THE THEOREMS OF BONY AND BREZIS ON FLOW-INVARIANT SETS

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Throughout this note Ω is a domain in real Euclidean space E_n, $X(x)$ is a function on Ω to E_n, and F is a closed subset of Ω. We shall be concerned with trajectories of the vector field X, that is, with solutions of

$$\frac{dx}{dt} = X[x(t)], \quad x(t) \in \Omega.$$

The set F is *flow invariant* for X if every trajectory $x(t)$ which meets F at t_0 must remain in F for $t > t_0$. Thus, in the case of flow invariance,

$$x(t_0) \in F \Rightarrow x(t) \in F \text{ for } t_0 \leqq t < t_1,$$

where $[t_0, t_1)$ is the interval of existence for the trajectory through the point $x(t_0)$. When the solution does not exist beyond t_0, the condition is considered to be vacuously fulfilled.

Our objective is to generalize a remarkable theorem for flow-invariant sets that was recently obtained by Bony [2] and to show its relation to another theorem of Brezis [3]. The proofs here are simpler than those given hitherto, and the results are stronger. However, this paper is expository.

1. The theorems of Bony. Let $y \in F$ and let S be a sphere which has y on its boundary but does not contain any point of F in its interior. If S is centered at x, the vector $v(y) = x - y$ is normal to F at y in the sense of Bony. The following hypotheses involving v are used only at points y admitting a normal in this sense. In other words, if there is no sphere S as described above, the hypotheses are considered to be vacuously fulfilled.

For a given real-valued function δ, the upper left and right Dini derivates are respectively.

$$D^-\delta(t) = \limsup_{h\to 0+} \frac{\delta(t) - \delta(t-h)}{h}, \quad D^+\delta(t) = \limsup_{h\to 0+} \frac{\delta(t+h) - \delta(t)}{h}.$$

The lower Dini derivates D_- and D_+ are defined similarly, with lim inf instead of lim sup.

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We say that a real-valued function ρ is a *uniqueness function* if the conditions

$$D^-\delta(t) \leqq \rho[\delta(t)], \quad D^+\delta(t) \leqq \rho[\delta(t)], \quad 0 < t < \varepsilon$$

together imply $\delta(t) = 0$, $0 < t < \varepsilon$, for every continuous function $\delta(t)$ satisfying

$$\delta(t) \geqq 0, \ \delta(0) = 0.$$

The uniqueness is required only for some positive ε.

THEOREM 1. (Bony). *Let X and F satisfy the following two conditions*:
(i) $(x-y)\cdot[X(x)-X(y)] \leqq |x-y|\rho(|x-y|)$ *for a uniqueness function* ρ;
(ii) $\nu(y)\cdot X(y) \leqq 0$ *whenever* $\nu(y)$ *is normal to* F *at* y.
Then F *is flow-invariant for* X.

Bony's theorem in its original form [2] is obtained when condition (i) is replaced by the familiar Lipschitz condition,

$$|X(x) - X(y)| \leqq K|x-y|, \ K \text{ constant}.$$

This corresponds to the choice $\rho(s) = Ks$, which is well known to be a uniqueness function in the above sense.

If Theorem 1 does not hold we can find t_0 such that $x(t_0) \in F$, but $x(t)$ is not in F on some interval $t_0 < t < t_1$ on which $x(t)$ exists. In all such cases we shall take $t_0 = 0$, as can be done without loss of generality. Let t be on $0 < t < t_1$ and let $\delta(t)$ denote the distance from $x(t)$ to F. Then

$$\delta(0) = 0, \quad \delta(t) > 0 \ \text{ for } \ 0 < t < t_1.$$

For fixed t on $(0, t_1)$ let $x_h = x(t+h)$, let $x = x(t)$, and let $y \in F$ be a nearest point to x. Evidently

$$\delta(t+h) \leqq |x_h - y|, \ \delta(t) = |x-y|,$$

and hence, by the identity $a - b = (a^2 - b^2)/(a+b)$,

$$\delta(t+h) - \delta(t) \leqq \frac{|x_h - y|^2 - |x-y|^2}{|x_h - y| + |x-y|}. \tag{1}$$

The differential equation $dx/dt = X(x)$ gives

$$x_h = x + hX(x) + o(h).$$

If we compute $x_h - y$ from this and dot the result with itself, the numerator in (1) is found to be

$$2h(x-y)\cdot X(x) + o(h).$$

Dividing (1) by h and letting $h \to 0+$ therefore gives

$$D^+\delta(t) \leqq \frac{(x-y)\cdot X(x)}{|x-y|}. \tag{2}$$

The vector $v(y) = x - y$ is normal to F at y in the sense of Bony, and hence $(x-y)\cdot X(y) \leqq 0$. If this term is subtracted from the numerator in (2) the resulting inequality is

$$D^+\delta(t) \leqq \frac{(x-y)\cdot[X(x)-X(y)]}{|x-y|} \leqq \rho(|x-y|) = \rho[\delta(t)].$$

A more difficult argument, which we omit, gives a corresponding inequality for $D^-\delta(t)$. Since ρ is a uniqueness function, it follows that $\delta(t) = 0$, and this is a contradiction.

