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ON PROBLEMS RELATED TO GROWTH, ENTROPY, AND SPECTRUM IN GROUP THEORY

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ABSTRACT. We review some known results and open problems related to the growth of groups. For a finitely generated group Γ, given whenever necessary *together with* a finite generating set, we discuss the notions of

(1) uniformly exponential growth,
(2) growth tightness,
(3) regularity of growth series,
(4) classical growth series (with respect to word lengths),
(5) growth series with respect to weights,
(6) complete growth series,
(7) spectral radius of simple random walks on Cayley graphs.

From the modern point of view, a dynamical system is a pair (G, X), where G is a group (or a semi-group) and X is a set on which G acts; introducing different structures on G and X, we get different directions for the Theory of Dynamical Systems. Many of the dynamical properties of the pair (G, X) depend on appropriate properties of the group G.

In this paper we discuss such notions for groups as growth, entropy, amenability and spectral radius of random walks, which are closely related to the notions of growth of manifolds and foliations, entropy of geodesic flows and of symbolic systems, spectral theory of Laplace operator and theory of harmonic functions on Riemannian manifolds.

It is a great pleasure for the first author to mention that one of the results discussed in this paper, namely the cogrowth criterion for amenability, was reported first at the seminar of Prof. D.V. Anosov on the Theory of

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Dynamical Systems; this was in the Steklov Mathematical Institute in the spring of 1975.

Growth considerations in group theory with motivations from differential geometry have been introduced in the early 1950's ([151], [52], as well as [60]) and again (independently!) in the late 1960's [125]. Nowadays, it is part of overlapping subjects with names such as *combinatorial group theory*, *geometric group theory*, or *asymptotic group theory* (the latter name being apparently first used in [71]). These theories have a rich history [33], for which milestones have been the 1966 book by W. Magnus, A. Karas, and D. Solitar [122], and the 1987 list of problems of R. Lyndon [118] (see also [119]). For the state of the art concerning infinite groups, see [85] for a quasi-isometric picture of the subject; for the asymptotic group theory of finite groups, see [98].

For growth of other objects, see the following papers, as well as the references therein.

- [73] and [157] for semi-groups,
- [171] for graphs,
- [73], [75], [80], and [146] for automata and languages,
- [6], [108], [172] for algebras,
- [101] for C*-algebras,
- papers by V. Jones, S. Popa, and others, including [96] and [64], for subrings of rings,
- [91], Ch. IX, for homogeneous spaces, pseudogroups, and foliations,
- [113] and [114] for subgroup growth.

For applications of group growth to other mathematical subjects, see among others

- [125] and [78] for geometry,
- [77], [102], and [123] for ergodic theory,
- [73] and [121] for automata and cellular automata theory,
- [173] for random walks,
- [74] and [78] for the theory of invariant means,
- [4] for ODE's and [8], [158] for PDE's. (In the paper [8], the only groups which appear with their growth properties are free groups; in private comments to the first author, A. Babin has added that all knot groups are also relevant.)

1. Uniformly exponential growth

Let Γ be a finitely generated group and let S be a finite set of generators of Γ. For $\gamma \in \Gamma$, the *word length* $\ell_S(\gamma)$ is the minimum of the integers $m \geq 0$ for which there exist $s_1, \dots, s_m \in S \cup S^{-1}$ with $\gamma = s_1 \dots s_m$. The *growth function* of the pair (Γ, S) associates to an integer $n \geq 0$ the number

$\beta(\Gamma, S; n)$ of elements $\gamma \in \Gamma$ such that $\ell_S(\gamma) \leq n$. The *exponential growth rate* of the pair (Γ, S) is the limit

$$\omega(\Gamma, S) = \lim_{n \to \infty} \sqrt[n]{\beta(\Gamma, S; n)}.$$

(As β is clearly submultiplicative, i.e., as $\beta(m+n) \leq \beta(m)\beta(n)$ for all $m, n \geq 0$, it is a classical fact that the limit exists (see [142], Problem 98 of Part I, p. 23).) Some authors introduce the *logarithm* of $\omega(\Gamma, S)$, and call it the *entropy* of the pair (Γ, S) [81]; the reason is that, if Γ is the fundamental group of a compact Riemannian manifold of unit diameter, and if S is an appropriate generating set (given by the geometry), then $\log \omega(\Gamma, S)$ is a lower bound for the topological entropy of the geodesic flow of the manifold [123].

For example, one has

$$\begin{aligned}
&\mathbb{Z} \supset S = \{1\} && \Longrightarrow \beta(n) = 2n+1 && \Longrightarrow \omega(\mathbb{Z}, S) = 1 \\
&F_k \supset S_k = \text{ free basis} && \Longrightarrow \beta(n) = \frac{k(2k-1)^n - 1}{k-1} && \Longrightarrow \omega(F_k, S_k) = \\
&\qquad = 2k-1
\end{aligned}$$

where, for an integer $k \geq 2$, we denote by F_k the free group with k generators.

The group Γ is said to be of *exponential growth* if $\omega(\Gamma, S) > 1$ (it is easy to check that this condition holds for one particular S if and only if it holds for all S as above). It is said to be of *uniformly exponential growth* if $\inf_S \omega(\Gamma, S) > 1$.

The *minimal growth rate* of Γ is

$$\omega(\Gamma) = \inf \omega(\Gamma, S)$$

where the infimum is taken over all S as above. The *main open problem* on minimal growth rates is to find out whether one can have $\omega(\Gamma, S) > 1$ for all S and $\omega(\Gamma) = 1$ (see [81], Remark 5.12). It could be easier to settle this problem first for restricted classes of groups, such as solvable groups of exponential growth, Coxeter groups of exponential growth, lattices in semi-simple Lie groups, or infinite groups with Kazhdan's property (T). One should also consider particular constructions, such as free products with amalgamation, HNN-extensions, or semi-direct products.

One possible approach for constructing groups of exponential growth but not of uniformly exponential growth is suggested in [79].

The following summarizes some known facts on $\omega(\cdot)$. Recall that a group is by definition *equally as large as* F_2 if it has a subgroup of finite index which has a quotient isomorphic to F_2.

Theorem. *Let Γ' be a subgroup of finite index in Γ and let Γ'' be a quotient of Γ'*

(i) $\omega(F_k) = 2k - 1$;
(ii) *if* $\omega(\Gamma'') > 1$, *then* $\omega(\Gamma') > 1$;
(iii) *if* $\omega(\Gamma') > 1$, *then* $\omega(\Gamma) > 1$;
(iv) *if* Γ *is equally as large as* F_2, *then* $\omega(\Gamma) > 1$;
(v) *if* Γ *has a presentation with* k *generators and* $l \leq k - 2$ *relations, then* $\omega(\Gamma) > 1$;
(vi) *if* Γ *has a presentation with* k *generators and* $k-1$ *relations* $r_1, \ldots, r_{k-1}$, *where* $r_1 = (r_0)^m$ *for some nonempty word* r_0 *and some* $m \geq 2$, *then* $\omega(\Gamma) > 1$;
(vii) *if* Γ *is a Gromov hyperbolic group which is torsion free and nonelementary, then* $\omega(\Gamma) > 1$;
(viii) *if* Γ *is a Coxeter group which is isomorphic to a lattice (possibly not uniform) in the group of isometries of the hyperbolic space* H^n *for some* $n \geq 2$, *then* $\omega(\Gamma) > 1$.

On proofs. (i) If S_k is a free generating set of F_k, we have already observed that $\omega(F_k, S_k) = 2k - 1$. Let now S be any finite generating set of F_k. The canonical image $\underline{S}$ of S in the abelianized group $(F_k)^{\mathrm{ab}} = \mathbb{Z}^k$ generates $\mathbb{Z}^k$. Thus $\underline{S}$ contains a subset $\underline{R}$ of k elements generating a subgroup of finite index in $\mathbb{Z}^k$. Let R be a subset of S projecting onto $\underline{R}$. The subgroup $\langle R \rangle$ of F_k generated by R is free (as a subgroup of a free group), of rank at most k (because $|R| = k$) and of rank at least k (because $(\langle R \rangle)^{\mathrm{ab}} \approx \mathbb{Z}^k$). Hence R is a free basis of $\langle R \rangle \approx F_k$, and it follows that $\omega(\Gamma, S) \geq \omega(\langle R \rangle, R) = \omega(F_k, S_k)$. (This argument appears already in Example 5.13 of [81].)

Claim (ii) is straightforward. Claim (iii) follows from the elementary (and smart!) Proposition 3.3 of [153], showing more precisely that $\omega(\Gamma)^{2[\Gamma:\Gamma']-1} \geq \omega(\Gamma')$. Claim (iv) follows from Claims (i) to (iii).

Digression from the proof: *if two finitely generated groups* Γ_1 *and* Γ_2 *are quasi-isometric, we do not know how to show that* $\omega(\Gamma_1) > 1$ *implies* $\omega(\Gamma_2) > 1$.

Claim (v) is due to B. Baumslag and S. Pride [14]; see also [83], pp. 82–83 and [17], Ch. IV, Theorem 8. For (vi), see [161], as well as [15] and the same reference of M. Gromov.

Observe that Statements (v) *and* (vi) *include the following: if* Γ *is a one-relator group, then* $\omega(\Gamma) > 1$ *as soon as* Γ *has rank* ≥ 3, *or has torsion; for other sufficient conditions, see* [35].

Claim (vii) follows from the following theorem. (Though its proof, due to T. Delzant, is unpublished, see [44].) In case Γ is moreover the fundamental group of a compact hyperbolic 3-manifold, it is in [153].

