

---

# PUMPING LEMMAS FOR WEIGHTED AUTOMATA

AGNISHOM CHATTOPADHYAY, FILIP MAZOWIECKI, ANCA MUSCHOLL,  
AND CRISTIAN RIVEROS

Chennai Mathematical Institute, India  
*e-mail address:* agnishom@cmi.ac.in

Max Planck Institute for Software Systems, Germany  
*e-mail address:* filipm@mpi-sws.org

Université de Bordeaux, France  
*e-mail address:* anca@labri.fr

Pontificia Universidad Católica de Chile, Santiago, Chile  
*e-mail address:* cristian.riveros@uc.cl

---

**ABSTRACT.** We present pumping lemmas for five classes of functions definable by fragments of weighted automata over the min-plus semiring, the max-plus semiring and the semiring of natural numbers. As a corollary we show that the hierarchy of functions definable by unambiguous, finitely-ambiguous, polynomially-ambiguous weighted automata, and the full class of weighted automata is strict for the min-plus and max-plus semirings.

## 1. INTRODUCTION

Weighted automata (WA for short) are a quantitative extension of finite state automata used to compute functions over words. They have been extensively studied since Schützenberger’s early works [31], in particular decidability questions [20, 1], model extensions [11], logical characterisations [11, 19], and various applications [25, 9] have been thoroughly investigated in recent years.

The class of functions computed by WA enjoys several equivalent representations in terms of automata and logics. Alur et al. introduced some years ago the expressive model of cost register automata (CRA for short) [3, 4], an alternative model for computing functions over words inspired by programming paradigms, that received a lot of attention recently [22,

---

\* A preliminary version of this paper appeared in the proceedings of the 35th Symposium on Theoretical Aspects of Computer Science (STACS) [23].

Agnishom Chattopadhyay was an intern at the University of Bordeaux, partially supported by the UMI ReLaX and the French-Indian project IoTTTA.

Filip Mazowiecki was partially supported by EPSRC grants EP/M011801/1 and EP/M012298/1. The work was partly done during a postdoctoral stay at the University of Bordeaux.

Anca Muscholl was partially supported by the ANR project DeLTA (ANR-16-CE40-0007).

Cristian Riveros was supported by FONDECYT grant 11150653 and the Millennium Nucleus Center for Semantic Web Research under grant NC120004.

24, 10, 2]. The idea is to enhance deterministic finite automata with registers, storing values that can be combined by using operations over a fixed semiring. In [3], it was shown that CRA are strictly more expressive than WA. Interestingly, it was also shown that a natural fragment of CRA is equally expressive to WA, which provides a new viewpoint on this class of functions.

Regarding logics, Droste and Gastin introduced in [11] the so-called Weighted Logic (WL), a natural extension of monadic second order logics (MSO) from the boolean semiring to a commutative semiring. The semantics of this logics maps an MSO formula over strings to one or zero in the semiring, depending whether the input satisfies the formula or not. Furthermore, WL uses sum and product quantifiers that allow to aggregate the output of boolean formulas producing an output value in the semiring. Although WL is far more expressive than WA, it was shown in [11] that a natural syntactic restriction of WL is equally expressive to WA, giving the first logical characterisation of WA. Weighted logics or, more generally, quantitative logics have found many applications in understanding WA [12, 19], verification [7] and computational complexity [5].

The decidability and complexity of various decision problems for WA have also been investigated, unfortunately often with negative results [20, 1]. For this reason, research has focused on various fragments of WA over different semirings. For example, over a one-letter alphabet, where WA are equivalent to linear recurrences, some new decidability results were recently shown for limited fragments [26, 27]. Further restrictions of WA involve bounding their numbers of runs. Among them, the most studied classes are *unambiguous automata*, *finitely-ambiguous automata*, and *polynomially-ambiguous automata*, where the numbers of accepting runs is bounded by one, a constant, a polynomial in the size of input, respectively [33, 18, 17]. These subclasses of WA turned out to be robust, enjoying equivalent characterisations in terms of cost register automata [3] and weighted logics [19].

Although functions defined by WA and its subclasses have been studied in terms of representations and decidability, little is known about expressibility issues. Indeed, we are not aware of any general techniques to show if a function is definable or not by some WA, or any of its subclasses. Results related to the expressiveness of WA usually require sophisticated arguments for each particular function [18, 22] and there is no clear path to generalise these techniques. As a matter of fact, the strict inclusions between unambiguous, finitely-ambiguous, polynomially-ambiguous, and the full class of WA are “well-known” to the community, but it is hard to find references to formal proofs (see related work below). In contrast, for regular languages or first-order logics there exist elegant and powerful techniques for showing inexpressibility, as for example, the Myhill-Nerode congruence for regular languages [15] or Ehrenfeucht-Fraïssé games [14, 13, 21] and aperiodic congruences for first-order logic [32]. One would like to have similar techniques in the quantitative world that simplify inexpressibility arguments for WA, CRA or WL. Such techniques would help to understand the inner structure of these functions and unveil their limits of expressibility.

In this paper, we embark in the work of building an expressibility toolbox for weighted automata. We present five pumping lemmas, each of them for a different class or subclass of functions defined by WA over the min-plus semiring, the max-plus semiring or the semiring of natural numbers. For each pumping lemma we show examples of functions that do not satisfy the lemma, giving very short inexpressibility proofs. Our results do not attempt to fully characterise the class or subclasses of weighted automata in terms of pumping properties, nor to provide conditions that can be verified by a computer. Our goal is to

devise a systematic way to reason about expressibility of weighted automata and to provide simple arguments to show that functions do not belong to a given class.

*Related work.* In [16], it is shown that over the min-plus semiring polynomially-ambiguous automata are strictly more expressive than finitely-ambiguous automata. In [18] strict inclusions between unambiguous automata, finitely-ambiguous automata, and the full class of WA are shown over the max-plus semiring. In both papers the strict inclusions are shown by analysing particular functions. Using results from [8] one can deduce that unambiguous automata are strictly included in the other classes over the min-plus and max-plus semirings. Gathering these results we obtain strict inclusions between unambiguous automata, finitely-ambiguous automata, and the full class of WA over the min-plus semiring. However, to our knowledge, there is no reference for a strict inclusion between polynomially-ambiguous automata and the full class of WA. Recently there has been some work on the semiring of rational numbers with the usual plus and product [24, 6]. In these papers the polynomial-ambiguous fragment over the one-letter alphabet is characterised in terms of a fragment of linear recurrence sequences. Both papers provide proofs that polynomially-ambiguous weighted automata are strictly contained in the full class of weighted automata over the semiring of rationals.

*Differences with the conference version.* Compared to [23], we present new pumping lemmas for the max-plus semiring (Section 6) regarding finitely ambiguous and polynomially ambiguous max-plus automata. As a corollary we obtain a strict hierarchy of functions similar to the one for min-plus automata.

*Organization.* In Section 2 we introduce weighted automata and some basic definitions. In Section 3 and Section 4 we present pumping lemmas for weighted automata over the semiring of natural numbers and its extension using the operation min. In Section 5 we show the pumping lemma for polynomially-ambiguous automata over the min-plus semiring, then we turn to the max-plus semiring in Section 6. Concluding remarks can be found in Section 7.

## 2. PRELIMINARIES

In this section, we recall the definitions of weighted automata. We start with the definitions that are standard in this area. A monoid  $\mathbb{M} = (M, \otimes, \mathbb{1})$  is a set  $M$  with an associative operation  $\otimes$  and a neutral element  $\mathbb{1}$ . Standard examples of monoids are: the set of words  $\Sigma^*$  with concatenation and empty word; or the set of matrices with multiplication and the identity matrix. A semiring is a structure  $\mathbb{S} = (S, \oplus, \odot, \mathbb{0}, \mathbb{1})$ , where  $(S, \oplus, \mathbb{0})$  is a commutative monoid,  $(S - \{\mathbb{0}\}, \odot, \mathbb{1})$  is a monoid, multiplication distributes over addition, and  $\mathbb{0} \odot s = s \odot \mathbb{0} = \mathbb{0}$  for each  $s \in S$ . If the multiplication is commutative, we say that  $\mathbb{S}$  is commutative. In this paper, we always assume that  $\mathbb{S}$  is commutative. We usually denote  $S$  or  $M$  by the name of the semiring or monoid  $\mathbb{S}$  or  $\mathbb{M}$ . We are interested mostly in the tropical semirings: the *min-plus semiring*  $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$  and the *max-plus semiring*  $(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ . We are also interested in the *semiring of natural numbers with infinity*  $(\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ , where  $\infty + n = \infty$  for every  $n \in \mathbb{N} \cup \{\infty\}$  and  $\infty \cdot n = \infty$  if  $n \neq 0$  and 0 otherwise. We denote the tropical semirings by  $\mathbb{N}_{\min,+}$  and  $\mathbb{N}_{\max,+}$ ; and the later semiring by  $\mathbb{N}_{+,x}$ . Note that  $\mathbb{N}_{+,x}$  is an extension of the standard semiring of natural numbers  $\mathbb{N}$  and all our results for  $\mathbb{N}_{+,x}$  also hold for  $\mathbb{N}$ . We use the extended version of  $\mathbb{N}$  to transfer some results from  $\mathbb{N}_{+,x}$  to  $\mathbb{N}_{\min,+}$  (see Section 4). Given a finite set  $Q$ , we denote by  $\mathbb{S}^{Q \times Q}$  ( $\mathbb{S}^Q$ ) the set of square matrices

(vectors resp.) over  $\mathbb{S}$  indexed by  $Q$ . The algebra induced by  $\mathbb{S}$  over  $\mathbb{S}^{Q \times Q}$  and  $\mathbb{S}^Q$  is defined as usual.

We also consider two finite semirings that will be useful during proofs. We consider the boolean semiring  $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$  and the extended boolean semiring  $\mathbb{B}_\infty = (\{0, 1, \infty\}, \vee, \wedge, 0, 1)$  such that  $\infty \vee n = \infty$  for every  $n \in \{0, 1, \infty\}$ ,  $\infty \wedge 0 = 0$ , and  $\infty \wedge n = \infty$  if  $n \in \{1, \infty\}$ . Both finite semirings will be used as *abstractions* of  $\mathbb{N}_{\min,+}$  and  $\mathbb{N}_{+,\times}$ , respectively.

In this paper, we study the specification of functions from words to values, namely, from  $\Sigma^*$  to  $\mathbb{S}$ . We say that a function  $f : \Sigma^* \rightarrow \mathbb{S}$  is definable by a computational system  $\mathcal{A}$  (e.g., by a weighted automaton) if  $f(w) = \llbracket \mathcal{A} \rrbracket(w)$  for any  $w \in \Sigma^*$ , where  $\llbracket \mathcal{A} \rrbracket$  is the semantics of  $\mathcal{A}$  over words.

**2.1. Weighted automata.** Fix a finite alphabet  $\Sigma$  and a commutative semiring  $\mathbb{S}$ . A *weighted automaton* (WA for short) over  $\Sigma$  and  $\mathbb{S}$  is a tuple  $\mathcal{A} = (Q, \Sigma, \{M_a\}_{a \in \Sigma}, I, F)$  where  $Q$  is a finite set of states,  $\{M_a\}_{a \in \Sigma}$  is a set of matrices such that  $M_a \in \mathbb{S}^{Q \times Q}$  and  $I, F \in \mathbb{S}^Q$  are the initial and the final vectors, respectively [30, 12]. We say that a state  $q$  is initial if  $I(q) \neq 0$  and accepting if  $F(q) \neq 0$ . We usually say that an entry  $M_a(p, q) = s$  is a transition and write  $p \xrightarrow{a/s} q$ . Furthermore, we say that a run  $\rho$  of  $\mathcal{A}$  over a word  $w = a_1 \dots a_n$  is a sequence of transitions:  $\rho = q_0 \xrightarrow{a_1/s_1} q_1 \xrightarrow{a_2/s_2} \dots \xrightarrow{a_n/s_n} q_n$ , where  $s_i \neq 0$  for all  $1 \leq i \leq n$  and  $I(q_0) \neq 0$ . We refer to  $q_i$  as the  $i$ -th state of the run  $\rho$ . The run  $\rho$  is accepting if  $F(q_n) \neq 0$ , and the weight of an accepting run  $\rho$  is defined by  $|\rho| = I(q_0) \odot (\odot_{i=1}^n s_i) \odot F(q_n)$ . We define  $\text{Run}_{\mathcal{A}}(w)$  as the set of all accepting runs of  $\mathcal{A}$  over  $w$ . Finally, the output of  $\mathcal{A}$  over a word  $w$  is defined by  $\llbracket \mathcal{A} \rrbracket(w) = I^t \cdot M_{a_1} \cdot \dots \cdot M_{a_n} \cdot F = \bigoplus_{\rho \in \text{Run}_{\mathcal{A}}(w)} |\rho|$  where  $I^t$  is the transpose of  $I$  and the second sum is equal to 0 if  $\text{Run}_{\mathcal{A}}(w)$  is empty. For a word  $w = a_1 \dots a_n$  we usually denote  $M_w = M_{a_1} \cdot \dots \cdot M_{a_n}$  and then  $\llbracket \mathcal{A} \rrbracket(w) = I^t \cdot M_w \cdot F$ . Note that  $M_w(p, q)$  provides the cost of moving from state  $p$  to state  $q$  reading the word  $w$ . Functions defined by weighted automata are called *regular functions*.

