Available online at www.sciencedirect.com

ScienceDirect

Theoretical Computer Science 388 (2007) 181–198

Theoretical Computer Science

www.elsevier.com/locate/tcs

Concurrent games with tail objectives☆

Krishnendu Chatterjee*

EECS, University of California, Berkeley, USA

Received 23 October 2006; received in revised form 6 July 2007; accepted 29 July 2007

Communicated by J. Esparza

Abstract

We study infinite stochastic games played by two players over a finite state space, with objectives specified by sets of infinite traces. The games are *concurrent* (players make moves simultaneously and independently), *stochastic* (the next state is determined by a probability distribution that depends on the current state and chosen moves of the players) and *infinite* (proceed for an infinite number of rounds). The analysis of concurrent stochastic games can be classified into: *quantitative analysis*, analyzing the optimum value of the game and ε-optimal strategies that ensure values within ε of the optimum value; and *qualitative analysis*, analyzing the set of states with optimum value 1 and ε-optimal strategies for the states with optimum value 1. We consider concurrent games with tail objectives, i.e., objectives that are independent of the finite-prefix of traces, and show that the class of tail objectives is strictly richer than that of the ω-regular objectives. We develop new proof techniques to extend several properties of concurrent games with ω-regular objectives to concurrent games with tail objectives. We prove the *positive limit-one* property for tail objectives. The positive limit-one property states that for all concurrent games if the optimum value for a player is positive for a tail objective Φ at some state, then there is a state where the optimum value is 1 for the player for the objective Φ. We also show that the optimum values of *zero-sum* (strictly conflicting objectives) games with tail objectives can be related to equilibrium values of *nonzero-sum* (not strictly conflicting objectives) games with simpler reachability objectives. A consequence of our analysis presents a polynomial time reduction of the quantitative analysis of tail objectives to the qualitative analysis for the subclass of one-player stochastic games (Markov decision processes).

Keywords: Concurrent games; Omega-regular objectives; Muller objectives; Tail objectives

1. Introduction

Stochastic games. Non-cooperative games provide a natural framework to model interactions between agents [13]. A wide class of games progress over time and in stateful manner, and the current game depends on the history of interactions. Infinite *stochastic games* [14,9] are a natural model for such dynamic games. A stochastic game is played over a finite *state space* and is played in rounds. In *concurrent games*, in each round, each player chooses an

☆ A preliminary version of the paper appeared in Computer Science Logic (CSL) 2006.

* Tel.: +1 510 524 1360.
E-mail address: c_krish@cs.berkeley.edu.

0304-3975/$ - see front matter
doi:10.1016/j.tcs.2007.07.047

action from a finite set of available actions, simultaneously and independently of the other player. The game proceeds to a new state according to a probabilistic transition relation (stochastic transition matrix) based on the current state and the joint actions of the players. Concurrent games (also known as Blackwell games) subsume the simpler class of *turn-based games*, where at every state at most one player can choose between multiple actions; and Markov decision processes (MDPs), where only one player can choose between multiple actions at every state. Concurrent games also provide the framework to model synchronous reactive systems [6]. In verification and control of finite state reactive systems such games proceed for an infinite number of rounds, generating an infinite sequence of states, called the *outcome* of the game. The players receive a payoff based on a payoff function that maps every outcome to a real number.

Objectives. Payoffs are generally Borel measurable functions [12]. That is, the payoff set for each player is a Borel set B_i in the Cantor topology on S^ω (where S is the set of states), and player i gets payoff 1 if the outcome of the game is a member of B_i, and 0 otherwise. In verification, payoff functions are usually index sets of *ω-regular languages*. The ω-regular languages generalize the classical regular languages to infinite strings, they occur in low levels of the Borel hierarchy (they are in $\mathbf{\Sigma}_3^0 \cap \mathbf{\Pi}_3^0$), and they form a robust and expressive language for determining payoffs for commonly used specifications. The simplest ω-regular objectives correspond to safety ("closed sets") and reachability ("open sets") objectives.

Zero-sum games, determinacy and nonzero-sum games. Games may be *zero-sum*, where two players have directly conflicting objectives and the payoff of one player is one minus the payoff of the other, or *nonzero-sum*, where each player has a prescribed payoff function based on the outcome of the game. The fundamental question for games is the existence of equilibrium values. For zero-sum games, this involves showing a *determinacy* theorem that states that the expected optimum value obtained by player 1 is exactly one minus the expected optimum value obtained by player 2. For one-step zero-sum games, this is von Neumann's minmax theorem [16]. For infinite games, the existence of such equilibria is not obvious, in fact, by using the axiom of choice, one can construct games for which determinacy does not hold. However, a remarkable result by Martin [12] shows that all stochastic zero-sum games with Borel payoffs are determined. For nonzero-sum games, the fundamental equilibrium concept is a *Nash equilibrium* [10], that is, a strategy profile such that no player can gain by deviating from the profile, assuming the other player continues playing the strategy in the profile.

Qualitative and quantitative analysis. The analysis of zero-sum concurrent games can be broadly classified into: (a) *quantitative analysis* and (b) *qualitative analysis*. The quantitative analysis involves the analysis of the optimum values of the games and ε-optimal strategies that ensure values within ε of the optimum value. The qualitative analysis involves the simpler analysis of the set of states where the optimum value is 1, and *ε-limit-sure* winning strategies that ensure satisfying the objective with value at least $1 - \varepsilon$. In general, the qualitative analysis of concurrent games is simpler as compared to quantitative analysis, as it only considers the case when the value is 1. Optimum values in concurrent games can be irrational even for reachability and safety objectives (with all rational transition probabilities) and hence quantitative analysis requires more involved analysis.

Properties of concurrent games. The result of Martin [12] established the determinacy of zero-sum concurrent games for all Borel objectives. The determinacy result sets forth the problem of study and closer understanding of properties and behaviors of concurrent games with different class of objectives. Several interesting questions related to concurrent games are: (1) characterizing certain zero–one laws for concurrent games; (2) relationship of qualitative and quantitative analysis; (3) relationship of zero-sum and nonzero-sum games. The results of [6,7,2] exhibited several interesting properties for concurrent games with ω-regular objectives specified as parity objectives. The result of [6] showed the positive limit-one property, that states if there is a state with positive optimum value, then there is a state with optimum value 1, for concurrent games with parity objectives. The positive limit-one property has been a key property to develop algorithms and improved complexity bound for quantitative analysis of concurrent games with parity objectives [2]. The above properties can possibly be the basic ingredients for the computational complexity analysis of quantitative analysis of concurrent games.

Outline of results. In this work, we consider *tail objectives*, the objectives that do not depend on any finite-prefix of the traces. Tail objectives subsume canonical ω-regular objectives such as parity objectives and Muller objectives, and we show that there exist tail objectives that cannot be expressed as ω-regular objectives. Hence the class of tail objectives is a strictly richer class of objectives than ω-regular objectives. Our result characterizes several properties of concurrent games with tail objectives. The results are as follows.

1. We show the positive limit-one property for concurrent games with tail objectives. Our result thus extends the result of [6] from parity objectives to a richer class of objectives that lies in the higher levels of Borel hierarchy. The result of [6] follows from a complementation argument of quantitative μ-calculus formula. Our proof technique is completely different: it uses certain strategy construction procedures and a convergence result from measure theory (Lévy's zero–one law). It may be noted that the positive limit-one property for concurrent games with Muller objectives follows from the positive limit-one property for parity objectives and the reduction of Muller objectives to parity objectives [15]. Since Muller objectives are tail objectives, our result presents a direct proof for the positive limit-one property for concurrent games with Muller objectives.
2. We relate the optimum values of zero-sum games with tail objectives with Nash equilibrium values of nonzero-sum games with reachability objectives. This establishes a relationship between the values of concurrent games with complex tail objectives and Nash equilibrium of nonzero-sum games with simpler objectives. From the above analysis we obtain a polynomial time reduction of quantitative analysis of tail objectives to qualitative analysis for the special case of MDPs. The above result was previously known for the subclass of ω-regular objectives specified as Muller objectives [4,5,1]. The proof techniques of [4,5,1] use different analysis of the structure of MDPs and is completely different from our proof techniques.
3. We also present construction of witnesses of ε-optimal strategies as witnesses of certain limit-sure winning strategies that respect some local conditions, for all Muller objectives.

2. Definitions

Notation. For a countable set A, a *probability distribution* on A is a function $\delta\colon A \to [0, 1]$ such that $\sum_{a\in A} \delta(a) = 1$. We denote the set of probability distributions on A by $\mathcal{D}(A)$. Given a distribution $\delta \in \mathcal{D}(A)$, we denote by $\mathrm{Supp}(\delta) = \{x \in A \mid \delta(x) > 0\}$ the *support* of δ.

Definition 1 (*Concurrent Games*). A (two-player) *concurrent game structure* $G = \langle S, \mathit{Moves}, \mathit{Mv}_1, \mathit{Mv}_2, \delta\rangle$ consists of the following components:

- A finite state space S and a finite set *Moves* of moves.
- Two move assignments $\mathit{Mv}_1, \mathit{Mv}_2\colon S \to 2^{\mathit{Moves}} \setminus \emptyset$. For $i \in \{1, 2\}$, assignment Mv_i associates with each state $s \in S$ the non-empty set $\mathit{Mv}_i(s) \subseteq \mathit{Moves}$ of moves available to player i at s.
- A probabilistic transition function $\delta : S \times \mathit{Moves} \times \mathit{Moves} \to \mathcal{D}(S)$, that gives the probability $\delta(s, a_1, a_2)(t)$ of a transition from s to t when player 1 plays move a_1 and player 2 plays move a_2, for all $s, t \in S$ and $a_1 \in \mathit{Mv}_1(s)$, $a_2 \in \mathit{Mv}_2(s)$. ■

An important special class of concurrent games is Markov decision processes (MDPs), where at every state s we have $|\mathit{Mv}_2(s)| = 1$, i.e., the set of available moves for player 2 is singleton at every state.

At every state $s \in S$, player 1 chooses a move $a_1 \in \mathit{Mv}_1(s)$, and simultaneously and independently player 2 chooses a move $a_2 \in \mathit{Mv}_2(s)$. The game then proceeds to the successor state t with probability $\delta(s, a_1, a_2)(t)$, for all $t \in S$. A state s is called an *absorbing state* if for all $a_1 \in \mathit{Mv}_1(s)$ and $a_2 \in \mathit{Mv}_2(s)$ we have $\delta(s, a_1, a_2)(s) = 1$. In other words, at s for all choices of moves of the players the next state is always s. We assume that the players act *non-cooperatively*, i.e., each player chooses her strategy independently and secretly from the other player, and is only interested in maximizing her own reward.

A *path* or a *play* ω of G is an infinite sequence $\omega = \langle s_0, s_1, s_2, \ldots\rangle$ of states in S such that for all $k \geq 0$, there are moves $a_1^k \in \mathit{Mv}_1(s_k)$ and $a_2^k \in \mathit{Mv}_2(s_k)$ with $\delta(s_k, a_1^k, a_2^k)(s_{k+1}) > 0$. We denote by Ω the set of all paths and by Ω_s the set of all paths $\omega = \langle s_0, s_1, s_2, \ldots\rangle$ such that $s_0 = s$, i.e., the set of plays starting from state s.

Strategies. A *selector* ξ for player $i \in \{1, 2\}$ is a function $\xi : S \to \mathcal{D}(\mathit{Moves})$ such that for all $s \in S$ and $a \in \mathit{Moves}$, if $\xi(s)(a) > 0$, then $a \in \mathit{Mv}_i(s)$. We denote by Λ_i the set of all selectors for player $i \in \{1, 2\}$. A *strategy* for player 1 is a function $\sigma : S^+ \to \Lambda_1$ that associates with every finite non-empty sequence of states, representing the history of the play so far, a selector, i.e., for all $w \in S^*$ and $s \in S$ we have $\mathrm{Supp}(\sigma(w \cdot s)) \subseteq \mathit{Mv}_1(s)$. Similarly we define strategies π for player 2. We denote by Σ and Π the set of all strategies for player 1 and player 2, respectively.

Once the starting state s and the strategies σ and π for the two players have been chosen, the game is reduced to an ordinary stochastic process. Hence the probabilities of events are uniquely defined, where an *event* $\mathcal{A} \subseteq \Omega_s$ is a measurable set of paths. For an event $\mathcal{A} \subseteq \Omega_s$ we denote by $\Pr_s^{\sigma,\pi}(\mathcal{A})$ the probability that a path belongs to $\mathcal{A}$ when the game starts from s and the players follows the strategies σ and π, and for a measurable function $f : \Omega \to \mathbb{R}$ we denote by $\mathbb{E}_s^{\sigma,\pi}[f]$ the expectation of the function f under the probability distribution $\Pr_s^{\sigma,\pi}(\cdot)$. For $i \geq 0$, we also denote by $\Theta_i : \Omega \to S$ the random variable denoting the i-th state along a path.

