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Fixed-Parameter Complexity and Approximability of Norm Maximization

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Abstract The problem of maximizing the pth power of a p-norm over a halfspace-presented polytope in $\mathbb{R}^d$ is a convex maximization problem which plays a fundamental role in computational convexity. Mangasarian and Shiau showed in 1986 that this problem is $\mathbb{NP}$-hard for all values $p \in \mathbb{N}$ if the dimension d of the ambient space is part of the input. In this paper, we use the theory of parameterized complexity to analyze how heavily the hardness of norm maximization relies on the parameter d. More precisely, we show that for $p = 1$ the problem is fixed-parameter tractable (in FPT for short) but that for all $p > 1$ norm maximization is W[1]-hard. Concerning approximation algorithms for norm maximization, we show that, for fixed accuracy, there is a straightforward approximation algorithm for norm maximization in FPT running time, but there is no FPT-approximation algorithm with a running time depending polynomially on the accuracy. As with the $\mathbb{NP}$-hardness of norm maximization, the W[1]-hardness immediately carries over to various radius computation tasks in computational convexity.

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1 Introduction and Preliminaries

The problem of computing geometric functionals of polytopes arises in many applications in mathematical programming, operations research, statistics, physics, chemistry, or medicine (see e.g. [17] for an overview). Hence, the question how efficiently these functionals can be computed or approximated has been studied extensively, e.g. in [2,3,16,18,22].

Of particular interest is the problem of maximizing (the p-th power of) a p-norm over a polytope. Despite its simple formulation, this problem already exhibits the combinatorial properties which are responsible for hardness or tractability of the computation of many important geometric functionals. As for most computational problems on polytopes, the presentation of the input polytope is crucial for the computational complexity of norm maximization: If the input polytope is presented as the convex hull of finitely many points, norm maximization is solvable in polynomial time by the trivial algorithm of computing and comparing the norm of all these points. The situation changes dramatically when the input polytope is presented as the intersection of halfspaces. The present paper is concerned with the investigation of the parameterized complexity of this problem.

For $p \in \mathbb{N} \cup \{\infty\}$, a precise formulation of the norm maximization problem that we consider is as follows:

Problem 1.1 (NORMMAX$_p$)

Input: $d \in \mathbb{N}$, $\gamma \in \mathbb{Q}$, rational $\mathcal{H}$-presentation of a symmetric polytope $P \subseteq \mathbb{R}^d$

Parameter: d

Question: Is $\max\{\|x\|_p^p : x \in P\} \geq \gamma$?

Here, a rational $\mathcal{H}$-presentation of a polytope is a presentation as intersection of finitely many halfspaces which are defined by inequalities that have only rational coefficients.

Mangasarian and Shiau showed in [22] that, for $p = \infty$ (with the understanding that $\|x\|_\infty^\infty = \|x\|_\infty$), NORMMAX$_\infty$ is solvable in polynomial time via Linear Programming. For all $p \in \mathbb{N}$, on the other hand, NORMMAX$_p$ is $\mathbb{NP}$-complete. (When speaking of $\mathbb{NP}$-hardness of parameterized problems, we mean the same decision problem, simply ignoring the parameter.) Moreover, Bodlaender et al. [2] showed that $\mathbb{NP}$-hardness persists for all $p \in \mathbb{N}$ even when the instances are restricted to full-dimensional parallelotopes presented as a Minkowski sum of d linearly independent line segments. Additionally, Brieden showed in [3] that there is no polynomial time approximation algorithm for norm maximization for any constant performance ratio, unless $\mathbb{P} = \mathbb{NP}$.

It is important to note that, as usual in the realm of computational convexity, the dimension d is part of the input and the hardness of NORMMAX$_p$ relies heavily on this fact, especially for the very restricted instances in [2]. Indeed, if d is a constant, the obvious brute force algorithm of converting the presentation of P yields a polynomial time algorithm with running time $n^{O(d)}$, where n denotes the number of halfspaces in the presentation of P. However, this algorithm quickly becomes impractical as n grows, even for moderate values of d. The main purpose of this paper is to close the gap between $\mathbb{NP}$-hardness for unbounded dimension and a theoretically polynomial, yet impractical algorithm for fixed dimension.

A suitable tool that allows us to analyze how strongly the hardness of NORMMAX$_p$ depends on the parameter d is the Parameterized Complexity Theory. For an introduction to Fixed-Parameter Tractability, we refer to the textbooks [9,10,24]. This theory has already been applied successfully to show the intractability of several problems in computational geometry even in low dimensions, see e.g. [5,6,12–14,20].

Our analysis of NORMMAX$_p$ shows that, although NORMMAX$_p$ is $\mathbb{NP}$-hard for all $p \in \mathbb{N}$, the hardness has a different flavor for different types of norms: Whereas hardness of NORMMAX$_1$ only comes with the growth of the dimension, NORMMAX$_p$ has to be considered intractable already in small dimensions for all other values of p.

More precisely, we prove the following theorem:

Theorem 1.2 (Fixed-parameter complexity of NORMMAX) NORMMAX$_1$ *is in FPT, whereas* NORMMAX$_p$ *is* W[1]*-hard for all* $p \in \mathbb{N} \setminus \{1\}$.

The presented reduction also shows that in the hard cases no $n^{o(d)}$ algorithm for NORMMAX$_p$ exists, unless the *Exponential Time Hypothesis*[1] is false. Thus, the brute force algorithm for NORMMAX$_p$ mentioned above already has the best achievable complexity, if $p \in \mathbb{N} \setminus \{1\}$.

In this case, one can also ask how strongly the inapproximability result of [3] relies on the fact that NORMMAX$_p$ is a problem in unbounded dimension. For this purpose, call an algorithm that produces an $\bar{x} \in P$ such that, for some $\beta \in \mathbb{N}$,

$$\|\bar{x}\|_p^p \geq \left(\frac{\beta - 1}{\beta}\right)^p \max\{\|x\|_p^p : x \in P\},$$

a β-approximation algorithm for NORMMAX$_p$. The proof that NORMMAX$_1$ is in FPT then suggests the following: Replace the unit ball of the p-norm by a suitable symmetric polytope which approximates it sufficiently well and use the maximum of this polytopal norm as an approximation for the maximum of the p-norm. Then the polytopal norm can be maximized by solving a linear program for every facet of the unit ball and linear programs can be solved in $T_{LP}(d, n) := O(2^{2^d} n)$ (see [23]), which, for fixed d, is polynomial in n, the number of facets of the input polytope. This yields an FPT-time approximation algorithm for fixed accuracy β.

Theorem 1.3 (Approximation complexity of NORMMAX) *Let* $p \in \mathbb{N} \setminus \{1\}$. *For every fixed* $\beta \in \mathbb{N}$*, there is a* β*-approximation algorithm for* NORMMAX$_p$ *which runs in time* $O(\beta^d T_{LP}(d, n))$. *Conversely, there is no scheme of* β*-approximation algorithms for* NORMMAX$_p$ *with running time* $O(f(d)q(\beta, d, n))$ *with a polynomial* q *and an arbitrary computable function* f.

[1] The Exponential Time Hypothesis conjectures that n-variable 3-CNFSAT cannot be solved in $2^{o(n)}$-time; cf. [19,21].

Hence, although the problem is not in APX, approximation of NORMMAX$_p$ is possible for moderate values of β and d. However, approximation tends to become costly as soon as the dimension or the desired accuracy grows.

Finally, analogous to the $\mathbb{NP}$-hardness of NORMMAX$_p$, the W[1]-hardness of NORMMAX$_p$ implies the intractability of various problems in computational convexity as immediate corollaries. In Sect. 4, we show that for the respective values of p, the problems CIRCUMRADIUS$_p$-$\mathcal{H}$, DIAMETER$_p$-$\mathcal{H}$, INRADIUS$_p$-$\mathcal{V}$, and WIDTH$_p$-$\mathcal{V}$ (all parameterized by the dimension) are W[1]-hard.

