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A Note on the Convergence of Multivariate Formal Power Series Solutions of Meromorphic Pfaffian Systems

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Abstract
Here, we present some complements to theorems of R. Gerard and Y. Sibuya, on the convergence of multivariate formal power series solutions of nonlinear meromorphic Pfaffian systems. Their most known results concern completely integrable systems with non-degenerate linear parts, whereas we consider some cases of non-integrability and degeneracy.

Keywords Meromorphic Pfaffian PDEs system · Formal power series solution · Convergence

1 Introduction

Consider a Pfaffian system:

$$dy = \mathbf{f}_1(x,y)\frac{dx_1}{x_1^{p_1}} + \ldots + \mathbf{f}_m(x,y)\frac{dx_m}{x_m^{p_m}}, \qquad x=(x_1,\ldots,x_m), \quad m>1, \qquad (1)$$
$$y=(y_1,\ldots,y_n)^\top,$$

where $\mathbf{f}_i : (\mathbb{C}^{m+n},0) \to (\mathbb{C}^n,0)$ are germs of holomorphic maps and $p_i \geqslant 1$ are integers. Equivalently, this system has the form:

$$\Theta := dy - \omega = 0,$$

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where ω is a ($\mathbb{C}^n$-valued) differential 1-form meromorphic in a neighborhood D of $0 \in \mathbb{C}^{m+n}$, with the polar locus:

$$\Sigma = \{(x, y) \in D \mid h(x, y) := x_1 \dots x_m = 0\}.$$

In the case $p_1 = \dots = p_m = 1$, we have the Pfaffian system (1) of *Fuchs type*; in this case, ω is a *logarithmic* 1-form in D, that is, the both $h\,\omega$ and $h\,d\omega$ are holomorphic in D.

Written in a PDEs form, Eq. 1 becomes:

$$x_1^{p_1}\,\frac{\partial y}{\partial x_1} = \mathbf{f}_1(x, y), \quad \dots, \quad x_m^{p_m}\,\frac{\partial y}{\partial x_m} = \mathbf{f}_m(x, y), \quad m > 1. \tag{2}$$

We study a question of the convergence of a formal power series solution

$$\varphi = \sum_{|k|>0} \mathbf{c}_k x^k \in \mathbb{C}[[x]]^n \quad (\mathbf{c}_k \in \mathbb{C}^n) \tag{3}$$

of such a system. Here, as is usual in a multivariate case,

$$k = (k_1, \dots, k_m) \in \mathbb{Z}_+^m, \quad |k| = k_1 + \dots + k_m, \quad x^k = x_1^{k_1} \dots x_m^{k_m}.$$

A basic work on this subject (as well as on the analytic and asymptotic properties of such series) is that by Gerard and Sibuya [3], in which Pfaffian systems are assumed to be *completely integrable* on $D \setminus \Sigma$. This means that for any $(x^0, y^0) \in D \setminus \Sigma$, there exists a unique solution $y = y(x)$ of Eq. 1 such that $y(x^0) = y^0$. Due to the Frobenius theorem, the complete integrability of Eq. 1 is equivalent to the relation

$$d\Theta = \Omega \wedge \Theta,$$

for some matrix differential 1-form Ω holomorphic in $D \setminus \Sigma$ (see [5, Ch.1, Th. 5.1] or [10]). However, in [3], there are obtained some assertions concerning the convergence of the series (3) that do not appeal to the complete integrability of Eq. 1 and thus hold for any Pfaffian system of the form (1). These are the following two theorems.

Theorem A *Any formal power series solution* (3) *of a Fuchsian system*

$$x_1\,\frac{\partial y}{\partial x_1} = \mathbf{f}_1(x, y), \quad \dots, \quad x_m\,\frac{\partial y}{\partial x_m} = \mathbf{f}_m(x, y) \tag{4}$$

converges near $0 \in \mathbb{C}^m$.

