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Proof Compression and NP Versus PSPACE

Abstract. We show that arbitrary tautologies of Johansson's minimal propositional logic are provable by "small" polynomial-size dag-like natural deductions in Prawitz's system for minimal propositional logic. These "small" deductions arise from standard "large" tree-like inputs by horizontal dag-like compression that is obtained by merging distinct nodes labeled with identical formulas occurring in horizontal sections of deductions involved. The underlying geometric idea: if the height, $h(\partial)$, and the total number of distinct formulas, $\phi(\partial)$, of a given tree-like deduction ∂ of a minimal tautology ρ are both polynomial in the length of ρ, $|\rho|$, then the size of the horizontal dag-like compression ∂^{C} is at most $h(\partial) \times \phi(\partial)$, and hence polynomial in $|\rho|$. That minimal tautologies ρ are provable by tree-like natural deductions ∂ with $|\rho|$-polynomial $h(\partial)$ and $\phi(\partial)$ follows via embedding from Hudelmaier's result that there are analogous sequent calculus deductions of sequent $\Rightarrow\rho$. The notion of dag-like provability involved is more sophisticated than Prawitz's tree-like one and its complexity is not clear yet. Our approach nevertheless provides a convergent sequence of NP lower approximations of PSPACE-complete validity of minimal logic (Savitch in J Comput Syst Sci 4(2):177–192, 1970); Statman in Theor Comput Sci 9(1):67–72, 1979; Svejdar in Arch Math Log 42(7):711–716, 2003).

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Proof Theoretic Background

We consider two types of proof theoretic formalism: Gentzen-style Sequent Calculus (abbr.: SC) and Prawitz's Natural Deduction (abbr.: ND). Both SC and ND admit standard tree-like interpretation, as well as generalized dag-like interpretation in which proofs (or deductions) are regarded as labeled rooted monoedge dags.[1] Our desired "small" deductions will arise from "large" standard tree-like inputs by appropriate dag-like compressing techniques. The compression in question is obtained by merging distinct nodes with identical labels, i.e. sequents or single formulas in the corresponding case of SC or ND, respectively.

[1] Recall that *'dag'* stands for *directed acyclic graph* (edges directed upwards).

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In our earlier SC related proof-compression research [4], [5], [1] dealing with sequent calculi[2] we obtained such basic result (et al):

Any tree-like deduction ∂ of any given sequent S is constructively compressible to a dag-like deduction ∂^{c} of S in which sequents occur at most once. I.e., in ∂^{c}, distinct nodes are supplied with distinct sequents (that occur in ∂).

However, even in the case of cutfree SC having good proof search and other nice properties (like Gentzen's subformula property), this result still gives us no polynomial control over the size of ∂^{c}. The reason is that sequents occurring in ∂^{c} can be viewed as collections of subformulas of S, which allows their total number to grow exponentially in the size of S, $|S|$. In contrast, ND deductions consist of single formulas, which gives hope to overcome this problem. On the other hand, in ND, full dag-like compression merging arbitrary nodes supplied with identical formulas is problematic, as there is a risk of confusion between deduced formulas and the same formulas used above as discharged assumptions. But we can try *horizontal dag-like compression* that should merge only the nodes occurring in horizontal sections of ND deductions involved. The underlying idea is explained in the abstract. Namely, if a tree-like input deduction ∂ of a given formula ρ has $|\rho|$-polynomial *height* (= maximal thread length), $h(\partial)$, and the *foundation* (= the total number of distinct formulas occurring in ∂), $\phi(\partial)$, is also polynomial in $|\rho|$, then so is the *size* (= total number of formulas) of the corresponding horizontal dag-like compression ∂^{c}, $|\partial^{\mathrm{c}}| \leq h(\partial) \times \phi(\partial)$. Moreover if maximal formula length in ∂, $\mu(\partial)$, is also polynomial in $|\rho|$, then so is the *weight* (= total number of characters occurring inside) of ∂^{c}.[3] It remains to show that every formula ρ that is valid in minimal logic admits a ND deduction ∂ with $|\rho|$-polynomial parameters $h(\partial)$ and $\phi(\partial)$. But this follows by a natural SC $\hookrightarrow$ ND embedding from Hudelmaier's result saying that there are analogous SC deductions of the corresponding sequent $\Rightarrow \rho$. To put it more precisely we argue along the following lines 1–4 in purely implicational propositional logic:

1. Formalize minimal logic as fragment $\mathrm{LM}_{\rightarrow}$ of Hudelmaier's tree-like cutfree intuitionistic sequent calculus. For any $\mathrm{LM}_{\rightarrow}$ proof ∂ of sequent $\Rightarrow \rho$:

[2] Also note [7] that shows a mimp-like formalization of ND that admits "explicit" and size-preserving strong normalization procedure.

[3] Actually, to estimate the complexity of ∂^{c}, polynomial size alone is sufficient, as we can enumerate the formulas and use those numbers instead of formulas themselves. Therefore we'll not refer to the weight of ∂^{c} anymore.

(a) $h(\partial)$ is polynomial (actually linear) in $|\rho|$,
(b) $\phi(\partial)$ and $\mu(\partial)$ are also polynomial in $|\rho|$.

2. Show that there exists a constructive (a)+(b) preserving embedding $\mathcal{F}$ of $\mathrm{LM}_{\to}$ into Prawitz's tree-like ND formalism $\mathrm{NM}_{\to}$ for minimal logic.
3. Elaborate the dag-like deducibility (provability) in $\mathrm{NM}_{\to}$.
4. Elaborate and apply *horizontal tree-to-dag proof compression* in $\mathrm{NM}_{\to}$. For any tree-like $\mathrm{NM}_{\to}$ input ∂, the size of dag-like output ∂^{C} is bounded by $h(\partial) \times \phi(\partial)$. Hence for any given tree-like $\mathrm{LM}_{\to}$ proof ∂ of ρ, the size of $(\mathcal{F}(\partial))^{\mathrm{C}}$ is polynomially bounded in $|\rho|$.

1. Detailed Exposition of Tree-Like Proof Systems

In the sequel we consider standard language $\mathcal{L}_{\to}$ of minimal logic whose formulas (α, β, γ, ρ etc.) are built up from propositional variables (p, q, r, etc.) using one propositional connective $\to$. The sequents are in the form $\Gamma \Rightarrow \alpha$ whose antecedents, Γ, are viewed as multisets of formulas; sequents $\Rightarrow \alpha$, i.e. $\emptyset \Rightarrow \alpha$, are identified with formulas α.

1.1. Sequent Calculus $\mathrm{LM}_{\to}$

$\mathrm{LM}_{\to}$ includes the following axioms ($\mathrm{M}A$) and inference rules ($\mathrm{M}I1 \to$), ($\mathrm{M}I2 \to$), ($\mathrm{M}E \to P$), ($\mathrm{M}E \to\to$) in the language $\mathcal{L}_{\to}$ (the constraints are shown in square brackets).[4]

$$\boxed{(\mathrm{M}A): \quad \Gamma, p \Rightarrow p}$$

$$\boxed{(\mathrm{M}I1\!\to): \quad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \quad [(\nexists \gamma) : (\alpha \to \beta) \to \gamma \in \Gamma]}$$

$$\boxed{(\mathrm{M}I2\!\to): \quad \frac{\Gamma, \alpha, \beta \to \gamma \Rightarrow \beta}{\Gamma, (\alpha \to \beta) \to \gamma \Rightarrow \alpha \to \beta}}$$

$$\boxed{(\mathrm{M}E\!\to\! P): \quad \frac{\Gamma, p, \gamma \Rightarrow q}{\Gamma, p, p \to \gamma \Rightarrow q} \quad [q \in \mathrm{VAR}(\Gamma, \gamma), p \neq q]}$$

$$\boxed{(\mathrm{M}E\!\to\to): \quad \frac{\Gamma, \alpha, \beta \to \gamma \Rightarrow \beta \quad \Gamma, \gamma \Rightarrow q}{\Gamma, (\alpha \to \beta) \to \gamma \Rightarrow q} \quad [q \in \mathrm{VAR}(\Gamma, \gamma)]}$$

[4]This is a slightly modified, equivalent version of the corresponding purely implicational and $\perp$-free subsystem of Hudelmaier's intuitionistic calculus LG, cf. [2]. The constraints $q \in VAR(\Gamma, \gamma)$ are added just for the sake of transparency.

CLAIM 1. $\mathrm{LM}_{\to}$ *is sound and complete with respect to minimal propositional logic* [3] *and tree-like deducibility. Thus any given formula* ρ *is valid in the minimal logic iff sequent* $\Rightarrow \rho$ *is tree-like deducible in* $\mathrm{LM}_{\to}$. *I.e., in symbols:* $(\mathrm{M}_{\to} \vdash \rho) \Longleftrightarrow (\mathrm{LM}_{\to} \vdash \Rightarrow \rho)$.

PROOF. Easily follows from [2]. ∎

Recall that for any (tree-like or dag-like) deduction ∂ we denote by $h(\partial)$ and $\phi(\partial)$ its height and foundation, respectively. Furthermore for any sequent (in particular, formula) S we denote by $|S|$ the total number of '$\to$'-occurrences in S and following [2] define the complexity degree $\deg(S)$:

1. $\deg(\Gamma, \alpha \to \beta \Rightarrow \alpha) := |\alpha \to \beta| + \sum_{\xi \in \Gamma} |\xi|$,
2. $\deg(\Gamma \Rightarrow \alpha) := |\alpha| + \sum_{\xi \in \Gamma} |\xi|$, *if* $(\nexists \beta) : \alpha \to \beta \in \Gamma$.

LEMMA 2.*1.* *Tree-like* $\mathrm{LM}_{\to}$ *deductions share the semi-subformula property, where semi-subformulas of* $(\alpha \to \beta) \to \gamma$ *include* $\beta \to \gamma$ *along with proper subformulas* $\alpha \to \beta$, α , β, γ. *In particular, any* α *occurring in a* $\mathrm{LM}_{\to}$ *deduction* ∂ *of* $\Rightarrow \rho$ *is a semi-subformula of* ρ, *and hence* $|\alpha| \leq |\rho|$. *Thus* $\mu(\partial) \leq |\rho|$.

2. *If* S' *occurs strictly above* S *in a given tree-like* $\mathrm{LM}_{\to}$ *deduction* ∂, *then* $\deg(S') < \deg(S)$.

3. *The height of any tree-like* $\mathrm{LM}_{\to}$ *deduction* ∂ *of* S *is linear in* $|S|$. *In particular if* S *is* $\Rightarrow \rho$, *then* $h(\partial) \leq 3|\rho|$.

4. *The foundation of any tree-like* $\mathrm{LM}_{\to}$ *deduction* ∂ *of* S *is at most quadratic in* $|S|$. *In particular if* S *is* $\Rightarrow \rho$, *then* $\phi(\partial) \leq (|\rho| + 1)^2$.

PROOF. 1: Obvious. Note that $\beta \to \gamma$ occurring in premises of $(\mathrm{M}I2 \to)$ and $(\mathrm{M}E \to\to)$ are semi-subformulas of $(\alpha \to \beta) \to \gamma$ occurring in the conclusions.

2–3: See [2].

4: Let $\mathrm{ssf}(\alpha)$ be the total number of distinct occurrences of semi-subformulas in a given formula α. It is readily seen that $\mathrm{ssf}(-)$ satisfies the following three conditions.

1. $\mathrm{ssf}(p) = 1$.
2. $\mathrm{ssf}(p \to \alpha) = 2 + \mathrm{ssf}(\alpha)$.
3. $\mathrm{ssf}((\alpha \to \beta) \to \gamma) = 1 + \mathrm{ssf}(\alpha \to \beta) + \mathrm{ssf}(\beta \to \gamma) - \mathrm{ssf}(\beta)$.

Moreover 1–3 can be viewed as recursive clauses defining $\mathrm{ssf}(\alpha)$, for any α. Having this we easily arrive at $\mathrm{ssf}(\alpha) \leq (|\alpha| + 1)^2$ (see Appendix A), which by the assertion 1 yields $\phi(\partial) \leq \mathrm{ssf}(\rho) \leq (|\rho|+1)^2$, as required, provided that $\Rightarrow \rho$ is the endsequent of ∂. ■

1.2. ND Calculus NM$_\rightarrow$ and Embedding ofLM$_\rightarrow$

Denote by NM$_\rightarrow$ a ND proof system for minimal logic that contains just two rules $(\rightarrow I)$, $(\rightarrow E)$ [6] (we write '$\rightarrow$' instead of '$\supset$').

$$\boxed{(\rightarrow I) : \dfrac{\begin{matrix}[\alpha] \\ \vdots \\ \beta\end{matrix}}{\alpha \rightarrow \beta}} \quad \boxed{(\rightarrow E) : \dfrac{\alpha \quad \alpha \rightarrow \beta}{\beta}}$$

CLAIM 3. (Prawitz). NM$_\rightarrow$ *is sound and complete with respect to minimal propositional logic and tree-like deducibility.*

PROOF. See [6]. ■

THEOREM 4. *There exists a recursive operator $\mathcal{F}$ that transforms any given tree-like* LM$_\rightarrow$ *deduction ∂ of $\Gamma \Rightarrow \rho$ into a tree-like* NM$_\rightarrow$ *deduction $\mathcal{F}(\partial)$ with root-formula ρ and assumptions occurring in Γ. Moreover ∂ and $\mathcal{F}(\partial)$ share linear (polynomial) upper bounds on the height (resp. foundation). If $\Gamma = \emptyset$, then $\mathcal{F}(\partial)$ is a tree-like* NM$_\rightarrow$ *proof of ρ such that the following holds.*

$$\boxed{h(\mathcal{F}(\partial)) \leq 18|\rho| \text{ and } \phi(\mathcal{F}(\partial)) < (|\rho|+1)^2(|\rho|+2) \text{ and } \mu(\mathcal{F}(\partial)) \leq 2|\rho|}$$

PROOF. $\mathcal{F}(\partial)$ is defined by straightforward recursion on $h(\partial)$ by standard pattern *sequent deduction* $\hookrightarrow$ *natural deduction*, where sequent deduction of $\Gamma \Longrightarrow \alpha$ is interpreted as a ND deduction of α from open assumptions occurring in Γ. The recursive clauses are as follows.

