

# Positive first-order logic on words

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## Abstract

We study  $\text{FO}^+$ , a fragment of first-order logic on finite words, where monadic predicates can only appear positively. We show that there is a FO-definable language that is monotone in monadic predicates but not definable in  $\text{FO}^+$ . This provides a simple proof that Lyndon’s preservation theorem fails on finite structures. We additionally show that given a regular language, it is undecidable whether it is definable in  $\text{FO}^+$ .

## 1 Introduction

Perservation theorems in first-order logic (FO) establish a link between semantics and syntactic properties [AG97, Ros08]. We will be particularly interested here in Lyndon’s theorem [Lyn59], which states that if a first-order formula is monotone in a predicate  $P$  (semantic property), then it is equivalent to a formula that is positive in  $P$  (syntactic property). This result was first shown to fail on finite structures in [AG87] with a very difficult proof, using a large array of techniques from different fields of mathematics such as probability theory, topology, lattice theory, and analytic number theory. A simpler but still quite intricate proof was later given by [Sto95], using Ehrenfeucht-Fraïssé games on grid-like structures equipped with two binary predicates.

The goal of this paper is to look at this problem through the lens of the theory of regular languages. We hope to show how this angle can bring new insights to both model theory and regular language theory.

## Contributions

We define a notion of closed language on ordered alphabets, requiring a monotonicity condition on monadic predicates. We define a negation-free logic  $\text{FO}^+$ , which can only define closed languages. We then proceed to show that there is a regular language that is closed and FO-definable, but not  $\text{FO}^+$  definable, thereby giving a proof of the failure of Lyndon’s theorem on finite structures that is much simpler than those in [AG87, Sto95].

This means that  $\text{FO}^+$ -definable languages form a strict subclass of closed FO-definable languages. We show that given a regular language, it is undecidable

whether it is  $\text{FO}^+$ -definable. To our knowledge, this is the first example of a natural subclass of regular languages for which membership is undecidable.

## Related works

### Membership in subclasses of regular languages

Related to our undecidability result, we can mention that there are subclasses of regular languages for which decidability of membership is an open problem. Such classes, also related to FO fragments, are the ones defined via quantifier-alternation: given a regular language, is it definable with a FO formula having at most  $k$  quantifier alternations? Recent works obtained decidability results for this question, but only for the first 3 levels of the quantifier alternation hierarchy [PZ19]. For higher levels, the problem remains open. Let us also mention the generalized star-height problem [PST89]: can a given regular language be defined in an extended regular expression (with complement allowed) with no nesting of Kleene star? In this case it is not even known whether all regular languages can be defined in this way.

### Monotone complexity

Positive fragments of first-order logic play a prominent role in complexity theory. Indeed, an active research program consists in studying positive fragments of complexity classes. This includes for instance trying to lift equivalent characterizations of a class to their positive versions, or investigating whether a semantic and a syntactic definition of the positive variant of a class are equivalent. See [GS92] for an introduction to monotone complexity, and [LSS96, STE94] for examples of characterizations of the positive versions of the classes P and NP through extensions of first-order logic. The aforementioned paper [AG87], which was the first to show the failure of Lyndon’s theorem on finite structures, does so by reproving in particular an important result on monotone circuit complexity first proved in [FSS81]:  $\text{Monotone-AC}^0 \neq \text{Monotone} \cap \text{AC}^0$ .

### Quantitative extensions

First-order logic on words has been extended to quantitative settings, which naturally yields a negation-free syntax, because complementation becomes problematic in those settings. This is the case in the theory of regular cost functions [Col12, KVB12], and in other quantitative extensions concerned with boundedness properties, such as  $\text{MSO}+\text{U}$  [Boj04] or Magnitude  $\text{MSO}$  [Col13]. We hope that the present work can shed a light on these extensions as well.

## Notations and prerequisites

If  $i, j \in \mathbb{N}$ , we note  $[i, j]$  the set  $\{i, i+1, \dots, j\}$ . We will note  $A$  a finite alphabet throughout the paper. The set of words on  $A$  is  $A^*$ . The length of  $u \in A^*$  is denoted  $|u|$ . We will note  $\text{dom}(u) = [0, |u| - 1]$  the set of positions of a word  $u$ . If  $u$  is a word and  $i \in \text{dom}(u)$ , we will note  $u[i]$  the letter at position  $i$ , and  $u[..i]$

the prefix of  $u$  up to position  $i$ . Similarly,  $u[i..j]$  is the infix of  $u$  from position  $i$  to  $j$  and  $u[i..]$  is the suffix of  $u$  starting in position  $i$ .

We will assume that the reader is familiar with the notion of regular languages of finite words, and with some ways to define such languages: non-deterministic finite automata (NFA), finite monoids, and first-order logic. See e.g. [DG08] for an introduction of all the needed material.

## 2 Monotonicity on words

### 2.1 Ordered alphabet

In this paper we will consider that the finite alphabet  $A$  is equipped with a partial order  $\leq_A$ . This partial order is naturally extended to words componentwise:  $a_1a_2 \dots a_n \leq_A b_1b_2 \dots b_m$  if  $n = m$  and for all  $i \in [1, n]$  we have  $a_i \leq_A b_i$ .

A special case that will be of interest here is when the alphabet is built as the powerset of a set  $P$  of *predicates*, i.e.  $A = 2^P$ , and the order  $\leq_A$  is inclusion. We will call this a *powerset alphabet*.

Taking  $A = 2^P$  is standard in settings such as verification and model theory, where several predicates can be considered independently of each other in some position.

Powerset alphabets constitute a particular case of ordered alphabets. The results obtained in this paper are valid for both the powerset case and the general case. Due to the nature of the results (existence of a counter-example and undecidability result), it is enough to show them in the particular case of powerset alphabets to cover both cases. Moreover, the powerset alphabet case allows us to directly establish a link with Lyndon's theorem, which is stated in the framework of model theory. For these reasons, we will keep the more general notion of ordered alphabet for generic definitions, but will prove our main results on powerset alphabets in order to directly obtain the stronger version of these results.

### 2.2 Closed languages

We fix  $A$  a finite ordered alphabet.

**Definition 1.** We say that a language  $L \subseteq A^*$  is *closed* if for all  $u \in L$  and  $v \geq_A u$ , we have  $v \in L$ .

**Example 2.** Let  $A = \{a, b\}$  with  $a \leq_A b$ . Then  $A^*bA^*$  is closed but  $A^*aA^*$  is not closed.

**Lemma 3.** A language  $L$  is closed if and only if for all  $(u, a, b, v) \in A^* \times A \times A \times A^*$  with  $a \leq_A b$ , we have  $uav \in L \Rightarrow ubv \in L$ .

*Proof.* The less trivial direction is the right-to-left one. Assume the right-hand side, and take  $u \leq_A v$  with  $u \in L$ . Recall that  $|u| = |v|$ , and positions where  $u$  and  $v$  differ are labelled with letters comparable for  $\leq_A$ , with the smaller being

in  $u$ . We can prove  $v \in L$ , by repetitively using the right-hand side hypothesis, once for each position where  $u$  and  $v$  differ.  $\square$

**Theorem 4.** *Given a regular language  $L \subseteq A^*$ , it is decidable whether  $L$  is closed.*

*Proof.* Let  $M$  be a finite monoid recognizing  $L$ , i.e. there exists a surjective monoid morphism  $h : A^* \rightarrow M$ , and a subset  $F \subseteq M$  such that  $L = h^{-1}(F)$ . We can compute such a  $(M, h, F)$  from any description of  $L$ . Recall that  $h$  is completely described by its behaviour on letters of  $A$ . To decide whether  $L$  is closed, it now suffices to verify that for all  $(x, a, b, y) \in M \times A \times A \times M$  with  $a \leq_A b$ , if  $x \cdot h(a) \cdot y \in F$  then  $x \cdot h(b) \cdot y \in F$ . Using the surjectivity of  $h$ , this is simply the characterization of Lemma 3 reflected in  $M$ .  $\square$

**Definition 5.** Let  $L \subseteq A^*$ , the *closure* of  $L$  is the language  $L^\uparrow = \{v \in A^* \mid \exists u \in L, u \leq_A v\}$ . It is the smallest closed language containing  $L$ .

In particular, if  $a \in A$ , we will note  $a^\uparrow$  the set  $\{b \in A \mid a \leq_A b\}$ .

**Lemma 6.** *Given a NFA for a language  $L$ , we can compute a NFA for  $L^\uparrow$ .*

*Proof.* Let  $\mathcal{A}$  be a NFA for  $L$ . We build a NFA  $\mathcal{A}^\uparrow$  from  $\mathcal{A}$ , by replacing every transition  $p \xrightarrow{a} q$  of  $\mathcal{A}$  by  $p \xrightarrow{a^\uparrow} q$ . We use here the standard convention where a transition  $p \xrightarrow{B} q$  with  $B \subseteq A$  stands for a set of transitions  $\{p \xrightarrow{b} q \mid b \in B\}$ . It is straightforward to verify that  $\mathcal{A}^\uparrow$  is a NFA for  $L^\uparrow$ : any run of  $\mathcal{A}^\uparrow$  on some word  $v$  can be mapped to a run of  $\mathcal{A}$  on some  $u \leq_A v$ .  $\square$

## 3 Positive first-order logic

### 3.1 Syntax and semantics

The main idea of positive FO, that we will note  $\text{FO}^+$ , is to guarantee via a syntactic restriction that it only defines closed languages.

Notice that since closed languages are not closed under complement, we cannot allow negation in the syntax of  $\text{FO}^+$ . This means we have to add dual versions of classical operators of first-order logic.

This naturally yields the following syntax for  $\text{FO}^+$ :

$$\varphi, \psi := a^\uparrow(x) \mid x \leq y \mid x < y \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \exists x. \varphi \mid \forall x. \varphi$$

As usual, variables  $x, y, \dots$  range over the positions of the input word. The semantics is the same as classical FO on words, with the notable exception that  $a^\uparrow(x)$  is true if and only if  $x$  is labelled by some  $b \in a^\uparrow$ . Unlike classical FO, it is not possible to require that a position is labelled by a letter  $a$ , except when  $a^\uparrow = \{a\}$ . This is necessary to guarantee that only closed languages can be defined. The formal definition of the semantics of  $\text{FO}^+$  can be found in Appendix A.1. The notation  $u \models \varphi$  denotes that  $u$  is a model of  $\varphi$ .

