Subgraphs and Well-Quasi-Ordering

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ABSTRACT

Let $\mathcal{G}$ be a class of graphs and let $\le$ be the subgraph or the induced subgraph relation. We call $\mathcal{G}$ an *ideal* (with respect to $\le$) if $G \le G' \in \mathcal{G}$ implies that $G \in \mathcal{G}$. In this paper, we study the ideals that are well-quasi-ordered by $\le$. The following are our main results. If $\le$ is the subgraph relation, we characterize the well-quasi-ordered ideals in terms of excluding subgraphs. If $\le$ is the induced subgraph relation, we present three well-quasi-ordered ideals. We also construct examples to disprove some of the possible generalizations of our results. The connections between some of our results and digraphs are considered in this paper too. ©1992 John Wiley & Sons, Inc.

1. INTRODUCTION

A binary relation $\le$ defined on a set Q is a *quasi-ordering* if $\le$ is reflexive and transitive. A sequence $q_1, q_2, \ldots$ of members of Q is called a *good sequence* (with respect to $\le$) if there exist $i < j$ such that $q_i \le q_j$. It is a *bad sequence* if otherwise. We call $(Q, \le)$ a *well-quasi-ordering* (or a *wqo*) if there is no infinite bad sequence. An *ideal* of Q (with respect to $\le$) is a subset Q' of Q such that $q \le q' \in Q'$ implies that $q \in Q'$. Clearly, if $(Q, \le)$ is a wqo and if Q' is an ideal of Q, then Q' can be characterized by a Kuratowski type theorem. Namely, Q' can be characterized by excluding finitely many members of Q.

Let Q be the class of all finite simple graphs and let $\le$ be the subgraph or induced subgraph relation. Then $(Q, \le)$ is not a wqo as shown by the bad sequence $C_3, C_4, \ldots$ of circuits. However, if Q is restricted to some smaller class of graphs, $(Q, \le)$ could be a wqo even if $\le$ is the induced subgraph

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relation (cf. [1] and [2]). In this paper, we are going to present a few more such classes.

Throughout this paper, $\leq$ is reserved for the induced subgraph relation. As usual, P_n, C_n, and K_n are respectively the path, circuit, and clique on n vertices. For $n = 1, 2, \ldots$, let $\mathcal{P}_n$ be the class of graphs without P_n subgraph (notice that this is a stronger restriction than "without P_n induced subgraph"). We shall prove in Section 2 that $(\mathcal{P}_n, \leq)$ is a wqo. In fact, we are able to prove an even stronger result. This stronger result is used to show that if Q is a graph ideal (with respect to the subgraph relation), then Q is well-quasi-ordered by the subgraph relation if and only if it is well-quasi-ordered by the induced subgraph relation. We characterize these graph ideals in terms of excluding subgraphs. In the same section, the connections between our results and digraphs are also considered. In Section 4, we restrict ourselves to bipartite graphs. We present two classes of bipartite graphs that are well-quasi-ordered by $\leq$. We also construct examples in Section 3 to disprove some of the possible generalizations of our results.

2. EXCLUDING LONG PATHS

Let $(Q, \leq)$ be a quasi-ordering and let G be a graph. A *Q-labeling* of G is a mapping f from $V(G)$ to Q. We call the pair (G, f) Q-*labeled graph*. If $\mathcal{F}$ is a class of graphs, we denote by $\mathcal{F}(Q)$ the class of Q-labeled graphs (G, f) such that $G \in \mathcal{F}$. For any two members (G, f) and (G', f') of $\mathcal{F}(Q)$, we define $(G', f') \leq_l (G', f')$ if $G \leq G'$ and $f(v) \leq f'(\sigma(v))$ for all $v \in V(G)$, where σ is an isomorphism from G to an induced subgraph of G'. The main result of this section is the following.

Theorem 2.1. $(\mathcal{P}_n(Q), \leq_l)$ is a wqo provided $(Q, \leq)$ is.

As a consequence, we have

Theorem 2.2. $(\mathcal{P}_n, \leq)$ is a wqo.

To prove Theorem 2.1, we need a few lemmas. We first define the *type* of a graph as follows. A single vertex has type 1, and inductively for $n > 1$, a graph G has type at most n if for some vertex $v \in V(G)$, every connected component of $G \backslash v$ has type at most $n - 1$. It is not difficult to show [5] that

Lemma 2.3. Every graph in $\mathcal{P}_n$ has type at most n.

For a quasi-ordering $(Q, \leq)$, let Q^* be the set of all finite sequences of members of Q. Suppose $(q_1, \ldots, q_s)$ and $(q'_1, \ldots, q'_t)$ are members of Q^*. We define $(q_1, \ldots, q_s) \leq^* (q'_1, \ldots, q'_t)$ if there exist indices $1 \leq i_1 < \cdots < i_s \leq t$ such that $q_1 \leq q'_{i_1}, \ldots, q_s \leq q'_{i_s}$. The following result is due to Higman [4]:

Lemma 2.4. $(Q^*, \leq^*)$ is a wqo provided $(Q, \leq)$ is.

Let $\mathcal{T}_n$ be the class of graphs of type at most n. From Lemma 2.3 it is clear that to prove Theorem 2.1, we only need to show

Lemma 2.5. $(\mathcal{T}_n(Q), \preceq_l)$ is a wqo provided $(Q, \leq)$ is.

Proof. We prove the lemma by induction on n. Since the only graph of type one is K_1, it follows that $(\mathcal{T}_1(Q), \preceq_l)$ is nothing but $(Q, \leq)$. Therefore the result is true for $n = 1$ and for every wqo $(Q, \leq)$.

