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Complexity Bounds for Some Finite Forms of Kruskal's Theorem

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Well-founded (partial) orders form an important and convenient mathematical basis for proving termination of algorithms. Well-partial orders provide a powerful method for proving the well-foundedness of partial orders (and hence for proving termination), since every partial ordering which extends a given *well-partial ordering* on the same domain is automatically well-founded. In this article it is shown by purely combinatorial means that the maximal order type of the homeomorphic embeddability relation on a given set of terms over a finite signature yields an appropriate ordinal recursive Hardy bound on the lengths of bad sequences which satisfy an effective growth rate condition. This result yields theoretical upper bounds for the computational complexity of algorithms, for which termination can be proved by Kruskal's theorem.

1. Introduction

The investigations in this article are motivated by the general question of how to extract bounds on the computational complexity of algorithms for which termination has been *proved* using a certain method. If a termination proof is formalizable in a reasonable formal system T (which contains primitive recursive arithmetic) and if a profound ordinal analysis [cf. Pohlers (1992)] of T is established, then (a folklore result of) classical proof theory [cf., for example, Buchholz (1991) or Weiermann (1993b) and others] yields that there is a Hardy function H_α (the Hardy functions will be defined explicitly in the text below) of level α less than the proof-theoretic ordinal of T such that H_α bounds the computational complexity of the algorithm under consideration. In Buchholz (1994) it

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has been shown how this proof-theoretic approach can be used to prove non trivial upper bounds for derivation lengths of certain classes of rewrite systems.

However, in this article the emphasis is put on direct (non-meta) mathematical evaluations *(which are based on purely combinatorial means) of a special type of well-*foundedness proof. One common method for proving the well-foundedness of a (countable) partial ordering $<$ consists in constructing an order preserving embedding of $<$ into a (countable) partial ordering $<'$ which is known *a priori* to be well-founded. If some reasonable smoothness conditions are satisfied, then it is well known [see, for example, Theorem 1 in Buchholz,Cichon and Weiermann (1994) for a precise formulation and proof of this fact] that the lengths of effectively given $<$-descending sequences can be bounded by a Hardy function of ordinal level determined by the set-theoretic height of $<'$. Moreover, in the case of rewrite systems where $<'$ is a multiset path ordering (or a lexicographic path ordering) which contains the rewrite relation under consideration it has been shown that (as conjectured by Cichon) the lengths of derivations are bounded by functions which correspond in growth rate to functions (of ordinal index determined by the set-theoretic height of $<'$) from the slow growing hierarchy [cf. Hofbauer (1990), Weiermann (1992) and Weiermann (1993a)].

The emphasis of this article is put on proving termination with well-partial orders. If $\leq$ is a partial ordering on a set X and if $\leq'$ is a well-partial ordering on X which is contained in $\leq$, then $\leq$ is automatically well-founded. Therefore well-partial orderings *are a very convenient tool for proving termination [cf., for example, Lescanne (1990)* for a discussion] and moreover they are the most general structures which can be used for proving termination in this way, since $\leq'$ is a well-partial ordering on X iff every partial ordering $\leq$ on X which extends $\leq'$ is well-founded. An important example of a well-partial ordering [due to Higman (1952) and Kruskal (1960)] is the homeomorphic embeddability relation on a fixed set of terms $\mathcal{T}$ over a finite (and more general well-partially ordered) signature. The well-partial orderedness of this relation yields, for example, the well-foundedness of Dershowitz's simplification orderings [cf. Dershowitz and Jouannaud (1990)].

In this article Hardy bounds for the lengths of (effectively given) bad (with respect to homeomorpic embeddability) sequences of terms (and hence bounds on the computational complexity of the algorithm under consideration) will be established. The proof given here is based on a combinatorial analysis of the finitary content of the (reification) proof of Kruskal's theorem from Rathjen and Weiermann (1993). (The proof is intended to be paradigmatic for well-partial orders for which the termination proof can be established by a construction of effective reifications [cf. Schütte and Simpson 1985].)

We will give an informal exposition how our combinatorial analysis of reifications proceeds. Assume that $\langle t_1, \ldots, t_m \rangle$ is a finite bad sequence with respect to the homeomorphic embeddability relation on the set $\mathcal{T}$ of terms (over a finite signature) under consideration, i.e. there are no natural numbers i and j such that $1 \leq i < j \leq m$ and t_i is homeomorphically embeddable into t_j. In this situation there is *a priori* no standard method to construct a function $f : \mathcal{T} \to ON$ so that $f(t_i) > f(t_{i+1})$ holds for all elements of such an arbitrary finite bad sequence $\langle t_1, \ldots, t_m \rangle$. All that we know by Kruskal's theorem is that every sequence $\langle t_i : i < \alpha \rangle$ $(\alpha \leq \omega)$ of terms which is bad for the homeomorphic embeddability relation is finite. But this means that the tree $Bad(\mathcal{T})$ of all finite bad sequences is well-founded and therefore there is a mapping, called *reification*, $o : Bad(\mathcal{T}) \to On$ such that if $\langle t_1, \ldots, t_m \rangle \in Bad(\mathcal{T})$ and $\langle t_1, \ldots, t_m, t_{m+1} \rangle \in Bad(\mathcal{T})$, then $o(\langle t_1, \ldots, t_m \rangle) > o(\langle t_1, \ldots, t_m, t_{m+1} \rangle)$. If we can define such a reification into an ordinal $\sigma + 1$ – which is represented by a standard ordinal notation system T – in a sufficiently smooth way [as it was done in Rathjen and Weiermann (1993)] than we can proceed in the following way. Assume that $\langle t_1, \dots, t_m \rangle$ is effectively given so that, say, the number of symbols in t_{i+1} is bounded by the number of symbols in t_1 plus the product of a *constant* with $(i+1)^2$. (This is for example the case for rewrite systems.) Then, for $1 \leq i \leq m$ a similar condition should hold (and this will be explicitly verified in this article) for the syntactical complexity of $o(\langle t_1, \dots, t_i, t_{i+1} \rangle)$ when compared with the syntactical complexity of $o(\langle t_1, \dots, t_i \rangle)$ and then using the theory of Hardy functions we can bound the lengths of such descending sequences of ordinals.

The canonical choice for σ will be the maximal order type of the homeomorphic embeddability relation [cf. Schmidt (1979)] where such ordinals are precisely computed), since it follows from results of de Jongh and Parikh (1977) that the maximal order type of an arbitrary well-partial order is always equal to the height of its associated tree of finite bad sequences [cf. Hasegawa (1994), Kříž and Thomas (1991)].

The definition of the Hardy hierarchy in this article (the exposition of this hierarchy given here will be concise and selfcontained) is *non standard.* It is taken from Buchholz, Cichon and Weiermann (1994) where it was shown that the new definition is equivalent to the classical one. However, the new approach – which does not refer to an underlying system of fundamental sequences – allows us to simplify some arguments considerably (as will be noticed by readers who are familiar with the classical approach [cf., for example, Rose (1984) for an exposition] to this hierarchy).

Our paper is organized as follows. In Section 2 we recall some basic facts from the theory of well-partial orders. In Section 3 we define an ordinal notation system T_V for the *small Veblen number* which is defined in Veblen (1908). Here any ordinal notation system of the same order type, which can be found in the literature of ordinal notations and proof theory [cf., for example, Ackermann (1951), Buchholz (1975), Dershowitz and Okada (1989), Pohlers (1989), Schütte (1954) and Schütte (1977)], can be used and will yield an equivalent Hardy hierarchy and hence an equivalent result. [This is a consequence of Buchholz, Cichon and Weiermann (1994).]

In Section 4 we introduce the *Hardy hierarchy* of number theoretic functions for the segment of ordinals below the small Veblen number. In the last three Sections we will utilize a combinatorial analysis of the fine structure of the (reification) proof of Kruskal's theorem from Rathjen and Weiermann (1994) to obtain our main theorem. We also indicate briefly how our approach can be applied to Higman's lemma.

Our paper is self-contained at least from a technical point of view. We presume no knowledge of proof-theory and the theory of Hardy functions. Only knowledge of a small amount of the theory of ordinals is assumed.

2. Well-Partial Orders

In this section we review some basic definitions and facts from the theory of well-partial orders The results are, of course, well known and proofs will skipped here since they are not needed in the rest of the article. Nevertheless we hope that the reader will find this selection informative. An introduction into the theory of well-partial orders can be found, for example, in Fraïssé (1986), de Jongh and Parikh (1977), Kříž and Thomas (1991), Schütte and Simpson (1985) and Simpson (1985). An informative survey about applications of ordinals and well-partial orders to the theory of rewrite systems is given, for example, in Gallier (1991).

A *partial order* $\mathcal{X}$ is an ordered pair $\langle X, \leq_X \rangle$ such that X $(:= \mathcal{D}om(\mathcal{X}))$, the *domain* of $\mathcal{X}$, is a set and $\leq_X$ $(:= po(\mathcal{X}))$, the *partial ordering* on $\mathcal{X}$, is a binary, reflexive, transitive and antisymmetric relation on X. The cardinality of X is denoted by $|\mathcal{X}|$.

A sequence $\langle x_i : i < \alpha \rangle$ $\quad (0 \leq \alpha \leq \omega)$ of elements in $\mathcal{D}om(\mathcal{X})$ is called *bad* (with respect to $po(\mathcal{X})$) if there are no indices i and j such that $i < j < \alpha$ and $\langle x_i, x_j \rangle \in po(\mathcal{X})$. Let $Bad(\mathcal{X})$ be the set of all finite bad sequences of elements in the domain of $\mathcal{X}$. $Bad(\mathcal{X})$ can be considered as a tree in the natural way. A partial order $\mathcal{X}$ is a *well-partial order*, if every bad (with respect to $po(\mathcal{X})$) sequence of elements in the domain of $\mathcal{X}$ is finite. A partial order $\mathcal{X}$ is a *well-founded order*, if every strictly decreasing (with respect to $po(\mathcal{X})$) sequence of elements in the domain of $\mathcal{X}$ is finite. A linear order which is well-founded is called *well-order*.

LEMMA 2.1. *Let $\mathcal{X}$ be a partial order. Then the following assertions are equivalent.*

1. *$\mathcal{X}$ is a well-partial order.*
2. *For every infinite sequence $\langle x_i : i \in \omega \rangle$ of elements in $\mathcal{D}om(\mathcal{X})$ there exists a strictly increasing sequence $\langle i_l : l < \omega \rangle$ of natural numbers so that $\langle x_{i_l} : l < \omega \rangle$ is weakly increasing with respect to $po(\mathcal{X})$.*
3. *There does neither exists an infinite strictly descending sequence (with respect to $po(\mathcal{X})$) of elements in $\mathcal{D}om(\mathcal{X})$ nor an infinite subset Y of $\mathcal{D}om(\mathcal{X})$ so that Y contains mutually $po(\mathcal{X})$-incomparable elements.*
4. *Every partial ordering $\leq'$ on $\mathcal{D}om(\mathcal{X})$ which extends $po(\mathcal{X})$ is well-founded.*
5. *Every linear ordering $\leq'$ on $\mathcal{D}om(\mathcal{X})$ which extends $po(\mathcal{X})$ is a well-ordering.*
6. *Every nonempty subset of $\mathcal{D}om(\mathcal{X})$ has at least one but at most finitely many minimal (with respect to $po(\mathcal{X})$) elements.*
7. *The tree $Bad(\mathcal{X})$ is well-founded.*
8. *There exists a* reification *$f : Bad(\mathcal{X}) \to On$ so that $f(\langle x_1, \ldots, x_m \rangle) > f(\langle x_1, \ldots, x_m, x_{m+1} \rangle)$ holds for all $\langle x_1, \ldots, x_m, x_{m+1} \rangle \in Bad(\mathcal{X})$.*

For a given finite sequence $\langle \mathcal{X}_i : i < \alpha \rangle$ $(0 < \alpha < \omega)$ of partial orders we define partial orders $\bigoplus\{\mathcal{X}_i : i < \alpha\}$ and $\bigotimes\{\mathcal{X}_i : i < \alpha\}$ as follows. The domain of $\bigoplus\{\mathcal{X}_i : i < \alpha\}$ is the set $\bigcup\{\{\langle i, x_i \rangle : x_i \in \mathcal{D}om(\mathcal{X}_i)\} : i < \alpha\}$. The partial ordering $\leq_{\bigoplus\{\mathcal{X}_i : i < \alpha\}}$ on $\bigoplus\{\mathcal{X}_i : i < \alpha\}$ is defined as follows:

$$\langle i, x_i \rangle \leq_{\bigoplus\{\mathcal{X}_i : i < \alpha\}} \langle j, x_j \rangle :\iff i = j \ \& \ \langle x_i, x_j \rangle \in po(\mathcal{X}_i).$$

The domain of $\bigotimes\{\mathcal{X}_i : i < \alpha\}$ is the set $\{\langle x_0, \ldots, x_{\alpha-1} \rangle : (\forall i < \alpha)[x_i \in \mathcal{D}om(\mathcal{X}_i)]\}$. The partial ordering $\leq_{\bigotimes\{\mathcal{X}_i : i < \alpha\}}$ on $\bigotimes\{\mathcal{X}_i : i < \alpha\}$ is defined as follows:

$$\langle x_0, \ldots, x_{\alpha-1} \rangle \leq_{\bigotimes\{\mathcal{X}_i : i < \alpha\}} \langle x'_0, \ldots, x'_{\alpha-1} \rangle :\iff (\forall i < \alpha)[\langle x_i, x'_i \rangle \in po(\mathcal{X}_i)].$$

LEMMA 2.2. *If $\langle \mathcal{X}_i : i < \alpha \rangle$ is a finite sequence of well-partial orders, then $\bigoplus\{\mathcal{X}_i : i < \alpha\}$ and $\bigotimes\{\mathcal{X}_i : i < \alpha\}$ are well-partial orders.*

For a given partial order $\mathcal{X}$ we define the *Higman order* $\mathcal{X}^{<\omega}$ as follows. The domain of $\mathcal{X}^{<\omega}$ is the set of finite sequences of elements in the domain of $\mathcal{X}$. The partial ordering $\leq_{\mathcal{X}^{<\omega}}$ on $\mathcal{X}^{<\omega}$, the *Higman ordering* on $\mathcal{X}^{<\omega}$, is defined as follows.

$$\langle x_1, \ldots, x_m \rangle \leq_{\mathcal{X}^{<\omega}} \langle x'_1, \ldots, x'_n \rangle$$

if there is a strictly monotonic increasing function $f : \{1, \ldots, m\} \to \{1, \ldots n\}$ such that $\langle x_i, x'_{f(i)} \rangle \in po(\mathcal{X})$ for $1 \leq i \leq m$.

