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# Universal partial order represented by means of oriented trees and other simple graphs

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**Abstract**

We present several simple representations of universal partially ordered sets and use them for the proof of universality of the class of oriented trees ordered by the graph homomorphisms. This (which we believe to be a surprising result) solves several open problems. It implies for example universality of cubic planar graphs. This is in sharp contrast with representing even groups (and monoids) by automorphisms (and endomorphisms) of a bounded degree and planar graph. Thus universal partial orders (thin categories) are representable by much simpler structures than categories in general.


## 1. Introduction

A countable partially ordered set is said to be *universal* if it contains any countable partial order as an (induced) suborder.

A universal partial order exist. This classical result has been proved several times [9, 14, 15] and in the context of category theory motivated the whole research area. Particularly, Hedrlín [6] found examples of universal partial orders with easy representation and this line culminated in proving that many frequent categories induce universal partial orders. These include e.g. oriented and undirected graphs with given chromatic number and girth ordered by the existence of homomorphism, see [21] for an extensive catalogue of such representations. In this catalogue are missing classes of graphs with bounded degrees and topologically restricted (say planar) graphs. In fact the category theory approach cannot be applied along the lines of [21] as these classes fail to represent all groups (in the case of

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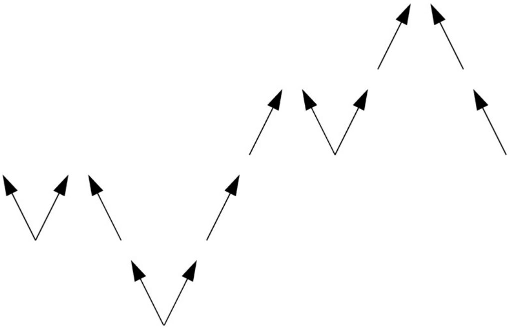

Fig. 1. Oriented path $P$.

topological restrictions, see [1]) and monoids (in the case of bounded degrees, see [2, 8]). A bit surprisingly for partially ordered sets all these problems have affirmative answer:

**Theorem 1.1.** *Denote by $\Delta_k$ the class of all finite graphs $G$ with maximal degree $\leq k$ ordered by the existence of a homomorphism. Then $\Delta_k$ is universal partial order iff $k \geq 3$.*

**Theorem 1.2.** *Denote by $\mathcal{K}$ the class of all cubic planar graphs then the class $\mathcal{K}$ ordered by the existence of a homomorphism is universal.*

Both these results follow from a result for oriented trees and linear forests. Let us state this in a greater detail:

An *oriented path* $P$ is any oriented graph $(V, E)$ where $V = \{v_0, v_1, \ldots, v_n\}$ and for every $i = 1, 2, \ldots, n$ either $(v_{i-1}, v_i) \in E$ or $(v_i, v_{i-1}) \in E$ (but not both), and there are no other edges. Thus an oriented path is any orientation of an undirected path. We denote the initial vertex $v_0$ and the terminal vertex $v_n$ of $p$ by $in(P)$ and $term(P)$ respectively. Examples are in Fig. 1, and here (as always) all arcs are oriented upwards.

A *(oriented) linear forest* is a disjoint union of finitely many paths (i.e. an orientation of a forest with path components). An *oriented tree* is an orientation of a tree.

The *length* $l(P)$ of a path $P$ is the number of edges in $P$. The *algebraic length* $al(P)$ of a path $P$ is the number of forwarding minus the number of backwarding arcs in the code of $P$. Thus the algebraic length of a path could be negative. The *level* $l_p(p_i)$ of $p_i$ is the algebraic length of the subpath $(p_0, p_1, \ldots, p_i)$ of $P$.

Denote by $\mathcal{P}$ the partial order generated by all finite paths and the existence of homomorphism (actually, we have to restrict ourselves to *cores*—minimal retracts—to obtain a partial order; otherwise we have a quasiorder which we can factorize by hom-equivalence, see [18] for details).

The class of all linear forests ordered by the existence of a homomorphisms will be denoted by $\mathcal{P}^*$. Finally let $\mathcal{T}$ denote the class of all orientations of finite trees ordered by the existence of homomorphisms.

It has been proved in [20] that $\mathcal{P}$ is a dense partial order (with the exception of a few gaps which were characterized; these gaps are formed by all core-paths of height $\leq$4). Let us remark that the problem of density of $\mathcal{T}$ is presently open (and in view of [19] this is an important problem). Reference [20] also posed (a seemingly too ambitious) question of whether $\mathcal{T}$ is universal partial order. A tree representation is found for finite partial orders and for all finitely dimensional infinite partial orders (the representation of countable chains follows from density) of paths. Here we give a solution of this problem:

**Theorem 1.3.** *$\mathcal{P}^*$ is universal partial order.*

**Theorem 1.4.** *$\mathcal{T}$ is universal partial order.*

These universality results are stated by means of embeddings: given two (finite or infinite) partial orders $(A, \leq_A)$ and $(A', \leq_{A'})$ a mapping $f : A \to A'$ is called an *embedding* if $f$ is injective and

$$x \leq_A y \qquad \text{iff } f(x) \leq_{A'} f(y)$$

for any pair $x, y \in A$. Thus a countable partial order $\mathcal{O}$ is universal if every countable partial order can be embedded into $\mathcal{O}$.

The universality of partial order $\mathcal{T}$ (or $\mathcal{P}^*$) may be also interpreted in finite terms by means of *on-line representation*.

