Universal Algebra for Generalised Metric Spaces

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1 Introduction

In [MPP16] the authors introduced the concepts of "quantitative algebras" and "quantitative theories", together with a deductive system for deriving valid entailment relations between "quantitative equations". In a nutshell, we can summarise the key ideas as follows:

• quantitative algebras: triplets $(A, d_A, \{op^A\}_{op \in \Sigma})$ where: (i) (A, d_A) is a metric space, (ii) $(A, \{op^A\}_{op \in \Sigma})$ is an algebra for the signature Σ , in

the sense of universal algebra i.e., a set A equipped with interpretations $op^A:A^n\to A$ for each n-ary operation op in the signature Σ , and (iii) such that $op^A:A^n\to A$ is nonexpansive $(A^n$ is equipped with the product metric).

- quantitative equation: a statement of the form $s =_{\epsilon} t$ between terms $s, t \in \operatorname{Terms}_{\Sigma}(X)$ built from a set of variables X, where $\epsilon \in [0, 1]$. The meaning of this statement is that, for every interpretation of the variables, the induced interpretations of the terms s and t have distance smaller or equal than ϵ .
- quantitative theory: a collection of quantitative equations (or, more generally, a collection of Horn implications between quantitative equations) that are valid in a certain class of quantitative algebras.

The entire research line consists in adapting some of the key ideas and results of universal algebra to the "quantitative" setting. A non-exhaustive list is: [Bac+18; MSV22; MSV21; MPP21; MV20; FMS21; MPP17; Bac+21; Adá22].

In this document we outline some key results that we have obtained developing the seminal ideas of [MPP16], but performing some modifications (sometimes nontrivial) and extensions to the original framework.

1.1 Our "Quantitative" Framework

The framework we present in this document differs from the original of [MPP16] on two key points:

1. quantitative algebras: triplets $(A, d_A, \{op^A\})$ where: (i) (A, d_A) is a generalised metric space (**GMet** space), (ii) $(A, \{op^A\}_{op \in \Sigma})$ is an algebra in the sense of universal algebra i.e., a set A equipped with interpretations $op^A: A^n \to A$ for each operation op in the signature Σ .

There are two relevant generalisations to appreciate in this definition.

- i. (A, d_A) is a space belonging to a category (**GMet**) of "generalised metric spaces". Examples of **GMet** categories include the familiar category **Met**, of metric spaces, and several others: pseudo-metric spaces, diffuse metric spaces, and many other classes. This allows more flexibility in the framework. It is always possible to restrict attention to metric spaces by fixing **GMet** = **Met**.¹
- ii. the interpretation op^A of each of the operations $op \in \Sigma$, is not required to be nonexpansive, and arbitrary set-theoretic functions $op^A:A^n \to A$ can be used, thus enlarging the class of models under consideration.

¹Indeed, it is possible to appreciate the other novelties of our framework by ignoring this level of generality and simply instantiating **GMet** to the well known category **Met**.

If one is only interested in the class of quantitative algebras whose operations are nonexpansive (wrt to the product metric on A^n), this class can be captured logically by a quantitative equation (see below), in the same way that commutative groups can be captured by the equation $x \cdot y = y \cdot x$ in universal algebra.

- 2. We replace the basic logical judgment of "quantitative equations" $(s =_{\epsilon} t)$ from [MPP16], with two separate types of judgments:
 - **GMet** equations: $\forall (A, d_A).s = t$,
 - **GMet** quantitative equations: $\forall (A, d_A).s =_{\epsilon} t$

where (A, d_A) is a **GMet** space and $s, t \in \text{Terms}_{\Sigma}(A)$. We only consider such judgments, that is, we do not investigate more complex logical statements built on top of those, such as, e.g., Horn implications.

The meaning of a **GMet** quantitative equation:

$$\forall (A, d_A).s =_{\epsilon} t$$

is that, for every nonexpansive interpretation of the **GMet** space "of variables" (A, d_A) into a quantitative algebra \mathbb{B} , the induced interpretations of s and t are at distance less than or equal to ϵ in \mathbb{B} (see Definition 3.5 for a formal definition).

The reader familiar with [MPP16] will notice that the semantics of **GMet** quantitative equations described above is in fact equivalent to the "basic quantitative inference" (in the terminology of [MPP16])

$$\left\{a_i =_{\epsilon_{ij}} a_j \mid a_i, a_j \in A, \epsilon_{ij} = d_A(a_i, a_j)\right\} \Rightarrow s =_{\epsilon} t$$

which has the key property of having, in the left premises, only $=_{\epsilon}$ judgments involving variables, and not complex terms. Hence **GMet** quantitative equations and "basic quantitative inferences" are essentially identical concepts.

What differs in our approach, though, is not merely a change in terminology, but a change in focus. We design the theory, and a Birkhoff–style deductive system, for reasoning about **GMet** equations $(\forall (A, d_A).s = t)$ and GMET quantitative equations $((\forall (A, d_A).s = t))$ only, and avoid discussing higher-level judgments such as Horn implications between them. By contrast, the deductive apparatus of [MPP16] (as subsequent literature) is designed around the concept of Horn implications. Consider for example the "Substitution" deductive rule from [MPP16].

$$\frac{x_1 =_{\epsilon_1} y_1, \dots, x_n =_{\epsilon_1} y_n \Rightarrow s =_{\epsilon} t}{\sigma(x_1) =_{\epsilon_1} \sigma(y_1), \dots, \sigma(x_n) =_{\epsilon_1} \sigma(y_n) \Rightarrow \sigma(s) =_{\epsilon} \sigma(t)}$$
 Substitution by σ

As it can be seen, even if the premise of the rule is a basic quantitiative inference because all $=_{\epsilon}$ judgments only involve variables, the conclusion is not, because the premises can be complex terms.

Our approach, therefore, can be considered as a restriction of that of [MPP16]. But from the restriction we can obtain simpler proofs and stronger results.

This document is organised as follows. In Section 2 we provide the necessary technical background. In Section 3 we present the main definition of our framework and state the main results.

2 Technical Background

In this section we provide all the necessary mathematical background needed to formally state our results and to verify the proofs. In Section 2.1 we cover material from Universal Algebra. In Section 2.2 from Category Theory. And in Section 2.3 from (generalised) Metric spaces.

2.1 Universal Algebra

We recall in this section some definitions of universal algebra we will build on, and we refer the reader to [Wec92] as reference specifically intended for computer scientists.

A signature Σ is a (possibly infinite) set of function symbols $op \in \Sigma$ each having a finite arity $ar(op) \in \mathbb{N}$. Operations of arity 0 are referred to as constants.

Definition 2.1. Given a signature $\Sigma = \{op_i\}_{i \in I}$, a Σ -algebra is a pair \mathbb{A} of the form $\mathbb{A} = (A, \{op^A\}_{op \in \Sigma})$ where:

- 1. $A \neq \emptyset$ is a nonempty set,
- 2. $\{op^A\}_{op\in\Sigma}$ is a collection of interpretations of the operations:

$$op^A: A^{ar(op)} \to A$$

of each function symbol in Σ .

Definition 2.2. Let $(A, \{op^A\}_{op \in \Sigma})$ and $(B, \{op^B\}_{op \in \Sigma})$ be two Σ -algebras, a function $f: A \to B$ is a homomorphism if:

$$f(op^A(a_1,...,a_n)) = op^B(f(a_1),...,f(a_n))$$

for all $a_1, \ldots, a_n \in A$ and $op \in \Sigma$.

We denote with $Alg(\Sigma)$ the collection of all Σ -algebras.

Definition 2.3 (Terms over Σ). Given a signature Σ and a set A, we denote with $\mathrm{Terms}_{\Sigma}(A)$ the collection of all Σ -terms built from A, i.e., the set inductively defined as follows:

$$a \in \operatorname{Terms}_{\Sigma}(A)$$
 $t_1, \ldots, t_n \in \operatorname{Terms}_{\Sigma}(A) \Longrightarrow op(t_1, \ldots, t_n) \in \operatorname{Terms}_{\Sigma}(A)$

for all $a \in A$ and $op \in \Sigma$.

Note that if $A = \emptyset$ and Σ does not contain any constants, then $\operatorname{Terms}_{\Sigma}(A) = \emptyset$.

Definition 2.4. We say that the pair (A, Σ) is *trivial* if $A = \emptyset$ and Σ does not contain any constants. Otherwise, we say that the pair (A, Σ) is *nontrivial*.

In what follows, if not otherwise explicitly stated, when writing $\operatorname{Terms}_{\Sigma}(A)$ we implicitly assume that the pair (A, Σ) is nontrivial.

Definition 2.5 (Equations). Given a signature Σ , a Σ -equation is a formal judgment ϕ of the form:

$$\forall A.s = t$$

where A is a set, (Σ, A) is nontrivial, and $s, t \in \text{Terms}_{\Sigma}(A)$. We denote with $\text{Eq}(\Sigma)$ the collection of all Σ -equations.

Often Σ is clear from the context and we will just talk about "equations" rather than Σ -equations.

Definition 2.6 (Interpretation). Given a Σ -algebra $\mathbb{A} = (A, \{op^A\}_{op \in \Sigma})$ and a set B, an interpretation of B in \mathbb{A} is a function $\tau : B \to A$. The interpretation τ extends uniquely to a function of type $[-]_{\tau}^A$: Terms $_{\Sigma}(B) \to A$ as follows:

$$[\![b]\!]_{\tau}^A = \tau(b)$$
 $[\![op(t_1, \dots, t_n)]\!]_{\tau}^A = op^A([\![t_1]\!]_{\tau}^A, \dots, [\![t_n]\!]_{\tau}^A).$

The following definition gives the semantics to equations and motivates the "\forall " syntactical notation we adopted (which is inspired by the treatment of equations in [Wec92]).

Definition 2.7 (Semantics of Equations). Given a Σ -algebra \mathbb{A} and an equation ϕ of the form $\forall B.s = t$, we say that \mathbb{A} satisfies ϕ (written $\mathbb{A} \models \phi$) if, for all interpretations $\tau : B \to A$, it holds that $[s]_{\tau}^{A} = [t]_{\tau}^{A}$:

$$\mathbb{A} \models \forall B.s = t \iff \text{ for all } \tau: B \to A, \, [\![s]\!]_\tau^A = [\![t]\!]_\tau^A.$$

Definition 2.8 (Equational Theory). Let $\mathcal{K} \subseteq \text{Alg}(\Sigma)$ be an arbitrary collection of Σ -algebras. The *equational theory* of \mathcal{K} is defined as the set of equations satisfied by all algebras in \mathcal{K} , formally:

$$\operatorname{Th}_{\Sigma}(\mathcal{K}) = \{ \phi \in \operatorname{Eq}(\Sigma) \mid \forall \mathbb{A} \in \mathcal{K}. \ \mathbb{A} \models \phi \}.$$

Definition 2.9 (Equationally defined classes). Let $\Phi \subseteq \text{Eq}(\Sigma)$ be an arbitrary collection of Σ -equations. The *models* of Φ are the Σ -algebras that satisfy all equations in Φ , formally:

$$\mathrm{Mod}_{\Sigma}(\Phi) = \{ \mathbb{A} \in \mathrm{Alg}(\Sigma) \mid \forall \phi \in \Phi. \ \mathbb{A} \models \phi \}$$

If $\mathcal{K} \subseteq Alg(\Sigma)$ is such that $\mathcal{K} = Mod_{\Sigma}(\Phi)$ for some $\Phi \subseteq Eq(\Sigma)$, we say that \mathcal{K} is an equationally defined (by Φ) class.

Definition 2.10 (Model Theoretic Entailment Relation). Let $\Phi \subseteq \text{Eq}(\Sigma)$ be an arbitrary collection of Σ -equations. We define a binary (consequence) relation $\Vdash_{\mathbf{Set}} \subseteq \mathcal{P}(\text{Eq}(\Sigma)) \times \text{Eq}(\Sigma)$ as follows:

$$\Phi \Vdash_{\mathbf{Set}} \phi \quad \iff \quad \phi \in \mathrm{Th}_{\Sigma}(\mathrm{Mod}_{\Sigma}(\Phi)).$$

Therefore, the meaning of $\Phi \Vdash_{\mathbf{Set}} \phi$ is that any Σ -algebra that satisfies Φ (i.e., all the equations in Φ) necessarily also satisfies the equation ϕ .

A fundamental result of Birkhoff establishes that $\Vdash_{\mathbf{Set}}$ coincides with the derivability relation $\Phi \vdash_{\mathbf{Set}} \phi$ (the binary relation $\vdash_{\mathbf{Set}}$ is defined by means of an inductive definition) of the deductive system of "equational logic". Thus, this celebrated result is a logical axiomatisation of the entailment relation $\Vdash_{\mathbf{Set}}$.

2.2 Category Theory

We assume basic knowledge on category theory. We recall here only some standard definitions and results. We refer to [Mac71] as a standard reference.

For a given signature Σ , we denote with $\mathbf{Alg}(\Sigma)$ the category having as objects Σ -algebras in $\mathrm{Alg}(\Sigma)$ and as arrows the homomorphisms of Σ -algebras.

For any $\Phi \subseteq \text{Eq}(\Sigma)$, the full subcategory of $\text{Alg}(\Sigma)$ whose objects are in $\text{Mod}_{\Sigma}(\Phi)$ is denoted by $\mathbf{Mod}_{\Sigma}(\Phi)$, with boldface notation. In other words, $\mathbf{Mod}_{\Sigma}(\Phi)$ is the category having as objects algebras \mathbb{A} such that $\mathbb{A} \models \phi$, for all $\phi \in \Phi$, and as morphisms all their homomorphisms of algebras. Note that, following the above definition, we simply have $\mathbf{Alg}(\Sigma) = \mathbf{Mod}_{\Sigma}(\emptyset)$.

There is a forgetful functor:

$$U_{\mathbf{Mod}_{\Sigma}(\Phi) \to \mathbf{Set}} : \mathbf{Mod}_{\Sigma}(\Phi) \to \mathbf{Set}$$

mapping an algebra in $\mathbf{Mod}_{\Sigma}(\Phi)$ to its carrier, and acting as identity on morphisms:

$$U_{\mathbf{Mod}_{\Sigma}(\Phi) \to \mathbf{Set}}(A, \{op^A\}_{op \in \Sigma}) = A$$

 $U_{\mathbf{Mod}_{\Sigma}(\Phi) \to \mathbf{Set}}(f) = f$

We often just write U when no confusion arises.

2.2.1 Monads and adjunctions

Definition 2.11 (Monad). Given a category \mathcal{C} , a monad on \mathcal{C} is a triple (M, η, μ) composed of a functor $M: \mathcal{C} \to \mathcal{C}$ together with two natural transformations: a unit $\eta: Id_{\mathcal{C}} \Rightarrow M$, where $Id_{\mathcal{C}}$ is the identity functor on \mathcal{C} , and a multiplication $\mu: M^2 \Rightarrow M$, satisfying $\mu \circ \eta_M = \mu \circ M \eta = id_M$ and $\mu \circ M \mu = \mu \circ \mu_M$.

Example 2.12. For any set $\Phi \subseteq \text{Eq}(\Sigma)$ of Σ -equations, we have an associated monad on **Set**, defined as follows:

• The functor M maps a set A to the set $\operatorname{Terms}_{\Sigma}(A)/\equiv$ of terms over A quotiented by the relation \equiv defined as follows:

$$s \equiv t \iff \Phi \Vdash_{\mathbf{Set}} \forall A.s = t$$

and maps a function $f:A\to B$ to the homomorphism

$$M(f): \operatorname{Terms}_{\Sigma}(A)/_{\equiv} \to \operatorname{Terms}_{\Sigma}(B)/_{\equiv}$$

defined by induction on terms t as follows:

$$M(f)([a]_{\equiv}) = [f(a)]_{\equiv}$$

$$M(f)([op(t_1,...,t_n)]_{\equiv}) = op^{F(A)}(M(f)([t_1]_{\equiv}),...,M(f)([t_n]_{\equiv}))$$
 where $op^{F(A)}([t_1]_{\equiv},...,[t_n]_{\equiv})$ is defined as $[op(t_1,...,t_n)]_{\equiv}$.

• For each set A, the unit $\eta_A: A \to \operatorname{Terms}_{\Sigma}(A)/_{\equiv}$ is defined as:

$$a \mapsto [a]_{=}$$
.