According to Bony the field X is *tangent to* F if $v(y)\cdot X(y) = 0$ for every $y \in F$ admitting a normal $v(y)$. In that case one can apply Theorem 1 as it stands and again with $-t$ replacing t. The result is the following, due also to Bony for the case $\rho(s) = Ks$:

THEOREM 2. (Bony). *Let X be tangent to F and let*

$$|X(x) - X(y)| \leqq \rho(|x-y|),$$

where ρ is a uniqueness function. Then any trajectory of $dx/dt = X(x)$ which meets F in one point must lie entirely in F.

The surprise in Theorems 1 and 2 is that F can fail to have a normal at a great many points, and it is by no means obvious *a priori* that the trajectory $x(t)$ could not escape from F at such a point. One of the main applications is to the sharp maximum principle [2], [5]. This application uses the full force of Bony's formulation, both as regards the one-sided condition (ii) and as regards the generality of the closed set F. At an opposite extreme, let F be the trace of a given solution-curve, $\tilde{x}(t)$. The statement that $x(t) \in F$ is then the familiar uniqueness theorem for autonomous systems.

2. The theorem of Brezis. To state the next result let $|x, F|$ denote the distance from any point x to the closed set F. We then have:

THEOREM 3. *Let X and F satisfy the following two conditions*:
(i) $(x-y)\cdot[X(x)-X(y)] \leqq |x-y|\rho(|x-y|)$ *for a uniqueness function* ρ;
(ii) $\liminf_{h\to 0+} \dfrac{|y + hX(y), F|}{h} = 0$ *for each* $y \in F$.
Then F is flow invariant for X.

The condition (ii) is needed only at each y which possesses a normal in the sense of Bony. If there exists a trajectory satisfying

$$\frac{dx}{dt} = X(x), \quad x(0) = y,$$

then $x(h) = y + hX(y) + o(h)$ and the hypothesis (ii) is indistinguishable from

$$\liminf_{h\to 0+} \frac{|x(h),F|}{h} = 0.$$

This formulation bears an interesting relation to the conclusion, since the latter means that $|x(h),F| = 0$ for all $h \geqq 0$ on the interval of existence.

To prove Theorem 3, let v be normal to F in Bony's sense at $y \in F$ and let the sphere associated with v have center x, so that $v = x - y$. For $h \geqq 0$ it is convenient to set

$$\varepsilon(h) = |y + hX(y), F|. \tag{3}$$

Clearly

$$|x - y| \leqq |x, F| \leqq |x - y - hX(y)| + \varepsilon(h), \tag{4}$$

where the first inequality follows from the fact that the sphere associated with $v(y)$ is free of points of F, and the second follows from

$$|x, F| \leqq |x - \tilde{x}| + |\tilde{x}, F| \tag{5}$$

with $\tilde{x} = y + hX(y)$. If the middle term is omitted from (4) and the resulting inequality is squared, we get

$$0 \leqq -2h(x - y) \cdot X(y) + o(h) + O[\varepsilon(h)].$$

Dividing by h and letting $h \to 0+$ through a suitable sequence, gives

$$X(y) \cdot (x - y) \leqq 0$$

which is Bony's condition (ii). Thus Theorem 3 follows from Theorem 1.

We want to formulate a weaker version of Theorem 3 which is very easy to prove, and yet generalizes the result of Brezis. To this end, ρ is called a *restricted uniqueness function* if the inequality

$$D_+\delta(t) \leqq \rho[\delta(t)], \quad 0 < t < \varepsilon,$$

implies $\delta(t) = 0$ for the same class of functions $\delta(t)$ as that considered above. Clearly, restricted uniqueness functions are also uniqueness functions.

THEOREM 4. (Brezis). *Let X and F satisfy the following two conditions:*
(i) $|X(x) - X(y)| \leqq \rho(|x-y|)$ *for a restricted uniqueness function* ρ;
(ii) $\liminf_{h\to 0+} \dfrac{|y + hX(y), F|}{h} = 0$ *for each* $y \in F$.
Then F is flow-invariant for X.

