Theoretical Computer Science 99 (1992) 291–299
Elsevier

Note

On the decidability of the equivalence problem for partially commutative rational power series

Stefano Varricchio

Dipartimento di Matematica Pura e Applicata, Università degli studi dell'Aquila, Via Vetoio, 67100 L'Aquila, Italy, and Laboratoire d'Informatique Théorique et Programmation, Université Paris VI, France

Communicated by A. Salomaa
Received February 1991
Revised October 1991

Abstract

Varricchio, S., On the decidability of the equivalence problem for partially commutative rational power series, Theoretical Computer Science 99 (1992) 291–299.

Starting from an idea of Harju and Karhumäki (1991), we prove the decidability of the equivalence problem for partially commutative power series and unambiguous rational sets of the free partially commutative monoids.

1. Introduction and preliminaries

In this paper we prove some decidability results about rational power series on a free partially commutative monoid. In particular, we show that for these series the equivalence problem is decidable; nevertheless, the same problem for rational sets of the free partially commutative monoid is undecidable in the general case.

As a consequence of our result, we prove that the equivalence problem for the unambiguous rational sets of the free partially commutative monoid is decidable for any concurrence relation. A similar result has been obtained in [5] in the particular case of a three letters alphabet $A = \{a, b, c\}$ and for the concurrence relation

0304-3975/92/$05.00

$\theta = \{(a,b),(a,c)\}$. Moreover, we find that both inclusion and disjointness problems remain undecidable for unambiguous rational sets.

In this work we use some techniques which have been introduced by Harju and Karhumäki [11] for proving the decidability of the equivalence problem for deterministic multitape automata. The most important fact they use is the decidability of the equivalence problem for recognizable series with coefficients in a division ring. In our construction we use an important result of Duchamp and Krob on the lower central series of the free partially commutative group [7].

In what follows, A denotes a finite alphabet and A^* the free monoid generated by A. The empty word is denoted by Λ. A *concurrence relation* θ is a subset of $A \times A$ and ρ_θ denotes the congruence relation of A^* generated by the set $C_\theta = \{(ab, ba) \mid (a,b) \in \theta\}$. The quotient $M(A,\theta) = A^*/\rho_\theta$ is the *free partially commutative monoid* associated with the concurrence relation θ. Similarly, one can define the *free partially commutative group* $F(A,\theta)$ as the quotient of the free group $F(A)$ with respect to congruence of $F(A)$ generated by C_θ.

A monoid M is called *finitely factored* if for any $m \in M$ the set $T(m) = \{(m_1, \ldots, m_k) \mid m = m_1 \ldots m_k,\ m_i \neq 1 \text{ for } 1 \leqslant i \leqslant k, \text{ and } k \geqslant 0\}$ is finite. Let K be a semiring and M a monoid; we consider the K-module $K[[M]]$ of all the applications $\alpha : M \to K$. An element $\alpha \in K[[M]]$ will be also denoted by $\sum_{m \in M} \alpha(m)m$ or by $\sum_{m \in M} (\alpha, m)m$, where (α, m) denotes $\alpha(m)$. $K[[M]]$ is called the K-module of the *formal power series* over M with coefficients in K. If M is a finitely factored monoid then $K[[M]]$ has a structure of K-algebra where the product of two series $S, T \in K[[M]]$ is defined by

$$ST = \sum_{m \in M} \left(\sum_{uv = m} (S,u)(T,v) \right) m.$$

Moreover, if S is a series such that $(S, \Lambda) = 0$, one can define the series S^* by

$$S^* = \sum_{m \in M} \left(\sum_{u_1 \ldots u_k = m} (S, u_1) \ldots (S, u_k) \right) m.$$

We denote by $K[M]$ the subalgebra of $K[[M]]$, which consists of all *polynomials* over M with coefficients in K, i.e. those series with finitely many nonzero coefficients. The K-algebra $K[M]$ can be also defined for a monoid which is not finitely factored. If G is a group then $K[G]$ is also called a *group-ring*.

A formal power series $S \in K[[M]]$ is called *recognizable* if there exist a positive integer n, $\lambda, \gamma \in K^n$ and a monoid morphism $\mu : M \to K^{n \times n}$ such that for any $m \in M$ one has

$$(S, m) = \lambda \mu(m) \gamma,$$

where λ, γ are to be considered as a row vector and a column vector, respectively; moreover, the triple (λ, μ, γ) is called a *linear representation* of order n.

