Learning Higher-Order Programs without Meta-Interpretive Learning *

Stanisław J. Purgał^{1†}, David M. Cerna^{2,3‡§}, Cezary Kaliszyk^{1†}

¹University of Innsbruck, Innsbruck, Austria

²Czech Academy of Sciences Institute of Computer Science (CAS ICS), Prague, Czechia ³Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria stanislaw.purgal@uibk.ac.at, dcerna@{cs.cas.cz, risc.jku.at}, cezary.kaliszyk@uibk.ac.at

Abstract

Learning complex programs through *inductive logic* programming (ILP) remains a formidable challenge. Existing higher-order enabled ILP systems show improved accuracy and learning performance, though remain hampered by the limitations of the underlying learning mechanism. Experimental results show that our extension of the versatile *Learning From Failures* paradigm by higher-order definitions significantly improves learning performance without the burdensome human guidance required by existing systems. Our theoretical framework captures a class of higher-order definitions preserving soundness of existing subsumption-based pruning methods.

1 Introduction

Inductive Logic Programming (ILP) [Mug91; NdW97] is a form of symbolic machine learning which learns a logic program from background knowledge (BK) predicates and sets of positive and negative example runs of the goal program.

Naively, learning a logic program which takes a positive integer n and returns a list of list of the form $[[1], [1, 2], \cdots, [1, \cdots, n]]$ would not come across as a formidable learning task. A logic program is easily constructed using conventional higher-order (HO) definitions.

$$\begin{split} \texttt{allSeqN}(N,L)\text{:-iterate}(succ,0,N,A), \ \texttt{map}(p,A,L). \\ \texttt{p}(A,B)\text{:-iterate}(succ,0,A,B). \end{split}$$

The first iterate produces the list $[1, \dots, N]$ and map applies a functionally equivalent iterate to each member of $[1, \dots, N]$, thus producing the desired outcome. However, this seemingly innocuous function requires 25 literals spread over five clauses when written as a function-free, first-order (FO) logic program, a formidable task for most if not all existing FO ILP approaches [CDEM21].

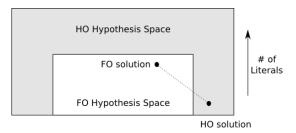


Figure 1: Inclusion of HO definitions increases the size of the search space, but can lead to the search space containing a shorter solutions.

Excessively large BK can, in many cases, lead to performance loss [Cro20; SKB03]. In contrast, adding HO definitions increases the overall size of the search space, but may result in the presence of significantly simpler solutions (see Figure 1). Enabling a learner, with a strong bias towards short solutions, with the ability to use HO definitions can result in improved performance. We developed an HO-enabled Popper [CM21a] (Hopper), a novel ILP system designed to learn optimally short solutions. Experiments show significantly better performance on hard tasks when compared with Popper and the best performing HO-enabled ILP system, $Metagol_{HO}$ [CMM20]. See Section 4.

Existing HO-enabled ILP systems are based on *Meta-inter-pretive Learning* (MiL) [MLPT14]. The efficiency and performance of MiL-based systems is strongly dependent on significant human guidance in the form of *metarules* (a restricted form of HO horn clauses). Choosing these rules is an art in all but the simplest of cases. For example, iterate, being ternary (w.r.t. FO arguments), poses a challenge for some systems, and in the case of *HEXMIL*_{HO} [CMM20], this definition cannot be considered as only binary definitions are allowed (w.r.t. FO arguments).

Limiting human participation when fine-tuning the search space is an essential step towards strong symbolic machine learning. The novel *Learning from Failures* (LFF) paradigm [CM21a], realized through *Popper*, prunes the search space as part of the learning process. Not only does this decrease human guidance, it also removes limitations on the structure of HO definitions allowing us to further exploit the above mentioned benefits.

Integrating HO concepts into MiL-based systems is quite seamless as HO definitions are essentially a special type of

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[†]Supported by the ERC starting grant no. 714034 SMART and §

[‡]Contact Author

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metarule. Thus, HO enabling MiL learners requires minimal change to the theoretical foundations. In the case of LFF learners, like *Popper*, the pruning mechanism influences which HO definitions may be soundly used (See page 813 of [CM21a]).

We avoid these soundness issues by indirectly adding HO definitions. *Hopper* uses FO instances of HO definitions each of which is associated with a set of unique predicates symbols denoting the HO arguments of the definition. These predicates symbols occur in the head literal of clauses occurring in the candidate program **iff** their associated FO instance occurs in the candidate program. Thus, only programs with matching structure may be pruned. We further examine this issue in Section 3 and provide a construction encapsulating the accepted class of HO definitions.

Succinctly, we work within the class of HO definitions that are *monotone* with respect to *subsumption* and *entailment*; $p_1 \leq_{\theta} (\models) \ p_2 \Rightarrow H(p_1) \leq_{\theta} (\models) \ H(p_2)$ where p_1 and p_2 are logic programs, and $H(\cdot)$ is an HO definition incorporating parts of p_1 and p_2 . Similar to classes considered in literature, our class excludes most cases of HO negation (see Section 3.4). However, our framework opens the opportunity to invent HO predicates during learning (an important open problem), though this remains too inefficient in practice and is left to future work.

2 Related Work

The authors of [CMM20] (Section 2) provide a literature survey concerning the synthesis of Higher-Order (HO) programs and, in particular, how existing ILP systems deal with HO constructions. Below we provide a brief summary of this survey and focus on introducing the state-of-the-art systems, namely, HO extensions of *Metagol* [CM16] and *HEXMIL* [KEI18]. We also introduce Popper [CM21a], the system *Hopper* is based on. Additionally, a detailed survey of the current state of ILP research has recently been published [CDEM21].

2.1 Predicate Invention and HO Synthesis

Effective use of HO predicates is intimately connected to auxiliary Predicate Invention (PI). The following illustrates how fold/4 can be used together with PI to provide a succinct program for reversing a list:

Including p in the background knowledge is unintuitive. It is reasonable to expect the synthesizer to produce it. Many of the well known, non-MiL based ILP frameworks do not support predicate invention, *Foil* [Qui90], *Progol* [Mug95], *Tilde* [Blo99], and *Aleph* [Sri01] to name a few. While there has been much interest, throughout ILP's long history, concerning PI, it remained an open problem discussed in "ILP turns 20" [MRP+12]. Since then, there have been a few successful approaches. Both *ILASP* [Law18] and δ *ILP* [EG18] can, in a restricted sense, introduce invented predicates, however neither handles infinite domains nor are suited for the task we are investigating, manipulation of trees and lists.

The best performing systems with respect to the aforementioned tasks are *Metagol* [CM16] and *HEXMIL* [KEI18]; both

are based on *Meta-interpretive Learning* (MiL) [MLPT14], where PI is considered at every step of program construction. However, a strong language bias is needed for an efficient search procedure. This language bias comes in the form of *Metarules* [CM14], a restricted form of HO horn clauses.

Definition 1 ([CT20]) A metarule is a second-order Horn clause of the form $A_0 \leftarrow A_1, \cdots, A_n$, where A_i is a literal $P(T_1, \cdots, T_m)$, s.t. P is either a predicate symbol or a HO variable and each T_i is either a constant or a FO variable.

For further discussion see Section 2.2. *Popper* [CM21a], does not directly support PI, though, it is possible to enforce PI through the language bias (*Poppi* is an PI-enabled extension [CM21b]). *Popper*'s language bias, while partially fixed, is essentially an arbitrary ASP program. The Authors of [CM21a] illustrate this by providing ASP code emulating the chain metarule¹ (see Appendix A of [CM21a]). We exploit this feature to extend *Popper*, allowing it to construct programs containing instances of HO definitions. *Hopper*, our extension, has drastically improved performance when compared with *Popper*. *Hopper* also outperforms the state-of-the-art MiL-based ILP systems extended by HO definitions. For further discussion of *Popper* see Section 2.3, and for *Hopper* see Section 3.

2.2 Metagol and HEXMIL

We briefly summarize existing HO-capable ILP systems introduced by A. Cropper *et al.* [CMM20].

Higher-order Metagol

In short, Metagol is an MiL-learner implemented using a Prolog meta-interpreter. As input, Metagol takes a set of predicate declarations PD of the form body_pred(P/n), sets of positive E^+ and negative E^- examples , compiled background knowledge BK_c , and a set of metarules M. The examples provide the arity and name of the goal predicate. Initially, Metagol attempts to satisfy E^+ using BK_c . If this fails, then Metagol attempts to unify the current goal atom with a metarule from $m \in M$. At this point Metagol tries to prove the body of metarule m. If successful, the derivation provides a Prolog program which can be tested on E^- . If the program entails some of E^- , Metagol backtracks and tries to find another program. Invented predicates are introduced while proving the body of a metarule when BK_c is not sufficient for construction of a program.

