Probabilistic Stable Functions on Discrete Cones are Power Series.

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Abstract

We study the category $\mathbf{Cstab_m}$ of measurable cones and measurable stable functions—a denotational model of an higher-order language with *continuous* probabilities and full recursion [7]. We look at $\mathbf{Cstab_m}$ as a model for *discrete* probabilities, by showing the existence of a cartesian closed, full and faithful functor which embeds probabilistic coherence spaces—a *fully abstract* denotational model of an higher language with full recursion and *discrete* probabilities [8]—into $\mathbf{Cstab_m}$. The proof is based on a generalization of Bernstein's theorem from real analysis allowing to see stable functions between discrete cones as generalized power series.

CCS Concepts • **Theory of computation → Probabilistic computation**; **Denotational semantics**;

Keywords Lambda calculus, Probabilistic computation, Denotational semantics

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1 Introduction

Probabilistic reasoning allows us to describe the behavior of systems with inherent uncertainty, or on which we have an incomplete knowledge. To handle statistical models, we can employ probabilistic programming languages: they give us tools to build, evaluate and transform those models. While it is sometimes enough to consider *discrete* probabilities, we may also want to model systems where the underlying space of events has inherent *continuous* aspects: for instance in hybrid control systems [1], as used e.g. in flight management. In the machine learning community [10, 12], statistical models are also used to express our *beliefs* about the world, that we may then update using *Bayesian inference*—the ability to condition values of variables via observations.

As a consequence, several probabilistic *continuous* languages have been introduced and studied, such as Church [11], Anglican [19], as well as formal operational semantics for them [2]. Giving a *fully abstract denotational* semantics to a *higher-order* probabilistic language with full recursion, however, has proved to be harder than in the non-probabilistic case. For discrete probabilities, there are two known such fully abstract models: in [5], Danos and Harmer introduced a fully abstract denotational semantics of a probabilistic extension of idealized Algol, based on game semantics; and in [4] Ehrhard, Pagani and Tasson showed that the category **Pcoh** of *probabilistic coherence spaces* gives a fully abstract model for $\mathsf{PCF}_\oplus$, a discrete probabilistic variant of Plotkin's PCF.

While we currently don't know any fully abstract denotational semantics for a higher-order language with full recursion and continuous probabilities, several denotational models have been introduced. The pioneering work of Kozen [13] gave a denotational semantics to a *first-order* while-language endowed with a random real number generator. In [18], Staton et al give a denotational semantics to an higher-order language: they first develop a distributive category based on measurable spaces as a model of the first-order fragment of their language, and then extend it into a cartesian closed category using a standard construction based on the functor category. Recently, Ehrhard, Pagani and Tasson introduced in [7] the category $\mathbf{Cstab_m}$, as a denotational model of an extension of PCF with continuous probabilities. It is presented as a refinement with *measurability constraints* of the category **Cstab** of abstract cones and so-called *stable* functions between cones, consisting in a generalization of *absolutely monotonic functions* from real analysis.

Here, we look at the category $\mathbf{Cstab_m}$ from the point of view of *discrete* probabilities. It was noted in [7] that there is a natural way to see any probabilistic coherent space as an object of **Cstab**. In this work, we show that this connection leads to a full and faithful functor $\mathcal{F}$ from $\mathbf{Pcoh_!}$—the Kleisli category of **Pcoh**—into **Cstab**. We do that by showing that every stable function between probabilistic coherent spaces can be seen as a power series, using McMillan's extension [15] to an abstract setting of Bernstein's theorem for absolutely monotonic functions. We then show that this functor $\mathcal{F}$ is cartesian closed, i.e. respects the cartesian closed structure of $\mathbf{Pcoh_!}$. In the last part, we turn $\mathcal{F}$ into a functor $\mathcal{F}^m$: $\mathbf{Pcoh_!} \to \mathbf{Cstab_m}$, and we show that $\mathcal{F}^m$ too is cartesian closed. To sum up, the contribution of this paper is the construction of a cartesian closed full embedding from $\mathbf{Pcoh_!}$ into $\mathbf{Cstab_m}$. Since $\mathbf{Pcoh_!}$ is known to be a fully abstract denotational model of $\mathsf{PCF}_\oplus$, an immediate corollary is that $\mathbf{Cstab_m}$ too is a fully abstract model of $\mathsf{PCF}_\oplus$.

An extended version of this paper with more details is available online [3].

2 Discrete and Continuous Probabilistic Extension of PCF: an Overview.

A simple way to add probabilities to a (higher-order) programming language is to add a fair probabilistic choice operator to the syntax.

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Such an approach has been applied to various extensions of the λ-calculus [14]. To fix ideas, we give here the syntax of a (minimal) probabilistic variant of Plotkin's PCF [16], that we will call $\mathsf{PCF}_\oplus$. It is a typed language, whose types are given by: $A ::= N \mid A \to A$, where N is the base type representing naturals numbers. The programs are generated by the following grammar:

$$M,N \in \mathsf{PCF}_\oplus ::= x \mid \lambda x^A \cdot M \mid (MN) \mid (YN) \mid \mathrm{let}(x, M, N)$$
$$\mid \mathrm{ifz}\ (M, N, L) \mid \underline{n} \mid \mathrm{succ}\ (M) \mid \mathrm{pred}\ (M) \mid M \oplus N.$$

The operator $\oplus$ is the fair probabilistic choice operator, Y is a recursion operator, and n ranges over natural numbers. The ifz construct tests if its first argument (of type N) is 0, reduces to its second argument if it is the case, and to its third otherwise. We endow this language with a natural operational semantics [8], that we choose to be call-by-name. However, for expressiveness we need to be able to simulate a call-by-value discipline on terms of ground type N: it is enabled by the let-construct.

We can see that the kind of probabilistic behavior captured by $\mathsf{PCF}_\oplus$ is *discrete*, in the sense that it manipulates distributions on countable sets. In [4], Ehrhard and Danos introduced a model of Linear Logic designed to model discrete higher-order probabilistic computations: the category **Pcoh** of *probabilistic coherence spaces (PCSs)*. It has been shown in [8] that $\mathbf{Pcoh}_!$, the Kleisli category of **Pcoh** is a *fully abstract* model of $\mathsf{PCF}_\oplus$, while the Eilenberg-Moore category of **Pcoh** is a *fully abstract* model of a probabilistic variant of Levy's call-by-push-value calculus.

We give here some examples to illustrate the denotational semantics of $\mathsf{PCF}_\oplus$ in $\mathbf{Pcoh}_!$. Basically, the denotation of a program consists of a vector on $\mathbb{R}_+^X$, where X is the countable set of possible outcomes. For instance, the denotation of the program $\underline{0} \oplus \underline{1}$ of type N is the vector $x \in \mathbb{R}_+^{\mathbb{N}}$, defined by $x_0 = \frac{1}{2}$, $x_1 = \frac{1}{2}$, and $x_k = 0$ for $k \notin \{0, 1\}$. Morphisms in $\mathbf{Pcoh}_!$, on the other hand, can be seen as analytic functions (i.e. power series) between real vector spaces. Let us look at the denotation of the simple $\mathsf{PCF}_\oplus$ program below:

$$M := \lambda x^N \cdot \left(\underline{0} \oplus \mathrm{ifz}(x, \underline{1}, \mathrm{ifz}(x, \underline{0}, \Omega))\right),$$

where Ω is the usual encoding of a never terminating term using the recursion operator. The denotation of M consists of the following function $\mathbb{R}_+^{\mathbb{N}} \to \mathbb{R}_+^{\mathbb{N}}$:

$$f(x)_k = \begin{cases} \frac{1}{2} + \frac{1}{2}\sum_{i \neq 0} x_i \cdot x_0 & \text{if } k = 0; \\ \frac{1}{2} x_0 & \text{if } k = 1; \\ 0 & \text{if } k \notin \{0, 1\}. \end{cases}$$

We can see that $f(x)_k$ coincides with the probability of obtaining $\underline{k}$ if we pass to M a term N with x as denotation. A monomial of the form $\alpha \cdot x_{i_1} \dots x_{i_n}$ expresses that the following event has probability α to occur: M extracts n samples from N, and the result of these (potentially distinct) samplings, seen as a multiset, is $[i_1, \dots, i_n]$. Observe that f here is a polynomial in x; however since we have recursion in our language, there are programs that do an unbounded number of calls to their arguments: then their denotations are not polynomials anymore, but they are still analytic functions. It has to be noted that the analytic nature of $\mathbf{Pcoh}_!$ morphisms plays a key role in the proof of full abstraction for $\mathsf{PCF}_\oplus$.

Observe that this way of building a model for $\mathsf{PCF}_\oplus$ is utterly dependent on the fact that we consider discrete probabilities over a countable sets of values. In recent years, however, there has been much focus on continuous probabilities in higher-order languages, with the aim of handling classical mathematical distributions on reals, as for instance normal or Gaussian distributions, that are widely used to build generic physical or statistical models, as well as expressing transformations over these distributions. An example of such language is $\mathsf{PCF}_{\mathtt{sample}}$, defined by Ehrhard, Pagani and Tasson in [7], that can be seen as a continuous counterpart to $\mathsf{PCF}_\oplus$. The base type of $\mathsf{PCF}_{\mathtt{sample}}$ is the real type R, and types are generated by: $A ::= R \mid A \to A$. The programs are generated by the grammar below:

$$M, N \in \mathsf{PCF}_{\mathrm{sample}} ::= x \mid \lambda x^A \cdot M \mid (MN) \mid (YN)$$
$$\mid \mathrm{let}(x, M, N) \mid \mathrm{ifz}\ (M, N, L) \mid \underline{r} \mid \mathtt{sample} \mid \underline{f}(M_1, \dots, M_n),$$

where r is any real number, and f is in a fixed countable set of measurable functions $\mathbb{R}^n \to \mathbb{R}$. The constant `sample` stands for the uniform distribution over $[0, 1]$. Observe that since we can choose arbitrarily the countable set of measurable functions we take as primitives, we can encode classical probability distributions as soon as they can be obtained in a measurable way from the uniform distribution, as verified for instance by Gaussian or normal distributions. This language is actually expressive enough to simulate other probabilistic features, as for instance Bayesian conditioning, as highlighted in [7]. Moreover, we can argue it is also *more general* than $\mathsf{PCF}_\oplus$: first it allows to encode integers (since $\mathbb{N} \subseteq \mathbb{R}$) and basic arithmetic operations over them. Secondly, since the orders operator $\geq: \mathbb{R} \times \mathbb{R} \to \{0, 1\} \subseteq \mathbb{R}$ is measurable, we can construct in $\mathsf{PCF}_{\mathrm{sample}}$ terms like this one:

$$\mathrm{ifz}(\underline{\geq}(\mathtt{sample}, \frac{1}{2}), M, N), \tag{1}$$

which encodes a fair choice between M and N.