Suppose now $n > 1$. Let $(G_1, f_1), (G_2, f_2), \ldots$ be an infinite sequence of members of $\mathcal{T}_n(Q)$. We need to show that there exist indices $i < j$ such that $(G_i, f_i) \preceq_l (G_j, f_j)$. For each $i = 1, 2, \ldots$, let v_i be a vertex of G_i such that every connected component of $G_i \backslash v_i$ has type at most $n - 1$. By taking an infinite subsequence if necessary, we may assume that $f_1(v_1) \leq f_2(v_2) \leq \cdots$ (we can do this since $(Q, \leq)$ is a wqo). Let $G_{i1}, \ldots, G_{ik_i}$ be the connected components of $G_i \backslash v_i$, and let $f_{i1}, \ldots, f_{ik_i}$ be their Q'-labelings respectively, where $Q' = Q \times \{0, 1\}$ and $f_{ij}(v) = (f_i(v), e_i(v))$ $(v \in V(G_{ij}))$, such that $e_i(v) = 1$ if and only if v is adjacent to v_i in G_i. Clearly, all the labeled graphs (G_{ij}, f_{ij}) are members of $\mathcal{T}_{n-1}(Q')$. For any $(q, e), (q', e') \in Q'$, we define $(q, e) \leq' (q', e')$ if $q \leq q'$ and $e = e'$. It is not difficult to see that $(Q', \leq')$ is a wqo provided $(Q, \leq)$ is. Thus from the hypothesis of our induction we deduce that $\mathcal{T}_{n-1}(Q')$ is a wqo. Now it follows from Lemma 2.4 that there exist indices $i < j$ such that $((G_{i1}, f_{i1}), \ldots, (G_{ik_i}, f_{ik_i})) \preceq_l^* ((G_{j1}, f_{j1}), \ldots, (G_{jk_j}, f_{jk_j}))$. We claim that $(G_i, f_i) \preceq_l (G_j, f_j)$. Let $1 \leq j_1 < \cdots < j_{k_i} \leq k_j$ such that $(G_{i1}, f_{i1}) \preceq_l (G_{jj_1}, f_{jj_1}), \ldots, (G_{is}, f_{is}) \preceq_l (G_{jj_s}, f_{jj_s})$, where $s = k_i$. From the definition of $\preceq_l$, we deduce that for each $k = 1, \ldots, s$, there exists an induced subgraph G'_{ik} of G_{jj_k}, and an isomorphism σ_k from G_{ik} to G'_{ik}, such that $f_{ik}(v) \leq' f_{jj_k}(\sigma_k(v))$ for all $v \in V(G_{ik})$. Let σ be the mapping from $V(G_i)$ to $V(G_j)$ such that $\sigma(v_i) = v_j$ and for every other vertex $v \in V(G_i)$, $\sigma(v) = \sigma_k(v)$ if $v \in V(G_{ik})$. Clearly. σ is an isomorphism from G_i to an induced subgraph of G_j and $f_i(v) \leq f_j(\sigma(v))$ for all $v \in V(G_i)$. Thus $(G_i, f_i) \preceq_l (G_j, f_j)$ as required. ∎

Remark. (1) It looks like Lemma 2.5 is stronger than Theorem 2.1 because of Lemma 2.3. But the fact is that they are equivalent. Readers are invited to prove the equivalence by showing that $\mathcal{T}_n \subseteq \mathcal{P}_{2^n}$ for all positive integers n.

(2) Let $\mathcal{F}$ be a class of graphs. We shall denote by $\mathcal{D}(\mathcal{F})$ the class of all digraphs D such that the underlying graph of $\mathcal{D}$ belongs to $\mathcal{F}$. It is not difficult to see that a minor modification of our proof above proves that $(\mathcal{D}(\mathcal{P}_n), \preceq_d)$ is a wqo, where the induced subdigraph relation $\preceq_d$ is defined in the natural way.

Let us denote in the rest of this section that $G_1 \subseteq G_2$ if G_1 is a subgraph of G_2, and $D_1 \subseteq_d D_2$ if D_1 is a subdigraph of D_2.

Theorem 2.6. Let $\mathcal{F}$ be an ideal of graphs with respect to $\subseteq$. Then the following are equivalent

(i) $(\mathcal{D}(\mathcal{F}), \leq_d)$ is a wqo;
(ii) $(\mathcal{D}(\mathcal{F}), \subseteq_d)$ is a wqo;
(iii) $\mathcal{F}$ is contained in $\mathcal{P}_n$ for some positive integer n.

Proof. The implications (iii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (ii) are clear. To prove (ii) $\Rightarrow$ (iii), we only need to exhibit a bad sequence of digraphs (with respect to $\subseteq_d$) $D_4, D_5, \ldots$ such that their underlying graphs are $P_4, P_5, \ldots$, respectively. Let D_n $(n = 4, 5, \ldots)$ be defined on $\{1, \ldots, n\}$ such that the arcs of D_n are $(1,2), (3,2), \ldots, (i, i-1), \ldots, (n-1, n-2), (n-1, n)$. Then it is clear that $D_4, D_5, \ldots$ is a sequence satisfying the requirements. ∎

In the rest of this section, we are going to characterize the ideals $\mathcal{F}$ of graphs (with respect to $\subseteq$), such that $(\mathcal{F}, \subseteq)$ is a wqo. We first observe that $C_3, C_4, \ldots$ is a bad sequence. Another such bad sequence is $F_1, F_2, \ldots,$ where for each $i = 1, 2, \ldots,$

$$V(F_i) = \{y_1, y_2, z_1, z_2, x_1, \ldots, x_i\}$$

and

$$E(F_i) = \{y_1x_1, y_2x_1, z_1x_i, z_2x_i, x_1x_2, x_2x_3, \ldots, x_{i-1}x_i\}.$$

Theorem 2.7. Let $\mathcal{F}$ be an ideal of graphs with respect to $\subseteq$. Then the following are equivalent:

(i) $(\mathcal{F}, \subseteq)$ is a wqo;
(ii) $(\mathcal{F}, \leq)$ is a wqo;
(iii) $\mathcal{F}$ contains only finitely many graphs C_n and F_n.

