Polynomial time computable functions over the reals characterized using discrete ordinary differential equations

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Abstract

The class of functions from the integers to the integers computable in polynomial time has been characterized recently using discrete ordinary differential equations (ODE), also known as finite differences. In the framework of ordinary differential equations, this is very natural to try to extend the approach to classes of functions over the reals, and not only over the integers. Recently, an extension of previous characterization was obtained for functions from the integers to the reals, but the method used in the proof, based on the existence of a continuous function from the integers to a suitable discrete set of reals, cannot extend to functions from the reals to the reals, as such a function cannot exist for clear topological reasons.

In this article, we prove that this is indeed possible to provide an elegant and simple algebraic characterization of functions from the reals to the reals: we provide a characterization of such functions as the smallest class of functions that contains some basic functions, and that is closed by composition, linear length ODEs, and a natural effective limit schema. This is obtained using an alternative proof technique based on the construction of specific suitable functions defined recursively, and a barycentric method.

Furthermore, we also extend previous characterizations in several directions: First, we prove that there is no need of multiplication. We prove a normal form theorem, with a nice side effect related to formal neural networks. Indeed, given some fixed error and some polynomial time t(n), our settings produce effectively some neural network that computes the function over its domain with the given precision, for any t(n)-polynomial time computable function f.

As far as we know, this is in particular the first time that polynomial time computable functions over the reals are characterized in an algebraic way, in a so simple discrete manner. Furthermore, we believe that no relation between

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polynomial time computable functions over the reals (in the sense of computable analysis) and formal neural networks has been obtained before. We believe that these characterizations share the above mentioned benefits of these approaches. In particular, compared to already existing characterizations of polynomial time (over the integers or reals), in the so-called field of implicit complexity, this is obtained without any explicit bound on the growth of function, and this is based on a very natural consideration in the framework of differential equations, namely assuming *linear* ordinary differential equations.

This also points out the possibility of using tools such as changes of variables to capture computability and complexity measures, or as a tool for programming, even over the discrete. We also believe that these results show some lights on recent characterizations of complexity classes based on (classical continuous) ordinary differential equations, providing possibly really simpler proof techniques and explanations.

1 Introduction

Ordinary differential equations can be considered as a universal language for modeling various phenomena in experimental sciences. They have been studied intensively in the last centuries, and there mathematical theory are well-understood: see e.g. [Arn78; BR89; CL55]. A series of recent articles, initially motivated by understanding analog models of computations have established some characterizations of classical discrete complexity classes from computability theory using ordinary differential equations. In particular, it has been proved that the length of trajectories provides a robust notion of time complexity that corresponds to classical time complexity for models such as Turing machines [BGP16; BGP17]: See [BP20] for most recent survey.

Unfortunately, while the above mentioned results are easy to state, their proofs are rather highly technical and mixing considerations about approximations, control of errors, and various constructions to emulate in a continuous fashion some discrete processes. There have been some recent attempts to go to simpler constructions in order to simplify their programming [Bou22], as these constructions have recently lead to solve various open problems, with very visible awarded outcomes: This includes the proof of the existence of a universal ordinary differential equation [BP17], the proof of the Turing completeness of chemical reactions [Fag+17], or hardness of problems related to dynamical systems [GZ18].

Initially motivated by trying to go to simpler proofs, several authors have considered their discrete counterparts, that are called discrete ODEs, also known as difference equations [BD19; BD18]. The basic principle is, for a function $\mathbf{f}(x)$, to consider its discrete derivative defined as $\Delta \mathbf{f}(x) = \mathbf{f}(x+1) - \mathbf{f}(x)$. As in these articles, we intentionally also write $\mathbf{f}'(x)$ for $\Delta \mathbf{f}(x)$ to help to understand statements with respect to their classical continuous counterparts. It turns out that this provided some algebraic characterizations of complexity classes, but that a key difference between the two frameworks is that there is no simple expression for the derivative of the composition of functions in the discrete settings, and hence that actually both approaches have some common aspects, but are at the end not yet directly connected.

Notice that the theory of discrete ordinary differential equations is widely used in some contexts such as function approximation [Gel71] and in *discrete calculus* [Gra+89; Gle05; IAB09; Lau] for combinatorial analysis, but is rather unknown. Actually, the similarities between discrete and continuous statements have been historically observed, under the terminology of *umbral* or *symbolic calculus* as early as in the 19th century, even if not yet fully understood, and often rediscovered in many fields, with various names.

Following [BD19], while the underlying computational content of finite differences theory is clear and has been pointed out many times, no fundamental connections with algorithms and complexity had been formally established before [BD19; BD18], where it was proved that many complexity and computability classes can be characterized algebraically using discrete ODEs.

In the context of algebraic classes of functions, a classical notation is the following: Call *operation* an operator that takes finitely many functions, and returns some new function defined from them. Then $[f_1, f_2, \ldots, f_k; op_1, op_2, \ldots, op_\ell]$ denotes the smallest set of functions containing functions f_1, f_2, \ldots, f_k that is closed under operations $op_1, op_2, \ldots op_\ell$. Call *discrete function* a function of type $f: S_1 \times \cdots \times S_d \to S_1' \times \ldots S_{d'}'$, where each S_i, S_i' is either \mathbb{N} , \mathbb{Z} . Write **FPTIME** for the class of functions computable in polynomial time. A main result of [BD19; BD18] is the following (\mathbb{LDL} stands for linear derivation on length):

Theorem 1 ([BD19]). For discrete functions, we have $\mathbb{LDL} = \mathbf{FPTIME}$ where $\mathbb{LDL} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \times, sg(x)]$; composition, linear length ODE].

That is to say, \mathbb{LDL} (and hence **FPTIME** for functions over the integers) is the smallest class of functions that contains the constant functions $\mathbf{0}$ and $\mathbf{1}$, the projections π_i^k , the length function $\ell(x)$ (that maps an integer to the length of its binary representation), the addition function x+y, the subtraction function x-y, the multiplication function $x \times y$ (that we will also often denote $x \cdot y$), the sign function $\mathrm{sg}(x)$ and closed under composition (when defined) and linear length-ODE scheme: The linear length-ODE scheme basically (a formal definition is provided in Definition 4) corresponds to define functions from linear ODEs with respect to derivation along the length of the argument, that is to say of the form $\frac{\partial \mathbf{f}(x,\mathbf{y})}{\partial \ell} = \mathbf{A}[\mathbf{f}(x,\mathbf{y}),x,\mathbf{y}] \cdot \mathbf{f}(x,\mathbf{y}) + \mathbf{B}[\mathbf{f}(x,\mathbf{y}),x,\mathbf{y}]$. Here, in the above description, we use the notation $\frac{\partial \mathbf{f}(x,\mathbf{y})}{\partial \ell}$, which corresponds to the derivation of \mathbf{f} along the length function: Given some function $\mathcal{L}: \mathbb{N}^{p+1} \to \mathbb{Z}$, and in particular for the case of where $\mathcal{L}(x,\mathbf{y}) = \ell(x)$,

$$\frac{\partial \mathbf{f}(x, \mathbf{y})}{\partial \mathcal{L}} = \frac{\partial \mathbf{f}(x, \mathbf{y})}{\partial \mathcal{L}(x, \mathbf{y})} = \mathbf{h}(\mathbf{f}(x, \mathbf{y}), x, \mathbf{y}) \tag{1}$$

is a formal synonym for $\mathbf{f}(x+1,\mathbf{y}) = \mathbf{f}(x,\mathbf{y}) + (\mathcal{L}(x+1,\mathbf{y}) - \mathcal{L}(x,\mathbf{y})) \cdot \mathbf{h}(\mathbf{f}(x,\mathbf{y}),x,\mathbf{y}).$

NB 1. This concept, introduced in [BD19; BD18], is motivated by the fact that the latter expression is similar to classical formula for classical continuous ODEs:

$$\frac{\delta f(x, \mathbf{y})}{\delta x} = \frac{\delta \mathcal{L}(x, \mathbf{y})}{\delta x} \cdot \frac{\delta f(x, \mathbf{y})}{\delta \mathcal{L}(x, \mathbf{y})},$$

and hence this is similar to a change of variable. Consequently, a linear length-ODE is basically a linear ODE over variable t, once the change of variable $t = \ell(x)$ is done.

Call *continuous function* a function of type $f: S_1 \times \cdots \times S_d \to S'_1 \times \dots S'_{d'}$, where each S_i, S'_i is either \mathbb{R} , \mathbb{N} or \mathbb{Z} . Considering that $\mathbb{N} \subset \mathbb{R}$, most of the basic functions and operations in this characterization (for example, $+, -, \dots$) have a clear meaning over the reals, i.e. are continuous functions. As ordinary differential equations are naturally living over the reals, it is rather natural to understand if one may go to computation theory for functions over the reals. We consider here computability and complexity over the reals in the most classical sense, that is to say, computable analysis (see e.g. [Wei00]). So, a very natural question is to understand whether we can characterize **FPTIME** for continuous functions, and not only discrete functions. This is one of the problem we solve in this article.

NB 2. As in [BB22], clearly, we can consider $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R}$, but as functions may have different type of outputs, composition is an issue. We simply admit that composition may not be defined in some cases. In other words, we consider that composition is a partial operator: for example, given $f: \mathbb{N} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$, the composition of g and f is defined as expected, but f cannot be composed with a function such as $h: \mathbb{N} \to \mathbb{N}$.

Actually, there has been a first attempt in [BB22], but the authors succeeded there only to characterize in an algebraic manner functions from the integers to the reals, while this would be more natural to talk about (or at least cover) functions from the reals to the reals ($\|.\|$ stands for the sup-norm): Namely, [BB22] considers $\mathbb{LDL}^{\bullet} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \times, \overline{\operatorname{cond}}(x), \frac{x}{2}; composition, linear length ODE]$, where $\ell : \mathbb{N} \to \mathbb{N}$

is the length function, mapping some integer to the length of its binary representation, $\frac{x}{2}$: $\mathbb{R} \to \mathbb{R}$ is the function that divides by 2, and all other basic functions are defined exactly as for \mathbb{LDL} , but considered here as functions from the reals to reals. $\overline{\operatorname{cond}}(x): \mathbb{R} \to \mathbb{R}$ is some piecewise affine function that takes value 1 for $x > \frac{3}{4}$ and 0 for $x < \frac{1}{4}$, and continuous piecewise affine: In particular, its restrictions to the integers is the function $\operatorname{sg}(x)$ considered in \mathbb{LDL} .

Theorem 2 ([BB22]). A function $\mathbf{f}: \mathbb{N}^d \to \mathbb{R}^{d'}$ is computable in polynomial time if and only if there exists $\tilde{\mathbf{f}}: \mathbb{N}^{d+1} \to \mathbb{R}^{d'} \in \mathbb{LDL}^{\bullet}$ such that for all $\mathbf{m} \in \mathbb{N}^d$, $n \in \mathbb{N}$, $\|\tilde{\mathbf{f}}(\mathbf{m}, 2^n) - \mathbf{f}(\mathbf{m})\| \leq 2^{-n}$.

A point is that their proof method is using some functions mapping in a continuous manner the integers into some suitable subsets of reals (namely the Cantor-like set \mathscr{I} corresponding to the reals whose radix 4 expansion is made of only 1 and 3). They observe this cannot be expended to functions over the reals, as such a continuous mapping $\mathbb{R} \to \mathscr{I}$ cannot exist, and leave open the question whether a similar result can be established for functions from the reals to the reals.

In the current article, we solve the issue, by using an alternative proof method (but using some of the constructions from [BB22] that we extend in several directions): First we observe that multiplication is not needed: Consider

$$\mathbb{LDL}^{\oslash} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \overline{\operatorname{cond}}(x), \frac{x}{2}; composition, linear \ length \ ODE],$$

that is \mathbb{LDL}^{\bullet} but without multiplication (when *I* is some interval, we write $\mathbf{x} \in I$ when this holds componentwise).

Theorem 3 (Main theorem 1, functions over the reals). A continuous function $\mathbf{f}: \mathbb{R}^d \to \mathbb{R}^{d'}$ is computable in polynomial time if and only if there exists $\tilde{\mathbf{f}}: \mathbb{R}^d \times \mathbb{N}^2 \to \mathbb{R}^{d'} \in \mathbb{LDL}^{\otimes}$ such that for all $\mathbf{x} \in \mathbb{R}^d$, $X \in \mathbb{N}$, $\mathbf{x} \in [-2^X, 2^X]$, $n \in \mathbb{N}$, $\|\tilde{\mathbf{f}}(\mathbf{x}, 2^X, 2^n) - \mathbf{f}(\mathbf{x})\| \leq 2^{-n}$.