We see, however, that $\mathbf{Pcoh}_!$ cannot be a model for $\mathsf{PCF}_{\mathtt{sample}}$: indeed it would only allow to represent *discrete* distributions over reals, but not continuous ones. In [7], Ehrhard, Pagani and Tasson introduced the cartesian closed category $\mathbf{Cstab_m}$ of measurable cones and measurable stables functions, and showed that it provides an adequate and sound denotational model for $\mathsf{PCF}_{\mathtt{sample}}$. The denotation of the base type R is taken as the set of finite measures over reals, and the denotation of higher-order types is then built naturally using the cartesian closed structure. From there, it is natural to ask ourselves: how *good* $\mathbf{Cstab_m}$ is as a model of probabilistic higher-order languages ?

The present paper is devoted to give a partial answer to this question: in the case where we restrict ourselves to a *discrete fragment* of $\mathsf{PCF}_{\mathtt{sample}}$. To make more precise what we mean, let us consider a continuous language with an *explicit* discrete fragment which has both R and N as base types: we consider the language $\mathsf{PCF}_{\oplus,\mathtt{sample}}$ with all syntactic constructs of both $\mathsf{PCF}_\oplus$ and $\mathsf{PCF}_{\mathtt{sample}}$, as well as an operator `real` with the typing rule:

$$\frac{\Gamma \vdash M : N}{\Gamma \vdash \mathtt{real}(M) : R}$$

designed to enable the continuous constructs to act on the discrete fragment, by giving a way to see any distribution on $\mathbb{N}$ as a distribution on $\mathbb{R}$. The language $\mathsf{PCF}_{\oplus,\mathtt{sample}}$ is designed to talk about *approximating by discrete tests* the programs in $\mathsf{PCF}_{\mathtt{sample}}$: for instance if we consider a $\mathsf{PCF}_{\mathtt{sample}}$ program $M : R \to R$, we can approximate its behavior by considering all programs $\lambda x \cdot KM(Lx) : N \to N$, where L and K are $\mathsf{PCF}_{\oplus,\mathtt{sample}}$ programs of type respectively $N \to R$, and $R \to N$. Observe that the program K may be built using for instance the order operators as in (1), while the existence of such programs L is guarantee by our `real(·)` construct. We can extend in a natural way the denotational semantics of $\mathsf{PCF}_{\mathtt{sample}}$ given in [7] to $\mathsf{PCF}_{\oplus,\mathtt{sample}}$: in the same way that the denotational semantics of R is taken as the set of all finite measures on $\mathbb{R}$, we take the denotational semantics of N as the set $\mathrm{Meas}(\mathbb{N})$ of all finite measures over $\mathbb{N}$. We take as denotational semantics of the operator `real` the function: $[\![\mathtt{real}]\!]_{\mathbf{Cstab_m}} : \mu \in \mathrm{Meas}(\mathbb{N}) \mapsto (A \in \Sigma_{\mathbb{R}} \mapsto \sum_{n \in \mathbb{N} \cap A} \mu(n)) \in \mathrm{Meas}(\mathbb{R})$. We will see later that this function is indeed a morphism in $\mathbf{Cstab_m}(\mathrm{Meas}(\mathbb{N}), \mathrm{Meas}(\mathbb{R}))$. What we would like to know is: what is the structure of the sub-category of $\mathbf{Cstab_m}$ given by the *discrete types* of $\mathsf{PCF}_{\oplus,\mathtt{sample}}$, i.e generated inductively by $[\![N]\!]_{\mathbf{Cstab_m}}, \Rightarrow, \times$?

The starting point of our work is the connection highlighted in [7] between PCSs and complete cones: every PCSs can be seen as a complete cone, in such a way that the denotational semantics of N in $\mathbf{Pcoh}_!$ becomes the set of finite measures over $\mathbb{N}$. We formalize this connection by a functor $F^m : \mathbf{Pcoh}_! \to \mathbf{Cstab_m}$. However, to be able to use $\mathbf{Pcoh}_!$ to obtain information about the discrete types sub-category of $\mathbf{Cstab_m}$, we need to know whether this connection is preserved at higher-order types: does the $\Rightarrow$ construct in $\mathbf{Cstab_m}$ make some wild functions not representable in $\mathbf{Pcoh}_!$ to appear, e.g. not analytic? In this paper, we show that this is not the case, meaning that the functor F^m is full and faithful, and cartesian closed. Since $\mathbf{Pcoh}_!$ is a fully abstract model of $\mathsf{PCF}_{\oplus}$, it means that the discrete fragment of $\mathsf{PCF}_{\oplus,\mathtt{sample}}$ is fully abstract in $\mathbf{Cstab_m}$. More generally, it tells us also that for any $\mathsf{PCF}_{\mathtt{sample}}$ program M, regardless of the continuous computations it does, every approximation of M by discrete tests has for denotation in $\mathbf{Cstab_m}$ a power series: hence we are able to gather information on the behavior of M by looking only at power series, that are much easier to handle that general functions between cones.

3 Cones and Stable Functions

The category of measurable cones and measurable, stable functions ($\mathbf{Cstab_m}$), was introduced by Ehrhard, Pagani, Tasson in [7] as a model for $\mathsf{PCF}_{\mathtt{sample}}$. They actually introduced it as a refinement of the category of complete cones and stable functions, denoted $\mathbf{Cstab}$. Stable functions on cones are a generalization of well-known *absolutely monotonic functions* in real analysis: they are those functions $f : [0, \infty) \to \mathbb{R}_+$ which are infinitely differentiable, and such that moreover all their derivatives are non-negative. The relevance of such functions comes from a result due to Bernstein: every absolutely monotonic function coincides with a power series. Moreover, it is possible to characterize absolutely monotonic functions without explicitly asking for them to be differentiable: it is exactly those functions such that all the so-called *higher-order differences*, which are quantities defined only by sum and subtraction of terms of the form $f(x)$, are non-negative. (see [20], chapter 4). The definition of *pre-stable functions* in [7] generalizes this characterization.

In this section, we first recall basic facts about cones and stable functions, all extracted from [7]. Then we will prove a generalization of Bernstein's theorem for pre-stable functions over a particular class of cones, which is the main technical contribution of this paper. We will do that following the work of McMillan on a generalization of Bernstein's theorem for functions ranging over abstract domains endowed with partition systems, see [15].

3.1 Cones

The use of a notion of cones in denotational semantics to deal with probabilistic behavior goes back to Kozen in [13]. We take here the same definition of cone as in [7].

Definition 3.1. A cone C is a $\mathbb{R}_+$-semimodule given together with an $\mathbb{R}_+$ valued function $\|\cdot\|_C$ called *norm of* C, and verifying:

$$(x + y = x + y') \Rightarrow y = y' \qquad \|\alpha x\|_C = \alpha \|x\|_C$$
$$\|x + x'\|_C \le \|x\|_C + \|x'\|_C \qquad \|x\|_C = 0 \Rightarrow x = 0$$
$$\|x\|_C \le \|x + x'\|_C$$

The most immediate example of cone is the non-negative real half-line, when we take as norm the identity. Another example is the positive quadrant in a 2-dimensional plan, endowed with the euclidian norm. In a way, the notion of cones generalizes spaces where all elements are *non-negative*. This analogy gives us a generic way to define a pre-order, using the + of the cone structure.

Definition 3.2. Let be C a cone. Then we define a partial order $\le_C$ on C by: $x \le_C y$ if there exists $z \in C$, with $y = x + z$.

We define $\mathscr{B}C$ as the set of elements in C of norm smaller or equal to 1. We will sometimes call it the *unit ball* of C. Moreover, we will also be interested in the *open unit ball* $\mathscr{B}^\circ C$, defined as the set of elements of C of norm smaller than 1.

In [7], the authors restrict themselves to cones verifying a completeness criterion: it allows them to define the denotation of the recursion operator in $\mathsf{PCF}_{\mathsf{sample}}$, thus enforcing the existence of fixpoints.

Definition 3.3. A cone C is said to be:

- *sequentially complete* if any non-decreasing sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $\mathscr{B}C$ has a least upper bound $\sup_{n \in \mathbb{N}} x_n \in \mathscr{B}C$;
- *directed complete* if for any directed subset D of $\mathscr{B}C$, D has a least upper bound $\sup D \in \mathscr{B}C$;
- *a lattice cone* if any two elements x, y of C have a least upper bound $x \vee y$.

Observe that a directed-complete cone is always sequentially complete. We illustrate Definition 3.3 by giving the cone used in [7] as the denotational semantics of the base type R in $\mathsf{PCF}_{\mathsf{sample}}$.

Example 3.4. We take $\mathrm{Meas}(\mathbb{R})$ as the set of *finite* measures over $\mathbb{R}$, and the norm as $\|\mu\|_{\mathrm{Meas}(\mathbb{R})} = \mu(\mathbb{R})$. $\mathrm{Meas}(\mathbb{R})$ is a directed-complete cone. For every $r \in \mathbb{R}$, the denotational semantics of the term $\underline{r}$ in [7] is δ_r, the *Dirac measure with respect to* r defined by taking $\delta_r(U) = 1$ if $r \in U$, and $\delta_r(U) = 0$ otherwise.

In a similar way, we define $\mathrm{Meas}(X)$ as the directed-complete cone of finite measures over X, for any measurable space X.

In [7], the authors ask for the cones they consider only to be sequentially complete. It is due to the fact they want to add measurability requirements to their cones, and as a rule, sequential completeness interacts better with measurability than directed completeness since measurable sets are closed under *countable* unions, but not *general unions.* In this work however, we are only interested in cones arising from *probabilistic coherence spaces* in a way we will develop in Section 4. Since those cones have an underlying *discrete* structure, we will be able to show that they are actually *directed complete.* We will need this information, since we will apply McMillan's results [15] obtained in the more general framework of abstract domains with partitions, in which he asks for directed completeness. That's because directed completness for cones also enforces the existence of *infinum*, as stated in the lemma below, whose proof can be found in the long version.

Lemma 3.5. *If a cone C is:*

- sequentially *complete, then every non-increasing sequence $(x_n)_{n\in\mathbb{N}}$ has a greatest lower bound* $\inf(x_n)_{n\in\mathbb{N}}$*;*
- directed *complete, then for every $D \subseteq C$ directed for the reverse order, D has a greatest lower bound* $\inf D$.

3.2 Pre-Stable Functions between Cones

As said before, the notion of pre-stable function is a generalization of the notion of *absolutely monotonic* real functions.

