Variations on Cops and Robbers

Alan Frieze,[1] Michael Krivelevich,[2] and Po-Shen Loh[3]

[1] DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY
PITTSBURGH, PENNSYLVANIA 15213
E-mail: alan@random.math.cmu.edu

[2] SCHOOL OF MATHEMATICAL SCIENCES
RAYMOND AND BEVERLY SACKLER FACULTY
OF EXACT SCIENCES, TEL AVIV UNIVERSITY
TEL AVIV 69978, ISRAEL
E-mail: krivelev@post.tau.ac.il

[3] DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY
PITTSBURGH, PENNSYLVANIA 15213
E-mail: ploh@cmu.edu

Received April 14, 2010; Revised November 24, 2010

Published online in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/jgt.20591

Abstract: We consider several variants of the classical Cops and Robbers game. We treat the version where the robber can move $R \geq 1$ edges at a time, establishing a general upper bound of $n/\alpha^{(1-o(1))\sqrt{\log_\alpha n}}$, where $\alpha = 1+1/R$, thus generalizing the best known upper bound for the classical case $R=1$ due to Lu and Peng, and Scott and Sudakov. We also show that in this case, the cop number of an n-vertex graph can be as large as

Contract grant sponsor: NSF; Contract grant number: DMS-0753472 (to A. F.); Contract grant sponsor: USA-Israel BSF; Contract grant number: 2006322 (to M. K.); Contract grant sponsor: Israel Science Foundation; Contract grant number: 1063/08 (to M. K.); Contract grant sponsor: Pazy memorial award (to M. K.).

$n^{1-1/(R-2)}$ for finite $R \geq 5$, but linear in n if R is infinite. For $R=1$, we study the directed graph version of the problem, and show that the cop number of any strongly connected digraph on n vertices is $O(n(\log\log n)^2/\log n)$. Our approach is based on expansion.

Keywords: *Cop number; Meyniel's conjecture; Games on graphs*

1. INTRODUCTION AND MAIN RESULTS

The game of *Cops and Robbers*, introduced by Nowakowski and Winkler [25] and independently by Quillot [26], is a perfect information game played on a fixed graph G. There are two players, a set of c cops, for some integer $c \geq 1$, and a robber. Initially, the cops are placed onto vertices of their choice in G (where more than one cop can be placed at a vertex). Then the robber, being fully aware of the cops' placement, positions himself on one of the vertices of G. Then the cops and the robber move in alternate rounds, with the cops moving first; however, players are permitted to remain stationary on their turn if they wish. The players use the edges of G to move from vertex to vertex. The cops win and the game ends if eventually a cop steps into the vertex currently occupied by the robber; otherwise, i.e. if the robber can elude the cops indefinitely, the robber wins.

The *cop number* of G, denoted by $c(G)$, is the minimum number of cops needed to win on G. This parameter was introduced by Aigner and Fromme [1], and there is now an extensive literature on this fascinating problem. We direct the reader to any of [4–6, 12, 16, 17, 19, 20, 23, 24, 28], or any of the surveys [3, 14, 18] for detailed accounts of the known results. All results focus on connected graphs, because the problem for a disconnected graph obviously decomposes into the sum of the answers for each connected component.

The most well-known open question in this area is Meyniel's conjecture, published by Frankl in [15]. It states that for every connected graph G on n vertices, $O(\sqrt{n})$ cops are enough to win. This conjecture, if true, is best possible, as projective plane graphs (n-vertex graphs without cycles of lengths 3 and 4, and with all degrees at least $c\sqrt{n}$, for a constant c) are easily seen to require at least $c\sqrt{n}$ cops. So far, the progress towards establishing Meyniel's conjecture has been rather slow. Frankl [15] proved the upper bound of $O(n\log\log n/\log n)$; some 20 years later Chiniforooshan [10] improved it to $O(n/\log n)$. Finally, the upper bound of $n/2^{(1-o(1))\sqrt{\log n}}$ was established independently by Lu and Peng [22], and Scott and Sudakov [27]. Bounds on the typical behavior of the cop number for the random graph $G_{n,p}$ have been obtained for various values of $p=p(n)$ by Bollobás et al. [8] and by Łuczak and Prałat [22]. Many other versions and ramifications of the above described classical setting have been studied, such as the ranged version in [9], limited visibility in [21], etc., but we do not pursue them here.

We employ an approach based on the notion of *expansion*, which has had many applications in mathematics and theoretical computer science. Recall that a graph is said to be a *c-expander* if every subset S of at most $n/2$ vertices has $|N(S)\setminus S| > c|S|$. Surprisingly, although our expansion-based method applies a different technique than that used by Lu-Peng and Scott-Sudakov, it produces the same bound.

Furthermore, our method also allows us to address some natural variants of the problem. First, we consider the directed graph version of the Cops and Robbers problem. The setting here is a straightforward adaptation of the undirected setting described above, with the only difference being that the players need to respect the direction of any edge while moving along it. Problems for directed graphs (digraphs) are usually much more difficult. To the best of our knowledge, there have been no results on this problem in this case. In Section 3, we observe that the essence of the problem is to consider only *strongly connected* digraphs, i.e. those which have directed paths from any vertex to any other vertex. We prove the following general upper bound.

Theorem 1.1. *Every strongly connected digraph on n vertices has cop number* $O(n \cdot (\log\log n)^2 / \log n)$.

We then use a purely expansion-based argument to provide an alternate proof of the best result for general graphs.

Theorem 1.2. *Every connected graph on n vertices has cop number at most* $n / 2^{(1-o(1))\sqrt{\log_2 n}}$.

Our approach also works in the case when the robber moves faster than the cops. Indeed, this setting was recently considered in [13]. The usual problem is less interesting if a cop can move faster than the robber, because then one cop is sufficient: he can chase down the robber. (One must introduce additional mechanisms to make that version nontrivial, as is done in [11].) Hence, we consider the case when the robber moves at speed $R > 1$ and the cop moves at speed 1; the robber can take any walk of length R from his current position, but he is not allowed to pass through any vertex occupied by a cop. With our alternate approach, we are able to extend Theorem 1.2 for a faster robber.

Theorem 1.3. *Let* $R \geq 1$ *be a given finite constant, and let* $\alpha = 1 + 1/R$. *For every connected graph on n vertices,* $n / \alpha^{(1-o(1))\sqrt{\log_\alpha n}}$ *cops are sufficient to catch any robber who moves at speed R.*

Remark. Observe that for the original case $R = 1$, the constant α is precisely 2. Therefore, this extends all current best results in the traditional setting.

It is also interesting to note that in the fast robber setting, the cop number can be drastically different. Indeed, Proposition 5.1 in Section 5 exhibits an n-vertex graph for which the cop number jumps from 2 to $\Theta(\sqrt{n})$ when the robber's speed increases from 1 to 2. For higher speeds, we also show that the general lower bound climbs beyond $n^{1/2}$, and even reaches $\Omega(n)$ for an infinite-speed robber.

Theorem 1.4. *For any given robber speed* $R \geq 5$, *the following hold for sufficiently large n.*

(i) *If* $R < \infty$, *there exists a connected n-vertex graph which requires at least* $n^{1-1/(R-2)}$ *cops.*
(ii) *If* $R = \infty$, *there exists a connected n-vertex graph which requires at least* $10^{-6} \cdot n$ *cops.*

Throughout our paper, we will omit floor and ceiling signs whenever they are not essential, to improve clarity of presentation. All logarithms are in base $e \approx 2.718$ unless otherwise specified. The following asymptotic notation will be utilized extensively. For two functions $f(n)$ and $g(n)$, we write $f(n) \ll g(n)$, $f(n) = o(g(n))$, or $g(n) = \omega(f(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$, and $f(n) = O(g(n))$ or $g(n) = \Omega(f(n))$ if there exists a constant M such that $|f(n)| \le M|g(n)|$ for all sufficiently large n. The number of vertices n is assumed to be sufficiently large where necessary.

