Zero-Reachability in Probabilistic Multi-Counter Automata

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***Abstract*—We study the qualitative and quantitative zero-reachability problem in probabilistic multi-counter systems. We identify the undecidable variants of the problems, and then we concentrate on the remaining two cases. In the first case, when we are interested in the probability of all runs that visit zero in *some* counter, we show that the qualitative zero-reachability is decidable in time which is polynomial in the size of a given pMC and doubly exponential in the number of counters. Further, we show that the probability of all zero-reaching runs can be effectively approximated up to an arbitrarily small given error $\varepsilon > 0$ in time which is polynomial in $\log(\varepsilon)$, exponential in the size of a given pMC, and doubly exponential in the number of counters. In the second case, we are interested in the probability of all runs that visit zero in some counter different from the last counter. Here we show that the qualitative zero-reachability is decidable and SQUAREROOTSUM-hard, and the probability of all zero-reaching runs can be effectively approximated up to an arbitrarily small given error $\varepsilon > 0$ (these result applies to pMC satisfying a suitable technical condition that can be verified in polynomial time). The proof techniques invented in the second case allow to construct counterexamples for some classical results about ergodicity in stochastic Petri nets.**

I. INTRODUCTION

A *probabilistic multi-counter automaton (pMC)* $\mathcal{A}$ of dimension $d \in \mathbb{N}$ is an abstract fully probabilistic computational device equipped with a finite-state control unit and d unbounded counters that can store non-negative integers. A *configuration* $p\boldsymbol{v}$ of $\mathcal{A}$ is given by the current control state p and the vector of current counter values $\boldsymbol{v}$. The dynamics of $\mathcal{A}$ is defined by a finite set of *rules* of the form (p, α, c, q) where p is the current control state, q is the next control state, α is a d-dimensional vector of counter changes ranging over $\{-1, 0, 1\}^d$, and c is a subset of counters that are tested for zero. Moreover, each rule is assigned a positive integer *weight*. A rule (p, α, c, q) is *enabled* in a configuration $p\boldsymbol{v}$ if the set of all counters with zero value in $\boldsymbol{v}$ is precisely c and no component of $\boldsymbol{v} + \alpha$ is negative; such an enabled rule can be *fired* in $p\boldsymbol{v}$ and generates a *probabilistic transition* $p\boldsymbol{v} \xrightarrow{x} q(\boldsymbol{v}+\alpha)$ where the probability x is equal to the weight of the rule divided by the total weight of all rules enabled in $p\boldsymbol{v}$. A special subclass of pMC are *probabilistic vector addition systems with states (pVASS)*, which are equivalent to (discrete-time) *stochastic Petri nets (SPN)*. Intuitively, a pVASS is a pMC where no subset of counters is tested for zero explicitly (see Section II for a precise definition).

The decidability and complexity of basic qualitative/quantitative problems for pMCs has so far been studied mainly in the one-dimensional case, and there are also some results about unbounded SPN (a more detailed overview of the existing results is given below). In this paper, we consider *multi-dimensional* pMC and the associated *zero-reachability* problem. That is, we are interested in the probability of all runs initiated in a given $p\boldsymbol{v}$ that eventually visit a "zero configuration". Since there are several counters, the notion of "zero configuration" can be formalized in various ways (for example, we might want to have zero in some counter, in all counters simultaneously, or in a given subset of counters). Therefore, we consider a general *stopping criterion* $\mathcal{Z}$ which consists of *minimal* subsets of counters that are required to be simultaneously zero. For example, if $\mathcal{Z} = \mathcal{Z}_{all} = \{\{1\}, \ldots, \{d\}\}$, then a run is stopped when reaching a configuration with zero in *some* counter; and if we put $\mathcal{Z} = \{\{1, 2\}\}$, then a run is stopped when reaching a configuration with zero in counters 1 and 2 (and possibly also in other counters). We use $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}))$ to denote the probability of all runs initiated in $p\boldsymbol{v}$ that reach a configuration satisfying the stopping criterion $\mathcal{Z}$. The main algorithmic problems considered in this paper are the following:

- *Qualitative $\mathcal{Z}$-reachability:* Is $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z})) = 1$?
- *Approximation:* Can $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}))$ be approximated up to a given absolute/relative error $\varepsilon > 0$?

We start by observing that the above problems are not effectively solvable in general, and we show that there are only two potentially decidable cases, where $\mathcal{Z}$ is equal either to $\mathcal{Z}_{all}$ (Case I) or to $\mathcal{Z}_{-i} = \mathcal{Z}_{all} \smallsetminus \{\{i\}\}$ (Case II). Recall that if $\mathcal{Z} = \mathcal{Z}_{all}$, then a run is stopped when some counter reaches

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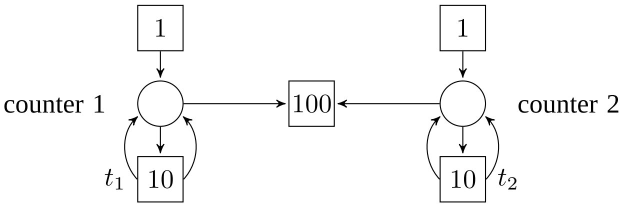

Fig. 1: Firing process may not be ergodic.

zero; and if $\mathcal{Z} = \mathcal{Z}_{-i}$, then a run is stopped when a counter different from i reaches zero. Cases I and II are analyzed independently and the following results are achieved:

Case I: We show that the qualitative $\mathcal{Z}_{all}$-reachability problem is decidable in time polynomial in $|\mathcal{A}|$ and doubly exponential in d. In particular, this means that the problem is decidable in *polynomial time for every fixed* d. Then, we show that $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all}))$ can be effectively approximated up to a given absolute/relative error $\varepsilon > 0$ in time which is polynomial in $|\varepsilon|$, exponential in $|\mathcal{A}|$, and doubly exponential in d (in the special case when $d = 1$, the problem is known to be solvable in time polynomial in $|\mathcal{A}|$ and $|\varepsilon|$, see [19]).

Case II: We analyze Case II only under a technical assumption that counter i is not critical; roughly speaking, this means that counter i has either a tendency to increase or a tendency to decrease when the other counters are positive. The problem whether counter i is critical or not is solvable in time polynomial in $|\mathcal{A}|$, so we can efficiently check whether a given pMC can be analyzed by our methods.

Under the mentioned assumption, we show how to construct a suitable martingale which captures the behaviour of certain runs in $\mathcal{A}$. Thus, we obtain a new and versatile tool for analyzing quantitative properties of runs in multi-dimensional pMC, which is more powerful than the martingale of [14] constructed for one-dimensional pMC. Using this martingale and the results of [8], we show that the qualitative $\mathcal{Z}_{-i}$-reachability problem is decidable. We also show that the problem is SQUARE-ROOM-SUM-hard, even for two-dimensional pMC satisfying the mentioned technical assumption. Further, we show that $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{-i}))$ can be effectively approximated up to a given absolute error $\varepsilon > 0$. The main reason why we do not provide any upper complexity bounds in Case II is a missing upper bound for coverability in VAS with one zero test (see [8]).

It is worth noting that the techniques developed in Case II reveal the existence of phenomena that should not exist according to the previous results about ergodicity in SPN. A classical paper in this area [23] has been written by Florin & Natkin in 80s. In the paper, it is claimed that if the state-space of a given SPN (with arbitrarily many unbounded places) is strongly connected, then the firing process is ergodic (see Section IV.B. in [23]). In the setting of discrete-time probabilistic Petri nets, this means that for almost all runs, the limit frequency of transitions performed along a run is defined and takes the same value. However, in Fig. 1 there is an example of a pVASS (depicted as SPN with weighted transitions) with two counters (places) and strongly connected state space where the limit frequency of transitions may take two eligible values (each with probability $1/2$). Intuitively, if both counters are positive, then both of them have a tendency to decrease (i.e., the trend of the only BSCC of $\mathcal{F}_{\mathcal{A}}$ is negative in both components, see Section III-A). However, if we reach a configuration where the first counter is zero and the second counter is sufficiently large, then the second counter starts to *increase*, i.e., it never becomes zero again with some positive probability (cf. the *oc-trend* of the only BSCC D of $\mathcal{B}_1$ introduced in Section III-B). The first counter stays zero for most of the time, because when it becomes positive, it is immediatelly emptied with a very large probability. This means that the frequency of firing t_2 will be much higher than the frequency of firing t_1. When we reach a configuration where the first counter is large and the second counter is zero, the situation is symmetric, i.e., the frequency of firing t_1 becomes much higher than the frequency of firing t_2. Further, almost every run eventually behaves according to one the two scenarios, and therefore there are two eligible limit frequencies of transitions, each of which is taken with probability $1/2$. So, we must unfortunately conclude that the results of [23] are not valid for general SPN.

Related Work. One-dimensional pMC and their extensions into decision processes and games were studied in [12], [20], [14], [19], [11], [21], [10]. In particular, in [19] it was shown that termination probability (a "selective" variant of zero-reachability) in one-dimensional pMC can be approximated up to an arbitrarily small given error in polynomial time. In [14], it was shown how to construct a martingale for a given one-dimensional pMC which allows to derive tail bounds on termination time (we use this martingale in Section III-A).

There are also many papers about SPN (see, e.g., [28], [5]), and some of these works also consider algorithmic aspects of unbounded SPN (see, e.g., [1], [22], [23]).

Considerable amount of papers has been devoted to algorithmic analysis of so called probabilistic lossy channel systems (PLCS) and their game extensions (see e.g. [24], [7], [2], [4], [3]). PLCS are a stochastic extension of lossy channel systems, i.e., an infinite-state model comprising several interconnected queues coupled with a finite-state control unit. The main ingredient, which makes results about PLCS incomparable with our results on pMCs, is that queues may lose messages with a fixed loss-rate, which substantially simplifies the associated analysis.

II. PRELIMINARIES

We use $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}^+$, $\mathbb{Q}$, and $\mathbb{R}$ to denote the set of all integers, non-negative integers, positive integers, rational numbers, and real numbers, respectively.

Let $\mathcal{V} = (V, L, \rightarrow)$, where V is a non-empty set of vertices, L a non-empty set of *labels*, and $\rightarrow \subseteq V \times L \times V$ a *total* relation (i.e., for every $v \in V$ there is at least one *outgoing* transition $(v, \ell, u) \in \rightarrow$). As usual, we write $v \xrightarrow{\ell} u$ instead of $(v, \ell, u) \in \rightarrow$, and $v \rightarrow u$ iff $v \xrightarrow{\ell} u$ for some $\ell \in L$. The reflexive and transitive closure of $\rightarrow$ is denoted by $\rightarrow^*$. A *finite path* in $\mathcal{V}$ of *length* $k \geq 0$ is a finite sequence of the form $v_0 \ell_0 v_1 \ell_1 \ldots \ell_{k-1} v_k$, where $v_i \xrightarrow{\ell_i} v_{i+1}$ for all $0 \leq i < k$.

The length of a finite path w is denoted by $length(w)$. A *run* in $\mathcal{V}$ is an infinite sequence w of vertices such that every finite prefix of w ending in a vertex is a finite path in $\mathcal{V}$. The individual vertices of w are denoted by $w(0), w(1), \ldots$. The sets of all finite paths and all runs in $\mathcal{V}$ are denoted by $FPath_{\mathcal{V}}$ and $Run_{\mathcal{V}}$, respectively. The sets of all finite paths and all runs in $\mathcal{V}$ that start with a given finite path w are denoted by $FPath_{\mathcal{V}}(w)$ and $Run_{\mathcal{V}}(w)$, respectively. A *strongly connected component (SCC)* of $\mathcal{V}$ is a maximal subset $C \subseteq V$ such that for all $v, u \in C$ we have that $v \to^* u$. A SCC C of $\mathcal{V}$ is a *bottom SCC (BSCC)* of $\mathcal{V}$ if for all $v \in C$ and $u \in V$ such that $v \to u$ we have that $u \in C$.

We assume familiarity with basic notions of probability theory, e.g., *probability space*, *random variable*, or the *expected value*. As usual, a *probability distribution* over a finite or countably infinite set A is a function $f : A \to [0,1]$ such that $\sum_{a \in A} f(a) = 1$. We call f *positive* if $f(a) > 0$ for every $a \in A$, and *rational* if $f(a) \in \mathbb{Q}$ for every $a \in A$.

Definition 1. *A* labeled Markov chain *is a tuple $\mathcal{M} = (S, L, \to, Prob)$ where $S \neq \emptyset$ is a finite or countably infinite set of* states, *$L \neq \emptyset$ is a finite or countably infinite set of* labels, *$\to \subseteq S \times L \times S$ is a total* transition relation, *and $Prob$ is a function that assigns to each state $s \in S$ a positive probability distribution over the outgoing transitions of s. We write $s \xrightarrow{\ell,x} t$ when $s \xrightarrow{\ell} t$ and x is the probability of (s, ℓ, t).*

If $L = \{\ell\}$ is a singleton, we say that $\mathcal{M}$ is *non-labeled*, and we omit both L and ℓ when specifying $\mathcal{M}$ (in particular, we write $s \xrightarrow{x} t$ instead of $s \xrightarrow{\ell,x} t$). To every $s \in S$ we associate the standard probability space $(Run_{\mathcal{M}}(s), \mathcal{F}, \mathcal{P})$ of runs starting at s, where $\mathcal{F}$ is the σ-field generated by all *basic cylinders* $Run_{\mathcal{M}}(w)$, where w is a finite path starting at s, and $\mathcal{P} : \mathcal{F} \to [0,1]$ is the unique probability measure such that $\mathcal{P}(Run_{\mathcal{M}}(w)) = \prod_{i=1}^{length(w)} x_i$ where x_i is the probability of $w(i{-}1) \xrightarrow{\ell_{i-1}} w(i)$ for every $1 \leq i \leq length(w)$. If $length(w) = 0$, we put $\mathcal{P}(Run_{\mathcal{M}}(w)) = 1$.

Now we introduce probabilistic multi-counter automata (pMC). For technical convenience, we consider *labeled* rules, where the associated finite set of labels always contains a distinguished element τ. The role of the labels becomes clear in Section III-B where we abstract a (labeled) one-dimensional pMC from a given multi-dimensional one.

Definition 2. *Let L be a finite set of labels such that $\tau \in L$, and let $d \in \mathbb{N}^+$. An L-labeled d-dimensional* probabilistic multi-counter automaton (pMC) *is a triple $\mathcal{A} = (Q, \gamma, W)$, where*

- *Q is a finite set of* states,
- *$\gamma \subseteq Q \times \{-1,0,1\}^d \times 2^{\{1,\ldots,d\}} \times L \times Q$ is a set of* rules *such that for all $p \in Q$ and $c \subseteq \{1, \ldots, d\}$ there is at least one outgoing rule of the form $(p, \boldsymbol{\alpha}, c, \ell, q)$,*
- *$W : \gamma \to \mathbb{N}^+$ is a* weight assignment.

The encoding size of $\mathcal{A}$ is denoted by $|\mathcal{A}|$, where the weights used in W and the counter indexes used in γ are encoded in binary.

A *configuration* of $\mathcal{A}$ is an element of $Q \times \mathbb{N}^d$, written as $p\boldsymbol{v}$. We use $Z(p\boldsymbol{v}) = \{i \mid 1 \leq i \leq d, \boldsymbol{v}[i] = 0\}$ to denote the set of all counters that are zero in $p\boldsymbol{v}$. A rule $(p, \boldsymbol{\alpha}, c, \ell, q) \in \gamma$ is *enabled* in a configuration $p\boldsymbol{v}$ if $Z(p\boldsymbol{v}) = c$ and for all $1 \leq i \leq d$ where $\boldsymbol{\alpha}[i] = -1$ we have that $\boldsymbol{v}[i] > 0$.

The semantics of a $\mathcal{A}$ is given by the associated L-labeled Markov chain $\mathcal{M}_{\mathcal{A}}$ whose states are the configurations of $\mathcal{A}$, and the outgoing transitions of a configuration $p\boldsymbol{v}$ are determined as follows:

- If no rule of γ is enabled in $p\boldsymbol{v}$, then $p\boldsymbol{v} \xrightarrow{\tau,1} p\boldsymbol{v}$ is the only outgoing transition of $p\boldsymbol{v}$;
- otherwise, for every rule $(p, \boldsymbol{\alpha}, c, \ell, q) \in \gamma$ enabled in $p\boldsymbol{v}$ there is a transition $p\boldsymbol{v} \xrightarrow{x,\ell} q\boldsymbol{u}$ such that $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{\alpha}$ and $x = W((p, \boldsymbol{\alpha}, c, \ell, q))/T$, where T is the total weight of all rules enabled in $p\boldsymbol{v}$.

When $L = \{\tau\}$, we say that $\mathcal{A}$ is *non-labeled*, and both L and τ are omitted when specifying $\mathcal{A}$. We say that $\mathcal{A}$ is a *probabilistic vector addition system with states (pVASS)* if no subset of counters is tested for zero, i.e., for every $(p, \boldsymbol{\alpha}, \ell, q) \in Q \times \{-1,0,1\}^d \times L \times Q$ we have that γ contains either all rules of the form $(p, \boldsymbol{\alpha}, c, \ell, q)$ (for all $c \subseteq \{1, \ldots, d\}$) with the same weight, or no such rule. For every configuration $p\boldsymbol{v}$, we use $state(p\boldsymbol{v})$ and $cval(p\boldsymbol{v})$ to denote the control state p and the vector of counter values $\boldsymbol{v}$, respectively. We also use $cval_i(p\boldsymbol{v})$ to denote $\boldsymbol{v}[i]$.

Qualitative zero-reachability. A *stopping criterion* is a non-empty set $\mathcal{Z} \subseteq 2^{\{1,\ldots,d\}}$ of pairwise incomparable non-empty subsets of counters. For every configuration $p\boldsymbol{v}$, let $Run(p\boldsymbol{v}, \mathcal{Z})$ be the set of all $w \in Run(p\boldsymbol{v})$ such that there exist $k \in \mathbb{N}$ and $\varrho \in \mathcal{Z}$ satisfying $\varrho \subseteq Z(w(i))$. Intuitively, $\mathcal{Z}$ specifies the minimal subsets of counters that must be *simultaneously* zero to stop a run. The *qualitative $\mathcal{Z}$-reachability problem* is formulated as follows:

Instance: A d-dimensional pMC $\mathcal{A}$ and a control state p of $\mathcal{A}$.
Question: Do we have $\mathcal{P}(Run(p\mathbf{1}, \mathcal{Z})) = 1$?