First, we want to be able to talk about those $\vec{u} = (u_1, \ldots, u_n)$, such that $\|x + \sum u_i\|_C \leq 1$ for a fixed $x \in \mathscr{B}C$, and $n \in \mathbb{N}$. To that end, we introduce a cone C^n_x whose unit ball is exactly such elements. It is an adaptation of the definition given in [7] for the case where $n = 1$, and we show in the same way that it is indeed a cone.

Definition 3.6 (Local Cone)**.** Let be C a cone, $n \in \mathbb{N}$, and $x \in \mathscr{B}^\circ C$. We call *$n$-local cone at x*, and we denote C^n_x the cone C^n endowed with the following norm:

$$\|(u_1, \ldots, u_n)\|_{C^n_x} = \inf\{\frac{1}{r} \mid x + r \cdot \sum_{1\leq i\leq n} u_i \in \mathscr{B}C \wedge r > 0\}.$$

We can show that whenever C is a directed-complete cone, C^n_x is also directed-complete. We first give an inductive definition of pre-stability, following [7].

Definition 3.7. A function $f : \mathscr{B}C \to D$ is n-pre-stable when:

- f is non-decreasing;
- if $n > 0$, $\forall x \in \mathscr{B}C$, $y \in \mathscr{B}C^1_x \mapsto f(x + y) - f(x)$ is $(n - 1)$ pre-stable.

f is *pre-stable* if it is n-pre-stable for every $n \in \mathbb{N}$.

In [7], the authors also give an alternative, handier to use, characterization of pre-stable functions, which is the one that we will use in the following. The idea, similarly to what is done in real analysis, is to define so-called *higher-order differences*, and force them to be non-negative. For $n \in \mathbb{N}$, we use $\mathcal{P}_+(n)$ (respectively $\mathcal{P}_-(n)$) for the set of all subsets I of $\{1, \ldots, n\}$ such that $n - \mathrm{card}(I)$ is even (respectively odd). Since we have only explicit addition, not subtraction, we define separately the positive part Δ^n_+ and the negative part Δ^n_- of those differences: For $f : \mathscr{B}C \to D$, $x \in \mathscr{B}C$, $\vec{u} \in \mathscr{B}C^n_x$, and $\epsilon \in \{-, +\}$, we define:

$$\Delta^n_\epsilon(f)(x \mid \vec{u}) = \sum_{I\in\mathcal{P}_\epsilon(n)} f(x + \sum_{i\in I} u_i).$$

Definition 3.8. We say that $f : \mathscr{B}C \to D$ is *pre-stable* if, for every $n \in \mathbb{N}$, for every $x \in \mathscr{B}C$, $\vec{u} \in \mathscr{B}C^n_x$, it holds that:

$$\Delta^n_-(f)(x \mid \vec{u}) \leq \Delta^n_+(f)(x \mid \vec{u}).$$

If f is pre-stable, we will set $\Delta^n f(x \mid \vec{u}) = \Delta^n_+ f(x \mid \vec{u}) - \Delta^n_- f(x \mid \vec{u})$. Observe that the quantity $\Delta^n f(x \mid \vec{u})$ is actually symmetric in $\vec{u}$, i.e. stable under permutations of the coordinates of $\vec{u}$.

Definition 3.9. A function $f : \mathscr{B}C \to D$ is called a *stable function from C to D* if it is pre-stable, sequentially Scott-continuous, and moreover there exists $\lambda \in \mathbb{R}_+$ such that $f(\mathscr{B}C) \subseteq \lambda \cdot \mathscr{B}D$.

Definition 3.10. **Cstab** is the category whose objects are sequentially complete cones, and morphisms from C to D are the stable functions f from C to D such that $f(\mathscr{B}C) \subseteq \mathscr{B}D$.

In [7], the authors endow **Cstab** with a cartesian closed structure. The product cone is defined as $\prod_{i\in I} C_i = \{(x_i)_{i\in I} \mid \forall i \in I, x_i \in C_i\}$, and $\|x\|_{\prod_{i\in I} C_i} = \sup_{i\in I} \|x_i\|_{C_i}$. The function cone $C \Rightarrow D$ is the set of all stable functions, with $\|f\|_{C\Rightarrow D} = \sup_{x\in\mathscr{B}C} \|f(x)\|_D$. It was also shown in [7] that these cones are indeed sequentially complete, and that the least upper bound in $C \Rightarrow D$ is computed pointwise. We will use also the cone of pre-stable functions from C to D, which is also sequentially complete.

3.3 A generalization of Bernstein's theorem for pre-stable functions

In [15], McMillan generalized Bernstein's Theorem on absolutely monotonic function from real analysis to general domains with partitions systems. Here, we present its result in the more restricted setting of pre-stable functions on directed-complete lattice cones. Its approach consists in first defining an analogue of derivatives for pre-stable functions, and then showing that pre-stable functions can be written as the infinite sum generated by an analogue of Taylor expansion on $\mathscr{B}^\circ C$. We give here the main steps of the construction directly on cones, and highlight some properties of the Taylor series which are true for cones, but not in the general framework McMillan considered.

3.3.1 Derivatives of a pre-stable function

We are now going, following McMillan [15], to construct derivatives for pre-stable functions on directed complete cones. This construction is based on the use of a notion of *partition*: a *partition* of $x \in \mathscr{B}C$ is a multiset $\pi = [u_1, \ldots, u_n] \in M_f(C)$ such that $x = \sum_{1\leq i\leq n} u_i$. We write $\pi \sim x$ when the multiset π is a partition of x. We will denote by + the usual union on multiset: $[y_1, \ldots, y_n] + [z_1, \ldots, z_m] = [y_1, \ldots, y_n, z_1, \ldots, z_m]$. We call $\mathcal{P}(x)$ the set of partitions of x.

Definition 3.11 (Refinement Preorder)**.** If π_1, π_2 are in $\mathcal{P}(x)$, we says that $\pi_1 \leq \pi_2$ if $\pi_1 = [u_1, \ldots, u_n]$, and $\pi_2 = \alpha_1 + \ldots + \alpha_n$ with each of the α_i a partition of u_i.

Observe that when π_1 and π_2 are partition of x, $\pi_2 \leq \pi_1$ means that π_1 is a more *finely grained* decomposition of x. If $\vec{u}$ is an n-tuple in $\mathscr{B}C$, we extend the refinement order to $\mathcal{P}(\vec{u}) = \mathcal{P}(u_1) \times \ldots \times \mathcal{P}(u_n)$.

Lemma 3.12. *Let C be a lattice cone, and $x \in C$. Then $\mathcal{P}(x)$ is a directed set.*

The proof of Lemma 3.12 may be found in the long version. Observe that, as a consequence, the refinement preorder turns also $\mathcal{P}(\vec{u})$ into a directed set.

Definition 3.13 (from [15])**.** Let C be a lattice cone, D a cone, and let $f : \mathscr{B}C \to D$ be a pre-stable function. Then for every $x \in \mathscr{B}C$, and $\vec{u} = (u_1, \ldots, u_n) \in \mathscr{B}C^n_x$, we define $\Phi^{f,n}_{x,\vec{u}} : \mathcal{P}(\vec{u}) \to D$ as:

$$\Phi^{f,n}_{x,\vec{u}}(\pi_1, \ldots \pi_n) = \sum_{y_1\in\pi_1} \ldots \sum_{y_n\in\pi_n} \Delta^n f(x \mid y_1, \ldots, y_n).$$

It holds (see [15] for more details) that $\Phi^{f,n}_{x,\vec{u}}$ is a non-increasing function whenever f is pre-stable. Since $\mathcal{P}(\vec{u})$ is a directed set (by

Lemma 3.12), $\Phi^{f,n}_{x,\vec{u}}$ has a greatest lower bound whenever D is a directed-complete cone.

Definition 3.14 (from [15])**.** Let be C a lattice cone, D a directed-complete lattice cone, and $f : \mathscr{B}C \to D$ a pre-stable function. Let be $\vec{u} \in \mathscr{B}C^n_x$. Then the *derivative of f in x at rank n towards the direction $\vec{u}$* is the function $\mathbf{D}^n f(x \mid \cdot) : \mathscr{B}C^n_x \to D$ defined as

$$\mathbf{D}^n f(x \mid \vec{u}) = \inf_{\vec{\pi} \in \mathcal{P}(\vec{u})} \Phi^f_{x,\vec{u}}(\vec{\pi}).$$

In order to highlight the link with differentiation in real analysis, we illustrate Definition 3.14 on the basic case where $f : \mathbb{R}_+ \to \mathbb{R}_+$,

Example 3.15. We take C and D as the positive real half-line, and $x \in [0, 1)$. Let be h such that $x + h \leq 1$. Then:

$$\mathbf{D}^1 f(x \mid h) = \inf_{\pi \text{ with } \pi \sim h} \sum_{y \in \pi} f(x + y) - f(x).$$

Since f is pre-stable hence an absolutely monotonic function on reals, f is convex, and moreover differentiable (see [20]). From there, by considering a particular family of partitions, we can show that $\mathbf{D}^1 f(x \mid h) = h \cdot f'(x)$ (the proof can be found in the long version).

Lemma 3.16 (from [15])**.** *Let C be a lattice cone, D a directed complete cone, f a pre-stable function from C to D. Let be $x \in \mathscr{B}^\circ C$. Then $\mathbf{D}^n f(x \mid \cdot)$ is a symmetric function $\mathscr{B}(C^n_x) \to D$ such that moreover:*

- $0 \leq \mathbf{D}^n f(x \mid \vec{u}) \leq \Delta^n f(x \mid \vec{u})$;
- *both $\vec{u} \mapsto \mathbf{D}^n f(x \mid \vec{u})$ and $\vec{u} \mapsto \Delta^n f(x \mid \vec{u}) - \mathbf{D}^n f(x \mid \vec{u})$ are pre-stable functions from C^n_x to D.*

We have seen in Example 3.15 that our so-called derivatives of pre-stable functions play the same role as the differential of a differentiable function, which are actually *linear* operators $df^n_x : \mathbb{R}^n \to \mathbb{R}$. While the abstract domains considered in [15] do not have to be $\mathbb{R}_+$ semi-modules, so have no notion of linearity, we are able to show in our case (see the long version for the proof) that the $\mathbf{D}^n f$ are linear in the sense of Lemma 3.17 below.

