Antichain Algorithms for Finite Automata*

Laurent Doyen[1] and Jean-François Raskin[2]

[1]LSV, ENS Cachan & CNRS, France
[2] U.L.B., Université Libre de Bruxelles, Belgium

Abstract. We present a general theory that exploits simulation relations on transition systems to obtain antichain algorithms for solving the reachability and repeated reachability problems. Antichains are more succinct than the sets of states manipulated by the traditional fixpoint algorithms. The theory justifies the correctness of the antichain algorithms, and applications such as the universality problem for finite automata illustrate efficiency improvements. Finally, we show that new and provably better antichain algorithms can be obtained for the emptiness problem of alternating automata over finite and infinite words.

1 Introduction

Finite state-transition systems are useful for the design and verification of program models. One of the essential model-checking questions is the *reachability problem* which asks, given an initial state s and a final state s', if there exists a (finite) path from s to s'. For reactive (non-terminating) programs, the *repeated reachability problem* asks, given an initial state s and a final state s', if there exists an infinite path from s that visits s' infinitely often.

The (repeated) reachability problem underlies important verification questions. For example, in the automata-based approach to model-checking [26, 27], the correctness of a program A with respect to a specification B (where A and B are finite automata) is defined by the language inclusion $L(A) \subseteq L(B)$, that is all traces of the program (executions) should be traces of the specification. The language inclusion problem is equivalent to the *emptiness problem* "is $L(A) \cap L^c(B)$ empty ?" where $L^c(B)$ is the complement of $L(B)$. If G is a transition system (or an automaton) defined as the product of A with an automaton B^c obtained by complementation of B, then the emptiness problem can be viewed as a reachability question on G for automata on finite words, and as a repeated reachability question for Büchi automata on infinite words. Note that complementation procedures resort to exponential subset constructions [18, 21, 17, 22]. Therefore,

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while the (repeated) reachability problem, which is NLogSpace-complete, can be solved in linear time in the size of G, the language inclusion problem, which is PSpace-complete, requires exponential time (in the size of B). In practice, implementations for finite words give reasonably good results (see e.g. [24]), while the complementation constructions for infinite words are difficult to implement and automata with more than around ten states are intractable [15, 25].

Recently, dramatic performance improvements have been obtained by so-called *antichain algorithms* for the reachability and repeated reachability problems on the subset construction and its variants for infinite words [8, 5, 11]. The idea is always to exploit the special structure of the subset constructions. As an example, consider the classical subset construction for the complementation of automata on finite words. States of the complement automaton are sets of states of the original automaton, that we call *cells* and denote by s_i. Set inclusion between cells is a partial order that turns out to be a simulation relation for the complement automaton: if $s_2 \subseteq s_1$ and there is a transition from s_1 to s_3, then there exists a transition from s_2 to some $s_4 \subseteq s_3$. This structural property carries over to the sets of cells manipulated by reachability algorithms: if $s_2 \subseteq s_1$ and a final cell can be reached from s_1, then a final cell can be reached from s_2. Therefore, in a breadth-first search algorithm with backward state traversal, if s_1 is visited by the algorithm, then s_2 is visited simultaneously; the algorithm manipulates $\subseteq$-downward closed sets of cells that can be canonically and compactly represented by the *antichain* of their $\subseteq$-maximal elements. Antichains serve as a symbolic data-structure on which efficient symbolic operations can be defined. Antichain algorithms have been implemented for automata on finite words [8], on finite trees [5], on infinite words [11, 14], and for other applications where exponential constructions are involved such as model-checking of linear-time logic [10], games of imperfect information [7, 4], and synthesis of linear-time specifications [12]. They outperform explicit and BDD-based algorithms by orders of magnitude [9, 3, 12].

In Section 3, we present an abstract theory to justify the correctness of antichain algorithms. For backward state traversal algorithms, we first show that forward simulation relations (such as set inclusion in the above example) are required to maintain closed sets in the algorithms. This corresponds to view antichains as a suitable symbolic data-structure to represent closed sets. Then, we develop a new approach in which antichains are sets of *promising states* in the (repeated) reachability analysis. This view is justified by mean of backward simulation relations. In our example, it turns out that set inclusion is also a backward simulation which implies that if $s_2 \subseteq s_1$ and s_2 is reachable, then s_1 is reachable. Therefore, an algorithm which traverses the state space in a backward fashion need not to explore the predecessors of s_2 if s_1 has been visited previously by the algorithm. We say that s_1 is more promising[1] than s_2. As a consequence, the algorithms can safely drop non-$\subseteq$-maximal cells, hence keeping $\subseteq$-maximal

[1] Note that this is not a heuristic: if s_1 is more promising that s_2, then the exploration of the predecessors of s_2 can be omitted without spoiling the correctness of the analysis.

cells only. While the two views coincide when set inclusion is used for finite automata, we argue that the promising state view provides better algorithms in general. This is illustrated on finite automata where algorithms in the symbolic view remain unchanged when coarser (hence improved) simulation relations are used, while in the promising state view, we obtain new antichain algorithms that are provably better: fixed points can be reached in fewer iterations, and the antichains that are manipulated are smaller. Dual results are obtained for forward state traversal algorithms.

In Section 4, we revisit classical problems of automata theory: the universality problem for nondeterministic automata, the emptiness problem for alternating automata on finite and infinite words, and the emptiness of a product of automata. In such applications, the transition systems are of exponential size and thus they are not constructed prior to the reachability analysis, but explored on-the-fly. And consequently, simulation relations needed by the antichain algorithms should be given without any computation on the transition system itself (which is the case of set inclusion for the subset construction). However, we show that by computing a simulation relation on the original automaton, coarser simulation relations can be induced on the exponential constructions. On the way, we introduce a new notion of backward simulation for alternating automata.

2 Preliminaries

Relations A *pre-order* over a finite set V is a binary relation $\preceq \subseteq V \times V$ which is reflexive and transitive. If $v_1 \preceq v_2$, we say that v_1 is smaller than v_2 (or v_2 is greater than v_1). A pre-order $\preceq'$ is *coarser* than $\preceq$ if for all $v_1, v_2 \in V$, if $v_1 \preceq v_2$, then $v_1 \preceq' v_2$. The $\preceq$*-upward closure* of a set $S \subseteq V$ is the set $\mathsf{Up}(\preceq, S) = \{v_1 \in V \mid \exists v_2 \in S : v_2 \preceq v_1\}$ of elements that are greater than some element in S. A set S is $\preceq$*-upward-closed* if it is equal to its $\preceq$-upward closure, and $\mathsf{Min}(\preceq, S) = \{v_1 \in S \mid \forall v_2 \in S : v_2 \preceq v_1 \rightarrow v_1 \preceq v_2\}$ denotes the minimal elements of S. Note that $\mathsf{Min}(\preceq, S) \subseteq S \subseteq \mathsf{Up}(\preceq, S)$. Analogously, define the $\preceq$*-downward closure* $\mathsf{Down}(\preceq, S) = \{v_1 \in V \mid \exists v_2 \in S : v_1 \preceq v_2\}$ of a set S, say that S is $\preceq$*-downward-closed* if $S = \mathsf{Down}(\preceq, S)$, and let $\mathsf{Max}(\preceq, S) = \{v_1 \in S \mid \forall v_2 \in S : v_1 \preceq v_2 \rightarrow v_2 \preceq v_1\}$ be the set of maximal elements[2] of S.

A set $S \subseteq V$ is a *quasi-antichain* if for all $v_1, v_2 \in S$, either v_1 and v_2 are $\preceq$-incomparable, or $v_1 \preceq v_2$ and $v_2 \preceq v_1$. The sets $\mathsf{Min}(\preceq, S)$ and $\mathsf{Max}(\preceq, S)$ are quasi-antichains. A *partial order* is a pre-order which is antisymmetric. For partial orders, the sets $\mathsf{Min}(\preceq, S)$ and $\mathsf{Max}(\preceq, S)$ are *antichains*, i.e., sets of pairwise $\preceq$-incomparable elements. By abuse of language, we call antichains the sets of minimal (or maximal) elements even if the pre-order is not a partial order, and denote by $\mathcal{A}$ the set of antichains over 2^V.

Antichains can be used as a symbolic data-structure to represent $\preceq$-upward-closed sets. Note that the union and intersection of $\preceq$-upward-closed sets is $\preceq$-upward-closed. The symbolic representation of an $\preceq$-upward-closed set S is the

[2] We also denote this set by $\mathsf{Max}(\succeq, S)$, and we equally say that a set is $\preceq$-downward-closed or $\succeq$-downward-closed, etc.

antichain $\widetilde{S} = \mathsf{Min}(\preceq, S)$. Operations on antichains are defined as follows. The membership question "given v and S, is $v \in S$?" becomes "given v and $\widetilde{S}$, is there $\tilde{v} \in \widetilde{S}$ such that $\tilde{v} \preceq v$?"; the emptiness question is unchanged as $S = \varnothing$ if and only if $\widetilde{S} = \varnothing$; the relation of set inclusion $S_1 \subseteq S_2$ becomes $\widetilde{S}_1 \sqsubseteq \widetilde{S}_2$ defined by $\forall v_1 \in \widetilde{S}_1 \cdot \exists v_2 \in \widetilde{S}_2 : v_2 \preceq v_1$. If $\langle V, \preceq \rangle$ is a semi-lattice with least upper bound lub, then $\langle \mathcal{A}, \sqsubseteq \rangle$ is a complete lattice (the *lattice of antichains*) where the intersection $S_1 \cap S_2$ is represented by $\widetilde{S}_1 \sqcap \widetilde{S}_2 = \mathsf{Min}(\preceq, \{\mathsf{lub}(v_1, v_2) \mid v_1 \in \widetilde{S}_1 \land v_2 \in \widetilde{S}_2\})$, and the union $S_1 \cup S_2$ by $\widetilde{S}_1 \sqcup \widetilde{S}_2 = \mathsf{Min}(\preceq, \widetilde{S}_1 \cup \widetilde{S}_2)$. Analogous definitions exist for antichains of $\preceq$-downward-closed sets if $\langle V, \preceq \rangle$ is a semi-lattice with greatest lower bound. Other operations mixing $\preceq$-upward-closed sets and $\preceq$-downward-closed sets can be defined over antichains (such as mixed set inclusion, or emptiness of mixed intersection).

Simulation relations Let $G = (V, E, \mathsf{Init}, \mathsf{Final})$ be a transition system with finite set of states V, transition relation $E \subseteq V \times V$, initial states $\mathsf{Init} \subseteq V$, and final states $\mathsf{Final} \subseteq V$. We define two notions of *simulation* [19]:

- a pre-order $\preceq_{\mathsf{f}}$ over V is a *forward simulation* for G ("$v_2 \preceq_{\mathsf{f}} v_1$" reads v_2 forward simulates v_1) if for all $v_1, v_2, v_3 \in V$, if $v_2 \preceq_{\mathsf{f}} v_1$ and $E(v_1, v_3)$, then there exists $v_4 \in V$ such that $v_4 \preceq_{\mathsf{f}} v_3$ and $E(v_2, v_4)$;
- a pre-order $\succeq_{\mathsf{b}}$ over V is a *backward simulation* for G, ("$v_2 \succeq_{\mathsf{b}} v_1$" reads v_2 backward simulates v_1), if for all $v_1, v_2, v_3 \in V$, if $v_2 \succeq_{\mathsf{b}} v_1$ and $E(v_3, v_1)$, then there exists $v_4 \in V$ such that $v_4 \succeq_{\mathsf{b}} v_3$ and $E(v_4, v_2)$.