Theorem B *If* $p_j = 1$ *for some* j (*that is, the system* (2) *is Fuchsian along the component* $\{x_j = 0\}$ *of its polar locus*) *and the corresponding Jacobi matrix* $\partial \mathbf{f}_j/\partial y(0, 0)$ *does not have non-negative integer eigenvalues, then any formal power series solution* (3) *of* Eq. 2 *converges near* $0 \in \mathbb{C}^m$ (*in the case of the complete integrability of such a system, there holds the existence and uniqueness of the solution*).

We will show that the sufficient condition of convergence from Theorem B can be weakened for a fixed formal solution φ in the following way (see the proof in Section 3).

Theorem 1 *Let* $p_1 = 1$ *in the system* (2) *and let* φ *be its formal power series solution. If the matrix*

$$A = \frac{\partial \mathbf{f}_1}{\partial y}(x, \varphi)|_{x_1=0} \in Mat(n, \mathbb{C}[[x_2, \dots, x_m]])$$

is such that $\det(A - jI) \not\equiv 0$ *for any non-negative integer* j, *then* φ *converges near* $0 \in \mathbb{C}^m$.

In the following theorem from [3], the complete integrability of the Pfaffian system is required.

Theorem C *In the non-Fuchsian case with all $p_i > 1$, if there are $j \neq l$ such that the Jacobi matrices $\partial \mathbf{f}_j/\partial y(0,0)$, $\partial \mathbf{f}_l/\partial y(0,0)$ are non-degenerate, then there exists a unique formal power series solution* (3) *of* Eq. 2 *and it converges near* $0 \in \mathbb{C}^m$.

Note that Theorem C highlights the essential difference between the case $m > 1$ (the multivariate case) and the case $m = 1$ (that of one independent variable x). In the latter, a (automatically) completely integrable case, even a scalar equation

$$x^p \frac{dy}{dx} = f(x, y), \qquad p > 1,$$

with $\partial f/\partial y(0,0) \neq 0$, generically has no convergent power series solution near $0 \in \mathbb{C}$. This follows from the theory of the Martinet–Ramis moduli [6] (see also [4]), due to which a generic two-dimensional holomorphic vector field near a degenerate singular point $(x, y) = (0, 0)$ of saddle-node type,

$$\begin{cases} \dot{x} = x^p, \\ \dot{y} = f(x, y), \end{cases}$$

has no holomorphic "horizontal" separatrix.

Another difference from the case of one independent variable x, which follows from Theorem C, is the coincidence of the formal and the analytic classifications of multivariate completely integrable *linear* (the case of $\mathbf{f}_i(x, y) = A_i(x)y$, $A_i(x) \in \mathrm{Mat}(n, \mathbb{C})$) Pfaffian systems whose matrices $A_i(0)$ have pairwise distinct eigenvalues. In other words, there is, surprisingly enough, no Stokes phenomenon for such systems (see [3, Ch. II, Th. 5.5] and also [2], where an alternative proof based on the k-summability theory is proposed).

In a more recent paper by Sibuya [9] (where $m = 2$), the assumption of complete integrability in the above Theorem C is omitted and there is proved that a formal power series solution (3) of Eq. 2 (if any) converges near $0 \in \mathbb{C}^m$, under the rest of the assumptions of the theorem.

In Section 2, we discuss the condition of complete integrability in more detail and study the convergence of formal power series solutions of Eq. 2 in the non-integrable case (Theorems 2, 3). In Section 3, we give the proof of Theorem 1, and in Section 4 we complement the above Gerard–Sibuya Theorem C for the non-Fuchsian case by some sufficient condition of the convergence of the series (3) satisfying (2) with the zero Jacobi matrices $\partial \mathbf{f}_i/\partial y(0,0)$ (Theorem 4).