Lemma 3.17. *Let C, D be two directed complete lattice cones, $x \in \mathscr{B}^\circ C$.*

- *Let $f : \mathscr{B}C \to D$ be a pre-stable function. Then $\mathbf{D}^n f(x \mid \cdot) : \mathscr{B}(C^n_x) \to D$ is n-linear, in the sense that:*

$$\mathbf{D}^n f(x \mid u_1, \ldots, \lambda \cdot v + w, \ldots u_n) = \lambda \cdot \mathbf{D}^n f(x \mid u_1, \ldots, v, \ldots u_n) + \mathbf{D}^n f(x \mid u_1, \ldots, w, \ldots u_n).$$

- *For any $\vec{u} \in \mathscr{B}(C^n_x)$, the function $f \in \mathbf{Cstab}(C, D) \mapsto \mathbf{D}^n f(x \mid \vec{u}) \in D$ is linear and directed Scott-continuous.*

The proof of Lemma 3.17 can be found in the long version. The linearity of the derivatives means that for every $x \in \mathscr{B}^\circ C$, we can extend $\mathbf{D}^n f(x \mid \cdot)$ to a function $C^x_n \to D$. We will use implicitly this extension in the following, especially in Definition 3.18.

3.3.2 Taylor Series for pre-stable functions

We have seen above that the $\mathbf{D}^n f$ are a notion of differential for pre-stable functions. Following further this idea, McMillan defined an analogue to the Taylor expansion. We give here a slightly different, but equivalent, formulation, taking advantage of the linearity of the derivatives for cones (the original formulation, as well as the proof of equivalence, are detailed in the long version). In all this section C and D are going to be directed complete lattice cones, and $f : \mathscr{B}C \to D$ a pre-stable function.

Definition 3.18. Let be $x \in \mathscr{B}^\circ C$. We call *Taylor partial sum of f in x at the rank N* the function $Tf^N(x \mid \cdot) : \mathscr{B}C^1_x \to D$ defined as:

$$Tf^N(x \mid y) = f(x) + \sum_{k=1}^{N} \frac{1}{k!} \mathbf{D}^k f(x \mid y, \ldots, y).$$

The next step consists in establishing that the $T^N f$ are actually a non-increasing bounded sequence in the cone of pre-stable functions from C to D, which will allow to define the *Taylor series of f*, as the supremum of the $T^N f$ (see the long version for more details on the proof).

Lemma 3.19. *Let be y is in $\mathscr{B}^\circ C$, and x in $\mathscr{B}C^1_y$. Then $\forall N \in \mathbb{N}$, $Tf^N(x \mid y) \leq f(x + y)$, and the function $(x \in \mathscr{B}C^1_y \mapsto f(x + y) - Tf^N(x \mid y))$ is pre-stable.*

Since we have shown that the partial sum of the Taylor series of f was a bounded non-decreasing sequence in the sequentially complete cone of pre-stable functions from C^1_x to D, we can now define the *Taylor series of f* as its supremum.

Definition 3.20. We define $Tf(x \mid \cdot) : \mathscr{B}C^1_x \to D$ *the Taylor series of f in x* as: $Tf(x \mid y) = \sup_{N \in \mathbb{N}} Tf^N(x \mid y)$.

3.3.3 Extended Bernstein's theorem

McMillan's main result is to show that f coincides with its Taylor series in 0 on the open unit ball of the cone. The proofs are done in [15], and a sketch can be found in the long version.

Proposition 3.21 (Extended Bernstein's Theorem)**.** *Let be C, D directed-complete lattice cones, and $f : \mathscr{B}C \to D$ a pre-stable function. Then for every $x \in \mathscr{B}^\circ C$, it holds that $f(x) = Tf(0 \mid x)$.*

4 Cstab is a conservative extension of $\mathbf{Pcoh}_!$

Probabilistic coherence spaces (PCS) were introduced by Ehrhard and Danos in [4] as a model of higher-order probabilistic computation. It was successful in giving a fully abstract model both of $\mathsf{PCF}_\oplus$, and of a discrete probabilistic extension of Levy's Call-by-Push-Value. In this section, we present briefly basic definitions from [4] and highlight an embedding from PCSs into cones.

4.1 Probabilistic Coherence Spaces

The definition of the PCS model of Linear Logic follows the tradition initiated by Girard with Coherence Spaces in [9], and followed for instance by Ehrhard in [6] when defining hypercoherence spaces. A coherent space interpreting a type can be seen as a symmetric graph, and the interpretation of a program of this type is a *clique* of this graph. Interestingly, such a graph A can be alternatively characterized by giving its set of vertices (that we will call *web*), and a family of subsets of this web, meant to be the family of the cliques of A. Then we know that an arbitrary family of subsets of a given web arises indeed as a family of cliques for some graph when some *duality criterion* is verified.

PCSs are designed to express probabilistic behavior of programs. As a consequence, a clique is not a subset of the web anymore, but a *quantitative* way to associate a non-negative real coefficient to every element in the web.

Definition 4.1 (Pre-Probabilistic Coherent Spaces)**.** A Pre-PCS is a pair $X = (|X|, PX)$, where $|X|$ is a countable set called *web of X*, PX is a subset of $\subseteq \mathbb{R}_+^{|X|}$ whose elements are called *cliques of X*.

We need here to introduce some notations to deal with infinite dimensional $\mathbb{R}$-vector spaces. Given a countable web A, and a an element of A, we denote e_a the vector in $\mathbb{R}_+^A$ which is 1 in a, and 0 elsewhere. We are also going to introduce a scalar product on vectors in $\mathbb{R}_+^A$: if $u, v \in \mathbb{R}_+^A$, we will denote $\langle u, v\rangle = \sum_{a\in|X|} u_a v_a \in \mathbb{R} \cup \{\infty\}$. Moreover, if A and B are countable sets, $x \in \mathbb{R}_+^{A\times B}$, and $u \in \mathbb{R}_+^A$, we denote by $x \cdot u$ the vector in $(\mathbb{R}_+ \cup \{\infty\})^B$ given by $(x \cdot u)_b = \sum_{a\in A} x_{a,b} u_a$ for every $b \in B$.

We are going to give examples of pre-PCS modeling discrete data-types. First, we define a pre-PCS $\mathbf{1}$ to correspond to unit type. Since unit-type programs have only one possible outcome (that they can reach or not), $\mathbf{1}$ has only one vertex: $|\mathbf{1}| = \{\star\}$. We want the denotation of a unit-type program to express its probability of termination: consequently, we take the set of cliques $P\mathbf{1}$ as the interval $[0, 1]$. Let us now look at what happens when we consider programs of type N: a program can now have a countable numbers of possible outcomes, so the web will consist of $\mathbb{N}$, and cliques will be sub-distributions on these vertices.

Example 4.2 (Pre-PCS of Natural Numbers)**.** We define the Pre-PCS $\mathbb{N}^{\mathbf{Pcoh}}$ by taking $|\mathbb{N}|^{\mathbf{Pcoh}} = \mathbb{N}$, and $P\mathbb{N}^{\mathbf{Pcoh}} = \{u \in \mathbb{R}_+^{\mathbb{N}} \mid \sum_{n\in\mathbb{N}} u_n \leq 1\}$. It corresponds to the denotational semantics of the base type N of $\mathrm{PCF}_\oplus$ in $\mathbf{Pcoh}_!$.

We now need to give a *quantitative* bi-duality criterion, to specify which one of the $PX \subseteq \mathbb{R}_+^{|X|}$ are indeed *valid* families of cliques. To do that, we first define a *duality operator*: if $X = (|X|, PX)$ is a pre-PCS, we define the pre-PCS $(X)^\perp = (|X|, \{u \in \mathbb{R}_+^{|X|} \mid \forall v \in PX, \langle u, v\rangle \leq 1\})$.

Definition 4.3 (Probabilistic Coherent Spaces)**.** A pre-PCS X is a *PCS* if $((X)^\perp)^\perp = X$ and moreover the following technical conditions hold:

- $\forall a \in |X|, \exists \lambda > 0$ such that $\lambda e_a \in PX$;
- $\forall a \in |X|, \exists M \geq 0$, such that $\forall u \in PX, u_a \leq M$.

We may see easily that both $\mathbf{1}$ and $\mathbb{N}^{\mathbf{Pcoh}}$ are indeed PCSs.

As highlighted in Example 4.4 from [7], we can associate in a generic way a cone to any PCS: we consider the extension of the space of cliques by all uniform scaling by positive reals. We formalize this idea in Definition 4.4 below.

Definition 4.4. Let be X a PCS. We define a cone C_X as the $\mathbb{R}_+$ semi-module $\{\alpha \cdot x \mid \alpha \geq 0, x \in PX\}$ where $+$ is the usual addition on vectors. We endow it with $\|\cdot\|_{C_X}$ defined by:

$$\|x\|_{C_X} = \sup_{y\in P(X)^\perp} \langle x, y\rangle = \inf\{\frac{1}{r} \mid r \cdot x \in PX\}.$$

It is easily seen that it is indeed a cone (the proof uses the so-called technical conditions from Definition 4.3); we will call *discrete cones* all cones obtained from some PCS. We can see that $\mathscr{B}C_X$ consists exactly of the set PX of cliques of X. Looking at the cone order $\preceq_{C_X}$, as defined in Definition 3.2, we see that it coincides on PX with the pointwise order in $\mathbb{R}_+^{|X|}$. It is relevant since we know already from [4] that PX is a bounded-complete and ω-continuous cpo with respect to this pointwise order.

Lemma 4.5. *For every PCS X, C_X is a directed-complete lattice cone.*

4.2 The Category Pcoh.

Intuitively a morphism in $\mathbf{Pcoh}(X, Y)$ is a linear map from $\mathbb{R}_+^{|X|}$ to $\mathbb{R}_+^{|Y|}$ *preserving* the cliques.

Definition 4.6 (Morphisms of PCSs)**.** Let X, Y be two PCSs. A *morphism of PCSs between X and Y* is a matrix $x \in \mathbb{R}_+^{|X|\times|Y|}$ such that for every $u \in PX$, it holds that $x \cdot u \in PY$.

We now illustrate Definition 4.6 by looking at the morphisms from **Bool** to itself: they are the $x \in \mathbb{R}_+^{\{\mathbf{t},\mathbf{f}\}\times\{\mathbf{t},\mathbf{f}\}}$ with $x_{\mathbf{t},\mathbf{t}} + x_{\mathbf{t},\mathbf{f}} \leq 1$, and similarly $x_{\mathbf{f},\mathbf{t}} + x_{\mathbf{f},\mathbf{f}} \leq 1$, i.e. those matrices specifying the transitions for a probabilistic Markov chain with two states $\mathbf{t}$ and $\mathbf{f}$.