The notations $\preceq_{\mathsf{f}}$ and $\succeq_{\mathsf{b}}$ are inspired by the fact that in the subset construction for finite automata, $\subseteq$ is a forward simulation and $\supseteq$ is a backward simulation (see also Section 4.1). Note that a forward simulation for G is a backward simulation for the transition system with transition relation $E^{-1} = \{(v_1, v_2) \mid (v_2, v_1) \in E\}$.

We say that a simulation over V is *compatible* with a set $S \subseteq V$ if for all $v_1, v_2 \in V$, if $v_1 \in S$ and v_2 (forward or backward) simulates v_1, then $v_2 \in S$. Note that a forward simulation $\preceq_{\mathsf{f}}$ is compatible with S if and only if S is $\preceq_{\mathsf{f}}$-downward-closed, and a backward simulation $\succeq_{\mathsf{b}}$ is compatible with S if and only if S is $\succeq_{\mathsf{b}}$-upward-closed. In the sequel, we will be interested in simulation relations that are compatible with Init, or Final, or with both.

Fixpoint algorithms Let $G = (V, E, \mathsf{Init}, \mathsf{Final})$ be a transition system and let $S, S' \subseteq V$ be sets of states. The sets of *predecessors* and *successors* of S in one step are denoted $\mathsf{pre}(S) = \{v_1 \mid \exists v_2 \in S : E(v_1, v_2)\}$ and $\mathsf{post}(S) = \{v_1 \mid \exists v_2 \in S : E(v_2, v_1)\}$ respectively. We denote by $\mathsf{pre}^*(S)$ the set $\bigcup_{i \geq 0} \mathsf{pre}^i(S)$ where $\mathsf{pre}^0(S) = S$ and $\mathsf{pre}^i(S) = \mathsf{pre}(\mathsf{pre}^{i-1}(S))$ for all $i \geq 1$, and by $\mathsf{pre}^+(S)$ the set $\bigcup_{i \geq 1} \mathsf{pre}^i(S)$. The operators post^* and post^+ are defined analogously. A finite *path* in G is a sequence $v_0 v_1 \dots v_n$ of states such that $E(v_i, v_{i+1})$ for all $0 \leq i < n$. Infinite paths are defined analogously. We say that S' is *reachable* from S if there exists a finite path $v_0 v_1 \dots v_n$ with $v_0 \in S$ and $v_n \in S'$.

The *reachability problem* for G asks if Final is reachable from Init, and the *repeated reachability problem* for G asks if there exists an infinite path starting from Init and passing through Final infinitely many times. To solve these problems, we can use the following classical fixpoint algorithms:

1. The *backward reachability algorithm* computes the sequence of sets: $\mathsf{B}(0) = \mathsf{Final}$ and $\mathsf{B}(i) = \mathsf{B}(i-1) \cup \mathsf{pre}(\mathsf{B}(i-1))$ for all $i \geq 1$.
2. The *backward repeated reachability algorithm* computes the sequence of sets: $\mathsf{BB}(0) = \mathsf{Final}$ and $\mathsf{BB}(i) = \mathsf{pre}^+(\mathsf{BB}(i-1)) \cap \mathsf{Final}$ for all $i \geq 1$.
3. The *forward reachability algorithm* computes the sequence of sets: $\mathsf{F}(0) = \mathsf{Init}$ and $\mathsf{F}(i) = \mathsf{F}(i-1) \cup \mathsf{post}(\mathsf{F}(i-1))$ for all $i \geq 1$.
4. The *forward repeated reachability algorithm* computes the sequence of sets: $\mathsf{FF}(0) = \mathsf{Final} \cap \mathsf{post}^*(\mathsf{Init})$ and $\mathsf{FF}(i) = \mathsf{post}^+(\mathsf{FF}(i-1)) \cap \mathsf{Final}$ for all $i \geq 1$.

The above sequences converge to a fixpoint because the operations involved are monotone. We denote by B^*, BB^*, F^*, and FF^* the respective fixpoints. Note that $\mathsf{B}^* = \mathsf{pre}^*(\mathsf{Final})$ and $\mathsf{F}^* = \mathsf{post}^*(\mathsf{Init})$. Call *recurrent* the states that have a cycle through them. The set BB^* contains the final states that can reach a recurrent final state, and FF^* contains the final states that are reachable from a reachable recurrent final state.

Theorem 1. *Let $G = (V, E, \mathsf{Init}, \mathsf{Final})$ be a transition system. Then,*

(a) the answer to the reachability problem for G is YES *if and only if* $\mathsf{B}^* \cap \mathsf{Init}$ *is nonempty if and only if* $\mathsf{F}^* \cap \mathsf{Final}$ *is nonempty;*
(b) the answer to the repeated reachability problem for G is YES *if and only if* BB^* *is reachable from* Init *if and only if* FF^* *is nonempty.*

3 Antichain fixpoint algorithms

In this section, we show that the sets in the sequences B, BB, F, and FF can be replaced by antichains for well chosen pre-orders. Two views can be developed: when backward algorithms are combined with forward simulation pre-orders (or forward algorithms with backward simulations), antichains are *symbolic representations* of closed sets; when backward algorithms are combined with backward simulation pre-orders (or forward algorithms with forward simulations), antichains are sets of *promising states*. It may be surprising to consider algorithms for the reachability problem (which can be solved in linear time), based on simulation relations (which can be computed in quadratic time). However, such algorithms are useful for applications where the transition systems have a special structure for which simulation relations *need not to be computed*. For example, the relation of set inclusion is always a forward simulation in the subset construction for finite automata (see Section 4 for details and other applications). We develop these two views below.

3.1 Antichains as a symbolic representation

Backward reachability First, we show that the sets computed by the backward algorithm B are $\preceq_{\mathsf{f}}$-downward-closed for *all* forward simulations $\preceq_{\mathsf{f}}$ of the transition system G compatible with Final.

Lemma 2. *Let $G = (V, E, \mathsf{Init}, \mathsf{Final})$ be a transition system. A pre-order $\preceq_{\mathsf{f}}$ over V is a forward simulation in G if and only if $\mathsf{pre}(S)$ is $\preceq_{\mathsf{f}}$-downward-closed for all $\preceq_{\mathsf{f}}$-downward-closed sets $S \subseteq V$.*

Proof. First, assume that $\preceq_{\mathsf{f}}$ is a forward simulation in G, and let $S \subseteq V$ be a $\preceq_{\mathsf{f}}$-downward-closed set. We show that $\mathsf{pre}(S)$ is $\preceq_{\mathsf{f}}$-downward-closed, i.e. that if $v_1 \in \mathsf{pre}(S)$ and $v_2 \preceq_{\mathsf{f}} v_1$, then $v_2 \in \mathsf{pre}(S)$. As $v_1 \in \mathsf{pre}(S)$, there exists $v_3 \in S$ such that $E(v_1, v_3)$. By definition of forward simulation, there exists v_4 such that $E(v_2, v_4)$ and $v_4 \preceq_{\mathsf{f}} v_3$. Since S is $\preceq_{\mathsf{f}}$-downward-closed and $v_3 \in S$, we conclude that $v_4 \in S$, and thus $v_2 \in \mathsf{pre}(S)$.

Second, assume that $\mathsf{pre}(S)$ is $\preceq_{\mathsf{f}}$-downward-closed when S is $\preceq_{\mathsf{f}}$-downward-closed. We show that $\preceq_{\mathsf{f}}$ is a forward simulation in G. Let $v_1, v_2, v_3 \in V$ such that $v_2 \preceq_{\mathsf{f}} v_1$ and $E(v_1, v_3)$. Let $S = \mathsf{Down}(\preceq_{\mathsf{f}}, \{v_3\})$ so that $\mathsf{pre}(S)$ is $\preceq_{\mathsf{f}}$-downward-closed. Since $v_1 \in \mathsf{pre}(S)$ and $v_2 \preceq_{\mathsf{f}} v_1$, we have $v_2 \in \mathsf{pre}(S)$ and thus there exists $v_4 \in S$ (i.e., $v_4 \preceq_{\mathsf{f}} v_3$) such that $E(v_2, v_4)$. This shows that $\preceq_{\mathsf{f}}$ is a forward simulation in G. $\square$

Assume that we have a forward simulation $\preceq_{\mathsf{f}}$ in G compatible with Final, and call this hypothesis **H1**.

Lemma 3. *Under* **H1**, *the sets $\mathsf{B}(i)$ and $\mathsf{BB}(i)$ are $\preceq_{\mathsf{f}}$-downward-closed for all $i \geq 0$.*

Proof. By induction, using Lemma 2 and the fact that $\mathsf{B}(0) = \mathsf{BB}(0) = \mathsf{Final}$ is $\preceq_{\mathsf{f}}$-downward-closed since $\preceq_{\mathsf{f}}$ is compatible with Final. $\square$

Since the sets in the backward algorithms B and BB are $\preceq_{\mathsf{f}}$-downward-closed, we can use the antichain of their maximal elements as a symbolic representation, and adapt the fixpoint algorithms accordingly. Given a forward simulation $\preceq_{\mathsf{f}}$ in G compatible with Final, the antichain algorithm for backward reachability is as follows:

- $\widetilde{\mathsf{B}}(0) = \mathsf{Max}(\preceq_{\mathsf{f}}, \mathsf{Final})$;
- $\widetilde{\mathsf{B}}(i) = \mathsf{Max}(\preceq_{\mathsf{f}}, \widetilde{\mathsf{B}}(i-1) \cup \mathsf{pre}(\mathsf{Down}(\preceq_{\mathsf{f}}, \widetilde{\mathsf{B}}(i-1))))$, for all $i \geq 1$.

Lemma 4. *Under* **H1**, *$\widetilde{\mathsf{B}}(i) = \mathsf{Max}(\preceq_{\mathsf{f}}, \mathsf{B}(i))$ and $\mathsf{B}(i) = \mathsf{Down}(\preceq_{\mathsf{f}}, \widetilde{\mathsf{B}}(i))$ for all $i \geq 0$.*

Corollary 5. *Under* **H1**, *for all $i \geq 0$, $\mathsf{B}(i+1) = \mathsf{B}(i)$ if and only if $\widetilde{\mathsf{B}}(i+1) = \widetilde{\mathsf{B}}(i)$.*

Theorem 6. *Under* **H1**, *$\mathsf{B}^* \cap \mathsf{Init} \neq \varnothing$ if and only if $\mathsf{Down}(\preceq_{\mathsf{f}}, \widetilde{\mathsf{B}}^*) \cap \mathsf{Init} \neq \varnothing$.*

So the antichain algorithm for backward reachability computes exactly the same information as the classical algorithm and the two algorithms reach their fixpoint after exactly the same number of iterations. However, the antichain algorithm can be more efficient in practice if the symbolic representation by antichains is significantly more succinct and if the computations on the antichains can be done efficiently. In particular, the predecessors of $\mathsf{Down}(\preceq_{\mathsf{f}}, \widetilde{\mathsf{B}}(i-1))$ needed to obtain $\widetilde{\mathsf{B}}(i)$ should be computed in a way that avoids constructing $\mathsf{Down}(\preceq_{\mathsf{f}}, \widetilde{\mathsf{B}}(i-1))$. For applications of the antichain algorithm in automata theory (see also Section 4), it can be shown that this operation can be computed efficiently (see e.g. [8, 11]).

