GENOCCHI POLYNOMIALS

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1. INTRODUCTION

The purpose of this investigation is to exhibit some of the fundamental properties of $G_n^{(k)}(x)$, the *generalized Genocchi polynomials of order k*.

When $k = 1$, we have the *Genocchi polynomials* $G_n^{(1)}(x) \equiv G_n(x)$.

If $x = 0$, we obtain the *generalized Genocchi numbers of order k*, $G_n^{(k)}(o) \equiv G_n^{(k)}$. When $k = 1$, the *Genocchi numbers* $G_n(o) \equiv G_n$ result.

Information on generalized Bernoulli numbers $B_n^{(k)}(x)$ and generalized Euler polynomials $E_n^{(k)}(x)$, to which $G_n^{(k)}(x)$ may be related, is to be found in [3], [4], [11], [13], [15], and [16]. Nörlund in particular has extensively investigated these polynomials. Several references to Genocchi's work occur in [10]. Also see, for example, Genocchi [5].

While many of the properties of Genocchi polynomials bear a close resemblance to the corresponding properties of Bernoulli and Euler polynomials, some properties are rather different. Indeed, Genocchi polynomials are worthy of an investigation *per se*. To the best of my knowledge, the material on Genocchi polynomials presented here has not previously appeared. Some of the data relating to Genocchi numbers may also be fresh.

Genocchi Numbers

Genocchi numbers may be defined - cf. [2] - by

$$\sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^t+1} = t\left(1 - \tanh \frac{t}{2}\right) \tag{1.1}$$

G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers Volume 4, 145–166.

so that

$$G_0 = 0. \tag{1.1)'$$

Since $\tanh \frac{t}{2}$ is an odd function, $t \tanh \frac{t}{2}$ is even in t. Hence we deduce that

$$\left.\begin{array}{ll} G_1 = 1 & \\ G_{2n+1} = 0 & (n = 1, 2, 3, \ldots) \end{array}\right\} \tag{1.2}$$

The first few Genocchi numbers are [2]:

n	0	1	2	4	6	8	10	12	14	16	18	20
G_n	0	1	−1	1	−3	17	−155	2073	−38227	929567	−28820619	1109652905

(1.3)

Designating Bernoulli numbers by B_n ($\equiv B_n^{(1)}(0)$) we know [2], [10] that

Genocchi's Theorem:

$$G_{2m} = 2(1 - 2^{2m})\, B_{2m} \tag{1.4}$$

Note that Euler's numbers E_n are given [2] by $E_n = 2^n E_n(\frac{1}{2})$ and not by $E_n(0)$ ($\equiv E_n^{(1)}(0)$). In fact,

$$G_{2m} = 2m\, E_{2m\text{-}1}(0) \tag{1.5}$$

A recent appearance of Genocchi numbers occurs in [6] where the author examines integers related to the Bessel function $J_1(z)$, and where Genocchi numbers are part of the value of $\sigma_{2n}(-\frac{1}{2})$, $\sigma_{2n}(v)$ being the Rayleigh function.

2. GENERALIZED GENOCCHI POLYNOMIALS

Definition and Basic Properties

Generalized Genocchi polynomials of order k, $G_n^{(k)}(x)$, are defined by

$$\sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{t^n}{n!} = \left(\frac{2t}{e^t + 1}\right)^k e^{tx} \quad (k = 0, 1, 2, 3, \ldots) \tag{2.1}$$

whence

$$G_n^{(k)}(x) = 0 \qquad (n < k) \tag{2.1)'$$

and

$$G_n^{(o)}(x) = x^n \tag{2.2}$$

The first few generalized Genocchi polynomials are:

$$
\begin{aligned}
G_k^{(k)}(x) &= k! \\
G_{k+1}^{(k)}(x) &= (k+1)!\left\{x-\frac{k}{2}\right\} \\
G_{k+2}^{(k)}(x) &= \frac{(k+2)!}{2!}\left\{x^2-kx+\frac{k(k-1)}{4}\right\} \\
G_{k+3}^{(k)}(x) &= \frac{(k+3)!}{3!}\left\{x^3-\frac{3k}{2}x^2+\frac{3k(k-1)}{4}x-\frac{k^2(k-3)}{8}\right\} \\
G_{k+4}^{(k)}(x) &= \frac{(k+4)!}{4!}\left\{x^4-\frac{4k}{2}x^3+\frac{6k(k-1)}{4}x^2-\frac{4k^2(k-3)}{8}x\right. \\
&\qquad \left. +\frac{k(k-1)(k^2-5k-2)}{16}\right\} \\
G_{k+5}^{(k)}(x) &= \frac{(k+5)!}{5!}\left\{x^5-\frac{5k}{2}x^4+\frac{10k(k-1)}{4}x^3-\frac{10k^2(k-3)}{8}x^2\right. \\
&\qquad \left. +\frac{5k(k-1)(k^2-5k-2)}{16}x-k^2\frac{k^3-10k^2+15k+10}{32}\right\}
\end{aligned}
\tag{2.3}
$$

In particular, when $k = 1$, so that we omit the superscript,

$$
\begin{aligned}
G_0(x) &= 0 & & \\
G_1(x) &= 1 & & \\
G_2(x) &= 2x-1 & &= 2(x-\tfrac{1}{2}) \\
G_3(x) &= 3x^2-3x & &= 3x(x-1) \\
G_4(x) &= 4x^3-6x^2+1 & &= 4(x-\tfrac{1}{2})(x^2-x-\tfrac{1}{2}) \\
G_5(x) &= 5x^4-10x^3+5x & &= 5x(x-1)(x^2-x-1) \\
G_6(x) &= 6x^5-15x^4+15x^2-3 & &= 6(x-\tfrac{1}{2})(x^4-2x^3-x^2+2x+1) \\
G_7(x) &= 7x^6-21x^5+35x^3-21x & &= 7x(x-1)(x^4-2x^3-2x^2+3x+3)
\end{aligned}
\tag{2.4}
$$

Putting $x = 0$ in (2.4) we obtain the simplest values (1.2) for G_n. It is worthwhile writing out the values of some Genocchi numbers of order k by putting $x = 0$ in (2.3).

Comparing (2.1) with the definition of $E_n^{(k)}(x)$ in, say, [13] we readily see that

$$G_n^{(k)}(x) = n(n-1)\ldots(n-k+1)\, E_{n-k}^{(k)}(x) \quad n \geq k \geq 0 \tag{2.5}$$

Because of this simple relationship, many results for $G_n^{(k)}(x)$ will parallel those of $E_n^{(k)}(x)$, and, to a lesser extent, those of $B_n^{(k)}(x)$. However, some results are unique to the theory of Genocchi polynomials.

