Programming and Computer Software, Vol. 26, No. 2, 2000, pp. 107–115.
Original Russian Text Copyright © 2000 by Khmel'nov.

Improved Algorithms for Solving Difference and q-Difference Equations

D. E. Khmel'nov
Department of Computational Mathematics and Cybernetics, Moscow State University, Vorob'evy gory, Moscow, 119899 Russia

Received April 7, 1999

Abstract—Improved algorithms for finding denominators of rational solutions of linear difference and q-difference equations with polynomial coefficients are proposed. The improved efficiency of these algorithms is achieved as a result of a more efficient implementation of the Abramov algorithm (due to the use of the Man and Wright algorithm for calculating the dispersion, which is extended for the case of q-dispersion) and of the improvement of this algorithm by using an additional procedure for minimizing the degree of the denominator (similar to the Migushov algorithm). The case of difference equations is analyzed in detail, whereas q-difference equations are considered by analogy with the first case. The algorithms described were implemented in Maple V.

1. STATEMENT OF THE PROBLEM

We use the statement of the problem described in [1].

Definition 1. *The field K of characteristic 0 equipped by an algorithm for finding integer-valued roots of equations of the form $p(x) = 0$, where $p(x) \in K[x]$, is called appropriate. By the integer-valued root, we mean the root of the form $1n$, where $n \in Z$ and 1 is the identity of K.*

Consider the difference equation

$$\sum_{v=0}^{n} a_v(x)F(x+v) = b(x), \tag{1}$$

where $a_0(x), \ldots, a_n(x), b(x) \in K[x]$. It is required to find $F(x) \in K(x)$.

The problem of finding $F(x) \in K(x)$ can be reduced to finding a polynomial $s(x) \in K[x]$ that is a multiple of the denominator of the irreducible form of any rational solution to (1) and to finding a polynomial solution $F(x) \in K[x]$.

This paper is dedicated to the analysis of the known and improved algorithms for solving the first of these two problems. The second problem can be solved by methods presented in [2–4].

2. STRUCTURE OF THE DENOMINATOR

Substitute $F(x) = \dfrac{t(x)}{s(x)}$ in equation (1) and multiply both sides by the denominator of the left-hand side to obtain

$$\sum_{v=0}^{n} a_v(x)t(x+v) \prod_{k=0,\, k\neq v}^{n} s(x+k) = b(x)\prod_{k=0}^{n} s(x+k). \tag{2}$$

Let us fix a k_0 and consider the division of equation (2) by $s(x+k_0)$. It follows from the structure of the left-hand side of (2) that, for the left-hand side of (2) to be divisible by $s(x+k_0)$, it is necessary (under the condition that $\dfrac{t(x)}{s(x)}$ is irreducible) that

$$s(x+k_0)\,|\,a_{k_0}(x)t(x+k_0) \prod_{k=0,\, k\neq k_0}^{n} s(x+k). \tag{3}$$

Consider this condition in terms of the roots of polynomials. Denote by $R_{p(x)}$ the set of the roots of the polynomial $p(x)$ in the algebraic closure $\bar{K}$ of K; by $E^nM(M \in \bar{K})$, we denote the set obtained from M by subtracting $1n$ from its every element. It is clear that

$$R_{p(x+n)} = E^n R_{p(x)}.$$

Now condition (3) is written as

$$E^{k_0}R_{s(x)} \subset R_{a_{k_0}(x)} \cup \left[\bigcup_{k=0,\, k\neq k_0}^{n} E^k R_{s(x+k)}\right],$$

where the union is interpreted as the augmentation; i.e., all elements are considered to be different so that $\{1, 1, 2\} \cup \{1, 2, 2, 4\} = \{1, 1, 1, 2, 2, 2, 4\}$.

Thus, we have proved the following proposition.

Proposition 1. *In order for $s(x)$ to be the denominator of the irreducible form of a rational solution to equation (1), it is necessary that*

$$\forall k_0\left(R_{s(x)} \subset E^{-k_0} R_{a_{k_0}(x)} \cup \left[\bigcup_{k=0, k\neq k_0}^{n} E^{k-k_0} R_{s(x)}\right]\right). \quad (4)$$

Definition 2.

1. $a, b \in \overline{K}$ are integer-connected $\Leftrightarrow \exists_{n \in Z}\, a = b + 1n$;

2. $A \subseteq \overline{K}$ is integer-connected $\Leftrightarrow {}_{a \in A, b \in A} a, b$ are integer-connected;

3. $A, B \subseteq \overline{K}$ are mutually integer-connected $\Leftrightarrow A \cup B$ is integer-connected;

4. A family of sets is integer-connected $\Leftrightarrow$ any pair of the sets of this family is mutually integer-connected.