• For each set A, the multiplication

$$\mu_A : \operatorname{Terms}_{\Sigma}(\operatorname{Terms}_{\Sigma}(A)/_{\equiv})/_{\equiv} \to \operatorname{Terms}_{\Sigma}(A)/_{\equiv}$$

is defined by the following "flattening" operation:

$$[s([t_1]_{\equiv},\ldots,[t_n]_{\equiv})]_{\equiv} \mapsto [s\{t_1/[t_1]_{\equiv},\ldots,t_n/[t_n]_{\equiv}\}]_{\equiv}$$

where $s([t_1]_{\equiv}, \ldots, [t_n]_{\equiv})$ denotes that $[t_1]_{\equiv}, \ldots, [t_n]_{\equiv}$ are all and only the elements of $\operatorname{Terms}_{\Sigma}(A)/_{\equiv}$ appearing in the term s, and $s\{t_1/[t_1]_{\equiv}, \ldots, t_n/[t_n]_{\equiv}\}$ denotes the simultaneous substitution in s of each of these equivalence classes with one representative.

It can be shown that, indeed, the above definitions do not depend on specific choices of representatives of the \equiv -equivalence classes.

Monads can be defined as arising from adjunctions.

Definition 2.13 (Adjunction²). Let $U: \mathcal{D} \to \mathcal{C}, F: \mathcal{C} \to \mathcal{D}$ be functors. F is a left adjoint of U (notation: $F \dashv U$) if there is a natural transformation

²See, e.g., [Awo10, Chapter 9] for several equivalent definitions.

 $\eta: Id_{\mathcal{C}} \Rightarrow U \circ F$ such that for any \mathcal{C} -object X, any \mathcal{D} -object Y and \mathcal{C} -morphism $f: X \to U(Y)$ there is an unique \mathcal{D} -morphism $g: F(X) \to Y$ such that $f = U(g) \circ \eta_X$. Diagrammatically:

$$F(X) \xrightarrow{g} Y$$

$$U(F(X)) \xrightarrow{U(g)} U(Y)$$

$$\uparrow \qquad \qquad \downarrow \qquad \qquad$$

The natural transformation η is called the unit of the adjunction. Given an adjunction $F \dashv U$, we also have a natural transformation $\epsilon : F \circ U \Rightarrow Id_{\mathcal{D}}$, which is called the counit of the adjunction and which satisfies the following identities:³

$$U\epsilon \circ \eta_U = id_U \qquad \epsilon_F \circ F\eta = id_F.$$

Proposition 2.14. ⁴ Every adjunction $F: \mathcal{C} \to \mathcal{D} \dashv U: \mathcal{D} \to \mathcal{C}$ defines a monad (M, η, μ) where:

- M is the functor $U \circ F$
- the unit $\eta: Id_{\mathcal{C}} \Rightarrow M$ of the monad is the unit of the adjunction
- the multiplication $\mu: M^2 \Rightarrow M$ is given by

$$\mu_X = U(\epsilon_{F(X)})$$

where $\epsilon: F \circ U \to Id_{\mathcal{D}}$ is the counit of the adjunction.

The monad of quotiented terms from Example 2.12 arises from the adjuction between the forgetful functor $U : \mathbf{Mod}_{\Sigma}(\Phi) \to \mathbf{Set}$ and the functor mapping a set to the free object in $\mathbf{Mod}_{\Sigma}(\Phi)$, as we discuss in the following section.

2.2.2 Free objects

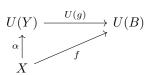
Definition 2.15 (Free object). Let $U: \mathcal{D} \to \mathcal{C}$ be a functor, $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and $\alpha: X \to U(Y)$. We say that Y is a U-free object generated by X with respect to α if the following **UMP** (Universal Mapping Property) holds: for every $B \in \mathcal{D}$ and every \mathcal{C} -morphism $f: X \to U(B)$, there exists a unique \mathcal{D} -morphism $g: Y \to B$ such that $f = U(g) \circ \alpha$, as indicated in the following

³See, e.g., [Awo10] Proposition 10.1.

⁴See, e.g., [Awo10] Chapter 10.2.

diagram.





We say that the category \mathcal{D} has U-free objects if for every $X \in \mathcal{C}$ there exist:

- 1. an object $D_X \in \mathcal{D}$, and
- 2. a function $\alpha_X: X \to U(D_X)$

such that D_X is a *U*-free object generated by *X* with respect to α_X .

If the functor U and the map α_X are clear from the context, we just refer to "the free object generated by X" instead of "U-free object generated by X with respect to α_X ".

Free objects, when they exist, are unique up to isomorphism.

Example 2.16. It is a standard result in universal algebra that the forgetful functor $U: \mathbf{Mod}_{\Sigma}(\Phi) \to \mathbf{Set}$ has U-free objects. For any set A, take the algebra $F(A) = (\mathrm{Terms}_{\Sigma}(A)/\equiv, \{op^{F(A)}\}_{op\in\Sigma})$, where $\mathrm{Terms}_{\Sigma}(A)/\equiv$ and $op^{F(A)}$ are defined as in Example 2.12. Then F(A) is the U-free object generated by A with respect to the function $\alpha: A \to \mathrm{Terms}_{\Sigma}(A)/\equiv$ defined as $\alpha(a) = [a]_{\equiv}$.

The following proposition states that if $U: \mathcal{D} \to \mathcal{C}$ is a functor such that \mathcal{D} has U-free objects, then there is a functor F, called the free functor, which assigns to elements of \mathcal{C} their corresponding U-free object, and which gives an adjunction $F \dashv U$.

Proposition 2.17. Let $U: \mathcal{D} \to \mathcal{C}$ be a functor such that free U-objects exist in \mathcal{D} , i.e., such that for every $X \in \mathcal{C}$ there exist an object $D_X \in \mathcal{D}$ and a function $\alpha_X: X \to U(D_X)$ such that D_X is the U-free object generated by X with respect to α_X . Then $U: \mathcal{D} \to \mathcal{C}$ has a left adjoint $F: \mathcal{C} \to \mathcal{D}$

$$F \dashv U$$

with F the functor mapping an object X to the U-free object D_X and mapping a morphism $f: X \to Y$ to the unique \mathcal{D} -morphism $F(f): F(X) \to F(Y)$ that makes the following diagram commute:

$$F(X) \xrightarrow{F(f)} F(Y)$$

A proof of this result can be found in Appendix A.

From this adjunction, we can then build the monad $M = (U \circ F)$ of U-free objects, as explained in Proposition 2.14.

Example 2.18. In Example 2.16 we have identified free U-objects, for U: $\mathbf{Mod}_{\Sigma}(\Phi) \to \mathbf{Set}$, as the algebras of quotiented terms. As explained in Proposition 2.17, we can then define a functor F such that we obtain an adjunction $F \dashv U$. By Proposition 2.14, we have a monad $M = (U \circ F)$, which can be see to be exactly the monad from Example 2.12.

2.2.3 Forgetful functors and monadicity

A monad M has an associated category of M-algebras.

Definition 2.19 (Eilenberg-Moore algebras for a monad). Let (M, η, μ) be a monad on \mathcal{C} . An algebra for M (or M-algebra) is a pair (A, α) where $A \in \mathcal{C}$ is an object and $\alpha: M(A) \to A$ is a morphism such that (1) $\alpha \circ \eta_A = id_A$ and (2) $\alpha \circ M\alpha = \alpha \circ \mu_A$ hold. Given two M-algebras (A, α) and (A', α') , an M-algebra morphism is an arrow $f: A \to A'$ in \mathcal{C} such that $f \circ \alpha = \alpha' \circ M(f)$. The category of M-algebras and their morphisms, denoted $\mathbf{EM}(M)$, is called the Eilenberg-Moore category for M.

Proposition 2.20 (Existence of the comparison functor). Let $F: \mathcal{C} \to \mathcal{D} \dashv U: \mathcal{D} \to \mathcal{C}$ be an adjunction, and let UF be the induced monad. Then there exists a functor

$$K: \mathcal{D} \to \mathbf{EM}(UF)$$

called the (canonical) comparison functor.⁵

There are interesting cases in which this comparison functor is an isomorphism. In this case, we say that the functor is strictly monadic.

Definition 2.21 (Monadic adjunction, monadic functor). Let $F: \mathcal{C} \to \mathcal{D} \dashv U: \mathcal{D} \to \mathcal{C}$ be an adjunction. We say that the adjunction is strictly *monadic* if the comparison functor is an isomorphism. Given a functor $U: \mathcal{D} \to \mathcal{C}$, we say that U strictly monadic if it has a left adjoint F such that the adjunction is strictly monadic.

In the rest of this paper, we often omit the adjective "strict", i.e., whenever we say that an adjunction or functor is monadic, we actually mean that the adjunction or functor is strictly monadic.

The following theorem, due to Beck, gives useful equivalent characterizations of monadicity. For a proof, see e.g. Theorem 1 of [Mac71, p. 151].

Proposition 2.22. [Beck's theorem]

Let $F: \mathcal{C} \to \mathcal{D} \dashv U: \mathcal{D} \to \mathcal{C}$ be an adjunction. The following are equivalent:

 $^{^5 \}rm See,~e.g.,~https://ncatlab.org/nlab/show/monadic+functor for the construction of the comparison functor.$

- 1. the comparison functor is an isomorphism.
- 2. $U: \mathcal{D} \to \mathcal{C}$ creates coequalizers for all \mathcal{D} -arrows f, g such that U(f), U(g) has an absolute coequalizer (in \mathcal{C}).
- 3. $U: \mathcal{D} \to \mathcal{C}$ creates coequalizers for all \mathcal{D} -arrows f, g such that U(f), U(g) has a split coequalizer (in \mathcal{C}).

where:

- an absolute coequalizer (in C) of C-arrows $f, g : A \to B$ is a C-arrow $e : B \to C$ such that for all functors F, F(e) is a coequalizer of F(f), F(g).
- a split coequalizer (in C) of C-arrows $f, g: A \to B$ is a C-arrow $e: B \to C$ such that $e \circ f = e \circ g$ and such that there exist arrows $s: C \to B$ and $t: B \to A$ such that $e \circ s = id_C$, $f \circ t = id_B$ and $g \circ t = s \circ e$.
- $U: \mathcal{D} \to \mathcal{C}$ creates coequalizers for the \mathcal{D} -arrows f, g if for all coequalizer $e: U(B) \to C$ of U(f), U(g) (in \mathcal{C}), there are unique D and $u: B \to D$ (in \mathcal{D}) such that U(D) = C, U(u) = e and u is a coequalizer of f, g.

The following result is well known.

Proposition 2.23. The functor $U : \mathbf{Mod}_{\Sigma}(\Phi) \to \mathbf{Set}$ is monadic.

A proof of this result, which indeed relies on the characterizations of monadicity given by Beck's theorem (Proposition 2.22), can be found in [Mac71, ch VI, §8, Thm 1].

As recalled in Example 2.18, the monad M of quotiented terms arises from the adjunction $F \vdash U$, where $U : \mathbf{Mod}_{\Sigma}(\Phi) \to \mathbf{Set}$ and F is the functor mapping sets to U-free objects. Hence, Proposition 2.23 allows us to conclude that the category $\mathbf{EM}(M)$ of Eilenberg-Moore algebras for M is isomorphic to the category $\mathbf{Mod}_{\Sigma}(\Phi)$ of models of Φ .

2.3 Metric Spaces and Generalised Metric spaces

Definition 2.24. A fuzzy relation on a set A is a map $d: A \times A \to [0,1]$. The pair (A, d_A) is called a fuzzy relation space (sometimes, we directly call (A, d_A) a fuzzy relation as well). A morphism between two fuzzy relation spaces (A, d_A) and (B, d_B) is a map $f: A \to B$ that is nonexpansive, namely,

$$\forall a, a' \in A, d_B(f(a), f(a')) < d_A(a, a').$$

We denote by **FRel** the category of fuzzy relation spaces and nonexpansive maps.

We will be interested in full subcategories of **FRel** obtained by restricting to fuzzy relations that satisfy certain constraints, such as the ones listed below (where $\epsilon, \delta \in [0, 1]$):

$$\forall a \in A, \quad d(a, a) \le 0 \tag{1}$$

$$\forall a, b \in A, \quad d(a, b) \le 0 \implies a = b \tag{2}$$

$$\forall a, b \in A, \quad d(a, b) \le \epsilon \Longrightarrow d(b, a) \le \epsilon$$
 (3)

$$\forall a, b, c \in A, \quad d(a, b) \le \epsilon \land d(b, c) \le \delta \Longrightarrow d(a, c) \le \epsilon + \delta \tag{4}$$

For example, the category **Met** of ordinary metric space and nonexpansive maps can be seen as the full subcategory of **FRel** satisfying (1,2,3,4). Indeed (1) expresses that the distance from a point to itself is zero and (2) states that points at distance zero must be equal, (3) expresses symmetry (d(a,b) = d(b,a)) of the fuzzy relation and (4) expresses the triangular inequality property.

We will abstractly denote with **GMet** any full subcategory of **FRel** obtained by restricting to objects satisfying a chosen set of implications expressible as Horn implications⁶ which are built from atomic statements of the form a = b or $d(a, b) \le \epsilon$.

We denote with $U_{\mathbf{GMet} \to \mathbf{Set}}$: $\mathbf{GMet} \to \mathbf{Set}$ the forgetful functor defined as expected, which we simply denote by U when no confusion arises.

3 Presentation of the Framework

In this section we present our framework. We will introduce it following the same patter of Section 2.1 "Universal Algebra", and we will have the following correspondence:

Σ -Algebra	Quantitative Σ -Algebra
Homomorphism of Σ -Algebras	Homomorphism of quantitative Σ -Algebras
Equations	Equations and Quantitative Equations
Equational Theory	Quantitative Equational Theory
Equationally defined class	Quantitative Equationally defined class
of Σ -algebras	of quantitative Σ -algebras

We begin with the central notion of this section, the concept of **GMet** quantitative algebras.

Definition 3.1 (Quantitative Algebra). Fix some **GMet** category and a signature Σ. A **GMet** quantitative Σ-algebra \mathbb{A} is a triple $\mathbb{A} = (A, d_A, \{op^A\}_{op \in \Sigma})$ where:

• (A, d_A) is a **GMet** space, i.e., A is a set and

$$d_A: A^2 \to [0,1]$$

is a fuzzy relation satisfying the conditions of GMet spaces.

⁶A Horn implication is a first-order expression of the form $\forall \vec{x}. ((F_1 \land \cdots \land F_n) \Rightarrow G)$, where F_1, \ldots, F_n, G are atomic statements whose variables are in \vec{x} .

• $(A, \{op^A\}_{op \in \Sigma})$ is a Σ -algebra, i.e.,

$$op^A: A^{ar(op)} \to A$$

is an interpretation of all the operation symbols in Σ .

We remark that, in contrast with the definition in [MPP16] (and with much subsequent literature [Bac+21; MV20; MPP17; MPP21]), under our definition the interpretations op^A of the operations in Σ are not required to be nonexpansive and can be arbitrary set-theoretical functions.

In what follows we assume that a given **GMet** category and signature Σ are chosen and we just write "quantitative algebra" in place of "**GMet** quantitative Σ -algebra".

Definition 3.2 (Homomorphisms). Given two quantitative algebras $\mathbb{A} = (A, d_A, \{op^A\}_{op \in \Sigma})$ and $\mathbb{B} = (B, d_B, \{op^B\}_{op \in \Sigma})$, a homomorphism (of quantitative algebras) is a function $f: A \to B$ such that:

• $f:(A,d_A)\to (B,d_B)$ is nonexpansive, i.e.,

$$d_B(f(a_1), f(a_2)) \le d_A(a_1, a_2)$$

for all $a_1, a_2 \in A$, and

• f is a homomorphism between the Σ -algebras $\mathbb{A} = (A, \{op^A\}_{op \in \Sigma})$ and $\mathbb{B} = (B, \{op^B\}_{op \in \Sigma})$, i.e.,

$$f(op^{A}(a_{1},...,a_{n})) = op^{B}(f(a_{1}),...,f(a_{n}))$$

for all $a_1, \ldots, a_n \in A$ and $op \in \Sigma$.

We denote with $\mathbf{QAlg}(\Sigma)$ the category of quantitative Σ -algebras and their homomorphisms.