This paper is organized as follows. In the remainder of this section, we explain our notation. In Sect. 2, we will analyze the parameterized complexity of NORMMAX$_p$, i.e., we prove Theorem 1.2 and prepare some technical lemmas, which we will also use in Sect. 3 where we prove Theorem 1.3. Finally, in Sect. 4, we prove the corollaries for the mentioned radius computation tasks.

Notation

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are used to denote the set of positive integers, integers, rational numbers, and real numbers, respectively.

For a positive integer $n \in \mathbb{N}$, we will abbreviate $[n] := \{1, \ldots, n\}$.

Throughout this paper, we work in d-dimensional real space $\mathbb{R}^d$ and for $A \subseteq \mathbb{R}^d$ we write $\text{lin}(A)$, $\text{aff}(A)$, $\text{conv}(A)$, $\text{pos}(A)$, $\text{int}(A)$, $\text{relint}(A)$, and $\text{bd}(A)$ for the linear, affine, convex and positive hull and the interior, relative interior and the boundary of A, respectively.

For a set $A \subseteq \mathbb{R}^d$, its dimension is $\dim(A) := \dim(\text{aff}(A))$. Furthermore, for any two sets $A, B \subset \mathbb{R}^d$ and $\rho \in \mathbb{R}$, let $\rho A := \{\rho a : a \in A\}$ and $A + B := \{a + b : a \in A, b \in B\}$ be the *ρ-dilatation* of A and the *Minkowski sum* of A and B, respectively. We abbreviate $A + (-B)$ by $A - B$ and $A + \{c\}$ by $A + c$. A set $K \subseteq \mathbb{R}^d$ is called *0-symmetric* if $-K = K$. If there is a $c \in \mathbb{R}^d$ such that $-(c + K) = c + K$, we call *K-symmetric*.

If a polytope $P \subseteq \mathbb{R}^d$ is described as a bounded intersection of halfspaces, we say that P is in $\mathcal{H}$-presentation. If P is given as the convex hull of finitely many points, we call this a $\mathcal{V}$-presentation of P. For a convex set $C \subseteq \mathbb{R}^d$, we let $\text{ext}(C)$ denote the set of *extreme points* of C.

For $1 \leq p < \infty$, the p-norm of a point $x = (x_1, \ldots, x_d)^{\mathrm{T}} \in \mathbb{R}^d$ is defined as

$$\|x\|_p := \Big(\sum_{i=1}^{d} |x_i|^p\Big)^{\frac{1}{p}}$$

for $p = \infty$, and we let $\|x\|_\infty := \max\{|x_i| : i \in [d]\}$.

For $p \in [1, \infty]$, we write $\mathbb{B}_p^d := \{x \in \mathbb{R}^d : \|x\|_p \leq 1\}$ for the unit ball of $\|\cdot\|_p$ and $\mathbb{S}_p^{d-1} := \{x \in \mathbb{R}^d : \|x\|_p = 1\}$ for the unit sphere in $\mathbb{R}^d$.

For two vectors $x, y \in \mathbb{R}^d$, we use the notation $x^{\mathrm{T}} y := \sum_{i=1}^{d} x_i y_i$ for the standard scalar/inner/dot product of x and y and by

$$H_{\leq}(a, \beta) := \{x \in \mathbb{R}^d : a^{\mathrm{T}} x \leq \beta\}$$

we denote the half-space induced by $a \in \mathbb{R}^d$ and $\beta \in \mathbb{R}$, bounded by the hyperplane $H_=(a, \beta) := \{x \in \mathbb{R}^d : a^T x = \beta\}$.

If X is a finite set and $k \in \mathbb{N}$, then $\binom{X}{k} := \{Y \subseteq X : |Y| = k\}$ denotes the set of all subsets of X of cardinality k.

The standard basis in $\mathbb{R}^d$ is denoted by $\{e_i : i \in [d]\}$ and the all-ones vector by $\mathbb{1} := (1, \ldots, 1)^T \in \mathbb{R}^d$.

We denote by $\mathbb{P}$ (and $\mathbb{NP}$, respectively) the classes of decision problems that are solvable (verifiable, respectively) in polynomial time. For an account on complexity theory, we refer to [1,11,25]. We write FPT for the class of fixed-parameter tractable problems and W[1] for the problems of the first level of the W-hierarchy in the theory of Fixed-Parameter Tractability. For a thorough introduction to Fixed-Parameter Tractability, we refer to the textbooks [9,10,24]. In addition, the next paragraph roughly sketches the main ideas of the theory.

1.1 Parameterized Complexity Theory

Parameterized complexity theory provides a framework for the study of algorithmic problems by measuring their complexity in terms of one or more parameters, explicitly or implicitly given by their underlying structure, in addition to the problem input size. The main idea is to devise exponential time algorithms for $\mathbb{NP}$-hard problems, confining exponentiality to the parameters (or to show that such algorithms do not exist).

A problem with input instance of size n and with a non-negative integer parameter k is fixed-parameter tractable if it can be solved by an algorithm that runs in $O(f(k)n^c)$ time, where f is a computable function depending only on k and c is a constant independent of k. Such an algorithm is called an FPT-algorithm and it is (informally) said to run in FPT-time. The class of all fixed-parameter tractable problems, denoted by FPT, is the parameterized complexity analog of $\mathbb{P}$. Megiddo's algorithm for linear programming [23] mentioned above is an example of an FPT-algorithm (with the dimension d being the parameter). Parameterized complexity theory provides a set of general algorithmic techniques for proving fixed-parameter tractability that have been successfully used for a variety of parameterized algorithmic problems in graph theory, logic, and computational biology. More importantly for our purposes, the theory also provides a framework for establishing fixed-parameter intractability.

To this end, an (infinite) hierarchy of complexity classes has been introduced, the W-hierarchy, with FPT being its lowest class. Its first level, W[1], can be thought of as the parameterized analog of $\mathbb{NP}$. Hardness is sought via FPT-reductions, i.e., an FPT-time many-one mapping from a problem Π, parameterized with k, to a problem Π_0, parameterized with k_0, such that $k_0 \leq g(k)$ for some computable function g. These reductions preserve fixed-parameter tractability between parameterized problems. Hardness for some level of the hierarchy can be thought as the parameterized complexity analog of $\mathbb{NP}$-hardness, and, as in classical complexity theory, intractability results are conditional and, thus, serve as relative lower bounds. The working assumption for parameterized complexity is that all levels of the W-hierarchy are pairwise distinct. For example, a problem that is W[1]-hard for some parameterization is not fixed-parameter tractable for this parameterization unless FPT = W[1]. Hence, for our purpose, the dimension d of the ambient space is a natural parameter for studying the parameterized complexity of NORMMAX. Proving NORMMAX to be W[1]-hard with respect to d gives thus strong evidence that an FPT-algorithm does not exist, under standard complexity-theoretic assumptions.

2 Fixed-Parameter Complexity of Norm Maximization

2.1 Intractability

We will first prove the hardness result for NORMMAX$_p$ for $p \geq 2$ via an FPT-reduction of the W[1]-complete problem CLIQUE to NORMMAX$_p$. The formal parameterized decision problem of CLIQUE is given in Problem 2.1; a proof of its W[1]-completeness can be found, e.g., in [10, Thm. 6.1].

Problem 2.1 (CLIQUE)

Input: $n, k \in \mathbb{N}$, $E \subseteq \binom{[n]}{2}$

Parameter: k

Question: Does $G = ([n], E)$ contain a clique of size k?

Moreover, Chen et al. showed in [7] that CLIQUE cannot be solved in time $n^{o(k)}$, unless the Exponential Time Hypothesis fails.