Claim (viii) follows from [115], where it is shown that the hypothesis of Claim (iv) does hold. *Does Claim* (viii) *carry over to any Coxeter group* *of exponential growth?* (It is known that any infinite Coxeter group has a subgroup of finite index which maps onto $\mathbb{Z}$; see [63] and [40].)

Observe that Claim (viii) holds for Coxeter groups which are *hyperbolic*, and that Moussong's thesis [127] provides a simple criterion for this. □

There are "exotic groups" to which similar arguments apply. For example, V. Guba [86] has shown that there exists a finitely generated *simple* group in which all the two-generator subgroups are free non-Abelian; one has clearly $\omega(\text{Guba group}) \geq 3$.

Theorem (M. Gromov, T. Delzant). *For a torsion-free nonelementary hyperbolic group Γ, there exists an integer n_Γ with the following properties:*

(i) *for all $x, y \in \Gamma$ such that $xy \neq yx$ and $n \geq n_\Gamma$, the elements x^n and yx^ny^{-1} generate freely a free subgroup of rank 2 in Γ;*

(ii) $\omega(\Gamma) \geq \sqrt[n_\Gamma]{3}$.

On proofs. Observe that (ii) is a straightforward consequence of (i) and of the previous Theorem. As for (i), it is a consequence of a theorem, first stated by Gromov (Sec. 5.3 in [84]), later made precise and proved by Delzant, which can be stated as follows. For any hyperbolic group Γ, there exists an integer n_Γ such that, for all $x \in \Gamma$, the *normal* subgroup generated by x^{n_Γ} is free.

Here is an open question on minimal growth rates, which generalizes the "main open problem" recalled above. Does there exist a finitely generated group Γ such that $\omega(\Gamma) < \omega(\Gamma, S)$ for all S? What about the Baumslag–Solitar group $\Gamma = \langle a, b \mid ab^2a^{-1} = b^3 \rangle$?

Let Γ_g denote the fundamental group of a closed orientable surface of genus $g \geq 2$. What is the exact value of $\omega(\Gamma_g)$? Here is an argument showing that $\omega(\Gamma_g) \geq 4g - 3$.

Let S be an *arbitrary* system of generators of Γ_g. Observe that S contains some subset R of $2g$ elements which generates a subgroup of finite index in the abelianized group $\mathbb{Z}^{2g}$ of Γ_g. If R_0 is the complement of one (arbitrary) element in R, then R_0 generates a subgroup $\langle R_0 \rangle$ of infinite index in Γ_g. Such a group is free (being the fundamental group of a noncompact surface) of rank exactly $2g - 1$ (because its abelianization is isomorphic to $\mathbb{Z}^{2g-1}$). Hence $\omega(\Gamma_g, S) \geq \omega(\langle R_0 \rangle, R_0) = 4g - 3$.

On the other hand, one has, for example, $5 \leq \omega(\Gamma_2) \leq \omega(\Gamma_2, \text{canonical}) \approx 6.9798$; for the upper numerical value, see the beginning of Sec. 4 below. More generally, numerical computations show that $\omega(\Gamma_g, \text{canonical}) \approx 4g - 1 - \epsilon_g$ with ϵ_g quite small, for all $g \geq 2$. It is an open problem to show that $\omega(\Gamma_g) = \omega(\Gamma_g, \text{canonical})$.

It is conjectured in [81], Sec. 5.14 that, for a group Γ which has a presentation with k generators and $\ell \leq k-1$ relations and for the corresponding set S of k generators, one has $\omega(\Gamma, S) \geq 2(k-\ell)-1$.

What are other values of $\omega(\Gamma)$? For example, for Coxeter groups? for one-relation groups? (progress on this in [35]).

Let $(\Gamma_n = \langle S \mid R_n \rangle)_{n\geq 1}$ be a sequence of one-relation groups, with the generating sets being identified with each other. Let k denote the number of generators in S; assume that the relations R_n's are cyclically reduced and that their lengths tend to infinity as $n \to \infty$. Under what conditions does one have $\lim_{n\to\infty} \omega(\Gamma_n, S) = 2k-1$ and $\lim_{n\to\infty} \omega(\Gamma_n) = 2k-1$? The computations of [39] show such a family with $\lim_{n\to\infty} \omega(\Gamma_n, \{x,y\}) \neq 3$.

Given an irreducible word w in the elements and their inverses of a free basis $S_2 = \{a, b\}$, representing an element $c = w(a,b)$ in the free group F_2 over S_2, what is in terms of w the value of the growth rate $\omega(F_2, \{a,b,c\})$?

Let Γ be a finitely generated group and let k_0 be its rank (namely, here the minimal cardinality of its generating sets). For each $k \geq k_0$ set

$$\underline{\omega}^{(k)}(\Gamma) = \inf \omega(\Gamma, S) \qquad \text{and} \qquad \overline{\omega}^{(k)}(\Gamma) = \sup \omega(\Gamma, S),$$

where the extrema are taken over all finite generating sets S with exactly k distinct elements, and such that $s, t \in S, s \neq t \Rightarrow s \neq t^{-1}$. How do these quantities depend on k? What are those G and S for which the infima and maxima are realized? For a free group F of rank k_0, one has $\underline{\omega}^{(k_0)}(F) = \overline{\omega}^{(k_0)}(F) = 2k_0 - 1$; what are the exact values of $\underline{\omega}^{(k)}(F)$ and $\overline{\omega}^{(k)}(F)$ for $k > k_0$? (Partial results have been obtained by Koubi [106], for example on $\underline{\omega}^{(k_0+1)}(F)$.)

Compare with the following question in differential geometry: given a compact manifold M, what are the Riemannian structures g (normalized by diameter$(M,g) = 1$) for which the entropy

$$\lim_{R\to\infty} R^{-1} \log \mathrm{Vol}\ \mathrm{Ball}_{\tilde{M},g}(x_0, R)$$

is minimum? See, e.g., [21]. (The volume is computed in the universal covering $\tilde{M}$ of M, for the metric lifted from M, around a point $x_0 \in \tilde{M}$, and the limit does not depend on the choice of the point x_0.) See also [145] for a generalization to more general measured metric spaces, and [87].

There are several other types of constants, depending on pairs (Γ, S), with related extrema over the S's, depending only on Γ, which give rise to interesting problems. One of these types is spectral radius (see Sec. 6 below), another one is Kazhdan constants [131].

2. Growth tightness

Define a pair (Γ, S) to be *growth tight* if $\omega(\Gamma, S) > \omega(\Gamma/N, \underline{S})$ for all normal subgroups N of Γ not reduced to $\{1\}$, with $\underline{S}$ denoting the canonical image of S in Γ/N.

Proposition. *For a free group F_k of rank $k \geq 2$ and for a free basis S_k of F_k, the pair (F_k, S_k) is growth tight.*

On the proof. For any proper quotient $F_k \to F_k/N$, choose a nonempty reduced word w representing an element of N. Let $\mathcal{L}$ denote the language of all reduced words in $S_k \cup S_k^{-1}$ and let $\mathcal{L}_w$ denote the sublanguage of those words which do not contain w as a subword. For each $n \geq 0$, let $\beta(\mathcal{L}_w, n)$ denote the number of words of length at most n in $\mathcal{L}_w$, and let $\omega(\mathcal{L}_w) = \limsup_{n\to\infty} \sqrt[n]{\beta(\mathcal{L}_w, n)}$ be the corresponding growth rate.

On the one hand, $\omega(\mathcal{L}_w)$ is strictly less than the growth rate $2k-1$ of $\mathcal{L}$. On the other hand, there is a natural map from $\mathcal{L}_w$ *onto* F_k/N and one has $\omega(\mathcal{L}_w) \geq \omega(F_k/N, \underline{S})$. The proposition follows. □

Observation. *Let Γ be a finitely generated group. If there exists a finite generating set S such that $\omega(\Gamma, S) = \omega(\Gamma)$ and if (Γ, S) is growth tight, then Γ is Hopfian.*

Proof. This is a straightforward consequence of the definitions. □

In particular, it follows that free groups are Hopfian.

Does the previous proposition extend to the following: *let Γ be a Gromov hyperbolic group, let N be an infinite normal subgroup of Γ, let S be a system of generators in Γ and denote by $\underline{S}$ its canonical image in Γ/N; then $\omega(\Gamma, S) > \omega(\Gamma/N, \underline{S})$*? A natural program is to extend as much as possible the proof above, expressing $\omega(\mathcal{L})$ as the spectral radius of the adjacency matrix of an appropriate finite state automaton, and using Perron–Frobenius theory to prove an inequality of the form $\omega(\mathcal{L}_w) < \omega(\mathcal{L})$. (See, e.g., Wielandt's Lemma in Sec. 2.3, Ch. XIII of [62].)

Z. Sela announced a proof that every Gromov hyperbolic group is Hopfian [144], Sec. 2, as much as we guess with quite different arguments.

The previous question is related to the following one. Let us call a finite state automaton *ergodic* if any state distinct from the initial state can be reached from any other state. The question is: does there exist an ergodic finite state automaton which recognizes the language of geodesic normal forms for the elements of a hyperbolic group? A variant of this was asked by D. Epstein in connection with the first term of asymptotic expansions for the growth functions of hyperbolic groups (see Sec. 3).

Let us finish this section with an example of a pair which is not growth tight. Let Γ be a direct product $F_k \times F_k$, let $S = (S_k \times \{1\}) \cup (\{1\} \times S_k)$, where S_k is a free basis of F_k, and let $\Gamma \to \Gamma/N = F_k$ be the first projection. An easy computation shows that $\omega(\Gamma, S) = 2k-1 = \omega(\Gamma/N, \underline{S})$ so that (Γ, S) is indeed not growth tight. (Observe that this group *is* nevertheless Hopfian!)