2 The Relations of Complete Integrability

If the system (2) is completely integrable, for any its solution $y(x)$, the coincidence of the second partial derivatives $\partial^2 y/\partial x_i \partial x_j$ and $\partial^2 y/\partial x_j \partial x_i$ implies:

$$\frac{1}{x_i^{p_i}} \frac{\partial}{\partial x_j} \mathbf{f}_i(x, y(x)) = \frac{1}{x_j^{p_j}} \frac{\partial}{\partial x_i} \mathbf{f}_j(x, y(x)),$$

whence,

$$\frac{1}{x_i^{p_i}}\frac{\partial \mathbf{f}_i}{\partial x_j}(x, y(x)) - \frac{1}{x_j^{p_j}}\frac{\partial \mathbf{f}_j}{\partial x_i}(x, y(x)) = \frac{1}{x_i^{p_i}x_j^{p_j}}\frac{\partial \mathbf{f}_j}{\partial y}(x, y(x))\mathbf{f}_i(x, y(x)) - \frac{1}{x_i^{p_i}x_j^{p_j}}\frac{\partial \mathbf{f}_i}{\partial y}(x, y(x))\mathbf{f}_j(x, y(x)).$$

Since in the case of complete integrability for any $(x^0, y^0) \in D \setminus \Sigma$ there exists a unique solution $y = y(x)$ of Eq. 2 such that $y(x^0) = y^0$, one has

$$\frac{1}{x_i^{p_i}}\frac{\partial \mathbf{f}_i}{\partial x_j} - \frac{1}{x_j^{p_j}}\frac{\partial \mathbf{f}_j}{\partial x_i} \equiv \frac{1}{x_i^{p_i}x_j^{p_j}}\left(\frac{\partial \mathbf{f}_j}{\partial y}\mathbf{f}_i - \frac{\partial \mathbf{f}_i}{\partial y}\mathbf{f}_j\right), \quad i, j = 1, \dots, m, \tag{5}$$

in $D \setminus \Sigma$. Or, equivalently, all the vector functions

$$\mathbf{F}_{ij}(x, y) = x_j^{p_j}\frac{\partial \mathbf{f}_i}{\partial x_j} - x_i^{p_i}\frac{\partial \mathbf{f}_j}{\partial x_i} + \frac{\partial \mathbf{f}_i}{\partial y}\mathbf{f}_j - \frac{\partial \mathbf{f}_j}{\partial y}\mathbf{f}_i, \quad i, j = 1, \dots, m,$$

are equal to 0 identically. Conversely, let us show that if all the vector functions $\mathbf{F}_{ij}(x, y)$, $i, j = 1, \dots, m$, equal 0 identically, then the system (2) is completely integrable. Indeed, in this case, (5) holds and for $\Theta = dy - \sum_{i=1}^m \mathbf{f}_i(x, y)dx_i/x_i^{p_i}$ we have:

$$\begin{aligned} d\Theta &= \sum_{i=1}^m\sum_{j=1}^m \frac{1}{x_i^{p_i}}\frac{\partial \mathbf{f}_i}{\partial x_j}dx_i \wedge dx_j + \left(\sum_{i=1}^m \frac{1}{x_i^{p_i}}\frac{\partial \mathbf{f}_i}{\partial y}dx_i\right) \wedge dy \\ &= \sum_{i<j}\left(\frac{1}{x_i^{p_i}}\frac{\partial \mathbf{f}_i}{\partial x_j} - \frac{1}{x_j^{p_j}}\frac{\partial \mathbf{f}_j}{\partial x_i}\right)dx_i \wedge dx_j + \\ &\quad +\Omega \wedge dy = \sum_{i<j}\frac{1}{x_i^{p_i}x_j^{p_j}}\left(\frac{\partial \mathbf{f}_j}{\partial y}\mathbf{f}_i - \frac{\partial \mathbf{f}_i}{\partial y}\mathbf{f}_j\right)dx_i \wedge dx_j + \Omega \wedge dy = \Omega \wedge \Theta, \end{aligned}$$

where,

$$\Omega = \sum_{i=1}^m \frac{1}{x_i^{p_i}}\frac{\partial \mathbf{f}_i}{\partial y}dx_i$$

is a matrix differential 1-form holomorphic in $D \setminus \Sigma$. Hence, the Frobenius integrability condition is fulfilled.

