Monadic chain logic over iterations and applications to push-down systems

Dietrich Kuske[1] and Markus Lohrey[2]

[1] Institut für Informatik, Universität Leipzig, Germany
kuske@informatik.uni-leipzig.de
[2] Markus Lohrey
Universität Stuttgart, FMI, Germany
lohrey@informatik.uni-stuttgart.de

Abstract. It is shown that the monadic chain theory of the iteration $\mathcal{A}^*$ of a structure $\mathcal{A}$ (in the sense of Shelah and Stupp) is decidable in case the first-order theory of the structure $\mathcal{A}$ is decidable. This result fail, if Muchnik's clone-predicate is also included in the iteration. A model of pushdown automata, where the stack alphabet is given by an arbitrary (possibly infinite) relational structure is given, is introduced. Using the decidability result for the monadic chain theory of the iteration $\mathcal{A}^*$, it is shown that the configuration graph of such a pushdown automaton has a decidable first-order theory with regular reachability predicates in case the stack structure $\mathcal{A}$ has decidable first-order theory with regular reachability predicates.

1 Introduction

In this paper, we study iterations of relational structures, their logical properties, and apply our results to the model checking of a powerful extension of pushdown systems.

The *local full iteration* $\mathcal{A}^*_{\text{loc}}$ of a relational base structure $\mathcal{A}$ with universe A consists of the set A^* of finite words over A. One of its relations is the immediate successor relation son. The sons of a word w carry the relations of the base structure $\mathcal{A}$. Furthermore, Muchnik's unary clone predicate collects all words whose final two letters are identical. Semenov [21] sketched a proof of what is now known as Muchnik's preservation theorem: The monadic second order (MSO for short) theory of the local full iteration $\mathcal{A}^*_{\text{loc}}$ can be reduced to the MSO-theory of the base structure $\mathcal{A}$ and, if two base structures have the same MSO-theory, then the same holds for their iterations. Hence, if the MSO-theory of a structure $\mathcal{A}$ is decidable, then also the MSO-theory of the local full iteration $\mathcal{A}^*_{\text{loc}}$ is decidable. A full proof of this result was given by Walukiewicz in [28]. A first-order variant of Muchnik's theorem for first-order logic follows from [14]. For modulo counting extensions of MSO and for guarded second order logic, it was shown by Blumensath & Kreutzer [3].

The *full iteration* $\mathcal{A}^*_{\text{fu}}$ differs from the local full iteration only in as far as it contains the prefix relation on A^* instead of the immediate successor relation son. Since this prefix relation is the transitive closure of son, there is an MSO-interpretation of the full iteration in the local full iteration. As an immediate consequence from [28, 3], one obtains a preservation theorem for MSO and its modulo counting extensions for this full iteration in place of the local full iteration. One can express in first-order logic that an element of the full iteration (i.e., a word over the base structure) represents a path in the base structure. This leads to the failure of both parts of the preservation theorem for the full iteration and first-order logic. More precisely, the transition graph $\mathcal{A}$ of a universal Turing machine has a decidable first-order theory, but the first-order theory of its full iteration is undecidable (Proposition 3.4). Furthermore, for any $n \in \mathbb{N}$, we present structures $\mathcal{A}_n$ and $\mathcal{B}_n$ that cannot be distinguished by first-order sentences of quantifier depth n, but their full iterations differ in some first-order sentence of quantifier depth 6 (Proposition 3.5).

To overcome these problems, Section 4 is devoted to the study of the *basic iteration* where one omits Muchnik's clone predicate but keeps the prefix order. For basic iterations, the preservation theorem for MSO was proved by Stupp [23] (cf. [22]). Rabin's seminal result on the decidability of the MSO-theory of the complete infinite binary tree [18] is an immediate corollary of this preservation theorem. For this basic iteration we are able to prove the preservation theorem for first-order logic. In fact, we can show even more: If a structure has a decidable first-order theory, then its basic iteration has a decidable $\mathrm{MSO}^{\mathrm{chain}}$-theory (Thm. 4.16). $\mathrm{MSO}^{\mathrm{chain}}$ is the fragment of MSO where second-order quantification is restricted to chains (i.e., ordered subsets) with respect to the tree structure of the iteration. $\mathrm{MSO}^{\mathrm{chain}}$ on trees was investigated in [24]. To reduce the $\mathrm{MSO}^{\mathrm{chain}}$-theory of the basic iteration to the first-order theory of the base structure, we proceed as follows: First, we show that quantification over chains can be restricted to ultimately periodic chains of bounded offset and period length (Thm. 4.13). Truth of $\mathrm{MSO}^{\mathrm{chain}}$-formulas with bounded quantification can be determined in a bounded prefix of the basic iteration. Finally, this bounded prefix can be interpreted in the base structure. Since all these bounds can be computed effectively, our preservation theorem follows.

Roughly speaking, the results from Section 3 and Section 4 show that, in order to have a first-order preservation theorem for the iteration, we are not allowed to copy an infinite amount of information between the levels of the tree structure — this is in some sense the essence of the clone-predicate. Thus, the clone-predicate has an immense effect on the expressive power of the basic iteration although it looks quite innocent at first glance. It should be also noted that the clone-predicate allows to define the unraveling of a graph G within the full iteration of G (cf. [6])

In Section 5 we present an application of our decidability result for $\mathrm{MSO}^{\mathrm{chain}}$ over basic iterations to pushdown systems. Pushdown systems were used to model the state space of sequential programs with nested procedure calls, see e.g. [9]. Model-checking problems for pushdown systems were studied for various temporal logics (LTL, CTL, modal μ-calculus) [1, 9, 13, 27]. When modeling recursive sequential programs via pushdown systems, it is necessary to abstract local variables (which have to be stored on the stack) with an infinite range (like for instance integers) to some finite range, in order to obtain a finite pushdown alphabet. This abstraction may lead to so called spurious counterexamples [8]. Here, we introduce pushdown systems where the stack alphabet is the (possibly infinite) universe of an arbitrary stack structure $\mathcal{A}$. With any change of the control state, our pushdown model associates one of three basic operations: (i) replacing the topmost symbol of the stack by another one according to some binary predicate of the stack structure, (ii) pushing or (iii) popping a symbol from some unary predicate of the stack structure. Such a pushdown system can directly model programs with nested procedure calls, where procedures use variables with an infinite domain. The configuration graph of such a pushdown system is defined as for finite stack alphabets. We study the logic FOREG for these configuration graphs. FOREG is the extension of first-order logic which allows to define new binary predicates by regular expressions over the binary predicates of the base structure $\mathcal{A}$. FOREG was studied in [15, 20, 25] and proposed as an expressive XML-query language. FOREG is a suitable language for the specification of reachability properties of reactive systems; its expressive power is between first-order logic and MSO. Based on our decidability result Thm. 4.16 we show that if FOREG is decidable for the base structure $\mathcal{A}$ of a pushdown system, then FOREG remains decidable for the configuration graph of the pushdown system (Thm. 5.1). For this result, it is important that in our pushdown model procedure calls and returns cannot transfer an infinite amount of information to another call level. This reflects our undecidability result Proposition 3.4 for the clone predicate.

2 Preliminaries

Let Σ be a (not necessarily finite) alphabet. With Σ^* we denote the set of all finite words over Σ. The empty word is denoted by ε. The set of all nonempty finite words over Σ is Σ^+. With $\preceq$ we denote the prefix relation on finite words. For $u, v \in \Sigma^*$ we write $u \prec v$ if $u \preceq v$ and $u \neq v$. For a subalphabet $\Gamma \subseteq \Sigma$ and a word $u \in \Sigma^*$ we denote with $|u|_\Gamma$ the number of positions in u, where a symbol from Γ occurs. In case Σ is finite, we denote with $\mathrm{REG}(\Sigma)$ the set of all regular languages over the alphabet Σ.

2.1 Iterations

Let σ be a finite relational signature and let $\mathcal{A} = (A, (R^{\mathcal{A}})_{R\in\sigma})$ be a relational structure over the signature σ. The *basic iteration* $\mathcal{A}^*_{\mathrm{ba}}$ *of* $\mathcal{A}$ is the structure

$$\mathcal{A}^*_{\mathrm{ba}} = (A^*, \preceq, (\widehat{R})_{R\in\sigma}, \varepsilon)$$

where

$$\widehat{R} = \{(ua_1, \dots, ua_n) \mid u \in A^*, (a_1, \dots, a_n) \in R^{\mathcal{A}}\}$$

for $R \in \sigma$.

Example 2.1. Suppose the structure $\mathcal{A}$ has two elements a and b and two unary relations $R_1 = \{a\}$ and $R_2 = \{b\}$. Then $\widehat{R}_1 = \{a, b\}^* a$ and $\widehat{R}_2 = \{a, b\}^* b$. Hence the basic iteration $\mathcal{A}^*_{\mathrm{ba}}$ can be visualized as a complete binary tree with unary predicates telling whether the current node is the first or the second son of its father. In addition, the root ε is a constant of the structure $\mathcal{A}^*_{\mathrm{ba}}$.

In the *full iteration* $\mathcal{A}^*_{\mathrm{fu}}$ of $\mathcal{A}$, we have the additional unary *clone predicate* $\mathrm{cl} = \{uaa \mid u \in A^*, a \in A\}$, i.e.,

$$\mathcal{A}^*_{\mathrm{fu}} = (A^*, \preceq, \mathrm{cl}, (\widehat{R})_{R\in\sigma}, \varepsilon) \ .$$

We will also consider a relaxation of the full iteration where the prefix relation is replaced by the direct successor relation son $= \{(u, ua) \mid a \in A^*, a \in A\}$, i.e.,

$$\mathcal{A}^*_{\mathrm{loc}} = (A^*, \mathrm{son}, \mathrm{cl}, (\widehat{R})_{R\in\sigma}, \varepsilon) \ .$$

We refer to this iteration as *local iteration*. Note that $\mathcal{A}^*_{\mathrm{fu}}$ is MSO-definable (but not first-order definable) in $\mathcal{A}^*_{\mathrm{loc}}$.

2.2 Logics

Let σ be some signature. *Atomic formulas* are $R(x_1, \dots, x_n)$, $x_1 = x_2$, and $x_1 \in X$ where $x_1, \dots, x_n$ are *individual variables*, $R \in \sigma$ is an n-are relational symbol, and X is a *set variable. Monadic second-order formulas* are obtained from atomic formulas by conjunction, negation, and quantification $\exists x$ and $\exists X$ for x an individual and X a set variable. The satisfaction relation $(\mathcal{A}, \bar{a}, \bar{C}) \models \varphi(\bar{x}, \bar{X})$ is defined as usual with the understanding that set variables range over subsets of A. A *first-order formula* is a monadic second-order formula without set variables.

Now let $\preceq$ be a designated binary relation symbol in σ. A *monadic second-order chain formula* or $\mathrm{MSO}^{\mathrm{chain}}$-formula is just a monadic second-order formula. For these $\mathrm{MSO}^{\mathrm{chain}}$-formulas, we define a new satisfaction relation $(\mathcal{A}, \bar{a}, \bar{C}) \models^{\mathrm{chain}} \varphi(\bar{x}, \bar{X})$: it is defined as $\models$ with the only difference that set variables range over those subsets C of A that satisfy $x \preceq y \vee y \preceq x$ for all $x, y \in C$. Note that if φ is a first-order formula, then $\mathcal{A} \models \varphi$ if and only if $\mathcal{A} \models^{\mathrm{chain}} \varphi$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-structures. Then we write $\mathcal{A} \equiv^{\mathrm{MSO}}_m \mathcal{B}$ if, for any MSO-formula φ of quantifier depth at most m, we have $\mathcal{A} \models \varphi \iff \mathcal{B} \models \varphi$. This relation is an equivalence relation. If we only consider first-order formulas φ of quantifier depth at most m, then we write $\mathcal{A} \equiv^{\mathrm{FO}}_m \mathcal{B}$. The notion $\mathcal{A} \equiv^{\mathrm{chain}}_m \mathcal{B}$ is to be understood similarly with respect to the monadic chain logic $\mathrm{MSO}^{\mathrm{chain}}$.

3 The theories of the full iteration

We first deal with MSO-theories. Muchnik's theorem sharpens an earlier result of Stupp. Its full proof can be found in [28] (cf. also [2]).

Theorem 3.1. *Let σ be some finite relational signature. There exists a computable function* red *from the set of MSO-formulas over the signature* $(\sigma, \mathrm{cl}, \mathrm{son})$ *into the set of MSO-formulas over the signature σ such that, for any σ-structure $\mathcal{A}$, we have $\mathcal{A} \models \mathrm{red}(\varphi)$ if and only if $\mathcal{A}^*_{\mathrm{loc}} \models \varphi$.*

One infers immediately:

Corollary 3.2. *If the MSO-theory of a structure $\mathcal{A}$ is decidable, then the MSO-theory of its local iteration $\mathcal{A}^*_{\mathrm{loc}}$ is decidable as well.*

To derive another corollary, let $m \in \mathbb{N}$ be arbitrary. Then, there is a finite set Φ of MSO-formulas such that any MSO-sentence of quantifier depth at most m is logically equivalent to some sentence from Φ. Let n be an upper bound for the quantifier depth of $\mathrm{red}(\varphi)$ for $\varphi \in \Phi$. This observation yields:

Corollary 3.3. *For any $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that, for any two σ-structures $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A} \equiv_n^{\mathrm{MSO}} \mathcal{B}$, we have $\mathcal{A}^*_{\mathrm{loc}} \equiv_m^{\mathrm{MSO}} \mathcal{B}^*_{\mathrm{loc}}$.*

Note that the MSO-theories of the local and the full encoding can be reduced onto each other. Hence Muchnik's Theorem 3.1 and Corollaries 3.2 and 3.3 hold for the full iteration $\mathcal{A}^*_{\mathrm{fu}}$ in place of the local iteration $\mathcal{A}^*_{\mathrm{loc}}$ equally well. Surprisingly, this is not the case for first-order logic as the following two examples show. First, we give a counterexample to the first-order version of Corollary 3.2 and therefore Theorem 3.1 with the full iteration taking the place of the local iteration.

