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Borel Chromatic Numbers

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We study in this paper graph coloring problems in the context of descriptive set theory. We consider graphs $\mathcal{G} = (X, R)$, where the vertex set X is a standard Borel space (i.e., a complete separable metrizable space equipped with its σ-algebra of Borel sets), and the edge relation $R \subseteq X^2$ is "definable", i.e., Borel, analytic, co-analytic, etc. A *Borel n-coloring* of such a graph, where $1 \leqslant n \leqslant \aleph_0$, is a Borel map $c\colon X \to Y$ with $\operatorname{card}(Y) = n$, such that $xRy \Rightarrow c(x) \neq c(y)$. If such a Borel coloring exists we define the *Borel chromatic number* of $\mathcal{G}$, in symbols $\chi_B(\mathcal{G})$, to be the smallest such n. Otherwise we say that $\mathcal{G}$ has *uncountable Borel chromatic number*, in symbols $\chi_B(\mathcal{G}) > \aleph_0$.

In Section 3 we discuss several interesting examples of Borel graphs $\mathcal{G}$ for which the usual chromatic number $\chi(\mathcal{G})$ is small while its Borel chromatic number $\chi_B(\mathcal{G})$ is large. For instance, there are examples of Borel graphs $\mathcal{G}$

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which are acyclic, so that $\chi(\mathcal{G})=2$, but $\chi_B(\mathcal{G})>\aleph_0$. This phenomenon is related to the absence of a "definable" transversal for the connected components of such graphs (i.e., a "definable" set containing exactly one element in each connected component of the graph).

Of particular interest to us here are graphs $\mathcal{G}_{\{F_n\}}=(X, R_{\{F_n\}})$ generated by a countable sequence of Borel functions $F_n\colon X\to X$ on a standard Borel space, where

$$xR_{\{F_n\}}y \Leftrightarrow \exists n(F_n(x)=y \text{ or } F_n(y)=x).$$

When each F_n is countable-to-1, the graph $\mathcal{G}_{\{F_n\}}$ is *locally countable* (i.e., every vertex has countably many neighbors) and conversely every locally countable Borel graph is of the form $\mathcal{G}_{\{F_n\}}$ for some sequence $\{F_n\}$ of Borel countable-to-1 functions.

In Section 4 we prove some basic facts about such graphs in relation to their Borel chromatic numbers. For example, we show that if $\mathcal{G}$ is a Borel locally countable graph with $\chi_B(\mathcal{G})\leqslant\aleph_0$, then $\mathcal{G}$ has a Borel kernel, i.e., a maximal discrete set which is Borel. This in turn implies that for any Borel graph $\mathcal{G}$ of degree $\leqslant k<\aleph_0$, $\chi_B(\mathcal{G})\leqslant k+1$. We also show that if $\chi'_B(\mathcal{G})$ is the Borel edge chromatic number of $\mathcal{G}$ (i.e., the Borel chromatic number $\chi_B(\check{\mathcal{G}})$ of the graph $\check{\mathcal{G}}$ whose vertices are the edges of $\mathcal{G}$ with the edges connected in $\check{\mathcal{G}}$ if they have a vertex in common), then $\chi'_B(\mathcal{G})\leqslant\aleph_0$ for any Borel locally countable graph $\mathcal{G}$.

In Section 5 we study the case of the graph $\mathcal{G}_F$ generated by a single Borel function F. In this case we can completely analyze $\chi_B(\mathcal{G}_F)$. We show that it can take only the values: 1, 2, 3, $\aleph_0$ (and all these are possible). It follows (from this and results of Section 3) that for finitely many Borel functions $F_1, ..., F_n$, $\chi_B(\mathcal{G}_{F_1, ..., F_n})$ is either $\leqslant 3^n$ or is equal to $\aleph_0$. In particular, $\chi_B(\mathcal{G}_{F_1, ..., F_n})\leqslant 3^n$ if each F_i is finite-to-1. (On the other hand $\chi_B(\mathcal{G}_{\{F_n\}})>\aleph_0$ can happen if $\{F_n\}$ is an infinite sequence of Borel automorphisms.)

In Section 6 we provide a complete analysis of the situation under which $\chi_B(\mathcal{G})>\aleph_0$. We show that there is a fixed (locally countable acyclic) Borel graph $\mathcal{G}_0=(2^{\mathbb{N}}, R_0)$ on the Cantor set $2^{\mathbb{N}}$ such that $\chi_B(\mathcal{G}_0)>\aleph_0$, and for any analytic graph $\mathcal{G}=(X, R)$ on a Polish space X we have the following dichotomy: Either $\chi_B(\mathcal{G})\leqslant\aleph_0$ or else there is a continuous map $f\colon 2^{\mathbb{N}}\to X$ with $xR_0y\Rightarrow f(x)\,Rf(y)$ (i.e., a continuous homomorphism of $\mathcal{G}_0$ into $\mathcal{G}$). In many interesting cases f can be taken to be 1-1, i.e., an embedding, for example when $\mathcal{G}$ is locally countable or acyclic. Of course exactly one of the preceding alternatives can hold.

We also show that one cannot extend, in any reasonable way, this result to co-analytic graphs in ZFC alone, but as we point out in Section 6, D) there is an appropriate extension to all graphs in the determinacy context (with countable coloring replaced by wellordered coloring).

In Section 7 we discuss universality results for various classes of graphs or functions, which are of interest independently of this work on chromatic numbers. We show, for example, that there exists a universal Borel locally countable graph or locally countable acyclic graph or locally finite graph, etc. Also there exists a universal Borel countable-to-1 function, finite-to-1 function, etc. Further we show, extending results in ergodic theory from group to semigroup actions, that every Borel action of a countable semigroup by countable-to-1 functions on a standard Borel space, which has only countably many finite invariant sets, admits a countable generator. In particular, this implies that any Borel countable-to-1 Borel function f with at most countably many periodic points can be embedded in the infinite shift map $s_\infty\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, and in particular $\mathscr{G}_f$ can be embedded in $\mathscr{G}_{s_\infty}$. We also obtain an appropriate result for embedding, in a weaker sense, $\leqslant n$-to-1 functions in the finite shift $s_n\colon n^{\mathbb{N}} \to n^{\mathbb{N}}$.

In Section 8 we primarily discuss some problems concerning a possible dichotomy characterizing when $\chi_B(\mathscr{G})$ is infinite.

Finally, in Section 9 we address another aspect, namely the possibilities for the *usual*, *non-Borel* chromatic number $\chi(\mathscr{G})$ of a *Borel graph* $\mathscr{G}$.

For an *open* graph $\mathscr{G} = (X, R)$ on a Polish space X one has that $\chi(\mathscr{G}) \leqslant \aleph_0$ or else the complete graph on $2^{\mathbb{N}}$ embeds in $\mathscr{G}$, so $\chi(\mathscr{G}) = \mathfrak{c}$. We show that $\chi(\mathscr{G})$ for *closed* graphs $\mathscr{G}$ can take any value in the set $\{1, 2, 3, ..., \aleph_0, \aleph_1, \mathfrak{c}\}$, but it is unknown if these are all the possible values. However we show that there is an F_σ graph $\mathscr{G}$ whose chromatic number is $\geqslant \theta$ for any cardinal θ for which we can find a chain in $(\mathbb{N}^{\mathbb{N}}, <^*)$ (where $<^*$ is the partial ordering of eventual dominance) of size θ, but in the forcing extension obtained by adding $\mathfrak{c}^+$ Cohen reals, $\chi(\mathscr{G})$ is bounded by the continuum of the ground model.

There are several interesting open problems that are suggested by the results in this paper. Here is a sample:

(i) Are there acyclic Borel graphs with $3 < \chi_B(\mathscr{G}) < \aleph_0$? (See 3.3)

(ii) Assume that $F_i\colon X \to X$, $i = 1, ..., n$, is $\leqslant k$-to-1 for some k. Then we show in Section 5 that $\chi_B(\mathscr{G}_{F_1, ..., F_n}) \leqslant 3^n$ and in Section 4 that $\chi_B(\mathscr{G}_{F_1, ..., F_n}) \leqslant (k+1)n + 1$. Is it true that actually $\chi_B(\mathscr{G}_{F_1, ..., F_n}) \leqslant 2n + 1$? (See 4.9).

(iii) It is interesting to compute $\chi_B(\mathscr{G})$ for various concrete $\mathscr{G}$. For example, related to (ii) we have the following problem: Consider $\mathscr{G}'$, the free part of the shift graph on $2^{\mathbb{Z} \times \mathbb{Z}}$ (a more detailed definition is given in 4.8). It is known that $3 \leqslant \chi_B(\mathscr{G}') \leqslant 5$. What is $\chi_B(\mathscr{G}')$? (See 4.8.)

(iv) Is there a dichotomy characterizing when $\chi_B(\mathscr{G})$ is infinite (for all analytic graphs $\mathscr{G}$ or any interesting subclass)? In the case $\mathscr{G} = \mathscr{G}_f$, using the results in Section 7, in some sense this problem reduces to characterizing when the shift graph restricted to a Borel $A \subseteq [\mathbb{N}]^{\mathbb{N}}$ has infinite χ_B. It has infinite χ_B if A contains a homogeneous set $[H]^{\mathbb{N}}$ ($H \subseteq \mathbb{N}$ infinite). Is the converse true? (See 8.3.)

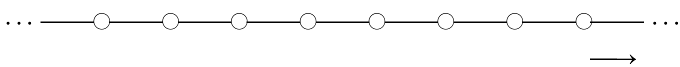
(v) Characterize exactly the chromatic numbers $\chi(\mathcal{G})$ of closed graphs on Polish spaces. (See Section 9.)

Miklos Laczkovich has answered problem (i) above affirmatively. His examples are given in an Appendix at the end of the paper and are included with his permission.

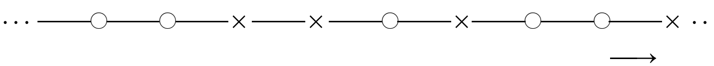
1. A MATCHING PROBLEM

We will first consider a problem, which although not directly related to our main topic of Borel chromatic numbers, gives some idea of the type of situation that can arise in our context.

Imagine a two-sided ordered discrete line L (i.e., a copy of $\mathbb{Z}$)



and divide the points in this line into two disjoint infinite sets A, B ($A \cap B = \emptyset$, $A \cup B = L$)



(○ means the point is in A, $\times$ means that it is in B). Is it possible to match "effectively" the points of A with those B (i.e., find "effectively" a 1-1 correspondence of A with B)? The answer is clearly affirmative if we are allowed to choose first an origin in the line L. But is that necessary? This is of course a rather vague problem but one can argue that the result we prove in this section (Theorem 1.1 below) provides a precise formulation which leads to the conclusion that the choice of an origin is indeed necessary.

Before we do that, however, let us consider the following more concrete situation that illustrates the difficulties with finding an "effective" matching. Let $Y = 2^{\mathbb{Z}}$ and S be the shift on Y: $S(y)(n) = y(n+1)$. Let $X \subseteq Y$ be the dense G_δ set of all aperiodic elements of Y, i.e., $x \in X \Leftrightarrow \forall n \neq 0,\ n \in \mathbb{Z}$ ($S^n(x) \neq x$). Clearly X is a shift-invariant subset of Y and every orbit of the shift on X is infinite and in fact it is a two-sided discrete line: If L is an orbit, we define an ordering on it by declaring that $x < y$ iff $\exists n > 0$ ($S^n(x) = y$). Now consider a partition $A \cup B = \emptyset$, $A \cup B = X$ of X in 2 Borel sets such that both meet each orbit in an infinite set. Thus, if L is an orbit, then $A^L = A \cap L$, $B^L = B \cap L$ is a partition of the discrete line L as before. Put

$$A \sim B \Leftrightarrow \text{there is a Borel bijection } f\colon A \to B \text{ such that for all } x \in A,\ f(x) \text{ is in the same orbit as } x.$$

Intuitively, $A \sim B$ means that one can find a matching of A^L with B^L for each orbit L "effectively". So if one could indeed have a method to match "effectively" the two infinite pieces of a partition of a discrete line, then one would always have $A \sim B$. However, it is easy to find examples of Borel A, B for which this fails. Indeed, notice that if μ is the usual coin-tossing measure on $2^{\mathbb{Z}}$ and $A \sim B$, then $\mu(A) = \mu(B)$. This is because, as it is easy to see,

$$A \sim B \Leftrightarrow \text{there are partitions } A = \bigcup_{i \in \mathbb{N}} A_i,\ B = \bigcup_{i \in \mathbb{N}} B_i \text{ into Borel sets, and } n_i \in \mathbb{Z} \text{ such that } S^{n_i}(A_i) = B_i.$$

Since μ is shift-invariant, this proves our assertion. However, it is easy to find a partition of X into Borel sets A, B meeting every orbit infinitely often, such that $\mu(A) = 1/4$, $\mu(B) = 3/4$ (for example), so that $A \not\sim B$.

The problem in this situation is of course that, as it is easy to check, there is no way to choose "effectively" an origin in each orbit of X, i.e., there is no Borel set $U \subseteq X$ having exactly one point in each orbit (a Borel transversal). We will indeed see below that this is a necessary condition for an "effective" matching. We will first set things up in context.

Suppose X is a standard Borel space and $T\colon X \to X$ an aperiodic Borel automorphism of X. (*Aperiodic* means, as before, that $\forall x\, \forall n \neq 0$, $n \in \mathbb{Z}$ $(T^n(x) \neq x)$.) Again each orbit of T in X can be viewed as a discrete line.

A *Borel transversal* for T is a Borel set $U \subseteq X$ meeting every orbit exactly once. This corresponds to choosing an origin in each orbit (viewed as a discrete line) "effectively". Thus it is clear that if a Borel transversal exists and $A \cap B = X$, $A \cup B = X$ is a Borel partition of X into 2 pieces each of which meets each orbit infinitely often, then $A \sim B$ ($\sim$ is defined exactly as before in this more general context). The next theorem (proved jointly with A. Ditzen) shows that the existence of a Borel transversal is necessary as well for this "effective" matching and thus makes precise our contention, in the beginning of this section, that the choice of an origin is necessary for "effectively" matching.

1.1. Theorem (with A. Ditzen). *Let X be a standard Borel space and $T: X \to X$ an aperiodic Borel automorphism. Then the following are equivalent*:

(i) *For every Borel partition $A \cap B = \emptyset$, $A \cup B = X$ such that A, B meet each orbit infinitely often, we have $A \sim B$.*

(ii) *There is a Borel transversal for the orbits of T.*

Proof. The direction (ii) $\Rightarrow$ (i) has been already noted.

Assume now (i) in order to prove (ii). We will need some results from the theory of hyperfinite Borel equivalence relations, for which a convenient reference is Dougherty–Jackson–Kechris [1994] (abbreviated DJK below).

Let E be a Borel equivalence relation on a standard Borel space Z. Given Borel sets $A, B \subseteq Z$ we write

$$A \sim B \Leftrightarrow \text{there is a Borel bijection}$$

$$f: A \to B \quad \text{with} \quad f(x)\, Ex, \quad \forall x \in A.$$

We say that E is *compressible* if there is a Borel set $A \subseteq Z$ with $Z \sim A$ such that $Z \backslash A$ meets every equivalence class. We call a Borel set $A \subseteq Z$ compressible if $E \mid A$ is compressible.

We are particularly interested here in the case $E = E_T$, the equivalence relation whose classes are the orbits of T (i.e., $xE_Ty \Leftrightarrow \exists n \in \mathbb{Z}(T^n(x) = y)$). In this case it follows from 2.1 of DJK that a Borel set $A \subseteq X$ is compressible (for E_T) iff there are Borel sets $B, C \subseteq A$ with $B \cap C = \emptyset$ and $A \sim B \sim C$. Also from 2.2 of DJK we have that if $A \subseteq X$ is a Borel set which is compressible, so is its saturation $[A] = \{T^n(x): n \in \mathbb{Z}, x \in A\}$.

We will first show that E_T is compressible. As usual we view every orbit L of T as ordered in order type $\mathbb{Z}$ by the relation

$$x <_L y \Leftrightarrow \exists n > 0(T^n(x) = y).$$

For each $N \in \mathbb{N}$, we can find a Borel set $M = M_N \subseteq X$ meeting each orbit unboundedly often in both directions of $<_L$, so that for any orbit L any two members of $M^L = L \cap M$ have distance at least N (in $<_L$). The existence of such M can be easily derived from the proof of (5) $\Rightarrow$ (1) in 5.1 of DJK (due to Slaman and Steel). Take $N = 3$. Then clearly $T: M \to X \backslash M$, $T^2: M \to X \backslash M$ and $T(M) \cap T^2(M) = \emptyset$. Since (i) holds, there is a Borel bijection $f: X \backslash M \to M$ such that $f(x)$ is in the same orbit as x. Then $T \circ f$, $T^2 \circ f: X \backslash M \to X \backslash M$ and $T \circ f(X \backslash M) \cap T^2 \circ f(X \backslash M) = \emptyset$, So $X \backslash M \sim T \circ f(X \backslash M) \sim T^2 \circ f(X \backslash M)$ and $T \circ f(X \backslash M)$, $T^2 \circ f(X \backslash M)$ are disjoint, thus $X \backslash M$ is compressible and so is $[X \backslash M] = X$, i.e., E_T is compressible.

Assume now that (ii) fails, i.e., there is no Borel transversal for the orbits of T. From Section 2 of DJK this means that E_T is not *smooth*, i.e., there is no Borel function $f: X \to W$, W a standard Borel space, with $xE_T x' \Leftrightarrow f(x) = f(x')$. By 3.4 of DJK (a special case of the Glimm–Effros dichotomy) there is a Borel set $Y \subseteq X$ such that $E_0 \cong_B E_T \mid Y$, where E_0 is the following equivalence relation on $2^{\mathbb{N}}$:

$$xE_0 y \Leftrightarrow \exists n\, \forall m \geqslant n(x(m) = y(m))$$

and $\cong_B$ means Borel isomorphism. Let $X' = [Y]$. Then clearly $E_T \mid X'$ is compressible, and we can also assume that $Y, X' \setminus Y$ meet every orbit contained in X' in an infinite set. So, by (i) again, we have $Y \sim X' \setminus Y$, say via g. In particular, $E_T \mid (X' \setminus Y) \cong_B E_T \mid Y \cong_B E_0$.

Given a Borel equivalence relation E on a standard Borel space Z, a Borel measure μ on Z is called *invariant* (for E) if $\mu(A) = \mu(B)$ for any two Borel sets $A, B \subseteq Z$ with $A \sim B$. Then it is clear that the usual coin-tossing measure on $2^{\mathbb{N}}$ is the unique invariant probability Borel measure for E_0. So let μ_1, μ_2 be the unique invariant probability measures for $E_T \mid Y$ and $E_T \mid (X' \setminus Y)$. Let $\mu = \mu_1 + \mu_2$. We will argue that μ is invariant for E_T, which contradicts immediately the compressibility of E_T.