2. PRELIMINARIES

Previous attempts to solve the general case of this problem have relied on the following two observations, which we restate here in forms that are convenient for our analysis. Recall that $c(G)$ denotes the cop number of G.

Lemma 2.1. *Let G be a connected n-vertex graph.*

(i) *If v is a vertex of maximum degree Δ, then $c(G) \le 1 + c(G')$, where G' is a connected graph with at most $n-1-\Delta$ vertices.*
(ii) *If $v_1 v_2 \dots v_t$ is a shortest path between v_1 and v_t, then $c(G) \le 1 + c(G')$, where G' is a connected graph with at most $n-t$ vertices.*

Part (i) follows by permanently stationing a cop at v, thereby prohibiting the robber from entering $\{v\} \cup N(v)$. Part (ii) uses a result of Aigner and Fromme [1] which establishes that a single cop can ensure that after finitely many moves, the robber never enters $\{v_1, \dots, v_t\}$. In both situations, if $G[U_1], \dots, G[U_k]$ are the connected components of the remainder of the graph, then only $\max_i c(G[U_i])$ more cops are sufficient to capture the robber.

It is worth noting that the result of Aigner and Fromme used in part (ii) relies critically on the bidirectionality of the edges, and on the robber moving only one edge per turn. In order to consider more general settings, we use a new approach based on expansion. Let ∂S denote the set of all vertices which are outside S, but adjacent to some vertex in S.

Lemma 2.2. *Let G be a connected n-vertex graph. Suppose that G has a set of vertices S with $|S| < n/2$, but $|\partial S| \le p|S|$ for some $0 < p < 1$. Then $c(G) \le p|S| + c(G')$, where G' is a connected graph with at most $n - |S|$ vertices.*

Remark. This result will still hold even when the robber is permitted to move at speed $R > 1$.

Proof. Permanently station one cop on each vertex in ∂S. Let $G[U_1], \dots, G[U_k]$ be the connected components of $G \setminus \partial S$. The barrier of cops will prevent the robber from ever entering ∂S, so in particular, he will be forced to remain within a single connected component $G[U_j]$. Therefore, as before, only $\max_i c(G[U_i])$ more cops are required to capture the robber.

It remains to show that $|U_i| \le n - |S|$ for all i. Observe that since $|S| < n/2$, every connected component spanned by S has size at most $|S| < n/2 < n - |S|$. On the other hand, every connected component spanned by $G \setminus (S \cup \partial S)$ obviously has size at most $n - |S| - |\partial S| \leq n - |S|$. This completes the proof. ■

Although all statements in this paper are about deterministic graphs, we will use the probabilistic method to develop strategies and provide constructions. We will repeatedly use the Chernoff bound in our analysis, so we record a version of it here. (See, e.g. [2].)

Fact 2.3. *For any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that any binomial random variable X with mean μ satisfies*

$$\mathbb{P}[X < (1-\varepsilon)\mu] < e^{-(\varepsilon^2/2)\mu} \quad \text{and} \quad \mathbb{P}[X > (1+\varepsilon)\mu] < e^{-c_\varepsilon \mu},$$

where c_ε is a constant determined by ε. When $\varepsilon = \frac{1}{2}$, we may take $c_{1/2} = \frac{1}{10}$.

3. DIRECTED GRAPHS

Recall that a digraph is *strongly connected* if there is a directed path from any vertex to any other vertex, *weakly connected* if its underlying undirected graph is connected, and *disconnected* if its underlying undirected graph is disconnected. We claim that the essence of the directed case of this problem is to investigate the cop number of an arbitrary strongly connected digraph D. Indeed, as in the case of ordinary graphs, if the underlying undirected graph G is disconnected, with connected components $G[V_1], \ldots, G[V_t]$, then the cop number of D is clearly the sum of the cop numbers of $D[V_1], \ldots, D[V_t]$. The following proposition shows that the weakly connected case reduces to solving the strongly connected case.

Proposition 3.1. *Let D be a directed graph, whose strongly connected components are $D[V_1], \ldots, D[V_t]$. Let c_i denote the cop number of $D[V_i]$, and construct a directed acyclic graph D' with vertex set $[t]$, where $\overrightarrow{ij}$ is an edge if and only if D has an edge from V_i to V_j. Then the problem of determining the cop number of D reduces to an optimization problem involving only D' and the c_i's, for $i = 1, 2, \ldots t$.*

Proof. In a directed acyclic graph, we call v a *source vertex* if it has in-degree zero, and we say that *v feeds into w* if there is a directed path from v to w. The first observation is that it is never useful to initially position cops in any V_i where i is a non-source vertex of D'. Indeed, consider positioning those cops in a strong component V_i', where i' is a source vertex of D' which feeds into i instead. Let V_j be the strong component containing the robber's initial vertex. If i does not feed into j, then this alternate placement makes no difference, because the cops in V_i would be useless anyway. Otherwise, if i feeds into j, let all cops initially stay stationary until the relocated cops move to their old positions in V_i. Then, run the old algorithm (which had those cops starting in V_i).

Therefore, we only need to choose the numbers of cops to place in the strong components corresponding to source vertices of D'. Assign an integer variable x_i to each source. Consider any vertex $j \in D'$, and let S be the set of source vertices which feed into it. We must have the inequality $\sum_S x_i \geq c_j$, because if the robber started in V_j, then the only sources of cops are from the V_i with $i \in S$. We thus introduce one constraint per vertex of D'.

It remains to show that if all constraints are satisfied, then the robber certainly can be caught. Let the robber's initial position be in V_j. Route all possible cops into V_j. By the constraints, we will be able to move at least c_j cops there. If the robber stays in V_j, he will certainly be caught eventually, so he must exit to another strong component V_k. In D', the set of sources which feed into k is a superset of those which feed into j, so we can move the cops in V_j to V_k, and move the cops from the other sources directly to V_k. This will force the robber out of V_k, and the process must terminate eventually because D' is finite. ■

Solving the optimization problem is outside of the scope of this article, since it has an entirely different flavor. Instead, we now proceed to prove Theorem 1.1. Diameter-based arguments break down completely, so previous bounds for general graphs (e.g. Frankl [15], Chiniforooshan [10], and Lu and Peng [22]) do not apply. However, expansion is immune to this difficulty. In the context of directed graphs, let us call a digraph a *c-in-expander* if every subset S of at most $n/2$ vertices has $|\partial^- S| \geq c|S|$, where $\partial^- S$ is the set of all vertices $v \notin S$ which have a directed edge into S. Let us bring some basic tools from the previous section to the directed setting. We omit the proof because it follows essentially the same lines as Lemmas 2.1(i) and 2.2.

Lemma 3.2. *Let D be a strongly connected n-vertex digraph.*

(i) *If v is a vertex of out-degree Δ, then $c(D) \leq 1 + c(D')$, where D' is a strongly connected digraph with at most $n-1-\Delta$ vertices.*
(ii) *Suppose that D has a set of vertices S with $|S| < n/2$, but $|\partial^- S| \leq p|S|$ for some $0 < p < 1$. Then $c(D) \leq p|S| + c(D')$, where D' is a strongly connected digraph with at most $n - |S|$ vertices.*

The previous lemma is cumbersome to apply by itself. However, it allows us to clean up our graph, at the cost of reserving a few cops for this purpose. We record the following statement, which is more convenient to use.