Thus, we see that the complete integrability of the system (2) is described by at most $m(m-1)/2$ vector ($nm(m-1)/2$ scalar) relations (since $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$). Further, we use the two results by Ploski [7] (a version with the detailed proof is [8]) following from his sharpened version of Artin's Approximation Theorem [1]. These are:

1. If $f(x_1, \dots, x_m, y)$ is a non-zero germ of a holomorphic function $(\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ and $\varphi \in \mathbb{C}[[x_1, \dots, x_m]]$ is a formal power series without a constant term such that $f(x_1, \dots, x_m, \varphi) = 0$, then φ converges near $0 \in \mathbb{C}^m$;
2. If $\mathbf{f}(x_1, \dots, x_m, \mathbf{y})$ is a germ of a holomorphic map $(\mathbb{C}^{m+n}, 0) \to (\mathbb{C}^n, 0)$, $\varphi \in \mathbb{C}[[x_1, \dots, x_m]]^n$ is a (vector) formal power series without a constant term such that $\mathbf{f}(x_1, \dots, x_m, \varphi) = 0$, and

   $$\det \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(x_1, \dots, x_m, \varphi) \not\equiv 0,$$

   then φ converges near $0 \in \mathbb{C}^m$.

Theorem 2 *In the case of the scalar unknown y (i.e., $n = 1$), if the system (2) is not completely integrable then any of its formal power series solution φ converges near $0 \in \mathbb{C}^m$.*

Proof Non-integrability implies that at least one of the (scalar in this case) functions $\mathbf{F}_{ij}(x, y)$ is not identically 0 (say, $\mathbf{F}_{12}$), while $\mathbf{F}_{12}(x, \varphi) = 0$. Thus, the assertion follows from the first result by Ploski. □

Theorem 3 *Let the system (2) have a formal power series solution $\varphi \in \mathbb{C}[[x_1, \dots, x_m]]^n$ and among all the vector functions $\mathbf{F}_{ij}(x, y)$ there are n vector components $g_1, \dots, g_n$ such that:*

$$\det\left(\frac{\partial g_i}{\partial y_j}\right)(x_1, \dots, x_m, \varphi) \not\equiv 0.$$

Then, φ converges near $0 \in \mathbb{C}^m$.

Proof Follows from the second result by Ploski. □

In the bivariate case of two independent variables x_1, x_2, the set $\{\mathbf{F}_{ij}\}$ consists of the one vector function $\mathbf{F}_{12}$, and thus Theorem 3 is simplified as follows.

Corollary 1 *Let the bivariate system (2) have a formal power series solution $\varphi \in \mathbb{C}[[x_1, x_2]]^n$ such that for the vector function $\mathbf{F}_{12}(x_1, x_2, y)$ there holds:*

$$\det\frac{\partial \mathbf{F}_{12}}{\partial y}(x_1, x_2, \varphi) \not\equiv 0.$$

Then, φ converges near $0 \in \mathbb{C}^2$.

Note that the assumptions of Theorem 3 and Corollary 1 imply the non-integrability of the involved systems.

3 Proof of Theorem 1

Clearly, the formal solution φ is represented in the form:

$$\varphi = \sum_{j=0}^{\infty} \mathbf{c}_j(x_2, \dots, x_m)\, x_1^j, \qquad \mathbf{c}_j \in \mathbb{C}[[x_2, \dots, x_m]]^n.$$

Let us prove the convergence of all $\mathbf{c}_j$ in some common polydisc $\Delta \subset \mathbb{C}^{m-1}$ centered at the origin.

From the equality:

$$x_1 \frac{\partial \varphi}{\partial x_1} = \mathbf{f}_1(x, \varphi) \tag{6}$$

it follows that:

$$\mathbf{f}_1(0, x_2, \dots, x_m, \mathbf{c}_0) = 0.$$

Since by the theorem assumption:

$$\det A = \det\frac{\partial \mathbf{f}_1}{\partial y}(0, x_2, \dots, x_m, \mathbf{c}_0) \not\equiv 0,$$

the series $\mathbf{c}_0$ converges near $0 \in \mathbb{C}^{m-1}$ due to the second result by Ploski, and thus the matrix A is holomorphic in some polydisc $\Delta \subset \mathbb{C}^{m-1}$ centered at the origin.

