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Cyclic Petri Net Reachability Sets are Semi-linear Effectively Constructible

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Abstract

We prove that cyclic Petri net reachability sets are finite unions of linear combinations of integer vectors whose sizes are exponential in the size of the cyclic Petri net.

1 Introduction

Petri nets form one of the most used models for the specification and the verification of parallel and concurrent systems and most recently for real-time systems.

Recall that verifying a Petri net often amounts to test the inclusion of a Petri net language (representing the implementation) in a finite automaton language (representing the specification). This is equivalent to decide whether a vector of nonnegative integers is a reachable state in the Petri net. Another important problem can be given by the equality problem i.e. the problem of deciding whether two different Petri nets have the same reachability set.

The former, the so called reachability problem, has been shown decidable, in two different ways by Mayr [13] and Kosaraju [12], 15 years ago. However, the proposed algorithms are not primitive recursive and, up to now, we don't even know whether the reachability problem for Petri nets is primitive recursive or not. The equality problem has been shown undecidable by Hack [8], in 1976. These two problems are probably the most difficult ones in Petri net theory. To the best of our knowledge, up to now, no tool implementing these algorithms has been provided.

There exist, however, some other algorithms for the verification of subclasses of Petri nets such as free choice Petri nets [6], conflict free Petri nets

[9], reversible Petri nets [3,4,11,14] and marked graphs [5,6] (See [7] for a recent panorama on the decidability questions in Petri nets).

We add to this list cyclic Petri nets. A Petri net is said to be cyclic if and only if its initial marking is a home state marking. Cyclic Petri nets are used in the design and control of production systems and in the design and verification of communicating protocols. They have been introduced by Araki and Kasami [1] in 1977. In [1], Araki and Kasami proved that the reachability sets of cyclic Petri nets are semi-linear. They also gave non primitive recursive algorithms to decide (1) the cyclicity problem (i.e. the problem of deciding whether a Petri net is cyclic), (2) the reachability and (3) the equality problem for cyclic Petri nets. We, however, gave in [2], $2^{c.n.\log(n)}$ space algorithms to decide the same problems.

In this paper, we show that cyclic Petri net reachability sets can be given in a form of finite unions of linear combinations of nonnegative vectors whose sizes are exponential in the size of the Petri nets.

Our proof is organized as follows. First, we note that what makes the reachability sets for reversible Petri nets semi-linear [2,3,11] is the fact that, for a reversible Petri net (N, M_0), we have $\mathcal{RS}(N, M_0) = \mathcal{RS}(\bar{N}, M_0)$ where $\bar{N}$ is the Petri net inverse of N ([1]). However, for cyclic Petri nets, we only have $\mathcal{RS}(N, M_0) \subseteq \mathcal{RS}(\bar{N}, M_0)$. Hopefully, we proved, in section 3, that for every Petri net (N, M_0) such that $\mathcal{RS}(N, M_0) \subseteq \mathcal{RS}(\bar{N}, M_0)$ there exists a Petri net N' such that $\mathcal{RS}(N, M_0) = \mathcal{RS}(N', M_0) = \mathcal{RS}(\bar{N}', M_0)$. Hence, cyclic Petri net reachability sets are semi-linear. By using the well known Rackoff coverability theorem (Theorem 2.6), we prove, in section 4, that cyclic Petri net reachability sets can be effectively given by computing two finite sets, $U, V \in \mathbb{N}^{|P|}$, such that, for every $M \in (U \cup V)$, we have $size(M) \leq 2^{c.size(N,M_0).\log(size(N,M_0))}$.