A weighted automaton  $\mathcal{A}$  is called *unambiguous* (U-WA) if  $|\text{Run}_{\mathcal{A}}(w)| \leq 1$  for every  $w \in \Sigma^*$ ; and  $\mathcal{A}$  is called *finitely-ambiguous* (FA-WA) if there exists a uniform bound  $N$  such that  $|\text{Run}_{\mathcal{A}}(w)| \leq N$  for every  $w \in \Sigma^*$  [33, 18]. Furthermore,  $\mathcal{A}$  is called *polynomially-ambiguous* (PA-WA) if the function  $|\text{Run}_{\mathcal{A}}(w)|$  is bounded by a polynomial in the length of  $w$  [17]. We call the classes of functions definable by such automata *unambiguous regular*, *finitely-ambiguous regular* and *polynomially-ambiguous regular*.

Note that every unambiguous WA over  $\mathbb{N}_{\min,+}$  and  $\mathbb{N}_{\max,+}$  can be defined by a WA over the semiring  $\mathbb{N}_{+,\times}$  (recall that  $\infty$  is in  $\mathbb{N}_{+,\times}$ ). This follows essentially from the fact that an unambiguous automaton over the min-plus (or max-plus) semiring can be translated into a cost-register automaton that uses only the  $+$  operation (see e.g. [3, Theorem 13]). Such a cost-register automaton can be seen as a linear cost-register automaton over the  $\mathbb{N}_{+,\times}$ , which itself translates easily into a WA over the semiring  $\mathbb{N}_{+,\times}$ .

Therefore, the class of unambiguous regular functions over  $\mathbb{N}_{\min,+}$  (and  $\mathbb{N}_{\max,+}$ ) is included in the class of regular functions over  $\mathbb{N}_{+,\times}$ . The inclusions are strict since regular functions over  $\mathbb{N}_{\min,+}$  (and  $\mathbb{N}_{\max,+}$ ) are always bounded by a linear function in the size of the word, and it is easy to define the function  $f(w) = 2^{|w|}$  over  $\mathbb{N}_{+,\times}$ . Below, we give several examples of functions defined by WA over  $\mathbb{N}_{+,\times}$  and  $\mathbb{N}_{\min,+}$  that will be used in paper. Recall that in the latter semiring  $0 = \infty$  and  $\odot = +$ . Transitions  $p \xrightarrow{a/s} q$ , where  $s = 0$  are omitted.

**Example 2.1.** Let  $\Sigma = \{a, b\}$ . Consider the function  $f_1$  that for given word  $w \in \Sigma^*$  outputs the length of the biggest suffix of  $a$ 's (and  $\infty$  if the word ends in  $b$ ). This is defined by the

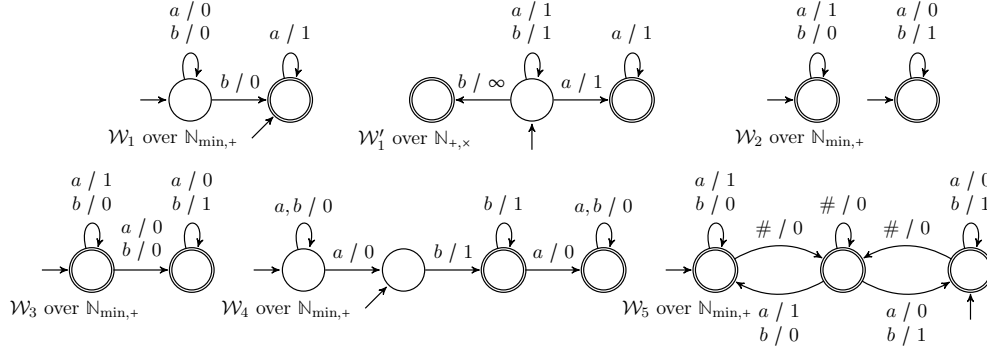


Figure 1: Examples of weighted automata. For WA over  $\mathbb{N}_{\min,+}$  the initial and accepting states are labelled by 0 in the corresponding vector, and  $\infty$  otherwise. Similarly, for WA over  $\mathbb{N}_{+,x}$  the initial and accepting states are labelled by 1 in the corresponding vector, and 0 otherwise.

WA  $\mathcal{W}_1$  over  $\mathbb{N}_{\min,+}$  in Figure 1. One can easily check that  $\mathcal{W}_1$  is unambiguous, hence  $f_1$  is an unambiguous regular function over  $\mathbb{N}_{\min,+}$ . In Figure 1, the WA  $\mathcal{W}_1'$  over  $\mathbb{N}_{+,x}$  also defines  $f_1$ .

**Example 2.2.** Let  $\Sigma = \{a, b\}$ . Consider the function  $f_2$  that for a given word  $w \in \Sigma^*$  outputs  $\min\{|w|_a, |w|_b\}$ , namely, counts the number of each letter and returns the minimum. This is defined by the WA  $\mathcal{W}_2$  in Figure 1. The WA  $\mathcal{W}_2$  is finitely-ambiguous, hence  $f_2$  is a finitely-ambiguous regular function.

**Example 2.3.** Let  $\Sigma = \{a, b\}$ . Consider the function  $f_3$  that for a given word  $w = a_1 \dots a_n \in \Sigma^*$  outputs  $\min_{0 \leq i \leq n} \{|a_1 \dots a_i|_a + |a_{i+1} \dots a_n|_b\}$ . This is defined by the WA  $\mathcal{W}_3$  in Figure 1. This WA is polynomially-ambiguous, hence  $f_3$  is a polynomially-ambiguous regular function.

**Example 2.4.** Let  $\Sigma = \{a, b\}$ . Consider the function  $f_4$  that for a given word  $w \in \Sigma^*$  computes the shortest subword of  $b$ 's (if there is none it outputs  $\infty$ ). This is defined by  $\mathcal{W}_4$  in Figure 1. The WA is polynomially-ambiguous, hence  $f_4$  belongs to polynomially-ambiguous functions.

**Example 2.5.** Let  $\Sigma = \{a, b, \#\}$ . Consider the function  $f_5$  such that, for any  $w \in \Sigma^*$  of the form  $w_0 \# w_1 \# \dots \# w_n$  with  $w_i \in \{a, b\}^*$ , it computes  $\min\{|w_i|_a, |w_i|_b\}$  for each subword  $w_i$  and then it sums these values over all subwords  $w_i$ , that is,  $f_5(w) = \sum_{i=0}^n \min\{|w_i|_a, |w_i|_b\}$ . This function is defined by the WA  $\mathcal{W}_5$  in Figure 1. Given that this WA has an exponential number of runs, the function  $f_5$  is a regular function, but not necessarily a polynomially-ambiguous regular function.

We will also discuss variants of the functions  $f_1$ ,  $f_2$ ,  $f_3$ ,  $f_4$  and  $f_5$  in the max-plus semiring in Section 6.

We assume that our weighted automata are always trim, namely, all their states are reachable from some initial state (accessible, for short) and they can reach some final state (co-accessible, for short). Verifying if a state is accessible or co-accessible is reduced to a reachability test in the transition graph [28] and this can be done in NLOGSPACE. Thus, we can assume without loss of generality that all our automata are trimmed.

**2.2. Finite monoids and idempotents.** We say that a monoid is finite if the set of its elements is finite. Let  $\mathbb{M} = (M, \otimes, \mathbb{1})$  be a finite monoid. We say that  $\iota \in \mathbb{M}$  is an idempotent if  $\iota \otimes \iota = \iota$ . The following lemma is a standard result for finite monoids and idempotents (see e.g. [29]).

**Lemma 2.6.** *Let  $\mathbb{M}$  be a finite monoid. There exists some  $N > 0$  such that every sequence  $m_1, \dots, m_n$ , with  $m_i \in \mathbb{M}$  and  $n \geq N$ , can be factorised as:*

$$(m_1 \otimes \dots \otimes m_i) \otimes (m_{i+1} \otimes \dots \otimes m_j) \otimes (m_{j+1} \dots \otimes m_n),$$

where  $i < j \leq n$  and  $(m_{i+1} \otimes \dots \otimes m_j)$  is an idempotent.

We will mainly use two finite monoids of matrices,  $\mathbb{B}^{Q \times Q}$  and  $\mathbb{B}_\infty^{Q \times Q}$ . For doing this we define abstractions, i.e., homomorphisms of  $\mathbb{N}_{\min,+}^{Q \times Q}$  to  $\mathbb{B}^{Q \times Q}$ ,  $\mathbb{N}_{\max,+}^{Q \times Q}$  to  $\mathbb{B}^{Q \times Q}$ , and  $\mathbb{N}_{+,x}^{Q \times Q}$  to  $\mathbb{B}_\infty^{Q \times Q}$ . They are obtained from the homomorphisms defined on elements of the matrices, namely  $h_1 : \mathbb{N}_{\min,+} \rightarrow \mathbb{B}$ ,  $h_2 : \mathbb{N}_{\max,+} \rightarrow \mathbb{B}$ , and  $h_3 : \mathbb{N}_{+,x} \rightarrow \mathbb{B}_\infty$ , by setting  $h_1(m) = 0$  iff  $m = \infty$ ;  $h_2(m) = 0$  iff  $m = -\infty$ ;  $h_3(0) = 0$ ,  $h_3(\infty) = \infty$ , and  $h_3(m) = 1$  if  $m \neq 0, \infty$ . For matrices  $M_1 \in \mathbb{N}_{\min,+}^{Q \times Q}$ ,  $M_2 \in \mathbb{N}_{\max,+}^{Q \times Q}$ , or  $M_3 \in \mathbb{N}_{+,x}^{Q \times Q}$  we denote by  $\bar{M}_1 = h_1(M_1)$ ,  $\bar{M}_2 = h_2(M_2)$ , or  $\bar{M}_3 = h_3(M_3)$  their abstractions in  $\mathbb{B}^{Q \times Q}$  or  $\mathbb{B}_\infty^{Q \times Q}$ . By abuse of language we denote a matrix  $M$  from  $\mathbb{N}_{\min,+}^{Q \times Q}$ ,  $\mathbb{N}_{\max,+}^{Q \times Q}$  or  $\mathbb{N}_{+,x}^{Q \times Q}$  as *idempotent*, if its abstraction  $\bar{M}$  is idempotent.

### 3. REGULAR FUNCTIONS OVER $\mathbb{N}_{+,x}$

In this section we consider regular functions over  $\mathbb{N}_{+,x}$ . As a corollary of the pumping lemma we show that FA-WA are strictly more expressive than U-WA over  $\mathbb{N}_{\min,+}$  and  $\mathbb{N}_{\max,+}$  (Example 3.2 and beginning of Section 6). Moreover, this shows that there are finitely-ambiguous regular functions over  $\mathbb{N}_{\min,+}$  and  $\mathbb{N}_{\max,+}$  that cannot be defined by any regular function over  $\mathbb{N}_{+,x}$ .

We introduce some notation to simplify the presentation. Given  $u \cdot v \cdot w = \hat{u} \cdot \hat{v} \cdot \hat{w}$ , where  $u, v, w, \hat{u}, \hat{v}, \hat{w} \in \Sigma^*$ , we say that  $\hat{u} \cdot \hat{v} \cdot \hat{w}$  is a *refinement* of  $u \cdot v \cdot w$  if there exist  $u', w'$  such that  $u \cdot u' = \hat{u}$ ,  $w' \cdot w = \hat{w}$ ,  $u' \cdot \hat{v} \cdot w' = v$ , and  $\hat{v} \neq \epsilon$ . We underline the infixes  $v$  and  $\hat{v}$  to emphasise the refined part.

**Theorem 3.1** (Pumping Lemma for regular functions over  $\mathbb{N}_{+,x}$ ). *Let  $f : \Sigma^* \rightarrow \mathbb{N} \cup \{\infty\}$  be a regular function over  $\mathbb{N}_{+,x}$ . There exists  $N$  such that for all words of the form  $u \cdot v \cdot w \in \Sigma^*$  with  $|v| \geq N$ ,  $v \neq \epsilon$ , there exists a refinement  $\hat{u} \cdot \hat{v} \cdot \hat{w}$  of  $u \cdot v \cdot w$  such that one of the following two conditions holds:*

- $f(\hat{u} \cdot \hat{v}^i \cdot \hat{w}) = f(\hat{u} \cdot \hat{v}^{i+1} \cdot \hat{w})$  for every  $i \geq N$ .
- $f(\hat{u} \cdot \hat{v}^i \cdot \hat{w}) < f(\hat{u} \cdot \hat{v}^{i+1} \cdot \hat{w})$  for every  $i \geq N$ .

