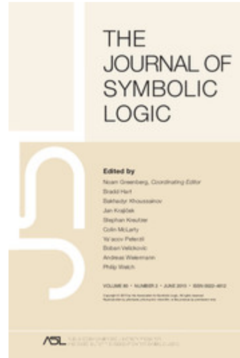

Gödel Kurt. Über die Vollständigkeit des Logikkalküls (1929). *Collected Works, Volume I, Publications 1929–1936*, by Kurt Gödel, edited by Feferman Solomon, Dawson John W. Jr., Kleene Stephen C., Moore Gregory H., Solovay Robert M., and van Heijenoort Jean, Clarendon Press, Oxford University Press, New York and Oxford 1986, even pp. 60–100.Gödel Kurt. On the completeness of the calculus of logic (1929). English translation by Stefan Bauer-Mengelberg and Jean van Heijenoort of the preceding. *Collected Works, Volume I, Publications 1929–1936*, by Kurt Gödel, edited by Feferman Solomon, Dawson John W. Jr., Kleene Stephen C., Moore Gregory H., Solovay Robert M., and van Heijenoort Jean, Clarendon Press, Oxford University Press, New York and Oxford 1986, odd pp. 61–101.Gödel Kurt. Die Vollständigkeit der Axiome des logischen Funktionenkalküls (1930). A reprint of 4182. *Collected Works, Volume I, Publications 1929–1936*, by Kurt Gödel, edited by Feferman Solomon, Dawson John W. Jr., Kleene Stephen C., Moore Gregory H., Solovay Robert M., and van Heijenoort Jean, Clarendon Press, Oxford University Press, New York and Oxford 1986, even pp. 102–122.Gödel Kurt. The completeness of the axioms of the functional calculus of logic (1930). A reprint of XL 475 (English translation by Stefan Bauer-Mengelberg of the preceding). *Collected Works, Volume I, Publications 1929–1936*, by Kurt Gödel, edited by Feferman Solomon, Dawson John W. Jr., Kleene Stephen C., Moore Gregory H., Solovay Robert M., and van Heijenoort Jean, Clarendon Press, Oxford University Press, New York and Oxford 1986, odd pp. 103– 123.Gödel Kurt. Über die Vollständigkeit des Logikkalküls (1930a). *Collected Works, Volume I, Publications 1929– 1936*, by Kurt Gödel, edited by Feferman

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REVIEWS

With this issue, the JOURNAL initiates its reviewing of computer software pertaining to symbolic logic (see the last two reviews in this issue). Authors, publishers, and users of such software are invited to bring it to the attention of the Reviews Editors. Information and review copies should be sent to *The Journal of Symbolic Logic, U.C.L.A., Los Angeles, California 90024.*

To be reviewed in the JOURNAL, logic software must be "published" in the sense of being available to the general public, through normal trade channels.

SOLOMON FEFERMAN. *Gödel's life and work.* ***Collected works***, Volume I, ***Publications 1929–1936***, by Kurt Gödel, edited by Solomon Feferman, John W. Dawson, Jr., Stephen C. Kleene, Gregory H. Moore, Robert M. Solovay, and Jean van Heijenoort, Clarendon Press, Oxford University Press, New York and Oxford 1986, pp. 1–36.