**Example 7.** On alphabet  $A = \{a, b, c\}$  with  $a \leq_A b$ .

- $\forall x. a^\uparrow(x)$  recognizes  $(a + b)^*$ .
- $\exists x. b^\uparrow(x)$  recognizes  $A^*bA^*$ .

**Remark 8.** In the powerset alphabet framework where  $A = 2^P$ , we can naturally view  $\text{FO}^+$  as the negation-free fragment of first-order logic, by having atomic predicates  $a^\uparrow(x)$  range directly over  $P$  instead of  $A = 2^P$ . We can then drop the  $a^\uparrow$  notation, as predicates from  $P$  are considered independently of each other. This way,  $p(x)$  will be true if and only if the letter  $S \in A$  labelling  $x$  contains  $p$ . A letter predicate  $S^\uparrow(x)$  in the former syntax can then be expressed by  $\bigwedge_{p \in S} p(x)$ , so  $\text{FO}^+$  based on predicates from  $P$  is indeed equivalent to  $\text{FO}^+$  based on  $A$ . We will take this convention when working on powerset alphabets.

**Example 9.** Let  $A = 2^P$  with  $P = \{a, b\}$ . The formula  $\exists x, y. x \leq y \wedge a(x) \wedge b(y)$  recognizes  $A^*\{a, b\}A^* + A^*\{a\}A^*\{b\}A^*$ .

### 3.2 Properties of $\text{FO}^+$

**Lemma 10.** Assume the order on  $A$  is trivial, i.e. no two distinct letters are comparable. Then all languages are closed, and any FO-definable language is  $\text{FO}^+$ -definable.

*Proof.* The fact that all languages are closed in this case follows from the fact that for two words  $u, v$  we have  $u \leq_A v$  if and only if  $u = v$ .

If  $L$  is definable by a FO formula  $\varphi$ , we can build a  $\text{FO}^+$  formula  $\psi$  from  $\varphi$  by pushing negations to the leaves using the usual rewritings such as  $\neg(\varphi \wedge \psi) = \neg\varphi \vee \neg\psi$  and  $\neg(\exists x. \varphi) = \forall x. \neg\varphi$ . For all letter  $a \in A$  and variable  $x$ , we then replace all occurrences of  $\neg a(x)$  by  $\bigvee_{b \neq a} b(x)$ .  $\square$

**Lemma 11.** The logic  $\text{FO}^+$  can only define closed languages.

*Proof.* By induction on formulas, see Appendix A.2 for details.  $\square$

It is natural to ask whether the converse of Lemma 11 holds: if a language is FO-definable and closed, then is it necessarily  $\text{FO}^+$ -definable? This will be the purpose of Section 4.

**Definition 12.** The quantifier rank of a formula  $\varphi$ , noted  $\text{qr}(\varphi)$  is its number of nested quantifiers. It can be defined by induction in the following way: if  $\varphi$  is atomic then  $\text{qr}(\varphi) = 0$ , otherwise,  $\text{qr}(\varphi \wedge \psi) = \text{qr}(\varphi \vee \psi) = \max(\text{qr}(\varphi), \text{qr}(\psi))$  and  $\text{qr}(\exists x. \varphi) = \text{qr}(\forall x. \varphi) = \text{qr}(\varphi) + 1$ .

### 3.3 Ordered Ehrenfeucht-Fraïssé games

We will explain here how  $\text{FO}^+$ -definability can be captured by an ordered variant of Ehrenfeucht-Fraïssé games, that we will call  $\text{EF}^+$ -games.

This notion was defined in [Sto95] for general structures, we will instantiate it here on words.

We define the  $n$ -round  $\text{EF}^+$ -game on two words  $u, v \in A^*$ , noted  $\text{EF}_n^+(u, v)$ . This game is played between two players, Spoiler and Duplicator.

If  $k \in \mathbb{N}$ , a  $k$ -position of the game is of the form  $(u, \alpha, v, \beta)$ , where  $\alpha : [1, k] \rightarrow \text{dom}(u)$  and  $\beta : [1, k] \rightarrow \text{dom}(v)$  are valuations for  $k$  variables in  $u$  and  $v$  respectively. We can think of  $\alpha$  and  $\beta$  as giving the position of  $k$  previously placed tokens in  $u$  and  $v$ .

A  $k$ -position  $(u, \alpha, v, \beta)$  is *valid* if for all  $i \in [1, k]$ , we have  $u[\alpha(i)] \leq_A v[\beta(i)]$ , and for all  $i, j \in [1, k]$ ,  $\alpha(i) \leq \alpha(j)$  if and only if  $\beta(i) \leq \beta(j)$ .

Notice the difference with usual EF-games: here we do not ask that tokens placed in the same round have same label, but that the label in  $u$  is  $\leq_A$ -smaller than the label in  $v$ . This feature is intended to capture  $\text{FO}^+$  instead of FO.

The game starts from the 0-position  $(u, \emptyset, v, \emptyset)$ .

At each round, starting from a  $k$ -position  $(u, \alpha, v, \beta)$ , the game is played as follows. If  $(u, \alpha, v, \beta)$  is not valid, then Spoiler wins. Otherwise, if  $k = n$ , then Duplicator wins. Otherwise, Spoiler chooses a position in one of the two words, and places pebble number  $k + 1$  on it. Duplicator answers by placing pebble number  $k + 1$  on a position of the other word. Let us call  $\alpha'$  and  $\beta'$  the extensions of  $\alpha$  and  $\beta$  with these new pebbles. If  $(u, \alpha', v, \beta')$  is not a valid  $(k + 1)$ -position, then Spoiler immediately wins the game, otherwise, the game moves to the next round with  $(k + 1)$ -position  $(u, \alpha', v, \beta')$ .

We will note  $u \preceq_n v$  when Duplicator has a winning strategy in  $\text{EF}_n^+(u, v)$ .

**Theorem 13** ([Sto95, Thm 2.4]). *We have  $u \preceq_n v$  if and only if for all formula  $\varphi$  of  $\text{FO}^+$  with  $\text{qr}(\varphi) \leq n$ ,  $u \models \varphi \Rightarrow v \models \varphi$ .*

Since the proof of Theorem 13 does not appear in [Sto95], it can be found in Appendix A.3 for completeness.

**Corollary 14.** *A language  $L$  is not  $\text{FO}^+$ -definable if and only if for all  $n \in \mathbb{N}$ , there exists  $(u, v) \in L \times \bar{L}$  such that  $u \preceq_n v$ .*

*Proof.*  $\Leftarrow$  : Let  $n \in \mathbb{N}$ , there exists  $(u, v) \in L \times \bar{L}$  such that  $u \preceq_n v$ . By Theorem 13, any formula of quantifier rank  $n$  accepting  $u$  must accept  $v$ , so no formula of quantifier rank  $n$  recognizes  $L$ . This is true for all  $n \in \mathbb{N}$ , so  $L$  is not  $\text{FO}^+$ -definable.

$\Rightarrow$  (contrapositive): Assume there exists  $n \in \mathbb{N}$  such that for all  $(u, v) \in L \times \bar{L}$ ,  $u \not\preceq_n v$ . By Theorem 13, this means for all  $(u, v) \in L \times \bar{L}$ , there exists a formula  $\varphi_{u,v}$  of quantifier rank  $n$  accepting  $u$  but not  $v$ . Since there are finitely many  $\text{FO}^+$  formulas of rank  $n$  up to logical equivalence [Lib04, Lem 3.13], the set of formulas  $F = \{\varphi_{u,v} \mid (u, v) \in L \times \bar{L}\}$  can be chosen finite. We define  $\psi = \bigvee_{u \in L} \bigwedge_{v \notin L} \varphi_{u,v}$ , where the intersections and union are finite since  $F$  is finite. For all  $u \in L$ ,  $u$  is accepted by  $\bigwedge_{v \notin L} \varphi_{u,v}$  hence by  $\psi$ , and conversely, a word accepted by  $\psi$  must be accepted by some  $\bigwedge_{v \notin L} \varphi_{u,v}$ , so it cannot be in  $\bar{L}$ .  $\square$

## 4 Syntax versus semantics

We will now answer the natural question posed in Section 3.2: is any FO-definable closed language  $\text{FO}^+$ -definable?

### 4.1 A counter-example language

This section is dedicated to the proof of the following theorem:

**Theorem 15.** *There is a FO-definable closed language  $K$  on a powerset alphabet that is not  $\text{FO}^+$ -definable.*

Let  $P = \{a, b, c\}$  and  $A = 2^P$ , ordered by inclusion.

We will note  $\binom{a}{b}, \binom{b}{c}, \binom{c}{a}$  for the letters  $\{a, b\}, \{b, c\}, \{a, c\}$  respectively, and  $\top$  for  $\{a, b, c\}$ . If  $x \in P$  we will often note  $x$  instead of  $\{x\}$  to lighten notations.

We now define the wanted language by:

$$K := (a^\uparrow b^\uparrow c^\uparrow)^* + A^* \top A^*.$$

We claim that  $K$  satisfies the requirements of Theorem 15.

**Lemma 16.**  *$K$  is closed and FO-definable.*

*Proof.* The fact that  $K$  is closed is straightforward from its definition, as the union of two closed languages.