By an *on-line representation* of a class $\mathcal{K}$ of partial orders, we mean that one can construct a representation of any partial order $R$ in class $\mathcal{K}$ under the circumstances that the elements of $R$ are revealed one by one. The on-line representation of a class of partial orders can be considered as a game between two players Alice and Bob. Bob chooses a partial order $(A, \leq_A)$ in the class $\mathcal{K}$, and reveals the elements of $A$ one by one to Alice (Bob is a bad guy). Whenever an element of $x$ of $A$ is revealed to Alice, the relations among $x$ and previously revealed elements are also revealed. Alice is required to construct an oriented tree to represent $x$ before the next element is revealed. Alice wins a game if he succeeds in constructing a representation of $A$. The class $\mathcal{K}$ of partial orders is on-line representable if Alice has a winning strategy.

For the benefit of the reader we include the following easy result (see e.g. [20]):

**Theorem 1.5.** *The following three statements are equivalent:*

1. *Every countable partial order is tree representable.*
2. *The class of all finite partial orders is on-line tree representable.*
3. *The class of all countable partial orders is on-line tree representable.*

**Proof.** It is obvious that $3 \Longrightarrow 1$. To see that $2 \Longrightarrow 3$, we note that if Bob has a winning strategy for the class of all finite partial orders, then this strategy can be applied to construct an on-line representation of any countable partial order, because at each step the revealed part induces a finite partial order.

We now prove that $1 \Longrightarrow 2$: let $\mathcal{O}$ be universal homogeneous partial order. Assume that $f : \mathcal{O} \to \mathcal{P}$ is a tree representation of $\mathcal{O}$. As $\mathcal{O}$ has the extension property the tree representation yields an on-line representation of any finite partial order. □

Of course a similar statement holds for representation by paths, linear forests etc.

The on-line formulation of universality clearly indicates why the usual representation of partially ordered sets by vectors ordered coordinatewise cannot succeed. Given two integer vectors $\vec{v} = (v_1, \dots, v_t) \leq \vec{v}' = (v'_1, \dots, v'_t)$ there are only finitely many vectors $w$ with $\vec{v} \leq \vec{w} \leq \vec{v}'$ (and we need infinitely many of them). Even if we allow rational numbers for coordinates then still the interval $\vec{v} \leq \vec{w} \leq \vec{v}'$ would be finite dimensional (and we need arbitrarily large dimension in this, and in any other interval). Alice does not know where the next move will be played!

The paper is organized as follows: the main result—on-line representability by linear forests—is proved in Section 4. The proof is advanced by two constructions of special partial orders: multicut structure $\mathcal{MC}$ (Section 2) and truncated vectors $\mathcal{V}^*$ (Section 3). Each of these steps is necessary for our proof. Of course we could start strictly with $\mathcal{V}^*$ but the structure $\mathcal{MC}$ is included as it serves as a warm-up for more complicated embeddings $\mathcal{V}^*$, $\mathcal{P}^*$ and $\mathcal{T}$. It also presents the link to our companion papers on generic (homogeneous universal) partial order [11]. (However note that none of the partial order $\mathcal{MC}$, $\mathcal{V}^*$, $\mathcal{P}^*$ and $\mathcal{T}$ is homogeneous.)

In Section 6 we use the linear forest representation to get universality of trees undirected graphs with strong local properties. There we prove Theorems 1.1, 1.2 and 1.4.

## 2. Universal structure $\mathcal{MC}$

We start by simple construction of the universal partial order which underlines and helps to understand the structure $\mathcal{V}^*$. Our structure $\mathcal{MC}$ is similar to one found by [6], however the way we embed any partial order into it is different and we believe it is easier to understand. In a way this is the most natural definition. It is based on methods discussed in [12, 17].

**Definition 2.1.** Each element of $\mathcal{MC}$ is a finite set of positive integers. For $X, Y \in \mathcal{MC}$ we put $X \leq_{\mathcal{MC}} Y$ if and only if for each $y \in Y$ there exist $x \in X$ such that $x \subseteq y$.

**Theorem 2.1.** *$\mathcal{MC}$ is universal partial order.*

**Proof.** We will show how to embed on-line any partially ordered set $(A, \leq_A)$ into $\mathcal{MC}$. This means that to each $x \in A$ we have assigned unique positive integer $t(x)$—the time of the creation of $x$. Additionally for $x \in A$ we define

$$D(x) = \{y; y \in A, t(y) \leq t(x), y \leq_A x\}. \tag{1}$$

Thus $D(x)$ is the down set of previously created elements of $A$ at the time $t(x)$.

Put

$$\phi_{\mathcal{MC}}(x) = \{D(y); y \in A, t(y) \leq t(x), x \leq_A y\}. \tag{2}$$

Clearly this is an on-line definition of a mapping $(A, \leq_A) \to \mathcal{MC}$. For instance the partially ordered set from Fig. 2 will be represented as follows:

$$\begin{array}{ll}
D(1) = \{1\}, & \phi_{\mathcal{MC}}(1) = \{\{1\}\} \\
D(2) = \{1, 2\}, & \phi_{\mathcal{MC}}(2) = \{\{1, 2\}\} \\
D(3) = \{3\}, & \phi_{\mathcal{MC}}(3) = \{\{1, 2\}, \{3\}\} \\
D(4) = \{4\}, & \phi_{\mathcal{MC}}(4) = \{\{1\}, \{1, 2\}, \{3\}, \{4\}\}.
\end{array}$$

Theorem 2.1 follows from the following lemma.

**Lemma 2.2.** *$\phi_{\mathcal{MC}}$ is an embedding of $\mathcal{P}$ into $\mathcal{MC}$.*

**Proof.** 1. Assume $x \leq_P y$. We prove $\phi_{\mathcal{MC}}(x) \leq_{\mathcal{MC}} \phi_{\mathcal{MC}}(y)$:

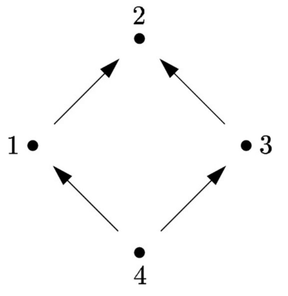

Fig. 2. Partially ordered set $P$.