Here $\mathbf{1} = (1, \ldots, 1)$ is a d-dimensional vector of 1's. We also use $Run(p\boldsymbol{v}, \neg\mathcal{Z})$ to denote $Run(p\boldsymbol{v}) \smallsetminus Run(p\boldsymbol{v}, \mathcal{Z})$, and we say that $w \in FPath(p\boldsymbol{v})$ is *$\mathcal{Z}$-safe* if for all $w(i)$ where $0 \leq i < length(w)$ and all $\varrho \in \mathcal{Z}$ we have that $\varrho \not\subseteq Z(w(i))$.

III. The Results

We start by observing that the qualitative zero-reachability problem is undecidable in general, and we identify potentially decidable subcases.

Observation 1. *Let $\mathcal{Z} \subseteq 2^{\{1,\ldots,d\}}$ be a stopping criterion satisfying one of the following conditions:*

(a) *there is $\varrho \in \mathcal{Z}$ with more than one element;*
(b) *there are $i, j \in \{1, \ldots, d\}$ such that $i \neq j$ and for every $\varrho \in \mathcal{Z}$ we have that $\{i, j\} \cap \varrho = \emptyset$.*

Then, the qualitative $\mathcal{Z}$-reachability problem is undecidable, *even if the set of instances is restricted to pairs $(\mathcal{A}, p)$ such that $\mathcal{P}(Run(p\mathbf{1}, \mathcal{Z}))$ is either 0 or 1 (hence, $\mathcal{P}(Run(p\mathbf{1}, \mathcal{Z}))$ cannot be effectively approximated up to an absolute error smaller than 0.5).*

A proof of Observation 1 is immediate. For a given Minsky machine M (see [27]) with two counters initialized to one, we construct pMCs $\mathcal{A}_a$ and $\mathcal{A}_b$ of dimension 2 and 3, respectively, and a control state p such that

- if M halts, then $\mathcal{P}(Run_{\mathcal{M}_{\mathcal{A}_a}}(p\mathbf{1},\{\{1,2\}\})) = 1$ and $\mathcal{P}(Run_{\mathcal{M}_{\mathcal{A}_b}}(p\mathbf{1},\{\{3\}\})) = 1$;
- if M does not halt, then $\mathcal{P}(Run_{\mathcal{M}_{\mathcal{A}_a}}(p\mathbf{1},\{\{1,2\}\})) = 0$ and $\mathcal{P}(Run_{\mathcal{M}_{\mathcal{A}_b}}(p\mathbf{1},\{\{3\}\})) = 0$.

The construction of $\mathcal{A}_a$ and $\mathcal{A}_b$ is trivial (and hence omitted). Note that $\mathcal{A}_b$ can faithfully simulate the instructions of M using the counters 1 and 2. The third counter is decreased to zero only when a control state corresponding to the halting instruction of M is reached. Similarly, $\mathcal{A}_a$ simulates the instructions of M using its two counters, but here we need to ensure that a configuration where *both* counters are simultaneously zero is entered iff a control state corresponding to the halting instruction of M is reached. This is achieved by increasing both counters by 1 initially, and then decreasing/increasing counter i before/after simulating a given instruction of M operating on counter i.

Note that the construction of $\mathcal{A}_a$ and $\mathcal{A}_b$ can trivially be adapted to pMCs of higher dimensions satisfying the conditions (a) and (b) of Observation 1, respectively. However, there are two cases not covered by Observation 1:

I. $\mathcal{Z}_{all} = \{\{1\},\ldots,\{d\}\}$, i.e., a run is stopped when *some* counter reaches zero.
II. $\mathcal{Z}_{-i} = \{\{1\},\ldots,\{d\}\} \smallsetminus \{\{i\}\}$ where $i \in \{1,\ldots,d\}$, i.e., a run is stopped when a counter different from i reaches zero. The counters different from i are called *stopping counters*.

These cases are analyzed in the following subsections.

A. Zero-Reachability, Case I

For the rest of this section, let us fix a (non-labeled) pMC $\mathcal{A} = (Q,\gamma,W)$ of dimension $d \in \mathbb{N}^+$ and a configuration $p\boldsymbol{v}$.

Our aim is to identify the conditions under which $\mathcal{P}(Run(p\boldsymbol{v},\neg\mathcal{Z}_{all})) > 0$. To achieve that, we first consider a (non-labeled) finite-state Markov chain $\mathcal{F}_\mathcal{A} = (Q, \hookrightarrow, Prob)$ where $q \overset{x}{\hookrightarrow} r$ iff

$$x \quad = \sum_{(q,\boldsymbol{\alpha},\emptyset,r)\in\gamma} P_\emptyset(q,\boldsymbol{\alpha},\emptyset,r) \quad > \quad 0.$$

Here $P_\emptyset : \gamma \to [0,1]$ is the probability assignment for the rules defined as follows (we write $P_\emptyset(q,\boldsymbol{\alpha},\emptyset,r)$ instead of $P_\emptyset((q,\boldsymbol{\alpha},\emptyset,r))$):

- For every rule $(p,\boldsymbol{\alpha},c,q)$ where $c \neq \emptyset$ we put $P_\emptyset(p,\boldsymbol{\alpha},c,q) = 0$.
- $P_\emptyset(p,\boldsymbol{\alpha},\emptyset,q) = W((p,\alpha,\emptyset,q))/T$, where T is the total weight of all rules of the form $(p,\boldsymbol{\alpha}',\emptyset,q')$.

Intuitively, a state q of $\mathcal{F}_\mathcal{A}$ captures the behavior of configurations $q\boldsymbol{u}$ where all components of $\boldsymbol{u}$ are positive.

Further, we partition the states of Q into SCCs $C_1,\ldots,C_m$ according to $\hookrightarrow$. Note that every run $w \in Run(p\boldsymbol{v})$ eventually *stays* in precisely one C_j, i.e., there is precisely one $1 \leq j \leq m$ such that for some $k \in \mathbb{N}$, the control state of every $w(k')$, where $k' \geq k$, belongs to C_i. We use $Run(p\boldsymbol{v},C_j)$ to denote the set of all $w \in Run(p\boldsymbol{v},\neg\mathcal{Z}_{all})$ that stay in C_j. Obviously,

$$Run(p\boldsymbol{v},\neg\mathcal{Z}_{all}) = Run(p\boldsymbol{v},C_1) \uplus \cdots \uplus Run(p\boldsymbol{v},C_m).$$

For any $n \in \mathbb{N}$ denote by P_n the probability that a run w initiated in $p\boldsymbol{v}$ satisfies the following for every $0 \leq i \leq n$: $state(w(i))$ does not belong to any BSCC of $\mathcal{F}_\mathcal{A}$ and $Z(w(i)) = \emptyset$. The following lemma shows that P_n decays exponentially fast.

Lemma 1. *For any $n \in \mathbb{N}$ we have*

$$P_n \leq (1 - p_{min}^{|Q|})^{\lfloor \frac{n}{|Q|} \rfloor},$$

where p_{min} is the minimal positive transition probability in $\mathcal{M}_\mathcal{A}$. In particular, for any non-bottom SCC C of $\mathcal{F}_\mathcal{A}$ we have $\mathcal{P}(Run(p\boldsymbol{v},C)) = 0$.

Proof: The lemma immediately follows from the fact that for every configuration $p\boldsymbol{v}$ there is a path (in $\mathcal{A}$) of length at most $|Q|$ to a configuration $q\boldsymbol{u}$ satisfying either $Z(q\boldsymbol{u}) \neq \emptyset$ or $q \in D$ for some BSCC D of $\mathcal{F}_\mathcal{A}$. ∎

Now, let C be a BSCC of $\mathcal{F}_\mathcal{A}$. For every $q \in C$, let $change^q$ be a d-dimensional vector of *expected counter changes* given by

$$change^q_i = \sum_{(q,\boldsymbol{\alpha},\emptyset,r)\in\gamma} P_\emptyset(q,\boldsymbol{\alpha},\emptyset,r)\cdot\boldsymbol{\alpha}[i]\,.$$

Note that C can be seen as a finite-state irreducible Markov chain, and hence there exists the unique *invariant distribution* μ on the states of C (see, e.g., [25]) satisfying

$$\mu(q) \quad = \sum_{r\overset{x}{\hookrightarrow}q} \mu(r)\cdot x\,.$$

The *trend* of C is a d-dimensional vector $\boldsymbol{t}$ defined by

$$\boldsymbol{t}[i] \quad = \sum_{q\in C} \mu(q)\cdot change^q_i\,.$$

Further, for every $i \in \{1,\ldots,d\}$ and every $q \in C$, we denote by $botfin_i(q)$ the *least* $j \in \mathbb{N}$ such that for every configuration $q\boldsymbol{u}$ where $\boldsymbol{u}[i] = j$, there is *no* $w \in FPath_{\mathcal{M}_\mathcal{A}}(q\boldsymbol{u})$ where counter i is zero in the last configuration of w and all counters stay positive in every $w(k)$, where $0 \leq k < length(w)$. If there is no such j, we put $botfin_i(q) = \infty$. It is easy to show that if $botfin_i(q) < \infty$, then $botfin_i(q) \leq |C|$; and if $botfin_i(q) = \infty$, then $botfin_i(r) = \infty$ for all $r \in C$. Moreover, if $botfin_i(q) < \infty$, then there is a $\mathcal{Z}_{-i}$-safe finite path of length at most $|C|-1$ from $q\boldsymbol{u}$ to a configuration with i-th counter equal to 0, where $\boldsymbol{u}[i] = botfin_i(q) - 1$ and $\boldsymbol{u}[\ell] = |C|$ for $\ell \neq i$. In particular, the number $botfin_i(q)$ is computable in time polynomial in $|C|$.

We say that counter i is *decreasing* in C if $botfin_i(q) = \infty$ for some (and hence all) $q \in C$.

Definition 3. *Let C be a BSCC of $\mathcal{F}_\mathcal{A}$ with trend $\boldsymbol{t}$, and let $i \in \{1,\ldots,d\}$. We say that counter i is* diverging *in C if either $\boldsymbol{t}[i] > 0$, or $\boldsymbol{t}[i] = 0$ and the counter i is not decreasing in C.*

Intuitively, our aim is to prove that $\mathcal{P}(Run(p\boldsymbol{v},C)) > 0$ iff all counters are diverging in C and $p\boldsymbol{v}$ can reach a configuration $q\boldsymbol{u}$ (via a $\mathcal{Z}_{all}$-safe finite path) where all components of $\boldsymbol{u}$ are "sufficiently large". To analyze the individual counters, for every $i \in \{1,\ldots,d\}$ we introduce a (labeled) *one-dimensional* pMC which faithfully simulates the behavior of counter i and "updates" the other counters just symbolically in the labels.

Definition 4. *Let $L = \{-1, 0, 1\}^{d-1}$, and let $\mathcal{B}_i = (Q, \hat{\gamma}, \hat{W})$ be an L-labeled pMC of dimension one such that*

- $(q, j, \emptyset, \boldsymbol{\beta}, r) \in \hat{\gamma}$ *iff* $(q, \langle \boldsymbol{\beta}, j\rangle_i, \emptyset, r) \in \gamma$;
- $(q, j, \{1\}, \boldsymbol{\beta}, r) \in \hat{\gamma}$ *iff* $(q, \langle \boldsymbol{\beta}, j\rangle_i, \{i\}, r) \in \gamma$;
- $\hat{W}(q, j, \emptyset, \boldsymbol{\beta}, r) = W(q, \langle \boldsymbol{\beta}, j\rangle_i, \emptyset, r)$.
- $\hat{W}(q, j, \{1\}, \boldsymbol{\beta}, r) = W(q, \langle \boldsymbol{\beta}, j\rangle_i, \{i\}, r)$.

Here, $\langle (j_1, \dots, j_{d-1}), j\rangle_i = (j_1, \dots, j_{i-1}, j, j_i, \dots, j_{d-1})$.

Observe that the symbolic updates of the counters different from i "performed" in the labels of $\mathcal{B}_i$ mimic the real updates performed by $\mathcal{A}$ in configurations where all of these counters are positive.

Given a run $w \equiv p_0(v_0)\,\boldsymbol{\alpha}_0\, p_1(v_1)\,\boldsymbol{\alpha}_1\, p_2(v_2)\,\boldsymbol{\alpha}_2 \dots$ in $Run_{\mathcal{M}_\mathcal{B}}(p_0(v_0))$ and $k \in \mathbb{N}$, we denote by $tot(w;k)$ the vector $\sum_{n=0}^{k-1} \boldsymbol{\alpha}_n$, and given $j \in \{1, \dots, d\} \smallsetminus \{i\}$, we denote by $tot_j(w;k)$ the number $\sum_{n=0}^{k-1} \boldsymbol{\alpha}_n[j]$ (i.e., the j-th component of $\sum_{n=0}^{k-1} \boldsymbol{\alpha}_n$).

Let Υ_i be a function which for a given run $w \equiv p_0\boldsymbol{v}_0\, p_1\boldsymbol{v}_1\, p_2\boldsymbol{v}_2 \dots$ of $Run_{\mathcal{M}_\mathcal{A}}(p\boldsymbol{v}, \neg\mathcal{Z}_{-i})$ returns a run $\Upsilon_i(w) \equiv p_0(\boldsymbol{v}_0[i])\,\boldsymbol{\alpha}_0\, p_1(\boldsymbol{v}_1[i])\,\boldsymbol{\alpha}_1\, p_2(\boldsymbol{v}_2[i])\,\boldsymbol{\alpha}_2 \dots$ of $Run_{\mathcal{M}_{\mathcal{B}_i}}(p(\boldsymbol{v}[i]))$ where the label $\boldsymbol{\alpha}_j$ corresponds to the update in the abstracted counters performed in the transition $p_j\boldsymbol{v}_j \rightarrow p_{j+1}\boldsymbol{v}_{j+1}$, i.e., $\boldsymbol{v}_{j+1} - \boldsymbol{v}_j = \langle \boldsymbol{\alpha}_j, \boldsymbol{v}_{j+1}[i] - \boldsymbol{v}_j[i]\rangle_i$. The next lemma is immediate.

Lemma 2. *For all $w \in Run_{\mathcal{M}_\mathcal{A}}(p\boldsymbol{v}, \neg\mathcal{Z}_{-i})$ and $k \in \mathbb{N}$ we have that*

- $state(w(k)) = state(\Upsilon_i(w)(k))$,
- $cval(w(k)) = \langle tot(\Upsilon_i(w);k), cval_1(\Upsilon_i(w)(k))\rangle_i$.

Further, for every measurable set $R \subseteq Run_{\mathcal{M}_\mathcal{A}}(p\boldsymbol{v}, \neg\mathcal{Z}_{-i})$ we have that $\Upsilon_i(R)$ is measurable and

$$\mathcal{P}(R) \;=\; \mathcal{P}(\Upsilon_i(R)) \tag{1}$$

Now we examine the runs of $Run(p\boldsymbol{v}, C)$ where C is a BSCC of $\mathcal{F}_\mathcal{A}$ such that some counter is not diverging in C. A proof of the next lemma can be found in Appendix A.

Lemma 3. *Let C be a BSCC of $\mathcal{F}_\mathcal{A}$. If some counter is not diverging in C, then $\mathcal{P}(Run(p\boldsymbol{v}, C)) = 0$.*

It remains to consider the case when C is a BSCC of $\mathcal{F}_\mathcal{A}$ where all counters are diverging. Here we use the results of [14] which allow to derive a bound on divergence probability in one-dimensional pMC. These results are based on designing and analyzing a suitable martingale for one-dimensional pMC.

Lemma 4. *Let $\mathcal{B}$ be a 1-dimensional pMC, let C be a BSCC of $\mathcal{F}_\mathcal{B}$ such that the trend t of the only counter in C is positive and let $\delta = 2|C|/x_{\min}^{|C|}$ where $x_{\min}$ is the smallest non-zero transition probability in $\mathcal{M}_\mathcal{B}$. Then for all $q \in C$ and $k > 2\delta/t$ we have that $\mathcal{P}(q(k), \neg\mathcal{Z}) \geq 1 - \left(a^k/(1+a)\right)$, where $\mathcal{Z} = \{1\}$ and $a = \exp\left(-t^2 \,/\, 8(\delta+t+1)^2\right)$.*

Proof: Denote by $[q(k){\downarrow}, \ell]$ the probability that a run initiated in $q(k)$ visits a configuration with zero counter value for the first time in exactly ℓ steps. By Proposition 7 of [15] we obtain for all $\ell \geq h = 2\delta/t$ [1],

$$[q(k){\downarrow}, \ell] \quad\leq\quad a^\ell$$

where $a = \exp\left(-t^2 \,/\, 8(\delta+t+1)^2\right)$ for $\delta \leq 2|C|/x_{\min}^{|C|}$ [2].

Thus

$$\mathcal{P}(q(k), \neg\mathcal{Z}) \geq 1 - \sum_{\ell=k}^{\infty} [q(k){\downarrow}, \ell] = 1 - \frac{a^k}{1+a}$$

■

Definition 5. *Let C be a BSCC of $\mathcal{F}_\mathcal{A}$ where all counters are diverging, and let $q \in C$. We say that a configuration $q\boldsymbol{u}$ is* above *a given $n \in \mathbb{N}$ if $\boldsymbol{u}[i] \geq n$ for every i such that $\boldsymbol{t}[i] > 0$, and $\boldsymbol{u}[i] \geq botfin_i(q)$ for every i such that $\boldsymbol{t}[i] = 0$.*

Lemma 5. *Let C be a BSCC of $\mathcal{F}_\mathcal{A}$ where all counters are diverging. Then $\mathcal{P}(Run(p\boldsymbol{v}, C)) > 0$ iff there is a $\mathcal{Z}_{all}$-safe finite path of the form $p\boldsymbol{v} \rightarrow^* q\boldsymbol{u} \rightarrow^* q\boldsymbol{z}$ where $q \in C$, $q\boldsymbol{u}$ is above 1, $\boldsymbol{z} - \boldsymbol{u} \geq \boldsymbol{0}$, and $(\boldsymbol{z} - \boldsymbol{u})[i] > 0$ for every i such that $\boldsymbol{t}[i] > 0$.*

Proof: We start with "⇒". Let $\boldsymbol{t}$ be the trend of C. We show that for almost all $w \in Run(p\boldsymbol{v}, C)$ and all $i \in \{1, \dots, d\}$, one of the following conditions holds:

(A) $\boldsymbol{t}[i] > 0$ and $\liminf_{k\to\infty} cval_i(w(k)) = \infty$,
(B) $\boldsymbol{t}[i] = 0$ and $cval_i(w(k)) \geq botfin_i(state(w(k)))$ for all k's large enough.

First, recall that C is also a BSCC of $\mathcal{F}_{\mathcal{B}_i}$, and realize that the trend of the (only) counter in the BSCC C of $\mathcal{F}_{\mathcal{B}_i}$ is $\boldsymbol{t}[i]$.

