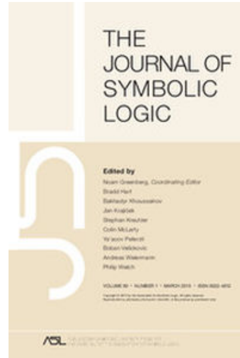

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AN UNDECIDABLE PROBLEM IN FINITE COMBINATORICS

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§0. Introduction. Problems of computing probabilities of statements about large, finite structures have become an important subject area of finite combinatorics. Within the last two decades many researchers have turned their attention to such problems and have developed a variety of methods for dealing with them. Applications of these ideas include nonconstructive existence proofs for graphs with certain properties by showing the properties occur with nonzero probabilities (for examples see Erdös and Spencer [ES] and Bollobás [Bo]), and determination of average running times for sorting algorithms by computing asymptotic probabilities of statements about permutations (see Knuth [Kn]). Two types of techniques recur in solutions to such problems: probabilistic techniques, such as those used by Erdös and Spencer [ES], and classical assymptotic techniques, such as those surveyed by Bender [Be] and de Bruijn [Br]. Studying this body of techniques, one notices that characteristics of these problems suggest certain methods of solution, in much the same way that the form of an integrand may suggest certain substitutions. The question arises, then, as to whether there is a systematic way to approach these problems: is there an algorithm for computing asymptotic probabilities? I will show that the answer is "no"—for any reasonable formulation, the problem of computing asymptotic probabilities is undecidable.

The main theorem of the paper is Theorem 1.6, which says that there is a finitely axiomatizable class in which every first order sentence has an asymptotic probability of 0 or 1—i.e., is almost always true or almost always false in finite structures—but for which the problem of deciding whether a sentence has asymptotic probability 0 or 1 is undecidable. Heretofore, classes known to have such a 0-1 law have had decidable asymptotic probability problems (see Lynch [Ly] for examples and a discussion of previous work in the area).

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§1. Definitions and statement of the result. In this section I make precise the ideas discussed in the Introduction, and state the main result.

I will assume the reader is familiar with the basics of model theory; consult Chang and Keisler [CK] on matters of notation and terminology. I will always denote a

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finite language containing only relation symbols, and $\mathscr{C}$ will be a class of L-structures closed under isomorphisms.

1.1. DEFINITION. $\mathscr{A}_n$ will denote the set of L-structures in $\mathscr{C}$ with universe $n = \{0, 1, \ldots, n-1\}$, and $|\mathscr{A}_n| = a_n$. For a sentence φ let c_n be the number of models of φ in $\mathscr{A}_n$ and define the *labeled asymptotic probability* of φ to be

$$\mu(\varphi) = \lim_{n\to\infty} c_n/a_n$$

whenever the limit exists.

$\mathscr{B}_n$ will denote a representative set from the isomorphism classes in $\mathscr{A}_n$, and $|\mathscr{B}_n| = b_n$. For a sentence φ, let d_n be the number of models of φ in $\mathscr{B}_n$ and define the *unlabeled asymptotic probability* of φ to be

$$\nu(\varphi) = \lim_{n\to\infty} d_n/b_n$$

whenever the limit exists.

The modifiers "labeled" and "unlabeled" come from combinatorics. Isomorphic structures in $\mathscr{A}_n$ may be distinguished by the labels of their elements (in this case, the ordinals $0, 1, \ldots, n-1$), whereas $\mathscr{B}_n$ contains just one element from each isomorphism class so labels serve no purpose.

I will be interested in cases where asymptotic probabilities are as simple as possible.

1.2. DEFINITION. $\mathscr{C}$ has a *labeled* (*unlabeled*) *0-1 law* if $\mu(\varphi)$ ($\nu(\varphi)$) is 0 or 1 for every first order sentence φ.

In Fagin [Fa1, Fa2], in Glebskiĭ, Kogan, Liogon'kiĭ, and Talanov [GKLT], and in Liogon'kiĭ [Li] it is shown that if $\mathscr{C}$ is the class of all L-structures, or the class of graphs, it has both a labeled and unlabeled 0-1 law. The approach in [Fa1] and [Li] is to show that a certain set of sentences, each with asymptotic probability 1, forms a complete theory. This set is in fact recursive, so the problem of computing asymptotic probabilities of first order sentences for the class of all L-structures, or all graphs, is decidable. Grandjean [Gr] has shown that these problems are PSPACE-complete. Compare this with the undecidability of the problem of determining which first order sentences hold in all finite L-structures or graphs (see Vaught [Va] and Theorem 2.4 below). It is much easier to determine the first order properties of *almost all* finite L-structures or graphs. Thus, to show the undecidability of the general problem, I must consider more complex classes of structures.

