

Stackelberg Mean-payoff Games with a Rationally Bounded Adversarial Follower

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Two-player Stackelberg games are non-zero sum strategic games between a leader (Player 0) and a follower (Player 1). Such games are played sequentially: first, the leader announces her strategy, second, the follower chooses his strategy, and then both players receive their respective payoff which is a function of the two strategies. The function that maps strategies to pairs of payoffs is known by the two players. As a consequence, if we assume that the follower is fully rational then we can deduce that the follower responds by playing a so-called best-response to the strategy of the leader in order to maximise his own payoff. In turn, the leader should choose a strategy that maximizes the value that she receives when the follower chooses a best-response to her strategy. If we cannot impose which best-response is chosen by the follower, we say that the setting is adversarial. But sometimes, a more realistic assumption is to consider that the follower has only bounded rationality: the follower responds with one of his ϵ -best-responses, for some fixed $\epsilon > 0$.

In this paper, we study the ϵ -optimal *Adversarial Stackelberg Value*, \mathbf{ASV}^ϵ for short, which is the value that the leader can obtain against any ϵ -best-response of a rationally bounded adversarial follower. The \mathbf{ASV}^ϵ of Player 0 is the supremum of the values that Player 0 can obtain by announcing her strategy to Player 1 who in turn responds with an ϵ -optimal strategy. We consider the setting of infinite duration games played on graphs with mean-payoff objectives.

Our results are as follows. First, we show that Player 0 may need an infinite memory strategy to achieve the \mathbf{ASV}^ϵ . Second, we show that the threshold problem, i.e. given a rational c , deciding whether $\mathbf{ASV}^\epsilon > c$, is in NP, and a finite memory strategy of Player 0 suffices to achieve this threshold c . Third, we study the effect on \mathbf{ASV}^ϵ when Player 0 is restricted to only finite memory strategies. We also improve upon some of the results related to the memory required by strategies of Player 0 obtained earlier in the framework of two-player Adversarial Stackelberg mean-payoff games where the ϵ is not fixed. Fourth, we provide an EXPTIME algorithm to compute the \mathbf{ASV}^ϵ . Finally, we prove that the \mathbf{ASV}^ϵ is always achievable, possibly with an infinite memory strategy. This is in contrast with the framework of two-player Adversarial Stackelberg mean-payoff games where the ϵ is not fixed.

1 Introduction

Stackelberg games [21] feature strategic interactions among rational agents in markets that comprise of a leader and followers. The leader starts the game by announcing her strategy and the followers respond by playing an optimal response to the leader's strategy.

Our work belongs to the framework of synthesis of reactive programs [19, 1]. These programs maintain a continuous interaction with the environment in which they operate; they are deterministic functions that given a history of interactions choose an action. We consider the framework of *rational synthesis* [10] of reactive programs where both the program and the environment have their own goals, and are thus rational, leading to a continuous *non-zero sum* interaction between the program and its environment. The work of [10] guides some of our design choices related to the study of infinite duration games on graphs for the synthesis problem [6, 19]. However, Boolean ω -regular payoff functions have been studied in [10] as opposed to the quantitative long-run average (mean-payoff) function that we study here. Mean-payoff function is not ω -regular, and the regular tree automata techniques used in [10] cannot be adapted to our setting.

To illustrate our formal setting, we start with an example of a game graph shown in **Figure 1** with mean-payoff objectives for both players. This example allows us to introduce the basic notions. The set V of vertices is partitioned into V_0 (represented by circles) and V_1 (represented by squares) that are owned by the leader (also called Player 0) and the follower (also called Player 1) respectively. The edges for which the weights are not written correspond to 0 payoff for both players. In the tuple on the edges, the first one is the payoff of the leader, while the second one is the payoff of the follower. The game starts in vertex v_0 and we put a token in this state. The game is then played for an infinite number of rounds as follows: the owner of the vertex in which the token lies chooses an edge starting from that vertex to move the token to an adjacent vertex. A new round is then started from the vertex that the token reaches. The infinite number of rounds of the game define an infinite path in the graph. The two players receive as payoffs the limit inferior of the average payoff along the prefixes for their respective dimension. Each player's objective is to maximize the payoff that she receives. For $i \in \{0, 1\}$, Player i plays strategies that are functions $\sigma_i : V^* \cdot V_i \rightarrow V$, and there are uncountably many strategies. In the illustrations of the introduction, for simplicity, we only use memoryless strategies, that are functions $\sigma_i : V_i \rightarrow V$, but in general to play optimally players may need complex strategies that use infinite memory. This will be illustrated later in the paper, and this makes all our problems challenging.

A *Stackelberg profile* is one where the follower cannot improve his payoff by unilateral deviation. The example in **Figure 1** illustrates the fact that a Stackelberg profile may produce a better payoff for the leader than all Nash equilibria (NE). We consider only pure strategies for each player. Consider the strategy σ_L^{Nash} of the leader in which she always plays $v_1 \rightarrow v_1$, $v_2 \rightarrow v_3$ and $v_3 \rightarrow v_3$. Let σ_F^{Nash} be a strategy of the follower in which he plays $v_0 \rightarrow v_1$. The strategy profile $(\sigma_L^{\text{Nash}}, \sigma_F^{\text{Nash}})$ is a Nash Equilibrium since none of the players can unilaterally deviate and improve their individual payoff, and this profile yields a payoff of 0 for the leader. However, if the leader announces a strategy $\sigma_L^{\text{Stackelberg}}$ where she plays $v_1 \rightarrow v_1$, $v_2 \rightarrow v_2$ and $v_3 \rightarrow v_3$, the follower will get a better payoff if he responds with a strategy $\sigma_F^{\text{Stackelberg}}$ where he plays $v_0 \rightarrow v_2$. Thus, the strategy profile $(\sigma_L^{\text{Stackelberg}}, \sigma_F^{\text{Stackelberg}})$ is a Stackelberg profile, and it yields a payoff of 1 for the leader. Note that this profile is not a Nash equilibrium since the leader can improve her payoff by switching to a strategy where she plays (v_2, v_3) thus increasing her payoff to 2. In the resultant profile, the follower receives a payoff of 0, and can increase her payoff to 1 by playing $v_0 \rightarrow v_1$ which thus results into the profile $(\sigma_L^{\text{Nash}}, \sigma_F^{\text{Nash}})$ which is an NE.

Thus, we can see that the leader with the power to communicate her strategy can influence the

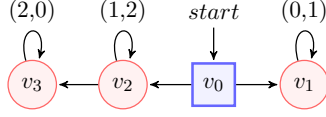


Figure 1: An example in which a Stackelberg profile gives a better payoff to Player 0 than every Nash equilibrium

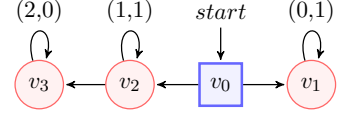


Figure 2: An example illustrating co-operative and adversarial followers in Stackelberg mean-payoff games

follower to play a strategy which she desires and thereby get a better payoff for herself. Both the leader and the follower aim at maximising their respective payoffs. However, the follower can have multiple optimal responses to the leader's strategy. Two different scenarios can be considered in this setting: either the optimal-response strategy is imposed by the leader (or equivalently chosen co-operatively by the two players), or the optimal-response strategy is chosen adversarially by the follower. We demonstrate the two scenarios with an example depicted in **Figure 2**. Here, the leader can announce her strategy σ_L where she plays $v_2 \rightarrow v_2$. However, in this example the follower has two optimal responses, i.e. he can play $v_0 \rightarrow v_1$ or $v_0 \rightarrow v_2$. If the follower is co-operative, then he will choose the strategy which also maximises the leader's payoff. In this example, the co-operative follower will choose to play the strategy $v_0 \rightarrow v_2$. Thus, in the co-operative setting, the leader receives a payoff of 1 and the follower receives a payoff of 1. On the other hand, if the follower is adversarial, he will choose the strategy which minimises the payoff of the leader. In this example, the adversarial follower will choose to play $v_0 \rightarrow v_1$. Thus, in the adversarial setting, the leader receives a payoff of 0 and the follower receives a payoff of 1.

The adversarial case is more robust: it allows us to model the situation in which the leader can choose her strategy and must be prepared to face any rational response of the follower, i.e. if the follower has several possible optimal responses then the leader's strategy should be designed to face all of them. This adversarial scenario has been recently introduced in [16] to model the synthesis of reactive systems in rational environments.

As illustrated above, due to the sequential nature of Stackelberg games, Player 1 knows the strategy that Player 0 will play before the game starts. Therefore, if Player 1 is rational, then he must choose a strategy that maximises his payoff in response to Player 0's strategy. Such a strategy is called a best-response to Player 0's strategy σ_0 . In this work, we assume that Player 1 has only bounded rationality: Player 1 may not always choose a best-response to Player 0's strategy σ_0 , we only assume that, for a given fixed $\epsilon > 0$, he plays a strategy that gives him a payoff that is up to ϵ less than the best payoff that he can achieve against the leader strategy. Considering ϵ -best-response may be more reasonable in practice as best-responses may be difficult to compute for the follower or difficult to execute for example. In this setting, Player 0 should announce a strategy σ_0 such that the payoff she receives is the supremum over all adversarial ϵ -best-responses of Player 1. We call the payoff obtained by the leader when she plays such a strategy as the ϵ -optimal Adversarial Stackelberg Value, or simply, the **ASV** $^\epsilon$. We demonstrate the difference between an adversarial fully rational follower and an adversarial bounded rational follower with the example depicted in **Figure 3**. The leader has two possible choices from v_0 ; she would either play $v_0 \rightarrow v_1$ or play $v_0 \rightarrow v_4$. If the follower is fully rational, he would play $v_1 \rightarrow v_3$ and $v_4 \rightarrow v_6$ which correspond to his best-responses to the two strategies of the leader. The leader in this case chooses $v_0 \rightarrow v_1$ to maximise her payoff to 6 which is the Adversarial Stackelberg Value (**ASV**). If the follower is rationally bounded, the leader should be prepared to receive the lowest payoff among the set of ϵ -best-responses of the follower for a given ϵ . For $\epsilon > 1$, this can be modelled by follower choosing $v_1 \rightarrow v_2$ and $v_4 \rightarrow v_6$ as responses to the two strategies of the leader respectively. The leader in this

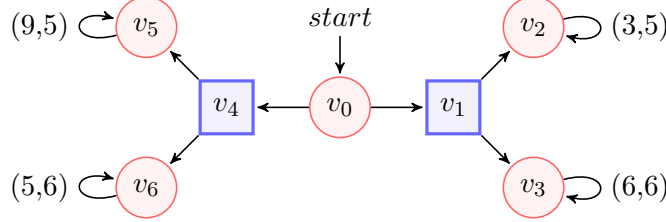


Figure 3: An example illustrating Stackelberg mean-payoff games with an adversarial fully rational follower and an adversarial rationally bounded follower

case chooses $v_0 \rightarrow v_4$ to maximize her payoff, and thus receives a payoff of 5 which is the ϵ -optimal Adversarial Stackelberg Value (\mathbf{ASV}^ϵ).

Our contributions The authors in [9] assume that the follower is fully rational, and will always play an adversarial best-response for a given strategy of leader (when it exists). In this work, we assume that the follower is bounded rational and will always play an ϵ -optimal adversarial best-response to the strategy of the leader, for a fixed $\epsilon > 0$. Our results are described below, and are summarized in **Table 1**. The results obtained in this work are in blue, while the results obtained in [9] are in green.

We begin by showing that infinite memory may be required in general for Player 0 to achieve the \mathbf{ASV}^ϵ (**Theorem 3.1**). We also consider the memory required by Player 1 to play ϵ -best-responses to a strategy of Player 0. We show that Player 1 too may require infinite memory to play an ϵ -best-response (**Theorem 3.3**).

For a given mean-payoff game, a rational value c and an $\epsilon > 0$, the *threshold problem* is to check if $\mathbf{ASV}^\epsilon > c$. Similar to the results obtained in [9], we introduce a notion of witness for proving that $\mathbf{ASV}^\epsilon > c$ (**Theorem 4.2**). However, we need different proof techniques that are more complex than the ones used in [9] due to challenges that appear because of considering the follower to be bounded rational. We show that the threshold problem is in NP, and if $\mathbf{ASV}^\epsilon > c$, then a finite memory strategy of Player 0 suffices (**Theorem 4.5**). We also show that the problem is at least as hard as the value problem in zero-sum mean-payoff games [8] whose precise complexity is a longstanding open problem [22] (**Theorem 4.13**). Additionally, we prove that a finite memory strategy is also sufficient to achieve $\mathbf{ASV} > c$ (**Theorem 4.11**).

We denote by $\mathbf{ASV}_{\text{FM}}^\epsilon$ the ϵ -optimal Adversarial Stackelberg Value obtained by Player 0 when she is limited to playing only finite memory strategies. We use \mathbf{ASV}_{FM} when ϵ is not fixed as has been studied in [9]. We show that $\mathbf{ASV}_{\text{FM}}^\epsilon = \mathbf{ASV}^\epsilon$ and $\mathbf{ASV}_{\text{FM}} = \mathbf{ASV}$ (**Corollary 4.12**).

Next we study the problem of computation of \mathbf{ASV}^ϵ . In [9], it has been established that the \mathbf{ASV} can be expressed as a formula in the theory of reals with addition, and can be computed with quantifier elimination. However, no precise complexity results have been provided. In this work, we show that we can adapt the methods used in [9] to express the \mathbf{ASV}^ϵ as a formula in the theory of reals with addition (**Theorem 5.5**). Further, this approach gives us the necessary intuition to formulate the problem of computing \mathbf{ASV}^ϵ using a set of linear programs, where each linear program has exponential number of constraints, thus giving us a EXPTIME algorithm to compute the \mathbf{ASV}^ϵ (**Theorem 5.8**).

Finally, we study the problem of achievability of \mathbf{ASV}^ϵ . The \mathbf{ASV}^ϵ is said to be achievable if there exists a strategy σ_0 for Player 0 such that Player 0 by choosing the strategy σ_0 receives a payoff that is at least equal to the \mathbf{ASV}^ϵ . While in [9], it has been shown that \mathbf{ASV} is not always achievable, here, in contrast that \mathbf{ASV}^ϵ is always achievable (**Theorem 6.1**).

| | Threshold Problem | Computing ASV | Achievability |
|---|---|--|---|
| Adversarial fully rational follower | NP Finite Memory Strategy [Theorem 4.11] | Theory Of Reals | No |
| Adversarial bounded rational follower | NP Finite Memory Strategy [Theorem 4.5] | Theory Of Reals [Theorem 5.5] Solving LP in EXPTIME [Theorem 5.8] | Yes [Theorem 6.1] (Requires Infinite Memory [Theorem 3.1]) |

Table 1: Summary of our results

Related Works Stackelberg games on graphs have been first considered in [10], where the authors study rational synthesis for ω -regular objectives in a setting where the followers are co-operative. In [9], Stackelberg mean-payoff Games in adversarial setting, and Stackelberg discounted sum games in both adversarial and co-operative setting have been considered. There, the authors postulate full rationality of the follower; we relax that hypothesis here. In [11], mean-payoff Stackelberg games in the co-operative setting have been studied. They did not considered the adversarial setting. In [14], the authors study the effects of limited memory on both Nash and Stackelberg (or leader) equilibria in multi-player discounted sum games. A Stackelberg equilibrium has been defined there as a profile that gives the highest payoff to the leader among all the Stackelberg profiles. Stackelberg (leader) equilibrium and incentive equilibrium over bi-matrix games have been studied in [12]. The authors consider mixed strategies, and the existence of Nash equilibrium in bi-matrix games [18, 17] implies the existence of leader and incentive equilibria. In [16], adversarial rational synthesis for ω -regular objectives have been studied. In [7], precise complexity results for various ω -regular objectives have been established for both adversarial and co-operative settings.

Incentive equilibrium has been studied in [13] for mean-payoff games in a co-operative setting. The ability of the leader to incentivise her followers provides the leader with more freedom in selecting strategy profiles, and thus can improve the payoff for the leader in such games even when compared with Stackelberg games. In [5], the authors study secure Nash equilibrium, where each player first maximises her own payoff, and then minimises the payoff of the other player. This setting was studied for mean-payoff games in [3]. Player 0 and Player 1 are symmetric and not asymmetric as in Stackelberg games.

Structure of the paper In **Section 2**, we introduce the necessary definitions and concepts used in the paper. In **Section 3**, we examine the memory requirements of both players for playing their strategies. In **Section 4**, we show that the threshold problem, i.e. given a rational c , checking if $\mathbf{ASV}^\epsilon > c$, can be decided in NP, and if $\mathbf{ASV}^\epsilon > c$, then a finite memory strategy suffices to achieve this threshold c . We present an improvement with respect to the memory requirement for a previous result established in [9]. We also study the effect of limiting Player 0 to playing finite memory strategies on the \mathbf{ASV}^ϵ . The section ends with a result that the threshold problem is at least as hard as the value problem of zero-sum mean-payoff games. In **Section 5**, we present an algorithm to compute the \mathbf{ASV}^ϵ in EXPTIME. In **Section 6**, we show that the \mathbf{ASV}^ϵ is always achievable, which is in contrast with the framework of two-player Adversarial Stackelberg mean-

payoff games where the ϵ is not fixed, as studied in [9]. We conclude in **Section 7**.

