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On the growth of linear languages

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Abstract

It is well known that the growth of a context-free language is either polynomial or exponential. However no algorithm for such an alternative is known. In this article we determine such an algorithm for the subclass of unambiguous linear languages.

Keywords: Context-free grammar; Linear language; Growth function; Polynomial and exponential growth; Oriented graph

1. Introduction

Let L be a language over the alphabet $\mathbf{\Sigma}$, that is, a subset of the free monoid $\mathbf{\Sigma}^*$ of all finite words over $\mathbf{\Sigma}$. We write ε for the empty word and $\mathbf{\Sigma}^+ = \mathbf{\Sigma}^* \setminus \{\varepsilon\}$. For a word $w \in \mathbf{\Sigma}^*$, its *length* (number of letters) is denoted by $|w|$. The *growth function* of L is the function

$$\gamma_L(n) = \left|\left\{w \in L\colon\ |w| \leqslant n\right\}\right|$$

which counts the words of L of length at most n.

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Recall that given two monotone functions $f, g : \mathbb{N} \to \mathbb{N}$ one says that f is *dominated* by g, and one writes $f \preccurlyeq g$, if there exist $c \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that $f(n) \leqslant g(cn)$ for all $n \geqslant n_0$. If $f \preccurlyeq g$ and $g \preccurlyeq f$ one says that f and g have the same growth and one writes $f \sim g$. Note that $\sim$ is an equivalence relation: we denote by $[f]$ the equivalence class containing f.

The *growth type* of a language L is the equivalence class $[\gamma_L]$ of its growth function. If $[\gamma_L(n)] = [n^d]$ for some $d \in \mathbb{N}$ one says that L has *polynomial growth* (*of degree* d). If $[\gamma_L(n)] = [\exp(n)]$ one says that L has *exponential growth*. Note that $[n^d] \neq [n^{d'}]$ if $d \neq d'$ and that, on the other hand, $[\exp(n)] = [2^n] = [a^n]$ for all $a > 1$.

A *context-free grammar* is a quadruple $\mathcal{C} = (\mathbf{V}, \mathbf{\Sigma}, \mathbf{P}, S)$, where $\mathbf{V}$ is a finite set of *variables,* disjoint from the finite *alphabet* $\mathbf{\Sigma}$, the variable S is the *start symbol,* and $\mathbf{P} \subset \mathbf{V} \times (\mathbf{V} \cup \mathbf{\Sigma})^*$ is a finite set of *production rules*. We write $T \vdash u$ or $(T \vdash u) \in \mathbf{P}$ if $(T, u) \in \mathbf{P}$. For $v, w \in (\mathbf{V} \cup \mathbf{\Sigma})^*$, we write $v \Rightarrow w$ if $v = v_1 T v_2$ and $w = v_1 u v_2$, where $T \vdash u$, $v_1 \in (\mathbf{V} \cup \mathbf{\Sigma})^*$ and $v_2 \in \mathbf{\Sigma}^*$. A *rightmost derivation* is a sequence $v = w_0, w_1, \ldots, w_k = w \in (\mathbf{V} \cup \mathbf{\Sigma})^*$ such that $w_{i-1} \Rightarrow w_i$; we then write $v \overset{*}{\Rightarrow} w$. For $T \in \mathbf{V}$, we consider the language $L_T = \{w \in \mathbf{\Sigma}^*\colon T \overset{*}{\Rightarrow} w\}$. The *language generated by* $\mathcal{C}$ is $L(\mathcal{C}) = L_S$.

A *context-free language* is a language generated by a context-free grammar.

A grammar and the language generated by it are called *linear,* if every production rule in $\mathbf{P}$ is of the form $T \vdash v_1 U v_2$ or $T \vdash v$, where $v, v_1, v_2 \in \mathbf{\Sigma}^*$ and $T, U \in \mathbf{V}$. If furthermore in this situation one always has $v_2 = \varepsilon$ (the empty word), then grammar and language are called *right linear*. Analogously, they are called *left linear* if instead one always has $v_1 = \varepsilon$. In both cases, language and grammar are also called *regular*. (It is well known that left and right linear languages are the same, i.e., every left linear language is also generated by a right linear grammar and conversely. Moreover a fundamental theorem of Kleene states that the class of regular languages coincides with the class of *rational* languages, that is, the smallest class of languages containing the finite languages and which is closed under unions, products and $*$-operation ($L_1 * L_2$ is the monoid generated by L_1 and L_2); see, for instance, [7] or [12].)

