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Transformational Systems and the Algebraic Structure of Atomic Formulas

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Abstract

If the set of atomic formulas is augmented by adding a 'universal formula' and a 'null formula', then the equivalence classes of this set under alphabetic variation form a complete non-modular lattice, with 'instance' as the partial ordering, 'greatest common instance' as the meet operation, and 'least common generalization' as the join operation. The greatest common instance of two formulas can be obtained from Robinson's Unification Algorithm. An algorithm is given for computing the least common generalization of two formulas, the covering relation of the lattice is determined, bounds are obtained on the length of chains from one formula to another, and it is shown that any formula is the least common generalization of its set of ground instances.

A transformational system is a finite set of clauses containing only units and transformations, which are clauses containing exactly one positive and one negative literal. It is shown that every unsatisfiable transformational system has a refutation where every resolution has at least one resolvend which is an initial clause. An algorithm is given for computing a common generalization of all atomic formulas which can be derived from a transformational system, and it is shown that there is no decision procedure for transformational systems.

INTRODUCTION

This paper is a collection of theoretical properties of the entities and operations used in the resolution approach to mechanical theorem-proving (Robinson 1965). Hopefully, this material will eventually be helpful in the development of more efficient proof procedures, but we are presently unable to formulate a complete procedure which takes full advantage of our results.

These results fall in two separate areas. The first is the algebraic structure of atomic formulas under instantiation. Robinson's Unification Algorithm allows the computation of the greatest common instance of any finite set of unifiable atomic formulas. This suggests the existence of a dual operation of 'least common generalization'. It turns out that such an operation does exist and can be computed by a simple algorithm.

As a result, if one adds a 'universal atomic formula' (whose ground instances are all ground formulas) and a 'null atomic formula' (with no ground instances), then the set of atomic formulas (more precisely, the equivalence classes of atomic formulas under alphabetic variation) forms a complete non-modular lattice, with 'instance' as the partial ordering, 'greatest common instance' as the meet operation, and 'least common generalization' as the join operation. The covering relation in this lattice determines the set of 'closest' instances of an atomic formula, and bounds can be obtained on the length of chains from one atomic formula to another.

The second area is the properties of transformational systems, which are sets of clauses containing only units and mixed two-clauses (clauses with one positive and one negative literal). The refutation of transformational systems seems to be the simplest non-trivial case of theorem-proving. Its simplicity is that a search for a refutation can be limited to resolutions in which one resolvend is a unit and the other is an initial clause; in effect one is doing path-searching rather than tree-searching. The non-triviality is that Church's theorem still holds: by an appropriate mapping of Post's correspondence problem it can be shown that no decision procedure exists for transformational systems.

A connection between these areas is provided by the least common generalization. It can be used to examine a subset S of a transformational system and to compute a single unit which is a 'super-consequence' of S. This unit is not necessarily a valid consequence of S, but each of the (usually infinite number of) consequences must be an instance of the super-consequence. Thus when the super-consequence can be shown to be irrelevant, all of the consequences of S must be irrelevant, and the generation of an infinite sequence of useless clauses can be avoided.

Throughout this paper, we will use the definitions and results given in Robinson's original paper on resolution (Robinson 1965). Also, a variety of well-known properties of lattices will be stated without proof; these properties are discussed in the opening chapters of a standard text such as Birkhoff (1967).

THE LATTICE STRUCTURE OF ATOMIC FORMULAS

In this section we will show that the operation of instantiation induces a lattice-like structure on the set of atomic formulas, and we will examine various properties of this structure. First, we must generalize the notion of an atomic formula to include a *universal formula* $\mathscr{A}$ and a *null formula* Ω; these formulas will be the greatest and least elements of the lattice.

Definition. A *generalized atomic formula* (GAF) is either a conventional atomic formula (CAF) as defined in Robinson (1965), or one of the special symbols $\mathscr{A}$, Ω. A GAF is called a *ground GAF* iff it is a CAF and it contains no occurrences of variables.

Given GAFs A and B, we write $A \geqslant B$ (read A is a *generalization* of B or B is an *instance* of A) iff A is $\mathscr{A}$, or B is Ω, or A and B are both CAFs and there exists a substitution θ such that $B = A\theta$. If $A \geqslant B$ and $B \geqslant A$, we write $A \simeq B$ (read A and B are *variants* or A and B are *equivalent*). If $A \geqslant B$ and not $B \geqslant A$, we write $A > B$ (read A is a *proper generalization* of B or B is a *proper instance* of A).

(We assume that the set of conventional atomic formulas is generated from a fixed but unspecified vocabulary containing at least one constant, one unary function symbol, and one binary predicate symbol. A second, distinct predicate symbol will be required in Theorem 8; a binary function symbol will be required in Theorem 11.)

Corollary 1

The relation $\geqslant$ is a quasi-ordering in which $\mathscr{A}$ and Ω are unique greatest and least elements, i.e., for all GAFs A, B, and C:

$A \geqslant A$
$A \geqslant B$ and $B \geqslant C$ implies $A \geqslant C$
$\mathscr{A} \geqslant A$
$A \geqslant \Omega$
$A \geqslant \mathscr{A}$ iff $A = \mathscr{A}$
$\Omega \geqslant A$ iff $A = \Omega$.

Lemma 1

$A \simeq B$ iff one of the following cases occurs:

(1) $A = B = \mathscr{A}$.

(2) $A = B = \Omega$.

(3) A and B are CAFs and there is a substitution θ such that: $B = A\theta$, for all variables X occurring in A, $X\theta$ is a variable, and for all pairs X, Y of distinct variables occurring in A, $X\theta \neq Y\theta$.

Proof. If either A or B is not a CAF, the lemma is trivial. If A and B are CAFs and $A \simeq B$, then there are substitutions θ and ψ such that $B = A\theta$ and $A = B\psi = A\theta\psi$. Then if X is any variable occurring in A, $X\theta\psi = X$, so that $X\theta$ must be a variable. If X and Y are distinct variables occurring in A, then $X\theta\psi = X$ and $Y\theta\psi = Y \neq X\theta\psi$, so that $X\theta \neq Y\theta$.

On the other hand, if $B = A\theta$, where θ meets condition (3), let $X_1, \ldots, X_n$ be the variables occurring in A. Then $\psi = \{X_1/X_1\theta, \ldots, X_n/X_n\theta\}$ is a substitution, and $B\psi = A\{X_1\theta/X_1, \ldots, X_n\theta/X_n\}\psi = A$. Thus $A \simeq B$.

