Information Sciences 275 (2014) 168–198

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Reduction of fuzzy automata by means of fuzzy quasi-orders ☆

Aleksandar Stamenković, Miroslav Ćirić *, Jelena Ignjatović

University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, P. O. Box 224, 18000 Niš, Serbia

ARTICLE INFO

Article history:
Received 11 March 2009
Received in revised form 20 September 2013
Accepted 9 February 2014
Available online 19 February 2014

Keywords:
Fuzzy automaton
Fuzzy quasi-order
State reduction
Afterset automaton
Fuzzy relation equation
Fuzzy discrete event system

ABSTRACT

In our recent paper we have established close relationships between state reduction of a fuzzy automaton and resolution of a particular system of fuzzy relation equations. In that paper we have also studied reductions by means of those solutions which are fuzzy equivalences. In this paper we will see that in some cases better reductions can be obtained using the solutions of this system that are fuzzy quasi-orders. Generally, fuzzy quasi-orders and fuzzy equivalences are equally good in the state reduction, but we show that right and left invariant fuzzy quasi-orders give better reductions than right and left invariant fuzzy equivalences. We also show that alternate reductions by means of fuzzy quasi-orders give better results than alternate reductions by means of fuzzy equivalences. Furthermore we study a more general type of fuzzy quasi-orders, weakly right and left invariant ones, and we show that they are closely related to determinization of fuzzy automatons. We also demonstrate some applications of weakly left invariant fuzzy quasi-orders in conflict analysis of fuzzy discrete event systems.

1. Introduction

Unlike deterministic finite automata (DFA), whose efficient minimization is possible, the state minimization problem for non-deterministic finite automata (NFA) is computationally hard (PSPACE-complete, [41,86]) and known algorithms like in [16,42,57,58,80] cannot be used in practice. For that reason, many researchers aimed their attention to NFA state reduction methods which do not necessarily give a minimal one, but they give "reasonably" small NFAs that can be constructed efficiently. The basic idea of reducing the number of states of NFAs by computing and merging indistinguishable states resembles the minimization algorithm for DFAs, but is more complicated. That led to the concept of a right invariant equivalence on an NFA, studied by Ilie and Yu [36,37], who showed that they can be used to construct small NFAs from regular expressions. In particular, both the partial derivative automaton and the follow automaton of a given regular expression are factor automata of the position automaton with respect to the right invariant equivalences (cf. [19,20,35,37,38]). Right invariant equivalences have been also studied in [10,11,18,37,39,40]. Moreover, the same concept was studied under the name "bisimulation equivalence" in many areas of computer science and mathematics, such as modal logic, concurrency theory, set theory, formal verification, and model checking, and numerous algorithms have been proposed to compute the greatest bisimulation equivalence on a given labeled graph or a labeled transition system (cf. [54,59–61,64,77,79]). The faster algorithms are based on the crucial equivalence between the greatest bisimulation equivalence and the relational coarsest partition problem (see [28,29,43,76,63]).

☆ Research supported by Ministry of Education and Science, Republic of Serbia, Grant No. 174013.
* Corresponding author. Tel.: +381 18224492; fax: +381 18533014.
E-mail addresses: aca@pmf.ni.ac.rs (A. Stamenković), miroslav.ciric@pmf.edu.rs (M. Ćirić), jelena.ignjatovic@pmf.edu.rs (J. Ignjatović).

http://dx.doi.org/10.1016/j.ins.2014.02.028
0020-0255/

Better results in the state reduction of NFAs can be achieved in two ways. The first one was also proposed by Ilie and Yu in [36,37,39,40] who introduced the dual concept of a left invariant equivalence on an NFA and showed that even smaller NFAs can be obtained alternating reductions by means of right invariant and left invariant equivalences. On the other hand, Champarnaud and Coulon in [17,18] proposed use of quasi-orders (preorders) instead of equivalences and showed that the method based on quasi-orders gives better reductions than the method based on equivalences. They gave an algorithm for computing the greatest right invariant and left invariant quasi-orders on an NFA working in a polynomial time, which was later improved in [39,40].

The above mentioned issues will be discussed here within the context of fuzzy automata. Fuzzy automata and languages were studied since late 1960s, and until early 2000s mainly fuzzy automata and languages with membership values in the Gödel structure have been discussed. The last ten years attention of researchers is increasingly focused on fuzzy automata and languages over some more general structures, such as complete residuated lattices, lattice ordered monoids, and other kinds of lattices. In this paper we study fuzzy automata and transition systems taking membership values in a complete residuated lattice. Such automata were first studied by Qiu in [70,71], where some basic concepts were discussed, and later, Qiu and his coworkers have carried out extensive research of these fuzzy automata (cf. [72,74,81–85]). From a different point of view, fuzzy automata taking membership values in a complete residuated lattices were studied by Ignjatović, Ćirić and their coworkers in [24,25,32–34] and other papers. It is essential in this paper that the structure of truth values is a complete residuated lattice, because the residuation plays a key role in solving the systems of fuzzy relation inequalities and equations, on which our reduction methods are based.

Fuzzy finite automata are generalizations of NFAs, and the mentioned problems concerning minimization and reduction of NFAs are also present in the work with fuzzy automata. Reduction of the number of states of fuzzy automata was studied in [2,21,46,55,62,69,81,85], and the algorithms given there were also based on the idea of computing and merging indistinguishable states. They were called minimization algorithms, but the term minimization is not adequate because it does not involve the usual construction of the minimal fuzzy automaton in the collection of all fuzzy automata recognizing the same fuzzy language, but just the procedure of computing and merging indistinguishable states. Therefore, these are essentially just state reduction algorithms.

In the deterministic case we can effectively detect and merge indistinguishable states, but in the non-deterministic case we have sets of states and it is seemingly very difficult to decide whether two states are distinguishable or not. What we shall do in this paper is to find a superset such that one is certain not to merge state that should not be merged. There can always be states which could be merged but detecting those is too computationally expensive. In the case of fuzzy automata, this problem is even worse because we work with fuzzy sets of states. However, it turned out that in the non-deterministic case indistinguishability can be successfully modeled by equivalences and quasi-orders. In [24,25] we have shown that in the fuzzy case the indistinguishability can be modeled by fuzzy equivalences, and here we show that this can be done by fuzzy quasi-orders. It is worth noting that in all previous papers dealing with reduction of fuzzy automata (cf. [2,21,46,55,62,69]) only reductions by means of crisp equivalences have been investigated. This paper, as well as [24,25], shows that better reductions can be achieved employing fuzzy relations, namely, fuzzy equivalences and fuzzy quasi-orders.

In contrast to [24,25], where we have performed reductions constructing factor fuzzy transition systems and factor fuzzy automata corresponding to fuzzy equivalences, here we construct afterset fuzzy transition systems and afterset fuzzy automata which are determined by fuzzy quasi-orders. To ensure that the afterset automaton is equivalent with the original one, the related fuzzy quasi-order is required to be a solution to some specific systems of fuzzy relation equations and inequalities, and to get better reductions we search for bigger solutions, preferably for the greatest one, if any. The most general of these systems is called just the *general system*. We describe some properties of this system, but we were not able to give some general method for computing the appropriate solutions, because it usually consists of an infinite number of inequalities or equations. For that reason we consider some of its instances which are easier to solve. In Section 4 we consider systems of fuzzy relation equations whose solutions (in the collection of fuzzy quasi-orders) are called right and left invariant fuzzy quasi-orders. We prove the existence of the greatest right invariant fuzzy quasi-order on a fuzzy automaton or a fuzzy transition system contained in a given fuzzy quasi-order, and provide certain methods for its computing (cf. Theorem 4.3 or Algorithm 4.5, and Theorem 4.9), which work if the underlying complete residuated lattice satisfies certain additional conditions. We show that for some special types of right invariant fuzzy quasi-orders (right invariant crisp quasi orders, strongly right invariant fuzzy quasi-orders), the greatest fuzzy quasi-orders of these types can be computed without any additional condition on the structure of truth-values. However, these special types of right invariant fuzzy quasi-orders give worse reductions.

We also demonstrate that better reductions can be obtained in two ways. The first one is to use some more general types of fuzzy quasi-orders, which are called weakly right and left invariant. Computation of the greatest weakly right invariant fuzzy quasi-order on a fuzzy finite automaton also requires certain additional conditions on the structure of truth-values. However, although in the general case the greatest weakly right invariant fuzzy quasi-order gives a better reduction than the greatest right invariant one, its computational time is theoretically exponential (cf. Algorithms 5.7 and 5.8), in contrast to the greatest right invariant fuzzy quasi-order, which can be computed in a polynomial time (under the above mentioned conditions on the structure of truth-values). Another way to ensure better reductions is to alternate reductions by means of the greatest right and left invariant fuzzy quasi-orders, or by means of the greatest weakly right and left invariant fuzzy quasi-orders. Certain properties of these alternate reductions are described in Section 5.4.

Finally, in Section 6 we demonstrate some applications of weakly left invariant fuzzy quasi-orders in the fuzzy discrete event systems theory. We show that every fuzzy automaton $\mathscr{A}$ is conflict-equivalent with the afterset fuzzy automaton $\mathscr{A}/R$ w.r.t. any weakly left invariant fuzzy quasi-order R on $\mathscr{A}$. For the sake of conflict analysis, this means that in the parallel composition of fuzzy automata every component can be replaced by such afterset fuzzy automaton, what results in a smaller fuzzy automaton to be analysed, and do not affect conflicting properties of the components. It is also interesting to study applications of fuzzy quasi-orders for reducing automaton states in other branches of the theory of discrete event systems, for example in the fault diagnosis, and these applications will be a subject of our future research.

Note again that the meaning of state reductions by means of fuzzy quasi-orders and fuzzy equivalences is in their possible effectiveness, as opposed to the minimization which is not effective. However, by Theorem 3.5 we show that there exists a fuzzy automaton such that no its state reduction by means of fuzzy quasi-orders or fuzzy equivalences provide a minimal fuzzy automaton.

2. Preliminaries

2.1. Fuzzy sets and relations

In this paper we will use complete residuated lattices as structures of membership values. A *residuated lattice* is an algebra $\mathscr{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ such that.

(L1) $(L, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,
(L2) $(L, \otimes, 1)$ is a commutative monoid with the unit 1,
(L3) $\otimes$ and $\rightarrow$ form an *adjoint pair*, i.e., they satisfy the *adjunction property*: for all $x, y, z \in L$,

$$x \otimes y \leqslant z \iff x \leqslant y \rightarrow z. \tag{1}$$

If, in addition, $(L, \wedge, \vee, 0, 1)$ is a complete lattice, then $\mathscr{L}$ is called a *complete residuated lattice*.

The operations $\otimes$ (called *multiplication*) and $\rightarrow$ (called *residuum*) are intended for modeling the conjunction and implication of the corresponding logical calculus, and supremum ($\bigvee$) and infimum ($\bigwedge$) are intended for modeling of the existential and general quantifier, respectively. An operation $\leftrightarrow$ defined by

$$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x), \tag{2}$$

called *biresiduum* (or *biimplication*), is used for modeling the equivalence of truth values. It can be easily verified that with respect to $\leqslant$, $\otimes$ is isotonic in both arguments, and $\rightarrow$ is isotonic in the second and antitonic in the first argument. Emphasizing their monoidal structure, in some sources residuated lattices are called integral, commutative, residuated ℓ-monoids [30]. It can be easily verified that with respect to $\leqslant$, $\otimes$ is isotonic in both arguments, $\rightarrow$ is isotonic in the second and antitonic in the first argument, and for any $x, y, z \in L$ and any $\{x_i\}_{i \in I}, \{y_i\}_{i \in I} \subseteq L$, the following hold:

$$x \rightarrow y \leqslant x \otimes z \rightarrow y \otimes z, \tag{3}$$

$$\left(\bigvee_{i \in I} x_i\right) \otimes x = \bigvee_{i \in I} (x_i \otimes x), \tag{4}$$

$$\bigwedge_{i \in I} (x_i \rightarrow y_i) \leqslant \left(\bigwedge_{i \in I} x_i\right) \rightarrow \left(\bigwedge_{i \in I} y_i\right) \tag{5}$$

$$\bigwedge_{i \in I} (x_i \rightarrow y_i) \leqslant \left(\bigvee_{i \in I} x_i\right) \rightarrow \left(\bigvee_{i \in I} y_i\right). \tag{6}$$

For other properties of complete residuated lattices one can refer to [3,4].

The most studied and applied structures of truth values, defined on the real unit interval $[0, 1]$ with $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$, are the *Łukasiewicz structure* ($x \otimes y = \max(x + y - 1, 0)$, $x \rightarrow y = \min(1 - x + y, 1)$), the *Goguen* (*product*) *structure* ($x \otimes y = x \cdot y, x \rightarrow y = 1$ if $x \leqslant y$ and $= y/x$ otherwise) and the *Gödel structure* ($x \otimes y = \min(x, y)$, $x \rightarrow y = 1$ if $x \leqslant y$ and $= y$ otherwise). More generally, an algebra $([0, 1], \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is a complete residuated lattice if and only if $\otimes$ is a left-continuous t-norm and the residuum is defined by $x \rightarrow y = \bigvee\{u \in [0, 1] \mid u \otimes x \leqslant y\}$. Another important set of truth values is the set $\{a_0, a_1, \ldots, a_n\}$, $0 = a_0 < \ldots < a_n = 1$, with $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. A special case of the latter algebras is the two-element Boolean algebra of classical logic with the support $\{0, 1\}$. The only adjoint pair on the two-element Boolean algebra consists of the classical conjunction and implication operations. This structure of truth values we call the *Boolean structure*. A residuated lattice $\mathscr{L}$ satisfying $x \otimes y = x \wedge y$ is called a *Heyting algebra*, whereas a Heyting algebra satisfying the prelinearity axiom $(x \rightarrow y) \vee (y \rightarrow x) = 1$ is called a *Gödel algebra*. If any finitelly generated subalgebra of residuated lattice $\mathscr{L}$ is finite, then $\mathscr{L}$ is called *locally finite*. For example, every Gödel algebra, and hence, the Gödel structure, is locally finite, whereas the product structure is not locally finite.

In the further text $\mathscr{L}$ will be a complete residuated lattice. A *fuzzy subset* of a set A *over* $\mathscr{L}$, or simply a *fuzzy subset* of A, is any mapping from A into L. Ordinary crisp subsets of A are considered as fuzzy subsets of A taking membership values in the set $\{0,1\} \subseteq L$. Let f and g be two fuzzy subsets of A. The *equality* of f and g is defined as the usual equality of mappings, i.e., $f = g$ if and only if $f(x) = g(x)$, for every $x \in A$. The *inclusion* $f \leqslant g$ is also defined pointwise: $f \leqslant g$ if and only if $f(x) \leqslant g(x)$, for every $x \in A$. Endowed with this partial order the set L^A of all fuzzy subsets of A forms a complete residuated lattice, in which the meet (intersection) $\bigwedge_{i\in I} f_i$ and the join (union) $\bigvee_{i\in I} f_i$ of an arbitrary family $\{f_i\}_{i\in I}$ of fuzzy subsets of A are mappings from A into L defined by

$$\left(\bigwedge_{i\in I} f_i\right)(x) = \bigwedge_{i\in I} f_i(x), \qquad \left(\bigvee_{i\in I} f_i\right)(x) = \bigvee_{i\in I} f_i(x),$$

and the *product* $f \otimes g$ is a fuzzy subset defined by $f \otimes g(x) = f(x) \otimes g(x)$, for every $x \in A$. The *crisp part* of a fuzzy subset f of A is a crisp subset $\hat{f} = \{a \in A \mid f(a) = 1\}$ of A. We will also consider $\hat{f}$ as a mapping $\hat{f} : A \to L$ defined by $\hat{f}(a) = 1$, if $f(a) = 1$, and $\hat{f}(a) = 0$, if $f(a) < 1$.

A *fuzzy relation* on a set A is any mapping from $A \times A$ into L, that is to say, any fuzzy subset of $A \times A$, and the equality, inclusion, joins, meets and ordering of fuzzy relations are defined as for fuzzy sets. The set of all fuzzy relations on A will be denoted by $\mathscr{R}(A)$.

For fuzzy relations $P, Q \in \mathscr{R}(A)$, their *composition* $P \circ Q$ is a fuzzy relation on A defined by

$$(P \circ Q)(a,b) = \bigvee_{c\in A} P(a,c) \otimes Q(c,b), \tag{7}$$

for all $a, b \in A$, and for a fuzzy subset f of A and a fuzzy relation $P \in \mathscr{R}(A)$, the *compositions* $f \circ P$ and $P \circ f$ are fuzzy subsets of A defined by

$$(f \circ P)(a) = \bigvee_{b\in A} f(b) \otimes P(b,a), \quad (P \circ f)(a) = \bigvee_{b\in A} P(a,b) \otimes f(b), \tag{8}$$

for any $a \in A$. Finally, for fuzzy subsets f and g of A we write

$$f \circ g = \bigvee_{a\in A} f(a) \otimes g(a). \tag{9}$$

The value $f \circ g$ can be interpreted as the "degree of overlapping" of f and g.

For any $P, Q, R \in \mathscr{R}(A)$ and any $\{P_i\}_{i\in I}, \{Q_i\}_{i\in I} \subseteq \mathscr{R}(A)$, the following hold:

$$(P \circ Q) \circ R = P \circ (Q \circ R), \tag{10}$$

$$P \leqslant Q \text{ implies } P \circ R \leqslant Q \circ R \text{ and } R \circ P \leqslant R \circ Q, \tag{11}$$

$$P \circ \left(\bigvee_{i\in I} Q_i\right) = \bigvee_{i\in I}(P \circ Q_i), \ \left(\bigvee_{i\in I} P_i\right) \circ Q = \bigvee_{i\in I}(P_i \circ Q) \tag{12}$$

$$P \circ \left(\bigwedge_{i\in I} Q_i\right) \leqslant \bigwedge_{i\in I}(P \circ Q_i), \ \left(\bigwedge_{i\in I} P_i\right) \circ Q \leqslant \bigwedge_{i\in I}(P_i \circ Q). \tag{13}$$

We can also easily verify that

$$(f \circ P) \circ Q = f \circ (P \circ Q), \quad (f \circ P) \circ g = f \circ (P \circ g), \tag{14}$$

for arbitrary fuzzy subsets f and g of A, and fuzzy relations P and Q on A, and hence, the parentheses in (10) can be omitted. For $n \in \mathbb{N}$, an n-th power of a fuzzy relation R on A is a fuzzy relation R^n on A defined inductively by $R^1 = R$ and $R^{n+1} = R^n \circ R$. We also define R^0 to be the equality relation on A.

Note also that if A is a finite set with n elements, then P and Q can be treated as $n \times n$ fuzzy matrices over $\mathscr{L}$ and $P \circ Q$ is the matrix product, whereas $f \circ P$ can be treated as the product of a $1 \times n$ matrix f and an $n \times n$ matrix P, and $P \circ f$ as the product of an $n \times n$ matrix P and an $n \times 1$ matrix f^t (the transpose of f).

A fuzzy relation R on A is said to be.

(R) *reflexive* (or *fuzzy reflexive*) if $R(a,a) = 1$, for every $a \in A$;
(S) *symmetric* (or *fuzzy symmetric*) if $R(a,b) = R(b,a)$, for all $a, b \in A$;
(T) *transitive* (or *fuzzy transitive*) if for all $a, b, c \in A$ we have

$$R(a,b) \otimes R(b,c) \leqslant R(a,c).$$

For a fuzzy relation R on a set A, a fuzzy relation R^∞ on A defined by

$$R^\infty = \bigvee_{n\in\mathbb{N}} R^n$$

is the least transitive fuzzy relation on A containing R, and it is called the *transitive closure* of R.

A fuzzy relation on A which is reflexive, symmetric and transitive is called a *fuzzy equivalence*. With respect to the ordering of fuzzy relations, the set $\mathcal{E}(A)$ of all fuzzy equivalences on A is a complete lattice, in which the meet coincide with the ordinary intersection of fuzzy relations, but in the general case, the join in $\mathcal{E}(A)$ does not coincide with the ordinary union of fuzzy relations.

For a fuzzy equivalence E on A and $a \in A$ we define a fuzzy subset E_a of A by $E_a(x) = E(a,x)$, for every $x \in A$. We call E_a an *equivalence class* of E determined by a. The set $A/E = \{E_a \mid a \in A\}$ is called the *factor set* of A w.r.t. E (cf. [3,22]). For an equivalence π on A, the related factor set will be denoted by A/π and the equivalence class of an element $a \in A$ by π_a. A fuzzy equivalence E on a set A is called a *fuzzy equality* if for all $x, y \in A$, $E(x,y) = 1$ implies $x = y$. In other words, E is a fuzzy equality if and only if its crisp part $\widehat{E}$ is a crisp equality.

A fuzzy relation on a set A which is reflexive and transitive is called a *fuzzy quasi-order*, and a reflexive and transitive crisp relation on A is called a *quasi-order*. In some sources quasi-orders and fuzzy quasi-orders are called preorders and fuzzy preorders (for example, see [17,18,39,40]). Note that a reflexive fuzzy relation R is a fuzzy quasi-order if and only if $R^2 = R$. With respect to the ordering of fuzzy relations, the set $\mathcal{Q}(A)$ of all fuzzy quasi-orders on A is a complete lattice, in which the meet coincide with the ordinary intersection of fuzzy relations. Nevertheless, in the general case, the join in $\mathcal{Q}(A)$ does not coincide with the ordinary union of fuzzy relations. Namely, if R is the join in $\mathcal{Q}(A)$ of a family $\{R_i\}_{i\in I}$ of fuzzy quasi-orders on A, then R can be represented by

$$R = \left(\bigvee_{i\in I} R_i\right)^{\infty} = \bigvee_{n\in\mathbb{N}} \left(\bigvee_{i\in I} R_i\right)^{n}. \tag{15}$$

If R is a fuzzy quasi-order on a set A, then a fuzzy relation E_R defined by $E_R = R \wedge R^{-1}$ is a fuzzy equivalence on A, and is called a *natural fuzzy equivalence* of R. A fuzzy quasi-order R on a set A is a *fuzzy order* if for all $a, b \in A$, $R(a,b) = R(b,a) = 1$ implies $a = b$, i.e., if the natural fuzzy equivalence E_R of R is a fuzzy equality. Clearly, a fuzzy quasi-order R is a fuzzy order if and only if its crisp part $\widehat{R}$ is a crisp order.