A contex-free grammar $\mathcal{C}$ is called *unambiguous* if for all $w \in L(\mathcal{C})$ there exists a *unique* leftmost derivation $S \overset{*}{\Rightarrow} w$. A contex-free language is *unambiguous* if it is generated by an unambiguous contex-free grammar. Note that there are contex-free languages that cannot be generated by unambiguous grammars, these are called *inherently ambiguous* languages. In our setting, the language

$$L = \left\{a^n b^n c^m\colon n, m > 0\right\} \cup \left\{a^n b^m c^m\colon n, m > 0\right\}$$

is linear and inherently ambiguous (using Ogden's iteration lemma [16] (see also Chapter 6 in [8]) it can be deduced that one always has two different derivations for the words of the form $a^n b^n c^n$). Thus there exist inherently ambiguous linear languages.

For context-free languages, Trofimov [17] has shown that the growth is either polynomial or exponential. This has also been proved independently by Incitti [13] and by Bridson and Gilman [1]. The context-free languages with polynomial growth are characterized as the *bounded* languages, i.e., those that are contained in $w_1^* w_2^* \cdots w_k^*$ for finitely many words $w_1, \ldots, w_k$ over $\mathbf{\Sigma}$. Trofimov [17] also gave an example of a *context-sensitive* language that has *intermediate* growth. Independently, Grigorchuk and Machì [10] have considered the same example, showing that it is even an *indexed* language (a specific form of being context-sensitive: see for instance [6]).

In [13] it is asked whether there exists an algorithm which, given a context-free grammar, determines whether the growth of the corresponding language is polynomial or exponential.

In this note we present such an algorithm for unambiguous linear languages. The idea is to relate the growth of a language generated by a given linear grammar to the growth of an oriented graph associated with the grammar, similar to the *dependency di-graph* considered by Kuich [14] (and later by Ceccherini-Silberstein and Woess [4,5]) and then to apply Ufnarovskii's criterion for the growth of oriented graphs [18] which was originally used for determining the growth of affine algebras (see also [19]).

2. A canonical form for linear grammars

Recall that two grammars are called *equivalent* if they generate the same language.

Now we present an algorithm which transforms any linear grammar into an equivalent one which is in a *canonical form*. Recall that a variable $A \in \mathbf{P}$ is *superfluous* if, either there is no derivation $S \overset{*}{\Rightarrow} w$ with $w \in (\mathbf{V} \cup \mathbf{\Sigma})^*$ containing A, or $L_A = \emptyset$, and that a grammar $\mathcal{C}$ is *reduced* if it has no superfluous variables.

Proposition 1 (Canonical form for linear grammars). *Let $\mathcal{C} = (\mathbf{V}, \mathbf{\Sigma}, \mathbf{P}, S)$ be a linear grammar. Then $\mathcal{C}$ is equivalent to a reduced grammar $\overline{\mathcal{C}} = (\overline{\mathbf{V}}, \mathbf{\Sigma}, \overline{\mathbf{P}}, S)$ where the productions are only of the form*:

$$A_1 \vdash a_1 B_1, \qquad A_2 \vdash B_2 a_2, \qquad A_3 \vdash a_3 \tag{1}$$

where $A_1, A_2, A_3, B_1, B_2 \in \overline{\mathbf{V}}$ and $a_1, a_2, a_3 \in \mathbf{\Sigma}$.

If $\varepsilon \in L(\mathcal{C})$, in addition to the previous productions one also has $(S \vdash \varepsilon) \in \overline{\mathbf{P}}$.

Moreover, if $\mathcal{C}$ is unambiguous then $\overline{\mathcal{C}}$ is also unambiguous.

Proof. We can suppose that $\mathcal{C}$ is reduced, otherwise we eliminate the superfluous variables and all productions involving them.

Next we transform the grammar $\mathcal{C}$ into a grammar $\mathcal{C}'$ which is ε-free, that is there is no rule of the form $A \vdash \varepsilon$. There is a simple algorithm for passing from $\mathcal{C}$ to $\mathcal{C}'$ that generates $L \setminus \{\varepsilon\}$; see, e.g., [11, Section 4.3].