Further, we substitute $\varphi = \mathbf{c}_0 + \psi := \mathbf{c}_0 + \sum_{j=1}^{\infty} \mathbf{c}_j\, x_1^j$ in the equality (6) and obtain

$$x_1 \frac{\partial}{\partial x_1} \left(\sum_{j=1}^{\infty} \mathbf{c}_j\, x_1^j \right) = \mathbf{f}_1(x, \mathbf{c}_0 + \psi),$$

that is,

$$\sum_{j=1}^{\infty} j \mathbf{c}_j\, x_1^j = \mathbf{f}_1(x, \mathbf{c}_0) + \sum_{j=1}^{\infty} \frac{\partial \mathbf{f}_1}{\partial y}(x, \mathbf{c}_0)\mathbf{c}_j\, x_1^j + O(\psi^2), \tag{7}$$

whence,

$$(A - I)\mathbf{c}_1 = -\frac{\partial \mathbf{f}_1}{\partial x_1}(0, x_2, \ldots, x_m, \mathbf{c}_0).$$

More generally, if we already have that $\mathbf{c}_0, \ldots, \mathbf{c}_{j-1}$ are holomorphic in Δ then Eq. 7 implies:

$$(A - jI)\mathbf{c}_j = \mathbf{h}_j(x_2, \ldots, x_m, \mathbf{c}_0, \ldots, \mathbf{c}_{j-1}),$$

where the right-hand side is holomorphic in Δ. To prove the holomorphicity of $\mathbf{c}_j$ in Δ, let us consider a polynomial (in λ):

$$P(x_2, \ldots, x_m, \lambda) := \det(A - \lambda I) = (-\lambda)^n + (\operatorname{tr} A)(-\lambda)^{n-1} + \ldots + \det A, \quad (x_2, \ldots, x_m) \in \Delta.$$

Since its coefficients are holomorphic in Δ, for any $(x_2, \ldots, x_m) \in \Delta$ the roots of $P(x_2, \ldots, x_m, \lambda)$ are close to those of $P(0, \ldots, 0, \lambda)$. Hence, there exists $j_0 \in \mathbb{N}$ such that $\det(A - jI) \neq 0$ for any $j > j_0$, $(x_2, \ldots, x_m) \in \Delta$. Therefore, $\mathbf{c}_j$ is holomorphic in Δ except, maybe (if $j \leqslant j_0$), along a complex hypersurface:

$$\{P(x_2, \ldots, x_m, j) = 0\} \subset \Delta.$$

But $\mathbf{c}_j$ is represented by a (formal) Taylor series at the origin, thus it is holomorphic in Δ (in fact, maybe in some smaller polydisc; however, this sequence of polydiscs of holomorphicity is stabilized for $j > j_0$).

Now, we can regard the (vector) formal series ψ as a formal solution of an ODE system of Fuchs type:

$$x_1 \frac{dy}{dx_1} = \mathbf{f}_1(x_1, z, \mathbf{c}_0(z) + y),$$

depending on the parameter $z = (x_2, \ldots, x_m) \in \mathbb{C}^{m-1}$: $\psi = \sum_{j=1}^{\infty} \mathbf{c}_j(z)\, x_1^j$, where the coefficients $\mathbf{c}_j$ are holomorphic in the common polydisc Δ. As $\psi(0) = 0$, the convergence of ψ follows from a natural generalization (for the multidimensional parameter z) of the lemma we present below. This finishes the proof of Theorem 1.

Lemma 1 [3, Ch. III, Lemme 1.1] *Consider an ODE system:*

$$x_1 \frac{dy}{dx_1} = \mathbf{f}(x_1, z, y),$$

where $z \in \mathbb{C}$ is a parameter and $\mathbf{f} : (\mathbb{C} \times \mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a germ of a holomorphic map. Then its any formal solution $\sum_{j=0}^{\infty} \mathbf{a}_j(z)\, x_1^j$, where all $\mathbf{a}_j$ are holomorphic in some common disc centered at the origin and $\mathbf{a}_0(0) = 0$, is convergent.