Objectives. We specify objectives for the players by providing the set of *winning plays* $\Phi \subseteq \Omega$ for each player. Given an objective Φ we denote by $\overline{\Phi} = \Omega \setminus \Phi$, the complementary objective of Φ. A concurrent game with objective Φ_1 for player 1 and Φ_2 for player 2 is *zero-sum* if $\Phi_2 = \overline{\Phi}_1$. A general class of objectives are the Borel objectives [11]. A *Borel objective* $\Phi \subseteq S^\omega$ is a Borel set in the Cantor topology on S^ω. In this paper we consider *ω-regular objectives* [15], which lie in the first 21/2 levels of the Borel hierarchy (i.e., in the intersection of $\mathbf{\Sigma}_3^0$ and $\mathbf{\Pi}_3^0$) and *tail objectives* which is a strict superset of ω-regular objectives. The ω-regular objectives, and subclasses thereof, and tail objectives are defined below. For a play $\omega = \langle s_0, s_1, s_2, \ldots \rangle \in \Omega$, we define $\mathrm{Inf}(\omega) = \{s \in S \mid s_k = s \text{ for infinitely many } k \geq 0\}$ to be the set of states that occur infinitely often in ω.

- *Reachability and safety objectives*. Given a set $T \subseteq S$ of "target" states, the reachability objective requires that some state of T be visited. The set of winning plays is thus $\mathrm{Reach}(T) = \{\omega = \langle s_0, s_1, s_2, \ldots \rangle \in \Omega \mid s_k \in T \text{ for some } k \geq 0\}$. Given a set $F \subseteq S$, the safety objective requires that only states of F be visited. Thus, the set of winning plays is $\mathrm{Safe}(F) = \{\omega = \langle s_0, s_1, s_2, \ldots \rangle \in \Omega \mid s_k \in F \text{ for all } k \geq 0\}$.
- *Büchi and coBüchi objectives.* Given a set $B \subseteq S$ of "Büchi" states, the Büchi objective requires that B is visited infinitely often. Formally, the set of winning plays is $\text{Büchi}(B) = \{\omega \in \Omega \mid \mathrm{Inf}(\omega) \cap B \neq \emptyset\}$. Given $C \subseteq S$, the coBüchi objective requires that all states visited infinitely often are in C. Formally, the set of winning plays is $\text{coBüchi}(C) = \{\omega \in \Omega \mid \mathrm{Inf}(\omega) \subseteq C\}$.
- *Parity objectives.* For $c, d \in \mathbb{N}$, we let $[c..d] = \{c, c+1, \ldots, d\}$. Let $p : S \to [0..d]$ be a function that assigns a *priority* $p(s)$ to every state $s \in S$, where $d \in \mathbb{N}$. The *Even parity objective* is defined as $\mathrm{Parity}(p) = \{\omega \in \Omega \mid \min(p(\mathrm{Inf}(\omega))) \text{ is even}\}$, and the *Odd parity objective* as $\mathrm{coParity}(p) = \{\omega \in \Omega \mid \min(p(\mathrm{Inf}(\omega))) \text{ is odd}\}$.
- *Muller objectives.* Given a set $\mathcal{M} \subseteq 2^S$ of subset of states, the *Muller objective* is defined as $\mathrm{Muller}(\mathcal{M}) = \{\omega \in \Omega \mid \mathrm{Inf}(\omega) \in \mathcal{M}\}$.
- *Tail objectives.* Informally the class of tail objectives is the subclass of Borel objectives that are independent of all finite-prefixes. An objective Φ is a tail objective, if the following condition hold: a path $\omega \in \Phi$ if and only if for all $i \geq 0$, $\omega_i \in \Phi$, where ω_i denotes the path ω with the prefix of length i deleted. Formally, let $\mathcal{G}_i = \boldsymbol{\sigma}(\Theta_i, \Theta_{i+1}, \ldots)$ be the $\boldsymbol{\sigma}$-field generated by the random variables $\Theta_i, \Theta_{i+1}, \ldots$.[1] The tail $\boldsymbol{\sigma}$-field $\mathcal{T}$ is defined as $\mathcal{T} = \bigcap_{i \geq 0} \mathcal{G}_i$. An objective Φ is a tail objective if and only if Φ belongs to the tail $\boldsymbol{\sigma}$-field $\mathcal{T}$, i.e., the tail objectives are indicator functions of events $\mathcal{A} \in \mathcal{T}$.

The Muller and parity objectives are canonical forms to represent ω-regular objectives [15]. Observe that Muller and parity objectives are tail objectives. Note that for a priority function $p : S \to \{0, 1\}$, an even parity objective $\mathrm{Parity}(p)$ is equivalent to the Büchi objective $\text{Büchi}(p^{-1}(0))$, i.e., the Büchi set consists of the states with priority 0. Büchi and coBüchi objectives are special cases of parity objectives. Reachability objectives are not necessarily tail objectives, but for a set $T \subseteq S$ of states, if every state $s \in T$ is an absorbing state, then the objective $\mathrm{Reach}(T)$ is equivalent to $\text{Büchi}(T)$ and hence is a tail objective. It may be noted that since $\boldsymbol{\sigma}$-fields are closed under complementation, the class of tail objectives are closed under complementation. We give an example to show that the class of tail objectives is richer than that of ω-regular objectives.[2]

Example 1. Let r be a reward function that maps every state s to a real-valued reward $r(s)$, i.e., $r : S \to \mathbb{R}$. For a constant $c \in \mathbb{R}$, consider the objective Φ_c defined as follows:

$$\Phi_c = \left\{ \omega \in \Omega \mid \omega = \langle s_1, s_2, s_3, \ldots \rangle, \lim \inf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} r(s_i) \geq c \right\}.$$

[1] We use σ for strategies and $\boldsymbol{\sigma}$ (boldface) for sigma-fields.

[2] Our example shows that there are $\mathbf{\Pi}_3^0$-hard objectives that are tail objectives. It is possible that the tail objectives can express objectives in even higher levels of Borel hierarchy than $\mathbf{\Pi}_3^0$, which would make our results stronger.

Intuitively, Φ_c accepts the set of paths such that the "long-run" average of the rewards in the path is at least the constant c. The "long-run" average condition is hard for the third level of the Borel hierarchy (i.e., $\mathbf{\Pi}_3^0$-hard (see Section 2.1 for $\mathbf{\Pi}_3^0$-hardness proof)) and cannot be expressed as an ω-regular objective. It may be noted that the "long-run" average of a path is independent of all finite-prefixes of the path. Formally, the objectives Φ_c are tail objectives. Since Φ_c are $\mathbf{\Pi}_3^0$-hard objectives, it follows that tail objectives lie in higher levels of Borel hierarchy than ω-regular objectives. ■

Values. The probability that a path satisfies an objective Φ starting from state $s \in S$, given strategies σ, π for the players is $\mathrm{Pr}_s^{\sigma,\pi}(\Phi)$. Given a state $s \in S$ and an objective Φ, we are interested in the maximal probability with which player 1 can ensure that Φ and player 2 can ensure that $\overline{\Phi}$ holds from s. We call such probability the *value of the game* G at s for player $i \in \{1, 2\}$. The value for player 1 and player 2 are given by the functions $\langle\langle 1\rangle\rangle_{val}(\Phi) : S \to [0, 1]$ and $\langle\langle 2\rangle\rangle_{val}(\overline{\Phi}) : S \to [0, 1]$, defined for all $s \in S$ by $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) = \sup_{\sigma\in\Sigma} \inf_{\pi\in\Pi} \mathrm{Pr}_s^{\sigma,\pi}(\Phi)$ and $\langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s) = \sup_{\pi\in\Pi} \inf_{\sigma\in\Sigma} \mathrm{Pr}_s^{\sigma,\pi}(\overline{\Phi})$. Note that the objectives of the player are complementary and hence we have a zero-sum game. Concurrent games satisfy a *quantitative* version of determinacy [12], stating that for all Borel objectives Φ and all $s \in S$, we have $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) + \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s) = 1$. A strategy σ for player 1 is *optimal* for objective Φ if for all $s \in S$ we have $\inf_{\pi\in\Pi} \mathrm{Pr}_s^{\sigma,\pi}(\Phi) = \langle\langle 1\rangle\rangle_{val}(\Phi)(s)$. For $\varepsilon > 0$, a strategy σ for player 1 is *ε-optimal* for objective Φ if for all $s \in S$ we have $\inf_{\pi\in\Pi} \mathrm{Pr}_s^{\sigma,\pi}(\Phi) \geq \langle\langle 1\rangle\rangle_{val}(\Phi)(s) - \varepsilon$. We define optimal and ε-optimal strategies for player 2 symmetrically. For $\varepsilon > 0$, an objective Φ for player 1 and $\overline{\Phi}$ for player 2, we denote by $\Sigma_\varepsilon(\Phi)$ and $\Pi_\varepsilon(\overline{\Phi})$ the set of ε-optimal strategies for player 1 and player 2, respectively. Even in concurrent games with reachability objectives optimal strategies need not exist [6], and ε-optimal strategies, for all $\varepsilon > 0$, is the best one can achieve. Note that the quantitative determinacy of concurrent games is equivalent to the existence of ε-optimal strategies for objective Φ for player 1 and $\overline{\Phi}$ for player 2, for all $\varepsilon > 0$, at all states $s \in S$, i.e., for all $\varepsilon > 0$, $\Sigma_\varepsilon(\Phi) \neq \emptyset$ and $\Pi_\varepsilon(\overline{\Phi}) \neq \emptyset$.

We refer to the analysis of computing the *limit-sure winning* states (the set of states s such that $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) = 1$) and ε-limit-sure winning strategies (ε-optimal strategies for the limit-sure winning states) as the *qualitative* analysis of objective Φ. We refer to the analysis of computing the values and the ε-optimal strategies as the *quantitative* analysis of objective Φ.

Notation for qualitative sets. We use the following notation for the qualitative sets for the rest of the paper:

$$W_1^1 = \{s \mid \langle\langle 1\rangle\rangle_{val}(\Phi)(s) = 1\}; \qquad W_2^1 = \{s \mid \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s) = 1\}.$$
$$W_1^{>0} = \{s \mid \langle\langle 1\rangle\rangle_{val}(\Phi)(s) > 0\}; \qquad W_2^{>0} = \{s \mid \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s) > 0\}.$$

By determinacy of concurrent games with tail objectives, we have $W_1^1 = S \setminus W_2^{>0}$ and $W_2^1 = S \setminus W_1^{>0}$.

2.1. *Hardness of* lim inf *objectives*

Borel hierarchy. For an (possibly infinite) alphabet A, let A^ω and A^* denote the set of infinite and finite words on A, respectively. The finite Borel hierarchy $(\mathbf{\Sigma}_1^0, \mathbf{\Pi}_1^0)$, $(\mathbf{\Sigma}_2^0, \mathbf{\Pi}_2^0)$, $(\mathbf{\Sigma}_3^0, \mathbf{\Pi}_3^0)$, ... is defined as follows:

- $\mathbf{\Sigma}_1^0 = \{W \cdot A^\omega \mid W \subseteq A^*\}$ is the set of open sets;
- for all $n \geq 1$, $\mathbf{\Pi}_n^0 = \{A^\omega \setminus L \mid L \in \mathbf{\Sigma}_n^0\}$ consists of the complement of sets in $\mathbf{\Sigma}_n^0$;
- for all $n \geq 1$, $\mathbf{\Sigma}_{n+1}^0 = \{\bigcup_{i\in\mathbb{N}} L_i \mid \forall i \in \mathbb{N}.\ L_i \in \mathbf{\Pi}_n^0\}$ is the set obtained by countable union of sets in $\mathbf{\Pi}_n^0$.

Definition 2 (*Wadge Game*). Let A and B be two (possibly infinite) alphabets. Let $X \subseteq A^\omega$ and $Y \subseteq B^\omega$. The Wadge game $G_W(X, Y)$ is a two-player game between player 1 and player 2 as follows. Player 1 first chooses a letter $a_0 \in A$ and then player 2 chooses a (possibly empty) finite word $b_0 \in B^*$, then player 1 chooses a letter $a_1 \in A$ and then player 2 chooses a word $b_1 \in B^*$, and so on. The play consists in writing a word $w_X = a_0 a_1 \ldots$ by player 1 and $w_Y = b_0 b_1 \ldots$ by player 2. Player 2 wins if and only if both w_Y is infinite and $w_X \in X$ iff $w_Y \in Y$.