Similarly one eliminates the *chain rules*, i.e., productions of the form $A \vdash B$, where $A, B \in \mathbf{V}$. Again there is a simple algorithm that transforms a reduced grammar $\mathcal{C}'$ into an equivalent reduced grammar $\mathcal{C}''$ without chain rules, see, e.g., [11, Section 4.3] or [15, Corollary 5.3].

Note that these transformations preserve unambiguity, namely $\mathcal{C}''$ is unambiguous if the original grammar $\mathcal{C}$ is such.

The generic production in $\mathcal{C}''$ is then of the form

$$A \vdash a_1 a_2 \cdots a_m B b_1 b_2 \cdots b_n \tag{2}$$

or

$$C \vdash c_1 c_2 \cdots c_s \tag{3}$$

where $A, B, C \in \mathbf{V}''$, $a_i, b_j, c_k \in \mathbf{\Sigma}$ and $m + n, s \geqslant 1$. Call the *length* of a production (2) or (3) the quantity $\ell = m + n + 1$ and $\ell = s$, respectively.

Suppose that in (2) one has $m \geqslant 2$. Then the length of the production (2) can be shortened by enlarging $\mathbf{V}''$ to $\mathbf{V}'' \cup \{B_1\}$, where $B_1 \notin \mathbf{V}''$ and then substituting (2) in $\mathbf{P}''$ with the two productions

$$A \vdash a_1 a_2 \cdots a_{m-1} B_1 b_1 b_2 \cdots b_n, \qquad B_1 \vdash a_m B. \tag{4}$$

Similarly one acts on the right part of (2) whenever $n \geqslant 2$.

If $m = n = 1$ then (4) is replaced by

$$A \vdash a_1 B_1, \qquad B_1 \vdash B b_1. \tag{5}$$

Finally, given a production of the form (3), one enlarges $\mathbf{V}''$ to $\mathbf{V}'' \cup \{C_1\}$, where $C_1 \notin \mathbf{V}''$, replaces it with

$$C \vdash c_1 c_2 \cdots c_{s-1} C_1, \qquad C_1 \vdash c_s \tag{6}$$

and then reduces to the previous cases. By recurrence one shortens the production lengths to minimality obtaining a grammar $\overline{\mathcal{C}}$ in the desired canonical form. □

3. The oriented graph associated with a linear grammar

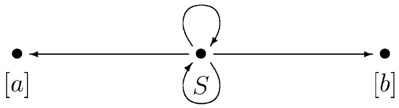
With a linear grammar $\mathcal{C} = (\mathbf{V}, \mathbf{\Sigma}, \mathbf{P}, S)$ we associate an *oriented graph* $\mathcal{G}(\mathcal{C}) = (V, E; S, \mathcal{F})$ with vertex set $V = \mathbf{V} \cup \mathbf{\Sigma}'$ where $\mathbf{\Sigma}' = \{[a]\colon\ a \in \mathbf{\Sigma}^* \text{ and } \exists (A \vdash a) \in \mathbf{P}\}$ and there is an edge from T to U (notation $T \to U$), $T, U \in V$, if $T \in \mathbf{V}$ and in $\mathbf{P}$ there is a production $T \vdash u$ with, either $u \in (\mathbf{V} \cup \mathbf{\Sigma})^*$ containing U, if $U \in \mathbf{V}$, or $u \in \mathbf{\Sigma}^*$, if $U = [u] \in \mathbf{\Sigma}'$. Also, S and $\mathcal{F} = \mathbf{\Sigma}' \subseteq V$ are the *initial* and *terminal* vertices, respectively. Note that $\mathcal{G}$ might have *loops* (a vertex may be joined to itself) and *multiple edges* (several edges may go from one vertex to another); in particular, there could also be multiple loops.

This is a variant of the *dependency di-graph* associated with a general context-free grammar considered by Kuich [14] and by Ceccherini-Silberstein and Woess [4,5]. For a richer structure on $\mathcal{G}$ see [2] where it is shown that *ergodic* unambiguous linear languages are *growth-sensitive*.[1]

[1] A language L over a finite alphabet $\mathbf{\Sigma}$ is called growth-sensitive if its (exponential) *growth rate* $\gamma(L) = \limsup_{n\to\infty} |\{w \in L\colon\ \ell(w) \leqslant n\}|^{1/n}$ is (strictly) greater than the growth rate $\gamma(L^F)$ of any sublanguage L^F obtained by forbidding any non-empty set F of subwords. Also, a context-free grammar (and associated language) is *ergodic* if its dependency di-graph is strongly connected. It is known that ergodic regular languages are growth-sensitive [3] and it was shown in [4,5] that also unambiguous *non-linear* context-free languages which are ergodic are growth-sensitive.

