

ONE DETERMINISTIC-COUNTER AUTOMATA

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ABSTRACT. We introduce one deterministic-counter automata (ODCA), which are one-counter automata where all runs labelled by a given word have the same counter effect, a property we call counter-determinacy. ODCAs are an extension of visibly one-counter automata - one-counter automata (OCA) where the input alphabet determines the actions on the counter. They are a natural way to introduce non-determinism/weights to OCAs while maintaining the decidability of crucial problems, that are undecidable on general OCAs. For example, the equivalence problem is decidable for deterministic OCAs whereas it is undecidable for non-deterministic OCAs. We consider both non-deterministic and weighted ODCAs. This work shows that the equivalence problem is decidable in polynomial time for weighted ODCAs over a field and polynomial space for non-deterministic ODCAs. As a corollary, we get that the regularity problem, i.e., the problem of checking whether an input weighted ODCA is equivalent to some weighted automaton, is also in polynomial time. Furthermore, we show that the covering and coverable equivalence problems for uninitialised weighted ODCAs are decidable in polynomial time.

We also introduce a few reachability problems that are of independent interest and show that they are in P. These reachability problems later help in solving the equivalence problem.

INTRODUCTION

Visibly pushdown automata (VPDA) was introduced by Alur and Madhusudan in 2004 [2]. They have received a lot of attention as they are a strict subclass of pushdown automata, suitable for program analysis. VPDAs enjoy tractable decidable properties, which are undecidable in the general case. The visibly restriction, in essence, is that the stack operations are *input-driven*, i.e., only depends on the letter read.

In this paper, we investigate a relaxation in the visibly constraint on one-counter automata (OCA): the counter actions are no longer input-driven, but are deterministic. We could summarise this as: “any run on a given word has a fixed counter effect”. We give a model satisfying this new restriction, which includes all visibly OCA.

Syntactically, one deterministic-counter automata (ODCA) contain two parts:

- (1) Counter structure: This is a deterministic OCA without epsilon transitions. The transitions are deterministic, and the state transitions depend only on the current state, the alphabet, and whether the counter is zero.
- (2) Finite state machine: This machine has finite states and no counters. It can be deterministic, non-deterministic, or weighted. The transitions of

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this machine depend on its current state, current counter structure state, input alphabet, and whether the counter value is zero.

An ODCA will be called deterministic, non-deterministic, or weighted depending on the type of the finite state machine. One can observe that the class of deterministic OCA and the class of visibly OCA are specific cases of ODCAs. In a visibly OCA, the input alphabet determines the counter structure.

An ODCA represents a function that maps words (over a finite alphabet) to a weight. The run of a word over an ODCA determines its accepting weight. In the case of weighted ODCA, the weights come from a field, and in the case of deterministic and non-deterministic ODCA, the weights come from the boolean semiring. Hence a deterministic or non-deterministic ODCA represents a language, which is the set of all words whose weight is 1.

A non-deterministic ODCA can have a succinct representation compared to the deterministic ODCA recognising the same language. For example, for any $k \in \mathbb{N}$, let \mathcal{L}_k denote the language $\{a^n(b+c)^mb(b+c)^k \mid m, n \in \mathbb{N} \text{ and } m > n\}$. The non-deterministic ODCA recognising the language \mathcal{L}_k guesses whether a b encountered after reading the string $a^n(b+c)^{n+1}$ for some $n \in \mathbb{N}$ is at the k^{th} position from the end of the string. An example of a non-deterministic ODCA that recognises \mathcal{L}_2 is shown in Figure 1. The deterministic ODCA that recognises the same language will have to check whether every b encountered after reading the string $a^n(b+c)^{n+1}$ is at the k^{th} position from the end. This will require an additional 2^k states.

Our results. Two ODCAs are *equivalent* if the functions they represent are equal. Observe that deterministic real-time OCAs are deterministic ODCAs. We also note that deterministic ODCAs are deterministic real-time OCAs. Böhm et al. [5] proved that the equivalence of deterministic OCA is in non-deterministic log space. We show that a non-deterministic ODCA is equivalent to an exponentially sized deterministic ODCA. Therefore, unlike non-deterministic OCAs, the equivalence of two non-deterministic ODCAs is decidable and can be determined by a PSPACE machine.

This paper also presents a polynomial time algorithm for deciding the equivalence problem of two weighted ODCAs. If the two ODCAs are non-equivalent, we output a word (whose length is polynomial in the size of the two ODCAs) that the two weighted ODCAs accept with different weights. We dedicate Section 4 to prove Theorem 1.

Theorem 1. *There exists a polynomial time algorithm that decides if two weighted ODCAs are equivalent and outputs a word that distinguishes them, if they are non-equivalent.*

To solve the equivalence problem for weighted ODCA, we introduce a few reachability problems. These problems are also of independent interest. The *complement to vector space (co-VS) reachability problem* takes a weighted ODCA, a vector space, and an initial configuration as input. It asks whether it is possible, starting from the initial configuration, to reach a configuration in the complement of the given vector space. We develop novel ideas to show that the unary (resp. binary) co-VS reachability problem is in P (resp. NP). Let us call a word a *witness* if the run of the word ‘reaches’ a configuration desired by the reachability problem. Through a series of lemmas, we identify two interesting properties of witnesses.

- (1) The witnesses satisfy a small model property - a witness that is longer than a polynomial can be ‘cut’ to get a shorter witness. We remove parts of a

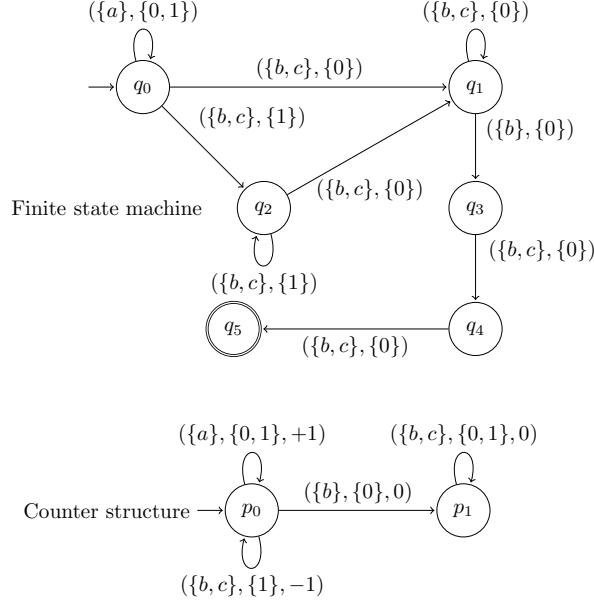


FIGURE 1. The figure shows a non-deterministic ODCA recognising the language $\mathcal{L} = \{a^n(b+c)^mb(b+c)^2 \mid m, n \in \mathbb{N} \text{ and } m > n\}$. Let $S \subseteq \Sigma, R \subseteq \{0, 1\}$ and $T \in \{-1, 0, +1\}$ are non-empty sets. For $i, j \in \mathbb{N}$, if a transition from q_i to q_j is labelled (S, R) and $(s, r) \in S \times R$, then there is a transition from q_i to q_j on reading the symbol s . The current counter value should be 0 if $r = 0$ and greater than 0 if $r = 1$. Similarly, if a transition from p_i to p_j is labelled (S, R, T) and $(s, r, t) \in S \times R \times \{T\}$, then there is a transition from p_i to p_j on reading the symbol s with counter action t . The current counter value should be 0 if $r = 0$ and greater than 0 if $r = 1$.

long run and join the remaining portions. The challenge is identifying cuts that preserve the counter actions during the run.

- (2) The lexicographically smallest word that witnesses the reachability is of the form $uy_1^{r_1}vy_2^{r_2}w$ where u, v, w, y_1 and y_2 are ‘small’ words and $r_1, r_2 \in \mathbb{N}$.

The reachability problems, along with the ideas developed in the context of real-time OCA by Böhm et al. [3] (also see [4] [6]), and Valiant, Paterson [21] help us solve the equivalence problem for ODCAs.

Next, we consider the regularity problem - the problem of deciding whether a weighted ODCA is equivalent to some weighted automata. In Theorem 36, we show that regularity of ODCA is decidable in polynomial time. This is done by showing the existence of infinitely many equivalence classes by “pumping up” some parts of a run.

Next, we look at uninitialised ODCAs - an ODCA without initial finite state distribution and initial counter state. We show that the “equivalence” problem for uninitialised ODCAs are in polynomial time.

Related work. Extensive studies have been conducted on weighted automata with weights from semirings. Tzeng [20] gave a polynomial time algorithm to decide the equivalence of two probabilistic automata. The result has been extended to weighted automata with weights over a field. On the other hand, the problem is undecidable if the weights are over the semiring $(\mathbb{N}, \min, +)$ [14]. Unlike the extensive literature on weighted automata, the study on weighted versions of pushdown or one-counter machines is limited [11] [12] [16]. One of the major bottlenecks is the undecidability of many interesting problems.

Probabilistic Pushdown Automata (PPDA) is equivalent to probabilistic recursive state machines (RSMs) or recursive Markov chains [8] [15]. These models have been studied extensively for the analysis and model checking of procedural programs [9]. PPDA can model probabilistic sequential programs with recursive procedure calls. They are also a generalisation of stochastic context-free grammars [1] used in natural language processing, molecular biology, and many variants of one-dimensional random walks [7]. Kucera et al. [16] have looked at model-checking of probabilistic pushdown systems and Brázdil et al. [8] studied temporal properties of probabilistic pushdown automata. The equivalence problem of PPDA was examined by Forejt et al. [10] and they showed that it is equivalent to the multiplicity equivalence of context-free grammars. The decidability of the latter problem is open. Kiefer et al. [13] show that the equivalence of probabilistic VPDA is logspace equivalent to polynomial identity testing. The later problem is known to be in coRP .

The bisimilarity problem of probabilistic VPDA (resp. probabilistic OCA) was shown to be EXPTIME-complete (resp. PSPACE-complete) by Forejt et al. [11]. They also proved the decidability of the bisimilarity problem of PPDA. Etessami et al. [9] show that probabilistic OCA and Quasi-Birth-Death processes are equivalent.

Moving on to the non-weighted models, for non-deterministic pushdown automata the equivalence problem is known to be undecidable. On the other hand, from the seminal result by Sénizergues [17], we know that the equivalence problem for deterministic pushdown automata is decidable. The lower bound, though, is primitive recursive [18]. The equivalence problem for deterministic one-counter automata (with and without ϵ transitions) is decidable in polynomial time. In fact, similar to that of deterministic finite automata, the problem is NL-complete [5].

Outline of the paper. The rest of this paper is organised as follows. Section 1 contains the basic definitions and some lemmas from linear algebra. We also give a formal definition of ODCA. In Section 2, we look at the special cases of non-deterministic and deterministic ODCAs and show the decidability of their equivalence problems. Section 3 analyses a few reachability problems of weighted ODCAs. In Section 4, we prove Theorem 1 and show that the equivalence of weighted ODCAs is in polynomial time. Section 5, gives a polynomial time algorithm for the regularity problem of weighted ODCA, and in Section 6, we prove that the covering problem for weighted ODCA is in polynomial time. Section 7 gives a short conclusion.

1. PRELIMINARIES

1.1. Basic notations. An alphabet is a non-empty finite set of letters. In this paper, we denote the alphabet by Σ . We use Σ^* to denote the set of finite length words over Σ , and for all $l \in \mathbb{N}$, we use $\Sigma^{\leq l}$ (resp. Σ^l) to denote the set of words over Σ having length less than or equal to l (resp. exactly equal to l). Given a

word $w \in \Sigma^*$, we use $|w|$ to denote the length of the word w . We use the notation $[i, j]$ to denote the interval $\{i, i+1, \dots, j\}$. We say that a word $u = a_1 \cdots a_k$ is a subword of a word w , if $w = u_0 a_1 u_1 a_2 \cdots a_k u_k$, where $a_i \in \Sigma$, $u_j \in \Sigma^*$ for all $i \in [1, k]$ and $j \in [0, k]$. We call u a proper subword of w if $u \neq w$. We say that a word u is a prefix of a word w if there exists $v \in \Sigma^*$ such that $w = uv$. Given a word $w = a_0 \cdots a_n$, we write $w[i \cdots j]$ to denote the factor $a_i \cdots a_j$. Given $d \in \mathbb{N}$, $\text{sign}(d) = 0$ if $d = 0$ and is 1 otherwise.

1.2. Linear algebra. A field $\mathcal{F} = (S, +, \cdot, 0, 1)$ is a set S with operations $+$ and \cdot and distinguished elements 0 and 1 such that $(S, +, 0)$ and $(S, \cdot, 1)$ are groups. In this paper, we use $\mathbf{x}, \mathbf{y}, \mathbf{z}$ to denote row vectors over a field \mathcal{F} , s, t, r to denote elements in a field \mathcal{F} and $\mathbb{A}, \mathbb{B}, \mathbb{M}$ to denote matrices over a field \mathcal{F} . We use \mathcal{U}, \mathcal{V} to denote vector spaces. We recall the following facts.

Lemma 2 ([19]). *The following are true for a field \mathcal{F} .*

- (1) *For any set X of n vectors in \mathcal{F}^m with $n > m$, there exists a vector $\mathbf{x} \in X$ that is a linear combination of the other vectors in X .*
- (2) *Given a set B of n vectors in \mathcal{F}^m and a vector $\mathbf{x} \in \mathcal{F}^m$, we can check if \mathbf{x} is a linear combination of vectors in B in time polynomial in m and n .*
- (3) *Let $k, r \in \mathbb{N}$ and $\mathbb{M} \in \mathcal{F}^{k \times k}$. The matrix \mathbb{M}^r can be computed in time polynomial in k and $\log(r)$.* \square

The following properties of vector spaces are important.

Lemma 3. *Let \mathcal{V} be a vector space, $k \in \mathbb{N}$ and for all $r \in [0, k]$ $\mathbf{z}_r \in \mathcal{F}^k$ and $\mathbb{M}_r \in \mathcal{F}^{k \times k}$. Then, there exists an $i \in [1, k]$ such that the following conditions are true:*

- (1) *\mathbf{z}_i is a linear combination of $\mathbf{z}_0, \dots, \mathbf{z}_{i-1}$, and*
- (2) *if $\mathbf{z}_i \mathbb{M}_i \notin \mathcal{V}$, then there exists $j < i$ such that $\mathbf{z}_j \mathbb{M}_i \notin \mathcal{V}$.*

Proof. Let $k \in \mathbb{N}, r \in [0, k]$, $\mathbf{z}_r \in \mathcal{F}^k, \mathbb{M}_r \in \mathcal{F}^{k \times k}$ be matrices over \mathcal{F} and \mathcal{V} be a vector space.

1. Consider the set $\{\mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_k\}$ of $k+1$ vectors of dimension k . It follows from Lemma 2 that there are at most k independent vectors of dimension k , and hence not all elements of the set can be independent.

2. Let $i \in [1, k]$ be such that \mathbf{z}_i is a linear combination of $\mathbf{z}_0, \dots, \mathbf{z}_{i-1}$ and $\mathbf{z}_i \mathbb{M}_i \notin \mathcal{V}$. Let us assume for contradiction that $\mathbf{z}_j \mathbb{M}_i \in \mathcal{V}$ for all $j \in [0, i-1]$. Since \mathbf{z}_i is a linear combination on $\mathbf{z}_0, \dots, \mathbf{z}_{i-1}$, there exists $s_0, \dots, s_{i-1} \in \mathcal{F}$ such that

$$\mathbf{z}_i = s_0 \cdot \mathbf{z}_0 + s_1 \cdot \mathbf{z}_1 + \cdots + s_{i-1} \cdot \mathbf{z}_{i-1}$$

Since $\mathbf{z}_i \mathbb{M}_i = \sum_{j=0}^{i-1} s_j \cdot \mathbf{z}_j \mathbb{M}_i$ and \mathcal{V} is closed under linear combinations, we get that $\mathbf{z}_i \mathbb{M}_i \in \mathcal{V}$ contradicting our initial assumption. \square

Lemma 4. *Let \mathcal{V} be a vector space, $k \in \mathbb{N}$ and for all $r \in [0, k^2]$ $\mathbb{A}_r, \mathbb{M}_r, \mathbb{B}_r \in \mathcal{F}^{k \times k}$. Then, there exists an $i \in [1, k^2]$ such that for all $\mathbf{x} \in \mathcal{F}^k$ the following conditions are true:*

- (1) *\mathbb{M}_i is a linear combination of $\mathbb{M}_0, \dots, \mathbb{M}_{i-1}$, and*
- (2) *if $\mathbf{x} \mathbb{A}_i \mathbb{M}_i \mathbb{B}_i \notin \mathcal{V}$, then there exists a $j < i$ such that $\mathbf{x} \mathbb{A}_i \mathbb{M}_j \mathbb{B}_i \notin \mathcal{V}$.*

Proof. Let $\mathbb{A}_r, \mathbb{M}_r, \mathbb{B}_r \in \mathcal{F}^{k \times k}$ for $r \in [0, k^2]$, be matrices over \mathcal{F} and \mathcal{V} a vector space.

1. Consider the set $\{\mathbb{M}_0, \mathbb{M}_1, \dots, \mathbb{M}_{k^2}\}$ of $k^2 + 1$ matrices of dimension k^2 . It follows from Lemma 2 that there are at most k^2 independent vectors of dimension k^2 , and hence not all elements of this set can be independent.

2. Let $i \in [1, k^2]$ be such that \mathbb{M}_i is a linear combination of $\mathbb{M}_0, \dots, \mathbb{M}_{i-1}$ and $\mathbf{x}\mathbb{A}_i\mathbb{M}_i\mathbb{B}_i \notin \mathcal{V}$. Since \mathbb{M}_i is dependent on $\mathbb{M}_0, \dots, \mathbb{M}_{i-1}$, we prove that there exists $j < i$ such that $\mathbf{x}\mathbb{A}_i\mathbb{M}_j\mathbb{B}_i \notin \mathcal{V}$. Let us assume for contradiction that this is not the case. Since \mathbb{M}_i is a linear combination on $\mathbb{M}_0, \dots, \mathbb{M}_{i-1}$, there exists $s_0, \dots, s_{i-1} \in \mathcal{F}$ such that

$$\mathbb{M}_i = s_0 \cdot \mathbb{M}_0 + s_1 \cdot \mathbb{M}_1 + \dots + s_{i-1} \cdot \mathbb{M}_{i-1}$$

Since $\mathbf{x}\mathbb{A}_i\mathbb{M}_j\mathbb{B}_i \in \mathcal{V}$ for all $j \in [0, i-1]$ we get that $\mathbf{x}\mathbb{A}_i\mathbb{M}_i\mathbb{B}_i = \sum_{j=0}^{i-1} s_j \cdot \mathbf{x}\mathbb{A}_i\mathbb{M}_j\mathbb{B}_i \in \mathcal{V}$, which is a contradiction. \square

Lemma 5. Let $k \in \mathbb{N}$, $\mathbb{A} \in \mathcal{F}^{k \times k}$ and $\mathcal{V} \subseteq \mathcal{F}^k$ be a vector space. Then the following set is a vector space,

$$\mathcal{U} = \{\mathbf{y} \in \mathcal{F}^k \mid \mathbf{y}\mathbb{A} \in \mathcal{V}\}.$$

Proof. To prove that \mathcal{U} is a vector space, it suffices to show that it is closed under vector addition and scalar multiplication. First, we prove that \mathcal{U} is closed under vector addition. Let $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{U}$ be two vectors, since $\mathbf{z}_1\mathbb{A}, \mathbf{z}_2\mathbb{A} \in \mathcal{V}$, $(\mathbf{z}_1 + \mathbf{z}_2)\mathbb{A} = \mathbf{z}_1\mathbb{A} + \mathbf{z}_2\mathbb{A} \in \mathcal{V}$. Therefore, $\mathbf{z}_1 + \mathbf{z}_2 \in \mathcal{U}$. Now we prove that \mathcal{U} is closed under scalar multiplication. For any vector $\mathbf{z}_1 \in \mathcal{U}$, we know that $\mathbf{z}_1\mathbb{A} \in \mathcal{V}$. Since \mathcal{V} is a vector space, for any scalar $r \in \mathcal{F}$, $(r \cdot \mathbf{z}_1)\mathbb{A} \in \mathcal{V}$, and therefore $r \cdot \mathbf{z}_1 \in \mathcal{U}$. This concludes the proof. \square

In particular, the above lemma holds for the vector space $\{\mathbf{0} \in \mathcal{F}^k\}$.

