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Bounds for the degrees in the Nullstellensatz

By W. Dale Brownawell*

I. Introduction

Let $P_1, \ldots, P_m \in \mathbf{C}[x_1, \ldots, x_n] = \mathbf{C}[\mathbf{x}]$ have degree at most $D \geq 1$. If $P_1, \ldots, P_m$ have no common zero in $\mathbf{C}^n$, then a special case of Hilbert's Nullstellensatz says that there are polynomials $A_1, \ldots, A_m \in \mathbf{C}[\mathbf{x}]$ such that

$$A_1 P_1 + \cdots + A_m P_m = 1. \tag{1.1}$$

When $P_1, \ldots, P_m$ have no common zeros at infinity either, the fundamental theorem of elimination theory (e.g. [La], p. 169) shows that one may choose A_i so that

$$\deg A_i \leq n(D-1). \tag{1.2}$$

In the general case much less has been known. D.W. Masser and G. Wüstholz in [Ma-Wü] used the classical methods of G. Hermann [He] to show that one may take

$$\deg A_i \leq 2(2D)^{2^{n-1}}.$$

Moreover E. Mayr and A. Meyer showed [Ma-Me] that such doubly exponential growth is in general unavoidable when expressing elements of polynomial ideals in terms of bases. Their procedure is explicit, but rather elaborate. (See e.g. [Ba-Sti] for an agreeable exposition.) For $k, D \in \mathbf{N}$, $D \geq 5$, Mayr and Meyer construct polynomials $P_1, \ldots, P_{n+1} \in \mathbf{Z}[x_1, \ldots, x_n]$, $n = 10k$, with $\max \deg P_i = D$ such that $x_1 - x_n \in (P_1, \ldots, P_{n+1})$. But they show that for any $A_1, \ldots, A_{n+1} \in \mathbf{Z}[x_1, \ldots, x_n]$ with

$$x_1 - x_n = A_1 P_1 + \cdots + A_{n+1} P_{n+1},$$

one must have $\max \deg A_i > (D-2)^{2^{k-1}}$.

Nevertheless, it has been suspected that the special case (1.1) is fundamentally different and that one should still be able to find A_i there with degrees

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much nearer the lower bound

$$\deg A_i \geq D^n - D^{n-1}$$

furnished by the following variant of an example of Masser and P. Philippon:

$$P_1 = x_1^D,\ P_2 = x_1 - x_2^D, \ldots, P_{n-1} = x_{n-2} - x_{n-1}^D,\ P_n = 1 - x_{n-1}x_n^{D-1}.$$

For these P_i, consideration of any solution of (1.1) along the curve

$$\mathbf{x}(t) := \left(t^{(D-1)D^{n-2}}, \ldots, t^{(D-1)D}, t^{D-1}, t^{-1}\right)$$

shows that the coefficient A_1 satisfies $\deg_{x_n} A_1 \geq D^n - D^{n-1}$. One can obtain such a solution A_i from the telescoping sum

$$\left(x_n^\delta\right)x_1^D - x_n^\delta\left(x_1^D - \left(x_2^D\right)^D\right) - x_n^\delta\left(x_2^{D^2} - \left(x_3^D\right)^{D^2}\right) - \cdots$$
$$- x_n^\delta\left(x_{n-2}^{Dn-2} - \left(x_{n-1}^D\right)^{D^{n-2}}\right) + \left(1 - \left(x_{n-1}x_n^{D-1}\right)^{D^{n-1}}\right) = 1,$$

where $\delta = D^n - D^{n-1}$, as noticed by Philippon and the author. The purpose of this paper is to establish the following result:

THEOREM 1. *Let the polynomials* $P_1, \ldots, P_m \in \mathbf{C}[\mathbf{x}]$ *with* $\deg P_i \leq D$ *have no common zero in* $\mathbf{C}^n$. *Then there are polynomials* $A_1, \ldots, A_m \in \mathbf{C}[\mathbf{x}]$ *with*

$$\deg A_i \leq \mu n D^\mu + \mu D,$$

where $\mu = \min\{m, n\}$, *such that*

$$A_1 P_1 + \cdots + A_m P_m = 1.$$

Of course, by the well-known "Rabinowitsch trick," Theorem 1 implies the following effective version of Hilbert's Nullstellensatz:

COROLLARY. *If* $Q, P_1, \ldots, P_m \in \mathbf{C}[\mathbf{x}]$ *have degrees* $\leq D$ *and if* Q *vanishes on all the common zeros of* $P_1, \ldots, P_m$ *in* $\mathbf{C}^n$, *then there are* $e \in \mathbf{N}$ *and polynomials* $B_1, \ldots, B_m \in \mathbf{C}[\mathbf{x}]$ *with*

$$e \leq e' := (\mu + 1)(n + 2)(D + 1)^{\mu+1},$$
$$\deg B_i P_i \leq eD + e',$$

where $\mu = \min\{m, n\}$, *such that*

$$B_1 P_1 + \cdots + B_m P_m = Q^e.$$

Proof. Let $P_0 = 1 - x_{n+1}Q$ for a new variable x_{n+1}. By Theorem 1, we find $A_0, \ldots, A_m \in \mathbf{C}[x_1, \ldots, x_{n+1}]$ with $\deg A_i \leq e'$, such that

$$A_0 P_0 + \cdots + A_m P_m = 1.$$

Now substitute $x_{n+1} = Q^{-1}$ and clear out the denominators. □

It will be clear from the proof that the constants can be improved somewhat. By the Lefschetz principle, Theorem 1 and Theorem 2 below hold over any algebraically closed field of characteristic zero and thus, if properly formulated (e.g. $(P_1, \ldots, P_m)$ has rank equal to $n + 1$ or has no zeros in an algebraic closure $\bar{k}$), over any field k of characteristic zero. For solvable systems of linear equations can be solved in the coefficient field.