Does there exist any S for which the pair $(F_k \times F_k, S)$ is growth tight? Is it clear that $\omega(F_k \times F_k) = \underline{\omega}^{(2k)}(F_k \times F_k) = 2k-1$? What is $\overline{\omega}^{(2k)}(F_k \times F_k)$?

3. Regularity of growth functions

For a pair (Γ, S) as in Sec. 1, it is convenient to consider both the growth function $n \mapsto \beta(n) = \beta(\Gamma, S; n)$ and the *spherical growth function* $n \mapsto \sigma(n) = \beta(n) - \beta(n-1)$. If Γ is infinite, one has

$$\omega(\Gamma, S) \;=\; \lim_{n\to\infty} \sqrt[n]{\beta(n)} \;=\; \lim_{n\to\infty} \sqrt[n]{\sigma(n)}. \tag{$*$}$$

Indeed, $\omega(\Gamma, S)$ is by definition the inverse of the radius of convergence of the series $B(z) = \sum_{n=0}^{\infty} \beta(n) z^n$. As σ is a submultiplicative function, the inverse of the radius of convergence of $\Sigma(z) = \sum_{n=0}^{\infty} \sigma(n) z^n$ is given similarly by the limit $\lim_{n\to\infty} \sqrt[n]{\sigma(n)}$, and this limit is at least 1 because $\sigma(n) \geq 1$ for all $n \geq 0$. As one has $\Sigma(z) = (1-z)B(z)$, it follows that $\Sigma(z)$ and $B(z)$ have the same radius of convergence, and $(*)$ follows.

A few easy examples may avoid rash conjectures about the functions β and σ. The first is the pair

$$\Gamma = \langle s, t | s^3 = t^3 = (st)^3 = 1 \rangle \supset S \;= \{s, t\},$$

for which Γ is a discrete group of orientation preserving isometries of the Euclidean plane, with fundamental domain the union of two isometric equilateral triangles glued along a common side. A computation shows that $\sigma(2k-1) = 8k-2$ and $\sigma(2k) = 10k-2$ for $k \geq 2$. In particular, $\sigma(10) = 48 > \sigma(11) = 46$, showing that *spherical growth functions need not be increasing*. Also

$$\sigma(2k) \qquad \text{is not always} \quad \leq \quad \text{than} \qquad \frac{\sigma(2k-1) + \sigma(2k+1)}{2}$$

and this disproves the conjecture formulated in 1976 by V. Beliayev and N. Sesekin (see Problem 5.2 in [107]).

The modular group provides a second example, due to A. Machi [120]; in particular, it shows that the limit $\lim_{n\to\infty} \beta(n+1)\beta(n)^{-1}$ may exist for one set of generators and not for another one. More precisely, let Γ be the free product of a group $\{1, s\}$ of order 2 and of a group $\{1, t, t^2\}$ of order 3. For $S = \{s, t\}$, a computation shows that $\beta(2k) = 7{\cdot}2^k - 6$ and $\beta(2k+1) = 10{\cdot}2^k - 6$ for $k \geq 1$; in particular, one has

$$\lim_{k\to\infty} \frac{\beta(2k)}{\beta(2k-1)} = \frac{7}{5} < \lim_{n\to\infty} \sqrt[n]{\beta(n)} = \sqrt{2} < \lim_{k\to\infty} \frac{\beta(2k+1)}{\beta(2k)} = \frac{10}{7}.$$

In this case the series

$$\sum_{n=0}^{\infty} \beta(n) z^n = \frac{1+3z+2z^2}{(1-2z^2)(1-z)}$$

defines a rational function with *two poles* on its circle of convergence. (The relation between the formula of the rational function and the formula for the coefficients $\beta(n)$'s is described, for example, in Sec. 7.3 of [65].) For the other generating set $S' = \{s, st\}$ of the same group and for the resulting growth function β', another computation shows that

$$\lim_{n\to\infty} \frac{\beta'(n+1)}{\beta'(n)} = \frac{\sqrt{5}-1}{2} = \lim_{n\to\infty} \sqrt[n]{\beta'(n)}.$$

In this case the series

$$\sum_{n=0}^{\infty} \beta'(n) z^n = \frac{1+2z+2z^2+z^3}{(1-z-z^2)(1-z)}$$

has a *unique pole* on its circle of convergence.

To investigate (ir)regularities of the growth functions, maybe one should think again about the relation between exponential growth and paradoxical behavior in the sense of [45].

In general, for a pair (Γ, S), little is known about when the quotients $\beta(n+1)/\beta(n)$ converge to the exponential growth rate ω, or when the quotients $\omega^{-n}\beta(n)$ converge to some constant, for $n \to \infty$.

If Γ is hyperbolic, M. Coornaert [41] has shown that there exist constants $c_1, c_2 > 0$ such that $c_1\omega^n \leq \beta(n) \leq c_2\omega^n$ for all $n \geq 0$. Machi's example with the modular group described above shows that one cannot expect $c_1 = c_2$ in general.

Say that $\gamma \in \Gamma$ is a *dead end* with respect to S if one has

$$\ell_S(\gamma s) \leq \ell_S(\gamma) \quad \text{for all} \quad s \in S,$$

namely, if a geodesic segment from 1 to γ cannot be extended beyond γ. The investigation of these dead ends was started independently by several researches (including O. Bogopolski, C. Champetier, and A. Valette; see also [89]).

As a first example, consider the direct product $\mathbb{Z} \times \{\epsilon, j\}$ of the integers with the group of order 2, and the generating set $\{(1, \epsilon), (1, j)\}$; then $(0, j)$ is a dead end. Consider then the one-relation presentation $\langle s, t | ststs \rangle$ and the generating set $\{s, t\}$ (the group is $\mathbb{Z}$ and one can take $s = 2, t = -3$); then st is a dead end. (These examples were shown to us by A. Valette and C. Champetier.) In the other direction, it is, for example, known that a group with presentation $\langle S | R \rangle$ satisfying a small cancellation hypothesis $C'(1/6)$ has no dead end (Lemma 4.19 in [32]).

Given a pair (Γ, S), denote by $D_S(\Gamma)$ the corresponding subset of dead ends and by $\delta(n)$ the number of dead end elements of S-length at most n. O. Bogopolski has asked the following questions [25].

- Given Γ, does there exist S with $D_S(\Gamma) = \emptyset$?
- The same question for Γ hyperbolic.
- Is it always true that $\lim_{n\to\infty} \frac{\delta(n)}{\beta(n)} = 0$?
- Given (Γ, S), does there exist an integer $L > 0$ such that, for any $\gamma \in \Gamma$, there exists $t \in \Gamma$ with $\ell_S(t) \leq L$ and $\ell_S(\gamma t) = \ell_S(\gamma) + 1$? (The answer is "yes" in case Γ is hyperbolic, or more generally, in case (Γ, S) has finitely many cone-types.)

Let (Γ, S) be a pair with Γ of *polynomial growth,* namely, such that the growth function $n \mapsto \beta(n) = \beta(\Gamma, S; n)$ satisfies

$$\beta(n) \leq cn^d$$

for some constants $c, d > 0$ and for all $n \geq 0$ (it is easy to check that this depends on Γ only, and not on S). By one of Gromov's famous theorems [82], Γ is then virtually nilpotent and the *polynomial growth rate*

$$d = \limsup_{n\to\infty} \frac{\log \beta(n)}{\log n}$$

is an integer; the latter is given by a formula of Wolf–Bass [13] and Guivarc'h [88], also found by B. Hartley (independently, in a work which has not been published). This theorem of Gromov has been extended to appropriate classes of semi-groups [73] and graphs [171].

For a pair (Γ, S) and constants $c > 0$, $d \geq 1$, the property $\beta(n) \geq cn^d$ for all $n \geq 0$ is equivalent to the isoperimetric estimate

$$|\Omega|^{1/d} \leq c|\partial\Omega|^{1/(d-1)}$$

for all finite subset Ω of Γ, where the boundary of Ω is by definition

$$\partial\Omega = \left\{x \in \Omega | \text{ there exists } s \in S \cup S^{-1} \text{ such that } xs \notin \Omega |\right\};$$

see [183].

Pansu has shown that the limit

$$c_1 = \lim_{n\to\infty} \frac{\beta(n)}{n^d}$$

exists for every virtually nilpotent group [134]. Grunewald has claimed that

$$\beta(n) = c_1 n^d + O\left(n^{d-\frac{1}{2}}\right),$$

but proofs are not known to us. Recently, for Γ a 2-step nilpotent group (this means that any commutator in Γ is central), M. Stoll [163] has shown that $\beta(n) = c_1 n^d + O\left(n^{d-1}\right)$. It is an open problem to find out whether or when the limit

$$\lim_{n\to\infty} \frac{\beta(n) - c_1 n^d}{n^{d-1}}$$

exists and is finite.

One should clearly investigate further the asymptotics of the growth functions $\beta(n)$ for nilpotent groups.

Other references on groups of polynomial growth include [48], [49], [170], [173], [180].

Groups of subexponential growth (and groups of polynomial growth in particular) are known to have various interesting properties. For example, they are amenable, a fact first observed in [1]. Also, they give rise to combinatorial Laplacians on their Cayley graphs, and thus to bounded operators on ℓ^p spaces, and these have spectra which are *independent of* $p \in [1, \infty]$ [158].