2 Definitions and previous results

A *Petri net* is a tuple $N = (P, T, F)$ where $P = \{p_1, p_2, \ldots, p_m\}$ is a finite set of places, $T = \{t_1, t_2, \ldots, t_n\}$ is a finite set of transitions and $F : (P \times T) \cup (T \times P) \longrightarrow \mathbb{N}$ is the flow function. A *marking* M is a nonnegative m-dimensional integer vector. A *(marked) Petri net* is a couple (N, M_0) where N is a Petri net and M_0 is the *initial marking*. Given a marking $M \in \mathbb{N}^m$ and a sequence $\sigma = t_1.t_2 \ldots t_k \in T^*$, we recall that σ is *fireable* at M if for every place $p \in P$ and for every prefix $\sigma_i = t_1.t_2 \ldots t_i$ of σ we have:

$$M(p) + \sum_{j=1}^{i-1} (F(t_j, p) - F(p, t_j)) \geq F(p, t_i)$$

[1] A Petri net is reversible if for every transition t there exists a transition $\bar{t}$ whose effect annuls the effect of t. The inverse $\bar{N}$ of N is given by inverting the direction of all the arcs in N.

A fireable sequence $\sigma = t_1.t_2.\ldots.t_k$ at a marking M may *fire* and it yields the marking M', $M \xrightarrow{\sigma} M'$, defined by:

$$M'(p) = M(p) + \sum_{i=1}^{k}(F(t_i,p) - F(p,t_i)) \; ; \forall p \in P$$

The *reachability set* $\mathcal{R}S(N, M_0)$ of a marked Petri net (N, M_0) is the set of all reachable markings:

$$\mathcal{R}S(N, M_0) = \{M \in \mathbb{N}^m \mid \exists \sigma \in T^* \, , \, M_0 \xrightarrow{\sigma} M\}$$

The *coverability set* $\mathcal{C}S(N, M_0)$ of a marked Petri net (N, M_0) is the set of all covered markings:

$$\mathcal{C}S(N, M_0) = \{M \in \mathbb{N}^m \mid \exists \sigma \in T^* \, , \, \exists M' \geq M \, , \, M_0 \xrightarrow{\sigma} M'\}$$

Given two finite subsets U and V of $\mathbb{N}^m$, $U + V^*$ is defined by:

$$U + V^* = \{u + \sum_{i=1}^{k} \lambda_i.v_i \mid u \in U \, , \, v_i \in V \, , \, \lambda_1, \ldots, \lambda_k \in \mathbb{N}\}$$

Since we will be considering the size of minimal markings and period vectors in Petri net reachability sets, it is necessary to have a precise idea of the size of Petri nets. For that we use the same definition of size as Valk and Vidal-Naquet.

Definition 2.1 *[16] Let $N = (P, T, F)$ be a Petri net and M_0 be a marking. The size of N is*

$$size(N) = \sum_{t \in T \; , \; p \in P} [\log(F(p,t)) + \log(F(t,p))]$$

The size of M_0 is

$$size(M_0) = \sum_{p \in P} \log(M_0(p))$$

The size of the marked Petri net (N, M_0) is

$$size(N, M_0) = size(N) + size(M_0)$$

Let us introduce *cyclic* Petri nets.

Definition 2.2 ***(Cyclic Petri nets)***
A marked Petri net (N, M_0) is cyclic *if from every reachable marking M it is possible to come back to M_0 (i.e. $M \in \mathcal{R}S(N, M_0) \Longrightarrow M_0 \in \mathcal{R}S(N, M)$).*

Definition 2.3 ***(The inverse of a Petri net)***
For a Petri net N, its inverse *$\bar{N} = (P, \bar{T}, \bar{F})$ is given by:*

- $\bar{T} = \{\bar{t} \mid t \in T\}$ *and*
- $\bar{F}(\bar{t}, p) = F(p, t)$ *and* $\bar{F}(p, \bar{t}) = F(t, p)$*, for every* $p \in P$ *and* $t \in T$*.*

From the definitions of cyclic Petri nets, we deduce the following lemma.

Lemma 2.4 *Let* (N, M_0) *be a Petri net.* (N, M_0) *is cyclic if and only if* $\mathcal{R}S(N, M_0) \subseteq \mathcal{R}S(\bar{N}, M_0)$*.*

From the work in [2], [3] and [11], we recall the following theorem.