To show that  $K$  is FO-definable, we can simply use the classical characterizations of first-order definable languages [DG08], by verifying for instance that its 5-state minimal automaton is counter-free, or that its 21-element syntactic monoid is aperiodic, see Appendix A.4 for details and representations of these objects. In addition, it is useful to give an intuition on how a FO formula can describe the language  $K$ , as we will later build on this understanding in Section 5. We describe the behaviour of such a formula in Appendix A.5.  $\square$

**Lemma 17.**  *$K$  is not  $\text{FO}^+$ -definable.*

*Proof.* We establish this using Corollary 14. Let  $n \in \mathbb{N}$ , and  $N = 2^n$ . We define  $u = (abc)^N$  and  $v = [\binom{a}{b} \binom{b}{c} \binom{c}{a}]^{N-1} \binom{a}{b} \binom{b}{c}$ . Notice that  $u \in K$ , and  $v \notin K$  because  $|v| \equiv 2 \pmod{3}$ . By Corollary 14, it suffices to prove that  $u \preceq_n v$  to conclude. We give a strategy for Duplicator in  $\text{EF}_n^+(u, v)$ . The strategy is an adaptation from the classical strategy showing that  $(aa)^*$  is not FO-definable [Lib04]. We consider that at the beginning, tokens  $first, last$  are placed on the first and last positions on  $u$ , and  $first', last'$  on the first and last position of  $v$ . The strategy of Duplicator during the game is then as follows: every time Spoiler places a token in one of the words, Duplicator answers in the other by replicating the closest distance (and direction) to an existing token. This strategy is illustrated in Figure 1, where move  $i$  of Spoiler (resp. Duplicator) is represented by  $\textcolor{red}{i}$  (resp.  $\textcolor{green}{i}$ ).

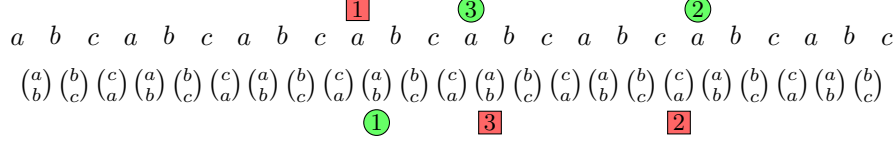


Figure 1: An example of Duplicator's strategy for  $n = 3$ .

We have to show that this strategy of Duplicator allows him to play  $n$  rounds without losing the game. This proof is similar to the classical one for  $(aa)^*$ , see e.g. [Lib04]. The main intuition is that the length of the non-matching intervals between  $u$  and  $v$  is at worst divided by 2 at each round, and it starts with a length of  $2^n$ , so Duplicator can survive  $n$  rounds. A detailed proof can be found in Appendix A.6.  $\square$

## 4.2 Lyndon's Theorem

In this section we consider first-order logic on arbitrary signatures and unconstrained structures.

**Definition 18.** A formula  $\varphi$  is *monotone* in a predicate  $P$  if whenever a structure  $S$  is a model of  $\varphi$ , any structure  $S'$  obtained from  $S$  by adding tuples to  $P$  is also a model of  $\varphi$ .

**Example 19.** On graphs, where the only predicate is the edge predicate, the formula asking for the existence of a triangle is monotone, but the formula stating that the graph is not a clique is not monotone.

**Definition 20.** A formula  $\varphi$  is *positive* in  $P$  if it never uses  $P$  under a negation.

Let us recall the statement of Lyndon's Theorem, which holds on general (possibly infinite) structures:

**Theorem 21** ([Lyn59, Cor 2.1]). *If  $\psi$  is a first-order formula monotone in predicates  $P_1, \dots, P_n$ , then it is equivalent to a formula positive in predicates  $P_1, \dots, P_n$ .*

We will now see explicitly how the language  $K$  from Section 4.1 can be used to show that Lyndon's Theorem fails on finite structures.

The failure of this theorem on finite structures was first shown in [AG87] with a very difficult proof, then reproved in [Sto95] with a simpler one, using the Ehrenfeucht-Fraïssé technique. Still, the proof from [Sto95] is quite involved compared to the one we present here.

### Several monotone predicates

In order to view our words as general finite structures, it suffices to axiomatize the fact that  $\leq$  is a total order. This can be done with a formula  $\psi_{tot} = (\forall x, y. x \leq y \vee y \leq x) \wedge (\forall x, y, z. x \leq y \wedge y \leq z \Rightarrow x \leq z) \wedge (\forall x, y. x \leq$



$y \wedge y \leq x \Rightarrow x = y) \wedge (\forall x. x \leq x)$ . Notice that  $\psi_{tot}$  is not monotone in the predicate  $\leq$ .

If we allow monotonicity on several predicates, then we directly obtain the failure of Lyndon's theorem of finite structures. Indeed, let  $\varphi$  be the FO-formula defining  $K$ , obtained in Lemma 16, and let  $\psi = \varphi \wedge \psi_{tot}$ . Then,  $\psi$  is monotone in predicates  $a, b, c$ , and finite structures satisfying  $\psi$  are exactly words of  $K$ . However, as we proved in Theorem 15, no first-order formula that is positive in predicates  $a, b, c$  can define the same class of structures.

### Single monotone predicate

Other formulations of Lyndon's Theorem use a single monotone predicate, as in [Sto95]. We can encode the language  $K$  in this framework, by using one binary predicate  $A$  to represent all letter predicates. Notice that apart from the empty word, all words of  $K$  have at least 3 positions. The three first positions are noted 0, 1, 2. Let  $\psi_3$  be a FO-formula stating that there are at least 3 elements 0, 1, 2, and that for all  $y \notin \{0, 1, 2\}$  and for all  $x$ ,  $A(x, y)$  holds. We build  $\varphi'$  from the formula  $\varphi$  recognizing the language  $K$  by replacing every occurrence of  $a(x)$  (resp.  $b(x), c(x)$ ) by  $A(x, 0)$  (resp.  $A(x, 1), A(x, 2)$ ).

Finally, we define the formula  $\psi' = \psi_{tot} \wedge \psi_3 \wedge \varphi'$ , which accepts exactly non-empty words of  $K$ . No formula positive in  $A$  can recognize this class of structures, otherwise we could obtain from it a  $\text{FO}^+$ -formula contradicting Theorem 15. This is done by replacing every occurrence of  $A(x, y)$  by  $(a(x) \wedge y = 0) \vee (b(x) \wedge y = 1) \vee (c(x) \wedge y = 2) \vee y \geq 3$ .

### Closure under surjective homomorphisms

Lyndon's theorem is also often stated in the following way: if a FO formula defines a class of structures closed under surjective homomorphisms, then it is equivalent to a positive formula. This formulation is equivalent to saying that the formula is monotone in all predicates. We can deal with this framework as well, by incorporating a predicate  $\preceq$  to the signature, and changing the formula  $\psi$  obtained above by  $\psi'' = (\exists x, y. x \leq y \wedge x \not\preceq y) \vee (\psi \wedge \forall x, y. (x \leq y \vee x \not\preceq y))$ .

This way, monotonicity constraints on  $\leq$  and  $\preceq$  become trivial: the only structures of interest, where  $\psi''$  is not trivially true or false, are the ones where  $\preceq$  is indeed the negation of  $\leq$ . We can therefore axiomatize in  $\psi_{tot}$  the fact that  $\leq$  is a total order, by using  $\leq$  and  $\preceq$  freely.

## 5 Undecidability of $\text{FO}^+$ -definability

This section is dedicated to the proof of the following Theorem:

**Theorem 22.** *The following problem is undecidable: given  $L$  a regular language on a powerset alphabet, is  $L$   $\text{FO}^+$ -definable?*

### 5.1 The Turing Machine Mortality problem

We will start by describing the problem we will reduce from, called Turing Machine (TM) Mortality.

The TM Mortality problem asks, given a deterministic TM  $M$ , whether there exists a bound  $n \in \mathbb{N}$  such that from any finite configuration (state of the machine, position on the tape, and content of the tape), the machine halts in at most  $n$  steps. We say that  $M$  is *mortal* if such an  $n$  exists.

**Theorem 23** ([Hoo66]). *The TM Mortality problem is undecidable.*

**Remark 24.** The standard mortality problem as formulated in [Hoo66] does not ask for a uniform bound on the halting time, and allows for infinite configurations, but it is well-known that the two formulations are equivalent using a compactness argument. Indeed, if for all  $n \in \mathbb{N}$ , the TM has a run of length at least  $n$  from some configuration  $C_n$ , then we can find a configuration  $C$  that is a limit of a subsequence of  $(C_n)_{n \in \mathbb{N}}$ , so that  $M$  has an infinite run from  $C$ .

Notice that the initial and final states of  $M$  play no role here, so we will omit them in the description of  $M$ . Indeed, we can assume that  $M$  halts whenever there is no transition from the current configuration.

Let  $M = (\Gamma, Q, \Delta)$  be a deterministic TM, where  $\Gamma$  is the alphabet of  $M$ ,  $Q$  its set of states, and  $\Delta \subseteq Q \times \Gamma \times Q \times \Gamma \times \{\leftarrow, \rightarrow\}$  its (deterministic) transition table. We want to build a regular language  $L$  such that  $L$  is  $\text{FO}^+$ -definable if and only if  $M$  is mortal.

We will also assume without loss of generality that  $Q$  is partitioned into  $Q_1, Q_2, Q_3$ , and that all possible successors of a state in  $Q_1$  (resp.  $Q_2, Q_3$ ) are in  $Q_2$  (resp.  $Q_3, Q_1$ ). We will say that  $p$  has *type*  $i$  if  $p \in Q_i$ . The *successor type* of 1 (resp. 2, 3) is 2 (resp. 3, 1).

Our goal is now to start from an instance  $M$  of TM Mortality, and define a regular language  $L$  such that  $L$  is  $\text{FO}^+$ -definable if and only if  $M$  is mortal.

## 5.2 The base language $L_{\text{base}}$

### The base alphabet

We define first a *base alphabet*, that will be used to encode configurations of the TM  $M$ :

$$A_{\text{base}} = \Gamma \cup (\Delta \times \Gamma) \cup (\Gamma \times \Delta) \cup (\Delta \times \Gamma \times \Delta) \cup (Q \times \Gamma) \cup \{\#\}.$$

We will note  $a_\delta$  (resp.  $a^{\delta'}, a_\delta^{\delta'}$ ) the letters from  $\Delta \times \Gamma$  (resp.  $\Gamma \times \Delta, \Delta \times \Gamma \times \Delta$ ), and  $[q.a]$  letters of  $Q \times \Gamma$ .

The letter  $[q.a]$  is used to encode the position of the reading head,  $q \in Q$  being the current state of the machine, and  $a \in \Gamma$  the letter it is reading.

A letter  $a_\delta$  will be used to encode a position of the tape that the reading head just left, via a transition  $\delta$  writing an  $a$  on this position. A letter  $a^{\delta'}$  will be used for a position of the tape containing  $a$ , and that the reading head is about to enter via a transition  $\delta'$ . We use  $a_\delta^{\delta'}$  if both are simultaneously true, i.e. the reading head is coming back to the position it just visited.