Let $A, B \subseteq X'$ be Borel sets with $A \sim B$ as witnessed by $h: A \to B$. We want to show that $\mu(A) = \mu(B)$. Split A into disjoint Borel parts $A_1 = \{x \in A \cap Y: h(x) \in B \cap Y\}$, $A_2 = \{x \in A \cap Y: h(x) \in B \setminus Y\}$, $A_3 = \{x \in A \setminus Y: h(x) \in B \setminus Y\}$, $A_4 = \{x \in A \setminus Y: h(x) \in B \cap Y\}$. Then

$$\mu(A_1) = \mu_1(A_1) = \mu_1(h(A_1)) = \mu(h(A_1))$$

and similarly $\mu(A_3) = \mu(h(A_3))$. Since $E_T \mid Y, E_T \mid (X' \setminus Y)$ each have a unique invariant probability Borel measure, and g witnesses that $Y \sim X' \setminus Y$, so that in particular g is a Borel isomorphism of $E_T \mid Y, E_T \mid (X' \setminus Y)$, it follows that $g\mu_1 = \mu_2$. So $\mu_1(A_2) = \mu_2(g(A_2))$. Also $g \circ h^{-1}: h(A_2) \to g(A_2)$ and witnesses that $h(A_2) \sim g(A_2)$, so $\mu_2(h(A_2)) = \mu_2(g(A_2))$, thus $\mu(A_2) = \mu_1(A_2) = \mu_2(g(A_2)) = \mu_2(h(A_2)) = \mu(h(A_2))$. Similarly, $\mu(A_4) = \mu(h(A_4))$. Thus $\mu(A) = \mu(h(A)) = \mu(B)$ and we are done. ∎

2. GRAPHS

Let X be a set. *A graph* on X is a binary relation $R \subseteq X^2$ which is symmetric ($xRy \Rightarrow yRx$) and irreflexive ($\neg xRx$). We denote this by $\mathcal{G} = (X, R)$. If xRy we say that $\{x, y\}$ is an *edge* of $\mathcal{G}$. The elements of X are the *vertices* of g. The *neighbors* of $x \in X$ are the $y \in X$ which are joined by edges to x.

A *path* of $\mathcal{G}$ is a finite sequence $x_0, x_1, ..., x_n$ with $n \geqslant 1$ such that $\{x_i, x_{i+1}\}$ is an edge if $i < n$ and $x_i \neq x_j$ except possibly for $\{i, j\} = \{0, n\}$.

If $x = x_0, x_n = y$ and $x \neq y$ we call this a *path from* x *to* y. If $n \geqslant 3$ and $x_0 = x_n$ we call this a *cycle*. A graph with no cycles is called *acyclic*.

The *connected component* of x is the set of all vertices y for which there is a path from x to y. These are the equivalence classes of the following equivalence relation on X:

$$xE_{\mathscr{G}}y \Leftrightarrow \text{there is a path from } x \text{ to } y.$$

If there is only one connected component, we call $\mathscr{G}$ *connected*. An acyclic connected graph is called a *tree*. Clearly every connected component of a graph is connected and if the graph is acyclic it is a tree. (An acyclic, but not necessarily connected, graph is called a *forest*.)

The *degree* of a vertex x, in symbols $d(x)$, is the cardinality of the set of its neighbors. If the degree of every vertex is countable (finite) we call $\mathscr{G}$ *locally countable* (*finite*).

A *coloring* of $\mathscr{G}$ is a map $c: X \to Y$, such that $xRy \Rightarrow c(x) \neq c(y)$. If $\operatorname{card}(Y) = k$ we call this a k-*coloring*. The smallest cardinal k for which $\mathscr{G}$ admits a k-coloring is called the *chromatic number* of $\mathscr{G}$, in symbols $\chi(\mathscr{G})$. (Sometimes in graph theory the term "good coloring" is used for what we simply call "coloring" here.)

A graph $\mathscr{G} = (X, R)$ on a standard Borel space X will be called *Borel*, *analytic*, etc. if the relation $R \subseteq X^2$ is Borel, analytic, etc. Note that if $\mathscr{G}$ is analytic, so is the associated equivalence relation $E_{\mathscr{G}}$. On the other hand if $\mathscr{G}$ is Borel and acyclic, then $E_{\mathscr{G}}$ is also co-analytic, thus Borel. This is because

$$xE_{\mathscr{G}}y \Leftrightarrow x = y \text{ or } \{x \neq y \;\&\; \exists n \geqslant 1\; \exists!(x_0, ..., x_n)\; [x = x_0 \;\&\; y = x_n \;\&$$
$$\forall i < n(x_i R x_{i+1}) \;\&\; \forall i \neq j(x_i \neq x_j)]\}.$$

We will be particularly interested in graphs on standard Borel spaces generated by functions. Let X be standard Borel and $\{F_i\}_{i \in I}$ a family of functions from X into X. We denote by $\mathscr{G}_{\{F_i\}} = (X, R_{\{F_i\}})$ the graph given by

$$xR_{\{F_i\}}y \Leftrightarrow x \neq y \;\&\; \exists i(F_i(x) = y \text{ or } F_i(y) = x).$$

We write $\mathscr{G}_F$ for $\mathscr{G}_{\{F\}}$ and $\mathscr{G}_{F_1, ..., F_n}$ for $\mathscr{G}_{\{F_1, ..., F_n\}}$. If I is countable and each F_i is Borel, then $\mathscr{G}_{\{F_i\}}$ is Borel. If moreover each F_i is $\leqslant \aleph_0$-to-1 (i.e., the preimage of every point is countable), then $\mathscr{G}_{\{F_i\}}$ is locally countable. Conversely, if $\mathscr{G}$ is a locally countable Borel graph, there is a sequence of Borel functions $\{F_i\}_{i \in \mathbb{N}}$ such that $\mathscr{G} = \mathscr{G}_{\{F_i\}}$. (This is because the relation $xR^*y \Leftrightarrow x = y$ or xRy is Borel with countable sections, so there is a sequence of Borel functions $\{F_i\}_{i\in\mathbb{N}}$ such that for all x, $R^*(x)=\{y: xR^*y\}=\{F_i(x): i\in\mathbb{N}\}$.) Similarly if $\mathcal{G}$ is locally finite of bounded degree k (i.e., $d(x)\leqslant k$ for all $x\in X$), then there are k Borel functions $F_1, ..., F_k$, with $\mathcal{G}=\mathcal{G}_{F_1, ..., F_k}$ (and each F_i is $\leqslant k$-to-1).

3. BOREL CHROMATIC NUMBERS

Let $\mathcal{G}=(X, R)$ be a graph on a standard Borel space X. Let $1\leqslant n\leqslant\aleph_0$. A *Borel n-coloring* of $\mathcal{G}$ is a Borel map $c: X\to Y$, where $\mathrm{card}(Y)=n$, such that $xRy\Rightarrow c(x)\neq c(y)$. Equivalently it is a partition $X=\bigcup_{i\in I} C_i$, $\mathrm{card}(I)=n$, C_i Borel, such that $x, y\in C_i$ implies that $\neg xRy$. If a Borel n-coloring exists with $1\leqslant n\leqslant\aleph_0$, then we write

$$\chi_B(\mathcal{G})\leqslant\aleph_0,$$

and we define

$$\chi_B(\mathcal{G})=\text{the smallest } 1\leqslant n\leqslant\aleph_0 \text{ for which a Borel } n\text{-coloring exists.}$$

We call $\chi_B(\mathcal{G})$ the *Borel chromatic number* of $\mathcal{G}$. If no such coloring exists, we write

$$\chi_B(\mathcal{G})>\aleph_0.$$

It is trivial to see that there are Borel graphs of any Borel chromatic number $1\leqslant n\leqslant\aleph_0$ and also of Borel chromatic number $>\aleph_0$. For example, let X be a Polish space, and put $\mathcal{G}_X=(X, R_X)$, where $R_X=X^2\setminus\{(x, x): x\in X\}$. If $\mathrm{card}(X)=n$, then $\chi_B(\mathcal{G}_X)=n$ and if $\mathrm{card}(X)>\aleph_0$, $\chi_B(\mathcal{G}_X)>\aleph_0$.

There are however more interesting examples that show that the *Borel* chromatic number of a graph is large even if its chromatic number is the smallest nontrivial possible, i.e., 2. For that recall the easy fact that the chromatic number of an acyclic graph is $\leqslant 2$.

3.1. EXAMPLE. There is a Polish space X and a sequence $\{F_n\}_{n\in\mathbb{N}}$ of Borel automorphisms of X such that $\mathcal{G}_{\{F_n\}}$ is acyclic, so that $\chi(\mathcal{G}_{\{F_n\}})\leqslant 2$, but $\chi_B(\mathcal{G}_{\{F_n\}})>\aleph_0$.

Proof. Let $X=S_\infty$ be the infinite symmetric group, i.e., the group of all permutations of $\mathbb{N}$. It is a G_δ subset of the Baire space $\mathcal{N}=\mathbb{N}^\mathbb{N}$ and with the relative topology it is a Polish group. Let $F=\{\{g_n\}\in S_\infty^\mathbb{N}: \{g_n\}$ is dense in S_∞ and generates a free subgroup of $S_\infty\}$. Then F is a dense G_δ in $S_\infty^\mathbb{N}$, so is nonempty. Fix $\{g_n\}\in F$. Let $F_n: S_\infty\to S_\infty$ be left-multiplication by g_n. Then clearly $\mathcal{G}_{\{F_n\}}$ is acyclic. Assume now $c: X\to\mathbb{N}$ was a Borel coloring. Then for some $i\in\mathbb{N}$, $c^{-1}(\{i\})=C_i$ is nonmeager and has the

Baire property, so $C_i C_i^{-1}$ has nonempty interior, thus it contains some g_n. So there are $x, y \in C_i$ with $g_n x = y$, i.e., $F_n(x) = y$ and so $x R_{\{F_n\}} y$, a contradiction. ∎

Another example of a locally countable Borel graph $\mathcal{G}$ with $\chi(\mathcal{G}) \leqslant 2$ but $\chi_B(\mathcal{G}) > \aleph_0$, can be found in Thomas [1986]. His graph is on the reals and two points x, y are connected by an edge exactly when $|x-y|$ is of the form 3^k for some $k \in \mathbb{Z}$. He shows that this graph has no odd cycles, so $\chi(\mathcal{G}) \leqslant 2$, but it is easy to check that $\chi_B(\mathcal{G}) > \aleph_0$ by using the fact that if $A \subseteq \mathbb{R}$ is measurable of positive measure, then $A - A$ contains an open interval around 0.

Notice that if $\mathcal{G} = (X, R)$ is a Borel acyclic graph and there is a Borel transversal for the connected components of $\mathcal{G}$, then $\chi_B(\mathcal{G}) \leqslant 2$. So the fact that $\chi_B(\mathcal{G}_{\{F_n\}}) > \aleph_0$ in the example before indicates a strong failure of the existence of Borel transversals for the connected components of this graph.

On the other hand, it is still possible to have a Borel graph, in fact one of the form $\mathcal{G}_F$ for a single aperiodic Borel automorphism F, such that $\chi_B(\mathcal{G}_F) = 2$, while $\mathcal{G}_F$ has no Borel transversal. To see this let $X = 2^{\mathbb{N}}$ and let $F\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the so-called *odometer map*; $F(x)$ is obtained from $x \in 2^{\mathbb{N}}$ as follows: if $x = (1, 1, 1, ...)$ then $F(x) = (0, 0, 0, ...)$; if $x = 1^n 0 \alpha$, with $n \geqslant 0$ and $\alpha \in 2^{\mathbb{N}}$, then $F(x) = 0^n 1 \alpha$. Thus $F(x)$ is obtained by adding 1 to $x(0)$ modulo 2 and carrying to $x(1), x(2), ...$ It is easy to see that $c\colon 2^{\mathbb{N}} \to \{0, 1\}$ given by $c(x) = x(0)$ is a Borel 2-coloring of $\mathcal{G}_F$. On the other hand the connected components of $\mathcal{G}_F$ are exactly the equivalence classes of E_0, except for the eventually constant sequences which form a single connected component of $\mathcal{G}_F$. So there is no Borel transversal for the connected components of $\mathcal{G}_F$.

We next consider an example of an acyclic graph with Borel chromatic number $\aleph_0$. It seems to be a folklore result. The earliest reference that we have found which explicitly proves it is Nesetril–Rodl [1985].

3.2. Example. Let $[\mathbb{N}]^{\mathbb{N}}$ be the set of strictly increasing sequences of nonnegative integers (viewed as a closed subset of the Baire space). Let S be the shift map on $X = [\mathbb{N}]^{\mathbb{N}}$. Then $\chi_B(\mathcal{G}_S) = \aleph_0$. (Again notice that $\mathcal{G}_S$ is acyclic.)

Proof. First we see that $\chi_B(\mathcal{G}_S) \leqslant \aleph_0$. For that just put $c(x) = x(0)$.

To see that $\chi_B(\mathcal{G}_S) = \aleph_0$, assume, towards a contradiction, that $c\colon X \to Y$, $\operatorname{card}(Y) = n < \aleph_0$ is a Borel coloring. Then, by the Galvin–Prikry Theorem, there is $y \in Y$ and an infinite set $A \subseteq \mathbb{N}$ such that for every increasing sequence $x \in [A]^{\mathbb{N}}$, $c(x) = y$. But if $x \in [A]^{\mathbb{N}}$, then $S(x) \in [A]^{\mathbb{N}}$, which is a contradiction. ∎

There are examples of aperiodic Borel automorphisms T for which $\chi_B(\mathcal{G}_T)=3$ (see Section 5). However the following is open:

3.3. *Open Problem.* Is there an acyclic Borel graph $\mathcal{G}$ with $\chi_B(\mathcal{G})=n$ for some $3<n<\aleph_0$? More generally, given $k\geqslant 2$ is there a Borel graph $\mathcal{G}$ with $\chi_B(\mathcal{G})=k+2$ but $\chi(\mathcal{G})\leqslant k$? (**Addendum:** This problem has now been solved by M. Laczkovich; see the Appendix.)

In connection with this question we would like to point out that for any $k\geqslant 2$ there is a graph $\mathcal{G}_k$ with $\chi(\mathcal{G}_k)=k$ but $\chi_B(\mathcal{G}_k)=k+1$. To see this let S be the shift map on $2^{\mathbb{Z}}$, let $X=\{x\in 2^{\mathbb{Z}}: \forall n(S^n(x)\neq x)\}$ be the aperiodic part of $2^{\mathbb{Z}}$, and let $\mathcal{G}_k=(X, R_k)$ be the following graph on X:

$$xR_k y \Leftrightarrow \exists j\in\mathbb{Z}(0<|j|<k \;\&\; S^j(x)=y).$$

It is easy to see that the chromatic number of $\mathcal{G}_k$ is exactly k. We will prove that $\chi_B(\mathcal{G}_k)=k+1$. First, toward a contradiction, assume that $\chi_B(\mathcal{G}_k)=k$. Let $X=A_0\cup\cdots\cup A_{k-1}$ be a Borel coloring. It is trivial that each A_i meets every connected component of $\mathcal{G}_k$, i.e., every orbit (in X) of S. It is also clear that for each such orbit L there is a permutation π: $\{1, ..., k-1\}\to\{1, ..., k-1\}$ such that if $x\in A_0\cap L$, then $S^i(x)\in A_{\pi(i)}$, $i=1, ..., k-1$. Since there are only finitely many such permutations, and any invariant under S Borel subset of X is either meager or comeager (since S is a homeomorphism with a dense orbit), it follows that there is a comeager invariant Borel subset $Y\subseteq X$ and a permutation π as above such that for every orbit L contained in Y, π is the corresponding permutation. Thus $S(A_0\cap Y)=A_{\pi(1)}\cap Y$, $S(A_{\pi(1)}\cap Y)=A_{\pi(2)}\cap Y, ...,$ $S(A_{\pi(k-2)}\cap Y)=A_{\pi(k-1)}\cap Y$, and $S^k(A_0\cap Y)=A_0\cap Y$. Thus $A_0\cap Y$ is either meager or comeager (again since S^k is a homeomorphism with a dense orbit) and thus so is every $A_i\cap Y$ $(i=0, ..., k-1)$, a contradiction.

It remains to prove that $\chi_B(\mathcal{G}_k)\leqslant k+1$. To see this we will use the following fact already stated in the proof of 1.1:

For any N there is a Borel set $M_N\subseteq X$ such that for each $x\in X$, $\{i\in\mathbb{Z}: S^i(x)\in M_N\}$ is unbounded in both directions, and if $S^i(x)$, $S^j(x)\in M_N(i, j\in\mathbb{Z})$ with $i<j$, then $j-i>N$.

We will show that there is a Borel set $A\subseteq X$ such that for each x, $\{i\in\mathbb{Z}: S^i(x)\in A\}$ is unbounded in both directions, and if $S^i(x)$, $S^j(x)\in A$, with $i<j$, then $j-i\in\{k, k+1\}$. Granting this, it is easy to define a Borel $(k+1)$-coloring of $\mathcal{G}_k$. Let L be an orbit of S, say $L=\{S^i(x): i\in\mathbb{Z}\}$ for some x. Consider $i<j$ such that $S^i(x), S^j(x)\in A$ but for no $i<n<j$ we have $S^n(x)\in A$. Color $S^{i+n}(x)$ by the color n if $0\leqslant n<j-i$.

It remains to define A. Choose $N>k^2+k$ in the preceding fact and consider $M=M_N$. Let $x\in X$, $i, j\in\mathbb{Z}$, be such that $S^i(x)$, $S^j(x)\in M$ and for no $i<n<j$ we have $S^n(x)\in M$. Then $j-i>k^2+k$. Put $j=i+mk+r$ for some $0\leqslant r<k$. If $r=0$ add to M the points $S^{i+pk}(x)$ for $1\leqslant p<m$. If $r>0$, then, noticing that $m > r$, add to M the points $S^{i+pk}(x)$, for $1 \leqslant p \leqslant m - r$ and the points $S^{i+(m-r)k+q(k+1)}(x)$, for $1 \leqslant q < r$. Call the set of points that have been added to the orbit of x by this process $B(x)$. (Note that $B(x)$ does not depend on the choice of x in its orbit.) Let $A = M \cup \bigcup_{x \in X} B(x)$.

4. SOME GENERAL FACTS

We will collect here some basic facts concerning Borel chromatic numbers.

Suppose $\mathcal{G} = (X, R)$ is a graph, where X is a standard Borel space. Call a set $Y \subseteq X (\mathcal{G}-)$ *discrete* if no two elements of Y are connected by an edge (i.e., $x, y \in Y \Rightarrow \neg xRy$). Then we have:

4.1. PROPOSITION. *If $\mathcal{G} = (X, R)$ is a graph on a standard Borel space, then $\chi_B(\mathcal{G}) \leqslant \aleph_0$ iff $X = \bigcup_n X_n$, with each X_n Borel and discrete.*

A maximal discrete set is called a *kernel*. Thus $Y \subseteq X$ is a kernel if no two elements of Y are connected by an edge and for any $x \in X \setminus Y$ there is an element of Y connected by an edge to x. We now have the following fact essentially noticed in Jackson–Kechris–Louveau.

4.2. PROPOSITION. *Assume X is standard Borel, and $\mathcal{G} = (X, R)$ a graph on X such that for any Borel set $Y \subseteq X$, $R(Y) = \{x \colon \exists y \in Y (xRy)\}$ is also Borel. (For example, this is true if $\mathcal{G}$ is Borel and locally countable.) Then if $\chi_B(\mathcal{G}) \leqslant \aleph_0$, $\mathcal{G}$ admits a Borel kernel.*

Proof. Let, by 4.1, $X = \bigcup_n X_n$ with each X_n Borel discrete and $X_n \cap X_m = \emptyset$ if $n \neq m$. Inductively define Y_n as follows: $Y_0 = X_0$, $Y_{n+1} = Y_n \cup (X_{n+1} \setminus R(Y_n))$. Let $Y = \bigcup_n Y_n$. By induction on n we see that each Y_n is discrete, so Y is discrete. Let now $x \in X \setminus Y$, and let m be such that $x \in X_m$. Clearly $m = n+1$ for some n. If $x \in R(Y_n)$ we are done. Else $x \in X_{n+1} \setminus R(Y_n) = Y_{n+1}$, which is a contradiction. ∎

Before we proceed, let us note the following alternative characterization of "$\chi_B(\mathcal{G}) \leqslant \aleph_0$."