Corollary 3.3. *Let D be a strongly connected digraph with n vertices, and let $0 < p < 1$ be arbitrary. Then $c(D) \leq pn + c(D')$, where D' is a strongly connected digraph with at most n vertices and maximum out-degree at most $1/p$, which is also a p-in-expander.*

Proof. We repeatedly apply Lemma 3.2. As long as there is a vertex v of out-degree at least $1/p$, part (i) shows that at the cost of one cop, we can reduce the number of vertices by at least $(1/p)+1$. Similarly, if there is a set S of at most half the vertices with $|\partial^- S| \leq p|S|$, we can reduce the number of vertices by at least $|S|$, at the cost of $p|S|$ cops. Note that in both cases, the number of cops expended is at most a p-fraction of the number of vertices discarded. Therefore, if we repeat this process until exhaustion, we will have a digraph D' with $m \leq n$ vertices, with the stated properties, such that $c(D) \leq p(n-m) + c(D') \leq pn + c(D')$, as claimed. ■

We are now ready to prove Theorem 1.1. Corollary 3.3 implies that it is an immediate consequence of the following final lemma.

Lemma 3.4. *Let* $p = 13\,(\log\log n)^2/\log n$. *Every strongly connected digraph* D *on* $m \le n$ *vertices with maximum out-degree at most* $1/p$ *and in-expansion at least* p *can be guarded by at most* $2pn$ *cops.*

Proof. Note that if $m \le pn$, we are trivially done by placing a cop on each vertex of D. So, we may assume that $m > pn$. Let $r = (6/p)\log(4/p)$. For each vertex v, let $B^+(v)$ denote the set of all vertices which are reachable from v by (directed) walks of length at most r. Similarly, for $S \subseteq V$, let $B^-(S)$ contain every vertex that can reach some vertex in S by a directed walk of length at most r.

Our first claim is that it is possible to position $2pn$ cops so that for every subset S of size $1 \le |S| \le 2p^{-r}$, the set $B^-(S)$ contains at least $|S|$ cops. Indeed, Inequality A.1 from the Appendix gives

$$2p^{-r}(1+p)^r \le np/2 < m/2,$$

so for such a set S, the in-expansion property ensures that $|B^-(S)| \ge |S|(1+p)^r$. Note that Inequality A.1 from the Appendix also shows that $(1+p)^r \ge (16/p)\log n$.

Therefore, if we position cops randomly, by independently placing a cop at each vertex with probability p, the expected number of cops in $B^-(S)$ is at least $|S| \cdot 16\log n$. The Chernoff bound (Fact 2.3) shows that the probability that this is below half its expectation is at most $e^{-(1/8)|S|\cdot 16\log n} \le n^{-2|S|}$.

Since the number of subsets of s vertices is at most n^s, a union bound over all S of size s shows that with probability at least $1-n^{-s}$, every such $B^-(S)$ contains at least $|S| \cdot 8\log n \ge |S|$ cops. Taking another union bound over all $s \in \{1, \ldots, 2p^{-r}\}$, we see that **whp**,[1] this holds for every $1 \le |S| \le 2p^{-r}$. Also, the Chernoff bound implies that **whp**, at most $2pn$ cops were placed by the random process. Putting these two together, we see that it is indeed possible to place only $2pn$ cops so that for every S of size $1 \le |S| \le 2p^{-r}$, the set $B^-(S)$ contains at least $|S|$ cops. (This is a non-constructive proof in the sense that we do not provide an efficient deterministic algorithm for cop placement; however, we have shown that the random algorithm succeeds nearly all of the time.)

Now assume that the cops are placed as above. Let the robber's position be v. By the maximum out-degree condition, $|B^+(v)| \le 1 + p^{-1} + \cdots + p^{-r} < 2p^{-r}$. We now use Hall's theorem to show that for each $w \in B^+(v)$, there is a distinct cop c_w which can reach it within r moves. Indeed, for this we consider an auxiliary bipartite graph where the left side is a copy of the vertex set of $B^+(v)$, and the right side contains one vertex for each of the $2pn$ cops placed above. We place edges between vertices w on the left and all vertices on the right corresponding to cops within distance r of w. By the previous argument, for every subset S of the left side, at least $|S|$ vertices on the right have neighbor(s) in S. This verifies the Hall condition. Therefore, we can send a distinct cop to each vertex of $B^+(v)$, so that after r moves the entire set $B^+(v)$ is occupied by cops. As this is the complete set of possible positions for the robber after r moves, this results in his capture. ■

---

[1]With probability tending to 1 as $n \to \infty$.

4. GENERAL GRAPHS

In the previous section, we used Hall's theorem to route distinct cops to each position which needed to be blocked. This argument can be improved by performing several iterations. The main idea is to draw conclusions from the failure of the Hall condition. The $k=1$ version of the following lemma essentially appears in [22], but our statement has an expansion flavor built in, and so is more amenable to our approach.

Lemma 4.1. *Given a bipartite graph with parts A and B and a number k, it is always possible to partition* $A=S\cup T$ *such that* $|N(S)|\le k|S|$, *and for every subset* $U\subseteq T$, *we have* $|N(U)|\ge k|U|$.

Proof. Start with $S=\emptyset$. As long as there is a subset $U\subseteq A$ with $|N(U)|\le k|U|$, add U to S, and delete U from A. It is clear that at the end of this process, if we consider the original graph, $|N(S)|\le k|S|$. However, in our modified graph, every subset of the remaining A expands by at least k times in B. ∎

Next, we isolate a component of our directed graph proof, so that we can use it in a modular form. Let $B_r(S)$ denote the set of all vertices which are within distance r from at least one vertex in S.

Lemma 4.2. *Let* n,p,r *be given, with np sufficiently large. In every n-vertex graph G, it is possible to distribute* $2pn$ *cops such that for every set S with* $|B_r(S)|\ge(16/p)|S|\log n$, *there are at least* $|S|$ *cops in* $B_r(S)$.

Proof. Position cops randomly, by independently placing a cop at each vertex with probability p. For each S in the statement, the expected number of cops in $B_r(S)$ is at least $16|S|\log n$. The Chernoff bound (Fact 2.3) shows that the probability that this is below half its expectation is $e^{-(1/8)\cdot 16|S|\log n}\le n^{-2|S|}$.

Since the number of subsets of s vertices is at most n^s, a union bound over all S of size s shows that with probability at least $1-n^{-s}$, every such $B_r(S)$ contains at least $8|S|\log n\ge|S|$ cops. Taking another union bound over all $s\in\{1,\ldots,n\}$, we see that with probability at least $1-2/n$, this holds for every $1\le|S|\le n$. Yet $\mathrm{Bin}[n,p]$ is at most $2np$ **whp** by the Chernoff bound, so we conclude that there is positive probability of our procedure giving all of the desired properties, using only $2pn$ cops. ∎

Next, we translate Corollary 3.3 to the case of undirected graphs, via Lemmas 2.1(i) and 2.2. The proof of the following statement is analogous to Corollary 3.3, so we do not record it again.

Corollary 4.3. *Let G be a connected graph with n vertices, and let* $0<p<1$ *be arbitrary. Then* $c(G)\le pn+c(G')$, *where* G' *is a connected graph with* $m\le n$ *vertices and maximum degree at most* $1/p$, *which is also a p-expander.*

In light of this corollary, it is clear that Theorem 1.2 is immediate from the following final lemma.

