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A structure to decide reachability in Petri nets

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Abstract

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A new structure to analyse Petri nets and decide reachability is presented. Originated in Mayr's regular constraint graphs with a consistent marking, it is simplified, cleared and made more flexible by the introduction of the new structure of *precovering graph*. With its help we prove new results for languages generated by a Petri net and initial and final markings constraints.

Introduction

Petri nets are a mathematical model used for the analysis of parallel processes [14]. The reachability problem for Petri nets was first mentioned in terms of vector addition systems in [6] and remained unsolved for a long time. The non-semi-linearity of the reachability set of some Petri nets [4] made it impossible to reduce the problem to the decidable emptiness of two computable semi-linear sets. It is only by the use of a singular technique that Mayr [10] proved that the problem is decidable. This technique was simplified later by Kosaraju [7, 11]. Unfortunately, the complexity of the two proofs (especially in [10]) wrapped the result in mystery and no use of their original ideas has been made until now.

In this article we extract from these two proofs a computable structure: *the perfect marked graph-transition sequences* which does not only allow to decide reachability but permits the general study of Petri nets with initial and final markings. We show the power of this new structure for the languages generated by a Petri net with initial and final markings.

This structure has its origin in Mayr's *regular constraint graphs with a consistent marking* [10] but the principle of the algorithm we use to build it is far more simple than Mayr's algorithm and essentially due to Kosaraju [7]. Technical steps are greatly simplified by the introduction of the precovering graphs. We completely suppressed the use of Presburger's arithmetic.

In the domain of Petri net languages, the main consequence of the perfect marked graph-transition sequences is a very general iteration lemma. As consequences we first get that the set $\{|u|_a, u \in L\}$ for an $a \in \Sigma$ and $L \subset \Sigma^*$ the language of a Petri net (with initial and final markings) whose transitions are arbitrary labelled is finite or contains an infinite arithmetic sequence. This leads us to prove that a wide family of languages are not Petri net languages in the sense of the most general definition [13]. We then obtain that the regularity of the unlabelled language of a Petri net with an initial and a final marking is decidable. We conclude by reproving that if Σ is a finite alphabet containing at least two letters, the language PAL(Σ) of the palindromes of Σ^* is not a Petri net language. This result has been proved already by Jantzen in [5] with a completely different method.

The Sections 1 and 2 of this article deal with notations and three well-known results that are used later. In Section 3 we present some structures on Petri nets. The first one is the well-known *covering graph* of which we slightly generalized the construction. The other two are the *precovering graphs* and the *marked graph-transition sequences*, the structure to decide reachability. The algorithm to decide reachability is presented in Section 4, it is based on one hand on a condition concerning the marked graph-transition sequences to compute a firable sequence from it; on the other hand on the decomposition of the marked graph-transition sequences which do not satisfy this condition. The announced results in Petri net languages theory are proved in Section 5.

1. Notations and elementary definitions

1.1. Petri nets and the reachability problem

$\mathbb{Z}$ is the set of integers, $\mathbb{N}$ is the set of positive integers. If P is a finite set let Card(P) be its cardinality. We define $\mathbb{N}^P$ and $\mathbb{Z}^P$, the sets of functions from P into $\mathbb{N}$ and $\mathbb{Z}$ (i.e. the set of the Card(P)-tuples indexed by P) and we define on $\mathbb{N}^P$ the component-wise order.

$$M \leq M' \Leftrightarrow \forall p \in P,\ M(p) \leq M'(p) \quad \text{(we say } M \text{ is smaller than } M'\text{)},$$

$$M < M' \Leftrightarrow M \leq M' \text{ and } M \neq M'.$$

A *Petri net* R is a 4-tuple (P, T, Pre, Post) where P is the finite set of *places*, T the finite set of *transitions* and Pre and Post two mappings from T in $\mathbb{N}^P$. The elements of $\mathbb{N}^P$ are called the markings of the Petri net R.

We say that $t \in T$ is *firable* at $m \in \mathbb{N}^P$ and the resulting markings is $m' \in \mathbb{N}^P$, denoted by $m[t\rangle m'$, iff $m \geqslant \mathrm{Pre}(t)$ and $m' = m - \mathrm{Pre}(t) + \mathrm{Post}(t)$. The firability at $m \in \mathbb{N}^P$ of a sequence of transitions $u = t_1 \ldots t_n$ is defined by induction.

$$m[u\rangle m' \Leftrightarrow u = u't_n,\ m[u'\rangle m'',\ m''[t_n\rangle m' \quad \text{and}$$

$$m[u\rangle \Leftrightarrow \exists m' \in \mathbb{N}^P,\ m[u\rangle m'.$$

We are now in a position to define the *reachability problem for Petri nets* (in terms of vector addition systems in [6]):

> *Given a Petri net R and two markings m_i and m_f find an algorithm to decide if there exists a sequence of transitions u such that $m_i[u\rangle m_f$.*

To present the algorithm we will need some additional definitions and concepts in formal language and graph theories. We introduce them now.

1.2. Some additional definitions

First we extend the definition of the markings to allow the value of some components to be arbitrarily large and define $\bar{\mathbb{N}} = \mathbb{N} \cup \{\omega\}$ where ω is the cardinality of $\mathbb{N}$ ($\omega + x = \omega$, $\omega - x = \omega$, $\omega \geqslant x\ \forall x \in \mathbb{N}$), $\bar{\mathbb{N}}^P$ the set of Card (P)-tuples in $\bar{\mathbb{N}}$ indexed by P.

On $\bar{\mathbb{N}}^P$ we define the component-wise order as in $\mathbb{N}^P$ and the ω-order defined by

$$M \leqslant_\omega M' \Leftrightarrow \forall p \in P,\ M'(p) \neq \omega \Rightarrow M(p) = M'(p)$$
(we say M is under M'),

$<_\omega$ being the corresponding strict order.

We trivially extend the definition of firability of sequences to markings in $\bar{\mathbb{N}}^P$. The following property is evident.

Proposition 1.1. *Let $R = (P, T, \mathrm{Pre}, \mathrm{Post})$ be a Petri net, $m \in \bar{\mathbb{N}}^P$ and u a sequence of transitions then $m|u\rangle$ implies that there exists $k \in \mathbb{N}$ such that for any $\mu \in \mathbb{N}^P$*

$$\mu \leqslant_\omega m \text{ and } \mu(p) \geqslant k \text{ if } m(p) = \omega \Rightarrow \mu[u\rangle.$$

It is useful to define the displacement of a sequence $u = t_1 \ldots t_n$ as $\|u\| = \sum_{i=1}^n \mathrm{Post}(t_i) - \mathrm{Pre}(t_i)$. For $x \in \bar{\mathbb{N}}^P$ the $p \in P$ such that $x(p) = \omega$ are called the ω-components of x. $\bar{x} \in \bar{\mathbb{N}}^P$ for $x \in \bar{\mathbb{N}}$ is defined for each $p \in P$ by $\bar{x}(p) = x$. For $x \in \bar{\mathbb{N}}^P$ and $T \subset P$, $x|_T$ is the vector of $\bar{\mathbb{N}}^T$ such that $x|_T(p) = x(p)$ for each $p \in T$.

Remark. In this article the letters m and $\mathcal{M}$ will designate members of $\bar{\mathbb{N}}^P$. μ will always be a member of $\mathbb{N}^P$. The relation between $\mu, m, \mathcal{M}$ will always be $m \geqslant_\omega \mathcal{M} \geqslant_\omega \mu$.

The announced concepts in formal language and graph theories are now introduced.

Formal languages

Let Σ be a finite alphabet. Σ^* denotes the set of finite words on Σ (i.e. the free monoid on Σ), λ is the empty word. For $u \in \Sigma^*$, $u \neq \lambda$, $u = a_1 a_2 a_3 \ldots a_n$ we define

- $\lg(u) = n$ its length,
- $|u| \in \mathbb{N}^{\Sigma}$ the Parikh image of u, $|u|(a)$ denoted by $|u|_a$ is the number of occurrences of a in u,
- $u^R = a_n \ldots a_1$ its reverse,
- $u(i) = a_i \in \Sigma$ the ith letter of u.

A language L on alphabet Σ is a subset of Σ^*. A morphism from a language L into another L' is a function ϕ such that $\phi(uv) = \phi(u)\phi(v)$.

Given a Petri net R, we will use the *language of the sequences firable at* $\mu_i \in \mathbb{N}^P$, and the *language of the sequences firable at* μ_i *for which the resulting marking is* $\mu_f \in \mathbb{N}^P$:

$$L(R, \mu_i) = \{u \in T^*,\ \mu_i[u\rangle\},$$

$$L(R, \mu_i, \mu_f) = \{u \in T^*,\ \mu_i[u\rangle\mu_f\}.$$

Some other concepts and definitions which are not necessary in the proof of the decidability of reachability will be introduced in Section 5.3.

Graphs

A *directed graph* G is a couple (V, E) where V is the finite set of *vertices* and E a multi-set of elements of $V \times V$, called the *directed edges* or *arcs*. $E(G)$ denotes the set of arcs of G, and $V(G)$ the set of vertices of G. The set of the paths in G from $x \in V(G)$ to $y \in V(G)$ will be considered as a language on $E$and denoted by $L_0(G, x, y)$. If we consider G as an automaton, $L_0(G, x, y)$ is the language of G with x as initial state and y as final state.

For any $x \in V$, the elements of $L_0(G, x, x)$ are called (*directed*) *circuits*. For $x \in V(G)$ we denote by

$$\omega^+(x) = \{(x, y) \in E(G)\} \text{ the set of arcs leaving } x,$$

$$\omega^-(x) = \{(y, x) \in E(G)\} \text{ the set of arcs entering } x.$$

For a vertex x we define the *strongly connected component* of x (SCC of x) by $G' = (V', E')$,

$$V' = \{y \in V \mid L_0(G, x, y) \neq \emptyset \text{ and } L_0(G, y, x) \neq \emptyset\},$$

$$E' = E \cap V' \times V'.$$

A graph is called *strongly connected* iff it is the SCC of one of its vertices (and it is then the SCC of any of its vertices). A *labelled graph* is a couple (G, t) where G is a graph and t is a mapping from E into a set T. If we select two vertices x and y, the language of (G, t) from x to y is $t(L_0(G, x, y)) \subset T^*$. Since there will be no ambiguity concerning the mapping t, this language will be denoted by $L(G, x, y)$. This language is the language of G considered as an automaton over alphabet T.