Theorem 2.5 *[2,3,11] For every Petri net* (N, M_0) *such that* $\mathcal{R}S(N, M_0) = \mathcal{R}S(\bar{N}, M_0)$*, we have* $\mathcal{R}S(N, M_0) = U + V^*$*, where* $U \min(\mathcal{R}S(N, M_0))$ *and* $V = \min((\mathcal{R}S(N, M_0) - M_0) \cap \mathbb{N}^{|P|})$*.*

The coverability problem for Petri nets is the problem of deciding for a marked Petri net (N, M_0) and a marking $M \in \mathbb{N}^{|P|}$ whether $M \in \mathcal{C}S(N, M_0)$. This problem has been shown decidable by Karp and Miller [10]. However, the first primitive recursive algorithm for this problem has been given by Rackoff [15]. Let us recall Rackoff's theorem on the coverability problem. This will help us to find a bound on the space complexity of the cyclicity problem (the problem of deciding whether a Petri net is cyclic).

Theorem 2.6 *[15] Let* $N = (P, T, F)$ *be a Petri net and* $M_0, M_f \in \mathbb{N}^{|P|}$ *be two markings. Then* $M_f \in \mathcal{C}S(N, M_0)$ *if and only if there exist a sequence* $\sigma \in T^*$ *and a marking* $M \in \mathbb{N}^{|P|}$ *such that:*

$$\begin{cases} M_0 \xrightarrow{\sigma} M \\ M \geq M_f \\ |\sigma| \leq 2^{2^{c_0 \cdot size(N, M_f) \cdot \log(size(N, M_f))}} \end{cases}$$

where $c_0 > 0$ *is some fixed constant.*

3 Cyclic Petri net reachability sets are semi-linear

In this section, we show that if a Petri net (N, M_0) is cyclic then there exists a Petri net N_c such that $\mathcal{R}S(N, M_0) = \mathcal{R}S(N_c, M_0) = \mathcal{R}S(\bar{N}_c, M_0)$. We first define such a Petri net N_c.

Definition 3.1 *Let* (N, M_0) *be a Petri net. The* ***complete*** *Petri net* N_c *associated to* (N, M_0) *is defined by* $N_c = (P, T_c, F_c)$*, where*

- $T_c = \{t_M \mid t \in T\ ,\ M \in \mathcal{R}S(N, M_0)\ ,\ M \xrightarrow{t}\ ;\ \forall M' \in \mathcal{R}S(N, M_0)\ ,\ M' < M\ ,\ \neg(M' \xrightarrow{t})\}$ *and*
- *for every* $p \in P$*, we have* $F_c(p, t_M) = M(p)$ *and* $F_c(t_M, p) = F(t, p) - F(p, t) + M(p)$*.*

Example 3.2 *For the Petri net defined in figure 1 - a - with the initial marking* $M = (0, 2)$*, we have* $\mathcal{R}S(N, M_0) = \{(x, y) \mid x + y \geq 2\}$ *and* $T_c =$

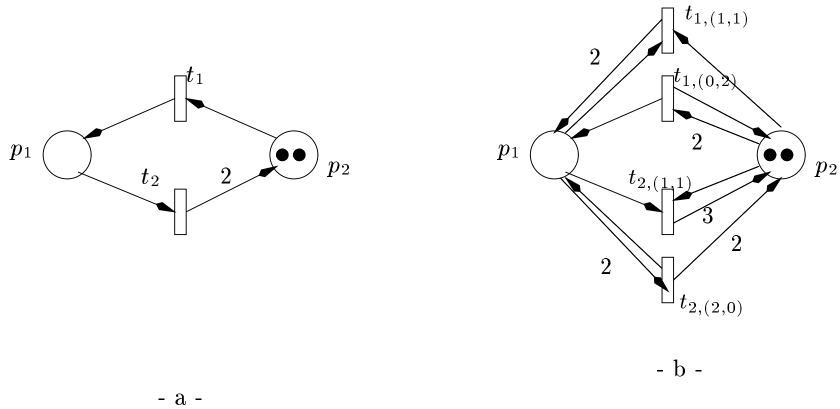

Fig. 1. - a - A Petri net (N, M_0) and - b - its completion N_c.