The difference between Metagol and $Metagol_{\rm HO}$ is the inclusion of interpreted background knowledge BK_{in} . For example, map/3 as BK_{in} takes the form:

```
ibk([map,[],[],_],[]).
ibk([map,[A|As],[B|Bs],F],[[F,A,B],[map,As,Bs,F]]).
```

Metagol handles BK_{in} as it handles metarules. When used, *Metagol* attempts to prove the body of map, i.e. F(A,B). Either F is substituted by a predicate contained in BK_c or replaced by an invented predicate that becomes the goal atom and is proven using metarules or BK_{in} .

A consequence of this approach is that substituting the goal atom by a predicate defined as BK_{in} cannot result in a derivation defining a Prolog program. Like with metarules,

 $^{{}^{1}}P(A, B):=Q(A, C), R(C, B).$

additional proof steps are necessary. The following program defining $\mathtt{half}_{1st}(A,B)$, which computes the last half of a list², illustrates why this may be problematic:

```
\begin{aligned} \operatorname{half}_{1\operatorname{st}}(A,B) &:\operatorname{-reverse}(A,C), \\ & \operatorname{case}_{1\operatorname{ist}}(p_{[\,]},p_{[H|T]},C,B). \\ \operatorname{p}_{[\,]}(A) &:\operatorname{-empty}(A). \\ \operatorname{p}_{[H|T]}(A,B,C) &:\operatorname{-empty}(B),\operatorname{empty}(C). \\ \operatorname{p}_{[H|T]}(A,B,C) &:\operatorname{-front}(B,D)^3, \\ & \underline{\operatorname{case}_{1\operatorname{ist}}(p_{[\,]},p_{[H|T]},D,E)}, \\ \operatorname{append}(E,A,C). \end{aligned}
```

The HO predicate $\mathtt{case_{list}}(p_{[\:]},p_{[H|T]},A,B)$ calls $p_{[\:]}$ if A is empty and $p_{[H|T]}$ otherwise. Our definition of $half_{\mathtt{lst}}(A,B)$ cannot be found using the standard search procedure as every occurrence of $\mathtt{case_{list}}$ results in a call to the metainterpreter's proof procedure. The underlined call to $\mathtt{case_{list}}$ results in PI for $\mathtt{p}_{[H|T]}$ ad infinitum. Similarly, the initial goal cannot be substituted unless it's explicitly specified.

As with $half_{1st}(A, B)$, The following program defining issubtree(A, B), which computes whether B is a subtree of A, requires recursively calling issubtree through any.

This can be resolved using *metatypes* (see Section 4), but this is non-standard, results in a strong language bias, and does not always work. *Hopper* successfully learns these predicates without any significant drawbacks.

Negation of invented predicates (HO arguments of BK_{in} definitions), to the best of our knowledge, is not fully supported by $Metagol_{\rm HO}$ (See Section 4.2 of [CMM20]). Hopper has similar issues which are discussed in Section 3.4.

Higher-order HEXMIL

HEXMIL is an ASP encoding of Meta-interpretive Learning [KEI18]. Given that ASP can be quite restrictive, HEXMIL exploits the HEX formalism for encoding MiL. HEX allows the ASP solver to interface with external resources [ERS16]. HEXMIL is restricted to forward-chained metarules:

Definition 2 Forward-chained metarules are of the form: $P(A,B): Q_1(A,C_1), Q_2(C_1,C_2), \cdots, Q_n(C_{n-1},B), R_1(D_1), \cdots, R_m(D_m)$ where $D_i \in \{A,C_1,\cdots,C_{n-1},B\}$.

Thus, only Dyadic learning task may be handled. Furthermore, many useful metarules are not of this form, i.e. P(A,B):- Q(A,B),R(A,B). $HEXMIL_{\rm HO}$, incorporates HO definitions into the forward-chained structure of Definition 2. For details concerning the encoding see Section 4.4 of [CMM20]. The authors of [CMM20] illustrated $HEXMIL_{\rm HO}$'s poor performance on list manipulation tasks, and its limitations make application to tasks of interest difficult. Thus, we focus on $Metagol_{\rm HO}$ in Section 4.

The LFF paradigm together with Popper provide a novel approach to inductive logic programming, based on counterexample guided inductive synthesis (CEGIS) [SL08]. Both LFF and the system implementing it were introduced by A. Cropper and R. Morel [CM21a]. As input, *Popper* takes a set of predicate declarations PD, sets of positive E^+ and negative E^- examples, and background knowledge BK, the typical setting for *learning from entailment* ILP [Rae08].

During the **generate phase**, candidate programs are chosen from the viable hypothesis space, i.e. the space of programs which have yet to be ruled out by generated constraints. The chosen program is then tested (**test phase**) against E^+ and E^- . If only some of E^+ and/or some of E^- is entailed by the candidate hypothesis, Popper builds constraints (**constrain phase**) which further restrict the viable hypothesis space searched during the **generate phase**. When a candidate program only entails E^+ , *Popper* terminates.

Popper searches through a finite hypothesis space, parameterized by features of the language bias (i.e. number of body predicates, variables, etc.). Importantly, if an optimal solution is present in this parameterized hypothesis space, Popper will find it (Theorem 11 [CM21a]). Optimal is defined as the solution containing the fewest literals [CM21a].

An essential aspect of this *generate*, *test*, *constrain* loop is the choice of constraints. Depending on how a candidate program performs in the **test phase**, Popper introduces constraints pruning specializations and/or generalizations of the candidate program. Specialization/generalization is defined via Θ-subsumption [Plo70; Rey70]. *Popper* may also introduce **Elimination** Constraints pruning *separable*⁴ sets of clauses. Details concerning the benefits of this approach are presented in [CM21a]. Essentially, *Popper* refines the hypothesis space, not the program [Sri01; Mug95; QCJ93].

In addition to constraints introduced during the search, like the majority of ILP systems, *Popper* incorporates a form of *language bias* [NCWSC97], that is predefined syntactic and/or semantic restrictions of the hypothesis space. *Popper* minimally requires *predicate declarations*, i.e. whether a predicate can be used in the head or body of a clause, and with what arities the predicate may appear. Popper accepts *mode declaration*-like hypothesis constraints [Mug95] which declare, for each argument of a given predicate, the type and direction. Additional hypothesis constraints can be formulated as ASP programs (mentioned in Section 2.1).

Popper implements the *generate, test, constrain* loop using a *multi-shot solving* framework [GKKS19] and an encoding of both definite logic programs and constraints within the *ASP* [Lif19] paradigm. The language bias together with generated constraints are encoded as an ASP program. The ASP solver is run on this program and the resulting model (if one exists) is an encoding of a candidate program.

²half_{lst}([1,2],[2]), half_{lst}([1,2,3],[3]), half_{fst}([1,2,3],[1,2]). ³front(A, B):-reverse(A, C), tail(C, D), reverse(D, B).

^{2.3} Popper: Learning From Failures (LFF)

⁴No head literal of a clause in the set occurs as a body literal of a clause in the set.

3 Theoretical Framework

We provide a brief overview of logic programming. Our exposition is far from comprehensive. We refer the interested reader to a more detailed source [Llo87].

3.1 Preliminaries

Let \mathcal{P} be a countable set of *predicate symbols* (denoted by p,q,r,p_1,q_1,\cdots), \mathcal{V}_f be a countable set of *first-order (FO)* variables (denoted by A,B,C,\cdots), and \mathcal{V}_h be a countable set of *HO variables* (denoted by P,Q,R,\cdots). Let \mathcal{T} denote the set of FO terms constructed from a finite set of function symbols and \mathcal{V}_f (denoted by s,t,s_1,t_1,\cdots).

An atom is of the form $p(T_1,\cdots,T_m,t_1,\cdots,t_n)$. Let us denote this atom by a, then sy(a)=p is the symbol of the atom, $ag_h(a)=\{T_1,\cdots,T_m\}$ are its HO-arguments, and $ag_f(a)=\{t_1,\cdots,t_n\}$ are its FO-arguments. When $ag_h(a)=\emptyset$ and $sy(a)\in\mathcal{P}$ we refer to a as FO, when $ag_h(a)\subset\mathcal{P}$ and $sy(a)\in\mathcal{P}$ we refer to a as HO-ground, otherwise it is HO. A literal is either an atom or its negation. A literal is HO if the atom it contains is HO.

A clause is a set of literals. A Horn clause contains at most one positive literal while a definite clause must have exactly one positive literal. The atom of the positive literal of a definite clause c is referred to as the head of c (denoted by hd(c)), while the set of atoms of negated literals is referred to as the body (denoted by bd(c)). A function-free definite (f.f.d) clause only contains variables as FO arguments. We refer to a finite set of clauses as a theory. A theory is considered FO if all atoms are FO. Replacing variables $P_1, \cdots, P_n, A_1, \cdots, A_m$ by predicate symbols p_1, \cdots, p_n and terms $t_1 \cdots t_m$ is a substitution (θ, σ, \cdots) denoted by $\theta = \{P_1 \mapsto p_1, \cdots, P_n \mapsto p_n, A_1 \mapsto t_1, \cdots, A_m \mapsto t_m\}$. A substitution θ unifies two atoms when $a\theta = b\theta$.

