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On the Decidability of the OL-DOL Equivalence Problem

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It is shown that decidability of equivalence between an FDOL language and a DOL language implies that of equivalence between a OL language and a DOL language. Another consequence of the first decidability is the decidability of 'DOLness' of a OL language.

1. INTRODUCTION

It has been known for some time that the equivalence problem for languages generated by iterated finite substitutions (the *OL equivalence problem*) is undecidable, see Blattner (1973) (or Rozenberg (1972) or Salomaa (1973b)). The problem is closely related to the equivalence problem for languages of sentential forms of context-free grammars, undecidable, too. On the other hand, the work of M. Nielsen (1974) and especially K. Čulik II and I. Friš (1977) has shown recently that the equivalence problem for languages generated by iterated morphisms (the *DOL equivalence problem*) is decidable (see also Ehrenfeucht and Rozenberg, 1978).

This situation renders some interest to the 'intermediate' problem of equivalence between two languages, one generated by iterated finite substitution and the other by iterated morphism (the *OL-DOL equivalence problem*, as we call it). Moreover, it is not difficult to see that the decidability of this problem implies that of the following question. Can a given language generated by iterated finite substitution be also generated by iterating some morphism?

The conjectures known to the author concerning the decidability of the OL-DOL equivalence problem are in the negative, e.g. in Salomaa (1975). We shall show that the problem can be reduced to the equivalence problem between two languages, one generated by iterated morphism and the other also generated by iterated morphism but using a finite number of axioms (the *FDOL-DOL equivalence problem*) which in our opinion is unlikely to turn out to be undecidable. Although not explicitly stated, the argumentations in Sections 3 and 4 can be used to prove that the OL-DOL equivalence problem is decidable if the DOL systems are restricted to those generating linearly growing sequences.

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The necessary background in formal language theory can be obtained e.g. from Salomaa (1973a) and in L systems theory from Herman and Rozenberg (1975) or Rozenberg and Salomaa (1976).

2. Notations and Preliminaries

We denote the length of the word P by $|P|$ and the Parikh vector of P by $[P]$. The empty word is denoted by Λ.

An *FOL system* is an ordered triple $H = (A, \sigma, X)$ where A is a finite alphabet, σ is a finite substitution $A^* \to 2^{A^*}$ and X is finite set of words over A (the *axioms*). If σ is an endomorphism on A^*, then H is called an *FDOL system*. If X consists of one word α only, then H is called a *OL system* (similarly a *DOL system*) and we write $H = (A, \sigma, \alpha)$. If $\Lambda \notin \sigma(a)$ for all $a \in A$, then H is called *propagating* or a *PFOL system* (similarly *POL system*, *PDOL system*, etc.). The relation $P \in \sigma(a)$, $a \in A$, is also written as $a \mapsto_H P$ and it is called a *production*. A production $a \mapsto_H P$ is called *deterministic* if the cardinality of $\sigma(a)$ equals 1, otherwise it is called *nondeterministic*. The *language generated by* H is

$$L(H) = \bigcup_{i=0}^{\infty} \sigma^i(X).$$

Two FOL systems H_1 and H_2 are called *equivalent* if $L(H_1) = L(H_2)$. By the *OL-DOL equivalence problem* we mean the problem of decidability of equivalence of a OL system and a DOL system. The *FDOL-DOL equivalence problem* and the *FOL-DOL equivalence problem* are defined similarly.

Let $G = (A, \delta, \omega)$ be a DOL system. The *sequence generated by* G is $(\delta^n(\omega))$ and it is denoted by $E(G)$. The mapping f_G from the set of natural numbers into itself defined by

$$f_G(n) = |\delta^n(\omega)| \qquad (n = 0, 1, \ldots)$$

is called the *growth function* of G. If the range of f_G is an infinite semilinear set, then we say that $E(G)$ *grows linearly*.

3. The Case of Nonlinear Growth

We shall need some auxiliary results before proceeding to the proof of the main assertion.

Lemma 3.1. *Let G be a DOL system such that $L(G)$ is infinite and $E(G)$ does not grow linearly. Then there exists an effectively obtainable natural number p such that*

$$\lim_{n\to\infty} \sum_{i=1}^{p} |f_G(n+i) - f_G(n)| = \infty.$$

Proof. Let G be (A, δ, ω) and denote $G_m = (A, \delta^m, \omega)$ $(m = 1, 2, \ldots)$. As is well known,

$$f_{G_m}(n) = Cn^l \rho^n + \text{terms of lower order}$$

for some positive constants C and ρ and natural numbers l and m such that either $\rho = 1$ and $l \geqslant 2$ or $\rho > 1$ and $l \geqslant 0$. It is shown in Berstel and Mignotte (1976) that we may take

$$m \leqslant \exp(2k(3 \log k)^{1/2})$$

where k is the cardinality of A.

We now choose

$$p = 2 \text{ entier}(\exp(2k(3 \log k)^{1/2})).$$

Denote $\bar{n} = \text{entier}(n/m) + 1$. Then

$$\sum_{i=1}^{p} |f_G(n+i) - f_G(n)| \geqslant |f_{G_m}(\bar{n}+1) - f_{G_m}(\bar{n})|$$
$$= C(\rho - 1)\, \bar{n}^l \rho^{\bar{n}} + \text{terms of lower order}$$

whence the claim is established for $\rho > 1$. On the other hand, if $\rho = 1$, we may further choose $m \leqslant \frac{1}{2}p$ in such a way that $f_{G_m}(n)$ is a polynomial of degree $l \geqslant 2$. Thus in this case

$$\sum_{i=1}^{p} |f_G(n+i) - f_G(n)| \geqslant |f_{G_m}(\bar{n}+1) - f_{G_m}(\bar{n})|$$
$$= Cl\bar{n}^{l-1} + \text{terms of lower degree}$$

which establishes the claim for $\rho = 1$ ∎

The following lemma shows that in order of a OL system H to be equivalent to a DOL system G, such that $E(G)$ does not grow linearly, it is the case that $L(H)$ can be generated by only a finite number of applications of nondeterministic productions. (Thus, in effect, we may assume H to be an FDOL system.)

