

Orbit-finite linear programming

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Abstract—An infinite set is orbit-finite if, up to permutations of the underlying structure of atoms, it has only finitely many elements. We study a generalisation of linear programming where constraints are expressed by an orbit-finite system of linear inequalities. As our principal contribution we provide a decision procedure for checking if such a system has a real solution, and for computing the minimal/maximal value of a linear objective function over the solution set. We also show undecidability of these problems in case when only integer solutions are considered. Therefore **orbit-finite linear programming is decidable, while orbit-finite integer linear programming is not.**

I. INTRODUCTION

Applications of (integer) linear programming are ubiquitous in computer science (see e.g. [1], [2], [3]), including recent and potential future applications to analysis of data-enriched models [4], [5], [6], [7]. This paper is a continuation of the study of *orbit-finite* systems of linear equations [8], i.e., systems which are infinite but finite up to permutations. In this setting one fixes a countably infinite set \mathbb{A} , whose elements are called *atoms* (or data values) [9], [10], assuming that atoms can only be accessed in a very limited way, namely can only be tested for equality. Starting from atoms one builds a hierarchy of sets which are *orbit-finite*: they are infinite, but finite up to permutations of atoms. Along these lines, we study orbit-finite sets of linear inequalities, over an orbit-finite set of unknowns.

The main result of [8] is a decision procedure to check if a given orbit-finite system of equations is solvable. This result is general and applies to systems over a wide range of commutative rings, in particular to real and integer solvability. In this paper we do a next step and extend the setting from equations to inequalities. Our goal is algorithmic solvability of orbit-finite systems of inequalities, but also optimisation (minimisation/maximisation) of linear objective functions over the solution sets of such systems.

We call this problem *orbit-finite (integer) linear programming* (depending on whether the solutions are real or integer).

Example 1. For illustration, consider the set \mathbb{A} as unknowns, and the infinite system of constraints given by an infinite matrix whose rows and columns are indexed by \mathbb{A} :

$$\begin{bmatrix} 0 & 1 & 1 & \cdots \\ 1 & 0 & 1 & \cdots \\ 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \mathbf{x} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} \quad (1)$$

Alternatively, one can write the infinite set of non-strict inequalities over unknowns $\alpha \in \mathbb{A}$, indexed by atoms $\beta \in \mathbb{A}$:

$$\sum_{\alpha \in \mathbb{A} \setminus \{\beta\}} \alpha \geq 1 \quad (\beta \in \mathbb{A}). \quad (2)$$

Any permutation π of \mathbb{A} induces a permutation of the inequalities by sending

$$\sum_{\alpha \in \mathbb{A} \setminus \{\beta\}} \alpha \geq 1 \quad \mapsto \quad \sum_{\alpha \in \mathbb{A} \setminus \{\pi(\beta)\}} \alpha \geq 1,$$

but the whole system (2) is invariant under permutations of atoms. Furthermore, up to permutations of atoms the system consists of just one equation – it is one *orbit*; in the sequel we consider orbit-finite systems (finite unions of orbits). Likewise, the matrix $\mathbb{A} \times \mathbb{A} \rightarrow \mathbb{R}$ in (1) consists, up to permutation of atoms, of just two entries. Indeed, its domain $\mathbb{A} \times \mathbb{A}$ is a union of two orbits: $\{(\alpha, \beta) \mid \alpha = \beta\}$ and $\{(\alpha, \beta) \mid \alpha \neq \beta\}$, and the matrix is defined by assigning a real number to each orbit. It is therefore invariant under permutations of atoms.

The system (2) is solvable. For example, given $n > 1$ atoms $S = \{\alpha_1, \dots, \alpha_n\} \subseteq \mathbb{A}$, the vector $\mathbf{x}_n : \mathbb{A} \rightarrow \mathbb{R}$ defined by:

$$\mathbf{x}_n(\alpha) = \frac{1}{n-1} \quad \text{if } \alpha \in S, \quad \mathbf{x}_n(\alpha) = 0 \quad \text{if } \alpha \notin S,$$

is a solution since the left-hand side of (2) sums up to 1 if $\beta \in S$, and to $\frac{n}{n-1}$ if $\beta \notin S$. \triangleleft

Orbit-finite system of equations are solvable by a reduction to solvability of a finite number of classical finite systems [8]. This strategy does not seem to extend to linear programming, i.e., to optimisation of a linear objective function subject to an orbit-finite system of inequality constraints. As illustrated in the next example, orbit-finite linear programming faces phenomena not present in the classical setting, for instance **the objective function may not achieve its optimum over solutions of non-strict inequalities.**

Example 2. Suppose that we aim at *minimizing*, with respect to the constraints (2), of the value of the objective function:

$$S(\mathbf{x}) = \sum_{\alpha \in \mathbb{A}} \mathbf{x}(\alpha). \quad (3)$$

The function is invariant under permutations of atoms, and its value is always greater than 1. Indeed, for every solution $\mathbf{x} : \mathbb{A} \rightarrow \mathbb{R}$ there is necessarily some $\beta \in \mathbb{A}$ such that $\mathbf{x}(\beta) > 0$, and hence

$$S(\mathbf{x}) > \sum_{\alpha \in \mathbb{A} \setminus \{\beta\}} \mathbf{x}(\alpha) \geq 1. \quad (4)$$

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What is the minimal value of the objective function? For solutions \mathbf{x}_n defined in Example 1, the value $S(\mathbf{x}_n) = \frac{n}{n-1}$ may be arbitrarily close to 1 but, according to (4), S never achieves 1. Surprisingly, this is **in contrast with classical linear programming** where, whenever constraints are specified by non-strict inequalities and are solvable, a linear objective function always achieves its minimum (or diverges to $-\infty$). \triangleleft

Contribution: As our main contribution, we provide decision procedures for orbit-finite linear programming, both for the decision problem of solvability and for the optimisation problem. The core ingredient of our approach is to **reduce solvability (resp. optimisation) of an orbit-finite system of inequalities to the analogous question on a finite system which is polynomially parametrised**, i.e., where coefficients are univariate polynomials in an integer variable n . Intuitively, n corresponds to the number of atoms involved in a solution. In this parametrised setting we ask for solvability for *some* $n \in \mathbb{N}$, or for optimisation when ranging over *all* $n \in \mathbb{N}$. In both cases we can compute an answer by encoding the problem into first-order real arithmetic [11], [12], [13], [14].

Example 3. For instance, the system (1) is transformed to the following two inequalities with one unknown x , which are polynomially (actually, linearly) parametrised in a parameter n (the details are exposed in Example 10 in Section VII-C):

$$\begin{bmatrix} n-1 \\ n \end{bmatrix} \cdot x \geq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (5)$$

The objective function S (3) is likewise transformed to a polynomially parametrised linear map $x \mapsto n \cdot x$. For all $n > 1$, the system (5) is solvable and the minimal solution is $x = \frac{1}{n-1}$. Therefore the minimal value of the objective function is $\frac{n}{n-1}$. Ranging over all $n \in \mathbb{N}$, the minimal value approaches arbitrarily closely to 1, but never reaches 1. \triangleleft

As our second main result we prove undecidability of orbit-finite *integer* linear programming, already for the decision problem of solvability. While the classical linear programming and integer linear programming are on the opposite sides of the feasibility border, in case of orbit-finite systems the two problems are on the opposite sides of the decidability border.

Related research: Our results generalise, or are closely related to, some earlier results on systems of linear equations. Systems studied in [4] have row indexes of atom dimension 1. In a more general but still restricted case studied in [15], all row indexes are assumed to have the same atom dimension. Furthermore, columns of a matrix are assumed to be finitary in [4], [15]. Both the papers investigate (nonnegative) integer solvability and are subsumed by [8], a starting point for our investigations.

The work [5] goes beyond [4] and investigates linear equations, in atom dimension 1, over ordered atoms. Nonnegative integer solvability is decidable and equivalent to VAS reachability (and hence ACKERMANN-complete [16], [17], [18]).

Systems in another related work [19] are over a finite field, contain only finite equations, and are studied as a special case

of orbit-finite constraint satisfaction problems. Furthermore, solutions sought are not restricted to be finitely-supported.

Orbit-finitely generated vector spaces were recently investigated in [7] and applied to orbit-finite *weighted* automata. In [20] the authors study cones in such spaces which are invariant under permutations of atoms, and extend accordingly theorems of Carathéodory and Minkowski-Weyl.

Outline: After preliminaries on orbit-finite sets in Section II, in Section III we introduce the setting of orbit-finite linear inequalities and in Section IV we state our results. The rest of the paper is devoted to proofs. In Sections V and VI we develop tools that are later used in decision procedures for linear programming in Sections VII and VIII. Finally, Section IX contains the proof of undecidability of integer linear programming. We conclude in Section X.

II. PRELIMINARIES ON ORBIT-FINITE SETS

Our definitions rely on basic notions and results of the theory of *sets with atoms* [9], also known as nominal sets [10], [21]. We only work with *equality atoms* which have no additional structure except for the equality.

Sets with atoms: We fix a countably infinite set \mathbb{A} whose elements we call *atoms*. Greek letters $\alpha, \beta, \gamma, \dots$ are reserved to range over atoms. The universe of sets with atoms is defined formally by a suitably adapted cumulative hierarchy of sets, by transfinite induction: the only set of *rank* 0 is the empty set; and for a cardinal i , a set of rank i may contain, as elements, sets of rank smaller than i as well as atoms. In particular, nonempty subsets $X \subseteq \mathbb{A}$ have rank 1.

The group AUT of all permutations of \mathbb{A} , called in this paper *atom automorphisms*, acts on sets with atoms by consistently renaming all atoms in a given set. Formally, by another transfinite induction, for $\pi \in \text{AUT}$ we define $\pi(X) = \{\pi(x) \mid x \in X\}$. Via standard set-theoretic encodings of pairs or finite sequences we obtain, in particular, the pointwise action on pairs $\pi(x, y) = (\pi(x), \pi(y))$, and likewise on finite sequences. Relations and functions from X to Y are considered as subsets of $X \times Y$.

We restrict to sets with atoms X that only depend on finitely many atoms, in the following sense. For $T \subseteq \mathbb{A}$, let $\text{AUT}_T = \{\pi \in \text{AUT} \mid \pi(\alpha) = \alpha \text{ for every } \alpha \in T\}$ be the set of atom automorphisms that *fix* T , called *T-atom automorphisms*. A finite set $T \subseteq_{\text{fin}} \mathbb{A}$ (we use the symbol \subseteq_{fin} for finite subsets) is a *support* of X if for all $\pi \in \text{AUT}_T$ it holds $\pi(X) = X$. We also say: T *supports* X , or X is *T-supported*. Thus a set is *T-supported* if and only if it is invariant under all *T-atom* automorphisms. As a special case, a function f is *T-supported* if $f(\pi(x)) = \pi(f(x))$ for every argument x and $\pi \in \text{AUT}_T$. A *T-supported* set is also T' -supported, assuming $T \subseteq T'$.

A set x is *finitely supported* if it has some finite support; in this case x always has the least (inclusion-wise) support, denoted $\text{sup}(x)$, called *the support of x* (cf. [9, Sect. 6]). Thus x is *T-supported* if, and only if $\text{sup}(x) \subseteq T$. Sets supported by \emptyset (i.e., invariant under all atom automorphisms) we call *equivariant*.

Example 4. Given $\alpha, \beta \in \mathbb{A}$, the support of the set $\mathbb{A} \setminus \{\alpha, \beta\}$ is $\{\alpha, \beta\}$. The set \mathbb{A}^2 and the projection function $\pi_1 : \mathbb{A}^2 \rightarrow \mathbb{A} : (\alpha, \beta) \mapsto \alpha$ are both equivariant; and the support of a tuple $\langle \alpha_1, \dots, \alpha_n \rangle \in \mathbb{A}^n$, encoded as a set in a standard way, is the set of atoms $\{\alpha_1, \dots, \alpha_n\}$ appearing in it. \triangleleft

From now on, we shall only consider sets that are hereditarily finitely supported, i.e., ones that have a finite support, whose every element has some finite support, and so on.

Orbit-finite sets: Let $T \subseteq_{\text{fin}} \mathbb{A}$. Two atoms or sets x, y are in the same T -orbit if $\pi(x) = y$ for some $\pi \in \text{AUT}_T$. This equivalence relation splits atoms and sets into equivalence classes, which we call T -orbits; \emptyset -orbits we call equivariant orbits, or simply *orbits*. By the very definition, every T -orbit U is T -supported: $\text{sup}(U) \subseteq T$.¹

T -supported sets are exactly unions of (necessarily disjoint) T -orbits. Finite unions of T -orbits, for any $T \subseteq_{\text{fin}} \mathbb{A}$, are called *orbit-finite* sets. Orbit-finiteness is stable under orbit-refinement: if $T \subseteq T' \subseteq_{\text{fin}} \mathbb{A}$, a finite union of T -orbits is also a finite union of T' -orbits (but the number of orbits may increase, cf. [9, Theorem 3.16]).

Example 5. Examples of orbit-finite sets are: \mathbb{A} (1 orbit); $\mathbb{A} \setminus \{\alpha\}$ for some $\alpha \in \mathbb{A}$ (1 $\{\alpha\}$ -orbit); \mathbb{A}^2 (2 orbits: diagonal and non-diagonal); \mathbb{A}^3 (5 orbits, corresponding to equality types of triples); non-repeating n -tuples of atoms (1 orbit)

$$\mathbb{A}^{(n)} = \{\alpha_1 \dots \alpha_n \in \mathbb{A}^n \mid \alpha_i \neq \alpha_j \text{ for } i \neq j\};$$

n -sets of atoms $\binom{\mathbb{A}}{n} = \{X \subseteq \mathbb{A} \mid |X| = n\}$ (1 orbit). The set $\mathcal{P}_{\text{fin}}(\mathbb{A})$ of all finite subsets of atoms is orbit-infinite as cardinality is an invariant of each orbit. \triangleleft

By the *equality type* of an n -tuple $a_1 \dots a_n \in \mathbb{A}^n$ we mean the set $\{(i, j) \mid a_i = a_j\}$. Each orbit $U \subseteq \mathbb{A}^n$ contains all n -tuples of the same equality type. In particular, each orbit included in $\mathbb{A}^{(n)} \times \mathbb{A}^{(m)} \subseteq \mathbb{A}^{n+m}$ is induced by a partial injection ι from $\{1 \dots n\}$ to $\{1 \dots m\}$:

Lemma 1. *Orbits $U \subseteq \mathbb{A}^{(n)} \times \mathbb{A}^{(m)}$ are exactly sets of the form:*

$$\left\{ (a, b) \in \mathbb{A}^{(n)} \times \mathbb{A}^{(m)} \mid \forall i, j : a(i) = b(j) \iff \iota(i) = j \right\},$$

where ι is a partial injection from $\{1 \dots n\}$ to $\{1 \dots m\}$.

Furthermore, for $T \subseteq_{\text{fin}} \mathbb{A}$, each T -orbit $U \subseteq \mathbb{A}^{(n)}$ is determined by fixing pairwise distinct atoms from T on a subset $I \subseteq \{1 \dots n\}$ of positions, while allowing arbitrary atoms from $\mathbb{A} \setminus T$ on remaining positions $\{1 \dots n\} \setminus I$:

Lemma 2. *Let $T \subseteq_{\text{fin}} \mathbb{A}$. T -orbits $U \subseteq \mathbb{A}^{(n)}$ are exactly sets of the form:*

$$\{a \in \mathbb{A}^{(n)} \mid \Pi_{n,I}(a) = u, \Pi_{n,\{1 \dots n\} \setminus I}(a) \in (\mathbb{A} \setminus T)^{(n-\ell)}\}, \quad (6)$$

where $I \subseteq \{1 \dots n\}$, $|I| = \ell$, and $u \in T^{(\ell)}$. The projection $\Pi_{n,I} : \mathbb{A}^{(n)} \rightarrow \mathbb{A}^{(\ell)}$ is defined in the expected way.