4.3. PROPOSITION. *Let (X, τ) be a Polish space with topology τ. Let $\mathcal{G} = (X, R)$ be a graph with $R(Y)$ Borel for each Borel $Y \subseteq X$. Then $\chi_B(\mathcal{G}) \leqslant \aleph_0$ iff there is a Polish topology $\sigma \supseteq \tau$ such that $\forall x (x \notin \overline{R(x)}^{\sigma})$ (where $R(x) = \{y \colon xRy\}$).*

Proof. $\Leftarrow$: Let $\{U_n\}$ be a basis for a. Let $c(x) =$ the least n such that $x \in U_n$ and $U_n \cap R(x) = \emptyset$. Equivalently, $c(x) =$ the least n with $x \in U_n$ and $x \notin R(U_n)$. So c is Borel and clearly a coloring.

$\Rightarrow$: If $c: X \to \mathbb{N}$ is a Borel coloring, then, by standard descriptive set theory, there is a Polish topology $\sigma \supseteq \tau$ in which c is continuous (where $\mathbb{N}$ is discrete). Thus each $A_n = c^{-1}[\{n\}]$ is σ-clopen. If $x \in A_n$, then $A_n \cap R(x) = \emptyset$, so $x \notin \overline{R(x)}^\sigma$. ∎

By a *directed graph* on a standard Borel space X we mean a binary relation $P \subseteq X^2$ which is just irreflexive. We denote this by $\mathscr{D} = (X, P)$. A pair (x, y) with $(x, y) \in P$ is an edge of this graph. To each such directed graph there corresponds a graph $\mathscr{G}_{\mathscr{D}} = (X, \bar{P})$, where $x\bar{P}y \Leftrightarrow xPy$ or yPx. The degree of x in $\mathscr{D}$ is the cardinality of $P(x) = \{y: xPy\}$. We also call this the *out-degree* of x in $\mathscr{G}_{\mathscr{D}}$.

4.4. PROPOSITION. *Assume (X, τ) is Polish and $\mathscr{D} = (X, P)$ is a directed graph on X. Let $\mathscr{G} = \mathscr{G}_{\mathscr{D}}$ be the corresponding graph. Assume that for every Borel set $Y \subseteq X$ the set $P^{-1}(Y) = \{x: \exists y(y \in Y \,\&\, xPy)\}$ is Borel. Then $\chi_B(\mathscr{G}) \leqslant \aleph_0$ iff there is a Polish topology $\sigma \supseteq \tau$ such that $x \notin \overline{P(x)}^\sigma$.*

Proof. As in 4.3. ∎

Using these facts we can easily prove the following.

4.5. PROPOSITION. *If X is standard Borel, $\mathscr{D} = (X, P)$ is a directed graph on X such that for each Borel $Y \subseteq X$ we have that $P^{-1}(Y)$ is Borel, then if $\mathscr{G} = \mathscr{G}_{\mathscr{D}}$ has finite out-degree at each point $x \in X$, $\chi_B(\mathscr{G}) \leqslant \aleph_0$. In particular, if $\mathscr{G}$ is a locally finite Borel graph or if $\mathscr{G} = \mathscr{G}_{F_1, F_2, ..., F_n}$ with each $F_i: X \to X$ Borel, then $\chi_B(\mathscr{G}) \leqslant \aleph_0$.*

Proof. Take $\sigma = \tau$ in 4.4. ∎

We can improve this in the case the degree is bounded.

4.6. PROPOSITION. *Let X be standard Borel and $\mathscr{G} = (X, R)$ a Borel graph which is locally finite of bounded degree k (i.e., $d(x) \leqslant k, \forall x \in X$). Then $\chi_B(\mathscr{G}) \leqslant k+1$.*

Proof. By induction on k. For $k = 0$ this is obvious. Assume it is true for k and assume $d(x) \leqslant k+1$ for all x. By 4.5, $\chi_B(\mathscr{G}) \leqslant \aleph_0$ and so by 4.2 let $Y \subseteq X$ be a Borel kernel for $\mathscr{G}$. Let $Z = X \setminus Y$ and consider $\mathscr{G}' = (Z, R \cap Z^2)$. Then $d'(z) \leqslant k$ for each $z \in Z$, where d' is the degree relative to $\mathscr{G}'$. Let then by induction hypothesis $c': Z \to \{1, ..., k\}$ be a Borel coloring for $\mathscr{G}'$. Define c by letting $c(x) = k+1$ if $x \in Y$. ∎

From this we immediately have:

4.7. PROPOSITION. *Let X be standard Borel and F_i, $1 \leqslant i \leqslant n$, be Borel functions each of which is $\leqslant k$-to-1. Then*

$$\chi_B(\mathcal{G}_{F_1, ..., F_n}) \leqslant (k+1)\, n + 1.$$

After this paper has been completed, we became aware that the case $k=1$ of 4.7 can be deduced from a result of Blaszczyk and Yong [1988] by increasing the topology of X making F continuous, X Polish zero-dimensional, and adding no new Borel sets.

By the trivial examples of Section 3 the estimate of 4.6 is best possible. However, the following is open:

4.8. *Open Problem.* Let $\mathcal{G} = (2^{\mathbb{Z}\times\mathbb{Z}}, R)$ be the shift graph on $2^{\mathbb{Z}\times\mathbb{Z}}$, i.e., $R = R_{F_1, F_2}$ where $F_1((x_{m,n})) = (x_{m+1,n})$, $F_2((x_{m,n})) = (x_{m,n+1})$. By 4.6, $\chi_B(\mathcal{G}) \leqslant 5$, and it is not hard to see that $\chi_B(\mathcal{G}) \geqslant 3$. R. Muchnik has pointed out that the complete graph with 5 vertices is contained in $\mathcal{G}$, so $\chi_B(\mathcal{G}) = 5$. Now consider the restriction of $\mathcal{G}$ to its free part, i.e., the graph $\mathcal{G}' = (X, R)$, where $X \subseteq 2^{\mathbb{Z}\times\mathbb{Z}}$ is defined by $X = \{(x_{m,n}) \in 2^{\mathbb{Z}\times\mathbb{Z}} : \forall m, n, k, \ell \in \mathbb{Z}(k \neq 0$ or $\ell \neq 0 \Rightarrow x_{m+k, n+\ell} \neq x_{m,n})\}$. Again $3 \leqslant \chi_B(\mathcal{G}') \leqslant 5$. What is $\chi_B(\mathcal{G}')$?

It is a standard fact of graph theory that if X is a finite set, and $\mathcal{D} = (X, P)$ a directed graph, so that the corresponding graph $\mathcal{G} = \mathcal{G}_{\mathcal{D}}$ has out-degree $\leqslant n$, then the chromatic number of $\mathcal{G}$ is $\leqslant 2n+1$. (For the convenience of the reader, we sketch here the proof of this fact. Assume that such a directed graph is given, and notice that there must be a vertex of degree $\leqslant 2n$ (in $\mathcal{G}$). Otherwise, if v is the number of vertices, and all of them have degree $> 2n$, then there must be $> 2nv/2 = nv$ edges in $\mathcal{G}$, while on the other hand, since the out-degree is $\leqslant n$, there must be $\leqslant nv$ edges, a contradiction. From this it easily follows that there is an enumeration $a_1, ..., a_v$ of the vertices, so that the degree of a_i in the induced graph on $\{a_i, a_{i+1}, ..., a_v\}$ is $\leqslant 2n$. Then the coloring of $\mathcal{G}$ into $2n+1$ colors proceeds as follows: Color a_v arbitrarily. Assuming $a_v, a_{v-1}, ..., a_{i+1}$ have been colored, notice that a_i is connected with at most $2n$ vertices from $\{a_{i+1}, ..., a_v\}$, so we can color a_i with one of the available $2n+1$ colors that is different from those assigned to $a_{i+1}, ..., a_v$.)

This suggests the following question on improving 4.7.

4.9. *Open Problem.* In 4.7 is it true that

$$\chi_B(\mathcal{G}_{F_1, ..., F_n}) \leqslant 2n+1?$$

We will see in Section 5 that this holds if we replace $2n+1$ by 3^n.

Finally, it is interesting to see what happens if we consider edge colorings instead of vertex colorings.

For any graph $\mathcal{G} = (X, R)$ we consider its dual graph $\check{\mathcal{G}} = (\check{X}, \check{R})$, whose vertices are the edges $\{x, y\}$ of $\mathcal{G}$ and $\{x, y\}\ \check{R}\{z, w\}$ iff $\{x, y\} \cap \{z, w\} \neq \emptyset$ (i.e., two edges of $\mathcal{G}$, viewed as vertices of $\check{\mathcal{G}}$, are connected by an edge in $\check{\mathcal{G}}$ iff they have a vertex in common). An *edge coloring* of $\mathcal{G}$ is a coloring of $\check{\mathcal{G}}$, i.e., a map $c\colon \check{X} \to Y$ such that $c(\{x, y\}) \neq c(\{z, w\})$ if the distinct edges $\{x, y\}, \{z, w\}$ have a vertex in common. Similarly we define *edge k-coloring* and the *edge chromatic number*, $\chi'(\mathcal{G})$, of $\mathcal{G}$.

If $\mathcal{G} = (X, R)$ is a Borel graph on a standard Borel space X, then we can view $\check{X}$ as being also a standard Borel space and $\check{\mathcal{G}}$ a Borel graph on it. For example, fixing a Borel linear ordering $<$ of X, we can identify every edge $\{x, y\}$ of X with the pair (x, y) if $x < y$ or with the pair (y, x) is $y < x$. Thus $\check{X} = \{(x, y) \in X^2 : x < y \mathbin{\&} (x, y) \in R\}$ is a Borel subset of X^2 and easily $\check{R}$ is a Borel graph on $\check{X}$. We denote by $\chi'_B(\mathcal{G}) = \chi_B(\check{\mathcal{G}})$ the *Borel edge chromatic number* of $\mathcal{G}$.

We have seen in Section 2 that there are locally countable Borel graphs $\mathcal{G}$ (even acyclic ones) with $\chi_B(\mathcal{G}) > \aleph_0$. However we have the following.

4.10. Proposition. *Let $\mathcal{G} = (X, R)$ be a Borel locally countable graph on a standard Borel space X. Then $\chi'_B(\mathcal{G}) \leqslant \aleph_0$.*

Proof. Let $E_{\mathcal{G}}$ be the equivalence relation associated with $\mathcal{G}$. Then $E_{\mathcal{G}}$ is a countable (i.e., every equivalence class is countable) Borel equivalence relation, so by a result of Feldman–Moore [1977] *and its proof*, we can find a sequence $\{g_n\}$ of Borel involutions $(g_n^2(x) = x)$ of X such that $xE_{\mathcal{G}}y \Leftrightarrow \exists n(g_n(x) = y)$. Given now an edge $\{x, y\}$ of $\mathcal{G}$, let $c(\{x, y\}) =$ the least n such that $g_n(x) = y$ (or equivalently $g_n(y) = x$). This is clearly a Borel edge coloring with $\leqslant \aleph_0$ colors. ∎

It is also straightforward from 4.6 that if $\mathcal{G} = (X, R)$ is a Borel graph which is locally finite of bounded degree k, then $\check{\mathcal{G}}$ is a Borel graph which is locally finite of degree $2(k-1)$ and so $\chi'_B(\mathcal{G}) = \chi_B(\check{\mathcal{G}}) \leqslant 2k - 1$. For finite graphs $\mathcal{G}$, which have bounded degree k, it is known (see Jensen–Toft [1995]) that $\chi'(\mathcal{G}) \leqslant k + 1$. Is this bound true for $\chi'_B(\mathcal{G})$?

In the case of directed Borel graphs it is more natural to consider the so-called shift-operation, which is even more extensively studied. Thus, for a directed Borel graph $\mathcal{G} = (X, R)$, the *shift graph* $s\mathcal{G}$ of $\mathcal{G}$ is the directed graph (R, sR) where sR is the set at all ordered pairs $((x, y), (y, z))$ such that $(x, y), (y, z) \in R$. The standard facts about shift are transferable into the Borel context. For example in case $k = \chi_B(\mathcal{G})$ is finite, $\chi_B(s\mathcal{G})$ is bounded by the minimal number n such that $k \leqslant \binom{n}{\lfloor n/2 \rfloor}$ but it bounds $\log_2 k$ (see Erdos–Hajnal [1968], Harner–Entringer [1972], Duffus–Lefmann–Rodl [1995]). In general we have the following form of Proposition 4.10.

4.11. PROPOSITION. *$\chi_B(s\mathcal{G}) \leqslant \aleph_0$ for every directed Borel graph $\mathcal{G} = (X, R)$.*

Proof. Clearly we may assume $X = \mathbb{N}^{\mathbb{N}}$. Let $\{d_n\}$ be a fixed countable dense subset of X and let $<_\ell$ be the lexicographical ordering of $\mathbb{N}^{\mathbb{N}}$. This splits the vertex set R of the shift graph in two pieces $R^+ = <_\ell \cap R$ and $R^- = R \setminus <_\ell$, so it is sufficient to color one of the sets, say, R^+ into countably many Borel colors: Let $f\colon R^+ \to \mathbb{N}$ be defined by $f(a, b) = n$ iff n is the minimal integer such that $a <_\ell d_n <_\ell b$. It is clear that f is Borel and good. ∎

5. ONE, TWO, THREE, INFINITY

We will calculate here the possible Borel chromatic numbers of Borel graphs of the form $\mathcal{G}_F$ for a single Borel function F. Equivalently, these are the graphs $\mathcal{G} = \mathcal{G}_{\mathcal{D}}$ corresponding to directed Borel graphs $\mathcal{D} = (X, P)$, which have out-degree $\leqslant 1$. Indeed if $\mathcal{D}$ is such a directed graph then for each $x \in X$ there is at most one y with xPy so there is a Borel function $F\colon X \to X$ such that for every $x, xPF(x)$ if $\exists y(xPy)$, while otherwise $F(x) = x$. Then $\mathcal{G}_{\mathcal{D}} = \mathcal{G}_F$.

5.1. THEOREM. *Let X be a standard Borel space and $F\colon X \to X$ a Borel function. Then $\chi_B(\mathcal{G}_F) \in \{1, 2, 3, \aleph_0\}$.*

Proof. Fix $n = 2, 3, \ldots$. Denote by s_n the shift map on $n^{\mathbb{N}}$, i.e., $s_n\colon n^{\mathbb{N}} \to n^{\mathbb{N}}$ is given by $s_n(x)(i) = x(i+1)$ (here $n = \{0, 1, \ldots, n-1\}$).

5.2. LEMMA. $\chi_B(\mathcal{G}_{s_n}) \leqslant 3$.

Proof. By induction on $n \geqslant 2$.
n = 2. Partition $2^{\mathbb{N}}$ as follows:

$$A_0 = \{x \in 2^{\mathbb{N}} : x \text{ starts with an odd number of 0's}\};$$

$$A_1 = \{x \in 2^{\mathbb{N}} : x \text{ starts with an odd number of 1's}\};$$

$$A_2 = 2^{\mathbb{N}} \setminus A_0 \cup A_1).$$

This is easily a coloring of $\mathcal{G}_{s_2}$.
n → n + 1. Let $c\colon n^{\mathbb{N}} \to 3$ be a Borel coloring of $\mathcal{G}_{s_n}$. We will find a Borel coloring $c^*\colon (n+1)^{\mathbb{N}} \to 3$. Put $A = \{0, \ldots, n-1\} = n$. We consider three cases for $x \in (n+1)^{\mathbb{N}}$:

Case 1. There are infinitely many i with $x(i) = n$ and infinitely many i with $x(i) \in A$. Then put

$$c^*(x)=0, \qquad \text{if } x \text{ starts with an odd number of } n\text{'s};$$

$$c^*(x)=1, \qquad \text{if } x \text{ starts with an odd number of elements of } A;$$

$$c^*(x)=2, \qquad \text{otherwise.}$$

If X_1 is the set of such x's, X_1 is shift invariant and $c^* \mid X_1 \colon X_1 \to 3$ is a Borel coloring.

Case 2. There are only finitely many i with $x(i) \in A$. The set of such x's, say X_2, is a single connected component, so we can easily define a Borel coloring $c^* \mid X_2 \colon X_2 \to 2$.

Case 3. There are only finitely many i with $x(i)=n$. For such x we can write it uniquely as $x = s \hat{}\, y$, where $s \in (n+1)^k$ for some $k \geqslant 0$, $s(k-1)=n$ (if $k>0$) and $y \in A^{\mathbb{N}}$. Then put

$$c^*(x) = (c(y)+k) \qquad \text{mod } 3.$$

The set of such x's, say X_3, is also shift invariant and $c^* \mid X_3 \colon X_3 \to 3$ is a Borel coloring.

Then $c^* = \bigcup_{i=1}^{3} c^* \mid X_i$ is a Borel coloring of $\mathscr{G}_{s_{n+1}}$ in $\leqslant 3$ colors. ∎

Consider now $\mathscr{G}_F$. By 4.5, $\chi_B(\mathscr{G}_F) \leqslant \aleph_0$. So assume that $\chi_B(\mathscr{G}_F) \leqslant n < \aleph_0$ in order to show that $\chi_B(\mathscr{G}_F) \leqslant 3$. Let $c \colon X \to n$ be a Borel coloring and define $p \colon X \to n^{\mathbb{N}}$ by

$$p(x)(i) = c(F^i(x)).$$

Then $p(F(x)) = s_n(p(x))$. Let $c' \colon n^{\mathbb{N}} \to 3$ be a Borel coloring of $\mathscr{G}_{s_n}$ by 5.2. Then $c'' = c' \circ p$ is a Borel coloring of $\mathscr{G}_F$. Indeed, if $x \neq y$ with $F(x)=y$, then $c(x) \neq c(y)$, thus $p(x)(0) = c(x) \neq c(y) = p(y)(0)$, so $p(x) \neq p(y)$ and $s_n(p(x)) = p(F(x)) = p(y)$, thus $c''(x) = c'(p(x)) \neq c'(p(y)) = c''(y)$. ∎

To see some examples, notice first that if $F \colon X \to X$ is a Borel automorphism with $F^2 = \text{identity}$, then $\chi_B(F) \in \{1, 2\}$ and both cases can easily occur. On the other hand, if s_n is the shift on $n^{\mathbb{N}}$ for $n \geqslant 2$, then $\chi_B(\mathscr{G}_{s_n}) = 3$. We have already seen in 5.2 that $\chi_B(\mathscr{G}_{s_n}) \leqslant 3$. To see that $\chi_B(\mathscr{G}_{s_n}) = 3$ assume, toward a contradiction, that $n^{\mathbb{N}} = B_0 \cup B_1$ is a Borel coloring. Then for x non-constant, $x \in B_0 \Rightarrow s_n(x) \in B_1$, $x \in B_1 \Rightarrow s_n(x) \in B_0$, so $x \in B_i \Rightarrow s_n^2(x) \in B_i$, $i=0, 1$. Thus B_0, B_1 are invariant under s_n^2 and thus each one of them is meager or comeager. Say B_0 is comeager. Since s_n is a homeomorphism of $0 \hat{}\, n^{\mathbb{N}}$ with $n^{\mathbb{N}}$ and $B_0 \cap (0 \hat{}\, n^{\mathbb{N}})$ is comeager in $0 \hat{}\, n^{\mathbb{N}}$, while $s_n(B_0 \cap (0 \hat{}\, n^{\mathbb{N}})) \subseteq B_1$, it follows that B_1 is also comeager, which is a contradiction.

Next notice that, by 4.7 and 5.1, if $F \colon X \to X$ is $\leqslant k$-to-1 for some k, then $\chi_B(F) \leqslant 3$. In particular, this happens if F is a Borel automorphism of X.

To see an example of a Borel automorphism F with $\chi_B(\mathcal{G}_F)=3$, let F be the restriction of the shift on $2^{\mathbb{Z}}$ to its aperiodic part (see Section 3).

Finally, recall that in 2.2 we showed that the graph of the shift on $[\mathbb{N}]^{\mathbb{N}}$ has Borel chromatic number equal to $\aleph_0$. Notice that this shift is finite-to-1.