$\{t_{1,(0,2)}, t_{1,(1,1)}, t_{2,(1,1)}, t_{2,(2,0)}\}$. *The complete Petri net N_c, associated to the marked Petri net (N, M_0), is given in figure 1 - b -.*

From the definition of a complete Petri net, associated to a Petri net (N, M_0), we deduce the following lemma.

Lemma 3.3 *Let (N, M_0) be a Petri net, N_c be the complete Petri net associated to N and $M_1, M_2 \in \mathcal{RS}(N, M_0)$. Then, $M_2 \in \mathcal{RS}(N, M_1)$ if and only if $M_2 \in \mathcal{RS}(N_c, M_1)$.*

Hence, by Lemma 3.3, (N, M_0) is cyclic if and only if (N_c, M_0) is cyclic. Moreover, we have the following lemma.

Lemma 3.4 *The Petri net (N, M_0) is cyclic if and only if $\mathcal{RS}(N, M_0) = \mathcal{RS}(\bar{N}_c, M_0)$.*

Proof. From Lemma 3.3, we have $\mathcal{RS}(N, M_0) = \mathcal{RS}(N_c, M_0)$ and (N, M_0) is cyclic if and only if (N_c, M_0) is cyclic. Hence, to prove this lemma, it suffices to prove that (N_c, M_0) is cyclic if and only if $\mathcal{RS}(N_c, M_0) = \mathcal{RS}(\bar{N}_c, M_0)$. Finally, by using Lemma 2.4, this is equivalent to prove that if (N_c, M_0) is cyclic then $\mathcal{RS}(N_c, M_0) \supseteq \mathcal{RS}(\bar{N}_c, M_0)$. For that, let $M \in \mathcal{RS}(\bar{N}_c, M_0)$. There exists, then, a sequence $\sigma_c \in \bar{T}_c^*$ such that $M_0 \xrightarrow{\sigma_c} M$ in $\bar{N}_c$. We prove, by induction on the length $k = |\sigma_c|$, that $M \in \mathcal{RS}(N_c, M_0)$.

- <u>*Base*</u>: Suppose that $k = 0$. Thus, $M = M_0$ and hence, $M \in \mathcal{RS}(N_c, M_0)$.
- <u>*Induction*</u>: Now let $k \geq 1$. The sequence σ_c is of the form $\sigma_c = \bar{t}_1.\bar{t}_2 \ldots \bar{t}_k$. Let $\sigma'_c = \bar{t}_1.\bar{t}_2 \ldots \bar{t}_{k-1}$. There exists, then, a markings M' such that

$$M_0 \quad \xrightarrow{\sigma'_c} \quad M' \quad \xrightarrow{\bar{t}_k} \quad M \qquad \text{in } \bar{N}_c$$

  By induction hypothesis, we have $M' \in \mathcal{RS}(N_c, M_0)$. Now, let t_k be the transition inverse of $\bar{t}_k$. Hence $t_k \in T_c$ and $M \xrightarrow{t_k} M'$ in N_c. However, by definition of N_c, there exist two markings $M_k, M'_k \in \mathcal{RS}(N, M_0) = \mathcal{RS}(N_c, M_0)$

such that:

$$\begin{cases} M_k \xrightarrow{t_k} M'_k & \text{in } N_c \\ M_k \leq M \end{cases}$$

Thus, $M'_k \leq M'$ and, because (N_c, M_0) is cyclic, there exists a sequence $\sigma'_k \in T^*$ such that $M'_k \xrightarrow{\sigma'_k} M_k$ in N_c. Consequently, we have:

$$M' \xrightarrow{\sigma'_k} M \qquad \text{in } N_c$$

Thus $M \in \mathcal{R}S(N_c, M_0)$.