¹The inclusion may be strict, for singleton T -orbits O .

Indeed, the set (6) is invariant under T -atom automorphisms, and each two of its elements are related by some T -atom automorphism.

Atom automorphisms preserve the size of the support: $|\text{sup}(X)| = |\text{sup}(\pi(X))|$ for every set X and $\pi \in \text{AUT}$. We define *atom dimension* of an orbit as the size of the support of its elements. For instance, atom dimension of $\mathbb{A}^{(n)}$ is n .

III. ORBIT-FINITE (INTEGER) LINEAR PROGRAMMING

We introduce now the setting of linear inequalities we work with, and formulate our main results. We are working in vector spaces over the real² field \mathbb{R} , generated by a fixed orbit-finite set B in the sense explained below.

Definition 3. By a *vector* over B we mean any finitely-supported function \mathbf{v} from B to \mathbb{R} , written $\mathbf{v} : B \rightarrow_{\text{fs}} \mathbb{R}$ (vectors are written using boldface). Vectors with integer range, $\mathbf{v} : B \rightarrow_{\text{fs}} \mathbb{Z}$, we call *integer* vectors.

The set of all vectors over B we denote by $\text{LIN}(B) = B \rightarrow_{\text{fs}} \mathbb{R}$. It is a vector space, with pointwise addition and scalar multiplication: for $\mathbf{v}, \mathbf{v}' \in \text{LIN}(B)$, $b \in B$ and $q \in \mathbb{R}$, we have $(\mathbf{v} + \mathbf{v}')(b) = \mathbf{v}(b) + \mathbf{v}'(b)$ and $(q \cdot \mathbf{v})(b) = q \cdot \mathbf{v}(b)$. These operations preserve the property of being finitely-supported, e.g., $\text{sup}(\mathbf{v} + \mathbf{v}') \subseteq \text{sup}(\mathbf{v}) \cup \text{sup}(\mathbf{v}')$. $\text{LIN}(B)$ may be considered as the vector space *generated* by B . We define the *domain* of a vector $\mathbf{v} \in \text{LIN}(B)$ as $\text{dom}(\mathbf{v}) = \{b \in B \mid \mathbf{v}(b) \neq 0\}$. A vector \mathbf{v} over B is *finitary*, written $\mathbf{v} : B \rightarrow_{\text{fin}} \mathbb{R}$, if $\text{dom}(\mathbf{v})$ is finite, i.e., $\mathbf{v}(b) = 0$ for almost all $b \in B$.

Example 6. Let $B = \mathbb{A}^{(2)}$. Let $\alpha, \beta \in \mathbb{A}$ be two fixed atoms. The function $\mathbf{v} : B \rightarrow \mathbb{R}$ defined, for $\chi, \gamma \in \mathbb{A} \setminus \{\alpha, \beta\}$, by

$$\begin{aligned} \mathbf{v}(\alpha\chi) &= \mathbf{v}(\chi\alpha) = -1 & \mathbf{v}(\alpha\beta) &= \mathbf{v}(\beta\alpha) = 3 \\ \mathbf{v}(\beta\chi) &= \mathbf{v}(\chi\beta) = -2 & \mathbf{v}(\chi\gamma) &= 0 \end{aligned}$$

is an $\{\alpha, \beta\}$ -supported integer vector over B . It is not finitary, as $\text{dom}(\mathbf{v}) = \{\delta\sigma \in \mathbb{A}^{(2)} \mid \{\delta, \sigma\} \cap \{\alpha, \beta\} \neq \emptyset\}$ is infinite. Finitary $\{\alpha, \beta\}$ -supported vectors over B assign 0 to all elements of B except for $\alpha\beta$ and $\beta\alpha$. \triangleleft

A finitary vector \mathbf{v} with domain $\text{dom}(\mathbf{v}) = \{b_1, \dots, b_k\}$ such that $\mathbf{v}(b_1) = q_1, \dots, \mathbf{v}(b_k) = q_k$, may be identified with a formal linear combination of elements of B :

$$\mathbf{v} = q_1 \cdot b_1 + \dots + q_k \cdot b_k. \quad (7)$$

The subspace of $\text{LIN}(B)$ consisting of all finitary vectors we denote by $\text{FIN-LIN}(B) = B \rightarrow_{\text{fin}} \mathbb{R}$. For finite B of size $|B| = n$, $\text{LIN}(B) = \text{FIN-LIN}(B)$ is isomorphic to \mathbb{R}^n .

For a subset $X \subseteq B$, we denote by $\mathbf{1}_X \in \text{LIN}(B)$ the characteristic function of X , i.e., the vector that maps each element of X to 1 and all elements of $B \setminus X$ to 0:

$$\mathbf{1}_X : b \mapsto \begin{cases} 1 & \text{if } b \in X \\ 0 & \text{otherwise.} \end{cases}$$

²All the results of the paper still hold if reals \mathbb{R} are replaced by rationals \mathbb{Q} in all the subsequent definitions and results.

We write $\mathbf{1}_b$ instead of $\mathbf{1}_{\{b\}}$, and $\mathbf{1}$ instead of $\mathbf{1}_B$.

Lemma 4. Let $T \subseteq_{\text{fin}} \mathbb{A}$ and $\mathbf{v} \in \text{LIN}(B)$ such that $\text{sup}(\mathbf{v}) \subseteq T$. Then

- (i) \mathbf{v} is constant, when restricted to every T -orbit $U \subseteq B$;
- (ii) \mathbf{v} is a linear combination of characteristic vectors $\mathbf{1}_U$ of T -orbits $U \subseteq B$.

Proof. The first part follows immediately as T supports \mathbf{v} . This allows us to write $\mathbf{v}(U) \in \mathbb{R}$ in place of $\mathbf{v}(x)$ for $x \in U$. As required in the second part, we have:

$$\mathbf{v} = \sum_U \mathbf{v}(U) \cdot \mathbf{1}_U, \quad (8)$$

where U ranges over finitely many T -orbits $U \subseteq B$. \square

We note that the inner product of vectors $\mathbf{x}, \mathbf{y} \in \text{LIN}(B)$,

$$\mathbf{x} \cdot \mathbf{y} = \sum_{b \in B} \mathbf{x}(b) \mathbf{y}(b),$$

is not always well-defined. We consider the right-hand side sum as well-defined when there are only finitely many $b \in B$ for which both $\mathbf{x}(b)$ and $\mathbf{y}(b)$ are non-zero (equivalently, the intersection $\text{dom}(\mathbf{x}) \cap \text{dom}(\mathbf{y})$ is finite).³

Orbit-finite systems of linear inequalities: Fix an orbit-finite set C (it can be thought of as the set of unknowns). By a linear inequality over C we mean a triple $e = (\mathbf{a}, t)$ where $\mathbf{a} : C \rightarrow_{\text{fs}} \mathbb{Z}$ is an integer vector of left-hand side coefficients and $t \in \mathbb{Z}$ is a right-hand side target value⁴. An \mathbb{R} -solution (real solution) of e is any vector $\mathbf{x} : C \rightarrow_{\text{fs}} \mathbb{R}$ such that the inner product $\mathbf{a} \cdot \mathbf{x}$ is well-defined and

$$\mathbf{a} \cdot \mathbf{x} \geq t;$$

\mathbf{x} is an \mathbb{Z} -solution (integer solution) if $\mathbf{x} : C \rightarrow_{\text{fs}} \mathbb{Z}$. We may also consider constrained solutions, e.g., *finitary* ones. A linear equality $\mathbf{a} \cdot \mathbf{x} = t$ may be encoded by two opposite inequalities:

$$\mathbf{a} \cdot \mathbf{x} \geq t \quad -\mathbf{a} \cdot \mathbf{x} \geq -t.$$

In this paper we investigate sets of inequalities indexed by an orbit-finite set. Formally, an orbit-finite system of linear inequalities (over C) is the pair (\mathbf{A}, \mathbf{t}) , where $\mathbf{A} : B \times C \rightarrow_{\text{fs}} \mathbb{Z}$ is an integer matrix with row index B and column index C , and $\mathbf{t} : B \rightarrow_{\text{fs}} \mathbb{Z}$ is an integer target vector:

$$\begin{array}{ccc} \cdots & c & \cdots \\ \vdots & \vdots & \\ b & \left[\begin{array}{ccc} \cdots & \mathbf{A}(b, c) & \cdots \end{array} \right] & \left[\begin{array}{c} \vdots \\ \mathbf{t}(b) \\ \vdots \end{array} \right] \\ \vdots & \vdots & \end{array}$$

For $b \in B$ we denote by $\mathbf{A}(b, _) \in \text{LIN}(C)$ the corresponding (row) vector. A solution of a system (\mathbf{A}, \mathbf{t}) is any vector $\mathbf{x} \in \text{LIN}(C)$ which is a solution of all inequalities $(\mathbf{A}(b, _), \mathbf{t}(b))$, $b \in B$. Equivalently, \mathbf{x} is a solution if $\mathbf{A} \cdot \mathbf{x} \geq \mathbf{t}$, where \geq is

³In particular, $\mathbf{x} \cdot \mathbf{y}$ is always well-defined when one of \mathbf{x}, \mathbf{y} is finitary.

⁴Rational coefficients and target are easily scaled up to integers.

the pointwise order on vectors, and the (partial) operation of multiplication of a matrix \mathbf{A} by a vector \mathbf{x} is defined in an expected way:

$$(\mathbf{A} \cdot \mathbf{x})(b) = \mathbf{A}(b, _) \cdot \mathbf{x}$$

for every $b \in B$. The result $\mathbf{A} \cdot \mathbf{x} \in \text{LIN}(B)$ is well-defined if $\mathbf{A}(b, _) \cdot \mathbf{x}$ is well-defined for all $b \in B$.

By the following examples, restricting to equivariant, finitary or integer solutions only has impact on solvability:

Example 7. Let columns be indexed by $C = \mathbb{A}$, and consider the system consisting of just one infinitary inequality $(\mathbf{1}_{\mathbb{A}}, 1)$ (B is thus a singleton). Identifying column indexes $\alpha \in \mathbb{A}$ with unknowns, the inequality may be written as:

$$\sum_{\alpha \in \mathbb{A}} \alpha \geq 1.$$

The inequality has an integer (finitary) solution, i.e., $\mathbf{x} = \mathbf{1}_{\alpha}$ for any $\alpha \in \mathbb{A}$, but no equivariant one. Indeed, equivariant vectors $\mathbf{x} : \mathbb{A} \rightarrow_{\text{fs}} \mathbb{R}$ are necessarily constant ones $\mathbf{x} = r \cdot \mathbf{1}_{\mathbb{A}}$ (cf. Lemma 4), and then the inner product

$$\mathbf{1}_{\mathbb{A}} \cdot \mathbf{x} = \sum_{\alpha \in \mathbb{A}} \mathbf{x}(\alpha) = \sum_{\alpha \in \mathbb{A}} r$$

is well-defined only if $r = 0$, i.e. $\mathbf{x}(\alpha) = 0$ for all $\alpha \in \mathbb{A}$. \triangleleft

Example 8. Let columns be indexed by $C = \mathbb{A}^{(2)}$ and rows by $B = \binom{\mathbb{A}}{2}$. Consider the system containing, for every $\{\alpha, \beta\} \in B$, the inequality $(\mathbf{1}_{\alpha\beta} + \mathbf{1}_{\beta\alpha}, 1)$. Using the formal-sum notation as in (7) it may be written as $(\alpha\beta + \beta\alpha, 1)$ or, identifying column indexes $\alpha\beta \in C$ with unknowns, as:

$$\alpha\beta + \beta\alpha \geq 1 \quad (\alpha, \beta \in \mathbb{A}, \alpha \neq \beta).$$

All the equations are thus finitary, and the target is $\mathbf{t} = \mathbf{1}_B$. The constant vector $\mathbf{x} = \frac{1}{2} : (\alpha, \beta) \mapsto \frac{1}{2}$ is a solution, even if we extend the system with symmetric inequalities

$$\alpha\beta + \beta\alpha \leq 1 \quad (\alpha, \beta \in \mathbb{A}, \alpha \neq \beta).$$

The extended system has no finitary solution. It has no integer solution either. Indeed, any solution \mathbf{x} necessarily satisfies, for every distinct atoms $\alpha, \beta \in \mathbb{A} \setminus \text{sup}(\mathbf{x})$, the equality $\mathbf{x}(\alpha\beta) = \mathbf{x}(\beta\alpha)$, which is incompatible with $\mathbf{x}(\alpha\beta) + \mathbf{x}(\beta\alpha) = 1$. \triangleleft

IV. RESULTS

Solvability problems: We investigate decision problems of solvability of orbit-finite systems of inequalities over the ring of reals or integers. Consequently, we use \mathbb{F} to stand either for \mathbb{R} or \mathbb{Z} . We identify a couple of variants. In the first one we ask about existence of a finitely-supported solution:

INEQ-SOLV(\mathbb{F}):

Input: An orbit-finite system of linear inequalities.

Question: Does it have a finitely-supported \mathbb{F} -solution?

A closely related variant is solvability of equalities, under the restriction to nonnegative solutions only:

NONNEG-EQ-SOLV(\mathbb{F}):

Input: *An orbit-finite system of linear equations.*

Question: *Does it have a finitely-supported nonnegative \mathbb{F} -solution?*

Furthermore, both the problems have “finitary” versions, where one seeks for finitary solutions only, denoted as FIN-INEQ-SOLV(\mathbb{F}) and FIN-NONNEG-EQ-SOLV(\mathbb{F}), respectively.

It turns out that three out of the four problems are inter-reducible, and hence equi-decidable, both for $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{Z}$ (the proof is in Appendix):

Theorem 5. *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{Z}\}$. The problems INEQ-SOLV(\mathbb{F}), FIN-INEQ-SOLV(\mathbb{F}) and NONNEG-EQ-SOLV(\mathbb{F}) are inter-reducible.*

The three problems listed in Theorem 5 deserve a shared name *orbit-finite linear programming* (in case of \mathbb{R}) and *orbit-finite integer linear programming* (in case of \mathbb{Z}). Diagram (9) below shows the reductions of Theorem 5 using dashed arrows.

As our two main results we prove that the linear programming is decidable, while the integer one is not:

Theorem 6. *FIN-INEQ-SOLV(\mathbb{R}) is decidable.*

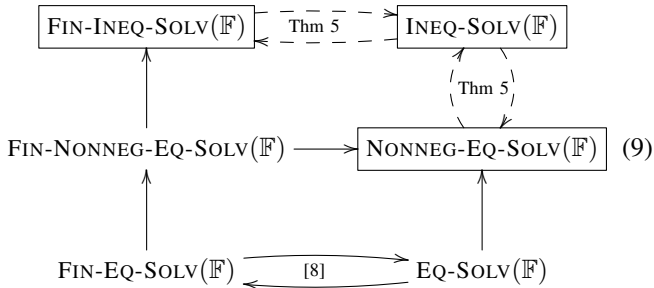
Theorem 7. *FIN-INEQ-SOLV(\mathbb{Z}) is undecidable.*

(The proofs occupy Sections VII and IX, respectively.) Additionally, we settle the status of the last variant (the proof, relying on the recent result of [15], is in Appendix):

Theorem 8. *FIN-NONNEG-EQ-SOLV(\mathbb{F}) is decidable, for $\mathbb{F} \in \{\mathbb{R}, \mathbb{Z}\}$.*

In consequence of Theorems 7 and 8, at least in case $\mathbb{F} = \mathbb{Z}$ the above diagram can not be completed by the two missing arrows to FIN-NONNEG-EQ-SOLV(\mathbb{Z}).