The following is an easy corollary of 5.1.

5.3. COROLLARY. *Let X be standard Borel and $F_1, ..., F_n$ be Borel functions on X. Then if $\chi_B(\mathcal{G}_{F_1, ..., F_n})<\aleph_0$ we have $\chi_B(\mathcal{G}_{F_1, ..., F_n})\leqslant 3^n$. In particular, if $F_1, ..., F_n$ are $\leqslant k$-to-1, for some finite k, then*

$$\chi_B(\mathcal{G}_{F_1, ..., F_n})\leqslant 3^n.$$

Proof. If $\chi_B(\mathcal{G}_{F_1, ..., F_n})<\aleph_0$, then each $\chi_B(\mathcal{G}_{F_i})<\aleph_0$, so $\chi_B(\mathcal{G}_{F_i})\leqslant 3$. Let $c_i\colon X\to 3$ be a Borel coloring of $\mathcal{G}_{F_i}$. Then $c(x)=(c_1(x), ..., c_n(x))$ is a Borel coloring of $\mathcal{G}_{F_1, ..., F_n}$. ∎

After the completion of this paper, we became aware that an alternate proof of Theorem 5.1 can be given using Corollary 2.2 of Krawczyk and Steprans [1993].

The following concept, implicit in the proof of 5.1 will also play an important role in the sequel.

5.4. DEFINITION. Let $\mathcal{G}=(X, R)$, $\mathcal{G}'=(X', R')$ be two graphs. A *homomorphism* of $\mathcal{G}$ into $\mathcal{G}'$ is a map $f\colon X\to X'$ such that $xRy \Rightarrow f(x)\, R' f(y)$. An *embedding* is an injective homomorphism, i.e., an isomorphism of $\mathcal{G}$ with a subgraph of $\mathcal{G}'$ (in the usual sense of graph theory).

If $\mathcal{F}$ is some class of functions we write

$$\mathcal{G}\leqslant_{\mathcal{F}}\mathcal{G}',$$

if there is a homomorphism of $\mathcal{G}$ into $\mathcal{G}'$ which belongs to $\mathcal{F}$, and

$$\mathcal{G}\sqsubseteq_{\mathcal{F}}\mathcal{G}'$$

if there is an embedding of $\mathcal{G}$ into $\mathcal{G}'$ which belongs to $\mathcal{F}$.

In particular, we use

$$\mathcal{G}\leqslant_B\mathcal{G}',\ \mathcal{G}\sqsubseteq_B\mathcal{G}',$$

when $\mathcal{F}=$ the class of Borel functions, and

$$\mathcal{G}\leqslant_c\mathcal{G}',\qquad \mathcal{G}\sqsubseteq_c\mathcal{G}',$$

when $\mathcal{F}=$ the class of continuous functions.

The main point about Borel homomorphisms in our context is the following simple observation:

If f is a Borel homomorphism of $\mathcal{G}$ into $\mathcal{G}'$ and $c: X' \to Y$ is a Borel coloring of $\mathcal{G}'$, then $c \circ f$ is a Borel coloring of $\mathcal{G}$. Thus

$$\mathcal{G} \leqslant_B \mathcal{G}' \Rightarrow \chi_B(\mathcal{G}) \leqslant \chi_B(\mathcal{G}').$$

In the course of the proof of 5.1 we have shown that if $\mathcal{G}_F$ has finite Borel chromatic number n, then $\mathcal{G}_F \leqslant_B \mathcal{G}_{s_n}$, the graph of the shift on $n^{\mathbb{N}}$, and thus, by 5.2, $\mathcal{G}_F \leqslant_B \mathcal{G}_{s_3}$. Thus we have for any Borel function $F: X \to X$:

$$\chi_B(\mathcal{G}_F) < \aleph_0 \Leftrightarrow \exists n(\mathcal{G}_F \leqslant_B \mathcal{G}_{s_n}) \Leftrightarrow \mathcal{G} \leqslant_B \mathcal{G}_{s_3}.$$

Similarly for any F we have

$$\mathcal{G}_F \leqslant_B \mathcal{G}_{s_\infty},$$

where s_∞ is the shift on $\mathbb{N}^{\mathbb{N}}$. (This is equivalent to saying that $\chi_B(\mathcal{G}_F) \leqslant \aleph_0$.) This will be strengthened in 7.8 for $\leqslant \aleph_0$-to-1 F.

We conclude by discussing an interesting example concerning the notion of Borel embeddability of graphs. Consider the group $G = \langle a, b; a^2 = b^2 = 1 \rangle$ generated by two generators a, b satisfying $a^2 = b^2 = 1$. Let G act on 2^G by shift, i.e., $g \cdot x(h) = x(g^{-1}h)$. Let $X_1 \subseteq 2^G$ be the free part of this action, i.e.,

$$x \in X_1 \Leftrightarrow \forall g \neq 1 (g \cdot x \neq x).$$

On X_1 we can define the graph

$$x R_1 y \Leftrightarrow a \cdot x = y \qquad \text{or} \quad b \cdot x = y.$$

In $\mathcal{G}_1 = \langle X_1, R_1 \rangle$ every vertex x has exactly two neighbors, $a \cdot x$ and $b \cdot x$. Thus $\chi'_B(\mathcal{G}_1) = 2$. Let us also notice that there is no Borel relation $< \subseteq X_1^2$ with the property that for each connected component C of $\mathcal{G}_1$, $< | C$ is a linear ordering of order type $\mathbb{Z}$ such that for $x \in C$, $\{a \cdot x, b \cdot x\} = \{y: y$ is the successor or predecessor of x in $< | C\}$. Indeed, if that was possible, define the following Borel 2-coloring of $\mathcal{G}_1$:

$$x \in A_0 \Leftrightarrow a \cdot x < b \cdot x$$

$$x \in A_1 \Leftrightarrow b \cdot x < a \cdot x.$$

Then notice that $b \cdot A_0 = A_1$ and $ba \cdot A_0 = A_0$, $ba \cdot A_1 = A_1$. Since the homeomorphism $x \mapsto ba \cdot x$ of X_1 has a dense orbit, it follows that both A_0, A_1 are meager or both are comeager, which is a contradiction.

Now consider the shift S on $2^{\mathbb{Z}}$. Let $X_2 \subseteq 2^{\mathbb{Z}}$ be its aperiodic part and let $\mathcal{G}_2 = \langle X_2, R_2 \rangle$ the corresponding graph on X_2. Then $\chi'_B(\mathcal{G}_2) > 2$, because if $\chi'_B(\mathcal{G}_2) = 2$ and $c\colon R_2 \to \{p, q\}$ is a Borel edge coloring, define the following Borel 2-coloring of $\mathcal{G}_2$:

$$x \in A_0 \Leftrightarrow c(x, S(x)) = p$$

$$x \in A_1 \Leftrightarrow c(x, S(x)) = q.$$

This shows that $\chi_B(\mathcal{G}_2) = 2$, which is a contradiction, as $\chi_B(\mathcal{G}_2) = 3$.

It is quite clear that $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic graphs. However, it is not hard to see that $\mathcal{G}_1 \not\sqsubseteq_B \mathcal{G}_2$ and $\mathcal{G}_2 \not\sqsubseteq_B \mathcal{G}_1$. Indeed, if $\mathcal{G}_1 \sqsubseteq_B \mathcal{G}_2$, say via f, let for $x, y \in X_1$

$$x < y \Leftrightarrow x, y \text{ are in the same connected component of } X, \text{ and}$$
$$\exists n > 0 (f(y) = S^n(f(x))).$$

This gives a Borel relation $<$ on X_1 with all the preceding properties, which is a contradiction. On the other hand if $\mathcal{G}_2 \sqsubseteq_B \mathcal{G}_1$, then $\chi'_B(\mathcal{G}_2) \leqslant \chi'_B(\mathcal{G}_1) = 2$, a contradiction too.

6. A MINIMAL GRAPH WITH UNCOUNTABLE CHROMATIC NUMBER

In this section we study the situation under which an analytic graph $\mathcal{G}$ has uncountable Borel chromatic number. The main result is the dichotomy theorem 6.3 below. First we need a definition.

6.1. Definition. Fix a sequence $\{t_n\} \subseteq 2^{<\mathbb{N}}$ (= the set of finite binary sequences), so that length$(t_n) = n$ and $\forall t \in 2^{<\mathbb{N}} \exists n (t \subseteq t_n)$ (i.e., $\{t_n\}$ is dense). Define then the graph $\mathcal{G}_0 = (2^{\mathbb{N}}, R_0)$ as follows

$$xR_0 y \Leftrightarrow \exists n [x \mid n = y \mid n = t_n \ \&\ x(n) = 1 - y(n) \ \&\ \forall m > n (x(m) = y(m))].$$

Concerning this graph we have:

6.2. Proposition. $\chi_B(\mathcal{G}_0) > \aleph_0$.

Proof. Assume not and let $\{A_i\}_{i \in \mathbb{N}}$ be a Borel coloring of $\mathcal{G}_0$. Then for some $i \in \mathbb{N}$, A_i is not meager, so there is $t \in 2^{<\mathbb{N}}$ such that A_i is comeager on $N_t = \{x \in 2^{\mathbb{N}} : t \subseteq x\}$. Fix n such that $t \subseteq t_n$. Then A_i is comeager on N_{t_n}. Let $\pi\colon N_{t_n {}^\frown 0} \to N_{t_n {}^\frown 1}$ be the homeomorphism $\pi(t_n {}^\frown 0 {}^\frown x) = t_n {}^\frown 1 {}^\frown x$. Then $\pi(A_i \cap N_{t_n {}^\frown 0})$ is comeager on $N_{t_n {}^\frown 1}$. But A_i is also comeager on $N_{t_n {}^\frown 1}$, thus

$\pi(A_i \cap N_{t_n{}^\wedge 0}) \cap A_i \neq \varnothing$. Let $x \in A_i \cap N_{t_n{}^\wedge 0}$ be such that $\pi(x) \in A_i$. Then $x R_0 \pi(x)$, which is a contradiction. ∎

We now have that $\mathcal{G}_0$ is the minimal analytic graph with uncountable Borel chromatic number, in the following sense.

6.3. THEOREM. *Let X be a Polish space and $\mathcal{G} = (X, R)$ an analytic graph (i.e., $R \subseteq X^2$ is analytic). Then exactly one of the following holds*:

(I) $\chi_B(\mathcal{G}) \leqslant \aleph_0$;

(II) $\mathcal{G}_0 \leqslant_c \mathcal{G}$.

This result is proved using methods of effective descriptive set theory, in particular the Gandy-Harrington topology. In fact one has the following effective version (which by standard arguments implies 6.3):

6.4. THEOREM. *Let $\mathcal{G} = (\mathbb{N}^{\mathbb{N}}, R)$ be a Σ^1_1 graph (i.e., $R \subseteq (\mathbb{N}^{\mathbb{N}})^2$ is Σ^1_1). Then exactly one of the following holds*;

(I) *There is a Δ^1_1 coloring c: $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ for $\mathcal{G}$*;

(II) $\mathcal{G}_0 \leqslant_c \mathcal{G}$.

Before we proceed to prove this result we would like to make a few comments.

(A) It is natural to ask whether in 6.3 and 6.4 we can replace (II) by the stronger $\mathcal{G}_0 \sqsubseteq_c \mathcal{G}$. We do not know the answer for a general analytic graph but we conjecture that it is positive. We can prove it though in many interesting cases, as for example when $\mathcal{G}$ is acyclic or is locally countable. We can in fact formulate a technical condition on $\mathcal{G}$, which includes both these cases, for which this is true, as follows:

6.5. DEFINITION. Let $\mathcal{G} = (X, R)$ be an analytic graph. We say that $\mathcal{G}$ is *almost acyclic* if we can write $R = \bigcup_n R_n$, with each R_n analytic, such that the following holds: For any xRy and any $n_1, ..., n_k \in \mathbb{N}$, $x_1, y_1, x_2, y_2, ..., x_k, y_k \in X$, if $R'_{n_1}(x, x_1)$, $R'_{n_1}(y, y_1)$, $R'_{n_2}(x_1, x_2)$, $R'_{n_2}(y_1, y_2)$, ..., $R'_{n_k}(x_{k-1}, x_k)$, $R'_{n_k}(y_{k-1}, y_k)$ all hold, where each R'_n is either R_n or $\check{R}_n = \{(x, y): (y, x) \in R_n\}$, then $x_k \neq y_k$.

EXAMPLES. (1) If $\mathcal{G} = (X, R)$ is an analytic acyclic graph, then it is almost acyclic as we can see by taking $R = R_n$, for each n.

(2) If $\mathcal{G} = (X, R)$ is an analytic locally countable graph, then it is almost acyclic. To see this first notice, that, by a standard reflection argument, there is a Borel locally countable graph $\mathcal{G}' = (X, R')$ with $R \subseteq R'$. Since for each $x \in X$, $R'(x)$ is countable, let F_n: $X \to X$ be a sequence of

Borel functions such that $R'(x) \cup \{x\} = \{F_n(x) : n \in \mathbb{N}\}$. Since each F_n is countable-to-1, it follows that $F_n(X) = Y_n$ is Borel and there is a Borel partition $Y_n = \bigcup_{i \leqslant \aleph_0} Y_{n,i}$ and for each $i = 1, 2, ..., \aleph_0$ Borel functions $f^j_{n,i}\colon Y_{n,i} \to X$, for $0 \leqslant j < i$, such that $Y_{n,i} = \{y \in Y_n : F_n^{-1}(\{y\}) \text{ has cardinality } i\}$, and for $y \in Y_n$, $F_n^{-1}(\{y\}) = \{f^j_{n,i}(y) : j < i\}$. Enumerate $\{f^j_{n,i}\}$ in a sequence $\{f_k\}$. Then each f_k is a partial Borel isomorphism, i.e., a Borel bijection between two Borel subsets of X, and $\{f_k\}$ generates the graph $\mathcal{G}'$ in the sense that $xR'y \Leftrightarrow x \neq y \;\&\; \exists k(f_k(x) = y)$. Let now

$$R_k = R \cap \{(x, y) : f_k(x) = y\}.$$

Then R_k is analytic, $R = \bigcup_k R_k$, and witnesses that $\mathcal{G}$ is almost acyclic.

We now have the following:

6.6. THEOREM. *Let X be a Polish space and $\mathcal{G} = (X, R)$ an analytic almost acyclic graph. Then exactly one of the following holds*:

(I) $\chi_B(\mathcal{G}) \leqslant \aleph_0$;

(II) $\mathcal{G}_0 \sqsubseteq_c \mathcal{G}$.

Again, this is a consequence of the effective version below.

6.7. THEOREM. *Let $\mathcal{G} = (\mathbb{N}^{\mathbb{N}}, R)$ be a Σ^1_1 graph and assume $R = \bigcup_n R_n$, with each R_n also Σ^1_1, witnesses that $\mathcal{G}$ is almost acyclic. Then exactly one of the following holds*:

(I) *There is a Δ^1_1 coloring $c\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ for $\mathcal{G}$*;

(II) $\mathcal{G}_0 \sqsubseteq_c \mathcal{G}$.

(B) The definition of $\mathcal{G}_0$ depends of course on the choice of $\{t_n\}$ (see 6.1). However, it follows from 6.3 that a different choice of $\{t_n\}$, say $\{t'_n\}$, will produce a graph $\mathcal{G}'_0$ such that $\mathcal{G}_0 \sqsubseteq_c \mathcal{G}'_0$ and $\mathcal{G}'_0 \sqsubseteq_c \mathcal{G}_0$, i.e., any such graph is, up to homeomorphism, a subgraph of any other one.

Notice also that the graph $\mathcal{G}_0$ is acyclic and that its connected components are exactly the equivalence classes of the equivalence relation E_0 defined in 1.1. In fact the proof of 6.3 is motivated by that of the Glimm–Effros type dichotomy involving E_0 given in Harrington–Kechris–Louveau [1990], abbreviated by HKL below. Also notice that if instead of choosing $\{t_n\}$ having the density property of 6.1, we take $t'_n = 0^n$, then the graph defined as in 6.1 for $\{t'_n\}$, when restricted to the $x \in 2^{\mathbb{N}}$ which have infinitely many 1's, is homeomorphically isomorphic to the graph of the shift on $[\mathbb{N}]^{\mathbb{N}}$, which as we saw in Example 3.2, has Borel chromatic number equal to $\aleph_0$. Thus this density property of 6.1 is responsible for the uncountability of the Borel chromatic number of the graph $\mathcal{G}_0$.

(C) Theorem 6.3 also implies that, within the definable context, the question of whether an analytic graph has countable Borel chromatic number is very robust. For example, it follows from that result that if $\mathcal{G}$ admits a countable coloring that is Baire measurable in the strong sense, i.e., its composition with a continuous function is Baire measurable, then it also admits one which is Borel. This is because if this weaker coloring exists, but (I) of 6.3 fails, then (II) must hold, so $\mathcal{G}_0$ admits a countable coloring that is Baire measurable, which contradicts the proof of 6.2 (we only used there that each A_i has the Baire property). Similarly, one can see by a simple modification of the proof of 6.2 (using a density argument) that $\mathcal{G}_0$ does not admit a countable coloring which is universally measurable, so this implies that if an analytic graph admits a universally measurable countable coloring, then it admits a Borel one.

Similarly, we can see that in alternative (II) of 6.3 one can replace $\leqslant_c$, by $\leqslant_{\mathrm{BM}}$, $\leqslant_{\mathrm{UM}}$, where BM (resp. UM) denotes the class of Baire measurable (resp. universally measurable) functions.

(D) There is also a version of 6.3 which holds for arbitrary graphs on Polish spaces if one assumes $\mathrm{AD}+\mathrm{V}=L(\mathbb{R})$ or $\mathrm{AD}_{\mathbb{R}}$ (the axiom of determinacy for games on the reals). Given any graph $\mathcal{G}=(X, R)$, with X Polish, we say that $\mathcal{G}$ has *well ordered chromatic number* if there is a coloring $c: X \to$ ordinals. Then under any of these hypotheses, every graph $\mathcal{G}$ has either wellordered chromatic number or $\mathcal{G}_0 \leqslant_c \mathcal{G}$.

Still under the same hypotheses, it also follows that for every analytic graph $\mathcal{G}=(X, R)$, if $\chi_B(\mathcal{G}) > \aleph_0$ then actually $\chi_B(\mathcal{G})=\chi(\mathcal{G})=2^{\aleph_0}$ in the following sense: If $c: X \to Y$ is a coloring, where Y is an arbitrary set, then there is an injection of $2^{\mathbb{N}}$ into Y. (Since there is obviously a Borel coloring $c: X \to 2^{\mathbb{N}}$, this justifies the assertion that $\chi_B(\mathcal{G})=\chi(\mathcal{G})=2^{\aleph_0}$.) To see this, notice that we can assume that $Y=X/E$ for some equivalence relation E on X. From our hypotheses it follows, by results of Woodin, that either X/E is wellorderable or else $2^{\mathbb{N}}$ embeds into X/E. But if X/E is wellorderable, then we cannot have $\mathcal{G}_0 \leqslant_c \mathcal{G}$, since $\mathcal{G}_0$ cannot have wellordered chromatic number. So it must be that $\chi_B(\mathcal{G}) \leqslant \aleph_0$, a contradiction.

(E) Theorem 6.3 is the best result that can be proved in ZFC. This follows from the next theorem that shows that 6.3 cannot be extended in any reasonable way, working in ZFC, to co-analytic graphs.