Definition. Let S be a set of GAFs. If I is a GAF such that, for all $A \in S$, $I \leqslant A$, then I is a *common instance* of S. If I is a common instance of S and, for all common instances I' of S, $I \geqslant I'$, then I is a *greatest common instance* of S. If G is a GAF such that, for all $A \in S$, $G \geqslant A$, then G is a *common* *generalization* of S. If G is a common generalization of S and, for all common generalizations G' of S, $G \leqslant G'$, then G is a *least common generalization* of S.

It is easily shown that:

Corollary 2

$\mathscr{A}$ and Ω are the only greatest common instance and least common generalization of the empty set. Ω and $\mathscr{A}$ are the only greatest common instance and least common generalization of the set of all GAFS.

If A is a greatest common instance (least common generalization) of S, then B is a greatest common instance (least common generalization) of S iff $B \simeq A$.

We now define two total, computable binary functions $\sqcap$ and $\sqcup$, and show that these functions produce a greatest common instance and a least common generalization of any pair of GAFS.

Definition. Given GAFS A and B, we define the GAF $A \sqcap B$ as follows:

(1) If $A = \mathscr{A}$, then $A \sqcap B = B$.

(2) If $B = \mathscr{A}$, then $A \sqcap B = A$.

(3) If $A = \Omega$ or $B = \Omega$, then $A \sqcap B = \Omega$.

(4) If A and B are both CAFS, then let A' and B' be variants of A and B respectively (chosen in some standard manner) such that no variable occurs in both A' and B'. If A' and B' are not unifiable, then $A \sqcap B = \Omega$. Otherwise $A \sqcap B = A'\sigma = B'\sigma$, where σ is the most general unifier of A' and B' [obtained from the Unification Algorithm given in Robinson (1965)].

Then the Unification Theorem in Robinson (1965) has the following direct consequence:

Theorem 1

For all GAFS A, B, and C, $A \geqslant A \sqcap B$, $B \geqslant A \sqcap B$, and if $A \geqslant C$ and $B \geqslant C$, then $A \sqcap B \geqslant C$. Thus $A \sqcap B$ is a greatest common instance of $\{A, B\}$.

Definition. Given GAFS A and B, we define the GAF $A \sqcup B$ as follows:

(1) If $A = \Omega$, then $A \sqcup B = B$.

(2) If $B = \Omega$, then $A \sqcup B = A$.

(3) If $A = \mathscr{A}$, or $B = \mathscr{A}$, or A and B begin with distinct predicate symbols, then $A \sqcup B = \mathscr{A}$.

(4) If A and B are CAFS beginning with the same predicate symbol, then let $Z_1, Z_2, \ldots$ be a sequence of variables which do not occur in A or B, and obtain $A \sqcup B$ by the following *Anti-unification Algorithm*:*

(a) Set the variables $\bar{A}$ to A, $\bar{B}$ to B, ζ and η to the empty substitution, and i to zero.

(b) If $\bar{A} = \bar{B}$, exit with $A \sqcup B = \bar{A} = \bar{B}$.

(c) Let k be the index of the first symbol position at which $\bar{A}$ and $\bar{B}$ differ,

* This algorithm has been discovered independently by Mr Gordon Plotkin of the University of Edinburgh.

and let S and T be the terms which occur, beginning in the kth position, in $\bar{A}$ and $\bar{B}$ respectively.

(d) If, for some j such that $1 \leqslant j \leqslant i$, $Z_j\zeta = S$ and $Z_j\eta = T$, then alter $\bar{A}$ by replacing the occurrence of S beginning in the kth position by Z_j, alter $\bar{B}$ by replacing the occurrence of T beginning in the kth position by Z_j, and go to step (b).

(e) Otherwise, increase i by one, alter $\bar{A}$ by replacing the occurrence of S beginning in the kth position by Z_i, alter $\bar{B}$ by replacing the occurrence of T beginning in the kth position by Z_i, replace ζ by $\zeta \cup \{S/Z_i\}$, replace η by $\eta \cup \{T/Z_i\}$, and go to step (b).

Each iteration of the Anti-unification Algorithm will either produce the termination condition $\bar{A} = \bar{B}$, or else it will increase k without increasing the length of $\bar{A}$ or $\bar{B}$. Thus, since k cannot exceed the length of $\bar{A}$ or $\bar{B}$, the algorithm must always terminate. The following lemma may be proved by induction on the number of iterations:

Lemma 2

After each iteration of the Anti-unification Algorithm:

(1) $\bar{A}\zeta = A$ and $\bar{B}\eta = B$.

(2) Each of the variables $Z_1, \ldots, Z_i$ occurs in both $\bar{A}$ and $\bar{B}$, but only to the left of the first symbol position at which $\bar{A}$ and $\bar{B}$ differ.

(3) There exist terms $S_1, \ldots, S_i, T_1, \ldots, T_i$ such that:

(a) $\zeta = \{S_1/Z_1, \ldots, S_i/Z_i\}$.

(b) $\eta = \{T_1/Z_1, \ldots, T_i/Z_i\}$.

(c) For $1 \leqslant j \leqslant i$, S_j and T_j differ in their first symbols.

(d) For $1 \leqslant j, k \leqslant i$, if $j \neq k$ then either $S_j \neq S_k$ or $T_j \neq T_k$.

Theorem 2

For all GAFs A, B, and C, $A \sqcup B \geqslant A$, $A \sqcup B \geqslant B$, and if $C \geqslant A$ and $C \geqslant B$, then $C \geqslant A \sqcup B$. Thus $A \sqcup B$ is a least common generalization of $\{A, B\}$.

Proof. The theorem is non-trivial only in the case where A and B are both CAFs beginning with the same predicate symbol. In this case, $A \sqcup B$ is defined by the Anti-unification Algorithm, and part 1 of Lemma 2 implies that $A \sqcup B \geqslant A$ and $A \sqcup B \geqslant B$.