This review and the following reviews will cover most of Gödel's ***Collected works***, Volume I. The volume is a beautifully prepared, meticulously edited collection of Gödel's publications through 1936, prepared under the auspices of the Association for Symbolic Logic. Gödel's writings in German are presented with page-facing English translations, many of them prepared especially for this volume. In every case, the even numbered pages contain the original German text, and the translation appears on the subsequent odd-numbered pages. Future volumes will include items from Gödel's extensive *Nachlass* as well as the remainder of his publications. In addition to Gödel's own work, there is a personal and scientific biography by Feferman, a chronology by Dawson, and an interesting collection of photographs. Each group of related articles is introduced by an extensive "Introductory note" by an expert. The notes are intended to put the articles in historical perspective, to explain their content, and to indicate further developments. The items belonging to the field of this JOURNAL are reviewed below in these same groups. In the 1930's, Gödel wrote many brief factual reviews for the ***Zentralblatt für Mathematik und ihre Grenzgebiete*** (abbreviated here ***Zentralblatt***) and a few for the ***Monatshefte für Mathematik und Physik*** (abbreviated here ***Monatshefte***). Most of these are not explicitly noted in the present reviews. Also in the 1930's, Gödel participated extensively in Karl Menger's colloquium. His contributions are reported in ***Ergebnisse eines mathematischen Kolloquiums*** (abbreviated here ***Ergebnisse***) in brief articles, essentially abstracts. The volume contains a fifty-three page bibliography. Items will be referenced here as in this bibliography: author's name followed by a year and an additional lower-case letter when necessary to remove ambiguities. The corresponding citation in Alonzo Church's *A bibliography of symbolic logic*, this JOURNAL, vol. 1 (1936), pp. 121–218 will also be given. The reviewer's anthology, XXXI 484, which overlaps slightly with the volume under review, will be referred to as *Davis 1965*.

Every logician must be profoundly aware of the immense change that Kurt Gödel's work has brought to our field. Work in logic pre-Gödel can be compared with pre-Galileo cosmology; despite technical brilliance, the logicians seem like Koestler's "sleepwalkers," prevented by ideological blinders from seeing connections that, in retrospect, it is hard for us to understand how they missed. Attending sessions of the famous "Vienna circle," but rarely participating, Gödel found himself totally out of sympathy with the positivist-empiricist mood that prevailed. Although he was certainly interested in the three schools of thought about the foundations of mathematics then contending (logicism, intuitionism, and Hilbert's formalist program), Gödel quite emphatically belonged to none of them. His philosophical views had been deeply influenced by his study of Kant going back to his adolescent years, and he later asserted that, already by 1925, he had adopted a "realist" view towards the existence of mathematical objects. To someone who was positivistically inclined, the only sense of mathematical "truth" that could be imagined was provability in some sense, but Gödel's realist view made it natural for him to distinguish truth from

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provability and to raise the question of whether these concepts are equivalent (already in his doctoral dissertation, *Gödel 1929*), and then (in his revolutionary *Gödel 1931*) to show that they are in fact distinct.

During Gödel's years in Princeton, he and Einstein became good friends. In arriving at his special theory of relativity, Einstein had frankly embraced a thoroughgoing empiricism, and had quite explicitly opposed the very philosophical tradition that Gödel was later to embrace. In fact the success of Einstein's ideas had helped create the *Zeitgeist* in which the Vienna circle later flourished. These two great thinkers both had an uncommon ability to think about matters in a fundamental way unencumbered by the accepted wisdom of their times; and, ironically, where Einstein achieved his success embracing positivism, Gödel achieved his opposing it.

Feferman's sensitive biographical essay portrays Gödel as a scientist, sure of his powers and fully aware that many of his views went against the "dominant philosophical prejudices" of his time. He was extremely cautious personally, meticulously correct in dealings with colleagues, deeply conservative, and distrustful of medical advice. Gödel's great achievements are of course well known to all logicians: the completeness and compactness theorems for first-order logic, the incompleteness of arithmetic, and the consistency of the axiom of choice and the generalized continuum hypothesis with the Zermelo-Fraenkel axioms. Gödel spent his later years thinking systematically about a great variety of subjects, including history, philosophy, theology, and even demonology. Feferman's brief indication of what is to be found in the Gödel *Nachlass* whets our appetite for what is to come in subsequent volumes.

MARTIN DAVIS

JOHN W. DAWSON, Jr. *A Gödel chronology.* Ibid., pp. 37–43.

This is a brief list of the key events in Gödel's life from his birth in Brünn, Moravia on April 28, 1906, to his death from "malnutrition and inanition" on January 14, 1978, and concluding with the death of Gödel's wife, Adele, and the bequest of Gödel's *Nachlass* to the Institute for Advanced Study on February 4, 1981.

MARTIN DAVIS

KURT GÖDEL. *Über die Vollständigkeit des Logikkalküls (1929).* Ibid., even pp. 60–100.