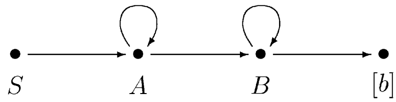
A path starting at S and terminating at a final vertex is termed *admissible*.

Example 1. Let $L_1 = \mathbf{\Sigma}^+$ be the language (over $\mathbf{\Sigma} = \{a, b\}$) of the free semigroup of rank two. It is a linear (indeed regular) language; it is generated by the (right linear) grammar $\mathcal{C}_1 = (\{S\}, \{a, b\}, \mathbf{P}_1, S)$ where $\mathbf{P}_1 = \{S \vdash aS, bS, a, b\}$. It is clearly a language of *exponential* growth. The corresponding oriented graph $\mathcal{G}(\mathcal{C}_1)$ is



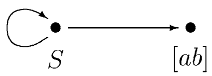
(7)

Example 2. Let $L_2 = \{a^n b^m\colon\ n, m \geqslant 1\}$. This is again a regular language; a (right linear) grammar for it is $\mathcal{C}_2 = (\{S, A, B\}, \{a, b\}, \mathbf{P}_2, S)$ where $\mathbf{P}_2 = \{S \vdash aA;\ A \vdash aA, B;\ B \vdash bB, b\}$. The growth is *quadratic*, namely $[\gamma_L(n)] = [n^2]$. The corresponding oriented graph $\mathcal{G}(\mathcal{C}_2)$ is



(8)

Example 3. Let $L_3 = \{a^n b^n\colon\ n \geqslant 1\}$. This is a linear (though *non-regular*) language; it is generated by the linear grammar $\mathcal{C}_3 = (\{S\}, \{a, b\}, \mathbf{P}_3, S)$ where $\mathbf{P}_3 = \{S \vdash aSb, ab\}$. Here the growth is *linear*: $[\gamma_L(n)] = [n]$. The corresponding oriented graph $\mathcal{G}(\mathcal{C}_3)$ is



(9)

Remark 2. In the proof of Proposition 1, the edge in $\mathcal{G}(\mathcal{C}'')$ corresponding to the production (2), after repeated applications of the transformations (4), becomes a path in $\mathcal{G}(\overline{\mathcal{C}})$ of the form

$$A \to B_1 \to B_2 \to \cdots \to B_{m+n-1} \to B.$$

Here $A \to B_1$ corresponds to $A \vdash a_1 B_1$, $B_i \to B_{i+1}$ to $B_i \vdash a_{i+1} B_{i+1}$, for $i = 1, 2, \ldots, m-1$, $B_{m+j} \to B_{m+j+1}$ correspond to $B_{m+j} \vdash B_{m+j+1} b_{n-j}$ for $j = 0, 1, \ldots, n-2$ and, finally $B_{m+n-1} \to B$ corresponds to $B_{m+n-1} \vdash B b_1$.

Analogously the edge corresponding to the production (3) becomes the path

$$C \to C_1 \to C_2 \to \cdots \to C_{n-1} \to [c_1]$$

where the edge $C \to C_1$ corresponds to $C \vdash C_1 c_n$, the edges $C_i \to C_{i+1}$ correspond to $C_i \vdash C_{i+1} c_{n-i}$, for $i = 1, 2, \ldots, n-2$ and $C_{n-1} \to [c_1]$ to $C_{n-1} \vdash c_1$.