2. Three elementary results

We present here three well-known and easy results that we will use in the following. The first one is Euler's classical theorem on the paths in a graph; it is used in every proof of the reachability theorem [10, 7]. The second one is a refinement of Presburger's arithmetic we need in the algorithm. We define a set Π easier to compute than the semi-linear sets of [10, 7]. The third one is implicit in [10, 7]. It tells that the paths in a graph have a canonical decomposition in circuits.

Theorem 2.1. *Let $G = (V, E)$ be a strongly connected graph and $x \in \mathbb{N}^E$ satisfying $x \geq \bar{1}$ and*

$$\forall q \in V: \quad \sum_{e \in \omega^+(q)} x(e) = \sum_{e \in \omega^-(q)} x(e)$$

then for any $q \in V$, there exists a path $u \in L_0(G, q, q)$ such that $|u| = x$. These equations are traditionally called Kirchoff's laws.

Theorem 2.2. *Let P be a finite set and $Ax = b$ a system of equations for $x \in \mathbb{N}^P$. The set*

$$R = \{p \in P \mid \forall x \in \mathbb{N}^P,\ Ax = 0 \Rightarrow x(p) = 0\}$$

is computable and if $R \neq \emptyset$ the set $S = \{x|_R,\ x \in \mathbb{N}^P,\ Ax = b\}$ is finite and computable. Moreover, a vector $x_0 \in \mathbb{N}^P$ such that $x_0|_R \geq \bar{1}$ and $Ax_0 = 0$ is called a maximal support solution of $Ax = 0$. We can compute one such solution.

An algorithm for computing the sets R and S and a maximal support solution is described in [8].

Theorem 2.3. *Let $G = (V, E)$ be a finite graph, $(r, q) \in V^2$. There exists a finite computable subset Σ of $L_0(G, r, q)$ such that for any $u \in L_0(G, r, q)$, there exists $s \in \Sigma$ and*

$$u = u_0 s(1) u_1 s(2) \ldots s(\lg(s)) u_{\lg(s)}$$

with $u_i \in L_0(G, q_i, q_i)$, where q_i is defined by $q_0 = r$, $s(i) = (q_{i-1}, q_i)$ and $q_{\lg(s)} = q$.

Proof. Σ is the set of the circuit-free words of $L_0(G, r, q)$. $\square$

3. The structures

3.1. The covering graphs

The structure of *covering graph* has been introduced first by Karp and Miller [6]. It allows to decide if the marking in a given place p can be arbitrarily increased. We will have to test this property during the algorithm for reachability.

Let $R = (P, T, \text{Pre}, \text{Post})$ be a Petri net and G a graph labelled by T. We choose $q_0 \in V(G)$ as the initial state of G. We define the covering graph by a simple generalization of its classical definitions (see [6]) to take into account the additional constraint that we are only interested in the firing sequences which are also a path leaving q_0 in G. It must be noticed that it is possible to simulate this constraint by adding a place for each vertex of G and by creating a transition for each arc $e=(q, q')$ of G whose action is to take a token in the place associated with q and to put it in the place assoicated with q' for the new places and the action of $t(e)$ for the remaining places. The initial new marking m_i' is: *one token in the place associated with* q_0, *zero in the other new places and* m_i *elsewhere.*

Definition. Let $R=(P, T, \text{Pre}, \text{Post})$ be a Petri net, $m_i \in \bar{\mathbb{N}}^P$, G a graph labelled by T and $q_0 \in V(G)$. Let $\mathcal{G}=(V, E)$ a graph labelled by T such that $V \subset V(G) \times \bar{\mathbb{N}}^P$. $\mathcal{G}$ is a *covering graph* for (R, G, q_0, m_i) and we write

$$\mathcal{G} \in \mathrm{CG}(R, G, q_0, m_i)$$

iff it is obtained as the result of an execution of the following algorithm:

begin
  % we begin by building the tree $\mathcal{G}'$ %
  % building $\mathcal{G}'$ %
  begin
    $\mathcal{G}'$ is a graph with only one unmarked vertex labelled (q_0, m_i) while there exists an unmarked vertex k labelled (q, m) in $\mathcal{G}'$
  do
  begin
    mark it
    for all $e=(q, q') \in E$ such that $m[t(e)\rangle \mathcal{M}'$
      define m' by
        $m'(p)=\omega$ if there exists an ancestor k'' of k labelled (q', m'') with
            $m'' < \mathcal{M}'$ and $\mathcal{M}'(p) > m''(p)$
        $m'(p)=\mathcal{M}'(p)$ else
      build a vertex k' labelled (q', m') and an arc (k, k') labelled $t(e)$.
      if there exists an ancestor of k' different from k' with the same label,
      mark k'.
    end
  end

  % building $\mathcal{G}$ %
  begin
    $V(\mathcal{G})$ is the set of the labels of the vertices of $\mathcal{G}'$ there exists an arc $((q, m), (q', m'))$ labelled t in $\mathcal{G}$ iff there exists an arc (k, k') in $E(\mathcal{G}')$ labelled t where k is labelled (q, m) and k' is labelled (q', m').
  end
end

It is clear that by construction $\mathrm{CG}(R, G, q_0, m_i) \neq \emptyset$. The other main properties of $\mathrm{CG}(R, G, q_0, m_i)$ are given in the following theorem.

Theorem 3.1. *Let* $\mathcal{G} \in \mathrm{CG}(R, G, q_0, m_i)$ *then the following assertions are true*:

(i) $\mathcal{G}$ *is finite and we can compute a member of* $\mathrm{CG}(R, G, q_0, m_i)$.

(ii) *For any* $e = ((q, m), (q', m')) \in E(\mathcal{G})$ *we have* $m[t(e)\rangle\mathcal{M}' \leq_\omega m'$.

(iii) $u \in L(R, m_i) \cap L(G, q_0, q)$ *implies* $u \in L(\mathcal{G}, (q_0, m_i), (q, m))$ *where* $m \geq_\omega m_i + \|u\|$.

(iv) *For each* $(q, m) \in V(\mathcal{G})$, *and each* $N \in \mathbb{N}$ *we can compute* $u_N \in L(G, q_0, q) \cap L(R, m_i)$ *such that* $m_i[u_N\rangle m_N$ *where* m_N *satisfies*

$$m_N \geq_\omega m \quad \text{and} \quad m(p) = \omega \Rightarrow m_N(p) \geq N.$$

(v) *There exists a vertex* $(q_0, m) \in V(\mathcal{G})$ *such that for each* $(q_0, m') \in V(\mathcal{G})$

$$m_i \leq_\omega m' \Rightarrow m' \leq_\omega m \text{ (i.e. } m \text{ is the largest marking over } m_i \text{ in } q_0)$$

m is called the covering of (R, G, q_0, m_i) *denoted by* $m = C(R, G, q_0, m_i)$. *m does not depend on the chosen* $\mathcal{G} \in \mathrm{CG}(R, G, q_0, m_i)$ *and* $m_i \leq_\omega m$.

Proof. Properties (i) to (iv) are classical. To prove (v) let $\mathcal{G}' \in \mathrm{CG}(R, G, q_0, m_i)$ and $(q_0, m') \in V(\mathcal{G})$, $(q_0, m'') \in V(\mathcal{G}')$ such that $m' \geq_\omega m_i$, $m'' \geq_\omega m_i$ we claim that there exists $(q_0, m) \in V(\mathcal{G})$ and $m \geq_\omega m'$, $m \geq_\omega m''$. By (iv) for any $N \in \mathbb{N}$ there exists two sequences u_1 and u_2 in $L(G, q_0, q_0)$ such that

$$m_i[u_1\rangle m_1, \ m_i[u_2\rangle m_2,$$

$$m_1(p) = m_i(p) \text{ if } m_i(p) = m'(p), \ m_1(p) > m_i(p) + N \text{ else,}$$

$$m_2(p) = m_i(p) \text{ if } m_i(p) = m''(p), \ m_2(p) > m_i(p) + N \text{ else.}$$

Now $m_i[u_1u_2\rangle m_3$ and by (iii) there exists $(q_0, m) \in V(\mathcal{G})$ such that $m \geq_\omega m_3$. Let us choose N greater than any finite component of a vertex of $\mathcal{G}$. We get $m'(p) = \omega$ or $m''(p) = \omega \Rightarrow m_3(p) > N \Rightarrow m(p) = \omega$ and $m_3(p) = m'(p) = m''(p)$ else. Thus $m \geq_\omega m'$ and $m \geq_\omega m''$.

Now let $(q_0, m) \in V(\mathcal{G})$ such that $m' \geq_\omega m$ is impossible for $(q_0, m) \in V(\mathcal{G})$, clearly m satisfies the conclusion of the theorem. □

We now give the important definition of *covering sequence*.

Definition. Let $R = (P, T, \mathrm{Pre}, \mathrm{Post})$ be a Petri net, G a directed graph labelled by T, $q_0 \in V(G)$, $m_i \in \bar{\mathbb{N}}^P$. Let $m = C(R, G, q_0, m_i)$ be the covering of m_i then any $u \in L(G, q_0, q_0) \cap L(R, m_i)$ such that

$$m_i(p) \neq \omega \ \Rightarrow \ (\|u\|(p) \geq 0 \text{ and } \|u\|(p) > 0 \text{ iff } m(p) = \omega)$$

is called a *covering sequence* of (R, G, q_0, m_i). The set of the covering sequences is denoted by $\mathrm{CS}(R, G, q_0, m_i)$.

We have an evident corollary of Theorem 3.1.

Corollary 3.1. $\mathrm{CS}(R, G, q_0, m_i) \neq \emptyset$ *and we can compute one of its elements.*

3.2. The precovering graphs

The *precovering graphs* are implicit in Mayr's and Kosaraju's proofs. In our proof they will simplify the technical steps with the help of their simple and elegant properties. In Section 3.2.3 we will establish that we can decompose this structure gracefully. These decomposition results are a fundamental part of the reachability algorithm.

3.2.1. Definition and elementary properties

Definition. Let R be a Petri net. A T-labelled directed strongly connected graph $G = (V, E)$ such that $V \subset \bar{\mathbb{N}}^P$ is a *precovering graph* on R iff

$$\forall e \in E(G), \quad e = (m, m') \Rightarrow m[t(e)\rangle \mathcal{M}' \leqslant_\omega m'.$$

We say that a precovering graph is initiated if one of its vertices is distinguished; an *initiated precovering graph* (*IPG*) is thus a couple (C, m) where $m \in V(C)$.

The fundamental properties of precovering graphs are listed in the following proposition.