Since the statement of the result is algebraic, it would be interesting to obtain a totally algebraic proof. Although the bound on the degree is nearly optimal, if one works over $\mathbf{Z}[\mathbf{x}]$, the resulting bounds (see [Ma-Wü]) on the height of the coefficients of the A_i are still not strong enough to deduce even the global inequalities of [Br]. Perhaps an algebraic proof would also give such bounds on the height.

We say that $Q_1, \ldots, Q_m \in \mathbf{C}[\mathbf{x}] =: R$ is a *regular sequence* if $Q_1 \neq 0$ and for $i = 2, \ldots, m$, Q_i is not a zero divisor of $R/(Q_1, \ldots, Q_{i-1})$. Note that we expressly allow the possibility that $(Q_1, \ldots, Q_m) = R$. We shall deduce Theorem 1 as a corollary of the following result:

THEOREM 2. *Let* $Q_1, \ldots, Q_m \in R = \mathbf{C}[\mathbf{x}]$ *be a regular sequence with* $\deg Q_i = D_i > 0$, $i = 1, \ldots, m$, *having no common zeros. Then there exist* $A_1, \ldots, A_m \in R$ *with*

$$\deg A_i < \mu n D_1 \cdots D_\mu + \mu D,$$

where $\mu = \min\{m, n\}$ *and* $D = \max_{i=1,\ldots,m} D_i$, *such that*

$$A_1 Q_1 + \cdots + A_m Q_m = 1.$$

Theorem 2 has a corollary which is analogous to the general Hilbert Nullstellensatz and which, up to certain constants, is of equivalent strength to Theorem 2 itself.

COROLLARY. *Let* $Q_1, \ldots, Q_m \in R$ *be a regular sequence with* $\deg Q_i > 0$, $i = 1, \ldots, m$. *If* $\operatorname{rank}(Q_1, \ldots, Q_m) = m \leq n$ *and* $P \in R$ *of degree* D_0 *vanishes on all the common zeros of* $Q_1, \ldots, Q_m$, *then for some* $e \in \mathbf{N}$ *and polynomials* $B_1, \ldots, B_m \in R$ *with*

$$e < e'' := (m+1)(n+1)(D_0+1)D_1 \cdots D_m + (m+1)D,$$

$$\deg B_i < eD_0 + e'',$$

where $D = \max\{1 + D_0, D_1, \ldots, D_m\}$,

$$B_1 Q_1 + \cdots + B_m Q_m = P^e.$$

Proof. $Q_1, \ldots, Q_m, 1 - x_{n+1}P$ is now a regular sequence. Apply Theorem 2 and the Rabinowitsch trick. □

If $\operatorname{rank}(Q_1, \ldots, Q_m) = n + 1$, then multiplying by P the equation given in Theorem 2 yields an expression with $e = 1$ and smaller e''. Moreover if we take $P = 1$ in the corollary we retrieve the case $m \le n$ of Theorem 2 with slightly increased constants.

The main task of this paper is to transpose to the appropriate setting some of the elimination techniques in the theory of transcendental numbers based on Yu. V. Nesterenko's use of the Chow form ([Ne1]–[Ne3]) in such a way that the dependence on the coordinates of the point in question is carefully controlled. (Recently Philippon has shown in [Ph2] how to deduce our main Proposition 8 efficiently from the development in [Ph1].) Then we appeal to powerful estimates from several complex variables to complete the proof.

I am deeply indebted to C. A. Berenstein and A. Yger. When I told them of the global inequalities following from [Br], they generously informed me of their elegant work [Be-Yg], [Be-St], partly with D. C. Struppa, establishing the crucial link between such asymptotic lower bounds on $\max\{|P_i(\omega)|\}$ and the existence of an explicit solution A_i of (1.1) with upper bounds on the degrees. I also want to thank D. W. Masser for many useful conversations. In particular he pointed out that the results of H. Skoda [Sk], to which he was alerted by R. Narasimhan, give somewhat sharper bounds on degrees than were obtained in the first version of this paper.

II. The Chow form

All ideals $\mathfrak{A}$ in this section will be homogeneous ideals of $k[x_0, \ldots, x_n]$, where k is a field of characteristic 0. For any natural number d, $1 \le d \le n$, introduce linear forms

$$L_j(\mathbf{x}) = u_{j0}x_0 + \cdots + u_{jn}x_n,$$

$j = 1, \ldots, d$, in the new variables u_{jl}, $0 \le l \le n$, and consider the ideal $\mathfrak{A}'(d)$ in $k[\mathbf{x}, \mathbf{u}_1, \ldots, \mathbf{u}_d]$ consisting of all polynomials G for which there is a natural number M with

$$(x_0, \ldots, x_n)^M \cdot G \subset (\mathfrak{A}, L_1, \ldots, L_d).$$

If $\mathfrak{A}$ is unmixed of rank $n + 1 - d$, then $\mathfrak{A}(d) = \mathfrak{A}'(d) \cap k[\mathbf{u}_1, \ldots, \mathbf{u}_d]$ is a non-zero principal ideal [Ne1], and we call any generator F a *Chow form* for $\mathfrak{A}$. The *Chow degree* $\delta_{\mathfrak{A}} = \delta_F$ will be the total degree of F with respect to, say, $u_{10}, \ldots, u_{1n}$. By the symmetry of the definition, one sees that F is homogeneous of the same degree δ_F with respect to each of the d sets of variables $\mathbf{u}_1, \ldots, \mathbf{u}_d$ and in fact is invariant, up to non-zero constant factors, under permutations of $\mathbf{u}_1, \ldots, \mathbf{u}_d$. If F_1 and F_2 are both Chow forms for the ideal $\mathfrak{A}$, then $F_1/F_2 \in k^*$, and we write $F_1 \sim F_2$.