We denote with $U_{\mathbf{QAlg}(\Sigma)\to\mathbf{GMet}}$ and $U_{\mathbf{QAlg}(\Sigma)\to\mathbf{Alg}(\Sigma)}$ the forgetful functors defined as expected, which make the following diagram (where, without ambiguity, we denote all functors involved just by U) commute:

$$egin{align*} \mathbf{QAlg}(\Sigma) & \stackrel{U}{\longrightarrow} \mathbf{GMet} \ & \downarrow U & \downarrow U \ & \mathbf{Alg}(\Sigma) & \stackrel{U}{\longrightarrow} \mathbf{Set} \ & \end{array}$$

Definition 3.3 (**GMet** equations and quantitative equations). Fix a given **GMet** category and a signature Σ . A **GMet** equation is a judgment of the form:

$$\forall (A, d_A).s = t$$

where (A, d_A) is a **GMet** space and $s, t \in \text{Terms}_{\Sigma}(A)$. Similarly, a **GMet** quantitative equation is a judgment of the form:

$$\forall (A, d_A).s =_{\epsilon} t$$

where (A, d_A) is a (A, d_A) is a **GMet**, $s, t \in \text{Terms}_{\Sigma}(A)$ and $\epsilon \in [0, 1]$.

We denote with $\operatorname{GEq}_{=}(\mathbf{GMet}, \Sigma)$ and $\operatorname{GEq}_{\epsilon}(\mathbf{GMet}, \Sigma)$ the set of \mathbf{GMet} equations and quantitative equations respectively. We denote with $\operatorname{GEq}(\mathbf{GMet}, \Sigma)$ their union: $\operatorname{GEq}(\mathbf{GMet}, \Sigma) = \operatorname{GEq}_{=}(\mathbf{GMet}, \Sigma) \cup \operatorname{GEq}_{\epsilon}(\mathbf{GMet}, \Sigma)$. When \mathbf{GMet} and Σ are clear from the context, we just write $\operatorname{GEq}_{=}$, $\operatorname{GEq}_{\epsilon}$ and $\operatorname{GEq}_{=}$.

Definition 3.4 (Interpretations). Given a **GMet** quantitative Σ-algebra $\mathbb{A} = (A, d_A, \{op^A\}_{op \in \Sigma})$ and a **GMet** space (B, d_B) , an interpretation of (B, d_B) into \mathbb{A} is a nonexpansive function $\tau : (A, d_A) \to (B, d_B)$. The interpretation τ extends uniquely to a (set-theoretic) function of type $\llbracket _ \rrbracket_{\tau}^A : \operatorname{Terms}_{\Sigma}(B) \to A$ specified as in Definition 2.6.

In accordance with the above definition, all interpretations of quantitative algebras are nonexpansive. While this prevents any confusion, we will sometimes stress the fact that the interpretations are nonexpansive as this is often crucial in some statements and proofs.

Definition 3.5 (Semantics of **GMet** equations and quantitative equations). Let $\mathbb{A} = (A, d_A, \{op^A\})$ be a **GMet** quantitative Σ -algebra. Let ϕ_1 and ϕ_2 be the following **GMet** equation and quantitative equation, respectively:

$$\phi_1 = \forall (B, d_B).s = t$$
 $\phi_2 = \forall (B, d_B).s =_{\epsilon} t.$

We say that \mathbb{A} satisfies ϕ_1 , written $\mathbb{A} \models \phi$, if for all (nonexpansive) interpretations $\tau: (B, d_B) \to (A, d_A)$ of (B, d_B) in \mathbb{A} , $[\![s]\!]_{\tau}^A = [\![t]\!]_{\tau}^A$ holds. Similarly, we say that \mathbb{A} satisfies ϕ_2 , written $\mathbb{A} \models \phi$, if or all (nonexpansive) interpretations $\tau: (B, d_B) \to (A, d_A)$ of (B, d_B) in \mathbb{A} , $d_B([\![s]\!]_{\tau}^A, [\![t]\!]_{\tau}^A) \leq \epsilon$ holds.

Definition 3.6 (Quantitative Equational Theory). Let $\mathcal{K} \subseteq \mathbf{QAlg}(\Sigma)$ be an arbitrary collection of **GMet** quantitative algebras of type Σ . The *quantitative* equational theory of \mathcal{K} is defined as the set of **GMet** equations and quantitative equations satisfied by all quantitative algebras in \mathcal{K} , formally:

$$QTh_{\Sigma}(\mathcal{K}) = \{ \phi \in GEq(\Sigma) \mid \forall \mathbb{A} \in \mathcal{K}. \ \mathbb{A} \models \phi \}.$$

Definition 3.7 (Equationally defined classes). Let $\Phi \subseteq \operatorname{GEq}(\Sigma)$ be an arbitrary collection of **GMet** equations and quantitative equations. The *models* of Φ are the quantitative algebras that satisfy all equations in Φ , formally:

$$\mathrm{QMod}_{\Sigma}(\Phi) = \{ \mathbb{A} \in \mathbf{QAlg}(\Sigma) \mid \forall \phi \in \Phi. \ \mathbb{A} \models \phi \}$$

If $\mathcal{K} \subseteq \mathbf{QAlg}(\Sigma)$ is such that $\mathcal{K} = \mathrm{QMod}_{\Sigma}(\Phi)$ for some $\Phi \subseteq \mathrm{GEq}(\Sigma)$, we say that \mathcal{K} is an equationally defined (by Φ) class.

Definition 3.8. For any $\Phi \subseteq \operatorname{GEq}(\Sigma)$, the full subcategory of $\operatorname{\mathbf{QAlg}}(\Sigma)$ whose objects are in $\operatorname{QMod}_{\Sigma}(\Phi)$ is denoted by $\operatorname{\mathbf{QMod}}_{\Sigma}(\Phi)$, with boldface notation. In other words, $\operatorname{\mathbf{QMod}}_{\Sigma}(\Phi)$ is the category having as objects quantitative algebras \mathbb{A} such that $\mathbb{A} \models \phi$, for all $\phi \in \Phi$, and as morphisms all their homomorphisms of quantitative algebras.

Note that, following the above definition, we simply have $\mathbf{QAlg}(\Sigma) = \mathbf{QMod}_{\Sigma}(\emptyset)$.

We denote with $U_{\mathbf{QMod}_{\Sigma}(\Phi)\to\mathbf{GMet}}$ the forgetful functor defined as the restriction of $U_{\mathbf{QAlg}(\Sigma)\to\mathbf{GMet}}$. As usual, this is most often just denoted by U when no confusion arises.

Definition 3.9 (Model Theoretic Entailment Relation). Let $\Phi \subseteq \operatorname{GEq}(\Sigma)$ be an arbitrary collection of **GMet** equations and quantitative equations. We define a binary (consequence) relation $\Vdash_{\mathbf{GMet}} \subseteq \mathcal{P}(\operatorname{GEq}) \times \operatorname{GEq}$ as follows:

$$\Phi \Vdash_{\mathbf{GMet}} \phi \quad \iff \quad \phi \in \mathrm{QTh}_{\Sigma}(\mathrm{QMod}_{\Sigma}(\Phi)).$$

Therefore, the meaning of $\Phi \Vdash_{\mathbf{GMet}} \phi$ is that any quantitative Σ -algebra that satisfies Φ (i.e., all the \mathbf{GMet} equations and quantitative equations in Φ) necessarily also satisfies ϕ .

4 Our Results

Our results can be summarised as follows.

I) The relation $\Vdash_{\mathbf{GMet}}$ can be axiomatised by means of a deductive system analogous to the deductive system of Birkhoff's equational logic. More formally, there is an inductively defined relation $\vdash_{\mathbf{GMet}} \subseteq \mathcal{P}(\mathrm{GEq}(\Sigma)) \times \mathrm{GEq}(\Sigma)$, specified as the smallest relation containing a given set of pairs and closed under a given set of deductive rules (see in Section 4.1), such that:

$$\Phi \vdash_{\mathbf{GMet}} \phi \Longleftrightarrow \Phi \Vdash_{\mathbf{GMet}} \phi.$$

The soundness of the deductive system (i.e., the implication $\Phi \vdash_{\mathbf{GMet}} \phi \Rightarrow \Phi \Vdash_{\mathbf{GMet}} \phi$) is proved in Section 4.1. The completeness (i.e., the implication $\Phi \Vdash_{\mathbf{GMet}} \phi \Rightarrow \Phi \vdash_{\mathbf{GMet}} \phi$) is proved in Section 4.3, as a consequence of our second result (II).

II) Let us denote by U the forgetful functor $U : \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{Met}$. For every $\Phi \subseteq \mathrm{GEq}(\Sigma)$, the subcategory $\mathbf{QMod}_{\Sigma}(\Phi) \subseteq \mathbf{QAlg}(\Sigma)$ has U-free objects, in the sense of Definition 2.15.

The *U*-free object $F(A, d_A)$ generated by $(A, d_A) \in \mathbf{GMet}$ can be identified (up-to isomorphism of quantitative algebras) as:

$$F(A, d_A) = (\operatorname{Terms}_{\Sigma}(A)/\equiv, \Delta^{F(A, d_A)}, \{op^{F(A, d_A)}\}_{op \in \Sigma})$$

where:

(a) the equivalence relation $\equiv \subseteq \operatorname{Terms}_{\Sigma}(A) \times \operatorname{Terms}_{\Sigma}(A)$ is defined as:

$$s \equiv t \iff \Phi \vdash_{\mathbf{GMet}} \forall (A, d_A).s = t$$

(b) the fuzzy relation $\Delta^{F(A,d_A)}: (\mathrm{Terms}_{\Sigma}(A)/\equiv)^2 \to [0,1]$ is defined as:

$$\Delta^{F(A,d_A)}([s]_{\equiv},[t]_{\equiv}) \leq \epsilon \iff \Phi \vdash_{\mathbf{Met}} \forall (A,d_A).s =_{\epsilon} t$$

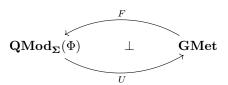
(c) The interpretation $op^{F(A,d_A)}$: $(\operatorname{Terms}_{\Sigma}(A)/\equiv)^n \to (\operatorname{Terms}_{\Sigma}(A)/\equiv)$ of any n-ary operation $op \in \Sigma$, is defined as:

$$op^{F(A,d_A)}([s_1]_{\equiv},\ldots,[s_n]_{\equiv}) = [op(s_1,\ldots,s_n)]_{\equiv}$$

It can be shown that the definitions of $\Delta^{F(A,d_A)}$ and $op^{F(A,d_A)}$ are well specified regardless of the choice of representatives s,t for the classes $[s]_{\equiv},[t]_{\equiv}$, and that indeed the quantitative algebra $F(A,d_A) \in \mathrm{QMod}_{\Sigma}(\Phi)$.

These results are formally stated and proved in Section 4.2, and they give us the analogous of the result mentioned in Example 2.16 for (non-quantitative) universal algebra.

III) As a corollary of the two results above and of Proposition 2.17, there is a functor $F: \mathbf{GMet} \to \mathbf{QMod}_{\Sigma}(\Phi)$ which associates to each \mathbf{GMet} space (A, d_A) the corresponding free object $F(A, d_A)$. The functor F is a left adjoint of U:



where we wrote U for $U_{\mathbf{QMod}_{\Sigma}(\Phi)\to\mathbf{GMet}}$ to improve readability.

This adjunction gives us a monad M on \mathbf{GMet} , which is defined analogously to the monad of quotiented terms discussed in Examples 2.12 and 2.18.

We concretely identify this adjunction and monad in Section 4.4.

IV) The functor $U : \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{GMet}$ is (strictly) monadic, i.e., there is an isomorphism of categories:

$$EM(M) \cong \mathbf{QMod}_{\Sigma}(\Phi)$$

where EM(M) is the category of Eilenberg–Moore algebras for the monad M.

This result is formally stated and proved in Section 4.5.

4.1 Presentation of the deductive system and proof of soundness

In this section, given a choice of **GMet** category (for example, **Met**), we introduce a deductive system which can be used to derive judgments of the form: $\Phi \vdash_{\mathbf{GMet}} \phi$, for $\Phi \in \mathcal{P}(\mathrm{GEq}(\Sigma))$ and $\phi \in \mathrm{GEq}(\Sigma)$. Thus, formally, we define by induction a relation $\vdash_{\mathbf{GMet}} \subseteq \mathcal{P}(\mathrm{GEq}(\Sigma)) \times \mathrm{GEq}(\Sigma)$.

Some of the axioms schemes (i.e., the base cases of the inductive definition) defining $\vdash_{\mathbf{GMet}}$ are derived in a systematic way by the set of Horn clauses defining the Category \mathbf{GMet} , which we recall are of the form:

$$\forall \vec{x}. (F_1 \wedge \cdots \wedge F_n) \Rightarrow G$$
 with F_1, \dots, F_n, G of the form: $x = y$ or $d(x, y) \leq \epsilon$.

We give concrete instances of the axiom schemes for the important case **GMet** = **Met** in Example 4.2.

Definition 4.1. The relation $\vdash_{\mathbf{GMet}} \subseteq \mathcal{P}(GEq(\Sigma)) \times GEq(\Sigma)$ is defined as the smallest relation such that:

1. Closure under the INIT rule: given any $\Phi \subseteq \mathcal{P}(GEq(\Sigma))$ and $\phi \in GEq(\Sigma)$, if $\phi \in \Phi$ then $\Phi \vdash_{\mathbf{GMet}} \phi$ holds. That is:

$$\frac{}{\Phi \vdash_{\mathbf{GMet}} \phi} \text{INIT (proviso: } \phi \in \Phi)$$

2. Closure under the CUT rule: given any $\Phi, \Phi' \subseteq \mathcal{P}(GEq(\Sigma))$ and $\psi \in GEq(\Sigma)$, if for all $\phi \in \Phi'$, $\Phi \vdash_{\mathbf{GMet}} \phi$ holds and if $\Phi \cup \Phi' \vdash_{\mathbf{GMet}} \psi$ holds, then $\Phi \vdash_{\mathbf{GMet}} \psi$ holds. That is:

$$\frac{\{\Phi \vdash_{\mathbf{GMet}} \phi\}_{\phi \in \Phi'} \qquad \Phi, \Phi' \vdash_{\mathbf{GMet}} \psi}{\Phi \vdash_{\mathbf{GMet}} \psi} \text{ CUT}$$

3. Closure under the WEAKENING rule: given any $\Phi, \Phi' \subseteq \mathcal{P}(GEq(\Sigma))$ and $\phi \in GEq(\Sigma)$, if $\Phi \vdash_{\mathbf{GMet}} \phi$ holds, then $\Phi \cup \Phi' \vdash_{\mathbf{GMet}} \phi$ holds. That is:

$$\frac{\Phi \vdash_{\mathbf{GMet}} \phi}{\Phi \cup \Phi' \vdash_{\mathbf{GMet}} \phi} \text{WEAKENING}$$

4. The relation $\vdash_{\mathbf{GMet}}$ contains all the basic pairs $\Phi \vdash_{\mathbf{GMet}} \phi$ listed below (a–k). These pairs are "axiom schemes", meaning that pairs are obtained from axiom schemes by instantiating the involved metric spaces (A, d_A) , terms $s, t \in \mathrm{Terms}_{\Sigma}(A)$, substitutions σ , etc., to concrete ones. In order to improve readability, we just write \vdash in place of $\vdash_{\mathbf{GMet}}$.