1.

$$\boxed{(\mathrm{M}A) : \Gamma, p \Rightarrow p} \overset{\mathcal{F}}{\hookrightarrow} \boxed{p}$$

2.

$$\boxed{\begin{matrix}(\mathrm{M}I1 \rightarrow) \ : \dfrac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} \\ [(\nexists\gamma) : (\alpha \rightarrow \beta) \rightarrow \gamma \in \Gamma]\end{matrix}} \overset{\mathcal{F}}{\hookrightarrow} \boxed{\dfrac{\begin{matrix}[\alpha] \\ \Downarrow \\ \beta\end{matrix}}{\alpha \rightarrow \beta}(\rightarrow I)}$$

3.

$$\boxed{(\mathrm{M}I2 \to) : \ \frac{\Gamma, \alpha, \beta \to \gamma \Rightarrow \beta}{\Gamma, (\alpha \to \beta) \to \gamma \Rightarrow \alpha \to \beta}} \overset{\mathcal{F}}{\hookrightarrow}$$

$$\boxed{\begin{array}{ccc}
 & \dfrac{\dfrac{[\beta]^2}{\alpha \to \beta}(\to I) \qquad (\alpha \to \beta) \to \gamma}{\dfrac{\gamma}{\beta \to \gamma^{[2]}}}\,(\to E) & (\to I) \\
[\alpha]^1 & & \\
\Downarrow & \Downarrow & \\
\searrow\!\!\searrow & \swarrow\!\!\swarrow & \\
 & \dfrac{\beta}{\alpha \to \beta^{[1]}}(\to I) &
\end{array}}$$

4.

$$\boxed{\begin{array}{c}(\mathrm{M}E \to P) : \dfrac{\Gamma, p, \gamma \Rightarrow q}{\Gamma, p, p \to \gamma \Rightarrow q} \\ \left[q \in \mathrm{VAR}\left(\Gamma, \gamma\right), p \neq q\right]\end{array}} \overset{\mathcal{F}}{\hookrightarrow}$$

$$\boxed{\begin{array}{cc}
 & \dfrac{p \quad p \to \gamma}{\gamma}(\to E) \\
p & \\
\Downarrow & \Downarrow \\
\searrow\!\!\searrow & \swarrow\!\!\swarrow \\
 q &
\end{array}}$$

5.

$$\boxed{(\mathrm{M}E \to\to) : \frac{\Gamma, \alpha, \beta \to \gamma \Rightarrow \beta \qquad \Gamma, \gamma \Rightarrow q}{\Gamma, (\alpha \to \beta) \to \gamma \Rightarrow q} \ \left[q \in \mathrm{VAR}\left(\Gamma, \gamma\right)\right]} \overset{\mathcal{F}}{\hookrightarrow}$$

$$\boxed{\dfrac{\dfrac{\begin{array}{cc}
 & \dfrac{\dfrac{[\beta]^2}{\alpha \to \beta} \quad (\alpha \to \beta) \to \gamma}{\dfrac{\gamma}{\beta \to \gamma^{[2]}}} \\
[\alpha]^1 & \\
\Downarrow & \Downarrow \\
\searrow\!\!\searrow & \swarrow\!\!\swarrow \\
\multicolumn{2}{c}{\dfrac{\beta}{\alpha \to \beta^{[1]}}}
\end{array} \qquad (\alpha \to \beta) \to \gamma}{\gamma} \qquad \dfrac{\begin{array}{c}[\gamma]_3 \\ \Downarrow \\ q\end{array}}{\gamma \to q^{[3]}}}{q}}$$

Note that each embedding clause increases the height at most by 6 (just as in the case (M$E\to\to$)), which yields $h(\mathcal{F}(\partial)) \leq 6\cdot h(\partial) \leq 18|\rho|$ according to Lemma 2 (3).[5] By the same token, formulas occurring in $\mathcal{F}(\partial)$ include the ones occurring in ∂ together with possibly new formulas $\gamma \to q$ (with old γ and q) shown on the right-hand side in the case (M$E\to\to$). There are at most $\phi(\partial)$ and $|\rho|+1$ such γ and q, respectively. Hence by Lemma 2 (1, 4) we arrive at $\phi(\mathcal{F}(\partial)) < (|\rho|+1)^2 + (|\rho|+1)^2(|\rho|+1) = (|\rho|+1)^2(|\rho|+2)$ and $\mu(\mathcal{F}(\partial)) \leq 2|\rho|$, as required. ∎

1.3. Tree-Like Extension NM$^*_{\to}$

For technical reasons we extend NM$_{\to}$ to an equivalent tree-like calculus NM$^*_{\to}$ that contains multipremise rules of inference[6] of the form

$$\boxed{(M) : \frac{\Gamma}{\gamma}}$$

instead of original NM$_{\to}$ rules $(\to I)$, $(\to E)$. Here Γ is a multiset containing γ, and/or β, if $\gamma = \alpha \to \beta$, and/or arbitrary δ_i together with $\delta_i \to \gamma$ $(i < |\Gamma|)$. Thus in particular, (M) includes repetition rules

$$\boxed{(R) : \frac{\gamma}{\gamma}} \quad \boxed{(R)^* : \frac{\gamma \quad \cdots \quad \gamma}{\gamma}}$$

as well as following inferences

$$\boxed{(\to I)^* : \frac{\begin{array}{c}[\alpha]\\ \vdots\\ \beta\end{array} \quad \cdots \quad \begin{array}{c}[\alpha]\\ \vdots\\ \beta\end{array}}{\alpha \to \beta}} \quad \boxed{(\to I, R)^* : \frac{\begin{array}{c}[\alpha]\\ \vdots\\ \beta\end{array} \quad \cdots \quad \begin{array}{c}[\alpha]\\ \vdots\\ \beta\end{array} \quad \alpha \to \beta \quad \cdots \alpha \to \beta}{\alpha \to \beta}}$$

$$\boxed{(\to E)^* : \frac{\delta_1 \quad \delta_1 \to \gamma \quad \cdots \quad \delta_m \quad \delta_m \to \gamma}{\gamma}}$$

$$\boxed{(\to E, R)^* : \frac{\delta_1 \quad \delta_1 \to \gamma \quad \cdots \quad \delta_m \quad \delta_m \to \gamma \quad \gamma \quad \cdots \quad \gamma}{\gamma}}$$

[5] Note that translation of $(ME \to\to)$ whose left-hand premise is axiom yields height growth rate at most 3, instead of 6.

[6] The multipremise rules are auxiliary tools, used in the definition of dag-to-tree unfolding (Chapter 5), that are eliminable by horizontal tree-to-dag compression (Chapter 4).

$$(\rightarrow I, E) : \frac{\begin{array}{c}[\alpha] \\ \vdots \\ \beta\end{array} \quad \delta \quad \delta \rightarrow (\alpha \rightarrow \beta)}{\alpha \rightarrow \beta}$$

$$(\rightarrow I, E, R) : \frac{\begin{array}{c}[\alpha] \\ \vdots \\ \beta\end{array} \quad \delta \quad \delta \rightarrow (\alpha \rightarrow \beta) \quad \alpha \rightarrow \beta}{\alpha \rightarrow \beta}$$

In $\mathrm{NM}^*_{\rightarrow}$, we consider ordinary tree-like deductions. Discharging is inherited from $\mathrm{NM}_{\rightarrow}$ via sub-occurrences of $(\rightarrow I)$.

LEMMA 5. *Tree-like provability in* $\mathrm{NM}^*_{\rightarrow}$ *is sound and complete with respect to minimal propositional logic.*

PROOF. Completeness follows from Claim 3, as $\mathrm{NM}_{\rightarrow}$ is contained in $\mathrm{NM}^*_{\rightarrow}$. Soundness is obvious, as each (M) strengthens valid rules (R), $(\rightarrow I)$ and/or $(\rightarrow E)$. ∎

2. Dag-Like Deducibility and Provability in $\mathrm{NM}_{\rightarrow}$

2.1. Introduction

Further on we upgrade $\mathrm{NM}^*_{\rightarrow}$ to a desired dag-like extension, $\mathrm{NM}^{\star}_{\rightarrow}$. Let us start with informal description (cf. formal definitions below). For purely technical reasons we'll consider only *regular dags* (abbr.: *redags*), which are specified as rooted monoedge dags D (the roots, $\varrho(D)$, being the bottoms) whose vertices (also called nodes) allow universal (i.e. path-invariant) height assignment such that all leaves x have the same height $h(x) = h(D)$. We also assume that the redags' nodes can have arbitrary many children and parents (the roots have no parents and the leaves have no children). Distinct children are either singletons or conjugate pairs (mutually separated by fixed partitions S). Furthermore, we supply nodes with formulas by a fixed assignment ℓ^{F}. The inferences (M) associated with D are determined by standard *local correctness* conditions on ℓ^{F} and S such that $\ell^{\mathrm{F}}(\varrho(D)) = \rho$, while children's ℓ^{F}-formulas either coincide with the conclusion's ones or are premises β of the conclusion's ℓ^{F}-formulas $\alpha \rightarrow \beta$, or else conjugate premises δ, $\delta \rightarrow \gamma$ of the conclusion's ℓ^{F}-formulas γ. These labeled redags, $\widetilde{D}$, are called dag-like *deduction frames*. Dag-like deductions extend deduction frames by adding *grandparent assignments* that are represented by appropriate boolean functions G.[7] Now G are defined for arbitrary descending chains $\overrightarrow{e_k}$ of edges $e_1 = \langle u_1, v_1 \rangle, \cdots, e_k = \langle u_k, v_k \rangle$, $k \geq 1$, in D, such that $G(\overrightarrow{e_1}) = 1$ and for all $1 \leq i < k$, $G(\overrightarrow{e_{i+1}}) = 1$ iff $G(\overrightarrow{e_i}) = 1$ and u_{i+1} are chosen "legitimate" parents of the nearest downward-branching nodes v_{i+1} that occur below or coincide with u_i.[8] Besides, G must satisfy certain conditions of *local coherence*. Pairs $\partial = \left\langle \widetilde{D}, G \right\rangle$ are called dag-like $\mathrm{NM}^{\star}_{\rightarrow}$ *deductions*. Genuine deducibility is determined by *deduction threads*, i.e. paths of nodes along $\overrightarrow{e_k}$ connecting top formulas with the roots, for all maximal $\overrightarrow{e_k}$ with $G(\overrightarrow{e_k}) = 1$.[9] The *discharging function* is defined as usual for every deduction thread with respect to introduction (rules) of its top formula (also called *assumption*). A deduction thread is *closed* if its assumption is discharged; otherwise it is called *open*. A given dag-like $\mathrm{NM}^{\star}_{\rightarrow}$ deduction ∂ is called a dag-like $\mathrm{NM}^{\star}_{\rightarrow}$ *proof* of D's root formula ρ (abbr.: $\partial \vdash \rho$) if every deduction thread in ∂ is closed. Note that in a tree-like deduction frame $\widetilde{T}$ (T being a rooted tree) any assumption determines a unique deduction thread to which it belongs; this makes G obsolete while reducing $\left\langle \widetilde{T}, G \right\rangle \vdash \rho$ to standard Prawitz's provability $\widetilde{T} \vdash \rho$. In the dag-like case, we regard $\widetilde{D} \vdash \rho$ as an abbreviation for $(\exists G) \left\langle \widetilde{D}, G \right\rangle \vdash \rho$, which enables us to redefine dag-like provability by reducing proofs ∂ to the underlying *proof frames* $\widetilde{D}$. Thus our *polysize proofs* are actually polysize proof frames.

2.2. Formal Definitions

Definition 6. Consider a rooted monoedge redag $D = \langle \mathrm{V}(D), \mathrm{E}(D) \rangle$, $\mathrm{E}(D) \subset \mathrm{V}(D)^2$. $\mathrm{V}(D)$ and $\mathrm{E}(D)$ are called the *vertices* (or *nodes*) and the *edges* (ordered), respectively; if $\langle u, v \rangle \in \mathrm{E}(D)$, then u and v are called *parents* and *children* of each other, respectively. For any $u \in \mathrm{V}(D)$ denote by $h(u, D) \geq 0$ the *height* of u and let $h(D) := \max\{h(u, D) : u \in \mathrm{V}(D)\}$ (the *height* of D). Any $u \in \mathrm{V}(D)$ has $\deg_{\uparrow}(u, D) \geq 0$ children $\mathrm{C}(u, D) := \left\{u^{(1)}, \cdots, u^{\left(\deg_{\uparrow}(u,D)\right)}\right\}$ and $\deg_{\downarrow}(u, D) \geq 0$ parents $\mathrm{P}(u, D) :=$

[7]In more familiar Frege-Hilbert-Bernays-Gentzen-style proof systems (both tree-like and dag-like) deductions are solely determined by the deduction frames.

[8]Loosely speaking, u_{i+1} with $G(\overrightarrow{e_{i+1}}) = 1$ are those parents of v_{i+1} which determine "legitimate" paths from leaves down to the root, when passing through $\overrightarrow{e_i}$ with $G(\overrightarrow{e_i}) = 1$.

[9]Contrary to the tree-like case where deduction threads are uniquely determined by top formulas, dag-like deduction frames admit several options depending on the assignments G.

$\left\{u_{(1)},\cdots,u_{\left(\deg_{\downarrow}(u,D)\right)}\right\}$ (both ordered).[10] A $u\in\mathrm{V}(D)$ with $\deg_{\downarrow}(u,D)>1$ is called an *inverse branching* node. The set $\mathrm{L}(D)$ denotes $\{u\in\mathrm{V}(D):\deg_{\uparrow}(u,D)=0\}$ (*leaves*), and $\varrho(D):=$ the root of D; thus $\mathrm{P}(u,D)=\emptyset\Leftrightarrow u=\varrho(D)\Leftrightarrow h(u,D)=0$ and $\mathrm{C}(u,D)=\emptyset\Leftrightarrow u\in\mathrm{L}(D)\Leftrightarrow h(u,D)=h(D)$. With $u\in\mathrm{V}(D)\setminus\mathrm{L}(D)$ we associate a fixed partition[11] $\mathrm{S}(u,D)\subset\mathrm{C}(u,D)\cup\mathrm{C}(u,D)^{2}$ such that $\mathrm{C}(u,D)=(\mathrm{S}(u,D)\cap\mathrm{V}(D))\cup\{x,y:\langle x,y\rangle\in\mathrm{S}(u,D)\}$. Set $\mathrm{S}(D):=\bigcup_{u\in\mathrm{V}(D)\setminus\mathrm{L}(D)}\mathrm{S}(u,D)$, to be abbreviated by S; analogously, we'll often drop 'D' in $\mathrm{V}(D)$, $\mathrm{E}(D)$, $h(D)$, $h(u,D)$, $\deg_{\uparrow}(u,D)$, $\deg_{\downarrow}(u,D)$, $\mathrm{C}(u,D)$, $\mathrm{P}(u,D)$, etc. (see below), if D is clear from the context.

Besides, let $x\prec_{D}y:\Leftrightarrow$ 'x *occurs strictly below* y *in* D' and $x\preceq_{D}y:\Leftrightarrow x\prec_{D}y\vee x=y$. Denote by $\mathrm{K}(D)$ the sets of ascending chains $\Theta=[x_0,\cdots,x_k]$, $k\geq 0$, $(\forall i<k)\,\langle x_i,x_{i+1}\rangle\in\mathrm{E}(D)$ and let $\Theta_{\mathrm{S}}:=x_0$, $\Theta_{\mathrm{T}}:=x_k$. So $x\preceq_{D}y\Leftrightarrow(\exists\Theta\in\mathrm{K}(D))(\Theta_{\mathrm{S}}=x\wedge\Theta_{\mathrm{T}}=y)$. Let $U(u,D):=\Theta_{\mathrm{S}}=x_0$ for uniquely determined $\Theta=[x_0,\cdots,x_k]\in\mathrm{K}(D)$ of maximal length such that $\Theta_{\mathrm{T}}=u$ and either $\deg_{\downarrow}(u)\neq 1$ and $k=0$, or else $k>0$ and $(\forall i<k)\deg_{\downarrow}(x_{i+1})=1$.