2 Preliminaries

We denote by \mathbb{N} , \mathbb{N}^+ , \mathbb{Q} , and \mathbb{R} the set of naturals, the set of naturals without 0, the set of rationals, and the set of reals respectively.

Arenas An (bi-weighted) arena $\mathcal{A} = (V, E, \langle V_0, V_1 \rangle, w_0, w_1)$ consists of a finite set V of vertices, a set $E \subseteq V \times V$ of edges such that for all $v \in V$ there exists $v' \in V$ such that $(v, v') \in E$, a partition $\langle V_0, V_1 \rangle$ of V , where V_0 (resp. V_1) is the set of vertices for Player 0 (resp. Player 1), and two edge weight functions $w_0 : E \rightarrow \mathbb{Z}$, $w_1 : E \rightarrow \mathbb{Z}$. In the sequel, we denote the maximum absolute value of a weight in \mathcal{A} by W . We refer to w_0 and w_1 as the weight functions assigning weights to the *first dimension* and the *second dimension* of an edge respectively. We assume in the sequel that these weights are given in binary.

Plays and histories A play in \mathcal{A} is an infinite sequence of vertices $\pi = \pi_0 \pi_1 \dots \in V^\omega$ such that for all $k \in \mathbb{N}$, we have $(\pi_k, \pi_{k+1}) \in E$. We denote by $\mathbf{Plays}_{\mathcal{A}}$ the set of plays in \mathcal{A} , omitting the subscript \mathcal{A} when the underlying arena is clear from the context. Given $\pi = \pi_0 \pi_1 \dots \in \mathbf{Plays}_{\mathcal{A}}$ and $k \in \mathbb{N}$, the prefix $\pi_0 \pi_1 \dots \pi_k$ of π (resp. suffix $\pi_k \pi_{k+1} \dots$ of π) is denoted by $\pi_{\leq k}$ (resp. $\pi_{\geq k}$). A *history* in \mathcal{A} is a (non-empty) prefix of a play in \mathcal{A} . The length $|h|$ of an history $h = \pi_{\leq k}$ is the number $|h| = k$ of its edges. We denote by $\mathbf{Hist}_{\mathcal{A}}$ the set of histories in \mathcal{A} ; the symbol \mathcal{A} is omitted when clear from the context. Given $i \in \{0, 1\}$ the set $\mathbf{Hist}_{\mathcal{A}}^i$ denotes the set of histories such that their last vertex belongs to V_i . We denote the first vertex and the last vertex of a history h by $\text{first}(h)$ and $\text{last}(h)$ respectively. We write $h \leq \pi$ whenever h is a prefix of π .

Games A *mean-payoff game* $\mathcal{G} = (\mathcal{A}, \langle \underline{\mathbf{MP}}_0, \underline{\mathbf{MP}}_1 \rangle)$ consists of a bi-weighted arena \mathcal{A} , a payoff function $\underline{\mathbf{MP}}_0 : \mathbf{Plays}_{\mathcal{A}} \rightarrow \mathbb{R}$ for Player 0 and a payoff function $\underline{\mathbf{MP}}_1 : \mathbf{Plays}_{\mathcal{A}} \rightarrow \mathbb{R}$ for Player 1 which are defined as follows. Given a play $\pi \in \mathbf{Plays}_{\mathcal{A}}$ and $i \in \{0, 1\}$, the payoff $\underline{\mathbf{MP}}_i(\pi)$ is given by $\underline{\mathbf{MP}}_i(\pi) = \liminf_{k \rightarrow \infty} \frac{1}{k} w_i(\pi_{\leq k})$, where the weight $w_i(h)$ of an history $h \in \mathbf{Hist}$ is the sum of the weights assigned by w_i to its edges. In our definition of the mean-payoff, we have used \liminf , we will also need the \limsup case for technical reasons. Here is the formal definition together with its notation: $\overline{\mathbf{MP}}_i(\pi) = \limsup_{k \rightarrow \infty} \frac{1}{k} w_i(\pi_{\leq k})$. The size of the game \mathcal{G} , denoted $|\mathcal{G}|$, is the sum of the number of vertices and edges appearing in the arena \mathcal{A} .

Let V and E be respectively the set of vertices and the set of edges of \mathcal{G} . The *unfolding* of the game \mathcal{G} starting from a vertex $v \in V$ is a tree $T_v(\mathcal{G})$ of infinite depth with its root v such that there is a one-to-one correspondence between the set of plays π of \mathcal{G} with $\text{first}(\pi) = v$ and the branches of $T_v(\mathcal{G})$. Every node $p \in V^+$ of $T_v(\mathcal{G})$ is a play $p = v_1 \dots v_n$ in \mathcal{G} , where $v_1 = v$. There is an edge from $p = v_1 \dots v_n$ to $p' = v_1 \dots v_n v'_n$ iff $(v_n, v'_n) \in E$.

Strategies and payoffs A strategy for Player $i \in \{0, 1\}$ in the game $\mathcal{G} = (\mathcal{A}, \langle \underline{\mathbf{MP}}_0, \underline{\mathbf{MP}}_1 \rangle)$ is a function $\sigma : \mathbf{Hist}_{\mathcal{A}}^i \rightarrow V$ that maps histories ending in a vertex $v \in V_i$ to a successor of v . The set of all strategies of Player $i \in \{0, 1\}$ in the game \mathcal{G} is denoted by $\Sigma_i(\mathcal{G})$, or Σ_i when \mathcal{G} is clear from the context.

A strategy has memory M if it can be realized as the output of a state machine with M states. A memoryless (or positional) strategy is a strategy with memory 1, that is, a function that only depends on the last element of the given partial play. We denote by Σ_i^{ML} the set of memoryless strategies of Player i , and Σ_i^{FM} her set of finite memory strategies. A *profile* is a pair of strategies $\bar{\sigma} = (\sigma_0, \sigma_1)$, where $\sigma_0 \in \Sigma_0(\mathcal{G})$ and $\sigma_1 \in \Sigma_1(\mathcal{G})$. As we consider games with perfect information and deterministic transitions, any profile $\bar{\sigma}$ yields, from any history h , a unique play or *outcome*, denoted $\mathbf{Out}_h(\mathcal{G}, \bar{\sigma})$. Formally, $\mathbf{Out}_h(\mathcal{G}, \bar{\sigma})$ is the play π such that $\pi_{\leq |h|-1} = h$ and $\forall k \geq |h| - 1$ it holds that $\pi_{k+1} = \sigma_i(\pi_{\leq k})$ if $\pi_k \in V_i$. The set of outcomes (resp. histories) compatible with a strategy

$\sigma \in \Sigma_{i \in \{0,1\}}(\mathcal{G})$ after a history h is $\mathbf{Out}_h(\mathcal{G}, \sigma) = \{\pi \mid \exists \sigma' \in \Sigma_{1-i}(\mathcal{G}) \text{ such that } \pi = \mathbf{Out}_h(\mathcal{G}, (\sigma, \sigma'))\}$ (resp. $\mathbf{Hist}_h(\sigma) = \{h' \in \mathbf{Hist}(\mathcal{G}) \mid \pi \in \mathbf{Out}_h(\mathcal{G}, \sigma), n \in \mathbb{N} : h' = \pi_{\leq n}\}$).

Each outcome $\pi \in \mathcal{G} = (\mathcal{A}, \langle \underline{\mathbf{MP}}_0, \underline{\mathbf{MP}}_1 \rangle)$ yields a payoff $\mathbf{MP}(\pi) = (\underline{\mathbf{MP}}_0(\pi), \underline{\mathbf{MP}}_1(\pi))$, where $\underline{\mathbf{MP}}_0(\pi)$ is the payoff for Player 0 and $\underline{\mathbf{MP}}_1(\pi)$ is the payoff for Player 1. We denote with $\mathbf{MP}(h, \sigma) = \mathbf{MP}(\mathbf{Out}_h(\mathcal{G}, \bar{\sigma}))$ the payoff of a profile of strategies $\bar{\sigma}$ after a history h .

Usually, we consider instances of games such that the players start playing at a fixed vertex v_0 . Thus, we call an initialized game a pair (\mathcal{G}, v_0) , where \mathcal{G} is a game and $v_0 \in V$ is the initial vertex. When the initial vertex v_0 is clear from context, we use \mathcal{G} , $\mathbf{Out}(\mathcal{G}, \bar{\sigma})$, $\mathbf{Out}(\mathcal{G}, \sigma)$, $\mathbf{MP}(\bar{\sigma})$ instead of \mathcal{G}_{v_0} , $\mathbf{Out}_{v_0}(\mathcal{G}, \bar{\sigma})$, $\mathbf{Out}_{v_0}(\mathcal{G}, \sigma)$, $\mathbf{MP}_{v_0}(\bar{\sigma})$. We sometimes simplify further the notation omitting \mathcal{G} when the latter is clear from the context.

Strongly Connected Components (SCC) In the mathematical theory of directed graphs, a graph is said to be strongly connected if every vertex is reachable from every other vertex. A Strongly Connected Component of a directed graph \mathcal{G} is a subgraph that is strongly connected. In the sequel, unless otherwise mentioned, we say that SCC is a strongly connected component of the graph \mathcal{G} which may or may not be maximal.

Best-responses and adversarial value Let $\mathcal{G} = (\mathcal{A}, \langle \underline{\mathbf{MP}}_0, \underline{\mathbf{MP}}_1 \rangle)$ be a two-dimensional mean-payoff game on the bi-weighted arena \mathcal{A} . Given a strategy σ_0 for Player 0, we define two sets of strategies for Player 1:

1. his best-responses to σ_0 , denoted by $\mathbf{BR}_1(\sigma_0)$, and defined as:

$$\{\sigma_1 \in \Sigma_1 \mid \forall v \in V. \forall \sigma'_1 \in \Sigma_1 : \underline{\mathbf{MP}}_1(\mathbf{Out}_v(\sigma_0, \sigma_1)) \geq \underline{\mathbf{MP}}_1(\mathbf{Out}_v(\sigma_0, \sigma'_1))\}$$

2. his ϵ -best-responses to σ_0 , for $\epsilon > 0$, denoted by $\mathbf{BR}_1^\epsilon(\sigma_0)$, and defined as:

$$\{\sigma_1 \in \Sigma_1 \mid \forall v \in V. \forall \sigma'_1 \in \Sigma_1 : \underline{\mathbf{MP}}_1(\mathbf{Out}_v(\sigma_0, \sigma_1)) > \underline{\mathbf{MP}}_1(\mathbf{Out}_v(\sigma_0, \sigma'_1)) - \epsilon\}$$

Note that for the ϵ -best-response when $\epsilon > 0$, we use $>$ instead of \geq .

We also introduce the following notation for zero-sum games (that are needed as intermediary steps in our algorithms). Let \mathcal{A} be an arena, $v \in V$ one of its states, and $\mathcal{O} \subseteq \mathbf{Plays}_{\mathcal{A}}$ be a set of plays (called objective), then we write $\mathcal{A}, v \models \ll i \gg \mathcal{O}$, if:

$$\exists \sigma_i \in \Sigma_i. \forall \sigma_{1-i} \in \Sigma_{1-i} : \mathbf{Out}_v(\mathcal{A}, (\sigma_i, \sigma_{1-i})) \in \mathcal{O}, \text{ for } i \in \{0, 1\}$$

Here the underlying interpretation is zero-sum: Player i wants to force an outcome in \mathcal{O} and Player $1 - i$ has the opposite goal. All the zero-sum games we consider in this paper are *determined* meaning that for all \mathcal{A} , for all objectives $\mathcal{O} \subseteq \mathbf{Plays}_{\mathcal{A}}$ we have that:

$$\mathcal{A}, v \models \ll i \gg \mathcal{O} \iff \mathcal{A}, v \not\models \ll 1 - i \gg \mathbf{Plays}_{\mathcal{A}} \setminus \mathcal{O}$$

We sometimes simplify the notation omitting \mathcal{A} when the arena being referenced is clear from the context.

Convex hull and F_{\min} Given a dimension d , a finite set $X \subset \mathbb{Q}^d$ of rational vectors, we define the convex hull $\mathbf{CH}(X) = \{v \mid v = \sum_{x \in X} \alpha_x \cdot x \wedge \forall x \in X : \alpha_x \in [0, 1] \wedge \sum_{x \in X} \alpha_x = 1\}$ as the set of all their convex combinations. Given a finite set of d -dimensional rational vectors $X \subset \mathbb{Q}^d$, let $f_{\min}(X)$ be the vector $v = (v_1, v_2, \dots, v_n)$ where $v_i = \min\{c \mid \exists x \in X : x_i = c\}$ i.e. the vector v is the pointwise minimum of the vectors in X . For $S \subseteq \mathbb{Q}^d$, we define $F_{\min}(S) = \{f_{\min}(P) \mid P \text{ is a finite subset of } S\}$.

Mean-payoffs induced by simple cycles Given a play $\pi \in \mathbf{Plays}_{\mathcal{A}}$, we denote by $\inf(\pi)$ the set of vertices v that appear infinitely many times along π , i.e., $\inf(\pi) = \{v \in V \mid \forall n \in \mathbb{N}. \exists j \in \mathbb{N}, j \geq n$

$i : \pi(j) = v\}$. It is easy to see that $\inf(\pi)$ forms an SCC in the underlying graph of the arena \mathcal{A} . A *cycle* c is a sequence of edges that starts and stops in a given vertex v , it is simple if it does not contain repetition of any other vertex. Given an SCC S , we write $\mathbb{C}(S)$ for the set of simple cycles inside S . Given a simple cycle c , for $i \in \{0, 1\}$, let $\mathbf{MP}_i(c) = \frac{w_i(c)}{|c|}^1$ be the mean of the weights in each dimension along the edges in the simple cycle c , and we call the pair $(\mathbf{MP}_0(c), \mathbf{MP}_1(c))$ the mean-payoff coordinate of the cycle c . We write $\text{CH}(\mathbb{C}(S))$ for the convex-hull of the set of mean-payoff coordinates of simple cycles of S .

Lemma 2.1. ([9, 4]) *Let S be an SCC in the arena \mathcal{A} with a set V of vertices, the following three properties hold:*

1. *for all $\pi \in \mathbf{Plays}_{\mathcal{A}}$, if $\inf(\pi) \subseteq S$, then $(\mathbf{MP}_0(\pi), \mathbf{MP}_1(\pi)) \in F_{\min}(\text{CH}(\mathbb{C}(S)))$*
2. *for all $(x, y) \in F_{\min}(\text{CH}(\mathbb{C}(S)))$, there exists a play $\pi \in \mathbf{Plays}_{\mathcal{A}}$ such that $\inf(\pi) = S$ and $(\mathbf{MP}_0(\pi), \mathbf{MP}_1(\pi)) = (x, y)$.*
3. *The set $F_{\min}(\text{CH}(\mathbb{C}(S)))$ is effectively expressible in $\langle \mathbb{R}, +, < \rangle$ as a conjunction of $\mathcal{O}(m^2)$ linear inequations, where m is the number of mean-payoff coordinates of simple cycles in S , which is $\mathcal{O}(W \cdot |V|)$. Hence this set of inequations can be pseudopolynomial in size.*

Adversarial Stackelberg Value for MP As the set of best-responses in mean-payoff games can be empty, we use the notion of ϵ -best-responses for the definition of **ASV** which are guaranteed to always exist. We now define:

$$\mathbf{ASV}(v) = \sup_{\sigma_0 \in \Sigma_0, \epsilon \geq 0 | \mathbf{BR}_1^\epsilon(\sigma_0) \neq \emptyset} \inf_{\sigma_1 \in \mathbf{BR}_1^\epsilon(\sigma_0)} \mathbf{MP}_0(\mathbf{Out}_v(\sigma_0, \sigma_1))$$

We also associate a (adversarial) value to a strategy $\sigma_0 \in \Sigma_0$ of Player 0:

$$\mathbf{ASV}(\sigma_0)(v) = \sup_{\epsilon \geq 0 | \mathbf{BR}_1^\epsilon(\sigma_0) \neq \emptyset} \inf_{\sigma_1 \in \mathbf{BR}_1^\epsilon(\sigma_0)} \mathbf{MP}_0(\mathbf{Out}_v(\sigma_0, \sigma_1))$$

Clearly, we have that:

$$\mathbf{ASV}(v) = \sup_{\sigma_0 \in \Sigma_0} \mathbf{ASV}(\sigma_0)(v)$$

Given an $\epsilon > 0$, we define an adversarial value for Player 0 as:

$$\mathbf{ASV}^\epsilon(v) = \sup_{\sigma_0 \in \Sigma_0} \inf_{\sigma_1 \in \mathbf{BR}_1^\epsilon(\sigma_0)} \mathbf{MP}_0(\mathbf{Out}_v(\sigma_0, \sigma_1))$$

Clearly, we have that:

$$\mathbf{ASV}(v) = \sup_{\epsilon > 0} \mathbf{ASV}^\epsilon(v)$$

We also define the adversarial Stackelberg value, where strategies of Player 0 are restricted to finite memory:

$$\mathbf{ASV}_{\text{FM}}^\epsilon(v) = \sup_{\sigma_0 \in \Sigma_0^{\text{FM}}} \inf_{\sigma_1 \in \mathbf{BR}_1^\epsilon(\sigma_0)} \mathbf{MP}_0(\mathbf{Out}_v(\sigma_0, \sigma_1))$$

where Σ_0^{FM} refers to the set of all finite memory strategies of Player 0.

