

# THE MEMBERSHIP AND THRESHOLD PROBLEMS FOR HYPERGEOMETRIC SEQUENCES WITH QUADRATIC PARAMETERS

GEORGE KENISON

**ABSTRACT.** The membership and threshold problems for recurrence sequences are fundamental open decision problems in automated verification. The former problem asks whether a chosen target is an element of a sequence, whilst the latter asks whether every term in a sequence is bounded from below by a given value.

A rational-valued sequence  $\langle u_n \rangle_n$  is hypergeometric if it satisfies a first-order linear recurrence of the form  $p(n)u_{n+1} = q(n)u_n$  with polynomial coefficients  $p, q \in \mathbb{Z}[x]$ . In this note we establish decidability results for the aforementioned problems for restricted classes of hypergeometric sequences. For example, we establish decidability for the aforementioned problems under the assumption that the polynomial coefficients  $p, q \in \mathbb{Z}[x]$  are monic and split over an imaginary rational extension of  $\mathbb{Q}$ . We also establish conditional decidability results; that is, conditional on Schanuel's conjecture, when the irreducible factors of the monic polynomial coefficients  $p, q \in \mathbb{Z}[x]$  are either linear or quadratic.

## 1. INTRODUCTION

**1.1. Motivation.** The membership and threshold problems for recurrence sequences pertain to fundamental aspects of verification, the analysis of algorithms, and computational modelling. Indeed, they frequently appear in discussions on topics such as weighted automated, loop termination and Markov processes. The membership problem asks whether a chosen target is in the orbit of (commonly, is *reached* by) a given sequence. Meanwhile, the threshold problem asks whether every term in a real-valued sequence is bounded from below by a given value (commonly, the *threshold*).

Arguably, the most famous such membership problem is the *Skolem Problem* for linear recurrence sequences with constant coefficients. Skolem asks whether a given recurrence sequence vanishes at some point; that is to say, is zero a term in the sequence? The decidability of Skolem is a long-standing open problem; in fact, decidability of Skolem is only known for linear recurrence sequences of order at most four [14, 25].

In this paper we restrict our focus to the membership and threshold problems for hypergeometric sequences (the *Hypergeometric Membership* and *Hypergeometric Threshold Problems*, respectively). Here a *hypergeometric sequence* is a rational-valued first-order linear recurrence sequence with polynomial coefficients; that is to say, a sequence  $\langle u_n \rangle_n$  of rationals that satisfies a relation of the form

$$(1) \quad p(n)u_{n+1} = cq(n)u_n$$

where  $p, q \in \mathbb{Z}[n]$  and  $c \in \mathbb{Q}$ .

For the avoidance of doubt, the input tuple  $(\langle u_n \rangle_n, t)$  for both the Hypergeometric Membership and Hypergeometric Threshold Problems is a hypergeometric sequence  $\langle u_n \rangle_n$  and a non-zero rational  $t$ . The former problem is to determine whether there is an  $n \in \mathbb{N}_0$  such that  $u_n = t$ . The latter problem is to determine whether  $u_n \geq t$  for each  $n \in \mathbb{N}_0$ . Both of the decision problems are open.

**1.2. Contributions.** In this paper we present decidability results for classes of hypergeometric sequences. These classes place restrictions on the polynomial coefficients  $p, q \in \mathbb{Z}[n]$  in (1). For a hypergeometric sequence that satisfies (1), we call the roots of the coefficients  $p, q \in \mathbb{Z}[n]$  the *parameters* of the sequence.

Our main contributions are decidability and conditional decidability results. Indeed, the Hypergeometric Membership and Threshold problems are decidable for the sequences whose polynomial coefficients are:

- monic and split over an imaginary quadratic number field (Theorem 3.5). If Schanuel's conjecture is true, then the aforementioned problems are decidable for the sequences where the polynomial coefficients in (1) are:
- monic and all the irreducible factors of  $p$  and  $q$  are either linear or quadratic (Corollary 4.9); or
- monic and the sequence parameters possess certain twofold symmetries (Theorem 4.5).

Recall that Schanuel's conjecture is a grand unifying prediction in transcendental number theory that subsumes several principal results (cf. [11, 3, 27]). We delay a formal statement of the conjecture (to the Preliminaries) and definition of the class of sequences, so-called class  $\mathfrak{C}$ , with twofold symmetries (to Section 4). As motivation to study the aforementioned class  $\mathfrak{C}$ , we discuss sequences with unnested radical and cyclotomic parameters in Subsection 4.3.