Definition 3 (*Wadge Reduction*). Given alphabets A and B, a set $X \subseteq A^\omega$ is Wadge reducible to a set $Y \subseteq B^\omega$, denoted as $X \leq_W Y$, if and only if there exists a continuous function $f : A^\omega \to B^\omega$ such that $X = f^{-1}(Y)$. If $X \leq_W Y$ and $Y \leq_W X$, then X and Y are Wadge equivalent and we denote this by $X \equiv_W Y$.

The notion of strategies in Wadge games and winners are defined similarly to the notion of games on graphs. The Wadge games and Wadge reduction are related by the following result.

Proposition 1 ([17]). *Player 2 has a winning strategy in the Wadge game $G_W(X, Y)$ iff $X \leq_W Y$.*

Wadge equivalence preserves Borel hierarchy and defines the natural notion of completeness.

Proposition 2. *If $X \equiv_W Y$, then X and Y belong to the same level of Borel hierarchy.*

Definition 4. A set $Y \in \mathbf{\Sigma}_n^0$ (resp. $Y \in \mathbf{\Pi}_n^0$) is $\mathbf{\Sigma}_n^0$-complete (resp. $\mathbf{\Pi}_n^0$-complete) if and only if $X \leq_W Y$ for all $X \in \mathbf{\Sigma}_n^0$ (resp. $X \in \mathbf{\Pi}_n^0$).

Our goal is to show that the lim inf objectives (defined in Example 1) are $\mathbf{\Pi}_3^0$-hard. We first present some notation.

Notation. Let A be an alphabet and $B = \{b_0, b_1\}$. For a word $w \in A^*$ or $w \in B^*$ we denote by $\mathsf{len}(w)$ the length of w. For an infinite word w or finite word w with $\mathsf{len}(w) \geq k$ we denote by $(w \upharpoonright k)$ the prefix of length k of w. For a word $w \in B^\omega$ or $w \in B^*$ with $\mathsf{len}(w) \geq k$, we denote by

$$\mathsf{avg}(w \upharpoonright k) = \frac{\text{number of } b_0 \text{ in } (w \upharpoonright k)}{k},$$

i.e., the average of b_0's in $(w \upharpoonright k)$. For a finite word w we denote by $\mathsf{avg}(w) = \mathsf{avg}(w \upharpoonright \mathsf{len}(w))$. Let

$$\begin{aligned} Y &= \left\{ w \in B^\omega \mid \lim \inf_{k \to \infty} \mathsf{avg}(w \upharpoonright k) = 1 \right\} \\ &= \bigcap_{i \geq 0} \bigcup_{j \geq 0} \bigcap_{k \geq j} \left\{ w \in B^\omega \mid \mathsf{avg}(w \upharpoonright k) \geq 1 - \frac{1}{i} \right\}. \end{aligned}$$

Hardness of Y. We will show that Y is $\mathbf{\Pi}_3^0$-hard. To prove the result we consider an arbitrary $X \in \mathbf{\Pi}_3^0$ and show that $X \leq_W Y$. A set $X \subseteq A^\omega$ in $\mathbf{\Pi}_3^0$ is obtained as the countable intersection of countable union of closed sets, i.e.,

$$X = \bigcap_{i \geq 0} \bigcup_{j \geq 0} (A^j \cdot (F_{ij})^\omega),$$

where $F_{ij} \subseteq A$, and A^j denotes the set of words of length j in A^*. We show such a X is Wadge reducible to Y, by showing that player 2 has a winning strategy in $G_W(X, Y)$. In the reduction we will use the following notation: given a word $w \in A^*$, let

$$\begin{aligned} &\mathsf{sat}(w) = \{i \mid \exists j \geq 0.\ w \in A^j \cdot (F_{ij})^*\}; \\ &d(w) = \max\{l \mid \forall l' \leq l.\ l' \in \mathsf{sat}(w)\} + 1. \end{aligned}$$

For example if $\mathsf{sat}(w) = \{0, 1, 2, 4, 6, 7\}$, then $d(w) = \max\{0, 1, 2\} + 1 = 3$. The play between player 1 and player 2 proceeds as follows:

$$\begin{array}{llllllll} \text{Player 1: } w_X & = & a_1 & a_2 & a_3 & \ldots; & & \forall i \geq 1.\ a_i \in A \\ \text{Player 2: } w_Y & = & w_Y(1) & w_Y(2) & w_Y(3) & \ldots; & & \forall i \geq 1.\ w_Y(i) \in B^+. \end{array}$$

A winning strategy for player 2 is as follows: let the current prefix of w_X of length k be $(w_X \upharpoonright k) = a_1 a_2 \ldots a_k$ and the current prefix of w_Y be $w_Y(1) w_Y(2) \ldots w_Y(k-1)$, then the word $w_Y(k)$ is generated satisfying the following conditions.

1. (*Condition* 1.) There exists $\ell \leq \mathsf{len}(w_Y(k))$ such that

$$\mathsf{avg}(w_Y(1) w_Y(2) \ldots w_Y(k-1)(w_Y(k) \upharpoonright \ell)) \geq 1 - \frac{1}{d(w_X \upharpoonright k)},$$

for all $\ell_1 \leq \ell$

$$\mathsf{avg}(w_Y(1) \ldots w_Y(k-1)(w_Y(k) \upharpoonright \ell)) \geq \mathsf{avg}(w_Y(1) \ldots w_Y(k-1)(w_Y(k) \upharpoonright \ell_1))$$

and for all ℓ_2 such that $\ell \leq \ell_2 \leq \mathsf{len}(w_Y(k))$ we have

$$\mathsf{avg}(w_Y(1) w_Y(2) \ldots w_Y(k-1)(w_Y(k) \upharpoonright \ell_2)) \geq 1 - \frac{1}{d(w_X \upharpoonright k)}.$$

2. (*Condition* 2.)

$$1 - \frac{1}{d(w_X \upharpoonright k)} \leq \mathsf{avg}(w_Y(1)w_Y(2)\ldots w_Y(k)) \leq 1 - \frac{1}{d(w_X \upharpoonright k)+1}.$$

Intuitively, player 2 plays as follows: (a) initially player 2 plays a sequence of b_0's to ensure that the average of b_0's crosses $(1 - \frac{1}{d(w_X \upharpoonright k)})$, and (b) then plays a sequence of b_0 and b_1's to ensure that the average of b_0 in $w_Y(1)w_Y(2)\ldots w_Y(k)$ is in the interval

$$\left[1 - \frac{1}{d(w_X \upharpoonright k)}, 1 - \frac{1}{d(w_X \upharpoonright k)+1}\right],$$

and the average never falls below $(1-\frac{1}{d(w_X \upharpoonright k)})$ while generating $w_Y(k)$ once it crosses $(1-\frac{1}{d(w_X \upharpoonright k)})$. Clearly, player 2 has such a strategy. Given a word $w_X \in A^\omega$, the corresponding word w_Y generated is an infinite word. Hence we need to prove $w_X \in X$ if and only if $w_Y \in Y$. We prove implications in both directions.

Claim 1 *($w_X \in X \Rightarrow w_Y \in Y$). Let $w_X \in X$ and we show that $w_Y \in Y$. Given $w_X \in X$, we have $\forall i \geq 0.\ \exists j \geq 0.\ w_X \in A^j \cdot (F_{ij})^\omega$. Given $i \geq 0$, let*

$$j(i) = \min\{j \geq 0 \mid w_X \in A^j \cdot (F_{ij})^\omega\}; \qquad \widehat{j}(i) = \max\{j(i') \mid i' \leq i\}.$$

Given $i \geq 0$, for $j = \widehat{j}(i)$, for all $k \geq j$ we have $(w_X \upharpoonright k) \in A^j \cdot (F_{ij})^*$. Consider the sequence $(w_X \upharpoonright j), (w_X \upharpoonright j+1), \ldots$: for all $k \geq j$ we have $\{i' \mid i' \leq i\} \subseteq \mathsf{sat}(w_X \upharpoonright k)$. Hence in the corresponding sequence of the word w_Y it is ensured that for all $\ell \geq \mathsf{len}(w_Y(1)w_Y(2)\ldots w_Y(j))$ we have $\mathsf{avg}(w_Y \upharpoonright \ell) \geq 1-\frac{1}{i+1}$. Hence $\liminf_{n\to\infty} \mathsf{avg}(w_Y \upharpoonright n) \geq 1 - \frac{1}{i+1}$. Since this holds for all $i \geq 0$, let $i \to \infty$ to obtain that $\liminf_{n\to\infty} \mathsf{avg}(w_Y \upharpoonright n) \geq 1 = 1$ (the equality follows as the average can never be more than 1). Hence $w_Y \in Y$.

Claim 2 *($w_Y \in Y \Rightarrow w_X \in X$). Let $w_Y \in Y$ and we show $w_X \in X$. Fix $i \geq 0$. Since $\liminf_{n\to\infty} \mathsf{avg}(w_Y \upharpoonright n) = 1$, it follows that from some point on average never falls below $1 - \frac{1}{i+1}$. Then there exists j such that for all $l \geq j$ we have $d(w_X \upharpoonright l) \geq i+1$ and hence $\{i' \mid i' \leq i\} \subseteq \mathsf{sat}(w_Y \upharpoonright l)$. Hence for all $l \geq j$ we have $(w_X \upharpoonright l) \in A^j \cdot (F_{ij})^*$ and thus we obtain that $w_X \in A^j \cdot (F_{ij})^\omega$, i.e., $\exists j \geq 0$ such that $w_X \in A^j \cdot (F_{ij})^\omega$. Since this holds for all $i \geq 0$, it follows that $w_X \in X$.*

From Claims 1 and 2 it follows that Y is $\mathbf{\Pi}_3^0$-hard, and as an easy consequence we have that the objectives Φ_c defined in Example 1 is $\mathbf{\Pi}_3^0$-hard. Hence tail objectives contain $\mathbf{\Pi}_3^0$-hard objectives and since tail objectives are closed under complementation it also follows that tail objectives contain $\mathbf{\Sigma}_3^0$-hard objectives.

3. Positive limit-one property

The *positive limit-one* property for concurrent games, for a class $\mathcal{C}$ of objectives, states that for all objectives $\Phi \in \mathcal{C}$, for all concurrent games G, if there is a state s such that the value for player 1 is positive at s for Φ, then there is a state s' where the value for player 1 is 1 for Φ. The property means if a player can win with positive value from some state, then from some state she can win with value 1. The positive limit-one property was proved for parity objectives in [6] and has been one of the key properties used in the algorithmic analysis of concurrent games with parity objectives [2]. In this section we prove the *positive limit-one* property for concurrent games with tail objectives, and thereby extend the positive limit-one property from parity objectives to a richer class of objectives that subsumes several canonical ω-regular objectives. Our proof uses a result from measure theory and certain strategy constructions, whereas the proof for the subclass of parity objectives [6] followed from complementation arguments of quantitative μ-calculus formulas. We first show an example that the positive limit-one property is not true for all objectives, even for simpler class of games.

Example 2. Consider the game shown in Fig. 1, where at every state s, we have $Mv_1(s) = Mv_2(s) = \{1\}$ (i.e., the set of moves is singleton at all states). From all states the next state is s_0 and s_1 with equal probability. Consider the objective $\bigcirc(s_1)$ which specifies the next state is s_1; i.e., a play ω starting from state s is winning if the first state of the play is s and the second state (or the next state from s) in the play is s_1. Given the objective $\Phi = \bigcirc(s_1)$ for player 1, we have $\langle\langle 1\rangle\rangle_{val}(\Phi)(s_0) = \langle\langle 1\rangle\rangle_{val}(\Phi)(s_1) = 1/2$. Hence though the value is positive at s_0, there is no state with value 1 for player 1. ■

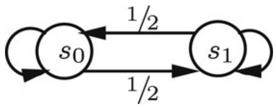

Fig. 1. A simple Markov chain.

Notation. In the setting of concurrent games the natural filtration sequence $(\mathcal{F}_n)$ for the stochastic process under any pair of strategies is defined as

$$\mathcal{F}_n = \boldsymbol{\sigma}(\Theta_1, \Theta_2, \ldots, \Theta_n)$$

i.e., the $\boldsymbol{\sigma}$-field generated by the random variables $\Theta_1, \Theta_2, \ldots, \Theta_n$.

Conditional expectations. Given a $\boldsymbol{\sigma}$-algebra $\mathcal{H}$, the *conditional expectation* $\mathbb{E}[f \mid \mathcal{H}]$ of a measurable function f is a random variable Z that satisfies the following properties: (a) Z is $\mathcal{H}$ measurable and (b) for all $\mathcal{A} \in \mathcal{H}$ we have $\mathbb{E}[f\mathbf{1}_{\mathcal{A}}] = \mathbb{E}[Z\mathbf{1}_{\mathcal{A}}]$, where $\mathbf{1}_{\mathcal{A}}$ is the indicator of event $\mathcal{A}$ (see [8] for details). Another key property of conditional expectation is as follows: $\mathbb{E}[\mathbb{E}[f \mid \mathcal{H}]] = \mathbb{E}[f]$ (again see [8] for details).

