Many Hard Examples for Resolution

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Abstract. For every choice of positive integers c and k such that $k \geq 3$ and $c2^{-k} \geq 0.7$, there is a positive number ϵ such that, with probability tending to 1 as n tends to ∞, a randomly chosen family of cn clauses of size k over n variables is unsatisfiable, but every resolution proof of its unsatisfiability must generate at least $(1 + \epsilon)^n$ clauses.

Categories and Subject Descriptors: F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems—*complexity of proof procedures*

General Terms: Algorithm, Theory, Verification

Additional Key Words and Phrases: Resolution, satisfiability

1. *Introduction*

A *truth assignment* is a mapping f that assigns 0 or 1 to each variable in its domain. For each such variable x, we define $f(\bar{x}) = 1 - f(x)$ and refer to both x and $\bar{x}$ as *literals*; a *clause* is a set of literals. A truth assignment f *satisfies* a clause C if and only if $f(w) = 1$ for at least one literal w in C; the assignment *satisfies* a family F of clauses if and only if it satisfies every clause in F. A family of clauses is called *satisfiable* if it is satisfied by at least one truth assignment; otherwise, it is called *unsatisfiable.*

If A, B are clauses and x is a variable such that $x \in A$, $\bar{x} \in B$, then the clause $(A - \{x\}) \cup (B - \{\bar{x}\})$ is called a *resolvent* of A and B. (It is often required that $x \notin B$, $\bar{x} \notin A$ and that there be no variable y other than x such that $y \in A \cup B$, $\bar{y} \in A \cup B$. The definition we use is simpler and its adoption certainly brings about no loss of generality in what we are about to do.) Obviously, every truth assignment satisfying both A and B satisfies all their resolvents.

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Now let F be a family of clauses and $C_1, C_2, \ldots, C_N$ be a sequence of clauses such that

—each C_k belongs to F or is a resolvent of some C_i, C_j such that $i < k, j < k$,
—C_N is the empty clause.

Straightforward induction on k shows that every truth assignment satisfying F must satisfy each C_k; since no truth assignment satisfies the empty clause C_N, it follows that F is unsatisfiable. For this reason, the sequence $C_1, C_2, \ldots, C_N$ is called a *resolution proof* of unsatisfiability of F. (It is also easy to show that every unsatisfiable F admits a resolution proof of unsatisfiability. This observation is often credited to Robinson [15], even though it follows instantly from an analysis of a procedure designed by Davis and Putnam [5]; Robinson defined the notion of resolution and proved a more general theorem concerning predicate calculus. Incidentally, the Davis–Putnam procedure was also proposed some fifty years earlier by Löwenheim [11–13]. An account of Löwenheim's work can be found in Hammer and Rudeanu [9]; we are grateful to Yves Crama for this information. We are also grateful to Wolfgang Bibel for informing us that resolution was introduced before Robinson by Blake [2]; see Bibel [1, p. 205].) By the *resolution complexity* of an unsatisfiable family F of clauses, we mean the smallest N such that there is a resolution proof $C_1, C_2, \ldots, C_N$ of unsatisfiability of F.

Haken [7] was the first to prove that there is an infinite sequence of families of clauses whose resolution complexity increases faster than every polynomial in the size of the input; his lower bound was exponential in the cube root of the input size. Later on, Urquhart [17] used a different construction to improve the lower bound to an exponential in the input size. (Urquhart's construction evolves from an idea of Tseitin [16], who constructed an infinite sequence of families of clauses for which the complexity of so-called regular resolution grows at least as fast as an exponential in the square root of the input size; Tseitin's lower bound was subsequently improved by Galil [6] to an exponential in the input size.) The aim of our paper is to prove that randomly generated sparse families of clauses are very likely to have the property of Urquhart's examples.

2. *The Theorem and An Outline of Its Proof*

Let us call a clause C *ordinary* if there is no variable x such that $x \in C$ and $\bar{x} \in C$; let n, m, k be positive integers and let X be a set of size n. The *random family of m clauses of size k over n variables* consists of independent random variables $C_1, C_2, \ldots, C_m$ such that each C_i is distributed uniformly over all $\binom{n}{k} 2^k$ ordinary clauses of size k with variables coming from X. (We have chosen this definition partly because it suggests a reasonably practical way of generating random families of clauses and partly because it simplifies our subsequent references to random hypergraphs. The proof of our result can be easily adapted to other definitions of a random family.)