In order to show the hardness result, we will first show how to construct a polytope P for a graph $G = ([n], E)$ with the property that

$$\max\{\|x\|_p^p : x \in P\} = k \quad \Longleftrightarrow \quad G \text{ contains a clique of size } k.$$

This "reduction" will be laid out as if irrational numbers were computable with infinite precision. The second part of this section will then show that the numbers can be rounded to a sufficiently rough grid in order to make the reduction suitable for the Turing machine model.

2.2 The Construction

Let (n, k, E) be an instance of CLIQUE and $p \in [1, \infty)$. Throughout this paper, we assume without loss of generality that n is an even number. (If not, we add an isolated vertex to the graph.)

We choose $d := 2k$ and consider

$$\mathbb{R}^{2k} = \mathbb{R}^2 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^2,$$

i.e., we will think of a vector $x \in \mathbb{R}^{2k}$ as k two-dimensional vectors stacked upon each other. Therefore, it will be convenient to use the following notation:

Notation 2.2 By indexing a vector $x \in \mathbb{R}^{2k}$, we refer to the k two-dimensional vectors $x_1, \ldots, x_k \in \mathbb{R}^2$ such that $x = (x_1^{\mathrm{T}}, \ldots, x_k^{\mathrm{T}})^{\mathrm{T}}$. Further, for $a \in \mathbb{R}^2$ and $\beta \in \mathbb{R}$, we let

$$H_{\leq}^{i}(a, \beta) := \{x \in \mathbb{R}^{2k} : a^{\mathrm{T}} x_i \leq \beta\}.$$

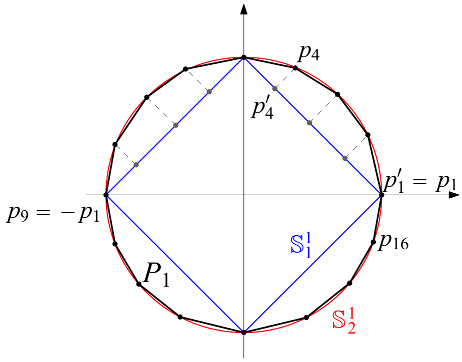

Fig. 1 Construction of P_1 in the case $p = 2, n = 8$

In order to construct an $\mathcal{H}$-presentation of a polytope $P \subseteq \mathbb{B}^2_p \times \mathbb{B}^2_p \times \cdots \times \mathbb{B}^2_p$, we will first construct a 2-dimensional polytope $P_1 \subseteq \mathbb{B}^2_p$ as our basic building block by placing vertices on the unit sphere $\mathbb{S}^1_p$ (compare Fig. 1):

For $v \in [\frac{n}{2}]$, let

$$p'_v := \binom{1}{0} + \frac{2(v-1)}{n}\binom{-1}{1} \quad \text{and} \quad \{p_v\} := \left(p'_v + [0,\infty)\binom{1}{1}\right) \cap \mathbb{S}^1_p; \tag{1}$$

for $v \in [n] \setminus [\frac{n}{2}]$, let

$$p'_v := \binom{0}{1} + \frac{2v-(n+2)}{n}\binom{-1}{-1} \quad \text{and} \quad \{p_v\} := \left(p'_v + [0,\infty)\binom{-1}{1}\right) \cap \mathbb{S}^1_p. \tag{2}$$

For $v \in [2n] \setminus [n]$, let

$$p_v := -p_{v-n}$$

and

$$P_1 := \operatorname{conv}\{p_1, \ldots, p_{2n}\} = \bigcap_{v \in [2n]} H_{\leq}(a_v, \beta_v) \subseteq \mathbb{R}^2. \tag{3}$$

Note that P_1 is 0-symmetric by construction and that the required $\mathcal{H}$-presentation of P_1 in (3) can be computed in time $O(n \log(n))$, see e.g. [8, Chap. 11]. For notational convenience, we also define

$$p_{2n+1} := p_1 \quad \text{and} \quad p_{-1} := p_{2n}.$$

Lemma 2.3 (Distance between the p_v) *Let $P_1 := \operatorname{conv}\{p_1, \ldots, p_{2n}\}$ be the polytope defined in (3) and $v \in [2n]$. The distance between two neighboring points on $\mathbb{S}^1_p$ satisfies*

$$\|p_v - p_{v+1}\|_2 \in \left[\frac{2\sqrt{2}}{n}, \frac{4}{n}\right].$$

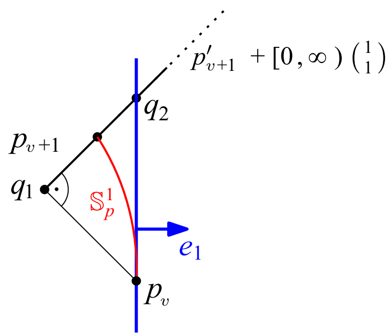

Fig. 2 The situation in the proof of Lemma 2.3

Proof Let $\Pi : \mathbb{R}^2 \to \mathbb{B}^2_1$ denote the projection onto $\mathbb{B}^2_1$. By the definitions in Eqs. (1) and (2), we have $\Pi(p_v) = p'_v$. Since Π is contracting, the equidistant placement of $p'_1, \ldots, p'_n$ yields $\|p_v - p_{v+1}\|_2 \geq \|p'_v - p'_{v+1}\|_2 = \frac{2\sqrt{2}}{n}$ for all $v \in [2n]$.

For the other bound, assume that $v \leq \frac{n}{4}$. (The other cases can be handled with the same arguments.) By elementary properties of $\mathbb{B}^2_p$, we have $e_1^{\mathrm{T}} p_{v+1} \leq e_1^{\mathrm{T}} p_v$ and $\mathbb{1}^{\mathrm{T}} p_{v+1} \geq \mathbb{1}^{\mathrm{T}} p_v$ and thus $p_{v+1} \in [q_1, q_2]$ with q_1, q_2 defined as in Fig. 2.

Inspection of the triangle $\operatorname{conv}\{p_v, q_1, q_2\}$ shows that it is equilateral with a right angle at q_1. Thus, $\|p_v - p_{v+1}\|_2 \leq \|p_v - q_2\|_2 = \sqrt{2\|p_v - q_1\|_2^2} = \frac{4}{n}$. □

Using Notation 2.2, we define a polytope $P_2 \subseteq \mathbb{R}^{2k}$ via

$$P_2 := \bigcap_{i\in[k]} \bigcap_{v\in[2n]} H^i_{\leq}(a_v, \beta_v) \subseteq \mathbb{R}^{2k}.$$

Observe that P_2 is 0-symmetric by construction and that any vertex x of P_2 is of the form $x = (p_{v_1}, \ldots, p_{v_k})^{\mathrm{T}}$ for suitable $v_1, \ldots, v_k \in [2n]$.

As for any $x = (x_1^{\mathrm{T}}, \ldots, x_k^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{2k}$, the identity

$$\|x\|_p^p = \sum_{i=1}^{k} \|x_i\|_p^p$$

holds, and as for $p \in \mathbb{N}\setminus\{1\}$, the unit sphere $\{x \in \mathbb{R}^2 : \|x\|_p^p = 1\}$ contains no straight line segments, and it follows that for $x \in P_2$,

$$\|x\|_p^p \geq k \iff x = \begin{pmatrix} p_{v_1} \\ \vdots \\ p_{v_k} \end{pmatrix} \text{ for some } v_1, \ldots, v_k \in [2n].$$

Hence, all points of maximal norm $\|x\|_p^p = k$ are the direct products of k vertices $p_{v_1}, \ldots, p_{v_k} \in \mathbb{R}^2$ of the 2-dimensional polytope P_1, which we will use to encode a clique that consists of the graph vertices $v_1, \dots, v_k$. In order to encode the requirement of an edge between two vertices, we proceed as follows:

For $v \in [2n]$, let $x_v, y_v \in \mathbb{R}$ be the coordinates of $p_v = (x_v, y_v)^{\mathrm{T}}$ and define

$$q_v := \begin{pmatrix} \operatorname{sgn}(x_v)|x_v|^{p-1} \\ \operatorname{sgn}(y_v)|y_v|^{p-1} \end{pmatrix}. \tag{4}$$

