THE UNIVERSAL FIELD OF FRACTIONS OF A SEMIFIR I. NUMERATORS AND DENOMINATORS

P. M. COHN

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Contents

Introduction

In Chapter 7 of [7] a criterion was given for a general ring to have a field of fractions, together with a construction of the field, in which the elements were obtained as equivalence classes of systems of matrix equations (for a simplification of the existence proof see Malcolmson [15]). Although the construction is quite explicit, the resulting fields are not easy to work with, on account of the 'determinantal sums' which enter into most calculations. But there is a special case in which the construction is more straightforward; a *semifir* R always has a universal field of fractions U, and the construction of U can be accomplished without the explicit use of determinantal sums. There are many questions that can be asked about U and our purpose in this paper and those that follow is to answer some of the easier ones.

Perhaps the most basic problem is to find a normal form for the elements of U, but one should not expect too much. Some guide as to what we can expect is provided by the commutative case. For a moment let R be a commutative integral domain and U its field of fractions. Then (as is well known) every $p \in U$ can be written in the form

$$(1) \qquad p = ab^{-1}, \quad \text{for } a, b \in R,\ b \neq 0,$$

and $p = ab^{-1} = a_1 b_1{}^{-1}$ if and only if $b_1 a = a_1 b$. This is not exactly what one usually understands by a normal form; what is more, there may not exist a particularly well suited pair a, b to represent p. To give an example, in the coordinate ring of the quadric $xt = yz$ (the simplest non-UFD) we have $x/y = z/t$, but there is no representation of this fraction which can be used for all specializations: x/y fails if we put $x = y = 0$, while z/t fails if we put $z = t = 0$. In a principal ideal domain, or more generally, a Bezout domain, the situation is rather better. Here we can find a 'reduced' representation (1), i.e. we can choose a, b in (1) to be coprime, and then a, b are unique up to a unit factor. Moreover, in this case a, b are comaximal, i.e. R contains c, d such that $ad - bc = 1$, and this shows that there is no homomorphism of R into a field which maps both a and b to 0, so b may be regarded as the 'universal denominator' of the element ab^{-1}.

Our first task will be to develop a parallel when R is a semifir (a notion which reduces to Bezout domain in the commutative case). In this case each element p of U is obtained as a component of the solution u of a system

$$(2) \qquad Au = a,$$

where A is a full matrix and a a column vector over R. In order to compare different systems defining p we need to examine the left R-module determined by the matrix (A, a).

The relation between matrices and the modules defined by them is examined for quite general rings in § 2. The basic notion is 'stable association', already implicit in the work of Fitting [14] and considered for elements (i.e. 1×1 matrices) in Chapter 3 of [7] under the name 'similarity'. A number of equivalent conditions for stable association of matrices are given that are valid in quite general rings; but in semifirs we shall find that the description of stably associated matrices can be further simplified. The results of this section are restated in terms of modules in § 3, which deals with prime modules, a generalization of the torsion modules over firs described in Chapter 5 of [7].

In § 4 we take a semifir R and its universal field of fractions U and for any $p \in U$ compare the different systems of equations which determine p. The precise results are somewhat technical (Theorem 4.7); they may be summed up by saying that we can pass from one system for p to any other by a series of (at most six) basic operations on the matrix of the system. We shall consider mainly systems with an $m \times (m+1)$ matrix; the first m columns form the *numerator* and the last m the *denominator* of the system (cf. [7, p. 251]), and under suitable finiteness conditions (holding, for example, in a fir) there is a 'reduced' system for p whose numerator and denominator are determined up to stable association. More precisely, the whole $m \times m+1$ matrix is determined up to 'stable biassociation'. The least value of m is the *depth* of p; this is a numerical invariant of p which has no analogue in the commutative case. In fact, in a commutative domain it is always 1 and in a right Ore domain at most 2. In a sequel to this paper (Part II) we shall examine the depth in detail and in particular show that in the universal field of fractions of a free algebra the depth is unbounded.

We now turn to applications. In the first place we prove a result on 'universal denominators' (Theorem 5.2). In essence this states that for a fir R with universal field of fractions U, if $p \in U$ and A is a reduced matrix describing p, then A can also be used to describe p in any other field formed from R (i.e. any specialization of U) in which p is defined. Secondly we answer a question raised by W. S. Martindale (in conversation): do rational identities in the free field hold universally? We shall find that such identities hold in any weakly finite ring (Theorem 5.5). The proof uses a refinement of the method of § 4, which associates a matrix to each formal expression in the generators, not merely to each element of the free field. It may be worth remarking that the result depends on nothing after Cramer's rule (see § 4, formula (3)).

In § 6 we examine the R-module structure of U and show that U contains a local system of prime submodules which can be described intrinsically. This observation can be used to give an independent construction of U as a direct limit of R-modules, as will be shown in a more general context in [4].

A major application of the methods developed here is to the determination of centralizers and normalizers in the universal field of fractions of a free algebra. This will form the subject of Part III, to follow (which to a large extent is independent of II).

Although some results of [7] are used in isolated places, most of the paper can be read independently. To facilitate the reader's task, essential notation is collected in § 1, while any results used are usually summarized in the text.

The three parts (of which this is the first) started out as one paper. I am most grateful to G. M. Bergman who as a referee with seemingly unending patience and enthusiasm provided me over a period of four years with over 100 pages of comments, correcting errors and suggesting illuminating examples as well as further results and problems; it is hoped that the readability has improved as a result. My thanks also go to W. Dicks for reading an early draft and suggesting a number of improvements, and to G. Révész for pointing out some inaccuracies.

1. *Notation and terminology*

This section explains the conventions used and recalls some terms from [7] that may be unfamiliar; other terms are explained on their first occurrence.

Throughout the paper all rings have a unit-element 1, which is inherited by subrings, preserved by homomorphisms, and acts unitally on modules. If R is a ring, the set of all $m \times n$ matrices over R is denoted by ${}^mR^n$, and we write R^n for ${}^1R^n$ and ${}^mR$ for ${}^mR^1$. The ring of all $n \times n$ matrices over R is denoted by R_n and the set of all invertible $n \times n$ matrices is written $\mathbf{GL}_n(R)$. To save space, columns are sometimes written as rows, with a superscript T to indicate transposition.

An element with a two-sided inverse is called a *unit*. An $m \times n$ matrix A is called a *left zerodivisor* if $Ab = 0$ for some $b \in {}^nR$, with $b \neq 0$; in the contrary case A is a *left non-zerodivisor*. *Right zerodivisors* are defined similarly. The set of non-zero elements of a ring R is denoted by R^*. If R^* contains 1 and is closed under multiplication, R is called an *integral domain* (as in the commutative case). We remind the reader that fields need not be commutative; the prefix 'skew' is sometimes added to stress this fact. The term *k-algebra* has its usual meaning (cf. for example, [11, Chapter 3]); since k is always a commutative field, a non-zero k-algebra is just a ring whose centre contains a copy of k. If R is any ring, by an *R-ring* we understand a ring S with a homomorphism λ: $R \to S$; when R is stated to be a k-algebra, λ is understood to be a k-algebra homomorphism. An R-ring which is also a field is called an *R-field*, *epic* in case it is generated as a field by the image of R.

A ring is said to have *IBN* (*invariant basis number*) if every invertible matrix over it is square, and to be *weakly finite* if for any square matrices A, B of the same order, $AB = I$ implies $BA = I$. In terms of modules, IBN means that the rank of a free R-module is uniquely determined, and R is weakly finite if and only if no free R-module of finite rank is isomorphic to a proper direct summand of itself. Either of these characterizations shows that every weakly finite ring which is non-trivial (that is, not equal to 0) has IBN (the converse is false, cf. [5]).

From [7] we recall that a *semifir* is a ring with IBN in which all finitely generated right (or equivalently, left) ideals are free, as modules over the ring. It is well known (and easily verified) that every semifir is weakly finite. In Chapter 1 of [7] a number of equivalent properties defining semifirs are given, of which the most useful for us here is the following: R is a semifir if and only if $R \neq 0$ and for $x \in R^n$, $y \in {}^nR$ (where $n \geqslant 1$), such that $xy = 0$, there exists $P \in \mathbf{GL}_n(R)$ such that $xP.P^{-1}y = 0$ is a *trivial relation*, i.e. for each $i = 1, \ldots, n$, the ith component of either xP or $P^{-1}y$ is 0.

By a *fir* (free ideal ring) one understands a ring with IBN in which all right ideals and all left ideals are free. Thus every fir is a semifir (but not conversely); moreover, a fir is always *atomic*, i.e. every element not zero or a unit is a product of *atoms* (unfactorable elements).

As examples of firs we have, besides the principal ideal domains, the free algebras over a field: $k\langle x, y, \ldots\rangle$, and coproducts of skew fields over a common subfield. More generally, any coproduct of firs over a common subfield is a fir (cf. [**1**, **6**, **10**]), for example, if K is a skew field which is a k-algebra, then the free K-ring on a set X (always understood to be a k-algebra) is a fir, for we may regard it as the coproduct of K and $k\langle X\rangle$ over k; it will be written $K_k\langle X\rangle$. An example of a semifir which is not a fir is the free power-series ring over a field: $k\langle\langle X\rangle\rangle$, where $|X| > 1$.

2. *The matrix of definition of a module*

Let R be a ring and M a finitely presented left R-module. If

$$R^m \xrightarrow{\alpha} R^n \longrightarrow M \longrightarrow 0$$

is a presentation of M, then M is determined up to isomorphism by the $m \times n$ matrix A describing the mapping α, and every $m \times n$ matrix defines a finitely presented left R-module in this way. We note that α is injective if and only if A is a right non-zerodivisor. Let us call two matrices over R *left equivalent* if the left modules they define are isomorphic; *right equivalent* matrices are defined correspondingly.

Our aim in this section is to give an explicit description of equivalence of matrices, viz. stable association, which in essence goes back to Fitting [**14**]. In weakly finite rings, this condition can be simplified to the notion of 'comaximal relation' and in semifirs this can be further simplified to 'coprime relation'.

To obtain a criterion for equivalence we shall use the following generalization of Theorem 3.3.2 of [**7**]. It is essentially a formulation of Schanuel's lemma; for the form given here I am indebted to W. Dicks (cf. also [**2**]). Two maps α: $Q \to P$, α': $Q' \to P'$ between R-modules are said to be *associated* if there is a commutative square

$$\begin{array}{ccc} Q & \xrightarrow{\alpha} & P \\ \downarrow & & \downarrow \\ Q' & \xrightarrow{\alpha'} & P' \end{array}$$

where the vertical maps are isomorphisms. If there are R-modules S, T such that $\alpha \oplus 1_S$ is associated to $1_T \oplus \alpha'$, then α, α' are said to be *stably associated.* For maps between free modules, represented by matrices over R, this reduces to the definition given in [**9**] or [**11**], as we see from Corollary 1 below.

THEOREM 2.1. *Let R be a ring, and let α: $Q \to P$ and α': $Q' \to P'$ be two homomorphisms of left R-modules. Then the following two conditions are equivalent:*

(a) *there is an isomorphism μ: $Q \oplus P' \to P \oplus Q'$ of the form*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \textit{with inverse} \quad \begin{pmatrix} \delta' & \beta' \\ \gamma' & \alpha' \end{pmatrix};$$

(b) *α is stably associated to α'.*

Further, these conditions imply
(c) $\operatorname{coker}\alpha \cong \operatorname{coker}\alpha'$,
and if P, P' are projective and α, α' are injective, then the converse holds.