LEMMA 3.2. *Let H be a OL system and G be a DOL system such that $E(G)$ does not grow linearly and $L(G) = L(H)$. Then the set of those words of $L(H)$ which can be directly derived from words of $L(H)$ using nondeterministic productions is finite.*

Proof. Let G be (A, δ, ω). Assume then, contrary to the claim, that we may find words PQ_1R, $PQ_2R \in L(H)$ as long as we please, which are directly derived from some word of $L(H)$ using some productions $a \mapsto_H Q_1$ and $a \mapsto_H Q_2$

where $Q_1 \neq Q_2$. Since $L(G) = L(H)$, there are natural numbers n_1 and n_2 such that

$$PQ_iR = \delta^{n_i}(\omega) \qquad (i = 1, 2).$$

We may assume that $n_1 < n_2$ without restricting our case. Let $m = n_2 - n_1$.

From now on we consider G only. For convenience we may assume that G is propagating. If this is not case originally we replace $E(G)$ by the PDOL sequence which is derived from $E(G)$ by erasing all symbols b such that $\Lambda \in \{\delta(b), \delta^2(b),...\}$ (cf. the proof of Theorem 5.1 where this trick is exploited more thoroughly). Denote

$$PQ_iR = \delta^{n_1+(i-1)m}(\omega) \qquad (i = 1, 2,...).$$

We say that a symbol c is k-growing if $|\delta^m(c)| = |\delta^{2m}(c)| = \cdots = |\delta^{(k-1)m}(c)| = 1$ and $|\delta^{km}(c)| \geqslant 2$. Let p be the number given by Lemma 3.1. We shall show that only a bounded number of symbols of PQ_1R are k-growing for $k = 1,..., p$ with a bound not depending on n_1. This clearly is in contradiction with Lemma 3.1 when it is applied to G.

Let $u = |Q_2| - |Q_1|$. We see that at most u symbols of P and R in PQ_1R contribute to Q_2 in PQ_2R. Thus at most $pu + |Q_1|$ symbols in PQ_1R contribute to symbols of Q_{p+1} in $PQ_{p+1}R$. Consider then the remaining symbols of PQ_1R. Since at most u symbols in P and R can be 1-growing we see that among these remaining symbols at most u of them can be k-growing for each $k = 1,..., p$. Thus the number of k-growing symbols ($k = 1,..., p$) in PQ_1R is at most $2pu + |Q_1|$. ∎

We are now ready to prove one of our main theorems.

THEOREM 3.1. *Suppose there is an algorithm for deciding the FDOL-DOL equivalence problem. Then there is an algorithm which of any given OL system H and DOL system G, such that E(G) does not grow linearly, decides whether or not* $L(H) = L(G)$.

Proof. Let $H = (A, \sigma, \alpha)$ be a OL system and $G = (A, \delta, \omega)$ a DOL system such that $L(H)$ and $L(G)$ are infinite and $E(G)$ does not grow linearly.

Let $B \subseteq A$ consist of exactly those symbols of A which have nondeterministic productions in H. Denote

$$L = \sigma(L(H) \cap A^*BA^*).$$

Now if L is infinite then by Lemma 3.2 $L(H) \neq L(G)$. Thus we may assume that L is finite. (Since L is an EOL language, its finiteness can be decided and, if finite, it can be effectively obtained, see e.g. Rozenberg and Salomaa, 1976.)

We transform H into an FDOL system $H' = (A, \tau, L \cup \{\alpha\})$ by taking exactly one production from H for each symbol of A to define τ. Then $L(H) = L(H')$. ∎

4. Typical Cases of Linear Growth

We state first some well known results concerning powers of words. For a word P we call the shortest word Q, such that $P = Q^i$ for some natural number i, the *root* of P and denote $Q = P^{1/2}$.

Lemma 4.1. (i) *If $P_1^i = P_2^j$ for some natural numbers $i, j \geqslant 1$, then $P_1^{1/2} = P_2^{1/2}$.*

(ii) *Let the lengths of the roots of P_1 and P_2 be $l_1 \geqslant 1$ and $l_2 \geqslant 1$, respectively. If P_1 and P_2 have a common subword of length $l_1 l_2$, then $l_1 = l_2$.*

Proof. For a proof of (i) see e.g. Lentin (1972) or Ruohonen (1977), Lemma 1. (ii) is a rather direct consequence of (i). (Note that for any words Q_1 and Q_2, $(Q_1Q_2)^{1/2}$ and $(Q_2Q_1)^{1/2}$ are of the same length.) ∎

Let l be a natural number. We say that a OL system $H = (A, \sigma, \alpha)$ is *l-bounded* if

(i) $L(H) = L \cup F$ where F is finite and

$$L \subseteq \bigcup_{\substack{P_1, Q, P_2 \in A^* - A^l A^+ \\ Q \neq A, R \in Q/A^+}} P_1 Q^{2l} Q^* R P_2 ; \tag{1}$$

(ii) for each P_1, Q, $P_2 \in A^* - A^l A^+$; $Q \neq A$ and $R \in Q/A^+$, $L \cap P_1 Q^{2l} Q^* R P_2$ is either empty or infinite; here we assume that we have chosen Q to be the shortest possible, i.e. we replace Q by $Q^{1/2}$ if necessary, and that thereafter we have chosen P_1 and P_2 to be the shortest possible, too, i.e. as much as possible of P_1 and P_2 is included in the 'periodic' part; by Lemma 4.1 this leads to unique P_1, Q, P_2 and R and we speak of the *initial part* P_1, *period* Q and *final part* RP_2 of a word $P_1 Q^i R P_2 \in L$.

The smallest possible F is called the *initial mess* of $L(H)$ and the largest possible L is called the *main body* of $L(H)$.