Proposition 3.4. *The exists a structure $\mathcal{A}$ with a decidable first-order theory such that the full iteration $\mathcal{A}^*_{\mathrm{fu}}$ has an undecidable first-order theory.*

Proof. Let $\mathcal{M}$ be a Turing machine that accepts a non-recursive set L (we assume that $\mathcal{M}$ accepts with empty tape). Let Σ be the set of tape symbols and states of $\mathcal{M}$. Then consider the following structure $\mathcal{A} = (A, E, (E_a)_{a \in \Sigma})$

- The universe A of $\mathcal{A}$ is the set of configurations of $\mathcal{M}$.
- For two configurations c_1, c_2, we have $(c_1, c_2) \in E$ if and only if c_2 can be obtained from c_1 by one step of the Turing machine.
- For two configurations c_1, c_2 and $a \in \Sigma$, we have $(c_1, c_2) \in E_a$ if and only if $c_2 = c_1 a$.

The first-order theory of $\mathcal{A}$ is decidable since $\mathcal{A}$ is automatic [12]. We can write a formula α with one free variable x such that $(\mathcal{A}, c) \models \alpha$ if and only if c is a configuration with empty tape. Furthermore, from a state q and an input word w, we can write a first-order formula φ_{qw} with one free variable x such that $(\mathcal{A}, c) \models \varphi_{qw}$ if and only if $c = qw$ for any configuration c.

Now consider the full iteration of $\mathcal{A}$. The formulas $\widehat{\alpha}$ and $\widehat{\varphi_{qw}}$ are obtained by replacing $E(y,z)$ by $\widehat{E}(y,z)$, $E_a(y,z)$ by $\widehat{E_a}(y,z)$, and by restricting the quantification to siblings of the free variable x. Furthermore, let w be some input word and let q_0 be the initial state of $\mathcal{M}$. Then w is accepted if and only if there exists a sequence of configurations $u = c_0 c_1 \ldots c_n \in A^*$ such that the following hold in the full iteration of $\mathcal{A}$:

- the minimal nonempty prefix c_0 of u satisfies $\widehat{\varphi_{q_0 w}}$
- u satisfies $\widehat{\alpha}$
- for all proper and non-empty prefixes v of u, we have

$$\exists v', v'' : v \lessdot v' \preceq u \land v \lessdot v'' \land \mathrm{cl}(v'') \land \widehat{E}(v'', v'),$$

where $x \lessdot y$ is shorthand for $x \prec y \land \forall z (x \preceq z \prec y \rightarrow x = z)$. Since the language of the Turing machine $\mathcal{M}$ is non-recursive, this proves that the first-order theory of the full iteration of $\mathcal{A}$ is undecidable. $\square$

Next, we present a counterexample to the first-order version of Corollary 3.3 with the full iteration taking the place of the local iteration.

Proposition 3.5. *For every $n \in \mathbb{N}$ there exist structures $\mathcal{A}_n$ and $\mathcal{B}_n$ such that $\mathcal{A}_n \equiv_n \mathcal{B}_n$ but $(\mathcal{A}_n)^*_{\mathrm{fu}} \not\equiv_6 (\mathcal{B}_n)^*_{\mathrm{fu}}$.*

Proof. Let the signature σ contain a binary relation symbol E and two constant symbols a and b. For $n \in \mathbb{N}$, we will work with the structures $\mathcal{A}_n = (\mathbb{Z}, \mathrm{succ}, 0, 2^{n+1})$ that consists of a copy of the integers with successor relation. Note that in $\mathcal{A}_n$ there is a path of length 2^{n+1} from the first to the second constant. We will also consider the structures $\mathcal{B}_n = (\mathbb{Z}, \mathrm{succ}, 2^{n+1}, 0)$ that differ from $\mathcal{A}_n$ only in the order of the constants. Then the structures $\mathcal{A}_n$ and $\mathcal{B}_n$ cannot be distinguished by any first-order sentence of quantifier rank at most n, i.e., $\mathcal{A}_n \equiv_n \mathcal{B}_n$.

Now consider the following sentence φ in the language of the full iteration of $\mathcal{A}_n$ and $\mathcal{B}_n$:

$$\begin{aligned}\exists x \exists z : & x \in \widehat{a} \wedge z \in \widehat{b} \wedge x \preceq z \wedge \\ & \forall y (x \preceq y \prec z \rightarrow \exists y' \exists y'' : y \lessdot y' \preceq z \wedge y \lessdot y'' \wedge \mathrm{cl}(y'') \wedge \widehat{E}(y'', y'))\end{aligned}$$

where $x \lessdot z$ abbreviates $x \prec z \wedge \forall y (x \preceq y \prec z \rightarrow x = y)$. We show that $\mathcal{A}_n$ satisfies φ. Take $x = 0$ and $z = 0\,1\,2 \ldots 2^{n+1}$. Since the last letters of these words are a and b, resp., they belong to $\widehat{a}$ and $\widehat{b}$, resp. Any word y with $x \preceq y \prec z$ has the form $0\,1\, \ldots i$ for some $0 \leq i < 2^{n+1}$. Then $y' = y\,(i+1)$ and $y'' = y\,i$ ensure that φ indeed holds.

On the other hand, $\mathcal{B}^*_n$ does not satisfy φ: Suppose it would, i.e., there are $x = x'a$ and $y = xa_1a_2 \ldots a_k b$ satisfying the second line of the formula φ. Then $a\,a_1\,a_2\, \ldots\, a_k\, b$ is a path in $\mathcal{B}$ from a to b - but such a path does not exist. Since φ has quantifier rank 6, we obtain $(\mathcal{A}_n)^*_{\mathrm{fu}} \not\equiv_6 (\mathcal{B}_n)^*_{\mathrm{fu}}$. ☐

In Proposition 3.4 and 3.5, the clone-predicate is crucial. Without the clone-predicate a first-order version of Muchnik's theorem and its corollaries hold. The following theorem is a direct corollary of a more general result on so called factorized unfoldings from our earlier paper [14].

Theorem 3.6 ([14]). *Let σ be a finite relational signature.*

- *There exists a computable function* red *from the set of first-order formulas over* $(\sigma, \mathrm{cl}, \mathrm{son})$ *into the set of first-order formulas over σ such that, for any σ-structure $\mathcal{A}$, we have* $\mathcal{A} \models \mathrm{red}(\varphi)$ *if and only if $\mathcal{A}^*_{\mathrm{loc}} \models \varphi$.*
- *If the first-order theory of a structure $\mathcal{A}$ is decidable, then the first-order theory of its local iteration $\mathcal{A}^*_{\mathrm{loc}}$ is decidable as well.*
- *For any $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that, for any two σ-structures $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A} \equiv^{\mathrm{FO}}_n \mathcal{B}$, we have $\mathcal{A}^*_{\mathrm{loc}} \equiv^{\mathrm{FO}}_m \mathcal{B}^*_{\mathrm{loc}}$.*

4 The $\mathrm{MSO}^{\mathrm{chain}}$-theory of the basic iteration

In this section, we will show that statements analogous to Muchnik's Theorem 3.1 and Corollaries 3.2 and 3.3 hold for basic iterations and first-order logic. In doing so, it turns out that we can even consider the $\mathrm{MSO}^{\mathrm{chain}}$-theory of the basic iteration. Let us fix a base structure $\mathcal{A} = (A, (R)_{R\in\sigma})$ over a signature σ. In the rest of this section, we will use the abbreviation

$$t = \mathcal{A}^*_{\mathrm{ba}}.$$

4.1 Preliminaries

For $i, \ell \in \mathbb{N}$, let $\tau_{i,\ell}$ be the extension of the signature $(\sigma, \preceq)$ by i individual and ℓ chain constants. We write τ_i for $\tau_{i,0}$. Using Hintikka-formulas (see [7] for the definition and properties of these formulas) one can show that for any signature τ and $m \in \mathbb{N}$, there are only finitely many equivalence classes of $\equiv^{\mathrm{chain}}_m$. An upper bound $N_i(\ell, m)$ for the number of equivalence classes of $\equiv_m$ on formulas over the signature $\tau_{i,\ell}$ can be effectively computed, see [10].

For $u \in A^*$, let t_u denote the τ_1-structure $(uA^*, \sqsubseteq, (\bar{R})_{R\in\sigma}, u)$ where

- the relation $\sqsubseteq$ is the restriction of $\preceq$ to uA^*
- $\bar{R}$ is the restriction of $\widehat{R}$ to uA^+ (the restriction to uA^* could contain tuples of the form $(u, u, \ldots, u)$ which are excluded from $\bar{R}$)

For any $u, v \in A^*$, the mapping $f : t_u \to t_v$ with $f(ux) = vx$ is an isomorphism – this is the reason to consider $\bar{R}$ and not the restriction of $\widehat{R}$ to uA^*. Similarly, the τ_2-structure $t_{u,v} = (uA^* \setminus vA^+, \sqsubseteq, (\bar{R})_{R\in\sigma}, u, v)$ is defined for $u, v \in A^*$ with $u \preceq v$. Here, again, $\bar{R}$ is the restriction of $\widehat{R}$ to $uA^+ \setminus vA^+$.

Example 2.1 (continued). In the case of Example 2.1, t_u is just the subtree rooted at the node u. On the other hand, $t_{u,v}$ is obtained from t_u by deleting all descendents of v and marking the node v as a constant. Thus, we can think of $t_{u,v}$ as a tree with a marked leaf. These *special trees* are fundamental in the work of Gurevich & Shelah [11] and in Thomas' study of the monadic second-order chain theory of the complete binary tree [24]. The following constructions generalize those from [11, 24] to the more general context of basic iterations as considered here.

In the following, fix some $\ell \in \mathbb{N}$. We then define the operations of product and infinite product of $\tau_{i,\ell}$-structures: If $\mathcal{A}$ is a $\tau_{2,\ell}$-structure with second individual constant v and $\mathcal{B}$ a disjoint $\tau_{i,\ell}$-structure with first individual constant u, then their *product* $\mathcal{A} \cdot \mathcal{B}$ is a $\tau_{i,\ell}$-structure. It is obtained from the union of $\mathcal{A}$ and $\mathcal{B}$ by identifying v and u and erasing it from the list of constants. In other words, the constants in $\mathcal{A} \cdot \mathcal{B}$ are the first constant from $\mathcal{A}$ and all but the first constant from $\mathcal{B}$. Now let $\mathcal{A}_n$ be disjoint $\tau_{2,\ell}$-structures with individual constants u_n and v_n for $n \in \mathbb{N}$. Then the *infinite product* $\prod_{n\in\mathbb{N}} \mathcal{A}_n$ is a $\tau_{1,\ell}$-structure. It is obtained from the union of the structures $\mathcal{A}_n$ by identifying v_n and u_{n+1}

for any $n \in \mathbb{N}$. The only individual constant of this infinite product is u_0. If $\mathcal{A}_n \cong \mathcal{A}_{n+1}$ for all $n \in \mathbb{N}$, then we write simply $\mathcal{A}_0^\omega$ for the infinite product of the structures $\mathcal{A}_n$.

Standard applications of Ehrenfeucht-Fraïssé-games (see [7]) yield:

Proposition 4.1. *Let $m, \ell \in \mathbb{N}$, $\mathcal{A}_n, \mathcal{A}'_n$ be $\tau_{2,\ell}$-structures for $n \in \mathbb{N}$ and let $\mathcal{B}, \mathcal{B}'$ be some $\tau_{i,\ell}$-structures such that $\mathcal{A}_n \equiv_m^{\text{chain}} \mathcal{A}'_n$ for $n \in \mathbb{N}$ and $\mathcal{B} \equiv_m^{\text{chain}} \mathcal{B}'$. Then*

$$\mathcal{A}_0 \cdot \mathcal{B} \equiv_m^{\text{chain}} \mathcal{A}'_0 \cdot \mathcal{B}' \quad \textit{and} \quad \prod_{n\in\mathbb{N}} \mathcal{A}_n \equiv_m^{\text{chain}} \prod_{n\in\mathbb{N}} \mathcal{A}'_n.$$

4.2 Ultimately periodic chains and their combinatorics

For a word $u \in A^\infty$, let $\downarrow u \subseteq A^*$ denote the set of finite prefixes of u, and $\Downarrow u = \downarrow u \setminus \{u\}$. Similarly, $\downarrow C = \bigcup\{\downarrow u \mid u \in C\}$ for $C \subseteq A^*$. Finally, $u^{-1}C = \{v \in A^* \mid uv \in C\}$ for $u \in A^*$ and $C \subseteq A^*$.