Consider arbitrary $y' \in P$ such that $D(y') \in \phi_{\mathcal{MC}}(y)$. From Definition 2 it follows easily that $x \leq_P y \leq_P y'$. To prove inequality $\phi_{\mathcal{MC}}(x) \leq_{\mathcal{MC}} \phi_{\mathcal{MC}}(y)$ it suffices to find a subset of $D(y')$ which belongs to the set $\phi_{\mathcal{MC}}(x)$. We examine two possibilities:

If $t(y') \leq t(x)$, then $D(y') \in \phi_{\mathcal{MC}}(x)$ (as follows directly from Definition 2 and from the inequality $x \leq_P y'$).

In the case $t(y') > t(x)$, consider any $z \in D(x)$. We have inequalities $z \leq_P x \leq_P y'$ and $t(z) \leq t(x) < t(y')$ and thus we get $t(z) \in D(y')$ directly from Definition 2.1. Thus $D(x) \subseteq D(y')$.

2. Assume $\phi_{\mathcal{MC}}(x) \leq_{\mathcal{MC}} \phi_{\mathcal{MC}}(y)$. We prove $x <_P y$:

We have $D(y) \in \phi_{\mathcal{MC}}(y)$ and thus there exists $x'$ such that $D(x') \in \phi_{\mathcal{MC}}(x)$ and $D(x') \subseteq D(y)$. Since $x' \in D(y)$ we get $x' \leq_P y$ by Definition 2.1. Inequality $x \leq_P x'$ follows from $D(x') \in \phi_{\mathcal{MC}}(x)$ and consequently $x \leq_P y$ follows from the transitivity of $\leq_P$. $\square$

**Remark.** It seems that the extension properties of partial orders (which underline any homogeneous universal structure (see e.g. [3])) are natural to handle using cuts: a *cut* in a partially ordered set $(A, \leq_A)$ is any downward closed subset of $X$. Cuts (with inclusion ordering) may be used to represent $(A, \leq_A)$ as well as to enlarge $(A, \leq_A)$ into the smallest complete partial order (Dedekind cuts construction of real numbers and of McNeille completion for arbitrary partial orders well documented by most textbook on set theory). But it is possible to say that these classical ideas are surprisingly pertinent to date (as nicely put by Rota [16]). One example of this is the number system ("surreal numbers") due to Conway [4, 13] which uses the cuts to generate *on line* the new numbers from the old one. Our results continue in the same direction, yet our task is more complicated: we want to construct a given (large) partial order in the situation when the partial order is given us step by step, one vertex at each step, by an adversary (or our enemy). Cuts do not suffice here, but multicuts do (compare also [11]).

## 3. $\mathcal{V}^*$

In this section we present the partial order of finite sets of vectors $\mathcal{V}^*$. This structure corresponds more closely to graph homomorphisms. In some way it presents a dual structure to the $\mathcal{MC}$.

We will consider nonempty 0–1 vectors $\vec{v}$ of finite length. The length of the vector $\vec{v}$ will be denoted as $|\vec{v}|$. The $n$th coordinate of $\vec{v}$ will be denoted as $\vec{v}_n$.

For two vectors $\vec{u}$ and $\vec{v}$ we write that $\vec{u} \leq \vec{v}$ iff $|\vec{u}| \geq |\vec{v}|$ for each $n \leq |\vec{v}|$ holds $\vec{u}_n \geq \vec{v}_n$. (I.e. when $|\vec{u}| \geq |\vec{v}|$ and in the overlapping part the elements of $\vec{u}$ are greater or equal to corresponding elements in $\vec{v}$.)

**Definition 3.1.** Denote by $\mathcal{V}^*$ the class of all finite sets of vectors of any length.

For $U, V \in \mathcal{V}^*$, we put $U \leq_{\mathcal{V}^*} V$, if for each $\vec{u} \in U$ there exists $\vec{v} \in V$ such that $\vec{u} \leq \vec{v}$.

$\leq_{\mathcal{V}^*}$ is obviously the partial order on $\mathcal{V}^*$.

**Theorem 3.1.** *Structure $\mathcal{V}^*$ is a countable universal partially ordered set.*

**Proof.** We construct an on-line embedding of a finite partially ordered set $(A, \leq_A)$. To each $x \in A$ we assign a unique positive integer $t(x)$ (creation time). Additionally we assign each $x \in A$ vector $\vec{v}(x) = (v_1, v_2, \ldots, v_{t(x)})$ where $v_i = 1$ iff $x \leq y$ and $t(y) = i \leq t(x)$. To every $x \in A$ assign a set of vectors $\phi_{\mathcal{V}^*}(x) \in \mathcal{V}^*$ as follows

$$\phi_{\mathcal{V}^*}(x) = \{\vec{v}(y); y \in A, t(y) \leq t(x), y \leq_A x\}.$$

**Example 3.1.** The partial order from Fig. 2 will be represented as follows:

$$\begin{array}{ll}
\vec{v}(1) = (1), & \phi_{\mathcal{V}^*}(1) = \{(1)\} \\
\vec{v}(2) = (0, 1), & \phi_{\mathcal{V}^*}(2) = \{(1), (0, 1)\} \\
\vec{v}(3) = (0, 1, 1), & \phi_{\mathcal{V}^*}(3) = \{(0, 1, 1)\} \\
\vec{v}(4) = (1, 1, 1, 1), & \phi_{\mathcal{V}^*}(4) = \{(1, 1, 1, 1)\}.
\end{array}$$

We prove that $\phi_{\mathcal{V}^*}\colon A \to \mathcal{V}^*$ is an embedding. We proceed by induction over $t(x)$ analogously as in the above proof of Theorem 2.1.