Proof. (a) $\Rightarrow$ (b). Take $S = P'$, $T = P$, and the vertical isomorphisms

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix}\begin{pmatrix}1 & -\beta'\\ 0 & 1\end{pmatrix}.$$

(b) $\Rightarrow$ (a). If

$$\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}p & q\\ r & s\end{pmatrix} = \begin{pmatrix}x & y\\ z & t\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \alpha'\end{pmatrix},$$

where

$$\begin{pmatrix}p & q\\ r & s\end{pmatrix}^{-1} = \begin{pmatrix}s' & q'\\ r' & p'\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}x & y\\ z & t\end{pmatrix}^{-1} = \begin{pmatrix}t' & y'\\ z' & x'\end{pmatrix},$$

then it is easily checked that

$$\begin{pmatrix}\alpha & y\\ -r' & p't\end{pmatrix}^{-1} = \begin{pmatrix}pt' & -q\\ z' & \alpha'\end{pmatrix}.$$

(b) $\Rightarrow$ (c). This is clear. Now let P, P' be projective, let α, α' be injective, and assume (c). Then there exist maps $\gamma\colon P' \to P$ and $\beta'\colon P \to P'$ making the following diagram commutative, and γ induces $-\gamma'\colon Q' \to Q$ and β' induces $-\beta\colon Q \to Q'$:

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r, "\alpha"] \arrow[d, shift right, "-\beta"'] & P \arrow[r, "\varphi"] \arrow[d, shift right, "\beta'"'] & \operatorname{coker}\alpha \arrow[r] \arrow[d, "\cong"] & 0\\
0 \arrow[r] & Q' \arrow[r, "\alpha'"] \arrow[u, shift right, "-\gamma'"'] & P' \arrow[r, "\varphi'"] \arrow[u, shift right, "\gamma"'] & \operatorname{coker}\alpha' \arrow[r] & 0
\end{tikzcd}$$

Further, $(1_P - \beta'\gamma)\varphi = 0$, whence $1 - \beta'\gamma = \delta'\alpha$ for some $\delta'\colon P \to Q$, because P is projective. Likewise $(1 - \gamma\beta')\varphi' = 0$, whence $1 - \gamma\beta' = \delta\alpha'$ for some $\delta\colon P' \to Q'$. Now it is easily verified that

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\colon Q \oplus P' \to P \oplus Q' \text{ has inverse } \begin{pmatrix}\delta' & \beta'\\ \gamma' & \alpha'\end{pmatrix}\colon P \oplus Q' \to Q \oplus P'.$$

The proof that (a) $\Leftrightarrow$ (b) shows that the definition of stable association can be made a little more precise:

COROLLARY 1. *If $\alpha\colon Q \to P$ is stably associated to $\alpha'\colon Q' \to P'$, then $\alpha \oplus 1_{P'}$ is associated to $\alpha' \oplus 1_P$. Hence two matrices $A \in {}^rR^m$ and $A' \in {}^sR^n$ are stably associated, qua maps, if and only if*

$$\begin{pmatrix}A & 0\\ 0 & I_n\end{pmatrix} \quad \textit{is associated to} \quad \begin{pmatrix}I_m & 0\\ 0 & A'\end{pmatrix}.$$

In terms of matrices we get the following criteria for equivalence by taking P, P', Q, Q' to be free:

COROLLARY 2. *Let $A \in {}^rR^m$, $A' \in {}^sR^n$ be two matrices which are right non-zerodivisors.*

Then the following are equivalent:

(a) *there exists* $M = \begin{pmatrix} A & * \\ * & * \end{pmatrix} \in {}^{r+n}R^{s+m}$ *with inverse of the form* $\begin{pmatrix} * & * \\ * & A' \end{pmatrix}$;

(b) A *and* A' *are stably associated;*

(c) A *and* A' *are left equivalent.*

Moreover, two non-zerodivisors are left equivalent if and only if they are right equivalent.

Here the last part follows by the symmetry of (b).

We note that if R has IBN, then $A \in {}^rR^m$ and $A' \in {}^sR^n$ can be stably associated only if $r+n = s+m$, that is if $n-s = m-r$. This is an expression of the well-known fact that for a ring with IBN the *characteristic* of a module M defined by a right non-zerodivisor $A \in {}^rR^m$, viz. $m-r$, is independent of the choice of presentation, a consequence of Schanuel's lemma. Let us define the *characteristic* of an $r \times m$ matrix as $m-r$. Then we can say that in a ring with IBN, left equivalent matrices which are right non-zerodivisors have the same characteristic. More generally, any stably associated matrices (over a ring with IBN) have the same characteristic. We observe that a non-zerodivisor matrix A of characteristic t defines a left module of characteristic t and a right module of characteristic $-t$; here we shall be concerned mainly with left modules.

In a weakly finite ring the notion of equivalence of matrices can be simplified as follows. Consider a relation

$$AB' = BA' \tag{1}$$

between matrices. This can also be written

$$(A \quad B)\begin{pmatrix} -B' \\ A' \end{pmatrix} = 0.$$

We shall call A, B *right comaximal* if the matrix $(A \quad B)$ has a right inverse, and we shall call A', B' *left comaximal* if $\begin{pmatrix} A' \\ B' \end{pmatrix}$ has a left inverse. Now (1) is called a *comaximal* relation if A, B are right comaximal and A', B' are left comaximal.

PROPOSITION 2.2. *Let R be any ring and let $A \in {}^rR^m$, $A' \in {}^sR^n$. Then the following two conditions are equivalent:*

(a) A, A' *satisfy a comaximal relation* (1);

(b) *there is an* $(r+n) \times (s+m)$ *matrix* $\begin{pmatrix} A & * \\ * & * \end{pmatrix}$ *with a right inverse of the form* $\begin{pmatrix} * & * \\ * & A' \end{pmatrix}$.

In particular, (a) *and* (b) *hold whenever*

(c) A *and* A' *are stably associated,*

and in a weakly finite ring (a)–(c) *are equivalent for matrices of the same characteristic.*

Proof. If A, A' satisfy a comaximal relation (1), say

$$AD' - BC' = I, \quad DA' - CB' = I, \tag{2}$$

then on writing

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} D' & -B' \\ -C' & A' \end{pmatrix}, \tag{3}$$

we have $MN = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix}$. Hence M has the right inverse

$$\begin{pmatrix} D' & -B' \\ -C' & A' \end{pmatrix}\begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} = \begin{pmatrix} * & * \\ * & A' \end{pmatrix}.$$

Conversely, if N in (3) is a right inverse of M, then (1) and (2) hold, and hence (1) is a comaximal relation. This shows that (a) $\Leftrightarrow$ (b).

Now (c) $\Rightarrow$ (b) by Theorem 2.1, Corollary 2; and (b) $\Rightarrow$ (c) under the given conditions, because when $m-r = n-s$, then $r+n = s+m$, and for a square matrix over a weakly finite ring, any right inverse is an inverse. Now another application of Theorem 2.1, Corollary 2, completes the proof.

For later use we note the explicit form of the relation of stable association between A and A':

(4)
$$\begin{pmatrix} A & B \\ 0 & I \end{pmatrix}\begin{pmatrix} D' & -B' \\ -C' & A' \end{pmatrix} = \begin{pmatrix} I & 0 \\ -C' & A' \end{pmatrix}.$$

By Corollary 2 of Theorem 2.1 we obtain the

COROLLARY. *In a weakly finite ring R, two matrices A and A' of the same characteristic which are right non-zerodivisors are left equivalent if and only if there is a comaximal relation*

$$AB' = BA'.$$

In a semifir the relation of equivalence can be still further simplified. We recall that for any matrix A or homomorphism α between free modules (over any ring R) the *inner rank* is defined as the least integer r for which A can be written as a product of a matrix of r columns by a matrix with r rows, or in terms of α, the least r for which α can be factored through R^r (cf. [7, Chapter 5]). Over a skew field this reduces to the usual rank. We denote the inner rank of A by $\operatorname{rk} A$ and note that for any $m \times n$ matrix A,

$$\operatorname{rk} A \leqslant \min\{m, n\}.$$

If $\operatorname{rk} A = m$, A is called *left full*; if $\operatorname{rk} A = n$, A is *right full*; and if A is left and right full, it is said to be *full*. Thus any full matrix is necessarily square, and if an $m \times n$ matrix is left full, then $m \leqslant n$.

An $m \times n$ matrix A over a ring R is said to be *left prime* if in any equation

$$A = PQ, \quad \text{where } P \in R_m,\ Q \in {}^mR^n,$$

P has necessarily a right inverse; *right prime* matrices are defined correspondingly. We note that if A is left prime, then so is (A, B), where B is any matrix with the same number of rows as A. Any right invertible matrix is left prime, for if $A = PQ$ and $AB = I$, then $P.QB = I$, so P is right invertible. Over a weakly finite ring any left prime matrix is left full, for if A is $m \times n$ and of inner rank less than m, then $A = PQ$, where $P \in R_m$ and P has a column of zeros; hence P is not a unit and so has no right inverse. Therefore A cannot be left prime. Thus over a weakly finite ring we have the implications

$$\text{right invertible} \Rightarrow \text{left prime} \Rightarrow \text{left full}.$$

It is clear that the only matrices that are both left and right prime are units. If a matrix is full and a non-unit, but cannot be written as a product of two square non-units, it is called an *atom.*

A relation (1) between matrices is called *coprime* if (A, B) is left prime and $\begin{pmatrix} B' \\ A' \end{pmatrix}$ is right prime. Our object will be to clarify the connexions between coprime and comaximal relations over a semifir. First, we prove an auxiliary result on the reduction of relations over a semifir.

LEMMA 2.3. *Let R be a semifir, and let $P \in {}^rR^t$, $Q \in {}^tR^s$ such that $PQ = 0$. Then*

(i) *there exist $T \in \mathbf{GL}_t(R)$ and an integer t', with $0 \leqslant t' \leqslant t$, such that the first t' rows of TQ are zero and all columns of PT^{-1} after the first t' are zero.*

Moreover, if P is left full and Q is right full, then

(ii) $r+s \leqslant t$,

and if equality holds in (ii), *then*

(iii) *there exist $P_1 \in R_r$, $Q_1 \in R_s$ such that $P = P_1P'$, $Q = Q'Q_1$, and $P'Q' = 0$ is a coprime relation.*

In that case,

(iv) *$T \in \mathbf{GL}_t(R)$ can be chosen so that*

$$P'T^{-1} = (I_r \quad 0) \quad \textit{and} \quad TQ' = \begin{pmatrix} 0 \\ I_s \end{pmatrix}.$$

Further,

(v) *there exist $P'' \in {}^sR^t$ and $Q'' \in {}^tR^r$ such that $\begin{pmatrix} P' \\ P'' \end{pmatrix}$ and $(Q'' \quad Q')$ are mutually inverse.*

Proof. The columns of P span a submodule of ${}^rR$. Since R is a semifir, there exists $T \in \mathbf{GL}_t(R)$ such that the first t' columns of PT^{-1} are linearly independent over R, while the remaining columns are zero. Thus $PT^{-1} = (P_1, 0)$; let $TQ = \begin{pmatrix} Q_2 \\ Q_1 \end{pmatrix}$ be a corresponding partitioning of Q. Then $0 = PQ = P_1Q_2$, whence $Q_2 = 0$ by the linear independence of the columns of P_1, and (i) is proved.

For the rest of the proof we may assume, by (i), that P, Q have already been brought to the forms $(P_1, 0)$, $\begin{pmatrix} 0 \\ Q_1 \end{pmatrix}$. Moreover, P is now left full and Q right full, whence

$$r \leqslant t', \quad s \leqslant t-t', \tag{5}$$

and (ii) follows. If equality holds in (ii), then $s \leqslant s+r-t'$, that is, $t' \leqslant r$, and hence $r = t'$, $s = t-t'$ (by (5)). Thus P_1 and Q_1 are square and $P = P_1P'$, where $P' = (I_r, 0)$; similarly $Q = Q'Q_1$, where $Q' = \begin{pmatrix} 0 \\ I_s \end{pmatrix}$ and now the remaining assertions follow.

The hypothesis of this lemma may be restated in terms of characteristics of matrices:

PROPOSITION 2.4. *Let R be a semifir. If A, A' are matrices of the same characteristic over R, satisfying a relation*

$$AB' = BA', \tag{6}$$

such that (A, B) is left full and $\begin{pmatrix} B' \\ A' \end{pmatrix}$ is right full, then we can cancel square left and right *factors so as to obtain a comaximal relation; i.e. there exist square matrices P_1, Q_1 such that $A = P_1A_0$, $B = P_1B_0$, $A' = A'_0Q_1$, $B' = B'_0Q_1$, and $A_0B'_0 = B_0A'_0$ is a comaximal relation.*

Proof. Write $(A, B) = P$, $\begin{pmatrix} -B' \\ A' \end{pmatrix} = Q$. Then (6) may be expressed as $PQ = 0$. Moreover, P is left full and Q is right full, and we have equality in (ii) because A and A' have the same characteristic. Thus we can find P_1, Q_1, T such that

$$(A, B)T^{-1} = P_1(I, 0), \quad T\begin{pmatrix} -B' \\ A' \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}Q_1.$$

On cancelling P_1 and Q_1 we obtain the desired comaximal relation.