When $\rho(s) = Ks$ and when the lim inf in (ii) is replaced by lim, the result is Brezis' theorem in its original form [3]. Theorem 4 follows from Theorem 3, which is stronger both as regards the class $\{\rho\}$ and as regards the condition (i).

To deduce Theorem 4 from first principles, let $\delta(t)$ and $x(t)$ be as in the proof of Theorem 1, and define $\varepsilon(h)$ by (3). Then by (5)

$$\delta(t+h) \leqq \left| x(t+h) - y - hX(y) \right| + \varepsilon(h).$$

Since $x(t+h) = x + hX(x) + o(h)$ this gives

$$\delta(t+h) \leqq \left| \delta(t) + hX(x) - hX(y) \right| + o(h) + \varepsilon(h)$$

and hence

$$\delta(t+h) - \delta(t) \leqq h \left| X(x) - X(y) \right| + o(h) + \varepsilon(h).$$

Upon dividing by h and letting $h \to 0+$ through a suitable sequence, we get

$$D_{+}\delta(t) \leqq \rho[\delta(t)].$$

The conclusion follows at once.

Instead of considering the point $y + hX(y)$ as above, Brezis considers the point $x(h)$ on the trajectory satisfying

$$\frac{dx}{dt} = X(x), \qquad x(0) = y.$$

This seemingly minor alteration makes quite a difference, because the proof now depends on the existence of the trajectory through y and on its stability with respect to the initial value, y. (The first step of Brezis' proof invokes the stability inequality, which was not used here.) Existence and stability are available in the case $\rho(s) = Ks$ considered by Brezis, but are less immediate for general ρ.

3. Osgood functions. Discussion of the first-order equation for ρ involves knowledge of Dini derivates, and some of their properties are given now. In a general way, it can be said that these properties resemble those of ordinary derivatives. For instance, if f and $\phi \geqq 0$ are continuous then

$$D \int_0^{f(t)} \phi(s)ds = \phi[f(t)]Df(t), \tag{6}$$

where D stands for any one of the four derivates. The proof for D^- and D_- follows from

$$\frac{1}{h} \int_{f(t-h)}^{f(t)} \phi(s)ds = \frac{f(t) - f(t-h)}{h} \phi(\xi),$$

where ξ is between $f(t)$ and $f(t-h)$. This, in turn, is just the first mean-value theorem for integrals. Proof for D^+ and D_+ is similar.

As another illustration, suppose the continuous function g satisfies

$$Dg(t) < 1, \ 0 < t \leqq t_1; \ g(0) = 0, \tag{7}$$

where D is one of the derivates. Then $g(t) \leqq t$ on this interval. We give the proof for D_-; the case D_+ is a little harder. If the conclusion fails, the function $G(t) = g(t) - t$ attains a positive maximum at some point t, $0 < t \leqq t_1$. Thus $G(t-h) \leqq G(t)$ for each small positive h or equivalently,

$$\frac{g(t) - g(t-h)}{h} \geqq 1.$$

Hence the lim inf is also $\geqq 1$ and this is a contradiction.

A function $\rho(s)$ is an *Osgood function* if ρ is continuous, nonnegative, and if

$$\int_0^\eta \frac{ds}{\rho(s)} = \infty$$

for each small positive η. Since the meaning of the integral is not clear when 0 is a limit point of zeros of ρ, we agree that the above equation means

$$\lim_{\varepsilon \to 0+} \int_0^\eta \frac{ds}{\varepsilon + \rho(s)} = \infty. \tag{8}$$

In other words, the integral is interpreted in the sense of Lebesgue.

THEOREM 5. *Every Osgood function is a uniqueness function for each of the four Dini derivates, hence is usable for ρ in Theorems* 1–4.

The fact that Osgood functions are uniqueness functions is well known, but the following proof, based on [**6**] and [7], is simpler than proofs sometimes given. For $\varepsilon > 0$ define

$$g(t) = \int_0^{\delta(t)} \frac{ds}{\varepsilon + \rho(s)}.$$

If D denotes D_- or D_+, then by (6) and by $D\delta \leqq \rho(\delta)$,

$$Dg(t) = \frac{D\delta(t)}{\varepsilon + \rho[\delta(t)]} \leqq \frac{\rho[\delta(t)]}{\varepsilon + \rho[\delta(t)]} < 1.$$

Since $g(0) = 0$ we get $g(t) \leqq t$ by (7) and hence

$$\int_0^{\delta(t)} \frac{ds}{\varepsilon + \rho(s)} \leqq t_1, \quad 0 < t \leqq t_1. \tag{9}$$

If $\delta(t) = \eta > 0$ at some point t, this choice of t in (9) contradicts (8).