The set of recognizable series is denoted by $\mathrm{Rec}(K[[M]])$. The set of *rational series* $\mathrm{Rat}(K[[M]])$ is defined as the smallest subalgebra of $K[[M]]$ containing

$K[M]$ and closed with respect to the star operation. In the sequel, we adopt the following equivalent notations: $K\langle\langle A\rangle\rangle = K[[A^*]]$, $K\langle\langle A,\theta\rangle\rangle = K[[M(A,\theta)]]$, $K\langle A\rangle = K[A^*]$, $K\langle A,\theta\rangle = K[M(A,\theta)]$. We observe that A^* and $M(A,\theta)$ are finitely factored; hence, product and star operation can be considered in $K\langle\langle A\rangle\rangle$ and $K\langle\langle A,\theta\rangle\rangle$. In this paper we consider formal power series with coefficients in the field $\mathbb{Q}$ of rational numbers, but all results are true if one considers any (computable) division ring.

Formal power series over the free monoid A^* have been extensively studied (cf. [2, 9, 15]) in the context of the theoretical computer science as the natural extension of the concept of formal language. Formal power series over the free partially commutative free monoid have been considered in [8] and they seem to be the natural object for studying multiplicity in trace languages theory.

2. Preliminary results

In order to prove our main result we need some preliminary lemmas and propositions.

Proposition 2.1. *Let K, K' be two rings. Then any morphism $\psi: K\to K'$ has a canonical extension to a morphism $\psi': K\langle\langle A\rangle\rangle \to K'\langle\langle A\rangle\rangle$ such that* $\mathrm{Rec}(K'\langle\langle A\rangle\rangle) \supseteq \psi'(\mathrm{Rec}(K\langle\langle A\rangle\rangle))$.

Proof. Let $S\in\mathrm{Rec}(K\langle\langle A\rangle\rangle)$. Then $\psi'(S)$ is defined by

$$(\psi'(S), w) = \psi(S, w), \quad w\in A^*.$$

Since ψ is a morphism, the same holds for ψ'. Moreover, if S is recognizable then there exist $\lambda, \gamma\in K^n$ and a morphism $\mu: A^*\to K^{n\times n}$ such that for any $w\in A^*$

$$(S, w) = \lambda\mu(w)\gamma.$$

We may consider $\lambda', \gamma'\in K'^n$ and a morphism $\mu': A^*\to K'^{n\times n}$ defined by $\lambda'_i = \psi(\lambda_i)$, $\gamma'_i = \psi(\gamma_i)$ for $1\leqslant i\leqslant n$, $(\mu'(a))_{ij} = \psi((\mu(a))_{ij})$ for $1\leqslant i\leqslant n$, $1\leqslant j\leqslant n$ and $a\in A^*$. Since ψ is a morphism, for any $w\in A^*$ one has

$$(\psi'(S), w) = \psi(S, w) = \psi(\lambda\mu(w)\gamma) = \lambda'\mu'(w)\gamma',$$

and $\psi'(S)$ is recognizable. $\square$

Definition 2.2. Let A be a finite alphabet, $S\in\mathbb{Q}\langle\langle A\rangle\rangle$ and x a letter ($x\notin A$). The series $S'\in\mathbb{Q}\langle A\rangle\langle\langle x\rangle\rangle$, defined by

$$(S', x^n) = \sum_{u\in A^n} (S, u)u, \quad n\geqslant 0,$$

is called the *single-variable projection* of S.

Clearly, for any $S, T \in \mathbb{Q}\langle\langle A \rangle\rangle$ one has that $S = T$ if and only if $S' = T'$. Moreover, the recognizability is preserved, as is stated in the following proposition.