3.2 Interpretable Theories and Groundings

Our hypothesis space consists a particular type of theory.

Definition 3 A clause c is proper⁶ if $ag_h(hd(c))$ are pairwise distinct, $ag_h(hd(c)) \subset \mathcal{V}_h$, and $\forall a \in bd(c)$,

- a) if $sy(a) \in \mathcal{V}_h$, then $sy(a) \in ag_h(hd(c))$, and
- b) if $p \in ag_{k}(a)$ and $p \in \mathcal{V}_{h}$, then $p \in ag_{k}(hd(c))$.

A finite set of proper clauses d with the same head (denoted hd(d)) is referred to as a HO definition. A set of distinct HO definitions is a library. Let $\mathcal{P}_{PI} \subset \mathcal{P}$ be a set of predicate symbols reserved for invented predicates.

Definition 4 A f.f.d theory \mathfrak{T} is interpretable if $\forall c \in \mathfrak{T}$, $ag_h(hd(c)) = \emptyset$ and $\forall l \in bd(c)$, l is higher-order ground,

- a) if $ag_h(l) \neq \emptyset$, then $\forall c' \in \mathfrak{T}$, $sy(hd(c')) \neq sy(l)$, and
- b) $\forall p \in aq_{b}(l), \exists c' \in \mathfrak{T}, s.t. \ sy(hd(c')) = p \in \mathcal{P}_{PI}.$

Atoms s.t. $ag_h(l) \neq \emptyset$ are external. The set of external atoms of an interpretable theory \mathfrak{T} is denoted by $ex(\mathfrak{T})$.

Let $S_{PI}(\mathfrak{T}) = \{p_i \mid p_i \in ag_h(a) \land a \in ex(\mathfrak{T})\}$, the set of predicates which need to be invented. During the **generate phase** we enforce invention of $S_{PI}(\mathfrak{T})$ by pruning programs which contain external literals, but do not contain clauses for their arguments. We discuss this in more detail in Section 3.3.

Otherwise, interpretable theories do not require significant

Otherwise, interpretable theories do not require significant adaption of *Popper's generate, test, constrain loop* [CM21a]. The HO arguments of external literals are ignored by the ASP solver, which searches for so-called *principle programs* (an FO representation of interpretable theories).

Example 1 Consider reverse, half_{lst}, and issubtree of Section 2.1 & 2.2. Each is an interpretable theory. The sets of external literals of these theories are $\{ fold(p,C,A,B) \}$, $\{ case_{list}(p_{[]},p_{[H|T]},C,B), case_{list}(p_{[]},p_{[H|T]},D,E) \}$, and $\{ any(cond,C,B) \}$, respectively.

Definition 5 Let L be a library, and $\mathfrak T$ an interpretable theory. $\mathfrak T$ is L-compatible if $\forall l \in ex(\mathfrak T), \exists d \in L$. s.t. $hd(d)\sigma = l$ for some substitution σ . Let df(L,l) = d and $\theta(L,l) = \sigma$.

Example 2 The program in Section 2.1 is L-compatible with

$$\begin{split} \operatorname{fold}(P,A,B,C)\text{:-}&\operatorname{empty}(B),C = A.\\ \operatorname{fold}(P,A,B,C)\text{:-}&\operatorname{head}(B,H),\operatorname{P}(H,D),\\ \operatorname{tail}(B,T),\operatorname{fold}(P,D,T,C). \end{split}$$

Let
$$l = fold(p, C, A, B)$$
: $def(L, l) = fold(P, A, B, C)$ and $\theta(L, l) = \{P \mapsto p, A \mapsto C, B \mapsto A, C \mapsto B\}$.

An L-compatible theory $\mathfrak T$ can be L-grounded. This requires replacing external literals of $\mathfrak T$ by FO literals (removal of all HO arguments), adding clauses which associate the first order literals with the appropriate $d \in L$ and argument instantiations. The principle program contains all clauses derived from $\mathfrak T$ (See Example 3).

Example 3 Using the library of Example 2 and a modified version of the program from Section 2.1, we get the following L-grounding:

$$\begin{split} & \texttt{reverse}(A,B)\text{:-}\,\texttt{empty}(C),\,\texttt{fold}_a(C,A,B).\\ & \texttt{fold}_{p.a}(A,B,C)\text{:-}\,\texttt{head}(C,B),\,\texttt{tail}(C,A).\\ & \texttt{fold}_a(A,B,C)\text{:-}\,\texttt{fold}(\texttt{fold}_{p.a},A,B,C).\\ & \texttt{fold}(P,A,B,C)\text{:-}\,\texttt{empty}(B),A=C.\\ & \texttt{fold}(P,A,B,C)\text{:-}\,\texttt{head}(B,H),P(H,D),\\ & \texttt{tail}(B,T),\texttt{fold}(P,D,T,C). \end{split}$$

The first two clauses form the principle program.

If an L-compatible theory contains multiple external literals whose symbol is fold, i.e. $fold(fold_{p.a}, C, A, B)$, $fold(fold_{p.a}, D, E, R)$ both are renamed to $fold_a$. However, if the higher-order arguments differ, then they are renamed to $fold_a$ and $fold_b$. When a definition takes more than one HO argument and arguments of instances partially overlap, duplicating clauses may be required during the construction of the L-grounding. Soundness of the pruning mechanism is preserved when the FO literals uniquely depend on the arguments fed to HO definitions.

 $[\]overline{}^5 sy(l)$, $ag_h(l)$, and $ag_f(l)$ apply to literals with similar affect. $\overline{}^6 Similar$ to definitional HO of W. Wadge [Wad91].

3.3 Interpretable Theories and Constraints

The constraints of Section 2.3 are based on Θ -subsumption:

Definition 6 (Θ -subsumption) An FO theory T_1 subsumes an FO theory T_2 , denoted by $T_1 \leq_{\theta} T_2$ iff, $\forall c_2 \in T_2 \exists c_1 \in T_1$ s.t. $c_1 \leq_{\theta} c_2$, where $c_1 \leq_{\theta} c_2$ iff, $\exists \theta$ s.t. $c_1 \theta \subseteq c_2$.

Importantly, the following property holds:

Proposition 1 if
$$T_1 \leq_{\theta} T_2$$
, then $T_1 \models T_2$

The pruning ability of *Popper*'s Generalization and specialization constraints follows from Proposition 1.

Definition 7 An FO theory T_1 is a generalization (specialization) of an FO theory T_2 iff $T_1 \leq_{\theta} T_2$ $(T_2 \leq_{\theta} T_1)$.

Given a library L and a space of L-compatible theories, we can compare L-groundings using Θ -subsumption and prune generalizations (specializations), based on the **Test phase**.

Groundings and Elimination Constraints

During the **generate phase**, elimination constraints prune separable programs (See Footnote 4). While L-groundings are non-separable, and thus avoid pruning in the presence of elimination constraints, it is inefficient to query the ASP solver for L-groundings. The ASP solver would have to know the library and how to include definitions. Furthermore, the library must to written in an ASP-friendly form [CM21a]. Instead we query the ASP solver for the *principle program*. The definitions from the library L are treated as BK. Consider Example 3, during the **generate phase** the ASP solver may return an encoding of the following clauses:

$$\label{eq:powerse} \begin{split} \texttt{reverse}(A,B)\text{:-}&\,\texttt{empty}(C),\,\,\texttt{fold}(C,A,B).\\ &\,\texttt{p}(A,B,C)\text{:-}\,\texttt{head}(C,B),\,\,\texttt{tail}(C,A). \end{split}$$

During the **test phase** the rest of the *L*-grounding is reintroduced. While this eliminates inefficiencies, the above program is now separable and may be pruned. To efficiently implement HO synthesis we relaxed the elimination constraint in the presence of a library. Instead we introduce so called *call graph constraints* defining the relationship between HO literals and auxiliary clauses. This is similar to the *dependency graph* introduced in [CM21b].

3.4 Negation, Generalization, and Specialization

Negation of HO literals can interfere with *Popper* constraints. Consider the ILP task and candidate programs:

$$\begin{split} E^+:f(b). & f(c). & \mathbf{E}^-:f(a). \\ BK: \left\{ \begin{array}{ll} p(a). & p(b). \\ q(a). & q(c). \end{array} \right\} & \mathbf{HO}:N(P,A)\text{:- not }P(A). \\ & \underline{prog_s} & \underline{prog_f} \\ \hline \mathbf{f}(A)\text{:- N}(p_1,A). \\ \mathbf{p}_1(A)\text{:- p}(A), \ \mathbf{q}(A). & \mathbf{p}_1(A)\text{:- p}(A). \\ \end{split}$$

The optimal solution is $prog_s$ and $prog_f \leq_{\theta} prog_s$. Note, $prog_f \models f(b) \land \neg f(a) \land \neg f(c)$, it does not entail all of E^+ . We should generalize $prog_f$ to find a solution, i.e. add literals to p_1 . The introduced constraints [CM21a] prune programs

extending p_1 , i.e. $prog_s$. Similar holds for specializations. Consider the ILP task and candidate programs:

The optimal solution is $prog_s$ and $prog_s \leq_{\theta} prog_f$. Note, $prog_f \models f(a) \land f(b) \land f(c)$, it entails some of E^- . We should specialize p_1 to find a solution, i.e. add clauses to $prog_f$. The introduced constraints [CM21a] prune programs that add clauses, i.e. $prog_s$.