Proposition 2. *Let $s(x)$ be the denominator of the irreducible form of a rational solution to equation (1). Then $r \in R_{s(x)} \Rightarrow \exists_{k_0, a \in R_{a_{k_0}(x)}}$: a, r are integer-connected.*

Proof. Let us represent $R_{s(x)}$ as a union of nonintersecting integer-connected subsets

$$R_{s(x)} = \bigcup_{p=1}^{m} R^p_{s(x)}.$$

Suppose that the assertion of Proposition 2 in not true. Since $R^p_{s(x)}$ is integer-connected, we have

$$\exists r(r \in R^p_{s(x)}) \wedge \forall k_0(R^p_{s(x)} \cap E^{-k_0} R_{a_{k_0}(x)} = \varnothing)$$

and condition (4) is written as

$$\forall k_0\left(R^p_{s(x)} \subset \bigcup_{k=0, k\neq k_0}^{n} E^{k-k_0} R^p_{s(x)}\right).$$

Let $k_0 = 0$. Then, the above condition is written as

$$R^p_{s(x)} \subset \bigcup_{k=1}^{n} E^k R^p_{s(x)}. \quad (5)$$

Since $R^p_{s(x)}$ is integer-connected, it is equivalent to a finite subset of integers and has the order induced by integer numbers; therefore, there is a maximal (and minimal) element in this subset. By the definition of $E^n M$, the maximal element of $R^p_{s(x)}$ is greater than that of $E^n R^p_{s(x)}$ for $n > 0$. Thus, condition (5) does not hold. Therefore, the assumption that the assertion of Proposition 2 does not hold cannot be true. □

Proposition 2 means that every root of the polynomial $s(x)$ is integer-connected with a root of at least one of the polynomials $a_0(x), \ldots, a_n(x)$.

With regard to this assertion, we obtain a decomposition of condition (4) with respect to integer-connected roots of the polynomials $a_0(x), \ldots, a_n(x)$, i.e., with respect to the integer-connected subsets $R_{a_0(x)}, \ldots, R_{a_n(x)}$. Namely, this condition falls apart into the independent conditions

$$\forall k_0\left(R^p_{s(x)} \subset E^{-k_0} R^p_{a_{k_0}(x)} \cup \left[\bigcup_{k=0, k\neq k_0}^{n} E^{k-k_0} R^p_{s(x)}\right]\right), \quad (6)$$

where $R^p_{s(x)}, R^p_{a_0(x)}, \ldots, R^p_{a_n(x)}$ are mutually integer-connected.

Using the reasoning similar to that used in the proof of Proposition 2, we obtain the following proposition.

Proposition 3. *Let $s(x)$ be the denominator of the irreducible form of the rational solution to equation (1). Then, for all p,*

1. $\max R^p_{s(x)} \in R^p_{a_0(x)}$;

2. $\min R^p_{s(x)} \in R^p_{a_n(x)}$.

Corollary 1. *Under condition (6), it is sufficient to consider only those p for which $R^p_{a_0(x)}$ and $R^p_{a_n(x)}$ are nonempty. In other words, we are interested only in the roots of the polynomials $a_0(x), \ldots, a_n(x)$ that are integer-connected with the roots of $a_0(x)$ and $a_n(x)$, which are integer-connected with each other. Indeed, by Proposition 3, $s(x) = 1$ for all other p.*

Recall one more definition from [1].

Definition 3. 1. *The dispersion of two polynomials is the magnitude*

$$\mathrm{dis}(s(x), t(x))$$
$$= \max\{r \mid r \in Z, r \geq 0, deg(gcd(s(x+r), t(x)) \geq 1\};$$

2. *For the irreducible form of the rational function $F(x) = \dfrac{t(x)}{s(x)}$, the dispersion is*

$$\mathrm{Dis}(F(x)) = \mathrm{dis}(s(x), s(x)).$$

It is clear that the following equalities hold:

$$\mathrm{dis}(s(x), t(x)) = \max_p\{\max R^p_{s(x)} - \min R^p_{t(x)}\} \quad (7)$$

and

$$\mathrm{Dis}(F(x)) = \max_p\{\max R^p_{s(x)} - \min R^p_{s(x)}\}.$$

Corollary 2. Proposition 3 entails two propositions proved in [1]; namely, the necessity of the existence of $\mathrm{dis}(a_0(x), a_n(x)) \geq n$ and the relation $\mathrm{Dis}(F(x)) \geq \mathrm{dis}(a_0(x), a_n(x-n)) = \mathrm{dis}(a_0(x), a_n(x)) - n$.