KURT GÖDEL. *On the completeness of the calculus of logic (1929).* English translation by Stefan Bauer-Mengelberg and Jean van Heijenoort of the preceding. Ibid., odd pp. 61–101.

KURT GÖDEL. *Die Vollständigkeit der Axiome des logischen Funktionenkalküls (1930).* A reprint of 4182. Ibid., even pp. 102–122.

KURT GÖDEL. *The completeness of the axioms of the functional calculus of logic (1930).* A reprint of XL 475 (English translation by Stefan Bauer-Mengelberg of the preceding). Ibid., odd pp. 103–123.

KURT GÖDEL. *Über die Vollständigkeit des Logikkalküls (1930a).* Ibid., p. 124. (Reprinted from ***Die Naturwissenschaften***, vol. 18 (1930), p. 1068.)

KURT GÖDEL. *On the completeness of the calculus of logic (1930a).* English translation by John Dawson of the preceding. Ibid., p. 125.

BURTON DREBEN and JEAN VAN HEIJENOORT. *Introductory note to 1929, 1930 and 1930a.* Ibid., pp. 44–59.

The question of the completeness of their axiomatization of first-order logic was posed explicitly by Hilbert and Ackermann in 1928. In Gödel's doctoral dissertation, he proved this completeness both for single formulas and for countably infinite sets of formulas; moreover he obtained these results for logic formulated either with or without equality. The published version (*Gödel 1930*) also noted the compactness theorem for countably infinite sets of formulas. *Gödel 1930a* is a one-paragraph abstract of a lecture given at a meeting at Königsberg at which Gödel quite unexpectedly announced his incompleteness theorem. See the review below of *Gödel 1931a.*

The techniques used in the proof of completeness were Skolem normal form together with what is today usually called the Herbrand expansion, although it really goes back to Löwenheim and Skolem. In their "Note," Dreben and van Heijenoort emphasize that the very question of completeness could not really have arisen in the principal schools of mathematical logic as they were in the early part of the twentieth century, and present a careful and very illuminating discussion of the relationships between the work of Skolem, Herbrand, and Gödel in the 1920's. It is quite clear that Skolem and Herbrand each had all the technical methods and results needed to obtain the completeness theorem. Their orientation pointed them in other directions, however, and indeed their tendency to regard non-constructive methods as questionable made it difficult for them even to pose the question.

Gödel's dissertation begins with a fascinating introduction that was omitted from the published version. This introduction contains the striking sentence: "We cannot at all exclude out of hand, however, a proof of the unsolvability of a problem if we observe that what is at issue here is only unsolvability by certain *precisely stated formal* means of inference" (p. 63). MARTIN DAVIS

KURT GÖDEL. *Einige metamathematische Resultate über Entscheidungsdefinitheit und Widerspruchsfreiheit (1930b).* A reprint of 418*1*. Ibid., pp. 140, 142.

KURT GÖDEL. *Some metamathematical results on completeness and consistency (1930b).* A reprint of XXXVII 405 (English translation by Stefan Bauer-Mengelberg of the preceding). Ibid., pp. 141, 143.

KURT GÖDEL. *Über formal unentscheidbare Sätze der Principia mathematica und verwandter Systeme I (1931).* A reprint of 418*3*. Ibid., even pp. 144–194.

KURT GÖDEL. *On formally undecidable propositions of Principia mathematica and related systems I (1931).* A reprint of XXXVII 405 (English translation by Jean van Heijenoort of the preceding). Ibid., odd pp. 145–195.

KURT GÖDEL. *Über Vollständigkeit und Widerspruchsfreiheit (1932b).* A reprint of 418*8*. Ibid., pp. 234, 236.

KURT GÖDEL. *On completeness and consistency (1932b).* A reprint of XXXII 405 (English translation by Jean van Heijenoort of the preceding). Ibid., pp. 235, 237.

STEPHEN C. KLEENE. *Introductory note to 1930b, 1931 and 1932b.* Ibid., pp. 126–141.