Finally, the letter  $\#$  is used as separator between different configurations.

### Configuration words

The encoding of a configuration of  $M$  is therefore a word of the form (for example)  $a_1 a_2 \dots (a_{i-1})^{\delta'} [q.a_i] (a_{i+1})_\delta \dots a_n$ . The letter  $(a_{i+1})_\delta$  indicates that the reading head came from the right via a transition  $\delta = (-, -, q, a_{i+1}, \leftarrow)$  (where  $-$  is a placeholder for an unknown element). The letter  $(a_{i-1})^{\delta'}$  indicates that it will go in the next step to the left via a transition  $\delta' = (q, a_i, -, -, \leftarrow)$ .

A word  $u \in (A_{base})^*$  is a *configuration word* if it encodes a configuration of  $M$  with no incoherences. More formally,  $u$  is a configuration word if  $u$  contains no  $\#$ , exactly one letter from  $Q \times \Gamma$  (the reading head), and either one  $a_\delta$  and one  $b^{\delta'}$  located on each side of the head, or just one letter  $a_\delta^{\delta'}$  adjacent to the head. Moreover, the labels  $\delta$  and  $\delta'$  both have to be coherent with the current content of the tape.

**Remark 25.** Because we ask these predecessor and successor labellings to be present, configuration words only encode TM configurations that have a predecessor and a successor configuration.

The *type* of a configuration word is simply the type in  $\{1, 2, 3\}$  of the unique state it contains.

Let us call  $C \subseteq (A_{base})^*$  the languages of configuration words. This language  $C$  is partitioned into  $C_1, C_2, C_3$  according to the type of the configuration word.

We can now define the language  $L_{base}$ . The basic idea is that we want  $L_{base}$  to be  $(C_1 \# C_2 \# C_3 \#)^*$ , but in order to avoid unnecessary bookkeeping later in the proof, we do not want to care about the endpoints being  $C_1$  and  $C_3$ . Let us also drop the last  $\#$  which is useless as a separator, and assume that  $C_1$  appears at least once. This gives for  $L_{base}$  the more complicated expression:

$$L_{base} := (\varepsilon + C_3 \# + C_2 \# C_3 \#) (C_1 \# C_2 \# C_3 \#)^* (C_1 + C_1 \# C_2 + C_1 \# C_2 \# C_3).$$

Notice that  $L_{base}$  cannot verify that the sequence is an actual run of  $M$ , since it just controls that the immediate neighbourhood of the reading head is valid, and that the types succeed each other according to the 1-2-3 cycle. The rest of the tape can be arbitrarily changed from one configuration word to the next.

### 5.3 The alphabet $A$

We now define another alphabet  $A_{amb}$  (*amb* for ambiguous), consisting of some unordered pairs of letters from  $A_{base}$ . An unordered pair  $\{a, b\}$  is in  $A_{amb}$  if  $a$  can be replaced by  $b$  in the encodings of two successive configurations of  $M$ . Thus, let  $A_{amb}$  be the following set of unordered pairs (we note the “predecessor” element first to facilitate the reading):

- $\{a_\delta, a\}$ ,  $a \in \Gamma$ ,  $\delta \in \Delta$
- $\{a, a^{\delta'}\}$ ,  $a \in \Gamma$ ,  $\delta' \in \Delta$
- $\{a^{\delta'}, [q.a]\}$ ,  $\delta' = (-, -, q, -, -) \in \Delta$
- $\{a_\delta^{\delta'}, [q.a]\}$ ,  $\delta' = (p, -, q, -, d) \in \Delta$ ,  $\delta = (q, a, p, -, -d) \in \Delta$

- $\{[p.a], b_\delta\}, \delta = (p, a, -, b, -) \in \Delta$
- $\{[p.a], b_\delta^{\delta'}\}, \delta = (p, a, q, b, d) \in \Delta, \delta' = (q, -, -, -, -d) \in \Delta$

Notice that all letters of  $A_{amb}$  have a clear “predecessor” element: even the possible ambiguity regarding letters  $a_\delta^{\delta'}$  are resolved thanks to the type constraint on transitions of  $M$ . For readability, we will use the notation  $\binom{a}{b}$  instead of  $\{a, b\}$ , where the upper letter is the predecessor element.

We can now define the alphabet  $A = A_{base} \cup A_{amb}$ , partially ordered by  $a <_A b$  if  $a \in A_{base}, b \in A_{amb}, a \in b$ . For now we use the general formalism of ordered alphabet for simplicity. We will later describe in Remark 42 how the construction is easily modified to fit in the powerset alphabet framework.

## 5.4 Superposing configuration words

**Lemma 26.** *If  $u_1, u_2 \in C$  encode two successive configurations of the same length, then there exists  $v \in A^*$  such that  $u_1 \leq_A v$  and  $u_2 \leq_A v$ .*

*Proof.* It suffices to take the letters in  $v$  to be the union of letters in  $u_1, u_2$  when these letters differ. For instance if  $u_1 = aab^{\delta'}[p.a]c_\delta c$  and  $u_2 = aa^{\delta'}[q.b]d_\delta cc$ , then  $v = a\binom{a}{a^{\delta''}}\binom{b^{\delta'}}{[q.b]}\binom{[p.a]}{d_\delta}\binom{c_\delta}{c}c$ .  $\square$

**Lemma 27.** *Let  $u_1, u_2 \in C$ , and  $v \in A^*$  such that  $u_1 \leq_A v$  and  $u_2 \leq_A v$ . Then either  $u_1 = u_2$ , or one is the successor configuration of the other.*

*Proof.* The alphabet  $A_{amb}$  is defined to enforce this.

Let  $[p.a]$  and  $[q.b]$  be the reading heads in  $u_1$  and  $u_2$ .

If  $p$  and  $q$  have the same type, then the  $a_\delta, a^{\delta'}$  extra labelling and the definition of  $A_{amb}$  are not compatible with the reading head changing position, so they must appear as the same  $[p.a]$  in the same position. In this case, this forces  $u_1 = u_2$ , as any difference would result in an incompatibility with the definition of  $C$  or  $A_{amb}$ .

If  $p$  and  $q$  do not have the same type, then one of them, say  $p$ , is the predecessor in the 1-2-3 cycle order. Then, the next transition  $\delta'$  labelling a letter adjacent to the reading head in  $u_1$  must yield state  $q$ , and the previous transition  $\delta$  next to  $[q.b]$  must be equal to  $\delta'$ , in order to avoid violation of local constraints imposed by  $C$  and  $A_{amb}$ . Therefore, locally there is a valid transition between  $u_1$  and  $u_2$ . On positions outside of the reading head in both  $u_1, u_2$  (labelled or not), the alphabet  $A$  ensure that the letter from  $\Gamma$  are the same in both words.  $\square$

**Lemma 28.** *It is impossible to have three distinct words  $u_1, u_2, u_3 \in C$  and  $v \in A^*$  such that for all  $i \in \{1, 2, 3\}$ ,  $u_i \leq_A v$ .*

*Proof.* By Lemma 27, any pair from  $\{u_1, u_2, u_3\}$  must consist in a configuration and its successor. However, since the reading head must move at each step, from  $u_1$  to  $u_2$  and from  $u_2$  to  $u_3$ , this means the reading head moves either 0 or 2 positions between  $u_1$  and  $u_3$ , which yields a contradiction.  $\square$

## 5.5 The language $L$

We finally define  $L$  to be the closure of  $L_{base}$  on alphabet  $A$ , so that  $L$  can contain letters from  $A_{amb}$ .

**Lemma 29.**  *$L$  is FO-definable and closed.*

*Proof.* The language  $L$  is closed by construction.

The fact that  $L$  is FO-definable can be obtained by combining the fact that  $C_1, C_2, C_3$  are all FO-definable, together with the fact that the language  $K$  from Section 4.1 is FO-definable as well, by Lemma 16. We also need Lemma 28 to guarantee that the equivalent of the letter  $\top$  from  $K$  never appears.  $\square$

## 5.6 The reduction

The goal of this section is to prove that  $L$  is  $FO^+$ -definable if and only if  $M$  is mortal, using Corollary 14.

The idea is that runs of  $M$  will allow us to build instances of  $EF^+$  games for  $L$ , with longer runs of  $M$  corresponding to Duplicator winning more rounds. Conversely, we will show that if  $M$  is mortal, then Spoiler wins any  $EF^+$  game in a fixed number of rounds.

### 5.6.1 $M$ not mortal $\implies L$ not $FO^+$ -definable

Let  $n \in \mathbb{N}$ , we aim to build  $(u, v) \in L \times \bar{L}$  such that  $u \preceq_n v$ .

There is a configuration from which  $M$  has a run of length  $N + 3$ , with  $N = 2^{n+1} + 1$ . Let  $u = u_0 \# u_1 \# \dots \# u_N$  be an encoding of this run where each  $u_i \in C$ , and where we omitted the first and last configurations of the run, which may not be representable in  $C$  by Remark 25. Here all the  $u_i$ 's are of the same length  $K$ , the size of the tape needed for this run.

By Lemma 26, for each  $i \in [0, N - 1]$ , there exists  $v_i \in A^*$  such that  $u_i \leq_A v_i$  and  $u_{i+1} \leq_A v_i$ .

We build  $v = u_0 \# v_1 \# \dots \# v_{N-2} \# u_N$ . Notice that  $v \notin L$ , because the types of  $u_0$  and  $u_N$  forces them to be separated by  $N - 1 \pmod 3$  configurations as in  $u$ , but in  $v$  they are separated by  $N - 2 \pmod 3$  configurations.

We describe a strategy for Duplicator witnessing  $u \preceq_n v$ . It is a simple adaptation from the proof of Lemma 17, so we will just sketch the idea.

Let us consider that initially, there is a pair of initial (resp. final) tokens at the beginning (resp. end) of  $u, v$ . We will consider that the initial tokens are “blue”, and the final ones are “yellow”. In the following, a pair of tokens will be blue if they are in the same position in  $u, v$ , and yellow if the token in  $v$  is shifted by  $K$  positions to the left compared to its corresponding token in  $u$ .