Proposition 3.1. *Let* $R = (P, T, \mathrm{Pre}, \mathrm{Post})$ *be a Petri net and* $G = (V, E)$ *a precovering graph on R. Then*

(i) $e = (m, m') \in E \Rightarrow m[t(e)\rangle m'$ *and the vertices of G have the same* ω*-components. We then may define* $\Omega(G) = \{p \in P \mid \forall m \in V, m(p) = \omega\}$, *the set of the* ω*-components of G.*

(ii) *For two vertices m and m' in V,* $u \in L(G, m, m') \Rightarrow m[u\rangle m'$.

(iii) *Every strongly connected sub-graph of a precovering graph is a precovering graph.*

(iv) *Let* $m_i \in V$, $\mathcal{M}_i \in \bar{\mathbb{N}}^P$ *such that* $\mathcal{M}_i \leqslant_\omega m_i$ *and* $\mathcal{G} \in \mathrm{CG}(R, G, m_i, \mathcal{M}_i)$, *then the vertices* $(m, \mathcal{M})$ *of* $\mathcal{G}$ *satisfy* $\mathcal{M} \leqslant_\omega m$ *and the projection*

$$\Pi_2 : V(\mathcal{G}) \to \bar{\mathbb{N}}^P, \qquad (m, \mathcal{M}) \to \mathcal{M}$$

is injective. Moreover, each SCC of the graph $\Pi_2(\mathcal{G})$ *is a precovering graph on R.*

Proof. (i) is due to the strong connectivity of the graph, (ii) is an iteration of (i), (iii) is evident. (iv) Let $(m, \mathcal{M}) \in V(\mathcal{G})$, by Theorem 3.1 (iv) for any $N \in \mathbb{N}$ there exists $u_N \in L(G, m_i, m)$ such that

$$\mathcal{M}[u_N\rangle \mathcal{M}_N, \qquad \lim_{N \to +\infty} \mathcal{M}_N = \mathcal{M}$$

but by (ii) $m_i[u_N\rangle m$ thus $\mathcal{M}_N = \|u_N\| + \mathcal{M}_i \leqslant_\omega m_i + \|u_N\| = m$ finally $\mathcal{M} \leqslant_\omega m$ which implies that Π_2 is injective since the vertices of G have the same ω-components.

By Theorem 3.1(ii), $e=((m,\mathcal{M}),(m',\mathcal{M}'))\in E(\mathcal{G})$ implies $m[t(e)\rangle\mathcal{M}'\leqslant_\omega m'$ and the SCC of $\Pi_2(\mathcal{G})$ are thus precovering graphs. □

Property (ii) tells that the precovering graphs describe the firability on bounded places. (iii) and (iv) show that when we decompose a precovering graph we get new precovering graphs.

3.2.2. The inversion of the precovering graphs

A useful operation will be to reverse the Petri net and the precovering graphs. The reason why we will have to do this is that we will increase the marking in some places to make some sequences firable and then we will have to decrease it. This second operation is equivalent to increase the marking in the reverse net.

Definitions. Let $R=(P,T,\text{Pre},\text{Post})$ be a Petri net. We define

(i) The Petri net R^{rev} as $R^{\text{rev}}=(P,T^{\text{rev}},\text{Pre},\text{Post})$ where $T^{\text{rev}}=\{t^{\text{rev}},\ t\in T\}$, $\text{Pre}(t^{\text{rev}})=\text{Post}(t)$, $\text{Post}(t^{\text{rev}})=\text{Pre}(t)$.

(ii) For $u\in T^*$ its reverse $u^{\text{rev}}\in(T^{\text{rev}})^*$ is defined as $u^{\text{rev}}=t_n^{\text{rev}}\ldots t_1^{\text{rev}}$.

(iii) For $C=(V,E)$ a precovering graph on R, $C^{\text{rev}}=(V,E^{\text{rev}})$ where $E^{\text{rev}}=\{e^{\text{rev}}$ for $e\in E\}$ *and* e^{rev} is defined for $e=(q,q')$ by $e^{\text{rev}}=(q',q)$, $t(e^{\text{rev}})=t(e)^{\text{rev}}$.

The following proposition is clear.

Proposition 3.2. (i) $m[u\rangle m'$ *in* $R\Leftrightarrow m'[u^{\text{rev}}\rangle m$ in R^{rev}.

(ii) *If C is a precovering graph on R, C^{rev} is a precovering graph on R^{rev}.*

3.2.3. The decomposition of the precovering graphs

We now apply the properties of the precovering graph (Proposition 3.1) to decompose them gracefully. We present three propositions corresponding to three situations that will occur in the decomposition algorithm presented at Theorem 4.2. The general principle of the decomposition is the same in each case and described in Fig. 1. A precovering graph is burst into a finite set of alternating sequences of transitions and precovering graphs. Each new precovering graph is in a sense smaller than the original one either because it contains less arcs or because its vertices have less ω-components.

Proposition 3.3. *Let $\mathcal{G}=(C,m)$ be an IPG on a Petri net R, $\mathcal{M}\leqslant_\omega m$.*

If $m=C(R,C,m,\mathcal{M})$ then let $u\in \mathrm{CS}(R,C,m,\mathcal{M})$ and $v\in L(C,m,m)$. There exists two integers k_v and k_v' such that

$$k\geqslant k_v \text{ implies } \mathcal{M}[u^kv\rangle,\qquad k\geqslant k_v' \text{ implies } u^kv\in \mathrm{CS}(R,C,m,\mathcal{M}).$$

If $m\neq C(R,C,m,\mathcal{M})$ then we can compute a finite (possibly empty) subset Σ of T^* and for every $s\in\Sigma$ a sequence of IPG: $(C_0^s,m_0^s)\ldots(C_{\lg(s)}^s,m_{\lg(s)}^s)$ such that

$$\mathcal{M}=m_0^s,\qquad \Omega(C_i^s)\subsetneq\Omega(C),\qquad m_i^s[s(i)\rangle m_i^s+\|s(i)\|\leqslant_\omega m_{i+1}^s.$$

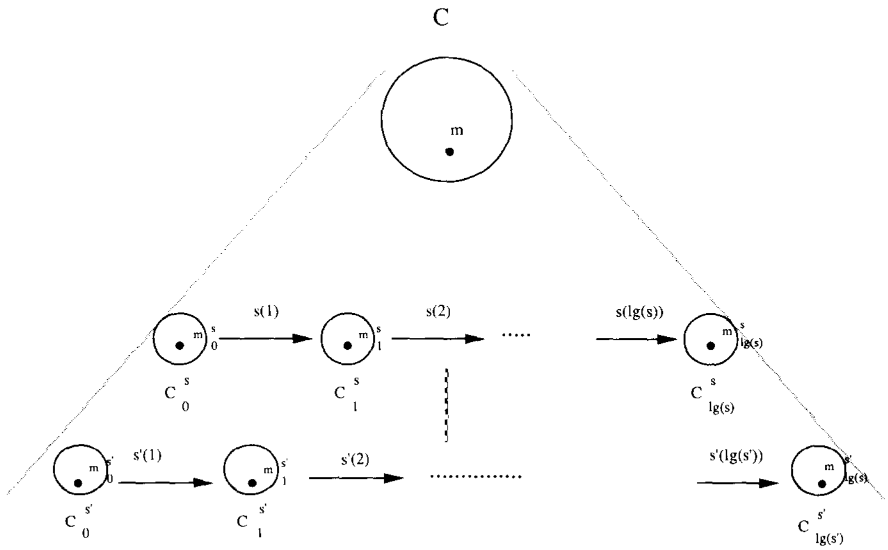

Fig. 1.

For every $u \in L(C, m, m)$ *and* $\mathcal{M}[u\rangle$, *there exists* $s \in \Sigma$ *such that*

$$u = u_0 s(1) \ldots s(\lg(s)) u_{\lg(s)} \text{ with } u_i \in L(C_i^s, m_i^s, m_i^s).$$

Proof. If $m = C(R, C, m, \mathcal{M})$ the conclusion is a consequence of $m[v\rangle$ by Proposition 1.1 and the definition of a covering sequence. If $m \neq C(R, C, m, \mathcal{M})$, compute a $\mathcal{G} \in \mathrm{CG}(R, C, m, \mathcal{M})$, let $u \in L(C, m, m)$ such that $\mathcal{M}[v\rangle$ by Theorem 3.1, $u \in L(\mathcal{G}, (m, \mathcal{M}), (m, \mathcal{M}'))$ and

$$\mathcal{M}' \leqslant_\omega C(R, C, m, \mathcal{M}) <_\omega m \quad (\text{since } (m, C(R, C, m, \mathcal{M})) \in V(\mathcal{G}) \text{ and } m \neq C(R, C, m, \mathcal{M})).$$

To conclude we just apply Theorem 2.3 to $L(\mathcal{G}, (m, \mathcal{M}), (m, \mathcal{M}'))$ and transform $\mathcal{G}$ by the projection Π_2. C_i^s is the SCC of m_i^s in $\Pi_2(\mathcal{G})$, $m_i^s[s(i)\rangle m_i^s + \|s(i)\| \leqslant_\omega m_{i+1}^s$ by Theorem 3.1(ii) and $\Omega(C_i^s) \subsetneq \Omega(C)$ because

$$m_0^s[s(1)\rangle m_0^s + \|s(1)\| \leqslant_\omega m_1^s \ldots [s(n)\rangle m_{n-1}^s + \|s(n)\| \leqslant_\omega m_n^s = \mathcal{M}' <_\omega m$$

and the m_i^s have strictly less ω-components than m. □

By applying the inversion operation, we get the reverse version of this proposition.

Proposition 3.4. *Let* $\mathcal{G} = (C, m)$ *be an IPG on a Petri net* R, $\mathcal{M} \leqslant_\omega m$. *If* $m = C(R^{\mathrm{rev}}, C^{\mathrm{rev}}, m, \mathcal{M})$ *then let* $u^{\mathrm{rev}} \in \mathrm{CS}(R^{\mathrm{rev}}, C^{\mathrm{rev}}, m, \mathcal{M})$ *and* $v \in L(C, m, m)$, *there exists two integers* k_v *and* k'_v *such that*

$$k \geqslant k_v \text{ implies } \mathcal{M}[(u^{\mathrm{rev}})^k v^{\mathrm{rev}}\rangle,$$

$$k \geqslant k'_v \text{ implies } (u^{\mathrm{rev}})^k v^{\mathrm{rev}} \in \mathrm{CS}(R^{\mathrm{rev}}, C^{\mathrm{rev}}, m, \mathcal{M}).$$

If $m \neq C(R^{\mathrm{rev}}, C^{\mathrm{rev}}, m, \mathcal{M})$ *then we can compute a finite (possibly empty) subset* Σ *of* T^* *and for every* $s \in \Sigma$ *a sequence of IPG*: $(C_0^s, m_0^s)\ldots(C_{\lg(s)}^s, m_{\lg(s)}^s)$ *such that*

$$\mathcal{M} = m_{\lg(s)}^s, \qquad \Omega(C_i^s) \subsetneq \Omega(C), \qquad m_i^s[s(i-1)^{\mathrm{rev}}\rangle m_i^s - \|s(i-1)\| \leq_\omega m_{i-1}^s.$$

For every $u \in L(C, m, m)$ *such that* $\mathcal{M}[u^{\mathrm{rev}}\rangle$, *there exists* $s \in \Sigma$ *and*

$$u = u_0 s(1)\ldots s(\lg(s)) u_{\lg(s)} \text{ with } u_i \in L(C_i^s, m_i^s, m_i^s).$$

The third case of decomposition will occur in a situation which does not depend on the precovering graph but on the structure it will belong to.