LEMMA 2.3. (HIGMAN'S LEMMA) *If $\mathcal{X}$ is a well-partial order, then $\mathcal{X}^{<\omega}$ is a well-partial order.*

Let $\langle \mathcal{X}_i : i < \alpha \rangle$ $(0 < \alpha < \omega)$ be a finite sequence of partial orders. If nothing else is stated we will assume from now on that $\mathcal{D}om(\mathcal{X}_0) \neq \emptyset$. We define a partial order $\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle)$ in the following way. The domain, $\mathcal{D}om(\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle))$, of $\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle)$ is given by the least set which is closed under the following rule:
If $0 \leq i < \alpha$, if $x_i \in \mathcal{D}om(\mathcal{X}_i)$ and if $t_1, \ldots, t_i \in \mathcal{D}om(\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle))$,
then $\langle \langle i, x_i \rangle, \langle t_1, \ldots, t_i \rangle \rangle \in \mathcal{D}om(\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle))$.
($\mathcal{D}om(\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle))$ can be interpreted in the obvious way as a set of terms where the function symbols are taken from the set $\mathcal{D}om(\mathcal{X}_0) \cup \ldots \cup \mathcal{D}om(\mathcal{X}_{\alpha-1})$.)

The partial ordering $\leq_{\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle)}$ on $\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle)$, the *homeomorphic embeddability relation* on $\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle)$, is by definition the least binary and reflexive relation on $\mathcal{D}om(\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle))$ such that,

1 if $0 \leq i \leq j$ and $x_j \in \mathcal{D}om(\mathcal{X}_j)$, then $t_i \leq_{\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle)} \langle \langle j, x_j \rangle, \langle t_1, \ldots, t_j \rangle \rangle$,
2 if $1 \leq j$, $x_j, x'_j \in \mathcal{D}om(\mathcal{X}_j)$, $\langle x_j, x'_j \rangle \in po(\mathcal{X}_j)$ and if $t_i \leq_{\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle)} t'_i$ for $1 \leq i \leq j$, then
$\langle \langle j, x_j \rangle, \langle t_1, \ldots, t_j \rangle \rangle \leq_{\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle)} \langle \langle j, x'_j \rangle, \langle t'_1, \ldots, t'_j \rangle \rangle$.

The homeomorphic embeddability relation can easily shown to be transitive. The following theorem is due to Higman (1952) and Kruskal (1960).

LEMMA 2.4. (KRUSKAL'S THEOREM (RESTRICTED VERSION)) *If $\langle \mathcal{X}_i : i < \alpha \rangle$ is a finite sequence of well-partial orders, then $\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle)$ is a well-partial order.*

For a partial order $\mathcal{X}$ and $x \in \mathcal{D}om(\mathcal{X})$ let $L_{\mathcal{X}}(x) := \{x' \in \mathcal{D}om(\mathcal{X}) : \neg \langle x, x' \rangle \in po(\mathcal{X})\}$ and $\mathcal{X}(x) := \langle L_{\mathcal{X}}(x), po(\mathcal{X}) \cap (L_{\mathcal{X}}(x) \times L_{\mathcal{X}}(x)) \rangle$.

LEMMA 2.5. *If $\mathcal{X}$ is a partial order, then $\mathcal{X}$ is a well-partial order iff $\mathcal{X}(x)$ is a well-partial order for every $x \in \mathcal{D}om(\mathcal{X})$.*

We close this section with a brief survey of the de Jongh and Parikh (1977) theory of maximal orders types. For every well-founded order $\mathcal{X} = \langle X, \leq \rangle$, there exists a well-ordering $\leq'$ on X such that $\leq \subseteq \leq'$. Therefore we can define $motype(\mathcal{X})$, the *maximal order type* of $\mathcal{X}$, in the following way:

$$motype(\mathcal{X}) := \sup\{otype(\langle X, \leq' \rangle) : \leq' \text{ is a well-ordering on } X \text{ so that } \leq \subseteq \leq'\}.$$

If $\mathcal{X}$ and $\mathcal{Y}$ are partial orders we set $\mathcal{X} \subseteq \mathcal{Y}$ iff $\mathcal{D}om(\mathcal{X}) \subseteq \mathcal{D}om(\mathcal{Y})$ and $po(\mathcal{X}) = po(\mathcal{Y}) \cap (\mathcal{D}om(\mathcal{X}) \times \mathcal{D}om(\mathcal{X}))$. We say that a function $e : \mathcal{D}om(\mathcal{X}) \to \mathcal{D}om(\mathcal{Y})$ is a *quasi-embedding* of a partial order $\mathcal{X}$ into a partial order $\mathcal{Y}$, if $\langle e(x), e(x') \rangle \in po(\mathcal{Y})$ implies $\langle x, x' \rangle \in po(\mathcal{X})$ for all $x, x' \in \mathcal{D}om(\mathcal{X})$.

LEMMA 2.6. *1 If $\mathcal{X}$ is a well-partial order and $\mathcal{Y}$ is a partial order such that $\mathcal{Y} \subset \mathcal{X}$, then $\mathcal{Y}$ is a well-partial order and $motype(\mathcal{Y}) \leq motype(\mathcal{X})$.*

2 If e is a quasi-embedding of a partial order $\mathcal{Y}$ into a well-partial order $\mathcal{X}$, then $\mathcal{Y}$ is a well-partial order and $motype(\mathcal{Y}) \leq motype(\mathcal{X})$.

LEMMA 2.7. *Let $\mathcal{X} = \langle X, \leq_X \rangle$ be a well-partial order.*

1 There exists a linear ordering $\leq'_X$ on X so that $\leq_X \subseteq \leq'_X$ and $motype(\mathcal{X}) = otype(\langle X, \leq_X \rangle)$.
2 $motype(\mathcal{X}) = \sup\{motype(\mathcal{X}(x)) + 1 : x \in X\}$.
3 The set-theoretic height of the tree $Bad(\mathcal{X})$ of finite bad sequences of elements in X is equal to $motype(\mathcal{X})$
4 $motype(\mathcal{X})$ is the least ordinal α such that there exists a function $f : Bad(\mathcal{X}) \to \alpha + 1$ such that $f(\langle x_1, \ldots, x_m \rangle) > f(\langle x_1, \ldots, x_m, x_{m+1} \rangle)$ holds for all $\langle x_1, \ldots, x_m, x_{m+1} \rangle \in Bad(\mathcal{X})$.

Note that assertion 3) follows from assertion 2) and that assertion 4) follows from assertion 3). As an interesting consequence of assertion 3) of this lemma we note that the maximal order type of a recursive well-partial order is always a recursive ordinal.

3. A Notation System for the small Veblen number

In this section [following the exposition given in Rathjen and Weiermann (1993)] we introduce the Howard-Bachmann hierarchy of ordinals and we single out a subsystem of ordinals which represents the small Veblen number [see, for example, Veblen (1908) for a definition] in a natural way. Our approach yields the intended results in a straightforward and selfcontained way but for a better understanding of the underlying intuitions of the subject it will perhaps also be useful to study Veblen's article Veblen (1908), Bachmann (1950), or a standard text book (which covers ordinal notations) in proof-theory [cf., for example, Pohlers (1989) or Schütte (1977)].

In the sequel small Greek letters range over ordinals. Let $AP := \{\xi : (\exists \eta)[\xi = \omega^\eta]\}$ be the class of *additive principal numbers* and let $E := \{\xi : \xi = \omega^\xi\}$ be the class of *ε-numbers* which is enumerated by the function $\lambda\xi.\varepsilon_\xi$. We write $\alpha =_{NF} \alpha_1 + \cdots + \alpha_n$, if $\alpha = \alpha_1 + \cdots + \alpha_n$, $\alpha_1, \ldots, \alpha_n \in AP$ and $\alpha > \alpha_1 \geq \ldots \geq \alpha_n$. We write $\alpha =_{NF} \omega^{\alpha_1}$, if $\alpha = \omega^{\alpha_1}$ and $\alpha > \alpha_1$. Let Ω denote the first uncountable ordinal. For any ordinal α less than $\varepsilon_{\Omega+1}$ we define the set $E_\Omega(\alpha)$ which contains the ε-numbers below Ω which are needed for the unique representation of α in normal-form recursively as follows.

1 $E_\Omega(0) := E_\Omega(\Omega) := \emptyset$,
2 $E_\Omega(\alpha) := \{\alpha\}$, if $\alpha \in E \cap \Omega$,
3 $E_\Omega(\alpha) := E_\Omega(\alpha_1) \cup \ldots \cup E_\Omega(\alpha_n)$, if $\alpha =_{NF} \alpha_1 + \cdots + \alpha_n$,
4 $E_\Omega(\alpha) := E_\Omega(\alpha_1)$, if $\alpha =_{NF} \omega^{\alpha_1}$.

Let $\alpha^* := \max(\{0\} \cup E_\Omega(\alpha))$. We define sets of ordinals $C(\alpha, \beta)$, $C_n(\alpha, \beta)$ and ordinals $\vartheta\alpha$ by main recursion on $\alpha < \varepsilon_{\Omega+1}$ and subsidiary recursion on $n < \omega$ (for $\beta < \Omega$) as follows.

(C1) $\{0, \Omega\} \cup \beta \subseteq C_n(\alpha, \beta)$,
(C2) $\xi, \eta \in C_n(\alpha, \beta) \Rightarrow \omega^\xi + \eta \in C_{n+1}(\alpha, \beta)$,
(C3) $\xi \in C_n(\alpha, \beta) \cap \alpha \Rightarrow \vartheta\xi \in C_{n+1}(\alpha, \beta)$,

(C4) $C(\alpha,\beta) := \bigcup\{C_n(\alpha,\beta) : n < \omega\}$,
(C5) $\vartheta\alpha := \min\{\xi < \Omega : C(\alpha,\xi)\cap\Omega \subseteq \xi \;\&\; \alpha \in C(\alpha,\xi)\}$.

LEMMA 3.1. *$\vartheta\alpha$ is defined for every $\alpha < \varepsilon_{\Omega+1}$.*

PROOF. By (C1) $E_\Omega(\alpha) \subseteq C(\alpha,\alpha^*+1)$. By (C2) $\alpha \in C(\alpha,\alpha^*+1)$. Note that for every $\beta < \Omega$ the set $C(\alpha,\beta)$ is countable. This is due to the fact that $C(\alpha,\beta)$ is the closure of a countable ordinal under countably many operations. Put $\gamma_0 := \alpha^*+1$. Assume recursively that $\gamma_n < \Omega$ is defined. Since $C(\alpha,\gamma_n)$ is countable there is a countable ordinal γ_{n+1} such that $C(\alpha,\gamma_n)\cap\Omega \subseteq \gamma_{n+1}$. Put $\gamma := \sup\{\gamma_n : n < \omega\}$. Then $\gamma < \Omega$ – since Ω is regular –, $\alpha \in C(\alpha,\gamma)$ and $C(\alpha,\gamma)\cap\Omega \subseteq \gamma$. Thus $\vartheta\alpha$ is well-defined. □

LEMMA 3.2. *1 $\vartheta\alpha \in E$,*
2 $\alpha \in C(\alpha,\vartheta\alpha)$,
3 $\vartheta\alpha = C(\alpha,\vartheta\alpha)\cap\Omega$, and $\vartheta\alpha \notin C(\alpha,\vartheta\alpha)$,
4 $\gamma \in C(\alpha,\beta) \iff E_\Omega(\gamma) \subseteq C(\alpha,\beta)$,
5 $E_\Omega(\alpha) < \vartheta\alpha$,
6 $\vartheta\alpha = \vartheta\beta \implies \alpha = \beta$,
7 $\vartheta\alpha < \vartheta\beta \iff (\alpha < \beta \wedge \alpha^* < \vartheta\beta) \vee (\beta < \alpha \wedge \vartheta\alpha \le \beta^*)$,
8 $\beta < \vartheta\alpha \iff \omega^\beta < \vartheta\alpha$,
9 $C(\alpha,\alpha^*+1)\cap\Omega \in On \Rightarrow C(\alpha,\alpha^*+1) = C(\alpha,\vartheta\alpha)$,
10 $C(\alpha,\beta)\cap\Omega \in On$.

PROOF. 1) and 8) issue from closure of $\vartheta\alpha$ under (C2).