Standard techniques, which are sometimes suppressed in this presentation to conserve space, allow us to produce a variety of results for $G_n^{(k)}(x)$. It is useful to put $k = 1$ in each case to see what happens for the Genocchi polynomials $G_n(x)$. Some theorems valid only when $k = 1$, i.e., for which no generalization exists, are considered in a separate section of this paper.

Differentiating both sides of (2.1) w.r.t. x and simplifying gives the "Appell property"

$$\frac{d}{dx} G_n^{(k)}(x) = n\, G_{n-1}^{(k)}(x) \quad (n-1 \geq 0) \tag{2.6}$$

whence

$$\frac{d^p}{dx^p} G_n^{(k)}(x) = n(n-1)(n-p+1)\, G_{n-p}^{(k)}(x) \quad (n-p \geq 0) \tag{2.7}$$

so that

$$\frac{d^n}{dx^n} G_n^{(k)}(x) = n! \tag{2.8}$$

Integration of (2.6) yields $G_n^{(k)}$ as the constant of integration. Polynomials A_n (x) are called *Appell polynomials* [4] if they have the property

$$\frac{d\, A_n(x)}{dx} = n\, A_{n-1}(x) \tag{2.9}$$

Thus, Bernoulli, Euler and Genocchi polynomials are Appell polynomials.

If in (2.1) we write

$$\Phi = \left(\frac{2t}{e^t + 1}\right)^k e^{tx} \tag{2.10}$$

then

$$\frac{\partial \Phi}{\partial x} = t\Phi \tag{2.11}$$

(which may be used to establish (2.6)) and

$$t\,\frac{\partial \Phi}{\partial t} - \left\{\frac{k+tx}{t} - \frac{ke^t}{e^t+1}\right\}\frac{\partial \Phi}{\partial x} = 0 \tag{2.12}$$

From (2.6), we have, on anticipating (2.22),

$$\int_0^1 G_n^{(k)}(x)\, dx = \frac{G_{n+1}^{(k)}(1) - G_{n+1}^{(k)}}{n+1}$$

$$= \begin{cases} 0 & \text{if n is even} \\ -\dfrac{2\, G_{n+1}^{(k)}}{n+1} & \text{if n is odd} \end{cases} \quad \text{by (1.2)} \tag{2.13}$$

More generally,

$$\int_{x}^{x+1} G_n^{(k)}(x)\,dx = \frac{G_{n+1}^{(k)}(1+x) - G_{n+1}^{(k)}(x)}{n+1} \tag{2.14}$$

Also from (2.6)

$$G_n^{(k)}(x) - G_n^{(k)} = n \int_0^x G_{n-1}^{(k)}(x)\,dx \tag{2.15}$$

If we know the $G_n^{(k)}$, then (2.15) used in conjunction with (2.3) provides us with a means of successively calculating the $G_n^{(k)}(x)$, though this becomes complicated when n is large.

Substitution of $k = 1$ in (2.13) – (2.15) yields important special cases.

Recurrence Relations

Using Cauchy's rule for the multiplication of power series, we derive from (2.1) the *summation formula*

$$G_n^{(k)}(x+y) = \sum_{j=0}^{n} \binom{n}{j} G_j^{(k)}(x)\, y^{n-j} \tag{2.16}$$

Special cases of (2.16) include, when $x = 0$ and y is replaced by x, the *explicit form*

$$G_n^{(k)}(x) = \sum_{j=0}^{n} \binom{n}{j} G_j^{(k)} x^{n-j} \tag{2.17}$$

and, when $y = k$, the *difference, or recurrence, relation*

$$G_n^{(k)}(k+x) - G_n^{(k)}(x) = \sum_{j=0}^{n-1} \binom{n}{j} G_j^{(k)}(x)\, k^{n-j} \tag{2.18}$$

In (2.18), $k+x$ may be replaced by $1+x$ without loss of generality, in which case the factor k^{n-j} becomes 1, so that

$$G_n^{(k)}(1+x) - G_n^{(k)}(x) = \sum_{j=0}^{n-1} \binom{n}{j} G_j^{(k)}(x) \tag{2.19}$$

that is

$$G_n^{(k)}(1+x) = \sum_{j=0}^{n} \binom{n}{j} G_j^{(k)}(x) \tag{2.19$'$}$$

Equivalence of (2.14) and (2.19) gives

$$\int_{x}^{x+1} G_{n-1}^{(k)}(x)\,dx = \frac{1}{n} \sum_{j=0}^{n-1} \binom{n}{j} G_j^{(k)}(x) \tag{2.14$'$}$$

From (2.17) or (2.19)

$$G_n^{(k)}(1) = \sum_{j=0}^{n} \binom{n}{j} G_j^{(k)} \tag{2.17$'$}$$

We now prove

Theorem 1: $G_n^{(k)}(1+x) + G_n^{(k)}(x) = 2n\, G_{n-1}^{(k-1)}(x) \quad (n \geq 1)$ (2.20)

Proof:
$$\sum_{n=1}^{\infty} \left[G_n^{(k)}(1+x) + G_n^{(k)}(x) \right] \frac{t^n}{n!}$$
$$= \left(\frac{2t}{e^t+1}\right)^k e^{t(1+x)} + \left(\frac{2t}{e^t+1}\right)^k e^{tx} \quad \text{by (2.1)}$$
$$= 2t \left(\frac{2t}{e^t+1}\right)^{k-1} e^{tx}$$
$$= 2n \sum_{n=1}^{\infty} G_{n-1}^{(k-1)}(x) \frac{t^n}{n!} \quad \text{by (2.1)}$$

whence the result follows on equating coefficients of $\frac{t^n}{n!}$.

Corollary (k = 1): $G_n(1+x) + G_n(x) = 2n\, x^{n-1} \quad (n \geq 1)$ (2.21)

by (2.20) and (2.2), whence

$$G_n(1) = -G_n \quad (n>1) \tag{2.22}$$

Taking (2.17)′, with k = 1, in conjunction with (2.22) we deduce the *recurrence relation* for Genocchi numbers

$$G_n = -\tfrac{1}{2} \sum_{j=0}^{n-1} \binom{n}{j} G_j \quad (n > 1) \tag{2.23}$$

Next, from (2.16), on simplification and use of (1.2), we observe that

$$G_n(x) + (-1)^{n-1} G_n(-x) = 2n\, x^{n-1} \tag{2.24}$$

and

$$G_n(x) + (-1)^n G_n(-x) = 2 \sum_{i=1}^{\left[\frac{N}{2}\right]} \binom{n}{2i} G_{2i}\, x^{n-2i} \tag{2.25}$$

where N = n or n − 1 according as n is even or odd.