Remark 1. Antichains as a data-structure have been used previously for representing the sets of backward reachable states in *well-structured transition systems* [1, 13]. So, the sequence $\widetilde{\mathsf{B}}$ converges also when the underlying state space is infinite and $\preceq_{\mathsf{f}}$ is a well-quasi order.

Backward repeated reachability Let $\preceq_{\mathsf{f}}$ be a forward simulation for G compatible with Final (**H1**). The antichain algorithm for repeated backward reachability is defined as follows:

- $\widetilde{\mathsf{BB}}(0) = \mathsf{Max}(\preceq_{\mathsf{f}}, \mathsf{Final})$;
- $\widetilde{\mathsf{BB}}(i) = \mathsf{Max}(\preceq_{\mathsf{f}}, \mathsf{pre}^{+}(\mathsf{Down}(\preceq_{\mathsf{f}}, \widetilde{\mathsf{BB}}(i-1))) \cap \mathsf{Final})$, for all $i \geq 1$.

Note that a symbolic representation of $\mathsf{pre}^{+}(\mathsf{Down}(\preceq_{\mathsf{f}}, \widetilde{\mathsf{BB}}(i-1))$ is computed by the antichain algorithm $\widetilde{\mathsf{B}}$ with $\widetilde{\mathsf{B}}(0) = \mathsf{Max}(\preceq_{\mathsf{f}}, \mathsf{pre}(\mathsf{Down}(\preceq_{\mathsf{f}}, \widetilde{\mathsf{BB}}(i-1))))$. Using Lemma 3, we get the following result and corollary.

Lemma 7. *Under* **H1***,* $\widetilde{\mathsf{BB}}(i) = \mathsf{Max}(\preceq_{\mathsf{f}}, \mathsf{BB}(i))$ *and* $\mathsf{BB}(i) = \mathsf{Down}(\preceq_{\mathsf{f}}, \widetilde{\mathsf{BB}}(i))$ *for all* $i \geq 0$*.*

Corollary 8. *Under* **H1***, for all* $i \geq 0$*,* $\mathsf{BB}(i+1) = \mathsf{BB}(i)$ *if and only if* $\widetilde{\mathsf{BB}}(i+1) = \widetilde{\mathsf{BB}}(i)$*.*

Theorem 9. *Under* **H1***,* BB^{*} *is reachable from* Init *if and only if* $\mathsf{Down}(\preceq_{\mathsf{f}}, \widetilde{\mathsf{BB}}^{*})$ *is reachable from* Init*.*

Forward algorithms We state the dual of Lemma 2 and Lemma 3 for the forward algorithms F and FF, and obtain antichain algorithms $\widetilde{\mathsf{F}}$ and $\widetilde{\mathsf{FF}}$ using backward simulations. The proofs and details are omitted as they are analogous to the backward algorithms.

Lemma 10. *Let* $G = (V, E, \mathsf{Init}, \mathsf{Final})$ *be a transition system. A pre-order* $\succeq_{\mathsf{b}}$ *over* V *is a backward simulation in* G *if and only if* $\mathsf{post}(S)$ *is* $\succeq_{\mathsf{b}}$*-upward-closed for all* $\succeq_{\mathsf{b}}$*-upward-closed sets* $S \subseteq V$*.*

Lemma 11. *Let* $G = (V, E, \mathsf{Init}, \mathsf{Final})$ *be a transition system and let* $\succeq_{\mathsf{b}}$ *be a backward simulation in* G*. If* $\succeq_{\mathsf{b}}$ *is compatible with* Init*, then* $\mathsf{F}(i)$ *is* $\succeq_{\mathsf{b}}$*-upward-closed for all* $i \geq 0$*. If* $\succeq_{\mathsf{b}}$ *is compatible with* Init *and* Final*, then* $\mathsf{FF}(i)$ *is* $\succeq_{\mathsf{b}}$*-upward-closed for all* $i \geq 0$*.*

3.2 Antichains of promising states

Traditionally, the antichain approaches have been presented as symbolic algorithms using forward simulations to justify backward algorithms, and vice versa (see above and e.g., [8, 10, 11]). In this section, we develop an original theory called *antichains of promising states* that uses backward simulations to justify backward algorithms, and forward simulations to justify forward algorithms. We obtain new antichain algorithms that do not compute the same information as the classical algorithms. In particular, we show that convergence is reached at least as soon as in the original algorithms, but it may be reached sooner. On this basis, we define in Section 4 new antichain algorithms that are provably better than the antichain algorithms of [8, 11].

Backward reachability Let $\succeq_{\mathsf{b}}$ be a backward simulation relation compatible with Init (**H2**). The *sequence of antichains of backward promising states* is defined as follows:

- $\widehat{\mathsf{B}}(0) = \mathsf{Max}(\succeq_{\mathsf{b}}, \mathsf{Final})$;
- $\widehat{\mathsf{B}}(i) = \mathsf{Max}(\succeq_{\mathsf{b}}, \widehat{\mathsf{B}}(i-1) \cup \mathsf{pre}(\widehat{\mathsf{B}}(i-1)))$, for all $i \geq 1$.

Note that while in the sequence $\widetilde{\mathsf{B}}$ we took the $\preceq_{\mathsf{f}}$-downward-closure of $\widetilde{\mathsf{B}}(i-1)$ before computing pre, this is not necessary here. And note that the original sets $\mathsf{B}(i)$ are $\preceq_{\mathsf{f}}$-downward-closed (and represented symbolically by $\widetilde{\mathsf{B}}(i)$), while they are not necessarily $\succeq_{\mathsf{b}}$-downward-closed (here, $\widehat{\mathsf{B}}(i) \subseteq \mathsf{B}(i)$ is a set of most promising states in $\mathsf{B}(i)$). The correctness of this algorithm is justified by monotonicity properties. Define the pre-order $\sqsubseteq_{\mathsf{b}} \subseteq 2^V \times 2^V$ as follows: $S_1 \sqsubseteq_{\mathsf{b}} S_2$ if $\forall v_1 \in S_1 \cdot \exists v_2 \in S_2 : v_2 \succeq_{\mathsf{b}} v_1$. We write $S_1 \approx_{\mathsf{b}} S_2$ if $S_1 \sqsubseteq_{\mathsf{b}} S_2$ and $S_2 \sqsubseteq_{\mathsf{b}} S_1$.

Lemma 12. *Under* **H2***, the operators* pre*,* $\mathsf{Max}(\succeq_{\mathsf{b}}, \cdot)$*, and* $\cup$ *(and their compositions) are* $\sqsubseteq_{\mathsf{b}}$*-monotone.*

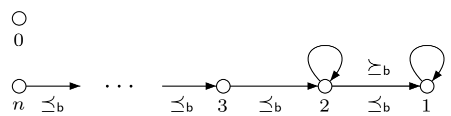
Proof. First, assume that $S_1 \sqsubseteq_{\mathsf{b}} S_2$ and show that $\mathsf{pre}(S_1) \sqsubseteq_{\mathsf{b}} \mathsf{pre}(S_2)$. For all $v_3 \in \mathsf{pre}(S_1)$, there exists $v_1 \in S_1$ such that $E(v_3, v_1)$ (by definition of pre). Since $S_1 \sqsubseteq_{\mathsf{b}} S_2$ and $v_1 \in S_1$, there exists $v_2 \in S_2$ with $v_2 \succeq_{\mathsf{b}} v_1$. By definition of $\succeq_{\mathsf{b}}$, there exists $v_4 \succeq_{\mathsf{b}} v_3$ with $E(v_4, v_2)$ hence $v_4 \in \mathsf{pre}(S_2)$.

Second, assume that $S_1 \sqsubseteq_{\mathsf{b}} S_2$ and show that $\mathsf{Max}(\succeq_{\mathsf{b}}, S_1) \sqsubseteq_{\mathsf{b}} \mathsf{Max}(\succeq_{\mathsf{b}}, S_2)$. For all $v_1 \in \mathsf{Max}(\succeq_{\mathsf{b}}, S_1)$, we have $v_1 \in S_1$ and thus there exists $v_2 \in S_2$ such that $v_2 \succeq_{\mathsf{b}} v_1$. Hence there exists $v_2' \in \mathsf{Max}(\succeq_{\mathsf{b}}, S_2)$ such that $v_2' \succeq_{\mathsf{b}} v_2 \succeq_{\mathsf{b}} v_1$.

Third, assume that $S_1 \sqsubseteq_{\mathsf{b}} S_2$ and $S_3 \sqsubseteq_{\mathsf{b}} S_4$, and show that $S_1 \cup S_3 \sqsubseteq_{\mathsf{b}} S_2 \cup S_4$. For all $v_{13} \in S_1 \cup S_3$, either $v_{13} \in S_1$ and then there exists $v_{24} \in S_2$ such that $v_{24} \succeq_{\mathsf{b}} v_{13}$, or $v_{13} \in S_3$ and then there exists $v_{24} \in S_4$ such that $v_{24} \succeq_{\mathsf{b}} v_{13}$. In all cases, $v_{24} \in S_2 \cup S_4$. □

Lemma 13. *Under* **H2***,* $\widehat{\mathsf{B}}(i) \approx_{\mathsf{b}} \mathsf{B}(i)$ *for all* $i \geq 0$*.*

Proof. By induction, using the fact that $\mathsf{B}(0) = \mathsf{Final} \approx_{\mathsf{b}} \mathsf{Max}(\succeq_{\mathsf{b}}, \mathsf{Final}) = \widehat{\mathsf{B}}(0)$ (which holds trivially since $S \approx_{\mathsf{b}} \mathsf{Max}(\succeq_{\mathsf{b}}, S)$ for all sets S) and Lemma 12. □



$\mathsf{B}(0) = \{1\}$	$\widehat{B}(0) = \{1\} = \widehat{\mathsf{B}}^{\natural}$
$\mathsf{B}(1) = \{1, 2\}$	$\widehat{B}(1) = \{1, 2\}$
$\mathsf{B}(2) = \{1, 2, 3\}$	$\widehat{B}(2) = \{1, 2\}$
$\dots$	$\dots$

Fig. 1. Backward reachability with $\mathsf{Final} = \{1\}$.