This actually works for general continuous functions (Theorem 2 is a special case):

Theorem 4 (Main theorem 1, general continuous functions). A function $\mathbf{f} : \mathbb{R}^d \times \mathbb{N}^{d''} \to \mathbb{R}^{d'}$ is computable in polynomial time iff there exists $\tilde{\mathbf{f}} : \mathbb{R}^d \times \mathbb{N}^{d''+2} \to \mathbb{R}^{d'} \in \mathbb{LDL}^{\odot}$ such that for all $\mathbf{x} \in \mathbb{R}^d$, $X \in \mathbb{N}$, $\mathbf{x} \in [-2^X, 2^X]$, $\mathbf{m} \in \mathbb{N}^{d''}$, $n \in \mathbb{N}$, $\|\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{m}, 2^X, 2^n) - \mathbf{f}(\mathbf{x}, \mathbf{m})\| \le 2^{-n}$.

This has also some strong links with formal neural networks:

Theorem 5 (Main theorem 3, Formal neural networks). Given some function $f : \mathbb{R}^d \to \mathbb{R}^{d'}$, for some given error and some polynomial time t(n), we can efficiently produce some formal neural network that computes the function over its domain with the given precision, for any polynomial time computable t(n) function f.

Furthermore, this improves the characterization of [BD19]. Indeed, given a function $\mathbf{f}: \mathbb{R}^d \to \mathbb{R}^{d'}$, we denote $\mathrm{DP}(f)$ for its discrete part: This is the function from $\mathbb{N}^d \to \mathbb{N}^{d'}$ whose value in $\mathbf{n} \in \mathbb{N}^d$ is the (componentwise) integer part of $\mathbf{f}(\mathbf{n})$. Given a class $\mathscr C$ of such functions, we write $\mathrm{DP}(\mathscr C)$ for the class of the discrete parts of the functions of $\mathscr C$.

Theorem 6. $DP(\mathbb{LDL}^{\odot}) = FPTIME$.

We can also extend the statements from [BB22]:

Definition 1 (Operation *ELim*). Given $\tilde{\mathbf{f}}: \mathbb{R}^d \times \mathbb{N}^{d''} \times \mathbb{N} \to \mathbb{R}^{d'} \in \mathbb{LDL}^{\oslash}$ such that for all $\mathbf{x} \in \mathbb{R}^d$, $X \in \mathbb{N}$, $\mathbf{x} \in [-2^X, 2^X]$, $\mathbf{m} \in \mathbb{N}^{d''}$, $n \in \mathbb{N}$, $\|\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{m}, 2^X, 2^n) - \mathbf{f}(\mathbf{x}, \mathbf{m})\| \leq 2^{-n}$, then $ELim(\tilde{\mathbf{f}})$ is the (clearly uniquely defined) corresponding function $\mathbf{f}: \mathbb{R}^d \to \mathbb{R}^{d'}$.

Theorem 7. A continuous function \mathbf{f} is computable in polynomial time if and only if all its components belong to $\overline{\mathbb{LDL}^{\odot}}$, where

$$\overline{\mathbb{LDL}^{\odot}} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \overline{cond}(x), \frac{x}{2}; composition, linear length ODE; ELim].$$
 In particular:

Theorem 8 (Main theorem 4). $\overline{\mathbb{LDL}^{\oslash}} \cap \mathbb{R}^{\mathbb{R}} = \mathbf{FPTIME} \cap \mathbb{R}^{\mathbb{R}}$

In Section 2, we recall the theory of discrete ODEs. In Section 3, we recall required concepts from computable analysis. In Section 4, we establish some properties about particular functions, required for our proofs. In Section 5, we prove that functions from LDL• are polynomial time computable, and then we prove a kind of reverse implication for functions over words. Section 6 then proves Theorems 7 and 8. Section 7 is a discussion about some of the consequences of our proofs. Appendix contains some complements, and missing proofs, and some complements on state of the art. Notice that some of the proofs in the main part of the documents are also repeated with more details in appendix.

1.1 Related work.

Recursion schemes constitutes a classical major approach of classical computability theory and, to some extent, of complexity theory. The foundational characterization of **FPTIME** due to Cobham [**cob65**], and then others based on safe recursion [BC92] or ramification ([LM93; Lei94]), or for other classes [LM95], gave birth to the very vivid field of *implicit complexity* at the interplay of logic and theory of programming: See [P C98; CK13] for monographs.

When considering continuous functions, various computability and complexity classes have been recently characterized using classical continuous ODEs: See survey [BP20].

Here we are considering an approach in between, where we characterize complexity classes using discrete ODEs. This approach is born from the attempt of [BD19; BD18] to explain some of the constructions for continuous ODEs in an easier way. At the end, both models turn out to be rather different. Indeed, a key aspect of the proofs over the continuum is based on some closure properties, based on the formula for the derivative of a composition, while there is no such equivalent for discrete ODEs. However, an unexpected side effect of the approach was the characterizations obtained in [BD19; BD18]. They provided a characterization of **FPTIME** for discrete functions that does not require to specify an explicit bound in the recursion, in contrast to Cobham's work [cob65], nor to assign a specific role or type to variables, in contrast to safe recursion or ramification [BC92; Lei95]. The characterization, like ours, happens to be very simple using only natural notions from the world of ODE. In particular, considering *linear* ordinary differential equations is something very natural in this context.

Our proof is based on some constructions of the very recent [BB22]. However, we improve several of the statements and the constructions as we avoid multiplications. This latter paper was not able to characterize functions from the reals to the reals, but only sequences of reals. Furthermore, our proof method, based on some adaptive barycenter is different, as their proof method cannot extend to the reals for topological reasons. Our

barycentric method is inspired from some constructions of [Bou+07], but once again the context of continuous ODEs and discrete ODEs is very different from the absence of a derivative formula for composition. This requires to construct explicitly some very particular functions as we do in Section 4.

Our ways of simulating Turing machines have some reminiscence of similar constructions used in other contexts such as Neural Networks [SS95; Sie99]. But with respect to all previous contexts, as far as we know, only a few papers have been devoted to characterize complexity, and even computability, classes in the sense of computable analysis. There have been some attempts the so-called \mathbb{R} -recursive functions [BP20]. For discrete schemata, we only know [Bra96] and [NRY21], focusing on computability and not complexity.

2 Some concepts from the theory of discrete ODEs

In this section, we recall some concepts and definitions from discrete ODEs, either well-known or established in [BD19; BD18; BB22]. Appendix B presents the theory with many more details.

Definition 2 ([BB22]). A \overline{cond} -polynomial expression $P(x_1,...,x_h)$ is an expression built-on $+,-,\times$ (often denoted \cdot) and \overline{cond} () functions over a set of variables $V=\{x_1,...,x_h\}$ and integer constants. The degree $\deg(x,P)$ of a term $x \in V$ in P is defined inductively as follows: $\deg(x,x)=1$ and for $x' \in V \cup \mathbb{Z}$ such that $x' \neq x$, $\deg(x,x')=0$; $\deg(x,P+Q)=\max\{\deg(x,P),\deg(x,Q)\}$; $\deg(x,P+Q)=\deg(x,P)+\deg(x,Q)$; $\deg(x,g(P))=0$. A \overline{cond} -polynomial expression P is essentially constant in x if $\deg(x,P)=0$.

Compared to the classical notion of degree in polynomial expression, all subterms that are within the scope of a sign (that is to say cond()) function contributes 0 to the degree. A vectorial function (resp. a matrix or a vector) is said to be a cond-polynomial expression if all its coordinates (resp. coefficients) are. It is said to be *essentially constant* if all its coefficients are.

Definition 3 ([BD19; BD18; BB22]). A \overline{cond} -polynomial expression $\mathbf{g}(\mathbf{f}(x,\mathbf{y}),x,\mathbf{y})$ is essentially linear in $\mathbf{f}(x,\mathbf{y})$ if it is of the form $\mathbf{g}(\mathbf{f}(x,\mathbf{y}),x,\mathbf{y}) = \mathbf{A}[\mathbf{f}(x,\mathbf{y}),x,\mathbf{y}] \cdot \mathbf{f}(x,\mathbf{y}) + \mathbf{B}[\mathbf{f}(x,\mathbf{y}),x,\mathbf{y}]$ where \mathbf{A} and \mathbf{B} are \overline{cond} -polynomial expressions essentially constant in $\mathbf{f}(x,\mathbf{y})$.

For example, the expression $P(x,y,z) = x \cdot \overline{\operatorname{cond}}((x^2-z) \cdot y) + y^3$ is essentially linear in x, essentially constant in z and not linear in y. The expression: $z + (1 - \overline{\operatorname{cond}}(x)) \cdot (1 - \overline{\operatorname{cond}}(-x)) \cdot (y-z)$ is essentially constant in x and linear in y and z.

Definition 4 (Linear length ODE [BD19; BD18]). Function \mathbf{f} is linear \mathcal{L} -ODE definable (from \mathbf{u} , \mathbf{g} and \mathbf{h}) if it corresponds to the solution of

$$f(0,\mathbf{y}) = \mathbf{g}(\mathbf{y})$$
 and $\frac{\partial \mathbf{f}(x,\mathbf{y})}{\partial \ell} = \mathbf{u}(\mathbf{f}(x,\mathbf{y}),\mathbf{h}(x,\mathbf{y}),x,\mathbf{y})$ (2)

where **u** is essentially linear in $\mathbf{f}(x, \mathbf{y})$.

A fundamental fact is that the derivation with respects to length provides a way to do a kind of change of variables: consequently, we will often define some functions by defining their value in 2^0 , and then 2^{n+1} from their value in 2^n as its corresponds to

some discrete ODE after this change of variable. We will also implicitly use that some basic functions such as $n \mapsto 2^n$ can easily be defined, and that we can produce $2^{T(\ell(\omega))}$ for any polynomial T: see [BD19; BD18].

Lemma 1 ([BD19; BD18]). Let $f: \mathbb{N}^{p+1} \to \mathbb{Z}^d$, $\mathcal{L}: \mathbb{N}^{p+1} \to \mathbb{Z}$ be some functions and assume that (1) holds considering $\mathcal{L}(x, \mathbf{y}) = \ell(x)$. Then $\mathbf{f}(x, \mathbf{y})$ is given by $\mathbf{f}(x, \mathbf{y}) = \mathbf{F}(\ell(x), \mathbf{y})$ where \mathbf{F} is the solution of initial value problem $F(1, \mathbf{y}) = \mathbf{f}(0, \mathbf{y})$, and $\frac{\partial \mathbf{F}(t, \mathbf{y})}{\partial t} = \mathbf{h}(\mathbf{F}(t, \mathbf{y}), 2^t - 1, \mathbf{y})$.

3 Concepts from computable analysis

When we say that a function $f: S_1 \times \cdots \times S_d \to \mathbb{R}^{d'}$ is (respectively: polynomial-time) computable this will always be in the sense of computable analysis: see e.g. [BHW08; Wei00]. We recall here the basic concepts and definitions, mostly following the book [Ko91], whose subject is complexity theory in computable analysis. This section is basically repeating the formalization proposed in [BB22] done to mix complexity issues dealing with integer and real arguments: a dyadic number d is a rational number with a finite binary expansion. That is to say $d = m/2^n$ for some integers $m \in \mathbb{Z}$, $n \in \mathbb{N}$, $n \geq 0$. Let \mathbb{D} be the set of all dyadic rational numbers. We denote by \mathbb{D}_n the set of all dyadic rationals d with a representation s of precision $\operatorname{prec}(s) = n$; that is, $\mathbb{D}_n = \{m \cdot 2^{-n} \mid m \in \mathbb{Z}\}$.

Definition 5 ([Ko91]). For each real number x, a function $\phi : \mathbb{N} \to \mathbb{D}$ is said to binary converge to x if for all $n \in \mathbb{N}$, $\operatorname{prec}(\phi(n)) = n$ and $|\phi(n) - x| \leq 2^{-n}$. Let CF_x (Cauchy function) denotes the set of all functions binary converging to x.