For n even in (2.24), we deduce that

$$G_n(x) - G_n(-x) = 2n\, x^{n-1} \tag{2.24$'$}$$

whence

$$G_n(1) - G_n(-1) = 2n \tag{2.24$''$}$$

Combining (2.21) and (2.24)′ by subtraction yields

$$G_n(1+x) + G_n(-x) = 0 \tag{2.26}$$

In evaluating the left-hand side of (2.24), the determining factor is $1-(-1)^j$, while in the case of (2.25) it is $1+(-1)^j$.

Finally, if we replace x by $x-1$ in (2.21), or use (2.1), it follows that

$$G_n(x)+G_n(x-1) = 2n(x-1)^{n-1} \tag{2.27}$$

(Cf. Nielsen [14] for $E_n(x)$.)

The Operators Δ, ∇

The symbols Δ and ∇ ("nabla") represent the operations of obtaining the *difference*, and taking the *mean*, of $f(1+x)$ and $f(x)$, thus:

$$\Delta f(x) = f(1+x)-f(x) \tag{2.28}$$

and

$$\nabla f(x) = \frac{f(1+x)+f(x)}{2} \tag{2.29}$$

Hence, by (2.21),

$$\nabla G_n(x) = nx^{n-1} \tag*{(2.29)$'$}$$

(so $\nabla G_n(0)=0$ as in (2.22), $\nabla G_n(1)=n$, $\nabla G_n(-1)=(-1)^{n-1}n$) while from (2.19)

$$\Delta G_n(x) = \sum_{j=0}^{n-1}\binom{n}{j} G_j(x) \tag*{(2.19)$''$}$$

More generally, from (2.20)

$$\nabla G_n^{(k)}(x) = n\, G_{n-1}^{(k-1)}(x) \qquad (n \geq 1) \tag*{(2.20)$'$}$$

i.e., the operator ∇ depresses the order k by one and the degree n by one.

Repeated application of ∇ on $G_n^{(k)}(x)$ $n-k$ times leads by (2.2) to

$$(\nabla)^{n-k} G_n^{(k)}(x) = n(n-1)\ldots(n-k+1)\, x^{n-k} = \frac{n!}{(n-k)!} x^{n-k} \tag{2.30}$$

Finally,

$$\sum_{n=k}^{\infty} \frac{t^n}{n!}(\nabla)^{n-k} G_n^{(k)}(x) = t^k e^{tx} \tag{2.31}$$

Use of Δ and ∇ will recur subsequently in this article.

Complementary Argument

The functions x and $k-x$ are called complementary arguments for the Genocchi polynomials. We now prove the "complementary argument theorem", which has a symmetry property.

Theorem 2: $G_n^{(k)}(k-x) = (-1)^{k+n} G_n^{(k)}(x)$ (2.32)

Proof: $$\sum_{n=0}^{\infty} G_n^{(k)}(k-x)\frac{t^n}{n!} = \left(\frac{2t}{1+e^t}\right)^k e^{t(k-x)} \quad \text{by (2.1)}$$

$$= (-1)^k \left(\frac{2(-t)}{e^{-t}+1}\right)^k e^{-tx}$$

$$= (-1)^k \sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{(-t)^n}{n!} \quad \text{by (2.1)}$$

$$= \sum_{n=0}^{\infty} (-1)^{k+n} G_n^k(x)\frac{t^n}{n!}$$

Equating coefficients of $\frac{t^n}{n!}$ we deduce the theorem.

Corollary 1: Putting $k = 1$ in (2.32) we derive

$$\left.\begin{array}{ll} G_n(1-x) = G_n(x) & \text{if n is odd} \\ G_n(1-x) = -G_n(x) & \text{if n is even} \end{array}\right\} \tag{2.33}$$

Corollary 2: Replace x by $x + \frac{k}{2}$ in (2.32). It follows that

$$\left.\begin{array}{ll} G_n^{(k)}\left(\frac{k}{2}+x\right) = G_n^{(k)}\left(\frac{k}{2}-x\right) & \text{if } n+k \text{ is even} \\ \qquad = -G_n^{(k)}\left(\frac{k}{2}-x\right) & \text{if } n+k \text{ is odd} \end{array}\right\} \tag{2.34}$$

so that

$$\left.\begin{array}{ll} G_n\left(\frac{1}{2}+x\right) = G_n\left(\frac{1}{2}-x\right) & \text{n odd} \\ \qquad = -G_n\left(\frac{1}{2}-x\right) & \text{n even} \end{array}\right\} \tag{2.35}$$

whence, with $x = \frac{1}{2}$, the results in (2.22), and (2.40) and (2.41) follow.

Two Summation Formulas

Theorem 3: $\sum_{s=1}^{m} (-1)^s s^n = \frac{(-1)^m G_{n+1}(1+m) - G_{n+1}(1)}{2(n+1)}$ (2.36)

Proof: $$\frac{1}{2}\frac{\left[(-1)^m G_{n+1}(1+m) - G_{n+1}(1)\right]}{n+1}$$

$$= \frac{1}{2}\sum_{s=1}^{m} (-1)^s \frac{\left[G_{n+1}(1+s) + G_{n+1}(s)\right]}{n+1}$$

$$= \sum_{s=1}^{m} (-1)^s \frac{\nabla G_{n+1}(s)}{n+1} \quad \text{by (2.29)}$$

$$= \sum_{s=1}^{m} (-1)^s s^n \quad \text{by (2.20)}', (2.2)$$

Miline-Thomson [13] remarks that this method can clearly be applied if the sth term in a finite series is a polynomial in s. In fact, by (2.20), we may generalize the result to read

$$\sum_{s=1}^{m} (-1)^s G_n^{(k-1)}(x) = \frac{(-1)^m G_{n+1}^{(k)}(1+m) - G_{n+1}^{(k)}(1)}{2(n+1)} \tag{2.36$'$}$$

<u>**Theorem 4:**</u> $G_n^{(k)}(x) = \sum_{r=0}^{n} \binom{n}{r} G_r^{(k)}\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)^{n-r}$ (2.37)

<u>**Proof:**</u> $\left(\frac{2t}{e^t+1}\right)^k e^{tx} = \left(\frac{2t}{e^t+1}\right)^k e^{\frac{t}{2}} e^{(x-\frac{1}{2})t}$

i.e., $\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!} = \sum_{r=0}^{\infty} G_r^{(k)}\left(\frac{1}{2}\right)\frac{t^r}{r!}\sum_{m=0}^{\infty}\left(x-\frac{1}{2}\right)^m \frac{t^m}{m!}$ by (2.1)

Cauchy's rule for multiplication of power series produces the result.