Consider the magnitude $c_p = \max R^p_{a_0(x)}$. By Proposition 2 and the Corollary 3, the structure of $s(x)$ is as follows:

$$s(x) = \prod_{q=1}^{Q}\prod_{p=0}^{N}(x-(c_q-p))^{S_{qp}}, \quad (8)$$

where $q = 1, \ldots, Q$ are the indexes of the roots mentioned in Corollary 1, $N = \mathrm{dis}(a_0(x), a_n(x-n))$, and S_{qp} are unknown multiplicities of the roots of polynomial (8). Thus, the initial problem is reduced to determining these multiplicities.

Basing on Corollary 1, we can write the following representation:

$$a_k(x) = f_k(x)\bar{a}_k(x), \quad (9)$$

where $\bar{a}_k(x) = \prod_{q=1}^{Q}\prod_{p=0}^{N}(x-(c_q-p))^{A_{qp}^k}$ and $f_k(x)$ and $\bar{a}_k(x)$ are relatively prime.

3. THE IMPORTANCE OF NECESSARY CONDITIONS

Let $A(\|S_{qp}\|)$ be a set of necessary conditions that are satisfied by the desired multiplicities $\|S_{qp}\|$ for which $s(x)$ in (8) is the denominator of the irreducible form of a rational solution to equation (1). To find the polynomial $\bar{s}(x)$ that is a multiple of the denominator of any irreducible solution to equation (1), it is required to find

$$\bar{S}_{qp} = \max\{S_{qp}\} \quad (10)$$

where $\Omega = \{S_{qp} | A(\|S_{qp}\|)\}$.

Thus, the function of the necessary conditions is to determine the feasible set Ω over which the maximization operation is performed. The stricter the condition, the fewer the elements of Ω and, therefore, the fewer the values $\bar{S}_{qp}$ and, thus, the degree of the desired polynomial $\bar{s}(x)$.

4. THE SECOND SET OF NECESSARY CONDITIONS

Substitute $F(x) = t(x)/s(x)$ in equation (1) with regard to (8) and (9) to obtain

$$\sum_{k=0}^{n}\left[\prod_{q=1}^{Q}\prod_{p=0}^{N}(x-(c_q-p))^{A_{qp}^k} \times \frac{f_k(x)t(x+k)}{\prod_{q=1}^{Q}\prod_{p=0}^{N}(x+k-(c_q-p))^{S_{qp}}}\right] = b(x).$$

The denominator of the kth term of the sum is

$$s_k(x) = \prod_{q=1}^{Q}\prod_{p=0}^{N}(x+k-(c_q-p))^{[S_{qp}-A_{q,p+k}^k]_+}, \quad (11)$$

where $[x]_+ = \max(x, 0)$.

Proposition 4. *In order for $s(x)$ to be the denominator of the reduced form of a rational solution to equation (1), it necessary that*

$$\forall q \forall k_0 \forall p([S_{qp} - A_{q,p+k_0}^{k_0}]_+ \le \max_{k \ne k_0}\{[S_{q,p+k_0-k} - A_{q,p+k_0}^{k}]_+\}). \quad (12)$$

Proof. Consider $(x + k - (c_q - p))$. By virtue of representation (11), the degree of this expression in the denominator of the k_0th term of the sum is $[S_{qp} - A_{q,p+k_0}^{k_0}]_+$ and its degree in the denominator of the kth term is $[S_{q,p+k_0-k} - A_{q,p+k_0}^{k}]_+$. Thus, condition (12) means that it is necessary that, for any root of the denominator of any term in the sum, another term exists with the denominator having the same root with the multiplicity greater or equal than the multiplicity of this root in the first term. This condition is evident from the decomposition into the sum of elementary fractions: it is necessary to guarantee the resultant zero in the numerator of the term of the decomposition corresponding to the given multplicity of the root under consideration. □

5. THE ABRAMOV ALGORITHM

Consider the algorithm presented in [5].

Let $A(x) = a_n(x-n)$, $B(x) = a_0(x)$.

Here is this algorithm written [5] in a pseudocode.

```
INPUT: nonzero polynomials A(x) and B(x)
OUTPUT: the polynomial s(x)
s(x) := 1;
R(m) := Res_x(A(x), B(x + m));
if R(m) has nonnegative integer roots then
N := the greatest nonnegative integer root of R(m);
   for i := N, N – 1, ..., 0 do
      d(x) := gcd(A(x), B(x + i));
      A(x) := A(x)/d(x);
      B(x) := B(x)/d(x – i);
      s(x) := s(x)d(x)d(x – 1) ... d(x – i)
   od
fi.
```

This algorithm implements a method for constructing the polynomial $s(x)$ satisfying certain conditions in (4) and condition (10). Let us prove this fact.