Let $\widetilde{D}=\langle D,\mathrm{S},\ell^{\mathrm{F}}\rangle$ extend $\langle D,\mathrm{S}\rangle$ by labeling function $\ell^{\mathrm{F}}:\mathrm{V}(D)\to\mathrm{F}(\mathcal{L}_{\to})$, where $\mathrm{F}(\mathcal{L}_{\to})$ is the set of $\mathcal{L}_{\to}$ formulas. $\widetilde{D}$ is called *dag-like* $\mathrm{NM}^{\star}_{\to}$ *deduction frame* iff for all $u\in\mathrm{V}$ and $x,y\in\mathrm{C}(u)$ the following conditions 1–3 of *local correctness* are satisfied (along with standard ones with regard to $\langle D,\mathrm{S}\rangle$).

1. $h(x)=h(y)=h(u)+1$.
2. If $x\in\mathrm{S}(u)$ then either $\ell^{\mathrm{F}}(u)=\ell^{\mathrm{F}}(x)$ or $\ell^{\mathrm{F}}(u)=\alpha\to\ell^{\mathrm{F}}(x)$ [abbr.: $\langle u,x\rangle\in(\to I)_{\alpha}$] for a (uniquely determined) $\alpha\in\mathrm{F}(\mathcal{L}_{\to})$.
3. If $\langle x,y\rangle\in\mathrm{S}(u)$ then $\langle y,x\rangle\in\mathrm{S}(u)$ and either $\ell^{\mathrm{F}}(y)=\ell^{\mathrm{F}}(x)\to\ell^{\mathrm{F}}(u)$ or $\ell^{\mathrm{F}}(x)=\ell^{\mathrm{F}}(y)\to\ell^{\mathrm{F}}(u)$.

The size of $\widetilde{D}$ is $\left|\widetilde{D}\right|:=|D|=|\mathrm{V}(D)|$. Let $\mathcal{R}^{\star}$ be the set of dag-like $\mathrm{NM}^{\star}_{\to}$ deduction frames.

Denote by $\overrightarrow{\mathrm{E}}(D)$ the set of chains $\overrightarrow{e_k}=e_1,\cdots,e_k$ for $e_i=\langle u_i,v_i\rangle\in\mathrm{E}(D)$, $1\leq i\leq k$, such that $v_1\in\mathrm{L}$ while if $1<i\leq k$, then $\deg_{\downarrow}(v_i)>1$ and $v_i\preceq u_{i-1}$. Let $\overrightarrow{e_k}=\overrightarrow{e_m}*\overrightarrow{e}$ be an abbreviation of $\overrightarrow{e_k}=\overrightarrow{e_{m+l}}\wedge\overrightarrow{e}=e_{m+1},\cdots,e_{m+l}$. For any $G:\overrightarrow{\mathrm{E}}(D)\to\{0,1\}$, a pair $\partial=\left\langle\widetilde{D},G\right\rangle$ is called *dag-like* $\mathrm{NM}^{\star}_{\to}$ *deduction* iff the following conditions 1–5 of *local coherence*

[10] That is, $\deg_{\downarrow}(u,D)$ (resp. $\deg_{\uparrow}(u,D)$) is the total number of targets with source u (resp. total number of sources with target u), in D.

[11] Not necessarily disjoint.

are satisfied, where $e_\imath = \langle u_\imath, v_\imath \rangle$ and $e'_\imath = \langle u'_\imath, v'_\imath \rangle$ while $\overrightarrow{e_k} = e_1, \cdots, e_k$ and $\overrightarrow{e'_r} := e'_1, \cdots, e'_r$.

1. $G(\overrightarrow{e_1}) = G(e_1) = 1$.
2. If $G(\overrightarrow{e_{k+1}}) = 1$ then $G(\overrightarrow{e_k}) = 1$ and $v_{k+1} = U(u_k)$.
3. If $G(\overrightarrow{e_k}) = 1$ and $U(u_k) \neq \varrho$, then $\sum\limits_{x \in \mathrm{P}(U(u_k))} G(\overrightarrow{e_k}, \langle x, U(u_k) \rangle) > 0.$[12]
4. If $G(\overrightarrow{e_k}) = 1$, $\overrightarrow{e_k} = \overrightarrow{e_m} * \overrightarrow{e}$, $v_{m+1} \preceq v \preceq u_m$, $z \preceq v_m$ and $\langle z, z' \rangle \in \mathrm{S}(v)$, then $\exists \overrightarrow{e'_t} = \overrightarrow{e'_s} * \overrightarrow{e'}$ with $\overrightarrow{e'} = \overrightarrow{e}$, $v'_{s+1} \preceq v \preceq u'_s$, $z' \preceq v'_s$ and $G\left(\overrightarrow{e'_t}\right) = 1$.
5. If $\deg_\uparrow(y) > 0$ and $\deg_\downarrow(y) > 1$, then $(\forall x \in \mathrm{P}(y)) (\exists \overrightarrow{e_k}) \, G(\overrightarrow{e_k}, \langle x, y \rangle) = 1$.

Note that identical assignment $\mathbf{1} : \overrightarrow{\mathrm{E}}(D) \to \{1\}$ is locally coherent. Let the size of ∂, $|\partial|$, be that of $\widetilde{D}$. Denote by $\mathcal{D}^\star$ the set of dag-like $\mathrm{NM}^\star_\to$ deductions.

EXAMPLE 7. Below $\boxed{v}$ indicates that $\deg_\downarrow(v) = 2$. Let $\widetilde{D}$ be given by

$$D = \boxed{\dfrac{\dfrac{\dfrac{1}{6}}{11} \qquad \dfrac{\boxed{y} \qquad \dfrac{2 \quad 3 \quad 4}{\boxed{x}} \qquad \dfrac{5}{10}}{14}}{\dfrac{15 \qquad\qquad 16}{17}}}$$

with $\mathrm{S}(x) = \{\langle 2,3 \rangle, \langle 3,2 \rangle, 4\}$, $\mathrm{S}(14) = \{\langle x,10 \rangle, \langle 10,x \rangle\}$, $\mathrm{S}(15) = \{\langle y,11 \rangle, \langle 11,y \rangle\}$, $\mathrm{S}(16) = \{\langle y,14 \rangle, \langle 14,y \rangle\}$, $\mathrm{S}(17) = \{\langle 15,16 \rangle, \langle 16,15 \rangle\}$ and $\ell^{\mathrm{F}}(1) = \ell^{\mathrm{F}}(6) = \ell^{\mathrm{F}}(11) = (\beta \to (\gamma \to \beta)) \to \gamma$, $\ell^{\mathrm{F}}(2) = \alpha$, $\ell^{\mathrm{F}}(3) = \alpha \to (\gamma \to \beta)$, $\ell^{\mathrm{F}}(4) = \ell^{\mathrm{F}}(14) = \ell^{\mathrm{F}}(17) = \beta$, $\ell^{\mathrm{F}}(5) = \ell^{\mathrm{F}}(10) = \ell^{\mathrm{F}}(15) = \gamma$, $\ell^{\mathrm{F}}(x) = \ell^{\mathrm{F}}(16) = \gamma \to \beta$, $\ell^{\mathrm{F}}(y) = \beta \to (\gamma \to \beta)$. So $\left|\widetilde{D}\right| = |D| = 14$.

Moreover let $\partial = \left\langle \widetilde{D}, G \right\rangle$, where $1 = G(\langle 6,1 \rangle) = G(\langle x,2 \rangle) = G(\langle x,3 \rangle)$ $= G(\langle x,4 \rangle) = G(\langle 10,5 \rangle) = G(\langle x,2 \rangle, \langle y,x \rangle) = G(\langle x,3 \rangle, \langle y,x \rangle)$ $= G(\langle x,4 \rangle, \langle y,x \rangle) = G(\langle x,4 \rangle, \langle 14,x \rangle) = G(\langle x,2 \rangle, \langle y,x \rangle, \langle 15,y \rangle)$ $= G(\langle x,3 \rangle, \langle y,x \rangle, \langle 15,y \rangle) = G(\langle x,4 \rangle, \langle y,x \rangle, \langle 16,y \rangle)$. Then $\partial \in \mathcal{D}^\star$, $|\partial| = 14$.

DEFINITION 8. Note that any maximal ascending chain of the form $\Theta = [\varrho = x_0, \cdots, x_h] \in \mathrm{K}$ uniquely determines a sequence $0 = f(0) < \cdots < f(m) = h$ such that $\left\{x_{f(j)} : 0 < j < m\right\} = \left\{x_i \in \Theta : i < h \wedge \deg_\downarrow(x_i) > 1\right\}$.

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[12] This condition is optional.

Define $\overrightarrow{e_m} = e_1, \cdots, e_m$ by $e_j := \left\langle x_{f(m-j+1)-1}, x_{f(m-j+1)} \right\rangle$, for all $1 \leq j \leq m$. Now let $\partial = \left\langle \widetilde{D}, G \right\rangle \in \mathcal{D}^\star$ and call Θ an *ascending deduction thread* if $G(\overrightarrow{e_m}) = 1$. Denote by $\mathrm{T}(\partial)$ the set of ascending deduction threads, in ∂.[13]

A given $\alpha \in \mathrm{F}(\mathcal{L}_\to)$ is called an *open* (or *undischarged*) *assumption* in ∂ if there exists $\Theta = [\varrho = x_0, x_1, \cdots, x_h] \in \mathrm{T}(\partial)$ with $\ell^{\mathrm{F}}(x_h) = \alpha$ that contains no $\langle x_i, x_{i+1} \rangle \in (\to I)_\alpha$ for $i < h$. Such Θ is called an *open thread*; other deduction threads are called *closed*. Denote by Γ_∂ the set of open assumptions in ∂. Call ∂ a *dag-like* $\mathrm{NM}^\star_\to$ *deduction* from Γ_∂. If $\Gamma_\partial = \emptyset$ then ∂ is called a *dag-like* $\mathrm{NM}^\star_\to$ *proof* of $\rho := \ell^{\mathrm{F}}(\varrho)$ (abbr.: $\partial \vdash \rho$). Denote by $\mathcal{P}^\star$ the set of $\mathrm{NM}^\star_\to$ proofs.

Remark 9. In Example 7 (above) we have

$$\mathrm{T}(\partial) = \left\{ \begin{array}{c} [17, 15, 11, 6, 1], [17, 15, y, x, 2], [17, 15, y, x, 3], \\ [17, 16, y, x, 4], [17, 16, 14, x, 4], [17, 16, 14, 10, 5] \end{array} \right\}.$$

Thus, in particular, ∂ has an open thread $[17, 16, 14, x, 4]$ and a closed thread $[17, 16, y, x, 4]$, both having the same assumption $\beta = \ell^{\mathrm{F}}(4)$. Hence $\beta \in \Gamma_\partial$ and in particular $\partial \notin \mathcal{P}^\star$.

In the sequel dag-like $\mathrm{NM}^\star_\to$ deductions (proofs) are also called *dag-like* $\mathrm{NM}_\to$ *deductions* (*proofs*). Note that in the tree-like domain such dag-like (actually redag-like) provability is equivalent to canonical tree-like $\mathrm{NM}_\to$ provability. Indeed, in any tree-like deduction, every leaf has exactly one deduction thread, and hence G can be dropped entirely. Thus in $\mathrm{NM}^\star_\to$, tree-like deductions are just deduction frames. Also note that $\mathrm{NM}^*_\to$ (and hence also $\mathrm{NM}_\to$) is tree-like embeddable into $\mathrm{NM}^\star_\to$ by iterating the repetition rule (R), if necessary, in order to fulfill the redag height condition $h(x) = h(\partial)$, for all leaves x. Obviously this operation preserves $h(\partial)$, $\phi(\partial)$ and $\mu(\partial)$.

3. Horizontal Tree-to-Dag Compression

Any given tree-like $\mathrm{NM}_\to$ deduction ∂ with root formula ρ can be compressed into a dag-like $\mathrm{NM}_\to$ deduction $\partial^{\mathrm{C}} = \left\langle \widetilde{D}, G \right\rangle$ of the same conclusion ρ such that the size of its compressed deduction frame $\widetilde{D}$ is at most $h(\partial) \times \phi(\partial)$. In particular, if $\partial = \mathcal{F}(\partial_0)$ for ∂_0 being a tree-like $\mathrm{LM}_\to$ deduction of $\Rightarrow \rho$ and $\mathcal{F}$ the embedding of Theorem 4, then ∂^{C} will be a

[13] By the local coherence condition 1, $G(\overrightarrow{e_m}) = 1$ holds if $m = 1$. Hence in a tree-like ∂, every maximal ascending chain is an ascending deduction thread.

desired dag-like $|\rho|$-polysize $\mathrm{NM}_{\rightarrow}$ deduction of ρ. The operation $\partial \hookrightarrow \partial^{\mathrm{C}}$ (called *horizontal compression*) runs by bottom-up recursion on $h(\partial)$ such that for any $n \leq h(\partial)$, the n^{th} horizontal section of $\widetilde{D}$ is obtained by merging all nodes with identical formulas occurring in the n^{th} horizontal section of ∂ (this operation we call *horizontal collapsing*). Thus the horizontal compression is obtained by bottom-up iteration of the horizontal collapsing. $|\partial^{\mathrm{C}}| \leq h(\partial) \times \phi(\partial)$ is obvious, as the size of every (compressed) n^{th} horizontal section of ∂^{C} can't exceed $\phi(\partial)$. Now let us take a closer look at the structure of ∂^{C}. For any $n \leq h = h(\partial)$, let $\partial_n^{\mathrm{C}} = \left\langle \widetilde{D}_n, G_n \right\rangle$ for $\widetilde{D}_n = \langle D_n, \mathrm{S}_n, \ell_n^{\mathrm{F}} \rangle$ be a deduction that is obtained after executing the n^{th} collapsing step in question. Note that $\partial = \partial_0^{\mathrm{C}} = \widetilde{D}_0$ and $\partial^{\mathrm{C}} = \partial_h^{\mathrm{C}} = \left\langle \widetilde{D}_h, G_h \right\rangle$, while $\widetilde{D}_{n+1}$ arises from $\widetilde{D}_n$ by merging distinct vertices $x \in L_{n+1}(D_n)$ labeled with identical formulas, $\ell^{\mathrm{F}}(x)$, and defining edges by corresponding homomorphism, where $L_k(D_m) := \{x \in \mathrm{V}(D_m) : h(x) = k\}$ (= the k^{th} section of D_m). Moreover, for any $i \leq n < j$ we have $L_{i+1}(D_{n+1}) = L_{i+1}\left(D_{h(\partial)}\right)$, $L_{j+1}(D_{n+1}) = L_{j+1}(D_0)$, while all $x \in L_j(D_{n+1})$ are roots of the corresponding tree-like subgraphs of ∂. Thus $L_{n+1}(D_{n+1}) \subseteq L_{n+1}(D_n)$, while $x \neq y \in L_{n+1}(D_{n+1})$ implies $\ell^{\mathrm{F}}(x) \neq \ell^{\mathrm{F}}(y)$. (If $L_{n+1}(D_{n+1}) = L_{n+1}(D_n)$, then $\widetilde{D}_{n+1} = \widetilde{D}_n$.) Having this we stipulate G_{n+1} and observe that $\partial_{n+1}^{\mathrm{C}} = \left\langle \widetilde{D}_{n+1}, G_{n+1} \right\rangle$ preserves the open (resp. closed) assumptions of ∂_n^{C}. The same conclusion with regard to ∂ and ∂^{C} follows immediately by induction on $n \leq h$. In particular, if ∂ is a tree-like $\mathrm{NM}_{\rightarrow}^{\star}$ proof of ρ, then ∂^{C} is a dag-like $\mathrm{NM}_{\rightarrow}$ proof of ρ. This completes our informal description of the required tree-to-dag horizontal compression $\partial \hookrightarrow \partial^{\mathrm{C}}$. Formal definitions are shown below.