A set $C \subseteq A^*$ is *ultimately p-periodic with offset q* if it can be written as $E \cup uv^*F$ with $E \subseteq \Downarrow u$ and $F \subseteq \Downarrow v$, $|u| = q$, and $|v| = p$. Since all elements of E are prefixes of u and all elements of F are prefixes of v, any ultimately periodic set is linearly ordered, i.e., a chain.

Note that $v = \varepsilon$ is possible, so finite chains are ultimately 0-periodic. Furthermore, if C is ultimately p-periodic with offset q, then it is also ultimately xp-periodic with offset $q+y$ for any $x, y \in \mathbb{N}$ with $x \geq 1$.

Lemma 4.2. *Let $u, v \in A^*$ and $E, F \subseteq A^*$ be finite such that $C = E \cup uv^*F$ is a chain. Then C is ultimately $|v|$-periodic.*

Proof. If $v = \varepsilon$, the set C is finite and is therefore ultimately 0-periodic. So let $v \neq \varepsilon$. Since $E \cup uv^*F$ is a chain, all words from E are prefixes of uv^ω and all words from F are prefixes of v^ω. Hence there is $n \in \mathbb{N}$ with $E \subseteq \Downarrow uv^n$ and $F \subseteq \Downarrow v^{n+1}$. Let $F_i = F \cap v^i A^* \cap \Downarrow v^{i+1}$. Then

$$\begin{aligned} uv^*F &= (uv^*F \cap \Downarrow uv^n) \cup (uv^*F \cap uv^n v^*) \\ &= (uv^*F \cap \Downarrow uv^n) \cup \left(uv^* \bigcup_{i=0}^{n} F_i \cap uv^n v^* \right) \\ &= (uv^*F \cap \Downarrow uv^n) \cup uv^n v^* \bigcup_{i=1}^{n} v^{-i} F_i. \end{aligned}$$

Hence, the chain C is indeed ultimately $|v|$-periodic. □

Lemma 4.3. *Let $C = E \cup uv^*F$ with $E \subseteq \Downarrow u$ and $F \subseteq \Downarrow v$. Let, furthermore, $x, y \in A^*$ with $|x| \geq |u|$ and $|y| = |v|$. Then the mapping $f_{x,y} : (t_{xy}, C \cap xyA^*) \to (t_{xyy}, C \cap xyyA^*)$ defined by $f_{x,y}(xyz) = xy^2z$ for $z \in A^*$ is an isomorphism.*

Proof. For $y = \varepsilon$, the statement is immediate since then $f_{x,y}$ is the identity on t_{xy}. So let $y \neq \varepsilon$. Then $f_{x,y}$ is clearly an isomorphism from t_{xy} onto t_{xy^2}. It remains to be shown that

$f_{x,y}$ preserves and reflects membership in C. So let $z \in A^*$ with $xyz \in C$. Since $|x| \geq |u|$ and $E \subseteq \Downarrow u$, we obtain $xyz \in uv^*F$. Hence there exist $w \in F$ and $n \in \mathbb{N}$ with $xyz = uv^n w$. Since $|u| \leq |x|$ and u is a prefix of xyz, it is also a prefix of x, i.e., there exists $x' \in A^*$ with $ux' = x$. Hence $x'yz = v^n w \preceq v^{n+1}$. Since $|y| = |v|$, this implies $x'y^2z = v^{n+1}w$ and therefore $xy^2z = ux'y^2z = uv^{n+1}w \in C$. The converse implication is shown similarly. $\square$

In the following, we will consider the structure t together with $\ell+1$ chains $C_1, \ldots, C_\ell, C$. To make the presentation more concise, write $\bar{C}$ for the ℓ-tuple $(C_1, \ldots, C_\ell)$. We will also meet structures t_u and $t_{u,v}$ together with the restriction of $\bar{C}, C$ to their domain. Again for simplicity, we write, e.g., (t_u, C) for $(t_u, C \cap uA^*)$.

Lemma 4.4. *Let $u, w \in A^*$ and, for $1 \leq i \leq \ell$, let $E_i \subseteq \Downarrow u$, $F_i \subseteq \Downarrow w$, and $C_i = E_i \cup uw^*F_i$. Let $a, b > 0$, $E \subseteq \Downarrow u$, $F \subseteq \Downarrow w^{a+b}$, and $C = E \cup u(w^{a+b})^*F$. Then there is an $a \cdot |w|$-periodic chain D such that*

$$(t, \bar{C}, D) \cong (t_{\varepsilon,uw^a}, \bar{C}, C) \cdot \prod_{n>0} (t_{uw^{n(a+b)},uw^{n(a+b)+a}}, \bar{C}, C) \ .$$

Proof. The mapping $f : uA^* \to uwA^* : ux \mapsto uwx$ is an isomorphism from t_u onto t_{uw}. Since $C_i \cap uA^* = uw^*F_i$, it is even an isomorphism from $(t_u, \bar{C})$ onto $(t_{uw}, \bar{C})$. Now consider its power $g = f^{a+b}$. Since $C \cap uA^* = u(w^{a+b})^*F$, the mapping g proves $(t_u, \bar{C}, C) \cong (t_{uw^{a+b}}, \bar{C}, C)$. Since $g(uw^a) = uw^{2a+b}$, we even have $(t_{u,uw^a}, \bar{C}, C) \cong (t_{uw^{a+b},uw^{2a+b}}, \bar{C}, C)$. For $n > 0$, the power g^n proves $(t_{u,uw^a}, \bar{C}, C) \cong (t_{uw^{n(a+b)},uw^{n(a+b)+a}}, \bar{C}, C)$. Hence we have

$$(t_{\varepsilon,uw^a}, \bar{C}, C) \cdot \prod_{n>0} (t_{uw^{n(a+b)},uw^{n(a+b)+a}}, \bar{C}, C) \cong (t_{\varepsilon,uw^a}, \bar{C}, C) \cdot (t_{uw^{a+b},uw^{2a+b}}, \bar{C}, C)^\omega \ .$$

Let $E' = C \cap {\downarrow} uw^a$ and $F' = F \setminus w^a A^+ \subseteq {\downarrow} w^a$ since $F \subseteq {\downarrow} w^\omega$. Now set $D = E' \cup uw^a\,(w^a)^* F'$. Since $E' \subseteq {\downarrow} uw^a$ and $F' \subseteq {\downarrow} w^a$, the set D is a chain. By Lemma 4.2, D is ultimately $a \cdot |w|$-periodic.

Then $(t_{\varepsilon,uw^a}, \bar{C}, C) = (t_{\varepsilon,uw^a}, \bar{C}, E') = (t_{\varepsilon,uw^a}, \bar{C}, D)$ since $C \subseteq {\downarrow} uw^\omega$. Furthermore,

$$\begin{aligned} f^b(uw^aF') &= uw^{a+b}(F \setminus w^aA^+) \\ &= uw^{a+b}F \setminus uw^{2a+b}A^+ \\ &= C \cap uw^{a+b}A^* \setminus uw^{2a+b}A^+ \end{aligned}$$

implies $(t_{uw^{a+b},uw^{2a+b}}, \bar{C}, C) \cong (t_{uw^a,uw^{2a}}, \bar{C}, uw^aF') = (t_{uw^a,uw^{2a}}, \bar{C}, D)$.

Finally, $f^{(n-1)a}$ is an isomorphism from $(t_{uw^a,uw^{2a}}, \bar{C}, D)$ onto $(t_{uw^{na},uw^{(n+1)a}}, \bar{C}, D)$ for any $n > 0$. Hence we have

$$\begin{aligned} (t_{\varepsilon,uw^a}, \bar{C}, C) \cdot \prod_{n>0} (t_{uw^{n(a+b)},uw^{n(a+b)+a}}, \bar{C}, C) &\cong (t_{\varepsilon,uw^a}, \bar{C}, C) \cdot (t_{uw^{a+b},uw^{2a+b}}, \bar{C}, C)^\omega \\ &\cong (t_{\varepsilon,uw^a}, \bar{C}, D) \cdot (t_{uw^a,uw^{2a}}, \bar{C}, D)^\omega \\ &\cong (t_{\varepsilon,uw^a}, \bar{C}, D) \cdot \prod_{n>0} (t_{uw^{na},uw^{(n+1)a}}, \bar{C}, D) \\ &= (t, \bar{C}, D) \ . \end{aligned}$$

$\square$

4.3 Shortening ultimately periodic chains

Suppose we are in the realm of Example 2.1 and let $C_i \subseteq A^*$ be regular and let $u_i \in A^*$. Then, as a corollary from Rabin's tree theorem, for any $C \subseteq A^*$, there exists a *regular* set $D \subseteq A^*$ that satisfies the same MSO-formulas of quantifier depth m in the structure $(t, C_1, \ldots, C_\ell, u_1, \ldots, u_n)$ as C does. In this section, we want to prove a similar result for basic iterations. Then, "regular set" is replaced by "ultimately periodic chain". In addition, we want to bound the offset and the period of the chain D.

We start showing that some ultimately periodic chain D exists that can take the role of C (Proposition 4.7). Proposition 4.12 will allow to bound the period of D (thereby possibly enlarging the offset). Finally, Lemma 4.8 bounds the size of the offset (without changing the period). Finally, Theorem 4.13 shows that we succeeded in our desire to find an equivalent ultimately periodic chain D of small period and offset.

Existence of ultimately periodic chains

Lemma 4.5. *Let $m \in \mathbb{N}$ and let $C \subseteq A^*$ be any chain. Then there exists an ultimately periodic chain D such that $(t, C) \equiv_m^{\text{chain}} (t, D)$.*

Proof. Assume C not to be ultimately periodic (and therefore infinite) and let $\alpha \in A^\omega$ with $C \subseteq \downarrow\alpha$. By Ramsey's theorem [19] (see [16] for this application), there is a strictly increasing sequence $u_1 \prec u_2 \prec u_3 \cdots$ of non-empty prefixes of α such that, for any $1 \le i < j$, we have

$$(t_{u_1,u_2}, C) \equiv_m^{\text{chain}} (t_{u_i,u_j}, C) \ .$$

This implies

$$\begin{aligned}(t, C) &= (t_{\varepsilon,u_1}, C) \cdot \prod_{n>0} (t_{u_n,u_{n+1}}, C)\\ &\equiv_m^{\text{chain}} (t_{\varepsilon,u_1}, C) \cdot (t_{u_1,u_2}, C)^\omega\end{aligned}$$

Now let $v \in A^+$ with $u_1 v = u_2$ and consider $E = C \cap \downarrow u_1$, $F = u_1^{-1}(C \cap \downarrow u_2) = u_1^{-1} C \cap \downarrow v$, and $D = E \cup u_1 v^* F$. Then we can continue:

$$\begin{aligned}(t_{\varepsilon,u_1}, C) \cdot (t_{u_1,u_2}, C)^\omega &= (t_{\varepsilon,u_1}, E) \cdot (t_{u_1,u_2}, F)^\omega\\ &\cong (t_{\varepsilon,u_1}, D) \cdot \prod_{n\ge 0} (t_{u_1 v^n, u_1 v^{n+1}}, D) = (t, D).\end{aligned}$$

Since $E \subseteq \downarrow u_1 v^\omega$ and $F \subseteq \downarrow v^\omega$, the set D is linearly ordered and therefore ultimately periodic by Lemma 4.2. ☐

Lemma 4.6. *Let $m \in \mathbb{N}$, $\ell > 0$, $C_1, \ldots, C_\ell \subseteq A^*$ be ultimately periodic chains with empty offset and let $C \subseteq A^*$ be any chain such that $\downarrow C = \downarrow C_i$ for $1 \le i \le \ell$. Then there exists an ultimately periodic chain D such that $(t, \bar{C}, C) \equiv_m^{\text{chain}} (t, \bar{C}, D)$.*

Proof. Assume C not to be ultimately periodic (and therefore infinite). There are $\tilde{v} \in A^*$ and $\emptyset \neq F_i \subseteq \Downarrow v$ with $C_i = \tilde{v}^* F_i$ for all $1 \leq i \leq \ell$ such that any element of C is a prefix of $\tilde{v}^\omega$. By Ramsey's theorem [19] (see [16] for this application), there are natural numbers $1 < n_1 < n_2 \cdots$ such that, for any $1 \leq i < j$, we have

$$(t_{u_1,u_2}, \bar{C}, C) \equiv_m^{\text{chain}} (t_{u_i,u_j}, \bar{C}, C)$$

where $u_i = \tilde{v}^{n_i}$. Now let $n = n_2 - n_1$ and $v = \tilde{v}^n$, i.e., $u_1 v = u_2$. Since $n_1 > 0$, the mapping $f_{\tilde{v}^{n_1-1},\tilde{v}}$ from Lemma 4.3 is an isomorphism from $(t_{u_1}, \bar{C})$ to $(t_{u_1\tilde{v}}, \bar{C})$. Its n^{th} power maps $u_1 v^i$ to $u_1 v^i \tilde{v}^n = u_1 v^{i+1}$ for all $i \in \mathbb{N}$. Hence,

$$(t_{u_1,u_2}, \bar{C}) = (t_{u_1,u_1v}, \bar{C}) \cong (t_{u_1v^i,u_1v^{i+1}}, \bar{C})$$

for any $i \in \mathbb{N}$. Furthermore, let

$$F = u_1^{-1}(C \cap u_1 A^* \setminus u_2 A^+) \text{ and } D = (C \cap \Downarrow u_1) \cup u_1 v^* F$$