Gödel 1931 is, of course, the revolutionary paper in which the discoveries about arithmetic incompleteness were expounded. *Gödel 1930b* is a short abstract of this work. The translator chose to translate the difficult "Entscheidungsdefinitheit" as "completeness." Certainly the notion is precisely what today is called the completeness of a theory. Gödel presumably had a reason, however, at the time he wrote, for avoiding the term "Vollständigkeit" which is the proper German for "completeness"; probably it was to distinguish this sense from that in his dissertation. An alternative would have been "decidability"; but that translation also has its difficulties.

Gödel 1931 is remarkable not only because of the sensational results it contains, but also for the clarity of exposition and for the sharpness of the results. A rigorous exposition of the incompleteness theorems must focus on particular formal systems and will obtain particular undecidable sentences. Gödel selected as the formal system on which to base his development, which he called *P*, a version of the simple theory of types with Peano's postulates for the base type, and with a powerful comprehension axiom. This was presumably to emphasize that, as was clear from ***Principia mathematica***, all of ordinary mathematics could be formalized in this system. To emphasize the definitive character of the undecidability results, he obtained them not only for *P*, but also for any ω-consistent primitive recursive extension of *P*. On the other hand, he used the Chinese remainder theorem in an ingenious manner to show that the sentences proved undecidable in these systems could all be taken to be arithmetic.

To go from the intuitive basis of Gödel's proof to a completely rigorous version, the main technical obstacle was to show that certain metamathematical relations (in particular, that between the Gödel numbers of a proof and of the formula that the particular proof establishes) were numeralwise representable in *P*. Gödel accomplished this by first showing that these relations are primitive recursive ("primitive" was added later by Kleene; Gödel used the term "rekursiv," which has been translated simply as "recursive"), and then that all primitive recursive relations are numeralwise representable in *P*. It is very interesting that while the proof of the second result is only very briefly sketched, that of the first is given in great detail. Gödel was presumably concerned to present his results in such a way as to obtain ready acceptance. Since it was generally accepted that a system like *P* was adequate for the formalization of ordinary mathematics, Gödel evidently felt that there was no need to present the tedious details for such an elementary part of mathematics as recursive definitions in arithmetic. The reduction of metamathematical notions to ordinary mathematical ones was, on the other hand, quite novel. Verging on paradox as it did, it would have been easy for doubters to conclude that there was a lacuna here. And so, Gödel presented a list of forty-five definitions of primitive recursive functions and relations culminating with xBy: "x is the number of a proof of the formula whose number is y." It can well be argued that in giving this proof, Gödel incidentally invented programming languages. An experienced programmer will see at a glance that this list of forty-five formulas is really a computer program.

Kleene's "Note" places Gödel's work in the historical context of the Hilbert program. Kleene also gives a very careful and detailed outline of *Gödel 1931*. He concludes by mentioning Rosser's improvement of

Gödel's incompleteness theorem in which the assumption of ω-consistency is replaced by simple consistency, and the more recent Paris–Harrington discovery of a form of Ramsey's theorem that is undecidable in Peano arithmetic.

One of the results in *Gödel 1931* (Satz X) that seems to have gone largely unnoticed is that one can associate with each primitive recursive predicate $P(x)$ a formula of the predicate calculus that is satisfiable if and only if $P(x)$ is true for all natural numbers x. This result made a positive solution of Hilbert's Entscheidungsproblem seem very remote. Indeed the unsolvability of the Entscheidungsproblem is an immediate consequence of Satz X and the unsolvability results found around 1936. In particular this can be seen by using Kleene's result that there is a primitive recursive predicate $R(a, x)$ such that $(\exists x)R(a, x)$ is not recursive. In his "Note," Kleene suggests that other unsolvability results then available, such as Church's and Turing's would not have sufficed, yielding only a general recursive R for which the above holds. But surely, it would have been perfectly plain in 1936 that applying Gödel arithmetization to the λ-calculus or to the Turing machine formalism would readily yield a primitive recursive R corresponding to the unsolvable problems of determining whether a formula has a normal form or whether a particular Turing machine halts, respectively. Of course, it is true that Kleene made the existence of such an R explicit at the time, and the others did not.