Thus, when passing from a general linear grammar $\mathcal{C}$ to an equivalent in canonical form, say $\overline{C}$, according to the algorithm from Proposition 1, the corresponding graph $\mathcal{G}(\mathcal{C})$ is transformed into $\mathcal{G}(\overline{\mathcal{C}})$. One should note that the (*oriented*) *algebraic-topological* structure of the two graphs is the same. For instance, as Example 3 is concerned, we have

$\bar{\mathcal{C}}_3 = (\{S, A, B\}, \{a, b\}, \bar{\mathbf{P}}_3, S)$ where $\bar{\mathbf{P}}_3 = \{S \vdash aA, aB;\ A \vdash Sb;\ B \vdash b\}$, so that the corresponding oriented graph $\mathcal{G}(\bar{\mathcal{C}}_3)$ is

$$\begin{tikzcd} A \arrow[r, bend left] & S \arrow[l, bend left] \arrow[r] & B \arrow[r] & {[b]} \end{tikzcd} \tag{10}$$

Note that also for $\mathcal{C}_3' = (\{S, A\}, \{a, b\}, \mathbf{P}', S)$ with $\mathbf{P}' = \{S \vdash aA;\ A \vdash Sb, b\}$, which is a grammar (in canonical form) equivalent to $\mathcal{C}$, the corresponding oriented graph $\mathcal{G}(\mathcal{C}_3')$ has the same (oriented) algebraic-topological structure as $\mathcal{G}(\mathcal{C}_3)$ and $\mathcal{G}(\bar{\mathcal{C}}_3)$:

$$\begin{tikzcd} S \arrow[r, bend left] & A \arrow[l, bend left] \arrow[r] & {[b]} \end{tikzcd} \tag{11}$$

4. Ufnarovskii's criterion for the growth of oriented graphs

Let $\mathcal{G} = (V, E)$ be a finite oriented graph possibly with multiple edges and loops. A *path* of length m is an alternating sequence $v_0, e_1, v_1, e_2, \ldots, v_{m-1}, e_m, v_m$ of vertices $v_i \in V$ and edges $e_i \in E$, such that v_{i-1} is the *start* vertex and v_i is the *end* vertex of the edge e_i, $i = 1, 2, \ldots, m$. The path is *closed* if $v_0 = v_m$, a *chain* if all *edges* are distinct, a *simple chain* if all vertices are distinct (and thus *a fortiori* also all edges are distinct), a *cycle* if it is a closed chain, and a *simple cycle* if $v_i \neq v_i$ for all $(i, j) \neq (0, m), (m, 0)$.

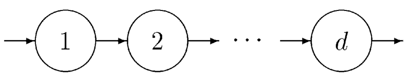
The *growth function* of $\mathcal{G}$ is the function $\gamma_{\mathcal{G}}(m)$ which counts the number of all paths of length at most m in $\mathcal{G}$. It is easy to see (Theorem 1 in [18]) that the *Poincaré series* associated with $\gamma_{\mathcal{G}}$, namely $F(X) = \sum_m^\infty \gamma_{\mathcal{G}}(m) X^m$, is a *rational function* so that the *growth* of $\mathcal{G}$, $[\gamma_{\mathcal{G}}]$, is either polynomial or exponential.

A vertex is termed *cyclic* it occurs in some cycle, *acyclic* otherwise and *doubly cyclic* if (at least) two distinct cycles (as graphs, not as paths) pass through it. $\mathcal{G}$ is *cyclically simple* if it has no doubly cyclic vertices.

We are now ready to state Ufnarovskii's criterion; for the sake of completeness and for the reader's convenience we shall also present the proof (even slightly simplified w.r.t. the original one).

Theorem 3. *The growth of any oriented graph $\mathcal{G}$ is either polynomial or exponential. More precisely,*

(1) *The growth is* exponential *if and only if $\mathcal{G}$ has a* doubly *cyclic vertex*;
(2) *The growth is* polynomial *if and only if $\mathcal{G}$ is cyclically* simple. *If this is the case, the polynomial degree is the length d of a maximal circuit of the form*



(12)

where the circles are simple cycles (*of any length*) *and the lines are simple chains* (*of any length*).

Proof. (1) Let m denote the maximum length of two distinct cycles passing through a doubly cyclic vertex v. With any binary word $w = \epsilon_1\epsilon_2\cdots\epsilon_n$ associate the path R_w, of length at most mn, consisting of the sequence of paths $R_{\epsilon_1}, R_{\epsilon_2}, \ldots, R_{\epsilon_n}$ where R_{ϵ_i} is a cycle from v to v around the first cycle if $\epsilon_i = 0$ and around the second one if $\epsilon_i = 1$. Thus, $\gamma_{\mathcal{G}}(mn) \geqslant 2^n$ and $[\gamma_{\mathcal{G}}] \geqslant [2^n]$, showing that the growth of $\mathcal{G}$ is exponential.[2]