Handling negation of invented predicates is feasible, but non-trivial as it would require significant changes to the constraint construction procedure. We leave it to future work.

4 Experiments

A possible, albeit very weak, program synthesizer is an enumeration procedure which orders all possible programs constructable from the BK by size, testing each until a solution is found. In [CM21a], this procedure was referred to as Enumerate. Popper extends Enumerate by pruning the hypothesis space based on the performance of previously tested programs.

The pruning mechanism will never prune the shortest solution. Thus, the important question when evaluating *Popper*, and *Hopper*, is not if *Popper* will find a solution, nor is it a high quality solution, but rather how long it takes *Popper* to find the solution. An extensive suite of experiments was presented in [CM21a], illustrating that *Popper* significantly outperforms *Enumerate* and existing ILP systems.

One way to improve the performance of *LFF*-based ILP systems, like *Popper*, is to introduce techniques which shorten or simplify the solution. The authors of [CMM20], in addition to introducing *Metagol*_{HO} and *HEXMIL*_{HO}, provided a comprehensive suite of experiments illustrating that the addition of HO predicates can improve accuracy and, most importantly, reduce learning time. Reduction in learning times results from HO predicates reducing the complexity/size of the solutions.

The experiments in [CM21a] thoroughly cover scalability issues and learning performance on simple list transformation task, but do not cover performance on complex task with large solutions. The experiments presented in [CMM20] illustrate performance gains when a HO library is used to solve many simple task and how the addition of HO predicates allow the synthesis of relatively complex predicates such as dropLast. When the solution is large *Popper*'s performance degrades significantly. When the solution requires complex interaction between predicates and clauses it becomes exceedingly difficult to find an set of metarules for *Metagol*_{HO} without being overly descriptive or suffering from long learning times.

Our experiments illustrate that combining *Popper* with HO predicates [CMM20] significantly improves *Popper*'s performance at learning complex programs. As in [CM21a], we

Task	Popper (Opt)	#Literals	PI?	Hopper	Hopper (Opt)	#Literals	HO-Predicates	$Metagol_{ m HO}$	Metatypes?
Learning Programs by learning from Failures [CM21a]									
dropK	1.1s	7	no	0.5s	0.1s	4	iterate	no	no
allEven	0.2s	7	no	0.2s	0.1s	4	all	yes	no
findDup	<u>0.25s</u>	7	no		<u>0.5s</u>	10	caseList	no	yes
length	0.1s	7	no	0.2s	0.1s	5	fold	yes	no
member	0.1s	5	no	0.2s	0.1s	4	any	yes	no
sorted	65.0s	9	no	46.3s	0.4s	6	fold	yes	no
reverse	11.2s	8	no	7.7s	0.5s	6	fold	yes	no
Learning Higher-Order Logic Programs [CMM20]									
dropLast	300.0s	10	no	300s	2.9s	6	map	yes	no
encryption	300.0s	12	no	300s	1.2s	7	map	yes	no
Additional Tasks									
repeatN	5.0s	7	no	0.6s	0.1s	5	iterate	yes	no
rotateN	300.0s	10	no	300s	2.6s	6	iterate	yes	no
allSeqN	300.0s	25	yes	300s	5.0s	9	iterate, map	yes	no
dropLastK	300.0s	17	yes	300s	37.7s	11	map	no	no
firstHalf	300.0s	14	yes	300s	0.2s	9	iterateStep	yes	no
lastHalf	300.0s	12	no	300s	155.2s	12	caseList	no	yes
of1And2	300.0s	13	no	300s	6.9s	13	try	no	no
isPalindrome	300.0s	11	no	157s	2.4s	9	condlist	no	yes
depth	300.0s	14	yes	300s	10.1s	8	fold	yes	yes
isBranch	300.0s	17	yes	300s	25.9s	12	caseTree, any	no	yes
isSubTree	2.9s	11	yes	1.0s	0.9s	7	any	yes	yes
addN	300.0s	15	yes	300s	1.4s	9	map, caseInt	yes	no
mulFromSuc	300.0s	19	yes	300s	1.2s	7	iterate	yes	no

Table 1: We ran Popper, Hopper, Optimized Hopper, and $Metagol_{HO}$ on a single core with a timeout of 300 second. Times denote the average of 5 runs. Evaluation time for Popper and Hopper was set to a thousandth of a second, sufficient time for all task involved.

use predicate declarations body_pred(head, 2), type declarations type(head, (list, element)), direction declarations direction(head, (in, out)), and the parameters required for *Popper*'s search mechanism, max_var, max_body, and max_clauses.

We reevaluated 7 of the tasks presented in [CM21a] and 2 presented in [CMM20]. Additionally, we added 8 list manipulation tasks, 3 tree manipulation tasks, and 2 arithmetic tasks (separated by type in Table 1). Our additional tasks are significantly harder than the tasks evaluated in previous work.

For each task, we guarantee that optimal solution is present in the hypothesis space and record how long Popper and Hopper take to find it. We ran Popper using optimal settings and miminal BK. In some cases, the tasks cannot be solved by Popper without Predicate Invention (See Column PI? of Table 1). We ran *Hopper* in two modes, Column *Hopper* concerns running *Hopper* with the same settings and a superset of the BK used by Popper (minus constructions used to force invention), while Column Hopper (Opt) concerns running Hopper with optimal settings and minimal BK. For Popper and Hopper, settings such as max_var significantly impact performance. This is why *Hopper* performs significantly better post-optimization. Using a comparably large BK incurs an insignificant performance impact compared to using unintuitively large parameter settings. The predicates used for a particular task are listed in Column HO-Predicate of Table 1.

Popper and Hopper timed out (300 seconds elapsed) when large clauses or many variables are required. Optimized Hopper solved every tasks. Note, Optimized Hopper closely emulates how such a system would be used as it is pragmatic to search assuming smaller clauses and fewer variables are suffi-

cient and expand the space as needed. *Popper*'s and *Hopper*'s performance degrades by just assuming the solution may be complex. For *findDup*, *Hopper* found the FO solution.

We attempted to solve each task using $Metagol_{\rm HO}$. Successful learning using $Metagol_{\rm HO}$ is highly dependent on the choice of the metarules. To simplify matters, we chose metarules that mimic the clauses found in the solution. In some cases, this requires explicitly limiting how certain variables are instantiated by adding declarations, i.e. metagol:type(Q,2,head_pred), to the body of a metarule (denoted by metatype in Table 1). This amounts to significant human-guidance thus simplifying learning.

5 Conclusion and Future Work

We extended the LFF-based ILP system *Popper* to effectively use user provided HO definitions during learning. Our experiments show *Hopper* (when optimized) is capable of outperforming *Popper* on most tasks, especially the harder tasks we introduced in this work (Section 4). *Hopper* requires minimal guidance compared to the top-performing MiL-based ILP system $Metagol_{HO}$. Our experiments test the theoretical possibility of $Metagol_{HO}$ finding a solution under significant guidance. However, given the sensitivity to metarules choice and the fact that many tasks have ternary and even 4-ary predicates, it is unreasonable to expect a impartial analysis.

We provide a theoretical framework encapsulating the accepted HO definitions, a fragment of the definitions monotone over subsumption and entailment, and discuss the limitations of this framework. A detailed account is provided in Section 3. The main limitation of this framework concerns HO-negation which we leave to future work. Our framework also allows for

invention of HO predicates during learning through constructions of the form ho(P,Q,x,y):- P(Q,x,y). We can verify that Hopper can, in principle, find the solution, but we have not managed to successfully invent an HO predicate during learning. We plan for further investigation of this problem.

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A Implementation

We implement our method by building on top of code provided by [CM21a]. The changes we applied include:

Processing HO predicates We allow user to declare some background knowledge predicates to be HO. Based on these declarations we generate ASP constraints discussed in subsection 3.2 and Prolog code that allows execution of programs generated with those constraints.

Generating context-passing versions of HO predicates Sometimes the HO argument predicates (referred to as $S_{PI}(\mathfrak{T})$ in subsection 3.2) require context that exists in the predicate that calls them, however is inaccessible to them in our framework. To make it accessible we support automating generation of *more contextual* versions of HO predicates. These predicates have higher arity and take more FO arguments. These arguments are only used in HO calls, and are simply passed as arguments to HO predicate calls. In [CMM20] this process is referred to as "currying" (though it is somewhat different to what *currying* is usually considered to be).