If $\mathfrak{A}$ is unmixed of rank r with primary decomposition

$$\mathfrak{A} = \mathfrak{Q}_1 \cap \cdots \cap \mathfrak{Q}_s,$$

where $\mathfrak{Q}_i$ is $\mathfrak{P}_i$-primary of exponent e_i and F_i is a Chow form for $\mathfrak{P}_i$, $i = 1, \ldots, s$, then any Chow form F for $\mathfrak{A}$ satisfies [Ne1]

$$F \sim F_1^{e_1} \cdots F_s^{e_s}.$$

Conversely we shall call any such product a Chow form of rank r, since it is the Chow form of the corresponding intersection of symbolic powers $\mathfrak{P}_1^{(e_1)} \cap \cdots \cap \mathfrak{P}_s^{(e_s)}$. A Chow form of a prime ideal $\mathfrak{P}$ of rank $r = n + 1 - d$, with $x_i \notin \mathfrak{P}$, can be written as

$$F = a \prod_{\gamma} (\alpha_0^{\gamma} u_{d0} + \cdots + \alpha_n^{\gamma} u_{dn}), \tag{2.1}$$

where $\alpha_i = 1$, $a \in k[\mathbf{u}_1, \ldots, \mathbf{u}_{d-1}] =: k_{d-1}$, $K := k_{d-1}(\alpha_0, \ldots, \alpha_n)$ has degree $[K : k_{d-1}] = \delta_{\mathfrak{P}}$, and γ runs through all k_{d-1}-embeddings of K into an algebraic closure $\overline{k_{d-1}}$ [Ne2]. These properties were known classically (see e.g. [VdW]).

We now turn to the basic tool of our investigation, which was introduced by Nesterenko in [Ne2]. Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_s$ be prime ideals of rank r with respective Chow forms

$$F_i = a_i \prod_{\gamma \in \Gamma_i} (\alpha_{i0}^{\gamma} u_{d0} + \cdots + \alpha_{in}^{\gamma} u_{dn}), \tag{2.2}$$

where for each $i = 1, \ldots, s$, some $\alpha_{ik}^{\gamma} = 1$, and Γ_i is the set of k_{d-1}-embeddings of some $k_{d-1}(\alpha_{i0}^{\gamma}, \ldots, \alpha_{in}^{\gamma})$ into $\overline{k_{d-1}}$. Let $k_1, \ldots, k_s \in \mathbf{N}$ and set $F = F_1^{k_1} \cdots F_s^{k_s}$. Let $Q \in R'[\mathbf{x}]$ be homogeneous in $\mathbf{x}$, where $R' \supset k[\mathbf{u}_1, \ldots, \mathbf{u}_{d-1}]$ is an integrally closed integral domain. Then we define the *resultant* $\operatorname{Res}(F, Q)$ of F and Q to be

$$\operatorname{Res}(F, Q) = \prod_{i=1}^{s} \left(a_i^{\deg Q} \prod_{\gamma \in \Gamma_i} Q(\alpha_{i0}^{\gamma}, \ldots, \alpha_{in}^{\gamma}) \right)^{k_i}.$$

We shall sometimes denote the degree of a homogeneous polynomial $Q \in R'[\mathbf{x}]$ also as δ_Q. The basic algebraic properties of $\operatorname{Res}(F, Q)$ that we shall need are contained in the following result.

PROPOSITION 3. *With the preceding notation,*

i) $\operatorname{Res}(F, Q) \in R'$.

ii) *If $R' = k[\mathbf{u}_1, \ldots, \mathbf{u}_{d-1}]$ and $Q \in k[\mathbf{x}] \setminus \cup \mathfrak{P}_i$, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be a complete list of the minimal prime ideals, with Chow forms $E_1, \ldots, E_t$, associated to the ideals $(\mathfrak{P}_i, Q)$, $i = 1, \ldots, s$. Then there are (positive) $f_1, \ldots, f_t \in \mathbf{N}$*

such that

$$\operatorname{Res}(F, Q) \sim E_1^{f_1} \cdots E_t^{f_t},$$

i.e. $\operatorname{Res}(F, Q)$ *is a Chow form of rank* $r + 1$.

Proof. Part i) follows from Lemma 4 of [Ne2]. Part ii) follows from the proof of Lemma 6 of [Ne3], although details are given there ostensibly over **Z**. □

III. Absolute value of Chow forms

In this section we take $k = \mathbf{C}$, although analogous considerations apply to any $\mathbf{C}_p$ as well. If $\mathfrak{A}$ is a homogeneous unmixed ideal of $\mathbf{C}[x_0, \ldots, x_n]$ with Chow form F and if $\omega = (\omega_0, \ldots, \omega_n) \in \mathbf{C}^{n+1}$ is non-zero, then the (*normalized*) *absolute value* of $\mathfrak{A}$ at ω is defined in the following way: For $j = 1, \ldots, d$, let $S^{(j)} = (s_{kl}^{(j)})$ be skew symmetric matrices in the new variables $s_{kl}^{(j)}$, $0 \leq k < l \leq n$. Then following Nesterenko [Ne3], we define

$$\|\mathfrak{A}\|_\omega := \|F\|_\omega := \frac{H\big(F(S^{(1)}\omega, \ldots, S^{(d)}\omega)\big)}{|\omega|^{d\delta_{\mathfrak{A}}} H(F)},$$

where $|\omega| := \max_i |\omega_i|$ and H denotes the height (= maximum absolute value) of the coefficients when the variables in the numerator are taken to be the $s_{kl}^{(j)}$. The term $H(F)$ appears in the denominator to make $\|\mathfrak{A}\|_\omega$ well-defined. We also use the notation that

$$\|Q\|_\omega := \frac{|Q(\omega)|}{|\omega|^{\delta_Q} H(Q)}$$

for non-zero homogeneous $Q \in \mathbf{C}[\mathrm{x}]$ and non-zero $\omega \in \mathbf{C}^{n+1}$.