**1.3. Background.** Given a *non-degenerate* hypergeometric sequence  $\langle u_n \rangle_n$  (definition in the Preliminaries), there is a single difficult instance of both the membership and threshold problems. That is to say, for all but at most one value of  $t$ , the instance  $(\langle u_n \rangle_n, t)$  of the membership (resp. threshold) problem can be decided by elementary means (Proposition 2.5). The problem instances when  $t$  is the critical value are both open. The object of this paper is to show that for restricted classes of hypergeometric sequences we can employ techniques from transcendental number theory to settle restricted variants of the hypergeometric membership and threshold problems.

Previous works by Kenison et al. [8] and Nosan et al. [17] established respective reductions from the threshold and membership problems for hypergeometric sequences to the problem of determining equality between gamma products, the *Gamma Product Equality Problem* (GPEP). Here a gamma product is an expression of the form  $\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_\ell)$  where  $\Gamma$  denotes the gamma function,  $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x} dx$  for  $z \in \mathbb{C}$  with  $\text{Re}(z) > 0$ . It is possible to analytically extend the domain of  $\Gamma$  to the whole complex plane minus the non-positive integers where the function has simple poles.

In [8], Kenison et al. establish decidability of the Hypergeometric Threshold Problem subject to a strengthening of the Period Conjecture (see [10, Conjecture 1]) posed by Kontsevich and Zagier. Previous works [8, 17] also consider the restriction of the Hypergeometric Membership Problem to the class of sequences with rational parameters. Deciding GPEP under this restriction reduces to determining all the multiplicative relations of the Gamma function for given inputs (for full details see [17, Theorem 5]). The latter problem is the subject of an older conjecture due to Rohrlich [11, 27]. Rohrlich's Conjecture predicts that any multiplicative relation of the form  $\prod_{j=1}^n (2\pi)^{-1/2} \Gamma(\alpha_j) = \theta$  for appropriate  $\alpha_j \in \mathbb{Q}$  and real-algebraic  $\theta$  can be derived from the standard functional identities for the gamma function (the translation, reflection, and multiplicative properties discussed in the Preliminaries).

**1.4. Related work.** Suppose that  $\langle u_n \rangle_n$  is a hypergeometric sequence that satisfies recurrence (1). Without loss of generality, we associate a unique pair of polynomial coefficients  $p, q \in \mathbb{Z}[n]$  to  $\langle u_n \rangle_n$  by assuming that both polynomials are of lowest degree and that  $q$  is monic. We call  $r(n) := cq(n)/p(n)$  the *shift quotient* of the sequence  $\langle u_n \rangle_n$ .

*The Hypergeometric Membership Problem.* In [17], Nosan et al. prove that the Hypergeometric Membership Problem with rational parameters is decidable; their  $p$ -adic argument considers the prime divisors of the terms  $u_n$  and the target  $t$ . Briefly, their strategy (for non-degenerate cases) shows that for sufficiently large  $n \in \mathbb{N}$ , there is a prime divisor  $p$  of  $u_n$  that cannot also be a prime divisor of  $t$ . This is sufficient to reduce such instances of the membership problem to an exhaustive search for  $t$  amongst the initial terms of the hypergeometric sequence.

On the one hand, the work by Nosan et al. can settle decidability of the membership problem for examples of the form

$$(2) \quad u_{n+1} = \frac{(n+1)(n+7/9)(n+5/9)}{(n+11/9)(n+8/9)(n+2/9)} u_n,$$

which we cannot handle herein. A heuristic argument as to why such examples are beyond our approach is given in Section 5. On the other hand, we establish decidability results for certain classes of hypergeometric sequences whose parameters are not necessarily rationals such as

$$v_{n+1} = \frac{n^4 - 4n^3 + 9n^2 - 10n + 5}{\Phi_{12}(n-1)} v_n$$

and

$$w_{n+1} = \frac{n^2 - 4n + 5}{n^2 - 4n + 13} w_n$$

(here  $\Phi_{12}(n)$  is the 12th cyclotomic polynomial). For sequence  $\langle w_n \rangle_n$  as above, we demonstrate the reduction of the aforementioned decision problems ( $\langle w_n \rangle_n, t$ ) to a finite search query in Example 3.2.

During the preparation of this note, there has been rapid progress on the decidability of the Hypergeometric Membership Problem with quadratic parameters (building on the  $p$ -adic methods of [17]).