Example 4 From a HO map predicate

$$\begin{split} & \max(P,[\;],[\;]). \\ & \max(P,[H_1|T_1],[H_2|T_2]) & :- P(H_1,H_2), \max(P,T_1,T_2). \end{split}$$

we automatically generate a more contextual version

$$\begin{split} & \max(P,[\;],[\;],V). \\ & \max(P,[H_1|T_1],[H_2|T_2],V) \text{:-} \ P(H_1,H_2,V), \\ & \max(P,T_1,T_2,V). \end{split}$$

which allows for construction of a program that adds a number to every element of a list using map

$$\begin{split} & \mathbf{f}(A,B,C)\text{:-} \ \mathrm{map}(\mathbf{p_1},B,C,A). \\ & \mathbf{p_1}(A,B,C)\text{:-} \ \mathrm{add}(A,C,B). \end{split}$$

Force all generated code to be used Since ASP can now generate many different predicates, some of them might not even be called in the main predicate. To avoid such useless code we make ASP keep track of a *call graph* – which predicates call which other predicates, and add a constraint that forces every defined defined predicate to be called (possibly indirectly) by the main predicate. This not only removes many variations of effectively the same program, but also significantly prunes the hypothesis space, pruning programs ignored by other constraints (explained in sec. 3.3).

Changes to separability and recursion We add a few small changes to solve the problems that appear when generating multiple predicates. We make sure that clauses that call HO predicates (and thus different predicates from the program) are not considered *separable*. We also change how recursion is handled – otherwise recursion would allow all invented arguments to be called everywhere in the program, needlessly increasing search space.

B Experimental details

Here we describe all tasks and HO predicates presented in Section 4.

B.1 Higher-Order predicates

al1/2

The argument predicate is true for all elements of the list

$$\boxed{ \begin{array}{c} \operatorname{all}(P,[\]). \\ \operatorname{all}(P,[H|T])\text{:-}\operatorname{call}(P,H),\operatorname{all}(P,T). \end{array}}$$

any/3

checks if for some element in a list and an input object if provided to a given predicate, then that predicate holds.

$$\begin{aligned} & \operatorname{any}(P,[H|_],B)\text{:-}\operatorname{call}(P,H,B).\\ & \operatorname{any}(P,[_|T],B)\text{:-}\operatorname{any}(P,T,B). \end{aligned}$$

caseList/4

a deconstructor for a list, calling first or second predicate depending on whether the list is empty

$$\begin{aligned} &\mathsf{caseList}(P, \mathsf{_}, [], B) \text{:-} \; \mathsf{call}(P, B). \\ &\mathsf{caseList}(\mathsf{_}, Q, [H|T], B) \text{:-} \; \mathsf{call}(Q, H, T, B). \end{aligned}$$

caseList/5

a deconstructor for a list, calling the first, second, third predicate depending on whether the list is empty or singleton. Takes an additional argument which is passed to the predicate arguments.

$$\begin{aligned} &\mathsf{caseList}(P, _, _, [], B)\text{:-}\; \mathsf{call}(P, B).\\ &\mathsf{caseList}(_, Q, _, [H], B)\text{:-}\; \mathsf{call}(Q, H, B).\\ &\mathsf{caseList}(_, _, R, [H|T], B)\text{:-}\; \mathsf{call}(R, H, T, B). \end{aligned}$$

caseTree/4

a deconstructor for a tree, calling first or second predicate depending on whether the tree is a leaf

caseInt/4

A deconstructor for a natural number, calling first or second predicate depending on whether the number is 0.

$$\begin{split} \mathsf{caseint}(P, _, 0, X, Y) &:- \mathsf{call}(P, X, Y). \\ &\mathsf{caseint}(_, Q, N, B) &:- \mathsf{less0}(N), \mathsf{pred}(N, M), \\ &\mathsf{call}(Q, M, X, Y). \end{split}$$

fold/4

combines all elements of a list using the argument predicate.

$$\begin{cases} \texttt{fold}(_, X, [], X). \\ \texttt{fold}(P, Acc, [H|T], Y) \text{:-} \ \texttt{call}(P, Acc, H, W), \\ \texttt{fold}(P, W, T, Y). \end{cases}$$

map/3

checks that the output is a list of the same length as input, such that the argument predicate holds between all their elements.

$$\begin{split} & \max(P,[\;],[\;]). \\ & \max(P,[H_1|T_1],[H_2|T_2])\text{:-} & \operatorname{call}(P,H_1,H_2), \max(P,T_1,T_2). \end{split}$$

try/3

checks whether at least one of argument predicates hold on the last argument

condList/2

true if argument is an empty list, otherwise call argument predicate on the head and the tail.

$$\begin{aligned} & \operatorname{condList}(P,[]). \\ & \operatorname{condList}(P,[H|T])\text{:-} \operatorname{call}(P,H,T). \end{aligned}$$

iterate/4

iterate the argument predicate n times

```
\begin{split} \texttt{iterate}(\_,X,0,X).\\ \texttt{iterate}(P,X,K,W)\text{:-}&\,\texttt{integer}(K),0 < K,!,\\ &\,\texttt{pred}(K,R),\\ &\,\texttt{call}(P,X,H),\\ &\,\texttt{iterate}(P,H,R,W). \end{split}
```

iterateStep/5

A variant of iterate/4, but the iterator instead of being modified by 1 is modified using an argument predicate. Also the output here is a list of all intermediate values.

```
\begin{aligned} &\texttt{iterate}(\_,\_,\_,0,[]).\\ &\texttt{iterate}(P,Q,X,K,[H|T])\text{:-} &\texttt{integer}(K)\\ &\texttt{call}(P,K,R),\\ &\texttt{call}(Q,X,H,W),\\ &\texttt{iterate}(P,Q,W,R,T). \end{aligned}
```

C Learning tasks

In this section we provide the experimental set up for all learning task discussed above. For each task we provide the following information:

- FO Background: BK used for the first order learning task.
- **FO Parameters**: The values for the essential parameters of *Popper* and any switches which where enabled during FO learning.
- **FO Solution**: The solution found by *Popper* or the solution we expected *Popper* to find if given enough time.

- HO Optimal Background: BK used for the Higher order learning task when ran in the optimal configuration. When ran in the non-optimal configuration we used the union of the FO BK with the HO optimal BK minus any HO predicates used for predicate invention during FO learning.
- **HO Optimal Parameters**: The values for the essential parameters of *Hopper* and any switches which where enabled during HO optimal learning. In the non-optimal case we used the setting of the first order learning task minus switches activated specifically for predicate invention during the first order task.
- HO Solution: The solution found by Hopper or the solution we expected Hopper to find if given enough time.

C.1 dropK/3

drop first k elements from a list

- FO Background: suc, pred, zero, tail, head, eq
- FO Parameters:

max_vars: 5 max_body: 3 max_clause: 2

Switches: enable_recursion

• FO Solution:

```
 \begin{vmatrix} \operatorname{dropK}(A,B,C)\text{:-} \operatorname{zero}(A), \operatorname{eq}(B,C), \\ \operatorname{dropK}(A,B,C)\text{:-} \operatorname{tail}(B,D), \operatorname{pred}(A,E), \operatorname{dropK}(E,D,C). \end{vmatrix}
```

- HO Optimal Background: suc, zero, tail, head
- HO Optimal Parameters:

max_vars: 3 max_body: 2 max_clause: 2 Switches:

• HO Solution:

```
\label{eq:dropk} \begin{split} \operatorname{dropk}(A,B,C)\text{:-}\operatorname{iterate\_a}(B,A,C).\\ \operatorname{iterate\_p\_a}(A,B)\text{:-}\operatorname{tail}(A,B). \end{split}
```

C.2 allEven/1

check whether all elements on a list are even

- FO Background: suc, zero, even, tail, head, empty
- FO Parameters:

max_vars: 3 max_body: 4 max_clause: 2

Switches: enable_recursion

- HO Optimal Background: even, all
- HO Optimal Parameters:

max_vars: 2 max_body: 2 max_clause: 2 Switches:

• HO Solution:

```
\begin{split} & \texttt{allEven}(A)\text{:-} \ & \texttt{all\_a}(A). \\ & \texttt{all\_p\_a}(A)\text{:-} \ & \texttt{even}(A). \end{split}
```

C.3 findDup/2

check whether an element is present on a list at least twice

- FO Background: member, head, tail, empty
- FO Parameters:

max_vars: 4 max_body: 4 max_clause: 2

Switches: enable_recursion

FO Solution:

```
\label{eq:findDup} \begin{split} & \mathtt{findDup}(A,B)\text{:-}\ \mathsf{tail}(A,C), \mathtt{empty}(B,C), \mathtt{head}(A,B). \\ & \mathtt{findDup}(A,B)\text{:-}\ \mathsf{tail}(A,C), \mathtt{findDup}(C,B). \end{split}
```

- HO Optimal Background: empty, eq, member
- HO Optimal Parameters:

max_vars: 3 max_body: 2 max_clause: 4

Switches:non_datalog , allow_singletons

• HO Solution:

```
\begin{split} & \mathsf{findDup}(A,B)\text{:-}\,\mathsf{caseList\_a}(A,B).\\ & \mathsf{caseList\_p\_a}(A)\text{:-}\,\mathsf{empty}(B),\mathsf{member}(A,B).\\ & \mathsf{caseList\_q\_a}(A)\text{:-}\,\mathsf{member}(C,B),\mathsf{eq}(C,A).\\ & \mathsf{caseList\_q\_a}(A)\text{:-}\,\mathsf{caseList\_a}(B,C). \end{split}
```

C.4 length/2

Finds the length of a list.