□

Lemma 3.4 forms an easy proof to the semi-linearity of cyclic Petri nets. The first proof of the semi-linearity of cyclic Petri nets is attributable to Araki and Kasami. It is based on Presburger arithmetic and very difficult to read (see [1]).
By Lemma 3.4 and Theorem 2.5, we have

Theorem 3.5 *For every cyclic Petri net (N, M_0) we have $\mathcal{R}S(N, M_0) = U + V^*$, where $U = \min(\mathcal{R}S(N, M_0))$ and $V = \min((\mathcal{R}S(N, M_0) - M_0) \cap \mathbb{N}^{|P|})$.*

4 Effective construction of cyclic Petri net reachability sets

In this section, we prove that if the Petri net (N, M_0) is cyclic then, for every $M \in \mathcal{R}S(N, M_0)$, we have $size(M) \leq 2^{c.size(N,M_0).\log(size(N,M_0))}$. Our proof is organized as follow: first we prove this result for bounded and cyclic Petri nets then we generalize it to (non bounded) cyclic Petri nets.

4.1 The reachability set for bounded cyclic Petri nets

Bounded Petri nets have finite reachability sets. However, their reachability sets may be very large. In fact, Valk and Vidal-Naquet [16] gave bounded Petri nets with non primitive recursive reachability sets. This is not true for cyclic Petri nets. The following lemma gives an exponential bound on the size of the reachable markings in a bounded and cyclic Petri net.

Lemma 4.1 *Let (N, M_0) be a bounded and cyclic Petri net. Then, for every marking $M \in \mathcal{R}S(N, M_0)$, we have:*

$$size(M) \leq 2^{c_1.size(N,M_0).\log(size(N,M_0))}$$

where $c_1 > 0$ is some fixed constant.

Proof. Because the Petri net (N, M_0) is bounded and cyclic, for every marking $M \in \mathcal{R}S(N, M_0)$, we have:

- (N, M) is bounded and cyclic,
- $M_0 \in \mathcal{R}S(N, M)$ and
- for every marking $M' > M_0$, $M' \notin \mathcal{R}S(N, M)$.

Hence, $M_0 \in \mathcal{C}S(N, M)$ and, for every marking $M' > M_0$, we have $M' \notin \mathcal{C}S(N, M)$.By Theorem 2.6, there is a sequence $\sigma \in T^*$ such that:

$$\begin{cases} M \xrightarrow{\sigma} M_0 \\ |\sigma| \leq 2^{2^{c_0 \cdot size(N,M_0) \cdot \log(size(N,M_0))}} \end{cases}$$

Consequently, we have:

$$size(M) \quad = \quad size(M_0 + \sum_{t \in T} |\sigma|_t.(F(.,t) - F(t,.)))$$

Note that, $|\sigma|_{t_i}$ is the number of occurrences of t_i in σ and $F(.,t)$ (resp. $F(t,.)$) is the vector in $\mathbb{N}^{|P|}$ whose i^{th} component is $F(p_i, t)$ (resp. $F(t, p_i)$).
Hence,

$$\begin{aligned} size(M) &\leq size(M_0) + size(\sum_{t \in T} |\sigma|_t.F(.,t)) + size(\sum_{t \in T} |\sigma|_t.F(t,.)) \\ &\leq size(M_0) + \log(|\sigma|). \sum_{t \in T} (size(F(.,t)) + size(F(t,.))) \\ &\leq size(M_0) + 2^{c_0.size(N,M_0).\log(size(N,M_0))}.size(N) \\ &\leq 2^{(c_0+1).size(N,M_0).\log(size(N,M_0))} \end{aligned}$$

We put $c_1 = c_0 + 1$, this ends the proof. □

4.2 The reachability set for unbounded cyclic Petri nets

Lemma 4.2 *Let (N, M_0) be a cyclic Petri net and $\mathcal{R}S(N, M_0) = U + V^*$ ($U = \min(\mathcal{R}S(N, M_0))$ and $V = \min((\mathcal{R}S(N, M_0) - M_0) \cap \mathbb{N}^{|P|})$). Then, for every marking $M \in (U \cup V)$, we have:*

$$size(M) \leq 2^{c_2.size(N,M_0).\log(size(N,M_0))}$$

where $c_2 > 0$ is some fixed constant.