Noting that for all $x \in P_1$ and $v \in [2n]$, $q_v^{\mathrm{T}} x = 1$ if and only if $x = p_v$, we define

$$\varepsilon := 1 - \max\{q_u^{\mathrm{T}} p_v : u, v \in [2n], u \neq v\} > 0 \tag{5}$$

and for $u, v \in [n]$ and $i, j \in [k]$,

$$E_{uv}^{ij} := \{x \in \mathbb{R}^{2k} : \varepsilon - 2 \leq q_u^{\mathrm{T}} x_i + q_v^{\mathrm{T}} x_j \leq 2 - \varepsilon\}$$

and

$$F_{uv}^{ij} := \{x \in \mathbb{R}^{2k} : \varepsilon - 2 \leq q_u^{\mathrm{T}} x_i - q_v^{\mathrm{T}} x_j \leq 2 - \varepsilon\}.$$

Thus, if x is a vertex of P_2 with $x_i = \pm p_u$ and $x_j = \pm p_v$ for some $u, v \in [n]$, then $x \notin E_{uv}^{ij} \cap F_{uv}^{ij}$. Hence if $u, v \in [n]$ and $\{u, v\} \notin E$, the constraints of $E_{uv}^{ij} \cap F_{uv}^{ij}$ make sure that P does not contain a vertex with $x_i = \pm p_u$ and $x_j = \pm p_v$.

Finally, to encode the Clique instance, we let $N := \binom{[n]}{2} \setminus E$, define

$$P := P_2 \cap \bigcap_{\substack{\{u,v\} \in N \\ i,j \in [k], i \neq j}} (E_{uv}^{ij} \cap F_{uv}^{ij}) \cap \bigcap_{\substack{v \in [n] \\ i,j \in [k], i \neq j}} (E_{vv}^{ij} \cap F_{vv}^{ij}),$$

and obtain the following lemma:

Lemma 2.4 (Reduction with infinite precision) *Let (n, k, E) be an instance of* Clique*, $p \in [1, \infty)$ and $P \subseteq \mathbb{R}^{2k}$ the polytope obtained by the construction above. Then*

$$\max\{\|x\|_p^p : x \in P\} = k \iff G = ([n], E) \text{ contains a clique of size } k.$$

2.3 Analysis of the Constructed Polytope

We will now investigate how much we can perturb the (possibly irrational) polytope P in order to make it suitable for an FPT-reduction without losing its ability to decide between Yes and No instances of Clique. For this purpose, we define the constant

$$U := \frac{1}{n^{2p} k^2}. \tag{6}$$

In the following, we show that rounding the vertices $p_1, \dots, p_{2n}$ of our initial polytope $P_1 \subseteq \mathbb{R}^2$ to the grid $\frac{U}{2}\mathbb{Z}^2$ preserves all important features of our reduction. Since the parameter p is a constant in NORMMAX$_p$, all the necessary computations can be carried out with a precision of $O(\log(nk))$ bits. Since we only need a polynomial number of computations, the whole reduction can be carried out in polynomial time.

Lemma 2.5 *Let $P_1 = \operatorname{conv}\{p_1, \dots, p_{2n}\} \subseteq \mathbb{R}^2$ with $p_1, \dots, p_{2n} \in \mathbb{S}^1_p$ be the polytope from (3). For $\varepsilon := 1 - \max\{q_u^T p_v : u, v \in [2n], u \neq v\}$ with q_u defined as in (4), we have*

$$\varepsilon \geq \frac{2^{p-1}}{pn^p}.$$

Proof Let $x := (x_1, x_2)^T \in \mathbb{S}^1_p$ and $y := (y_1, y_2)^T \in \mathbb{S}^1_1$ with $x, y \geq 0$, $\|x - e_1\|_2 \geq \frac{2\sqrt{2}}{n}$, and $\|y - e_1\|_2 \geq \frac{2\sqrt{2}}{n}$. Since $\mathbb{B}^2_1 \subseteq \mathbb{B}^2_p$, $x_2 \geq y_2 \geq \frac{2}{n}$. Combining this inequality with $x \in \mathbb{S}^1_p$ yields

$$x_1 = (1 - x_2^p)^{\frac{1}{p}} \leq \left(1 - \left(\frac{2}{n}\right)^p\right)^{\frac{1}{p}} \leq 1 - \frac{2^p}{pn^p}, \tag{7}$$

where the last inequality follows by bounding the concave function $x \mapsto x^{\frac{1}{p}}$ from above by a linear approximation at $x = 1$.

Now, let $u, v \in [2n]$ with $u \neq v$. Then

$$q_u^T p_v = q_u^T p_u + q_u^T (p_v - p_u) = 1 + \cos(q_u, p_v - p_u)\, \|q_u\|_2\, \|p_v - p_u\|_2. \tag{8}$$

Since the points of lowest curvature on $\mathbb{S}^1_p$ are $\pm e_1$ and $\pm e_2$, and since $e_1 = p_1 = q_1$, we obtain $\cos(q_u, p_v - p_u) \leq \cos(e_1, p_2 - e_1)$, which in turn can be bounded by

$$\cos(e_1, p_2 - e_1) \leq \frac{x_1 - 1}{\|p_2 - e_1\|_2}$$

with $x_1 = e_1^T x$ for the point $x \in \mathbb{S}^p_1$ defined above. Further, $q_u \in \mathbb{S}^1_{\frac{p}{p-1}}$ implies $\|q_u\|_2 \geq \frac{\sqrt{2}}{2}$, and $\|p_v - p_u\|_2 \geq \frac{2\sqrt{2}}{n}$ by Lemma 2.3. Using (7), we can continue (8) to

$$q_u^T p_v \leq 1 - \frac{2^p}{pn^p \|p_2 - e_1\|} \cdot \frac{\sqrt{2}}{2} \cdot \frac{2\sqrt{2}}{n} \leq 1 - \frac{2^{p-1}}{pn^p},$$

where the last inequality follows again from Lemma 2.3. □

For $v \in [n]$, let $\bar{p}_v$ be the rounding of p_v to the grid $\frac{U}{2}\mathbb{Z}^2$ and define $\bar{p}_v = -\bar{p}_{v-n}$ for $v \in [2n] \setminus [n]$ and further

$$\bar{P}_1 := \operatorname{conv}\{\bar{p}_1, \dots, \bar{p}_{2n}\}. \tag{9}$$

For $\bar{p}_v = (\bar{x}_v, \bar{y}_v)^{\mathrm{T}} \in \mathbb{R}^2$, define

$$\bar{q}_v := \begin{pmatrix} \operatorname{sgn}(\bar{x}_v)\,|\bar{x}_v|^{p-1} \\ \operatorname{sgn}(\bar{y}_v)\,|\bar{y}_v|^{p-1} \end{pmatrix}.$$

By choice of our grid, we get

$$\|p_v - \bar{p}_v\|_{p'} \le U \quad \text{for all } p' \ge 1. \tag{10}$$

Moreover, if $q \in [1, \infty)$ is such that $\frac{1}{p} + \frac{1}{q} = 1$, then $\|q_v\|_q = 1$ for all $v \in [2n]$ and since $x \mapsto x^{p-1}$ is Lipschitz continuous on $[-1, 1]$ with Lipschitz constant $L = p-1$, we obtain

$$\|q_v - \bar{q}_v\|_1 \le (p-1)U. \tag{11}$$

First, we show that the points $\bar{p}_1, \ldots, \bar{p}_{2n}$ are still in convex position, which is binned into a separate lemma.