Corollary 14 (Early convergence). *Under* **H2***, for all* $i \geq 0$*, (a) if* $\mathsf{B}(i+1) = \mathsf{B}(i)$*, then* $\widehat{\mathsf{B}}(i+1) \approx_{\mathsf{b}} \widehat{\mathsf{B}}(i)$*, and (b)* $\mathsf{B}(i) \cap \mathsf{Init} \neq \varnothing$ *if and only if* $\widehat{\mathsf{B}}(i) \cap \mathsf{Init} \neq \varnothing$*.*

Denote by $\widehat{\mathsf{B}}^{\natural}$ the value $\widehat{\mathsf{B}}(i)$ for the smallest $i \geq 0$ such that $\widehat{\mathsf{B}}(i) \approx_{\mathsf{b}} \widehat{\mathsf{B}}(i+1)$. Corollary 14 ensures that convergence (modulo $\approx_{\mathsf{b}}$) on the sequence $\widehat{\mathsf{B}}$ occurs at the latest when B converges. Also, as $\succeq_{\mathsf{b}}$ is compatible with Init, if $\mathsf{B}(i)$ intersects Init then we know that $\widehat{\mathsf{B}}(i)$ also intersects Init. So, for both positive and negative instances of the reachability problem, we never need to compute more iterations in the sequence $\widehat{\mathsf{B}}$ than in the sequence B. We establish the correctness of the sequence $\widehat{\mathsf{B}}$ to decide the reachability problem.

Theorem 15 (Correctness). *Under* **H2***,* $\mathsf{B}^* \cap \mathsf{Init} \neq \varnothing$ *if and only if* $\widehat{\mathsf{B}}^{\natural} \cap \mathsf{Init} \neq \varnothing$*.*

Proof. Assume that $v \in \mathsf{B}^* = \mathsf{B}(i)$ and $v \in \mathsf{Init}$. Since $\widehat{\mathsf{B}}(i) \approx_{\mathsf{b}} \mathsf{B}(i)$ by Lemma 13, there exists $v' \in \widehat{\mathsf{B}}(i) \cap \mathsf{Init}$ by Corollary 14(b). By Corollary 14(a), we have $\widehat{\mathsf{B}}^{\natural} \approx_{\mathsf{b}} \widehat{\mathsf{B}}(j)$ for some $j \leq i$, and by Lemma 12 all sets $\widehat{\mathsf{B}}(k)$ for $k \geq j$ are $\approx_{\mathsf{b}}$-equivalent. In particular (for $k = i$), $\mathsf{B}(i) \approx_{\mathsf{b}} \widehat{\mathsf{B}}(i) \approx_{\mathsf{b}} \widehat{\mathsf{B}}^{\natural}$, and thus there exists $v'' \in \widehat{\mathsf{B}}^{\natural}$ such that $v'' \succeq_{\mathsf{b}} v'$, yielding $v'' \in \mathsf{Init}$ since $\succeq_{\mathsf{b}}$ is compatible with Init. Hence $\widehat{\mathsf{B}}^{\natural} \cap \mathsf{Init} \neq \varnothing$. For the other direction, we use the fact that $\widehat{\mathsf{B}}(i) \subseteq \mathsf{B}(i)$ for all $i \geq 0$. $\square$

Example 1. Consider the transition system in Fig. 1 where $\mathsf{Final} = \{1\}$ and $\mathsf{Init} = \{0\}$. The classical backward reachability algorithm computes the sequence $\mathsf{B}(0) = \{1\}, \mathsf{B}(1) = \{1, 2\}, \dots, \mathsf{B}(i) = \{1, 2, \dots, i+1\}$ and converges to $\{1, \dots, n\}$ after $O(n)$ iterations. Consider the backward simulation $\succeq_{\mathsf{b}}$ as depicted on Fig. 1. States 1 and 2 are mutually simulated by each other, and $i \succeq_{\mathsf{b}} i+1$ for all $1 \leq i < n$. The antichain algorithm for backward reachability based on $\succeq_{\mathsf{b}}$ computes the sequence $\widehat{B}(0) = \{1\}$, $\widehat{B}(1) = \{1, 2\}$ and the algorithm halts since $\widehat{B}(0) \approx_{\mathsf{b}} \widehat{B}(1)$, i.e. $\widehat{\mathsf{B}}^{\natural} = \widehat{\mathsf{B}}(0)$. We get early convergence because state 1 is more promising than all other states, yet is not reachable from Init.

Backward repeated reachability Let $\succeq_{\mathsf{b}}$ be a backward simulation relation compatible with both Final and Init (**H3**). Using such a relation, we define the *sequence of antichains of backward repeated promising states* as follows:

– $\widehat{\mathsf{BB}}(0) = \mathsf{Max}(\succeq_{\mathsf{b}}, \mathsf{Final})$;

– $\widehat{\mathsf{BB}}(i) = \mathsf{Max}(\succeq_{\mathsf{b}}, \mathsf{pre}^{+}(\widehat{\mathsf{BB}}(i-1)) \cap \mathsf{Final})$, for all $i \geq 1$.

Note that the computation of $S_i = \mathsf{pre}^{+}(\widehat{\mathsf{BB}}(i-1))$ can be replaced by algorithm $\widehat{\mathsf{B}}$ with $\widehat{\mathsf{B}}(0) = \mathsf{Max}(\succeq_{\mathsf{b}}, \mathsf{pre}(\widehat{\mathsf{BB}}(i-1)))$. This yields $\widehat{\mathsf{B}}^{\natural} \approx_{\mathsf{b}} S_i$ which is sufficient to ensure correctness of the algorithm. We have required that $\succeq_{\mathsf{b}}$ is compatible with Final to have the following property.

Lemma 16. *Under* **H3***, the operator $\lambda S \cdot S \cap \mathsf{Final}$ is $\sqsubseteq_{\mathsf{b}}$-monotone.*

Proof. Assume that $S_1 \sqsubseteq_{\mathsf{b}} S_2$ and show that $S_1 \cap \mathsf{Final} \sqsubseteq_{\mathsf{b}} S_2 \cap \mathsf{Final}$. For all $v_1 \in S_1$, there exists $v_2 \in S_2$ such that $v_2 \succeq_{\mathsf{b}} v_1$. In particular, for $v_1 \in S_1 \cap \mathsf{Final}$ there exists $v_2 \in S_2$ such that $v_2 \succeq_{\mathsf{b}} v_1$, and $v_2 \in \mathsf{Final}$ since $\succeq_{\mathsf{b}}$ is compatible with Final, hence $v_2 \in S_2 \cap \mathsf{Final}$. □

Lemma 17. *Under* **H3***, for all $i \geq 0$, $\widehat{\mathsf{BB}}(i) \approx_{\mathsf{b}} \mathsf{BB}(i)$.*

Proof. By induction, using Lemma 16, Lemma 12 (since **H3** implies **H2**), and the fact that $\mathsf{BB}(0) = \mathsf{Final} \approx_{\mathsf{b}} \mathsf{Max}(\succeq_{\mathsf{b}}, \mathsf{Final}) = \widehat{\mathsf{BB}}(0)$. □

Corollary 18 (Early convergence). *Under* **H3***, for all $i \geq 0$, if $\mathsf{BB}(i+1) = \mathsf{BB}(i)$ then $\widehat{\mathsf{BB}}(i+1) \approx_{\mathsf{b}} \widehat{\mathsf{BB}}(i)$.*

Denote by $\widehat{\mathsf{BB}}^{\natural}$ the value $\widehat{\mathsf{BB}}(i)$ for the smallest $i \geq 0$ such that $\widehat{\mathsf{BB}}(i) \approx_{\mathsf{b}} \widehat{\mathsf{BB}}(i+1)$.

Theorem 19 (Correctness). *Under* **H3***, BB^{*} is reachable from Init if and only if $\widehat{\mathsf{BB}}^{\natural}$ is reachable from Init.*

Proof. We know that $\mathsf{BB}^{*} \approx_{\mathsf{b}} \widehat{\mathsf{BB}}^{\natural}$. This is a consequence of Lemma 17 and the fact that pre^{+}, $\lambda S \cdot S \cap \mathsf{Final}$, and $\mathsf{Max}(\succeq_{\mathsf{b}}, \cdot)$ are $\sqsubseteq_{\mathsf{b}}$-monotone operators (by Lemma 12 and Lemma 16). Assume that BB^{*} is reachable from Init and let $v_0 v_1 \ldots v_n$ be a path in G such that $v_0 \in \mathsf{Init}$, $v_n \in \mathsf{BB}^{*}$. We show by induction that there exists a path $v'_0 v'_1 \ldots v'_n$ in G such that $v'_i \succeq_{\mathsf{b}} v_i$ for all i, $0 \leq i \leq n$. Base case: $i = n$. By lemma 17, as $v_n \in \mathsf{BB}^{*}$, there exists $v'_n \in \widehat{\mathsf{BB}}^{\natural}$ such that $v'_n \succeq_{\mathsf{b}} v_n$. Inductive case $0 \leq i < n$. By induction hypothesis, we know that there exists a path $v'_{i+1} \ldots v'_n$ in G such that $v'_j \succeq_{\mathsf{b}} v_j$ for all j such that $i+1 \leq j \leq n$. As $v'_{i+1} \succeq_{\mathsf{b}} v_{i+1}$, by properties of $\succeq_{\mathsf{b}}$, we know that there exists v' such that $v' \succeq_{\mathsf{b}} v_i$ and $E(v', v'_{i+1})$, so we take $v'_i = v'$. As $\succeq_{\mathsf{b}}$ is compatible with Init, we conclude that as $v_0 \in \mathsf{Init}$, we have $v'_0 \in \mathsf{Init}$ as well, and we are done. For the other direction, we use the fact that $\widehat{\mathsf{BB}}(i) \subseteq \mathsf{BB}(i)$ for all $i \geq 0$. □

Forward reachability algorithm Let $\preceq_{\mathsf{f}}$ be a forward simulation relation compatible with Final (**H4**). Using such a relation, we define the sequence of *antichains of forward reachable promising states* as follows:

– $\widehat{\mathsf{F}}(0) = \mathsf{Min}(\preceq_{\mathsf{f}}, \mathsf{Init})$;
– $\widehat{\mathsf{F}}(i) = \mathsf{Min}(\preceq_{\mathsf{f}}, \widehat{\mathsf{F}}(i-1) \cup \mathsf{post}(\widehat{\mathsf{F}}(i-1)))$, for all $i \geq 1$.

The following results are proved in an analogous way as the ones for the backward algorithms in the previous paragraphs. Let $S_1, S_2 \subseteq V$, we define the pre-order $\sqsubseteq_{\mathsf{f}} \subseteq 2^V \times 2^V$ as follows: $S_1 \sqsubseteq_{\mathsf{f}} S_2$ if $\forall v_1 \in S_1 \cdot \exists v_2 \in S_2 : v_2 \preceq_{\mathsf{f}} v_1$. We write $S_1 \approx_{\mathsf{f}} S_2$ if $S_1 \sqsubseteq_{\mathsf{f}} S_2$ and $S_2 \sqsubseteq_{\mathsf{f}} S_1$.

Lemma 20. *Under* **H4***, the operators* post*,* $\mathsf{Min}(\preceq_{\mathsf{f}}, \cdot)$*,* $\lambda S \cdot S \cap \mathsf{Final}$*, and* $\cup$ *(and their compositions) are* $\sqsubseteq_{\mathsf{f}}$*-monotone.*

Lemma 21. *Under* **H4***,* $\widehat{\mathsf{F}}(i) \approx_{\mathsf{f}} \mathsf{F}(i)$ *for all* $i \geq 0$*.*

Corollary 22 (Early convergence). *Under* **H4***, for all* $i \geq 0$*, (a) if* $\mathsf{F}(i+1) = \mathsf{F}(i)$*, then* $\widehat{\mathsf{F}}(i+1) \approx_{\mathsf{f}} \widehat{\mathsf{F}}(i)$*, and (b)* $\mathsf{F}(i) \cap \mathsf{Final} \neq \varnothing$ *if and only if* $\widehat{\mathsf{F}}(i) \cap \mathsf{Final} \neq \varnothing$*.*

Denote by $\widehat{\mathsf{F}}^{\natural}$ the set $\widehat{\mathsf{F}}(i)$ for the smallest $i \geq 0$ such that $\widehat{\mathsf{F}}(i) \approx_{\mathsf{b}} \widehat{\mathsf{F}}(i+1)$.