In *Gödel 1932b*, the claim is made that the undecidability results already hold for the system Z consisting of the Peano axioms and the scheme of (primitive) recursion with first-order logic as the underlying logic. Moreover, the results continue to hold for any ω-consistent system that contains Z for which the set of numbers of axioms as well as the relations of immediate consequence under the rules of inference are decidable (that is numeralwise representable) in Z. No proofs are given. A note added by Gödel in 1966 when he saw the translation states that for the claimed results to hold the notion of a system "containing" another needs to be defined carefully (and not necessarily in the most obvious way).

MARTIN DAVIS

KURT GÖDEL. *Diskussion zur Grundlegung der Mathematik (1931a).* A reprint of 4184. Ibid., pp. 200, 202.

KURT GÖDEL. *Discussion on providing a foundation for mathematics (1931a).* English translation by John Dawson of the preceding. Ibid., pp. 201, 203. (Reprinted from ***History and philosophy of logic***, vol. 5 (1984), pp. 125–126.)

KURT GÖDEL. *Nachtrag.* A reprint of 4185. Ibid., pp. 202, 204.

KURT GÖDEL. *Postscript.* English translation by John Dawson of the preceding. Ibid., pp. 203, 205. (Reprinted from ***History and philosophy of logic***, vol. 5 (1984), pp. 127–128.)

JOHN W. DAWSON, Jr. *Introductory note to 1931a, 1932e, f and g.* Ibid., pp. 196–199.

In September 1930, a conference on the "epistemology of the exact sciences" was held in Königsberg. A key feature of the conference consisted of one-hour addresses on the three contending schools on the foundations of mathematics: Carnap on logicism, Heyting on intuitionism, and von Neumann on formalism. (*Gödel 1932e, f,* and *g* are his reviews for the ***Zentralblatt*** of the printed versions of their respective lectures. They are brief and factual and are not otherwise listed here.) At the conference, Gödel gave a twenty-minute talk on his dissertation (an abstract of which was his *1930a* already mentioned). The conference concluded with a roundtable discussion of the foundations of mathematics. Into this discussion Gödel dropped a bombshell: first tentatively and then quite explicitly he broached the subject of undecidability. He began by discussing the issue of the sufficiency of consistency as a criterion for accepting a formal system for mathematics, stating that it was indeed quite possible that a false statement "of the form $(Ex)F(x)$, where F is a finitary property of natural numbers" could be provable in a consistent system. Perhaps (as Dawson suggests) emboldened by a comment by von Neumann, Gödel continued with the definite assertion: "one can even give examples of propositions... that, while contentually [inhaltlich] true, are unprovable in the formal system of classical mathematics. Therefore, if one adjoins the negation of such a proposition to the axioms of classical mathematics, one obtains a consistent system in which a contentually false proposition is provable" (p. 203).

Gödel 1931a consists of his comments at this discussion together with a postscript (requested by the editors) in which he outlines his results. Dawson's "Note" gives a careful and illuminating description of the events surrounding the Königsberg conference.

MARTIN DAVIS

KURT GÖDEL. Review of Hilbert's *Die Grundlegung der elementaren Zahlentheorie* (108***18***). Ibid., pp. 212, 214. (Reprinted from ***Zentralblatt für Mathematik und ihre Grenzgebiete***, vol. 1 (1931), p. 260.)

KURT GÖDEL. English translation by John Dawson of this review. Ibid., pp. 213, 215.

SOLOMON FEFERMAN. *Introductory note to 1931c.* Ibid., pp. 208–213.

In the paper, reviewed by Gödel for the ***Zentralblatt***, Hilbert proposed adjoining to first-order arithmetic with primitive recursive definitions a new rule stated by Gödel as follows: "in case it is demonstrated that, whenever z is a particular numeral, the formula $A(z)$ is a correct numerical formula, then the formula $(x)A(x)$ may be taken as a premise" (p. 215). The "demonstration" was to be by finitary means. Feferman notes that it is impossible to tell whether Hilbert introduced this rule in response to Gödel's results. Gödel's cautious review emphasizes that the rule is "structurally ... of an entirely new kind." Feferman gives a very interesting account of the role this new rule was seen as playing at the time, especially in connection with the great hopes then held out for the consistency proofs that had been given by Ackermann and von Neumann. He points out that "the main defect" of Hilbert's paper is that a "precisely defined formal system ... is replaced by an imprecisely defined system ... using the vague ... concept of finitary proof in an essential way" (p. 212). Finally, Feferman discusses the relation between Hilbert's proposed rule and various forms of the ω-rule. MARTIN DAVIS

KURT GÖDEL. *Zum intuitionistische Aussagenkalkül (1932).* A reprint of 418*6*. Ibid., pp. 222, 224.