A recent preprint by the author and colleagues [9], establishes decidability results for the Hypergeometric Membership Problem for restricted classes of sequences. Therein, we establish unconditional decidability for the membership problem for hypergeometric sequences with parameters drawn from the integers of a single quadratic field. For comparison, this note establishes unconditional decidability of both the membership and threshold problems for hypergeometric sequences when the parameters are drawn from the integers of a single imaginary quadratic field ([Theorem 3.5](#)). On the other hand, this notes establishes decidability (conditional on Schanuel's conjecture) of these two decision problems when the parameters are drawn from the integers of any number of quadratic fields ([Corollary 4.9](#)).

*The Hypergeometric Threshold Problem.* One of the main strengths of the approach in this paper is the consideration of the Hypergeometric Threshold Problem; by comparison, the  $p$ -adic techniques employed by Nosan et al. [17] are only applicable to the Hypergeometric Membership Problem. Note, for example, that our approach establishes decidability results for the threshold problem to sequences such as  $\langle v_n \rangle_n$  and  $\langle w_n \rangle_n$  above.

In [8], the author and colleagues consider the related *Hypergeometric Inequality Problem* as follows: given two hypergeometric sequences  $\langle s_n \rangle_n$  and  $\langle t_n \rangle_n$ , determine whether  $s_n \geq t_n$  for each  $n \in \mathbb{N}_0$ . Therein we gave a Turing reduction from the Hypergeometric Inequality Problem to the Hypergeometric Threshold Problem. As a consequence of [Proposition 3.1](#) herein, the decidability of the Hypergeometric Inequality Problem with Gaussian integer parameters is decidable (since the work in [8] reduces the problem to deciding the Hypergeometric Threshold Problem with Gaussian integer parameters).

**1.5. Wider research context.** There is a large corpus of research into infinite product identities related to the gamma function (see, for example, [5, 1, 7]). This is particularly relevant to our approach, via transcendence theory, to resolving equality testing problems. An illustrative example is given by Borwein et al. [4, pages 4–6]:

$$(3) \quad \prod_{k=2}^{\infty} \frac{k^5 - 1}{k^5 + 1} = \frac{2 \cdot \Gamma(-\omega_{10}) \Gamma(\omega_{10}^2) \Gamma(-\omega_{10}^3) \Gamma(\omega_{10}^4)}{5 \cdot \Gamma(\omega_{10}) \Gamma(-\omega_{10}^2) \Gamma(\omega_{10}^3) \Gamma(-\omega_{10}^4)}$$

where  $\omega_{10} = e^{2\pi i/10}$ . It is not known whether the limit in [Equation 3](#) is algebraic.

Identities for (short) gamma products and quotients are also the subject of much research interest (a non-exhaustive list includes [23, 26, 13, 16, 5, 18]). The identity

$$\frac{\Gamma(1/14)\Gamma(9/14)\Gamma(11/14)}{\Gamma(3/14)\Gamma(5/14)\Gamma(13/14)} = 2,$$

is an application of the functional properties of the gamma function (see [Subsection 2.1](#)) [5]. This example illustrates that even when exact values of the gamma function at rational points are not known (indeed it is not even known whether  $\Gamma(1/14)$  is algebraic or transcendental), there is a wealth of interest in the multiplicative relations between said values.

**1.6. Structure.** This paper is structured as follows. In the next section we gather together relevant preliminary material. In [Section 3](#), we establish decidability of the Hypergeometric Membership and Threshold Problems with Gaussian integer parameters ([Proposition 3.1](#)). We then push the techniques further and establish decidability of the Hypergeometric Membership and Threshold Problems with parameters drawn from the integers of any imaginary quadratic number field ([Theorem 3.5](#)). In [Section 4](#), we first define a class of hypergeometric sequences  $\mathfrak{C}$  whose parameters possess certain twofold symmetries. Then we show that subject to the truth of Schanuel's conjecture, the Hypergeometric Membership and Threshold Problems for sequences in  $\mathfrak{C}$  are both decidable ([Theorem 4.5](#)). Finally, we summarise our work and make suggestions for future avenues of research in the conclusion ([Section 5](#)).

## 2. PRELIMINARIES

**2.1. The gamma function.** The gamma function has been studied by luminaries such as Euler, Gauss, Legendre, and Weierstrass (amongst many others). We briefly recall standard results for the gamma function. Further details and historical accounts are given in a number of sources (cf. [\[28, 2\]](#)).