Before going into the details of the proof let us show how to use the lemma.

**Example 3.2.** We show that  $f_2$  from Example 2.2 is not definable by any WA over  $\mathbb{N}_{+,x}$ . Indeed, suppose it is definable and fix  $N$  from Theorem 3.1. Consider the word  $w = a^{(N+1)^2} \underline{b^N}$  and notice that  $f_2(w) = N$ . By refining  $w$  we get  $\hat{u} \cdot \hat{v} \cdot \hat{w} = a^{(N+1)^2} \underline{b^n} \underline{b^m} \underline{b^l}$  for some  $n, m, l$  such that  $1 \leq m \leq N$  and  $n + m + l = N$ . Since  $n + m \cdot N + l < n + m \cdot (N+1) + l < (N+1)^2$  it must be the case that  $f_2(\hat{u} \cdot \hat{v}^i \cdot \hat{w}) < f_2(\hat{u} \cdot \hat{v}^{i+1} \cdot \hat{w})$  for all  $i \geq N$ . However,  $f_2(\hat{u} \cdot \hat{v}^i \cdot \hat{w}) = (N+1)^2$  for  $i$  sufficiently large, which is a contradiction.

**Example 3.3.** On the other hand, the function  $f_1$  from Example 2.1 satisfies Theorem 3.1. Consider a word  $u \cdot \underline{v} \cdot w \in \Sigma^*$  and its refinement  $\hat{u} \cdot \hat{\underline{v}} \cdot \hat{w}$ . If  $\hat{w}$  or  $\hat{v}$  contain  $b$  then  $f(\hat{u} \cdot \hat{\underline{v}}^i \cdot \hat{w}) = f(\hat{u} \cdot \hat{\underline{v}}^{i+1} \cdot \hat{w})$  because the suffix of  $a$ 's remains the same. Otherwise,  $f(\hat{u} \cdot \hat{\underline{v}}^i \cdot \hat{w}) < f(\hat{u} \cdot \hat{\underline{v}}^{i+1} \cdot \hat{w})$  since the suffix of  $a$ 's increases when pumping. Moreover, it is straightforward to generalise this argument and prove Theorem 3.1 for all U-WA over  $\mathbb{N}_{\min,+}$ .

To prove Theorem 3.1 we use the following definitions. For a matrix  $M \in \mathbb{N}_{+,x}^{Q \times Q}$  recall that  $\bar{M}$  is its homomorphic image in  $\mathbb{B}_{\infty}^{Q \times Q}$  (see Section 2.2). We write that  $M$  and  $N$  in  $\mathbb{N}_{+,x}^{Q \times Q}$  are equivalent, denoted  $M \equiv_{\mathbb{B}_{\infty}} N$ , iff  $\bar{M} = \bar{N}$ . We also extend the homomorphic image and equivalence relation from matrices to vectors. We say that  $D \in \mathbb{N}_{+,x}^{Q \times Q}$  is an *idempotent* if  $\bar{D}$  is an idempotent in the finite monoid  $\mathbb{B}_{\infty}^{Q \times Q}$ .

**Lemma 3.4.** *If  $M \equiv_{\mathbb{B}_{\infty}} N$ , then  $x^T \cdot M \cdot y > 0$  if and only if  $x^T \cdot N \cdot y > 0$  for every  $x, y \in \mathbb{N}_{+,x}^Q$ .*

*Proof.* Suppose that  $x^T \cdot M \cdot y > 0$ . By definition  $x^T \cdot M \cdot y = \sum_{p,q} x(p) \cdot M(p,q) \cdot y(q)$ . Then there exist  $p, q \in Q$  such that  $x(p) \cdot M(p,q) \cdot y(q) > 0$  and, in particular,  $M(p,q) > 0$ . Given that  $M \equiv_{\mathbb{B}_{\infty}} N$  we conclude  $N(p,q) > 0$  and  $x(p) \cdot N(p,q) \cdot y(q) > 0$ , which proves  $x^T \cdot N \cdot y > 0$ .  $\square$

*Proof of Theorem 3.1.* Let  $\mathcal{A} = (Q, \Sigma, \{M_a\}_{a \in \Sigma}, I, F)$  be a WA over  $\mathbb{N}_{+,x}$  such that  $f = \llbracket \mathcal{A} \rrbracket$ . Without loss of generality, we assume that  $I(q) \neq \infty$  and  $M_a(p,q) \neq \infty$  for every  $p, q \in Q$  and  $a \in \Sigma$ , namely,  $\infty$  can only appear in the final vector  $F$ . Indeed, if  $\infty$  is used in  $I$  or some  $M_a$ , we can construct two weighted automata  $\mathcal{A}', \mathcal{A}^\infty$  such that  $\mathcal{A}'$  is the same as  $\mathcal{A}$  but each  $\infty$ -initial state or each  $\infty$ -transition is replaced with 0, and  $\mathcal{A}^\infty$  outputs  $\infty$  if there exists some run in  $\mathcal{A}$  that outputs  $\infty$  and 0 otherwise. Note that  $\mathcal{A}'$  has no  $\infty$ -transition or  $\infty$ -initial state and  $\mathcal{A}^\infty$  can be constructed in such a way that only the final vector contains  $\infty$ -values. The disjoint union of  $\mathcal{A}'$  and  $\mathcal{A}^\infty$  is equivalent to  $\mathcal{A}$ .

Let  $N = \max\{|Q|, K\}$  where  $K$  is the constant from Lemma 2.6 for the finite monoid  $\mathbb{B}_{\infty}^{Q \times Q}$ . For every word  $u \cdot v \cdot w \in \Sigma^*$  such that  $v = a_1 \dots a_n$  with  $n \geq N$ , consider the output  $I^T \cdot M_u \cdot M_v \cdot M_w \cdot F$  of  $\mathcal{A}$  over  $u \cdot v \cdot w$ . By Lemma 2.6, there exists a factorisation of the form:

$$M_v = (M_{a_1} \cdot \dots \cdot M_{a_i}) \cdot (M_{a_{i+1}} \cdot \dots \cdot M_{a_j}) \cdot (M_{a_{j+1}} \cdot \dots \cdot M_{a_n})$$

for some  $i < j$  where  $M_{a_{i+1} \dots a_j}$  is an idempotent (i.e.,  $\bar{M}_{a_{i+1} \dots a_j}$  is an idempotent). We define the refinement  $\hat{u} \cdot \hat{\underline{v}} \cdot \hat{w}$  of  $u \cdot \underline{v} \cdot w$  such that  $\hat{u} = u \cdot (a_1 \dots a_i)$ ,  $\hat{\underline{v}} = a_{i+1} \dots a_j$ , and  $\hat{w} = (a_{j+1} \dots a_n) \cdot w$ . Furthermore, define  $x = I \cdot M_{ua_1 \dots a_i}$ ,  $D = M_{a_{i+1} \dots a_j}$ , and  $y = M_{a_{j+1} \dots a_n w} \cdot F$ . Note that  $f(\hat{u} \cdot \hat{\underline{v}}^i \cdot \hat{w}) = x^T \cdot D^i \cdot y$  for every  $i \geq 0$  and  $D$  is an idempotent (i.e.,  $\bar{D}$  is an idempotent). It remains to show the following lemma.

**Lemma 3.5.** *For every idempotent  $D \in \mathbb{N}_{+,x}^{Q \times Q}$  and  $x, y \in \mathbb{N}_{+,x}^Q$  where  $D$  and  $x$  do not contain  $\infty$ -values, one of the conditions holds:*

$$x^T \cdot D^i \cdot y = x^T \cdot D^{i+1} \cdot y \quad \text{for every } i \geq |Q|, \quad \text{or} \quad (3.1)$$

$$x^T \cdot D^i \cdot y < x^T \cdot D^{i+1} \cdot y \quad \text{for every } i \geq |Q|. \quad (3.2)$$

We start showing that Lemma 3.5 holds when  $y = e_p$  for some  $p \in Q$ , where  $e_p(q) = 1$  if  $q = p$  and 0 otherwise. Note that  $z = \sum_{p \in Q} z(p) \cdot e_p$  for every vector  $z$ .

We say that  $p$  is *D-stable* (or just *stable*) if  $D(p,p) > 0$ . Note that if  $p$  is stable, then  $D^i(p,p) > 0$  for every  $i > 0$  (recall that  $D$  is idempotent). Furthermore,  $D \cdot e_p = e_p + z$  for

some  $z \in \mathbb{N}_{+, \times}^Q$ . Suppose that  $p$  is stable and  $D \cdot e_p = e_p + z$  for some vector  $z$ . Then for  $i > 0$ :

$$x^T \cdot D^{i+1} \cdot e_p = x^T \cdot D^i \cdot (e_p + z) = x^T \cdot D^i \cdot e_p + x^T \cdot D^i \cdot z$$

Given that  $D$  is idempotent and  $D^i \equiv_{\mathbb{B}_\infty} D$ , by Lemma 3.4 we have that  $x^T \cdot D^i \cdot z > 0$  if, and only if,  $x^T \cdot D \cdot z > 0$ . Therefore, if  $x^T \cdot D \cdot z > 0$ , we get that  $x^T \cdot D^i \cdot e_p < x^T \cdot D^{i+1} \cdot e_p$  for every  $i > 0$ , in particular, for every  $i \geq |Q|$ . Otherwise,  $x^T \cdot D \cdot z = 0$  and  $x^T \cdot D^i \cdot e_p = x^T \cdot D^{i+1} \cdot e_p$  for every  $i > 0$ , in particular, for every  $i \geq |Q|$ .

Let  $P \subseteq Q$  be the set of all non-stable states in  $D$ . Consider the relation  $\leq_D \subseteq P \times P$  such that  $p \leq_D q$  if  $p = q$  or  $D(p, q) > 0$ . One can easily check that  $\leq_D$  forms a partial order over  $P$ , namely, that  $\leq_D$  is reflexive, antisymmetric, and transitive. Indeed, transitivity holds because  $D$  is idempotent. To prove antisymmetry, note that for every non-stable states  $p$  and  $q$ , if  $p \leq_D q$ ,  $q \leq_D p$  and  $p \neq q$  hold, then  $D(p, p) > 0$ . This is a contradiction since  $p$  is non-stable.

Since  $\leq_D$  is a partial order, we prove the lemma for  $y = e_p$  by induction over  $\leq_D$ . Formally, we strengthen the inductive hypothesis such that conditions (3.1) and (3.2) hold for every  $i \geq N_q$ , where  $N_q = |\{q' \in P \mid q' \leq_D q\}|$  (notice that  $N_q \leq |Q|$  for every  $q$ ). The base case is for  $N_p = 0$ , which means that  $p$  is stable. In the inductive case  $N_p > 0$  the state  $p$  is non-stable. Then

$$x^T \cdot D^{i+1} \cdot e_p = x^T \cdot D^i \cdot (c_1 \cdot e_{q_1} + \dots + c_k \cdot e_{q_k}) = c_1(x^T \cdot D^i \cdot e_{q_1}) + \dots + c_k(x^T \cdot D^i \cdot e_{q_k})$$

for pairwise different states  $q_1, \dots, q_k$  and positive values  $c_1, \dots, c_k \in \mathbb{N}$  such that  $q_j$  is either stable or  $q_j <_D p$ . Thus all states  $q_1, \dots, q_k$  satisfy our inductive hypothesis.

Consider the partition of  $q_1, \dots, q_k$  into sets  $C_ =$  and  $C_ < =$  such that  $C_ =$  and  $C_ < =$  satisfy condition (3.1) and (3.2), respectively. If  $C_ < = = \emptyset$ , then for every  $i \geq N_p$  we have:

$$\begin{aligned} x^T \cdot D^{i+1} \cdot e_p &= c_1(x^T \cdot D^i \cdot e_{q_1}) + \dots + c_k(x^T \cdot D^i \cdot e_{q_k}) \\ &= c_1(x^T \cdot D^{i-1} \cdot e_{q_1}) + \dots + c_k(x^T \cdot D^{i-1} \cdot e_{q_k}) \\ &= x^T \cdot D^i \cdot e_p. \end{aligned} \tag{3.3}$$

Note that  $x^T \cdot D^i \cdot e_{q_j} = x^T \cdot D^{i-1} \cdot e_{q_j}$  holds by the inductive hypothesis and because  $N_p > N_{q_j}$  for every  $q_j$ . Suppose otherwise, that  $C_ < = \neq \emptyset$  and there exists a state  $q_j$  that satisfies  $x^T \cdot D^i \cdot e_{q_j} < x^T \cdot D^{i+1} \cdot e_{q_j}$  for every  $i \geq N_{q_j}$ . Then it is straightforward that equality (3.3) becomes a strict inequality and condition (3.2) holds.

