mathscr{R}$ if they generate the same principal right ideal of S:

$$a \mathscr{R} b \Longleftrightarrow aS^1 = bS^1.$$

The relations $\mathscr{L}$ and $\mathscr{J}$ are similarly defined in terms of principal left and two-sided ideals, respectively. The relation $\mathscr{H}$ is the intersection of $\mathscr{R}$ and $\mathscr{L}$, whilst the final relation $\mathscr{D}$ is the *join* of $\mathscr{R}$ and $\mathscr{L}$, i.e., the smallest equivalence relation on S to contain both $\mathscr{R}$ and $\mathscr{L}$ — see Howie (1995, Chap. 2). We will see that Clifford (1941) employed a precursor of the relation $\mathscr{J}$.

Ever since their introduction, Green's relations have proved to be immensely useful in the study of the structure of semigroups. Indeed, their fundamental importance to semigroup theory has led Howie to comment:

> ... on encountering a new semigroup, almost the first question one asks is 'What are the Green relations like?' (Howie 2002, p. 9)

Furthermore, Green's relations have provided a model for subsequent techniques in semigroup theory, as evidenced by the number of generalisations of these relations which have found a use in the literature; see Wallace (1963), Márki and Steinfeld (1974) and Pastijn (1975), to name but a few. See also Cripps (1982).

Also introduced in Green's 1951 paper was the concept of a *regular* semigroup: a semigroup S is said to be *regular* if, for each $a \in S$, there exists $x \in S$ such that $axa = a$. This notion was introduced for semigroups, at the suggestion of David Rees, by analogy with that of *(von Neumann) regularity* in rings; the concept of a regular ring had been introduced by von Neumann (1936) as an algebraic tool for the study of complemented modular lattices (Murray and von Neumann 1936), which at that time were finding an application in the recasting of projective geometry in terms of lattices (Goodearl 1981). The study of the various classes of regular semigroups has proved particularly fruitful over the years (Howie 1995); perhaps the first of these to make an appearance was the class of *completely regular semigroups* in Clifford (1941), as we shall see.

The third and final concept which we must record here for future reference is that of an *inverse semigroup*. Inverse semigroups arose from the study of systems of partial one-one mappings of a set and the desire to find an abstract structure which corresponds to such a system, in much the same way that the study of permutations of a set had yielded the concept of an abstract group. As we have seen, they were introduced independently by Wagner in 1952, and by Preston in 1954. Wagner called these semigroups *generalised groups*; it was Preston who coined the term 'inverse semigroup'. An abstract inverse semigroup is defined to be a semigroup S in which every element $s \in S$ has a unique *generalised inverse*, i.e., an element $s' \in S$ such that $ss's = s$ and $s'ss' = s'$. Note that a traditional group inverse is an inverse in this generalised sense, but not conversely. This is just one of several equivalent ways of defining an inverse semigroup; we may, for example, define an inverse semigroup to be a regular semigroup whose idempotents (elements e with $e^2 = e$) commute (Howie 1995, Theorem 5.1.1). In the terminology to be used in Sect. 7, an inverse semigroup is a regular semigroup in which the idempotents form a semilattice. The theory of inverse semigroups forms a major part of modern semigroup theory. Much more has been written on the history of inverse semigroups than on the history of any other aspect of semigroup theory; see Preston (1991), Schein (1981, 1986, 2002), and also Lawson (1998, Chap. 1). An early appearance of inverse semigroups in the literature will be seen in Clifford (1941) in Sect. 7.

3 Group-theoretic beginnings

We begin by considering the group-theoretic origins of the term ‘semigroup’, and therefore start our story in the first decade of the twentieth century. As related by Kleiner (1986), group theory had developed over the previous 150 years from a variety of sources. In the second half of the nineteenth century, the notion of an *abstract* group had gradually emerged and by the beginning of the twentieth century, a definition of a group had been developed which was essentially that used today, if a little wordier and more tentative; see Neumann (1999).

The first textbook to survey groups purely in the abstract was *Éléments de la Théorie des Groupes Abstraits* by de Séguier (1904) (Kleiner 1986, p. 212). Although the book dealt largely with finite groups, attempts were made to extend certain general theorems to the infinite case—see Dickson (1904). It was in this context that de Séguier discovered that certain systems which form groups when finite do not form groups when infinite. The desire to give these ‘non-groups’ a name led de Séguier to the definition of a new concept: that of a ‘semigroup’.

The notion of a semigroup was introduced to the English-speaking mathematical world later that same year in a review of de Séguier’s book by Dickson (1904). Dickson followed this review a few months later with the first ever paper to feature the word ‘semigroup’ (Dickson 1905b). In this brief paper, Dickson explored some of the motivation for the definition of a semigroup. In that particular volume of *Transactions of the American Mathematical Society*, this semigroup paper appears immediately after another paper by Dickson (1905a) and, indeed, the two papers appear to be complementary; for example, both treat the independence of the postulates used to define the abstract systems under consideration. It seems likely also that the studies found in Dickson (1905a,b) were inspired by the axiomatic investigations of abstract groups by other American mathematicians which had been appearing in the early years of the twentieth century; see, for example, Huntington (1901–1902) and Moore (1902).

From Dickson’s review, we have (the English translation of) de Séguier’s original definition:

Definition 1 A set G, which has generating set $S \subseteq G$ with respect to a given binary operation, forms a *semigroup* if the following postulates hold:

(1) $(ab)c = a(bc)$, for all $a, b, c \in G$;
(2) for any $a \in S$ and any $b \in G$, there is at most one solution, $x \in G$, of $ax = b$;
(3) similarly for $xa = b$.

It is immediately clear that this is not the modern definition of a semigroup, for which only postulate (1) is required; de Séguier suggested the term *corps* for a system satisfying postulate (1) only—see Dubreil (1981, p. 59).

De Séguier went on to state (without proof) that it follows that, for any $a \in G$, $ax = ax'$ implies $x = x'$ (left cancellation). In his review, Dickson suggested that de Séguier’s argument must have run thus: we express a in terms of generators $a = a_1 \cdots a_n$ then repeatedly apply postulate (2) to the equality $ax = ax'$, eventually obtaining $x = x'$. However, there is one small technical point here over which Dickson showed some concern: with the definition of a ‘semigroup’ taken as above, there is no guarantee that $a_2 \cdots a_n x$, $a_3 \cdots a_n x$, etc. belong to G; closure of the binary operation is not demanded explicitly. Of course, in defining a binary operation on a set S to be a mapping $S \times S \to S$, we do in fact ensure that closure is built into the definition, and so we can hardly doubt that de Séguier simply regarded closure as being implicit. However, as Dickson commented: "Most readers, I think, would find it more natural to have [closure] as a postulate..." (Dickson 1904, p. 160). He therefore modified the definition of a 'semi-group' accordingly (inserting a hyphen in the process). He also removed all reference to the generating set S. Indeed, in his subsequent paper, Dickson (1905b) tweaked his definition of 'semi-group' once more, to include the above left cancellation condition and its dual as postulates:

Definition 2 A set G forms a *semi-group* under a given binary operation if the following postulates hold:

(0′) if $a, b \in G$, then $ab \in G$;
(1′) $(ab)c = a(bc)$, for all $a, b, c \in G$;
(2′) for any $a, x, x' \in G$, if $ax = ax'$, then $x = x'$;
(3′) for any $a, x, x' \in G$, if $xa = x'a$, then $x = x'$.

It is now clear that what Dickson defined as his 'semi-group' is what we would today call a *cancellative semigroup*. It is known that every finite cancellative semigroup is a group (Howie 1995, p. 61) and, in fact, Definition 2 is precisely that which had been given previously by Heinrich Weber (1882) as the definition of a (finite) group. Note that a similar definition had been given by Kronecker some years earlier in the commutative case—see Neumann (1999). De Séguier's 'semigroup' was therefore only of special interest when it was infinite and did not form a group.

After giving the above definition, Dickson went on to demonstrate the independence of the postulates (0′)–(3′), taking the finite and countably and uncountably infinite cases separately, by constructing examples of systems in which three postulates hold whilst one fails. A similar demonstration for both groups and fields can be found in Dickson (1905a).

Dickson's main concern in his 'semi-group' paper was the question: when is a 'semi-group' a group? As we have seen, a cancellative semigroup is certainly a group in the finite case, but not, in general, in the infinite case. The most obvious example of a cancellative semigroup which is not a group is $\mathbb{N}$ under addition. However, Dickson did not give this as an example. He instead went through a rather involved method in order to construct a 'semi-group' which is not a group. We provide a taste of that method. Let G and H be groups and suppose that G and H are 'homomorphic' in the sense that there exists a binary relation $R \subseteq G \times H$ such that

(i) for each $g \in G$, there is at least one $h \in H$ with $g\,R\,h$;
(ii) for each $h \in H$, there is at least one $g \in G$ with $g\,R\,h$;
(iii) if $g\,R\,h$ and $g'\,R\,h'$, for $g, g' \in G$ and $h, h' \in H$, then $gg'\,R\,hh'$.

Now define the subsets $G' \subseteq G$ and $H' \subseteq H$ by

$$G' = \{g \in G : g\,R\,1\} \quad \text{and} \quad H' = \{h \in H : 1\,R\,h\}.$$

In the case where both G and H are finite, it is an easy exercise to show that G' and H' are both subgroups. De Séguier had attempted, without success, to show that this is still true in the infinite case. In fact, it is not, as Dickson demonstrated with suitably constructed counterexamples. In the infinite case, G' and H' form 'semi-groups' in the sense of Definition 2; it was to describe such objects as G' and H' that de Séguier introduced his notion of a 'semi-group' in the first place.

One thing which is clear about Dickson's work on semigroups is that it certainly cannot be called semigroup *theory*—it was simply an exploration of the group axioms and of the relatively new idea of an infinite abstract group, and therefore comes firmly under the heading of 'group theory'. For a more detailed overview of this work of Dickson, see Schmidt (1966, Chap. 4).

The word 'semigroup' did not, of course, become fixed with Dickson's definition. Indeed, as can be seen from Appendix A, it was used in a number of different senses before it finally settled down with the modern definition. This, however, is not the only potential source of confusion for modern readers where earlier terminology is concerned. As observed in Sect. 2, there were other researchers in the early decades of the twentieth century who unwittingly studied semigroups and referred to the systems under consideration simply as 'groups'. To give an example, we have the work of Frobenius and Schur (1906), in which it was proved that the irreducible components of the representation of a 'group' on a finite dimensional vector space are completely determined by the character of the representation. However, in their introduction, Frobenius and Schur make the explicit comment that the presence of neither identity nor inverses in the 'group' need be demanded; nevertheless, they use the term 'group' throughout.