The standard relations for the gamma function give the functional identities: the *recurrence* or *translation property*  $\Gamma(z+1) = z\Gamma(z)$  for  $z \notin \mathbb{Z}$ ; the *reflection property*  $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$  for  $z \notin \mathbb{Z}$ ; and the *Gauss multiplication formula*

$$\prod_{k=0}^{n-1} \Gamma(z + k/n) = (2\pi)^{(n-1)/2} n^{1/2-nz} \Gamma(nz)$$

is valid so long as none of the  $n$  functions encounter a pole. (The case  $n = 2$  is more commonly known as Legendre's duplication formula.) In the domain of the Gamma function, repeated application of the translation property gives the following 'rising factorial' identity. For  $n \in \{1, 2, \dots\}$ , we have

$$\frac{\Gamma(z+n)}{\Gamma(z)} = z(z+1) \cdots (z+n-1).$$

Similarly, the 'falling factorial' identity is given by

$$\frac{\Gamma(z+1)}{\Gamma(z-n+1)} = z(z-1) \cdots (z-n+1).$$

The next technical lemma is derived from the aforementioned properties of the gamma function.

**Lemma 2.1.** *Suppose that  $\rho, w \in \mathbb{C}$  are algebraic numbers such that both  $\rho + w, \rho - w$  lie in the domain of the gamma function and  $w$  is neither an integer nor a half-integer. Up to multiplication by an algebraic number, we have the following equalities:*

$$\Gamma(\rho+w)\Gamma(\rho-w) = \begin{cases} \frac{2\pi i e^{\pi w i}}{w(1 - e^{2\pi w i})} & \text{if } \rho \text{ is an integer, or} \\ \frac{2\pi e^{\pi w i}}{e^{2\pi w i} + 1} & \text{if } \rho \text{ is a half-integer.} \end{cases}$$

*Proof.* Let us apply the rising and falling factorial identities (as appropriate to the sign of  $\rho$ ). Then, up to multiplication by an algebraic number, we have the following equalities:

$$\Gamma(\rho + w)\Gamma(\rho - w) = \begin{cases} \Gamma(w)\Gamma(-w) & \text{if } \rho \text{ is an integer, or} \\ \Gamma(1/2 + w)\Gamma(1/2 - w) & \text{if } \rho \text{ is a half-integer.} \end{cases}$$

Consider the first of the two cases above. The reflection and recurrence formula leads to

$$\Gamma(w)\Gamma(-w) = \frac{\Gamma(w)\Gamma(1-w)}{-w} = \frac{\pi}{-w \sin(\pi w)} = -\frac{2\pi i}{w(e^{\pi w i} - e^{-\pi w i})}.$$

For the second case, we employ the cosine variant of Euler's reflection formula to obtain

$$\Gamma(1/2 + w)\Gamma(1/2 - w) = \frac{\pi}{\cos(\pi w)} = \frac{2\pi}{e^{\pi w i} + e^{-\pi w i}}.$$

The equalities in the statement of the lemma quickly follow.  $\square$

The following straightforward corollary will be useful in the sequel.

**Corollary 2.2.** *Let  $\rho$  be an integer or half-integer. Subject to our previous assumptions and up to multiplication by an algebraic number, we have the following:*

$$\Gamma(\rho + bi)\Gamma(\rho - bi) = \begin{cases} \frac{2\pi i e^{\pi b}}{b(1 - e^{2\pi b})} & \text{if } \rho \text{ is an integer, or} \\ \frac{2\pi i e^{\pi b}}{1 + e^{2\pi b}} & \text{if } \rho \text{ is a half-integer.} \end{cases}$$

**2.2. Shift quotients and infinite products.** From the definition of a hypergeometric sequence, it is clear that we have a product formulation of the  $n$ th term  $u_n = u_0 \cdot \prod_{k=0}^n r(k)$ . Thus the membership and threshold problems reduce to analysing the sequence of partial products  $\langle \prod_{k=0}^n r(k) \rangle_n$ . In the sequel, we shall assume that  $r(n) \neq 0$  for each  $n \in \mathbb{N}_0$ .

Let us sketch a high-level overview of our approach. Given an instance  $(\langle u_n \rangle_n, t)$  of the Hypergeometric Membership Problem we want to show there is an effectively computable upper bound  $N$  that depends only on the sequence and target such that  $u_n = t$  only if  $n < N$ . Thus we reduce the Hypergeometric Membership Problem to an exhaustive search that asks whether  $t \in \{u_0, u_1, \dots, u_{N-1}\}$ . The reduction from the Hypergeometric Threshold Problem to a finite search problem follows similarly.

The following straightforward lemma appears in previous works [8, 17].