Proposition 2.3. *Let $S \in \mathrm{Rec}(\mathbb{Q}\langle\langle A \rangle\rangle)$. Then $S' \in \mathrm{Rec}(\mathbb{Q}\langle A \rangle\langle\langle x \rangle\rangle)$.*

Proof. Let $\lambda, \gamma \in \mathbb{Q}^n$ and $\mu : A^* \to \mathbb{Q}^{n \times n}$ such that for any $w \in A^*$

$$(S, w) = \lambda\mu(w)\gamma.$$

We consider $\lambda', \gamma' \in \mathbb{Q}\langle A \rangle^n$ and $\mu' : \{x\}^* \to \mathbb{Q}\langle A \rangle^{n \times n}$, defined by $\lambda' = \lambda$, $\gamma' = \gamma$ and $(\mu'(x))_{ij} = \sum_{a \in A} (\mu(a))_{ij} a$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$. By induction, one can prove that for any $m \geqslant 1$

$$(\mu'(x^m))_{ij} = \sum_{u \in A^m} (\mu(u))_{ij} u, \quad \text{for } 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n. \tag{2.1}$$

For $m = 1$ the equality holds by definition. Let $m > 1$ and suppose that the equality is true for $m - 1$. Thus, for any $i, j \in [1, n]$ one has

$$\begin{aligned}
(\mu'(x^m))_{ij} &= (\mu'(x^{m-1})\mu'(x))_{ij} = \sum_{k=1}^{n} (\mu'(x^{m-1}))_{ik}(\mu'(x))_{kj} \\
&= \sum_{k=1}^{n} \left(\sum_{u \in A^{m-1}} (\mu(u))_{ik} u \right) \left(\sum_{a \in A} (\mu(a))_{kj} a \right) \\
&= \sum_{u \in A^{m-1}, a \in A} \left(\sum_{k=1}^{n} (\mu(u))_{ik} (\mu(a))_{kj} \right) ua = \sum_{u \in A^{m-1}, a \in A} (\mu(ua))_{ij} ua \\
&= \sum_{v \in A^m} (\mu(v))_{ij} v.
\end{aligned}$$

By (2.1), one can write

$$\begin{aligned}
\lambda'\mu'(x^m)\gamma' &= \sum_{i,j=1}^{n} \lambda_i \left(\sum_{u \in A^m} (\mu(u))_{ij} u \right) \gamma_j = \sum_{u \in A^m} \left(\sum_{i,j=1}^{n} \lambda_i (\mu(u))_{ij} \gamma_j \right) u \\
&= \sum_{u \in A^m} (\lambda\mu(u)\gamma) u = \sum_{u \in A^m} (S, u) u = (S', x^m).
\end{aligned}$$

Therefore, S' is recognizable and the statement is true.

Proposition 2.4. *Let $\varphi : A^* \to M(A, \theta)$ be the canonical epimorphism. Then φ can be extended to an epimorphism $\varphi' : \mathbb{Q}\langle\langle A \rangle\rangle \to \mathbb{Q}\langle\langle A, \theta \rangle\rangle$ such that for any $S \in \mathbb{Q}\langle\langle A \rangle\rangle$, with $(S, 1) = 0$, one has $(\varphi'(S), 1) = 0$ and $\varphi'(S^*) = (\varphi'(S))^*$.*

Proof. Let $S \in \mathbb{Q}\langle\langle A \rangle\rangle$. We define $\varphi'(S)$ by

$$(\varphi'(S), m) = \sum_{w \in \varphi^{-1}(m)} (S, w), \quad m \in M(A, \theta).$$

Let $S, T \in \mathbb{Q}\langle\langle A \rangle\rangle$. Then $\varphi'(S+T) = \varphi'(S) + \varphi'(T)$ follows immediately from the definition of φ'. On the other hand, for any $m \in M(A, \theta)$ one has

$$\begin{aligned}(\varphi'(ST), m) &= \sum_{w \in \varphi^{-1}(m)} (ST, w) = \sum_{w \in \varphi^{-1}(m)} \sum_{uv = w} (S, u)(T, v) \\ &= \sum_{st = m} \sum_{u \in \varphi^{-1}(s), v \in \varphi^{-1}(t)} (S, u)(T, v) = \sum_{st = m} (\varphi'(S), s)(\varphi'(T), t) \\ &= (\varphi'(S)\varphi'(T), m).\end{aligned}$$

If $(S, \Lambda) = 0$ then, by definition, one has $(\varphi'(S), 1) = (S, \Lambda) = 0$. Moreover,

$$\begin{aligned}(\varphi'(S^*), m) &= \sum_{w \in \varphi^{-1}(m)} (S^*, w) = \sum_{w \in \varphi^{-1}(m)} \sum_{u_1 \ldots u_k = w} (S, u_1) \ldots (S, u_k) \\ &= \sum_{t_1 \ldots t_k = m} \sum_{u_1 \in \varphi^{-1}(t_1), \ldots, u_k \in \varphi^{-1}(t_k)} (S, u_1) \ldots (S, u_k) \\ &= \sum_{t_1 \ldots t_k = m} (\varphi'(S), t_1) \ldots (\varphi'(S), t_k) = ((\varphi'(S))^*, m).\end{aligned}$$ □