1.3. One deterministic-counter automata. A one deterministic-counter automata (ODCA) consists of two parts, a finite state machine, which is a weighted automaton over a semiring, and a counter structure, which is a deterministic OCA. An ODCA is defined as follows:

Definition 6. A one deterministic-counter automata (ODCA), \mathcal{A} over an alphabet Σ and a semiring \mathcal{S} is as defined below:

$$\mathcal{A} = (C, \delta, p_0; Q, \lambda, \Delta, \eta)$$

- C is a non-empty finite set of counter states.
- $\delta : C \times \Sigma \times \{0, 1\} \rightarrow C \times \{-1, 0, +1\}$ is the deterministic counter transition.
- $p_0 \in C$ is the start state for counter transition.
- Q is a non-empty finite set of states of the finite state machine. We assume $|C| = |Q|$ and use \mathbf{K} to denote $|Q|$.
- $\lambda \in \mathcal{S}^{\mathbf{K}}$ is the initial distribution where the i^{th} component of λ indicates the initial weight on state $q_i \in Q$.
- $\Delta : C \times \Sigma \times \{0, 1\} \rightarrow \mathcal{S}^{\mathbf{K} \times \mathbf{K}}$ gives the transition matrix for all $p \in C$, $a \in \Sigma$ and $d \in \{0, 1\}$. The component in the i^{th} row and j^{th} column of $\Delta(p, a, d)$ denotes the weight on the transition from state $q_i \in Q$ to state $q_j \in Q$ on reading symbol a from counter state p and counter value n with $\text{sign}(n) = d$.

- $\boldsymbol{\eta} \in \mathcal{S}^K$ is the final distribution, where the i^{th} component of $\boldsymbol{\eta}$ indicates the output weight on state $q_i \in Q$.

A configuration \mathbf{c} of an ODCA is of the form $(\mathbf{x}_c, p_c, n_c) \in \mathcal{S}^K \times C \times \mathbb{N}$. The configuration $(\boldsymbol{\lambda}, p_0, 0)$ is the initial configuration of \mathcal{A} . A *transition* is a tuple $\tau = (\iota_\tau, d_\tau, a_\tau, \mathbf{ce}_\tau, A_\tau, \theta_\tau)$ where $\iota_\tau, \theta_\tau \in C$ are counter states, $d_\tau \in \{0, 1\}$ is to denote whether the current counter value is zero or not, $a_\tau \in \Sigma$, $\mathbf{ce}_\tau \in \{-1, 0, 1\}$ is the *counter-effect*, $\mathbb{A}_\tau \in \mathcal{S}^{K \times K}$ such that $\Delta(\iota_\tau, a_\tau, d_\tau) = \mathbb{A}_\tau$, and $\delta(c\iota_\tau, a_\tau, d_\tau) = \theta_\tau$.

Given a transition τ and a configuration \mathbf{c} , we denote the application of τ to \mathbf{c} as $\tau(\mathbf{c}) = (\mathbf{x}_c \mathbb{A}_\tau, \theta_\tau, n_c + \mathbf{ce}_\tau)$ if $p_c = \iota_\tau$ and $d_\tau = 0$ if and only if $n_c = 0$, and is undefined otherwise. Note that the counter values always stay positive, implying that we cannot perform a decrement operation on the counter from a configuration with a counter value of zero.

Given a sequence of transitions $T = \tau_0 \cdots \tau_{\ell-1}$, we denote $\mathbf{word}(T) = a_{\tau_0} \cdots a_{\tau_{\ell-1}}$ the word labelling it, $\mathbf{we}(T) = \mathbb{A}_{\tau_0} \cdots \mathbb{A}_{\tau_{\ell-1}}$ its weight-effect matrix, and $\mathbf{ce}(T) = \mathbf{ce}_{\tau_0} + \cdots + \mathbf{ce}_{\tau_{\ell-1}}$ its counter-effect. For all $0 \leq i < j \leq |\ell - 1|$, we use $T_{i \dots j}$ to denote the sequence of transitions $\tau_i \cdots \tau_j$ and $|T|$ to denote ℓ .

We call a sequence of transition $T = \tau_0 \cdots \tau_\ell$ *floating* if for all $i \in [0, \ell - 1]$ $d_{\tau_i} = 1$ and *non-floating* otherwise. We denote $\min_{\mathbf{ce}}(T) = \min_i(\mathbf{ce}(\tau_0 \cdots \tau_i))$ the minimal effect of its prefixes and call it its *decrease* and $\max_{\mathbf{ce}}(T) = \max_i(\mathbf{ce}(\tau_0 \cdots \tau_i))$ is the maximal effect of its prefixes. We say that the sequence of transitions T is *valid* if for every $i \in [0, \ell - 2]$, $\theta_{\tau_i} = \iota_{\tau_{i+1}}$. We will only consider valid sequences of transitions.

A *run* π is an alternate sequence of configurations and transitions denoted as $\pi = \mathbf{c}_0 \tau_0 \mathbf{c}_1 \cdots \tau_{\ell-1} \mathbf{c}_\ell$ such that for every i , $\mathbf{c}_{i+1} = \tau_i(\mathbf{c}_i)$. Given a sequence of transition T and a configuration \mathbf{c} , we denote $T(\mathbf{c})$ the run obtained by applying T to \mathbf{c} sequentially (if it is defined). The word labelling it, its length, weight effect, and counter-effect are those of its underlying sequence of transitions.

Observe that, for a valid floating sequence of transitions, $T(\mathbf{c})$ is defined if and only if $n_c > -\min_{\mathbf{ce}}(T)$, and for a valid non-floating sequence of transitions, $T(\mathbf{c})$ is defined if and only if $n_c = -\min_{\mathbf{ce}}(T)$ and for every i , $d_{\tau_i} = 0$ if and only if $\mathbf{ce}(\tau_0 \cdots \tau_{i-1}) = \min_{\mathbf{ce}}(T)$. In particular, observe that if a valid floating sequence of transition T is applicable to a configuration (\mathbf{x}_c, p_c, n_c) , then for every $n' \geq n_c$ and vector $\mathbf{x}' \in \mathcal{S}^K$, it is applicable to (\mathbf{x}', p_c, n') .

For any word w , there is at most one run labelled by w starting from a given configuration \mathbf{c}_0 . We denote this run $\pi(w, \mathbf{c}_0)$. A run $\pi(w, \mathbf{c}_0) = \mathbf{c}_0 \tau_0 \mathbf{c}_1 \cdots \tau_{\ell-1} \mathbf{c}_\ell$ is also represented as $\mathbf{c}_0 \xrightarrow{w} \mathbf{c}_\ell$. We use the notation $\mathbf{c}_0 \rightarrow^* \mathbf{c}_\ell$ to denote the existence of some word w such that $\mathbf{c}_0 \xrightarrow{w} \mathbf{c}_\ell$. The counter effect of a word w on a floating run $\mathbf{c}_0 \xrightarrow{w} \mathbf{c}_\ell$ is $n_{\mathbf{c}_\ell} - n_{\mathbf{c}_0}$. The weight with which a word w is accepted by \mathcal{A} along the run $\mathbf{c}_0 \xrightarrow{w} \mathbf{c}_\ell$ is denoted by $f_{\mathcal{A}}(w, \mathbf{c}_0) = \boldsymbol{\lambda} \mathbf{we}(\pi(w, \mathbf{c}_0)) \boldsymbol{\eta}^\top$. We use the notation $f_{\mathcal{A}}(w)$ to denote $f_{\mathcal{A}}(w, (\boldsymbol{\lambda}, p_0, 0))$.

Let \mathcal{A} and \mathcal{B} be two ODCA's. Consider the configurations \mathbf{c} of \mathcal{A} and \mathbf{d} of \mathcal{B} . We say that $\mathbf{c} \equiv_l \mathbf{d}$ if and only if for all $w \in \Sigma^{\leq l}$, $f_{\mathcal{A}}(w, \mathbf{c}) = f_{\mathcal{B}}(w, \mathbf{d})$ otherwise $\mathbf{c} \not\equiv_l \mathbf{d}$. We say that the configurations \mathbf{c} and \mathbf{d} are equivalent if and only if $\mathbf{c} \equiv_l \mathbf{d}$ for all $l \in \mathbb{N}$ and we denote this by $\mathbf{c} \equiv \mathbf{d}$. We say that \mathcal{A} and \mathcal{B} are equivalent if for all $w \in \Sigma^*$, $f_{\mathcal{A}}(w) = f_{\mathcal{B}}(w)$.

If the ODCA is defined over a semiring which is also a field, then we call the model a weighted ODCA, and if it is the boolean semiring then we call it a non-deterministic/deterministic ODCA. Note that the equivalence problem of ODCA

defined over an arbitrary semiring is undecidable because of the undecidability of equivalence of weighted automata over semirings. The class of weighted ODCA includes deterministic OCA, visibly weighted OCA, and deterministic weighted OCA. We have to bring in known examples from literature. Have others considered weighted oca? Does our model capture it? Also, note that the δ need not be a function and A is always a function. In that case, the control states are non-deterministic. Like in the previous section, we can determinise it (with an exponential blow-up). The question is, Is equivalence checking in this model (as well as the weighted case) in PTIME?

Given a weighted ODCA \mathcal{A} over the alphabet Σ and a field \mathcal{F} , we define its M -unfolding weighted automata \mathcal{A}^M as a finite state weighted automaton that recognises the same function as \mathcal{A} for all runs where the counter value does not exceed M . A formal definition is given below.

Definition 7 (M -unfolding weighted automata). *Let $\mathcal{A} = (C, \delta, p_0; Q, \lambda, \Delta, \eta)$ be a weighted ODCA over the alphabet Σ and a field \mathcal{F} , let $K = |Q| = |C|$. For a given $M \in \mathbb{N}$, we define an M -unfolding weighted automata \mathcal{A}^M of \mathcal{A} as follows, $\mathcal{A}^M = (C', \delta', p'_0; Q', \lambda', \Delta', \eta'_F)$ where,*

- $C' = C \times [0, M]$ is the finite set of counter states.
- $\delta' : C' \times \Sigma \rightarrow C'$ is the deterministic counter transition. Let $p, q \in C, m \in \mathbb{N}, a \in \Sigma$ and $d \in \{-1, 0, +1\}$. $\delta'((p, m), a) = (q, m + d)$, if $\delta(p, a, \text{sign}(m)) = (q, d)$.
- $p'_0 = (p_0, 0)$ is the initial counter state.
- $Q' = Q \times [0, M]$ is the finite set of states.
- $\lambda' \in \mathcal{F}^{|Q'|}$ is the initial distribution.

$$\lambda'[i] = \begin{cases} \lambda[i], & \text{if } i < K \\ 0, & \text{otherwise} \end{cases}$$

- $\Delta' : C' \times \Sigma \rightarrow \mathcal{F}^{|Q'| \times |Q'|}$ gives the transition matrix. For $i, j \in |Q'|, p \in C, m \in \mathbb{N}$ and $a \in \Sigma$,

$$\Delta'((p, m), a)[i][j] = \begin{cases} \Delta(p, a, 0)[i][j], & \text{if } i, j < K \\ \Delta(p, a, 1)[i \bmod K][j \bmod K], & \text{if } \frac{i}{K} = \frac{j}{K} \\ 0, & \text{otherwise} \end{cases}$$

- $\eta'_F \in \mathcal{F}^{|Q'|}$ is the final distribution.

$$\eta'_F[i] = \eta[i \bmod K]$$

An uninitialised weighted ODCA \mathcal{A} is a weighted ODCA without an initial counter state and initial distribution. Formally, $\mathcal{A} = (C, \delta; Q, \Delta, \eta)$. Given an uninitialised weighted ODCA \mathcal{A} and an initial configuration $c_0 = (\mathbf{x}, p, 0)$, we define the weighted ODCA $\mathcal{A}\langle c_0 \rangle = (C, \delta, p; Q, \mathbf{x}, \Delta, \eta)$.

Weighted automata (**WA**) is a restricted form of an ODCA where the counter value is fixed at zero. The above notions of transitions, runs, acceptance, etc. are used for **WA** also. We also use the classical notion and represent weighted automata as $\mathcal{A} = (Q, \lambda, \Delta, \eta)$, without counter states.

2. NONDETERMINISTIC / DETERMINISTIC ODCA

A deterministic/non-deterministic ODCA \mathcal{A} is an ODCA over the boolean semiring $\mathcal{S} = (\{0, 1\}, \vee, \wedge)$. The language recognised by \mathcal{A} is given by $\mathcal{L}(\mathcal{A}) = \{w \mid f_{\mathcal{A}}(w) = 1\}$. We say an ODCA $\mathcal{A} = (C, \delta, p_0; Q, \lambda, \Delta, \eta)$ is *deterministic* if for every transition sequence $T = \tau_0 \cdots \tau_{\ell-1}$, the vector $\lambda \mathbf{we}(T)$ contains exactly one 1 and non-deterministic otherwise.

The following theorem is ‘analogous’ to the case of finite automata. The idea is a simple subset construction.

Theorem 8. *For every language recognised by a non-deterministic ODCA, there is a deterministic ODCA of at most exponential size that recognise it.*

Proof. Let $\mathcal{A} = (C, \delta, p_0; Q, \lambda, \Delta, \eta)$ be a non-deterministic ODCA. Given a vector $\mathbf{x} \in \mathcal{S}^k$ for some $k \in \mathbb{N}$, we define the function $\text{IsDet}: \mathcal{S}^k \rightarrow \{\text{true}, \text{false}\}$ as follows:

$$\text{IsDet}(\mathbf{x}) = \begin{cases} \text{true, if } \exists i < k \text{ s.t. } \mathbf{x}[i] = 1 \text{ and } \forall j \neq i, \mathbf{x}[j] = 0 \\ \text{false, otherwise.} \end{cases}$$

Given a transition matrix \mathbb{A} corresponding to the states Q , we define its determination $\det(\mathbb{A})$ as follows. There are rows and columns corresponding to each set in 2^Q . For any $q_i \in Q$, let $\mathcal{M}(q_i, \mathbb{A}) = \{q_j \mid \mathbb{A}[i][j] = 1\}$ be the set of all states in the row of q_i whose entries are 1. With the notation that $\det(\mathbb{A})[s][s']$ corresponds to the entry of the cell corresponding to the sets $s, s' \in 2^Q$, we let $\det(\mathbb{A})[s][s'] = 1$ if and only if $s' = \bigcup_{q \in s} \mathcal{M}(q_i, \mathbb{A})$. We claim that $\mathcal{A}_{\det} = (C, \delta, p_0; Q, \lambda, \Delta', \eta')$, with η' such that for any $S \in 2^Q, \eta'[S] = \bigvee_{s \in S} \eta[s]$ and for all $p \in C, a \in \Sigma$ and $d \in \{0, 1\}$, $\Delta'(p, a, d) = \det(\Delta(p, a, d))$ is such that it is deterministic and $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_{\det})$.

For this, for any sequence of operations $T = \tau_0 \cdots \tau_{\ell-1}$, let $\mathbf{v}_T, \mathbf{v}'_T$ be the vectors corresponding to $\lambda \mathbf{we}(T)$ in \mathcal{A} and \mathcal{A}_{\det} respectively. Then we have $\text{IsDet}(\mathbf{v}'_T) = 1$ and for any $S \in 2^Q, \mathbf{v}'_T[S] = 1$ if and only if for all $q_i \in S, \mathbf{v}_T[i] = 1$. \square

The equivalence of deterministic ODCAs can be decided in non-deterministic log space [3]. From the above theorem, it follows that a PSPACE machine can decide on the equivalence of non-deterministic ODCAs.

Theorem 9. *Equivalence of non-deterministic ODCA is in PSPACE.*

As a corollary, we get the following.

Corollary 10. *The emptiness and the universality problems of non-deterministic ODCA are in PSPACE.*

3. REACHABILITY PROBLEMS IN WEIGHTED ODCA

In this section, we examine two reachability problems of weighted ODCAs. In the subsequent section, we develop the techniques that play a key role in proving the equivalence of weighted ODCA.

A weighted ODCA $\mathcal{A} = (C, \delta, p_0; Q, \lambda, \Delta, \eta)$. Without loss of generality, assume $|C| = |Q|$ and denote $|Q|$ by K . We use $\mathcal{V} \subseteq \mathcal{F}^K$ to denote a vector space and $\overline{\mathcal{V}} = \mathcal{F}^K \setminus \mathcal{V}$ to denote the set complement of \mathcal{V} . Let $S \subseteq C$ be a subset of the set of counter states, $X \subseteq \mathbb{N}$ a set of counter values and $w \in \Sigma^*$. The notation $\mathbf{c} \xrightarrow{w} \overline{\mathcal{V}} \times S \times X$ denotes the run $\mathbf{c} \xrightarrow{w} \mathbf{d}$ where $\mathbf{d} \in \overline{\mathcal{V}} \times S \times X$. We call $z \in \Sigma^*$ a *reachability witness* for $(\mathbf{c}, \overline{\mathcal{V}}, S, X)$ if $\mathbf{c} \xrightarrow{z} \overline{\mathcal{V}} \times S \times X$. Moreover, we say z is a

minimal reachability witness for $(c, \overline{\mathcal{V}}, S, X)$ if $c \xrightarrow{z} \overline{\mathcal{V}} \times S \times X$ and for all $u \in \Sigma^*$ with $c \xrightarrow{u} \overline{\mathcal{V}} \times S \times X$, $|u| \geq |z|$. We use $c \xrightarrow{*} \overline{\mathcal{V}} \times S \times X$ to denote that there exists a word $u \in \Sigma^*$ such that $c \xrightarrow{u} \overline{\mathcal{V}} \times S \times X$.