<u>Some Numerical Results</u>

Let $x = 0$, $n = 2m$ in (2.32). Then

$$G_{2m}^{(k)}(k) = (-1)^k G_{2m}^{(k)} \tag{2.38}$$

while

$$G_{2m+1}^{(k)}(k) = (-1)^{k+1} G_{2m+1}^{(k)} \tag{2.38$'$}$$

Furthermore, $x = \frac{k}{2}$, $n = 2m$ in (2.32) yield

$$G_{2m}^{(k)}\left(\frac{1}{2}k\right) = (-1)^k G_{2m}^{(k)}\left(\frac{1}{2}k\right) \tag{2.39}$$

Now, let $k = 1$. Results (2.33) – (2.35) then lead to the following numerical values for $n > 1$.

<u>n even:</u>

$$\left.\begin{array}{l} G_n\left(\frac{1}{2}\right) = 0 \\ G_n(1) = -G_n \\ G_n\left(\frac{1}{4}\right) = G_n\left(\frac{3}{4}\right),\ G_n\left(\frac{1}{3}\right) = -G_n\left(\frac{2}{3}\right) \end{array}\right\} \tag{2.40}$$

<u>n odd:</u>

$$\left.\begin{array}{l} G_n(1) = G_n = 0 \\ G_n\left(\frac{1}{4}\right) = G_n\left(\frac{3}{4}\right),\ G_n\left(\frac{1}{3}\right) = G_n\left(\frac{2}{3}\right) \end{array}\right\} \tag{2.41}$$

From the above, we conclude that $G_{2n}(x)$ has a factor $x-\frac{1}{2}$ whereas $G_{2n+1}(x)$ has a factor $x(x-1)$. See (2.4). Next,

$$\begin{aligned} \int_0^{\frac{1}{2}} G_n(x)\,dx &= \frac{G_{n+1}\left(\frac{1}{2}\right) - G_{n+1}}{n+1} && \text{from (2.6)} \\ &= -\frac{G_{n+1}}{n+1} && \text{n even by (2.40)} \\ &= \frac{G_{n+1}\left(\frac{1}{2}\right)}{n+1} && \text{n odd by (1.2)} \end{aligned} \tag{2.42}$$

Relationships of use in computation include

$$G_n\left(\frac{1}{2}\right) = \frac{n}{2^{n-1}} E_{n-1} = n\, E_{n-1}\left(\frac{1}{2}\right) \tag{2.43}$$

and

$$G_{n+1} = \frac{n+1}{2^n} C_n = (n+1)\, E_n(0) = (-1)^n (n+1)\, E_n(1) \tag{2.44}$$

in which $C_n = 2^n E_n(0)$ are the *voisins* ("cousins") of E_n in Nörlund's phraseology.

Applying these, we may substitute appropriately in various formulas in [3], [11] and [16] to obtain, for example,

$$\sum_{m=0}^{n} \binom{2n}{2m} \frac{2^m}{2m+1} G_{2m+1}\left(\frac{1}{2}\right) = 0 \tag{2.45}$$

$$G_{n+1}^{(k)}(x) = (n+1) \sum_{m=k-1}^{n} \binom{n}{m} \frac{G_{m+1}^{(k)}}{m+1} x^{n-m} \tag{2.46}$$

$$\sum_{m=0}^{n} \binom{n}{m} \frac{G_{m+1}}{m+1} + \frac{G_{n+1}}{n+1} = 0 \quad \text{by (2.40), (2.46)} \tag{2.46$'$}$$

$$G_{2m} = 2^{4m} \frac{2^{2m}-1}{2^{2m-1}-1} B_{2m}\left(\frac{1}{4}\right). \tag{2.47}$$

Integral formulas adaptable form [16] include

$$G_{n+1}(2x) = 2^{n+1}(n+1) \int_x^{x+\frac{1}{2}} B_n(x)\,dx \tag{2.48}$$

whence

$$G_{n+1} = 2^{n+1}(n+1) \int_0^{\frac{1}{2}} B_n\,dx \quad (x=0 \text{ in } (2.48)) \tag{2.48$'$}$$

and

$$G_{n+1}\left(\frac{1}{2}\right) = 2^{n+1}(n+1) \int_{\frac{1}{4}}^{\frac{3}{4}} B_n\left(\frac{1}{4}\right) dx \quad (x=\tfrac{1}{4} \text{ in } (2.48)) \tag{2.48$''$}$$

Relationships between G_n and E_n are, e.g.,

$$E_{n-1} = \frac{1}{2n} \sum_{j=0}^{n} \binom{n}{j} 2^j G_j \quad \text{by (2.16), (2.43)} \tag{2.49}$$

and

$$G_n = \frac{2n}{2^n} \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} E_{n-1-j} \quad \text{by [16], (2.44)} \tag{2.50}$$

while another connection between G_n and $E_n(0)$ is

$$G_{n+1} = -(n+1) \sum_{s=0}^{n} \binom{n}{s} E_{n-s}(0). \tag{2.51}$$

Graphs of $G_n(x)$ in the Interval (0,1)

As we have seen in (2.40) and (2.41), $G_{2n}(x)$ has a zero at $x = \frac{1}{2}$ whereas $G_{2n+1}(x)$ has zeros at 0,1. Following a routine argument for $B_n(x)$ (cf. [7], [13]), we may establish that these are the only zeros of $G_n(x)$ in (0,1).

Because of the connection (2.5) between $G_n(x)$ and $E_n(x)$, the graphs for $G_n(x)$ will be similar to those for $E_n(x)$ and are therefore not reproduced here. For the graphs of $-E_1(x)$, $-E_2(x)$, $(-1)^n E_{2n-1}(x)$ and $(-1)^n E_{2n}(x)$, we refer the reader to [16]. One may compare these graphs with those of $B_n(x)$ given in [7].