Proposition 3.5. *Let* $\mathcal{G} = (C, m)$ *be an IPG on a Petri net* $R = (P, T, \mathrm{Pre}, \mathrm{Post})$. *Let* $\mathscr{E} \subset E(C)$ *be a non-empty subset of arcs and* $F \subset \mathbb{N}^{\mathscr{E}}$ *a finite set of* Card($\mathscr{E}$)*-tuples. There exists a finite (possibly empty) computable subset* Σ *of* T^* *and for every* $s \in \Sigma$ *a sequence of IPG*: $(C_0^s, m_0^s)\ldots(C_{\lg(s)}^s, m_{\lg(s)}^s)$ *such that*

$$m = m_0^s = m_{\lg(s)}^s, \qquad \Omega(C_i^s) = \Omega(C),$$

$$\mathrm{Card}(E(C_i^s)) < \mathrm{Card}(E(C)), \qquad m_i^s[s(i)\rangle m_{i+1}^s.$$

For every $u \in L(C, m, m)$ *satisfying* $u = t(u^0)$, $|u^0||_{\mathscr{E}} \in F$ *there exists* $s \in \Sigma$ *and*

$$u = u_0 s(1) u_1 s(2)\ldots s(\lg(s)) u_{\lg(s)} \text{ with } u_i \in L(C_i^s, m_i^s, m_i^s).$$

Proof. Let $C' = (V(C), E(C) - \mathscr{E})$ and $u^0 \in L_0(C, m, m)$ such that $|u^0|_{\mathscr{E}} \in F$, then u^0 is of the form

$$u^0 = u_{0s(1)}^0 u_1^0 s(2)\ldots s(\lg(s)) u_{\lg(s)}^0$$

where $|s||_{\mathscr{E}} = |u^0||_{\mathscr{E}} \in F$, $|s||_{E-\mathscr{E}} = \bar{0}$, $u_i^0 \in L(C, q_i, q_i')$ where q_i and q_i' are defined by $s(i) = (q_{i-1}', q_i)$, $q_0 = q_{\lg(s)} = m$. Since F is finite, there exists a finite set of words $s \in E$ such that $|s||_{\mathscr{E}} \in F$ and $|s||_{E-\mathscr{E}} = \bar{0}$. Applying the result of Theorem 2.3 to $L_0(C', q_i, q_i')$ we get the result with C_i^s the SCC of m_i^s in C' and

$$\mathrm{Card}(E(C_i^s)) \leq \mathrm{Card}(E(C')) = \mathrm{Card}(E(C)) - \mathrm{Card}(\mathscr{E}).$$

Moreover, $m = m_0^s = m_{\lg(s)}^s$ because $q_0 = q_{\lg(s)} = m$, $m_i^s[s(i)\rangle m_{i+1}^s$ is by Proposition 3.1(i). $\square$

3.3. The marked graph-transition sequences

3.3.1. Definitions

The *graph-transition sequences* are the result of the decomposition of the precovering graphs. Their definition is deduced from the general decomposition scheme of Propositions 3.3–3.5.

Definition. Let $R = (P, T, \text{Pre}, \text{Post})$ be a Petri net. *A graph-transition sequence* (*GTS*) $\mathcal{U}$ on R is a finite alternating sequence of initiated precovering graphs and transitions of the Petri net:

$$(C_0, m_0)t_1(C_1, m_1)\ldots t_n(C_n, m_n).$$

A graph-transition sequence is *marked* (*MGTS*) if there exists a function φ which associates a couple $(\mathcal{M}_i, \mathcal{M}'_i) \in (\bar{\mathbb{N}}^P)^2$ to each initiated precovering graph (C_i, m_i) of the GTS such that

$$\mathcal{M}_i \leqslant_\omega m_i \quad \text{and} \quad \mathcal{M}'_i \leqslant_\omega m_i.$$

$\mathcal{M}_i$ will be called the *input marking* of C_i and $\mathcal{M}'_i$ the *output marking* of C_i.

The marked graph-transition sequence will be denoted $(\mathcal{U}, \varphi)$, $\mathcal{M}^{\text{in}}(\mathcal{U}, \varphi) = \mathcal{M}_0$ is the *input marking* of $(\mathcal{U}, \varphi)$; $\mathcal{M}^{\text{out}}(\mathcal{U}, \varphi) = \mathcal{M}'_n$ is the *output marking* of $(\mathcal{U}, \varphi)$.

The language of a MGTS is the set of the sequences firable in the Petri net R which are made of paths in the IPG of the GTS and respect the initial and the final markings of each IPG.

Definition. Let $(\mathcal{U}, \varphi)$ be a MGTS on a Petri net R. Its language, denoted by $L(\mathcal{U}, \varphi)$, is the set of the sequences $u = u_0t_1u_1\ldots t_nu_n$ such that

(i) $u_i \in L(C_i, m_i, m_i)$,

(ii) there exists $\mu_0, \mu'_0, \ldots, \mu_n, \mu'_n$ in $\mathbb{N}^P$ such that $\mu_i \leqslant_\omega \mathcal{M}_i$, $\mu'_i \leqslant_\omega \mathcal{M}'_i$ and

$$\mu_0[u_0\rangle\mu'_0[t_1\rangle\mu_1[u_1\rangle\ldots[t_n\rangle\mu_n[u_n\rangle\mu'_n.$$

These definitions are illustrated in Fig. 2.

The principle of the algorithm is to find a MGTS having some properties which allow us to compute a sequence belonging to its language. To do that we will approximate the constraints given in the previous definition by decidable constraints from which we will deduce a sequence of $L(\mathcal{U}, \varphi)$. The characteristic equation we introduce now is the "linear" part of these constraints.

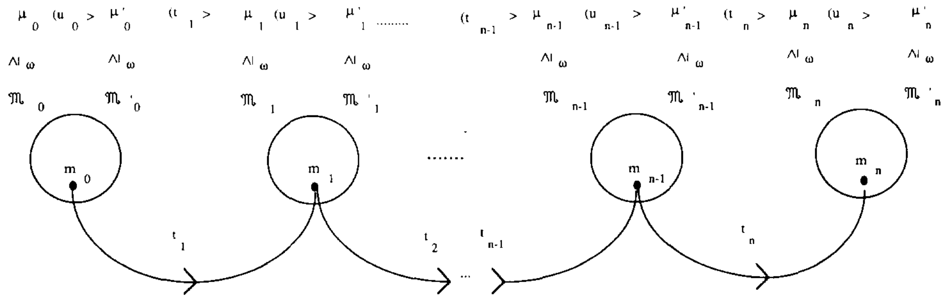

Fig. 2.

3.3.2. *The characteristic equation of a MGTS*

Definition. Let $u \in L(\mathcal{U}, \varphi)$, where $(\mathcal{U}, \varphi)$ is a MGTS on a Petri net R. We define the set of components by $\mathscr{C} = \{c_i(p),\ c'_i(p)$ for $0 \leq i \leq n$ and $p \in P\}$ (this is the set of variables for the components of the input and output markings of the IPG of $(\mathcal{U}, \varphi)$) and let $E = \bigcup_{i=0}^{n} E(C_i)$. A vector $x \in \mathbb{N}^{E \cup \mathscr{C}}$ is *associated* with $u = u_0 t_1 u_1 \ldots t_n u_n \in L(\mathcal{U}, \varphi)$ iff there exist

$$\text{for } 0 \leq i \leq n: \quad u_i^0 \in L_0(C_i, m_i, m_i) \text{ such that } u_i = t(u_i^0),$$

$$\text{for } 0 \leq i \leq n: \quad \mu_i \in \mathbb{N}^P,\ \mu'_i \in \mathbb{N}^P \text{ such that } \mu_i \leq_\omega \mathcal{M}_i,\ \mu'_i \leq_\omega \mathcal{M}'_i \text{ and}$$

$$\mu_0[u_0\rangle \mu'_0[t_1\rangle \mu_1[u_1\rangle \ldots [t_n\rangle \mu_n[u_n\rangle \mu'_n$$

and $x|_{E(C_i)} = |u_i^0|$, $x(c_i(p)) = \mu_i(p)$, $x(c'_i(p)) = \mu'_i(p)$.

This vector, which exists but is not unique for a given $u \in L(\mathcal{U}, \varphi)$, satisfies a system of equations which is called the characteristic equation of $(\mathcal{U}, \varphi)$. We now present this system.

Theorem and Definition. Let $(\mathcal{U}, \varphi)$ a MGTS on a Petri net R. The *characteristic equation* of $(\mathcal{U}, \varphi)$ is the following system of equations satisfied by every vector $x \in \mathbb{N}^{E \cup \mathscr{C}}$ associated with a $u \in L(\mathcal{U}, \varphi)$:

for every $0 \leq i \leq n$ and $p \in P$,

$$x(c_i(p)) = \mathcal{M}_i(p) \qquad \text{if } \mathcal{M}_i(p) \neq \omega,$$

$$x(c'_i(p)) = \mathcal{M}'_i(p) \qquad \text{if } \mathcal{M}'_i(p) \neq \omega,$$

$$x(c_{i+1}(p)) - x(c'_i(p)) = \|t_{i+1}\|(p) \qquad \text{if } i \leq n-1,$$

$$x(c_i(p)) + \sum_{e \in E(C_i)} x(e)\|t(e)\|(p) - x(c'_i(p)) = 0.$$

for every $0 \leq i \leq n$ and $m \in V(C_i)$,

$$\sum_{e \in \omega^+(m)} x(e) - \sum_{e \in \omega^-(m)} x(e) = 0.$$

The fact that any x associated with an element of $L(\mathcal{U}, \varphi)$ satisfies these equations is straightforward and left to the reader.