Theorem 23 (Correctness). *Under* **H4***,* $\mathsf{F}^* \cap \mathsf{Final} \neq \varnothing$ *if and only if* $\widehat{\mathsf{F}}^{\natural} \cap \mathsf{Final} \neq \varnothing$*.*

Forward repeated reachability algorithm Let $\preceq_{\mathsf{f}}$ be a forward simulation relation which is compatible with Final. The *forward repeated reachability sequence of promising states* is defined as follows:

- $\widehat{\mathsf{FF}}(0) = \mathsf{Final} \cap \widehat{\mathsf{F}}^{\natural}$;
- $\widehat{\mathsf{FF}}(i) = \mathsf{Min}(\preceq_{\mathsf{f}}, \mathsf{post}^+(\widehat{\mathsf{FF}}(i-1)) \cap \mathsf{Final})$, for all $i \geq 1$.

Lemma 24. *Under* **H4***,* $\widehat{\mathsf{FF}}(i) \approx_{\mathsf{f}} \mathsf{FF}(i)$ *for all* $i \geq 0$*,*

Proof. By induction, using the fact that $\mathsf{FF}(0) = \mathsf{Final} \cap \mathsf{F}^* \approx_{\mathsf{f}} \mathsf{Final} \cap \mathsf{F}^{\natural} = \widehat{\mathsf{FF}}(0)$ because $\mathsf{F}^* \approx_{\mathsf{f}} \mathsf{F}^{\natural}$ (using Lemma 21 and monotonicity of $\lambda S \cdot S \cap \mathsf{Final}$) and Lemma 20. □

We denote by $\widehat{\mathsf{FF}}^{\natural}$ the set $\widehat{\mathsf{FF}}(i)$ for the smallest $i \geq 0$ such that $\widehat{\mathsf{FF}}(i) \approx_{\mathsf{f}} \widehat{\mathsf{FF}}(i+1)$.

Corollary 25 (Early convergence). *Under* **H4***, for all* $i \geq 0$*, if* $\mathsf{FF}(i+1) = \mathsf{FF}(i)$ *then* $\widehat{\mathsf{FF}}(i+1) \approx_{\mathsf{f}} \widehat{\mathsf{FF}}(i)$*.*

Theorem 26 (Correctness). *Under* **H4***,* FF^* *is nonempty if and only if* $\widehat{\mathsf{FF}}^{\natural}$ *is nonempty.*

Remark 2. Note that here the relation $\preceq_{\mathsf{f}}$ needs only to be compatible with Final (and not with Init). This is in contrast with the relation $\succeq_{\mathsf{b}}$ that needs to be both compatible with Init and Final to ensure correctness of the sequence of backward repeated promising states.

Remark 3. In antichain algorithms of promising states, if $\preceq^1$ is coarser than $\preceq^2$, then the induced relation $\approx^1$ on sets of states is coarser than $\approx^2$ which entails that convergence modulo $\approx^1$ occurs at the latest when convergence modulo $\approx^2$ occurs, and possibly earlier. This is illustrated in the next section.

4 Applications

In this section, we present applications of the antichain algorithms to solve classical (and computationally hard) problems in automata theory. We consider automata running on finite and infinite words.

An *alternating automaton* [6] is a tuple $A = (Q, q_\iota, \Sigma, \delta, \alpha)$ where:

- Q is a finite set of states;
- $q_\iota \in Q$ is the initial state;
- Σ is a finite alphabet;
- $\delta : Q \times \Sigma \to 2^{2^Q}$ is the transition relation that maps each state q and letter σ to a set $\{C_1, \dots, C_n\}$ where each $C_i \subseteq Q$ is a *choice*;
- $\alpha \subseteq Q$ is the set of accepting states.

In an alternating automaton, the (finite or infinite) input word $w = \sigma_0\sigma_1 \dots$ over Σ is processed by two players in a turn-based game played in rounds. Each round starts in a state of the automaton, and the first round starts in q_ι. In round i, the first player makes a choice $C \in \delta(q_i, \sigma_i)$ where q_i is the state in round i and σ_i is the i^{th} letter of the input word. Then, the second player chooses a state $q_{i+1} \in C$, and the next round starts in q_{i+1}. A finite input word is accepted by A if the first player has a strategy to force an accepting state of A in the last round; an infinite input word is accepted by A if the first player has a strategy to force infinitely many rounds to be in an accepting state of A. A run of an alternating automaton corresponds to a fixed strategy of the first player.

Formally, a *run* of A over a (finite or infinite) word $w = \sigma_0\sigma_1 \dots$ is a tree $\langle T_w, r\rangle$ where $T_w \subseteq \mathbb{N}^*$ is a prefix-closed subset of $\mathbb{N}$, and $r : T_w \to Q$ is a labelling function such that $r(\epsilon) = q_\iota$ and for all $x \in T_w$, there exists $C = \{q_1, \dots, q_c\} \in \delta(r(x), \sigma_{|x|})$ such that $x \cdot i \in T_w$ and $r(x \cdot i) = q_i$ for each $i = 1, \dots, k$.

A run $\langle T_w, r\rangle$ of A on an a finite word w is *accepting* if $r(x) \in \alpha$ for all nodes $x \in T_w$ of length $|w|$ reachable from ϵ; and a run $\langle T_w, r\rangle$ of A on an infinite word w is *accepting* if all paths from ϵ visit nodes labeled by accepting states infinitely often (i.e., all paths satisfy a Büchi condition). A (finite or infinite) word w is *accepted* by A if there exists an accepting run on w. Alternating automata on finite words are called AFA, and alternating automata on infinite words are called ABW. The *language* of an AFA (resp., ABW) A is the set $L(A)$ of finite (resp., infinite) words accepted by A.

The *emptiness problem* for alternating automata is to decide if the language of a given alternating automaton (AFA or ABW) is empty. This problem is PSpace-complete for both AFA and ABW [18, 23]. For finite words, we also consider the *universality problem* which is to decide if the language of a given AFA with alphabet Σ is equal to Σ^*, which is PSpace-complete even for the special case of nondeterministic automata. A *nondeterministic automaton* (NFA) is an AFA such that $\delta(q, \sigma)$ is a set of singletons for all states q and letters σ.

We use antichain algorithms to solve the emptiness problem of AFA and ABW, as well as the universality problem for NFA, and the emptiness problem for NFA specified by a product of automata. In the case of NFA, it is more convenient to represent the transition relation as a function $\delta : Q \times \Sigma \to 2^Q$ where $\delta(q, \sigma) = \{q_1, \ldots, q_n\}$ represents the set of singletons $\{\{q_1\}, \ldots, \{q_n\}\}$.

4.1 Universality problem for NFA

Let $A = (Q, q_\iota, \Sigma, \delta, \alpha)$ be an NFA, and define the subset construction $G(A) = (V, E, \mathsf{Init}, \mathsf{Final})$ as follows: $V = 2^Q$, $\mathsf{Init} = \{v \in V \mid q_\iota \in v\}$, $\mathsf{Final} = \{v \in V \mid v \subseteq Q \setminus \alpha\}$, and $E(v_1, v_2)$ if there exists $\sigma \in \Sigma$ such that $\delta(q, \sigma) \subseteq v_2$ for all $q \in v_1$. A classical result shows that $L(A) \neq \Sigma^*$ if and only if Final is reachable from Init in $G(A)$, and thus we can solve the universality problem for A using antichain algorithms for the reachability problem on $G(A)$.

Antichains as symbolic representation Consider the relation $\preceq_{\mathsf{F}}$ on the states of $G(A)$ defined by $v_2 \preceq_{\mathsf{F}} v_1$ if and only if $v_2 \subseteq v_1$. Note that $\preceq_{\mathsf{F}}$ is a partial order.

Lemma 27. *$\preceq_{\mathsf{F}}$ is a forward simulation in $G(A)$ compatible with* Final.

Proof. First, if $v_1 \in \mathsf{Final}$ and $v_2 \preceq_{\mathsf{F}} v_1$, then $v_2 \subseteq v_1 \subseteq Q \setminus \alpha$ i.e., $v_2 \in \mathsf{Final}$. Second, if $v_2 \preceq_{\mathsf{F}} v_1$ and $E(v_1, v_3)$, then for some $\sigma \in \Sigma$, we have $\delta(q, \sigma) \subseteq v_3$ for all $q \in v_1$, and thus also for all $q \in v_2$ i.e., $E(v_2, v_4)$ for $v_4 = v_3$, and trivially $v_4 \preceq_{\mathsf{F}} v_3$. ☐

The antichain algorithm for backward reachability is instantiated as follows:

- $\widetilde{\mathsf{B}}(0) = \mathsf{Max}(\subseteq, \mathsf{Final}) = \{Q \setminus \alpha\}$;
- $\widetilde{\mathsf{B}}(i) = \mathsf{Max}(\subseteq, \widetilde{\mathsf{B}}(i-1) \cup \mathsf{pre}(\mathsf{Down}(\subseteq, \widetilde{\mathsf{B}}(i-1))))$, for all $i \geq 1$.

Details about efficient computation of this sequence as well as experimental comparison with the classical algorithm based on determinization can be found in [8].

Antichains of promising states Consider the relation $\succeq_{\mathsf{B}}$ such that $v_2 \succeq_{\mathsf{B}} v_1$ if $v_2 \supseteq v_1$. Note that $v_2 \succeq_{\mathsf{B}} v_1$ if and only if $v_1 \preceq_{\mathsf{F}} v_2$.

Lemma 28. *$\succeq_{\mathsf{B}}$ is a backward simulation in $G(A)$ compatible with* Init.

Proof. First, if $v_1 \in \mathsf{Init}$ and $v_2 \succeq_{\mathsf{B}} v_1$, then $q_\iota \in v_1 \subseteq v_2$ i.e., $v_2 \in \mathsf{Init}$. Second, if $v_2 \succeq_{\mathsf{B}} v_1$ and $E(v_3, v_1)$, then for some $\sigma \in \Sigma$, we have $\delta(q, \sigma) \subseteq v_1 \subseteq v_2$ for all $q \in v_3$, and thus $E(v_4, v_2)$ for $v_4 = v_3$, and trivially $v_4 \succeq_{\mathsf{B}} v_3$. ☐

The corresponding antichain algorithm for backward reachability is instantiated as follows:

- $\widehat{\mathsf{B}}(0) = \mathsf{Max}(\supseteq, \mathsf{Final}) = \{Q \setminus \alpha\}$;
- $\widehat{\mathsf{B}}(i) = \mathsf{Max}(\supseteq, \widehat{\mathsf{B}}(i-1) \cup \mathsf{pre}(\widehat{\mathsf{B}}(i-1)))$, for all $i \geq 1$.