Recurrence Relation between Polynomials of Successive Orders

Theorem 5: $$G_n^{(k+1)}(x) = \frac{2n(k-x)}{k} G_{n-1}^{(k)}(x) + \frac{2}{k}(n-k) G_n^{(k)}(x) \tag{2.52}$$

Proof: Differentiate both sides of (2.1) w.r.t. t, and then multiply by t. We have

$$\sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{nt^{n-1}}{n!} \cdot t$$

$$= \left(\frac{2t}{e^t+1}\right)^k xe^{tx} \cdot t + kt^{k-1} \left(\frac{2}{e^t+1}\right)^k e^{tx} \cdot t - \frac{k}{2}\left(\frac{2}{e^t+1}\right)^{k+1} \cdot t^k e^{t(1+x)} \cdot t$$

$$= x \sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{t^{n+1}}{n!} + k \sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{t^n}{n!} - \frac{k}{2} \sum_{n=0}^{\infty} G_n^{(k+1)}(1+x) \frac{t^n}{n!}$$

Equate coefficients of t^{n+1}. Then

$$G_{n+1}^{(k)}(x) = x\, G_n^{(k)}(x) + k \frac{G_{n+1}^{(k)}}{n+1}(x) - \frac{k}{2} \frac{G_{n+1}^{(k+1)}(1+x)}{n+1} \quad \ldots\ldots (\alpha)$$

Now, by (2.20),

$$G_{n+1}^{(k+1)}(1+x) = 2(n+1) G_n^{(k)}(x) - G_{n+1}^{(k+1)}(x) \quad \ldots\ldots (\beta)$$

Hence, by (α) and (β),

$$G_{n+1}^{(k)}(x) = x\,G_n^{(k)}(x) + \frac{k}{n+1}\left\{G_{n+1}^{(k)}(x) - \tfrac{1}{k}\left(2(n+1)\,G_n^{(k)}(x) - G_{(n+1)}^{(k+1)}(x)\right)\right\}$$

On tidying up these expressions, we obtain

$$G_{n+1}^{(k+1)}(x) = \frac{2(n+1)}{k}(k-x)\,G_n^{(k)}(x) + \frac{2}{k}\left\{n+1-k\right\}G_{n+1}^{(k)}(x) \;\ldots\ldots\;(\gamma)$$

Replacing n + 1 by n, we get

$$G_n^{(k+1)}(x) = \frac{2n(k-x)}{k}\,G_{n-1}^{(k)}(x) + \frac{2}{k}(n-k)\,G_n^{(k)}(x) \qquad \ldots\ldots\;(\delta)$$

This corresponds to the recurrence relations for k^{th} order Euler polynomials and Bernoulli polynomials which are given in [11], [13] and [16]. Observe the existence of an extra term on the right-hand side in the Genocchi case. This is due to the necessity to differentiate the extra factor t^k in (2.1) which is absent in the Euler and Bernoulli cases.

When $x = 0$ in (δ), we have

$$G_n^{(k+1)} = 2n\,G_{n-1}^{(k)} + \frac{2}{k}(n-k)\,G_n^{(k)} \qquad (2.52)'$$

Connection with Bernoulli Polynomials

Theorem 6: $G_n^{(k)}(x) = 2^n \sum_{r=0}^{k} (-1)^{r+k}\binom{k}{r} B_n^{(k)}\left(\frac{x+r}{2}\right)$ (2.53)

Proof: $\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!} = 2^k\left(\dfrac{\frac{t}{2}}{e^{\frac{t}{2}}+1}\right)^k e^{\frac{t}{2}x}$ by (2.1)

$$= \frac{t^k\,e^{\frac{t}{2}x}\,(e^{\frac{t}{2}}-1)^k}{(e^t-1)^k} \qquad (A)$$

$$= (-1)^k \sum_{r=0}^{k} \frac{(-1)^r\binom{k}{r} e^{\frac{(x+r)t}{2}}\,t^k}{(e^t-1)^k}$$

$$= 2^n \sum_{n=0}^{\infty}\sum_{r=0}^{k} (-1)^{r+k}\binom{k}{r} B_n^{(k)}\left(\frac{x+r}{2}\right)\frac{t^n}{n!}$$

on using the definition of $B_n^{(k)}(x)$ [16]. Equating the coefficients of $\frac{t^n}{n!}$ in the summation, we obtain the formula (2.53).

When $k = 1$, we may proceed quickly to the formula from (A). Alternatively, substitute $k = 1$ in the formula. In either case, we have

$$G_n(x) = 2^n\left[B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right)\right] \qquad (2.53)'$$

There does not appear to be a corresponding result in the literature for $E_n^{(k)}(x)$, though the case $k = 1$ is given in Erdelyi [3].

Result (2.53)$'$ also follows from (2.5) and [4, p.41].

Equivalently, (2.53)$'$ may be expresses as

$$G_n(x) = 2\left[2^n B_n\left(\frac{x+1}{2}\right) - B_n(x)\right] \tag{2.53$''$}$$

Lastly, we mention that the method of [12] for reciprocals of the k^{th} power of *Pell-Lucas numbers* Q_n allows us to determine that

$$\frac{1}{Q_n^k} = \frac{1}{2^k m^k (\delta^x \gamma^{1-x})^n} \sum_{r=0}^{\infty} G_r^{(k)}(x) \log\left(\frac{\delta}{\gamma}\right)^r \cdot \frac{n^r}{r!} \tag{2.54}$$

where $m = n \log(\delta/\gamma)$ and $\gamma = 1+\sqrt{2}$, $\delta = 1-\sqrt{2}$. Specialization involving Genocchi numbers occurs in (2.54) when $k = 1$, $x = 0$.

3. GENOCCHI POLYNOMIALS OF THE FIRST ORDER

While many results for Genocchi polynomials $G_n^{(1)}(x)$ can be generalized, some cannot. An important instance of this is the *multiplication theorem* for $G_n(mx)$ for which we must consider two cases, m even and n odd. In the case m even, Bernoulli polynomials occur.