3.1. Horizontal Collapsing

Recall that horizontal compression $\partial \hookrightarrow \partial^{\mathrm{C}}$ is obtained by bottom-up iteration of the *horizontal collapsing* that merges distinct nodes labeled with identical formulas occurring in the same horizontal section of a given dag-like deduction ∂. Our next definition formalizes the latter operation, where for any D and $x \in \mathrm{V}(D)$ we let $(D)_x := \langle \mathrm{V}((D)_x), \mathrm{E}((D)_x) \rangle$ for $\mathrm{V}((D)_x) = \{y \in \mathrm{V}(D) : x \preceq y\}$ and $\mathrm{E}((D)_x) = \mathrm{E}(D) \cap \mathrm{V}((D)_x)^2$. For any $n > 0$ we let $\mathcal{D}_n^{\star} \subseteq \mathcal{D}^{\star}$ be the set of $\partial = \left\langle \widetilde{D}, G \right\rangle \in \mathcal{D}^{\star}$ such that $(D)_x$ are pairwise disjoint trees (i.e. subtrees of D), for all $x \in L_n(D)$. Note that $\mathcal{D}_n^{\star} = \mathcal{D}^{\star}$ for $n > h(D)$, while $\mathcal{D}_1^{\star}$ consists of all tree-like $\mathrm{NM}_{\rightarrow}^{*}$ deductions.

So in the sequel we'll rename $\mathcal{D}_1^{\star}$ to $\mathcal{T}^*$ and denote its elements by $\langle T, \text{ S}, \ell^{\text{F}}\rangle$, rather than $\langle D, \text{S}, \ell^{\text{F}}\rangle$ (recall that G is irrelevant in the tree-like case).

Definition 10. (horizontal collapsing). Let $\partial = \left\langle \widetilde{D}, G\right\rangle \in \mathcal{D}_n^{\star}$, $\widetilde{D} = \langle D, \text{S}, \ell^{\text{F}}\rangle$, $n \le h := h(D)$, $\alpha \in \text{F}(\mathcal{L}_{\rightarrow})$ and $S_{n,\alpha} = \{y \in L_n(D) : \ell^{\text{F}}(y) = \alpha\}$, $|S_{n,\alpha}| > 1$. Moreover let $r \in S_{n,\alpha}$ be fixed. Let $C_\alpha = \bigcup_{y \in S_{n,\alpha}} \text{C}(y, D)$ and denote by $(D)_{\alpha,r}$ a tree extending upper subtrees $\bigcup_{z \in C_\alpha} (D)_z$ by a new root r. We construct a dag-like deduction $\partial^{\text{C}}_{n,\alpha} = \left\langle \widetilde{D}_{n,\alpha}, G_{n,\alpha}\right\rangle$, $\widetilde{D}_{n,\alpha} = \langle D_{n,\alpha}, \text{S}_{n,\alpha}, \ell^{\text{F}}_{n,\alpha}\rangle$, by collapsing $S_{n,\alpha}$ to $\{r\}$. To begin with we stipulate $\widetilde{D}_{n,\alpha}$.

1. $D_{n,\alpha}$ arises from D by substituting $(D)_{\alpha,r}$ for $(D)_r$ and deleting $(D)_y$ for all $r \ne y \in S_{n,\alpha}$. That is, in the formal terms, we have

$$\text{V}(D_{n,\alpha}) = \left(\text{V}(D) \setminus \bigcup_{y \in S_{n,\alpha}} \text{V}\left((D)_y\right)\right) \cup \text{V}\left((D)_{\alpha,r}\right) \textit{ and } \text{E}(D_{n,\alpha}) =$$

$$\left(\text{E}(D) \cap \text{V}(D_{n,\alpha})^2\right) \cup \left\{\langle r, v\rangle : v \in \bigcup_{y \in S_{n,\alpha}} \text{C}(y,D)\right\} \cup \left\{\langle u, r\rangle : u \in \bigcup_{y \in S_{n,\alpha}} \text{P}(y,D)\right\}.$$

2. For any $u \in \text{V}(D_{n,\alpha})$ we define $\text{S}_{n,\alpha}(u, D_{n,\alpha})$ by cases as follows.
   (a) If $u \notin \{r\} \cup \bigcup_{y \in S_{n,\alpha}} \text{P}(y, D)$, then $\text{S}_{n,\alpha}(u, D_{n,\alpha}) := \text{S}(u, D)$.
   (b) $\text{S}_{n,\alpha}(r, D_{n,\alpha}) := \bigcup_{y \in S_{n,\alpha}} \text{S}(y, D)$.
   (c) Suppose $u \in \bigcup_{y \in S_{n,\alpha}} \text{P}(y, D)$. We let $\text{S}_{n,\alpha}(u, D_{n,\alpha}) := X \cup Y$, where
   $X = (\text{S}(u, D) \cap L_n(D_{n,\alpha})) \cup \{r\}$ and
   $Y = \left\{\langle y_0, y_1\rangle \in L_n(D_{n,\alpha})^2 : \begin{array}{c}(\exists \langle x_0, x_1\rangle \in \text{S}(u, D))(\forall i \le 1)\\ (x_i = y_i \vee (r \ne x_i \in S_{n,\alpha} \wedge y_i = r))\end{array}\right\}$.
3. For any $u \in \text{V}(D_{n,\alpha})$ we let $\ell^{\text{F}}_{n,\alpha}(u) := \ell^{\text{F}}(u)$.

This completes $\widetilde{D}_{n,\alpha}$. To stipulate $G_{n,\alpha}$ suppose $\overrightarrow{e_k} = e_1, \cdots, e_k \in \overrightarrow{\text{E}}(D_{n,\alpha})$, $1 \le k \le h = h(D_{n,\alpha}) = h(D)$. If $k = 1$ then let $G_{n,\alpha}(\overrightarrow{e_k}) = G_{n,\alpha}(e_1) := 1$. Otherwise define $G_{n,\alpha}(\overrightarrow{e_k})$ by cases as follows, where $u_2 \in L_j(D_{n,\alpha})$, $v_2 = U(u_1) \in L_{j+1}(D_{n,\alpha})$, $j + 1 < h$. Note that $j + 1 \le n$, as $\deg_{\downarrow}(v_2, D_{n,\alpha}) > 1$.

1. Suppose $n = h$ and $v_1 \ne r$. Then let $G_{n,\alpha}(\overrightarrow{e_k}) := G(\overrightarrow{e_k})$.
2. Suppose $n = h$ and $v_1 = r$. Then let $G_{n,\alpha}(\overrightarrow{e_k}) := \max_{v' \in \text{C}(u_1,D) \cap S_{n,\alpha}} G\left(\overrightarrow{e'_k}\right)$, where $e'_1 = \langle u_1, v'\rangle$, $(\forall \iota \ne 1)\, e'_\iota = e_\iota$.

3. Suppose $j+1<n<h$. Then let $G_{n,\alpha}(\overrightarrow{e_k}) := G(\overrightarrow{e_k})$.
4. Suppose $j+1=n<h$ and $v_2 \neq r$. Then let $G_{n,\alpha}(\overrightarrow{e_k}) := G(\overrightarrow{e_k})$.
5. Suppose $j+1=n<h$ and $v_2=r$. Let v' be (uniquely) determined by $\mathrm{C}(u_2,D) \cap S_{n,\alpha} \ni v' \preceq u_1$. Then let

$$G_{n,\alpha}(\overrightarrow{e_k}) := \begin{cases} G\left(\overrightarrow{e'_k}\right), & \text{if } \deg_{\downarrow}(v',D)>1, \\ G\left(\overrightarrow{e''_{k-1}}\right), & \text{if } \deg_{\downarrow}(v',D)=1, \end{cases}$$

where $e'_2 = \langle u_2, v'\rangle, (\forall \iota \neq 2)\, e'_\iota = e_\iota$ and $e''_1 = e_1, (\forall \iota \neq 1)\, e''_\iota = e_{\iota+1}$.

This completes our definition of $\partial^{\mathrm{C}}_{n,\alpha}$ under the assumption $|S_{n,\alpha}|>1$.

To complete the (n,α)-*collapsing operation* $\partial \hookrightarrow \partial^{\mathrm{C}}_{n,\alpha}$, let $\partial^{\mathrm{C}}_{n,\alpha} := \partial$ in the case $|S_{n,\alpha}|=1$. Now let ∂^{C}_n arise by applying (n,α)-collapsing successively to all $\alpha = \ell^{\mathrm{F}}(x)$, $x \in L_n(D)$, and arbitrary $r \in S_{n,\alpha}$. Thus $\partial_n{}^{\mathrm{C}}$ is the iteration of $\partial^{\mathrm{C}}_{n,\alpha}$ with respect to all α occurring in the n^{th} section of D. The operation $\partial \hookrightarrow \partial^{\mathrm{C}}_n$ is called the *horizontal collapsing on level* n, in $\mathrm{NM}^{\star}_{\rightarrow}$.

Lemma 11. *The following conditions 1–6 hold for any* $\partial = \left\langle \widetilde{D}, G \right\rangle \in \mathcal{D}^{\star}_n$, $\widetilde{D} = \langle D, \mathrm{S}, \ell^{\mathrm{F}}\rangle$, $n \leq h$, *and* $\partial^{\mathrm{C}}_n = \left\langle \widetilde{D}_n, G_n \right\rangle$, $\widetilde{D}_n = \langle D_n,\ \mathrm{S}_n, \ell^{\mathrm{F}}_n\rangle$.

1. $\partial^{\mathrm{C}}_n \in \mathcal{D}^{\star}_n$.
2. $\mathrm{V}(D_n) \subseteq \mathrm{V}(D)$, $\mathrm{L}(D_n) \subseteq \mathrm{L}(D)$, $\varrho(D_n) = \varrho$ *and* $h(D_n) = h$.
3. *For any* $i \neq n$, $L_i(D_n) = L_i(D)$, *whereas* $L_n(D_n) \subseteq L_n(D)$.
4. *For any* $i \leq n$, $|L_i(D_n)| \leq \phi(\partial)$.
5. *For any* $i \leq h$, $\ell^{\mathrm{F}}(L_i(D_n)) = \ell^{\mathrm{F}}(L_i(D))$. *Thus* ∂^{C}_n *and* ∂ *have the same formulas, and hence* $\phi(\partial^{\mathrm{C}}_n) = \phi(\partial)$.
6. $\Gamma_{\partial^{\mathrm{C}}_n} = \Gamma_{\partial}$.

Proof. By iteration, it will suffice to prove analogous assertions with respect to every (n,α)-collapsing involved. We skip trivial conditions 2–5 and verify 1. It is clear that $\widetilde{D}_{n,\alpha}$ is a (locally correct) deduction frame. Consider $G_{n,\alpha}$ and corresponding local coherence conditions 1–5. Let us verify the only nontrivial condition 5. Suppose $\deg_{\uparrow}(y, D_{n,\alpha}) > 0$, $\deg_{\downarrow}(y, D_{n,\alpha}) > 1$ and $x \in \mathrm{P}(y, D_{n,\alpha})$. So $h > h(y) \leq n$. Moreover, if $h(y) = n$ for $y \neq r$, then $\mathrm{P}(y,D_{n,\alpha}) = \mathrm{P}(y,D)$ and we are done by the assumption $\partial \in \mathcal{D}^{\star}_n$. Suppose $h(y) = n$ and $y = r$. Then x determines a $y' \in \mathrm{C}(x,D) \cap S_{n,\alpha}$, and hence by Definition 10 ($G_{n,\alpha}$: 5), $G_{n,\alpha}(e_1, \langle x, y\rangle) = 1$ holds for any $e_1 = \langle u_1, v_1\rangle \in \overrightarrow{\mathrm{E}}(D)$ satisfying $y' \preceq_D u_1$. This also yields $e_1 \in \overrightarrow{\mathrm{E}}(D_{n,\alpha})$ and $y = U(u_1,D_{n,\alpha})$, as required. Case $h(y) < n$ is treated analogously, except expanding e_1 to corresponding $\overrightarrow{e_\imath}$. This completes condition 1 of the lemma.

Now consider condition 6 (with respect to every (n,α) -collapsing involved). In order to prove crucial inclusion $\Gamma_{\partial^{\mathrm{C}}_{n,\alpha}} \subseteq \Gamma_\partial$, it will suffice to show that there is an assumption-preserving embedding of the open threads in $\partial^{\mathrm{C}}_{n,\alpha}$ into the open threads in ∂. So let $\Theta_{n,\alpha} = [\varrho = x_0, \cdots, x_h] \in \mathrm{T}(\partial^{\mathrm{C}}_{n,\alpha})$ be any given open thread in $\partial^{\mathrm{C}}_{n,\alpha}$ together with sequence $0 < f_{n,\alpha}(1) < \cdots < f_{n,\alpha}(m) = h$ and chain $\overrightarrow{e_m}$ for $(\forall \imath \in [1,m])\, e_\imath = \langle x_{f_{n,\alpha}(m-\imath+1)-1}, x_{f_{n,\alpha}(m-\imath+1)} \rangle$ (cf. Definition 8). A desired open thread in ∂, $\Theta = [\varrho = y_0, \cdots, y_h] \in \mathrm{T}(\partial)$ and correlated sequence $0 < f(0) < \cdots < f(l) = h$ are defined by cases as follows, where $r \in L_n(D_{n,\alpha}) \setminus L_n(D)$, $n \le h$.