Linear equations vs inequalities: Solvability of orbit-finite systems of *equations* (EQ-SOLV(\mathbb{F})) easily reduces to INEQ-SOLV(\mathbb{F}), by replacing each equation with two opposite inequalities, but also to NONNEG-EQ-SOLV(\mathbb{F}), by replacing each unknown with a difference of two unknowns. Likewise does the variant FIN-EQ-SOLV(\mathbb{F}), where one only seeks for finitary solutions. Here is the diagram of all reductions:



Theorem 9 ([8] Thms 4.4 and 6.1). *EQ-SOLV(\mathbb{F}) and FIN-EQ-SOLV(\mathbb{F}) are inter-reducible and decidable⁵.*

⁵The results of [8] apply to system of equations where coefficients and solutions are from any fixed commutative and effective ring \mathbb{F} . This includes integers \mathbb{Z} or rationals \mathbb{Q} (and hence also to applies to real solutions).

Optimisation problems: We consider $\mathbb{F} = \mathbb{R}$, due the undecidability of Theorem 7. All variants of linear programming mentioned above have corresponding *maximisation* problems. In each variant input contains, except for a system (\mathbf{A}, \mathbf{t}) , an integer vector $\mathbf{s} : C \rightarrow_{\text{fs}} \mathbb{Z}$ that represents a (partial) linear *objective* function $S : \text{LIN}(C) \rightarrow_{\text{fs}} \mathbb{R}$, defined by

$$S(\mathbf{x}) = \mathbf{s} \cdot \mathbf{x}.$$

The maximisation problem asks to compute the supremum of the objective function over all (finitary, nonnegative) solutions of (\mathbf{A}, \mathbf{t}) . A symmetrical *minimisation* problem is easily transformed to a maximisation one by replacing \mathbf{s} with $-\mathbf{s}$. This yields three optimisation problems INEQ-MAX(\mathbb{R}), FIN-INEQ-MAX(\mathbb{R}) and NONNEG-EQ-MAX(\mathbb{R}) which are, as before, inter-reducible (the proof is in Appendix):

Theorem 10. *The problems INEQ-MAX(\mathbb{R}), FIN-INEQ-MAX(\mathbb{R}) and NONNEG-EQ-MAX(\mathbb{R}) are inter-reducible.*

As our last main result we strengthen Theorem 6 to the optimisation setting (the proof is in Section VIII):

Theorem 11. *FIN-INEQ-MAX(\mathbb{R}) is computable.*

Representation of input: There are several possible ways of representing input $(\mathbf{A}, \mathbf{t}, \mathbf{s})$ to our algorithms. One possibility is to rely on the equivalence between (hereditary) orbit-finite sets and *definable* sets [9, Sect. 4]. We choose another standard possibility, and assume that a vector $\mathbf{v} : D \rightarrow_{\text{fs}} \mathbb{Z}$ over an orbit-finite set D is represented by:

- (1) its support $T = \text{sup}(\mathbf{v})$,
- (2) list L of all T -orbits $U \subseteq D$,
- (3) list of values $\mathbf{v}(U) \in \mathbb{Z}$ for all $U \in L$ (cf. Lemma 4).

We apply this representation to \mathbf{A} , \mathbf{t} , and \mathbf{s} . Concerning point (2), each S -orbit is finitely representable, cf. [9, Thm 6.3].

Strict inequalities: In this paper we consider system of *non-strict* inequalities, for the sake of presentation. The decision procedures of Theorems 6 and 11, work equally well if both *non-strict* and *strict* inequalities are allowed. Reductions between FIN-INEQ-SOLV(\mathbb{F}) and INEQ-SOLV(\mathbb{F}) work as well, but not the reductions from (FIN-)NONNEG-EQ-SOLV(\mathbb{F}) to (FIN-)INEQ-SOLV(\mathbb{F}) as we can not simulate equalities with strict inequalities.

V. FINITELY SEMI-SUPPORTED SETS

We introduce the novel concept of *semi-support* which plays a central role in the proofs of Theorems 6 and 11.

For any $T \subseteq_{\text{fin}} \mathbb{A}$ consider the set of all atom automorphisms that preserve T as a set only:

$$\text{AUT}_{\{T\}} = \{ \pi \in \text{AUT} \mid \pi(T) = T \}.$$

Elements of $\text{AUT}_{\{T\}}$ we call *semi- T -atom automorphisms*. Accordingly, we define *semi- T -orbits* as equivalence classes with respect to the action of $\text{AUT}_{\{T\}}$: two sets (elements) x, y are in the same semi- T -orbit if $\pi(x) = y$ for some $\pi \in \text{AUT}_{\{T\}}$. We have

$$\text{AUT}_T \subseteq \text{AUT}_{\{T\}} \subseteq \text{AUT},$$

and hence every equivariant orbit splits into finitely many semi- T -orbits, each of which splits in turn into finitely many T -orbits. A set x is *semi- T -supported* if $\pi(x) = x$ for all $\pi \in \text{AUT}_{\{T\}}$. Equivalently, x is a union of semi- T -orbits. Note that each semi- T -supported set is T -supported, but the opposite implication is not true. When T is irrelevant, we speak of finitely semi-supported sets. Finally notice that a semi- T -supported set is not necessarily semi- T' -supported for $T \subseteq T'$, which distinguishes semi-support from standard support.

Example 9. Let $T = \{\alpha, \beta\} \subseteq \mathbb{A}$. The vector \mathbf{v} , defined in Example 6 in Section III, is not semi- T -supported. Indeed,

$$\pi(\mathbf{v})(\alpha, \chi) = \mathbf{v}(\beta, \chi) \neq \mathbf{v}(\alpha, \chi)$$

for any $\chi \notin T$ and $\pi \in \text{AUT}_{\{T\}}$ that swaps α and β but preserves all other atoms. The *averaged* vector \mathbf{v}' defined by

$$\begin{aligned} \mathbf{v}'(\alpha\chi) &= \mathbf{v}'(\chi\alpha) = -1.5 & \mathbf{v}'(\alpha\beta) &= \mathbf{v}'(\beta\alpha) = 3 \\ \mathbf{v}'(\beta\chi) &= \mathbf{v}'(\chi\beta) = -1.5 & \mathbf{v}'(\chi\gamma) &= 0, \end{aligned}$$

for $\chi, \gamma \in \mathbb{A} \setminus \{\alpha, \beta\}$, is semi- T -supported. \triangleleft

Clearly, with the size of T increasing towards infinity, the number of T -orbits included in one equivariant orbit may increase towards infinity as well. The crucial property of semi- T -supported sets is that they do not suffer from this unbounded growth: the number of semi- T -orbits included in a fixed equivariant orbit is bounded, no matter how large T is. We will need this property for semi- T -orbits $U \subseteq \mathbb{A}^{(n)}$, $n \in \mathbb{N}$, and it follows immediately by Lemma 12. Intuitively speaking, each such semi- T -orbit is determined by a subset $I \subseteq \{1 \dots n\}$ of positions which is filled by arbitrary pairwise different atoms from T , the remaining positions $\{1 \dots n\} \setminus I$ are filled by arbitrary atoms from $\mathbb{A} \setminus T$ (cf. Lemma 2 in Section II).

Lemma 12. Let $T \subseteq_{\text{fin}} \mathbb{A}$. Each semi- T -orbit $O \subseteq \mathbb{A}^{(n)}$ is of the form

$$\begin{aligned} O = \{ a \in \mathbb{A}^{(n)} \mid \Pi_{n,I}(a) \in T^{(\ell)}, \\ \Pi_{n,\{1\dots n\} \setminus I}(a) \in (\mathbb{A} \setminus T)^{(n-\ell)} \}, \end{aligned} \quad (10)$$

for some $I \subseteq \{1 \dots n\}$ of size ℓ .

A key observation is that a solvable equivariant system of inequalities necessarily has a finitely semi-supported solution:

Lemma 13. If an equivariant system of inequalities (\mathbf{A}, \mathbf{t}) has a finitely supported (finitary) solution \mathbf{x} then it also has a finitely semi-supported (finitary) one \mathbf{y} such that $S(\mathbf{x}) = S(\mathbf{y})$ for every equivariant partial linear map $S : \text{LIN}(C) \rightarrow \mathbb{R}$.

Proof. Let $\mathbf{x} : C \rightarrow \mathbb{R}$ be a solution of the system, namely

$$\mathbf{A} \cdot \mathbf{x} \geq \mathbf{t}.$$

Let $T = \text{sup}(\mathbf{x})$ and $n = |T|$. As (\mathbf{A}, \mathbf{t}) is equivariant, atom automorphisms preserve being a solution, namely for every $\rho \in \text{AUT}$, the vector $\rho(\mathbf{x})$ is also a solution:

$$\mathbf{A} \cdot \rho(\mathbf{x}) \geq \mathbf{t}.$$

Consider $\text{AUT}_{\mathbb{A} \setminus T}$, the subgroup of atom automorphisms that only permute T and preserve all other atoms. Knowing that the size of $\text{AUT}_{\mathbb{A} \setminus T}$ is $n!$, we have

$$\mathbf{A} \cdot \left(\sum_{\rho \in \text{AUT}_{\mathbb{A} \setminus T}} \rho(\mathbf{x}) \right) \geq n! \cdot \mathbf{t},$$

and hence the vector \mathbf{y} defined by averaging (cf. Example 9)

$$\mathbf{y} = \frac{1}{n!} \cdot \sum_{\rho \in \text{AUT}_{\mathbb{A} \setminus T}} \rho(\mathbf{x}) \quad (11)$$

is also a solution of the system, namely $\mathbf{A} \cdot \mathbf{y} \geq \mathbf{t}$. We notice that for finitary \mathbf{x} , the vector \mathbf{y} is finitary as well. Furthermore, we claim that the vector \mathbf{y} is semi- T -supported. To prove this, we fix an arbitrary semi- T -atom automorphism $\pi \in \text{AUT}_{\{T\}}$, aiming at showing that $\pi(\mathbf{y}) = \mathbf{y}$. It factors through

$$\pi = \sigma \circ \rho$$

for some $\rho \in \text{AUT}_{\mathbb{A} \setminus T}$ and $\sigma \in \text{AUT}_T$. Indeed, ρ acts as π on T but is identity elsewhere, while σ acts as π outside of T but is identity on T . A crucial but simple observation is that, by the very construction of \mathbf{y} , we have

$$\rho(\mathbf{y}) = \mathbf{y}. \quad (12)$$

Indeed, as \mathbf{y} is defined by averaging over all $\rho' \in \text{AUT}_{\mathbb{A} \setminus T}$,

$$\rho \left(\sum_{\rho' \in \text{AUT}_{\mathbb{A} \setminus T}} \rho'(\mathbf{x}) \right) = \sum_{\rho' \in \text{AUT}_{\mathbb{A} \setminus T}} \rho \circ \rho'(\mathbf{x}) = \sum_{\rho' \in \text{AUT}_{\mathbb{A} \setminus T}} \rho'(\mathbf{x})$$

which implies $\rho(\mathbf{y}) = \mathbf{y}$. Moreover, as action of atom automorphisms preserves the support, we have

$$\text{sup}(\rho'(\mathbf{x})) = \rho'(\text{sup}(\mathbf{x}))$$

for every $\rho' \in \text{AUT}$, and therefore

$$\text{sup}(\rho'(\mathbf{x})) = \text{sup}(\mathbf{x})$$

for every $\rho' \in \text{AUT}_{\mathbb{A} \setminus T}$. Therefore T supports the right-hand side of (11), which means that $\text{sup}(\mathbf{y}) \subseteq T$ and implies

$$\sigma(\mathbf{y}) = \mathbf{y}. \quad (13)$$

By (12) and (13) we obtain $\pi(\mathbf{y}) = \mathbf{y}$, as required.

Finally, let $S : \text{LIN}(C) \rightarrow \mathbb{R}$. By equivariance of S we get $S(\rho(\mathbf{x})) = S(\mathbf{x})$ for every $\rho \in \text{AUT}$. By (11) and linearity of S we obtain $S(\mathbf{y}) = S(\mathbf{x})$. \square

VI. POLYNOMIALLY-PARAMETRIZED INEQUALITIES

We now introduce a core problem that will serve as a target of reductions in the proofs of Theorems 6 and 11 in Sections VII and VIII. Consider a finite inequality of the form:

$$p_1(n) \cdot x_1 + \dots + p_k(n) \cdot x_k \geq q(n), \quad (14)$$

where p_1, \dots, p_n and q are univariate polynomials with integer coefficients, and x_1, \dots, x_n are unknowns. The special unknown n plays a role of a nonnegative integer parameter, and that is why we call such an inequality *polynomially-parametrised*. In the sequel we study solvability of a finite system P of such inequalities (14) with the same unknowns

x_1, \dots, x_k . For every fixed value $n \in \mathbb{N}$, by evaluating all polynomials in n we get an ordinary finite system $P(n)$ of linear inequalities with unknowns x_1, \dots, x_k . We use the matrix form $P(n) = (\mathbf{A}(n), \mathbf{t}(n))$ when convenient. We are interested in checking if for some value $n \in \mathbb{N}$, the system $P(n)$ has a real solution:

POLY-INEQ-SOLV:

Input: A finite system of polynomially-parametrised inequalities P .

Question: Does $P(n)$ have a real solution for some $n \in \mathbb{N}$?

We mostly care about a special case of the problem when the input system P is *monotonic*: whenever $P(n)$ has a real solution, for some $n \in \mathbb{N}$, then $P(n+1)$ also does.

We show decidability of POLY-INEQ-SOLV (cf. Theorem 15) by encoding the problem into real arithmetic, i.e., first-order theory of $(\mathbb{R}, +, \cdot, 0, 1, \leq)$. We say that a real arithmetic formula $\varphi(x_1, \dots, x_k)$ with free variables x_1, \dots, x_k defines the set of all valuations $\{x_1, \dots, x_k\} \rightarrow \mathbb{R}$ satisfying it. Consider a macro $\forall^\omega x : \varphi$ saying that a formula φ holds for all sufficiently large values of its free variable x :

$$\forall^\omega x : \varphi(x, \vec{y}) \equiv \exists \tilde{x} : \forall x : x > \tilde{x} \implies \varphi(x, \vec{y}).$$

We will use a variant of the above formula, where the variable x ranges over all sufficiently large *integers* instead of reals:

$$\forall_{\mathbb{Z}}^\omega x : \varphi(x, \vec{y}) \equiv \exists \tilde{x} : \forall x : x > \tilde{x} \wedge x \in \mathbb{Z} \implies \varphi(x, \vec{y}).$$

Note that $\forall_{\mathbb{Z}}^\omega x : \varphi$ is no more a shorthand but rather a new semantic quantifier, as it considers only integer values of x . Therefore we distinguish the macro from the semantic quantifier using the subscript \mathbb{Z} , and consider real arithmetic extended with this new quantifier. Nevertheless, an observation crucial for further reasoning is that the semantic quantifier is actually equivalent to the syntactic macro (proof in Appendix):

Lemma 14. For every real arithmetic formula φ , the formulas

$$\forall^\omega x : \varphi \quad \text{and} \quad \forall_{\mathbb{Z}}^\omega x : \varphi$$

are equivalent, i.e., define the same set.

Theorem 15. POLY-INEQ-SOLV is decidable. For monotonic instances, POLY-INEQ-SOLV is decidable in EXPTIME.

Proof. The problem is easily encodable into real arithmetic.

Consider a fixed system P of polynomially-parametrised inequalities over unknowns x_1, \dots, x_k , and let n range over *reals*, not just over nonnegative integers. For each $n \in \mathbb{R}$, we get the system $P(n)$ of linear inequalities with *real* coefficients. Let

$$\sigma_P(n, x_1, \dots, x_k) \quad (15)$$

be the conjunction of inequalities in P , each of the form (14); it is thus a quantifier-free real arithmetic formula which says that a tuple $\vec{x} = x_1, \dots, x_k$ is a solution of $P(n)$. The existential real arithmetic formula $\psi(n) \equiv \exists \vec{x} : \sigma_P(n, \vec{x})$ with one free variable n , says that $P(n)$ has a real solution. Thus

POLY-INEQ-SOLV has positive answer exactly when $\psi(n)$ is true for some $n \in \mathbb{N}$ (the proof of decidability is in Appendix). Likewise, monotonic POLY-INEQ-SOLV has positive answer exactly when $\psi(n)$ is true for all sufficiently large $n \in \mathbb{N}$. Indeed, if $\psi(n)$ for some $n \in \mathbb{N}$ then $\psi(n')$ for all $n' \in \mathbb{N}$ greater than n . By Lemma 14, solving monotonic POLY-INEQ-SOLV reduces to evaluating the real arithmetic formula

$$\forall^\omega n : \exists \vec{x} : \sigma_P(n, \vec{x}). \quad (16)$$

Evaluating formulas of fixed quantifier alternation depth is doable in EXPTIME [13], [14, Theorem 14.16], which yields the complexity for monotonic instances. \square

VII. DECIDABILITY OF REAL SOLVABILITY

We prove Theorem 6 by showing a reduction of FIN-INEQ-SOLV(\mathbb{R}) to POLY-INEQ-SOLV (cf. Example 3 in Section I).