When Spoiler plays a token in  $u_i$  (resp.  $v_i$ ), Duplicator will look at the color of the closest token in  $u_i$ , (resp.  $v_i$ ), and answer with a token of the same color, i.e. by playing in  $v_i$  (resp.  $u_i$ ) for blue, and in  $v_{i-1}$  (resp.  $u_{i+1}$ ) for yellow. Of course, the same strategy applies to tokens played on  $\#$  positions.

This strategy preserves the following invariant: after  $k$  rounds, the number of  $\#$  between the last blue token and the first yellow token on the same word ( $u$  or  $v$ ) is at least  $2^{n-k}$ . This invariant guarantees that Duplicator wins the  $n$ -round game, since this gap will never be empty.

### 5.6.2 $M$ mortal $\implies L$ $\text{FO}^+$ -definable

Let  $M$  be a mortal  $TM$ , and  $n$  be the length of a maximal run of  $M$ , starting from any configuration.

We will show that  $L$  is  $\text{FO}^+$ -definable, by giving a strategy for Spoiler in  $EF_{f(n)}^+(u, v)$  for any  $(u, v) \in L \times \bar{L}$ , where the number of rounds  $f(n)$  depends only on  $n$ , and not on  $u, v$ .

Let us start by some auxiliary definitions on configuration words.

If  $u \in C$  is a configuration word, let us define its *height*  $h(u)$  to be the length of the run starting in  $u$ , and not going outside of the tape specified in  $u$ .

If  $u \in C$ , let us also define its  $n$ -approximation as the maximal word in  $(A_{\text{base}})^{\leq n} \cdot (Q \times \Gamma) \cdot (A_{\text{base}})^{\leq n}$  that is an infix on  $u$ . I.e. we remove letters whose distance to the reading head is bigger than  $n$ . Let us note  $\alpha_n(u)$  this  $n$ -approximation of  $u$ .

Here are a few properties of the height:

**Lemma 30.**     • for all  $u \in C$ , we have  $0 < h(u) < n$ .

- for all  $u \in C$  and  $x, y \in \Gamma^*$ , we have  $h(xuy) \geq h(u)$ .
- for all  $u \in C$ , we have  $h(u) = h(\alpha_n(u))$ .
- if  $v \in C$  is the successor configuration of  $u \in C$ , then  $h(v) = h(u) - 1$ .

*Proof.* The first item is a consequence of the fact that  $M$  is mortal with bound  $n$ , and moreover we ask that all words from  $C$  have a predecessor configuration and a successor one (Remark 25). The second item comes from the fact that the run of length  $h(u)$  starting in  $u$  is still possible when adding a context  $x, y$ , which is not affected by this run. The third item uses the fact that a run can only visit the  $n$ -approximation of  $u$ , so the context outside of  $\alpha_n(u)$  does not affect the height  $h(u)$ . The fourth item is a basic consequence of the definition of the height.  $\square$

**Corollary 31.** *The height of a configuration word  $u$  is a  $\text{FO}^+$ -definable property, i.e for all  $k \in \mathbb{N}$  there exists a  $\text{FO}^+$  formula  $h_k$  such that  $h_k$  accepts a configuration word  $u \in C$  if and only if  $h(u) = k$ .*

*Proof.* From Lemma 30, the formula  $h_k$  can simply use a lookup table to verify that  $\alpha_n(u)$  is of height  $k$ . Using  $\text{FO}^+$  instead of  $\text{FO}$  is not a restriction when we assume the input to be in  $(A_{\text{base}})^*$ . When evaluated on  $A^*$ , the formula  $h_k$  will accept the closure of configuration words of height  $k$ .  $\square$

**Remark 32.** We use here the fact that computation is done locally around the reading head to obtain Corollary 31. This seems to make Turing Machines more suited to this reduction than e.g. cellular automata, where computation is done in parallel on the whole tape.

Thanks to the height abstraction, we will show that we can focus on playing a special kind of abstracted EF-game.

### The integer game

Let  $\Sigma_{base} = [0, n]$  and  $\Sigma_{amb} = \{ \binom{i}{i-1} \mid 1 \leq i \leq n \}$ . Let  $\Sigma = \Sigma_{base} \cup \Sigma_{amb}$ , ordered by  $i \leq_\Sigma \binom{i}{i-1}$  and  $i-1 \leq_\Sigma \binom{i}{i-1}$  for all  $\binom{i}{i-1} \in \Sigma_{amb}$ .

We define the  $n$ -integer game as follows: It is played on an arena  $(u, v)$  with  $u \in (\Sigma_{base})^*$  and  $v \in (\Sigma_{amb})^*$ . If we note  $i$  (resp.  $j$ ) the first (resp. last) letter of  $u$ , then the first (resp. last) letter of  $v$  is  $\binom{i}{i-1}$  (resp.  $\binom{j+1}{j}$ ).

The rest of the rules is very close to those of  $EF^+(u, v)$ : in each round, Spoiler plays a token in  $u$  or  $v$ , Duplicator has to answer with a token in the other word, while maintaining the order between tokens, and the constraint that the label of a token in  $u$  is  $\leq_\Sigma$ -smaller than the label of its counterpart in  $v$ . We add an additional *neighbouring constraint* for Duplicator: consecutive tokens in one word must be related to consecutive tokens in the other, and in this case, if two tokens of  $v$  are in consecutive positions labelled  $\binom{i}{i-1} \binom{j}{j-1}$ , the corresponding tokens in  $u$  must be either labelled  $i, j$  or  $i-1, j-1$ . A mix  $i, j-1$  or  $i-1, j$  is not allowed.

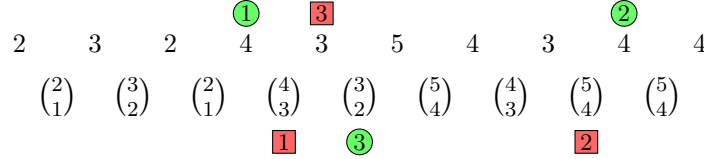


Figure 2: A position of the integer game.

**Lemma 33.** For all  $n \in \mathbb{N}$ , Spoiler can win any  $n$ -integer game in  $2n$  rounds.

*Proof.* We proceed by induction on  $n$ .

For  $n = 1$ , the constraints on the game forces  $u \in 1(0+1)^*0$  and  $v \in \binom{1}{0}^*$ .

We can have Spoiler play on the last occurrence of 1 in  $u$ , and on the successor position labelled 0. Duplicator cannot respond to these two moves while respecting the neighbouring constraint, so Spoiler wins in 2 moves.

Assume now that for some  $n \geq 1$ , Spoiler wins any  $n$ -integer game in  $2n$  moves, and consider on  $(n+1)$ -integer game arena  $(u, v)$ . If the letters  $n+1$  and  $\binom{n+1}{n}$  do not appear in  $u, v$  respectively, then Spoiler can win in  $2n$  moves by induction hypothesis.

If the letter  $n+1$  do not appear in  $u$ , then let  $y$  be the first position labelled  $\binom{n+1}{n}$  in  $v$ . By definition of the integer game  $y$  cannot be the first position of  $v$ , otherwise  $u$  should start with  $n+1$ . We will choose position  $y$  in  $v$  for the first move of Spoiler, let  $x$  be the position in  $u$  answered by Duplicator. We can

assume that  $x$  is not the first position of  $u$ , otherwise Spoiler can win in the next move. If Spoiler were to play  $x - 1$  in  $u$ , with  $u[x - 1] = i$ , by the neighbouring constraint Duplicator would be forced to answer  $y - 1$  in  $v$ , with label  $\binom{i+1}{i}$ . This shows that the words  $u[..x - 1]$  and  $v[..y - 1]$  form a correct  $n$ -integer arena, as the integer  $n + 1$  is not present anymore, and all other constraints are respected. Therefore, Spoiler can win by playing  $2n$  moves in these prefixes. This gives a total of  $2n + 1$  moves in the original  $(n + 1)$ -integer game.

Finally, if the letter  $n + 1$  does appear in  $u$ , Spoiler starts by playing the last occurrence of  $n + 1$  in  $u$  at position  $x$ . Duplicator must answer a position  $y$  labelled  $\binom{n+1}{n}$ . As before, using the neighbouring constraint, we know that if  $i = u[x + 1]$ , then  $v[y + 1] = \binom{i}{i-1}$ . Therefore, the words  $u[x + 1..]$  and  $v[y + 1..]$  form an  $(n + 1)$ -integer game arena, and moreover the letter  $n + 1$  does not appear in  $u[x + 1..]$  (by choice of  $x$ ). Using the precedent case, we know that Spoiler can win from there in  $2n + 1$  moves, playing only on  $u[x + 1..]$  and  $v[y + 1..]$ . This gives a total of  $2n + 2$  moves in the original  $(n + 1)$ -integer game, thereby completing the induction proof.  $\square$

**Remark 34.** Lemma 33 still holds if the definition of  $n$ -integer game is generalized to include the symmetric case where, if we note  $i, j$  the first and last letters of  $u$  respectively,  $v$  starts with  $\binom{i+1}{i}$  and ends with  $\binom{j}{j-1}$ . Indeed, it suffices to consider the mirrored images of  $u$  and  $v$  to show that Spoiler wins in the same amount of rounds.

#### From the integer game to the original $EF^+$ -game.

Let us now show how we can use this integer game to describe a strategy for Spoiler in the original  $EF^+$ -game.

Let  $(u, v) \in L \times \bar{L}$ , and recall that  $n$  is the length of a maximal run of the Turing Machine  $M$ . We will show that Spoiler wins  $EF_{f(n)}^+(u, v)$ , for some  $f(n)$  depending solely on  $n$ .

Without loss of generality we can assume that  $u \in L_{base}$ . This is because there exists  $u' \in L_{base}$  with  $u' \leq_A u$ , and we can consider the pair  $(u', v)$  instead of  $(u, v)$ . Indeed, if Spoiler wins on  $(u', v)$ , then the same strategy is winning on  $(u, v)$ , since the winning condition is only easier for him in  $(u, v)$ .