Now suppose C is a GAF such that $C \geqslant A$ and $C \geqslant B$, and let $D = C \sqcap (A \sqcup B)$. By Theorem 1, $C \geqslant D$, $(A \sqcup B) \geqslant D$, and, since A and B are both common instances of C and $(A \sqcup B)$, $D \geqslant A$ and $D \geqslant B$. Moreover, since D is both an instance of a CAF and a generalization of a CAF, D must be a CAF. Thus there exist substitutions θ, ψ, ϕ such that $D = (A \sqcup B)\theta$, $A = D\psi = (A \sqcup B)\theta\psi$, and $B = D\phi = (A \sqcup B)\theta\phi$.

Let $\zeta = \{S_1/Z_1, \ldots, S_i/Z_i\}$ and $\eta = \{T_1/Z_1, \ldots, T_i/Z_i\}$ be the final values of the variables ζ and η in the execution of the Anti-unification Algorithm used to compute $A \sqcup B$. By part 1 of Lemma 2, $(A \sqcup B)\zeta = A = (A \sqcup B)\theta\psi$, and $(A \sqcup B)\eta = B = (A \sqcup B)\theta\phi$. Thus if X is any variable occurring in $A \sqcup B$, then $X\zeta = X\theta\psi$ and $X\eta = X\theta\phi$.

We now use Lemma 1 to show that $D=(A\sqcup B)\theta$ is a variant of $A\sqcup B$. Let X be any variable which occurs in $A\sqcup B$. Then:

(1) If X is one of the Z_j, then $X\theta\psi=X\zeta=S_j$ and $X\theta\phi=X\eta=T_j$. If $X\theta$ were not a variable then S_j and T_j would begin with the same symbol, which contradicts part 3c of Lemma 2.

(2) If X is not one of the Z_j, then $X\theta\psi=X\zeta=X$, so that $X\theta$ must be a variable.

Let X and Y be any pair of distinct variables which occur in $A\sqcup B$. Then:

(1) If X is some Z_j and Y is some Z_k, then $X\theta\psi=S_j$, $Y\theta\psi=S_k$, $X\theta\phi=T_j$, and $Y\theta\phi=T_k$. By part 3d of Lemma 2, either $S_j\neq S_k$ or $T_j\neq T_k$. Either case implies that $X\theta\neq Y\theta$.

(2) If neither X nor Y is one of the Z_j, then $X\theta\psi=X$ and $Y\theta\psi=Y$, so that $X\theta\neq Y\theta$.

(3) If X is some Z_j and Y is not one of the Z_j, then $X\theta\psi=S_j$ and $X\theta\phi=T_j$, so that part 3c of Lemma 2 implies $X\theta\psi\neq X\theta\phi$. But $Y\theta\psi=Y\theta\phi=Y$. Thus $X\theta\neq Y\theta$.

(4) If Y is some Z_j and X is not one of the Z_j, the argument is similar to (3) with X and Y interchanged.

Thus, by Lemma 1, D is a variant of $A\sqcup B$. Then, since $C\geqslant D$, we have $C\geqslant A\sqcup B$.

Theorems 1 and 2 establish the existence of a lattice structure. However, since $\geqslant$ is a quasi-ordering rather than a partial ordering, the lattice elements are not the atomic formulas themselves, but the equivalence classes induced by the relation $\simeq$.

Corollary 3

The set of equivalence classes of GAFs under the relation $\simeq$ is a lattice, in which $\{\mathscr{A}\}$ is the greatest element, $\{\Omega\}$ is the least element, and (the appropriate generalizations to the equivalence classes of) $\geqslant$, $\sqcap$, and $\sqcup$ are the partial ordering, meet, and join operations. We call this lattice the *GAF lattice.*

The existence of the GAF lattice implies the following results, which hold for lattices in general:

Corollary 4

For all GAFs A, B, and C;

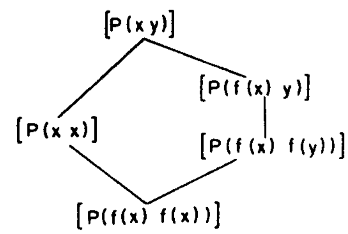
$$A\sqcap B\simeq B\sqcap A$$
$$(A\sqcap B)\sqcap C\simeq A\sqcap(B\sqcap C)$$
$$A\sqcup B\simeq B\sqcup A$$
$$(A\sqcup B)\sqcup C\simeq A\sqcup(B\sqcup C)$$
$$A\sqcap(B\sqcup C)\geqslant(A\sqcap B)\sqcup(A\sqcap C)$$
$$A\sqcup(B\sqcap C)\leqslant(A\sqcup B)\sqcap(A\sqcup C)$$
$$\text{If } A\leqslant C \text{ then } A\sqcup(B\sqcap C)\leqslant(A\sqcup B)\sqcap C.$$

Unfortunately, the equivalences which correspond to the last three statements of this corollary are (in general) false, since:

Theorem 3

The GAF lattice is non-modular.

Proof. Let P be a binary predicate symbol, let f be a unary function symbol, and let $[A]$ denote the equivalent class of A under $\simeq$. Then the GAF lattice contains the following non-modular sublattice:



In the remainder of this section we will determine various special properties of the GAF lattice. We begin by defining a relation $A \rightarrow B$ which will eventually be shown to be the covering relation of the lattice.

Definition. For GAFs A and B, the relation $A \rightarrow B$ holds iff one of the following cases occurs:

(1) A is $\mathscr{A}$ and B is a CAF containing no function symbols and no repeated occurrences of a variable (e.g., $B = PX_1 \ldots X_k$).

(2) A and B are CAFs such that for some function symbol F of degree k, $B \simeq A\{FZ_1 \ldots Z_k/X\}$, where X is a variable occurring in A and $Z_1, \ldots, Z_k$ are distinct variables not occurring in A.

(3) A and B are CAFs such that $B \simeq A\{Y/X\}$, where X and Y are distinct variables occurring in A.

(4) A is a ground GAF and $B = \Omega$.

It is evident that $A \rightarrow B$ implies $A > B$.

Definition. For $n \geqslant 0$, a sequence $A_0, \ldots, A_n$ of GAFs is called a *chain* of length n from A to B iff $A = A_0 > A_1 > \ldots > A_n \simeq B$. An infinite sequence $A_0, A_1, \ldots$ is called an *infinite descending* (*or ascending*) *chain* from A_0 iff $A_0 > A_1 > \ldots$ (or $A_0 < A_1 < \ldots$). In each of these definitions, if the relation $>$ can be replaced by the stronger relation $\rightarrow$, then the (infinite descending or ascending) chain is called *total.*

Theorem 4

If $A \geqslant B$, then there exists a total chain $A = A_0 \rightarrow A_1 \rightarrow \ldots \rightarrow A_n \simeq B$ from A to B. If $A > B$, then the length n is at least one. [This is a variant of the factorization theorem proved in Reynolds (1968).]