Concerning (A), it follows, e.g., from the results of [14], that almost all runs $w' \in Run_{\mathcal{M}_{\mathcal{B}_i}}(p(1))$ that stay in C and do not visit a configuration with zero counter satisfy $\liminf_{k\to\infty} cval_1(w'(k)) = \infty$. In particular, this means that almost all $w' \in \Upsilon_i(Run(p\boldsymbol{v}, C))$ satisfy this property. Hence, by Lemma 2, for almost all $w \in Run(p\boldsymbol{v}, C)$ we have that $\liminf_{k\to\infty} cval_i(w(k)) = \infty$.

Concerning (B), note that almost all runs $w \in Run(p\boldsymbol{v}, C)$ satisfying $cval_i(w'(k)) < botfin_i(state(w(k)))$ for infinitely many k's eventually visit zero in some counter (there is a path of length at most $|C|$ from each such $w(k)$ to a configuration with zero in counter i, or in one of the other counters).

The above claim immediately implies that for every $k \in \mathbb{N}$, almost every run of $Run(p\boldsymbol{v}, C)$ visits a configuration $q\boldsymbol{u}$ above k. Hence, there must be a $\mathcal{Z}_{all}$-safe path of the form $p\boldsymbol{v} \rightarrow^* q\boldsymbol{u} \rightarrow^* q\boldsymbol{z}$ with the required properties.

"⇐": If there is a $\mathcal{Z}_{all}$-safe path of the form $p\boldsymbol{v} \rightarrow^* q\boldsymbol{u} \rightarrow^* q\boldsymbol{z}$ where $q \in C$, $q\boldsymbol{u}$ is above 1, $\boldsymbol{z} - \boldsymbol{u} \geq \boldsymbol{0}$, and $(\boldsymbol{z} - \boldsymbol{u})[i] > 0$ for every i such that $\boldsymbol{t}[i] > 0$, then $p\boldsymbol{v}$ can a reach a configuration $q\boldsymbol{y}$ above k for an arbitrarily large $k \in \mathbb{N}$ via a $\mathcal{Z}_{all}$-safe path.

By Lemma 4, there exists $k \in \mathbb{N}$ such that for every $i \in \{1, \dots, d\}$ where $\boldsymbol{t}[i] > 0$ and every $n \geq k$, the probability of all $w \in Run_{\mathcal{M}_{\mathcal{B}_i}}(q(n))$ that visit a configuration with zero counter is strictly smaller than $1/d$. Let $q\boldsymbol{y}$ be a configuration

[1] The precise bound on h is given in Proposition 7 [15].

[2] The bound on δ is given in Proposition 6 [15].

above k reachable from $p\boldsymbol{v}$ via a $\mathcal{Z}_{all}$-safe path (the existence of such a $q\boldsymbol{y}$ follows from the existence of $p\boldsymbol{v} \to^* q\boldsymbol{u} \to^* q\boldsymbol{z}$). It suffices to show that $\mathcal{P}(Run(q\boldsymbol{y}, \mathcal{Z}_{all})) < 1$. For every $i \in \{1, \ldots, d\}$ where $\boldsymbol{t}[i] > 0$, let R_i be the set of all $w \in Run(q\boldsymbol{y}, \mathcal{Z}_{all})$ such that $cval_i(w(k)) = 0$ for some $k \in \mathbb{N}$ and all counters stay positive in all $w(k')$ where $k' < k$. Clearly, $Run(q\boldsymbol{y}, \mathcal{Z}_{all}) = \bigcup_i R_i$, and thus we obtain

$$\mathcal{P}(Run(q\boldsymbol{y}, \mathcal{Z}_{all})) \leq \sum_i \mathcal{P}(R_i) = \sum_i \mathcal{P}(\Upsilon_i(R_i)) < d \cdot \frac{1}{d} = 1$$

■

The following lemma shows that it is possible to decide, whether for a given $n \in \mathbb{N}$ a configuration above n can be reached via a $\mathcal{Z}_{all}$-safe path. Its proof uses the results of [9] on the coverability problem in (non-stochastic) VASS.

Lemma 6. *Let C be a BSCC of $\mathcal{F}_{\mathcal{A}}$ where all counters are diverging and let $q \in C$. There is a $\mathcal{Z}_{all}$-safe finite path of the form $p\boldsymbol{v} \to^* q\boldsymbol{u}$ with $q\boldsymbol{u}$ is above some $n \in \mathbb{N}$ iff there is a $\mathcal{Z}_{all}$-safe finite path of length at most $(|Q|+|\gamma|)\cdot(3+n)^{(3d)!+1}$ of the form $p\boldsymbol{v} \to^* q\boldsymbol{u'}$ with $q\boldsymbol{u'}$ is above n. Moreover, the existence of such a path can be decided in time $(|\mathcal{A}| \cdot n)^{c' \cdot 2^{d \log(d)}}$ where c' is a fixed constant independent of d and $\mathcal{A}$.*

Proof: We employ a decision procedure of [9] for VASS coverability. Since we need to reach $q\boldsymbol{u'}$ above n via a $\mathcal{Z}_{all}$-safe finite path, we transform $\mathcal{A}$ into a (non-probabilistic) VASS $\mathcal{A}'$ whose control states and rules are determined as follows: for every rule $(p, \boldsymbol{\alpha}, \emptyset, q)$ of $\mathcal{A}$, we add to $\mathcal{A}'$ the control states p, q together with two auxiliary fresh control states q', q'', and we also add the rules $(p, \boldsymbol{-1}, q')$, $(q', \boldsymbol{1}, q'')$, $(q'', \boldsymbol{\alpha}, q)$. Hence, $\mathcal{A}'$ behaves like $\mathcal{A}$, but when some counter becomes zero, then $\mathcal{A}'$ is stuck (i.e., no transition is enabled except for the self-loop). Now it is easy to check that $p\boldsymbol{v}$ can reach a configuration $q\boldsymbol{u}$ above n via a $\mathcal{Z}_{all}$-safe finite path in $\mathcal{A}$ iff $p\boldsymbol{v}$ can reach a configuration $q\boldsymbol{u}$ above n via *some* finite path in $\mathcal{A}'$, which is exactly the coverability problem for VASS. Theorem 1 in [9] shows that such a configuration can be reached iff there is configuration $q\boldsymbol{u'}$ above n reachable via some finite path of length at most $m = (|Q| + |\gamma|) \cdot (3 + n)^{(3d)!+1}$. (The term $(|Q| + |\gamma|)$ represents the number of control states of $\mathcal{A}'$.) This path induces, in a natural way, a $\mathcal{Z}_{all}$-safe path from $p\boldsymbol{v}$ to $q\boldsymbol{u'}$ in $\mathcal{A}$ of length at most $m/2$. Moreover, Theorem 2 in [9] shows that the existence of such a path in $\mathcal{A}'$ can be decided in time $(|Q| + |\gamma|) \cdot (3 + n)^{2^{\mathcal{O}(d \log(d))}}$, which proves the lemma. ■

Theorem 1. *The qualitative $\mathcal{Z}_{all}$-reachability problem for d-dimensional pMC is decidable in time $|\mathcal{A}|^{\kappa \cdot 2^{d \log(d)}}$, where κ is a fixed constant independent of d and $\mathcal{A}$.*

Proof: Note that the Markov chain $\mathcal{F}_{\mathcal{A}}$ is computable in time polynomial in $|\mathcal{A}|$ and d, and we can efficiently identify all diverging BSCCs of $\mathcal{F}_{\mathcal{A}}$. For each diverging BSCC C, we need to check the condition of Lemma 5. By applying Lemma 2.3. of [30], we obtain that if there exist *some* $q\boldsymbol{u}$ above 1 and a $\mathcal{Z}_{all}$-safe finite path of the form $q\boldsymbol{u} \to^* q\boldsymbol{z}$ such that $\boldsymbol{z} - \boldsymbol{u} \geq \boldsymbol{0}$ and $(\boldsymbol{z} - \boldsymbol{u})[i] > 0$ for every i where $\boldsymbol{t}[i] > 0$, then such a path exists for *every* $q\boldsymbol{u}$ above $|\mathcal{A}|^{c \cdot d}$ and its length is bounded by $|\mathcal{A}|^{c \cdot d}$. Here c is a fixed constant independent of $|\mathcal{A}|$ and d (let us note that Lemma 2.3. of [30] is formulated for vector addition systems without states and a non-strict increase in every counter, but the corresponding result for VASS is easy to derive; see also Lemma 15 in [13]). Hence, the existence of such a path for a given $q \in C$ can be decided in $\mathcal{O}(|\mathcal{A}|^{c \cdot d})$ time. It remains to check whether $p\boldsymbol{v}$ can reach a configuration $q\boldsymbol{u}$ above $|\mathcal{A}|^{c \cdot d}$ via a $\mathcal{Z}_{all}$-safe finite path. By Lemma 6 this can be done in time $(|\mathcal{A}| \cdot |\mathcal{A}|^{c \cdot d})^{c' \cdot 2^{d \log(d)}}$ for another constant c'. This gives us the desired complexity bound. ■

Note that for every fixed dimension d, the qualitative $\mathcal{Z}_{all}$-reachability problem is solvable in polynomial time.

Now we show that $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all}))$ can be effectively approximated up to an arbitrarily small absolute/relative error $\varepsilon > 0$. A full proof of Theorem 2 can be found in Appendix B.

Theorem 2. *For a given d-dimensional pMC $\mathcal{A}$ and its initial configuration $p\boldsymbol{v}$, the probability $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all}))$ can be approximated up to a given absolute error $\varepsilon > 0$ in time $(\exp(|\mathcal{A}|) \cdot \log(1/\varepsilon))^{\mathcal{O}(d \cdot d!)}$.*

Proof sketch: First we check whether $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all})) = 1$ (using the algorithm of Theorem 1) and return 1 if it is the case. Otherwise, we first show how to approximate $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all}))$ under the assumption that p is in some diverging BSCC of $\mathcal{F}_{\mathcal{A}}$, and then we show how to drop this assumption.

So, let C be a diverging BSCC of $\mathcal{F}_{\mathcal{A}}$ such that $\mathcal{P}(Run(p\boldsymbol{v}, C)) < 1$, and let us assume that $p \in C$. We show how to compute $\nu > 0$ such that $|\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all})) - \nu| \leq d \cdot \varepsilon$ in time $(\exp(|\mathcal{A}|) \cdot \log(1/\varepsilon))^{\mathcal{O}(d!)}$. We proceed by induction on d. The key idea of the inductive step is to find a sufficiently large constant K such that if some counter reaches K, it can be safely "forgotten", i.e., replaced by ∞, without influencing the probability of reaching zero in some counter by more than ε. Hence, whenever we visit a configuration $q\boldsymbol{u}$ where some counter value in $\boldsymbol{u}$ reaches K, we can apply induction hypothesis and approximate the probability or reaching zero in some counter from $q\boldsymbol{u}$ by "forgetting" the large counter a thus reducing the dimension. Obviously, there are only finitely many configurations where all counters are below K, and here we employ the standard methods for finite-state Markov chains. The number K is computed by using the bounds of Lemma 4.

Let us note that the base (when $d = 1$) is handled by relying only on Lemma 4. Alternatively, we could employ the results of [19]. This would improve the complexity for $d = 1$, but not for higher dimensions.

Finally, we show how to approximate $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all}))$ when the control state p does not belong to a BSCC of $\mathcal{F}_{\mathcal{A}}$. Here we use the bound of Lemma 1. ■

Note that if $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all})) > 0$, then this probability is at least $p_{min}^{m \cdot |Q|}$ where p_{min} is the least positive transition probability in $\mathcal{M}_{\mathcal{A}}$ and m is the maximal component of $\boldsymbol{v}$. Hence, Theorem 2 can also be used to approximate $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all}))$ up to a given *relative* error $\varepsilon > 0$.

B. Zero-Reachability, Case II

Let us fix a (non-labeled) pMC $\mathcal{A} = (Q, \gamma, W)$ of dimension $d \in \mathbb{N}^+$ and $i \in \{1, \ldots, d\}$. As in the previous section, our aim is to identify the conditions under which $Run(p\mathbf{1}, \neg\mathcal{Z}_{-i}) > 0$. Without restrictions, we assume that $i = d$, i.e., we consider $\mathcal{Z}_{-d} = \{\{1\}, \ldots, \{d-1\}\}$. Also, for technical reasons, we assume that $Run(p\mathbf{1}, \neg\mathcal{Z}_{-d}) = Run(p\boldsymbol{u}^{in}, \neg\mathcal{Z}_{-d})$ where $\boldsymbol{u}^{in}_i = 1$ for all $i \in \{1, \ldots, d-1\}$ but $\boldsymbol{u}^{in}_d = 0$. (Note that every pMC can be easily modified in polynomial time so that this condition is satisfied.)

To analyze the runs of $Run(p\boldsymbol{u}^{in}, \neg\mathcal{Z}_{-d})$, we re-use the finite-state Markov chain $\mathcal{F}_{\mathcal{A}}$ introduced in Section III-A. Intuitively, the chain $\mathcal{F}_{\mathcal{A}}$ is useful for analyzing those runs of $Run(p\boldsymbol{u}^{in}, \neg\mathcal{Z}_{-d})$ where *all* counters stay positive. Since the structure of $Run(p\boldsymbol{u}^{in}, \neg\mathcal{Z}_{-d})$ is more complex than in Section III-A, we also need some new analytic tools.

We also re-use the L-labeled 1-dimensional pMC $\mathcal{B}_d$ to deal with runs that visit zero in counter d infinitely many times. To simplify notation, we use $\mathcal{B}$ to denote $\mathcal{B}_d$. The behaviour of $\mathcal{B}$ is analyzed using the finite-state Markov chain $\mathcal{X}$ (see Definition 6 below) that has been employed already in [14] to design a model-checking algorithm for linear-time properties and one-dimensional pMC.

Let us denote by $[q{\downarrow}r]$ the probability that a run of $\mathcal{M}_{\mathcal{B}}$ initiated in $q(0)$ visits the configurations $r(0)$ without visiting any configuration of the form $r'(0)$ (where $r' \neq r$) in between. Given $q \in Q$, we denote by $[q{\uparrow}]$ the probability $1 - \sum_{r \in Q}[q{\downarrow}r]$ that a run initiated in $q(0)$ never visits a configuration with zero counter value (except for the initial one).

Definition 6. *Let $\mathcal{X}_{\mathcal{B}} = (X, \rightarrow, Prob)$ be a non-labelled finite-state Markov chain where $X = Q \cup \{q{\uparrow} \mid q \in Q\}$ and the transitions are defined as follows:*

- $q \xrightarrow{x} r$ *iff* $0 < x = [q{\downarrow}r]$;
- $q \xrightarrow{x} q{\uparrow}$ *iff* $0 < x = [q{\uparrow}]$;
- *there are no other transitions.*

The correspondence between the runs of $Run_{\mathcal{M}_{\mathcal{B}}}(p(0))$ and $Run_{\mathcal{X}_{\mathcal{B}}}(p)$ is formally captured by a function $\Phi : Run_{\mathcal{M}_{\mathcal{B}}}(p(0)) \rightarrow Run_{\mathcal{X}_{\mathcal{B}}}(p) \cup \{\bot\}$, where $\Phi(w)$ is obtained from a given $w \in Run_{\mathcal{M}_{\mathcal{B}}}(p(0))$ as follows:

- First, each *maximal* subpath in w of the form $q(0), \ldots, r(0)$ such that the counter stays positive in all of the intermediate configurations is replaced with a single transition $q \rightarrow r$.
- Note that if w contained infinitely many configurations with zero counter, then the resulting sequence is a run of $Run_{\mathcal{X}_{\mathcal{B}}}(p)$, and thus we obtain our $\Phi(w)$. Otherwise, the resulting sequence takes the form $v\,\hat{w}$, where $v \in FPath_{\mathcal{X}_{\mathcal{B}}}(p)$ and $\hat{w}$ is a suffix of w initiated in a configuration $r(1)$. Let q be the last state of v. Then, $\Phi(w)$ is either $v\,(q{\uparrow})^\omega$ or $\bot$, depending on whether $[q{\uparrow}] > 0$ or not, respectively (here, $(q{\uparrow})^\omega$ is a infinite sequence of $q{\uparrow}$).

Lemma 7. *For every measurable subset $R \subseteq Run_{\mathcal{X}_{\mathcal{B}}}(p)$ we have that $\Phi^{-1}(R)$ is measurable and $\mathcal{P}(R) = \mathcal{P}(\Phi^{-1}(R))$.*

A proof of Lemma 7 is straightforward (it suffices to check that the lemma holds for all basic cylinders $Run_{\mathcal{X}_{\mathcal{B}}}(w)$ where $w \in FPath_{\mathcal{X}_{\mathcal{B}}}(p)$). Note that Lemma 7 implies $\mathcal{P}(\Phi{=}\bot) = 0$.

Let $D_1, \ldots, D_k$ be all BSCCs of $\mathcal{X}_{\mathcal{B}}$ reachable from p. Further, for every D_j, we use $Run(p\boldsymbol{u}^{in}, D_j)$ to denote the set of all $w \in Run_{\mathcal{M}_{\mathcal{A}}}(p\boldsymbol{u}^{in}, \neg\mathcal{Z}_{-d})$ such that $\Phi(\Upsilon_d(w)) \neq \bot$ and $\Phi(\Upsilon_d(w))$ visits D_j. Observe that

$$\mathcal{P}(Run_{\mathcal{M}_{\mathcal{A}}}(p\boldsymbol{u}^{in}, \neg\mathcal{Z}_{-d})) \;=\; \sum_{j=1}^{k} \mathcal{P}(Run(p\boldsymbol{u}^{in}, D_j)) \quad (2)$$

Indeed, note that almost all runs w of $Run_{\mathcal{X}_{\mathcal{B}}}(p)$ visit some D_j, and hence by Lemma 7, we obtain that $\Phi(w)$ visits some D_j for almost all $w \in Run_{\mathcal{M}_{\mathcal{B}}}(p(1))$. In particular, for almost all w of $\Upsilon_d(Run_{\mathcal{M}_{\mathcal{A}}}(p\boldsymbol{u}^{in}, \neg\mathcal{Z}_{-d}))$ we have that $\Phi(w)$ visits some D_j. By Lemma 2, for almost all $w \in Run_{\mathcal{M}_{\mathcal{A}}}(p\boldsymbol{u}^{in}, \neg\mathcal{Z}_{-d})$, the run $\Phi(\Upsilon_d(w))$ visits some D_j, which proves Equation (2).