4. The algorithm

4.1. *Finding a firable sequence*

4.1.1. *Some necessary conditions on the MGTS*

We are going to list the conditions on a MGTS that will allow us to get a firable sequence from it. We begin with some explanations.

The firable sequence will be constructed from a solution of the characteristic equation. This solution will have to satisfy the following additional properties:

- $u_i^0 \in L_0(C_i, m_i, m_i)$ is realised if $|u_i^0| = x|_{E(C_i)} \geq \bar{1}$, (Theorem 2.1).
- $\mu_i[u_i\rangle$ is partially realised if we can increase the $\mu_i(p)$ when $\mathcal{M}_i(p) = \omega$.
- $\mu_i[u_i^{\text{rev}}\rangle$ is partially realised if we can increase the $\mu_i'(p)$ when $\mathcal{M}_i'(p) = \omega$.

In addition we will require the following properties for the MGTS:

- $\mu_i'[t_{i+1}\rangle$ imposes $\mathcal{M}_i'[t_{i+1}\rangle$.
- $\mu_i[u_i\rangle$ is achieved if we can increase $\mu_i(p)$ when $m_i(p) = \omega$ and $\mathcal{M}_i(p) \neq \omega$.
- $\mu_i[u_i^{\text{rev}}\rangle$ is achieved if we can increase $\mu_i'(p)$ when $m_i(p) = \omega$ and $\mathcal{M}_i'(p) \neq \omega$.
- $\mu_{\text{i}} + \|u\| = \mu_{\text{f}}$ is realised if $\mathcal{M}^{\text{in}}(\mathcal{U}, \varphi) = \mu_{\text{i}}$ and $\mathcal{M}^{\text{out}}(\mathcal{U}, \varphi) = \mu_{\text{f}}$.

Some of these conditions are realised by decomposing the MGTS into perfect MGTS that we are just going to define. The remaining conditions will be checked after or realised before decomposing the MGTS.

Definition. Let $(\mathcal{U}, \varphi)$ be a MGTS on a Petri net R, $Ax = b$ its characteristic equation. We say that $(\mathcal{U}, \varphi)$ is perfect iff

(i) for $0 \leq i \leq n$,

$$m_i = C(R, C_i, m_i, \mathcal{M}_i), \qquad m_i = C(R^{\text{rev}}, C_i^{\text{rev}}, m_i, \mathcal{M}_i')$$

(ii) there exists a solution $x \in \mathbb{N}^{E \cup \mathscr{C}}$ of the equation $Ax = 0$ such that

$$x|_E \geq \bar{1}, \qquad x(c_i(p)) \geq 1 \text{ if } \mathcal{M}_i(p) = \omega, \qquad x(c_i'(p)) \geq 1 \text{ if } \mathcal{M}_i'(p) = \omega.$$

We now prove that if $(\mathcal{U}, \varphi)$ is perfect, we can, with two additional assumptions, find an element of $L(\mathcal{U}, \varphi)$.

4.1.2. The iteration lemma

We begin with a very useful definition.

Definition. Let $(\mathcal{U}, \varphi)$ be a MGTS on a Petri net R. A sequence $((u_i, v_i))_{0 \leq i \leq n}$ where

$$u_i \in \text{CS}(R, C_i, m_i, \mathcal{M}_i), \qquad v_i^{\text{rev}} \in \text{CS}(R^{\text{rev}}, C_i^{\text{rev}}, m_i, \mathcal{M}_i')$$

is called a *sequence of covering sequences for* $(\mathcal{U}, \varphi)$.

We now present the iteration lemma.

Lemma 4.1. *Let $(\mathcal{U}, \varphi)$ be a perfect MGTS on a Petri net $R = (P, T, \text{Pre}, \text{Post})$, $Ax = b$ its characteristic equation. Suppose that $\mathcal{M}_i'[t_{i+1}\rangle$ then*

- *Let $x_1 \in \mathbb{Z}^{E \cup \mathscr{C}}$ be a solution in (nonnecessarily positive) integers of $Ax = b$,*
- *let $x_0 \in \mathbb{N}^{E \cup \mathscr{C}}$ be a maximal support solution of $Ax = 0$,*
- *let $((u_i, v_i))_{0 \leq i \leq n}$ a sequence of covering sequences for $(\mathcal{U}, \varphi)$.*

We can compute

$$\alpha \in \mathbb{N}, \qquad k_0 \in \mathbb{N}, \qquad w_i \in L(C_i, m_i, m_i), \qquad \beta_i \in L(C_i, m_i, m_i)$$

such that

(i) $\sum_{i=0}^{j} \|u_i w_i v_i\|(p) = \alpha(x_0(c_j'(p)) - x_0(c_0(p)))$,

(ii) *for* $k \geq k_0$: $(u_0)^k \beta_0 (w_0)^k (v_0)^k t_1 (u_1)^k \beta_1 (w_1)^k (v_1)^k \ \ldots \ (u_n)^k \beta_n (w_n)^k (v_n)^k \in L(\mathcal{U}, \varphi)$.

Proof. We first define β_i and w_i. Compute $n \in \mathbb{N}$ such that $x_1 + nx_0 \geq 0$ and $x_1|_E + nx_0|_E \geq \bar{1}$ (such an n exists since $(\mathcal{U}, \varphi)$ is perfect); let $x_1' = x_1 + nx_0$. We may compute by Theorem 2.1 $\beta_i^0 \in L_0(C_i, m_i, m_i)$ such that $|\beta_i^0| = x_1'|_{E(C_i)}$; let $\beta_i = t(\beta_i^0)$. Take $(u_i^0, v_i^0) \in L_0(C_i, m_i, m_i)^2$ such that $u_i = t(u_i^0)$ and $v_i = t(v_i^0)$. $(\mathcal{U}, \varphi)$ is complete so we can compute an $\alpha \in \mathbb{N}$ satisfying

$$\alpha x_0(c_i(p)) + \|u_i\|(p) > 0, \qquad \alpha x_0(c_i'(p)) - \|v_i\|(p) > 0 \quad \text{when } m_i(p) = \omega$$

and

$$\alpha x_0|_{E(C_i)} - |u_i^0| - |v_i^0| \geq \bar{1}.$$

By Theorem 2.1 we compute $w_i^0 \in L_0(C_i, m_i, m_i)$ such that $|w_i^0| = \alpha x_0|_{E(C_i)} - |u_i^0| - |v_i^0|$; let $w_i = t(w_i^0)$.

(i) is an easy consequence of $Ax_0 = 0$ and of the definition of w_i, we leave the proof for the reader.

(ii) We first remark that for any $u \in T^*$ and $\mu \in \mathbb{N}^P$,

$$\mu[u\rangle \text{ and } \mu + k\|u\|[u^{\mathrm{rev}}\rangle \Rightarrow \mu[u^k\rangle \mu + k\|u\|.$$

We then write

$$\mu_i(k)(p) = x_1'(c_i(p)) + k\alpha x_0(c_i(p)),$$

$$\mu_i'(k)(p) = x_1'(c_i'(p)) + k\alpha x_0(c_i'(p)),$$

$\mu_i(k) \leq_\omega \mathcal{M}_i$ and $\mu_i'(k) \leq_\omega \mathcal{M}_i'$ by the characteristic equation. It is now easy to check that by the choice of α and for k large enough,

$$\mu_i(k)[u_i^k\rangle \mu_i(k) + k\|u_i\|,$$

$$\mu_i(k) + k\|u_i\|[\beta_i\rangle \mu_i(k) + \|\beta_i\| + k\|u_i\|[w_i^k\rangle \mu_i'(k) - k\|v_i\|,$$

$$\mu_i'(k) - k\|v_i\|[v_i^k\rangle \mu_i'(k)[t_{i+1}\rangle \mu_{i+1}(k). \quad \square$$

We deduce from this lemma the following.

Corollary 4.1. *Let* $(\mathcal{U}, \varphi)$ *be a perfect MGTS on a Petri net* $R = (P, T, \text{Pre}, \text{Post})$, $Ax = b$ *its characteristic equation. Then*

$L(\mathcal{U}, \varphi) \neq \emptyset$

$\Leftrightarrow \mathcal{M}_i'[t_{i+1}\rangle$ *for every* $0 \leq i \leq n-1$

and the equation $Ax = b$ *has an integer (not necessarily positive) solution.*

Now it is clear that these properties are decidable. If w can compute for a Petri net R and two markings $\mu_{\rm i}$ and $\mu_{\rm f}$ a finite set Γ of perfect MGTS such that

$$L(R, \mu_{\rm i}, \mu_{\rm f}) = \bigcup_{(\mathcal{U}, \varphi) \in \Gamma} L(\mathcal{U}, \varphi)$$

we shall have proved that reachability is decidable. We show in the next section how to compute these perfect MGTS.

4.2. *The decomposition of the MGTS*

We begin by defining the decomposition scheme as it appeared in Section 3.2.3.

Definition. A MGTS on a Petri net R is said to be *decomposed into* a finite (possibly empty) set Γ of MGTS iff there exists $0 \leq j \leq n$ such that

(i) for every $(\mathcal{U}', \varphi') \in \Gamma$, $\mathcal{U}'$ is obtained by replacing in $\mathcal{U}$ C_j by a GTS $(C_0, m_0)t_1 \ldots t_k(C_k, m_k)$

(ii) $L(\mathcal{U}, \varphi) = \bigcup_{(\mathcal{U}', \varphi') \in \Gamma} L(\mathcal{U}', \varphi')$.

Now we can introduce the *decomposition theorem.*

Theorem 4.2. *Let* $(\mathcal{U}, \varphi)$ *be a MGTS on a Petri net* $R = (P, T, \text{Pre}, \text{Post})$. *We can decompose* $(\mathcal{U}, \varphi)$ *into a finite (possibly empty) computable set of perfect MGTS* Γ *satisfying for any* $(\mathcal{U}', \varphi') \in \Gamma$

$$\mathcal{M}^{\text{in}}(\mathcal{U}', \varphi') \leq_\omega \mathcal{M}^{\text{in}}(\mathcal{U}, \varphi), \qquad \mathcal{M}^{\text{out}}(\mathcal{U}', \varphi') \leq_\omega \mathcal{M}^{\text{out}}(\mathcal{U}, \varphi).$$

Proof.

Let $Ax = b$ be the characteristic equation of $(\mathcal{U}, \varphi)$. If $(\mathcal{U}, \varphi)$ is not perfect then one of these four cases occurs.