There are also basic problems which are still open about *groups of intermediate growth,* namely about finitely generated groups which are neither of polynomial nor of exponential growth. After J. Milnor [124] asked in 1968 whether these groups exist at all, examples have been discovered in the early 1980's; for this progress, we refer to [74]. Let us however repeat that the following most important question is still open: *does there exist finitely presented groups of intermediate growth?*

It is appropriate to quote here Problem 12 of [118]: *There is clearly much to be done in determining the possible growth functions of groups and in relating them to properties of groups.*

4. Classical growth series

The usual *growth series* of (Γ, S) is the formal power series

$$\Sigma(\Gamma, S; z) \;=\; \sum_{n=0}^{\infty} \sigma(n) z^n \;=\; \sum_{\gamma \in \Gamma} z^{\ell_S(\gamma)} \;\in\; \mathbb{Z}[[z]]$$

and is denoted by $\Sigma(z)$ when the pair (Γ, S) is clear from the context. Its radius of convergence is $\omega(\Gamma, S)^{-1}$ when Γ is infinite. One has also

$$\frac{\Sigma(\Gamma, S; z)}{1 - z} \;=\; \sum_{n=0}^{\infty} \beta(n) z^n$$

because $\beta(n) = \sigma(0) + \ldots + \sigma(n)$ for all $n \geq 0$ (as already observed in Sec. 3 above).

For example, one has

$$
\begin{array}{lll}
\mathbb{Z} \supset S = \{1\} & \Longrightarrow \Sigma(z) & = \dfrac{1+z}{1-z}, \\
\mathbb{Z} \supset S = \{2,3\} & \Longrightarrow \Sigma(z) & = -5 - 2z + 2z^2 + \dfrac{6}{1-z}, \\
\mathbb{Z}^k \supset S = \text{ usual basis} & \Longrightarrow \Sigma(z) = \left(\dfrac{1+z}{1-z}\right)^k, & \\
F_k \supset S_k = \text{ free basis} & \Longrightarrow \Sigma(z) = \dfrac{1+z}{1-(2k-1)z}. &
\end{array}
$$

As one more example, the fundamental group Γ_g of a compact closed orientable surface of genus $g \geq 2$ and its usual set S_g of $2g$ generators provide the growth series

$$\Sigma(\Gamma_g, S_g; z) = \frac{1 + 2z + \ldots + 2z^{2g-1} + z^{2g}}{1 - (4g-2)z - \ldots - (4g-2)z^{2g-1} + z^{2g}}$$

[28]. This rational function has exactly two poles outside the unit circle, which are positive real numbers, say $\omega > 1$ and $\omega^{-1} < 1$; see [31] and [139]. With the notations of Sec. 1, one has of course $\omega = \omega(\Gamma_g, S_g)$. Other generating sets provide dramatically different functions; for example,

$$\Sigma(\Gamma_g, S_g^{\mathrm{Alo}}; z) = \frac{1 + 2z + z^2}{1 - (8g-6) + z^2}$$

for the interesting set S_g^{Alo} of J. Alonso [3].

It is remarkable that many of these series are *rational functions*: besides the examples above, this is also known to be the case for

- Coxeter groups with standard S (see [26], p. 45, Exerc. 26, and [135], the argument being that of [160], Sec. 3),
- free abelian groups with arbitrary S (see [103], [104]),
- virtually abelian groups with arbitrary S [19],
- the Heisenberg group $\langle x, y \mid xyx^{-1}y^{-1} \text{ central} \rangle$ with $S = \{x, y\}$ (see [20], [155], and [165]),
- Gromov hyperbolic groups with arbitrary S ([84], see also [28], [29] and [61], Ch. 9),
- appropriate kinds of automatic groups [54], Sec. 2.5, and [129],
- solvable Baumslag–Solitar groups $\langle x, y | x^{-1}yx = y^b \rangle$ with $S = \{x, y\}$ [39],
- two-step nilpotent group Γ with $[\Gamma, \Gamma] \approx \mathbb{Z}$, with appropriate S [165] (sic!),
- fundamental groups of those quotients of triangular buildings studied by S. Barré (see [9] and [10], Sec. 3.1),

and many other cases ([3], [20], [27], [93], [94], [95], [164], [174],...). The zeros, poles, symmetries, and some special values of these rational functions have been investigated in several cases ([31], [56]–[59], [139]. Observe that, for the growth series attached to a pair (Γ, S) to be rational, it is *sufficient* that the pair has "finitely many cone types" (in the sense of [61], Ch. 9), but this is *not necessary*, as the Heisenberg group demonstrates (see [155], as well as [162]).

It is curious that the rationality of the growth series of finitely generated abelian groups, stated and proved by D. Klarner in 1981, has been "implicitly known" long before. On the one hand, it follows immediately from the rationality of the Hilbert–Poincaré series of a finitely generated commutative graded alebra (see, for example, Theorem 11.9 in Ch. 3 of [105]); this was observed in several papers, e.g., [22]. On the other hand, it is a straightforward consequence of structure results for the so-called rational subsets of abelian monoids, due to S. Eilenberg and M. Schützenberg and going back to 1969 [53], as F. Liardet has observed to us.

Pairs (Γ, S) providing growth series which are *not* rational are more difficult to find. They include

- finitely presented groups with unsolvable word problem (see [28], quoting an observation of W. Thurston),
- groups of intermediate growth [69], [71], [72],
- two-step nilpotent group Γ with $[\Gamma, \Gamma] \approx \mathbb{Z}$, with appropriate S [165] (sic!!),
- examples of W. Parry which are restricted wreath products [138].

(About the first of these classes, it is easy to extend the proof to the case of recursively presented groups.) Of the four classes of growth series above, the first three are even transcendental, while examples of Parry may be nonrational and algebraic.

For a finitely generated group of subexponential growth, observe that growth series are either rational, or nonalgebraic (see [142], Part VIII, No. 167, or [55], p. 368).

Computations of growth series are often interesting challenges, and we list some open cases:

- the Richard Thompson group $\langle s, t | [st^{-1}, s^{-1}ts] = [st^{-1}, s^{-2}ts^2] = 1 \rangle$ [30],
- the Baumslag–Solitar group $\Gamma = \langle a, b \mid ab^2a^{-1} = b^3 \rangle$ [16],
- the Burnside groups.

Other related references include [159] and [174].

We would like to single out the following particular case of results of M. Stoll.

Let

$$H_2 = \left\langle x_1, x_2, y_1, y_2 \;\middle|\; \begin{array}{ll} [x_i, x_j] = [y_i, y_j] = 1 & \text{for all } \; i, j \\ [x_i, y_j] = 1 & \text{for } \; i \neq j \\ [x_1, y_1] = [x_2, y_2] & \text{is central} \end{array} \right\rangle =$$

$$= \left\{ \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & 0 & \mathbb{Z} \\ 0 & 0 & 1 & \mathbb{Z} \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

be the second Heisenberg group and set

$$S_{\text{standard}} = \{ x_1, x_1^{-1}, x_2, x_2^{-1}, y_1, y_1^{-1}, y_2, y_2^{-1} \}$$

$$S_{\text{Stoll}} = S_{\text{standard}} \cup \left\{ ab \;\middle|\; a \in \{x_1, x_1^{-1}, y_1, y_1^{-1}\} \text{ and } b \in \{x_2, x_2^{-1}, y_2, y_2^{-1}\} \right\}.$$

One has the following result; see [165], in particular, Corollary 5.11 and Theorem 6.1.

Theorem (M. Stoll). (i) *The growth series of H_2 with respect to S_{standard} is transcendental;*

(ii) *the growth series of H_2 with respect to S_{Stoll} is rational.*

Remark. The growth series for

$$H_1 = \langle x, y \mid [x, y] \quad \text{is central} \rangle$$

with respect to $\{x, x^{-1}, y, y^{-1}\}$ is

$$\Sigma(z) = \frac{1 + z + 4z^2 + 11z^3 + 8z^4 + 21z^5 + 6z^6 + 9z^7 + z^8}{(1-z)^4 (1 + z + z^2)(1 + z^2)}$$

(see [20] (beware of a mistake in the formula for Σ!) and [155]).

Open problem: does there exist a generating set of H_1 for which the corresponding growth series is *not* rational?

There are several other kinds of growth-like notions for groups Γ or for pairs (Γ, S) that we have not mentionned so far. Let us only quote some of them.

- The growth of the number of subgroups of finite index. About this, let us mention the remarkable result of A. Lubotzky, A. Mann, and D. Segal [116]: for a finitely generated and residually finite group Γ, the growth of the number of subgroups of finite index is polynomial if and only if the group is virtually solvable of finite rank (a group is of *finite rank* if there is a bound on the numbers of generators of its finitely generated subgroups). See also [112], [113], and [114].

- The growth of the number of conjugacy classes of elements, which is related to spectra of closed geodesics in Riemannian manifolds (see Secs. 5.2 and 8.5 in [84] and [7]).
- The growth of the ranks of the factors of the lower central series of a group [78].
- The growth of Dehn functions, which measure the complexity of the word problem, and of related functions (see, e.g., p. 82 in [85]). A recent result of J.-C. Birget, E. Rips, and M. Sapir shows that Dehn functions can be "almost anything" (in a precise meaning!); see [24].
- The growth of the number of orbits in product actions of appropriate permutation groups (actions on finite sets, or more generally the so-called *oligomorphic* actions studied by P. Cameron). Among other references, see [187].
- The growth of the minimal number of generators for direct products of a group. There are many papers on this problem by J. Wiegold and others, of which we quote [188] and [189].
- Growth of particular finite subsets of a group, such as, for the automorphism group of a shift of finite type, the *doubly exponential* growth of the number of automorphisms of so-called *finite range* [186].