Achievability of \mathbf{ASV}^ϵ Given $\epsilon > 0$, we have that $\mathbf{ASV}^\epsilon(v) = c$ is achievable in a mean-payoff game \mathcal{G} from a vertex v , if there exists a strategy σ_0 for Player 0 such that

$$\forall \sigma_1 \in \mathbf{BR}_1^\epsilon(\sigma_0) : \mathbf{MP}_0(\mathbf{Out}_v(\sigma_0, \sigma_1)) \geq c$$

In the sequel, unless otherwise mentioned, we refer to a two-dimensional non-zero sum two-player mean-payoff game simply as a mean-payoff game.

¹We do not use \mathbf{MP}_i since \liminf and \limsup are the same for a finite sequence of edges

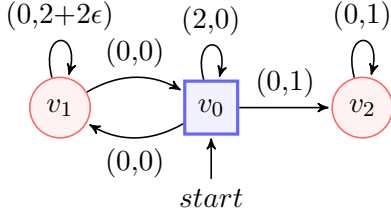


Figure 4: Finite memory strategy of Player 0 may not achieve $\mathbf{ASV}^\epsilon(v_0)$.

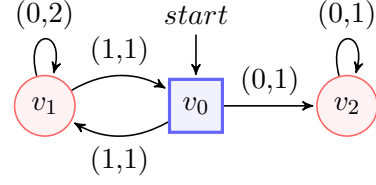


Figure 5: An example in which a finite memory strategy for Player 0 suffices to achieve $\mathbf{ASV}^\epsilon(v_0)$.

3 \mathbf{ASV}^ϵ for a fixed ϵ

In this section, we study the memory requirements for the strategies of both Player 0 and Player 1.

First we show that there exists a mean-payoff game \mathcal{G} in which Player 0 needs an infinite memory strategy to achieve the \mathbf{ASV}^ϵ .

We also show that there exists a mean-payoff game \mathcal{G} in which the \mathbf{ASV}^ϵ can be achieved using a finite memory (but not memoryless) strategy for Player 0.

Theorem 3.1. *There exists a mean-payoff game \mathcal{G} , a vertex v in \mathcal{G} , and an $\epsilon > 0$ such that Player 0 needs an infinite memory strategy to achieve the $\mathbf{ASV}^\epsilon(v)$.*

Proof. Consider the example in **Figure 4**. We show that in this example the $\mathbf{ASV}^\epsilon(v_0) = 1$, and that this value can only be achieved using an infinite memory strategy. Assume a strategy σ_0 for Player 0 such that the game is played in rounds. In round k

- if Player 1 plays $v_0 \rightarrow v_0$ repeatedly at least k times before playing $v_0 \rightarrow v_1$, then from v_1 , play $v_1 \rightarrow v_1$ repeatedly k times and then play $v_1 \rightarrow v_0$ and move to round $k + 1$;
- else, if Player 1 plays $v_0 \rightarrow v_0$ less than k times before playing $v_0 \rightarrow v_1$, then from v_1 , play $v_1 \rightarrow v_0$.

The best-response for Player 1 to strategy σ_0 would be to choose k sequentially as $k = 1, 2, 3, \dots$, to get a play $\pi = ((v_0)^i(v_1)^i)_{i \in \mathbb{N}}$. We have that $\mathbf{MP}_1(\pi) = 1 + \epsilon$ and $\mathbf{MP}_0(\pi) = 1$. Player 1 can sacrifice an amount that is less than ϵ to minimize the mean-payoff of Player 0, and thus he would not like to play $v_0 \rightarrow v_2$. In particular, a strategy σ_1 of Player 1 that prescribes playing the edge $v_0 \rightarrow v_2$ some time yields a mean-payoff of 1 for Player 1, and hence we conclude that $\sigma_1 \notin \mathbf{BR}_1^\epsilon(\sigma_0)$. Player 1 cannot play any other strategy without increasing the mean-payoff of Player 0 and/or decreasing his own payoff. We can see that Player 1 does not have a finite memory best-response strategy. Thus, the $\mathbf{ASV}^\epsilon(\sigma_0)(v_0) = 1$.

We claim that $\mathbf{ASV}^\epsilon(\sigma_0)(v_0) = \mathbf{ASV}^\epsilon(v_0)$. For every strategy σ_1 of Player 1 such that $\sigma_1 \in \mathbf{BR}_1^\epsilon(\sigma_0)$, we note that the higher the payoff Player 1 has, the lower is the payoff for Player 0. For every other strategy σ'_0 of Player 0, if best-response of Player 1 to σ'_0 gives a mean-payoff less than $1 + \epsilon$, then Player 1 will switch to v_2 , thus giving Player 0 a payoff of 0. If best-response of Player 1 to σ'_0 gives a mean-payoff greater than $1 + \epsilon$, then Player 0 will have a lower $\mathbf{ASV}^\epsilon(\sigma'_0)$.

Now we show that a finite memory strategy of Player 0 cannot achieve an $\mathbf{ASV}^\epsilon(v_0)$ of 1. Consider a finite memory strategy of Player 0. If Player 1 has an infinite memory ϵ -best-response which cannot be encoded by finite memory, it can only lead to looping over v_0 more and more, this gives him a payoff which is eventually 0. Thus consider a finite memory response of Player 1 to the finite memory strategy of Player 0. Note that Player 0 would choose a finite memory strategy such

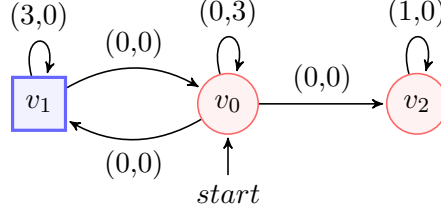


Figure 6: No finite memory ϵ -best-response of Player 1 exists for a strategy σ_0 of Player 0

that the best-response of Player 1 gives him a value of at least $1 + \epsilon$. Also since both players have finite memory strategies, the resultant outcome is a regular play over vertices v_0 and v_1 . In every such regular play, the effect of the edge from v_0 to v_1 and the edge from v_1 to v_0 is non-negligible, and hence if the payoff of Player 1 is at least $1 + \epsilon$, the payoff of Player 0 will be less than 1. Thus no finite memory strategy can achieve an \mathbf{ASV}^ϵ that is equal to 1. \square

The following example shows the existence of mean-payoff games in which Player 0 can achieve the adversarial value with finite memory (but not memoryless) strategies.

Theorem 3.2. *There exists a mean-payoff game \mathcal{G} , a vertex v in \mathcal{G} , and an $\epsilon > 0$ such that a finite memory strategy of Player 0 suffices to achieve $\mathbf{ASV}^\epsilon(v)$.*

Proof. Consider the example in **Figure 5**. We show that $\mathbf{ASV}^\epsilon(v_0) = 1 - \epsilon$. Assume a strategy σ_0 for Player 0 defined as: repeat forever, from v_1 play j times $v_1 \rightarrow v_1$, and then repeat playing $v_1 \rightarrow v_0$ for k times, with j and k chosen such that mean-payoff for Player 0 is equal to $1 - \epsilon$. For every rational ϵ , such a k always exists. In this example, we have that $k = \frac{1-\epsilon}{2\epsilon}j$. The ϵ -best-response of Player 1 to σ_0 is to always play $v_0 \rightarrow v_1$ as by playing this edge forever, Player 1 gets a mean-payoff equal to $1 + \epsilon$, whereas if Player 1 plays $v_0 \rightarrow v_2$, then Player 1 receives a payoff of 1. Since $1 \not\geq (1 + \epsilon) - \epsilon$, a strategy of Player 1 that chooses $v_0 \rightarrow v_2$ is not an ϵ -best-response for Player 1, thus forcing Player 1 to play $v_0 \rightarrow v_1$. Thus $\mathbf{ASV}^\epsilon(v_0)$ is achieved with a finite memory strategy of size k for Player 0. Note that this size k is a function of ϵ . \square

We can now show that there exists a mean-payoff game such that for a given strategy σ_0 of Player 0, and for a given $\epsilon > 0$, there may not exist a finite memory strategy of Player 1 that is an ϵ -best-response to σ_0 .

Theorem 3.3. *There exists a mean-payoff game \mathcal{G} and an $\epsilon > 0$ such that for some Player 0 strategy σ_0 , for every Player 1 strategy $\sigma_1 \in \mathbf{BR}_1^\epsilon(\sigma_0)$, we have that σ_1 is not a finite memory strategy.*

Proof. Consider the example in **Figure 6**, and the following strategy σ_0 of Player 0. Player 0 loops over v_0 i times, and then sends the token back to v_1 . Player 1 loops over v_1 k times, and then sends the token to v_0 . If $k \geq i$, then Player 0 increases i by 1, and repeats the above, otherwise she sends the token to v_2 . Clearly for all $\epsilon \leq 1.5$, no finite memory strategy is an ϵ -best-response to σ_0 . \square

4 Threshold Problem for \mathbf{ASV}^ϵ

In this section, given a rational c , we study the threshold problem of determining if $\mathbf{ASV}^\epsilon(v) > c$ for a mean-payoff game \mathcal{G} and where v is a vertex in \mathcal{G} .

We start by showing that if $\mathbf{ASV}^\epsilon(v) > c$, then there exists a strategy σ_0 for Player 0 that enforces $\mathbf{ASV}^\epsilon(\sigma_0)(v) > c$.

Lemma 4.1. *For all mean-payoff games \mathcal{G} , for all vertices v in \mathcal{G} , for all $\epsilon > 0$, and for all rationals c , we have that $\mathbf{ASV}^\epsilon(v) > c$ iff there exists a strategy $\sigma_0 \in \Sigma_0$ such that $\mathbf{ASV}^\epsilon(\sigma_0)(v) > c$.*

Proof. The right to left direction of the proof is trivial as σ_0 can play the role of witness for $\mathbf{ASV}^\epsilon(v) > c$, i.e., if there exists a strategy σ_0 of Player 0 such that $\mathbf{ASV}^\epsilon(\sigma_0)(v) > c$, then $\mathbf{ASV}^\epsilon(v) > c$.

For the left to right direction of the proof, let $\mathbf{ASV}^\epsilon(v) = c'$. By definition of $\mathbf{ASV}^\epsilon(v)$, we have that $c' = \sup_{\sigma_0 \in \Sigma_0} \mathbf{ASV}^\epsilon(\sigma_0)(v)$.

By definition of sup, for all $\delta > 0$, there exists σ_0^δ such that $\mathbf{ASV}^\epsilon(\sigma_0^\delta)(v) > c' - \delta$.

Let us consider a $\delta > 0$ such that $c' - \delta > c$. Such a δ exists as $c' > c$. Then we have that there exists σ_0 such that $\mathbf{ASV}^\epsilon(\sigma_0)(v) > c' - \delta > c$. \square

Witnesses for \mathbf{ASV}^ϵ For a mean-payoff game \mathcal{G} and an $\epsilon > 0$, we associate with each vertex v in \mathcal{G} , the following set of pairs of real numbers:

$$\Lambda^\epsilon(v) = \{(c, d) \in \mathbb{R}^2 \mid v \models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 > d - \epsilon\}$$

We say that a vertex v is $(c, d)^\epsilon$ -bad if $(c, d) \in \Lambda^\epsilon(v)$. Let $c' \in \mathbb{R}$. A play π in \mathcal{G} is called a $(c', d)^\epsilon$ -witness of $\mathbf{ASV}^\epsilon(v) > c$ if $(\underline{\mathbf{MP}}_0(\pi), \underline{\mathbf{MP}}_1(\pi)) = (c', d)$ where $c' > c$, and π does not contain any $(c, d)^\epsilon$ -bad vertex. A play π is called a witness of $\mathbf{ASV}^\epsilon(v) > c$ if it is a $(c', d)^\epsilon$ -witness of $\mathbf{ASV}^\epsilon(v) > c$ for some c', d . The following theorem states the existence of a witness.

A similar result has been established in [9] where the authors show that $\mathbf{ASV}(v) > c$ if and only if there exists a witness for $\mathbf{ASV}(v) > c$. However, proving the direction that if $\mathbf{ASV}^\epsilon(v) > c$, then there exists a witness for $\mathbf{ASV}^\epsilon(v) > c$ is more challenging in our case, and requires a different proof technique.

Theorem 4.2. *For all mean-payoff games \mathcal{G} , for all vertices v in \mathcal{G} , for all $\epsilon > 0$, and for all rationals c , we have that $\mathbf{ASV}^\epsilon(v) > c$ if and only if there exists a $(c', d)^\epsilon$ -witness of $\mathbf{ASV}^\epsilon(v) > c$, where d is some rational.*

Proof. First we prove the right to left direction, i.e., we are given a play π in \mathcal{G} that starts from v and the play π is such that $(\underline{\mathbf{MP}}_0(\pi), \underline{\mathbf{MP}}_1(\pi)) = (c', d)$ for $c' > c$ and does not cross a $(c, d)^\epsilon$ -bad vertex. We need to prove that $\mathbf{ASV}^\epsilon(v) > c$. We do this by defining a strategy σ_0 for Player 0, such that $\mathbf{ASV}^\epsilon(\sigma_0)(v) > c$:

1. $\forall h \leq \pi$, if $\text{last}(h)$ is a Player 0 vertex, the strategy σ_0 is such that $\sigma_0(h)$ follows π .
2. $\forall h \not\leq \pi$, where there has been a deviation from π by Player 1, we assume that Player 0 switches to a *punishing* strategy defined as follows: In the subgame after history h' where $\text{last}(h')$ is the first vertex from which Player 1 deviates from π , we know that Player 0 has a strategy to enforce the objective: $\underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 \leq d - \epsilon$. This is true because π does not cross any $(c, d)^\epsilon$ -bad vertex and since n -dimensional mean-payoff games are determined.

Let us now establish that the strategy σ_0 satisfies $\mathbf{ASV}^\epsilon(\sigma_0)(v) > c$. First note that, since $\underline{\mathbf{MP}}_1(\pi) = d$, we have that $\sup_{\sigma_1 \in \mathbf{BR}_1^\epsilon(\sigma_0)} \underline{\mathbf{MP}}_1(\text{Out}_v(\sigma_0, \sigma_1)) \geq d$. Now consider some strategy $\sigma'_1 \in \mathbf{BR}_1^\epsilon(\sigma_0)$ and let $\pi' = \text{Out}_v(\sigma_0, \sigma'_1)$. Clearly, π' is such that $\underline{\mathbf{MP}}_1(\pi') > \sup_{\sigma_1 \in \mathbf{BR}_1^\epsilon(\sigma_0)} \underline{\mathbf{MP}}_1(\text{Out}_v(\sigma_0, \sigma_1)) - \epsilon \geq d - \epsilon$. If $\pi' = \pi$, we know that $\underline{\mathbf{MP}}_0(\pi') > c$. If $\pi' \neq \pi$, then when π' deviates from π we know that Player 0 employs the punishing strategy, thus making sure that $\underline{\mathbf{MP}}_0(\pi') > c \vee$

$\underline{\mathbf{MP}}_1(\pi') \leq d - \epsilon$. Since $\sigma'_1 \in \mathbf{BR}_1^\epsilon(\sigma_0)$, it must be true that $\underline{\mathbf{MP}}_0(\pi') > c$. Thus, $\forall \sigma'_1 \in \mathbf{BR}_1^\epsilon(\sigma_0)$, we have $\underline{\mathbf{MP}}_0(\mathbf{Out}_v(\sigma_0, \sigma'_1)) > c$. Therefore, $\mathbf{ASV}^\epsilon(\sigma_0)(v) > c$, which implies $\mathbf{ASV}^\epsilon(v) > c$.

Now we consider the left to right direction of the proof, i.e., we are given that $\mathbf{ASV}^\epsilon(v) > c$. Hence by **Lemma 4.1**, there exists a strategy σ_0 for Player 0 such that $\mathbf{ASV}^\epsilon(\sigma_0)(v) > c$. Thus, there exists a $\delta > 0$, such that

$$\inf_{\sigma_1 \in \mathbf{BR}_1^\epsilon(\sigma_0)} \underline{\mathbf{MP}}_0(\mathbf{Out}_v(\sigma_0, \sigma_1)) = c' = c + \delta$$

Let $d = \sup_{\sigma_1 \in \mathbf{BR}_1^\epsilon(\sigma_0)} \underline{\mathbf{MP}}_1(\mathbf{Out}_v(\sigma_0, \sigma_1))$. We first prove that for all $\sigma_1 \in \mathbf{BR}_1^\epsilon(\sigma_0)$, we have that $\mathbf{Out}_v(\sigma_0, \sigma_1)$ does not cross a $(c, d)^\epsilon$ -bad vertex. For every $\sigma_1 \in \mathbf{BR}_1^\epsilon(\sigma_0)$, we let $\pi_{\sigma_1} = \mathbf{Out}_v(\sigma_0, \sigma_1)$. We note that $\underline{\mathbf{MP}}_1(\pi_{\sigma_1}) > d - \epsilon$ and $\underline{\mathbf{MP}}_0(\pi_{\sigma_1}) > c$. For every $\pi' \in \mathbf{Out}_v(\sigma_0)$, we know that if $\underline{\mathbf{MP}}_1(\pi') > d - \epsilon$, then there exists a strategy $\sigma'_1 \in \mathbf{BR}_1^\epsilon(\sigma_0)$ such that $\pi' = \mathbf{Out}_v(\sigma_0, \sigma'_1)$. This means that $\underline{\mathbf{MP}}_0(\pi') > c$. Thus we can see that every deviation from π_{σ_1} either gives Player 1 a mean-payoff that is at most $d - \epsilon$ or Player 0 a mean-payoff greater than c . Therefore, we conclude that π_{σ_1} does not cross any $(c, d)^\epsilon$ -bad vertex.