We have shown that either (3.1) or (3.2) holds for  $y = e_p$ . It remains to extend this to any vector  $y \in \mathbb{N}_{+, \times}^Q$  (possibly with  $\infty$ ). Note that

$$x^T \cdot D^{i+1} \cdot y = y(q_1) \cdot (x^T \cdot D^{i+1} \cdot e_{q_1}) + \dots + y(q_k) \cdot (x^T \cdot D^{i+1} \cdot e_{q_k})$$

for some states  $q_1, \dots, q_k$  such that  $y(q_j) > 0$  for every  $j \leq k$ . We consider two cases. First, if there exists  $j$  such that  $y(q_j) = \infty$  and  $x^T \cdot D^i \cdot e_{q_j} > 0$  for  $i \geq N$ , then  $x^T \cdot D^i \cdot y = \infty$  for every  $i \geq 0$ . Thus,  $x^T \cdot D^i \cdot y$  satisfies condition (3.1). Second, suppose that for every  $j$  we have  $y(q_j) \neq \infty$  or  $x^T \cdot D^i \cdot e_{q_j} = 0$  for  $i \geq N$ . It suffices to consider the case when  $y(q_j) \neq \infty$  for all  $j$ . Then if some  $x^T \cdot D^i \cdot e_{q_j}$  satisfies condition (3.2) we have that  $x^T \cdot D^i \cdot y$  satisfies condition (3.2). Conversely, if every  $x^T \cdot D^i \cdot e_{q_j}$  satisfies condition (3.1) we have that  $x^T \cdot D^i \cdot y$  satisfies condition (3.1).  $\square$

One could try to simplify Theorem 3.1 changing the condition  $i \geq N$  to  $i \geq 0$ . Unfortunately, we do not know if the theorem would remain true. A naive approach would be



to use a generalisation of Lemma 2.6, but intuitively, the behaviour of non-stable states is problematic. We conclude with the following remarks, straightforward from the proof. We will use them in Section 4.

**Remark 3.6.** Changing  $y$  to  $y'$  such that  $y \equiv_{\mathbb{B}_\infty} y'$  does not influence whether condition (3.1) or condition (3.2) holds in Lemma 3.5 (notice that here we need that the abstractions have values in  $\mathbb{B}_\infty$  not in  $\mathbb{B}$ ). Similarly, changing  $x$  to  $x'$  such that  $x \equiv_{\mathbb{B}_\infty} x'$  does not influence whether condition (3.1) or (3.2) holds.

**Remark 3.7.** The constant  $N$  and the refinement of  $w$  depend only on the finite monoid  $\mathbb{B}_\infty^{Q \times Q}$ . In particular they are independent of the initial vectors  $I$  and  $F$ .

#### 4. FINITE-MIN REGULAR FUNCTIONS

In this section we focus on regular functions over  $\mathbb{N}_{+, \times}$  with some min operations allowed. Formally, we say that  $f : \Sigma^* \rightarrow \mathbb{N} \cup \{\infty\}$  is a finite-min regular function, if there exist regular functions  $f_1, \dots, f_m$  over  $\mathbb{N}_{+, \times}$  such that  $f(w) = \min\{f_1(w), \dots, f_m(w)\}$ . It is known that over  $\mathbb{N}_{\min, +}$ , FA-WA are equivalent to a finite sum of U-WA [33], hence the functions defined by FA-WA are included in the class of finite-min regular functions. As a corollary of the pumping lemma in this section we show that PA-WA are strictly more expressive than FA-WA over  $\mathbb{N}_{\min, +}$  (Example 4.2 and Example 4.3).

We start by introducing some notation. Generalising the notation used in the previous section, we define for  $n > 0$  an  $n$ -pumping representation for a word  $w \in \Sigma^*$  as a factorisation of the form

$$w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \dots \cdot u_{n-1} \cdot \underline{v_n} \cdot u_n,$$

where  $w = u_0 \cdot v_1 \cdot u_1 \cdot v_2 \cdot \dots \cdot v_n \cdot u_n$  and  $v_k \neq \epsilon$  for all  $k$ . A refinement of an  $n$ -pumping representation for  $w$  is given by

$$w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \dots \cdot u'_{n-1} \cdot \underline{y_n} \cdot u'_n,$$

if  $v_k = x_k \cdot y_k \cdot z_k$ ,  $u'_k = z_k \cdot u_k \cdot x_{k+1}$ ; where  $z_0 = x_{n+1} = \epsilon$  and  $y_k \neq \epsilon$  for every  $k$ . Let  $S \subseteq \{1, \dots, n\}$  such that  $S \neq \emptyset$ . Let  $\underline{y_k}$  be a factor of the refined  $n$ -pumping representation of  $w$ . By  $\underline{y_k}(S, i)$  we denote the word  $y_k^i$  if  $k \in S$  and  $y_k$  otherwise. By  $w(S, i)$  we denote the word

$$w = u'_0 \cdot \underline{y_1}(S, i) \cdot u'_1 \cdot \underline{y_2}(S, i) \cdot \dots \cdot u'_{n-1} \cdot \underline{y_n}(S, i) \cdot u'_n.$$

In other words, in  $w(S, i)$  we pump  $i$  times each factor  $y_k$ , for all  $k \in S$ . Note that the pumping always refers to the *refinement* of the  $n$ -pumping representation.

**Theorem 4.1** (Pumping Lemma for finite-min regular functions). *Let  $f : \Sigma^* \rightarrow \mathbb{N} \cup \{\infty\}$  be a finite-min regular function. There exists  $N$  such that for all  $n$ -pumping representations*

$$w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \dots \cdot u_{n-1} \cdot \underline{v_n} \cdot u_n,$$

where  $|v_i| \geq N$  for all  $i$ , there exists a refinement

$$w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \dots \cdot u'_{n-1} \cdot \underline{y_n} \cdot u'_n,$$

such that for every sequence of non-empty, pairwise different subsets  $S_1, \dots, S_k \subseteq \{1, \dots, n\}$  with  $k \geq N$  one of the following holds:

- there exists  $j$  such that  $f(w(S_j, i)) < f(w(S_j, i+1))$  for all but finitely many  $i$ ;

- there exist  $j_1 \neq j_2$  such that  $f(w(S_{j_1} \cup S_{j_2}, i)) = f(w(S_{j_1} \cup S_{j_2}, i+1))$  for all but finitely many  $i$ .

Before proving Theorem 4.1, we show how to use it with two examples.

**Example 4.2.** We show that  $f_3$  from Example 2.3 is not definable by finite-min regular functions. Indeed, fix  $N$  from Theorem 4.1 and consider the  $n$ -pumping representation  $w = (\underline{b}^N \cdot \underline{a}^N)^N$ . We index each pumping factor with a pair  $(s, j)$ , where  $j \leq N$  denotes the block  $b^N a^N$ , and  $s \in \{1, 2\}$  denotes the factor in the block. First, notice that  $f_3(w) = N \cdot (N-1)$  because runs minimising the value for  $\mathcal{W}_3$  change the state after reading the last  $b$  in one of the blocks. We define the sets  $S_j = \{(1, j), (2, j)\}$  for  $j \in \{1, \dots, N\}$ . Clearly  $f_3(w(S_j, i)) = f_3(w)$  for all  $j$  and  $i$ , because the run minimising the value changes the state after the last  $b$  in the  $j$ -th block. On the other hand  $f_3(w(S_{j_1} \cup S_{j_2}, i)) < f_3(w(S_{j_1} \cup S_{j_2}, i+1))$  for all  $i$  and  $j_1 \neq j_2$ . Hence  $f_3$  does not satisfy the pumping lemma for finite-min regular functions.

**Example 4.3.** We show that  $f_4$  from Example 2.4 is not definable by finite-min regular functions. Indeed, fix  $N$  from Theorem 4.1. Consider the  $N$ -pumping representation  $w = (\underline{b}^N a)^N$ . Then by definition  $f_4(w) = N$ . In the refinement all pumping parts will be of the form  $b^n$  for  $1 \leq n \leq N$ . We define the sets  $S_j = \{1, \dots, N\} \setminus \{j\}$  for all  $1 \leq i \leq N$ . Clearly  $f_4(w(S_j, i)) = N$  for all  $j$  and  $i$ . On the other hand  $f_4(w(S_{j_1} \cup S_{j_2}, i)) < f_4(w(S_{j_1} \cup S_{j_2}, i+1))$  for all  $i$  and  $j_1 \neq j_2$ . Hence  $f_4$  does not satisfy the pumping lemma for finite-min regular functions.

*Proof of Theorem 4.1.* Let  $f_1, \dots, f_m$  be regular functions over  $\mathbb{N}_{+, \times}$  such that  $f(w) = \min\{f_1(w), \dots, f_m(w)\}$  for every  $w$ . Furthermore, consider  $\mathcal{A}_j = (Q_j, \Sigma, \{M_{j,a}\}_{a \in \Sigma}, I_j, F_j)$  the corresponding WA for  $f_j$ . Let  $Q = \bigcup_j Q_j$  (we assume that  $Q_1, \dots, Q_m$  are pairwise disjoint) and consider the set of matrices  $\{U_a\}_{a \in \Sigma}$  where  $U_a \in \mathbb{N}_{+, \times}^{Q \times Q}$  such that  $U_a(p, q) = M_{j,a}(p, q)$  whenever  $p, q \in Q_j$  and 0 otherwise. Then  $f_j(w) = (I'_j)^t \cdot U_w \cdot F'_j$  for every  $j$  and  $w \in \Sigma^*$  where  $I'_j$  and  $F'_j$  are the extensions of  $I_j$  and  $F_j$  from  $Q_j$  into  $Q$  such that  $I'_j(q) = I_j(q)$  and  $F'_j(q) = F_j(q)$  whenever  $q \in Q_j$  and 0 otherwise. Notice that  $\{U_a\}_{a \in \Sigma}$  synchronise the behaviour of  $f_1, \dots, f_m$  in a single set of matrices and project the output of  $f_j$  with  $I'_j$  and  $F'_j$ . Let  $N = \max\{K, m+1\}$  such that  $K$  is the constant from Lemma 2.6 applied to  $\mathbb{B}_{\infty}^{Q \times Q}$ .

Let  $w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \dots \cdot u_{n-1} \cdot \underline{v_n} \cdot u_n$  be an  $n$ -pumping representation as in the statement of the theorem. For every  $i$  we apply Theorem 3.1 to  $u_{\leq i} \cdot \underline{v_i} \cdot t_{\geq i}$ , where  $u_{\leq i} = u_0 \cdot v_1 \cdot \dots \cdot u_{i-1}$  and  $t_{\geq i} = u_i \cdot v_{i+1} \cdot \dots \cdot u_n$ , and  $\{U_a\}_{a \in \Sigma}$  (recall that the refinement of  $u_{\leq i} \cdot \underline{v_i} \cdot t_{\geq i}$  depends only on  $\{U_a\}_{a \in \Sigma}$ , and not on the initial or final vector, see Remark 3.7). So by Theorem 3.1 we obtain a refinement

$$w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \dots \cdot u'_{n-1} \cdot \underline{y_n} \cdot u'_n,$$

where each  $y_i$  is idempotent w.r.t.  $\{U_a\}_{a \in \Sigma}$ .

Note that the refinement is the same for each function  $f_j$ . Therefore, we obtain

$$f_j(w) = (I'_j)^t \cdot U_{u'_0} \cdot D_1 \cdot \dots \cdot U_{u'_{n-1}} \cdot D_n \cdot U_{u'_n} \cdot F'_j$$

where all  $D_i = U_{y_i}$  are idempotents.

**Lemma 4.4.** Let  $S \subseteq \{1, \dots, n\}$  be a non-empty set and fix one function  $f_j$ . Then  $f_j(w(S, i)) < f_j(w(S, i+1))$  for every  $i \geq N$  iff there exists  $k \in S$  such that  $f_j(w(\{k\}, i)) < f_j(w(\{k\}, i+1))$  for every  $i \geq N$ .