2) and 3) follow from Lemma 3.1 and the definition of $\vartheta\alpha$.

4). If $E_\Omega(\gamma) \subseteq C(\alpha,\beta)$, then $\gamma \in C(\alpha,\beta)$ by (C2). On the other hand, $\gamma \in C_n(\alpha,\beta) \implies E_\Omega(\gamma) \subseteq C_n(\alpha,\beta)$ is easily seen by induction on n.

5). $\alpha^* \in C(\alpha,\vartheta\alpha)$ holds by 4). As $\alpha^* < \Omega$, this implies $\alpha^* < \vartheta\alpha$ by 3).

6). Suppose, aiming at a contradiction, that $\vartheta\alpha = \vartheta\beta$ and $\alpha < \beta$. Then $C(\alpha,\vartheta\alpha) \subseteq C(\beta,\vartheta\beta)$; hence $\alpha \in C(\beta,\vartheta\beta)\cap\beta$ by 2); thence $\vartheta\alpha = \vartheta\beta \in C(\alpha,\vartheta\beta)$, contradicting 3).

7). Suppose $\alpha < \beta$. Then $\vartheta\alpha < \vartheta\beta$ implies $\alpha^* < \vartheta\beta$ by 5). If $\alpha^* < \vartheta\beta$, then $\alpha \in C(\beta,\vartheta\beta)$; hence $\vartheta\alpha \in C(\beta,\vartheta\beta)$; thus $\vartheta\alpha < \vartheta\beta$. This shows

$$(a)\quad \alpha < \beta \implies (\vartheta\alpha < \vartheta\beta \iff \alpha^* < \vartheta\beta).$$

By interchanging the roles of α and β, and employing 5), one obtains

$$(b)\quad \beta < \alpha \implies (\vartheta\alpha < \vartheta\beta \iff \vartheta\alpha \le \beta^*).$$

(a) and (b) yield 7).

9). "$\subseteq$": This follows from 5). "$\supseteq$": Let $\beta := C(\alpha,\alpha^*+1)\cap\Omega$. Since $\alpha^* < \Omega$ we see that the cardinality of $C(\alpha,\alpha^*+1))$ is countable, hence $\beta < \Omega$. From $\alpha^*+1 \subseteq C(\alpha,\beta)$ we see $\alpha \in C(\alpha,\beta)$. $\beta \subseteq C(\alpha,\alpha^*+1)$ yields (by definition of $C(\cdot,\cdot)$) $C(\alpha,\beta) \subseteq C(\alpha,\alpha^*+1)$, hence $C(\alpha,\beta)\cap\Omega \subseteq C(\alpha,\alpha^*+1)\cap\Omega = \beta$. The definition of $\vartheta\alpha$ therefore yields $\vartheta\alpha \le \beta$, hence $C(\alpha,\vartheta\alpha) \subseteq C(\alpha,\beta) \subseteq C(\alpha,\alpha^*+1)$.

10). Proof by induction on α: We prove $\gamma \in C(\alpha,\beta)\cap\Omega \Rightarrow \gamma \subseteq C(\alpha,\beta)$ by subsidiary induction on γ.

1. $\gamma \in \{0\}\cup\beta$. This case is trivial.
2. $\gamma = \omega^\xi + \eta$ with $\gamma > \xi,\eta \in C(\alpha,\beta)$: Then $\xi < \gamma$, hence $\xi+1 \subseteq C(\alpha,\beta)$ by induction hypothesis. Now assume $\delta < \gamma$. Then $\delta = \omega^{\xi_0} + \cdots + \omega^{\xi_{n-1}}$ where $\xi \ge \xi_0 \ge \ldots \ge \xi_{n-1}$.

$\xi + 1 \subseteq C(\alpha, \beta)$ therefore yields $\xi_0, \ldots, \xi_{n-1} \in C(\alpha, \beta)$, hence $\delta \in C(\alpha, \beta)$.
3. $\gamma = \vartheta\xi$ where $\xi \in C(\alpha, \beta) \cap \alpha$. Then $\xi^* \in C(\alpha, \beta) \cap \Omega$ & $\xi^* < \gamma$, hence $\xi^* + 1 \subseteq C(\alpha, \beta)$ follows by subsidiary induction hypothesis. The main induction hypothesis yields $C(\xi, \xi^* + 1) \cap \Omega \in On$. By 8) and $\xi^* + 1 \subseteq C(\alpha, \beta)$. We therefore obtain $\gamma = \vartheta\xi \subseteq C(\xi, \vartheta\xi) = C(\xi, \xi^* + 1) \subseteq C(\alpha, \xi^* + 1) \subseteq C(\alpha, \beta)$. □

The *small Veblen number* is by definition $\vartheta\Omega^\omega$. The *big Veblen number*, which is also called first E-number [cf. Veblen (1908)], is by definition $\vartheta\Omega^\Omega$.

DEFINITION 3.1. *Inductive definition of a set of terms T_V and a natural number $N\alpha$, called* norm *of α (cf.*Buchholz, Cichon and Weiermann (1994)*), for $\alpha \in T_V$.*

1 $0 \in T_V$, $N0 := 0$,
2 $\vartheta 0 \in T_V$, $N\vartheta 0 := 1$,
3 $\alpha =_{NF} \alpha_1 + \cdots + \alpha_n$, $\alpha_1, \ldots, \alpha_n \in T_V \Rightarrow \alpha \in T_V$ *and* $N\alpha := \max\{N\alpha_1, \ldots, N\alpha_n, n\} + 1$,
4 $\alpha =_{NF} \omega^{\alpha_1}$, $\alpha_1 \in T_V \Rightarrow \alpha_1 \in T_V$ & $N\alpha := N\alpha_1 + 1$,
5 $\alpha = \vartheta(\Omega^n \cdot \alpha_0 + \cdots + \Omega^0 \cdot \alpha_n)$, $\alpha_0, \ldots, \alpha_n \in T_V$, $\alpha_0 \neq 0 \Rightarrow \alpha \in T_V$ *and* $N\alpha := \max\{N\alpha_0, \ldots, N\alpha_n, n+1\} + 1$.

It follows from assertions 9 and 10 of Lemma 3.2 that $\vartheta\Omega^\omega = C(\Omega^\omega, 0) \cap \Omega = T_V$. Thus every ordinal less than $\vartheta\Omega^\omega$ is denoted by an ordinal term of T_V and vice versa. In the sequel small Greek letters range over T_V. The following lemma shows, that T_V and the less than relation on T_V can be considered as primitive recursive relations (after a straightforward coding in the natural numbers).

LEMMA 3.3. *If $\alpha = \vartheta(\Omega^m \cdot \alpha_0 + \cdots + \Omega^0 \cdot \alpha_m)$, $\beta = \vartheta(\Omega^n \cdot \beta_0 + \cdots + \Omega^0 \cdot \beta_n)$ and $\alpha_0 \neq 0 \neq \beta_0$, then $\alpha < \beta$ if and only if one of the following conditions holds:*

1 there exists a natural number $i \in \{0, \ldots, n\}$ such that $\alpha \leq \beta_i$, or
2 $\alpha_i < \beta$ for every $i < m$ and $m < n$, or
3 $m = n$ and there exists a natural number $j \in \{0, \ldots, n\}$ such that $\alpha_j < \beta_j$ and $\alpha_k = \beta_k$ for every natural number $k \in \{0, \ldots, j-1\}$.

PROOF. The assertion follows from Lemma 3.2 7).□

For convenience we recall the definition of the *natural sum*, $\oplus$, and the *natural product*, $\otimes$, of ordinals. Let $\alpha \oplus 0 := 0 \oplus \alpha := \alpha$. If $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_m}$ and $\beta =_{NF} \omega^{\beta_1} + \cdots + \omega^{\beta_n}$, then $\alpha \oplus \beta := \omega^{\gamma_1} + \cdots + \omega^{\gamma_{n+m}}$, where $\langle \gamma_1, \ldots, \gamma_{m+n} \rangle$ is a rearrangement of $\langle \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \rangle$ such that $\gamma_1 \geq \ldots \geq \gamma_{m+n}$.

LEMMA 3.4. *1* $\alpha \oplus \beta = \beta \oplus \alpha$,
2 $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$,
3 $\alpha, \beta < \omega^\gamma \Rightarrow \alpha \oplus \beta < \omega^\gamma$.

Let $\alpha \otimes 0 := 0 \otimes \alpha := 0$. If $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_m}$ and $\beta =_{NF} \omega^{\beta_1} + \cdots + \omega^{\beta_n}$, then $\alpha \otimes \beta := \bigoplus\{\omega^{\alpha_i \oplus \beta_j} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$.

LEMMA 3.5. *1* $\alpha \otimes \beta = \beta \otimes \alpha$,
2 $(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$,
3 $\alpha, \beta < \omega^{\omega^\gamma} \Rightarrow \alpha \otimes \beta < \omega^{\omega^\gamma}$.

4. A hierarchy of Hardy-functions for T_V

For every natural number n the set $\{\alpha \in T_V : N\alpha \leq n\}$ is a *finite* set. Based on this simple observation we can introduce numbertheoretic functions, the *Hardy functions*, in the following way.

$H_0(x) = x$, and if $\alpha \neq 0$, then
$H_\alpha(x) := \max\{H_\beta(x+1) : \beta < \alpha \,\&\, N\beta \leq 2^{N\alpha+x}\}$.

The definition of H_α proceeds by recursion on α. Thus, for $\beta < \alpha$ the functions H_β are defined by recursion hypothesis, and since there are only finitely many $\beta < \alpha$ so that $N\beta \leq 2^{N\alpha+x}$ holds there will be only finitely many values $H_\beta(x+1)$ over which the maximum is taken. In this definition the use of the exponential function is convenient for our purposes but it does not play a crucial role. It could be replaced by any other reasonable strictly monotonic increasing primitive recursive function [cf. Buchholz, Cichon and Weiermann (1994) for an analysis of this dependency]. Since T_V and $< \cap (T_V \times T_V)$ are primitive recursive relations H_α is clearly a recursive function. It is also important to notice that the definition of H_α can be carried out in the same way for any system of ordinal notations (known in the literature of proof theory) since a norm function is by symbol-counting always definable for all such systems.

A complete list of the ordertheoretic properties which will be used in this article of the Hardy functions is given in the following lemma.

LEMMA 4.1. *1* $x < y \Rightarrow H_\alpha(x) < H_\alpha(y)$,
2 $\alpha < \beta \,\&\, N\alpha \leq 2^{N\beta+x} \Rightarrow H_\alpha(x) \leq H_\beta(x)$,
3 $x \leq H_\alpha(x)$,
4 $H_\alpha(x) < H_{\alpha+1}(x)$.

PROOF. The lemma follows immediately from the definition of the Hardy functions as will be shown now.
1) If $\alpha = 0$, then H_α is strictly monotonic increasing. Assume that $\alpha > 0$ and that H_β is strictly monotonic increasing for $\beta < \alpha$. Assume $x < y$. We have $H_\alpha(x) = H_\beta(x+1) < H_\beta(y+1)$ for some $\beta < \alpha$ so that $N\beta \leq 2^{N\alpha+x}$. Since $N\beta \leq 2^{N\alpha+y}$ the definition of H_α yields $H_\beta(y+1) \leq H_\alpha(y)$.
2) The definition of H_β and 1) yield $H_\beta(x) \geq H_\alpha(x+1) > H_\alpha(x)$.
3) follows from 1).
4) follows from 2). □

Note that assertion 2) implies that the Hardy-hierarchy does not collapse, i.e. we have that $\alpha < \beta$ implies the existence of a natural number x_0 so that $H_\alpha(x) < H_\beta(x)$ holds for all $x \geq x_0$. But, and that is even more important for our intended applications, by assertion 2) we can effectively compute a natural number x_0 from which on upwards the strict majorization holds. Simply choose x_0 so that $N\alpha \leq 2^{N\beta+x_0}$ holds. In Buchholz, Cichon and Weiermann (1994) it has been shown that this approach to Hardy hierarchies is equivalent to the classical one. To get some familiarity with (and some insight into the tremendous growth rate of) the Hardy functions the reader may prove the following assertions about Hardy functions: $H_{\alpha+1}(x) = H_\alpha(x+1)$; $H_\alpha(H_\beta(x)) \leq H_{\alpha\oplus\beta}(x)$; $\underbrace{H_{\omega^\alpha}(\dots(H_{\omega^\alpha})}_{x+1-times}(x)\dots) \leq H_{\omega^{\alpha+1}}(x)$.

5. The Bounding Lemma

In the sequel we fix a set of terms and we are going to analyze the lengths of effectively given bad sequences of terms. No bounding result in terms of recursive functions is provable if there are no effectiveness restrictions imposed on the bad sequences under consideration. To have a simple measure for the syntactical complexity of the terms and to have a simple control over bad sequences of function symbols we therefore assume that the underlying signature is finite. Nevertheless we cover – simultaneously – slightly more general situations where the signature is (infinite but) well-partially ordered and where some reasonable assumptions on the signature are presumed. These more general sets of terms are covered by clause (C3) of the definition of $Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$ (see below).