Proof. Let $N_b = (P_b, T, F_b)$ be the Petri net defined by:

- P_b is the set of bounded places in (N, M_0),
- for every $t \in T$ and for every $p \in P_b$, $F_b(p, t) = F(p, t)$ and $F_b(t, p) = F(t, p)$.

Let $M_{b0} \in \mathbb{N}^{|P_b|}$ such that, for every $p \in P_b$, $M_{b0}(p) = M_0(p)$.
The Petri net (N_b, M_{b0}) is cyclic and bounded. Hence, by Lemma 4.1, for every $M_b \in \mathcal{R}S(N_b, M_{b0})$, we have $size(M_b) \leq 2^{c_1.size(N_b,M_{b0}).\log(size(N_b,M_{b0}))}$.

Let us first compute the size of the elements in V. For that, we define the incidence matrix of N, $C \in \mathbb{Z}^{|P|\times|T|}$ and the set of reachable sequences $L(N_b, M_{b0}, M_{b0})$ as follows:

- $C(i,j) = F(t_j, p_i) - F(p_i, t_j)$ and
- $L(N_b, M_{b0}, M_{b0}) = \{\sigma \in T^* \mid M_{b0} \xrightarrow{\sigma} M_{b0}\}$.

For a sequence $\sigma \in T^*$, we note $\Phi(\sigma)$, the Parikh vector of σ, $\Phi(\sigma) \in \mathbb{N}^{|T|}$ and $\Phi(\sigma)(i) = |\sigma|_{t_i}$([2]). For a set of sequences $L \subseteq T^*$, we note $\Phi(L) = \{\Phi(\sigma) \mid \sigma \in L\}$.

Claim 1. $V = \min\{v \in \mathbb{N}^{|P|} \mid v \neq 0 \ , \exists x \in \Phi(L(N_b, M_{b0}, M_{b0})) \ , C.x = v\}$.

Proof of claim 1.

Let $I = \min\{v \in \mathbb{N}^{|P|} \mid v \neq 0 \ , \exists x \in \Phi(L(N_b, M_{b0}, M_{b0})) \ , C.x = v\}$. We prove by double inclusions that $I = V$:

$I \subseteq V$: Let $v \in I$. There is then a sequence $\sigma \in L(N_b, M_{b0}, M_{b0})$ such that $C.\Phi(\sigma) = v$. Hence, there is $K \in \mathbb{N}$ and $M \in \mathcal{RS}(N, M_0)$ such that $M(p) = M_0(p)$ if $p \in P_b$, $M(p) > K$ if $p \in P - P_b$ and $M \xrightarrow{\sigma} (M + v)$. (N, M_0) is cyclic and $M \in \mathcal{RS}(N, M_0)$ imply that there is $\sigma' \in T^*$ such that $M \xrightarrow{\sigma'} M_0$. Hence, $(M+v) \xrightarrow{\sigma'} (M_0+v)$. Thus $M_0+v \in \mathcal{RS}(N, M_0)$ and for every M such that $M_0 < M < M_0+v$ we have $M \notin \mathcal{RS}(N, M_0)$. Hence, $v \in V$. Thus, $I \subseteq V$.

$V \subseteq I$: Let $v \in V$. Hence $v \neq 0$ and there is a sequence $\sigma \in T^*$ such that $M_0 \xrightarrow{\sigma} (M_0 + v)$ and for every M such that $M_0 < M < M_0 + v$ we have $M \notin \mathcal{RS}(N, M_0)$. Thus, $\sigma \in L(N_b, M_{b0}, M_{b0})$, $C.\Phi(\sigma) = v$ and for every $v' < v$ we have $(M_0 + v') \notin \mathcal{RS}(N, M_0)$. Hence, $v \in I$. Consequently, $V \subseteq I$.