Consider step $n$. Let $x \in A$, $t(x) = n$ and $y$ any element of $A$, where $t(y) < t(x)$.

1. Assume $x \leq_A y$. We prove $\phi_{\mathcal{V}^*}(x) \leq_{\mathcal{V}^*} \phi_{\mathcal{V}^*}(y)$:

Consider arbitrary $x' \in A$ such that $\vec{v}(x') \in \phi_{\mathcal{V}^*}(x)$. From the definition of $\phi_{\mathcal{V}^*}$ it follows easily that $x' \leq_A x$. To prove the inequality $\phi_{\mathcal{V}^*}(x) \leq_{\mathcal{V}^*} \phi_{\mathcal{V}^*}(y)$ it suffices to find the vector $\vec{u} \in \phi_{\mathcal{V}^*}(y)$ such that $\vec{v}(x') \leq \vec{u}$. We examine two possibilities:

If $t(x') < t(y)$, then $\vec{v}(x') \in \phi_{\mathcal{V}^*}(y)$ follows directly from the definition of $\phi_{\mathcal{V}^*}$ and inequality $x' \leq_A y$.

In the case $t(x') > t(y)$ put $t = t(y)$ and consider any $z \in A$ such that $\vec{v}(y)_{t(z)} = 1$. We have inequalities $x' \leq_A x \leq_A y \leq_A z$. Further $t(z) \leq t(y) < t(x')$ and thus $\vec{v}(x')_{t(z)} = 1$. Consequently $\vec{v}(x') \leq \vec{v}(y)$.

2. Assume $\phi_{\mathcal{V}^*}(x) \leq_{\mathcal{V}^*} \phi_{\mathcal{V}^*}(y)$. We prove $x \leq_A y$:

There exist $z, \vec{v}(z) \in \phi_{\mathcal{V}^*}(y)$ such that $\vec{v}(x) \leq \vec{v}(z)$. We have $|\vec{v}(z)| \leq |\vec{v}(x)|$ and thus $t(z) \leq t(x)$. The vector $\vec{v}(z)$ has on the $t(z)$th position 1 and so the $t(z)$th position of $\vec{v}(x)$ is equal to 1. From the definition of $\vec{v}(x)$ we get the inequality $x \leq_A z$. From the definition of $\phi_{\mathcal{V}^*}(y)$ we get $z \leq_A y$ and thus $x \leq_A y$. $\square$

Note the similarity of this proof and the above proof of Lemma 2.2. In fact we can derive Theorem 3.1 from Lemma 2.2 as follows: for a finite set $x$ of positive integers let $\chi(x)$ be a characteristic vector of $A$ (of length $\max x$).

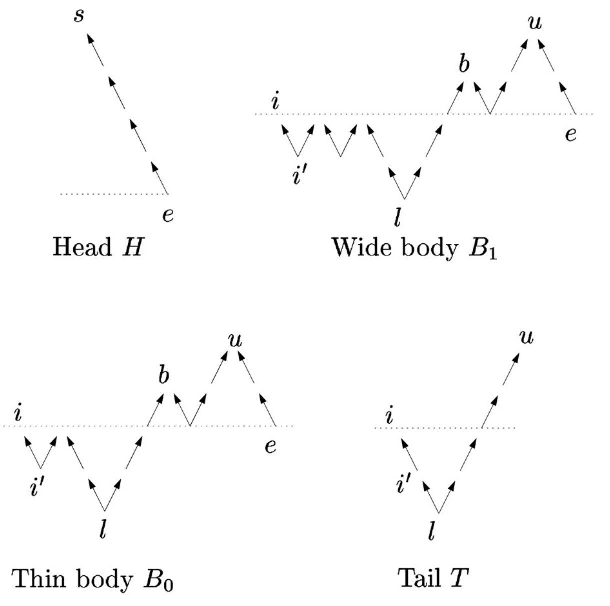


Fig. 3. Paths used to construct multipedes.

For $X \in \mathcal{MC}$ put $\overline{\chi}(X) = \{\chi(x); x \in X\}$. The mapping of $\overline{\chi}$ is 1–1 and it satisfies $X \leq_{\mathcal{MC}} Y$ iff $\overline{\chi}(X) \geq_{\mathcal{V}^*} \overline{\chi}(Y)$ and thus $\overline{\chi}$ is an embedding of $\mathcal{MC}$ into the dual order of $\mathcal{V}^*$.

## 4. Universality of $\mathcal{P}^*$

In this section we will construct the embedding of $\mathcal{V}^*$ into finite disjoint unions of finite oriented paths ordered by graph homomorphisms. This partial order will be denoted by $(\mathcal{P}^*, \leq_{\mathcal{P}^*})$.

First, we assign to every 0–1 vector $\vec{v}$ an oriented path $M(\vec{v})$.

The oriented paths $M(\vec{v})$, called *multipedes*, will be constructed by a concatenation of paths defined by Fig. 3. Some vertices of the paths are labeled to be easily referred to in the text and thus the sets of vertices of individual graphs are partly overlapping.