1. Suppose $x_n \ne r$. Then let $\Theta := \Theta_{n,\alpha}$, i.e. $(\forall i \le h)\, y_i := x_i$. Moreover let $l := m$ and $f := f_{n,\alpha}$. Hence $G_{n,\alpha}(\overrightarrow{e_m}) = G(\overrightarrow{e_m})$.
2. Suppose $x_n = r$ and $n = h$. Hence $r = x_n = x_h = x_{f_{n,\alpha}(m)}$. Moreover by Definition 10 $(G_{n,\alpha} : 2)$ we have $G_{n,\alpha}(\overrightarrow{e_m}) = \max_{v' \in \mathrm{C}(x_{h-1},D) \cap S_{n,\alpha}} G\left(\overrightarrow{e'_m}\right)$, where $e'_1 = \langle x_{h-1}, v' \rangle \in \mathrm{E}(D)$ and $(\forall i \ne 1)\, e'_i = e_i$. Hence there is $v' \in \mathrm{C}(u_1, D) \cap S_{n,\alpha}$ such that $G\left(\overrightarrow{e'_m}\right) = G_{n,\alpha}(\overrightarrow{e_m})$. Now let $y_n = y_h := v', (\forall i < h)\, y_i := x_i$ together with $l := m$ and $f := f_{n,\alpha}$.
3. Suppose $x_n = r$ and $n = f_{n,\alpha}(m-1) < h$. Consider $e_1 = \langle x_{h-1}, x_h \rangle \in \mathrm{E}(D_{n,\alpha})$ and suppose $\mathrm{C}(x_{f_{n,\alpha}(m-1)-1}, D) \cap S_{n,\alpha} \ni v' \preceq_D x_{h-1}$ as in Definition 10 $(G_{n,\alpha} : 5)$. Let $y_n := v', (\forall i \ne n)\, y_i := x_i$. To stipulate f consider two subcases:
   (a) $\deg_\downarrow(v', D) > 1$,
   (b) $\deg_\downarrow(v', D) = 1$.

   Now in the subcase (a) let $l := m$ with $f := f_{n,\alpha}$ and in the subcase (b) we let $l := m-1$ with $f(m-1) := h$, $(\forall j < m-1)\, f(j) := f_{n,\alpha}(j)$. This yields $G_{n,\alpha}(\overrightarrow{e_m}) = G\left(\overrightarrow{e'_m}\right)$ and $G_{n,\alpha}(\overrightarrow{e_m}) = G\left(\overrightarrow{e''_{m-1}}\right)$ in (a) and (b), respectively, where $e'_\imath$ and $e''_\imath$ are as in Definition 10 $(G_{n,\alpha} : 5)$.

This completes Θ and f. It remains to prove that it is an open thread in ∂. Obviously $\Theta_{n,\alpha} \hookrightarrow \Theta$ preserves formulas. Thus $\Theta \in \mathrm{K}(D)$ with $\Theta_{\mathrm{S}} = y_0 = \varrho$ and $\Theta_{\mathrm{T}} = y_h \in \mathrm{L}(D)$, while $(\forall i \le h)\, \ell^{\mathrm{F}}_{n,\alpha}(x_i) = \ell^{\mathrm{F}}(y_i)$. Moreover, for any $i < h$, $\deg_\downarrow(x_i) > 1$ iff $i \in Rng(f)$. To complete the proof it will suffice to show that $G_{n,\alpha}(\overrightarrow{e_m}) = 1$ implies $G(\overrightarrow{e_m}) = 1$, $G\left(\overrightarrow{e'_m}\right) = 1$ and/or $G\left(\overrightarrow{e''_{m-1}}\right) = 1$, in every case 1–3 under consideration. But this is obvious, since $G_{n,\alpha} \hookrightarrow G$ follows the same pattern as Definition 10. Thus $\Gamma_{\partial^{\mathrm{C}}_{n,\alpha}} \subseteq \Gamma_{\partial}$. $\Gamma_{\partial} \subseteq \Gamma_{\partial^{\mathrm{C}}_{n,\alpha}}$ is proved analogously by inversion $\Theta \hookrightarrow \Theta_{n,\alpha}$. This completes the whole proof. ■

3.2. Horizontal Compression

As mentioned above, horizontal compression $\partial \hookrightarrow \partial^{\mathrm{C}}$ is obtained by bottom-up iteration of horizontal collapsing $\partial \hookrightarrow \partial_n^{\mathrm{C}}$, $n \leq h(\partial)$. For the sake of brevity we consider tree-like inputs $\partial \in \mathcal{T}^*$.

Definition 12. (horizontal compressing). For any given $\partial \in \mathcal{T}^*$ denote by $\partial^{\mathrm{C}} \in \mathcal{D}^{\star}$ the last deduction in the following iteration chain

$$\partial = \partial^{\mathrm{C}}_{(0)},\ \partial^{\mathrm{C}}_{(1)},\ \cdots,\ \partial^{\mathrm{C}}_{(h(\partial))} = \partial^{\mathrm{C}}$$

where for every $i < h(\partial)$ we let $\partial^{\mathrm{C}}_{(i+1)} := \left(\partial^{\mathrm{C}}_{(i)}\right)^{\mathrm{C}}_{i+1}$. It's clear that all ∂^{C} in question are mutually isomorphic (actually equal up to the choice of $r \in S_{n,\alpha}$). The operation $\partial \hookrightarrow \partial^{\mathrm{C}}$ is called the *horizontal dag-like compression*, in $\mathrm{NM}^{\star}_{\rightarrow}$.

Example 13. Below $\boxed{v}$ abbreviates that $\overleftarrow{\deg}(v) = 2$.

Let $\partial = \widetilde{D} = \boxed{\dfrac{\dfrac{\dfrac{\dfrac{1}{7}}{12} \quad \dfrac{\dfrac{2 \quad 3}{8}}{13}}{16} \quad \dfrac{\dfrac{\dfrac{4}{9}}{14} \quad \dfrac{\dfrac{5}{10} \quad \dfrac{6}{11}}{15}}{17}}{18}}$ with

$\mathrm{s}(8) = \{\langle 2,3\rangle, \langle 3,2\rangle\}$, $\mathrm{s}(15) = \{\langle 10,11\rangle, \langle 11,10\rangle\}$, $\mathrm{s}(16) = \{\langle 12,13\rangle, \langle 13,12\rangle\}$, $\mathrm{s}(17) = \{\langle 14,15\rangle, \langle 15,14\rangle\}$, $\mathrm{s}(18) = \{\langle 16,17\rangle, \langle 17,16\rangle\}$ and $\ell^{\mathrm{F}}(4) = \ell^{\mathrm{F}}(5)$, $\ell^{\mathrm{F}}(8) = \ell^{\mathrm{F}}(9) = \ell^{\mathrm{F}}(10)$, $\ell^{\mathrm{F}}(13) = \ell^{\mathrm{F}}(14)$.

Then $\partial = \partial^{\mathrm{C}}_{(0)} = \partial^{\mathrm{C}}_{(1)} \hookrightarrow \partial^{\mathrm{C}}_{(2)} = \left\langle \widetilde{D}_2, G_2 \right\rangle$, where

$\widetilde{D}_2 = \boxed{\dfrac{\dfrac{\dfrac{\dfrac{1}{7}}{12} \quad \dfrac{\dfrac{2 \quad 3}{8} \quad \dfrac{4}{9}}{\boxed{y}} \quad \dfrac{\dfrac{5}{10} \quad \dfrac{6}{11}}{15}}{16 \quad 17}}{18}}$ with

$\mathrm{s}(y) = \{8,9\}$, $\mathrm{s}(16) = \{\langle 12,y\rangle, \langle y,12\rangle\}$, $\mathrm{s}(17) = \{\langle y,15\rangle, \langle 15,y\rangle\}$ and $\ell^{\mathrm{F}}(y) = \ell^{\mathrm{F}}(13) = \ell^{\mathrm{F}}(14)$, while $G_2(\overrightarrow{e_2}) = 1 \Leftrightarrow (e_1 \in \{\langle 8,2\rangle, \langle 8,3\rangle\} \wedge e_2 = \langle 16,y\rangle) \vee (e_1 = \langle 9,4\rangle \wedge e_2 = \langle 17,y\rangle)$.

Furthermore $\partial^{\mathrm{C}}_{(2)} \hookrightarrow \partial^{\mathrm{C}}_{(3)} = \left\langle \widetilde{D}_3, G_3 \right\rangle$, where

$$\widetilde{D}_3 = \boxed{\begin{array}{c} \begin{array}{ccc} \dfrac{\dfrac{1}{7}}{12} & \quad & \dfrac{\dfrac{2 \quad 3 \quad 4 \quad 5}{\boxed{x}} \qquad \dfrac{6}{11}}{\boxed{y} \qquad\qquad 15} \end{array} \\ \hline \dfrac{16 \qquad\qquad 17}{18} \end{array}} \text{ with}$$

$\mathrm{s}(x) = \{\langle 2,3\rangle, \langle 3,2\rangle, 4, 5\}$, $\mathrm{s}(15) = \{\langle 11,x\rangle, \langle x,11\rangle\}$ and $\ell^{\mathrm{F}}(x) = \ell^{\mathrm{F}}(8) = \ell^{\mathrm{F}}(9) = \ell^{\mathrm{F}}(10)$, while $G_3(\overrightarrow{e_2}) = 1 \Leftrightarrow (e_1 \in \{\langle x,2\rangle, \langle x,3\rangle, \langle x,4\rangle\} \wedge e_2 = \langle y,x\rangle) \vee (e_1 = \langle x,5\rangle \wedge e_2 = \langle 15,x\rangle)$ and $G_3(\overrightarrow{e_3}) = 1 \Leftrightarrow (e_1 = \langle x,4\rangle \wedge e_2 = \langle y,x\rangle \wedge e_3 = \langle 17,y\rangle) \vee (e_1 \in \{\langle x,2\rangle, \langle x,3\rangle\} \wedge e_2 = \langle y,x\rangle \wedge e_3 = \langle 16,y\rangle)$.

Finally $\partial^{\mathrm{C}}_{(3)} \hookrightarrow \partial^{\mathrm{C}}_{(4)} = \left\langle \widetilde{D}_4, G_4 \right\rangle = \partial^{\mathrm{C}}$, where

$$\widetilde{D}_4 = \boxed{\begin{array}{c} \begin{array}{ccc} \dfrac{\dfrac{1}{5}}{12} & \quad & \dfrac{\dfrac{2 \quad 3 \quad u}{\boxed{x}} \qquad \dfrac{6}{11}}{\boxed{y} \qquad\qquad 15} \end{array} \\ \hline \dfrac{16 \qquad\qquad 17}{18} \end{array}} \text{ with}$$

$\mathrm{s}(x) = \{\langle 2,3\rangle, \langle 3,2\rangle, u\}$ and $\ell^{\mathrm{F}}(u) = \ell^{\mathrm{F}}(4) = \ell^{\mathrm{F}}(5)$, while $G_4(\overrightarrow{e_2}) = 1 \Leftrightarrow (e_1 \in \{\langle x,2\rangle, \langle x,3\rangle, \langle x,u\rangle\} \wedge e_2 = \langle y,x\rangle) \vee (e_1 = \langle x,u\rangle \wedge e_2 = \langle 15,x\rangle)$ and $G_4(\overrightarrow{e_3}) = 1 \Leftrightarrow (e_1 \in \{\langle x,2\rangle, \langle x,3\rangle\} \wedge e_2 = \langle y,x\rangle \wedge e_3 = \langle 16,y\rangle) \vee (e_1 = \langle x,u\rangle \wedge e_2 = \langle y,x\rangle \wedge e_3 = \langle 17,y\rangle)$.

The latter yields

$$\mathrm{T}(\partial^{\mathrm{C}}) = \left\{ \begin{array}{c} [18,16,12,5,1], [18,16,y,x,2], [18,16,y,x,3], \\ [18,17,y,x,u], [18,17,15,x,u], [18,17,15,11,6] \end{array} \right\}.$$

Note that ∂^{C} is isomorphic to dag-like deduction ∂ shown in Example 7.

THEOREM 14. *For any tree-like deduction $\partial \in \mathcal{T}^*$ with root-formula ρ, ∂^{C} is a dag-like* $\mathrm{NM}_{\rightarrow}$ *deduction of ρ from the same assumptions $\Gamma_{\partial^{\mathrm{C}}} = \Gamma_{\partial}$. Moreover $|\partial^{\mathrm{C}}| \leq h(\partial) \times \phi(\partial)$ and $\mu(\partial^{\mathrm{C}}) = \mu(\partial)$. In particular, if $\Gamma_{\partial} = \emptyset$ and $h(\partial)$, $\phi(\partial)$ are polynomial in $|\rho|$, then ∂^{C} is a dag-like* $\mathrm{NM}_{\rightarrow}$ *proof of ρ whose size is polynomial in $|\rho|$.*

PROOF. Let $\partial = \widetilde{T} = \langle T, \mathrm{s}, \ell^{\mathrm{F}}\rangle \in \mathcal{T}^*$ and $\partial_n^{\mathrm{C}} = \left\langle \widetilde{D}_n, G_n \right\rangle$, $\widetilde{D}_n = \langle D_n, \mathrm{s}_n, \ell_n^{\mathrm{F}}\rangle$, for $n \leq h(D)$. By Lemma 11 we have

$$\begin{aligned} |\partial^{\mathrm{C}}| &= \sum_{n=0}^{h(T)} |L_n(D_n)| \leq 1 + 2 + \sum_{n=2}^{h(T)} |L_n(D_n)| \\ &\leq 3 + (h(T) - 1) \cdot \phi(\partial) < h(T) \cdot \phi(\partial) = h(\partial) \times \phi(\partial) \end{aligned}$$

as required. The rest follows from Lemma 11 by induction on $n \leq h(T)$. ■

Together with Theorem 4 and Lemma 5 this yields

COROLLARY 15. *Any minimal tautology ρ has a dag-like* $\mathrm{NM}_{\rightarrow}$ *proof* ∂^{C} *whose size is polynomial in* $|\rho|$. *Actually the following holds.*

$$\boxed{|\partial^{\mathrm{C}}| < 18\,|\rho|\,(|\rho|+1)^2\,(|\rho|+2) = \mathcal{O}\left(|\rho|^4\right) \textit{ and } \mu\left(\partial^{\mathrm{C}}\right) \leq 2\,|\rho|}$$

4. Dag-to-Tree Unfolding

We learned that all minimal propositional tautologies are provable by dag-like $\mathrm{NM}_{\rightarrow}$ deductions of "small" size, but at the moment we don't know whether underlying dag-like provability infers validity in minimal logic. The affirmative answer follows by dag-to-tree unfolding, to be thought of as inversion of the tree-to-dag compression under consideration. The unfolded tree-like deduction $\partial^{\mathrm{U}} = \widetilde{T}$ is defined by descending recursion on the height of a given dag-like deduction $\partial = \left\langle \widetilde{D}, G \right\rangle$ such that for any $n \leq h(\partial)$, the n^{th} horizontal section of $\widetilde{T}$ is obtained by splitting previously obtained nodes v, $h(v) = n$, having p parents, $u_1, \cdots u_p$, $p > 1$, into p new copies $v_1, \cdots v_p$. Previously obtained (tree-like!) successors of v are separated according to the underlying assignment G such that for every $1 \leq i < p$, u_i becomes the only parent of v_i. Except for the G-related separation this is just standard graph theoretic dag-to-tree unfolding (see below a precise definition).