**Lemma 2.3.** *Consider the class of hypergeometric sequences  $\langle u_n \rangle_n$  whose shift quotients  $r(n)$  either diverge to  $\pm\infty$  or converge to a limit  $\ell$  with  $|\ell| \neq 1$ . For this class, the Hypergeometric Membership and Threshold Problems are both decidable.*

*Proof.* Let us first assume that  $r(n)$  diverges to  $\pm\infty$ . In this case it is easily seen that, for each  $t \in \mathbb{Q}$ , there exists a computable  $N \in \mathbb{N}$  such that if  $n \geq N$  then  $|u_n| = |u_0 \cdot \prod_{k=0}^n r(k)| > |t|$ . Thus the Membership Problem in such instances reduces to an exhaustive search that asks whether  $t \in \{u_0, u_1, \dots, u_{N-1}\}$ . Mutatis mutandis, decidability is similarly established for instances of the Membership Problem where  $r(n)$  converges to a limit  $\ell$  with  $|\ell| \neq 1$ .



Similar elementary reasoning proves that the Hypergeometric Threshold Problem for the aforementioned class of sequences is also decidable.  $\square$

Thus, for both of the decision problems, a single challenging case remains: the case where the shift quotient  $r(n) \in \mathbb{Q}(n)$  of the sequence converges to  $\pm 1$  as  $n \rightarrow \infty$ . We say an infinite product  $\prod_{k=0}^{\infty} r(k)$  *converges* if the sequence of partial products converges to a finite non-zero limit (otherwise the product is said to diverge). Recall the following classical theorem [28, 5].

**Theorem 2.4.** *Consider the rational function*

$$r(k) := \frac{c(k + \alpha_1) \cdots (k + \alpha_m)}{(k + \beta_1) \cdots (k + \beta_{m'})}$$

where we suppose that each  $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_{m'}$  is a complex number that is neither zero nor a negative integer. The infinite product  $\prod_{k=0}^{\infty} r(k)$  converges to a finite non-zero limit only if  $c = 1$ ,  $m = m'$ , and  $\sum_j \alpha_j = \sum_j \beta_j$ . Further, the value of the limit is given by

$$\prod_{k=0}^{\infty} r(k) = \prod_{j=1}^m \frac{\Gamma(\beta_j)}{\Gamma(\alpha_j)}.$$

With Theorem 2.4 in mind, it is useful to introduce the following terminology for shift quotients. We call a shift quotient  $r(k)$  (as above) *harmonious* if  $r(k)$  satisfies the assumptions  $c = 1$ ,  $m = m'$ , and  $\sum_j \alpha_j = \sum_j \beta_j$ . From Theorem 2.4 it is immediately apparent that a hypergeometric sequence  $\langle u_n \rangle_n$  with shift quotient  $r$  converges to a finite non-zero limit only if  $r$  is harmonious.

Nosan et al. [17] prove a specialisation of the following result. We include a proof (which is all but identical to the proof of Proposition 2 in [17]) here for the sake of completeness.

**Proposition 2.5.** *Let  $\langle u_n \rangle_n$  be a hypergeometric sequence whose shift quotient is given by a ratio of two polynomials with real coefficients. For such sequences, the Hypergeometric Membership and Threshold Problems are both Turing-reducible to the following decision problem. Given  $d \in \mathbb{N}$ ,  $\alpha_1, \dots, \alpha_d \in \mathbb{C} \setminus \mathbb{Z}_{<0}$  (the roots of some  $P(x) \in \mathbb{R}[x]$ ), and  $\beta_1, \dots, \beta_d \in \mathbb{C} \setminus \mathbb{Z}_{<0}$  (the roots of some  $Q(x) \in \mathbb{R}[x]$ ), determine whether*

$$\frac{\Gamma(\beta_1) \cdots \Gamma(\beta_d)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_d)} = t$$

for  $t \in \mathbb{Q} \setminus 0$ .

*Proof.* From Lemma 2.3, we need only consider cases where the associated shift quotient  $r(k)$  converges to  $\pm 1$  and, by Theorem 2.4, we can assume without loss of generality that  $r(k)$  is harmonious. We treat the case that the sequence of partial products  $\langle \prod_{k=0}^n r(k) \rangle_n$  is eventually strictly increasing. The case where the sequence of partial products is eventually strictly decreasing follows *mutatis mutandis*.

Consider an instance  $(\langle u_n \rangle_n, t)$  of the Hypergeometric Membership Problem with  $r(k)$  as above. Let  $\tau := \prod_{k=0}^{\infty} r(k)$ . Assume that the sequence of partial products  $\langle \prod_{k=0}^n r(k) \rangle_n$  is eventually strictly increasing. Then there exists a computable  $N \in \mathbb{N}$  such that  $u_n < \tau$  for each  $n \geq N$ . There are

two subcases to consider. First, if  $\tau \leq t$  then it is clear that  $u_n < t$  for each  $n \geq N$  and so decidability in this instance reduces to an exhaustive search that asks whether  $t \in \{u_0, u_1, \dots, u_{N-1}\}$ . Second, if  $\tau > t$  then there exists an  $N_1 \in \mathbb{N}$  such that  $u_n > t$  for each  $n \geq N_1$ . Thus, again, decidability in this instance reduces to an exhaustive search.