Now we examine the runs of $Run(p\boldsymbol{u}^{in}, D_j)$ in greater detail and characterize the conditions under which $\mathcal{P}(Run(p\boldsymbol{u}^{in}, D_j)) > 0$. Note that for every BSCC D in $\mathcal{X}_{\mathcal{B}}$ we have that either $D = \{q{\uparrow}\}$ for some $q \in Q$, or $D \subseteq Q$. We treat these two types of BSCCs separately, starting with the former.

Lemma 8. $\mathcal{P}(\bigcup_{q \in Q} Run(p\boldsymbol{u}^{in}, \{q{\uparrow}\})) > 0$ *iff there exists a BSCC C of $\mathcal{F}_{\mathcal{A}}$ with* all *counters diverging and a $\mathcal{Z}_{-d}$-safe finite path of the form $p\boldsymbol{v} \rightarrow^* q\boldsymbol{u} \rightarrow^* q\boldsymbol{z}$ where the subpath $q\boldsymbol{u} \rightarrow^* q\boldsymbol{z}$ is $\mathcal{Z}_{all}$-safe, $q \in C$, $q\boldsymbol{u}$ is above 1, $\boldsymbol{z} - \boldsymbol{u} \geq \mathbf{0}$, and $(\boldsymbol{z} - \boldsymbol{u})[i] > 0$ for every i such that $\boldsymbol{t}[i] > 0$.*

A proof of Lemma 8 can be found in Appendix C. Now let D be a BSCC of $\mathcal{X}_{\mathcal{B}}$ reachable from p such that $D \subseteq Q$ (i.e., $D \neq \{q{\uparrow}\}$ for any $q \in Q$). Let $\boldsymbol{e} \in [1, \infty)^D$ where $\boldsymbol{e}[q]$ is the expected number of transitions needed to revisit a configuration with zero counter from $q(0)$ in $\mathcal{M}_{\mathcal{B}}$.

Proposition 1 ([14], Corollary 6)**.** *The problem whether $\boldsymbol{e}[q] < \infty$ is decidable in polynomial time.*

From now on, we assume that $\boldsymbol{e}[q] < \infty$ for all $q \in D$.

In Section III-A, we used the trend $\boldsymbol{t} \in \mathbb{R}^d$ to determine tendency of counters either to diverge, or to reach zero. As defined, each $\boldsymbol{t}[i]$ corresponds to the long-run average change per transition of counter i as long as all counters stay positive. Allowing zero value in counter d, the trend $\boldsymbol{t}[i]$ is no longer equal to the long-run average change per transition of counter i and hence it does not correctly characterize its behavior. Therefore, we need to redefine the notion of trend in this case.

Recall that $\mathcal{B}$ is $L = \{-1, 0, 1\}^{d-1}$-labeled pMC. Given $i \in \{1, \ldots, d{-}1\}$, we denote by $\boldsymbol{\delta}_i \in \mathbb{R}^Q$ the vector where $\boldsymbol{\delta}_i[q]$ is the i-th component of the expected total reward accumulated along a run from $q(0)$ before revisiting another configuration with zero counter. Formally, $\boldsymbol{\delta}_i[q] = \mathbb{E}T_i$ where T_i is a random variable which to every $w \in Run_{\mathcal{M}_{\mathcal{B}}}(q(0))$ assigns $tot_i(w; \ell)$ such that $\ell > 0$ is the least number satisfying $w(\ell) = r(0)$ for some $r \in D$.

Let $\boldsymbol{\mu}_{oc} \in [0,1]^D$ be the invariant distribution of the BSCC D of $\mathcal{X}_\mathcal{B}$, i.e., $\boldsymbol{\mu}_{oc}$ is the unique solution of

$$\boldsymbol{\mu}_{oc}[q] \quad = \quad \sum_{r \in D, r \xrightarrow{x} q} \boldsymbol{\mu}_{oc}[r] \cdot x$$

The *oc-trend* of D is a $(d-1)$-dimensional vector $\boldsymbol{t}_{oc} \in [-1,1]^{d-1}$ defined by

$$\boldsymbol{t}_{oc}[i] \quad = \quad (\boldsymbol{\mu}_{oc}^T \cdot \boldsymbol{\delta}_i) \,/\, (\boldsymbol{\mu}_{oc}^T \cdot \boldsymbol{e})$$

The following lemma follows from the standard results about ergodic Markov chains (see, e.g., [29]).

Lemma 9. *For almost all $w \in Run_{\mathcal{M}_\mathcal{B}}(q(0))$ we have that*

$$\boldsymbol{t}_{oc}[i] \quad = \quad \lim_{k \to \infty} \frac{tot_i(w;k)}{k}$$

That is, $\boldsymbol{t}_{oc}[i]$ is the i-th component of the expected long-run average reward per transition in a run of $Run_{\mathcal{M}_\mathcal{B}}(q(0))$, and as such, determines the long-run average change per transition of counter i as long as all counters of $\{1,\ldots,d-1\}$ remain positive.

Further, for every $i \in \{1,\ldots,d-1\}$ and every $q \in D$, we denote by $botinf_i(q)$ the *least* $j \in \mathbb{N}$ such that every $w \in FPath_{\mathcal{M}_\mathcal{B}}(q(0))$ ending in $q(0)$ where $w(n) \neq q(0)$ for all $1 \leq n < length(w)$ satisfies $tot_i(w; length(w)) \geq -j$. If there is no such j, we put $botinf_i(q) = \infty$. It is easy to show that if $botinf_i(q) = \infty$, then $botinf_i(r) = \infty$ for all $r \in D$.

Lemma 10. *If $botinf_i(q) < \infty$, then $botinf_i(q) \leq 3|Q|^3$ and the exact value of $botinf_i(q)$ is computable in time polynomial in $|\mathcal{A}|$.*

A proof Lemma 10 can be found in Appendix C. We say that counter i is *oc-decreasing* in D if $botinf_i(q) = \infty$ for some (and hence all) $q \in D$.

Definition 7. *For a given $i \in \{1,\ldots,d-1\}$, we say that the i-th reward is* oc-diverging *in D if either $\boldsymbol{t}_{oc}[i] > 0$, or $\boldsymbol{t}_{oc}[i] = 0$ and counter i is not oc-decreasing in D.*

Lemma 11. *If some reward is not oc-diverging in D, then $\mathcal{P}(Run(p\boldsymbol{u}^{in}, D)) = 0$.*

A proof of Lemma11 can be found in Appendix C. It remains to analyze the case when all rewards are oc-diverging in D. Similarly to Case I, we need to obtain a bound on probability of divergence of an arbitrary counter $i \in \{1,\ldots,d-1\}$ with $\boldsymbol{t}_{oc}[i] > 0$. The following lemma (an analogue of Lemma 4) is crucial in the process.

Lemma 12. *Let $\mathcal{D}$ be a $\{-1,0,1\}$-labeled one-dimensional pMC, let D be a BSCC of $\mathcal{X}_\mathcal{D}$ such that the oc-trend t_{oc} of the only reward in D is positive. Then for all $q \in D$, there exist computable constants h' and A_0 where $0 < A_0 < 1$, such that for all $h \geq h'$ we have that the probability that a run $w \in Run_{\mathcal{M}_\mathcal{D}}(q(0))$ satisfies*

$$\inf_{k \in \mathbb{N}} tot_1(w;k) \quad \geq \quad -h$$

is at least $1 - A_0^h$.

A proof of Lemma 12 is the most involved part of this paper, where we need to construct new analytic tools. A sketch of the proof is included at the and of this section.

Definition 8. *Let D be a BSCC of $\mathcal{X}_\mathcal{B}$ where all rewards are oc-diverging, and let $q \in D$. We say that a configuration $q\boldsymbol{u}$ is* oc-above *a given $n \in \mathbb{N}$ if $\boldsymbol{u}[i] \geq n$ for every $i \in \{1,\ldots,d-1\}$ such that $\boldsymbol{t}_{oc}[i] > 0$, and $\boldsymbol{u}[i] \geq botinf_i(q)$ for every $i \in \{1,\ldots,d-1\}$ such that $\boldsymbol{t}_{oc}[i] = 0$.*

The next lemma is an analogue of Lemma 5 and it is proven using the same technique, using Lemma 12 instead of Lemma 4. A full proof can be found in Appendix C.

Lemma 13. *Let D be a BSCC of $\mathcal{X}_\mathcal{B}$ where all rewards are diverging. Then there exists a computable constant $n \in \mathbb{N}$ such that $\mathcal{P}(Run(p\boldsymbol{u}^{in}, D)) > 0$ iff there is a $\mathcal{Z}_{-d}$-safe finite path of the form $p\boldsymbol{u}^{in} \to^* q\boldsymbol{u}$ where $\boldsymbol{u}$ is oc-above n and $\boldsymbol{u}[d] = 0$.*

A direct consequence of Lemma 13 and the results of [8] is the following:

Theorem 3. *The qualitative $\mathcal{Z}_{-d}$-reachability problem for d-dimensional pMC is decidable (assuming $\boldsymbol{e}[q] < \infty$ for all $q \in D$ in every BSCC of $\mathcal{X}_\mathcal{B}$).*

A proof of Theorem 3 is straightforward, since we can effectively compute the structure of $\mathcal{X}_\mathcal{B}$ (in time polynomial in $|\mathcal{A}|$, express its transition probabilities and oc-trends in BSCCs of $\mathcal{X}_\mathcal{B}$ in the existential fragment of Tarski algebra, an thus effectively identify all BSCCs of $\mathcal{X}_\mathcal{B}$ where all rewards are oc-diverging. To check the condition of Lemma 13, we use the algorithm of [8] for constructing finite representation of filtered covers in VAS with one zero test. This is the only part where we miss an upper complexity bound, and therefore we cannot provide any bound in Theorem 3. It is worth noting that the qualitative $\mathcal{Z}_{-d}$-reachability problem is SQUARE-ROOT-SUM-hard (see below), and hence it cannot be solved efficiently without a breakthrough results in the complexity of exact algorithms. For more comments and a proof of the next Proposition, see Appendix C.

Proposition 2. *The qualitative $\mathcal{Z}_{-d}$-reachability problem is* SQUARE-ROOT-SUM*-hard, even for two-dimensional pMC where $\boldsymbol{e}[q] < \infty$ for all $q \in D$ in every BSCC of $\mathcal{X}_\mathcal{B}$.*

Using Lemma 13, we can also approximate $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{-d}))$ up to an arbitrarily small absolute error $\varepsilon > 0$ (due to the problems mentined above, we do not provide any complexity bounds). The procedure mimics the one of Theorem 2. The difference is that now we eventually use methods for one-dimensional pMC instead of the methods for finite-state Markov chains. The details are given in Appendix E.

Theorem 4. *For a given d-dimensional pMC $\mathcal{A}$ and its initial configuration $p\boldsymbol{v}$, the probability $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{-d}))$ can be effectively approximated up to a given absolute error $\varepsilon > 0$.*

A Proof of Lemma 12. The lemma differs from Lemma 4 in that it effectively bounds the probability of not reaching zero in one of the counters of a *two-dimensional* pMC (the second counter is encoded in the labels). Hence, the results on one-dimensional pMCs are not sufficient here. Below, we sketch a stronger method that allows us to prove the lemma. The method is again based on analyzing a suitable martingale; however, the construction and structure of the martingale is much more complex than in the one-dimensional case.

Before we show how to construct the desired martingale, let us mention the following useful lemma:

Lemma 14. *Let $r \in D$. Given a run $w \in Run_{\mathcal{M}_\mathcal{B}}(r(0))$, we denote by $E(w) = \inf\{\ell > 0 \mid cval_1(w(\ell)) = 0\}$, i.e., the time it takes w to re-visit zero counter value. Then there are constants $c' \in \mathbb{N}$ and $a \in (0,1)$ computable in polynomial space such that for all $k \geq c'$ we have*

$$\mathcal{P}(E \geq k) \quad \leq \quad a^k$$

Proof: This follows immediately from Proposition 6 and Theorem 7 in [16]. ■

Let us fix an 1-dimensional pMC $\mathcal{D}$ with the set of states Q and let us assume, for simplicity, that $\mathcal{X}_\mathcal{D}$ is strongly connected (assume that the set of states of $\mathcal{X}_\mathcal{D}$ is $D \subseteq Q$). Let us summarize notation used throughout the proof.

- Let $\boldsymbol{e}_\downarrow \in [1,\infty)^Q$ be the vector such that $\boldsymbol{e}_\downarrow[q]$ is the expected total time of a run from $q(1)$ to the first visit of $r(0)$ for some $r \in Q$. By our assumptions, $\boldsymbol{e}_\downarrow$ is finite.
- Recall that $\boldsymbol{e} \in [1,\infty)^D$ is the vector such that $\boldsymbol{e}[q]$ is the expected total time of a nonempty run from $q(0)$ to the first visit of $r(0)$ for some $r \in Q$. Since $\boldsymbol{e}_\downarrow$ is finite, also $\boldsymbol{e}$ is finite.
- Let $\boldsymbol{\delta}_\downarrow \in \mathbb{R}^Q$ be the vector such that $\boldsymbol{\delta}_\downarrow[q]$ is the expected total reward accumulated during a run from $q(1)$ to the first visit of $r(0)$ for some $r \in Q$. Since $|\boldsymbol{\delta}_\downarrow[q]| \leq |\boldsymbol{e}_\downarrow[q]|$ holds for all $q \in Q$, the vector $\boldsymbol{\delta}_\downarrow$ is finite.
- Recall that $\boldsymbol{\delta}_1 \in \mathbb{R}^D$ is the vector such that $\boldsymbol{\delta}_1[q]$ is the expected total reward accumulated during a nonempty run from $q(0)$ to the first visit of $r(0)$ for some $r \in Q$. Similarly as before, $\boldsymbol{\delta}_1$ is finite.
- Let $G \in \mathbb{R}^{Q \times Q}$ denote the matrix such that $G[q,r]$ is the probability that starting from $q(1)$ the configuration $r(0)$ is visited before visiting any configuration $r'(0)$ for any $r' \neq r$. By our assumptions the matrix G is stochastic, i.e., $G\mathbf{1} = \mathbf{1}$.
- Let us denote by $A \in \mathbb{R}^{D \times D}$ transition matrix of the chain $\mathcal{X}_\mathcal{D}$, i.e., $A[q,r]$ is the probability that starting from $q(0)$ the configuration $r(0)$ is visited before visiting any configuration $r'(0)$ for any $r' \neq r$. By our assumptions the matrix A is stochastic and irreducible.
- Recall that $\boldsymbol{\mu}_{oc}^T = \boldsymbol{\mu}_{oc}^T A \in [0,1]^D$ denotes the invariant distribution of the finite Markov chain $\mathcal{X}_\mathcal{D}$ induced by A.
- Recall that $t = (\boldsymbol{\mu}_{oc}^T \boldsymbol{\delta}_1)/(\boldsymbol{\mu}_{oc}^T \boldsymbol{e}) \in [-1,+1]$ is the oc-trend of D, so intuitively t is the expected average reward per step accumulated during a run started from $q(0)$ for some $q \in D$.
- Let $\boldsymbol{r}_\downarrow := \boldsymbol{\delta}_\downarrow - t\boldsymbol{e}_\downarrow \in \mathbb{R}^Q$ and let $\boldsymbol{r}_0 := \boldsymbol{\delta}_1 - t\boldsymbol{e} \in \mathbb{R}^D$.

Lemma 15. *There exists a vector $\boldsymbol{g}(0) \in \mathbb{R}^Q$ such that*

$$\boldsymbol{g}(0)[D] = \boldsymbol{r}_0 + A\boldsymbol{g}(0)[D]\,, \tag{3}$$

where $\boldsymbol{g}(0)[D]$ denotes the vector obtained from $\boldsymbol{g}(0)$ by deleting the non-D-components.

Extend $\boldsymbol{g}(0)$ to a function $\boldsymbol{g} : \mathbb{N} \to \mathbb{R}^Q$ inductively with

$$\boldsymbol{g}(n+1) = \boldsymbol{r}_\downarrow + G\boldsymbol{g}(n) \qquad \text{for all } n \in \mathbb{N}. \tag{4}$$

Lemma 16. *There is $\boldsymbol{g}(0)$ satisfying (3) for which we have the following: There exists a constant c effectively computable in polynomial space such that for every $r \in D$ and $n \geq 1$ we have $|\boldsymbol{g}(0)[r]| \leq c$ and $|\boldsymbol{g}(n)[r]| \leq c \cdot n$.*

Let us fix $q \in D$ and $h \in \mathbb{N}$ such that $(t \cdot \sqrt[4]{h})/c \geq c'$, where c is from the previous lemma and c' from Lemma 14. For a run $w \in Run_{\mathcal{M}_\mathcal{D}}(q(0))$ and all $\ell \in \mathbb{N}$ let $p^{(\ell)} \in Q$ and $x_1^{(\ell)}, x_2^{(\ell)} \in \mathbb{N}$ be such that $p^{(\ell)} = state(w(\ell))$, $x_2^{(\ell)} = cval(w(\ell))$ and $x_1^{(\ell)} = h + tot(w;\ell)$.

Now let us define

$$m^{(\ell)} := x_1^{(\ell)} - t\ell + \boldsymbol{g}\big(x_2^{(\ell)}\big)[p^{(\ell)}] \qquad \text{for all } \ell \in \mathbb{N}. \tag{5}$$

Then we have:

Proposition 3. *Write $\mathcal{E}$ for the expectation with respect to $\mathcal{P}$. We have for all $\ell \in \mathbb{N}$:*

$$\mathcal{E}\left(m^{(\ell+1)} \;\middle|\; w(\ell)\right) = m^{(\ell)}\,.$$

In other words, the stochastic process $\{m^{(\ell)}\}_{\ell=0}^\infty$ is a martingale. Unfortunately, this martingale may have unbounded differences, i.e. $|m_i^{(\ell+1)} - m_i^{(\ell)}|$ may become arbitrarily large with increasing ℓ, which prohibits us from applying standard tools of martingale theory (such as Azuma's inequality) directly on $\{m^{(\ell)}\}_{\ell=0}^\infty$. We now show how to overcome this difficulty.

Let us now fix $i \in \mathbb{N}$ such that $i \geq h$ and denote $K = (t \cdot \sqrt[4]{i})/c$. We define a new stochastic process as follows:

$$m_i^{(\ell)} := \begin{cases} m^{(\ell)} & \text{if } x_2^{(\ell')} \leq K \text{ for all } \ell' \leq \ell \\ m_i^{(\ell-1)} & \text{otherwise.} \end{cases} \tag{6}$$

Observe that $\{m_i^{(\ell)}\}_{\ell=0}^\infty$ is also a martingale. Moreover, using the bound of Lemma 16 we have for every $\ell \in \mathbb{N}$ that $|m_i^{(\ell+1)} - m_i^{(\ell)}| \leq 1 + t + 2cK \leq 4t\sqrt[4]{i}$, i.e., $\{m_i^{(\ell)}\}_{\ell=0}^\infty$ is a bounded-difference martingale.