6.8. THEOREM. *If $\aleph_1^{L[a]}=\aleph_1$ for some $a \subseteq \mathbb{N}$, then there exist two co-analytic graphs $\mathcal{H}_0$ and $\mathcal{H}_1$, such that*

(i) *$\mathcal{H}_0$ and $\mathcal{H}_1$ are both uncountably-chromatic, but*

(ii) *every graph $\mathcal{H}$ such that $\mathcal{H} \leqslant \mathcal{H}_0$ and $\mathcal{H} \leqslant \mathcal{H}_1$ must be countably-chromatic.*

Proof. Let $\mathbb{Q}$ denote the set of rationals and let $\sigma\mathbb{Q}$ be the set of all wellordered subsets of $\mathbb{Q}$. We consider $\sigma\mathbb{Q}$ as a subset of the Cantor set $\{0, 1\}^{\mathbb{Q}}$ which is identified with $\mathscr{P}(\mathbb{Q})$. Our assumption allows us to pick a co-analytic set $\Omega \subseteq \sigma\mathbb{Q} \times \mathscr{P}(\mathbb{Q})$ such that for every $\alpha < \omega_1$, there is exactly one element (x_0, x_1) in Ω such that $\operatorname{otp}(x_0) = \alpha$. Let $\mathscr{G} = \langle V, EE \rangle$, where

$$V = \{(x_0, x_1, x_2) \in \sigma\mathbb{Q} \times \mathscr{P}(\mathbb{Q}) \times \sigma\mathbb{Q} \colon (x_0, x_1) \in \Omega \text{ and } \operatorname{otp}(x_0) = \operatorname{otp}(x_2)\},$$

and

$$\{(x_0, x_1, x_2), (y_0, y_1, y_2)\} \in EE \qquad \text{iff} \quad x_2 \text{ end-extends } y_2 \quad \text{or } y_2 \text{ end-extends } x_2.$$

For a basic clopen set $C \subseteq \{0, 1\}^{\mathbb{Q}}$, let $\mathscr{G}_C = \langle V_C, EE \rangle$ be the subgraph of $\mathscr{G}$, where

$$V_C = \{(x_0, x_1, x_2) \in V \colon x_0 \in C\}.$$

6.9. Lemma. *If C and D are two disjoint clopen subsets of $\{0, 1\}^{\mathbb{Q}}$, then every graph $\mathscr{H}$ such that $\mathscr{H} \leqslant \mathscr{G}_C$ and $\mathscr{H} \leqslant \mathscr{G}_D$ must be countably-chromatic.*

Proof. Suppose $\mathscr{H} = \langle Y, E \rangle$ and that $f\colon Y \to V_C$ and $g\colon Y \to V_D$ witness $\mathscr{H} \leqslant \mathscr{G}_C$ and $\mathscr{H} \leqslant \mathscr{G}_D$, respectively. Define $h\colon Y \to \mathbb{Q}$ as follows: For a given $y \in Y$, if $\alpha = \operatorname{otp}(f(y)_0) < \beta = \operatorname{otp}(g(y)_0)$, let $h(y)$ be the αth element of $g(y)_2$; if $\beta = \operatorname{otp}(g(y)_0) < \alpha = \operatorname{otp}(f(y)_0)$, let $h(y)$ be the βth element of $f(y)_2$. Then it is easily checked that $h(x) \neq h(y)$, whenever $\{x, y\} \in E$. ∎

Let U be the union of all clopen subsets C of $\{0, 1\}^{\mathbb{Q}}$ for which $\mathscr{G}_C$ is countably chromatic.

6.10. Lemma. *$\{0, 1\}^{\mathbb{Q}} \setminus U$ has at least two elements.*

Proof. For this it suffices to show that $\mathscr{G}$ itself is not countably chromatic. This can be easily deduced from the well-known fact due to Kurepa [1956] that $\sigma\mathbb{Q}$ is not $\mathbb{Q}$-embeddable, but we shall give the simple direct argument. Let $V = \bigcup_{n \in \mathbb{N}} V_n$ be a given decomposition and let q_1 be an arbitrarily chosen rational. If there is $(x, y, z) \in V_1$ such that $\sup(z) < q_1$ choose one and call it (x^1, y^1, z^1); otherwise let (x^1, y^1, z^1) be an arbitrary element of V such that $\sup(z^1) < q_1$. Pick a rational $q_2 < q_1$ such that $q_2 > \sup(z^1)$. If there is $(x, y, z) \in V_2$ such that z end-extends z^1 and $\sup(z) < q_2$, choose one and call it (x^2, y^2, z^2); otherwise choose (x^2, y^2, z^2) to be any $(x, y, z) \in V$ such that z end-extends z^1 and $\sup(z) < q_2$, and so on. Let $z^* = \bigcup_{n \in \mathbb{N}} z^n$ and let (x^*, y^*) be the unique element of Ω such that $\operatorname{otp}(x^*) = \operatorname{otp}(z^*)$. Fix an n such that $(x^*, y^*, z^*) \in V_n$. Then by the construction, $(x^n, y^n, z^n) \in V_n$ and $\{(x^n, y^n, z^n), (x^*, y^*, z^*)\} \in EE$ so this shows that the V_n is not $\mathscr{G}$-independent. ∎

By Lemma 6.10, pick disjoint clopen $C, D \subseteq \{0,1\}^{\mathbb{Q}}$ which are *not* subsets of U. Then $\mathcal{G}_C$ and $\mathcal{G}_D$ satisfy the conclusion of the theorem. ∎

(F) The product $\mathcal{H}_0 \times \mathcal{H}_1$ of two graphs $\mathcal{H}_0, \mathcal{H}_1$ is the graph whose vertex set is the cartesian product of the vertex sets of $\mathcal{H}_0, \mathcal{H}_1$ and where a pair (x_0, x_1) is connected by an edge to (y_0, y_1) iff $\{x_0, y_0\}$ is an edge of $\mathcal{H}_0$ and $\{x_1, y_1\}$ an edge of $\mathcal{H}_1$. A well-known conjecture of Hedetniemi [1966] asserts that

$$\chi(\mathcal{H}_0 \times \mathcal{H}_1) = \min\{\chi(\mathcal{H}_0), \chi(\mathcal{H}_1)\}$$

at least when $\mathcal{H}_0, \mathcal{H}_1$ are finite. The best known result about this conjecture is due to El-Zahar–Sauer [1985] and verifies it for chromatic numbers $\leqslant 4$. Note that the proof of 6.8 above shows that Hedetniemi's conjecture fails for the two co-analytic graphs $\mathcal{H}_0, \mathcal{H}_1$. We should mention here that the existence of two uncountably chromatic graphs whose product is countably chromatic was first made explicit by Hajnal [1985], though this phenomenon was known much earlier, but in a slightly different guise (see the references of Hajnal [1985]). The following consequence of 6.4 shows that this pathology does not occur in the case of Borel chromatic numbers of analytic graphs, i.e., in this context Hedetniemi's conjecture is true in the uncountable case.

6.11. Corollary. *If $\mathcal{H}_0, \mathcal{H}_1$ are two analytic graphs with uncountable Borel chromatic number, then their product $\mathcal{H}_0 \times \mathcal{H}_1$ also has uncountable Borel chromatic number.*

Proof. By 6.4 choose continuous homomorphisms f_0, f_1 of $\mathcal{G}_0$ into $\mathcal{H}_0, \mathcal{H}_1$, resp. Then $f(x) = (f_0(x), f_1(x))$ is a continuous homomorphism of $\mathcal{G}_0$ into $\mathcal{H}_0 \times \mathcal{H}_1$, so $\chi_B(\mathcal{H}_0 \times \mathcal{H}_1) > \aleph_0$. ∎

We will now give the proof of 6.4.

Proof of 6.4. Let $\Phi \subseteq \mathcal{P}(\mathbb{N}^{\mathbb{N}})$ be defined as follows: For $A \subseteq \mathbb{N}^{\mathbb{N}}$, let

$$\begin{aligned} \Phi(A) &\Leftrightarrow A \text{ is } \mathcal{G}\text{-discrete} \\ &\Leftrightarrow \forall x, y (x, y \in A \Rightarrow (x, y) \notin R) \\ &\Leftrightarrow A^2 \cap R = \varnothing. \end{aligned}$$

Then Φ is Π^1_1 on Σ^1_1, so if $\Phi(A)$ holds for some $A \in \Sigma^1_1$, then by the First Reflection Theorem, there is a Δ^1_1 set $B \supseteq A$ with $\Phi(B)$, i.e., every Σ^1_1 discrete set is contained in a Δ^1_1 discrete set.

We now have two cases:

(I) $\bigcup\{A \in \Sigma^1_1 : A$ is $\mathcal{G}$-discrete$\} = \mathbb{N}^{\mathbb{N}}$. Then, by the above, $\bigcup\{A \in \Delta^1_1 : A$ is $\mathcal{G}$-discrete$\} = \mathbb{N}^{\mathbb{N}}$, i.e., $\forall x \in \mathbb{N}^{\mathbb{N}}\, \exists A \in \Delta^1_1$ (A is $\mathcal{G}$-discrete &

$x \in A$). Let (C, D) as in 3.3.1 of HKL be a standard system of coding of Δ^1_1 subsets of $\mathbb{N}^{\mathbb{N}}$. Then $\forall x \in \mathbb{N}^{\mathbb{N}}\, \exists n(n \in C \,\&\, D_n$ is $\mathcal{G}$-discrete $\&\, x \in D_n)$, so by Δ^1_1-selection there is a Δ^1_1 map $c\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ with $c(x) \in C$, $D_{c(x)}$ $\mathcal{G}$-discrete, and $x \in D_{c(x)}$. This c is clearly a Δ^1_1 coloring of $\mathcal{G}$.

(II) $\bigcup\{A \in \Sigma^1_1\colon A$ is $\mathcal{G}$-discrete$\}$ $Y_1 \subsetneqq \mathbb{N}^{\mathbb{N}}$. Since $Y_1 = \bigcup\{A \in \Delta^1_1\colon A$ is $\mathcal{G}$-discrete$\}$, we have

$$y \in Y_1 \Leftrightarrow \exists n(n \in C \,\&\, D_n \text{ is } \mathcal{G}\text{-discrete } \&\, y \in D_n),$$

so Y_1 is Π^1_1. Let $X_1 = \mathbb{N}^{\mathbb{N}} \setminus Y_1$, so that X_1 is Σ^1_1, nonempty, and

$$\forall A \in \Sigma^1_1(\emptyset \neq A \subseteq X_1 \Rightarrow A^2 \cap R \neq \emptyset). \tag{$*$}$$

We will then show that $\mathcal{G}_0 \leqslant_c \mathcal{G}$. The argument is a variant of that of the proof of 1.4 of HKL.

Let $X = X_1 \cap \{x \in \mathbb{N}^{\mathbb{N}}\colon \omega^x_1 = \omega^{CK}_1 = \text{the first non-recursive ordinal})\}$. Then X is Σ^1_1 non-$\emptyset$ and the Gandy–Harrington topology restricted to X, which we will denote by τ_X, is Polish. Fix also a compatible metric d_X for τ_X for which we can clearly assume that $d_X \geqslant \delta$, with δ the usual metric of $\mathbb{N}^{\mathbb{N}}$ (restricted to X). Fix also a strategy Σ for player α in the strong Choquet game of (R, τ_2), where τ_2 is the Gandy–Harrington topology on $(\mathbb{N}^{\mathbb{N}})^2$. We can assume that in his nth move α plays a set of diameter $\leqslant 2^{-n}$, in the usual metric of $(\mathbb{N}^{\mathbb{N}})^2$.

Note that $(*)$ implies that if $U \subseteq X$ is nonempty τ_X-open, then $U^2 \cap R \neq \emptyset$.

We will now define for each $s \in 2^{<\mathbb{N}}$, $s \neq \emptyset$, a point $x_s \in X$ and a τ_X-open set U_s containing x_s such that $\overline{U_{s^\wedge i}}^{\tau_X} \subseteq U_s$ and $d_X(U_s) \leqslant 2^{-\ell h(s)}$. Moreover, we will define for $k \in \mathbb{N}$, $m \in \mathbb{N}$, $s \in 2^m$, nonempty τ_2-open sets $R_{k,s}$, $\tilde{R}_{k,s}$ such that

(i) $R_{k,\emptyset} = R$;
(ii) $(x_{t_k{}^\wedge 0^\wedge s}, x_{t_k{}^\wedge 1^\wedge s}) \in R_{k,s}$;
(iii) The play

$\underline{\beta}$	$\underline{\alpha}$
$R_{k,\emptyset}, (x_{t_k{}^\wedge 0}, x_{t_k{}^\wedge 1})$	
	$\tilde{R}_{k,\emptyset}$
$R_{k,s(0)}, (x_{t_k{}^\wedge 0^\wedge s(0)}, x_{t_k{}^\wedge 1^\wedge s(0)})$	
	$\tilde{R}_{k,s}(0)$
$\cdots$	
$R_{k,s}, (x_{t_k{}^\wedge 0^\wedge s}, x_{t_k{}^\wedge 1^\wedge s})$	
	$\tilde{R}_{k,s}$

is a play in the strong Choquet game for (R, τ_2) in which α follows Σ.

Assuming this has been done, let for $x \in 2^{\mathbb{N}}$, $f(x)$ be the unique point of $\bigcap_n U_{x|n}$. Then $f: 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is continuous. Also if xR_0y, so that $x = t_k{}^\wedge 0^\wedge z$, $y = t_k{}^\wedge 1^\wedge z$, for some k, z, then $\bigcap_{n\in\mathbb{N}} \tilde{R}_{k, z|n}$ is a singleton, say (a, b), and $(x_{t_k{}^\wedge 0^\wedge z|n}, x_{t_k{}^\wedge 1^\wedge z|n}) \to (a, b)$. Since $x_{t_k{}^\wedge 0^\wedge z|n} \to f(x)$ and $x_{t_k{}^\wedge 1^\wedge z|n} \to f(y)$, it follows that $(a, b) = (f(x), f(y)) \in R$, so $f(x)\, Rf(y)$, i.e., f shows that $\mathscr{G}_0 \leqslant_c \mathscr{G}$.

We now proceed to the construction: First there are $x_0, x_1 \in X$ with $x_0 R x_1$. Choose then small enough τ_X-open nbhds U_0, U_1 of x_0, x_1. Put $R_{0,\varnothing} = R$, $\tilde{R}_{0,\varnothing} = \Sigma(R_{0,\varnothing}(x_0, x_1))$.

Suppose now x_s, U_s, have been found for all $s \in \bigcup_{i\leqslant n} 2^i$ and $R_{k,u}$, $\tilde{R}_{k,u}$ for $k, m, u \in 2^m$ with $k+1+m \leqslant n$ (i.e., $t_{k^\wedge 0^\wedge u}, t_{k^\wedge 1^\wedge u} \in \bigcup_{i\leqslant n} 2^i$). We proceed to the stage $n+1$. Let

$$V = \{ y_{t_n} : \exists (y_s)_{s\in 2^n, s\neq t_n} [\forall s \in 2^n (y_s \in U_s) \;\&\; \forall k\, \forall m\, \forall u \in 2^m (k+1+m = n \Rightarrow (y_{t_k{}^\wedge 0^\wedge u}, y_{t_k{}^\wedge 1^\wedge u}) \in \tilde{R}_{k,u})]\}.$$

This is a τ_X-open subset of X, which is nonempty as $x_{t_n} \in V$, so we can find $y^0_{t_n} R y^1_{t_n}$, $y^i_{t_n} \in V$. Let (y^i_s), $s \in 2^n$, $s \neq t_n$ verify that $y^i_{t_n} \in V$. Put $x_{s^\wedge i} = y^i_s$. Then we have $x_{s^\wedge i} \in U_s (s \in 2^n, i \in \{0, 1\})$; $(x_{t_k{}^\wedge 0^\wedge u^\wedge i}, x_{t_k{}^\wedge 1^\wedge u^\wedge i}) \in \tilde{R}_{k,u}$ for all $k, m, u \in 2^m$ with $k+1+m = n$, $i \in \{0, 1\}$; $(x_{t_n{}^\wedge 0} x_{t_n{}^\wedge 1}) \in R$. Now let $U_{s^\wedge i}$ be a small enough τ_X-nbhd of $x_{s^\wedge i}$. Put for $k, m, u \in 2^m$ with $k+1+m = n$, $R_{k, u^\wedge i} = \tilde{R}_{k,u}$, so that $(x_{t_k{}^\wedge 0^\wedge u^\wedge i}, x_{t_k{}^\wedge 1^\wedge u^\wedge i}) \in R_{k, u^\wedge i}$ and let $\tilde{R}_{k, u^\wedge i} = \Sigma(R_{k,\varnothing}, (x_{t_k{}^\wedge 0^\wedge}, {}_{t_k{}^\wedge 1}), ..., R_{k, u^\wedge i}, (x_{t_k{}^\wedge 0^\wedge u^\wedge i}, x_{t_k{}^\wedge 1^\wedge u^\wedge i}))$. Finally, put $R_{n,\varnothing} = R$ and $\tilde{R}_{n,\varnothing} = \Sigma(R_{n,\varnothing}, (x_{t_n{}^\wedge 0}, x_{t_n{}^\wedge 1}))$. This completes the construction and the proof. ∎

Finally, we indicate the changes needed to prove 6.7. To achieve $\mathscr{G}_0 \sqsubseteq_c \mathscr{G}$ instead of just $\mathscr{G}_0 \leqslant_c \mathscr{G}$ it is clear that, in the preceding proof, we need to make sure that $U_{s^\wedge 0} \cap U_{s^\wedge 1} = \varnothing$, for all $s \in 2^{<\mathbb{N}}$. Again for that it is enough to make sure that the points x_s, $s \in 2^n$, are distinct for each $n \geqslant 1$. To achieve this we make the following modification in the proof: Instead of requiring that $R_{k,\varnothing} = R (k \in \mathbb{N})$ we require that $R_{k,\varnothing} = R_{p_k}$ for some $p_k \in \mathbb{N}$, depending on k.

It is clear that in the beginning of the construction, since $(x_0, x_1) \in R$, there is p_0 with $(x_0, x_1) \in R_{p_0}$ and we let $R_{0,\varnothing} = R_{p_0}$. At stage $n+1$ we have $(x_{t_n{}^\wedge 0}, x_{t_n{}^\wedge 1}) \in R$, so again we can find p_n such that $(x_{t_n{}^\wedge 0}, x_{t_n{}^\wedge 1}) \in R_{p_n}$. Put $R_{n,\varnothing} = R_{p_n}$. It remains to check that if at stage $n+1$ the points $x_{s^\wedge i}$ are found as before, then $x_{s^\wedge 0} \neq x_{s^\wedge 1}$, $\forall s \in 2^n$, so we can choose $U_{s^\wedge 0}$, $U_{s^\wedge 1}$ with $U_{s^\wedge 0} \cap U_{s^\wedge 1} = \varnothing$. But, by construction, it is easy to check that there is a sequence $n_1, ..., n_k \in \mathbb{N}$ and $x_1, y_1, ..., x_{k-1}, y_{k-1}$ such that $R'_{p_{n_1}}(x_{t_n{}^\wedge 0}, x_1)$, $R'_{p_{n_1}}(x_{t_n{}^\wedge 1}, y_1)$, $R'_{p_{n_2}}(x_1, x_2)$, $R'_{p_{n_2}}(y_1, y_2)$, ..., $R'_{p_{n_k}}(x_{k-1}, x_{s^\wedge 0})$, $R'_{pn_k}(y_{k-1}, x_{s^\wedge 1})$, where each $R'_{p_{n_i}}$ is either $R_{p_{n_i}}$ or $\check{R}_{p_{n_i}}$. Thus $x_{s^\wedge 0} \neq x_{s^\wedge 1}$. (For example, if

$n=2$, $s=(00)$, $t_1=(0)$, $t_2=(0,1)$, then $k=3$, $x_1=x_{110}$, $x_2=x_{100}$, $y_1=x_{111}$, $y_2=x_{101}$, $n_1=0$, $R'_{p_{n_1}}=R_{p_0}$, $n_2=1$, $R'_{p_{n_1}}=\check{R}_{p_1}$, $n_3=0$, $R'_{p_{n_3}}=\check{R}_{p_0}$.)