All that remains is to decide whether  $\tau \leq t$ . It is clear that, by computing  $\tau$  to sufficient precision, the problem of determining whether  $\tau < t$  or  $\tau > t$  is recursively enumerable. Thus we need only test whether the equality  $\tau = t$  holds. By [Theorem 2.4](#), we know that  $\tau = \prod_{j=1}^m \Gamma(\beta_j)/\Gamma(\alpha_j)$ , from which we deduce the desired result.

For the sake of brevity, we omit the argument for reduction from the Hypergeometric Threshold Problem. The argument is near identical to the reasoning displayed in the reduction from the Hypergeometric Membership Problem.  $\square$

**2.3. Number fields.** We recall standard results for quadratic fields below (cf. [\[24, Chapter 3\]](#)). A number field  $K$  is *quadratic* if  $[K : \mathbb{Q}] = 2$ . A field  $K$  is quadratic if and only if there is a square-free integer  $d$  such that  $K = \mathbb{Q}(\sqrt{d})$ . Further, a quadratic field  $\mathbb{Q}(\sqrt{d})$  is *imaginary* if  $d < 0$ .

**Theorem 2.6.** *Suppose that  $d \in \mathbb{Z}$  is square-free. Then the algebraic integers of  $\mathbb{Q}(\sqrt{d})$  are given by  $\mathbb{Z}[\sqrt{d}]$  if  $d \not\equiv 1 \pmod{4}$  or  $\mathbb{Z}[1/2 + \sqrt{d}/2]$  if  $d \equiv 1 \pmod{4}$ .*

We include the following straightforward lemma for ease of reference.

**Lemma 2.7.** *Let  $\mathbb{L}/\mathbb{Q}$  be a finite Galois extension. We suppose that  $\hat{\mathcal{P}} \in \mathbb{L}(X_1, \dots, X_m)$  is a polynomial such that  $\hat{\mathcal{P}}(s_1, \dots, s_m) = 0$  with  $(s_1, \dots, s_m) \in \mathbb{C}^m$ . Then there is a polynomial  $Q \in \mathbb{Q}(X_1, \dots, X_m)$  such that  $Q(s_1, \dots, s_m) = 0$ .*

*Proof.* Let  $\hat{\mathcal{P}} = \sum_{(t_1, \dots, t_m)} c_{(t_1, \dots, t_m)} X_1^{t_1} X_2^{t_2} \cdots X_m^{t_m}$  and for each  $\sigma \in G$  (the Galois group of  $\mathbb{L}/\mathbb{Q}$ ) let

$$\sigma(\hat{\mathcal{P}}) = \sum_{(t_1, \dots, t_m)} \sigma(c_{(t_1, \dots, t_m)}) X_1^{t_1} X_2^{t_2} \cdots X_m^{t_m}.$$

Let  $Q = N_{\mathbb{L}/\mathbb{Q}}(\hat{\mathcal{P}}) := \prod_{\sigma \in G} \sigma(\hat{\mathcal{P}})$ . It is clear that each of the coefficients of the polynomial  $Q$  is rational since the coefficients are invariant under the action of the group  $G$ . Further,

$$Q(s_1, \dots, s_m) = \hat{\mathcal{P}}(s_1, \dots, s_m) \prod_{\sigma \in G \setminus e_G} \sigma(\hat{\mathcal{P}})(s_1, \dots, s_m) = 0,$$

as desired.  $\square$

**2.4. Transcendental number theory.** The transcendence degree of a field extension is a measure of the size of the extension. In fact, for finitely generated extensions of  $\mathbb{L}/\mathbb{Q}$  (such as those that we consider), the transcendence degree indicates the largest cardinality of an algebraically independent subset of  $\mathbb{L}$  over  $\mathbb{Q}$ . For a field extension  $\mathbb{L}/\mathbb{Q}$ , a subset  $\{\xi_1, \dots, \xi_n\} \subset \mathbb{L}$  is *algebraically independent* over  $\mathbb{Q}$  if for each polynomial

$$P(X_1, \dots, X_n) \in \mathbb{Q}[X_1, \dots, X_n]$$

we have that  $P(\xi_1, \dots, \xi_n) = 0$  only if  $P$  is identically zero.