A comaximal relation (6) over any ring is clearly coprime; conversely, if (6) is coprime, then (A, B) is left prime and hence left full, similarly on the right and, applying Proposition 2.4, we therefore get

COROLLARY 1. *Any comaximal relation* (6) *between matrices over a ring R is coprime; conversely, if R is a semifir and A, A' are matrices of the same characteristic, then any coprime relation is comaximal.*

Thus over a semifir, coprime and comaximal relations (for matrices A, A' of the same characteristic) mean the same thing. The hypothesis on the characteristic cannot be omitted: in the free algebra $k\langle x, y\rangle$ (a semifir) we have

$$(x, y)\begin{pmatrix} x \\ y \end{pmatrix} = (x, y)\begin{pmatrix} x \\ y \end{pmatrix},$$

and this is a coprime relation which is not comaximal.

COROLLARY 2. *Over a semifir, two matrices A, A' which are right non-zerodivisors are left equivalent if and only if they have the same characteristic and there is a coprime relation* (6).

We now come to an important consequence of Lemma 2.3 much used later, the partition lemma:

LEMMA 2.5. *In a semifir R, suppose that a product PQ of matrices has an $r' \times s''$ block of zeros as shown:*

$$PQ = \begin{array}{c} \begin{pmatrix} A' & 0 \\ A & A'' \end{pmatrix} \begin{array}{l} r' \\ r'' \end{array} \\ \begin{array}{cc} s' & s'' \end{array} \end{array}, \tag{7}$$

where r', r'', s', s'' indicate the number of rows and columns respectively. Then there is an invertible matrix T such that

$$PT^{-1} = \begin{array}{c} \begin{pmatrix} B' & 0 \\ B & B'' \end{pmatrix} \begin{array}{l} r' \\ r'' \end{array} \\ \begin{array}{cc} t' & t'' \end{array} \end{array} \quad \text{and} \quad TQ = \begin{array}{c} \begin{pmatrix} C' & 0 \\ C & C'' \end{pmatrix} \begin{array}{l} t' \\ t'' \end{array} \\ \begin{array}{cc} s' & s'' \end{array} \end{array}. \tag{8}$$

Proof. Partition P, Q as $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$, $Q = (Q_1, Q_2)$, where P_1, P_2 have r', r'' rows respectively and Q_1, Q_2 have s', s'' columns respectively. Then $P_1Q_2 = 0$. Hence, by

Lemma 2.3, there exists $T \in \mathbf{GL}_{t}(R)$ such that the first t' rows in TQ_2 are 0 and all columns after the first t' in P_1T^{-1} are 0, whence PT^{-1}, TQ have the form (8) and the proof is complete.

The following remark will be used later, in the proof of Theorem 4.7. If in Lemma 2.5, Q is square, $A' = I$, and A'' is right full, then T can be chosen so that $B' = C' = I$. For we have $B'C' = I$, $B''C'' = A''$, and hence $t' \geqslant s'$, $t'' \geqslant s''$. Since $t'+t'' = s'+s''$, it follows that $t' = s'$, $t'' = s''$, and B', C' are invertible. Now by modifying T we can ensure that $B' = C' = I$, as required.

We also note

PROPOSITION 2.6. *Any left full matrix over a semifir R is a right non-zerodivisor.*

Proof. Let $A \in {}^mR^n$ be left full and suppose that $PA = 0$, with $P \neq 0$. By Lemma 2.3 there exists $T \in \mathbf{GL}_m(R)$ such that $PT.T^{-1}A = 0$ is a trivial relation. Thus $A_1 = T^{-1}A$ has a row of zeros, and since $A = TA_1$, we can, by omitting the row of zeros of A_1 and the corresponding column of T, write A as a product of an $m \times m-1$ by an $m-1 \times n$ matrix, which contradicts the fact that A was left full.

Since every left prime matrix is left full, we have the

COROLLARY. *Over a semifir every left prime matrix is a right non-zerodivisor.*

In the presence of suitable chain conditions the connexion between full and prime can be made more explicit. We recall that a ring is said to satisfy *left ACC_n* if its n-generator left ideals satisfy the ascending chain condition; *right ACC_n* is defined correspondingly. It is well known (and easily checked, cf. the proof of Proposition 2.7) that if a semifir has left and right ACC_n then every full $n \times n$ matrix can be written as a product of atoms. For brevity we shall call a semifir *fully atomic* if every full matrix is a product of atoms. For example, every fir is fully atomic, and the free power series ring $k\langle\langle X\rangle\rangle$ is an example of a fully atomic semifir which is not a fir (cf. [7, Chapters 1,2]).

We remark that over a semifir R with left ACC_n every free left R-module has ACC on n-generator submodules. This follows as in the proof of Corollary 1 to Theorem 1.2.3 of [7, p. 49].

PROPOSITION 2.7. *Let R be a semifir and A an $m \times n$ matrix of inner rank t over R. Suppose that A can be written in the form*

$$A = PA', \quad \textit{with } P \in {}^mR^t,\ A' \in {}^tR^n, \tag{9}$$

where A' is left prime. Then in any other factorization $A = P'A''$ of A into an $m \times t$ matrix P' by a $t \times n$ matrix A'', P' is a left factor of P and A' is a right factor of A''.

Such an expression (9) *for A (with A' left prime) exists whenever R is a semifir with left ACC_m, or R is fully atomic.*

Proof. Let $A = PA' = P'A''$, where P, P' have t columns and A' is left prime; then the submodule of R^n spanned by the rows of A, viz. R^mA, is not contained in any t'-generator submodule of R^n for $t' < t$ (because $t = \mathrm{rk}\,A$) and R^tA' is a maximal t-generator submodule containing it, because A' is left prime.

We have $R^tA' \cap R^tA'' \supseteq R^mA$. Hence the familiar exact sequence

$$0 \to R^tA' \cap R^tA'' \to R^tA' \oplus R^tA'' \to R^tA' + R^tA'' \to 0$$

shows that

$$\mathrm{rk}(R^tA' + R^tA'') + \mathrm{rk}(R^tA' \cap R^tA'') = 2t.$$

Since each summand is at least t, each must equal t. By the maximality of R^tA' this means that $R^tA' \supseteq R^tA''$, and hence A' is a right factor of A'' and P' is a left factor of P.

Now assume left ACC_m and write $A = P_0A_0$, where $P_0 \in {}^mR^t$, $A_0 \in {}^tR^n$, and A_0 is left full. Since $t \leqslant m$, we can choose a maximal t-generator submodule of R^n containing R^tA_0; this is of the form R^tA', where the rows of A' are the generators. The equation (9) follows, and A' is left prime by the maximality of R^tA'.

When R is fully atomic, write $A = P_1A_1$, where P_1 is $m \times t$ and A_1 is $t \times n$. Then A_1 is left full. Let A_1' be a full $t \times t$ submatrix of A_1. Then in any factorization

$$A_1 = P_2A_2, \quad \text{with } P_2 \in R_t,\ A_2 \in {}^tR^n,$$

P_2 is a left factor of A_1' and so the number of terms in a complete factorization of P_2 is bounded by the corresponding number for A_1'. Taking P_2 with a maximal number of factors we ensure that A_2 is left prime. This completes the proof.

Without ACC_m the final assertion of Proposition 2.7 need not hold. For example, let kF be the group algebra over k of the free group F on x, y, z and let R be the subalgebra of kF generated by z and all elements $z^{-n}x, z^{-n}y\ (n = 1, 2, \ldots)$. As in [7, p. 108] it follows that R is a right fir, and hence a semifir, but not a fir. Suppose that

(10) $$(x, y) = u(x', y'),$$

where (x', y') is left prime. Consider the endomorphism $p \mapsto \bar{p}$ of R which maps x and all $z^{-n}x$ to 0 but leaves y, z fixed. Applying this map to (10) we find that $\bar{u}\bar{x}' = \bar{x} = 0$, $\bar{u}\bar{y}' = \bar{y} \neq 0$, whence $\bar{u} \neq 0$ and so $\bar{x}' = 0$. It follows that x' has zero constant term, and since all the generators $z^{-n}x, z^{-n}y, z$ in R are left divisible by z, so is x'. By symmetry the same is true of y', so (x', y') cannot be left prime.

3. *Prime modules over semifirs*

We pause to examine briefly the modules defined by prime or full matrices over a semifir (see also [12]). Since every finitely generated submodule of a free module over a semifir R is again free, every finitely presented left R-module M has a presentation

(1) $$0 \longrightarrow R^m \xrightarrow{\alpha} R^n \longrightarrow M \longrightarrow 0,$$

where the $m \times n$ matrix defining α is a right non-zerodivisor. The characteristic of M is denoted by $\chi(M)$; thus $\chi(M) = n - m$. If $\mathrm{Hom}_R(M, R) = 0$, M is said to be *bound*; by applying $\mathrm{Hom}(-, R)$ to (1) we see that M is bound if and only if it can be presented by a matrix which is a left non-zerodivisor.

Let R be a semifir. A finitely presented left R-module M is said to be *positive* if M has a presentation (1) in which the matrix A representing α is left full; we recall that in this case $m \leqslant n$, so that $\chi(M) = n - m \geqslant 0$. If moreover, A is left prime, M is said to be *positive prime*. We note that A need not be a left non-zerodivisor; by what has been said this is so precisely if M is bound.

A left R-module M with presentation (1) is said to be *negative* if the matrix A representing α is right full (so that $n \leqslant m$). If moreover, A is right prime, M is said to be *negative prime*; by Proposition 2.6 every negative module is necessarily bound. By a *prime* module we understand a positive prime or negative prime module and a *full* module is a positive or negative module; thus a full module is defined by a (left or right) full matrix and a prime module by a (left or right) prime matrix.

A module which is both positive and negative is called a *torsion* module; this agrees with the terminology in [7, p. 184]. Thus a torsion module is defined by a square full matrix. Of course a prime torsion module is necessarily zero; we shall call a torsion module *simple* if it is defined by a matrix which is an atom. It should be observed that this is not the same as a simple R-module; it is a simple object in the abelian category of torsion modules.

By examining the definitions we obtain the following intrinsic description of full and prime modules; for reference we also include torsion modules.

PROPOSITION 3.1. *Let M be a finitely presented module over a semifir R. Then*
(i) *M is positive if and only if $\chi(M') \geqslant 0$ for all finitely generated submodules M' of M,*
(ii) *M is positive prime if and only if $\chi(M') > 0$ for all finitely generated non-zero submodules M' of M,*
(iii) *M is negative if and only if $\chi(M'') \leqslant 0$ for all finitely presented quotients M'' of M,*
(iv) *M is negative prime if and only if $\chi(M'') < 0$ for all finitely presented non-zero quotients M'' of M,*
(v) *M is a torsion module if and only if $\chi(M) = 0$ and $\chi(M') \geqslant 0$ for all finitely generated submodules M' of M.*

Any submodule of characteristic 0 of a positive module and any quotient of characteristic 0 of a negative module is a torsion module. Hence conditions (ii) and (iv) may also be expressed by saying: M is prime if and only if either M is positive and has no non-zero torsion submodules or M is negative and has no non-zero torsion quotients.

We saw that whereas a negative module is necessarily bound, a positive module need not be. Let us consider any positive module M; if M is not bound, there is a non-zero homomorphism $f\colon M \to R$. Since the image is free, M splits over the kernel of f; thus we have

(2) $$M = M_0 \oplus M_1,$$

where M_0 is free and M_1 is again positive. If M_1 is not bound, we can split off another factor R, but this process must stop eventually because the rank of any free direct summand of M is bounded by the number of generators of M. Thus by taking M_0 to be free of maximal rank in (2) we ensure that M_1 is bound. This proves the first assertion below; the second is an immediate consequence.