By Corollary 1, we may consider $\bar{a}_k(x)$ from (9) instead of $a_k(x)$. With regard to (6), condition (4) turns into

$$\forall k_0 \forall p\left(S_{p-k_0} \le A_p^{k_0} + \sum_{k=0,\, k\ne k_0}^{n} S_{p-k}\right)$$

(this fact can be verified by the direct substitution of (8) and (9) in (1)). Here $S_p = 0$ for $p < 0$ and $p > N$ and the first index in the multiplicities is omitted for convenience (by virtue of decomposition (6)).

The above condition is written as

$$\forall k_0 \forall p\left(S_p \le A_{p+k_0}^{k_0} + \sum_{k=0,\, k\ne k_0}^{n} S_{p+k-k_0}\right). \quad (13)$$

Let us derive a set of necessary conditions from conditions (13) for $k_0 = 0$ and $k_0 = n$: $\forall p(S_p \le A_p^0 + \sum_{k=1}^{n} S_{p+k})$, $\forall p(S_p \le A_{p+n}^n + \sum_{k=1}^{n} S_{p-k})$. We relax these conditions (with regard to the fact that $S_p = 0$ for $p < 0$ and $p > N$, and $S_p \ge 0$ otherwise) and write them as

$$\forall p\left(S_p \le A_p^0 + \sum_{k=p+1}^{N} S_k\right),$$
$$\forall p\left(S_p \le A_{p+n}^n + \sum_{k=0}^{p-1} S_k\right). \quad (14)$$

In terms of multiplicities, the core of the algorithm under consideration is written as

$$D_p^i := \min\{A_{p+n}^{n,i}, A_{p-i}^{0,i}\}$$
$$A_{p+n}^{n,i-1} := A_{p+n}^{n,i} - D_p^i$$
$$A_p^{0,i-1} := A_p^{0,i} - D_{p+i}^i$$
$$S^i := S^{i+1} + \sum_{k=0}^{i} D_{p+k}^i$$

(every line is executed for p from 0 to N).

In terms of multiplicities, this algorithm is interpreted as follows. The maximum of S_p is sought over all simultaneous solutions of two systems of linear inequalities (14) using the special form of these systems and of their independent solutions $S_p = A_p^0 + \sum_{k=p+1}^{N} S_k$ for the first system and $S_p = A_{p+n}^n + \sum_{k=0}^{p-1} S_k$ for the second one. For this purpose, S_p are sought as sums of increments. At each step of the algorithm, every system includes a single inequality of the form $S_m \le A$. This is achieved by the downward direction of the loop from N to 0. The increment is determined by the inequality with the smaller right-hand side (i.e., the smaller of the two possible increments is used). The systems are modified so that they remain true for the remainders of S_p.

6. THE MIGUSHOV ALGORITHM

In 1996, S.V. Migushov proposed an algorithm [6] based on the necessary conditions (12) and (10). The algorithm consists of two stages—preliminary and computational.

At the preliminary stage, every coefficient $a_0(x), \ldots, a_n(x)$ of equation (1) is represented as a product of certain polynomials having only the roots of interest (Corollary 1 to Proposition 3).

The second stage is subdivided into two steps: first, on the basis of the polynomials obtained at the first stage, a polynomial is obtained that is a multiple of the denominator of any irreducible solution to equation (1); i.e., this polynomial can be used to find rational solutions to the initial equation (1). Then, the multiplicity of the roots of this polynomial is reduced to ensure that conditions (12) and (10) are satisfied (the initial polynomial gives upper bounds of the multiplicities). Thus, the degree of the resultant polynomial is reduced, which provides a more accurate estimate of the denominator of the rational solution to equation (1).

By virtue of the structure of conditions (12), all the multiplicities are maximized simultaneously by condition (10); therefore, the maximal multiplicities satisfy conditions (12).

We do not present the full text of the Migushov algorithm in this paper for two reasons. On the one hand, it is rather lengthy, and, on the other hand, the algorithm described below is a variation of the Migushov algorithm.

7. THE SCHEME OF THE ALGORITHM PROPOSED

As was mentioned above, our algorithm is a variation of the Migushov algorithm [6]. The new algorithm is based on the same necessary conditions (12) and condition (10); thus, both algorithms give identical results. The structure of our algorithm is also similar to the structure the Migushov algorithm [6]: at the first stage, a polynomial $s(x) \in K[x]$ is obtained that is a multiple of the irreducible form of any rational solution to equation (1); at the second stage, the degree of this polynomial is reduced by using a stronger necessary condition.