We call **Pcoh** the category of PCSs and morphisms of PCSs. In [4], it is endowed with the structure of a model of linear logic. We are only going to recall here partly the exponential structure, since our main focus will be on the Kleisli category associated to **Pcoh**. The construction of the exponential structure is done by defining a functor !, as well as dereliction and digging making **Pcoh** a Seely category, and consequently a model of linear logic. Here, we are only going to recall explicitly the effect of ! on PCSs. We denote by $M_f(|X|)$ the set of finite multisets over the web of X, and we take it as the web of the PCS $!X$. If $\mu \in M_f(A)$, we call *support of* μ, and we denote $\mathrm{Supp}(\mu)$, the set of elements a is A such that a appears in μ. Moreover, we will use the following notation: for every $x \in \mathbb{R}_+^{|X|}$, and $\mu \in M_f(|X|)$, we denote $x^\mu = \prod_{a\in\mathrm{Supp}(\mu)} x_a^{\mu(a)} \in \mathbb{R}_+$.

Definition 4.7. Let be X a PCS. We define the *promotion* of $x \in PX$, as the element $x^! \in \mathbb{R}_+^{M_f(|X|)}$ given by $x^!_\mu = x^\mu$. We define $!X = (M_f(|X|), \{x^! \mid x \in X\}^{\perp\perp})$.

4.3 The Kleisli Category of Probabilistic Coherence Spaces

Morphisms in the Kleisli category represent programs that can use several times their argument, while morphisms in the original category are *linear*. The Kleisli category for **Pcoh**, denoted $\mathbf{Pcoh}_!$, has also PCSs for objects, while $\mathbf{Pcoh}_!(X, Y) = \mathbf{Pcoh}(!X, Y)$. We give here a direct characterization of $\mathbf{Pcoh}_!$ morphisms.

Lemma 4.8 (from [4])**.** *Let be $f \in \mathbb{R}_+^{M_f(|X|)\times|Y|}$. Then f is a morphism in $\mathbf{Pcoh}_!(X, Y)$, if and only if for every $x \in PX$, $f \cdot x^! \in PY$.*

What Lemma 4.8 tells us is that any $f \in \mathbf{Pcoh}_!(X, Y)$ is entirely characterized by the map $\widetilde{f} : x \in PX \to f \cdot x^! \in PY$. We call $\widetilde{f}$ the *functional interpretation of the morphism* f, and denote by $\mathcal{E}^{X,Y}$ the set of all functional interpretations of morphisms in $\mathbf{Pcoh}_!(X, Y)$. It has been shown in [4] that $\widetilde{(\cdot)}$ is actually a bijection from $\mathbf{Pcoh}_!(X, Y)$ to $\mathcal{E}^{X,Y}$; we will denote by $\widetilde{(\cdot)}^{-1}$ its inverse. Observe that we can see the maps in $\mathcal{E}^{X,Y}$ as *entire series*, in the sense that they can be written as the supremum of a sequence of polynomials. Indeed, for any morphism f, and $x \in PX$, we can write:

$$\widetilde{f}(x) = \sup_{N\in\mathbb{N}} \sum_{b\in|Y|} \Big(\sum_{\mu \text{ with } \mathrm{card}(\mu)\leq N} f_{\mu,b} \cdot x^\mu \Big) \cdot e_b$$

As the Kleisli category of the comonad ! in a Seely category, $\mathbf{Pcoh}_!$ is a cartesian closed category. We give here explicitly the construction of the product and arrow constructs: if X and Y are PCSs, $X \Rightarrow Y$ is defined by $|X \Rightarrow Y| = M_f(|X|) \times |Y|$ and $P(X \Rightarrow Y) = \mathbf{Pcoh}_!(X, Y)$. If $(X_i)_{i\in I}$ is a family of PCSs, $\prod_{i\in I} X_i$ is defined by $|\prod_{i\in I} X_i| = \cup_{i\in I}\{i\} \times |X_i|$ and $PX = \{x \in \mathbb{R}_+^{|\prod_{i\in I} X_i|} \mid \forall i \in I,\ \pi_i(x) \in PX_i\}$, where $\pi_i(x)_a = x_{(i,a)}$.

4.4 A fully faithful functor $\mathcal{F} : \mathbf{Pcoh}_! \to \mathbf{Cstab}$.

Recall that Definition 4.4 gave a way to transform a PCS into a cone. Moreover, as stated in Proposition 4.9 below, a morphism in $\mathbf{Pcoh}_!$ can also be transformed into a stable function:

Proposition 4.9. *For every $f \in \mathbf{Pcoh}_!(X, Y)$, $\widetilde{f}$ is a stable function from C_X to C_Y.*

Proof. We know from [4] that $\widetilde{f} : PX \to PY$ is sequentially Scott-continuous with respect to the orders $\leq_{C_X}$, $\leq_{C_Y}$. Moreover $\widetilde{f}$ is pre-stable: it comes from the fact that $\widetilde{f}$ can be written as a power series with all its coefficients non-negative. Finally, we have to show that $\widetilde{f}(\mathscr{B}C_X) \subseteq \mathscr{B}C_Y$. Since $\mathscr{B}C_X = PX$, $\mathscr{B}C_Y = PY$, and moreover $f \in \mathbf{Pcoh}_!(X, Y)$, the result holds by Lemma 4.8. □

Thus we can define a functor $\mathcal{F} : \mathbf{Pcoh}_! \to \mathbf{Cstab}$, by taking $\mathcal{F}X = C_X$, and $\mathcal{F}f = \widetilde{f}$. As mentioned before, it was shown in [4] that $\widetilde{\cdot}$ is an injection from $\mathbf{Pcoh}_!(X, Y)$, hence $\mathcal{F}$ is a *faithful* functor. In the remainder of this section, we are going to show that $\mathcal{F}$ is actually also *full*, hence makes $\mathbf{Cstab}$ a conservative extension of $\mathbf{Pcoh}_!$.

In the following, we fix X and Y two PCSs, and a stable function $g \in \mathbf{Cstab}(\mathcal{F}X, \mathcal{F}Y)$. Our goal is to show that there exists a morphism $f \in \mathbf{Pcoh}_!(X, Y)$, with g as functional interpretation. First, recall that we have shown in Lemma 4.5 that for every PCS Z, the cone $\mathcal{F}Z$ is a directed complete lattice cone. It means that all results in Section 3.3 can be used here: in particular, g has higher-order derivatives $D^n g$, which makes Definition 4.10 below valid.

Definition 4.10. We define $f \in \mathbb{R}_+^{M_f(|X|)\times|Y|}$ by taking:

$$f_{[a_1,\dots,a_k],b} = \frac{\alpha_{[a_1,\dots,a_k]}}{k!}\left(\mathbf{D}^k g(0 \mid e_{a_1}, \dots e_{a_k})\right)_b \in \mathbb{R}^+,$$

where $\alpha_\mu = \#\{(c_1, \dots, c_k) \in |X|^k \text{ with } \mu = [c_1, \dots, c_k]\}$.

We want to show now that $f \in \mathbf{Pcoh}_!(X, Y)$, and that f has g as functional interpretation. The key observation here is that f has been built in such a way that its functional interpretation coincides with $Tg(0 \mid \cdot)$—the Taylor series of g defined in Definition 3.20—on all elements in PX with *finite support*.

Lemma 4.11. *Let be $x \in PX$, such that $Supp(x) = \{a \in PX \mid x_a > 0\}$ is finite. Then it holds that $f \cdot x^!$ is finite (i.e for every $b \in |Y|$, $(f \cdot x^!)_b < \infty$), and moreover $f \cdot x^! = Tg(0 \mid x)$.*

Proof. Let $A = \{a_1, \dots, a_m\} \subseteq |X|$ be the set $\mathrm{Supp}(x)$. For any $b \in |Y|$, we can deduce from the definition of f that:

$$(f \cdot x^!)_b = \sum_{k=0}^{\infty} \sum_{\mu=[c_1,\dots,c_k]\in M_f^k(A)} \frac{\alpha_\mu}{k!} \cdot \mathbf{D}^k g(0 \mid e_{c_1}, \dots e_{c_k})_b \cdot x^\mu,$$

where $M_f^k(A)$ stands for the set of multisets over A of cardinality k. Looking at the definition of α_μ, we see that this implies:

$$(f \cdot x^!)_b = \sum_{k=0}^{\infty} \sum_{(c_1,\dots,c_k)\in A^k} \frac{1}{k!}\mathbf{D}^k g(0 \mid e_{c_1}, \dots e_{c_k})_b \cdot \prod_{i=1}^{k} x_{c_i}. \quad (2)$$

By Lemma 3.17, we know that $\mathbf{D}^k g(0 \mid \cdot)$ is k-linear. As a consequence, and since $x = \sum_{i=1}^m x_{c_i} \cdot e_{c_i}$ and that moreover A is *finite*, we see that (2) implies the result:

$$(f \cdot x^!)_b = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{D}^k g(0 \mid x, \dots x)_b = (Tg(0 \mid x))_b.$$

□

Since g is a stable function between directed complete lattice cones, we can apply the generalized Bernstein's Theorem as stated in Proposition 3.21.

Lemma 4.12. *$\forall x \in \mathscr{B}^\circ C_X$ such that $Supp(()x)$ is finite, $f \cdot x^! = g(x)$.*

Proof. Proposition 3.21 tells us: $\forall x \in \mathscr{B}^\circ C_X$, $g(x) = Tg(0 \mid x)$. We conclude by using Lemma 4.11. □

Using Lemma 4.12, we show now that $\widetilde{f}$ and g coincide on PX.

Lemma 4.13. *$\forall x \in PX$, $f \cdot x^! = g(x)$, and moreover $f \in \mathbf{Pcoh}_!(X, Y)$.*

Proof. Let be $x \in PX$. We define a sequence $(y_n)_n \in \mathbb{N}$ of elements in PX, by taking: $(y_n)_a = \begin{cases} (1 - \frac{1}{2^n}) \cdot x_a \text{ if } \lambda(a) \leq n \\ 0 \text{ otherwise,} \end{cases}$, where we have fixed λ an arbitrary enumeration of the elements of $|X|$—λ exists since $|X|$ is a countable set. Observe that the sequence $(y_n)_{n\in\mathbb{N}}$ is non-decreasing, with $x = \sup_{n\in\mathbb{N}} y_n$. Since g is a morphism in $\mathbf{Cstab}$, g is sequentially Scott-continuous, hence $g(x) = \sup_{n\in\mathbb{N}} g(y_n)$. Moreover, for every n, y_n has finite support and $\|y_n\|_{C_X} < 1$. So by using Lemma 4.12: we see that $g(y_n) = f \cdot y_n^!$. As a consequence, $g(x) = \sup_{n\in\mathbb{N}} f \cdot y_n^!$. Since moreover, we know from [4] that both $x \mapsto x^!$ and $x \mapsto u \cdot x$ are Scott continuous, $\sup_{n\in\mathbb{N}} f \cdot y_n^! = f \cdot x^!$, and finally we obtain $f \cdot x^! = g(x)$. Since $g(\mathscr{B}C_X) \subseteq \mathscr{B}C_Y$, it implies also that $\widetilde{f}(PX) \subseteq PY$. Thus by Lemma 4.8 $f \in \mathbf{Pcoh}_!(X, Y)$. □

Lemma 4.13 shows that for any morphism g in $\mathbf{Cstab}(\mathcal{F}X, \mathcal{F}Y)$, there exists an $f \in \mathbf{Pcoh}_!(X, Y)$ such that $\mathcal{F}f = g$, thus it shows that $\mathcal{F}$ is full.