A. Preliminaries

Consider an orbit-finite system of inequalities given by a matrix $\mathbf{A} : B \times C \rightarrow_{\text{fs}} \mathbb{Z}$ and a target vector $\mathbf{t} : B \rightarrow_{\text{fs}} \mathbb{Z}$.

Lemma 16. W.l.o.g. we can assume that B and C are disjoint unions of equivariant orbits $\mathbb{A}^{(k)}$, $k \in \mathbb{N}$:

$$B = \mathbb{A}^{(n_1)} \uplus \dots \uplus \mathbb{A}^{(n_s)} \quad C = \mathbb{A}^{(m_1)} \uplus \dots \uplus \mathbb{A}^{(m_r)} \quad (17)$$

(see the figure below), and that \mathbf{A} and \mathbf{t} are equivariant.

$$\mathbf{A} = \begin{matrix} & \mathbb{A}^{(m_1)} & \mathbb{A}^{(m_2)} & \dots & \mathbb{A}^{(m_r)} \\ \begin{matrix} \mathbb{A}^{(n_1)} \\ \mathbb{A}^{(n_2)} \\ \dots \\ \mathbb{A}^{(n_s)} \end{matrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} & \end{matrix} \quad \mathbf{t} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

The proof is in Appendix. Note that this includes the case of finite systems, namely $n_1 = \dots = n_s = m_1 = \dots = m_r = 0$.

B. Idea of the reduction

Suppose only finitary T -supported solutions are sought, for a fixed $T \subseteq_{\text{fin}} \mathbb{A}$. FIN-INEQ-SOLV(\mathbb{R}) reduces then to a finite system of inequalities $(\mathbf{A}', \mathbf{t}')$ obtained from (\mathbf{A}, \mathbf{t}) as follows:

- (1) Keep only columns indexed by T -tuples (= elements of *finite* T -orbits) $c \in C$, discarding all other columns.
- (2) Pick arbitrary representatives of *all* T -orbits included in B , and keep only rows of \mathbf{A} and entries of \mathbf{t} indexed by the representatives, discarding all others.

The system $(\mathbf{A}', \mathbf{t}')$ is solvable if, and only if the original one (\mathbf{A}, \mathbf{t}) has a finitary T -supported solution. Indeed, (1) is justified as a finitary T -supported solution of (\mathbf{A}, \mathbf{t}) forcedly assigns 0 to each non- T -tuple. Furthermore, (2) is justified as every solution of $(\mathbf{A}', \mathbf{t}')$ is preserved by T -atom automorphisms, and hence is a solution of the original system.

The above reduction yields no algorithm, as we do not know a priori any bound on size of support of solutions of (\mathbf{A}, \mathbf{t}) , and the size of $(\mathbf{A}', \mathbf{t}')$ depends on the number of T -orbits and hence grows unboundedly when T grows. We overcome this difficulty by using semi- T -orbits instead of T -orbits, and

relying on Lemmas 13 and 12. The latter one guarantees that the number of semi- T -orbits is constant - independent of T . Once we additionally merge (sum up) all columns indexed by elements of the same semi- T -orbit, we get \mathbf{A}' of constant size.

This still does not yield an algorithm, as entries of \mathbf{A}' change when T grows. We however crucially discover that the growth of the entries of \mathbf{A}' is *polynomial* in $n = |T|$, for sufficiently large n . Therefore, \mathbf{A}' is a matrix of polynomials in one unknown n , and solvability of (\mathbf{A}, \mathbf{t}) is equivalent to solvability of $(\mathbf{A}', \mathbf{t}')$ for some value $n \in \mathbb{N}$. As argued in Section VI, solvability is expressible in real arithmetic even if n ranges over integers and not reals.

C. Reduction of FIN-INEQ-SOLV(\mathbb{R}) to monotonic POLY-INEQ-SOLV

Let us fix an equivariant system (\mathbf{A}, \mathbf{t}) . We are going to construct a finite system P of polynomially-parametrised inequalities such that (\mathbf{A}, \mathbf{t}) has a finitary real solution if and only if $P(n)$ has a real solution for some $n \in \mathbb{N}$.

Let us denote by $d = \max\{n_1, \dots, n_s, m_1, \dots, m_r\}$ the maximal atom dimension of orbits included in B and C .

Let $T \subseteq_{\text{fin}} \mathbb{A}$ be an arbitrary finite subset of atoms. Both B and C split into semi- T -orbits, refining (17):

$$B = B_1 \uplus \dots \uplus B_N \quad C = C_1 \uplus \dots \uplus C_{M'}. \quad (18)$$

Let C_1, \dots, C_M be the *finite* semi- T -orbits among $C_1, \dots, C_{M'}$. Importantly, by Lemma 12, N and M do not depend on T as long as $|T| \geq d$. In fact $M = r$, the number of orbits included in C , as by Lemma 12 we deduce:

Lemma 17. *Assuming $|T| \geq \ell$, the equivariant orbit $\mathbb{A}^{(\ell)}$ includes exactly one finite semi- T -orbit, namely $T^{(\ell)}$.*

Let b_1, \dots, b_N be arbitrarily chosen representatives of semi- T -orbits included in B . Given \mathbf{A} and \mathbf{t} , we define the $N \times M$ matrix $\mathbf{A}(T)$ and a vector $\mathbf{t}(T) \in \mathbb{Z}^N$ as follows:

- (1) Pick N rows of \mathbf{A} , indexed by b_1, \dots, b_N , and discard other rows; this yields a matrix $\mathbf{A}'(T)$ with N rows.
- (2) Likewise pick the corresponding entries of \mathbf{t} and discard others, thus yielding a finite vector $\mathbf{t}(T) \in \mathbb{Z}^N$.
- (3) Pick columns of $\mathbf{A}'(T)$ indexed by elements of all finite semi- T -orbits included in C , and discard other columns; this yields a finite matrix $\mathbf{A}''(T)$.
- (4) Merge (sum up) columns of $\mathbf{A}''(T)$ indexed by elements of the same semi- T -orbit; this yields an $N \times M$ matrix $\mathbf{A}(T)$.

Formally, for $b \in B$ and $C_j \subseteq C$, $j \in \{1 \dots M\}$, we denote by $\sum \mathbf{A}(b, C_j)$ the finite sum

$$\sum \mathbf{A}(b, C_j) = \sum_{c \in C_j} \mathbf{A}(b, c),$$

and define the $N \times M$ matrix $\mathbf{A}(T)$ and a vector $\mathbf{t}(T) \in \mathbb{Z}^N$:

$$\mathbf{A}(T)(i, j) = \sum \mathbf{A}(b_i, C_j) \quad \mathbf{t}(T)(i) = \mathbf{t}(b_i). \quad (19)$$

Example 10. We explain how the system (5) in Example 3 in Section I is obtained. Fix a non-empty $T \subseteq_{\text{fin}} \mathbb{A}$. The set \mathbb{A}

includes two semi- T -orbits, namely T and $(\mathbb{A} \setminus T)$. Therefore the system $(\mathbf{A}(T), \mathbf{t}(T))$ has two inequalities. Only one of these semi- T -orbits is finite, namely $C_1 = T$. Therefore the system has just one unknown. Pick arbitrary representatives of the semi- T -orbits, $b_1 \in T$ and $b_2 \in (\mathbb{A} \setminus T)$. We have

$$\begin{aligned} \mathbf{A}(T)(1, 1) &= \sum_{c \in T} \mathbf{A}(b_1, c) = |T| - 1 \\ \mathbf{A}(T)(2, 1) &= \sum_{c \in T} \mathbf{A}(b_2, c) = |T|. \end{aligned}$$

Replacing $|T|$ with n yields the system (5). \triangleleft

The choice of representatives b_i is irrelevant, and therefore $\mathbf{A}(T)$ and $\mathbf{t}(T)$ are well defined, due to (proof in Appendix):

Lemma 18. *If $b, b' \in B$ are in the same semi- T -orbit, then $\mathbf{t}(b) = \mathbf{t}(b')$ and $\sum \mathbf{A}(b, C_j) = \sum \mathbf{A}(b', C_j)$ for every $j \in \{1 \dots M\}$.*

Lemma 19. *(\mathbf{A}, \mathbf{t}) has a finitary T -supported real solution if, and only if $P(T) = (\mathbf{A}(T), \mathbf{t}(T))$ has a real solution.*

Proof. In one direction, assuming existence of a finitary T -supported real solution of (\mathbf{A}, \mathbf{t}) , we take any finitary semi- T -supported real solution $\mathbf{x}' : C \rightarrow_{\text{fin}} \mathbb{R}$ (as guaranteed by Lemma 13). It is automatically a solution of the system $(\mathbf{A}'(T), \mathbf{t}(T))$, as the system is obtained by discarding inequalities. As \mathbf{x}' is finitary, we have $\mathbf{x}'(c) = 0$ for all $c \notin C' = C_1 \uplus \dots \uplus C_M$. Therefore the restriction \mathbf{x}'' of \mathbf{x}' to C' is a solution of $(\mathbf{A}''(T), \mathbf{t}(T))$, since

$$\mathbf{A}'(T) \cdot \mathbf{x}' = \mathbf{A}''(T) \cdot \mathbf{x}'' \quad (20)$$

Furthermore, \mathbf{x}'' is constant on every C_j (the value on elements of C_j we denote as $\mathbf{x}''(C_j)$). Therefore, the vector $\mathbf{x} : \{1 \dots M\} \rightarrow \mathbb{R}$ defined by $\mathbf{x}(j) = \mathbf{x}''(C_j)$ is a solution of $(\mathbf{A}(T), \mathbf{t}(T))$, since the definition (19) implies

$$\mathbf{A}''(T) \cdot \mathbf{x}'' = \mathbf{A}(T) \cdot \mathbf{x}. \quad (21)$$

In the opposite direction, take an arbitrary solution $\mathbf{x} : \{1 \dots M\} \rightarrow \mathbb{R}$ of $P(T) = (\mathbf{A}(T), \mathbf{t}(T))$. We define a finitary vector \mathbf{x}' that maps each element of each finite semi- T -orbit C_j to $\mathbf{x}(j)$. By definition, this vector is semi- T -supported, and hence also T -supported; the equalities (20) and (21) holds again, and hence \mathbf{x}' and is a solution of $(\mathbf{A}'(T), \mathbf{t}(T))$. In order to prove that \mathbf{x}' is also a solution of (\mathbf{A}, \mathbf{t}) we recall that each inequality in this system is obtained by applying a semi- T -atom automorphism π to some inequality e in $(\mathbf{A}'(T), \mathbf{t}(T))$. As \mathbf{x}' solves e , $\pi(\mathbf{x}')$ solves $\pi(e)$ by equivariance of (\mathbf{A}, \mathbf{t}) . Finally, as $\pi(\mathbf{x}') = \mathbf{x}'$, we deduce that \mathbf{x}' solves $\pi(e)$. As the semi- T -atom automorphism π was chosen arbitrarily, \mathbf{x}' is a solution of (\mathbf{A}, \mathbf{t}) . \square

As an immediate corollary of Lemma 19 we get:

Lemma 20. *If $P(T)$ has a real solution and $T \subseteq T'$ then $P(T')$ has a real solution as well.*

The function $T \mapsto P(T)$ is equivariant, i.e., invariant under action of atom automorphisms. In consequence, the entries of

$\mathbf{A}(T)$ and $\mathbf{t}(T)$ do not depend on the set T itself, but only on its size $|T|$. Indeed, if $|T| = |T'|$ then $\pi(T) = T'$ for some atom automorphism π , and hence $\pi(P(T)) = P(T')$. Since the system $P(T)$ is atom-less we have also $\pi(P(T)) = P(T)$, which implies $P(T) = P(T')$. We may thus meaningfully write $P(|T|) = (\mathbf{A}(|T|), \mathbf{t}(|T|))$, i.e., $P(n) = (\mathbf{A}(n), \mathbf{t}(n))$ for $n \in \mathbb{N}$ (cf. Example 10).

It remains to argue that the dependence on $|T|$ is polynomial, as long as $|T| \geq 2d$:

Lemma 21. *There are univariate polynomials $p_{ij}(n) \in \mathbb{Z}[n]$ such that $\mathbf{A}(n)(i, j) = p_{ij}(n)$ for $n \geq 2d$.*

Proof. Let $n = |T|$. Fix two semi- T -orbits B_i and C_j included in B and C , respectively. Each of them is included in a unique equivariant orbit, say:

$$B_i \subseteq B' = \mathbb{A}^{(p)} \quad C_j \subseteq C' = \mathbb{A}^{(\ell)}$$

(cf. the partitions (17)). Recall Lemma 12: B_i is determined by the subset $I \subseteq \{1 \dots p\}$ of positions where atoms of T appear in tuples belonging to B_i . Let $m = |I|$. On the other hand $C_j = T^{(\ell)}$ (cf. Lemma 17). Note that $m = |T \cap \text{sup}(b_i)|$.

We are going to demonstrate that the value $\sum \mathbf{A}(b_i, C_j)$ is polynomially depending on $n = |T|$. We will use the polynomials $q_{v,w}$ of degree w , for $v, w \leq d$, defined by

$$q_{v,w}(n) = (n-v) \cdot (n-v-1) \cdot \dots \cdot (n-v-w+1). \quad (22)$$

The value $q_{v,w}(n)$ can be interpreted as follows:

Claim 21.1. *For $n - v \geq w$, $q_{v,w}(n)$ is equal to the number of arrangements of w items chosen from $n - v$ objects.*

Denote by \mathcal{D} the set of equivariant orbits $U \subseteq B' \times C'$. For $U \in \mathcal{D}$, we put $U(b_i) := \{c \in C_j \mid (b_i, c) \in U\}$. As \mathbf{A} is equivariant, the value $\mathbf{A}(b_i, c)$ depends only on the orbit to which (b_i, c) belongs. We write $\mathbf{A}(U)$, for $U \in \mathcal{D}$, and get:

Claim 21.2. $\sum \mathbf{A}(b_i, C_j) = \sum_{U \in \mathcal{D}} \mathbf{A}(U) \cdot |U(b_i)|$.

By Lemma 1 in Section II, orbits $U \subseteq B' \times C'$ are in one-to-one correspondence with partial injections $\iota : \{1 \dots p\} \rightarrow \{1 \dots \ell\}$. We write U_ι for the orbit corresponding to ι . Let $\text{dom}(\iota) = \{x \mid \iota(x) \text{ is defined}\}$ denote the domain of ι .

Claim 21.3. $U_\iota(b_i) \neq \emptyset$ if, and only if $\text{dom}(\iota) \subseteq I$.

Indeed, recall again Lemma 1 which yields

$$U_\iota(b_i) = \{c \in C_j \mid \forall x, y : b_i(x) = c(y) \iff \iota(x) = y\}.$$

If $\text{dom}(\iota) \subseteq I$, the set $U_\iota(b_i)$ contains tuples $c \in C_j$ with fixed values on positions $J = \{\iota(x) \mid x \in \text{dom}(\iota)\}$, namely

$$b_i(x) = c(\iota(x)), \quad (23)$$

and arbitrary other atoms from T elsewhere, and therefore is nonempty. If there is $x \in \text{dom}(\iota) \setminus I$ then $b_i(x) \notin T$ and therefore no $c \in C_j$ satisfies (23). Claim 21.3 is thus proved.