KURT GÖDEL. *On the intuitionistic propositional calculus (1932).* English translation by John Dawson of the preceding. Ibid., pp. 223, 225.

A. S. TROELSTRA. *Introductory note to 1932.* Ibid., pp. 222–223.

In this brief note, Gödel outlines a proof of the fact that the Heyting (intuitionistic) propositional calculus cannot be viewed as a many-valued logic for any finite number of truth values. A construction is given of a monotonically increasing sequence of propositional calculus intermediate in strength between the intuitionistic and classical propositional calculi. Troelstra's "Note" discusses subsequent developments. MARTIN DAVIS

KURT GÖDEL. *Ein Spezialfall des Entscheidungsproblems der theoretischen Logik (1932a).* A reprint of 418*7*. Ibid., even pp. 230–234.

KURT GÖDEL. *A special case of the decision problem for theoretical logic (1932a).* English translation by John Dawson of the preceding. Ibid., odd pp. 231–235.

KURT GÖDEL. *Zum Entscheidungsproblem des logischen Funktionenkalküls (1933i).* A reprint of 418*13*. Ibid., even pp. 306–326.

KURT GÖDEL. *On the decision problem for the functional calculus of logic (1933i).* English translation by John Dawson of the preceding. Ibid., odd pp. 307–327.

KURT GÖDEL. Review of Kalmár's *Über die Erfüllbarkeit derjenigen Zählausdrücke, welche in der Normalform zwei benachbarte Allzeichen enthalten* (3847). Ibid., pp. 330, 332. (Reprinted from ***Zentralblatt für Mathematik und ihre Grenzgebiete***, vol. 6 (1933), pp. 385–386.)

KURT GÖDEL. English translation by John Dawson of this review. Ibid., pp. 331, 333.

WARREN D. GOLDFARB. *Introductory note to 1932a, 1933i and l.* Ibid., pp. 226–231.

While Gödel's work on the decision problem for first-order logic lacked the spectacular quality of his completeness and incompleteness theorems, he did achieve what were, in a sense, the best possible results. Research had focused on *prefix classes* of prenex formulas, defined as having particular quantificational prefixes. Decision algorithms were found for certain prefix classes called *decidable*; for other prefix classes, called *reduction classes*, algorithms were found that "reduce" the satisfiability of an arbitrary formula to that of a formula that belongs to the class. The goal (which of course we now know was unattainable) was to find a prefix class that was both decidable and a reduction class. Gödel showed that the $\exists\cdots\exists\forall\forall\exists\cdots\exists$ class was decidable whereas the $\forall\forall\forall\exists\cdots\exists$ class was a reduction class. The gap was thus reduced to a single universal quantifier. Gödel succeeded in showing that formulas belonging to the $\exists\cdots\exists\forall\forall\exists\cdots\exists$ class had the additional property of being satisfiable if and only if they were satisfiable in a finite domain.

Gödel 1932a is a brief ***Ergebnisse*** report on his decidability result. *Gödel 1933i* is the complete paper containing both results. *Gödel 1933l* is a review of a paper by Kalmár who had independently discovered essentially the same technique for establishing decidability (as Schütte also did).

Gödel 1933i has the unique distinction of being the only one of his publications in which he claimed a result that subsequently turned out to be false. Gödel claimed that his decidability result (as well as the fact about finite satisfiability just mentioned) was also true for first-order logic with identity. Difficulties in supplying a proof of this gradually convinced investigators that the problem had to be regarded as still open. Warren Goldfarb settled the matter in 1983 by showing that even the $\forall\forall\exists$ class with identity is undecidable.