PROPOSITION 3.2. *Any positive module M over a semifir R is a direct sum of a free module and a bound positive module. A positive prime module M is a direct sum of a free module and a bound positive prime module.*

Next we turn to chain conditions. Let R be a semifir satisfying left and right ACC_n for all n. If M is a left R-module with presentation (1), then the r-generator submodules of M correspond to $(m+r)$-generator submodules of R^n and so satisfy the ascending chain condition. Now for bound modules we have a duality provided by $\mathrm{Ext}^1_R(-,R)$

[7, Chapter 5], under which r-generator submodules of M correspond to quotients of $\mathrm{Ext}_R^1(M,R)$ by $(m+r)$-generator submodules, with bounded characteristic, provided that the characteristics of the submodules are bounded. Hence we have.

PROPOSITION 3.3. *Let M be a finitely presented module over a semifir with left and right ACC_n, for all n. Then any set of bound submodules on at most r generators (for any fixed r) and of bounded characteristics satisfies both chain conditions.*

We observe that over a fir every finitely presented module satisfies both chain conditions for bound submodules [7, Theorem 5.2.3, p. 181]; in particular, this applies to bound prime modules, since these are finitely presented.

A related result is obtained by restating Proposition 2.7 in terms of modules. In essence this states that every finitely presented module over a fully atomic semifir has a largest positive prime quotient. More precisely we have

PROPOSITION 3.4. *Let R be a semifir with left ACC_m or a fully atomic semifir, and let M be an m-generator submodule of R^n (for some n). Denote by t the least integer for which there is a t-generator submodule between M and R^n (thus $t \leqslant \min(m,n)$). Then there is a greatest t-generator submodule N between M and R^n, that is, $M \subseteq N \subseteq R^n$ and N contains every t-generator module with this property.*

In [7, Chapter 5] it was proved that the torsion modules over a semifir form an abelian category and it follows that the endomorphism ring of a simple torsion module is a skew field [7, p. 186]. For more general prime modules this need not hold. For example, let M be a prime module of characteristic $r > 0$; then $M^2 = M \oplus M$ is again prime (by Proposition 3.1 applied to a 3×3 diagram with M^2 at the centre), but the endomorphism ring clearly has nilpotents. There is however a partial result in this direction which will be useful to us later.

PROPOSITION 3.5. *Let R be a semifir and let M, N be any prime R-modules of characteristic 1. Then any non-zero homomorphism $f\colon M \to N$ is injective.*

Proof. We have an exact sequence

$$0 \longrightarrow \ker f \longrightarrow M \xrightarrow{f} N \longrightarrow \operatorname{coker} f \longrightarrow 0.$$

If $\ker f \neq 0$, then by Proposition 3.1, $\chi(\ker f) > 0$, whence $\chi(\operatorname{coker} f) = \chi(\ker f) > 0$, $\chi(\operatorname{im} f) = 1 - \chi(\operatorname{coker} f) \leqslant 0$; therefore $\operatorname{im} f = 0$.

In particular, taking $N = M$, we obtain the

COROLLARY. *Let R be a semifir and M a prime R-module of characteristic 1. Then $\mathrm{End}_R(M)$ is an integral domain.*

This corollary actually holds for a somewhat larger class of modules, as follows from the next two results (proved by similar arguments), for which I am indebted to the referee.

LEMMA 3.6. *Let M, N be non-zero bound modules over a semifir R.*
(i) *If, for all non-zero bound submodules $M' \subseteq M$, $N' \subseteq N$,*

$$\chi(M')+\chi(N') > \chi(M), \tag{3}$$

then every non-zero homomorphism f: $M \to N$ has a free kernel.
(ii) *If* (3) *holds for all proper bound submodules $M' \subset M, N' \subset N$, then every non-zero homomorphism f: $M \to N$ is surjective.*

In case (i) we see by taking $M' = M$ that N must be positive prime; similarly in (ii) M is negative prime.

PROPOSITION 3.7. *Let R be a semifir and M a finitely presented non-zero R-module such that for every proper non-zero finitely generated submodule M' of M,*

$$\chi(M') > \tfrac{1}{2}\chi(M). \tag{4}$$

Then $\mathrm{End}_R(M)$ *is an integral domain. In particular this holds for every prime module of characteristic* ± 1.

We remark that the modules of characteristic ± 1 satisfying (4) are just the prime modules of these characteristics. For $|\chi(M)| > 1$ every module M satisfying (4) is prime, but not every prime module need satisfy (4), since the conclusion of Proposition 3.7 does not hold for all prime modules, as we have seen.

4. *Numerators and denominators*

We now come to the main topic of this paper, an examination of the form taken by 'fractions' over a semifir. We begin by recalling some facts about the universal field of fractions.

If R is any ring, an epic R-field in which R is embedded is called a *field of fractions* of R. By a *universal field of fractions* of R one understands a field of fractions U of R which is 'universal' for specializations (cf. [7, Chapter 7] 'strict specializations of R-sfields' in the terminology of [3]). When such U exists, it is unique up to natural isomorphism. If R is a semifir, it is known [7, p. 283] that R always has a universal field of fractions U, and that U is characterized uniquely among epic R-fields by the property that any square matrix A over R becomes invertible over U if and only if A is full over R. The homomorphism $R \to U$ is injective because it preserves full matrices; in fact it is inner rank preserving (cf. [8, Theorem 2]). In particular, any matrix A over R is a left non-zerodivisor over U if and only if A is right full over R (this provides another proof of Proposition 2.6). It follows that if A, B are both right full (or both left full), then so is AB; this need not hold in more general rings (cf. also [13]). In the other direction, if AB is right full, then so is B, as is easily checked.

For the moment let R be any ring and K an epic R-field. Any element of K may be obtained as the last component u_m of the solution u of a system of equations (left system)

$$Au = a, \quad \text{with } A \in {}^rR^m,\ a \in {}^rR, \tag{1}$$

where A has rank m over K. We note for reference that given any such system (1) we can, without affecting the solution, replace A by an $m \times m$ matrix which is invertible over K. For we have $r \geqslant m$, because A has rank m, and by picking out m suitably chosen rows of

A we can reduce (1) to a system with a square matrix invertible over K and having the same solution.

For the remainder of this section R will be a semifir and U its universal field of fractions. Our object will be to compare the different systems defining a given element $p \in U$. It will be helpful to introduce a more systematic notation. We shall take as our basic matrix A the augmented matrix which in the notation of (1) would be written $(-a, A)$. Thus we shall write our system in the form

$$(2) \qquad Au = 0, \quad \text{with } A \in {}^{r}R^{m+1}.$$

The columns of A will be indicated by suffixes, thus

$$A = (A_0, A_1, \ldots, A_m) = (A_0, A_*, A_m),$$

where A_* is the $r \times m-1$ matrix $(A_1, \ldots, A_{m-1})$. We shall call (2) an *admissible system* and A an *admissible matrix* of *order m* for the element p of U if (2) has a unique solution $u \in {}^{m+1}U$, normalized by the condition $u_0 = 1$, and this solution satisfies $u_m = p$.

From what has been said it is clear that a matrix $A \in {}^{r}R^{m+1}$ is admissible if and only if (A_*, A_m) is right full but A is not. For this will ensure that (2) has a non-zero solution, but that any solution with 0-component 0 is itself 0; thus it will have a unique normalized solution. (Corresponding definitions could be given for an admissible system over any epic R-field K, for any ring R, replacing inner rank by rank over K.)

Given a system (2) with normalized solution u, we have

$$(3) \qquad (A_0, A_*) = (A_*, A_m)\begin{pmatrix} -u_* & I \\ -u_m & 0 \end{pmatrix},$$

where $u = (u_0, u_*, u_m)^{\mathrm{T}}$. We shall call (A_0, A_*) the *numerator* and (A_*, A_m) the *denominator* of u_m in the system (2); as in the commutative case they depend on the actual system (2), not merely on u_m. For (2) to be an admissible system for p, the denominator of p in this system must be right full, and when this is so, (3) shows that $p \neq 0$ if and only if its numerator in (2) is right full. This may be regarded as the non-commutative analogue of Cramer's rule (cf. [7, p. 251]).

We remark that the numerator and denominator are square matrices whenever $r = m$; the terminology will be used mainly in that case.

In order to compare different systems we shall describe some operations which can be carried out on a system without changing its solution. Let A be an admissible matrix for p. If u satisfies $Au = 0$, then it also satisfies $PAu = 0$ for any matrix P for which the product is defined, but the solutions of $Au = 0$ and $AQu = 0$ are generally not the same. However, we are only concerned with the ratio of the first and last components of u and this ratio is preserved when Q is suitably restricted. Let us define a *bordered* matrix as a matrix Q^* which agrees with the unit matrix in the first and last row; thus

$$(4) \qquad Q^* = (Q_0^*, Q_*^*, Q_n^*) = \begin{pmatrix} 1 & 0 & 0 \\ Q_0 & Q_* & Q_n \\ 0 & 0 & 1 \end{pmatrix} \in {}^{m+1}R^{n+1}.$$

We now define the following two operations on an admissible matrix A for p:

λ. *replace A by PA such that PA is again admissible for p,*

ρ. *replace A by AQ^*, where Q^* is bordered, such that AQ^* is again admissible for p.*

The inverse operations, cancelling a left or a bordered right factor from A so as to obtain again an admissible matrix, are denoted by λ', ρ' respectively, and an operation is called *trivial* or *full* if the matrix P or Q^* involved is invertible or full, respectively. An admissible matrix A is said to *admit* an operation if the result of performing it on A is again admissible.

We shall require a notion of equivalence between admissible matrices, analogous to stable association between arbitrary matrices, but taking into account the (distinguished) first and last columns of the former. The following definition and its uses below were suggested by the referee.

Let R be a weakly finite ring. Two matrices A, A' of the same characteristic are said to be *stably biassociated* if there is a comaximal relation

$$AB' = BA', \tag{5}$$

where B' is a bordered matrix.

Evidently two stably biassociated matrices are also stably associated. If we consider the left modules M, M' defined by A, A' respectively, then stable biassociation corresponds to an isomorphism in which two distinguished elements of M (corresponding to the first and last elements of the solution u of (2)) transform to two distinguished elements of M' (cf. § 6).

Let A, A' be two admissible matrices for the elements p, p' of U respectively. Then $A'u' = 0$, where $u' = (1, u'_*, p')^{\mathrm{T}}$. Suppose now that A and A' are stably biassociated, say (5) holds, with B' bordered. Then $AB'u' = BA'u' = 0$ and here $B'u' = (1, *, p')^{\mathrm{T}}$, and hence $p' = p$. This shows that two admissible matrices that are stably biassociated determine the same element of U. One of our tasks will be to find conditions under which the converse holds.

We first prove an analogue of Proposition 2.2, giving equivalent conditions for stable biassociation. For technical reasons we shall make a change in the sign of the blocks composing the inverse matrix.

PROPOSITION 4.1. *Let R be a weakly finite ring and let $A \in {}^rR^{m+1}$, $A' \in {}^sR^{n+1}$, where $r+n = s+m$. Then the following conditions are equivalent:*

(a) *A and A' satisfy a comaximal relation* (5) *with a bordered matrix B', and so are stably biassociated;*

(b) *there is an $(r+n+1)\times(s+m+1)$ matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with inverse $\begin{pmatrix} -D' & B' \\ C' & -A' \end{pmatrix}$, where C, B' are bordered and D, D' have their first and last rows zero;*

(c) *there is a relation*

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} U = V \begin{pmatrix} I & 0 \\ 0 & A' \end{pmatrix}, \tag{6}$$

where U, V are invertible and U has the partitioned form

$$U = \begin{pmatrix} -D' & B' \\ C' & -A' \end{pmatrix},$$

where B' is bordered and D' has first and last row zero.

Proof. (a) $\Rightarrow$ (b). By Proposition 2.2 we have two mutually inverse matrices

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} -D' & B' \\ C' & -A' \end{pmatrix}. \tag{7}$$

Since the first row of B' is $e_1 = (1, 0, \ldots, 0)$, we can subtract multiples of the $(r+1)$th column of M^{-1} from the first r columns so as to reduce the first row of D' to zero, and make the corresponding change in M. This will only affect C and D, but not A and B, in M. Similarly, since the last row of B' is e_{m+1} we can reduce the last row of D' to zero (at the expense of further changes to C and D). Since the relation $M^{-1}M = I$ has been preserved, we see that the first and last rows of D are zero, while those of C agree with the unit matrix, i.e. C is bordered. Thus (b) holds; now the converse follows as in Proposition 2.2, that is, (a) $\Leftrightarrow$ (b).