THEOREM. *For every choice of positive integers c and k such that $k \geq 3$ and $c2^{-k} \geq 0.7$ there is a positive number ϵ such that, with probability tending to one as n tends to infinity, the random family of cn clauses of size k over n variables is unsatisfiable and its resolution complexity is at least* $(1 + \epsilon)^n$.

Proving that, with probability tending to 1 as n tends to ∞, the random family F is unsatisfiable is a straightforward exercise: every truth assignment f whose domain consists of all n variables satisfies each C_i with probability $1 - 2^{-k}$, and so it satisfies

F with probability $(1 - 2^{-k})^{cn}$. Since there are precisely 2^n choices of f, the probability that F is satisfiable is at most t^n with

$$t = 2(1 - 2^{-k})^c < 2\exp(-c2^{-k}) \leq 2e^{-0.7} < 0.999,$$

and the desired conclusion follows. (Here, the assumption that $k \geq 3$ is superfluous.)

The remainder of this paper is devoted to proving that, with probability tending to 1 as n tends to ∞, the resolution complexity of F is at least $(1 + \epsilon)^n$. (Here, the assumption that $c2^{-k} \geq 0.7$ becomes superfluous, but the assumption that $k \geq 3$ becomes essential: as pointed out by Cook [4], the Davis–Putnam procedure requires only a polynomial time to test satisfiability of any family of clauses of size at most 2; in case the family turns out to be unsatisfiable, the procedure produces a resolution proof of length at most n.) Our proof comes in two parts: first we show that the random family is very likely to have certain properties (Lemma 4) and then we show that these properties impose a lower bound on resolution complexity (Lemma 5).

Lemma 4 guarantees that there are numbers a, d such that $0 < d < a/8$ and such that F is very likely to have the following properties:

—For every family F' of at most an clauses from F, there are at least $\frac{1}{2}|F'|$ variables x such that x is involved in precisely one clause from F'.

—There is a "dense" collection of sets D of dn variables such that, for every family F' of at most an clauses from F, every truth assignment with domain D has an extension that satisfies F'.

To prove this lemma, we show that all "locally sparse" families of clauses have these properties (Lemma 3) and check by a routine computation that F is very likely to be locally sparse (Lemma 1).

The strategy used in our proof of Lemma 5 goes back to Haken: select a large family of "special assignments" and find a not-too-many-to-one mapping from this family to the set of clauses in the resolution proof. Actually, it was Urquhart's variation on Haken's theme that inspired our proof: if there is a reasonably uniformly distributed family of linear-size sets D such that every truth assignment with domain D is special and if, for every special assignment f, the resolution proof includes a linear-size clause C that is not satisfied by f, then the mapping that sends f onto C is not-too-many-to-one. To find C, Urquhart seemed to rely heavily on a special property of his families of clauses: these families include many clauses whose deletion produces a satisfiable family. Unfortunately, the random F is far from having this property; fortunately, Urquhart's rule may be reformulated in terms that make sense in any F: choose the first C in the resolution proof such that deriving C from F subject to f requires a large (but not *too* large) number of clauses from F.

3. *Random Hypergraphs Are Locally Sparse*

A *hypergraph* is a set X along with a family of not necessarily distinct subsets E_i of X; elements of X are called the *vertices* of the hypergraph and the sets E_i are called the *edges*. (We consider edges E_i, E_j to be distinct if $i \neq j$, even though we may have $E_i = E_j$ in this case.) The hypergraph is said to be *k-uniform* if all its edges have size k. We say that a hypergraph with n vertices is *(x, y)-sparse* if every set of s vertices such that $s \leq xn$ contains at most ys edges.

Let n, m, k be positive integers; let X be a set of size n. The *random k-uniform hypergraph with n vertices and m edges* consists of X along with independent random variables $E_1, E_2, \ldots, E_m$ such that each E_i is distributed uniformly over all $\binom{n}{k}$ subsets of X that have size k.