7. UNIVERSAL GRAPHS

In this section we will discuss a number of universality results for classes of graphs or functions. Some of these results are of interest independently of chromatic numbers. In particular, 7.6 below generalizes to countable semigroup actions results on the existence of countable generators, proved earlier for countable group actions by Weiss [1989], DJK, and Jackson–Kechris–Louveau [199?] (see also Kechris [1994]). Also 7.8 and 7.12 establish some important universality properties of the shift maps. We will consider the implication of these results to chromatic numbers in Section 8.

A binary relation $R \subseteq X^2$ is called *locally countable* if $\forall x \in X$ both $\{y: yRx\}$ and $\{y: xRy\}$ are countable. A set $Z \subseteq X$ is called *R-invariant* if $\forall x \in Z$ $\forall y \in X(xRy$ or $yRx \Rightarrow y \in Z)$.

7.1. PROPOSITION. *There is a locally countable, Borel relation U on a Polish space X which is universal for locally countable Borel relations in the following sense: if $R \subseteq Y^2$ is Borel, locally countable with Y standard Borel, then there is a Borel, U-invariant set $Z \subseteq X$ such that $U|Z$ is Borel isomorphic to R.*

Proof. The underlying space of the universal relation is $X=(2^{\mathbb{N}})^{F_\omega} \times 2^{F_\omega^2}$, where F_ω is the free group on $\aleph_0$ generators. Define the universal relation by

$$((x_\sigma)_{\sigma \in F_\omega}, \alpha)\ U((y_\sigma)_{\sigma \in F_\omega}, \beta) \Leftrightarrow \exists \sigma_0 \in F_\omega(y_\sigma = x_{\sigma\sigma_0}, \text{ for any } \sigma \in F_\omega,$$
$$\text{and } \alpha(e, \sigma_0)=1, \beta=\sigma_0\alpha),$$

where

$$\sigma\alpha(\tau_1, \tau_2)=1 \qquad \text{iff} \quad \alpha(\tau_1\sigma, \tau_2\sigma)=1$$

for $(\tau_1, \tau_2) \in F_\omega^2$, and $e=$ identity of F_ω. U is clearly Borel and locally countable.

Now, let Y be a standard Borel space and R a locally countable Borel relation on Y. We can assume that $Y=2^{\mathbb{N}}$. Let F_ω act on Y in a Borel fashion so that the equivalence relation induced by R coincides with the equivalence relation induced by the action (see Feldman–Moore [1977]). Define a Borel embedding φ of Y into X: $\varphi(y)=((\sigma \cdot y)_{\sigma \in F_\omega}, \alpha^y)$, where $\alpha^y(\tau_1, \tau_2)=1$ iff $\tau_1 \cdot y R \tau_2 \cdot y$. φ is clearly Borel and 1-to-1.

Now, we have to show that xRy iff $\varphi(x)U\varphi(y)$. Let xRy. Then there is $\sigma_0 \in F_\omega$ with $y=\sigma_0 \cdot x$. But then $\alpha^x(e, \sigma_0)=1$ and $(\sigma \cdot y)_{\sigma \in F_\omega}=(\sigma\sigma_0 \cdot x)_{\sigma \in F_\omega}$. So, it remains to check that $\alpha^y=\sigma_0\alpha^x$. But $\alpha^y(\tau_1, \tau_2)=1$ iff $\tau_1 \cdot yR\tau_2 \cdot y$ iff $\tau_1\sigma_0 \cdot xR\tau_2\sigma_0 \cdot x$ iff $\alpha^x(\tau_1\sigma_0, \tau_2\sigma_0)=1$ iff $\sigma_0\alpha^x(\tau_1, \tau_2)=1$.

If $\varphi(x)U\varphi(y)$, then $\exists\sigma_0((\sigma \cdot y)_{\sigma \in F_\omega}=(\sigma\sigma_0 \cdot x)_{\sigma \in F_\omega})$, so $y=\sigma_0 \cdot x$. Moreover, $\alpha^x(e, \sigma_0)=1$ hence xRy.

We also have to show that $\varphi(Y)$ is U-invariant. If $\varphi(x)\ U((z_\sigma)_{\sigma \in F_\omega}, \beta)$, then for some σ_0, $(z_\sigma)_{\sigma \in F_\omega}=(\sigma\sigma_0 \cdot x)$ and $\beta=\sigma_0\alpha^x$. It is easy to check that $\varphi(z_e)=((z_\sigma)_{\sigma \in F_\omega}, \beta)$. Similarly for $((z_\sigma)_{\sigma \in F_\omega}, \beta)\ U\varphi(x)$. ∎

Let $\mathscr{C}$ be a class of graphs in standard Borel spaces. We say that $\mathscr{G}_0=(X_0, R_0) \in \mathscr{C}$ is *universal* in $\mathscr{C}$ if for every $\mathscr{G}=(Y, R) \in \mathscr{C}$ there is a Borel set $A \subseteq X_0$ which is $E_{\mathscr{G}_0}$-invariant, so that $\mathscr{G}$ is Borel isomorphic to $\mathscr{G}_0 \mid A$. From the preceding Proposition 7.1 one can obtain universal elements for many classes of locally countable graphs. For example we have

7.2. Corollary. (i) *There is a universal Borel locally countable graph. There is a universal Borel locally countable acyclic graph.*

(ii) *There is a universal Borel locally finite graph* (*resp., of bounded degree* k). *Similarly for acyclic graphs.*

We can also apply 7.1 to obtain universal functions. Let $\mathscr{K}$ be a class of Borel functions in standard Borel spaces. We say that $f \in \mathscr{K}$, $f\colon X \to X$ is *universal in* $\mathscr{K}$ if for any $g \in \mathscr{K}$, $g\colon Y \to Y$, there is a Borel injection $\varphi\colon Y \to X$ such that $\varphi \circ g=f \circ \varphi$.

7.3. Corollary. *There is a universal Borel* $\leqslant\aleph_0$-*to*-1 *function. Similarly for finite to*-1 *or* $\leqslant k$-*to*-1 *functions.*

This follows easily from 7.1 by identifying a function $f\colon X \to X$ with the relation $R(x, y) \Leftrightarrow f(x)=y$.

Let $\mathscr{F}$ be a semigroup of Borel mappings on a standard Borel space X. A countable partition $\{A_n : n \in \mathbb{N}\}$ of X consisting of Borel sets is called a *countable generator* for $\mathscr{F}$ provided for any $x, y \in X$ the condition $\forall f \in \mathscr{F}$ $\forall n \in \mathbb{N}(f(x) \in A_n$ iff $f(y) \in A_n)$ implies that $x=y$. Sometimes we view the generator $\{A_n\}$ as a Borel function $\varphi\colon X \to \mathbb{N}$, where $\varphi^{-1}[\{n\}]=A_n$.

7.4. Lemma. *Let X be standard Borel. Let $f\colon X \to X$ be Borel,* $\leqslant\aleph_0$-*to*-1, *and without periodic points. Then f has a countable generator.*

Proof. *Claim.* Let $T\colon X \to X$ be a Borel, aperiodic automorphism. Then there exists a Borel function $\psi\colon X \to \mathbb{N}$ such that if $x, y \in X$ and $x \neq y$, then

$\exists n_1, n_2 \in \mathbb{Z}$ such that $n_1 \geqslant 0, n_2 \leqslant 0$, and $\psi(T^{n_1}(x)) \neq \psi(T^{n_1}(y))$ and $\psi(T^{n_2}(x)) \neq \psi(T^{n_2}(y))$.

Proof. Let $\varphi: X \to \mathbb{N}$ be a countable generator for T whose existence is guaranteed by [DJK, Prop. 11.3]. By the proof of (5) $\Rightarrow$ (1) in Section 1 of DJK (that we already used in 1.1), we can find a sequence $(M_n)_{n \in \mathbb{N}}$ of Borel subsets of X such that each M_n is unbounded in both directions in each T-orbit, $M_{n+1} \subseteq M_n$, and $\bigcap_n M_n = \emptyset$. Now, let $x \in X$, and let n_x be the smallest $n \in \mathbb{N}$ with $x \notin M_n$. Find $n_1 \in \mathbb{N}$ smallest such that $T^{-n_1}(x) \in M_{n_x}$ and $n_2 \in \mathbb{N}$ smallest such that $T^{n_2}(x) \in M_{n_x}$. Put

$$\psi'(x) = (n_1, n_2, \varphi(T^{-n_1}(x)), \varphi(T^{-n_1+1}(x)), ..., \varphi(T^{n_2}(x))).$$

Let $\psi = \chi \circ \psi'$ where $\chi: \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ is a bijection.

To check that ψ has the required property, let $x \neq y$, and assume towards a contradiction that $\forall k \geqslant 0, \psi'(T^k(x)) = \psi'(T^k(y))$. (The case $\forall k \leqslant 0(\psi'(T^k(x)) = \psi'(T^k(y)))$ is handled in a similar fashion.) Since φ is a generator for T, there is $k_0 \in \mathbb{Z}$ with $\varphi(T^{k_0}(x)) \neq \varphi(T^{k_0}(y))$. Recursively define a sequence (m_k^x) by putting $m_0^x =$ the second coordinate of $\psi'(x)$, and $m_{k+1}^x =$ the second coordinate of $\psi'(T^{m_0^x + \cdots + m_k^x}(x))$. Define (m_k^y) in a similar way. Since $\forall k \geqslant 0(\psi'(T^k(x)) = \psi'(T^k(y)))$, it follows by induction that $m_k^x = m_k^y$ for all $k \in \mathbb{N}$. Put $p_k = m_0^x + \cdots + m_k^x = m_0^y + \cdots + m_k^y$. As in the proof of Theorem 1.1, we view the T-orbits of x and y as ordered in type $\mathbb{Z}$ by $x_1 < x_2$ iff $\exists n > 0(T^n(x_1) = x_2)$. Note that for any k there are n_k^x and n_k^y such that $T^{p_k}(x) \in M_{n_k^x}$, $T^{p_k}(y) \in M_{n_k^y}$, and there are no elements of $M_{n_k^x}$ between (with respect to $<$) x and $T^{p_k}(x)$ and no elements of $M_{n_k^y}$ between y and $T^{p_k}(y)$. Since $m_k^x = m_k^y > 0$, $p_k \to \infty$, whence $n_k^x, n_k^y \to \infty$ as $k \to \infty$. Thus, since $\bigcap_n M_n = \emptyset$, it follows that from some k on x, $T^{k_0}(x) \notin M_{n_k^y}$ and there are no points from $M_{n_k^x}$ between x and $T^{k_0}(x)$. Similarly for y, $T^{k_0}(y)$, and $M_{n_k^y}$. Thus, for k big enough $\varphi(T^{k_0}(x)) =$ the $(p_{k+1} - k_0 + 1)$'th (counting from the right to the left!) coordinate of $\psi'(T^{p_k}(x))$ and $\varphi(T^{k_0}(y)) =$ the $(p_{k+1} - k_0 + 1)$'th (counting from the right to the left) coordinate of $\psi'(T^{p_k}(y))$. But since $\psi'(T^{p_k}(x)) = \psi'(T^{p_k}(y))$, $\varphi(T^{k_0}(x)) = \varphi(T^{k_0}(y))$, a contradiction, which finishes the proof of the claim.

Let E_f be the tail equivalence relation induced by f, that is, xE_fy iff $\exists n, m \in \mathbb{N}(f^n(x) = f^m(y))$. By [DJK, Corollary 8.2], there exists $T: X \to X$ a Borel bijection such that the equivalence relation E_T induced by T, xE_Ty iff $\exists n \in \mathbb{Z}(y = T^n(x))$, coincides with E_f. Since f does not have periodic points, all equivalence classes of E_f are infinite, so T is aperiodic. Let $\psi: X \to \mathbb{N}$ be constructed for T as in the claim. Define $\varphi': X \to \mathbb{N}^{<\mathbb{N}}$ as follows: For $x \in X$, let k_x be the unique $k \in \mathbb{Z}$ with $T^k(x) = f(x)$. Put

$$\varphi'(x) = (k_x, \psi(x), ..., \psi(T^{k_x}(x))).$$

Let $\varphi = \chi \circ \varphi'$, where $\chi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ is a bijection. It is easy to see that φ is Borel. We show that φ is a generator. Let $x \neq y$, and assume towards a contradiction that $\varphi'(f^n(x)) = \varphi'(f^n(y))$ for all $n \in \mathbb{N}$. Let $k_n =$ the first coordinate of $\varphi'(f^n(x)) =$ the first coordinate of $\varphi'(f^n(y))$. Let $p_n = k_0 + k_1 + \cdots + k_n$ and $p_{-1} = 0$. Since neither x nor y are preperiodic points for f, $p_n \neq p_{n'}$ if $n \neq n'$. Thus, $\limsup p_n = \infty$ or $\liminf p_n = -\infty$. Assume $\limsup p_n = \infty$ (The other case is dealt with similarly.) Let $m \geqslant 0$ be such that $\psi(T^m(x)) \neq \psi(T^m(y))$. Now find $n_0 \in \mathbb{N}$ such that $p_{n_0 - 1} \leqslant m \leqslant p_{n_0}$. Since $\varphi'(f^{n_0}(x)) = \varphi'(f^{n_0}(y))$, we have

$$\begin{aligned}(k_{n_0}, \psi(f^{n_0}(x)), ..., \psi(T^{k_{n_0}}(f^{n_0}(x))))\\ = (k_{n_0}, \psi(f^{n_0}(y)), ..., \psi(T^{k n_0}(f^{n_0}(y)))).\end{aligned}$$

Also since $f^{n_0}(x) = T^{p_{n_0-1}}(x)$, $f^{n_0}(y) = T^{p_{n_0-1}}(y)$, we get $\psi(T^m(x)) = \psi(T^m(y))$, a contradiction. ∎

7.5. Lemma. *Let X be a standard Borel space. Let $\mathscr{F}$ be a countable semigroup of Borel functions mapping X into X. Assume there is no finite set $K \subseteq X$ such that for all $f \in \mathscr{F}$, $f(K) \subseteq K$. Then there exists a Borel function $F\colon X \to X$ such that*

(i) $\forall x \in X\, \exists f \in \mathscr{F}(F(x) = f(x))$;

(ii) *F does not have periodic points.*

Proof. Define the equivalence relation xEy iff $x = y$ or $\exists f, g \in \mathscr{F}$ $(x = f(y)\ \&\ y = g(x))$. Since $\mathscr{F}$ is countable, E is Borel and has countable equivalence classes. Let $Y_1, Y_2 \subseteq X$ be disjoint, Borel, E-invariant, and such that $X = Y_1 \cup Y_2$, all E-equivalence classes contained in Y_1 are infinite and all E-equivalence classes contained in Y_2 are finite. By Jackson–Kechris–Louveau [199?] (see also Kechris [1994]), we can find a Borel aperiodic bijection $T\colon Y_1 \to Y_1$, with $xET(x)$ for any $x \in Y_1$. Define $F \mid Y_1 = T$.

To define F on Y_2, consider

$$Z = \{x \in Y_2 \colon \exists f \in \mathscr{F}(\neg xEf(x))\}.$$

Then $Z \subseteq Y_2$ is Borel. Since each E-equivalence class contained in Y_2 is finite and no finite set is invariant under all functions from $\mathscr{F}$, Z meets every equivalence class of $E \mid Y_2$. Let $S \subseteq Z$ be a Borel transversal of $E \mid Y_2$. For $x \in Y_2 \backslash S$, let $F(x)$ be the unique element of S with $xEF(x)$. To define F for $x \in S$, enumerate $\mathscr{F} = \{f_n \colon n \in \mathbb{N}\}$ and put $F(x) = f_{n_x}(x)$, where $n_x = \min\{n \in \mathbb{N}\colon \neg f_n(x)\ Ex\}$. Such an n_x exists since $x \in S \subseteq Z$.

It is easy to verify (i). To check (ii), assume, toward a contradiction, that $x = F^n(x)$ for some $x \in X$ and $n \geqslant 1$. Clearly $x \in Y_2$ since T is aperiodic. By construction, $\neg xEF(x)$ or $\neg xEF^2(x)$, depending on whether $x \in S$ or not.

But by (i), $xEF^k(x)$ for $k \leqslant n$; thus $n=1$. It follows that $x=F^k(x)$, for all $k \in \mathbb{N}$ thus $xEF(x)$ and $xEF^2(x)$, so we have a contradiction. ▮

7.6. THEOREM. *Let $\mathcal{F}$ be a countable semigroup of $\leqslant\aleph_0$-to-1 Borel functions on a standard Borel space X. Assume that the family of finite sets $K \subseteq X$ such that for every $f \in \mathcal{F}$, $f(K) \subseteq K$ is countable. Then $\mathcal{F}$ has a countable generator.*

Proof. We define a countable generator $\varphi: X \to \mathbb{N}$. Let $A = \{x \in X: \exists f \in \mathcal{F} \exists \text{ finite } K \subseteq X (\forall g \in \mathcal{F} (g(K) \subseteq K) \,\&\, f(x) \in K)\}$. Then A is countable, and $X \backslash A$ is invariant under mappings from $\mathcal{F}$. Let $\varphi \mid A$ be any bijection between A and the even natural numbers. Let $Y = X \backslash A$. It remains to define $\varphi \mid Y$. Let $\{f_n: n \in \mathbb{N}\} = \{f \mid Y: f \in \mathcal{F}\}$. Let $F: Y \to Y$ be as in Lemma 7.5 defined for $\{f_n: n \in \mathbb{N}\}$. Finally, let $\psi: Y \to \mathbb{N}$ be a countable generator for F, whose existence is guaranteed by Lemma 7.4. Define $\varphi': y \to \mathbb{N}^2$ as follows: for $x \in Y$, let n_x be the smallest $n \in \mathbb{N}$ with $F(x) = f_n(x)$; put $\varphi'(x) = (\psi(x), n_x)$. Let $\varphi \mid Y = \chi \circ \varphi'$, where $\chi: \mathbb{N}^2 \to \{2m+1: m \in \mathbb{N}\}$ is a bijection. It is easy to see that φ is Borel. To check that φ is a generator, it is enough to show that for $x, y \in Y$ if $\varphi'(f(x)) = \varphi'(f(y))$ for all $f \in \mathcal{F}$, then $x = y$, since the other cases are trivial. Recursively define a sequence: $n_1 =$ the second coordinate of $\varphi'(x) =$ the second coordinate of $\varphi'(y)$, and $n_{k+1} =$ the second coordinate of $\varphi'(f_{n_k} \circ \cdots \circ f_{n_1}(x)) =$ the second coordinate of $\varphi'(f_{n_k} \circ \cdots \circ f_{n_1}(y))$. Then $F^k(x) = f_{n_k} \circ \cdots \circ f_{n_1}(x)$ and $F^k(y) = f_{n_k} \circ \cdots \circ f_{n_1}(y)$ and moreover for all k, $\psi(F^k(x)) =$ the first coordinate of $\varphi'(f_{n_k} \circ \cdots \circ f_{n_1}(x)) =$ the first coordinate of $\varphi'(f_{n_k} \circ \cdots \circ f_{n_1}(y)) = \psi(F^k(y))$. Thus, since ψ is a generator, $x = y$. ▮

Let $\mathcal{F}$ be a semigroup. Define the shift action of $\mathcal{F}$ on $\mathbb{N}^{\mathcal{F}}$ as follows: for $x \in \mathbb{N}^{\mathcal{F}}$ and $f \in \mathcal{F}$ put

$$f \cdot x(g) = x(gf) \qquad \text{for} \quad g \in \mathcal{F}.$$

7.7. COROLLARY. *Assume a countable semigroup $\mathcal{F}$ acts by $\leqslant\aleph_0$-to-1 Borel functions on X in such a way that the set of finite $K \subseteq X$ with $\forall f \in \mathcal{F} (f \cdot K \subseteq K)$ is countable. Then there is a Borel embedding $\alpha: X \to \mathbb{N}^{\mathcal{F}}$ such that $\alpha(f \cdot x) = f \cdot \alpha(x)$, $\forall f \in \mathcal{F}$.*

Proof. Let $\varphi: X \to \mathbb{N}$ be a countable generator for $\mathcal{F}' = \{x \mapsto f \cdot x: f \in \mathcal{F}\}$. Put $\alpha(x)(f) = \varphi(f \cdot x)$. ▮

The next corollary follows immediately from the above corollary applied to $\mathcal{F} = \mathbb{N}$. It gives an important universality property of the graph of the shift on $\mathbb{N}^{\mathbb{N}}$.