Lemma 4.4. *There is a function* $p=p(n)=2^{-(1-o(1))\sqrt{\log_2 n}}$ *for which the following holds. Every connected graph G on* $m\le n$ *vertices with maximum degree less than* $1/p$ *and expansion at least p can be guarded by at most* $(1+o(1))\sqrt{\log_2 n}\cdot 2pn$ *cops.*

Proof. As in the proof for directed graphs, we may assume that $m>pn$, or else we are trivially done. Inequality A.2 shows that there is a function $p=2^{-(1-o(1))\sqrt{\log_2 n}}$ and a positive integer $l=(1+o(1))\sqrt{\log_2 n}$ such that when we define $k=(16/p)\log n$, we have the inequalities

$$k^{l+1} \le (1+p)^{-2^l} np/2 \quad \text{and} \quad (1+p)^{2^l} \ge k. \tag{1}$$

We will split the cops into $l+1$ groups $C_0, C_1, \ldots, C_l$, each of size $2pn$. Choose the initial positions of the cops in C_i by applying Lemma 4.2 with parameter $r=2^i$. Let the robber's initial position be v.

Let $N_0=B_1(v)$ be the set of vertices that the robber can reach in 1 move. By the maximum degree condition, $|N_0| \le (1/p)<k$. Consider the auxiliary bipartite graph in which $A=N_0$, $B=V$, and a is adjacent to b if and only if they are at distance at most 1 in G. Then Lemma 4.1 implies that we can partition $N_0=S_0 \cup T_0$ such that (in G) $|B_1(S_0)| \le k|S_0|$ and every subset $U \subseteq T_0$ has $|B_1(U)| \ge k|U|$. Therefore, by construction of C_0, Hall's theorem shows us how to send a distinct cop from C_0 to each vertex of T_0 in the first move, preventing the robber from ever occupying a vertex of T_0.

Thus the robber's position after his second move is restricted to $N_1=B_1(S_0)$, and $|B_1(S_0)| \le k|S_0| \le k^2$. Repeating the same trick, we can partition $N_1=S_1 \cup T_1$ and use the cops in C_1 to prevent the robber from ever entering T_1, yet $|B_2(S_1)| \le k|S_1| \le k^3$. Hence the robber's position after his fourth move is restricted to $N_2=B_2(S_1)$.

The radii of the balls double at each iteration of this procedure, so we eventually conclude that after his 2^lth move, the robber is still contained within a set $N_l = B_{2^{l-1}}(S_{l-1})$ of size at most k^{l+1}. However, when we iterate the argument a final time, the partition $N_l=S_l \cup T_l$ must have $S_l=\emptyset$. Indeed, since G is a p-expander, every non-empty set S of size at most $(1+p)^{-(r-1)}m/2>(1+p)^{-(r-1)}np/2$ has $|B_r(S)| \ge (1+p)^r|S|$. As Inequality (1) ensures that $|N_l| \le k^{l+1} \le (1+p)^{-2^l}np/2$ and $(1+p)^{2^l} \ge k$, we conclude that S_l is indeed empty. Therefore, the cops in C_l can completely cover N_l within 2^l moves. Since N_l was the set of possible positions for the robber after his 2^lth move, the robber is captured. ■

5. FAST ROBBER

In this section, we assume that the robber can traverse up to R edges in a single move. Cops may only move by a single edge per move. We begin by observing that the cop number of a graph can dramatically increase even if the robber's speed only grows to $R=2$.

Proposition 5.1. *Let G be the 1-subdivision of K_n, where a vertex is added on each edge. The ordinary cop number of G is 2, but if the robber can move at speed 2, then the cop number rises to $\lceil n/2 \rceil = \Theta(\sqrt{|V(G)|})$.*

Proof. Call a vertex of a 1-subdivision an *internal vertex* if it was added to subdivide an edge, and a *join vertex* otherwise. In the ordinary setting, by placing two cops on arbitrary join vertices a, b, they can catch the robber within three moves. Indeed, if the robber starts on the internal vertex between two join vertices u, w, then both cops move toward u, w on their first move. Regardless of which of u, w the robber moves to, a cop will be adjacent, and can catch him on the next turn. Otherwise, if the robber starts on a join vertex v, then the cop at a moves to the internal vertex between a and v. The robber must move to an internal vertex, say between v and w. The cop at a follows him to v, and the other cop moves to the internal vertex between b and w. The robber will now be caught in the next round.

On the other hand, if the robber moves at speed 2, then $\lceil n/2 \rceil$ cops are required to catch him. To see this, note that any $m < \lceil n/2 \rceil$ cops can be immediately adjacent to only at most $2m < n$ join vertices. So, the robber can choose a non-dominated join vertex, say v, to start on and wait. When a cop moves adjacent to him, there will be at least one join vertex w with no adjacent cop. Importantly, the vertex between v and w is unoccupied, since otherwise a cop would be adjacent to w. So, the robber can advance to w in a single move, and again be nonadjacent to any cop. He can repeat this indefinitely, eluding $\lceil n/2 \rceil - 1$ cops.

Note that if $\lceil n/2 \rceil$ cops are used, they can initially sit on internal vertices so that all join vertices are dominated. The robber must then select an internal vertex for his initial position, say between join vertices v and w. These two vertices are not dominated by the same cop, because the only vertex which does is occupied by the robber. So, the two cops that dominate v and w can advance to occupy v and w in their first turn. This traps the robber, and he will be captured in the next round. ■

Let us now turn our attention to upper bounds. Unfortunately, diameter-based arguments completely break down, because Lemma 2.1(ii) does not hold for fast robbers. However, since even a fast robber cannot pass through vertices occupied by cops, Lemma 2.2 still applies. Therefore, we can adapt the proof from the previous section to this case. The first step is to extend Lemma 2.1(i) to this setting.

Lemma 5.2. *Let n,p,R be given, with np sufficiently large. Every n-vertex graph G has a set U of $2pn$ vertices such that the following holds. Place R cops on each vertex of U, and let the robber choose a starting position. Then there is a set S of size at most $((16/p)\log n)^{2^R}$ such that the robber's position after his first move must lie in S.*

Proof. Let $k = (16/p)\log n$. We construct U such that every vertex v with $|B_1(v)| \geq k$ has at least $|B_1(v)| \cdot p/2$ vertices of U in $N(v)$. By independently including each vertex with probability p, the probability that this property fails at a fixed v is at most $e^{-kp/8} = n^{-2}$ by the Chernoff bound. Combining a union bound over all v with the fact that $\mathrm{Bin}[n,p]$ is at most $2np$ **whp**, we see that we have positive probability of obtaining the desired construction.

Now let $C_1, \ldots, C_R$ be R sets of cops, where each set has one cop on each vertex of U. The robber cannot select an initial vertex with degree at least k, or else he will be adjacent to a cop in C_1 (who will catch him immediately, since cops move first). So, assume that the robber's initial vertex v has $|B_1(v)| \leq k$.

We will simultaneously dispatch the cops in $C_2, \ldots, C_R$, so that in their first move, they occupy high-degree vertices in the vicinity of v. Consider the vertices of $B_1(v)$ which have degree at least $(2/p)k$. By construction, each such vertex will have at least $k \geq |B_1(v)|$ cops in C_2 in its neighborhood, so by the greedy algorithm, we may send cops in C_2 to occupy these vertices before the robber has a chance to move. Since the robber cannot pass through any cops, he must avoid these vertices forever.

Let $S_1 \subseteq B_1(v)$ be the remaining vertices, and let $S_2 = B_1(S_1)$. These are the potential positions that the robber can reach within distance 2, and we have $|S_2| \leq (2/p)k^2$.

Repeating this argument again, we see that we may send cops in C_3 to occupy vertices in S_2 that have degree at least $(2/p)\cdot(2/p)k^2 = (2/p)^2k^2$. Then, $S_3 = B_1(S_2)$ is the set of potential positions that the robber can reach within distance 3, and $|S_3| \leq (2/p)^2k^2 \cdot |S_2| \leq (2/p)^3k^4$. Continuing in this way, we see that after dispatching C_R, we have restricted the set of positions that the robber can reach within distance R to a set of size at most

$$\left(\frac{2}{p}\right)^{2^{R-1}-1} k^{2^{R-1}} < k^{2^R}.$$

This is the maximum distance he can cover in his first move. ■

Now we are ready to extend our earlier proof to the fast robber setting. Since Lemma 2.2 holds for fast robbers, the obvious translation of Corollary 3.3 implies that Theorem 1.3 is a consequence of the following lemma.