Corollary 2.5. *For any $T \in \mathrm{Rat}(\mathbb{Q}\langle\langle A, \theta \rangle\rangle)$ there exists $S \in \mathrm{Rat}(\mathbb{Q}\langle\langle A \rangle\rangle)$ such that $\varphi'(S) = T$. Moreover, S can be effectively obtained from T.*

Proof. Suppose that a rational expression E of T is given. Then we can construct a rational expression E' by substituting any $m \in M(A, \theta)$ which appears in E with a representative $w \in \varphi^{-1}(m)$. If $S \in \mathrm{Rat}(\mathbb{Q}\langle\langle A \rangle\rangle)$ is the series associated with E' then Proposition 2.4 assures that $\varphi'(S) = T$.

3. Decidability results

In this section, using techniques introduced by Harju and Karhumäki [11] for proving some decidability results about multitape finite automata, we show the decidability of the equivalence problem in $\mathrm{Rat}(\mathbb{Q}\langle\langle A, \theta \rangle\rangle)$. The following theorem gives an algorithm for testing the equivalence of two series in $\mathrm{Rec}(K\langle\langle A \rangle\rangle)$, when K is a subring of a division ring (cf. [9, pp. 143–145]).

Theorem 3.1 (Eilenberg equality theorem). *Let K be a subsemiring of a division ring. Let $S, T \in \mathrm{Rec}(K\langle\langle A \rangle\rangle)$ and n, m the dimensions of two linear representations of S and T. Then $S = T$ if and only if S and T coincide on all words w with $|w| \leqslant n + m$.*

The previous theorem is usually proved when K is a subsemiring of a field by using simple theorems of linear algebra. But the same arguments are also valid for proving the statement when K is contained in a division ring.

We recall that a group G is called an *ordered group* if there exists a total ordering $\leqslant$ on G such that for any h, g, x, y in G, $h \leqslant g$ implies $xhy \leqslant xgy$. The following theorem has been recently proved by Duchamp and Krob [7].

Theorem 3.2. *Let $F(A, \theta)$ be a free partially commutative group. Then $F(A, \theta)$ is an ordered group.*

The importance of ordered groups in the study of formal series is due to the following result of Neumann (cf. [13]).

Theorem 3.3. *Let D be a division ring and G an ordered group. Then the group ring $D[G]$ is embeddable in a division ring.*

Corollary 3.4. *The ring $\mathbb{Q}\langle A, \theta\rangle$ is embeddable in a division ring.*

Proof. $M(A, \theta)$ is a subsemigroup of $F(A, \theta)$; therefore, $\mathbb{Q}\langle A, \theta\rangle$ is a subsemiring of $\mathbb{Q}[F(A, \theta)]$. The statement is a consequence of Theorems 3.2 and 3.3.

Theorem 3.5. *Let $S, T \in \mathrm{Rec}(\mathbb{Q}\langle\langle A\rangle\rangle)$ and $\varphi' : \mathbb{Q}\langle\langle A\rangle\rangle \to \mathbb{Q}\langle\langle A, \theta\rangle\rangle$ be the natural projection. Then it is decidable whether $\varphi'(S) = \varphi'(T)$.*

Proof. Let $S', T' \in \mathbb{Q}\langle A\rangle\langle\langle x\rangle\rangle$ be the single variable projections of S and T. By Proposition 2.4, $\varphi' : \mathbb{Q}\langle\langle A\rangle\rangle \to \mathbb{Q}\langle\langle A, \theta\rangle\rangle$ is a ring morphism and also its restriction $\varphi' : \mathbb{Q}\langle A\rangle \to \mathbb{Q}\langle A, \theta\rangle$ is a ring morphism. Therefore, by Proposition 2.1, φ' has a natural extension to a morphism $\varphi'' : \mathbb{Q}\langle A\rangle\langle\langle x\rangle\rangle \to \mathbb{Q}\langle A, \theta\rangle\langle\langle x\rangle\rangle$ which preserves the recognizability. By construction, one has

$$\varphi'(S) = \varphi'(T) \Leftrightarrow \varphi''(S') = \varphi''(T'). \tag{3.1}$$