This (to modern eyes) confusing usage of the word 'group' is due largely to the reasons we have seen in this section: the notion of an axiomatically-defined group was still quite new, and there was also a degree of uncertainty in the move to the infinite case. The groups which many of these early researchers were used to dealing with were groups of permutations of a finite set, for which only closure need be postulated to ensure that it is a group, since a subsemigroup of a finite group (in this case, the symmetric group $\mathscr{S}_n$, where n is the cardinality of the set in question) is necessarily a group. Klein (1979, p. 315) commented that when infinite Lie groups were first considered, the intention had been to define them simply as systems which are closed under composition; it was found later that the presence of inverses must be demanded explicitly.

> Thus at this early stage the distinction between a semigroup and a group was quite blurred, and what we today would call a semigroup often went under the name of a group, well into the 20th century. (Lawson 1992, p. 265)

We will see in Sect. 6 that this 'blurring' persisted even as far as the birth of the 'true' theory of algebraic semigroups in the work of Anton Suschkewitsch.

4 Brandt groupoids

In this section, we survey some work of Heinrich Brandt. Although no semigroups appeared in this work, we include Brandt's researches here as they will tie into the material of Sect. 6. Brandt's 1927 paper has a greater significance in the wider setting of mathematics, however, as it may be regarded as an early paper on category theory.

Our particular object of interest in Brandt's work is the notion of a so-called *Brandt groupoid*. This is an algebraic structure which arose from Brandt's work on quaternary quadratic forms, in which he set out to generalise some results of Gauss (1801) in the binary case. An overview of Brandt's work in general, and of groupoids in particular, may be found in Fritzsche and Hoehnke (1986).

A *(rational) binary quadratic form* is simply a second degree homogeneous polynomial over $\mathbb{Q}$ in two variables: $f = ax^2 + bxy + cy^2$; this may of course also be expressed in matrix form:

$$f = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We say that the form f is *integral* if $a, b, c \in \mathbb{Z}$, and *primitive* if $\gcd(a, b, c) = 1$. The *discriminant* of f is $b^2 - 4ac$.

Gauss studied the composition of binary quadratic forms, for which he gave the following definition: an integral form

$$f_3 = a_3x_3^2 + b_3x_3y_3 + c_3y_3^2$$

is the *composition* of two other integral forms

$$f_1 = a_1x_1^2 + b_1x_1y_1 + c_1y_1^2 \quad \text{and} \quad f_2 = a_2x_2^2 + b_2x_2y_2 + c_2y_2^2$$

if there exists a bilinear transformation

$$\begin{aligned} x_3 &= px_1x_2 + qx_1y_2 + rx_2y_1 + sy_1y_2, \\ y_3 &= p'x_1x_2 + q'x_1y_2 + r'x_2y_1 + s'y_1y_2 \end{aligned}$$

such that $f_3 = f_1 f_2$ and the six determinants

$$\begin{aligned} P &= pq' - qp', \quad & R &= ps' - sp', \quad & T &= qs' - sq', \\ Q &= pr' - rp', \quad & S &= qr' - rq', \quad & U &= rs' - sr' \end{aligned}$$

are all coprime (Butts and Pall 1968, p. 24). (We note that in the process of deriving this composition, Gauss generalised the well-known identity for products of sums of two squares (see Hollings 2006) to products of expressions of the form $ax^2+bxy+cy^2$.) In a similar way, we say that two binary quadratic forms f and g are *(properly) equivalent* if there is a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$f\left(A\begin{pmatrix} x \\ y \end{pmatrix}\right) = g(x, y).$$

Thus we may sort forms into equivalence classes and consider the composition of classes in a natural way. In effect, Gauss demonstrated that, for a fixed discriminant, the classes of primitive integral binary quadratic forms have the structure of a finite Abelian group. Gauss also studied the composition of forms of different discriminants, although the subsequent literature has focussed largely on the case of a fixed discriminant, seemingly due to the influence of Dirichlet—see Butts and Pall (1968). For an excellent elementary account of composition of binary quadratic forms, see Schoenebeck (2004). See also Kneser (1982).

As noted above, Heinrich Brandt's principal research interest was in *quaternary* quadratic forms, i.e., second degree homogeneous polynomials in four variables, and the possible extension of Gauss' results on binary forms to the quaternary case. Indeed, the first significant results on quaternary forms are due to Brandt (Kneser et al. 1986). In particular, beginning with his doctoral dissertation (Strasbourg 1912; main results published in Brandt 1913), Brandt investigated the composition of quaternary quadratic forms. However, adopting a composition for quaternary forms analogous to that employed by Gauss in the binary case, it immediately became clear to Brandt that not every quaternary form admits a composition. Gauss had shown that any two integral binary forms with the same discriminant may be composed to obtain a new integral binary form. In the quaternary case, however, an integral form is composable only if its discriminant is a square D^2 and its adjoint form is divisible by D. Brandt called such composable forms *K-forms* (Brandt 1924). *Primitive* K-forms (i.e., K-forms with coprime coefficients) are precisely those integral quaternary quadratic forms which admit a composition. However, of particular significance for our present purposes is the fact that, given two such K-forms, there is no guarantee that we can compose them *with eachother*. Thus, unlike in the binary case, the equivalence classes (defined analogously in the quaternary case) of primitive integral quaternary quadratic forms, with fixed discriminant, do not form a group. Indeed, the composition in the quaternary case is only partially defined.

Pursuing the analogy with Gauss' work, Brandt set out to describe the structure of this partial composition of quaternary forms. His crucial observation in this regard was the fact that every quaternary quadratic form possesses both a left and a right identity; any two such forms are composable if, and only if, the right identity of one coincides with the left identity of the other (Brown 1999). To the eyes of a modern algebraist, a category structure is beginning to emerge. However, the introduction of the concept of a category was still some years ahead (see below) and so it was left to Brandt to determine the abstract structure formed by the composition of quaternary forms; this would necessarily be some type of partial magma, i.e., a magma in which not all products are defined. It was quite natural for Brandt to seek the abstract structure of composable quaternary forms, as the abstract point of view is one which had gained a lot of ground in the early decades of the twentieth century (Birkhoff 1973). Hot on the heels of the concept of an abstract group had come other abstract notions—that of a ring, for example (Kleiner 1996). Furthermore, Birkhoff's notion of a '(universal) algebra' was just around the corner (Birkhoff 1935) and, beginning in the 1930s, attempts to find an abstract version of the 'pseudogroup' of Veblen and Whitehead (1932) would eventually lead to the notions of both inverse semigroups and so-called *inductive groupoids*—see Schein (1986, 2002).

Brandt's first steps towards such an axiomatisation were taken in Brandt (1925) and also in Brandt (1926a), where he observed that wherever the composition of quaternary forms is defined, it is associative. These increasingly abstract considerations culminated in Brandt's introduction of the notion of a *Gruppoid* (Brandt 1926b), now termed a *Brandt groupoid* (Clifford 1942):

Definition 3 A partial magma B forms a *Brandt groupoid* if the following postulates hold:

(1) for $a, b, c \in B$, if $ab = c$, then each of the elements a, b, c is uniquely determined by the other two;
(2) if we write $\exists xy$ to mean "the product xy is defined," then for $a, b, c \in B$:
   (i) if $\exists ab$ and $\exists bc$, then $\exists(ab)c$, $\exists a(bc)$ and $(ab)c = a(bc)$;
   (ii) if $\exists ab$ and $\exists(ab)c$, then $\exists bc$, $\exists a(bc)$ and $a(bc) = (ab)c$;
   (iii) if $\exists bc$ and $\exists a(bc)$, then $\exists ab$, $\exists(ab)c$ and $(ab)c = a(bc)$;
(3) for each $a \in B$, there exist the following uniquely defined elements:
   (i) a left identity e;
   (ii) a right identity f;
   (iii) a left inverse $\overline{a}$, with respect to f;
(4) if $e, f \in B$ are idempotents, then there exists an element $a \in B$ for which e is a left identity and f is a right identity.

We note that Brandt confined his attention to the finite case. In condition (3), we clearly see the abstraction of Brandt's observation concerning left and right identities for quaternary forms. Some years later, Brandt obtained other axiomatisations of his groupoids (Brandt 1940); on this subject, see also Stolt (1958). It is observed in Fritzsche and Hoehnke (1986, Sect. 4) that a partial composition had already appeared for bilinear forms in Study (1923), under which the forms in question have the structure of a groupoid in the sense of Definition 3; however, Study's composition appeared in a geometrical form, and no attempt had been made to obtain an abstract characterisation.

At this point, we sound a note of caution: just like 'semigroup', the term 'groupoid' has been used in a number of different senses over the years. Birkhoff (1934) used it mean an associative magma with identity (see Appendix A). A few years later, Hausmann and Ore (1937) were using it simply to mean a magma; indeed, it is used in this sense in the introductory chapters of both Clifford and Preston (1961) and Howie (1995). However, the sense in which 'groupoid' is perhaps most often used these days is in connection with category theory, where a 'groupoid' is a small category (i.e., a category based upon a set, rather than a class) in which all arrows are invertible. The systematic study of such objects seems to have been initiated by Ehresmann (1965). We note that whilst a Brandt groupoid is clearly not a groupoid in the sense of Hausmann and Ore, it *is* however a groupoid in the category-theoretic sense. More specifically, a Brandt groupoid G is a groupoid in which the following additional condition holds: for all objects x, y of G, there exists $a \in G$ such that $\mathbf{d}(a) = x$ and $\mathbf{r}(a) = y$, where $\mathbf{d}(\cdot)$ and $\mathbf{r}(\cdot)$ denote the domain and range functions, respectively. In modern terminology, the term *Brandt groupoid* has largely been superceded by *connected groupoid* or *transitive groupoid*—see, for example, Lawson (1998, p. 105). It is also worth noting that Brandt himself was rather critical of the other uses of the term 'groupoid' and the potential for confusion that this created—see, for example, his *Zentralblatt* reviews of Richardson (1940) and Borůvka (1941).

The similarities between the above definition of a Brandt groupoid and the definition later given for a category by Eilenberg and Mac Lane (1945, Chap. I) are quite striking; indeed, it has been alleged that Eilenberg and Mac Lane were influenced by Brandt's axioms (Brown 1987, p. 114).