Intuitively, a Turing machine M computes a real function f the following way: 1. The input x to f, represented by some $\phi \in CF_x$, is given to M as an oracle; 2. The output precision 2^{-n} is given in the form of integer n as the input to M; 3. The computation of M usually takes two steps, though sometimes these two steps may be repeated for an indefinite number of times; 4. M computes, from the output precision 2^{-n} , the required input precision 2^{-m} ; 5. M queries the oracle to get $\phi(m)$, such that $\|\phi(m) - x\| \leq 2^{-m}$, and computes from $\phi(m)$ an output $d \in \mathbb{D}$ with $\|d - f(x)\| \leq 2^{-n}$. More formally:

Definition 6 ([Ko91]). A real function $f: \mathbb{R} \to \mathbb{R}$ is computable if there is a function-oracle TM M such that for each $x \in \mathbb{R}$ and each $\phi \in CF_x$, the function ψ computed by M with oracle ϕ (i.e., $\psi(n) = M^{\phi}(n)$) is in $CF_{f(x)}$.

Assume that M is an oracle machine which computes f on a domain G. For any oracle $\phi \in CF_x$, with $x \in G$, let $T_M(\phi, n)$ be the number of steps for M to halt on input n with oracle ϕ , and $T'_M(x,n) = \max \{T_M(\phi,n) \mid \phi \in CF_x\}$. The time complexity of f is defined as follows:

Definition 7 ([Ko91]). Let G be bounded closed interval [a,b]. Let $f: G \to \mathbb{R}$ be a computable function. Then, we say that the time complexity of f on G is bounded by a function $t: G \times \mathbb{N} \to \mathbb{N}$ if there exists an oracle TM M which computes f such that for all $x \in G$ and all n > 0, $T'_M(x,n) \le t(x,n)$.

In other words, the idea is to measure the time complexity of a real function based on two parameters: input real number x and output precision 2^{-n} . Sometimes, it is more convenient to simplify the complexity measure to be based on only one parameter, the

output precision. For this purpose, we say the uniform time complexity of f on G is bounded by a function $t': \mathbb{N} \to \mathbb{N}$ if the time complexity of f on G is bounded by a function $t: G \times \mathbb{N} \to \mathbb{N}$ with the property that for all $x \in G$, $t(x, n) \le t'(n)$.

However, if we do so, it is important to realize that if we had taken $G = \mathbb{R}$ in previous definition, for unbounded functions f, the uniform time complexity would not have existed, because the number of moves required to write down the integral part of f(x) grows as x approaches $+\infty$ or $-\infty$. Therefore, the approach of [Ko91] is to do as follows (The bounds -2^X and 2^X are somewhat arbitrary, but are chosen here because the binary expansion of any $x \in (-2^n, 2^n)$ has n bits in the integral part).

Definition 8 (Adapted from [Ko91]). For functions f(x) whose domain is \mathbb{R} , we say that the (non-uniform) time complexity of f is bounded by a function $t': \mathbb{N}^2 \to \mathbb{N}$ if the time complexity of f on $[-2^X, 2^X]$ is bounded by a function $t: \mathbb{N}^2 \to \mathbb{N}$ such that $t(x,n) \le t'(X,n)$ for all $x \in [-2^X, 2^X]$.

As we want to talk about general functions in \mathscr{F} , we extend the approach to more general functions. (for conciseness, when $\mathbf{x} = (x_1, \dots, x_p)$, $\mathbf{X} = (X_1, \dots, X_p)$, we write $\mathbf{x} \in [-2^{\mathbf{X}}, 2^{\mathbf{X}}]$ as a shortcut for $x_1 \in [-2^{X_1}, 2^{X_1}], \dots, x_p \in [-2^{X_p}, 2^{X_p}]$).

Definition 9 (Complexity for real functions: general case). *Consider a function* $f(x_1,...,x_p, n_1,...,n_q)$ whose domain is $\mathbb{R}^p \times \mathbb{N}^q$. We say that the (non-uniform) time complexity of f is bounded by a function $t': \mathbb{N}^{p+q+1} \to \mathbb{N}$ if the time complexity of $f(\cdot,...,\cdot,\ell(n_1),...,\ell(n_q))$ on $[-2^{X_1},2^{X_1}] \times ... [-2^{X_p},2^{X_p}]$ is bounded by a function $t(\cdot,...,\cdot,\ell(n_1),...,\ell(n_q),\cdot): \mathbb{N}^p \times \mathbb{N} \to \mathbb{N}$ such that $t(\mathbf{x},\ell(n_1),...,\ell(n_q),n) \leq t'(\mathbf{X},\ell(n_1),...,\ell(n_q),n)$ whenever $\mathbf{x} \in [-2^{\mathbf{X}},2^{\mathbf{X}}]$. We say that f is polynomial time computable if t' can be chosen as a polynomial. We say that a vectorial function is polynomial time computable iff all its components are.

We do so that this measures of complexity extends the usual complexity for functions over the integers, where complexity of integers is measured with respects of their lengths, and over the reals, where complexity is measured with respect to their approximation. In particular, in the specific case of a function $f: \mathbb{N}^d \to \mathbb{R}^{d'}$, that basically means there is some polynomial $t': \mathbb{N}^{d+1} \to \mathbb{N}$ so that the time complexity of producing some dyadic approximating $f(\mathbf{m})$ at precision 2^{-n} is bounded by $t'(\ell(m_1), \dots, \ell(m_d), n)$.

In other words, when considering that a function is polynomial time computable, it is in the length of all its integer arguments, as this is the usual convention. However, we need sometimes to consider also polynomial dependency directly in one of some specific integer argument, say n_i , and not on its length $\ell(n_i)$. We say that the function is polynomial time computable, with respect to the value of n_i when this holds (keeping possible other integer arguments n_j , $j \neq i$, measured by their length).

4 Some results about various functions

Call affine function a function $f: \mathbb{R}^n \to \mathbb{R}$ of the form $f(x_1, \dots, x_n) = w_1 x_1 + \dots + w_n x_n + h$, for some real w_1, \dots, w_n or a function $f: \mathbb{R}^n \to \mathbb{R}^m$ whose m components are of this form. We call *neural function* the smallest class of functions that is obtained by considering an affine function, or an affine function where one of several of its variable has been replaced by $\overline{\operatorname{cond}}(g)$ where g is inductively a neural function.

NB 3. The idea is to capture the class of functions computed by some (non-recurrent) formal neural network: every neural function can clearly be interpreted as a formal

neural network where the activation function is the function $\overline{cond}()$. As an example, $x + \overline{cond}(x + 2y + 3\overline{cond}(x))$ is a neural function, which can be considered as a depth 2 formal neural network. A function such as $x^2 + 2$ is not a neural function, as it involves a multiplication.

A key of our proofs is the construction of very specific functions in \mathbb{LDL}° : we write $\{x\}$ for the fractional part of real x, i.e. $\{x\} = x - \lfloor x \rfloor$. We provide more details and show some graphical representations of most of them in appendix, in order in particular to show that these functions are sometimes highly non-trivial.

Lemma 2. There exists $\xi_1, \xi_2 : \mathbb{N} \times \mathbb{R} \mapsto \mathbb{R} \in \mathbb{LDL}^{\odot}$ such that, for all $n \in \mathbb{N}$ and $x \in [-2^n, 2^n]$, whenever $x \in [\lfloor x \rfloor - \frac{1}{2}, \lfloor x \rfloor + \frac{1}{4}]$, $\xi_1(2^n, x) = \{x\}$, and whenever $x \in [\lfloor x \rfloor, \lfloor x \rfloor + \frac{3}{4}]$, $\xi_2(2^n, x) = \{x\}$.

Proof. Consider $\xi_1(N,x) = \xi(N,x-\frac{5}{8}) - \frac{1}{4}$ and $\xi_2(N,x) = \xi(N,x+\frac{1}{8})$, where $\xi(N,x) = \xi'(N+1,x) - \xi'(N+1,-x) + \frac{3}{4} - \frac{3}{4}\overline{\operatorname{cond}}(\frac{1}{4}+4x)$ where $\xi'(2^0,x) = \frac{3}{4}\overline{\operatorname{cond}}(\frac{1}{6}+\frac{2}{3}x)$, and $\xi'(2^{n+1},x) = \xi'(2^n,F(2^n,x))$, with $F(K,x) = x - K.\overline{\operatorname{cond}}(\frac{1}{4}+4(x-K))$, considering $F(0,x) = \xi'(2^0,x)$. A proof by induction (more intuitions and details in appendix) shows that it satisfies our claims. It remains to prove that this corresponds to a function in LDL $^{\circlearrowleft}$, but the key is to observe that, from an easy induction, $\xi'(2^n,x) = F(2^0,F(2^1,F(2^2(\dots,F(2^{n-1},x)))))$, and hence can be obtained as $H(2^{n-1},2^n,x)$ with H defined by some linear length ordinary differential equation using the derivative on its first variable expressing the recurrence $H(2^0,2^n,x) = F(2^{n-1},x)$ and $H(2^{t+1},2^n,x) = F(2^{n-1-t},H(2^t,2^n,x))$. □

Considering $\sigma_i(2^n, x) = x - \xi_i(2^n, x)$, we obtain next lemma. Using recursive constructions, we can also get (details and graphical representations in appendix).

Lemma 3. There exists $\sigma_1, \sigma_2 : \mathbb{N} \times \mathbb{R} \mapsto \mathbb{R} \in \mathbb{LDL}^{\oslash}$ such that, for all $n \in \mathbb{N}$ and $x \in [-2^n, 2^n]$, whenever $x \in I_1 = [\lfloor x \rfloor - \frac{1}{2}, \lfloor x \rfloor + \frac{1}{4}]$, $\sigma_1(2^n, x) = \lfloor x \rfloor$, and whenever $x \in I_2 = [\lfloor x \rfloor, \lfloor x \rfloor + \frac{3}{4}]$, $\sigma_2(2^n, x) = \lfloor x \rfloor$.

Lemma 4. There exists $\operatorname{mod}_2 : \mathbb{N} \times \mathbb{R} \mapsto [0,1] \in \mathbb{LDL}^{\odot}$ such that for all $n \in \mathbb{N}$, $x \in [-2^n, 2^n]$, whenever $x \in [\lfloor x \rfloor - \frac{1}{4}, \lfloor x \rfloor + \frac{1}{2}]$, $\operatorname{mod}_2(x)$ is $\lfloor x \rfloor$ modulo 2.

Lemma 5. There exists $\div_2 : \mathbb{N} \times \mathbb{R} \mapsto [0,1] \in \mathbb{LDL}^{\oslash}$ such that for all $n \in \mathbb{N}$, $x \in [-2^n, 2^n]$, whenever $x \in [\lfloor x \rfloor, \lfloor x \rfloor + \frac{1}{2}]$, $\div_2(x)$ is the integer division of $\lfloor x \rfloor$ by 2.

Lemma 6. There exists $\lambda : \mathbb{N} \times \mathbb{R} \mapsto [0,1] \in \mathbb{LDL}^{\odot}$ such that for all $n \in \mathbb{N}$, $x \in [-2^n, 2^n]$, whenever $x \in [\lfloor x \rfloor + \frac{1}{4}, \lfloor x \rfloor + \frac{1}{2}]$, $\lambda(2^n, x) = 0$ and whenever $x \in [\lfloor x \rfloor + \frac{3}{4}, \lfloor x \rfloor + 1]$, $\lambda(2^n, x) = 1$.

Lemma 7. Consider $T(d,l) = \overline{cond}(d-3/4+l/2)$. For $l \in [0,1]$, we have T(0,l) = 0, and T(1,l) = l.

Lemma 8. Assume you are given some integers $\alpha_1, \alpha_2, ..., \alpha_n$, and some values $V_1, V_2, ..., V_n$. Then there is some neural function, that we write $\operatorname{send}(\alpha_i \mapsto V_i)_{i \in \{1, ..., n\}}$, that maps any $x \in [\alpha_i - 1/4, \alpha_i + 1/4]$ to V_i , for all $i \in \{1, ..., n\}$.

Proof. Sort the α_i so that $\alpha_1 < \alpha_2 < \dots, \alpha_n$. Then consider $T_1 + \overline{\text{cond}}(q - \alpha_1)(T_2 - T_1) + \overline{\text{cond}}(q - \alpha_2)(T_3 - E_3) + \dots + \overline{\text{cond}}(q - \alpha_{n-1})(T_n - T_{n-1})$.