By Proposition 2.3, $S', T' \in \mathrm{Rec}(\mathbb{Q}\langle A\rangle\langle\langle x\rangle\rangle)$ and, by Proposition 2.1, $\varphi''(S'), \varphi''(T') \in \mathrm{Rec}(\mathbb{Q}\langle A, \theta\rangle\langle\langle x\rangle\rangle)$. Moreover, the linear representation of $\varphi''(S')$ and $\varphi''(T')$ may be effectively constructed from those of S and T. By Theorem 3.1 and Corollary 3.4, we can decide whether $\varphi''(S') = \varphi''(T')$. Therefore, by (3.1), it is decidable if $\varphi'(S) = \varphi'(T)$.

Theorem 3.6. *Let T_1, T_2 be two rational series of $\mathbb{Q}\langle\langle A, \theta\rangle\rangle$. Then it is decidable whether $T_1 = T_2$.*

Proof. By Corollary 2.5, we may effectively construct $S_1, S_2 \in \mathrm{Rat}(\mathbb{Q}\langle\langle A\rangle\rangle)$ such that $\varphi'(S_1) = T_1$ and $\varphi'(S_2) = T_2$. By Shützenberger Theorem (cf. [2]), $S_1, S_2 \in \mathrm{Rec}(\mathbb{Q}\langle\langle A\rangle\rangle)$ and two linear representations of S_1, S_2 can be constructed. Indeed if P is a polynomial then a linear representation of P can be given. Moreover, if $S, T \in \mathrm{Rec}(\mathbb{Q}\langle\langle A\rangle\rangle)$ and $k \in \mathbb{Q}$ then we may construct the linear representations of $S + T, ST, kS$ and S^*

from those of S and T (cf. [9]). By Theorem 3.5, we may decide whether $\varphi'(S_1)=\varphi'(S_2)$ and, consequently, also $T_1=T_2$. $\square$

Now we give some applications of the previous results to the theory of trace languages. We recall that a *trace language* is a subset of the free partially commutative monoid $M(A,\theta)$. These languages have been extensively studied and they were introduced by Mazurkiewicz [12] as mathematical models for describing the behaviour of concurrent systems.

Definition 3.7. Let $M(A,\theta)$ be a free partially commutative monoid. The family $R(M(A,\theta))$ of the *regular trace languages* is defined as the smallest family of subsets of $M(A,\theta)$ containing the finite sets and closed with respect to the operations of union, product and star.

Many decidability problems have been considered for regular trace languages. Bertoni et al. [3] proved that for a transitive concurrence relation θ the equivalence problem in $R(M(A,\theta))$ is decidable. Aalbersbeg and Welzel [1] proved that the only case in which this problem is decidable is when θ is transitive. Subsequently, many undecidable problems have been discovered for trace languages (cf. [6]).

Now we recall the definition of unambiguous trace languages introduced and studied by Bertoni et al. [4] and Sakarovitch [14].

Definition 3.8. Let L, N be two trace languages of $M(A,\theta)$. The product LN is called *unambiguous if any element $m\in LN$ can be uniquely factorized as a product $m=ln$* with $l\in L$ and $n\in N$. The generated submonoid N^* is called *unambiguous* if any element $m\in N^*\setminus 1$ can be uniquely factorized as a product $m=n_1\ldots n_k$, with $n_i\in N$ for $1\leqslant i\leqslant k$. The union $L\cup N$ is *unambiguous* if $L\cap N$ is empty.

Definition 3.9. The family $\mathrm{UR}(M(A,\theta))$ of the *unambiguous regular trace languages* is defined as the smallest family of subsets of $M(A,\theta)$ containing the finite sets and closed with respect to the unambiguous operations of union, product and star.

The decidability of the equivalence problem in $\mathrm{UR}(M(A,\theta))$ has been proved in the particular case when $A=\{a,b,c\}$ and $\theta=\{(a,b),(a,c)\}$ [5]. We can prove the following more general result.

Theorem 3.10. *For any finite alphabet A and concurrence relation θ, the equivalence problem is decidable in* $\mathrm{UR}(M(A,\theta))$.