Goldfarb's note gives the history leading up to Gödel's results, outlines the proofs, and explains how in the 1960's and 1970's it became ever more doubtful that Gödel had really obtained his decidability result for the case with identity. Finally he outlines the proof of his own undecidability result.

MARTIN DAVIS

KURT GÖDEL. *Eine Eigenschaft der Realisierungen des Aussagenkalküls (1932c).* A reprint of 418*9*. Ibid., pp. 238, 240.

KURT GÖDEL. *A property of the realizations of the propositional calculus (1932c).* English translation by John Dawson of the preceding. Ibid., pp. 239, 241.

W. V. QUINE. *Introductory note to 1932c.* Ibid., pp. 238–239.

Gödel proves a form of what has become known as Lindenbaum's theorem. Specifically, let negation and implication operations be defined on an arbitrary set. Let a set of these elements contain all instances of a particular axiom set for the propositional calculus and be closed under modus ponens. Gödel shows that this set can be extended to one that for each element contains either it or its negation (but not both), and such that an implication belongs to the set if and only either its consequent does or its antecedent does not belong to the set. The proof is a straightforward transfinite induction. As Quine points out, what is noteworthy about this ***Ergebnisse*** paper is the consideration of systems of logic with a possibly uncountable number of elements.

MARTIN DAVIS

KURT GÖDEL. Review of Church's *A set of postulates for the foundation of logic* (359*4*). Ibid., pp. 256, 258. (Reprinted from ***Zentralblatt für Mathematik und ihre Grenzgebiete***, vol. 4 (1932), pp. 145–146.)

KURT GÖDEL. English translation by John Dawson of this review. Ibid., pp. 257, 259.

KURT GÖDEL. Review of Church's *A set of postulates for the foundation of logic (second paper)* (359*6*). Ibid., pp. 380, 382. (Reprinted from ***Zentralblatt für Mathematik und ihre Grenzgebiete***, vol. 8 (1934), p. 289.)

KURT GÖDEL. English translation by John Dawson of this review. Ibid., pp. 381, 383.

KURT GÖDEL. Review of Church's *A proof of freedom from contradiction* (359*8*). Ibid., pp. 398, 400. (Reprinted from ***Zentralblatt für Mathematik und ihre Grenzgebiete***, vol. 12 (1936), pp. 241–242.)

KURT GÖDEL. English translation by John Dawson of this review. Ibid., pp. 399, 401.

STEPHEN C. KLEENE. *Introductory note to 1932k, 1934e and 1936b.* Ibid., pp. 256–257.

The λ-calculus originated in Church's attempt to create a comprehensive logical system in which function abstraction was used to obtain the existence of abstract entities and paradoxes were to be avoided by restricting the law of the excluded middle. As successive attempts turned out to be inconsistent, the λ-calculus was extracted as a provably consistent subsystem. Gödel's brief reviews of Church's papers were for the ***Zentralblatt***. Church's thesis was first developed for λ-definability; these reviews are interesting, although they are Gödel's typical abstract-like factual accounts, because they make it clear that Gödel was familiar with Church's systems. Kleene's "Note" recounts the history of these systems.

In his review of 359*4*, Gödel carelessly states: "while in this system the set of all sets that do not contain themselves... can be formed, it cannot be proved that it either does or does not contain itself [von ihr nicht beweisen, daß sie sich entweder selbst enthält oder nicht enthält]." Of course, Gödel was entitled to assert merely that there was no evident way to prove this.

MARTIN DAVIS

KURT GÖDEL. *Über Unabhängigkeitsbeweise im Aussagenkalküls (1933a).* A reprint of 418*10*. Ibid., pp. 268, 270.

KURT GÖDEL. *On independence proofs in the propositional calculus (1933a).* English translation by John Dawson of the preceding. Ibid., pp. 269, 271.