LEMMA 1. *For every choice of positive integers c and k and for every number y such that $(k-1)y > 1$, there is a positive x such that, with probability tending to 1 as n tends to ∞, the random k-uniform hypergraph with n vertices and cn edges is (x, y)-sparse.*

PROOF. Write

$$\begin{aligned} \epsilon &= y - (k-1)^{-1}, \\ x &= \left(\frac{1}{2e}\left(\frac{y}{ce}\right)^{y}\right)^{1/(k-1)\epsilon}, \\ f(n) &= e\left(\frac{ce}{y}\right)^{y} n^{-(k-1)\epsilon/2}, \\ g(n) &= n^{1/2}. \end{aligned}$$

The probability that the hypergraph is not (x, y)-sparse is at most

$$\sum_{k \le s \le xn} a(n, s)$$

with

$$a(n, s) = \binom{n}{s} \sum_{i \ge ys} \binom{cn}{i} \left(\binom{s}{k}\binom{n}{k}^{-1}\right)^{i} \left(1 - \binom{s}{k}\binom{n}{k}^{-1}\right)^{cn-1}.$$

Using the inequality

$$\sum_{i \ge tm} \binom{m}{i} p^{i}(1-p)^{m-i} \le \left(\frac{ep}{t}\right)^{tm},$$

valid whenever $p < t \le 1$ (a simple proof can be found, for instance, in Chvátal [3]), with $m = cn$, $p = \binom{s}{k}\binom{n}{k}^{-1}$, and $t = ys/cn$, we find that

$$a(n, s) \le \binom{n}{s}\left(\frac{ces^{k-1}}{yn^{k-1}}\right)^{ys};$$

next, using the elementary bound $\binom{n}{s} < (en/s)^s$, we obtain

$$a(n, s) \le \left(e\left(\frac{ce}{y}\right)^{y}\left(\frac{s}{n}\right)^{(k-1)\epsilon}\right)^{s}.$$

Our choice of x guarantees that $a(n, s) \le (\frac{1}{2})^s$ whenever $s \le xn$, and so

$$\sum_{g(n) \le s \le xn} a(n, s) \le 2^{1-g(n)} \to 0. \tag{3.1}$$

On the other hand, we have $a(n, s) \le (f(n))^s$ whenever $s \le g(n)$, and so

$$\sum_{k \le s \le g(n)} a(n, s) \le f(n)^{k}(1 - f(n))^{-1} \to 0. \tag{3.2}$$

The desired conclusion follows from (3.1) and (3.2). □

4. *A Lemma on Systems of Distinct Representatives*

A *system of distinct representatives* (*SDR*) of a family $(E_i : i \in I)$ of not necessarily distinct sets is a set of distinct points $x_i (i \in I)$ such that each x_i belongs to E_i. The classical theorem of Hall [8] asserts that $(E_i : i \in I)$ has an SDR if and only if

$$\left| \bigcup_{i \in J} E_i \right| \geq |J| \tag{4.1}$$

for all subsets J of I. In the next section, we use the "if" part of the following corollary of Hall's theorem.

LEMMA 2. *A family $(E_i : i \in I)$ has an SDR with at most t points in a set S if and only if it has an SDR and*

$$|J| - \left| \bigcup_{i \in J} (E_i - S) \right| \leq t \tag{4.2}$$

for all subsets J of I.

PROOF. The "only if" part is obvious; to prove the "if" part, we may assume that $|S| > t$ (otherwise the conclusion is trivial). Now enlarge I into a set I^* of size $|I| + |S| - t$ and set $E_i = S$ whenever $i \in I^* - I$. Clearly $(E_i : i \in I)$ has an SDR with at most t points in S if and only if $(E_i : i \in I^*)$ has an SDR. Hence Hall's theorem reduces our task to proving that (4.1) holds for all subsets J of I^*. By assumption, $(E_i : i \in I)$ has an SDR, and so (4.1) holds whenever $J \subseteq I$. On the other hand, if $J \not\subseteq I$, then

$$\left| \bigcup_{i \in J} E_i \right| = \left| \bigcup_{i \in J \cap I} (E_i - S) \right| + |S|,$$

and so assumption (4.2) with $J \cap I$ in place of J implies

$$\left| \bigcup_{i \in J} E_i \right| \geq |J \cap I| - t + |S| = |J \cap I| + |I^* - I| \geq |J|;$$

hence (4.1) holds again. □

Lemma 2 is a special case of a theorem of Hoffman and Kuhn [10]; in turn, as shown by Welsh [18], the Hoffman–Kuhn theorem may be seen as a special case of an earlier theorem of Rado [14].