A partial order is *flat* if $\mathcal{D}om(\mathcal{X})$ is a finite set and $po(\mathcal{X})$ is the equality relation on $\mathcal{D}om(\mathcal{X})$. Every flat partial order is a well-partial order. Let $\langle \mathcal{X}_i : i < \alpha \rangle$ $(0 < \alpha < \omega)$ be a finite sequence of flat partial orders where it is tacitly assumed, as we agreed before, that $\mathcal{D}om(\mathcal{X}_0) \neq \emptyset$. Let $Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$ be the least set of ("names" for) partial orders such that,

(C 1) if $0 \leq i < \alpha$ and $\mathcal{Y} \subseteq \mathcal{X}_i$, then $\mathcal{Y} \in Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$,
(C 2) If $\langle\langle \mathcal{Y}_{jk} : j < 2\rangle : k < \gamma\rangle$ is a finite array of elements in $Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$, then $\bigoplus\{\bigotimes\{\mathcal{Y}_{jk} : j < 2\} : k < \gamma\} \in Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$.
(C 3) if $0 < \beta \leq \alpha$ and if $\langle \mathcal{Y}_j : j < \beta \rangle$ is a finite sequence of partial orders in $Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$,
then $\mathcal{T}(\langle \mathcal{Y}_j : j < \beta \rangle) \in Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$.

We define a natural number $c(\mathcal{Y})$ for every $\mathcal{Y} \in Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$.

1 $c(\mathcal{Y}) := 0$, if $\mathcal{Y} \in Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$ by (C 1).
2 $c(\bigoplus\{\bigotimes\{\mathcal{Y}_{jk} : j < 2\} : k < \gamma\}) := \max\{c(\mathcal{Y}_{jk}) : j < 2 \;\&\; k < \gamma\} + 1$,
3 $c(\mathcal{T}(\langle \mathcal{Y}_j : j < \beta \rangle)) := \max\{c(\mathcal{Y}_0), \dots, c(\mathcal{Y}_{\beta-1})\} + 1$.

We define an ordinal $o(\mathcal{Y}) \in T_V$ for every $\mathcal{Y} \in Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$.

1 $o(\mathcal{Y}) := \mid \mathcal{Y} \mid$, if $\mathcal{Y} \in Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$ by (C 1).
2 $o(\bigoplus\{\bigotimes\{\mathcal{Y}_{jk} : j < 2\} : k < \gamma\}) := \bigoplus\{\bigotimes\{\omega^{\omega^{o(\mathcal{Y}_{jk})}} : j < 2\} : k < \gamma\}$,
3 $o(\mathcal{T}(\langle \mathcal{Y}_j : j < \beta \rangle)) := \vartheta(\Omega^{\beta-1} \cdot o(\mathcal{Y}_{\beta-1}) + \dots + \Omega^0 \cdot o(\mathcal{Y}_0))$.

We define for every $y \in \mathcal{D}om(\mathcal{T}(\langle \mathcal{Y}_j : j < \beta \rangle))$ (where $\mathcal{T}(\langle \mathcal{Y}_j : j < \beta \rangle) \in Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$ by (C3)) a natural number $dp_{\mathcal{T}(\langle \mathcal{Y}_j : j<\beta \rangle)}(y)$ as follows:

1 if y has the form $\langle\langle 0, y_0\rangle, \langle\rangle\rangle$, then $dp_{\mathcal{T}(\langle \mathcal{Y}_j : j<\beta \rangle)}(y) := 0$,
2 if y has the form $\langle\langle i, y_i\rangle, \langle t_1, \dots, t_i\rangle\rangle$ where $i \neq 0$,
then $dp_{\mathcal{T}(\langle \mathcal{Y}_j : j<\beta \rangle)}(y) := \max\{dp_{\mathcal{T}(\langle \mathcal{Y}_j : j<\beta \rangle)}(t_1), \dots, dp_{\mathcal{T}(\langle \mathcal{Y}_j : j<\beta \rangle)}(t_i)\} + 1$.

We define for every $y \in \mathcal{D}om(\mathcal{Y})$ (where $\mathcal{Y} \in Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$) a natural number $rk_{\mathcal{Y}}(y)$ by recursion on $c(\mathcal{Y})$ as follows:

1 $rk_{\mathcal{Y}}(y) := 0$, if $c(\mathcal{Y}) = 0$.
2 if y has the form $\langle k, \langle y_{0k}, y_{1k} \rangle \rangle$ where $y \in \bigoplus\{\bigotimes\{\mathcal{Y}_{jk} : j < 2\} : k < \gamma\}$, then $rk_{\bigoplus\{\bigotimes\{\mathcal{Y}_{jk}:j<2\}:k<\gamma\}}(y) := \max\{rk_{\mathcal{Y}_{0k}}(y_{1k}), rk_{\mathcal{Y}_{1k}}(y_{1k})\} + 1$,
3 if y has the form $\langle \langle i, y_i \rangle, \langle t_1, \ldots, t_i \rangle \rangle$, where $y \in \mathcal{T}(\langle \mathcal{Y}_j : j < \beta \rangle)$, then we define $rk_{\mathcal{T}(\langle \mathcal{Y}_j:j<\beta \rangle)}(y)$ by primitive recursion on $dp_{\mathcal{T}(\langle \mathcal{Y}_j:j<\beta\rangle)}(y)$:
If $i = 0$, then $rk_{\mathcal{T}(\langle \mathcal{Y}_j:j<\beta\rangle)}(y) := rk_{\mathcal{Y}_0}(y_0)$.
If $i \neq 0$, then $rk_{\mathcal{T}(\langle \mathcal{Y}_j:j<\beta\rangle)}(\langle \langle i, y_i \rangle, \langle t_1, \ldots, t_i \rangle \rangle) :=$
$\max\{rk_{\mathcal{Y}_i}(y_i), rk_{\mathcal{T}(\langle \mathcal{X}_i:i<\alpha\rangle)}(t_1) + 1, \ldots, rk_{\mathcal{T}(\langle \mathcal{X}_i:i<\alpha\rangle)}(t_i) + 1\}$.

The following lemma gives a purely combinatorial analysis of the finitary content of the ordinal computations done in Rathjen and Weiermann (1993) and Schmidt (1979). It looks technical and therefore an informal explanation of the underlying mathematical background may be helpful. Let $\mathcal{Y}$ be a well-partial order. Assume that we want to compute an upper bound $o(\mathcal{Y})$ for $motype(\mathcal{Y})$. By assertion 2) of Lemma 2.7 it is sufficient to show that $o(\mathcal{Y}(y)) < o(\mathcal{Y})$ holds for all $y \in \mathcal{D}om(\mathcal{Y})$ where $o(\mathcal{Y}(y))$ is an upper bound for $motype(\mathcal{Y}(y))$. (Based on a transfinite induction along a certain specific well-ordering this method has been applied in Schmidt (1979) for computing the precise maximal order type of $\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle)$). This abstract mathematical argumentation has to be converted into a purely combinatorial statement – which can be proved without transfinite induction - about appropriate ordinal assignments, quasi-embeddings and reifications (cf. lemma 2.6 and 2.7). How appropriate reifications can be defined will be seen later in the proof of theorem 6.1.

LEMMA 5.1. *Let $\langle \mathcal{X}_i : i < \alpha \rangle$ be a finite sequence of flat partial orders. Let $\mathcal{Y} \in Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$ and $z \in \mathcal{D}om(\mathcal{Y})$. Then there is a partial order $\mathcal{Y}^z \in Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$ and a quasi-embedding $e(\mathcal{Y}, z)$ of $\mathcal{Y}(z)$ into $\mathcal{Y}^z$ such that:*

($\mathcal{Y}^z$1) $rk_{\mathcal{Y}^z}(e(\mathcal{Y}, z)(u)) \leq rk_{\mathcal{Y}}(u) + 1$ *for every* $u \in \mathcal{D}om(\mathcal{Y}(z))$,
($\mathcal{Y}^z$2) $N(o(\mathcal{Y}^z)) \leq N(o(\mathcal{Y})) + 6 \cdot (rk_{\mathcal{Y}}(z) + 1)$,
($\mathcal{Y}^z$3) $o(\mathcal{Y}^z) < o(\mathcal{Y})$.

PROOF. The proof is by main induction on $c(\mathcal{Y})$ and subsidiary induction on $rk_{\mathcal{Y}}(z)$.
CASE A: $\mathcal{Y} \subset \mathcal{X}_{i_0}$ and $z \in \mathcal{D}om(\mathcal{Y})$. Let $\mathcal{Y}^z$ be the flat partial order with domain $\mathcal{D}om(\mathcal{Y}) \setminus \{z\}$ and let $e(\mathcal{Y}, z)$ be the restriction of the identy map to $\mathcal{D}om(\mathcal{Y}(z))$. Then $\mathcal{Y}^z$ and $e(\mathcal{Y}, z)$ satisfy ($\mathcal{Y}^z$1), ($\mathcal{Y}^z$2) and ($\mathcal{Y}^z$3).
CASE B: $\mathcal{Y} = \bigoplus\{\bigotimes\{\mathcal{Y}_{jk} : j < 2\} : k < \gamma\}$. Let

$$z = \langle k_0, \langle y_{0k_0}, y_{1k_0} \rangle \rangle.$$

By our main induction hypothesis we can pick for $j \in \{0, 1\}$ a partial order

$$\mathcal{Y}_{jk_0}^{y_{jk_0}}$$

and a quasi-embedding

$$e(\mathcal{Y}_{jk_0}, y_{jk_0})$$

of $\mathcal{Y}_{jk_0}(y_{jk_0})$ into $\mathcal{Y}_{jk_0}^{y_{jk_0}}$ which satisfy ($\mathcal{Y}_{jk_0}^{y_{jk_0}}$1), ($\mathcal{Y}_{jk_0}^{y_{jk_0}}$2) and ($\mathcal{Y}_{jk_0}^{y_{jk_0}}$3).

Put

$$\mathcal{Y}^z := \bigoplus\{\bigotimes\{\mathcal{Y}_{jk} : j < 2\} : k_0 \neq k < \alpha\} \oplus (\mathcal{Y}_{0k_0}^{y_{0k_0}} \otimes \mathcal{Y}_{1k_0}) \oplus (\mathcal{Y}_{0k_0} \otimes \mathcal{Y}_{1k_0}^{y_{1k_0}}).$$

Then

$$\mathcal{Y}^z \in Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$$

and

$$o(\mathcal{Y}^z) = \bigoplus\{\omega^{\omega^{o(\mathcal{Y}_{0k})} \oplus \omega^{o(\mathcal{Y}_{1k})}} : k_0 \neq k < \alpha\} \oplus \omega^{\omega^{o(\mathcal{Y}_{0k_0}^{y_{0k_0}})} \oplus \omega^{o(\mathcal{Y}_{1k_0})}} \oplus \omega^{\omega^{o(\mathcal{Y}_{0k_0})} \oplus \omega^{o(\mathcal{Y}_{1k_0}^{y_{1k_0}})}}.$$

Thus by Lemma 3.4

$$o(\mathcal{Y}^z) < o(\mathcal{Y}).$$

Furthermore

$$N(o(\mathcal{Y}^z)) \leq \max\big\{\alpha - 1 + 2, 1 + \max\{N(\omega^{o(\mathcal{Y}_{0k})} \oplus \omega^{o(\mathcal{Y}_{1k})}) : k_0 \neq k < \alpha\},$$

$$1 + N(\omega^{o(\mathcal{Y}_{0k_0}^{y_{0k_0}})} \oplus \omega^{o(\mathcal{Y}_{1k_0})}), 1 + N(\omega^{o(\mathcal{Y}_{0k_0})} \oplus \omega^{o(\mathcal{Y}_{1k_0}^{y_{1k_0}})})\big\} + 1.$$

Therefore

$$N(o(\mathcal{Y}^z)) \leq \max\big\{\alpha + 2, 2, 4 + N(o(\mathcal{Y}_{00})), 4 + N(o(\mathcal{Y}_{10})), \ldots,$$

$$4 + N(o(\mathcal{Y}_{0\alpha-1}), 4 + N(o(\mathcal{Y}_{1\alpha-1})), 4 + N(o(\mathcal{Y}_{0k_0}^{y_{0k_0}})), 4 + N(o(\mathcal{Y}_{1k_0}^{y_{1k_0}}))\big\}.$$

Since

$$N(o(\mathcal{Y}_{0k_0}^{y_{0k_0}})) \leq N(o(\mathcal{Y}_{0k_0})) + 6 \cdot (rk_{\mathcal{Y}_{0k_0}}(y_{0k_0}) + 1)$$

and

$$N(o(\mathcal{Y}_{1k_0}^{y_{1k_0}})) \leq N(o(\mathcal{Y}_{1k_0})) + 6 \cdot (rk_{\mathcal{Y}_{1k_0}}(y_{1k_0}) + 1)$$

and

$$rk_{\mathcal{Y}_{0k_0}}(y_{0k_0}), rk_{\mathcal{Y}_{1k_0}}(y_{1k_0}) < rk_{\mathcal{Y}}(z)$$

and

$$\alpha \leq N(o(\mathcal{Y}))$$

and

$$N(o(\mathcal{Y}_{0k})), N(o(\mathcal{Y}_{1k})) < N(o(\mathcal{Y}))$$

for $0 \leq k < \gamma$ we see

$$N(o(\mathcal{Y}^z)) \leq N(o(\mathcal{Y})) + 6 \cdot (rk_{\mathcal{Y}}(z) + 1).$$

We define $e(\mathcal{Y}, z)$ as follows. Let

$$u = \langle k, \langle u_0, u_1 \rangle\rangle \in \mathcal{D}om(\mathcal{Y}(z)).$$

If $k \neq k_0$, then put $e(\mathcal{Y}, z)(u) := u$. If $k = k_0$, then there is a minimal $j < 2$ such that $\langle y_{jk_0}, u_{jk_0}\rangle \in po(\mathcal{Y}_{jk_0})$ does not hold. (Otherwise $u \in \mathcal{D}om(\mathcal{Y}(z))$ would not hold.)
If $j = 0$, then put

$$e(\mathcal{Y}, z)(u) := \langle \alpha, \langle e(\mathcal{Y}_{0k_0}, y_{0k_0})(u_{0k_0}), u_{1k_0}\rangle\rangle.$$

Here α is by definition the index of $\mathcal{Y}_{0k_0}^{y_{0k_0}} \otimes \mathcal{Y}_{1k_0}$ in $\mathcal{Y}^z$.
If $j = 1$, then put

$$e(\mathcal{Y}, z)(u) := \langle \alpha + 1, \langle u_{0k_0}, e(\mathcal{Y}_{1k_0}, y_{1k_0})(u_{1k_0})\rangle\rangle.$$

Here $\alpha + 1$ is by definition the index of $\mathcal{Y}_{0k_0} \otimes \mathcal{Y}_{1k_0}^{y_{1k_0}}$ in $\mathcal{Y}^z$. In both cases

$$rk_{\mathcal{Y}^z}(e(\mathcal{Y}, z)(u)) \leq rk_{\mathcal{Y}}(u) + 1$$

follows from $(\mathcal{Y}_{0k_0}^{y_{0k_0}} 1)$ resp. $(\mathcal{Y}_{1k_0}^{y_{1k_0}} 1)$.