Claim 21.4. *Let $k = |\text{dom}(\iota)|$ be the number of pairs related by ι . If $U_\iota \neq \emptyset$ then $|U_\iota(b_i)| = q_{m, \ell-k}(n)$.*

According to (23), tuples $c \in U_\iota(b_i)$ have fixed values on k positions in J . The remaining $\ell - k$ positions in tuples $c \in U_\iota(b_i)$ are filled arbitrarily using $n - m$ atoms from $T \setminus \text{sup}(b_i)$. Due to the assumption that $n \geq 2d$, we have $n - m \geq d \geq \ell - k$, and therefore using Claim 21.1 (for $w = \ell - k$) we deduce $|U_\iota(b_i)| = q_{m, \ell-k}(n)$, thus proving Claim 21.4.

Once $b_i \in B_i$ and $U \in \mathcal{D}$ are fixed, the values k, ℓ and m are fixed too, and the formula of Claim 21.4 is an univariate polynomial of degree $\ell - k$. The formula of Claim 21.2 yields the required polynomial⁶ $\mathbf{A}(n)(i, j) = p_{ij}(n)$ and hence the proof of Lemma 21 is completed. \square

Relying on Lemma 21 we get a polynomially-parametrised system $P(n) = (\mathbf{A}(n), \mathbf{t}(n))$. It remains to argue that:

Lemma 22. *The system P is computable from (\mathbf{A}, \mathbf{t}) .*

Proof. Indeed, it is enough to range over representations of semi- T -orbits B_i and C_j of B and C , respectively (such representations are given by Lemma 12), and for each pair of such orbits proceed with computations outlined in the proof of Lemma 21, applied to an arbitrarily chosen representative $b_i \in B_i$. \square

The system P is monotonic due to Lemma 20. By Lemma 19, (\mathbf{A}, \mathbf{t}) has a finitary real solution if, and only if $P(n)$ has a real solution for some integer $n \geq 2d$. As the last step, by substituting $n + 2d$ in place of n to all polynomials appearing in P , we get a system P' such that $P(n)$ has a real solution for some integer $n \geq 2d$ if, and only if $P'(n)$ has a real solution for some $n \in \mathbb{N}$. Therefore FIN-INEQ-SOLV(\mathbb{R}) reduces to monotonic POLY-INEQ-SOLV.

Remark 1 (Complexity). The size of the polynomially-parametrised system depends exponentially on atom dimension d of (\mathbf{A}, \mathbf{t}) , but polynomially on the number of orbits included in $B \times C$. In consequence, for fixed atom dimension we get a polynomial-timed reduction, and hence the decision procedure for FIN-INEQ-SOLV(\mathbb{R}) in EXPTIME. Without fixing atom dimension, the procedure is in 2-EXPTIME. The same applies to the solution of the optimisation problem in Section VIII. \triangleleft

VIII. OPTIMISATION PROBLEMS

In this section we prove Theorem 11: we introduce a maximisation variant of POLY-INEQ-SOLV and adapt of the reduction of Section VII-C to the maximisation setting.

A. Polynomially parametrized maximisation problem

We naturally consider a maximisation problem, whose instance $(P, S) = (P(n), S(n))$ consists of a finite system of polynomially-parametrized inequalities P as in (14), plus an *objective* function S . The objective function is a linear map which, similarly as P , is assumed to be polynomially-parametrized by n ranging over non-negative integers:

$$S(n, x_1, \dots, x_k) = p_1(n) \cdot x_1 + \dots + p_k(n) \cdot x_k. \quad (24)$$

⁶This confirms, in particular, that $\mathbf{A}(T)(i, j)$ is independent from the actual set T , and only depend on its size $n = |T|$.

Given $n \in \mathbb{R}$ (not necessarily $n \in \mathbb{N}$), we define $s_n \in \mathbb{R} \cup \{-\infty, +\infty\}$ as the supremum of the objective function over all solutions of the system $P(n)$:

$$s_n = \text{supremum of } \{S(n, \vec{x}) \mid \vec{x} \text{ is a solution of } P(n)\},$$

under a proviso that $s_n = -\infty$ if the solution set of $P(n)$ is empty. In the following we apply the convention that $-\infty < r < +\infty$ for all $r \in \mathbb{R}$. An instance (P, S) is called *monotonic* if its supremum sequence $(s_n)_{n \in \mathbb{N}}$ is non-decreasing:

$$s_n \leq s_{n+1} \text{ for every } n \in \mathbb{N}.$$

In consequence, for monotonic instances (P, S) the limit $\lim_{n \in \mathbb{N}} s_n$ is always defined and belongs to $\mathbb{R} \cup \{-\infty, +\infty\}$. The value of $\lim_{n \in \mathbb{N}} s_n$ we call the *supremum* of (P, S) .

For the sake of formulating a computation problem, we use the following representation of real numbers: a number $r \in \mathbb{R}$ is *represented* by a real arithmetic formula $\varphi(x)$ with one free variable if the formula defines the singleton set $\{r\}$.⁷

POLY-INEQ-MAX:

Input: A monotonic instance (P, S) .

Output: (Representation of) the supremum of (P, S) .

The following result is a strengthening of Theorem 15:

Theorem 23. POLY-INEQ-MAX is in EXPTIME.

Proof. Consider a fixed input (P, S) . Let $\sigma_P(n, \vec{x})$ be the quantifier-free arithmetic formula (15) saying that \vec{x} is a solution of $P(n)$. We start by observing that the set $\{(n, m) \in \mathbb{R}^2 \mid s_n \geq m\}$ is definable in real arithmetic (cf. (16) in the proof of Theorem 15)⁸:

$$s_n \geq m \equiv \forall \ell < m : \exists \vec{x} : \sigma_P(n, \vec{x}) \wedge S(n, \vec{x}) > \ell. \quad (25)$$

This allows us to define also " $s_n < m$ " using negation, and " $s_n \leq m$ " $\equiv \forall \ell > m : s_n < \ell$. Then we formulate two useful claims (the proofs are immediate using Lemma 14):

Claim 23.1. $\lim_{n \in \mathbb{N}} s_n = +\infty$ (resp. $-\infty$) if, and only if the following formula evaluates to true:

$$\forall m : \forall^\omega n : s_n \geq m \quad (\text{resp. } \forall m : \forall^\omega n : s_n < m).$$

Claim 23.2. If $\lim_{n \in \mathbb{N}} s_n \in \mathbb{R}$ then it is definable by the formula $\varphi(m) \equiv \forall \ell < m : \forall^\omega n : \ell \leq s_n \leq m$.

The procedure to compute the supremum of (P, S) simply invokes the EXPTIME decision procedure for real arithmetic formulas of bounded quantifier alternation depth [14, Theorem 14.16]. If any of the two formulas of Claim 23.1 evaluates to true, the procedure returns $+\infty$ (resp. $-\infty$), otherwise the procedure returns the formula of Claim 23.2. \square

Interestingly, the supremum can not be irrational (the proof is in Appendix):

Theorem 24. The supremum of a monotonic instance (P, S) belongs to $\mathbb{Q} \cup \{-\infty, +\infty\}$.

⁷Using quantifier elimination, the formula can be further transformed to the standard representation as a pair (p, i) , where p is a univariate polynomial and i a positive integer, such that r is the i th real root of p .

⁸Notably, this approach works for strict inequalities as well.

B. Reduction of FIN-INEQ-MAX(\mathbb{R}) to POLY-INEQ-MAX

We only sketch the reduction as it amounts to slightly adapting the reduction of Section VII-C. The input of FIN-INEQ-MAX(\mathbb{R}) consists of a system (\mathbf{A}, \mathbf{t}) and an integer vector $\mathbf{s} : C \rightarrow_{\text{fs}} \mathbb{Z}$ representing the objective function

$$S(\mathbf{x}) = \mathbf{s} \cdot \mathbf{x},$$

and we ask for the supremum of values $S(\mathbf{x})$, for \mathbf{x} ranging over finitary solutions of (\mathbf{A}, \mathbf{t}) .

In addition to Lemma 16 we show (in Appendix):

Lemma 25. W.l.o.g. we may assume that \mathbf{s} is equivariant.

We proceed by adapting the reduction of Section VII-C: given an instance $(\mathbf{A}, \mathbf{t}, \mathbf{s})$ of FIN-INEQ-MAX(\mathbb{R}) we compute an instance (P', S') of POLY-INEQ-MAX with the same supremum. The finite system $P'(n) = (\mathbf{A}'(n), \mathbf{t}'(n))$ of polynomially-parametrised inequalities is exactly as in Section VII-C. Furthermore, we compute the objective function

$$S'(n, x_1, \dots, x_k) = p_1(n) \cdot x_1 + \dots + p_M(n) \cdot x_M \quad (26)$$

as follows. As \mathbf{s} is equivariant, it necessarily assigns the same value $\mathbf{s}(C_j)$ to all elements of C_j . Assuming $|T| = n \geq 2d$ ($n \geq d$ is sufficient here), the finite semi- T -orbit $C_j = T^{(\ell)}$ contains $q_{0,\ell}(n)$ elements, which allows us to define $p_j(n)$ as

$$p_j(n) = \mathbf{s}(C_j) \cdot q_{0,\ell}(n)$$

(cf. Claims 21.2 and 21.4, and Example 10 in Section VII-C). With this S' we extend the proof of Lemma 19 to obtain:

Lemma 26. Let $n = |T|$. The values of S on finitary T -supported real solutions of (\mathbf{A}, \mathbf{t}) are the same as the values of $S'(n)$ on real solutions of $P'(n)$.

Lemma 27. The instance (P', S') is monotonic.

(Proofs in Appendix.) This completes the reduction of FIN-INEQ-MAX(\mathbb{R}) to monotonic POLY-INEQ-MAX.

Example 11. For illustration of the reduction, let the row and column indexing sets be

$$B = \mathbb{A}^{(2)} \uplus \mathbb{A}^{(0)} \quad C = \mathbb{A}^{(2)} \uplus \mathbb{A}$$

(we identify $\mathbb{A}^{(1)}$ with just \mathbb{A} , and note that $\mathbb{A}^{(0)} = \{\varepsilon\}$ is a singleton), and suppose we maximise the objective function

$$S(\mathbf{x}) = 3 \cdot \mathbf{1}_{\mathbb{A}^{(2)}} \cdot \mathbf{x} = 3 \cdot \sum_{\alpha\beta \in \mathbb{A}^{(2)}} \mathbf{x}(\alpha\beta)$$

over finitary solutions $\mathbf{x} : \mathbb{A}^{(2)} \uplus \mathbb{A} \rightarrow_{\text{fin}} \mathbb{R}$ of the following system of inequalities (\geq is reversed for convenience):

$$\begin{aligned} \alpha\beta &\leq 2\alpha + 2\beta & (\alpha\beta \in \mathbb{A}^{(2)}) \\ \sum_{\alpha \in \mathbb{A}} \alpha &\leq 1. \end{aligned} \quad (27)$$

Intuitively, unknowns correspond to vertices α and edges $\alpha\beta$ of an infinite directed clique, and the value assigned by a solution of (27) to an edge $\alpha\beta$ should be at most twice the sum

of values assigned to its ends. In matrix form $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{t}$, where \mathbf{A} and \mathbf{t} split into blocks corresponding to orbits included in B and C as follows:

$$\begin{array}{c} \mathbb{A}^{(2)} \quad \quad \mathbb{A} \\ \mathbb{A}^{(2)} \left[\begin{array}{ccc|ccc} 1 & & & -2 & -2 & \\ & 1 & & & -2 & -2 \\ & & \ddots & & & \\ 0 & \dots & 0 & 1 & \dots & 1 \end{array} \right] \quad \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\ \mathbb{A}^{(0)} \end{array}$$

According to Lemma 12, for any $T \subseteq_{\text{fin}} \mathbb{A}$ the set B includes 5 semi- T -orbits: 4 of them included in $\mathbb{A}^{(2)}$:

$$T^{(2)} \quad T \times (\mathbb{A} \setminus T) \quad (\mathbb{A} \setminus T) \times T \quad (\mathbb{A} \setminus T)^2 \quad (28)$$

and the remaining one being the singleton $\mathbb{A}^{(0)} = \{\varepsilon\}$. Therefore the reduction produces a system of 5 inequalities. The set C includes exactly 2 finite semi- T -orbits, namely $T^{(2)} \subseteq \mathbb{A}^{(2)}$ and $T \subseteq \mathbb{A}$. Let x_2 and x_1 be the corresponding unknowns. Here are the inequalities: the four ones on the left corresponding to the semi- T -orbits (28), and the one on the right corresponding the singleton one:

$$\begin{aligned} x_2 - 4x_1 &\leq 0 & nx_1 &\leq 1 \\ -2x_1 &\leq 0 \\ -2x_1 &\leq 0 \\ 0 &\leq 0 \end{aligned} \quad (29)$$

For instance, the coefficient -4 in the first one arises as:

$$\mathbf{A}(U_1) \cdot |U_1(\alpha\beta)| + \mathbf{A}(U_2) \cdot |U_2(\alpha\beta)| = -2 \cdot 1 - 2 \cdot 1 = -4$$

(cf. Claim 21.2), for some arbitrary $\alpha\beta \in T^{(2)}$ and the following two orbits included in $\mathbb{A}^{(2)} \times \mathbb{A}$:

$$U_1 = \{(\alpha\beta, \alpha) \mid \alpha\beta \in \mathbb{A}^{(2)}\}, \quad U_2 = \{(\alpha\beta, \beta) \mid \alpha\beta \in \mathbb{A}^{(2)}\}.$$

We omit $\mathbf{A}(U_3) \cdot |U_3(\alpha\beta)|$ above, as $A(U_3) = 0$ for the orbit

$$U_3 = \{(\alpha\beta, \gamma) \mid \alpha\beta \in \mathbb{A}^{(2)}, \gamma \notin \{\alpha, \beta\}\}.$$

Likewise, the coefficient $n = |T|$ in the last one arises as

$$\mathbf{A}(U) \cdot |O(\varepsilon)| = 1 \cdot n = n,$$

for $\varepsilon \in \mathbb{A}^{(0)}$ and the orbit $U = \{\varepsilon\} \times \mathbb{A}$. Finally, the objective function (26) produced by the reduction is

$$S'(x_2, x_1) = S(\mathbb{A}^{(2)}) \cdot q_{0,2}(n) \cdot x_2 = 3 \cdot \frac{n(n-1)}{2} \cdot x_2$$

and it achieves arbitrarily large values when n grows, since the solutions of (29) are only restricted by $x_2 \leq 4x_1 \leq \frac{4}{n}$. \triangleleft

IX. UNDECIDABILITY OF INTEGER SOLVABILITY

We prove Theorem 7 by showing undecidability of FIN-INEQ-SOLV(\mathbb{Z}). We proceed by reduction from the reachability problem of counter machines.

We conveniently define a d -counter machine as a finite set of instructions I , where each instruction is a function

$$i : \{1 \dots d\} \rightarrow \mathbb{Z} \cup \{\text{ZERO}\}$$

that specifies, for each counter $k \in \{1, \dots, d\}$, either the additive update of k (if $i(k) \in \mathbb{Z}$) or the zero-test of k (if $i(k) = \text{ZERO}$). Configurations of M are nonnegative vectors $c \in \mathbb{N}^d$, and each instruction induces steps between configurations: $c \xrightarrow{i} c'$ if $c'(k) = c(k) + i(k)$ whenever $i(k) \in \mathbb{N}$, and $c'(k) = c(k) = 0$ whenever $i(k) = \text{ZERO}$. A run of M is defined as a finite sequence of steps

$$c_0 \xrightarrow{i_1} c_1 \xrightarrow{i_2} \dots \xrightarrow{i_n} c_n. \quad (30)$$

The reachability problem asks, given a machine M and two its configurations, a source c_0 and a target c_f , if M admits a run from c_0 to c_f . The problem is undecidable, as counter machines can easily simulate classical Minsky machines.⁹

For $k \in \{1, \dots, d\}$ we denote by $\text{ZERO}(k) = \{i \in I \mid i(k) = \text{ZERO}\}$ the set of instructions that zero-test counter k , and $\text{UPD}(k) = \{i \in I \mid i(k) \in \mathbb{Z}\}$ the set of instructions that update counter k .