It is worth noting that different concepts of a fuzzy order have been discussed in literature concerning fuzzy relations (for example, see [5–8] and other sources cited there). In particular, fuzzy orders defined here differ from fuzzy orderings defined in [5,6,8].

For more information about fuzzy sets and fuzzy logic we refer to the books [3,4,45], as well as to recent papers [87,88], which review fuzzy logic and uncertainty in a much broader perspective.

2.2. Fuzzy automata and languages

In the rest of the paper, if not noted otherwise, let $\mathscr{L}$ be a complete residuated lattice.

A *fuzzy transition system over* $\mathscr{L}$, or simply a *fuzzy transition system*, is defined as a triple $\mathscr{A} = (A, X, \delta^A)$, where A and X are the *set of states* and the *input alphabet*, and $\delta^A : A \times X \times A \to L$ is a fuzzy subset of $A \times X \times A$, called the *fuzzy transition function*. We can interpret $\delta^A(a,x,b)$ as the degree to which an input letter $x \in X$ causes a transition from a state $a \in A$ into a state $b \in A$. The input alphabet X will be always finite, but for methodological reasons we will allow the set of states A to be infinite. A fuzzy transition system whose set of states is finite is called a *fuzzy finite transition system*. Cardinality of a fuzzy transition system $\mathscr{A} = (A, X, \delta^A)$, denoted as $|\mathscr{A}|$, is defined as the cardinality of its set of states A.

Let X^* denote the free monoid over the alphabet X, and let $e \in X^*$ be the empty word. The mapping δ^A can be extended up to a mapping $\delta^A_* : A \times X^* \times A \to L$ as follows: If $a, b \in A$, then

$$\delta^A_*(a,e,b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}, \tag{16}$$

and if $a, b \in A$, $u \in X^*$ and $x \in X$, then

$$\delta^A_*(a,ux,b) = \bigvee_{c\in A} \delta^A_*(a,u,c) \otimes \delta^A(c,x,b). \tag{17}$$

By (4) and Theorem 3.1 [47] (see also [70,71,74]), we have that

$$\delta^A_*(a,uv,b) = \bigvee_{c\in A} \delta^A_*(a,u,c) \otimes \delta^A_*(c,v,b), \tag{18}$$

for all $a, b \in A$ and $u, v \in X^*$, i.e., if $w = x_1 \cdots x_n$, for $x_1, \ldots, x_n \in X$, then

$$\delta^A_*(a,w,b) = \bigvee_{(c_1,\ldots,c_{n-1})\in A^{n-1}} \delta^A(a,x_1,c_1) \otimes \delta^A(c_1,x_2,c_2) \otimes \cdots \otimes \delta^A(c_{n-1},x_n,b). \tag{19}$$

Intuitively, the product $\delta^A(a,x_1,c_1) \otimes \delta^A(c_1,x_2,c_2) \otimes \cdots \otimes \delta^A(c_{n-1},x_n,b)$ represents the degree to which the input word w causes a transition from a state a into a state b through the sequence of intermediate states $c_1, \ldots, c_{n-1} \in A$, and $\delta^A_*(a,w,b)$ represents the supremum of degrees of all possible transitions from a into b caused by w.

For any $u \in X^*$, and any $a, b \in A$ define a fuzzy relation δ^A_u on A by

$$\delta^A_u(a,b) = \delta^A_*(a,u,b), \tag{20}$$

called the *fuzzy transition relation* determined by u. Then (18) can be written as

$$\delta^A_{uv} = \delta^A_u \circ \delta^A_v, \tag{21}$$

for all $u, v \in X^*$.

An *initial fuzzy transition system* is defined as a quadruple $\mathscr{A} = (A, X, \delta^A, \sigma^A)$, where (A, X, δ^A) is a fuzzy transition system and $\sigma^A \in L^A$ is the fuzzy set of *initial states*, and a *fuzzy automaton* is defined as a five-tuple $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$, where $(A, X, \delta^A, \sigma^A)$ is as above, and $\tau^A \in L^A$ is the fuzzy set of *terminal states*. We also say that $\mathscr{A}$ is a *fuzzy automaton belonging to the fuzzy transition system* (A, X, δ^A).

A *fuzzy language* in X^* over $\mathscr{L}$, or briefly a *fuzzy language*, is any fuzzy subset of X^*, i.e., any mapping from X^* into L. A *fuzzy language recognized by a fuzzy automaton* $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$, denoted as $[\![\mathscr{A}]\!]$, is a fuzzy language in L^{X^*} defined by

$$[\![\mathscr{A}]\!](u) = \bigvee_{a,b\in A} \sigma^A(a) \otimes \delta^A_*(a, u, b) \otimes \tau^A(b), \tag{22}$$

or equivalently,

$$\begin{aligned} [\![\mathscr{A}]\!](e) &= \sigma^A \circ \tau^A, \\ [\![\mathscr{A}]\!](u) &= \sigma^A \circ \delta^A_{x_1} \circ \delta^A_{x_2} \circ \cdots \circ \delta^A_{x_n} \circ \tau^A, \end{aligned} \tag{23}$$

for any $u = x_1x_2\ldots x_n \in X^+$, where $x_1, x_2, \ldots, x_n \in X$. In other words, the equality (22) means that the membership degree of the word u to the fuzzy language $[\![\mathscr{A}]\!]$ is equal to the degree to which $\mathscr{A}$ recognizes or accepts the word u.

The *reverse fuzzy transition system* of a fuzzy transition system $\mathscr{A} = (A, X, \delta^A)$ is a fuzzy transition system $\overline{\mathscr{A}} = (A, X, \bar{\delta}^A)$ with the fuzzy transition function defined by $\bar{\delta}^A(a, x, b) = \delta^A(b, x, a)$, for all $a, b \in A$ and $x \in X$. A *reverse fuzzy automaton* of a fuzzy automaton $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ is $\overline{\mathscr{A}} = (A, X, \bar{\delta}^A, \bar{\sigma}^A, \bar{\tau}^A)$, a fuzzy automaton with the fuzzy transition function $\bar{\delta}^A$ defined as above, and fuzzy sets of initial and terminal states defined by $\bar{\sigma}^A = \tau^A$ and $\bar{\tau}^A = \sigma^A$.

Fuzzy transition systems $\mathscr{A} = (A, X, \delta^A)$ and $\mathscr{A}' = (A', X, \delta^{A'})$ are *isomorphic* if there exists a bijective mapping $\phi : A \to A'$ such that $\delta^A(a, x, b) = \delta^{A'}(\phi(a), x, \phi(b))$, for all $a, b \in A$ and $x \in X$. It is easy to check that in this case we also have that $\delta^A_*(a, u, b) = \delta^{A'}_*(\phi(a), u, \phi(b))$, for all $a, b \in A$ and $u \in X^*$. Similarly, fuzzy automata $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ and $\mathscr{A}' = (A', X, \delta^{A'}, \sigma^{A'}, \tau^{A'})$ are *isomorphic* if there is a bijective mapping $\phi : A \to A'$ such that $\delta^A(a, x, b) = \delta^{A'}(\phi(a), x, \phi(b))$, for all $a, b \in A$ and $x \in X$, and also, $\sigma^A(a) = \sigma^{A'}(\phi(a))$ and $\tau^A(a) = \tau^{A'}(\phi(a))$, for every $a \in A$.

If $\mathscr{A} = (A, X, \delta^A)$ is a fuzzy transition system such that δ^A is a crisp relation, then $\mathscr{A}$ is an ordinary crisp *non-deterministic transition system*, while if δ^A is a mapping of $A \times X$ into A, then $\mathscr{A}$ is an ordinary *deterministic transition system*. Evidently, in these two cases we have that δ^A_* is also a crisp subset of $A \times X^* \times A$, and a mapping of $A \times X^*$ into A, respectively. In other words, non-deterministic transition systems are fuzzy transition systems over the Boolean structure. If $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ such that δ^A is a crisp relation and σ^A and τ^A are crisp subsets of A, then $\mathscr{A}$ is called a *non-deterministic automaton*.

For undefined notions and notation one can refer to [3,4,62].

3. Afterset and foreset fuzzy automata

Let R be a fuzzy quasi-order on a set A. For each $a \in A$, the *R-afterset* of a is the fuzzy set $R_a \in L^A$ defined by $R_a(b) = R(a, b)$, for any $b \in A$, while the *R-foreset* of a is the fuzzy set $R^a \in L^A$ defined by $R^a(b) = R(b, a)$, for any $b \in A$ (see [1,26,27]). The set of all R-aftersets will be denoted by A/R, and the set of all R-foresets will be denoted by $A \setminus R$. Clearly, if R is a fuzzy equivalence, then $A/R = A \setminus R$ is the set of all equivalence classes of R.

If f is an arbitrary fuzzy subset of A, then fuzzy relations R_f and R^f on A defined by

$$R_f(a, b) = f(a) \to f(b), \; R^f(a, b) = f(b) \to f(a), \tag{24}$$

for all $a, b \in A$, are fuzzy quasi-orders on A. In particular, if f is a normalized fuzzy subset of A, then it is an afterset of R_f and a foreset of R^f.

Theorem 3.1. *Let R be a fuzzy quasi-order on a set A and E the natural fuzzy equivalence of R. Then*

(a) *For arbitrary $a, b \in A$ the following conditions are equivalent:*
  (i) $E(a, b) = 1$;
  (ii) $E_a = E_b$;
  (iii) $R_a = R_b$;
  (iv) $R^a = R^b$.
(b) *Functions $R_a \mapsto E_a$ of A/R to A/E, and $R_a \mapsto R^a$ of A/R to $A \setminus R$, are bijective functions.*

Proof. (a) Consider arbitrary $a,b\in A$.

(i) ⇒ (ii). Let $E(a,b)=1$, that is $R(a,b)=R(b,a)=1$. Then for every $c\in A$ we have that

$$R_b(c)=R(b,c)=R(a,b)\otimes R(b,c)\leqslant R(a,c)=R_a(c),$$

whence $R_b\leqslant R_a$. Analogously we prove that $R_a\leqslant R_b$, and therefore, $R_a=R_b$.

(ii) ⇒ (i). Let $R_a=R_b$. Then

$$R(a,b)=R_a(b)\geqslant R_b(b)=R(b,b)=1,$$

which yields $R(a,b)=1$. Analogously we prove that $R(b,a)=1$, and hence, $E(a,b)=1$.

Equivalence (i) ⇔ (iii) can be proved similarly as (i) ⇔ (ii).

The assertion (b) follows immediately by (a). □

Let us consider the Gödel structure and a fuzzy quasi-order R on a set A given by

$$R=\begin{bmatrix}1 & 0.3 & 0.3\\ 0 & 1 & 0.2\\ 0 & 1 & 1\end{bmatrix}.$$

The natural fuzzy equivalence E_R of R is calculated by $E_R(a,b)=R(a,b)\wedge R^{-1}(a,b)=R(a,b)\wedge R(b,a)$, i.e.

$$E_R=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0.2\\ 0 & 0.2 & 1\end{bmatrix}.$$

If A is a finite set with n elements and a fuzzy quasi-order R on A is treated as an $n\times n$ fuzzy matrix over $\mathscr{L}$, then R-aftersets are row vectors, whereas R-foresets are column vectors of this matrix. The previous theorem says that ith and jth row vectors of this matrix are equal if and only if its ith and jth column vectors are equal, and vice versa. Moreover, we have that R is a fuzzy order if and only if all its row vectors are different, or equivalently, if and only if all its column vectors are different.

Let $\mathscr{A}=(A,X,\delta^A)$ be a fuzzy transition system and let R be a fuzzy quasi-order on A. We can define the fuzzy transition function $\delta^{A/R}:A/R\times X\times A/R\to L$ by

$$\delta^{A/R}(R_a,x,R_b)=\bigvee_{a',b'\in A}R(a,a')\otimes\delta^A(a',x,b')\otimes R(b',b), \tag{25}$$

or equivalently

$$\delta^{A/R}(R_a,x,R_b)=(R\circ\delta^A_x\circ R)(a,b)=R_a\circ\delta^A_x\circ R^b, \tag{26}$$

for all $a,b\in A$ and $x\in X$. According to the statement (a) of Theorem 3.1, $\delta^{A/R}$ is well-defined, and we have that $\mathscr{A}/R=(A/R,X,\delta^{A/R})$ is a fuzzy transition system, called the *afterset fuzzy transition system* of $\mathscr{A}$ w.r.t. R.

In addition, if $\mathscr{A}=(A,X,\delta^A,\sigma^A,\tau^A)$ is a fuzzy automaton, then we define the fuzzy transition function $\delta^{A/R}$ as in (25), and we also define a fuzzy set $\sigma^{A/R}\in L^{A/R}$ of initial states and a fuzzy set $\tau^{A/R}\in L^{A/R}$ of terminal states by

$$\sigma^{A/R}(R_a)=\bigvee_{a'\in A}\sigma^A(a')\otimes R(a',a)=(\sigma^A\circ R)(a)=\sigma^A\circ R^a, \tag{27}$$

$$\tau^{A/R}(R_a)=\bigvee_{a'\in A}R(a,a')\otimes\tau^A(a')=(R\circ\tau^A)(a)=R_a\circ\tau^A, \tag{28}$$

for any $a\in A$. According to (a) of Theorem 3.1, $\sigma^{A/R}$ and $\tau^{A/R}$ are well-defined functions, and we have that $\mathscr{A}/R=(A/R,X,\delta^{A/R},\sigma^{A/R},\tau^{A/R})$ is a fuzzy automaton, which is called the *afterset fuzzy automaton* of $\mathscr{A}$ w.r.t. R.

Analogously, for a fuzzy transition system $\mathscr{A}=(A,X,\delta^A)$, the *foreset fuzzy transition system* of $\mathscr{A}$ w.r.t. R is a fuzzy transition system $\mathscr{A}\setminus R=(A\setminus R,X,\delta^{A\setminus R})$ with the fuzzy transition function $\delta^{A\setminus R}$ defined by

$$\delta^{A\setminus R}(R^a,x,R^b)=\bigvee_{a',b'\in A}R(a,a')\otimes\delta^A(a',x,b')\otimes R(b',b)=\left(R\circ\delta^A_x\circ R\right)(a,b)=R_a\circ\delta^A_x\circ R^b, \tag{29}$$

for all $a,b\in A$ and $x\in X$. In addition, for a fuzzy automaton $\mathscr{A}=(A,X,\delta^A,\sigma^A,\tau^A)$, the *foreset fuzzy automaton* of $\mathscr{A}$ w.r.t. R is a fuzzy automaton $\mathscr{A}\setminus R=(A\setminus R,X,\delta^{A\setminus R},\sigma^{A\setminus R},\tau^{A\setminus R})$ with a fuzzy set $\sigma^{A\setminus R}\in L^{A\setminus R}$ of initial states and a fuzzy set $\tau^{A\setminus R}\in L^{A\setminus R}$ of terminal states by

$$\sigma^{A\backslash R}(R^a) = \bigvee_{a'\in A} \sigma^A(a') \otimes R(a',a) = (\sigma^A \circ R)(a) = \sigma^A \circ R^a, \tag{30}$$

$$\tau^{A\backslash R}(R^a) = \bigvee_{a'\in A} R(a,a') \otimes \tau^A(a') = (R \circ \tau^A)(a) = R_a \circ \tau^A, \tag{31}$$

for any $a \in A$.

We can easily prove the following:

Theorem 3.2. *For any fuzzy quasi-order R on a fuzzy automaton (transition system) $\mathscr{A}$ the afterset fuzzy automaton (transition system) $\mathscr{A}/R$ and the foreset fuzzy automaton (transition system) $\mathscr{A}\setminus R$ are isomorphic.*

Proof. This follows immediately by (25) and (29) and (b) of Theorem 3.1. □

In view of Theorem 3.2, in the remainder of this paper we will consider only afterset fuzzy automata and transition systems. We will see in Example 4.8 that the factor fuzzy automaton (transition system) $\mathscr{A}/E_R$ of $\mathscr{A}$, w.r.t. the natural fuzzy equivalence E_R of R, is not necessarily isomorphic to fuzzy automata $\mathscr{A}/R$ and $\mathscr{A}\setminus R$, but by (b) of Theorem 3.1, it has the same cardinality as $\mathscr{A}/R$ and $\mathscr{A}\setminus R$, and if $[\![\mathscr{A}]\!] = [\![\mathscr{A}/R]\!] (= [\![\mathscr{A}\setminus R]\!])$, then we also have that $[\![\mathscr{A}]\!] = [\![\mathscr{A}/E_R]\!]$.

If $\mathscr{A} = (A, X, \delta^A)$ is a fuzzy transition system and R is a fuzzy quasi-order on A, then we also define a new fuzzy transition function $\delta^{A|R} : A \times X \times A \to L$ by

$$\delta^{A|R}(a,x,b) = (R \circ \delta^A_x \circ R)(a,b), \text{ for all } a,b \in A \text{ and } x \in X,$$

i.e., $\delta^{A|R}_x = R \circ \delta^A_x \circ R$, for each $x \in X$, and we obtain a new fuzzy transition system $\mathscr{A}|R = (A, X, \delta^{A|R})$ with the same set of states and input alphabet as the original one. Furthermore, if $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ is a fuzzy automaton, then we also set $\sigma^{A|R} = \sigma^A$ and $\tau^{A|R} = \tau^A$, and we have that $\mathscr{A}|R = (A, X, \delta^{A|R}, \sigma^{A|R}, \tau^{A|R})$ is a fuzzy automaton.

The following theorem can be conceived as a version of the well-known Second Isomorphism Theorem, concerning fuzzy transition systems and fuzzy quasi-orders on them. (cf. [9], §2.6).

Theorem 3.3. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton and let R and S be fuzzy quasi-orders on $\mathscr{A}$ such that $R \leqslant S$. Then a fuzzy relation S/R on A/R defined by*

$$S/R(R_a, R_b) = S(a,b), \qquad \text{for all } a,b \in A, \tag{32}$$

is a fuzzy quasi-order on A/R and fuzzy automata $\mathscr{A}/S$, $(\mathscr{A}/R)/(S/R)$ and $(\mathscr{A}|R)/S$ are isomorphic.

Proof. Let $a, a', b, b' \in A$ such that $R_a = R_{a'}$ and $R_b = R_{b'}$, i.e., $E_R(a,a') = E_R(b,b') = 1$. Since $R \leqslant S$, we also have that $R^{-1} \leqslant S^{-1}$, whence $E_R \leqslant E_S$, and by this it follows that $E_S(a,a') = E_S(b,b') = 1$, so $S(a,b) = S(a',b')$. Therefore, S/R is a well-defined fuzzy relation, and clearly, S/R is a fuzzy quasi-order.

For the sake of simplicity set $S/R = Q$. Define a mapping $\phi : A/S \to (A/R)/Q$ by

$$\phi(S_a) = Q_{R_a}, \qquad \text{for every } a \in A.$$

According to Theorem 3.1, for arbitrary $a, b \in A$ we have that

$$S_a = S_b \iff S(a,b) = S(b,a) = 1 \iff Q(R_a,R_b) = Q(R_b,R_a) = 1 \iff Q_{R_a} = Q_{R_b} \iff \phi(S_a) = \phi(S_b),$$

and hence, ϕ is a well-defined and injective function. It is clear that ϕ is also a surjective function. Thus, ϕ is a bijective function of A/S onto $(A/R)/Q$.

Since $R \leqslant S$ implies $R \circ S = S \circ R = S$, for arbitrary $a, b \in A$ and $x \in X$ we have that

$$\delta_x^{(A/R)/Q}(\phi(S_a),\phi(S_b)) = \delta_x^{(A/R)/Q}(Q_{R_a},Q_{R_b}) = \left(Q \circ \delta_x^{A/R} \circ Q\right)(R_a,R_b) = \bigvee_{c,d\in A} Q(R_a,R_c) \otimes \delta_x^{A/R}(R_c,R_d) \otimes Q(R_d,R_b)$$
$$= \bigvee_{c,d\in A} S(a,c) \otimes \left(R \circ \delta_x^A \circ R\right)(c,d) \otimes S(d,b) = \left(S \circ R \circ \delta_x^A \circ R \circ S\right)(a,b) = \left(S \circ \delta_x^A \circ S\right)(a,b) = \delta_x^{A/S}(S_a,S_b).$$

Moreover, for any $a \in A$ we have that

$$\sigma^{(A/R)/Q}(\phi(S_a)) = \sigma^{(A/R)/Q}(Q_{R_a}) = \sigma^{A/R}(R_a) = \sigma^A(a) = \sigma^{A/S}(S_a),$$

and similarly, $\tau^{(A/R)/Q}(\phi(S_a)) = \tau^{A/S}(S_a)$. Therefore, ϕ is an isomorphism of the fuzzy automaton $\mathscr{A}/S$ onto the fuzzy automaton $(\mathscr{A}/R)/(S/R)$.

Next, for all $a, b \in A$ and $x \in X$ we have that

$$\delta^{(A|R)/S}(S_a,x,S_b) = \left(S \circ \delta_x^{A|R} \circ S\right)(a,b) = \left(S \circ R \circ \delta_x^A \circ R \circ S\right)(a,b) = \left(S \circ \delta_x^A \circ S\right)(a,b) = \delta^{A/S}(S_a,x,S_b),$$

and $\sigma^{(A|R)/S} = \sigma^{A/S}$, $\tau^{(A|R)/S} = \tau^{A/S}$, so fuzzy automata $(\mathscr{A}|R)/S$ and $\mathscr{A}/S$ are isomorphic. □

If in the proof of the previous theorem we disregard fuzzy sets of initial and terminal states, we see that the theorem also hold for fuzzy transition systems.