DEFINITION 16. Consider any $\partial = \left\langle \widetilde{D}, G \right\rangle \in \mathcal{D}_n^{\star}$, $\widetilde{D} = \langle D, \mathrm{S}, \ell^{\mathrm{F}} \rangle$, $n \leq h := h(D)$, and a fixed $r \in L_n(D)$ with $p := |\mathrm{P}(r, D)| > 1$. Let $\varepsilon : [p] \rightarrow \mathrm{P}(r, D)$ be a fixed 1–1 enumeration of $\mathrm{P}(r, D)$. We define (n, r)-unfolded deduction $\partial_{n,r}^{\mathrm{U}} = \left\langle \widetilde{D}_{n,r}, G_{n,r} \right\rangle$, $\widetilde{D}_{n,r} = \left\langle D_{n,r}, \mathrm{S}_{n,r}, \ell_{n,r}^{\mathrm{F}} \right\rangle$, that arises by tree-like unfolding of r, as follows. Let R be a fixed set of new vertices $r_1, \cdots, r_p \notin \mathrm{V}(D)$ and $(D)_{r_1}, \cdots, (D)_{r_p}$ a collection of pairwise disjoint (tree-like!) copies of $(D)_r$. Then for any $1 \leq i \leq p$ we denote by $(D)_i^-$ a subtree of $(D)_{r_i}$ whose top edges are copies of top edges $e_1 \in \mathrm{E}(D)$ for which there are (uniquely determined) chains $\overrightarrow{e_k} \in \overrightarrow{\mathrm{E}}(D)$, $k = h - n + 1$, with $G\left(\overrightarrow{e}_k, \langle \varepsilon(i), r \rangle\right) = 1$ (thus $\varrho\left((D)_i^-\right) = r_i$). Having this we stipulate:

1. $D_{n,r}$ arises from D by substituting $(D)_i^-$ for $(D)_r$, for every $1 \leq i \leq p$. That is,

$\mathrm{V}(D_{n,r}) :=\ (\mathrm{V}(D) \setminus \mathrm{V}((D)_r)) \cup \bigcup_{i=1}^{p} \mathrm{V}\left((D)_i^-\right)$. The edges are given by $\mathrm{E}(D_{n,r}) := (\mathrm{E}(D) \setminus \mathrm{E}((D)_r)) \cup \bigcup_{i=1}^{p} \left(\mathrm{E}\left((D)_i^-\right) \cup \langle \varepsilon(i), r_i\rangle\right)$.

2. For any $u \in \mathrm{V}(D_{n,r})$ we define $\mathrm{S}_{n,r}(u, D_{n,r})$ by cases as follows.
   (a) If $u \notin \bigcup_{i=1}^{p} \mathrm{V}\left((D)_i^-\right) \cup \mathrm{P}(r, D)$, then $\mathrm{S}_{n,r}(u, D_{n,r}) := \mathrm{S}(u, D)$.
   (b) If $u \in \bigcup_{i=1}^{p} \mathrm{V}\left((D)_i^-\right)$, then $\mathrm{S}_{n,r}(u, D_{n,r}) := \mathrm{S}(u, D)$ (modulo isomorphism).
   (c) For any $1 \le i \le p$ we let $\mathrm{S}_{n,r}(\varepsilon(i), D_{n,r}) := X_i \cup Y_i$, where $X_i = \left\{ y \in L_n(D_{n,r}) : \begin{array}{c} (\exists x \in \mathrm{S}(\varepsilon(i), D)) \\ (x = y \vee (x = r \wedge y = r_i)) \end{array} \right\}$ and $Y_i = \left\{ \langle y_0, y_1\rangle \in L_n(D_{n,r})^2 : \begin{array}{c} (\exists \langle x_0, x_1\rangle \in \mathrm{S}(\varepsilon(i), D))(\forall j \le 1) \\ (x_j = y_j \vee (x_j = r \wedge y_j = r_i)) \end{array} \right\}$.
3. For any $u \in \mathrm{V}(D_{n,r})$ we let $\ell_{n,r}^{\mathrm{F}}(u) := \ell^{\mathrm{F}}(\widehat{u})$, where $\widehat{u} \in \mathrm{V}(D)$ is a preimage of u in D (thus $\widehat{u} = u$ iff $u \notin \bigcup_{i=1}^{p} \mathrm{V}\left((D)_i^-\right)$).

This completes $\widetilde{D}_{n,r}$. To stipulate $G_{n,r}$ let $\overrightarrow{e_k} = e_1, \cdots, e_k \in \overrightarrow{\mathrm{E}}(D_{n,r})$, $1 < k$, and define $G_{n,r}(\overrightarrow{e_k})$ by cases as follows, where $\widehat{e}_1 = \langle \widehat{u}_1, \widehat{v}_1\rangle \in \mathrm{E}(D)$ is the (uniquely determined) preimage of e_1 in D (thus $\widehat{e}_1 = e_1$ iff $v_1 \notin \bigcup_{i=1}^{p} \mathrm{V}\left((D)_i^-\right)$), while $u_2 \in L_j(D_{n,r})$, $v_2 \in L_{j+1}(D_{n,r})$, $j + 1 < h = h(D) = h(D_{n,r})$. Note that $j + 1 < n$, as $\deg_{\downarrow}(v_2, D_{n,r}) > 1$.

1. Suppose $n = h$ and $v_1 \notin R$. Then let $G_{n,r}(\overrightarrow{e_k}) := G(\overrightarrow{e_k})$.
2. Suppose $n = h$ and $v_1 \in R$. Then let $G_{n,r}(\overrightarrow{e_k}) := G\left(\overrightarrow{e'_k}\right)$, where $e'_1 = \langle u_1, r\rangle, (\forall \iota \ne 1)\, e'_\iota = e_\iota$.
3. Suppose $n < h$ and $e_1 \in \mathrm{E}(D)$. Then let $G_{n,r}(\overrightarrow{e_k}) := G(\overrightarrow{e_k})$.
4. Suppose $n < h$ and $\widehat{e}_1 \ne e_1 \in \mathrm{E}\left((D)_i^-\right)$. Then let $G_{n,r}(\overrightarrow{e_k}) := G\left(\overrightarrow{e''_{k+1}}\right)$, where $e''_1 = \widehat{e}_1, e''_2 = \langle \varepsilon(i), r\rangle, (\forall \iota > 2)\, e''_\iota = e_{\iota-1}$.

This completes our definition of $\partial_{n,r}^{\mathrm{U}}$ under the assumption $|\mathrm{P}(r, D)| > 1$.

To complete the (n, r)-*unfolding operation* $\partial \hookrightarrow \partial_{n,r}^{\mathrm{U}}$, we let $\partial_{n,r}^{\mathrm{U}} := \partial$ in the case $|\mathrm{P}(r, D)| = 1$. Now let ∂_n^{U} arise from ∂ by applying (n, r)-unfolding successively to all $r \in L_n(D)$. That is, ∂_n^{U} is the iteration of $\partial_{n,r}^{\mathrm{U}}$ with respect to all nodes r occurring in the n^{th} horizontal section of D. The operation $\partial \hookrightarrow \partial^{\mathrm{U}}_{n,r}$ is called the *horizontal unfolding on level* n, in $\mathrm{NM}^{\star}_{\rightarrow}$.

LEMMA 17. *For any* $\partial = \left\langle \widetilde{D}, G \right\rangle \in \mathcal{D}^{\star}_{n}$, $\widetilde{D} = \langle D, \mathrm{S}, \ell^{\mathrm{F}} \rangle$, $\partial^{\mathrm{U}}_{n} = \left\langle \widetilde{D}_n, G_n \right\rangle$, $\widetilde{D}_n = \langle D_n, \mathrm{S}_n, \ell^{\mathrm{F}}_n \rangle$, $n \le h(D)$, *the following conditions 1–5 hold.*

1. $\partial^{\mathrm{U}}_0 = \partial$ *and* $\partial^{\mathrm{U}}_n \in \mathcal{D}^{\star}_{n-1}$ *for* $n > 0$.
2. $\varrho(D_n) = \varrho(D)$ *and* $h(D_n) = h(D)$.
3. *For any* $i < n$, $L_i(D_n) = L_i(D)$, *while* $L_n(D_n) \supseteq L_n(D)$.
4. *For any* $i \le n < j$, $\ell^{\mathrm{F}}(L_i(D_n)) = \ell^{\mathrm{F}}(L_i(D))$ *and* $\ell^{\mathrm{F}}(L_j(D_n)) \subseteq \ell^{\mathrm{F}}(L_j(D))$. *Hence* $\phi(\partial^{\mathrm{U}}_n) \subseteq \phi(\partial)$.
5. $\Gamma_{\partial^{\mathrm{U}}_n} = \Gamma_{\partial}$.

PROOF. By iteration, it will suffice to prove analogous assertions with respect to every (n, r)-unfolding involved. We skip trivial conditions 2–4 and verify 1, while reducing 1 to sufficient weakening $\partial^{\mathrm{U}}_{n,r} \in \mathcal{D}^{\star}_{n}$. First of all we observe that every subtree $(D)^{-}_{i}$ represents a (tree-like) $\mathrm{NM}^{*}_{\rightarrow}$ deduction of $\ell^{\mathrm{F}}(r_i) = \ell^{\mathrm{F}}(r)$ such that $h\left((D)^{-}_{i}\right) = h((D)_r)$. This follows from local coherence of ∂ (see Definition 6). Actually it suffices to verify that $(D)^{-}_{i}$ is not empty and locally correct. That $(D)^{-}_{i} \neq \emptyset$ obviously follows from condition 5 (for $k = 1$) of local coherence. To show that $(D)^{-}_{i}$ is locally correct suppose that $r_i \preceq v \preceq u_1$ in $(D)_{r_i}$ and $e_1 = \langle u_1, v_1 \rangle$ ($\widehat{e}_1 = \langle \widehat{u}_1, \widehat{v}_1 \rangle$) is a top edge in $(D)_{r_i}$ (resp. $(D)_r$) such that $G(\widehat{e}_1, \langle \varepsilon(i), r \rangle) = 1$. So v and u_1 are both in $(D)^{-}_{i}$. Clearly so is the (only) parent of $v \neq r_i$, while condition 4 (for $m = 1$) of local coherence shows that the same holds true for all adjacent pairs of children of v in $(D)_{r_i}$. The rest of local correctness is easily inherited from that of $(D)_r$. Thus $(D)^{-}_{i}$ is locally correct, and hence so is $\widetilde{D}_{n,r}$.

To complete the proof of $\partial^{\mathrm{U}}_{n,r} \in \mathcal{D}^{\star}_{n}$ consider nontrivial conditions 4, 5 of corresponding local coherence.

4: Suppose $G_{n,r}(\overrightarrow{e_k}) = 1$, $\overrightarrow{e_k} = \overrightarrow{e_m} * \overrightarrow{e}$, $v_{m+1} \preceq_{D_{n,r}} v \preceq_{D_{n,r}} u_m$, $z \preceq_{D_{n,r}} v_m$ and $\langle z, z' \rangle \in \mathrm{S}(v, D_{n,r})$, where $n < h$ (cf. Definition 16; other cases are trivial). We look for a $\overrightarrow{e'_t} = \overrightarrow{e'_s} * \overrightarrow{e} \in \overrightarrow{\mathrm{E}}(D_{n,r})$ with $v'_{s+1} \preceq_{D_{n,r}} v \preceq_{D_{n,r}} u'_s$, $z' \preceq_{D_{n,r}} v'_s$ and $G\left(\overrightarrow{e'_t}\right) = 1$. If $m = 1$ then let $\overrightarrow{e'_t} := \overrightarrow{e'_1} * \overrightarrow{e}$, $s = 1$, where $e'_1 = \langle u'_1, v'_1 \rangle$ is a top edge in $D_{n,r}$ such that $z' \preceq_{D_{n,r}} v'_1$. This yields $G_{n,r}\left(\overrightarrow{e'_t}\right) = 1$ by local coherence of ∂. If $m > 1$ then $v \in \mathrm{V}(D)$. Furthermore, if $e_1 \in \mathrm{E}(D)$ then $\overrightarrow{e_k} \in \overrightarrow{\mathrm{E}}(D)$ and $G_{n,r}(\overrightarrow{e_k}) = G(\overrightarrow{e_k})$. Moreover, by local coherence of ∂, there exists $\overrightarrow{e''_t} = \overrightarrow{e''_s} * \overrightarrow{e''} \in \overrightarrow{\mathrm{E}}(D)$ with $\overrightarrow{e''} = \overrightarrow{e}$, $v''_{s+1} \preceq_D v \preceq_D u''_s$, $z' \preceq_D v''_s$

and $G\left(\overrightarrow{e''_t}\right) = 1$. Now if $r \npreceq_D u''_1$ then $\overrightarrow{e''_t} \in \overrightarrow{\mathrm{E}}(D_{n,r})$ and we are done by $\overrightarrow{e'_t} := \overrightarrow{e''_t}$. Otherwise, $s > 2$ and there exists $i \in [p]$ such that $e''_2 = \langle \varepsilon(i), r\rangle$ and $e''_1 = \widehat{e}_0$ for some $e_0 \in \mathrm{E}\Big(\left(D\right)_i^-\Big)$. Then let $\overrightarrow{e'_t} := \overrightarrow{e'_{s-1}} * \overrightarrow{e} \in \overrightarrow{\mathrm{E}}(D_{n,r})$ where $e'_1 = e_0, e'_2 = e''_3, \cdots, e'_{s-1} = e''_s$. Finally suppose $e_1 \in \mathrm{E}\Big(\left(D\right)_i^-\Big)$. So by Definition 16 we have $G_{n,r}(\overrightarrow{e_k}) = G\left(\overrightarrow{e''_{k+1}}\right) = 1$ for $e''_1 = \widehat{e}_1, e''_2 = \langle \varepsilon(i), r\rangle, (\forall \iota > 2)\, e''_\iota = e_{\iota-1}$, while $\overrightarrow{e''_{k+1}} = \overrightarrow{e''_{m+1}} * \overrightarrow{e''} \in \overrightarrow{\mathrm{E}}(D)$ with $\overrightarrow{e''} = \overrightarrow{e}$, $v_{m+2} \preceq_D v \preceq_D u_{m+1}$ and $z \preceq_D v_{m+1}$. Hence by local coherence of ∂, there exists $\overrightarrow{e'''_t} = \overrightarrow{e'''_s} * \overrightarrow{e'''} \in \overrightarrow{\mathrm{E}}(D)$ with $\overrightarrow{e'''} = \overrightarrow{e''} = \overrightarrow{e}$, $v'''_{s+1} \preceq_D v \preceq_D u'''_s$, $z' \preceq_D v'''_s$ and $G\left(\overrightarrow{e'''_t}\right) = 1$. The rest follows the previous pattern. That is, if $r \npreceq_D u'''_1$ then $\overrightarrow{e''_t} \in \overrightarrow{\mathrm{E}}(D_{n,r})$ and we let $\overrightarrow{e'_t} := \overrightarrow{e'''_t}$. Otherwise, $s > 2$ and there exists $i \in [p]$ such that $e'''_2 = \langle \varepsilon(i), r\rangle$ and $e'''_1 = \widehat{e}_0$ for some $e_0 \in \mathrm{E}\Big(\left(D\right)_i^-\Big)$. Then let $\overrightarrow{e'_t} := \overrightarrow{e'''_{s-1}} * \overrightarrow{e} \in \overrightarrow{\mathrm{E}}(D_{n,r})$ where $e'_1 = e_0, e'_2 = e'''_3, \cdots, e_{s-1} = e'''_s$.