For $s \in F$, we get $u_1 s \in C \setminus u_2 A^+$. Since any element of C as well as u_2 are prefixes of v^ω, we get $u_1 s \preceq u_2 = u_1 v$ and therefore $s \preceq v$. Hence D is an ultimately periodic chain by Lemma 4.2. Now let $i \geq 0$. Then the restriction of D to $t_{u_1v^i,u_1v^{i+1}}$ equals $u_1 v^i F$. Thus, we get

$$(t_{u_i,u_{i+1}}, \bar{C}, C) \equiv_m^{\text{chain}} (t_{u_1,u_2}, \bar{C}, C) = (t_{u_1,u_1v}, \bar{C}, D) \cong (t_{u_1v^i,u_1v^{i+1}}, \bar{C}, D)\ .$$

From Proposition 4.1, we get

$$\begin{aligned}
(t, \bar{C}, C) &= (t_{\varepsilon,u_1}, \bar{C}, C) \cdot \prod_{i \in \mathbb{N}} (t_{u_i,u_{i+1}}, \bar{C}, C) \\
&\equiv_m^{\text{chain}} (t_{\varepsilon,u_1}, \bar{C}, C) \cdot \prod_{i \in \mathbb{N}} (t_{u_1v^i,u_1v^{i+1}}, \bar{C}, D) \\
&= (t, \bar{C}, D).
\end{aligned}$$

□

Proposition 4.7. *Let $m \in \mathbb{N}$, $C_1, \ldots, C_\ell \subseteq A^*$ be ultimately periodic chains and let $C \subseteq A^*$ be any chain. Then there exists an ultimately periodic chain D such that $(t, \bar{C}, C) \equiv_m^{\text{chain}} (t, \bar{C}, D)$*

Proof. Assume C not to be ultimately periodic (and therefore infinite). Renumbering the chains $C_1, \ldots, C_\ell$ if necessary, we can assume $\downarrow C_i = \downarrow C$ for $1 \leq i \leq k$ and $\downarrow C_i \neq \downarrow C$ for $k < i \leq \ell$. Then there exists $u \in \downarrow C$ such that $C_i \cap uA^*$ is periodic for all $1 \leq i \leq k$ and empty for $k < i \leq \ell$.

If $k = 0$, Lemma 4.5 yields an ultimately periodic chain $D_2 \subseteq uA^*$ with $(t_u, C) \equiv_m^{\text{chain}} (t_u, D_2)$ which implies $(t_u, C_1, \ldots, C_\ell, C) \equiv_m (t_u, C_1, \ldots, C_\ell, D_2)$ since the restriction of C_i to t_u is empty for all i.

Now suppose $k > 0$. Then $(t_u, C_1, \ldots, C_k, C) \cong (t, u^{-1}C_1, \ldots, u^{-1}C_k, u^{-1}C)$ satisfies the assumptions of Lemma 4.6. Hence also in this case there exists an ultimately periodic chain $D_2 \subseteq uA^*$ with

$$(t_u, C_1, \ldots, C_k, C) \equiv_m^{\text{chain}} (t_u, C_1, \ldots, C_k, D_2)$$

which implies

$$(t_u, C_1, \ldots, C_\ell, C) \equiv_m^{\text{chain}} (t_u, C_1, \ldots, C_\ell, D_2)$$

since the restriction of C_i to t_u is empty for $k < i \leq \ell$. Since $D_2 \subseteq uA^*$ and $C \setminus uA^+ \subseteq {\downarrow}u$, the set $D = (C \setminus uA^+) \cup D_2$ is an ultimately periodic chain. Now Proposition 4.1 implies

$$\begin{aligned}(t, \bar{C}, C) &= (t_{\varepsilon,u}, \bar{C}, C) \cdot (t_u, \bar{C}, C) \\ &\equiv_m^{\text{chain}} (t_{\varepsilon,u}, \bar{C}, C) \cdot (t_u, \bar{C}, D_2) \\ &= (t, \bar{C}, D).\end{aligned}$$

□

Ultimately periodic chains with small offset

Lemma 4.8. *Let $m > 0$, $C_i \subseteq A^*$ be an ultimately p_i-periodic chain with offset q_i for $1 \leq i \leq \ell$ and let $C \subseteq A^*$ be an ultimately p-periodic chain with offset $q > \max(q_1, \ldots, q_\ell) + \operatorname{lcm}(p_1, \ldots, p_\ell) \cdot (N_1(\ell+1, m)+2)$. Then there exists an ultimately p-periodic chain D with offset $q - \operatorname{lcm}(p_1, \ldots, p_\ell)$ such that $(t, \bar{C}, C) \equiv_m^{\text{chain}} (t, \bar{C}, D)$.*

Proof. Let $C = E \cup uv^*F$ with $E \subseteq {\Downarrow}u$, $F \subseteq {\Downarrow}v$, $|u| = q$ and $|v| = p$. Let u' be the prefix of u of length $\max(q_1, \ldots, q_\ell)$. Since the length of the corresponding suffix exceeds $\operatorname{lcm}(p_1, \ldots, p_\ell) \cdot (N_1(\ell+1, m) + 2)$, it can be factorized into xyz such that $|x|, |y| > 0$ are multiples of $\operatorname{lcm}(p_1, \ldots, p_\ell)$, $z \neq \varepsilon$, and $(t_{u'x}, \bar{C}, C) \equiv_m^{\text{chain}} (t_{u'xy}, \bar{C}, C)$. We have $u = u'xyz$.

Define $f : t_{u'x} \to t_{u'xy}$ by $f(u'xw) = u'xyw$. This mapping is clearly an isomorphism from $t_{u'x}$ to $t_{u'xy}$ – we show that it preserves and reflects membership in C_i as well:

So let $1 \leq i \leq \ell$. If $t_{u'x}$ does not contain any element of C_i, then no such element can belong to $t_{u'xy}$, hence f preserves and reflects membership in C_i. Now suppose $t_{u'x}$ contains some element of C_i, i.e., $(t_{u'x}, \bar{C}) \models \exists x : x \in C_i$. Since the quantifier rank of this formula is $1 \leq m$ and $(t_{u'x}, \bar{C}, C) \equiv_m^{\text{chain}} (t_{u'xy}, \bar{C}, C)$, there is some element in $C_i \cap u'xyA^*$. Since $|u'| \geq q_i$, we can write $C_i = E' \cup u'v'^*F'$ with $|v'| = p_i$, $E' \subseteq {\Downarrow}u'$ and $F' \subseteq {\Downarrow}v'$. Hence $C_i \cap u'xyA^* \neq \emptyset$ implies $v'^*F' \cap xyA^* \neq \emptyset$. Since the lengths of x and of y are nonzero multiples of $p_i = |v'|$, we obtain $x, y \in v'^+$. Hence f is a power of the isomorphism from Lemma 4.3. This proves that $f(w) \in C_i$ if and only if $w \in C_i$ for any $w \in u'xA^*$ and any $1 \leq i \leq \ell$. In other words, $f : (t_{u'x}, \bar{C}) \to (t_{u'xy}, \bar{C})$ is an isomorphism.

Now consider the function $g : t \to t_{\varepsilon,u'x} \cdot t_{u'xy}$ defined by

$$g(w) = \begin{cases} f(w) & \text{if } w \in u'xA^* \\ w & \text{otherwise.} \end{cases}$$

Then g is clearly an isomorphism. It equals the union of the identity on $t_{\varepsilon,u'x}$ and the isomorphism f. Hence it reflects and preserves membership in C_i for $1 \le i \le \ell$. Hence it is an isomorphism from $(t, \bar{C})$ to $(t_{\varepsilon,u'x} \cdot t_{u'xy}, \bar{C})$.

Now let $C' = (C \setminus u'xA^+) \cup (C \cap u'xyA^*)$ be the restriction of C to $t_{\varepsilon,u'x} \cdot t_{u'xy}$ and let D be the preimage of C' under g. Then we have

$$\begin{aligned}(t, \bar{C}, C) &= (t_{\varepsilon,u'x}, \bar{C}, C) \cdot (t_{u'x}, \bar{C}, C)\\ &\equiv_m^{\text{chain}} (t_{\varepsilon,u'x}, \bar{C}, C) \cdot (t_{u'xy}, \bar{C}, C)\\ &\cong (t, \bar{C}, D).\end{aligned}$$

It remains to be shown that D is an ultimately p-periodic chain with offset $q - \mathrm{lcm}(p_1, \dots, p_\ell)$. Since $u'xy \preceq u$ and $E = C \setminus uA^*$ we have $C \setminus u'xA^+ = E \setminus u'xA^+$. Thus,

$$\begin{aligned}C' &= C \setminus u'xA^+ \cup (C \cap u'xyA^*)\\ &= E \setminus u'xA^+ \cup (C \cap u'xyA^*) \setminus uA^* \cup (C \cap u'xyA^* \cap uA^*)\\ &= E \setminus u'xA^+ \cup (C \setminus uA^* \cap u'xyA^*) \cup (C \cap uA^*)\\ &= E \setminus u'xA^+ \cup (E \cap u'xyA^*) \cup uv^*F,\end{aligned}$$

where the last equality follows from $C \setminus uA^* = E$ and $C \cap uA^* = uv^*F$. Note that

$$\begin{aligned}g^{-1}(E \setminus u'xA^+) &\subseteq g^{-1}(\downarrow u'x) = \downarrow u'x \subseteq \Downarrow u'xz\\ g^{-1}(E \cap u'xyA^*) &\subseteq g^{-1}(\Downarrow u \cap u'xyA^*) \subseteq \Downarrow u'xz\\ g^{-1}(uv^*F) &= u'xzv^*F\ .\end{aligned}$$

Hence $D = g^{-1}(C')$ is the union of a subset of $\Downarrow u'xz$ and $(u'xz)v^*F$, i.e., it is ultimately p-periodic with offset $|u'xz| = |u| - |y| \le q - \mathrm{lcm}(p_1, \dots, p_\ell)$. $\square$

The above arguments can also be used to prove the following lemma.

Lemma 4.9. *Let $m > 0$, $C_i \subseteq A^*$ be an ultimately p_i-periodic chain with offset q_i for $1 \le i \le k$ and let $u_i \in A^*$ be words with $|u_i| = q_i$ for $k < i \le \ell$ and let $u \in A^*$ with $|u| \ge \max(q_1, \dots, q_\ell) + \mathrm{lcm}(p_1, \dots, p_k) \cdot (N_1(\ell+1, m) + 2)$. Then there exists a word $v \in A^*$ with $|v| \le q - \mathrm{lcm}(p_1, \dots, p_k)$ and $(t, \bar{C}, \bar{u}, u) \equiv_m^{\text{chain}} (t, \bar{C}, \bar{u}, v)$.*

Ultimately periodic chains with small period

Lemma 4.10. *Let $m, \ell \in \mathbb{N}$, $v, w \in A^*$ and $F, F_1, \dots, F_\ell \subseteq A^*$ such that*

(a) $p = |v|$ is a multiple of $|w|$
(b) $p > 2|w|(N_2(\ell + 1, m) + 2)$
(c) $\emptyset \ne F_i \subseteq \Downarrow w$ and $C_i = w^*F_i$ for $1 \le i \le \ell$
(d) $\emptyset \ne F \subseteq \Downarrow v$ and $C = v^*F$
(e) $v^\omega = w^\omega$

Then there exists an ultimately p'-periodic chain $D \subseteq A^*$ (with offset q') such that
(A) $(t, \bar{C}, C) \equiv_m^{\text{chain}} (t, \bar{C}, D)$
(B) p' *is a multiple of* $|w|$
(C) $p' < p$

Proof. Let $v = v_1 v_2$ with $|v_2| = |w|(N_2(\ell+1, m) + 2)$. Then the word v_2 can be factorized as $v_2 = x_1' x_2 x_3$ such that $|x_1'|$ and $|x_2|$ are multiples of $|w|$, $x_2 \neq \varepsilon$, and

$$(t_{v_1 x_1', v v_1 x_1'}, \bar{C}, C) \equiv_m^{\text{chain}} (t_{v_1 x_1' x_2, v v_1 x_1'}, \bar{C}, C). \tag{1}$$

Since the lengths of x_1', x_2, and $v_2 = x_1' x_2 x_3$ are multiples of the length of w, so is the length of x_3. Similarly, the length of w divides the lengths of v_2 and of $v = v_1 v_2$ and therefore that of v_1 and of $x_1 = v_1 x_1' \neq \varepsilon$. Furthermore, $|x_1| = |v_1| + |x_1'| \geq |v_1| > |v_2|$ because of (b). Hence $|x_2| \leq |v_2| < |x_1|$. From (1) we get

$$(t_{x_1, v x_1}, \bar{C}, C) \equiv_m^{\text{chain}} (t_{x_1 x_2, v x_1}, \bar{C}, C).$$

Since $v^\omega = w^\omega$ and since $|v|$ is a positive multiple of $|w|$, we get $v \in w^+$. Hence, also $x_1, x_2 \in w^+$ and $x_3 \in w^*$.