- FO Background: head, tail, succ, empty, zero.
- FO Parameters:

max_vars: 4 max_body: 3 max_clause: 2 Switches: enable_recursion

• FO Solution:

```
\begin{aligned} & \mathsf{length}(A,B) \text{:-} \; \mathsf{empty}(A), \mathsf{zero}(B). \\ & \mathsf{length}(A,B) \text{:-} \; \mathsf{tail}(A,C), \mathsf{length}(C,D), \\ & \mathsf{suc}(D,B). \end{aligned}
```

- HO Optimal Background: succ, zero, fold.
- HO Optimal Parameters:

max_vars: 3
max_body: 2
max_clause: 2

Switches: non_datalog, allow_singletons.

• HO Solution:

```
\label{eq:length} \begin{split} & \mathsf{length}(A,B)\text{:-}\;\mathsf{zero}(C), \mathsf{fold\_a}(C,A,B). \\ & \mathsf{fold\_p\_a}(A,B,C)\text{:-}\;\mathsf{suc}(A,C). \end{split}
```

C.5 member/2

check whether an element is on a list

- FO Background: head, tail, empty.
- FO Parameters:

max_vars: 3 max_body: 2 max_clause: 2

Switches: enable_recursion

• FO Solution:

```
 \left| \begin{array}{l} \mathtt{member}(A,B) \text{:-} \ \mathtt{head}(A,B). \\ \mathtt{member}(A,B) \text{:-} \ \mathtt{tail}(A,C), \mathtt{member}(C,B). \end{array} \right|
```

- HO Optimal Background: head, tail, empty, any.
- HO Optimal Parameters:

max_vars: 2 max_body: 2 max_clause: 2 Switches:

$$\begin{aligned} & \texttt{member}(A,B)\text{:- any_a}(A,B).\\ & \texttt{any_p_a}(A,B)\text{:- head}(A,B). \end{aligned}$$

Check whether a list is sorted (non-decreasing)

- FO Background: empty, geq, head, tail, suc, zero, pred.
- FO Solution:

```
\begin{split} \mathsf{sorted}(A) &:= \mathsf{tail}(A,B), \mathsf{empty}(B). \\ &\mathsf{sorted}(A) &:= \mathsf{head}(A,D), \mathsf{tail}(A,B), \\ &\mathsf{head}(B,C), \mathsf{geq}(C,D), \\ &\mathsf{sorted}(B). \end{split}
```

- HO Background: empty, geq, head, tail, suc, zero, pred, fold.
- HO Solution:

```
\begin{split} \mathbf{sorted}(A)\text{:-}\ \mathbf{zero}(C), \mathbf{fold\_a}(C,A,B). \\ \mathbf{fold\_p\_a}(A,B,C)\text{:-}\ \mathbf{pred}(B,C), \mathbf{geq}(C,A). \end{split}
```

C.7 reverse/2

Reverse a list

- FO Background: empty, app, head, tail.
- FO Parameters:

max_vars: 5 max_body: 5 max_clause: 2

Switches: enable_recursion

• FO Solution:

```
\begin{aligned} \texttt{reverse}(A,B)&:=\texttt{empty}(A),\texttt{empty}(B).\\ \texttt{reverse}(A,B)&:=\texttt{head}(A,D),\texttt{tail}(A,E),\\ \texttt{reverse}(E,C),\texttt{app}(C,D,B). \end{aligned}
```

- HO optimal Background: empty, head, tail, fold.
- HO optimal Parameters:

max_vars: 5 max_body: 5 max_clause: 2

Switches: enable_recursion

• HO Solution:

```
\begin{split} \texttt{reverse}(A,B)&:=\texttt{empty}(C),\\ &\texttt{fold\_a}(C,A,B).\\ \texttt{fold\_p\_a}(A,B,C)&:=\texttt{head}(C,B),\texttt{tail}(C,A). \end{split}
```

C.8 dropLast/2

given a list of lists, drop the last element from each list

- FO Background:con, tail, empty, reverse
- FO Parameters:

max_vars: 8 max_body: 6 max_clause: 2

Switches: enable_recursion

• FO Solution:

```
 \begin{aligned} \mathsf{dropLast}(A,B)\text{:-}\ \mathsf{empty}(A), \mathsf{empty}(B). \\ \mathsf{dropLast}(A,B)\text{:-}\ \mathsf{con}(A,B,C), \mathsf{reverse}(C,E), \\ \mathsf{tail}(E,F), \mathsf{reverse}(F,G), \\ \mathsf{con}(B,G,H), \mathsf{dropLast}(D,H). \end{aligned}
```

- HO optimal Background: reverse, tail, map
- HO optimal Parameters:

max_vars: 5 max_body: 3 max_clause: 3 Switches:

• HO Solution:

```
 \begin{vmatrix} \mathsf{dropLast}(A,B) \text{:-} & \mathsf{map}(\mathsf{map\_p\_a},A,B) \text{.} \\ & \mathsf{map\_p\_a}(A,B) \text{:-} & \mathsf{reverse}(A,C), \mathsf{tail}(C,D), \\ & \mathsf{reverse}(D,B) \text{.} \end{vmatrix}
```

C.9 encryption/2

convert characters to integers, add 2 to each of them, then convert them back

- FO Background: pred, char_to_int, int_to_char, head, con, tail, empty
- FO Parameters:

max_vars: 9 max_body: 10 max_clause: 2

Switches: enable_recursion

```
\begin{tabular}{ll} encryption(A,B):=&empty(A),empty(B).\\ encryption(A,B):=&head(A,C),head(A,D),\\ &cons(B,E,F),\\ &char\_to\_int(C,G),pred(G,H),\\ &pred(H,I),int\_to\_char(I,E),\\ &encryption(D,F).\\ \end{tabular}
```

- HO optimal Background: char_to_int, int_to_char, pred, map
- HO optimal Parameters:

max_vars: 5 max_body: 4 max_clause: 2 Switches:

• HO Solution:

```
\begin{split} & \texttt{encryption}(A,B)\text{:-}\,\texttt{map\_a}(A,B).\\ & \texttt{map\_p\_a}(A,B)\text{:-}\,\texttt{char\_to\_int}(A,E),\texttt{pred}(E,D),\\ & \texttt{pred}(D,C),\texttt{int\_to\_char}(C,B). \end{split}
```

C.10 repeatN/3

Construct a list made of the input list argument repeated n times.

- FO Background: empty, app1, zero, pred.
- FO Parameters:

max_vars: 5 max_body: 3 max_clause: 2

Switches: enable_recursion, non_datalog,

allow_singletons

• FO Solution:

```
\begin{split} \texttt{repeatN}(A,B,C)\text{:-}&\,\texttt{empty}(C),\texttt{zero}(B).\\ \texttt{repeatN}(A,B,C)\text{:-}&\,\texttt{pred}(B,E),\texttt{repeatN}(A,E,D),\\ &\,\texttt{app2}(D,A,C). \end{split}
```

- HO Optimal Background: empty, app, zero, pred, iterate.
- HO Optimal Parameters:

max_vars: 4 max_body: 2 max_clause: 2 Switches:

HO Solution:

```
\begin{split} \texttt{repeatN}(A,B,C)\text{:-}&\,\texttt{empty}(D),\\ &\,\texttt{iterate\_a}(D,B,C,A).\\ \\ &\,\texttt{iterate\_p\_a}(A,B,C)\text{:-}&\,\texttt{app}(A,C,B). \end{split}
```

C.11 rotateN/3

Move first element of a list to the end n times.

- FO Background: empty, app2, head, tail, less0, zero, eq, pred.
- FO Parameters:

max_vars: 7 max_body: 6 max_clause: 2

Switches: enable_recursion

• FO Solution:

- HO Optimal Background: empty, app2, head, tail, iterate, zero, eq, pred.
- HO Optimal Parameters:

max_vars: 4 max_body: 3 max_clause: 2 Switches:

HO Solution:

```
\begin{split} \operatorname{rotateN}(A,B,C)\text{:-iterate\_a}(B,A,C).\\ \operatorname{iterate\_p\_a}(A,B,C)\text{:-head}(A,D), \operatorname{tail}(A,C),\\ \operatorname{app2}(C,D,B). \end{split}
```

C.12 allSeqN/2

Construct a list of lists, consisting of all sequences from 1 to i with i < n.