5: Suppose $y \in \mathrm{V}(D_{n,r})$ and $x \in \mathrm{P}(y, D_{n,r})$ where $\deg_\uparrow(y, D_{n,r}) > 0$ and $\deg_\downarrow(y, D_{n,r}) > 1$ (hence $y \in \mathrm{V}(D)$). If $(\forall i \in [p])\, y \nprec_D \varepsilon(i)$ then we are done by the assumption $\partial \in \mathcal{D}^\star$. Otherwise $y \prec_D \varepsilon(i)$ for some $i \in [p]$, while by the assumption $\partial \in \mathcal{D}^\star$, there exists $\overrightarrow{a_l} \in \overrightarrow{\mathrm{E}}(D)$ such that $G(\overrightarrow{a_l}, \langle x, y\rangle) = 1$. Moreover $\overrightarrow{a_l}$ determines a $\overrightarrow{e_k} \in \overrightarrow{\mathrm{E}}(D_{n,r})$ such that $G_{n,r}(\overrightarrow{e_k}, \langle x, y\rangle) = G(\overrightarrow{a_l}, \langle x, y\rangle) = 1$ and either $\overrightarrow{a_l} = \overrightarrow{e_l}$ or $\overrightarrow{a_l} = \overrightarrow{e'_k}$ or $\overrightarrow{a_l} = \overrightarrow{e''_{k+1}}$ as in the definition of $G_{n,r}$ (see Definition 16). This completes the proof of $\partial^{\mathrm{U}}_{n,r} \in \mathcal{D}^\star_n$, which by iteration yields $\partial^{\mathrm{U}}_n \in \mathcal{D}^\star_{n-1}$ for $n > 0$, as required.

Now consider the last condition 5 of the lemma (with respect to every (n,r)-unfolding involved). To prove the inclusion $\Gamma_{\partial^{\mathrm{U}}_{n,r}} \subseteq \Gamma_\partial$, it will suffice to show that there is an assumption-preserving embedding of open threads in $\partial^{\mathrm{U}}_{n,r}$ into the open threads in ∂. So let $\Theta_{n,r} = [\varrho = x_0, \cdots, x_h] \in \mathrm{T}\big(\partial^{\mathrm{U}}_{n,r}\big)$ be an open thread in $\partial^{\mathrm{U}}_{n,r}$ together with sequence $0 < f_{n,r}(1) < \cdots < f_{n,r}(m) = h$ and chain $\overrightarrow{e_m}$ for $(\forall \iota \in [1,m])\, e_\iota = \big\langle x_{f_{n,r}(m-\iota+1)-1}, x_{f_{n,r}(m-\iota+1)}\big\rangle$ (cf. Definition 8). A desired open thread in ∂, $\Theta = [\varrho = y_0, \cdots, y_h] \in \mathrm{T}(\partial)$, and correlated sequence $0 < f(0) < \cdots < f(l) = h$ are defined by cases as follows, where $r \in L_n(D)$, $r_i \in L_n\big(D^{\mathrm{U}}_{n,r}\big)$, $n \le h$, $1 \le i \le p$.

1. Suppose $n = h$ and $x_n \notin R$. Then let $\Theta := \Theta_{n,r}$, i.e. $(\forall \iota \le h)\, y_\iota := x_\iota$. Moreover let $l := m$ and $f := f_{n,r}$. Hence $G_{n,r}(\overrightarrow{e_m}) = G(\overrightarrow{e_m})$.

2. Suppose $n = h$ and $x_n = r_i$. Then $G_{n,r}(\overrightarrow{e_k}) = G\left(\overrightarrow{e'_k}\right)$, where $e'_1 = \langle u_1, r\rangle, (\forall \iota \neq 1)\, e'_\iota = e_\iota$. So let $y_n = y_h := r, (\forall \iota < h)\, y_\iota := x_\iota$ together with $l := m$ and $f := f_{n,r}$.

3. Suppose $n < h$ and $x_n \notin \bigcup_{i=1}^{p} \mathrm{V}\left((D)_i^-\right)$. Then let $\Theta := \Theta_{n,r}$ together with $l := m$ and $f := f_{n,r}$. Hence $G_{n,r}(\overrightarrow{e_m}) = G(\overrightarrow{e_m})$.

4. Suppose $n < h$ and $x_n \in \mathrm{V}\left((D)_i^-\right)$. Then $G_{n,r}(\overrightarrow{e_k}) = G\left(\overrightarrow{e''_{k+1}}\right)$, where $e''_1 = \widehat{e}_1, e''_2 = \langle \varepsilon(i), r\rangle, (\forall \iota > 2)\, e''_\iota = e_{\iota-1}$. So let $(\forall \iota > n)\, y_\iota := \widehat{x}_\iota, y_n := r, (\forall \jmath < r)\, y_\jmath := x_\jmath$, where $\widehat{x}_\iota$ is the preimage of x_ι in D. Also let $l := m+1$ and $(\forall \iota < m)\, f(\iota) := f_{n,r}(\iota), f(m) := n, f(m+1) := h$.

This completes Θ and f. It remains to prove that it is an open thread in ∂. Obviously $\Theta_{n,r} \hookrightarrow \Theta$ preserves formulas. Thus $\Theta \in \mathrm{K}(D)$ with $\Theta_{\mathrm{S}} = y_0 = \varrho$ and $\Theta_{\mathrm{T}} = y_h \in \mathrm{L}(D)$, while $(\forall \iota \leq h)\, \ell^{\mathrm{F}}_{n,r}(x_\iota) = \ell^{\mathrm{F}}(y_\iota)$. Moreover, for any $\iota < h$, $\deg_\downarrow(x_\iota) > 1$ iff $\iota \in \mathit{Rng}(f)$. To complete the proof we observe that $G_{n,r}(\overrightarrow{e_m}) = 1$ implies $G(\overrightarrow{e_m}) = 1$, $G\left(\overrightarrow{e'_m}\right) = 1$ and/or $G\left(\overrightarrow{e''_{m-1}}\right) = 1$ in every case 1–4 under consideration. Hence $\Gamma_{\partial^{\mathrm{U}}_{n,r}} \subseteq \Gamma_\partial$. $\Gamma_\partial \subseteq \Gamma_{\partial^{\mathrm{U}}_{n,r}}$ is proved analogously by inversion $\Theta \hookrightarrow \Theta_{n,r}$. This completes the whole proof. ■

Definition 18. (horizontal unfolding). For any given $\partial \in \mathcal{D}^\star$ denote by $\partial^{\mathrm{U}} \in \mathcal{T}^*$ the last deduction in the following iteration chain

$$\partial = \partial^{\mathrm{U}}_{(0)}, \partial^{\mathrm{U}}_{(1)} \cdots, \partial^{\mathrm{U}}_{(h(\partial))} = \partial^{\mathrm{U}}$$

where for every $i < h(\partial)$ we let $\partial^{\mathrm{U}}_{(i+1)} := \left(\partial^{\mathrm{U}}_{(i)}\right)^{\mathrm{U}}_{i+1}$. It is readily seen that all ∂^{U} in question are mutually isomorphic (actually equal up to the enumerations ε). The operation $\partial \hookrightarrow \partial^{\mathrm{U}}$ is called the *horizontal unfolding*, in $\mathrm{NM}^\star_{\rightarrow}$.

Theorem 19. *For any dag-like* $\mathrm{NM}_{\rightarrow}$ *deduction ∂ with root-formula ρ, ∂^{U} is a tree-like* $\mathrm{NM}^*_{\rightarrow}$ *deduction of ρ such that $\Gamma_{\partial^{\mathrm{U}}} = \Gamma_\partial$. In particular, if ∂ is a dag-like* $\mathrm{NM}_{\rightarrow}$ *proof of ρ, then ∂^{U} is a tree-like* $\mathrm{NM}^*_{\rightarrow}$ *proof of ρ.*

Proof. The assertions follow by iteration from Lemma 17, as $\Gamma_{\partial^{\mathrm{U}}} \subseteq \Gamma_\partial = \emptyset$ obviously implies $\Gamma_{\partial^{\mathrm{U}}} = \emptyset$. ■

Together with Lemma 5 the latter assertion yields

Corollary 20. *Dag-like* $\mathrm{NM}_{\rightarrow}$ *provability is sound and complete with respect to minimal propositional logic.*

Together with Corollary 15 this yields

Corollary 21. *A given formula ρ is valid in minimal propositional logic iff there exists a dag-like* $\mathrm{NM}_{\rightarrow}$ *proof of ρ whose size is $\mathcal{O}\left(|\rho|^4\right)$. I.e.:* $\mathrm{M}_{\rightarrow} \vdash \rho \Longleftrightarrow \left(\exists \widetilde{D}\right)_{|\widetilde{D}|=\mathcal{O}\left(|\rho|^4\right)} (\exists G) \left\langle \widetilde{D}, G \right\rangle \vdash \rho$.

5. Complexity of Verifications

Consider the right-hand side of the last equivalence

$$\boxed{\mathrm{M}_{\rightarrow} \vdash \rho \Longleftrightarrow \left(\exists \widetilde{D}\right)_{|\widetilde{D}|=\mathcal{O}\left(|\rho|^4\right)} \left(\exists G : \overrightarrow{\mathrm{E}}(D) \rightarrow \{0,1\}\right) \left\langle \widetilde{D}, G \right\rangle \vdash \rho}$$

Problem 22. *Let* $\widetilde{D} = \langle D, \mathrm{S}, \ell^{\mathrm{F}} \rangle$, $\left|\widetilde{D}\right| = \mathcal{O}\left(|\rho|^4\right)$, $\rho = \ell^{\mathrm{F}}(\varrho(D))$. *Suppose there exists a* $G : \overrightarrow{\mathrm{E}}(D) \rightarrow \{0,1\}$ *such that* $\left\langle \widetilde{D}, G \right\rangle \vdash \rho$. *What is the complexity of the corresponding* $\left(\exists G : \overrightarrow{\mathrm{E}}(D) \rightarrow \{0,1\}\right) \left\langle \widetilde{D}, G \right\rangle \vdash \rho$ *(abbr.:* $\mathrm{PROOF}(\widetilde{D})$ *)?*

To begin with let $\mathrm{FRAME}(\widetilde{D})$ denote a weaker assertion $\widetilde{D} \in \mathcal{R}^{\star}$.

Lemma 23. $\mathrm{FRAME}(\widetilde{D})$ *is in* **P**.

Proof. We have to show that local correctness of $\widetilde{D}$ is verifiable by a deterministic TM in $\left|\widetilde{D}\right|$-polynomial (i.e. $|D|$-polynomial) time. But this is obvious by standard polysize encoding of the underlying parameters. Thus in particular, if $|D|$ is polynomial in $|\rho|$ (like in the case of Corollary 21), then $\mathrm{FRAME}(\widetilde{D})$ is verifiable by a deterministic TM in $|\rho|$-polynomial time. ∎

It is not clear, however, whether there is any polynomial encoding of an assignment G that is defined for all chains of edges $\overrightarrow{e_k}$ in question. Consequently, it is unclear whether local coherence of G and the rest of proposition $\left\langle \widetilde{D}, G \right\rangle \vdash \rho$ is also verifiable by a deterministic TM in $|D|$ -polynomial time. So at the moment the complexity of $\mathrm{PROOF}(\widetilde{D})$ remains unclear.[14]

[14] In some cases, however, $\mathrm{PROOF}(\widetilde{D})$ easily follows from $\mathrm{FRAME}(\widetilde{D})$(see Appendix B).

Nevertheless $(\exists G)\left\langle \widetilde{D}, G\right\rangle \vdash \rho$ permits a convergent sequence of lower NP approximations of $\mathrm{M}_{\rightarrow} \vdash \rho$ by reducing the domain of G. Namely we replace $\overrightarrow{\mathrm{E}}(D)$ by polynomial approximations $\overrightarrow{\mathrm{E}_q}(D) := \left\{\overrightarrow{e}_k \in \overrightarrow{\mathrm{E}}(D) : 0 < k \leq q\right\}$, for every fixed $0 < q \leq h_{\downarrow}(D)$, where $h_{\downarrow}(D)$ is maximum number of downward-branching nodes occurring in a path of D, and consider NP-approximations $\mathrm{PROOF}_q(\widetilde{D})$ with respect to $G_{\leq q} : \overrightarrow{\mathrm{E}_q}(D) \rightarrow \{0,1\}$, instead of G. That is, instead of G we consider boolean assignments $G_{\leq q}$ that are defined for all $\overrightarrow{e_k} = \langle u_1, v_1\rangle, \cdots, \langle u_k, v_k\rangle$, $0 < k \leq q$, such that for every $0 < i < k$, u_{i+1} is a chosen "legitimate" parent of some downward-branching node v_{i+1} that occurs below or coincides with u_i. The rest of $G_{\leq q}$ and $\langle D, G_{\leq q}\rangle \vdash \rho$ is specified accordingly, where $G_{\leq q}$-deduction threads are maximal paths of nodes Θ such that $G_{\leq q}(\overrightarrow{e_k}) = 1$ for every $\overrightarrow{e_k} \in \overrightarrow{\mathrm{E}_q}(D)$ with $u_i, v_i \in \Theta$ for all $0 < i < k$. Note that in the general case of $0 < q < h_{\downarrow}(D)$, $G_{\leq q}$ might allow more threads than G does. Now for any fixed q, $G_{\leq q}$ has an obvious $|D|$-polysize encoding. Using this observation, by previous considerations we arrive at

Claim 24. *For any deduction frame D with root formula ρ the following holds, where* $\mathrm{PROOF}_q(\widetilde{D})$ *abbreviates* $\left(\exists G_{\leq q} : \overrightarrow{\mathrm{E}_q}(D) \rightarrow \{0,1\}\right)\left\langle \widetilde{D}, G_{\leq q}\right\rangle \vdash \rho$.

1. $\mathrm{PROOF}_q(\widetilde{D}) \Longleftrightarrow \mathrm{PROOF}_{h_{\downarrow}(D)}(\widetilde{D})$ *and for any* $q \leq h_{\downarrow}(D)$, $\left\langle \widetilde{D}, G_{\leq q}\right\rangle \vdash \rho$ *implies* $\mathrm{M}_{\rightarrow} \vdash \rho$.
2. *For any fixed q, validity of* $\mathrm{PROOF}_q(\widetilde{D})$ *is verifiable by a TM, non-deterministically, in $|\rho|$-polynomial time, provided that the weight of $\widetilde{D}$ is polynomial in $|\rho|$.*
3. *For any fixed q, validity of* $\left(\exists \widetilde{D}\right)\mathrm{PROOF}_q(\widetilde{D})$ *is verifiable by a non-deterministic TM in $|\rho|$-polynomial time. Thus in order to show that ρ is valid in the minimal logic it would suffice to confirm* $\left(\exists \widetilde{D}\right)\mathrm{PROOF}_q(\widetilde{D})$ *by a non-deterministic TM in $|\rho|$ -polynomial time, for any chosen fixed q. The bigger q the better chance for success (provided that ρ is valid indeed).*

The precise definitions and proof of this claim will appear elsewhere and are thus out of scope of this paper.