When (b) holds we have equation (4), from § 2 (p. 7), which (after changing the sign of B') yields (c), and assuming (c), we obtain a relation (5) where B' is bordered. By (4), § 2, we have $AD' - BC' = I$, whence (5) is right comaximal, and it is left comaximal because U in (6) is invertible.

Corollary 1. *The relation of stable biassociation is an equivalence relation.*

Proof. The relation is clearly reflexive, and by (b) also symmetric; to prove transitivity we use (c). Thus let

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} -D' & B' \\ C' & -A' \end{pmatrix} = V\begin{pmatrix} I & 0 \\ 0 & A' \end{pmatrix}, \quad \begin{pmatrix} A' & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} -D'' & B'' \\ C'' & -A'' \end{pmatrix} = W\begin{pmatrix} I & 0 \\ 0 & A'' \end{pmatrix},$$

where B' and B'' are bordered and the first and last rows of D' are zero. Then

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} B' & -D' \\ -A' & C' \end{pmatrix} = V\begin{pmatrix} 0 & I \\ A' & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & I \\ A' & 0 \end{pmatrix}\begin{pmatrix} -D'' & B'' \\ C'' & -A'' \end{pmatrix} = W_1\begin{pmatrix} I & 0 \\ 0 & A'' \end{pmatrix},$$

where W_1 is obtained from W by an appropriate interchange of rows, and

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} B' & -D' \\ -A' & C' \end{pmatrix}\begin{pmatrix} -D'' & B'' \\ C'' & -A'' \end{pmatrix} = VW_1\begin{pmatrix} I & 0 \\ 0 & A'' \end{pmatrix}.$$

Here $B'B'' + D'A''$ is again bordered, as is easily verified; therefore A is indeed stably biassociated to A'' and the proof is complete.

Next we show that stable biassociation implies a strong form of stable association. Consider a square matrix of the form

$$P = \begin{pmatrix} 1 & 0 \\ 0 & P_1 \end{pmatrix}.$$

Clearly P is stably associated to P_1; hence if P is invertible, then so is P_1. More generally, if an invertible matrix has a row consisting of zeros except for a single entry 1, then the 'cofactor' of the entry 1 is again invertible. We apply this remark to M in (7). The first row of D is 0 and that of C is e_1. Hence we obtain an invertible matrix by omitting the first column and the $(r+1)$th row:

$$M_1 = \begin{pmatrix} A_* A_m & B \\ C_1 & D_1 \end{pmatrix}, \quad M_1{}^{-1} = \begin{pmatrix} -D_1' & B' \\ C_1' & -(A_*' \ A_n') \end{pmatrix}.$$

We thus find that $(A_* \ A_m)$ is stably associated to $(A_*' \ A_n')$. A similar argument applies to the last row of C, D. Hence we obtain

Corollary 2. *If A, A' are stably biassociated, where $A = (A_0 \ A_* \ A_m)$, $A' = (A_0' \ A_*' \ A_n')$, then the pairs $(A_0 \ A_*)$ and $(A_0' \ A_*')$, $(A_* \ A_m)$ and $(A_*' \ A_n')$, A_* and A_*' are each stably associated.*

We now return to the operations $\lambda, \lambda', \rho, \rho'$ and determine conditions under which they preserve admissibility.

PROPOSITION 4.2. *Let R be a semifir with universal field of fractions U.*

(i) *The matrix $A \in {}^rR^{m+1}$ is admissible if and only if $(A_* \ A_m)$ is right full but A is not; in particular this entails $r \geq m$.*

(ii) *Suppose that $B = PA$, where $A \in {}^rR^{m+1}$, $B \in {}^sR^{m+1}$, $P \in {}^sR^r$. If A is admissible for $p \in U$, then B is admissible for p if and only if $P(A_* \ A_m)$ is right full. If B is admissible for p, then A is admissible for p if and only if A is not right full.*

(iii) *Suppose that $B = AQ^*$, where $A \in {}^rR^{m+1}$, $B \in {}^rR^{n+1}$, and $Q^* \in {}^{m+1}R^{n+1}$ is bordered. If A is admissible for p, then B is admissible for p if and only if Q^* is right full but B is not; when this is so, $m \geq n$. If B is admissible for p, then A is admissible for p if and only if its denominator $(A_* \ A_m)$ is right full.*

(iv) *For a right full matrix P, A is admissible if and only if PA is, and for a full bordered matrix Q^*, A is admissible if and only if AQ^* is.*

Proof. We have already seen (i). To prove the first assertion of (ii), take an admissible system $Au = 0$ for p and multiply on the left by P; we obtain

$$PAu = 0. \tag{8}$$

This is admissible for p if and only if $P(A_* \ A_m)$ is right full, for then (8) will have a unique normalized solution u, necessarily the same as that of (2). For the operation λ', let B be admissible. Then $(A_* \ A_m)$ is right full because $(B_* \ B_m) = P(A_* \ A_m)$ is. Hence A will be admissible if and only if A is not right full. Further, A cannot be admissible for any $q \neq p$, for if it were, B would also be admissible for q, which contradicts the uniqueness.

(iii) Suppose first that A is admissible for p. Let Q^* be a bordered matrix (4) and consider the system

$$AQ^*v = 0. \tag{9}$$

By hypothesis, (2) has a unique normalized solution u. Hence the solution of (9) satisfies

$$Q^*v = uv_0.$$

In particular, if v is normalized, the form of Q^* shows that $v_n = u_m$, so when (9) is admissible, it will be admissible for p. By (i), (9) is admissible if and only if $A(Q^*_* \ Q^*_n)$ is right full, but AQ^* is not. Since $A(Q^*_* \ Q^*_n)$ is right full, so is $(Q^*_* \ Q^*_n)$ and now the form of the first row of Q^* shows that Q^* is right full. Clearly when this is so, then $n \leq m$. For the inverse operation, let B be admissible for p. Then A is admissible for p precisely when $(A_* \ A_m)$ is right full; for $Au = 0$ has the normalized solution $u = Q^*v$ and under the given assumptions this is unique.

(iv) When P is right full, $Au = 0$ and $PAu = 0$ have the same solutions, and this proves the first part. Suppose now that Q^*, given by (4), is full. Then $n = m$, $\begin{pmatrix} Q_* & Q_m \\ 0 & 1 \end{pmatrix}$ is full, and we have

$$A(Q^*_* \ Q^*_m) = A\begin{pmatrix} 0 & 0 \\ Q_* & Q_m \\ 0 & 1 \end{pmatrix} = (A_* \ A_m)\begin{pmatrix} Q_* & Q_m \\ 0 & 1 \end{pmatrix},$$

therefore $A(Q^*_* \ Q^*_m)$ is full. Secondly, if u is any solution of (2), then $Q^*v = u$ has a solution v because Q^* is full, and so $AQ^*v = 0$, which shows that AQ^* is not right full.

Thus ρ can always be applied with a full bordered matrix Q^*. Next, if $B = AQ^*$ with a full bordered matrix Q^*, then

$$(B_* \ B_m) = (A_* \ A_m)\begin{pmatrix} Q_* & Q_m \\ 0 & 1 \end{pmatrix}.$$

Now the same argument as before shows that $(A_* \ A_m)$ is right full, while A is not because AQ^* is not right full, so we can always cancel a full bordered matrix Q^*. This completes the proof.

The next three lemmas deal with the possibility of simplifying an admissible system by operations $\lambda, \lambda', \rho, \rho'$. First we give conditions for an admissible matrix to be left full:

LEMMA 4.3. *Let R be a semifir and $A \in {}^rR^{m+1}$ an admissible matrix. Then the characteristic of A is at most* 1, *that is, $m \leqslant r$, and the following conditions are equivalent:*

(a) *the characteristic of A is exactly* 1, *that is, $m = r$;*

(b) *A is left full;*

(c) *the number of rows of A cannot be decreased by an operation λ';*

(d) *the number of rows of A cannot be decreased by an operation λ.*

Further, any admissible matrix $A \in {}^rR^{m+1}$ can be transformed by an operation λ, or also by an operation λ', to a left full admissible matrix $A' \in {}^rR^{m+1}$, and every operation λ or λ' admitted by a full admissible matrix which does not increase the number of rows is full.

Proof. Since $Au = 0$ has a unique normalized solution u over U, the rank of A is m, and hence $r \geqslant m$, where equality holds if and only if the rank of A is r, that is, A is left full. This shows that (a) $\Leftrightarrow$ (b).

Now since A is of rank m, we can write

$$A = PA', \quad \text{with } P \in {}^rR^m,\ A' \in {}^mR^{m+1},$$

where P is right full and A' is left full. Clearly A' is not right full, so by Proposition 4.2(ii), A' is admissible; moreover P is full precisely when $r = m$. This shows that (a) $\Leftrightarrow$ (c).

Next we can put

$$A'' = EA, \quad E \in {}^mR^r, \quad A'' \in {}^mR^{m+1},$$

where E consists of m rows of the $r \times r$ unit matrix, chosen so that A'' is left full. Again A'' is admissible; further, E is full if and only if $r = m$. This shows that (a) $\Leftrightarrow$ (d), and it also proves the last assertion.

Next we examine when an admissible matrix is left prime; we recall that in particular such a matrix is left full.

LEMMA 4.4. *Let R be a semifir and let $A \in {}^rR^{m+1}$ be an admissible matrix. Then the following conditions are equivalent:*

(a) *A is left prime (hence $r = m$);*

(b) *every non-trivial operation λ' admitted by A increases the number of rows.*

Further, if R has left ACC_m then any admissible matrix $A \in {}^rR^{m+1}$ can be transformed by an operation λ': $A = PA'$ to a left prime admissible matrix $A' \in {}^mR^{m+1}$.

Proof. Suppose that A is left prime; then A is left full and no operation λ' can decrease the number of rows. Moreover, in any factorization of A,

$$A = PB, \quad \text{with } P \in R_r,\ B \in {}^rR^{m+1}, \tag{10}$$

P is a unit, so only trivial operations λ' do not increase the number of rows, i.e. (b) holds. Conversely, if (b) holds, then by Lemma 4.3, A is left full and in any equation (10), P is a unit, and hence A is left prime, that is (a) holds.

If R has left ACC_m we can apply Proposition 2.7 to obtain an equation (10), where B is left prime.

We now turn to the operations ρ, ρ'; we recall that the *order* of an admissible $r \times m+1$ matrix is m. An admissible matrix $A = (A_0\ A_*\ A_m)$ will be called right **-prime* if A_* is right prime.

LEMMA 4.5. *Let R be a semifir and let $A \in {}^rR^{m+1}$ be an admissible matrix. An operation ρ admitted by A cannot increase, nor an operation ρ' decrease, the order of A. An operation ρ or ρ' preserves the order of A precisely when the operation is full.*

The following conditions are equivalent:

(a) *A is right *-prime;*

(b) *every bordered square right factor of A is a unit;*

(c) *all full operations ρ' admitted by A are trivial.*

*Further, if R has right ACC_{m-1}, then any admissible matrix $A \in {}^rR^{m+1}$ can be reduced by a full operation ρ': $A = A'Q^*$ to one which is right *-prime.*

Proof. Consider an operation ρ'. If

$$B = AQ^*, \quad B \in {}^rR^{n+1}, \quad Q^* \in {}^{m+1}R^{n+1}, \tag{11}$$

and B is admissible, then Q^* is right full (Proposition 4.2(iii)), and hence $n \leqslant m$. This shows that ρ cannot increase, nor ρ' decrease, the order, and that it preserves the order if and only if Q^* is full.

Now let $A = A'Q^*$, where Q^* is a square bordered matrix. Then $A_* = A'_*Q_*$, and clearly Q^* is a unit if and only if Q_* is. It follows that A_* is right prime if and only if Q^* is a unit in every factorization (11), that is, (a) $\Leftrightarrow$ (b), and (b) $\Leftrightarrow$ (c) is clear because any full operation ρ' preserves the order and so has a square matrix Q^*; conversely, when Q^* is square, ρ' preserves the order and so is full.