Although it is not particularly easy to see from Definition 3, we observe that if a zero element 0 is adjoined to a Brandt groupoid B and each hitherto undefined product is defined to be 0, then $B \cup \{0\} := B^0$ forms a semigroup, called a *Brandt semigroup*, which we will deal with below. Brandt groupoids, and this adjunction of a zero element in particular, were explored by Clifford and Preston (1961, Sect. 3.3). Brandt, on the other hand, did not go this far. In a single sentence, he commented that a zero *could* be adjoined but that, in general, there is little advantage in doing so! The first person to study the effects of this adjunction of a zero element seems to have been Clifford (1942) in a paper on matrix representations of semigroups, where the obtained results are applied to Brandt semigroups in the final section. Although Clifford gave no indication as to why he chose to work with Brandt semigroups at this point, what is clear is that these semigroups provided him with an excellent example to work with and allowed him to demonstrate his results in action. Moreover, despite the fact that Clifford made no comment to this effect, Brandt semigroups are a special case of the semigroups which he had studied in an earlier paper (Clifford (1941): see Sect. 7), and are also a special case of the semigroups which had only recently been studied by Rees (1940) (see below). We can only speculate that Clifford's attention had been drawn to Brandt's work and his comments on the possible adjunction of a zero to a Brandt groupoid, and that he recognised the derived semigroups as being prime examples of the various interrelated classes of semigroups whose study was 'in the air' at that time.

The axiomatic definition of a Brandt groupoid given in Definition 3 is perhaps somewhat unedifying and so it is probably for this reason that Brandt went on to give a complete and transparent description of the structure of a Gruppoid B. It is a particularly simple structure and may be represented as follows. Let G be a group and I be a nonempty set. The elements of B can be represented by triples (i, g, j), where $i, j \in I$ and $g \in G$. The product $(i, g, j)(k, h, l)$ of two triples is defined to be (i, gh, l) if $j = k$ and is undefined otherwise. Notice that this is in fact a representation of G by matrices: the triples (i, g, j) may be replaced by matrices (g_{ij}). The representation is still applicable in the case of a Brandt semigroup B^0, provided we effect the natural modification of putting $(i, g, j)(k, h, l) = 0$ whenever $j \neq k$.

This latter description of B^0 is, in fact, a special case of that of a Rees matrix semigroup, which we will see in Sect. 6. As we will also see, Rees matrix semigroups provide a concrete description of a class of semigroups termed *completely 0-simple semigroups* (Rees 1940); a Brandt semigroup B^0 is therefore completely 0-simple. Moreover, B^0 is also an inverse semigroup, thus placing Brandt semigroups firmly within the purview of (and making them an excellent source of examples for) two of the major strands in the early study of semigroups. Indeed, combining results of Clifford (1942) and Munn (1957), Clifford and Preston (1961, Theorem 3.9) demonstrated that a semigroup is completely 0-simple and inverse if, and only if, it is a Brandt semigroup. The following is a concrete example, in matrix form, of the Brandt semigroup which arises when G is the trivial group and $|I| = 2$:

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

(Howie 1995, p. 32).

It seems that other mathematicians were slow to adopt Brandt's notion of a Gruppoid; Higgins has suggested that this was "probably because of a general distaste for partial operations" (Higgins 1971, p. 171). Indeed, it is possible to see this same distaste in the development of the notion of an inverse semigroup—see Schein (1986, 2002). Nevertheless, it is worth mentioning that Brandt's groupoid was not the only abstract structure with a partially-defined multiplication to emerge in the 1920s. Another was the *mixed group* (*Mischgruppe*) of Loewy (1927), introduced to describe isomorphisms between conjugate field extensions (Brown 1987). To see an example of a mixed group, we take a group G with a (not necessarily normal) subgroup H. Then the collection of all right cosets of H in G is a mixed group under the operation given by $(Hx)(Hy) = Hxy$ whenever x lies in the normaliser of H in G, and undefined otherwise (Dixon 1963). On the surface, mixed groups appear to be rather different from Brandt groupoids (Bruck 1966, Sect. II.5). However, they turn out to be very closely related—see Fritzsche and Hoehnke (1986, Sect. 4).

We see then that although Brandt came very close to studying specific instances of both (finite) inverse and completely 0-simple semigroups, he did not, in the end, deal with semigroups directly. As Preston has commented, from the semigroup theorist's perspective, Brandt "seems to have missed a great chance" (Preston 1991, p. 20). Whilst Brandt went on to find far-reaching applications for the groupoid concept in the study of the arithmetic of ideals in rings of algebraic integers (Brandt 1928a) and rational Dedekind algebras (Brandt 1928b), Brandt *semigroups* have turned out to be objects of considerable interest in semigroup theory: they appear quite naturally, for example, in connection with matrices over rings (Lawson 1998, p. 86) and have also found an application in formal language theory to so-called 'Brandt codes' (Fritzsche and Hoehnke 1986, Sect. 8.3). At first glance, this makes Brandt's comment on there being little advantage in adjoining a zero to a groupoid seem somewhat short-sighted. But of course we should not judge Brandt so harshly—there can be little wonder that he saw no advantage in studying semigroups, as there was hardly any semigroup theory to speak of at that time! Brandt regarded his introduction of groupoids as a "natural and even necessary supplement to ordinary group theory" (Brandt 1926b, p. 360). Indeed, some modern authors contend that it is in fact the notion of a groupoid which is the more natural, rather than that of a group (see, for example, Brown (1999) or Corfield (2001))—the debate will doubtless continue!

5 Anton Kazimirovich Suschkewitsch

In Sect. 2, we cited the Russian mathematician Anton Kazimirovich Suschkewitsch as having been the first 'proper' semigroup theorist. Certainly, he was the first to attempt a systematic study of various classes of generalised groups, semigroups included. As we have already observed, Suschkewitsch's work was not widely known, particularly in the West, and so its influence on subsequent semigroup theory was not as great as it might have been. Nevertheless, we will devote this section to a brief biography of Suschkewitsch and a short discussion of his semigroup-theoretic work, before looking at his early contribution to the development of the Rees Theorem in Sect. 6.

Suschkewitsch was born in Borisoglebsk, in southern Russia, in 1889. Pursuing an apparent early aptitude, he went to Berlin to study mathematics, where he attended the lectures of Frobenius, Schur and Schwarz; Frobenius' approach to group theory was later to influence Suschkewitsch's own algebraic investigations. It was probably also in Berlin that Suschkewitsch met Emmy Noether, with whom he corresponded for many years afterwards.

After leaving Berlin, Suschkewitsch continued his studies at St. Petersburg University, where he graduated in 1913. The First World War prevented him from returning to Germany, so he instead moved to Kharkov (or Kharkiv) in the Ukraine, where he taught at various high schools, and ultimately submitted his Master's dissertation to Kharkov University in 1917. In 1918, he became an assistant professor at Kharkov University.

It was against the backdrop of the social upheaval following the October Revolution of 1917 that Suschkewitsch wrote his doctoral dissertation, *The Theory of Operations as the General Theory of Groups* (Suschkewitsch 1922). In his introduction, he noted that papers published elsewhere in the world after 1914 were not available to him, and that, owing to the difficult times, his research had often been interrupted, for reasons beyond his control (Gluskin and Schein 1972, pp. 367–368).

From 1921 to 1929, Suschkewitsch was a professor at Voronezh University, and it was here that he successfully defended his doctoral dissertation in 1926. In 1929, Suschkewitsch returned to Kharkov as a member of the newly-established Ukrainian Mathematics and Mechanics Research Institute, and became the first head of Kharkov University's Department of Algebra and Number Theory in 1933, a position he held until his death in 1961.

Suschkewitsch's systematic study of generalised groups began with his doctoral dissertation, the results of which were later published in a series of papers, not to mention his monograph (Suschkewitsch 1937). Preston, quoting Paul Cohn, has observed that Suschkewitsch had the "exhilarating advantage" of being the first to tackle this topic: "the major basic results were just waiting there to be discovered" (Preston 1991, p. 20). Abstract algebra in general was not widely studied in Russia/the Soviet Union in the first decades of the twentieth century, but Suschkewitsch was one of those teachers and researchers who helped to give algebra greater standing (Malcev 1971). Not only did he obtain some of the first results in the theory of semigroups, but also in the theory of loops and quasigroups—see Pflugfelder (2000). Furthermore, Suschkewitsch's interests extended to the history of mathematics (see, for example, Suschkewitsch (1951)) and it seems that he even dabbled in logic, as evidenced by the presence of a paper (Suschkewitsch 1935) in two logic bibliographies: Church (1938) and Küng (1962). He was the author of a well-regarded textbook on number theory (Suschkewitsch 1956).

Returning to his work in semigroup theory, as we will see in the next section, Suschkewitsch studied minimal two-sided ideals ('kernels') of semigroups and showed that every finite semigroup possesses such a kernel. This led naturally to the study of finite simple semigroups (i.e., finite semigroups with no proper, non-zero ideals) and to a finite version of the Rees Theorem (Suschkewitsch 1928). Suschkewitsch's kernels were subsequently studied by Schwarz (1943, 1951) and Clifford (1948).

The initial impetus for Suschkewitsch's consideration of semigroups (and also of other types of generalised groups) was provided by the simple observation that the study of groups is simply the study of a special type of transformation: permutations. In his dissertation, he remarked that the next step in the theory must surely be the investigation of more general types of transformation. He then set about the study of various different kinds of 'generalised groups' (i.e., magmas) and began to piece together the requisite building blocks for a general theory of magmas: notions of submagmas, isomorphism of magmas, etc.

In a specific instance of the mathematician's standard trick of reducing a given question to a known one, Suschkewitsch considered solutions to semigroup problems 'modulo group theory': a semigroup-theoretic problem would be considered solved if it could be reduced to a problem in group theory, or if the semigroup in question could somehow be described in terms of groups. We will see this approach at work in the next section, where the kernel of a finite semigroup will be written as the disjoint union of mutually isomorphic groups. The 'reduction to group theory' approach has been applied many times in subsequent semigroup theory; from a semigroup theorist's point of view, a group is regarded as an object of known structure. Indeed, the Rees Theorem follows this pattern: a completely 0-simple semigroup is described in terms of a group, two nonempty sets and a matrix.

Knauer (1980) has observed that Suschkewitsch's semigroup investigations departed from what little semigroup theory had come before in that they considered questions in semigroup theory without the motivation of applications to other areas, such as group theory. Nevertheless, Suschkewitsch was interested in semigroups which are somehow 'close' to groups. He even tried to quantify this notion of 'closeness'. We know from Sect. 3 that a finite group G may be defined as a system in which the equations $ax = b$ and $ya = b$ each have a unique solution for any $a, b \in G$. In an arbitrary semigroup, of course, the solutions to these equations need not be unique, if they exist at all. Suschkewitsch used the number of solutions to each of these equations as a measure of the 'closeness' of a given semigroup to a group (Gluskin and Lyapin 1959).