Let $n > 0$. Then $v^n x_1 = x_1 (x_2 x_3 x_1)^n$. Furthermore, let $g : t_{x_1} \to t_{v^n x_1}$ be given by $g(x_1 z) = v^n x_1 z$. Since x_1 and $x_2 x_3 x_1$ are nonempty powers of w, the mapping g is a power of the mapping f from Lemma 4.3 (with $u = \varepsilon$, $v = y = w$, $xy = x_1$, $E = \emptyset$ and $F = F_i$ for any $1 \leq i \leq \ell$). Hence it is an isomorphism from $(t_{x_1}, \bar{C})$ to $(t_{v^n x_1}, \bar{C})$. Since $C = v^* F$, this mapping preserves and reflects membership in C as well, i.e., it is even an isomorphism from $(t_{x_1}, \bar{C}, C)$ to $(t_{v^n x_1}, \bar{C}, C)$. Since $g(v x_1) = v^{n+1} x_1$, we obtain

$$(t_{x_1, v x_1}, \bar{C}, C) \cong (t_{v^n x_1, v^{n+1} x_1}, \bar{C}, C).$$

Using the same arguments, we also get

$$(t_{x_1 x_2, v x_1}, \bar{C}, C) \cong (t_{v^n x_1 x_2, v^{n+1} x_1}, \bar{C}, C).$$

Hence we have

$$\begin{aligned}
(t, \bar{C}, C) &= (t_{\varepsilon, v x_1}, \bar{C}, C) \cdot \prod_{n>0} (t_{v^n x_1, v^{n+1} x_1}, \bar{C}, C) \\
&\cong (t_{\varepsilon, v x_1}, \bar{C}, C) \cdot (t_{x_1, v x_1}, \bar{C}, C)^\omega \\
&\equiv_m^{\text{chain}} (t_{\varepsilon, v x_1}, \bar{C}, C) \cdot (t_{x_1 x_2, v x_1}, \bar{C}, C)^\omega \\
&\cong (t_{\varepsilon, v x_1}, \bar{C}, C) \cdot \prod_{n>0} (t_{v^n x_1 x_2, v^{n+1} x_1}, \bar{C}, C) \ .
\end{aligned}$$

Now let $u = x_1 x_2$ and $a, b \in \mathbb{N}$ with $w^a = x_3 x_1$ and $w^b = x_2$. Then $v^n x_1 x_2 = x_1 (x_2 x_3 x_1)^n x_2 = x_1 x_2 (x_3 x_1 x_2)^n = u w^{(a+b)n}$ and $v^{n+1} x_1 = (x_1 x_2 x_3)^{n+1} x_1 = x_1 x_2 (x_3 x_1 x_2)^n x_3 x_1 = u w^{(a+b)n+a}$. Thus, by Lemma 4.4, there exists an ultimately $a \cdot |w|$-periodic chain D with

$$(t, \bar{C}, D) \cong (t_{\varepsilon, v x_1}, \bar{C}, C) \cdot \prod_{n>0} (t_{v^n x_1 x_2, v^{n+1} x_1}, \bar{C}, C) \ ,$$

i.e., such that $(t, \bar{C}, C) \equiv_m^{\text{chain}} (t, \bar{C}, D)$. With $p' = a \cdot |w|$, the result follows. $\square$

Lemma 4.11. *Let $m \in \mathbb{N}$, $C_i \subseteq A^*$ be an ultimately p_i-periodic chain with offset q_i for $1 \le i \le \ell$ and let $C \subseteq A^*$ be an ultimately p-periodic chain with offset q. Suppose furthermore*

(a) p is a multiple of p_i for all $1 \le i \le \ell$ and
(b) $p > 2\operatorname{lcm}(p_1, \ldots, p_\ell)(N_2(\ell+1, m) + 2)$

Then there exists an ultimately p'-periodic chain $D \subseteq A^*$ such that

(A) $(t, \bar{C}, C) \equiv_m^{\text{chain}} (t, \bar{C}, D)$
(B) p' is a multiple of p_i for all $1 \le i \le \ell$
(C) $p' < p$

Proof. If C is finite, then it is ultimately 0-periodic, so setting $D = C$ and $p' = 0$ proves the lemma. So let $C = E \cup uv^*F$ be infinite with $E \subseteq \Downarrow u$ and $F \subseteq \Downarrow v$ such that $p = |v|$ and $q = |u|$. Since p is a multiple of any of the values p_i, there is $n \in \mathbb{N}$ with $p = n \cdot \operatorname{lcm}(p_1, \ldots, p_\ell)$.

Let $\tilde{u}$ be the prefix of uv^ω of length $\max\{q_1, q_2, \ldots, q_\ell, q\} + p$ and choose $\tilde{v}$ such that $\tilde{u}\tilde{v}^\omega = uv^\omega$ and $|\tilde{v}| = |v|$. Then there exist $\tilde{E} \subseteq \Downarrow\tilde{u}$ and $\tilde{F} \subseteq \Downarrow\tilde{v}$ with $C = \tilde{E} \cup \tilde{u}\tilde{v}^*\tilde{F}$.

Now let $1 \le i \le \ell$. Then let, similarly to above, $\tilde{u}_i$ be the prefix of $u_i v_i^\omega$ of length $|\tilde{u}| - p \ge q_i$ and chose $\tilde{v}_i$ such that $\tilde{u}_i\tilde{v}_i^\omega = u_i v_i^\omega$ and $|\tilde{v}_i| = \operatorname{lcm}(p_1, \ldots, p_\ell)$. Then there are $\tilde{E}_i \subseteq \Downarrow\tilde{u}_i$ and $\tilde{F}_i \subseteq \Downarrow\tilde{v}_i$ such that $C_i = \tilde{E}_i \cup \tilde{u}_i\tilde{v}_i^*\tilde{F}_i$.

We consider the chains

$$\begin{aligned} C_i' &= \tilde{u}^{-1}C_i = \{x \in A^* \mid \tilde{u}x \in C_i\} \text{ and} \\ C' &= \tilde{u}^{-1}C = \{x \in A^* \mid \tilde{u}x \in C\} \end{aligned}$$

and the values $\tilde{q} = |\tilde{u}|$, $\tilde{p} = |\tilde{v}|$, $\tilde{q}_i = |\tilde{u}_i|$, and $\tilde{p}_i = |\tilde{v}_i|$. Renumbering the chains if necessary, we can assume $C_1', \ldots, C_k' \ne \emptyset$ and $C_{k+1}' = \cdots = C_\ell' = \emptyset$.

Let $1 \le i \le k$. Then $\tilde{u}^{-1}C_i \ne \emptyset$ implies that $\tilde{u}$ is a prefix of $\tilde{u}_i\tilde{v}_i^\omega$. Furthermore, $|\tilde{u}| = |\tilde{u}_i| + p = |\tilde{u}_i| + n\operatorname{lcm}(p_1, \ldots, p_\ell) = |\tilde{u}_i\tilde{v}_i^n|$ ensures $\tilde{u} = \tilde{u}_i\tilde{v}_i^n$. Hence $\tilde{u}_i$ is a prefix of $\tilde{u}$ of length $|\tilde{u}| - p$. Since this holds for all $1 \le i, j \le k$, we obtain $\tilde{u}_i = \tilde{u}_j$ and $\tilde{v}_i = \tilde{v}_j =: w$.

To apply Lemma 4.10, we verify the following:

(a') $\tilde{p} = |\tilde{v}| = |v| = p$ is a multiple of $\operatorname{lcm}(p_1, \ldots, p_\ell) = |\tilde{v}_i| = \tilde{p}_i = |w|$ for all $1 \le i \le k$.
(b') $\tilde{p} = p > 2\operatorname{lcm}(p_1, \ldots, p_\ell) \cdot (N_2(\ell+1, m) + 2) = 2\operatorname{lcm}(\tilde{p}_1, \ldots, \tilde{p}_\ell) \cdot (N_2(\ell+1, m) + 2) = 2|w| \cdot (N_2(\ell+1, m) + 2)$.
(c') Since $C_i' \ne \emptyset$, the set C_i contains some word of length at least $|\tilde{u}| > |\tilde{u}_i|$. Hence $\emptyset \ne \tilde{F}_i \subseteq \Downarrow\tilde{v}_i = \Downarrow w$. Furthermore, $C_i' = \tilde{u}^{-1}C_i = (\tilde{u}_i\tilde{v}_i^n)^{-1}(\tilde{E}_i \cup \tilde{u}_i\tilde{v}_i^*\tilde{F}_i) = w^*\tilde{F}_i$ since $\tilde{E}_i \subseteq \Downarrow\tilde{u}_i$ and $\tilde{v}_i = w$.
(d') Since C is infinite, we have $\emptyset \ne \tilde{F} \subseteq \Downarrow\tilde{v}$. Furthermore, $C' = \tilde{u}^{-1}C = \tilde{u}^{-1}(\tilde{E} \cup \tilde{u}\tilde{v}^*\tilde{F}) = \tilde{v}^*\tilde{F}$ since $\tilde{E} \subseteq \Downarrow\tilde{u}$.
(e') Recall that $\tilde{u} = \tilde{u}_i\tilde{v}_i^n = \tilde{u}_i w^n$, i.e., w^n is a suffix of $\tilde{u}$ of length $n\operatorname{lcm}(p_1, \ldots, p_\ell) = p$. We also have $uv^\omega = \tilde{u}\tilde{v}^\omega$, $|\tilde{u}| \ge |uv|$, and $|\tilde{v}| = |v| = p$. Hence $\tilde{v}$ is also a suffix of $\tilde{u}$ of length p. But this implies $\tilde{v} = w^n$ and therefore $\tilde{v}^\omega = w^\omega$.

Hence, by Lemma 4.10, there is an ultimately p'-periodic chain D' such that

(A) $(t, C'_1, \ldots, C'_k, C) \equiv_m^{\text{chain}} (t, C'_1, \ldots, C'_k, D)$
(B) p' is a multiple of $|w| = \operatorname{lcm}(p_1, \ldots, p_\ell)$
(C) $p' < \tilde{p} = p$.

Since $C'_{k+1}, \ldots, C'_\ell = \emptyset$, we can sharpen (A) to $(t, C'_1, \ldots, C'_\ell, C) \equiv_m^{\text{chain}} (t, C'_1, \ldots, C'_\ell, D)$. Hence we have:

$$\begin{aligned}(t, \bar{C}, C) &= (t_{\varepsilon,\tilde{u}}, \bar{C}, C) \cdot (t_{\tilde{u}}, \bar{C}, C)\\ &\cong (t_{\varepsilon,\tilde{u}}, \bar{C}, C) \cdot (t, \bar{C}', C')\\ &\equiv_m^{\text{chain}} (t_{\varepsilon,\tilde{u}}, \bar{C}, C) \cdot (t, \bar{C}', D')\\ &\cong (t, \bar{C}, D)\end{aligned}$$

with $D = (C \cap \Downarrow\tilde{u}) \cup \tilde{u}D'$. Since the period length of D' equals p', the chain D is ultimately p'-periodic. □

Proposition 4.12. *Let $m \in \mathbb{N}$, $C_i \subseteq A^*$ be an ultimately p_i-periodic chain with offset q_i for $1 \le i \le \ell$ and let $C \subseteq A^*$ be an ultimately p-periodic chain. Then there exists an ultimately p'-periodic chain D such that $(t, \bar{C}, C) \equiv_m^{\text{chain}} (t, \bar{C}, D)$ and $p' \le 2\operatorname{lcm}(p_1, \ldots, p_\ell) \cdot (N_2(\ell + 1, m) + 2)$.*

Proof. Suppose $p'_0 := p > 2\operatorname{lcm}(p_1, \ldots, p_\ell) \cdot (N_2(\ell+1, m)+2)$, where w.l.o.g. p'_0 is a multiple of p_i for $1 \le i \le \ell$. Iterative applications of Lemma 4.11 yield ultimately p'_i-periodic chains satisfying (A) and (B) from Lemma 4.11, and $p'_0 > p'_1 > \cdots p'_n$. This process terminates once $p'_n \le 2\operatorname{lcm}(p_1, \ldots, p_\ell) \cdot (N_2(\ell + 1, m) + 2)$. □

Theorem 4.13. *Let $m \in \mathbb{N}$, $C_i \subseteq A^*$ be ultimately p_i-periodic chains with offset q_i for $1 \le i \le \ell$ and let $C \subseteq A^*$ be a chain. Then there exists an ultimately p'-periodic chain D with offset $q' \le \max(q_1, \ldots, q_\ell) + \operatorname{lcm}(p_1, \ldots, p_\ell) \cdot (N_1(\ell+1, m)+2)$ and $p' \le 2\operatorname{lcm}(p_1, \ldots, p_\ell) \cdot (N_2(\ell + 1, m) + 2)$ such that $(t, \bar{C}, C) \equiv_m^{\text{chain}} (t, \bar{C}, D)$.*

Proof. By Prop. 4.7, we can assume C to be ultimately periodic. Prop. 4.12 allows to bound its period by $2\operatorname{lcm}(p_1, \ldots, p_\ell) \cdot (N_2(\ell+1, m)+2)$. Although this increases the offset, an iterative application of Lemma 4.8 shortens the offset again to a value of at most $\max(q_1, \ldots, q_\ell) + \operatorname{lcm}(p_1, \ldots, p_\ell) \cdot (N_1(\ell + 1, m) + 2)$ without increasing the period. □

4.4 Bounded MSO$^{\text{chain}}$-theory

Suppose we are in the realm of Example 2.1. Then Rabin [17] has shown that one can restrict set-quantification in MSO-formulas to *regular* sets without changing the truth of the sentence. In this section, we show the corresponding result for basic iterations and MSO$^{\text{chain}}$-sentences where "regular set" is replaced by "ultimately periodic chain". In addition, we can bound the size of the period and offset of these chains.