In this case we are left with verifying that $e(\mathcal{Y}, z)$ is a quasi-embedding. Let $u, v \in \mathcal{D}om(\mathcal{Y}(z))$ and assume

$$\langle e(\mathcal{Y}, z)(u), e(\mathcal{Y}, z)(v)\rangle \in po(\mathcal{Y}^z). \tag{5.1}$$

Let $u = \langle k, \langle u_0, u_1\rangle\rangle$ where $k < \alpha$, $u_0 \in \mathcal{D}om(\mathcal{Y}_{0k})$ and $u_1 \in \mathcal{D}om(\mathcal{Y}_{1k})$.

If $k \neq k_0$, then $e(\mathcal{Y}, z)(u) = u$. Assume that v has the form $\langle k', \langle v_0, v_1\rangle\rangle$. By (5.1) we see $k = k'$. Thus $e(\mathcal{Y}, z)(v) = v$ and $\langle u, v\rangle \in po(\mathcal{Y}(z))$.

Now we assume $k = k_0$. If $e(\mathcal{Y}, z)(u)$ has the form $\langle \alpha, \langle e(\mathcal{Y}_{0k_0}, y_{0k_0})(u_0), u_1\rangle\rangle$, then by (5.1) $e(\mathcal{Y}, z)(v)$ has necessarily the form $\langle \alpha, \langle e(\mathcal{Y}_{0k_0}, y_{0k_0})(v_0), v_1\rangle\rangle$ and

$$\langle\langle e(\mathcal{Y}_{0k_0}, y_{0k_0})(u_0), u_1\rangle, \langle e(\mathcal{Y}_{0k_0}, y_{0k_0})(v_0), v_1\rangle\rangle \in po(\mathcal{Y}_{0k_0}^{y_{0k_0}} \otimes \mathcal{Y}_{1k_0})$$

must hold. Thence

$$\langle e(\mathcal{Y}_{0k_0}, y_{0k_0})(u_0), e(\mathcal{Y}_{0k_0}, y_{0k_0})(v_0)\rangle \in po(\mathcal{Y}_{0k_0}^{y_{0k_0}})$$

and

$$\langle u_1, v_1\rangle \in po(\mathcal{Y}_{1k_0}).$$

Since $e(\mathcal{Y}_{0k_0}, y_{0k_0})$ is a quasi-embedding we conclude

$$\langle u_0, v_0\rangle \in po(\mathcal{Y}_{0k_0}(y_{0k_0})) \subseteq po(\mathcal{Y}_{0k_0}).$$

Thus $\langle u, v\rangle \in po(\mathcal{Y}(z))$.

Similarly $\langle u, v\rangle \in po(\mathcal{Y}^z)$ can be shown if $e(\mathcal{Y}, z)(u)$ has the form

$$\langle \alpha + 1, \langle u_0, e(\mathcal{Y}_{1k_0}, y_{1k_0})(u_1)\rangle\rangle.$$

CASE C: $\mathcal{Y} = \mathcal{T}(\langle \mathcal{Y}_j : j < \beta\rangle)$ and $z \in \mathcal{D}om(\mathcal{Y})$.

Case 1: z has the form $\langle\langle 0, y_0\rangle, \langle\rangle\rangle$ where $y_0 \in \mathcal{D}om(\mathcal{Y}_0)$.
By induction hypothesis there is a partial order $\mathcal{Y}_0^{y_0} \in Comp(\langle \mathcal{X}_i : i < \alpha\rangle)$ and a quasi-embedding $e(\mathcal{Y}_0, y_0)$ of $\mathcal{Y}_0(y_0)$ into $\mathcal{Y}_0^{y_0}$ so that $(\mathcal{Y}_0^{y_0} 1)$, $(\mathcal{Y}_0^{y_0} 2)$ and $(\mathcal{Y}_0^{y_0} 3)$ are satisfied. Put

$$\mathcal{Y}^z := \mathcal{T}(\langle \mathcal{Y}_0^{y_0}, \mathcal{Y}_1, \ldots, \mathcal{Y}_{\beta-1}\rangle).$$

Then $\mathcal{Y}^z \in Comp(\langle \mathcal{X}_i : i < \alpha\rangle)$.

Furthermore

$$o(\mathcal{Y}^z) = \vartheta(\Omega^{\beta-1} \cdot o(\mathcal{Y}_{\beta-1}) + \cdots + \Omega^0 \cdot o(\mathcal{Y}_0^{y_0})).$$

Thence by Lemma 3.3 and $(\mathcal{Y}_0^{y_0} 3)$

$$o(\mathcal{Y}^z) < o(\mathcal{Y}).$$

Furthermore

$$N(o(\mathcal{Y}^z)) = \max\{N(o(\mathcal{Y}_{\beta-1})), \ldots, N(o(\mathcal{Y}_1)), N(o(\mathcal{Y}_0^{y_0})), \beta\} + 1.$$

Thus $(\mathcal{Y}_0^{y_0}2)$ yields

$$N(o(\mathcal{Y}^z)) \le \max\{N(o(\mathcal{Y}_{\beta-1})), \ldots, N(o(\mathcal{Y}_0)) + 6 \cdot (rk_{\mathcal{Y}_0}(y_0) + 1), \beta\} + 1.$$

Therefore

$$N(o(\mathcal{Y}^z)) \le \max\{N(o(\mathcal{Y}_{\beta-1})), \ldots, N(o(\mathcal{Y}_0)), \beta\} + 1 + 6 \cdot (rk_{\mathcal{Y}_0}(y_0) + 1).$$

Thence

$$N(o(\mathcal{Y}^z)) \le N(o(\mathcal{Y})) + 6 \cdot (rk_{\mathcal{Y}}(z) + 1).$$

We define $e(\mathcal{Y}, z)$ as follows. Let

$$u = \langle\langle q, y\rangle, \langle u_1, \ldots, u_q\rangle\rangle \in \mathcal{D}om(\mathcal{Y}(z))$$

where $y \in \mathcal{D}om(\mathcal{Y}_q)$. We will define $e(\mathcal{Y}, z)(u)$ by recursion on the depth $dp_{\mathcal{Y}}(u)$ of u such that

$$rk_{\mathcal{Y}^z}(e(\mathcal{Y}, z)(u)) \le rk_{\mathcal{Y}}(u) + 1.$$

If $q = 0$, then

$$y \in \mathcal{D}om(\mathcal{Y}(y_0)).$$

(Otherwise $u \in \mathcal{D}om(\mathcal{Y}(z))$ would not hold.) So we can put

$$e(\mathcal{Y}, z)(u) := \langle\langle 0, e(\mathcal{Y}_0, y_0)(y)\rangle, \langle\rangle\rangle.$$

Then $e(\mathcal{Y}, z)(u) \in \mathcal{D}om(\mathcal{Y}^z)$ and

$$rk_{\mathcal{Y}^z}(e(\mathcal{Y}, z)(u)) \le rk_{\mathcal{Y}}(u) + 1$$

follows from $(\mathcal{Y}_0^{y_0}1)$.

If $q > 0$, then $y \in \mathcal{D}om(\mathcal{Y}_q)$. We may assume recursively that

$$e(\mathcal{Y}, z)(u_1), \ldots, e(\mathcal{Y}, z)(u_q)$$

are defined and that

$$rk_{\mathcal{Y}^z}(e(\mathcal{Y}, z)(u_l)) \le rk_{\mathcal{Y}}(u_l) + 1 \tag{5.2}$$

holds for $1 \le l \le q$. Then we may put

$$e(\mathcal{Y}, z)(u) := \langle\langle q, y\rangle, \langle e(\mathcal{Y}, z)(u_1), \ldots, e(\mathcal{Y}, z)(u_q)\rangle\rangle.$$

Then $e(\mathcal{Y}, z)(u) \in \mathcal{D}om(\mathcal{Y}^z)$ and

$$rk_{\mathcal{Y}^z}(e(\mathcal{Y}, z)(u)) \le rk_{\mathcal{Y}}(u) + 1$$

holds by (5.2).

By induction on $dp_{\mathcal{Y}}(u) + dp_{\mathcal{Y}}(v)$ we show that

$$\langle e(\mathcal{Y}, z)(u), e(\mathcal{Y}, z)(v)\rangle \in po(\mathcal{Y}^z) \tag{5.3}$$

implies $\langle u, v\rangle \in po(\mathcal{Y}(z))$ for all $u, v \in \mathcal{D}om(\mathcal{Y}(z))$. Assume

$$u = \langle\langle q, y\rangle, \langle u_1, \ldots, u_q\rangle\rangle$$

and

$$v = \langle\langle q', y'\rangle, \langle v_1, \ldots, v_{q'}\rangle\rangle.$$

Subcase 1.1. $q' > 0$ and $\langle e(\mathcal{Y}, z)(u), e(\mathcal{Y}, z)(v_l)\rangle \in po(\mathcal{Y}^z)$ holds for some $l \in \{1, \ldots, q'\}$. The induction hypothesis yields

$$\langle u, v_l\rangle \in po(\mathcal{Y}(z)).$$

Since $\langle v_l, v\rangle \in po(\mathcal{Y}(z))$ we see $\langle u, v\rangle \in po(\mathcal{Y}(z))$.

Subcase 1.2. $q' = 0$ or $q' > 0$ and there is no $l \in \{1, \ldots, q'\}$ such that

$$\langle e(\mathcal{Y}, z)(u), e(\mathcal{Y}, z)(v_l)\rangle \in po(\mathcal{Y}^z)$$

holds. By (5.3) we see that $q = q'$ must hold.

Subcase 1.2.1. $q' = 0$. Then $e(\mathcal{Y}, z)(u) = \langle 0, e(\mathcal{Y}_0, y_0)(y)\rangle$ and $e(\mathcal{Y}, z)(v) = \langle 0, e(\mathcal{Y}_0, y_0)(y')\rangle$ and by (5.3)

$$\langle e(\mathcal{Y}_0, y_0)(y), e(\mathcal{Y}_0, y_0)(y')\rangle \in po(\mathcal{Y}_0^{y_0}).$$

Since $e(\mathcal{Y}_0, y_0)$ is a quasi-embedding we conclude

$$\langle y, y'\rangle \in po(\mathcal{Y}_0(y_0)).$$

Hence $\langle u, v\rangle \in po(\mathcal{Y}(z))$.