(2) Suppose now that $\mathcal{G}$ is cyclically simple. We can suppose that $\mathcal{G}$ is connected and, moreover, that given any two edges there is an oriented path passing through them (in some order): indeed, denoting by e_1 and e_2 two edges such that no path passes through them simultaneously one gets $\gamma_{\mathcal{G}}(n) = \gamma_{\mathcal{G}\setminus e_1}(n) + \gamma_{\mathcal{G}\setminus e_2}(n) - \gamma_{\mathcal{G}\setminus\{e_1,e_2\}}(n)$ so that $\gamma_{\mathcal{G}} \leqslant \gamma_{\mathcal{G}\setminus e_1} + \gamma_{\mathcal{G}\setminus e_2} \leqslant 2\gamma_{\mathcal{G}}$. This shows that $\gamma_{\mathcal{G}} = \max\{\gamma_{\mathcal{G}\setminus e_1}, \gamma_{\mathcal{G}\setminus e_2}\}$.

If there is no cycle then there are only finitely many paths. We thus suppose that there exists at least one cycle, which, by our assumptions, is simple. It is obvious that there is at most one edge entering and at most one edge exiting from it: otherwise, with the previous assumptions, we would contradict our hypothesis of simplicity of $\mathcal{G}$. We continue such edges to a maximal simple chain, all of whose vertices except the first, and possibly the last, are acyclic. If one of these extrema is cyclic we construct the corresponding cycle and we continue. We eventually arrive to a path as in (12).

Set

$$Y(n,d) = \bigl\{\mathbf{y} = (y_1, y_2, \ldots, y_d)\colon\ y_i \in \mathbb{N} \cup \{0\} \text{ and } y_1 + y_2 + \cdots + y_d \leqslant n\bigr\}$$

and observe that $|Y(n,d)| = \sum_{i=1}^{d} \binom{d}{i}\binom{n}{i} = \binom{n+d}{d}$ so that

$$\bigl[\bigl|Y(n,d)\bigr|\bigr] = \bigl[n^d\bigr]. \tag{13}$$

With a tuple $(y_1, y_2, \ldots, y_d) \in Y(n,d)$ we then associate the unique path, of length at most $m_1 n + m_2$ (where m_1 is the maximum length of the cycles and $m_2 = |E|$), which starts (respectively ends) at the leftmost (respectively rightmost) vertex in (12) and consists in turning around the first cycle y_1 times, the second one y_2 times and so on. Thus we have an embedding of $Y(n,d)$ into the set of all paths in $\mathcal{G}$ of length at most $m_1 n + m$, which gives

$$\bigl|Y(n,d)\bigr| \leqslant \gamma_{\mathcal{G}}(m_1 n + m_2). \tag{14}$$

Conversely, with a path p of length at most $m_3 n$ (where m_3 is the minimal length of the cycles) we associate an element $\mathbf{y}(p) \in Y(n,d)$ by setting y_i the number of times the path goes around the ith cycle. It is clear that the map $p \to \mathbf{y}(p)$ is in general non-injective. However, note that $\mathbf{y}(p) = \mathbf{y}(p')$ for two paths p and p' starting at the same vertex and of the same length (or which is equivalent, with same end vertex) infers $p = p'$. Thus, the full

[2] Observe that as $\mathcal{G}$ is finite, it is of *bounded degree*, that is there exists $K > 0$ such that $\partial(v) \leqslant K$ for all $v \in V$, where $\partial(v) = |\{w \in V\colon\ v \sim w\}|$ is the degree of v. Thus the growth of $\gamma_{\mathcal{G}}$ is at most exponential (it cannot be *super-exponential*).

invariants for the paths p are $\mathbf{y}(p)$ and the starting and ending vertices p^+ and p^- (there are at most $|V|^2$ of this latter) and we get

$$\gamma_{\mathcal{G}}(m_3 n) \leqslant |V|^2 \cdot \big|Y(n,d)\big|. \tag{15}$$

Finally, observing that given a monotone function $f:\mathbb{N} \to \mathbb{N}$ and positive numbers c_1, c_2, c_3 and c_4, one has

$$\big[c_1 f(c_2 n + c_3) + c_4\big] = \big[f(n)\big], \tag{16}$$

from (13), (14) and (15) one deduces $[\gamma_{\mathcal{G}}(n)] = [n^d]$ and the proof is complete. $\square$

5. The algorithm for the growth of unambiguous linear languages

We are now ready to present the algorithm for determining whether a given unambiguous linear language is of polynomial or exponential growth.