On the other hand, our algorithm is an extension of the Abramov algorithm [5], since the latter is used as the first stage of the former. Due to this property, our algorithm may be used as an optional complement to the algorithm from [5]. That is, while implementing the algorithm for finding the denominator of the rational solution, we can specify whether it is required to minimize the result or not. This option is useful because such a minimization does not always yield increased speed in computations when finding the rational solution (for details, see Section 9).

We have already described the first part of the algorithm; the description of the second part follows.

INPUT: the polynomials $a_0(x), \ldots, a_n(x)$ from equation (1) and the polynomial $s(x)$ multiple of the denominator of any irreducible solution to equation (1).

OUTPUT: the polynomial $\bar{s}(x)$ multiple of the denominator of the irreducible form of any rational solution to equation (1), maybe of a reduced degree.

```
s̄(x) := s(x);
for k := 0 to n do
    s_k(x) := s(x + k)/gcd(s(x + k), a_k(x));
od;
repeat
    for k := 0 to n do
        m_k(x) := lcm(s_0(x), ..., s_{k-1}(x), s_{k+1}(x), ..., s_n(x));
        d_k(x) := s_k(x)/gcd(m_k(x), s_k(x));
    od;
    ds(x) := lcm(d_0(x), ..., d_k(x - k), ..., d_n(x - n));
    if ds(x) ≠ 1 then
        s̄(x) := s̄(x)/ds(x);
        for k := 0 to n do
            s_k(x) := s_k(x)/gcd(ds(x + k), s_k(x));
        od;
    fi;
until deg(ds(x)) = 0.
```

(here *lcm* denotes the least common multiple of polynomials).

The idea of this algorithm is as follows. First, the denominators $s_k(x)$ of all terms of the sum are calculated. Then, for each of them, a polynomial $d_k(x)$ with the same roots is calculated; however, the multiplicity of each of these roots corresponds to the excess of the initial multiplicity over the maximal multiplicity of the same root in the remaining denominators. Then, the polynomial $ds(x)$ corresponding to the factor of $s(x)$ that is the cause this excess is calculated; the polynomial $s(x)$ is reduced by this factor; and the denominators are recalculated. The process is repeated until a balanced (in terms of condition (12)) polynomial $s(x)$ is obtained; that is, until the excess factor becomes 1. This process is finite, since the initial multiplicities are finite and nonnegative and the multiplicity of at least one of the roots decrease at every step but cannot become less than zero.

8. COMPUTER IMPLEMENTATION

The algorithms described were implemented in the computer algebra system MAPLE V. The program [7] implementing the algorithm from [5] and the algorithm for determining the polynomial solution after the substitution of the denominator found was used as the basis. The implementation retains the opportunity provided by program from [7] to work not only over the field of rationals, but also over simple algebraic extensions of this field.

All algorithms described above require the calculation of the dispersion $\mathrm{dis}(a_0(x), a_n(x-n))$. As a rule, the dispersion is calculated by finding integer-valued roots of the polynomial obtained as the resultant is determined. This polynomial rapidly grows both in terms of degree and the absolute magnitude of the coefficients as the degree of the coefficients in the recurrent relation increases. In our experiments, the degree was as much as 100 and the coefficients reached values of the order of 10^{100}. It is clear that such large values slow down the algorithm. For this reason, it was very important to use an efficient algorithm for dispersion computation.

We used the algorithm from [8], which uses the factoring of polynomials.

9. EFFICIENCY ESTIMATE

First of all, it must be noted that the algorithms for finding a polynomial multiple of the denominator of any irreducible solution to equation (1) are intended for solving an auxiliary problem necessary for finding rational solutions to this equation. Thus, when analyzing an auxiliary algorithm in terms of time, we must keep in mind the execution time of the algorithm for solving the general problem. If this latter algorithm is based on the auxiliary algorithm, a slightly increased execution time of the auxiliary algorithm that gives a polynomial of a smaller degree can lead to a reduced total execution time by economizing on time for solving the equation obtained after substitution of the denominator of a smaller degree.

We consider the same example as in [6]:

$$(x+m+n)^k Y(x+n) + x^k y(x) = b(x).$$

Here is the result given by the program in MAPLE V.

```
n = 3 m = 3 k = 5
Equation is
y(x + 3)*(x + 6)^5 + y(x)*x^5 = 1
Solve using simple ...
dis = 3
Degree of denominator is 20
Solution using simple is
1/4*(2*x^5 + 15*x^4 - 135*x^2 + 243)/x^5/(x + 3)^5
Time used: 6.000
Solve using minimize ...
dis = 3
Degree of denominator is 10
Solution using minimize is
1/4*(2*x^5 + 15*x^4 - 135*x^2 + 243)/x^5/(x + 3)^5
Time used: 1.000
```

The first solution was found by the algorithm from [5]; the second solution was given by our improved algorithm. It is seen that the execution time was reduced from six seconds to one second, due to a decrease in the degree of the denominator from 20 to 10. This is in spite of the second algorithm requiring more time than the first one, since the second algorithm includes the first one as its part.