Lemma 4.2. *There is an algorithm which of any natural number l and OL system H decides whether or not H is l-bounded and in the positive case produces the initial mess and all those initial parts P_1, periods Q and final parts RP_2 for which the intersection of $P_1 Q^{2l} Q^* R P_2$ and the main body of $L(H)$ is infinite.*

Proof. The language K at the right hand side of (1) is regular. Since $L(H) - K$ is an EOL language the finiteness of which is known to be decidable we have an algorithm for testing l-boundedness of H. The rest of our algorithm amounts to testing finiteness of $L(H) \cap P_1Q^{2l}Q^*RP_2$ (an EOL language) and and a straightforward search of the initial mess. ∎

LEMMA 4.3. *Let $H = (A, \sigma, \alpha)$ be an l-bounded OL system for some natural number l. Then, for each period Q of a word in the main body of $L(H)$, $\sigma(Q) \neq \{\Lambda\}$ and there is a uniquely determined word $\bar{Q}$ such that $\sigma(Q) \subseteq \bar{Q}^*$ and $\bar{Q} = Q_1Q_2$ where Q_2Q_1 is a period of some word in the main body of $L(H)$.*

Proof. Let S_1 and S_2 be any nonempty words of $\sigma(Q)$. We show that $S_1^{1/2} = S_2^{1/2}$. Now let $P_1Q^iRP_2$ be a word in the main body of $L(H)$ with initial part P_1, period Q and final part RP_2. We may assume i to be as large as we please. But then a word $\bar{P}_1S_1^{i_1}S_2^{i_2}\bar{R}\bar{P}_2$, where $\bar{P}_1 \in \sigma(P_1)$ and $\bar{R}\bar{P}_2 \in \sigma(RP_2)$, $i_1 = i_2 = \frac{1}{2}i$ if i is even and $i_1 = \frac{1}{2}(i-1)$, $i_2 = \frac{1}{2}(i+1)$ if i is odd, is also in the main body of $L(H)$ having, say, the period Q'. By Lemma 4.1 (ii), $|S_1^{1/2}| = |Q'| = |S_2^{1/2}|$ and hence $S_1^{1/2} = S_2^{1/2} =_{\text{def}} \bar{Q}$. Thus in this case $\sigma(Q) \subseteq \bar{Q}^*$ and clearly we can find words Q_1, Q_2 such that $\bar{Q} = Q_1Q_2$ and $Q' = Q_2Q_1$.

It remains to be shown that $\sigma(Q) \neq \{\Lambda\}$. Assume the contrary. We know that for arbitrarily large j words

$$X \underset{\text{def}}{=} P_1Q^jRP_2$$

(initial part P_1, period Q and final part RP_2) appear in the main body of $L(H)$. If j is sufficiently large we know also that counting backwards any (fixed) number t of steps in a derivation of X symbols of Q are derived from symbols of periods in the words of the particular derivation. However, because of what we have shown above, for t 'large enough' (with a bound not depending on j) the sequence of alphabets of symbols appearing in the periods of the words in the drivation will be periodic. But this means that the symbols appearing in Q have already appeared as the symbols of some period in one of the t 'ancestors' of X and thus j can not be very large unless $\sigma(Q) \neq \{\Lambda\}$, a contradiction. ∎

LEMMA 4.4. *Let l be a natural number and G be an l-bounded DOL system such that $E(G)$ grows linearly. Then $L(G)$ is regular (with an effectively obtainable regular expression).*

Proof. Denote $G = (A, \delta, \omega)$. We note first that if any two words belonging to the main body of $L(G)$ have common prefixes and suffixes of length $l^2 + 2l$, then their initial parts, periods and final parts are the same (cf. Lemma 4.1). On the other hand, prefixes and suffixes of any length appear periodically in DOL sequences (see Herman and Rozenberg, 1975). Thus we can find natural numbers t and s such that the words $\delta^{nt+s}(\omega)$ $(n = 0, 1,...)$ all belong to the main body of $L(G)$ and have a common initial part P_1, period Q and final part

RP_2. Moreover, after replacing t by a multiple of it if necessary (cf. the proof of Lemma 3.1) we may assume that there are natural numbers u and v such that

$$| \delta^{nt+s}(\omega)| = un + v \qquad (n = 0, 1,...).$$

But then $| Q |$ divides u and hence

$$L \underset{\text{def}}{=} \{\delta^{nt+s}(\omega) \mid n = 0, 1,...\} = P_1(Q^x)^* Q^y RP_2,$$

a regular language, for some natural numbers x and y. Finally

$$L(G) = \{\omega,..., \delta^{s-1}(\omega)\} \cup \left(\bigcup_{i=0}^{t-1} \delta^i(L)\right)$$

is regular.

The effectiveness of the lemma should be evident. ∎

LEMMA 4.5. *Let l be a natural number and $H = (A, \sigma, \alpha)$ be an l-bounded OL system such that, for any period Q of a word in the main body of $L(H)$, $\Lambda \in \sigma(Q) \cup \sigma^2(Q) \cup \cdots$ Then $L(H)$ is regular (with an effectively obtainable regular expression).*

Proof. The case where $L(H)$ is finite is trivial. So let $L(H)$ be infinite.

Let the natural number m be so large that $\Lambda \in \sigma^m(Q)$ for any period Q of a word in the main body of $L(H)$.

Consider now an initial part P_1, period Q and a final part RP_2 of a word in the main body of $L(H)$. Denote

$$L = L(H) \cap P_1 Q^{2l} Q^* RP_2,$$

an infinite language. It suffices to show that $\sigma^m(L)$ is regular. But this follows immediately, since $\Lambda \in \sigma^m(Q)$ and L is infinite and hence

$$\sigma^m(L) = \sigma^m(P_1)(\sigma^m(Q))^* \sigma^m(RP_2).$$

The effectiveness of the lemma follows from Lemma 4.2. ∎

We are now in the position to prove the first of the two main results in this section.

THEOREM 4.1. *There is an algorithm which given any natural number l, an l-bounded OL system H and an l-bounded DOL system G, such that $E(G)$ grows linearly, decides whether $L(H) = L(G)$.*

Proof. Let $G = (A, \delta, \omega)$ and $H = (A, \sigma, \alpha)$. By Lemma 4.4, $L(G)$ is regular. Since $L(H) - L(G)$ is then an EOL language (the emptiness of which is decidable) we may assume that $L(H) \subseteq L(G)$.

We divide the periods of words of the main body of $L(H)$ into two classes:

(I) periods Q such that $\Lambda \notin \sigma(Q) \cup \sigma^2(Q) \cup \cdots$;

(II) the other periods, i.e. periods Q such that for any $P \in \sigma(Q) \cup \sigma^2(Q) \cup \cdots$, $\Lambda \in \sigma(P) \cup \sigma^2(P) \cup \cdots$

Note that these are indeed the only kinds of periods that can appear in words of the main body of $L(H)$, cf. Lemma 4.3. Clearly words with periods of type (I) (resp. (II)) derive only words with periods of the same type.