Put $t = |\vec{v}|$. The path $M(\vec{v})$ is constructed as concatenation $M_0 M_1 \ldots M_{t+1}$ of paths defined as follows

$$\begin{aligned} M_0 &= H, \\ M_n &= B_0 \qquad \text{iff } 1 \leq n \leq t \text{ and } \vec{v}_n = 0, \\ M_n &= B_1 \qquad \text{iff } 1 \leq n \leq t \text{ and } \vec{v}_n = 1, \\ M_{t+1} &= T. \end{aligned}$$

In order to allow explicit references of individual vertices of path $M(\vec{v})$ we define the vertices of $M(\vec{v})$ to be pairs $(n, v)$ where $0 \leq n \leq t+1$ and $v$ is a vertex of $M_n$ different

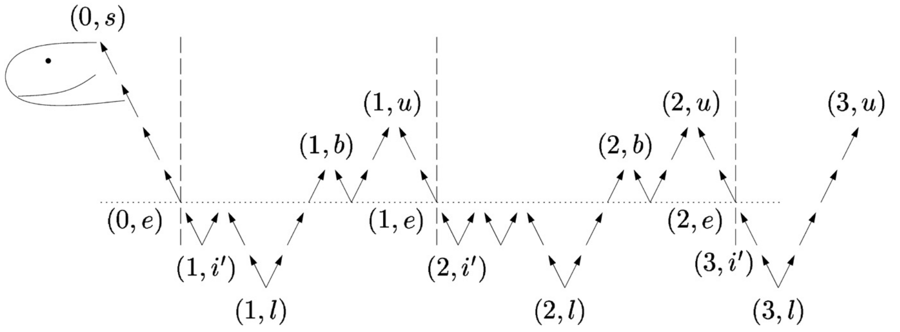


Fig. 4. Multipede $M(0, 1)$.

from $i$ (because the vertex $e$ of $M_{n-1}$ is unified with the vertex $i$ of $M_n$). Formally we can define $M(\vec{v})$ as follows:

The vertices of graph $M(\vec{v})$ are pairs $(n, u)$, where $0 \le n \le |\vec{v}|, u \in V(M_n)$ and $u \ne i$.

There is an edge from $(n, u)$ to $(n, u')$ if and only if there is an edge from $u$ to $u'$ in $M_n$. There is an edge from $(n + 1, i')$ to $(n, i)$ for each $0 \le n \le t$.

**Example 4.1.** For the vector (0,1) the resulting multipede is shown in Fig. 4.

For a set $\vec{V}$ of vectors put $\phi_{\mathcal{P}^*}(\vec{V}) = \{M(\vec{v}); \vec{v} \in \vec{V}\}$. We prove

**Theorem 4.1.** *$\phi_{\mathcal{P}^*}$ is an embedding of $\mathcal{V}^*$ into $\mathcal{P}^*$.*

**Proof.** $\phi_{\mathcal{P}^*}$ is clearly injective. The embedding property will follow from embedding of singleton sets:

**Claim 4.1.** $\vec{u} \le \vec{v}$ iff $M(\vec{u}) \le_{\mathcal{P}^*} M(\vec{v})$.

**Proof.** Assume that $f : M(\vec{u}) \to M(\vec{v})$ is a path homomorphism. We shall prove that the multipedes $M(\vec{u})$, $M(\vec{v})$ are chosen so that $f$ corresponds to $\vec{u} \le \vec{v}$. First, we make several observations:

1. $f(0, s) = (0, s)$.

   This follows directly from the fact that the only monotonic subpath of length 5 of $M(\vec{u})$ is between vertices $(0, s)$ and $(1, i')$.
2. $f(n, l) = (n', l)$.

   This follows from the fact that each homomorphism preserves algebraic distances. It follows from 1 that the algebraic distance of any vertex $x$ in $M(\vec{u})$ to $(0, s)$ must be equivalent to the algebraic distance of $f(x)$ to $(0, s)$ in $M(\vec{v})$. Thus $f$ preserves the levels of vertices. There are no other vertices at level $-6$.
3. $f(n, l) = (n', l) \Longrightarrow f(n, b) = (n', b)$ for each $n \le |\vec{u}|,\ n' \le |\vec{v}|$.

   Similarly to 2, there no other vertices at level $-3$ in $M(\vec{v})$ whose distance from $(n', l)$ is at most 3.
4. $f(n, l) = (n', l) \Longrightarrow f(n, u) = (n', u)$ for each $n \le |\vec{u}|,\ n' \le |\vec{v}|$.

   $f(n, b) = (n', b)$ follows from 3. There no other vertices in $M(\vec{v})$ at level $-2$ whose distance from $(n', l)$ is at most 3.

5. $f(n,l) = (n',l) \Longrightarrow f(n+1,i') = (n'+1,i')$ for each $n \le |\vec{u}| - 1$, $n' \le |\vec{v}| - 1$.

   From 4 we get that $f(n,u) = (n',u)$. The distance of $(n,u)$ from $(n+1,i')$ in $M(\vec{u})$ is 3. $(n'+1,l)$ is the only vertex having level $-5$ and whose distance from $(n',u)$ in $M(\vec{v})$ is at most 3.

6. $f(n,l) = (n',l) \Longrightarrow f(n+1,l) = (n'+1,l)$ for each $n \le |\vec{u}| - 1$, $n' \le |\vec{v}| - 1$.

   It follows from 5 that $f(n,i') = (n',i')$. The distance of $(n+1,i')$ from $(n+1,l)$ in $M(\vec{u})$ is at most $r$. $(n'+1,l)$ is the only vertex having level $-6$ and whose distance from $(n'+1,i')$ in $M(\vec{v})$ is at most 5.

It follows from 1 that $f(1,l) = (1,l)$. From 5 we get that $f(n,l) = (n,l) \Longrightarrow f(n+1,l) = (n+1,l)$ for each $n \le |\vec{v}| - 1$. By induction $f(n,l) = (n,l)$ for $n \le |\vec{v}|$. We also have that $|\vec{u}| \ge |\vec{v}|$. It is easy to see that $u_n \ge v_n$ for each $n \le |\vec{v}|$ (the distances of $(n,l)$ and $(n+1,l)$ in $M(\vec{v})$ must be shorter or equal to the distances of the corresponding lowest vertices in $M(\vec{u})$). It follows that the existence of a homomorphism $f\colon M(\vec{u}) \to M(\vec{v})$ implies that $\vec{u} \le \vec{v}$.