The classes I will consider have interesting combinatorial properties. To state them I need the following definition.

1.3. DEFINITION. Let $\mathfrak{A}$ be an L-structure. For elements a and b in the universe of $\mathfrak{A}$, define $a \sim b$ if for some relation symbol R in L and sequences $\vec{x}, \vec{y}, \vec{z}$, of variables

$$\mathfrak{A} \models (\exists \vec{x}, \vec{y}, \vec{x}) R(\vec{x}, a, \vec{y}, b, \vec{z}).$$

Let $\sim^*$ be the least equivalence relation extending $\sim$. The $\sim^*$ equivalence classes are called *components* of $\mathfrak{A}$. An L-structure $\mathfrak{K}$ is called a *component* of $\mathfrak{A}$ if it is a substructure of $\mathfrak{A}$ and its universe is a component of $\mathfrak{A}$. $\mathfrak{A}$ is *connected* if it has just one component. $\mathfrak{B}$ is a *closed extension* of $\mathfrak{A}$ (written $\mathfrak{B} \supseteq_{\mathrm{cl}} \mathfrak{A}$) if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and is a union of components of $\mathfrak{B}$.

If L contains only one relation symbol, which interprets an edge relation on graphs, then the definitions of *component* and *connectivity* correspond to the graph theoretic notions.

I will consider classes that are closed under disjoint unions and components—i.e., those classes $\mathscr{C}$ with the following properties.

(i) if $\mathfrak{A}, \mathfrak{B} \in \mathscr{C}$ then $\mathfrak{A} \amalg \mathfrak{B}$, the disjoint union of $\mathfrak{A}$ and $\mathfrak{B}$, is in $\mathscr{C}$.

(ii) If $\mathfrak{A} \in \mathscr{C}$ and $\mathfrak{K}$ is a component of $\mathfrak{A}$ then $\mathfrak{K} \in \mathscr{C}$.

Classes with these closure properties have figured prominently in combinatorial research because they include many familiar examples (graphs, permutations, unary functions, etc.) and because enumerative problèms of such classes often yield to methods involving generating series (see [Co1] or [Co2] for a more comprehensive discussion of these classes). The utility of generating series in these problems rests on the relationships given in Theorem 1.5 below. Its statement requires the following definition.

1.4. Definition. Let $\mathscr{C}$ be a class of L-structures, $a_n = |\mathscr{A}_n|$, $b_n = |\mathscr{B}_n|$, as in Definition 1.1. The *exponential generating series* for $\mathscr{C}$ is

$$a(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

The *ordinary generating series* for $\mathscr{C}$ is

$$b(x) = \sum_{n=0}^{\infty} b_n x^n.$$

By convention $a_0 = b_0 = 1$.

1.5. Theorem. *Let $\mathscr{C}$ be closed under disjoint unions and components.*

(i) *If k_n is the number of connected structures in $\mathscr{A}_n$ and $k(x) = \sum_{n=0}^{\infty} k_n x^n/n!$ then*

$$a(x) = \sum_{n=0}^{\infty} \frac{k(x)^n}{n!} = \exp(k(x)).$$

These operations should be interpreted formally, without any considerations of convergence.

(ii) *If l_n is the number of connected structures in $\mathscr{B}_n$ then*

$$b(x) = \prod_{n=1}^{\infty} (1 - x^n)^{-l_n}.$$

Again these operations should be interpreted formally.

Theorem 1.5 is well known to combinatorialists (see Bender and Goldman [BG]). I do not include the proof, which is quite straightforward (see [Co1] or [Co2] for details).

I now state the main theorem of the paper.