Remark 3.4. For any given fuzzy quasi-order R on a fuzzy automaton $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$, the rule $a \mapsto R_a$ defines a surjective function of A onto A/R. This means that the afterset fuzzy automaton $\mathscr{A}/R$ has smaller or equal cardinality than the fuzzy automaton $\mathscr{A}$.

Now, if R and S are fuzzy quasi-orders on $\mathscr{A}$ such that $R \leqslant S$, according to Theorem 3.3, the afterset fuzzy automaton $\mathscr{A}/S$ has smaller or equal cardinality than $\mathscr{A}/R$. This fact will be frequently used in the rest of the paper.

Let us note that if $\mathscr{A}$ is a fuzzy automaton or a fuzzy transition system, A is its set of states, and R, S and T are fuzzy quasi-orders on A such that $R \leqslant S$ and $R \leqslant T$, then

$$S \leqslant T \iff S/R \leqslant T/R, \tag{33}$$

and hence, a mapping $\Phi : \mathscr{Q}_R(A) = \{S \in \mathscr{Q}(A) | R \leqslant S\} \to \mathscr{Q}(A/R)$, given by $\Phi : S \mapsto S/R$, is injective (in fact, it is an order isomorphism of $\mathscr{Q}_R(A)$ onto a subset of $\mathscr{Q}(A/R)$). In particular, for a fuzzy quasi-order R on A, the fuzzy relation R/R on A/R will be denoted by $\widetilde{R}$. It can be easily verified that $\widetilde{R}$ is a fuzzy order on A/R, and if E is a fuzzy equivalence on A, then $\widetilde{E}$ is a fuzzy equality on A/E.

For a fuzzy automaton $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ and a fuzzy quasi-order R on A we have that the fuzzy language $[\![\mathscr{A}/R]\!]$ recognized by the afterset fuzzy automaton $\mathscr{A}/R$ is given by

$$\begin{aligned} &[\![\mathscr{A}/R]\!](e) = \sigma^A \circ R \circ \tau^A, \\ &[\![\mathscr{A}/R]\!](u) = \sigma^A \circ R \circ \delta^A_{x_1} \circ R \circ \delta^A_{x_2} \circ R \circ \cdots \circ R \circ \delta^A_{x_n} \circ R \circ \tau^A, \end{aligned} \tag{34}$$

whereas the fuzzy language $[\![\mathscr{A}]\!]$ recognized by $\mathscr{A}$ is given by

$$\begin{aligned} &[\![\mathscr{A}]\!](e) = \sigma^A \circ \tau^A, \\ &[\![\mathscr{A}]\!](u) = \sigma^A \circ \delta^A_{x_1} \circ \delta^A_{x_2} \circ \cdots \circ \delta^A_{x_n} \circ \tau^A, \end{aligned}$$

for any $u = x_1 x_2 \ldots x_n \in X^+$, where $x_1, x_2, \ldots, x_n \in X$. Let us note that Eq. (34) follows immediately by definition of the afterset fuzzy automaton $\mathscr{A}/R$ (Eqs. (26)–(28)), by Eqs. (10) and (14), and the fact that $R \circ R = R$, for every fuzzy quasi-order R. Hence, the fuzzy automaton $\mathscr{A}$ and the afterset fuzzy automaton $\mathscr{A}/R$ are *equivalent*, i.e., they recognize the same fuzzy language, if and only if the fuzzy quasi-order R is a solution to a system of fuzzy relation equations

$$\begin{aligned} &\sigma^A \circ \tau^A = \sigma^A \circ R \circ \tau^A, \\ &\sigma^A \circ \delta^A_{x_1} \circ \delta^A_{x_2} \circ \cdots \circ \delta^A_{x_n} \circ \tau^A = \sigma^A \circ R \circ \delta^A_{x_1} \circ R \circ \delta^A_{x_2} \circ R \circ \cdots \circ R \circ \delta^A_{x_n} \circ R \circ \tau^A, \end{aligned} \tag{35}$$

for all $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in X$. We will call (35) the *general system*.

The general system has at least one solution in $\mathscr{Q}(A)$, the equality relation on A. It will be called the *trivial solution*. To attain the best possible reduction of $\mathscr{A}$, we have to find the greatest solution to the general system in $\mathscr{Q}(A)$, if it exists, or to find as big a solution as possible. However, the general system does not necessarily have the greatest solution (see Example 3.2), and also, it may consist of infinitely many equations, and finding its nontrivial solutions may be a very difficult task. For that reason we will aim our attention to some instances of the general system. These instances have to be as general as possible, but they have to be easier to solve. From a practical point of view, these instances have to consist of finitely many equations.

The following theorem describes some properties of the set of all solutions to the general system.

Theorem 3.5. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton.*

The set of all solutions to the general system in $\mathscr{Q}(A)$ is an order ideal of the lattice $\mathscr{Q}(A)$.

Consequently, if a fuzzy quasi-order R on A is a solution to the general system, then its natural fuzzy equivalence E_R is also a solution to the general system.

Proof. Consider arbitrary $n \in \mathbb{N}$, $x_1, x_2, \ldots, x_n \in X$, and fuzzy quasi-orders R and S on A such that S is a solution to the general system and $R \leqslant S$. By the facts that S is a solution to the general system and $R \leqslant S$, by reflexivity of R, and by (11) we obtain that

$$\begin{aligned} \sigma^A \circ \delta^A_{x_1} \circ \delta^A_{x_2} \circ \cdots \circ \delta^A_{x_n} \circ \tau^A &\leqslant \sigma^A \circ R \circ \delta^A_{x_1} \circ R \circ \delta^A_{x_2} \circ R \circ \cdots \circ R \circ \delta^A_{x_n} \circ R \circ \tau^A \leqslant \sigma^A \circ S \circ \delta^A_{x_1} \circ S \circ \delta^A_{x_2} \circ S \circ \cdots \circ S \circ \delta^A_{x_n} \circ S \circ \tau^A \\ &= \sigma^A \circ \delta^A_{x_1} \circ \delta^A_{x_2} \circ \cdots \circ \delta^A_{x_n} \circ \tau^A, \end{aligned}$$

and hence, R is a solution to the general system. By this it follows that solutions to the general system in $\mathscr{Q}(A)$ form an order ideal of the lattice $\mathscr{Q}(A)$.

The second part of the theorem follows immediately by the fact that $E_R = R \wedge R^{-1} \leqslant R$. $\square$

The following example shows that there are fuzzy quasi-orders which are not solutions to the general system, but their natural fuzzy equivalences are solutions to this system.

Example 3.6. Let $\mathscr{L}$ be the Boolean structure, let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton over $\mathscr{L}$, where $A = \{1,2,3\}$, $X = \{x, y\}$, and δ_x^A, δ_y^A, σ^A and τ^A are given by

$$\delta_x^A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \delta_y^A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \sigma^A = [\,1 \quad 1 \quad 1\,], \quad \tau^A = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

and consider a fuzzy quasi-order R on A given by

$$R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then we have that

$$\sigma^A \circ R \circ \delta_x^A \circ R \circ \delta_y^A \circ R \circ \tau^A = 1 \neq 0 = \sigma^A \circ \delta_x^A \circ \delta_y^A \circ \tau^A,$$

so R is not a solution to the general system, but its natural fuzzy equivalence E_R is the equality relation on A, and hence, it is a solution to the general system.

The next example shows that the general system does not necessarily have the greatest solution.

Example 3.7. Let $\mathscr{L}$ be the Boolean structure, let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton over $\mathscr{L}$, where $A = \{1,2,3\}$, $X = \{x\}$, and δ_x^A, σ^A and τ^A are given by

$$\delta_x^A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \sigma^A = [\,1 \quad 1 \quad 1\,], \quad \tau^A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and consider fuzzy quasi-orders (in fact, fuzzy equivalences) E and F on A given by

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

We have that both E and F are solutions to the general system (since E is right invariant and F is left invariant, see the next section for details). On the other hand, the join of E and F in the lattice $\mathscr{Q}(A)$ is a fuzzy quasi-order U given by

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

and it is not a solution to the general system, since

$$\sigma^A \circ U \circ \delta_x^A \circ U \circ \delta_x^A \circ U \circ \tau^A = 1 \neq 0 = \sigma^A \circ \delta_x^A \circ \delta_x^A \circ \tau^A.$$

If the general system would have the greatest solution R in $\mathscr{Q}(A)$, then $E \leqslant R$ and $F \leqslant R$ would imply $U \leqslant R$, and by Theorem 3.5 we would obtain that U is a solution to the general system. Hence, we conclude that the general system does not have the greatest solution in $\mathscr{Q}(A)$.

The next theorem demonstrates one shortcoming of state reductions by means of fuzzy quasi-orders and fuzzy equivalences. Namely, we show that for some fuzzy automata no reduction will result in its minimal automaton.

Theorem 3.8. *There exists a fuzzy automaton $\mathscr{A}$ such that no reduction of $\mathscr{A}$ by means of fuzzy quasi-orders provide a minimal fuzzy automaton.*

Proof. Let $\mathscr{L}$ be the Boolean structure and $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ a fuzzy automaton over $\mathscr{L}$, where $|A| = 4$, $X = \{x\}$, and δ^A, σ^A, and τ^A are given by

$$\delta_x^A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \sigma^A = [\,0 \quad 1 \quad 0 \quad 0\,], \qquad \tau^A = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

It is easy to check that for each $u \in X^*$ the following is true:

$$[\![\mathscr{A}]\!](u) = \begin{cases} 0 & \text{if } u = e \text{ or } u = x^n, \text{for } n \geqslant 2, \\ 1 & \text{if } u = x, \end{cases}$$

(in fact, $\mathscr{A}$ is a nondeterministic automaton and $[\![\mathscr{A}]\!]$ is an ordinary crisp language consisting only of the letter x). If $\mathscr{B} = (B, X, \delta^B, \sigma^B, \tau^B)$ is a fuzzy automaton over $\mathscr{L}$ with $|B| = 2$, and

$$\delta_x^B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \sigma^B = [\,1 \quad 0\,], \qquad \tau^B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

then $\mathscr{B}$ recognizes $[\![\mathscr{A}]\!]$, and it is a minimal fuzzy automaton of $[\![\mathscr{A}]\!]$, since $[\![\mathscr{A}]\!]$ can not be recognized by a fuzzy automaton with only one state.

Consider now an arbitrary fuzzy equivalence

$$E = \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ a_{12} & 1 & a_{23} & a_{24} \\ a_{13} & a_{23} & 1 & a_{34} \\ a_{14} & a_{24} & a_{34} & 1 \end{bmatrix}$$

on A, and suppose that E is a solution to the general system corresponding to the fuzzy automaton $\mathscr{A}$. We will show that E can not reduce $\mathscr{A}$ to a fuzzy automaton with two states.

First, by $\sigma^A \circ E \circ \tau^A = a_{23} \vee a_{24}$ and $\sigma^A \circ E \circ \tau^A = \sigma^A \circ \tau^A = [\![\mathscr{A}]\!](e) = 0$ it follows $a_{23} = a_{24} = 0$. Next, reflexivity and transitivity of E yield $E \circ E = E$, what implies

$$a_{12} \wedge a_{13} = 0, \qquad a_{12} = 0 \text{ or } a_{13} = 0 \tag{36}$$
$$a_{12} \wedge a_{14} = 0, \qquad a_{12} = 0 \text{ or } a_{14} = 0 \tag{37}$$
$$a_{13} \vee (a_{14} \wedge a_{34}) = a_{13}, \quad \text{i.e.,} \quad a_{13} = 0 \text{ implies } a_{14} = 0 \text{ or } a_{34} = 0, \tag{38}$$
$$a_{14} \vee (a_{13} \wedge a_{34}) = a_{14}, \qquad a_{14} = 0 \text{ implies } a_{13} = 0 \text{ or } a_{34} = 0, \tag{39}$$
$$a_{34} \vee (a_{13} \wedge a_{14}) = a_{34}, \qquad a_{34} = 0 \text{ implies } a_{13} = 0 \text{ or } a_{14} = 0. \tag{40}$$

If $a_{12} = 1$, then by (36) and (37) we obtain $a_{13} = a_{14} = 0$, and hence

$$E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{or} \qquad E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

However, none of these two matrices is a solution to the general system. Therefore, we conclude that $a_{12} = 0$. According to (38)–(40), we distinguish the following five cases

$$\begin{aligned} & a_{13} = a_{14} = a_{34} = 0, \\ & a_{13} = a_{14} = 0, \quad a_{34} = 1, \\ & a_{13} = a_{34} = 0, \quad a_{14} = 1, \\ & a_{14} = a_{34} = 0, \quad a_{13} = 1, \\ & a_{13} = a_{14} = a_{34} = 1, \end{aligned}$$

and we obtain that E has one of the following forms

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}. \tag{41}$$

In the first case, E is the equality relation, and it does not provide any reduction of $\mathscr{A}$, and in the second and fourth case, it can be easily verified that E is a solution to the general system, but it reduces $\mathscr{A}$ to a fuzzy automaton with three states. Finally, in the third and fifth case, E is not a solution to the general system, since

$$\sigma^A \circ E \circ \delta_x^A \circ E \circ \delta_x^A \circ E \circ \tau_x^A = 1 \neq 0 = \sigma^A \circ \delta_x^A \circ \delta_x^A \circ \tau_x^A.$$

Therefore, any state reduction of $\mathscr{A}$ by means of fuzzy equivalences does not provide fuzzy automaton with less than three states. According to (b) of Theorem 3.1, the same conclusion also holds for fuzzy quasi-orders. This completes the proof of the theorem. $\square$

4. Right and left invariant fuzzy quasi-orders

As in [24,25], where similar questions concerning fuzzy equivalences have been considered, here we study the following two instances of the general system. Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy transition system. If a fuzzy quasi-order R on A is a solution to system

$$R \circ \delta_x^A \circ R = \delta_x^A \circ R, \text{ for every } x \in X, \tag{42}$$

then it will be called a *right invariant fuzzy quasi-order* on $\mathscr{A}$, and if it is a solution to system

$$R \circ \delta_x^A \circ R = R \circ \delta_x^A, \text{ for every } x \in X, \tag{43}$$

then it will be called a *left invariant fuzzy quasi-order* on $\mathscr{A}$. A crisp quasi-order on A which is a solution to (42) is called a *right invariant quasi-order* on $\mathscr{A}$, and a crisp quasi-order which is a solution to (43) is called a *left invariant quasi-order* on $\mathscr{A}$. Let us note that a fuzzy quasi-order on A is both right and left invariant if and only if it is a solution to system

$$R \circ \delta_x^A = \delta_x^A \circ R, \text{ for every } x \in X, \tag{44}$$

and then it is called an *invariant fuzzy quasi-order*.

If $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ is a fuzzy automaton, then by a *right invariant fuzzy quasi-order* on $\mathscr{A}$ we mean a fuzzy quasi-order R on A which is a solution to (42) and

$$R \circ \tau^A = \tau^A, \tag{45}$$

and a *left invariant fuzzy quasi-order* on $\mathscr{A}$ is a fuzzy quasi-order R on A which is a solution to (43) and

$$\sigma^A \circ R = \sigma^A. \tag{46}$$

It is clear that all right and left invariant fuzzy quasi-orders on a fuzzy automaton $\mathscr{A}$ are solutions of the general system (35), and hence, the corresponding afterset fuzzy transition systems are equivalent to $\mathscr{A}$.

In other words, right (resp. left) invariant fuzzy quasi-orders on the fuzzy automaton $\mathscr{A}$ are exactly those right (resp. left) invariant fuzzy quasi-orders on the fuzzy transition system (A, X, δ^A) which are solutions to the fuzzy relation Eq. (45) (resp. (46)). It is well-known (see [23,66–68,78]) that solutions to (45) (resp. (46)) in $\mathscr{Q}(A)$ form a principal ideal of $\mathscr{Q}(A)$ whose greatest element is a fuzzy quasi-order R^τ(resp. R_σ) defined by (24) (here we write $\tau^A = \tau$ and $\sigma^A = \sigma$). This means that right (resp. left) invariant fuzzy quasi-orders on the fuzzy automaton $\mathscr{A}$ are those right (resp. left) invariant fuzzy quasi-orders on the fuzzy transition system (A, X, δ^A) which are contained in R^τ(resp. R_σ).

Let us note that fuzzy equivalences satisfying (42) and (43) have been studied in [24,25]. They are respectively called right and left invariant fuzzy equivalences. Right and left invariant quasi-orders have been used for the state reduction of nondeterministic transition systems by Champarnaud and Coulon [17,18], Ilie et al. [39], and Ilie et al. [40] (see also [36,37]).

By the following theorem we give a characterization of right invariant fuzzy quasi-orders:

Theorem 4.1. *Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy transition system and R a fuzzy quasi-order on A. Then the following conditions are equivalent:*

(i) *R is a right invariant fuzzy quasi-order;*
(ii) $R \circ \delta_x^A \leqslant \delta_x^A \circ R$, for every $x \in X$;
(iii) *for all $a, b \in A$ we have*

$$R(a,b) \leqslant \bigwedge_{x\in X} \bigwedge_{c\in A} \left(\delta_x^A \circ R\right)(b,c) \to \left(\delta_x^A \circ R\right)(a,c). \tag{47}$$

Proof. (i) $\Leftrightarrow$ (ii). Consider an arbitrary $x \in X$. If $R \circ \delta_x^A \circ R = \delta_x^A \circ R$, then by reflexivity of R it follows

$$R \circ \delta_x^A \leqslant R \circ \delta_x^A \circ R = \delta_x^A \circ R.$$

Conversely, if $R \circ \delta_x^A \leqslant \delta_x^A \circ R$ then $R \circ \delta_x^A \circ R \leqslant \delta_x^A \circ R \circ R = \delta_x^A \circ R$, and since the opposite inequality follows by reflexivity of R, we conclude that $R \circ \delta_x^A \circ R = \delta_x^A \circ R$.

(i) $\Rightarrow$ (iii). Let R be a right invariant fuzzy equivalence. Then for all $x \in X$ and $a, b, c \in A$ we have that

$$R(a,b) \otimes \left(\delta_x^A \circ R\right)(b,c) \leqslant \left(R \circ \delta_x^A \circ R\right)(a,c) = \left(\delta_x^A \circ R\right)(a,c),$$

and by the adjunction property we obtain that $R(a,b) \leqslant \left(\delta_x^A \circ R\right)(b,c) \to \left(\delta_x^A \circ R\right)(a,c)$. Hence,

$$R(a,b) \leqslant \left(\delta_x^A \circ R\right)(b,c) \to \left(\delta_x^A \circ R\right)(a,c). \tag{48}$$

Since (48) is satisfied for all $c \in A$ and $x \in X$, we conclude that (47) holds.

(iii) $\Rightarrow$ (i). If (iii) holds, then for arbitrary $x \in X$ and $a, b, c \in A$ we have that

$$R(a,b) \leqslant \left(\delta_x^A \circ R\right)(b,c) \to \left(\delta_x^A \circ R\right)(a,c),$$

and by the adjunction property we obtain that $R(a,b) \otimes \left(\delta_x^A \circ R\right)(b,c) \leqslant \left(\delta_x^A \circ R\right)(a,c)$. Now,

$$\left(R \circ \delta_x^A \circ R\right)(a,c) = \bigvee_{b \in A} R(a,b) \otimes \left(\delta_x^A \circ R\right)(b,c) \leqslant \left(\delta_x^A \circ R\right)(a,c),$$

whence $R \circ \delta_x^A \circ R \leqslant \delta_x^A \circ R$, and since the opposite inequality follows immediately by reflexivity of R, we conclude that $R \circ \delta_x^A \circ R = \delta_x^A \circ R$, for every $x \in X$, i.e., R is a right invariant fuzzy quasi-order. □

Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy transition system and R a fuzzy quasi-order on A. Let us define a fuzzy relation R^r on A by

$$R^r(a,b) = \bigwedge_{x \in X} \bigwedge_{c \in A} (\delta_x^A \circ R)(b,c) \to (\delta_x^A \circ R)(a,c), \tag{49}$$

for all $a, b \in A$. Since R^r is an intersection of a family of fuzzy quasi-orders defined as in (24), we have that R^r is also a fuzzy quasi-order. According to Theorem 4.1, R is a right invariant fuzzy quasi-order if and only if $R \leqslant R^r$.

Moreover, we have the following:

Lemma 4.2. *Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy transition system, and let R and S be fuzzy quasi-orders on A.*

If $R \leqslant S$, then $R^r \leqslant S^r$.s

Proof. Consider arbitrary $a, b \in A$ and $x \in X$. By $R \leqslant S$ it follows $R \circ S = S$, and by (3), for arbitrary $c, d \in A$ we have that

$$\left(\delta_x^A \circ R\right)(b,c) \to \left(\delta_x^A \circ R\right)(a,c) \leqslant (\delta_x \circ R)(b,c) \otimes S(c,d) \to \left(\delta_x^A \circ R\right)(a,c) \otimes S(c,d).$$

Now, by (6) we obtain that

$$\begin{aligned} R^r(a,b) &\leqslant \bigwedge_{c \in A} \left(\delta_x^A \circ R\right)(b,c) \to \left(\delta_x^A \circ R\right)(a,c) \leqslant \bigwedge_{c \in A} \left[\left(\delta_x^A \circ R\right)(b,c) \otimes S(c,d) \to \left(\delta_x^A \circ R\right)(b,c) \otimes S(c,d)\right] \\ &\leqslant \left[\bigvee_{c \in A} \left(\delta_x^A \circ R\right)(b,c) \otimes S(c,d)\right] \to \left[\bigvee_{c \in A} \left(\delta_x^A \circ R\right)(a,c) \otimes S(c,d)\right] = \left(\delta_x^A \circ R \circ S\right)(b,d) \to \left(\delta_x^A \circ R \circ S\right)(a,d) \\ &= \left(\delta_x^A \circ S\right)(b,d) \to \left(\delta_x^A \circ S\right)(a,d). \end{aligned}$$

Since this holds for all $x \in X$ and $d \in A$, we conclude that

$$R^r(a,b) \leqslant \bigwedge_{x \in X} \bigwedge_{d \in A} \left(\delta_x^A \circ S\right)(b,d) \to \left(\delta_x^A \circ S\right)(a,d) = S^r(a,b),$$

and hence, $R^r \leqslant S^r$. □

Now we prove the following:

Theorem 4.3. *For any fuzzy transition system or fuzzy automaton $\mathscr{A}$ and any fuzzy quasi-order R on the set of states of $\mathscr{A}$ there exists the greatest right invariant fuzzy quasi-order on $\mathscr{A}$ contained in R.*

Proof. First we note that the family $\{R_i\}_{i \in I}$ of all right invariant fuzzy quasi-orders on $\mathscr{A}$ contained in R is non-empty, since it contains the equality relation on $\mathscr{A}$. Let Q be the join of this family in the lattice $\mathscr{Q}(A)$ of all fuzzy quasi-orders on $\mathscr{A}$. Then Q is a fuzzy quasi-order contained in R, and we will show that it is also right invariant.