Almost-sure convergence. Given a random variable X and a sequence $(X_n)_{n\geq 0}$ of random variables we write $X_n \to X$ almost-surely if $\lim_{n\to\infty} \Pr(\{\omega \mid X_n(\omega) = X(\omega)\}) = 1$, i.e., with probability 1 the sequence converges to X.

Lemma 1 (*Lévy's 0–1 Law*). *Suppose $\mathcal{H}_n \uparrow \mathcal{H}_\infty$, i.e.,$\mathcal{H}_n$ is a sequence of increasing $\boldsymbol{\sigma}$-fields and $\mathcal{H}_\infty = \boldsymbol{\sigma}(\cup_n \mathcal{H}_n)$. For all events $\mathcal{A} \in \mathcal{H}_\infty$ we have*

$$\mathbb{E}[\mathbf{1}_{\mathcal{A}} \mid \mathcal{H}_n] = \Pr(\mathcal{A} \mid \mathcal{H}_n) \to \mathbf{1}_{\mathcal{A}} \textit{ almost-surely, (i.e., with probability } 1),$$

where $\mathbf{1}_{\mathcal{A}}$ is the indicator function of event $\mathcal{A}$.

The proof of the lemma is available in the book of Durrett (page 262–263) [8]. An immediate consequence of Lemma 1 in the setting of concurrent games is the following lemma.

Lemma 2 (*0–1 Law in Concurrent Games*). *For all concurrent game structures G, for all events $\mathcal{A} \in \mathcal{F}_\infty = \boldsymbol{\sigma}(\cup_n \mathcal{F}_n)$, for all strategies $(\sigma, \pi) \in \Sigma \times \Pi$, for all states $s \in S$, we have*

$$\Pr_s^{\sigma,\pi}(\mathcal{A} \mid \mathcal{F}_n) \to \mathbf{1}_{\mathcal{A}} \textit{ almost-surely}.$$

Intuitively, the lemma means that the probability $\Pr_s^{\sigma,\pi}(\mathcal{A} \mid \mathcal{F}_n)$ converges almost-surely (i.e., with probability 1) to 0 or 1 (since indicator functions take values in the range $\{0, 1\}$). Note that the tail $\boldsymbol{\sigma}$-field $\mathcal{T}$ is a subset of $\mathcal{F}_\infty$, i.e., $\mathcal{T} \subseteq \mathcal{F}_\infty$, and hence the result of Lemma 2 holds for all $\mathcal{A} \in \mathcal{T}$.

Objectives as indicator functions. Objectives Φ are indicator functions $\Phi : \Omega \to \{0, 1\}$ defined as follows:

$$\Phi(\omega) = \begin{cases} 1 & \text{if } \omega \in \Phi \\ 0 & \text{otherwise.} \end{cases}$$

Notation. Given strategies σ and π for player 1 and player 2, a tail objective $\overline{\Phi}$, and a state s, for $\beta > 0$, let

$$\begin{aligned} H_n^{1,\beta}(\sigma, \pi, \overline{\Phi}) &= \{\langle s_1, s_2, \ldots, s_n, s_{n+1}, \ldots\rangle \mid \Pr_s^{\sigma,\pi}(\overline{\Phi} \mid \langle s_1, s_2, \ldots, s_n\rangle) \geq 1 - \beta\} \\ &= \{\omega \mid \Pr_s^{\sigma,\pi}(\overline{\Phi} \mid \mathcal{F}_n)(\omega) \geq 1 - \beta\}; \end{aligned}$$

denote the set of paths ω such that the probability of satisfying $\overline{\Phi}$ given the strategies σ and π, and the prefix of length n of ω is at least $1 - \beta$. Similarly, let

$$\begin{aligned} H_n^{0,\beta}(\sigma, \pi, \overline{\Phi}) &= \{\langle s_1, s_2, \ldots, s_n, s_{n+1}, \ldots\rangle \mid \Pr_s^{\sigma,\pi}(\overline{\Phi} \mid \langle s_1, s_2, \ldots, s_n\rangle) \leq \beta\} \\ &= \{\omega \mid \Pr_s^{\sigma,\pi}(\overline{\Phi} \mid \mathcal{F}_n)(\omega) \leq \beta\}; \end{aligned}$$

denote the set of paths ω such that the probability of satisfying $\overline{\Phi}$ given the strategies σ and π, and the prefix of length n of ω is at most β. We often refer to prefixes of paths in $H_n^{1,\beta}$ as histories in $H_n^{1,\beta}$, and analogously for $H_n^{0,\beta}$.

Proposition 3. *For all concurrent game structures G, for all strategies σ and π for player 1 and player 2, respectively, for all tail objectives $\overline{\Phi}$, for all states $s \in S$, for all $\beta > 0$ and $\varepsilon > 0$, there exists n, such that* $\Pr_s^{\sigma,\pi}(H_n^{1,\beta}(\sigma,\pi,\overline{\Phi}) \cup H_n^{0,\beta}(\sigma,\pi,\overline{\Phi})) \geq 1-\varepsilon$.

Proof. Let $f_n = \Pr_s^{\sigma,\pi}(\overline{\Phi} \mid \mathcal{F}_n)$. By Lemma 2, we have $f_n \to \overline{\Phi}$ almost-surely as $n \to \infty$. Since almost-sure convergence implies convergence in probability we have

$$\begin{aligned}&\forall \beta > 0.\ \lim_{n\to\infty} \Pr_s^{\sigma,\pi}(\{\omega \mid |f_n(\omega) - \overline{\Phi}(\omega)| \geq \beta\}) = 0\\ &\quad\Rightarrow \qquad \forall \beta > 0.\ \lim_{n\to\infty} \Pr_s^{\sigma,\pi}(\{\omega \mid |f_n(\omega) - \overline{\Phi}(\omega)| \leq \beta\}) = 1.\end{aligned}$$

Since $\overline{\Phi}$ is an indicator function we have

$$\begin{aligned}&\forall \beta > 0.\ \lim_{n\to\infty} \Pr_s^{\sigma,\pi}(\{\omega \mid f_n(\omega) \geq 1-\beta \text{ or } f_n(\omega) \leq \beta\}) = 1\\ &\quad\Rightarrow \qquad \forall \beta > 0.\ \lim_{n\to\infty} \Pr_s^{\sigma,\pi}(H_n^{1,\beta}(\sigma,\pi,\overline{\Phi}) \cup H_n^{0,\beta}(\sigma,\pi,\overline{\Phi})) = 1.\end{aligned}$$

Hence we have

$$\forall \beta > 0.\ \forall \varepsilon > 0.\ \exists n_0.\ \forall n \geq n_0.\ \Pr_s^{\sigma,\pi}(H_n^{1,\beta}(\sigma,\pi,\overline{\Phi}) \cup H_n^{0,\beta}(\sigma,\pi,\overline{\Phi})) \geq 1-\varepsilon.$$

The result follows. ■

Lemma 3 (*Always-Positive Implies Probability 1*). *Let $\alpha > 0$ be a real constant greater than 0. For all objectives Φ, for all strategies σ and π, and for all states s, if*

$$f_n = \Pr_s^{\sigma,\pi}(\Phi \mid \mathcal{F}_n) > \alpha, \quad \forall n, \qquad \textit{i.e., } f_n(\omega) > \alpha \textit{ almost-surely for all } n;$$

then $\Pr_s^{\sigma,\pi}(\Phi) = 1$.

Proof. We show that for all $\varepsilon > 0$ we have $\Pr_s^{\sigma,\pi}(\Phi) \geq 1-2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, the result follows. Given $\varepsilon > 0$ and $\alpha > 0$, we chose β such that $0 < \beta < \alpha$ and $0 < \beta < \varepsilon$. By Proposition 3 there exists a n_0 such that for all $n > n_0$ we have

$$\Pr_s^{\sigma,\pi}(\{\omega \mid f_n(\omega) \geq 1-\beta \text{ or } f_n(\omega) \leq \beta\}) \geq 1-\varepsilon.$$

Since $f_n(\omega) \geq \alpha > \beta$ almost-surely for all n, we have $\Pr_s^{\sigma,\pi}(\{\omega \mid f_n(\omega) \geq 1-\beta\}) \geq 1-\varepsilon$, i.e., we have $\Pr_s^{\sigma,\pi}(\Phi \mid \mathcal{F}_n) \geq 1-\beta$ with probability at least $1-\varepsilon$. Hence we have

$$\Pr_s^{\sigma,\pi}(\Phi) = \mathbb{E}_s^{\sigma,\pi}[\Phi] = \mathbb{E}_s^{\sigma,\pi}[\mathbb{E}_s^{\sigma,\pi}[\Phi \mid \mathcal{F}_n]] \geq (1-\beta)\cdot(1-\varepsilon) \geq 1-2\varepsilon.$$

Observe that we have used the property of conditional expectation to infer that $\mathbb{E}_s^{\sigma,\pi}[\Phi] = \mathbb{E}_s^{\sigma,\pi}[\mathbb{E}_s^{\sigma,\pi}[\Phi \mid \mathcal{F}_n]]$. The desired result follows. ■

Theorem 1 (*Positive Limit-One Property*). *For all concurrent game structures G, for all tail objectives Φ, if there exists a state $s \in S$ such that $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) > 0$, then there exists a state $s' \in S$ such that $\langle\langle 1\rangle\rangle_{val}(\Phi)(s') = 1$.*

The basic idea of the proof. We prove the desired result by contradiction. We assume towards contradiction that from some state s we have $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) = \alpha > 0$ and for all states s_1 we have $\langle\langle 1\rangle\rangle_{val}(\Phi)(s_1) \leq \eta < 1$. We fix ε-optimal strategies σ and π for player 1 and player 2, for sufficiently small $\varepsilon > 0$. By Proposition 3, for all $0 < \beta < 1$, there exists n such that $\Pr_s^{\sigma,\pi}(H_n^{1,\beta}(\sigma,\pi,\overline{\Phi}) \cup H_n^{0,\beta}(\sigma,\pi,\overline{\Phi})) \geq 1-\frac{\varepsilon}{4}$. The strategy π is modified to a strategy $\widetilde{\pi}$ as follows: on histories in $H_n^{0,\beta}(\sigma,\pi,\overline{\Phi})$, the strategy $\widetilde{\pi}$ ignores the history of length n and switches to an $\frac{\varepsilon}{4}$-optimal strategy, and otherwise plays as π. By suitable choice of β (depending on ε) we show that player 2 can ensure that the probability of satisfying Φ from s given σ is less than $\alpha - \varepsilon$. This contradicts that σ is an ε-optimal strategy and $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) = \alpha$. The idea is illustrated in Fig. 2. We formally prove the result now.

Proof (*Of Theorem 1*). Assume towards contradiction that there exists a state s such that $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) > 0$, but for all states s' we have $\langle\langle 1\rangle\rangle_{val}(\Phi)(s') < 1$. Let $\alpha = 1 - \langle\langle 1\rangle\rangle_{val}(\Phi)(s) = \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s)$. Since $0 < \langle\langle 1\rangle\rangle_{val}(\Phi)(s) < 1$, we have $0 < \alpha < 1$. Since $\langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s') = 1 - \langle\langle 1\rangle\rangle_{val}(\Phi)(s')$ and for all states s' we have $\langle\langle 1\rangle\rangle_{val}(\Phi)(s') < 1$, it follows that $\langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s') > 0$, for all states s'. Fix η such that $0 < \eta = \min_{s'\in S}\langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s')$. Also observe that

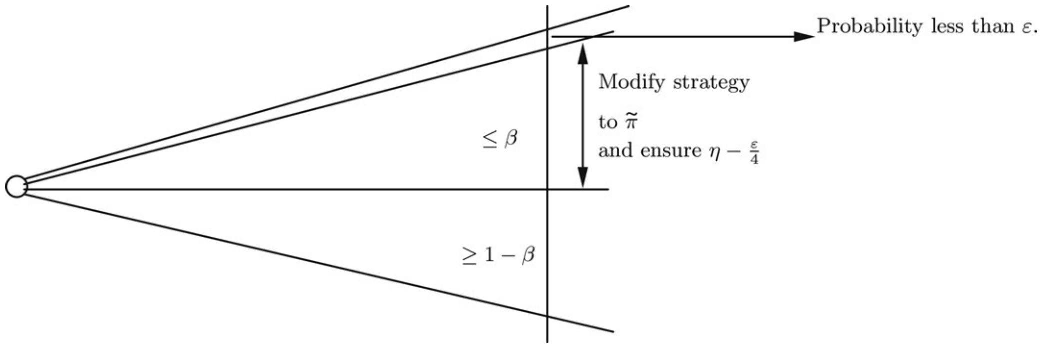

Fig. 2. An illustration of idea of Theorem 1.