10. THE CASE OF q-DIFFERENCE EQUATIONS

The algorithm presented in [5] for finding the polynomial multiple of the denominator of the irreducible form of any rational solution to the q-difference equation is similar to the corresponding algorithm for difference equations. The reasoning used for the substantiation of the algorithm for difference equations in terms of the roots of the coefficients remains the same for q-difference equations; however, additional analysis of the zero root is required because it goes to itself as the operator $Q : x \longrightarrow qx$ acts on the polynomial. For this reason, the algorithm used for solving q-difference equations consists of two parts. The first part is the full analog of the algorithm used in the case of difference equations, and the second part analyzes the case of the zero root. The overall result is the product of the results given by the first and second parts of the algorithm.

As in the case of difference equations, the improvement of the algorithm proposed in this paper complements the algorithm described in [5]. Our algorithm is divided into three stages. The first two stages coincide with the first two stages of the algorithm in [5]. The third stage is an extension of the second stage of our algorithm used for solving difference equations to the case of q-difference equations.

Similarly to the case of difference equations, we introduce the notion of q-dispersion:

$$q - \mathrm{dis}(s(x), t(x))$$
$$= \max\{r | r \in Z, r \geq 0, deg(gcd(s(q^r x), t(x))) \geq 1\}.$$

The algorithm presented in [8] was extended for this case.

Procedures that implement the algorithm for finding rational solutions to q-difference equations were developed in MAPLE V. Consider an example of using these procedures.

The example is taken from paper [5]. However, the result given in [5] is inaccurate. We included the check of the result in our MAPLE V session.

```
> eq:=q^3*(q*x+1)*y(q^2*x)-2*q^2*(x+1)*y(q*x)+ y(x)*(x+q)
=(q^5-2*q^3+1)*x^2=x*(q^4-2*q^3+1):
> sol:='LQEtools/q_ratpolysols' (eq, y(x), minimize):
> lprint (sol);
(q^7*x^2+_C[1]*q^7+q^6*x^3+3*q^6*_C[1]+
x^2*q^6+4*_C[1]*q^5+3*x^3*q^5+2*x^3*q^4+2*q^4*_C[1]-
4*q^4*x^2-2*q^3*q^3*x^3-6*q^3*x^2-*q^3*_C[1]-4*x^3*q^2-4*q^2*_C[1]-
6*q^2*x^2-3*_C[1]*q-3*x^2*q-3*x^2*q-3*q-x^2-_C[1]-x^3_/
(x+q)/x/(q^3+q^2-q-1)/(q^4+2*q^3+3*q^2+2*q+1)
> ss:=simplify (sudu(y(q^2*x)=suds(x=q^2*x, sol),
```

```
(y(q*x)=suds(x=q*x, sol), (y(x)=sol, eq)))):
> lprint(simplify(lhs(ss)-rhs(ss)));
0
```

11. VAN HOEIJ'S ALGORITHM

11.1. The Case of Systems of Difference Equations

In 1998, van Hoeij proposed a new algorithm for finding rational solution to linear difference equations [9].

In that paper, much effort is focused on finding denominators of such solutions. The case of the system of linear difference first-order equations

$$Y(x+1) = A(x)Y(x), \tag{15}$$

is considered, where $A \in GL_n(K(x))$ and $Y(x) \in K(x)^n$. The author suggests that the scalar case be considered by reducing to the corresponding system of linear difference first-order equations.

The algorithm is based on determining a finite set of points that can be roots of the denominator and on the analysis of system's singularities at these points. This analysis can be outlined as follows.

1. From the matrix $A(x)$, the finite set $\bar{S}$ is determined that contains all possible roots of the denominator (poles) of the solution to system (15).

2. Let $p \in \bar{S}$. Consider a nonnegative integer N such that $p - N \notin \bar{S}$. Then $Y(x) = A(x-1)Y(x-1) = A(x-1)A(x-2)Y(x-2) = \ldots$; that is,

$$Y(x) = A_N(x)Y(x-N), \tag{16}$$

where $A_N(x) = A(x-1)\ldots A(x-N)$.

3. By virtue of the choice of p and N, $Y(x-N)$ has no pole at the point p; i.e., p is not a root of the denominator of $Y(x-N)$. Thus, if $Y(x)$ has a pole at p, i.e., if p is a root of the denominator of $Y(x)$, then its multiplicity is determined by the order of the poles of the elements of $A(x)$ at the point p, i.e., by the multiplicity of the root p of the denominators of this matrix elements (this is evident from equation (16)).