We assume that the vector space $\mathcal{V} \subseteq \mathcal{F}^K$ will be provided by giving a suitable basis for \mathcal{V} .

(1) *co-VS reachability* problem:

INPUT: a weighted ODCA \mathcal{A} , an initial configuration c , a vector space \mathcal{V} , set of counter states S and counter value m .

OUTPUT: *Yes*, if there exists a run $c \xrightarrow{*} \overline{\mathcal{V}} \times S \times \{m\}$ in \mathcal{A} . *No*, otherwise.

(2) *co-VS coverability* problem:

INPUT: a weighted ODCA \mathcal{A} , an initial configuration c , a vector space \mathcal{V} , and set of counter states S .

OUTPUT: *Yes*, if there exists a run $c \xrightarrow{*} \overline{\mathcal{V}} \times S \times \mathbb{N}$ in \mathcal{A} . *No*, otherwise.

Note that in the second problem, the counter value of the final configuration is not part of the input. We consider the cases where the counter values of the initial configuration and the final counter value, if part of the input, are given in unary or in binary notation separately. Note that the size of the unary representation is exponentially larger than the binary representation for the same value.

First, we look at the particular case of co-VS reachability problem for weighted automata. Note that for weighted automata, the counter value is always zero. Given a weighted automata \mathcal{B} , with k states, an initial configuration \bar{c} , a vector space $\mathcal{U} \subseteq \mathcal{F}^k$ and a set of counter states S , the co-VS reachability problem asks whether there exists a run $\bar{c} \xrightarrow{*} \overline{\mathcal{U}} \times S \times \{0\}$.

Theorem 11. *There is a polynomial time algorithm that decides the co-VS reachability problem for weighted automata and outputs a minimal reachability witness if it exists.*

Proof. Tzeng [20] gives a polynomial time algorithm for the equivalence of two probabilistic automata by reducing the problem to the co-VS reachability problem where $\mathcal{V} = \{0\}$. The same algorithm can be modified to solve the general co-VS reachability problem. \square

The following lemma will help us break down both the reachability problems into smaller sub-problems.

Lemma 12. *Let $z \in \Sigma^*$ be a minimal reachability witness for $(c, \overline{\mathcal{V}}, S, X)$. Consider arbitrary $z_1 z_2 \in \Sigma^*$ such that $z = z_1 z_2$. Let d, e be configurations such that $c \xrightarrow{z_1} d \xrightarrow{z_2} e$ and $A \in \mathcal{F}^{K \times K}$ be such that $x_d A = x_e$. Then z_1 is a minimal reachability witness for $(c, \overline{\mathcal{U}}, \{p_d\}, \{n_d\})$, where $\mathcal{U} = \{y \in \mathcal{F}^K \mid yA \in \mathcal{V}\}$.*

Proof. Let $z \in \Sigma^*$ be a minimal reachability witness for $(c, \overline{\mathcal{V}}, S, X)$, d, e be configurations such that $c \xrightarrow{z_1} d \xrightarrow{z_2} e$ where $z_1, z_2 \in \Sigma^*$ with $z = z_1 z_2$ and $A \in \mathcal{F}^{K \times K}$ be such that $x_d A = x_e$. Let $\mathcal{U} = \{y \in \mathcal{F}^K \mid yA \in \mathcal{V}\}$. Assume for contradiction that there exists $z'_1 \in \Sigma^*$ smaller than z_1 and $c \xrightarrow{z'_1} f$ for some configuration $f \in \overline{\mathcal{U}} \times \{p_d\} \times \{n_d\}$. Note that for all $y \in \overline{\mathcal{U}}$, the vector $yA \in \mathcal{V}$. Since $n_f = n_d$ and $p_f = p_d$, the run $c \xrightarrow{z'_1} f \xrightarrow{z_2} \overline{\mathcal{V}} \times \{p_e\} \times \{n_e\}$ is a valid run and the word $z'_1 z_2$ contradicts the minimality of z . \square

The following subsection shows that the unary version of co-VS reachability and coverability are in P. In the subsection after, we show that binary version of both problems are in NP.

3.1. Unary reachability in P. In this subsection, we show that both the reachability problems of weighted ODCA are solvable in polynomial time when the counter values are given in unary representation.

Theorem 13. *Unary co-VS reachability and co-VS coverability problems are decidable in polynomial time.*

The theorem is proved by showing a small model property. i.e., the length of a minimal witness of reachability is bounded by a polynomial in the number of states K and the input counter value(s). This is proved by showing that the maximum and minimum counter values encountered during the run of a minimal reachability witness do not exceed some bound. Assume this is not true. In this case, there are two sub-runs of the run which satisfy the following conditions. In the first part, the counter values increases and reaches a maximum counter value. In the second part, the counter values decreases. We show that in such a scenario, we can cut parts from both the sub-runs by maintaining the reachability conditions. This is proved in Lemma 15.

Now, we prove that if the number of distinct counter values encountered during the run of a minimal reachability witness is polynomially bounded, then we can bound the length of that witness.

Lemma 14. *Let $z \in \Sigma^*$ be a minimal reachability witness for (c, \bar{V}, S, X) . If the number of distinct counter values encountered during the run $c \xrightarrow{z} \bar{V} \times S \times X$ is t , then $|z| \leq K^2 \cdot t$.*

Proof. Let $c = c_1$ and $T(c_1) = c_1 \tau_1 c_2 \cdots \tau_{h-1} c_h$ be the run on word z from c_1 and T the corresponding sequence of transitions. Let t be the number of distinct counter values encountered during this run. Now assume for contradiction that $h > K^2 \cdot t$, then by Pigeon-hole principle, there are $K + 1$ many configurations $c_{i_0}, c_{i_1}, \dots, c_{i_K}$ with the same counter state and counter value during this run. Let \mathbb{A}_j denote the matrix such that $\mathbf{x}_{c_{i_j}} \mathbb{A}_j = \mathbf{x}_{c_h}$ for all $j \in [0, K]$. From Lemma 4 we get that there exists $r \leq K$, and $t \in [0, r - 1]$ such that $\mathbf{x}_{c_{i_t}} \mathbb{A}_r \in \bar{V}$. Consider the sequence of transitions $T' = \tau_1 \dots \tau_{i_t} \tau_{r \dots \ell-1}$ and $v = \text{word}(T')$. The run $\pi(v, c_1) = T'(c_1)$ is a valid run since $n_{c_t} = n_{c_r}$ and $p_{c_t} = p_{c_r}$. This is a shorter run than $\pi(z, c_1)$ and $c_1 \xrightarrow{v} \bar{V} \times S \times X$. This is a contradiction since z was assumed to be minimal. \square

It now suffices to show that the number of distinct counter values encountered during the run of a minimal witness is polynomially bounded. We first show that if the run of a minimal reachability witness of $(c, \bar{V}, S, \{m\})$ is a floating run, then the maximum and minimum counter values encountered during this run are bounded by a polynomial in K and the initial and final counter values.

Lemma 15. *Let $z \in \Sigma^*$ be a minimal reachability witness for $(c, \bar{V}, S, \{m\})$. If $c \xrightarrow{z} \bar{V} \times S \times \{m\}$ is a floating run, then the maximum counter value during this run is less than $\max(n_c, m) + K^4$.*

Proof. Let $z \in \Sigma^*$ be a reachability witness for $(c, \bar{V}, S, \{m\})$ and $c \xrightarrow{z} \bar{V} \times S \times \{m\}$ be a floating run. Let $f \in \bar{V} \times S \times \{m\}$, such that $c \xrightarrow{z} f$. We prove that the

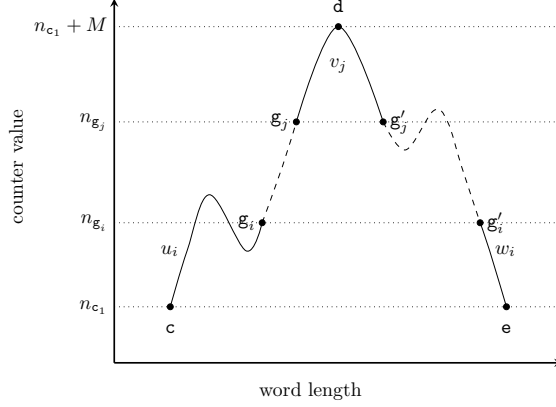


FIGURE 2. The figure shows the floating run from a configuration c to e such that $\mathbf{x}_e \in \overline{\mathcal{U}}$. The configurations g_i and g_j (resp. g'_i and g'_j) are where the counter values n_{g_i} and n_{g_j} are encountered for the last (resp. first) time before (resp. after) reaching counter value $n_c + M$. Also, $p_{g_i} = p_{g'_i} = p_{g_j} = p_{g'_j}$. The dashed line denotes the part of the run which can be removed to get a shorter reachability witness for $(c, \overline{\mathcal{U}}, \{p_e\}, \{n_e\})$.

maximum counter value encountered during this run are bounded. Let us assume that $\max(n_c, m) = n_c$. The case where $\max(n_c, m) = m$ can be proven analogously. Assume for contradiction that the maximum counter value encountered during this run is greater than $n_c + K^4$. There exists $z_1, z_2, z_3 \in \Sigma^*$ such that $z = z_1 z_2 z_3$ and configurations d, e such that the run on z from c can be written as follows:

$$c \xrightarrow{z_1} d \xrightarrow{z_2} e \xrightarrow{z_3} f$$

where $n_e = n_c$ and $n_d = n_c + \max_{ce}(\pi(z, c))$ (see Figure 2). Let $\mathbb{M} \in \mathcal{F}^{K \times K}$ such that $\mathbf{x}_f = \mathbf{x}_e \mathbb{M}$. From Lemma 5 we know that the set $\mathcal{U} = \{\mathbf{y} \in \mathcal{F}^K \mid \mathbf{y} \mathbb{M} \in \mathcal{V}\}$ is a vector space and hence the vector $\mathbf{x}_e \in \overline{\mathcal{U}}$. From Lemma 12 we know that $z_1 z_2$ is a minimal reachability witness for $(c, \overline{\mathcal{U}}, \{p_e\}, \{n_e\})$. We contradict the minimality of $z_1 z_2$.

Let $c_1 = c$ and $T(c_1) = c_1 \tau_1 c_2 \cdots \tau_{\ell-1} c_\ell$ denote the run on word $z_1 z_2$ from the configuration c_1 and T the corresponding sequence of transitions. Let $M = \max_{ce}(\pi(z, c))$. Note that $M = n_d - n_c$. For any $i \in [0, M]$, we denote by l_i and d_i the indices such that the counter value $n_{c_1} + i$ is encountered for the last (resp. first) time before (resp. after) reaching counter value $n_{c_1} + M$ in $T(c_1)$. That is, $\text{ce}(T_{1 \dots l_i-1}) = \text{ce}(T_{1 \dots d_i-1}) = i$, and for any $j \in [l_i, d_i - 2]$, $\text{ce}_{T_{1 \dots j}} > i$. We call $g_i = T_{1 \dots l_i-1}(c_1)$ and $g'_i = T_{1 \dots d_i-1}(c_1)$.

Let $r = K^2 + 1$. Since $M > K^4$, by Pigeonhole principle, there exists set of indices $X = \{i_1, i_2, \dots, i_r\} \subseteq [0, M]$ such that for any $k < r$, we have $i_k < i_r$ and for all $h, j \in X$ $p_{g_h} = p_{g'_h} = p_{g_j} = p_{g'_j}$. For all $j \in X$, let u_j, v_j, w_j be words such that $c_1 \xrightarrow{u_j} g_j \xrightarrow{v_j} g'_j \xrightarrow{w_j} g'_1$ as depicted in Figure 2. For all $j \in X$, let matrix \mathbb{A}_j and \mathbb{B}_j be such that $\mathbf{x}_{g'_j} = \mathbf{x}_{g_j} \mathbb{A}_j$ and $\mathbf{x}_{g'_1} = \mathbf{x}_{g'_j} \mathbb{B}_j$. We know that for all $j \in X$, $\mathbf{x}_{g_j} \mathbb{A}_j \mathbb{B}_j \in \overline{\mathcal{U}}$. Now we list the matrices in the following sequence

$\mathbb{A}_{i_r}, \mathbb{A}_{i_{r-1}}, \dots, \mathbb{A}_{i_1}$. From Lemma 4, it follows that, there exists $h, j \in X$ with $h < j$ such that $\mathbf{x}_{\mathbf{g}_h} \mathbb{A}_j \mathbb{B}_h \in \overline{\mathcal{U}}$.

Consider the sequence of transitions $T' = \tau_1 \dots \tau_{l_h-1} \tau_{l_j} \dots \tau_{d_j-1} \tau_{d_h} \dots \tau_\ell$. The word $u_h v_j w_h = \text{word}(T')$ is a proper subword of $z_1 z_2$ and the run $\pi(u_h v_j w_h, \mathbf{c}_1) = T'(\mathbf{c}_1)$ is a valid floating run shorter than $\pi(w, \mathbf{c}_1)$ and $\mathbf{c}_1 \xrightarrow{u_h v_j w_h} \mathbf{e}'$ such that $\mathbf{e}' \in \overline{\mathcal{U}} \times \{p_{\mathbf{e}}\} \times \{n_{\mathbf{e}}\}$. This contradicts the minimality of $z_1 z_2$. \square

Now, we prove that for any run (need not necessarily be a floating run) of a minimal reachability witness z for $(\mathbf{c}, \overline{\mathcal{V}}, S, m)$, the maximum counter value encountered during the run $\mathbf{c} \xrightarrow{z} \overline{\mathcal{V}} \times S \times \{m\}$ is bounded by a polynomial in the number of states of the machine and the initial and final counter values. This is achieved by applying Lemma 15 multiple times on the run of the minimal witness (refer Figure 3).

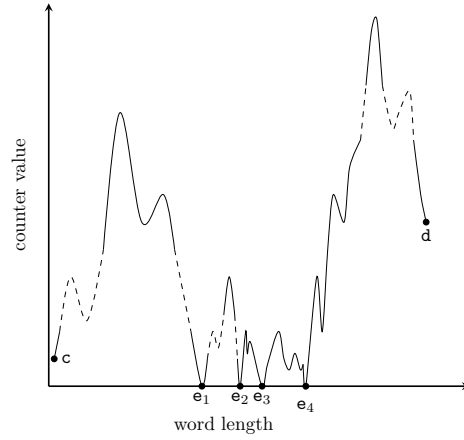


FIGURE 3. The figure shows a run from configuration \mathbf{c} to \mathbf{d} such that $\mathbf{x}_{\mathbf{d}} \in \overline{\mathcal{V}}$. Configurations $\mathbf{e}_1, \dots, \mathbf{e}_4$ denote the configurations where counter value zero is encountered during the run. The dashed lines denote the portions that can be removed to get a shorter reachability witness for $(\mathbf{c}, \overline{\mathcal{V}}, \{p_{\mathbf{d}}\}, \{n_{\mathbf{d}}\})$.

Lemma 16. *If $z \in \Sigma^*$ is a minimal reachability witness for $(\mathbf{c}, \overline{\mathcal{V}}, S, \{m\})$ then the maximum counter value encountered during the run $\mathbf{c} \xrightarrow{z} \overline{\mathcal{V}} \times S \times \{m\}$ is less than $\max(n_{\mathbf{c}}, m) + K^4$.*

Proof. Let $z \in \Sigma^*$ be a minimal reachability witness for $(\mathbf{c}, \overline{\mathcal{V}}, S, \{m\})$. Consider the run of word z from \mathbf{c} . Let $\mathbf{d} \in \overline{\mathcal{V}} \times S \times \{m\}$ such that $\mathbf{c} \xrightarrow{z} \mathbf{d}$. Assume for contradiction that the maximum counter value encountered during the run $\mathbf{c} \xrightarrow{z} \mathbf{d}$ is greater than $\max(n_{\mathbf{c}}, m) + K^4$. Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_t$ be all the configurations in this run such that $n_{\mathbf{e}_i} = 0$ for all $i \in [1, t]$. There exists words $u_1, u_2, \dots, u_{t+1} \in \Sigma^*$ such that $z = u_1 u_2 \dots u_{t+1}$ and

$$\mathbf{c} \xrightarrow{u_1} \mathbf{e}_1 \xrightarrow{u_2} \mathbf{e}_2 \xrightarrow{u_3} \dots \xrightarrow{u_t} \mathbf{e}_t \xrightarrow{u_{t+1}} \mathbf{d}$$

Note that $\mathbf{c} \xrightarrow{u_1} \mathbf{e}_1$, $\mathbf{e}_t \xrightarrow{u_{t+1}} \mathbf{d}$ and $\mathbf{e}_i \xrightarrow{u_{i+1}} \mathbf{e}_{i+1}$ for all $i \in [1, t-1]$ are floating runs (refer Figure 3).

We show that the counter values are bounded during each of these floating runs. First, we consider the floating run $c \xrightarrow{u_1} e_1$. Let $\mathbb{A} \in \mathcal{F}^{K \times K}$ be such that $\mathbf{x}_d = \mathbf{x}_{e_1} \mathbb{A}$. From Lemma 5 we know that the set $\mathcal{U} = \{\mathbf{y} \in \mathcal{F}^K \mid \mathbf{y} \mathbb{A} \in \mathcal{V}\}$ is a vector space and hence the vector $\mathbf{x}_{e_1} \in \overline{\mathcal{U}}$. From Lemma 12, we know that u_1 is a minimal reachability witness for $(c, \overline{\mathcal{U}}, \{p_{e_1}\}, \{0\})$ and therefore by Lemma 15 we know that the maximum counter value encountered during the run $\pi(u_1, c)$ is less than $n_c + K^4$.

Similarly for the floating run $e_t \xrightarrow{u_{t+1}} d$, the maximum counter value is bounded by $n_d + K^4$. Now consider the floating runs $e_i \xrightarrow{u_{i+1}} e_{i+1}$ for all $i \in [1, t-1]$. Again by applying Lemma 15 we get that the maximum counter value encountered during each of these sub-runs is less than K^4 . Therefore, the maximum counter value encountered during the run $c \xrightarrow{z} \overline{\mathcal{V}} \times S \times \{m\}$ is less than $\max(n_c, m) + K^4$. \square

We have shown that the counter values are polynomially bounded during the run of a minimal reachability witness for the co-VS reachability problem. Our next objective is to prove an analogous result for the co-VS coverability problem. The problem is similar to co-VS reachability, except that now we are not given a final counter value. A crucial ingredient in proving this is Lemma 17 where we prove that if the run of a minimal reachability witness z for $(c, \overline{\mathcal{V}}, S, \mathbb{N})$ is a floating run, then the number of distinct counter values encountered during the run $c \xrightarrow{z} \overline{\mathcal{V}} \times S \times \mathbb{N}$ is polynomially bounded in K and n_c . Using this and the ideas presented earlier for co-VS reachability, we can prove the existence of a polynomial length witness for the co-VS coverability problem.