Now assume $\vec{u} \le \vec{v}$. Put $\vec{u} = (u_1, \dots, u_r)$, put $\vec{v} = (v_1, \dots, v_s)$. We have $r \ge s$ and $u_i \ge v_i$ for $i = 1, \dots, s$. Let $M(\vec{u})$ be constructed by concatenation of paths $M_0, M_1, \dots, M_{s+1}$ and $M(\vec{v})$ be constructed by concatenation of paths $M'_0, M'_1, \dots, M'_{t+1}$. We can construct the homomorphism $f : M(\vec{u}) \to M(\vec{v})$ as follows:

Put $f(n,x) = (n,x)$ for each $n$ such that $M_n = M'_n$. In the case $M_n = B_1$ and $M'_n = B_0$ we put $f(n,x) = (n,h(x))$, where $h$ is a homomorphism of $B_1$ to $B_0$ such that $h(i) = i$ and $h(e) = e$. If $n > r$ and either $M_n = B_0$ or $M_n = B_1$ we put $f(n,x) = (s+1,h'(x))$ where $h'$ is a homomorphism of $B_0 \to T$ (or $B_1 \to T$) such that $h'(i) = i$. Finally we put $f(r+1,x) = (s+1,x)$.

It is easy to verify that $f$ is a homomorphism $M(\vec{u}) \to M(\vec{v})$. □

## 5. Applications

The words over finite paths (i.e. elements of the class $\mathcal{P}^*$) have a very simple combinatorial structure and this in turn implies that very special classes of graphs are universal. Here we state some of the consequences of our main construction (Theorem 4.1).

### 5.1. *Planar graphs of degree* $\le 3$

We use the indicator technique ("arrow construction") which allows us to replace arcs of a graph by copies of a gadget ("indicator") in such a way that the (global) homomorphism properties are preserved, see [18, 21]. More precisely this can be done as follows:

Any graph $I$ with two distinguished vertices $a, b$ is called an *indicator* (we use the indicator defined by Fig. 5). Given a graph $G = (V,E)$ we denote by $G * (I,a,b)$ the following graph $(W,F)$:

$$W = (E \times V(I))/\sim.$$

Thus the vertices of $(V,E)$ are equivalence classes of the equivalence $\sim$. For a pair $(e,x) \in E \times V(I)$ its equivalence class will be denoted by $[e,x]$.

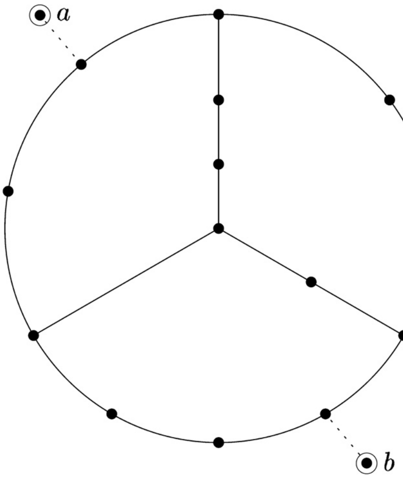


Fig. 5. $(I, a, b)$.

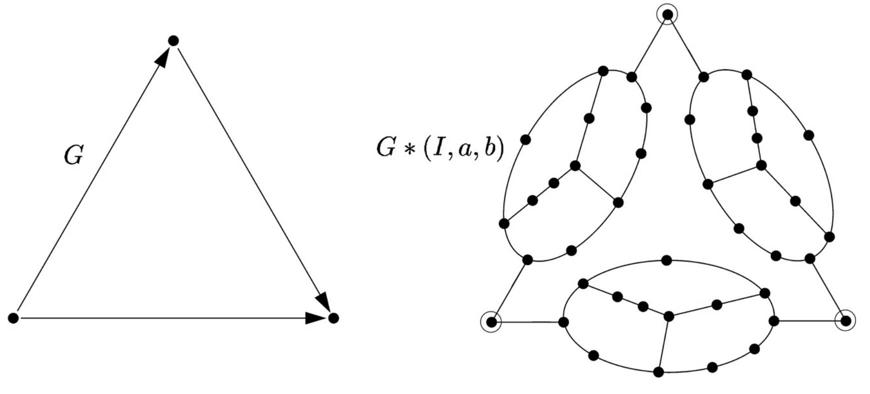


Fig. 6. Construction of $G * (I, a, b)$.

The equivalence $\sim$ is generated by the following pairs:

$$((x, y), a) \sim ((x, y'), a),$$
$$((x, y), b) \sim ((x', y), b),$$
$$((x, y), b) \sim ((y, z), a).$$

We put $\{[e, x], [e', x']\} \in F$ iff $e = e'$ and $\{x, x'\} \in E(I)$.
The indicator construction is schematically shown in Fig. 6.
We have the following properties:

**Claim 5.1.**

1. $P * (I, a, b)$ *is a planar graph with all its degrees* $\leq 3$ *for every path* $P$.