4. Further discussion of uniqueness. So far, we have required uniqueness for arbitrary continuous functions $\delta(t)$. However, the function $\delta(t)$ for which uniqueness is actually needed is somewhat restricted; it is the composition of a Lipschitzian function with the differentiable function $x(t)$. To see this, note that (5) as it stands and (5) with x and $\tilde{x}$ interchanged gives

$$(10) \qquad |L(x) - L(\tilde{x})| \leqq |x - \tilde{x}|,$$

where $L(x) = |x, F|$. Since $\delta(t) = |x(t), F| = L[x(t)]$, the above remark is verified.

If X is locally bounded, then by (10)

$$|\delta(t) - \delta(\tilde{t})| \leqq |x(t) - x(\tilde{t})| \leqq M|t - \tilde{t}|,$$

where M is a bound for $|dx/dt| = |X(x)|$ in the relevant neighborhood, and hence, $\delta(t)$ is locally Lipschitzian. If, in addition, X is continuous, then $\delta(t) = o(t)$ as $t \to 0+$. This holds under Brezis' hypothesis whether X is continuous or not. To get it under Bony's hypothesis, note that the equation below (2) implies

$$(11) \qquad D^{+}\delta(t) \leqq |X(x) - X(y)|.$$

As $t \to 0+$ clearly $x \to x(0) \in F$, hence the nearest point y approaches $x(0)$ also, and the right side of (11) is less than ε near $0+$ for each positive ε. Applying (7) to $\delta(t)/\varepsilon$ gives $\delta(t) \leqq \varepsilon t$ near 0, as desired.

The reader familiar with uniqueness theorems of Kamke will know that the condition $\delta(t) = o(t)$ at $0+$ usually extends the class of functions ρ for which uniqueness holds. Accordingly, we call ρ a *generalized uniqueness function* if the conditions

$$(12) \qquad D^{-}\delta(t) \leqq \rho[\delta(t)], \quad D^{+}\delta(t) \leqq \rho[\delta(t)], \quad 0 < t < \varepsilon$$

imply $\delta(t) = 0$, $0 < t < \varepsilon$, for every function $\delta(t)$ on $0 \leqq t < \varepsilon$ which satisfies

$$\delta(t) \geqq 0, \ \delta(t) \in \mathrm{Lip}\,1, \quad \lim_{t \to 0+} \frac{\delta(t)}{t} = 0.$$

So far we have required that $dx/dt = X(x)$ hold for all t. It is usually sufficient, however, to have $x(t)$ continuous and to have the differential equation hold except perhaps on a countable set. When such is the case it is said that the differential equation holds $\mathrm{mod}\,E$.

By considering the integral of a Cantor function one sees that the hypothesis $\mathrm{mod}\,E$ cannot be replaced by a similar hypothesis $\mathrm{mod}\,N$, where N denotes an arbitrary null set. However, the extension can be made if x is required to be absolutely continuous. In that case the differential equation can be interpreted as an integral equation,

$$x(t) = \int_{t_0}^{t} X[x(s)]ds + x(t_0).$$

Clearly $\delta(t)$ is continuous if $x(t)$ is. To check for absolute continuity one would consider

$$|x(t_1) - x(t_2)| + |x(t_3) - x(t_4)| + \cdots + |x(t_{m-1}) - x(t_m)| \leqq \eta.$$

This gives a similar inequality for $\delta(t) = L[x(t)]$ and hence, L maps the absolutely continuous functions on E_n into absolutely continuous functions on E_1. It is also true that the above analysis gives (12) at each point t, where $dx/dt = X(x)$. Hence if the latter holds $\bmod E$ or $\bmod N$, as the case may be, so does the former.

It is left for the reader to formulate what is meant by a uniqueness function $\bmod E$ or $\bmod N$. The results of this discussion are then summarized as follows:

THEOREM 6. *In Theorems* 1–3 *suppose the hypothesis is changed in one of the following three ways*:

(i) X is *continuous and* ρ *is a generalized uniqueness function*; *or*

(ii) $dx/dt = X(x) \bmod E$, *and* ρ *is a uniqueness function* $\bmod E$; *or*

(iii) $dx/dt = X(x) \bmod N$, *and* ρ *is a uniqueness function* $\bmod N$.

Then the conclusions still hold.

The most important special case is given by the following:

THEOREM 7. *The conclusions of Theorems* 1–3 *hold for every Osgood function* ρ, *even if the differential equation* $dx/dt = X(x)$ *is given only* $\bmod E$ *or* $\bmod N$.

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