As we observed in Sect. 2, Suschkewitsch was the first to prove that any semigroup may be embedded in a full transformation monoid (Suschkewitsch 1926) and, indeed, semigroups of transformations continued to interest Suschkewitsch for the rest of his career (see, for example, Suschkewitsch (1940)). According to Clifford and Preston (1961, p. 52), Suschkewitsch even gave a characterisation of Green's relations in a full transformation monoid (Suschkewitsch, 1937, Chap. 3, Sect. 31). His other topics of investigation into semigroups included the study of matrix semigroups and that of the representation of semigroups by matrices. Two papers in particular (Suschkewitsch 1933, 1935) dealt with matrix representations. The former aimed to determine all representations of finite simple semigroups by means of matrices, whilst the latter provided an abstract version of the material of the earlier paper, as well as lifting the requirement of finiteness. This work was later extended by Clifford (1942) and others; see Preston (1974, p. 38).

Given the relative unavailability of Suschkewitsch's papers in the West, it is difficult to assess fully his contribution to semigroup theory. Nevertheless, it is clear that it was Suschkewitsch who initiated a number of lines of inquiry into semigroups, whether later researchers were aware of his contributions or not.

6 The Rees theorem

In this section, the main section of this article, we will look at the development of semigroup theory's first major structure theorem: the Rees Theorem. This result, in its final form, appeared in a 1940 paper by David Rees and gives the structure of a class of semigroups termed *completely* (0-)*simple semigroups*. These semigroups arose in Rees' work through consideration of semigroup analogues of certain properties for rings; they also emerged quite naturally around the same time in the (independent) work of Clifford (1941), which we will examine in the next section. Two earlier papers had given the structure of certain special classes of simple semigroups. As we observed in the previous section, Suschkewitsch had determined the structure of *finite* simple semigroups (Suschkewitsch 1928). In particular, he studied, and named, (finite) *left groups* and *right groups*. Alfred Clifford then extended this work to infinite right groups (Clifford 1933b). We will come to each of these papers in turn but first, for reference, let us define the classes of semigroups which are to be considered.

Definition 4 (Howie, 1995, Sect. 3.1) A semigroup S is called *simple* if its only (two-sided) ideal is itself. A semigroup S with 0 is called *0-simple* if its only ideals are itself and $\{0\}$, and $S^2 \neq \{0\}$.

A semigroup S without 0 (respectively, with 0) is *completely simple (completely 0-simple)* if the following conditions hold:

(CS1) S is simple (0-simple);

(CS2) S has a *primitive* idempotent, i.e., a non-zero idempotent e such that, for all non-zero idempotents $f \in S$, $ef = fe = f \Rightarrow e = f$.

Another term which will be used below is *left simple*: by analogy with the above definition, a semigroup S is termed *left simple* if its only left ideal is itself. *Right simplicity* is defined dually. We note that a standard partial ordering is often imposed upon the idempotents of a semigroup: for idempotents i, j, we have $i \leq j$ if, and only if, $i = ij = ji$. Condition (CS2) therefore says that an idempotent e is primitive if there are no idempotents below e in this ordering.

By our comments in Sect. 4, the example of a Brandt semigroup which was given in that section serves as an example of a completely 0-simple semigroup. An example of a 0-simple semigroup which is not *completely* 0-simple is the *bicyclic monoid*: the monoid $B = \mathbb{N}^0 \times \mathbb{N}^0$ with multiplication

$$(a, b)(c, d) = (a - b + \max\{b, c\}, d - c + \max\{b, c\}).$$

Our claim that B is not completely 0-simple will be justified later in the section.

Condition (CS2) is sometimes replaced by the equivalent *descending chain conditions* (Clifford and Preston 1961, Theorem 2.48):

(CS2′) Any descending chain of left or right principal ideals must be finite, i.e., any chain of either of the forms

$$a_1S \supset a_2S \supset a_3S \supset \cdots ;$$
$$Sa_1 \supset Sa_2 \supset Sa_3 \supset \cdots ,$$

where $a_1, a_2, a_2, \ldots \in S$, must terminate.

To put this another way, a semigroup S satisfies (CS2′) if it possesses minimal left and right ideals; by our comments in Sect. 2, a minimal left/right ideal is necessarily principal. A finite semigroup must clearly satisfy condition (CS2′), and so a finite simple semigroup is necessarily completely simple. Of the three main papers (Suschkewitsch 1928; Clifford 1933b; Rees 1940) to be examined in this section, it is only that of Suschkewitsch which makes any explicit mention of minimal one-sided ideals, and then only in the finite case. Minimal one-sided ideals of semigroups were subsequently studied in a more general setting by Schwarz (1943). In this paper, which was completed largely without access to outside sources (Grošek, et. al. 1994), Schwarz also studied both maximal subgroups and radicals of semigroups—see Jakubík and Kolibiar (1974). The presence of (non-zero) minimal one-sided ideals in infinite completely (0-)simple semigroups was established by Clifford (1948, 1949).

Before we can state the Rees Theorem, we first need the following:

Definition 5 (Howie, 1995, pp. 70–71) Let G be a group, let I, Λ be non-empty sets and let P be a matrix over the 0-group $G^0 := G \cup \{0\}$ which is 'regular' in the sense that each row and each column contains at least one non-zero entry. The $I \times \Lambda$ *Rees matrix semigroup over* G^0 *with sandwich matrix* P, which we denote by $\mathscr{M}^0(G; I, \Lambda; P)$, is the set $I \times G \times \Lambda \cup \{0\}$, together with the binary operation given by:

$$0(i, g, \lambda) = 0 = (i, g, \lambda)0 = 00, \quad \text{for all} \quad (i, g, \lambda) \in I \times G \times \Lambda \quad \text{and}$$

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, gp_{\lambda j}h, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0 \end{cases}$$

We define a Rees matrix semigroup *without* 0 simply by deleting all reference to 0 in the above definition, dropping the insistence on P being 'regular' and defining multiplication simply by $(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda j}h, \mu)$ (Howie 1995, p. 77).

We can finally state the Rees Theorem. This result was originally due to Rees (1940); we give the formulation of Howie (1995, Theorem 3.2.3):

Theorem 1 *Let S be a Rees matrix semigroup over a 0-group with regular sandwich matrix. Then S is completely 0-simple. Conversely, every completely 0-simple semigroup is isomorphic to such a Rees matrix semigroup.*

For clarity, we state the completely simple version of this theorem separately; again, this is a result of Rees (1940), but we give the formulation of Howie (1995, Theorem 3.3.1):

Theorem 2 *Let S be a Rees matrix semigroup without* 0. *Then S is completely simple. Conversely, every completely simple semigroup is isomorphic to such a Rees matrix semigroup.*

As an example of the complementary nature of the topological and algebraic theories of semigroups, we note that this latter theorem was subsequently adapted to the compact case by Wallace (1956). The significance of the Rees Theorem for semigroup theory is reflected in the fact that Clifford and Preston gave it such a prominent place in the first volume of their *The Algebraic Theory of Semigroups* (Clifford and Preston 1961, Sect. 3.2).

We are now ready to see how these theorems developed. We begin with the 1928 paper of Suschkewitsch, mentioned in the previous section; this paper may be regarded as the first 'proper' paper on semigroups, since it was the first paper to consider semigroups in their own right, without a view to applying the obtained results to any other area (cf. the work of Frobenius and Schur (1906) mentioned at the end of Sect. 3). Clifford and Preston (1961) devoted an appendix to a summary of this paper.

Suschkewitsch's stated aim for the paper was the construction of a theory for certain finite abstract semigroups which was analogous to the theory given by Wedderburn (1907) for linear algebras; it is interesting to note that these questions seem to have been suggested to Suschkewitsch by Emmy Noether—see Suschkewitsch (1928, footnote 1). His objects of study were finite 'groups' in which the elements do not necessarily have unique inverses: the 'endliche Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit' of his title. Suschkewitsch's own rendering into English of 'das Gesetz der eindeutigen Umkehrbarkeit' was 'the rule of uniform reversibility' (Suschkewitsch 1929); in modern terminology, this rule is simply cancellativity.

In line with our comments at the end of Sect. 3, Suschkewitsch referred to the systems under consideration simply as *groups*. In the cases when the system in question did form a group in the traditional sense, Suschkewitsch called it an *ordinary group*; some time later, he suggested the name *classical group* (Suschkewitsch 1937, p. 5). In modern terminology, Suschkewitsch completely determined the structure of finite simple semigroups and showed that every finite semigroup contains such a semigroup as a minimal ideal. Perhaps because he was approaching the problem from a group-theoretic angle, Suschkewitsch did not consider the 0-simple case. While Suschkewitsch's terminology was rooted largely in group theory (he drew upon the notation of his erstwhile lecturer Frobenius (1895)), he did, very briefly, make a connection with ring theory in his introduction (in reference to his work being analogous to, and motivated by, that of Wedderburn), where he referred to the generalisation of a ring in which the operation of addition is discarded.

The system with which Suschkewitsch began was simply, in modern terminology, a finite semigroup $\mathfrak{G}$. Bearing in mind Weber's definition of a finite group (see the comments following Definition 2), it was natural for Suschkewitsch to drop the cancellation conditions and study the resulting object. Coining the term along the way, he showed that each minimal left ideal of $\mathfrak{G}$ is a *left group*: a semigroup which is both left simple and right cancellative. Dually, every minimal right ideal is a *right group*. Moreover, without using the term 'direct product', Suschkewitsch proved that every (finite) left group is the direct product of a group and a left zero semigroup (i.e., a semigroup in which $ab = a$, for all elements a, b); in fact, Suschkewitsch's proof carries over quite easily to the infinite case (Suschkewitsch 1937). Dually, any right group is the direct product of a group and a right zero semigroup. This result is yet another example of Suschkewitsch's 'reduction to group theory' approach to semigroups and variants of it have been rediscovered by a range of authors over the years—see Clifford and Preston (1961, p. 38) for references. That any right group is indeed such a direct product will be easier to see when we discuss Clifford's 1933 extension of Suschkewitsch's results.

Any two minimal left ideals (or any two minimal right ideals) of $\mathfrak{G}$ are isomorphic; Suschkewitsch denoted the minimal left ideals of $\mathfrak{G}$ by $\mathfrak{A}_1, \ldots, \mathfrak{A}_r$ and the minimal right ideals by $\mathfrak{B}_1, \ldots, \mathfrak{B}_s$. He showed that each $\mathfrak{A}_\kappa$ is the union of s disjoint, pairwise isomorphic (ordinary) groups $\mathfrak{C}_{\kappa\lambda}$:

$$\mathfrak{A}_\kappa = \mathfrak{C}_{\kappa 1} \cup \mathfrak{C}_{\kappa 2} \cup \cdots \cup \mathfrak{C}_{\kappa s},$$

and that each $\mathfrak{B}_\lambda$ is the union of r such groups:

$$\mathfrak{B}_\lambda = \mathfrak{C}_{1\lambda} \cup \mathfrak{C}_{2\lambda} \cup \cdots \cup \mathfrak{C}_{r\lambda}.$$

Each $\mathfrak{C}_{\kappa\lambda}$ is in fact the intersection of a minimal left ideal with a minimal right ideal: $\mathfrak{C}_{\kappa\lambda} = \mathfrak{A}_\kappa \cap \mathfrak{B}_\lambda$.