2. *If $f : P \to P'$ is a path homomorphism then the mapping $\phi(f)$ defined by*

$$\phi(f)[(u, v), x] = [(f(u), f(v)), x]$$

*is a homomorphism $\phi(P) \to \phi(P')$.*

3. *If $g : \phi(P) \to \phi(P')$ then there exists $f : P \to P'$.*

**Proof.** Only the last claim needs explanation. Put $I' = I - \{a, b\}$ (thus $I'$ is the main block of $I$). Observe that the only cycles in the graph $P * (I, a, b)$ of length $\leq 7$ belong to the set $\{[a, z]; z \in V(I')\}$ from an edge $e \in P$. In fact all non-trivial blocks of $P * (I, a, b)$ are isomorphic to $I'$. It is well known that $I'$ is rigid (see e.g. [18]). This in turn means that for any homomorphism $g : P * (I, a, b) \to P' * (I, a, b)$ there exists a mapping $f : V(P) \to V(P')$ such that for every edge $e = (x, y) \in E$ and $z \in V(I')$ holds $g([e, z]) = [(f(x), f(y)), z]$. This $f$ is a desired homomorphism $P \to P'$. (Note that this correspondence of $g$ and $f$ is not functorial; the graph $I$ fails to be rigid.) $\square$

Put $\phi(P) = P * (I, a, b)$. We have proved $P \to P'$ iff $\phi(P) \to \phi(P')$. Note that $\phi(P)$ is planar and that all degrees $\leq 3$. It is a graph theory routine to extend $\phi(P)$ to planar cubic graphs such that the following holds:

**Theorem 5.1.** *The class of all finite planar cubic graphs is universal.*

This implies Theorem 1.2.

### *5.2. Series parallel graphs*

We can use the indicator construction to obtain the following

**Theorem 5.2.** *For any $l$ the class of all series parallel graphs of girth $>l$ is universal.*

Fix $l \geq 2$. Theorem 5.2 is proved similarly as 5.1 by means of the indicator $I_l$ defined on Fig. 7. The vertices of $I_l$ are $a, b, c_1, c_2, c_3$ together with

$$\begin{aligned}
&v_{12}(1), \ldots, v_{12}(l-1), v'_{12}(1), \ldots, v'_{12}(l),\\
&v_{13}(1), \ldots, v_{13}(l-1), v'_{13}(1), \ldots, v'_{13}(l),\\
&v_{23}(1), \ldots, v_{23}(l-1), v'_{23}(1), \ldots, v'_{23}(l).
\end{aligned}$$

The edges of $I_l$ form pairs $\{a, v'_{12}(2)\}$, $\{v'_{23}(l-2), b\}$ and edges of paths joining vertices $c_1, c_2, c_3$:

$$\begin{array}{lll}
\{c_1, v_{12}\}, & \{c_1, v'_{12}\}, & \{c_1, v_{13}\},\\
\{v_{ij}(k), v_{ij}(k+1)\} & \text{for } 1 \leq i < j \leq 3,\ k = 1, \ldots, l-2, & \\
\{v'_{ij}(k), v'_{ij}(k+1)\} & \text{for } 1 \leq i < j \leq 3,\ k = 1, \ldots, l-2, & \\
\{c_1, v_{1i}(1)\}, & \{c_1, v'_{1i}(1)\} & \text{for } i = 2, 3,\\
\{c_2, v_{12}(l-1)\}, & \{c_2, v'_{12}(l)\}, & \\
\{c_2, v_{23}(1)\}, & \{c_2, v'_{23}(1)\}, & \\
\{c_1, v_{i3}(l-1)\}, & \{c_3, v'_{i3}(l)\} & \text{for } i = 2, 3.
\end{array}$$

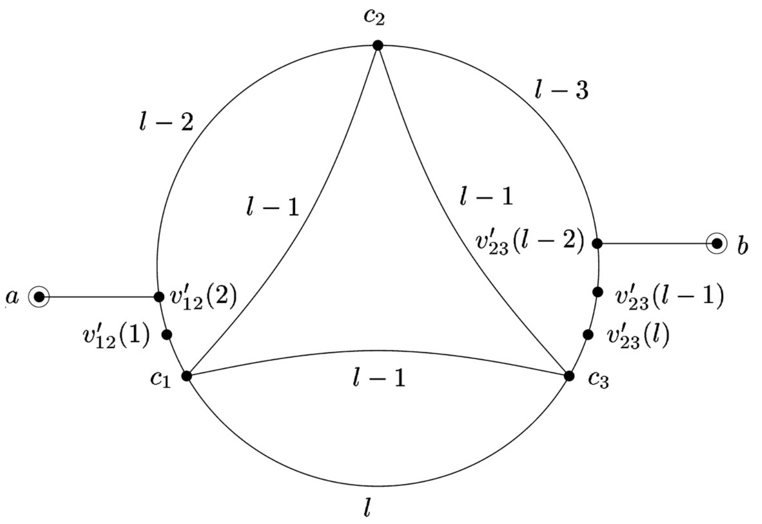


Fig. 7. $(I_l, a, b)$.

The graph $I_l$ has girth $>2l+1$. Put $I' = I - \{a, b\}$. $I'$ is not rigid but it is a core graph and it has no automorphism which maps $v'_{12}(2)$ to $v'_{23}(l-2)$. It follows that we may argue similarly as in the proof of Theorem 5.1. We omit the details.

## 5.3. *Connected graphs and oriented trees*

Examples of universal classes given by Theorems 5.1 and 5.2 are (highly) disconnected. However it is easy to modify our construction of $\mathcal{P}^*$ universality to obtain the following:

**Theorem 5.3.** *The class $\mathcal{T}$ of all finite oriented trees ordered by the existence of homomorphism is universal.*

**Proof.** Recall the main ingredients of the proof of Theorem 4.1. To every vector $\vec{v}$ we assigned a multipede $M(\vec{v})$ in such a way that $\vec{v} \le_{\mathcal{V}^*} \vec{v}'$ iff $M(\vec{v}) \to M(\vec{v}')$.