Finally, when R has right ACC_{m-1}, we can by Proposition 2.7 write

$$A_* = A'_*Q_*, \quad \text{with } A'_* \in {}^rR^{m-1},\ Q_* \in R_{m-1},$$

where A'_* is right *-prime and Q_* is full. Hence on writing $A = A'Q^*$, where

$$A' = (A_0\ A'_*\ A_m), \quad Q^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & Q_* & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we find that Q^* is full, whence A' is admissible and right *-prime.

We note without proof that the equivalent conditions of Lemma 4.3 are preserved by all full but by no non-full operations $\lambda, \lambda', \rho, \rho'$; those of Lemma 4.4 are preserved by all full operations ρ', but not generally by all full operations ρ; and those of Lemma 4.5 are preserved by all full operations λ' but not generally by full operations λ.

The conditions studied in Lemmas 4.3–4.5 are invariant under stable biassociation; they may thus be regarded as properties of the module with two distinguished elements defined by the admissible matrix. But there are other properties of interest that are not invariant under stable biassociation. For example, consider the condition for all operations ρ admitted by the admissible matrix A to be full (that is, order-preserving, by Lemma 4.5). This condition concerns not merely the module, but also the given presentation (saying that it does not admit a certain sort of simplification). Let us give two examples to illustrate the behaviour of ρ.

Firstly, we observe that from any admissible matrix $A = (A_0\ A_*\ A_m)$ we can get a stably biassociated one by adding a row and column:

$$A' = \begin{pmatrix} A_0 & A_* & 0 & A_m \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and the latter always admits an order-decreasing operation ρ:

$$\begin{pmatrix} A_0 & A_* & 0 & A_m \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_0 & A_* & A_m \\ 0 & 0 & 0 \end{pmatrix}.$$

Secondly, an admissible matrix which does not admit a non-full operation ρ may come to admit one after the application of a full operation ρ'. Thus take $R = k\langle x, y, z\rangle$ and

$$A = \begin{pmatrix} y & -x & 0 \\ 0 & x & -z \end{pmatrix}.$$

This is an admissible matrix for $z^{-1}xx^{-1}y = z^{-1}y$, as is easily checked. No operation ρ will decrease the order, but we can write

$$A = \begin{pmatrix} y & -1 & 0 \\ 0 & 1 & -z \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} y & -1 & 0 \\ 0 & 1 & -z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y & -z \\ 0 & 0 \end{pmatrix},$$

so that we obtain a system of order 1. Here the operation ρ' clearly represents the cancellation of xx^{-1}.

Before we can describe the process of reducing the order of an admissible matrix by operations including ρ, we state the conditions under which two admissible systems determine the same element of U:

LEMMA 4.6. *Let R be a semifir and let U be its universal field of fractions. Then two admissible matrices $A \in {}^rR^{m+1}$ and $B \in {}^sR^{n+1}$ determine the same element p of U if and only if there exist matrices P, P', bordered matrices Q^*, Q'^*, and admissible matrices A', B' such that*

$$(12) \qquad A = PA', \quad B = B'Q^*,$$

$$(13) \qquad P'A' = B'Q'^*.$$

In the above situation we can further take P right full, Q^* full, and A' left full.

Proof. Suppose that (12) and (13) hold; by (12), A and A' determine the same element p of U, and B, B' determine the same element q. If $A'u = 0$, we apply (13); Q'^*u has the form $(1, v_*, p)^{\mathrm{T}}$, so if $p \neq q$, then $B'Q'^*u \neq 0$ and we have a contradiction.

Conversely, assume that A and B determine the same element p. By hypothesis we have the admissible systems

$$Au = 0, \quad A \in {}^rR^{m+1}, \tag{14}$$

$$Bv = 0, \quad B \in {}^sR^{n+1}. \tag{15}$$

It follows that

$$\begin{pmatrix} A_0 & A_* & 0 & A_m & A_m \\ B_0 & 0 & B_* & B_n & 0 \end{pmatrix}$$

is an admissible system for $p - p = 0$, so its numerator is not full, i.e.

$$\begin{pmatrix} A_0 & A_* & 0 & A_m \\ B_0 & 0 & B_* & B_n \end{pmatrix}$$

has inner rank $k < m+n$. We interchange the last two column blocks and apply Lemma 2.5 to an expression of this matrix as a product of an $(m+n) \times k$ by a $k \times (m+n)$ matrix, and obtain an equation

$$\begin{pmatrix} A_0 & A_* & A_m & 0 \\ B_0 & 0 & B_n & B_* \end{pmatrix} = \begin{pmatrix} P & 0 \\ P' & B'_* \end{pmatrix} \begin{pmatrix} A' & 0 \\ -Q'_* & Q_* \end{pmatrix}. \tag{16}$$

On equating blocks we find

$$(A_0 \quad A_* \quad A_m) = PA', \tag{17}$$

$$(B_0 \quad 0 \quad B_n) = P'A' - B'_*Q'_*, \tag{18}$$

$$B_* = B'_*Q_*. \tag{19}$$

Here (17) is the first equation of (12), and (19) yields the second equation of (12) if we set

$$B' = (B_0 \quad B'_* \quad B_n), \quad Q^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & Q_* & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, (18) may be written

$$P'A' = (B_0 \quad B'_* \quad B_n) \begin{pmatrix} 1 & 0 & 0 \\ 0 & Q'_* & 0 \\ 0 & 0 & 1 \end{pmatrix} = B'Q'^*,$$

where

$$B' = (B_0 \quad B'_* \quad B_n), \quad Q'^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & Q'_* & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and this is of the form (13).

It remains to prove that P is right full, A' is left full, and Q^* is full. This will also show (by Proposition 4.2) that A', B' are admissible.

We denote by t the number of columns of P, P', equivalently, the number of rows in A' (cf. (16)), and by t' the number of columns in B'_*, equivalently, the number of rows in

Q'_*, Q_*. From the way (16) was chosen we have

$$t+t' = k < m+n. \tag{20}$$

But $t \geqslant \operatorname{rk} P \geqslant \operatorname{rk} A = m$ (cf. (17)) and $t' \geqslant \operatorname{rk} Q_* \geqslant \operatorname{rk} B_* = n-1$ (cf. (19)). A comparison with (20) shows that $t = \operatorname{rk} P = m$ and $t' = \operatorname{rk} Q_* = n-1$. Hence P is right full, A' has characteristic 1 and so is left full (Lemma 4.3), and Q_*, and therefore also Q^*, is full, and the proof is complete.

Note that although we proved P to be right full and Q^* full, we have not proved this for P' and Q'^*. Hence the common value of the two sides of (13) need not itself be an admissible matrix, and we have not obtained a chain of operations $\lambda, \lambda', \rho, \rho'$ connecting A and B.

To study the situation where Q'^* may not be full, let the rank of Q'_* be $l-1$. Then we can write $Q'_* = R_* S_*$, where R_* has $l-1$ columns and S_* has $l-1$ rows. Putting

$$R^* = \begin{pmatrix} 1 & 0 & 0 \\ Q'_0 & R_* & Q'_n \\ 0 & 0 & 1 \end{pmatrix}, \quad S^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & S_* & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have

$$Q'^* = R^* S^*, \tag{21}$$

where R^* is bordered and right full and S^* is bordered and left full. By the first part of the above proof, $A'u = 0$; hence on combining this with (13) and (21) we get

$$0 = P'A'u = B'Q'^*u = (B'R^*)(S^*u).$$

Since S^* is bordered, the vector S^*u has first entry 1, therefore $B'R^*$ is not right full. Since R^* is right full and B' is admissible, Proposition 4.2 (iii) tells us that $B'R^*$ is admissible. Thus $B' \mapsto B'R^*$ is an operation ρ taking B' to an admissible matrix of order l. This still does not lead to a 'path' from A to B by admissible operations: although $B'' = B'R^*$ is so obtainable from B, $P'A' = B''S^*$ is still not admissible unless S^* is full. In fact, as we shall now show, we can ensure that S^* is full by taking the order of A to be small enough. Specifically we shall assume the following condition suggested by the referee:

O(B, A). *B cannot be transformed by any full operation ρ' followed by any operation ρ to an admissible matrix of smaller order than A.*

Since $B \mapsto B'$ by a full operation ρ' and $B' \mapsto B'R^*$ by an operation ρ, it then follows that $l = \text{order } B'R^* \geqslant \text{order } A' = m$. Now S^* is $(l+1) \times (m+1)$ and is left full, so $l \leqslant m$, and hence $l = m$. It follows that $Q'^* = R^*S^*$ is right full, and so $P'A' = B'Q'^*$ is admissible; now by Proposition 4.2 (iii), P' is right full. This proves

COROLLARY 1. *In the situation of Lemma 4.5, if O(B, A) holds, then the common value of the two sides of (13) is an admissible matrix. Hence we can then pass from A to B by a chain of four operations $\lambda', \lambda, \rho', \rho$, in that order. Here ρ is full, and if A is left full, then λ' is full.*

Let us return to Lemma 4.6. If A is left prime, P must be a unit, and if B is right $*$-prime, Q^* must be a unit; hence we obtain

COROLLARY 2. *In the situation of Lemma* 4.6, *if A is left prime and B right $*$-prime, then there exist a matrix P and a bordered matrix Q^* such that $PA = BQ^*$. If, moreover, A, B are reduced, so that they have the same characteristic* 1, *then they are stably biassociated.*

Here A is called *reduced* if A is left prime and right $*$-prime.

The last part follows because the relation $PA = BQ^*$ is necessarily coprime, and hence comaximal, by Proposition 2.4, Corollary 1.

To state the next result we need the following definition. Let $A \in {}^{r}R^{m+1}$ be an admissible matrix over a semifir R; A will be called *minimal* if it is left full and cannot be transformed by a full operation ρ', followed by an operation ρ, to a matrix of smaller order, in other words, $\mathrm{O}(A, A)$ holds. By Lemma 4.3 and induction on the order we see that every element has a minimal admissible matrix.

We now come to the main result of this section, whose content and proof owe considerable improvements to the referee.

THEOREM 4.7. *Let R be a semifir, U its universal field of fractions, and p an element of U.*

(i) *We can pass between any two admissible matrices A, B for p by some sequence of operations $\lambda, \lambda', \rho, \rho'$ (at most six in all).*

(ii) *The minimal admissible matrices for p are just the admissible matrices of least order for p, and we can pass from any one minimal admissible matrix for p to any other by four full operations, viz. $\lambda', \lambda, \rho', \rho$, in that order.*

(iii) *If A and B are left full admissible matrices for p, then we can pass by full operations λ', ρ' from A to an admissible matrix A'' and from B to an admissible matrix B'' such that A'', B'' are stably biassociated.*

(iv) *Any two reduced matrices A, B for p are stably biassociated, and if there exists a reduced matrix for p, then there exists a reduced minimal matrix for p.*

Proof. We first prove (ii). We have seen that minimal admissible matrices for p exist. Let A, B be two such; without loss of generality we may assume that order $A \leqslant$ order B. Then $\mathrm{O}(B, A)$ certainly holds, so by Lemma 4.6, Corollary 1, the two sides of (13) represent an admissible matrix and we have a path from A to B by our four operations. By Lemma 4.6, P is right full, and since A is left full, P must be full; by the lemma, Q^* is also full. Because B is minimal, Q'^* must be full, and since A' and $P'A' = B'Q'^*$ are left full, P' must be full, by Lemma 4.3. So the four operations used are full. In particular, this implies that order A = order B, and this proves the last part of (ii).

Since all minimal admissible matrices for p have the same order, and since from any admissible matrix A for p we can construct a minimal admissible matrix by repeated operations ρ', ρ, λ' that do not increase the order, it follows that the common order of all minimal admissible matrices for p is the least order of any admissible matrix for p; this proves the rest of (ii).

We now let B be any admissible matrix for p, and we let A be any minimal admissible matrix for p; then $\mathrm{O}(B, A)$ will certainly hold and we can pass from B to A by four operations $\rho', \rho, \lambda', \lambda$. Hence, using any minimal admissible matrix A as a connecting link, we can pass between any two admissible matrices by a sequence of at most eight operations: $\rho', \rho, \lambda', \lambda, \lambda', \lambda, \rho', \rho$. By allowing the choice of the 'connecting link' to depend on one of the given matrices, we can in fact eliminate the middle two operations, getting a chain of six. This proves (i).