Proof. The implications (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iii) are clear. To prove (iii) $\Rightarrow$ (ii), we assume that there exists a positive integer N such that $C_n, F_n \notin \mathcal{F}$ for all $n \geq N$. It follows that no graph in $\mathcal{F}$ has a subgraph C_n or F_n for $n \geq N$.

Let G be a graph. As usual, the degree of a vertex $v \in V(G)$, denoted by $d(v)$, is the number of edges incident with v. An edge $e = xy$ of G is called a *bridge* if (1) $G \backslash e$ has one more connected components than G has; (2) the component of $G \backslash e$ containing x is a path on at least two vertices such that x is an end of this path; and (3) the degree of y in G is at least three. It is clear that the degree of x in G is exactly two. Let Q be the set of positive integers. We define the *condensation* of a graph G to be a Q-labeled graph $(c(G), f)$ as follows. If G is disconnected, then the condensation of G is the disjoint union of the condensations of its connected components. If G is a path P_k, then $c(G) = K_1$ and $f(v) = k$ where $\{v\} = V(K_1)$. If G is connected and it is not a path, we assume that $e_1 = x_1y_1, \ldots, e_k = x_ky_k$ are all the bridges of G, where $k \geq 0$. We also assume that $e_1', \ldots, e_k'$ are the other edges incident with $x_1, \ldots, x_k$, respectively. Then $c(G)$ is the connected component of $G\backslash\{e_1', \ldots, e_k'\}$ that contains $x_1, \ldots, x_k$. For each $i = 1, \ldots, k$, $f(x_i)$ is defined to be n if the connected component of $G\backslash e_i$ that contains x_i is P_n; and for every other vertex v of $c(G)$, $f(v)$ is defined to be 1. It is clear that for any two graphs G and G', $(c(G), f) \leq_l (c(G'), f')$ implies $G \leq G'$. Let $\mathscr{C}$ be the class of graphs $c(G)$ for all $G \in \mathscr{F}$. Then our result follows if we can prove that $(\mathscr{C}(Q), \leq_l)$ is a wqo. By Theorem 2.1, it is enough for us to prove that $\mathscr{C}$ is a subset of $\mathscr{P}_{3N}$.

Let $G \in \mathscr{C}$. We need to show that $G \in \mathscr{P}_{3N}$. Without loss of generality, we may assume that G is connected and $|V(G)| \geq 3$. We first observe, from the definition of $\mathscr{C}$, that for every edge xy of G, $d(x) + d(y) \geq 4$. Now suppose that G has a path P on $k \geq 3N$ vertices $x_1, \ldots, x_k$ such that $E(P) = \{x_1x_2, x_2x_3, \ldots, x_{k-1}x_k\}$. We chose P with k maximum. Since $d(x_1) + d(x_2) \geq 4$, there exists an edge $e \in E(G) - E(P)$ such that e is incident with either x_1 or x_2. Similarly, there is an edge $e' \in E(G) - E(P)$ incident with either x_{k-1} or x_k. It is not difficult to see from the maximality of k that the subgraph of G formed by the edge set $E(P) \cup \{e, e'\}$ contains either C_n or F_n for some $n \geq N$, contradicting the assumption. ∎

Remark. For an ideal $\mathscr{D}$ of digraphs (with respect to $\subseteq_d$), we do not know any necessary and sufficient condition to make $(\mathscr{D}, \subseteq_d)$ a wqo.

3. COUNTEREXAMPLES

As we have seen from the previous section that $(\mathscr{P}_n, \leq)$ is a wqo. If $\mathscr{Q}_n$ is the class of graphs G such that G has no induced subgraph P_n, then a natural question is: is $(\mathscr{Q}_n, \leq)$ a wqo? It has been shown in [1] that the answer is yes if n is four. For n bigger than four, the following bad sequence (which is also mentioned in [1]) shows that the answer is no.

Example 1. For each $n = 3, 4, \ldots$, let S_n be the graph on $\{x_1, \ldots, x_{2n}\}$ such that $x_1x_2, x_2x_3, \ldots, x_{2n-1}x_{2n}, x_{2n}x_1$ are edges of S_n, and the other edges of S_n are all the pairs of the form $x_{2i}x_{2j}$. It is not difficult to see that $S_3, S_4, \ldots$ is a bad sequence. Moreover, for each $n = 3, 4, \ldots$, S_n has no induced subgraphs $2K_2$ (the disjoint union of two copies of K_2) and $\overline{2K_2}$ (the complement of $2K_2$).

It was conjectured in [1] that if P_5, $\overline{P_5}$, S_3, and $\overline{S_3}$ are excluded, then we end up with a class of graphs well-quasi-ordered by $\leq$. However, our next example shows that even if $2K_2$, $\overline{2K_2}$, C_5, and S_4 are also excluded (which is equivalent to excluding $2K_2$, C_5, S_3, $S_4, \ldots$ and their complements), there still is a bad sequence.