Lemma 2.6 *Let $\bar{P}_1 = \operatorname{conv}\{\bar{p}_1, \ldots, \bar{p}_{2n}\} \subseteq \mathbb{R}^2$ be the polytope from* (9). *Then $\operatorname{ext}(\bar{P}_1) = \{\bar{p}_1, \ldots, \bar{p}_{2n}\}$ and the coding length of an $\mathcal{H}$-presentation of $\bar{P}_1$ is polynomially bounded in the coding length of $\bar{p}_1, \ldots, \bar{p}_{2n}$.*

Proof For $v \in [2n]$, we have $q_v^{\mathrm{T}} \bar{p}_v \ge 1 - \|q_v\|_2 U \ge 1 - \|q_v\|_q U = 1 - U$, since $p \ge 2$ and therefore $q \le 2$. For $u \in [2n] \setminus \{v\}$, we get $q_v^{\mathrm{T}} \bar{p}_u \le 1 - \varepsilon + U$. Since $1 - \varepsilon + U < 1 - U$, the hyperplane $H_=(q_v, 1 - \varepsilon + U)$ separates $\bar{p}_v$ from $\operatorname{conv}(\{\bar{p}_1, \ldots, \bar{p}_{2n}\} \setminus \{\bar{p}_v\})$ and hence $\bar{p}_v \in \operatorname{ext}(\bar{P}_1)$.

Assume now that $\bar{P}_1 := \{x \in \mathbb{R}^2 : \bar{a}_v^{\mathrm{T}} x \le 1 \text{ for all } v \in [2n]\}$ is an $\mathcal{H}$-presentation of $\bar{P}_1$. Applying Cramer's Rule, we see that, for all $v \in [2n]$, the entries of $\bar{a}_v$ are quotients of polynomials in $\bar{p}_1, \ldots, \bar{p}_{2n}$ and so the coding length of the $\mathcal{H}$-presentation of $\bar{P}_1$ is bounded by a polynomial in the coding length of $\bar{p}_1, \ldots, \bar{p}_{2n}$. □

Since the coding length of $\bar{P}_1$ is polynomially bounded, we also get that the coding length of

$$\bar{P}_2 := \bigcap_{i \in [k]} \bigcap_{v \in [2n]} H^i_{\le}(\bar{a}_v, \bar{\beta}_v) \subseteq \mathbb{R}^{2k}$$

is polynomially bounded.

Now, let $\bar{\varepsilon} := 1 - \max\{\bar{q}_u^{\mathrm{T}} \bar{p}_v : u, v \in [2n], u \ne v\}$. By expanding the expression $\bar{q}_u^{\mathrm{T}} \bar{p}_v = \big(q_u + (\bar{q}_u - q_u)\big)^{\mathrm{T}} \big(p_v + (\bar{p}_v - p_v)\big)$ and using (10) and (11), we obtain

$$\bar{\varepsilon} \ge \varepsilon - 3pU > 0. \tag{12}$$

Finally, define

$$\bar{E}^{ij}_{uv} := \{x \in \mathbb{R}^{2k} : \bar{\varepsilon} - 2 \le \bar{p}_u^{\mathrm{T}} x_i + \bar{p}_v^{\mathrm{T}} x_j \le 2 - \bar{\varepsilon}\},$$

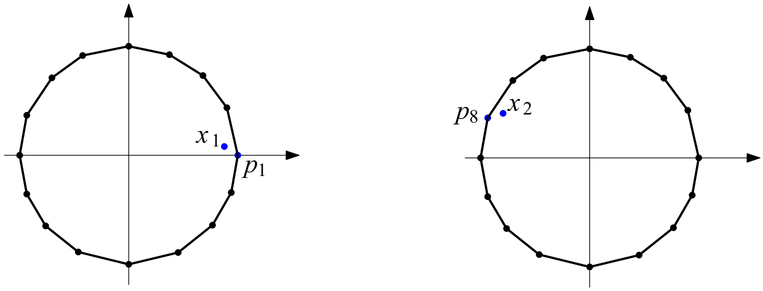

Fig. 3 Illustration of Notation 2.7. The figure shows a point $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^4$ with $m_1(x) = 1$ and $m_2(x) = 8$

and

$$\bar{F}_{uv}^{ij} := \{x \in \mathbb{R}^{2k} : \bar{\varepsilon} - 2 \leq \bar{p}_u^{\mathrm{T}} x_i -, \bar{p}_v^{\mathrm{T}} x_j \leq 2 - \bar{\varepsilon}\},$$

and for $N := \binom{[n]}{2} \setminus E$, let

$$\bar{P} := \bar{P}_2 \cap \bigcap_{\substack{\{u,v\} \in N \\ i,j \in [k], i \neq j}} (\bar{E}_{uv}^{ij} \cap \bar{F}_{uv}^{ij}) \cap \bigcap_{\substack{v \in [n] \\ i,j \in [k], i \neq j}} (\bar{E}_{vv}^{ij} \cap \bar{F}_{vv}^{ij}). \tag{13}$$

The following two lemmas will now prepare the proof that we can still reduce CLIQUE to norm maximization over $\bar{P}$. To be able to state them in a concise way, we introduce the following notation:

Notation 2.7 Let $\bar{P} \subseteq \mathbb{R}^{2k}$ be the polytope from (13) and $x = \left(x_1^{\mathrm{T}}, \ldots, x_k^{\mathrm{T}}\right)^{\mathrm{T}} \in \bar{P}$. By letting

$$m_i(x) \in \arg\max\{\bar{q}_v^{\mathrm{T}} x_i : v \in [2n]\},$$

we can refer to the index of a vertex which is "closest" to x in the sense that $\bar{q}_{m_i(x)}^{\mathrm{T}} x \geq \bar{q}_v^{\mathrm{T}} x$ for all $v \in [2n]$. This is illustrated in Fig. 3.

First, we show that if $\bar{P}$ contains a point which is "close" (in the sense specified in Notation 2.7) to a clique vertex, then $\bar{P}$ contains the clique vertex itself.

Lemma 2.8 *Let $\bar{P} \subseteq \mathbb{R}^{2k}$ be the polytope constructed above in (13). If there exists $\bar{x} \in \bar{P}$ such that $\bar{q}_{m_i(\bar{x})}^{\mathrm{T}} \bar{x} > 1 - \frac{\bar{\varepsilon}}{2}$ for all $i \in [k]$, then $(\bar{p}_{m_1(\bar{x})}^{\mathrm{T}}, \ldots, \bar{p}_{m_k(\bar{x})}^{\mathrm{T}})^{\mathrm{T}} \in \bar{P}$.*

Proof Since $\bar{q}_{m_i(\bar{x})}^{\mathrm{T}} \bar{x}_i > 1 - \frac{\bar{\varepsilon}}{2}$ for all $i \in [k]$, for no pair $(i, j) \in [k]^2$ the inequalities $\bar{q}_{m_i(\bar{x})}^{\mathrm{T}} x_i + \bar{q}_{m_j(\bar{x})}^{\mathrm{T}} x_j \leq 2 - \bar{\varepsilon}$ can be present in the description of $\bar{P}$. Since, by definition of $\bar{\varepsilon}$, we have $\bar{q}_v^{\mathrm{T}} \bar{p}_{m_i(\bar{x})} \leq 1 - \bar{\varepsilon}$ for all $v \in [2n] \setminus \{m_i(\bar{x})\}$ and $i \in [k]$, we can conclude that

$$\left(\bar{p}_{m_1(\bar{x})}^{\mathrm{T}}, \ldots, \bar{p}_{m_k(\bar{x})}^{\mathrm{T}}\right)^{\mathrm{T}} \in \bar{P}.$$

$\square$

In view of Lemma 2.8, it remains to show that the norm of a vertex which is "far" from a clique vertex is sufficiently small:

Lemma 2.9 *Let* $v \in [2n]$ *and* $Q := \operatorname{conv}\{0, \bar{p}_v, \bar{p}_{v+1}\} \cap H_{\leq}\left(\bar{q}_v, 1 - \frac{\bar{\varepsilon}}{2}\right) \cap H_{\leq}\left(\bar{q}_{v+1}, 1 - \frac{\bar{\varepsilon}}{2}\right)$. *Then, for n sufficiently large,*

$$\max\{\|x\|_p^p : x \in Q\} \leq 1 - \frac{2^{p-3}}{pn^p}.$$

Proof Let $Q' := \operatorname{conv}\{0, e_1, \bar{p}_2\} \cap H_{\leq}\left(e_1, 1 - \frac{\bar{\varepsilon}}{2}\right) \cap H_{\leq}\left(\bar{q}_2, 1 - \frac{\bar{\varepsilon}}{2}\right)$. Since e_1 is a point of lowest curvature on the boundary of $\mathbb{B}_p^2$, we have $\max\{\|x\|_p^p : x \in Q\} \leq \max\{\|x\|_p^p : x \in Q'\} = \|x^*\|_p^p$, where x^* fulfills $e_1^{\mathrm{T}} x^* = 1 - \frac{\bar{\varepsilon}}{2}$ and $x^* = \lambda e_1 + (1-\lambda)\bar{p}_2$ for some $\lambda \in [0, 1]$. From the first property, we can deduce $\lambda = 1 - \frac{\bar{\varepsilon}}{2}$, which implies $e_2^{\mathrm{T}} x^* = \frac{\bar{\varepsilon}}{2} e_2^{\mathrm{T}} \bar{p}_2$. By Lemma 2.3, $e_2^{\mathrm{T}} \bar{p}_2 \leq \frac{2}{n} + U$. Putting things together, we obtain

$$\|x^*\| \leq \left(1 - \frac{\bar{\varepsilon}}{2}\right)^p + \left(\frac{\bar{\varepsilon}}{2}\left(\frac{2}{n} + U\right)\right)^p \leq \left(1 - \frac{\bar{\varepsilon}}{2}\right) + \left(\frac{\bar{\varepsilon}}{2}\left(\frac{2}{n} + U\right)\right)^p. \tag{14}$$

By Lemma 2.5 and Inequality (12), $\bar{\varepsilon} \geq \frac{2^{p-1}}{pn^p} - 3pU$. By the choice of U and the assumption that n is sufficiently large, we can therefore continue (14) and obtain

$$\left(1 - \frac{\bar{\varepsilon}}{2}\right) + \left(\frac{\bar{\varepsilon}}{2}\left(\frac{2}{n} + U\right)\right)^p \leq 1 - \frac{2^{p-3}}{pn^p}.$$

□

2.4 Hardness Part of Theorem 1.2

The following lemma shows that it is sufficient to carry out the reduction described by Lemma 2.4 with finite precision as described in this section. It completes the proof of the hardness part of Theorem 1.2. For notational convenience, we use the clique number $\omega(G)$ to denote the size of the biggest clique in a graph $G = ([n], E)$.

Lemma 2.10 (Reduction with finite precision) *Let* (n, k, E) *be an instance of* CLIQUE, $G = ([n], E)$ *and* $\bar{P} \subseteq \mathbb{R}^{2k}$ *be the polytope with rounded coordinates constructed above in* (13). *Then*

$$\omega(G) \geq k \iff \max\{\|x\|_p^p : x \in \bar{P}\} \geq k(1-U)^p \tag{15}$$

and

$$\omega(G) < k \iff \max\{\|x\|_p^p : x \in \bar{P}\} \leq (k-1)(1+U)^p + 1 - \frac{2^{p-3}}{pn^p}. \tag{16}$$

Proof Since $(k-1)(1+U)^p+1-\frac{2^{p-3}}{pn^p} < k(1-U)^p$, if suffices to show the "forward" direction in both (15) and (16).

If $\omega(G) \geq k$ and $\{v_1, \ldots, v_k\} \subseteq [n]$ is the vertex set of a k-clique in G, then $\bar{P}$ contains the vertex $x^* = (\bar{p}_{v_1}^{\mathrm{T}}, \ldots, \bar{p}_{v_k}^{\mathrm{T}})^{\mathrm{T}}$ and $\|x^*\|_p^p \geq k(1-U)^p$ by (10).

Assume now that $\omega(G) < k$ and let $x^* \in \bar{P}$ be a vertex of maximal norm in $\bar{P}$. If $\bar{q}_{m_i(x^*)}^{\mathrm{T}} x^* > 1 - \frac{\bar{\varepsilon}}{2}$ for all $i \in [k]$, Lemma 2.8 would imply that $(\bar{p}_{m_1(x^*)}^{\mathrm{T}}, \ldots, \bar{p}_{m_k(x^*)}^{\mathrm{T}})^{\mathrm{T}}$ is a vertex of $\bar{P}$ and therefore contradict $\omega(G) < k$. Hence, there is some $i \in [k]$ such that $\bar{q}_{m_i(x^*)}^{\mathrm{T}} x^* \leq 1 - \frac{\bar{\varepsilon}}{2}$. By adding a constant number of vertices to G, we can assume that n is sufficiently large and apply Lemma 2.9 in order to obtain $\|x_i^*\|_p^p \leq 1 - \frac{2^{p-3}}{pn^p}$. As $\|x_j^*\|_p^p \leq (1+U)^p$ for all $j \in [k] \setminus \{i\}$, the right-hand side of (16) follows. □

The construction of the polytope P (or $\bar{P}$) relies on the fact that, for $p \geq 2$, the boundary of the unit ball of a p-norm contains no straight line segment. This is not the case for $p = 1$ and we show in the next section that NORMMAX$_1$ is indeed in FPT.

2.5 Tractability

This section completes the proof of Theorem 1.2 by showing that NORMMAX$_1$ is fixed-parameter tractable.

The statement of Theorem 2.12 is formulated for positive homogeneous functions of degree 1 and therefore slightly more general than needed for Theorem 1.2 but will be of use in Sect. 3. Here, a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is called positive homogeneous of degree 1, if for any $\lambda \geq 0$, $f(\lambda x) = \lambda f(x)$. Hence, the result for NORMMAX$_1$ can be obtained from Theorem 2.12 by choosing $\varphi_d : \mathbb{R}^d \to \mathbb{R}; x \mapsto \|x\|_1$ in Problem 2.11.

Problem 2.11 (MAX-Φ) Suppose that for each $d \in \mathbb{N}$, $\varphi_d : \mathbb{R}^d \to \mathbb{R}$ is positive homogeneous of degree 1 and let $\Phi := (\varphi_d)_{d \in \mathbb{N}}$. The problem MAX-$\Phi$ is defined as follows:

Input:	$d \in \mathbb{N}$, $\gamma \in \mathbb{Q}$, rational $\mathcal{H}$-presentation of a polytope $P \subseteq \mathbb{R}^d$
Parameter:	d
Question:	Is $\max\{\varphi_d(x) : x \in P\} \geq \gamma$?

Theorem 2.12 (Tractability of MAX-Φ) *For each $d \in \mathbb{N}$, let $\varphi_d : \mathbb{R}^d \to \mathbb{R}$ be positive homogeneous of degree 1 and $\Phi := (\varphi_d)_{d \in \mathbb{N}}$. Suppose that, for $d \in \mathbb{N}$, the set $\mathbb{B}^d := \{x \in \mathbb{R}^d : \varphi_d(x) \leq 1\}$ is a full-dimensional polytope, a rational $\mathcal{H}$-presentation of which can be computed in time $f(d)$ for a computable function $f : \mathbb{N} \to \mathbb{N}$. Then* MAX-$\Phi$ *is in FPT.*

Proof Let $\mathbb{B}^d = \bigcap_{i=1}^m H_{\leq}(a_i, 1)$ be an $\mathcal{H}$-presentation of $\mathbb{B}^d$. Then $m \in O(f(d))$. Because of the homogeneity of φ_d, $\{x \in \mathbb{R}^d : \varphi_d(x) \leq \lambda\} = \lambda \mathbb{B}^d$ and $\varphi_d(x) = \max_{i \in [m]} a_i^{\mathrm{T}} x$. Hence,

$$\max\{\varphi_d(x) : x \in P\} = \max_{i \in [m]} \max\{a_i^{\mathrm{T}} x : x \in P\}.$$

Thus, MAX-Φ can be decided by the following algorithm:

(1) Compute an $\mathcal{H}$-presentation of $\mathbb{B}^d$ in time $f(d)$.
(2) Solve m linear programs $\max\{a_i^{\mathrm{T}} x : x \in P\}$ in time $T_{LP}(d, n)$.
(3) Compare the biggest objective value to γ.