4. By analyzing all elements of $\bar{S}$ in such a way, we can determine the upper bounds of their multiplicities in the denominators of the elements of $Y(x)$ and, thus, the upper bounds of the denominators themselves (by virtue of the properties of the set $\bar{S}$).

Similar to equation (16), the equation

$$Y(x) = A_{-N}(x)Y(x+N), \tag{17}$$

can be obtained, where $A_{-N}(x) = A_N^{-1}(x+N)$, which gives another set of upper bounds of the denominators. The estimates based on equation (16) are called "left" and based on equation (17) are called "right."

It has been proved that the denominators obtained by this algorithm are exact in the case when all solutions to the given system are rational.

11.2. The Version of the Algorithm for the Scalar Case

As was mentioned above, the author of [9] suggests that the scalar case be considered by reducing it to the corresponding system of linear difference first-order equations. However, this approach has certain drawbacks:

(i) The algorithm uses a considerable number of costly matrix operations (shift and multiplication operations).

(ii) When the scalar equation (1) is reduced to system (15), the elements of $Y(x)$ are $F(x), \ldots, F(x+n-1)$ and the set $\bar{S}$ includes "superfluous" elements related to the roots of the denominators of $F(x+1), \ldots, F(x+n-1)$, although only the denominator of $F(x)$ is of interest for us. This leads to some extra work.

Basing ourselves on the ideas of van Hoeij's algorithm, the following modification can be proposed for solving scalar equations directly, without reducing them to systems of linear difference first-order equations.

1. Let $N = \mathrm{dis}(a_0(x), a_n(x-n))$. From equation (1), we can write $F(x)$ as

$$F(x) = f_{00} + \sum_{v=1}^{n} f_{0v}(x)F(x-v), \tag{18}$$

where $f_{0k} \in K(x)$. Let us substitute $F(x-1) = f_{00} + \sum_{v=1}^{n} f_{0v}(x-1)F(x-v-1)$ in equation (18). Then,

$$F(x) = f_{10} + \sum_{v=2}^{n+1} f_{1v}(x)F(x-v), \tag{19}$$

where $f_{1k} \in K(x)$. Substituting successively $F(x-1), \ldots, F(x-N)$ expressed from (18), we obtain

$$F(x) = f_{N0} + \sum_{v=N+1}^{n+N} f_{Nv}(x)F(x-v), \tag{20}$$

where $f_{Nk} \in K(x)$.

2. By Corollary 2 to Proposition 3, the roots of the denominator of $F(x)$ are different from the roots of the denominators of $F(x-(N+1)), \ldots, F(x-(n-N))$. Thus, by virtue of (20), these roots and their multiplicities are determined by the coefficients $f_{N0}, f_{N,N+1}, \ldots, f_{N,n+N}$; namely, the least common multiple of the denominators

of these coefficients can be used as the upper bound of the denominator of $F(x)$.

Similar to the original algorithm for systems of equations, in addition to the "left" estimate described above, we can consider the "right" estimate based on the expression

$$F(x) = f_{-00} + \sum_{v=1}^{n} f_{-0v}(x)F(x+v), \qquad (21)$$

where $f_{-0k} \in K(x)$.

Note that the version of the algorithm presented here differs from the original one in that it does not consider the possible roots of the denominators one by one, but analyzes all possible roots simultaneously. A similar method may be used for systems of equations as well, since the separate consideration of points affects only the magnitude of N; however, the number that suits all points simultaneously must be considered all the same for a certain point; therefore, all the necessary transformations must be made N times. In the process, the estimate of the denominator may include the roots that are not contained in $\bar{S}$ (in the scalar case, these roots are not involved in the structure of the denominator (8)); these roots must be discarded.

11.3. Example

Consider the following example:

$$(x+10)(x+8)^2(x+5)y(x+4)$$
$$+(-x^3-12x^2-50x-76)y(x+2)$$
$$+(x+2)xy(x) = 0.$$

This equation has no rational solutions except for the trivial one.

For this equation, $N = \mathrm{dis}((x+2)x, (x+6)(x+4)^2(x+1)) = 6$. It is seen from (8) that the denominator can have the roots $\{-6, -5, -4, -3, -2, -1, 0\}$.