Example 1 Consider a Pfaffian system:

$$\left.\begin{aligned} x_1^2\frac{\partial y_1}{\partial x_1} &= y_1^2+y_2^2-x_2^2,\\ x_1^2\frac{\partial y_2}{\partial x_1} &= y_1y_2-x_1x_2, \end{aligned}\right\} = \mathbf{f}_1(x,y)$$

$$\left.\begin{aligned} x_2\frac{\partial y_1}{\partial x_2} &= y_1y_2-x_1x_2,\\ x_2\frac{\partial y_2}{\partial x_2} &= y_1^2+x_2-x_1^2, \end{aligned}\right\} = \mathbf{f}_2(x,y)$$

which is Fuchsian along $\{x_2 = 0\}$. It is not completely integrable: one can verify that $\mathbf{F}_{12}$ has non-zero vector components. The system also has the following properties:

1. It possesses a solution
$$\varphi = (x_1, x_2)^\top;$$
2. The Jacobi matrix $\frac{\partial \mathbf{f}_2}{\partial y}(0,0)$ is a zero matrix, but
$$\frac{\partial \mathbf{f}_2}{\partial y} = \begin{pmatrix} y_2 & y_1 \\ 2y_1 & 0 \end{pmatrix},$$
which after the substitution $y_1 = x_1$, $y_2 = x_2 = 0$ has no non-negative integer eigenvalues;
3. There holds:
$$\det \frac{\partial \mathbf{F}_{12}}{\partial y}(x,\varphi) \not\equiv 0.$$

4 Complement to Theorem C: the Bivariate Case

If the assumption of Theorem C on the non-degeneracy of the matrices $\partial\mathbf{f}_i/\partial y(0,0)$ is not satisfied, the system (2) still can have a formal power series solution (for example, if one of these matrices is non-degenerate, see [3, Ch. II, Lemme 3.3]). In this section, we will consider the case of two independent variables x_1, x_2 (that is, $m = 2$) and propose a sufficient condition of the convergence of such a solution.

For any f from $\mathbb{C}[[x]]$ (respectively, from $\mathbb{C}[[x]]^n$ or $\mathrm{Mat}(n, \mathbb{C}[[x]])$), we will say that

$$\mathrm{ord}_{x_i} f \geqslant p \in \mathbb{Z}_+,$$

if $f = x_i^p\, g$, with g from $\mathbb{C}[[x]]$ (respectively, from $\mathbb{C}[[x]]^n$ or $\mathrm{Mat}(n, \mathbb{C}[[x]])$).

Theorem 4 *If the Jacobi matrices $\partial\mathbf{f}_i/\partial y$ on the formal solution* (3) *of* Eq. 2 *satisfy:*

$$ord_{x_i}\frac{\partial \mathbf{f}_i}{\partial y}(x_1,x_2,\varphi) \geqslant p_i - 1, \qquad i = 1,\ 2,$$

then, the series φ converges near $0 \in \mathbb{C}^2$.

Proof We start the proof with a lemma for an ODE system:

$$x_1^p\frac{dy}{dx_1} = \mathbf{f}(x_1, z, y), \qquad y = (y_1,\ldots,y_n)^\top, \tag{8}$$

depending on the parameter $z \in \mathbb{C}$, where $\mathbf{f} : (\mathbb{C}\times\mathbb{C}\times\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a germ of a holomorphic map and $p \geqslant 1$ is an integer. □