Now consider a sequence $(\sigma_i)_{i \in \mathbb{N}}$ of Player 1 strategies such that $\sigma_i \in \mathbf{BR}_1^\epsilon(\sigma_0)$ for all $i \in \mathbb{N}$, and $\lim_{i \rightarrow \infty} \underline{\mathbf{MP}}_1(\mathbf{Out}_v(\sigma_0, \sigma_i)) = d$. Let $\pi_i = \mathbf{Out}_v(\sigma_0, \sigma_i)$. Let $\inf(\pi_i)$ be the set of vertices that occur infinitely often in π_i , and let V_{π_i} be the set of vertices appearing along the play π_i . Since there are finitely many SCCs, w.l.o.g., we can assume that for all $i, j \in \mathbb{N}$, we have that $\inf(\pi_i) = \inf(\pi_j)$, that is, all the plays end up in the same SCC, say S , and also $V_{\pi_i} = V_{\pi_j} = V_\pi$ (say). Note that $S \subseteq V_\pi$.

Note that for every $\epsilon \geq \delta > 0$, there is a strategy $\sigma_1^\delta \in \mathbf{BR}_1^\epsilon(\sigma_0)$ of Player 1, and a corresponding play $\pi' = \mathbf{Out}_v(\sigma_0, \sigma_1^\delta)$ such that $\underline{\mathbf{MP}}_1(\pi') > d - \delta$, and $\underline{\mathbf{MP}}_0(\pi') \geq c'$. Also the set $V_{\pi'}$ of vertices appearing in π' be such that $V_{\pi'} \subseteq V_\pi$, and $\inf(\pi') \subseteq S$.

Now since $F_{\min}(\text{CH}(\mathbb{C}(S)))$ is a closed set, we have that $(\hat{c}, d) \in F_{\min}(\text{CH}(\mathbb{C}(S)))$ for some \hat{c} where $\hat{c} \geq c' > c$. By **Lemma 2.1**, there exists a play π^* such that $(\underline{\mathbf{MP}}_0(\pi^*), \underline{\mathbf{MP}}_1(\pi^*)) = (c'', d)$. Also $\inf(\pi^*) \subseteq S$, and $V_{\pi^*} \subseteq V_\pi$. The proof follows since for all vertices $v \in V_\pi$, we have that v is not $(c, d)^\epsilon$ -bad. \square

Now, we establish a small witness property to establish that the threshold problem is in NP.

Lemma 4.3. *For all mean-payoff games \mathcal{G} , for all vertices v in \mathcal{G} , for all $\epsilon > 0$, and for all rationals c , we have that $\mathbf{ASV}^\epsilon(v) > c$ if and only if there exist three acyclic plays π_1, π_2, π_3 , and two simple cycles l_1, l_2 such that:*

1. $\text{first}(\pi_1) = v$, $\text{first}(\pi_2) = \text{last}(\pi_1)$, $\text{first}(\pi_3) = \text{last}(\pi_2)$, $\text{first}(\pi_2) = \text{last}(\pi_3)$, and $\text{first}(\pi_2) = \text{first}(l_1)$, and $\text{first}(\pi_3) = \text{first}(l_2)$.

2. there exist $\alpha, \beta \in \mathbb{Q}^+$, where $\alpha + \beta = 1$, such that:

$$(a) \alpha \cdot \mathbf{MP}_0(l_1) + \beta \cdot \mathbf{MP}_0(l_2) = c' > c$$

$$(b) \alpha \cdot \mathbf{MP}_1(l_1) + \beta \cdot \mathbf{MP}_1(l_2) = d, \text{ for some rational } d$$

Furthermore, α , β , and d can be chosen so that they can be represented with a polynomial number of bits.

3. there is no $(c, d)^\epsilon$ -bad vertex v' along π_1, π_2, π_3, l_1 and l_2 .

Proof. This proof is similar to the proof of **Lemma 8** in [9]. For the right to left direction of the proof, where we are given finite acyclic plays π_1, π_2, π_3 , simple cycles l_1 and l_2 and constants α, β , we consider the witness $\pi = \pi_1 \rho_1 \rho_2 \rho_3 \dots$ where, for all $i \in \mathbb{N}$, we let $\rho_i = l_1^{[\alpha \cdot i]} \cdot \pi_2 \cdot l_2^{[\beta \cdot i]} \cdot \pi_3$. We know that $\underline{\mathbf{MP}}_1(\pi) = \alpha \cdot \mathbf{MP}_1(l_1) + \beta \cdot \mathbf{MP}_1(l_2) = d$ and $\underline{\mathbf{MP}}_0(\pi) = \alpha \cdot \mathbf{MP}_0(l_1) + \beta \cdot \mathbf{MP}_0(l_2) > c$. For all vertices v in π_1, π_2, π_3, l_1 and l_2 , it is given that v is not $(c, d)^\epsilon$ -bad. Therefore, π is a suitable witness thus proving from **Theorem 4.2** that $\mathbf{ASV}^\epsilon(v) > c$.

For the left to right direction of the proof, we are given $\mathbf{ASV}^\epsilon(v) > c$. Using **Theorem 4.2**, we can construct a play π such that $\underline{\mathbf{MP}}_0(\pi) > c$ and $\underline{\mathbf{MP}}_1(\pi) = d$, and π does not cross a $(c, d)^\epsilon$ -bad vertex, i.e., for all vertices v' appearing in π , we have that $v' \not\ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 > d - \epsilon$. First, the value d must be chosen so that the vertices v' appearing in π do not belong to the $(c, d)^\epsilon$ -bad vertices. Let $\text{inf}(\pi) = S$ be the set of vertices appearing infinitely often in π . Note that S forms an SCC. By abuse of notation, we also denote this SCC by S here. By **Lemma 2.1**, we have that $(\underline{\mathbf{MP}}_0(\pi), \underline{\mathbf{MP}}_1(\pi)) \in \mathbf{F}_{\min}(\mathbf{CH}(\mathbb{C}(S)))$. From **Proposition 1** of [4], for a bi-weighted arena, we have that $\mathbf{F}_{\min}(\mathbf{CH}(\mathbb{C}(S))) = \mathbf{CH}(\mathbf{F}_{\min}(\mathbb{C}(S)))$. Since $\mathbf{CH}(\mathbf{F}_{\min}(\mathbb{C}(S)))$ can be expressed using conjunctions of linear inequations whose coefficients have polynomial number of bits, the same also follows for $\mathbf{F}_{\min}(\mathbf{CH}(\mathbb{C}(S)))$ in a bi-weighted arena. In addition, it is proven in [2] that the set $\Lambda^\epsilon(v')$ is definable by a disjunction of conjunctions of linear inequations whose coefficients have polynomial number of bits in the descriptions of the game \mathcal{G} and of ϵ . Hence $\overline{\Lambda}^\epsilon(v')$ is also definable by a disjunction of conjunctions of linear inequations whose coefficients have polynomial number of bits in the descriptions of the game \mathcal{G} and of ϵ . As a consequence of **Theorem 2** in [2] which states that given a system of linear inequations that is satisfiable, there exists a point with polynomial representation that satisfies the system, we have that d can be chosen such that $(c, d) \in \overline{\Lambda}^\epsilon(v')$ and $(\underline{\mathbf{MP}}_0(\pi), d) \in \mathbf{F}_{\min}(\mathbf{CH}(\mathbb{C}(S)))$, and hence d can be represented with a polynomial number of bits.

Second, following the proof of **Lemma 8** in [9], and by applying the Carathéodory baricenter theorem, we can find two simple cycles l_1, l_2 in the SCC S and acyclic finite plays π_1, π_2 and π_3 from π , and two positive rational constants $\alpha, \beta \in \mathbb{Q}^+$, such that $\text{first}(\pi_1) = v$, $\text{first}(\pi_2) = \text{last}(\pi_1)$, $\text{first}(\pi_3) = \text{last}(\pi_2)$, $\text{first}(\pi_2) = \text{last}(\pi_3)$, and $\text{first}(\pi_2) = \text{first}(l_1)$, and $\text{first}(\pi_3) = \text{first}(l_2)$, and $\alpha + \beta = 1$, $\alpha \cdot \mathbf{MP}_0(l_1) + \beta \cdot \mathbf{MP}_0(l_2) > c$ and $\alpha \cdot \mathbf{MP}_1(l_1) + \beta \cdot \mathbf{MP}_1(l_2) = d$. Again, using **Theorem 2** in [2], we can assume that α and β are rational values that can be represented using a polynomial number of bits. We note that for all vertices v in π_1, π_2, π_3, l_1 and l_2 , we have that v is not $(c, d)^\epsilon$ -bad. \square

In this work, we show that we can further improve the witness for $\mathbf{ASV}^\epsilon(v) > c$ by showing the existence of a regular witness.

Theorem 4.4. *For all mean-payoff games \mathcal{G} , for all vertices v in \mathcal{G} , for all $\epsilon > 0$, and for all rationals c , we have that $\mathbf{ASV}^\epsilon(v) > c$ if and only if there exists a regular $(c', d)^\epsilon$ -witness of $\mathbf{ASV}^\epsilon(v) > c$, where d is some rational.*

Proof. We only focus on the left to right direction since the proof of the other direction is exactly the same as in the proof of **Theorem 4.2**.

Consider the witness π in the proof of **Lemma 4.3**. We construct a regular-witness π' for $\mathbf{ASV}^\epsilon(v) > c$ where $\pi' = \pi_1 \cdot (l_1^{[\alpha' \cdot k]} \cdot \pi_2 \cdot l_2^{[\beta' \cdot k]} \cdot \pi_3)^\omega$ and α', β' are constants in \mathbb{Q} and k is some large integer. We construct π' by modifying π as follows. We need to consider the following cases.

Case 1: $\mathbf{MP}_0(l_1) > \mathbf{MP}_0(l_2)$ and $\mathbf{MP}_1(l_1) < \mathbf{MP}_1(l_2)$

Here, one simple cycle, l_1 , increases Player 0's mean-payoff while the other simple cycle, l_2 , increases Player 1's mean-payoff. We can build a witness $\pi' = \pi_1 \cdot (l_1^{[\alpha \cdot k]} \cdot \pi_2 \cdot l_2^{[(\beta + \tau) \cdot k]} \cdot \pi_3)^\omega$ for some very large

$k \in \mathbb{N}$ and for some small $\tau > 0$ such that $\underline{\mathbf{MP}}_0(\pi') > c$ and $\underline{\mathbf{MP}}_1(\pi') = d$.² We note that k and τ are polynomial in the size of \mathcal{G} , and the largest weight W appearing on the edges of \mathcal{G} .

Case 2: $\mathbf{MP}_0(l_1) < \mathbf{MP}_0(l_2)$ and $\mathbf{MP}_1(l_1) > \mathbf{MP}_1(l_2)$

This is analogous to **case 1**, and proceeds as mentioned above.

Case 3: $\mathbf{MP}_0(l_1) > \mathbf{MP}_0(l_2)$ and $\mathbf{MP}_1(l_1) > \mathbf{MP}_1(l_2)$

One cycle, l_1 , increases both Player 0 and Player 1's mean-payoffs, while the other, l_2 , decreases it. In this case, we can just omit one of the cycles and consider the one that gives a larger mean-payoff, to get a finite memory strategy. Thus, $\pi' = \pi_1.l_1^\omega$ and we get $\underline{\mathbf{MP}}_0(\pi') > c$, $\underline{\mathbf{MP}}_1(\pi') \geq d$. Suppose $\underline{\mathbf{MP}}_1(\pi') = d' \geq d$. Since no vertex in π_1, π_2, π_3, l_1 , and l_2 is $(c, d)^\epsilon$ -bad, we also have that they are not $(c, d')^\epsilon$ -bad, and thus π' is a witness for $\mathbf{ASV}^\epsilon(v) > c$.

Case 4: $\mathbf{MP}_0(l_1) < \mathbf{MP}_0(l_2)$ and $\mathbf{MP}_1(l_1) < \mathbf{MP}_1(l_2)$

This is analogous to **case 3**, and proceeds as mentioned above.

In each of these cases, we have that $\mathbf{ASV}^\epsilon(v) > c$: $\underline{\mathbf{MP}}_0(\pi') > c$ and $\underline{\mathbf{MP}}_1(\pi') \geq d$. Since we know that π does not cross a $(c, d)^\epsilon$ -bad vertex, and the vertices of the play π' are a subset of the vertices of the play π , we have that π' is a witness for $\mathbf{ASV}^\epsilon(v) > c$. \square

Now, we state the following theorem that establishes the NP-membership of $\mathbf{ASV}^\epsilon(v) > c$.

Theorem 4.5. *For all mean-payoff games \mathcal{G} , for all vertices v in \mathcal{G} , for all $\epsilon > 0$, and for all rationals c , it can be decided in non-deterministic polynomial time if $\mathbf{ASV}^\epsilon(v) > c$, and a pseudopolynomial memory strategy of Player 0 suffices for this threshold.*

For proving **Theorem 4.5**, we start by stating a property of multi-dimensional mean-payoff games proved in [20] that we rephrase here for a two-dimensional mean-payoff game. This property expresses a relation between mean-payoff limsup and mean-payoff liminf objectives. We recall that in [20], the objective of Player 1 is to maximize the payoff in each dimension, i.e., for two-dimensional setting, given two rational c and d , Player 1 wins if he has a winning strategy for $\underline{\mathbf{MP}}_0 \geq c \wedge \underline{\mathbf{MP}}_1 \geq d$; otherwise Player 0 wins due to determinacy of multi-dimensional mean-payoff games. We call the mean-payoff game setting in [20] 2D-max mean-payoff games to distinguish it from the mean-payoff games that we consider here. Later we will relate the two settings.

Proposition 4.6. (Lemma 14 in [20]) *For all mean-payoff games \mathcal{G} , for all vertices v in the game \mathcal{G} , and for all rationals c, d , we have*

$$v \models \ll 1 \gg \overline{\mathbf{MP}}_0 \geq c \wedge \underline{\mathbf{MP}}_1 \geq d$$

if and only if

$$v \models \ll 1 \gg \underline{\mathbf{MP}}_0 \geq c \wedge \underline{\mathbf{MP}}_1 \geq d$$

We now recall another property of multi-dimensional mean-payoff games proved in [20] that we rephrase here for a 2D-max mean-payoff game. This property expresses a bound on the weight of every finite play $\pi^f \in \mathbf{Out}_v(\sigma_0)$ where σ_0 is a memoryless winning strategy for Player 0.

Lemma 4.7. (Lemma 10 in [20]) *For all 2D-max mean-payoff games \mathcal{G} , for all vertices v in \mathcal{G} , and for all rationals c, d , if Player 0 ³ wins $\underline{\mathbf{MP}}_0 < c \vee \underline{\mathbf{MP}}_1 < d$ from v then she has a memoryless winning strategy σ_0 to do so, and there exist three constants $m_{\mathcal{G}}, c_{\mathcal{G}}, d_{\mathcal{G}} \in \mathbb{R}$ such that:*

²For more details, we refer the reader to the Appendix.

³Player 0 is called Player 2 in [20]

$c_{\mathcal{G}} < c, d_{\mathcal{G}} < d$, and for all finite plays $\pi^f \in \mathbf{Out}_v(\sigma_0)$, i.e. starting from v and compatible with σ_0 , we have that

$$w_0(\pi^f) \leq m_{\mathcal{G}} + c_{\mathcal{G}} \cdot |\pi^f|$$

or

$$w_1(\pi^f) \leq m_{\mathcal{G}} + d_{\mathcal{G}} \cdot |\pi^f|$$

We now relate the 2D-max mean-payoff game in [20] where the objective of Player 1 is to maximize the payoff in both dimensions to our setting where in a game \mathcal{G} , Player 1 maximizes the payoff in the second dimension, and minimizes the payoff on the first dimension from his set of available responses to a strategy of Player 0. The objective of Player 0 then is to maximize the payoff in the first dimension and minimize the payoff in the second dimension, i.e. given two rationals c and d , Player 0's objective is to ensure $v \models \ll 0 \gg \underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 < d$. We now state a modification of **Lemma 4.7** as follows:

Lemma 4.8. *For all mean-payoff games \mathcal{G} , for all vertices v in \mathcal{G} , and for all rationals c, d , if Player 0 wins $\underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 < d$ from v , then she has a memoryless winning strategy σ_0 to do so, and there exist three constants $m_{\mathcal{G}}, c_{\mathcal{G}}, d_{\mathcal{G}} \in \mathbb{R}$ such that $c_{\mathcal{G}} > c, d_{\mathcal{G}} < d$, and for all finite plays $\pi^f \in \mathbf{Out}_v(\sigma_0)$, i.e. starting in v and compatible with σ_0 , we have that*

$$w_0(\pi^f) \geq -m_{\mathcal{G}} + c_{\mathcal{G}} \cdot |\pi^f|$$

or

$$w_1(\pi^f) \leq m_{\mathcal{G}} + d_{\mathcal{G}} \cdot |\pi^f|$$

Proof. We show this by a reduction to a 2D-max mean-payoff game where Player 0's objective is to ensure $v \models \ll 0 \gg \underline{\mathbf{MP}}_0 < c \vee \underline{\mathbf{MP}}_1 < d$ in a 2D-max mean-payoff game.