With the structure of the ideals of $\mathfrak{G}$ so determined, Suschkewitsch was able to define the *kernel* $\mathfrak{K}$ of $\mathfrak{G}$, by analogy with a definition of Wedderburn (1907). This is a particular ideal of $\mathfrak{G}$, whose construction was the main aim of the paper and is defined thus:

$$\mathfrak{K} = \bigcup_{\kappa=1}^{r} \mathfrak{A}_\kappa = \bigcup_{\lambda=1}^{s} \mathfrak{B}_\lambda = \bigcup_{\kappa=1}^{r} \bigcup_{\lambda=1}^{s} \mathfrak{C}_{\kappa\lambda}.$$

Suschkewitsch gave a schematic representation of $\mathfrak{K}$ in the form of a grid, with the $\mathfrak{A}_\kappa$ labelling the columns and the $\mathfrak{B}_\lambda$ the rows; the entry appearing in the grid in the $\mathfrak{A}_\kappa$-column and the $\mathfrak{B}_\lambda$-row was $\mathfrak{C}_{\kappa\lambda}$. This diagram was the precursor of the 'egg-box picture' of Clifford and Preston (1961, p. 48): a useful device, familiar to semigroup theorists everywhere, used to visualise the Green's relations of a given semigroup.

Taking the 'reduction to group theory' approach mentioned in the previous section, Suschkewitsch showed that $\mathfrak{K}$ is completely determined as an ideal of $\mathfrak{G}$ by the following:

(1) the structure of the groups $\mathfrak{C}_{\kappa\lambda}$;
(2) the numbers r and s;
(3) the $(r-1)(s-1)$ products $E_{11}E_{\kappa\lambda}$ ($\kappa = 2, \ldots, r$; $\lambda = 2, \ldots, s$), where $E_{\kappa\lambda}$ is the identity of $\mathfrak{C}_{\kappa\lambda}$.

In this case, Suschkewitsch referred to $\mathfrak{K}$ as the *Kern* of $\mathfrak{G}$. He also showed that the above three conditions may be chosen arbitrarily in order to construct a kernel which is no longer regarded as a subsemigroup of another semigroup. He referred to such a 'stand-alone' kernel as a *Kerngruppe*. The paper concludes with a concrete example of the construction of a Kerngruppe as a subsemigroup of a full transformation monoid.

Although it is not obvious from the notation used, Suschkewitsch's conditions (1)–(3) above gave rise to a finite version of the Rees Theorem. However, his methods were rather involved and his structure theorem was "not in a readily usable form" (Clifford and Preston 1961, p. 208). As we have seen, a more useful formulation of the theorem came from Rees (1940), who, in introducing his 'matrix semigroups', gave a practical, coordinate-based recipe for constructing completely (0-)simple semigroups. Suschkewitsch's approach was entirely coordinate-free.

We move now to an extension of Suschkewitsch's work, due to Clifford (1933b). Clifford was one of the major pioneers of semigroup theory, and his work will also have a prominent position in the next section. His 1933 paper concerned various systems of postulates first given for groups by Dickson (1905a) and was very 'group-axiomatic' in spirit. Clifford's approach was probably inspired by his Ph.D. supervisors Morgan Ward and E.T. Bell, both of whom had carried out 'axiomatic' investigations at around that time—see, for example, Ward (1930) and Bell (1933). At the time of writing his 1933 paper, Clifford was not aware of the prior work of Suschkewitsch (Preston 1991, p. 21).

Given a set G upon which a binary operation is defined, Clifford listed the following postulates:

(I) if $a, b \in G$, then $ab \in G$;
(II) for all $a, b, c \in G$, $a(bc) = (ab)c$;
(III) for each $a \in G$, there exists at least one left identity $e \in G$: $ea = a$;
($\mathrm{IV_L}$) for each $a \in G$ and each left identity e of a, there exists at least one left inverse b of a, with respect to e: $ba = e$;
($\mathrm{IV_R}$) for each $a \in G$ and each left identity e of a, there exists at least one right inverse b of a, with respect to e: $ab = e$;
($\mathrm{V_L}$) for each $a \in G$, there exists at least one left identity e of a and at least one left inverse b, with respect to e: $ba = e$;
($\mathrm{V_R}$) for each $a \in G$, there exists at least one left identity e of a and at least one right inverse b, with respect to e: $ba = e$.

As Dickson had shown, a system satisfying postulates I, II, III and $\mathrm{IV_L}$ is simply a group; the systems (I, II, III, X), where X is $\mathrm{IV_R}$, $\mathrm{V_L}$ or $\mathrm{V_R}$, however, are not. The determination of the nature of these latter systems was Clifford's goal. He was motivated by an apparently ambiguous statement in the definition of a group given by van der Waerden (1930): his fourth postulate can be interpreted either as $\mathrm{IV_L}$ or $\mathrm{V_L}$. However, as Clifford commented:

> He evidently intended the former, judging from subsequent deductions; nevertheless, I thought it would be of interest to see what the weaker postulate would lead to. (Clifford 1933b, p. 866)

The first result of Clifford's paper was the perhaps slightly surprising theorem that the systems (I, II, III, $\mathrm{IV_R}$), (I, II, III, $\mathrm{V_L}$) and (I, II, III, $\mathrm{V_R}$) are, in fact, equivalent. To such systems, Clifford gave the name *multiple group*. Later in the paper, he gave an alternative definition which, in modern terminology, runs thus: a *multiple group* is a semigroup G in which, for all $a, b \in G$, the equation $ax = b$ has a unique solution in G.

This last condition had in fact appeared briefly in Suschkewitsch (1928), and we know from Sect. 3 that it implies left cancellation in G. Moreover, it is easy to deduce that such a G is also right simple. In other words, G is a right group, in Suschkewitsch's sense. Clifford's multiple groups therefore represent a partial extension of Suschkewitsch's work to the infinite case; they do not constitute a full description of completely (0-)simple semigroups, however, since not every such semigroup is a left/right group.

We note that a related axiomatic study was carried out by Baer and Levi (1932). We have seen that a left/right group is a semigroup which is both right/left cancellative and left/right simple. Baer and Levi studied the relationship between the conditions of being left/right simple and left/right cancellative. A semigroup which is both left and right simple is of course a group; Baer and Levi described those semigroups which are both right simple and right cancellative (dually, left simple and left cancellative). Such semigroups are now known as *Baer–Levi semigroups*; see Clifford and Preston (1961, p. 39) and Clifford and Preston (1967, Sect. 8.1).

The name 'multiple group' was not chosen arbitrarily: as Clifford determined, a multiple group is a semigroup which is a union of groups. Let G be a multiple group. Clifford gave the name *index* to the number of left identities in G, and denoted this number by α; the left identities were denoted $e_1, e_2, \ldots$. We note that α may be infinite and observe a slightly enigmatic feature of Clifford's paper: his notation suggests that he was restricting his attention to the countable case (see below), and yet his concluding example was uncountable. Despite the notation, Clifford's arguments still stand in the uncountable case.

Clifford partitioned G into subsets of the form

$$K_i = \{a \in G : a \text{ has left inverses with respect to } e_i\}.$$

The K_i form isomorphic groups, which Clifford called the *groups of composition* of G. The abstract group K isomorphic to each of the K_i was termed the *composition group* of G. Thus Clifford's α corresponded to Suschkewitsch's r and Clifford's K_i $(i = 1, \ldots, \alpha)$ to Suschkewitsch's $\mathfrak{C}_{\kappa\lambda}$ $(\kappa = 1, \ldots, r)$.

The goal of Clifford's paper was the following theorem:

Theorem 3 *Let H be a group and α be a cardinal. We can construct a multiple group G with index α and with H as its composition group. Conversely, a multiple group G is completely determined by its index α and composition group H.*

The construction given by Clifford works as follows. We select the group which we wish to be the composition group of our multiple group, say $H = \{e, a, b, c, \ldots\}$, where e is the identity of H. We put $K_i = \{e_i, a_i, b_i, c_i, \ldots\}$ for our groups of composition, where i ranges from 1 up to our chosen index α. The multiple group is then the union of all of the K_i, with multiplication of elements from different K_i given by $a_i b_j = (ab)_j$. Note that if we instead write the element a_i as a pair (a, i), then a multiple group with composition $(a, i)(b, j) = (ab, j)$ is easily seen to be the direct product of a group H and a right zero semigroup $\{1, \ldots, \alpha\}$. Suschkewitsch's description of left/right groups was thereby extended to the infinite case. Clifford concluded with the following numerical example of a multiple group. Let $\mathbb{C}^*$ be the collection of all non-zero complex numbers, and define a binary operation on $\mathbb{C}^*$ by $a * b = |a|b$. Then, under this operation, $\mathbb{C}^*$ forms a multiple group whose index is the continuum c and whose composition group is the group of non-zero real numbers under ordinary multiplication (Clifford 1933b, p. 871).

It has been commented that

> ... *good* algebraic theories never originate axiomatically, ... they rather spring out of applications, they are *discovered*, (Schein 1997, p. 267)

and whilst this is undoubtedly true in many instances, Clifford's 1933 paper demonstrates that it is not true in general: in this case, an axiomatic starting point gave rise to a major strand in the theory of semigroups. Preston says of Clifford (1933b) that it

> ... must have appeared at the time to its author as an incidental piece of work ... In fact it turned out to be a remarkably fruitful start for the structure theory of semigroups. (Preston 1974, p. 33)

Not only did Clifford (1933b) extend the work of Suschkewitsch, and build towards the later Rees Theorem, it also sparked the study of semigroups which are unions of groups (or of other semigroups), which was to provide semigroup theory with its first major independent theorem—more on this in the final section.

We have not yet reached a full description of completely 0-simple semigroups, as given at the beginning of the section. In order to tie up these loose ends, we turn at last to Rees (1940). This paper, which was written in the summer vacation shortly after Rees had completed his undergraduate course (Preston 1991, p. 18), generalised the preceding results of Suschkewitsch and Clifford, and gave the now-familiar Rees Theorem. Unlike these papers, Rees used the term *semigroup* and used it in its modern sense—indeed, we can speculate that this is *why* it is the modern sense. The paper was communicated by Philip Hall, who also receives an acknowledgement for his encouragement and for his help in the preparation of the paper. Preston (1991, p. 19) comments that Hall's lectures on general algebra in Cambridge exerted a strong influence on the study of abstract algebra at that time, and speculates that Birkhoff had also received inspiration for his work on universal algebra from Hall (the paper in which Birkhoff introduced the notion of a universal algebra (Birkhoff 1935) was also communicated by Hall). Indeed, Hall seems to have taken quite a broad, 'non-traditional' view of algebra: Preston also comments that semigroups were "used quite naturally" in Hall's lectures a few years later, around 1950.