Subcase 1.2.2. $q' > 0$. Then

$$e(\mathcal{Y}, z)(u) = \langle\langle q, y\rangle, \langle e(\mathcal{Y}, z)(u_1), \ldots, e(\mathcal{Y}, z)(u_q)\rangle\rangle$$

and

$$e(\mathcal{Y}, z)(v) = \langle\langle q, y'\rangle, \langle e(\mathcal{Y}, z)(v_1), \ldots, e(\mathcal{Y}, z)(v_q)\rangle\rangle.$$

By (5.3) we see

$$\langle e(\mathcal{Y}, z)(u_l), e(\mathcal{Y}, z)(v_l)\rangle \in po(\mathcal{Y}^z)$$

for $1 \le l \le q$. The induction hypothesis yields

$$\langle u_l, v_l\rangle \in po(\mathcal{Y}(z))$$

for $1 \le l \le q$. Since $\langle y, y'\rangle \in po(\mathcal{Y}_q)$ holds by (5.3) we see

$$\langle u, v\rangle \in po(\mathcal{Y}(z)).$$

Case 2: z has the form

$$\langle\langle r, y_r\rangle, \langle z_1, \ldots, z_r\rangle\rangle$$

for some $r > 0$ where $y_r \in \mathcal{D}om(\mathcal{Y}_r)$. By our main induction hypothesis we can pick a partial order

$$\mathcal{Y}_r^{y_r}$$

and a quasi-embedding

$$e(\mathcal{Y}_r, y_r)$$

of $\mathcal{Y}_r(y_r)$ into $\mathcal{Y}_r^{y_r}$ so that $(\mathcal{Y}_r^{y_r}1)$, $(\mathcal{Y}_r^{y_r}2))$ and $(\mathcal{Y}_r^{y_r}3)$ are satisfied. By the subsidiary induction hypothesis we can pick for $1 \le j \le r$ a partial order

$$\mathcal{Y}^{z_j}$$

and a quasi-embedding

$$e(\mathcal{Y}, z_j)$$

of $\mathcal{Y}(z_j)$ into $\mathcal{Y}^{z_j}$ such that $(\mathcal{Y}_j^{z_j}1)$, $(\mathcal{Y}_j^{z_j}2)$ and $(\mathcal{Y}_j^{z_j}3)$ are true. Let $\underline{y_r}$ be the flat partial order with domain $\{y_r\}$. Let

$$\mathcal{Y}^z := \mathcal{T}(\langle \mathcal{Y}_0, ..., \mathcal{Y}_{r-2}, \bigoplus\{\mathcal{Y}_r \times \mathcal{Y}^{z_j} : 1 \le j \le r\} \oplus (\mathcal{Y}_{r-1} \otimes \underline{y_r}), \mathcal{Y}_r^{y_r}, \mathcal{Y}_{r+1}, ..., \mathcal{Y}_{\beta-1}\rangle).$$

Then

$$\mathcal{Y}^z \in Comp(\langle \mathcal{X}_i : i < \alpha \rangle).$$

Furthermore
$o(\mathcal{Y}^z) = \vartheta(\Omega^{\beta-1} \cdot o(\mathcal{Y}_{\beta-1}) + \cdots + \Omega^{r+1} \cdot o(\mathcal{Y}_{r+1}) + \Omega^r \cdot o(\mathcal{Y}_r^{y_r}) + \Omega^{r-1} \cdot o(\bigoplus\{\mathcal{Y}_r \otimes \mathcal{Y}^{z_j} : 1 \le j \le r\} \oplus (\mathcal{Y}_{r-1} \otimes \underline{y_r})) + \Omega^{r-2} \cdot o(\mathcal{Y}_{r-2}) + \cdots + \Omega^0 \cdot o(\mathcal{Y}_0))$.
Thus

$$o(\mathcal{Y}^z) < o(\mathcal{Y})$$

holds by Lemma 3.3 and $(\mathcal{Y}^{z_j}3)$ $\quad (1 \le j \le r)$. Furthermore

$$\begin{aligned} N(\mathcal{Y}^z) = \quad & \max\{N(o(\mathcal{Y}_{\beta-1})), \ldots, N(o(\mathcal{Y}_{r+1})), N(o(\mathcal{Y}_r^{y_r})), \\ & N(o(\bigoplus\{\omega^{\omega^{o(\mathcal{Y}_r)} \oplus \omega^{o(\mathcal{Y}^{z_j})}} : 1 \le j \le r\} \oplus \omega^{\omega^{o(\mathcal{Y}_{r-1})} \oplus \omega^{o(\underline{y_r})}})), \\ & N(o(\mathcal{Y}_{r-2})), \ldots, N(o(\mathcal{Y}_0)), \beta\} + 1. \end{aligned}$$

By $(\mathcal{Y}_r^{y_r}2)$ we know

$$N(o(\mathcal{Y}_r^{y_r})) \le N(o(\mathcal{Y}_r)) + 6 \cdot (rk_{\mathcal{Y}_r}(y_r) + 1).$$

Furthermore

$$N(\bigoplus\{\omega^{\omega^{o(\mathcal{Y}_r)} \oplus \omega^{o(\mathcal{Y}^{z_j})}} : 1 \le j \le r\} \oplus \omega^{\omega^{o(\mathcal{Y}_{r-1})} + \omega}) \le$$

$$\max\Big\{N(o(\mathcal{Y}_{r-1})) + 3, 3, 3 + \max\{N(o(\mathcal{Y}_r)), N(o(\mathcal{Y}^{z_1}))\}, \ldots$$

$$3 + \max\{N(o(\mathcal{Y}_r)), N(o(\mathcal{Y}^{z_r}))\}, r+1\Big\} + 1$$

$$\le \max\Big\{N(o(\mathcal{Y}_{r-1})) + 3, 3 + \max\{N(o(\mathcal{Y}_r)), N(o(\mathcal{Y})) + 6 \cdot (rk_{\mathcal{Y}}(z_1) + 1)\}, \ldots$$

$$3 + \max\{N(o(\mathcal{Y}_r)), N(o(\mathcal{Y})) + 6 \cdot (rk_{\mathcal{Y}}(z_r) + 1)\}, r+1\Big\} + 1 =: (*).$$

We know that

$$N(o(\mathcal{Y}_i)) < N(o(\mathcal{Y}))$$

is true for $0 \le i \le r$. Furthermore

$$r \le \beta.$$

Thus there is some $i_0 \le r$ such that

$$(*) \le \max\{\beta + 2, 4 + N(o(\mathcal{Y})) + 6 \cdot (rk_{\mathcal{Y}}(z_{i_0}) + 1)\}.$$

Therefore

$$N(o(\mathcal{Y}^z)) \le$$

$$\max\{N(o(\mathcal{Y}_r)) + 6 \cdot (rk_{\mathcal{Y}_r}(y_r) + 1) + 1, 5 + N(o(\mathcal{Y})) + 6 \cdot (rk_{\mathcal{Y}}(z_{i_0}) + 1), \beta + 3\}.$$

We see

$$N(o(\mathcal{Y})) = \max\{N(o(\mathcal{Y}_{\beta-1})), \ldots, N(o(\mathcal{Y}_0)), \beta\} + 1.$$

Therefore

$$N(o(\mathcal{Y})) + 6 \cdot (rk_{\mathcal{Y}}(z) + 1) > \beta + 3$$

and

$$N(o(\mathcal{Y})) + 6 \cdot (rk_{\mathcal{Y}}(z) + 1) \geq N(o(\mathcal{Y}_r)) + 6 \cdot (rk_{\mathcal{Y}_r}(y_r)) + 1) + 1.$$

Since

$$rk_{\mathcal{Y}}(z) > rk_{\mathcal{Y}}(z_{i_0})$$

we conclude

$$N(o(\mathcal{Y})) + 6 \cdot (rk_{\mathcal{Y}}(z) + 1) \geq 5 + N(o(\mathcal{Y})) + 6 \cdot (rk_{\mathcal{Y}}(z_{i_0}) + 1).$$

Thus

$$N(o(\mathcal{Y})) + 6 \cdot (rk_{\mathcal{Y}}(z) + 1) \geq N(o(\mathcal{Y}^z)).$$

We define $e(\mathcal{Y}, z)$ as follows. Let

$$u = \langle\langle q, y\rangle, \langle u_1, \ldots, u_q\rangle\rangle \in \mathcal{D}om(\mathcal{Y}(z))$$

where $y \in \mathcal{D}om(\mathcal{Y}_q)$. We will define $e(\mathcal{Y}, z)(u)$ by recursion on the depth $dp_{\mathcal{Y}}(u)$ of u such that

$$rk_{\mathcal{Y}^z} e(\mathcal{Y}, z)(u) \leq rk_{\mathcal{Y}}(u) + 1.$$

We assume that $e(\mathcal{Y}, z)(u_1), \ldots, e(\mathcal{Y}, z)(u_q)$ are defined for $q \geq 0$ and that

$$rk_{\mathcal{Y}^z}(e(\mathcal{Y}, z)(u_l) \leq rk_{\mathcal{Y}}(u_l) + 1 \tag{5.4}$$

holds for $1 \leq l \leq q$.

If $q \neq r$ and if $q \neq r - 1$ we can set

$$e(\mathcal{Y}, z)(u) := \langle\langle q, y\rangle, \langle e(\mathcal{Y}, z)(u_1), \ldots, e(\mathcal{Y}, z)(u_q)\rangle\rangle.$$

Then $e(\mathcal{Y}, z)(u) \in \mathcal{D}om(\mathcal{Y}^z)$ and

$$rk_{\mathcal{Y}^z}(e(\mathcal{Y}, z)(u)) \leq rk_{\mathcal{Y}}(u) + 1$$

holds by (5.4).

If $q = r - 1$ put

$$e(\mathcal{Y}, z)(u) := \langle r - 1, \langle\langle r + 1, \langle y, y_r\rangle\rangle\rangle, \langle e(\mathcal{Y}, z)(u_1), \ldots, e(\mathcal{Y}, z)(u_q)\rangle\rangle.$$

Here $r + 1$ is the index of $\mathcal{Y}_{r-1} \otimes \underline{y_r}$ in $\bigoplus\{\mathcal{Y}_r \otimes \mathcal{Y}^{z_j} : 1 \leq j \leq r\} \oplus (\mathcal{Y}_{r-1} \otimes \underline{y_r})$.

Then

$$e(\mathcal{Y}, z)(u) \in \mathcal{D}om(\mathcal{Y}^z).$$

Since

$$rk_{\mathcal{Y}_{r-1} \otimes \underline{y_r}}(\langle y, y_r\rangle) \leq rk_{\mathcal{Y}_{r-1}}(y) + 1$$

the inequation

$$rk_{\mathcal{Y}^z}(e(\mathcal{Y}, z)(u)) \leq rk_{\mathcal{Y}}(u) + 1$$

holds by (5.4).

Finally assume $q = r$. If $y \in \mathcal{D}om(\mathcal{Y}_r(y_r))$, then we may put

$$e(\mathcal{Y}, z)(u) := \langle\langle r, e(\mathcal{Y}_r, y_r)(y)\rangle, \langle e(\mathcal{Y}, z)(u_1), \ldots, e(\mathcal{Y}, z)(u_q)\rangle\rangle \in \mathcal{D}om(\mathcal{Y}^z),$$

and

$$rk_{\mathcal{Y}^z}(e(\mathcal{Y},z)(u)) \leq rk_{\mathcal{Y}}(u)+1$$

holds by $(\mathcal{Y}_r^{y_r}1)$ and (5.4).

Now assume $\langle y_r, y\rangle \in po(\mathcal{Y}_r)$. Then there is a minimal $j \leq q$ such that $\langle z_j, u_j\rangle \in po(\mathcal{Y})$ does not hold. (Otherwise $u \in \mathcal{D}om(\mathcal{Y}(z))$ would not hold.) Put $e(\mathcal{Y},z)(u) := \langle\langle r-1, \langle j, \langle y, e(\mathcal{Y},z_j)(u_j)\rangle\rangle\rangle,$
$\langle e(\mathcal{Y},z)(u_1),\ldots,e(\mathcal{Y},z)(u_{j-1}),e(\mathcal{Y},z)(u_{j+1}),\ldots,e(\mathcal{Y},z)(u_q)\rangle\rangle$.

Then

$$e(\mathcal{Y},z)(u) \in \mathcal{D}om(\mathcal{Y}^z).$$

Since by (5.4)

$$rk_{\mathcal{Y}_r\otimes\mathcal{Y}^{z_j}}(\langle y, e(\mathcal{Y}_j,z_j)(u_j)\rangle) \leq \max\{rk_{\mathcal{Y}_r}(y)+1, rk_{\mathcal{Y}}(u_j)+2\},$$

we see

$$rk_{\mathcal{Y}^z}(e(\mathcal{Y},z)(u)) \leq rk_{\mathcal{Y}}(u)+1$$

again using (5.4). By induction on $dp_{\mathcal{Y}}(u)+dp_{\mathcal{Y}}(v)$ we finally show that

$$\langle e(\mathcal{Y},z)(u), e(\mathcal{Y},z)(v)\rangle \in po(\mathcal{Y}^z) \tag{5.5}$$

implies $\langle u,v\rangle \in po(\mathcal{Y}(z))$ for all $u,v \in \mathcal{D}om(\mathcal{Y}(z))$. Let

$$u = \langle\langle q,y\rangle, \langle v_1,\ldots,v_q\rangle\rangle$$

and

$$v = \langle\langle q',y'\rangle, \langle v_1,\ldots,v_{q'}\rangle\rangle.$$

Subcase 2.1. $q'>0$ and there exists an $l \in \{1,\ldots,q'\}$ such that

$$\langle e(\mathcal{Y},z)(u), e(\mathcal{Y},z)(v_l)\rangle \in po(\mathcal{Y}^z).$$

Then $\langle u,v_l\rangle \in po(\mathcal{Y}(z))$ holds by the induction hypothesis. Since $\langle v_l,v\rangle \in po(\mathcal{Y}(z))$ we see $\langle u,v\rangle \in po(\mathcal{Y}(z))$.

Subcase 2.2. $q'=0$, or $q'\neq 0$ and there is no $l \in \{1,\ldots,q'\}$ such that

$$\langle e(\mathcal{Y},z)(u), e(\mathcal{Y},z)(v_l)\rangle \in po(\mathcal{Y}^z)$$

holds. By (5.5) and the definition of $e(\mathcal{Y},z)$ we see that $q=q'$ must hold.