Proof. Consider the case where A and B are CAFs, and let θ be the substitution such that $B = A\theta$ and every variable of θ occurs in A. Then the following algorithm will produce the sequence $A_0, \ldots, A_n$:

(1) Let $A_0 = A$, $\theta_0 = \theta$, $i = 0$.

(2) If no term of θ_i contains a function symbol, go to step 3. Otherwise, let $FT_1 \ldots T_k/X$ be some component of θ_i whose term contains a function

symbol, and let $Z_1, \ldots, Z_k$ be distinct variables not occurring in A_i. Then set

$$A_{i+1} = A_i\{FZ_1 \ldots Z_k/X\}$$
$$\theta_{i+1} = (\theta_i - \{FT_1 \ldots T_k/X\}) \cup \{T_1/Z_1, \ldots, T_k/Z_k\}.$$

Increase i by 1 and repeat step 2.

(3) If there is no pair X, Y of distinct variables occurring in A_i such that $X\theta_i = Y\theta_i$, then go to step 4. Otherwise, let X, Y be such a pair and set

$$A_{i+1} = A_i\{Y/X\}$$
$$\theta_{i+1} = \theta_i - \{X\theta_i/X\}.$$

Increase i by 1 and repeat step 3.

(4) Set $n = i$ and terminate.

Step 2 must terminate since each iteration decreases the total number of function symbol occurrences in the terms of θ_i. Step 3 must terminate since each iteration decreases the number of distinct variables occurring in A_i. By induction on i, one can show that:

$$B = A_i\theta_i (0 \leqslant i \leqslant n).$$

All variables of θ_i occur in $A_i (0 \leqslant i \leqslant n)$.

$$A_i \rightarrow A_{i+1} \quad (0 \leqslant i \leqslant n-1).$$

When termination is reached, all of the terms of θ_n will be variables, and for any pair X, Y of distinct variables occurring in A_n, $X\theta_n \neq Y\theta_n$. Thus $A_n \simeq B$.

We now consider the cases where A or B is not a CAF. If $A = \mathscr{A}$ and B is a CAF, let P be the predicate symbol beginning B, let k be the degree of P, and let $A' = PZ_1 \ldots Z_k$, where $Z_1, \ldots, Z_k$ are distinct variables. Then $A \rightarrow A' \geqslant B$ and A' and B are CAFs, so that a chain from A' to B can be constructed as above and joined to the link $A \rightarrow A'$.

If A is $\mathscr{A}$ or a CAF and B is Ω, let B' be any ground instance of A. Then $A \geqslant B' \rightarrow B$ and B' is a CAF, so that a chain from A to B' can be constructed as above and joined to the link $B' \rightarrow B$.

The remaining cases are $A = B = \mathscr{A}$ and $A = B = \Omega$, which are trivial. The assertion that $n \geqslant 1$ when $A > B$ is also trivial, since $n = 0$ implies $A \simeq B$.

Definition. The *size* of a GAF is defined as follows: size $(\mathscr{A}) = 0$. size $(\Omega) = \infty$. If A is a CAF, then size (A) is the number of symbol occurrences in A minus the number of distinct variables occurring in A.

Corollary 5

If A and B are GAFs, then $A \simeq B$ implies size $(B) =$ size (A), and $A \rightarrow B$ implies size $(B) >$ size (A). More generally (by Theorem 4) $A > B$ implies size $(B) >$ size (A).

We will now use Theorem 4 and Corollary 5 to obtain bounds on the length of chains, to determine the covering relation, and to show that the GAF lattice is complete.

Theorem 5

(1) If $B \neq \Omega$, then there is no chain from A to B whose length is greater than size $(B) -$ size (A).

(2) There are no infinite ascending chains from any GAF.

(3) If A is a ground GAF, then the only chain from A to Ω is $A \rightarrow \Omega$, and there are no infinite descending chains from A.

(4) If A is not a ground GAF and is not Ω, then there is no bound on the length of chains from A to Ω, and there is an infinite descending total chain from A.

Proof. (1) Let $A = A_0 > A_1 > \ldots > A_n \simeq B$ be any chain from A to B. Then by Corollary 5, size $(A) =$ size $(A_0) <$ size $(A_1) < \ldots <$ size $(A_n) =$ size (B). Thus $n <$ size $(B) -$ size (A).

(2) An infinite ascending chain $A_0 < A_1 < A_2 < \ldots$ would imply that size $(A_0) >$ size $(A_1) >$ size $(A_2) > \ldots$. But this is impossible, since $A_1 \neq \Omega$ implies that A_1 has finite size.

(3) If A_0 is a ground GAF, then $A_0 > A_1$ implies $A_1 = \Omega$, and there is no A_2 such that $\Omega > A_2$.

(4) Suppose A is a non-ground CAF. Let X be a variable occurring in A, let F be a unary function symbol, and let $A_i = A\{FX/X\}^i$. Then $A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \ldots$ is an infinite descending total chain from A, and for any $n \geqslant 0$, $A_0 \rightarrow A_1 \ldots \rightarrow A_n \rightarrow \Omega$ is a chain of length $n+1$ from A to Ω.

If A is $\mathscr{A}$, similar chains can be constructed by taking A_1 to be $PZ_1 \ldots Z_k$, where P is a predicate symbol of degree $k \geqslant 1$ and $Z_1, \ldots, Z_k$ are distinct variables, and then continuing as above.

Definition. A covers B iff $A > B$ and there is no GAF C such that $A > C > B$.

Theorem 6

A covers B iff $A \rightarrow B$.

Proof. Suppose A covers B. By Theorem 4, there is a total chain $A = A_0 \rightarrow A_1 \rightarrow \ldots \rightarrow A_n \simeq B$ with some length $n \geqslant 1$. But if $n > 1$, then $A > A_1 > B$, which contradicts the definition of covering. Thus $n = 1$ and $A \rightarrow A_1 \simeq B$. By the definition of $\rightarrow$, this implies $A \rightarrow B$.