Unlike those of Suschkewitsch and Clifford, Rees' paper was quite ring-theoretic in spirit. Other than in his opening paragraph, however, Rees gave little indication as to his motivation:

> This paper is an attempt to apply the methods of the theory of algebras to the more general problem of the structure of semi-groups ... (Rees 1940, p. 387)

His notation was very similar to that used by Albert (1939) for linear associative algebras and, since Albert's book had appeared in print only a year before Rees' first paper, we might suspect that this provided the impetus for Rees to investigate the situation where the operation of addition is dropped. However, when asked about this by Gordon Preston, Rees stated that this was not the case (Preston 1991, p. 18). At any rate, the notion of a simple ring is an object of wide-spread interest in ring theory and, since many ring-theoretic questions can be put into terms of multiplication alone (see also the comments on unique factorisation in Appendix B), it is perhaps not so surprising that Rees would investigate simple semigroups, and also other ring-theoretic ideas, without addition. For example, early on in the paper, in an obvious analogue of ring theory, Rees defined an *ideal* of a semigroup and constructed what is now known as the *Rees quotient*: the quotient of a semigroup by an ideal—see Howie (1995, Sect. 1.7). Using this quotient, Rees was able to prove semigroup analogues of the Second and Third Isomorphism Theorems for groups and for rings (Cameron 1998, pp. 41–42, 82–83), thereby forging links with established algebra. Despite this, Rees' paper does feel strikingly original; this impression is no doubt created in part by the fact that there are very few references in the paper: Albert (1939) and Poole (1937) are alluded to, though full references are not given. The only complete reference is to Suschkewitsch (1928). Of course, there was little semigroup literature to which Rees could refer.

After giving his analogues of the isomorphism theorems, Rees proved further technical results on ideals, before arriving at his definition for a *simple* semigroup, adapting Albert's usage for linear algebras (Albert 1939). Rees' definition differed slightly from that given in Definition 4: in the event that a semigroup S has a zero 0, he did not admit the zero ideal $\{0\}$ as a 'proper' ideal of S. He therefore defined a 'simple' semigroup to be a semigroup which has no 'proper' ideals (in his sense) and which is not the zero semigroup of order 2. Nevertheless, with a little help from a result of Clifford and Preston (1961, Lemma 2.26), it can be seen that Rees' notion of 'simple' covered both the modern notions of simple and 0-simple. Arguably, Rees' terminology leads to more neatly phrased results than that in use today, since it captures all cases under a single heading. However, in our discussions below, we will adopt the modern usage. We note also that Rees' acknowledgement of the possible presence of a zero element, something not touched upon by either Suschkewitsch or Clifford, inspired as they were by group theory, is indicative of Rees' coming from the directions of rings.

Rees related his definition to the earlier work of Suschkewitsch. He constructed Suschkewitsch's Kerngruppe by much more transparent methods than those originally employed by Suschkewitsch himself (though, to be perfectly fair to Suschkewitsch, it is probable that Rees' methods are easier to follow only because they are phrased in terms that are more familiar to modern eyes). The first part of the paper ends with the conclusion that the class of Kerngruppen is precisely the class of finite simple semigroups without zero, together with the zero semigroup.

Part 2 of Rees' paper builds towards the eventual statement and proof of the Rees Theorem. In the process, it takes in some elementary results on ideals as well as some basic definitions, including that of a primitive idempotent. With these definitions made, Rees defined the all-important concept—that of a *completely* (0-)simple semigroup:

Definition 6 A semigroup S is *completely (0-)simple* if

(R1) S is (0-)simple;
(R2) if $x \in S$, then there exist idempotents $e, f \in S$ such that $xe = x$ and $fx = x$, or, equivalently, x belongs to at least one set of the form fSe, where e and f are idempotents;
(R3) every idempotent of S is primitive.

Condition (R3) as given above was not the condition initially adopted by Rees. He first gave the condition as: *there exists a primitive idempotent f under every non-primitive idempotent e* (i.e., for idempotents e, f with $f = ef = fe$, it follows that $e = f$). However, Rees then immediately proved that *every* idempotent of a completely (0-)simple semigroup is in fact primitive. It is quite clear that Rees' definition of the notion of a primitive idempotent in a semigroup was directly inspired by that in a ring, where the notion is used, for example, in the block decomposition of rings (Karpilovsky 2001). A primitive idempotent in a ring is a non-zero idempotent e such that if $e = e_1 + e_2$, where e_1, e_2 are orthogonal idempotents, then it follows that one of e_1, e_2 is 0. It is an easy exercise to show that in the multiplicative semigroup of a ring, the two notions of 'primitive idempotent' coincide.

Definition 6 clearly differs a little from our Definition 4. Condition (CS2) has its origins in Rees' very brief *Note on semi-groups* (Rees 1941). In this follow-up note, Rees proved that his original condition (R2) is in fact a consequence of (R3). Moreover, it turns out that we need only demand the existence of at least one primitive idempotent, as per our condition (CS2). Note that in order to demonstrate that the bicyclic monoid B is not completely 0-simple, we need only show that it contains a non-primitive idempotent, thanks to (R3). The idempotent $(1, 1)$ serves as a suitable example: $(1, 1)(2, 2) = (2, 2)(1, 1) = (2, 2)$.

Returning to Rees' original paper, we note that his goal was a full description of completely (0-)simple semigroups. Certain classes of rings, simple rings in particular, may be exhibited as rings of matrices, and so it was natural for Rees to seek to represent completely (0-)simple semigroups by matrices. To this end, he defined his *regular matrix semigroups* as follows. Let L and M be non-empty sets and let G be a group. We adjoin a zero 0 to G to obtain the 0-group $G^0 := G \cup \{0\}$. The matrix $A = (x_{ij})$, where $x \in G^0$, i ranges over L and j ranges over M, will be called an (L, M)*-matrix over* G^0. The (L, M)-matrix over G^0 with just one non-zero entry $x \in G^0$ in the ijth position will be denoted by $(x)_{ij}$.

Definition 7 (Rees 1940, Sect. 2.9) Let S be the semigroup consisting of (L, M)-matrices $(x)_{ij}$ over a 0-group G^0, together with the zero matrix O. Multiplication in S is defined by:

$$\mathrm{O}(x)_{ij} = (x)_{ij}\mathrm{O} = \mathrm{O}\mathrm{O} = \mathrm{O};$$
$$(x)_{ij}(y)_{kl} = (x)_{ij}P(y)_{kl} = (xp_{jk}y)_{il},$$

where $P = (p_{qr})$ is an (M, L)-matrix over G^0. The semigroup S is called the *matrix semigroup over* G^0 *with defining matrix* P.

If P is regular, in the sense that each row and each column contains a non-zero entry, then S is called the *regular matrix semigroup over* G^0 *with defining matrix* P.

Note that in the above definition of multiplication in S, it is possible that $p_{jk} = 0$, in which case we see that $(xp_{jk}y)_{il} = (0)_{il} = \mathrm{O}$. Rees' regular matrix semigroups and the Rees matrix semigroups introduced at the beginning of the section are of course one and the same. It is easy to see that Rees' original formulation in Definition 7 is equivalent to the 'triples' version found in Definition 5, which was first given by Clifford (1941). Then, despite some small cosmetic differences, Rees' concluding theorems clearly give the result which is now hailed as the Rees Theorem. Just like the methods and results of Clifford which we will consider in the next section, the Rees Theorem, and the Rees matrix construction in particular, have provided a model for subsequent researches—see Meakin (1985).

As has been pointed out, the Rees Theorem is a partial analogue of the Wedderburn–Artin Theorem for rings, which states, among other things, that any right Artinian ring (that is, a ring which satisfies the descending chain condition for principal right ideals) is simple and isomorphic to some ring of square matrices over a division ring (Cohn 1977). Thus, although the Rees Theorem was semigroup theory's first important (and complete) structure theorem, it was guided by earlier results on rings. In the next section, we will finally encounter semigroup theory's first 'independent' theorem.

7 Unions of groups and semigroups

We saw in the previous section that, on the way to the Rees Theorem, Alfred Clifford studied semigroups which are unions of groups (Clifford 1933b). He revisited this topic a number of years later in a paper published in 1941, and, in so doing, arrived at semigroup theory's first independent structure theorem. Moreover, Clifford's results and methods would provide a framework for many subsequent researchers. It is the 1941 paper of Clifford which will occupy us in this final section. In particular, we will point out the remarkable influence that this paper has had upon the subsequent theory of semigroups. For surveys concerning unions of groups, see Clifford (1972) and Preston (1996).

Of key importance in Clifford (1941) was the notion of a completely simple semigroup, as we will see below. It seems that Clifford developed the material of his 1941 paper independently of that of Rees, and it was only when he was preparing it for publication that he became aware of Rees' prior work, as described in the previous section. He therefore rewrote his paper accordingly. It was also at this point that Clifford became aware of the work of Suschkewitsch for the first time, thanks to a reference in Rees (1940) (see Howie (2002) and Rhodes (1996)).

Recall that Suschkewitsch had considered finite right groups and had found these to be disjoint unions of groups (Suschkewitsch 1928). Clifford had then made a similar observation in the infinite case (Clifford 1933b). Perhaps in the spirit of studying semigroups which are 'close' to groups, Clifford's first semigroup paper of the 1940s picked up this theme once more, and began with the following definition, the motivation for which can be seen in Theorem 4 below.

Definition 8 A semigroup S is said to *admit relative inverses* if, for every $a \in S$,

(1) there exists $e \in S$ such that $ae = a = ea$, and
(2) there exists $a' \in S$ such that $aa' = e = a'a$,

i.e., for every element there is a two-sided identity, and a two-sided inverse with respect to that identity.

Semigroups which admit relative inverses are now termed *completely regular* semigroups, a term introduced by Petrich (1973); in what follows, we will adopt the modern terminology. Let e be the identity element corresponding to some $a \in S$, as defined by (1) and (2). Then e is idempotent, since $e^2 = aa'aa' = aea' = aa' = e$. Following Clifford, we will say that *a belongs to e*. The following result is reasonably easy to see, and originally generalised a result of Poole (1937):

Theorem 4 (Clifford 1941, Theorem 1) *A semigroup S is completely regular if, and only if, it is the disjoint union of groups S_e, where S_e is a group with identity e, consisting of all elements belonging to e.*

According to this theorem, we may take a number of disjoint groups and form their union, thereby constructing a completely regular semigroup. However, if we do this, we will run into difficulties when defining products (in the semigroup) of elements from different groups. In his 1933 paper, Clifford had made use of the underlying structure of his composition group K, as well as the fact that all of the K_i were isomorphic, to define the product of $a_i \in K_i$ and $b_j \in K_j$ as $(ab)_j \in K_j$ (see Sect. 6). In general, however, the problem of defining the multiplication of elements from different groups is not so easily solved; the groups S_e need not be isomorphic.