Thus we can write  $u = u_0 \# u_1 \# \dots \# u_N$ , where each  $u_i$  is in  $C$ . Let us also write  $v = v_0 \# v_1 \# \dots \# v_T$ , where each  $v_i$  does not contain  $\#$ .

We will now describe a strategy for Spoiler in  $EF^+(u, v)$ , that is winning in a number  $f(n)$  of rounds only depending on  $n$ .

We will reuse ideas from the explicit formula from Lemma 16, developed in Appendix A.5.

Let us call *local factor* a factor of the form  $v_i \# v_{i+1}$ . A local factor is *forbidden* if it is not a factor of any word in  $L$ . If  $v$  contains a forbidden local factor, Spoiler can win in a constant number of moves (at most 5), by pointing the problematic positions in this local factor, that Duplicator will not be able to replicate in  $u$ . We therefore assume from now on that  $v$  does not contain a forbidden local factor.



**Definition 35.** A factor  $v_i$  of  $v$  is *compatible* with type  $j \in \{1, 2, 3\}$  if there exists  $u' \in C_j$  with  $u' \leq_A v_i$ . The *set-type* of  $v_i$  is  $\{j \mid v_i \text{ is compatible with } j\}$ .

By Lemma 28, each  $v_i$  is compatible with at most 2 distinct types in  $\{1, 2, 3\}$ . If  $v_i$  is compatible with 2 types, then one is the predecessor (resp. successor) of the other in the 1-2-3 cycle order, and we call it the *first type* (resp. *second type*) of  $v_i$ . We will consider that  $v_0$  (resp.  $v_T$ ) is only compatible with  $\text{type}(u_0)$  (resp.  $\text{type}(u_N)$ ). Indeed, if Duplicator matches  $v_0$  to a word  $u_i$  with  $i \neq 0$ , Spoiler can win the game in the next round, by choosing a  $\#$  position before  $u_i$  (and same argument for  $v_T$ ).

**Definition 36.** A factor of the form  $v_i \# v_{i+1} \# \dots \# v_j$  of  $v$  is called *ambiguous* if each  $v_i$  is compatible with two types, and the set-types succeed each other in the cycle order  $\{1, 2\} \rightarrow \{2, 3\} \rightarrow \{3, 1\} \rightarrow \{1, 2\}$ . For instance if the set-type of  $v_i$  is  $\{2, 3\}$ , then  $v_{i+1}$  must have set-type  $\{3, 1\}$ , etc. An ambiguous factor is *maximal* if it is not contained in a strictly larger ambiguous factor.

**Definition 37.** A factor  $v_i$  of  $v$  is called an *anchor* if either  $i = 0, i = T$  or if  $v_{i-1} \# v_i \# v_{i+1}$  is not ambiguous.

If  $v_i$  is an anchor, we can uniquely define its *anchor type*. It is simply its type if  $i = 0$  or  $T$ , and otherwise since  $v_{i-1} \# v_i \# v_{i+1}$  is not ambiguous, we define the anchor type of  $v_i$  to be the only possible type for  $v_i$  that is compatible with its two neighbours.

**Example 38.** Assume  $v_5$  has set-type  $\{2, 3\}$ ,  $v_6$  has set-type  $\{3, 1\}$ , and  $v_7$  has set-type  $\{2, 3\}$ . Then  $v_6$  is an anchor, and its anchor type is 1. The type 3 is indeed impossible for  $v_6$ , since  $v_7$  is not compatible with its successor type 1.

Notice that if Duplicator maps an anchor  $v_i$  to a word  $u_j$  such that  $\text{type}(u_j)$  is not the anchor type of  $v_i$ , then Spoiler can win in at most 5 moves, by pointing to a contradiction with the immediate neighbourhood of  $v_i$ .

**Definition 39.** A maximal ambiguous factor  $v_i \# v_{i+1} \# \dots \# v_j$  is *coherent* if  $v_{i-1} \# v_{i+1} \# \dots \# v_{j+1} \in L$ , and this is witnessed by the anchor types of  $v_{i-1}$  and  $v_{j+1}$ .

In other words,  $v_i \# v_{i+1} \# \dots \# v_j$  if the anchor types at the extremities are either both compatible with the first type of both  $v_i, v_j$ , or are both compatible with the second type. Here “compatible” is taken in the sense of the 1-2-3 cycle order.

**Example 40.** Let  $w = v_i \# v_{i+1} \# \dots \# v_j$  be a maximal ambiguous factor, where  $v_i$  has set-type  $\{1, 2\}$  and  $v_j$  has set-type  $\{2, 3\}$ . Assume  $v_{i-1}$  has anchor type 1, so it is compatible with the second type of  $v_i$ . This means that for  $w$  to be coherent, we need  $v_{j+1}$  to have anchor type 1, in order to be compatible with the second type of  $v_j$  as well.

**Lemma 41.**  $v$  contains a maximal ambiguous factor  $w$  that is not coherent.

*Proof.* Assume that all maximal ambiguous factors of  $v$  are coherent. Since  $v$  does not contain forbidden local factors, we have that the anchor types of two consecutive anchors follow the 1-2-3 order. This means that the anchor types, together with the coherence of maximal ambiguous factors, witness that  $v \in L$ . Since we know that  $v \notin L$ , this is a contradiction.  $\square$

We are now ready to describe the strategy of Spoiler. Spoiler will place a token at the beginning of  $w$ , and a token at the end of  $w$ . Because  $w$  is not coherent, Duplicator is forced to answer with the first type for one of these tokens, and with the second type for the other: otherwise Spoiler immediately wins by exposing the incoherence with the anchors delimiting  $w$ .

Spoiler can now play only between these existing tokens, and import the strategy from the integer game, by abstracting each word  $u_i$  by its height and each word  $v_j \geq_A u', u''$  (where  $u', u'' \in C$ ,  $h(u') = 1 + h(u'')$ ) by  $\binom{h(u')}{h(u'')}$ . Each factor of  $u, v$  delimited by  $\#$  corresponds to a single position in the abstracted integer game. Spoiler can for instance mimic a move of the integer game by playing in the first position of the corresponding factor in  $u$  or  $v$ , i.e. just after a  $\#$ -labelled position.

If Duplicator does not comply with the rules of the integer game, then Spoiler can punish it in a constant number of rounds (at most 5), because local constraints have been failed. The neighbourhood rule of the integer game is enforced by the fact that two consecutive words  $u_i, u_{i+1}$  in  $u$  must have types following the 1-2-3 order, so they must be matched both in the first type or both in the second type of corresponding factors  $v_j, v_{j+1}$  of the ambiguous factor  $w$ . Notice that Corollary 31 can be reflected here by the fact that if Duplicator tries to match two words of different height, Spoiler can detect it in  $\log n$  rounds, by inspecting only the  $n$ -neighbourhood of the reading heads, and using a dichotomy strategy (or in  $n$  rounds with a more naive strategy, inspecting the whole  $n$ -neighbourhood).

Combining these arguments and by Lemma 33, we obtain that following this strategy, Spoiler will win in at most  $f(n) = 2 + 2n + \log n + 5$  rounds, by punishing Duplicator as soon as Duplicator loses the  $n$ -integer game.

Using Theorem 13, we obtain that  $L$  is  $\text{FO}^+$ -definable, with a formula of quantifier rank at most  $f(n)$ . This concludes the proof of Theorem 22.

**Remark 42.** The alphabet  $A$  can be turned into a powerset alphabet, by adding all subsets of  $A_{base}$  absent from  $A_{amb}$ , rejecting any word containing  $\emptyset$  but no new non-empty subset, and accepting any word containing a new non-empty subset. This shows that this undecidability result still holds in the special case of powerset alphabet.

## Conclusion

We believe this paper gives an example of fruitful interaction between automata theory and model theory. Indeed, a classical result of model theory, the failure of Lyndon's theorem on finite structures, has been greatly simplified by considering

regular languages. Conversely, this question coming from model theory led to considering the logic  $\text{FO}^+$  on words, which defines the first (to our knowledge) natural fragment of regular languages with undecidable membership. We hope that the tools developed in this paper can be further used in both fields, and that this will encourage more interactions of this form in the future.

In the short term, we are interested in extending these techniques to the framework of cost functions, see [Kup14, Kup], and to other extensions of regular languages.

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## A Appendix

### A.1 Formal semantics of $\text{FO}^+$

We define here formally the semantics of  $\text{FO}^+$ . If  $\varphi$  is a formula with free variables  $\text{FV}(\varphi)$ , its semantics is a set  $\llbracket \varphi \rrbracket$  of pairs of the form  $(u, \alpha)$ , where  $u \in A^*$  and  $\alpha : \text{FV}(\varphi) \rightarrow \text{dom}(u)$  a valuation for the free variables.

We write indistinctively  $u, \alpha \models \varphi$  or  $(u, \alpha) \in \llbracket \varphi \rrbracket$ , to signify that  $(u, \alpha)$  is accepted by  $\varphi$ .

We define  $\llbracket \varphi \rrbracket$  by induction on  $\phi$ :

- $u, \alpha \models a^\dagger(x)$  iff  $u[\alpha(x)] \geq_A a$ .
- $u, \alpha \models x \leq y$  iff  $\alpha(x) \leq \alpha(y)$ .
- $u, \alpha \models x < y$  iff  $\alpha(x) < \alpha(y)$ .
- $\llbracket \varphi \vee \psi \rrbracket = \llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket$ .
- $\llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket$ .
- $u, \alpha \models \exists x. \varphi$  iff there exists  $i \in \text{dom}(u)$  such that  $(u, \alpha[x \mapsto i]) \in \llbracket \varphi \rrbracket$ .
- $u, \alpha \models \forall x. \varphi$  iff for all  $i \in \text{dom}(u)$ , we have  $(u, \alpha[x \mapsto i]) \in \llbracket \varphi \rrbracket$ .

### A.2 Proof of Lemma 11

We prove here that any language definable by  $\text{FO}^+$  is closed.

This is done by induction on the  $\text{FO}^+$  formula  $\varphi$ , where the induction property is strengthened to include possible free variables: for all  $(u, \alpha) \in \llbracket \varphi \rrbracket$  and  $v \geq_A u$ , we have  $(v, \alpha) \in \llbracket \varphi \rrbracket$ .