Multiplication Theorem

Theorem 7 (Multiplication Theorem):

$$\left.\begin{aligned} G_n(mx) &= m^{n-1} \sum_{s=0}^{m-1} (-1)^s G_n\left(x+\frac{s}{m}\right) && \text{m odd} \\ G_n(mx) &= -2\, m^{n-1} \sum_{s=0}^{m-1} (-1)^s B_n\left(x+\frac{s}{m}\right) && \text{m even} \end{aligned}\right\} \tag{3.1}$$

Proof: m odd

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{s=0}^{m-1} (-1)^s G_n\left(x+\frac{s}{m}\right) = \sum_{s=0}^{m-1} \frac{2t}{e^t+1} \cdot (-1)^s e^{tx} e^{\frac{s}{m}t} \qquad \text{by (2.1)}$$

$$= \frac{2t}{e^t+1} \cdot e^{tx}\left\{1 - e^{\frac{t}{m}} + e^{\frac{2t}{m}} - \ldots + (-1)^{m-1} e^{\frac{m-1}{m}t}\right\}$$

$$= \frac{2t}{e^t+1} e^{tx} \frac{\{1-(-1)^m e^t\}}{1+e^{\frac{t}{m}}} \qquad \text{by the geometric progression formula}$$

$$= \frac{2t\, e^{tx}}{1+e^{\frac{t}{m}}} \qquad \text{since m is odd}$$

$$= \frac{2m \cdot \frac{t}{m} e^{mx\frac{t}{m}}}{1+e^{\frac{t}{m}}}$$

$$= \sum_{n=0}^{\infty} 2m \frac{t^n}{m^n \cdot n!} G_n(mx) \qquad \text{by (2.1)}$$

Hence, $\quad G_n(mx) = m^{n-1} \sum_{s=0}^{m-1} (-1)^s G_n\left(x+\frac{s}{m}\right)$

m even

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{s=0}^{m-1} (-1)^s B_n\left(x+\frac{s}{m}\right) = \sum_{s=0}^{m-1} \frac{t}{e^t-1} (-1)^s e^{\left(x+\frac{s}{m}\right)t} = \frac{t\, e^{tx}}{e^t-1} \cdot \frac{1-(-1)^m e^t}{1+e^{\frac{t}{m}}}$$

as in the case m odd

$$= -\frac{t\, e^{tx}}{1+e^{\frac{t}{m}}} \qquad \text{for m even}$$

$$= -\frac{2m}{2} \cdot \frac{t}{m} \frac{e^{mx \cdot \frac{t}{m}}}{1+e^{\frac{t}{m}}}$$

$$= -\sum_{n=0}^{\infty} \frac{1}{2} m \frac{t^n}{m^n \cdot n!} G_n(mx) \qquad \text{by (2.1)}$$

Hence, $\quad G_n(mx) = -2m^{n-1} \sum_{s=0}^{m-1} (-1)^s B_n\left(x+\frac{s}{m}\right) \qquad$ m even

For example, with $m = 3$, $x = 0$,

$$G_n = 3^{n-1}\left(G_n - G_n\left(\tfrac{1}{3}\right) + G_n\left(\tfrac{2}{3}\right)\right)$$

Our multiplication theorem corresponds to *Raabe's theorem* for multiplication for Bernoulli polynomials. Setting $m = 2$ in (3.1), we revisit (2.53)$'$ for $G_n(2x)$.

Orthogonality - Type Relation

Theorem 8: $\displaystyle \int_0^1 G_u(x)\, G_v(x)\, dx = \begin{cases} 2(-1)^u \dfrac{u!\, v!}{(u+v)!} G_{u+v} & \text{if } u+v \text{ is even} \\ 0 & \text{.......... odd} \end{cases}$ (3.2)

Proof:

$$\int_0^1 G_u(x)\, G_v(x)\, dx = \frac{1}{v+1}\left[G_u(x)\, G_{v+1}(x)\right]_0^1 - \frac{1}{v+1}\int_0^1 G_{v+1}(x) \cdot u\, G_{u-1}\, dx \qquad \text{by (2.6)}$$

$$= -\frac{u}{v+1}\int_0^1 G_{u-1}(x)\, G_{v+1}(x)\, dx \qquad \text{by (2.40), (2.41)} \quad \text{.... (i)}$$

$$= (-1)^{u-1} \frac{u!\, v!}{(u+v)!}\left[G_{u+v}(x)\right]_0^1 \qquad \text{by repeated application of (i)}$$

$$= (-1)^{u-1} \frac{u!\,v!}{(u+v)!}\left(G_{u+v}(1) - G_{u+v}\right)$$

$$\left.\begin{aligned} &= 2(-1)^{u} \frac{u!\,v!}{(u+v)!} G_{u+v} \quad \text{if } u+v \text{ is even} \\ &= 0 \qquad\qquad\qquad\qquad \ldots\ldots\ldots \text{odd} \end{aligned}\right\} \quad \text{by (2.40), (2.41)}$$

Some interesting special cases arise when $u = v = 2w$; $u = v = 2w + 1$; $u = 2$, $v = 2w$; $u = 1$, $v = 2w - 1$ (the last one leading to the value $-\frac{G_{2w}}{w}$ for the integral). See Jordan [7] for corresponding results for the Euler polynomials.

Interchanging u and v in (3.2) we have $(-1)^u\, G_{u+v} = (-1)^v\, G_{u+v}$. Next, if $u = v + 1$ we then have $G_{2u+1} = -G_{2u+1}$, that is $G_{2u+1} = 0$ as we know (1.2).

Theorem 8 may be generalized for $G_u^{(k)}(x)$ and $G_v^{(k)}(x)$, the limits of integration then being k and 0). Recall - cf. (2.38), (2.38)′ – that generally $G_{u+v}^{(k)} \neq 0$ when $u+v$ is of odd parity.

Fourier Expansions in the Interval (0,1)

Two cases for the Fourier development of Genocchi polynomials over the interval (0,1) need to be considered, namely, n even and n odd.

Following standard methods ([4], [7]), we eventually obtain

$$G_{2n}(x) = (-1)^n\, 4\,(2n)! \sum_{m=0}^{\infty} \frac{\cos(2m+1)\pi x}{[(2m+1)\pi]^{2n}} \tag{3.3}$$

and

$$G_{2n+1}(x) = (-1)^n\, 4\,(2n+1)! \sum_{m=0}^{\infty} \frac{\sin(2m+1)\pi x}{[(2m+1)\pi]^{2n+1}} \tag{3.4}$$

Fourier expansions (3.3) and (3.4) are in agreement with the corresponding Fourier developments for Euler polynomials given in, say, [9], [11] and [16], which Jordan [7] attributes to Lindelöf [9]. See also Jordan [7] for Fourier expansions for $E_n(x)$ different from those in the sources quoted.

Using (1.4) and [8, p. 238] with $x = 0$, we deduce that

$$\frac{(-1)^m\, \pi^{2m}\, G_{2m}}{4(2m)!} = 1 + \frac{1}{3^{2m}} + \frac{1}{5^{2m}} + \frac{1}{7^{2m}} + \cdots \tag{3.5}$$

(giving known series for $\frac{\pi^2}{8}$, $\frac{\pi^4}{96}$, and $\frac{\pi^6}{980}$ when $m = 1, 2, 3$, respectively).