Another issue which arises, assuming we can surmount the multiplication problem, is that the set product $S_e S_f = \{ab : a \in S_e, b \in S_f\}$ of two groups is not necessarily contained within a third group but will, in general, be scattered throughout several. This is not a problem as such, but if the product $S_e S_f$ were contained in a third group S_g, then it would lend a certain extra elegance to the theory. Clifford investigated the possibility of decomposing a completely regular semigroup S into the disjoint union of subsemigroups S_e belonging to some class of semigroups (other than groups) in such a way that this last property holds. He found a complete description of the structure of the required subsemigroups, before recognising that these were precisely the completely simple semigroups studied by Rees. Thus, we *can* have $S_e S_f \subseteq S_g$ if, instead of splitting S into groups, we split it into completely simple semigroups. We see then that completely simple semigroups arose very naturally for Clifford, giving him good reason to pounce upon the concrete examples of such which were provided by Brandt semigroups, as we saw in Sect. 4.

Clifford devoted Sect. 2 of his paper to the investigation of the ideals of S. As was indicated in our Sect. 2, the notion of an ideal is a fundamental concept in the modern theory of semigroups, so much so that the definition is almost always assumed in current papers; the fact that Clifford included the definition of an ideal in a semigroup (as had Rees before him) is indicative of the fact that this was still relatively new mathematical ground (although ideals had already seen extensive use in the earlier work of Clifford and others on factorisation in semigroups—see Appendix B). It is worth noting that whilst studying principal ideals, Clifford worked with the set of generators, J_a, of the principal (two-sided) ideal generated by an element a. This foreshadowed the later introduction of Green's relation $\mathscr{J}$, since the set J_a is, of course, the $\mathscr{J}$-class of a.

Clifford determined that the collection $\mathfrak{B}$ of principal ideals of S forms a *semilattice*: a commutative semigroup in which every element is idempotent, or, equivalently, a partially ordered set in which every pair of elements has a greatest lower bound. In the case of Clifford's $\mathfrak{B}$, multiplication of ideals $\mathfrak{a}$ and $\mathfrak{b}$ is defined by $\mathfrak{ab} = \mathfrak{a} \cap \mathfrak{b}$. It is clear that this operation is commutative and that $\mathfrak{a}^2 = \mathfrak{a}$, for every ideal $\mathfrak{a} \in \mathfrak{B}$; if we were to employ the alternative characterisation of a semilattice as a partially ordered set, then we would of course have $\mathfrak{a} \leq \mathfrak{b}$ if, and only if, $\mathfrak{a} \subseteq \mathfrak{b}$. Clifford denoted by P the abstract semilattice isomorphic to $\mathfrak{B}$. In reference to the decomposition of semigroups into unions of subsemigroups of better-known structure, Lyapin commented:

> Success in the application of such a method depends first on how well we know the properties of the semigroups which are components of such a union, and, second, on the character of the interrelations between the components in that union. (Lyapin 1963, p. 307)

In this respect, the following theorem of Clifford ticks all the boxes:

Theorem 5 (Clifford 1941, Theorem 2) *Every completely regular semigroup S determines a semilattice P such that each $\alpha \in P$ corresponds to a subsemigroup S_α of S with the following properties:*

(1) *S is the disjoint union of the S_α;*
(2) *each S_α is a completely simple semigroup;*
(3) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$.

Conversely, any semigroup S with this structure is completely regular; we say that S is a semilattice of completely simple semigroups.

Thus Clifford not only managed to decompose S in such a way that the desired property (3) holds, but he also decomposed S into semigroups of known structure (thanks to Rees): the goal of any decent structure theorem. This theorem has no analogue in either group or ring theory and is therefore our much-vaunted first structure theorem of an independent semigroup theory. It is also important in another respect: it employs a construction which has been the basis for semigroup structure theories ever since. This is particularly true of the special case embodied in Theorem 6 below, which has provided a framework for subsequent researchers; see, for example, Fountain (1977), where certain semigroups are characterised as semilattices of left cancellative monoids.

We observe that something interesting happens when P is the trivial semilattice $\{1\}$. This corresponds to the case when S has just one principal ideal, necessarily itself. Property (1) of Theorem 5 becomes simply: $S = S_1$, where S is completely regular and S_1 is completely simple. Indeed, it can be shown that a simple semigroup is completely regular if, and only if, it is completely simple (Howie 1995, Theorem 4.1.2). Completely *0-simple* semigroups, however, fail to be completely regular; see Howie (1995, p. 103).

As the terminology suggests, a completely regular semigroup S is also *regular*, in the sense defined in Sect. 2. To see this, let $a \in S$ belong to the identity $e \in S$, and denote by a' the inverse of a with respect to e. Then, clearly, $aa'a = a$.

Clifford went on to consider two special cases of completely regular semigroups. The first of these was the case when idempotents commute with each other (i.e., the idempotents form a semilattice). The modern term for a completely regular semigroup in which idempotents commute is a *Clifford semigroup* (Howie 1995, Sect. 4.2). Clifford considered this special case because it enabled him to make the S_α groups once more; in much the same way that he had shown that a completely regular semigroup is a semilattice of completely simple semigroups, Clifford proved that a Clifford semigroup is a semilattice of *groups*; he gave the following neat recipe for the construction of such a semigroup:

Theorem 6 (Clifford 1941, Theorem 3) *Let P be a semilattice. To each $\alpha \in P$, we assign a group S_α in such a way that distinct groups S_α are disjoint. For each pair*

$\alpha > \beta$ (i.e., $\alpha\beta = \beta$), let $\varphi_{\alpha\beta} : S_\alpha \to S_\beta$ be a morphism such that $\varphi_{\alpha\beta}\varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$ if $\alpha > \beta > \gamma$, and let $\varphi_{\alpha\alpha}$ be the identity automorphism of S_α. We let S be the union of the S_α and define the product of $a_\alpha \in S_\alpha$ and $b_\beta \in S_\beta$ to be

$$a_\alpha b_\beta = (a_\alpha \varphi_{\alpha\gamma})(b_\beta \varphi_{\beta\gamma}),$$

where $\gamma = \alpha\beta$. Then S is a Clifford semigroup. Conversely, every Clifford semigroup is isomorphic to a semigroup constructed in this way.

Again, we have here a theorem with no direct analogue in either group or ring theory; if Theorem 5 was the first theorem of an independent semigroup theory, then Theorem 6 is surely the second—in line with our comments following Theorem 5, Preston comments that Theorem 6

> ... exhibits a kind of structure theorem unique to semigroup theory, and has provided a pattern which has frequently acted as a guide to the possibilities in other more complicated situations. (Preston 1974, p. 37)

It transpired that there are a number of equivalent characterisations of a Clifford semigroup (Howie 1995, Theorem 4.2.1). A Clifford semigroup is certainly regular and so, by comments in Sect. 2, it is necessarily an inverse semigroup, since its idempotents commute. In particular, a Clifford semigroup may be regarded as an inverse semigroup which is a union of groups (Clifford and Preston 1961, Theorem 4.11). Theorem 6 therefore represents a first theorem on inverse semigroups.

It seems that the requirement that idempotents commute is a crucial one, since in the contrary case, no recipe can be found which is as neat as the one above. In fact, idempotents play a crucial role in semigroup theory more generally. Many classes of semigroups are studied in terms of the properties of their idempotents; for example, inverse semigroups may be characterised as semigroups in which every $\mathscr{R}$-class and every $\mathscr{L}$-class contains precisely one idempotent, where $\mathscr{R}$ and $\mathscr{L}$ are Green's relations, as in Sect. 2 (Howie 1995, Theorem 5.1.1). Indeed, idempotents still play an important role in current semigroup theory, as the author can attest from personal experience; see, for example, Hollings (2007a). Clifford was thus the first to highlight the important role played by idempotents in semigroups. As we saw in the previous section, Rees had also used idempotents, but perhaps not to the extent of showing their pivotal position in the theory. We can speculate that Clifford was maybe looking for new methods in the burgeoning theory of semigroups which were simply not applicable in group theory. It is likely that he drew inspiration from the theory of rings and algebras, where idempotents had been useful tools ever since the coining of the term by Peirce (1881): for example, in the study of von Neumann regular rings (Sect. 2)—see Skornjakov (2001). See also the comments in Sect. 6 on primitive idempotents.

The importance of idempotents is high-lighted further by the fact that the problem of characterising completely regular semigroup whose idempotents do not commute is far less tractable. Clifford therefore only considered a special case in this situation. This was the case when the semilattice P has just two elements, $P = \{\alpha, \beta\}$. We have already observed that in the case where P is trivial, a completely simple semigroup results. Thus the two-element case was the simplest (new) case that Clifford could study. In a bid to simplify matters further, rather than taking the semigroup S to be the disjoint union of subsemigroups of arbitrary structure, Clifford instead took S to be the disjoint union of semigroups S_α, S_β ($\alpha > \beta$), where S_α may be a semigroup of any type, but S_β is specifically a Rees matrix semigroup (without zero) and, as such, is a completely simple ideal of S. In spite of Clifford's valiant attempts to simplify the situation, the resulting structure theorem was rather messier than Theorem 6 and is perhaps a little unedifying to look at. Consequently, we will not record it here; see Clifford (1941, Theorem 4). However, this second special case set the scene for some of Clifford's later work on *ideal extensions* (Clifford 1950)—a semigroup analogy of Schreier's theory of group extensions, and just one way of constructing new semigroups from old; see Clifford (1950) and Clifford and Preston (1961, Sects. 4.4–4.5). We note also that in the special case $S = S_\alpha \,\dot{\cup}\, S_\beta$, Clifford briefly considered the subcase where the idempotents of S_β form a subsemigroup; a regular semigroup in which idempotents form a subsemigroup is termed an *orthodox semigroup* and has seen extensive study (see, for example, Hall 1970).

Out of all the papers that we have surveyed in this article, I feel that those of Alfred Clifford must be singled out for special mention: even today, his work remains wonderfully lucid; Gordon Preston has referred to Clifford's "limpid clarity" (Preston 1991, p. 29) and Douglas Munn said of Clifford's papers that

> [their] clarity appealed to me greatly and this consolidated my decision to work in the field. (Munn 2008)

Elegant results and interesting problems are certainly very important for the development of any mathematical theory, but the quality of the exposition of those early results should not be underestimated. Semigroup theory was lucky to have someone like Alfred Clifford not only to carry out some of the early investigations in the field, but also to present and promote those results and thereby help the study of semigroups to grow.