5. *Properties of Locally Sparse Hypergraphs*

The *boundary* of a family F of edges in a hypergraph is the set of all the vertices that belong to precisely one edge in F.

We say that a hypergraph with n vertices has *property* $P(a)$ if every family of m edges such that $m \leq an$ has boundary of size at least $m/2$. We say that a hypergraph with n vertices has *property* $Q(a, b)$ if at least 50% of all sets S of $\lfloor bn \rfloor$ vertices have the following property: there is a subset D of S such that $|S - D| \leq (a/32) \cdot |S|$ and such that every family of at most an edges has an SDR disjoint from D.

LEMMA 3. *Let H be a k-uniform hypergraph with n vertices and cn edges. If H is $(ak, 4/(2k + 1))$-sparse and $(x, 0.5 + (a/512))$-sparse, then it has properties $P(a)$ and $Q(a, b)$ with $b = \min(x/2k, a/64ck^3, a/8)$.*

PROOF. Showing that H has property $P(a)$ is a straightforward exercise: consider any family of m edges of H and let pm denote the size of its boundary. Clearly, the family covers at most $(k + p)m/2$ vertices; if $m \leq an$, then it covers at most akn vertices, and so $p \geq \frac{1}{2}$ since H is $(ak, 4/(2k + 1))$-sparse.

To prove that H has property $Q(a, b)$, set $s = \lfloor bn \rfloor$ and, for each set S of s vertices, let $N(S)$ denote the number of edges that intersect S in at least two vertices. Then let $\bar{N}$ denote the average $N(S)$ and call S *normal* if $N(S) \leq 2\bar{N}$. Clearly, at least 50% of all sets of s vertices are normal; hence we only need to find, in each normal S, a subset D such that $|(S - D)| \leq as/32$ and such that every family of at most an edges has an SDR disjoint from D.

Note that

$$\binom{n}{s}\bar{N} \leq cn\binom{k}{2}\binom{n-2}{s-2},$$

and so

$$\bar{N} \leq \frac{ck^2s^2}{2n} \leq \left(\frac{ck^2b}{2}\right) \cdot s \leq \left(\frac{a}{128k}\right)s. \tag{5.1}$$

Now consider an arbitrary but fixed normal S. By a *cluster*, we mean any family of edges whose boundary is contained in S. Clearly, for each subset D of S, every minimal family of edges with no SDR disjoint from D is a cluster. Hence we only need find a subset D of S such that $|S - D| \leq as/32$ and such that every cluster of at most an edges has an SDR disjoint from D. For this purpose, let $(E_i : i \in I)$ be the union of all clusters that have at most an edges; we only need prove that

$$(E_i : i \in I) \text{ has an SDR with at most } \frac{as}{32} \text{ points in } S. \tag{5.2}$$

To prove (5.2), note first that

$$\text{every cluster of at most } an \text{ edges has at most } 2s \text{ edges} \tag{5.3}$$

because H has property $P(a)$, and then that

$$4s \leq an \tag{5.4}$$

because $b \leq a/8$. Since the union of any two clusters is a cluster, (5.3) and (5.4) imply that

$$|I| \leq 2s. \tag{5.5}$$

Trivially, property $P(a)$ guarantees that every family of at most an edges has an SDR; in particular, $(E_i: i \in I)$ has an SDR. Hence, Lemma 2 reduces proving (5.2) to proving that

$$|J| - \left|\bigcup_{i \in J} (E_i - S)\right| \leq \frac{as}{32} \tag{5.6}$$

for all subsets J of I.