(a)
$$(REFL \text{ of } =)$$
:

$$\emptyset \vdash \forall (A, d_A).s = s$$

(b) (SYMM of =):

$$\forall (A, d_A).s = t \vdash \forall (A, d_A).t = s$$

(c) (TRANS of =):

$$\forall (A, d_A).s = t, \forall (A, d_A).t = u \vdash \forall (A, d_A).s = u$$

(d) (CONG of =): given $op \in \Sigma$ of arity n,

$$\forall (A, d_A).s_1 = t_1, \dots, \forall (A, d_A).s_n = t_n \vdash \forall (A, d_A).op(s_1, \dots, s_n) = op(t_1, \dots, t_n)$$

(e) (SUBSTITUTION of = and $=_{\epsilon}$):

Given

- (A, d_A) and (B, d_B) , metric spaces:
- $\sigma: A \to Terms_{\Sigma}(B)$, a substitution,

we have the following two similar axiom schemes: one allowing substitution on conclusions that are equations (=) and the other on conclusions that are quantitative equations (=_{ϵ}):

$$\Psi, \forall (A, d_A).s = t \vdash \forall (B, d_B).\sigma(s) = \sigma(t)$$

and

$$\Psi, \forall (A, d_A).s =_{\epsilon} t \vdash \forall (B, d_B).\sigma(s) =_{\epsilon} \sigma(t)$$

where in both axiom schemes, the set Ψ is defined as:

$$\Psi = \{ \forall (B, d_B) . \sigma(a_i) =_{\epsilon_{i,j}} \sigma(a_j) \mid a_i, a_j \in A, \ \epsilon_{i,j} := d_A(a_i, a_j) \}$$

(f) (USE VARIABLES): For a metric space (A,d_A) and $a,a'\in A$ and $\epsilon=d_A(a,a')$:

$$\emptyset \vdash \forall (A, d_A).a =_{\epsilon} a'$$

(g) (Max/Up-closure): for all $\epsilon \leq \delta$:

$$\forall (A, d_A).s =_{\epsilon} t \vdash \forall (A, d_A).s =_{\delta} t$$

(h) (1-Max):

$$\emptyset \vdash \forall (A, d_A).s =_1 t$$

(i) (Order Completeness): For an index set I,

$$\{\forall (A, d_A).s =_{\epsilon_i} t\}_{i \in I} \vdash \forall (A, d_A).s =_{\inf\{\epsilon_i\}_{i \in I}} t$$

(j) (Left and right) congruence of = with respect to $=_{\epsilon}$:

$$\forall (A, d_A).s = t, \forall (A, d_A).t =_{\epsilon} u \vdash \forall (A, d_A).s =_{\epsilon} u$$

and

$$\forall (A, d_A).s = t, \forall (A, d_A).u =_{\epsilon} s \vdash \forall (A, d_A).u =_{\epsilon} t$$

(k) For each Horn clause defining the given category **GMet**, of the form:

$$\forall X.(F_1 \land \cdots \land F_n) \Rightarrow G$$
 with F_1, \ldots, F_n, G of the form: $x = x'$ or $d(x, x') \leq \epsilon$.

and for any **GMet** space (A, d_A) and instantiation of the variables X appearing in the Horn implications to Σ -terms:

$$\sigma: X \to \mathrm{Terms}_{\Sigma}(A)$$

there is a corresponding axiom scheme:

$$\phi_{F_1},\ldots,\phi_{F_n}\vdash\phi_G$$

where the equation or quantitative equation ϕ_F , for $F \in \{F_1, \dots, F_n, G\}$, is defined as follows:

Atomic formula F	Equation/Quantitative Equation ϕ_F
x = y	$\forall (A, d_A). \sigma(x) = \sigma(y)$
$d(x,y) \le \epsilon$	$\forall (A, d_A).\sigma(x) =_{\epsilon} \sigma(y)$

We refer to Example 4.2 for the complete list of axiom schemes corresponding to the Horn implications defining the category **Met**.

Example 4.2. The relation $\vdash_{\mathbf{GMet}}$ specified in Definition 4.1 includes some Axiom schemes (see 4k) associated with the Horn implications defining the category **GMet**. Here we explicitly present these axioms for the category **Met** which is defined by the following Horn implications:

• Distance zero implies equality:

$$\forall x, y. \ d(x, y) \leq 0 \Rightarrow x = y$$
 Horn implication

which gives:

$$\forall (A, d_A).s =_0 t \vdash_{\mathbf{Met}} \forall (A, d_A).s = t$$
 Axiom Scheme

• Equality implies distance zero:

$$\forall x, y. \ x = y \Rightarrow d(x, y) \leq 0$$
 Horn implication

which gives:

$$\forall (A, d_A).s = t \vdash_{\mathbf{Met}} \forall (A, d_A).s =_0 t$$
 Axiom Scheme

• Symmetry: for $\epsilon \in [0, 1]$,

$$\forall x, y. \ d(x, y) \le \epsilon \Rightarrow d(y, x) \le \epsilon$$
 Horn implication

which gives:

$$\forall (A, d_A).s =_{\epsilon} t \vdash_{\mathbf{Met}} \forall (A, d_A).t =_{\epsilon} s$$
 Axiom Scheme

• Triangular Inequality: for $\epsilon_1, \epsilon_2, \delta \in [0, 1]$ with $\delta = \min\{1, \epsilon_1 + \epsilon_2\}$,

$$\forall x, y, z. \ d(x, y) \leq \epsilon_1 \land d(y, z) \leq \epsilon_2 \Rightarrow d(x, z) \leq \delta$$

which gives:

$$\forall (A, d_A).s =_{\epsilon_1} t, \ \forall (A, d_A).t =_{\epsilon_2} u \vdash_{\mathbf{Met}} \forall (A, d_A).s =_{\delta} u$$
 Axiom Scheme.

Remark 4.3. The two axiom scheme (Left/Right congruences, see 4j) express that equality (=) is a left/right congruence with respect to the relation (= $_{\epsilon}$). We note that, in the specific setting of **Met** and $\vdash_{\mathbf{Met}}$, in presence of the Symmetry axiom (expressing the implication $x =_{\epsilon} y \Rightarrow y =_{\epsilon} x$), each of the two congruences implies the other. However for arbitrary **GMet** categories not validating the Symmetry axiom, the two congruences axioms are not necessarily mutually derivable from each other.

The first basic result regarding the deductive system is the soundness theorem.

Theorem 4.4 (Soundness). For every chosen category **GMet**, the inclusion $\vdash_{\mathbf{GMet}} \subseteq \Vdash_{\mathbf{GMet}} holds$.

Proof. Assume $\Phi \vdash_{\mathbf{GMet}} \phi$. We prove that $\Phi \Vdash_{\mathbf{GMet}} \phi$ holds by induction of the derivation tree used to derive $\Phi \vdash_{\mathbf{GMet}} \phi$.

Most cases are straightforward, including the occurrences of INIT, CUT and WEAKNENING rules and most axiom schemes.

The only non-obvious case is the "Substitution" axiom scheme (see 4e).

In this proof, for illustrative purposes, we consider one easy case, the axiom scheme associated to the Symmetry Horn implication of the category **Met** (see 4k). Then we discuss the more involved case of the "Substitution" axiom scheme (4e).

Case (4k) (Symmetry of $=_{\epsilon}$).

Suppose the derivation ends with an instance of the (4k) axiom of the form:

$$\forall (A, d_A).s =_{\epsilon} t \vdash_{\mathbf{Met}} \forall (A, d_A).t =_{\epsilon} s$$

which corresponds (see Example 4.2) to the Horn implication:

$$\forall x, y. \ d(x, y) \le \epsilon \Rightarrow d(y, x) \le \epsilon$$

of the category Met of metric spaces.

Then the pair $\Phi \vdash_{\mathbf{GMet}} \phi$ is:

$$\Phi = \{ \forall (A, d_A).s =_{\epsilon} t \}$$
 and $\phi = \forall (A, d_A).t =_{\epsilon} s$

We need to show that

$$\forall (A, d_A).s =_{\epsilon} t \Vdash_{\mathbf{GMet}} \forall (A, d_A).t =_{\epsilon} s$$

By definition of $\Vdash_{\mathbf{Met}}$ this coincides with the following statement:

Statement:
$$\forall (A, d_A).t =_{\epsilon} s \in \mathrm{QTh}_{\Sigma}(\mathrm{QMod}_{\Sigma}(\{ \forall (A, d_A).s =_{\epsilon} t \})).$$

This statement can be verified by simple unfolding of the definitions. Indeed, take any quantitative algebra $\mathbb{B} \in \mathrm{QMod}_{\Sigma}(\{ \ \forall (A,d_A).s =_{\epsilon} t \ \})$. We need to show that $\mathbb{B} \models \forall (A,d_A).t =_{\epsilon} s$.

Let $\tau: (A, d_A) \to (B, d_B)$ be an arbitrary (nonexpansive) interpretation of the **GMet** space (A, d_A) in \mathbb{B} . Since, by hypothesis,

$$\mathbb{B} \in \mathrm{QMod}_{\Sigma}(\{ \ \forall (A, d_A).s =_{\epsilon} t \ \})$$

we know that

$$d_B(\llbracket s \rrbracket_{\tau}^{\mathbb{B}}, \llbracket t \rrbracket_{\tau}^{\mathbb{B}}) \le \epsilon$$

And since (B, d_B) is a **Met** space satisfying the Horn implication

$$\forall x, y. \ d(x, y) \le \epsilon \Rightarrow d(y, x) \le \epsilon$$

we derive that

$$d_B(\llbracket t \rrbracket_{\tau}^{\mathbb{B}}, \llbracket s \rrbracket_{\tau}^{\mathbb{B}}) \le \epsilon$$

Since the interpretation τ is arbitrary, this implies that

$$\mathbb{B} \models \forall (A, d_A).t =_{\epsilon} s$$

as desired.

Case "Substitution" (see 4e).

Let $\sigma: A \to Terms_{\Sigma}(B)$ be an arbitrary substitution. We need to show that an arbitrary quantitative algebra $\mathbb{C} = (C, d_C, \{op^C\})$ satisfying all the left-side premises of the Substitution axiom, i.e.,

1.

$$\mathbb{C} \models \forall (A, d_A).s =_{\epsilon} t$$

2.

$$\mathbb{C} \models \{ \forall (B, d_B).\sigma(a_i) =_{\epsilon_{i,j}} \sigma(a_j) \mid a_i, a_j \in A, \ \epsilon_{i,j} := d_A(a_i, a_j) \}$$

necessarily also satisfies the right-side equation or quantitative equation (we just consider the quantitative equation case as the two are similar):

$$\mathbb{C} \models \forall (B, d_B).\sigma(s) =_{\epsilon} \sigma(t)$$

Towards this end, take an arbitrary nonexpansive interpretation:

$$\tau: (B, d_B) \to (C, d_C)$$

From τ , we define a new interpretation $\hat{\sigma}$ defined as:

$$\hat{\sigma}: (A, d_A) \to (C, d_C) \qquad \hat{\sigma}(a) := \llbracket \sigma(a) \rrbracket_{\tau}^{\mathbb{C}}$$

Before proceeding further, we need to show that $\hat{\sigma}$ is nonexpansive. So, take any $a, a' \in A$ and assume $d_A(a, a') = \delta$. Since (from the second hypothesis):

$$\mathbb{C} \models \forall (B, d_B). \sigma(a) =_{\delta} \sigma(a')$$

it holds (taking the interpretation τ) that:

$$d_C(\llbracket \sigma(a) \rrbracket_{\tau}^{\mathbb{C}}, \llbracket \sigma(a') \rrbracket_{\tau}^{\mathbb{C}}) \leq \delta$$

But this, by definition of $\hat{\sigma}$, means that

$$d_C(\hat{\sigma}(a), \hat{\sigma}(a')) < \delta$$

which concludes the proof that $\hat{\sigma}$ is nonexpansive.

From the first hypothesis (i.e., $\mathbb{C} \models \forall (A, d_A).s =_{\epsilon} t$) and taking as interpretation $\hat{\sigma}$, we know that

$$d_C(\llbracket s \rrbracket_{\hat{\sigma}}^{\mathbb{C}}, \llbracket t \rrbracket_{\hat{\sigma}}^{\mathbb{C}}) \leq \epsilon.$$

It is now sufficient to observe that:

$$\llbracket \sigma(s) \rrbracket_{\tau}^{\mathbb{C}} = \llbracket s \rrbracket_{\hat{\sigma}}^{\mathbb{C}} \qquad \llbracket \sigma(t) \rrbracket_{\tau}^{\mathbb{C}} = \llbracket t \rrbracket_{\hat{\sigma}}^{\mathbb{C}}$$

from which we derive that

$$d_C(\llbracket \sigma(s) \rrbracket_{\tau}^{\mathbb{C}}, \llbracket \sigma(t) \rrbracket_{\tau}^{\mathbb{C}}) \le \epsilon$$

Since τ was chosen as an arbitrary nonexpansive interpretation, we can derive the desired:

$$\mathbb{C} \models \forall (A, d_A).\sigma(s) =_{\epsilon} \sigma(t).$$

The above proof of soundness is rather direct and simple. Proving the opposite direction, the completeness theorem (Theorem 4.22 below) requires instead more work. Indeed we will use the proof system $\Vdash_{\mathbf{GMet}}$ to construct free objects in $\mathbf{QMod}_{\Sigma}(\Phi)$, prove some results about such free objects and finally derive the completeness results.

Our results regarding free objects in $\mathbf{QMod}_{\Sigma}(\Phi)$ are presented in Subsection 4.2. The completeness Theorem 4.22 is also established in that section, as a corollary.

4.2 The free quantitative algebra

In the following, we fix a signature Σ , a **GMet** category and a quantitative theory $\Phi \subseteq \text{GEq}(\Sigma)$. Let us denote by U the forgetful functor

$$U:\mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{GMet}$$

which acts on objects as:

$$(A, d_A, \{op^A\}_{op \in \Sigma}) \mapsto (A, d_A)$$

and as identity on morphisms.

We are going to prove the following statement using the deductive system $\vdash_{\mathbf{GMet}}$ as main tool. Recall that free objects are defined as in Definition 2.15.

Theorem 4.5. The category $\mathbf{QMod}_{\Sigma}(\Phi)$ has *U*-free objects.

The proof of this statement occupies the rest of this section.

For a given **GMet** space (A, d_A) , which is fixed in what follows, we are going to explicitly construct the U-free object in $\mathbf{QMod}_{\Sigma}(\Phi)$, denoted by $F(A, d_A)$, with respect to a nonexpansive map

$$\alpha: (A, d_A) \to U(F(A, d_A))$$

defined later on in Definition 4.18.

The main tool of the construction of $F(A, d_A)$ is the deductive system $\vdash_{\mathbf{GMet}}$.

We are going to proceed as follows:

1. (Subsection 4.2.1) Formally define the quantitative algebra $F(A, d_A)$.

- 2. (Subsection 4.2.2) Prove that, indeed, $F(A, d_A)$ belongs to $\mathbf{QMod}_{\Sigma}(\Phi)$. In other words, we show that $F(A, d_A)$ satisfies all the \mathbf{GMet} (quantitative) equations in Φ .
- 3. (Subsection 4.2.3) Finally, define the map

$$\alpha: (A, d_A) \to U(F(A, d_A))$$

and show that $F(A, d_A)$ satisfies the universal property defining the (unique, up to isomorphism) free algebra generated by (A, d_A) .

4.2.1 Definition of $F(A, d_A)$

In the following, a signature Σ , a **GMet** category and a quantitative theory $\Phi \subseteq \text{GEq}(\Sigma)$ are fixed, together with an arbitrary **GMet** space (A, d_A) .

To improve readability we just write \vdash in place of $\vdash_{\mathbf{GMet}}$.

We start by defining a binary relation (\equiv):

$$\equiv \subset Terms_{\Sigma}(A) \times Terms_{\Sigma}(A)$$

and a fuzzy relation (d):

$$d: (Terms_{\Sigma}(A) \times Terms_{\Sigma}(A)) \rightarrow [0,1]$$

on the set of terms $Terms_{\Sigma}(A)$ built from A, as follows.

Definition 4.6. We define \equiv as follows, for all $s, t \in Terms_{\Sigma}(A)$:

$$s \equiv t \Leftrightarrow \Phi \vdash \forall (A, d_A).s = t$$

We define d as follows, for all $s, t \in Terms_{\Sigma}(A)$:

$$d(s,t) = \inf_{\epsilon} \big\{ \Phi \vdash \forall (A,d_A).s =_{\epsilon} t \big\}.$$

Lemma 4.7. The relation \equiv is an equivalence relation.

Proof. This is due to the presence in the system of axioms: (**REFL of** =) (see 4a), (**SYMM of** =) (see 4b) and (**TRANS of** =) (see 4c)

Lemma 4.8. The relation \equiv is a congruence relation:

$$s_1 \equiv t_1, \dots, s_n \equiv t_n \Rightarrow op(s_1, \dots, s_n) \equiv op(t_1, \dots, t_n)$$

for all $op \in \Sigma$.

Proof. This is due to the presence in the system of axiom (**CONG of** =) (see 4d). \Box

Lemma 4.9. The function d is a fuzzy relation on $\operatorname{Terms}_{\Sigma}(A)$. Furthermore it satisfies:

$$d(s,t) = \epsilon \quad \Rightarrow \quad \Phi \vdash \forall (A,d_A).s =_{\epsilon} t.$$

Proof. The fact that d is a fuzzy relation (i.e., a function of type $(Terms_{\Sigma}(A))^2 \to [0,1]$) follows from the axioms $(\mathbf{1}\text{-}\mathbf{MAX})$ 4h $(d(x,y) \le 1)$ and the fact that all ϵ 's are positive (hence $d(x,y) \ge 0$).