One can also imagine other types of generating series, such as

- exponential series of the form

$$\sum_{n=0}^{\infty} \frac{\sigma(\Gamma, S; n)}{n!} z^n,$$

- Dirichlet series of the form

$$\sum_{n=0}^{\infty} \sigma(\Gamma, S; n)\, n^{-s},$$

- and Newtonian series of the form

$$\sum_{n=0}^{\infty} \sigma(\Gamma, S; n) \binom{z}{n}$$

(see [65] or [176]).

5. Growth series with weights

The setting above carries over to the case of *proper weights*, namely to functions $\lambda : \Gamma \to \mathbb{R}_+$ such that

$$\begin{array}{ll} \{\gamma \in \Gamma \,|\, \lambda(\gamma) \leq K\} & \text{is finite for all} \quad K \geq 0 \\ \lambda(\gamma_1\gamma_2) \leq \lambda(\gamma_1)\lambda(\gamma_2) & \text{for all} \quad \gamma_1, \gamma_2 \in \Gamma \end{array}$$

(proper submultiplicative functions). Each proper weight λ gives rise to a left-invariant pseudo-metric $(\gamma_1, \gamma_2) \mapsto \lambda(\gamma_1^{-1}\gamma_2)$ on Γ which is proper (i.e., any ball of finite radius is finite), and vice versa.

One denotes by $\beta(\Gamma, \lambda; K)$ the number of elements γ in Γ of weight $\lambda(\gamma) \leq K$ and by

$$\omega(\Gamma, \lambda) = \lim_{K \to \infty} \sqrt[K]{\beta(\Gamma, \lambda; K)}$$

the corresponding growth rate. The previous considerations correspond to the particular case $\lambda = \ell_S$ defined by a finite generating set S of Γ, and items (5.1) to (5.6) below to other interesting cases. More on weights on groups in [76].

5.1. A first class of examples consists of *relative growth* for subgroups. More precisely, consider a finitely generated group Γ, a finite system S of generators of Γ and the corresponding word length function ℓ_S, an arbitrary subgroup Γ_0 of Γ (not necessarily finitely generated), and the restriction $\lambda : \Gamma_0 \to \mathbb{N}$ of ℓ_S. We then write $\beta(\Gamma_0 \text{ rel } \Gamma, S; n)$ instead of $\beta(\Gamma_0, \lambda; n)$, and similarly for the growth rate $\omega(\Gamma_0 \text{ rel } \Gamma, S)$.

Here is a sample of questions which it is natural to ask in this context (from a list in the talk of A. Lubotzky – July 1996).

- In case Γ is solvable, is the relative growth of a subgroup Γ_0 either polynomial or exponential? (Compare with [180] and [124].)
- Same questions for Γ linear. (Compare with [169]; see also [154].)
- In case the subgroup Γ_0 is infinite cyclic, can one have intermediate growth?
- Does there exist pairs $\Gamma_0 < \Gamma$ giving rise to growth functions $\beta(\Gamma_0 \text{ rel } \Gamma, S; n) \approx n^d$ with $d \in \mathbb{R}_+$ and $d \notin \mathbb{N}$, or even $d \notin \mathbb{Q}$? (See the comments in $3.K_4$ of [85] about the similarity with Kolmogorov's complexity function.)

In case the subgroup Γ_0 of Γ is itself generated by some finite set S_0, one may compare the relative growth of $\beta(\Gamma_0 \text{ rel } \Gamma, S; n)$ and the net growth $\beta(\Gamma_0, S_0; n)$ of Γ_0. This gives one notion of "distorsion" of Γ_0 in Γ (*not* that discussed in Chs. 3 and 4 of [85]!).

Coming back to an arbitrary subgroup Γ_0 of a group Γ generated by a finite set S, one has the formal power series

$$\Sigma(\Gamma_0 \text{ rel } \Gamma, S; z) = \sum_{n=1}^{\infty} \sigma(\Gamma_0 \text{ rel } \Gamma, S; n)\, z^n = \sum_{\gamma \in \Gamma_0} z^{\ell_S(\gamma)} \in \mathbb{Z}[[z]]$$

as in Sec. 4 above, where $\sigma(\dots; n) = \beta(\dots; n) - \beta(\dots; n-1)$ denotes again the cardinality of the appropriate *sphere*.

Observation. *For an integer $k \geq 2$, consider the word length $\ell : F_k \longrightarrow \mathbb{N}$ defined by a free basis S_k of the free group F_k, and its subgroup of commutators. Then the corresponding series $\Sigma([F_k, F_k]$ rel $F_k, S_k; z)$ is not rational.*

On the proof. Let N be a normal subgroup of F_k, set $\Gamma = F_k/N$ and denote by $\pi : F_k \longrightarrow \Gamma$ the canonical projection. Consider on Γ the simple random walk with respect to the generators $\pi(S_k)$, for which the probability of walking in one step from x to $xt \in \Gamma$ is zero for $t \notin \pi(S_k \cup S_k^{-1})$ and is $\frac{j}{2k}$ for $t \in \pi(S_k \cup S_k^{-1})$, where j is the cardinality of $\pi^{-1}(t) \cap (S_k \cup S_k^{-1})$; in particular, it may be that $j > 1$. Let first

$$G_1(z) = \sum_{n=0}^{\infty} p^{(n)}(1,1) z^n$$

denote the Green function of this random walk (more on this notion in Sec. 7 below). Second, write N instead of $\Sigma(N$ rel $F_k, S_k)$ the series

$$N(z) =$$
$$= \sum_{n=0}^{\infty} \sharp\Big(N \cap \{\text{sphere of radius } n \text{ around } 1 \text{ in } F_k \text{ with respect to } S_k\}\Big) z^n$$

for the relative growth. And third, consider the rational function with rational coefficients in two variables defined by

$$R(x,y) = \frac{1}{y}\left[\frac{2k-1}{2k} - 2(k-1)\frac{(1-y)^2}{\left(\frac{2k-1}{k}\right)^2 x^2 - (1-y)^2}\right] \in \mathbb{Q}(x,y).$$

The formula

$$G_1(z) = R\left(z, \sqrt{1 - \frac{2k-1}{k^2} z^2}\right) N\left(k\, \frac{1 - \sqrt{1 - \frac{2k-1}{k^2} z^2}}{(2k-1)z}\right)$$

was proved in the first author's thesis [66]. Observe, in particular, that G_1 and N are together algebraic or not.

Let us now assume that $k = 2$ and $N = [F_k, F_k]$. Then $\Gamma = \mathbb{Z}^2$ and $G_1(z)$ is the Green function of the standard simple random walk on the integer plane (drunkard's walk). It is well known that

$$G_1(z) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 \left(\frac{z}{4}\right)^{2n} = F\left(\frac{1}{2}, \frac{1}{2}, 1 \; ; \; z^2\right)$$

(see, for example, Sec. 7.3 in [47] for the first equality, and recall that $F(\alpha, \beta, \gamma \; ; \; x)$ is the usual notation for hypergeometric series). For the values $\frac{1}{2}, \frac{1}{2}, 1$ of the parameters α, β, γ the monodromy group of the corresponding hypergeometric equation is known to be generated by the matrices (well defined up to conjugacy)

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$$

in $GL(2, \mathbb{C})$; see [92], Formulas (1.1.5) and (4.3.4) of Ch. 2. As this monodromy group has no finite orbit in $\mathbb{C}^2 \setminus \{0\}$, the function $F\left(\frac{1}{2}, \frac{1}{2}, 1 \; ; \; z^2\right)$ is not algebraic. $\square$

We confess here that we have not looked for a formal proof of the fact that $\Sigma([F_k, F_k] \text{ rel } F_k, S_k; z)$ is not a rational function when $k \geq 3$ (although we believe in this).

Similar non-rationality results were established for the fundamental group of a closed orientable surface of genus at least 2 and its group of commutators [141], as well as for some 2-step nilpotent group and its center [175].

5.2. There is a notion of growth for coset spaces. About this, let us only mention one result due to Kazhdan (quoted on p. 18 of [85], see also [166]) and one due to Rosset [149].

Theorem (Kazhdan, Rosset). (i) *If Γ has Kazhdan's Property* (T) *and if Γ_0 is a subgroup such that the growth of Γ/Γ_0 is subexponential, then Γ_0 is necessarily of finite index in Γ;*

(ii) *if Γ is finitely generated and has non-exponential growth, and if Γ_0 is a normal subgroup such that Γ/Γ_0 is solvable, then Γ_0 is finitely generated.*

5.3. An important particular case of relative growth is that of *cogrowth*. For a pair (Γ, S) and a subgroup $\Gamma_0 < \Gamma$ as above, observe first that one has obviously $\omega(\Gamma, S) \geq \omega(\Gamma_0 \text{ rel } \Gamma, S)$. Assume moreover that $\Gamma = F_k$ is a free group, that S_k is a free basis in F_k, that $\Gamma_0 = N$ is not reduced to $\{1\}$, and that N is a *normal* subgroup in F_k. It is straightforward to check that $2k-1 \geq \omega(N \text{ rel } F_k, S_k) \geq \sqrt{2k-1}$. (For the second inequality, it is enough to consider *one* reduced word $w = avb \in N \setminus \{1\}$, where $a, b \in S_k \cup S_k^{-1}$, and to estimate from below the growth of the numbers of conjugates $uwu^{-1} \in N$, where the word $u \in F_k$ does not end with the letter a^{-1} or with the letter b.) But one has moreover the *strict* inequality in

$$2k - 1 \geq \omega(N \text{ rel } F_k, S_k) > \sqrt{2k-1} \qquad (*)$$

by [66], [68]. (Compare with [99], Lemma 3.1 and Theorem 3.)