For each $i \in I$ we have that $R_i \leqslant Q$, and by Lemma 4.2 it follows that $R_i \leqslant R_i^r \leqslant Q^r$, whence $Q \leqslant Q^r$. Now, according to Theorem 4.1, we obtain that Q is a right invariant fuzzy quasi-order, and thus, it is the greatest right invariant fuzzy quasi-order on $\mathscr{A}$ contained in R. □

The previous theorem proves the existence of the greatest right invariant fuzzy quasi-order contained in a given fuzzy quasi-order, but it does not provide a way to construct it. In the sequel we discuss the procedures for computing the greatest right invariant fuzzy quasi-order contained in a given fuzzy quasi-order.

Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy transition system, let R be a fuzzy quasi-order on $\mathscr{A}$. By $\mathscr{L}(\mathscr{A}, R)$ we will denote the subalgebra of $\mathscr{L}$ generated by the set $\delta^A(A \times X \times A) \cup R(A \times A)$.

As we have noted before, the problem of computing the greatest right invariant fuzzy quasi-order on a fuzzy automaton $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ one reduces to the problem of computing the greatest right invariant fuzzy quasi-order on a fuzzy transition system (A, X, δ^A) contained in the fuzzy quasi-order $R^\tau(\tau = \tau^A)$. For that reason, in the sequel we consider the problem how to construct the greatest right invariant fuzzy quasi-order R^{ri} contained in a given fuzzy quasi-order R on a fuzzy transition system.

Theorem 4.4. *Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy transition system, let R be a fuzzy quasi-order on $\mathscr{A}$ and let R^{ri} be the greatest right invariant fuzzy quasi-order on $\mathscr{A}$ contained in R.*

Define inductively a sequence $\{R_k\}_{k\in\mathbb{N}}$ of fuzzy quasi-orders on $\mathscr{A}$ as follows:

$$R_1 = R, \; R_{k+1} = R_k \wedge R_k^r, \text{ for each } k \in \mathbb{N}. \tag{50}$$

Then

(a) $R^{\mathrm{ri}} \leqslant \cdots \leqslant R_{k+1} \leqslant R_k \leqslant \cdots \leqslant R_1 = R$;
(b) *If $R_s = R_{s+t}$, for some $s, t \in \mathbb{N}$, then $R_s = R_{s+1} = R^{\mathrm{ri}}$;*
(c) *If $\mathscr{A}$ is finite and $\mathscr{L}(\mathscr{A}, R)$ satisfies DCC, then $R_s = R^{\mathrm{ri}}$ for some $s \in \mathbb{N}$.*

Proof.

(a) Clearly, $R_{k+1} \leqslant R_k$, for each $k \in \mathbb{N}$, and $R^{\mathrm{ri}} \leqslant R_1$. Suppose that $R^{\mathrm{ri}} \leqslant R_k$, for some $k \in \mathbb{N}$. Then $R^{\mathrm{ri}} \leqslant (R^{\mathrm{ri}})^r \leqslant R_k^r$, so $R^{\mathrm{ri}} \leqslant R_k \wedge R_k^r = R_{k+1}$. Therefore, by induction we obtain that $R^{\mathrm{ri}} \leqslant R_k$, for every $k \in \mathbb{N}$.
(b) Let $R_s = R_{s+t}$, for some $s, t \in \mathbb{N}$. Then $R_s = R_{s+t} \leqslant R_{s+1} = R_s \wedge R_s^r \leqslant R_m^r$, what means that R_s is a right invariant fuzzy quasi-order. Since R^{ri} is the greatest right invariant fuzzy quasi-order contained in R, we conclude that $R_s = R_{s+1} = R^{\mathrm{ri}}$.
(c) Let $\mathscr{A}$ be a finite fuzzy transition system and let $\mathscr{L}(\mathscr{A}, R)$ satisfy DCC. Then fuzzy relations $\{R_k\}_{k\in\mathbb{N}}$ can be considered as fuzzy matrices with entries in $\mathscr{L}(\mathscr{A}, R)$, and for any pair $(a, b) \in A \times A$, the (a, b)-entries of these matrices form a decreasing sequence $\{R_k(a, b)\}_{k\in\mathbb{N}}$ of elements of $\mathscr{L}(\mathscr{A}, R)$. By the hypothesis, all these sequences stabilize, and since there is a finite number of these sequences, there exists $s \in \mathbb{N}$ such that after s steps all these sequences are stabilized. This means that the sequence $\{R_k\}_{k\in\mathbb{N}}$ of fuzzy quasi-orders also stabilizes after s steps, i.e., $R_s = R_{s+1} = R^{\mathrm{ri}}$. □

The previous theorem can be transformed into the following algorithm.

Algorithm 4.5. *Computation of the greatest right invariant fuzzy quasi-order*

The input of this algorithm is either a fuzzy finite transition system $\mathscr{A} = (A, X, \delta^A)$ and a fuzzy quasi-order R on $\mathscr{A}$, or a fuzzy finite automaton $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$, and in this case $R = R^\tau$, where $\tau = \tau^A$. The algorithm computes the greatest right invariant fuzzy quasi-order R^{ri} on $\mathscr{A}$ contained in R.

The procedure constructs the sequence of fuzzy quasi-orders $\{R_k\}_{k\in\mathbb{N}}$, in the following way:

(A1) In the first step we set $R_1 = R$.
(A2) After the kth step let R_k be the fuzzy quasi-order that has been constructed.
(A3) In the next step we construct the fuzzy quasi-order R_{k+1} by means of the formula (50).
(A4) Simultaneously, we check whether $R_{k+1} = R_k$.
(A5) When we find the smallest number s such that $R_{s+1} = R_s$, the procedure of constructing the sequence $\{R_k\}_{k\in\mathbb{N}}$ terminates and R_s is the greatest right invariant fuzzy quasi-order on $\mathscr{A}$ contained in R.

According to Theorem 4.4, if the subalgebra $\mathscr{L}(\mathscr{A}, R)$ of $\mathscr{L}$, generated by all values taken by fuzzy transition relations of $\mathscr{A}$ and R, satisfies DCC, the algorithm terminates in a finite number of steps.

Consider the computational time of this algorithm. Let n denote the number of states of $\mathscr{A}$ and m the number of letters in the input alphabet X, and let $c_\vee$, $c_\wedge$, $c_\otimes$ and $c_\to$ be respectively computational costs of the operations $\vee$, $\wedge$, $\otimes$ and $\to$ in $\mathscr{L}$. In particular, if $\mathscr{L}$ is linearly ordered, we can assume that $c_\vee = c_\wedge = 1$, and when $\mathscr{L}$ is the Gödel structure, we can also assume that $c_\otimes = c_\to = 1$.

When $\mathscr{A} = (A, X, \delta^A)$ is a fuzzy transition system and the fuzzy quasi-order R is given, in the step (A1) we have no anything to compute, but in the case when $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ is a fuzzy automaton, then in (A1) we compute R^τ, which can be done in time $O(n^2 c_\to)$.

In (A3) we first compute the composition of fuzzy relations δ_x^A and R, and if this computation is performed according to the definition of composition od fuzzy relations, its computational time is $O(n^3(c_\otimes + c_\vee))$. Then we compute R_k^r and R_{k+1} by means of (49) and (50), and the computational time of this part is $O(mn^3(c_\to + c_\wedge))$. Thus, the total computational time of (A3) is $O(n^3(m(c_\to + c_\wedge) + c_\otimes + c_\vee))$.

In (A4) computational time to check whether $R_{k+1} = R_k$ is $O(n^2)$.

The hardest problem is to estimate the number of steps, in the case when it is finite. Consider fuzzy relations R_k as fuzzy matrices. After each step in the construction of the sequence $\{R_k\}_{k\in\mathbb{N}}$ we check whether some entry has changed its value, and the algorithm terminates after the first step in which there was no change. Suppose that $\mathscr{L}(\mathscr{A}, R)$ satisfies DCC. Then $\{\{R_k(a, b)\}_{k\in\mathbb{N}} | (a, b) \in A^2\}$ is a finite collection of finite sequences, so there exists $l \in \mathbb{N}$ such that the number of different elements in each of these sequences is less than or equal to l. As the sequence $\{R_k\}_{k\in\mathbb{N}}$ is descending, each entry can change its value at most $l - 1$ times, and the total number of changes is less than or equal to $(l - 1)(n^2 - n)$ (the diagonal values must always be 1). Therefore, the algorithm terminates after at most $(l-1)(n^2-n)+2$ steps (in the first and last step values do not change).

Summing up, we get that the total computation time for the whole algorithm is $O(ln^5(m(c_{\to}+c_{\wedge})+c_{\otimes}+c_{\vee}))$, and hence, the algorithm is polynomial-time.

Let us note that the number l is characteristic of the sequence $\{R_k\}_{k\in\mathbb{N}}$, and in general it is not characteristic of the algebra $\mathscr{L}(\mathscr{A},R)$. However, in some cases the number of different elements in all descending chains in $\mathscr{L}(\mathscr{A},R)$ may have an upper bound l. For example, if the algebra $\mathscr{L}(\mathscr{A},R)$ is finite, then we can assume that l is the number of elements of this algebra. In particular, if $\mathscr{L}$ is the Gödel structure, then the only values that can be taken by fuzzy relations $\{R_k\}_{k\in\mathbb{N}}$ are 1 and those taken by fuzzy relations $\{\delta_x\}_{x\in X}$ and R. In this case, if j is the number of all values taken by fuzzy relations $\{\delta_x\}_{x\in X}$ and R, then the algorithm terminates after at most $j(n^2-n)+2$ steps, and total computation time is $O(jmn^5)$. Since $j\leqslant mn^2+n^2$, total computation time can also be roughly expressed as $O(m^2n^7)$.

According to (c) of Theorem 4.4, if the structure $\mathscr{L}$ is locally finite, then for every fuzzy transition system $\mathscr{A}$ over $\mathscr{L}$ we have that every sequence of fuzzy quasi-orders defined by (50) is finite. However, this does not necessarily hold if $\mathscr{L}$ is not locally finite, as the following example shows:

Example 4.6. Let $\mathscr{L}$ be the Goguen (product) structure and $\mathscr{A}=(A,X,\delta^A)$ a fuzzy transition system over $\mathscr{L}$, where $A=\{1,2\},\ X=\{x\}$, and δ_x^A is given by

$$\delta_x^A=\begin{bmatrix}0.2 & 0\\ 0 & 0.1\end{bmatrix},$$

and let R be the universal relation on A. Applying to R the procedure from Theorem 4.4, we obtain a sequence $\{R_k\}_{k\in\mathbb{N}}$ of fuzzy quasi-orders given by

$$R_k=\begin{bmatrix}1 & 1\\ \frac{1}{2^{k-1}} & 1\end{bmatrix},\ k\in\mathbb{N},$$

whose all members are different, i.e., this sequence is infinite. We also have that the greatest right invariant fuzzy quasi-order contained in R is given by

$$R^{\mathrm{ri}}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}.$$

For a fuzzy transition system $\mathscr{A}=(A,X,\delta^A)$ over a complete residuated lattice $\mathscr{L}$, the greatest left invariant fuzzy quasi-order R^{li} contained in a given fuzzy quasi-order R on A can be computed in a similar way as R^{ri}. Indeed, inductively we define a sequence $\{R_k\}_{k\in\mathbb{N}}$ of fuzzy quasi-orders on A by

$$R_1=R,\ R_{k+1}=R_k\wedge R_k^l,\ \text{for each } k\in\mathbb{N}, \tag{51}$$

where R_k^l is a fuzzy quasi-order on A defined by

$$R_k^l(a,b)=\bigwedge_{x\in X}\bigwedge_{c\in A}(R_k\circ\delta_x^A)(c,a)\to(R_k\circ\delta_x^A)(c,b),\qquad \text{for all } a,b\in A.$$

If $\mathscr{L}$ is locally finite, then this sequence is necessarily finite and R^{li} equals the least element of this sequence.

It is worth noting that the greatest right and left invariant fuzzy quasi-orders are calculated using iterative procedures, but these calculations are not approximative. Whenever these procedures terminate in a finite number of steps, exact solutions to the considered systems of fuzzy relation equations are obtained.

Note also that for a fuzzy transition system $\mathscr{A}=(A,X,\delta^A)$ over a complete residuated lattice $\mathscr{L}$, in [24,25] we gave a procedure for computing the greatest right invariant fuzzy equivalence E^{rie} contained in a given fuzzy equivalence E on A. This procedure is similar to the procedure given in Theorem 4.4 for fuzzy quasi-orders, and it also works for all fuzzy finite transition systems over a locally finite complete residuated lattice. Namely, inductively we define a sequence $\{E_k\}_{k\in\mathbb{N}}$ of fuzzy equivalences on A by

$$E_1=E,\ E_{k+1}=E_k\wedge E_k^{req},\ \text{for each } k\in\mathbb{N}, \tag{52}$$

where E_k^{req} is a fuzzy equivalence defined by

$$E_k^{req}(a,b)=\bigwedge_{x\in X}\bigwedge_{c\in A}\left(\delta_x^A\circ E_k\right)(a,c)\leftrightarrow\left(\delta_x^A\circ E_k\right)(b,c),\qquad \text{for all } a,b\in A.$$

It was proved in [24,25] that if $\mathscr{L}$ is locally finite, then this sequence is necessarily finite and E^{rie} equals the least element of this sequence.

By the next example we show that it is possible that the sequence of fuzzy equivalences defined by (52) is infinite, but the sequence of fuzzy quasi-orders defined by (50) is finite.

Example 4.7. Let $\mathscr{L}$ be the Goguen (product) structure and $\mathscr{A} = (A, X, \delta^A)$ a fuzzy transition system over $\mathscr{L}$, where $A = \{1, 2, 3\}$, $X = \{x\}$, and δ_x^A is given by

$$\delta_x^A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

If we start from the universal relation on A, applying the rule (52) we obtain an infinite sequence $\{E_k\}_{k\in\mathbb{N}}$ of fuzzy equivalences on A, where

$$E_k = \begin{bmatrix} 1 & 1 & \frac{1}{2^{k-1}} \\ 1 & 1 & \frac{1}{2^{k-1}} \\ \frac{1}{2^{k-1}} & \frac{1}{2^{k-1}} & 1 \end{bmatrix}, \quad k \in \mathbb{N}.$$

On the other hand, if we also start from the universal relation, the rule (50) gives a finite sequence $\{R_k\}_{k\in\mathbb{N}}$ of fuzzy quasi-orders on A, where

$$R_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}, \quad R_k = R_2, \text{ for each } k \in \mathbb{N},\ k \geqslant 3.$$

Reduction of fuzzy transition systems by means of right and left invariant fuzzy equivalences has been studied in [24,25]. Since the set of all right invariant fuzzy equivalences is a subset of the set of all right invariant fuzzy quasi-orders, the greatest element of this subset (the greatest right invariant fuzzy equivalence) is less or equal than the greatest element of the whole set (the greatest right invariant fuzzy quasi-order). The next example shows that this inequality can be strict. Thus, reduction of a fuzzy transition system by using the greatest right invariant fuzzy quasi-order gives better results than its reduction by using the greatest right invariant fuzzy equivalence, according to Remark 3.4.

Furthermore, as we have shown by Theorem 3.5, if a fuzzy quasi-order R on a fuzzy transition system $\mathscr{A}$ is a solution to the general system, then its natural fuzzy equivalence E_R is also a solution to the general system. But, the next example also shows that if R is a right invariant fuzzy quasi-order, then E_R is not necessarily a right invariant fuzzy equivalence.

Example 4.8. Let $\mathscr{L}$ be the Boolean structure, and let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy transition system over $\mathscr{L}$, where $A = \{1, 2, 3\}$, $X = \{x, y\}$, and fuzzy transition relations δ_x^A and δ_y^A are given by

$$\delta_x^A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \delta_y^A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The greatest right invariant fuzzy quasi order R^{ri} on $\mathscr{A}$, its natural fuzzy equivalence $E_{R^{\mathrm{ri}}}$, and the greatest right invariant fuzzy equivalence E^{ri} on $\mathscr{A}$ are given by

$$R^{\mathrm{ri}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad E_{R^{\mathrm{ri}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad E^{\mathrm{ri}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, E^{ri} do not reduce the number of states of $\mathscr{A}$, but R^{ri} reduces $\mathscr{A}$ to a fuzzy transition system with two states.

Moreover, R^{ri} is a right invariant fuzzy quasi-order, but its natural fuzzy equivalence $E_{R^{\mathrm{ri}}}$ is not a right invariant fuzzy equivalence, because $E^{\mathrm{ri}} < E_{R^{\mathrm{ri}}}$. We also have that the afterset fuzzy transition system $\mathscr{A}/R^{\mathrm{ri}}$ is not isomorphic to the factor fuzzy transition system $\mathscr{A}/E_{R^{\mathrm{ri}}}$, since

$$R^{\mathrm{ri}} \circ \delta_y^A \circ R^{\mathrm{ri}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad E_{R^{\mathrm{ri}}} \circ \delta_y^A \circ E_{R^{\mathrm{ri}}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Next we consider the case when $\mathscr{L}$ is a complete residuated lattice satisfying the following conditions:

$$x \vee \left(\bigwedge_{i\in I} y_i \right) = \bigwedge_{i\in I} (x \vee y_i), \tag{53}$$

$$x \otimes \left(\bigwedge_{i\in I} y_i \right) = \bigwedge_{i\in I} (x \otimes y_i), \tag{54}$$

for all $x \in L$ and $\{y_i\}_{i\in I} \subseteq L$. Let us note that if $\mathscr{L} = ([0,1], \wedge, \vee, \otimes, \to, 0, 1)$, where $[0,1]$ is the real unit interval and $\otimes$ is a left-continuous t-norm on $[0,1]$, then (53) follows immediately by linearity of $\mathscr{L}$, and $\mathscr{L}$ satisfies (54) if and only if $\otimes$ is a continuous t-norm, i.e., if and only if $\mathscr{L}$ is a BL-algebra (cf. [3,4]). Therefore, conditions (53) and (54) hold for every BL-algebra on the real unit interval. In particular, the Łukasiewicz, Goguen (product) and Gödel structures fulfill (53) and (54).