We check first, by Lemma 4.2, that the initial messes of $L(G)$ and $L(H)$ are the same and that for any initial part P_1, period Q and final part RP_2 the intersections of $P_1Q^{2l}Q^*RP_2$ with the main bodies of $L(G)$ and $L(H)$ are either both empty or both infinite. (These intersections are EOL languages.)

Let L_1 and L_2 be the subsets of the main bodies of $L(G)$ and $L(H)$, respectively, the periods of the words of which belong to class (II). It follows easily from Lemmata 4.4–5 that L_1 and L_2 are regular and hence we may assume that $L_1 = L_2$.

Let us assume that $P_1Q^iRP_2$ with initial part P_1, period Q and final part RP_2 is a word (or one of them, if any) in the main body of $L(G)$ which is not in $L(H)$ and has the smallest value of i among all such words. Necessarily Q belongs to class (I). We shall show that i does not exceed a certain computable upper bound.

There is a computable natural number $r \geqslant 1$ such that for any $U_1V^kWU_2$ (initial part U_1, period V and final part WU_2) in the main body of $L(G)$, with k greater than a computable lower bound, also the words $U_1V^{k\pm r}WU_2$ are in this mean body (cf. the proof of Lemma 4.4). Now denote

$$u = \text{l.c.m. } \{n \mid T = (T^{1/2})^n \in \sigma(Q') \text{ and } Q' \text{ is a period of class (I) in the main body of } L(H)\}$$

(cf. Lemma 4.3). Thus, if i exceeds a certain computable bound, we know that $P_1Q^{i-ur}RP_2$ is in the main body of $L(G)$ and hence also in the main body of $L(H)$ by the minimality of i. Let $P_1Q^{i-ur}RP_2$ be directly derived by H from the word $\bar{P}_1\bar{Q}^j\bar{R}\bar{P}_2$ (initial part $\bar{P}_1$, period $\bar{Q}$ and final part $\bar{R}\bar{P}_2$) belonging to the main bodies of $L(G)$ and $L(H)$ (we may again assume i to exceed some computable bound and obviously $j \leqslant i - ur$; cf. Lemma 4.1). For any natural number v such that $T = (T^{1/2})^v \in \sigma(\bar{Q})$ we may assume that $\bar{P}_1\bar{Q}^{j+ur/v}\bar{R}\bar{P}_2$ is in the main body of $L(G)$. Now, if $j + (u/v)r < i$, $\bar{P}_1\bar{Q}^{j+ur/v}\bar{R}\bar{P}_2$ belongs to the main body of $L(H)$, too, and hence $P_1Q^iRP_2 \in L(H)$ and we are through. On the other hand, if $j + (u/v)r = i$, i.e. $j = i - ur$ and $v = 1$, either $\bar{P}_1\bar{Q}^i\bar{R}\bar{P}_2 \in L(H)$ (whence also $P_1Q^iRP_2 \in L(H)$ and again we are through) or $\bar{P}_1\bar{Q}^i\bar{R}\bar{P}_2 \notin L(H)$ in which case we repeat the above argumentation starting from $\bar{P}_1\bar{Q}^i\bar{R}\bar{P}_2$. The process started will eventually stop and we obtain a computable upper bound for i. ∎

Let l be a natural number. We say that a OL system $H = (A, \sigma, \alpha)$ is *doubly l-bounded* if

(i) $L(H) = L \cup F$ where F is finite and L is included in the union of all languages

$$P_1Q_1^{2l}Q_1^*R_1P_2R_2Q_2^{2l}Q_2^*P_3$$

where $P_1, Q_1, P_2, Q_2, P_3 \in A^* - A^lA^+$; $Q_1, Q_2, P_2 \neq \Lambda$; $R_1 \in Q_1/A^+$; $R_2 \in A^+\backslash Q_2$ and—as in the definition of l-boundedness—we require that P_1, Q_1, P_2, Q_2 and P_3 are the shortest possible (especially we require that, even after including all possible symbols of P_2 in the 'periodic' parts, P_2 must be nonempty); we call P_1 the *initial part*, Q_1 and Q_2 the *periods*, $R_1P_2R_2$ the *middle part* and P_3 the *final part* and these are uniquely determined;

(ii) for each $P_1, Q_1, R_1, P_2, R_2, Q_2, P_3$ satisfying the requirements of (i), the intersection

$$K \underset{\text{def}}{=} L \cap P_1Q_1^{2l}Q_1^*R_1P_2R_2Q_2^{2l}Q_2^*P_3$$

is either empty or then both $K/P_2R_2Q^*P_3$ and $P_1Q^*R_1P_2\backslash K$ are infinite;

(iii) only symbols appearing in the words of F but not in words of L or in the middle parts of words of L but not in other parts of these words may have nondeterministic productions.

The *initial mess* of $L(H)$ equals the smallest F and *the main body* of $L(H)$ is the largest L.

LEMMA 4.6. *There is an algorithm which of any natural number l and OL system H decides whether or not H is doubly l-bounded and in the positive case produces the initial mess and those initial parts P_1, periods Q_1 and Q_2, middle parts $R_1P_2R_2$ and final parts P_3 for which*

$$L(H) \cap P_1Q_1^{2l}Q_1^*R_1P_2R_2Q_2^{2l}Q_2^*P_3$$

is infinite.