1.6. Theorem. *There is a class $\mathscr{C}$ such that*

(i) *$\mathscr{C}$ is finitely axiomatizable;*

(ii) *$\mathscr{C}$ has an unlabeled 0-1 law;*

(iii) *$\mathscr{C}$ is closed under disjoint unions and components; and*

(iv) *the set T' of first order sentences φ such that $\mu(\varphi) = 1$ is not recursive.*

§2. Three theorems. The proof of Theorem 1.6 rests on three theorems. The first gives a necessary and sufficient condition for a non-fast-growing class closed under disjoint unions and components to have an unlabeled 0-1 law; I use only sufficiency but state the theorem in full generality. The second provides an easy means to construct classes that satisfy this condition. The third will allow me to show that the set of sentences with probability 1 is not recursive.

2.1. THEOREM. *Let $\mathscr{C}$ be closed under disjoint unions and components, and suppose that its ordinary generating series $\sum_{n=0}^{\infty} b_n x^n$ has radius of convergence greater than* 0. *Then $\mathscr{C}$ has an unlabeled* 0-1 *law iff*

$$\lim_{n\to\infty} \frac{b_{n-1}}{b_n} = 1.$$

Moreover, when this condition holds the set of first order sentences φ such that $v(\varphi) = 1$ consists of the sentences which hold in all finite closed extensions in $\mathscr{C}$ of some finite structure in $\mathscr{C}$. That is,

$$T_{\mathrm{fin}} = \bigcup_{\mathfrak{A}\in\mathscr{C}} \bigcap_{\mathfrak{B}\supseteq_{\mathrm{cl}}\mathfrak{A}} \mathrm{Th}(\mathfrak{B}),$$

where the union and intersection range over finite structures only.

The proof may be found in [Co1] or [Co2]. In the latter it is shown that the statement of the theorem may be strengthened to cover monadic second order sentences, but only the first order case is needed here.

The next theorem is a special case of a more general theorem due to Bateman and Erdös; see [BE] for a proof.

2.2. THEOREM (BATEMAN AND ERDÖS). *If*

$$\sum_{n=0}^{\infty} b_n x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-l_n},$$

where $l_n = 0$ or 1 for each n and g.c.d. $\{n \in \omega : l_n \neq 0\}$ is 1, *then*

$$\lim_{n\to\infty} \frac{b_{n-1}}{b_n} = 1.$$

To state the final theorem I need the following definitions and notation.

2.3. DEFINITION. For a theory T,

$$T_{\mathrm{fin}} = \bigcap_{\mathfrak{A}\models T} \mathrm{Th}(\mathfrak{A}),$$

where the intersection again ranges over just finite models; i.e., T_{fin} is the theory consisting of sentences true in all finite models of T. T is *finitely inseparable* if the deductive closure of T and the complement of T_{fin} (within the set of first order sentences in the language of T) are recursively inseparable—that is to say, if there is not a recursive set A such that $T \subseteq A$ and $\mathrm{comp}(T_{\mathrm{fin}}) \cap A = \varnothing$.

L_{bin} will denote the language consisting of one binary relation.

2.4. THEOREM (VAUGHT). *The set of valid sentences in the language L_{bin} is finitely inseparable.*

The proof may be found in [Va].

§3. Proof of the main theorem. The proof of Theorem 1.6 will be easier to understand broken into two parts. The first part, Lemma 3.1, specifies a set of conditions on a theory T that will insure that the class of models of T satisfies conditions (i)–(iv) of Theorem 1.6. The second part is to list the sentences in such a T.

3.1. LEMMA. *The class of models of a theory T in a language L_{code} satisfies conditions* (i)–(iv) *of Theorem* 1.6 *if the following are true.*

(i) *T is finite and preserved under disjoint unions and components.*

(ii) *T has at most one connected model of each finite cardinality, and the g.c.d. of the cardinalities at which they occur is* 1.

(iii) *Each finite connected model of T codes a finite L_{bin}-structure, and each finite L_{bin}-structure is coded by at least one finite connected model of T.*

(iv) *There is a recursive function taking L_{bin} sentences φ to L_{code} formulas $\hat{\varphi}(x)$ with one free variable x such that for any finite model $\mathfrak{A}$ of T, $\mathfrak{A} \models \hat{\varphi}(a)$ iff the component of $\mathfrak{A}$ containing a codes an L_{bin}-structure satisfying φ.*

PROOF. The coding mentioned in (iii) and (iv) will be made explicit later.

Conditions (i) and (iii) of Theorem 1.6 follow immediately: by hypothesis, the class of models of T is finitely axiomatizable and closed under disjoint unions and components.