Lemma 5.3. *Let R be a positive integer, and let $\alpha = 1 + 1/R$. There is a function $p = p(n) = \alpha^{-(1-o(1))\sqrt{\log_\alpha n}}$ for which the following holds. In every connected graph G on $m \leq n$ vertices with maximum degree less than $1/p$ and expansion at least p, $(1+o(1))\sqrt{\log_\alpha n} \cdot 2pn$ cops can always capture a speed-R robber.*

Proof. As usual, we may assume that $m > pn$, or else we are trivially done. Define the sequence $d_0, d_1, \ldots$ via the recursion $d_0 = R$, $d_{i+1} = d_i + \lceil d_i/R \rceil$. Let $r_i = \lceil d_i/R \rceil$. Inequality A.3 shows that there is a function $p = \alpha^{-(1-o(1))\sqrt{\log_\alpha n}}$ and a positive integer $l = (1+o(1))\sqrt{\log_\alpha n}$ such that when we define $k = (16/p)\log n$, we have the inequalities

$$k^{2^R} \cdot k^l \leq (1+p)^{-r_l} \frac{np}{2} \quad \text{and} \quad (1+p)^{r_l} \geq k. \tag{2}$$

We use Lemma 5.2 to distribute $2Rpn$ cops such that the robber's position after his first move will always be contained in a set N_0 of size at most k^{2^R}. We split the remaining cops into $l+1$ groups $C_0, C_1, \ldots, C_l$. Choose the initial positions of the cops in C_i by applying Lemma 4.2 with parameter r_i. Let the robber's initial position be v.

The rest of the proof is nearly identical to that of Lemma 4.4. At each step, we consider the robber's set of possible intermediate positions in $B_{d_i}(v)$, which he occupies on or after his $\lfloor d_i/R \rfloor$th move, but strictly before the completion of his $(\lfloor d_i/R \rfloor + 1)$-st move. We let this set be N_i, and inductively assume it has size at most $k^{2^R} \cdot k^i$.

Since the cops move first, they can travel by distance $\lfloor d_i/R \rfloor + 1 \geq r_i$ by this time. Lemma 4.1 partitions $N_i = S_i \cup T_i$ such that $|B_{r_i}(S_i)| \leq k|S_i|$ and every subset $U \subseteq T_i$ has $|B_{r_i}(U)| \geq k|U|$. By construction of C_i, Hall's theorem shows us how to send a distinct cop from C_i to each vertex of T_i. Hence the robber actually cannot occupy T_i anytime after his $\lfloor d_i/R \rfloor$th move. Therefore, the set of positions in $B_{d_{i+1}}(v)$ which the robber may occupy between his $\lfloor d_{i+1}/R \rfloor$th and $(\lfloor d_{i+1}/R \rfloor + 1)$-st moves is restricted to some N_{i+1}, of size at most $k^{2^R} \cdot k^{i+1}$.

This procedure terminates because when we partition $N_l = S_l \cup T_l$, the expansion property ensures that $S_l = \emptyset$. Indeed, the inequalities in (2) are precisely what are required to show that the sets are small enough to expand, and that their radii are large enough for the expansion factor to exceed k. Therefore, the cops in C_l can completely cover N_l within r_l moves. Since N_l was the set of possible positions for the robber within distance d_l, the robber is captured. ■

6. LOWER BOUND FOR INFINITELY FAST ROBBER

We now proceed to prove that the $\Omega(\sqrt{n})$ general lower bound can be sharpened considerably in the setting when the robber moves faster than the cops. As a warm-up, we start with the second part of Theorem 1.4, which states that there are n-vertex graphs on which an infinitely fast robber can always evade cn cops, for an absolute constant c. The graphs will be instances of $G_{n,p}$ with $p=200/n$. We will need some routine lemmas about $G_{n,p}$.

Lemma 6.1. *Let $p=200/n$. Then, **whp** every set of $s \le 0.6n$ vertices in $G_{n,p}$ has average degree at most $0.9np$.*

Proof. Note that the average degree condition is equivalent to enforcing that each such set spans at most $(s/2)\cdot 0.9np$ edges. We will take a union bound, but we split the range for s into three parts. First, consider any set S of $0.3n<s\le 0.6n$ vertices. The number of edges in S is $\text{Bin}[\binom{s}{2},p]$, so $(s/2)\cdot 0.9np$ exceeds its mean by a factor of at least 50%. Therefore, the Chernoff bound (Fact 2.3) implies that the probability that S fails is at most $e^{-0.1\cdot(s^2/2)\cdot p} \le e^{-3s}$. Taking a union bound over all such S with $0.3n<s<0.6n$, we accumulate a failure probability of at most

$$\sum_{s=0.3n}^{0.6n} \binom{n}{s} e^{-3s} \le \sum_{s=0.3n}^{0.6n} \left(\frac{en}{s} e^{-3}\right)^s \le \sum_{s=0.3n}^{0.6n} \left(\frac{e}{0.3} e^{-3}\right)^s \le \sum_{s=0.3n}^{0.6n} 2^{-s}. \tag{3}$$

Next, we consider s in the range $\log n<s\le 0.3n$. Here, we use a simpler bound for the probability that a given set of s vertices spans more than $(s/2)\cdot 0.9np$ edges. Combining this with a union bound over all S of these sizes, we bound the total failure probability in this range by:

$$\begin{aligned}\sum_{s=\log n}^{0.3n} \binom{n}{s} \binom{\frac{s^2}{2}}{\frac{s}{2}\cdot 0.9np} p^{\frac{s}{2}\cdot 0.9np} &\le \sum_{s=\log n}^{0.3n} \left(\frac{en}{s}\right)^s \left(\frac{es}{0.9np}\right)^{\frac{s}{2}\cdot 0.9np} p^{\frac{s}{2}\cdot 0.9np} \\ &\le \sum_{s=\log n}^{0.3n} \left[\frac{en}{s}\cdot\left(\frac{es}{0.9n}\right)^{90}\right]^s. \end{aligned} \tag{4}$$

Since the quantity in the square brackets increases with s (its exponent is $+89$), we may replace s with its maximum value in this range, to obtain an upper bound of:

$$\sum_{s=\log n}^{0.3n} \left[\frac{en}{0.3n}\cdot\left(\frac{e\cdot 0.3n}{0.9n}\right)^{90}\right]^s = \sum_{s=\log n}^{0.3n} \left[\frac{e}{0.3}\cdot\left(\frac{e}{3}\right)^{90}\right]^s < \sum_{s=\log n}^{0.3n} 2^{-s}. \tag{5}$$

For the final range $1 \le s \le \log n$, we may substitute $\log n$ for s into Inequality (4), for the same reason as above. So, the total failure probability in that range is at most

$$\sum_{s=1}^{\log n} \left[\frac{en}{\log n} \cdot \left(\frac{e \log n}{0.9n} \right)^{90} \right]^s \le \log n \cdot \frac{en}{\log n} \cdot \left(\frac{e \log n}{0.9n} \right)^{90} = O\left(\frac{\log^{90} n}{n^{89}} \right). \tag{6}$$

Combining Inequalities (3), (5), and (6), we obtain the desired result. ■

Our next lemma allows us to delete small (but linear-size) vertex subsets without destroying too many edges. Let us say that an edge is *covered* by a vertex subset if one of its endpoints is in the subset.