Subcase 2.2.1. $q\neq r$ and $q \neq r-1$. (The case $q=0$ is allowed.) Then

$$e(\mathcal{Y},z)(u) = \langle\langle q,y\rangle, \langle e(\mathcal{Y},z)(u_1),\ldots,e(\mathcal{Y},z)(u_q)\rangle\rangle,$$

$$e(\mathcal{Y},z)(v) = \langle\langle q,y'\rangle\rangle, \langle e(\mathcal{Y},z)(v_1),\ldots,e(\mathcal{Y},z)(v_q)\rangle\rangle$$

and by (5.5)

$$\langle e(\mathcal{Y},z)(u_l), e(\mathcal{Y},z)(v_l)\rangle \in po(\mathcal{Y}^z)$$

for $1\leq l\leq q$. By induction hypothesis

$$\langle u_l, v_l\rangle \in po(\mathcal{Y}(z))$$

for $1 \leq l \leq q$. By (5.5)

$$\langle y,y'\rangle \in po(\mathcal{Y}_q)$$

and thus

$$\langle u, v\rangle \in po(\mathcal{Y}(z)).$$

Subcase 2.2.2. $q = q' = r - 1$. Then

$$e(\mathcal{Y}, z)(u) = \langle\langle r-1, \langle r+1, \langle y, y_r\rangle\rangle\rangle, \langle e(\mathcal{Y}, z)(u_1), \ldots, e(\mathcal{Y}, z)(u_q)\rangle\rangle,$$

$$e(\mathcal{Y}, z)(v) = \langle\langle r-1, \langle r+1, \langle y', y_r\rangle\rangle\rangle, \langle e(\mathcal{Y}, z)(v_1), \ldots, e(\mathcal{Y}, z)(v_q)\rangle\rangle$$

and by (5.5)

$$\langle e(\mathcal{Y}, z)(u_l), e(\mathcal{Y}, z)(v_l)\rangle \in po(\mathcal{Y}^z)$$

for $1 \le l \le q$. By induction hypothesis

$$\langle u_l, v_l\rangle \in po(\mathcal{Y}(z))$$

for $1 \le l \le q$. Since by (5.5)

$$\langle\langle y, y_r\rangle, \langle y', y_r\rangle\rangle \in po(\mathcal{Y}_{r-1} \otimes \underline{y_r})$$

we see

$$\langle y, y'\rangle \in po(\mathcal{Y}_{r-1}).$$

Thus

$$\langle u, v\rangle \in po(\mathcal{Y}(z)).$$

Subcase 2.2.3. $q = q' = r$.

Subsubcase 2.2.3.1. $y \in \mathcal{D}om(\mathcal{Y}_r(y_r))$. By (5.5) we see $y' \in \mathcal{D}om(\mathcal{Y}_r(y_r))$. Then

$$e(\mathcal{Y}, z)(u) = \langle\langle q, e(\mathcal{Y}_q, y_q)(y)\rangle, \langle e(\mathcal{Y}, z)(u_1), \ldots, e(\mathcal{Y}, z)(u_q)\rangle\rangle,$$

$$e(\mathcal{Y}, z)(v) = \langle\langle q, e(\mathcal{Y}_q, y_q)(y')\rangle, \langle e(\mathcal{Y}, z)(v_1), \ldots, e(\mathcal{Y}, z)(v_q)\rangle\rangle$$

and by (5.5)

$$\langle e(\mathcal{Y}_q, y_q)(y), e(\mathcal{Y}_q, y_q)(y')\rangle \in po(\mathcal{Y}_r^{y_r})$$

and

$$\langle e(\mathcal{Y}, z)(u_l), e(\mathcal{Y}, z)(v_l)\rangle \in po(\mathcal{Y}^z)$$

for $1 \le l \le q$. Since $e(\mathcal{Y}_q, y_q)$ is a quasi-embedding we see

$$\langle y, y'\rangle \in po(\mathcal{Y}_q(y_q)).$$

By induction hypothesis

$$\langle u_l, v_l\rangle \in po(\mathcal{Y}(z))$$

for $1 \le l \le q$. Thus

$$\langle u, v\rangle \in po(\mathcal{Y}(z)).$$

Subsubcase 2.2.3.2. $y \notin \mathcal{D}om(\mathcal{Y}_r(y_r))$. Then by (5.5) we see $y' \notin \mathcal{D}om(\mathcal{Y}_r(y_r))$. Then $e(\mathcal{Y}, z)(u)$ has the form

$$\langle\langle j, \langle y, e(\mathcal{Y}, z_j)(u_j)\rangle\rangle, \langle e(\mathcal{Y}, z)(u_1), \ldots, e(\mathcal{Y}, z)(u_{j-1}), e(\mathcal{Y}, z)(u_{j+1}), \ldots, e(\mathcal{Y}, z)(u_q)\rangle\rangle$$

and $e(\mathcal{Y}, z)(v)$ has the form

$$\langle\langle j', \langle y', e(\mathcal{Y}, z_{j'})(v_{j'})\rangle\rangle, \langle e(\mathcal{Y}, z)(v_1), \ldots, e(\mathcal{Y}, z)(v_{j'-1}), e(\mathcal{Y}, z)(v_{j'+1}), \ldots, e(\mathcal{Y}, z)(v_q)\rangle\rangle.$$

By (5.5) necessarily $j = j'$. Furthermore (5.5) yields

$$\langle\langle y, e(\mathcal{Y}, z_j)(u_j)\rangle, \langle y', e(\mathcal{Y}, z_j)(v_j)\rangle\rangle \in po(\mathcal{Y}_r \otimes \mathcal{Y}^{z_j}).$$

Hence

$$\langle y, y'\rangle \in po(\mathcal{Y}_r)$$

and

$$\langle e(\mathcal{Y}, z_j)(u_j), e(\mathcal{Y}, z_j)(v_j)\rangle \in po(\mathcal{Y}^{z_j}).$$

Since $e(\mathcal{Y}, z_j)$ is a quasi-embedding we see

$$\langle u_j, v_j\rangle \in po(\mathcal{Y}(z_j)) \subseteq po(\mathcal{Y}(z)).$$

By (5.5)

$$\langle e(\mathcal{Y}, z)(u_l), e(\mathcal{Y}, z)(v_l)\rangle \in po(\mathcal{Y}(z))$$

for $1 \leq l \leq q$ and $l \neq j$. By induction hypothesis

$$\langle u_l, v_l\rangle \in po(\mathcal{Y}(z))$$

for $1 \leq l \leq q$ and $l \neq j$. Thus

$$\langle u, v\rangle \in po(\mathcal{Y}(z)).$$

□

A similar result holds for the Higman order as can be seen by a careful inspection of Schütte and Simpson (1985). Furthermore a careful inspection of the corresponding proof given in Rathjen and Weiermann (1993) yields a similar bounding lemma for trees with unbounded outdegree, i.e. for terms over a signature with variadic function symbols. In fact, the bounding lemma is intended to be paradigmatic for all well-partial orders for which effective reifications can be constructed in a way which corresponds to the constructions given in this article or in Rathjen and Weiermann (1993) and Schütte and Simpson (1985).

6. The Bounding Theorem

THEOREM 6.1. *Let $\langle \mathcal{X}_i : i < \alpha\rangle$ be a finite sequence of flat partial orders so that $\mathcal{D}om(\mathcal{X}_0) \neq \emptyset$. Let $\mathcal{Y} \in Comp(\langle \mathcal{X}_i : i < \alpha\rangle)$.*

1 *If $s = \langle s_i : i < \beta\rangle$ $(\beta \leq \omega)$ is a bad sequence of elements in the domain of $\mathcal{Y}$, then $\beta < \omega$.*
2 *Let $k < \omega$. If $s = \langle s_i : i < \beta\rangle$ $(\beta \leq \omega)$ is a bad sequence of elements in the domain of $\mathcal{Y}$ such that*

$$rk_{\mathcal{Y}}(s_j) \leq rk_{\mathcal{Y}}(s_0) + k \cdot j^2$$

holds for $0 \leq j < \beta$, then

$$\beta \leq H_{\omega \cdot o(\mathcal{Y})}(15 \cdot (rk_{\mathcal{Y}}(s_0) + 2 + k) + N(o(\mathcal{Y})) + 17).$$

PROOF. 1) This follows exactly as in Rathjen and Weiermann (1993). For convenience we will repeat the argument. Let $\mathcal{Y}_0 := \mathcal{Y}$, $z_0 := s_0$, $\sigma_0 := o(\mathcal{Y}_0)$ and let e_0 be the identity map restricted to $\mathcal{D}om(\mathcal{Y}_0)$. For $0 \leq i < \beta$ define by (primitive) recursion

$$\mathcal{Y}_{i+1} := \mathcal{Y}_i^{z_i},$$

$$e_{i+1} := e(\mathcal{Y}_i, z_i) \circ e_i,$$

$$z_{i+1} := e_{i+1}(s_{i+1})$$

and

$$\sigma_{i+1} := o(\mathcal{Y}_{i+1}).$$

Then $\sigma_{i+1} < \sigma_i$ for $i < \beta$ and thus $\beta < \omega$ since there is no infinite strictly descending sequence of ordinals, provided the whole construction is well defined which we are going to prove now. By the Bounding lemma we have only to prove that $z_i \in \mathcal{D}om(\mathcal{Y}_i)$ holds for $0 \leq i < \beta$. By induction on $n < \beta$ we show slightly more general that $\langle e_n(s_n), e_n(s_{n+1}), \ldots \rangle \in Bad(\mathcal{Y}_n)$. This claim holds for $n = 0$, since $\langle s_0, s_1, \ldots \rangle \in Bad(\mathcal{Y}_0)$. Assume now that $\langle e_n(s_n), e_n(s_{n+1}), \ldots \rangle \in Bad(\mathcal{Y}_n)$. Then $e_n(s_n) = z_n \in \mathcal{D}om(\mathcal{Y}_n)$ and $\langle e_n(s_{n+1}), \ldots \rangle \in Bad(\mathcal{Y}_n(z_n))$. Since $e(\mathcal{Y}_n, z_n)$ is a quasi-embedding of $\mathcal{Y}_n$ into $\mathcal{Y}_{n+1} = \mathcal{Y}_n^{z_n} \in Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$ we see that $\langle e_{n+1}(s_{n+1}), \ldots \rangle \in Bad(\mathcal{Y}_{n+1})$ holds.
2) Here we show how one can extract from the (reification) proof of Kruskal's theorem above (and the Bounding Lemma) a bound for the lengths of bad sequences for which the growth rate is effectively bounded. Let $\mathcal{Y}_0 := \mathcal{Y}$, $z_0 := s_0$, $\tau_0 := \omega \cdot o(\mathcal{Y}_0)$ and let e_0 be the identity map restricted to $\mathcal{D}om(\mathcal{Y}_0)$. Then $o(\mathcal{Y}_0) \neq 0$. Let $d := rk_{\mathcal{Y}}(s_0)$. For $0 \leq i < \beta$ define by (primitive) recursion

$$\mathcal{Y}_{i+1} := \mathcal{Y}_i^{z_i},$$

$$e_{i+1} := e(\mathcal{Y}_i, z_i) \circ e_i,$$

$$z_{i+1} := e_{i+1}(s_{i+1})$$

and

$$\tau_{i+1} := \omega \cdot o(\mathcal{Y}_{i+1}) + 6 \cdot \sum_{0 \leq j \leq i} (d + 1 + (k+1) \cdot j^2).$$

Let

$$R := 15 \cdot (d + 2 + k) + N(o(\mathcal{Y})) + 17.$$

We prove

$$H_{\tau_0}(R) > \ldots > H_{\tau_{\beta-1}}(R)$$

which will obviously imply

$$\beta \leq H_{\tau_0}(R).$$

Indeed, the first assertion of the Bounding Lemma yields

$$rk_{\mathcal{Y}_{i+1}}(e_i(s_j)) \leq rk_{\mathcal{Y}_0}(s_j) + i$$

for $0 \leq i \leq j < \beta$. The assumption on the growth rate of the sequence under consideration yields

$$rk_{\mathcal{Y}_0}(s_j) \leq d + k \cdot j^2$$

for $0 \leq i < \beta$. This discussion yields

$$rk_{\mathcal{Y}_{i+1}}(e_j(s_j)) \leq rk_{\mathcal{Y}_0}(s_j) + j \leq d + (k+1) \cdot (j+1)^2$$

for $0 \leq j < \beta$. The second assertion of the Bounding Lemma and a straightforward induction on $j < \beta$ yield

$$N(o(\mathcal{Y}_{i+1})) \leq N(o(\mathcal{Y}_0)) + 6 \cdot \sum_{0 \leq j \leq i} (d + 1 + (k+1) \cdot (j+1)^2)$$

for $0 \leq i < \beta$. This assertion can be used to show that

$$N(\tau_{i+1}) < 2^{R + N(\tau_i)}$$

holds for $0 \leq i < \beta$.
$\sum_{0 \leq j \leq i Thethirdassertionofthe Bounding Lemma yields} o(Y_{i+1}) < o(\mathcal{Y}_i)$, hence

$$\tau_{i+1} < \tau_i$$

for $0 \leq i < \beta$. Assertion 2 of Lemma 4.1 yields $H_{\tau_{i+1}}(R) < H_{\tau_i}(R)$ for $0 \leq i < \beta$, hence, finally

$$\beta < H_{\tau_0}(R)$$

is verified, which was to be shown. □

We obtain immediately the following application.

Corollary 6.1. (Kruskal's Theorem (weak finite form I)) *Let $\langle \mathcal{X}_i : i < \alpha \rangle$ be a finite sequence of flat partial orders so that $\mathcal{D}om(\mathcal{X}_0) \neq \emptyset$.*
Let $\delta := \Omega^{\alpha-1} \cdot | \mathcal{X}_{\alpha-1} | + \cdots + \Omega^0 \cdot | \mathcal{X}_0 |$.