Now let H_i be the set of all runs w that satisfy $x_1^{(i)} = 0$ and $x_1^{(\ell)} > 0$ for all $0 \leq \ell < i$. Moreover, denote by $Over_i$ the set of all runs w such that $x_2^{(\ell)} \geq K$ for some $0 \leq \ell \leq i$, and by $\neg Over_i$ the complement of $Over_i$.

Note that every run can perform at most i-revisits of zero counter value during the first i steps. By Lemma 14 the probability that counter value at least K is reached between to visits of zero counter is at most a^K. It follows that $\mathcal{P}(Over_i) \leq i \cdot a^{(t \cdot \sqrt[4]{i})/c}$.

Next, for every run $w \in \neg Over_i \cap H_i$ it holds

$$\begin{aligned}(m_i^{(i)} - m_i^{(0)})(w) &= (m^{(i)} - m^{(0)})(w) \\ &= -it + \boldsymbol{g}(x_2^{(i)})[p^{(i)}] - h - \boldsymbol{g}(0)[p^{(0)}] \\ &\leq -it + 2cK = -it + t \cdot \sqrt[4]{i} \leq -i\frac{t}{2},\end{aligned}$$

where the first inequality follows from the bound on $\boldsymbol{g}(n)$ in Lemma 16 and the last inequality holds since $\sqrt[4]{i} \leq i/2$ for all $i \geq 3$.

Using the Azuma's inequality, we get

$$\begin{aligned}\mathcal{P}(Over_i \cap H_i) &\leq \mathcal{P}(m_i^{(i)} - m_i^{(0)} \leq -it/2) \\ &\leq \exp\left(-\frac{i^2 \cdot t^2}{8i(4t\sqrt[4]{i})^2}\right) = \exp\left(-\frac{\sqrt{i}}{128}\right).\end{aligned}$$

Altogether, we have

$$\begin{aligned}\mathcal{P}(H_i) &= \mathcal{P}(H_i \cap Over_i) + \mathcal{P}(H_i \cap \neg Over_i) \\ &\leq i \cdot a^{(t \cdot \sqrt[4]{i})/c} + e^{-\sqrt{i}/128} \leq i \cdot A^{\sqrt[4]{i}},\end{aligned}$$

where $A = \max\{a^{t/c}, 2^{-1/128}\}$. Note that A is also computable in polynomial space.

We now have all the tools needed to prove Lemma 12. We have

$$\begin{aligned}\mathcal{P}(\liminf_{k\to\infty} tot_1(w;k) \leq -h) &\leq \mathcal{P}(\inf_{k\in\mathbb{N}} tot_1(w;k) \leq -h) \\ &= \sum_{i\geq h} \mathcal{P}(H_i) \leq \sum_{i \geq h} i \cdot A^{\sqrt[4]{i}}.\end{aligned}$$

Note that $\sum_{\ell=h}^{\infty} \ell \cdot A^{\sqrt[4]{\ell}} = \sum_{j=\lfloor \sqrt[4]{h} \rfloor}^{\infty} \sum_{\ell=j^4}^{(j+1)^4-1} \ell \cdot A^{\sqrt[4]{\ell}} \leq \sum_{j=\lfloor \sqrt[4]{h} \rfloor}^{\infty} \sum_{\ell=j^4}^{(j+1)^4-1} (j+1)^4 A^j \leq \sum_{j=\lfloor \sqrt[4]{h} \rfloor}^{\infty} 8(j+1)^7 A^j$. Using standard methods of calculus we can bound the last sum by $(c'' \cdot h^7 \cdot A^h)/(1-A)^8$ for some known constant c'' independent of $\mathcal{B}$. Thus, from the knowledge of A and c'' we can easily compute, again in polynomial space, numbers $h_0 \in \mathbb{N}$, $A_0 \in (0,1)$ such that for all $h \geq h_0$ it holds

$$\mathcal{P}(\liminf_{k\to\infty} tot_1(w;k) \geq h) \geq 1 - A_0^h.$$

IV. Conclusions

We have shown that the qualitative zero-reachability problem is decidable in Case I and II, and the probability of all zero-reaching runs can be effectively approximated. Let us not when the technical condition adopted in Case II is not satisfied, than the oc-trends may be undefined and the problem requires a completely different approach. An important technical contribution of this paper is the new martingale defined in Section III-B, which provides a versatile tool for attacking other problems of pMC analysis (model-checking, expected termination time, constructing (sub)optimal strategies in multi-counter decision processes, etc.) similarly as the martingale of [14] for one-dimensional pMC.

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APPENDIX A
PROOFS OF SECTION III-A

Lemma 3 *Let C be a BSCC of $\mathcal{F}_{\mathcal{A}}$. If some counter is not diverging in C, then $\mathcal{P}(Run(p\boldsymbol{v}, C)) = 0$.*

Proof: Assume that counter i is not diverging, and consider the one-dimensional pMC $\mathcal{B}_i$. Observe that $\mathcal{F}_{\mathcal{B}_i}$ is the same as $\mathcal{F}_{\mathcal{A}}$, and hence $\mathcal{F}_{\mathcal{B}_i}$ has the same transition probabilities and BSCCs as $\mathcal{F}_{\mathcal{A}}$. In particular, the only counter of $\mathcal{B}_i$ is not diverging in the BSCC C of $\mathcal{F}_{\mathcal{B}_i}$. By the results of [14], almost all runs of $Run_{\mathcal{M}_{\mathcal{B}_i}}(p(\boldsymbol{v}[i]))$ that stay in C eventually visit zero value in the only counter. Since all runs of $\Upsilon_i(Run(p\boldsymbol{v}, C))$ stay in C but none of them ever visits a configuration with zero counter value, we obtain that

$$\mathcal{P}(Run(p\boldsymbol{v}, C)) = \mathcal{P}(\Upsilon_i(Run(p\boldsymbol{v}, C)) = 0$$

■

APPENDIX B
APPROXIMATION ALGORITHM FOR $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all}))$

We show that $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all}))$ can be effectively approximated up to an arbitrarily small absolute/relative error $\varepsilon > 0$. First we solve this problem under the assumption that p is in some BSCC of $\mathcal{F}_{\mathcal{A}}$. Then we show how to drop this assumption.

Proposition 4. *There is an algorithm which, for a given d-dimensional pMC $\mathcal{A}$, its initial configuration $p\boldsymbol{v}$ such that p is in a BSCC of $\mathcal{F}_{\mathcal{A}}$, and a given $\varepsilon > 0$ computes a number ν such that $|\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all})) - \nu| \leq d \cdot \varepsilon$. The algorithm runs in time $(\exp(|\mathcal{A}|) \cdot \log(1/\varepsilon))^{\mathcal{O}(d!)}$.*

Proof: In the following, we denote by C the BSCC of $\mathcal{A}$ containing the initial state p. Note that we may assume that $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all})) < 1$. From the proof of Lemma 6 it follows that checking this condition boils down to checking the existence of a certain path of length at most $|\mathcal{A}'|^{\mathcal{O}(d!)}$ in a suitable VASS $\mathcal{A}'$ of size polynomial in $|\mathcal{A}|$. This can be done it time $(\exp(|\mathcal{A}|)^{\mathcal{O}(d!)}$.

We can check this condition using an algorithm of Theorem 1, and if it does not hold we may output $\nu = 1$. In particular, we may assume that the trend of every counter in C is non-negative.

We proceed by induction on d. For technical convenience we slightly change the statement about the complexity: we show that the running time of the algorithm is $(\exp(|\mathcal{A}|^c) \cdot \log(\boldsymbol{v}_{\max}/\varepsilon))^{d!}$, for some constants c, c' independent of $\mathcal{A}$. Clearly, this new statement implies the one in the proposition.

Before we present the algorithm, let us make an important observation. Recall the number a defined in Lemma 4 for an arbitrary one-dimensional pMC $\mathcal{B}$ with a positive trend of the counter. Now suppose that for a given $\mathcal{B}$ and given $\varepsilon > 0$ we want to find some K such that $\frac{a^K}{1-a} < \varepsilon$. Note that it suffices to pick any

$$K > \frac{\log(1/\varepsilon)}{(1-a)\log(1/a)}.$$

From the definition of a we have $K \in \exp(\mathcal{B}^{O(1)}) \cdot \log(1/\varepsilon)$ and that K can be computed in time polynomial in $|\mathcal{B}|$. In particular there is a constant c independent of $\mathcal{B}$ such that $K \leq \exp(|\mathcal{B}|^c) \cdot \log(1/\varepsilon)$ and we choose c as the desired constant.

Now let us prove the proposition.

$d = 1$: First let us assume that the trend of the single counter in C is 0. Then, by Lemma 5 it must be the case that $\mathcal{P}(Run(r(\ell), \mathcal{Z}_{all})) = 0$ for every $r \in C$ and every $\ell \geq |C|$. Thus, if the initial counter value is $\geq |Q|$, we may output $\nu = 0$. Otherwise, we may approximate the probability by constructing a finite-state polynomial-sized Markov chain $\mathcal{M}_{|C|}$ whose states are those configurations of $\mathcal{A}$ where the counter is bounded by $|C|$ and whose transitions are naturally derived from $\mathcal{A}$. Formally, $\mathcal{M}_{|C|}$ is obtained from $\mathcal{M}_A$ by removing all configurations $r(\ell)$ with $\ell > |C|$ and replacing all transitions outgoing from configurations of the form $r(|C|)$ with a self loop of probability 1. Clearly, the value $\mathcal{P}(Run(p\boldsymbol{\ell}, \mathcal{Z}_{all}))$ is equal to the probability of reaching a configuration with a zero counter from $p(\ell)$ in $\mathcal{M}_{|C|}$, which can be computed in polynomial time by standard methods.

If the trend of the counter in C is positive, then let us consider the number a from Lemma 4 computed for $\mathcal{A}$ and C. As discussed above, we may compute, in time polynomial in $|\mathcal{A}|$, a number $K \leq \exp(|\mathcal{A}|^c) \cdot \log(1/\varepsilon)$ such that $\frac{a^K}{1-a} < \varepsilon$. We can now again construct a finite-state Markov chain $\mathcal{M}_K$ by discarding all configurations in $\mathcal{M}_{\mathcal{A}}$ where the counter surpasses K and replacing the transitions outgoing from configurations of the form $r(K)$ with self-loops.

Now let us consider an initial configuration $q(\ell)$ with $\ell \leq K$ and denote $P(q(\ell))$ the probability of reaching a configuration with zero counter in from $q(\ell)$ in $\mathcal{M}_K$. We claim that $|\mathcal{P}(Run(r(\ell), \mathcal{Z}_{all})) - P(q(\ell))| \leq \varepsilon$. Indeed, from the construction of $\mathcal{M}_K$ we get that $|\mathcal{P}(Run(r(\ell), \mathcal{Z}_{all})) - P(q(\ell))|$ is bounded by the probability, that a run initiated in $q(\ell)$ in $\mathcal{A}$ reaches a configuration of the form $r(K)$ via a $\mathcal{Z}_{all}$-safe path *and then* visits a configuration with zero counter. This value is in turn bounded by a probability that a run initiated in $r(K)$ decreases the counter to 0, which is at most $\frac{a^K}{1+a} \leq a^K$ by Lemma 4, and thus at most ε by the choice of K. Thus, it suffices to compute $P(q(\ell))$ via standard algorithms and return it as ν.

The same argument shows that if the initial counter value ℓ is greater than K, we can output $\nu = 0$ as a correct ε-approximation.

Note that the construction of $\mathcal{M}_K$ and computing the reachability probability in it can be performed in time $(|\mathcal{A}| \cdot K)^{c'}$ for a suitable constant c' independent of $\mathcal{A}$. This finishes the proof of a base case of our induction.

$d > 1$: Here we will use the algorithm for the $(d-1)$-dimensional case as a sub-procedure. For any counter i and any vector $\boldsymbol{\beta} \in \{-1, 0, 1\}^d$ we denote by $\boldsymbol{\beta}_{-i}$ the $(d-1)$-dimensional vector obtained from $\boldsymbol{\beta}$ by deleting its i-component. Moreover, we define a $(d-1)$-dimensional pMC $\mathcal{A}_{-i}$ obtained from $\mathcal{A}$ by "forgetting" the i-th counter. I.e., $\mathcal{A} = (Q, \gamma_{-i}, W_{-i})$, where $(p, \boldsymbol{\alpha}, c, q) \in \gamma_{-i}$ iff there is $(p, \boldsymbol{\beta}, c, q) \in \gamma$ such that $\boldsymbol{\beta}_{-i} = \boldsymbol{\alpha}$; and where $W_{-i}(p, \boldsymbol{\alpha}, c, q) = \sum W(p, \boldsymbol{\beta}, c, q)$ with the summation proceeding over all $\boldsymbol{\beta}$ such that $\boldsymbol{\beta}_{-i} = \boldsymbol{\alpha}$.

Now let us prove the proposition. Let $\boldsymbol{t}$ be the trend of C. For every counter i such that $\boldsymbol{t}[i] > 0$ we denote by a_i the number a of Lemma 4 computed for C in $\mathcal{B}_i$ (note that C is a BSCC of every $\mathcal{B}_i$). We put $a_{max} = \max\{a_i \mid \boldsymbol{t}[i] > 0\}$. We again compute, as discussed above, in time polynomial in $|\mathcal{A}|$ a number $K \leq \exp(|\mathcal{A}|^c) \cdot \log(1/\varepsilon)$ such that $\frac{a_{max}^K}{1-a_{max}} < \varepsilon$. (If $\boldsymbol{t} = \boldsymbol{0}$, we do not need to define K at all, as will be shown below.) For any configuration $q\boldsymbol{u}$ we denote by $mindiv(q\boldsymbol{u})$ the smallest i such that either $\boldsymbol{t}[i] > 0$ and $\boldsymbol{u}[i] \geq K$ or $\boldsymbol{t}[i] = 0$ and $\boldsymbol{u}[i] \geq |C|$ (if such i does not exist, we put $mindiv(q\boldsymbol{u}) = \perp$).

Consider a finite-state Markov chain $\mathcal{M}_K^d$ which can be obtained from $\mathcal{M}_{\mathcal{A}}$ as follows:

- We remove all configurations where at least one of the counters with positive trend is greater than K, together with adjacent transitions.
- We remove all configurations where at least one of the counters with zero trend is greater than $|C|$, together with adjacent transitions.
- We add new states q_{down} and q_{up}, both of them having a self-loop as the only outgoing transition.
- For every $1 \leq i \leq d$ and every remaining configuration $q\boldsymbol{u}$ with $mindiv(q\boldsymbol{u}) = i$ we remove all transitions outgoing from $q\boldsymbol{u}$ and replace them with the following transitions:
  - A transition leading to q_{down}, whose probability is equal to some $((d-1)\cdot\varepsilon)$-approximation of $\mathcal{P}_{\mathcal{A}_{-i}}(Run(q\boldsymbol{u}_{-i}, \mathcal{Z}_{all}))$ (which can be computed using the algorithm for dimension $d-1$).
  - A transition leading to q_{up}, with probability $1-x$, where x is such that $q\boldsymbol{u} \xrightarrow{x} q_{down}$.

Above, $\mathcal{P}_{\mathcal{A}_{-i}}(X)$ represents the probability of event X in pMC $\mathcal{A}_{-i}$.

Now for an initial configuration $p\boldsymbol{v}$ belonging to the states of $\mathcal{M}_K^d$ let $P(p\boldsymbol{v})$ be the probability of reaching, when starting in $p\boldsymbol{v}$ in $\mathcal{M}_k^d$, either the state q_{down} or a configuration in which at least one of the counters is 0. Note that $P(p\boldsymbol{v})$ can be computed in time polynomial in $|\mathcal{M}_K^d|$. We claim that $|\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all})) - P(p\boldsymbol{v})| \leq d \cdot \varepsilon$.

Indeed, let us denote Div the set of all configurations $q\boldsymbol{u}$ such that $q\boldsymbol{u}$ is a state of $\mathcal{M}_K^d$ and $mindiv(q\boldsymbol{u}) \neq \perp$. For every $q\boldsymbol{u} \in Div$ we denote by $x_{q\boldsymbol{u}}$ the probability of the transition leading from $q\boldsymbol{u}$ to q_{down} in $\mathcal{M}_K^d$. Then $|\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all})) - P(p\boldsymbol{v})| \leq \max_{q\boldsymbol{u}\in Div} |\mathcal{P}(Run(q\boldsymbol{u}, \mathcal{Z}_{all}) - x_{q\boldsymbol{u}}|$. Now $\mathcal{P}(Run(q\boldsymbol{u}, \mathcal{Z}_{all}) \leq P_1(q\boldsymbol{u}) + P_2(q\boldsymbol{u})$, where $P_1(q\boldsymbol{u})$ is the probability that a run initiated in $q\boldsymbol{u}$ in $\mathcal{A}$ visits a configuration with i-th counter 0 via a $\mathcal{Z}_{-i}$-safe path, and $P_2(q\boldsymbol{u})$ is the probability that a run initiated in $q\boldsymbol{u}$ in $\mathcal{A}$ visits a configuration with some counter equal to 0 via an $\{i\}$-safe path.

So let us fix $q\boldsymbol{u} \in Div$ and denote $i = mindiv(q\boldsymbol{u})$. If $\boldsymbol{t}[i] = 0$, then we have $P_1(q\boldsymbol{u}) = 0$, since this counter is not decreasing in C and thus it cannot decrease by more than $|C|$. Otherwise $P_1(q\boldsymbol{u})$ is bounded by the probability that a run initiated in $q(K)$ in $\mathcal{B}_i$ reaches a configuration where the counter is 0. From Lemma 4 we get that $\mathcal{P}_{\mathcal{B}_i}(Run(q(K), \mathcal{Z}_{all})) \leq \frac{a_i^K}{1-a_i} \leq \frac{a_{max}^K}{1-a_{max}} \leq \varepsilon$, where the last inequality follows from the choice of K.

For $P_2(q\boldsymbol{u})$ note that $P_2(\boldsymbol{u}) = \mathcal{P}_{\mathcal{A}_{-i}}(Run(q\boldsymbol{u}_{-i}, \mathcal{Z}_{all}))$ and thus by the construction of $\mathcal{M}_K^d$ we have $|P_2(q\boldsymbol{u}) - x_{q\boldsymbol{u}}| \leq (d-1) \cdot \varepsilon$.