*Proof.* By definition  $f_j(w(S, i)) = (I'_j)^t \cdot U_{u'_0} \cdot D_1^{s_1} \cdot \dots \cdot U_{u'_{n-1}} \cdot D_n^{s_n} \cdot U_{u'_n} \cdot F'_j$  where  $s_k = i$  if  $k \in S$  and  $s_k = 1$  otherwise. Since all  $D_i$  are idempotents then for all  $k$ :

$$\begin{aligned} (I'_j)^t \cdot U_{u'_0} \cdot D_1^{s_1} \cdot \dots \cdot D_{k-1}^{s_{k-1}} \cdot U_{u'_{k-1}} &\equiv_{\mathbb{B}_\infty} (I'_j)^t \cdot U_{u'_0} \cdot D_1 \cdot \dots \cdot D_{k-1} \cdot U_{u'_{k-1}} \\ U_{u'_k} \cdot D_{k+1}^{s_{k+1}} \cdot \dots \cdot D_n^{s_n} \cdot U_{u'_n} \cdot F'_j &\equiv_{\mathbb{B}_\infty} U_{u'_k} \cdot D_{k+1} \cdot \dots \cdot D_n \cdot U_{u'_n} \cdot F'_j. \end{aligned}$$

Hence, the lemma follows from Remark 3.6.  $\square$

To finish the proof we analyse  $f(w(S, i)) = \min\{f_1(w(S, i)), \dots, f_m(w(S, i))\}$ . Consider a sequence of subsets  $S_1, \dots, S_k$  with  $k \geq N$ . Suppose there is a set  $S_l$  such that for every  $1 \leq j \leq m$ , we have  $f_j(w(S_l, i)) < f_j(w(S_l, i+1))$  for every  $i \geq N$ . In this case,  $f(w(S_l, i)) < f(w(S_l, i+1))$  holds for all  $i \geq N$ , so the first condition of the theorem is met.

Suppose otherwise that no such  $S_l$  exists. In particular, for every  $S_l$  there is at least one  $j$  such that  $f_j(w(\{s\}, i)) = f_j(w(\{s\}, i+1))$  for all  $i \geq N$  and all  $s \in S_l$ , hence  $f_j(w(S_l, i)) = f_j(w(S_l, i+1))$  for all  $i \geq N$ . For every  $S_l$  let  $X_l \subseteq \{1, \dots, m\}$  be the set of indices  $j$  such that  $f_j(w(S_l, i)) = f_j(w(S_l, i+1))$  for all  $i \geq N$ . By the above assumptions, every  $X_l$  is non-empty. Since  $k \geq N > m$  there exists  $l_1, l_2$  such that  $X_{l_1} \cap X_{l_2} \neq \emptyset$ . From Lemma 4.4 it follows that for  $i \geq N$  it holds:  $f_j(w(S_{l_1} \cup S_{l_2}, i)) = f_j(w(S_{l_1} \cup S_{l_2}, i+1))$  for all  $j \in X_{l_1} \cap X_{l_2}$ ; and  $f_j(w(S_{l_1} \cup S_{l_2}, i)) < f_j(w(S_{l_1} \cup S_{l_2}, i+1))$  for all  $j \in \{1, \dots, m\} \setminus (X_{l_1} \cap X_{l_2})$ . Hence for  $i$  sufficiently large  $f(w(S_{l_1} \cup S_{l_2}, i)) = \min_{j \in X_{l_1} \cap X_{l_2}} (f_j(w(S_{l_1} \cup S_{l_2}, i))) = f(w(S_{l_1} \cup S_{l_2}, i+1))$ , which concludes the proof.  $\square$

## 5. POLY-AMBIGUOUS REGULAR FUNCTIONS OVER THE MIN-PLUS SEMIRING

In this section we focus on polynomially-ambiguous regular functions over  $\mathbb{N}_{\min,+}$ . We expect that there is a wider class of functions, definable like in the previous section, where Theorem 5.1 holds, but this is left for future work. A consequence of this section is that WA are strictly more expressive than PA-WA (see Examples 5.2 and 5.3).

We will use in the following the notation of  $n$ -pumping representations from Section 4. A sequence of non-empty sets  $S_1, \dots, S_m$  over  $\{1, \dots, n\}$  is called a *partition* if the sets are pairwise disjoint and their union is  $\{1, \dots, n\}$ . Furthermore, we say that  $S \subseteq \{1, \dots, n\}$  is a *selection set* for  $S_1, \dots, S_m$  if  $|S \cap S_i| = 1$  for every  $i$ .

**Theorem 5.1** (Pumping Lemma for polynomially-ambiguous automata). *Let  $f : \Sigma^* \rightarrow \mathbb{N} \cup \{\infty\}$  be a polynomially-ambiguous regular function over  $\mathbb{N}_{\min,+}$ . There exist  $N$  and a function  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $n$ -pumping representation:*

$$w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \dots \cdot u_{n-1} \cdot \underline{v_n} \cdot u_n,$$

where  $|v_i| \geq N$  for every  $i \leq n$ , there exists a refinement:

$$w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \dots \cdot u'_{n-1} \cdot \underline{y_n} \cdot u'_n,$$

such that for every partition  $\pi = S_1, \dots, S_m$  of  $\{1, \dots, n\}$  with  $m \geq \varphi(\max_j(|S_j|))$ , one of the following holds:

- there exists  $j$  such that  $f(w(S_j, i)) = f(w(S_j, i+1))$  for all but finitely many  $i$ ;
- there exists a selection set  $S \subseteq \{1, \dots, n\}$  for  $\pi$  such that  $f(w(S, i)) < f(w(S, i+1))$  for all but finitely many  $i$ .

**Example 5.2.** We show that  $f_5$  from Example 2.5 is not definable by any PA-WA. Indeed, let  $N$  and  $\varphi$  be the constant and the function from Theorem 5.1. Consider the following  $2m$ -pumping representation:  $w = (\underline{a^N} \cdot \underline{b^N} \#)^m$  where  $m \geq \varphi(2)$  (here  $\max_i(|S_i|)$  will be equal to 2). We index the  $j$ -th block of  $a$ 's with  $j$  and the  $j$ -th block of  $b$ 's with  $j'$ . We define the subsets  $S_1, \dots, S_m$  as  $S_j = \{j, j'\}$ . Clearly, for all  $j$  we have  $f_5(w(S_j, i)) < f_5(w(S_j, i+1))$ . On the other hand for every selection set  $S$  we have  $f_5(w(S, i)) = f_5(w(S, i+1))$ . Hence  $f_5$  does not satisfy Theorem 5.1.

**Example 5.3.** The function  $f_5$  in Example 2.5 is essentially the function  $f_2$  from Example 2.2 applied to the subwords between the symbols  $\#$ , where the outputs are aggregated with  $+$ . In a similar way one can define a min-plus automaton recognising  $f_6(w) = \sum_i f_4(w_i)$  for any  $w \in \Sigma^*$  of the form  $w_0 \# w_1 \# \dots \# w_n$  with  $w_i \in \{a, b\}^*$ , where  $f_4$  is the function computing the minimal block of  $b$ 's from Example 2.4. We show that  $f_6$  is not definable by PA-WA over  $\mathbb{N}_{\min,+}$ . Consider the following  $2m$ -pumping representation:  $w = (\underline{b^N} \cdot \underline{a} \cdot \underline{b^N} \#)^m$  where  $m \geq \varphi(2)$  (here  $\max_i(|S_i|)$  is again 2). As in Example 5.2, we index the first  $j$ -th block of  $b$ 's with  $j$  and the second  $j$ -th block of  $b$ 's with  $j'$ , and we set  $S_j = \{j, j'\}$ , for  $1 \leq j \leq m$ . Clearly, for all  $j$  we have  $f_6(w(S_j, i)) < f_6(w(S_j, i+1))$ . On the other hand for every selection set  $S$  we have  $f_6(w(S, i)) = f_6(w(S, i+1))$ . Hence  $f_6$  does not satisfy Theorem 5.1 either.

Consider the set of matrices  $\mathbb{N}_{\min,+}^{Q \times Q}$  over the min-plus semiring. Recall that here  $\oplus = \min$ ,  $\odot = +$ ,  $\mathbb{0} = \infty$ ,  $\mathbb{1} = 0$ , and the product of matrices  $M, N \in \mathbb{N}_{\min,+}^{Q \times Q}$  is defined by  $M \cdot N(p, q) = \min_r (M(p, r) + N(r, q))$ . Also, recall that for any  $M \in \mathbb{N}_{\min,+}^{Q \times Q}$  we denote by  $\bar{M}$  the homomorphic image of  $M$  into the finite monoid  $\mathbb{B}^{Q \times Q}$  (see Section 2.2). Similar as in Section 3 and Section 4, we say that  $D \in \mathbb{N}_{\min,+}^{Q \times Q}$  is an idempotent if  $\bar{D}$  is an idempotent in the finite monoid  $\mathbb{B}^{Q \times Q}$ .

The following lemma states a special property of polynomially-ambiguous automata that we exploit in the proof of Theorem 5.1.

**Lemma 5.4.** *Let  $\mathcal{A} = (Q, \Sigma, \{M_a\}_{a \in \Sigma}, I, F)$  be a polynomially-ambiguous weighted automaton over the min-plus semiring. For every idempotent  $D \in \{M_w \mid w \in \Sigma^*\}$  and for every  $p, q \in Q$ , there exist constants  $c, d \in \mathbb{N}_{\min,+}$  and  $b \in \mathbb{N}$  such that  $D^{b+i}(p, q) = c \cdot i + d$  for all  $i \geq 0$ .*

*Proof.* We can view  $\bar{D} \in \mathbb{B}^{Q \times Q}$  as the adjacency matrix of a graph. Now we show that the cycles of the directed graph defined by  $\bar{D}$  can be only self-loops. Indeed, assume by contradiction that there exists a cycle passing through  $r, s \in Q$  with  $r \neq s$  then  $\bar{D}(r, s) = \bar{D}(s, r) = \bar{D}(r, r) = \bar{D}(s, s) = 1$  (because  $\bar{D}$  is an idempotent). Since  $D \in \{M_w \mid w \in \Sigma^*\}$ , it can be checked using a simple pumping argument as in [34] that  $\mathcal{A}$  cannot be polynomially-ambiguous. Therefore,  $\bar{D}$  forms an acyclic graph with some self-loops and the states in  $Q$  can be ordered as  $p_1, \dots, p_n$ , such that  $\bar{D}(p_j, p_i) = +\infty$  for every  $i < j$ .

If  $D(p, q) = +\infty$  then it suffices to take  $c = d = +\infty$  since  $D$  is an idempotent. Otherwise,  $D^i(p, q) = \min(\sum_{1 \leq k \leq i} D(p_{j_{k-1}}, p_{j_k}))$ , where the minimum is taken over all sequences  $(j_k)_k$  such that  $p_{j_0} = p$ ,  $p_{j_i} = q$  and  $1 \leq j_{k-1} \leq j_k \leq n$  for all  $k$ . Notice that there are at most  $n$  distinct elements  $D(p_{j_{k-1}}, p_{j_k})$  with  $j_{k-1} < j_k$ . If a sequence  $(j_k)_k$  sums up to the minimal value  $D^i(p, q)$  then we can assume without restriction that the remaining elements of this sequence correspond to a single self-loop  $D(p_j, p_j)$ , for some  $j = j_k$  such that  $D(p_j, p_j) \leq D(p_{j_l}, p_{j_l})$  for every  $0 \leq l \leq i$ . A simple computation shows that there exists some  $i_0$  such that

for all  $i \geq i_0$  the unique self-loop in the sequence has the minimal value  $c = \min_j D(p_j, p_j)$  among all self-loops.

Consider now all sequences  $(j_k)_k$  such that  $1 \leq j_{k-1} < j_k \leq n$ , that contain an index  $j$  with  $c = D(p_j, p_j)$ . Fix such a sequence  $(j_k)_k$  such that  $\sum_k D(p_{j_{k-1}}, p_{j_k})$  is minimal and let  $m$  be its length. The lemma follows by taking  $b = m + i_0$  and  $d = \sum_k D(p_{j_{k-1}}, p_{j_k}) + i_0 \cdot c$ .  $\square$

*Proof of Theorem 5.1.* Consider a polynomially-ambiguous WA  $\mathcal{A} = (Q, \Sigma, \{M_a\}_{a \in \Sigma}, I, F)$  over  $\mathbb{N}_{\min,+}$  such that  $f = \llbracket \mathcal{A} \rrbracket$ . We take for  $N$  the constant from Lemma 2.6 for the finite monoid  $\mathbb{B}^{Q \times Q}$ . The function  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  will be determined later in the proof.

Consider an  $n$ -pumping representation  $w$  like in the statement of the lemma. Recall that the output  $|\rho|$  of a run  $\rho$  over the word  $w$  is defined as  $I \cdot M_w \cdot F$ . By Lemma 2.6, for every  $v_k$  there exists a factorisation  $v_k = x_k y_k z_k$  such that  $M_{y_k}$  is an idempotent and  $|y_k| \leq N$ . We denote  $D_k = M_{y_k}$  and define:

$$w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \dots \cdot u'_{n-1} \cdot \underline{y_n} \cdot u'_n$$

such that each word  $y_k$  is the factor of  $v_k$  corresponding to the idempotent  $D_k$ . In the remaining of the proof we denote  $w_{\leq k} = u'_0 \cdot y_1 \cdot \dots \cdot u'_{k-1}$ . For every  $S \subseteq \{1 \dots n\}$  we denote by  $w_{\leq k}(S, i)$  the word  $w_{\leq k}$  with all  $y_j$  pumped  $i$  times for all  $j < k$  such that  $j \in S$ .