For an MSO$^{\text{chain}}$-formula ψ and $q, p \in \mathbb{N}$ let $\exists C \le (q, p) : \psi$ stand for "there exists an ultimately p'-periodic chain C with offset at most q and $p' \le p$ such that ψ holds". Similarly, $\exists x \le q : \psi$ means "there exists a word x of length at most q such that ψ holds". The formulas $\forall C \le (q, p) : \psi$ and $\forall x \le q : \psi$ should be understood similarly.

Definition 4.14. *A* bounded $\mathrm{MSO}^{\mathrm{chain}}$-sentence *is an expression of the form*

$$Q_1 C_1 \leq (q_1, p_1) \cdots Q_\ell C_\ell \leq (q_\ell, p_\ell)\, Q'_1 x_1 \leq r_1 \cdots Q'_k x_k \leq r_k : \psi$$

where ψ is a Boolean combination of atomic formulas and $Q_i, Q'_j \in \{\exists, \forall\}$.

Proposition 4.15. *From a $\mathrm{MSO}^{\mathrm{chain}}$-sentence φ, one can effectively compute a bounded $\mathrm{MSO}^{\mathrm{chain}}$-sentence ψ such that, for any structure $\mathcal{A}$, we have $\mathcal{A}^*_{\mathrm{ba}} \models \varphi$ if and only if $\mathcal{A}^*_{\mathrm{ba}} \models \psi$.*

Proof. We can assume φ to be in prenex normal form. Suppose φ contains some elementary quantifier that is followed by a chain quantifier, i.e., $\varphi = \cdots Q_1 x\, Q_2 C \cdots : \psi$. Then we can rewrite φ into $\cdots \exists C'\, Q_2 C \cdots \exists x : C' = \{x\} \wedge \psi$ or $\cdots \forall C'\, Q_2 C \cdots \forall x : C' = \{x\} \rightarrow \psi$ depending on whether Q_1 was existential or universal. Here $C' = \{x\}$ is an abbreviation for $x \in C' \wedge \forall y : y \in C \rightarrow y = x$. The quantifier prefix and the quantifier free kernel of the resulting expression is of the form required for bounded formulas. Now Theorem 4.13 and Lemma 4.9 allows to bound the chain and word quantifiers effectively. $\square$

4.5 Reduction of the $\mathrm{MSO}^{\mathrm{chain}}$-theory to the first-order theory

Theorem 4.16. *From a $\mathrm{MSO}^{\mathrm{chain}}$-sentence φ, one can effectively compute a first-order sentence φ' such that, for any structure $\mathcal{A}$, we have $\mathcal{A}^*_{\mathrm{ba}} \models \varphi$ if and only if $\mathcal{A} \models \varphi'$.*

Proof. Let $\mathcal{A}$ be some structure. For $n \in \mathbb{N}$ let $\mathcal{A}^{\leq n} = (A^{\leq n}, \preceq, (\widehat{R})_{R \in \sigma}, \mathrm{eq})$ where

- $A^{\leq n}$ is the set of words in A^* of length at most n
- $(u, v) \in \mathrm{eq}$ if and only if there exist $a \in A$ and $u', v' \in A^*$ with $u = ua$ and $v = va$.

Now let φ be a bounded $\mathrm{MSO}^{\mathrm{chain}}$-sentence with first-order kernel ψ and let $n \in \mathbb{N}$ be the maximal number appearing in the bounds in φ. Note that ψ does not relate the chains C_i directly, but only indirectly via the variables x_j. This allows to write a first-order formula α in the language of $\mathcal{A}^{\leq n}$ such that $\mathcal{A} \models \varphi$ if and only if $\mathcal{A}^{\leq n} \models \alpha$. Here, the predicate eq is necessary in order to express the periodicity of a chain.

Note that the first-order theory of $\mathcal{A}^{\leq n}$ can be reduced to that of $\mathcal{A}$. There is even such a reduction that works uniformly in n and $\mathcal{A}$. Hence the proof is complete. $\square$

Since Theorem 4.16 parallels Muchnik's Theorem 3.1, we can derive similar corollaries:

Corollary 4.17. *Let σ be some finite relational signature.*

- *If the first-order theory of a σ-structure $\mathcal{A}$ is decidable, then the $\mathrm{MSO}^{\mathrm{chain}}$-theory of its basic iteration $\mathcal{A}^*_{\mathrm{ba}}$ is decidable as well.*
- *For any $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that, for any two σ-structures $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A} \equiv^{\mathrm{FO}}_n \mathcal{B}$, we have $\mathcal{A}^*_{\mathrm{ba}} \equiv^{\mathrm{MSO}^{\mathrm{chain}}}_m \mathcal{B}^*_{\mathrm{ba}}$.*

5 FOREG over pushdown systems

In this section we apply our decidability result for $\mathrm{MSO}^{\mathrm{chain}}$ over basic iterations to pushdown systems. We introduce pushdown systems where the stack alphabet is the (possibly infinite) universe of an arbitrary base structure G. Push- and pop operations are triggered via the relations of the base structure G and a finite set of control states. The configuration graph of such a pushdown system is defined as for finite stack alphabets. We study the logic FOREG for these configuration graphs. FOREG is the extension of first-order logic which allows to define new binary predicates by regular expressions over the binary predicates of the base structure $\mathcal{A}$. Based on our decidability result Corollary 4.17 we show that if FOREG is decidable for the base structure G of a pushdown system, then FOREG remains decidable for the configuration graph of the pushdown system (Theorem 5.1).

5.1 The logic FOREG

Let Σ be a finite alphabet of labels. Let $G = (A, (E_\sigma)_{\sigma\in\Sigma}, R_1, \ldots, R_m)$ be a relational structure, where $E_\sigma \subseteq A\times A$ are binary relations and R_i are additional non-binary relations. For a word $w = \sigma_1 \cdots \sigma_n$ with $\sigma_i \in \Sigma$ we define the binary relation $\xrightarrow{w}_G = E_{\sigma_1} \circ \cdots \circ E_{\sigma_n}$. We have $\xrightarrow{\varepsilon}_G = \mathrm{id}_A$ and $\xrightarrow{\sigma}_G = E_\sigma$ for $\sigma \in \Sigma$. For a regular language $L \subseteq \Sigma^*$ we define $\mathrm{reach}_L = \bigcup_{w\in L} \xrightarrow{w}_G$. A *FOREG-formula* over the structure G is simply an first-order formula over the extended structure $(A, (\mathrm{reach}_L)_{L\in\mathrm{REG}(\Sigma)}, R_1, \ldots, R_m)$.

5.2 Pushdown systems over infinite stack alphabets

A *pushdown system* $S = (Q, G, \tau)$ over a stack structure G is given by the following data:

- G is a relational structure of the form $G = (A, (\mathrm{eq}_\alpha)_{\alpha\in\Sigma_1}, (\mathrm{push}_\beta)_{\beta\in\Sigma_2}, (\mathrm{pop}_\gamma)_{\gamma\in\Sigma_3}, \bot)$, where Σ_1, Σ_2, Σ_3 are finite and mutually disjoint alphabets (let $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ in the following), $\mathrm{eq}_\alpha \subseteq A \times A$, $\mathrm{push}_\beta, \mathrm{pop}_\gamma \subseteq A$, and $\bot \in A$.
- Q is a finite set of states such that $Q \cap A = \emptyset$.
- $\tau : \Sigma \to Q \times Q$

With S we associate the configuration graph $\mathcal{C}(S) = (A^+Q, (E_\sigma)_{\sigma\in\Sigma})$, where:

- $E_\alpha = \{(wap, wbq) \mid w \in A^*, (a,b) \in \mathrm{eq}_\alpha, \tau(\alpha) = (p,q)\}$ for $\alpha \in \Sigma_1$
- $E_\beta = \{(wp, waq) \mid w \in A^+, a \in \mathrm{push}_\beta, \tau(\beta) = (p,q)\}$ for $\beta \in \Sigma_2$
- $E_\gamma = \{(wap, wq) \mid w \in A^+, a \in \mathrm{pop}_\gamma, \tau(\gamma) = (p,q)\}$ for $\gamma \in \Sigma_3$

The following theorem is the main result of this section:

Theorem 5.1. *Let G be a stack structure with decidable FOREG-theory. Then the configuration graph $\mathcal{C}(S)$ has a decidable FOREG-theory.*

The rest of this section is devoted to the proof of this theorem. The idea is to define, using the logic FOREG, a suitable structure $\mathcal{A}$ in the stack structure G. Since we assume that the FOREG-theory of G is decidable, it follows that the first-order theory of $\mathcal{A}$ is decidable. Thus, by Corollary 4.17 the $\mathrm{MSO}^{\mathrm{chain}}$-theory of the basic iteration $\mathcal{A}^*_{\mathrm{ba}}$ is decidable. To obtain the claimed result, we will give a $\mathrm{MSO}^{\mathrm{chain}}$-interpretation of the configuration graph $\mathcal{C}(S)$ in $\mathcal{A}^*_{\mathrm{ba}}$.

For a finite automaton T and states μ and ν of T, we write $L(T, \mu, \nu)$ for the set of words that label some path from μ to ν in T.

Let us fix regular languages $L_1, \dots, L_k \subseteq \Sigma^*$. Then there exists a finite automaton $T = (\Theta, \Sigma, \delta)$ with state set Θ such that every language L_i is a union of languages of the form $L(T, \mu, \nu)$ for certain states $\mu, \nu \in \Theta$. For $\mu, \nu \in \Theta$ let

$$\mathrm{reach}_{\mu,\nu} = \{(up, vq) \in A^+Q \times A^+Q \mid \exists w \in L(T, \mu, \nu) : up \xrightarrow{w}_{\mathcal{C}(S)} vq\}.$$

Thus, $\mathrm{reach}_{\mu,\nu} = \mathrm{reach}_L$ for $L = L(T, \mu, \nu)$. We will show that the first-order theory of the structure

$$\mathcal{B} = (A^+Q, (\mathrm{reach}_{\mu,\nu})_{\mu,\nu \in \Theta}) \tag{2}$$

is decidable. Since the decision procedure for the first-order theory of $\mathcal{B}$ will be uniform in the automaton T, this proves Theorem 5.1.

Let $\mu, \nu \in \Theta$, $p, q \in Q$, $u \in A^+$, and $a \in A$. We write $(up, uaq) \in \mathrm{reach}^{(+)}_{\mu,\nu}$ if and only if there exist $\beta \in \Sigma_2$ and $x \in \Sigma^*$ such that $\beta x \in L(T, \mu, \nu)$, $up \xrightarrow{\beta x}_{\mathcal{C}(S)} uaq$, $|y|_{\Sigma_2} \geq |y|_{\Sigma_3}$ for every prefix y of x, and $|x|_{\Sigma_2} = |x|_{\Sigma_3}$. Thus, (up, uaq) belongs to the relation $\mathrm{reach}^{(+)}_{\mu,\nu}$ if there exists a path from up to uaq in the configuration graph $\mathcal{C}(S)$ whose label belongs to $L(T, \mu, \nu)$ such that all the configurations along this path except the very first one up are of the form uvr for some $v \in A^+$, and $r \in Q$. Note that $(up, uaq) \in \mathrm{reach}^{(+)}_{\mu,\nu}$ implies $(vp, vaq) \in \mathrm{reach}^{(+)}_{\mu,\nu}$ for all $v \in A^+$. Symmetrically, we write $(uap, uq) \in \mathrm{reach}^{(-)}_{\mu,\nu}$ if and only if there exist $\gamma \in \Sigma_3$ and $x \in \Sigma^*$ such that $x\gamma \in L(T, \mu, \nu)$, $uap \xrightarrow{x\gamma}_{\mathcal{C}(S)} uq$, $|y|_{\Sigma_2} \geq |y|_{\Sigma_3}$ for every prefix y of x, and $|x|_{\Sigma_2} = |x|_{\Sigma_3}$. Finally, for $\mu, \nu \in \Theta$, $p, q \in Q$, $u, v \in A^+$ we write $(up, vq) \in \mathrm{reach}^{(=)}_{\mu,\nu}$ if and only if there exists $w \in L(T, \mu, \nu)$ such that $up \xrightarrow{w}_{\mathcal{C}(S)} vq$, $|y|_{\Sigma_2} \geq |y|_{\Sigma_3}$ for every prefix y of w, and $|w|_{\Sigma_2} = |w|_{\Sigma_3}$. Thus, (up, vq) belongs to the relation $\mathrm{reach}^{(+)}_{\mu,\nu}$ if there exists a path from up to vq in the configuration graph $\mathcal{C}(S)$ whose label belongs to $L(T, \mu, \nu)$ such that all the configurations along this path are of the form wr for some $w \in A^+$ and $r \in Q$, and such that $|w| = |u| = |v|$. Hence $(uap, uaq) \in \mathrm{reach}^{(=)}_{\mu,\nu}$ for some (and hence every) $u \in A^*$ if and only if $(ap, bq) \in \mathrm{reach}_{\mu,\nu}$.