Altogether we have

$$|\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all})) - P(p\boldsymbol{v})| \leq |P_1(q\boldsymbol{u}) + P_2(q\boldsymbol{u}) - x_{q\boldsymbol{u}}| \leq \varepsilon + (d-1) \cdot \varepsilon = d \cdot \varepsilon.$$

Therefore it suffices to compute $P(p\boldsymbol{v})$ via standard methods and output is as ν. Finally, if the initial configuration $p\boldsymbol{v}$ does not belong to the state space of $\mathcal{M}_K^d$ let us denote $i = mindiv(p\boldsymbol{v})$. Then it suffices to output some $((d-1)\cdot\varepsilon)$-approximation of $\mathcal{P}_{\mathcal{A}_{-i}}(Run(p\boldsymbol{v}_{-i}, \mathcal{Z}_{all}))$ as ν. If $\boldsymbol{t}[i] = 0$, then ν is also an $((d-1)\cdot\varepsilon)$-approximation of $\mathcal{P}_(Run(p\boldsymbol{v}, \mathcal{Z}_{all}))$, otherwise $|\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all})) - \nu| \leq (d-1)\cdot\varepsilon + P_1(p\boldsymbol{v})$ where P_1 is defined in the same way as above. Since the probability of reaching zero counter in $\mathcal{B}_i$ with initial counter value $> K$ can be only smaller than the probability for initial value K, the bound on P_1 above applies and we get $|\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all})) - \nu| \leq d \cdot \varepsilon$.

Now let us discuss the complexity of the algorithm. Note that for any d we have $K \leq \exp(|\mathcal{A}|^c) \cdot \log(1/\varepsilon)$, and the construction of $\mathcal{M}_K^d$ (or $\mathcal{M}_K$) and the computation of the reachability probabilities can be done in time $(|\mathcal{A}|\cdot K^d)^{c'} \cdot T(d-1) \leq (\exp|\mathcal{A}|^{c+1} \cdot \log(1/\varepsilon))^{dc'}$ for some constant c' independent of $\mathcal{A}$ and d, where $T(d-1)$ is the running time of the algorithm on a $(d-1)$-dimensional pMC of size $\leq |\mathcal{A}|$ (the pMCs $|\mathcal{A}_{-i}|$ examined during the recursive call of the algorithm are of size $\leq |\mathcal{A}|$). Solving this recurrence we get that the running time of the algorithm is $(\exp(|\mathcal{A}|) \cdot \log(1/\varepsilon))^{\mathcal{O}(d!)}$.

Lemma 4 we get that $\mathcal{P}_{\mathcal{B}_i}(Run(q(K), \mathcal{Z}_{all})) \leq \frac{a_i^K}{1+a_i} \leq a_{max}^K \leq \varepsilon$, where the last inequality follows from the choice of K. ■

With the help of algorithm from Proposition 4 we can easily approximate $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all}))$ even if p is not in any BSCC of $\mathcal{A}$.

Theorem 2 *For a given d-dimensional pMC $\mathcal{A}$ and its initial configuration $p\boldsymbol{v}$, the probability $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all}))$ can be approximated up to a given absolute error $\varepsilon > 0$ in time $(\exp(|\mathcal{A}|) \cdot \log(1/\varepsilon))^{\mathcal{O}(d \cdot d!)}$.*

Proof: First we compute an integer $n \in \exp(|\mathcal{A}|^{\mathcal{O}(1)}) \cdot \log(1/\varepsilon)$ such that $(1 - p_{min}^{|Q|})^{\lfloor \frac{n}{|Q|} \rfloor} \leq \varepsilon/2$. This can be done in time polynomial in $|\mathcal{A}|$ and $\log(1/\varepsilon)$. By Lemma 1 the probability that a run does not visit, in at most n steps, a configuration $q\boldsymbol{u}$ with either $Z(q\boldsymbol{u}) \neq \emptyset$ or q being in some BSCC of $\mathcal{A}$ is at most $\varepsilon/2$. Now we construct an n-step unfolding of $\mathcal{A}$ from $p\boldsymbol{v}$, i.e. we construct a finite-state Markov chain $\mathcal{M}$ such that

- its states are tuples of the form $(q\boldsymbol{u}, j)$, where $0 \leq j \leq n$ and $q\boldsymbol{u}$ is reachable from $p\boldsymbol{v}$ in $\leq n$ steps in $\mathcal{A}$,
- for every $0 \leq j < n$ we have $(q\boldsymbol{u}, j) \xrightarrow{y} (q'\boldsymbol{u}', j+1)$ iff $q\boldsymbol{u} \xrightarrow{y} q'\boldsymbol{u}'$ in $\mathcal{M}_{\mathcal{A}}$,
- there are no other transitions in $\mathcal{M}$.

We add to this $\mathcal{M}$ new states q_{up} and q_{down}, and for every state $(q\boldsymbol{u}, j)$ with q in some BSCC of $\mathcal{A}$ we replace the transitions outgoing from this state with two transitions $(q\boldsymbol{u}, j) \xrightarrow{x} q_{down}$, $(q\boldsymbol{u}, j) \xrightarrow{1-x} q_{up}$, where x is some $(\varepsilon/2)$-approximation of $\mathcal{P}(Run(q\boldsymbol{u}, \mathcal{Z}_{all}))$, which can be computed using the algorithm from Proposition 4. Moreover, for every state $(q\boldsymbol{u}, j)$ with $Z(q\boldsymbol{u}) \neq \emptyset$ we replace all its outgoing transitions with a single transition leading to q_{down}. It is immediate that the probability of reaching q_{down} from $p\boldsymbol{v}$ is an ε-approximation of $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{all}))$.

The number of states of $\mathcal{M}$ is at most $m = n \cdot |Q| \cdot (2n)^d$ and the algorithm of Proposition 4 is called at most m times, which gives us the required complexity bound. ∎

APPENDIX C
PROOFS OF SECTION III-B

Lemma 8 *$\mathcal{P}(\bigcup_{q \in Q} Run(p\boldsymbol{u}^{in}, \{q\uparrow\})) > 0$ iff there exists a BSCC C of $\mathcal{F}_{\mathcal{A}}$ with* all *counters diverging and a $\mathcal{Z}_{-d}$-safe finite path of the form $p\boldsymbol{v} \to^* q\boldsymbol{u} \to^* q\boldsymbol{z}$ where the subpath $q\boldsymbol{u} \to^* q\boldsymbol{z}$ is $\mathcal{Z}_{all}$-safe, $q \in C$, $q\boldsymbol{u}$ is above 1, $\boldsymbol{z} - \boldsymbol{u} \geq \boldsymbol{0}$, and $(\boldsymbol{z} - \boldsymbol{u})[i] > 0$ for every i such that $\boldsymbol{t}[i] > 0$.*

Proof: "⇒" Note that $\mathcal{P}(Run(p\boldsymbol{u}^{in}, \{q\uparrow\})) > 0$ for some $q \in Q$. By Lemma 1, almost every run of $Run(p\boldsymbol{u}^{in}, \{q\uparrow\})$ stays eventually in some BSCC of $\mathcal{F}_{\mathcal{A}}$. Let C be a BSCC such that the probability of all $w \in Run(p\boldsymbol{u}^{in}, \{q\uparrow\})$ that stay is C is positive, and let $\boldsymbol{t}$ be the trend of C. We use R to denote the set of all $w \in Run(p\boldsymbol{u}^{in}, \{q\uparrow\})$ that stay in C.

We claim that each counter i must be diverging in C. First, let us consider $1 \leq i \leq d-1$. Consider the one-counter pMC $\mathcal{B}_i$. Note that the trend of C in $\mathcal{B}_i$ is to $\boldsymbol{t}[i]$. For the sake of contradiction, assume that counter i is not diverging, i.e., we have either $t_i < 0$, or $t_i = 0$ and counter i is decreasing in C. Then, by [14], starting in a configuration $p(k)$ of $\mathcal{B}_i$ where $p \in C$, a configuration with zero counter value is reached from $p(k)$ with probability one. However, then, due to Equation (1) and Proposition 2, almost every run of R visits a configuration with zero in one of the counters of $\{1, \ldots, d-1\}$ (note that zero may be reached in some counter before inevitably reaching zero in counter i). As $R \subseteq Run(p\boldsymbol{u}^{in}, \{q\uparrow\}) \subseteq Run_{\mathcal{M}_{\mathcal{A}}}(p\boldsymbol{u}^{in}, \neg\mathcal{Z}_{-d})$, we obtain that $\mathcal{P}(R) = 0$, which is a contradiction. Now consider $i = d$. Similarly as above, starting in a configuration $p(k)$ of $\mathcal{B}_d$ where $p \in C$, a configuration with zero counter value is reached from $p(k)$ with probability one. This implies that almost all runs w of R reach configurations with zero counter value in the counter d infinitely many times, and hence, by Proposition 2, $\Phi(\Upsilon_d(w))$ does not reach $\bigcup_{q \in Q}\{q\uparrow\}$ at all. It follows that $\mathcal{P}(R) = 0$, a contradiction.

Now we prove that for almost all runs $w \in R$ and for all counters i, one of the following holds:

(A) $t_i > 0$ and $\liminf_{k \to \infty} cval_i(w(k)) = \infty$,

(B) $t_i = 0$ and $cval_i(w(k)) \geq -botfin_i(state(w(k)))$ for all k's large enough.

The argument is the same as in the proof of Lemma 5. From (A) and (B), we immediately obtain the existence of a finite path $p\boldsymbol{v} \to^* q\boldsymbol{u} \to^* q\boldsymbol{z}$ with the required properties.

"⇐" We argue similarly as in Lemma 5.

∎

Lemma 10 *If $botinf_i(q) < \infty$, then $botinf_i(q) \leq 3|Q|^3$ and the exact value of $botinf_i(q)$ is computable in time polynomial in $|\mathcal{A}|$.*

Proof sketch: We show that if $botinf_i(q) < \infty$, then there is $w \in FPath_{\mathcal{M}_{\mathcal{B}}}(q(0))$ ending in $q(0)$ where $w(n) \neq q(0)$ for all $1 \leq n < length(w)$, $tot_i(w; length(w)) = -botinf_i(q)$, and the counter is bounded by $2|Q|^2$ along w. From this we immediately obtain that w visits at most $3|Q|^3$ different configurations, and we can safely assume that no configuration is visited twice (if the reward accumulated between two consecutive visits to the same configuration is non-negative, we can remove the cycle and thus produce a path whose total accumulated reward can be only smaller; and if the the reward accumulated between two consecutive visits to the same configuration is negative, we have that $botinf_i(q) = \infty$, which is a contradiction).

To see that there is such a path w where the counter is bounded by $2|Q|^2$, it suffices to realize that if it was not the case, we could always decrease the number of configurations visited by w where the counter value is above $2|Q|^2$ by removing some subpaths of w such that the total reward accumulated in these subpaths in non-negative. More precisely, we show that there exist configurations $r(i_1)$, $r(i_2)$, $s(i_2)$ and $s(i_1)$ consecutively visited by w where $0 < i_1 < i_2 \leq 2|Q|^2$, the counter stays positive in all configurations between $r(i_1)$ and $s(i_1)$, the finite path from $r(i_2)$ to $s(i_2)$ visits at least one configuration with counter value above $2|Q|^2$, and the finite path from $r(i_2)$ to $s(i_2)$ can be "performed" also from $r(i_1)$ without visiting a configuration with zero counter. If the total reward accumulated in the paths from $r(i_1)$, $r(i_2)$ and from $s(i_2)$ to $s(i_1)$ is negative, we obtain that $\mathit{botinf}_i(q) = \infty$ because we can "iterate" the two subpaths. If it is non-negative, we can remove the subpaths from $r(i_1)$ to $r(i_2)$ and from $s(i_2)$ to $s(i_1)$ from w, and thus decrease the number of configuration with counter value above $2|Q|^2$, making the total accumulated reward only smaller.

Using the above observations, one can easily compute $\mathit{botinf}_i(q)$ in polynomial time. ■

Lemma 11 *If some reward is not oc-diverging in D, then $\mathcal{P}(Run(p\boldsymbol{u}^{in}, D)) = 0$.*

Proof: Assume that counter i is not diverging in D. Let us fix some $q \in D$. Let w be a run in $\mathcal{M}_{\mathcal{B}}$ initiated in $q(0)$ and let $I_1 < I_2 < \cdots$ be non-negative integers such that w_{I_k} is the k-th occurrence of $q(0)$ in w. Given $i \in \{1, \ldots, d-1\}$ and $k \geq 1$, we denote by $T_i^k(w) = tot_i(w; I_{k+1} - 1) - tot_i(w; I_k)$ the i-th component of the total reward accumulated between the k-th visit (inclusive) and the k+1-st visit to $q(0)$ (non-inclusive). We denote by $\mathbb{E}T_i^k$ the expected value of T_i^k.

Observe that $T_i^1, T_i^2, \ldots$ are mutually independent and identically distributed. Thus $T_i^1, T_i^2, \ldots$ determines a random walk $S_i^1, S_i^2, \ldots$, here $S_i^k = \sum_{j=1}^{k} T_i^j$, on $\mathbb{Z}$. Note that $S_i^k = tot_i(w; k+1)$. By the strong law of large numbers, for almost all $w \in Run_{\mathcal{M}_{\mathcal{B}}}(q(0))$,

$$\begin{aligned}
\mathbb{E}T_i^1 &= \lim_{k\to\infty} \frac{S_i^k(w)}{k} \\
&= \lim_{k\to\infty} \frac{S_i^k(w)}{E^k(w)} \frac{E^k(w)}{k} \\
&= \lim_{k\to\infty} \frac{S_i^k(w)}{E^k(w)} \lim_{k\to\infty} \frac{E^k(w)}{k} \\
&= \lim_{k\to\infty} \frac{tot_i(w;k)}{k} \lim_{k\to\infty} \boldsymbol{e}[q] \\
&= \boldsymbol{t}_{oc}[q] \\
&\leq 0
\end{aligned}$$

(Here $E^k(w)$ denotes the number of steps between the k-th and $k+1$-st visit to $q(0)$ in w.) Also, $\mathcal{P}(T_i^1 < 0) > 0$. By Theorem 8.3.4 [17], for almost all $w \in Run_{\mathcal{M}_{\mathcal{B}}}(q(0))$ we have that $\liminf_{k\to\infty} S_i^k(w) = -\infty$.

However, this also means that almost every run $w \in Run_{\mathcal{M}_{\mathcal{B}}}(q(0))$ satisfies that $\lim_{\ell\to\infty} tot_i(w;\ell) = -\infty$. Subsequently, as all runs of $\Upsilon_d(Run(p\boldsymbol{u}^{in}, D))$ visit $q(0)$, almost all runs w of $\Upsilon_d(Run(p\boldsymbol{u}^{in}, D))$ satisfy $\lim_{\ell\to\infty} tot_i(w;\ell) = -\infty$. Thus, by Lemma 2, almost all runs of $Run(p\boldsymbol{u}^{in}, D)$ visit zero in one of the counters in $\{1, \ldots, d-1\}$. This means, that $Run(p\boldsymbol{u}^{in}, D) = 0$. ■

Lemma 13 *Let D be a BSCC of $\mathcal{X}_{\mathcal{B}}$ where all rewards are diverging. Then there exists a computable constant $n \in \mathbb{N}$ such that $\mathcal{P}(Run(p\boldsymbol{u}^{in}, D)) > 0$ iff there is a $\mathcal{Z}_{-d}$-safe finite path of the form $p\boldsymbol{u}^{in} \to^* q\boldsymbol{u}$ where $\boldsymbol{u}$ is oc-above n and $\boldsymbol{u}[d] = 0$.*

Proof: The constant n is computed using Lemma 12. We choose a sufficiently large n such that the probability of Lemma 12 is smaller than $1/d$ for every $q \in D$.

$\Leftarrow$: Assume that counter i satisfies $\boldsymbol{t}_{oc}[i] > 0$. By Lemma 9, almost every run w of $\mathcal{M}_{\mathcal{B}}$ initiated in $q(0)$ satisfies

$$\lim_{k\to\infty} tot_i(w;k) \,/\, k = \boldsymbol{t}_{oc}[i] > 0$$

It follows that there is $c > 0$ such that for a sufficiently large $k \in \mathbb{N}$ we have $tot_i(w;k) \,/\, k \geq c$. It follows that $tot_i(w;k) \geq ck$ for all sufficiently large k. Thus for all counters i satisfying $\boldsymbol{t}_{oc}[i] > 0$ and for almost all runs w of $\mathcal{M}_{\mathcal{B}}$ initiated in $q(0)$ we have that $\lim_{k\to\infty} tot_i(w;k) = \infty$.

For every $n \in \mathbb{N}$ we denote by R_n the set of all runs w initiated in $q(0)$ such that $tot_i(w;k) > -n$ for all k and all i satisfying $\boldsymbol{t}_{oc}[i] > 0$. By the above argument, $\mathcal{P}(\bigcup_n R_n) = 1$. Hence, there must be n such that $\mathcal{P}(R_n) > 0$.

Let $q\boldsymbol{u}$ be any configuration that is above n and satisfies $\boldsymbol{u}[d] = 0$. Then $\Upsilon_d(Run(q\boldsymbol{u}, \mathcal{Z}_{-d})) \supseteq R_n$ and hence $\mathcal{P}(Run(q\boldsymbol{u}, \mathcal{Z}_{-d})) \geq \mathcal{P}(R_n) > 0$. By our assumption, such a configuration $q\boldsymbol{u}$ is reachable from $p\boldsymbol{u}^{in}$ via a $\mathcal{Z}_{-d}$-safe path, and thus $\mathcal{P}(Run(p\boldsymbol{u}^{in}, D)) > 0$.

$\Rightarrow$: We show that for almost all $w \in Run(p\boldsymbol{u}^{in}, D)$ and all $i \in \{1, \ldots, d-1\}$, one of the following conditions holds:

(A) $\boldsymbol{t}_{oc}[i] > 0$ and $\liminf_{k\to\infty} cval_i(w(k)) = \infty$,

(B) $\boldsymbol{t}_{oc}[i] = 0$ and $cval_i(w(k)) \geq \mathit{botinf}_i(state(w(k)))$ for all k's large enough.

Concerning (A), note that for almost all runs w of $\mathcal{M}_\mathcal{B}$ initiated in $q(0)$ where $q \in D$ we have that

$$\lim_{k\to\infty} tot_i(w;k) \,/\, k = \boldsymbol{t}_{oc}[i] > 0$$

which implies, as above, that $\lim_{k\to\infty} tot_i(w;k) = \infty$. Let $q\boldsymbol{u}$ be a configuration of $\mathcal{A}$ which is oc-above 1 and satisfies $\boldsymbol{u}[d] = 0$. Then almost all runs w of $\Upsilon_d(Run(q\boldsymbol{u}, \mathcal{Z}_{-d}))$ satisfy $\lim_{k\to\infty} tot_i(w;k) = \infty$, and hence also almost all runs w of $Run(q\boldsymbol{u}, \mathcal{Z}_{-d})$ satisfy $\liminf_{k\to\infty} cval_i(w(k)) = \infty$. As almost every run of $Run(p\boldsymbol{u}^{in}, D)$ visits $q\boldsymbol{u}$ for some $\boldsymbol{u}$ that is oc-above 1 nad satisfying $\boldsymbol{u}[d] = 0$, almost all runs w in $Run(p\boldsymbol{u}^{in}, D)$ satisfy $\liminf_{k\to\infty} cval_i(w(k)) = \infty$.