- FO Background:suc, pred, zero, head, head2, tail, tail2, empty, less0
- FO Parameters:

max_vars: 7 max_body: 7 max_clause: 5

Switches: enable_recursion, enable_ho_recursion,

non_datalog, allow_singletons

• FO Solution:

```
\begin{split} \text{allSeqN}(A,B)\text{:-}&\,\operatorname{zero}(C), \text{h\_a}(C,A,D), \text{g\_a}(D,B).\\ \text{g\_p\_a}(A,B)\text{:-}&\,\operatorname{empty}(A), \text{empty}(B).\\ \text{g\_p\_a}(A,B)\text{:-}&\,\operatorname{head}(A,C), \text{tail}(A,F), \text{zero}(E),\\ \text{h\_p\_a}(E,C,D), \text{g\_p\_a}(F,G),\\ \text{tail2}(B,G), \text{head2}(B,D).\\ \text{h\_p\_a}(A,B,C)\text{:-}&\,\operatorname{zero}(B), \text{empty}(C).\\ \text{h\_p\_a}(A,B,C)\text{:-}&\,\operatorname{pred}(B,D), \text{suc}(A,E), \text{less0}(B),\\ \text{h\_p\_a}(E,D,F), \text{tail2}(C,F),\\ \text{head}(C,E). \end{split}
```

- HO Optimal Background: suc, zero, ite, map
- HO Optimal Parameters:

max_vars: 4 max_body: 3 max_clause: 3 Switches: • HO Solution:

```
\begin{split} \operatorname{allSeqN}(A,B)\text{:-}\operatorname{zero}(C), \operatorname{iterate\_a}(C,A,D), \\ \operatorname{map\_a}(D,B). \\ \operatorname{iterate\_p\_a}(A,B)\text{:-}\operatorname{suc}(A,B). \\ \operatorname{map\_p\_a}(A,B)\text{:-}\operatorname{zero}(C), \operatorname{iterate\_a}(C,A,B). \end{split}
```

C.13 dropLastK/3

Given a list of lists, drop the last k elements from each list

- FO Background: pred, eq, zero, tail, reverse, cons, empty
- FO Parameters:

max_vars: 8 max_body: 6 max_clause: 4

Switches: enable_recursion, enable_ho_recursion

• FO Solution:

```
 \begin{split} \mathsf{dropLastK}(A,B,C) &:= \mathsf{zero}(A), \mathsf{eq}(C,B), \mathsf{eq}(B,C). \\ \mathsf{dropLastK}(A,B,C) &:= \mathsf{pred}(A,E), \mathsf{g\_a}(B,D), \\ \mathsf{dropLastK}(E,D,C). \\ \mathsf{g\_p\_a}(A,B) &:= \mathsf{empty}(A), \mathsf{empty}(B). \\ \mathsf{g\_p\_a}(A,B) &:= \mathsf{cons}(A,C,D), \mathsf{reverse}(C,E), \\ \mathsf{tail}(E,F), \mathsf{reverse}(F,G), \\ \mathsf{cons}(B,G,H), \mathsf{g\_p\_a}(D,H). \end{split}
```

- HO Optimal Background: pred, eq, zero, tail, reverse.map
- HO Optimal Parameters:

max_vars: 5 max_body: 3 max_clause: 3

Switches: enable_recursion

· HO Solution:

```
 \begin{split} & \operatorname{dropLastK}(A,B,C)\text{:-}\operatorname{eq}(C,B),\operatorname{zero}(A). \\ & \operatorname{dropLastK}(A,B,C)\text{:-}\operatorname{pred}(A,E),\operatorname{map}(B,D), \\ & \operatorname{dropLastK}(E,D,C). \\ & \operatorname{map\_p\_a}(A,B)\text{:-}\operatorname{reverse}(A,D),\operatorname{tail}(D,C), \\ & \operatorname{reverse}(C,B). \end{split}
```

C.14 firstHalf/2

Check whether the second argument is equal to the first half of the first argument

- FO Background: head, tail, cons, app, true, empty, suc, pred, zero, len
- FO Parameters:

max_vars: 8 max_body: 7 max_clause: 3 Switches:

• FO Solution:

```
\begin{split} & \texttt{firstHalf}(A,B)\text{:-} \ \texttt{len}(A,C), \texttt{g\_a}(A,C,B). \\ & \texttt{g\_p\_a}(A,B,C)\text{:-} \ \texttt{true}(A), \texttt{zero}(B), \texttt{empty}(C). \\ & \texttt{g\_p\_a}(A,B,C)\text{:-} \ \texttt{pred}(B,D), \texttt{head}(A,F), \\ & \texttt{tail}(A,G), \texttt{pred}(D,E), \\ & \texttt{g\_a}(G,E,H), \texttt{cons}(C,F,H). \end{split}
```

- HO Optimal Background: head, tail, pred, len, iterateStep
- HO Optimal Parameters:

max_vars: 3 max_body: 2 max_clause: 3 Switches:

• HO Solution:

```
\begin{aligned} & \mathsf{firstHalf}(A,B)\text{:-} \mathsf{len}(A,C), \\ & & \mathsf{iterateStep\_a}(A,C,B). \\ & \mathsf{iterateStep\_p\_a}(A,B)\text{:-} \mathsf{pred}(A,C), \mathsf{pred}(C,B). \\ & \mathsf{iterateStep\_q\_a}(A,B,C)\text{:-} \mathsf{head}(A,B), \mathsf{tail}(A,C). \end{aligned}
```

C.15 lastHalf/2

check whether the second argument is equal to the last half of the first argument

- FO Background: empty, tail, front, last, app
- FO Parameters:

max_vars: 6 max_body: 5 max_clause: 3 Switches:

• FO Solution:

```
\label{eq:lastHalf} \begin{split} & \mathsf{lastHalf}(A,B)\text{:-}\ \mathsf{empty}(B), \mathsf{empty}(A). \\ & \mathsf{lastHalf}(A,B)\text{:-}\ \mathsf{tail}(A,B), \mathsf{empty}(B). \\ & \mathsf{lastHalf}(A,B)\text{:-}\ \mathsf{front}(A,C), \mathsf{last}(A,F), \\ & \mathsf{tail}(C,E), \mathsf{lastHalf}(E,D), \\ & \mathsf{app}(D,F,B). \end{split}
```

- HO Optimal Background: front, app2, reverse, empty, caseList
- HO optimal Parameters:

max_vars: 5 max_body: 3 max_clause: 4 Switches:

```
\label{eq:lastHalf} \begin{split} \mathsf{lastHalf}(A,B) &:- \mathsf{reverse}(A,C), \\ &\quad \mathsf{caseList\_a}(C,B). \\ \mathsf{caseList\_p\_a}(A) &:- \mathsf{empty}(A). \\ \mathsf{caseList\_r\_a}(A,B,C) &:- \mathsf{front}(B,E), \\ &\quad \mathsf{caseList\_a}(E,D), \\ &\quad \mathsf{app}(D,A,C). \end{split}
```

C.16 of 1And 2/1

check whether a list consists only of $1\mathrm{s}$ and 2

• FO Background: suc, zero, cons1, empty.

• FO Parameters:

max_vars: 5 max_body: 5 max_clause: 3

Switches: enable_recursion

• FO Solution:

```
\begin{array}{c} \texttt{of1And2}(A) \text{:-} \ \texttt{empty}(A). \\ \texttt{of1And2}(A) \text{:-} \ \texttt{cons1}(A,B,C), \texttt{zero}(D), \texttt{suc}(D,B), \\ \texttt{of1And2}(B). \\ \texttt{of1And2}(A) \text{:-} \ \texttt{cons1}(A,B,C), \texttt{zero}(D), \texttt{suc}(D,E), \\ \texttt{suc}(E,B), \texttt{of1And2}(B). \end{array}
```

- HO Optimal Background: suc, zero, cons, empty. try.
- HO Optimal Parameters:

max_vars: 3 max_body: 4 max_clause: 3

Switches: enable recursion

HO Solution:

```
\begin{split} & \mathsf{of1And2}(A)\text{:-}\;\mathsf{empty}(A). \\ & \mathsf{of1And2}(A)\text{:-}\;\mathsf{cons}(A,C,B),\mathsf{try\_a}(C), \\ & \mathsf{of1And2}(B). \\ & \mathsf{try\_p\_a}(A)\text{:-}\;\mathsf{zero}(B),\mathsf{suc}(B,A). \\ & \mathsf{try\_q\_a}(A)\text{:-}\;\mathsf{zero}(B),\mathsf{suc}(B,C),\mathsf{suc}(C,A). \end{split}
```

C.17 isPalindrome/1

Check whether a list is a palindrome (the same read normally and in reverse)

- FO Background: head, tail, empty, front, last.
- FO Parameters:

max_vars: 4 max_body: 5 max_clause: 3

Switches: enable_recursion

• FO Solution:

```
\begin{split} & \texttt{isPalindrome}(A)\text{:-} \texttt{empty}(A). \\ & \texttt{isPalindrome}(A)\text{:-} \texttt{front}(A,B), \texttt{empty}(B). \\ & \texttt{isPalindrome}(A)\text{:-} \texttt{head}(A,D), \texttt{tail}(A,B), \\ & \texttt{front}(B,C), \texttt{last}(B,D), \\ & \texttt{isPalindrome}(C). \end{split}
```