Conversely, suppose $A \rightarrow B$. Since this implies that $A > B$, we need only show that $A > C > B$ is impossible. We distinguish three cases:

(1) A is a ground GAF and $B = \Omega$. This prevents $A > C > B$ since Ω is the only proper instance of A.

(2) Either (i) A is $\mathscr{A}$ and B is a CAF containing no function symbols and no repeated occurrences of a variable, or (ii) A and B are GAFs such that $B \simeq A\{Y/X\}$, where X and Y are distinct variables occurring in A. In either situation, size $(B) =$ size $(A) + 1$, while $A > C > B$ would imply size $(B) >$ size $(A) + 1$.

(3) A and B are CAFs such that for some function symbol F of degree k, $B \simeq A\{FZ_1 \ldots Z_k/X_1\}$, where X_1 is a variable occurring in A and $Z_1, \ldots, Z_k$ are distinct variables not occurring in A. Suppose there exists a GAF C such that $A > C > B$. Then by Theorem 4, there exists a CAF A_1 such that $A \rightarrow A_1 \geqslant C > B$. Since $A \rightarrow A_1$ and A is a CAF, we have $A_1 \simeq A\theta$, where θ is a substitution which meets the criteria of part 2 or 3 of the definition of the relation $\rightarrow$.

Since $A\theta \simeq A_1 \geqslant C > B \simeq A\{FZ_1 \ldots Z_k / X_1\}$, there is a substitution ψ such that $A\theta\psi = A\{FZ_1 \ldots Z_k / X_1\}$.

Let $X_1, \ldots, X_n$ be the variables occurring in A. Then $X_1\theta\psi = FZ_1 \ldots Z_k$, and for $2 \leqslant i \leqslant n$, $X_i\theta\psi = X_i$. This implies that either (i) $X_1\theta$ begins with the symbol F and $X_2\theta, \ldots, X_n\theta$ are distinct variables, or (ii) $X_1\theta, \ldots, X_n\theta$ are distinct variables. In order to satisfy both this condition and part 2 or 3 of the definition of $\rightarrow$, θ must have the form $\{FZ_1' \ldots Z_k' / X_1\}$, where $Z_1', \ldots, Z_k'$ are distinct variables not occurring in A. But then $A\theta = A\{FZ_1' \ldots Z_k' / X_1\}$ is a variant of $A\{FZ_1 \ldots Z_k / X_1\}$, which contradicts $A\theta \simeq A_1 \geqslant C > B \simeq A\{FZ_1 \ldots Z_k / X_1\}$.

Theorem 7

Any set S of GAFs possesses a greatest common instance and a least common generalization, i.e., the GAF lattice is *complete*.

Proof. If $S = \{A_1, \ldots, A_n\}$ is finite, then the theorem is trivial. Let

$$I_0 = \mathscr{A}$$
$$I_{i+1} = I_i \sqcap A_{i+1} \quad (0 \leqslant i \leqslant n-1)$$
$$G_0 = \Omega$$
$$G_{i+1} = G_i \sqcup A_{i+1} \quad (0 \leqslant i \leqslant n-1).$$

Then it is easily shown that I_n and G_n are a greatest common instance and a least common generalization of S.

If S is infinite, let $A_1, A_2, \ldots$ be an enumeration of the members of S, and let

$$G_0 = \Omega$$
$$G_{i+1} = G_i \sqcup A_{i+1} \quad (i \geqslant 0).$$

The G_is satisfy $G_0 \leqslant G_1 \leqslant G_2 \leqslant \ldots$. But there cannot be an infinite sequence of is such that $G_i < G_{i+1}$, for this would imply that some sub-sequence of the G_is is an infinite ascending chain. Thus there exists an integer i_0 such that $G_i \simeq G_{i_0}$ for all $i \geqslant i_0$. It is easily shown that G_{i_0} is a least common generalization of S.

A similar argument cannot be used for the greatest common instance, since the GAF lattice does contain infinite descending chains. Instead let

$$T = \{B \mid B \leqslant A \textit{ for all } A \in S\},$$

and let I be the least common generalization of T. Then if A is any member of S, $A \geqslant B$ for all $B \in T$, and therefore $A \geqslant I$; thus I is a common instance of S. On the other hand, if B is a common instance of S, then $B \in T$, and therefore $B \leqslant I$; thus I is a greatest common instance.

Since all greatest common instances of the same set are equivalent, we will refer to 'the' greatest common instance of a set, assuming that a particular member of the equivalence class can be selected in some well-defined manner. A similar situation holds for the least common generalization.

Definition. If S is a set of GAFs, we write $\sqcap S$ for the greatest common instance of S and $\sqcup S$ for the least common generalization of S.

It is easily shown that:

Corollary 6

If S and T are sets of GAFs such that $S \supseteq T$, then $\sqcup S \geqslant \sqcup T$ and $\sqcap S \leqslant \sqcap T$.

So far, our investigation of the GAF lattice has been limited to relations between the GAFs themselves. We now consider the relation between GAFs and their sets of ground instances:

Definition. For any GAF A, $\mathscr{G}(A)$ denotes the set of all ground GAFs which are instances of A.

Theorem 8

For all GAFs A and B:

$$\mathscr{G}(A) \text{ is empty iff } A = \Omega.$$
$$A \simeq \sqcup \mathscr{G}(A).$$
$$A \geqslant B \text{ iff } \mathscr{G}(A) \supseteq \mathscr{G}(B).$$
$$\mathscr{G}(A \sqcap B) = \mathscr{G}(A) \cap \mathscr{G}(B).$$
$$\mathscr{G}(A \sqcup B) \supseteq \mathscr{G}(A) \cup \mathscr{G}(B).$$

Proof. The crux of the theorem is $A \simeq \sqcup \mathscr{G}(A)$, which implies that the set of ground instances of A is 'rich' enough to determine A within an equivalence. Once this assertion has been established, the rest of the theorem is a straightforward application of elementary lattice theory.

Suppose A is a non-ground CAF, and let $B = \sqcup \mathscr{G}(A)$. Since every ground instance is an instance, A is a common generalization of $\mathscr{G}(A)$, so that $A \geqslant B$. Since A has at least one ground instance, $B \neq \Omega$. Thus there is a substitution θ such that $B = A\theta$.