Proof. For any subset L of $M(A,\theta)$ we denote by $C(L)\in\mathbb{Q}\langle\langle A,\theta\rangle\rangle$ the characteristic series of L, i.e. $(C(L),m)=1$ if $m\in L$ and $(C(L),m)=0$ if $m\notin L$. Let L,N be two trace languages. If LN, $L\cup N$ and L^* are unambiguous, then it immediately follows that

$$C(LN)=C(L)C(N),\qquad C(L\cup N)=C(L)+C(N)$$

and if $1 \notin L$

$$C(L^*) = C(L)^*.$$

From this it is easy to see that the characteristic series of an unambiguous regular trace language is a rational series. Moreover, for any $L, N \in \mathrm{UR}(M(A,\theta))$ one has

$$L = N \Leftrightarrow C(L) = C(N).$$

By Theorem 3.6, we can decide if $C(L) = C(N)$ and the same holds for $L = N$.

We recall that *inclusion problem* (*disjointness problem*) in a given family of languages $\mathscr{L}$ consists in deciding, for any two languages L, N in $\mathscr{L}$, if $L \supseteq N$ $\{L \cap N = \emptyset\}$. In [3] the undecidability of the equivalence problem in $R(M(A,\theta))$ is proved by means of the following proposition.

Proposition 3.11. *If the equivalence problem is decidable in $R(M(A,\theta))$ then the equivalence problem is decidable for multitape (nondeterministic) automata.*

Using the same techniques, up to simple modifications, one can prove the following proposition.

Proposition 3.12. *If the inclusion problem (disjointness problem) is decidable in $\mathrm{UR}(M(A,\theta))$ then the inclusion problem (disjointness problem) is decidable for multitape deterministic automata.*

Since the inclusion and disjointness problems are undecidable for multitape deterministic automata (cf. [10]), we have the following proposition.

Proposition 3.13. *The inclusion and disjointness problems are undecidable in $\mathrm{UR}(M(A,\theta))$.*

References

[1] IJ.J. Aalbersberg and E. Welzel, Trace languages defined by regular string languages, *RAIRO Inform. Théor. Appl.* **20** (1986) 103–119.

[2] J. Berstel and C. Reutenauer, *Rational Series and Their Languages* (Springer, Berlin, 1988).

[3] A. Bertoni, G. Mauri and N. Sabadini, Equivalence and membership problems for regular trace languages, in: *ICALP '82*, Lecture Notes in Computer Science, Vol. 140 (Springer, Berlin, 1982) 61–72.

[4] A. Bertoni, G. Mauri and N. Sabadini, Unambiguous regular trace languages, in: *Algebra, Combinatorics and Logic in Computer Science*, Colloq. Math. Soc. János Bolyai, Vol. 42 (North-Holland, Amsterdam, 1985) 113–123.

[5] D. Bruschi, G. Pighizzini and N. Sabadini, On the existence of minimal asynchronous automata and on the equivalence problem for unambiguous regular trace languages, in: *STACS '88*, Lectures Notes in Computer Science, Vol. 294 (Springer, Berlin, 1988) 334–346.

[6] C. Choffrut, Free partially commutative monoids, Research Report LITP 86–20, Paris, 1986.

[7] G. Duchamp and D. Krob, The lower central series of the free partially commutative group, LITP Report No. 90.64, Paris, 1990.

[8] G. Duchamp and D. Krob, Partially commutative formal power series, LITP Report no. 90.64, Paris, 1990.

[9] S. Eilenberg, *Automata, Languages and Machines, Vol. A* (Academic Press, New York, 1974).

[10] P.C. Fisher and A.L. Rozemberg, Multitape one-way nonwriting automata, *J. Comput. System Sci.* **2** (1968) 88–101.

[11] T. Harju and J. Karhumäki, Decidability of the multiplicity equivalence of multitape finite automata, *Theoret. Comput. Sci.* **78** (1991) 347–355.

[12] A. Mazurkiewicz, Concurrent program schemes and their interpretations, DAIMI Report PB-78, Aarhus University, 1977.

[13] D.S. Passman, *The Algebraic Structure of Group Rings* (Wiley, New York, 1977).

[14] J. Sakarovitch, On regular trace languages, *Theoret. Comput. Sci.* **52** (1987) 59–75.

[15] A. Salomaa and M. Soittola, *Automata-Theoretic Aspects of Formal Power Series* (Springer, New York, 1978).