Satisfiability of CTL* with constraints[1]

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HIGHLIGHTS
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Overview

- Introduction: CTL* with constraints
- WMSO+B
- Satisfiability of CTL* with constraints
- Constraints over $\mathbb{Z}$

P - countable set of atomic propositions

Definition. CTL* formulas

state $\varphi := p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathsf{E}\psi \qquad (p \in \mathsf{P})$

path $\psi := \varphi \mid \neg\psi \mid (\psi \wedge \psi) \mid \mathsf{X}\psi \mid \psi\mathsf{U}\psi$

CTL* with constraints

P - countable set of atomic propositions
V - countable set of variables
$\mathcal{S}$ - finite set of relation symbols (*Signature*)

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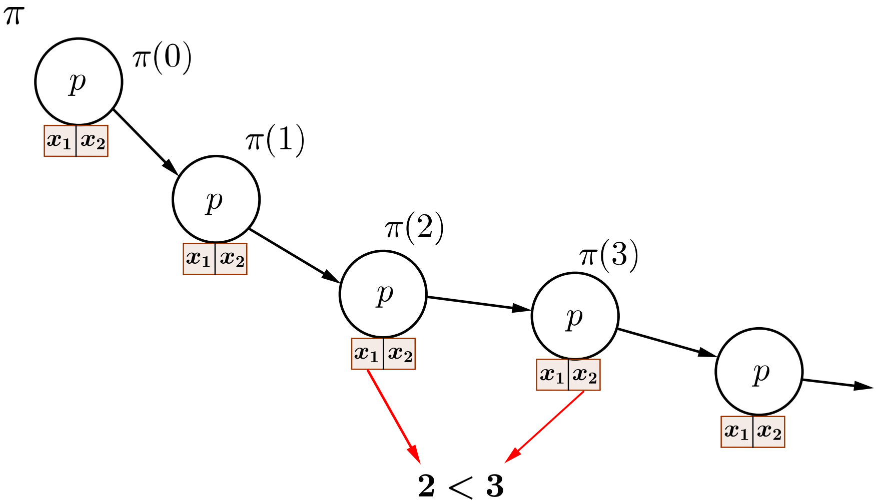
path $\quad \psi \ := \ \varphi \mid \neg\psi \mid (\psi \wedge \psi) \mid \mathsf{X}\psi \mid \psi\mathsf{U}\psi \mid r(\mathsf{X}^{i_1}x_1, \ldots, \mathsf{X}^{i_k}x_k)$

Atomic Constraint

$r \in \mathcal{S}, \quad k = \mathrm{ar}(r), \quad x_1, \ldots, x_k \in \mathsf{V}, \quad i_1, \ldots, i_k \geq 0$

CTL* with constraints

$$\mathsf{G}p \wedge <(\mathsf{X}^2 x_1, \mathsf{X}^3 x_2)$$



CTL* with constraints

V - variables

$\mathcal{S}$ - signature

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Atomic Constraints

$$(\mathcal{C}, \pi) \models r(\mathrm{X}^{i_1} x_1, \ldots, \mathrm{X}^{i_k} x_k) \text{ iff}$$
$$r(\gamma(\pi(i_1), x_1), \ldots, \gamma(\pi(i_k), x_k))$$

WMSO+B

$\mathcal{S}$ - signature, $r \in \mathcal{S}$ of arity k

$x_1, \ldots, x_k, x, y$ - FO variables

X - **Monadic** SO variable

Definition. WMSO+B formulas over $\mathcal{S}$

$$\varphi \ := \ r(x_1, \ldots, x_k) \mid x = y \mid x \in X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x\ \varphi \mid \exists X\ \varphi \mid \mathrm{B}X : \varphi$$

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$(A, r_1, \ldots, r_n) \models \mathsf{B}X : \varphi(X)$ iff $\exists b \in \mathbb{N}$ s.t. for all finite $C \subseteq A$ with $\mathcal{A} \models \varphi(C)$ we have $|C| \leq b$.