(i) There exists $c_i(p)$, $(c_i'(p))$ such that $Ax = 0 \Rightarrow x(c_i(p)) = 0$. In this case, by Theorem 2.2, we can compute the finite set of values taken by those $x(c_i(p))$, $(x(c_i'(p)))$ in $Ax = b$ and we substitute to the corresponding $\mathcal{M}_i(p) = \omega$, $(\mathcal{M}_i'(p) = \omega)$ those values. The desired property will be then obtained in one step. This decomposition is shown in Fig. 3.

(ii) There exists $e \in E(C_i)$ such that $Ax = 0 \Rightarrow x(e) = 0$. We then compute the finite set of values taken by $x(e)$ in $Ax = b$ and using Proposition 3.5, we substitute in $\mathcal{U}$ GTS to C_i. It remains to mark the new IPG. To do that we remark that if $(C_0^s, m_0^s)t_1 \ldots t_{\lg(s)}(C_{\lg(s)}^s, m_{\lg(s)}^s)$ is one of these IPG and if u_i is such that

$$\mu_i[u_i\rangle\mu_i', \ \mu_i \leq_\omega \mathcal{M}_i, \ \mu_i \leq_\omega \mathcal{M}_i', \ u_i = u_i^0 t_1 \ldots t_{\lg(s)} u_i^{\lg(s)}$$

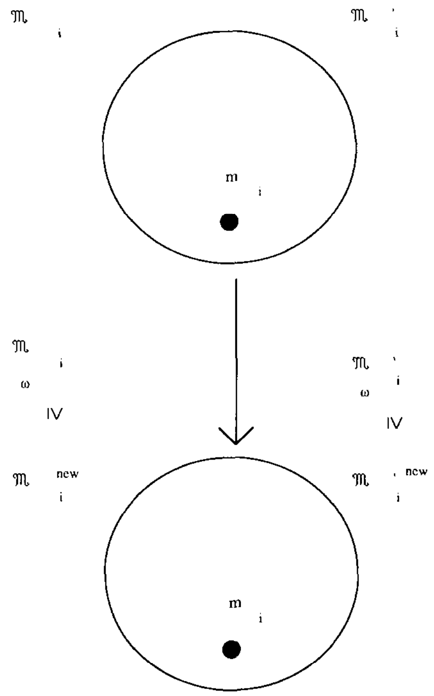

Fig. 3.

then

$$\mu_i[u_i^0 \dots t_j\rangle \mu_i^j \leqslant_\omega m_j^s$$

$$\Rightarrow \mu_i[u_i^0 \dots t_j u_i^j\rangle \mu_i^{j'} \leqslant_\omega m_j^s \qquad \text{(since } C_j \text{ is a precovering graph)}$$

$$\Rightarrow \mu_i[u_i^0 \dots t_j u_i^j t_{j+1}\rangle \mu_i^{j+1} \leqslant_\omega m_{j+1}^s \quad \text{(since } m_j^s[t_j\rangle m_{j+1}^s)$$

and the new markings will be

$$(\mathcal{M}_i, m_0^s) \qquad \text{for } C_0^s,$$

$$(m_j^s, m_j^s) \qquad \text{for } C_j^s,\ 0<j<\lg(s),$$

$$(m_{\lg(s)}^s, \mathcal{M}_i') \qquad \text{for } C_{\lg(s)}^s.$$

Of course $\mathcal{M}_i \leqslant_\omega m_0^s = m_i$, $\mathcal{M}_i' \leqslant_\omega m_{\lg(s)}^s = m_i$. This is illustrated byFig. 4.

(iii) There exists i such that $m_i \neq C(R, C_i, m_i, \mathcal{M}_i)$. We apply Proposition 3.3 and substitute GTS to C_i. Now we must mark the new IPG. Let $(C_0^s, m_0^s)t_1 \dots t_{\lg(s)}(C_{\lg(s)}^s, m_{\lg(s)}^s)$ be one of those IPG similarly to the previous case we will mark

$$C_0^s \text{ by } (\mathcal{M}_i, m_0^s), \qquad C_j^s \text{ by } (m_j^s, m_j^s) \quad \text{for } 0<j<\lg(s)$$

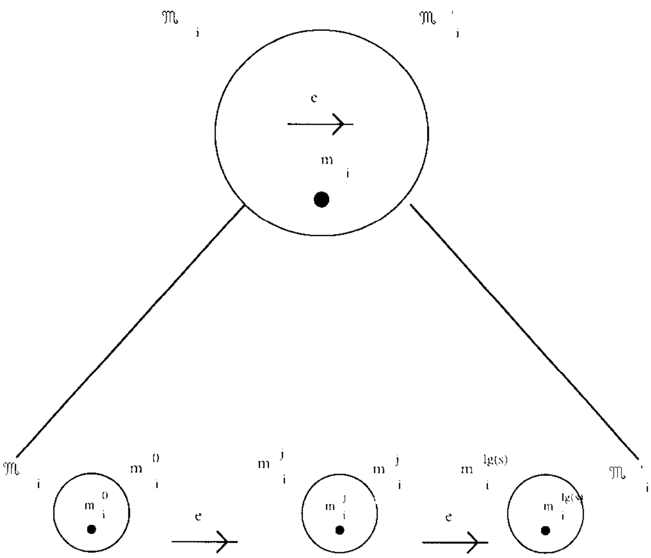

Fig. 4.

The problem is for the output marking of $C^s_{\lg(s)}$. If $u_i \in T^*$ satisfies $\mu_i[u_i\rangle\mu'_i$ for $\mu_i \leqslant_\omega \mathcal{M}_i$ and $\mu'_i \leqslant_\omega \mathcal{M}'_i$ and if u_i is decomposed into $u_i = u_i^0 t_1 \ldots t_{\lg(s)} u_i^{\lg(s)}$ where $u_i^j \in L(C^s_j, m^s_j, m^s_j))$, we get $\mu'_i \leqslant_\omega \mathcal{M}'_i$ and $\mu'_i \leqslant_\omega m^s_{\lg(s)}$ thus

$$m^s_{\lg(s)}(p) \neq \omega \text{ and } \mathcal{M}'_i(p) \neq \omega \implies m^s_{\lg(s)}(p) = \mathcal{M}'_i(p).$$

We then restrict the substitution of C_i to the IPG in which $m^s_{\lg(s)}$ satisfies the previous property and we mark $C^s_{\lg(s)}$ by $(m^s_{\lg(s)}, \mathcal{M}'^{\text{new}}_i)$ where $\mathcal{M}'^{\text{new}}_i$ is defined by $\mathcal{M}'^{\text{new}}_i \leqslant_\omega \mathcal{M}'_i$, $\mathcal{M}'^{\text{new}}_i \leqslant_\omega m^s_{\lg(s)}$ and $\mathcal{M}'^{\text{new}}_i(p) = \omega$ iff $\mathcal{M}'_i(p) = m^s_{\lg(s)}(p) = \omega$, (see Fig. 5).

(iv) There exists i such that $m_i \neq C(R^{\text{rev}}, C^{\text{rev}}_i, m_i, \mathcal{M}'_i)$. We apply Proposition 3.4 and invert the process of case (iii).

To prove the termination of this algorithm it suffices to remark that in cases (ii),

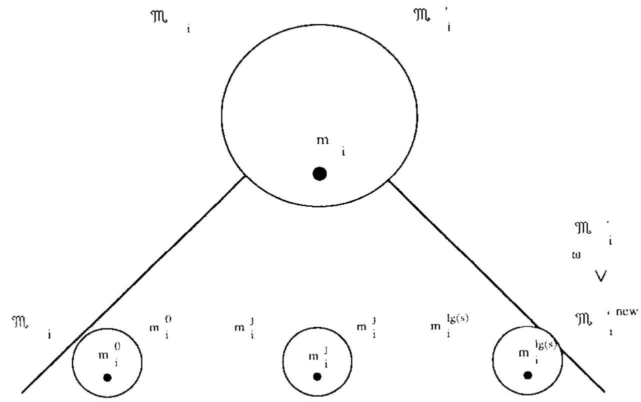

Fig. 5.

(iii) and (iv), we diminish in the new IPG the number of ω-components or the number of arcs when the number of ω-components remains unchanged. Thus we strictly decrease the couple $(\mathrm{Card}(\Omega(C)), \mathrm{Card}(E(C)))$ according to the lexicographic order. The convergence of the algorithm is then a consequence of the well-foundedness of multiset ordering [2].

The conditions $\mathcal{M}^{\mathrm{in}}(\mathcal{U}', \varphi') \leqslant_\omega \mathcal{M}^{\mathrm{in}}(\mathcal{U}, \varphi)$ and $\mathcal{M}^{\mathrm{out}}(\mathcal{U}', \varphi') \leqslant_\omega \mathcal{M}^{\mathrm{out}}(\mathcal{U}, \varphi)$ are realised since each time we substitute at the same place a marking $\mathcal{M}'$ to another $\mathcal{M}$ we take care that $\mathcal{M}' \leqslant_\omega \mathcal{M}$, (see Figs. 3–5).

4.3. *Proving the decidability of the reachability*

Let $R = (P, T, \mathrm{Pre}, \mathrm{Post})$ be a Petri net and μ_{i}, μ_{f} two markings in $\mathbb{N}^P$. We apply the decomposition Theorem on the MGTS defined by $\mathcal{U}_0 = (C, \bar{\omega})$ where C is a graph with one vertex $\bar{\omega}$ and Card(T) arcs $(\bar{\omega}, \bar{\omega})$ each labelled by a different transition of T. The marking φ_0 of $(C, \bar{\omega})$ is $(\mu_{\mathrm{i}}, \mu_{\mathrm{f}})$. Then

$$L(\mathcal{U}_0, \varphi_0) = L(R, \mu_{\mathrm{i}}, \mu_{\mathrm{f}})$$

and we get the desired theorem.

Theorem 4.3. *Let $R = (P, T, \mathrm{Pre}, \mathrm{Post})$ be a Petri net and μ_{i} and μ_{f} an initial and a final marking. We can compute a finite (possibly empty) set Γ of perfect MGTS having μ_{i} and μ_{f} as input and output marking such that*

$$L(R, \mu_{\mathrm{i}}, \mu_{\mathrm{f}}) = \bigcup_{(\mathcal{U}, \varphi) \in \Gamma} L(\mathcal{U}, \varphi).$$

The direct corollary is the following.

Corollary 4.3. *Let $R = (P, T, \mathrm{Pre}, \mathrm{Post})$ a Petri net and μ_{i} and μ_{f} an initial and a final marking. We can decide if $L(R, \mu_{\mathrm{i}}, \mu_{\mathrm{f}}) \neq \emptyset$ and if it is the case we can compute a reachability sequence in it.*

Proof. "$\mathcal{M}'_i[t_{i+1}\rangle$" and "$Ax = b$ has an integer solution" are two decidable properties. □

5. Some consequences of the new structure in Petri net language theory

The structure we have presented permits to prove new results in Petri net language theory and gives new techniques to study Petri net languages with a final marking. In this section we begin by proving that wide classes of languages are not Petri net languages. Then we prove that regularity is decidable for unlabelled Petri net languages. We conclude by a new proof of the fact that PAL(Σ) is not a Petri net language. This result is due to Jantzen [5] who proved it in a fully different way.