4.5 $\mathcal{F}$ preserves the cartesian structure.

We want now to show that the functor $\mathcal{F}$ is *cartesian closed*, meaning that it embeds the cartesian closed category $\mathbf{Pcoh}_!$ into the cartesian closed category $\mathbf{Cstab}$ in such a way that:

- $\mathcal{F}$ preserves the product: for every family $(X_i)_{i\in I}$ of PCSs, $\mathcal{F}(\prod_{i\in I}^{\mathbf{Pcoh}_!} X_i)$ is isomorphic to $\prod_{i\in I}^{\mathbf{Cstab_m}} \mathcal{F}X_i$;
- $\mathcal{F}$ preserves function spaces: for every X, Y PCSs, $\mathcal{F}(X \Rightarrow Y)$ is isomorphic to $\mathcal{F}X \Rightarrow \mathcal{F}Y$.

Lemma 4.14. *$\mathcal{F}$ preserves cartesian products.*

Proof. We fix a family $\mathscr{I} = (X_i)_{i\in I}$ of PCSs. In order to construct an isomorphism, we have a canonical candidate, given by:

$$\Psi^{\mathscr{I}} = \langle \mathcal{F}(\pi_i) \mid i \in I \rangle \in \mathbf{Cstab}(\mathcal{F}(\prod_{i\in I}^{\mathbf{Pcoh}_!} X_i), \prod_{i\in I}^{\mathbf{Cstab_m}} \mathcal{F}X_i), \quad (3)$$

where $\langle\cdot\rangle$ is the cartesian product on morphisms in $\mathbf{Cstab}$. We show that $\Psi^{\mathscr{I}}$ is an isomorphism, i.e. that it has an inverse. The only candidate is $\Theta^{\mathscr{I}} : y \in \mathscr{B}(\prod_{i\in I}^{\mathbf{Cstab}} \mathcal{F}X_i) \mapsto \Theta(y) \in (\mathcal{F}(\prod_{i\in I}^{\mathbf{Pcoh}_!} X_i))$, defined by: $\forall i \in I, a \in |X_i|, \Theta(y)_{i,a} = (y_i)_a$. We see immediately that $\Theta^{\mathscr{I}}$ is linear, hence pre-stable, and that moreover it is Scott-continuous. Besides, it is also preserves the unit ball, since $\forall y \in \mathscr{B}D$, $\|\Theta^{\mathscr{I}}(y)\|_{\mathcal{F}(\prod_{i\in I}^{\mathbf{Pcoh}_!} X_i)} = \|y\|_{\prod_{i\in I}^{\mathbf{Cstab}} \mathcal{F}X_i}$ (the proof can be found in the long version). Thus $\Theta^{\mathscr{I}}$ is a morphism in $\mathbf{Cstab}$. □

Lemma 4.15. *$\mathcal{F}$ preserves function spaces.*

Proof. Let X, Y two PCSs. As previously, there is a canonical candidate for the isomorphism: we define $\Upsilon^{X,Y}$ as the currying in **Cstab** of the morphism:

$$\mathcal{F}(X \Rightarrow Y) \times \mathcal{F}X \xrightarrow{\Theta^{X\Rightarrow Y,X}} \mathcal{F}(X \Rightarrow Y \times X) \xrightarrow{\mathcal{F}(\mathrm{eval}_{X,Y})} \mathcal{F}Y,$$

where $\Theta^{X\Rightarrow Y,X}$ is as defined in the proof of Lemma 4.14 above. Unfolding the definition, we see that actually: $\Upsilon^{X,Y} : f \in \mathscr{B}\mathcal{F}(X \Rightarrow Y) \mapsto \widetilde{f} \in (\mathcal{F}X \Rightarrow \mathcal{F}Y)$. In section 4.4 we have shown that $\widetilde{(\cdot)}$ is a bijection from $\mathbf{Pcoh}_!(X, Y)$ into $\mathbf{Cstab}(C_X, C_Y)$: it means we can define $\Xi^{X,Y} = \widetilde{(\cdot)}^{-1}$ the inverse function of $\Upsilon^{X,Y}$. From there, we only have to show that $\Xi^{X,Y}$ is a morphism in $\mathbf{Cstab}(\mathcal{F}X \Rightarrow \mathcal{F}Y, \mathcal{F}X \Rightarrow Y)$. Recall from Section 4.4 that for every multiset $\mu = [a_1, \ldots, a_k] \in M_f(|X|)$, and $b \in |Y|$:

$$\Xi^{X,Y}(f)_{\mu,b} = \frac{\alpha_\mu}{k!} \cdot \left(\mathbf{D}^k f(0 \mid e_{a_1}, \ldots e_{a_k})\right)_b .$$

By Lemma 3.17, it holds that for any $\vec{u} \in \mathscr{B}(\mathcal{F}X)_0^k$, the function $f \in \mathbf{Cstab}(\mathcal{F}X, \mathcal{F}Y) \mapsto \mathbf{D}^k f(0 \mid \vec{u}) \in \mathcal{F}Y$ is linear and Scott-continuous. As a consequence, $\Xi^{X,Y}$ too is linear and Scott-continuous. Moreover, it also preserves the unit ball (see the proof in the long version), hence is in $\mathbf{Cstab}(\mathcal{F}X \Rightarrow \mathcal{F}Y, \mathcal{F}(X \Rightarrow Y))$. □

As a direct consequence of Lemma 4.14 and Lemma 4.15, we can state the following theorem:

Theorem 4.16. *$\mathcal{F}$ is full and faithful, and it respects the cartesian closed structures.*

5 Adding Measurability Requirements

In [7], the authors developed a sound model of $\mathsf{PCF}_{\mathsf{sample}}$ based on stable functions. However, as explained in more details in [7], they need to add to their morphisms some *measurability requirements*, both on cones and on functions between them, since the denotational semantics of the `let`(x, M, N) construct uses an integral, to model the fact that M is evaluated before being passed as argument to N.

We call Borel-measurable functions $\mathbb{R}^n \to \mathbb{R}^k$ those functions which are measurable when both $\mathbb{R}^n$ and $\mathbb{R}^k$ are endowed with the Borel Σ-algebra associated with the standard topology of $\mathbb{R}$. The relevant properties of the class of measurable functions $\mathbb{R}^n \to \mathbb{R}^k$ is that they are closed by arithmetic operations, composition, and pointwise limit, see for example Chapter 21 of [17].

5.1 The category $\mathbf{Cstab_m}$

$\mathbf{Cstab_m}$ is built as a *refinement* of the category **Cstab**. The objects of $\mathbf{Cstab_m}$ are going to be complete cones, endowed with a family of *measurability tests*.

If C is a complete cone, we denote by C' the set of linear and Scott-continuous functions $C \to \mathbb{R}_+$.

Definition 5.1. A *measurable cone* (MC) is a pair consisting of a cone C, and a collection of *measurability tests* $(\mathcal{M}^n(C))_{n\in\mathbb{N}}$, where for every n, $\mathcal{M}^n(C) \subseteq {C'}^{\mathbb{R}^n}$, such that:

- for every $n \in \mathbb{N}$, $0 \in \mathcal{M}^n(C)$;
- for every $n, p \in \mathbb{N}$, if $l \in \mathcal{M}^n(C)$, and $h : \mathbb{R}^p \to \mathbb{R}^n$ is Borel-measurable, then $l \circ h \in \mathcal{M}^p(C)$;
- for any $l \in \mathcal{M}^n(C)$, and $x \in C$, the function $u \in \mathbb{R}^n \mapsto l(u)(x) \in \mathbb{R}$ is Borel-measurable.

Example 5.2 (from [7])**.** Let X be a measurable space. We endow the cone of finite measures $\mathrm{Meas}(X)$ with the family $\mathcal{M}(X)$ of measurable tests defined as:

$$\mathcal{M}^n(X) = \{\epsilon_U \mid U \in \Sigma_X\} \quad \text{where} \quad \epsilon_U(\vec{r})(\mu) = \mu(U),$$

where Σ_X is the set of all measurable subsets of X. Observe that in this case, the measurable tests correspond to the measurable sets. In the following, we will denote $\overline{\mathrm{Meas}(X)}$ the measurable cone $(\mathrm{Meas}(X), (\mathcal{M}^n(X))_{n\in\mathbb{N}})$.

We define now *measurable paths*, which are meant to be the *admissible* ways to send $\mathbb{R}^n$ into a MC C.

Definition 5.3 (Measurable Paths)**.** Let $(C, (\mathcal{M}^n(C))_{n\in\mathbb{N}})$ be a MC. A *measurable path (MP) of arity n on C* is a function $\gamma : \mathbb{R}^n \to C$, such that $\gamma(\mathbb{R}^n)$ is bounded in C, and $\forall k \in \mathbb{N}$, $\forall l \in \mathcal{M}^k(C)$, the function $(\vec{r}, \vec{s}) \in \mathbb{R}^{k+n} \mapsto l(\vec{r})(\gamma(\vec{s})) \in \mathbb{R}_+$ is Borel-measurable.

We denote $\mathrm{Paths}^n(C)$ the set of MPs of arity n for the MC C. When a MP γ verifies $\gamma(\mathbb{R}^n) \subseteq \mathscr{B}C$, we say it is *unitary*. In [7], the authors add *measurability requirements* to their definition of stable functions: they ask them to preserve measurable paths.

Definition 5.4. Let be C, D two MCs. A stable function $f : \mathscr{B}C \to D$ is *measurable* if for all unitary $\gamma \in \mathrm{Paths}^n(C)$, $f \circ \gamma \in \mathrm{Paths}^n(D)$.

The category $\mathbf{Cstab_m}$ is therefore the category whose objects are MCs, and whose morphisms are measurable stable functions between MCs.