WMSO+B

Example

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G is acyclic $\text{Path}_E(a, b, Z)$

$$\text{BPaths}_E(x, y) = \mathsf{B}Z : \text{Path}_E(x, y, Z)$$

Theorem (Bojanczyk, Torunczyk, 2012)

Satisfiability over infinite trees is decidable for Bool(MSO, WMSO+B).

SATISFIABILITY

- Fix a signature $\mathcal{S}$
- Fix a concrete domain $\mathcal{A}$ ($\mathcal{S}$-structure)
- Let φ be a CTL$^*(\mathcal{S})$-state formula

φ is $\mathcal{A}$-satisfiable if there is an $\mathcal{A}$-constraint graph $\mathcal{C} = (\mathcal{A}, \mathcal{K}, \gamma)$ and a node v of $\mathcal{K}$ such that $(\mathcal{C}, v) \models \varphi$.

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Satisfiability of $\mathrm{CTL}^*(\mathcal{A})$

INPUT: $\varphi \in \mathrm{CTL}^*(\mathcal{S})$.

QUESTION: Is φ $\mathcal{A}$-satisfiable?

Satisfiability of CTL$^*(\mathcal{A})$

Theorem 1

If $\mathcal{A}$ is negation-closed and has the property EHomDef(Bool(MSO, WMSO+B)) then

satisfiability of CTL$^*(\mathcal{A})$ is decidable.

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Example

$(\mathbb{Z}, <, =)$ is negation-closed.

- $\neg\, x < y$ if and only if $x = y \vee y < x$.

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Definition. EHomDef

Given a signature $\mathcal{S}$ and a logic $\mathcal{L}$,
an $\mathcal{S}$-structure $\mathcal{A}$ has the property EHomDef($\mathcal{L}$) iff

there exists a $\mathcal{L}$-sentence ψ such that
for every $\mathcal{S}$-structure $\mathcal{B}$ with countable domain the following holds:

there exists a homomorphism $h : \mathcal{B} \to \mathcal{A}$ if and only if $\mathcal{B} \models \psi$.

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We write $\mathcal{B} \leq \mathcal{A}$ if a homomorphism from $\mathcal{B}$ to $\mathcal{A}$ exists.

EHomDef

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Characterization:

For a countable structure $\mathcal{B} = (B, r_<, r_=)$
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Translation:

$\psi := \neg \exists x \, \exists y (\text{reach}_{\leq}(x, y) \wedge y < x)$

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Translation:

$\psi := \neg\exists x\, \exists y(\text{reach}_\leq(x, y) \wedge y < x)$

$$\mathcal{B} \models \psi \text{ iff } \mathcal{B} \text{ has no } <\text{-cycles} \quad \text{iff } \mathcal{B} \leq \mathcal{Q}$$

Satisfiability of $\mathrm{CTL}^*(\mathcal{A})$

Theorem 1

If $\mathcal{A}$ is negation-closed and has the property $\mathrm{EHomDef}(\mathrm{Bool}(\mathrm{MSO}, \mathrm{WMSO}{+}\mathrm{B}))$ then

satisfiability of $\mathrm{CTL}^*(\mathcal{A})$ is decidable.

Satisfiability of CTL$^*(\mathcal{A})$

Theorem 1

If $\mathcal{A}$ is negation-closed and has the property EHomDef(Bool(MSO, WMSO+B)) then

$$\text{satisfiability of CTL}^*(\mathcal{A}) \text{ is decidable.}$$

- tree model property of CTL$^*(\mathcal{S})$ [Gascon, 2008]

Theorem 1

If $\mathcal{A}$ is negation-closed and has the property EHomDef(Bool(MSO, WMSO+B)) then

satisfiability of $\mathrm{CTL}^*(\mathcal{A})$ is decidable.

- tree model property of $\mathrm{CTL}^*(\mathcal{S})$ [Gascon, 2008]
- decidability result for Bool(MSO, WMSO+B) [Bojanczyk and Torunczyk, 2012]

Concrete domains over $\mathbb{Z}$

How do we use all this?