As $T_{LP}(d, n) = O(2^{2^d} n)$ by [23], the above algorithm has the claimed FPT running time. □

We can also establish fixed-parameter tractability for the two problems [0, 1]-$\textsc{ParMax}_p$ and [−1, 1]-$\textsc{ParMax}_p$ as considered by Bodlaender et al. in [2].

Problem 2.13 ([0, 1]-$\textsc{ParMax}_p$)

Input:	$d \in \mathbb{N}, \gamma \in \mathbb{Q}, v_1, \ldots, v_n \in \mathbb{Q}^d$ linearly independent
Parameter:	d
Question:	Is $\max\{\|x\|_p^p : x \in \sum_{i=1}^d [0, 1] v_i\} \geq \gamma$?

Problem 2.14 ([−1, 1]-$\textsc{ParMax}_p$)

Input:	$d \in \mathbb{N}, \gamma \in \mathbb{Q}, v_1, \ldots, v_n \in \mathbb{Q}^d$ linearly independent
Parameter:	d
Question:	Is $\max\{\|x\|_p^p : x \in \sum_{i=1}^d [-1, 1] v_i\} \geq \gamma$?

Bodlaender et al. showed in [2] that Problems 2.13 and 2.14 are both $\mathbb{NP}$-hard, so that the $\mathbb{NP}$-hardness of $\textsc{NormMax}_p$ persists even on very restricted instances. However, the following theorem shows that these problems are fixed-parameter tractable, when parameterized by the dimension. So in this case, the hardness of $\textsc{ParMax}_p$ is really a phenomenon of high dimensions.

Theorem 2.15 (Tractability of $\textsc{ParMax}_p$) *For all $p \in \mathbb{N}$, Problems 2.13 and 2.14 are in FPT.*

Proof We only consider Problem 2.13; the argument for Problem 2.14 is exactly the same. The vertices of the polytope $P := \sum_{i=1}^d [0, 1] v_i$ are all of the form $\sum_{i=1}^d \lambda_i v_i$ for some vector $\lambda = (\lambda_1, \ldots, \lambda_d)^{\mathrm{T}} \in \{0, 1\}^d$. As the maximum of $\|\cdot\|_p^p$ is attained at a vertex of P, it suffices to compute the norm of all 2^d possible choices of $\lambda \in \{0, 1\}^d$. This clearly is an FPT-algorithm for Problem 2.13. □

3 Approximation

3.1 FPT-Approximation for Fixed Accuracy

In [3], Brieden showed that, for all $p \in \mathbb{N}$, $\textsc{NormMax}_p$ is not contained in APX (i.e., there is no polynomial time approximation algorithm with a constant approximation ratio). As norm maximization with a polytopal unit ball is in FPT, we can give a straightforward approximation algorithm that has FPT running time for any fixed accuracy by replacing the unit ball $\mathbb{B}_p^d$ by an approximating polytope. The following proposition concerning the complexity of such a polytope can be obtained from [4, Lemmas 3.7 and 3.8].

Proposition 3.1 (Approximation of balls by polytopes) *Let $p \in \mathbb{N}$ and $\beta \in \mathbb{N}$ be fixed. There is a symmetric polytope $B \subseteq \mathbb{R}^d$ with a rational $\mathcal{H}$-presentation and at most $O(\beta^d)$ facets such that*

$$\mathbb{B}_p^d \subseteq B \subseteq \frac{\beta}{\beta - 1}\mathbb{B}_p^d, \tag{17}$$

and B can be computed in time $O(\beta^d)$.

Lemma 3.2 (FPT-Approximation algorithm for fixed accuracy) *Let $p \in \mathbb{N}$ and $\beta \in \mathbb{N}$ be fixed. There is an algorithm which for every $\mathcal{H}$-presented polytope $P \subseteq \mathbb{R}^d$ runs in time $O(\beta^d T_{LP}(d, n))$ and produces an $\bar{x} \in P$ such that*

$$\|\bar{x}\|_p^p \geq \left(\frac{\beta - 1}{\beta}\right)^p \max\{\|x\|_p^p : x \in P\}.$$

Proof The following algorithm has the desired properties:

(1) Compute an $\mathcal{H}$-presentation of a symmetric polytope $B \subseteq \mathbb{R}^d$ with the properties of Proposition 3.1 and let $\|\cdot\|_B : \mathbb{R}^d \to \mathbb{R}; x \mapsto \|x\|_B := \min\{\lambda \geq 0 : x \in \lambda B\}$
(2) Choose $\bar{x} \in \arg\max\{\|x\|_B : x \in P\}$.

It follows from Proposition 3.1 that step (1) can be accomplished in time $O(\beta^d)$. As the number of facets of B is in $O(\beta^d)$, it follows from Theorem 2.12 that the maximization of $\|\cdot\|_B$ over P can be done in time $O(\beta^d T_{LP}(d, n))$.

In order to show the performance ratio of the above algorithm, observe that Property (17) of B implies that $\frac{\beta-1}{\beta}\|x\|_p \leq \|x\|_B \leq \|x\|_p$ for all $x \in \mathbb{R}^d$. Hence, if $x^* \in \arg\max\{\|x\|_p^p : x \in P\}$, we get

$$\|\bar{x}\|_p^p \geq \|\bar{x}\|_B^p \geq \|x^*\|_B^p \geq \left(\frac{\beta - 1}{\beta}\right)^p \|x^*\|_p^p = \left(\frac{\beta - 1}{\beta}\right)^p \max\{\|x\|_p^p : x \in P\}.$$

□

3.2 No FPT-Approximation for Variable Accuracy

Finally, we will show that the straightforward approximation of the previous section is already best possible in the sense that there is no algorithm with polynomial dependence on the approximation quality and exponential dependence only on the dimension. Hence, combined with Lemma 3.2, Lemma 3.3 completes the proof of Theorem 1.3. In fact, the basis for this has already been established in Lemma 2.10 and we can give the result right away.

Lemma 3.3 (No polynomial dependence on β) *Let $f : \mathbb{N} \to \mathbb{R}$ be a computable function and $q : \mathbb{R}^3 \to \mathbb{R}$ a polynomial function. If* W[1]$\neq$*FPT, there is no algorithm which for every $\mathcal{H}$-presented polytope $P \subseteq \mathbb{R}^d$ runs in time $O(f(d)q(\beta, d, n))$ and produces an $\bar{x} \in P$ such that*

$$\|\bar{x}\|_p^p \geq \left(\frac{\beta-1}{\beta}\right)^p \max\{\|x\|_p^p : x \in P\}.$$

Proof Let (n, k, E) be an instance of the W[1]-hard problem CLIQUE and $\bar{P} \subseteq \mathbb{R}^{2k}$ the polytope constructed in (13). By Lemma 2.10, it can be decided if $G = ([n], E)$ has a clique of size k by determining whether

$$\begin{aligned} \textbf{either } &\max\{\|x\|_p^p : x \in \bar{P}\} \geq k(1-U)^p \\ \textbf{or } &\max\{\|x\|_p^p : x \in \bar{P}\} \leq (k-1)(1+U)^p + 1 - \tfrac{2^{p-3}}{pn^p} \end{aligned} \tag{18}$$

Assume that an algorithm with the claimed properties exists and call it $\mathcal{A}$. One easily checks that there is a suitable constant $C > 0$ such that it suffices to choose $\beta \geq \frac{pn^p k}{C}$ in order to fulfill

$$\left(\frac{\beta}{\beta-1}\right)^p \left((k-1)(1+U)^p + 1 - \frac{2^{p-3}}{pn^p}\right) < k(1-U)^p.$$

Hence, we can run the following algorithm $\mathcal{A}'$ in order to decide (18):

(1) Choose $\beta := \left\lceil \frac{pn^p k}{C} \right\rceil$.
(2) Run $\mathcal{A}$ on the polytope $\bar{P}$ and obtain an approximate norm maximal vertex $\bar{x} \in \bar{P}$.
(3) If $\|\bar{x}\|_p^p > (k-1)(1+U)^p + 1 - \frac{2^{p-3}}{pn^p}$, decide $\max\{\|x\|_p^p : x \in \bar{P}\} \geq k(1-U)^p$.
Else, decide $\max\{\|x\|_p^p : x \in P\} \leq (k-1)(1+U)^p + 1 - \frac{2^{p-3}}{pn^p}$.