Does this generalize to hyperbolic groups? That is, for a nonelementary hyperbolic group Γ and a finite set S of generators of Γ, does one have

$$\omega(\Gamma, S) \geq \omega(N \text{ rel } \Gamma, S) > \sqrt{\omega(\Gamma, S)}$$

for any normal subgroup N of Γ which is not finite?

One motivation for the notion of cogrowth comes from the study of random walks. Anticipating on the notation of Sec. 7 below, we now describe a relation between spectral radius and cogrowth.

Let first (Γ, S) be a pair as above and let Γ_0 be a subgroup of Γ. Recall that the *Cayley graph* $Cay(\Gamma_0 \setminus \Gamma, S)$ is the graph with the vertex set $\Gamma_0 \setminus \Gamma$, and with an edge between two vertices $\Gamma_0\gamma_1, \Gamma_0\gamma_2$ if and only if $\gamma_1^{-1}\gamma_2 \in S \cup S^{-1}$ (this is also called the *Schreier graph* of Γ modulo Γ_0 with respect to S). We denote by $\mu(\Gamma_0 \setminus \Gamma, S)$ the spectral radius of the self-adjoint bounded operator T defined on $\ell^2(\Gamma_0 \setminus \Gamma)$ by $(Tf)(\Gamma_0\gamma) = \sum_{s \in S} \left(f(\Gamma_0\gamma s) + f(\Gamma_0\gamma s^{-1})\right)$; in case $\Gamma_0 = \{1\}$, this definition of $\mu(\Gamma, S)$ coincides with that of Sec. 7 below; see [179].

Let now F_k be a free group on a free basis S_k of k elements, and let Γ_0 be a subgroup of F_k (*not* necessarily normal). Denoting by α the relative growth $\omega(\Gamma_0 \text{ rel } F_k, S_k)$, one has the formula

$$\mu(\Gamma_0 \setminus F_k, S_k) = \begin{cases} \frac{\sqrt{2k-1}}{k} & \text{if } \; 1 \leq \alpha \leq \sqrt{2k-1} \\ \frac{\sqrt{2k-1}}{2k}\left(\frac{\sqrt{2k-1}}{\alpha} + \frac{\alpha}{\sqrt{2k-1}}\right) & \text{if } \; \sqrt{2k-1} < \alpha \leq 2k-1 \end{cases}$$

which shows a "phase transition" for the dependence of the spectral radius $\mu(\Gamma_0 \setminus F_k, S_k)$ in terms of the relative growth α [68].

In the special case of a *normal* subgroup of F_k, we write N intead of Γ_0; we denote by Γ the quotient group of F_k by N, by S the canonical image of S_k in Γ, and we write $\mu(\Gamma, S)$ instead of $\mu(N \setminus F_k, S_k)$. (Observe that S may be a "set with multiplicity," but we leave here this discussion to the reader.) The previous formula relates now the spectral radius $\mu(\Gamma, S)$ and the *cogrowth* α. One has $\alpha = 1$ if and only if $N = \{1\}$, and in this case the computation of $\mu(\Gamma, S) = \mu(F_k, S_k)$ is that of H. Kesten [99]. One cannot have $1 < \alpha \leq \sqrt{2k-1}$. One has $\alpha = 2k-1$ if and only if Γ is amenable. (Brief history: this criterion of amenability was established in 1974, written for publication in 1976, published in 1978, and published in its English translation in 1980 [68]!)

The previous criterion of amenability was used in [133] and [2], where it is shown that there exist non-amenable groups without subgroups isomorphic to F_2 (an answer to a question going back to J. von Neumann [130] and M. Day [43]), and in [67], where it is shown that there exist Γ-homogeneous spaces without Γ-invariant means and without freely acting subgroups of Γ isomorphic to F_2.

Do the formulas relating spectral radius and cogrowth carry over in some form to subgroups of hyperbolic groups?

More on cogrowth in [68], as well as in [32], [37], [42], [70], [147], [168], and [177].

5.4. Consider a finitely generated group Γ, a finite set $S = \{s_1, \dots, s_k\}$ of generators of Γ, and a sequence $\{\alpha_1, \beta_1, \dots, \alpha_k, \beta_k\}$ of strictly positive numbers (with $\beta_j = \alpha_j$ in case $s_j^{-1} = s_j$). Define the corresponding *Bernoulli weight* $\lambda : \Gamma \to \mathbb{R}_+$ by

$$\lambda'(s_j) = \alpha_j \quad \text{and} \quad \lambda'(s_j^{-1}) = \beta_j \quad \text{for} \quad j \in \{1, \dots, k\},$$

$$\lambda(\gamma) = \inf \left\{ \sum_{j=1}^{n} \lambda'(t_i) \in \mathbb{R}_+ \,\middle|\, \gamma = t_1 t_2 \dots t_n \text{ and } t_1, \dots, t_n \in S \cup S^{-1} \right\}.$$

Here are two examples of the corresponding growth series $\sum_{\gamma \in \Gamma} z^{\lambda(\gamma)}$: one has

$$\Sigma(\mathbb{Z}^k, \lambda; z) \;=\; \prod_{j=1}^{k} \frac{1 - z^{\alpha_j + \beta_j}}{(1 - z^{\alpha_j})\,(1 - z^{\beta_j})}$$

for $\Gamma = \mathbb{Z}^k$ and S a standard basis, and

$$\frac{1}{\Sigma(F_k, \lambda; z)} - 1 = \sum_{j=1}^{n} \left(\frac{(1 - z^{\alpha_j})\,(1 - z^{\beta_j})}{1 - z^{\alpha_j + \beta_j}} - 1 \right)$$

for $\Gamma = F_k$ and S a free basis. These formulas appear in [159], pp. 528–528. Once the formula is established for $\Gamma = \mathbb{Z} \supset S = \{1\}$, the other cases follow because the series are multiplicative for direct products, and the "reciprocal of the series minus one" are additive over free products (as observed in [93]).

Consider again a pair (Γ, S) with $S = \{s_1, \dots, s_k\}$ and a function $\lambda' : S \cup S^{-1} \to \mathbb{R}_+^*$ as above, and let moreover $(\mu(s,t))_{s,t \in S \cup S^{-1}}$ be a matrix of strictly positive numbers (with appropriate conditions if $s^2 = 1$ for some $s \in S$). For a finite sequence $(t_1, \dots, t_n)$ of letters in $S \cup S^{-1}$, set

$$\lambda'\big((t_1, \dots, t_n)\big) \;=\; \lambda'(t_1) \;+\; \sum_{j=1}^{n-1} \mu(t_j, t_{j+1})$$

and define a *Markov semi-weight* $\lambda : \Gamma \to \mathbb{R}_+$ by

$$\lambda(\gamma) \;=\; \inf\left\{ \lambda'\big((t_1,\ldots,t_n)\big) \;\big|\; \gamma = t_1\ldots t_n \quad\text{and}\quad t_1,\ldots,t_n \in S\cup S^{-1} \right\}.$$

If $\mu(s,t)$ depends on t only, this is a Bernoulli weight. In general, λ is a *semi*-weight, namely there exists a constant $C > 0$ such that

$$\lambda(\gamma_1\gamma_2) \;\leq\; \lambda(\gamma_1) \;+\; \lambda(\gamma_2) \;+\; C$$

for all $\gamma_1, \gamma_2 \in \Gamma$ (and thus $\gamma \mapsto \lambda(\gamma) + C$ is a weight), but λ needs not be a weight. Such Markov semiweights were used in [68] to show that the relative growth series $\Sigma(\Gamma_0 \text{ rel } F_k, S_k; z)$ is rational for a finitely generated subgroup Γ_0 of a free group F_k.

5.5. Another class of weights comes from geometry. For a group Γ acting properly and isometrically on a metric space (X,d) with a base point x_0, there is a naturally associated weight defined by

$$\lambda(\gamma) \;=\; d(\gamma x_0, x_0)$$

for all $\gamma \in \Gamma$. If Γ is an irreducible lattice in a connected semi-simple real Lie group G of rank at least 2, it is remarkable that such weights are always equivalent to word-length weights. This is straightforward in case G/Γ is compact (Lemma 2 in [125]), but it is a profound result otherwise [117].

5.6. Here is one more class of weights, associated again to a group Γ generated by a finite set S. For each $\gamma \in \Gamma$, denote by $I(\gamma)$ the number of geodesic paths from 1 to γ in the Cayley graph of (Γ, S) and set

$$\Sigma_k(z) \;=\; \sum_{\gamma\in\Gamma} I(\gamma)^k z^{\ell_S(\gamma)} \;\in\; \mathbb{Z}[[z]]$$

for all $k \geq 0$. The rationality of $\Sigma_1(z)$ for hyperbolic groups goes back to [84], Corollary 5.2.A; see also [129].

In the particular case of surface groups

$$\Gamma_g = \langle a_1, b_1, \ldots, a_g, b_g | [a_1,b_1]\ldots[a_g,b_g] = 1\rangle$$
$$\text{with } S_g = \{a_1, b_1, \ldots, a_g, b_g\}$$

as above, L. Bartholdi has recently extended a computation due to Cannon [28] for $k = 0$ and has shown that the $\Sigma_k(z)$'s are rational functions which one can write down. For example, for $k = 2$,

$$\Sigma_2(z) = 1 + 4g \times$$
$$\times \frac{z + z^2 + \ldots z^{2g-1}}{1 - (4g-2)z - \ldots - (4g-2)z^{2g-1} + z^{2g} - z^{2g-1}(1-z)(3-2z^{2g-1})}.$$

As $\Sigma_2(z)$ encodes the numbers of unimodal closed paths of any given length, these computations provide lower estimates for the corresponding spectral radius (see Sec. 7 below).