Given a d -counter machine M and two configurations c_0, c_f , we construct an orbit-finite system of inequalities $S = (\mathbf{A}, \mathbf{t})$ such that M admits a run from c_0 to c_f if and only if S has a finitary nonnegative integer solution. (Nonnegativeness is not an additional constraint, as it may be enforced by adding inequalities $x \geq 0$ for all unknowns x .) We describe construction of S gradually, on the way giving intuitive explanations and sketching the proof of the if direction.

The system S has unknowns $e_{\alpha\beta}$ indexed by pairs of distinct atoms $\alpha\beta \in \mathbb{A}^{(2)}$, and contains the following inequalities:

$$e_{\alpha\beta} \leq 1 \quad (\alpha\beta \in \mathbb{A}^{(2)}). \quad (31)$$

Therefore, in every solution the unknowns $e_{\alpha\beta}$ define a directed graph G , where atoms are vertices, $e_{\alpha\beta} = 1$ encodes an edge from α to β and $e_{\alpha\beta} = 0$ encodes a non-edge. In case of a finitary solution, the graph G is finite (when atoms with no adjacent edges are dropped). Let us fix two distinct atoms $\iota, \zeta \in \mathbb{A}$. The system S contains the following further equations and inequalities:

$$\begin{aligned} \sum_{\beta \neq \alpha} e_{\beta\alpha} &= \sum_{\beta \neq \alpha} e_{\alpha\beta} \leq 1 & (\alpha \in \mathbb{A} \setminus \{\iota, \zeta\}) \\ \sum_{\beta \neq \iota} e_{\beta\iota} &= 0 & \sum_{\beta \neq \iota} e_{\iota\beta} = 1 \\ \sum_{\beta \neq \zeta} e_{\beta\zeta} &= 1 & \sum_{\beta \neq \zeta} e_{\zeta\beta} = 0 \end{aligned} \quad (32)$$

enforcing that in-degree of every vertex, except for ι and ζ , is the same as its out-degree, and equal 0 or 1. Furthermore, in-degree of ι and out-degree of ζ are 0, while out-degree of ι and in-degree of ζ are 1. Thus atoms split into three categories: inner nodes (with in- and out-degree equal 1), end nodes (ι and ζ) and non-nodes (with in- and out-degree equal 0). Therefore, the graph G defined by a finitary solution forcedly consists of a directed path from ι to ζ plus a number of vertex disjoint directed cycles. The path will be used below to encode a run

⁹The model of d -counter machines resembles vector addition systems with zero tests, and has undecidable reachability already for $d = 5$.

of M : each its edge, intuitively speaking, will be assigned a configuration of M , while each its inner node will be assigned an instruction of M .

The system S has also unknowns $t_{i\alpha}$ indexed by instructions $i \in I$ of M and atoms $\alpha \in \mathbb{A}$, and the following equations:

$$\sum_{i \in I} t_{i\alpha} = \sum_{\beta \neq \alpha} e_{\alpha\beta} \quad (\alpha \in \mathbb{A} \setminus \{\iota, \zeta\}). \quad (33)$$

Therefore in every finitary solution, for each inner node α of the above-defined graph G , there is exactly one instruction $i \in I$ such that $t_{i\alpha}$ equals 1 (intuitively, this instruction i is assigned to node α), and $t_{i\alpha}$ equals 0 for all other instructions. (This applies to *all* inner nodes of G , both those on the path as well as those on cycles.) For non-nodes α , all $t_{i\alpha}$ are necessarily equal 0. Note that the values of unknowns $t_{i\iota}$ and $t_{i\zeta}$ are unrestricted, as they are irrelevant.

Finally, the system S contains unknowns $c_{\alpha\beta\gamma k}$ indexed by $\alpha\beta\gamma \in \mathbb{A}^{(3)}$ and $k \in \{1 \dots d\}$. The following inequalities:

$$c_{\alpha\beta\gamma k} \leq e_{\alpha\beta} \quad (\alpha\beta\gamma \in \mathbb{A}^{(3)}, k \in \{1 \dots d\}) \quad (34)$$

would enforce that, whatever atom γ is, the value of unknown $c_{\alpha\beta\gamma k}$ may be 0 or 1 when $\alpha\beta$ is an edge (i.e., when $e_{\alpha\beta} = 1$), but is forcedly 0 when $\alpha\beta$ is a non-edge (i.e., when $e_{\alpha\beta} = 0$). Intuitively, we represent the k th coordinate of the configuration assigned to the edge $\alpha\beta$ by the (necessarily finite) sum

$$\sum_{\gamma \notin \{\alpha, \beta\}} c_{\alpha\beta\gamma k}. \quad (35)$$

In order to deal with zero tests, we add to S not just the inequalities (34), but the following strengthening thereof:

$$c_{\alpha\beta\gamma k} + \sum_{i \in \text{ZERO}(k)} t_{i\alpha} \leq e_{\alpha\beta} \quad (\alpha\beta\gamma \in \mathbb{A}^{(3)}, k \in \{1 \dots d\}). \quad (36)$$

In consequence, for every edge $\alpha\beta$, if the instruction i executed at α zero-tests counter k , the sum

$$\sum_{i \in \text{ZERO}(k)} t_{i\alpha}$$

equals 1 and therefore the k th coordinate of the configuration outgoing from α , encoded by (35), is necessarily 0 (the same applies also to the configuration incomming to α , due to inequalities (37) below). Otherwise, if the instruction i does not zero-test counter k , the k th coordinate of the configuration is unrestricted. (Again, this applies to *all* edges of G , both those on the path as well as those on cycles.) As a further consequence, for a non-edge $\alpha\beta$, the configuration stored at $\alpha\beta$, encoded by (35), is necessarily the zero configuration.

Recall that, due to (32) and (34), for every $\alpha \in \mathbb{A} \setminus \{\iota, \zeta\}$, unknowns $c_{\beta\alpha\gamma k}$ may be positive for at most one $\beta \in \mathbb{A}$; likewise unknowns $c_{\alpha\beta\gamma k}$. In order to enforce correctness of encoding of a run of M , we add to S the following equations:

$$\sum_{\beta, \gamma \neq \alpha} c_{\beta\alpha\gamma k} + \sum_{i \in \text{UPD}(k)} i(k) \cdot t_{i\alpha} = \sum_{\beta, \gamma \neq \alpha} c_{\alpha\beta\gamma k} \quad (37)$$

($\alpha \in \mathbb{A} \setminus \{\iota, \zeta\}, k \in \{1 \dots d\}$). These equalities say that for every inner node or non-node α (i.e., every atom except for the

end nodes ι and ζ), on every coordinate k , the configuration incomming to α differs from the configuration outgoing from α exactly by the sum

$$\sum_{i \in \text{UPD}(k)} i(k) \cdot t_{i\alpha}$$

ranging over those instructions i of M that update counter k . We can safely further restrict the range of i to those instructions for which $t_{i\alpha} \neq 0$. Remembering that for each α there is at most one instruction i satisfying the latter requirement, we get that the configurations differ on coordinate k by exactly $i(k)$ (if i updates counter k) or the configuration are equal on coordinate k (if i zero-tests counter k , or there is no instruction i such that $t_{i\alpha} \neq 0$).

As the last bit we add to S the requirement that the configuration assigned to the edge outgoing from ι is the source one c_0 , and the configuration assigned to the edge incomming to ζ is the target one c_f :

$$\begin{aligned} \sum_{\beta, \gamma \neq \iota} c_{\iota\beta\gamma k} &= c_0(k) & (k \in \{1, \dots, d\}) \\ \sum_{\beta, \gamma \neq \zeta} c_{\beta\zeta\gamma k} &= c_f(k) & (k \in \{1, \dots, d\}). \end{aligned} \quad (38)$$

The construction of S is thus completed, and it remains to argue towards its correctness:

Lemma 28. *M admits a run from c_0 to c_f if and only if S has a finitary nonnegative integer solution.*

Remark 2. The proof does not adapt to FIN-NONNEG-EQ-SOLV(\mathbb{Z}). Indeed, the standard way of transforming inequalities into equations involves adding an infinite set of additional unknowns, that may be all non-zero in a solution. \triangleleft

X. CONCLUSIONS

As two main contributions, we show two contrasting results: decidability of orbit-finite linear programming, and undecidability of orbit-finite integer linear programming. For decidability, we invent a novel concept of semi-orbit, and provide a reduction to a finite but polynomially-parametrised linear programming problem. We consider non-strict inequalities for simplicity of presentation only, as our decision procedure, relying on real arithmetic, may be applied to mixed systems of strict and non-strict inequalities equally well.

Concerning integer linear programming, an intriguing research task is to identify the decidability borderline. We suspect decidability in case when all inequalities are finitary, and in case of atom dimension 1 (along the lines of [4]).

In this paper only finitely supported solutions are considered. We do not know the decidability status of linear programming when this restriction is dropped (like in [19]). It is decidable for finitary inequalities, where existence of a solution implies existence of an equivariant one [22].

In this paper we exclusively consider equality atoms, and extension to richer structures seems highly non-trivial. Our initial results suggest however decidability of EQ-SOLV(\mathbb{F}), for any commutative ring \mathbb{F} , over ordered atoms.

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APPENDIX

A. Spans

We introduce notation useful in proving Theorems 5 and 10. For subsets $P \subseteq \text{LIN}(B)$ and $\mathbb{F} \subseteq \mathbb{R}$, we define $\text{FIN-SPAN}_{\mathbb{F}}(P) \subseteq \text{LIN}(B)$ as the set of all linear \mathbb{F} -combinations of vectors from P :

$$\text{FIN-SPAN}_{\mathbb{F}}(P) = \{ q_1 \cdot \mathbf{p}_1 + \dots + q_k \cdot \mathbf{p}_k \mid k \geq 0, q_1, \dots, q_k \in \mathbb{F}, \mathbf{p}_1, \dots, \mathbf{p}_k \in P \}.$$

Recall that given a matrix $\mathbf{A} \in \text{LIN}(B \times C)$ with rows B and columns C , we can define a partial operation of multiplication of \mathbf{A} by a vector $\mathbf{v} \in \text{LIN}(C)$ in an expected way:

$$(\mathbf{A} \cdot \mathbf{v})(b) = \mathbf{A}(b, _) \cdot \mathbf{v}$$

for every $b \in B$. The result $\mathbf{A} \cdot \mathbf{v} \in \text{LIN}(B)$ is well-defined if $\mathbf{A}(b, _) \cdot \mathbf{v}$ is well-defined for all $b \in B$. For $c \in C$ we denote by $\mathbf{A}(_, c) \in \text{LIN}(B)$ the corresponding (column) vector. The multiplication $\mathbf{A} \cdot \mathbf{v}$ can be also seen as an *orbit-finite* linear combination of column vectors $\mathbf{A}(_, c)$, for $c \in C$, with coefficients given by \mathbf{v} . This allows us to define the *span* of \mathbf{A} seen as a C -indexed orbit-finite set of vectors $\mathbf{A}(_, c) \in \text{LIN}(B)$:

$$\text{SPAN}_{\mathbb{F}}(\mathbf{A}) := \{ \mathbf{A} \cdot \mathbf{v} \mid \mathbf{v} : C \rightarrow_{\text{fs}} \mathbb{F}, \mathbf{A} \cdot \mathbf{v} \text{ well-def.} \}.$$

Therefore, a system of inequalities (\mathbf{A}, \mathbf{t}) has a solution if $\text{SPAN}_{\mathbb{F}}(\mathbf{A})$ contains some vector $\mathbf{u} \geq \mathbf{t}$. When \mathbf{v} is finitary, well-definedness is vacuous, and we may define:

$$\text{FIN-SPAN}_{\mathbb{F}}(\mathbf{A}) := \{ \mathbf{A} \cdot \mathbf{v} \mid \mathbf{v} : C \rightarrow_{\text{fin}} \mathbb{F} \} = \text{FIN-SPAN}_{\mathbb{F}}(P)$$

for $P = \{ \mathbf{A}(_, c) \mid c \in C \}$ the set of column vectors of \mathbf{A} . Therefore, a system of inequalities (\mathbf{A}, \mathbf{t}) has a finitary solution if $\text{FIN-SPAN}_{\mathbb{F}}(\mathbf{A})$ contains some vector $\mathbf{u} \geq \mathbf{t}$.

B. Proofs missing in Section III

Proof of Theorems 5 and 10. Recall that we consider supremum of a maximisation problem to be $-\infty$ if the constraints in the problem are infeasible. Therefore proving that two maximisation problems have the same supremum also proves that the underlying systems of inequalities are equisolvable. In consequence, Theorem 10 implies 5, and we may concentrate in the sequel on proving the former one.

The proof of mutual reductions between $\text{INEQ-MAX}(\mathbb{F})$ and $\text{NONNEG-EQ-MAX}(\mathbb{F})$ amounts to lifting of standard arguments from finite to orbit-finite systems, and checking that all constructed objects are finitely supported. We include the reductions here mostly in order to get acquainted with orbit-finite systems. One of the remaining two reductions builds on results of [8].

Reduction of $\text{INEQ-MAX}(\mathbb{F})$ to $\text{NONNEG-EQ-MAX}(\mathbb{F})$: Consider an instance $(\mathbf{A}, \mathbf{t}, \mathbf{s})$ of $\text{INEQ-MAX}(\mathbb{F})$, supported by S , where $\mathbf{A} : B \times C \rightarrow_{\text{fs}} \mathbb{F}$, $\mathbf{t} : B \rightarrow_{\text{fs}} \mathbb{F}$ and $\mathbf{s} : C \rightarrow_{\text{fs}} \mathbb{F}$. We construct an instance $(\mathbf{A}', \mathbf{t}, \mathbf{s}')$ of $\text{NONNEG-EQ-MAX}(\mathbb{F})$, with the same target vector \mathbf{t} , and $\mathbf{A}' : B \times (C \uplus C \uplus B) \rightarrow_{\text{fs}} \mathbb{F}$, $\mathbf{s}' : (C \uplus C \uplus B) \rightarrow_{\text{fs}} \mathbb{F}$, such that

$$\text{supremum}(\mathbf{A}', \mathbf{t}, \mathbf{s}') = \text{supremum}(\mathbf{A}, \mathbf{t}, \mathbf{s}).$$

In the new system, we double each variable x into x_+ and x_- , and we add a fresh variable per each equation. The matrix \mathbf{A}' of the new system is a composition of \mathbf{A} , $-\mathbf{A}$, and the diagonal matrix $B \times B \rightarrow_{\text{fs}} \mathbb{F}$ with -1 in the diagonal:

$$\mathbf{A}' = \left[\begin{array}{c|c|c} & & -1 \\ \mathbf{A} & -\mathbf{A} & \\ \hline & & \ddots \\ & & -1 \end{array} \right]$$

Similarly, s' is defined as the composition of s , $-s$ and the zero vector $B \rightarrow_{fs} \mathbb{F}$:

$$s' = \begin{bmatrix} s & | & -s & | & 0 & \cdots & 0 \end{bmatrix}$$

A' and s' are thus supported by S .