- HO Optimal Background: empty, any, front, last, condList.
- HO Optimal Parameters:

max_vars: 3 max_body: 3 max_clause: 3 Switches:

• HO Solution:

```
\begin{split} \textbf{isPalindrome}(A) &:- \textbf{condlist\_a}(A). \\ \textbf{condlist\_p\_a}(A, B) &:- \textbf{any}(A), \textbf{empty}(B). \\ \textbf{condlist\_p\_a}(A, B) &:- \textbf{last}(B, A), \textbf{front}(B, C), \\ \textbf{condlist\_a}(C). \end{split}
```

C.18 depth/2

find the depth of a tree

- FO Background: eq, children, zero, suc, empty, tail, head, max, g_a
- FO Parameters:

max_vars: 7 max_body: 5 max_clause: 3

Switches: enable_recursion,enable_ho_recursion

• FO Solution:

```
\begin{split} \operatorname{depth}(A,B)\text{:-}&\operatorname{zero}(D),\operatorname{children}(A,C),\\ &\operatorname{g\_a}(D,C,E),\operatorname{suc}(E,B).\\ \operatorname{g\_p\_a}(A,B,C)\text{:-}&\operatorname{eq}(A,C),\operatorname{empty}(B).\\ \operatorname{g\_p\_a}(A,B,C)\text{:-}&\operatorname{tail}(B,E),\operatorname{head}(B,D),\operatorname{depth}(D,F),\\ &\operatorname{max}(A,F,G),\operatorname{g\_p\_a}(G,E,C). \end{split}
```

- HO Optimal Background: children, max, zero, suc, fold
- HO Optimal Parameters:

max_vars: 5 max_body: 4 max_clause: 2

Switches: enable_recursion

```
\begin{split} \operatorname{depth}(A,B)\text{:-}\operatorname{children}(A,C),\operatorname{zero}(E),\\ \operatorname{fold\_a}(E,C,D),\operatorname{suc}(D,B).\\ \operatorname{fold\_p\_a}(A,B,C)\text{:-}\operatorname{depth}(B,D),\operatorname{max}(D,A,C). \end{split}
```

C.19 isBranch/2

check whether a given list is a branch of the tree

• FO Background: children, root, head, head2, tail, q_a

• FO Parameters:

max_vars: 5 max_body: 5 max_clause: 4

Switches: enable_recursion,enable_ho_recursion

• FO Solution:

```
\label{eq:isBranch} \begin{split} \mathsf{isBranch}(A,B) &:- \mathsf{tail}(B,D), \mathsf{children}(A,D), \\ & \mathsf{root}(A,C), \mathsf{head2}(B,C). \\ \mathsf{isBranch}(A,B) &:- \mathsf{root}(A,C), \mathsf{children}(A,E), \\ & \mathsf{tail}(B,D), \mathsf{g\_a}(E,D), \\ & \mathsf{head2}(B,C). \\ & \mathsf{g\_p\_a}(A,B) &:- \mathsf{head}(A,C), \mathsf{isBranch}(C,B). \\ & \mathsf{g\_p\_a}(A,B) &:- \mathsf{tail}(A,C), \mathsf{g\_p\_a}(C,B). \end{split}
```

- HO Optimal Background: head, tail, empty, any, caseTree
- HO Optimal Parameters:

max_vars: 4 max_body: 3 max_clause: 4

Switches: enable_recursion

• HO Solution:

```
\label{eq:isBranch} \begin{split} & \text{isBranch}(A,B)\text{:-} \ \text{casetree\_a}(A,B).\\ & \text{casetree\_p\_a}(A,B)\text{:-} \ \text{tail}(B,C), \text{empty}(C),\\ & \text{head}(B,A).\\ & \text{any\_p\_a}(A,B)\text{:-} \ \text{tail}(B,C), \text{casetree\_a}(A,C).\\ & \text{casetree\_q\_a}(A,B,C)\text{:-} \ \text{head}(C,A), \text{any\_a}(B,C). \end{split}
```

C.20 isSubTree/2

check whether the second argument is a sub-tree of the first argument.

- FO Background: head, tail, empty, children, g_a.
- FO Parameters:

max_vars: 4 max_body: 2 max_clause: 4

Switches: enable_recursion,enable_ho_recursion

• FO Solution:

```
\begin{split} & \texttt{g\_p\_a}(A,B)\text{:-}\ \mathsf{children}(B,A). \\ & \texttt{g\_p\_a}(A,B)\text{:-}\ \mathsf{head}(A,B). \\ & \texttt{g\_p\_a}(A,B)\text{:-}\ \mathsf{tail}(A,C), \texttt{g\_a}(C,B). \\ & \mathsf{isSubTree}(A,B)\text{:-}\ \mathsf{children}(A,C), \texttt{g\_p\_a}(C,B). \end{split}
```

- HO Optimal Background: children, any, eq1.
- HO Optimal Parameters:

max_vars: 3 max_body: 2 max_clause: 3

Switches: enable_recursion

• HO Solution:

```
\begin{split} \texttt{isSubTree}(A,B)&:= \texttt{eq}(A,B).\\ \texttt{isSubTree}(A,B)&:= \texttt{children}(A,C),\\ \texttt{any\_a}(C,B).\\ \texttt{any\_p\_a}(A,B)&:= \texttt{isSubTree}(A,B). \end{split}
```

C.21 addN/3

add n to every element of a list (with no addition predicate in the background knowledge)

- FO Background: suc, pred, zero, eq, less0, cons, cons2, empty, g_a
- FO Parameters:

max_vars: 6 max_body: 4 max_clause: 4

Switches: enable_recursion,enable_ho_recursion

• FO Solution:

```
\label{eq:addN} \begin{array}{l} \operatorname{addN}(A,B,C)\text{:-}\operatorname{zero}(A),\operatorname{eq}(B,C).\\ \operatorname{addN}(A,B,C)\text{:-}\operatorname{lessO}(A),\operatorname{g_a}(B,D),\\ \operatorname{pred}(A,E),\operatorname{addN}(E,D,C).\\ \operatorname{g_p_a}(A,B)\text{:-}\operatorname{empty}(B),\operatorname{empty}(A).\\ \operatorname{g_p_a}(A,B)\text{:-}\operatorname{cons}(A,C,D),\operatorname{suc}(C,E),\\ \operatorname{g_p_a}(D,F),\operatorname{cons2}(B,E,F). \end{array}
```

- HO Optimal Background: suc, eq, caseint, map
- HO Optimal Parameters:

max_vars: 4 max_body: 2 max_clause: 4 Switches:

```
\begin{array}{c} \operatorname{addN}(A,B,C)\text{:-}\operatorname{caseint\_a}(A,B,C).\\ \operatorname{map\_p\_a}(A,B)\text{:-}\operatorname{suc}(A,B).\\ \operatorname{caseint\_q\_a}(A,B,C)\text{:-}\operatorname{map\_a}(B,D),\\ \operatorname{caseint\_a}(A,D,C).\\ \operatorname{caseint\_p\_a}(A,B)\text{:-}\operatorname{eq}(A,B). \end{array}
```

Multiply two numbers (with no addition predicate in the background knowledge).

- FO Background: suc, pred, eq, zero, less0, g_a, h_a.
- FO Parameters:

max_vars: 6 max_body: 4 max_clause: 5

Switches: allow_singletons, non_datalog.

• FO Solution:

```
\begin{split} & \mathsf{mulFromSuc}(A,B,C)\text{:-}\,\mathsf{zero}(D), \mathsf{g\_a}(D,A,C,B). \\ & \mathsf{g\_p\_a}(A,B,C,D)\text{:-}\,\mathsf{eq}(A,C), \mathsf{zero}(B). \\ & \mathsf{g\_p\_a}(A,B,C,D)\text{:-}\,\mathsf{pred}(B,E), \mathsf{less0}(B), \\ & & \mathsf{h\_a}(A,D,F), \mathsf{g\_a}(F,E,C,D). \\ & \mathsf{h\_p\_a}(A,B,C)\text{:-}\,\mathsf{eq}(A,C), \mathsf{zero}(B). \\ & & \mathsf{h\_p\_a}(A,B)\text{:-}\,\mathsf{suc}(A,E), \mathsf{less0}(B), \\ & & \mathsf{pred}(B,D), \mathsf{h\_a}(E,D,C). \end{split}
```

- HO Optimal Background: suc, zero1, iterate.
- HO Optimal Parameters:

max_vars: 4 max_body: 2 max_clause: 3 Switches:

```
\begin{split} & \texttt{mulFromSuc}(A,B)\text{:-}\, \texttt{zero1}(D), \\ & \texttt{iterate\_a}(D,A,C,B). \\ & \texttt{iterate\_p\_a}(A,B,C)\text{:-}\, \texttt{iterate\_a}(A,C,B). \\ & \texttt{iterate\_p\_a}(A,B)\text{:-}\, \texttt{suc}(A,B). \end{split}
```