Proof. The proof is rather the same as that of Lemma 4.2. To check (ii) we need to know that EOL languages are effectively closed under quotients with regular languages. ∎

LEMMA 4.7. *Let $H = (A, \sigma, \alpha)$ be a doubly l-bounded OL system for some natural number l. Then for each word in the main body of $L(H)$ with periods Q_1 and Q_2 there are uniquely determined words $\bar{Q}_1$ and $\bar{Q}_2$ such that $\sigma(Q_1) = \{\bar{Q}_1^{i_1}\}$ and $\sigma(Q_2) = \{\bar{Q}_2^{i_2}\}$ for some natural numbers $i_1, i_2 \geqslant 1$ and $\bar{Q}_1 = Q_1'Q_1''$, $\bar{Q}_2 = Q_2'Q_2''$ where $Q_1''Q_1'$ and $Q_2''Q_2'$ are periods of some word in the main body of $L(H)$.*

Proof. The proof goes in the lines of that of Lemma 4.3 and is left to the reader. ▮

It is now easy to see that only symbols of middle parts (or words in the initial mess) can directly derive symbols of the P_2's in the middle parts: Because of the minimality of P_2's, symbols of periodical parts can not contribute to them in direct derivations. By Lemma 4.7, symbols of middle parts can not contribute to initial parts or final parts (what we need here is that periodical parts are not erased). Therefore, by (iii) and the lemma, the initial parts and the final parts follow each other periodically in successive derivations and so do also the periods. If then an initial part, say, derives symbols of a middle part, a contradiction with (ii) immediately arises.

In analogy to Theorem 4.1 we have now

THEOREM 4.2. *There is an algorithm which given any natural number l, a doubly l-bounded OL system H and a doubly l-bounded DOL system G, such that $E(G)$ grows linearly, decides whether $L(H) = L(G)$.*

Proof. Denote $G = (A, \delta, \omega)$ and $H = (A, \sigma, \alpha)$.

In view of our basic goal, that is to reduce the OL-DOL equivalence problem to the FDOL-DOL equivalence problem, the case where only finitely many words of $L(H)$ contain occurrences of symbols having nondeterministic productions in H does not actually interest us because it can be treated as in the proof of Theorem 3.1. A reader wishing to make the proof of the present theorem complete in this respect may verify that this case can be proved much in the same way as in the proof of Theorem 4.1 and the rest of this proof.

Thus we may assume that certain symbols in some middle parts of words of the main body of $L(H)$ have nondeterministic productions in H. This means, of course, that these symbols can not appear in any initial parts, periods or final parts of words in the main body of $L(H)$. Since we may assume that the initial messes of $L(G)$ and $L(H)$ are the same and that for any initial part P_1, periods Q_1 and Q_2, middle part $R_1P_2R_2$ and final part P_3 the intersections of

$$P_1Q_1^{2l}Q_1^*R_1P_2R_2Q_2^{2l}Q_2^*P_3$$

with the main bodies of $L(G)$ and $L(H)$ are either both empty or both infinite (cf. Lemma 4.6), we know in fact that *all* middle parts of words of the main bodies of $L(H)$ (and $L(G)$) contain occurrences of symbols which do not appear elsewhere in the words of the main body (cf. also the second and the fifth properties of linearly growing DOL sequences listed at the beginning of the next section). Using these symbols as a 'spine' (see Herman and Rozenberg, 1975) and proceeding as in the proof of Lemma 4.4 we can effectively find natural numbers $r_1 \geqslant 1$ $r_2 \geqslant 1$ such that for any word $P_1Q_1^{j_1}R_1P_2R_2Q_2^{j_2}P_3$ (with obvious notation) in the main body of $L(G)$, where j_1 and j_2 exceed a certain computable bound, also

$$P_1 Q_1^{j_1+r_1} R_1 P_2 R_2 Q_2^{j_2+r_2} P_3 \quad \text{and} \quad P_1 Q_1^{j_1-r_1} R_1 P_2 R_2 Q_2^{j_2-r_2} P_3$$

are in this main body.

Since symbols of periods of words in the main body of $L(H)$ have deterministic productions it follows that there exists a natural number $t \geqslant 1$ such that

$$[(\sigma^t(Q))^{1/2}] = [Q]$$

for any such period Q (cf. Lemma 4.7). Let now

$$X \underset{\text{def}}{=} P_1 Q_1^{j_1} R_1 P_2 R_2 Q_2^{j_2} P_3$$

(with obvious notation) be a word of the main body of $L(H)$. If $X \in L(G)$, then the ratio

$$\max(j_1, j_2)/(\min(j_1, j_2) + 1)$$

is bounded (cf. the fourth property of linearly growing DOL sequences in the beginning of Section 5). On the other hand, if

$$\sigma^t(Q_1) = ((\sigma^t(Q_1))^{1/2})^{i_1} \quad \text{and} \quad \sigma^t(Q_2) = ((\sigma^t(Q_2))^{1/2})^{i_2}$$

and $i_1 \neq i_2$, some such ratios certainly are not bounded. Thus $i_1 = i_2$.

Now let

$$Y \underset{\text{def}}{=} P_1 Q_1^{s_1} R_1 P_2 R_2 Q_2^{s_2} P_3$$

(with obvious notation) be a word in the symmetric difference of $L(G)$ and $L(H)$ with the least value of $\max(s_1, s_2)$ (or one of them, if any). We shall show that $\min(s_1, s_2)$ does not exceed a certain computable upper bound.

Suppose first that $Y \in L(H)$, let it be derived in t steps from

$$\bar{P}_1 \bar{Q}_1^{u_1} \bar{R}_1 \bar{P}_2 \bar{R}_2 \bar{Q}_2^{u_2} \bar{P}_3 \in L(H)$$

(with obvious notation) by H. Then

$$[Q_1] = [\bar{Q}_1], [Q_2] = [\bar{Q}_2], u_1 \leqslant s_1, u_2 \leqslant s_2$$

and

$$\frac{|\sigma^t(\bar{Q}_1)|}{|\bar{Q}_1|} = \frac{|\sigma^t(\bar{Q}_2)|}{|\bar{Q}_2|} \underset{\text{def}}{=} i.$$

Arguing as in the proof of Theorem 4.1 we may assume that $\max(u_1, u_2) < \max(s_1, s_2)$. Then, if s_1 and s_2 are 'large enough', we know that

$$\bar{P}_1 \bar{Q}_1^{u_1-r_1} \bar{R}_1 \bar{P}_2 \bar{R}_2 \bar{Q}_2^{u_2-r_2} \bar{P}_3 \in L(G) \cap L(H).$$

But then also

$$P_1Q_1^{s_1-ir_1}R_1P_2R_2Q_2^{s_2-ir_2}P_3 \in L(G) \cap L(H).$$

Thus if s_1 and s_2 exceed a certain computable bound we know that $P_1Q_1^{s_1}R_1P_2R_2Q_2^{s_2}P_3 \in L(G)$.