We prove this lemma in two parts. If Player 0 wins $\underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 < d$, we show (i) the existence of a memoryless strategy σ_0 for Player 0, and (ii) that there exist three constants $m_{\mathcal{G}}, c_{\mathcal{G}}, d_{\mathcal{G}} \in \mathbb{R}$ such that $c_{\mathcal{G}} > c, d_{\mathcal{G}} < d$, and for all finite plays $\pi^f \in \mathbf{Out}_v(\sigma_0)$, i.e. starting from v and compatible with σ_0 , we have that either $w_0(\pi^f) \geq -m_{\mathcal{G}} + c_{\mathcal{G}} \cdot |\pi^f|$ or $w_1(\pi^f) \leq m_{\mathcal{G}} + d_{\mathcal{G}} \cdot |\pi^f|$.

Assume that Player 0 has a winning strategy from vertex v in \mathcal{G} for $\underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 < d$. To prove (i), we subtract $2c$ from the weights on the first dimension of all the edges, followed by multiplying them with -1 . We call the resultant 2D-max mean-payoff game \mathcal{G}' , and we have that $v \models \ll 0 \gg \underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 < d$ in \mathcal{G} if and only if $v \models \ll 0 \gg \overline{\mathbf{MP}}_0 < c \vee \underline{\mathbf{MP}}_1 < d$ in \mathcal{G}' . Using **Proposition 4.6** and determinacy of multi-dimensional mean-payoff games, it follows that $v \models \ll 0 \gg \underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 < d$ in \mathcal{G} if and only if $v \models \ll 0 \gg \underline{\mathbf{MP}}_0 < c \vee \underline{\mathbf{MP}}_1 < d$ in \mathcal{G}' . Also from [20], we have that if Player 0 has a winning strategy in \mathcal{G}' for $\underline{\mathbf{MP}}_0 < c \vee \underline{\mathbf{MP}}_1 < d$, then she has a memoryless strategy σ_0 for the same, and the proof of **Lemma 14** in [20] shows that same memoryless strategy σ_0 is also winning for $\overline{\mathbf{MP}}_0 < c \vee \underline{\mathbf{MP}}_1 < d$, thus concluding that if Player 0 wins $\underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 < d$ in \mathcal{G} from vertex v , then she has a memoryless winning strategy.

We prove (ii) by contradiction. Assume that Player 0 wins $\underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 < d$ from vertex v in \mathcal{G} , and by part (i), she has a memoryless winning strategy σ_0 . Assume for contradiction, that there does not exist three constants $m_{\mathcal{G}}, c_{\mathcal{G}}, d_{\mathcal{G}} \in \mathbb{R}$ such that $c_{\mathcal{G}} > c, d_{\mathcal{G}} < d$, such that for all finite plays $\pi^f \in \mathbf{Out}_v(\sigma_0)$, i.e. starting in v and compatible with σ_0 , we have either $w_0(\pi^f) \geq -m_{\mathcal{G}} + c_{\mathcal{G}} \cdot |\pi^f|$ or $w_1(\pi^f) \leq m_{\mathcal{G}} + d_{\mathcal{G}} \cdot |\pi^f|$.

Consider the steps in the construction of \mathcal{G}' as defined above. As we subtract $2c$ from the weights on the first dimension of each edge, and multiply the resultant weights on the first dimension by -1 , we have that there does not exist three constants $m_{\mathcal{G}}, c_{\mathcal{G}}, d_{\mathcal{G}} \in \mathbb{R}$ such that $c_{\mathcal{G}} > c, d_{\mathcal{G}} < d$, and for all finite plays $\pi^f \in \mathbf{Out}_v(\sigma_0)$ in \mathcal{G} , i.e. starting from v and compatible with σ_0 , we have either

$$w_0(\pi^f) \geq -m_{\mathcal{G}} + c_{\mathcal{G}} \cdot |\pi^f|$$

or

$$w_1(\pi^f) \leq m_G + d_G \cdot |\pi^f|$$

if and only if there does not exist three constants $m_G, c_G, d_G \in \mathbb{R}$ such that $c_G > c, d_G < d$, and for all finite plays $\pi^f \in \mathbf{Out}_v(\sigma_0)$ in the 2D-max mean-payoff game \mathcal{G}' for the objective $\underline{\mathbf{MP}}_0 < c \vee \underline{\mathbf{MP}}_1 < d$, i.e. starting in v and compatible with σ_0 , we have either

$$w_0(\pi^f) \leq m_G + (2c - c_G) \cdot |\pi^f|$$

or

$$w_1(\pi^f) \leq m_G + d_G \cdot |\pi^f|$$

Let $2c - c_G = c'_G$, and we have that $c'_G < c$. Now since σ_0 is a winning for Player 0 for the objective $\underline{\mathbf{MP}}_0 < c \vee \underline{\mathbf{MP}}_1 < d$ in \mathcal{G}' from v , we reach a contradiction by **Lemma 4.7**, and due to determinacy of multi-dimensional mean-payoff games. \square

Using **Lemma 4.8**, we can now prove that Player 0 can ensure from a vertex v that $v \not\ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 > d$ if and only if she can also ensure that $v \not\ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 \geq d'$ for all $d' > d$. This is established in the following lemma.

Lemma 4.9. *For all mean-payoff games \mathcal{G} , for all vertices $v \in \mathcal{G}$, and for all rationals c, d , we have that:*

$$v \models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 > d$$

if and only if there exists a $d' \in \mathbb{R}$, where $d' > d$ such that

$$v \models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 \geq d'$$

Proof. For the right to left direction of the proof, it is trivial to see that if $v \models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 \geq d'$ for some $d' > d$, then we have that $v \models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 > d$.

For the left to right direction of the proof, we prove the contrapositive, i.e., we assume that $\forall d' > d$, we have $v \not\models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 \geq d'$. Now we prove that $v \not\models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 > d$.

Since $\forall d' > d$, Player 1 loses $\underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 \geq d'$ from a given vertex v , due to determinacy of multi-dimensional mean-payoff games, Player 0 wins $\underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 < d'$ from vertex v . By **Lemma 4.8**, Player 0 has a memoryless strategy σ_0 to achieve the objective $\underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 < d'$ from vertex v . Note that Player 0 has only finitely many memoryless strategies. Therefore there exists a strategy σ_0^* that achieves the objective $v \models \ll 0 \gg \underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 < d'$ for all $d' > d$. Now from **Lemma 4.7**, for every $d' > d$, there exists three constants $m_G, c_G, d'_G \in \mathbb{R}$ such that $c_G > c, d'_G < d'$, and for all finite plays $\pi^f \in \mathbf{Out}_v(\sigma_0^*)$, we have that

$$w_0(\pi^f) \geq -m_G + c_G \cdot |\pi^f|$$

or

$$w_1(\pi^f) \leq m_G + d'_G \cdot |\pi^f|$$

Note that since the above is true for every $d' > d$, we can indeed consider a $d_G \in \mathbb{R}$, where $d_G \leq d$, such that for all $d' > d$, and for all finite plays $\pi^f \in \mathbf{Out}_v(\sigma_0^*)$, we have that

$$w_0(\pi^f) \geq -m_G + c_G \cdot |\pi^f|$$

or we have that

$$w_1(\pi^f) \leq m_G + d_G \cdot |\pi^f|$$

Hence, for every play $\pi \in \mathbf{Out}_v(\sigma_0^*)$, we have that

$$\overline{\mathbf{MP}}_0(\pi) \geq c_G \vee \underline{\mathbf{MP}}_1(\pi) \leq d_G$$

Thus, we get

$$\begin{aligned} v \models \ll 0 \gg \overline{\mathbf{MP}}_0 \geq c_G \vee \underline{\mathbf{MP}}_1 \leq d_G \\ \iff v \models \ll 0 \gg \overline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 < d' \quad \text{for every } d' > d, \text{ since } c_G > c, d_G < d' \end{aligned}$$

We now construct a 2D-max mean-payoff game \mathcal{G}' from the given game \mathcal{G} by multiplying the first dimension of the weights of all the edges by -1 . Thus, in the game \mathcal{G}' , we get

$$\begin{aligned} v \models \ll 0 \gg \underline{\mathbf{MP}}_0 < -c \vee \underline{\mathbf{MP}}_1 < d' \quad \text{for every } d' > d \\ \iff v \models \ll 0 \gg \overline{\mathbf{MP}}_0 < -c \vee \underline{\mathbf{MP}}_1 < d' \quad \text{for every } d' > d \text{ (from Proposition 4.6)} \end{aligned}$$

We now construct a game \mathcal{G}'' from the game \mathcal{G}' by multiplying the first dimension of the weights of all the edges by -1 . Note that, we get back the original game \mathcal{G} after this modification, i.e. \mathcal{G}'' has the same arena as that of \mathcal{G} . Thus, in the game \mathcal{G} , we have that

$$v \models \ll 0 \gg \underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 < d' \quad \text{for every } d' > d$$

Recall by **Lemma 4.8**, if Player 0 wins $\underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 < d'$, then she has a memoryless strategy for this objective, and since there are finitely many memoryless strategies, there exists a memoryless strategy σ_0^* of Player 0 that wins for all $d' > d$. This also implies that by using σ_0^* , from vertex v , Player 0 can ensure $\underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 \leq d$, that is,

$$\begin{aligned} v \models \ll 0 \gg \underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 \leq d \\ \iff v \not\models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 > d \quad (\text{by determinacy of multi-dimensional mean-payoff games}). \end{aligned}$$

□

We now have all the ingredients to prove **Theorem 4.5**.

Proof of Theorem 4.5. According to **Lemma 4.3**, we consider a non-deterministic Turing machine that establishes the membership to NP by guessing a reachable SCC S , a finite play π_1 to reach S from v , two simple cycles l_1, l_2 , along with two finite plays π_2 and π_3 that connects the two simple cycles, and parameters $\alpha, \beta \in \mathbb{Q}^+$. Additionally, for each vertex v' that appear along the plays π_1, π_2 and π_3 , and on the simple cycles l_1 and l_2 , the Turing machine guesses a memoryless strategy $\sigma_0^{v'}$ for Player 0 that establishes $v' \not\models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 > d - \epsilon$ which implies by determinacy of multi-dimensional mean-payoff games, that $v' \models \ll 0 \gg \underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 \leq d - \epsilon$.

Besides, from **Theorem 4.4**, we can obtain a regular witness π' . Using π' , we build a finite memory strategy σ_0^{FM} for Player 0 as stated below:

1. Player 0 follows π' if Player 1 does not deviate from π' . The finite memory strategy stems from the finite k as required in the proof of **Theorem 4.4**.
2. For each vertex $v' \in \pi'$, Player 0 employs the memoryless strategy $\sigma_0^{v'}$ that establishes $v' \not\models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 > d - \epsilon$. The existence of such a memoryless strategy follows from the proof of **Lemma 4.9**.

It remains to show that all the guesses can be verified in polynomial time. The only difficult part concerns the memoryless strategies of Player 0 to punish deviations of Player 1 from the witness play π' . These memoryless strategies are used to prove that the witness does not cross $(c, d)^\epsilon$ -bad vertices. For vertex $v' \in \pi'$, we consider a memoryless strategy $\sigma_0^{v'}$, and we need to establish that it can enforce $\underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 \leq d - \epsilon$. Towards this, we adapt the proof of **Lemma 10** in [20], which in turn is based on the polynomial time algorithm of Kosaraju and Sullivan [15] for detecting zero-cycles in multi-weighted directed graphs.

Consider the bi-weighted graph obtained from \mathcal{G} by fixing the choices of Player 0 according to the memoryless strategy $\sigma_0^{v'}$. We first compute the set of maximal SCCs that are reachable from v' in this bi-weighted graph. This can be done in linear time. For each SCC S , we need to check that Player 1 cannot achieve $\underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 > d - \epsilon$.

We first recall the definition of multi-cycles from [20], which is a multi-set of simple cycles from the SCC S . For a simple cycle $C = (e_1, \dots, e_n)$, let $w(C) = \sum_{e \in C} w(e)$. For a multi-cycle \mathcal{C} , let $w(\mathcal{C}) = \sum_{C \in \mathcal{C}} w(C)$ (note that in this summation, a cycle C may appear multiple times in \mathcal{C}). A non-negative multi-cycle is a non-empty multi-set of simple cycles \mathcal{C} such that $w(\mathcal{C}) \geq 0$ (i.e., in both the dimensions, the weight is non-negative). In [20], it has been shown that the problem of deciding if S has a non-negative multi-cycle can be solved in polynomial time by solving a set of linear inequations. In our case, we are interested in multi-cycles such that $w_0(\mathcal{C}) \leq c$ and $w_1(\mathcal{C}) > d - \epsilon$. As in the proof of **Lemma 10** in [20], this can be checked by defining the following set of linear constraints. Let V_S and E_S respectively denote the set of vertices and the set of edges in S . For every edge $e \in E_S$, we consider a variable χ_e .

- (a) For $v \in V_S$, let $\text{In}(v)$ and $\text{Out}(v)$ respectively denote the set of incoming edges to v and the set of outgoing edges from v . For every $v \in V_S$, we define the linear constraint $\sum_{e \in \text{In}(v)} \chi_e = \sum_{e \in \text{Out}(v)} \chi_e$ which intuitively models flow constraints.
- (b) For every $e \in E_S$, we define the constraint $\chi_e \geq 0$.
- (c) We also add the constraint $\sum_{e \in E_S} \chi_e \cdot w_0(e) \leq c$ and $\sum_{e \in E_S} \chi_e \cdot w_1(e) > d - \epsilon$.
- (d) Finally, we define the constraint $\sum_{e \in E_S} \chi_e \geq 1$ that ensures that the multi-cycle is non-empty.

This set of linear constraints can be solved in polynomial time, and formally following the arguments from [15], it has a solution if and only if there exists a multi-cycle \mathcal{C} such that $w_0(\mathcal{C}) \leq c$ and $w_1(\mathcal{C}) > d - \epsilon$. The NP-membership follows since we have linearly many maximal SCCs from each vertex v' in the bi-weighted graph that is obtained from \mathcal{G} by fixing the choices of Player 0 according to the memoryless strategy $\sigma_0^{v'}$, and there are linearly many vertices v' for which we need to check that v' is not $(c, d)^\epsilon$ -bad.

Now we show that the memory required by the strategy σ_0^{FM} as described above is pseudopolynomial in the input size. Recall from the proof of **Theorem 4.4** that k and τ are polynomial in the size of \mathcal{G} , and the largest weight W appearing on the edges of \mathcal{G} . Assuming that the weights are given in binary, the number of states in the finite state machine realizing this strategy is thus $\text{poly}(|\mathcal{G}|, W)$, and hence pseudopolynomial in the input size, assuming that the weights are given in binary. \square

In [9], it has been shown that given a mean-payoff game \mathcal{G} , a vertex v in \mathcal{G} , and a rational c , one can decide in non-deterministic polynomial time if $\mathbf{ASV}(v) > c$. The use of an infinite memory strategy σ_0 for Player 0 such that $\mathbf{ASV}(\sigma_0)(v) > c$ has been shown in [9]. Here we give an improvement to that result in [9] showing that if there exists a strategy σ_0 for Player 0 such that $\mathbf{ASV}(\sigma_0)(v) > c$, then there exists a finite memory strategy σ_0^{FM} such that $\mathbf{ASV}(\sigma_0^{\text{FM}})(v) > c$.

Towards this, we first define the notion of a witness for **ASV** as it appears in [9].

Witnesses for ASV For a mean-payoff game \mathcal{G} , we associate with each vertex v in \mathcal{G} , the following set of pairs of real numbers:

$$\Lambda(v) = \{(c, d) \in \mathbb{R}^2 \mid v \models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 \geq d\}$$

A vertex v is said to be (c, d) -bad if $(c, d) \in \Lambda(v)$. Let $c' \in \mathbb{R}$. A play π in \mathcal{G} is called a (c', d) -witness of $\mathbf{ASV}(v) > c$ if $(\underline{\mathbf{MP}}_0(\pi), \underline{\mathbf{MP}}_1(\pi)) = (c', d)$ where $c' > c$, and π does not contain any (c, d) -bad vertex. A play π is called a witness of $\mathbf{ASV}^\epsilon(v) > c$ if it is a (c', d) -witness of $\mathbf{ASV}(v) > c$ for some c', d .

Now we state the following theorem which is similar to **Theorem 4.4**, but in the context of **ASV** instead of \mathbf{ASV}^ϵ .

Theorem 4.10. *For all mean-payoff games \mathcal{G} , for all vertices v in \mathcal{G} , and for all rationals c , we have that $\mathbf{ASV}(v) > c$ if and only if there exists a regular (c', d) -witness of $\mathbf{ASV}(v) > c$.*

The proof of this theorem is exactly the same as that of **Theorem 4.4**, and hence omitted.

Now using **Lemma 8** in [9] (which is similar to **Lemma 4.3**, but in the context of **ASV** instead of \mathbf{ASV}^ϵ), and using **Theorem 4.10**, we obtain the following theorem.

Theorem 4.11. *For all mean-payoff games \mathcal{G} , for all vertices $v \in V$, and for all rationals c , if $\mathbf{ASV}(v) > c$, then there exists a pseudopolynomial memory strategy σ_0 for Player 0 such that $\mathbf{ASV}(\sigma_0)(v) > c$.*

The proof follows since as in the proof of **Theorem 4.4**, the values of k and τ are polynomial in the size of \mathcal{G} , and the weights on the edges which are assumed to be given in binary.