It is natural to ask for sufficient conditions of pairs (Γ, S) ensuring that all Σ_k's are rational. For example, it is so for all S in case Γ is Gromov hyperbolic. But it is *not* sufficient that (Γ, S) has finitely many cone types; indeed, for $\Gamma = \mathbb{Z}^2$ and S the standard basis, a simple calculation shows that $\Sigma_2(z)$ is algebraic and not rational (L. Bartholdi).

6. Complete growth series

For a (not necessarily commutative) ring A with unit, let $A[[z]]$ denote the ring of formal power series in z with coefficients in A. Recall that an element of $A[[z]]$ is *rational* if it is in the smallest subring $A[[z]]^{rat}$ of $A[[z]]$ which contains the ring $A[z]$ of polynomials and the inverse of any of its elements which is invertible in $A[[z]]$. This definition coincides with the usual one in case A is a commutative field. For all this, see, e.g., [150].

Consider a group Γ generated as a *monoid* by a finite set T, and denote by $\ell_T : \Gamma \to \mathbb{N}$ the corresponding word-length function. (Observe that, if Γ is generated as a group by a set S, it is generated as a monoid by $S \cup S^{-1}$.) The *complete growth series* of (Γ, T) is the formal power series

$$\Sigma_{\mathrm{com}}(\Gamma, T; z) = \Sigma_{\mathrm{com}}(z) = \sum_{n=0}^{\infty} \left(\sum_{\substack{\gamma \in \Gamma \\ \ell_T(\gamma) = n}} \gamma \right) z^n = \sum_{\gamma \in \Gamma} \gamma z^{\ell_T(\gamma)} \in \mathbb{Z}[\Gamma][[z]].$$

The augmentation map $\epsilon : \mathbb{Z}[\Gamma] \to \mathbb{Z}$ induces a morphism of rings $\mathbb{Z}[\Gamma][[z]] \to \mathbb{Z}[[z]]$ again denoted by ϵ, and one has clearly $\epsilon\left(\Sigma_{\mathrm{com}}(z)\right) = \Sigma(z) \in \mathbb{Z}[[z]]$.

Complete growth series were introduced by F. Liardet in his thesis [111].

Before, they have appeared implicitly in the following disguise: given a pair (Γ, S), denote for any integer $n \geq 0$ by χ_n the characteristic function of the sphere of radius n centered at $1 \in \Gamma$. Given a unitary representation π of Γ, the spectra of the self-adjoint operators $\pi(\chi_n)$ enter into various problems of "uniform distribution"; see, e.g., [4].

Let us also mention the growth series in more than one variable which appear in [152], Proposition 26, and in [136].

The easiest example of a complete growth series is probably the following:

$$\mathbb{Z} \supset T = \{1, -1\} \quad \Longrightarrow \quad \Sigma_{\mathrm{com}}(z) = 1 + \sum_{n=1}^{\infty} (\delta_n + \delta_{-n})\, z^n = \frac{1 - z^2}{1 - (\delta_1 + \delta_{-1})\, z + z^2}$$

(where, here as later, we write δ_γ for γ viewed in $\mathbb{Z}[\Gamma]$). Compare with the well-known generating function

$$1 + \sum_{n=1}^{\infty} 2T_n(x)z^n = \frac{1-z^2}{1-2xz+z^2}$$

for Chebyshev polynomials. (Recall that $T_n(\cos\theta) = \cos n\theta$ for all $n \geq 0$ and that $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for $n \geq 1$.) For $\Gamma = F_k$ and $T = S_k \coprod S_k^{-1}$, with S_k a free basis of F_k, one has similarly

$$\Sigma_{\mathrm{com}}(z) = \frac{1-z^2}{1-\left(\sum_{t\in T}\delta_t\right)z + (2k-1)z^2}.$$

It is a result of F. Liardet [111] that, *for any finitely generated group* Γ *which is virtually Abelian and for any finite set* T *which generates* Γ *as a monoid, the corresponding series* $\Sigma_{\mathrm{com}}(z) \in \mathbb{Z}[\Gamma][[z]]$ *is rational.* The precise result is too long to be quoted here; but here is a corollary for the Abelian case, formulated as in [111] (groups hardly appear explicitly in Klarner's papers).

Theorem (Klarner, Liardet). *Let* Γ *be an abelian group and let* T *be a finite subset generating* Γ *as a monoid. Then the corresponding complete growth series is of the form*

$$\Sigma_{\mathrm{com}}(z) = \frac{P(z)}{\prod_{t\in T}(1-\delta_t z)},$$

where $P(z)$ *is a polynomial in* $\mathbb{Z}[\Gamma][z]$.

There does not seem to exist an equally simple statement in the virtually Abelian case.

One crucial ingredient of the proof is the following classical result already used by D. Klarner: for any integer $k \geq 1$, any ideal $E \lhd \mathbb{N}^k$ is finitely generated. (Recall that E is an ideal if $u \in \mathbb{N}^k$, $v \in E \implies u + v \in E$, and that E is finitely generated if there exists $v_1, \ldots, v_j$ such that $E = \bigcup_{1\leq i\leq j}\left(v_i + \mathbb{N}^k\right)$. This classical result is sometimes attributed to Hilbert and to Gordan [167], p. 10, and sometimes known as *Dickson's Lemma* [18], p. 184.)

The result of M. Benson [19] quoted in Sec. 4 is a straightforward consequence of the previous theorem and of its generalization to virtually Abelian groups, because the image $\Sigma(z) = \epsilon\left(\Sigma_{\mathrm{com}}(z)\right)$ by the augmentation ϵ is rational in $\mathbb{Z}[[z]]$ as soon as $\Sigma_{\mathrm{com}}(z)$ is rational in $\mathbb{Z}[\Gamma][[z]]$.

As already observed in a different context (Subsec. 5.4 above), complete growth series are multiplicative for direct products, and the "reciprocal of the series minus one" is additive over free products. This provides further rational complete growth series from known ones.

Complete growth series have also been computed recently for various other cases, including Coxeter systems and pairs (Γ_g, S_g) associated to closed orientable surfaces of genus $g \geq 2$. (See [80]; this uses rewriting systems and a formula obtained in [75]. See also [11].) More precisely, the series $\Sigma_{\mathrm{com}}(z)$ for this case is rational and equals

$$(1-z^2)(1-z^{4g}) \times \Big(1+(4g-1)z^2+(4g-1)z^{4g}+z^{4g+2} - \\ - Az(1+z^{4g}) - Cz^{2g+1} + Dz^{2g}(1+z^2)\Big)^{-1},$$

where A, C, D are as follows. First, A is the sum of the $4g$ elements in $S_g \cup (S_g)^{-1}$. Then, denoting by $r_g = a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$ the relator of the presentation $\Gamma_g = \langle S_g | r_g \rangle$, the term C is the sum of the $8g$ distinct subwords of length $2g-1$ which appear in cyclic conjugates of r_g and r_g^{-1}, and D is the sum of the $4g$ distinct subwords of length $2g$ which appear in cyclic conjugates of r_g and r_g^{-1}, and which begin by a letter in $\{b_1, b_1^{-1}, \dots, b_g, b_g^{-1}\}$.

More generally, *for Γ a hyperbolic group and for S an arbitrary finite generating set, the resulting complete growth series is rational.* This was checked in [80], Proposition 6 and independently by L. Bartholdi.

Does there exist a pair (Γ, S) such that the corresponding series $\Sigma_{\mathrm{com}}(z)$ is *not* rational in $\mathbb{Z}[\Gamma][[z]]$ and such that its image $\Sigma(z)$ by the augmentation map *is* rational in $\mathbb{Z}[[z]]$? The answer seems to be "yes," and L. Bartholdi has good reasons to believe that the Heisenberg group H_1 generated by $S = \{x, y\}$ qualifies, where

$$H_1 = \left\{ \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix} \right\}, \qquad x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(See Sec. 4 above for the standard growth series.)

More generally, for a nilpotent group Γ and a finite set of generators S, does the rationality of the corresponding *complete* growth series imply that Γ is virtually Abelian?

For a pair (Γ, S), one may also view the complete growth series as an *operator growth series*

$$\Sigma_{\mathrm{op}}(z) = \sum_{\gamma \in \Gamma} \gamma z^{\ell_S(\gamma)} \in A(\Gamma)[[z]],$$

where $A(\Gamma)$ is a Banach algebra containing $\mathbb{C}[\Gamma]$ as a subalgebra. Natural candidates are the group algebra $\ell^1(\Gamma)$ and the reduced C*-algebra $C_r^*(\Gamma)$ of Γ (both are natural completions of $\mathbb{C}[\Gamma]$), as well as the C*-algebra of all bounded operators on the Hilbert space

$$\ell^2(\Gamma) = \left\{ \phi : \Gamma \to \mathbb{C} \mid \sum_{\gamma \in \Gamma} |\phi(\gamma)|^2 < \infty \right\}.$$

A formal power series $\sum_{n \geq 0} a_n z^n$ with coefficients in a Banach algebra A has a *radius of convergence* R defined by $R^{-1} = \limsup_{n \to \infty} \sqrt[n]{\|a_n\|}$.