Recall that  $\text{Run}_{\mathcal{A}}(w)$  is the set of all accepting runs on  $w$ , and let  $\rho \in \text{Run}_{\mathcal{A}}(w)$ . Every run induces two states for each  $1 \leq k \leq n$ : states preceding and following each word  $y_k$ . In the rest of the proof these will be the most important parts of a run. To work with them, we define the abstraction of  $\rho$ , denoted by  $\bar{\rho} : \{1, \dots, n\} \rightarrow Q \times Q$ , such that  $\bar{\rho}(k) = (p, q)$  where  $p$  and  $q$  are the states of  $\rho$  reached after  $w_{\leq k}$  and  $w_{\leq k} \cdot y_k$ , respectively. Similarly, for  $S \subseteq \{1, \dots, n\}$ ,  $i \geq 1$ , and  $\rho \in \text{Run}_{\mathcal{A}}(w(S, i))$  we define  $\bar{\rho} : \{1, \dots, n\} \rightarrow Q \times Q$  such that  $\bar{\rho}(k) = (p, q)$  where  $p$  and  $q$  are the states of  $\rho$  reached after  $w_{\leq k}(S, i)$  and  $w_{\leq k}(S, i) \cdot y_k(S, i)$ , respectively. We denote by  $\overline{\text{Run}_{\mathcal{A}}}(w)$  the set of all  $\bar{\rho}$  with  $\rho \in \text{Run}_{\mathcal{A}}(w)$ , and same for  $\overline{\text{Run}_{\mathcal{A}}}(w(S, i))$ . Observe that since all  $D_k$  are idempotents,  $\overline{\text{Run}_{\mathcal{A}}}(w(S, i)) = \overline{\text{Run}_{\mathcal{A}}}(w)$  for all subsets  $S$  and  $i \geq 1$ .

The next step is to prove that the cardinality of  $\overline{\text{Run}_{\mathcal{A}}}(w)$  is bounded by a polynomial  $P(\cdot)$  depending only on  $\mathcal{A}$ , namely such that  $|\overline{\text{Run}_{\mathcal{A}}}(w)| \leq P(n)$ . Let  $w'$  be the word obtained from  $w$  where each  $u'_i$  is replaced with a word  $u''_i$  of length at most  $|\mathbb{B}^{Q \times Q}|$  such that  $\overline{M_{u'_i}} = \overline{M_{u''_i}}$  (it is straightforward to prove that  $u''_i$  exists by the pigeonhole principle). Then  $|\text{Run}_{\mathcal{A}}(w')| \geq |\text{Run}_{\mathcal{A}}(w)|$ . Recall that  $|y_i| \leq N$  and that  $N$  depends only on  $|\mathbb{B}^{Q \times Q}|$ . Then by definition  $|w'| \leq (N + |\mathbb{B}^{Q \times Q}|) \cdot (n + 1)$  and thus  $|\text{Run}_{\mathcal{A}}(w')| \leq R((N + |\mathbb{B}^{Q \times Q}|) \cdot (n + 1))$ , where  $R$  is the polynomial bounding the number of runs in  $\mathcal{A}$ . The claim follows with  $P(n) = R((N + |\mathbb{B}^{Q \times Q}|) \cdot (n + 1))$ .

Fix a non-empty set  $S \subseteq \{1, \dots, n\}$  and a run  $\rho \in \text{Run}_{\mathcal{A}}(w)$ . For every  $k \in S$  let  $b_{\bar{\rho}(k)}^k$ ,  $c_{\bar{\rho}(k)}^k$  and  $d_{\bar{\rho}(k)}^k$  be the constants from Lemma 5.4 such that  $D_k^{b_{\bar{\rho}(k)}^k + i}[\bar{\rho}(k)] = c_{\bar{\rho}(k)}^k \cdot i + d_{\bar{\rho}(k)}^k$  for  $i \geq 0$ . Since  $\rho$  is accepting,  $c_{\bar{\rho}(k)}^k, d_{\bar{\rho}(k)}^k < +\infty$ . We show that:

- (1)  $\llbracket \mathcal{A} \rrbracket(w(S, i)) = \llbracket \mathcal{A} \rrbracket(w(S, i + 1))$  for all sufficiently large  $i$  iff there exists a run  $\rho \in \text{Run}_{\mathcal{A}}(w)$  such that  $c_{\bar{\rho}(k)}^k = 0$  for every  $k \in S$ ;
- (2)  $\llbracket \mathcal{A} \rrbracket(w(S, i)) < \llbracket \mathcal{A} \rrbracket(w(S, i + 1))$  for all sufficiently large  $i$  iff for every run  $\rho \in \text{Run}_{\mathcal{A}}(w)$  there exists  $k$  such that  $c_{\bar{\rho}(k)}^k > 0$ .

Since  $|y_k| \leq N$ , the set of idempotents in Lemma 5.4 is finite. So we may assume a common bound  $i_0$  such that  $b_{\bar{\rho}(k)}^k = i_0$  for all  $k$  in Lemma 5.4. Let  $\rho \in \text{Run}_{\mathcal{A}}(w(S, i + 1))$  be

a run realising the minimum value for  $i \geq i_0$ . Given that  $D_k$  is idempotent one can obtain a run  $\rho' \in \text{Run}_{\mathcal{A}}(w(S, i))$  such that  $\bar{\rho}' = \bar{\rho}$  by removing one copy of each  $y_k$ . In particular  $|\rho'| \leq |\rho|$ , which proves  $\llbracket \mathcal{A} \rrbracket(w(S, i)) \leq \llbracket \mathcal{A} \rrbracket(w(S, i+1))$ . It follows that it suffices to show (1) above.

To prove (1) suppose first that  $\llbracket \mathcal{A} \rrbracket(w(S, i)) = \llbracket \mathcal{A} \rrbracket(w(S, i+1))$  for all sufficiently large  $i$ . Let  $\rho \in \mathcal{A}(w(S, i+1))$  and  $\rho' \in \mathcal{A}(w(S, i))$  be the previous runs realising the minimum on  $w(S, i+1)$  and its shortening, respectively. By Lemma 5.4  $D_k^{i_0+i+1}[\bar{\rho}(k)] = c_{\bar{\rho}(k)}^k \cdot (i+1) + d_{\bar{\rho}(k)}^k$ . If  $c_{\bar{\rho}(k)}^k > 0$  for some  $k$  then the inequality  $\llbracket \mathcal{A} \rrbracket(w(S, i_0 + i)) \leq \llbracket \mathcal{A} \rrbracket(w(S, i_0 + i+1))$  would be sharp, which is a contradiction. For the other direction suppose there exists a run  $\rho \in \text{Run}_{\mathcal{A}}(w)$  such that  $c_{\bar{\rho}(k)}^k = 0$  for every  $k \in S$ . Then for every  $i \geq 0$  there exists a run  $\rho_i \in \text{Run}_{\mathcal{A}}(w(S, i_0 + i))$  such that  $|\rho_i| \leq |\rho| + \sum_k d_{\bar{\rho}(k)}^k$ . Since  $\llbracket \mathcal{A} \rrbracket(w(S, i_0 + i)) \leq \llbracket \mathcal{A} \rrbracket(w(S, i_0 + i+1)) \leq |\rho| + \sum_k d_{\bar{\rho}(k)}^k$  it follows that  $\llbracket \mathcal{A} \rrbracket(w(S, i_0 + i)) = \llbracket \mathcal{A} \rrbracket(w(S, i_0 + i+1))$  for all sufficiently large  $i$ .

Given the previous discussion, let  $\bar{R}_k = \{\bar{\rho} \in \overline{\text{Run}}_{\mathcal{A}}(w) \mid c_{\bar{\rho}(k)}^k > 0\}$  for every  $k \in \{1, \dots, n\}$ . The set  $\bar{R}_k$  represents the abstractions of the runs over  $w$  that will grow when pumping  $w(\{k\}, i)$ . Then, we can restate (2) as:  $\llbracket \mathcal{A} \rrbracket(w(S, i)) < \llbracket \mathcal{A} \rrbracket(w(S, i+1))$  for all sufficiently large  $i$  iff  $\bigcup_{k \in S} \bar{R}_k = \overline{\text{Run}}_{\mathcal{A}}(w)$ .

We are now ready to prove the theorem. Fix a partition  $S_1, \dots, S_m$  of  $\{1, \dots, n\}$  for some  $m \geq \varphi(\max |S_l|)$  ( $\varphi$  will be defined below). Suppose the first condition is not true, namely, for all  $1 \leq j \leq m$  there exists arbitrarily big values  $i$  such that  $f(w(S_j, i)) \neq f(w(S_j, i+1))$ . From (2) it follows that  $f(w(S_j, i)) < f(w(S_j, i+1))$  for all sufficiently large  $i$  and  $\bigcup_{k \in S_j} \bar{R}_k = \overline{\text{Run}}_{\mathcal{A}}(w)$  for every  $j \leq m$ . Let  $L = \max |S_l|$ . We assume that  $L > 1$ , otherwise every selection  $S$  contains a whole set  $S_k$  for some  $k$  and we are done.

To construct the selection set  $S = \{k_1, \dots, k_m\}$  we define by induction the sets  $G_j$ . For every  $j \in \{1, \dots, m\}$  let  $G_j = \overline{\text{Run}}_{\mathcal{A}}(w) \setminus \bigcup_{l \leq j} \bar{R}_{k_l}$  (where  $k_0$  is undefined, so  $G_0 = \overline{\text{Run}}_{\mathcal{A}}(w)$ ). Intuitively,  $G_j$  correspond to runs that are not covered by the set  $\{k_1, \dots, k_j\}$ . For the inductive case, suppose that  $j \geq 0$  and  $G_j \neq \emptyset$ . Since  $\bigcup_{k \in S_{j+1}} \bar{R}_k = \overline{\text{Run}}_{\mathcal{A}}(w)$ , by the pigeonhole principle there exist  $k_{j+1} \in S_{j+1}$  such that  $|\bar{R}_{k_{j+1}} \cap G_j| \geq |G_j|/|S_{j+1}|$ . We add  $k_{j+1}$  to  $S$  and so  $|G_{j+1}| \leq |G_j| - |G_j|/|S_{j+1}| = |G_j| \cdot (|S_{j+1}| - 1)/|S_{j+1}| \leq |G_j| \cdot (L - 1)/L$ . Suppose this procedure continues until  $j = m$  and  $G_m \neq \emptyset$ . Then  $1 \leq |\overline{\text{Run}}_{\mathcal{A}}(w)| \cdot ((L - 1)/L)^m$ , and  $|\overline{\text{Run}}_{\mathcal{A}}(w)| \geq (L/(L - 1))^m$ . However, we know that  $|\overline{\text{Run}}_{\mathcal{A}}(w)|$  is bounded by a polynomial function  $P(n)$  depending on  $|\mathcal{A}|$ . Thus, it suffices to choose  $\varphi$  such that  $m \geq \varphi(L)$  implies  $(L/(L - 1))^m > P(L \cdot m) \geq P(n) \geq |\overline{\text{Run}}_{\mathcal{A}}(w)|$  (recall that  $S_1, \dots, S_m$  is a partition of  $\{1, \dots, n\}$  and  $L \cdot m \geq n$ ). Therefore,  $G_m = \emptyset$  and thus  $\bigcup_{k \in S} \bar{R}_k = \overline{\text{Run}}_{\mathcal{A}}(w)$ , which concludes the proof.  $\square$

## 6. PUMPING LEMMAS FOR THE MAX-PLUS SEMIRING

In this section, we consider finitely ambiguous and polynomially ambiguous weighted automata over the  $\mathbb{N}_{\max,+}$  semiring. Notice that U-WA over  $\mathbb{N}_{\max,+}$  is the same class of functions as U-WA over  $\mathbb{N}_{\min,+}$  and thus Theorem 3.1 also holds for this class. For this reason, here we focus on the ambiguous cases, dividing the section into two parts to deal separately with the finitely ambiguous and polynomially ambiguous cases.

### 6.1. Pumping Lemma for Finitely Ambiguous Weighted Automata over $\mathbb{N}_{\max,+}$ .

We use the definitions of *refinements* and *n-pumping representations* from Section 3 and Section 4. In order to formulate the pumping lemma for finitely-ambiguous functions over the  $\mathbb{N}_{\max,+}$  semiring, we define a few more notations.