since $\langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s) = \alpha < 1$, we have $\eta < 1$. Let c be a constant such that $c > 0$, and $\alpha \cdot (1+c) = \gamma < 1$ (such a constant exists as $\alpha < 1$). Also let $c_1 > 1$ be a constant such that $c_1 \cdot \gamma < 1$ (such a constant exists since $\gamma < 1$). Hence we have $1 - c_1 \cdot \gamma > 0$ and $1 - \frac{1}{c_1} > 0$. Fix $\varepsilon > 0$ and $\beta > 0$ such that

$$0 < 2\varepsilon < \min\left\{\frac{\eta}{4}, 2c\cdot\alpha, \frac{\eta}{4}\cdot(1 - c_1\cdot\gamma)\right\}; \qquad \beta < \min\left\{\varepsilon, \frac{1}{2}, 1 - \frac{1}{c_1}\right\}. \tag{1}$$

Fix ε-optimal strategies σ_ε for player 1 and π_ε for player 2. Let $H_n^{1,\beta} = H_n^{1,\beta}(\sigma_\varepsilon, \pi_\varepsilon, \overline{\Phi})$ and $H_n^{0,\beta} = H_n^{0,\beta}(\sigma_\varepsilon, \pi_\varepsilon, \overline{\Phi})$. Consider n such that $\Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta} \cup H_n^{0,\beta}) \geq 1 - \frac{\varepsilon}{4}$ (such a n exists by Proposition 3). Also observe that since $\beta < \frac{1}{2}$ we have $H_n^{1,\beta} \cap H_n^{0,\beta} = \emptyset$. Let

$$val = \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{1,\beta}) \cdot \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta}) + \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \cdot \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{0,\beta}).$$

We have

$$val \leq \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi}) \leq val + \frac{\varepsilon}{4}. \tag{2}$$

The first inequality follows since $H_n^{1,\beta} \cap H_n^{0,\beta} = \emptyset$, and the second inequality follows since $\Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta} \cup H_n^{0,\beta}) \geq 1 - \frac{\varepsilon}{4}$. Since σ_ε and π_ε are ε-optimal strategies we have $\alpha - \varepsilon \leq \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi}) \leq \alpha + \varepsilon$. This along with (2) yield that

$$\alpha - \varepsilon - \frac{\varepsilon}{4} \leq val \leq \alpha + \varepsilon. \tag{3}$$

Observe that $\Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{1,\beta}) \geq 1 - \beta$ and $\Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \leq \beta$. Let $q = \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta})$. Since $\Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{1,\beta}) \geq 1 - \beta$; by ignoring the term $\Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \cdot \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{0,\beta})$ in val, and from the second inequality of (3), we obtain that $(1-\beta)\cdot q \leq \alpha + \varepsilon$. Since $\varepsilon < c \cdot \alpha$, $\beta < 1 - \frac{1}{c_1}$, and $\gamma = \alpha \cdot (1+c)$, we have

$$q \leq \frac{\alpha + \varepsilon}{1-\beta} < \frac{\alpha\cdot(1+c)}{1 - \left(1 - \frac{1}{c_1}\right)} = c_1 \cdot \gamma \tag{4}$$

We construct a strategy $\widehat{\pi}_\varepsilon$ as follows: the strategy $\widehat{\pi}_\varepsilon$ follows the strategy π_ε for the first $n-1$-stages; if a history in $H_n^{1,\beta}$ is generated it follows π_ε, and otherwise it ignores the history and switches to an ε-optimal strategy. Formally, for a history $\langle s_1, s_2, \ldots, s_k\rangle$ we have

$$\widehat{\pi}_\varepsilon(\langle s_1, s_2, \ldots, s_k\rangle) = \begin{cases} \pi_\varepsilon(\langle s_1, s_2, \ldots, s_k\rangle) & \text{if } k < n; \\ & \text{or } \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid \langle s_1, s_2, \ldots, s_n\rangle) \geq 1 - \beta; \\ \widetilde{\pi}_\varepsilon(\langle s_n, \ldots, s_k\rangle) & k \geq n, \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid \langle s_1, s_2, \ldots, s_n\rangle) < 1 - \beta, \\ & \text{ where } \widetilde{\pi}_\varepsilon \text{ is an } \varepsilon\text{-optimal strategy} \end{cases}$$

Since $\widehat{\pi}_\varepsilon$ and π_ε coincide for $n-1$-stages we have $\Pr_s^{\sigma_\varepsilon,\widehat{\pi}_\varepsilon}(H_n^{1,\beta}) = \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta})$ and $\Pr_s^{\sigma_\varepsilon,\widehat{\pi}_\varepsilon}(H_n^{0,\beta}) = \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{0,\beta})$. Moreover, since $\overline{\Phi}$ is a tail objective that is independent of the prefix of length n, $\eta \leq \min_{s' \in S} \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s')$, and $\widetilde{\pi}_\varepsilon$

is an ε-optimal strategy, we have $\Pr_s^{\sigma_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \geq \eta - \varepsilon$. Also observe that

$$\begin{aligned}\Pr_s^{\sigma_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) &\geq (\eta-\varepsilon) = \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) + (\eta - \varepsilon - \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}))\\ &\geq \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) + (\eta-\varepsilon-\beta),\end{aligned} \tag{5}$$

since $\Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \leq \beta$. Hence we have the following inequality

$$\begin{aligned}
\Pr_s^{\sigma_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi}) &\geq \Pr_s^{\sigma_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi} \mid H_n^{1,\beta}) \cdot \Pr_s^{\sigma_\varepsilon,\widehat{\pi}_\varepsilon}(H_n^{1,\beta}) && + \Pr_s^{\sigma_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \cdot \Pr_s^{\sigma_\varepsilon,\widehat{\pi}_\varepsilon}(H_n^{0,\beta})\\
&= \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{1,\beta}) \cdot \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta}) && + \Pr_s^{\sigma_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \cdot \Pr_s^{\sigma_\varepsilon,\widehat{\pi}_\varepsilon}(H_n^{0,\beta})\\
&\geq \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{1,\beta}) \cdot \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta}) && + \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \cdot \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{0,\beta})\\
&\quad + (\eta-\varepsilon-\beta)\cdot\left(1-q-\tfrac{\varepsilon}{4}\right) && \left(\text{since } \Pr_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{0,\beta}) \geq 1-q-\tfrac{\varepsilon}{4}\right)\\
&= val + (\eta-\varepsilon-\beta)\cdot(1-q-\tfrac{\varepsilon}{4})\\
&\geq \alpha-\varepsilon-\tfrac{\varepsilon}{4} + (\eta-\varepsilon-\beta)\cdot\left(1-q-\tfrac{\varepsilon}{4}\right) && \text{(recall first inequality of (3))}\\
&> \alpha-\varepsilon-\tfrac{\varepsilon}{4} + (\eta-2\varepsilon)\cdot\left(1-q-\tfrac{\varepsilon}{4}\right) && \text{(since } \beta < \varepsilon \text{ by (1))}\\
&> \alpha-\varepsilon-\tfrac{\varepsilon}{4} + \tfrac{\eta}{2}\cdot\left(1-q-\tfrac{\varepsilon}{4}\right) && \text{(since } 2\varepsilon < \tfrac{\eta}{2} \text{ by (1))}\\
&> \alpha-\varepsilon-\tfrac{\varepsilon}{4} + \tfrac{\eta}{2}\cdot(1-c_1\cdot\gamma) - \tfrac{\eta}{2}\cdot\tfrac{\varepsilon}{4} && \text{(since } q < c_1\cdot\gamma \text{ by (4))}\\
&> \alpha-\varepsilon-\frac{\varepsilon}{4} + 4\varepsilon - \frac{\varepsilon}{8} && \text{(since } 2\varepsilon < \tfrac{\eta}{4}\cdot(1-c_1\cdot\gamma) \text{ by (1),}\\
& && \text{and } \eta \leq 1)\\
&> \alpha+\varepsilon.
\end{aligned}$$

The first equality follows since for histories in $H_n^{1,\beta}$, the strategies π_ε and $\widehat{\pi}_\varepsilon$ coincide; and the second inequality uses (5). Hence we have $\Pr_s^{\sigma_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi}) > \alpha + \varepsilon$ and $\Pr_s^{\sigma_\varepsilon,\widehat{\pi}_\varepsilon}(\Phi) < 1-\alpha-\varepsilon$. This is a contradiction to the fact that $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) = 1-\alpha$ and σ_ε is an ε-optimal strategy. The desired result follows. ■

Recall that by determinacy of concurrent games with tail objectives, we have $W_1^1 = S \setminus W_2^{>0}$ and $W_2^1 = S \setminus W_1^{>0}$. We now present a finer characterization of the qualitative sets.

Corollary 1. *For all concurrent game structures G, with tail objectives Φ for player 1, the following assertions hold:*

1. (a) *if $W_1^{>0} \neq \emptyset$, then $W_1^1 \neq \emptyset$; and* (b) *if $W_2^{>0} \neq \emptyset$, then $W_2^1 \neq \emptyset$.*
2. (a) *if $W_1^{>0} = S$, then $W_1^1 = S$; and* (b) *if $W_2^{>0} = S$, then $W_2^1 = S$.*

Proof. The first result is a direct consequence of Theorem 1. The second result is derived as follows: if $W_1^{>0} = S$, then by determinacy we have $W_2^1 = \emptyset$. If $W_2^1 = \emptyset$, it follows from part 1 that $W_2^{>0} = \emptyset$, and hence $W_1^1 = S$. The result of part 2 shows that if a player has positive optimum value at every state, then the optimum value is 1 at all states. ■

Extension to countable state space. We first present an example to show that Corollary 1 (and hence also Theorem 1) does not extend directly to concurrent games with countable state space. Then we present the appropriate extension of Theorem 1 to concurrent games with countable state space.

Example 3. Consider a concurrent game defined on a countable state space S as follows: $S = S_N \cup \{t\}$, where $S_N = \{s_i \mid i = 0, 1, 2, \ldots\}$. For every state $s \in S$ we have $Mv_1(s) = Mv_2(s) = \{1\}$. The transition probabilities are specified as follows: the state t is an absorbing state; and from state s_i the next state is s_{i+1} with probability $(\frac{1}{2})^{\frac{1}{2^i}}$, and the next state is t with the rest of the probability. Consider the tail objective $\Phi = \text{Büchi}(\{t\})$. For a state s_i we have $\langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s_i) = (\frac{1}{2})^{\sum_{j=i}^{\infty}\frac{1}{2^j}} = (\frac{1}{2})^{\frac{1}{2^{i-1}}} < 1$. That is, we have $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) > 0$, for all $s \in S$. Hence $W_1^{>0} = S$, however, $W_1^1 \neq S$. It follows that Corollary 1 does not extend to concurrent games with countable state space. ■

We now present the appropriate extension of Theorem 1 to countable state spaces.

Theorem 2. *For all concurrent game structures G with countable state space, for all tail objectives Φ, if there exists a state $s \in S$ such that $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) > 0$, then $\sup_{s' \in S}\langle\langle 1\rangle\rangle_{val}(\Phi)(s') = 1$.*

Proof. The key difference to the proof of Theorem 1 is to fix the constants. Assume towards contradiction that there exists a state s such that $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) > 0$, but $\sup_{s' \in S} \langle\langle 1\rangle\rangle_{val}(\Phi)(s') < 1$. Let $\alpha = 1 - \langle\langle 1\rangle\rangle_{val}(\Phi)(s) = \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s)$. Since $0 < \langle\langle 1\rangle\rangle_{val}(\Phi)(s) < 1$, we have $0 < \alpha < 1$. Let $\eta = \inf_{s' \in S} \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s')$. Since $\sup_{s' \in S} \langle\langle 1\rangle\rangle_{val}(\Phi)(s') < 1$ and $\langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s') = 1 - \langle\langle 1\rangle\rangle_{val}(\Phi)(s')$ for all $s' \in S$, we have $0 < \eta$. Also observe that since $\langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s) = \alpha < 1$, we have $\eta < 1$. Once the constant η is fixed, we can essentially follow the proof of Theorem 1 to obtain the desired result. ■

4. Zero-sum tail games to nonzero-sum reachability games

In this section we relate the values of zero-sum games with tail objectives with the Nash equilibrium values of nonzero-sum games with reachability objectives. The result shows that the values of a zero-sum game with complex objectives can be related to equilibrium values of a nonzero-sum game with simpler objectives. We also show that for MDPs, the value function for a tail objective Φ can be computed by computing the maximal probability of reaching the set of states with value 1. As an immediate consequence of the above analysis, we obtain a polynomial time reduction of the quantitative analysis of MDPs with tail objectives to the qualitative analysis. We first prove a *limit-reachability* property of ε-optimal strategies: the property states that for tail objectives, if the players play ε-optimal strategies, for small $\varepsilon > 0$, then the game reaches $W_1^1 \cup W_2^1$ with high probability.