Example

The structure $\mathcal{Q} = (\mathbb{Q}, <, =)$ has the property EHomDef(WMSO) and it is negation-closed.

$\Rightarrow$ satisfiability of $\mathrm{CTL}^*(\mathcal{Q})$ is decidable!

How do we use all this?

Example

The structure $\mathcal{Q} = (\mathbb{Q}, <, =)$ has the property EHomDef(WMSO) and it is negation-closed.

$\Rightarrow$ satisfiability of CTL$^*(\mathcal{Q})$ is decidable!

Theorem 2

$\mathcal{Z} = (\mathbb{Z}, <, =)$ has the property EHomDef(WMSO+B)

($\Rightarrow$ satisfiability of CTL$^*(\mathcal{Z})$ is decidable!)

STEP 1: $(\mathbb{Z}, <)$

Lemma - Characterization

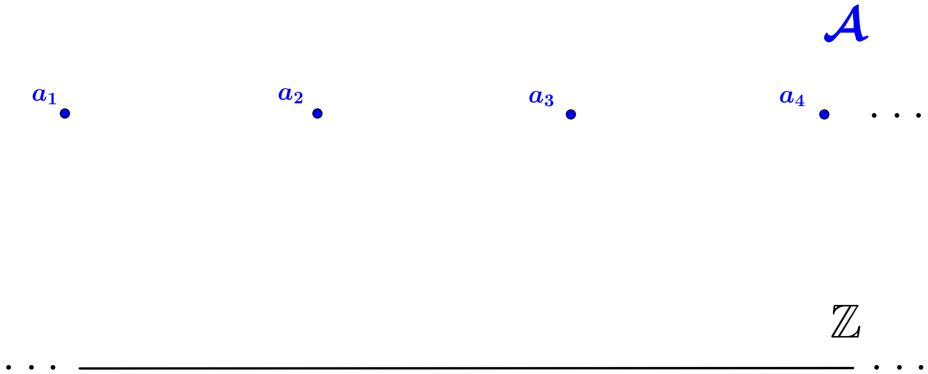
Let $\mathcal{A} = (A, <)$ be a countable structure.

$\mathcal{A} \preceq (\mathbb{Z}, <)$ if and only if:

(H1) $\mathcal{A}$ does not contain cycles

(H2) for all $x, y \in A$ there is a $b \in \mathbb{N}$ such that the length of all paths from x to y is bounded by b.

Concrete domains over $\mathbb{Z}$



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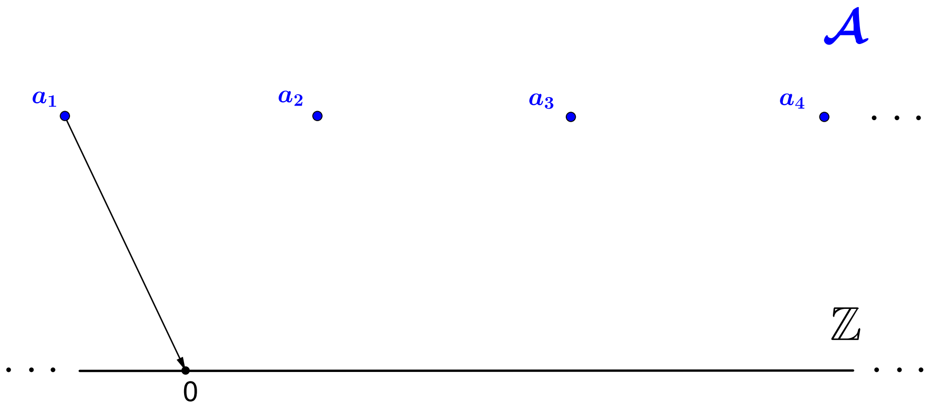