It has been shown in [9] that given a mean-payoff game \mathcal{G} , a vertex v , and a rational $c \in \mathbb{Q}$, checking if $\mathbf{ASV}(v) > c$ is in NP. The use of \leq for establishing the existence of memoryless strategies for Player 0 for $\underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 \leq d - \epsilon$ in the case of \mathbf{ASV}^ϵ instead of $<$ in $\underline{\mathbf{MP}}_0 > c \vee \underline{\mathbf{MP}}_1 < d$ in the case of **ASV** makes the proof for \mathbf{ASV}^ϵ more involved.

Now we establish that in a mean-payoff game \mathcal{G} , the \mathbf{ASV}^ϵ and the **ASV** from every vertex v in the game \mathcal{G} do not change even if Player 0 is restricted to using only finite memory strategies. Formally, we state this below.

Corollary 4.12. *For all mean-payoff games \mathcal{G} , for all vertices v in \mathcal{G} , for all $\epsilon > 0$, and for all rationals c , if $\mathbf{ASV}^\epsilon(v) = c$ ($\mathbf{ASV}(v) = c$), then for every $c' < c$, there exists a finite memory strategy σ_0^{FM} for Player 0 such that $\mathbf{ASV}^\epsilon(\sigma_0^{FM})(v) > c'$ ($\mathbf{ASV}(\sigma_0^{FM})(v) > c'$). This implies that $\sup_{\sigma_0 \in \Sigma_0^{FM}} \mathbf{ASV}^\epsilon(\sigma_0)(v) = \mathbf{ASV}^\epsilon(v) = c$ ($\sup_{\sigma_0 \in \Sigma_0^{FM}} \mathbf{ASV}(\sigma_0)(v) = \mathbf{ASV}(v) = c$), that is, $\mathbf{ASV}_{FM}^\epsilon(v) = \mathbf{ASV}^\epsilon(v)$ ($\mathbf{ASV}_{FM}(v) = \mathbf{ASV}(v)$).*

We conclude this section by showing that the threshold problem is at least as hard as the value problem in the zero-sum mean-payoff game which is known to be in $\mathbf{NP} \cap \mathbf{coNP}$, and whose precise complexity is not known [22]. The zero-sum mean-payoff game is played between two players, Player 0 and Player 1, for an infinite duration and on a finite (single) weighted arena $\mathcal{A} = (V, E, \langle V_0, V_1 \rangle, w)$, where V is a set of vertices partitioned into V_0 and V_1 belonging to Player 0 and Player 1 respectively, E is a set of edges, and $w : E \rightarrow \mathbb{Q}$ assigns a rational weight to the edges of \mathcal{A} . We denote the zero-sum mean-payoff game by $\mathcal{G}_0 = (\mathcal{A}, \mathbf{MP})$. Initially, a token is put on some vertex of \mathcal{G}_0 . At each step of the play, the player controlling the vertex where the token is present chooses an outgoing edge and moves the token along the edge to the next vertex. Players interact in this way an infinite number of times and a play π of the game is simply an infinite

path traversed by the token. At each step, the objective of Player 0 is to choose an outgoing edges from the vertices she owns in a way so as to maximise the \liminf of the mean of the play π , denoted $\mathbf{MP}(\pi)$, while the objective of Player 1 is the opposite. Given a rational c , the *value* problem in the zero-sum mean-payoff game is to decide whether Player 0 has a strategy to get a mean-payoff greater than c against all possible strategies of Player 1. Zero-sum mean-payoff games are determined, and optimal memoryless strategies are known to exist for both players.

We now provide the following hardness result.

Theorem 4.13. *For all mean-payoff games \mathcal{G} , for all vertices v in \mathcal{G} , for all $\epsilon > 0$, and for all rationals c , the problem of deciding if $\mathbf{ASV}^\epsilon(v) > c$ or $\mathbf{ASV}(v) > c$ is at least as hard as solving the value problem in zero-sum mean-payoff games.*

Proof. We show the proof for $\mathbf{ASV}^\epsilon(v) > c$. The proof for the case of $\mathbf{ASV}(v) > c$ is exactly the same. Consider a zero-sum mean-payoff game $\mathcal{G}_0 = (\mathcal{A}, \mathbf{MP})$, where $\mathcal{A} = (V, E, \langle V_0, V_1 \rangle, w)$. We construct a bi-weighted mean-payoff game $\mathcal{G} = (\mathcal{A}', \mathbf{MP}_0, \mathbf{MP}_1)$ from \mathcal{G}_0 simply by adding to the arena \mathcal{A} a weight function w_1 that assigns a weight 0 to each edge. Stated formally, $\mathcal{A}' = (V, E, \langle V_0, V_1 \rangle, w, w_1)$ such that for all $e \in E$, we have that $w_1(e) = 0$.

Now consider that from a vertex $v \in V$, Player 0 has a winning strategy σ_0 in \mathcal{G}_0 such that $\mathbf{MP}(\sigma_0, \sigma_1) > c$ for all Player 1 strategies σ_1 , and where c is a rational. We show that by playing σ_0 from v in \mathcal{G} , we have that $\mathbf{ASV}^\epsilon(v) > c$. For every play $\pi \in \mathbf{Out}_v(\sigma_0)$ in \mathcal{G} , we have that $\mathbf{MP}_1(\pi) = 0$, and Player 1 has a response σ_π such that $\mathbf{Out}_v(\sigma_0, \sigma_\pi) = \pi$. In \mathcal{G} , Player 1 thus chooses a strategy that minimizes the mean-payoff of Player 0, and since in \mathcal{G}_0 , we have that $\mathbf{MP}(\sigma_0, \sigma_1) > c$ for all strategies σ_1 of Player 1, it follows that $\mathbf{ASV}^\epsilon(v) > c$.

Now in the other direction, consider that in \mathcal{G} , we have $\mathbf{ASV}^\epsilon(v) > c$. Thus from **Lemma 4.1**, there exists a strategy σ_0 for Player 0 such that $\mathbf{ASV}^\epsilon(v)(\sigma_0) > c$. Using similar arguments as above, we see that σ_0 is also a winning strategy in \mathcal{G}_0 giving a mean-payoff greater than c to Player 0. \square

5 Computation of the \mathbf{ASV}^ϵ

Given a two-dimensional mean-payoff game \mathcal{G} , a vertex v in \mathcal{G} , in this section, we show how to compute $\mathbf{ASV}^\epsilon(v)$. We first show a method to compute $\mathbf{ASV}^\epsilon(v)$ using the theory of reals with additions. This approach is very similar to the one followed in [9] with the difference that we use the formula Ψ_v^ϵ over first order theory of reals with addition instead of Ψ_v as has been described below. We describe this approach for completeness, and also because it gives us the necessary intuition to compute $\mathbf{ASV}^\epsilon(v)$, using a second method, by solving a set of linear programs (LP) leading to establishing an EXPTIME algorithm. This second method is new to this work, and does not appear in [9]. Moreover, [9] does not provide any complexity for the computation of \mathbf{ASV} . One can show that, with small modifications to our method involving solving linear programs, it is possible to compute $\mathbf{ASV}(v)$ as well in EXPTIME.

In the previous section, we established the existence of a notion of witness for $\mathbf{ASV}^\epsilon(v) > c$, for a rational c , leading to an NP algorithm for the threshold problem. We now show how to use this notion to effectively compute the $\mathbf{ASV}^\epsilon(v)$. To do this, we refer to the following lemma which has been established in [9].

Lemma 5.1. [Lemma 9 in [9]] *For all mean-payoff games \mathcal{G} , for all vertices v in \mathcal{G} , and for all rationals c, d , we can effectively construct a formula $\Psi_v(x, y)$ of $\langle \mathbb{R}, +, < \rangle$ with two free variables such that $(c, d) \in \Lambda(v)$ if and only if the formula $\Psi_v(x, y)[x/c, y/d]$ is true.*

Using the above lemma we can now compute an effective representation of the infinite set of pairs $\Lambda^\epsilon(v)$ for each vertex v of the mean-payoff game. This is stated in the following lemma.

Lemma 5.2. *For all mean-payoff games \mathcal{G} , for all vertices v in \mathcal{G} , for all $\epsilon > 0$, and for all rationals c, d , we can effectively construct a formula $\Psi_v^\epsilon(x, y)$ of $\langle \mathbb{R}, +, < \rangle$ with two free variables such that $(c, d) \in \Lambda^\epsilon(v)$ if and only if the formula $\Psi_v^\epsilon(x, y)[x/c, y/d]$ is true.*

Proof. From the definition of $\Lambda(v)$ from [9], we know that a pair of real values $(c, d) \in \Lambda(v)$ if $v \models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 \geq d$. We now recall from the definition of $\Lambda^\epsilon(v)$ that $(c, d) \in \Lambda^\epsilon(v)$ if $v \models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 \geq d - \epsilon$. From this, we can see that $\Psi_v^\epsilon(x, y) \equiv \exists e > 0 \cdot \Psi_v(x, y - \epsilon + e)$ \square

Extended mean-payoff game Similar to the approach of [9], we modify the arena \mathcal{A} in the given mean-payoff game $\mathcal{G} = (\mathcal{A}, \langle \underline{\mathbf{MP}}_0, \underline{\mathbf{MP}}_1 \rangle)$ with $\mathcal{A} = (V, E, \langle V_0, V_1 \rangle, w_0, w_1)$, and construct an extended mean-payoff game $\mathcal{G}^{\text{ext}} = (\mathcal{A}^{\text{ext}}, \langle \underline{\mathbf{MP}}_0, \underline{\mathbf{MP}}_1 \rangle)$, where the new arena \mathcal{A}^{ext} is defined as $\mathcal{A}^{\text{ext}} = (V^{\text{ext}}, E^{\text{ext}}, \langle V_0^{\text{ext}}, V_1^{\text{ext}} \rangle, w_0^{\text{ext}}, w_1^{\text{ext}})$, and whose vertices and edges are defined as follows. The set of vertices is $V^{\text{ext}} = V \times 2^V$. With a history h in \mathcal{G} , we associate a vertex in \mathcal{G}^{ext} which is a pair (v, P) , where $v = \text{last}(h)$ and P is the set of the vertices traversed along h . Accordingly the set of edges and the weight functions are respectively defined as $E^{\text{ext}} = \{((v, P), (v', P')) \mid (v, v') \in E \text{ and } P' = P \cup \{v'\}\}$ and $w_i^{\text{ext}}((v, P), (v', P')) = w_i(v, v')$, for $i \in \{0, 1\}$. We can see that there exists a bijection between the plays π in \mathcal{G} and the plays π^{ext} in \mathcal{G}^{ext} which start in vertices of the form $(v, \{v\})$, i.e. π^{ext} is mapped to the play π in \mathcal{G} that is obtained by removing the second dimension of its vertices.

We write the following proposition which is the same as **Proposition 10** in [9] with the difference that we use here \mathbf{ASV}^ϵ instead of \mathbf{ASV} .

Proposition 5.3. *For all mean-payoff games \mathcal{G} , the following holds:*

- *Let π^{ext} be an infinite play in the extended mean-payoff game and π be its projection on the original mean-payoff game \mathcal{G} (over the first component of each vertex); the following properties hold:*
 - *For all $i < j$, if $\pi^{\text{ext}}(i) = (v_i, P_i)$ and $\pi^{\text{ext}}(j) = (v_j, P_j)$, then $P_i \subseteq P_j$*
 - *$\underline{\mathbf{MP}}_i(\pi^{\text{ext}}) = \underline{\mathbf{MP}}_i(\pi)$, for $i \in \{0, 1\}$.*
- *The unfolding of \mathcal{G} from v and the unfolding of \mathcal{G}^{ext} from $(v, \{v\})$ are isomorphic and so $\mathbf{ASV}^\epsilon(v) = \mathbf{ASV}^\epsilon(v, \{v\})$*

By the first point of the above proposition and since the set of vertices of the mean-payoff game is finite, the second component of any play π^{ext} , that keeps track of the set of vertices visited along π^{ext} , stabilises into a set of vertices of \mathcal{G} which we denote by $V^*(\pi^{\text{ext}})$. We now show how to characterize $\mathbf{ASV}^\epsilon(v)$ with the notion of witness introduced above and the decomposition of \mathcal{G}^{ext} into SCC. This is formalized in the following lemma which is similar to **Lemma 11** in [9] where it characterizes \mathbf{ASV} instead of \mathbf{ASV}^ϵ .

Lemma 5.4. *For all mean-payoff games \mathcal{G} and for all vertices v in \mathcal{G} , let $\text{SCC}^{\text{ext}}(v)$ be the set of strongly-connected components in \mathcal{G}^{ext} which are reachable from $(v, \{v\})$. Then we have*

$$\mathbf{ASV}^\epsilon(v) = \max_{S \in \text{SCC}^{\text{ext}}(v)} \sup\{c \in \mathbb{R} \mid \exists \pi^{\text{ext}} : \pi^{\text{ext}} \text{ is a witness for } \mathbf{ASV}^\epsilon(v, \{v\}) > c \text{ and } V^*(\pi^{\text{ext}}) = S\}$$

Proof. First, we note the following sequence of inequalities:

$$\begin{aligned}
\mathbf{ASV}^\epsilon(v) &= \sup\{c \in \mathbb{R} \mid \mathbf{ASV}^\epsilon(v) \geq c\} \\
&= \sup\{c \in \mathbb{R} \mid \mathbf{ASV}^\epsilon(v) > c\} \\
&= \sup\{c \in \mathbb{R} \mid \exists \pi : \pi \text{ is a witness for } \mathbf{ASV}^\epsilon(v) > c\} \\
&= \sup\{c \in \mathbb{R} \mid \exists \pi^{\text{ext}} : \pi^{\text{ext}} \text{ is a witness for } \mathbf{ASV}^\epsilon(v, \{v\}) > c\} \\
&= \max_{S \in \text{SCC}^{\text{ext}}(v)} \sup\{c \in \mathbb{R} \mid \exists \pi^{\text{ext}} : \pi^{\text{ext}} \text{ is a witness for } \mathbf{ASV}^\epsilon(v, \{v\}) > c \text{ and } V^*(\pi^{\text{ext}}) = S\}
\end{aligned}$$

The first two equalities follow from the definition of the supremum and that $\mathbf{ASV}^\epsilon \in \mathbb{R}$. The third equality follows from **Theorem 4.2** that guarantees the existence of witnesses for strict inequalities. The fourth equality is due to the second point in **Proposition 5.3**. The last equality is a consequence of first point in **Proposition 5.3**. \square

By definition of \mathcal{G}^{ext} , for every SCC S of \mathcal{G}^{ext} , there exists a set of vertices of \mathcal{G} which we denote by $V^*(S)$ such that every vertex of S is of the form $(v', V^*(S))$, where v' is a vertex in \mathcal{G} . Now for the SCC S , we define $\Lambda_S^{\text{ext}} = \bigcup_{v \in V^*(S)} \Lambda^\epsilon(v)$ as the set of (c, d) such that Player 1 can ensure $v \models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 > d - \epsilon$ from some vertex $v \in S$. Applying **Lemma 5.2**, we can now construct a formula $\Psi_S^\epsilon(x, y)$ which symbolically encodes the set Λ_S^{ext} .

Now, we can prove that $\mathbf{ASV}^\epsilon(v)$ is effectively computable.

Theorem 5.5. *For all mean-payoff games \mathcal{G} , for all vertices v in \mathcal{G} and for all $\epsilon > 0$, the value $\mathbf{ASV}^\epsilon(v)$ can be effectively expressed by a formula in $\langle \mathbb{R}, +, < \rangle$ and can be computed from this formula.*

Proof. To prove this theorem, we build a formula in $\langle \mathbb{R}, +, < \rangle$ that is true iff $\mathbf{ASV}^\epsilon(v) = z$. Recall from **Lemma 5.4** that

$$\mathbf{ASV}^\epsilon(v) = \max_{S \in \text{SCC}^{\text{ext}}(v)} \sup\{c \in \mathbb{R} \mid \exists \pi^{\text{ext}} : \pi^{\text{ext}} \text{ is a witness for } \mathbf{ASV}^\epsilon(v, \{v\}) > c \text{ and } V^*(\pi^{\text{ext}}) = S\}$$

Since it is easy to express $\max_{S \in \text{SCC}^{\text{ext}}(v)}$ in $\langle \mathbb{R}, +, < \rangle$, we concentrate on one SCC S reachable from $(v, \{v\})$, and we show how to express

$$\sup\{c \in \mathbb{R} \mid \exists \pi^{\text{ext}} : \pi^{\text{ext}} \text{ is a witness for } \mathbf{ASV}^\epsilon(v, \{v\}) > c \text{ and } V^*(\pi^{\text{ext}}) = S\}$$

in $\langle \mathbb{R}, +, < \rangle$.