Lemma 6.2. *Let λ, c be positive real constants such that $c > e \cdot (e/4)^{4\lambda}$. Then, for $p = \lambda/n$, **whp** every set of cn vertices in $G_{n,p}$ covers at most $4np \cdot cn$ edges.*

Proof. Any given set of cn vertices is potentially incident to $\binom{cn}{2} + c(1-c)n^2 \le cn^2$ edges, each of which is independently present with probability p. So, the probability that over $4np \cdot cn$ appear is at most

$$\binom{cn^2}{4np \cdot cn} p^{4np \cdot cn} \le \left(\frac{e}{4p} \right)^{4np \cdot cn} p^{4np \cdot cn} = \left(\frac{e}{4} \right)^{4\lambda \cdot cn}.$$

Therefore, taking a union bound over all subsets of size cn, the total failure probability is at most

$$\binom{n}{cn} \cdot \left(\frac{e}{4} \right)^{4\lambda \cdot cn} \le \left[\frac{e}{c} \cdot \left(\frac{e}{4} \right)^{4\lambda} \right]^{cn} = \alpha^{cn},$$

for some constant $\alpha < 1$. Hence this probability tends to zero, as claimed. ■

We are now ready for the proof of Theorem 1.4(ii).

Proof of Theorem 1.4(ii). Let $c = 800^{-2}$, and let G be an instance of $G_{n,p}$ with $p = 200/n$. The previous lemmas, together with a classical result regarding the giant component, show that we can ensure that G has the following properties **whp**:

(i) G has at least $99n$ edges (Chernoff).
(ii) Every subset of cn vertices covers at most $800cn$ edges.
(iii) Every subset of $800cn$ vertices covers at most $800^2 cn$ edges.
(iv) Every subset of at most $0.6n$ vertices has average degree at most $0.9np$.
(v) G has a connected component H which contains at least $0.99n$ vertices [7, Theorem 6.11].

We claim that these properties are enough to allow the robber to escape cn cops indefinitely on the connected graph H. This will establish Theorem 1.4(ii) because $c = 800^{-2}$ and H has at least $0.99n$ vertices by (v).

Indeed, suppose that there are only cn cops, and let C be the set of vertices that they initially occupy. Let C^+ be the union of C with all immediate neighbors of vertices in C, and let U be the complement of C^+. Property (ii) shows that $|C^+| \le 800cn$, so by (iii), the total number of edges covered by C^+ is always at most $800^2 cn = n$. So, $G[U]$ induces at least $98n$ edges, and hence has average degree at least $0.98np$. Some connected component of $G[U]$ must have at least that average degree, and (iv) shows that it must then have size at least $0.6n$. Therefore, $G[U]$ always has a connected component of at least this size.

The robber's strategy is to initially place himself in an arbitrary vertex v of the largest connected component of $G[U]$, which has size at least $0.6n$. After the cops move, let U' be the complement of the new C^+. There must still be a connected component of size at least $0.6n$ in $G[U']$; the robber selects an arbitrary vertex x in it. Since these two large components both have size at least $0.6n$, they must overlap in some vertex w. Therefore, there is a path P_1 from v to w entirely contained in U, and a path P_2 from w to x entirely contained in U'. Yet even though the cops have moved, by definition of U, their current positions are still outside of U, since U excluded their old immediate neighborhoods. Therefore, both paths P_i completely avoid all cops, so the robber can indeed move to x in his turn. This preserves the condition that he is always in the largest connected component outside C^+, so he can repeat this indefinitely. ■

Remark. A more careful implementation of the above argument allows the robber to escape when his speed R is not infinite, but rather at least $C\log n$ for some large enough constant $C>0$. This is due to the fact that the large connected subgraph of $G[U]$ in the above argument can be chosen in addition to be of logarithmic diameter, allowing the robber to escape from it to U' in a logarithmic number of steps. A similar argument is presented in more detail in the next section.

7. LOWER BOUNDS FOR FINITE-SPEED ROBBER

We now extend the ideas of the previous section to prove the first part of Theorem 1.4. In the last section, connectivity alone was enough, since the robber could move infinitely quickly. Here, we also need to control the lengths of the paths involved. The graph will still be an instance of $G_{n,p}$, but this time p will be of order $n^{(1/(R-2))-1}$. As usual, we begin by stating some routine facts about $G_{n,p}$.

Lemma 7.1. *In $G_{n,p}$, **whp** there is an edge between every pair of disjoint sets of size $s_0=(3/p)\log n$.*

Proof. For any fixed pair of disjoint sets of size s_0, the probability that all crossing edges are absent is at most $(1-p)^{s_0^2}$. There are at most $\binom{n}{s_0}^2$ ways to choose these sets, so a union bound implies that the probability that this property does not hold in $G_{n,p}$ is at most

$$\binom{n}{s_0}^2(1-p)^{s_0^2}\le\left(\frac{en}{s_0}\right)^{2s_0}e^{-ps_0^2}=\left[\left(\frac{en}{s_0}\right)^2e^{-ps_0}\right]^{s_0}$$

$$\le[n^2e^{-3\log n}]^{s_0}=n^{-s_0}=o(1).$$ ■

Lemma 7.2. *In $G_{n,p}$, **whp** every set of size $s\le s_0=(3/p)\log n$ spans at most $s\cdot 6\log n$ edges.*

Proof. For fixed s, the number of sets of s vertices is $\binom{n}{s}$. The probability that a particular set of s vertices spans at least $k=s\cdot 6\log n$ edges is at most $\binom{s^2/2}{k}p^k$. Therefore, the probability that our property fails for a certain fixed s is at most:

$$\begin{aligned}\binom{n}{s}\cdot\binom{s^2/2}{s\cdot 6\log n}p^{s\cdot 6\log n} &\leq \left(\frac{en}{s}\right)^s\cdot\left(\frac{esp}{12\log n}\right)^{s\cdot 6\log n}\\ &=\left[\frac{en}{s}\cdot\left(\frac{esp}{12\log n}\right)^{6\log n}\right]^s\\ &\leq\left[en\cdot\left(\frac{e}{4}\right)^{6\log n}\right]^s.\end{aligned}$$

Since $(e/4)^6\approx e^{-2.3}$, this probability is at most n^{-s} for large n. Taking a final union bound over all $s\leq s_0$, we see that the total failure probability is still $o(1)$, as claimed. ■

Lemma 7.3. *Let $\gamma>0$ be fixed, and suppose $np\to\infty$. Then $G_{n,p}$ has the following property **whp**. For every integer t between γnp and $(\gamma^3/2e^5)n$, every subset U of t vertices has the property that the number of vertices $v\notin U$ with $d_U(v)\geq\gamma np$ is at most $3\cdot t/\gamma np$.*

Proof. Let $k=3\cdot(t/\gamma np)$. For each fixed t, there are $\binom{n}{t}$ ways to choose the set U, and at most $\binom{n}{k}$ ways to choose k vertices outside U. For each of these vertices, the number of neighbors in U is distributed as $\text{Bin}[t,p]$, and the probability that this Binomial random variable exceeds γnp is at most $\binom{t}{\gamma np}p^{\gamma np}$. Putting this all together, the probability that our property fails for a certain fixed value of t is at most:

$$\begin{aligned}\binom{n}{t}\cdot\binom{n}{k}\cdot\left[\binom{t}{\gamma np}p^{\gamma np}\right]^k &\leq\binom{n}{t}^2\cdot\left[\binom{t}{\gamma np}p^{\gamma np}\right]^k\\ &\leq\left(\frac{en}{t}\right)^{2t}\cdot\left[\frac{etp}{\gamma np}\right]^{\gamma np\cdot k}\\ &=\left(\frac{en}{t}\right)^{2t}\cdot\left[\frac{et}{\gamma n}\right]^{3t}\\ &=\left[\frac{e^5}{\gamma^3}\cdot\frac{t}{n}\right]^t.\end{aligned}$$

Since t is at most $T=(\gamma^3/2e^5)n$, the final bound is at most 2^{-t}. Summing over all t from γnp to T, we see that since $np\to\infty$, the total failure probability is still $o(1)$, as desired. ■

Now we move to Theorem 1.4(i), which is more convenient to prove in the following (equivalent) reparameterized form.