The property:

$$d(s,t) = \epsilon \quad \Rightarrow \quad \Phi \vdash \forall (A,d_A).s =_{\epsilon} t$$

follows from the presence of the axiom (**ORDER COMPLETENESS**) (see 4i). \Box

Lemma 4.10. The fuzzy relation d satisfies all instances of any of the implication axioms defining the category **GMet**, when equality is replaced by the equivalence relation \equiv . Formally, given any implication axiom defining the category **GMet**,

$$x_1 = y_1, \dots, x_n = y_n, d(v_1, w_1) \le \epsilon_1, \dots, d(v_m, w_m) \le \epsilon_m \implies x = y$$

or

$$x_1 = y_1, \dots, x_n = y_n, d(v_1, w_1) \le \epsilon_1, \dots, d(v_m, w_m) \le \epsilon_m \Rightarrow d(x, y) \le \epsilon$$

and given any mapping $\sigma: X \to \operatorname{Terms}_{\Sigma}(A)$ from the variables X used in the implication to terms in $\operatorname{Terms}_{\Sigma}(A)$, the fuzzy relation d satisfies the implication:

$$\sigma(x_1) \equiv \sigma(y_1), \dots, \sigma(x_n) \equiv \sigma(y_n), d(\sigma(v_1), \sigma(w_1)) \leq \epsilon_1, \dots, d(\sigma(v_m), \sigma(w_m)) \leq \epsilon_m \implies \sigma(x) \equiv \sigma(y)$$

or

$$\sigma(x_1) \equiv \sigma(y_1), \dots, \sigma(x_n) \equiv \sigma(y_n), d(\sigma(v_1), \sigma(w_1)) \leq \epsilon_1, \dots, d(\sigma(v_m), \sigma(w_m)) \leq \epsilon_m \implies d(\sigma(x), \sigma(y)) \leq \epsilon_m$$
respectively.

Proof. By Lemma 4.9 we have

$$d(s,t) = \epsilon \quad \Rightarrow \quad \Phi \vdash \forall (A,d_A).s =_{\epsilon} t$$

By definition of \equiv , it follows from

$$\sigma(x_1) \equiv \sigma(y_1), \dots, \sigma(x_n) \equiv \sigma(y_n)$$

that

$$\Phi \vdash \forall (A, d_A) . \sigma(x_1) = \sigma(y_1), \dots, \Phi \vdash \forall (A, d_A) . \sigma(x_n) = \sigma(y_n)$$

By Lemma 4.9 it follows from

$$d(\sigma(v_1), \sigma(w_1)) \le \epsilon_1, \dots, d(\sigma(v_m), \sigma(w_m)) \le \epsilon_m$$

that

$$\Phi \vdash \forall (A, d_A).\sigma(v_1) =_{\epsilon_1} \sigma(w_1), \dots, \Phi \vdash \forall (A, d_A).\sigma(v_m) =_{\epsilon_m} \sigma(w_m)$$

Hence, by applying the (**CUT**) rule (see 2) using as hypothesis all the statements above together with the appropriate instantiation of the axiom of **GMet** (see 4k), we conclude

$$\Phi \vdash \forall (A, d_A). \sigma(x) = \sigma(y)$$

or

$$\Phi \vdash \forall (A, d_A).\sigma(x) =_{\epsilon} \sigma(y)$$

From these we obtain, by definition of \equiv and of d respectively, that

$$\sigma(x) \equiv \sigma(y)$$

or

$$d(\sigma(x), \sigma(y)) \le \epsilon$$

The following technical lemma relates the proof system (\vdash) with the definition of the fuzzy relation d.

Lemma 4.11. $\Phi \vdash \forall (A, d_A).s =_{\epsilon} t \iff d(s, t) \leq \epsilon.$

Proof. The (\Rightarrow) direction follows immediately from the definition of d as an infimum.

For the (\Leftarrow) direction, assume $d(s,t) \leq \epsilon$. Let $d(s,t) = \delta$ with $\delta \leq \epsilon$.

As we already established,

$$d(s,t) = \delta \quad \Rightarrow \quad \Phi \vdash \forall (A, d_A).s =_{\delta} t$$

holds and therefore we deduce that:

$$\Phi \vdash \forall (A, d_A).s =_{\delta} t$$

holds. From this, using the axiom (MAX/Upclosure) (see 4g), we can derive

$$\Phi \vdash \forall (A, d_A).s =_{\epsilon} t$$

as desired. \Box

The following technical lemma shows that the fuzzy relation d is compatible with the equivalence relation \equiv .

Lemma 4.12. The equivalence relation \equiv is a left and right congruence with respect to the fuzzy relation d, in the following sense: for all $s, t, u \in \text{Terms}_{\Sigma}(A)$:

$$s \equiv t$$
 and $d(t, u) \le \epsilon \implies d(s, u) \le \epsilon$

and

$$s \equiv t$$
 and $d(u,s) < \epsilon \implies d(u,t) < \epsilon$

.

Proof. We just consider the first implication (left-congruence) as the other case is similar. Assume $s \equiv t$ and $d(t,u) \leq \epsilon$. By definition of \equiv and by Lemma 4.11 this means that:

$$\Phi \vdash \forall (A, d_A).s = t \qquad \Phi \vdash \forall (A, d_A).t =_{\epsilon} u$$

Using the presence in the deductive system of the axiom ((Left-)congruence of = with respect to $=_{\epsilon}$) (see 4j) we obtain that:

$$\Phi \vdash \forall (A, d_A).s =_{\epsilon} u$$

and from this we derive, by definition of d, that

$$d(s, u) \le \epsilon$$
.

Summarising, we have established that \equiv is an equivalence relation. Hence, the quotient $Terms_{\Sigma}(A)/\equiv$, consisting of \equiv -equivalence classes, is well defined. Furthermore, the equivalence \equiv is a left and right congruence for the fuzzy-relation d. Finally the fuzzy-relation d satisfies all axioms defining the category **GMet** when equality is interpreted as \equiv . This implies the following Corollary.

Corollary 4.13. The fuzzy relation $\Delta: (\operatorname{Terms}_{\Sigma}(A) / \equiv \times \operatorname{Terms}_{\Sigma}(A) / \equiv) \rightarrow [0,1]$ defined as:

$$\Delta([s]_{\equiv}, [t]_{\equiv}) = d(s, t)$$

is well defined (i.e., regardless of the choice of representatives). Furthermore, $(TermsA/\equiv, \Delta)$ is a **GMet** space.

Moreover, since we have already established in Lemma 4.8 that \equiv is a congruence on $\mathrm{Terms}_{\Sigma}(A)$, the interpretation op $op^{F(A,d_A)}$ of each operation $op \in \Sigma$ specified as:

$$op^{F(A,d_A)}([s_1]_{\equiv},\ldots,[s_n]_{\equiv})=[op(s_1,\ldots,s_n)]_{\equiv}$$

is well defined and does not depend on a specific choice of representatives for the equivalence classes.

We can collect the results of this subsection as follows:

Corollary 4.14. The structure $(\operatorname{Terms}_{\Sigma}(A)/\equiv, \Delta, \{op^{F(A,d_A)}\})$ is a quantitative algebra of type Σ .

The quantitative algebra identified above is our definition of $F(A, d_A)$.

Definition 4.15. The quantitative algebra $F(A, d_A)$ is defined as:

$$F(A, d_A) = (\text{Terms}_{\Sigma}(A)/\equiv, \Delta, \{op^{F(A, d_A)}\}_{op \in \Sigma}).$$

Sometimes, to improve notation, we will explicitly identify some relevant parameters involved in the construction of the quantitative algebra (such as Φ , (A, d_A) , Σ and the choice of **GMet** space). For example we will write:

$$F(B, d_B) = (\operatorname{Terms}_{\Sigma}(B)/\equiv, \Delta^{F(B, d_B)}, \{op^{F(B, d_B)}\}_{op \in \Sigma})$$

to highlight that the fuzzy relation $\Delta^{F(B,d_B)}$ is the one associated with (B,d_B) . Note that also the relation \equiv is parametric with respect to the generating metric space.

4.2.2 Proof that $F(A, d_A) \in \mathbf{QMod}_{\Sigma}(\Phi)$

We show in Lemma 4.17 that the quantitative algebra $F(A, d_A)$ constructed from (A, d_A) as in the previous section (for a fixed **GMet** category, signature Σ and quantitative theory Φ) satisfies all equations and quantitative equations in Φ .

The proof exploits the following general characterization of the interpretation of terms in $F(A, d_A)$.

Lemma 4.16. Let $s \in \operatorname{Terms}_{\Sigma}(X)$ and let $\tau : X \to \operatorname{Terms}_{\Sigma}(A)/\equiv$. Let $\sigma_{\tau} : X \to \operatorname{Terms}_{\Sigma}(A)$ be a choice function for τ , i.e., taking one representative for each equivalence class in $\operatorname{Terms}_{\Sigma}(A)/\equiv$. Then

$$[\![s]\!]_{\tau}^{F(A,d_A)} = [\sigma_{\tau}(s)]_{\equiv}$$

where $\sigma_{\tau}(s)$ is defined as usual as the homomorphic extension of σ_{τ} to terms over X.

Proof. The proof is by induction on s. For s = x, the result is immediate by definition of σ_{τ} , i.e.,

$$[\![x]\!]_{\tau}^{F(A,d_A)} = \tau(x) = [\sigma_{\tau}(s)]_{\equiv}$$

For $s = op(s_1, ...s_n)$, we have

$$[\![s]\!]_{\tau}^{F(A,d_A)} = op^{F(A,d_A)}([\![s_1]\!]_{\tau}^{F(A,d_A)}, ..., [\![s_n]\!]_{\tau}^{F(A,d_A)}) \qquad \text{(by definition of semantics)}$$

$$= op^{F(A,d_A)}([\![\sigma_{\tau}(s_1)]\!]_{\equiv}, ..., [\![\sigma_{\tau}(s_n)]\!]_{\equiv}) \qquad \text{(by inductive hypothesis)}$$

$$= [(op(\sigma_{\tau}(s_1), ..., \sigma_{\tau}(s_n))]_{\equiv} \qquad \text{(by definition of } op^{F(A,d_A)})$$

$$= [\sigma_{\tau}(op(s_1, ..., s_n))]_{\equiv} \qquad \text{(by definition of } \sigma_{\tau} \text{ on terms)}$$

Lemma 4.17. It holds that: $F(A, d_A) \in \mathrm{QMod}_{\Sigma}(\Phi)$.

Proof. Let

$$F(A, d_A) = \left(\text{Terms}_{\Sigma}(A) / \equiv, \Delta, \{ op^{F(A, d_A)} \}_{op \in \Sigma} \right)$$

as specified in the previous section.

We need to show that if $\phi \in \Phi$ with:

$$\phi = \forall (X, d_X).s = t$$
 equation

or

$$\phi = \forall (X, d_X).s =_{\epsilon} t$$
 quantitative equation

for some **GMet** space (X, d_X) and terms $s, t \in \text{Terms}_{\Sigma}(X)$, then

$$F(A, d_A) \models \phi$$
.

Equivalently, we need to show that for every nonexpansive interpretation $\tau:(X,d_X)\to (\mathrm{Terms}_\Sigma(A)/\equiv,\Delta)$, it holds that:

$$[s]_{\tau}^{F(A,d_A)} = [t]_{\tau}^{F(A,d_A)}$$
 equation

or

$$\Delta\Big([\![s]\!]_\tau^{F(A,d_A)},[\![t]\!]_\tau^{F(A,d_A)}\Big) \leq \epsilon \quad \text{ quantitative equation}$$

or equivalently, by applying Lemma 4.16 where we take σ_{τ} defined as in the lemma, we need to show that:

$$[\sigma_{\tau}(s)]_{\equiv} = [\sigma_{\tau}(t)]_{\equiv}$$
 equation

or

$$\Delta\Big([\sigma_{\tau}(s)]_{\equiv}\ ,\ [\sigma_{\tau}(t)]_{\equiv}\Big) \leq \epsilon$$
 quantitative equation

which in turn, by definition of the equivalence relation \equiv and of Δ (which is defined in terms of the fuzzy relation d), means that:

$$\sigma_{\tau}(s) \equiv \sigma_{\tau}(t)$$
 equation

or

$$d(\sigma_{\tau}(s), \sigma_{\tau}(t)) \leq \epsilon$$
 quantitative equation

Hence, by unfolding the definitions of \equiv and d, we need to show that:

$$\Phi \vdash \forall (A, d_A). \ \sigma_{\tau}(s) = \sigma_{\tau}(t)$$
 equation

or

$$\Phi \vdash \forall (A, d_A). \ \sigma_{\tau}(s) =_{\epsilon} \sigma_{\tau}(t)$$
 quantitative equation

is derivable.

We just show below the case when ϕ is an quantitative equation, the other case (equation) is identical, just using the appropriate version of the (**SUBSTITUTION**) axiom (see 4e).

The assumption that τ is nonexpansive means that for all $x_i, x_j \in X$:

$$d_X(x_i, x_j) \le \epsilon_{ij} \implies \Delta(\llbracket x_i \rrbracket_{\tau}^{F(A, d_A)}, \llbracket x_j \rrbracket_{\tau}^{F(A, d_A)}) \le \epsilon_{ij}$$

which by Lemma 4.16 is equivalent to

$$d_X(x_i, x_j) \le \epsilon_{ij} \implies \Delta([\sigma_\tau(x_i)]_{\equiv}, [\sigma_\tau(x_j)]_{\equiv}) \le \epsilon_{ij}$$

By definition of Δ , d and Lemma 4.11, this gives us:

$$d_X(x_i, x_j) \le \epsilon_{\epsilon_{ij}} \quad \Longrightarrow \quad \Phi \vdash \forall (A, d_A). \sigma_\tau(x_i) =_{\epsilon_{ij}} \sigma_\tau(x_j). \tag{5}$$

We know that:

(I) Since $\forall (X, d_X).s =_{\epsilon} t \in \Phi$, by (INIT)(see 4e):

$$\Phi \vdash \forall (X, d_X).s =_{\epsilon} t$$

(II) By (5) we have all of the following judgments:

$$\{\Phi \vdash \forall (A, d_A).\sigma_{\tau}(x_i) =_{\epsilon_{i,i}} \sigma_{\tau}(x_i) \mid x_i, x_i \in X, \quad \epsilon_{i,i} := d_X(x_i, x_i)\}$$

Note that, using σ_{τ} as substitution, (I) and (II) above constitute the premises of the (**SUBSTITUTION**) axiom (see 4e quantitative equation instance), which we restate here for convenience⁷:

$$\big\{\forall (A,d_A).\sigma_\tau(x_i) =_{\epsilon_{ij}} \sigma_\tau(x_j) \mid x_i,x_j \in X, \ \epsilon_{ij} := d_X(x_i,x_j)\big\}, \forall (X,d_X).s =_\epsilon t \vdash \forall (A,d_A).\sigma_\tau(s) =_\epsilon \sigma_\tau(t)$$

From which we can derive the desired:

$$\Phi \vdash \forall (A, d_A).\sigma_\tau(s) =_{\epsilon} \sigma_\tau(t)$$

as follows:

(I) (II)
$$\overline{(I) + (II) \vdash \forall (A, d_A).\sigma_{\tau}(s) =_{\epsilon} \sigma_{\tau}(t)}$$
 Sub $\Phi \vdash \forall (A, d_A).\sigma_{\tau}(s) =_{\epsilon} \sigma_{\tau}(t)$

⁷Compared to the version presented in the list of axioms of the proof system, apply the renaming $A \to X$ and $B \to A$.

4.2.3 Proof of freeness of $F(A, d_A)$

In this section a **GMet** cateogory, a signature Σ , a quantitative theory Φ and a **GMet** space (A, d_A) are fixed. The quantitative algebra $F(A, d_A)$ is constructed as specified in Definition 4.15:

$$F(A, d_A) = (\operatorname{Terms}_{\Sigma}(A)/\equiv, \Delta, \{op^{F(A, d_A)}\}_{op \in \Sigma}).$$

We first observe that the map $(a \mapsto [a]_{\equiv})$ is nonexpansive.