Let us now compute the size of the elements in V. As (N_b, M_{b0}) is cyclic and bounded, we have $\Phi(L(N_b, M_{b0}, M_{b0})) = U' + V'^*$, where, for every $x \in (U' \cup V')$, we have

$$\text{size}(x) \leq 2^{c.size(N_b, M_{b0}).\log(size(N_b, M_{b0}))}$$

Moreover, for every $i \in [1, m]$ and $j \in [1, n]$, we have $|C(i,j)| \leq 2^{size(N, M_0)}$. Thus, for every element $v \in V$, we have $size(v) \leq 2^{c'.size(N, M_0).\log(size(N, M_0))}$.

Let $L(N_b, M_{b0}) = \{\sigma \in T^* \mid M_{b0} \xrightarrow{\sigma}\}$. For a cyclic Petri net (N, M_0), $(\bar{N}, M_0)$ denotes the inverse of (N, M_0) and for a marking $M \in \mathbb{N}^{|P|}$, $\bar{M}$ denotes the marking defined by $\bar{M}(p) = M(p)$, for $p \in P_b$, and $\bar{M}(p) = M(p) + 1$, for $p \in P - P_b$.

Claim 2. $U = \min\{M \in \mathbb{N}^{|P|} \mid \bar{M} \in \mathcal{CS}(\bar{N}, M) \ , \exists \sigma \in L(N_b, M_{b0}) \ , M_0 + \Phi(\sigma) = M\}$.

Proof of claim 2.

Let $J = \min\{M \in \mathbb{N}^{|P|} \mid \bar{M} \in \mathcal{CS}(\bar{N}, M) \ , \exists \sigma \in L(N_b, M_{b0}) \ , M_0 +$

[2] $|\sigma|_{t_i}$ is the number of occurrences of t_i in σ

$\Phi(\sigma) = M\}$. We prove by double inclusions that $J = U$:

$J \subseteq U$: Let $M \in J$. There exist then a sequence $\bar{\sigma}' \in \bar{T}^*$ and an integer $K \in \mathbb{N}$ such that $M \xrightarrow{\bar{\sigma}'} M'$ in $\bar{N}$, where $M'(p) = M(p)$ if $p \in P_b$, $M'(p) > K$ if $p \in P - P_b$, $M'_0 \in \mathcal{RS}(N, M_0)$ and $M'_0 \xrightarrow{\sigma} M'$ ($\sigma \in L(N_b, M_{b0})$. Hence, $M \in \mathcal{RS}(N, M_0)$. Moreover, for every $M'' < M$ either $\bar{M}'' \notin \mathcal{CS}(N, M_0)$ or there is no $\sigma \in T^*$ such that $M_0 + \Phi(\sigma) = M''$. Thus, $M \in U$. Consequently, $J \subseteq U$.

$U \subseteq J$: Let $M \in U$. Then $M \in \mathcal{RS}(N, M_0)$ and for every $M' < M$ we have $M' \notin \mathcal{RS}(N, M_0)$. As (N, M_0) is cyclic, we have (N, M) is cyclic and $\mathcal{RS}(N, M_0) = \mathcal{RS}(N, M)$. Hence, $\bar{M} \in \mathcal{CS}(\bar{N}, M)$ and there is $\sigma \in L(N_b, M_{b0})$ such that $M_0 + \Phi(\sigma) = M$. Thus, $M \in J$. Consequently, $U \subseteq J$.

Let us now compute the size of the elements in U. The coverability problem is decidable in space $2^{c.n.\log(n)}$ (Theorem 2.6) and $\Phi(L(N_b, M_{b0})) = U'' + V''^*$, where, for every $x \in (U'' \cup V'')$, we have $\text{size}(x) \leq 2^{c''.size(N_b,M_{b0}).\log(size(N_b,M_{b0}))}$. Hence, for every element $u \in U$, we have $size(u) \leq 2^{c'''.size(N,M_0).\log(size(N,M_0))}$. Let $c_2 = \max\{c', c'''\}$. We have then, for every $M \in (U \cup V)$,

$$size(M) \leq 2^{c_2.size(N,M_0).\log(size(N,M_0))}$$

□

4.3 The reachability set for cyclic Petri nets

From Lemma 4.1 and Lemma 4.2 we have the following theorem.