Proposition 7.4. *Let $c>0$ be fixed, and let $p=(c^3/30000)n^{c-1}$. Then **whp** $G_{n,p}$ has the property that a robber with speed $(1/c)+2$ can always escape from n^{1-c} cops.*

Proof. Condition on the high-probability properties in Lemmas 7.1, 7.2, and 7.3 with $\gamma = c/4$. Also condition on the high-probability event that all degrees of $G_{n,p}$ are between $0.9np$ and $1.1np$. Note that $np = (c^3/30000)n^c$.

Let us specify the robber's winning strategy. The cops place themselves first. Let C be the set of vertices occupied by cops, and let C^+ be the union of C with all immediate neighbors of vertices in C. Since $|C| \leq n^{1-c}$ and all vertices have degree at most $1.1 \cdot (c^3/30000)n^c$, it follows that $|C^+| \leq (\gamma^3/2e^5)n$, where $\gamma = c/4$.

The robber's strategy uses the notion of the *k-core* of a graph, which is the largest induced subgraph that has all degrees at least k. It is well known that the k-core can always be obtained by repeatedly deleting all vertices of degree less than k, and the result is independent of the order in which these deletions are performed. Let H be the $np/3$-core of the graph induced by vertices outside C^+. Our first claim is that H always has size at least $(1-c^3)n$.

Indeed, since we conditioned on all degrees exceeding $0.9np$, as well as on the result of Lemma 7.3 with $\gamma = c/4$, the deletion of C^+ cannot hurt our minimum degree condition by very much. To be precise, the resulting graph has minimum degree at least $(0.9-\gamma)np$, except for some small set of vertices U_1 of size at most $(3/\gamma np) \cdot |C^+| \leq (10^6/c^4)n^{-c} \cdot |C^+|$. Applying the same result again, we find that after deleting U_1, the resulting graph has minimum degree at least $(0.9-2\gamma)np$, except for some even smaller set U_2 of size at most $((10^6/c^4)n^{-c})^2 \cdot |C^+|$. Repeatedly applying this result, we see that since $|C^+| \leq n$, this procedure must certainly terminate within $2/c$ iterations, giving a subgraph with all degrees at least $(0.9-(2/c)\gamma)np > np/3$. The total number of deleted vertices is at most

$$|C^+| \cdot \left[1 + \left(\frac{10^6}{c^4}n^{-c}\right) + \left(\frac{10^6}{c^4}n^{-c}\right)^2 + \cdots\right] < |C^+| \cdot 2 < c^3 n,$$

as claimed.

The robber's strategy is to choose an arbitrary vertex in H for his initial position. The cops then make their move, and occupy a new set of vertices C'. Let H' be the new $np/3$-core of the graph induced by all vertices except those in C' and its immediate neighborhood. Our final claim is that the robber can always move to a vertex in H'. Clearly, this will imply that the robber can evade the cops indefinitely.

We must show that there exists a path of length at most $(1/c)+2$ from the robber's current position to a vertex in H', which completely avoids C'. The main observation is that $C' \subseteq C^+$, because C^+ was defined to include all possible positions of cops in their next turn. Therefore, since the robber is in H (the $np/3$-core of $G \setminus C^+$), he has quite a lot of freedom to move without running into any of the cops positioned at C'.

More precisely, we will show that by traversing at most $(1/c)+1$ edges in H, the robber can already reach $s_0 = (3/p)\log n$ vertices. Indeed, let $S_0 = \{x\}, S_1, S_2, \ldots,$ be the sequence of sets in a breadth-first search performed in H from the robber's current position x. Let $T_i = S_0 \cup S_1 \cup \cdots \cup S_i$. Thus $T_{i+1} = T_i \cup N_H(T_i)$. Now suppose that $|T_{i+1}| \leq s_0$. It follows from our conditioning on Lemma 7.2 that

$$e(T_{i+1}) \geq |T_i| \left(\frac{np}{3} - 6\log n\right).$$

Applying this same result once again we see that

$$|T_{i+1}| \geq \frac{e(T_{i+1})}{6\log n} \geq \frac{np|T_i|}{20\log n}.$$

Since $np = (c^3/30000)n^c$, it follows that if $i_0 = (1/c)+1$ then $|T_{i_0}| \geq s_0$.

Yet we also conditioned on Lemma 7.1, so since $|H'| \geq (1-c^3)n$, there is an edge between T_{i_0} and H'. Therefore, since the robber is permitted to traverse $(1/c)+2$ edges in a single move, he can indeed land on a vertex in H' without passing through any vertex (in C') currently occupied by a cop. ■

8. CONCLUDING REMARKS

We have considered the directed version of the classical Cops and Robbers game, and also the version where the robber moves R edges at a time, but the cops move only one edge at a time. Our approach generalized the best known upper bound to the fast robber setting, and coincidentally reproved the same asymptotic in the original setting. However, for directed graphs, our general upper bound is weaker than the corresponding bound for the undirected case. It would be nice to obtain an upper bound for directed graphs with asymptotics similar to our other upper bounds in this paper. On the other hand, it may also be interesting to study the lower bound for directed graphs.

On the topic of lower bounds, the fast robber lower bound of $n^{1-1/(R-2)}$ we derived is only interesting for $R \geq 5$. It would be nice to know whether or not an $\omega(\sqrt{n})$ lower bound can already be achieved for $R=2$. Another possible version to address is when the cops and the robber both move at the *same* speed $R>1$. Our upper bound on the number of cops in the fast robber scenario still carries over, since faster cops are more powerful. It would be interesting to decide whether there is a better lower bound of $\omega(\sqrt{n})$ for this case.

APPENDIX A: ROUTINE INEQUALITIES

Inequality A.1. *Let* $p = 13(\log\log n)^2/\log n$, *and let* $r = (6/p)\log(4/p)$. *Then*

$$2p^{-r}(1+p)^r \leq \frac{np}{2} \quad \text{and} \quad (1+p)^r \geq \frac{16}{p}\log n.$$

Proof. The left-hand side of the first inequality is at most

$$2p^{-r}(1+p)^r \leq 2p^{-r}e^{pr} = 2p^{-r}\left(\frac{4}{p}\right)^6 \leq \left(\frac{4}{p}\right)^{r+6}.$$

Therefore, it suffices to show that $(4/p)^{r+7} \leq n$, or equivalently, that

$$(r+7)\log\frac{4}{p} \leq \log n.$$

Yet $r+7 \le 2r$, and $2r\log(4/p) = (12/p)(\log(4/p))^2$, so plugging in the definition of p, we see that this is indeed less than $\log n$.