**Base cases:**

Let  $(u, \alpha) \in \llbracket a^\dagger(x) \rrbracket$  and  $v \geq_A u$ , we have  $v[\alpha(x)] \geq_A u[\alpha(x)] \geq_A a$ , so  $(v, \alpha) \in \llbracket a^\dagger(x) \rrbracket$ .

Let  $(u, \alpha) \in \llbracket x \leq y \rrbracket$  and  $v \geq_A u$ . We have  $\alpha(x) \leq \alpha(y)$  so  $(v, \alpha) \in \llbracket x \leq y \rrbracket$ . The argument for  $<$  instead of  $\leq$  is identical.

**Induction cases:**

Let  $(u, \alpha) \in \llbracket \varphi \vee \psi \rrbracket$  and  $v \geq_A u$ . We have  $(u, \alpha) \in \llbracket \varphi \rrbracket$  or  $(u, \alpha) \in \llbracket \psi \rrbracket$ . Therefore, by induction hypothesis,  $(v, \alpha) \in \llbracket \varphi \rrbracket$  or  $(v, \alpha) \in \llbracket \psi \rrbracket$ , hence  $(v, \alpha) \in \llbracket \varphi \vee \psi \rrbracket$ . The argument for  $\varphi \wedge \psi$  is identical.

Let  $(u, \alpha) \in \llbracket \exists x. \varphi \rrbracket$  and  $v \geq_A u$ . There exists  $i \in \text{dom}(u)$  such that  $(u, \alpha[x \mapsto i]) \in \llbracket \varphi \rrbracket$ . By induction hypothesis,  $(v, \alpha[x \mapsto i]) \in \llbracket \varphi \rrbracket$ . Hence,  $(v, \alpha) \in \llbracket \exists x. \varphi \rrbracket$ . The argument for  $\forall$  is identical.

### A.3 Proof of Theorem 13

The proof is an adaptation of the classical proof for correctness of EF-games, see e.g. [Lib04].

Since  $\text{FO}^+$  is a fragment of FO, we can directly use the following Lemma:

**Lemma 43** ([Lib04, Lem 3.13]). *Let  $n, k \in \mathbb{N}$ . Up to logical equivalence, there are finitely many formulas of quantifier rank at most  $n$  using  $k$  free variables.*

We will now show a strengthening of Theorem 13, where free variables are incorporated in the statement:

**Theorem 44.** *Let  $n, k \in \mathbb{N}$ ,  $u, v \in A$ ,  $\alpha : [1, k] \rightarrow \text{dom}(u)$  and  $\beta : [1, k] \rightarrow \text{dom}(v)$  be valuations for  $k$  variables  $x_1, \dots, x_k$  in  $u, v$  respectively. Then Duplicator wins  $\text{EF}_n^+(u, \alpha, v, \beta)$  if and only if for any  $\text{FO}^+$  formula  $\varphi$  with  $\text{qr}(\varphi) \leq n$  using  $k$  free variables  $x_1 \dots x_k$ , we have  $u, \alpha \models \varphi \Rightarrow v, \beta \models \varphi$ .*

*Proof.* We prove this by induction on  $n$ .

**Base case**  $n = 0$ :

Notice that quantifier-free formulas of  $\text{FO}^+$  are just positive boolean combinations of atomic formulas, that either compare the values of the free variables, or assert that the label of a free variable is  $\leq_A$ -greater than some letter  $a \in A$ . Consider that there is a quantifier-free formula  $\varphi$  with  $k$  free variables accepting  $u, \alpha$  but rejecting  $v, \beta$ . This happens if and only if there is a variable  $x_i$  such that  $u[\alpha(x_i)] \not\leq_A v[\beta(x_i)]$ , or if two variables  $x_i, x_j$  are not in the same order according to  $\alpha$  and  $\beta$ . That is, this happens if and only if  $(u, \alpha, v, \beta)$  is not a valid  $k$ -position, i.e. if and only if Spoiler wins the 0-round game  $\text{EF}_0^+(u, \alpha, v, \beta)$ .

**Induction case:** Assume there is a  $\text{FO}^+$  formula  $\varphi$  with  $\text{qr}(\varphi) \leq n$ , accepting  $u, \alpha$  but not  $v, \beta$ . The formula  $\varphi$  is a positive combination of atomic formulas, formulas of the form  $\exists x.\psi$ , and formulas of the form  $\forall x.\psi$ . Therefore, one of these formulas accepts  $u, \alpha$  but not  $v, \beta$ . If it is an atomic formula, then Spoiler immediately wins  $\text{EF}_n^+(u, \alpha, v, \beta)$  as in the base case.

If it is a formula of the form  $\exists x.\psi$ , then Spoiler can use the following strategy: pick a position  $p$  witnessing that the formula is true for  $u, \alpha$ , and play the position  $p$  in  $u$ . Duplicator will answer a position  $p'$  in  $v$ , and the game will move to  $(u, \alpha', v, \beta')$ , where  $\alpha' = \alpha[x \mapsto p]$  and  $\beta' = \beta[x \mapsto p']$ . Since the formula  $\psi$  has quantifier rank at most  $n - 1$ , and accepts  $u, \alpha'$  but not  $v, \beta'$ , by induction hypothesis Spoiler can win in the remaining  $n - 1$  rounds of the game.

Now if it is a formula of the form  $\forall x.\psi$ , then Spoiler can do the following: pick a position  $p'$  witnessing that the formula is false for  $v, \beta$ , and play the position  $p'$  in  $v$ . Duplicator will answer a position  $p$  in  $u$ , and the game will move to  $(u, \alpha', v, \beta')$ , where  $\alpha' = \alpha[x \mapsto p]$  and  $\beta' = \beta[x \mapsto p']$ . Since the formula  $\psi$  has quantifier rank at most  $n - 1$ , and accepts  $u, \alpha'$  but not  $v, \beta'$ , by induction hypothesis Spoiler can win in the remaining  $n - 1$  rounds of the game.

Let us now show the converse implication. We assume any formula of quantifier rank at most  $n$  accepting  $u, \alpha$  must accept  $v, \beta$ , and we give a strategy for Duplicator in  $\text{EF}_n^+(u, \alpha, v, \beta)$ .

Suppose Spoiler places pebble  $x$  at position  $p$  in  $u$ . Let  $\alpha' = \alpha[x \mapsto p]$ . By Lemma 43, up to logical equivalence, there is only a finite set  $F$  of  $\text{FO}^+$  formulas of rank at most  $n - 1$  with  $k + 1$  free variables accepting  $u, \alpha'$ . Let  $\psi = \bigwedge_{\varphi \in F} \varphi$ . Then  $u, \alpha$  is accepted by the formula  $\exists x. \psi$  of rank  $n$  (as witnessed by  $p$ ), so by assumption we also have  $v, \beta \models \exists x. \psi$ . This means there is a  $p' \in \text{dom}(v)$  such that  $v, \beta' \models \psi$ , where  $\beta' = \beta[x \mapsto p']$ . Duplicator can answer position  $p'$  in  $v$ , and by induction hypothesis he will win the remaining of the game, since every formula of  $F$  accepts  $v, \beta'$ .

Suppose now that Spoiler places pebble  $x$  at position  $p'$  in  $v$ . Let  $\beta' = \beta[x \mapsto p']$ . Let  $F$  be the finite set of formulas (up to equivalence) of quantifier rank at most  $n - 1$  and with  $k + 1$  free variables, that reject  $v, \beta'$ . Let  $\psi = \bigvee_{\varphi \in F} \varphi$ , and  $\psi' = \forall x. \psi$ . By construction,  $x = p'$  witnesses that  $\psi'$  does not accept  $v, \beta$ . Our assumption implies that it does not accept  $u, \alpha$  either. So there is  $p \in \text{dom}(u)$  such that  $u, \alpha' \not\models \forall x. \psi$ , where  $\alpha' = \alpha[x \mapsto p]$ . Duplicator can answer position  $p$  in  $u$ . If a formula  $\varphi$  of rank at most  $n - 1$  is true in  $u, \alpha'$ , then by construction it cannot appear in  $F$ , therefore it is also true in  $v, \beta'$ . By induction hypothesis, Duplicator wins the remaining  $(n - 1)$ -round game starting from  $(u, \alpha', v, \beta')$ .  $\square$

#### A.4 Automaton and monoid for the language $K$

Recall that  $K = (a^\dagger b^\dagger c^\dagger)^* + A^* \top A^*$ , with  $A = 2^{\{a, b, c\}}$ .

We show here that  $K$  is FO-definable, using the characterizations of [DG08] on both the minimal automaton and the syntactic monoid.

##### Minimal automaton

The minimal deterministic finite automaton (DFA)  $\mathcal{A}$  recognizing  $K$  is depicted in Figure 3. We note  $\neg a = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$  the sub-alphabet of  $A$  of letters not containing  $a$ , similarly for  $\neg b$  and  $\neg c$ . The edges going to rejecting state  $\perp$  are grayed and dashed, and the ones going to accepting sink state  $q_\top$  are grayed, for readability. We also note  $a' = a^\dagger \setminus \{\top\} = \{\{a\}, \binom{a}{b}, \binom{c}{a}\}$ , and similarly for  $b', c'$ .

To show that  $K$  is FO-definable, it suffices to show that  $\mathcal{A}$  is counter-free, i.e. that there is no word  $u \in A^*$  such that there are two distincts states  $p, q$  of  $\mathcal{A}$  and  $k \in \mathbb{N}$  such that  $p \xrightarrow{u} q$  and  $q \xrightarrow{u^k} p$ . Assume such an  $u$  exists, since the only non-trivial strongly connected component in  $\mathcal{A}$  is  $\{q_a, q_b, q_c\}$ , these states are the only candidates for  $p, q$ . Since  $p, q$  are distinct, it means  $|u|$  is not a multiple of 3, and  $u$  induces a 3-cycle, either  $q_a \xrightarrow{u} q_b \xrightarrow{u} q_c \xrightarrow{u} q_a$  if  $|u| \equiv 1 \pmod 3$  or in the reverse order if  $|u| \equiv 2 \pmod 3$ . Thus, the first letter of  $u$  can be read from all states from  $\{q_a, q_b, q_c\}$ , while staying in this component. Such a letter does not exist, so we reach a contradiction. The DFA  $\mathcal{A}$  is counter-free, so  $K$  is FO-definable [DG08].