Example 2. Let $n \geq 2$ be an integer and let $V = \{a_1, \ldots, a_n, b_1, \ldots, b_{n+3}\}$. We first define two linear orderings on V

$$b_1 <' b_2 <' b_3 <' a_1 <' b_4 <' \cdots <' a_i$$
$$<' b_{i+3} <' \cdots <' a_{n-1} <' b_{n+2} <' b_{n+3} <' a_n$$

and

$$b_{n+3} <'' b_{n+2} <'' b_{n+1} <'' a_n <'' b_n <'' \cdots <'' a_i$$
$$<'' b_i <'' \cdots <'' a_2 <'' b_2 <'' b_1 <'' a_1.$$

We will write $x \leq' y$ ($x \leq'' y$) if $x = y$ or $x <' y$ ($x <'' y$). Let $V' = \{a'_1, \ldots, a'_n, b'_1, \ldots, b'_{n+3}\}$ and let $V'' = \{a''_1, \ldots, a''_n, b''_1, \ldots, b''_{n+3}\}$. We then define a graph T_n on $V \cup V' \cup V''$ such that $E(T_n) = E \cup E' \cup E''$ where $E = \{uv\colon u \neq v \in V\}$, $E' = \{vu'\colon v \in V, u' \in V'$, and $u \leq' v\}$ and $E'' = \{vu''\colon v \in V, u'' \in V''$, and $u \leq'' v\}$.

Claim 1. $T_2, T_3, \ldots$ is a bad sequence.

Proof. To prove this claim, we first observe that

(1) $V' \cup V''$ is a stable set and V is a clique.
(2) For each $x \in V(T_n)$, let $N(x)$ be the set of vertices of T_n that are adjacent to x. Then for $v' \in V'$ and $v'' \in V''$, we have $N(v') = \{u \in V\colon v \leq' u\}$ and $N(v'') = \{u \in V\colon v \leq'' u\}$.
(3) For every $x \in V' \cup V''$, $N(x)$ is a clique, but for every $x \in V$, $N(x)$ is not a clique.
(4) For any $u, v \in V' \cup V''$, $N(u) = N(v)$ only if $u = v$ or $u = b'_1$, $v = b''_{n+3}$.
(5) If $x_1 <' x_2 <' \cdots <' x_k$ is a consecutive (with respect to $<'$) set of vertices of V, then in the graph $T_n \backslash \{x_1, \ldots, x_{k-1}\}$, $N(x'_1) = \cdots = N(x'_k)$. The similar property is also valid for $<''$.

Now suppose there are indices $n < m$ such that $T_n \preceq T_m$. To avoid the notation problem, we refer the vertices of T_m as $U = \{c_1, \ldots, c_m, d_1, \ldots, d_{m+3}\}$, $U' = \{u'\colon u \in U\}$ and $U'' = \{u''\colon u \in V\}$. We also assume that T_n is a real induced subgraph of T_m, not just isomorphic to an induced subgraph of T_m.

It follows from observation (3) that $V \subseteq U$. Next, we shall show that $V' \cup V''$ can be chosen properly so that it is a subset of $U' \cup U''$. For suppose there is a vertex $x \in (V' \cup V'') \cap U$, from observation (1) we know that x is the only element contained in $(V' \cup V'') \cap U$. We shall prove that x can be replaced by a vertex in $(U' \cup U'') - (V' \cup V'')$. Since U is a clique, x is adjacent to all vertices in V and thus $x = b'_1$ or b''_{n+3}. Notice that T_n has only two vertices with V as neighborhood, it follows that at least one of d_1' and d_{m+3}'' is not included in $V' \cup V''$ (say d_1') and hence x can be replaced by d_1' as we wanted.

Let $V_0' = \{c_i' : c_i \in V\} \cup \{d_i' : d_i \in V\}$ and let $V_0'' \subseteq U''$ be defined similarly. We now show that $V' \cup V''$ can be replaced by $V_0' \cup V_0''$. We shall say two vertices x and y of a graph G are *similar* if $N(x) = N(y)$ in G. It is easy to check that this similar relation is an equivalence relation. Let $m' = |U|$ and $n' = |V|$. Then from observation (4) we deduce that in the graph T_m, both U' and U'' have m' equivalence classes and $U' \cup U''$ has $2m' - 1$ equivalence classes. Let $W = U - V$. It is clear from observation (5) that in the graph $T_m \backslash W$, U' and U'' have $m' - |W| = n'$ equivalence classes $X_1', \ldots, X_{n'}'$ and $X_1'', \ldots, X_{n'}''$ respectively, and $U' \cup U''$ has at most $2m' - 1 - 2|W| = 2n' - 1$ equivalence classes, namely $X_1', X_1'', \ldots, X_{n'-1}', X_{n'-1}'', X_{n'}' \cup X_{n'}''$, where $X_{n'}'$ and $X_{n'}''$ are the equivalence classes containing d_1' and d_{m+3}'' respectively. On the other hand, in the graph $T_m \backslash W$, $V' \cup V''$ has exactly $2n' - 1$ equivalence classes. Thus $U' \cup U''$ has exactly $2n' - 1$ equivalence classes in $T_m \backslash W$, namely $X_1', X_1'', \ldots, X_{n'-1}', X_{n'-1}'', X_{n'}' \cup X_{n'}''$. It follows that (i) the intersection of $V' \cup V''$ with each of these equivalence classes is a singleton except for $X_{n'}' \cup X_{n'}''$, which is a set of size two; and (ii) for any $x, y \in U - W$, x' and y'' must be contained in different of these equivalence classes unless $x = b_1$ and $y = b_{n+3}$. We also know from observation (5) that (iii) for any $x \neq y \in U - W$, x' and y', x'' and y'' are also respectively contained in different of these equivalence classes. From (ii), (iii), and the fact that $U' \cup U''$ has only $2n' - 1 = |V_0'| + |V_0''| - 1$ equivalence classes in $T_m \backslash W$ we deduce that (iv) the intersection of $V_0' \cup V_0''$ with each of these equivalence classes is also a singleton except for $X_{n'}' \cup X_{n'}''$, which is a set of size two. Thus we conclude from (i) and (iv) that $V' \cup V''$ can be replaced by $V_0' \cup V_0''$ as we wanted.