We have that the following is true:

Theorem 4.9. *Let $\mathscr{L}$ be a complete residuated lattice satisfying (53) and (54), let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy finite transition system over $\mathscr{L}$, let R be a fuzzy quasi-order on A, let R^{ri} be the greatest right invariant fuzzy quasi-order on $\mathscr{A}$ contained in R, and let $\{R_k\}_{k\in\mathbb{N}}$ be the sequence of fuzzy quasi-orders on A defined by (50). Then*

$$R^{\mathrm{ri}} = \bigwedge_{k\in\mathbb{N}} R_k. \tag{55}$$

Proof. It was proved in [25] that if (53) holds, then for all non-increasing sequences $\{x_k\}_{k\in\mathbb{N}}, \{y_k\}_{k\in\mathbb{N}} \subseteq L$ we have

$$\bigwedge_{k\in\mathbb{N}} (x_k \vee y_k) = \left(\bigwedge_{k\in\mathbb{N}} x_k\right) \vee \left(\bigwedge_{k\in\mathbb{N}} y_k\right). \tag{56}$$

For the sake of simplicity set

$$S = \bigwedge_{k\in\mathbb{N}} R_k.$$

Clearly, S is a fuzzy quasi-order. To prove (55) it is enough to prove that S is a right invariant fuzzy quasi-order on $\mathscr{A}$. First, we have that

$$S(a,b) \leqslant R_{k+1}(a,b) \leqslant R_k^{\mathrm{r}}(a,b) \leqslant \left(\delta_x^A \circ R_k\right)(b,c) \to \left(\delta_x^A \circ R_k\right)(a,c), \tag{57}$$

holds for all $a,b,c \in A$, $x \in X$ and $k \in \mathbb{N}$. Now, by (57) and (5) we obtain that

$$S(a,b) \leqslant \bigwedge_{k\in\mathbb{N}} \left(\left(\delta_x^A \circ R_k\right)(b,c) \to \left(\delta_x^A \circ R_k\right)(a,c)\right) \leqslant \bigwedge_{k\in\mathbb{N}} \left(\delta_x^A \circ R_k\right)(b,c) \to \bigwedge_{k\in\mathbb{N}} \left(\delta_x^A \circ R_k\right)(a,c), \tag{58}$$

for all $a,b,c \in A$ and $x \in X$. Next,

$$\begin{aligned} \bigwedge_{k\in\mathbb{N}} (\delta_x^A \circ R_k)(b,c) = & \bigwedge_{k\in\mathbb{N}} \left(\bigvee_{d\in A} \delta_x^A(b,d) \otimes R_k(d,c)\right) \\ = & \bigvee_{d\in A} \left(\bigwedge_{k\in\mathbb{N}} \delta_x^A(b,d) \otimes R_k(d,c)\right) \qquad \text{(by (56))} \\ = & \bigvee_{d\in A} \left(\delta_x^A(b,d) \otimes \left(\bigwedge_{k\in\mathbb{N}} R_k(d,c)\right)\right) \qquad \text{(by (54))} \\ = & \bigvee_{d\in A} \delta_x^A(b,d) \otimes S(d,c) = \left(\delta_x^A \circ S\right)(b,c). \end{aligned} \tag{59}$$

Use of condition (56) is justified by the facts that A is finite, and that $\{R_k(d,c)\}_{k\in\mathbb{N}}$ is a non-increasing sequence, so $\{\delta_x^A(b,d) \otimes R_k(d,c)\}_{k\in\mathbb{N}}$ is also a non-increasing sequence. In the same way we prove that

$$\bigwedge_{k\in\mathbb{N}} \left(\delta_x^A \circ R_k\right)(a,c) = \left(\delta_x^A \circ S\right)(a,c). \tag{60}$$

Therefore, by (58)–(60) we obtain that

$$S(a,b) \leqslant \left(\delta_x^A \circ S\right)(b,c) \to \left(\delta_x^A \circ S\right)(a,c).$$

Since this inequality holds for all $x \in X$ and $c \in A$, we have that

$$S(a,b) \leqslant \bigwedge_{x\in X} \bigwedge_{c\in A} \left(\delta_x^A \circ S\right)(b,c) \to \left(\delta_x^A \circ S\right)(a,c),$$

and by (iii) of Theorem 4.1 we obtain that S is a right invariant fuzzy quasi-order on $\mathscr{A}$. □

5. Some special cases, generalizations, and alternate reductions

For a given fuzzy quasi-order R on a fuzzy transition system $\mathscr{A}$, Theorem 4.4 gives a procedure for computing R^{ri} in case when the subalgebra $\mathscr{L}(\mathscr{A}, R)$ of $\mathscr{L}$ satisfies DCC, and Theorem 4.9 characterizes R^{ri} in case when $\mathscr{L}$ satisfies some additional distributivity conditions. But, what to do if $\mathscr{L}$ do not satisfy any of these conditions? In that case we could consider some subset of $\mathcal{Q}^{\rm ri}(\mathscr{A})$ whose greatest element can be effectively computed when $\mathscr{L}$ is any complete residuated lattice. Here we consider two such subsets. The first one is the set $\mathcal{Q}^{\rm cri}(\mathscr{A})$ of all right invariant crisp quasi-orders on $\mathscr{A}$, and the second one is the set $\mathcal{Q}^{\rm sri}(\mathscr{A})$ of strongly right invariant fuzzy quasi-orders, which will be defined latter.

5.1. Right and left invariant crisp quasi-orders

Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy finite transition system and R a fuzzy quasi-order on A. As we have seen, the main problem concerning the computation of the greatest right invariant fuzzy quasi-order on $\mathscr{A}$ contained in R is that the descending sequence $\{R_k\}_{k\in\mathbb{N}}$ of fuzzy quasi-orders on A defined by the rule (50) may be infinite. In that case, instead of $R^{\rm ri}$ we could compute the greatest right invariant crisp quasi-order $\varrho^{\rm ri}$ on $\mathscr{A}$, which could be done replacing the sequence $\{R_k\}_{k\in\mathbb{N}}$ by an appropriate sequence of crisp quasi-orders. Namely, it is not hard to prove the following theorem which gives a procedure for computing the greatest right invariant crisp quasi-order on a fuzzy transition system, contained in a given quasi-order.

Theorem 5.1. *Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy transition system, let ϱ be a quasi-order on $\mathscr{A}$ and let $\varrho^{\rm ri}$ be the greatest right invariant quasi-order on $\mathscr{A}$ contained in ϱ.*

Define inductively a sequence $\{\varrho_k\}_{k\in\mathbb{N}}$ of quasi-orders on $\mathscr{A}$ as follows:

$$\varrho_1 = \varrho, \ \varrho_{k+1} = \varrho_k \cap \widehat{\varrho_k^r}, \ \text{for each } k \in \mathbb{N}.$$

Then

(a) $\varrho^{\rm ri} \subseteq \cdots \subseteq \varrho_{k+1} \subseteq \varrho_k \subseteq \cdots \subseteq \varrho_1 = \varrho$;
(b) *If $\varrho_k = \varrho_{k+m}$, for some $k, m \in \mathbb{N}$, then $\varrho_k = \varrho_{k+1} = \varrho^{\rm ri}$;*
(c) *If $\mathscr{A}$ is finite, then $\varrho_k = \varrho^{\rm ri}$ for some $k \in \mathbb{N}$.*

It is worth mentioning that for any quasi-order ϱ on A we can compute the quasi-order $\widehat{\varrho^r}$ according to the following rule:

$$(a,b) \in \widehat{\varrho^r} \iff (\forall x \in X)(\forall c \in A)\ \left(\delta_x^A \circ \varrho\right)(b,c) \leqslant \left(\delta_x^A \circ \varrho\right)(a,c), \qquad \text{for all } a,b \in A. \tag{61}$$

Using the previous theorem we can easily construct an algorithm which computes $\varrho^{\rm ri}$, analogous to Algorithm 4.5. As well as its computational cost is concerned, the computational cost of any single step is the same as in Algorithm 4.5, except that here we can omit $c_\otimes$, $c_\to$ and $c_\wedge$, because in the computation of $\delta_x^A \circ \varrho_k$ we multiply only with 0 and 1, and the operations $\to$ and $\wedge$ are replaced by inequality in $\mathscr{L}$ (which can be realized through $c_\vee$) and conjunction in the two-element Boolean algebra. On the other hand, the upper bound of the number of steps is smaller ($l = 2$). Therefore, the computational cost of this algorithm is $O(mn^5 c_\vee)$.

Theorem 5.1 shows that the greatest right invariant crisp quasi-order can be effectively computed for any fuzzy finite transition system, and without imposing any additional requirements for the underlying complete residuated lattice (and even for any fuzzy finite transition system over an arbitrary lattice-ordered monoid). However, in cases when we are able to effectively compute the greatest right invariant fuzzy quasi-order, using it we can attain better reduction than using the greatest right invariant crisp quasi-order, as the next example shows. Namely, we have that $\varrho^{\rm ri} \leqslant R^{\rm ri}$, whence $|\mathscr{A}/R^{\rm ri}| \leqslant |\mathscr{A}/\varrho^{\rm ri}|$, according to Remark 3.4, and the example shows that this inequality can be strict.

Example 5.2. Let $\mathscr{L}$ be the Gödel structure, and let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy transition system over $\mathscr{L}$, where $A = \{1,2,3\}$, $X = \{x\}$, and δ_x^A is given by

$$\delta_x^A = \begin{bmatrix} 0 & 0.1 & 0 \\ 0.2 & 0 & 0 \\ 0.1 & 0 & 0 \end{bmatrix}.$$

Then the greatest right invariant fuzzy quasi-order $R^{\rm ri}$ and the greatest right invariant crisp quasi-order $\varrho^{\rm ri}$ on $\mathscr{A}$ are given by

$$R^{\rm ri} = \begin{bmatrix} 1 & 0.1 & 1 \\ 1 & 1 & 1 \\ 1 & 0.1 & 1 \end{bmatrix}, \qquad \varrho^{\rm ri} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, $\varrho^{\rm ri}$ do not reduce the number of states of $\mathscr{A}$, whereas $R^{\rm ri}$ reduces $\mathscr{A}$ to a fuzzy transition system with two states.

5.2. Strongly right and left invariant fuzzy quasi-orders

Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy transition system. If a fuzzy quasi-order R on A is a solution to system

$$R \circ \delta_x^A = \delta_x^A, \ \text{for every } x \in X, \tag{62}$$

then it will be called a *strongly right invariant fuzzy quasi-order* on $\mathscr{A}$, and if it is a solution to system

$$\delta_x^A \circ R = \delta_x^A, \text{ for every } x \in X, \tag{63}$$

then it is a *strongly left invariant fuzzy quasi-order* on $\mathscr{A}$. Clearly, every strongly right (resp. left) invariant fuzzy quasi-order is right (resp. left) invariant. Let us note that a fuzzy quasi-order on A is both strongly right and left invariant if and only if it is a solution to system

$$R \circ \delta_x^A \circ R = \delta_x^A, \text{ for every } x \in X, \tag{64}$$

and then it is called a *strongly invariant fuzzy quasi-order*.

When $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ is a fuzzy automaton, then a *stongly right invariant fuzzy quasi-order* on $\mathscr{A}$ is required to be solution both to (62) and an additional equation

$$R \circ \tau^A = \tau^A, \tag{65}$$

and a *strongly left invariant fuzzy quasi-order* on $\mathscr{A}$ is required to solution both to (63) and an equation

$$\sigma^A \circ R = \sigma^A. \tag{66}$$

It is not hard to see that every strongly right invariant fuzzy quasi-order on a fuzzy transition system or a fuzzy automaton is right invariant, and every strongly left invariant fuzzy quasi-order is left invariant.

If $\mathscr{A} = (A, X, \delta^A)$ is a fuzzy transition system, then we define a fuzzy relation R^{sri} on A as follows

$$R^{\mathrm{sri}}(a,b) = \bigwedge_{x \in X} \bigwedge_{c \in A} \delta_x^A(b,c) \to \delta_x^A(a,c), \text{ for all } a, b \in A. \tag{67}$$

The same definition can also be used for fuzzy automata.

We can easily show that the following statements are true.

Theorem 5.3.

(a) *Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy transition system and R a fuzzy quasi-order on A.*
Then R is strongly right invariant if and only if $R \leqslant R^{\mathrm{sri}}$, and consequently, R^{sri} is the greatest strongly right invariant fuzzy quasi-order on $\mathscr{A}$.
(b) *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton, $\tau = \tau^A$, and R a fuzzy quasi-order on A.*
Then R is strongly right invariant if and only if $R \leqslant R^{\mathrm{sri}} \wedge R^{\tau}$, and consequently, $R^{\mathrm{sri}} \wedge R^{\tau}$ is the greatest strongly right invariant fuzzy quasi-order on $\mathscr{A}$.

An analogous theorem can be stated and proved for strongly left invariant fuzzy quasi-orders.

According to (67), the greatest strongly right invariant fuzzy quasi-order can be effectively computed for any fuzzy finite transition system or fuzzy finite automaton, without imposing any additional requirements for the underlying complete residuated lattice. Clearly, the cost of its computing is $O(mn^3 c_{\to}(c_{\to} + c_{\wedge}))$. However, in cases when we are able to effectively compute the greatest right invariant fuzzy quasi-order, using it we can attain better reduction than using the greatest strongly right invariant quasi-order. Indeed, the following example presents a fuzzy transition system whose number of states can be reduced by means of right invariant fuzzy quasi-orders, but it can not be reduced using strongly right invariant ones.

Example 5.4. Consider again the fuzzy transition system $\mathscr{A}$ from Example 4.8. In this example we proved that the greatest right invariant fuzzy quasi-order R^{ri} on $\mathscr{A}$ reduces $\mathscr{A}$ to a fuzzy transition system with two states. On the other hand, the greatest strongly right invariant fuzzy quasi-order R^{sri} on $\mathscr{A}$ is given by

$$R^{\mathrm{sri}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and it reduces $\mathscr{A}$ to the fuzzy transition system $\mathscr{A}_2 = (A_2, X, \delta^{A_2})$ having also three states, whose fuzzy transition relations $\delta_x^{A_2}$ and $\delta_y^{A_2}$ are given by

$$\delta_x^{A_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \delta_y^{A_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Further, the greatest strongly right invariant fuzzy quasi-order R_2^{sri} on $\mathscr{A}_2$ is given by

$$R_2^{\mathrm{sri}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and the afterset fuzzy transition system $\mathscr{A}_2/R_2^{\mathrm{sri}}$ is isomorphic to $\mathscr{A}_2$. Therefore, the number of states of $\mathscr{A}$ can not be reduced by means of strongly right invariant fuzzy quasi-orders.

5.3. Weakly right and left invariant fuzzy quasi-orders

Here we consider certain generalizations of right and left invariant fuzzy quasi-orders. We start with the following definitions.

Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton. For any $u \in X^*$ we define fuzzy sets $\sigma_u^A, \tau_u^A \in L^A$ by

$$\sigma_u^A(a) = \bigvee_{b \in A} \sigma^A(b) \otimes \delta_*^A(b, u, a), \qquad \tau_u^A(a) = \bigvee_{b \in A} \delta_*^A(a, u, b) \otimes \tau^A(b), \tag{68}$$

for each $a \in A$. In other words, $\sigma_u^A = \sigma^A \circ \delta_u^A$ and $\tau_u^A = \delta_u^A \circ \tau^A$. Evidently, for the empty word $e \in X^*$ we have that $\sigma_e^A = \sigma^A$ and $\tau_e^A = \tau^A$. Note that fuzzy sets $\{\sigma_u^A\}_{u \in X^*}$ are states of the *Nerode automaton* $\mathscr{A}_N$ of $\mathscr{A}$, which plays a key role in determinization of the fuzzy automaton $\mathscr{A}$ (cf. [32]), and fuzzy sets $\{\tau_u^A\}_{u \in X^*}$ are states of the Nerode automaton of the reverse fuzzy automaton of $\mathscr{A}$, which is denoted by $\mathscr{A}_{\overline{N}}$ and called the *reverse Nerode automaton* of $\mathscr{A}$.

By the same rule (68), for any $a \in A$ we define fuzzy languages $\sigma_a^A, \tau_a^A \in L^{X^*}$, namely $\sigma_a^A(u) = \sigma_u^A(a)$ and $\tau_a^A(u) = \tau_u^A(a)$, for every $u \in X^*$. Following terminology used in [18] for non-deterministic automata, σ_a^A and τ_a^A are called respectively the *left fuzzy language* and the *right fuzzy language* associated with the state a. Left fuzzy languages have been already studied in [32,34].

A fuzzy quasi-order R on A which is a solution to a system of fuzzy relation equations

$$R \circ \tau_u^A = \tau_u^A, \text{ for every } u \in X^*, \tag{69}$$

is called a *weakly right invariant fuzzy quasi-order* on the fuzzy automaton $\mathscr{A}$, and if R is a solution to

$$\sigma_u^A \circ R = \sigma_u^A, \text{ for every } u \in X^*, \tag{70}$$

then it is called a *weakly left invariant fuzzy quasi-order* on $\mathscr{A}$. Fuzzy equivalences on $\mathscr{A}$ which are solutions to (69) will be called *weakly right invariant fuzzy equivalences*, and those which are solutions to (70) will be called *weakly left invariant fuzzy equivalences*.

Note that inequalities $\tau_u^A \leqslant R \circ \tau_u^A$ and $\sigma_u^A \leqslant R \circ \sigma_u^A$ hold for every reflexive fuzzy relation R, so system (69) is equivalent to the system of fuzzy relation inequalities $R \circ \tau_u^A \leqslant \tau_u^A$, for every $u \in X^*$, and system (70) is equivalent to the system of fuzzy relation inequalities $R \circ \sigma_u^A \leqslant \sigma_u^A$, for every $u \in X^*$.

The following theorem demonstrates that weakly right/left invariant fuzzy quasi-orders are a generalization of the right/left invariant ones.

Theorem 5.5. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton. Then*

(a) *Every weakly right invariant fuzzy quasi-order on $\mathscr{A}$ is a solution to the general system.*
(b) *Every right invariant fuzzy quasi-order on $\mathscr{A}$ is weakly right invariant.*

Proof.

(a) Let R be an arbitrary weakly right invariant fuzzy quasi-order on $\mathscr{A}$. By induction on n it can be proved that $R \circ \delta_{x_1}^A \circ R \circ \delta_{x_2}^A \circ R \circ \cdots \circ R \circ \delta_{x_n}^A \circ R \circ \tau^A = \delta_{x_1}^A \circ \delta_{x_2}^A \circ \cdots \circ \delta_{x_n}^A \circ \tau^A$, for every $n \in \mathbb{N}$ and all $x_1, x_2, \ldots, x_n \in X$, which implies that R is a solution to the general system.
(b) Let R be a right invariant fuzzy quasi-order on $\mathscr{A}$. For any $u \in X^*$ we have that $R \circ \delta_u^A \circ R = \delta_u^A \circ R$, and also $R \circ \tau^A = \tau^A$, which yields $R \circ \tau_u^A = R \circ \delta_u^A \circ \tau^A = R \circ \delta_u^A \circ R \circ \tau^A = \delta_u^A \circ R \circ \tau^A = \delta_u^A \circ \tau^A = \tau_u^A$. Thus, R is weakly right invariant. □

If $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ is a fuzzy automaton, then we define a fuzzy relation R^{wri} on A as follows

$$R^{\mathrm{wri}}(a, b) = \bigwedge_{u \in X^*} \tau_u^A(b) \to \tau_u^A(a) = \bigwedge_{u \in X^*} \tau_b^A(u) \to \tau_a^A(u), \qquad \text{for all } a, b \in A. \tag{71}$$

Now we state and prove the following theorem.

Theorem 5.6. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton.*

A fuzzy quasi-order R on A is weakly right invariant if and only if $R \leqslant R^{\mathrm{wri}}$. Consequently, R^{wri} is the greatest weakly right invariant fuzzy quasi-order on $\mathscr{A}$.

Proof. It is well-known that $R \circ \tau_u^A \leqslant \tau_u^A$ if and only if $R(a,b) \leqslant \tau_u^A(b) \to \tau_u^A(a)$, for all $a, b \in A$, and therefore, a fuzzy quasi-order R is weakly right invariant, i.e., $R \circ \tau_u^A \leqslant \tau_u^A$, for each $u \in X^*$, if and only if $R \leqslant R^{\mathrm{wri}}$.

Seeing that R^{wri} is a fuzzy quasi-order, as the intersection of a family of fuzzy quasi-orders defined as in (24), we conclude that R^{wri} is the greatest weakly right invariant fuzzy quasi-order on $\mathscr{A}$. □

It can be easily shown that R^{wri} is not only the greatest solution to system (69) in the set of all fuzzy quasi-orders on $\mathscr{A}$, but it is also the greatest solution to this system in the set of all fuzzy relations on $\mathscr{A}$. Let us also note that $R^{\mathrm{wri}}(a,b)$ can be interpreted as the degree of inclusion of the fuzzy language τ_b^A in the fuzzy language τ_a^A.

According to (71), to compute the greatest weakly right invariant fuzzy quasi-order R^{wri} on a fuzzy automaton $\mathscr{A}$, first we need to compute fuzzy sets τ_u^A, for all $u \in X^*$. As these fuzzy sets are states of the reverse Nerode automaton of $\mathscr{A}$, they could be computed using a procedure similar to the determinization procedure developed in [32].

Algorithm 5.7. Computation of all members of the collection $\{\tau_u^A\}_{u\in X^*}$

The input of this algorithm is a fuzzy finite automaton $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$. The algorithm computes all members of the collection $\{\tau_u^A\}_{u\in X^*}$.

The procedure is to construct the tree of the collection $\{\tau_u^A\}_{u\in X^*}$. It is constructed inductively in the following way:

(A1) The root of the tree is τ_ε^A, and we put $T_0 = \{\tau_\varepsilon^A\}$.
(A2) After the ith step let a tree T_i have been constructed, and vertices in T_i have been labelled either 'closed' or 'non-closed'. The meaning of these two terms will be made clear in the sequel.
(A3) In the next step we construct a tree T_{i+1} by enriching T_i as follows: for each non-closed leaf τ_u^A occuring in T_i, where $u \in X^*$, and each $x \in X$ we add a vertex $\tau_{xu}^A = \delta_x^A \circ \tau_u^A$ and an edge from τ_u^A to τ_{xu}^A. Simultaneously, we check whether τ_{xu}^A coincides with some fuzzy set that has already been constructed, and if it is true, then we mark τ_{xu}^A as closed. The procedure terminates when all leaves are marked closed.
(A4) When the tree of the collection $\{\tau_u^A\}_{u\in X^*}$ is constructed, its internal vertices correspond to all different members of this collection.

This procedure does necessarily terminates in a finite number of steps, since the collection $\{\tau_u^A\}_{u\in X^*}$ may be infinite. However, in cases when this collection is finite, the procedure will terminate in a finite number of steps, after computing all its members. For instance, this holds if the subsemiring $\mathscr{L}^*(\mathscr{A})$ of $\mathscr{L}^*$ generated by all truth values taken by the fuzzy transition relations $\left\{\delta_x^A\right\}_{x\in X}$ and the fuzzy set τ^A is finite (but not only in this case). If k denotes the number of elements of this subsemiring and n the number of states of $\mathscr{A}$, then the collection $\left\{\tau_u^A\right\}_{u\in X^*}$ can have at most k^n different members.

The tree that is constructed by this algorithm is a full m-ary tree, where m is the number of letters in the input alphabet X. At the end of the algorithm, the tree can contain at most k^n internal vertices, and according to the well-known theorem on full m-ary trees, the total number of vertices is at most $mk^n + 1$. Thus, in the construction of the tree we have performed at most mk^n compositions of the form $\delta_x^A \circ \tau_u^A$. As the computational cost of any single composition is $O(n^2(c_\otimes + c_\vee))$, the computational cost for all performed compositions is $O(mn^2k^n(c_\otimes + c_\vee))$. Time-consuming part of the procedure is the check whether the just computed fuzzy set is a copy of some previously computed fuzzy set. After we have constructed the jth fuzzy set, for some $j \in \mathbb{N}$ such that $2 \leqslant j \leqslant mk^n$+1, we compare it with the previously constructed fuzzy sets which correspond to non-closed vertices, whose number is at most $j - 1$. Therefore, the total number of performed checks does not exceed $1 + 2 + \cdots + mk^n = \frac{1}{2}mk^n(mk^n + 1)$. As the computational cost of any single check is $O(n)$, the computational cost for all performed checks is $O(m^2nk^{2n})$. Summarizing all the above, and supposing that the computational costs of $\otimes$ and $\vee$ are not exponential, we conclude that the computational cost of the whole algorithm is $O(m^2nk^{2n})$, the same as the cost of the part in which for any newly-constructed fuzzy set we ckeck whether it is a copy of some previously computed fuzzy set.