Corollary 4 (The algorithm). *Let L be an unambiguous linear language. Let $\mathcal{C}$ be an unambiguous linear grammar such that $L(\mathcal{C}) = L$. Using the algorithm in Proposition* 1 *transform $\mathcal{C}$ into an equivalent reduced grammar* (*that we still denote by $\mathcal{C}$*) *in canonical form. Let $\mathcal{G} = \mathcal{G}(\mathcal{C})$ be the corresponding oriented graph and apply Ufnarovskii's criterion* (*Theorem* 3) *to it*:

- *L has* exponential *growth if and only if $\mathcal{G}$ has a* doubly *cyclic vertex*;
- *L has* polynomial *growth if and only if $\mathcal{G}$ is cyclically* simple. *If this is the case, the polynomial degree of $[\gamma_L]$ is the length d of a maximal circuit in $\mathcal{G}$ of the form* (12).

Proof. Let L be an unambiguous linear language. By Proposition 1 we can suppose that L is generated by an unambiguous reduced grammar $\mathcal{C}$ in standard form. Let $\mathcal{G} = (V, E; S, \mathcal{F})$ be the associated oriented graph. By unambiguity, we have a one-to-one length-preserving correspondence between the words in L and the admissible paths in $\mathcal{G}$ so that, denoting by $\gamma_{\mathcal{G}}^a$ the growth function of admissible paths in $\mathcal{G}$, we have $\gamma_L \equiv \gamma_{\mathcal{G}}^a$.

We are thus only left to show that $\gamma_{\mathcal{G}} \sim \gamma_{\mathcal{G}}^a$.

Clearly

$$\gamma_{\mathcal{G}}^a(n) \leqslant \gamma_{\mathcal{G}}(n). \tag{17}$$

Conversely, given a vertex $v \in V$ denote by $p(S, v)$ a minimal path starting at S and terminating at v and, similarly, denote by $p(v, f_v)$ a minimal path starting at v and ending at some terminal vertex $f_v \in \mathcal{F}$: the existence of such paths is guaranteed by the fact that $\mathcal{C}$ is reduced. Set $M = \max\{|p(S, v)|, |p(v, f_v)|\colon v \in V\}$.

This way, with any not necessarily admissible path p starting at a vertex v and terminating at w, we associate a unique admissible path $p' = p(S, v) \cdot p \cdot p(v, f_v)$ whose length is bounded by $|p'| \leqslant |p| + 2M$.

Observing that, by definition of $\mathcal{F}$, all $f \in \mathcal{F}$ have degree $\partial(f) = 1$ in $\mathcal{G}$, we have that given an admissible path p of length n, there are at most $\binom{n}{2} = n(n-1)/2$ distinct not necessarily admissible subpaths of p (if S is acyclic (equivalently there is no derivation $S \overset{*}{\Rightarrow} w$ with $w \in (\mathbf{V} \cup \mathbf{\Sigma})^*$ containing S), then the not necessarily admissible subpaths are exactly $\binom{n}{2}$). Thus we have

$$\gamma_{\mathcal{G}}(n) \leqslant n^2 \gamma_{\mathcal{G}}^a(n + 2M). \tag{18}$$

If $\mathcal{G}$ has exponential growth then from (16), (17) and (18) one immediately deduces that $\gamma_{\mathcal{G}}^a$ is also of exponential type.

If $\mathcal{G}$ has polynomial growth, say $[\gamma_{\mathcal{G}}(n)] = [n^d]$, then, by Theorem 3, we can suppose that $\mathcal{G}$ is of the form given in (12) and the same arguments as in the proof of that theorem show that in fact $[\gamma_{\mathcal{G}}^a(n)] = [n^d]$, completing the proof. □

Remark 5. In Example 1 the oriented graph $\mathcal{G}(\mathcal{C}_1)$ has a *doubly* cyclic vertex, namely the start point S. Thus $\mathcal{G}(\mathcal{C}_1)$ and $L_1 = \mathbf{\Sigma}^*$ have *exponential* growth.