Condition (ii) of Theorem 1.6 follows almost as easily. Theorem 2.2, the theorem of Bateman and Erdös, implies that $\lim_{n\to\infty} b_{n-1}/b_n = 1$ (recall b_n is the number of unlabeled models of T of cardinality n). From Theorem 2.1, the models of T have an unlabeled 0-1 law.

Condition (iv) is the remaining part of Theorem 1.6 to be verified. I will show that the map $\varphi \to \forall x\, \hat{\varphi}(x)$ takes the set T^* of valid sentences in L_{bin} into the set T' of L_{code} sentences ψ with unlabeled asymptotic probability 1, and the complement of T^*_{fin} into the complement of T'. Hence, if T' were recursive then its preimage under this map would separate T^* from the complement of T^*_{fin}, contradicting Theorem 2.4.

Suppose φ is in T^*. Then φ holds in all finite L_{bin}-structures, so for any finite model $\mathfrak{A}$ of T and element a from $\mathfrak{A}$, $\mathfrak{A} \models \hat{\varphi}(a)$. Therefore $\forall x\, \hat{\varphi}(x)$ is true in all finite models of T and thus certainly is in T'.

Suppose φ is not in T^*_{fin}. Then φ fails in some finite L_{bin}-structure. This structure is coded by some connected L_{bin}-structure $\mathfrak{K}$. Now I must show that $\forall x\, \hat{\varphi}(x)$ is not in T'. By Theorem 2.1 a sentence is in T' iff for some finite model $\mathfrak{A}$ of T, the sentence is true in all finite closed extensions of $\mathfrak{A}$ that satisfy T. But $\mathfrak{A} \amalg \mathfrak{K}$ is a closed extension of $\mathfrak{A}$ that satisfies T, and $\mathfrak{A} \amalg \mathfrak{K} \not\models \hat{\varphi}(a)$ for any a from $\mathfrak{K}$, so

$$\mathfrak{A} \amalg \mathfrak{K} \not\models \forall x\, \hat{\varphi}(x).$$

Thus $\forall x\, \hat{\varphi}(x)$ is not in T'. I conclude that T' is not recursive. ⊣

Now I undertake the second part of the proof of Theorem 1.6—to construct a theory T satisfying conditions (i)–(iv) of Lemma 3.1. The following notation will be useful.

3.2. DEFINITION. Let $\mathfrak{A}$ be an L-structure and suppose that R is a binary relation symbol in L that interprets an equivalence relation on $\mathfrak{A}$. Define quantifiers $(\forall x.R(x, y))$ and $(\exists x.R(x, y))$ by

$$\models (\forall x.R(x, y))\varphi \leftrightarrow \forall x(R(x, y) \to \varphi),$$
$$\models (\exists x.R(x, y))\varphi \leftrightarrow \exists x(R(x, y) \wedge \varphi).$$

For any sentence φ and variable y not occurring in φ, define $\varphi^{R,y}$, the *relativization of φ to R and y*, by replacing every universal quantifier $\forall x$ in φ with $(\forall x. R(x, y))$ and every existential quantifier $\exists x$ in φ with $(\exists x. R(x, y))$.

Listed below are the axioms for T. For clarity I will state them in English rather than first order logic. Commentary on the role played by each axiom is interpersed. The symbols of L_{code} will be introduced along the way.

(1) $K(\cdot, \cdot)$ is an equivalence relation.

Components of models of T will be K-equivalence classes. In the sequel an axiom of the form "every component satisfies φ" should be understood to mean $\forall x\, \varphi^{K,x}$.

(2) $B(\cdot, \cdot)$ is an equivalence relation and a refinement of K.

Each B-equivalence will be a Boolean algebra (abbreviated BA—the one-element algebra will be included in this category). Since L_{code} does not contain constant or function symbols, the identity and zero elements will be designated by unary relations $I(\cdot)$ and $Z(\cdot)$, and the inf, sup, and complementation operations by relations $\wedge(\cdot,\cdot,\cdot)$, $\vee(\cdot,\cdot,\cdot)$, and $-(\cdot,\cdot)$.

(3) There are unique elements x and y in each B-equivalence class such that $I(x)$ and $Z(y)$; $\wedge(x, y, z)$ and $\vee(x, y, z)$ are binary operations on each B-equivalence class, $-(x, y)$ is a unary operation on each B-equivalence class, and $\wedge$, $\vee$ and $-$ may not relate elements from different B-equivalence classes.