Therefore, from a theoretical point of view, the algorithm is exponential in the number of states of $\mathscr{A}$, since the number of members of the collection $\{\tau_u^A\}_{u\in X^*}$ may grow exponentially. However, this number is usually much smaller than its theoretical upper bound k^n, which in practice makes this algorithm much faster.

Note that the number k is characteristic of the fuzzy finite automaton $\mathscr{A}$, and it is not a general characteristic of the semiring $\mathscr{L}^*$ and its finitely generated subsemirings. However, if we consider fuzzy automata over a finite lattice, the we can assume that k is the number of elements of this lattice. Moreover, if $\mathscr{L}$ is the Gödel structure, then the set of all truth values taken by the fuzzy transition relations $\left\{\delta_x^A\right\}_{x\in X}$ and the fuzzy set τ^A is a subsemiring of $\mathscr{L}^*$, and the number of these values does not exceed $mn^2 + n$, so we can use that number instead of k.

Now we can construct the following algorithm which computes the greatest weakly right invariant fuzzy quasi-order on a fuzzy finite automaton.

Algorithm 5.8. *Computation of the greatest weakly right invariant fuzzy quasi-order*

The input of this algorithm is a fuzzy finite automaton $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$. The algorithm computes the greatest weakly right invariant fuzzy quasi-order R^{wri} on $\mathscr{A}$.

The procedure consists of two parts:

(A1) First we compute fuzzy sets τ_u^A, for all $u \in X^*$, using Algorithm 5.7.
(A2) Then we compute R^{wri} by means of formula (71).

With the same requirement that the subsemiring $\mathscr{L}^*(\mathscr{A})$ of $\mathscr{L}^*$ is finite and has k elements, we again have that the collection $\{\tau_u^A\}_{u\in X^*}$ contains at most k^n different members. Therefore, for any pair $(a, b) \in A^2$, the value $R^{\mathrm{wri}}(a, b)$ can be computed performing the operation $\to$ at most k^n times and the operation $\wedge$ at most $k^n - 1$ times, which means that the computational cost of (A2) is $O(n^2 k^n (c_\to + c_\wedge))$. Assuming again that the computational costs of $\to$ and $\wedge$ are not exponential, we obtain that the computational cost of the whole algorithm is $O(m^2 n k^{2n})$, the same as for Algorithm 5.7.

Analogously we define a fuzzy quasi-order R^{wli}, and fuzzy equivalences E^{wrie} and E^{wlie} on $\mathscr{A}$ by

$$R^{\mathrm{wli}}(a, b) = \bigwedge_{u\in X^*} \sigma_u^A(a) \to \sigma_u^A(b) = \bigwedge_{u\in X^*} \sigma_a^A(u) \to \sigma_b^A(u), \ \text{for all } a, b \in A. \tag{72}$$

$$E^{\mathrm{wrie}}(a, b) = \bigwedge_{u\in X^*} \tau_u^A(a) \leftrightarrow \tau_u^A(b) = \bigwedge_{u\in X^*} \tau_a^A(u) \leftrightarrow \tau_b^A(u), \ \text{for all } a, b \in A, \tag{73}$$

$$E^{\mathrm{wlie}}(a, b) = \bigwedge_{u\in X^*} \sigma_u^A(a) \leftrightarrow \sigma_u^A(b) = \bigwedge_{u\in X^*} \sigma_a^A(u) \leftrightarrow \sigma_b^A(u), \ \text{for all } a, b \in A. \tag{74}$$

We have that R^{wli} is the greatest weakly left invariant fuzzy quasi-order on $\mathscr{A}$, and E^{wrie} and E^{wlie} are respectively the greatest weakly right and left invariant fuzzy equivalences. We can also state and prove theorems analogous to Theorems 5.5 and 5.6, and provide algorithms analogous to Algorithm 5.8. Clearly, E^{wrie} and E^{wlie} are respectively the natural fuzzy equivalences of fuzzy quasi-orders R^{wri} and R^{wlie}.

Consider again the tree constructed by Algorithm 5.7, under the assumption that the subsemiring $\mathscr{L}^*(\mathscr{A})$ is finite and has k elements. As we have already said, the tree has at most k^n non-closed vertices, so the height of this tree also does not exceed k^n, and hence, all different members of the collection $\{\tau_u^A\}_{u\in X^*}$ will be involved when we compute τ_u^A for all $u \in X^*$ such that $|u| \leqslant k^n$. Left out the part of Algorithm 5.7 in which we check whether the newly-computed fuzzy set is a copy of some previously computed fuzzy set. Then the tree T_i constructed after the ith step is a complete m-ary tree of height i, whose vertices correspond to fuzzy sets of the form τ_u^A, for all $u \in X^*$ such that $|u| \leqslant i$, and the leaves correspond to all fuzzy sets of the form τ_u^A, for $u \in X^*$ with $|u| = i$. Clearly, in the next step we enrich T_i adding vertices which correspond to all fuzzy sets of the form τ_u^A, for $u \in X^*$ with $|u| = i + 1$. This modified algorithm will be terminated when construction of the tree T_{k^n} of height k^n is complete. Since T_{k^n} is a complete m-ary tree of height k^n, the number of its vertices is $(m^{k^n+1} - 1)/(m - 1)$, and when we subtract 1 (for the root of the tree), we get $m(m^{k^n} - 1)/(m - 1)$, which is the number of all computed compositions of the form $\delta_x^A \circ \tau_u^A$, for $x \in X$ and $u \in X^*$. Hence, the computational cost of the modified algorithm is $O(n^2 m^{k^n} (c_\otimes + c_\vee))$, which is obviously worse than $O(m^2 n k^{2n})$, the computational cost of the original version of Algorithm 5.7.

Note that the main idea underlying this modification of Algorithm 5.7 has been exploited by Wu and Qiu in [82] (see also [46,65,86]). They considered a fuzzy finite transition system $\mathscr{A} = (A, X, \delta^A)$ with n states, over a finite bounded lattice with k elements, and constructed a sequence $\{M_i\}_{0\leqslant i\leqslant k^n}$ of the so-called behavior matrices of $\mathscr{A}$. For any i, the behavior matrix M_i is a matrix whose columns are vectors (fuzzy sets) of the form τ_u^A, for all words $u \in X^*$ such that $|u| \leqslant i$ (the starting vector $\tau_\varepsilon^A = \tau^A$ is taken to be the vector whose all entries are 1, i.e., τ_ε^A is the whole set A). The procedure of computing matrices M_i, for $0 \leqslant i \leqslant k^n$, provided by Wu and Qiu in [82], is exactly the same as what we have called here the modification of Algorithm 5.7. Simultaneously, the procedure of Wu and Qiu computes a sequence of crisp equivalence relations $\{\varrho_i\}_{0\leqslant i\leqslant k^n}$, which are defined so that two states $a, b \in A$ are ϱ_i-equivalent if and only if the rows in M_i corresponding to a and b coincide. The equivalence ϱ_{k^n} provides a reduction of $\mathscr{A}$, i.e., a fuzzy finite transition system with a reduced number of states, which is, in certain sense, equivalent to $\mathscr{A}$. In our terminology, the equivalence relation ϱ_{k^n} is actually the crisp part of the greatest weakly right invariant fuzzy equivalence E^{wrie} on $\mathscr{A}$.

It is worth mentioning that, according to Theorem 5.6, both the fuzzy equivalence E^{wrie} on a fuzzy finite automaton $\mathscr{A}$, and its crisp part $\widehat{E}^{\mathrm{wrie}}$, are weakly right invariant. Thus, both of them, as well as the fuzzy quasi-order R^{wri}, provide factor and afterset fuzzy automata which are equivalent to the original fuzzy automaton $\mathscr{A}$. On the other hand, according to Theorem 3.1, the fact that E^{wrie} is the natural fuzzy equivalence of R^{wri}, and the fact that a fuzzy equivalence and its crisp part have the same number of equivalence classes, we have that the afterset fuzzy automaton with respect to R^{wri} and the factor fuzzy automata with respect to E^{wrie} and $\widehat{E}^{\mathrm{wrie}}$ have the same number of states (although they may be non-isomorphic). As the computation of R^{wri}, E^{wrie} and $\widehat{E}^{\mathrm{wrie}}$ requires the same computational time, we conclude that it does not matter which of these three relations will be used in the reduction of the number of states of $\mathscr{A}$.

The above mentioned algorithm of Wu and Qiu also has another termination criterion – it will terminate whenever the first computed relation ϱ_i is equal to the equality relation. This criterion can sometimes significantly reduce the computation time, but it can be applied only in the rare cases when the greatest weakly right invariant fuzzy equivalence is a fuzzy equality (or the greatest weakly right invariant fuzzy quasi-order is a fuzzy order). This criterion may be useful in the detection of a certain kind of irreducibility, but it can not help in the reduction of the number of states.

Next, let us compare reductions of a fuzzy automaton $\mathscr{A}$ by means of the weakly right invariant fuzzy quasi-orders with those by means of the right invariant ones. According to (b) of Theorem 5.5, the set of all right invariant fuzzy quasi-orders on a fuzzy automaton $\mathscr{A}$ is contained in the set of all weakly right invariant fuzzy quasi-orders on $\mathscr{A}$. Therefore, for the greatest weakly right invariant fuzzy quasi-order R^{wri} on $\mathscr{A}$ and the greatest right invariant fuzzy quasi-order R^{ri} on $\mathscr{A}$ we have that $R^{\mathrm{wri}} \geqslant R^{\mathrm{ri}}$, and consequently, for the numbers of states of the corresponding afterset fuzzy automata the following holds: $|\mathscr{A}/R^{\mathrm{wri}}| \leqslant |\mathscr{A}/R^{\mathrm{ri}}|$. The example below shows that this inequality may be strict, which means that in general, weakly right invariant fuzzy quasi-orders provide better reductions than the right invariant ones. However, as we have seen, from the aspect of computation time, the advantage is on the side of right invariant fuzzy quasi-orders.

Example 5.9. Let $\mathscr{L}$ be the Boolean structure and $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ a fuzzy automaton over $\mathscr{L}$, where $A = \{1,2,3,4\}$, $X = \{x\}$, σ^A is any fuzzy subset of A and δ^A_x, and τ^A are given by

$$\delta^A_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tau^A = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then the greatest right invariant fuzzy equivalence R^{ri} and the greatest weakly right invariant fuzzy equivalence R^{wri} on $\mathscr{A}$ are given by

$$R^{\mathrm{ri}} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R^{\mathrm{wri}} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

Hence, R^{ri} is strictly smaller than R^{wri}, and R^{ri} do not reduce the number of states of $\mathscr{A}$, while R^{wri} reduces $\mathscr{A}$ to a fuzzy automaton with two states.

5.4. Alternate reductions

As shown in [25], where the state reduction by means of right and left invariant fuzzy equivalences has been discussed, better reductions can be achieved alternating reductions by means of the greatest right and left invariant fuzzy equivalences. Here we show that this also holds for right and left invariant fuzzy quasi-orders, as well as for weakly right and left invariant fuzzy quasi-orders.

Let $\mathscr{A}$ be a fuzzy finite transition system or a fuzzy finite automaton, and R^{ri} the greatest right invariant fuzzy quasi-order on $\mathscr{A}$. As in Theorem 5.2 of [25], we can prove that the greatest right invariant fuzzy quasi-order on the afterset fuzzy automaton $\mathscr{A}/R^{\mathrm{ri}}$ is a fuzzy order, and that the number of states of the afterset fuzzy automaton $\mathscr{A}/R^{\mathrm{ri}}$ can not be reduced by means of right invariant fuzzy quasi-orders. In other words, we can not perform more consecutive reductions using only right invariant fuzzy quasi-orders. This also holds for left invariant fuzzy quasi-orders, as well as for weakly right and left invariant fuzzy quasi-orders (but not for strongly right and left invariant fuzzy quasi-orders; see Example 6.2 in [25], where the same has been proved in the framework of fuzzy equivalences).

However, consecutive reductions are possible if we perform alternately reductions by means of the greatest right invariant and the greatest left invariant fuzzy quasi-orders, or more generaly, if we perform alternately reductions by means of the greatest weakly right invariant and the greatest weakly left invariant fuzzy quasi-orders. Such reductions we call *alternate reductions*.

The following is a very illustrative example.

Example 5.10. Let $\mathscr{L}$ be the Boolean structure and let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton over $\mathscr{L}$, where $A = \{1,2,3\}$, $X = \{x, y\}$, and δ^A_x, δ^A_y, σ^A and τ^A are given by

$$\delta^A_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \delta^A_y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \sigma^A = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \tau^A = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

It can be easily checked that the reduction of $\mathscr{A}$ by means of the greatest weakly right invariant fuzzy quasi-order on $\mathscr{A}$ gives a fuzzy automaton $\mathscr{A}_1$ which also has three states. As we have already said, further reduction of $\mathscr{A}_1$ by means of weakly right invariant fuzzy quasi-order can not reduce the number of states, but we can perform reduction of $\mathscr{A}_1$ by means of the greatest left invariant fuzzy quasi-order on $\mathscr{A}_1$, which will give a fuzzy automaton $\mathscr{A}_2$ with two states. Next we find that both the greatest weakly right invariant fuzzy quasi-order and the greatest weakly left invariant fuzzy quasi-order on $\mathscr{A}_2$ coincide with the equality relation, and the afterset fuzzy automata of $\mathscr{A}_2$ w.r.t. these fuzzy quasi-orders are isomorphic to $\mathscr{A}_2$, so no further reduction of the number of states of $\mathscr{A}_2$ by means of fuzzy quasi-orders of the considered types is possible.

On the other hand, the reduction of $\mathscr{A}$ by means of the greatest weakly left invariant fuzzy quasi-order on $\mathscr{A}$ produces a fuzzy automaton $\mathscr{A}'_1$ having also three states. Further, the greatest weakly right invariant fuzzy quasi-order on $\mathscr{A}'_1$, which is identical with the greatest weakly left invariant fuzzy quasi-order on $\mathscr{A}'_1$, determines an afterset fuzzy automaton which is isomorphic to $\mathscr{A}'_1$. Therefore, no further state reduction of $\mathscr{A}'_1$ by means of fuzzy quasi-orders of the considered types is possible.

It is important to note that all the mentioned greatest weakly right and left invariant fuzzy quasi-orders are also the greatest right and left invariant fuzzy quasi-orders, so all that has been said here for weakly right and left invariant fuzzy quasi-orders is also true for the right and left invariant ones.

Example 5.10 demonstrates several important things. First, the number of states of the fuzzy automaton $\mathscr{A}$ can not be reduced by any single reduction, neither by weakly right invariant fuzzy quasi-orders, nor by the weakly left invariant ones, but it can be reduced by means of an alternate reduction. In the first alternate reduction, in its first step, we were not able to reduce the number of states, but we did in the second step. Although the greatest weakly left invariant fuzzy quasi-order is not able to reduce the number of states of $\mathscr{A}$, after it has been slightly modified by reduction by means of the greatest weakly right invariant fuzzy quasi-order, further reduction by means of the greatest weakly left invariant fuzzy quasi-order was successful. And second, we see that the alternate reduction which starts with reduction by means of the greatest weakly right invariant fuzzy quasi-order, and the one which starts with reduction by means of the greatest weakly left invariant fuzzy quasi-order, do not necessarily give equally good results. One of them was able to reduce the fuzzy automaton $\mathscr{A}$ to a fuzzy automaton with two states, and the other has failed to reduce the number of states. However, there is no general rule that can help us to determine what is better, what type of reduction should start alternating reduction in a particular case.

The state reduction of non-deterministic automata by means of right invariant and left invariant quasi-orders has been studied by Champarnaud and Coulon [17,18], Ilie et al. [39], and Ilie et al. [40] (see also [36,37]). The mentioned authors used factor automata with respect to natural equivalences of the greatest right and left invariant quasi-orders R^{ri} and R^{li}, but none of them has considered afterset automata with respect to these quasi-orders. As we have noted earlier, afterset automata corresponding to these quasi-orders and factor automata corresponding to their natural equivalences are not necessarily isomorphic, but they have the same number of states and are equivalent to the original automaton $\mathscr{A}$. Thus, from the point of view of single reductions it does not matter whether we will use afterset automata or the corresponding factor automata, but from the point of view of alternate reductions there is a difference. If the quasi-orders R^{ri} and R^{li} are orders (i.e., anti-symmetric), as in the case of the automaton $\mathscr{A}$ in Example 5.10, their natural equivalences will be equality relations, so neither single nor alternate reductions by means of these equivalences will be successful. This also holds for alternate reductions by means of the greatest right and left invariant equivalences. However, alternate reductions by means of the greatest right and left invariant quasi-orders can be successful even in such cases, as Example 5.10 shows.

Another interesting issue is the following: Whether, after a certain number of steps in an alternate reduction, we can know (and how) that we have come in such a situation when further reduction of this type will not reduce the number of states? The answer to this question is affirmative if we perform an alternate reduction of a non-deterministic automaton by means of the greatest right and left invariant equivalences, or the greatest weakly right and left invariant equivalences. Indeed, if after two successive steps the number of states did not changed, then we can be sure that we have reached the smallest number of states and this alternate reduction is finished. However, this does not hold for alternate reductions by means of the greatest right and left invariant quasi-orders, or the greatest weakly right and left invariant quasi-orders. Forming the afterset automaton with respect to some order relation (anti-symmetric quasi-order) on a non-deterministic automaton we do not reduce the number of states, but we change transitions and we obtain an automaton which is not necessarily isomorphic with the original one, what makes possible to continue an alternate reduction and reduce the number of states (see again Example 5.10). The same conclusion can be drawn for alternate reductions of fuzzy transition systems and automata.

Finally, let us mention that an example provided by Ilie and Yu in [37] shows that single reductions of a non-deterministic automaton by means of the greatest right and left invariant equivalences can cause a polynomial decrease in the number of states, whereas an alternate reduction can cause an exponential decrease.

6. An example demonstrating some applications to fuzzy discrete event systems

In this section we give an example demonstrating some applications of weakly left invariant fuzzy quasi-orders to fuzzy discrete event systems. A more complete study of fuzzy discrete event systems will be a subject of our further work.

A *discrete event system* (*DES*) is a dynamical system whose state space is described by a discrete set, and states evolve as a result of asynchronously occurring discrete events over time [15,31]. Such systems have significant applications in many fields of computer science and engineering, such as concurrent and distributed software systems, computer and communication networks, manufacturing, and transportation and traffic control systems. Usually, a discrete event system is modeled by a finite state transition system (deterministic or nondeterministic), with events modeled by input letters, and the behavior of a discrete event system is described by the language generated by the transition system. However, in many situations states and state transitions, as well as control strategies, are somewhat imprecise, uncertain and vague. To take this kind of uncertainty into account, Lin and Ying extended classical discrete event systems to *fuzzy discrete event systems* (*FDES*) by proposing a fuzzy finite automaton model [48,49]. A basic framework of supervisory control theory of fuzzy discrete event was first initiated by Qiu [73], and then, fuzzy discrete event systems have been studied in a number of papers [12–14,44,48–53,73,75], and they have been successfully applied to biomedical control for HIV/AIDS treatment planning, robotic control, intelligent vehicle control, waste-water treatment, examination of chemical reactions, and in other fields.

In [48,49], and later in [14,44,73,75], fuzzy discrete event systems have been modeled by transition systems with fuzzy states and fuzzy inputs, whose transition function is defined over the sets of fuzzy states and fuzzy inputs in a deterministic way. In fact, such a transition system can be regarded as the determinization of a fuzzy transition system (defined as in this paper) by means of the accessible fuzzy subset construction (see [32,34]). On the other hand, in [12,13,53] fuzzy discrete event systems have been modeled by fuzzy transition systems with single crisp initial states. In all mentioned papers membership values have been taken in the Gödel or product structure.

Here, a fuzzy discrete event system will be modeled by a fuzzy finite automaton $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ over a complete residuated lattice $\mathscr{L}$, defined as in Section 2.2. Two kinds of fuzzy languages associated with this fuzzy automaton play a key role in study of fuzzy discrete event systems. The first one is the *fuzzy language* $L(\mathscr{A})$ *recognized by* $\mathscr{A}$, which is defined as in (22) (or (23)), and the second one is the *fuzzy language* $L_g(\mathscr{A})$ *generated by* $\mathscr{A}$, which is defined by

$$L_g(\mathscr{A})(u) = \bigvee_{a,b\in A} \sigma^A(a) \otimes \delta^A_*(a,u,b) = \bigvee_{b\in A}\left(\sigma^A \circ \delta^A_u\right)(b) = \bigvee_{b\in A}\sigma^A_u(b), \tag{75}$$

for every $u \in X^*$. Intuitively, $L_g(\mathscr{A})(u)$ represents the degree to which the input word u causes a transition from some initial state to any other state. Two fuzzy automata $\mathscr{A}$ and $\mathscr{B}$ are called *language-equivalent* if $L(\mathscr{A}) = L(\mathscr{B})$ and $L_g(\mathscr{A}) = L_g(\mathscr{B})$.

Discrete event models of complex dynamic systems are built rarely in a monolithic manner. Instead, a modular approach is used where models of individual components are built first, followed by the composition of these models to obtain the model of the overall system. In the automaton modeling formalism the composition of individual automata (that model interacting system components) is usually formalized by the *parallel composition* of automata. Once a complete system model has been obtained by parallel composition of a set of automata, the resulting monolithic model can be used to analyze the properties of the system.

Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ and $\mathscr{B} = (A, Y, \delta^B, \sigma^B, \tau^B)$ be fuzzy automata. The *product* of $\mathscr{A}$ and $\mathscr{B}$ is a fuzzy automaton $\mathscr{A}\times\mathscr{B} = (A\times B, X\cap Y, \delta^{A\times B}, \sigma^{A\times B}, \tau^{A\times B})$, defined by

$$\begin{aligned} &\delta^{A\times B}((a,b),x,(a',b')) = \delta^A(a,x,a') \otimes \delta^B(b,x,b'),\\ &\sigma^{A\times B}(a,b) = \sigma^A(a)\otimes\sigma^B(b), \quad \tau^{A\times B}(a,b) = \tau^A(a)\otimes\tau^B(b), \end{aligned} \tag{76}$$

for all $a,a' \in A,\ b,b' \in B$ and $x \in X \cap Y$, and the *parallel composition* of $\mathscr{A}$ and $\mathscr{B}$ is a fuzzy automaton $\mathscr{A}\|\mathscr{B} = (A\times B, X\cup Y, \delta^{A\|B}, \sigma^{A\|B}, \tau^{A\|B})$, defined by

$$\begin{aligned} &\delta^{A\|B}((a,b),x,(a',b')) = \begin{cases} \delta^A(a,x,a')\otimes\delta^B(b,x,b') & \text{if } x\in X\cap Y\\ \delta^A(a,x,a') & \text{if } x\in X\setminus Y \text{ and } b=b'\\ \delta^B(b,x,b') & \text{if } x\in Y\setminus X \text{ and } a=a'\\ 0 & \text{otherwise} \end{cases},\\ &\sigma^{A\|B}(a,b) = \sigma^A(a)\otimes\sigma^B(b), \quad \tau^{A\|B}(a,b) = \tau^A(a)\otimes\tau^B(b), \end{aligned} \tag{77}$$

for all $a,a' \in A,\ \ b,b' \in B$. Associativity is used to extend the definition of parallel composition to more than two automata.

In the parallel composition of fuzzy automata $\mathscr{A}$ and $\mathscr{B}$, a common input letter from $X\cap Y$ is executed in both automata simultaneously, what means that these two automata are synchronized on the common input letter. On the other hand, a private input letter from $X\setminus Y$ is executed in $\mathscr{A}$, while $\mathscr{B}$ is staying in the same state, and similarly for private letters from $Y\setminus X$. Clearly, if $X = Y$, then the parallel composition reduces to the product. However, even if $X \neq Y$, the parallel composition of fuzzy automata can be regarded as the product of suitable input extensions of these fuzzy automata, what will be shown in the sequel. If $X\cap Y = \emptyset$, then no synchronized transitions occur and $\mathscr{A}\|\mathscr{B}$ is the *concurrent behavior* of $\mathscr{A}$ and $\mathscr{B}$. This behavior is often termed the *shuffle* of $\mathscr{A}$ and $\mathscr{B}$.

Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton and let Y be an alphabet such that $X \subseteq Y$. Let us define a new transition function $\delta^{A_Y} : A\times Y\times A \to L$ by

$$\delta^{A_Y}(a,x,a') = \begin{cases} \delta^A(a,x,a') & \text{if } x\in X\\ 1 & \text{if } x\in Y\setminus X \text{ and } a=a'\\ 0 & \text{otherwise} \end{cases}, \tag{78}$$

for all $a, a' \in A$ and $x \in Y$. Then a fuzzy automaton $\mathscr{A}_Y = (A, Y, \delta^{A_Y}, \sigma^A, \tau^A)$ is called a *Y-input extension* of $\mathscr{A}$. In other words, input letters from X cause in $\mathscr{A}_Y$ the same transitions as in $\mathscr{A}$, while those from $Y \setminus X$ cause $\mathscr{A}_Y$ to stay in the same state. Evidently, $\delta_u^{A_Y}$ is the equality relation on A, for each $u \in (Y \setminus X)^*$.

An operation frequently performed on words and languages is the so-called natural projection, which transforms words over an alphabet Y to words over a smaller alphabet $X \subseteq Y$. Formally, a *natural projection*, or briefly a *projection*, is a mapping $\pi_X : Y^* \to X^*$, where $X \subseteq Y$, defined inductively by

$$\pi_X(w) = \begin{cases} e & \text{if } w \in (Y \setminus X)^* \\ w & \text{if } w \in X^* \\ \pi_X(u)\pi_X(v) & \text{if } w = uv, \text{ for some } u, v \in Y^* \end{cases}, \tag{79}$$

for each $w \in Y^*$ (cf. [15]). In other words, the word $\pi_X(w) \in X^*$ is obtained from w by deleting all appearances of letters from $Y \setminus X$.

First we prove the following:

Lemma 6.1. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton, let Y be an alphabet such that $X \subseteq Y$, and let $\mathscr{A}_Y = (A, Y, \delta^{A_Y}, \sigma^A, \tau^A)$ be the Y-input extension of $\mathscr{A}$. Then for every $u \in Y^*$ we have that*

$$L_g(\mathscr{A}_Y)(u) = L_g(\mathscr{A})(\pi_X(u)) \text{ and } L(\mathscr{A}_Y)(u) = L(\mathscr{A})(\pi_X(u)).$$

Proof. An arbitrary word $u \in Y^*$ can be represented in the form $u = u_1 v_1 u_2 v_2 \cdots u_n v_n u_{n+1}$, where $n \in \mathbb{N}, u_1, u_2, \ldots, u_{n+1} \in (Y \setminus X)^*$, and $v_1, v_2, \ldots, v_n \in X^*$, and clearly, $\pi_X(u) = v$, where $v = v_1 v_2 \cdots v_n$. Since $\delta_p^{A_Y}$ is the equality relation on A and $\delta_q^{A_Y} = \delta_q^A$, for all $p \in (Y \setminus X)^*$ and $q \in X^*$, then we have that

$$\begin{aligned} L_g(\mathscr{A}_Y)(u) &= \bigvee_{a \in A} (\sigma^A \circ \delta_u^{A_Y})(a) = \bigvee_{a \in A} \left(\sigma^A \circ \delta_{u_1}^{A_Y} \circ \delta_{v_1}^{A_Y} \circ \delta_{u_2}^{A_Y} \circ \delta_{v_2}^{A_Y} \circ \cdots \circ \delta_{u_n}^{A_Y} \circ \delta_{v_n}^{A_Y} \circ \delta_{u_{n+1}}^{A_Y}\right)(a) = \bigvee_{a \in A} \left(\sigma^A \circ \delta_{v_1}^A \circ \delta_{v_2}^A \circ \cdots \circ \delta_{v_n}^A\right)(a) \\ &= \bigvee_{a \in A} \left(\sigma^A \circ \delta_v^A\right)(a) = L_g(\mathscr{A})(v) = L_g(\mathscr{A})(\pi_X(u)), \end{aligned}$$

and similarly, $L(\mathscr{A}_Y)(u) = L(\mathscr{A})(\pi_X(u))$. $\square$

Now we prove the following:

Theorem 6.2. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ and $\mathscr{B} = (B, Y, \delta^B, \sigma^B, \tau^B)$ be fuzzy automata, let $Z = X \cup Y$, and let $\mathscr{A}_Z = (A, Z, \delta^{A_Z}, \sigma^A, \tau^A)$ and $\mathscr{B}_Z = (B, Z, \delta^{B_Z}, \sigma^B, \tau^B)$ be respectively their Z-input extensions.*

Then fuzzy automata $\mathscr{A} \| \mathscr{B}$ and $\mathscr{A}_Z \| \mathscr{B}_Z$ are isomorphic, and for each $u \in Z^*$ we have that

$$L_g(\mathscr{A} \| \mathscr{B})(u) = L_g(\mathscr{A}_Z)(u) \otimes L_g(\mathscr{B}_Z)(u) = L_g(\mathscr{A})(\pi_X(u)) \otimes L_g(\mathscr{B})(\pi_Y(u)), \tag{80}$$

$$L(\mathscr{A} \| \mathscr{B})(u) = L(\mathscr{A}_Z)(u) \otimes L(\mathscr{B}_Z)(u) = L(\mathscr{A})(\pi_X(u)) \otimes L(\mathscr{B})(\pi_Y(u)). \tag{81}$$

Proof. According to (78) and (77), for every $x \in Z = X \cup Y$ we have that

$$\delta^{A_Z \| B_Z}((a,b),x,(a',b')) = \delta^{A_Z}(a,x,a') \otimes \delta^{B_Z}(b,x,b') = \begin{cases} \delta^A(a,x,a') \otimes \delta^B(b,x,b'), & \text{if } x \in X \cap Y \\ \delta^A(a,x,a') \otimes 1, & \text{if } x \in X \setminus Y \text{ and } b = b' \\ 1 \otimes \delta^B(b,x,b'), & \text{if } x \in Y \setminus X \text{ and } a = a' \end{cases} = \delta^{A \| B}((a,b),x,(a',b')),$$

for every $a, a' \in A,\ b, b' \in B$. Since fuzzy automata $\mathscr{A}$ and $\mathscr{A}_Z$, as well as $\mathscr{B}$ and $\mathscr{B}_Z$, have the same fuzzy sets of initial and terminal states, we conclude that $\mathscr{A} \| \mathscr{B}$ and $\mathscr{A}_Z \| \mathscr{B}_Z$ are isomorphic. Moreover, according to Lemma 6.1, for each $u \in Z^* = (X \cup Y)^*$ we have that

$$\begin{aligned} L_g(\mathscr{A} \| \mathscr{B})(u) &= L_g(\mathscr{A}_Z \| \mathscr{B}_Z)(u) = \bigvee_{(a,b),(a',b') \in A \times B} \sigma^{A \| B}(a,b) \otimes \delta^{A_Z \| B_Z}((a,b),u,(a',b')) = \left(\bigvee_{a,a' \in A} \sigma^A(a) \otimes \delta^{A_Z}(a,u,a')\right) \otimes \left(\bigvee_{b,b' \in B} \sigma^B(b) \otimes \delta^{B_Z}(b,u,b')\right) \\ &= L_g(\mathscr{A}_Z)(u) \otimes L_g(\mathscr{B}_Z)(u) = L_g(\mathscr{A})(\pi_X(u)) \otimes L_g(\mathscr{B})(\pi_Y(u)). \end{aligned}$$

The rest of the assertion can be proved in a similar way. $\square$

In particular, if $X = Y$, i.e., if $\mathscr{A} \| \mathscr{B} = \mathscr{A} \times \mathscr{B}$, then by (80) and (81) it follows that

$$L_g(\mathscr{A} \times \mathscr{B})(u) = L_g(\mathscr{A})(u) \otimes L_g(\mathscr{B})(u), \tag{82}$$

$$L(\mathscr{A} \times \mathscr{B})(u) = L(\mathscr{A})(u) \otimes L(\mathscr{B})(u), \tag{83}$$

for every $u \in X^*$.

One of the key reasons for using automata to model discrete event systems is their amenability to analysis for answering various questions about the structure and behavior of the system, such as safety properties, blocking properties, and diagnosability. In the context of fuzzy automata we will consider blocking properties, which are originally concerned with the presence of deadlock and/or livelock in the transition system, i.e., with the problem of checking whether a terminal state can be reached from every reachable state.

A *prefix-closure* of a fuzzy language $f \in L^{X^*}$, denoted by $\bar{f}$, is a fuzzy language in L^{X^*} defined by

$$\bar{f}(u) = \bigvee_{v \in X^*} f(uv), \tag{84}$$

for any $u \in X^*$. It is easy to verify that the mapping $f \mapsto \bar{f}$ is a closure operator on L^{X^*}, i.e., for arbitrary $f, f_1, f_2 \in L^{X^*}$ we have that

$$f \leqslant \bar{f},\ \bar{\bar{f}} = \bar{f} \text{ and } f_1 \leqslant f_2 \text{ implies } \overline{f_1} \leqslant \overline{f_2}. \tag{85}$$

A fuzzy language $f \in L^{X^*}$ is called *prefix-closed* if $f = \bar{f}$.

We have that the following is true:

Lemma 6.3. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton. Then*

$$L(\mathscr{A}) \leqslant \overline{L(\mathscr{A})} \leqslant L_g(\mathscr{A}) = \overline{L_g(\mathscr{A})}. \tag{86}$$

Proof. According to $L(\mathscr{A}) \leqslant L_g(\mathscr{A})$ and (85), it is enough to prove $\overline{L_g(\mathscr{A})} \leqslant L_g(\mathscr{A})$. Indeed, for arbitrary $a, b, c \in A$ and $u, v \in X^*$ we have that

$$\sigma^A(a) \otimes \delta^A_u(a,c) \otimes \delta^A_v(c,b) \leqslant \sigma^A(a) \otimes \delta^A_u(a,c) \leqslant L_g(\mathscr{A})(u),$$

what implies that

$$\begin{aligned}
\overline{L_g(\mathscr{A})}(u) &= \bigvee_{v \in X^*} L_g(\mathscr{A})(uv) = \bigvee_{v \in X^*} \bigvee_{a,b \in A} \sigma^A(a) \otimes \delta^A_{uv}(a,b) \\
&= \bigvee_{v \in X^*} \bigvee_{a,b \in A} \bigvee_{c \in A} \sigma^A(a) \otimes \delta^A_u(a,c) \otimes \delta^A_v(c,b) = \bigvee_{a,c \in A} \left(\sigma^A(a) \otimes \delta^A_u(a,c)\right) \otimes \left(\bigvee_{v \in X^*} \bigvee_{b \in A} \delta^A_v(c,b)\right) \\
&\leqslant \bigvee_{a,c \in A} \sigma^A(a) \otimes \delta^A_u(a,c) = L_g(\mathscr{A})(u),
\end{aligned}$$

for every $u \in X^*$. Therefore, $\overline{L_g(\mathscr{A})} \leqslant L_g(\mathscr{A})$. □

It is worth noting that the fuzzy language $\overline{L(\mathscr{A})}$ can be represented by

$$\overline{L(\mathscr{A})}(u) = \bigvee_{v \in X^*} L(\mathscr{A})(uv) = \bigvee_{v \in X^*} \sigma^A \circ \delta^A_{uv} \circ \tau^A = \bigvee_{v \in X^*} \sigma^A \circ \delta^A_u \circ \delta^A_v \circ \tau^A = \bigvee_{v \in X^*} \sigma^A_u \circ \tau^A_v,$$

for every $u \in X^*$.

A fuzzy automaton $\mathscr{A}$ is said to be *blocking* if $\overline{L(\mathscr{A})} < L_g(\mathscr{A})$, where the inequality is proper, and otherwise, if $\overline{L(\mathscr{A})} = L_g(\mathscr{A})$, then $\mathscr{A}$ is referred to as *nonblocking*. These concepts generalize related concepts of the crisp discrete event systems theory, where a crisp transition system is considered to be blocking if it can reach a state from which no terminal state can be reached anymore. This includes both the possibility of a deadlock, where a transition system is stuck and unable to continue at all, and a livelock, where a transition system continues to run forever without achieving any further progress.

When we work with parallel compositions, the term conflicting is used instead of blocking. Namely, fuzzy automata $\mathscr{A}$ and $\mathscr{B}$ are said to be *nonconflicting* if their parallel composition $\mathscr{A} \| \mathscr{B}$ is nonblocking, and otherwise they are said to be *conflicting*. The parallel composition of a set of automata may be blocking even if each of the individual components is nonblocking (cf. [15]), and hence, it is necessarily to examine the transition structure of the parallel composition to answer blocking properties. But, the size of the state set of the parallel composition may in the worst case grow exponentially in the number of automata that are composed. This process is known as the curse of dimensionality in the study of complex systems composed of many interacting components.

The mentioned problems in analysis of large discrete event models may be mitigated if we adopt modular reasoning, which can make it possible to replace components in the parallel composition by smaller equivalent automata, and then to analyse a simpler system. Such an approach has been used in [56] in study of conflicting properties of crisp discrete event systems. Here we will show that every fuzzy automaton $\mathscr{A}$ is conflict-equivalent with the afterset fuzzy automaton $\mathscr{A}/R$ w.r.t. any weakly left invariant fuzzy quasi-order R on $\mathscr{A}$. This means that in the parallel composition of fuzzy automata every component can be replaced by such afterset fuzzy automaton, what results in a smaller fuzzy automaton to be analysed, and do not affect conflicting properties of the components.

Two fuzzy automata $\mathscr{A}$ and $\mathscr{B}$ are said to be *conflict-equivalent* if for every fuzzy automaton $\mathscr{C}$ we have that $\mathscr{A}\|\mathscr{C}$ is nonblocking if and only if $\mathscr{B}\|\mathscr{C}$ is nonblocking, i.e., if $\mathscr{A}$ and $\mathscr{B}$ are nonconflicting (conflicting) with the same fuzzy automata (cf. [56]).

Now we are ready to state and prove the main results of this section.

Theorem 6.4. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton and let R be a weakly left invariant fuzzy quasi-order on $\mathscr{A}$. Then fuzzy automata $\mathscr{A}$ and $\mathscr{A}/R$ are language-equivalent, and consequently, they are conflict-equivalent.*

Proof. As we already know, $L(\mathscr{A}) = L(\mathscr{A}/R)$. Moreover, by the dual of the equality proved in (a) of Theorem 5.5, for an arbitrary $u = x_1 \cdots x_n \in X^*$, where $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, we have that

$$\begin{aligned} L_g(\mathscr{A}/R)(u) = & \bigvee_{b\in A}\left(\sigma^{A/R}\circ\delta_u^{A/R}\right)(R_b) = \bigvee_{b\in A}\left(\sigma^{A/R}\circ\delta_{x_1}^{A/R}\circ\cdots\circ\delta_{x_n}^{A/R}\right)(R_b) \\ = & \bigvee_{a_1,a_2,\ldots,a_n,b\in A}\sigma^{A/R}(R_{a_1})\otimes\delta_{x_1}^{A/R}(R_{a_1},R_{a_2})\otimes\cdots\otimes\delta_{x_n}^{A/R}(R_{a_n},R_b) \\ = & \bigvee_{a_1,a_2,\ldots,a_n,b\in A}(\sigma^A\circ R)(a_1)\otimes\left(R\circ\delta_{x_1}^A\circ R\right)(a_1,a_2)\otimes\cdots\otimes\left(R\circ\delta_{x_n}^A\circ R\right)(a_n,b) \\ = & \bigvee_{b\in A}\left(\sigma^A\circ R\circ\delta_{x_1}^A\circ R\circ\cdots\circ R\circ\delta_{x_n}^A\circ R\right)(b) \\ = & \bigvee_{b\in A}\left(\sigma^A\circ\delta_{x_1}^A\circ\cdots\circ\delta_{x_n}^A\right)(b) = \bigvee_{b\in A}\left(\sigma^A\circ\delta_u^A\right)(b) = L_g(\mathscr{A})(u), \end{aligned}$$

and therefore, $L_g(\mathscr{A}/R) = L_g(\mathscr{A})$. Hence, $\mathscr{A}$ and $\mathscr{A}/R$ are language-equivalent.

Next, let $\mathscr{B} = (B, Y, \delta^B, \sigma^B, \tau^B)$ be an arbitrary fuzzy automaton, and let $Z = X\cup Y$. By the language-equivalence of $\mathscr{A}$ and $\mathscr{A}/R$ and Theorem 6.2, for every $u\in Z^* = (X\cup Y)^*$ we have that

$$\begin{aligned} L_g((\mathscr{A}/R)\|\mathscr{B})(u) &= L_g((\mathscr{A}/R)_Z)(u)\otimes L_g(\mathscr{B}_Z)(u) = L_g((\mathscr{A}/R))(\pi_X(u))\otimes L_g(\mathscr{B})(\pi_Y(u)) = L_g(\mathscr{A})(\pi_X(u))\otimes L_g(\mathscr{B})(\pi_Y(u) \\ &= L_g(\mathscr{A}_Z)(u)\otimes L_g(\mathscr{B}_Z)(u) = L_g(\mathscr{A}\|\mathscr{B})(u), \end{aligned}$$

and hence, $L_g((\mathscr{A}/R)\|\mathscr{B}) = L_g(\mathscr{A}\|\mathscr{B})$. Similarly we prove that $L((\mathscr{A}/R)\|\mathscr{B}) = L(\mathscr{A}\|\mathscr{B})$, and by this it follows that $\overline{L((\mathscr{A}/R)\|\mathscr{B})} = \overline{L(\mathscr{A}\|\mathscr{B})}$.

Hence, we have that $\overline{L(\mathscr{A}\|\mathscr{B})} = L_g(\mathscr{A}\|\mathscr{B})$ if and only if $\overline{L((\mathscr{A}/R)\|\mathscr{B})} = L_g((\mathscr{A}/R)\|\mathscr{B})$, what means that $\mathscr{A}$ and $\mathscr{A}/R$ are conflict-equivalent. □

The following example shows that the previous theorem do not hold for weakly right invariant fuzzy quasi-orders, i.e., a fuzzy automaton and its afterset fuzzy automaton w.r.t. a weakly right invariant fuzzy quasi-order are not necessarily language-equivalent nor conflict-equivalent.

Example 6.5. Let $\mathscr{L}$ be the Boolean structure and let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy automaton over $\mathscr{L}$, where $A = \{1, 2, 3, 4\}$, $X = \{x\}$, and δ_x^A, σ^A and τ^A are given by

$$\delta_x^A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \sigma^A = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}, \quad \tau^A = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

For every $u\in X^*$ we have that

$$\overline{L(\mathscr{A})}(u) = L_g(\mathscr{A})(u) = \begin{cases} 1 & \text{if } u = e \text{ or } u = x \\ 0 & \text{if } u = x^n, \text{ for some } n \geqslant 2 \end{cases},$$

and hence, the fuzzy automaton $\mathscr{A}$ is nonblocking.