On the other hand, all the oriented graphs in (7)–(11) are cyclically *simple*. Graphs and languages all have *polynomial* growth. In particular, as $\mathcal{G}(\mathcal{C}_2)$ has a maximal circuit of length $d = 2$, we have quadratic growth, while, in the remaining cases, $d = 1$ and we have linear growth.

6. On the growth of ambiguous linear languages

Let $\mathcal{C} = (\mathbf{V}, \mathbf{\Sigma}, \mathbf{P}, S)$ be a context-free grammar. For a variable $T \in \mathbf{V}$ we define the *ambiguity degree* $d_T(w)$ of a word $w \in \mathbf{\Sigma}^*$ as the number of all different righmost derivations $T \overset{*}{\Rightarrow} w$. We have $d_T(w) > 0$ if and only if $w \in L_T$. If the grammar is reduced, without chain rules, i.e., productions of the form $T \vdash U$, where $T, U \in \mathbf{V}$, and ε-free, then $d_T(w) < \infty$ always. We remark that there is a simple algorithm that transforms a (reduced) grammar into an equivalent (reduced) grammar without chain rules (see, for instance [11, Section 4.3], or [15]). Thus, with this terminology, the grammar is unambiguous if and only if $d_S(w) = 1$ for all $w \in L(\mathcal{C})$.

From now on all context-free grammars considered shall be reduced, with no chain rules and ε-free.

Let $L = L(\mathcal{C})$ be a context free language generated by a context-free grammar $\mathcal{C}$. The *weighted growth function* of L with respect to $\mathcal{C}$ is the function

$$\gamma_{L,\mathcal{C}}(n) = \sum_{w \in L:\ |w| \leqslant n} d_S(w). \tag{1}$$

Note that if $\mathcal{C}$ is unambiguous, then $\gamma_{L,\mathcal{C}}(n) = \gamma_L(n)$ the (standard) growth function of L defined in the Introduction. We then say that the *weighted growth type* of a context-free grammar $\mathcal{C}$ and of its associated language L is the equivalence class $[\gamma_{L,\mathcal{C}}]$ of its weighted growth function.

With this terminology we can state the following version of Corollary 4 for possibly ambiguous linear grammars and associated languages.

Proposition 6. *Let $\mathcal{C}$ be a linear grammar and $L = L(\mathcal{C})$ the associated language. Then the weighted growth type $[\gamma_{L,\mathcal{C}}]$ is either polynomial or exponential. More precisely, denoting by $\mathcal{G} = \mathcal{G}(\mathcal{C})$ the corresponding oriented graph, one has*:

- $[\gamma_{L,\mathcal{C}}]$ *is* exponential *if and only if $\mathcal{G}$ has a* doubly *cyclic vertex*;
- $[\gamma_{L,\mathcal{C}}]$ *is* polynomial *if and only if $\mathcal{G}$ is cyclically* simple. *If this is the case, the polynomial degree of $[\gamma_{L,\mathcal{C}}]$ is the length d of a maximal circuit in $\mathcal{G}$ of the form* (12).

Proof. The proof is the same as for Corollary 4. We only observe that in the present setting there is a one-to-one length-preserving correspondence between the set of different derivations $S \stackrel{*}{\Rightarrow} w$ for all the words w in L and all admissible paths in $\mathcal{G}$, so that, denoting by $\gamma_{\mathcal{G}}^a$ the growth function of admissible paths in $\mathcal{G}$, we have $\gamma_{L,\mathcal{C}} \equiv \gamma_{\mathcal{G}}^a$. □

Note added in proof

After a preliminary version of this article was circulating, it was pointed out to me that in fact an algorithm for the (standard) growth of context-free languages does exist.

Indeed Ginsburg and Spanier [9, Theorem 5.2(a)] proved that it is decidable whether or not a given context-free language is bounded or, equivalently, it has polynomial growth. As the non-bounded context-free languages are exactly those context-free languages of exponential growth (by the results of Trofimov, Incitti, and Bridson and Gilman cited in the Introduction), our assertion follows. However in [9] there is no indication on the degree of polynomial growth for bounded languages, but only an upper bound. Indeed by [9, Theorem 5.2(b)] if L is bounded, one can effectively find words $w_1, w_2, \ldots, w_t \in \mathbf{\Sigma}^*$ so that $L \subseteq w_1^* w_2^* \cdots w_t^*$ and, clearly, $[\gamma_L = [n^d]$ with $d \leqslant t$.

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