(4) For each of the (finitely many) axioms φ for Boolean algebras, excepting $0 \neq 1$, replace the constant and function symbols using the appropriate relation symbols to obtain $\bar{\varphi}$; include $\forall x\, \bar{\varphi}^{B,x}$ in T.

An axiom of the form "every BA satisfies φ" means $\forall x\, \bar{\varphi}^{B,x}$, where $\bar{\varphi}$ is defined as above.

According to condition (ii) of Lemma 3.1 there is at most one connected model of T of each finite cardinality. Since each BA has cardinality some power of 2, I can achieve this by stipulating that no two BAs within the same component have the same cardinality (it will also be necessary to check that the number of connected models does not increase as new relation symbols are added to L_{code}; this poses no difficulty). This will be accomplished by insuring that no two BAs within the same component have the same number of atoms. Introduce a predicate for atoms and zero elements.

(5) $A(x)$ holds iff x is an atom or a zero element in one of the BAs.

Next, linearly order the atoms and zero element of each Boolean algebra, the zero element being least.

(6) $\leq(x, y)$ holds only between elements x, y in the same BA such that $A(x)$ and $A(y)$. It is reflexive, antisymmetric, and transitive. For x and y in the same BA, if $A(x)$ and $A(y)$ then either $\leq(x, y)$ or $\leq(y, x)$. If $Z(x)$ then there are no elements less than x in the $\leq$ order.

Then pick out a BA (call it a *special* BA) from each component.

(7) $S(x)$ is true of at least two elements in each component. If $S(x)$ holds then $S(y)$ holds for all y in the same BA as x. If $S(x)$ and $S(y)$, and x and y are in the same component, then x and y are in the same BA.

Introduce a set of $\leq$-order isomorphisms from initial segments in the special BA onto the ordered elements of each of the other BAs in the component. The set is indexed by atoms in the special BA.

(8) $F(x, y, z)$ holds only if $S(x)$ and x is an atom (but not a zero element). For each x, either there are no elements y and z such that $F(x, y, z)$, or for each $y \leq x$ but not equal to x, there is a unique z such that $F(x, y, z)$; there are no other y and z for which $F(x, y, z)$ holds. The mapping so defined is an order isomorphism onto the elements z satisfying $A(z)$ in one of the other BAs in the same component as x. Every x satisfying $A(x)$ is either in a special BA or in the range of one of these maps.

Now there is at most one unlabeled, connected model of T of each finite cardinality.

It remains to code each L_{bin}-structure in connected models of T and define the map $\varphi \to \hat{\varphi}(x)$ described in Lemma 3.1. Do this by providing the atoms of the special BA within each component with a square matrix structure and then using the relation $F(\cdot,\cdot,\cdot)$ to define an incidence matrix for a binary relation.

(9) $R(x, y)$ and $C(x, y)$ (the row and column relations) are equivalence relations on the atoms of each special BA and do not relate any other elements. Each row and column of a special BA have precisely one element in common.

This defines a matrix structure on the atoms of each special BA. To make it a square matrix, specify a diagonal.

(10) $D(x)$ is a unary relation holding only if x is an atom of a special BA. Each row and each column contains a unique x such that $D(x)$.

Finally, provide that the matrix structures on the atoms of the special BAs are unique by requiring that the order $\leq$ extends, in a natural way, the order on the diagonal. This guarantees that there is at most one connected structure of each finite cardinality.

(11) For any two atoms x and y in the same special BA, $x \leq y$ iff the diagonal element in the same row as x is strictly less than the diagonal element in the same column as y, or x and y belong to the same row and the diagonal element in the same column as x is strictly less than the diagonal element in the same column as y, or x and y are equal.

This completes the list of axioms of T.

Let $\mathfrak{K}$ be a connected model of T. $\mathfrak{K}$ codes an L_{bin}-structure on the diagonal elements of the special BA in $\mathfrak{K}$. The L_{bin}-structure satisfies $R(x, y)$ iff for the unique element x' in the same row as x and same column as y there are y' and z' such that $F(x', y', z')$. Observe that this condition on x, y is first order expressable in the language L_{code} and that the universe of the L_{bin}-structure coded by $\mathfrak{K}$ is definable in $\mathfrak{K}$. Hence, it is a simple matter to define a recursive map $\varphi \to \hat{\varphi}(x)$ such that $\hat{\varphi}(a)$ holds in a finite model of T iff φ is true in the L_{bin}-structure coded by the component containing a.