Theorem 3 (*Limit-Reachability*). *For all concurrent game structures G, for all tail objectives Φ for player* 1*, for all $\varepsilon' > 0$, there exists $\varepsilon > 0$, such that for all states $s \in S$, for all ε-optimal strategies σ_ε and π_ε, we have*

$$\mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(\mathrm{Reach}(W_1^1 \cup W_2^1)) \geq 1 - \varepsilon'.$$

Proof. By determinacy it follows that $W_1^1 \cup W_2^1 = S \setminus (W_1^{>0} \cap W_2^{>0})$. For a state $s \in W_1^1 \cup W_2^1$ the result holds trivially. Consider a state $s \in W_1^{>0} \cap W_2^{>0}$ and let $\alpha = \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s)$. Observe that $0 < \alpha < 1$. Let $\eta_1 = \min_{s \in W_2^{>0}} \langle\langle 1\rangle\rangle_{val}(\Phi)(s)$ and $\eta_2 = \max_{s \in W_2^{>0}} \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s)$. Let $\eta = \min\{\eta_1, 1 - \eta_2\}$, and note that $0 < \eta < 1$. Given $\varepsilon' > 0$, fix ε such that $0 < 2\varepsilon < \min\{\frac{\eta}{2}, \frac{\eta \cdot \varepsilon'}{12}\}$. Fix any ε-optimal strategies σ_ε and π_ε for player 1 and player 2, respectively. Fix β such that $0 < \beta < \varepsilon$ and $\beta < \frac{1}{2}$. Let $H_n^{1,\beta} = H_n^{1,\beta}(\sigma_\varepsilon, \pi_\varepsilon, \overline{\Phi})$ and $H_n^{0,\beta} = H_n^{0,\beta}(\sigma_\varepsilon, \pi_\varepsilon, \overline{\Phi})$. Consider n such that $\mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta} \cup H_n^{0,\beta}) = 1 - \frac{\varepsilon}{4}$ (such a n exists by Proposition 3). As $\beta < \frac{1}{2}$, we have $H_n^{1,\beta} \cap H_n^{0,\beta} = \emptyset$. Let us denote by

$$val = \mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{1,\beta}) \cdot \mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta}) + \mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \cdot \mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{0,\beta}).$$

Similar to inequality (2) of Theorem 1 we obtain that

$$val \leq \mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi}) \leq val + \frac{\varepsilon}{4}.$$

Since σ_ε and π_ε are ε-optimal strategies, similar to inequality (3) of Theorem 1 we obtain that $\alpha - \varepsilon - \frac{\varepsilon}{4} \leq val \leq \alpha + \varepsilon$.

For $W \subseteq S$, let $\mathrm{Reach}^n(W) = \{\langle s_1, s_2, s_3 \ldots\rangle \mid \exists k \leq n.\ s_k \in W\}$ denote the set of paths that reach W in n steps. We use the following notation: $\overline{\mathrm{Reach}(W_1^1)} = \Omega \setminus \mathrm{Reach}^n(W_1^1)$, and $\overline{\mathrm{Reach}(W_2^1)} = \Omega \setminus \mathrm{Reach}^n(W_2^1)$. Consider a strategy $\widehat{\sigma}_\varepsilon$ defined as follows: for histories in $H_n^{1,\beta} \cap \overline{\mathrm{Reach}(W_2^1)}$, the strategy $\widehat{\sigma}_\varepsilon$ ignores the history after stage n and follows an ε-optimal strategy $\widetilde{\sigma}_\varepsilon$ (i.e., $\widetilde{\sigma}_\varepsilon$ is an ε-optimal strategy); and for all other histories it follows σ_ε. Let $z_1 = \mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta} \cap \overline{\mathrm{Reach}(W_2^1)})$. Since $\eta_2 = \max_{s \in W_2^{>0}} \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s)$, player 1 switches to an ε-optimal strategy for histories of length n in $H_n^{1,\beta} \cap \overline{\mathrm{Reach}(W_2^1)}$, and $\overline{\Phi}$ is a tail objective, it follows that for all $\omega = \langle s_1, s_2, \ldots, s_n, s_{n+1}, \ldots\rangle \in H_n^{1,\beta} \cap \overline{\mathrm{Reach}(W_2^1)}$, we have $\mathrm{Pr}_s^{\widehat{\sigma}_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid \langle s_1, s_2 \ldots, s_n\rangle) \leq \eta_2 + \varepsilon$; where as $\mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid \langle s_1, s_2 \ldots, s_n\rangle) \geq 1 - \beta$. Hence we have

$$val_2 = \mathrm{Pr}_s^{\widehat{\sigma}_\varepsilon,\pi_\varepsilon}(\overline{\Phi}) \leq \mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi}) - z_1 \cdot (1 - \beta - \eta_2 - \varepsilon) \leq val + \frac{\varepsilon}{4} - z_1 \cdot (1 - \beta - \eta_2 - \varepsilon),$$

since with probability z_1 the decrease is at least by $1 - \beta - \eta_2 - \varepsilon$. Since π_ε is an ε-optimal strategy, we have $val_2 \geq \alpha - \varepsilon$. Since $val \leq \alpha + \varepsilon$, we have the following inequality

$$z_1 \cdot (1 - \eta_2 - \beta - \varepsilon) \le 2\varepsilon + \frac{\varepsilon}{4} < 3\varepsilon$$
$$\Rightarrow z_1 < \frac{3\varepsilon}{\eta - \beta - \varepsilon} \quad (\text{since } \eta \le 1 - \eta_2)$$
$$\Rightarrow z_1 < \frac{3\varepsilon}{\eta - 2\varepsilon} < \frac{6\varepsilon}{\eta} < \frac{\varepsilon'}{4} \quad \left(\text{since } \beta < \varepsilon; \varepsilon < \frac{\eta}{4}; \varepsilon < \frac{\eta \cdot \varepsilon'}{24}\right).$$

Consider a strategy $\widehat{\pi}_\varepsilon$ defined as follows: for histories in $H_n^{0,\beta} \cap \overline{\mathrm{Reach}(W_1^1)}$, the strategy $\widehat{\pi}_\varepsilon$ ignores the history after stage n and follows an ε-optimal strategy $\widetilde{\pi}_\varepsilon$; and for all other histories it follows π_ε. Let $z_2 = \mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{0,\beta} \cap \overline{\mathrm{Reach}(W_1^1)})$. Since $\eta_1 = \min_{s \in W_2^{>0}} \langle\langle 2 \rangle\rangle_{val}(\overline{\Phi})(s)$, player 2 switches to an ε-optimal strategy for histories of length n in $H_n^{0,\beta} \cap \overline{\mathrm{Reach}(W_1^1)}$, and $\overline{\Phi}$ is a tail objective, it follows that for all $\omega = \langle s_1, s_2, \ldots, s_n, s_{n+1}, \ldots \rangle \in H_n^{1,\beta} \cap \overline{\mathrm{Reach}(W_1^1)}$, we have $\mathrm{Pr}_s^{\sigma_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi} \mid \langle s_1, s_2 \ldots, s_n \rangle) \ge \eta_1 - \varepsilon$; where as $\mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid \langle s_1, s_2 \ldots, s_n \rangle) \le \beta$. Hence we have

$$val_1 = \mathrm{Pr}_s^{\sigma_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi}) \ge \mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(\overline{\Phi}) + z_2 \cdot (\eta_1 - \varepsilon - \beta) \ge val + z_2 \cdot (\eta_1 - \varepsilon - \beta),$$

since with probability z_2 the increase is at least by $\eta_1 - \varepsilon - \beta$. Since σ_ε is an ε-optimal strategy, we have $val_1 \le \alpha + \varepsilon$. Since $val \ge \alpha - \varepsilon + \frac{\varepsilon}{4}$, we have the following inequality

$$z_2 \cdot (\eta_1 - \beta - \varepsilon) \le 2\varepsilon + \frac{\varepsilon}{4} < 3\varepsilon$$
$$\Rightarrow z_2 < \frac{3\varepsilon}{\eta - \beta - \varepsilon} \quad (\text{since } \eta \le \eta_1)$$
$$\Rightarrow z_2 < \frac{\varepsilon'}{4} \quad \left(\text{similar to the inequality for } z_1 < \frac{\varepsilon'}{4}\right).$$

Hence $z_1 + z_2 \le \frac{\varepsilon'}{2}$; and then we have

$$\begin{aligned}
\mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(\mathrm{Reach}(W_1^1 \cup W_2^1)) &\ge \mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(\mathrm{Reach}^n(W_1^1 \cup W_2^1) \cap (H_n^{1,\beta} \cup H_n^{0,\beta}))\\
&= \mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(\mathrm{Reach}^n(W_1^1 \cup W_2^1) \cap H_n^{1,\beta}) + \mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(\mathrm{Reach}^n(W_1^1 \cup W_2^1) \cap H_n^{0,\beta})\\
&\ge \mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(\mathrm{Reach}^n(W_1^1) \cap H_n^{1,\beta}) + \mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(\mathrm{Reach}^n(W_2^1) \cap H_n^{0,\beta})\\
&\ge \mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta}) + \mathrm{Pr}_s^{\sigma_\varepsilon,\pi_\varepsilon}(H_n^{0,\beta}) - (z_1 + z_2)\\
&\ge 1 - \frac{\varepsilon}{4} + \frac{\varepsilon'}{2} \ge 1 - \varepsilon' \quad (\text{since } \varepsilon \le \varepsilon').
\end{aligned}$$

The result follows. ■

Theorem 3 proves the limit-reachability property for tail objectives, under ε-optimal strategies, for small ε. We present an example to show that Theorem 3 is not true for all objectives, or for tail objectives with arbitrary strategies.

Example 4. Observe that in the game shown in Example 2, the objective was not a tail objective and we had $W_1^1 \cup W_2^1 = \emptyset$. Hence Theorem 3 need not necessarily hold for all objectives. Also consider the game shown in Fig. 3. In the game shown s_1 and s_2 are absorbing states. At s_0 the available moves for the players are as follows: $Mv_1(s_0) = \{a\}$ and $Mv_2(s_0) = \{1, 2\}$. The transition function is as follows: if player 2 plays move 2, then the next state is s_1 and s_2 with equal probability, and if player 2 plays move 1, then the next state is s_0. The objective of player 1 is $\Phi = \text{Büchi}(\{s_0, s_1\})$, i.e., to visit s_0 or s_1 infinitely often. We have $W_1^1 = \{s_1\}$ and $W_2^1 = \{s_2\}$. Given a strategy π that chooses move 1 always, the set $W_1^1 \cup W_2^1$ of states is reached with probability 0; however π is not an optimal or ε-optimal strategy for player 2 (for $\varepsilon < \frac{1}{2}$). This shows that Theorem 3 need not hold if ε-optimal strategies are not considered. In the game shown, for an optimal strategy for player 2 (e.g., a strategy to choose move 2) the play reaches $W_1^1 \cup W_2^1$ with probability 1. ■

The following example further illustrates Theorem 3.

Example 5 (*Concurrent Büchi Game*). Consider the concurrent game shown in Fig. 4. The available moves for the players at state s_0 and s_3 are $\{0, 1\}$ and $\{0, 1, q\}$, respectively. At all other states the available moves for both the

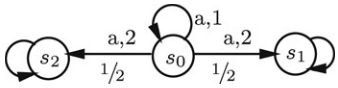

Fig. 3. A game with Büchi objective.