Let $X_1, \ldots, X_n$ be the variables occurring in A, let C be a constant, and let F be a unary function symbol. Then $\mathscr{G}(A)$ must contain the following ground instances:

$$G_C = A\{C/X_1, \ldots, C/X_n\}$$
$$G_F = A\{FC/X_1, \ldots, FC/X_n\}$$
$$G_N = A\{C/X_1, FC/X_2, \ldots, \underbrace{F \ldots F}_{n \text{ times}}C/X_n\}.$$

Since each of these ground GAFs are instances of B, there are substitutions ϕ_C, ϕ_F, and ϕ_N such that

$$G_C = B\phi_C = A\theta\phi_C$$
$$G_F = B\phi_F = A\theta\phi_F$$
$$G_N = B\phi_N = A\theta\phi_N.$$

From the equations for G_C and G_F we have, for $1 \leqslant i \leqslant n$, $X_i\theta\phi_C = C$ and $X_i\theta\phi_F = FC$; thus each $X_i\theta$ must be a variable. From the equations for G_N we have, for $1 \leqslant i \leqslant n$, $X_i\theta\phi_N = \underbrace{F \ldots F}_{i \text{ times}}C$; thus each $X_i\theta$ must be distinct. Therefore $B = A\theta$ is a variant of A.

If $A = \Omega$, then $\sqcup \mathscr{G}(A) = \sqcup\{\} = \Omega$. If A is a ground GAF, then $\sqcup \mathscr{G}(A) = \sqcup\{A\} \simeq A$. If $A = \mathscr{A}$, then $\mathscr{G}(\mathscr{A})$ is the set of all ground GAFs. Since this set contains GAFS which begin with different predicate symbols, its only common generalization is $\mathscr{A}$.

It should be noted that the existence of different predicate symbols is a necessary condition for this theorem; if all CAFS begin with the same symbol P, then $\bigsqcup\mathscr{G}(\mathscr{A})=PZ_1\ldots Z_k$.

Finally, we consider the interaction of our lattice relations with the operation of resolution. Since the lattice relations are relations between atomic formulas rather than clauses, it is difficult to make any significant statements about resolution in general. But useful results can be obtained for the case where a unit clause is resolved against a mixed two-clause (which we will call a *transformation*) to yield another unit.

Definition. A *transformation* is a clause containing exactly one positive and one negative literal; i.e., it is a clause of the form $\{\neg P, Q\}$, where P and Q are CAFS. If A is a GAF and $t=\{\neg P, Q\}$ is a transformation, then $r_t(A)$ is defined as follows: If $A=\mathscr{A}$ then $r_t(A)=Q$, else if $A=\Omega$ or $\{A\}$ and t have no resolvent, then $r_t(A)=\Omega$, otherwise $r_t(A)$ is the CAF which is the only member of the unique resolvent of $\{A\}$ and t.

Lemma 3

Let A be a GAF and $t=\{\neg P, Q\}$ be a transformation. If $A\sqcap P=\Omega$, then $r_t(A)=\Omega$. If $A\sqcap P\neq\Omega$, then there is a substitution σ such that $r_t(A)\simeq Q\sigma$, $A\sqcap P\simeq P\sigma$, and for all variables X, if X is a variable of σ or occurs in any term of σ, then either X occurs in P or X does not occur in Q.

Proof. If $A=\mathscr{A}$, then $A\sqcap P\neq\Omega$ and σ is the empty substitution. If $A=\Omega$, then $A\sqcap P=\Omega$ and $r_t(A)=\Omega$. Otherwise, let A' be a variant of A which has no variables in common with P or Q. If A' and P are not unifiable, then $A\sqcap P=\Omega$, $\{A\}$ and t have no resolvent, and $r_t(A)=\Omega$. If A' and P are unifiable, then $\{A\}$ and t have the single resolvent (whose only member is) $r_t(A)\simeq Q\sigma$, where σ is the most general unifier of A' and P. Then the definition of $\sqcap$ implies that $A\sqcap P\simeq P\sigma\neq\Omega$. Moreover, the nature of the Unification Algorithm insures that every variable of σ and every variable occurring in the terms of σ is a variable which occurs in A' or P. Thus if such a variable does not occur in P, it does not occur in Q.

Theorem 9

If A and B are GAFS and $t=\{\neg P, Q\}$ is a transformation, then

$$\begin{aligned}
&\text{if } A\geqslant B\sqcap P \text{ then } r_t(A)\geqslant r_t(B)\\
&r_t(A)\simeq r_t(A\sqcap P)\\
&Q\geqslant r_t(A)\\
&r_t(A\sqcap B)\leqslant r_t(A)\sqcap r_t(B)\\
&r_t(A\sqcup B)\geqslant r_t(A)\sqcup r_t(B).
\end{aligned}$$

Proof. Suppose $A\geqslant B\sqcap P$. Then $B\sqcap P$ is a common instance of A and P, so that $A\sqcap P\geqslant B\sqcap P$. By Lemma 3, if $B\sqcap P=\Omega$, then $r_t(A)\geqslant r_t(B)=\Omega$. Otherwise, $A\sqcap P\neq\Omega$ and $B\sqcap P\neq\Omega$, so that there are substitutions σ_A and σ_B such that $r_t(A)\simeq Q\sigma_A$, $r_t(B)\simeq Q\sigma_B$, $A\sqcap P\simeq P\sigma_A$, $B\sqcap P\simeq P\sigma_B$, and for all variables

X, if X is a variable of σ_A or σ_B or occurs in any term of σ_A or σ_B, then either X occurs in P or X does not occur in Q.

Since $P\sigma_A \geqslant P\sigma_B$, there is a substitution ϕ such that $P\sigma_B = P\sigma_A\phi$ and every variable of ϕ either occurs in P or in some term of σ_A. Let X be any variable occurring in Q. If X occurs in P then $X\sigma_B\phi = X\sigma_A$, while if X does not occur in P, then X is not a variable of σ_A, ϕ, or σ_B, so that $X\sigma_B = X\sigma_A\phi = X$. Thus $Q\sigma_B = Q\sigma_A\phi$, and $r_t(A) \simeq Q\sigma_A \geqslant Q\sigma_B \simeq r_t(B)$.

The remaining parts of the theorem are direct consequences of this result and the basic properties of $\sqcap$ and $\sqcup$.