It should be noted that $\widetilde{\mathsf{B}}(i) = \widehat{\mathsf{B}}(i)$, for all $i \geq 0$. In this particular case, the two views coincide due to the special structure of the transition system $G(A)$ (namely $\subseteq$ is a forward simulation and its inverse $\supseteq$ is a backward simulation).

In the rest of the paper, we establish the existence of simulation relations for various constructions in automata theory, and we omit the instantiation of the corresponding antichain algorithms in the promising state view.

Coarser simulations We show that the algorithms based on antichains of promising states can be improved using coarser simulations (obtained by exploiting the structure of the NFA before subset construction). We illustrate this below for backward algorithms and coarser backward simulations. Then we show that coarser forward simulations do not improve the backward antichain algorithms (in the symbolic view).

We construct a backward simulation coarser than $\succeq_{\mathsf{B}}$, using a pre-order $\gg_{\mathsf{b}} \subseteq Q \times Q$ on the state space of A such that for all $\sigma \in \Sigma$, for all $q_1, q_2, q_3 \in Q$, if $q_2 \gg_{\mathsf{b}} q_1$, then

(i) if $q_1 = q_\iota$, then $q_2 = q_\iota$, and
(ii) if $q_1 \in \delta(q_3, \sigma)$, then there exists $q_4 \in Q$ such that $q_2 \in \delta(q_4, \sigma)$ and $q_4 \gg_{\mathsf{b}} q_3$.

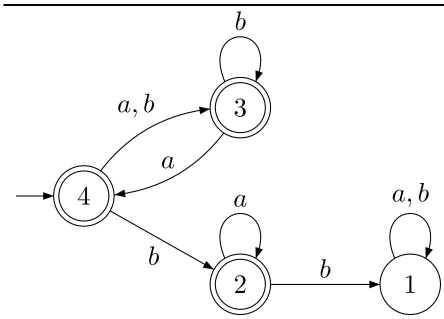
Such a relation $\gg_{\mathsf{b}}$ is usually called a *backward simulation relation* for the NFA A, and a maximal backward simulation relation (which is unique) can be computed in polynomial time (see e.g. [16]). Given $\gg_{\mathsf{b}}$, define the relation $\succeq_{\mathsf{B}^+}$ on $G(A)$ as follows: $v_2 \succeq_{\mathsf{B}^+} v_1$ if $\forall q_2 \notin v_2 \cdot \exists q_1 \notin v_1 : q_1 \gg_{\mathsf{b}} q_2$.

Lemma 29. *$\succeq_{\mathsf{B}^+}$ is a backward simulation for $G(A)$ compatible with* Init.

Proof. Let $v_2 \succeq_{\mathsf{B}^+} v_1$. First, if $v_2 \notin \mathsf{Init}$, then $q_\iota \notin v_2$ and by definition of $\succeq_{\mathsf{B}^+}$, there exists $q \notin v_1$ such that $q \gg_{\mathsf{b}} q_\iota$, thus $q = q_\iota$. Therefore $q_\iota \notin v_1$ and thus $v_1 \notin \mathsf{Init}$. Second, if $E(v_3, v_1)$, then for some $\sigma \in \Sigma$, we have $\delta(q, \sigma) \subseteq v_1$ for all $q \in v_3$. Let $v_4 = \{q \in Q \mid \delta(q, \sigma) \subseteq v_2\}$. We have $E(v_4, v_2)$ and we show that $v_4 \succeq_{\mathsf{B}^+} v_3$ i.e., for all $q_4 \notin v_4$, there exists $q_3 \notin v_3$ such that $q_3 \gg_{\mathsf{b}} q_4$. If $q_4 \notin v_4$, then there exists $q_2 \in \delta(q_4, \sigma)$ with $q_2 \notin v_2$. Since $v_2 \succeq_{\mathsf{B}^+} v_1$, there exists $q_1 \notin v_1$ such that $q_1 \gg_{\mathsf{b}} q_2$. Then, by definition of $\gg_{\mathsf{b}}$ there exists $q_3 \in Q$ such that $q_1 \in \delta(q_3, \sigma)$ and $q_3 \gg_{\mathsf{b}} q_4$. Since $q_1 \notin v_1$, we have $q_3 \notin v_3$. $\square$

Note that $\succeq_{\mathsf{B}^+}$ is coarser than $\succeq_{\mathsf{B}}$ because $v_2 \supseteq v_1$ is equivalent to say that for all $q_2 \notin v_2$, there exists $q_1 \notin v_1$ such that $q_1 = q_2$ (which implies that $q_1 \gg_{\mathsf{b}} q_2$ since $\gg_{\mathsf{b}}$ is a pre-order). Therefore, the antichains in the antichain algorithm based on $\succeq_{\mathsf{B}^+}$ are subsets of those based on $\succeq_{\mathsf{B}}$. By Corollary 14, the number of iterations of the algorithms based on $\succeq_{\mathsf{B}^+}$ and $\succeq_{\mathsf{B}}$ is the same when $L(A) \neq \Sigma^*$, and Example 2 below shows that the algorithm based on $\succeq_{\mathsf{B}^+}$ may converge faster when $L(A) = \Sigma^*$.

Example 2. Consider the nondeterministic finite automaton A with alphabet $\Sigma = \{a, b\}$ in Fig. 2. Note that every word is accepted by A i.e., $L(A) = \Sigma^*$ (it suffices to always go to state 3 from state 4). The backward antichain algorithm applied to the subset construction $G(A)$ (using $\succeq_{\mathsf{B}}$) converges after 3 iterations, and the intersection of $\widehat{B}^{\natural} = \{\{1, 2\}\}$ with the initial states of $G(A)$ is empty. Now, let $\gg_{\mathsf{b}}$ be the maximal backward simulation relation for A. We have $3 \gg_{\mathsf{b}} 2$, $3 \gg_{\mathsf{b}} 1$, and $q \gg_{\mathsf{b}} q$ for all $q \in \{1, 2, 3, 4\}$. The induced relation $\succeq_{\mathsf{B}^+}$ is such that $\{1\} \succeq_{\mathsf{B}^+} \{1, 2\}$ and $\{1, 2\} \succeq_{\mathsf{B}^+} \{1\}$. Therefore, using the relation $\succeq_{\mathsf{B}^+}$, we get $\widehat{\mathsf{B}}(0) \approx_{\mathsf{b}} \widehat{\mathsf{B}}(1)$ and the backward antichain algorithm based on $\succeq_{\mathsf{B}^+}$ converges faster, namely after 2 iterations.



	using $\succeq_{\mathsf{B}}$	using $\succeq_{\mathsf{B}+}$
$\widehat{\mathsf{B}}(0) =$	$\{\{1\}\}$	$\{\{1\}\}$
$\widehat{\mathsf{B}}(1) =$	$\{\{1,2\}\}$	$\{\{1\},\{1,2\}\}$
$\widehat{\mathsf{B}}(2) =$	$\{\{1,2\}\}$	end
	end	

Fig. 2. Improved antichain algorithm for the universality problem of NFA (Example 2).

Now, we consider coarser forward simulations (induced by pre-orders on the original NFA as above) and we show that they do not improve the algorithm based on antichains as symbolic data-structure. We prove this surprising result as follows. A *forward simulation relation* $\ll_{\mathsf{f}} \subseteq Q \times Q$ for A is a pre-order such that for all $\sigma \in \Sigma$, for all $q_1, q_2, q_3 \in Q$, if $q_2 \ll_{\mathsf{f}} q_1$, then

(*i*) if $q_1 \in \alpha$, then $q_2 \in \alpha$, and
(*ii*) if $q_3 \in \delta(q_1, \sigma)$, then there exists $q_4 \in \delta(q_2, \sigma)$ such that $q_4 \ll_{\mathsf{f}} q_3$.

Given a forward simulation relation $\ll_{\mathsf{f}}$ for A, define the relation $\preceq_{\mathsf{F}+}$ on $G(A)$ as follows: $v_2 \preceq_{\mathsf{F}+} v_1$ if $\forall q_2 \in v_2 \cdot \exists q_1 \in v_1 : q_1 \ll_{\mathsf{f}} q_2$.

Lemma 30. *$\preceq_{\mathsf{F}+}$ is a forward simulation for $G(A)$ compatible with* Final*.*

Proof. Let $v_2 \preceq_{\mathsf{F}+} v_1$. First, if $v_2 \notin \mathsf{Final}$, then $v_2 \cap \alpha \neq \varnothing$ and let $q_2 \in v_2 \cap \alpha$. By definition of $\preceq_{\mathsf{F}+}$, there exists $q_1 \in v_1$ such that $q_1 \ll_{\mathsf{f}} q_2$, thus $q_1 \in \alpha$. Therefore $v_1 \cap \alpha \neq \varnothing$ and $v_1 \notin \mathsf{Final}$. Second, if $E(v_1, v_3)$, then for some $\sigma \in \Sigma$, we have $\delta(q, \sigma) \subseteq v_3$ for all $q \in v_1$. Let $v_4 = \bigcup_{q \in v_2} \delta(q, \sigma)$. We have $E(v_2, v_4)$ and we show that $v_4 \preceq_{\mathsf{F}+} v_3$ i.e., for all $q_4 \in v_4$, there exists $q_3 \in v_3$ such that $q_3 \ll_{\mathsf{f}} q_4$. If $q_4 \in v_4$, then there exists $q_2 \in \delta(q_4, \sigma)$ with $q_2 \in v_2$. Since $v_2 \preceq_{\mathsf{F}+} v_1$, there exists $q_1 \in v_1$ such that $q_1 \ll_{\mathsf{f}} q_2$. Then, by definition of $\ll_{\mathsf{f}}$ there exists $q_3 \in \delta(q_1, \sigma)$ (such that $q_3 \ll_{\mathsf{f}} q_4$). Since $q_1 \in v_1$, we have $q_3 \in v_3$. $\square$

Lemma 31. *For all $i \geq 0$, all sets $v \in \widetilde{\mathsf{B}}(i)$ are $\ll_{\mathsf{f}}$-upward-closed (where $\widetilde{\mathsf{B}}$ is computed using $\preceq_{\mathsf{F}+}$).*