A fuzzy relation R on A given by

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix},$$

is a weakly right invariant fuzzy quasi-order on $\mathscr{A}$ (it is just the greatest one), and the related afterset fuzzy automaton is $\mathscr{A}/R = (A/R, X, \delta^{A/R}, \sigma^{A/R}, \tau^{A/R})$, where $\delta_x^{A/R}$, $\sigma^{A/R}$ and $\tau^{A/R}$ are given by

$$\delta_x^{A/R} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \sigma^{A/R} = [\,1 \quad 1 \quad 1\,], \quad \tau^{A/R} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

For every $u \in X^*$ we have that

$$\overline{L(\mathscr{A}/R)}(u) = \begin{cases} 1 & \text{if } u = e \text{ or } u = x \\ 0 & \text{if } u = x^n, \text{ for some } n \geqslant 2 \end{cases},$$

and $L_g(\mathscr{A}/R)(u) = 1$, for each $u \in X^*$. Hence, $\overline{L(\mathscr{A}/R)} < L_g(\mathscr{A}/R)$, and we have that the fuzzy automaton $\mathscr{A}/R$ is blocking. We also have that $L_g(\mathscr{A}) \neq L_g(\mathscr{A}/R)$, what means that $\mathscr{A}$ and $\mathscr{A}/R$ are not language-equivalent.

Next, let $\mathscr{B} = (B, X, \delta^B, \sigma^B, \tau^B)$, where $B = \{b\}$, $\delta^B(b, x, b) = 1$, for each $x \in X$, and $\sigma^B(b) = \tau^B(b) = 1$. Then $\mathscr{A}\|\mathscr{B} = \mathscr{A} \times \mathscr{B}$, and by (82) and (83) it follows that

$$L_g(\mathscr{A}\|\mathscr{B}) = L_g(\mathscr{A}) \otimes L_g(\mathscr{B}) = L_g(\mathscr{A}),\ L(\mathscr{A}\|\mathscr{B}) = L(\mathscr{A}) \otimes L(\mathscr{B}) = L(\mathscr{A}),$$
$$L_g((\mathscr{A}/R)\|\mathscr{B}) = L_g(\mathscr{A}/R) \otimes L_g(\mathscr{B}) = L_g(\mathscr{A}/R),\ L((\mathscr{A}/R)\|\mathscr{B}) = L(\mathscr{A}/R) \otimes L(\mathscr{B}) = L(\mathscr{A}/R).$$

Therefore,

$$\overline{L(\mathscr{A}\|\mathscr{B})} = \overline{L(\mathscr{A})} = L_g(\mathscr{A}) = L_g(\mathscr{A}\|\mathscr{B}),\ \overline{L((\mathscr{A}/R)\|\mathscr{B})} = \overline{L(\mathscr{A}/R)} < L_g(\mathscr{A}/R) = L_g((\mathscr{A}/R)\|\mathscr{B}),$$

what means that $\mathscr{A}\|\mathscr{B}$ is nonblocking and $(\mathscr{A}/R)\|\mathscr{B}$ is blocking, and hence, $\mathscr{A}$ and $\mathscr{A}/R$ are not conflict-equivalent.

7. Concluding remarks

In our recent paper we have established close relationships between the state reduction of a fuzzy automaton and the resolution of a particular system of fuzzy relation equations. We have studied reductions by means of those solutions which are fuzzy equivalences. In this paper we demonstrated that in some cases better reductions can be obtained using the solutions of this system that are fuzzy quasi-orders. Although by Theorem 3.5 we have proved that in the general case fuzzy quasi-orders and fuzzy equivalences are equally good in the state reduction, we have shown that in some cases fuzzy quasi-orders give better reductions. The meaning of state reductions by means of fuzzy quasi-orders and fuzzy equivalences is in their possible effectiveness, as opposed to the minimization which is not effective. By Theorem 3.5 we have shown that minimization of some fuzzy automata can not be realized as its state reduction by means of fuzzy quasi-orders or fuzzy equivalences.

We gave a procedure for computing the greatest right invariant fuzzy quasi-order on a fuzzy transition system or fuzzy automaton, which works if the underlying structure of truth values is a locally finite, but not only in this case. We also gave procedures for computing the greatest right invariant crisp quasi-order and the greatest strongly right invariant fuzzy quasi-order. They work for fuzzy automata over any complete residuated lattice. However, although these procedures are applicable to a larger class of fuzzy automata, we have proved that right invariant fuzzy quasi-orders give better reductions than right invariant crisp quasi-orders and strongly right invariant fuzzy quasi-orders. We also have studied a more general type of fuzzy quasi-orders, weakly right and left invariant ones. These fuzzy quasi-orders give better reductions than right and left invariant ones, but are harder to compute. In fact, weakly right and left invariant fuzzy quasi-orders on a fuzzy automaton are defined by means of two systems of fuzzy relation equations whose resolution include determinization of this fuzzy automaton and its reverse fuzzy automaton.

Finally, we have shown that better results in the state reduction can be achieved if we alternate reductions by means of right and left invariant fuzzy quasi-orders, or weakly right and left invariant fuzzy quasi-orders. Furthermore, we show that alternate reductions by means of fuzzy quasi-orders give better results than those by means of fuzzy equivalences. It is worth noting that the presented state reduction methods are based on the construction of the afterset fuzzy automaton w.r.t. a fuzzy quasi-order, and we have proved that such approach gives better results in alternate reductions than approach by Champarnaud and Coulon, Ilie, Navarro and Yu, and Ilie, Solis-Oba and Yu, whose state reduction methods are based on the construction of the factor automaton w.r.t. the natural equivalence of a quasi-order.

At the end of the paper we have demonstrated some applications of weakly left invariant fuzzy quasi-orders in conflict analysis of fuzzy discrete event systems. Another interesting problem is application of state reductions by means of fuzzy quasi-orders in fault diagnosis of discrete event systems. Since this problem is very complex and deserves special attention, it will be discussed in a separate paper.

Several questions remained unsolved, too. They include determining more precise conditions under which our iterative procedures for computing the greatest right and left invariant fuzzy quasi-orders terminate in a finite number of steps, finding alternative algorithms for computing the greatest right and left invariant fuzzy quasi-orders for use in cases where the mentioned iterative procedures do not terminate in a finite number of steps, as well as finding even faster algorithms for computing such fuzzy quasi-orders, and general procedures to decide whether we have reached the smallest number of states in alternate reductions, and so forth. All these issues will be topics of our future research.

References

[1] W. Bandler, L.J. Kohout, Fuzzy relational products as a tool for analysis and synthesis of the behaviour of complex natural and artificial systems, in: S.K. Wang, P.P. Chang (Eds.), Fuzzy Sets: Theory and Application to Policy Analysis and Information Systems, Plenum Press, New York, 1980, pp. 341–367.
[2] N.C. Basak, A. Gupta, On quotient machines of a fuzzy automaton and the minimal machine, Fuzzy Sets Syst. 125 (2002) 223–229.
[3] R. Bělohlávek, Fuzzy Relational Systems: Foundations and Principles, Kluwer, New York, 2002.
[4] R. Bělohlávek, V. Vychodil, Fuzzy Equational Logic, Springer, Berlin-Heidelberg, 2005.
[5] U. Bodenhofer, A similarity-based generalization of fuzzy orderings preserving the classical axioms, Int. J. Uncertain. Fuzz. Knowl.-Based Syst. 8 (2000) 593–610.
[6] U. Bodenhofer, Representations and constructions of similarity-based fuzzy orderings, Fuzzy Sets Syst. 137 (2003) 113–136.
[7] U. Bodenhofer, B. De Baets, J. Fodor, A compendium of fuzzy weak orders: representations and constructions, Fuzzy Sets Syst. 158 (2007) 811–829.
[8] U. Bodenhofer, F. Klawonn, A formal study of linearity axioms for fuzzy orderings, Fuzzy Sets Syst. 145 (2004) 323–354.
[9] S. Burris, H.P. Sankappanavar, A Course in Universal Algebra, Springer-Verlag, New York, 1981.
[10] C.S. Calude, E. Calude, B. Khoussainov, Finite nondeterministic automata: simulation and minimality, Theor. Comput. Sci. 242 (2000) 219–235.
[11] C. Câmpeanu, N. Sântean, S. Yu, Mergible states in large NFA, Theor. Comput. Sci. 330 (2005) 23–34.
[12] Y.Z. Cao, M.S. Ying, Supervisory control of fuzzy discrete event systems, IEEE Trans. Syst. Man Cybern. – Part B 35 (2005) 366–371.
[13] Y.Z. Cao, M.S. Ying, Observability and decentralized control of fuzzy discrete-event systems, IEEE Trans. Fuzzy Syst. 14 (2006) 202–216.
[14] Y.Z. Cao, M.S. Ying, G.Q. Chen, State-based control of fuzzy discrete-event systems, IEEE Trans. Syst. Man Cybern. – Part B 37 (2007) 410–424.
[15] C.G. Cassandras, S. Lafortune, Introduction to Discrete Event Systems, Springer, 2008.
[16] J.-M. Champarnaud, F. Coulon, Theoretical Study and Implementation of the Canonical Automaton, Technical Report AIA 2003.03, LIFAR, Université de Rouen, 2003.
[17] J.-M. Champarnaud, F. Coulon, NFA reduction algorithms by means of regular inequalities, in: Z. Ésik, Z. Fülöp, (Eds.), DLT 2003, Lecture Notes in Computer Science, vol. 2710, 2003, pp. 194–205.
[18] J.-M. Champarnaud, F. Coulon, NFA reduction algorithms by means of regular inequalities, Theor. Comput. Sci. 327 (2004) 241–253 (erratum: Theoretical Computer Science 347 (2005) 437–40)..
[19] J.-M. Champarnaud, D. Ziadi, New finite automaton constructions based on canonical derivatives, in: S. Yu, A. Paun (Eds.), CIAA 2000, Lecture Notes in Computer Science, vol. 2088, Springer, Berlin, 2001, pp. 94–104.
[20] J.-M. Champarnaud, D. Ziadi, Computing the equation automaton of a regular expression in $\mathcal{O}(s^2)$ space and time, in: A. Amir, G. Landau (Eds.), CPM 2001, Lecture Notes in Computer Science, vol. 2089, Springer, Berlin, 2001, pp. 157–168.
[21] W. Cheng, Z. Mo, Minimization algorithm of fuzzy finite automata, Fuzzy Sets Syst. 141 (2004) 439–448.
[22] M. Ćirić, J. Ignjatović, S. Bogdanović, Fuzzy equivalence relations and their equivalence classes, Fuzzy Sets Syst. 158 (2007) 1295–1313.
[23] M. Ćirić, J. Ignjatović, S. Bogdanović, Uniform fuzzy relations and fuzzy mappings, Fuzzy Sets Syst. 160 (2009) 1054–1081.
[24] M. Ćirić, A. Stamenković, J. Ignjatović, T. Petković, Factorization of fuzzy automata, in: E. Csuhaj-Varju, and Z. Ésik, (Eds.), FCT 2007, Lecture Notes in Computer Science, vol. 4639, 2007, pp. 213–225.
[25] M. Ćirić, A. Stamenković, J. Ignjatović, T. Petković, Fuzzy relation equations and reduction of fuzzy automata, J. Comput. Syst. Sci. 76 (2010) 609–633.
[26] B. De Baets, E. Kerre, The cutting of compositions, Fuzzy Sets Syst. 62 (1994) 295–309.
[27] M. De Cock, E.E. Kerre, Fuzzy modifiers based on fuzzy relations, Inf. Sci. 160 (2004) 173–199.
[28] A. Dovier, C. Piazza, A. Policriti, An efficient algorithm for computing bisimulation equivalence, Theor. Comput. Sci. 311 (2004) 221–256.
[29] R. Gentilini, C. Piazza, A. Policriti, From bisimulation to simulation: coarsest partition problems, J. Autom. Reasoning 31 (2003) 73–103.
[30] U. Höhle, Commutative, residuated ℓ-monoids, in: U. Höhle, E.P. Klement (Eds.), Non-Classical Logics and Their Applications to Fuzzy Subsets, Kluwer Academic Publishers, Boston, Dordrecht, 1995, pp. 53–106.
[31] B. Hrúz, M.C. Zhou, Modeling and Control of Discrete-Event Dynamical Systems: with Petri Nets and Other Tools, Springer, 2007.
[32] J. Ignjatović, M. Ćirić, S. Bogdanović, Determinization of fuzzy automata with membership values in complete residuated lattices, Inf. Sci. 178 (2008) 164–180.
[33] J. Ignjatović, M. Ćirić, S. Bogdanović, Fuzzy homomorphisms of algebras, Fuzzy Sets Syst. 160 (2009) 2345–2365.
[34] J. Ignjatović, M. Ćirić, S. Bogdanović, T. Petković, Myhill–Nerode type theory for fuzzy languages and automata, Fuzzy Sets Syst. 161 (2010) 1288–1324.
[35] L. Ilie, S. Yu, Constructing NFAs by optimal use of positions in regular expressions, in: A. Apostolico, M. Takeda (Eds.), CPM 2002, Lecture Notes in Computer Science, vol. 2373, Springer, Berlin, 2002, pp. 279–288.
[36] L. Ilie, S. Yu, Algorithms for computing small NFAs, in: K. Diks, et al., (Eds.), MFCS 2002, Lecture Notes in Computer Science, vol. 2420, 2002, pp. 328–340.
[37] L. Ilie, S. Yu, Reducing NFAs by invariant equivalences, Theor. Comput. Sci. 306 (2003) 373–390.
[38] L. Ilie, S. Yu, Follow automata, Inf. Comput. 186 (2003) 140–162.
[39] L. Ilie, G. Navarro, S. Yu, On NFA reductions, in: J. Karhumäki et al., (Eds.), Theory is Forever, Lecture Notes in Computer Science, vol. 3113, vol. 2004, pp. 112–124.
[40] L. Ilie, R. Solis-Oba, S. Yu, Reducing the size of NFAs by using equivalences and preorders, in: A. Apostolico, M. Crochemore, K. Park, (Eds.), CPM 2005, Lecture Notes in Computer Science, vol. 3537, 2005, pp. 310–321.
[41] T. Jiang, B. Ravikumar, Minimal NFA problems are hard, SIAM J. Comput. 22 (6) (1993) 1117–1141.
[42] T. Kameda, P. Weiner, On the state minimization of nondeterministic finite automata, IEEE Trans. Comput. C-19 (7) (1970) 617–627.
[43] P.C. Kannellakis, S.A. Smolka, CCS expressions, finite state processes, and three problems of equivalence, Inf. Comput. 86 (1990) 43–68.
[44] E. Kilic, Diagnosability of fuzzy discrete event systems, Inf. Sci. 178 (2008) 858–870.
[45] G. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic, Prentice-Hall PTR, 1995.
[46] H. Lei, Y.M. Li, Minimization of states in automata theory based on finite lattice-ordered monoids, Inf. Sci. 177 (2007) 1413–1421.
[47] Y.M. Li, W. Pedrycz, Fuzzy finite automata and fuzzy regular expressions with membership values in lattice ordered monoids, Fuzzy Sets Syst. 156 (2005) 68–92.
[48] F. Lin, H. Ying, Fuzzy discrete event systems and their observability, in: Proceedings of the 2001 IFSA/NAFIP Conference, 2001, pp. 1271–1276.
[49] F. Lin, H. Ying, Modeling and control of fuzzy discrete event systems, IEEE Trans. Man Syst. Cybern. – Part B 32 (2002) 408–415.
[50] F. Lin, H. Ying, R.D. MacArthur, J.A. Cohn, D. Barth-Jones, L.R. Crane, Decision making in fuzzy discrete event systems, Inf. Sci. 177 (2007) 3749–3763.
[51] F. Liu, D.W. Qiu, Decentralized supervisory control of fuzzy discrete event systems, Eur. J. Control 3 (2008) 234–243.
[52] F. Liu, D.W. Qiu, Diagnosability of fuzzy discrete event systems: a fuzzy approach, IEEE Trans. Fuzzy Syst. 17 (2) (2009) 372–384.
[53] J.P. Liu, Y.M. Li, The relationship of controllability between classical and fuzzy discrete-event systems, Inf. Sci. 178 (2008) 4142–4151.
[54] N. Lynch, F. Vaandrager, Forward and backward simulations: Part I. Untimed systems, Inf. Comput. 121 (1995) 214–233.
[55] D.S. Malik, J.N. Mordeson, M.K. Sen, Minimization of fuzzy finite automata, Inf. Sci. 113 (1999) 323–330.
[56] R. Malik, D. Streader, S. Reeves, fair testing revisited: a process-algebraic characterisation of conflicts, in: F. Wang, (Ed.), ATVA 2004, Lecture Notes in Computer Science, vol. 3299, 2004, pp. 120–134.
[57] B.F. Melnikov, A new algorithm of the state-minimization for the nondeterministic finite automata, Korean J. Comput. Appl. Math. 6 (2) (1999) 277–290.
[58] B.F. Melnikov, Once more about the state-minimization of the nondeterministic finite automata, Korean J. Comput. Appl. Math. 7 (3) (2000) 655–662.
[59] R. Milner, A calculus of communicating systems, in: G. Goos, J. Hartmanis (Eds.), Lecture Notes in Computer Science, vol. 92, Springer, 1980.
[60] R. Milner, Communication and Concurrency, Prentice-Hall International, 1989.

[61] R. Milner, Communicating and Mobile Systems: The π-Calculus, Cambridge University Press, Cambridge, 1999.
[62] J.N. Mordeson, D.S. Malik, Fuzzy Automata and Languages: Theory and Applications, Chapman & Hall/CRC, Boca Raton, London, 2002.
[63] R. Paige, R.E. Tarjan, Three partition refinement algorithms, SIAM J. Comput. 16 (6) (1987) 973–989.
[64] D. Park, Concurrency and automata on infinite sequences, in: P. Deussen (Ed.), Proc. 5th GI Conf, Lecture Notes in Computer Science, vol. 104, Springer-Verlag, Karlsruhe, Germany, 1981, pp. 167–183.
[65] K. Peeva, Finite L-fuzzy machines, Fuzzy Sets Syst. 141 (2004) 415–437.
[66] I. Perfilieva, Fuzzy function as an approximate solution to a system of fuzzy relation equations, Fuzzy Sets Syst. 147 (2004) 363–383.
[67] I. Perfilieva, S. Gottwald, Solvability and approximate solvability of fuzzy relation equations, Int. J. Gen. Syst. 32 (2003) 361–372.
[68] I. Perfilieva, V. Novák, System of fuzzy relation equations as a continuous model of IF-THEN rules, Inf. Sci. 177 (2007) 3218–3227.
[69] T. Petković, Congruences and homomorphisms of fuzzy automata, Fuzzy Sets Syst. 157 (2006) 444–458.
[70] D.W. Qiu, Automata theory based on completed residuated lattice-valued logic (I), Sci. China Ser. F 44 (6) (2001) 419–429.
[71] D.W. Qiu, Automata theory based on completed residuated lattice-valued logic (II), Sci. China Ser. F 45 (6) (2002) 442–452.
[72] D.W. Qiu, Characterizations of fuzzy finite automata, Fuzzy Sets Syst. 141 (2004) 391–414.
[73] D.W. Qiu, Supervisory control of fuzzy discrete event systems: a formal approach, IEEE Trans. Syst. Man Cybern – Part B 35 (2005) 72–88.
[74] D.W. Qiu, Pumping lemma in automata theory based on complete residuated lattice-valued logic: a note, Fuzzy Sets Syst. 157 (2006) 2128–2138.
[75] D.W. Qiu, F.C. Liu, Fuzzy discrete-event systems under fuzzy observability and a test algorithm, IEEE Trans. Fuzzy Syst. 17 (3) (2009) 578–589.
[76] F. Ranzato, F. Tapparo, Generalizing the PaigeTarjan algorithm by abstract interpretation, Inf. Comput. 206 (2008) 620–651.
[77] M. Roggenbach, M. Majster-Cederbaum, Towards a unified view of bisimulation: a comparative study, Theor. Comput. Sci. 238 (2000) 81–130.
[78] E. Sanchez, Resolution of composite fuzzy relation equations, Inf. Control 30 (1976) 38–48.
[79] D. Sangiorgi, On the Origins of Bisimulation, Coinduction, and Fixed Points, Technical Report UBLCS-2007-24, Department of Computer Science, University of Bologna, 2007.
[80] H. Sengoku, Minimization of Nondeterministic Finite Automata, Master Thesis, Kyoto University, 1992.
[81] L.H. Wu, D.W. Qiu, Automata theory based on complete residuated lattice-valued logic: reduction and minimization, Fuzzy Sets Syst. 161 (2010) 1635–1656.
[82] H. Xing, D.W. Qiu, Pumping lemma in context-free grammar theory based on complete residuated lattice-valued logic, Fuzzy Sets Syst. 160 (2009) 1141–1151.
[83] H. Xing, D.W. Qiu, Automata theory based on complete residuated lattice-valued logic: a categorical approach, Fuzzy Sets Syst. 160 (2009) 2416–2428.
[84] H. Xing, D.W. Qiu, F.C. Liu, Automata theory based on complete residuated lattice-valued logic: pushdown automata, Fuzzy Sets Syst. 160 (2009) 1125–1140.
[85] H. Xing, D.W. Qiu, F.C. Liu, Z.J. Fan, Equivalence in automata theory based on complete residuated lattice-valued logic, Fuzzy Sets Syst. 158 (2007) 1407–1422.
[86] S. Yu, Regular languages, in: G. Rozenberg, A. Salomaa (Eds.), Handbook of Formal Languages, vol. 1, Springer-Verlag, Berlin - Heidelberg, 1997, pp. 41–110.
[87] L.A. Zadeh, Toward a generalized theory of uncertainty (GTU) – an outline, Inf. Sci. 172 (2005) 1–40.
[88] L.A. Zadeh, Is there a need for fuzzy logic?, Inf Sci. 178 (2008) 2751–2779.