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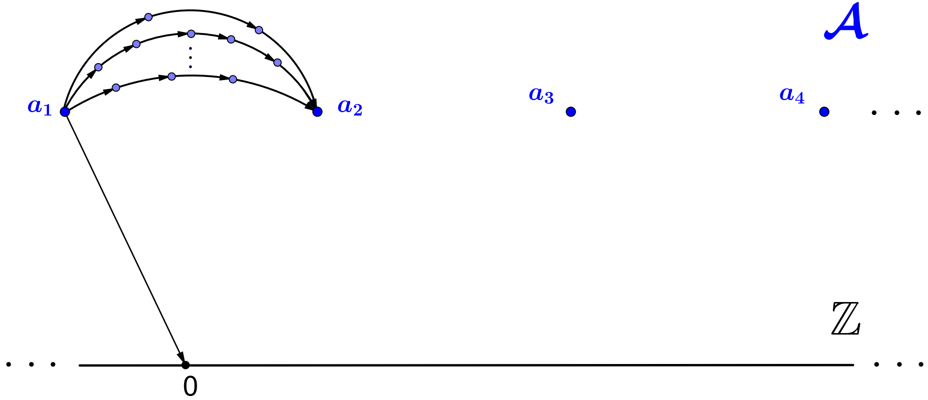

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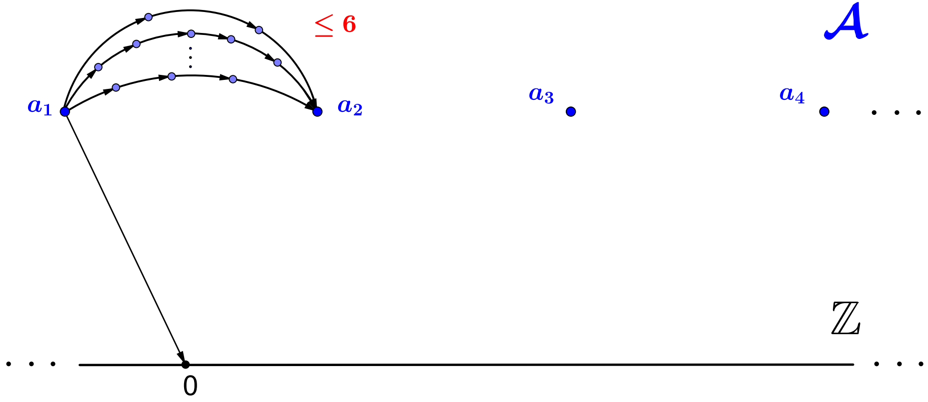

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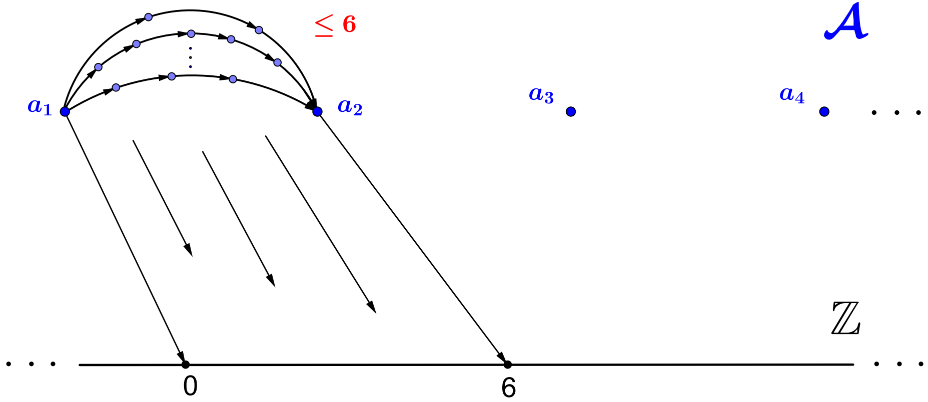

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Lemma - Translation

We can express H1 and H2 in WMSO+B.

(H1) $\neg \mathrm{ECycle}_<$

(H2) $\forall x \forall y\ \mathrm{BPaths}_<(x, y)$

STEP 2: $(\mathbb{Z}, <, =)$

Concrete domains over $\mathbb{Z}$

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STEP 3: $(\mathbb{Z}, <, =, (=_a)_{a\in\mathbb{Z}}, (\equiv_{a,b})_{0\leq a<b})$