For the second inequality, since p is small,

$$(1+p)^r \ge e^{pr/2} = \left(\frac{4}{p}\right)^3 > \frac{16}{p}\log n,$$

since $1/p = \log n/(13\ (\log\log n)^2)$. This completes the proof. ∎

Inequality A.2. *There is a function* $p = p(n) = 2^{-(1-o(1))\sqrt{\log_2 n}}$ *and a positive integer* $l = (1+o(1))\sqrt{\log_2 n}$ *such that when we define* $k = (16/p)\log n$, *we have the inequalities*

$$k^{l+1}(1+p)^{2^l} \le np/2 \quad \textit{and} \quad (1+p)^{2^l} \ge k.$$

Proof. First observe that we will have $\log(1/p) = \Theta(\sqrt{\log n})$, so $\log k = (1+o(1))\log(1/p)$. Let l be the smallest positive integer for which the second inequality is satisfied. This immediately gives $(1+p)^{2^l} \le k^2$. Also,

$$l = \left\lceil \log_2\left(\frac{\log k}{\log(1+p)}\right)\right\rceil = \left\lceil \log_2\left((1+o(1))\frac{1}{p}\log\frac{1}{p}\right)\right\rceil = (1+o(1))\log_2\frac{1}{p}.$$

To establish the first inequality, we have $k^{l+1}(1+p)^{2^l} \le k^{l+1}k^2$, so it suffices to show that

$$(l+3)\log_2 k \le \log_2\frac{np}{2}. \tag{A.1}$$

From the asymptotics of p, we have

$$l+3 = (1+o(1))\log_2\frac{1}{p},$$

$$\log_2 k = (1+o(1))\log_2\frac{1}{p},$$

$$\log_2\frac{np}{2} = (1-o(1))\log_2 n,$$

so it is clear that Inequality (A.1) is satisfied for an appropriate choice of $p = 2^{-(1-o(1))\sqrt{\log_2 n}}$. ∎

Inequality A.3. *Let* $R > 1$ *be given, and define the sequence* $d_0, d_1, \ldots$ *via the recursion* $d_0 = R$, $d_{i+1} = d_i + \lceil d_i/R \rceil$. *Let* $r_i = \lceil d_i/R \rceil$. *Then there is a function*

$$p = p(n) = \left(1+\frac{1}{R}\right)^{-(1-o(1))\sqrt{\log_{1+\frac{1}{R}} n}}$$

and a positive integer $l = (1+o(1))\sqrt{\log_{1+1/R} n}$ *such that when we define* $k = (16/p)\log n$, *we have the inequalities*

$$k^{2^R} \cdot k^l (1+p)^{r_l} \le \frac{np}{2} \quad \textit{and} \quad (1+p)^{r_l} \ge k.$$

Proof. The proof is nearly identical to the previous lemma. We will have $\log(1/p)=\Theta(\sqrt{\log n})$, so $\log k=(1+o(1))\log(1/p)$. Let l be the smallest positive integer for which the second inequality is satisfied. Since $r_{l+1}\le 2r_l$, this immediately gives $(1+p)^{r_l}\le k^2$.

Let us estimate an asymptotic upper bound for l. Observe that $d_i\ge(1+1/R)d_{i-1}$, so $r_l\ge(1+1/R)^l$. Hence if we let l' satisfy

$$(1+p)^{(1+\frac{1}{R})^{l'}}=k,$$

then $l\le l'$. Yet

$$l'=\log_{1+\frac{1}{R}}\frac{\log k}{\log(1+p)}=\log_{1+\frac{1}{R}}\left((1+o(1))\frac{1}{p}\log\frac{1}{p}\right)=(1+o(1))\log_{1+\frac{1}{R}}\frac{1}{p}.$$

To establish the first inequality, we initially noted that $(1+p)^{r_l}\le k^2$, so $k^{2^R}\cdot k^l(1+p)^{r_l}\le k^{2^R+2+l}$. Thus it suffices to show that

$$(2^R+2+l)\log_{1+\frac{1}{R}}k\le\log_{1+\frac{1}{R}}\frac{np}{2}. \tag{A.2}$$

From the asymptotics of p, we have

$$2^R+2+l\le(1+o(1))l'=(1+o(1))\log_{1+\frac{1}{R}}\frac{1}{p},$$

$$\log_{1+\frac{1}{R}}k=(1+o(1))\log_{1+\frac{1}{R}}\frac{1}{p},$$

$$\log_{1+\frac{1}{R}}\frac{np}{2}=(1-o(1))\log_{1+\frac{1}{R}}n,$$

so Inequality (A.2) is clearly satisfied by appropriately choosing $\log_{1+1/R}1/p=(1-o(1))\sqrt{\log_{1+1/R}n}$. This is precisely the asymptotic claimed in our statement. ■

ACKNOWLEDGMENTS

The authors thank the anonymous referees for their careful reading of this paper.

REFERENCES

[1] M. Aigner and M. Fromme, A game of cops and robbers, Discrete Appl Math 8 (1984), 1–12.
[2] N. Alon and J. Spencer, The Probabilistic Method, 3rd edition, Wiley, New York, 2007.
[3] B. Alspach, Searching and sweeping graphs, Le Matematiche 59 (2004), 5–37.
[4] T. Andreae, Note on a pursuit game played on graphs, Discrete Appl Math 9 (1984), 111–115.
[5] T. Andreae, On a pursuit game played on graphs for which a minor is excluded, J Combin Theory Ser B 41 (1986), 37–47.

[6] A. Berarducci and B. Intrigila, On the cop number of a graph, Adv Appl Math 14 (1993), 389–403.
[7] B. Bollobás, Random Graphs, Academic Press, London, 1985.
[8] B. Bollobás, G. Kun, and I. Leader, Cops and robbers in a random graph, submitted.
[9] A. Bonato, E. Chiniforooshan, and P. Prałat, Cops and Robbers from a Distance, Theoret Comput Sci 411 (2010), 3834–3844.
[10] E. Chiniforooshan, A better bound for the cop number of general graphs, J Graph Theory 58 (2008), 45–48.
[11] N. Clarke, A witness version of the Cops and Robber game, Discrete Math 309 (2009), 3292–3298.
[12] S. Fitzpatrick and R. Nowakowski, Copnumber of graphs with strong isometric dimension two, Ars Combin 59 (2001), 65–73.
[13] F. Fomin, P. Golovach, J. Kratochvíl, N. Nisse, and K. Suchan, Pursuing a fast robber on a graph, Theoret Comput Sci 411 (2010), 1167–1181.
[14] F. Fomin and D. Thilikos, An annotated bibliography in guaranteed graph searching, Theoret Comput Sci 399 (2008), 236–245.
[15] P. Frankl, Cops and robbers in graphs of large girth and Cayley graphs, Discrete Appl Math 17 (1987), 301–305.
[16] P. Frankl, On a pursuit game on Cayley graphs, Combinatorica 7 (1987), 67–70.
[17] A. Goldstein and E. Reingold, The complexity of pursuit on a graph, Theoret Comput Sci 143 (1995), 93–112.
[18] G. Hahn, Cops, robbers, and graphs, Tatra Mt Math Publ 36 (2007), 163–176.
[19] G. Hahn and G. MacGillivray, A note on k-cop, l-robber games on graphs, Discrete Math 306 (2006), 2492–2497.
[20] Y. Hamidoune, On a pursuit game on Cayley digraphs, European J Combin 8 (1987), 289–295.
[21] V. Isler, S. Kannan, and S. Khanna, Randomized pursuit-evasion with local visibility, SIAM J Discrete Math 20 (2006), 26–41.
[22] L. Lu and X. Peng, On Meyniel's conjecture of the cop number, submitted.
[23] T. Łuczak and P. Prałat, Chasing robbers on random graphs: zigzag theorem, Random Struct Algorithms 37 (2010), 516–524.
[24] M. Maamoun and H. Meyniel, On a game of policemen and robber, Discrete Appl Math 17 (1987), 307–309.
[25] R. Nowakowski and P. Winkler, Vertex to vertex pursuit in a graph, Discrete Math 43 (1983), 235–239.
[26] A. Quilliot, A short note about pursuit games played on a graph with a given genus, J Combin Theory Ser B 38 (1985), 89–92.
[27] A. Scott and B. Sudakov, A new bound for the cops and robbers problem, ArXiv:1004.2010.
[28] P. Seymour and R. Thomas, Graph searching and a min–max theorem for tree-width, J Combin Theory Ser B 58 (1993), 22–33.