Finally, we associate each graph T_n with a quasi-ordering $Q_n = (V, <)$ as follows: For distinct $x, y \in V$, we define $x < y$ if and only if $N(x) - V \subseteq N(y) - V$. It is not difficult to see that $x < y$ if and only if $x <' y$ and $x <'' y$. This ordering is illustrated by a diagram in Figure 1. Clearly, what we have shown actually is that Q_n can be obtained from Q_m by deleting $m - n$ elements. This is obviously impossible. Thus we conclude that T_n is not an induced subgraph of T_m and hence $T_2, T_3, \ldots$ is a bad sequence. ∎

Claim 2. For all integers $n \geq 2$,

(i) $V(T_n)$ can be partitioned into a clique and a stable set;
(ii) T_n has no induced subgraph $2K_2, \overline{2K_2}$, or C_5;
(iii) the two bipartite subgraphs of T_n formed by E' and E'' have no induced subgraph $2K_2$;
(iv) the bipartite subgraph of T_n formed by $E' \cup E''$ has no induced subgraph C_6, $3K_2$. or $\overline{C_8}$;
(v) T_n has no induced subgraph S_k or $\overline{S_k}$ for any $k \geq 3$.

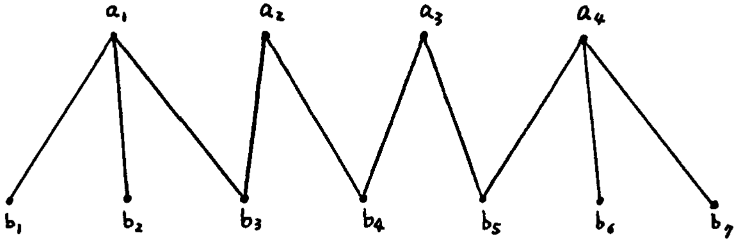

FIGURE 1. Q_4.

Proof. Claim 2(i) is clear because $(V, V' \cup V'')$ is such a partition. Thus (ii) follows since $2K_2$, $C_4 = \overline{2K_2}$ and C_5 do not have such a partition. Claim 2(iii) is trivial, and (iv) is a consequence of (iii). To prove (v) we observe that for each S_k ($k \geq 3$), there is only one way to partition the vertices into a clique and a stable set. Thus (iv) implies that T_n does not have induced subgraphs S_3, $\overline{S_3}$, and $S_4 = \overline{S_4}$. Since for all $k \geq 3$, S_k has at least one of these three graphs as an induced subgraph, (v) follows. ∎

Remark. Graphs satisfying (i) are called *split graphs* (see [3] for more information on these graphs). Example 2 shows that split graphs without induced S_3, $\overline{S_3}$, and S_4 are not well-quasi-ordered by $\leq$.

Incidentally, a simple modification of Example 2 gives a negative answer to another question in [1]. Namely, there exists a bad sequence $T_2', T_3', \ldots$ such that no T_i' has an induced subgraph K_3 or P_8.

Example 3. For every integer $n \geq 2$, let T_n' be the subgraph of T_n formed by $E' \cup E''$. Clearly T_n' is a bipartite graph (not just K_3 free). Moreover, from Claim 2(iv) we know that T_n' does not have an induced subgraph $3K_2$ (and hence P_8), C_6, or C_8.

Claim 3. $T_2', T_3', \ldots$ is a bad sequence.

Proof. We first observe that

(1) T_n' is a connected bipartite graph with the bipartition (X, Y), where $X = V$ and $Y = V' \cup V''$.
(2) Y has a partition (Y_1, Y_2) where $Y_1 = \{y_{11}, \ldots, y_{1k}\}$ and $Y_2 = \{y_{21}, \ldots, y_{2k}\}$ such that $N(y_{11}) \subseteq N(y_{12}) \subseteq \cdots \subseteq N(y_{1k})$ and $N(y_{21}) \subseteq N(y_{22}) \subseteq \cdots \subseteq N(y_{2k})$. But X does not have such a partition.

Now if there exist indices $n < m$ such that $T_n' \leq T_m'$, by adopting the notation in the proof of Claim 1, we deduce from the above observations that $V \subseteq U$ and $V' \cup V'' \subseteq U' \cup U''$. The rest of the proof is the same as that of Claim 1. ∎

Example 3 shows that it is not enough to exclude K_3 and P_8 to make a class of graphs to be well-quasi-ordered under $\leq$. However, it is enough to exclude K_3 and P_5 as shown in [1]. We do not know whether it is enough to exclude K_3 and P_7 (or even P_6). Partial results will be given in the next section. Finally, we close this section by making the following conjecture.

Conjecture. Let $n \geq 5$ be an integer and let Σ_n be the class of permutation graphs without induced P_n or $\overline{P_n}$. Then $(\Sigma_n, \leq)$ is a wqo.

We refer the reader to [3] for the definition and basic properties of permutation graphs. We would like to point out that an affirmative answer to this conjecture (even just for the case $n = 5$) would generalize the main result in [1], which says that the P_4-reducible graphs are well-quasi-ordered by $\leq$.

4. BIPARTITE GRAPHS

In the last section of this paper, we are going to present two classes of bipartite graphs that are well-quasi-ordered by $\leq$. Because of the close relationship between split graphs and bipartite graphs, our results can be converted very easily to propositions on split graphs. However, we shall not make the translations here because they are straightforward.