Suppose then that $Y \in L(G)$. If s_1 and s_2 are 'large enough', then we know that

$$P_1Q_1^{s_1-jr_1}R_1P_2R_2Q_2^{s_2-jr_2}P_3 \in L(G) \cap L(H),$$

where j equals the l.c.m. of the numbers $|\sigma^t(Q)|/|Q|$ (Q is a period of some word in the main body of $L(H)$), let it be derived in t steps from

$$\tilde{P}_1\tilde{Q}_1^{v_1}\tilde{R}_1\tilde{P}_2\tilde{R}_2\tilde{Q}_2^{v_2}\tilde{P}_3 \in L(G) \cap L(H)$$

(with obvious notation) by H. Clearly $v_1 \leqslant s_1 - jr_1$ and $v_2 \leqslant s_2 - jr_2$. But then we may assume that

$$\tilde{P}_1\tilde{Q}_1^{v_1+jr_1/h}\tilde{R}_1\tilde{P}_2\tilde{R}_2\tilde{Q}_2^{v_2+jr_2/h}\tilde{P}_3 \in L(G) \cap L(H),$$

where

$$\frac{|\sigma^t(\tilde{Q}_1)|}{|\tilde{Q}_1|} = \frac{|\sigma^t(\tilde{Q}_2)|}{|\tilde{Q}_2|} = h,$$

whence also $P_1Q_1^{s_1}R_1P_2R_2Q_2^{s_2}P_3 \in L(H)$ (the case where

$$\max(v_1 + jr_1/h, v_2 + jr_2/h) = \max(s_1, s_2)$$

can be dismissed as the corresponding case in the proof of Theorem 4.1).

So we obtain a computable upper bound for $\min(s_1, s_2)$. The emptiness of the symmetric difference of $L(G)$ and $L(H)$ is then easily decided using this upper bound since the set of possible words Y in the main body of $L(G)$ is finite and straightforward to obtain (consult the fourth property of linearly growing DOL sequences at the beginning of the next section) and the corresponding set for $L(H)$ is an EOL language. ▌

5. Reductions in the Case of Linear Growth

We first state some well known characterization results for linearly growing DOL sequences; cf. e.g. Karhumäki (1976). Let $G = (A, \delta, \omega)$ be a DOL system such that $E(G)$ grows linearly. Then

(i) for some natural numbers m and k we can write

$$\delta^n(\omega) = P_1^{(n)}Q_1^{(n)}P_2^{(n)}Q_2^{(n)} \cdots P_{k-1}^{(n)}Q_{k-1}^{(n)}P_k^{(n)} \qquad (n \geqslant m)$$

where

(ii) all symbols of A the numbers of occurrences of which are bounded in words of $L(G)$ appear only in the words $P_1^{(n)},..., P_k^{(n)}$, the endmost symbols of these words being such symbols (with the possible exception of the initial symbol of $P_1^{(n)}$ and/or the final symbol of $P_k^{(n)}$);

(iii) the lengths of $P_1^{(n)},..., P_k^{(n)}$ are bounded and the words $P_2^{(n)},..., P_{k-1}^{(n)}$ are nonempty whereas $P_1^{(n)}$ and $P_k^{(n)}$ can be empty;

(iv) the sequences $(|\, Q_i^{(n)} \,|)$ are not bounded, furthermore there are positive numbers r_1 and r_2 such that

$$r_1 \leqslant \frac{|\, Q_i^{(n)} \,|}{|\, Q_j^{(n)} \,|} \leqslant r_2 \qquad (n \geqslant m \text{ and } i, j = 1,..., k - 1);$$

(v) $P_i^{(n+1)}$ is a subword of $\delta(P_i^{(n)})$ and $\delta(Q_i^{(n)})$ is a subword of $Q_i^{(n+1)}$; the sequences $(P_i^{(n)})$ are periodical;

(vi) each symbol a occurring in the words $Q_i^{(n)}$ is nongrowing, i.e. the sequence $(|\, \delta^n(a)|)$ is bounded, and derives either Λ or an occurrence of itself in some number of steps.

We call the words $P_i^{(n)}$ and $Q_i^{(n)}$ the *bones* and the *flesh* of $\delta^n(\omega)$, respectively. It should be mentioned that these characterization results are effective.

An easy lemma shows that if a DOL system G, such that $E(G)$ grows linearly, is equivalent to some OL system H then either words of $L(G)$ do not have 'too many' bones or otherwise H is 'almost' deterministic.

LEMMA 5.1. *Let G be a DOL system such that $E(G)$ grows linearly. If $L(G) = L(H)$ for some OL system H, such that symbols having nondeterministic productions in H occur in infinitely many words of $L(H)$, then words of $L(G)$ have at most three bones. Moreover, in the case where there are three bones symbols having nondeterministic productions in H can appear only in the middle bone.*

Proof. Let $G = (A, \delta, \omega)$. Suppose that some $\delta^n(\omega)$ has at least three bones and that some symbol in its flesh, say in $Q_1^{(n)}$, has a nondeterministic production in H. Then for some words $\delta^{n'}(\omega)$ and $\delta^{n''}(\omega)$ we have $P_i^{(n')} = P_i^{(n'')}$ $(i = 1,..., k)$, $Q_i^{(n')} = Q_i^{(n'')}$ $(i = 2,..., k - 1)$ and $Q_1^{(n')} \neq Q_1^{(n'')}$. By (v) this implies that $(|\, Q_2^{(n)} \,|)$ is bounded, a contradiction with (iv).

A similar argument proves the rest of the lemma. (Note: If symbols in the flesh have only deterministic productions in H, then it is not the case that arbitrarily long subwords of the flesh can be erased by H, as is easily seen, cf. the proof of Lemma 4.3. This fact guarantees that the characterization given in (i)–(vi) can be used.) ∎

The main theorem in this section effectively reduces the case of linear growth to bounded or doubly bounded systems.