Lemma 4.18. The map $\alpha: (A, d_A) \to (\operatorname{Terms}_{\Sigma}(A)/\equiv, \Delta)$ defined as:

$$\alpha(a) = [a]_{\equiv}$$

for all $a \in A$, is nonexpansive.

Proof. Fix arbitrary $a_1, a_2 \in A$ and assume $d_A(a_1, a_2) = \epsilon$. We need to prove that:

$$\Delta([a_1]_{\equiv}, [a_2]_{\equiv}) \le \epsilon$$

This follows from the definition of Δ and the presence in the proof system of the (USE VARIABLES) axiom (see 4f)

Remark 4.19. We note that the map α is generally not an isometry (i.e., distance preserving) nor an injection. For example consider the case when **GMet** is the category of metric spaces and (A, d_A) is of the form:

$$A = \{a_1, a_2\}$$
 $d_A(a_1, a_2) = \frac{1}{2}$

i.e., it is the metric space consisting of two points at distance $\frac{1}{2}$ and consider the set Φ of quantitative equations:

$$\Phi = \{ \forall (A, d_A). a_1 =_0 a_2 \}$$

consisting of just one quantitative equation. In this case it easy to check that $\operatorname{Terms}_{\Sigma}(A)/\equiv \text{consists}$ of a single element (i.e., all terms in $\operatorname{Terms}_{\Sigma}(A)$ are \equiv -equivalent) and thus the map α is not an injection and it is not an isometry.

Recall that $U: \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{Met}$ is the forgetful functor which maps a quantitative algebra to its underlying metric space. We are now ready to proceed with the proof that $F(A, d_A)$ is the U-free object generated by (A, d_A) relative to the map $\alpha: (A, d_A) \to (\mathrm{Terms}_{\Sigma}(A)/\equiv, \Delta)$ defined earlier, in the sense of Definition 2.15.

Theorem 4.20. Let $U : \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{Met}$. The quantitative algebra $F(A, d_A)$ is the U-free object generated by (A, d_A) relative to the map $\alpha : (A, d_A) \to (\mathrm{Terms}_{\Sigma}(A)/\equiv, \Delta)$

$$a \stackrel{\alpha}{\mapsto} [a]_{\equiv}.$$

Proof. We need to show that for every quantitative algebra $\mathbb{B} \in \mathbf{QMod}_{\Sigma}(\Phi)$

$$\mathbb{B} = (B, d_B, op^{\mathbb{B}})$$

and nonexpansive map

$$f:(A,d_A)\to(B,d_B)$$

there is a unique homomorphism of quantitative algebras

$$\hat{f}: F(A, d_A) \to \mathbb{B}$$

which extends f, i.e., which satisfies $f = U(\hat{f}) \circ \alpha$.

In the remainder of the proof we will generally omit the explicit use of the forgetful functors on morphisms, i.e., we will often write the same symbol to denote a function f seen as a set function, or as a morphism of metric spaces, or as a morphism of quantitative algebras.

We proceed as follows. First (Existence) we exhibit a nonexpansive homomorphism $\hat{f}: F(A, d_A) \to \mathbb{B}$, then (Extension) we show that \hat{f} extends f, and lastly (Uniqueness) we show that \hat{f} is the unique such homomorphism.

Existence. Recall that, by definition, we have

$$F(A, d_A) = (\text{Terms}_{\Sigma}(A)/\equiv, \Delta, \{op^{F(A, d_A)}\}_{op \in \Sigma}).$$

For all terms $s \in Terms_{\Sigma}(A)$, define \hat{f} as

$$\hat{f}([s]_{\equiv}) := [\![s]\!]_f^{\mathbb{B}}$$

Before moving on, we need to establish that this is a good definition, i.e., that it does not depend on any choice of representative s for the class $[s]_{\equiv}$.

To see this, observe that if $s \equiv t$ then, by definition of the relation (\equiv), it holds that that $\Phi \vdash \forall (A, d_A).s = t$. By the soundness (Theorem 4.4) of the deductive system, we have that

$$\forall (A, d_A).s = t \in \mathrm{QTh}_{\Sigma}(\mathrm{QMod}_{\Sigma}(\Phi))$$

Since $\mathbb{B} \in \mathrm{QMod}_{\Sigma}(\Phi)$ by hypothesis, this means that

$$\mathbb{B} \models \forall (A, d_A).s = t$$

and, in particular taking $f:(A,d_A)\to (B,d_B)$ as (nonexpansive) interpretation, it holds that $[\![s]\!]_f^{\mathbb{B}}=[\![t]\!]_f^{\mathbb{B}}$. Hence \hat{f} is well defined as a function.

It remains to show that:

1. \hat{f} is a homomorphism and,

2. \hat{f} is nonexpansive.

The first follows from the interpretation $op^{F(A,d_A)}$ of the operations in $F(A,d_A)$ as follows:

$$\hat{f}(op^{F(A,d_A)}([s_1]_{\equiv},...,[s_n]_{\equiv})) = \hat{f}([op(s_1,...,s_n)]_{\equiv})
= [op(s_1,...,s_n)]_f^{\mathbb{B}}
= op^{\mathbb{B}}([s_1]_f^{\mathbb{B}},...,[s_n]_f^{\mathbb{B}})
= op^{\mathbb{B}}(\hat{f}([s_1]_{\equiv}),...,\hat{f}([s_n]_{\equiv})).$$

Regarding the second point (nonexpansiveness), take two arbitrary $[s]_{\equiv}$, $[t]_{\equiv} \in \text{Terms}_{\Sigma}(A)/_{\equiv}$ and let $\Delta([s]_{\equiv},[t]_{\equiv}) = \epsilon$ be their distance in $F(A,d_A)$. We need to show that

$$d_B(\hat{f}([s]_{\equiv}), \hat{f}([t]_{\equiv})) \le \epsilon.$$

As established in Lemma 4.11, the hypothesis $\Delta([s]_{\equiv}, [t]_{\equiv}) = \epsilon$ implies that:

$$\Phi \vdash \forall (A, d_A).s =_{\epsilon} t$$

From the soundness of the deductive system (Theorem 4.4), we therefore know that:

$$\forall (A, d_A).s =_{\epsilon} t \in \mathrm{QTh}_{\Sigma}(\mathrm{QMod}_{\Sigma}(\Phi))$$

and since $\mathbb{B} \in \mathrm{QMod}_{\Sigma}(\Phi)$ by hypothesis, we deduce that:

$$\mathbb{B} \models \forall (A, d_A).s =_{\epsilon} t$$

Taking as nonexpansive interpretation $f:(A,d_A)\to (B,d_B)$ we therefore obtain that: $d_B(\llbracket s\rrbracket_f^{\mathbb{F}}, \llbracket t\rrbracket_f^{\mathbb{F}}) \leq \epsilon$.

By the definition of \hat{f} we have

$$\hat{f}([s]_{\equiv}) = [\![s]\!]_f^{\mathbb{B}} \quad \text{ and } \hat{f}([t]_{\equiv}) = [\![t]\!]_f^{\mathbb{B}}$$

so we conclude as desired that

$$d_B(\hat{f}([s]_{\equiv}), \hat{f}([t]_{\equiv})) \le \epsilon$$

Extension. We need to show that $f = \hat{f} \circ \alpha$.

$$\begin{array}{rcl} \big(\hat{f}\circ\alpha\big)(a) & = & \hat{f}(\alpha(a)) \\ & = & \hat{f}([a]_{\equiv}) & \text{Definition of } i \\ & = & [\![a]\!]_f^{\mathbb{B}} & \text{Definition of } \hat{f} \\ & = & f(a) & \text{Definition of } [\![-]\!]_f^{\mathbb{B}} \end{array}$$

Uniqueness.

Let $g: F(A, d_A) \to \mathbb{B}$ be another nonexpansive homomorphism extending $f: (A, d_A) \to (B, d_B)$, i.e., $f = g \circ \alpha$.

We now prove that for all $s \in \text{Terms}_{\Sigma}(A)$ it holds that: $g([s]_{\equiv}) = \hat{f}([s]_{\equiv})$.

Again, this follows from the interpretation of the operations in $F(A, d_A)$. Formally, the proof goes by induction on the structure of s as follows:

The base case (s = a) is immediate, as both g and \hat{f} extend f, i.e.,

$$g([a]_{\equiv}) = g \circ \alpha(a) = f(a) = \hat{f} \circ \alpha(a) = \hat{f}([a]_{\equiv})$$

For the inductive case we use the fact that g and \hat{f} are homomorphisms, together with the inductive hypothesis:

$$\begin{split} g([op(s_1,...,s_n)]_{\equiv}) &= g(op^{F(A,d_A)}([s_1]_{\equiv},...,[s_n]_{\equiv})) \\ &= op^{\mathbb{B}}(g([s_1]_{\equiv}),...,g([s_n]_{\equiv})) \\ &= op^{\mathbb{B}}(\hat{f}([s_1]_{\equiv}),...,\hat{f}([s_n]_{\equiv})) \\ &= \hat{f}(op^{F(A,d_A)}([s_1]_{\equiv},...,[s_n]_{\equiv})) \\ &= \hat{f}([op(s_1,...,s_n)]_{\equiv}). \end{split}$$

4.3 Completeness of the deductive system

In this section we establish the completeness of the deductive system $\vdash_{\mathbf{GMet}}$, Theorem 4.22 below.

The proof relies on the following property of $F(A,d_A)$, which, as we have seen in the previous section, is the $U:\mathbf{QMod}_{\Sigma}(\Phi)\to\mathbf{GMet}$ -free object generated by a \mathbf{GMet} space (A,d_A) relative to the nonexpansive map $\alpha:(A,d_A)\to(\mathrm{Terms}_{\Sigma}(A)/_{\equiv},\Delta)$ defined as:

$$\alpha(a) = [a]_{=}$$

Lemma 4.21. For all $s, t \in \text{Terms}_{\Sigma}(A)$,

if
$$[s]_{\alpha}^{F(A,d_A)} = [t]_{\alpha}^{F(A,d_A)}$$
 then $\Phi \vdash \forall (A,d_A).s = t$

and

$$if \ \Delta(\llbracket s \rrbracket_{\alpha}^{F(A,d_A)}, \llbracket t \rrbracket_{\alpha}^{F(A,d_A)}) \leq \epsilon \ then \ \Phi \vdash \forall (A,d_A).s =_{\epsilon} t$$

Proof. For the equation case, suppose that $[\![s]\!]^{F(A,d_A)}_{\alpha}=[\![t]\!]^{F(A,d_A)}_{\alpha}$. Note that by Lemma 4.16, instantiating τ with α and σ_{τ} with the embedding $A\to \operatorname{Terms}_{\Sigma}(A)$, we have $[\![s]\!]^{F(A,d_A)}_{\alpha}=[\![s]\!]_{\equiv}$ and $[\![t]\!]^{F(A,d_A)}_{\alpha}=[\![t]\!]_{\equiv}$. Hence, $[\![s]\!]_{\equiv}=[\![t]\!]_{\equiv}$, which by definition of Ξ means $\Phi \vdash \forall (A,d_A).s=t$.

For quantitative equations, we analogously have that $\Delta(\llbracket s \rrbracket_{\alpha}^{F(A,d_A)}, \llbracket t \rrbracket_{\alpha}^{F(A,d_A)}) \leq \epsilon$ implies (by Lemma 4.16) $\Delta([s]_{\equiv}, [t]_{\equiv}) \leq \epsilon$. Then, by the definition of Δ as d and by Lemma 4.8, we conclude $\Phi \vdash \forall (A, d_A).s =_{\epsilon} t$.

We can now prove the completeness theorem.

Theorem 4.22 (Completeness of the deductive system). Fix a **GMet** category and signature Σ and a set $\Phi \subseteq \text{GEq}(\Sigma)$. For all equations and quantitative equations $\phi \in \text{GEq}(\Sigma)$:

If
$$\Phi \Vdash \phi$$
 then $\Phi \vdash \phi$.

Proof. Let us consider first the case of equations. So let ϕ be of the form $\phi := \forall (A, d_A).s = t$, for some metric space (A, d_A) and terms $s, t \in \text{Terms}_{\Sigma}(A)$. Note that since $s, t \in \text{Terms}_{\Sigma}(A)$, it must be the case that $\text{Terms}_{\Sigma}(A) \neq \emptyset$. In other words, the pair $(\Sigma, (A, d_A))$ is nontrivial in the sense of Definition 2.4.

Since we have established nontriviality, by the free algebra Theorem 4.20 we know that

$$F(A, d_A) = (\operatorname{Terms}_{\Sigma}(A) / \equiv, \Delta, \{op^{F(A, d_A)}\}_{op \in \Sigma})$$

is the $U: \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{GMet}$ -free object generated by (A, d_A) relative to the nonexpansive map $\alpha: (A, d_A) \to (\mathrm{Terms}_{\Sigma}(A)/\equiv, \Delta)$ defined as $\alpha(a) = [a]_{\equiv}$.

Recall that, by definition, the hypothesis $\Phi \Vdash \phi$ means that for all $\mathbb{B} \in \mathrm{QMod}_{\Sigma}(\Phi)$ and for all nonexpansive interpretations $\tau: (A,d_A) \to (B,d_B)$, $[\![s]\!]_{\tau}^{\mathbb{B}} = [\![t]\!]_{\tau}^{\mathbb{B}}$. Hence, since $F(A,d_A) \in \mathrm{QMod}_{\Sigma}(\Phi)$ and α is nonexpansive, we have $[\![s]\!]_{\alpha}^{F(A,d_A)} = [\![t]\!]_{\alpha}^{F(A,d_A)}$. Then by Lemma 4.21 we conclude that $\Phi \vdash \forall (A,d_A).s = t$.

Analogously, for quantitative equations, if ϕ is of the form $\forall (A, d_A).s =_{\epsilon} t$ then we derive from $\Phi \Vdash \phi$ that $\Delta(\llbracket s \rrbracket_{\alpha}^{F(A,d_A)}, \llbracket t \rrbracket_{\alpha}^{F(A,d_A)}) \leq \epsilon$, and by Lemma 4.21 we conclude $\Phi \vdash \forall (A, d_A).s =_{\epsilon} t$.

4.4 The free-forgetful adjuction and the monad of quotiented terms

Recall (see Section 2.2 Proposition 2.17) that given a functor $U: \mathcal{D} \to \mathcal{C}$ such that \mathcal{D} has U-free objects, then there is a functor $F: \mathcal{C} \to \mathcal{D}$ which assigns to each element of \mathcal{C} its corresponding U-free object, and which gives an adjunction $F \dashv U$ and a monad with functor $M = (U \circ F)$.