Fix a function  $f : \Sigma^* \rightarrow \mathbb{N}$  and a word  $w \in \Sigma^*$ . Suppose that we have an  $n$ -pumping representation for  $w$ , and a refinement thereof. Let  $\{1, \dots, n\}$  be the set of all indices in the refinement. We say that a refinement is *linear* if for every subset  $S \subseteq \{1, \dots, n\}$ , there exists  $K$  such that  $f(w(S, i+1)) = K + f(w(S, i))$  for all sufficiently large  $i$ . For linear refinements we let  $\Delta(S)$  denote the above value  $K$  (note that  $\Delta$  depends on  $f$  and  $w$ , which are fixed). Furthermore, we say that  $S \subseteq \{1, \dots, n\}$  is *decomposable* if

$$\Delta(S) = \sum_{j \in S} \Delta(\{j\}).$$

**Theorem 6.1.** *Let  $f : \Sigma^* \rightarrow \mathbb{N}$  be a finitely ambiguous function over the semiring  $\mathbb{N}_{\max,+}$ . There exists  $N \in \mathbb{N}$  such that for every  $n$ -pumping representation*

$$w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \dots \cdot \underline{v_n} \cdot u_n,$$

*where  $n \geq N$  and  $|v_i| \geq N$  for all  $i$ , there exists a linear refinement*

$$w = x_0 \cdot \underline{y_1} \cdot x_1 \cdot \underline{y_2} \cdot \dots \cdot \underline{y_n} \cdot x_n$$

*such that for every sequence of pairwise different, non-empty sets  $S_1, S_2, \dots, S_k \subseteq \{1, \dots, n\}$  with  $k \geq N$ , one of the following holds:*

- *there exists  $j$  such that  $S_j$  is not decomposable;*
- *there exist  $j_1$  and  $j_2$  such that  $\{l_1, l_2\}$  is decomposable for every  $l_1 \in S_{j_1}$  and  $l_2 \in S_{j_2}$ .*

Before proving the theorem we show how to use the pumping lemma on two examples.

**Example 6.2.** Consider the function  $g_3$  which computes

$$\max_{0 \leq i \leq n} |a_1 \dots a_i|_a + |a_{i+1} \dots a_n|_b,$$

for any  $w = a_1 \dots a_n \in \{a, b\}^*$ . This is defined by  $\mathcal{W}_3$  in Figure 1 if we change the semiring of the automaton from  $\mathbb{N}_{\min,+}$  to  $\mathbb{N}_{\max,+}$ . We show that this function cannot be expressed by any finitely ambiguous WA over  $\mathbb{N}_{\max,+}$ . Towards a contradiction fix  $N$  from Theorem 6.1 and consider the  $(2N+2)$ -pumping representation  $(\underline{a^{N+1}b^{N+1}})^{N+1}$ . In the refinement, we index the  $j$ -th block of  $a$ 's with  $j$  and the  $j$ -th block of  $b$ 's with  $j'$ . Let  $x_1, \dots, x_{N+1}$  and  $y_1, \dots, y_{N+1}$  be the lengths of all blocks of  $a$ 's and  $b$ 's in the refinement. We define the sets  $S_j = \{j, j'\}$  for all  $1 \leq j \leq N+1$ . We show that none of the conditions of the pumping lemma hold. First, it is easy to see that all sets  $S_j$  are decomposable. Indeed  $\Delta(S_j) = x_j + y_j = \Delta(\{j\}) + \Delta(\{j'\})$ . For the second condition we can assume that  $j_2 > j_1$ . Since the function counts  $a$ 's before  $b$ 's, the set  $\{j'_1, j_2\}$  is not decomposable because  $\Delta(\{j'_1, j_2\}) = \max(y_{j_1}, x_{j_2})$ . Thus, any  $S_{j_1}$  and  $S_{j_2}$  will not satisfy the second condition either.

**Example 6.3.** Consider the function  $g_4$  which computes the length of longest block of  $b$ 's. This is defined by  $\mathcal{W}_4$  in Figure 1 if we change the semiring of the automaton from  $\mathbb{N}_{\min,+}$  to  $\mathbb{N}_{\max,+}$ . We shall show that  $g_4$  cannot be expressed by a FA-WA over  $\mathbb{N}_{\max,+}$ . Towards a contradiction let  $N$  be the constant from Theorem 6.1. Consider the  $(N+1)$ -pumping representation  $(\underline{b^{N+1}a})^{N+1}$ . We define the sets  $S_j = \{j\}$  for all  $1 \leq j \leq N+1$ . First, each set is also decomposable for trivial reasons. Second, every index is not decomposable with any other since the function takes into account the value of at most one block of  $b$ 's.

*Proof of Theorem 6.1.* Let  $\mathcal{A}$  be the finitely ambiguous WA that computes  $f$ . Suppose that  $\mathcal{A}$  has ambiguity at most  $m$  and has  $r$  states. We set the value of  $N$  in the statement of the lemma to be  $\max(r^m, m) + 1$ . Now, consider a word  $w$  and an  $n$ -pumping representation of  $w$  according to the lemma. Since each  $v_i$  in the representation has length more than  $r^m$  we can refine the  $v_i$ 's such that in all the  $\leq m$  accepting runs over  $w$ , all subruns corresponding to the subwords  $y_i$  are cycles in  $\mathcal{A}$ , for all  $1 \leq i \leq n$ . It is readily verified that such a refinement is linear, and that  $\Delta(S)$  is determined by the cycles with maximal weight in the blocks belonging to  $S$ .

Consider the language of words which can be obtained by pumping the indices of the above refinement. Without loss of generality, assume that  $w$  has exactly  $m$  runs and denote these runs by  $\rho_1, \dots, \rho_m$ . Notice that if pumping a cycle in a run would increase the number of runs then we would immediately get a contradiction with the assumption of the automaton being finitely ambiguous. Therefore, for all words  $w(S_j, i)$  the number of runs is  $m$  and these runs are obtained from  $\rho_1, \dots, \rho_m$ . For every  $j \in \{1, \dots, n\}$  we denote by  $\rho_l[j]$  the part of the run corresponding to  $y_j$  in the  $l$ -th run. Note that by construction, each  $\rho_l[j]$  is a cycle. We define the weight of  $\rho_l[j]$  as the sum of all weights on transitions in  $\rho_l[j]$  and denote it by  $\text{wt}(\rho_l[j])$ . We say that a cycle  $\rho_l[j]$  is dominant if  $\text{wt}(\rho_{l'}[j]) \leq \text{wt}(\rho_l[j])$  for all  $l'$ .

The rest of the proof involves reasoning about the dominant cycles. First we make a simple observation. Suppose  $\rho_l[j]$  is dominant, then  $\text{wt}(\rho_l[j]) = \Delta(\{j\})$ . We make one more observation.

**Claim 6.4.** Assume that we have a linear refinement and  $S \subseteq \{1, \dots, n\}$  is a subset of indices of the refinement. Then  $S$  is decomposable if and only if there exists some run  $\rho$  such that for all  $j \in S$ ,  $\rho[j]$  is dominant.

*Proof.* Assume first that  $S$  is decomposable, and consider some run  $\rho$  such that  $\sum_{j \in S} \text{wt}(\rho[j])$  is maximal among all runs on  $w$ . We claim that  $\rho[j]$  is dominant for all  $j \in S$ . Assume this is not the case. Notice that the value computed by the automaton increases by  $\sum_{j \in S} \text{wt}(\rho[j])$  when  $S$  is pumped in  $\rho$ . By the choice of  $\rho$  and the fact that the refinement is linear, we have that  $\Delta(S) = \sum_{j \in S} \text{wt}(\rho[j])$ . By assumption we know that there is some  $j^* \in S$  such that  $\rho$  is not dominant for  $j^*$ , so  $\text{wt}(\rho[j^*]) < \Delta(\{j^*\})$ . But this means that  $\Delta(S) = \sum_{j \in S} \text{wt}(\rho_l[j]) < \sum_{j \in S} \Delta(\{j\})$ , which is a contradiction to  $S$  being linear.

For the reverse implication, consider a run  $\rho$  such that  $\rho[j]$  is dominant for all  $j \in S$ . In particular,  $\sum_{j \in S} \text{wt}(\rho[j]) \geq \sum_{j \in S} \text{wt}(\rho'[j])$  for any other run  $\rho'$ . This means that when the set  $S$  is pumped, the value computed by the automaton increases by  $\sum_{j \in S} \text{wt}(\rho_l[j])$ , which also happens to be  $\sum_{j \in S} \Delta(\{j\})$  since the cycles in consideration are dominant. ■

To conclude we show that if the first condition of the pumping lemma does not hold then the second condition must hold. Indeed, suppose that all sets are decomposable. Then by Claim 6.4 for every set  $S_j$  there is some run  $\rho_{l_j}$  in which all the cycles corresponding to  $S_j$  are dominant. But since there are more sets than runs, there must be some  $j_1 \neq j_2$  such that  $l_{j_1} = l_{j_2}$ , namely, two sets which have the same corresponding runs. However, by Claim 6.4 this means that  $\{k_1, k_2\}$  is decomposable for every  $k_1 \in S_{j_1}$  and  $k_2 \in S_{j_2}$ . □

**6.2. Pumping Lemma for Polynomially Ambiguous Weighted Automata over  $\mathbb{N}_{\max, +}$ .** In this section, we will re-use the definition of linear refinement and decomposability from the previous section. We will also re-use the definition of selection set from Section 5.



**Theorem 6.5.** *Let  $f : \Sigma^* \rightarrow \mathbb{N}$  be a polynomially-ambiguous regular function over  $\mathbb{N}_{\max,+}$ . There exist  $N$  and a function  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $n$ -pumping representations*

$$w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \dots \cdot u_{n-1} \cdot \underline{v_n} \cdot u_n,$$

where  $|v_i| \geq N$  for every  $1 \leq i \leq n$ , there exists a linear refinement

$$w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \dots \cdot u'_{n-1} \cdot \underline{y_n} \cdot u'_n,$$

such that for every partition  $\pi = S_1, S_2, \dots, S_m$  of  $\{1, \dots, n\}$  with  $m \geq \varphi(\max_j(|S_j|))$  one of the following holds:

- there exists  $j$  such that  $S_j$  is decomposable;
- there exists a selection set  $S$  for  $\pi$  such that  $S$  is not decomposable.

Before proving this lemma, we show how to use it on examples.

**Example 6.6.** Consider the function  $g_5$  such that, for any  $w$  of the form  $w_0 \# w_1 \# \dots \# w_n$  with  $w_i \in \{a, b\}^*$  it computes  $g_5(w) = \sum_{i=0}^n \max\{|w_i|_a, |w_i|_b\}$ . This is defined by  $\mathcal{W}_5$  in Figure 1 if we change the semiring of the automaton to  $\mathbb{N}_{\max,+}$ . We show that  $g_5$  cannot be expressed by a PA-WA. Assume the contrary and let  $N$  and  $\varphi$  be the constant and the function from Theorem 6.5. Let  $m$  be a number larger than  $\varphi(2)$ . We consider the refinements of the pumping representation  $(\underline{a^N b^N \#})^m$ . In the refinement, we refer to the  $j$ -th block of  $a$ 's as  $j$  and to the  $j'$ -th block of  $b$ 's as  $j'$ . We define the sets in the partition as  $S_j = \{j, j'\}$  for all  $1 \leq j \leq m$ . It is clear from the definition of the function  $g_5$  that no set  $S_j$  is decomposable, since only one of the blocks  $j$  and  $j'$  is relevant for the outer sum. Now, consider any selection set  $S$ . Since any two elements of  $S$  belong to different blocks of the word separated by  $\#$ 's,  $S$  is a decomposable set of indices. Therefore, the function  $g_5$  is not polynomially ambiguous over  $\mathbb{N}_{\max,+}$ .

**Example 6.7.** Consider the function  $g_6$  that given a word of the form  $w_0 \# w_1 \# \dots \# w_n$  with  $w_i \in \{a, b\}^*$  computes  $g_6(w) = \sum_{i=0}^n g_3(w_i)$ , where  $g_3$  from Example 6.2. We show now that  $g_6$  cannot be expressed as a PA-WA over  $\mathbb{N}_{\max,+}$ . Assume the contrary and let  $N$  and  $\varphi$  be the constant and the function from the lemma above. Let  $m$  be a number larger than  $\varphi(2)$ . We consider the refinements of the pumping representation  $(\underline{b^N a^N \#})^m$ . Like in Example 6.6 in the refinement we refer to the  $j$ -th block of  $b$ 's as  $j$  and to the  $j$ -th block of  $a$ 's as  $j'$ . Let  $x_j$  and  $y_j$  be the lengths of the block of  $b$ 's and the block of  $a$ 's, respectively. We define the sets in the partition as  $S_j = \{j, j'\}$  for all  $1 \leq j \leq m$ . Every set  $S_j$  is not decomposable since  $\Delta(S_j) = \max(x_j, y_j)$ . Consider any selection set  $S$  of  $\pi$ . It is easy to see that  $S$  is decomposable given that all elements belong to different blocks. We conclude that  $g_6$  cannot be defined by PA-WA over  $\mathbb{N}_{\max,+}$ .