Lemma 2 *Let*

$$\varphi=\sum_{j=0}^{\infty}\mathbf{a}_j(z)\,x_1^j,\qquad \mathbf{a}_0(0)=0,$$

be a formal power series solution of Eq. 8 *with the vector coefficients* $\mathbf{a}_j$ *holomorphic in some common disc* $\Delta\subset\mathbb{C}$ *of the parameter space centered at the origin. If*

$$ord_{x_1}\frac{\partial\mathbf{f}}{\partial y}(x_1,z,\varphi)\geqslant p-1,$$

then φ *converges in a neighborhood of zero.*

Proof Let us represent φ in the form:

$$\varphi=\sum_{j=0}^{N}\mathbf{a}_j(z)\,x_1^j+x_1^N(\mathbf{a}_{N+1}(z)x_1+\ldots)=:\varphi_N+x_1^N\psi,$$

with $N\geqslant p-1$. Then, the formal power series ψ will satisfy:

$$x_1^p\,\frac{d\varphi_N}{dx_1}+x_1^{N+p}\frac{d\psi}{dx_1}+Nx_1^{N+p-1}\psi=\mathbf{f}(x_1,z,\varphi_N)+x_1^N\frac{\partial\mathbf{f}}{\partial y}(x_1,z,\varphi_N)\psi+x_1^{2N}O(\psi^2).\quad(9)$$

Since

$$\frac{\partial\mathbf{f}}{\partial y}(x_1,z,\varphi)=\frac{\partial\mathbf{f}}{\partial y}(x_1,z,\varphi_N)+x_1^N O(\psi),\qquad \mathrm{ord}_{x_1}\psi\geqslant 1,$$

the assumption of the lemma also implies:

$$\mathrm{ord}_{x_1}\frac{\partial\mathbf{f}}{\partial y}(x_1,z,\varphi_N)\geqslant p-1,$$

hence, due to Eq. 9, the order of $\mathbf{f}(x_1,z,\varphi_N)-x_1^p\,d\varphi_N/dx_1$ is not less than $N+p-1$. This means that we can divide the relation (9) by x_1^{N+p-1} and obtain the ODE system of Fuchs type satisfied by the formal power series ψ:

$$x_1\,\frac{d\psi}{dx_1}=\mathbf{h}(x_1,z,\psi),$$

where $\mathbf{h}:\ (\mathbb{C}\times\mathbb{C}\times\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ is a germ of a holomorphic map. Since $\psi(0)=0$, the convergence of ψ, as in Theorem 1, again follows from Lemma 1. This proves Lemma 2. □

Let us show that the formal power series solution φ can be represented in the form:

$$\varphi=\sum_{j=0}^{\infty}\mathbf{c}_j(x_2)\,x_1^j,$$

where $\mathbf{c}_j\in\mathbb{C}\{x_2\}^n$ are holomorphic vector functions in some common disc $\Delta\subset\mathbb{C}$ centered at the origin. As in Theorem 1, φ is represented in such a form with $\mathbf{c}_j\in\mathbb{C}[[x_2]]^n$, and the main task is to prove the holomorphicity (convergence) of $\mathbf{c}_j$. The first one represents φ as:

$$\varphi=\mathbf{c}_0+\psi,\qquad \psi=\sum_{j=1}^{\infty}\mathbf{c}_j(x_2)\,x_1^j.$$

Then, in view of Eq. 2, $\mathbf{c}_0$ satisfies the relation:

$$x_2^{p_2}\,\frac{d\mathbf{c}_0}{dx_2}=\mathbf{f}_2(0,x_2,\mathbf{c}_0),$$

moreover, the assumption

$$\mathrm{ord}_{x_2} \frac{\partial \mathbf{f}_2}{\partial y}(x_1, x_2, \varphi) \geqslant p_2 - 1$$

clearly implies

$$\mathrm{ord}_0 \frac{\partial \mathbf{f}_2}{\partial y}(0, x_2, \mathbf{c}_0) = \mathrm{ord}_0 \frac{\partial \mathbf{f}_2}{\partial y}(x_1, x_2, \varphi)|_{x_1=0} \geqslant p_2 - 1.$$

Thus, by Lemma 2, the formal power series $\mathbf{c}_0$ converges near the origin.