Lemma 5.2. *For $c, d \in A^+Q$ and $\mu, \nu \in \Theta$ we have $(c, d) \in \mathrm{reach}_{\mu,\nu}$ if and only if there exist $m, n \geq 0$, $\mu_m, \dots, \mu_0, \nu_0, \dots, \nu_n \in \Theta$, and configurations $c_m, \dots, c_0, d_0, \dots, d_n \in A^+Q$ such that:*

- $c_m = c$, $d_n = d$
- $\mu_m = \mu$, $\nu_n = \nu$
- $(c_i, c_{i-1}) \in \mathrm{reach}^{(-)}_{\mu_i,\mu_{i-1}}$ *for* $1 \leq i \leq m$
- $(c_0, d_0) \in \mathrm{reach}^{(=)}_{\mu_0,\nu_0}$
- $(d_{j-1}, d_j) \in \mathrm{reach}^{(+)}_{\nu_{j-1},\nu_j}$ *for* $1 \leq j \leq n$

Proof. The if-direction of the lemma is obvious. For the only if-direction define the height $\mathrm{height}(uq)$ of a configuration uq ($u \in A^+$, $q \in Q$) as $|u|$. Now assume that $(c,d) \in \mathrm{reach}_{\mu,\nu}$. Thus, there exist configurations $e_0, \ldots, e_k \in A^+Q$, states $\rho_0, \ldots, \rho_k \in \Theta$, and $\sigma_1, \ldots, \sigma_k \in \Sigma$ such that $c = e_0$, $d = e_k$, $\mu = \rho_0$, $\nu = \rho_k$, $(e_{i-1}, e_i) \in E_{\sigma_i}$, and $\rho_{i-1} \xrightarrow{\sigma_i}_T \rho_i$ ($1 \le i \le k$). Let $h = \min\{\mathrm{height}(e_i) \mid 0 \le i \le k\}$, $i_\ell = \min\{i \mid \mathrm{height}(e_i) = h + \ell\}$ ($0 \le \ell \le \mathrm{height}(c) - h =: m$), and $j_\ell = \max\{i \mid \mathrm{height}(e_i) = h + \ell\}$ ($0 \le \ell \le \mathrm{height}(d) - h =: n$). Then define:

- $c_\ell = e_{i_\ell}$, $\mu_\ell = \rho_{i_\ell}$ ($0 \le \ell \le m$)
- $d_\ell = e_{j_\ell}$, $\nu_\ell = \rho_{j_\ell}$ ($0 \le \ell \le n$)

We clearly have $(c_0, d_0) \in \mathrm{reach}^{(=)}_{\mu_0,\nu_0}$, because c_0 and d_0 have the same height h, and between c_0 and d_0 the height is always at least h, i.e., $\mathrm{height}(e_i) \ge h$ for all $i_0 \le i \le j_0$. Similarly, we obtain $(c_\ell, c_{\ell-1}) \in \mathrm{reach}^{(-)}_{\mu_\ell,\mu_{\ell-1}}$ for $1 \le \ell \le m$: The configuration $c_{\ell-1}$ must be reached by a pop-operation from a configuration c' such that $\mathrm{height}(c') = \mathrm{height}(c_\ell)$ and between c_ℓ and c' all configurations have at least height $\mathrm{height}(c')$. The same arguments show that $(d_{\ell-1}, d_\ell) \in \mathrm{reach}^{(+)}_{\nu_{\ell-1},\nu_\ell}$ for $1 \le \ell \le n$. ◻

For all $p, q \in Q$ and $\mu, \nu \in \Theta$ define a binary predicate $H(p,\mu,q,\nu) \subseteq A \times A$ as follows, where $a, b \in A$:

$$(a,b) \in H(p,\mu,q,\nu) \iff (ap, bq) \in \mathrm{reach}_{\mu,\nu}$$

Lemma 5.3. *The relation $H(p,\mu,q,\nu)$ is effectively* FOREG*-definable over the stack structure G.*

Proof. We will construct effectively a finite automaton B with state set $Q \times \Theta$ and alphabet Σ_1 such that

$$(ap, bq) \in \mathrm{reach}_{\mu,\nu} \iff \exists u \in L(B, (p,\mu), (q,\nu)) : a \xrightarrow{u}_G b,$$

which proves the lemma. For this, we will construct a finite sequence of automata B_i ($i \ge 0$) with state set $Q \times \Theta$ and alphabet Σ_1, which converges to the automaton B. The finite state automaton B_0 contains the transition

$$(p,\mu) \xrightarrow{\alpha} (q,\nu)$$

if and only if $\tau(\alpha) = (p,q)$ and $\mu \xrightarrow{\alpha}_T \nu$ for some $\alpha \in \Sigma_1$.

Next, assume that B_i is already constructed. Assume that

(1) in T there are transitions $\mu \xrightarrow{\beta}_T \mu'$, ($\beta \in \Sigma_2$) and $\nu' \xrightarrow{\gamma}_T \nu$ ($\gamma \in \Sigma_3$),
(2) $\tau(\beta) = (p,p')$, $\tau(\gamma) = (q',q)$, and
(3) there exist $a \in \mathrm{push}_\beta, b \in \mathrm{pop}_\gamma$, and $u \in L(B_i, (p',\mu'), (q',\nu'))$ such that $a \xrightarrow{u}_G b$.

Note that (3) is decidable, since the FOREG-theory of G is decidable. In this situation we add the ε-transition $(p,\mu) \xrightarrow{\varepsilon} (q,\nu)$ to B_i and call the resulting automaton B_{i+1}. We repeat this process as long as we will add new ε-transitions. Note that in each step the state set is not changed. Let B be the resulting automaton. We claim that

$$(ap, bq) \in \mathrm{reach}_{\mu,\nu} \Leftrightarrow \exists u \in L(B, (p,\mu), (q,\nu)) : a \xrightarrow{u}_G b.$$

First assume that $a \xrightarrow{u}_G b$ for some word $u \in L(B, (p,\mu), (q,\nu))$. Let i be the minimal number such that $u \in L(B_i, (p,\mu), (q,\nu))$. We will prove $(ap, bq) \in \mathrm{reach}_{\mu,\nu}$ by an induction over i. If $i = 0$, then by construction of B_0 we have $ap \xrightarrow{u}_{\mathcal{C}(S)} bq$ and $u \in L(T, \mu, \nu)$, i.e., $(ap, bq) \in \mathrm{reach}_{\mu,\nu}$. Now assume that $i > 0$ and that B_i results from B_{i-1} by adding the ε-transition $(p', \mu') \xrightarrow{\varepsilon} (q', \nu')$. Thus,

(A) in T there are transitions $\mu' \xrightarrow{\beta}_T \mu''$, $(\beta \in \Sigma_2)$ and $\nu'' \xrightarrow{\gamma}_T \nu'$ $(\gamma \in \Sigma_3)$.
(B) $\tau(\beta) = (p', p'')$, $\tau(\gamma) = (q'', q')$, and
(C) there exist $a' \in \mathrm{push}_\beta, b' \in \mathrm{pop}_\gamma$ and $u' \in L(B_{i-1}, (p'', \mu''), (q'', \nu''))$ such that $a' \xrightarrow{u'}_G b'$.

Consider a path in B_i from (p,μ) to (q,ν), which is labeled with the word u. Since $u \notin L(B_{i-1}, (p,\mu), (q,\nu))$, this path has to use the new ε-transition at least once. Thus, there exist $k \geq 2$ and words $u_1, \ldots, u_k \in \Sigma_1^*$ with $u = u_1 u_2 \ldots u_k$ such that:

- $u_1 \in L(B_{i-1}, (p,\mu), (p', \mu'))$
- $u_j \in L(B_{i-1}, (q', \nu'), (p', \mu'))$ for $1 < j < k$
- $u_k \in L(B_{i-1}, (q', \nu'), (q, \nu))$

Moreover, since $a \xrightarrow{u}_G b$, there exist $c_1, c_2, \ldots, c_k \in A$ such that $a \xrightarrow{u_1}_G c_1$, $c_{j-1} \xrightarrow{u_j}_G c_j$ for $2 \leq j \leq k-1$, and $c_{k-1} \xrightarrow{u_k}_G b$. The induction hypothesis shows that

- $(ap, c_1 p') \in \mathrm{reach}_{\mu,\mu'}$,
- $(c_j q', c_{j+1} p') \in \mathrm{reach}_{\nu',\mu'}$ for $2 \leq j < k-1$, and
- $(c_{k-1} q', bq) \in \mathrm{reach}_{\nu',\nu}$.

Next, from (C) and the induction hypothesis for B_{i-1} we obtain

$$(a'p'', b'q'') \in \mathrm{reach}_{\mu'',\nu''}.$$

Moreover,

$$\forall c \in A : (cp', ca'p'') \in \mathrm{reach}_{\mu',\mu''} \text{ and } (cb'q'', cq') \in \mathrm{reach}_{\nu'',\nu'},$$

follow from (A)–(C). Altogether, this implies $(ap, bq) \in \mathrm{reach}_{\mu,\nu}$.

For the other direction, assume that $(ap, bq) \in \mathrm{reach}_{\mu,\nu}$. Hence, there exists a word $w \in L(T, \mu, \nu)$ such that $ap \xrightarrow{w}_{\mathcal{C}(S)} bq$. Since the configurations ap and bq have the same height, we obtain $|w|_{\Sigma_2} = |w|_{\Sigma_3}$. By induction on $|w|_{\Sigma_2}$, we show the existence of $u \in L(B, (p,\mu), (q,\nu))$ with $a \xrightarrow{u} b$. If $w \in \Sigma_1^*$, i.e., w does not contain any push or pop operations, then we have $w \in L(B_0, (p,\mu), (q,\nu))$ and $a \xrightarrow{w}_G b$. If w contains at least on push operation, then we can write $w = x\beta y\gamma z$ for some $\beta \in \Sigma_2$ and $\gamma \in \Sigma_3$ such that

$$ap \xrightarrow{x}_{\mathcal{C}(S)} cr \xrightarrow{\beta}_{\mathcal{C}(S)} ca'p' \quad a'p' \xrightarrow{y}_{\mathcal{C}(S)} b'q' \quad cb'q' \xrightarrow{\gamma}_{\mathcal{C}(S)} cr' \xrightarrow{z}_{\mathcal{C}(S)} bq, \tag{3}$$

where $c, a', b', c' \in A$ and $r, p', q', r' \in Q$. Moreover there exist states $\xi, \mu', \nu', \xi' \in \Theta$ such that

$$x \in L(T, \mu, \xi), \quad y \in L(T, \mu', \nu'), \quad z \in L(T, \xi', \nu) \tag{4}$$

and

$$\xi \xrightarrow{\beta}_T \mu' \qquad \nu' \xrightarrow{\gamma}_T \xi'. \tag{5}$$

Since the words x, y, z contain properly less push operations than w does, the induction hypothesis implies from (3) and (4):

- $a \xrightarrow{u}_G c$ for some $u \in L(B, (p, \mu), (r, \xi))$
- $c \xrightarrow{v}_G b$ for some $v \in L(B, (r', \xi'), (q, \nu))$
- $a' \xrightarrow{w'}_G b'$ for some $w' \in L(B, (p', \mu'), (q', \nu'))$

From the last line, together with (5), $a' \in \mathrm{push}_\beta$, $b' \in \mathrm{pop}_\gamma$, $\tau(\beta) = (r, p')$, and $\tau(\gamma) = (q', r')$ it follows that in the automaton B there exists an ε-transition $(r, \xi) \xrightarrow{\varepsilon} (r', \xi')$. Thus, $a \xrightarrow{uv}_G b$ and $uv \in L(B, (p, \mu), (q, \nu))$. This finishes the proof of the lemma. □

Next, we define for all $p, q \in Q$ and $\mu, \nu \in \Theta$ the following unary predicates on A:

$$a \in D(p,\mu,q,\nu) \quad \Leftrightarrow \quad \bigvee_{p' \in Q, \mu' \in \Theta, \gamma \in \Sigma_3} \exists b \in A : \left\{ \begin{array}{l} (a,b) \in H(p,\mu,p',\mu') \wedge \\ \tau(\gamma) = (p',q) \ \wedge \ b \in \mathrm{pop}_\gamma \wedge \\ \mu' \xrightarrow{\gamma}_T \nu \end{array} \right\}$$

$$a \in U(p,\mu,q,\nu) \quad \Leftrightarrow \quad \bigvee_{p' \in Q, \mu' \in \Theta, \beta \in \Sigma_2} \exists b \in A : \left\{ \begin{array}{l} \tau(\beta) = (p,p') \ \wedge \ b \in \mathrm{push}_\beta \wedge \\ \mu \xrightarrow{\beta}_T \mu' \wedge \\ (b,a) \in H(p',\mu',q,\nu) \end{array} \right\}$$

By Lemma 5.3, the unary predicates $D(p, \mu, q, \nu)$ and $U(p, \mu, q, \nu)$ are FOREG-definable in the stack structure G. The next lemma follows directly from the definition of the predicates $D(p, \mu, q, \nu)$, $H(p, \mu, q, \nu)$, and $U(p, \mu, q, \nu)$.