We have seen that Clifford's 1941 paper proved to be a starting point not only for much of his subsequent work—on semigroups which are unions of groups, ideal extensions, etc.—but also for the ensuing theory of semigroups. Some of the themes touched upon only briefly in Clifford (1941) have been expanded greatly by later researchers: the theories of inverse semigroups, orthodox semigroups, completely regular semigroups and Green's relations, for example. In this way, we see the justification for our earlier remark that Clifford was one of the pioneers of semigroup theory. The effect of Clifford (1941) on subsequent semigroup theory has been tremendous, as Preston enthuses:

> [It] was immensely influential. It contained definitive results that have been in continual use since. It introduced new concepts that provided powerful new tools for semigroup theory. (Preston 1996, p. 6)

One other thing is clear from Clifford (1941): an independent theory of semigroups had arrived.

As observed in Sect. 2, one of the major driving forces for the coalescence of a coherent theory of semigroups was Clifford and Preston's *The Algebraic Theory of*

Semigroups (1961, 1967), and it was the early work of Suschkewitsch, Clifford and Rees (as described in Sects. 5–7 of the present article) which provided the foundation for the first volume. To this was added the subsequent work of the likes of P. Dubreil, J.A. Green, E.S. Lyapin, A.I. Malcev, W.D. Munn, G.B. Preston, M.-P. Schützenberger, Š. Schwarz, and many others. Much of this work (for example, that of Schwarz) was carried out in ignorance of the work of earlier investigators but others were greatly influenced by the preliminary work which we have described in this article. The investigations of Suschkewitsch on finite simple semigroups, of Rees on completely (0-)simple semigroups, and of Clifford on unions of groups served not only as a solid starting-point for the theory of semigroups, providing elegant methods and a framework for subsequent research, but also as a source of further interesting problems. Thanks to the early boost that these early researches provided, the theory of semigroups continues to go from strength to strength.

Appendix A: The term 'semigroup'

As can hopefully be seen from the ideas explored in the present article, the theory of semigroups is somewhat different in spirit to that of groups; the name 'semigroup' exists for purely historical reasons, which we saw in Sect. 3. However, it is a term which has been used in a number of different senses. In this appendix, we will briefly examine the various different definitions. To avoid confusion, we will continue to use Bourbaki's term *magma* to mean a set which is closed under a given binary operation; all definitions in this appendix will be given in terms of magmas.

We have seen (Sect. 3) that the word 'semigroup' was first defined to correspond to the notion of an associative, cancellative magma. However, this did not mean that the terminology was fixed. On the contrary, between 1905 and 1940, the term 'semigroup' was used with at least four slightly different meanings. These were:

- an associative magma (Hilton 1908);
- an associative, cancellative magma (Schmidt 1966);
- an associative magma with left cancellation (Bell 1930);
- an associative, commutative, cancellative magma with identity (Clifford 1938).

Fortunately, each author began by defining what they meant by a 'semigroup'!

As far as I can determine, the earliest source to give the modern definition of a semigroup is in fact very close, chronologically speaking, to the initial definition of 1904. This source is Harold Hilton's 1908 book *An Introduction to the Theory of Groups of Finite Order*, in which we find:

> A set of elements is said to form a *group*, if (1) the product of any two (or the square of any one) of the elements is an element of the set; (2) the set contains the inverse of each element of the set. If the set satisfies condition (1) but not (2), it is called a *semi-group*. (Hilton 1908, p. 51)

(On page 1, Hilton had defined *elements* to be "things" whose composition is always associative. The existence of an identity element follows from the given conditions.) We know from his preface that Hilton consulted de Séguier's book but there is no definite indication of why he changed the definition of a semigroup. One possible solution is found in the preface:

> The nomenclature of the subject is by no means settled. I have tried to select definitions which have the advantage of being self-explanatory . . . (Hilton 1908, v)

The way in which the definitions of a group and a semigroup are given does seem to emphasise the fact that a semigroup requires half as many postulates as a group. The hyphen further emphasises the group connection. A later source addresses this question more explicitly. In Mulcrone (1962), after a consideration of the number of postulates required each by groups and by semigroups (at least in Mulcrone's formulation of their definitions), we find:

> Thus appears the appropriateness of the term *semigroup*, meaning "half a group" . . . (Mulcrone 1962, p. 299)

The hyphen in 'semi-group' originated with Dickson (1904), and continued to be used in most of the early sources on semigroups, perhaps to emphasise the connection with groups; after all, many of these sources concerned group theory, not semigroup theory. Clifford (1938, 1941) did not use the hyphen, but it was employed by Rees (1940), and persisted as far as such papers as Preston (1954a). As semigroup theory has taken on a life of its own, however, the hyphen has gradually been dropped.

Alternative names for semigroups abound:

> Many authors, including most of those writing in French, use the term 'demi-group' for an associative [magma]; these authors reserve 'semigroup' for what we shall call a cancellation semigroup. Other terms are 'monoid' (Bourbaki) and 'associative system' (Russian authors). (Bruck 1966, pp. 23–24)

In the paper of Leonard Dickson which we considered in Sect. 3 (Dickson 1905b), the name *algebra* was suggested for the modern notion of a semigroup.

Casting the net a little wider and considering other languages, we find the term *Semi-gruppe* being used in German to mean an associative, cancellative magma (Suschkewitsch 1936). However, this usage does not seem to have been very widespread, indeed, Suschkewitsch is the only author I can find who used this term. Arnold (1929) used the term *Halbgruppe* (with reference to Schmidt (1966)) to mean an associative, commutative, cancellative magma; nowadays, the German *Halbgruppe* corresponds to the modern English sense of *semigroup*. Rather than the term *associative system*, Russian authors now use *polugruppa* for an associative magma (the Russian prefix *polu-* corresponds to the Latinate *semi-*). Indeed, most modern terms for *semigroup* follow the 'half a group' pattern, e.g., the Portuguese *semigrupo*, the Hungarian *félcsoport* and the Japanese *hangun*. Hebrew, with the term *aguda* (meaning 'group' or 'society'), seems to be the only language which does not follow this pattern (although it does have the alternative term *chavura lemechetza*: 'half a group') (Schein 2008b).

By way of concluding this appendix, it is also worth commenting on the related term *monoid*, meaning an associative magma with identity. This term is a little more recent than *semigroup*, and seems to originate with Bourbaki (1943). Before this, Birkhoff

(1934) was using the term *groupoid* for an associative magma with identity. Other uses of the term 'groupoid' were discussed in Sect. 4.

Appendix B: Other developments

This paper is not an attempt to give an exhaustive account of all algebraic semigroups appearing in the mathematical literature in the first four decades of the twentieth century. I have attempted to pick out those threads which shaped the subsequent theory of semigroups. However, out of an instinct for something approaching completeness, it is worth pointing out some other semigroup-theoretic ideas which emerged in our chosen time span.

First of all, we have the work of Wilhelm Specht on what are now termed *wreath products*: just one way of constructing new semigroups from old. According to Wells (1976, p. 334), "the wreath product in something like its present form" appeared in Loewy (1927). This paper came to the attention of Issai Schur, who then suggested related questions to his student Specht. Petrich (1970) states that Specht went on to construct all irreducible matrix representations of the wreath product $M \wr P_n$ in terms of those of M and P_n, where M is a monoid and P_n is a subgroup of the symmetric group $\mathscr{S}_n$ (Specht 1933). Further details on the history of wreath products can be found in Kerber (1971).

One other strand of research which emerged in the 1930s, and which was inspired by the ring analogy, was the study of unique factorisation in semigroups. In particular, we have the work of Alfred Clifford. In the early 1930s, Clifford was a research student at the California Institute of Technology and worked towards his Ph.D. under the supervision of both Morgan Ward and E.T. Bell. Clifford completed his thesis *Arithmetic and Ideal Theory of Abstract Multiplication* in 1933 and, shortly thereafter, published a summary of his results (without proof) in a paper of the same name (Clifford 1934). A more detailed exploration followed a few years later (Clifford 1938).

The questions initially posed in Clifford's thesis were questions concerning rings:

1. Is every element of a ring R uniquely decomposable into prime elements?
2. If not, can we introduce 'ideal' elements in order to achieve this property?

Clifford commented:

> Since these questions can be put in terms involving only the operation of multiplication, it is natural to attempt a solution in the same terms. (Clifford 1934, p. 326)

Thus Clifford considered a ring with the operation of addition stripped away, leaving him with the 'abstract multiplication' of his title. In the series of publications mentioned above (Clifford 1933a, 1934, 1938), Clifford developed the notion of an *abstract arithmetic*: essentially, a commutative semigroup (with certain extra properties) in which every element can be decomposed uniquely as a product of irreducibles; such systems had previously been considered by Arnold (1929) in the non-commutative case, and by Ward (1928, 1935) in the cancellative case. In the case where such a commutative semigroup does not admit unique factorisation, Clifford obtained necessary and sufficient conditions for it to be *embedded* in a system which does; these conditions were analogous to those obtained for rings by Noether (1927). An overview of Clifford's work on unique decomposition can be found in Preston (1974). However, the study of abstract arithmetics does not seem to have continued as a major strand in semigroup theory. Indeed, this subject gets no mention in Clifford and Preston (1961, 1967) whatsoever. On the other hand, Clifford's 1938 paper is listed in the references of Lyapin (1963) but gets only a fleeting mention in the text. However, a brief overview of factorisation in semigroups versus factorisation in integral domains is given in Chapter 4 of Jacobson (1951), where semigroups which admit unique factorisation are called *Gaussian semigroups*; Clifford receives an acknowledgement in the preface to this book.

Finally, it is also worth mentioning the results obtained in the 1930s on the embeddability of semigroups in groups, by analogy with results on the embeddability of rings in fields. A commutative semigroup may be embedded in a group if, and only if, it is cancellative (Clifford and Preston 1961, Sect. 1.10). In the non-commutative case, however, cancellativity is a necessary condition but it is not sufficient. Indeed, Malcev (1937), following up on some earlier work by Suschkewitsch (1935), provided an example of a cancellative semigroup which is not group-embeddable. Two years later, he provided necessary and sufficient conditions for a semigroup to be embeddable in a group (Malcev 1939), though his set of conditions was countably infinite in size; Malcev proved further that no finite set of conditions would suffice (Malcev 1940). Questions of embeddability were subsequently considered by Dubreil (1943, 1954), Pták (1949), Lambek (1951), and also Tamari (1949, 1951, 1954), who generalised Malcev's embedding results to the more general case of 'semigroupoids', i.e., partial magmas (as in Sect. 4). The relationship between the sets of conditions given by Malcev and Lambek is described by Bush (1963).

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