We assume that all the graphs mentioned in this section are bipartite graphs unless otherwise stated. If $G = (X, Y, E)$ is a bipartite graph, we define the *bipartite complement* (with respect to the bipartition (X, Y)) of G, denoted by $\overline{G}$, to be the bipartite graph $(X, Y, X \times Y - E)$. Let J_1 and J_2 be the graphs illustrated in Figure 2. It is not difficult to see that the bipartite complement of J_1 and J_2 is still J_1 and J_2, respectively. Let $\mathcal{H}$ be the class of bipartite graphs G such that G does not have an induced subgraph P_7, J_1, or J_2. We shall prove that

Theorem 4.1. $(\mathcal{H}, \leq)$ is a wqo.

To prove Theorem 4.1, we first prove the following structure theorem on $\mathcal{H}$:

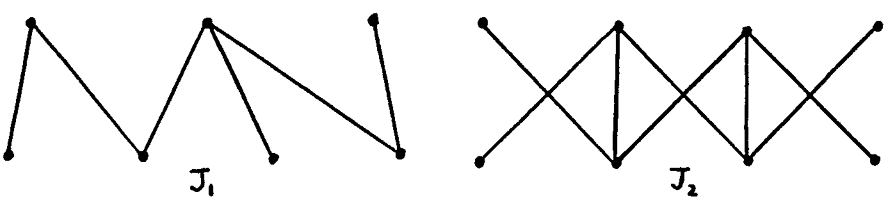

FIGURE 2. J_1 and J_2.

Theorem 4.2. Let G be a connected bipartite graph with $|V(G)| > 1$. If the bipartite complement $\overline{G}$ of G is also connected, then G has an induced P_7, J_1, or J_2.

Proof. Without loss of generality, we assume that (1) both G and $\overline{G}$ are connected; (2) $|V(G)| > 1$; and (3) for every vertex $v \in V(G)$, either $G\backslash v$ or $\overline{G}\backslash v$ is disconnected. We shall prove that if $G = (X, Y, E)$ has no induced P_7 and J_1, then G has an induced J_2.

It is clear from (1) that $|X|, |Y| \geq 3$. Therefore from (3) we may assume, by taking the bipartite complement if necessary, that there are two vertices $y_1, y_2 \in Y$ such that both $G\backslash y_1$ and $G\backslash y_2$ are disconnected. Naturally, these two cut vertices y_1 and y_2 separate G into three parts as follows: Let T be a spanning tree of G and let P be the unique y_1-y_2 path in T. Let e_i $(i = 1,2)$ be the edge in P adjacent to y_i and let T_0, T_1, and T_2 be the three connected components of $T\backslash\{e_1, e_2\}$ such that $y_i \in V(T_i)$ $(i = 1,2)$. Let (X_0, X_1, X_2) and (Y_0, Y_1, Y_2) be the partitions of X and $Y - \{y_1, y_2\}$ respectively such that $X_i \cup Y_i = V(T_i) - \{y_i\}$ $(i = 1,2)$. It is clear from the choice of y_1, y_2 that X_1 and X_2 are not empty. Let $G_0 = G\backslash(X_1 \cup X_2 \cup Y_1 \cup Y_2)$. It is obvious by looking at the spanning tree T that G_0, $G_0\backslash y_1$, and $G_0\backslash y_2$ are connected.

Claim 1. All the induced paths between y_1 and y_2 are P_3; and in addition, there exists a vertex $x_0 \in X_0$ such that x_0 is adjacent to both y_1 and y_2.

For suppose there exists an induced path P_k between y_1 and y_2 with $k \geq 5$. It is clear that this path is contained in G_0. Since X_1 and X_2 are not empty, it follows that this path can be extended to an induced path P_{k+2} of G that contains an induced P_7. The existence of x_0 follows from the fact that G_0 is connected.

Claim 2. For $i = 1, 2$, y_i is adjacent to all the vertices in X_i.

For if there is a vertex (say) $x_1 \in X_1$ that is not adjacent to y_1, then it is not difficult to see that the shortest path from x_1 to X_2 contains an induced P_7.

Claim 3. $Y_1 = Y_2 = \varnothing$.

Suppose that $Y_1 \neq \varnothing$. Then $Y_2 = \varnothing$ for otherwise the shortest path between Y_1 and Y_2 contains an induced P_7. Consequently, $X_2 = \{x_2\}$ is a singleton because we know from (1), (2), and (3) that G has no similar pairs. If there are vertices $y_1' \in Y_1$ and $x_1' \in X_1$ that are not adjacent, let $x_1 \in X_1$ such that y_1' is adjacent to x_1. Then it is clear that G induces a J_1 on $\{x_0, x_1, x_1', x_2, y_1, y_1', y_2\}$. Therefore $X_1 = \{x_1\}$ and $Y_1 = \{y_1'\}$ are singletons since G has no similar pairs. If there is a path P_k between y_1 and y_2 with $k \geq 5$, it follows from Claim 1 that there exists a vertex $x \in V(P_k)$ such that x is adjacent to both y_1 and y_2. Consequently, there is an induced J_1 of G contained in $V(P) \cup \{x_1, x_2, y_1'\}$. Thus every path (not just induced paths) from y_1 to y_2 is a P_3. From the fact that $G_0 \backslash y_1$ and $G_0 \backslash y_2$ are connected, we deduce that y_1 and y_2 have the same set X_0' of neighbors in G_0. If $Y_0 = \emptyset$, then $X_0' = X_0$ (since G is connected) is a singleton (since G has no similar pairs), which implies that $G = P_6$, contradicting (1) since $\overline{P_6}$ is disconnected. Therefore there exist vertices $y_0 \in Y_0$ and $x_0' \in X_0'$ that are adjacent. It follows that G has an induced J_1 on $\{x_0', x_1, x_2, y_0, y_1, y_1', y_2\}$.