Lemma 5.4. *We have:*

- $a \in D(p, \mu, q, \nu)$ *if and only if* $(uap, uq) \in \mathrm{reach}^{(-)}_{\mu,\nu}$ *for some (and hence every)* $u \in A^+$.
- $a \in U(p, \mu, q, \nu)$ *if and only if* $(up, uaq) \in \mathrm{reach}^{(+)}_{\mu,\nu}$ *for some (and hence every)* $u \in A^+$.

By combining Lemma 5.2 with Lemma 5.4 we obtain:

Lemma 5.5. *We have* $(up, vq) \in \mathrm{reach}_{\mu,\nu}$ *if and only if there exist*

- $m, n \geq 0$,
- $w \in A^*$,
- $a_0, \ldots, a_m, b_0, \ldots, b_n \in A$,
- $p_0, \ldots, p_m, q_0, \ldots, q_n \in Q$,
- $\mu_0, \ldots, \mu_m, \nu_0, \ldots, \nu_n \in \Theta$

such that:

- $\mu = \mu_m$, $\nu = \nu_n$, $p = p_m$, $q = q_n$,
- $u = wa_0 \cdots a_m$, $v = wb_0 \cdots b_n$,
- $a_i \in D(p_i, \mu_i, p_{i-1}, \mu_{i-1})$ *for all* $1 \leq i \leq m$,

– $(a_0, b_0) \in H(p_0, \mu_0, q_0, \nu_0)$,
– $b_j \in U(q_{j-1}, \nu_{j-1}, q_j, \nu_j)$ *for all* $1 \leq j \leq n$

Next, we define the binary relations $D(p, \mu, q, \nu)^*, U(p, \mu, q, \nu)^* \subseteq A^+ \times A^+$ as follows:

– $(u, v) \in D(p, \mu, q, \nu)^*$ if and only if there exist $m \geq 0$, $a_1, \dots, a_m \in A$, $p_0, \dots, p_m \in Q$, $\mu_0, \dots, \mu_m \in \Theta$ such that
  - $u = va_1 \cdots a_m$,
  - $p = p_m$, $q = p_0$, $\mu = \mu_m$, $\nu = \mu_0$, and
  - $a_i \in D(p_i, \mu_i, p_{i-1}, \mu_{i-1})$ for $1 \leq i \leq m$.
– $(u, v) \in U(p, \mu, q, \nu)^*$ if and only if there exist $n \geq 0$, $b_1, \dots, b_n \in A$, $q_0, \dots, q_n \in Q$, $\nu_0, \dots, \nu_n \in \Theta$ such that
  - $v = ub_1 \cdots b_n$,
  - $p = q_0$, $q = q_n$, $\mu = \nu_0$, $\nu = \nu_n$, and
  - $b_j \in U(q_{j-1}, \nu_{j-1}, q_j, \nu_j)$ for $1 \leq j \leq n$.

These definitions and Lemma 5.5 imply:

Lemma 5.6. *We have* $(up, vq) \in \mathrm{reach}_{\mu,\nu}$ *if and only if there exist* $u', v' \in A^+$, $p', q' \in Q$, $\mu', \nu' \in \Theta$ *such that*

– $(u, u') \in D(p, \mu, p', \mu')^*$
– $(u', v') \in \{(xy, xz) \mid x \in A^*, (y, z) \in H(p', \mu', q', \nu')\}$
– $(v', v) \in U(q', \nu', q, \nu)^*$.

Now, let us consider the structure

$$\begin{aligned}\mathcal{A} = (A \cup Q, (q)_{q\in Q}, &(D(p, \mu, q, \nu))_{p,q\in Q,\mu,\nu\in\Theta},\\ &(H(p, \mu, q, \nu))_{p,q\in Q,\mu,\nu\in\Theta},\\ &(U(p, \mu, q, \nu))_{p,q\in Q,\mu,\nu\in\Theta}).\end{aligned}$$

Since each of its relations is FOREG-definable in the stack structure G, and G has a decidable FOREG-theory, it follows that $\mathcal{A}$ has a decidable first-order theory. Thus, by Theorem 4.16,

$$\begin{aligned}\mathcal{A}^*_{\mathrm{ba}} = ((A \cup Q)^*, \preceq, &(\widehat{q})_{q\in Q},\\ &(D(\widehat{p, \mu, q}, \nu))_{p,q\in Q,\mu,\nu\in\Theta},\\ &(H(\widehat{p, \mu, q}, \nu))_{p,q\in Q,\mu,\nu\in\Theta},\\ &(U(\widehat{p, \mu, q}, \nu))_{p,q\in Q,\mu,\nu\in\Theta})\end{aligned}$$

has a decidable $\mathrm{MSO}^{\mathrm{chain}}$-theory. We finally show that the structure $\mathcal{B}$ from (2), for which we have to show decidability of the first-order theory, is $\mathrm{MSO}^{\mathrm{chain}}$-interpretable in $\mathcal{A}^*_{\mathrm{ba}}$, which proves Theorem 5.1. Clearly, the universe A^+Q of $\mathcal{B}$ is first-order definable within $\mathcal{A}^*_{\mathrm{ba}}$ using the prefix relation $\preceq$ and the unary relations $(A \cup Q)^*q$ for $q \in Q$. In order to define the relations $\mathrm{reach}_{\mu,\nu}$ of $\mathcal{B}$ it suffices by Lemma 5.6 to define the binary relations $D(p,\mu,q,\nu)^*, U(p,\mu,q,\nu)^*, \{(xy,xz) \mid x \in A^*, (y,z) \in H(p,\mu,q,\nu)\} \subseteq A^+ \times A^+$ in $\mathcal{A}$ using $\mathrm{MSO}^{\mathrm{chain}}$. The relation $\{(xy,xz) \mid x \in A^*, (y,z) \in H(p,\mu,q,\nu)\}$ can be defined as $H(\widehat{p,\mu,q},\nu) \cap A^+ \times A^+$; note that A^+ is first-order definable in $\mathcal{A}$ using the prefix relation $\preceq$ and the unary relations $(A \cup Q)^* q$ for $q \in Q$. The $\mathrm{MSO}^{\mathrm{chain}}$-definition of $D(p,\mu,q,\nu)^*$ follows Büchi's technique for expressing the existence of a successful run of an automaton on a finite word in MSO: In order to express $(u,v) \in D(p,\mu,q,\nu)^*$ for $u,v \in A^+$, we say that either $u = v$, $p = q$, and $\mu = \nu$, or v is a proper prefix of u and for the path P from v to u in the tree over A there exists a partition

$$P \setminus \{v\} = \biguplus_{\substack{r,s \in Q,\\ \xi,\rho \in \Theta}} X_{r,\xi,s,\rho}$$

such that:

- $X_{r,\xi,s,\rho} \subseteq A^* D(r,s,\xi,\rho)$,
- $u \in X_{p,\mu,r,\xi}$ for some $r \in Q$ and $\xi \in \Theta$,
- if w is the unique successor of v in the path P, then $w \in X_{r,\xi,q,\nu}$ for some $r \in Q$ and $\xi \in \Theta$,
- for all $x, xa \in P \setminus \{v\}$ (with $x \in A^+$, $a \in A$) there exist states $p', q', r' \in Q, \mu', \nu', \xi' \in \Theta$ such that $xa \in X_{p',\mu',q',\nu'}$ and $x \in X_{q',\nu',r',\xi'}$.

The predicate $U(p,\mu,q,\nu)^*$ can be defined analogously to $U(p,\mu,q,\nu)^*$. This completes the proof of Theorem 5.1.

6 Open problems

We have shown that the $\mathrm{MSO}^{\mathrm{chain}}$-theory (i.e., the fragment of the full MSO-theory where set quantification is restricted to chains) of the basic iteration $\mathcal{A}^*_{\mathrm{ba}}$ of a structure $\mathcal{A}$ can be reduced to the first-order theory of $\mathcal{A}$. Using this result, we have shown that the FOREG-theory of the configuration graph $\mathcal{C}(S)$ of a pushdown system S over an infinite stack structure $\mathcal{A}$ can be reduced to the FOREG-theory of $\mathcal{A}$. We plan to investigate whether similar preservation theorems can be shown also for temporal logics like CTL, CTL*, or the modal μ-calculus. Another interesting candidate for investigations of this kind is TC^2 [25], i.e., first-order logic extended by the transitive closure operator for binary relations.

References

1. R. Alur, M. Benedikt, K. Etessami, P. Godefroid, T. W. Reps, and M. Yannakakis. Analysis of recursive state machines. *ACM Trans. Program. Lang. Syst*, 27(4):786–818, 2005.
2. D. Berwanger and A. Blumensath. The monadic theory of tree-like structures. In *Automata, logics, and infinite games*, Lecture Notes in Comp. Sci. vol. 2500, pages 285–301. Springer, 2002.
3. A. Blumensath and S. Kreutzer. An extension of muchnik's theorem. *Journal of Logic and Computation*, 15:59–64, 2005.

4. A. Carayol and S. Wöhrle. The Caucal hierarchy of infinite graphs in terms of logic and higher-order pushdown automata. In *FSTTCS 2003*, Lecture Notes in Comp. Science vol. 2914, pages 112–123. Springer, 2003.
5. D. Caucal. On the transition graphs of Turing machines. *Theoretical Computer Science*, 296(2):195–223, 2003.
6. B. Courcelle and I. Walukiewicz. Monadic second-order logic, graph coverings and unfoldings of transition systems. *Ann. Pure Appl. Logic*, 92(1):35–62, 1998.
7. H.-D. Ebbinghaus and J. Flum. *Finite Model Theory*. Springer, 1991.
8. J. Esparza, S. Kiefer, and S. Schwoon. Abstraction refinement with Craig interpolation and symbolic pushdown systems. In *Proceedings of the 12th International Conference on Tools and Algorithms for the Construction and Analysis of Systems (TACAS 06), Vienna (Austria)*, 2006. to appear.
9. J. Esparza, A. Kucera, and S. Schwoon. Model checking LTL with regular valuations for pushdown systems. *Information and Computation*, 186(2):355–376, 2003.
10. Y. Gurevich. Monadic second-order theories. In J. Barwise and S. Feferman, editors, *Model-Theoretic Logics*, pages 479–506. Springer, 1985.
11. Y. Gurevich and S. Shelah. Rabin's uniformization problem. *J. of Symb. Logic*, 48:1105–1119, 1983.
12. B. Khoussainov and A. Nerode. Automatic presentations of structures. In *Logic and Computational Complexity*, Lecture Notes in Comp. Science vol. 960, pages 367–392. Springer, 1995.
13. O. Kupferman and M. Y. Vardi. An automata-theoretic approach to reasoning about infinite-state systems. In E. A. Emerson and A. P. Sistla, editors, *Proceedings of the 12th International Conference on Computer Aided Verification (CAV 2000), Chiacago (USA)*, number 1855 in Lecture Notes in Computer Science, pages 36–52. Springer, 2000.
14. D. Kuske and M. Lohrey. Logical aspects of Cayley-graphs: The monoid case. *International Journal of Algebra and Computation*, 2005. Accepted.
15. F. Neven and T. Schwentick. Expressive and efficient pattern languages for tree-structured data. In *PODS'00*, pages 145–156. ACM, 2000.
16. D. Perrin and J.-E. Pin. *Infinite Words*. Pure and Applied Mathematics vol. 141. Elsevier, 2004.
17. M. Rabin. *Automata on infinite objects and Church's problem*. American Mathematical Society, Providence, R.I., 1972. Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 13.
18. M. O. Rabin. Decidability of second-order theories and automata on infinite trees. *Transactions of the American Mathematical Society*, 141:1–35, 1969.
19. F. Ramsey. On a problem of formal logic. *Proc. London Math. Soc.*, 30:264–286, 1930.
20. T. Schwentick. On diving in trees. In *MFCS'00*, Lecture Notes in Comp. Science vol. 1893, pages 660–669. Springer, 2000.
21. A. L. Semenov. Decidability of monadic theories. In M. Chytil and V. Koubek, editors, *Proceedings of the 11th International Symposium of Mathematical Foundations of Computer Science (MFCS'84), Praha (Czechoslovakia)*, number 176 in Lecture Notes in Computer Science, pages 162–175. Springer, 1984.
22. S. Shelah. The monadic theory of order. *Annals of Mathematics, II. Series*, 102:379–419, 1975.
23. J. Stupp. The lattice-model is recursive in the original model. The Hebrew University, Jerusalem, 1975.
24. W. Thomas. On chain logic, path logic, and first-order logic over infinite trees. In *LICS'87*, pages 245–256. IEEE Computer Society Press, 1987.
25. W. Thomas and S. Wöhrle. Model checking synchronized products of infinite transition systems. In *LICS'04*, pages 2–11. IEEE Computer Society Press, 2004.
26. I. Walukiewicz. Model checking ctl properties of pushdown systems. In S. Kapoor and S. Prasad, editors, *Proceedings of the 20th Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2000), New Delhi (India)*, number 1974 in Lecture Notes in Computer Science, pages 127–138. Springer, 2000.
27. I. Walukiewicz. Pushdown processes: Games and model-checking. *Information and Computation*, 164(2):234–263, 2001.
28. I. Walukiewicz. Monadic second-order logic on tree-like structures. *Theoretical Computer Science*, 275(1–2):311–346, 2002.