Lemma 17. *If $z \in \Sigma^*$ is a minimal reachability witness for $(c, \overline{\mathcal{V}}, S, \mathbb{N})$ and $c \xrightarrow{z} \overline{\mathcal{V}} \times S \times \{m\}$ is a floating run then $|m - n_c| \leq K^2$.*

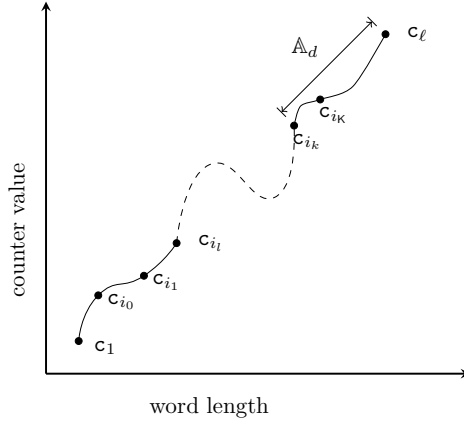


FIGURE 4. The figure shows a run from c_1 to c_ℓ such that $\mathbf{x}_{c_\ell} \in \overline{\mathcal{V}}$. The configurations c_{i_l} and c_{i_k} are where the counter values $n_{c_{i_l}}$ and $n_{c_{i_k}}$ are encountered for the last time. Also $p_{c_{i_l}} = p_{c_{i_k}}$. The dashed lines denote a part that can be removed to get a shorter reachability witness for $(c, \overline{\mathcal{V}}, \{p_{c_\ell}\}, \mathbb{N})$.

Proof. Let $z \in \Sigma^*$ be a minimal reachability witness for $(c, \overline{\mathcal{V}}, S, \mathbb{N})$ and $c \xrightarrow{z} \overline{\mathcal{V}} \times S \times \{m\}$ is a floating run. Assume for contradiction that $m > n_c + K^2$. The

case where $n_c > m + K^2$ can be proven analogously. Let $\mathbf{c}_1 = \mathbf{c}$ and $\pi(z, \mathbf{c}_1) = \mathbf{c}_1\tau_1\mathbf{c}_2\cdots\tau_{\ell-1}\mathbf{c}_\ell$ be such that $m = n_{\mathbf{c}_\ell} > n_{\mathbf{c}_1} + K^2$. Consider the sequence of transitions $T = \tau_0\tau_1\cdots\tau_{\ell-1}$ in $\pi(z, \mathbf{c}_1)$. Since there are only K counter states, by Pigeon-hole principle, there exists a strictly increasing sequence $I = 0 < i_0 < i_1 < \cdots < i_K \leq \ell$ such that for all $j, j' \in I$ (Condition 1) $p_{\mathbf{c}_j} = p_{\mathbf{c}_{j'}}$ and (Condition 2) if $j < j'$ then $n_{\mathbf{c}_j} < n_{\mathbf{c}_{j'}}$ and for all $d \in [j+1, j'-1]$, $n_{\mathbf{c}_j} < n_{\mathbf{c}_d} < n_{\mathbf{c}_{j'}}$.

Consider the set of configurations $\mathbf{c}_{i_0}, \mathbf{c}_{i_1}, \dots, \mathbf{c}_{i_K}$. For any $j \in [0, K]$, let \mathbb{A}_j denote the matrix such that $\mathbf{x}_{\mathbf{c}_{i_j}} \mathbb{A}_j = \mathbf{x}_{\mathbf{c}_\ell}$. Since $\mathbf{x}_{\mathbf{c}_{i_d}} \mathbb{A}_d \in \overline{\mathcal{V}}$ for all $d \in [0, K]$, from Lemma 3 we get that there exists $l, k \in [0, K]$ with $l < k$ such that $\mathbf{x}_{\mathbf{c}_{i_l}} \mathbb{A}_k \in \overline{\mathcal{V}}$.

Consider a configuration $\mathbf{e} = (\mathbf{x}, p, n)$. If $\pi(u, \mathbf{e})$ is a valid floating run with $\min_{\mathbf{c}\mathbf{e}}(\pi(u, \mathbf{e})) > 0$, then for all $m \in \mathbb{N}$ and $\mathbf{y} \in \mathcal{F}^K$, $\pi(u, (\mathbf{y}, p, m))$ is a valid run. Consider the sequence of transitions $T' = \tau_{i_k}\cdots\tau_{\ell-1}$ and let $u = \text{word}(T')$. Because of Condition 2, $\min_{\mathbf{c}\mathbf{e}}(\pi(u, \mathbf{c}_{i_k})) > 0$. Therefore the run $T''(\mathbf{c}_1)$ where $T'' = \tau_1\cdots\tau_{i_l-1}\tau_{i_k}\cdots\tau_{\ell-1}$ is a valid run shorter than $\pi(z, \mathbf{c}_1)$. This contradicts the minimality of z . \square

Now we show that for any run (need not be floating) of a minimal reachability witness z for $(\mathbf{c}, \overline{\mathcal{V}}, S, \mathbb{N})$, the maximum counter value encountered during the run $\mathbf{c} \xrightarrow{z} \overline{\mathcal{V}} \times S \times \mathbb{N}$ is bounded by a polynomial in K and the initial counter value.

Lemma 18. *If $z \in \Sigma^*$ is a minimal reachability witness for $(\mathbf{c}, \overline{\mathcal{V}}, S, \mathbb{N})$ then the maximum counter value encountered during the run $\mathbf{c} \xrightarrow{z} \overline{\mathcal{V}} \times S \times \mathbb{N}$ is less than $\max(n_c, K^2) + K^4$.*

Proof. Let $z \in \Sigma^*$ be a minimal reachability witness for $(\mathbf{c}, \overline{\mathcal{V}}, S, \mathbb{N})$. Consider the run of word z from \mathbf{c} . Let $\mathbf{d} \in \overline{\mathcal{V}} \times S \times \mathbb{N}$ such that $\mathbf{c} \xrightarrow{z} \mathbf{d}$. If $\mathbf{c} \xrightarrow{z} \mathbf{d}$ is a floating run, then by Lemma 17 the maximum counter value encountered during this run will be less than $n_c + K^2$. Now if $\mathbf{c} \xrightarrow{z} \mathbf{d}$ is not a floating run, then there exists $u_1, u_2 \in \Sigma^*$ such that $z = u_1u_2$ and $\mathbf{c} \xrightarrow{u_1} \mathbf{e} \xrightarrow{u_2} \mathbf{d}$ where, $n_{\mathbf{e}_i} = 0$ and $\mathbf{e} \xrightarrow{u_2} \mathbf{d}$ is a floating run.

Let $\mathbb{A} \in \mathcal{F}^{K \times K}$ be such that $\mathbf{x}_{\mathbf{d}} = \mathbf{x}_{\mathbf{e}} \mathbb{A}$. From Lemma 5, we know that the set $\mathcal{U} = \{\mathbf{y} \in \mathcal{F}^K \mid \mathbf{y} \mathbb{A} \in \mathcal{V}\}$ is a vector space and hence the vector $\mathbf{x}_{\mathbf{e}} \in \overline{\mathcal{U}}$. Note that for all $\mathbf{y} \in \overline{\mathcal{U}}$, the vector $\mathbf{y} \mathbb{A} \in \overline{\mathcal{V}}$. From Lemma 12, we know that u_1 is a minimal reachability witness for $(\mathbf{c}, \overline{\mathcal{U}}, \{p_{\mathbf{e}}\}, \{0\})$ and therefore by Lemma 16, we know that the maximum counter value encountered during the run $\pi(u_1, \mathbf{c})$ is less than $n_c + K^4$. Now since $\mathbf{e} \xrightarrow{u_2} \mathbf{d}$ is a floating run and u_2 is the minimal such word, from Lemma 17, we get that $n_{\mathbf{d}} \leq K^2$, and by Lemma 15, we know that the maximum counter value encountered during this run is less than $K^2 + K^4$. Therefore, we get that the maximum counter value encountered during the run $\mathbf{c} \xrightarrow{z} \mathbf{d}$ is less than $\max(n_c, K^2) + K^4$. \square

Proof of Theorem 13. For solving the co-VS reachability problem when the weighted ODCA $\mathcal{A} = (C, \delta, p_0; Q, \lambda, \Delta, \eta)$ with $K = |Q| = |C|$ states, initial configuration \mathbf{c} , vector space \mathcal{V} , set of counter states S and counter value m are given as inputs, we first consider the $\max(n_c, m) + K^4$ -unfolding weighted automata $\mathcal{A}^{\max(n_c, m) + K^4} = (C', \delta', p'_0; Q', \lambda', \Delta', \eta'_F)$ of \mathcal{A} as described in Definition 7. From Lemma 16, we know that the maximum counter value encountered during the run of the minimal reachability witness z for $(\mathbf{c}, \overline{\mathcal{V}}, S, \{m\})$ is less than $\max(n_c, m) + K^4$. We define a vector space $\mathcal{U} \subseteq \mathcal{F}^{|Q'|}$ as follows: A vector $\mathbf{x} \in \mathcal{F}^{|Q'|}$ is in \mathcal{U} if there

exists $\mathbf{y} \in \mathcal{V}$ such that for all $i \in [0, K-1]$, $\mathbf{x}[K \cdot m + i] = \mathbf{y}[i]$ and for all $n \neq m$ and $i \in [0, K-1]$, $\mathbf{x}[K \cdot n + i] = 0$.

Given a configuration \mathbf{c} of a weighted ODCA, we define the vector $\mathbf{z}_{\mathbf{c}} \in \mathcal{F}^{|Q'|}$.

$$\mathbf{z}_{\mathbf{c}}[i] = \begin{cases} \mathbf{x}_{\mathbf{c}}[i \bmod K], & \text{if } \frac{i}{K} = n_{\mathbf{c}} \\ 0, & \text{otherwise} \end{cases}$$

Now, consider the configuration $\bar{\mathbf{c}} = (\mathbf{z}_{\mathbf{c}}, (p_{\mathbf{c}}, n_{\mathbf{c}}))$ of $\mathcal{A}^{\max(n_{\mathbf{c}}, m) + K^4}$ and check whether $\bar{\mathbf{c}} \xrightarrow{*} \bar{\mathcal{U}} \times S \times \{0\}$. This is a co-VS reachability problem of weighted automata. Using Theorem 11, this can be solved in polynomial time.

For solving the co-VS coverability problem when the weighted ODCA \mathcal{A} with K states, an initial configuration \mathbf{c} , a vector space \mathcal{V} and a set of counter states S are given as inputs, we consider the $\max(n_{\mathbf{c}}, K^2) + K^4$ -unfolding weighted automata $\mathcal{A}^{\max(n_{\mathbf{c}}, K^2) + K^4} = (C', \delta', p'_0; Q', \lambda', \Delta', \eta'_F)$ of \mathcal{A} . From Lemma 18, we know that the maximum counter value encountered during the run of a minimal reachability witness z for $(\mathbf{c}, \bar{\mathcal{V}}, S, \mathbb{N})$ is less than $\max(n_{\mathbf{c}}, K^2) + K^4$. We define a vector space $\mathcal{U} \subseteq \mathcal{F}^{|Q'|}$ as follows: A vector $\mathbf{x} \in \mathcal{F}^{|Q'|}$ is in \mathcal{U} if there exists $\mathbf{y} \in \mathcal{V}$ and $m \in \mathbb{N}$ such that for all $i \in [0, K-1]$, $\mathbf{x}[K \cdot m + i] = \mathbf{y}[i]$ and for all $n \neq m$ and $i \in [0, K-1]$, $\mathbf{x}[K \cdot n + i] = 0$. Now, consider the configuration $\bar{\mathbf{c}} = (\mathbf{z}_{\mathbf{c}}, (p_{\mathbf{c}}, n_{\mathbf{c}}))$ of $\mathcal{A}^{\max(n_{\mathbf{c}}, K^2) + K^4}$ and check whether $\bar{\mathbf{c}} \xrightarrow{*} \bar{\mathcal{U}} \times S \times \{0\}$. This is a co-VS reachability problem of a weighted automaton. From Theorem 11, we know that this can be solved in polynomial time. \square

3.2. Binary reachability in NP. Consider the case where the counter values are specified in binary. Theorem 13 can still be applied to get an algorithm whose running time is polynomial in the input counter values. Since the counter values are represented in binary, their values can be exponentially large compared to their size. Therefore, we only get an exponential time algorithm for reachability from Theorem 13. This section shows that co-VS reachability can be tested in NP. The technically challenging part of the proof is proved in Lemma 22. It shows that the “lexicographically minimal” reachability witness z is of the form $uy_1^{r_1}vy_2^{r_2}w$, where the length of the words u, y_1, y_2, v and w are polynomially bounded in K and r_1, r_2 are polynomial values dependent on K and the input counter values. This is a polynomial sized representation of the witness (r_1, r_2 in binary) whose reachability can be verified in polynomial time. A non-deterministic machine guesses the words u, y_1, y_2, v , and w and verifies reachability in polynomial time.

Theorem 19. *Binary co-VS reachability and co-VS coverability problems are in NP.*

We aim to show that there is an “encoding” of a minimal reachability witness of polynomial size with respect to the input size. The following lemma shows that the length of a minimal reachability witness is bounded by a polynomial in the input counter values. Note that this can be exponential in size with respect to the input size when the counter values are represented in binary.

Lemma 20. (1) *If z is a minimal reachability witness for $(\mathbf{c}, \bar{\mathcal{V}}, S, \{m\})$ then $|z| \leq K^2 \cdot (\max(n_{\mathbf{c}}, m) + K^4)$.*
 (2) *If z is a minimal reachability witness for $(\mathbf{c}, \bar{\mathcal{V}}, S, \mathbb{N})$ then $|z| \leq K^2 \cdot (\max(n_{\mathbf{c}}, K^2) + K^4)$.*

Proof. 1. Let $z \in \Sigma^*$ be a minimal reachability witness for $(c, \bar{V}, S, \{m\})$. From Lemma 16, we know that the maximum counter value encountered during the run $c \xrightarrow{z} \bar{V} \times S \times \{m\}$ is less than $\max(n_c, m) + K^4$. Therefore, there are at most $\max(n_c, m) + K^4$ many distinct counter values encountered during this run. Now from Lemma 14 we get that $|z| \leq K^2 \cdot (\max(n_c, m) + K^4)$.

2. Let $z \in \Sigma^*$ be a minimal reachability witness for $(c, \bar{V}, S, \mathbb{N})$. From Lemma 18, we know that the maximum counter value encountered during the run $c \xrightarrow{z} \bar{V} \times S \times \mathbb{N}$ is less than $\max(n_c, K^2) + K^4$. Therefore, there are at most $\max(n_c, K^2) + K^4$ many distinct counter values encountered during this run. Now from Lemma 14 we get that $|z| \leq K^2 \cdot (\max(n_c, K^2) + K^4)$. \square

We define the counter effect of a word w with respect to a counter state $q \in C$ as $\text{ce}(\pi(w, c))$ where c is any configuration with $n_c = |w|$ and $p_c = q$. Note that for any two configuration c, d , $\text{ce}(\pi(u, c)) = \text{ce}(\pi(u, d))$ if $n_c = n_d = |w|$ and $p_c = p_d$. First, we consider the case of the run of a minimal reachability witness from c , which is a floating run. The following lemma is required for the special case where $|n_c - m|$ is bounded by a polynomial in K .

Lemma 21. *If $z \in \Sigma^*$ is a minimal reachability witness for $(c, \bar{V}, S, \{m\})$ and $c \xrightarrow{z} \bar{V} \times S \times \{m\}$ is a floating run, then*

- (1) *the minimum counter value during this run is greater than $\min(n_c, m) - K^4$, and*
- (2) *$|z| \leq K^2 \cdot (|n_c - m| + 2K^4)$.*

Proof. Let $z \in \Sigma^*$ be a minimal reachability witness for $(c, \bar{V}, S, \{m\})$ and $c \xrightarrow{z} \bar{V} \times S \times \{m\}$ is a floating run. We prove the claims in the lemma one by one.

1. *This case is symmetric to that of Lemma 15 and can be proven analogously.*

2. From Lemma 15 and Point 1, we get that the counter values encountered during the run $c \xrightarrow{z} \bar{V} \times S \times \{m\}$ lies between $\max(n_c, m) + K^4$ and $\min(n_c, m) - K^4$. Let $t = |n_c - m|$. There are at most $t + 2 \cdot K^4$ distinct counter values during this run. Now from Lemma 14 we get that $|z| \leq K^2 \cdot (t + 2 \cdot K^4)$. \square

We assume a total order on the symbols in Σ . Given two words $u, v \in \Sigma^*$, we say that u precedes v in the *lexicographical ordering* if $|u| < |v|$ or if $|u| = |v|$ and there exists an $i \in [0, |u| - 1]$ such that $u[0, i - 1] = v[0, i - 1]$ and $u[i]$ precedes $v[i]$ in the total ordering assumed on Σ . A word $z \in \Sigma^*$ is called the lexicographically minimal reachability witness for $(c, \bar{V}, S, \{m\})$, if $c \xrightarrow{z} \bar{V} \times S \times X$ and for all $u \in \Sigma^*$ with $c \xrightarrow{u} \bar{V} \times S \times X$, z precedes u in the lexicographical ordering. We show that the lexicographically minimal reachability witness z for $(c, \bar{V}, S, \{m\})$ has a special form. First, we consider the case of floating runs.

Lemma 22. *If $z \in \Sigma^*$ is the lexicographically minimal reachability witness for $(c, \bar{V}, S, \{m\})$ and $c \xrightarrow{z} \bar{V} \times S \times \{m\}$ is a floating run, then there exists $u, y, w \in \Sigma^*$ and $r \in \mathbb{N}$ such that $z = uy^r w$ and for all configurations d_k , $k \in [0, r]$, where $c \xrightarrow{uy^k} d_k$ the following conditions hold:*

- (1) $|u|, |y| \leq 3K^7$ and $|w| < 6K^7$,
- (2) either $n_{d_i} > n_{d_j}$ for all i, j such that $0 \leq i < j \leq r$ or $n_{d_i} < n_{d_j}$ for all i, j such that $0 \leq i < j \leq r$,
- (3) for all $i, j \in [0, r]$, $p_{d_i} = p_{d_j}$, and

$$(4) \ r \in [0, K^2 \cdot |n_c - m| + K^6].$$

Proof. Let z be the lexicographically minimal reachability witness for $(c, \overline{V}, S, \{m\})$ such that $c \xrightarrow{z} \overline{V} \times S \times \{m\}$ is a floating run. Let $t = |n_c - m|$. If $t \leq K^4$, then from Lemma 21, Item 2, we get that $|z| \leq 3K^6$ and the claim is trivially true. Consider the case where $n_c > m$. The case where $m > n_c$ can be proven analogously. Let us assume $t > K^4$ and let $d \in \mathbb{Z}$ be such that $d = -t + K^4 + 1$.