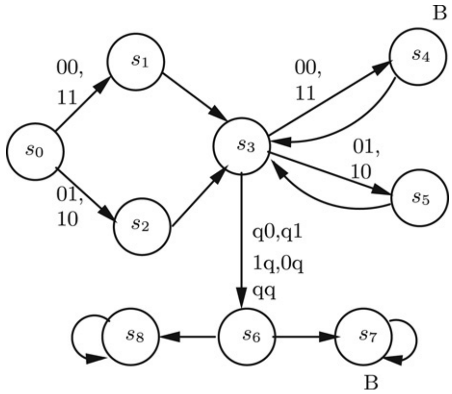

Fig. 4. A concurrent Büchi game.

players are singleton. The transitions are shown as labeled edges in the figure. The objective of player 1 is to visit s_4 or s_7 infinitely often, i.e., Büchi$(\{s_4, s_7\})$. The value for player 1 is 1 at state s_7 and 0 at state s_8. Observe that since at state s_3 each player can choose move q, it follows that the values for the players at state s_3 (and hence at states s_0, s_1, s_2, s_4, s_5 and s_6) is $1/2$. Consider the strategy σ for player 1 as follows: (a) at state s_0 it plays 0 and 1, each with probability $1/2$, and remembers the move played as the move b; (b) at state s_3, player 1 remembers the move c played by player 2 (since player 1 knows whether the state s_1 or s_2 was visited, it can infer the move played by player 2 at s_0); (c) at state s_3 player 1 plays move b as long as player 2 plays move c, otherwise player 1 plays the move q. Informally, player 1 plays both its move uniformly at random at s_0, and discloses to player 2, and remembers the move of player 2. As long as player 2 follows her move, player 1 follows her move chosen in the first round, else if player 2 deviates, then player 1 quits the game by playing q. A strategy π for player 2 can be defined similarly. Given strategies σ and π, and the starting state s_0, the play reaches s_7 and s_8 with probability 0, and the play satisfies Büchi$(\{s_4, s_8\})$ with probability $1/2$. However, observe that the strategy σ is not an optimal strategy. Given the strategy σ, consider the strategy $\overline{\pi}$ as follows: the strategy $\overline{\pi}$ chooses 0 and 1 with probability $1/2$ at s_0, and at s_3 if the chosen move c at s_0 matches with the move b for player 1, then player 2 plays q (i.e., quits the game) and otherwise it follows π. Given the strategy $\overline{\pi}$, if player 1 follows σ, then Büchi$(\{s_4, s_7\})$ is satisfied with only probability $\frac{1}{4}$. In the game shown, if both the players follow any pair of optimal strategies, then the game reaches s_7 and s_8 with probability 1. ■

Lemma 4 is immediate from Theorem 3.

Lemma 4. *For all concurrent game structures G, for all tail objectives Φ for player* 1 *and $\overline{\Phi}$ for player* 2*, for all states $s \in S$, we have*

$$\lim_{\varepsilon \to 0} \sup_{\sigma \in \Sigma_\varepsilon(\Phi), \pi \in \Pi_\varepsilon(\overline{\Phi})} \Pr_s^{\sigma,\pi}(\mathrm{Reach}(W_1^1 \cup W_2^1)) = 1;$$

$$\lim_{\varepsilon \to 0} \sup_{\sigma \in \Sigma_\varepsilon(\Phi), \pi \in \Pi_\varepsilon(\overline{\Phi})} \Pr_s^{\sigma,\pi}(\mathrm{Reach}(W_1^1)) = \langle\langle 1 \rangle\rangle_{val}(\Phi)(s);$$

$$\lim_{\varepsilon \to 0} \sup_{\sigma \in \Sigma_\varepsilon(\Phi), \pi \in \Pi_\varepsilon(\overline{\Phi})} \Pr_s^{\sigma,\pi}(\mathrm{Reach}(W_2^1)) = \langle\langle 2 \rangle\rangle_{val}(\overline{\Phi})(s).$$

Consider a nonzero-sum reachability game G_R such that the states in $W_1^1 \cup W_2^1$ are transformed to absorbing states and the objectives of both players are reachability objectives: the objective for player 1 is Reach(W_1^1) and the objective for player 2 is Reach(W_2^1). Note that the game G_R is not zero-sum in the following sense: there are infinite paths ω

such that $\omega \notin \text{Reach}(W_1^1)$ and $\omega \notin \text{Reach}(W_2^1)$ and each player gets a payoff 0 for the path ω. We define ε-Nash equilibrium of the game G_R and relate some special ε-Nash equilibrium of G_R with the values of G.

Definition 5 (*ε-Nash Equilibrium in G_R*). A strategy profile $(\sigma^*, \pi^*) \in \Sigma \times \Pi$ is an ε-Nash equilibrium at state s if the following two conditions hold:

$$\Pr_s^{\sigma^*,\pi^*}(\text{Reach}(W_1^1)) \geq \sup_{\sigma\in\Sigma} \Pr_s^{\sigma,\pi^*}(\text{Reach}(W_1^1)) - \varepsilon$$

$$\Pr_s^{\sigma^*,\pi^*}(\text{Reach}(W_2^1)) \geq \sup_{\pi\in\Pi} \Pr_s^{\sigma^*,\pi}(\text{Reach}(W_2^1)) - \varepsilon. \quad \blacksquare$$

Theorem 4 (*Nash Equilibrium of Reachability Game G_R*). *The following assertion holds for the game G_R.*

- *For all $\varepsilon > 0$, there is an ε-Nash equilibrium $(\sigma_\varepsilon^*, \pi_\varepsilon^*) \in \Sigma_\varepsilon(\Phi) \times \Pi_\varepsilon(\overline{\Phi})$ such that for all states s we have*

$$\lim_{\varepsilon\to 0} \Pr_s^{\sigma_\varepsilon^*,\pi_\varepsilon^*}(\text{Reach}(W_1^1)) = \langle\langle 1\rangle\rangle_{val}(\Phi)(s)$$

$$\lim_{\varepsilon\to 0} \Pr_s^{\sigma_\varepsilon^*,\pi_\varepsilon^*}(\text{Reach}(W_2^1)) = \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s).$$

Proof. It follows from Lemma 4. $\blacksquare$

Note that in case of MDPs the strategy for player 2 is trivial, i.e., player 2 has only one strategy. Hence in context of MDPs we drop the strategy π of player 2. A specialization of Theorem 4 in case of MDPs yields Theorem 5.

Theorem 5. *For all MDPs G, for all tail objectives Φ, we have*

$$\langle\langle 1\rangle\rangle_{val}(\Phi)(s) = \sup_{\sigma\in\Sigma} \Pr_s^{\sigma}(\text{Reach}(W_1^1)) = \langle\langle 1\rangle\rangle_{val}(\text{Reach}(W_1^1))(s).$$

Since the values in MDPs with reachability objectives can be computed in polynomial time (by linear programming) [3,9], our result presents a polynomial time reduction of quantitative analysis of tail objectives in MDPs to qualitative analysis.

5. Construction of ε-optimal strategies for Muller objectives

In this section we show that for Muller objectives witnesses of ε-optimal strategies can be constructed as witnesses of certain limit-sure winning strategies that respect certain local conditions. A key notion that will play an important role in the construction of ε-optimal strategies is the notion of *local optimality*. Informally, a selector function ξ is *locally optimal* if it is optimal in the one-step matrix game where each state is assigned a reward value $\langle\langle 1\rangle\rangle_{val}(\Phi)(s)$. A *locally optimal strategy* is a strategy that consists of locally optimal selectors. A *locally ε-optimal strategy* is a strategy that has a total deviation from locally optimal selectors of at most ε. We note that *local ε-optimality* and *ε-optimality* are very different notions. *Local ε-optimality* consists in the approximation of local optimal selectors; a locally ε-optimal strategy provides no guarantee of yielding a probability of winning the game close to the optimal one.

Definition 6 (*Locally ε-Optimal Selectors and Strategies*). A selector ξ is *locally optimal* for objective Φ if for all $s \in S$ and $a_2 \in Mv_2(s)$ we have

$$\mathbb{E}_s^{\xi(s),a_2}[\langle\langle 1\rangle\rangle_{val}(\Phi)(\Theta_1)] \geq \langle\langle 1\rangle\rangle_{val}(\Phi)(s).$$

We denote by $\Lambda^\ell(\Phi)$ the set of locally optimal selectors for objective Φ. A strategy σ is *locally optimal* for objective Φ if for every history $\langle s_0, s_1, \ldots, s_k\rangle$ we have $\sigma(\langle s_0, s_1, \ldots, s_k\rangle) \in \Lambda^\ell(\Phi)$, i.e., player 1 plays a locally optimal selector at every round of the play. We denote by $\Sigma^\ell(\Phi)$ the set of locally optimal strategies for objective Φ. A strategy σ_ε is *locally ε-optimal* for objective Φ if for every strategy $\pi \in \Pi$, for all $k \geq 1$, for all states s we have

$$\langle\langle 1\rangle\rangle_{val}(\Phi)(s) - \mathbb{E}_s^{\sigma,\pi}[\langle\langle 1\rangle\rangle_{val}(\Phi)(\Theta_k)] \leq \varepsilon.$$

Observe that a strategy that at each round i chooses a locally optimal selector with probability at least $(1 - \varepsilon_i)$, with $\sum_{i=0}^{\infty} \varepsilon_i \leq \varepsilon$, is a locally ε-optimal strategy. We denote by $\Sigma_\varepsilon^\ell(\Phi)$ the set of locally ε-optimal strategies for objective Φ. $\blacksquare$

We first show that for all tail objectives, for all $\varepsilon > 0$, there exist strategies that are ε-optimal and locally ε-optimal as well.

Lemma 5. *For all tail objectives Φ, for all $\varepsilon > 0$,*

1. $\Sigma_{\frac{\varepsilon}{2}}(\Phi) \subseteq \Sigma_{\varepsilon}^{\ell}(\Phi)$,
2. $\Sigma_{\varepsilon}(\Phi) \cap \Sigma_{\varepsilon}^{\ell}(\Phi) \neq \emptyset$.

Proof. For $\varepsilon > 0$, fix an $\frac{\varepsilon}{2}$-optimal strategy σ for player 1. By definition σ is an ε-optimal strategy as well. We argue that $\sigma \in \Sigma_{\varepsilon}^{\ell}(\Phi)$. Assume towards contradiction that $\sigma \notin \Sigma_{\varepsilon}^{\ell}(\Phi)$, i.e., there exists a player 2 strategy π, a state s, and k such that

$$\langle\langle 1\rangle\rangle_{val}(\Phi)(s) - \mathbb{E}_s^{\sigma,\pi}[\langle\langle 1\rangle\rangle_{val}(\Phi)(\Theta_k)] > \varepsilon.$$

Fix a strategy $\pi^* = (\pi + \widetilde{\pi})$ for player 2 as follows: play π for k steps, then switch to an $\frac{\varepsilon}{4}$-optimal strategy $\widetilde{\pi}$. Formally, for a history $\langle s_1, s_2, \ldots, s_n\rangle$ we have

$$\pi^*(\langle s_1, s_2, \ldots, s_n\rangle) = \begin{cases} \pi(\langle s_1, s_2, \ldots, s_n\rangle) & \text{if } n \leq k \\ \widetilde{\pi}(\langle s_{k+1}, s_{k+2}, \ldots, s_n\rangle) & \text{if } n > k, \\ & \text{where } \widetilde{\pi} \text{ is an } \frac{\varepsilon}{4}\text{-optimal strategy.} \end{cases}$$

Since Φ is a tail objective, we have $\Pr_s^{\sigma,\pi^*}(\Phi) = \sum_{t\in S} \Pr_t^{\sigma,\widetilde{\pi}}(\Phi) \cdot \Pr_s^{\sigma,\pi^*}(\Theta_k = t)$. Hence we obtain the following inequality

$$\begin{aligned} \Pr_s^{\sigma,\pi^*}(\Phi) &= \sum_{t\in S} \Pr_t^{\sigma,\widetilde{\pi}}(\Phi) \cdot \Pr_s^{\sigma,\pi^*}(\Theta_k = t) \\ &= \sum_{t\in S} \Pr_t^{\sigma,\widetilde{\pi}}(\Phi) \cdot \Pr_s^{\sigma,\pi}(\Theta_k = t) \\ &\leq \sum_{t\in S} \left(\langle\langle 1\rangle\rangle_{val}(\Phi)(t) + \frac{\varepsilon}{4}\right) \cdot \Pr_s^{\sigma,\pi}(\Theta_k = t) \quad \left(\text{since } \widetilde{\pi} \text{ is an } \frac{\varepsilon}{4}\text{-optimal strategy}\right) \\ &= \mathbb{E}_s^{\sigma,\pi}[\langle\langle 1\rangle\rangle_{val}(\Phi)(\Theta_k)] + \frac{\varepsilon}{4}. \end{aligned}$$

Hence we have

$$\Pr_s^{\sigma,\pi^*}(\Phi) < (\langle\langle 1\rangle\rangle_{val}(\Phi)(s) - \varepsilon) + \frac{\varepsilon}{4} = \langle\langle 1\rangle\rangle_{val}(\Phi)(s) - \frac{3\varepsilon}{4} < \langle\langle 1\rangle\rangle_{val}(\Phi)(s) - \frac{\varepsilon}{2}.$$

Since by assumption σ is an $\frac{\varepsilon}{2}$-optimal strategy we have a contradiction. This establishes the desired result. ■

Definition 7 (*Perennial ε-Optimal Strategies*). A strategy σ is a perennial ε-optimal strategy for objective Φ, if it is ε-optimal for all states s, and for all histories $\langle s_1, s_2, \ldots, s_k\rangle$, for all strategies $\pi \in \Pi$ for player 2, $\Pr_s^{\sigma,\pi}(\Phi \mid \langle s_1, s_2, \ldots, s_k\rangle) \geq \langle\langle 1\rangle\rangle_{val}(\Phi)(s_k) - \varepsilon$. In other words, for every history $\langle s_1, s_2, \ldots, s_k\rangle$, given the history the probability to satisfy Φ is within ε of the value at s_k. We denote by $\Sigma_{\varepsilon}^{PL}(\Phi)$ the set of perennial ε-optimal strategies for player 1, for objective Φ. The set of perennial ε-optimal strategies for player 2 is defined similarly and we denote them by $\Pi_{\varepsilon}^{PL}(\overline{\Phi})$. ■

Existence of perennial ε-optimal strategies. The results of [7] proves existence of perennial ε-optimal strategies for concurrent games with parity objectives, for all $\varepsilon > 0$. Since Muller objectives can be reduced to parity objectives, the following proposition follows.