PROPOSITION 4. *If* $k = \mathbf{C}$ *in Proposition* 3 *and if* $\omega \in \mathbf{C}^{n+1}$ *is non-zero, then*

$$\|\operatorname{Res}(F, Q)\|_\omega \leq c(F, Q) \max\{\|F\|_\omega, \|Q\|_\omega\},$$

where $c(F, Q)$ *is a constant depending only on* F *and* Q, *and if* $d > 1$, *then*

$$\deg_{\mathbf{u}_1} \operatorname{Res}(F, Q) = \delta_F \delta_Q.$$

Of course if $d = 1$, then $\operatorname{Res}(F, Q) \in \mathbf{C}^*$. The first assertion is essentially a transposition to $\mathbf{C}$ of Proposition 3 of [Ne3], whose proof we follow closely, although here we do not have to be so careful with the height of coefficients. On the other hand, since our main concern is growth as $|\omega| \to \infty$ (with ω_0 normalized to be 1), while Nesterenko takes $|\omega| = 1$ from the beginning, it seems reassuring to retrace the argument with $|\omega|$ appearing explicitly. Our efforts are repaid somewhat by the greater precision of the second assertion, where Nesterenko had only an inequality (Lemma 5 of [Ne3]).

Proof. In the argument we use the substitution

$$\sigma_\omega\colon \mathbf{u}_j \mapsto S^{(j)}\omega,$$

$j = 1, \dots, d$, where the $S^{(j)}$ are the skew symmetric matrices introduced earlier. By the continuity of $\|\cdot\|_\omega$ in ω, it suffices to establish our inequality off the closed set of ω's defined by

$$\sigma_\omega(a_1) \cdots \sigma_\omega(a_s) \equiv 0.$$

Now fix ω. We can specialize the $S^{(j)}$, $1 \le j \le d-1$, to the open set of skew symmetric matrices $T^{(j)} = (t_{kl}^{(j)})$ with the $t_{kl}^{(j)}$, $0 \le k < l \le n$, arbitrary elements of the unit polydisk $|t_{kl}^{(j)}| \le 1$ such that under the substitution τ_ω given on each $\mathbf{u}_j$ by $\mathbf{u}_j \mapsto T^{(j)}\omega$,

$$\tau_\omega(a_1) \cdots \tau_\omega(a_s) \neq 0. \tag{3.1}$$

Then since by (2.2) all the elements α_{ik}^γ are integral over

$$\mathbf{C}\left[\mathbf{u}_1, \dots, \mathbf{u}_{d-1}, a_1^{-1}, \dots, a_s^{-1}\right],$$

the induced map

$$\tau_\omega\colon \mathbf{C}[\mathbf{u}_1, \dots, \mathbf{u}_{d-1}] \to \mathbf{C}$$

can be extended to a homomorphism, also denoted τ_ω,

$$\tau_\omega\colon \mathbf{C}\left[\mathbf{u}_1, \dots, \mathbf{u}_{d-1}, \dots, \alpha_{ik}^\gamma, \dots\right] \to \mathbf{C}.$$

Set $\beta_{ik}^\gamma := \tau_\omega(\alpha_{ik}^\gamma)$ and $h_i^\gamma := \max\{|\beta_{i0}^\gamma|, \dots, |\beta_{in}^\gamma|\}$, for $i = 1, \dots, s$ and $\gamma \in \Gamma_i$. Gelfond proved (see equation (120), p. 135 of [Ge]) that for any polynomials $f_1, \dots, f_p$ over $\mathbf{C}$,

$$H(f_1) \cdots H(f_p) \le H(f_1 \cdots f_p)\exp(\Delta),$$

where Δ is the sum of the degrees in each variable. Applying Gelfond's inequality to the product

$$\tau_\omega(F) = \prod_{i=1}^{s} \left\{ \tau_\omega(a_i) \prod_\gamma (\beta_{i0}^\gamma u_{d0} + \cdots + \beta_{in}^\gamma u_{dn}) \right\}^{k_i}$$

gives that

$$\prod_{i=1}^{s} \left(|\tau_\omega(a_i)| \prod_\gamma h_i^\gamma \right)^{k_i} \le c_1(F)|\omega|^{(d-1)\delta_F}, \tag{3.2}$$

where $c_1(F) > 0$ depends only on F, since $|\tau_\omega(u_{jk})| \le (n+1)|\omega|$ and F is homogeneous of degree δ_F in each of $\mathbf{u}_1, \dots, \mathbf{u}_{d-1}$ (as well as $\mathbf{u}_d$).

Notice that since

$$\beta_{i0}^{\gamma} u_{d0} + \cdots + \beta_{in}^{\gamma} u_{dn}|_{\mathbf{u}_d = S^{(d)}\omega} = \sum_{0 \le j < k \le n} s_{jk}^{(d)} \left(\beta_{ij}^{\gamma} \omega_k - \beta_{ik}^{\gamma} \omega_j \right),$$

if we set $\rho_i^{\gamma} := \max_{j,k} |\beta_{ij}^{\gamma} \omega_k - \beta_{ik}^{\gamma} \omega_j|$, then Gelfond's inequality gives that

$$(3.3) \qquad \prod_{i=1}^{s} \left(|\tau(a_i)| \prod_{\gamma} \rho_i^{\gamma} \right)^{k_i} \le H\left(\tau_{\omega}(F)|_{\mathbf{u}_d = S^{(d)}\omega} \right) c_2(F) \le H(\sigma_{\omega}(F)) c_3(F).$$