THEOREM 5.1. *Let G be a DOL system, such that $E(G)$ grows linearly, and let H be a OL system such that symbols having nondeterministic productions occur in infinitely many words of $L(H)$. If $L(G) = L(H)$, then G and H are l-bounded or doubly l-bounded for some effectively computable natural number l. (In other words, there is an algorithm which starting with G and H either produces the result $L(G) \neq L(H)$ or then a natural number l such that if $L(G) = L(H)$ then G and H are l-bounded or doubly l-bounded.)*

Proof. Let $G = (A, \delta, \omega)$. We distinguish between three cases (cf. Lemma 5.1).

(1) Assume first that words of $L(G)$ have two bones and some symbol, say a, having a nondeterministic production in H appears in the flesh of infinitely many words of $L(G)$. Let $\delta^n(\omega) \in L(G)$ be a word containing s occurrences of a in its flesh. By enlargening n we may assume s to be as large as we please. Let

$$a \mapsto_H T_1 \qquad \text{and} \qquad a \mapsto_H T_2$$

be two different productions for a in H. We may assume T_1 to be nonempty.

We now apply the production $a \mapsto_H T_1$ to all occurrences of a in the flesh of $\delta^n(\omega)$ (which by our assumption is a word of $L(H)$, too) except for one occurrence to which we apply the production $a \mapsto_H T_2$. As a result we get s words of $L(H)$ (and $L(G)$) of equal length. Since the number of words of equal length in $L(G)$ is bounded by some computable constant c depending on A only (see Lee and Rozenberg, 1974) we see that $|T_1| \neq |T_2|$ (we may assume that $|T_1| > |T_2|$) and that at least entier(s/c) of these words are equal. So, for some words U, V, $W \in A^*$,

$$UT_1VT_2W = UT_2VT_1W \in L(G)$$

and we may assume V to be as long as we wish. But this implies that V is periodic with period $X =_{\text{def}} (T_2\backslash T_1)^{1/2}$, say $V = X^tY$ where $Y \in X/A^+$. Also the word UT_1VT_1W is in $L(G)$. Thus we have in $L(G)$ the words

$$UT_2X^tYT_1W \qquad \text{and} \qquad UT_2X^{t+q}YT_1W \tag{1}$$

for a natural number t which we can choose to be arbitrarily large and a natural number q such that $T_2\backslash T_1 = X^q$.

For the rest of this part of the proof we consider $E(G)$ only. Denote by B the subalphabet of A which consists of all symbols b such that $\delta^n(b) \neq \Lambda$ for all natural numbers n. Further denote by γ the morphism $A^* \to B^*$ given by

$$\gamma(d) = \begin{cases} d, & \text{if } d \in B \\ \Lambda, & \text{if } d \in A - B. \end{cases}$$

Denote $\bar{P} = \gamma(P)$ $(P \in A^*)$. Then $\bar{\delta} =_{\text{def}} \gamma\delta$ is an endomorphism on B^* and $\bar{G} = (B, \bar{\delta}, \bar{\omega})$ is a PDOL system. Furthermore

$$\bar{\delta}^n(\bar{\omega}) = \gamma\delta^n(\omega) \qquad (n = 0, 1, \ldots) \tag{2}$$

and for some natural number m

$$\delta^n(\omega) = \delta^m \bar{\delta}^{n-m}(\bar{\omega}) \qquad (n \geqslant m). \tag{3}$$

Clearly, by (3), it suffices to show that $\bar{G}$ is l-bounded for some natural number l. By (1) and (2) we have in $L(\bar{G})$ the words

$$\bar{U}\bar{T}_2\bar{X}^t\bar{Y}\bar{T}_1\bar{W} \qquad \text{and} \qquad \bar{U}\bar{T}_2\bar{X}^{t+q}\bar{Y}\bar{T}_1\bar{W},$$

let them be $\bar{\delta}^{n_1}(\bar{\omega})$ and $\bar{\delta}^{n_2}(\bar{\omega})$, respectively. Note that $\bar{X} \neq \Lambda$ because, as is well known, the number of consecutive symbols of $A - B$ in the words of $L(G)$ is effectively bounded and we may let t be larger than this bound. Obviously $n_2 > n_1$.

By (vi) and the fact that $\bar{G}$ is propagating each symbol in the flesh of words of $L(\bar{G})$ derives directly exactly one symbol (clearly $E(\bar{G})$ grows linearly). Since we may choose t to be as large as we wish, it follows that we may write $\bar{\delta}^{n_2-n_1}(\bar{X}) = X_1X_2$ where $\bar{X} = X_2X_1$. But then

$$\bar{\delta}^{s(n_2-n_1)}(\bar{U}\bar{T}_2\bar{X}^{t+q}\bar{Y}\bar{T}_1\bar{W}) = \bar{U}\bar{T}_2\bar{X}^{t+(s+1)q}\bar{Y}\bar{T}_1\bar{W}$$

($s = 0, 1, \ldots$) whence $L(\bar{G})$ is l-bounded for some natural number l. The computability of l should be obvious although it may be somewhat tedious to obtain explicit bounds.

(2) In our second case we assume that words of $L(G)$ have two bones but only in a finite number of words symbols of flesh have nondeterministic productions in H. Thus in infinitely many words of $L(G)$ the second bone, say, contains a symbol which has a nondeterministic production in H.

So, for some natural number p we have in $L(G)$ the words $\delta^{n_1}(\omega)$ and $\delta^{n_2}(\omega)$ ($n_1 \neq n_2$) where

$$Q_1^{(n_1)} = QS_1 \qquad \text{and} \qquad Q_1^{(n_2)} = QS_2$$

and $|S_1|$, $|S_2| \leqslant p$ and we may assume Q to be as long as we please (cf. the note in the proof of Lemma 5.1).

Again, as in (1), we consider $\bar{G}$ instead of G and show that $\bar{G}$ is l-bounded for some computable natural number l. In $L(\bar{G})$ we have the words $\bar{\delta}^{n_1}(\bar{\omega})$ and $\bar{\delta}^{n_2}(\bar{\omega})$ which are not identical (otherwise $L(\bar{G})$ is finite). We may assume that $n_2 > n_1$. Now, instead of n_2 we may consider any number $n_2 + j(n_2 - n_1)$ ($j \geqslant 0$). So we choose j in such a way that $\bar{\delta}^{(j+1)(n_2-n_1)}(a) = a$ for each symbol a in the flesh of words of $L(\bar{G})$ and that the bones of $\bar{\delta}^{n_1}(\bar{\omega})$ and $\bar{\delta}^{n_2'}(\bar{\omega})$, $n_2' = n_2 + j(n_2 - n_1)$, are the same.