We have seen in Section 4.2 that the forgetful functor $U : \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{GMet}$ has U-free objects, identified (up to isomorphism) as the quantitative algebras of quotiented terms. Hence, using the recipe from Proposition 2.17 we obtain the adjunction $F \dashv U$, where F is the functor assigning each \mathbf{GMet} space (A, d_A) to the free quantitative algebra of quotiented terms $F(A, d_A) = (\operatorname{Terms}_{\Sigma}(A) / \equiv$

 $,\Delta^{F(A,d_A)}, \operatorname{op}^{F(A,d_A)}).$ For a nonexpansive function $f:(A,d_A)\to(B,d_B),$ the functor F gives the nonexpansive homomorphism of quantitative algebras $F(f):F(A,d_A)\to F(B,d_B)$ which is the unique homomorphic extension of f. This means that once the forgetful functor U is applied, i.e., when seen as a (nonexpansive) function on terms, it can be inductively characterized as:

$$UF(f)([a]_{\equiv}) = [f(a)]_{\equiv}$$

and

$$UF(f)([op(t_1,...,t_n)]_{\equiv}) = op^{F(A,d_A)}(UF(f)([t_1]_{\equiv}),...,UF(f)([t_n]_{\equiv})).$$

The monad M we obtain is characterized as follows:

- The functor $M = U \circ F$ maps an object (A, d_A) to $M(A, d_A) = (\operatorname{Terms}_{\Sigma}(A)/\equiv, \Delta^{F(A, d_A)})$, and a morphism $f: (A, d_A) \to (B, d_B)$ to $M(f): (\operatorname{Terms}_{\Sigma}(A)/\equiv, \Delta^{F(A, d_A)}) \to (\operatorname{Terms}_{\Sigma}(B)/\equiv, \Delta^{F(B, d_B)})$ where $M(f)([t]_{\equiv}) = U \circ F([t]_{\equiv})$ is the nonexpansive homomorphism which can be specified by induction on terms t as above.
- The unit $\eta_{(A,d_A)}$ is the unit of the adjunction, i.e., the function $\alpha_{(A,d_A)}$ given together with the free object $F(A,d_A)$ in Section 4.2. This is indeed the function assigning to $a \in A$ the equivalence class $[a]_{\equiv}$.
- the multiplication $\mu_{(A,d_A)}$ is given as $\mu_{(A,d_A)} = U(\epsilon_{F(A,d_A)})$, where ϵ is the counit of the adjunction. We now show that, for $[t]_{\equiv} \in \operatorname{Terms}_{\Sigma}(\operatorname{Terms}_{\Sigma}(A)/\equiv)/\equiv$, the multiplication behaves as the function substituting each occurrence of an equivalence class of terms in t with a representative of the class, thus "flattening" the term as follows:

$$\mu_{(A,d_A)}\Big(\big[s\big([s_1]_{\equiv},\ldots,[s_n]_{\equiv}\big)\big]_{\equiv}\Big)=[s(s_1,\ldots,s_n)]_{\equiv}.$$

This can be see as follows. For $t = [t']_{\equiv}$ we have

$$\begin{split} \mu_{(A,d_A)}([[t']_{\equiv}]_{\equiv}) &= U(\epsilon_{F(A,d_A)})([[t']_{\equiv}]_{\equiv}) \\ &= U(\epsilon_{F(A,d_A)}) \circ \eta_{UF(A,d_A)}([t']_{\equiv}) \\ &= id_{UF(A,d_A)}([t']_{\equiv}) \\ &= [t']_{\equiv} \end{split}$$

where the second to last equation follows from the properties of the unitcounit triangle identities of the adjunction (see section 2.2). For t = $[op(t_1,...t_n)]_{\equiv}$, where $t_1,\ldots,t_n\in \mathrm{Terms}_{\Sigma}(\mathrm{Terms}_{\Sigma}(A)/\equiv)$, we have

$$\begin{split} \mu_{(A,d_A)}\Big([op(t_1,...t_n)]_{\equiv}\Big) &= U(\epsilon_{F(A,d_A)})\Big([op(t_1,...t_n)]_{\equiv}\Big) \\ &= U(\epsilon_{F(A,d_A)})\Big(op^{F(A,d_A)}([t_1]_{\equiv},...,[t_n]_{\equiv})\Big) \\ &= op^{F(A,d_A)}\Big(U(\epsilon_{F(A,d_A)})([t_1]_{\equiv}),...,U(\epsilon_{F(A,d_A)})([t_n]_{\equiv})\Big) \\ &= op^{F(A,d_A)}\Big(\mu_{(A,d_A)}([t_1]_{\equiv}),...,\mu_{(A,d_A)}([t_n]_{\equiv})\Big) \end{split}$$

where we exploited the fact that $\epsilon_{F(A,d_A)}$ is a homomorphism.

4.5 Monadicity

We have seen in the previous section that the functor $U : \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{GMet}$ has a left adjoint F, given by the functor associating to a \mathbf{GMet} space (A, d_A) the quantitative algebra $F(A, d_A)$ of quotiented terms generated by (A, d_A) , and we have identified the monad given by the adjunction.

We prove in this section that, as long as the **GMet** category satisfies a simple property, then the forgetful functor $U : \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{GMet}$ is strictly monadic (see Definition 2.21), i.e., that there is an isomorphism of categories:

$$\mathbf{EM}(M) \cong \mathbf{QMod}_{\Sigma}(\Phi)$$

where $\mathbf{EM}(M)$ is the *Eilenberg-Moore* category of M (see Definition 2.19).

We introduce some technical definitions required to state, in Definition 4.25, the property on the **GMet** category that will be required.

Definition 4.23. For a given category **GMet**, a functor $L : \mathbf{GMet} \to \mathbf{GMet}$ is called a *lifting of the n-ary set-theoretic product*, for some $n \in \mathbb{N}$, if the following diagram commutes:

$$egin{aligned} \mathbf{GMet} & \stackrel{L}{\longrightarrow} \mathbf{GMet} \ U_{\mathbf{GMet}
ightarrow \mathbf{Set}} & & \downarrow U_{\mathbf{GMet}
ightarrow \mathbf{Set}} \ \mathbf{Set} & \stackrel{(\square)^n}{\longrightarrow} \mathbf{Set} \end{aligned}$$

where $(_)^n$ denotes the *n*-ary (categorical) product in **Set**.

This means that, for an object $(A, d_A) \in \mathbf{GMet}$, the object $L(A, d_A)$ is a \mathbf{GMet} space of the form:

$$L(A, d_A) = (A^n, L(d_A))$$

where, with some abuse of notation, we have denoted its fuzzy relation with $L(d_A): A^n \times A^n \to [0,1]$. For an arrow $f: A \to B$, the arrow $L(f): F(A,d_A) \to L(B,d_B)$ is defined as:

$$L(f)(\langle a_1,\ldots,a_n\rangle) = \langle f(a_1),\ldots,f(a_n)\rangle$$

for all $\langle a_1, \ldots, a_n \rangle \in A^n$.

Definition 4.24. For a given category **GMet**, we say that a functor L: **GMet** \rightarrow **GMet** is a discrete lifting of the n-ary product if:

- 1. L is a lifting of the n-ary product (see Definition 4.23)
- 2. for any **GMet** spaces (A, d_A) and (B, d_B) and any set-theoretic function $f: A^n \to B$, there exists a nonexpansive map \hat{f} in **GMet** of type:

$$\hat{f}: L(A, d_A) \to (B, d_B)$$

such that $U_{\mathbf{GMet} \to \mathbf{Set}}(\hat{f}) = f$, i.e., \hat{f} is set-theoretically defined as f and moreover it is nonexpansive.

Definition 4.25. We say that a **GMet** category has discrete finite products if, for every $n \in \mathbb{N}$, there exists a functor $L^n : \mathbf{GMet} \to \mathbf{GMet}$ which is a discrete lifting of the n-ary product (see Definition 4.24).

Remark 4.26. Most **GMet** categories of interest have discrete finite products. For example, in **FRel**, for every $n \in \mathbb{N}$ a discrete lifting of the *n*-ary product L^n exists and is defined on objects as mapping (A, d_A) to $(A^n, L^n(d_A))$ where $L^n(d_A)(\vec{a}, \vec{a'}) = 1$ (constant 1 function), for all $\vec{a}, \vec{a'} \in A^n$.

In **Met**, the functor L^n maps (A, d_A) to $(A^n, L^n(d_A))$ where $L^n(d_A): A^n \times A^n \to [0, 1]$ is defined as:

$$L^{n}(d_{A})(\vec{a}, \vec{a'}) = \begin{cases} 0 & \text{if } \vec{a} = \vec{a'} \\ 1 & \text{otherwise.} \end{cases}$$

for all $\vec{a}, \vec{a'} \in A^n$.

At the present of moment, we do not know if all **GMet** categories have discrete finite products and, when they do, we do not know if they are unique.

In what follows, if not otherwise stated, we assume that the chosen **GMet** category has discrete finite products in the sense of Definition 4.25.

We will prove in Theorem 4.29 that $U: \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{GMet}$ satisfies the condition (3) of Beck's theorem (Proposition 2.22), from which we conclude by Beck's theorem that it is strictly monadic.

In order to do so, we first consider the special case when $\Phi = \emptyset$. In this case we recall that $\mathbf{QMod}_{\Sigma}(\emptyset) = \mathbf{QAlg}(\Sigma)$ consists of the category of all quantitative algebras of type Σ . We show that the forgetful functor $U_{\emptyset} : \mathbf{QAlg}(\Sigma) \to \mathbf{GMet}$ is strictly monadic.

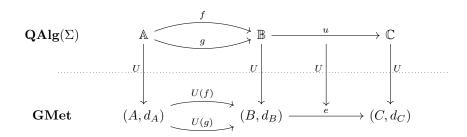
Theorem 4.27. Assume the category **GMet** has discrete finite products. Then the forgetful functor $U_{\emptyset} : \mathbf{QAlg}(\Sigma) \to \mathbf{GMet}$ is strictly monadic.

Proof. First, note that we have the results (I, II, III) of Sections 4.1, 4.2 and 4.4, instantiated to the special case $\Phi = \emptyset$. In particular, from result (III), we know that U_{\emptyset} has a left adjoint $F_{\emptyset} : \mathbf{GMet} \to \mathbf{QAlg}(\Sigma)$ $(F_{\emptyset} \dashv U_{\emptyset})$.

We then prove that the forgetful functor $U_{\emptyset}: \mathbf{QAlg}(\Sigma) \to \mathbf{GMet}$ creates coequalizers for all $\mathbf{QAlg}(\Sigma)$ -arrows f, g such that $U_{\emptyset}(f), U_{\emptyset}(g)$ has an absolute coequalizer (in \mathbf{GMet}), from which we derive by Beck's theorem (Proposition 2.22) that U_{\emptyset} is monadic.

The proof is analogous to that of Theorem 1 in [Mac71, Ch. VI, §8]. To ease notation, in this proof we write U instead of U_{\emptyset} . Let $f, g : \mathbb{A} \to \mathbb{B}$ be $\mathbf{QAlg}(\Sigma)$ -arrows such that $U(f), U(g) : (A, d_A) \to (B, d_B)$ have an absolute coequalizer $e : (B, d_B) \to (C, d_C)$. We show that there exists a quantitative algebra \mathbb{C} in $\mathbf{QAlg}(\Sigma)$ and an arrow $u : \mathbb{B} \to \mathbb{C}$ in $\mathbf{QAlg}(\Sigma)$ such that:

- i) $U(\mathbb{C}) = (C, d_C)$
- ii) e = U(u),
- iii) \mathbb{C} and u are unique with the properties (i) and (ii) above,
- iv) u is a coequalizer of f, g in $\mathbf{QAlg}(\Sigma)$.



We now prove all points (i - iv):

i) We define \mathbb{C} as $\mathbb{C} = (C, d_C, \{op^{\mathbb{C}}\}_{op \in \Sigma})$ which ensures, by definition, that $U(\mathbb{C}) = (C, d_C)$.

We need to define the interpretations $\{op^{\mathbb{C}}\}_{op\in\Sigma}$ of the operations in Σ . This is Equation 8 below.

Let $op \in \Sigma$ be an operation in Σ having arity n = ar(op).

Let $L^n : \mathbf{GMet} \to \mathbf{GMet}$ be some discrete lifting of the *n*-ary product functor in \mathbf{GMet} . It will be observed later that the definition of $op^{\mathbb{C}}$ does not depend on any specific choice of L^n .

The interpretations $op^{\mathbb{A}}$ and $op^{\mathbb{B}}$ of the operation $op \in \Sigma$ in \mathbb{A} and \mathbb{B} are set theoretic functions of type: $op^{\mathbb{A}} : A^n \to A$ and $op^{\mathbb{B}} : B^n \to B$, respectively.

By the properties of L^n , we therefore have that the following maps are in **GMet**, i.e., they are nonexpansive:

$$\hat{op}^{\mathbb{A}}: (A^n, L(d_A)) \to (A, d_A) \quad \hat{op}^{\mathbb{B}}: (B^n, L(d_B)) \to (B, d_B)$$

and defined set-theretically as $op^{\mathbb{A}}$ and $op^{\mathbb{B}}$, i.e., : $U_{\mathbf{GMet} \to \mathbf{Set}}(\hat{op}^{\mathbb{A}}) = op^{\mathbb{A}}$ and $U_{\mathbf{GMet} \to \mathbf{Set}}(\hat{op}^{\mathbb{B}}) = op^{\mathbb{B}}$.

We have the following identities in **GMet**:

$$e \circ \hat{op}^{\mathbb{B}} \circ L^n U(f) = e \circ U(f) \circ \hat{op}^{\mathbb{A}}$$
 (by f an homomorphism and definition of L^n on arrows)
$$= e \circ U(g) \circ \hat{op}^{\mathbb{A}}$$
 (by e a coequalizer)
$$= e \circ \hat{op}^{\mathbb{B}} \circ L^n U(g)$$
 (by g an homomorphism and definition of L^n on arrows)

Note that, since the arrow $e \in \mathbf{GMet}$ is an absolute coequalizer of U(f), U(g), by definition of absolute equalizers (see Proposition 2.22) instantiated with the functor L^n , we know that $L^n(e): (B^n, L^n(d_B)) \to (C^n, L^n(d_C))$ is a coequalizer of $L^nU(f), L^nU(g)$ in \mathbf{GMet} .

Hence we have the following diagram in **GMet**:

$$(A^n, L^n(d_A)) \xrightarrow{L^n U(g)} (B^n, L^n(d_B)) \xrightarrow{L^n(e)} (C^n, L^n(d_C))$$

$$e \circ \hat{op}^{\mathbb{B}} \qquad (C, d_C)$$

Since, as we observed, $L^n(e)$ is a coequalizer for $L^nU(f), L^nU(g)$, there exists an unique $h: (C^n, L^n(d_C)) \to (C, d_C)$ such that

$$h \circ L^n(e) = e \circ \hat{op}^{\mathbb{B}} \tag{6}$$

Diagrammatically:

$$(A^{n}, L^{n}(d_{A})) \xrightarrow{L^{n}U(g)} (B^{n}, L^{n}(d_{B})) \xrightarrow{L^{n}(e)} (C^{n}, L^{n}(d_{C}))$$

$$\downarrow h$$

$$\downarrow h$$

$$(C, d_{C})$$

$$(7)$$

We define $op^{\mathbb{C}}$, the interpretation of $op \in \Sigma$ in \mathbb{C} as such h. More formally, $op^{\mathbb{C}} = U_{\mathbf{GMet} \to \mathbf{Set}}(h)$ since $op^{\mathbb{C}}$ is just required to be a set-theoretic function of type $op^{\mathbb{C}} : C^n \to C$.

$$op^{\mathbb{C}} = U_{\mathbf{GMet} \to \mathbf{Set}}(h)$$
 (8)

Hence, the definition of $\mathbb{C} = (C, d_C, \{op^{\mathbb{C}}\}_{op \in \Sigma})$ is now complete.

We remark that the definition of $op^{\mathbb{C}}$ (and therefore of \mathbb{C}) remains the same even if another discrete n-ary product P^n is chosen instead of L^n . By using P^n , and following the same construction process, we would formally obtain a different h (of type $h: (C^n, P^n(d_C)) \to (C, d_C)$ rather than $h: (C^n, L^n(d_C)) \to (C, d_C)$). Still, the definition of $op^{\mathbb{C}} = U_{\mathbf{GMet} \to \mathbf{Set}}(h)$ (see Equation 8) is unaffected as it only depends on the set-theoretic definitions of all other maps involved. In short, our construction of $\mathbb{C} = (C, d_C, \{op^{\mathbb{C}}\}_{op \in \Sigma})$ is independent of any specific choice of discrete lifting of the n-ary product L^n .

- ii) We observe that the arrow $e:(A,d_A)\to (B,d_B)$ in **GMet** (i.e., a non-expansive map) further satisfies the required property for being a homomorphism (by Equation 6) and therefore it gives an arrow $u:\mathbb{B}\to\mathbb{C}$ in $\mathbf{QAlg}(\Sigma)$ with U(u)=e.
- iii) It remains to show that \mathbb{C} and u so defined are the unique possible ones satisfying properties (i) and (ii).

So consider any other quantitative algebra \mathbb{D} satisfying (i) and (ii). By property (i) $U(\mathbb{D}) = (C, d_C)$, it must be of the form:

$$\mathbb{D} = (C, d_C, \{op^{\mathbb{D}}\}_{op \in \Sigma})$$

for some choice of interpretations $op^{\mathbb{D}}: \mathbb{C}^n \to \mathbb{C}$ of the operations in Σ .