*Proof of Theorem 6.5.* The first part of the proof is the same as in the proof of Theorem 5.1 which only depends on the (polynomial) ambiguity of the automaton, and not on the semiring. We will use the same notation for the refinement  $(y_k)_k$  and in particular, all  $D_k = M_{y_k}$  are idempotents. We will also use the notation  $\overline{\text{Run}}_{\mathcal{A}}(w)$  of abstractions of runs and the notation  $\bar{\rho} : \{1, \dots, n\} \rightarrow Q \times Q$ . Finally recall from the proof of Theorem 5.1 that  $|\overline{\text{Run}}_{\mathcal{A}}(w)| \leq P(n)$  for some polynomial  $P(\cdot)$ .

We will also reuse Lemma 5.4, but over the  $\mathbb{N}_{\max,+}$  semiring. One can easily check that this lemma continues to hold over the max-plus semiring. The difference is that then  $c, d \in \mathbb{N}_{\max,+}$ . Therefore, for every  $k \in S$  let  $b_{\bar{\rho}(k)}^k$ ,  $c_{\bar{\rho}(k)}^k$  and  $d_{\bar{\rho}(k)}^k$  be the constants from

Lemma 5.4 such that:

$$D_k^{b_{\bar{\rho}(k)}^k + i}[\bar{\rho}(k)] = c_{\bar{\rho}(k)}^k \cdot i + d_{\bar{\rho}(k)}^k$$

for  $i \geq 0$ . Since  $\rho$  is accepting, we have  $c_{\bar{\rho}(k)}^k, d_{\bar{\rho}(k)}^k \neq -\infty$ .

First, we argue that the refinement defined by  $(y_k)_k$  is linear. Fix a non-empty set  $S \subseteq \{1, \dots, n\}$ . For  $\bar{\rho} \in \overline{\text{Run}}_{\mathcal{A}}(w)$  let  $c_{\bar{\rho}} = \sum_{k \in S} c_{\bar{\rho}(k)}^k$ . Recall that every run in  $\text{Run}_{\mathcal{A}}(w(S, i))$  has some abstraction in  $\overline{\text{Run}}_{\mathcal{A}}(w)$  and  $\mathcal{A}$  outputs the maximal value among all runs. It follows by considering  $i$  big enough that  $\Delta(S) = \max\{c_{\bar{\rho}} \mid \bar{\rho} \in \overline{\text{Run}}_{\mathcal{A}}(w)\}$ .

Let  $k \in \{1, \dots, n\}$ . We say that  $\bar{\rho} \in \overline{\text{Run}}_{\mathcal{A}}(w)$  is  $k$ -dominant if  $c_{\bar{\rho}(k)}^k \geq c_{\bar{\sigma}(k)}^k$  for every  $\bar{\sigma} \in \overline{\text{Run}}_{\mathcal{A}}(w)$ . Given  $k \in \{1, \dots, n\}$  we define  $\bar{R}_k = \{\bar{\rho} \in \overline{\text{Run}}_{\mathcal{A}}(w) \mid \bar{\rho} \text{ is not } k\text{-dominant}\}$ .

**Claim 6.8.** Assume that we have a linear refinement and  $S \subseteq \{1, \dots, n\}$  is a subset of indices of the refinement. Then  $S$  is decomposable if and only if there exists  $\bar{\rho}$  such that for all  $j \in S$ ,  $\bar{\rho}[j]$  is  $j$ -dominant.

*Proof.* Follows the same steps as the proof of Claim 6.4. ■

By Claim 6.8,  $S$  is not decomposable if and only if  $\bigcup_{k \in S} \bar{R}_k = \overline{\text{Run}}_{\mathcal{A}}(w)$ .

We are ready to prove the theorem. Fix a partition  $S_1, \dots, S_m$  for some  $m \geq \varphi(\max_l |S_l|)$ . Suppose the first condition is not true, namely, for every  $j$ , the set  $S_j$  is not decomposable. Let  $L = \max_l |S_l|$ . Since no set  $S_l$  is decomposable we know that  $L > 1$ . By the observations in the previous paragraph it suffices to construct a selection set  $S$  such that  $\bigcup_{k \in S} \bar{R}_k = \overline{\text{Run}}_{\mathcal{A}}(w)$ , which will imply that  $S$  is not decomposable.

The remaining part of the proof follows the same steps as the last part in the proof of Theorem 5.1. To construct the selection set  $S = \{k_1, \dots, k_m\}$  we define by induction the sets  $G_j$ . For every  $j \in \{1, \dots, m\}$  let  $G_j = \overline{\text{Run}}_{\mathcal{A}}(w) \setminus \bigcup_{l \leq j} \bar{R}_{k_l}$  (where  $k_0$  is undefined, so that  $G_0 = \overline{\text{Run}}_{\mathcal{A}}(w)$ ). Intuitively,  $G_j$  correspond to runs that are not covered by the set  $\{k_1, \dots, k_j\}$ . For the inductive case, suppose that  $j \geq 0$  and  $G_j \neq \emptyset$ . Since  $\bigcup_{k \in S_{j+1}} \bar{R}_k = \overline{\text{Run}}_{\mathcal{A}}(w)$ , by the pigeonhole principle there exist  $k_{j+1} \in S_{j+1}$  such that  $|\bar{R}_{k_{j+1}} \cap G_j| \geq |G_j|/|S_{j+1}|$ . We add  $k_{j+1}$  to  $S$  and so  $|G_{j+1}| \leq |G_j| - |G_j|/|S_{j+1}| = |G_j| \cdot (|S_{j+1}| - 1)/|S_{j+1}| \leq |G_j| \cdot (L - 1)/L$ . Suppose this procedure continues until  $j = m$  and  $G_m \neq \emptyset$ . Then  $1 \leq |\overline{\text{Run}}_{\mathcal{A}}(w)| \cdot ((L - 1)/L)^m$ , and  $|\overline{\text{Run}}_{\mathcal{A}}(w)| \geq (L/(L - 1))^m$ . However, we know that  $|\overline{\text{Run}}_{\mathcal{A}}(w)|$  is bounded by a polynomial function  $P(n)$  depending on  $|\mathcal{A}|$ . Thus, it suffices to choose  $\varphi$  such that  $m \geq \varphi(L)$  implies  $(L/(L - 1))^m > P(L \cdot m) \geq P(n) \geq |\overline{\text{Run}}_{\mathcal{A}}(w)|$  (recall that  $S_1, \dots, S_m$  is a partition of  $\{1, \dots, n\}$  and  $L \cdot m \geq n$ ). Therefore,  $G_m = \emptyset$  and thus  $\bigcup_{k \in S} \bar{R}_k = \overline{\text{Run}}_{\mathcal{A}}(w)$ , which concludes the proof. □

## 7. CONCLUSION

We have shown five pumping lemmas for five different classes of functions. We believe that the pumping lemmas in Section 5 and in Section 6 could be proved for a wider class of functions that would contain the class  $\mathbb{N}_{+, \times}$ , but this is left for future work. As a corollary of our results, we showed that regular functions over  $\mathbb{N}_{\min, +}$  and  $\mathbb{N}_{\max, +}$  form a strict hierarchy, namely:

$$\text{U-WA} \not\subseteq \text{FA-WA} \not\subseteq \text{PA-WA} \not\subseteq \text{WA}.$$

All strict inclusions, except for  $\text{PA-WA} \not\subseteq \text{WA}$ , could be extracted from the analysis of examples in [18]. However, our results provide a general machinery to prove such results.

Acknowledgments. We thank Shaull Almagor and the anonymous referees of STACS 2018 for their helpful comments.

## REFERENCES

- [1] S. Almagor, U. Boker, and O. Kupferman. What’s decidable about weighted automata? In *ATVA*, pages 482–491, 2011.
- [2] S. Almagor, M. Cadilhac, F. Mazowiecki, and G. A. Pérez. Weak cost register automata are still powerful. In *Developments in Language Theory - 22nd International Conference, DLT 2018, Tokyo, Japan, September 10-14, 2018, Proceedings*, pages 83–95, 2018.
- [3] R. Alur, L. D’Antoni, J. Deshmukh, M. Raghothaman, and Y. Yuan. Regular functions and cost register automata. In *LICS*, pages 13–22. IEEE Computer Society, 2013.
- [4] R. Alur and M. Raghothaman. Decision problems for additive regular functions. In *Automata, Languages, and Programming*, pages 37–48. Springer, 2013.
- [5] M. Arenas, M. Munoz, and C. Riveros. Descriptive complexity for counting complexity classes. In *Logic in Computer Science (LICS), 2017 32nd Annual ACM/IEEE Symposium on*, pages 1–12. IEEE, 2017.
- [6] C. Barloy, N. Fijalkow, N. Lhote, and F. Mazowiecki. A Robust Class of Linear Recurrence Sequences. *arXiv e-prints*, page arXiv:1908.03890, 2019.
- [7] U. Boker, K. Chatterjee, T. A. Henzinger, and O. Kupferman. Temporal specifications with accumulative values. *ACM Transactions on Computational Logic (TOCL)*, 15(4):27, 2014.
- [8] T. Colcombet, D. Kuperberg, A. Manuel, and S. Toruńczyk. Cost functions definable by min/max automata. In *33rd Symposium on Theoretical Aspects of Computer Science, STACS 2016, February 17-20, 2016, Orléans, France*, pages 29:1–29:13, 2016.
- [9] K. Culik II and J. Kari. Image compression using weighted finite automata. In *Mathematical Foundations of Computer Science 1993*, pages 392–402. Springer, 1993.
- [10] L. Daviaud, P. Reynier, and J. Talbot. A generalised twinning property for minimisation of cost register automata. In *LICS*, 2016.
- [11] M. Droste and P. Gastin. Weighted automata and weighted logics. *Theor. Comput. Sci.*, 380(1-2):69–86, 2007.
- [12] M. Droste, W. Kuich, and H. Vogler. *Handbook of Weighted Automata*. Springer, 1st edition, 2009.
- [13] A. Ehrenfeucht. An application of games to the completeness problem for formalized theories. *Fund. Math*, 49(129-141):13, 1961.
- [14] R. Fraïssé. Sur quelques classification des systemes de relations. *Université d’Alger, Publications Scientifiques, Série A*, 1:35–182, 1984.
- [15] J. E. Hopcroft and J. D. Ullman. *Introduction to Automata Theory, Languages and Computation*. Addison-Wesley, 1979.
- [16] D. Kirsten. A Burnside approach to the termination of Mohri’s algorithm for polynomially ambiguous min-plus-automata. *ITA*, 42(3):553–581, 2008.
- [17] D. Kirsten and S. Lombardy. Deciding unambiguity and sequentiality of polynomially ambiguous min-plus automata. In *STACS*, pages 589–600, 2009.
- [18] I. Klimann, S. Lombardy, J. Mairesse, and C. Prieur. Deciding unambiguity and sequentiality from a finitely ambiguous max-plus automaton. *Theor. Comput. Sci.*, 327(3):349–373, 2004.
- [19] S. Kreutzer and C. Riveros. Quantitative monadic second-order logic. In *LICS*, pages 113–122. IEEE Computer Society, 2013.
- [20] D. Krob. The equality problem for rational series with multiplicities in the tropical semiring is undecidable. In *ICALP*, pages 101–112, 1992.
- [21] L. Libkin. *Elements of finite model theory*. Springer Science & Business Media, 2013.
- [22] F. Mazowiecki and C. Riveros. Maximal partition logic: towards a logical characterization of copyless cost register automata. In *CSL*, 2015.

- [23] F. Mazowiecki and C. Riveros. Pumping lemmas for weighted automata. In *35th Symposium on Theoretical Aspects of Computer Science, STACS 2018, February 28 to March 3, 2018, Caen, France*, pages 50:1–50:14, 2018.
- [24] F. Mazowiecki and C. Riveros. Copyless cost-register automata: Structure, expressiveness, and closure properties. *J. Comput. Syst. Sci.*, 100:1–29, 2019.
- [25] M. Mohri. Finite-state transducers in language and speech processing. *Computational Linguistics*, 23(2):269–311, 1997.
- [26] J. Ouaknine and J. Worrell. On the positivity problem for simple linear recurrence sequences,. In *ICALP*, pages 318–329, 2014.
- [27] J. Ouaknine and J. Worrell. Ultimate positivity is decidable for simple linear recurrence sequences. In *ICALP*, pages 330–341, 2014.
- [28] C. H. Papadimitriou. *Computational complexity*. Addison-Wesley, 1993.
- [29] J.-É. Pin. Mathematical foundations of automata theory. *Lecture notes LIAFA, Université Paris*, 7, 2010.
- [30] J. Sakarovitch. *Elements of Automata Theory*. Cambridge University Press, 2009.
- [31] M.-P. Schützenberger. On the definition of a family of automata. *Information and Control*, 4:245–270, 1961.
- [32] M.-P. Schützenberger. On finite monoids having only trivial subgroups. *Information and Control*, 8:190–194, 1965.
- [33] A. Weber. Finite-valued distance automata. *Theor. Comput. Sci.*, 134(1):225–251, 1994.
- [34] A. Weber and H. Seidl. On the degree of ambiguity of finite automata. *Theoretical Computer Science*, 88(2):325–349, 1991.