More generally:

Lemma 9. Let N be some integer. Assume we are given some integers $\alpha_1, \alpha_2, \ldots, \alpha_n$, and some values $V_{i,j}$ for $1 \le i \le n$, and $0 \le j < N$. Then there is some neural function, that we write $send((\alpha_i, j) \mapsto V_{i,j})_{i \in \{1, \ldots, n\}, j \in \{0, \ldots, N-1\}}$, that maps any $x \in [\alpha_i - 1/4, \alpha_i + 1/4]$ and $y \in [j-1/4, j+1/4]$ to $V_{i,j}$, for all $i \in \{1, \ldots, n\}$, $j \in \{0, \ldots, N-1\}$.

Proof. If we define the function by $\operatorname{send}((\alpha_i,j)\mapsto V_{i,j})_{i\in\{1,\dots,n\},j\in\{1,\dots,N\}}(x,y) = \operatorname{send}(N\alpha_i+j\mapsto V_{i,j})_{i\in\{1,\dots,n\},j\in\{1,\dots,N\}}(Nx+y)$ this works when $x=\alpha_i$ for some i. Considering instead $\operatorname{send}(N\alpha_i+j\mapsto V_{i,j})_{i\in\{1,\dots,n\},j\in\{1,\dots,N\}}(N\operatorname{send}(\alpha_i\mapsto\alpha_i)_{i\in\{1,\dots,n\}}(x)+y)$ works for any $x\in[\alpha_i-1/4,\alpha_i+1/4]$.

5 Simulating Turing machines with functions of \mathbb{LDL}^{\bigcirc}

This section is devoted to prove a kind of reverse implication of the following proposition, whose proof follows by induction from standard arguments exactly as in [BB22].

Proposition 1 ([BB22]). All functions of \mathbb{LDL}^{\odot} are computable (in the sense of computable analysis) in polynomial time.

We now basically prove that for any polynomial time computable function $\mathbf{f}: \mathbb{R}^d \to \mathbb{R}^{d'}$, we can construct some function $\tilde{\mathbf{f}} \in \mathbb{LDL}^{\oslash}$ that simulates the computation of f. This basically requires to be able to simulate the computation of a Turing machine using some functions from $\mathbb{LDL}^{\circlearrowleft}$. We basically use the same ideas as in [BB22], but with some improvements, as we need to avoid multiplications, and even get neural functions.

Consider without loss of generality some Turing machine $M = (Q, \{0, 1\}, q_{init}, \delta, F)$ using the symbols 0, 1, 3, where B = 0 is the blank symbol. The reason of the choice of symbols 1 and 3 will be made clear latter. We assume $Q = \{0, 1, ..., |Q| - 1\}$. Let

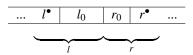
$$\ldots l_{-k}l_{-k+1}\ldots l_{-1}l_0r_0r_1\ldots r_n\ldots$$

denote the content of the tape of the Turing machine M. In this representation, the head is in front of symbol r_0 , and $l_i, r_i \in \{0,1,3\}$ for all i. Such a configuration C can be denoted by C = (q, l, r), where $l, r \in \Sigma^{\omega}$ are (possibly infinite, if we consider that the tape can be seen as a non finite word, in the case there is no blank on it) words over alphabet $\Sigma = \{1,3\}$ and $q \in Q$ denotes the internal state of M.

The idea is that such a configuration C can also be encoded by some element $\gamma_{config}(C)=(q,\bar{l},\bar{r})\in\mathbb{N}\times\mathbb{R}^2$, by considering $\bar{r}=\sum_{n\geq 0}r_n4^{-(n+1)},\bar{l}=\sum_{n\geq 0}l_{-n}4^{-(n+1)}$. Basically, in other words, we encode the configuration of bi-infinite tape Turing machine M by real numbers using their radix 4 encoding, but using only digits 1,3. If we write: $\gamma_{config}:\Sigma^\omega\to\mathbb{R}$ for the function that maps word $w=w_0w_1w_2\dots$ to $\gamma_{word}(w)=\sum_{n\geq 0}w_n4^{-(n+1)}$, we can also write $\gamma_{config}(C)=\gamma_{config}(q,l,r)=(q,\gamma_{word}(l),\gamma_{word}(r))$. Notice that this lives in $Q\times[0,1]^2$. Denoting the image of $\gamma_{word}:\Sigma^\omega\to\mathbb{R}$ by \mathscr{I} , this even lives in $Q\times\mathscr{I}^2$.

Lemma 10. We can construct some neural function \overline{Next} in \mathbb{LDL}^{\odot} that simulates one step of M, i.e. that computes the Next function sending a configuration C of Turing machine M to the next one.

Proof. We can write $l = l_0 l^{\bullet}$ and $r = r_0 r^{\bullet}$, where l_0 and r_0 are the first letters of l and r, and l^{\bullet} and r^{\bullet} corresponding to (possibly infinite) word $l_{-1} l_{-2} \dots$ and $r_1 r_2 \dots$ respectively.



The function *Next* is basically of the form

 $Next(q, l, r) = Next(q, l^{\bullet}l_0, r_0r^{\bullet}) = (q', l', r')$ defined as a definition by case of type:

$$(q',l',r') = \begin{cases} (q',l^{\bullet}l_0x,r^{\bullet}) & \text{whenever } \delta(q,r_0) = (q',x,\rightarrow) \\ (q',l^{\bullet},l_0xr^{\bullet}) & \text{whenever } \delta(q,r_0) = (q',x,\leftarrow) \end{cases}$$

This rewrites as a function \overline{Next} which is similar, working over the representation of the configurations as reals, considering $r_0 = \lfloor 4\overline{r} \rfloor$

$$\overline{Next}(q, \overline{l}, \overline{r}) = \overline{Next}(q, \overline{l^{\bullet}l_{0}}, \overline{r_{0}r^{\bullet}}) = (q', \overline{l'}, \overline{r'})$$

$$= \begin{cases}
(q', \overline{l^{\bullet}l_{0}x}, \overline{r^{\bullet}}) & \text{whenever } \delta(q, r_{0}) = (q', x, \rightarrow) \\
(q', \overline{l^{\bullet}}, \overline{l_{0}xr^{\bullet}}) & \text{whenever } \delta(q, r_{0}) = (q', x, \leftarrow)
\end{cases}$$
• in the first case " \rightarrow ": $\overline{l'} = 4^{-1}\overline{l} + 4^{-1}x$ and $\overline{r'} = \overline{r^{\bullet}} = \{4\overline{r}\}$
• in the second case " \leftarrow ": $\overline{l'} = \overline{l^{\bullet}} = \{4\overline{l}\}$ and $\overline{r'} = 4^{-2}\{4\overline{r}\} + 4^{-1}x + \lfloor 4\overline{l}\rfloor$
(3)

We introduce the following functions: $\rightarrow: Q \times \{0,1,3\} \mapsto \{0,1\}$ and $\leftarrow: Q \times \{0,1,3\} \mapsto \{0,1\}$ such that $\rightarrow (q,a)$ (respectively: $\leftarrow (q,a)$) is 1 when $\delta(q,a) = (_,_,\to)$ (resp. $(_,_,\leftarrow)$), i.e. the head moves right (resp. left), and 0 otherwise.

We can rewrite $\overline{\textit{Next}}(q, \bar{l}, \bar{r}) = (q', \bar{l}', \bar{r}')$ as

$$\begin{split} \overline{l'} &= \sum_{q,r_0} \left[\rightarrow (q,r_0) \left(\frac{\overline{l}}{4} + \frac{x}{4} \right) + \leftarrow (q,r_0) \left\{ 4\overline{l} \right\} \right] \\ \text{and } \overline{r'} &= \sum_{q,r_0} \left[\rightarrow (q,r_0) \left\{ 4\overline{r} \right\} + \leftarrow (q,r_0) \left(\frac{\left\{ 4r_0 \right\}}{4^2} + \frac{x}{4} + \lfloor 4\overline{l} \rfloor \right) \right]. \end{split}$$

The problem about such expressions is that we cannot expect the integer part and the fractional part function to be in \mathbb{LDL}^{\oslash} (as functions of this class are computable, and hence continuous, unlike the fractional part). But, a key point is that from our trick of using only symbols 1 and 3, we are sure that in an expression like $\lfloor \bar{r} \rfloor$, either it values 0 (this is the specific case where there remain only blanks in r), or that $4\bar{r}$ lives in interval [1,2) or in interval [3,3). That means that we could replace $\lfloor 4\bar{r} \rfloor$ by $\sigma(4\bar{r})$ where σ is some (piecewise affine) function obtained by composing in a suitable way the basic functions of \mathbb{LDL}^{\oslash} . In particular, $\sigma(x) = \overline{\operatorname{cond}}(x) + 2\overline{\operatorname{cond}}(x-2)$, $\xi(x) = x - \sigma(x)$, then $\xi(4\bar{r})$ would be the same as $\{4\bar{r}\}$, and $\sigma(4\bar{r})$ would be the same as $\lfloor 4\bar{r} \rfloor$ in our context in above expressions.

In other words, considering $r_0 = \sigma(4\bar{r})$ we could replace the above expression by

$$\begin{split} \overline{l'} &= \sum_{q,r_0} \left[\rightarrow (q,r_0) \left(\frac{\overline{l}}{4} + \frac{x}{4} \right) + \leftarrow (q,r_0) \xi(4\overline{l}) \right] \\ \text{and } \overline{r'} &= \sum_{q,r_0} \left[\rightarrow (q,r_0) \xi(4\overline{r}) + \leftarrow (q,r_0) \left(\frac{\xi(4r)}{4^2} + \frac{x}{4} + \sigma(4\overline{l}) \right) \right], \text{ and get something} \end{split}$$

that would be still work exactly, but using only piecewise continuous functions.

We could then write:

$$q' = \operatorname{send}((q, r) \mapsto \operatorname{next} q_r^q)_{q \in Q, r \in \{0, 1, 3\}}(q, \sigma(4\overline{r})),$$

using notation of Lemma 9, where $nextq_{r_0}^q = q'$ if $\delta(q, r_0) = (q', x, m)$ for $m \in \{\leftarrow, \rightarrow\}$.

We could also replace every $\to (q,r)$ in above expressions for \overline{l}' and \overline{l}' by $\operatorname{send}((q,r) \mapsto \to (q,r))(q,\sigma(4\overline{r}))$ and symmetrically for $\leftarrow (q,r)$. However, if we do so, we still might have some multiplications in the above expressions.

The key is to use Lemma 7: We can also write the above expressions as

$$\begin{split} \overline{l'} &= \sum_{q,r} \quad \left[\overline{\operatorname{cond}} \left(\operatorname{send}((q,r) \mapsto \to (q,r))(q,\sigma(4\overline{r})) - \frac{3}{4} + \frac{1}{2} \left(\frac{\overline{l}}{4} + \frac{x}{4} \right) \right) \\ &+ \overline{\operatorname{cond}} \left(\operatorname{send}((q,r) \mapsto \leftarrow (q,r))(q,\sigma(4\overline{r})) - \frac{3}{4} + \frac{1}{2} \xi(4\overline{l}) \right] \right) \\ \overline{r'} &= \sum_{q,r} \quad \left[\overline{\operatorname{cond}} \left(\operatorname{send}((q,r) \mapsto \to (q,r))(q,\sigma(4\overline{r})) - \frac{3}{4} + \frac{1}{2} \xi(4\overline{r}) \right) \right] \end{split}$$

 $+ \overline{\operatorname{cond}} \left(\operatorname{send}((q,r) \mapsto \leftarrow (q,r))(q,\sigma(4\overline{r})) - \frac{3}{4} + \frac{1}{2} \left(\frac{\xi(4r)}{4^2} + \frac{x}{4} + \sigma(4\overline{l}) \right) \right] \right)$

Once we have one step, we can simulate some arbitrary computation of a Turing machine, using some linear length ODE:

Proposition 2. Consider some Turing machine M that computes some function $f: \Sigma^* \to \Sigma^*$ in some time $T(\ell(\omega))$ on input ω . One can construct some function $\tilde{\mathbf{f}}: \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ in \mathbb{LDL}^{\odot} that does the same, with respect to the previous encoding: We have $\tilde{\mathbf{f}}(2^{T(\ell(\omega))}, \gamma_{word}(\omega))$ provides $f(\omega)$.

Proof. The idea is to define the function \overline{Exec} that maps some time 2^t and some initial configuration C to the configuration at time t. This can be obtained using some linear length ODE using previous Lemma.

$$\overline{Exec}(0,C) = C$$
 and $\frac{\partial \overline{Exec}}{\partial \ell}(t,C) = \overline{Next}(\overline{Exec}(t,C))$

We can then get the value of the computation as $\overline{Exec}(2^{T(\ell(\omega))}, C_{init})$ on input ω , considering $C_{init} = (q_0, 0, \gamma_{word}(\omega))$. By applying some projection, we get the following function $\tilde{\mathbf{f}}(x, y) = \pi_3^3(\overline{Exec}(x, q_0, 0, y))$ that satisfies the property.