Concerning (B), note that almost all runs $w \in Run(p\boldsymbol{u}^{in}, D)$ satisfying $cval_i(w'(k)) < botinf_i(state(w(k)))$ for infinitely many k's eventually visit zero in some counter (there is a path of length at most $3|Q|^3$ from each such $w(k)$ to a configuration with zero in counter i, or in one of the other counters).

The above claim immediately implies that for every $n \in \mathbb{N}$, almost every run of $Run(p\boldsymbol{u}^{in}, D)$ visits a configuration $q\boldsymbol{u}$ oc-above n.

The other implication is proven similarly as in Lemma 5. ■

Following [6] the SQUARE-ROOT-SUM problem is defined as follows. Given natural numbers $d_1, \ldots, d_n \in \mathbb{N}$ and $k \in \mathbb{N}$, decide whether $\sum_{i=1}^n \sqrt{d_i} \geq k$. Membership of square-root-sum in NP has been open since 1976. It is known that SQUARE-ROOT-SUM reduces to PosSLP and hence lies in the counting hierarchy, see [6] and the references therein for more information on square-root-sum, PosSLP, and the counting hierarchy.

Proposition 2 *The qualitative $\mathcal{Z}_{-d}$-reachability problem is* SQUARE-ROOT-SUM-*hard, even for two-dimensional pMC where $\mathbf{e}[q] < \infty$ for all $q \in D$ in every BSCC of $\mathcal{X}_\mathcal{B}$.*

Proof: We adapt a reduction from [21]. Let $d_1, \ldots, d_n, k \in \mathbb{N}$ be an instance of theSQUARE-ROOT-SUM problem. Let $m := \max\{d_1, \ldots, d_n, k\}$. Define $c_i := \frac{1}{2}(1 - d_i/m^2)$ for $i \in \{1, \ldots, n\}$.

We construct a pMC $\mathcal{A} = (Q, \gamma, W)$ as follows. Take $Q := \{q, r_1, \ldots, r_n, s_+, s_-\}$ and set of rules γ as listed below (we omit labels and some irrelevant rules). The weight assignment W is, for better readability, specified in terms of probabilities rather than weights, with the obvious intended meaning.

$$\begin{aligned}
\tfrac{1}{2n} &: (q, (0,0), \emptyset, r_i) && \text{for all } i \in \{1, \ldots, n\} \\
\tfrac{1}{2} &: (q, (0,-1), \emptyset, s_+) && \\
\tfrac{1}{2} &: (r_i, (0,+1), \emptyset, r_i) && \text{for all } i \in \{1, \ldots, n\} \\
c_i &: (r_i, (0,-1), \emptyset, r_i) && \text{for all } i \in \{1, \ldots, n\} \\
\tfrac{1}{2} - c_i &: (r_i, (0,0), \emptyset, s_-) && \text{for all } i \in \{1, \ldots, n\} \\
1 &: (r_i, (0,+1), \{2\}, q) && \text{for all } i \in \{1, \ldots, n\} \\
1 &: (s_-, (0,-1), \emptyset, s_-) && \\
1 &: (s_-, (-1,+1), \{2\}, q) && \\
\tfrac{k}{nm} &: (s_+, (+1,+1), \{2\}, q) && \\
1 - \tfrac{k}{nm} &: (s_+, (0,+1), \{2\}, q) &&
\end{aligned}$$

We claim that $\mathcal{P}(Run(q\mathbf{1}, \{\{1\}\})) = 1$ holds if and only if $\sum_i \sqrt{d_i} \geq k$ holds. It is shown in [21] that $r_i(1,1)$ reaches, with probability 1, the configuration $r_i(1,0)$ or $s_-(1,0)$ before reaching any other configuration with 0 in the second counter. In fact, it is shown there that the probability of reaching $s_-(1,0)$ is $\sqrt{d_i}/m$, and of reaching $r_i(1,0)$ is $1 - \sqrt{d_i}/m$. The only BSCC D of $\mathcal{X}_\mathcal{B}$ is $\{r_1, \ldots, r_n, s_+, s_-\}$. It follows for the invariant distribution $\boldsymbol{\mu}_{oc}$ that $\boldsymbol{\mu}_{oc}[s_+] = \frac{1}{2}$ and $\boldsymbol{\mu}_{oc}[s_-] = \frac{1}{2nm} \sum_i \sqrt{d_i}$. From the construction it is clear that $\boldsymbol{\delta}_1[s_+] = +\frac{k}{nm}$ and $\boldsymbol{\delta}_1[s_-] = -1$ and $\boldsymbol{\delta}_1[r_i] = 0$ for all $i \in \{1, \ldots, n\}$. Hence we:

$$\begin{aligned}
\boldsymbol{t}_{oc}[i] &= \left(\boldsymbol{\mu}_{oc}^T \cdot \boldsymbol{\delta}_i\right) / \left(\boldsymbol{\mu}_{oc}^T \cdot \boldsymbol{e}\right) \\
&= \left(\tfrac{1}{2} \cdot \tfrac{k}{nm} - \tfrac{1}{2nm} \textstyle\sum_i \sqrt{d_i}\right) / \left(\boldsymbol{\mu}_{oc}^T \cdot \boldsymbol{e}\right)
\end{aligned}$$

So we have $\boldsymbol{t}_{oc}[i] \leq 0$ if and only if $\sum_i \sqrt{d_i} \geq k$ holds. The statement then follows from Lemma 13. ■

APPENDIX D
MARTINGALE

A. Matrix Notation

In the following, Q will denote a finite set (of control states). We view the elements of $\mathbb{R}^Q$ and $\mathbb{R}^{Q\times Q}$ as vectors and matrices, respectively. The entries of a vector $\boldsymbol{v} \in \mathbb{R}^Q$ or a matrix $M \in \mathbb{R}^{Q\times Q}$ are denoted by $\boldsymbol{v}[p]$ and $M[p,q]$ for $p, q \in Q$. Vectors are column vectors by default; we denote the transpose of a vector $\boldsymbol{v}$ by $\boldsymbol{v}^T$, which is a row vector. For vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^Q$ we write $\boldsymbol{u} \leq \boldsymbol{v}$ (resp. $\boldsymbol{u} < \boldsymbol{v}$) if the respective inequality holds in all components. The vector all whose entries are 0 (or 1) is denoted by $\mathbf{0}$ (or $\mathbf{1}$, respectively). We denote the identity matrix by $I \in \{0,1\}^Q$ and the zero matrix by 0. A matrix $M \in [0,1]^{Q\times Q}$ is called *stochastic* (*substochastic*), if each row sums up to 1 (at most 1, respectively). A nonnegative matrix $M \in [0,\infty)^Q$ is called *irreducible*, if the directed graph $(Q, \{(p,q) \in Q^2 \mid M[p,q] > 0\})$ is strongly connected. We denote the spectral radius (i.e., the largest among the absolute values of the eigenvalues) of a matrix M by $\rho(M)$.

B. Proof of Lemma 15

The proof of Lemma 15 is based on the notion of *group inverses* for matrices [18]. Close connections of this concept to (finite) Markov chains are discussed in [26]. We have the following lemma:

Lemma 17. *Let P be a nonnegative irreducible matrix with $\rho(P) = 1$. Then there is a matrix, denoted by $(I-P)^{\#}$, such that $(I-P)(I-P)^{\#} = I - W$, where W is a matrix whose rows are scalar multiples of the dominant left eigenvector of P.*

Proof: In [26] the case of a stochastic matrix P is considered. In the following we adapt proofs from [26, Theorems 2.1 and 2.3]. For a square matrix M, a matrix $M^{\#}$ is called *group inverse* of M, if we have $MM^{\#}M = M$ and $M^{\#}MM^{\#} = M^{\#}$ and $MM^{\#} = M^{\#}M$. It is shown in [18, Lemma 2] that a matrix M has a group inverse if and only if M and M^2 have the same rank. As P is irreducible, the Perron-Frobenius theorem implies that the eigenvalue 1 has algebraic multiplicity equal to one. So 0 is an eigenvalue of $M := (I-P)$ with algebraic multiplicity 1. This implies that the Jordan form for M can be written as

$$\begin{pmatrix} 0 & 0 \\ 0 & J' \end{pmatrix}$$

where the square matrix J' is invertible. It follows that M and M^2 have the same rank, so $M^{\#}$ exists. Using the definition of group inverse, we have $(I - MM^{\#})P = (I - MM^{\#})$. In other words, the rows of $I - (I-P)(I-P)^{\#}$ are left eigenvectors of P with eigenvalue 1. The statement then follows by the Perron-Frobenius theorem. ■

Now we can prove Lemma 15.

Proof: Recall that the matrix A is stochastic and irreducible. Also recall from the main body of the paper that $\alpha^T A = \alpha^T$. It follows from the Perron-Frobenius theorem that $\rho(A) = 1$. Define $\boldsymbol{g}(0)[D] := (I-A)^{\#}\boldsymbol{r}_0$, where $(I-A)^{\#}$ is the matrix from Lemma 17. The non-D-components can be set arbitrarily, for instance, they can be set to 0. So we have $\boldsymbol{g}(0)[D] = \boldsymbol{r}_0 + A\boldsymbol{g}(0)[D] - W\boldsymbol{r}_0$, where the rows of W are multiples of $\boldsymbol{\alpha}^T$. We have:

$$\begin{aligned} \boldsymbol{\alpha}^T \boldsymbol{r}_0 &= \boldsymbol{\alpha}^T \left(\boldsymbol{\delta}_1 - \frac{\boldsymbol{\alpha}^T \boldsymbol{\delta}_1}{\boldsymbol{\alpha}^T \boldsymbol{e}} \boldsymbol{e} \right) && \text{by the definitions of } \boldsymbol{r}_0 \text{ and } t \\ &= 0\,. \end{aligned}$$

So (3) follows. ■

C. Proof of Proposition 3

For notational convenience, we assume in the following that $\mathcal{A}$ is a 2-dimensional pMC corresponding to the labelled 1-dimensional pMC $\mathcal{D}$ from the main body; i.e., the first counter of $\mathcal{A}$ encodes the rewards of $\mathcal{D}$, the second counter of $\mathcal{A}$ encodes the unique counter of $\mathcal{D}$.

Define the substochastic matrices $Q_{\rightarrow} \in [0,1]^{D\times D}$, $Q_{\uparrow} \in [0,1]^{D\times Q}$, $P_{\downarrow}, P_{\rightarrow}, P_{\uparrow} \in [0,1]^{Q\times Q}$ as follows:

$$Q_{\rightarrow}[p,q] := \sum \{y \mid \exists x_1 : p(1,0) \xrightarrow{y} q(x_1,0)\} \tag{7}$$

$$Q_{\uparrow}[p,q] := \sum \{y \mid \exists x_1 : p(1,0) \xrightarrow{y} q(x_1,1)\} \tag{8}$$

$$P_{\downarrow}[p,q] := \sum \{y \mid \exists x_1 : p(1,1) \xrightarrow{y} q(x_1,0)\} \tag{9}$$

$$P_{\rightarrow}[p,q] := \sum \{y \mid \exists x_1 : p(1,1) \xrightarrow{y} q(x_1,1)\} \tag{10}$$

$$P_{\uparrow}[p,q] := \sum \{y \mid \exists x_1 : p(1,1) \xrightarrow{y} q(x_1,2)\}\,, \tag{11}$$

where the transitions $p(1,0) \xrightarrow{y} q(x_1,0)$, etc. are in the Markov chain $\mathcal{M}_{\mathcal{A}}$. Note that $Q_{\rightarrow} + Q_{\uparrow}$ and $P_{\downarrow} + P_{\rightarrow} + P_{\uparrow}$ are stochastic. Observe that we have, e.g., that $Q_{\rightarrow}[p,q] = \sum \{y \mid p(0) \xrightarrow{y} q(0)\}$, where the transition $p(0) \xrightarrow{y} q(0)$ is in the Markov chain $\mathcal{M}_{\mathcal{D}}$.

The matrix G from the main body of the paper is (see e.g. [21]) the least (i.e., componentwise smallest) matrix with $G \in [0,1]^{Q\times Q}$ and

$$G = P_{\downarrow} + P_{\rightarrow} G + P_{\uparrow} G G\,. \tag{12}$$

Recall from the main body that G is stochastic.

For the matrix A defined in the main body we have

$$A = Q_{\rightarrow} + Q_{\uparrow} G[D]\,, \tag{13}$$

where $G[D] \in [0,1]^{Q\times D}$ denotes the matrix obtained from G by deleting the columns with indices in $Q \setminus D$. Recall from the main body that A is stochastic and irreducible.

Define

$$B := P_{\rightarrow} + P_{\uparrow}G + P_{\uparrow} \in [0,1]^{Q\times Q} . \tag{14}$$

Define the vectors $\boldsymbol{\delta}_{=0!} \in [-1,1]^D$, $\boldsymbol{\delta}_{>0!} \in [-1,1]^Q$ with

$$\boldsymbol{\delta}_{=0!}[p] := \sum\{yx_1 \mid \exists q \in Q\ \exists x_2 : p(1,0) \xrightarrow{y} q(1+x_1,x_2)\} \tag{15}$$

$$\boldsymbol{\delta}_{>0!}[p] := \sum\{yx_1 \mid \exists q \in Q\ \exists x_2 : p(1,1) \xrightarrow{y} q(1+x_1,x_2)\} , \tag{16}$$

where the transitions $p(1,0) \xrightarrow{y} q(1+x_1,x_2)$ and $p(1,1) \xrightarrow{y} q(1+x_1,x_2)$ are in the Markov chain $\mathcal{M}_{\mathcal{A}}$. We have that $\boldsymbol{\delta}_{=0!}[p]$ is the expected reward incurred in the next step when starting in $p(0)$. Similarly, $\boldsymbol{\delta}_{>0!}[p]$ is the expected reward incurred in the next step when starting in $p(x_2)$ for $x_2 \geq 1$.

Lemma 18. *The following equalities hold:*

$$\boldsymbol{e}_{\downarrow} = \mathbf{1} + B\boldsymbol{e}_{\downarrow} \tag{17}$$

$$\boldsymbol{\delta}_{\downarrow} = \boldsymbol{\delta}_{>0!} + B\boldsymbol{\delta}_{\downarrow} \tag{18}$$

Proof: Define the following vectors:

$$\begin{aligned}
\boldsymbol{e}_1 &:= P_{\downarrow}\mathbf{1} \\
\boldsymbol{e}_2 &:= P_{\rightarrow}(\mathbf{1} + \boldsymbol{e}_{\downarrow}) \\
\boldsymbol{e}_3 &:= P_{\uparrow}(\mathbf{1} + \boldsymbol{e}_{\downarrow}) \\
\boldsymbol{e}_4 &:= P_{\uparrow}G\boldsymbol{e}_{\downarrow}
\end{aligned}$$

Observe that $\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 + \boldsymbol{e}_4$ is the right-hand side of (17), so we have to show that $\boldsymbol{e}_{\downarrow} = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 + \boldsymbol{e}_4$. Let $q \in Q$. For concreteness we consider the configuration $q(1)$. We have that $\boldsymbol{e}_1[q]$ is the probability that the first step decreases the counter by 1. Note that we can view $\boldsymbol{e}_1[q]$ also as the probability that the first step decreases the counter by 1 (namely, to 0), multiplied with the conditional expected time to reach the 0-level from $q(1)$, conditioned under the event that the first step decreases the counter by 1. We have that $\boldsymbol{e}_2[q]$ is the probability that the first step keeps the counter constant (at 1), multiplied with the conditional expected time to reach the 0-level from $q(1)$, conditioned under the event that the first step keeps the counter constant. We have that $\boldsymbol{e}_3[q]$ is the probability that the first step increases the counter by 1 (namely, to 2), multiplied with the conditional expected time to reach the 1-level (again) from $q(1)$, conditioned under the event that the first step increases the counter by 1. Finally, $\boldsymbol{e}_4[q]$ is the probability that the first step increases the counter by 1 (namely, to 2), multiplied with the conditional expected time to reach the 0-level *after* having returned to the 1-level, conditioned under the event that the first step increases the counter by 1. So, $(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 + \boldsymbol{e}_4)[q]$ is the expected time to reach the 0-level. Hence (17) is proved. The proof of (18) is similar, with reward replacing time. ■

By combining (17) and (18) with the definition of $\boldsymbol{r}_{\downarrow}$ we obtain:

$$\boldsymbol{r}_{\downarrow} = \boldsymbol{\delta}_{>0!} - t\mathbf{1} + B\boldsymbol{r}_{\downarrow} \tag{19}$$

From the definitions we obtain

$$\boldsymbol{\delta}_1 := \boldsymbol{\delta}_{=0!} + Q_{\uparrow}\boldsymbol{\delta}_{\downarrow} \tag{20}$$

$$\boldsymbol{e} := \mathbf{1} + Q_{\uparrow}\boldsymbol{e}_{\downarrow} . \tag{21}$$

By combining (20) and (21) with the definition of $\boldsymbol{r}_0$ we obtain:

$$\boldsymbol{r}_0 = \boldsymbol{\delta}_{=0!} - t\mathbf{1} + Q_{\uparrow}\boldsymbol{r}_{\downarrow} . \tag{22}$$

Now we can prove Proposition 3.