Note that every finite L_{bin}-structure is coded by some finite connected model of T.

Consider the L_{bin}-structure with universe $\{0,1\}$ such that the binary relation consists of the pair $(0,0)$. The L_{code}-structure coding it has precisely one component, which consists of two BAs: a special BA (every component has exactly one special BA) and a BA corresponding to the pair $(0,0)$. The special BA has four atoms divided into rows and columns to form a two-by-two matrix. Denote these atoms (i, j), where $i \in \{0,1\}$ is the row index and $j \in \{0,1\}$ is the column index. By axiom (11), $(0,0) \leq (0,1) \leq (1,0) \leq (1,1)$. The zero of the special BA is less than each of these atoms. Now $F((0,0), y, z)$ is a one-to-one mapping from the predecessors y of $(0,0)$

(i.e., just the zero of the special BA) onto the elements z which are atoms or a zero of the other BA in the component. Hence, the other BA has no atoms; $F((0,0), y, z)$ holds only when y is the zero of the special BA and z is the zero of the other BA. The component therefore has $2^4 + 2^0 = 17$ elements.

Now consider the L_{bin}-structure with universe $\{0, 1\}$ such that the binary relation consists of the pairs $(0,0)$ and $(0,1)$. The L_{code}-structure coding it has precisely one component, which consists of three BAs: as before, a special BA with four atoms, and BAs corresponding to the pairs $(0,0)$ and $(0,1)$. The component therefore has $2^4 + 2^1 + 2^0 = 19$ elements.

It follows that the g.c.d. of the cardinalities of connected models of T is 1. All the conditions of Lemma 3.1 have been met. This concludes the proof of Theorem 1.6.

§4. Related matters. This section contains some observations and questions suggested by Theorem 1.6 and its proof.

The proof of Theorem 1.6 does not require the full strength of Theorem 2.2 (the Bateman-Erdös theorem). Arranging for T to have a connected model of every finite cardinality requires only a slight modification of axiom (10) in the last section. Then it is only necessary to know that b_{n-1}/b_n approaches 1 as n approaches ∞ when $\sum_{n=0}^{\infty} b_n x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-1}$. This is well known from the theory of partitions (see Andrews [An]; the simplest proof, however, is probably one based on techniques used in [BE] to prove the more general theorem).

A more compelling reason for the inclusion of Theorem 2.2 than a slight simplication of the proof is that it provides an easy means for verifying that a class has an unlabeled 0-1 law. Hence, it may prove useful in constructing more natural examples of classes with undecidable asymptotic properties. After all, no combinatorialist cares about the theory T described in §3. The undecidability in this case comes from the rich structure in models of T; this allows coding. It seems likely that more interesting combinatorial classes display this richness of structure.

What about the labeled case? By modifying axiom (10) as just described, modifying axiom (6) so that $\leq$ is a linear order on BAs rather than atoms, and adding a condition asserting that $\leq$ extends the order on atoms in a unique way, T will have exactly one connected model of each finite cardinality and each connected model has only the trivial automorphism. Thus, T has $n!$ labeled connected models of cardinality n. By Theorem 1.5(i), the models of T have exponential generating series $\exp(\sum_{n=1}^{\infty} x^n) = \exp(x/(1 - x))$. It follows from Theorem 6.8 of [Co2] that for a class closed under disjoint unions and components, with this exponential generating series, every first order (in fact, every monadic second order) sentence has a labeled asymptotic probability. Using the analogue of Theorem 2.1 for labeled structures (see [Co2]), the arguments of §3 show that the set of sentences with labeled asymptotic probability 1 is not recursive. I have not been able to construct a class that has a labeled 0-1 law but no algorithm for deciding labeled asymptotic probabilities; very likely one exists, but the asymptotic problems involved are difficult.

Finally, I pose the question of how undecidable asymptotic problems can be. Theorem 2.1 shows that the set of sentences with unlabeled asymptotic probability 1 (for models of theory T in §3) is Σ_2 in the arithmetical hierarchy (see Rogers [Ro]). Is this set Π_1? Is it complete in its level of the arithmetical hierarchy? These questions are of interest not so much in the case of the examples given here, but for asymptotic problems in general.

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