Furthermore, from (1.4) and the "remarkable relation" [4, p. 37], we derive

$$\frac{(-1)^{m-1}(2\pi)^{2m}G_{2m}}{4\,(2m)!\,(1-2^{2m})} = \zeta\,(2m) \tag{3.6}$$

where ζ (2m) is the *Riemann ζ - function* $\sum_{n=1}^{\infty} \frac{1}{n^{2m}}$. With m = 1, 2, 3 in turn there appear the known infinite series summations for $\frac{\pi^2}{6}$, $\frac{\pi^4}{90}$, and $\frac{\pi^6}{945}$ respectively.

Use of Boole's Theorem for Polynomials

A polynomial P(t) may be expanded in terms of Euler polynomials by Boole's Theorem thus ([13], see [1] also):

Boole's Theorem: $P(x+y) = \nabla P(x) + E_1(y)\ \nabla P'(x) + \frac{1}{2!} E_2(y)\ \nabla P''(x) + \ldots\ldots$ (3.7)

(a) Genocchi Polynomials in terms of Euler's E_n (0)

Put y = 0 in (3.7) and take $P(x) \equiv G_n(x)$.

Then Boole's expansion for $G_n(x)$ is

$$G_n(x) = \nabla G_n(x) + E_1(0)\ \nabla G_n'(x) + \frac{1}{2!} E_2(0)\ \nabla G_n''(x) + \ldots \tag{3.8}$$

The series on the right-hand side terminates after a finite number of terms.

Using (2.29)′, (1.3) and (1.5) we verify, for example, that $G_4(x) = 4x^3 - 6x^2 + 1$, as we know from (2.4).

(b) Bernoulli Polynomials in terms of Genocchi Polynomials

In (3.7), put x = 0 and replace y by x and P by B_n. Then

$$\left.\begin{aligned} B_n(x) &= \nabla B_n(0) + E_1(x)\ \nabla B_n'(0) + \tfrac{1}{2!} E_2(x)\ \nabla B_n''(x) + \ldots \\ &= \nabla B_n(0) + \tfrac{1}{2} G_2(x)\ \nabla B_n'(0) + \tfrac{1}{3!} G_3(x)\ \nabla B_n''(x) + \ldots \end{aligned}\right\} \tag{3.9}$$

by (2.5).

For example, $\nabla B_5(x) = x^5 + \frac{5}{3}x^3 - \frac{1}{6}x$ on calculation, whence

$B_5(x) = \frac{1}{6} G_6(x) + \frac{5}{12} G_4(x) - \frac{1}{12} G_2(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$ on calculation

Genocchi Polynomials in terms of Bernoulli Polynomials

The derivative of a polynomial may be expanded in terms of Bernoulli polynomials by the Euler-Maclaurin Theorem thus [13]:

Euler-Maclaurin Theorem: $P'(x+y) = \Delta P(x) + B_1(y)\ \Delta P'(x) + \frac{1}{2!} B_2(y)\ \Delta P''(x) + \ldots$ (3.10)

Putting $x = 0$ and replacing y by x and P′ by G_n in (3.10), we derive the Euler-Maclaurin expansion for Genocchi polynomials

$$G'_n(x) = \Delta G_n(0) + B_1(x)\ \Delta G'_n(0) + \frac{1}{2!} B_2(x)\ \Delta G''_n(0) + \frac{1}{3!} B_3(x)\ \Delta G'''_n(0) + \ldots \quad (3.11)$$

where $G'_n(x) = n\, G_{n-1}(x)$ by (2.6).

Accordingly, with $n = 5$ (for example), $\Delta G_5(x) = 20x^3 - 10x$ on calculation, so $G_4(x) = 4\, B_3(x) - 2\, B_1(x)$.

Inverse Mean ∇^{-1} of the Genocchi Polynomial

Defining the inverse mean function ∇^{-1} symbolically by $\nabla\, \nabla^{-1} = 1$, we proceed to give some brief information about $\nabla^{-1} G_n(x)$. [Refer to (2.29)′ for $\nabla G_n(x)$.]

Following the method used by Jordan [7] we obtain, after several steps in the manipulation, two equivalent forms

$$\left.\begin{aligned} \nabla^{-1} G_n(x) &= 2\left[(1 - x\, G_n(x) + \frac{G_{n+1}(x)}{n+1}\right] \\ &= \sum_{j=0}^{n} \binom{n}{j} \frac{G_j\, G_{n+1-j}(x)}{n+1-j} \qquad \text{by (2.17), (2.29)}' \end{aligned}\right\} \quad (3.12)$$

Combining these two forms when $x = 0$, we establish that

$$G_n + \frac{n}{n+1} G_{n+1} = \sum_{j=0}^{n} \binom{n}{j} \frac{G_j\, G_{n+1-j}}{2(n+1-j)} \quad (3.13)$$

This illustration shows how ∇^{-1} may be used to discover relationships among the $G_n(x)$, and the G_n.

Calculation in (3.13) when $n = 7, 8$ (say) leads to the common values $29\frac{3}{4}$, 34 respectively of each side of the equation.

4. DIFFERENTIAL EQUATIONS ASSOCIATED WITH $G_n(X)$

Earlier in (2.12), a partial differential equation was seen to appear naturally in the theory of $G_n^{(k)}(x)$. We now establish some differential equations which arise in a more special way when $k = 1$.

Descending Diagonal Functions

Imagine the information in (2.4) is slant-wise re-tabulated so that the following functions (*descending diagonal functions*) are emphasized $\left(|x| < 1\right)$:

$$\left.\begin{aligned}
g_1(x) &= G_1(1+2x+3x^2+4x^3+5x^4+\ldots\ldots\ldots\ldots) &&= G_1(1-x)^{-2} \\
g_2(x) &= G_2(1+3x+6x^2+10x^3+15x^4+\ldots\ldots\ldots) &&= G_2(1-x)^{-3} \\
g_3(x) &= G_3(1+4x+10x^2+20x^3+35x^4+\ldots\ldots\ldots) &&= G_3(1-x)^{-4} = 0 \\
g_4(x) &= G_4(1+5x+15x^2+35x^3+70x^4+\ldots\ldots\ldots) &&= G_4(1-x)^{-5} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
g_n(x) &= G_n\left(1+\binom{n+1}{1}x+\binom{n+2}{2}x^2+\binom{n+3}{3}x^3+\ldots\right) &&= G_n(1-x)^{-(n+1)}
\end{aligned}\right\} \tag{4.1}$$

Clearly

$$g_{2n+1}(x) = 0 \qquad (n = 1, 2, 3, \ldots) \quad \text{by (1.2)} \tag{4.1$'$}$$