It is important to recall here that the most general definition for a Petri net language is the following [13]:

Definition. A language L on an alphabet Σ is a Petri net language iff there exist a Petri net $R=(P, T, \text{Pre}, \text{Post})$, two marking μ_i and μ_f in $\mathbb{N}^P$ and a morphism h from T in Σ such that

$$L = h(L(R, \mu_i, \mu_f)).$$

(The definition in [13] is more restrictive: μ_i and μ_f may have only one nonzero component but we will not need a such restrictive definition.)

5.1. The iteration lemma

Lemma 4.1 is easily translated into the following lemma valid for any Petri net language.

Lemma 5.1. *Let $R=(P, T, \text{Pre}, \text{Post})$ a Petri net, $\mu_i \in \mathbb{N}^P$, $\mu_f \in \mathbb{N}^P$ an initial and a final marking. Let h be a morphism from T^* in Σ^*. There exists a finite (possibly empty) computable set of perfect MGTS Γ such that $\mathcal{M}^{\text{in}}(\mathcal{U}, \varphi)=\mu_i$, $\mathcal{M}^{\text{out}}(\mathcal{U}, \varphi)=\mu_f$, $L(\mathcal{U}, \varphi) \neq \emptyset$ for any $(\mathcal{U}, \varphi) \in \Gamma$ and*

$$h(L(R, \mu_i, \mu_f)) = \bigcup_{(\mathcal{U}, \varphi) \in \Gamma} h(L(\mathcal{U}, \varphi)).$$

Moreover, if $((u_i, v_i))_{0 \leqslant i \leqslant n}$ is a sequence of covering sequences for $(\mathcal{U}, \varphi) \in \Gamma$, there exists $k_0 \in \mathbb{N}$ and for every $0 \leqslant i \leqslant n$, $w_i \in L(C_i, m_i, m_i)$, $\beta_i \in L(C_i, m_i, m_i)$ such that for every $k \geqslant k_0$:

$$U_k = h(u_0)^k h(\beta_0) h(w_0)^k h(v_0)^k h(t_1) \ldots h(u_n)^k h(\beta_n) h(w_n)^k h(v_n)^k \in h(L(R, \mu_i, \mu_f)).$$

The following simplification of Lemma 5.1 permits to prove that a wide family of languages are not Petri net languages.

Theorem 5.1. *Let $R=(P, T, \text{Pre}, \text{Post})$ be a Petri net, $\mu_i \in \mathbb{N}^P$, $\mu_f \in \mathbb{N}^P$ an initial and a final marking, h a morphism from T^* in Σ^*. Let $a \in \Sigma$. We define*

$$\mathcal{L}(a) = \{|u|_a, \text{ for } u \in h(L(R, \mu_i, \mu_f))\}.$$

Then, $\mathcal{L}(a)$ is infinite $\Leftrightarrow \mathcal{L}(a)$ contains an arithmetic sequence with a nonzero ratio.

Proof. Let $v \in L(C_i, m_i, m_i)$ such that $|h(v)|_a \neq 0$ and $u_i' \in \text{CS}(R, C_i, m_i, \mathcal{M}_i)$ since $(\mathcal{U}, \varphi)$ is a perfect MGTS, for a k great enough $u_i = u_i'^k v$ is a covering sequence of $\text{CS}(R, C_i, m_i, \mathcal{M}_i)$ (Proposition 3.3) such that $|h(u_i)|_a \neq 0$. We then apply Lemma 5.1 for an arbitrary sequence $((u_j, v_j))_{0 \leqslant j \leqslant n}$ of covering sequences for $(\mathcal{U}, \varphi)$ containing u_i then for $l \geqslant k_0$,

$$|h(\beta_0)h(t_1)\ldots h(t_n)h(\beta_n)|_a + l|h(u_0)h(w_0)\ldots h(w_n)h(v_n)|_a \in \mathcal{L}(a). \quad \square$$

This theorem implies that the languages for which a set $\mathscr{L}(a)$ for an $a \in \Sigma$ is not dense enough are not Petri net languages. Some examples of such languages are given in the following corollary.

Corollary 5.1. *The languages*

$$\{a^{n^2},\ n \in \mathbb{N}\}, \qquad \{abab^2ab^3 \ldots ab^n,\ n \in \mathbb{N}\},$$
$$\{a^p,\ p \text{ prime}\}, \qquad \{a^{[n \operatorname{Log}(n)]},\ n \in \mathbb{N}\}$$

are not Petri net languages.

The fact that $\{abab^2ab^3 \ldots ab^n,\ n \in \mathbb{N}\}$ is not a Petri net language has already been used by Pelz and Parigot in [12].

5.2. The regularity of $L(R, \mu_i, \mu_f)$ is decidable

Before proving Theorem 5.2 we recall the iteration for regular languages [1].

Lemma 5.2. *Let L be a regular language. There exists a constant N_L such that for any word $u \in L$ in which N_L letters are marked, u can be written $u = \alpha x \beta$ where α, x, β contains at least one marked letter and $\alpha x^n \beta \in L$ for any $n \in \mathbb{N}$.*

We now prove Theorem 5.2.

Theorem 5.2. *Let $R = (P, T, \text{Pre}, \text{Post})$ be a Petri net, $\mu_i \in \mathbb{N}^P$, $\mu_f \in \mathbb{N}^P$ an initial and a final marking. The regularity of the language $L(R, \mu_i, \mu_f)$ is decidable.*

Proof. Let Γ be a set of MGTS computed as in Lemma 5.1. We are going to prove that $L(R, \mu_i, \mu_f)$ is regular iff for any $(\mathscr{U}, \varphi) \in \Gamma$ and any precovering graph C_i of $(\mathscr{U}, \varphi)$, $\Omega(C_i) = \emptyset$.

First it is clear that if $\Omega(C_i) = \emptyset$, $L(\mathscr{U}, \varphi) = L(C_0, m_0, m_0)t_1 \ldots t_n L(C_n, m_n, m_n)$ and is then regular; if this is true for every $(\mathscr{U}, \varphi) \in \Gamma$, $L(R, \mu_i, \mu_f)$ is regular as a union of regular languages.

Conversely, let A be the characteristic matrix of $(\mathscr{U}, \varphi)$ and x_0 any maximal support solution of $Ax = 0$. By Lemma 5.1 there exists β_i, u_i, w_i, v_i such that

(i) $x_0(c_j'(p)) = \sum_{i=0}^{j} \|u_i w_i v_i\|(p) \quad (x_0(c_0(p)) = 0)$,

(ii) $u_0^k \beta_0 w_0^k v_0^k t_1 \ldots u_n^k \beta_n w_n^k v_n^k \in L(R, \mu_i, \mu_f) \quad$ for $k \geq k_0$,

Let a_i, b_i, c_i, d_i, e_i be arbitrary letters and consider the morphism

$$\theta: \qquad a_i \to u_i, \quad b_i \to \beta_i, \quad c_i \to w_i, \quad d_i \to v_i, \quad e_i \to t_i.$$

The language $\theta^{-1}(L(R, \mu_i, \mu_f)) \cap a_0^* b_0 c_0^* d_0^* e_1 \ldots a_n^* b_n c_n^* d_n^*$ is regular and the words $a_0^k b_0 c_0^k d_0^k e_1 \ldots a_n^k b_n c_n^k d_n^k$ belong to it for $k \geq k_0$. We choose k large enough and apply Lemma 5.2, marking successively a_i, c_i, d_i and we get immediately $\|u_i\| = 0$, $\|w_i\| = 0$, $\|v_i\| = 0$. Now $(\mathscr{U}, \varphi)$ is perfect, then $\mathscr{M}_i = \mathscr{M}_i' = m_i$ and $x_0(c_i'(p)) = 0$ thus $m_i \in \mathbb{N}^P$ and $\Omega(C_i) = \emptyset$. $\square$

5.3. The set of palindromes is not a Petri net language

We now reprove Jantzen's result with the help of Lemma 5.1. Before doing the proof we recall some well-known definitions and results from combinatoric word theory.

Definitions. (i) $\mathrm{PAL}(\Sigma)=\{u\in\Sigma^* \mid u^R=u\}$.

(ii) Two words u and v of Σ^* are called *conjugate* iff there exists $(x,y)\in(\Sigma^*)^2$ such that $u=xy$ and $v=yx$.

(iii) v is a *prefix* of u, denoted by $v\ll u$ iff $u=vw$.

(iv) $v\in\Sigma^*$ is a *sub-word* of $u\in\Sigma^*$ iff $u=avb$.

(v) $u\in\Sigma^*$ is said to be *cube-free* if it does not contain a subword v such that $u=av^3b$.

(vi) A *primitive root* of $u\in\Sigma^*$ is a sub-word v of u such that $u=v^k$ and $\lg(v)$ is minimal.

The notions of conjugate and primitive root are essential for the forthcoming proofs. We lists their properties in the following lemma.

Lemma 5.3.1 *Let $(u,v)\in(\Sigma^*)^2$. If there exists one unique primitive root for u we will denote it by $\rho(u)$. Moreover, $v=u^k\Rightarrow\rho(u)=\rho(v)$, u and v are conjugate $\Rightarrow\rho(u)$ and $\rho(v)$ are conjugate, if for an infinity of k there exists k' such that $u^k\ll v^{k'}$ then $\rho(u)=\rho(v)$.*

The proof of this lemma is for one part proposed for exercises and for the other part a consequence of [3, Theorem 1.3.3].

In addition we will need two other results.

Lemma 5.3.2 *$abcde\in\mathrm{PAL}(\Sigma)\Rightarrow b$ is a sub-word of $(de)^R$ or d^R is a sub-word of ab*

Lemma 5.3.3 *If $\mathrm{Card}(\Sigma)\geq 2$ then there are arbitrary long cube-free words in Σ^*.*

Lemma 5.3.2 is trivial, Lemma 5.3.3 is a very classical result proved for example in [9].

We are now in a position to prove Theorem 5.3.