Next let A and B be left full admissible matrices for p, as in (iii). By Lemma 4.6, (12), we can transform A by a full operation λ', and B by a full operation ρ', to matrices A', B' satisfying

(22) $$P'A' = B'Q'^* \quad (Q'^* \text{ bordered}).$$

Since B is left full and Q^* is full, B' is left full, and hence the matrix $(P' \ B')$ is also left full. To show that $\begin{pmatrix} A' \\ Q'^* \end{pmatrix}$ is right full, suppose that $\begin{pmatrix} A' \\ Q'^* \end{pmatrix} x = 0$ for some $x \in {}^mU$. Since A' is admissible for p, the vector x is a right multiple of a vector $u = (1, u_*, p)^{\mathrm{T}}$, so if $x \neq 0$, its first entry must be non-zero. But $Q'^*x = 0$, so by the bordered form of Q'^*, the first entry of x is 0. This shows that $x = 0$, so $\begin{pmatrix} A' \\ Q'^* \end{pmatrix}$ is right full. Further, A' and B', being left full admissible matrices, both have characteristic 1 (Lemma 4.3(a),(b)), so by Proposition 2.4 we can cancel full square left and right factors from the two sides of (22) so as to obtain a comaximal relation. Cancelling a full square left factor from B' clearly corresponds to an admissible operation λ'. Now any full square right factor of Q'^* can be taken to be bordered, by the remark following Lemma 2.5, and cancelling this factor from A' corresponds to an admissible operation ρ'. The resulting comaximal relation

$$P''A'' = B''Q''^*$$

shows that A'' and B'' are stably biassociated.

Now the first statement of (iv) follows from (iii) by the definition of 'reduced'. For the second assertion we apply the reduction in (iii) to a minimal matrix A and a reduced matrix B. Of the resulting matrices A'' and B'', the first will again be minimal, because the operations applied to A were full, while the second can be taken equal to B, because B admits no non-trivial full operations λ' or ρ'. Hence A'', being stably biassociated to the reduced matrix B, will be a minimal reduced matrix for p. This completes the proof.

The least value of the order of any admissible matrix for p is called the *left depth* of p. By Theorem 4.7 (ii) it can be obtained as the order of any minimal admissible matrix for p. The *right depth* is defined analogously in terms of matrices A defining p over R by *right* linear relations $vA = 0$, where $v = (1, v_*, p) \in U^{m+1}$ and m is considered the order of this system. The properties of these functions will be the subject of Part II of this paper.

We note that whereas a minimal admissible matrix for a given $p \in U$ always exists, there may be no reduced matrix (in the absence of ACC). Moreover, a reduced matrix for p need not be minimal, as the example $\begin{pmatrix} y & -1 & 0 \\ 0 & 1 & -z \end{pmatrix}$ given earlier shows. But when there is a reduced matrix, there is also a minimal reduced matrix, by (iv) of Theorem 4.7. Some of the consequences of the existence of a reduced admissible matrix will be examined in the next section.

Sometimes a slight generalization of this set-up is needed, namely the equation corresponding to (2) which determines a $t \times t$ matrix over U. This is handled most simply by assuming R to be, not a semifir, but a full matrix ring over a semifir. Thus let T be a semifir with universal field of fractions V and write $R = T_t$, $U = V_t$. An $m \times n$ matrix A over R is now an $mt \times nt$ matrix over T; if its inner rank over T is r, then r/t will be called the *inner rank* of A, as matrix over R. So the inner rank need no longer be an integer, but it will be a rational number with denominator dividing t. By a left full matrix we understand as before an $m \times n$ matrix over R of inner rank m; clearly this will be left full as matrix over T. Right full and full matrices are defined correspondingly. The definitions of left and right prime, minimal, reduced, and of the basic operations

$\lambda, \rho, \lambda', \rho'$ are all as before. An admissible system for $p \in U$ can again be defined as a system (2) with matrix A and denominator both of inner rank m. It is clear from (3) that if the numerator (A_0, A_*) has rank s, then the rank of $p = u_m$ over U is $s-m+1$, a rational number between 0 and 1.

With these conventions Lemmas 4.3–4.6, and Corollaries, and Theorem 4.7 hold as stated, but for full matrix rings over semifirs. The proofs are similar to those given above and will be left to the reader. Likewise the partition lemma (Lemma 2.5) still applies, provided we interpret the 'number' of rows (or columns) appropriately as a fraction.

5. *Specializations*

For general R-fields (even when R is a semifir) we cannot of course expect such precise information as that obtained for the universal R-field in § 4. But the elements of any epic R-field, for any ring R, can be obtained as components of solutions of matrix equations and we begin our discussion in this general setting.

Let R be any ring and K an epic R-field. Given $p \in K$, we know that p may be obtained as the last component of a normalized solution $u = (1, u_*, p)^{\mathrm{T}}$ of an equation

$$Au = 0, \quad \text{with } A \in {}^rR^{m+1}, \tag{1}$$

where A has rank m over K. Any such matrix A is said to be *K-admissible* (for p). Clearly an $r \times (m+1)$ matrix $A = (A_0\ A_*\ A_m)$ is K-admissible if and only if the images of A and of $(A_*\ A_m)$ both have rank m over K. Further, it is clear that for any $p \in K$ we can find a K-admissible matrix of characteristic 1.

Now let L be another epic R-field and suppose that we have a specialization from L to K. Essentially this is a homomorphism $f\colon L_f \to K$ from a local R-subring L_f of L to K, compatible with R. If $\mathscr{P}_L, \mathscr{P}_K$ denote the sets of square matrices over R which become singular over L, K respectively, then $\mathscr{P}_L \subseteq \mathscr{P}_K$; in fact this condition is necessary and sufficient for a specialization from L to K to exist [7, p. 257]. It follows that the ranks over L and K of any matrix A over R are related by the inequality

$$\mathrm{rk}_L A \geqslant \mathrm{rk}_K A. \tag{2}$$

For the rank of a matrix A over a field may be described as the largest of the orders of non-singular minors of A.

Let A be a K-admissible $m \times (m+1)$ matrix. Then

$$\mathrm{rk}_K A = \mathrm{rk}_K(A_*\ A_m) = m,$$

and hence by (2), $\mathrm{rk}_L A = \mathrm{rk}_L(A_*\ A_m) = m$ (because the ranks of A and $(A_*\ A_m)$ cannot exceed m). Hence A is also L-admissible, and the remaining assertions of the following result are also easily verified.

PROPOSITION 5.1. *Let R be any ring, let K, L be any epic R-fields, and let $f\colon L_f \to K$ be a specialization between them. Then any K-admissible matrix A of characteristic* 1 *is also L-admissible, and if A defines the element p of L, then $p \in L_f$ and the element of K defined by A is pf.*

When this conclusion holds, we shall also say that the element p of L is *defined in K*.

In particular, if R has a universal field of fractions U, the proposition holds (with $L = U$) for any epic R-field K and the canonical specialization $f\colon U_f \to K$. Of course when R is a semifir, so that a universal field of fractions U exists, then 'U-admissible' as defined here is the same as 'admissible' in §4. We can now state

THEOREM 5.2 (theorem on universal denominators). *Let R be a semifir and let U be its universal field of fractions. If an element p of U can be defined by a reduced admissible matrix A, and there is a matrix for p which remains admissible over an epic R-field K, then A is also K-admissible.*

If moreover, R is a fully atomic semifir, then every element of U can be defined by a reduced admissible matrix.

Proof. Let $p \in U$ and let A be a reduced admissible matrix defining p. By hypothesis there is a U-admissible matrix B for p, of characteristic 1, which is also K-admissible. By Theorem 4.7 (iii) we can pass by full operations λ', ρ' from A to A'' and from B to B'' such that A'' and B'' are stably biassociated. Since A is left prime, λ' is trivial and we may take $A'' = A$. Thus A is stably biassociated to B''. Now since B is K-admissible, the system $Bz = 0$ has a normalized solution, and $B = B''Q^*$, so $B''y = 0$ has a normalized solution $y = Q^*z$ in K; therefore its denominator is invertible over K. Thus B'' is K-admissible, and hence so is A.

If R is fully atomic, and $p \in U$ is defined by an admissible matrix A, then A can be transformed to a left prime matrix A' by an operation λ', using Proposition 2.7, and A' can be transformed to a minimal reduced matrix by an operation ρ', using the left-right dual of Proposition 2.7 and Theorem 4.7 (iv).

In Theorem 5.2 we had to assume the existence of a reduced matrix (in the absence of chain conditions). This just corresponds to the fact that atomic factorizations need not exist in a Bezout domain. However, we do have highest common factors, and correspondingly we have

PROPOSITION 5.3. *Let R be a semifir with a universal field of fractions U. Given $p \in U$, let A, B be two admissible matrices for p. Then there is an admissible matrix C for p which is admissible over (at least) all R-fields where A or B is admissible.*

Proof. By operations λ' we can pass from A, B to left full matrices A', B'. By further full operations λ' applied to A', and ρ' applied to B', we pass to admissible matrices A'' and B'' which are stably biassociated (Theorem 4.7 (iii)). Put $C = A''$; then A is obtained from C by an operation λ. Hence if A is K-admissible, then so is C. Similarly when B is K-admissible, then so is B'' and hence C ($= A''$). This completes the proof.

We conclude this section with an application to rational identities using Cramer's rule, but none of the later results of §4.

Let us consider formal *rational expressions* in a family of symbolic indeterminates $\Xi = \{\xi_1, \dots, \xi_r\}$ and the elements of the base field k. We shall not identify expressions such as $(\xi_1 \xi_2)^{-1}$ and $\xi_2^{-1} \xi_1^{-1}$ (since if we did we would have no formal way of stating that $(\xi_1\xi_2)^{-1} = \xi_2^{-1}\xi_1^{-1}$ is an identity), nor exclude such expressions as $(\xi_1 - \xi_1)^{-1}$ (since we cannot specify the expressions f^{-1} to be excluded until we have studied identities). Of course an expression f^{-1} will be undefined if applied to a non-invertible element.

More generally, if K is a skew field which is a k-algebra, a rational expression in $\Xi \cup K$ will be called a *generalized* rational expression in Ξ. (It is understood that K is fixed; when only coefficients from k are involved, the qualifier 'generalized' is omitted.)

Given a generalized rational expression $f(\xi_1,\dots,\xi_r)$, a K-ring R which is a k-algebra, and $a_1,\dots,a_r \in R$, we may evaluate f in R by substituting a_i for ξ_i. The result will be either an element $f(a)=f(a_1,\dots,a_r)$ in R or undefined. We shall say that f is a *generalized rational identity* for R if for every choice of $a_1,\dots,a_r \in R$, $f(a)$ is either zero or undefined. Here an expression is undefined if at any stage in its evaluation we have to form the inverse of a non-invertible element in R. For this reason the concept of a rational identity behaves unexpectedly in a ring with few units. For example, in $\mathbf{Z}$ the rational identity $\xi-\xi^{-1}=0$ holds, though it fails in homomorphic images of $\mathbf{Z}$ such as $\mathbf{Z}/5\mathbf{Z}$. And even an algebra over a field, say a free k-algebra $k\langle X\rangle$, satisfies the identity $\xi\eta^{-1}-\eta^{-1}\xi=0$ though it has homomorphic images which do not. To obtain a correct statement on the specialization of rational identities we need to consider identities holding in skew fields.

Thus we shall call the expression $f(\xi_1,\dots,\xi_r)$ an *absolute generalized rational identity* if in $K_k \langle\!\langle x_1,\dots,x_r\rangle\!\rangle$, the universal field of fractions of $K_k\langle x_1,\dots,x_r\rangle$, the element $f(x_1,\dots,x_r)$ is either zero or undefined. From the universal property of $K_k\langle\!\langle X\rangle\!\rangle$ for specializations (Proposition 5.1) it follows that any absolute generalized rational identity f is in fact a generalized rational identity for all K-fields L(understood as k-algebras). Below we shall give a direct proof that such an f is a generalized rational identity for any weakly finite k-algebra containing K. We first reduce the question to matrix form; I am indebted to the referee for observing and correcting an error in the original formulation of this result.