To prove (5.6), write

$$P = \bigcup_{i \in J} (E_i \cap S), \qquad Q = \bigcup_{i \in J} (E_i - S).$$

Trivially, (5.5) guarantees that $|P| + |Q| \leq 2sk$; since H is $(x, 0.5 + (a/512))$-sparse, it follows that

$$|J| \leq \left(\frac{1}{2} + \frac{a}{512}\right)(|P| + |Q|),$$

and so

$$|P| + |Q| \geq \left(2 - \frac{a}{128}\right)|J|. \tag{5.7}$$

Clearly, $|J| \geq |P| - kN(S)$; since S is normal, it follows that

$$|J| \geq |P| - 2k\bar{N}. \tag{5.8}$$

Finally, (5.7), (5.8), (5.1), and (5.5) imply

$$|J| - |Q| \leq |P| - \left(1 - \frac{a}{128}\right)|J| \leq 2k\bar{N} + \frac{a}{128}|J| \leq \frac{as}{32},$$

and so (5.6) is proved. □

We use the following consequence of Lemmas 1 and 3; the assumption that $k \geq 3$ is used here and only here.

LEMMA 4. *For every choice of positive integers c and k such that $k \geq 3$, there are positive numbers a, b with $b \leq a/8$ such that, with probability tending to one as n tends to ∞, the k-uniform hypergraph with n vertices and cn edges has properties $P(a)$ and $Q(a, b)$.*

PROOF. By Lemma 1, there is a positive x' such that, with probability tending to 1 as n tends to ∞, the hypergraph is $(x', 4/(2k + 1))$-sparse; set $a = x'/k$. By Lemma 1 again, there is a positive x such that, with probability tending to 1 as n tends to ∞, the hypergraph is $(x, 0.5 + (a/512))$-sparse. The rest follows from Lemma 3. □

6. *A Lower Bound on Resolution Complexity*

When C is a clause, let $E(C)$ denote the set of all variables x such that precisely one of x and $\bar{x}$ belongs to C. Next, let H be a hypergraph with edges $E_1, E_2, \ldots, E_m$. We say that a family $C_1, C_2, \ldots, C_m$ of clauses is *based on H* if each C_i is an ordinary clause with $E(C_i) = E_i$.

LEMMA 5. *Let H be a hypergraph with n vertices and let F be an unsatisfiable family of clauses based on H. If H has properties $P(a)$ and $Q(a, b)$ with $b \leq a/8$, then F has resolution complexity at least*

$$\frac{1}{4}\left(\frac{e}{2}\right)^{a\lfloor bn\rfloor/16}.$$

PROOF. Write

$$s = \lfloor bn \rfloor, \qquad d = s - \left\lfloor \frac{as}{32} \right\rfloor,$$

and call a set S of s variables *special* if there is a subset D of S such that $|D| = d$ and such that every family of at most an clauses from F can be satisfied by a truth assignment whose domain is disjoint from D. For each special S, choose one D with this property and call it $D(S)$. By a *special pair*, we mean any pair (S, f) such that S is a special set and f is a truth assignment with domain $D(S)$. Since H has property $Q(a, b)$, there are at least $\frac{1}{2}\binom{n}{s}$ special sets; it follows that there are at least

$$\frac{1}{2}\binom{n}{s}2^d \tag{6.1}$$

special pairs.

Now consider any resolution proof $C_1, C_2, \ldots, C_N$ of unsatisfiability of F. Our plan is to assign one C_k to each special pair in such a way that each C_k is assigned to at most

$$\left(\frac{2}{e}\right)^{as/16}\binom{n}{s}2^d + \left(\frac{1}{2}\right)^{as/32}\binom{n}{s}2^d \tag{6.2}$$

special pairs. As soon as we do this, the desired lower bound on N follows at once: since $e^2 < 8$, quantity (6.2) is at most

$$2\left(\frac{2}{e}\right)^{as/16}\binom{n}{s}2^d,$$

and so (6.1) divided by (6.2) is at least

$$\frac{1}{4}\left(\frac{e}{2}\right)^{as/16}.$$

To implement this plan, we only need prove that

$$\text{for every special pair } (S, f)\text{, some } C_k \text{ not satisfied by } f \text{ has size at least } an/8; \tag{6.3}$$

then we can assign to each (S, f) a clause C_k with these properties. To verify that each C_k is assigned to at most (6.2) special pairs, let N_i denote the number of special pairs (S, f) such that C_k is assigned to (S, f) and $|D(S) \cap E(C_k)| = i$; we claim that