Let $c = c_1$ and $T(c_1) = c_1 \tau_1 c_2 \cdots \tau_{\ell-1} c_\ell$ denote the run on word z from the configuration c_1 . For any $i \in [0, K^4]$, we denote by l_i the index such that the counter value $n_{c_1} - i$ is encountered for the first time and r_i the index such that the counter value $n_{c_1} - i + d$ is encountered for the last time in $T(c_1)$ (see Figure 5). Since $t > K^4$, there are at least $K^4 + 1$ pairs of positions $(l_i, r_i), i \in [0, K^4]$ such that for all $i \in [0, K^4]$ the factor $z[l_i, r_i]$ has counter effect d with respect to counter state $p_{c_{l_i}}$. Note that these factors need not be all distinct. Let $X = \{(l_i, r_i)\}_{i \in [0, K^4]}$ be the set containing these pairs of positions and $W = \{z[l, r] \mid (l, r) \in X\}$ be the set containing the corresponding factors. Note that $|X| > K^4$.

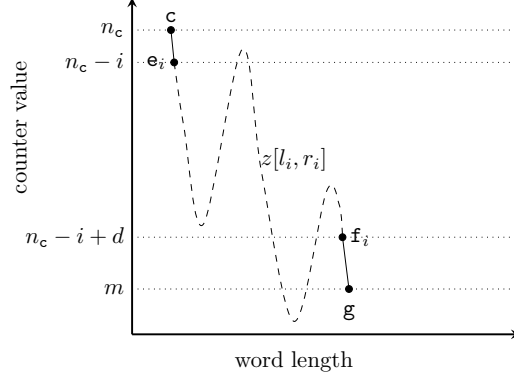


FIGURE 5. The figure shows the floating run from a configuration c to g such that $x_g \in \overline{V}$. The configurations c_{l_i} and c_{r_i} are where the counter values $n_c - i$ and $n_c - i + d$ are encountered for the first (resp. last) time during this run. The dashed line denotes the part of the run due to the factor $z[l_i, r_i]$ and has a counter effect d .

Claim 1. $|W| \leq K^4$.

Proof: Assume for contradiction that $|W| > K^4$. Let $g \in \overline{V} \times S \times \{m\}$ be such that $c \xrightarrow{z} g$. Since number of counter states is K , by Pigeon-hole principle there exists $Y \subseteq X$ with $|Y| = K^2 + 1$ such that for all $(l, r), (l', r') \in Y$, $p_{c_l} = p_{c_{l'}}$, $p_{c_r} = p_{c_{r'}}$, and $z[l, r] \neq z[l', r']$. We say $(l, r) < (l', r')$ if $z[l, r]$ precedes $z[l', r']$ in the lexicographical order. Therefore, the elements in Y have an ordering as follows: $(l_0, r_0) < (l_1, r_1) < \cdots < (l_{K^2}, r_{K^2})$. For all $i \in [0, K^2]$, let $u_i = z[1, l_i]$, $x_i = z[l_i, r_i]$, $w_i = z[r_i, \ell]$, configurations e_i, f_i be such that $c \xrightarrow{u_i} e_i \xrightarrow{x_i} f_i \xrightarrow{w_i} g$ and matrices A_i, M_i, B_i be such that $x_{e_i} = x_c A_i$, $x_{f_i} = x_{e_i} M_i$, $x_g = x_{f_i} B_i$.

We know that for all $i \in [0, K^2]$, $x_c A_i M_i B_i \in \overline{V}$. Consider the sequence of matrices M_0, M_1, \dots, M_{K^2} . From Lemma 4 we know that there exists an $i \in [0, K^2 -$

1] and $j < i$ such that $\mathbf{x}_c \mathbb{A}_i \mathbb{M}_j \mathbb{B}_i \in \bar{\mathcal{V}}$. Note that the word $u_i x_j w_i$ precedes z in the lexicographical ordering. Therefore the run $\mathbf{c} \xrightarrow{u_i x_j w_i} \bar{\mathcal{V}} \times S \times \{m\}$ contradicts minimality of z . $\square_{\text{Claim:1}}$

Since $|W| \leq K^4$ and $|X| > K^4$, there exists $i, j \in [0, K^4]$, with $i < j$ and $x \in \Sigma^*$ such that $(l_i, r_i) \in X$, $(l_j, r_j) \in X$ and $x = w[l_i, r_i] = w[l_j, r_j]$. Let $u_1, w_1, u_2, w_2 \in \Sigma^*$ such that $z = u_1 x w_1 = u_2 x w_2$. Since $u_1 \neq u_2$, either u_1 is a prefix of u_2 or u_2 a prefix of u_1 . Without loss of generality, let us assume u_1 is prefix of u_2 . Therefore, there exists $v \in \Sigma^*$ such that $u_2 = u_1 v$. Let \mathbf{e} be a configuration such that $\mathbf{c} \xrightarrow{u_1} \mathbf{e}$.

Claim 2. $|u_1|, |v|, |w_1| \leq 3K^6$.

Proof: Consider the set X . For any $i, j \in [0, K^4]$, with $i < j$, $n_{c_{l_i}} - n_{c_{l_j}} \leq K^4 + 1$ and $n_{c_{r_j}} - n_{c_{r_i}} \leq K^4 + 1$. Therefore the counter-effect of u_2 and w_2 can be at most K^4 . So the counter-effect of v with respect to counter state $p_{\mathbf{e}}$ can be at most K^4 . Since it is a minimal floating run from Lemma 21, we get that $|v| \leq 3K^6$. By similar arguments, the counter-effect of u_1 and w_1 can be at most K^4 , and again by Lemma 21, we get that their lengths are at most $3K^6$. $\square_{\text{Claim:2}}$

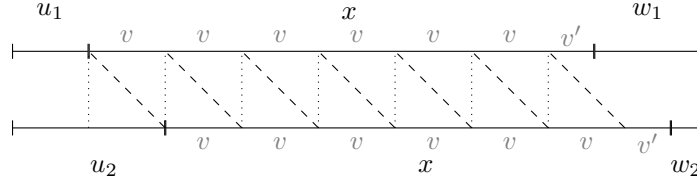


FIGURE 6. The figure shows the factorisation of a word $z = u_1 x w_1 = u_2 x w_2$, where x is an overlapping factor. The factor v is a prefix of x such that $u_2 = u_1 v$. The word z can be written as $u_1 v^i v' w_2$ for some $i \in \mathbb{N}$ and v' prefix of v .

Claim 3. *There exists $v' \in \Sigma^*$ and $r \in [0, K^2 \cdot |n_c - m| + K^6]$ such that $x = v^r v'$ with $|v'| \leq |v|$.*

Proof: Let $r \in \mathbb{N}$ be the largest number such that x is of the form $v^r v'$ for some $v' \in \Sigma^*$ (see Figure 6). We know that $z = u_2 x w_2$ and $u_2 = u_1 v$. Therefore, $z = u_1 v x w_2 = u_1 v v^r v' w_2 = u_1 v^r v v' w_2$. Furthermore, $z = u_1 x w_1 = u_1 v^r v' w_1$. Now since $u_1 v^r v v' w_2 = u_1 v^r v' w_1$, we get that $v v' w_2 = v' w_1$. Hence, if $|v'| \geq |v|$, then v is a prefix of v' . This is a contradiction since r was chosen to be the largest number such that x is of the form $v^r v'$.

In order to show the bound on the value r , we observe the following. We know that the counter effect of the run $\pi(x, \mathbf{e})$ is d . Therefore from Lemma 21 Point 2, we get that $|x| \leq K^2 \cdot (|d| + 2K^4)$. Therefore, the value of r is less than or equal to $K^2 \cdot (|d| + 2K^4)$. $\square_{\text{Claim:3}}$

From Claim 3 and Claim 2, we get that $|u_1 v' w_1| \leq 9K^6$ and $z = u_1 v^r v' w_1$ for some $r \in [0, K^2 \cdot (|d| + 2K^4)]$. Note that the factor v might start and end in different counter states during the run and, therefore need not always have a negative counter effect. However, we also know that the word v^r has a negative counter effect. For $i \in [1, 2K]$, let \mathbf{g}_i be the configuration such that $\mathbf{e} \xrightarrow{v^i} \mathbf{g}_i$. By

Pigeon-hole principle there exists $j, k \in [1, 2K]$ with $j < K$ and $k - j \leq K$ such that $p_{g_j} = p_{g_k}$. Also, note that the word $y = v^{k-j}$ has a negative counter-effect from the counter state p_{g_j} . Let $r' = \frac{r-j}{k-j}$ and $j' = (r - j) \pmod{(k - j)}$. Now consider the word $z = u_1 v^j y^{r'} v^{j'} v' w_1$. Since $|u_1|, |w_1|, |v| \leq 3K^6$, $j < K$ and $k - j \leq K$, we get that $|u_1 v^j| \leq 3K^7$, $|v^{j'} v' w_1| < 6K^7$, $|y| \leq 3K^7$ and $r' \in [0, K^2 \cdot (|d| + 2K^4)]$. \square

Lemma 23. *If $z \in \Sigma^*$ is the lexicographically minimal reachability witness for $(c, \overline{V}, S, \{m\})$, then there exists $u, y_1, v_1, v_2, v_3, y_2, w \in \Sigma^*$ and $r_1, r_2 \in \mathbb{N}$ such that*

- (1) $z = uy_1^{r_1} v_1 v_2 v_3 y_2^{r_2} w$,
- (2) $|uy_1 v_1 v_2 v_3 y_2 w| \leq 25K^7$,
- (3) $r_1, r_2 \leq \max\{m, n_c\} \cdot K^2 + K^6$,
- (4) $\pi(uy_1^{r_1} v_1, c)$ and $\pi(v_3 y_2^{r_2} w, d)$ are floating runs for configuration d where $c \xrightarrow{uy_1^{r_1} v_1 v_2} d$, and
- (5) $ce(\pi(uy_1^{r_1} v_1, c)) = ce(\pi(v_3 y_2^{r_2} w, c)) = -n_c$.

Proof. Let $z \in \Sigma^*$ be the lexicographically minimal reachability witness for $(c, \overline{V}, S, \{m\})$. Consider the run of word z from c . Let $d \in \overline{V} \times S \times \{m\}$ such that $c \xrightarrow{z} d$. Let $c = c_1$ and $T(c_1) = c_1 \tau_1 c_2 \cdots \tau_{\ell-1} c_\ell$ denote the run on word z from the configuration c_1 and T the corresponding sequence of transitions. Let e_1 be the first configuration with counter value zero and e_2 be the last configuration with counter value zero during this run. Let $z_1, z_2, z_3 \in \Sigma^*$ be such that $c \xrightarrow{z_1} e_1 \xrightarrow{z_2} e_2 \xrightarrow{z_3} c_\ell$ and $z = z_1 z_2 z_3$. Observe that $c \xrightarrow{z_1} e_1$ and $e_2 \xrightarrow{z_3} c_\ell$ are floating runs.

From Lemma 22, we know that there exists $u_1, u_3, v_1, v_3, y_1, y_3 \in \Sigma^*$ and $r_1, r_3 \in \mathbb{N}$ such that $z_1 = u_1 y_1^{r_1} v_1$, $z_3 = u_3 y_3^{r_3} v_3$, $|u_1|, |u_3| \leq 3K^7$, $|v_1|, |v_3| \leq 6K^7$, $|y_1|, |y_3| \leq 3K^7$, $r_1 \in [0, n_c \cdot K^2 + K^6]$ and $r_3 \in [0, m \cdot K^2 + K^6]$. Also, from Lemma 20 we get that $|z_2| \leq K^6$. \square

We now prove that the binary co-VS reachability and co-VS coverability problems are in NP. From Lemma 23 we observe there is a polynomial-size encoding of the lexicographically minimal word (where r_1 and r_2 are in binary). A non-deterministic machine can guess this encoding and verify the reachability in polynomial time since \mathbb{M}^r can be computed in $\log(r)$ time (see Lemma 2). A detailed proof is given below.

Proof of Theorem 19. Let us first look at the binary co-VS reachability problem. Let $z \in \Sigma^*$ be the lexicographically minimal reachability witness for $(c, \overline{V}, S, \{m\})$. Consider the run of word z from c . Let $d \in \overline{V} \times S \times \{m\}$ such that $c \xrightarrow{z} d$. Let $c = c_1$ and $T(c_1) = c_1 \tau_1 c_2 \cdots \tau_{\ell-1} c_\ell$ denote the run on word z from the configuration c_1 and T the corresponding sequence of transitions. Let e_1 be the first configuration with counter value zero and e_2 be the last configuration with counter value zero during this run. Let $z_1, z_2, z_3 \in \Sigma^*$ be such that $c \xrightarrow{z_1} e_1 \xrightarrow{z_2} e_2 \xrightarrow{z_3} c_\ell$ and $z = z_1 z_2 z_3$. Observe that $c \xrightarrow{z_1} e_1$ and $e_2 \xrightarrow{z_3} c_\ell$ are floating runs.

From Lemma 22, we know that there exists $u_1, u_3, v_1, v_3, y_1, y_3 \in \Sigma^*$ and $r_1, r_3 \in \mathbb{N}$ such that $z_1 = u_1 y_1^{r_1} v_1$, $z_3 = u_3 y_3^{r_3} v_3$, $|u_1|, |u_3| \leq 3K^7$, $|v_1|, |v_3| \leq 6K^7$, $|y_1|, |y_3| \leq 3K^7$, $r_1 \in [0, n_c \cdot K^2 + K^6]$ and $r_3 \in [0, m \cdot K^2 + K^6]$. Also, from Lemma 20 we get that $|z_2| \leq K^6$.

Our NP algorithm starts by guessing the words $u_1, y_1, v_1, z_2, u_3, y_3, v_3$, the values r_1, r_2 , and the configurations e_1 and e_2 . We first show how to verify if $c \xrightarrow{u_1 y_1^{r_1} v_1} e_1$.

The algorithm computes configuration \mathbf{f}_0 such that $\mathbf{c} \xrightarrow{u_1} \mathbf{f}_0$. Now it constructs the matrix \mathbb{M}_{y_1} and computes the configuration \mathbf{f}_1 such that $\mathbf{f}_0 \xrightarrow{y_1} \mathbf{f}_1$ and $\mathbf{x}_{\mathbf{f}_1} = \mathbf{x}_{\mathbf{f}_0} \mathbb{M}_{y_1}$. From Lemma 2, we know that $(\mathbb{M}_{y_1})^{r_1}$ can be computed by repeated powering in time polynomial in $\log(r_1)$ and K . Let \mathbf{f}_{r_1} be a configuration such that $\mathbf{f}_0 \xrightarrow{y^{r_1}} \mathbf{f}_{r_1}$. From Lemma 22, we know that $p_{\mathbf{f}_0} = p_{\mathbf{f}_{r_1}}$ and $n_{\mathbf{f}_{r_1}} = p_{\mathbf{f}_0} - r_1 \cdot (n_{\mathbf{f}_0} - n_{\mathbf{f}_1})$. Since $\mathbf{x}_{\mathbf{f}_{r_1}} = \mathbf{x}_{\mathbf{f}_0} (\mathbb{M}_{y_1})^{r_1}$, we can construct the configuration \mathbf{f}_{r_1} in polynomial time. We now verify in polynomial time whether $\mathbf{f}_{r_1} \xrightarrow{v_1} \mathbf{e}_1$ or not.

We can verify if $\mathbf{e}_2 \xrightarrow{u_3 y_3^{r_3} v_3} \mathbf{d}$ in a similar manner. The algorithm can also check whether $\mathbf{e}_1 \xrightarrow{z_2} \mathbf{e}_2$ in polynomial time since $|z_2| \leq K^6$. It finally checks whether $\mathbf{d} \in \bar{\mathcal{V}} \times S \times \{m\}$. Hence the binary co-VS reachability problem is decidable in NP.

As for the binary co-VS coverability problem, either the run of a minimal witness is a floating run or is not. In the former case where the run is floating, from Lemma 17, we know that the difference between the final and initial counter values is at most K^2 . In the latter case where the run is grounded, by Lemma 17, we get that the final value is at most K^2 . In both the cases, the algorithm guesses the final counter value, and the problem is reduced to the co-VS reachability problem, which is in NP. Hence the binary co-VS coverability problem is decidable in NP. \square

4. EQUIVALENCE OF WEIGHTED ODCA

In this section, we give a polynomial time algorithm to check the equivalence of two weighted ODCA (Theorem 1). The algorithm returns a minimal distinguishing word if the ODCA are non-equivalent. We use the reachability results presented in the previous section to show that the length of a minimal distinguishing word is short. The idea here is to prove that the maximum counter value encountered during the run of a minimal witness is polynomially bounded. We use this to reduce the equivalence problem to that of weighted automata.

In the remainder of this section, we fix two weighted ODCA \mathcal{A}_1 and \mathcal{A}_2 over an alphabet Σ and a field \mathcal{F} . For $i \in \{1, 2\}$,

$$\mathcal{A}_i = (C_i, \delta_i, p_{0_i}; Q_i, \lambda_i, \Delta_i, \eta_i).$$

Without loss of generality assume $K = |C_1| = |Q_1| = |C_2| = |Q_2|$. We will reason on the synchronised runs on pairs of configurations. Given two weighted ODCA, \mathcal{A}_1 and \mathcal{A}_2 and $i \in \mathbb{N}$, we denote a *configuration pair* as $\mathbf{h}_i = \langle \mathbf{c}_i, \mathbf{d}_i \rangle$ where \mathbf{c}_i is a configuration of \mathcal{A}_1 and \mathbf{d}_i is a configuration of \mathcal{A}_2 . We similarly consider *transition pairs* of \mathcal{A}_1 and \mathcal{A}_2 , and consider *synchronised runs* as the application of a sequence of transition pairs to a configuration pair. We fix a minimal word z (also called witness) that distinguishes \mathcal{A}_1 and \mathcal{A}_2 and $\ell = |z|$. Henceforth we will denote by

$$\Pi = \mathbf{h}_0 \tau_0 \mathbf{h}_1 \cdots \tau_{\ell-1} \mathbf{h}_\ell$$

the run pair of z from the initial configuration pair. We denote by $T = \tau_0 \cdots \tau_{\ell-1}$ the sequence of transition pairs of this run pair. To prove Theorem 1, we use the following lemma, which states that the counter values in Π are bounded by a polynomial $\text{poly}_0(K)$.

Lemma 24. *There is a polynomial $\text{poly}_0 : \mathbb{N} \rightarrow \mathbb{N}$ such that if two weighted ODCA \mathcal{A}_1 and \mathcal{A}_2 are not equivalent, then there exists a witness z such that the counter values encountered during the run of z are less than $\text{poly}_0(K)$.*

We use Lemma 24 to show that the length of the witness z is bounded by a polynomial $\text{poly}_1(K) = 2K^5 \text{poly}_0(K)$.