Appendix A: Proof of Lemma 2 (4)

A required loose upper bound $\text{ssf}(\xi) \leq (|\xi| + 1)^2$ is proved by induction on $|\xi|$, as follows. Recall the recursive clauses 1–3:

1. $\text{ssf}(p) := 1.$
2. $\text{ssf}(p \to \alpha) := 2 + \text{ssf}(\alpha).$
3. $\text{ssf}((\alpha \to \beta) \to \gamma) := 1 + \text{ssf}(\alpha \to \beta) + \text{ssf}(\beta \to \gamma) - \text{ssf}(\beta).$

- Basis of induction. Suppose $|\xi| = 0$. Hence $\xi = p$ and $\text{ssf}(\xi) = 1 = (|\xi| + 1)^2$, since $|p| = 0$.
- Induction step. Suppose $|\xi| > 0$. Hence $\xi = \alpha \to \beta$.
  - If $|\alpha| = 0$, then $\alpha = p$ and $\text{ssf}(\xi) = 2 + \text{ssf}(\beta) \underset{I.H.}{\leq} 2 + (|\beta| + 1)^2 < (|\beta| + 2)^2 = (|\xi| + 1)^2.$
  - Otherwise $\alpha = \gamma \to \delta$ and $\xi = (\gamma \to \delta) \to \beta$. If $|\delta| = 0$, then $\delta = p$ and $\text{ssf}(\xi) = 1 + \text{ssf}(\alpha) + \text{ssf}(p \to \beta) - \text{ssf}(p) = 2 + \text{ssf}(\alpha) + \text{ssf}(\beta) \underset{I.H.}{\leq} 2 + (|\alpha| + 1)^2 + (|\beta| + 1)^2 < (|\alpha| + |\beta| + 2)^2 = (|\xi| + 1)^2.$
  - Otherwise $\delta = \zeta \to \eta$ and $\xi = (\gamma \to (\zeta \to \eta)) \to \beta$. If $|\eta| = 0$, then $\eta = p$ and $\text{ssf}(\xi) = 1 + \text{ssf}(\alpha) + \text{ssf}((\zeta \to p) \to \beta) - \text{ssf}(\zeta \to p) = 2 + \text{ssf}(\alpha) + \text{ssf}(p \to \beta) - \text{ssf}(p) = 3 + \text{ssf}(\alpha) + \text{ssf}(\beta) \underset{I.H.}{\leq} 3 + (|\alpha| + 1)^2 + (|\beta| + 1)^2 < (|\alpha| + |\beta| + 2)^2 = (|\xi| + 1)^2.$
  - $\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$
  - Eventually we arrive at $\alpha = \gamma_1 \to \cdots \to \gamma_n \to p$ (right-associative) and $\text{ssf}(\xi) = \text{ssf}(\alpha \to \beta) = n + 1 + \text{ssf}(\alpha) + \text{ssf}(\beta) \underset{I.H.}{\leq} n + 1 + (|\alpha| + 1)^2 + (|\beta| + 1)^2 < (|\alpha| + |\beta| + 2)^2 = (|\xi| + 1)^2.$

This completes the proof of Lemma 2 (4).

Appendix B: Compressions of Huge Proofs

Sometimes dag-like compression alone provides a desired "fast" verification of minimal validity in $\mathcal{L}_{\to}$. For example, this is the case of the Fibonacci tautology problem, as follows. Consider the formulas: 1) $\eta = \alpha_1 \to \alpha_2$ and 2) $\sigma_k = \alpha_{k-2} \to (\alpha_{k-1} \to \alpha_k)$ for $k > 2$. Note that in minimal logic $\alpha_1 \to \alpha_n$ follows from assumptions $\eta, \sigma_3, \ldots, \sigma_n$ and the size of standard tree-like normal deduction ∂_n of this statement exceeds $Fibonacci(n)$. For $n = 5$ we have:

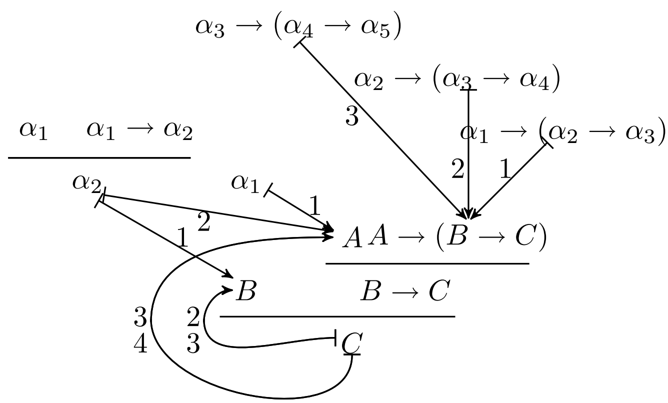
$$\dfrac{\dfrac{\dfrac{\begin{matrix}[\alpha_1]\\ \alpha_1\to\alpha_2\\ \alpha_1\to(\alpha_2\to\alpha_3)\\ \Pi_3\\ \alpha_3\end{matrix} \quad \dfrac{\dfrac{[\alpha_1]\quad \alpha_1\to\alpha_2}{\alpha_2}\quad \alpha_2\to(\alpha_3\to\alpha_4)}{\alpha_3\to\alpha_4}}{\alpha_4}\quad \dfrac{\begin{matrix}[\alpha_1]\\ \alpha_1\to\alpha_2\\ \alpha_1\to(\alpha_2\to\alpha_3)\\ \Pi_3\\ \alpha_3\end{matrix}\quad \alpha_3\to(\alpha_4\to\alpha_5)}{\alpha_4\to\alpha_5}}{\alpha_5}}{\alpha_1\to\alpha_5}$$

Generally, for each $5 \leq n$ we arrive at a tree-like deduction ∂_n like this:

$$\dfrac{\dfrac{\begin{matrix}[\alpha_1]\\ \eta\\ \sigma_3,\ldots,\sigma_{n-1}\\ \Pi_{n-1}\\ \alpha_{n-1}\end{matrix}\quad \dfrac{\begin{matrix}[\alpha_1]\\ \eta\\ \sigma_3,\ldots,\sigma_{n-2}\\ \Pi_{n-2}\\ \alpha_{n-2}\end{matrix}\quad \alpha_{n-2}\to(\alpha_{n-1}\to\alpha_n)}{\alpha_{n-1}\to\alpha_n}}{\alpha_n}}{\alpha_1\to\alpha_n}$$

$$|\Pi_2| = 1$$
$$|\Pi_3| = |\Pi_2| + 1$$
$$|\Pi_n| = |\Pi_{n-2}| + |\Pi_{n-1}| + 2$$
$$\text{Hence } Fibonacci(n) \leq |\Pi_n|$$

It is hardly possible to obtain polynomial tree-like deductions with the same conclusions. However, graph/dag representations could help, as mentioned in [4] (see below).



Towards polynomial representation

Our horizontal dag-like compressions provide a polynomial solution by successively merging distinct occurrences of identical formulas α_{n-2}, α_{n-3}, $\cdots$, α_1, which in the case $n = 5$ and tree-like $\mathrm{NM}_{\to}$ deduction ∂_5:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[\alpha_1]\\ \alpha_1\to\alpha_2\\ \alpha_1\to(\alpha_2\to\alpha_3)\\ \Pi_3\\ \alpha_3\end{matrix} \quad \dfrac{\dfrac{[\alpha_1] \quad \alpha_1\to\alpha_2}{\alpha_2} \quad \alpha_2\to(\alpha_3\to\alpha_4)}{\alpha_3\to\alpha_4}}{\alpha_4} \quad \dfrac{\begin{matrix}[\alpha_1]\\ \alpha_1\to\alpha_2\\ \alpha_1\to(\alpha_2\to\alpha_3)\\ \Pi_3\\ \alpha_3\end{matrix} \quad \alpha_3\to(\alpha_4\to\alpha_5)}{\alpha_4\to\alpha_5}}{\alpha_5}}{\alpha_1\to\alpha_5}$$

which yields the following compressed dag-like deduction frame $\widetilde{D}_5^{\mathrm{c}}$ (corresponding merging steps are shown in Figs. 1, 2 and 3).

Obviously $\widetilde{D}_5^{\mathrm{c}}$ is smaller than ∂_5. Generally, we obtain dag-like deduction frames $\widetilde{D}_n^{\mathrm{c}}$ of $\alpha_1 \to \alpha_n$ whose size and weight is smaller than $\sum_{i=1,n} i = \mathcal{O}(n^2)$ and $\mathcal{O}(n^3)$, respectively. A desired polynomial dag-like $\mathrm{NM}_{\to}$ deduction $\partial_n^{\mathrm{c}} = \left\langle \widetilde{D}_n^{\mathrm{c}}, G_n \right\rangle$ of $\alpha_1 \to \alpha_n$ from the open assumptions $\Gamma_n = \eta, \sigma_3, \ldots, \sigma_n$ is easily obtained by setting $G_n(\overrightarrow{e_k}) \equiv 1$, i.e. $G_n := \mathbf{1} : \overrightarrow{\mathrm{E}}(D_n^{\mathrm{c}}) \to \{1\}$. Now let $\zeta_n := \eta \to (\sigma_3 \to \cdots (\sigma_n \to (\alpha_1 \to \alpha_n)))$. Note that $\partial_n^{\mathrm{c}} = \left\langle \widetilde{D}_n^{\mathrm{c}}, \mathbf{1} \right\rangle$ easily extends to a dag-like proof $\partial_n^{+} = \left\langle \widetilde{D}_n^{+}, \mathbf{1} \right\rangle$ of ζ_n, in $\mathrm{NM}_{\to}$, by adding the corresponding introduction rules in the end-piece. Moreover, these conclusion hold true for arbitrary locally coherent assignment G_n instead of $\mathbf{1}$. This enables us to reduce the complexity of

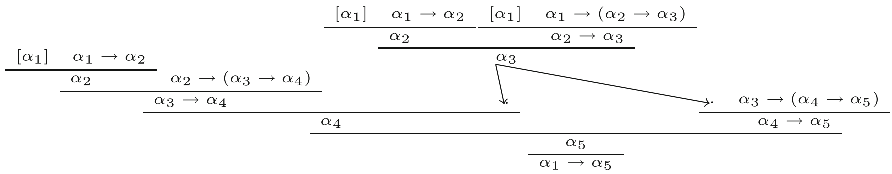


Figure 1. Collapsing the subderivation that proves α_3

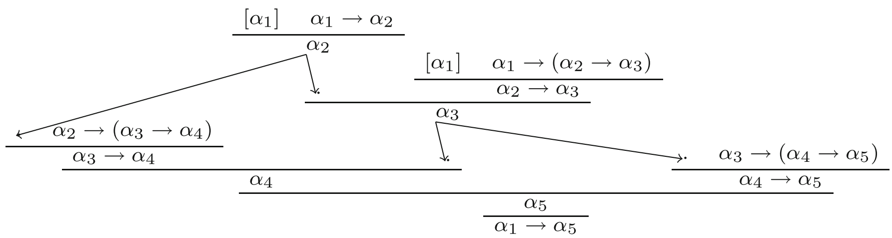


Figure 2. Collapsing the subderivation that proves α_2

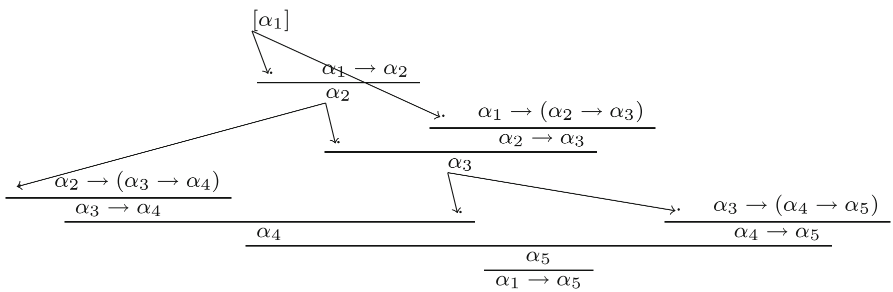
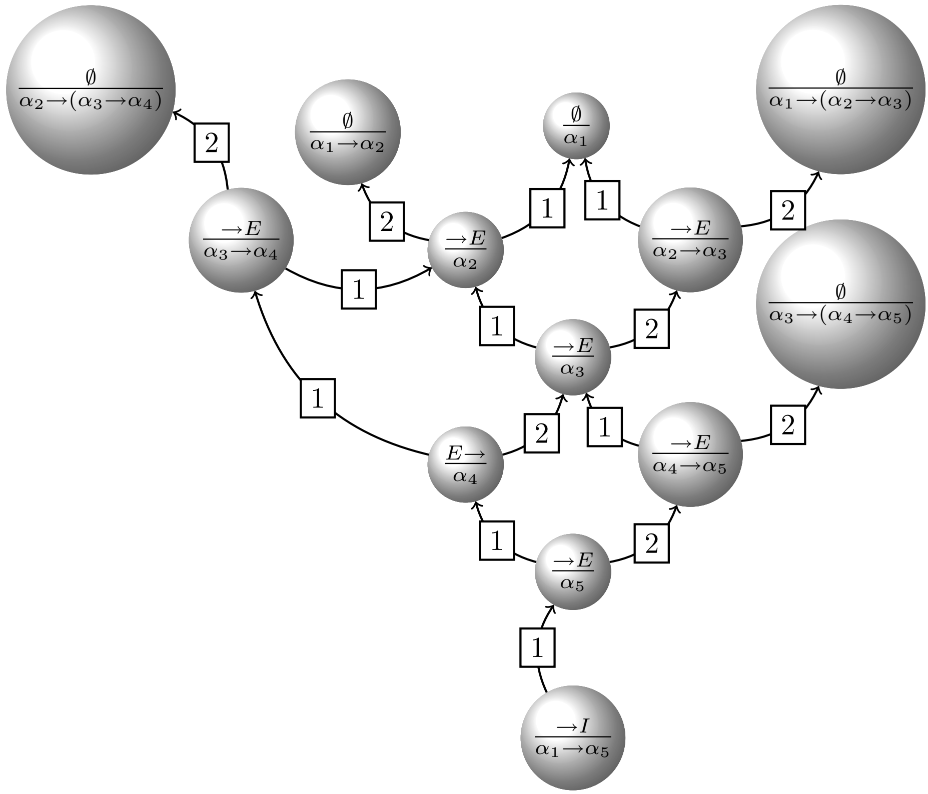


Figure 3. Collapsing the subderivation that proves α_1

PROOF($\widetilde{D}_n^+$) to the polynomial complexity of FRAME($\widetilde{D}_n^+$). The dag-like is shown below.



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