We remark that if $P, Q \in \mathbf{C}[\mathrm{x}]$ are homogeneous of degree ν, then clearly for any non-zero $\omega, \theta \in \mathbf{C}^{n+1}$,

$$(3.4) \qquad |Q(\omega)P(\theta) - Q(\theta)P(\omega)| \le \|\theta - \omega\| c'(P, Q) |\omega|^{\nu} |\theta|^{\nu}$$

where

$$\|\theta - \omega\| := \left(\max_{j,k} |\theta_j \omega_k - \theta_k \omega_j| \right) / \{ |\omega|\, |\theta| \}.$$

For j such that $|\omega_j| = |\omega|$, consider the polynomial

$$T_j(\mathrm{x}) = \prod_{i=1}^{s} \left((a_i)^{\delta_Q} \prod_{\gamma} \left\{ Q(\alpha_{i0}^{\gamma}, \ldots, \alpha_{in}^{\gamma}) x_j^{\delta_Q} - Q(\mathrm{x}) \left(\alpha_{ij}^{\gamma} \right)^{\delta_Q} \right\} \right)^{k_i}.$$

Letting $c = c'(x_j^{\delta_Q}, Q)$, we find from (3.4) that

$$|\tau_{\omega}(T_j)(\omega)| \le \prod_{i=1}^{s} \left(|\tau_{\omega}(a_i)|^{\delta_Q} \prod_{\gamma} \left(\rho_i^{\gamma} c (h_i^{\gamma})^{\delta_Q - 1} |\omega|^{\delta_Q - 1} \right) \right)^{k_i}$$
$$= \prod_{i=1}^{s} \left(|\tau_{\omega}(a_i)| \prod_{\gamma} \rho_i^{\gamma} \right)^{k_i} c^{\delta_F} |\omega|^{(\delta_Q - 1)\delta_F}$$
$$\times \prod_{i=1}^{s} \left(|\tau_{\omega}(a_i)| \prod_{\gamma} h_i^{\gamma} \right)^{k_i(\delta_Q - 1)}.$$

Thus by (3.2) and (3.3),

$$(3.5) \qquad |\tau_{\omega}(T_j)(\omega)| \le H(\sigma_{\omega} F) |\omega|^{d\delta_F(\delta_Q - 1)} c_4(F, Q).$$

However if we simply multiply out the expression for $T_j(\mathrm{x})$, we see from Proposition 3 that

$$(3.6) \qquad T_j(\mathrm{x}) = x_j^{\delta_F \delta_Q} \operatorname{Res}(F, Q) + Q(\mathrm{x}) C(\mathrm{x})$$

with $C(\mathrm{x}) \in \mathbf{C}[\mathbf{u}_1, \ldots, \mathbf{u}_{d-1}, \mathrm{x}]$ homogeneous of degree $\delta_F \delta_Q - \delta_Q$ in x. Thus using (3.2), one sees that

$$(3.7) \qquad |\tau_{\omega}(C)(\omega)| \le c_5(F, Q) |\omega|^{d\delta_F \delta_Q - \delta_Q}.$$

Therefore from (3.5), (3.6), (3.7), we see that

$$|\tau_{\omega}(\operatorname{Res}(F, Q)) \omega_j^{\delta_F \delta_Q}| \le c(F, Q) |\omega|^{d\delta_F \delta_Q} \max\{ \|F\|_{\omega}, \|Q\|_{\omega} \}.$$

But since by the Cauchy integral formula,

$$H\big(\sigma_\omega(\mathrm{Res}(F,Q))\big) \le \sup_{\tau_\omega}|\tau_\omega(\mathrm{Res}(F,Q))|,$$

where τ_ω is restricted to the open subset of $T^{(j)}$'s with $|t_{kl}^{(j)}| \le 1$ satisfying (3.1), we find that

$$\text{(3.8)} \qquad \|\mathrm{Res}(F,Q)\|_\omega \le c(F,Q)|\omega|^{(d-1)\Delta_0}\max\{\|F\|_\omega, \|Q\|_\omega\},$$

where $\Delta_0 = \delta_F\delta_Q - \deg_{\mathbf{u}_1}\mathrm{Res}(F,Q)$. Now all but one term of (3.8) is non-zero and independent of $|\omega|$. So as long as $d > 1$, $\Delta_0 = 0$. This completes the proof of Proposition 4. □

We remark that by the Lefschetz principle, the second assertion of Proposition 4 holds over all fields of characteristic zero.

LEMMA 5. *Let $\mathfrak{P}$ be a homogeneous prime ideal in $\mathbf{C}[x_0,\dots,x_n]$ of rank r with $x_0 \in \mathfrak{P}$. Then for $d = n - r + 1$ and any non-zero $\omega = (\omega_0,\dots,\omega_n)$ in $\mathbf{C}^{n+1}$,*

$$\|\mathfrak{P}\|_\omega \ge \big(|\omega_0|/|\omega|\big)^{d\delta_{\mathfrak{P}}}.$$

Proof. Since $x_0 \in \mathfrak{P}$, then in the notation of (2.1), each $\alpha_0^\gamma = 0$. Thus u_{d0} does not actually occur in the Chow form F of $\mathfrak{P}$. By the symmetry of F in $\mathbf{u}_1,\dots,\mathbf{u}_d$, the variables $u_{10},\dots,u_{d-1,0}$ do not occur either. Thus

$$F(\mathbf{u}_1,\dots,\mathbf{u}_d) = F(u_{11},\dots,u_{1n};\dots;u_{d1},\dots,u_{dn}).$$

Now all terms of $\sigma_\omega F$ of degree $d\delta_{\mathfrak{P}}$ in the various $s_{j0}^{(i)}$ occur in

$$\omega_0^{d\delta_{\mathfrak{P}}}F\big(s_{10}^{(1)},\dots,s_{n0}^{(1)};\cdots;s_{10}^{(d)},\dots,s_{n0}^{(d)}\big).$$