6 Towards functions over the reals

The purpose of this section is to prove Theorems 2, 3 and 4. Actually, the first two are a special case of Theorem 4, so we focus on the latter. The reverse implication of Theorem 4 mostly follows from Proposition 1 and arguments from computable analysis.

For the direct implication, the difficulty is that we know from previous section how to simulate Turing machines working over the Cantor-like set \mathscr{I} , while we want functions that work directly over the integers and over the reals. A first key is to be able to convert from integers/reals to representations using only symbols 1 and 3, that is to say, to map integers to \mathscr{I} , and \mathscr{I} to reals as in [BB22]. However, we need a stronger statement than the one of [BB22] to be able to do both the convention and simultaneously some product (but avoiding to use the multiplication in its definition).

Lemma 11 (From \mathscr{I} to \mathbb{R} , and multiplying in parallel). We can construct some function $EncodeMul: \mathbb{N} \times [0,1] \times \mathbb{R} \to \mathbb{R}$ in $\mathbb{LDL}^{\varnothing}$ that maps $\gamma_{word}(\overline{d})$ and λ with $\overline{d} \in \{1,3\}^*$ to real λd . It is surjective over the dyadic, in the sense that for any dyadic $d \in \mathbb{D}$, there is some (easily computable) such \overline{d} with $EncodeMul(2^{\ell(\overline{d})}, \overline{d}, \lambda) = \lambda d$.

Proof. Consider the following transformation: Every digit in the binary expansion of d is encoded by a pair of symbols in the radix 4 encoding of $\overline{d} \in [0,1]$: digit 0 (respectively: 1) is encoded by 11 (resp. 13) if before the "decimal" point in d, and digit 0 (respectively: 1) is encoded by 31 (resp. 33) if after. For example, for d=101.1 in base $2, \overline{d}=0.13111333$ in base 4. The transformation from \overline{d} to λd can be done by considering function $F:\mathbb{R}^3\to\mathbb{R}^3$ given by send $\left(5\mapsto (\sigma(16\overline{r_1}),2\mapsto\overline{l_2}+0,\lambda), \quad 7\mapsto (\sigma(16\overline{r_1}),2\overline{l_2}+\lambda,\lambda), \quad 13\mapsto (\sigma(16\overline{r_1}),(\overline{l_2}+0)/2,\lambda), \quad 15\to (\sigma(16\overline{r_1}),(\overline{l_2}+\lambda)/2,\lambda)\right)$ ($16\overline{r_1}$) with σ and ξ constructed as suitable approximation of the integer part and the fractional part, as in the previous section. We then just need to apply $\ell(\overline{d})$ times F on $(\overline{d},0,\lambda)$, and then project on the second component to get a function Encode that does the job. That is $EncodeMul(x,y,\lambda)=\pi_2^3(G(x,y,\lambda))$ with $G(0,y,\lambda)=(\overline{d},0,\lambda)$ and $\frac{\partial G}{\partial \ell}(t,\overline{d},\overline{l},\lambda)=F(G(\overline{d},\overline{l},\lambda))$.

In a symmetric way:

Lemma 12 (From \mathbb{N} to \mathscr{I} , [BB22]). We can construct some function Decode : $\mathbb{N}^d \to \mathbb{R}$ in \mathbb{LDL}^{\odot} that maps $n \in \mathbb{N}^d$ to some (easily computable) encoding of n in \mathscr{I} .

We now go to the the proof of the direct implication of Theorem 3. By lack of space, we discuss only the case d = d' = d'' = 1, i.e. of a polynomial time computable function $f: \mathbb{R} \times \mathbb{N} \to \mathbb{R}$. The general case is easy to adapt, by adding suitable arguments, and considering multi-tape Turing machines. From standard arguments from computable analysis (see e.g. [Corollary 2.21][Ko91]), the following holds¹.

Lemma 13. Assume $f: \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ is computable in polynomial time. There exists some polynomial $m: \mathbb{N}^2 \to \mathbb{N}$ and some $\tilde{f}: \mathbb{N}^3 \to \mathbb{Z}$ computable in polynomial time such that for all $x \in \mathbb{R}$, $\|2^{-n}\tilde{f}(\lfloor 2^{m(n,M)}x\rfloor, u, 2^M, 2^n) - f(x, u)\| \le 2^{-n}$ whenever $\frac{x}{2^{m(n,M)}} \in [-2^M, 2^M]$.

Assume we consider an approximation σ_i (with either i=1 or i=2) of the integer part function given by Lemma 3. Then, given n,M, when $2^{m(n,M)}x$ falls in some suitable interval I_i for σ_i , we are sure that $\sigma_i(2^{m(n,M)}x) = \lfloor 2^{m(n,M)}x \rfloor$. Consequently, $2^{-n}\tilde{f}(\sigma_i(2^{m(n,M)}x), u, 2^M, 2^n)$ provides some 2^{-n} -approximation of f(x).

Now, we can compute the value of \tilde{f} on these arguments by simulating the Turing machine that computes f, using functions from \mathbb{LDL}^{\oslash} . Namely, we just need to map all arguments (expected to be integers) to Cantor-like set \mathscr{I} , then use Proposition 2 to compute the (encoding over \mathscr{I}) of the corresponding integer, and then maps it back to some integer value. In other words, we can get an approximation of f(x,u) of the form $Formula_i(x,u,M,n) = EncodeMul(T_1,\tilde{f}(T_2,Decode(T_3,(\sigma_i(2^{m(n,M)}x),u,2^M,2^n))),2^{-n})$ for some suitable T_1 , T_2 and T_3 of polynomial size, big enough to cover (up to some constant) the time required by the Turing machine: here \tilde{f} is the function obtained from \tilde{f} by Proposition 2. This works when $2^{m(n,M)}x$ falls in the suitable interval I_i . Setting T_1 , T_2 and T_3 can be done exactly as we did previously (e.g. in Proposition 2).

The problem is that it might also be the case that $2^{m(n,M)}x$ falls in the complement of the intervals I_i . In that case, we have no clear idea of what could be the value of $\sigma_i(2^{m(n,M)}x)$, and of what might be the value of above expression $Formula_i(x,u,M,n)$. But the point is that when it happens for an x for σ_1 , we could have used σ_2 , and this would work, as one can check that the intervals of type I_1 covers the complements of

¹The idea is that m corresponds to the polynomial modulus of continuity of the function, and \tilde{f} gives some approximation of the function, requiring only requests on real argument x with precision m(n, M).

the intervals of type I_2 and conversely. They also overlap, but when x is both in some I_1 and I_2 , $Formula_1(x,u,M,n)$ and $Formula_2(x,u,M,n)$ may differ, but they are both 2^{-n} approximation of f(x).

The key is then to compute some suitable "adaptive" barycenter, using function λ , provided by Lemma 6. Observe from the statements of Lemma 3, from the statement of Lemma 6 that whenever $\lambda(2^n,x)=0$, we know that $\sigma_2(2^n,x)=\lfloor x\rfloor$; whenever $\lambda(2^n,x)=1$ we know that $\sigma_1(2^n,x)=\lfloor x\rfloor$; whenever $\lambda(2^n,x)\in(0,1)$, we know that $\sigma_1(2^n,x)=\lfloor x\rfloor+1$ and $\sigma_2(2^n,x)=\lfloor x\rfloor$. That means that if we consider $\lambda(2^n,x)$ Formula₁ $(x,u,M,n)+(1-\lambda(2^n,n))$ Formula₂(x,u,M,n) we are sure to get some 2^{-n} approximation of f(x).

There remains that this requires some multiplication with λ . But from the form of $Formula_i(x, u, M, n)$, this could be also be written as follows, and hence remain in \mathbb{LDL}° :

$$EncodeMul(T_1, \tilde{\tilde{f}}(T_2, Decode(T_3, (\sigma_1(2^{m(n,M)}x), u, 2^M, 2^n))), \lambda(2^n, x)2^{-n}) + \\ EncodeMul(T_1, \tilde{\tilde{f}}(T_2, Decode(T_3, (\sigma_2(2^{m(n,M)}x), u, 2^M, 2^n))), 1 - \lambda(2^n, x)2^{-n})$$
 (4)

7 Normal form and some of its consequences

From the proofs we also get a normal form theorem, namely formula (4). In particular,

Theorem 9. Any function $f: \mathbb{N}^d \to \mathbb{R}^{d'}$ can be obtained from the class $\overline{\mathbb{LDL}^{\odot}}$ using only one schema ELim.

We now go the discussion and proof of Theorem 5: Observing Formula 4, we see that when n and M are fixed, the expression is depending on u, T_1 , T_2 and T_3 . From our hypothesis that the function is of type $f: \mathbb{R}^d \to \mathbb{R}^{d'}$, we are in the case d'' = 0, i.e, where there is no such u. These T_i 's functions correspond basically to the (polynomial) time required by the Turing machine to compute the function f. From the previous constructions, it turns out that when this time is fixed to some polynomial t(n), the function is some neural function: i.e. formula 4 is providing some neural function that is guaranteed to be at precision 2^{-n} of f(x) over $[-2^M, 2^M]$. This corresponds to the statement of Theorem 5. In other works, Formula 4 can be seen as a function that generates uniformly a family of circuits/formal neurons approximating a given function at some given precision over some given domain.

Notice that we believe that $\overline{\text{cond}}()$ function can actually be replaced by tanh function, at the price of a discussion of the involved errors. We leave this as future work.

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A Extended state of the art

We provide some complements on the state of the art (Section 1.1).

As we wrote, our ways of simulating Turing machines have some reminiscence of similar constructions used in other contexts such as Neural Networks [SS95; Sie99]. In particular, we use Cantor-like encoding set $\mathscr I$ in a similar way to what is done in these references. These references use some particular sigmoid function σ (called sometimes *saturated linear function*) that values 0 when $x \le 0$, x for $0 \le x \le 1$, 1 for $x \ge 1$. Clearly, this is equivalent to $\overline{\operatorname{cond}}(\frac{1}{4}+\frac{1}{2}x)$, and hence their constructions can be reformulated using the $\overline{\operatorname{cond}}()$ function. However, first, the models considered in these references are recurrent, while our constructions are not recurrent neural networks, and second, their models are restricted to live on the compact domain [0,1], which forbids to get functions from $\mathbb R \to \mathbb R$, while our settings allows more general functions. Our proofs also require functions taking some integer arguments, that would be impossible to consider in their settings (unless at the price of an artificial reencoding). In some sense, our constructions can be seen as operators that maps to family of neural networks in the spirit of these models, instead of considering a fixed recurrent neural networks.

While there have been several characterizations of complexity classes over the discrete (see e.g. the monograph [Sie99] about the above discussed approach, but not only), but as far as we know, the relation between formal neural networks with classes of computable analysis has never been established before. We believe that our original settings allows to do so, while this is unclear with the above mentioned models.

Furthermore, once again, we were motivated by (discrete) ordinary differential equations, and the relations to formal neural networks is a side effect, but not the main goal that we wanted to obtain. And our settings is indeed a characterization in terms of classes of discrete ODEs.

Notice that today's formal neural networks are often built with the so-called ReLU (that stands for $Rectified\ Linear\ Unit$) function, that maps $x \le 0$ to 0, and $x \ge 0$ to x. This could be taken as a basis function instead of the function cond() by rexpressing the latter with a suitable expression with ReLU's functions. Notice also that our concept of neural function is not assuming that the last layer of the network is made of neurons, and the result may be output by some linear combination of the neurons in the last layer.

If we do not restrict to neural network related models, with respect to all previous contexts, as far as we know, only a few papers have been devoted to characterizations of complexity, and even computability, classes in the sense of computable analysis. There have been some attempts using continuous ODEs [Bou+07], that we already mentioned, or the so-called \mathbb{R} -recursive functions [BP20]. For discrete schemata, we only know [Bra96] and [NRY21], focusing on computability and not complexity.

B Some results from [BD19; BD18]

B.1 Some general statements

In order to be as self-contained as possible, we recall in this section some results and concepts from [BD19; BD18; BB22]. All the statements in this section are already present in [BD19; BD18; BB22]: We are just repeating them here in case this helps. We provide some of the proofs, when they are not in the preliminary ArXiv version.