TRANSFORMATIONAL SYSTEMS

In Theorem 9, we have developed certain properties of the resolution operation which hold only when one resolvend is a unit clause and the other is a transformation. This leads naturally to the investigation of sets of clauses, called *transformational systems*, which can be refuted by using this limited form of resolution.

Definition. A *transformational system* is a finite set of clauses each member of which is either a unit clause or a transformation. If S is a transformational system, then $\mathscr{P}(S)$ is the set of CAFs which occur in the positive units of S, $\mathscr{N}(S)$ is the set of CAFs which occur in the negative units of S, and $\mathscr{T}(S)$ is the set of transformations in S.

We will call a resolution a *cross-resolution* if one resolvend is a clause containing no negative literals and the other resolvend is a clause containing at least one negative literal. In Robinson (1965a), it is shown that a set of clauses S is unsatisfiable iff the empty clause can be derived by repeated cross-resolutions.

But if S is a transformational system, then any cross-resolution of members of S will either resolve a positive unit against a negative unit, yielding the empty clause, or a positive unit against a transformation, yielding another positive unit. Thus any refutation of a transformational system by cross-resolution will have the form shown in figure1 where the ps are positive units, the ts are transformations, and n is a negative unit.

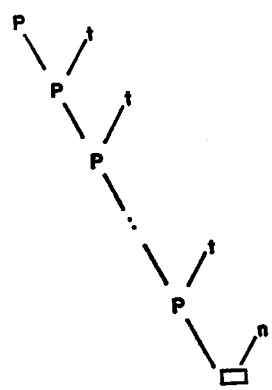


Figure 1

Definition. For any set P of GAFs and any set T of transformations, we define:

$$R_T(P)=\{r_t(A)\mid t\in T,\ A\in P\}$$
$$R_T^0(P)=P$$
$$R_T^{n+1}(P)=R_T(R_T^n(P))\quad (n\geqslant 0).$$

If S is a transformational system, it is evident that $\bigcup_{n=0}^{\infty} R^n_{\mathscr{T}(S)}(\mathscr{P}(S))-\{\Omega\}$ is the set of (the members of) all positive units which can be derived from S by cross-resolution. Moreover, a positive unit $\{A\}$ and a negative unit $\{\neg B\}$ will resolve to give the empty clause iff $A\sqcap B\neq\Omega$. Thus:

Corollary 7

A transformational system is unsatisfiable iff there exists an integer $n\geqslant 0$, and GAFs A and B such that $A\in R^n_{\mathscr{T}(S)}(\mathscr{P}(S))$, $B\in\mathscr{N}(S)$, and $A\sqcap B\neq\Omega$. Thus one can refute a transformational system by path-searching rather than tree-searching. In effect, the dyadic inference rule of resolution can be replaced by a finite set of monadic rules, one for each transformation.

The path-searching aspect is even more evident on the ground level. A transformational system S is unsatisfiable iff the empty clause can be derived from some finite set of ground instances of S by cross-resolution. Again, the refutation will have the form shown in figure 1, but now a positive unit $\{A\}$ will resolve against a transformation $\{\neg P,\ Q\}$ iff $A=P$. Thus:

Corollary 8

Let S be a transformational system and D be the directed graph whose set of nodes is the set of ground GAFs, and in which there is an arc from node P to node Q iff $\{\neg P,\ Q\}$ is a ground instance of some member of $\mathscr{T}(S)$. Then S is unsatisfiable iff there is a path in D from some ground instance of a member of $\mathscr{P}(S)$ to some ground instance of a member of $\mathscr{N}(S)$.

In the context of transformational systems, we can illustrate the notion of a 'super-consequence'. For a transformational system S, suppose that $P\subseteq\mathscr{P}(S)$ and $T\subseteq\mathscr{T}(S)$, Then a search procedure may eventually generate each member of the (usually infinite) set $S'=\bigcup_{n=0}^{\infty} R_T^n(P)$. We will call a clause C a *super-consequence* of P and T if C is a common generalization of S'. Usually C itself will not be a valid consequence of P and T, but if C can be shown to be irrelevant (e.g., by subsumption or purity), then the generation of all members of S' can be avoided.

The following theorem gives a method for computing super-consequences:

Theorem 10

Let P be a finite set of GAFs, let T be a finite set of transformations, and let:

$$C_0=\bigsqcup P$$
$$C_{n+1}=C_n\sqcup\bigsqcup R_T(\{C_n\})\quad (n\geqslant 0).$$

Then there is an integer n such that $C_{n+1}\simeq C_n$. If n_0 is the least such integer, then C_{n_0} is a common generalization of $\bigcup_{n=0}^{\infty} R_T^n(P)$.

Proof. The Cs satisfy $C_0 \leqslant C_1 \leqslant C_2 \leqslant \ldots$. Thus, since there are no infinite ascending chains of GAFs, there is an integer n such that $C_{n+1} \simeq C_n$. Let n_0 be the least such integer. Then by induction on i, $C_{n_0+1+i} \simeq C_{n_0}$ for all $i \geqslant 0$. Thus $C_n \leqslant C_{n_0}$ for all $n \geqslant 0$.

To complete the proof we will show, by induction on n, that C_n is a common generalization of $R_T^n(P)$. The assertion is obvious for $n=0$. Assuming it is true for n, let A be any member of $R_T^{n+1}(P)$. Then $A = r_t(B)$ for some $t \in T$ and $B \in R_T^n(P)$. Then $B \leqslant C_n$, so that $r_t(B) \leqslant r_t(C_n)$, by the first part of Theorem 9. Thus $A \leqslant r_t(C_n) \leqslant \bigsqcup R_T(\{C_n\}) \leqslant C_{n+1}$.

As an example, if $P = \{P(f(c)c)\}$ and $T = \{\{\neg P(x, y), P(f(f(y))x)\}\}$, then $n_0 = 1$, and $C_{n_0} \simeq P(f(x)x)$.

It is evident that the refutation of transformational systems is significantly simpler than the refutation of arbitrary sets of clauses. We conclude by showing that the problem is still non-trivial, in the precise sense that there is no decision procedure for transformational systems. This suggests that transformational systems may be a useful 'initial case' for the development of more efficient proof procedures.

Our proof will be accomplished by mapping a known unsolvable problem, the Post correspondence problem, into the decision problem for transformational systems.

