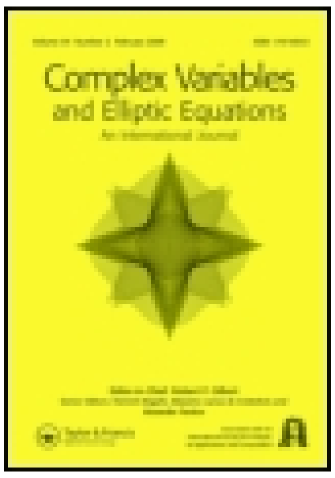

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Differentially transcendental formal power series

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Differentially Transcendental Formal Power Series

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We prove that a formal power series in $1/x$, whose coefficients are in a field extension of $\mathbf{Q}$ and are algebraically independent over $\mathbf{Q}$, is differentially transcendental (i.e. not differentially algebraic) over this field extension. This is stated without proof in [2]. This result provides a source of functions analytic at ∞ that are not differentially algebraic over $\mathbf{R}$. Such functions are of particular interest, because their germs belong to Hardy fields, but not to the class E of [1]—the intersection of all maximal Hardy fields.

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Suppose F is a field extension of the field of rational numbers $\mathbf{Q}$. Let x be an indeterminate and let $u = 1/x$. Let $F[[u]] = F[[1/x]]$ denote the ring of formal power series in u with coefficients in F. Then its field of quotients $F((u))$ is a differential field with at least two possible derivations: formal differentiation with respect to x and u denoted by D_x and D_u.

DEFINITION 1 (see e.g. §VI.1 [7]) *Suppose F is an extension field of K and $S \subseteq F$.*

(i) *S is algebraically dependent over K if a nonzero polynomial in finitely many variables with coefficients in K is annulled by elements of S.*
(ii) *S is a transcendence basis over K means that S is algebraically independent (i.e. not algebraically dependent) over K and is maximal with respect to this property (i.e. F is algebraic over $K(S)$).*
(iii) *A transcendence basis has unique cardinality (Theorem* VI.1.9 [7]) *which is called the transcendence degree of F over K and is denoted by* $\mathrm{tr.deg}_K F$.

DEFINITION 2 (see e.g. §I.6 [8]) *Suppose F is a field, K is a differential field, $F \subseteq K$, and $f \in K$. To say that f is differentially algebraic over F means that $\{f, f', f'', \ldots\}$ is algebraically dependent over K, i.e. f is a root of a nonzero differential polynomial with coefficients in F.*

PROPOSITION 1 (Proposition 7.4, [2]) *If the set $\{a_i \in F,\ i = 0, 1, \ldots\}$ is algebraically independent over $\mathbf{Q}$ then*

$$f = \sum_{i=0}^{\infty} a_i x^{-i} \in F\left(\left(\frac{1}{x}\right)\right)$$

is not differentially algebraic over F with respect to D_x.

Proof Suppose f is a root of a non-zero differential polynomial p over F with respect to D_x. Let $b_j \in F$ $(j = 0,1,\ldots \bar{j})$ be the coefficients of p. Then f is differentially algebraic with respect to D_x over K, where $K = Q(\{b_j\})$, so

$$\operatorname{tr.deg}_K K(f, D_x f, D_x^2 f, \ldots) < \infty.$$

Since $\operatorname{tr.deg}_Q K < \infty$, we have

$$\operatorname{tr.deg}_Q K(f, D_x f, D_x^2 f, \ldots) = \operatorname{tr.deg}_Q K + \operatorname{tr.deg}_K K(f, D_x f, D_x^2 f, \ldots) < \infty.$$

Therefore

$$\operatorname{tr.deg}_Q \mathbf{Q}(f, D_x f, D_x^2 f, \ldots) = \operatorname{tr.deg}_Q K(f, D_x f, D_x^2 f, \ldots)$$

$$- \operatorname{tr.deg}_{Q(f, D_x f, D_x^2 f, \ldots)} K(f, D - xf, D_x^2 f, \ldots) < \infty.$$

Note that

$$D_x f = D_u f\left(-\frac{1}{x^2}\right) = D_u F(-u^2),$$

$$D_x^2 f = (D_u^2 f(-u^2) + D_u f(-2u))(-u^2)$$

and so on, so $\mathbf{Q}(x, f, D_x f, D_x^2 f, \ldots) = \mathbf{Q}(u, f, D_u f, D_u^2 f, \ldots)$.