As said in the introduction:

Definition 10 (Discrete Derivative). The discrete derivative of $\mathbf{f}(x)$ is defined as $\Delta \mathbf{f}(x) = \mathbf{f}(x+1) - \mathbf{f}(x)$. We will also write \mathbf{f}' for $\Delta \mathbf{f}(x)$ to help readers not familiar with discrete differences to understand statements with respect to their classical continuous counterparts.

Several results from classical derivatives generalize to the settings of discrete differences: this includes linearity of derivation $(a \cdot f(x) + b \cdot g(x))' = a \cdot f'(x) + b \cdot g'(x)$, formulas for products and division such as $(f(x) \cdot g(x))' = f'(x) \cdot g(x+1) + f(x) \cdot g'(x) = f(x+1)g'(x) + f'(x)g(x)$. Notice that, however, there is no simple equivalent of the chain rule, in other words, there is no simple formula for the derivative of the composition of two functions.

A fundamental concept is the following:

Definition 11 (Discrete Integral). Given some function f(x), we write

$$\int_{a}^{b} \mathbf{f}(x) \delta x$$

as a synonym for $\int_a^b \mathbf{f}(x) \delta x = \sum_{x=a}^{x=b-1} \mathbf{f}(x)$ with the convention that it takes value 0 when a = b and $\int_a^b \mathbf{f}(x) \delta x = -\int_b^a \mathbf{f}(x) \delta x$ when a > b.

The telescope formula yields the so-called Fundamental Theorem of Finite Calculus:

Theorem 10 (Fundamental Theorem of Finite Calculus). *Let* $\mathbf{F}(x)$ *be some function. Then*,

$$\int_{a}^{b} \mathbf{F}'(x) \delta x = \mathbf{F}(b) - \mathbf{F}(a).$$

A classical concept in discrete calculus is the one of falling power defined as

$$x^{\underline{m}} = x \cdot (x-1) \cdot (x-2) \cdots (x-(m-1)).$$

This notion is motivated by the fact that it satisfies a derivative formula $(x^{\underline{m}})' = m \cdot x^{\underline{m-1}}$ similar to the classical one for powers in the continuous setting. In a similar spirit, we introduce the concept of falling exponential.

Definition 12 (Falling exponential). Given some function $\mathbf{U}(x)$, the expression \mathbf{U} to the falling exponential x, denoted by $\overline{2}^{\mathbf{U}(x)}$, stands for

$$\overline{2}^{\mathbf{U}(x)} = (1 + \mathbf{U}'(x-1)) \cdots (1 + \mathbf{U}'(1)) \cdot (1 + \mathbf{U}'(0))
= \prod_{t=0}^{t=x-1} (1 + \mathbf{U}'(t)),$$

with the convention that $\Pi_0^0 = \Pi_0^{-1} = id$, where id is the identity (sometimes denoted 1 hereafter).

This is motivated by the remarks that $2^x = \overline{2}^x$, and that the discrete derivative of a falling exponential is given by

$$\left(\overline{2}^{\mathbf{U}(x)}\right)' = \mathbf{U}'(x) \cdot \overline{2}^{\mathbf{U}(x)}$$

for all $x \in \mathbb{N}$.

Lemma 14 (Derivation of an integral with parameters). Consider

$$\mathbf{F}(x) = \int_{a(x)}^{b(x)} \mathbf{f}(x,t) \delta t.$$

Then

$$\mathbf{F}'(x) = \int_{a(x)}^{b(x)} \frac{\partial \mathbf{f}}{\partial x}(x,t) \delta t + \int_{0}^{-a'(x)} \mathbf{f}(x+1,a(x+1)+t) \delta t + \int_{0}^{b'(x)} \mathbf{f}(x+1,b(x)+t) \delta t.$$

In particular, when a(x) = a and b(x) = b are constant functions, $\mathbf{F}'(x) = \int_a^b \frac{\partial \mathbf{f}}{\partial x}(x,t) \delta t$, and when a(x) = a and b(x) = x, $\mathbf{F}'(x) = \int_a^x \frac{\partial \mathbf{f}}{\partial x}(x,t) \delta t + \mathbf{f}(x+1,x)$.

Proof.

$$\mathbf{F}'(x) = \mathbf{F}(x+1) - \mathbf{F}(x)$$

$$= \sum_{t=a(x+1)}^{b(x+1)-1} \mathbf{f}(x+1,t) - \sum_{t=a(x)}^{b(x)-1} \mathbf{f}(x,t)$$

$$= \sum_{t=a(x)}^{b(x)-1} (\mathbf{f}(x+1,t) - \mathbf{f}(x,t)) + \sum_{t=a(x+1)}^{t=a(x)-1} \mathbf{f}(x+1,t) + \sum_{t=b(x)}^{b(x+1)-1} \mathbf{f}(x+1,t)$$

$$= \sum_{t=a(x)}^{b(x)-1} \frac{\partial \mathbf{f}}{\partial x}(x,t) + \sum_{t=a(x+1)}^{t=a(x)-1} \mathbf{f}(x+1,t) + \sum_{t=b(x)}^{b(x+1)-1} \mathbf{f}(x+1,t)$$

$$= \sum_{t=a(x)}^{b(x)-1} \frac{\partial \mathbf{f}}{\partial x}(x,t) + \sum_{t=a(x+1)+a(x)-1}^{t=-a(x+1)+a(x)-1} \mathbf{f}(x+1,a(x+1)+t)$$

$$+ \sum_{t=0}^{b(x+1)-b(x)-1} \mathbf{f}(x+1,b(x)+t).$$

Lemma 15 (Solution of linear ODE). For matrices **A** and vectors **B** and **G**, the solution of equation $\mathbf{f}'(x, \mathbf{y}) = \mathbf{A}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}) \cdot \mathbf{f}(x, \mathbf{y}) + \mathbf{B}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y})$ with initial conditions $\mathbf{f}(0, \mathbf{y}) = \mathbf{G}(\mathbf{y})$ is

$$\mathbf{f}(x,\mathbf{y}) = \left(\overline{2}^{\int_0^x \mathbf{A}(\mathbf{f}(t,\mathbf{y}),\mathbf{h}(t,\mathbf{y}),t,\mathbf{y})\delta t}\right) \cdot \mathbf{G}(\mathbf{y})$$

$$+ \int_0^x \left(\overline{2}^{\int_{u+1}^x \mathbf{A}(\mathbf{f}(t,\mathbf{y}),\mathbf{h}(t,\mathbf{y}),t,\mathbf{y})\delta t}\right) \cdot \mathbf{B}(\mathbf{f}(u,\mathbf{y}),\mathbf{h}(u,\mathbf{y}),u,\mathbf{y})\delta u.$$

Proof. Denoting the right-hand side by $\mathbf{rhs}(x, \mathbf{y})$, we have

$$\overline{\mathbf{rhs}}'(x,\mathbf{y}) = \mathbf{A}(\mathbf{f}(x,\mathbf{y}),\mathbf{h}(x,\mathbf{y}),x,\mathbf{y}) \cdot \left(\overline{2}^{\int_0^x \mathbf{A}(\mathbf{f}(t,\mathbf{y}),\mathbf{h}(t,\mathbf{y}),t,\mathbf{y})\delta t}\right) \cdot \mathbf{G}(\mathbf{y})
+ \int_0^x \left(\overline{2}^{\int_{u+1}^x \mathbf{A}(\mathbf{f}(t,\mathbf{y}),\mathbf{h}(t,\mathbf{y}),t,\mathbf{y})\delta t}\right)' \cdot \mathbf{B}(\mathbf{f}(u,\mathbf{y}),\mathbf{h}(u,\mathbf{y}),u,\mathbf{y})\delta u
+ \left(\overline{2}^{\int_{x+1}^{x+1} \mathbf{A}(\mathbf{f}(t,\mathbf{y}),\mathbf{h}(t,\mathbf{y}),t,\mathbf{y})\delta t}\right) \cdot \mathbf{B}(\mathbf{f}(x,\mathbf{y}),\mathbf{h}(x,\mathbf{y}),x,\mathbf{y})
= \mathbf{A}(\mathbf{f}(x,\mathbf{y}),\mathbf{h}(x,\mathbf{y}),x,\mathbf{y}) \cdot \left(\overline{2}^{\int_0^x \mathbf{A}(\mathbf{f}(t,\mathbf{y}),\mathbf{h}(t,\mathbf{y}),t,\mathbf{y})\delta t}\right) \cdot \mathbf{G}(\mathbf{y})
+ \mathbf{A}(\mathbf{f}(x,\mathbf{y}),\mathbf{h}(x,\mathbf{y}),x,\mathbf{y}) \cdot \\
\int_0^x \left(\overline{2}^{\int_{u+1}^x \mathbf{A}(\mathbf{f}(t,\mathbf{y}),\mathbf{h}(t,\mathbf{y}),t,\mathbf{y})\delta t}\right) \mathbf{B}(\mathbf{f}(u,\mathbf{y}),\mathbf{h}(u,\mathbf{y}),u,\mathbf{y})\delta u
+ \mathbf{B}(\mathbf{f}(x,\mathbf{y}),\mathbf{h}(x,\mathbf{y}),x,\mathbf{y}) \\
= \mathbf{A}(\mathbf{f}(x,\mathbf{y}),\mathbf{h}(x,\mathbf{y}),x,\mathbf{y}) \cdot \mathbf{rhs}(x,\mathbf{y}) + \mathbf{B}(\mathbf{f}(x,\mathbf{y}),\mathbf{h}(x,\mathbf{y}),x,\mathbf{y})$$

where we have used linearity of derivation and definition of falling exponential for the first term, and derivation of an integral (Lemma 14) providing the other terms to get the first equality, and then the definition of falling exponential. This proves the property by unicity of solutions of a discrete ODE, observing that $\overline{\mathbf{rhs}}(0, \mathbf{y}) = \mathbf{G}(\mathbf{y})$.

We write also 1 for the identity.

NB 4. Notice that this can be rewritten as

$$\mathbf{f}(x,\mathbf{y}) = \sum_{u=-1}^{x-1} \left(\prod_{t=u+1}^{x-1} \left(1 + \mathbf{A}(\mathbf{f}(t,\mathbf{y}), \mathbf{h}(t,\mathbf{y}), t, \mathbf{y}) \right) \right) \cdot \mathbf{B}(\mathbf{f}(u,\mathbf{y}), \mathbf{h}(u,\mathbf{y}), u, \mathbf{y}), \quad (5)$$

with the (not so usual) conventions that for any function $\kappa(\cdot)$, $\prod_x^{x-1} \kappa(x) = 1$ and $\mathbf{B}(-1,\mathbf{y}) = \mathbf{G}(\mathbf{y})$. Such equivalent expressions both have a clear computational content. They can be interpreted as an algorithm unrolling the computation of $\mathbf{f}(x+1,\mathbf{y})$ from the computation of $\mathbf{f}(x,\mathbf{y})$, $\mathbf{f}(x-1,\mathbf{y})$, ..., $\mathbf{f}(0,\mathbf{y})$.

A fundamental fact is that the derivation with respect to length provides a way to do a kind of change of variables:

Lemma 16 (Alternative view, case of Length ODEs). Let $f: \mathbb{N}^{p+1} \to \mathbb{Z}^d$, $\mathcal{L}: \mathbb{N}^{p+1} \to \mathbb{Z}^d$ be some functions and assume that (1) holds considering $\mathcal{L}(x, \mathbf{y}) = \ell(x)$. Then $\mathbf{f}(x, \mathbf{y})$ is given by $\mathbf{f}(x, \mathbf{y}) = \mathbf{F}(\ell(x), \mathbf{y})$ where \mathbf{F} is the solution of initial value problem

$$\mathbf{F}(1,\mathbf{y}) = \mathbf{f}(0,\mathbf{y}),$$

$$\frac{\partial \mathbf{F}(t,\mathbf{y})}{\partial t} = \mathbf{h}(\mathbf{F}(t,\mathbf{y}), 2^t - 1, \mathbf{y}).$$

C On proofs

We provide here more details on proofs, that we had to put in appendix by lack of space.

C.1 Proof of reverse implication of Theorem 3

Assume there exists $\tilde{\mathbf{f}}: \mathbb{R}^d \times \mathbb{N}^{d''+2} \to \mathbb{R}^{d'} \in \mathbb{LDL}^{\odot}$ such that for all $\mathbf{x} \in \mathbb{R}^d$, $X \in \mathbb{N}$, $\mathbf{x} \in [-2^X, 2^X]$, $\mathbf{m} \in \mathbb{N}^{d''}$, $n \in \mathbb{N}$, $\|\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{m}, 2^X, 2^n) - \mathbf{f}(\mathbf{x}, \mathbf{m})\| \leq 2^{-n}$.