Since G has no similar pairs, Claim 3 implies that $X_1 = \{x_1\}$ and $X_2 = \{x_2\}$ are singletons. Now $G \backslash x_1$ and $G \backslash x_2$ are connected, thus $\overline{G} \backslash x_1$ and $\overline{G} \backslash x_2$ are disconnected. Apply Claim 3 to $\overline{G}$ and x_1, x_2, we conclude that y_1 and y_2 are adjacent to all the vertices in X_0. Choose a vertex $x_3 \in X_0$ of maximum degree. Since $\overline{G}$ is connected, there is a vertex $y_3 \in Y_0$ not adjacent to x_3. Let x_4 by a vertex adjacent to y_3. It is clear that $x_4 \in X_0$. From the choice of x_3, there exists a vertex y_4 adjacent to x_3 but not to x_4. Therefore G has an induced J_2 on $\{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$ as we wanted. ∎

Let $\mathcal{G}$ be a class of graphs. A sequence $G_1, G_2, \dots$ of graphs in $\mathcal{G}$ is called a *minimal bad sequence* if it is a bad sequence (with respect to $\le$) and there is no bad sequence $G_1', G_2', \dots$ of graphs in $\mathcal{G}$ with $|G_1| = |G_1'|, \dots, |G_{i-1}| = |G_{i-1}'|$ and $|G_i| > |G_i'|$ for some $i \ge 1$. It is clear that if $(\mathcal{G}, \le)$ has a bad sequence, then it has a minimal bad sequence.

Proof of Theorem 4.1. For if $(\mathcal{H}, \le)$ is not a wqo, we take a minimal bad sequence $G_1, G_2, \dots$ from $\mathcal{H}$. We say G is a *component* of G_i if G_i is disconnected and G is a connected component of G_i, or if $\overline{G_i}$ is disconnected and $\overline{G}$ is a connected component of $\overline{G_i}$. Let $\mathcal{H}'$ be the class of components of these graphs G_i. Then we claim that $(\mathcal{H}', \le)$ is not a wqo. For suppose it is. We define $\mathcal{G}$ to be the set of pairs (G, g), where $G \in \mathcal{H}'$ and $g \in \{0, 1\}$. We also define $(G, g) \le (G', g')$ if $G \le G'$ and $g = g'$. It is clear that $(\mathcal{G}, \le)$ is a wqo since $(\mathcal{H}', \le)$ is. For every positive integer i, let $G_{i1}, \dots, G_{ik_i}$ be the components of G_i. Let $g_i = 1$ if G_i is connected and let $g_i = 0$ if otherwise. From Lemma 2.4 we conclude that there are indices $i < j$ such that $((G_{i1}, g_i), \dots, (G_{ik_i}, g_i)) \le^* ((G_{j1}, g_j), \dots, (G_{jk_j}, g_j))$, and hence $G_i \le G_j$, contradicting the assumption that $G_1, G_2, \dots$ is a bad sequence. Therefore, $(\mathcal{H}', \le)$ has a bad sequence $G_{i_1 j_1}, G_{i_2 j_2}, \dots$. Choose i such that $i = i_k = \min\{i_1, i_2, \dots\}$. Then $G_1, \dots, G_{i-1}, G_{i_k j_k}, G_{i_{k+1} j_{k+1}}, \dots$ is a bad sequence, contradicting the assumption that $G_1, G_2, \dots$ is a minimal bad sequence. ∎

Corollary 4.3 ([1]). (i) Let $\mathcal{G}_1$ be the class of graphs (not necessarily bipartite) without induced K_3 and $K_2 + 2K_1$ (the disjoint union of K_2 and two copies of K_1). Then $(\mathcal{G}_1, \le)$ is a wqo.

(ii) Let $\mathcal{G}_2$ be the class of graphs (not necessarily bipartite) without induced K_3 and P_5. Then $(\mathcal{G}_2, \le)$ is a wqo.

Proof. Let $\mathcal{B}$ be the class of bipartite graphs. As observed in [1], $G \in \mathcal{G}_1 - \mathcal{B}$ if and only if $G = C_5$; and $G \in \mathcal{G}_2 - \mathcal{B}$ if and only if every connected component G' of G is a multiple C_5 (i.e., $V(G')$ can be partitioned into nonempty stable sets $V_1, \dots, V_5$ such that the edges of G' are precisely those $e = xy$ with $x \in V_i$ and $y \in V_{i+1}$ $(i = 1, \dots, 5)$ where $V_6 = V_1$). Since P_7, J_1, and J_2 have induced subgraphs P_5 and $K_2 + 2K_1$, the results follow from Theorem 4.1. ∎

In the following, we are going to present another class of bipartite graphs that are well-quasi-ordered under $\leq$. Let $\mathcal{H}_n$ be the class of bipartite graphs without induced P_n and its bipartite complement $\overline{P_n}$. A consequence of Example 3 is that $(\mathcal{H}_8, \leq)$ is not a wqo. However, we will show that

Theorem 4.4. $(\mathcal{H}_6, \leq)$ is a wqo.

We have no idea if $(\mathcal{H}_7, \leq)$ is a wqo. We remark it here that $\overline{P_7}$ is again a P_7.

To prove Theorem 4.4, we shall need the next lemmas, which follow from Theorem 4.1 immediately.

Lemma 4.5. $(\mathcal{H}_4, \leq)$ is a wqo.

Lemma 4.6. If $\mathcal{F}_0$ is the class of bipartite graphs without induced $2K_2$, then $(\mathcal{F}_0, \leq)$ is a wqo.

For any two bipartite graphs $G_1 = (X_1, Y_1, E_1)$ and $G_2 = (X_2, Y_2, E_2)$, we define the *join* of these two graphs to be the bipartite graph $G_1 + G_2 = (X_1 \cup X_2, Y_1 \cup Y_2, E_0 \cup E_1 \cup E_2)$ where $E_0 = \{xy : x \in X_1, y \in Y_2\}$. Clearly, $\overline{G_1 + G_2} = \overline{G_2} + \overline{G_1}$. Let $\mathcal{F}$ be the class of bipartite graphs G such that either $G \in \mathcal{F}_0$, or G or $\overline{G} \in \mathcal{H}_4$.