Such a value of c can be encoded by the following formula

$$\rho_v^S(c) \equiv \exists x, y \cdot x > c \wedge \Phi_S(x, y) \wedge \neg \Psi_S^\epsilon(c, y)$$

where $\Phi_S(x, y)$ is the symbolic encoding of $F_{\min}(\text{CH}(\mathbb{C}(S)))$ in $\langle \mathbb{R}, +, < \rangle$ as defined in **Lemma 2.1**. This states that the pair of values (x, y) are the mean-payoff values realisable by some play in S . By **Lemma 5.2**, the formula $\neg \Psi_S^\epsilon(c, y)$ expresses that the play does not cross a $(c, y)^\epsilon$ -bad vertex. So the conjunction $\exists x, y \cdot x > c \wedge \Phi_S(x, y) \wedge \neg \Psi_S^\epsilon(c, y)$ establishes the existence of a witness with mean-payoff values (x, y) for the threshold c , and hence satisfying this formula implies that $\mathbf{ASV}^\epsilon(v) > c$. Now we consider the formula

$$\rho_{\max, v}^S(z) \equiv \forall e > 0 \cdot \rho_v^S(z - e) \wedge \forall c \cdot \rho_v^S(c) \implies c < z$$

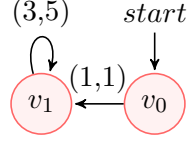


Figure 7: Example to calculate $\mathbf{ASV}^\epsilon(v)$

which is satisfied by a value that is the supremum over the set of values c such that c satisfies the formula ρ_v^S , and hence the formula $\rho_{\max,v}^S(z)$ expresses

$$\sup\{c \in \mathbb{R} \mid \exists \pi^{\text{ext}} : \pi^{\text{ext}} \text{ is a witness for } \mathbf{ASV}^\epsilon(v, \{v\}) > c \text{ and } V^*(\pi^{\text{ext}}) = S\}$$

From the formula $\rho_{\max,v}^S$, we can compute the $\mathbf{ASV}^\epsilon(v)$ by quantifier elimination in

$$\max_{S \in \text{SCC}^{\text{ext}}(v)} \exists z \cdot \rho_{\max,v}^S(z)$$

and obtain the unique value of z that makes this formula true, and equals $\mathbf{ASV}^\epsilon(v)$. \square

Example 5.6. We illustrate the computation of \mathbf{ASV}^ϵ with an example. Consider the mean-payoff game \mathcal{G} depicted in **Figure 7**. We note that in the mean-payoff game there exists exactly one SCC which is $S = \{v_1\}$, and it contains exactly one cycle $v_1 \rightarrow v_1$. Thus, $F_{\min}(\text{CH}(\mathbb{C}(S))) = \{(3, 5)\}$. Thus, we get that $\Phi_S(x, y) \equiv x = 3 \wedge y = 5$. Now $\Lambda^\epsilon(v) = \{(c, d) \mid c \geq 3 \wedge d < 5 + \epsilon\}$, and hence $\Psi_S^\epsilon(x, y) \equiv x \geq 3 \wedge y < 5 + \epsilon$. We note that the formula $\rho_{v_0}^S(c)$ holds for values of c less than 3, and by assigning 3 to x , and 5 to y , and thus $\rho_{\max,v_0}^S(z)$ holds true for $z = 3$. It follows that $\mathbf{ASV}^\epsilon(v_0) = 3$ since we have a single SCC in the example.

5.1 An EXPTIME algorithm for computing $\mathbf{ASV}^\epsilon(v)$

Now we provide another approach for computing $\mathbf{ASV}^\epsilon(v)$. We use linear programming to solve the problem and obtain an EXPTIME upper bound for the computation of $\mathbf{ASV}^\epsilon(v)$. Note that no complexity upper bound for computation of \mathbf{ASV} was reported in [9]. As described earlier, we first create the extended game \mathcal{G}^{ext} , followed by expressing the formula ρ_{\max,v_0}^S , for each SCC $S \in \text{SCC}^{\text{ext}}(v)$, as a set of linear programs such that the maximum value over the set of solutions over all SCCs corresponds to $\mathbf{ASV}^\epsilon(v)$. We first illustrate our approach with the help of the following example.

Example 5.7. Consider again the mean-payoff game \mathcal{G} in **Example 5.6**. We previously showed that the $\mathbf{ASV}^\epsilon(v_0)$ can be computed by quantifier elimination of a formula in the theory of reals with addition. Now, we compute $\mathbf{ASV}^\epsilon(v_0)$ by constructing \mathcal{G}^{ext} , and then by solving a set of linear programs for every SCC in \mathcal{G}^{ext} . In this example, there exists only one SCC $S = \{(v_1, \{v_0, v_1\})\}$ in \mathcal{G}^{ext} . From **Lemma 2.1**, we have that $F_{\min}(\text{CH}(\mathbb{C}(S)))$ can be defined using a set of linear inequations. Now recall from **Example 5.6** that $F_{\min}(\text{CH}(\mathbb{C}(S))) = \{(3, 5)\}$, and $\Lambda^\epsilon = \{(c, y) \mid c \geq 3 \wedge y < 5 + \epsilon\}$. Thus the complement of $\Lambda^\epsilon(v_0)$, that is, $\overline{\Lambda}^\epsilon(v_0) = \mathbb{R} \times \mathbb{R} - \Lambda^\epsilon(v_0) = \{(c, y) \mid c < 3 \vee y \geq 5 + \epsilon\}$. Also $\Phi_S(x, y)$ is satisfied by $x = 3 \wedge y = 5$. Now we express the formula $\rho_{v_0}^S(c)$ as $\exists x, y \models x > c \wedge x = 3 \wedge y = 5 \wedge (c < 3 \vee y \geq 5 + \epsilon)$. We maximise the value of c , which gives us the following two linear programs: *maximise c in $\exists x, y \models (x > c \wedge x = 3 \wedge y = 5 \wedge c < 3)$* and *maximise c in $\exists x, y \models (x > c \wedge x = 3 \wedge y = 5 \wedge y \geq (5 + \epsilon))$* which yields the set of solutions: $\{3\}$. Thus, we conclude that $\mathbf{ASV}^\epsilon(v_0) = 3$. Note that in an LP, the strict inequalities are replaced with non-strict inequalities, and computing the supremum in the objective function is replaced by maximizing the objective function.

Now we state the main result of this section.

Theorem 5.8. *For all mean-payoff games \mathcal{G} , for all vertices v in \mathcal{G} and for all $\epsilon > 0$, the $\mathbf{ASV}^\epsilon(v)$ can be computed in EXPTIME.*

Proof. The algorithm to compute $\mathbf{ASV}^\epsilon(v)$ is defined as follows:

1. First we note that using **Lemma 2.1**, for each SCC S in \mathcal{G}^{ext} , the set $F_{\min}(\text{CH}(\mathbb{C}(S)))$ can be expressed as a set of exponentially many inequations.
2. Recall that for every SCC S of \mathcal{G}^{ext} , there exists a set of vertices of \mathcal{G} which we denote by $V^*(S)$ such that every vertex of S is of the form $(v, V^*(S))$. Now note that the formula Ψ_S^ϵ corresponds to $\bigcup_{v \in V^*(S)} \Lambda^\epsilon(v)$, and hence $\neg \Psi_S^\epsilon$ in the formula $\rho_v^S(c)$ corresponds to the set $\bigcap_{v \in V^*(S)} \bar{\Lambda}^\epsilon(v)$. We show below that this set can be expressed as a union of exponentially many systems of strict and non-strict inequations.

From **Lemma 4** of [2], we have that $\Lambda^\epsilon(v)$ can be represented as a finite union of polyhedra. Considering a d -dimensional space, the set of points that satisfy the same set of linear inequations forms an equivalence class, also called *cells* [2]. Let $V_{\mathcal{G}}$ denote the set of mean-payoff coordinates of simple cycles in \mathcal{G} , and we have that $|V_{\mathcal{G}}| = \mathcal{O}(W \cdot |V|)^{\text{poly}(d)}$. Let $B(V_{\mathcal{G}})$ denote the set of geometric centres where each geometric centre is a centre of at most $d + 1$ points from $V_{\mathcal{G}}$. Thus $|B(V_{\mathcal{G}})| = \mathcal{O}(|V_{\mathcal{G}}|^{d+1})$. From **Lemma 6** of [2] that uses Carathéodory baricenter theorem in turn, we have that $\Lambda^\epsilon(v)$ can be represented as a union of all cells that contain a point from $B(V_{\mathcal{G}})$ which is in $\Lambda^\epsilon(v)$. Each cell is a polyhedron that can be represented by $\mathcal{O}(V_{\mathcal{G}})$ extremal points, or equivalently, by **Theorem 3** of [2], by $\mathcal{O}(V_{\mathcal{G}}) \cdot 2^d$ inequations.

It follows that $\bar{\Lambda}^\epsilon(v)$ can also be represented as a union of $\mathcal{O}(|V_{\mathcal{G}}|^{d+1})$ polyhedra. Hence $\bigcap_{v \in V^*(S)} \bar{\Lambda}^\epsilon(v)$ can also be represented as a union of $\mathcal{O}(|V_{\mathcal{G}}|^{d+1})$ polyhedra. Thus we have exponentially many linear programs corresponding to $\bigcap_{v \in V^*(S)} \bar{\Lambda}^\epsilon(v)$, since the weights on the edges are given in binary. In our case, we have $d = 2$.

Further, from **Lemma 2.1**, for bi-weighted arena, we have that $F_{\min}(\text{CH}(\mathbb{C}(S)))$ can be represented by $\mathcal{O}(|V_{\mathcal{G}}|^2)$ linear inequations, and hence $F_{\min}(\text{CH}(\mathbb{C}(S))) \cap \bigcap_{v \in V^*(S)} \bar{\Lambda}^\epsilon(v)$ can be represented by exponentially many linear programs. Further, these LPs can be constructed in exponential time.

3. For each SCC S in the mean-payoff game \mathcal{G}^{ext} , we see that the value satisfying the formula $\rho_{\max, v}^S$ can be expressed as a set of linear programs in the following manner:
 - For each linear program, we have two variables x and y that represent the mean-payoff values of Player 0 and Player 1 respectively, corresponding to plays in $F_{\min}(\text{CH}(\mathbb{C}(S)))$ and a third variable c represents the c in the formula $\rho_v^S(c)$.
 - The formula $\rho_v^S(c)$ can be expressed by a disjunction of a set of linear inequations, i.e., $\bigvee_i \text{Cst}_i$ where each Cst_i is a conjunction of linear inequations. The disjunctions arise due to the representation of $\bigcap_{v \in V^*(S)} \bar{\Lambda}^\epsilon(v)$ as stated above. We use the variables (c, d) in the linear inequations which represent the set $\bigcap_{v \in V^*(S)} \bar{\Lambda}^\epsilon(v)$, and the variables (x, y) in the

linear inequations representing the set $F_{\min}(\text{CH}(\mathbb{C}(S)))$. We also include the inequation $x > c$. Further, to express the formula $\rho_v^S(c)$, the variable d should assume the value of y that corresponds to the mean-payoff of Player 1 as stated earlier. Note that, we would need to represent the difference between $F_{\min}(\text{CH}(\mathbb{C}(S)))$ that uses variables c, y and $\bigcap_{v \in V^*(S)} \Lambda^\epsilon(v)$ that uses variables x, y . In the LP formulation, each strict inequation is replaced by a non-strict inequation, and supremum in the objective function is replaced with maximizing the objective.

- The set of linear programs we solve is *maximise c under the linear constraints $\exists x, y \cdot (c, x, y) \models \text{Cst}_i$ for each $\text{Cst}_i \in \bigvee_i \text{Cst}_i$* . The value satisfying the formula $\rho_{\max, v}^S$ is the maximum over the set of solutions of the linear programs obtained.

We solve the above sets of linear programs for each SCC S present in the mean-payoff game \mathcal{G} to get a set of values satisfying the formula $\rho_{\max, v}^S$. We choose the maximum of this set of values which is the $\mathbf{ASV}^\epsilon(v)$. Note that there can be exponentially many SCCs. Thus our algorithm runs in EXPTIME. \square

6 Achievability of the \mathbf{ASV}^ϵ

Here we show that \mathbf{ASV}^ϵ is achievable which is in contrast to [9] where the follower is fully rational.

Theorem 6.1. *For all mean-payoff games \mathcal{G} , for all vertices v in \mathcal{G} , and for all $\epsilon > 0$, we have that the $\mathbf{ASV}^\epsilon(v)$ is achievable.*

The rest of this section is devoted to proving **Theorem 6.1**. We start by defining the notion of a witness for $\mathbf{ASV}^\epsilon(\sigma_0)$.

Witness for $\mathbf{ASV}^\epsilon(\sigma_0)$ Given a mean-payoff game \mathcal{G} , a vertex v in \mathcal{G} , and an $\epsilon > 0$, we say that a play π is a witness for $\mathbf{ASV}^\epsilon(\sigma_0)(v) > c$ for a strategy σ_0 of Player 0 if $\pi \in \mathbf{Out}_v(\sigma_0)$, and π is a witness for $\mathbf{ASV}^\epsilon(v) > c$ when Player 0 uses strategy σ_0 that is defined as follows.

- σ_0 follows π if Player 1 does not deviate from π .
- If Player 1 deviates π , then for each vertex $v \in \pi$, we have that σ_0 consists of a memoryless strategy that establishes $v \not\ll 1 \gg \underline{\mathbf{MP}}_0 \leq c \wedge \underline{\mathbf{MP}}_1 > d - \epsilon$, where $d = \underline{\mathbf{MP}}_1(\pi)$. The existence of such a memoryless strategy for Player 0 has been established in **Section 4**.

From **Corollary 4.12**, we know that $\mathbf{ASV}^\epsilon(v) = \mathbf{ASV}_{\text{FM}}^\epsilon(v)$, i.e., the Adversarial Stackelberg Value for fixed ϵ from vertex v of a game \mathcal{G} remains the same even if we restrict Player 0 to using only finite memory strategies. Note that however it is possible that the $\mathbf{ASV}^\epsilon(v)$ is not achievable by a finite memory strategy of Player 0 as shown in **Theorem 3.1**.

We consider below the interesting case, where $\mathbf{ASV}^\epsilon(v)$ cannot be achieved by a finite memory strategy. We show that for such cases, it can indeed be achieved by an infinite memory strategy.

Let $\mathbf{ASV}^\epsilon(v) = c$. For every $c' < c$, from **Theorem 4.5**, there exists a finite memory strategy σ_0 such that $\mathbf{ASV}^\epsilon(\sigma_0)(v) > c'$. Now, consider a sequence of increasing real numbers $c_1 < c_2 < c_3 < \dots < c$ for which there exist finite memory strategies $\sigma_0^1, \sigma_0^2, \sigma_0^3, \dots$ such that for each c_i , we have that $\mathbf{ASV}^\epsilon(\sigma_0^i)(v) > c_i$.

We note from **Theorem 4.4** that there exists a regular witness π^i for $\mathbf{ASV}^\epsilon(\sigma_0^i)(v) > c_i$ for each σ_0^i where $\pi^i = \pi_{1i}(l_{1i}^{\alpha \cdot k_i} \cdot \pi_{2i} \cdot l_{2i}^{\beta \cdot k_i} \cdot \pi_{3i})^\omega$.

We now describe a sequence of plays which serve as witnesses for the sequence of finite memory strategies mentioned above. We show that these plays in the sequence differ from each other only in the value of k_i they use. This is stated in the following proposition.

Proposition 6.2. *There exists a sequence of increasing real numbers, $c_1 < c_2 < c_3 < \dots < c$, and finite memory strategies $\sigma_0^1, \sigma_0^2, \sigma_0^3, \dots$ of Player 0 such that for each c_i , we have $\mathbf{ASV}^\epsilon(\sigma_0^i)(v) > c_i$, and there exists a play π^i that is a witness for $\mathbf{ASV}^\epsilon(\sigma_0^i)(v) > c_i$, where $\pi^i = \pi_1(l_1^{\alpha \cdot k_i} \cdot \pi_2 \cdot l_2^{\beta \cdot k_i} \cdot \pi_3)^\omega$, and π_1, π_2 and π_3 are simple finite plays, and l_1, l_2 are simple cycles in the arena of the game \mathcal{G} .*

Proof. Consider the play $\pi^i = \pi_1(l_1^{\alpha \cdot k_i} \cdot \pi_2 \cdot l_2^{\beta \cdot k_i} \cdot \pi_3)^\omega$ which is a witness for $\mathbf{ASV}^\epsilon(\sigma_0^i)(v) > c_i$ for the strategy σ_0^i . Let $\mathbf{MP}_0(\pi^i) = c'_i > c_i$.

We have that $\mathbf{MP}_0(\pi^i)$ increases proportionally with i as $\alpha \cdot k_i$ and $\beta \cdot k_i$ increase with increasing k_i . This follows because we disregard the cases where $\mathbf{ASV}^\epsilon(v)$ is achievable with some finite memory strategy of Player 0, i.e., we only consider the case where $\mathbf{ASV}^\epsilon(v)$ is not achievable by a finite memory strategy of Player 0.