From Proposition 1, we know that $\tilde{\mathbf{f}}$ is computable in polynomial time (in the binary length of its arguments). Then $\mathbf{f}(\mathbf{x}, \mathbf{m})$ is computable: indeed, given \mathbf{x} , \mathbf{m} and n, we can approximate $\mathbf{f}(\mathbf{x}, \mathbf{m})$ at precision 2^{-n} on $[-2^X, 2^X]$ as follows: Approximate $\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{m}, 2^X, 2^{n+1})$ at precision $2^{-(n+1)}$ by some rational q, and output q. We will then have

$$||q - \mathbf{f}(\mathbf{x}, \mathbf{m})|| \leq ||q - \tilde{\mathbf{f}}(\mathbf{x}, \mathbf{m}, 2^X, 2^{n+1})|| + ||\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{m}, 2^X, 2^{n+1}) - \mathbf{f}(\mathbf{x}, \mathbf{m})|| \leq 2^{-(n+1)} + 2^{-(n+1)} < 2^{-n}.$$

All of this is done in polynomial time in n and the size of \mathbf{m} , and hence we get that \mathbf{f} is polynomial time computable from definitions.

C.2 Proof of Theorem 6

We know that a function $\mathbf{f}: \mathbb{R}^d \to \mathbb{R}^{d'}$ from \mathbb{LDL}^{\odot} is polynomial time computable by Proposition 1. That means that we can approximate it with arbitrary precision, in particular precision $\frac{1}{4}$ in polynomial time. Given such an approximation \mathbf{q} , it is easy to determine the corresponding integer part: return (componentwise) the closest integer to \mathbf{q} .

Conversely, if we have a function $\mathbf{f}: \mathbb{N}^d \to \mathbb{N}^{d'}$ that is polynomial time computable, our previous simulations of Turing machines provide a function in \mathbb{LDL}^{\odot} that computes it

C.3 Proof of Lemma 2

Observe that $sig(a,b,x) = \overline{cond}(\frac{1}{4} + (x-a)/(2(b-a)))$ corresponds to the piecewise continuous sigmoid function sig(a,b,x) given by

$$\operatorname{sig}(a,b,x) = \begin{cases} 0 & \text{if } x \le a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } b \le x \end{cases}$$

Proof. It is sufficient to construct some function ξ such that for all $n \in \mathbb{N}$ and $x \in [-2^n, 2^n]$, whenever $x \in [\lfloor x \rfloor + \frac{1}{8}, \lfloor x \rfloor + \frac{7}{8}]$, $\xi(2^n, x) = \{x - \frac{1}{8}\}$. Indeed, then $\xi_1(N, x) = \xi(N, x - \frac{5}{8}) - \frac{1}{4}$ and $\xi_2(N, x) = \xi(N, x + \frac{1}{8})$ would be solution.

Actually, if we take ξ' that satisfies the constraint only when $x \ge 0$, and that values

Actually, if we take ξ' that satisfies the constraint only when $x \ge 0$, and that values 0 for $x \le 0$, then $\frac{3}{4} - \xi'(-x)$ would satisfy the constraint when $x \le 0$, but values 3/4 for $x \ge 0$. So, $\xi(N,x) = \xi'(N+1,x) - \xi'(N+1,-x) + \frac{3}{4} - \frac{3}{4} \operatorname{sig}(0,\frac{1}{8},x)$ would work for all x

So it remains to construct ξ' such that for all $n \in \mathbb{N}$ and $x \in [0, 2^n]$, whenever $x \in [\lfloor x \rfloor + \frac{1}{8}, \lfloor x \rfloor + \frac{7}{8}]$, $\xi'(2^n, x) = \{x - \frac{1}{8}\}$, and $\xi'(N, x) = 0$ for $x \leq 0$.

It suffices to define ξ' by induction by $\xi'(2^0, x) = \frac{3}{4} \operatorname{sig}(\frac{1}{8}, \frac{7}{8}, x)$, and $\xi'(2^{n+1}, x) = \xi'(2^n, F(2^n, x))$, where $F(K, x) = x - K \cdot \operatorname{sig}(K, K + \frac{1}{8}, x)$.

Let *I* be $[\lfloor x \rfloor + \frac{1}{8}, \lfloor x \rfloor + \frac{7}{8}], x \in I$, and let us first study the value of $F(2^n, x)$:

- If $x \le 2^n$, by definition of sig, $F(2^n, x) = x$, then $F(2^n, x) \in I$.
- The case $2^n < x < 2^n + \frac{1}{8}$ cannot happen as we assume $x \in I$.
- If $2^n + \frac{1}{8} \le x$ then $F(2^n, x) = x 2^n$ and $F(2^n, x) \in [\lfloor x \rfloor 2^n + \frac{1}{8}, \lfloor x \rfloor 2^n + \frac{7}{8}]$

Now, the property is true by induction. Indeed, it is true for n=0 by the expression of $\xi'(2^0,x)$. We now assume that it is true for some $n\in\mathbb{N}$. We have $\xi'(2^{n+1},x)=\xi'(2^n,F(2^n,x))$. Thus, by induction hypothesis, $\xi'(2^{n+1},x)=\{F(2^n,x)-1/8\}$.

- If $x \le 2^n$, by definition of sig, $F(2^n, x) = x$, then $\xi'(2^{n+1}, x) = \{F(2^n, x) 1/8\} = \{x 1/8\}$
- The case $2^n < x < 2^n + \frac{1}{8}$ cannot happen with our constraint $x \in I$.
- If $2^n + \frac{1}{8} \le x$ then $F(2^n, x) = x 2^n$ and $\xi'(2^{n+1}, x) = \{F(2^n, x) 1/8\} = \{x 2^n 1/8\} = \{x\}.$

Thus the property is proved for all n, and from above expressions, we get ξ_1 and ξ_2 . A graphical representation of ξ_1 and ξ_2 can be found in Figure 1.

There remain to prove that the function ξ' is in \mathbb{LDL}^{\odot} . Unfortunately, this is not clear from the recursive definition, but this can be written in another way, from which this follows. Indeed, we have from an easy induction that $\xi'(2^n,x) = F(2^0,F(2^1,F(2^2(\dots,F(2^{n-1},x)))))$, if we define $F(2^0,x)$ as $F(2^0,x) = \xi'(2^0,x) = \frac{3}{4} \operatorname{sig}(\frac{1}{8},\frac{7}{8},x)$.

Then, we can obtain $\xi'(2^n, x) = H(2^{n-1}, 2^n, x)$ with

$$\begin{split} H(2^{0},2^{n},x) &= F(2^{n-1},x) \\ H(2^{t+1},2^{n},x) &= F(2^{n-1-t},H(2^{t},2^{n},x)) \\ &= H(2^{t},2^{n},x) - 2^{n-1-t}. \operatorname{sig}(2^{n-1-t},2^{n-1-t} + \frac{1}{8},H(2^{t},2^{n},x)) \end{split}$$

Such a recurrence can be then seen as a linear length ordinary differential equation, in the length of its first argument. It follows that ξ' , and hence ξ_1 and ξ_2 are in \mathbb{LDL}^{\oslash} . \square

C.4 Proof of Lemma 3

Proof. Consider $\sigma_i(2^n, x) = x - \xi_i(2^n, x)$ with the function defined in Lemma 2. A graphical representation of σ_1 and σ_2 can be found in Figure 2.

C.5 Proof of Lemma 4

Proof. We can take $\text{mod}_2(N,x) = \lambda(N, \frac{1}{2}x - \frac{3}{4})$ where λ is the function given by Lemma 6.

A graphical representation of mod 2 can be found in Figure 3.

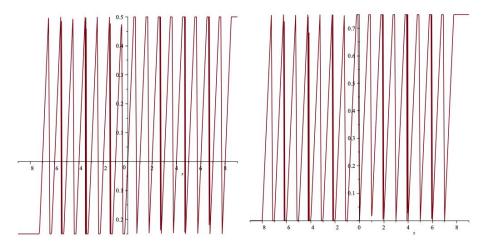


Figure 1: Graphical representation of $\xi_1(4,x)$ and $\xi_2(4,x)$ obtained with maple.

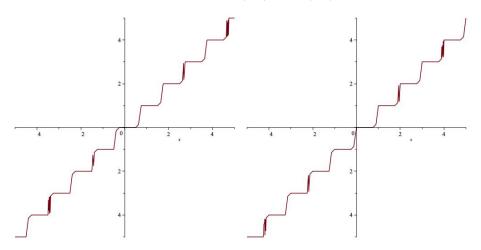


Figure 2: Graphical representation of $\sigma_1(4,x)$ and $\sigma_2(4,x)$ obtained with maple.

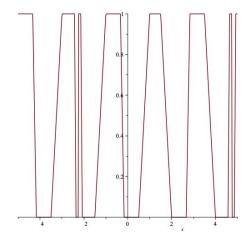


Figure 3: Graphical representation of $mod_2(4,x)$ obtained with maple.

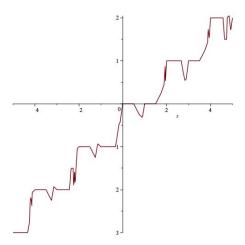


Figure 4: Graphical representation of $\div_2(4,x)$ obtained with maple.

C.6 Proof of Lemma 5

Proof. We can take $\div 2(N,x) = \frac{1}{2}(\sigma_2(N,x) - \text{mod}_2(N,x))$ where mod₂ is the function given by Lemma 4, and σ_2 is the function given by Lemma 3.

A graphical representation of mod 2 can be found in Figure 4.

C.7 Proof of Lemma 6

The idea is basically to use a technique similar to the one use for Lemma 2.

Proof. It is sufficient to construct a function λ' that works for $x \ge 0$, and that values 1 for $x \le 0$. That is to say, such that for all $n \in \mathbb{N}$, $x \in [0, 2^n]$, whenever $x \in [\lfloor x \rfloor + \frac{1}{4}, \lfloor x \rfloor + \frac{1}{2}]$, $\lambda'(2^n, x) = 0$, and whenever $x \in [\lfloor x \rfloor + \frac{3}{4}, \lfloor x \rfloor + 1]$, $\lambda'(2^n, x) = 1$, and whenever $x \le 0$, then $\lambda'(2^n, x) = 0$.

Indeed, then $\lambda(N,x) = \lambda'(N,x) + \lambda'(N,-1/4-x) - 1$ would be a solution, working for all x.

To solve the latter problem, we define λ' by induction by $\lambda'(2^0, x) = \text{sig}(\frac{1}{2}, \frac{3}{4}, x)$ $sig(0, \frac{1}{4}, x) + 1$, and $\lambda'(2^{n+1}, x) = \lambda'(2^n, G(2^n, x))$ where G(N, x) = x - D(N, x) and $D(N,x) = N \operatorname{sig}(N - \frac{1}{2}, N - \frac{1}{4}, x).$

By a reasoning similar to the proof of Lemma 2, it satisfies by induction the required properties.

A graphical representation of λ can be found in Figure 5.

There remain to prove that the function λ' is in \mathbb{LDL}° . Unfortunately, this is not clear from the recursive definition, but this can be written in another way, from which this follows. Indeed, we have from an easy induction that $\xi'(2^n,x) = G(2^0,G(2^1,G(2^2(\dots,G(2^{n-1},x)))))$, if we define $G(2^0,x)$ as $G(2^0,x) = \lambda'(2^0,x) = \text{sig}(\frac{1}{2},\frac{3}{4},x) - \text{sig}(0,\frac{1}{4},x) + 1$. Then, we can obtain $\xi'(2^n,x) = H(2^{n-1},2^n,x)$ with

$$\begin{split} H(2^{0},2^{n},x) &= G(2^{n-1},x) \\ H(2^{t+1},2^{n},x) &= G(2^{n-1-t},H(2^{t},2^{n},x)) \\ &= H(2^{t},2^{n},x) - 2^{n-1-t}\operatorname{sig}(2^{n-1-t} - \frac{1}{2},2^{n-1-t} - \frac{1}{4},H(2^{t},2^{n},x)) \end{split}$$

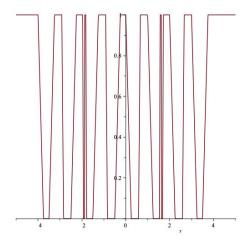


Figure 5: Graphical representation of $\lambda(4,x)$ obtained with maple.