7.8. COROLLARY. *Let s_∞: $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the shift. For any Borel f: $X \to X$ which is $\leqslant \aleph_0$-to-1 and has at most countably many periodic points, there is a Borel embedding α: $X \to \mathbb{N}^{\mathbb{N}}$ such that $\alpha \circ f = s_n \circ \alpha$. In particular, we have $\mathcal{G}_f \sqsubseteq_B \mathcal{G}_{s_\infty}$.*

It is clear that 7.8 fails if f has more than countably many periodic points. Also even if f is an aperiodic Borel automorphism (on X) it is not in general possible to find a Borel embedding α: $X \to n^{\mathbb{N}}$ with $\alpha \circ f = s_n \circ \alpha$, for finite n (see, e.g., Weiss [1984]). The next result establishes a weaker alternative. We need some notation and terminology first.

Let f: $X \to X$. Recall that E_f is the equivalence relation defined on X by $xE_f y \Leftrightarrow \exists m, n \in \mathbb{N}(f^m(x) = f^n(y))$. A set $C \subseteq X$ is *f-invariant* if $x \in C \Rightarrow f(x) \in C$; C is *E_f-invariant* if $x \in C \Rightarrow f(x) \in C$ and $f^{-1}(x) \subseteq C$.

7.9. DEFINITION. Let f: $X \to X$, g: $Y \to Y$. Then ψ: $X \to Y$ is called a *quasiembedding* if $\psi \circ f = g \circ \psi$ and for any x, $y \in X$ with $xE_f y$, if $x \neq y$, then $\psi(x) \neq \psi(y)$.

NOTATION. Let f: $X \to X$. Put

$$P^*(f) = \{x\colon \exists n, m \geqslant 0, n \neq m, (f^n(x) = f^m(x))\}$$

$$P(f) = \{x\colon \exists n > 0 (f^n(x) = x)\}.$$

If s_∞: $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is the shift, we put $P(s_\infty) = P$, $P^*(s_\infty) = P^*$. We consider $n^{\mathbb{N}}$ as included in the natural way in $m^{\mathbb{N}}$, if $2 \leqslant n \leqslant m \leqslant \aleph_0$, and the shift s_m as an extension of s_n.

Note that $P^*(f)$ is E_f-invariant, $P(f)$ is f-invariant and $P(f)$ is a *full section* of $E_f \mid P^*(f)$, i.e., meets every equivalence class of $E_f \mid P^*(f)$.

7.10. THEOREM. *Let X be a standard Borel space and let f: $X \to X$ be a Borel $\leqslant n$-to-1 function, $2 \leqslant n \leqslant \aleph_0$. Then there exists a Borel quasi-embedding ψ: $(X, f) \to (n^{\mathbb{N}}, s_n)$.*

Proof. Claim 1. There exists Borel ψ: $X \to n^{\mathbb{N}}$ such that

(i) $s_n \circ \psi = \psi \circ f$;

(ii) $\psi \mid (X \setminus \psi^{-1}(P^*))$ is a quasiembedding;

(iii) if $f(x) = f(y)$ and $\psi(x)(0) = \psi(y)(0)$, then $x = y$.

Proof. Fix $g_0, ..., g_{n-1}$, if n is finite, and $g_0, g_1, ...$, if $n = \aleph_0$, partial Borel functions such that $\forall x \in X\, \forall i < n$ if $x \in \mathrm{dom}(g_i)$, then $f(g_i(x)) = x$, and $\forall x \in X\, \exists! i \leqslant n f(x) \in \mathrm{dom}(g_i)$ and $g_i(f(x)) = x$. The existence of the g_i's follows from the fact the Borel set $\{(x, y) \in X \times X\colon x = f(y)\}$ has vertical sections of cardinality $\leqslant n \leqslant \aleph_0$, and hence can be represented as the union of the graphs of not more than n partial Borel functions. (If f is actually $<n$-to-1, then some of the g_i's may be empty.) Define $\psi(x)=(i_0, i_1, i_2, ...)$ where i_k is the unique $i<n$ with $g_i(f^{k+1}(x))=f^k(x)$. Now, (i) and (iii) follow immediately from the construction. To check (ii), let $x, y \in X$ be such that $x \neq y$ and $\psi(x)=\psi(y)$. First note that for $\ell \in \mathbb{N}$, $f^\ell(x) \neq f^\ell(y)$. Now additionally assume that xE_fy. Then there are $\ell, m \in \mathbb{N}$ with $f^\ell(x)=f^m(y)$. By the above observation $\ell \neq m$, and we can assume that $\ell > m$. Put $k=\ell-m$. For $y'=f^m(y)$ and $x'=f^m(x)$, we have

$$\psi(y')=\psi(f^k(x'))=s_n^{k+m}(\psi)(x))=s_n^{k+m}(\psi(y))=s_n^k(\psi(y')),$$

that is, $\psi(y') \in P$ whence $\psi(y) \in P^*$.

A similar method gives the following claim.

Claim 2. Let $Y \subseteq X$ be f-invariant, and let $f \mid Y$ be 1-to-1. Let ψ: $(Y, f) \to (2^{\mathbb{N}}, s_2)$ be a Borel quasiembedding. Then there is a Borel quasiembedding $\tilde{\psi}$: $([Y]_{E_f}, f) \to (n^{\mathbb{N}}, s_n)$ which extends ψ.

Proof. Define partial Borel functions h_1, h_2 on Y as follows: For $x \in Y$ put $x \in \mathrm{dom}(h_i)$ if there is $y \in Y$ with $f(y)=x$ and $\psi(y)(0)=i$; then put $h_i(x)=y$. (The h's are well-defined, since if $f(y_1)=f(y_2)$ and $\psi(y_1)(0)=\psi(y_2)(0)$, then $y_1 E_f y_2$ and $\psi(y_1)=\psi(y_2)$, whence $y_1=y_2$.)

Now, find partial Borel functions $g_0, ..., g_{n-1}$ on $[Y]_{E_f}$ as in the proof of Claim 1 so that, moreover, g_0 extends h_0 and g_1 extends h_1.

Define $\tilde{\psi}$: $[Y]_{E_f} \to n^{\mathbb{N}}$ as follows: let $x \in [Y]_{E_f}$, and let ℓ be minimal such that $f^\ell(x) \in Y$; for $k<\ell$ let g_{i_k} be such that $g_{i_k}(f^{k+1}(x))=f^k(x)$; put

$$\tilde{\psi}(x)=(i_0, i_1, ..., i_{\ell-1}, \psi(f^\ell(x))).$$

Again, it is straightforward to check that $\tilde{\psi}([Y]_{E_f})$ is s_n-invariant and that $\tilde{\psi} \circ f = s_n \circ \tilde{\psi}$. To see that $\tilde{\psi}$ is 1-to-1 on E_f-equivalence classes, let $x, y \in [Y]_{E_f}$ be such that $x \neq y$, $\tilde{\psi}(x)=\tilde{\psi}(y)$, and xE_fy. As in the proof of Claim 1 we show that $f^\ell(x) \neq f^\ell(y)$ for all $\ell \in \mathbb{N}$. Let $\ell \in \mathbb{N}$ be such that $f^\ell(x), f^\ell(y) \in Y$. Then since $f^\ell(x) \neq f^\ell(y)$,

$$s_n^\ell(\tilde{\psi}(x))=\psi(f^\ell(x)) \neq \psi(f^\ell(y))=s_n^\ell(\tilde{\psi}(y)),$$

whence $\tilde{\psi}(x) \neq \tilde{\psi}(y)$, which proves Claim 2.

We will define ψ: $X \to n^{\mathbb{N}}$ in stages. First we define it on $P^*(f)$. For each $\ell \in \mathbb{N} \setminus \{0, 1\}$ pick $\alpha_\ell \in 2^{\mathbb{N}}$ with period ℓ. Let S be a Borel transversal for $E_f \mid P(f)$. Each $x \in P(f)$ can be uniquely represented as $x=f^k(y)$ for some $y \in S$ and $0<k<\ell_y$, where ℓ_y is the smallest ℓ with $f^\ell(y)=y$. Put $\psi_0(x)=s_2^k(\alpha_{\ell_y})$. It is easy to see that ψ_0: $(P(f), f) \to (2^{\mathbb{N}}, s_2)$ is a Borel quasiembedding. Moreover, $f \mid P(f)$ is 1-to-1 and $P(f)$ is f-invariant. Thus, by Claim 2, we can extend ψ_0 to a Borel quasiembedding $\tilde{\psi}_0\colon ([P(f)]_{E_f}, f) \to (n^{\mathbb{N}}, s_n)$. But $[P(f)]_{E_f} = P^*(f)$. We define our ψ to be equal to $\tilde{\psi}_0$ on $P^*(f)$.

By considering $X \setminus P^*(f)$ and $f | (X \setminus P^*(f))$, we can assume that $P^*(f) = \emptyset$. From this point on we make this assumption. Let $\psi_1\colon X \to n^{\mathbb{N}}$ be as in Claim 1. Define our ψ to be equal to ψ_1 on $X \setminus \psi_1^{-1}(P^*)$. (Note that $X \setminus \psi_1^{-1}(P^*)$ is E_f-invariant.) Thus it is enough to define ψ on $\psi_1^{-1}(P^*)$. Note that $\psi_1^{-1}(P^*) = [\psi_1^{-1}(P)]_{E_f}$. Also, $\psi_1^{-1}(P)$ is f-invariant, and $f | \psi_1^{-1}(P)$ is 1-to-1. (Indeed, if $f(x) = f(y)$, $x, y \in \psi_1^{-1}(P)$, then $s_n(\psi_1(x)) = \psi_1(f(x)) = \psi_1(f(y)) = s_n(\psi_1(y))$, but, since s_n is 1-to-1 on P, $\psi_1(x) = \psi_1(y)$ whence $\psi_1(x)(0) = \psi_1(y)(0)$ and $x = y$ by Claim 1 (iii).) Thus, by Claim 2, the following claim finishes the proof.

Claim 3. Let $f\colon Y \to Y$ be Borel, 1-to-1 with $P(f) = \emptyset$. Then there is a Borel quasiembedding $\psi\colon (Y, f) \to (2^{\mathbb{N}}, s_2)$.

Proof. Note that any Borel $\psi\colon Y \to 2^{\mathbb{N}}$ with $\psi(Y)$ s_2-invariant, $\psi \circ f = s_2 \circ \psi$ and $\psi(Y) \subseteq 2^{\mathbb{N}} \setminus P$ is a quasiembedding.

First, we observe that if $A \subseteq Y$ is Borel, E_f-invariant and $E_f | A$ is smooth, then such a ψ exists on A. Simply choose $\alpha \in 2^{\mathbb{Z}}$ non-periodic to the right and fix a Borel transversal $S \subseteq A$ for $E_f | A$. Then put $\psi(x) = (\alpha(n), \alpha(n+1), \ldots)$, where $n \in \mathbb{Z}$ is such that $f^n(x) \in S$.

Thus, it is enough to define ψ outside of a smooth, E_f-invariant Borel set. To this end, it suffices to find a Borel set $D \subseteq Y$ such that $Y \setminus [D]_{E_f}$ is smooth and for each $x \in [D]_{E_f}$ the sequence $(\chi_D(f(x)), \chi_D(f^2(x)), \ldots)$ is not periodic. (The mapping $x \mapsto (\chi_D(f(x)), \chi_D(f^2(x)), \ldots)$ gives the required quasiembedding.) A set $C \subseteq Y$ is called *periodic* in $\mathcal{O} \subseteq Y$, $\mathcal{O}$ an f-orbit, if $\exists x \in \mathcal{O}\ \exists m \in \mathbb{N},\ m > 0\ \forall k \in \mathbb{N} (f^k(x) \in C \Leftrightarrow f^{k+m}(x) \in C)$.

As in the proof of Theorem 1.1, we view each orbit $\mathcal{O}$ of f as ordered in order type $\mathbb{Z}$ by the relation $x <_{\mathcal{O}} y$ iff $y = f^n(x)$ for some $n > 0$. Let $(M_n)_{n \in \mathbb{N}}$ be a sequence of Borel sets such that $M_{n+1} \subseteq M_n$, in each orbit $\mathcal{O}$ of f, M_n is unbounded in both directions (with respect to $<_{\mathcal{O}}$), $\bigcap_n M_n = \emptyset$ (see the proof of (5) $\Rightarrow$ (1) in 5.1 of DJK). We can find E_f-invariant Borel sets $C_1, C_2 \subseteq Y$, $Y = C_1 \cup C_2$ such that $\forall \mathcal{O} \subseteq C_1$ an f-orbit $\exists n (M_n$ is not periodic in $\mathcal{O})$, and $\forall \mathcal{O} \subseteq C_2$ an f-orbit $\forall n (M_n$ is periodic in $\mathcal{O})$. On C_1 we are done. On C_2 we can modify the sequence (M_n) (on each orbit the modification amounts to taking a subsequence) so that there is a Borel E_f-invariant set $C_3 \subseteq C_2$ such that $C_2 \setminus C_3$ is smooth and $\forall \mathcal{O} \subseteq C_3$ an f-orbit there are unboundedly many to the right $x \in \mathcal{O}$ for which we have $\{x, f(x), \ldots, f^{nk_n - 1}(x)\} \cap M_{n+1} = \emptyset$, where k_n is the smallest period of M_n on $\mathcal{O}$. Now, for $x \in C_3$ put

$$x \in D \qquad \text{iff the smallest } n \text{ with } x \notin M_n \text{ is odd.}$$

Then D is Borel and $[D]_{E_f} = C_3$. We claim that D is not periodic in any $\mathcal{O} \subseteq C_3$. Otherwise, let $m \in \mathbb{N}$ be its period in $\mathcal{O} \subseteq C_3$ an f-orbit. Then $M_m \cap D \cap \mathcal{O}$ is periodic with period mk_m, where k_m, is the smallest period of M_m in $\mathcal{O}$. But also

(∗) $\exists$ unboundedly many to the right $x \in \mathcal{O}$ with $\{x, f(x), ..., f^{mk_m - 1}(x)\} \cap M_{m+1} = \varnothing$.

Now, if m is even, $M_n \backslash M_{m+1} \subseteq D$ whence from (∗) $M_m \cap \{x, f(x), ..., f^{mk_m - 1}(x)\} \subseteq D$ for unboundedly many to the right $x \in \mathcal{O}$. But since $M_m \cap \mathcal{O} \cap D$ has period mk_m, it follows that $M_m \cap \{f^k(x): k \geqslant 0\} \subseteq D$, for some $x \in \mathcal{O}$, which contradicts the fact that M_{m+1} is unbounded in both directions in $\mathcal{O}$. If m is odd, $(M_m \backslash M_{m+1}) \cap D = \varnothing$, and again using (∗) we get $M_m \cap \{f^k(x): k \geqslant 0\} \cap D = \varnothing$ for some $x \in \mathcal{O}$, a contradiction. ∎

8. FINITE VS INFINITE CHROMATIC NUMBERS

In Section 6 we have proved a dichotomy result characterizing when a graph has uncountable Borel chromatic number. It is natural to inquire whether an analogous result might hold for infinite Borel chromatic numbers.

Recall Example 3.2, which shows that if S is the shift map on $[\mathbb{N}]^{\mathbb{N}}$ then $\chi_B(\mathcal{G}_S) = \aleph_0$. Also in (B) of Section 6 we pointed out that $\mathcal{G}_S$ is essentially the same as the graph corresponding to $t'_n = 0^n$, restricted to $x \in 2^{\mathbb{N}}$ with infinitely many 1's.

8.1. *Open Problem.* Is the following true? If X is a Polish space and $\mathcal{G} = (X, R)$ an analytic graph, then exactly one of the following holds:

(I) $\chi_B(\mathcal{G}) < \aleph_0$;

(II) $\mathcal{G}_S \leqslant_c \mathcal{G}$.

The simplest case is when $\mathcal{G} = \mathcal{G}_f$, for some Borel countable-to-1 function $f: X \to X$, and not much is known even in this case. We have however the following interesting fact, noticed by Louveau, which generalizes the remark following 5.1.

8.2. Proposition (Louveau). *Let X be Polish and $f: X \to X$ be Borel and countable-to-*1. *If for every $x \in X$ there is $k \in \mathbb{N}$ such that for infinitely many n, card$(f^{-1}(\{f^n(x)\})) \leqslant k$, then $\chi_B(\mathcal{G}_f) < \aleph_0$ (and so $\chi_B(\mathcal{G}_f) \leqslant 3$).*

Proof. We can clearly assume that there is a fixed $k \in \mathbb{N}$ such that for all $x \in X$ there are infinitely many n with card$(f^{-1}(\{f^n(x)\})) \leqslant k$. Let

$\{f_i\}_{i \geqslant 1}$ be a sequence of Borel functions such that $\mathrm{dom}(f_i) = \{x \in X: f^{-1}(\{x\}) \geqslant i\}$ and for $x \in X$,

$$f^{-1}(\{x\}) = \{f_i(x): 1 \leqslant i, x \in \mathrm{dom}(f_i)\}.$$

Let (X, R) be the graph generated by $f_1, ..., f_k$, i.e.,

$$xRy \Leftrightarrow x \neq y \;\&\; \exists i \leqslant k (f_i(x) = y \text{ or } f_i(y) = x).$$

Since each f_i is 1-to-1 it follows from 4.5 that there is a Borel coloring $c_1: X \to F$ of (X, R), with F finite.

Next let $U: X \to (k+1)^{\mathbb{N}}$ be defined as follows:

$$U(x)(n) = \begin{cases} 0, & \text{if } \mathrm{card}(f^{-1}(\{f^{n+1}(x)\})) > k; \\ i, & \text{otherwise, where } f_i(f^{n+1}(x)) = f^n(x). \end{cases}$$

If s_{k+1} is the shift on $(k+1)^{\mathbb{N}}$, then clearly $U(f(x)) = s_{k+1}(U(x))$. Let $c_2: (k+1)^{\mathbb{N}} \to 3$ be a Borel coloring of $\mathcal{G}_{s_{k+1}}$ (by 5.2). Define $c: X \to (k+1) \times 3$ by $c(x) = (c_1(x), c_2(U(x)))$. We claim that this is a (clearly Borel) coloring of $\mathcal{G}_f$. Indeed, if $x \neq f(x)$, either $U(x) \neq U(f(x))$ in which case $c_2(U(x)) \neq c_2(U(f(x)))$, or else $U(x) = U(f(x))$ and thus $U(x) = s_{k+1}(U(x))$, so $U(x)$ is constant. Our hypothesis implies that $U(x)$ cannot be constantly 0, so for some $1 \leqslant i \leqslant k$, $U(x) = i$, and thus $f_i(f(x)) = x$, i.e., $xRf(x)$ and so $c_1(x) \neq c_1(f(x))$. ∎

Next let us notice that if $f: X \to X$ is as before, then the periodic part $P^*(f) = \{x \in X: \exists n > 0 \,\exists m > 0 (f^n(x) = f^{n+m}(x))\}$ has finite Borel chromatic number (in $\mathcal{G}_f$). So we may as well assume that f has no periodic points in trying to characterize when $\mathcal{G}_f$ is infinite. Then by 7.8 there is a Borel embedding $\alpha: X \to \mathbb{N}^{\mathbb{N}}$ with $\alpha \circ f = s_\infty \circ \alpha$. Thus $\alpha(X)$ is a (shift-invariant) Borel subset of $\mathbb{N}^{\mathbb{N}}$ and $\mathcal{G}_f$ is Borel isomorphic to the shift graph restricted to $\alpha(X)$.