Lemma 25. *There is a polynomial $\text{poly}_1 : \mathbb{N} \rightarrow \mathbb{N}$ such that if two weighted ODCAs \mathcal{A}_1 and \mathcal{A}_2 are not equivalent, then there exists a witness z such that $|z|$ is less than or equal to $\text{poly}_1(K)$.*

Proof. Assume for contradiction that the length of a minimal witness z is greater than $\text{poly}_1(K)$. From Lemma 24, we know that the counter values encountered during the run Π in less than $\text{poly}_0(K)$. Since $|z| > \text{poly}_1(K)$, by the Pigeonhole principle, we get that there exist indices $0 \leq i_0 < i_2 < \dots < i_{2K} \leq \ell$ such that for all configuration pairs $\mathbf{h}_{i_j}, j \in [1, 2K]$, $n_{c_{i_j}} = n_{c_{i_{j-1}}}$, $n_{d_{i_j}} = n_{d_{i_{j-1}}}$, $p_{c_{i_j}} = p_{c_{i_{j-1}}}$ and $p_{d_{i_j}} = p_{d_{i_{j-1}}}$.

For all $j \in [0, 2K]$ we define the vector $\mathbf{x}_j \in \mathcal{F}^{2K}$ such that $\mathbf{x}_j[r] = \mathbf{x}_{c_{i_j}}[r]$, if $r < K$ and $\mathbf{x}_{d_{i_j}}[r - K]$, otherwise. We also define the vector $\boldsymbol{\eta} \in \mathcal{F}^{2K}$ such that $\boldsymbol{\eta}[r] = \boldsymbol{\eta}_1[r]$, if $r < K$ and $\boldsymbol{\eta}_2[r - K]$, otherwise. For all $j \in [0, 2K]$, let \mathbb{A}_j denote the matrix such that $\mathbf{x}_j \mathbb{A}_j = \mathbf{x}_\ell$. Since z is a minimal witness, we know that for all $j \in [0, 2K]$, $\mathbf{x}_j \mathbb{A}_j \boldsymbol{\eta}^\top \neq 0$. From Lemma 4, we get that there exists $r, r' \in [0, 2K]$, with $r' < r$ such that $\mathbf{x}_{r'} \mathbb{A}_r \boldsymbol{\eta}^\top \neq 0$. The sequence of transitions $\tau_{i_{r+1}} \dots \tau_\ell$ can be taken from $\mathbf{h}_{i_{r'}}$ since the counter values and counter states are the same for both configurations. Consider the sequence of transitions $T' = \tau_0 \dots \tau_{i_{r'}} \tau_{i_{r+1}} \dots \tau_\ell$ and let $w = \text{word}(T')$. The word w is a shorter witness than z and contradicts its minimality. \square

Lemma 25 helps us to reduce the equivalence problem of weighted ODCA to that of weighted automata by “simulating” the runs of weighted ODCAs up to length $\text{poly}_1(K)$ by two weighted automata. The naive algorithm will only give us a PSPACE procedure, but there is a polynomial time procedure to do this, and the proof is given below.

Proof of Theorem 1. We consider the two weighted ODCAs \mathcal{A}_1 and \mathcal{A}_2 . From Lemma 25, we know that the length of the minimal witness z is less than $\text{poly}_1(K)$. Let $M = \text{poly}_1(K)$. We construct the M -unfolding weighted automata \mathcal{A}_1^M and \mathcal{A}_2^M as described in Definition 7. It follows that, \mathcal{A}_1 is non-equivalent to \mathcal{A}_2 if and only if there exists a word $w \in \Sigma^{\leq M}$ such that $f_{\mathcal{A}_1^M}(w) \neq f_{\mathcal{A}_2^M}(w)$. Tzeng [20, Lemma 3.4] gives a polynomial time algorithm to output a minimal word that distinguishes two probabilistic automata. We conclude the proof by noting that the algorithm can be extended to the case of weighted automata. \square

The rest of this section is dedicated to proving Lemma 24. We adapt techniques developed by Böhm et al. [3] for OCAs. We start by labelling some configuration pairs as background points (see Figure 7). Consider the case where there is no background point in Π . By reducing the problem to co-VS reachability/coverability we show that the counter values in Π are polynomially bounded. Now consider the case where there is a background point \mathbf{h}_j in Π . We show that the counter values encountered during the run of Π till \mathbf{h}_j is polynomially bounded. This is shown by Lemma 29 and Lemma 35. We conclude by arguing that the length of the run from

h_j is polynomially bounded.

$$\overbrace{h_0 \tau_0 h_1 \tau_1 h_2 \cdots h_{j-1} \tau_{j-1}}^{\text{counters poly-bounded}} \quad \underbrace{h_j = \langle c_j, d_j \rangle}_{\substack{1^{st} \text{ configuration pair in} \\ \text{background space}}} \quad \overbrace{\tau_j \cdots \tau_{\ell-1} h_\ell}^{\text{counters poly-bounded}}$$

4.1. Configuration Space. Each pair of configuration $h = \langle c, d \rangle$ is mapped to a point in the space $\mathbb{N} \times \mathbb{N} \times (C_1 \times C_2) \times \mathcal{F}^K \times \mathcal{F}^K$, henceforth referred to as the *configuration space*. Here, the first two dimensions represent the two counter values, the third dimension $C_1 \times C_2$ corresponds to the pair of counter states, and the remaining dimensions represent the weight vector. The projection of the configuration space onto the first two dimensions is depicted in Figure 7. We partition the configuration space into three: initial space, belt space, and background space. The size of the initial space and, thickness and number of belts will be polynomially bounded in K . This partition is indexed on two polynomials, $\text{poly}_2 : \mathbb{N} \rightarrow \mathbb{N}$ and $\text{poly}_3 : \mathbb{N} \rightarrow \mathbb{N}$ chosen so that all belts are disjoint outside the initial space. We use some properties of these partitions to show that the length of a minimal witness is bounded. We assume $\text{poly}_2(K) = 516K^{21}$ and $\text{poly}_3(K) = 42K^{14}$. The precise polynomials are required in the proofs of Lemma 26 and Lemma 32.

- *initial space*: All configuration pairs $\langle c, d \rangle$ such that $n_c, n_d < \text{poly}_2(K)$.
- *belt space*: Let $\alpha, \beta \in [1, 3K^7]$ be co-prime. A belt of slope $\frac{\alpha}{\beta}$ consists of those configuration pairs $\langle c, d \rangle$ outside the initial space that satisfies $|\alpha \cdot n_c - \beta \cdot n_d| \leq \text{poly}_3(K)$. The belt space contains all configuration pairs $\langle c, d \rangle$ that is inside belts with slope $\frac{\alpha}{\beta}$.
- *background space*: All remaining configuration pairs.

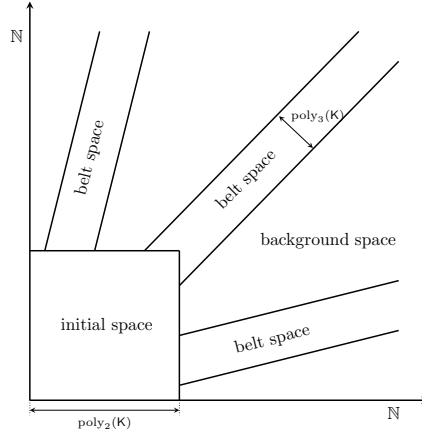


FIGURE 7. Projection of configuration space

The proof of the following lemma is similar to that of the non-weighted case presented in [3].

Lemma 26. *If $\langle c, d \rangle$ and $\langle e, f \rangle$ are configuration pairs inside two distinct belts and lie outside the initial space, then there is no $a \in \Sigma$ such that $\langle c, d \rangle \xrightarrow{a} \langle e, f \rangle$.*

Proof. Recall $\text{poly}_2(K) = 516K^{21}$ and $\text{poly}_3(K) = 42K^{14}$. Let B and B' be two distinct belts with μ being the slope of the belt B and μ' the slope of the belt B' . Hence $\mu \neq \mu'$. Without loss of generality, let us assume that $\mu' > \mu$. It suffices to show that for all $x > \text{poly}_2(K)$, we have

$$\mu x + \text{poly}_3(K) + 1 < \mu' x - \text{poly}_3(K) - 1.$$

We know that $\mu' - \mu \geq \frac{1}{3K^7}$ and $x > 516K^{21}$.

$$\text{Therefore, } \frac{516K^{21}}{6K^7} < (\mu' - \mu) \cdot x.$$

$$\begin{aligned} \implies \mu x + \frac{86K^{14}}{2} &< \mu' x - \frac{86K^{14}}{2} \\ \implies \mu x + 42K^{14} + K^{14} &< \mu' x - 42K^{14} - K^{14} \\ \implies \mu x + 42K^{14} + 1 &< \mu' x - 42K^{14} - 1 \end{aligned}$$

□

Lemma 26 ensures that the belts are disjoint outside the initial space and that no run can go from one belt to another without passing through the initial space or background space.

4.2. Belt space. We look at two scenarios in this section. First, we show that if a sub-run of the run of a minimal witness enters and exists a belt from the initial space, then the counter values encountered during this sub-run are polynomially bounded in K . Secondly, we show that if the run of a minimal witness never enters the background space, then the counter values encountered during this run are polynomially bounded in K . This is shown by reducing to co-VS reachability of an ODCA.

Let $\Pi_b = \mathbf{h}_i \tau_i \mathbf{h}_{i+1} \cdots \tau_{j-1} \mathbf{h}_j$ be a sub-run of the run of z inside a belt with slope $\frac{\alpha}{\beta}$. Similar to the technique mentioned in [5], each configuration pair \mathbf{h}_r , where $r \in [i, j]$ can alternatively be presented as $((\mathbf{x}_{c_r}, \mathbf{x}_{d_r}), p_{c_r}, p_{d_r}, l_r)$ where l_r denotes a line with slope $\frac{\alpha}{\beta}$ inside the given belt that contains the point (n_{c_r}, n_{d_r}) . Let L be the set of all lines with slope $\frac{\alpha}{\beta}$ inside the given belt. Note that $|L| = \text{poly}_3(K)$. The run Π_b is similar to the run of a weighted ODCA \mathcal{D} that has the tuple (p_{c_r}, p_{d_r}, l_r) as the state of the finite state machine and $\mathbf{x}_r \in \mathcal{F}^{2K}$ as its weight vector where $\mathbf{x}_r[i] = \mathbf{x}_{c_r}[i]$, if $i < K$ and $\mathbf{x}_r[i] = \mathbf{x}_{d_r}[i - K]$, otherwise. A formal definition of the ODCA \mathcal{D} is given below.

Definition 27. Let $\mathcal{A}_i = (C_i, \delta_i, p_{0_i}; Q_i, \lambda_i, \Delta_i, \eta_i)$ for $i \in \{1, 2\}$, be the two ODCAs given. Let L be the set of all lines with slope $\frac{\alpha}{\beta}$ inside the given belt. We define the ODCA $\mathcal{D} = (C, \delta, p_0; Q, \lambda, \Delta, \eta)$, where the initial state p_0 and the initial distribution λ are arbitrarily chosen.

- $C = C_1 \times C_2 \times L$ is a non-empty finite set of states.
- $\delta : C \times \Sigma \rightarrow C \times \{-1, 0, +1\}$ is the deterministic counter transition. Let $p_1, q_1 \in C_1, p_2, q_2 \in C_2, a \in \Sigma$ and $d_1, d_2 \in \{-1, 0, +1\}$. Let $l_1, l_2 \in L$ and $m_1, m_2 \in \mathbb{N}$, such that the point (m_1, m_2) lies on the line l_1 . $\delta((p_1, p_2, l_1), a) = ((q_1, q_2, l_2), d_1)$, if $\delta_1(p_1, a, 1) = (q_1, d_1)$ and $\delta_2(p_2, a, 1) = (q_2, d_2)$ and the point $(m_1 + d_1, m_2 + d_2)$ lies on the line l_2 . It is undefined otherwise.
- $Q = Q_1 \cup Q_2$ is a non-empty finite set of states of the finite state machine.

- $\Delta : C \times \Sigma \times \{0, 1\} \rightarrow \mathcal{F}^{2K \times 2K}$ gives the transition matrix for all $p \in C$, $a \in \Sigma$ and $d \in \{0, 1\}$. For $p_1 \in C_1, p_2 \in C_2, l \in L, m \in \mathbb{N}$ and $a \in \Sigma$,

$$\Delta((p_1, p_2, l), a)[i][j] = \begin{cases} \Delta(p_1, a, 1)[i][j], & \text{if } i, j < K \\ \begin{matrix} \Delta(p_2, a, 1)[i] & - \\ K[j] & - & K \end{matrix}, & \text{if } i, j > K \\ 0, & \text{otherwise} \end{cases}$$

- $\eta \in \mathcal{F}^{2K}$ is the final distribution.

$$\eta[i] = \begin{cases} \eta_1[i], & \text{if } i < K \\ \eta_2[i - K], & \text{otherwise} \end{cases}$$

The sub-run Π_b can now be seen as a floating run of a weighted ODCA \mathcal{D} . If the run Π ends inside a belt, then $\Pi_b = \mathbf{h}_i \tau_i \cdots \tau_{\ell-1} \mathbf{h}_\ell$. In this case, we show that the difference between the counter values of the first and last configuration pairs is smaller than a polynomial in K .

Lemma 28. *There is a polynomial $\text{poly} : \mathbb{N} \rightarrow \mathbb{N}$, such that if $\Pi_b = \mathbf{h}_i \tau_i \cdots \tau_{\ell-1} \mathbf{h}_\ell$ lies inside a belt, then $|n_{c_\ell} - n_{c_i}| \leq \text{poly}(K)$ and $|n_{d_\ell} - n_{d_i}| \leq \text{poly}(K)$.*

Proof. Let $\Pi_b = \mathbf{h}_i \tau_i \mathbf{h}_{i+1} \cdots \tau_{\ell-1} \mathbf{h}_\ell$ be a sub-run of the run of a minimal witness inside a belt and ends in the belt. As mentioned in Definition 27, we consider this as the run of the weighted ODCA \mathcal{D} . Since it is the run of a witness, $\mathbf{x}_j \boldsymbol{\eta}^\top \neq 0$. Consider the vector space $\mathcal{U} = \{\mathbf{y} \in \mathcal{F}^{2K} \mid \mathbf{y} \boldsymbol{\eta}^\top = 0\}$. Our problem now reduces to the co-VS coverability problem in machine \mathcal{D} and asks whether $(\mathbf{x}_i, (p_{c_i}, p_{d_i}, l_i), n_{c_i}) \xrightarrow{*} \overline{\mathcal{U}} \times \{(p_{c_\ell}, p_{d_\ell}, l_\ell)\} \times \mathbb{N}$. From Lemma 20, we know that the length of a minimal reachability witness for $((\mathbf{x}_i, (p_{c_i}, p_{d_i}, l_i), n_{c_i}), \overline{\mathcal{U}}, (p_{c_\ell}, p_{d_\ell}, l_\ell), \mathbb{N})$ is polynomially bounded in n_{c_i} and K . Hence proved. \square

In the following lemma, we show that if $\Pi_b = \mathbf{h}_i \tau_i \mathbf{h}_{i+1} \cdots \tau_{j-1} \mathbf{h}_j$ is a sub-run of Π inside a belt and either $n_{c_i} = n_{c_j}$ or $n_{d_i} = n_{d_j}$, then the counter values in Π_b cannot increase more than a polynomial in K from n_{c_i} and n_{d_i} .

Lemma 29. *There is a polynomial $\text{poly} : \mathbb{N} \rightarrow \mathbb{N}$ such that, if $\Pi_b = \mathbf{h}_i \tau_i \mathbf{h}_{i+1} \cdots \tau_{j-1} \mathbf{h}_j$ is a run inside a belt with $n_{c_i} = n_{c_j}$ or $n_{d_i} = n_{d_j}$, then the counter effect of any sub-run of Π_b is less than or equal to $\text{poly}(K)$.*

Proof. Let $\Pi_b = \mathbf{h}_i \tau_i \mathbf{h}_{i+1} \cdots \tau_{j-1} \mathbf{h}_j$ be a sub-run of the run of a minimal witness inside a belt such that $n_{c_i} = n_{c_j}$. We consider this as the run of the weighted ODCA \mathcal{D} as mentioned in Definition 27. Since it is the run of a witness, we know that there exists $\mathbb{A} \in \mathcal{F}^{2K \times 2K}$ such that $\mathbf{x}_j \mathbb{A} \boldsymbol{\eta}^\top \neq 0$. Consider the vector space $\mathcal{U} = \{\mathbf{y} \in \mathcal{F}^{2K} \mid \mathbf{y} \mathbb{A} \boldsymbol{\eta}^\top = 0\}$.

Our problem now reduces to the co-VS reachability problem in machine \mathcal{D} and asks whether $(\mathbf{x}_i, (p_{c_i}, p_{d_i}, l_i), n_{c_i}) \xrightarrow{*} \overline{\mathcal{U}} \times \{(p_{c_j}, p_{d_j}, l_j)\} \times \{n_{c_i}\}$. From Lemma 20, length of a minimal reachability witness for $((\mathbf{x}_i, (p_{c_i}, p_{d_i}, l_i), n_{c_i}), \overline{\mathcal{U}}, (p_{c_j}, p_{d_j}, l_j), \{n_{c_i}\})$ is bounded by a polynomial in n_{c_i} and K . Hence proved. \square

We have now shown that if the run of a minimal witness does not enter the background space, then the counter values in this run are polynomially bounded in K . Now we look at the case where the run enters the background space.

4.3. Background space. In this subsection, we consider the case where the run of a minimal witness enters the background space. We show that during the run of the minimal witness the counter values of the first configuration pair in the background space and the remaining length of the run is polynomially bounded in K .

Floating runs of a weighted ODCA are isomorphic to runs of a weighted automaton obtained by ignoring counter values. In order to bound the length of the run of a minimal witness in the background space, we introduce the notion of an *underlying uninitialised weighted automaton*.

Definition 30. For $l \in \{1, 2\}$, the underlying uninitialised weighted automaton of \mathcal{A} is the uninitialised weighted automaton $U(\mathcal{A}_l) = (Q'_l, \Delta'_l, \eta'_l)$, where $Q'_l = C_l \times Q_l$ and $\eta'_l \in \mathcal{F}^{K^2}$ is the final distribution. For $i < K^2$, $\eta'_l[i] = \eta_l[i \bmod K]$. The transition matrix is given by $\Delta'_l : \Sigma \rightarrow \mathcal{F}^{K^2 \times K^2}$. Let $a \in \Sigma$, $d \in \{-1, 0, +1\}$, $i, j < K^2$,

$$\Delta'_l(a)[i][j] = \begin{cases} \Delta_l(p_{\frac{i}{K}}, a, 1)[i \bmod K][j \bmod K], \\ \quad \text{if } \delta_l(p_{\frac{i}{K}}, a, 1) = (p_{\frac{j}{K}}, d) \\ 0 \text{ otherwise} \end{cases}$$

Note that a configuration of $U(\mathcal{A}_l)$ is a vector of dimension K^2 . A configuration \mathbf{c} of a weighted ODCA \mathcal{A} is said to be k -equivalent to a configuration $\bar{\mathbf{c}}$ of an uninitialised weighted automata \mathcal{B} , denoted $\mathbf{c} \sim_k \bar{\mathbf{c}}$, if for all $w \in \Sigma^{\leq k}$, $f_{\mathcal{A}}(w, \mathbf{c}) = f_{\mathcal{B}}(w, \bar{\mathbf{c}})$. We say that \mathbf{c} is not k -equivalent to $\bar{\mathbf{c}}$ otherwise and denote this as $\mathbf{c} \not\sim_k \bar{\mathbf{c}}$.