Proposition 4. *For all concurrent game structures, for all Muller objectives Φ, for all $\varepsilon > 0$, $\Sigma_{\varepsilon}^{PL}(\Phi) \neq \emptyset$ and $\Pi_{\varepsilon}^{PL}(\overline{\Phi}) \neq \emptyset$.*

Lemma 6. *For all concurrent game structures G, for all Muller objectives Φ for player 1 and $\overline{\Phi}$ for player 2, we have*

$$\inf_{\sigma\in\Sigma_\varepsilon^{PL}(\Phi)}\sup_{\pi\in\Pi}\Pr_s^{\sigma,\pi}(\overline{\Phi}\cap\mathrm{Safe}(W_1^{>0}\cap W_2^{>0}))=0;$$
$$\inf_{\sigma\in\Sigma_\varepsilon(\Phi)}\sup_{\pi\in\Pi}\Pr_s^{\sigma,\pi}(\overline{\Phi}\cap\mathrm{Safe}(W_1^{>0}\cap W_2^{>0}))=0;$$
$$\inf_{\pi\in\Pi_\varepsilon^{PL}(\overline{\Phi})}\sup_{\sigma\in\Sigma}\Pr_s^{\sigma,\pi}(\Phi\cap\mathrm{Safe}(W_1^{>0}\cap W_2^{>0}))=0;$$
$$\inf_{\pi\in\Pi_\varepsilon(\overline{\Phi})}\sup_{\sigma\in\Sigma}\Pr_s^{\sigma,\pi}(\Phi\cap\mathrm{Safe}(W_1^{>0}\cap W_2^{>0}))=0.$$

Proof. We show that

$$\inf_{\sigma\in\Sigma_\varepsilon^{PL}(\Phi)}\sup_{\pi\in\Pi}\Pr_s^{\sigma,\pi}(\overline{\Phi}\cap\mathrm{Safe}(W_1^{>0}\cap W_2^{>0}))=0.$$

Since for all $\varepsilon>0$ we have $\Sigma_\varepsilon^{PL}(\Phi)\subseteq\Sigma_\varepsilon(\Phi)$, this is sufficient to prove the first two claims. The results for the last two claims are symmetric. We prove the first claim as follows. Let $W^{>0}=W_1^{>0}\cap W_2^{>0}$. Let $\eta=\min_{s\in W^{>0}}\langle\langle 1\rangle\rangle_{val}(\Phi)(s)$, and observe that $0<\eta<1$. Fix $0<2\varepsilon<\eta$, and fix a perennial ε-optimal strategy $\sigma\in\Sigma_\varepsilon^{PL}(\Phi)$. Consider a strategy $\pi\in\Pi$ for player 2. Since $\sigma\in\Sigma_\varepsilon^{PL}(\Phi)$, for all $k\geq 1$, for all histories $\langle s_1,s_2,\ldots,s_k\rangle$ such that $s_i\in W^{>0}$ for all $i\leq k$, we have $\Pr_s^{\sigma,\pi}(\Phi\mid\langle s_1,s_2,\ldots,s_k\rangle)\geq\eta-\varepsilon>\frac{\eta}{2}$. For a history $\langle s_1,s_2,\ldots,s_k\rangle$ such that there exists $i\leq k$ and $s_i\notin W^{>0}$ we have $\Pr_s^{\sigma,\pi}(\mathrm{Reach}(W_1^1\cup W_2^1)\mid\langle s_1,s_2,\ldots,s_k\rangle)=1$. Hence it follows that for all n we have $\Pr_s^{\sigma,\pi}(\Phi\cup\mathrm{Reach}(W_1^1\cup W_2^1)\mid\mathcal{F}_n)>\frac{\eta}{2}$. Since $\frac{\eta}{2}>0$, by Lemma 3 we have $\Pr_s^{\sigma,\pi}(\Phi\cup\mathrm{Reach}(W_1^1\cup W_2^1))=1$, i.e., $\Pr_s^{\sigma,\pi}(\overline{\Phi}\cap\mathrm{Safe}(W^{>0}))=0$. The desired result follows. ■

Theorem 6. *Given a concurrent game structure G, with a tail objective Φ for player 1. Let $\sigma_\varepsilon\in\Sigma_\varepsilon^\ell(\Phi)$ be a locally ε-optimal strategy, and ε-optimal from W_1^1 (i.e., for all $s\in W_1^1$ and for all strategies π we have $\Pr_s^{\sigma_\varepsilon,\pi}(\Phi)\geq 1-\varepsilon$). If for all strategies π for player 2 we have $\Pr_s^{\sigma_\varepsilon,\pi}(\overline{\Phi}\cap\mathrm{Safe}(W_1^{>0}\cap W_2^{>0}))\leq\varepsilon$, then σ_ε is an 3ε-optimal strategy.*

Proof. Let $\eta_1=\max_{s\in W_2^{>0}}\langle\langle 1\rangle\rangle_{val}(\Phi)(s)$. Without loss of generality we assume that the states in W_2^1 are converted to absorbing states and player 2 wins if the play reaches W_2^1. Consider an arbitrary strategy π for player 2, and consider a state $s\in W_1^{>0}\cap W_2^{>0}$. Let $\alpha=\langle\langle 1\rangle\rangle_{val}(\Phi)(s)$. By local ε-optimality of σ_ε, for all $k\geq 1$, we have $\alpha-\varepsilon\leq\mathbb{E}_s^{\sigma_\varepsilon,\pi}[\langle\langle 1\rangle\rangle_{val}(\Phi)(\Theta_k)]$. Since for all $s\in S$ we have $\langle\langle 1\rangle\rangle_{val}(\Phi)(s)\leq 1$, we have $\mathbb{E}_s^{\sigma_\varepsilon,\pi}[\langle\langle 1\rangle\rangle_{val}(\Phi)(\Theta_k)]\leq\Pr_s^{\sigma_\varepsilon,\pi}(\Theta_k\in W_1^{>0})$. Hence we obtain the following inequality:

$$\begin{aligned}\alpha-\varepsilon\leq\mathbb{E}_s^{\sigma_\varepsilon,\pi}[\langle\langle 1\rangle\rangle_{val}(\Phi)(\Theta_k)]&\leq\Pr_s^{\sigma_\varepsilon,\pi}(\Theta_k\in W_1^{>0})\\&\leq\Pr_s^{\sigma_\varepsilon,\pi}(\mathrm{Safe}(W_1^{>0}))=1-\Pr_s^{\sigma_\varepsilon,\pi}(\mathrm{Reach}(W_2^1)).\end{aligned}$$

Hence we have $\Pr_s^{\sigma_\varepsilon,\pi}(\mathrm{Reach}(W_2^1))\leq 1-\alpha+\varepsilon$. Thus we obtain that $\Pr_s^{\sigma_\varepsilon,\pi}(\overline{\Phi}\cap\mathrm{Reach}(W_2^1))\leq 1-\alpha+\varepsilon$. Since σ_ε is ε-optimal from W_1^1, we have $\Pr_s^{\sigma_\varepsilon,\pi}(\overline{\Phi}\cap\mathrm{Reach}(W_1^1))\leq\varepsilon$. The above inequalities and along with the assumption of the lemma yield the following inequality:

$$\begin{aligned}\Pr_s^{\sigma_\varepsilon,\pi}(\overline{\Phi})&=\Pr_s^{\sigma_\varepsilon,\pi}(\overline{\Phi}\cap\mathrm{Safe}(W_1^{>0}\cap W_2^{>0}))+\Pr_s^{\sigma_\varepsilon,\pi}(\overline{\Phi}\cap\mathrm{Reach}(W_2^1))+\Pr_s^{\sigma_\varepsilon,\pi}(\overline{\Phi}\cap\mathrm{Reach}(W_1^1))\\&\leq\varepsilon+1-\alpha+\varepsilon+\varepsilon\leq 1-\alpha+3\varepsilon.\end{aligned}$$

Thus $\Pr_s^{\sigma_\varepsilon,\pi}(\Phi)\geq\alpha-3\varepsilon$. Since the above inequality holds for all π we obtain that σ_ε is an 3ε-optimal strategy. ■

Lemma 6 shows that the ε-optimal strategies for player 1 are limit-sure winning against objective $\overline{\Phi}\cap\mathrm{Safe}(W_1^{>0}\cap W_2^{>0})$, for Muller objectives Φ. Theorem 6 shows that if a strategy is ε-limit-sure winning for player 1 against objective $\overline{\Phi}\cap\mathrm{Safe}(W_1^{>0}\cap W_2^{>0})$ for player 2, then local ε-optimality guarantees 3ε-optimality. This characterizes ε-optimal strategies as local ε-optimal and ε-limit-sure winning strategies. We believe our results would be useful in the quantitative analysis of concurrent games with Muller objectives. Results similar to Lemma 6 and Theorem 6 along with the qualitative analysis for concurrent parity games lead to improved complexity results for quantitative analysis of concurrent parity games. Similarly the qualitative analysis of concurrent Muller games along with our results could lead to improve complexity results for quantitative analysis of concurrent Muller games.

6. Conclusion

In this work we studied concurrent games with tail objectives. We proved the positive limit-one property and also related the values of zero-sum tail games with Nash equilibria of nonzero-sum reachability games. We also presented construction of ε-optimal strategies for Muller objectives. The computation of the sets W_1^1, $W_1^{>0}$ and the corresponding sets for player 2 for concurrent games and its subclasses for tail objectives remain open. The more general problem of computing the value functions also remain open. We believe that algorithms for computing W_1^1, $W_1^{>0}$ and the properties we prove in the paper could lead to algorithms for computing value functions. The exact characterization of tail objectives in the Borel hierarchy also remains open.

Acknowledgments

This research was supported in part by the ONR grant N00014-02-1-0671, the AFOSR MURI grant F49620-00-1-0327, and the NSF ITR grant CCR-0225610. I thank Tom Henzinger for his support, encouragement and several insightful discussions. I am grateful to Luca de Alfaro for several key insights on concurrent games that I received from him. I thank David Aldous for a useful discussion. I thank Hugo Gimbert for many useful comments on the earlier version of the paper.

References

[1] K. Chatterjee, L. de Alfaro, T.A. Henzinger, Trading memory for randomness, in: QEST'04, IEEE Computer Society Press, 2004, pp. 206–217.
[2] K. Chatterjee, L. de Alfaro, T.A. Henzinger, The complexity of quantitative concurrent parity games, in: SODA'06, SIAM, 2006.
[3] A. Condon, The complexity of stochastic games, Information and Computation 96 (2) (1992) 203–224.
[4] C. Courcoubetis, M. Yannakakis, The complexity of probabilistic verification, Journal of the ACM 42 (4) (1995) 857–907.
[5] L. de Alfaro, Formal verification of probabilistic systems, Ph.D. Thesis, Stanford University, 1997.
[6] L. de Alfaro, T.A. Henzinger, Concurrent omega-regular games, in: LICS 00, IEEE Computer Society Press, 2000, pp. 141–154.
[7] L. de Alfaro, R. Majumdar, Quantitative solution of omega-regular games, in: STOC 01, ACM Press, 2001, pp. 675–683.
[8] Richard Durrett, Probability: Theory and Examples, Duxbury Press, 1995.
[9] J. Filar, K. Vrieze, Competitive Markov Decision Processes, Springer-Verlag, 1997.
[10] J.F. Nash Jr., Equilibrium points in n-person games, Proceedings of the National Academy of Sciences USA 36 (1950) 48–49.
[11] A. Kechris, Classical Descriptive Set Theory, Springer, 1995.
[12] D.A. Martin, The determinacy of Blackwell games, The Journal of Symbolic Logic 63 (4) (1998) 1565–1581.
[13] G. Owen, Game Theory, Academic Press, 1995.
[14] L.S. Shapley, Stochastic games, Proceedings of the National Academy of Sciences USA 39 (1953) 1095–1100.
[15] W. Thomas, Languages, automata, and logic, in: Beyond Words, in: Handbook of Formal Languages, vol. 3, Springer, 1997, pp. 389–455 (Chapter 7).
[16] J. von Neumann, O. Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, 1947.
[17] W.W. Wadge, Reducibility and determinateness of Baire spaces, Ph.D. Thesis, UC Berkeley, 1984.