PROPOSITION 5.4. *Let K be a skew field which is a k-algebra. To every generalized rational expression f over K in the set of indeterminates $\Xi=\{\xi_1,\dots,\xi_r\}$ one can associate a matrix $A=A(x)$ over $K_k\langle X\rangle$ of characteristic* 1, *say $m\times m+1$, with the following properties:*

(i) *at any point $a=(a_1,\dots,a_r)$ over any weakly finite K-ring R which is a k-algebra, $f(a)$ is defined if and only if the denominator of $A(a)$ is invertible over R;*

(ii) *if the point a satisfies the equivalent conditions of* (i), *then $f(a)$ may be obtained as the last component u_m of the unique normalized solution $u=(1,u_*,u_m)^{\mathrm{T}}$ of the equation $Au=0$.*

Proof. We shall use induction on the complexity of f. If $f=\xi_i$ we can take $A=(x_i,-1)$; similarly, if $f\in K$, we take $A=(f,-1)$. Here condition (i) is vacuous and (ii) is immediate.

If $f=g+h$ and the matrices associated to g,h are B,C of orders p,q respectively, then to f we associate the matrix

$$A=\begin{pmatrix} B_0 & 0 & -B_p & B_* & B_p \\ C_0 & C_* & C_q & 0 & 0\end{pmatrix}.$$

When we evaluate this matrix at a point a of a weakly finite ring R, we see that its denominator will be invertible if and only if those of B,C are (the 'only if' uses the weak finiteness of R and the fact that the denominators $(B_*\ B_p)$ and $(C_*\ C_q)$ are square matrices). This proves the induction step (i) for this case, while the verification of the induction step for (ii) is straightforward. Similarly, if $f=gh$, we use

$$A=\begin{pmatrix} 0 & 0 & B_0 & B_* & B_p \\ C_0 & C_* & C_q & 0 & 0\end{pmatrix}$$

with the same reasoning as before. When $g=-1$, this gives a matrix for $-h$.

There remains the case when $f = g^{-1}$. If the matrix associated to g is B; of order p, it would seem natural to take

$$A = (B_p \quad B_* \quad B_0).$$

This has almost the desired properties. For if $f(a)$ is defined, this means that $g(a)$ is defined and invertible, and hence, by the induction hypothesis, $B(a)$ defines $g(a)$ and, by Cramer's rule (§4, (3)), $B(a)$ will have an invertible numerator. Hence $A(a)$ has an invertible denominator and so defines $f(a)$. However, the converse may not hold; namely if $g(a)$ itself is undefined, so that $f(a) = g(a)^{-1}$ also cannot be defined, the denominator of A may still be invertible. (The simplest such case is that where $f = g^{-1} = (h^{-1})^{-1}$ and $h(a)$ is defined but not invertible.) Thus for $f = g^{-1}$ to be defined it is necessary for g to be defined and invertible; this will be so if the denominator of A contains both the numerator and the denominator of B. To achieve this we put

$$A = \begin{pmatrix} B_p & 0 & 0 & B_* & B_0 \\ 0 & B_* & B_p & 0 & 0 \end{pmatrix};$$

now it is easily checked that (i) and (ii) hold and this completes the proof.

We recall that a square matrix over $K_k\langle X\rangle$ is invertible over $K_k \not\langle X \not\rangle$ if and only if it is full. Hence an expression f is an absolute generalized rational identity if and only if the corresponding matrix A constructed in Proposition 5.4 has either a non-full denominator (making $f(x)$ undefined) or a non-full numerator (making $f(x) = 0$, if it is defined).

THEOREM 5.5. *Let K be a skew field which is a k-algebra and let R be a K-ring which is a k-algebra. Then every absolute generalized rational identity is a generalized rational identity for R if and only if R is weakly finite.*

Proof. Assume first that R is weakly finite; let f be an absolute generalized rational identity, let A be the associated matrix, and let $a \in R^r$. Since $A(x)$ has a non-full numerator or denominator, the same is true of $A(a)$. Now a non-full matrix over a weakly finite ring cannot be invertible, so either (i) $A(a)$ has a non-invertible denominator, or (ii) $A(a)$ has an invertible denominator and a non-full numerator. By Proposition 5.4, in case (i) $f(a)$ is not defined. In case (ii) the numerator is non-full, so by Cramer's rule (§4, (3)), we find that $f(a) \dotplus I$ is not full, say

$$\begin{pmatrix} f(a) & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} p \\ P \end{pmatrix} (q \quad Q),$$

where P, Q are square. Thus $PQ = I$, $pQ = 0 = Pq$, $pq = f(a)$. Since R is weakly finite, $QP = I$, so $p = 0 = q$ and $f(a) = 0$, as we wished to show.

Conversely, suppose that R is not weakly finite. Then R contains square matrices P, Q such that $PQ = I$ and $QP \neq I$. Let P, Q be $n \times n$ say; writing S, T for $n \times n$ matrices with indeterminate entries, consider the matrix equation

$$T(ST)^{-1}S - I = 0.$$

Written out in full, the left-hand side consists of n^2 expressions in the entries of S, T, and $(ST)^{-1}$; thus they are rational expressions which are (defined and) zero in the free field $K_k \not\langle X \not\rangle$, and so are absolute rational identities, but not all of them hold when we set $S = P$, $T = Q$, though all are defined.

6. *The R-module structure of U*

Let A be a left full matrix of characteristic 1 over a semifir R, say A is $m \times m+1$, and denote by M the left R-module defined by A. We obtain a presentation of M by taking as a generating set the components of a column $g = (g_0, \ldots, g_m)^{\mathrm{T}}$ with defining relations

$$Ag = 0. \tag{1}$$

In particular, if A is an admissible matrix for $p \in U$, with the admissible system $Au = 0$, then we have a module homomorphism

$$M \to U, \quad g_i \mapsto u_i \quad (i = 0, 1, \ldots, m). \tag{2}$$

Our aim will be to show that this mapping is an embedding and that the images in U form a local system, as A varies over all admissible matrices. To establish this fact we need a result on matrices with infinitely many rows.

LEMMA 6.1. *Let R be a fully atomic semifir, and let U be its universal field of fractions. If A is a matrix over R with possibly infinitely many rows and n columns, of rank $t \leqslant n$ over U, then $A = PA'$, where $A' \in {}^tR^n$.*

Proof. Let us choose t rows of A that are linearly independent over U and denote by M the submodule of R^n generated by these rows. Then M is not contained in any submodule of R^n generated by fewer than t elements. Hence by Proposition 3.4 there is a largest t-generated submodule, N say, containing M. Now for any other row c of A, the matrix consisting of the given t rows and c still has rank t over U; hence it has inner rank t over R, and so the module spanned by M and c lies in a t-generator submodule, and therefore it lies in N. Thus all rows of A lie in N and we obtain the required factorization by writing $N = R^tA'$.

The following special case does not require atomicity and is sometimes of interest.

COROLLARY. *Let R be a semifir and let U be its universal field of fractions. Given $a_i, b_i \in R$ $(i = 0, 1, \ldots, n)$, with $a_0, b_0 \neq 0$, such that*

$$a_i a_0^{-1} = b_i b_0^{-1} \quad \text{in } U, \tag{3}$$

there exist $a, b, c_i \in R$ such that $a_i = c_i a$, $b_i = c_i b$.

For, by (3), the $n+1 \times 2$ matrix C with columns $(a_i), (b_i)$ *has rank 1 over U. Therefore*

$$C = (c_0, \ldots, c_n)^{\mathrm{T}}.(a, b),$$

for some $a, b, c_i \in R$ and the result follows.

We apply Lemma 6.1 to establish the following property of U:

PROPOSITION 6.2. *Let R be a fully atomic semifir and let U be its universal field of fractions. Then any non-zero finitely generated submodule of U, qua left R-module, is finitely related and has positive characteristic.*

Proof. Let $M \subseteq U$ be generated by $u_1, \ldots, u_n$. Write $u = (u_1, \ldots, u_n)^{\mathrm{T}}$ and express the defining relations of M in matrix form

$$Au = 0, \tag{4}$$

where A is a matrix over R with n columns and possibly infinitely many rows. Since (4) has a non-zero solution in U, we have $\operatorname{rk} A = t < n$. By Lemma 6.1, A can be written in the form

$$(5) \qquad A = PA', \quad \text{with } A' \in {}^tR^n,\ t < n,$$

where $\operatorname{rk} A = \operatorname{rk} A' = t$. Then A and A' have the same rank t over U and so by (5) have the same right annihilator in ${}^nU$. Thus

$$(6) \qquad A'u = 0;$$

it follows that the module, M' say, defined by A' has M as a homomorphic image. But by (5) all the relations (4) are consequences of (6), whence M and M' are isomorphic. Thus M is a finitely presented module of characteristic $n-t > 0$.

Let us now return to U and consider any element $p \in U$ with a left prime admissible system $Au = 0$. Denote by M the left R-module defined by A and by M'' the submodule of U (qua left R-module) generated by the components of u. Clearly M'' is a homomorphic image of M, and M is prime of characteristic 1, because A is left prime. By Proposition 6.2, M'' has positive characteristic, and hence the kernel of the map $M \to M''$ has characteristic $1-\chi(M'') \leqslant 0$, and by Proposition 3.1 it must be 0; therefore $M'' \cong M$. Moreover, if B is any other left prime admissible matrix for p, then by Theorem 4.7 (iii) we can pass, by full operations ρ', from A to A'' and from B to B'' such that A'' is stably biassociated to B''. In particular, if A and B are any *reduced* admissible matrices for p, then by Theorem 4.7 (iv) they are stably biassociated; the transformation from A to B is then simply a change of presentation for M'' but leaving $1, p$ (as part of the generating set) fixed. Thus M'' is a module invariantly associated with p; we shall call it the module *associated* with p. The next result shows how we can associate a module with each finite family in U.

THEOREM 6.3. *Let R be a semifir with left and right ACC_n, for all n, and let U be its universal field of fractions. Then any finitely generated submodule of U (as R-module) is positive prime, and any (non-empty) finite family of non-zero elements of U is contained in a unique least left R-submodule of characteristic* 1 *of U.*

Proof. Let M be a submodule of U; by Proposition 6.2, any finitely generated non-zero submodule of M has positive characteristic. Hence M is positive prime, by Proposition 3.1. Now let $\{p_1, \ldots, p_k\}$ be a family of non-zero elements of U. We begin by finding a submodule of characteristic 1 to contain our family. On right multiplication by p_1^{-1} we may assume that $p_1 = 1$. When $k = 2$, we know that the module associated with p_2 is of characteristic 1 and it contains $p_1 = 1, p_2$. We shall use induction on k; thus assume $k > 2$ and let $1, p_2 \in M$ and $1, p_3, \ldots, p_k \in N$, where M, N are submodules of U of characteristic 1. Then $M \oplus N$ and $M+N$ are finitely presented, and hence so is $M \cap N$, by Proposition 5.3.1 of [7, p. 183]. Moreover, $M \cap N \neq 0$ because $1 \in M \cap N$; therefore $\chi(M \cap N) > 0$, by Proposition 6.2, and so

$$0 < \chi(M+N) = \chi(M)+\chi(N)-\chi(M \cap N) < \chi(M)+\chi(N) = 2.$$

It follows that $\chi(M+N) = 1$, and $M+N$ clearly contains $p_1, p_2, \ldots, p_k$.

Now the submodules of U of characteristic 1 containing the given family correspond to submodules of U/Rp_1 of characteristic 0 containing the images of the given family.

Since the modules of characteristic 0 are bound, there is a least such submodule, by Proposition 3.3, and the corresponding submodule of U is the required module.

We note that in contrast to the commutative case the family need not be contained in a cyclic submodule of U, for example, $R = k\langle x, y\rangle$, $p_1 = x^{-1}$, $p_2 = y^{-1}$.

As the referee has observed, the methods of this paper show that over any fir R the category of prime left R-modules of characteristic 1 extending R is a preorder (between any two such modules there is at most one homomorphism, necessarily an injection, by Proposition 3.5) and this preorder has faithful pushouts. By taking the direct limit one obtains again the universal field of fractions of R, in a particularly transparent form. For details of this construction (in the somewhat wider context of semihereditary rings) we refer the reader to [4].

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Department of Mathematics
Bedford College
Regent's Park
London NW1 4NS