As we need to test the equivalence of configurations from \mathcal{A}_1 and \mathcal{A}_2 , we consider the uninitialised weighted automata \mathcal{B} , which is a disjoint union of $U(\mathcal{A}_1)$ and $U(\mathcal{A}_2)$. This gives us a single automaton with which we can compare their configurations. Let $i \in \{1, 2\}$ and \mathbf{c} be a configuration of \mathcal{A}_i . For all $p \in C_i$ and $m < 2K^2$, we define the sets $\mathcal{W}_i^{p,m}$. The set $\mathcal{W}_i^{p,m}$ contains vectors $\mathbf{x} \in \mathcal{F}^K$ such that the configuration (\mathbf{x}, p, m) is $2K^2$ -equivalent to some configuration of \mathcal{B} . The set $\overline{\mathcal{W}}_i^{p,m}$ is the set $\mathcal{F}^K \setminus \mathcal{W}_i^{p,m}$. Formally,

$$\mathcal{W}_i^{p,m} = \{\mathbf{x} \in \mathcal{F}^K \mid \exists \bar{\mathbf{c}} \in \mathcal{F}^{2K^2}, \mathbf{c} = (\mathbf{x}, p, m) \sim_{2K^2} \bar{\mathbf{c}}\}$$

Lemma 31. For any $i \in \{1, 2\}$, $p \in C_i$ and $m < 2K^2$, the set $\mathcal{W}_i^{p,m}$ is a vector space.

Proof. To prove this, it suffices to show that it is closed under vector addition and scalar multiplication. We fix a set $\mathcal{W}_i^{p,m}$. First, we prove that it is closed under scalar multiplication. For any vector $\mathbf{z}_1 \in \mathcal{W}_i^{p,m}$, we know that there exists a configuration $\mathbf{c} = (\mathbf{z}_1, p, m)$ and $\bar{\mathbf{c}} \in \mathcal{F}^{2K^2}$ such that $\mathbf{c} \sim_{2K^2} \bar{\mathbf{c}}$. Now, for any scalar $r \in \mathcal{F}$, the configuration $(r \cdot \mathbf{z}_1, p, m) \sim_{2K^2} r \cdot \bar{\mathbf{c}}$. Therefore $r \cdot \mathbf{z}_1 \in \mathcal{W}_i^{p,m}$. Now, we show that it is closed under vector addition. Let $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{W}_i^{p,m}$ be two vectors. Therefore, there exists configurations $\mathbf{c}_1 = (\mathbf{z}_1, p, m)$, $\mathbf{c}_2 = (\mathbf{z}_2, p, m)$, $\bar{\mathbf{c}}_1 \in \mathcal{F}^{2K^2}$ and $\bar{\mathbf{c}}_2 \in \mathcal{F}^{2K^2}$, such that $\mathbf{c}_1 \sim_{2K^2} \bar{\mathbf{c}}_1$ and $\mathbf{c}_2 \sim_{2K^2} \bar{\mathbf{c}}_2$. Consider the configuration $\mathbf{c}_3 = (\mathbf{z}_1 + \mathbf{z}_2, p, m)$, $\mathbf{c}_3 \sim_{2K^2} \bar{\mathbf{c}}_1 + \bar{\mathbf{c}}_2$. Therefore, $\mathbf{z}_1 + \mathbf{z}_2 \in \mathcal{W}_i^{p,m}$. \square

The distance of a configuration \mathbf{c} of \mathcal{A}_i is the length of a minimal word that takes you from \mathbf{c} to a configuration (\mathbf{x}, p, m) for some $m < 2K^2$ and $p \in C_i$ such that $\mathbf{x} \in \overline{\mathcal{W}}_i^{p,m}$. We define $\text{dist}_{\mathcal{A}_i}(\mathbf{c})$ as:

$$\min\{|w| \mid \mathbf{c} \xrightarrow{w} (\mathbf{x}, p, m) \exists p \in C_i, m < 2K^2, \mathbf{x} \in \overline{\mathcal{W}}_i^{p,m}\}$$

The notion of distance play a key role in determining which parts of the run of a witness can be pumped out if it is not minimal. The following lemma is a crucial component in proving the equivalence of weighted ODCA. By Lemma 22, the lexicographically minimal reachability witness has a special form and this plays a crucial role in proving Lemma 32.

Lemma 32. *Let \mathbf{c} be a configuration of weighted ODCA \mathcal{A} . If $\text{dist}_{\mathcal{A}}(\mathbf{c}) < \infty$ then, $\text{dist}_{\mathcal{A}}(\mathbf{c}) = \frac{a}{b}n_c + d$ where $a, b \in [0, 3K^7]$ and $|d| < 42K^{14}$.*

Proof. Without loss of generality, let us consider the weighted ODCA \mathcal{A}_1 and a configuration \mathbf{c} of \mathcal{A}_1 . Let us assume that $\text{dist}_{\mathcal{A}_1}(\mathbf{c}) < \infty$. This means that $\mathbf{c} \rightarrow^* \mathbf{d}$ with $\mathbf{x}_d \in \overline{W}_1^{p,m}$ for some $p \in C_1$ and $m < 2K^2$. Since $n_d = m$, by Lemma 22, we know that there is a word $u = u_1 u_2^r u_3$ (with $r \geq 0$) such that that $\mathbf{c} \xrightarrow{u} \mathbf{d}$ where $|u| = \text{dist}_{\mathcal{A}_1}(\mathbf{c})$, $|u_1 u_3| \leq 9K^7$, $|u_2| \leq 3K^7$ and u_2 has a negative counter effect ℓ . Let g be the combined counter effect of u_1, u_3 and $\alpha = \frac{|u_2|}{\ell}$. Since $|u_1 u_3| \leq 9K^7$, we have $|g| \leq 9K^7$.

$$\begin{aligned} \text{dist}_{\mathcal{A}_1}(\mathbf{c}) &= \frac{n_c - n_d - g}{\ell} |u_2| + |u_1 u_3| \\ &= \alpha n_c - \underbrace{\alpha(n_d + g)}_d + |u_1 u_3| \end{aligned}$$

Since $1 \leq \alpha \leq 3K^7$ it follows that $-42K^{14} < d < 42K^{14}$. Hence proved. \square

The polynomials poly_1 and poly_2 were picked so that the configuration pairs with equal distance always lie in the belt space. Therefore, the background space points either have unequal or infinite distances.

Lemma 33. *For any configuration pair $\langle \mathbf{c}, \mathbf{d} \rangle$, in the background space, either $\text{dist}_{\mathcal{A}_1}(\mathbf{c}) \neq \text{dist}_{\mathcal{A}_2}(\mathbf{d})$ or $\text{dist}_{\mathcal{A}_1}(\mathbf{c}) = \text{dist}_{\mathcal{A}_2}(\mathbf{d}) = \infty$.*

Proof. Assume for contradiction that there is a configuration pair $\langle \mathbf{c}, \mathbf{d} \rangle$, in the background space such that $\text{dist}_{\mathcal{A}_1}(\mathbf{c}) = \text{dist}_{\mathcal{A}_2}(\mathbf{d}) < \infty$. Since $\text{dist}_{\mathcal{A}_1}(\mathbf{c}) = \text{dist}_{\mathcal{A}_2}(\mathbf{d})$. From Lemma 32, there exists $a_1, b_1, a_2, b_2 \in [0, 3K^7]$ and $d_1, d_2 < 42K^{14}$ such that

$$\frac{a_1}{b_1} n_c + d_1 = \text{dist}_{\mathcal{A}_1}(\mathbf{c}) = \text{dist}_{\mathcal{A}_1}(\mathbf{d}) = \frac{a_2}{b_2} n_d + d_2$$

Therefore $|\frac{a_1}{b_1} n_c - \frac{a_2}{b_2} n_d| \leq |d_2 - d_1| < 42K^{14}$. This satisfies the belt condition and is a configuration pair in the belt space. This contradicts our initial assumptions. \square

The following lemma shows that the length of the run Π in the background space is polynomially bounded in K and the counter values of the first background point in Π . The proof is similar to that in [3] and is given below.

Lemma 34. *If $\mathbf{h}_j = \langle \mathbf{c}_j, \mathbf{d}_j \rangle$ is the first configuration pair in the background space during Π , then $\ell - j$ is bounded by a polynomial in n_{c_j}, n_{d_j} and K .*

Proof. Let $\mathbf{h}_j = \langle \mathbf{c}_j, \mathbf{d}_j \rangle$ be the first configuration pair in the background space during the run Π , then from Lemma 33, either $\text{dist}_{\mathcal{A}_1}(\mathbf{c}_j) = \text{dist}_{\mathcal{A}_2}(\mathbf{d}_j) = \infty$ or $\text{dist}_{\mathcal{A}_1}(\mathbf{c}_j) \neq \text{dist}_{\mathcal{A}_2}(\mathbf{d}_j)$. We separately consider the two cases.

Case-1, $\text{dist}_{\mathcal{A}_1}(\mathbf{c}_j) = \text{dist}_{\mathcal{A}_2}(\mathbf{d}_j) = \infty$: then we prove that the remaining length of the witness from $\langle \mathbf{c}_j, \mathbf{d}_j \rangle$ is bounded by $2K^2$. Assume for contradiction that this is not the case and $\mathbf{c}_j \sim_{2K^2} \mathbf{d}_j$ but $\mathbf{c}_j \not\equiv \mathbf{d}_j$. Let $v \in \Sigma^{>2K^2}$ be the word which

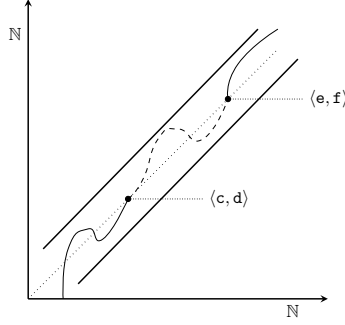


FIGURE 8. The figure depicts an α - β repetition inside a belt with slope $\frac{\alpha}{\beta}$. The configuration pairs $\langle c, d \rangle$ and $\langle e, f \rangle$ are α - β -related. i.e., they lie on a line with slope $\frac{\alpha}{\beta}$, $p_c = p_e$ and $p_d = p_f$.

distinguishes c and d . Therefore, there exists a prefix of v , $u \in \Sigma^{|v|-2K^2}$, and $i = \ell - 2K^2$ such that $\langle c_j, d_j \rangle \xrightarrow{u} \langle c_i, d_i \rangle$ and $c_i \not\equiv_{2K^2} d_i$.

Since v is a minimal witness $c_i \equiv_{2K^2-1} d_i$ and $c_i \not\equiv_{2K^2} d_i$. Since $\text{dist}_{A_1}(c_j) = \text{dist}_{A_2}(d_j) = \infty$, there exists configurations \bar{c}_i and \bar{d}_i in the underlying automaton \mathcal{B} such that $c_i \sim_{2K^2} \bar{c}_i$ and $d_i \sim_{2K^2} \bar{d}_i$. Since $c_i \equiv_{2K^2-1} d_i$, it follows that $\bar{c}_i \sim_{2K^2-1} \bar{d}_i$. From the equivalence result of weighted automata, we know that if two configurations of a weighted automata with k states are non-equivalent, then there is a word of length less than k which distinguishes them. Therefore, this is sufficient to prove that the underlying weighted automata with \bar{c}_i and \bar{d}_i as initial distributions are equivalent, and thus $\bar{c}_i \sim_{2K^2} \bar{d}_i$. This allows us to deduce that $c_i \equiv_{2K^2} d_i$, which is a contradiction. Therefore, the remaining length of the witness from $\langle c_j, d_j \rangle$ is bounded by $2K^2$.

Case-2, $\text{dist}_{A_1}(c_j) \neq \text{dist}_{A_2}(d_j)$: Without loss of generality, we suppose $\text{dist}_{A_1}(c_j) > \text{dist}_{A_2}(d_j)$. By definition of dist_{A_2} , there exists $u \in \Sigma^{\text{dist}_{A_2}(d_j)}$, $i > j$ and a configuration \bar{c} of the underlying automaton \mathcal{B} such that $c_j \xrightarrow{u} c_i$, $d_j \xrightarrow{u} d_i$, $c_i \sim_{2K^2} \bar{c}_i$ and $d_i \not\sim_{2K^2} \bar{c}_i$. Therefore $c_i \not\equiv_{2K^2} d_i$. By definition, there exists $v \in \Sigma^{\leq 2K^2}$ such that $f_{A_1}(v, c_i) \neq f_{A_2}(v, d_i)$ and hence $f_{A_1}(uv, c_j) \neq f_{A_2}(uv, d_j)$. As $uv \in \Sigma^{\text{dist}_{A_2}(d_j)+2K^2}$, we get that $c_j \not\equiv_{\text{dist}_{A_2}(d_j)+2K^2} d_j$. Therefore, there is $w \in \Sigma^{\leq \min\{\text{dist}_{A_1}(c_j), \text{dist}_{A_2}(d_j)\}+2K^2}$ that distinguishes c_j and d_j . \square

Let $\alpha, \beta \in [1, 3K^7]$ be co-prime. We say configuration pairs $\langle c, d \rangle$ and $\langle e, f \rangle$ are α - β related if $p_c = p_e$, $p_d = p_f$ and $\alpha \cdot n_c - \beta \cdot n_d = \alpha \cdot n_e - \beta \cdot n_f$. Roughly speaking, two configuration pairs are α - β related if they have the same state pairs and lie on a line with slope $\frac{\alpha}{\beta}$. An α - β repetition is a run $\pi_1 = c_i \tau_i c_{i+1} \tau_{i+1} \cdots \tau_{j-1} c_j$ that lies inside a belt with slope $\frac{\alpha}{\beta}$ such that c_i and c_j are α - β related. The following lemma bounds the counter values of the first configuration in the background space, if it exists, during the run Π .

Lemma 35. *If h_j is the first background point in Π then, counter values of h_j are less than $K^5 \cdot 42K^{14}$.*

Proof. Let h_j be the first point in the background space during the run Π . Assume for contradiction that n_{c_j} is greater than $K^5 \cdot 42K^{14}$. Let $\Pi = h_0 \tau_0 \cdots h_{j-1} \tau_{j-1} h_j \cdots h_\ell$ be a run of a minimal witness. Since h_j is the first point in the background space

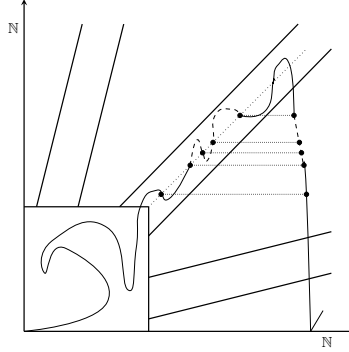


FIGURE 9. The figure shows the run of a word that enters the background space from the belt such that the counter values of the first configuration pair in the background space exceed a polynomial bound.

in this run and $n_{c_j} > K^5 \cdot 42K^{14}$, there exists $0 < i < j$ such that the sub-run $\Pi_b = \mathbf{h}_i \tau_i \mathbf{h}_{i+1} \cdots \tau_{j-2} \mathbf{h}_{j-1}$ lies inside a belt B with slope $\frac{\alpha}{\beta}$ for some $\alpha, \beta \in [1, 3K^7]$. Since we are looking at the run of a minimal witness, from Lemma 33 either $c_j \not\equiv_{2K^2} d_j$ or $\text{dist}(c_j) \neq \text{dist}(d_j)$. We separately consider the two cases.

Case-1: $\text{dist}_{A_1}(c_j) \neq \text{dist}_{A_2}(d_j)$: Without loss of generality, let us assume $\text{dist}_{A_1}(c_j) < \text{dist}_{A_2}(d_j)$. Therefore there exists $t \in \mathbb{N}$ with $j < t \leq \ell$ and configuration pair \mathbf{h}_t such that $m = n_{c_t} < 2K^2$, $p = p_{c_t}$ and $\mathbf{x}_{c_t} \in \overline{\mathcal{W}}_1^{p,m}$. We show that we can pump some portion out from Π_b to reach a configuration in the background space with unequal distance and smaller counter values.

Since $n_{c_j} > K^5 \cdot 42K^{14}$, by Pigeonhole principle, there exists indices $i_0 < i_1 < i_2 < \cdots, i_{K^2} < i'_0 < i'_1 < i'_2, \cdots, i'_{K^2}$ such that for all $r \in [1, K^2]$, (1) $\mathbf{h}_{i_{r-1}}$ and \mathbf{h}_{i_r} are α - β related and lie in belt B , (2) $n_{c_{i_{r-1}}} < n_{c_{i_r}} = n_{c_{i'_r}}$, (3) $p_{c_{i'_r}} = p_{c_{i'_{r-1}}}$, (4) for all $t \in \mathbb{N}$ with $i_r < t < j$, $n_{c_t} > n_{c_{i_r}}$, and (5) for all $t \in \mathbb{N}$ with $j < t < i'_r$, $n_{c_t} < n_{c_{i'_r}}$.

For $r \in [0, K^2]$ let $\mathbb{A}_r \in \mathcal{F}^{K \times K}$ denote the matrix such that $\mathbf{x}_{c_{i_r}} \mathbb{A}_r = \mathbf{x}_{c_{i'_r}}$ and $\mathbb{B}_r \in \mathcal{F}^{K \times K}$ denote the matrix such that $\mathbf{x}_{c_{i'_r}} \mathbb{B}_r = \mathbf{x}_{c_t} \in \overline{\mathcal{W}}_1^{p,m}$. Therefore for all $r \in [0, K^2]$, we have $\mathbf{x}_{c_{i_r}} \mathbb{A}_r \mathbb{B}_r \in \overline{\mathcal{W}}_1^{p,m}$. From Lemma 4, we have that there exists $s, r \in [0, K^2]$ with $s < r$ such that $\mathbf{x}_{c_{i_s}} \mathbb{A}_r \mathbb{B}_s \in \overline{\mathcal{W}}_1^{p,m}$. Consider the sequence of transitions $T' = \tau_0, \cdots, \tau_{i_s-1} \tau_{i_r}, \cdots, \tau_{j-1}$ and let $w = \text{word}(T')$. Let $\mathbf{h}_{j'}$ be the configuration such that $\mathbf{h}_0 \xrightarrow{w} \mathbf{h}_{j'}$. Since we have removed an α - β repetitions inside the belt, the configuration pair $\mathbf{h}_{j'}$ is a point in the background space (see Figure 9). Moreover, $n_{c_{j'}} < n_{c_j}$ and $\text{dist}_{A_1}(c_{j'}) < \infty$. Since it is a point in the background space, from Lemma 33, we get that $\text{dist}_{A_1}(c_{j'}) \neq \text{dist}_{A_2}(d_{j'})$. Therefore, there is a shorter path to a configuration in background space with smaller counter values and unequal distance. This is a contradiction.