By the properties of $\mathcal{A}$, the running time of the algorithm $\mathcal{A}'$ is $O(f(d)q(n^p k, d, n))$, and by Lemma 2.10 and the choice of β, $\mathcal{A}'$ decides (18) correctly. $\mathcal{A}'$ is thus an FPT-algorithm for CLIQUE. Unless FPT = W[1], this is a contradiction to the fact that CLIQUE is W[1]-hard. □

4 Some Implications

As stated in the introduction, norm maximization over polytopes plays a fundamental role in computational convexity. This section gives corollaries concerning the hardness of determining four important geometric functionals on polytopes.

If $P \subseteq \mathbb{R}^d$ is a polytope, we denote by $R(P, \mathbb{B}_p^d)$ ($r(P, \mathbb{B}_p^d)$, respectively) the circumradius (inradius) of P with respect to the p-norm. Further, we write $R_1(P, \mathbb{B}_p^d)$ ($r_1(P, \mathbb{B}_p^d)$) for half of the width (diameter) of P, i.e., half the radius of a smallest slab containing P (half the length of the longest line segment contained in P).

For $p \in \mathbb{N} \cup \{\infty\}$, we consider the following problems:

Problem 4.1 (CIRCUMDADIUS$_p$-$\mathcal{H}$)

Input: $d \in \mathbb{N}$, $\gamma \in \mathbb{Q}$, rational $\mathcal{H}$-presentation of a 0-symmetric polytope $P \subseteq \mathbb{R}^d$
Parameter: d
Question: Is $R(P, \mathbb{B}_p^d)^p \geq \gamma$?

Problem 4.2 (DIAMETER$_p$-$\mathcal{H}$)

Input: $d \in \mathbb{N}$, $\gamma \in \mathbb{Q}$, rational $\mathcal{H}$-presentation of a 0-symmetric polytope $P \subseteq \mathbb{R}^d$
Parameter: d
Question: Is $r_1(P, \mathbb{B}_p^d)^p \geq \gamma$?

In [16], Gritzmann and Klee showed that Problems 4.1 and 4.2 are solvable in polynomial time if $p = \infty$ and, using an identity for symmetric polytopes from their companion paper [15], that both problems are $\mathbb{NP}$-hard when $p \in \mathbb{N}$. Using the same identity, we can establish (in-)tractability for both problems when parameterized by the dimension:

Corollary 4.3 (Circumradius & Diameter) *For* $p = 1$, *Problems* 4.1 *and* 4.2 *are in FPT. For* $p \in \mathbb{N} \setminus \{1\}$, *both problems are* W[1]*-hard.*

Proof As shown in [15, (1.3)], for a 0-symmetric polytope $P \subseteq \mathbb{R}^d$, we have

$$R(P, \mathbb{B}_p^d)^p = r_1(P, \mathbb{B}_p^d)^p = \max\{\|x\|_p^p : x \in P\}.$$

Thus, tractability or hardness of Problems 4.1 and 4.2 follows from Theorem 1.2. □

Additionally, let $q \in [1, \infty]$ be such that $1/p + 1/q = 1$ (with $1/\infty = 0$).

Problem 4.4 (INRADIUS$_p$-$\mathcal{V}$)

Input: $d \in \mathbb{N}$, $\gamma \in \mathbb{Q}$, rational $\mathcal{V}$-presentation of a 0-symmetric polytope $P \subseteq \mathbb{R}^d$
Parameter: d
Question: Is $r(P, \mathbb{B}_q^d)^p \leq \gamma$?

Problem 4.5 (WIDTH$_p$-$\mathcal{V}$)

Input: $d \in \mathbb{N}$, $\gamma \in \mathbb{Q}$, rational $\mathcal{V}$-presentation of a 0-symmetric polytope $P \subseteq \mathbb{R}^d$
Parameter: d
Question: Is $R_1(P, \mathbb{B}_q^d)^p \leq \gamma$?

As for the previous two problems, the question of $\mathbb{NP}$-hardness of INRADIUS$_p$-$\mathcal{V}$ and WIDTH$_p$-$\mathcal{V}$ has been studied in [16]. It is shown that Problems 4.4 and 4.5 are solvable in polynomial time if $p = 1$ and using an identity for symmetric polytopes from [15] that both problems are $\mathbb{NP}$-hard when $p \in \mathbb{N}$. Here again, we can use the same identity to establish (in-)tractability for both problems when parameterized by the dimension:

Corollary 4.6 (Inradius & Width) *For* $p = 1$, *Problems* 4.4 *and* 4.5 *are in FPT. For* $p \in \mathbb{N} \setminus \{1\}$, *both problems are* W[1]*-hard.*

Proof It is shown in [15] that if $P \subseteq \mathbb{R}^d$ is a 0-symmetric polytope and P° is its polar, the identities

$$R_j(P, \mathbb{B}_q^d) r_j(P^\circ, \mathbb{B}_p^d) = 1$$

hold for all $j \in [d]$. As an $\mathcal{H}$-presentation of P° is readily translated into a $\mathcal{V}$-presentation of P, tractability or hardness of Problems 4.4 and 4.5 follows from Corollary 4.3. □

The reductions of Corollaries 4.3 and 4.6 also show that the algorithm in the proof of Lemma 3.2 can be used to compute the respective radii of a symmetric polytope $P \subseteq \mathbb{R}^d$ in the respective presentation. Lemma 3.3, in turn, shows that in these cases the given running time is also best possible.

5 Conclusion

In this paper, we investigated the parameterized complexity of finding the maximum of a p-norm over an $\mathcal{H}$-presented polytope in unbounded dimension. The $\mathbb{NP}$-hardness of this problem being known for all $p \in \mathbb{N}$ and choosing the dimension d as parameter, the central question of the paper is thus to find out to which extent this hardness is a phenomenon of high dimensions only. We showed that, in the case $p = 1$, norm maximization is fixed-parameter tractable. The hardness of NORMMAX$_1$ can thus indeed be confined to the dimension. In the other cases, the W[1]-hardness of NORMMAX$_p$ shows that, loosely speaking, the dimension is not the only one to blame and that the problem has to be considered intractable already in low dimensions. The same is true for the classic radii computation tasks investigated in the previous section.

Hence, for this basic class of problems, which—being geometric problems in unbounded dimension—are all $\mathbb{NP}$-hard, there is a qualitative cut through the class which demonstrates a possible way of tackling some of these problems. At this point, it is thus only natural to ask how good (or best possible) FPT running times can be achieved or if successful kernelization techniques can shed new light on geometric problems. In the cases where it is not only a problem of high dimensions, also the search for other parameters (the circumradius of the input polytope or the actual value of the norm maximum just being two natural candidates) might lead to a better understanding of what makes these problems actually hard. In this sense, we hope that this paper is just a first step towards a deeper investigation of the complexity of geometric problems in unbounded dimension that eventually allows to identify many tractable instances in practice.

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