Proof. First, for $\widetilde{\mathsf{B}}(0) = \{Q \setminus \alpha\}$ we show that $Q \setminus \alpha$ is $\ll_{\mathsf{f}}$-upward-closed. Let $q_1 \in Q \setminus \alpha$ and $q_1 \ll_{\mathsf{f}} q_2$. Then $q_2 \notin \alpha$ (as if $q_2 \in \alpha$, then we would have $q_1 \in \alpha$) and thus $q_2 \in Q \setminus \alpha$. Second, by induction assume that all sets $v \in \widetilde{\mathsf{B}}(i)$ are $\ll_{\mathsf{f}}$-upward-closed, and let $v \in \widetilde{\mathsf{B}}(i+1)$. Either $v \in \widetilde{\mathsf{B}}(i)$ and then v is $\ll_{\mathsf{f}}$-upward-closed, or $v \in \mathsf{pre}(\mathsf{Down}(\subseteq, \widetilde{\mathsf{B}}(i)))$ and for some $\sigma \in \Sigma$ and $v' \in \mathsf{Down}(\subseteq, \widetilde{\mathsf{B}}(i))$, we have $\delta(q, \sigma) \subseteq v'$ for all $q \in v$. Without loss of generality, we can assume that $v' \in \widetilde{\mathsf{B}}(i)$ and thus v' is $\ll_{\mathsf{f}}$-upward-closed (by induction hypothesis). In this case, assume towards contradiction that v is not $\lll_{\mathsf{f}}$-upward-closed i.e., there exist $q_2 \in v$ and $q_1 \notin v$ such that $q_2 \lll_{\mathsf{f}} q_1$. We consider two cases: (i) if $\delta(q_1, \sigma) \subseteq v'$, then $v \cup \{q_1\} \in \mathsf{pre}(\mathsf{Down}(\subseteq, \widetilde{\mathsf{B}}(i-1)))$ and v is a strict subset of $v \cup \{q_1\}$ showing that v is not $\subseteq$-maximal in $\widetilde{\mathsf{B}}(i)$, a contradiction; (ii) if there exists $q_3 \in \delta(q_1, \sigma)$ with $q_3 \notin v'$, then since $q_2 \lll_{\mathsf{f}} q_1$ there exists $q_4 \in \delta(q_2, \sigma)$ such that $q_4 \lll_{\mathsf{f}} q_3$. Since $q_2 \in v$, we have $\delta(q_2, \sigma) \subseteq v'$ and $q_4 \in v'$. Hence $q_4 \in v'$, $q_3 \notin v'$ and $q_4 \lll_{\mathsf{f}} q_3$ i.e., v' is not $\lll_{\mathsf{f}}$-upward-closed, a contradiction. □

Lemma 32. *For all $\lll_{\mathsf{f}}$-upward-closed sets v_1, v_2, we have $v_2 \preceq_{\mathsf{F}+} v_1$ if and only if $v_2 \preceq_{\mathsf{F}} v_1$.*

Proof. Let v_1, v_2 be $\lll_{\mathsf{f}}$-upward-closed sets. First, if $v_2 \preceq_{\mathsf{F}} v_1$, then $v_2 \subseteq v_1$ and for all $q_2 \in v_2$ there exists $q_1 \in v_1$ such that $q_2 = q_1$, and thus $q_1 \lll_{\mathsf{f}} q_2$. Hence $v_2 \preceq_{\mathsf{F}+} v_1$. Second, if $v_2 \preceq_{\mathsf{F}+} v_1$, then for all $q_2 \in v_2$ there exists $q_1 \in v_1$ such that $q_1 \lll_{\mathsf{f}} q_2$. Since v_1 is $\lll_{\mathsf{f}}$-upward-closed, $q_1 \in v_1$ implies $q_2 \in v_1$. Hence, for all $q_2 \in v_2$ we have $q_2 \in v_1$ i.e., $v_2 \subseteq v_1$ and $v_2 \preceq_{\mathsf{F}} v_1$. □

Corollary 33. *The antichain algorithms for backward reachability $\widetilde{\mathsf{B}}$ based on $\preceq_{\mathsf{F}+}$ and $\preceq_{\mathsf{F}}$ compute exactly the same sequences of sets.*

4.2 Emptiness problem for AFA

In this section, we use a new definition of backward simulation for alternating automata on finite words to construct an induced backward simulation on the subset construction for AFA.

Let $A = (Q, q_\iota, \Sigma, \delta, \alpha)$ be an AFA. Define the subset construction $G(A) = (V, E, \mathsf{Init}, \mathsf{Final})$ where $V = 2^Q$, $E = \{(v_1, v_2) \in V \times V \mid \exists \sigma \in \Sigma \cdot \forall q \in v_1 \cdot \exists C \in \delta(q, \sigma) : C \subseteq v_2\}$, $\mathsf{Init} = \{v \in V \mid q_\iota \in v\}$, and $\mathsf{Final} = \{v \in V \mid q \subseteq \alpha\}$.

As before, it is easy to see that $L(A) \neq \varnothing$ if and only if Final is reachable from Init in $G(A)$, and the emptiness problem for A can be solved using antichain algorithms for the reachability problem in $G(A)$ e.g., using the relation $\succeq_{\mathsf{B}}$ such that $v_2 \succeq_{\mathsf{B}} v_1$ if $v_2 \supseteq v_1$ which is a backward simulation in $G(A)$ compatible with Init.

As in the case of the universality problem for NFA, the relation $\succeq_{\mathsf{B}}$ can be improved using an appropriate notion of backward simulation relation defined on the AFA A. We introduce such a new notion as follows. An *backward alternating simulation relation* for an alternating automaton $A = (Q, q_\iota, \Sigma, \delta, \alpha)$ is a pre-order $\ggg_{\mathsf{b}}$ which is the reflexive closure of a relation $>_{\mathsf{b}}$ such that for all $\sigma \in \Sigma$, for all $q_1, q_2, q_3 \in Q$, if $q_2 >_{\mathsf{b}} q_1$, then

(i) if $q_1 = q_\iota$, then $q_2 = q_\iota$, and
(ii) if there exists $C \in \delta(q_3, \sigma)$ such that $q_1 \in C$, then there exists $q_4 \in Q$ such that (a) $q_2 \in C'$ for all $C' \in \delta(q_4, \sigma)$, and (b) $q_4 >_{\mathsf{b}} q_3$.

It can be shown that a unique maximal backward simulation relation exists for AFA (because the union of two backward simulation relations is again a backward simulation relation), and it can be computed in polynomial time using analogous fixpoint algorithms for computing standard simulation relations [16], e.g. the fixpoint iterations defined by $R_0 = \{(q_1, q_2) \in Q \times Q \mid q_1 = q_\iota \to q_2 = q_\iota\}$ and $R_i = \{(q_1, q_2) \in R_{i-1} \mid \forall q_3 \in Q : (\exists C \in \delta(q_3, \sigma) : q_1 \in C) \to \exists q_4 \in Q : (\forall C' \in \delta(q_4, \sigma) : q_2 \in C') \wedge (q_3, q_4) \in R_{i-1}\}$ for all $i \geq 1$. Note that for so-called *universal finite automata* (UFA) which are AFA where $\delta(q, \sigma)$ is a singleton for all $q \in Q$ and $\sigma \in \Sigma$, our definition of backward alternating simulation coincides with ordinary backward simulation for the dual of the UFA (which is an NFA with transition relation $\delta'(q, \sigma) = \big\{q \in C \mid \delta(q, \sigma) = \{C\}\big\}$).

As before, given a backward alternating simulation relation $\gg_{\mathsf{b}}$ for A, we define the relation $\succeq_{\mathsf{B}+}$ on $G(A)$ as follows: $v_2 \succeq_{\mathsf{B}+} v_1$ if $\forall q_2 \notin v_2 \cdot \exists q_1 \notin v_1 : q_1 \gg_{\mathsf{b}} q_2$.

Lemma 34. *$\succeq_{\mathsf{B}+}$ is a backward simulation in $G(A)$ compatible with* Init.

Proof. Let $v_1 \succeq_{\mathsf{B}+} v_2$. First, if $v_2 \notin \mathsf{Init}$, then $q_\iota \notin v_2$ and there exists $q_1 \notin v_1$ such that $q_1 \gg_{\mathsf{b}} q_\iota$, hence either $q_1 = q_\iota$, or $q_1 >_{\mathsf{b}} q_\iota$ implying $q_1 = q_\iota$. In both cases $q_\iota = q_1 \notin v_1$ i.e., $v_1 \notin \mathsf{Init}$. Second, assume $E(v_3, v_1)$ and $\sigma \in \Sigma$ is such that for all $q \in v_3$, there exists $C \in \delta(q, \sigma)$ such that $C \subseteq v_1$. Let $v_4 = \{q \mid \exists C' \in \delta(q, \sigma) : C' \subseteq v_2\}$. By definition of $G(A)$, we have $E(v_4, v_2)$. We show that $v_4 \succeq_{\mathsf{B}+} v_3$. To do this, pick an arbitrary $q_4 \notin v_4$ and show that there exists $q_3 \notin v_3$ such that $q_3 \gg_{\mathsf{b}} q_4$. Note that if $q_4 \notin v_3$, then we take $q_3 = q_4$ and we are done. So, we can assume that $q_4 \in v_3$. Hence there exists $C \in \delta(q_4, \sigma)$ such that $C \subseteq v_1$. And since $q_4 \notin v_4$, there exist $q_2 \in C$ and $q_2 \notin v_2$. As $v_2 \succeq_{\mathsf{B}+} v_1$, we know that there exists $q_1 \notin v_1$ such that $q_1 \gg_{\mathsf{b}} q_2$. Since $q_2 \in C$ and $C \subseteq v_1$, we have $q_2 \in v_1$ and therefore we cannot have $q_2 = q_1$, thus we have $q_1 >_{\mathsf{b}} q_2$. Since $q_2 \in C \in \delta(q_4, \sigma)$, and by definition of $>_{\mathsf{b}}$, there exists q_3 such that $q_3 >_{\mathsf{b}} q_4$ (and thus $q_3 \gg_{\mathsf{b}} q_4$) and $q_1 \in C'$ for all $C' \in \delta(q_3, \sigma)$. Since $q_1 \notin v_1$, this implies that $q_3 \notin v_3$. $\square$

4.3 Emptiness problem for ABW

The emptiness problem for ABW can be solved using a subset construction due to Miyano and Hayashi [20, 10, 11].

Given an ABW $A = (Q, q_\iota, \Sigma, \delta, \alpha)$, define the Miyano-Hayashi transition system $\mathsf{MH}(A) = (V, E, \mathsf{Init}, \mathsf{Final})$ where $V = 2^Q \times 2^Q$, and

- $\mathsf{Init} = \{\langle s, \varnothing \rangle \mid q_\iota \in s \subseteq V\}$,
- $\mathsf{Final} = 2^Q \times \{\varnothing\}$, and
- for all $v_1 = \langle s_1, o_1 \rangle$, and $v_2 = \langle s_2, o_2 \rangle$, we have $E(v_1, v_2)$ if there exists $\sigma \in \Sigma$ such that $\forall q \in s_1 \cdot \exists C \in \delta(q, \sigma) : C \subseteq s_2$, and either (i) $o_1 \neq \varnothing$ and $\forall q \in o_1 \cdot \exists C \in \delta(q, \sigma) : C \subseteq o_2 \cup (s_2 \cap \alpha)$, or (ii) $o_1 = \varnothing$ and $o_2 = s_2 \setminus \alpha$.

A classical result shows that $L(A) \neq \varnothing$ if and only if there exists an infinite path from Init in $\mathsf{MH}(A)$ that visits Final infinitely many times. Therefore, the emptiness problem for ABW can be reduced to the repeated reachability problem, and we can use an antichain algorithm (e.g., based on forward simulation) for repeated reachability to solve it. We construct a forward simulation for $\mathsf{MH}(A)$ using a classical notion of alternating simulation.