To obtain (4.1), the values of G_n in (1.3) are utilized along with the combinatorial formula (2.17) for $k = 1$. Observe that

$$g_n(x) = G_n \sum_{j=0}^{\infty} \binom{n+j}{j} x^j \tag{4.2}$$

In particular

$$\left.\begin{aligned}
&g_n(0) = G_n \\
&g_n\left(\tfrac{1}{2}\right) = 2^{n+1}\, G_n \left(= 2^{n+1}\, g_n(0)\right) \\
&g_n\left(\tfrac{3}{4}\right) = 2^{n+1}\, g_n\left(\tfrac{1}{4}\right) \\
&g_n\,(1) \text{ is not defined.}
\end{aligned}\right\} \tag{4.3}$$

Write

$$D \equiv D(x,y) = \sum_{n=1}^{\infty} g_n(x)\, y^{n-1} = \sum_{n=1}^{\infty} G_n\, (1-x)^{-(n+1)}\, y^{n-1} \tag{4.4}$$

Then we can derive the partial differential equation

$$(n+1)y\, \frac{\partial D}{\partial y} - (n-1)\,(1-x)\, \frac{\partial D}{\partial x} = 0 \tag{4.5}$$

From (4.1), it follows that

$$(1-x)\, \frac{d\, g_n(x)}{dx} = (n+1)\, g_n\,(x) \tag{4.6}$$

whence, on integrating by parts,

$$\int g_n(x)\,dx = \frac{1-x}{n} g_n(x) + c \tag{4.7}$$

Extending the preceding theory to general k, we find that

$$g_n^{(k)}x = G_{n+k-1}^{(k)} (1-x)^{-(n+k)} \tag{4.1a}$$

leading to a differential equation which is identical with (4.4) except that $n+1$ is replaced by $n+k$, but with D now standing for $\sum_{n=1}^{\infty} g_n^{(k)}(x)\, y^{n-1}$.

Rising Diagonal Functions

Suppose next that in the re-tabulation of (2.4) we concentrate on the *rising diagonal functions* $h_n(x)$:

$$\left.\begin{aligned}
h_1(x) &= G_1 \\
h_2(x) &= G_2 + x\,G_0 \\
h_3(x) &= G_3 + 2x\,G_1 \\
h_4(x) &= G_4 + 3x\,G_2 + x^2\,G_0 \\
h_5(x) &= G_5 + 4x\,G_3 + 3x^2\,G_1 \\
h_6(x) &= G_6 + 5x\,G_4 + 6x^2\,G_2 + x^3\,G_0 \\
h_7(x) &= G_7 + 6x\,G_5 + 10x^2\,G_3 + 4x^3\,G_1 \\
&\dots\dots\dots\dots\dots\dots
\end{aligned}\right\} \tag{4.8}$$

Generally

$$h_n(x) = \sum_{j=0}^{[n/2]} \binom{n-j}{j} G_{n-2j}\, x^j \tag{4.9}$$

Obviously

$$\left.\begin{aligned}
h_n(0) = G_n = g_n(0) &\quad \text{by (4.3)} \\
= 0 &\quad \text{for n odd, } > 1
\end{aligned}\right\} \tag{4.10}$$

Now let

$$\left.\begin{aligned}
R \equiv R(x,y) &= \sum_{n=1}^{\infty} h_n(x)\, y^{n-1} \\
&= (1-xy^2)^{-2} G_1 + y\,(1-xy^2)^{-3} G_2 + y^2\,(1-xy^2)^{-4} G_3 + \dots
\end{aligned}\right\} \tag{4.11}$$

from (4.8).

Then

$$\frac{\partial R}{\partial x} = y^2 \psi \tag{4.12}$$

and

$$\frac{\partial R}{\partial y} = 2xy \psi + \phi \tag{4.13}$$

where

$$\psi \equiv 2(1-xy^2)^{-3} G_1 + 3y(1-xy^2)^{-4} G_2 + 4y^2(1-xy^2)^{-5} G_3 + \ldots \tag{4.14}$$

and

$$\phi \equiv (1-xy^2)^{-3} G_2 + 2y(1-xy^2)^{-4} G_3 + 3y^2(1-xy^2)^{-5} G_4 + \ldots \tag{4.15}$$

Using (4.12) and (4.13), we are led to

$$\frac{\partial^2 R}{\partial y \partial x} = 2y \psi + y^2 \frac{\partial \psi}{\partial y} \tag{4.16}$$

and

$$\frac{\partial^2 R}{\partial x \partial y} = 2y \psi + 2xy \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial x} \tag{4.17}$$

Applying Bernoulli's Theorem, we derive

$$\frac{\partial \phi}{\partial x} = y^2 \frac{\partial \psi}{\partial y} - 2xy \frac{\partial \psi}{\partial x} \tag{4.18}$$

For general k,

$$h_n^{(k)}(x) = \sum_{j=0}^{[n/2]} \binom{n+k-1-j}{j} G_{n+k-1-2j}^{(k)} x^j \tag{4.19}$$

and

$$R^{(k)}(x,y) = (1-xy^2)^{-(k+1)} G_1 + y(1-xy^2)^{-(k+2)} G_2 + y^2 \cdot (1-xy^2)^{-(k+3)} G_3 + \ldots \tag{4.20}$$

The theory delineated above then produces (4.18) again, though ϕ and ψ are now fairly obvious extensions of the expressions in (4.15) and (4.14). Notice that in general $G_m^{(k)} \neq 0$ ($m = k, k+1, \ldots$).

An alternative approach for the case $k = 1$ is to treat n even and n odd separately whence differential equations are obtained for each parity of n.

5. CONCLUDING REMARKS

Other developments germane to the material in this paper are suppressed here in order to conserve space: e.g., the theory of *Genocchi polynomials of negative order* $G_n^{(-k)}(x)$ a treatment of which will be presented in a separate submission.

Finally, it would be nice to know something more about the life and mathematical works of Angelo Genocchi whose researches have prompted these investigations. Various snippets of mathematical information are available, e.g., in [10] and in L.E. Dickson's "History of the Theory of Numbers", Chelsea (1952). Furthermore, one finds in an 1885 *Encyclopaedia Britannica* article by Moritz Cantor an adverse comment on Genocchi's belief that Fibonacci's sexagesimal solution in his book *Flos* of the John of Palermo problem (namely: solve $x^3 + 2x^2 + 10x = 20$) involved knowledge of a method associated with Cardan in the sixteenth century.

Any new information of a biographical nature about Genocchi would be appreciated.

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