Case-2: $c_j \not\equiv_{2K^2} d_j$: Consider the sub-run Π_b . Since it is a run inside a belt, we can consider this as the run of the weighted ODCA \mathcal{D} . Since $n_{c_j} > K^4 \cdot 42K^{14}$, by Pigeon-hole principle, there exists indices $i_0, i_1, i_2, \cdots, i_{K^2}$ such that for all $r \in$

$[1, K^2]$, $\mathbf{h}_{i_{r-1}}$ and \mathbf{h}_{i_r} are α - β related with $n_{c_{i_{r-1}}} < n_{c_{i_r}}$ and for all $t \in \mathbb{N}$ with $i_r < t < j$, $n_{c_t} > n_{c_{i_r}}$.

Since it is the run of a minimal witness, we know that there exists $\mathbb{A} \in \mathcal{F}^{2K \times 2K}$ such that $\mathbf{x}_{j-1} \mathbb{A} \eta_F^\top \neq 0$. Consider the vector space $\mathcal{U} = \{\mathbf{y} \in \mathcal{F}^{2K} \mid \mathbf{y} \mathbb{A} \eta_F^\top = 0\}$. For $r \in [0, K^2]$, let \mathbb{A}_r denote the matrices such that $\mathbf{x}_{i_r} \mathbb{A}_r = \mathbf{x}_j \in \mathcal{U}$. Since $\mathbf{x}_{i_r} \mathbb{A}_r \in \overline{\mathcal{U}}$ for all $r \in [0, K^2]$, from Lemma 4, we get that there exists $r' \in [0, r-1]$ such that $\mathbf{x}_{c_{i'_r}} \mathbb{A}_r \in \overline{\mathcal{V}}$. The sequence of transitions $\tau_{i_r+1} \cdots \tau_\ell$ can be taken from $\mathbf{h}_{i'_r}$ since the counter values always stay positive. Consider the sequence of transitions $T' = \tau_0 \cdots \tau_{i'_r} \tau_{i_r+1} \cdots \tau_\ell$ and let $w = \text{word}(T')$. The word w is a shorter witness than z and contradicts its minimality. \square

Finally, we prove that the counter values encountered during the run Π are polynomially bounded in K using above lemmas.

Proof of Lemma 24. Consider the run Π . From Lemma 28, Lemma 29 and Lemma 35, we get that the counter values of configuration pairs inside the belt space during this run is polynomially bounded in K . Therefore, if it exists, the first background point in Π has polynomially bounded counter values. From Lemma 34, the length of Π after the first background point is polynomially bounded in K . Since initial space is already bounded by a polynomial in K , the maximum counter value in Π is polynomially bounded in K . \square

5. REGULARITY OF ODCA IS IN P

We say that a weighted ODCA $\mathcal{A} = (C, \delta, p_0; Q, \lambda, \Delta, \eta)$ is regular if there is a weighted automaton \mathcal{B} that is equivalent to it. We look at the regularity problem - the problem of deciding whether a weighted ODCA is regular. We fix a weighted ODCA $\mathcal{A} = (C, \delta, p_0; Q, \lambda, \Delta, \eta)$ and use N to denote $|C| \cdot |Q|$.

The proof technique is adapted from the ideas developed by Böhm et al. [6] in the context of real-time OCA. The crucial idea in proving regularity is to check for the existence of infinitely many equivalence classes. The proof relies on the notion of distance of configurations. Distance of a configuration is the length of a minimal word to be read to reach a configuration that does not have an N equivalent configuration in the underlying automata. The challenge is to find infinitely many configurations reachable from the initial configuration, so that no two of them have same distance. Our main contribution is in designing a “pumping” like argument to show this.

Theorem 36. *There is a polynomial time algorithm to decide whether a weighted ODCA is equivalent to some weighted automata.*

Recall the definition of $U(\mathcal{A})$ from Definition 30. We use \mathbf{c} to denote some configuration of \mathcal{A} and $\bar{\mathbf{c}}$ to denote some configuration of $U(\mathcal{A})$. For a $p \in C$ and $m \in \mathbb{N}$, we define

$$\mathcal{W}^{p,m} = \{\mathbf{x} \in \mathcal{F}^{|Q|} \mid \exists \bar{\mathbf{c}} \in \mathcal{F}^N, \mathbf{c} = (\mathbf{x}, p, m) \sim_N \bar{\mathbf{c}}\}.$$

The set $\overline{\mathcal{W}}^{p,m}$ is $\mathcal{F}^{|Q|} \setminus \mathcal{W}^{p,m}$. The distance of a configuration \mathbf{c} (denoted by $\text{dist}(\mathbf{c})$) is

$$\min\{|w| \mid \mathbf{c} \xrightarrow{w} (\mathbf{x}, p, m) \exists p \in C, m < N, \text{ and } \mathbf{x} \in \overline{\mathcal{W}}^{p,m}\}.$$

The following lemma shows when \mathcal{A} is not regular. Clean up, move to appendix, proof sketch.

Lemma 37. *Let c be an initial configuration of an ODCA \mathcal{A} . Then the following are equivalent.*

- (1) \mathcal{A} is not regular.
- (2) for all $t \in \mathbb{N}$, there exists configurations d, e s.t. $n_e < N, c \xrightarrow{*} d \xrightarrow{*} e$, $\mathbf{x}_e \in \overline{\mathcal{W}}^{p_e, n_e}$ and $t < \text{dist}(d) < \infty$.
- (3) there exists configurations d, e and a run $c \xrightarrow{*} d \xrightarrow{*} e$ s.t. $N^2 + N \leq n_d \leq 2N^2 + N$, $\mathbf{x}_e \in \overline{\mathcal{W}}^{p_e, n_e}$ with $n_e < N$.

Proof. $3 \rightarrow 2$: Consider an arbitrary $q \in C$, $m < N$ and vector space $\mathcal{V} = \mathcal{W}^{q, m}$. Let us assume for contradiction the complement of Point 2. That is, there exists a $t \in \mathbb{N}$ such that for all configurations d' where $c \xrightarrow{*} d' \xrightarrow{*} \overline{\mathcal{V}} \times \{q\} \times \{m\}$, $\text{dist}(d') \leq t$. Note that for all d' where $n_{d'} > N$, $\text{dist}(d') \geq n_{d'} - N$. Hence there exists an $M \in \mathbb{N}$ such that for all d' where $c \xrightarrow{*} d' \xrightarrow{*} \overline{\mathcal{V}} \times \{q\} \times \{m\}$, $n_{d'} \leq M$.

Consider a configuration d where $n_d > N^2 + N$ and a run $c \xrightarrow{*} d \xrightarrow{*} \overline{\mathcal{V}} \times \{q\} \times \{m\}$. Point 3 shows the existence of such a run. For contradiction, it suffices to show there exists a d' such that $c \xrightarrow{*} d' \xrightarrow{*} \overline{\mathcal{V}} \times \{q\} \times \{m\}$ and $n_{d'} > n_d$.

Let $m = |Q|^2 + 1$. Since $n_d > N^2 + N$, by Pigeonhole principle, there exists set of indices $X = \{i_1, i_2, \dots, i_m\}$ such that for $k, l \in [1, m]$ with $k < l$, we have $i_k < i_l$, and for all $h, j \in X$ $p_{c_h} = p_{c'_h} = p_{c_j} = p_{c'_j}$. Let u_j, v_j, w_j be words such that for all $j \in X$, $c \xrightarrow{u_j} c_j \xrightarrow{v_j} c'_j \xrightarrow{w_j} e$. For all $j \in X$, let matrix \mathbb{A}_j and \mathbb{B}_j be such that $\mathbf{x}_{c'_j} = \mathbf{x}_{c_j} \mathbb{A}_j$ and $\mathbf{x}_e = \mathbf{x}_{c'_j} \mathbb{B}_j$. We know that for all $j \in X$, $\mathbf{x}_{c_j} \mathbb{A}_j \mathbb{B}_j \in \overline{\mathcal{V}}$. List the matrices $\mathbb{A}_{i_1}, \mathbb{A}_{i_2}, \dots, \mathbb{A}_{i_m}$ in sequence. From Lemma 4, it follows that, there exists $i, j \in X$ with $i < j$ such that $\mathbf{x}_{c_j} \mathbb{A}_i \mathbb{B}_j \in \overline{\mathcal{V}}$. Consider the run $\pi(u_j v_i w_j, c_1)$. It contains a configuration d' where $n_{d'} > n_d$.

$2 \rightarrow 1$: Assume for contradiction that for all $t \in \mathbb{N}$, there exists configurations d, e such that $c \xrightarrow{*} d \xrightarrow{*} e$, $\mathbf{x}_e \in \overline{\mathcal{W}}^{p_e, n_e}$, $n_e < N$ and $t < \text{dist}(d) < \infty$ but \mathcal{A} is regular. Let \mathcal{B} be the weighted automaton equivalent to \mathcal{A} . We use $|\mathcal{B}|$ to represent the number of states of \mathcal{B} .

Let $t_1, t_2, \dots, t_{|\mathcal{B}|+1} \in \mathbb{N}$ such that for $i \in [1, |\mathcal{B}|]$, $t_i < t_{i+1}$, and \mathbf{d}_{t_i} be such that $c \xrightarrow{*} \mathbf{d}_{t_i} \xrightarrow{*} (\mathbf{x}_i, p_e, n_e)$, $\mathbf{x}_i \in \overline{\mathcal{W}}^{p_e, n_e}$ and $t_i < \text{dist}(\mathbf{d}_{t_i}) < t_{i+1} < \infty$. Clearly $\mathbf{d}_{t_i} \neq \mathbf{d}_{t_j}$ for $i \neq j$ and hence corresponds to two different states of \mathcal{B} . Since we can find more than $|\mathcal{B}|$ pairwise non-equivalent configurations, it contradicts the assumption that \mathcal{B} is equivalent to \mathcal{A} .

$1 \rightarrow 3$: We prove the contrapositive of the statement. Let us assume that there is no configurations d, e and a run $c \xrightarrow{*} d \xrightarrow{*} e$ such that $N^2 + N \leq n_d \leq 2N^2 + N$, $\mathbf{x}_e \in \overline{\mathcal{W}}^{p_e, n_e}$ with $n_e < N$. This implies that there does not exist a configuration d' such that $n_{d'} > 2N^2$, $c \xrightarrow{*} d' \xrightarrow{*} (\mathbf{y}, p_e, n_e)$ for some $\mathbf{y} \in \overline{\mathcal{W}}^{p_e, n_e}$. Assume for contradiction that there is such a run, then there exists a portion in this run that can be “pumped down” to get a run $c \xrightarrow{*} d'' \xrightarrow{*} (\mathbf{y}', p_e, n_e)$ for some configuration d'' such that $N^2 + N \leq n_{d''} \leq 2N^2 + N$ and $\mathbf{y}' \in \overline{\mathcal{W}}^{p_e, n_e}$. This is a contradiction. Therefore all runs starting from configuration with counter value greater than or equal to $N^2 + N$ “looks” similar to a run on a weighted automaton. This allows us to simulate the runs of \mathcal{A} using a weighted automaton. \square We now prove that the regularity problem for weighted ODCA is decidable in polynomial time.

Proof of Theorem 36. Let \mathcal{A} be a weighted ODCA. Lemma 37 shows that if \mathcal{A} is not regular, then there are words $u, v \in \Sigma^*$ and configurations d, e such that there

is a run of the form $c \xrightarrow{u} d \xrightarrow{v} e$ such that $N^2 + N \leq n_d \leq 2N^2 + N$, $\mathbf{x}_e \in \overline{\mathcal{W}}^{p_e, n_e}$ with $n_e < N$. The existence of such words u and v can be decided in polynomial time since the minimal length of such a path if it exists, is polynomially bounded in the number of states of the weighted ODCA by Lemma 20. This concludes the proof. \square

6. COVERING

Let \mathcal{A}_1 and \mathcal{A}_2 be two uninitialised weighted ODCA's. We say \mathcal{A}_2 *covers* \mathcal{A}_1 if for all initial configurations c_0 of \mathcal{A}_1 there exists an initial configuration d_0 of \mathcal{A}_2 such that $\mathcal{A}_1\langle c_0 \rangle$ and $\mathcal{A}_2\langle d_0 \rangle$ are equivalent. We say \mathcal{A}_1 and \mathcal{A}_2 are *coverable equivalent* if \mathcal{A}_1 covers \mathcal{A}_2 and \mathcal{A}_2 covers \mathcal{A}_1 . We show that the covering and coverable equivalence problems for uninitialised weighted ODCA's are decidable in polynomial time. The proof relies on the algorithm to check the equivalence of two weighted ODCA's and is given below.

Theorem 38. *Covering and coverable equivalence problems of uninitialised weighted ODCA's are in P.*

Proof. We fix two uninitialised weighted ODCA's $\mathcal{A}_1 = (C_1, \delta_1; Q_1, \Delta_1, \boldsymbol{\eta}_1)$ and $\mathcal{A}_2 = (C_2, \delta_2; Q_2, \Delta_2, \boldsymbol{\eta}_2)$ for this section. Without loss of generality, assume $K = |C_1| = |Q_1| = |C_2| = |Q_2|$. For $i \in [1, K]$ we define the vector $\mathbf{e}_i \in \mathcal{F}^K$ as follows:

$$\mathbf{e}_i[j] = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

For $j \in [1, K]$, $q \in C_1$, we use $\mathbf{h}_{j,q}$ to denote the configuration $(\mathbf{e}_j, q, 0)$ of \mathcal{A}_1 and for $i \in [1, K]$, $p \in C_2$, we use $\mathbf{g}_{i,p}$ to denote the configuration $(\mathbf{e}_i, p, 0)$ of \mathcal{A}_2 .

Claim 1. *There is a polynomial time algorithm to decide whether \mathcal{A}_2 covers $\mathcal{A}_1\langle \mathbf{h}_{j,q} \rangle$ for any $j \in [1, K]$ and $q \in C_1$.*

Proof: First, we check, in polynomial time (equivalence with a zero machine), whether $\mathcal{A}_1\langle \mathbf{h}_{j,q} \rangle$ accepts all words with weight $0 \in \mathcal{F}$. If that is the case, then $\mathcal{A}_1\langle \mathbf{h}_{j,q} \rangle$ and $\mathcal{A}_2\langle \mathbf{g}_0 \rangle$ are equivalent for the configuration $\mathbf{g}_0 = (\{0\}^K, p, 0)$, for any $p \in C_2$. Otherwise, there is some word w_0 accepted by $\mathcal{A}_1\langle \mathbf{h}_{j,q} \rangle$ with non-zero weight s (returned by the previous equivalence check). Without loss of generality, we consider the smallest one, whose size is polynomial in K .

We pick a $p \in C_2$ and check whether there exists an initial distribution from counter state p that makes the two machines equivalent. Assume that such an initial distribution exists and for all $i \in [1, K]$, let α_i denote the initial weight on state $q_i \in Q_2$. We use $\boldsymbol{\alpha}$ to denote the resultant initial distribution. We initialise an empty set B to store a system of linear equations.

The following steps will be repeated at most K times to check the existence of an initial distribution with initial state $p \in C_2$. Let w be the counter-example returned by the equivalence query in the previous step. For all $i \in [1, K]$, we compute $f_{\mathcal{A}_2\langle \mathbf{g}_{i,p} \rangle}(w)$. We add the linear equation $\sum_{i=1}^K \alpha_i \cdot f_{\mathcal{A}_2\langle \mathbf{g}_{i,p} \rangle}(w) = f_{\mathcal{A}_1\langle \mathbf{h}_{j,q} \rangle}(w)$ to B and compute values for α_i , $i \in [1, K]$, such that it satisfies the system of linear equations in B . We check whether $\mathcal{A}_1\langle \mathbf{h}_{j,q} \rangle$ and $\mathcal{A}_2\langle (\boldsymbol{\alpha}, p, 0) \rangle$ are equivalent or not. If they are not equivalent, we get a new counter example that distinguishes them. Now we repeat the procedure to compute a new initial distribution.

Note that the above procedure is executed at most K times to find an initial distribution if it exists. This is because we can find only K many linearly independent linear equations in K variables. Suppose the above procedure fails to find an initial distribution for which the machines are equivalent. In that case, there is an initial distribution of \mathcal{A}_1 with initial counter state q , for which \mathcal{A}_2 with initial counter state p does not have an equivalent initial distribution. We now pick a different counter state of C_2 and repeat the process until we find a $p \in C_2$ for which the algorithm finds an equivalent initial distribution. If for all $p \in C_2$, the algorithm returns false, then \mathcal{A}_2 does not cover $\mathcal{A}_1\langle h_{j,q} \rangle$. $\square_{\text{Claim:1}}$

First, we show the existence of a polynomial time procedure to check whether \mathcal{A}_2 covers \mathcal{A}_1 . For all $j \in [1, K]$, we check whether there exists an initial state $p \in C_2$ such that \mathcal{A}_2 with initial counter state p has an initial distribution that makes it equivalent to $\mathcal{A}_1\langle h_{j,q} \rangle$ using Claim 1. If we fail to find such a state in C_2 then we return false. We repeat this procedure for all $q \in C_1$. If for all $q \in C_1$ there exists a $p \in C_2$ such that \mathcal{A}_2 with initial counter state p has an initial distribution that makes it equivalent to $\mathcal{A}_1\langle h_{j,q} \rangle$ for all $j \in [1, K]$, then we say that \mathcal{A}_2 covers \mathcal{A}_1 otherwise we say that \mathcal{A}_2 does not cover \mathcal{A}_1 . Let us see why this is true. For simplifying the arguments we fix a $q \in C_1$. Assume that for all $j \in [1, K]$, there exists $p \in C_2$ such that $\mathcal{A}_1\langle h_{j,q} \rangle$ is equivalent to the configuration $(\mathbf{x}_{j,q}, p, 0)$ for some $\mathbf{x}_{j,q} \in \mathcal{F}^K$. Now, any initial configuration $(\lambda, q, 0)$ of \mathcal{A}_1 is equivalent to the configuration $(\sum_{j=1}^K \lambda[j] \cdot \mathbf{x}_{j,q}, p, 0)$ of \mathcal{A}_2 .

The coverable equivalence problem can now be solved by checking whether \mathcal{A}_1 covers \mathcal{A}_2 and \mathcal{A}_2 covers \mathcal{A}_1 , which can be done in time polynomial in K . \square

7. CONCLUSION

We introduced a new model called ODCA. The equivalence problem for non-deterministic ODCAs is in PSPACE. This is in contrast to non-deterministic OCA, where the equivalence problem is undecidable. We observe that undecidability is a consequence of non-determinism occurring in counter actions. In the case of weighted ODCAs, we show that the reachability, equivalence, regularity, and covering problems are in P.

The natural way to extend the work is to consider epsilon transitions in the ODCA. We conjecture that the equivalence, regularity, and covering problems will be polynomial time decidable. Another possible direction is to look at pushdown systems partitioned into a deterministic stack and a finite state machine. In this case, a non-deterministic model can be determinized (similar to Theorem 8). Even though all our algorithms are in polynomial time, they may not be ‘practical’. Considerable effort is required to find faster algorithms. We also leave open questions on learning and approximate equivalence of ODCAs.

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