For a Borel set $A \subseteq \mathbb{N}^{\mathbb{N}}$ let $\chi_B^{s_\infty}(A)$ be the chromatic number of the shift graph restricted to A. In some sense we have reduced our problem to understanding when $\chi_B^{s_\infty}(A)$ is infinite.

Next we verify that $\chi_B^{s_\infty}(\mathbb{N}^{\mathbb{N}} \setminus [\mathbb{N}]^{\mathbb{N}})$ is finite. To see this, define a finite Borel coloring of $\mathbb{N}^{\mathbb{N}} \setminus [\mathbb{N}]$ as follows:

(i) If x is constant, let $c(x) = 0$.

(ii) If x is not constant, but $x(0) = x(1)$, let ℓ be least with $x(\ell) \neq x(\ell+1)$. Put then

$$c(x) = (1, \ell \bmod 2).$$

(iii) If $x(0) < x(1)$, let k be such that $x(0) < x(1) < \cdots < x(k) \geqslant x(k+1)$. Then put

$$c(x) = (2, k \bmod 2).$$

(iv) If $x(0) > x(1)$, let k be such that $x(0) > x(1) > \cdots > x(k) \leqslant x(k+1)$. Put

$$c(x) = (3, k \bmod 2).$$

It follows that it is enough to consider $\chi_B^{s_\infty}(A)$ for $A \subseteq [\mathbb{N}]^{\mathbb{N}}$. As in 3.2, if there is infinite $H \subseteq \mathbb{N}$ with $[H]^{\mathbb{N}} \subseteq A$ clearly $\chi_B^{s_\infty}(A)$ is infinite. This leads naturally to the following question:

8.3. *Open Problem.* Let $A \subseteq [\mathbb{N}]^{\mathbb{N}}$ be Borel. Is it true that

$$\chi_B^{s_\infty}(A) = \aleph_0 \qquad \text{iff} \quad \exists H \in [\mathbb{N}]^{\mathbb{N}}([H]^{\mathbb{N}} \subseteq A)?$$

An affirmative answer to this problem implies, by our preceding remarks, a positive answer to 8.1 in the case $\mathcal{G} = \mathcal{G}_f$ for a Borel countable-to-1 function $f\colon X \to X$ on some Polish space.

9. CHROMATIC NUMBERS OF BOREL GRAPHS

In this section we give some examples of Borel graphs in order to get some idea about the behaviour of the (non-Borel) chromatic number on this class. We start by recalling the definition of the poset $\pi\mathbb{Q}$ of Todorcevic [1991]: The domain of $\pi\mathbb{Q}$ is the power-set of the rationals and the ordering is defined by letting $x \lessdot y$ if there is $q \in y$ such that $x = \{p \in y\colon p < q\}$.

9.1. Proposition. *The chromatic number of the comparability graph of $\pi\mathbb{Q}$ is equal to $\aleph_0$.*

Proof. By an argument as in 6.10, we see that the chromatic number is uncountable, so it remains to show that it is no more than $\aleph_1$. Let $<_w$ be a well-ordering of $\pi\mathbb{Q}$ and let $\{q_n\}$ be an enumeration of $\mathbb{Q}$. For $t \in \pi\mathbb{Q}$ which has proper extensions in $\pi\mathbb{Q}$, let $s(t)$ denote the $<_w$-minimal such extension. For $n \in \mathbb{N}$, let

$$\pi_n\mathbb{Q} = \{t \in \pi\mathbb{Q}\colon q_n = \min(s(t) \backslash t)\}.$$

Since $\pi_n\mathbb{Q}(n \in \mathbb{N})$ cover $\{t \in \pi\mathbb{Q}\colon \sup(t) < \infty\}$ it suffices to show that $\pi_n\mathbb{Q}$ is the union of $\aleph_1$ antichains.

Claim. Each $\pi_n\mathbb{Q}$ is a well-founded subset of $\pi\mathbb{Q}$.

Proof. Suppose some $\pi_n\mathbb{Q}$ contains an infinite decreasing sequence $t_0 > t_1 > t_2 > \cdots$. Then $s(t_0) \geqslant_w s(t_1) \geqslant_w \cdots$. So the sequence must stabilize from some point j on. Let s be the constant value of $s(t_i)$ $(i \geqslant j)$. By the definition of $\pi_n\mathbb{Q}$ we have that $q_n = \min(s \backslash t_i)$ for all $i \geqslant j$ contradicting the assumption that $\{t_i\}$ is a strictly decreasing sequence.

Thus, each $\pi_n\mathbb{Q}$ is a subtree of $\pi\mathbb{Q}$. Since elements of $\pi\mathbb{Q}$ have only countably many predecessors the height of $\pi_n\mathbb{Q}$ is $\leqslant \omega_1$ for all n. Thus each $\pi_n\mathbb{Q}$ can be covered by no more than $\aleph_1$ levels which themselves are antichains of $\pi_n\mathbb{Q}$. This finishes the proof. ∎

The second Borel graph $\mathcal{G}_c = (X, R_c)$ also appears in the paper Todorcevic [1991] and also has chromatic number equal to $\aleph_1$. It is in some sense better than the previous one, since it is an example of a *closed* graph, i.e., R_c is a closed subset of $X^2 \backslash \Delta (\Delta = \{(x, x) : x \in X\})$. (In the previous example the edge relation is F_σ rather than closed.) So let us reproduce the description of $\mathcal{G}_c$. The space X is equal to $\mathbb{N}^{\mathbb{N}}$. To define R_c we first associate to every $f \in \mathbb{N}^{\mathbb{N}}$ a sequence $\{f_i\}$ as follows: Let $n_0 < n_1 < \cdots$ be the list of all n such that $f(2n+1) \neq 0$ and for a given i let f_i be determined by $f_i \restriction n_k = f \restriction n_k$ and

$$f_i(n_k + j) = f(2^{i+1}(2n_k + 2j + 1)),$$

where $k = k(i)$ is minimal with the property

$$f(2n_0 + 1) + \cdots + f(2n_k + 1) > i;$$

if such a k does not exist, let $f_i = f$. Finally define R_c by letting $(f, g) \in R_c$ iff there is i such that either $f = g_i$ or $g = f_i$.

9.2. Proposition. *The chromatic number of* $(\mathbb{N}^{\mathbb{N}}, R_c)$ *is equal to* $\aleph_1$.

Proof. The uncountability of the chromatic number of $(\mathbb{N}^{\mathbb{N}}, R_c)$ is the content of Theorem 3 of Todorcevic [1991], so we concentrate on proving that it is not more than $\aleph_1$. This is an immediate consequence of Fodor's set-mapping theorem (see [Erdos–Hajnal–Mate–Rado [1984]; Thm 44.1]) applied to $f \mapsto \{f_i\}$. ∎

The following result tells us that the example of Proposition 9.2 is essentially best possible.

9.3. Proposition. *Let* $\mathcal{G} = (X, R)$ *be a graph such that* X *is a Polish space and* R *is open in* X^2. *Then the chromatic number at* $\mathcal{G}$ *is either countable or equal to the continuum (or more precisely* $(2^{\mathbb{N}}, (2^{\mathbb{N}})^2 \backslash \Delta) \sqsubseteq_c \mathcal{G}$).

Proof. This is an immediate consequence of the Open Coloring Axiom for such spaces (see Todorcevic [1989]). ∎

We finish the discussion with a class of Borel graphs where the chromatic number function has a much more complex behaviour than in the previous two examples. Let $<^*$ be the relation of eventual dominance in $\mathbb{N}^{\mathbb{N}}$, i.e., $f <^* g$ if $f(n) < g(n)$ for almost all n. Let

$$X = \{(f_0, f_1, f_2) \in (\mathbb{N}^{\mathbb{N}})^3 \colon f_0 <^* f_1 <^* f_2\},$$

and let $(f_0, f_1, f_2) \prec_s (g_0, g_1, g_2)$ iff

$$f_0 <^* f_1 <^* g_0 <^* f_2 <^* g_1 <^* g_2.$$

Consider the graph $\mathcal{G}_s = (X, R_s)$, where $(f_0, f_1, f_2)\, R_s (g_0, g_1, g_2)$ iff $(f_0, f_1, f_2) \prec_s (g_0, g_1, g_2)$ or $(g_0, g_1, g_2) \prec_s (f_0, f_1, f_2)$. Note that $\mathcal{G}_s$ is a Borel graph without triangles. The following fact shows that nevertheless the chromatic number of $\mathcal{G}_s$ is rather large.

9.4. PROPOSITION. *The chromatic number of $\mathcal{G}_s$ is $\geqslant \theta$ for any cardinal θ for which we can find a chain in $(\mathbb{N}^{\mathbb{N}}, <^*)$ of size θ.*

Proof. Clearly we may assume θ is regular. Fix a chain $W \subseteq \mathbb{N}^{\mathbb{N}}$ which has order type θ under $<^*$. Let $[W]^3$ be the set of all $(f, g, h) \in W^3$ such that $f <^* g <^* h$. It suffices to show that $\mathcal{G}_s \mid [W]^3$ has chromatic number θ. But this is one of the standard facts in combinatorial set theory (see [Williams [1977]; Thm 5.1.9]). ∎

Note that $\mathcal{G}_s$ is an F_σ-graph, the next in complexity to closed ones. For closed graphs we know (by Proposition 9.2) that we can realize every chromatic number from the set $\{1, 2, 3, ...\} \cup \{\aleph_0, \aleph_1, \mathfrak{c}\}$. We don't know whether these are all the possible values of the chromatic numbers of closed graphs, but we shall now see, using the graph $\mathcal{G}_s$, that they are not all the possibilities realized by the class of all F_σ-graphs.

9.5. THEOREM. *Let $\mathcal{C}$ be the standard poset for adding $\mathfrak{c}^+$ Cohen reals. Then the chromatic number of the graph $\mathcal{G}_s$ as defined in the forcing extension by $\mathcal{C}$ is bounded by the continuum of the ground model.*

Proof. Every $\mathcal{C}$-name $\dot{f} = \langle \dot{f}_0, \dot{f}_1, \dot{f}_2 \rangle$ for an element of the vertex set of $\dot{\mathcal{G}}_s$, has a countable support $S(\dot{f})$ which we take to have a limit order-type. Note that there are only continuum many isomorphism types of structures of the form

$$\mathcal{S}_{\dot{f}} = \langle S(\dot{f}), \dot{f}, \in, e \restriction [S(\dot{f})]^2 \rangle$$

where e: $[\mathfrak{c}^+]^2 \to \mathfrak{c}$ is a fixed mapping such that $e(\cdot, \alpha)$: $\alpha \to \mathfrak{c}$ is 1-to-1 for all α and where we assume (increasing $S(\dot{f})$ if necessary) that

$$e([S(\dot{f})]^2) \subseteq S(\dot{f}).$$

In fact there exist only $\mathfrak{c}$ many types even if we require that the resulting isomorphisms between two $S_{\dot{f}}$ and $S_{\dot{g}}$ have to be the identity on

$$S(\dot{f}) \cap \mathfrak{c} = S(\dot{g}) \cap \mathfrak{c}.$$

So fix one such type τ and let F_τ be the set of all $\dot{f}$'s whose $S_{\dot{f}}$ is of type τ. It is sufficient to show that it is forced that there is a coloring of F_τ with $\check{\mathfrak{c}}$-many colors. Let δ be the common order-type of all $S(\dot{f})$'s for $f \in F_\tau$. For $\gamma < \delta$ and $f \in F_\tau$ let $S_\gamma(\dot{f})$ be the subset of $S(\dot{f})$ consisting of the first γ members of $S(\dot{f})$ in the increasing enumeration. For a countable set $D \subseteq \mathfrak{c}^+$ let $\dot{h}_\xi(D)(\xi < \mathfrak{c})$ be a fixed enumeration of all $\mathscr{C}$-names of members of $\mathbb{N}^{\mathbb{N}}$ whose support is included in D.

Working in the generic extension by $\mathscr{C}$, to every element $\mathrm{int}_{\dot{G}}(\dot{f})$ for $f \in F_\tau$ we associate a pair $(\gamma_{\dot{f}}, \xi_{\dot{f}}) \in \check{\delta} \times \check{\mathfrak{c}} \cup \{(\infty, \infty)\}$ as follows: We let $\gamma_{\dot{f}}$ be the minimal ordinal $\gamma < \delta$ for which there is some $\xi < \check{\mathfrak{c}}$ such that

$$\mathrm{int}_{\dot{G}}(\dot{f}_0) \leqslant^* \mathrm{int}_{\dot{G}}(\dot{h}_\xi(S_\gamma(\dot{f}))) <^* \mathrm{int}_{\dot{G}}(\dot{f}_1)$$

and let $\xi_{\dot{f}}$ be the minimal ξ which works for $\gamma_{\dot{f}}$. If there is no such ordinal $< \delta$ we let $\gamma = \xi = \infty$. We claim $\mathrm{int}_{\dot{G}}(\dot{f}) \mapsto (\gamma_{\dot{f}}, \xi_{\dot{f}})$ is a coloring. For suppose that in the ground model some condition $p \in \mathscr{C}$ forces for some $\dot{f}$, $\dot{g} \in F$ and $(\gamma, \xi) \in \delta \times \mathfrak{c} \cup \{\infty, \infty)\}$ that (γ, ξ) is the pair associated to both $\dot{f}$, and $\dot{g}$ but $\dot{f} \prec_s \dot{g}$. Moreover, we assume that p decides the place from which point on the dominance between the members of $\dot{f}$ and $\dot{g}$ happens. Note that

$$D = S(\dot{f}) \cap S(\dot{g})$$

is an initial part of both $S(\dot{f})$ and $S(\dot{g})$, so let $\beta < \delta$ be such that (for $S_\beta(\dot{f}) = S(\dot{f}) \cap \beta)$

$$D = S_\beta(\dot{f}) = S_\beta(\dot{g})$$

Let $D_{\dot{f}} = S(\dot{f}) \backslash D$ and $D_{\dot{g}} = S(\dot{g}) \backslash D$. Working in the forcing extension by $\mathscr{C}_D$ (= all the conditions in $\mathscr{C}$ supported by D) below $p \restriction D$ we have that some condition (namely, $p \backslash (p \restriction D)$) of the product

$$C_{D_{\dot{f}}} \times C_{D_{\dot{g}}} \cong C_{D_{\dot{f}} \cup D_{\dot{g}}}$$

forces that

$$\dot{g}_0 <^* \dot{f}_2 <^* \dot{g}_1.$$

So working in the forcing extension by $\mathscr{C}_D$ we can define $h \in \mathbb{N}^{\mathbb{N}}$ by letting $h(n)$ to be the minimal m for which there is some extension of $p \restriction D_{\dot{f}}$ inside the poset $\mathscr{C}_{D_{\dot{f}}}$ which forces $\dot{f}_2(n) = m$. It is easily seen that $p \backslash (p \restriction D)$ forces that

$$\dot{g}_0 \leqslant^* \check{h} \leqslant^* \dot{g}_1.$$

It follows that the condition p forces that in the definition of $(\gamma_{\dot{g}}, \xi_{\dot{g}})$ we entered into the nontrivial case and that $\gamma_{\dot{f}} = \gamma \leqslant \beta$. But of course it forces the same thing about $\dot{f}$ since it forces $(\gamma_{\dot{g}}, \xi_{\dot{f}}) = (\gamma_{\dot{g}}, \xi_{\dot{g}}) = (\check{\gamma}, \check{\xi})$. Let $E = S_\gamma(\dot{f}) = S_\gamma(\dot{g})$. Then the condition p forces both that

$$\dot{f}_0 \leqslant^* \dot{h}_\xi(E) \leqslant^* \dot{f}_1,$$

and that

$$\dot{g}_0 \leqslant^* \dot{h}_\xi(E) \leqslant^* \dot{g}_1,$$

contradicting the fact that it also forces $\dot{f} \dot{\lessdot}_s \dot{g}$. ∎

APPENDIX

We present here M. Laczkovich's proof that for each $2 \leqslant n$ there is an acyclic Borel graph $\mathscr{G}$ with $\chi_B(\mathscr{G}) = n$. From this it also follows easily that for every $2 \leqslant k, 0 \leqslant n$ there is a Borel graph $\mathscr{G}$ with $\chi_B(\mathscr{G}) = k + n$ but $\chi(\mathscr{G}) = k$. So this solves Problem 3.3.

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. The linear fractional transformations are the functions $(ax + b)/(cx + d)$, where $a, b, c, d \in \mathbb{R}$, $ad - bc \neq 0$, mapping $\overline{\mathbb{R}}$ into itself (assuming that $1/0 = \infty$, $1/\infty = 0$). The set of linear fractional transformations will be denoted by LFT. It is a group under composition. By a theorem of J. von Neumann, Zur allgemeinen Theorie des Masses, *Fund. Math.* **13** (1929), 73–116, if the real numbers $a_i, b_i, c_i, d_i (i \in I)$ are algebraically independent over the rationals, the corresponding linear fractional transformations generate a free group.

Let n be a fixed positive integer and put

$$U = \{(a, b, c, d) \colon (ax + b)/(cx + d) > x + 1/n, \forall x \in [0, 1]\}.$$

Then U is open in $\mathbb{R}^4$ and so we can find a sequence a_k, b_k, c_k, d_k of elements of U, which are algebraically independent over the rationals and dense in U. Denote by α_k the element of LFT corresponding to (a_k, b_k, c_k, d_k) and by H the subgroup of LFT generated by the α_k. Let X be the co-countable set of all reals in $[0,1)$ which are not fixed points of any non-identity element of H. Define a graph $\mathcal{G}$ on X by connecting x, y by an edge if for some k, $\alpha_k(x) = y$ or $\alpha_k(y) = x$.

Clearly $\mathcal{G}$ is a Borel acyclic graph. If x, y are connected by an edge, then $|x - y| > 1/n$, so $X = \bigcup_n (X \cap [(i-1)/n, i/n))$ is a Borel partition showing that the Borel chromatic number of $\mathcal{G}$ is at most n. To show that it is at least n it is enough to prove that for any Lebesgue measurable set $A \subseteq X$, if no two members of A are connected by an edge, then the Lebesgue measure $\lambda(A)$ of A is at most $1/n$.

So suppose that for such an A, $\lambda(A) > 1/n$, toward a contradiction. Then we can find points x_0, y_0 in A such that x_0, y_0 are density points of A and $y_0 - x_0 > 1/n$. Find $\varepsilon > 0$ such that for $0 < h < \varepsilon$ we have

$$\lambda(A \cap [x, x+h]) > 0.9h$$

for $x = x_0, y_0$. Then for f a C^1 function on $[0, 1]$ such that

$$|f' - 1| < \varepsilon/10, \qquad |f(x_0) - y_0| < \varepsilon/10, \tag{1}$$

we have that $f(A) \cap A \neq \emptyset$. Find $\delta > 0$ such that whenever (a, b, c, d) satisfy $|a-1| < \delta$, $|b - (y_0 - x_0)| < \delta$, $|c| < \delta$, $|d - 1| < \delta$, then $(a, b, c, d) \in U$ and the associated fractional linear transformation f satisfies (1) above. So there is k such that the same holds for α_k instead of f, thus $\alpha_k(A) \cap A \neq \emptyset$, which contradicts the fact that no two elements of A are connected by an edge.

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