It is useful to recall the Gelfond–Schneider Theorem that establishes the transcendental of  $\alpha^\beta$  for algebraic numbers  $\alpha$  and  $\beta$  except for the cases where  $\alpha = 0, 1$  or  $\beta$  is rational.

Schanuel’s conjecture is a unifying prediction in transcendental number theory. If Schanuel’s conjecture is true, then it generalises several of the principal results in transcendental number theory such as: the Gelfond–Schneider Theorem, the Lindemann–Weierstrass Theorem, and Baker’s theorem (cf. [11, 3, 27]). The conjecture predicts that given  $\xi_1, \dots, \xi_n$  rationally linearly independent complex numbers, then there is a subset of

$$\{\xi_1, \dots, \xi_n, e^{\xi_1}, \dots, e^{\xi_n}\}$$

of size at least  $n$  that is algebraically independent over  $\mathbb{Q}$ .

**Conjecture 2.8** (Schanuel). *Suppose that  $\xi_1, \dots, \xi_n$  are complex numbers that are linearly independent over the rationals  $\mathbb{Q}$ . Then the transcendence degree of the field extension*

$$\mathbb{Q}(\xi_1, \dots, \xi_n, e^{\xi_1}, \dots, e^{\xi_n})$$

*over  $\mathbb{Q}$  is at least  $n$ .*

### 3. HYPERGEOMETRIC SEQUENCES WITH QUADRATIC PARAMETERS

The proof of the next proposition demonstrates the approach we shall take to establish decidability results in more abstract settings. We include a worked example in [Example 3.2](#). Recall that the *Gaussian integers*  $\mathbb{Z}[i]$  are those complex numbers of the form  $a + bi$  such that  $a, b \in \mathbb{Z}$ .

**Proposition 3.1.** *The Hypergeometric Membership and Threshold Problems with Gaussian integer parameters are both decidable.*

*Proof.* By [Theorem 2.4](#) and [Proposition 2.5](#), we need only consider hypergeometric sequences with harmonious shift quotients. Given such a shift quotient  $r$ , its roots and poles are either rational integers or appear as irrational complex conjugate pairs of the form  $a_m \pm b_m i \in \mathbb{Z}[i]$ . By [Corollary 2.2](#), each instance  $(\langle u_n \rangle_n, t)$  of the Hypergeometric Membership (resp. Threshold) Problem with Gaussian integer parameters reduces to testing an equality of the form

$$(4) \quad \theta \pi^\ell \prod_m (\sinh(b_m \pi))^{\varepsilon_m} = t.$$

Here  $\theta$  is rational and non-zero,  $\ell \in \mathbb{Z}$ , each pair  $\Gamma(a_m + b_m i)\Gamma(a_m - b_m i)$  from [Proposition 2.5](#) contributes a term  $(\sinh(b_m \pi))^{\varepsilon_m}$  in the finite product, and  $\varepsilon_m = \pm 1$ . We break the remainder of the proof into several subcases. Without loss of generality, we can assume that not all the roots and poles of  $r$  are rational integers, for otherwise testing (4) reduces to testing equality between two rational numbers.

We continue under the assumption that not all the roots and poles of  $r$  are rational integers. Let us now consider the product  $\prod_m (\sinh(b_m \pi))^{\varepsilon_m}$ . Up to multiplication by a rational, the following equalities hold

$$\prod_m (\sinh(b_m \pi))^{\varepsilon_m} = \prod_m (e^{b_m \pi} - e^{-b_m \pi})^{\varepsilon_m} = \frac{f(e^\pi)}{g(e^\pi)}$$

where  $f, g \in \mathbb{Q}[X]$  are non-trivial polynomials.

Observe that  $e^\pi = (e^{\pi i})^{-i} = (-1)^{-i}$ ; thus, by the Gelfond–Schneider theorem,  $e^\pi$  is transcendental. It follows that  $f(e^\pi)/g(e^\pi)$  is rational-valued only if  $g$  is a rational multiple of  $f$ . We divide this case into two further subcases:

- if  $\ell = 0$ , then, once again, testing whether (4) holds reduces to deciding whether two rationals are equal, and
- if  $\ell \neq 0$ , then testing whether (4) holds reduces to testing whether  $\pi^\ell$  is equal to a given rational number, which cannot hold for then  $\pi$  is necessarily algebraic.

All that remains is to consider the case where  $f(e^\pi)/g(e^\pi) \notin \mathbb{Q}$ , which we again split into two subcases.