Further, substituting $\varphi = \mathbf{c}_0 + \psi$ in the second equation of Eq. 2, we obtain the equality:

$$x_2^{p_2} \frac{\partial \psi}{\partial x_2} = \mathbf{h}(x_1, x_2) + \frac{\partial \mathbf{f}_2}{\partial y}(x_1, x_2, \mathbf{c}_0)\psi + O(\psi^2),$$

which implies the relations for each coefficient $\mathbf{c}_j$, $j \geqslant 1$:

$$x_2^{p_2} \frac{d\mathbf{c}_j}{dx_2} = \frac{\partial \mathbf{f}_2}{\partial y}(0, x_2, \mathbf{c}_0)\mathbf{c}_j + \mathbf{g}_j(x_2, \mathbf{c}_0, \dots, \mathbf{c}_{j-1}).$$

As shown above:

$$\mathrm{ord}_0 \frac{\partial \mathbf{f}_2}{\partial y}(0, x_2, \mathbf{c}_0) \geqslant p_2 - 1,$$

hence by Lemma 2, all the $\mathbf{c}_j$ converge near $0 \in \mathbb{C}$. Since they satisfy the linear ODE systems with the same homogeneous part, they are also holomorphic in some common disc $\Delta \subset \mathbb{C}$ centered at the origin (for instance, Δ may be a common disc of holomorphicity of $\partial \mathbf{f}_2/\partial y(0, x_2, \mathbf{c}_0)$ and $\mathbf{g}_1(x_2, \mathbf{c}_0)$).

We have that the series:

$$\varphi = \sum_{j=0}^{\infty} \mathbf{c}_j(x_2)\, x_1^j,$$

with the coefficients $\mathbf{c}_j$ holomorphically depending on the parameter $z = x_2 \in \Delta$, satisfies the ODE system:

$$x_1^{p_1} \frac{dy}{dx_1} = \mathbf{f}_1(x_1, z, y),$$

furthermore,

$$\mathrm{ord}_{x_1} \frac{\partial \mathbf{f}_1}{\partial y}(x_1, z, \varphi) \geqslant p_1 - 1.$$

Applying Lemma 2 one proves the convergence of φ near $0 \in \mathbb{C}^2$.

Corollary 2 *Let the germs $\mathbf{f}_1$, $\mathbf{f}_2$ have an expansion of the form:*

$$\mathbf{f}_i(x_1, x_2, y) = \mathbf{h}_i(x_1, x_2) + \sum_{|k| \geqslant p_i} \mathbf{f}_{i,k}(x_1, x_2)\, y^k, \quad \mathbf{h}_i, \mathbf{f}_{i,k} \in \mathbb{C}\{x_1, x_2\}^n, \qquad i = 1, 2.$$

Then if a formal power series solution of Eq. 2 *belongs to $x_1 x_2 \mathbb{C}[[x_1, x_2]]^n$, it converges near* $0 \in \mathbb{C}^2$.

Example 2 Consider a Pfaffian system:

$$\left.\begin{aligned} x_1^2 \tfrac{\partial y_1}{\partial x_1} &= (1 + x_1) y_1 y_2 + x_1^2 x_2 (1 - x_2), \\ x_1^2 \tfrac{\partial y_2}{\partial x_1} &= y_1^2 - y_2^2 + x_1^2 x_2 \tfrac{1 - 2x_1 x_2 - x_1^2 x_2}{(1 + x_1)^2}, \end{aligned}\right\} = \mathbf{f}_1(x, y)$$

$$\left.\begin{aligned} x_2^2 \tfrac{\partial y_1}{\partial x_2} &= (1 + x_1) y_1 y_2 + x_1 x_2^2 (1 - x_1), \\ x_2^2 \tfrac{\partial y_2}{\partial x_2} &= y_1 y_2 + x_1 x_2^2 \tfrac{1 - x_1}{1 + x_1}. \end{aligned}\right\} = \mathbf{f}_2(x, y)$$

It possesses a solution:

$$\varphi = \left(x_1 x_2, \frac{x_1 x_2}{1 + x_1} \right)^\top .$$

Note that both the Jacobi matrices $\frac{\partial \mathbf{f}_1}{\partial y}(0,0)$, $\frac{\partial \mathbf{f}_2}{\partial y}(0,0)$ are zero matrices, whereas $\varphi \in x_1 x_2 \mathbb{C}\{x_1\}^2$ and thus the assumptions of Corollary 2 are fulfilled.

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