Consider in particular, for a pair (Γ, S) as above, the radius of convergence R of its usual growth series $\Sigma(z) \in \mathbb{C}[[z]]$, the radius of convergence $R(\ell^1)$ of its operator growth series $\Sigma_{\text{op}}(z) \in \ell^1(\Gamma)[[z]]$, and the radius of convergence $R(C_r^*)$ of its operator growth series $\Sigma_{\text{op}}(z) \in C_r^*(\Gamma)[[z]]$.

Theorem ([80]). *With the above notations, one has*
(i) $R(\ell^1) = R$;
(ii) *if* Γ *is amenable, then* $R(C_r^*) = R(\ell^1)$;
(iii) *if* Γ *is Gromov hyperbolic and non elementary, then*
$1 > R(C_r^*) = \sqrt{R} > R$.

Moreover it is conjectured that $R(C_r^*) > R$ for all pairs (Γ, S) with Γ not amenable.

7. Spectral radius of simple random walks on Cayley graphs

The *spectral radius* of a locally finite connected graph X is the number

$$\mu(X) = \limsup_{n \to \infty} \sqrt[n]{p^{(n)}(x, y)} = \lim_{n \to \infty} \sqrt[2n]{p^{(2n)}(x, x)},$$

where, for $n \geq 0$ and for vertices x, y of X, the probability $p^{(n)}(x, y)$ is the ratio of the number of paths of length n from x to y by the number of all paths of length n starting at x; this is independent of the choice of the vertices x and y. In other words, $\mu(X)$ is the spectral radius of the *simple random walk* on this Cayley graph.

There are many equivalent definitions [178]. For viewing $\mu(X)$ as the spectral radius of an appropriate bounded operator on a Hilbert space, see also [99], as well as more recent works including [84] and [185]. (Though we consider only simple walks here, there are good motivations to study more general walks; see among others [51], [97], and [173].)

In case X is the Cayley graph of a finitely generated group Γ with respect to a finite set S of generators, we write

$$\mu(\Gamma, S)$$

instead of $\mu(X)$. There is the simple formula expressing $\mu(\Gamma, S)$ in terms of the *cogrowth* of (Γ, S) recalled in Subsec. 5.3 above, and a formula expressing $\mu(\Gamma, S)$ in terms of the *critical exponent* of the Poincaré series for the action of the fundamental group of $\mathrm{Cay}(\Gamma, S)$ on the universal cover of the same Cayley graph [42].

In connection with our Sec. 1, observe that

$$\mu(\Gamma, S) \geq \frac{1}{\sqrt{\omega(\Gamma, S)}}$$

(a very special case of inequalities due to A. Avez [5]). Indeed, denoting for each $n \geq 1$ by $B(n)$ the ball in Γ centered at 1 of radius n, of size $\beta(n)$, one has

$$1 = \left(\sum_{\gamma \in B(n)} p^{(n)}(1, \gamma) \right)^2 \leq \beta(n) \sum_{\gamma \in B(n)} p^{(n)}(1, \gamma)^2 = \beta(n) p^{(2n)}(1, 1)$$

by the Bouniakovski–Cauchy–Schwarz inequality, and the previous inequality follows.

As a particular case of results of H. Kesten ([99] and [100]), one has the basic inequalities

$$\frac{\sqrt{2k-1}}{k} \leq \mu(\Gamma, S) \leq 1,$$

where $k = |S|$. Moreover, the right-hand inequality is an equality if and only if Γ is amenable; also, if $k \geq 2$, the left-hand inequality is an equality if and only if Γ is free on S.

Very few exact values of $\mu(\Gamma, S)$ are known, besides cases where the group is amenable or virtually free. In particular, one does not know the value $\mu_g = \mu(\Gamma_g, S_g)$ for the fundamental group of an orientable surface. Is it transcendental ? (a question of P. Sarnak).

One should investigate the infimum of $\mu(\Gamma, S)$ when S varies (compare with Sec. 1 above).

One may also consider the *Green function*

$$G_1(z) = \sum_{n=0}^{\infty} p^{(n)}(1, 1) z^n \in \mathbb{R}[[z]]$$

of which $\mu(\Gamma, S)^{-1}$ is precisely the radius of convergence. If Γ contains a finitely generated free subgroup of finite index, then $G_1(z)$ is algebraic [178]; it is conjectured that an appropriate phrasing of the converse should be true (see again [178]).

Short of *computing* μ_g, there has been progress in *estimating* μ_g. The group Γ_g being neither free on S_g nor amenable, Kesten's inequalities imply

$$\frac{\sqrt{4g-1}}{2g} < \mu_g < 1 \qquad \text{and in particular} \qquad 0.6614 \approx \frac{\sqrt{7}}{4} < \mu_2 < 1.$$

We know three methods to *estimate* μ_g *from above* (see [12], [34], [128], [181], and [182]).

One is based on the existence of a spanning forest of degree $4g-1$ in the Cayley graph X_g of (Γ_g, S_g). (A spanning forest is a subgraph without circuits containing all vertices.) It shows that

$$\mu_g \leq \frac{\sqrt{4g-2}}{2g} + \frac{1}{4g}.$$

Another one involves restrictions, to the graph X_g appropriately embedded in the hyperbolic plane H^2, of eigenfunctions of the hyperbolic Laplacian on H^2. (The idea is to choose a number $\beta > 0$ and a function $F : H^2 \to \mathbb{R}_+^*$ such that $\Delta_{\text{smooth}} F = \beta F$, where Δ_{smooth} is the smooth Laplacian on H^2, to consider the restriction f of F to X_g, to estimate α such that $f(x) - \big(1/\text{degree}(x)\big) \sum_{x \sim x} f(y) \geq \alpha f(x)$ for all $x \in X_g$, the summation being over the $\deg(x)$ neighbors of x in X_g, and to conclude that $\mu_g \leq 1 - \alpha$.) This provides interesting numerical estimates for μ_g, and could probably be exploited further.

The third method uses the following easy fact, observed by Gabber. For a graph X, we denote by $\mathbb{X}^1$ the set of all *oriented* edges of X and by $e \mapsto \overline{e}$ the reversal of orientations (one has $e \neq \overline{e}$ and $\overline{\overline{e}} = e$ for all $e \in \mathbb{X}^1$); moreover X^0 denotes the set of vertices of X, and e_+, e_- the head and tail of an edge $e \in \mathbb{X}^1$.

Lemma (Gabber). *Let X be a connected regular graph of degree k.*

(i) *Suppose there exists a constant c and a function $\omega : \mathbb{X}^1 \to \mathbb{R}_+^*$ such that $\omega(\overline{e}) = \omega(e)^{-1}$ for all $e \in \mathbb{X}^1$ and such that*

$$\frac{1}{k} \sum_{e \in \mathbb{X}^1, e_+ = x} \omega(e) \leq c \qquad \textit{for all} \qquad x \in X^0.$$

Then $\mu(X) \leq c$;

(ii) *there exists a function $\omega_0 : \mathbb{X}^1 \to \mathbb{R}_+^*$ such that $\omega_0(\overline{e}) = \omega_0(e)^{-1}$ for all $e \in \mathbb{X}^1$ and such that*

$$\frac{1}{k} \sum_{e \in \mathbb{X}^1, e_+ = x} \omega_0(e) = \mu(X) \qquad \textit{for all} \qquad x \in X^0.$$

On the proof. Though there is apparently no *published* proof of (i), there is a simple argument which can be found in [38], and which is also in [12].

For (ii), one uses the existence of a function $f_0 : X^0 \to \mathbb{R}_+^*$ such that

$$\frac{1}{k} \sum_{\substack{y \in X^0 \\ y \sim x}} f(y) = \mu(X) f(x) \qquad \text{for all} \qquad x \in X^0$$

(where $y \sim x$ indicates summation over all neighbors y of x in X). It is then sufficient to set

$$\omega_0(e) = \frac{f_0(e_-)}{f_0(e_+)}$$

for all $e \in \mathbb{X}^1$.

There is a proof of the existence of f_0 in terms of graphs in [46], Proposition 1.5. But there are earlier proofs in the literature on irreducible stationary discrete Markov chains; see [90] and [143]. □

The most primitive use of this lemma, with a function ω taking only 2 distinct values, shows that

$$\mu_g \leq \frac{\sqrt{2g-1}}{g}$$

for all $g \geq 2$ [12]. More refined computations are due to A. Zuk [181], who shows in particular that

$$\mu_g \leq \frac{1}{\sqrt{g}}$$

for all $g \geq 2$, and to T. Nagnibeda [128]. In the case $g = 2$, one has, for example,

estimate from [12] :	$\mu_2 \leq 0.7373$
estimate from [181] :	$\mu_2 \leq 0.6909$
estimate from [128] :	$\mu_2 \leq 0.6629.$

There was also some work to *estimate* μ_g *from below*. Let

$$\Sigma_2(z) = \sum_{\gamma \in \Gamma_g} I(\gamma)^2 z^{\ell(\gamma)} = \sum_{n=0}^{\infty} \alpha(n) z^n$$

be the formal series introduced in Subsec. 5.6 above. The growth rate of the $\alpha(n)$'s is a lower bound for the growth rate of the number of all closed loops in X_g. Numerically, this gives, for example,

$$\mu_2 \geq 0.6614389$$

(compage with Kesten's estimate $\mu_2 \geq 0.6614378$!). More precise estimates for numbers of closed loops lead to

$$\mu_2 \geq 0.6624.$$

Both computations above are due to L. Bartholdi [11]. There are also lower estimates improving Kesten's estimates and not restricted to the groups Γ_g's, due to W. Paschke [140].

The pair (Γ_g, S_g) is a tempting test case. But most of the considerations above carry over in some form or another to other pairs such as those associated to one-relator presentations and to small cancellation presentations of groups.

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