Theorem 4.3 *For every cyclic Petri net (N, M_0), there exist two finite subsets U and V of $\mathbb{N}^m$ such that:*
1. *$\mathcal{RS}(N, M_0) = U + V^*$ and*
2. *$size(M) < 2^{c.size(N,M_0).\log(size(N,M_0))}$, for every $M \in (U \cup V)$,*
where c is a constant independent of N, M_0 and M.

Consequently, we can conclude by the following theorem.

Theorem 4.4 *The cyclicity problem and the reachability, the coverability, the boundedness, the liveness, the regularity and the inclusion problems for cyclic Petri nets are decidable in space $2^{c.n.\log(n)}$.*

References

[1] **Araki T., Kasami T.** *"Decidability problems on the strong connectivity of Petri net reachability sets"*, TCS 4, pp. 99-119, 1977.

[2] **Bouziane Z.** *"Algorithmes primitifs récursifs et problèmes EXPSPACE-complets dans les réseaux de Petri cycliques"*, Thèse de Doctorat de l'ENS de Cachan, November 1996.

[3] **Bouziane Z., Finkel F.** *"The equivalence problem for commutative semigroups and reversible Petri nets is complete in exponential space under log-lin reducibility"*, Presented at the International Conference in Semigroups and Related Topics at Kunming, China, August 1995 (Springer Verlag Proceeding 1997).

[4] **Cardoza E. Lipton R. Meyer A.** *"Exponential space complete problems for Petri nets and commutative semigroups"*, Proc. of the 8^{th} annual ACM Symposium on Theory of Computing, pp. 50-54, 1976.

[5] **Commoner F. Holt A.W. Even S. Pnueli A.** *"Marked directed graphs"*, Journal of Computer and System Sciences, 5, pp. 511-523, 1971.

[6] **Desel J. Esparza J.** *"Free choice Petri nets"*, Cambridge University Press, 1995.

[7] **Esparza J. Nielson M.** *"Decidability issues for Petri nets"*, Bulletin EATCS, 52, 1994.

[8] **Hack M.** *"The equality problem for vector addition systems is undecidable"*, Computation structures Group Memo 121, Project MAC, M.I.T. April 1975. Also TCS 2 1976 (77-95).

[9] **Howell R. Rosier L.** *"Completeness results for conflict-free vector replacement systems"*, Journal of Computer and System Sciences 37, pp. 349-366, 1988.

[10] **Karp R., Miller R.** *"Parallel program schemata"*, J. of Computer and System Science, Vol. 3, No. 4, May 1969.

[11] **Koppenhagen E., Mayr E.** "*The complexity of the equivalence problem for commutative semigroups"*, Technical Report TUM-I9603, 1996.

[12] **Kosaraju R.** *"Decidability of reachability in Vector Addition Systems"*, Proc. of the 14^{th} Annual ACM Symposium on Theory of Computing pp. 267-280, 1982.

[13] **Mayr E.** *"An algorithm for the general Petri net reachability problem"*, SIAM J. Compu. 13, 3, pp. 441-46-, 1984.

[14] **Mayr E. Meyer A.** *"The complexity of the word problems for commutative semigroups and polynomial ideals"*, Advances in Mathematics 46, pp. 305-329, 1982.

[15] **Rackoff C.** *"The covering and boundedness problems for vector addition systems"*, in Theoretical Computer Science 6, pp. 223-231, North Holland, 1978.

[16] **Valk R., Vidal-Naquet G.** *"Petri nets and regular languages"*, J. of Computer and System Sciences, Vol. 23, No. 3, pp. 299-325, December 1981.