For each $op \in \Sigma$, denote with $\hat{op}^{\mathbb{D}}$ the nonexpansive map

$$\hat{op}^{\mathbb{D}}: (C^n, L(d_C)) \to (C, d_C).$$

By property (ii), U(u) = e, it must be the case that $e : (B, d_B) \to (C, d_C)$ is a homormphism, which means it satisfies (cf. Equation 6, where h is replaced by $\hat{op}^{\mathbb{D}}$):

$$\hat{op}^{\mathbb{D}} \circ L^n(e) = e \circ \hat{op}^{\mathbb{B}} \tag{9}$$

This means that the nonexpansive map $\hat{op}^{\mathbb{D}}$ is a solution of the commuting diagram (7) and therefore we derive, by the uniqueness of h as a solution, that:

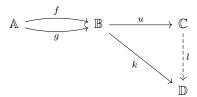
$$\hat{op}^{\mathbb{D}} = h$$

Finally, knowing that by definition

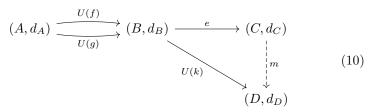
$$op^{\mathbb{C}} = U_{\mathbf{GMet} \to \mathbf{Set}}(h) \quad \text{and} \quad op^{\mathbb{D}} = U_{\mathbf{GMet} \to \mathbf{Set}}(\hat{op}^{\mathbb{D}})$$

we conclude that $op^{\mathbb{C}} = op^{\mathbb{D}}$. Hence, $\mathbb{C} = \mathbb{D}$ as desired.

iv) It remains to prove that u is a coequalizer for f, g in the category of quantitative algebras $\mathbf{QAlg}(\Sigma)$, i.e., that for any quantitative algebra \mathbb{D} in $\mathbf{QAlg}(\Sigma)$ and for all homomorphism $k : \mathbb{B} \to \mathbb{D}$, if $k \circ f = k \circ g$ then there is one and only one homomorphism $l : \mathbb{C} \to \mathbb{D}$ such that $l \circ u = k$.



From $k \circ f = k \circ g$ we derive that $U(k) \circ U(f) = U(k) \circ U(g)$. Hence, since U(u) = e and e is a coequalizer for U(f), U(g), we have by the universal property that there is one and only one **GMet** arrow $m: (C, d_C) \to (D, d_D)$ such that $m \circ e = U(k)$.



It remains to prove that m is an homomorphism from \mathbb{C} to \mathbb{D} , thus allowing us to define $l:\mathbb{C}\to\mathbb{D}$ (set-theoretically defined as m, i.e., U(l)=m) and thus to conclude our proof.

Towards this end, as already done previously, for every $op \in \Sigma$ and interpretations of the operations $op^{\mathbb{B}} : B^n \to B$ we denote with $\hat{op}^{\mathbb{B}}$ the **GMet** arrow (i.e., nonexpansive map):

$$\hat{op}^{\mathbb{B}}:(B^n,L(d_B))\to(B,d_B)$$

defined set-theoretically as $U(\hat{op}^{\mathbb{B}}) = op^{\mathbb{B}}$. Similarly, we have $U(\hat{op}^{\mathbb{C}}) = op^{\mathbb{C}}$ and $U(\hat{op}^{\mathbb{D}}) = op^{\mathbb{D}}$.

Proving that m is an homomorphism from \mathbb{C} to \mathbb{D} , amounts to showing that the following identity holds in **GMet**:

$$m \circ \hat{op}^{\mathbb{C}} = \hat{op}^{\mathbb{D}} \circ L^n(m).$$
 (11)

To prove this, first note that we have:

$$\begin{split} m \circ \hat{op}^{\mathbb{C}} \circ L^n(e) &= m \circ e \circ \hat{op}^{\mathbb{B}} & \text{(by e an homomorphism, see point (ii))} \\ &= U(k) \circ \hat{op}^{\mathbb{B}} & \text{(by e a coequalizer, see Diagram 10)} \\ &= \hat{op}^{\mathbb{D}} \circ L^n(U(k)) & \text{(by k an homomorphism)} \\ &= \hat{op}^{\mathbb{D}} \circ L^n(m) \circ L^n(e) & \text{(by L^n a functor and $U(k) = m \circ e$)} \end{split}$$

$$(B^{n}, L^{n}(d_{B})) \xrightarrow{L^{n}(e)} (C^{n}, L^{n}(d_{C})) \xrightarrow{L^{n}(m)} (D^{n}, L^{n}(d_{D}))$$

$$\hat{op}^{\mathbb{C}} \downarrow \qquad \qquad \downarrow \hat{op}^{\mathbb{D}}$$

$$(C, d_{C}) \xrightarrow{m} (D, d_{D})$$

Now recall that $L^n(e)$ is a coequalizer for $L^nU(f), L^nU(g)$, and thus it is an epimorphism. Therefore, we derive Equation 11 as desired: $m \circ \hat{op}^{\mathbb{C}} = \hat{op}^{\mathbb{D}} \circ L^n(m)$.

We are now going to use the above result, which deals with the special case $\Phi = \emptyset$, to prove Theorem 4.29 in its full generality, showing that the functor $U : \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{GMet}$ is monadic for arbitrary Φ . The proof is a generalization of the analogous result in [Adá22, Thm 2.17] which proves monadicity in the framework of [MPP16].

In particular, we prove that (as long as the **GMet** category has finite discrete products) U satisfies the condition (3) of Beck's theorem, i.e., in contrast with Theorem 4.27, we use split coequalizers instead of absolute coequalizers. We do so since split coequalizers guarantee the existence of a right inverse of the coequalizer, which allows us to apply the following fact (Lemma 4.28): $\mathbf{QMod}_{\Sigma}(\Phi)$ is closed under the images of homomorphisms that have a right inverse in \mathbf{GMet} . We remark that the result of Lemma 4.28 is valid for all \mathbf{GMet} categories, regardless of whether or not they have discrete finite products.

Lemma 4.28. Let $\mathbb{A} = (A, d_A, \{op^{\mathbb{A}}\}_{op \in \Sigma})$ be a quantitative algebra in $\mathbf{QMod}_{\Sigma}(\Phi)$, for some set of equantions and quantitative equations $\Phi \subseteq \mathrm{GEq}(\Sigma)$, and let $\mathbb{B} = (B, d_B, \{op^{\mathbb{B}}\}_{op \in \Sigma})$ be in $\mathbf{QAlg}(\Sigma)$. If there is a homomorphism of quantitative algebras $f : \mathbb{A} \to \mathbb{B}$ such that U(f) has a nonexpansive right inverse $g : (B, d_B) \to (A, d_A)$ then \mathbb{B} is in $\mathbf{QMod}_{\Sigma}(\Phi)$.

Proof. We need to show that, under the hypothesis of the statement, for every $\phi \in \Phi$:

$$\mathbb{B} \models \phi$$

holds. We first consider the case of ϕ being an equation:

$$\phi = \forall (X, d_X).s = t$$

for some **GMet** space (X, d_X) and terms $s, t \in \text{Terms}_{\Sigma}(X)$.

By definition (of $\mathbb{B} \models \phi$), we need to show that $\llbracket s \rrbracket_{\tau}^{\mathbb{B}} = \llbracket t \rrbracket_{\tau}^{\mathbb{B}}$, for all nonexpansive interpretations $\tau : (X, d_X) \to \mathbb{B}$.

Let $\tau:(X,d_X)\to (B,d_B)$ be such an interpretation. Then $g\circ \tau:(X,d_X)\to (A,d_A)$ is an interpretation in $\mathbb A$ (which is nonexpansive as **GMet** is a category

and thus the composition of nonexpansive functions is nonexpansive). We prove that for any term $r \in \operatorname{Terms}_{\Sigma}(X)$ (and so, in particular, s and t) it holds that:

$$U(f)(\llbracket r \rrbracket_{q \circ \tau}^{\mathbb{A}}) = \llbracket r \rrbracket_{\tau}^{\mathbb{B}} \tag{12}$$

The proof is by induction on r:

- if r = x then we have $U(f)(\llbracket x \rrbracket_{g \circ \tau}^{\mathbb{A}}) = U(f) \circ g \circ \tau(x) = \tau(x) = \llbracket x \rrbracket_{\tau}^{\mathbb{B}}$ since g is a right inverse of U(f), i.e., $U(f) \circ g = id_B$.
- if $r = op(r_1, ..., r_n)$ then we have

$$\begin{split} U(f)(\llbracket r \rrbracket_{g \circ \tau}^{\mathbb{A}}) &= U(f)(op^{\mathbb{A}}(\llbracket r_1 \rrbracket_{g \circ \tau}^{\mathbb{A}}, ..., \llbracket r_n \rrbracket_{g \circ \tau}^{\mathbb{A}})) \\ &= op^{\mathbb{B}}(U(f)(\llbracket r_1 \rrbracket_{g \circ \tau}^{\mathbb{A}}), ..., U(f)(\llbracket r_n \rrbracket_{g \circ \tau}^{\mathbb{A}})) \\ &\qquad \qquad \qquad \qquad \text{(by } U(f) \text{ a homomorphism)} \\ &= op^{\mathbb{B}}(\llbracket r_1 \rrbracket_{\tau}^{\mathbb{B}}, ..., \llbracket r_n \rrbracket_{\tau}^{\mathbb{B}})) \qquad \text{(by inductive hypothesis)} \\ &= \llbracket r \rrbracket_{\tau}^{\mathbb{B}} \end{split}$$

We now conclude the proof that \mathbb{B} satisfies the equation under the interpretation τ . Since, by hypothesis, \mathbb{A} is a model of Φ , we know that $\mathbb{A} \models \phi$ and therefore:

$$[s]_{q \circ \tau}^{\mathbb{A}} = [t]_{q \circ \tau}^{\mathbb{A}} \tag{13}$$

Then, we derive:

$$[\![s]\!]^{\mathbb{B}}_{\tau} = U(f)([\![s]\!]^{\mathbb{A}}_{g \circ \tau})$$
 (by (12))

$$= U(f)(\llbracket t \rrbracket_{q \circ \tau}^{\mathbb{A}})$$
 (by (13))

$$= [\![t]\!]^{\mathbb{B}}_{\tau} \tag{by (12)}$$

We now consider the case of quantitative equations $\phi \in \Phi$ of the form:

$$\phi = \forall (X, d_X).s =_{\epsilon} t.$$

Since, by hypothesis A is a model of Φ , we know that $A \models \phi$ and therefore:

$$d_A(\llbracket s \rrbracket_{g \circ \tau}^{\mathbb{A}}, \llbracket t \rrbracket_{g \circ \tau}^{\mathbb{A}}) \le \epsilon.$$

From which we derive:

$$d_{B}(\llbracket s \rrbracket_{\tau}^{\mathbb{B}}, \llbracket t \rrbracket_{\tau}^{\mathbb{B}}) = d_{B}(U(f)(\llbracket s \rrbracket_{g \circ \tau}^{\mathbb{B}}), U(f)(\llbracket t \rrbracket_{g \circ \tau}^{\mathbb{B}})) \qquad \text{(by (12))}$$

$$\leq d_{A}(\llbracket s \rrbracket_{g \circ \tau}^{\mathbb{A}}, \llbracket t \rrbracket_{g \circ \tau}^{\mathbb{A}}) \qquad \text{(by } U(f) \text{ nonexpansive)}$$

$$\leq \epsilon \qquad \qquad \text{(by A a model of } \Phi)$$

Theorem 4.29 (Monadicity of $U : \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{GMet}$). Assume the \mathbf{GMet} category has discrete finite products. The forgetful functor $U : \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{GMet}$ is monadic.

Proof. From result (III) (see Section 4.4), we know that there is an adjunction $F \dashv U$. We now prove that the forgetful functor $U : \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{GMet}$ creates coequalizers for all $\mathbf{QMod}_{\Sigma}(\Phi)$ -arrows f, g such that U(f), U(g) has a split coequalizer (in \mathbf{GMet}). From this, it immediately follows by Beck's theorem (Proposition 2.22) that U is monadic.

Let $f, g : \mathbb{A} \to \mathbb{B}$ be $\mathbf{QMod}_{\Sigma}(\Phi)$ -arrows such that $U(f), U(g) : (A, d_A) \to (B, d_B)$ have a split coequalizer $e : (B, d_B) \to (C, d_C)$. We show that there exists a unique algebra \mathbb{C} in $\mathbf{QMod}_{\Sigma}(\Phi)$ such that $U(\mathbb{C}) = (C, d_C)$, such that e = U(u) for $u : \mathbb{B} \to \mathbb{C}$ an arrow in $\mathbf{QMod}_{\Sigma}(\Phi)$, and such that u is a coequalizer of f, g in $\mathbf{QMod}_{\Sigma}(\Phi)$.

Recall that U_{\emptyset} is the forgetful functor $U_{\emptyset}: \mathbf{QAlg}(\Sigma) \to \mathbf{GMet}$ from Theorem 4.27. Since U_{\emptyset} is monadic (by Theorem 4.27), it satisfies condition (3) of Proposition 2.22. Since $\mathbf{QAlg}(\Sigma)$ -arrows between objects in $\mathbf{QMod}_{\Sigma}(\Phi)$ coincide with $\mathbf{QMod}_{\Sigma}(\Phi)$ -arrows, i.e., they are both defined as nonexpansive homomorphisms of quantitative Σ -algebras, the condition (3) of Proposition 2.22 implies that there is a unique algebra $\mathbb C$ in $\mathbf{QAlg}(\Sigma)$ such that $U_{\emptyset}(\mathbb C) = C$, such that $e = U_{\emptyset}(u)$ for $u : \mathbb B \to \mathbb C$ an arrow in $\mathbf{QAlg}(\Sigma)$, and such that u is a coequalizer of U(f), U(g) in $\mathbf{QAlg}(\Sigma)$.

Since e is a split coequalizer of U(f), U(g), it has a right inverse $r: (C, d_C) \to (B, d_B)$. Hence, since \mathbb{B} is an model of Φ and since $u: \mathbb{B} \to \mathbb{C}$ is a homomorphism, we derive by Lemma 4.28 that \mathbb{C} is also a model of Φ , i.e., $\mathbb{C} \in \mathbf{QMod}_{\Sigma}(\Phi)$.

We conclude by noticing that u is also a coequalizer in $\mathbf{QMod}_{\Sigma}(\Phi)$, since the universal property holds for u in the larger class $\mathbf{QAlg}(\Sigma)$.

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A Appendix to Section 2.2

Proof of Proposition 2.17

First, we show that F is indeed a functor:

• Since U is a functor we have that $U(id_{F(X)}) = id_{UF(X)}$ and the following commutes

$$U(F(X)) \xrightarrow{id_{UF(X)}} U(F(X))$$

$$\iota_X \uparrow \qquad \qquad \iota_X$$

By definition of $F(id_X)$ we have that $F(id_X)$ is the unique unique \mathbb{D} -morphism that makes the following diagram commute:

$$U(F(X)) \xrightarrow{U(F(id_X))} U(F(X))$$

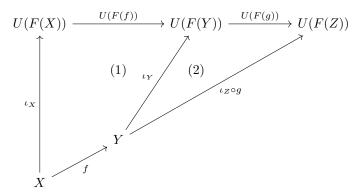
$$\iota_X \uparrow \qquad \qquad \qquad \downarrow_{X \circ id_X}$$

By unicity and by $\iota_X \circ id_X = \iota_X$, we conclude $id_{F(X)} = F(id_X)$.

• we now prove that for any $f: X \to Y$ and $g: Y \to Z$ it holds $F(g) \circ F(f) = F(g \circ f)$.

The following diagram commutes by the definition of U(F(f)) and U(F(g)),

i.e., the subdiagrams (1) and (2) respectively.



and moreover we have, since U is a functor, $U(F(g)) \circ U(F(f)) = U(F(g)) \circ F(f)$.

By definition, $F(g \circ f)$ is the unique morphism that makes the following diagram commute:

$$U(F(X)) \xrightarrow{U(F(g \circ f))} U(F(X))$$

$$\iota_X \uparrow \qquad \qquad \iota_Z \circ g \circ f$$

By unicity, we conclude that $F(g) \circ F(f) = F(g \circ f)$

We now prove that F is the left adjoint of U, i.e., that there is a natural transformation $\eta: Id_{\mathcal{C}} \Rightarrow U \circ F$ such that for any \mathcal{C} -object C, any \mathcal{D} -object D and \mathcal{C} -morphism $f: C \to U(D)$ there is an unique \mathcal{D} -morphism $g: F(C) \to D$ such that $f = g \circ \eta_C$.

We define η as $\eta_X = \iota_X$. We indeed have a natural transformation, since by definition of F on morphisms the following commutes:

$$U(F(X)) \xrightarrow{U(F(f))} U(F(Y))$$

$$\downarrow_{X} \qquad \qquad \downarrow_{Y} \qquad \downarrow_{Y} \qquad \qquad \downarrow_{Y} \qquad$$

Then the conditions of the definition of adjoint immediately follow by the definition of F.