Definition. A *correspondence problem* is a finite non-empty sequence of pairs, $\rho = (A_1, B_1), \ldots, (A_m, B_m)$, where each A_i and each B_i is a string over some finite set V of characters. The problem ρ is called *solvable* iff there exist integers $n \geqslant 0$, $i_0, \ldots, i_n$ such that $1 \leqslant i_j \leqslant m$ (for $0 \leqslant j \leqslant n$) and $A_{i_n} \ldots A_{i_0} = B_{i_n} \ldots B_{i_0}$, where the juxtaposition of string variables indicates string concatenation.

It is known (Floyd 1966, Post 1946) that there is no algorithm which will accept an arbitrary correspondence problem ρ and determine whether ρ is solvable.

In the following definition, corollary, and lemma, we assume that the vocabulary V is fixed, that σ is some one-to-one mapping from V into a set of ground terms, and that E is a constant, F is a binary function symbol, P is a binary predicate symbol, and X and Y are distinct variables.

Definition. Let $A = a_1 \ldots a_k$ be a string of length k over V, and T be a term. Then $\Sigma(A, T)$ denotes the term $F\sigma(a_1)F\sigma(a_2) \ldots F\sigma(a_k)T$.

Corollary 9

Let A and B be strings over V, T be a term, and θ be a substitution. Then

$$\Sigma(A, T)\theta = \Sigma(A, T\theta)$$
$$\Sigma(A, \Sigma(B, E)) = \Sigma(AB, E)$$
$$\Sigma(A, E) = \Sigma(B, E) \text{ iff } A = B.$$

Lemma 4

Let $\rho = (A_1, B_1), \ldots, (A_m, B_m)$ be a correspondence problem, and let S be the transformational system such that:

$$\mathscr{P}(S)=\{P\Sigma(A_i, E)\Sigma(B_i, E)\,|\,1\leqslant i\leqslant m\}$$
$$\mathscr{T}(S)=\{\{\neg PXY, P\Sigma(A_i, X)\Sigma(B_i, Y)\}\,|\,1\leqslant i\leqslant m\}$$
$$\mathscr{N}(S)=\{PXX\}.$$

Then S is unsatisfiable iff ρ is solvable.

An intuitive grasp of this lemma may be obtained by the following interpretation of S: E denotes the empty string. Each ground term $\sigma(a)$ denotes the character a. The function $F(a, s)$ denotes the string obtained by adding the character a to the beginning of the string s. The predicate $P(s, t)$ asserts that $s=A_{i_n}\ldots A_{i_0}$ and $t=B_{i_n}\ldots B_{i_0}$ for some $i_0,\ldots, i_n$. $\mathscr{P}(S)$ and $\mathscr{T}(S)$ are axioms which define P, and $\mathscr{N}(S)$ (whose negation occurs in S) is a theorem that ρ is solvable.

Proof. We first show, by induction on n, that

$$R^n_{\mathscr{T}(S)}(\mathscr{P}(S))=\bigcup_{i_0=1}^{m}\ldots\bigcup_{i_n=1}^{m}\{P\Sigma(A_{i_n}\ldots A_{i_0}, E)\,\Sigma(B_{i_n}\ldots B_{i_0}, E)\}.$$

The assertion is obvious for $n=0$. Assuming it is true for n,

$$R^{n+1}_{\mathscr{T}(S)}(\mathscr{P}(S))=\{r_{\{\neg PXY,P\Sigma(A_i,X)\Sigma(B_i,Y)\}}(x)\,|\,1\leqslant i\leqslant m \textit{ and } x\in R^n_{\mathscr{T}(S)}(\mathscr{P}(S))\}$$
$$=\bigcup_{i_0=1}^{m}\ldots\bigcup_{i_{n+1}=1}^{m}\{r\}$$

where

$$r=r_{\{\neg PXY,P\Sigma(A_{i_{n+1}},X)\Sigma(B_{i_{n+1}},Y)\}}(P\Sigma(A_{i_n}\ldots A_{i_0}, E)\,\Sigma(B_{i_n}\ldots B_{i_0}, E))$$
$$=P\Sigma(A_{i_{n+1}},X)\,\Sigma(B_{i_{n+1}},Y)\{\Sigma(A_{i_n}\ldots A_{i_0}, E)/X,$$
$$\Sigma(B_{i_n}\ldots B_{i_0}, E)/Y\}.$$

Then by the first two parts of Corollary 9,

$$r=P\Sigma(A_{i_{n+1}},\Sigma(A_{i_n}\ldots A_{i_0}, E))\,\Sigma(B_{i_{n+1}},\Sigma(B_{i_n}\ldots B_{i_0}, E))$$
$$=P\Sigma(A_{i_{n+1}}\ldots A_{i_0}, E)\,\Sigma(B_{i_{n+1}}\ldots B_{i_0}, E)$$

which completes the induction.

By Corollary 7, S is unsatisfiable iff there is some $n\geqslant 0$ and some $P\Sigma(A, E)\Sigma(B, E)\in R^n_{\mathscr{T}(S)}(\mathscr{P}(S))$, such that $P\Sigma(A, E)\Sigma(B, E)\sqcap PXX\neq\Omega$. But by the definition of $\sqcap$ and Corollary 9,

$$P\Sigma(A, E)\Sigma(B, E)\sqcap PXX\neq\Omega \text{ iff } \Sigma(A, E)=\Sigma(B, E) \text{ iff } A=B.$$

Thus S is unsatisfiable iff ρ is solvable.

Now suppose D were a decision procedure for transformational systems. Then, given any correspondence problem ρ, we could use the mapping of Lemma 4 to convert ρ into a transformational system S, and then determine whether ρ is solvable by applying D to S. Since it is known that this is impossible:

Theorem 11

There is no algorithm which will accept an arbitrary transformational system S and determine whether S is unsatisfiable.

In addition to establishing the non-triviality of transformational systems, this theorem is pertinent to Wos's unit preference strategy (Wos, Carson and Robinson 1964). Since cross-resolution is a special case of resolution, it is evident from figure 1 that a transformational system S is unsatisfiable iff the empty clause can be derived from S by repeated resolutions in which at least one resolvend is a unit. Thus:

Corollary 10

There is no algorithm which will accept an arbitrary finite set of clauses S and determine whether the empty clause can be derived from S by repeated resolutions in which at least one resolvend is a unit.

In effect, there is no decision procedure for the unit section of the unit preference strategy.

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