Such a recurrence can then be seen as a linear length ordinary differential equation, in the length of its first argument. It follows that λ' , and hence λ are in \mathbb{LDL}^{\oslash} .

C.8 Proof of Lemma 7

Proof. Just check that this is true for d = 0, and then for d = 1, from the definition of $\overline{\text{cond}}()$.

C.9 Proof of Proposition 1

We repeat here the idea of the proof of Proposition 1, taken from [BB22].

Proof. This is proved by induction. This is true for basis functions, from basic arguments from computable analysis. In particular as $\overline{\text{cond}}(.)$ is a continuous piecewise affine function with rational coefficients, it is computable in polynomial time from standard arguments.

Now, the class of polynomial time computable functions is preserved by composition: see e.g. [Ko91]: If this helps, the idea of the proof for COMP(f,g), is that by induction hypothesis, there exists M_f and M_g two Turing machines computing in polynomial time $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$. In order to compute COMP(f,g)(x) with precision 2^{-n} , we just need to compute g(x) with a precision $2^{-m(n)}$, where m(n) is the polynomial modulus of continuity of f. Then, we compute f(g(x)), which, by definition of M_f takes a polynomial time in n. Thus, since $P_{\mathbb{R}}^{P_{\mathbb{R}}} = P_{\mathbb{R}}$, COMP(f,g) is computable in polynomial time, so the class of polynomial time computable functions is preserved under composition.

It only remains to prove that the class of polynomial time computable functions is preserved by the linear length ODE schema: This is Lemma 17. \Box

Lemma 17 ([BB22]). The class of polynomial time computable functions is preserved by the linear length ODE schema.

We write \vec{x} for $2^x - 1$ for conciseness. We write $\| | \cdots | |$ for the sup norm of integer part: given some matrix $\mathbf{A} = (A_{i,j})_{1 \le i \le n, 1 \le j \le m}$, $\| | \mathbf{A} | | = \max_{i,j} \lceil A_{i,j} \rceil$. In particular, given a vector \mathbf{x} , it can be seen as a matrix with m = 1, and $\| | \mathbf{x} | \|$ is the sup norm of the integer part of its components.

Proof. Using Lemma 16 (This lemma is repeated from [BD19; BD18]), when the schema of Definition 4 holds, we can do a change of variable to consider $\mathbf{f}(x,\mathbf{y}) = \mathbf{F}(\ell(x),\mathbf{y})$, with \mathbf{F} solution of a discrete ODE of the form $\frac{\partial \mathbf{F}(t,\mathbf{y})}{\partial t} = \mathbf{A}(\mathbf{F}(t,\mathbf{y}),\mathbf{h}(\vec{t},\mathbf{y}),\vec{t},\mathbf{y}) \cdot \mathbf{F}(t,\mathbf{y}) + \mathbf{B}(\mathbf{F}(t,\mathbf{y}),\mathbf{h}(\vec{t},\mathbf{y}),\vec{t},\mathbf{y})$, that is to say, of the form (6) below. It then follows from:

Lemma 18 (Fundamental observation, [BB22]). Consider the ODE

$$\mathbf{F}'(x,\mathbf{y}) = \mathbf{A}(\mathbf{F}(x,\mathbf{y}), \mathbf{h}(\vec{x},\mathbf{y}), \vec{x}, \mathbf{y}) \cdot \mathbf{F}(x,\mathbf{y}) + \mathbf{B}(\mathbf{F}(x,\mathbf{y}), \mathbf{h}(\vec{x},\mathbf{y}), \vec{x}, \mathbf{y}). \tag{6}$$

Assume: 1. The initial condition $\mathbf{G}(\mathbf{y}) = {}^{def} \mathbf{F}(0,\mathbf{y})$, as well as $\mathbf{h}(\vec{x},\mathbf{y})$ are polynomial time computable with respect to the value of x. 2. $\mathbf{A}(\mathbf{F}(x,\mathbf{y}),\mathbf{h}(\vec{x},\mathbf{y}),\vec{x},\mathbf{y})$ and $\mathbf{B}(\mathbf{F}(x,\mathbf{y}),\mathbf{h}(\vec{x},\mathbf{y}),\vec{x},\mathbf{y})$ are sg-polynomial expressions essentially constant in $\mathbf{F}(x,\mathbf{y})$.

Then, there exists a polynomial p such that $\ell(\left\| |\mathbf{F}(x,\mathbf{y})| \right\|) \leq p(x,\ell(\left\| |\mathbf{y}| \right\|))$ and $\mathbf{F}(x,\mathbf{y})$ is polynomial time computable with respect to the value of x.

Proof. The fact that there exists a polynomial p such that $\ell(\|\mathbf{f}(x,\mathbf{y})\|) \le p(x,\ell(\|\mathbf{y}\|))$, follows from the fact that we can write some explicit formula for the solution of (6): This is Lemma 15, repeated from [BD19; BD18]. Now, bounding the size of the right hand side of formula (5) provides the statement.

Now the fact that $\mathbf{F}(x, \mathbf{y})$ is polynomial time computable, follows from a reasoning similar to the one of following lemma (the lemma below restricts the form of the recurrence, but the more general recurrence of (6) would basically not lead to any difficulty): The fact that the modulus of continuity of a linear expression of the form of the right hand side of (6) is necessarily affine in its first argument follows from the hypothesis and from previous paragraph, using the fact that $\overline{\text{cond}}()$ has a linear modulus of convergence.

Lemma 19 ([BB22]). Suppose that function $\mathbf{f}: \mathbb{N} \times \mathbb{R}^d \to \mathbb{R}^{d'}$ is such that for all x, y,

$$\mathbf{f}(0,\mathbf{y}) = \mathbf{g}(\mathbf{y})$$
 and $\mathbf{f}(x+1,\mathbf{y}) = \mathbf{h}(\mathbf{f}(x,\mathbf{y}),x,\mathbf{y}))$

for some functions $\mathbf{g}: \mathbb{R}^d \to \mathbb{R}^{d'}$ and $\mathbf{h}: \mathbb{R}^{d'} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{d'}$ both computable in polynomial time with respect to the value of x. Suppose that the modulus m_h of continuity of \mathbf{h} is affine in its first argument: For all $\mathbf{f}, \mathbf{f}' \in [-2^{\mathbf{F}}, 2^{\mathbf{F}}]$, $\mathbf{y} \in [-2^{\mathbf{Y}}, 2^{\mathbf{Y}}]$, $\|\mathbf{f} - \mathbf{f}'\| \leq 2^{-m_h(\mathbf{F}, \ell(x), \mathbf{Y}, n)}$ implies $|\mathbf{h}(\mathbf{f}, x, \mathbf{y}) - \mathbf{h}(\mathbf{f}', x, \mathbf{y})| \leq 2^{-n}$ with $m_h(\mathbf{F}, \ell(x), \mathbf{Y}, n) = \alpha n + p_h(\mathbf{F}, \ell(x), \mathbf{Y})$ for some α . Suppose there exists a polynomial p such that $\ell(\|\mathbf{f}(x, \mathbf{y})\|) \leq p(x, \ell(\|\mathbf{y}\|))$.

Then $\mathbf{f}(x, \mathbf{y})$ is computable in polynomial time with respect to the value of x.

Proof. The point is that we can compute $\mathbf{f}(n, \mathbf{y})$ by $\mathbf{z}_0 = \mathbf{f}(0, \mathbf{y}) = \mathbf{g}(\mathbf{y})$, then $\mathbf{z}_1 = \mathbf{f}(1, \mathbf{y}) = \mathbf{h}(\mathbf{z}_0, 0, \mathbf{y})$, then $\mathbf{z}_2 = \mathbf{f}(2, \mathbf{y}) = \mathbf{h}(\mathbf{z}_1, 1, \mathbf{y})$, then ..., then $\mathbf{z}_m = \mathbf{f}(m, \mathbf{y}) = \mathbf{h}(\mathbf{z}_{m-1}, m-1)$

 $(1, \mathbf{y})$. One needs to do so with some sufficient precision so that the result given by $\mathbf{f}(l, \mathbf{y})$ is correct, and so that the whole computation can be done in polynomial time.

Given y, we can determine Y such that $y \in [-2^Y, 2^Y]$. Assume for now that for all m,

$$z_m \in [-2^{Z_m}, 2^{Z_m}] \tag{7}$$

For
$$i = 0, 1, ... l$$
, consider $p(i) = \alpha^{l-i} n + \sum_{k=1}^{l-1} \alpha^{k-i} p_k(\mathbf{Z}_k, \ell(k), \mathbf{Y})$.

Using the fact that **g** is computable, approximate $\mathbf{z}_0 = \mathbf{g}(\mathbf{y})$ with precision $2^{-p(0)}$. This is doable polynomial time with respect to the value of p(0).

Then for i = 0, 1, ..., l, using the approximation of \mathbf{z}_i with precision $2^{-p(i)}$, compute an approximation of \mathbf{z}_{i+1} with precision $2^{-p(i+1)}$: this is feasible to get precision $2^{-p(i+1)}$ of \mathbf{z}_{i+1} , as $\mathbf{z}_{i+1} = \mathbf{f}(i+1,\mathbf{y}) = \mathbf{h}(\mathbf{z}_i,i,\mathbf{y})$, it is sufficient to consider precision $m_h(\mathbf{Z}_i,\ell(i),\mathbf{Y},p(i+1)) = \alpha p(i+1) + p_h(\mathbf{Z}_i,\ell(i),\mathbf{Y}) = \alpha^{l-i-1+1}n + \sum_{k=i+1}^{l-1} \alpha^{k-i-1+1}p_h(\mathbf{Z}_k,\ell(k),\mathbf{Y}) + p_h(\mathbf{Z}_i,\ell(i),\mathbf{Y}) = p(i)$. Observing that p(l) = n, we get z_l with precision 2^{-n} . All of this is is indeed feasible in polynomial time with respect to the value of l, under the condition that all the Z_i remain of size polynomial, that is to say, that we have indeed

(7). But this follows from our hypothesis on $\ell(|||\mathbf{f}(x,\mathbf{y})|||)$.

C.10 **Proof of Lemma 11**

We provide more details and intuition on the proof of Lemma 11.

To compute d, given \overline{d} , the intuition is to consider a three-tapes Turing machine $(Q, \Sigma, q_{init}, \delta, F)$ the first tape contains the input (\overline{d}) , and is read-only, the second and third one are write-only and empty at the beginning. We just use a different encoding on the second tape that the previous one: For the first tape, we do restrict to digits 0,1,3, while for the second, we use binary encoding.

Writing the natural Turing machine that does the transformation, this would basically do the following (in terms of real numbers), if we forget the encoding of the internal state.

$$F(\overline{r_1}, \overline{l_2}, \lambda) = \begin{cases} (\xi(16\overline{r_1}), 2\overline{l_2} + 0, \lambda) & \text{whenever } \sigma(16\overline{r_1}) = \overline{11} = 5\\ (\xi(16\overline{r_1}), 2\overline{l_2} + \lambda, \lambda) & \text{whenever } \sigma(16\overline{r_1}) = \overline{13} = 7\\ (\xi(16\overline{r_1}), (\overline{l_2} + 0)/2, \lambda) & \text{whenever } \sigma(16\overline{r_1}) = \overline{31} = 13\\ (\xi(16\overline{r_1}), (\overline{l_2} + \lambda)/2, \lambda) & \text{whenever } \sigma(16\overline{r_1}) = \overline{33} = 15 \end{cases}$$

Here we write \overline{ab} for the integer whose base 4 radix expansion is ab.

This is how we got the function F considered in the main part of the paper. Then the previous reasoning applies on the iterations of function F that would provide some encoding function.

Concerning the missing details on the choice of function σ and ξ . From the fact that we have only 1 and 3 in \overline{r} , the reasoning is valid as soon as $16\overline{r}$ is in the neighborhood of $16\bar{r} \in \{\overline{11}, \overline{13}, \overline{31}, \overline{33}\}.$

Proof of Lemma 12 C.11

Proof. We discuss only the case d=1. The generalization to the general case is easy to obtain.

We then do something similar as in Lemma 11 but now with function $\operatorname{send}(0 \mapsto (\operatorname{div}_2(\overline{r_1}), (\overline{l_2} + 0)/2), \quad 1 \mapsto (\operatorname{div}_2(\overline{r_1}), (\overline{l_2} + 1)/2))(\operatorname{mod}_2(\overline{r_1})). \qquad \qquad \square$