But the $s_{j0}^{(i)}$, $j \ne 0$, are algebraically independent, and $F \not\equiv 0$. Thus for every non-zero coefficient a of F, $a\omega_0^{d\delta_{\mathfrak{P}}}$ actually occurs as a coefficient in $\sigma_\omega F$. Hence by definition, on choosing a with $|a| = H(F)$,

$$\|\mathfrak{P}\|_\omega \ge \frac{|a|\,|\omega_0|^{d\delta_{\mathfrak{P}}}}{|\omega|^{d\delta_{\mathfrak{P}}}H(F)} = \frac{|\omega_0|^{d\delta_{\mathfrak{P}}}}{|\omega|^{d\delta_{\mathfrak{P}}}},$$

as required. □

LEMMA 6. *Under the conditions of Proposition 4, denote by $R^*(F,Q)$ the Chow form obtained from* $\mathrm{Res}(F,Q)$ *by omitting all factors whose corresponding prime ideals $\mathfrak{P}$ contain x_0. Then*

i) $\deg_{\mathbf{u}_1}R^*(F,Q) \le \delta_F\delta_Q$, and

ii) *if in addition $\omega_0 \ne 0$,*

$$\|R^*(F,Q)\|_\omega \le c_6(F,Q)\big(|\omega|/|\omega_0|\big)^{(d-1)\Delta}\max\{\|F\|_\omega, \|Q\|_\omega\},$$

where $\Delta = \delta_F\delta_Q - \deg_{\mathbf{u}_1}R^*(F,Q)$ and $c_6(F,Q) > 0$ depends only on F and Q.

Proof. This follows from Proposition 4, Lemma 5 and Gelfond's inequality applied to $\sigma_\omega(R^*(F, Q))$. □

LEMMA 7. *Let* $\mathfrak{A} = (P)$ *with* P *homogeneous. Then* $\delta_{\mathfrak{A}} = \delta_P$ *and*

$$\|\mathfrak{A}\|_\omega \leq \|P\|_\omega (n+1)^{2n\delta_P}.$$

Proof. The proof of Proposition 1 of [Ne3] establishes this result, although it is stated there only for $P \in \mathbf{Z}[\mathbf{x}]$. □

IV. Proof of theorems

In this section we apply the results of the preceding two sections to obtain a lower bound on the maximum modulus of a regular sequence having no common zeros. Then estimates from several complex variables will be invoked to obtain an upper bound on the degrees of one solution.

PROPOSITION 8. *If* $Q_1, \dots, Q_m \in \mathbf{C}[x_1, \dots, x_n]$ *is a regular sequence with no common zeros in* $\mathbf{C}^n$ *and* $\deg Q_i = D_i > 0$, $i = 1, \dots, m$, *then there exists a constant* $C > 0$, *depending only on* $Q_1, \dots, Q_m$ *such that for all non-zero* $\omega \in \mathbf{C}^n$ *with* $|\omega| := \max\{|\omega_1|, \dots, |\omega_n|\} \geq 1$, *we have*

$$\max|Q_i(\omega)| \geq C|\omega|^{1-(n-1)D_1 \cdots D_\mu},$$

where $\mu = \min\{m, n\}$.

Proof. For each k in the range $1 \leq k \leq \mu$ we consider a certain assertion H_k. We show that for $k < \mu$, $H_k \Rightarrow H_{k+1}$ and that H_μ implies the claim of our proposition. For a polynomial $P \in \mathbf{C}[x_1, \dots, x_n]$, let ${}^hP$ denote the homogenization ${}^hP := x_0^{\delta_P} P(x_1/x_0, \dots, x_n/x_0)$ and similarly for ideals. For $1 \leq k \leq m$, let F_k^* be defined inductively by letting F_1^* be the Chow form of the principal ideal $({}^hQ_1)$ and F_k^* be a Chow form obtained from $\operatorname{Res}(F_{k-1}^*, {}^hQ_k)$ by deleting those factors arising from prime ideals containing x_0. For $\omega = (\omega_1, \dots, \omega_n) \in \mathbf{C}^n$, let $\omega' := (1, \omega_1, \dots, \omega_n) \in \mathbf{C}^{n+1}$. Our assertion H_k consists of three parts:

i) F_k^* is a product (actually involving all) of the Chow forms of the prime ideals of ${}^h(Q_1, \dots, Q_k)$,

ii) $\Delta_k := \deg_{\mathbf{u}_1} F_k^* \leq \Delta_{k-1} D_k$, and

iii) There is a constant c_k, depending only on $Q_1, \dots, Q_k$, such that

$$\log\|F_k^*\|_{\omega'} \leq \sum_{j=1}^{k} (n-j+1)(D_j\Delta_{j-1} - \Delta_j)\log|\omega'| + \log\max\{\|{}^hQ_1\|_{\omega'}, \dots, \|{}^hQ_k\|_{\omega'}\} + c_k,$$

where we set $\Delta_0 := 1$.

$k = 1$: Here F_1^* is the Chow form of the principal ideal ${}^hQ_1$, and H_1 holds by Lemma 7, with $\Delta_1 = D_1$.
$H_{k-1} \Rightarrow H_k$: Standard facts on the correspondence between inhomogeneous ideals and those homogeneous ideals whose prime components do not contain x_0 (e.g. Cor., p. 184 of [Za-Sa] with $\mathfrak{A} = ({}^hQ_1, \ldots, {}^hQ_{k-1})$) show that

$$\left({}^hQ_1, \ldots, {}^hQ_{k-1}\right) = {}^h\left(Q_1, \ldots, Q_{k-1}\right) \cap \mathfrak{H}_k,$$

where every prime component of $\mathfrak{H}_k$ contains x_0. By hypothesis Q_k does not lie in any associated prime ideal of $(Q_1, \ldots, Q_{k-1})$; so ${}^hQ_k$ does not lie in any associated prime ideal of ${}^h(Q_1, \ldots, Q_{k-1})$. Thus by the induction hypothesis, ${}^hQ_k$ is not contained in any associated prime ideal of F_{k-1}^*. Therefore we can apply Proposition 3, Proposition 4 and Lemma 6 to find that $H_{k-1} \Rightarrow H_k$, as desired. Consequently we have established H_μ by induction.