Proof: We have

$$\begin{aligned}
&\mathcal{E}\left(m^{(\ell+1)} - m^{(\ell)} \;\middle|\; w(\ell),\ x_2^{(\ell)} = 0\right) \\
&= \left(\boldsymbol{\delta}_{=0!} - t\mathbf{1} + Q_{\rightarrow}\boldsymbol{g}(0) + Q_{\uparrow}\boldsymbol{g}(1) - \boldsymbol{g}(0)[D]\right)[p^{(\ell)}] && \text{by (5), (15), (7), (8)} \\
&= \left(\boldsymbol{\delta}_{=0!} - t\mathbf{1} + Q_{\rightarrow}\boldsymbol{g}(0) + Q_{\uparrow}\left(\boldsymbol{r}_{\downarrow} + G\boldsymbol{g}(0)\right) - \boldsymbol{g}(0)[D]\right)[p^{(\ell)}] && \text{by (4)} \\
&= \left(\boldsymbol{r}_0 + (A - I)\boldsymbol{g}(0)[D]\right)[p^{(\ell)}] && \text{by (22), (13)} \\
&= 0 && \text{by (3)}
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{E}\left(m^{(\ell+1)} - m^{(\ell)} \,\middle|\, w(\ell),\ x_2^{(\ell)} > 0\right) \\
&= \left(\boldsymbol{\delta}_{>0!} - t\mathbf{1} + P_{\downarrow}\boldsymbol{g}\big(x_2^{(\ell)} - 1\big) + P_{\rightarrow} \underbrace{\boldsymbol{g}\big(x_2^{(\ell)}\big)}_{\overset{(4)}{=}\ \boldsymbol{r}_{\downarrow} + G\boldsymbol{g}(x_2^{(\ell)} - 1)} + P_{\uparrow} \underbrace{\boldsymbol{g}\big(x_2^{(\ell)} + 1\big)}_{\overset{(4)}{=}\ \boldsymbol{r}_{\downarrow} + G\boldsymbol{g}(x_2^{(\ell)})} - \boldsymbol{g}\big(x_2^{(\ell)}\big)\right)[p^{(\ell)}] && \text{by (5), (16), (9)–(11)} \\
&= \left(\boldsymbol{\delta}_{>0!} - t\mathbf{1} + (P_{\rightarrow} + P_{\uparrow}G + P_{\uparrow})\boldsymbol{r}_{\downarrow} + (P_{\downarrow} + P_{\rightarrow}G + P_{\uparrow}GG)\,\boldsymbol{g}\big(x_2^{(\ell)} - 1\big) - \boldsymbol{g}\big(x_2^{(\ell)}\big)\right)[p^{(\ell)}] && \text{by (4)} \\
&= \left(\boldsymbol{r}_{\downarrow} + G\boldsymbol{g}\big(x_2^{(\ell)} - 1\big) - \boldsymbol{g}\big(x_2^{(\ell)}\big)\right)[p^{(\ell)}] && \text{by (14), (19), (12)} \\
&= 0 && \text{by (4)}\,.
\end{aligned}$$

■

D. Proof of Lemma 16

Define $e_{max} := 1 + \max_{q \in Q} \boldsymbol{e}_{\downarrow}[q] \geq 2$.

We first prove the following lemma:

Lemma 19. *There exists a vector $\boldsymbol{g} \in \mathbb{R}^D$ with $\boldsymbol{g} = \boldsymbol{r}_0 + A\boldsymbol{g}$ and*

$$0 \leq \boldsymbol{g}[q] \leq \frac{e_{max}|D|}{y_{min}^{|D|}} \qquad \text{for all } q \in D,$$

where y_{min} denotes the smallest nonzero entry in the matrix A.

Proof: Recall that by Lemma 15 there is a vector $\boldsymbol{g}(0)[D] \in \mathbb{R}^D$ with

$$\boldsymbol{g}(0)[D] = \boldsymbol{r}_0 + A\boldsymbol{g}(0)[D]\,.$$

Since A is stochastic, we have $A\mathbf{1} = \mathbf{1}$. So there is $\kappa \in \mathbb{R}$ such that with $\boldsymbol{g} := \boldsymbol{g}(0)[D] + \kappa\mathbf{1}$ we have

$$\boldsymbol{g} = \boldsymbol{r}_0 + A\boldsymbol{g} \tag{23}$$

and $g_{max} = e_{max}|D|/y_{min}^{|D|}$, where we denote by g_{min} and g_{max} the smallest and largest component of $\boldsymbol{g}$, respectively. We have to show $g_{min} \geq 0$. Let $q \in D$ such that $\boldsymbol{g}[q] = g_{max}$. Define the *distance* of a state $p \in D$, denoted by η_p, as the distance of p from q in the directed graph induced by A. Note that $\eta_q = 0$ and all $p \in D$ have distance at most $|D| - 1$, as A is irreducible. We prove by induction that a state p with distance i satisfies $\boldsymbol{g}[p] \geq g_{max} - e_{max}i/y_{min}^i$. The claim is obvious for the induction base ($i = 0$). For the induction step, let p be a state such that $\eta_p = i + 1$. Then there is a state r such that $A[r, p] > 0$ and $\eta_r = i$. We have

$$\begin{aligned}
\boldsymbol{g}[r] &= (A\boldsymbol{g})[r] + \boldsymbol{r}_0[r] && \text{by (23)} \\
&\leq (A\boldsymbol{g})[r] + e_{max} && \text{as } \boldsymbol{r}_0 \leq e_{max}\mathbf{1} \\
&= \big(A[r,p] \cdot \boldsymbol{g}[p] + \sum_{p' \neq p} A[r,p'] \cdot \boldsymbol{g}[p']\big) + e_{max} \\
&\leq A[r,p] \cdot \boldsymbol{g}[p] + (1 - A[r,p]) \cdot g_{max} + e_{max} && \text{as } A \text{ is stochastic.}
\end{aligned}$$

By rewriting the last inequality and applying the induction hypothesis to $\boldsymbol{g}[r]$ we obtain

$$\boldsymbol{g}[p] \geq g_{max} - \frac{g_{max} - \boldsymbol{g}[r] + e_{max}}{A[r,p]} \geq g_{max} - \frac{g_{max} - (g_{max} - e_{max}i/y_{min}^i) + e_{max}}{y_{min}} \geq g_{max} - \frac{e_{max}(i+1)}{y_{min}^{i+1}}\,.$$

This completes the induction step. Hence we have $g_{min} \geq 0$ as desired. ■

Now we prove Lemma 16:

Proof: We need the following explicit expression for $\boldsymbol{g}$:

$$\boldsymbol{g}(n) = G^n\boldsymbol{g}(0) + \sum_{i=0}^{n-1} G^i\boldsymbol{r}_{\downarrow} \qquad \text{for all } n \geq 0 \tag{24}$$

Let us prove (24) by induction on n. For the induction base note that the cases $n = 0, 1$ follow immediately from the definition (4) of $\boldsymbol{g}$. For the induction step let $n \geq 1$. We have:

$$
\begin{aligned}
\boldsymbol{g}(n+1) &= \boldsymbol{r}_{\downarrow} + G\boldsymbol{g}(n) && \text{by (4)} \\
&= G^{n+1}\boldsymbol{g}(0) + \sum_{i=0}^{n} G^i \boldsymbol{r}_{\downarrow} && \text{by the induction hypothesis}
\end{aligned}
$$

So (24) is proved. In the following we assume that $\boldsymbol{g}(0)$ is chosen as in Lemma 19. We then have:

$$
\begin{aligned}
|\boldsymbol{g}(n)| &\leq |\boldsymbol{g}(0)| + n|\boldsymbol{r}_{\downarrow}| && \text{by (24) and as } G \text{ is stochastic} \\
&\leq \frac{e_{max}|D|}{y_{min}^{|D|}} + ne_{max} && \text{by Lemma 19 and as } |\boldsymbol{r}_{\downarrow}| \leq |\boldsymbol{e}_{\downarrow}| \leq e_{max}
\end{aligned}
$$

■

APPENDIX E
PROOF OF THEOREM 4

We show that $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{-d}))$ can be effectively approximated up to an arbitrarily small absolute error $\varepsilon > 0$.

We will use the fact than the probability of reaching a specific set of states in a 1-dimensional pMC can be effectively approximated.

Lemma 20. *Let $\mathcal{A}'$ be any one-dimensional pMC and let Q be its set of states. Given an initial configuration $q(k)$, a set $S \subseteq Q$ and $\varepsilon > 0$ we can effectively approximate, up to the absolute error ε, the probability of reaching a configuration $r(j)$ with $r \in S$ from $q(k)$.*

Proof: The crucial observation is that if there is a path from a state t to S in $\mathcal{F}_{\mathcal{A}'}$, then for every $j \geq |Q|$ there is a path of length at most n from $q(j)$ to a configuration with the control state in S. If there is no path from t to S in $\mathcal{F}_{\mathcal{A}'}$, then a configuration with the control state in S cannot be reached from $q(j)$ for any j. Thus, the probability that a run initiated in $q(k)$ visits a counter value $q(k+i)$ without visiting S *and then* visits S is at most $(1 - p_{min}^{|Q|})^{\frac{i}{|Q|}}$, where p_{min} is the minimal non-zero probability in $\mathcal{A}'$. For a given ε, we can effectively compute i such that $(1 - p_{min}^{|Q|})^{\frac{i}{|Q|}} \leq \varepsilon$ and effectively construct a finite-state Markov chain $\mathcal{M}$ in which the configurations of $\mathcal{A}'$ with counter value $\leq i + k$ are encoded in the finite-state control unit (i.e., $\mathcal{M}$ can be defined as a Markov chain obtained from $\mathcal{M}_{\mathcal{A}'}$ by removing all configurations with counter height $> i + k$ together with their adjacent transitions and replace all transitions outgoing from configurations of the form $r(i + k)$ with self-loops on $r(i + k)$).

Using standard methods for finite-state Markov chains we can compute the probability of reaching the set $S' = \{r(j) \mid r \in S\}$ from $q(k)$ in $\mathcal{M}$. From the discussion above it follows that this value is an ε-approximation of the probability that $r(j)$ with $r \in S$ is reached in $\mathcal{A}'$. ■

The proof closely follows the proof of Theorem 2. We first show how to approximate the probability under the assumption that p is in some BSCC D of $\mathcal{X}_{\mathcal{B}}$. It is then easy to drop this assumption.

Proposition 5. *There is an algorithm which, for a given d-dimensional pMC $\mathcal{A}$, its initial configuration $p\boldsymbol{v}$ such that p is in a BSCC of $\mathcal{X}_{\mathcal{B}}$, and a given $\varepsilon > 0$ computes a number ν such that $|\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{-d})) - \nu| \leq d \cdot \varepsilon$.*

Proof: Clearly we need to consider only $d \geq 2$. We proceed by induction on d. The base case and the induction step are solved in almost identical way (which was the case also in the proof of Proposition 4). Therefore, below we present the proof of the induction step and only highlight the difference between the induction step and the base case when needed.

We again assume that $\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{-d})) < 1$. This can be checked effectively due to Theorem 3 and if the condition does not hold, we may output $\nu = 1$. In particular we assume that all rewards in D are oc-diverging.

Recall from the proof of Proposition 4 that for any counter i and any vector $\boldsymbol{\beta} \in \{-1, 0, 1\}^d$ we denote by $\boldsymbol{\beta}_{-i}$ the $(d-1)$-dimensional vector obtained from $\boldsymbol{\beta}$ by deleting its i-component; and by $\mathcal{A}_{-i}$ the $(d-1)$-dimensional pMC $\mathcal{A}_{-i}$ obtained from $\mathcal{A}$ by "forgetting" the i-th counter. (See the proof of Proposition 4 for a formal definition).

Let $\boldsymbol{t}_{oc}$ be the oc-trend of D. For every counter $i \in \{1, \ldots, d-1\}$ such that $rt[i] > 0$ we compute the number A_0 of Lemma 4 for D in $\mathcal{X}_{\mathcal{D}}$, and denote this number by A_i. We put $A_{max} = \max\{A_i \mid \boldsymbol{t}_{oc}[i] > 0\}$. Then we compute a number such that $\frac{A_{max}^K}{<}\varepsilon/2$. For any $(d-1)$-dimensional vector $\boldsymbol{x}$ we denote by $mindiv(\boldsymbol{x})$ the smallest $i \in \{1, \ldots, d-1\}$ such that either $\boldsymbol{t}_{oc}[i] > 0$ and $\boldsymbol{x}[i] \geq K$ or $\boldsymbol{t}[i] = 0$ and $\boldsymbol{x}[i] \geq 3|Q|^3$ (if such i does not exist, we put $mindiv(\boldsymbol{x}) = \bot$).

Consider a 1-dimensional pMC $\mathcal{A}_K = (Q', \gamma', W')$ which can be obtained from $\mathcal{A}$ as follows:

- Q' consists of all tuple $(q, \boldsymbol{u})$, where $q \in Q$ and $\boldsymbol{u}$ is an arbitrary $(d-1)$-dimensional vector of non-negative integers whose every component is bounded by K; additionally, Q' contains two special states $q\uparrow$ and $q\downarrow$.
- $((q, \boldsymbol{u}), j, c, (r, \boldsymbol{z})) \in \gamma'$ iff $mindiv(\boldsymbol{u}) = \bot$ and $(q, \langle \boldsymbol{z} - \boldsymbol{u}, j \rangle_d, c, r) \in \gamma$.

- For every $1 \leq i \leq d-1$ and every $(q, \boldsymbol{u}) \in |Q|$ such that $mindiv(\boldsymbol{u}) \neq \bot$ we have rules $((q, \boldsymbol{u}), 0, \emptyset, q\uparrow)$ and $((q, \boldsymbol{u}), 0, \emptyset, q\uparrow)$ in γ'.
- $W'((q, \boldsymbol{u}), j, c, (r, \boldsymbol{z})) = W(q, \langle \boldsymbol{z} - \boldsymbol{u}, j\rangle_d, c, r)$ for all rules in γ' of this shape.
- $W'((q, \boldsymbol{u}), 0, \emptyset, q\downarrow) = x$, where x is some $((d-1) \cdot \varepsilon)$-approximation of $\mathcal{P}_{\mathcal{A}_{-i}}(Run(q\boldsymbol{u}_{-i}, \mathcal{Z}_{-d}))$ (which can be computed using the algorithm for dimension $d-1$).
- $W'((q, \boldsymbol{u}), 0, \emptyset, q\uparrow) = 1 - W'((q, \boldsymbol{u}), 0, \emptyset, q\downarrow)$.

In other words $\mathcal{A}_K$ is obtained from $\mathcal{A}$ by encoding all configurations where all of the first $d-1$ counters are bounded by K explicitly into the state space. If one of these counters surpasses K, we "forget" about this counter and approximate the 0-reachability in the resulting configuration recursively.

By induction $\mathcal{A}_K$ can be effectively constructed.

Now for an initial configuration $p\boldsymbol{v}$ in which the first $d-1$ counters are bounded by K let $P(p\boldsymbol{v})$ be the probability of reaching, when starting in $p\boldsymbol{v}$ in $\mathcal{A}_K$, either the state q_{down} or a state in which at least one of the first $d-1$ counters is 0. Due to Lemma 20 we can approximate $P(p\boldsymbol{v})$ effectively up to $\varepsilon/2$. We claim that $|\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{-d})) - P(p\boldsymbol{v})| \leq d \cdot \varepsilon$.

Indeed, let us denote Div the set of all configurations $q\boldsymbol{y}$ of $\mathcal{A}$ such that $\boldsymbol{y}_{-d}$ is bounded by K and $mindiv(\boldsymbol{y}_{-d}) \neq \bot$. For every $q\boldsymbol{u} \in Div$ we denote by $x_{q\boldsymbol{u}}$ the probability of the transition leading from some $(q(k), \boldsymbol{u}_{-d})$ to q_{down} in $\mathcal{M}_{\mathcal{A}_K}$ (note that this probability is independent of k and is equal to the weight of the corresponding rule in $\mathcal{A}_K$). Then $|\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{-d})) - P(p\boldsymbol{v})| \leq \max_{q\boldsymbol{u} \in Div} |\mathcal{P}(Run(q < \boldsymbol{u}, \mathcal{Z}_{-d}) - x_{q\boldsymbol{u}}|$. Now $\mathcal{P}(Run(q\boldsymbol{u}, \mathcal{Z}_{-d}) \leq P_1(q\boldsymbol{u}) + P_2(q\boldsymbol{u})$, where $P_1(q\boldsymbol{u})$ is the probability that a run initiated in $q\boldsymbol{u}$ in $\mathcal{A}$ visits a configuration with i-th counter 0 via a $\mathcal{Z}_{-i,d}$-safe path, and $P_2(q\boldsymbol{u})$ is the probability that a run initiated in $q\boldsymbol{u}$ in $\mathcal{A}$ visits a configuration with some counter equal to 0 via an $\{i\}$-safe path.

So let us fix $q\boldsymbol{u} \in Div$ and denote $i = mindiv(\boldsymbol{u}_{-d})$. If $\boldsymbol{t}[i] = 0$, then we have $P_1(q\boldsymbol{u}) = 0$, by Lemma 10. Otherwise $P_1(q\boldsymbol{u})$ is bounded by the probability that a run w initiated in $q(K)$ in $\mathcal{B}$ satisfies $\inf_{j \geq 0} tot_i(w; j) \leq -K$ From Lemma 12 we get that this is bounded by $A_{max}^K \leq \varepsilon/2$, where the last inequality follows from the choice of K.

For $P_2(q\boldsymbol{u})$ note that $P_2(q\boldsymbol{u}) = \mathcal{P}_{\mathcal{A}_{-i}}(Run(q\boldsymbol{u}_{-i}, \mathcal{Z}_{-d}))$ and thus by the construction of $\mathcal{A}_K$ we have $|P_2(q\boldsymbol{u}) - x_{q\boldsymbol{u}}| \leq (d-1) \cdot \varepsilon$.

Altogether we have

$$|\mathcal{P}(Run(p\boldsymbol{v}, \mathcal{Z}_{-d})) - P(p\boldsymbol{v})| \leq |P_1(q\boldsymbol{u}) + P_2(q\boldsymbol{u}) - x_{q\boldsymbol{u}}| \leq \varepsilon/2 + (d-1) \cdot \varepsilon.$$

Now it is clear that approximating $P(p\boldsymbol{v})$ up to $\varepsilon/2$ and returning this value as ν yields the desired result. As in case 1, if some component of $\boldsymbol{v}$ surpasses K, we can immediately reduce the problem to the approximation for $(d-1)$-dimensional case.

Note that for the base case $d = 2$ the same approach can be used, the only difference that the weight of the rule $((q, \boldsymbol{u}), 0, \emptyset, q\uparrow)$ in $\mathcal{A}_K$ is 1 and the weight of $((q, \boldsymbol{u}), 0, \emptyset, q\downarrow)$ is 0.

∎

To prove Theorem 4 in its full generality it suffices to note, that we can effectively compute a constant $b \in (0,1)$ such that the probability that a run does not visit a configuration $q\boldsymbol{u}$ with q in some BSCC of $\mathcal{X}_{\mathcal{B}}$ or $Z(\boldsymbol{u}) \neq \emptyset$ in at most i steps is bounded by b^i (see Lemma 1 and Lemma 20). Therefore, to approximate the probability for $p\boldsymbol{v}$ with $\boldsymbol{v}$ not belonging to a BSCC of $\mathcal{X}_{\mathcal{B}}$ we can use the same approach as in case 1: we unfold $\mathcal{A}$ into a suitable number of steps and approximate the termination value in configurations where the state belongs to some D using the algorithm from the previous proposition. See the proof of Theorem 2 for further details.