Let us write

$$y(x) = -\frac{(-x^3-2*x-4)y(x-2)+(x-4)(x-2)y(x-4)}{(x+6)(x+4)^2(x+1)}$$

$$y(x) = -(y(x-8)(-1462272+460864x^2$$
$$+2066688x+184912x^4-1507008x^3+632432x^5$$
$$+117780x^7-424760x^6-13496x^8-444x^9-29x^{11}$$
$$+289x^{10}+x^{12})+y(x-10)(-522240+279616x^2$$
$$+632064x-20944x^4-463520x^3+24480x^7$$
$$-109472x^6+209112x^5-2156x^8+22x^{10}$$
$$-60x^9-x^{11}))/(x^3(x-2)^2(x-5)(x+2)^3(x-3)$$
$$\times(x+4)^3(x-1)(x+6)(x+1)).$$

Thus, the upper left estimate is obtained by discarding the superfluous roots of the polynomial $x^3(x-2)^2(x-5)(x+2)^3(x-3)(x+4)^3(x-1)(x+6)(x+1)$; i.e., this estimate is

$$x^3(x+2)^3(x+4)^3(x+6)(x+1). \qquad (22)$$

The similar procedure yields the upper right estimate

$$(x+6)^2(x+4)^2(x+2)^2x. \qquad (23)$$

The combination of these estimates (their least common divisor) yields

$$(x+6)(x+4)^2(x+2)^2x. \qquad (24)$$

Note that, as expected, the passage to the system of first-order equations and application of our algorithm to this system yields the same result.

The algorithm from [5] yields

$$(x+6)(x+5)(x+4)^2(x+3)^2(x+2)^2(x+1)x.$$

The minimization procedure applied to this polynomial gives the estimate

$$(x+4)^2(x+2)^2x.$$

The same result is obtained by the application of the minimization procedure to estimates (22), (23), and (24).

11.4.Alternative Scheme

In spite of the optimality of van Hoeij's algorithm for the case of the full set of rational solutions, in other cases, the results obtained are sometimes worse than those given by our scheme. As is seen from the above example, the minimization procedure can be applied to the results given by van Hoeij's algorithm. Thus, an alternative scheme is possible in which the minimization stage is performed as described in Section 7, and the first stage is replaced by van Hoeij's algorithm, modified for the scalar case.

12. "IDEAL" ALGORITHM

In conclusion, we note that, at least theoretically, the following exact algorithm for finding the denominator of the rational solution can be considered.

1. Find a polynomial multiple of the denominator of any irreducible solution to the initial equation using one of the algorithms described above.

2. Substitute this polynomial in the initial equation and find the polynomial solution to the equation obtained using the algorithm from [2, 4]; then the rational solution to the initial equation can be easily found.

3. Produce the denominator of this solution as a polynomial multiple of the denominator of any irreducible solution to the initial equation.

It is clear that the execution time of this algorithm does not pay for the accuracy of the result.

REFERENCES

1. Abramov, S.A., Rational Solutions to Linear Differential and Difference Equations with Polynomial Coefficients, *Zh. Vychisl. Mat. Mat. Fiz.*, 1989, vol. 29, no. 11, pp. 1611–1620.
2. Abramov, S.A., Problems in Computer Algebra Related to Constructing Solutions to Linear Differential and Difference Equations with Polynomial Coefficients, *Vestn. Mosk. Univ., Ser. 15, Vychisl. Mat. Kibern.*, 1989, no. 3, pp. 55–60.
3. Abramov, S.A. and Kvashenko, K.Yu., Fast Algorithm to Search for the Rational Solutions of Linear Equations, *Proc. 1991 Int. Symposium on Symbolic and Algebraic Computation, ISSAC'91*, New York, ACM, 1991, pp. 267–270.
4. Abramov, S.A., Bronstein, M., and Petkovšek, M., On Polynomial Solutions of Linear Operator Equations, *Proc. 1995 Int. Symposium on Symbolic and Algebraic Computation, ISSAC'95 (Montreal)*, New York, ACM, 1995, pp. 290–296.
5. Abramov, S.A., Rational Solutions to Linear Difference and q-Difference Equations with Polynomial Coefficients, *Programmirovanie*, 1995, no. 6, pp. 3–11.
6. Migushov, S.V., A New Algorithm for Constructing Rational Solutions to Difference Equations, *Fund. Appl. Math.* (in press).
7. Ryabenko, A.A., Programs for Constructing Rational Solutions to Linear Difference and q-Difference Equations with Polynomial Coefficients, *Programmirovanie*, 1997, no. 3.
8. Man, Y.-K. and Wright, F.J., Fast Polynomial Dispersion Computation and Its Application to Indefinite Summation, *Proc. 1995 Int. Symposium on Symbolic and Algebraic Computation, ISSAC'94*, New York, ACM, 1994, pp. 175–180.
9. Van Hoeij, M., Rational Solutions of Linear Difference Equations, *Proc. 1998 Int. Symposium on Symbolic and Algebraic Computation, ISSAC'98*, New York, ACM, 1998, pp. 120–123.