To conclude the proof we consider two cases:

a) $m = n + 1$. In this case, $\operatorname{Res}(F_n^*, {}^hQ_{n+1}) \in \mathbf{C}^*$ and we find directly from Proposition 4 and H_n that for some constant c_{n+1},

$$c_{n+1} \leq \sum_{j=1}^{n} (n - j + 1)(D_j\Delta_{j-1} - \Delta_j)\log|\omega'| + \log\max\left\{ \|{}^hQ_1\|_{\omega'}, \ldots, \|{}^hQ_{n+1}\|_{\omega'} \right\}.$$

b) $m \leq n$. In this case we see that $Q_1, \ldots, Q_m$ must be in the dehomogenization of any associated prime ideal of F_m^*. But since $Q_1, \ldots, Q_m$ have no common zeros in $\mathbf{C}^n$, there cannot be any such primes; i.e. F_m^* is a non-zero constant, and we conclude from H_m that there is a constant c_m' such that

$$c_m' \leq \sum_{j=1}^{m} (n - j + 1)\left(D_j\Delta_{j-1} - \Delta_j\right)\log|\omega'| + \log\max\left\{ \|{}^hQ_1\|_{\omega'}, \ldots, \|{}^hQ_m\|_{\omega'} \right\}.$$

Since $\Delta_1 = D_1$,

$$\sum_{j=1}^{\mu} (n - j + 1)\left(D_j\Delta_{j-1} - \Delta_j\right) \leq \sum_{j=2}^{\mu} (n - 1)\left(D_j\Delta_{j-1} - \Delta_j\right)D_{j+1} \cdots D_\mu$$
$$\leq (n - 1)D_1 \cdots D_\mu,$$

and, since $|\omega| \geq 1$,

$$\|{}^hQ_i\|_{\omega'} = \left|Q_i(\omega)\right| / \left\{H(Q_i)|\omega|^{\delta Q_i}\right\}.$$

Applying the last two displayed lines to the inequalities of a), b) completes the proof of Proposition 8. □

Remark. When the coefficients of the Q_i lie in $\mathbf{Z}$ or even in an algebraic number field, then the constants can be controlled and Proposition 8 holds in a rather sharp local form [Br] which leads to proofs, not involving infinite sequences of ideals, for algebraic independence results of [Ph1].

Proof of Theorem 2. The bridge from Proposition 8 to Theorem 2 comes from the theory of several complex variables. Our original proof followed the suggestion of Berenstein and Yger to invoke deconvolution formulas based on [Bern-An] (see below). For tighter control of the degree however, we appeal here to a result of Skoda [Sk] based on the work of L. Hörmander on the $\bar{\partial}$-equation and L^2-estimates. Let $K \geq 0$ be large enough so that the following integral converges:

$$\int_{\mathbf{C}^n} |Q|^{-2(1+\varepsilon)q-2} \|x\|^{-2K} \, d\lambda = I < \infty,$$

where $\varepsilon > 0$, $q = \min\{n, m-1\}$, $|Q|^2 = |Q_1|^2 + \cdots + |Q_m|^2$, and similarly $\|x\|^2 = 1 + |x_1|^2 + \cdots + |x_n|^2$ for $x = (x_1, \ldots, x_n) \in \mathbf{C}^n$. Then it follows as a special case of Theorem 1 of [Sk] that there are holomorphic $A_1, \ldots, A_m$ such that

$$1 = A_1 Q_1 + \cdots + A_m Q_m$$

and

$$\int_{\mathbf{C}^n} |A|^2 |Q|^{-2(1+\varepsilon)q} \|x\|^{-2K} \, d\lambda \leq \frac{1+\varepsilon}{\varepsilon} I < \infty.$$

Letting $-B$ denote the exponent occurring in Proposition 8, we see that convergence is guaranteed when

$$K > \max\{(q + q\varepsilon + 1)B + n, 0\}.$$

Arguing now as for example on page 143 of [Wa], we see that since each $|A_i|^2$ is plurisubharmonic, there are constants C_n, C_n' such that for any $\xi \in \mathbf{C}^n$ with $|\xi| = r$,

$$\begin{aligned} |A_i(\xi)|^2 &\leq C_n r^{-2n} \int_{B(\xi, r)} |A_i(x)|^2 \, d\lambda \\ &\leq C_n r^{-2n} \sup_{B(\xi, r)} \left\{ |Q|^{2(1+\varepsilon)q} \|x\|^{2K} \right\} \int_{B(\xi, r)} |A_i|^2 |Q|^{-2(1+\varepsilon)q} \|x\|^{-2K} \, d\lambda \\ &\leq C_n' r^{-2n + 2(1+\varepsilon)qD + 2K}, \end{aligned}$$

where $d\lambda$ denotes Lebesgue measure and C_n' depends on $Q_1, \ldots, Q_n$, ε, K. Thus each A_i is a polynomial of degree at most $(1+\varepsilon)qD + K - n$. Taking the infimum over K and ε, we find that the A_i are polynomials with

$$\begin{aligned} \text{(4.1)} \qquad \deg A_i &\leq \max\{qD - n, qD + (q+1)B\} \\ &< m(n-1) D_1 \cdots D_\mu + q(D-1). \end{aligned}$$