Example 5.5. Recall the function $[\![\mathtt{real}]\!]_{\mathbf{Cstab_m}} : \mathrm{Meas}(\mathbb{N}) \to \mathrm{Meas}(\mathbb{R})$, defined in Section 2 as:

$$[\![\mathtt{real}]\!]_{\mathbf{Cstab_m}} : \mu \mapsto (U \in \Sigma_{\mathbb{R}} \mapsto \sum_{n\in\mathbb{N}\cap U} \mu(n)).$$

$[\![\mathtt{real}]\!]_{\mathbf{Cstab_m}}$ is linear, Scott-continuous, and norm-preserving, and moreover it is a measurable function from $\overline{\mathrm{Meas}(\mathbb{N})}$ into $\overline{\mathrm{Meas}(\mathbb{R})}$, hence it is a morphism in $\mathbf{Cstab_m}$. In the same way, taking $\overline{\mathrm{Meas}(\mathbb{N})}$ as the denotational semantics of type N, we could complete the denotational semantics given in [7] for $\mathsf{PCF}_{\mathtt{sample}}$ in $\mathbf{Cstab_m}$ into a denotational semantics for $\mathsf{PCF}_{\oplus,\mathtt{sample}}$.

Observe that $[\![\mathtt{real}]\!]_{\mathbf{Cstab_m}}$ would not be measurable, if we endowed $\mathrm{Meas}(\mathbb{N})$ with for instance $\{0\}$ as measurability tests instead of $\mathcal{M}(\mathbb{N})$: indeed in that case, every $\gamma : \mathbb{R}^n \to \mathrm{Meas}(\mathbb{N})$ would be a MP. As a consequence, to be a measurable function, $[\![\mathtt{real}]\!]_{\mathbf{Cstab_m}}$ should verify: for every arbitrary function $\gamma : \mathbb{R}^n \to \mathrm{Meas}(\mathbb{N})$, $[\![\mathtt{real}]\!]_{\mathbf{Cstab_m}} \circ \gamma$ is a MP on $\overline{\mathrm{Meas}(\mathbb{R})}$. However, we can see this is not the case, for instance by considering γ of the form $\gamma(s) = \alpha(s) \cdot \{0\}^1$, where $\alpha : \mathbb{R} \to \mathbb{R}_+$ is not Borel-measurable.

In [7], the cartesian closed structure of $\mathbf{Cstab_m}$ is derived from the one of **Cstab** by endowing its exponentials and products with the measurability tests presented in Figure 1.

$$\mathcal{M}^n(\prod_{i\in I} \overline{C}_i) = \{\bigoplus_{i\in I} l_i \mid \forall i \in I, l_i \in \mathcal{M}^n(\overline{C}_i)\} \quad \text{with } I \text{ finite set.}$$

$$\mathcal{M}^n(\overline{C} \Rightarrow_m \overline{D}) = \{\gamma \rhd m \mid \gamma \in \mathrm{Paths}^n(\overline{C}), m \in \mathcal{M}^n(\overline{D})\},$$

with $(\bigoplus_{i\in I} l_i(\vec{r}))((x_i)_{i\in I}) = \sum_{i\in I} l_i(\vec{r})(x_i) \in \mathbb{R}_+$;
and $(\gamma \rhd m)(\vec{r})(f) = m(\vec{r})(f(\gamma(\vec{r})))$.

Figure 1. Cartesian Closed structure of $\mathbf{Cstab_m}$.

5.2 $\mathbf{Pcoh}_!$ is a full subcategory of $\mathbf{Cstab_m}$

We want now to convert the functor $\mathcal{F} : \mathbf{Pcoh}_! \to \mathbf{Cstab}$ into a functor $\mathcal{F}^m : \mathbf{Pcoh}_! \to \mathbf{Cstab_m}$. To build $\mathcal{F}^m$, we are going to endow each $\mathcal{F}X$ with measurability tests, in such a way that for any morphism $f \in \mathbf{Pcoh}_!$, $\mathcal{F}(f)$ will become *measurable*. Observe that we have an additional requirement, which would not be verified if we take for instance $\{0\}$ as measurability tests for every $\mathcal{F}X$: as explained in Section 2, we want also $\mathcal{F}^m(\mathbb{N}^{\mathbf{Pcoh}})$ to be isomorphic to $[\![N]\!]_{\mathbf{Cstab_m}} = \overline{\text{Meas}(\mathbb{N})}$, and (as highlighted in Example 5.5), we need to have enough measurable tests on $\overline{\text{Meas}(\mathbb{N})}$ to guarantee the existence of $[\![\texttt{real}]\!]_{\mathbf{Cstab_m}}$ as a morphism.

Definition 5.6. For any $X \in \mathbf{Pcoh}$, we define $\overline{C}_X$ as the measurable cone C_X endowed with the family $\mathcal{M}^n(X)_{n\in\mathbb{N}}$ of measurability tests defined as $\mathcal{M}^n(X) = \{0\} \cup \{\epsilon_a \mid a \in |X|\}$, where $\epsilon_a(\vec{r}, x) = x_a$.

We see that the ϵ_a are indeed linear (i.e commuting with linear combinations), and moreover Scott-continuous: hence they are indeed elements of C'_X. It is easy to verify that the other conditions are verified, and so $\overline{C}_X$ is indeed a MC. We give below a characterization of measurable paths on $\overline{C}_X$.

Lemma 5.7. *Let be X a PCS. Then Paths$^n(\overline{C}_X)$ is the set of all $\gamma : \mathbb{R}^n \to C_X$ such that:*

- $\exists \lambda \in \mathbb{R}, \gamma(\mathbb{R}^n) \subseteq \lambda \mathcal{B} C_X$;
- $\forall a \in |X|, \gamma_a : \vec{r} \in \mathbb{R}^n \mapsto \gamma(\vec{r})_a \in \mathbb{R}_+$ *is Borel-measurable.*

Two MCs with the same underlying cone, but different measurability tests may be isomorphic in **Cstab**: it is enough for them to have the same *measurable paths*. It is what happens in the example below, where we consider $\overline{C}_{\mathbb{N}^{\mathbf{Pcoh}}}$ and $\overline{\text{Meas}(\mathbb{N})}$. It is actually also what happens at higher-order types, as we will explain in Section 5.3.

Example 5.8. The two measurable cones $\overline{C}_{\mathbb{N}^{\mathbf{Pcoh}}}$ and $\overline{\text{Meas}(\mathbb{N})}$ have the same underlying cone, but they do not have the same measurability tests. Indeed $\mathcal{M}^n(\overline{C}_{\mathbb{N}^{\mathbf{Pcoh}}}) = \{\epsilon_n \mid n \in \mathbb{N}\}$, while $\mathcal{M}^n(\overline{\text{Meas}(\mathbb{N})}) = \{\epsilon_U \mid U \subseteq \mathbb{N}\}$. However, we can prove that they have the same MPs. First, $\text{Paths}^n(\overline{\text{Meas}(\mathbb{N})}) \subseteq \text{Paths}^n(\overline{C}_{\mathbb{N}^{\mathbf{Pcoh}}})$, since $\mathcal{M}^n(\overline{C}_{\mathbb{N}^{\mathbf{Pcoh}}})$ is a subset of $\mathcal{M}^n(\overline{\text{Meas}(\mathbb{N})})$. We detail now the proof of the reverse inclusion. Let $\gamma \in \text{Paths}^n(\overline{C}_{\mathbb{N}^{\mathbf{Pcoh}}})$. We have to show that for every $U \subseteq \mathbb{N}$, the function $(\vec{r}, \vec{s}) \in \mathbb{R}^{k+n} \mapsto \epsilon_U(\vec{r})(\gamma(\vec{s}))$ is Borel measurable. The key observation here is that $\epsilon_U(\vec{r})(\gamma(\vec{s}) = \sum_{m\in U} \epsilon_m(\vec{r})(\gamma(\vec{s}))$. Since $\gamma \in \text{Paths}^n(\overline{C}_{\mathbb{N}^{\mathbf{Pcoh}}})$ it holds that for every $m \in \mathbb{N}$, the function $((\vec{r}, \vec{s}) \in \mathbb{R}^{k+n} \mapsto \epsilon_m(\vec{r}, \gamma(\vec{s})) \in \mathbb{R}_+)$ is Borel measurable. Since the class of Borel measurable functions is closed by finite sum and pointwise limit, it leads to the result.

Lemma 5.9. *Let X, Y be two PCSs, and $f \in \mathbf{Pcoh}_!(X, Y)$. Then $\mathcal{F}f$ is measurable from $\overline{C}_X$ into $\overline{C}_Y$.*

The proof can be found in the long version. It uses the characterization of $\text{Paths}^n(\overline{C}_X)$ given in Lemma 5.7.

Theorem 5.10. *The functor $\mathcal{F}^m : \mathbf{Pcoh}_! \to \mathbf{Cstab_m}$ defined as $\mathcal{F}^m X = \overline{C}_X$, and $\mathcal{F}^m f = \mathcal{F} f$, is full and faithful.*

Proof. Observe that we can decompose the functor $\mathcal{F}$ as $\mathcal{F} = \texttt{Forget} \circ \mathcal{F}^m$, where `Forget` is the forgetful functor from $\mathbf{Cstab_m}$ to **Cstab**. As shown in Section 4.4, $\mathcal{F}$ is full and faithful. Moreover, `Forget` is faithful. From there, we are able to deduce the result (see the long version for more details). □

5.3 $\mathcal{F}^m$ is cartesian closed.

We want now to show that $\mathcal{F}^m$ is cartesian closed too. Since `Forget` is cartesian closed, and is the identity function on morphisms, it is enough to show that the **Cstab**-morphisms $\Psi^{\mathscr{I}}, \Theta^{\mathscr{I}}, \Upsilon^{X,Y}$ and $\Xi^{X,Y}$ defined in Lemmas 4.14 and Lemma 4.15 proofs, are also morphisms in $\mathbf{Cstab_m}$.

Lemma 5.11. *Let X be a PCS, $\overline{C}$ any MC, and f a morphism in $\mathbf{Cstab}(\texttt{Forget}(\overline{C}), \mathcal{F}X)$. We suppose that for every γ a unitary MP on $\overline{C}$, $\forall a \in |X|$ it holds that $(f \circ \gamma)_a$ is Borel-measurable. Then $f \in \mathbf{Cstab_m}(\overline{C}, \mathcal{F}^m X)$.*

Proof. We already know that f is a morphism in **Cstab**, hence we have only to show that it preserves MPs. Let γ be unitary in $\text{Paths}^n(\overline{C})$. Since both f and γ are bounded, $f \circ \gamma$ is bounded too. Using the characterization of MPs on $\mathcal{F}^m X$ given in Lemma 5.7, we obtain that $f \circ \gamma \in \text{Paths}^n(\mathcal{F}^m X)$. □

Lemma 5.12. *For all $\mathscr{I} = (X_i)_{i\in I}$ a finite family of PCSs:*

- $\Psi^{\mathscr{I}} \in \mathbf{Cstab_m}(\mathcal{F}^m(\prod^{\mathbf{Pcoh}_!}_{i\in I} X_i), \prod^{\mathbf{Cstab_m}}_{i\in I} \mathcal{F}^m X_i)$;
- $\Theta^{\mathscr{I}} \in \mathbf{Cstab_m}(\prod^{\mathbf{Cstab_m}}_{i\in I} \mathcal{F}^m X_i, \mathcal{F}^m(\prod^{\mathbf{Pcoh}_!}_{i\in I} X_i))$.