##### Syntactic monoid

It is also instructive to see what the syntactic monoid of  $K$  looks like, in particular to get a first intuition on how a FO formula can be defined for  $K$ .

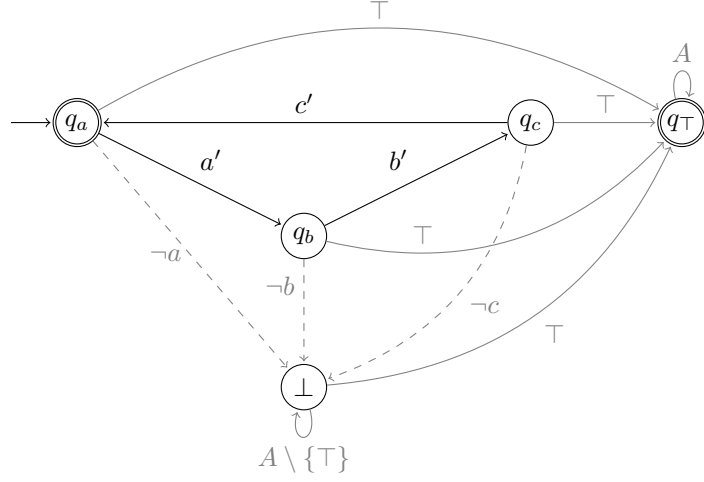


Figure 3: The minimal DFA  $\mathcal{A}$  of  $K$

1

$\begin{pmatrix} a \\ b \end{pmatrix}$	$\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}$	$\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} \begin{pmatrix} c \\ a \end{pmatrix}$
$\begin{pmatrix} b \\ c \end{pmatrix} \begin{pmatrix} c \\ a \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$	$\begin{pmatrix} b \\ c \end{pmatrix}$	$\begin{pmatrix} b \\ c \end{pmatrix} \begin{pmatrix} c \\ a \end{pmatrix}$
$\begin{pmatrix} c \\ a \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$	$\begin{pmatrix} c \\ a \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}$	$\begin{pmatrix} c \\ a \end{pmatrix}$

$a$	$ab$	$abc$
$bca$	$b$	$bc$
$ca$	$abc$	$c$

$\emptyset$

$\top$

Figure 4: The syntactic monoid  $M$  of  $K$



We depict this monoid  $M$  in Figure 4, using the eggbox representation based on Green's relations: boxes are  $\mathcal{J}$ -classes, lines are  $\mathcal{R}$ -classes, columns are  $\mathcal{L}$ -classes, and cells are  $\mathcal{H}$ -classes. See [Col11] for an introduction to Green's relations and eggbox representation.

The syntactic morphism  $h : A^* \rightarrow M$  is easily inferred, as elements of the monoid in  $h(A)$  are directly named after the letter mapping to them. The accepting part of  $M$  is  $F = \{1, \binom{a}{b}\binom{b}{c}\binom{c}{a}, \binom{c}{a}\binom{a}{b}\binom{b}{c}, abc, \top\}$ .

To show that  $K$  is FO-definable, it suffices to verify that  $M$  is aperiodic, which is directly visible on Figure 4, as all  $\mathcal{H}$ -classes are singletons (see [Col11]).

### A.5 An explicit FO formula for the language $K$

Recall that  $K = (a^\uparrow b^\uparrow c^\uparrow)^* + A^* \top A^*$ . We describe here the behaviour of a formula witnessing that  $K$  is FO-definable.

The  $A^* \top A^*$  part of  $K$  is just to rule out words containing  $\top$  by accepting them, which can be done by a formula  $\exists x. \top(x)$ . So we just need to design a formula  $\varphi$  for  $K' = (a^\uparrow b^\uparrow c^\uparrow)^*$ , assuming the letter  $\top$  does not appear, the final formula will then be  $\varphi \vee \exists x. \top(x)$ .

We will call *forbidden pattern* any word that is not an infix of a word in  $K'$ . Let us call *anchor* a position  $x$  such that either  $x$  is labelled by a singleton, or  $x$  is labelled by  $\binom{a}{b}$  (resp.  $\binom{b}{c}, \binom{c}{a}$ ) with  $x+1$  labelled by a letter different from  $\binom{b}{c}$  (resp.  $\binom{c}{a}, \binom{a}{b}$ ). The idea is that if  $x$  is an anchor position of  $u \in K'$ , then there is only one possibility for the value of  $x \bmod 3$ . If the first position is labelled by a letter from  $a^\uparrow$ , we will consider that it is an anchor labelled  $a$ , otherwise we will reject the input word. Similarly, the last position is either a  $c$  anchor or causes immediate rejection of the word. If  $x, y$  are successive anchor positions (i.e. with no other anchor positions between them), the word  $u[x+1..y-1]$  is necessarily an infix of  $((\binom{a}{b}\binom{b}{c}\binom{c}{a}))^*$ . We say that an anchor  $x$  *goes right-up* (resp. *right-down*) if we can replace the letter  $\binom{\alpha}{\beta}$  by  $\alpha$  (resp.  $\beta$ ) at position  $x+1$  without having a forbidden pattern in the immediate neighbourhood of  $x$ . Notice that  $x$  can not go both right-up and right-down. We define in the same way the left-up and left-down property by replacing  $x+1$  with  $x-1$ . For instance consider  $u = \binom{a}{b}\binom{b}{c}\binom{c}{a}\binom{a}{b}\binom{b}{c}\binom{c}{a}\binom{a}{b}\binom{b}{c}\binom{c}{a}\binom{a}{b}\binom{b}{c}\binom{c}{a}\binom{a}{b}\binom{b}{c}$ , then apart from the first and last position there are two anchors:  $x = 5$  labelled  $c$  and  $y = 10$  labelled  $\binom{b}{c}$ , because it is followed by another  $\binom{b}{c}$ .

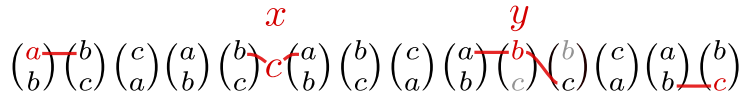


Figure 5: A visualization of anchors

The anchor  $x$  goes left-up and right-up, while the anchor  $y$  goes left-up and right-down. If  $d \in \{\text{up}, \text{down}\}$  is a direction, we say that two successive anchors  $x < y$  *agree on  $d$*  if  $x$  goes right- $d$  and  $y$  goes left- $d$ . We say that  $x$  and  $y$  agree if they agree on some  $d$ .

Now, the formula  $\varphi$  will express the following properties:

- for all  $x, x + 1$  consecutive anchors, the letters at positions  $x, x + 1, x + 2$  do not form a forbidden pattern (omit  $x + 2$  if  $x + 1$  is the last position).
- all non-consecutive successive anchors agree.

For instance the formula will accept the word  $u$  above, as the anchors  $0, x$  agree on up,  $x, y$  agree on up, and  $y, last$  agree on down.

It is routine to verify that these properties can be expressed in FO, and that they indeed characterize the language  $K'$ .

## A.6 Detailed proof of Lemma 17

We show here that the strategy of Duplicator defined in the proof of Theorem 15 of Section 4.1 indeed guarantees that Duplicator wins  $EF_n^+(u, v)$ .

We will generally write  $p, p'$  for related tokens,  $p$  being the position in  $u$  and  $p'$  the position in  $v$ .

The proof works by showing that the following invariant holds: after  $i$  rounds where Duplicator did not lose, if tokens in positions  $p < q$  in  $u$  are related to tokens  $p' < q'$  in  $v$ , and  $u[p..q] \not\leq_A v[p'..q']$ , let us note  $d = q - p, d' = q' - p'$ ; then  $d = d' + 1$  and  $d \geq 2^{n-i}$ . In other words, if we call *wrong interval* a factor  $u[p..q]$  or  $v[p'..q']$  such that  $u[p..q] \not\leq_A v[p'..q']$ , the invariant states that after  $i$  rounds, the length of the smallest wrong interval in  $u$  is at least  $2^{n-i}$ , and corresponding wrong intervals differ by 1, the one in  $u$  being longer. Before the first round, this invariant is true, as the only tokens are at the endpoints of  $u$  and  $v$ , and we have  $|u| = |v| + 1$  and  $|u| \geq 2^n$ . Now, assume the invariant true at round  $i$ , and consider round  $i + 1$ . When Spoiler plays a token in one of the words, two cases can happen. If it is played between previous tokens  $p, q$  (resp.  $p', q'$ ) such that  $u[p..q] \leq_A v[p'..q']$ , then Duplicator will simply answer the corresponding position in the other word, and the smallest wrong interval is not affected. If on the contrary, the new token is played in a minimal wrong interval, say  $u[p, q]$  on position  $r$ , then Duplicator will answer by preserving the closest distance between  $r - p$  and  $q - r$ . For instance if  $r - p < q - r$ , Duplicator will answer  $r' = p' + (r - p)$ . We can notice that by definition of the words  $u$  and  $v$ , and since  $u[p] \leq v[p']$  by the rules of the game, we have  $u[p..r] \leq_A v[p'..r']$ , and in particular  $u[r] \leq_A v[r']$ , so the move of Duplicator is legal. Moreover, since  $q - r > r - p$ , we have  $q - r \geq \frac{q - p}{2}$ , so using the induction hypothesis,  $q - r \leq 2^{n-(i+1)}$ . Moreover, since we had  $(q - p) = (q' - p') + 1$ , we now have  $(q - r) = (q - p) - (r - p) = (q' - p') + 1 - (r' - p') = (q' - r') + 1$ , so the invariant is preserved. The case where  $r - p \geq q - r$  is symmetrical. If on the other hand Spoiler plays in  $v$  a position  $r'$  in a wrong interval  $v[p'..q']$ , then  $\min(r' - p', q' - r')$  will be strictly smaller than  $2^{n-(i+1)}$ , and will be replicated by the answer  $r$  of Duplicator in  $u[p..q]$ . This means that the new smallest wrong interval created in  $u$  will have length at least  $2^{n-(i+1)}$ , thereby guaranteeing that the invariant is also preserved in this case.