This establishes Theorem 2. □

Deduction of Theorem 1 *from Theorem* 2. Let $Q_1, \dots, Q_i$ be a regular sequence of polynomials from $V = \mathbf{Z}P_1 + \cdots + \mathbf{Z}P_m$. It is easily seen that the Q_j are $\mathbf{C}$-linearly independent $\mathbf{Z}$-linear combinations of $P_1, \dots, P_m$. Thus $i \le m$. Assume for the moment that $\mathfrak{A}_i = (Q_1, \dots, Q_i)$ has a zero in $\mathbf{C}^n$. Then we see by the usual induction argument, using the principal ideal theorem ([No], p. 217) and Macaulay's Theorem ([No], p. 272), that rank $\mathfrak{A}_i = i$ and that $\mathfrak{A}_i$ is unmixed. Since $P_1, \dots, P_m$ have no common zero, they cannot all lie in any proper ideal. Thus by, say, Lemma 1, page 438 of [Ma-Wü], there is a $Q_{i+1} \in V$ such that $Q_1, \dots, Q_{i+1}$ is a longer regular sequence.

Now let $Q_1, \dots, Q_i$ be a maximal regular sequence from V. Then $i \le m$ and $Q_1, \dots, Q_i$ have no common zero. We apply Theorem 2 to $Q_1, \dots, Q_i$ and express the Q_j in terms of the P_k to obtain the bounds of Theorem 1. $\square$

Final remarks

Skoda's theorem demonstrates the existence of the desired solution of (1.1). However A_i satisfying (1.1) are given as explicit integrals in a remarkable formula of [Be-St] (based on the work of [Bern-An]; see also Eqn. (22) of [Bern]). As pointed out by Berenstein and Yger, our Proposition 8 can be used to ensure the convergence of those integrals. For clarity, we recall formula (2.11) of [Be-St]. Let ζ denote the new complex variables $(\zeta_1, \dots, \zeta_n)$, and for $k = 1, \dots, m$, define the 1-forms

$$h_k(\zeta, x) := \sum_{j=1}^{n} h_{kj}(\zeta, x)\, d\zeta_j$$

in terms of the functions

$$h_{kj}(\zeta, x) := \int_0^1 \frac{\partial Q_k}{\partial \zeta_j}(\zeta_1 + t(x_1 - \zeta_1), \dots, \zeta_n + t(x_n - \zeta_n))\, dt.$$

Since in the setting of Theorem 2, $Q_1, \dots, Q_m$ have no common zeros in $\mathbf{C}^n$, the differential form

$$\eta := \left(\sum_{k=1}^{m} \overline{Q_k(\zeta)} h_k(\zeta, x) \right) / |Q(\zeta)|^2,$$

is defined on $\mathbf{C}^n$. If we let $Q(\zeta) \cdot Q(x) := \sum_k \overline{Q_k(\zeta)} Q_k(x)$ and $\bar{\zeta} \cdot x := \sum_j \bar{\zeta}_j x_j$, then the formula of [Be-St] becomes

$$1 = \int_{\mathbf{C}^n} \sum_{k=0}^{m-1} C_k \left(\frac{1 + \bar{\zeta} \cdot x}{\|\zeta\|^2} \right)^{N-n+k} \left(\frac{\overline{Q(\zeta)} \cdot Q(x)}{|Q(\zeta)|^2} \right)^{m-k} (\bar{\partial} \partial \log \|\zeta\|^2)^{n-k} \wedge (\bar{\partial} \eta)^k,$$

where the C_k are explicit constants. One verifies the convergence of the integral if

$$N = (2m-1)B + 2(m-1)D + n - m + 2,$$

where $-B$ is the exponent occurring in Proposition 8. One expands the products and collects coefficients of the Q_i in an arbitrary manner to obtain an explicit solution A_i to (1.1) with

$$\deg A_i \leq (2m-1)(n-1)D^{\mu} + 3(m-1)(D-1) \tag{4.2}$$

and coefficients which are represented as convergent integrals. In the setting of Theorem 1, one extracts a maximal regular sequence $Q_1, \dots, Q_i$ as before and applies (4.2) to obtain an explicit solution to (1.1).

For a further illustration of the power of these results from several complex variables, we require the following easy remark:

PROPOSITION 8′. *If $P_1, \dots, P_m \in \mathbf{C}[x_1, \dots, x_n] \setminus \mathbf{C}$ have neither common zeros in $\mathbf{C}^n$ nor at infinity, then there is a constant $C' > 0$ such that for all $\omega = (\omega_1, \dots, \omega_n) \in \mathbf{C}^n$,*

$$\max\{|P_i(\omega)|\} \geq C'|\omega|,$$

where $|\omega| := \max\{1, |\omega_1|, \dots, |\omega_n|\}$.

Proof. We are assuming that ${}^hP_1, \dots, {}^hP_m$ have no common zero in $\mathbf{P}^n$. Thus by the compactness of $\mathbf{P}^n$, there is a $C' > 0$ with

$$\max\left\{\frac{|{}^hP_i(\theta)|}{|\theta|^{\deg P_i}}\right\} \geq C'.$$

When $\theta := (1, \omega_1, \dots, \omega_n)$, we find that

$$\max\{|P_i(\omega)|\} \geq C'|\omega|^{\min \deg P_i},$$

which implies the result. □

Note that $m \geq n-1$, so that $q = n$. Thus for such P_i's we find that (4.1) applies with $B = -1$ to give polynomials A_i satisfying (1.2):

PROPOSITION 9. *If $P_1, \dots, P_m$ have neither common zeros in $\mathbf{C}^n$ nor at infinity, then we can solve* (1.1) *with*

$$\deg A_i \leq n(D-1).$$

The proof of Theorem 1 shows clearly that the super-linear growth of our bounds for $\deg A_i$ of a solution of (1.1) is controlled by the degrees of certain homogeneous primary ideals whose associated prime ideals contain $(x_0, {}^hP_1, \dots, {}^hP_m)$.

PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA

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