$$\sum_{i \le as/32} N_i \le \left(\frac{2}{e}\right)^{as/16}\binom{n}{s}2^d \tag{6.4}$$

and

$$\sum_{i \ge as/32} N_i \le \binom{n}{s}2^{d-(as/32)}. \tag{6.5}$$

To justify these claims, write $m = |E(C_k)|$. We may assume that $m \ge an/8$, for otherwise every N_i is zero by definition. Now note that $|D(S) \cap E(C_k)| \le as/32$ implies $|S \cap E(C_k)| \le as/16$, and so

$$\sum_{i \le as/32} N_i \le \sum_{j \le as/16}\binom{m}{j}\binom{n-m}{s-j}2^d.$$

Hence (6.4) follows from a bound on the tail of the hypergeometric distribution (a simple proof of this bound can be found in Chvátal [3]). Next, if C_k is assigned to (S, f), then f does not satisfy C_k, and so $f(x) = 0$ whenever $x \in D(S)$, $x \in C_k$ and $f(x) = 1$ whenever $x \in D(S)$, $\bar{x} \in C_k$. Hence (6.5) follows at once.

To prove (6.3), consider an arbitrary but fixed special pair (S, f). We say that a family F' of clauses *implies a clause C subject to f* if every extension of f satisfying F' satisfies C; we say that a clause C is *complex* if no family of at most $an/2$ clauses from F implies C subject to f. Since (S, f) is a special pair, every family of at most $an/2$ clauses from F is satisfied by some extension of f; hence the empty clause is complex. It follows that there is a smallest subscript k such that C_k is complex; since no complex clause is satisfied by f, we only need prove that

$$|C_k| \geq \frac{an}{8}. \tag{6.6}$$

To prove (6.6), note first that no clause in F is complex as each clause implies itself subject to f (we may assume that $an/2 \geq 1$, for otherwise the desired lower bound on resolution complexity of F is less than 1). Hence C_k is a resolvent of some C_i, C_j such that $i < k, j < k$. By minimality of k, neither of C_i, C_j is complex. That is to say, there are families F_i, F_j of clauses in F such that $|F_i| \leq an/2$, $|F_j| \leq an/2$, every extension of f satisfying F_i satisfies C_i, and every extension of f satisfying F_j satisfies C_j. Since every truth assignment satisfying both C_i and C_j satisfies C_k, it follows that $F_i \cup F_j$ implies C_k subject to f. Now let F_k be a smallest family of clauses from F such that F_k implies C_k subject to f. Since $|F_k| \leq |F_i \cup F_j| \leq an$ and since H has property $P(a)$, there is a set W of at least $\frac{1}{2}|F_k|$ variables such that for each x in W there is precisely one C in F_k with $x \in E(C)$. Since C_k is complex, we have $|F_k| > an/2$, and so $|W| > an/4$; since $b \leq a/8$, it follows that $|W - S| > an/8$. We complete the proof of (6.6) by proving that

$$W - S \subseteq E(C_k). \tag{6.7}$$

To prove (6.7), consider any variable x in $W - S$. By definition of W, there is precisely one clause C in F_k with $x \in E(C)$. By minimality of F_k, some extension g of f satisfies $F_k - \{C\}$ without satisfying C_k. Setting $h(x) = 1 - g(x)$ and $h(y) = g(y)$ whenever $y \neq x$, observe that

$$g \text{ does not satisfy } C$$

(since it satisfies $F_k - \{C\}$ without satisfying C_k), and so

$$h \text{ satisfies } C$$

(since g and h differ on x and since $x \in E(C)$). Next, observe that

$$h \text{ satisfies } F_k - \{C\}$$

(since g and h agree on all variables involved in these clauses), and so

$$h \text{ satisfies } C_k$$

(since F_k implies C_k subject to f and since $x \notin D(S)$). Finally, since g and h differ only on x, and since precisely one of them satisfies C_k, we conclude that $x \in E(C_k)$. □

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RECEIVED JULY 1987; REVISED JANUARY 1988; ACCEPTED JANUARY 1988