1. *If $s = \langle s_i : i < \gamma \rangle$ $(\gamma \leq \omega)$ is a bad sequence of elements in the domain of $\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle)$, then $\gamma < \omega$.*
2. *If $k < \omega$ and if $s = \langle s_i : i < \gamma \rangle$ is a (necessarily finite) bad sequence of elements in the domain of $\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle)$ such that*

$$rk_{\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle)}(s_i) \leq rk_{\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle)}(s_0) + k \cdot i^2$$

holds for $0 \leq i < \gamma$, then

$$\gamma \leq H_{\vartheta\delta}(15 \cdot (rk_{\mathcal{T}(\langle \mathcal{X}_i : i < \alpha \rangle)}(s_0) + 2 + k) + \max\{\alpha, |\mathcal{X}_0|, \ldots, |\mathcal{X}_{\alpha-1}|\} + 30).$$

Corollary 6.2. (Kruskal's Theorem (weak finite form II)) *Let $\langle \mathcal{X}_i : i < \alpha \rangle$ be a finite sequence of flat partial orders so that $\mathcal{D}om(\mathcal{X}_0) \neq \emptyset$. Let $\mathcal{Y}_0, \ldots, \mathcal{Y}_{\beta-1} \in Comp(\langle \mathcal{X}_i : i < \alpha \rangle)$ $(\beta \leq \alpha)$. Let $\delta := \Omega^{\beta-1} \cdot o(\mathcal{Y}_{\beta-1}) + \cdots + \Omega^0 \cdot o(\mathcal{Y}_0)$.*

1. *If $s = \langle s_i : i < \gamma \rangle$ $(\gamma \leq \omega)$ is a bad sequence of elements in the domain of $\mathcal{T}(\langle \mathcal{Y}_j : j < \beta \rangle)$, then $\gamma < \omega$.*
2. *If $k < \omega$ and if $s = \langle s_i : i < \gamma \rangle$ is a (necessarily finite) bad sequence of elements in the domain of $\mathcal{T}(\langle \mathcal{Y}_j : j < \beta \rangle)$ such that*

$$rk_{\mathcal{T}(\langle \mathcal{Y}_j : j < \beta \rangle)}(s_i) \leq rk_{\mathcal{T}(\langle \mathcal{Y}_j : j < \beta \rangle)}(s_0) + k \cdot i^2$$

holds for $0 \leq i < \gamma$, then

$$\gamma \leq H_{\vartheta\delta}(15 \cdot (rk_{\mathcal{T}(\langle \mathcal{Y}_j : j < \beta \rangle)}(s_0) + 2 + k) + \max\{\beta, N(o(\mathcal{Y}_0)), \ldots, N(o(\mathcal{Y}_{\beta-1}))\} + 30).$$

An inspection of Hasegawa (1994), Rathjen and Weiermann (1993) and Schütte and Simpson (1985) shows that a similar result can be proved for the general Kruskal theorem and also for Higman's Lemma, for which we fix the result. Let $\mathcal{X}$ be a flat partial order. Let $t = \langle t_0, \ldots, t_{m-1}\rangle \in \mathcal{D}om(\mathcal{X}^{<\omega})$. Put $rk_{\mathcal{X}^{<\omega}}(t) := m$.

COROLLARY 6.3. (HIGMAN'S LEMMA (WEAK FINITE FORM)) *Let* $t = \langle t_0, \ldots, t_{m-1}\rangle$ *be a finite sequence of elements in the domain of* $\mathcal{X}^{<\omega}$ *such that*

$$rk_{\mathcal{X}^{<\omega}}(t_{j+1}) \leq rk_{\mathcal{X}^{<\omega}}(t_j) + d$$

holds for $0 \leq j < m$. *Then*

$$m \leq H_{\omega^{\omega^{|\mathcal{X}|-1}}}((4 + |\mathcal{X}| + 12 \cdot (rk_{\mathcal{X}^{<\omega}}(t_0) + 2 + d))^3).$$

PROOF. The proof can be read off from the proof of the Bounding Lemma and the proof above [or also from Hasegawa (1994)]. □

Due to the Extension Lemma of Cichon and Tahhan Bittar (1994), which will not be reproved here, a more general result can be shown for upper bounds on the lengths of sequences which necessarily contain a weakly increasing (with respect to the well-partial order in question) subsequence of a prescribed length $r < \omega$.

LEMMA 6.1. (THE TAHHAN-BITTAR EXTENSION LEMMA) *Let* $\langle \mathcal{X}_i : i < \alpha\rangle$ *be a finite sequence of flat partial orders so that* $\mathcal{D}om(\mathcal{X}_0) \neq \emptyset$. *Let* $k, r < \omega$. *Let* $\mathcal{Y} \in Comp(\langle \mathcal{X}_i : i < \alpha\rangle)$. *Assume that* $s = \langle s_i : i < \omega\rangle$ *is an infinite sequence of elements in the domain of* $\mathcal{Y}$ *such that*

$$rk_{\mathcal{Y}}(s_i) \leq rk_{\mathcal{Y}}(s_0) + k \cdot i^2$$

holds for $0 \leq i < \omega$. *Let*

$$A := H_{\omega^{\omega^{\omega^{o(\mathcal{Y})+1}}}, r}(15 \cdot (rk_{\mathcal{Y}}(s_0) + 2 + k) + no(o(\mathcal{Y})) + 17).$$

Then there exist natural numbers $i_0, \ldots, i_r \leq A$ *so that* $i_0 < \ldots < i_r$ *and* $\langle s_{i_l}, s_{i_{l+1}}\rangle \in po(\mathcal{Y})$ *holds for* $0 \leq l < r$.

Using this lemma we can show the following extension of the bounding theorem.

COROLLARY 6.4. (KRUSKAL'S THEOREM (STRONG FINITE FORM)) *Let* $\langle \mathcal{X}_i : i < \alpha\rangle$ *be a finite sequence of flat partial orders so that* $\mathcal{D}om(\mathcal{X}_0) \neq \emptyset$. *Let* $\mathcal{Y}_0, \ldots, \mathcal{Y}_{\beta-1} \in Comp(\langle \mathcal{X}_i : i < \alpha\rangle)$ $(\beta \leq \alpha)$. *Let* $\delta := \Omega^{\beta-1} \cdot o(\mathcal{Y}_{\beta-1}) + \cdots + \Omega^0 \cdot o(\mathcal{Y}_0)$.
Let $k, r < \omega$. *Assume that* $s = \langle s_i : i < \omega\rangle$ *is an infinite sequence in the domain of* $\mathcal{T}(\langle \mathcal{Y}_j : j < \beta\rangle)$ *such that*

$$rk_{\mathcal{T}(\langle \mathcal{Y}_j : j < \beta\rangle)}(s_i) \leq rk_{\mathcal{T}(\langle \mathcal{Y}_j : j < \beta\rangle)}(s_0) + k \cdot i^2$$

holds for $0 \leq i < \omega$, *Let*

$$A = H_{\vartheta\delta}(15 \cdot (rk_{\mathcal{T}(\langle \mathcal{Y}_j : j < \beta\rangle)}(s_0) + 4 + k) + \max\{\beta, N(o(\mathcal{Y}_0)), \ldots, N(o(\mathcal{Y}_{\beta-1}))\} + 30 + r).$$

Then there exist natural numbers $i_0, \ldots, i_r < A$ *so that* $i_0 < \ldots < i_r$ *and* $\langle s_{i_l}, s_{i_{l+1}}\rangle \in po(\mathcal{Y})$ *holds for* $0 \leq l < r$.

PROOF. Consider the sequence $\langle e(s_i) : 0 < i < \omega\rangle$ of elements in $\mathcal{Dom}(\mathcal{Y}^{s_0})$. Let $B := H_{\vartheta_{\omega^{\omega^{\omega^{o(\mathcal{Y}^{s_0})+1}}} \cdot r}}(15 \cdot (rk_{\mathcal{Y}^{s_0}}(e_1(s_1)) + 2 + k) + no(o(\mathcal{Y})) + 17)$. Then there exist by Lemma 6.1 $i_0, \ldots, i_r \leq B$ so that $1 \leq i_0 < \ldots < i_r$ and so that $\langle e(s_l), e_1(s_{l+1}) \in po(\mathcal{Y}^{s_0})$ holds for $0 \leq l < r$. Let $R := 15 \cdot (rk_{\mathcal{T}(\langle \mathcal{Y}_j : j < \beta\rangle)}(s_0) + 4 + k) + \max\{\beta, N(o(\mathcal{Y}_0)), \ldots, N(o(\mathcal{Y}_{\beta-1}))\} + 30$. Put $A := H_{\vartheta_{o(\mathcal{Y})}}(R + r)$. Then $A > B$, since $N\omega^{\omega^{\omega^{o(\mathcal{Y}^{s_0})+1}} \cdot r} \leq 2^{N(o(\mathcal{Y}))+R+r}$. □

In the particular case that the signature contains one constant symbol and one binary functions symbol, then an appropriate bounding function can be expressed in terms of H_{ε_0} and in the case that the signature contains one constant symbol and two binary function symbols, then an appropriate bounding function can be expressed in terms of H_{Γ_0}. These results will not be proved here. They follow by similar considerations using a slightly improved definition of $o(\mathcal{Y})$ [cf., for example, Hasegawa (1994)].

We close this section with a discussion of the optimality of the obtained Hardy bounds. Theorem 6.1 is optimal in the following sense. Assume that $\mathcal{Dom}(\mathcal{X}_0) \neq \emptyset$ and $\mathcal{Dom}(\mathcal{X}_i) \neq \emptyset$ for some $i < \omega$ so that $3 \leq i < \beta$. Then there does *not* exists a $\gamma < o(\mathcal{T}(\langle \mathcal{X}_i : i < \alpha\rangle))$ and no primitive recursive function p so that the following assertion holds:

Let $k < \omega$. Assume that $\langle s_i : i < \beta\rangle$ is a bad sequence in $\mathcal{Dom}(\mathcal{T}(\langle \mathcal{X}_i : i < \alpha\rangle))$ so that $rk_{\mathcal{T}(\langle \mathcal{X}_i : i<\alpha\rangle)}(s_i) \leq rk_{\mathcal{T}(\langle \mathcal{X}_i : i<\alpha\rangle)}(s_0) + k \cdot i^2$. Then $\beta < H_\gamma(p(rk_{\mathcal{T}(\langle \mathcal{X}_i : i<\alpha\rangle)}(s_0) + k))$.

This follows by considering the lengths of appropriate descending ordinal sequences of ordinal less than $o(\mathcal{T}(\langle \mathcal{X}_i : i < \alpha\rangle))$ [cf. Buchholz, Cichon and Weiermann (1994)].

7. Some theoretical implications on the computational complexity of algorithms

Theorem 6.1 and its corollaries have its own combinatorial interest. Nevertheless it can also be used for a general theoretical analysis of the computational complexity of algorithms for which the termination can be proved (in a sense made precise below) by so called simplification orderings on a given set of terms $\mathcal{T}$ which is in this section identified with an element of $Comp(\langle \mathcal{X}_i : i < \alpha\rangle)$. By definition a partial ordering $\preceq$ on $\mathcal{T}$ is a *simplification ordering* on $\mathcal{T}$ if its extends the homeomorphic embeddability relation on $\mathcal{T}$. Therefore simplification orderings are well-founded by Kruskal's theorem. Assume we have a computation procedure which performs computation sequences $\langle x_0, x_1, \ldots\rangle$ of elements of a given (countable) set X. If $\prec$ is the strict part of an arbitrary simplification ordering $\preceq$ on $\mathcal{T}$ and if we have proved the termination of the computation procedure, i.e. the finiteness of every computation sequence $\langle x_0, x_1, \ldots\rangle$, by constructing a function $f : X \to \mathcal{T}$ so that $f(x_{i+1}) \succ f(x_i)$ holds for any such sequence $\langle x_0, x_1, \ldots\rangle$ and if additionally the growth rate of these sequences is effectively controlled so that, say, $rk_{\mathcal{T}}(f(x_i) \leq rk_{\mathcal{T}}(f(x_0)) + k \cdot i^2$ is satisfied, then we have $m \leq H_{o(\mathcal{T})}((rk_{\mathcal{T}}(f(x_0)) + k) \cdot const)$ for every finite computation sequence $\langle x_0, x_1, \ldots, x_m\rangle$. More generally, if we have proved the termination of the computation procedure under consideration by showing that every sequence $\langle f(x_0), f(x_1), \ldots\rangle$ is bad for the homeomorphic embeddability relation, then under the mentioned effectiveness assumption mentioned before the same bound holds.

Applied to rewrite systems we get the following application. If R is a finite rewrite system over a set of terms $\mathcal{T}$ (over a finite signature) so that $\to_{\mathcal{R}}$ is contained in a simplification ordering, then the derivation length can be bounded in terms of the Hardy function $H_{o(\mathcal{T})}$. This is due to the fact that every rewrite step increases the rank of a given term only by a constant which depends on the rewrite system and the signature. This theoretical result yields for example that for α larger than the small Veblen number the termination of a finite rewrite system (over a finite signature) which can be used for computing the Hardy function H_α in a honest way – i.e. every output value is less than or equal to the number of reductions which are needed for computing it – cannot be proved by a simplification ordering.

For specific examples of simplification orderings our theoretical bounding result could be improved considerably – as mentioned in the introduction – in the following way. If the termination of R can be shown by using a multiset (lexicographic) path ordering, then the derivation lengths can by bounded in terms of a primitive (multiply) recursive function [cf. Hofbauer (1990) and Weiermann (1992)]. It is an *open problem* to prove or to disprove that there are always multiply recursive bounds on the derivation lengths of a finite rewrite system R over a finite signature, for which the rewrite relation $\rightarrow_{\mathcal{R}}$ is contained in a simplification ordering on the set of terms under consideration.

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