Then, if $\bar{\delta}^{n_2'-n_1}(\bar{P}_1^{(n_1)}) = \bar{P}_1^{(n_1)}$ we are through. So let us assume that

$$\bar{\delta}^{n_2'-n_1}(\bar{P}_1^{(n_1)}) = \bar{P}_1^{(n_1)}R_1, \qquad \bar{\delta}^{n_2'-n_1}(\bar{P}_2^{(n_1)}) = R_2\bar{P}_2^{(n_1)}$$

where $R_1 \neq \Lambda$. We first observe that then

$$\delta^{n_2'}(\bar{\omega}) = \bar{P}_1^{(n_1)} R_1 \bar{Q} \bar{S}_1 R_2 \bar{P}_2^{(n_1)} = \bar{P}_1^{(n_1)} \bar{Q} \bar{S}_2 \bar{P}_2^{(n_1)} \tag{4}$$

and since we can choose $\bar{Q}$ to be as long as we please we have $\bar{Q} = R_1{}^u R$, $R \in R_1/A^+$, for some natural number u. Further we see that

$$\delta^{n_1+i(n_2'-n_1)}(\bar{\omega}) = \bar{P}_1^{(n_1)} R_1^{u+i-1} R \bar{S}_2 R_2^{i-1} \bar{P}_2^{(n_1)} \qquad (i \geqslant 1).$$

If $R_2 = \Lambda$ we are through. So suppose that $R_2 \neq \Lambda$. Being able to choose the length of Q as large as we please, we note that at some later stage in the sequence $E(\bar{G})$ (4) will be valid. But this implies that $R_1^{u+i-1} R \bar{S}_2 R_2^{i-1}$ is periodical with period $R_1^{1/2}$ and, by Lemma 4.1, that $| R_1^{1/2} | = | R_2^{1/2} |$. Hence $L(\bar{G})$ is l-bounded for some natural number l. Again it is obvious that l can be computed effectively.

(3) In the last case we assume that words of $L(G)$ have three bones. This case is rather analogous to (2) and so we leave it to the reader. ▌

6. Conclusions

Combining the results of the previous sections we are able to present.

Theorem 6.1. *Suppose there is an algorithm for deciding the FDOL-DOL equivalence problem. Then there is an algorithm which of any OL system H and any DOL system G decides whether $L(H) = L(G)$.*

Proof. If only finitely many words of $L(H)$ contain occurrences of symbols having nondeterministic productions in H, we proceed as in the proof of Theorem 3.1. In the remaining cases we first decide whether or not $E(G)$ grows linearly (there are well known algorithms for this). If the answer is positive we use Theorems 4.1–2 and 5.1 and Lemmata 4.2 and 4.6. On the other hand, if the answer is negative we use Theorem 3.1. ▌

Corollary 6.1. *The FOL-DOL equivalence problem and the FDOL-DOL equivalence problem are either both decidable or both undecidable.*

Proof. Let $H = (A, \sigma, X)$ be an FOL system and $G = (A, \delta, \omega)$ be a DOL system. Define $H' = (A \cup \{c\}, \sigma', c)$ and $G' = (A \cup \{c\}, \delta', c)$, where $c \notin A$, by $\sigma'(a) = \sigma(a)$ for $a \in A$, $\sigma'(c) = X$ and $\delta'(a) = \delta(a)$ for $a \in A$, $\delta'(c) = \omega$. Then $L(H) = L(G)$ iff $L(H') = L(G')$. ▌

As another corollary we get the result that 'DOLness' of OL languages is decidable if the FDOL-DOL equivalence problem is decidable.

COROLLARY 6.2. *Suppose that the FDOL-DOL equivalence problem is decidable. Then there is an algorithm which of any OL system H decides whether or not there exists a DOL system G such that $L(H) = L(G)$ and in the positive case produces all such DOL systems (there will be a finite number of them).*

Proof. The following well known observation is a basic one for us. For any sequence (ω_n) of words over the alphabet A with cardinality k each of the sets $\{\omega_0, ..., \omega_l\}$ $(l \geqslant k)$ determines effectively a finite set of DOL systems which includes all systems G, if any, such that $(\omega_n) = E(G)$.

Let $H = (A, \sigma, \alpha)$. The case where $L(H)$ is finite is more or less trivial and is omitted.

For any subalphabet B of A we denote by γ_B the morphism $A^* \to (A - B)^*$ given by

$$\gamma_B(a) = \begin{cases} a, & \text{if } a \in A - B \\ \Lambda, & \text{if } a \in B. \end{cases}$$

We are interested in subalphabets B of A such that

(i) $\gamma_B(L(H))$ is infinite;

(ii) the set

$$K_B \underset{\text{def}}{=} L(H) \cap \left(\bigcup_{i=0}^{m} (B^*(A - B))^i B^*\right),$$

where m is the smallest number such that the cardinality of $\gamma_B(L(H)) - (A - B)^m(A - B)^+$ exceeds k, is finite.

Note that both (i) and (ii) are effectively testable and K_B can be found effectively.

Suppose now that $L(H) = L(G)$ for some DOL system $G = (A, \delta, \omega)$. Let B be the set of those symbols b of A for which $\Lambda \in \{\delta(b), \delta^2(b), ...\}$. Then necessarily (i) and (ii) hold true. On the other hand, it is evident that the $k + 1$ first words of $E(G)$ must be in K_B and K_B is some initial segment of $E(G)$ (because the system $(A - B, \gamma_B\delta, \gamma_B(\omega))$ is propagating).

Thus we simply go through all subalphabets B of A (including $B = \varnothing$) satisfying (i) and (ii), construct K_B and the DOL systems given by this set as indicated above and use Theorem 6.1 to find those of these DOL systems (if any) that are equivalent to H. ∎

A similar problem, namely the problem of 'DOLness' of context-free languages, is shown to be decidable in Linna (1976) and our techniques in the above proof bear resemblance to those used there.

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