Now since $\operatorname{tr.deg}_Q \mathbf{Q}(u) = 1$ and $\operatorname{tr.deg}_Q \mathbf{Q}(f, D_x f, D_x^2 f, \ldots) < \infty$, we have

$$\operatorname{tr.deg}_Q \mathbf{Q}(u, f, D_u f, D_u^2 f, \ldots) < \infty \qquad \text{and} \qquad \operatorname{tr.deg}_Q \mathbf{Q}(f, D_u f, D_u^2 f, \ldots) < \infty.$$

Therefore, f satisfies some non-zero differential polynomial q over $\mathbf{Q}$ with respect to D_u. Since the field $\mathbf{Q}$ is infinite, there exist $t_k \in \mathbf{Q}$ $(k = 0,1,\ldots k)$ such that $q(t_0, t_1, \ldots t_{\bar{k}}) \neq 0$. Let

$$g(u) = \sum_{k=0}^{\infty} \frac{t_k}{k!} u^k.$$

Then $q(g, D_u g, \ldots) \neq 0$, because it is non-zero when evaluated at $u = 0$ (it equals $q(\{t_k\})$).

If h is a formal sum $\sum_{i=0}^{\infty} y_i u^i$ with y_i indeterminate, then $q(h, D_u h, \ldots)$ is a power series in u, whose coefficients are polynomials in y_i over $\mathbf{Z}$. Since evaluating y_i at $(t_0, t_1, \ldots t_{\bar{k}}, 0, 0 \cdots)$ produces a non-zero answer, one of the above coefficients is a non-zero polynomial in y_i over $\mathbf{Q}$. Let r denote this polynomial. Evaluating y_i at a_i gives $q(f, D_u f, \ldots) = 0$, so, in particular, $r(a_i) = 0$, which contradicts the assumption that a_i are algebraically independent over $\mathbf{Q}$. ∎

The value of this result lies in providing a class of functions analytic at ∞ that are differentially transcendental over $\mathbf{R}$. The germs of such functions necessarily belong to Hardy fields (see the definition below), but not to Boshernitzan's class $E[1]$—the intersection of all maximal Hardy fields.[1]

[1] E is an extension of Hardy's class L of logarithmico-exponential functions [5] and is the maximal scale for functions in Hardy fields (functions of *regular growth*).

DEFINITION 3 (see [4]) *A differential field of continuous germs*[2] *of real functions at* $+\infty$, *where the derivation is ordinary differentiation, is called a Hardy field.*

COROLLARY 1 *Suppose* f *is a real function that is analytic at* ∞ *and the set of coefficients of its Taylor series at* ∞ *is algebraically independent over* $\mathbf{Q}$. *Then*

(i) f *is differentially transcendental over* $\mathbf{R}$,
(ii) $\mathbf{R}(f, f' \cdots)$ *is a Hardy field,*
(iii) $f \notin E$, *where* E *is the intersection of all maximal Hardy fields.*

Proof (i) is a special case of Proposition 1 with $F = \mathbf{R}$ and the series actually convergent. (ii) is a consequence of the fact that the zeros of an analytic function are necessarily isolated and, thus, cannot have ∞ as an accumulation point. This means that every nonzero element of $\mathbf{R}[f, f' \cdots]$, i.e. a nonzero differential polynomial of f, is invertible in a (punctured) neighborhood of ∞ (see Theorem 7.1[1]). (iii) is a consequence of the fact that E is a differentially algebraic extension of $\mathbf{R}$ (Theorem 14.4 [2]). ■

The few other known examples of classes of functions satisfying the conditions of Corollary 1 include

(i) Euler's Γ-function, which is not differentially algebraic over $\mathbf{R}$ by Hölder's theorem [6] and generates a Hardy field [9];
(ii) Functions represented by a Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^x},$$

where $a_n \in \mathbf{R}$ are subpolynomial in n and the set of all prime divisors of n in the support of a_n[3] is infinite, e.g. the Riemann ζ-function on a positive half-line [9];
(iii) The function

$$\sum_{n=1}^{\infty} \frac{1}{e_n(x)},$$

where $e_1(x) = e^x$ and $e_n(x) = e^{e_{n-1}(x)}$ for $n > 1$ [9];
(iv) Certain fractional iterates of e^x [3];
(v) Certain ultimately[4] $\mathcal{C}^\infty$ transexponential solutions of two difference equations: $f(x+1) = e^{f(x)}$ and $f(x+1) = e^{f(x)} - 1$ [3].

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[2]A *germ* is an equivalence class of functions, where two functions are equivalent, exactly when they agree in a neighborhood of the point of interest, in our case $+\infty$.
[3]The *support* of a_n is the set of all n such that $a_n \neq 0$.
[4]A property is said to hold *ultimately*, exactly when it holds in a neighborhood of $+\infty$.

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