However note that the latter condition holds if and only if there exist a homomorphism $f : M(\vec{v}) \to M(\vec{v}')$ such that the initial vertex $(0, s)$ of $M(\vec{v})$ is mapped to the initial vertex of $M(\vec{v}')$. This follows from the definition of $M(\vec{v})$ (see Section 4). Thus given set $S = \{\vec{v}_i; i \in I\} \in \mathcal{V}^*$ we can consider the tree $T(M(\vec{v}_i); i \in I)$ which we get from the disjoint union

$$\sum_{\vec{v} \in S} M(\vec{v})$$

by identifying the initial vertices $(0, s)$ in all multipedes $M(\vec{v}_i)$, $i \in I$. $\square$

This solves a problem of [20]. Note that applying the indicator construction to the Theorem 5.2 we get e.g.

**Corollary 5.1.** *The class of all connected series parallel graphs of given girth is universal.*

Our trees have large degrees so we do not have a direct analgoue for graphs of bounded degree. In the companion paper [10] we strengthen our results and we prove even that the class of finite oriented paths $\mathcal{P}$ forms a universal class.

## 6. Concluding remarks

1. One can extend the representation of $\mathcal{MC}$ and $\mathcal{V}^*$ to large (proper class) partial orders, see [17]. However for our other results (locally planar graphs) the proper class universality is open. A related problem was asked by Babai [2]: is it true that on every (infinite) set there exists a rigid locally planar graph? (Recall: a graph is locally planar if any of its finite subgraph is planar.)

One can see easily that the following is a weaker version of this question.

**Problem 6.1.** Is it true that any partial order of any cardinality is representable by locally planar graphs.

In fact the problem is open even for a graph not containing a large complete minor.

2. Bounded degree graphs cannot be class universal as any component of a graph with finite degrees is countable; see [7] for results in this direction.

3. All our graph embedding theorems (i.e. universalities of $\mathcal{T}$, $\mathcal{P}^*$, of planar cubic graphs and of series parallel graphs) can be made functorial and yield an embedding of categories. As an example we formulate this for the case of planar graphs only: let $\mathcal{K}$ be the class of all finite planar graphs. Let $(A, \leq_A)$ be a countable partial order. Then there exists a mapping $\phi$ which assigns to every $x \in A$ a finite planar graph $\phi(x)$ such that $x \leq_A y$ iff there exists a homomorphism $\phi(x) \to \phi(y)$. Moreover, between any two graphs $\phi(x), \phi(y)$ there exists at most one homomorphism (particularly $\phi(x)$ is a rigid graph for any $x$).

4. Our embedding represents any partial order by bounded degree graphs. Particularly any semilattice is represented as a small category. It is an open problem to represent a semilattice by endomorphism of a (single) bounded degree graph (see [2]).

5. Our structures (paths, linear forests, trees, series parallel graphs) are easy from the algorithmic point of view: for any of them the existence of a homomorphism can be tested polynomially (see [5, 7, 20]).

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## References

[1] L. Babai, Automorphism groups of graphs and edge contraction, Discrete Math. 8 (1974) 13–22.
[2] L. Babai, A. Pultr, Endomorphism monoids and topological subgraphs of graphs, J. Combin. Theory Ser. B 28, 3 (1980) 278–283.
[3] P.J. Cameron, The random graph, in: R.L. Graham, J. Nešetřil (Eds.), The Mathematics of Paul Erdős, Springer Verlag, 1998, pp. 333–351.

[4] J.H. Conway, On numbers and games, in: London Math. Soc. Monographs, Academic Press, 1976.
[5] A.M.H. Gerards, Homomorphisms of graphs into odd cycles, J. Graph Theory 12, 1 (1998) 73–83.
[6] Z. Hedrlín, On universal partly ordered sets and classes, J. Algebra 11 (1969) 503–509.
[7] P. Hell, J. Nešetřil, X. Zhu, Duality and polynomial testing of tree homomorphisms, Trans. Amer. Math. Soc. 348 (1996) 1281–1297.
[8] P. Hell, J. Nešetřil, Groups and monoids of regular graphs and of graphs with bounded degree, Canad. J. Math. XXV 2 (1973) 239–251.
[9] R. Fraïssé, Theory of Relations, North-Holland, Amsterdam, 1953.
[10] J. Hubička, J. Nešetřil, Finite paths are universal, ITI Series 2003, vol. 129, Charles University, 2003.
[11] J. Hubička, J. Nešetřil, On homogeneous graphs and posets, KAM-DIMATIA Series, vol. 606, Charles University, 2003.
[12] J. Hubička, Ramsey properties of universal sets, Diploma Thesis, Charles University, 2002.
[13] D.E. Knuth, Surreal Numbers, Addison Wesley, 1974.
[14] B. Jónson, Universal relational systems, Math. Scand. 4 (1956) 193–208.
[15] J.B. Johnston, Universal infinite partially ordered sets, Proc. Amer. Math. Soc. 7 (1956) 507–514.
[16] M. Kac, G.-C. Rota, J.T. Schwartz, Indiscrete Thoughts, Birkhäuser, 1997.
[17] J. Nešetřil, On universality of set systems, KAM-DIMATIA Series, vol. 491, Charles University, 2000.
[18] J. Nešetřil, Aspects of structural combinatorics, Taiwanese J. Math. 3–4 (1999) 381–424.
[19] J. Nešetřil, C. Tardif, Duality theorems for finite structures (characterising gaps and good characterizations), J. Combin. Theory Ser. B 80 (2000) 80–97.
[20] J. Nešetřil, X. Zhu, Path homomorphisms, Proc. Cambridge Philos. Soc. (1996) 207–220.
[21] A. Pultr, V. Trnková, Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories, North-Holland, 1980.