Lemma 4.7. $G \in \mathcal{H}_6$ if and only if G can be constructed from graphs in $\mathcal{F}$ by a series of join operations.

Proof. The "if" part is clear, so we only need to show the "only if" part. Let $G \in \mathcal{H}_6 - \mathcal{F}$. We want to show that $G = G_1 + G_2$ for some graphs G_1 and G_2. Let Z be the set of isolated vertices of G. If $Z \neq \varnothing$, it is clear that $G = G_1 + G_2$, where $G_1 = G \backslash Z$ and $G_2 = (Z, \varnothing, \varnothing)$. Thus we may assume that $Z = \varnothing$.

Let $G' = (X', Y', E')$ be an induced subgraph of G such that its connected components $G_i' = (X_i', Y_i', E_i')$ $(i = 1, \dots, n)$ are complete bipartite graphs. We choose this G' with the property that (i) n is maximized; and (ii) subject to (i), $|V(G')|$ is maximized. From the choice of G we know that $n \geq 2$. Let $X_0 \subseteq X - X'$ be the set of vertices that are adjacent to at least one vertex in Y' and let Y_0 be defined analogously. Let $A = X - (X' \cup X_0)$, $B = Y - (Y' \cup Y_0)$. It follows from the maximality of n that there is no edge between A and B.

Claim 1. If $x \in X_0$ is adjacent to a vertex $y \in Y_i'$, then x is adjacent to all the vertices in Y_i'.

For if x is not adjacent to $y' \in Y_i'$, let $x' \in X_i'$. Since G_i' is a complete bipartite graph, x' is adjacent to both y and y'. Let $j \neq i$ (this j exists since $n \geq 2$) and let $x'' \in X_j'$, $y'' \in Y_j'$. Then dependent on if x is adjacent to y'', G has an induced P_6 or $\overline{P_6}$ on $\{x, x', x'', y, y', y''\}$, contradicting the assumption.

Claim 2. For every $x \in X_0$, x is adjacent to all the vertices in Y'.

It follows from Claim 1 and the maximality of $V(G')$ that there exist $i \neq j$ such that x is adjacent to all the vertices in $Y_i' \cup Y_j'$. If x is not adjacent to some vertex $y \in Y'$, then we may assume that y is contained in Y_k' for some $k \neq i, j$. Take $x_i \in X_i', x_k \in X_k', y_i \in Y_i'$ and $y_j \in Y_j'$. It is clear that G has an induced $\overline{P_6}$ on $\{x, x_i, x_k, y, y_i, y_j\}$, contradicting the assumption again.
Similarly, we have

Claim 2′. For every $y \in Y_0$, y is adjacent to all the vertices in X'.

Claim 3. If $n \geq 3$, then G is the join of two smaller graphs.

From the choice of G and the assumption $Z = \varnothing$, we deduce that $X_0 \cup Y_0 \neq \varnothing$. By the symmetry between X and Y, we may assume that $X_0 \neq \varnothing$. Let G_1 be the subgraph of G induced on $X_0 \cup B$ and let $G_2 = G\backslash(X_0 \cup B)$. We want to show that $G = G_1 + G_2$. Because of Claim 2, we only need to show that for every $x \in X_0$, x is adjacent to all the vertices in Y_0. For if there exists a vertex $y \in Y_0$ that is not adjacent to x, let $x_1 \in X_1', x_3 \in X_3'$, $y_2 \in Y_2'$, and $y_3 \in Y_3'$. Then G has an induced P_6 on $\{x, x_1, x_3, y, y_2, y_3\}$, contradicting the assumption.
Finally, we consider the case that $n = 2$. If for every $x \in X_0, x$ is adjacent to all the vertices in Y_0, then $G = G_1 + G_2$, where G_1 and G_2 are the same as defined in the proof of Claim 3. If there exist $x \in X_0$ and $y \in Y_0$ that are not adjacent, let $x_1 \in X_1', x_2 \in X_2', y_1 \in Y_1'$, and $y_2 \in Y_2'$. Then it is clear that $\overline{G}$ has an induced $C_6 = \overline{3K_2}$ on $\{x, x_1, x_2, y, y_1, y_2\}$. Therefore, from Claim 3 we deduce that $\overline{G}$ is the join of two smaller graphs G_1 and G_2. It follows that G is the join of $\overline{G_2}$ and $\overline{G_1}$. ∎

Proof of Theorem 4.4. It follows from Lemma 4.5 and Lemma 4.6 that we only need to show $(\mathcal{H}_6 - \mathcal{F}, \leq)$ is a wqo. For suppose it is not. Take a minimal bad sequence $G_1, G_2, \ldots$ from $\mathcal{H}_6 - \mathcal{F}$. Because of Lemma 4.7, we may assume that each G_i is the join of G_{i1} and G_{i2}. Let $\mathcal{H}$ be the set of all these graphs G_{ij}. Then from Lemma 2.4 we deduce that $(\mathcal{H}, \leq)$ is not a wqo and thus there is a bad sequence $G_{i_1j_1}, G_{i_2j_2}, \ldots$ contained in $\mathcal{H}$. Choose $k \geq 1$ with i_k as small as possible. It is not difficult to see that $G_1, \ldots, G_{i_k-1}, G_{i_kj_k}, G_{i_{k+1}j_{k+1}}, \ldots$ is a bad sequence, contradicting the assumption that $G_1, G_2, \ldots$ is a minimal bad sequence. ∎

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