Any vector $x' : (C \uplus C \uplus B) \rightarrow_{fs} \mathbb{F}$ can be written as

$$x' = (x_+ | x_- | y),$$

where $x_+, x_- : C \rightarrow_{fs} \mathbb{F}$ and $y : B \rightarrow_{fs} \mathbb{F}$. If any such non-negative vector x' satisfies the above constructed system of constraints, i.e. if we have

$$A' \cdot (x_+ | x_- | y) = t, \quad (39)$$

then the vector $x_+ - x_-$, supported by $\text{sup}(x')$, is a solution of (A, t) , namely

$$A \cdot (x_+ - x_-) \geq A \cdot (x_+ - x_-) - y = A' \cdot (x_+ | x_- | y) = t.$$

Furthermore, by the very definition of s' we have

$$s' \cdot (x_+ | x_- | y) = s \cdot (x_+ - x_-), \quad (40)$$

which implies

$$\text{supremum}(A', t, s') \leq \text{supremum}(A, t, s).$$

In the opposite direction, given a finitely supported vector x such that $A \cdot x \geq t$, we define a non-negative vector $x' = (x_+ | x_- | y)$ supported by $\text{sup}(x) \cup S$ as follows:

$$\begin{aligned} x_+(c) &= \begin{cases} x(c) & \text{if } x(c) \geq 0, \\ 0 & \text{otherwise;} \end{cases} \\ x_-(c) &= \begin{cases} -x(c) & \text{if } x(c) < 0, \\ 0 & \text{otherwise;} \end{cases} \\ y(c) &= (A \cdot x - t)(c). \end{aligned}$$

Then $x = x_+ - x_-$ and

$$A' \cdot (x_+ | x_- | y) = A \cdot x - y = t.$$

The equality (40) holds again, which implies

$$\text{supremum}(A, t, s) \leq \text{supremum}(A', t, s').$$

Reduction of NONNEG-EQ-MAX(\mathbb{F}) to INEQ-MAX(\mathbb{F}):

For any orbit-finite system of linear equations supported by S :

$$A \cdot x = t,$$

its nonnegative solutions are exactly solutions of the following system of linear inequalities, also supported by S :

$$A \cdot x \geq t \quad A \cdot x \leq t \quad x \geq 0.$$

This implies an easy reduction from NONNEG-EQ-MAX(\mathbb{F}) to INEQ-MAX(\mathbb{F}).

Remark 3. The above two reductions preserve row-finiteness, i.e., transform a system of finite equations (inequalities) to a system of finite inequalities (equations). \triangleleft

Reduction of FIN-INEQ-MAX(\mathbb{F}) to INEQ-MAX(\mathbb{F}): Consider an instance (A, t, s) of FIN-INEQ-MAX(\mathbb{F}) supported by S , where $A : B \times C \rightarrow_{fs} \mathbb{F}$. We construct an instance (A', t', s') of INEQ-MAX(\mathbb{F}) as follows. The new system of inequalities $A' \cdot x' \geq t'$ is obtained by extending the column index C by one additional variable y and extending the system by one inequality:

$$A' = \begin{bmatrix} & & & 0 \\ & & & \vdots \\ & A & & 0 \\ 1 & \cdots & 1 & -1 \end{bmatrix} \quad t' = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

and the new objective function s' is defined as expected:

$$s' = \begin{bmatrix} s & | & 0 \end{bmatrix}.$$

The so constructed instance is supported by S , and its solutions have the form $x' = (x, y)$, where

$$A \cdot x \geq t \quad \sum_{c \in C} x(c) \geq y.$$

Any such finitely supported solution is necessarily finitary. This implies

$$\text{supremum}(A, t, s) = \text{supremum}(A', t', s').$$

Reduction of INEQ-MAX(\mathbb{F}) to FIN-INEQ-MAX(\mathbb{F}): We rely on the following result of [8]¹⁰:

Claim 28.1 ([8] Claim 20). *Let $\mathbb{F} \in \{\mathbb{Z}, \mathbb{R}\}$. Given an S -supported orbit-finite matrix M one can effectively construct an S -supported orbit-finite matrix \widetilde{M} such that $\text{SPAN}_{\mathbb{F}}(M) = \text{FIN-SPAN}_{\mathbb{F}}(\widetilde{M})$.*

Consider an instance (A, t, s) of INEQ-MAX(\mathbb{F}), and apply the above claim to the matrix

$$M = \begin{bmatrix} A \\ s \end{bmatrix}$$

in order to get the matrix

$$\widetilde{M} = \begin{bmatrix} A' \\ s' \end{bmatrix}$$

such that

$$\text{SPAN}_{\mathbb{F}}(M) = \text{FIN-SPAN}_{\mathbb{F}}(\widetilde{M}). \quad (41)$$

This yields an instance (A', t, s') of FIN-INEQ-MAX(\mathbb{F}), supported by $\text{sup}(A, t, s)$.

By the equality (41), for every $r \in \mathbb{R}$ we have the following: there exists a finitely supported vector x such that $A \cdot x \geq t$

¹⁰The result, as shown in [8], holds for any commutative ring \mathbb{F} .

and $s \cdot x = r$ if, and only if there exists a finitary vector x' such that $A' \cdot x' \geq t$ and $s' \cdot x' = r$. In consequence,

$$\text{supremum}(A, t, s) = \text{supremum}(A', t, s').$$

Theorem 10 is thus proved. \square

Proof of Theorem 8. We consider cases $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{Z}$ separately.

Case $\mathbb{F} = \mathbb{R}$: Decidability of $\text{FIN-NONNEG-EQ-SOLV}(\mathbb{R})$ follows by a direct reduction of $\text{FIN-NONNEG-EQ-SOLV}(\mathbb{R})$ to $\text{NONNEG-EQ-SOLV}(\mathbb{R})$ (similar to the reduction of $\text{FIN-INEQ-MAX}(\mathbb{F})$ to $\text{INEQ-MAX}(\mathbb{F})$) and Theorem 6.

Case $\mathbb{F} = \mathbb{Z}$: Decidability of $\text{FIN-NONNEG-EQ-SOLV}(\mathbb{Z})$ follows by results of [8] and [15].

Let (A, t) be an instance of $\text{FIN-NONNEG-EQ-SOLV}(\mathbb{Z})$, where $A : B \times C \rightarrow_{\text{fs}} \mathbb{Z}$, and consider the set of column vectors

$$P = \{ A(-, c) \mid c \in C \} \subseteq \text{LIN}(B)$$

of A . Then the system of equations $A \cdot x = t$ has a finitary non-negative integer solution if, and only if

$$t \in \text{FIN-SPAN}_{\mathbb{N}}(P). \quad (42)$$

We rely on Theorem 3.3 of [8] which says that $\text{LIN}(B)$ has an orbit-finite basis. Let $\hat{B} \subseteq \text{LIN}(B)$ be such a basis. This implies that there exists a linear isomorphism

$$\varphi : \text{LIN}(B) \rightarrow \text{FIN-LIN}(\hat{B}).$$

In consequence, (42) is equivalent to

$$\varphi(t) \in \text{FIN-SPAN}_{\mathbb{N}}(\varphi(P)). \quad (43)$$

By Remark 11.16 of [15] we can compute a finite set of vectors $\{t'_1, \dots, t'_k\} \subseteq \text{FIN-LIN}(\hat{B})$ and an orbit-finite subset $P' \subseteq \varphi(P)$ such that (43) holds if, and only if

$$t'_i \in \text{FIN-SPAN}_{\mathbb{Z}}(P') \quad (44)$$

for some $i \in \{1 \dots k\}$. The question (44) is nothing but finitary integer solvability of an orbit-finite system of equations, which is decidable using Theorem 6.1 of [8]. \square

C. Proofs missing in Section V

Proof of Lemma 12. Consider any tuple $t = (\alpha_1, \dots, \alpha_n) \in \mathbb{A}^{(n)}$. Let $I = \{i \in \{1 \dots n\} \mid \alpha_i \in T\}$ denote the positions in t filled by atoms from T . By applying all semi- T -automorphisms to t , we obtain all tuples, where positions from I are arbitrarily filled by elements of T , and positions outside of I are arbitrarily filled by elements of $\mathbb{A} \setminus T$. \square

D. Proofs missing in Section VI

We need some preparatory facts. We say that a formula $\varphi(x_1, \dots, x_k)$ with free variables x_1, \dots, x_k *defines* the set of all valuations $\{x_1, \dots, x_k\} \rightarrow \mathbb{R}$ satisfying it. When the order of free variables is fixed, we naturally identify the set defined by φ with a subset of \mathbb{R}^k . For technical reasons we consider arithmetic formulas *with constants*, which can be arbitrary reals.¹¹ E.g., the formula using the constant $\sqrt{3}$,

$$\exists y : x^2 - y^2 > \sqrt{3},$$

defines the set $\{x \in \mathbb{R} \mid x < -\sqrt[4]{3}\} \cup \{x \in \mathbb{R} \mid x > \sqrt[4]{3}\}$.

Lemma 29. *Every real arithmetic formula $\varphi(x)$ with one free variable, possibly with constants, defines a finite union of (possibly infinite) disjoint intervals.*

Proof. By quantifier elimination [11], the formula $\varphi(x)$ is equivalent to a quantifier-free formula $\bar{\varphi}(x)$ with constants, namely $\varphi(x)$ and $\bar{\varphi}(x)$ define the same set. Therefore $\bar{\varphi}(x)$ is a Boolean combination of inequalities

$$p(x) \geq 0,$$

for univariate polynomials $p \in \mathbb{R}[x]$, and validity of $\bar{\varphi}(x)$ depends only on the sign of $p(x)$, for (finitely many) polynomials that appear in $\bar{\varphi}(x)$. This implies the claim. \square

Proof of Lemma 14. Consider a formula $\varphi(x, y_1, \dots, y_\ell)$ and an arbitrary tuple R_1, \dots, R_ℓ of reals. Denote by $\varphi(x, R_1, \dots, R_\ell)$ the formula φ with each variable x_i replaced by the constant R_i . We need to prove equivalence of the closed formulas:

$$\forall^\omega x : \varphi(x, R_1, \dots, R_\ell) \iff \forall^\omega_{\mathbb{Z}} x : \varphi(x, R_1, \dots, R_\ell).$$

As the left-to-right implication follows by definition, it is sufficient to prove the opposite one. By Lemma 29, the set $D \subseteq \mathbb{R}$ defined by $\varphi(x, R_1, \dots, R_\ell)$ is a finite union of intervals. Therefore, if D contains all sufficiently large integers then it also necessarily contains all sufficiently large reals. This means that the right formula implies the left one, as required. \square

Proof of Theorem 15 for unrestricted input. Without the assumption of monotonicity of input, we also evaluate the formula

$$\forall^\omega n : \psi(n).$$

and answer positively if the formula is true. Otherwise, we know that the set defined by ψ , being a finite union of intervals (cf. Lemma 29), is bounded from above. We can thus compute an integer upper bound m_0 by evaluating closed existential formulas

$$\varphi_m \equiv \exists n : n > m \wedge \psi(n),$$

¹¹As the first order theories of reals and *algebraic* reals coincide, we could equivalently restrict to algebraic reals and use only algebraic real constants, which are definable in arithmetic.

for increasing nonnegative integer constants $m = 0, 1, \dots$, until φ_m eventually evaluates to false. Finally, we check the formula $\psi(m)$ for all nonnegative integer constants m between 0 and m_0 , and answer positively if $\psi(m)$ is true for some such m ; otherwise we answer negatively. \square

E. Proofs missing in Section VII

Proofs of Lemmas 16 and 25. We sketch the proofs only, as they amount to a slightly tedious but entirely standard exercise in sets with atoms.

Consider an instance $(\mathbf{A}, \mathbf{t}, \mathbf{s})$ of the maximisation problem FIN-INEQ-MAX(\mathbb{R}). Let $S = \text{sup}(\mathbf{A}, \mathbf{t}, \mathbf{s})$, and let $\mathbf{A} : B \times C \rightarrow_{\text{fs}} \mathbb{Z}$. Thus the row and column index sets B and C are necessarily supported by S . We want to effectively transform the instance into another one $(\tilde{\mathbf{A}}, \tilde{\mathbf{t}}, \tilde{\mathbf{s}})$, where the row and column index sets are disjoint unions of sets of the form $\mathbb{A}^{(\ell)}$ (non-repeating tuples of atoms of a fixed length), as in (17) in Section VII-A. Moreover, the transformation should preserve the supremum:

$$\text{supremum}(\mathbf{A}, \mathbf{t}, \mathbf{s}) = \text{supremum}(\tilde{\mathbf{A}}, \tilde{\mathbf{t}}, \tilde{\mathbf{s}}). \quad (45)$$

Recall that we consider supremum of a maximisation problem to be $-\infty$ if the constraints are infeasible. Therefore proving that two maximisation problems have the same supremum also proves that the underlying systems of inequalities are equisolvable.

We rely on the following result (which can be shown similarly as Lemma 6.2 in [9]):

Lemma 30. *Let $S \subseteq_{\text{fin}} \mathbb{A}$. For every S -orbit U there are some $k, n \in \mathbb{N}$ and an S -supported surjective function*

$$f : (\mathbb{A} \setminus S)^{(k)} \rightarrow U$$

such that for all $x \in U$, $|f^{-1}(x)| = n$.

In the sequel we write $n(U)$ for the constant n , in order to indicate which S -orbit U is considered.

We proceed in two steps. As the first one we show that the row and column index sets B and C may be assumed to be disjoint unions of sets of the form $(\mathbb{A} \setminus S)^{(\ell)}$. Towards this, consider the partition of B and C into S -orbits:

$$B = B_1 \uplus \dots \uplus B_k \quad C = C_1 \uplus \dots \uplus C_\ell,$$

and apply Lemma 30 to the S -orbits B_i and C_j to get:

$$f_i : (\mathbb{A} \setminus S)^{p_i} \rightarrow B_i \quad g_j : (\mathbb{A} \setminus S)^{q_j} \rightarrow C_j$$

for $i \in \{1 \dots k\}$, $j \in \{1 \dots \ell\}$ and $p_i, \dots, p_i, q_1, \dots, q_\ell \in \mathbb{N}$. We put:

$$\begin{aligned} B' &= (\mathbb{A} \setminus S)^{p_1} \uplus \dots \uplus (\mathbb{A} \setminus S)^{p_k} \\ C' &= (\mathbb{A} \setminus S)^{q_1} \uplus \dots \uplus (\mathbb{A} \setminus S)^{q_\ell} \end{aligned} \quad (46)$$

and define maps $f : B' \rightarrow B$ and $g : C' \rightarrow C$ by disjoint unions of f_1, \dots, f_k and g_1, \dots, g_ℓ , respectively:

$$f = f_1 \uplus \dots \uplus f_k \quad g = g_1 \uplus \dots \uplus g_\ell.$$

Both the maps are surjective. We write $(f, g) : B' \times C' \rightarrow B \times C$ for the product of the two maps. Finally, we define a matrix $\mathbf{A}' : (B' \times C') \rightarrow_{\text{fs}} \mathbb{Z}$ and vectors $\mathbf{t}' \in \text{LIN}(B')$ and $\mathbf{s}' \in \text{LIN}(C')$ by pre-composing with the above-defined maps:

$$\mathbf{A}' = \mathbf{A} \circ (f, g) \quad \mathbf{t}' = \mathbf{t} \circ f \quad \mathbf{s}' = \mathbf{s} \circ g, \quad (47)$$

and claim:

Lemma 31. $\text{supremum}(\mathbf{A}, \mathbf{t}, \mathbf{s}) = \text{supremum}(\mathbf{A}', \mathbf{t}', \mathbf{s}')$.