A pre-order $\ll_{\mathsf{f}} \subseteq Q \times Q$ is an *alternating forward simulation relation* [2] for an alternating automaton A if for all $\sigma \in \Sigma$, for all $q_1, q_2, q_3 \in Q$, if $q_2 \ll_{\mathsf{f}} q_1$, then

(i) if $q_1 \in \alpha$, then $q_2 \in \alpha$, and
(ii) for all $C_1 \in \delta(q_1, \sigma)$, there exists $C_2 \in \delta(q_2, \sigma)$ such that for all $q_4 \in C_2$, there exists $q_3 \in C_1$ such that $q_4 \ll_{\mathsf{f}} q_3$.

Given a forward alternating simulation relation $\ll_{\mathsf{f}}$ for A, define the relation $\preceq_{\mathsf{F+}}$ on $\mathsf{MH}(A)$ such that $\langle s_2, o_2 \rangle \preceq_{\mathsf{F+}} \langle s_1, o_1 \rangle$ if the following conditions hold: (a) $\forall q_2 \in s_2 \cdot \exists q_1 \in s_1 : q_2 \ll_{\mathsf{f}} q_1$, (b) $\forall q_2 \in o_2 \cdot \exists q_1 \in o_1 : q_2 \ll_{\mathsf{f}} q_1$, and (c) $o_1 = \varnothing$ if and only if $o_2 = \varnothing$.

Lemma 35. *$\preceq_{\mathsf{F+}}$ is a forward simulation in $\mathsf{MH}(A)$ compatible with Final.*

Proof. Let $\langle s_2, o_2 \rangle \preceq_{\mathsf{F+}} \langle s_1, o_1 \rangle$. First, if $\langle s_1, o_1 \rangle \in \mathsf{Final}$, then $o_1 = \varnothing$ and thus $o_2 = \varnothing$ by definition of $\preceq_{\mathsf{F+}}$. Hence $\langle s_2, o_2 \rangle \in \mathsf{Final}$. Second, assume $E(\langle s_1, o_1 \rangle, \langle s_3, o_3 \rangle)$ and $\sigma \in \Sigma$ is such that for all $q \in s_1$, there exists $C \in \delta(q, \sigma)$ such that $C \subseteq s_3$, and either (i) $o_1 \neq \varnothing$ and $\forall q \in o_1 \cdot \exists C \in \delta(q, \sigma) : C \subseteq o_3 \cup (s_3 \cap \alpha)$, or ($ii$) $o_1 = \varnothing$ and $o_3 = s_3 \setminus \alpha$.

In the first case (i), we construct $\langle s_4, o_4 \rangle$ such that $E(\langle s_2, o_2 \rangle, \langle s_4, o_4 \rangle)$ and $\langle s_4, o_4 \rangle \preceq_{\mathsf{F+}} \langle s_3, o_3 \rangle$, using the following intermediate constructions.

(1) For each $q_2 \in s_2$, we construct a set $\mathsf{succ}(q_2)$ as follows. By definition of $\preceq_{\mathsf{F+}}$, for $q_2 \in s_2$, there exists $q_1 \in s_1$ such that $q_2 \ll_{\mathsf{f}} q_1$. Since $q_1 \in s_1$, there exists $C_1 \in \delta(q_1, \sigma)$ with $C_1 \subseteq s_3$, and since $q_2 \ll_{\mathsf{f}} q_1$, there exists $C_2 \in \delta(q_2, \sigma)$ such that for all $q_4 \in C_2$, there exists $q_3 \in C_1$ such that $q_4 \ll_{\mathsf{f}} q_1$. We take $\mathsf{succ}(q_2) = C_2$.
(2) For each $q_2 \in o_2$, we construct two sets $\mathsf{succ}^{\alpha}(q_2)$ and $\mathsf{succ}^{\neg\alpha}(q_2)$ as follows. By definition of $\preceq_{\mathsf{F+}}$, for $q_2 \in o_2$, there exists $q_1 \in o_1$ such that $q_2 \ll_{\mathsf{f}} q_1$. Since $q_1 \in o_1$, there exists $C_1 \in \delta(q_1, \sigma)$ with $C_1 \subseteq o_3 \cup (s_3 \cap \alpha)$, and since $q_2 \ll_{\mathsf{f}} q_1$, there exists $C_2 \in \delta(q_2, \sigma)$ such that for all $q_4 \in C_2$, there exists $q_3 \in C_1$ such that $q_4 \ll_{\mathsf{f}} q_3$. We take $\mathsf{succ}^{\alpha}(q_2) = \{q \in C_2 \cap \alpha \mid \exists q' \in s_3 : q \ll_{\mathsf{f}} q'\}$ and $\mathsf{succ}^{\neg\alpha}(q_2) = C_2 \setminus \mathsf{succ}^{\alpha}(q_2)$.

Let $s_4 = \bigcup_{q_2 \in s_2} \mathsf{succ}(q_2) \cup \bigcup_{q_2 \in o_2} \mathsf{succ}^{\alpha}(q_2)$, and $o_4 = o_3 \cup \bigcup_{q_2 \in o_2} \mathsf{succ}^{\neg\alpha}(q_2)$. To prove that $E(\langle s_2, o_2 \rangle, \langle s_4, o_4 \rangle)$, we can check that for all $q_2 \in s_2$ there exists $C_2 \in \delta(q_2, \sigma)$ such that $C_2 = \mathsf{succ}(q_2) \subseteq s_4$, and that $o_2 \neq \varnothing$ (because $o_1 \neq \varnothing$ and $\langle s_2, o_2 \rangle \preceq_{\mathsf{F+}} \langle s_1, o_1 \rangle$) and for all $q_2 \in o_2$ there exists $C_2 \in \delta(q_2, \sigma)$ such that $C_2 \subseteq o_4 \cup (s_4 \cap \alpha)$ (because $\mathsf{succ}^{\neg\alpha}(q_2) \subseteq o_4$ and $\mathsf{succ}^{\alpha}(q_2) \subseteq s_4 \cap \alpha$). To prove that $\langle s_4, o_4 \rangle \preceq_{\mathsf{F+}} \langle s_3, o_3 \rangle$, we can check that

(a) for all $q_4 \in s_4$, there exists $q_3 \in s_3$ such that $q_4 \ll_{\mathsf{f}} q_3$. This holds since either $q_4 \in \mathsf{succ}(q_2)$ for some $q_2 \in s_2$ and by part (1) of the construction, there exists $q_3 \in s_3$ such that $q_4 \ll_{\mathsf{f}} q_3$, or $q_4 \in \mathsf{succ}^{\alpha}(q_2)$ for some $q_2 \in o_2$ and by definition of succ^{α} there exists $q' \in s_3$ such that $q_4 \ll_{\mathsf{f}} q'$;
(b) for all $q_4 \in o_4$, there exists $q_3 \in o_3$ such that $q_4 \ll_{\mathsf{f}} q_3$. This holds since either $q_4 \in o_3$ and we can take $q_3 = q_4$, or $q_4 \in \mathsf{succ}^{\neg\alpha}(q_2)$ for some $q_2 \in o_2$

and by part (2) of the construction, there exists $q_3 \in o_3 \cup (s_3 \cap \alpha)$ such that $q_4 \ll_{\mathsf{f}} q_3$. Now, either $q_4 \in \alpha$ and then $q_3 \notin s_3$ by definition of $\mathsf{succ}^{\neg\alpha}$, thus $q_3 \in o_3$; or $q_4 \notin \alpha$ and then $q_3 \notin \alpha$ by definition of $\ll_{\mathsf{f}}$, thus again $q_3 \in o_3$;

(c) if $o_3 \neq \varnothing$, then $o_4 \neq \varnothing$ since $o_3 \subseteq o_4$. And by (ii), if $o_4 \neq \varnothing$, then $o_3 \neq \varnothing$. Hence $o_3 = \varnothing$ if and only if $o_4 = \varnothing$.

In the second case (ii), we construct the sets $\mathsf{succ}(q_2)$ for each $q_2 \in s_2$ as in part (1) of the construction above, and define $s_4 = s_3 \cup \bigcup_{q_2 \in s_2} \mathsf{succ}(q_2)$ and $o_4 = s_4 \setminus \alpha$. We can check that $E(\langle s_2, o_2\rangle, \langle s_4, o_4\rangle)$ since for all $q_2 \in s_2$ there exists $C_2 \in \delta(q_2, \sigma)$ such that $C_2 = \mathsf{succ}(q_2) \subseteq s_4$, and that $o_2 = \varnothing$ (since $o_1 = \varnothing$ and $\langle s_2, o_2\rangle \preceq_{\mathsf{F+}} \langle s_1, o_1\rangle$) and $o_4 = s_4 \setminus \alpha$. We prove that $\langle s_4, o_4\rangle \preceq_{\mathsf{F+}} \langle s_3, o_3\rangle$ as follows: first, as in (i) above, we have for all $q_4 \in s_4$, there exists $q_3 \in s_3$ such that $q_4 \ll_{\mathsf{f}} q_3$; second, by definition of $\ll_{\mathsf{f}}$ if $q_4 \notin \alpha$, then $q_3 \notin \alpha$ thus for all $q_4 \in o_4$, there exists $q_3 \in o_3$ such that $q_4 \ll_{\mathsf{f}} q_3$; third, this implies that if $o_4 \neq \varnothing$, then $o_3 \neq \varnothing$. And since $o_3 \subseteq o_4$, if $o_3 \neq \varnothing$, then $o_4 \neq \varnothing$. Hence $o_3 = \varnothing$ if and only if $o_4 = \varnothing$. $\square$

4.4 Emptiness problem for a product of NFA

Consider NFAs $A_i = (Q_i, q_\iota^i, \Sigma \cup \{\tau_i\}, \delta_i, \alpha_i)$ for $1 \leq i \leq n$ where $\tau_1, \dots, \tau_n$ are internal actions, and Σ is a shared alphabet. The *synchronized product* $A_1 \otimes A_2 \otimes \dots \otimes A_n$ is the transition system $(V, E, \mathsf{Init}, \mathsf{Final})$ where

- $V = Q_1 \times Q_2 \times \dots \times Q_n$;
- $E(v_1, v_2)$ if $v_1 = (q_1^1, q_1^2, \dots, q_1^n)$, $v_2 = (q_2^1, q_2^2, \dots, q_2^n)$ and either $q_2^i \in \delta_i(q_1^i, \tau_i)$ for all $1 \leq i \leq n$, or there exists $\sigma \in \Sigma$ such that $q_2^i \in \delta_i(q_1^i, \sigma)$ for all $1 \leq i \leq n$;
- $\mathsf{Init} = \{(q_\iota^1, q_\iota^2, \dots, q_\iota^n)\}$;
- $\mathsf{Final} = \alpha_1 \times \alpha_2 \times \dots \times \alpha_n$.

For each $i = 1 \dots n$, let $\ll_{\mathsf{f}}^i \subseteq Q_i \times Q_i$ be a forward simulation relation for A_i. Define the relation $\preceq_{\mathsf{F+}}$ such that $(q_2^1, q_2^2, \dots, q_2^n) \preceq_{\mathsf{F+}} (q_1^1, q_1^2, \dots, q_1^n)$ if $q_2^i \ll_{\mathsf{f}}^i q_1^i$ for all $1 \leq i \leq n$.

Lemma 36. *$\preceq_{\mathsf{F+}}$ is a forward simulation in $A_1 \otimes \dots \otimes A_n$ compatible with* Final.

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