Note that there are finitely many possible simple plays and simple cycles. Thus w.l.o.g. we can assume that in the sequence $(\pi^i)_{i \in \mathbb{N}^+}$, the finite plays $\pi_{1i}, \pi_{2i}, \pi_{3i}$, and the simple cycles l_{1i}, l_{2i} are the same for different values of i . Thus, $\mathbf{MP}_0(\pi_{1i}(l_{1i}^{\alpha \cdot k_i} \cdot \pi_{2i} \cdot l_{2i}^{\beta \cdot k_i} \cdot \pi_{3i})^\omega) = c'_i > c_i$, $\mathbf{MP}_0(\pi_{1i}(l_{1i}^{\alpha \cdot k_{i+1}} \cdot \pi_{2i} \cdot l_{2i}^{\beta \cdot k_{i+1}} \cdot \pi_{3i})^\omega) = c'_{i+1} > c'_i$, $\mathbf{MP}_0(\pi_{1i}(l_{1i}^{\alpha \cdot k_{i+2}} \cdot \pi_{2i} \cdot l_{2i}^{\beta \cdot k_{i+2}} \cdot \pi_{3i})^\omega) = c'_{i+2} > c'_{i+1}$, and so on, and the only difference in the strategies σ_0^i as i changes is the value of k_i , i.e, we increase the value of $\alpha \cdot k_i$ and $\beta \cdot k_i$ with increasing k_i such that the effect of π_2 and π_3 on the mean-payoff is minimised. Thus, at the limit, as $i \rightarrow \infty$, the sequence $(c_i)_{i \in \mathbb{N}^+}$ converges to $\alpha \cdot \mathbf{MP}_0(l_1) + \beta \cdot \mathbf{MP}_0(l_2) = c$. \square

To show that $\lim_{i \rightarrow \infty} \mathbf{ASV}^\epsilon(\sigma_0^i)(v) = c$, we construct a play π^* as follows. The play π^* starts from v , follows π^1 until the mean-payoff of Player 0 over the prefix becomes greater than c_1 . Then for $i \in \{2, 3, \dots\}$, starting from $\text{first}(l_1)$, it follows π^i , excluding the initial simple finite play π_1 , until the mean-payoff of the prefix of π^i becomes greater than c_i . Then the play π^* follows the prefix of the play π^{i+1} , excluding the initial finite play π_1 , and so on.

Clearly, we can see that $\mathbf{MP}_1(\pi^*) = c$. We let $\mathbf{MP}_1(\pi^*) = d = \alpha \cdot \mathbf{MP}_1(l_1) + \beta \cdot \mathbf{MP}_1(l_2)$.

For the sequence of plays $(\pi^i)_{i \in \mathbb{N}^+}$ which are witnesses for $(\mathbf{ASV}^\epsilon(\sigma_0^i)(v) > c_i)_{i \in \mathbb{N}^+}$ for the strategies $(\sigma_0^i)_{i \in \mathbb{N}^+}$, we let $\mathbf{MP}_1(\pi_i) = d_i$. We state the following proposition.

Proposition 6.3. *The sequence $(d_i)_{i \in \mathbb{N}^+}$ is monotonic, and it converges to d in the limit.*

Proof. Recall that $\mathbf{MP}_0(\pi^i)$ increases monotonically with increasing i . Since the effect of the finite simple plays π_2 and π_3 decreases with increasing $\alpha \cdot k_i$ and $\beta \cdot k_i$, the mean-payoff on the second dimension also changes monotonically. If $\alpha \cdot \mathbf{MP}_1(l_1) + \beta \cdot \mathbf{MP}_1(l_2) \geq \frac{w_1(\pi_2) + w_1(\pi_3)}{|\pi_2| + |\pi_3|}$, then the sequence $(d_i)_{i \in \mathbb{N}^+}$ is monotonically non-decreasing. Otherwise, the sequence is monotonically decreasing.

The fact that this sequence converges to d in the limit can be seen from the construction of π^* as described above. \square

Now, we have the tools to prove **Theorem 6.1**.

Proof of Theorem 6.1. We start by constructing a sequence of increasing numbers $c_1 < c_2 < c_3 < \dots < c$ such that:

- For every $i \in \mathbb{N}^+$, we consider a strategy σ_0^i of Player 0 that ensures $\mathbf{ASV}^\epsilon(\sigma_0^i)(v) > c_i$. [Follows from **Theorem 4.5**]
- The strategy σ_0^i follows a play $\pi^i = \pi_1(l_1^{\alpha \cdot k_i} \cdot \pi_2 \cdot l_2^{\beta \cdot k_i} \cdot \pi_3)^\omega$ where π_1, π_2 and π_3 are simple finite plays and l_1, l_2 are simple cycles in the game \mathcal{G} . [Follows from **Proposition 6.2**]

- We let $\underline{\mathbf{MP}}_0(\pi^i) = d_i$. The sequence $(d_i)_{i \in \mathbb{N}^+}$ is monotonic. [Follows from **Proposition 6.3**]
- We construct a play π^* from $(\pi^i)_{i \in \mathbb{N}^+}$ as described above such that $\underline{\mathbf{MP}}_0(\pi^*) = c$ and $\underline{\mathbf{MP}}_1(\pi^*) = d$.

If the \mathbf{ASV}^ϵ is not achievable, then there exists a strategy of Player 1 to enforce some play π' such that $\underline{\mathbf{MP}}_0(\pi') = c' < c$ and $\underline{\mathbf{MP}}_1(\pi') = d' > d - \epsilon$. Now, we use the monotonicity of the sequence $(d_i)_{i \in \mathbb{N}^+}$ to show a contradiction.

Since the sequence $(d_i)_{i \in \mathbb{N}^+}$ is monotonic, there can be two cases.

1. The sequence $(d_i)_{i \in \mathbb{N}^+}$ is monotonically non-decreasing.
2. The sequence $(d_i)_{i \in \mathbb{N}^+}$ is monotonically decreasing.

We start with the first case where the sequence $(d_i)_{i \in \mathbb{N}^+}$ is non-decreasing. Assume for contradiction that $\mathbf{ASV}^\epsilon(v)$ is not achievable, i.e. Player 1 deviates from π^* to enforce the play π' such that $\underline{\mathbf{MP}}_0(\pi') = c' < c$ and $\underline{\mathbf{MP}}_1(\pi') = d'$.

Since $(d_i)_{i \in \mathbb{N}^+}$ is non-decreasing, and thus $d \geq d_i$ for all $i \in \mathbb{N}^+$, and since Player 1 can let his payoff to be reduced by an amount that is less than ϵ in order to reduce the payoff of Player 0, for all $i \in \mathbb{N}^+$ we have that $d' > d_i - \epsilon$. We know that the sequence $(c_i)_{i \in \mathbb{N}^+}$ is increasing. Thus, there exists a $j \in \mathbb{N}$ such that $c' < c_j$. Note that if $\pi' = \pi^r$ for some index r , we consider some j which is also greater than r .

Now, consider the strategy σ_0^j of Player 0 which follows the play π_j . We know that $\underline{\mathbf{MP}}_0(\pi_j) > c_j$ and $\underline{\mathbf{MP}}_1(\pi_j) = d_j$. We also know from **Lemma 4.3** that the play π_j does not cross a $(c_j, d_j)^\epsilon$ -bad vertex. Since by the construction of π^* , and by **Proposition 6.2**, the set of vertices appearing in the play π^* is the same as the set of vertices appearing in the play π_j , for every vertex v in π^* , we have that Player 1 does not have a strategy such that $v \models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c_j \wedge \underline{\mathbf{MP}}_1 > d_j - \epsilon$. Since $c' < c_j$ and $d' > d_j - \epsilon$, it also follows that Player 1 does not have a strategy such that $v \models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c' \wedge \underline{\mathbf{MP}}_1 \geq d'$. Stated otherwise, from the determinacy of multi-player mean-payoff games, we have that Player 0 has a strategy to ensure $v \not\models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c' \wedge \underline{\mathbf{MP}}_1 \geq d'$ for every vertex v appearing in π^* . In fact, Player 0 can ensure $v \not\models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c' \wedge \underline{\mathbf{MP}}_1 \geq d'$ by choosing the strategy $\sigma_{j'}$ for some $j' \geq j$. Since $\mathbf{ASV}^\epsilon(v) = \sup_{\sigma_0 \in \Sigma_0} \mathbf{ASV}^\epsilon(\sigma_0)(v)$, and the sequence

$(c_i)_{i \in \mathbb{N}^+}$ is increasing, and we have that $\mathbf{ASV}^\epsilon(\sigma_i)(v) > c_i$ for all $i \in \mathbb{N}^+$, it follows that the existence of π' is a contradiction.

Now, we consider the case where the sequence $(d_i)_{i \in \mathbb{N}^+}$ is monotonically decreasing. Again, assume for contradiction that $\mathbf{ASV}^\epsilon(v)$ is not achievable, i.e., Player 1 deviates from σ_0^* to enforce the play π' such that $\underline{\mathbf{MP}}_0(\pi') = c' < c$ and $\underline{\mathbf{MP}}_1(\pi') = d'$. Since the sequence $(d_i)_{i \in \mathbb{N}^+}$ is monotonically decreasing, we know that there must exist a $j \in \mathbb{N}$ such that (i) $d' > d_j - \epsilon$, and $\forall i \geq j$, we have that $d' > d_i - \epsilon$, and (ii) $c_j > c'$, which follows since $(c_i)_{i \in \mathbb{N}^+}$ is a strictly increasing sequence. Thus for every vertex v in π^* , Player 1 does not have a strategy such that there exists a play π in $\mathbf{Out}_v(\sigma_0^j)$, and $v \models \ll 1 \gg \underline{\mathbf{MP}}_0 \leq c_j \wedge \underline{\mathbf{MP}}_1 > d_j - \epsilon$.

Finally, using the fact that $c' < c_j$ and $d' > d_j - \epsilon$, the contradiction follows exactly as above where the sequence $(d_i)_{i \in \mathbb{N}^+}$ is monotonically non-decreasing. \square

7 Conclusion and Future Work

In this work, we considered two-player non-zero sum infinite duration mean-payoff Stackelberg games played on graph arenas. We also assumed that the follower is bounded rational, and hence would always choose an ϵ -best-response to the leader's strategy.

In [9], the authors assume that the follower is fully rational, and thus will always play an adversarial best-response for a given strategy of the leader (when it exists). However, the assumption that the follower is fully rational may not always be realistic.

We considered the threshold problem, i.e, checking if $\mathbf{ASV}^\epsilon > c$, and showed that the problem is in NP, and it is at least as hard as the value problem in zero-sum mean-payoff games. Additionally, we showed that if $\mathbf{ASV}^\epsilon > c$, we can construct a pseudopolynomial memory strategy for the leader to achieve this threshold. We also showed that both \mathbf{ASV}^ϵ and \mathbf{ASV} remain unaffected when the leader is allowed to use only finite memory strategies. We gave an EXPTIME algorithm to compute the \mathbf{ASV}^ϵ , and showed that the \mathbf{ASV}^ϵ is always achievable, possibly with an infinite memory strategy, which is in contrast with the framework where the follower is fully rational [9].

Several problems related to the results presented here can be studied. We conjecture that the threshold problem is at least as hard as deciding the existence of a winning strategy in zero-sum mean-payoff games in a bi-weighted game graph [20] whose precise complexity is also an open problem. We would like to study this relative hardness result. Another interesting direction is to explore possible characterization of Stackelberg mean-payoff games where a finite memory strategy of Player 0 suffices to achieve the \mathbf{ASV}^ϵ . This will allow us to get an estimate of how “complex” a game is in terms of the memory required by Player 0 in order to achieve the \mathbf{ASV}^ϵ .

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A Missing details in the proof of Lemma 4.3

Below we compute the expressions for k and τ for case 1 in the proof of **Theorem 4.4**. We know that for play $\pi = \pi_1 \rho_1 \rho_2 \rho_3 \dots$, where $\rho_i = l_1^{[\alpha i]} \cdot \pi_2 \cdot l_2^{[\beta i]} \cdot \pi_3$, constants $\alpha, \beta \in \mathbb{R}$ are chosen such that:

$$\begin{aligned}\alpha \cdot \underline{\mathbf{MP}}_0(l_1) + \beta \cdot \underline{\mathbf{MP}}_0(l_2) &= c' \\ \alpha \cdot \underline{\mathbf{MP}}_1(l_1) + \beta \cdot \underline{\mathbf{MP}}_1(l_2) &= d \\ \alpha + \beta &= 1\end{aligned}$$

We assume here that $\underline{\mathbf{MP}}_0(l_1) > \underline{\mathbf{MP}}_0(l_2)$ and $\underline{\mathbf{MP}}_1(l_1) < \underline{\mathbf{MP}}_1(l_2)$. This implies that one simple cycle, l_1 , increases Player 0's mean-payoff while the other simple cycle, l_2 , increases Player 1's mean-payoff. We build a play $\pi' = \pi_1 \cdot (l_1^{[\alpha]k} \cdot \pi_2 \cdot l_2^{[k+\tau]\beta} \cdot \pi_3)^\omega$ where we choose constant $k \in \mathbb{N}$ and constant $\tau > 0$ such that $\underline{\mathbf{MP}}_0(\pi') = c'$ and $\underline{\mathbf{MP}}_1(\pi') = d$. We try to express the conditions for k and τ below:

$$\begin{aligned}\underline{\mathbf{MP}}_0(\pi') &= \frac{k \cdot \alpha \cdot w_0(l_1) + (k + \tau) \cdot \beta \cdot w_0(l_2) + w_0(\pi_2) + w_0(\pi_3)}{k \cdot \alpha \cdot |l_1| + (k + \tau) \cdot \beta \cdot |l_2| + |\pi_2| + |\pi_3|} \\ &= \frac{k \cdot (\alpha \cdot w_0(l_1) + \beta \cdot w_0(l_2)) + \tau \cdot \beta \cdot w_0(l_2) + w_0(\pi_2) + w_0(\pi_3)}{k \cdot (\alpha \cdot |l_1| + \beta \cdot |l_2|) + \tau \cdot \beta + |\pi_2| + |\pi_3|} \\ \underline{\mathbf{MP}}_1(\pi') &= \frac{k \cdot \alpha \cdot w_1(l_1) + (k + \tau) \cdot \beta \cdot w_1(l_2) + w_1(\pi_2) + w_1(\pi_3)}{k \cdot \alpha \cdot |l_1| + (k + \tau) \cdot \beta \cdot |l_2| + |\pi_2| + |\pi_3|} \\ &= \frac{k \cdot (\alpha \cdot w_1(l_1) + \beta \cdot w_1(l_2)) + \tau \cdot \beta \cdot w_1(l_2) + w_1(\pi_2) + w_1(\pi_3)}{k \cdot (\alpha \cdot |l_1| + \beta \cdot |l_2|) + \tau \cdot \beta + |\pi_2| + |\pi_3|}\end{aligned}$$

Let $|\pi_2| + |\pi_3| = v$, $\alpha \cdot w_0(l_1) + \beta \cdot w_0(l_2) = x_0$, $\alpha \cdot w_1(l_1) + \beta \cdot w_1(l_2) = x_1$, $\alpha \cdot |l_1| + \beta \cdot |l_2| = y$, $w_0(\pi_2) + w_0(\pi_3) = z_0$ and $w_1(\pi_2) + w_1(\pi_3) = z_1$. We simplify the inequalities above to get:

$$\begin{aligned}\underline{\mathbf{MP}}_0(\pi') &= \frac{k \cdot x_0 + \tau \cdot \beta \cdot w_0(l_2) + z_0}{k \cdot y + \tau \cdot \beta + v} \\ \underline{\mathbf{MP}}_1(\pi') &= \frac{k \cdot x_1 + \tau \cdot \beta \cdot w_1(l_2) + z_1}{k \cdot y + \tau \cdot \beta + v}\end{aligned}$$

We know that $\underline{\mathbf{MP}}_0(\pi') = c'$ and $\underline{\mathbf{MP}}_1(\pi') = d$. Thus,

$$\begin{aligned}\frac{k \cdot x_0 + \tau \cdot \beta \cdot w_0(l_2) + z_0}{k \cdot y + \tau \cdot \beta + v} &= c' \\ \frac{k \cdot x_1 + \tau \cdot \beta \cdot w_1(l_2) + z_1}{k \cdot y + \tau \cdot \beta + v} &= d \\ k \cdot x_0 + \tau \cdot \beta \cdot w_0(l_2) + z_0 &= c' \cdot (k \cdot y + \tau \cdot \beta + v) \\ k \cdot x_1 + \tau \cdot \beta \cdot w_1(l_2) + z_1 &= d \cdot (k \cdot y + \tau \cdot \beta + v)\end{aligned}$$

Simplifying the above inequalities we get:

$$\begin{aligned}k \cdot (x_0 - c' \cdot y) &= c' \cdot v + \tau \cdot \beta \cdot (c' - w_0(l_2)) - z_0 \\ \tau \cdot \beta \cdot (w_1(l_2) - d) &= v \cdot d + k \cdot (d \cdot y - x_1) - z_1\end{aligned}$$

Finally, after substitution of τ in the first inequality expression and further simplification of both expressions, we finally get:

$$k = \frac{(c' \cdot v - z_0)(w_1(l_2) - d) + (c' - w_0(l_2))(v \cdot d - z_1)}{(x_0 - c' \cdot y)(w_1(l_2) - d) - (d \cdot y - x_1)}$$

$$\tau = \frac{v \cdot d - z_1}{\beta \cdot (w_1(l_2) - d)} + \frac{d \cdot y - x_1}{w_1(l_2) - d} \cdot \frac{k}{\beta}$$

The above two inequalities specify the range from which we can choose a suitable k and τ , such that the requirements $\underline{\mathbf{MP}}_0(\pi') = c'$ and $\underline{\mathbf{MP}}_1(\pi') = d$ are met. We note that k and τ are polynomial in size of the game and the weights on the edges, i.e., $\mathcal{O}(k) = |W|^2 \cdot |V|^3$ and $\mathcal{O}(\tau) = |W|^3 \cdot |V|^5$.