W. V. QUINE. *Introductory note to 1933a.* Ibid., pp. 268–269.

In this ***Ergebnisse*** note, Gödel gives an example of an independence result for a propositional calculus fragment that can be established by using infinitely many truth values, but not by using any finite set of truth values. MARTIN DAVIS

KURT GÖDEL. *Zur intuitionistischen Arithmetik und Zahlentheorie (1933e).* A reprint of 418*11*. Ibid., even pp. 286–294.

KURT GÖDEL. *On intuitionistic arithmetic and number theory (1933e).* English translation by Stefan Bauer-Mengelberg and Jean van Heijenoort of the preceding. Ibid., odd pp. 287–295.

A. S. TROELSTRA. *Introductory note to 1933e.* Ibid., pp. 282–287.

It had been supposed that intuitionistic mathematics as proposed by Brouwer was a subsystem of classical mathematics since it interdicted classical methods of proof regarded as incorrect. Although Brouwer had shown no interest in formal systems, his disciple Heyting had proposed logical systems intended to formalize intuitionistically acceptable reasoning. Even before Gödel, however, a number of researchers had realized that there was an important sense in which the classical propositional calculus could be regarded as a subsystem of the Heyting propositional calculus. Thus, a formula involving only negation and conjunction is derivable in Heyting's calculus if and only if it is a tautology. But this is effectively an embedding of the classical proposition calculus in Heyting's calculus, since the other connectives are (classically) definable in terms of negation and conjunction.

Gödel 1933e is an ***Ergebnisse*** paper in which the embedding is extended to arithmetic. Unusual for one of Gödel's ***Ergebnisse*** papers, full proofs are given. The specific classical system for arithmetic with which Gödel works was proposed by Herbrand, and as Troelstra points out in his "Note," it "is not quite a formal system as the term is usually understood today" (p. 282). For the intuitionistic system, Gödel uses the same axioms together with Heyting's formalization of intuitionistic mathematics. This latter involves a little syntactic juggling because Heyting's variables range over all "mathematical objects," not just natural numbers. Nowadays, one works with Peano arithmetic (PA) as the classical system and then HA, the intuitionistic version, is simply PA with the classical propositional calculus replaced by Heyting's. In this context, Gödel's result is essentially that a formula involving only negation, conjunction, and universal quantification is derivable in PA if and only if it is derivable in HA. Troelstra's "Note" gives a very complete historical discussion of related work, both before and after Gödel's.

An earlier English translation (by the present reviewer, and approved by Gödel with a few modifications) appeared in *Davis 1965*. The present translation does seem to be better.

MARTIN DAVIS

KURT GÖDEL. *Eine Interpretation des intuitionistischen Aussagenkalküls (1933f).* A reprint of 418*12*. Ibid., pp. 300, 302.

KURT GÖDEL. *An interpretation of the intuitionistic propositional calculus (1933f).* English translation by John Dawson of the preceding. Ibid., pp. 301, 303.

A. S. TROELSTRA. *Introductory note to 1933f.* Ibid., pp. 296–299.

In this short ***Ergebnisse*** note, Gödel considers a system obtained from the ordinary propositional calculus by adding a "modal" operator B with the intended interpretation "is provable" ("B" for "beweisbar"), the axioms

$$Bp \rightarrow p, \quad Bp \rightarrow (B(p \rightarrow q) \rightarrow Bq), \quad Bp \rightarrow BBp,$$

and the rule of inference: from α derive $B\alpha$. Heyting's propositional calculus is translated into Gödel's system by interpreting negation and conjunction as themselves and implication and disjunction of operands α, β as the implication and disjunction, respectively, of $B\alpha$ and $B\beta$. Gödel's system, regarded as a modal logic with B as a necessity operator, is equivalent to S4. Gödel notes that the translation of a formula provable in Heyting's propositional calculus is derivable in his system and conjectures the converse. Troelstra's "Note" points out that this converse was proved by McKinsey and Tarski. Troelstra goes on to outline the very fruitful subsequent history of these ideas. MARTIN DAVIS

KURT GÖDEL. Review of Hahn's ***Reelle Funktionen***. Ibid., even pp. 332–336. (Reprinted from ***Monatshefte für Mathematik und Physik***, vol. 40 (1933), Literaturberichte, pp. 20–22.)