Theorem 5.3. *Let $R=(P,T,\mathrm{Pre},\mathrm{Post})$ be a Petri net, μ_i and μ_f two markings in $\mathbb{N}^P$. Let h be any labelling on T in Σ^* then if $\mathrm{Card}(\Sigma)\geq 2$,*

$$\mathrm{PAL}(\Sigma)\neq h(L(R,\mu_i,\mu_f))$$

Proof. The principle of the proof is the following. We use Lemma 5.1 and prove that if

$$\mathrm{PAL}(\Sigma)=\bigcup_{(\mathcal{U},\varphi)\in\Gamma} h(L(\mathcal{U},\varphi))$$

then for any $(\mathcal{U}, \varphi) \in \Gamma$ and any IPG $\mathcal{G}_i = (C_i, m_i)$ of $(\mathcal{U}, \varphi)$, $L(C_i, m_i, m_i) = x_i^*$. This contradicts the existence of arbitrary long cube-free words in Σ^*.

Claim 1. *Let $(\mathcal{U}, \varphi)$ be a MGTS on R such that $h(L(\mathcal{U}, \varphi)) \subset \mathrm{PAL}(\Sigma)$. Let $((u_i, v_i))_{1 \leq i \leq n}$ be any sequence of covering sequences for $(\mathcal{U}, \varphi)$ and let $(w_i)_{0 \leq i \leq n}$ satisfies the conclusion of Lemma 5.1. Then for any integers l and m such that $0 \leq l \leq m \leq n$ one of the following two cases occurs:*

(i) *$\rho(h(u_l))$ is a conjugate of $\rho(x)$ for an*

$$x \in \bigcup_{i=m+1}^{n} \{h(u_i)^R, h(w_i)^R, h(v_i)^R\} \cup \{\lambda, h(v_m)^R\},$$

(ii) *$\rho(h(v_m)^R)$ is a conjugate of $\rho(x)$ for an*

$$x \in \bigcup_{i=0}^{l-1} \{h(u_i), h(w_i), h(v_i)\} \cup \{\lambda, h(u_l)\}.$$

Proof. Suppose $h(u_l) \neq \lambda$ and $h(v_m) \neq \lambda$. We have

$$h(u_0)^k h(\beta_0) h(w_0)^k h(v_0)^k h(t_l) \ldots h(u_n)^k h(\beta_n) h(w_n)^k h(v_n)^k \in \mathrm{PAL}(\Sigma)$$

then by Lemma 5.2 we obtain that either

$$h(u_l)^k \text{ is a sub-word of } (h(v_m)^k h(t_{m+1}) \ldots h(u_n)^k h(\beta_n) h(w_n)^k h(v_n)^k)^R$$

or

$$(h(v_m)^k)^R \text{ is a sub-word of } h(u_0)^k h(\beta_0) h(w_0)^k h(v_0)^k \ldots h(t_l) h(u_l)^k.$$

Suppose we are in the first case (the second one is symmetrical). We claim that $h(u_l)^k$ is of the form $ab^{k'}c$ with

$$b \in S = \bigcup_{i=m+1}^{n} \{h(u_i)^R, h(w_i)^R, h(v_i)^R\} \cup \{h(v_m)^R\}$$

and $\lim_{k \to +\infty} k' = +\infty$. Indeed if $h(u_l)^k$ does not contain a sub-word b^k with b in S, it has the form (see Fig. 6) $\alpha b^{k_1} \beta c^{k_2} \gamma$ where a and b are in S and α, β, γ have

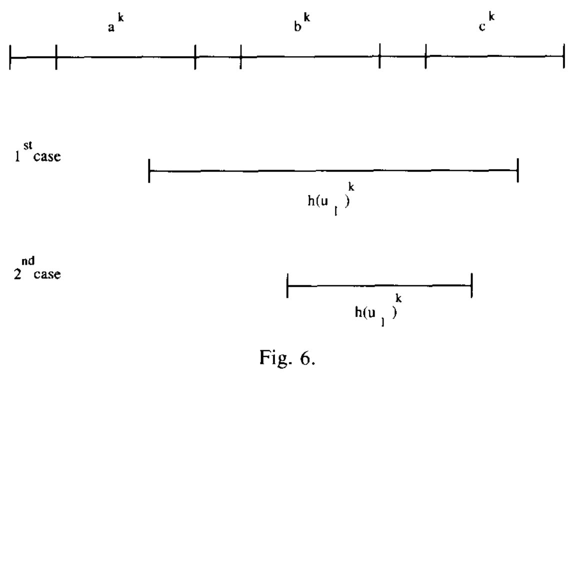

Fig. 6.

bounded length, moreover, the length of $h(u_l)^k$ tends to infinity with k which establishes our claim.

Now this property can be translated into "there exists x conjugate of $h(u_l)$ such that $x^k = b^{k'}ca$". The set S is finite and $h(u_l)$ has a finite set of conjugates thus there exist x conjugate of $h(u_l)$, $b \in S$ and an infinity of k such that $b^{k'} \ll x^k$ and $\lim_{k \to +\infty} k' = +\infty$. This implies by Lemma 5.3.1 that $\rho(b) = \rho(x)$ but $\rho(x)$ is a conjugate of $\rho(h(u_l))$ thus $\rho(h(u_l))$ is a conjugate of $\rho(b)$ for a $b \in S$ which is just our claim. □

Claim 2. *$h(L(\mathcal{U}, \varphi)) \subset \mathrm{PAL}(\Sigma)$ implies that for any IPG (C_i, m_i) of $(\mathcal{U}, \varphi)$ there exists $x_i \in \Sigma$ such that $L(C_i, m_i, m_i) \subset x_i^*$.*

Proof. Suppose that the property is true for $k < n+1$ IPG of $(\mathcal{U}, \varphi)$. We will prove it is true for $k+1$ IPG. Let l and m, such that $0 \leqslant l \leqslant m \leqslant n$, be the numbers of the first and the last IPG for which the property is not proved (see Fig. 7). If $h(L(C_l, m_l, m_l)) = \emptyset$ or $h(L(C_m, m_m, m_m)) = \emptyset$ the result is trivial. Suppose the contrary. Let u_l and v_m be two covering sequences of $\mathrm{CS}(R, C_l, m_l, \mathcal{M}_l)$ and $\mathrm{CS}(R, C_m^{\mathrm{rev}}, m_m, \mathcal{M}'_m)$ respectively. By Proposition 3.3 we may suppose $h(u_l) \neq \lambda$ and $h(v_m) \neq \lambda$. Moreover, since u_l^* and v_m^* are sets of covering sequences we may suppose $\lg(h(u_l)) = \lg(h(v_m))$.

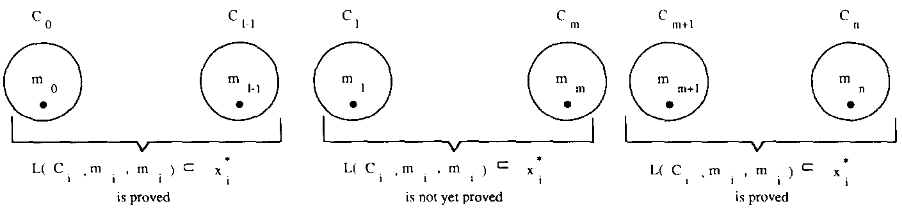

Fig. 7.

If the property is false for (C_l, m_l) and (C_m, m_m) there exists $u \in L(C_l, m_l, m_l)$ and $v \in L(C_m, m_m, m_m)$ such that $h(u) \neq \lambda$, $h(v) \neq \lambda$ and $\rho(h(u)) \neq \rho(h(u_l))$, $\rho(h(v)) \neq \rho(h(v_m))$. By replacing u by u^2 if necessary we will suppose $\lg(h(u)) \neq \lg(h(v))$.

We define $u_l(k) = u_l^k u$ and $v_m(k) = v v_m^k$. By Propositions 3.3 and 3.4 $u_l(k)$ and $v_m(k)$ are covering sequences for k large enough. We begin by proving that $\rho(h(u_l(k)))$ or $\rho(h(v_m(k)))$ take the same value for an infinity of k. Let $((u_i, v_i))_{0 \leqslant i \leqslant n}$ be an arbitrary sequence of covering sequences for $(\mathcal{U}, \varphi)$. We substitute $u_l(k))$ to u_l and $v_m(k)$ to v_m, then we apply the result of Claim 1. Since by the recursion hypothesis $\rho(h(u_i)) = \rho(h(w_i)) = \rho(h(v_i)) = \rho(x_i)$ for $i \leqslant l-1$ or $i \geqslant m+1$ the only case where the result is not trivial is when $\rho(h(u_l(k)))$ is a conjugate of $\rho(h(v_m(k)))$ for any k large enough. Let then for those k's $l(k) = \lg(\rho(h(u_l(k)))) = \lg(\rho(h(v_m(k))))$ (since two conjugate words have the same length). We have

$$l(k) \text{ divides } k \lg(h(u_l)) + \lg(h(u)), \qquad l(k) \text{ divides } k \lg(h(v_m)) + \lg(h(v)).$$

Thus $l(k)$ divides $\lg(h(u)) - \lg(h(v))$ and since $\lg(h(u)) \neq \lg(h(v))$, $l(k)$ may only take a finite set of values, so does $\rho(h(u_l(k)))$ (Σ is finite).

The conclusion is now easy. Let $\rho(h(u_l(k))) = x$ for an infinity of k. Since $h(u_l) \neq \lambda$ for an infinity of k's $h(u_l)^k \preccurlyeq h(u_l(k)) = x^{k'}$, so by Lemma 5.3.1 $\rho(h(u_l)) = \rho(x) = x$ and $h(u_l) = x^{k_0}$ so $h(u) = x^{k'-kk_0}$ and $\rho(h(u)) = \rho(x) = \rho(h(u_l))$, a contradiction. $\square$

By Lemma 5.3.3 there exists $u \in \Sigma^*$, a cube-free word such that

$$\lg(u) > \sup_{(\mathcal{U},\varphi)\in \Gamma} \left(\sum_{i=0}^{n} (\lg(h(t_i)) + 3\lg(x_i)) \right).$$

Then $uu^R \in \bigcup_{(\mathcal{U},\varphi)\in\Gamma} x_0^* h(t_1) \ldots x_{n-1}^* h(t_n) x_n^*$ thus $u = xy^3z$ with $y \neq \lambda$ which is impossible since u is cube-free. This completes the proof of Theorem 5.3. $\square$

Conclusion

In this article we tried to present the fundamental structure which permits to decide the reachability problem and extracted it from the proofs it was used by [10, 7]. The MGTS are now the analogue of the covering graph in the case where we add a final marking to the Petri net. As the precovering graph it leads to substantial progress in the study of firable sequences and Petri net languages. Now we hope that this new structure will be exploited to solve numerous problems in Petri net theory.

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