Proof.

- Recall that $\Psi^{\mathscr{I}} = \langle \mathcal{F}(\pi_i) \mid i \in I\rangle_{\mathbf{Cstab}}$. Since the cartesian product on morphisms in $\mathbf{Cstab_m}$ is the same as the one in **Cstab** (see [7]), and moreover $\mathcal{F}(\pi_i) = \mathcal{F}^m(\pi_i)$, we see that $\Psi^{\mathscr{I}}$ is also a morphism of $\mathbf{Cstab_m}$.
- Let γ be any unitary MP on $\prod^{\mathbf{Cstab_m}}_{i\in I} \mathcal{F}^m X_i$, and $(i, a_i) \in |\prod^{\mathbf{Pcoh}_!}_{i\in I} X_i| = \cup_{i\in\mathscr{I}} \{i\} \times |X_i|$. We see that $(\Theta^{\mathscr{I}} \circ \gamma)_{(i,a_i)}(\vec{r}) = (\gamma(\vec{r})_i)_a$. We consider the measurability test of arity 0 defined by $m = \oplus_{j\in I} l_j : \prod^{\mathbf{Cstab_m}}_{i\in I} \mathcal{F}^m X_i \to \mathbb{R}_+$, with $l_j = 0$ if $j \neq i$, and $l_i = \epsilon_{a_i}$. Since γ is a MP on $\prod^{\mathbf{Cstab_m}}_{i\in I} \mathcal{F}^m X_i$, and m a measurability test on the same cone, the function $(\vec{r} \in \mathbb{R}^n \mapsto m(\gamma(\vec{r})) \in \mathbb{R}_+)$ is Borel-measurable. Since $m(\gamma(\vec{r})) = (\gamma(\vec{r})_i)_a = (\Theta^{\mathscr{I}} \circ \gamma)_{(i,a_i)}(\vec{r})$, it holds that $(\Theta^{\mathscr{I}} \circ \gamma)_{(i,a_i)}$ is Borel-measurable, and by Lemma 5.11 the result holds.

□

Lemma 5.12 allows us to see that $\mathcal{F}^m$ is a cartesian functor. We want now to show that it also respects the $\Rightarrow$ construct. First, we show that the **Cstab** morphism $\Upsilon^{X,Y}$ is also a morphism in $\mathbf{Cstab_m}$.

Lemma 5.13. *For all X, Y PCSs,*

$$\Upsilon^{X,Y} \in \mathbf{Cstab_m}(\mathcal{F}^m(X \Rightarrow Y), \mathcal{F}^m X \Rightarrow \mathcal{F}^m Y)$$

Proof. Recall that $\Upsilon^{X,Y}$ is defined using currying in **Cstab**, $\Theta^{X\Rightarrow Y,X}$, and the eval morphism in **Cstab**. Since currying and structural morphisms are the same in $\mathbf{Cstab_m}$ as in **Cstab**, and moreover we have shown in Lemma 5.12 that $\Theta^{X\Rightarrow Y,X}$ is a morphism in **Cstab**, we have the result. □

Our goal now is to show that $\Xi^{X,Y}$ also is a $\mathbf{Cstab_m}$ morphism, again by using Lemma 5.11. However, the proof is going to be more involved as in the previous cases. By looking at the definition of $\Xi^{X,Y}$, we see that we need to show that for every unitary MP γ on $\mathcal{F}^m X \Rightarrow \mathcal{F}^m Y$, and any $(\mu, b) \in M_f(|X|) \times |Y|$, the function $\vec{r} \mapsto (\widetilde{\gamma(\vec{r})}^{-1})_{\mu,b}$ is Borel-measurable. But contrary to what we've done in the proof of Lemma 5.12 for $\Theta^{\mathscr{I}}$, the function $f \in (\mathcal{F}^m X \Rightarrow \mathcal{F}^m Y) \mapsto (\widetilde{f}^{-1})_{\mu,b} \in \mathbb{R}_+$ cannot be written as a measurability test. We solve this problem by showing that we can write it as some *partial derivative* of a measurability test, as expressed in Lemma 5.14 below.

Lemma 5.14. *Let X be a PCS, $\mu \in M_f(|X|)$, $b \in |Y|$, and $\{a_1, \dots, a_p\}$ the support of μ. There exists a measure-ability test m of arity p on $\mathcal{F}^m X \Rightarrow \mathcal{F}^m Y$, and $c, \beta > 0$ such that $\forall f \in \mathscr{B}(\mathcal{F}^m X \Rightarrow \mathcal{F}^m Y)$, the function $\phi^f : \vec{r} \in \mathbb{R}^p \mapsto m(\vec{r})(f) \in \mathbb{R}_+$ admits a partial derivative $\frac{\partial (\phi^f)^{card(\mu)}}{\partial {r_1}^{\mu(a_1)} \dots \partial {r_p}^{\mu(a_p)}}$ on $[0, c]^p$, and moreover its value in $\vec{0}$ is $\beta \cdot (\widetilde{f}^{-1})_{\mu,b}$.*

Proof. We take $m = \delta \rhd \epsilon_b$, where δ is the MP on $\mathcal{F} X$ defined as:

$$\delta : \vec{t} \in \mathbb{R}^p \mapsto \begin{cases} \sum_{1 \leq i \leq m} t_i \cdot e_{a_i} \text{ if } t_i \geq 0 \forall i \text{ and } \sum_{1 \leq i \leq m} t_i \leq 1; \\ 0 \text{ otherwise.} \end{cases}$$

ϵ_b is a measurability test on $\mathcal{F}^m Y$, and moreover by Lemma 5.7, we can see that δ is indeed in $\mathrm{Paths}^p(\mathcal{F}^m X)$, hence m is a measurability test on $\mathcal{F}^m X \Rightarrow \mathcal{F}^m Y$ built following the rules of Figure 1. We see that $\phi^f(\vec{r}) = \sum_{\nu \mid \mathrm{Supp}(\nu) \subseteq \{a_1, \dots, a_n\}} f_{\nu,b} \cdot \vec{r}^{\nu}$. From there, by using theorems of real analysis for normally convergent series of functions, we can deduce the result (the complete proof may be found in the long version). □

Lemma 5.15. *For every unitary $\gamma \in Paths^n(\mathcal{F}^m X \Rightarrow \mathcal{F}^m Y)$, and $(\mu, b) \in |X \Rightarrow Y|$, it holds that $(\Xi^{X,Y} \circ \gamma)_{\mu,b}$ is Borel measurable for every $(\mu, b) \in |X \Rightarrow Y|$.*

Proof. Let $\{a_1, \dots, a_p\}$ the support of μ. We take m, c as given by Lemma 5.14. Since m is a measurability test for $(\mathcal{F}^m X \Rightarrow \mathcal{F}^m Y)$, the function $(\vec{r}, \vec{s}) \in \mathbb{R}^{p+n} \mapsto \phi^{\gamma(\vec{s})}(\vec{r}) \in \mathbb{R}_+$ is Borel-measurable (where $\phi^{\gamma(\vec{s})}$ was defined in Lemma 5.14). Since $K = [0, c)^p \times \mathbb{R}^n$ is an element of the Borel algebra over $\mathbb{R}^{p+n}$, the restriction (that we denote ψ) of this function to K is measurable too. Since γ is unitary, $\gamma(\mathbb{R}^n) \subseteq \mathscr{B}(\mathcal{F}^m X \Rightarrow \mathcal{F}^m Y)$, and as a consequence Lemma 5.14 tells us that $\frac{\partial \psi^{\mathrm{card}(\mu)}}{\partial {r_1}^{\mu(a_1)} \dots \partial {r_p}^{\mu(a_p)}}$ exists, and moreover it holds that: $\widetilde{\gamma(\vec{s})}^{-1}_{\mu,b} = \frac{1}{\beta} \cdot \frac{\partial \psi^{\mathrm{card}(\mu)}}{\partial {r_1}^{\mu(a_1)} \dots \partial {r_p}^{\mu(a_p)}}(\vec{0}, \vec{s})$. From there, we can deduce that $\vec{s} \in \mathbb{R}^n \mapsto \widetilde{\gamma(\vec{s})}^{-1}_{\mu,b} \in \mathbb{R}_+$ is Borel-measurable. It comes from the fact that whenever a Borel-measurable function has a partial derivative, this derivative is measurable too, since the class of real-valued measurable functions is closed by addition, multiplication by a scalar and pointwise limit (more details can be found in the long version). Since $(\Xi^{X,Y} \circ \gamma)_{\mu,b} = \widetilde{\gamma(\vec{s})}^{-1}_{\mu,b}$, the result holds. □

Lemma 5.16. *For all X, Y PCSs,*

$$\Xi^{X,Y} \in \mathbf{Cstab_m}(\mathcal{F}^m X \Rightarrow \mathcal{F}^m Y, \mathcal{F}^m (X \Rightarrow Y)).$$

Theorem 5.17. *$\mathcal{F}^m$ is a cartesian closed full and faithful functor.*

6 Conclusion

Our full embedding of $\mathbf{Pcoh}_!$ into $\mathbf{Cstab}$ implies that every stable function f from PX to PY can be characterized by an element $\Xi^{X,Y}(f) \in \mathbb{R}^{M_f(|X|) \times |Y|}$, that has to be seen as a power series. It gives us a *concrete representation* of stable functions on discrete cones, similar to the notion of trace introduced by Girard in [9] for stable functions on quantitative domains. There are well-known real analysis results on power series, as for instance the *uniqueness theorem*—any power series which is null on an open subset has all its coefficients equal to 0—on which is based the proof of full abstraction for $\mathsf{PCF}_\oplus$ in $\mathbf{Pcoh}_!$ [8]. While we have not been able to extend such a concrete representation to cones which are not *directed-complete*, as for instance the cone $\mathrm{Meas}(\mathbb{R}) \Rightarrow_m \mathrm{Meas}(\mathbb{R})$, our result could hopefully be a first step in this direction. This kind of characterization could lead to a way towards a full abstraction result for the continuous language $\mathsf{PCF}_{\mathsf{sample}}$ in $\mathbf{Cstab_m}$, and more generally gives us new tools to reason about continuous probabilistic programs.

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