Proof. Define two functions $F : \text{LIN}(C) \rightarrow \text{LIN}(C')$ and $G : \text{LIN}(C') \rightarrow \text{LIN}(C)$ as follows:

$$\begin{aligned} F(\mathbf{x}) &: c' \mapsto \frac{\mathbf{x}(g(c'))}{n(C_i)}, \text{ where } g(c') \in C_i \\ G(\mathbf{x}') &: c \mapsto \sum_{g(c')=c} \mathbf{x}'(c'). \end{aligned}$$

Both F and G are supported by S . By the very definition of F and G , together with Lemma 30 we deduce the following two facts, assuming either $\mathbf{x}' = F(\mathbf{x})$ or $\mathbf{x} = G(\mathbf{x}')$, where $\mathbf{x} \in \text{LIN}(C)$ and $\mathbf{x}' \in \text{LIN}(C')$. First, the value of $\mathbf{A} \cdot \mathbf{x}$ is well-defined if, and only if the value of $\mathbf{A}' \cdot \mathbf{x}'$ is so, and in such case

$$\mathbf{A} \cdot \mathbf{x} \geq \mathbf{t} \iff \mathbf{A}' \cdot \mathbf{x}' \geq \mathbf{t}'.$$

Second, the value of $\mathbf{s} \cdot \mathbf{x}$ is well defined if, and only if the value of $\mathbf{s}' \cdot \mathbf{x}'$ is so, and in such case

$$\mathbf{s} \cdot \mathbf{x} = \mathbf{s}' \cdot \mathbf{x}'.$$

The two facts prove the lemma. \square

The instance $(\mathbf{A}', \mathbf{t}', \mathbf{s}')$ is supported by S .

As the second step we transform the instance $(\mathbf{A}', \mathbf{t}', \mathbf{s}')$ further so that the row and column index sets B and C are disjoint unions of sets of the form $\mathbb{A}^{(\ell)}$. Let $h : \mathbb{A} \rightarrow \mathbb{A} \setminus S$ be an arbitrarily chosen bijection. Since atoms from S do not appear in tuples belonging to B' or C' , the map h induces two further bijective maps

$$f : \tilde{B} \rightarrow B' \quad g : \tilde{C} \rightarrow C',$$

where

$$\tilde{B} = \mathbb{A}^{(p_1)} \uplus \dots \uplus \mathbb{A}^{(p_k)} \quad \tilde{C} = \mathbb{A}^{(q_1)} \uplus \dots \uplus \mathbb{A}^{(q_\ell)}$$

(cf. (46)). We define a matrix $\tilde{\mathbf{A}} : \tilde{B} \times \tilde{C} \rightarrow_{\text{fs}} \mathbb{Z}$ and two vectors $\tilde{\mathbf{t}} : \tilde{B} \rightarrow_{\text{fs}} \mathbb{Z}$ and $\tilde{\mathbf{s}} : \tilde{C} \rightarrow_{\text{fs}} \mathbb{Z}$ by pre-composing with the two above-defined maps, similarly as in (47):

$$\tilde{\mathbf{A}} = \mathbf{A}' \circ (f, g) \quad \tilde{\mathbf{t}} = \mathbf{t}' \circ f \quad \tilde{\mathbf{s}} = \mathbf{s}' \circ g.$$

Knowing that \mathbf{A}' , \mathbf{t}' and \mathbf{s}' are all supported by S , we deduce that the so defined instance $(\tilde{\mathbf{A}}, \tilde{\mathbf{t}}, \tilde{\mathbf{s}})$ is equivariant and independent from the choice of the bijection $h : \mathbb{A} \rightarrow \mathbb{A} \setminus S$.

Lemma 32. $\text{supremum}(\mathbf{A}', \mathbf{t}', \mathbf{s}') = \text{supremum}(\tilde{\mathbf{A}}, \tilde{\mathbf{t}}, \tilde{\mathbf{s}})$.

Proof. Similarly as before, assuming $\mathbf{x}' = g(\tilde{\mathbf{x}})$ for some vectors $\mathbf{x}' \in \text{LIN}(C')$ and $\tilde{\mathbf{x}} \in \text{LIN}(\tilde{C})$, we deduce the

following two facts. First, the value of $\mathbf{A}' \cdot \mathbf{x}'$ is well-defined if, and only if the value of $\tilde{\mathbf{A}} \cdot \tilde{\mathbf{x}}$ is so, and in such case

$$\mathbf{A}' \cdot \mathbf{x}' \geq \mathbf{t}' \iff \tilde{\mathbf{A}} \cdot \tilde{\mathbf{x}} \geq \tilde{\mathbf{t}}.$$

Second, the value of $\mathbf{s}' \cdot \mathbf{x}'$ is well defined if, and only if the value of $\tilde{\mathbf{s}} \cdot \tilde{\mathbf{x}}$ is so, and in such case

$$\mathbf{s}' \cdot \mathbf{x}' = \tilde{\mathbf{s}} \cdot \tilde{\mathbf{x}}.$$

The two facts prove the lemma. \square

The last two lemmas prove Lemmas 16 and 25. \square

Proof of Lemma 18. Let $\pi \in \text{AUT}_{(T)}$ be such that $\pi(b) = b'$. As \mathbf{t} is equivariant, it is necessarily constant on the whole equivariant orbit to which b and b' belong (cf. Lemma 4), and hence $\mathbf{t}(b') = \mathbf{t}(b)$.

For the second point fix $j \in \{1 \dots M\}$. As \mathbf{A} is equivariant, it is constant over the orbit included in $B \times C$ to which (b, c) belongs, for every $c \in C$, and hence $\mathbf{A}(b, c) = \mathbf{A}(\pi(b), \pi(c))$. This implies

$$\sum_{c \in C_j} \mathbf{A}(b, c) = \sum_{c \in C_j} \mathbf{A}(\pi(b), \pi(c)) = \sum_{c \in C_j} \mathbf{A}(b', \pi(c)).$$

Since π is a semi- T -atom automorphism, when restricted to the semi- T -orbit C_j it is a bijection $C_j \rightarrow C_j$, and hence the two sums below differ only by the order of summation and are thus equal:

$$\sum_{c \in C_j} \mathbf{A}(b', \pi(c)) = \sum_{c \in C_j} \mathbf{A}(b', c).$$

The above equalities imply

$$\sum_{c \in C_j} \mathbf{A}(b, c) = \sum_{c \in C_j} \mathbf{A}(b', c),$$

as required. \square

F. Proofs missing in Section VIII

Proof of Theorem 24 (sketch). In order to prove Theorem 24 we propose another method of solving polynomially parametrised system of inequalities, by an adaptation of the classical Fourier-Motzkin variable elimination procedure for systems of linear inequalities. This yields another proof of decidability of POLY-INEQ-MAX (and of POLY-INEQ-SOLV), but the complexity is worst than EXPTIME of Theorem 23.

We start by introducing a standard order on rational functions with integer coefficients, i.e. on (partial) functions of the form

$$r(n) = \frac{p(n)}{q(n)}$$

where p and q are polynomials with integer coefficients in one variable n , and not all coefficients of q are zero. The argument n ranges over reals. We write $r \prec r'$ when $r(n) < r'(n)$ for all sufficiently large $n \in \mathbb{R}$, i.e., when there is some $N \in \mathbb{N}$ such that $r(n) < r'(n)$ for all $n \geq N$. The order \prec is readily shown to be equivalent to the one in [23, Example 5.2(c)], and

therefore rational functions with integer coefficients, ordered by \prec , are an ordered field.

We present now a symbolic extension of Fourier-Motzkin variable elimination: we transform a given system P of polynomially parametrised system of inequalities with $k + 1$ unknowns, say x, y_1, \dots, y_k , into another polynomially parametrised system of inequalities P' with unknowns y_1, \dots, y_k such that, for all sufficiently large $n \in \mathbb{R}$, the solution set of $P'(n)$ is the same as the projection of the solution set of $P(n)$ to $\{y_1, \dots, y_k\}$ (we briefly say that $P(n)$ and $P'(n)$ coincide). This is done in two steps, through an intermediate system P_1 where the coefficient of x in each inequality is either 1 or 0.

Towards construction of P_1 , suppose P consists of inequalities of the following form:

$$p(n) \cdot x + \sum_{j=1}^k q_j(n) \cdot y_j + r(n) \geq 0. \quad (48)$$

Inequalities (48) of P where the unknown x does not appear ($p(n) = 0$) are automatically included in P_1 . Inequalities of P where the unknown x appears with non-zero coefficient ($p(n) \neq 0$ for all sufficiently large n) are transformed as follows: if $p \succ 0$ then we add to P_1 the following inequality

$$x \geq -\frac{1}{p(n)} \cdot \left(\sum_{j=1}^k q_j(n) \cdot y_j + r(n) \right)$$

(we call these inequalities *lower bounds* for x); and if $p \prec 0$ then we add to P_1 the following one:

$$x \leq -\frac{1}{p(n)} \cdot \left(\sum_{j=1}^k q_j(n) \cdot y_j + r(n) \right)$$

(we call these inequalities *upper bounds* for x). Each transformation preserves the solution set for sufficiently large $n \in \mathbb{R}$.

As the second step, we define P' . Inequalities of P_1 where the unknown x does not appear are automatically included in P' . Furthermore, for each lower bound and upper bound in P_1 , say

$$\begin{aligned} x &\geq -\frac{1}{p(n)} \cdot \left(\sum_{j=1}^k q_j(n) \cdot y_j + r(n) \right) \\ x &\leq -\frac{1}{p'(n)} \cdot \left(\sum_{j=1}^k q'_j(n) \cdot y_j + r'(n) \right), \end{aligned}$$

we add to P' the following inequality (without x)

$$\frac{1}{p'(n)} \cdot \left(\sum_{j=1}^k q'_j(n) \cdot y_j + r'(n) \right) \leq \frac{1}{p(n)} \cdot \left(\sum_{j=1}^k q_j(n) \cdot y_j + r(n) \right)$$

which is easily transformed to the right form (48) by multiplying both sides by $p(n) \cdot p'(n)$, and reversing \leq (since $p \cdot p' \prec 0$).

The systems P_1 and P' coincide for all sufficiently large $n \in \mathbb{R}$, and hence so do P and P' as well. Furthermore, from

the above transformations one can compute an $N \in \mathbb{N}$ such that $P(n)$ and $P'(n)$ coincide for all $n > N$.

By iterated application of the variable elimination we get:

Lemma 33. *Given a polynomially parametrised system P with unknowns $x_1, \dots, x_k, y_1, \dots, y_\ell$, one can compute a polynomially parametrised system P' with unknowns y_1, \dots, y_ℓ and $N \in \mathbb{N}$ such that $P(n)$ and $P'(n)$ coincide for all $n \geq N$.*

Using Lemma 33, we derive a method of computing the supremum of a monotonic instance of the polynomially parametrised maximisation problem. On the way we prove Theorem 24: the supremum is either rational, or belongs to $\{-\infty, +\infty\}$. We will rely on the following fact:

Lemma 34. *The supremum of a rational function over a subset of non-negative integers is rational.*

Consider a monotonic instance (P, S) with unknowns x_1, \dots, x_k . We extend the system P by one fresh unknown z and one inequality:

$$z \leq S(n, x_1, \dots, x_k)$$

(this yields a system P_1) and then compute, using Lemma 33, a polynomially parametrised system of inequalities P' with z as the only unknown, and an $N \in \mathbb{N}$ such that $P_1(n)$ and $P'(n)$ coincide for all $n \geq N$. Every inequality in P' is of one of the following forms:

$$p(n) \leq q(n) \cdot z \quad q(n) \cdot z \leq p(n) \quad p(n) \leq q(n)$$

where p, q are polynomials, which may be rewritten into

$$\frac{p(n)}{q(n)} \leq z \quad z \leq \frac{p(n)}{q(n)} \quad p(n) \leq q(n).$$

The rational functions on the left of the first type of inequality are called *lower bounds* for z , and symmetrically, the rational functions on the right of the second type of inequality are called *upper bounds* for z . Let $a(n)$ be the largest lower bound with respect to \prec , and symmetrically, let $b(n)$ be the smallest upper bound with respect to \prec (for simplicity of presentation we omit the border cases where there is no lower/upper bounds at all). Let C be the set of pairs of polynomials (p, q) related by some inequality of the third type. For every $n \geq N$, the maximum attainable value of the objective function $S(n)$ over solutions of $P(n)$ is exactly $b(n)$, assuming $P'(n)$ is solvable, i.e., assuming

- 1) $p(n) \leq q(n)$ for all $(p, q) \in C$.
- 2) $a(n) \leq b(n)$.

We now argue that the supremum of (P, S) is rational (or belongs to $\{-\infty, +\infty\}$). We will use the fact that for every $n \in \mathbb{N}$, the supremum of $(P(n), S(n))$, the instance of the classical linear programming, is rational (or belongs to $\{-\infty, +\infty\}$). We distinguish three cases:

- 1) There exists $(p, q) \in C$ such that $q \prec p$. Let $N_1 \in \mathbb{N}$ be the smallest non-negative integer such that $q(n) < p(n)$

for all $n \geq N_1$. Then $P'(n)$ is not solvable for $n \geq \max(N, N_1)$, and the supremum of (P, S) is

$$\max \{ \supremum(P(n), S(n)) \mid n < \max(N, N_1) \},$$

the maximum of a finite set of rational numbers, and hence forcedly rational.

- 2) $b \prec a$. Let $N_2 \in \mathbb{N}$ be the smallest non-negative integer such that $b(n) < a(n)$ for all $n \geq N_2$. Then $P'(n)$ is not solvable for $n \geq \max(N, N_2)$, and the supremum of (P, S) is

$$\max \{ \supremum(P(n), S(n)) \mid n < \max(N, N_2) \},$$

forcedly rational as before.

- 3) None of the above happens. Let $N_3 \in \mathbb{N}$ be the smallest non-negative integer such that $a(n) < b(n)$ for all $n \geq N_3$. Then the supremum of the maximisation problem $(P(n), S(n))$ is simply is the largest of

$$\max \{ \supremum(P(n), S(n)) \mid n < \max(N, N_3) \}$$

and the supremum of values of the rational function b over integers $n > \max(N, N_3)$:

$$\supremum \{ b(n) \mid n \geq \max(N, N_3) \},$$

which is rational by Lemma 34.

This completes the proof of Theorem 24. \square

Proof of Lemma 26. By Lemma 13, each value of S on a finitary T -supported solution of (\mathbf{A}, \mathbf{t}) is also achieved on some finitary semi- T -supported solution. This allows us to restrict to finitely semi-supported solutions only.

We observe that the mutual transformations between finitary semi- T -supported real solutions \mathbf{x} of (\mathbf{A}, \mathbf{t}) and real solutions \mathbf{x}' of $P'(n)$, used in the proof of Lemma 19, preserve the value of the objective functions: $S(\mathbf{x}) = S'(\mathbf{x}')$. \square

Proof of Lemma 27. Indeed, suppose $T \subseteq T'$ differ by one element, and let $|T| = n$ (therefore $|T'| = n + 1$). By Lemma 26 the supremum s_n is achieved on a finitary T -supported solution, which is also T' -supported. By Lemma 26 again, s_{n+1} is the maximal value achieved on a finitary T' -supported solution, which implies $s_n \leq s_{n+1}$. \square

G. Proofs missing in Section IX

Proof of Lemma 28. For the 'if' direction, given a finitary nonnegative integer solution of S , we consider the graph G determined by values of unknowns $e_{\alpha\beta}$, as discussed in the course of construction, consisting of inner nodes and two end nodes, and having the form of a finite directed path plus (possibly) a number of directed cycles. By the construction of S , each edge of G has assigned a configuration of M , and each inner node has assigned an instruction of M , so that the configuration on the edge outgoing from an inner node is exactly the result of executing its instruction on the

configuration assigned to the incoming edge. (As above, this applies to *all* inner nodes and edges of G , both those on the path as well as those on cycles.) Ignoring the cycles of G , we conclude that the sequence of configurations and instructions along the path of G is a run of M from c_0 to c_f .

For the 'only if' direction, given a run of M from c_0 to c_n as in (30), one constructs a solution of S in the form of a sole path involving end nodes $\alpha_0 = \iota, \alpha_{n+1} = \zeta$, n inner nodes $\alpha_1, \dots, \alpha_n$, and $n+1$ edges $\alpha_j \alpha_{j+1}$. Thus unknowns $e_{\alpha_j, \alpha_{j+1}}$ are equal 1. The values of unknowns $t_{i\alpha_j}$ are determined by instructions i_j used in the run, and the values of the unknowns $c_{\alpha_j \alpha_{j+1} \gamma k}$, for sufficiently many fresh atoms γ , are determined by configurations c_j . All other unknowns are equal 0. \square