- If  $\ell = 0$ , then it is trivial to see that (4) cannot hold as the right-hand side is rational.
- If  $\ell \neq 0$  and we assume, for a contradiction, that (4) holds, then a simple rearrangement of (4) shows that there is a non-trivial polynomial  $\mathcal{P} \in \mathbb{Q}[X, Y]$  such that  $\mathcal{P}(\pi, e^\pi) = 0$ . This contradicts Nesterenko’s theorem [15] that  $\pi$  and  $e^\pi$  are algebraically independent.

We have dispatched each of the subcases and conclude the desired result.  $\square$

**Example 3.2.** Suppose that  $\langle u_n \rangle_{n=0}^\infty$  is a hypergeometric sequence with Gaussian integer parameters defined as follows:

$$u_n = \frac{n^2 - 4n + 5}{n^2 - 4n + 13} u_{n-1} \text{ with } u_0 = 1.$$

Let us consider the membership (resp. threshold) problem  $(\langle u_n \rangle_n, t)$  with  $t \in \mathbb{Q}$ . First, we evaluate the associated infinite product as follows:

$$\prod_{k=0}^{\infty} \frac{k^2 - 4k + 5}{k^2 - 4k + 13} = \frac{\Gamma(-2 - 3i)\Gamma(-2 + 3i)}{\Gamma(-2 - i)\Gamma(-2 + i)} = \frac{\sinh(\pi)}{39 \sinh(3\pi)} = \frac{e^{3\pi}(e^{2\pi} - 1)}{39e^\pi(e^{6\pi} - 1)}.$$

Second, as directed in the proof of [Proposition 3.1](#), decidability of the membership (resp. threshold) problem in this instance reduces to determining whether the equality

$$\frac{e^{3\pi}(e^{2\pi} - 1)}{39e^\pi(e^{6\pi} - 1)} = t$$

holds.

A simple rearrangement shows that if the above equality holds, then there is a non-trivial polynomial  $p \in \mathbb{Q}[X]$  such that  $p(e^\pi) = 0$ , from which deduce that  $e^\pi$  is algebraic. However, by the Gelfond–Schneider Theorem,  $e^\pi$  is transcendental. We have reached a contradiction and deduce that the aforementioned equality cannot hold.

Finally, as described in [Proposition 2.5](#), the Hypergeometric Membership (resp. Threshold) Problem reduces to an exhaustive search of a computable number of initial terms in the sequence  $\langle u_n \rangle_n$ . For a given instance, we note the size of the search is determined by the value of  $t$ .

**Remark 3.3.** Subject to appropriate changes and by employing [Lemma 2.7](#), we can extend the result in [Proposition 3.1](#) from instances of the membership and threshold problems  $(\langle u_n \rangle_n, t)$  with  $u_0, t \in \mathbb{Q}$  to problem instances with

$u_0, t \in \mathbb{L}(\pi, e^\pi)$  where  $\mathbb{L}$  is any finite Galois extension of  $\mathbb{Q}$ . This extension similarly holds for [Theorem 3.5](#) (below).

**Remark 3.4.** Analogous decision procedures to the proof of [Proposition 3.1](#) also hold for other famous rings of integers: the Eisenstein, Kummer, and Kleinian integers. Recall that the *Eisenstein integers* are the elements of  $\mathbb{Z}[\zeta_3] = \{a + b\zeta_3 : a, b \in \mathbb{Z}\}$  where  $\zeta_3 := e^{2\pi i/3}$ . Similarly, the *Kummer integers* are the elements of  $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$ . Finally, the *Kleinian integers* are the elements of  $\mathbb{Z}[\mu] = \{a + b\mu : a, b \in \mathbb{Z}\}$  where  $\mu = -1/2 + \sqrt{-7}/2$ .

The claims for decidability in [Remark 3.4](#), follow from the next theorem. We establish decidability of the membership and threshold problems for hypergeometric sequences whose parameters are drawn from the ring of integers of an imaginary quadratic number field.

**Theorem 3.5.** *The Hypergeometric Membership and Threshold Problems for the sequences whose polynomial coefficients are monic and split over an imaginary quadratic number field.*

*Proof.* First, we observe that the assumptions on the polynomial coefficients can be written in terms of assumptions on the parameters of the sequences. Indeed, we can freely assume that the parameters of the sequences are drawn from the integers of an imaginary quadratic number field.

Mutatis mutandis, the proof of [Theorem 3.5](#) follows the approach in [Proposition 3.1](#). For the sake of brevity, we shall indicate only the major changes to [Proposition 3.1](#) here. Consider the ring of integers of an imaginary quadratic field  $\mathbb{Q}(\sqrt{d})$  where  $-d \in \mathbb{N}$  is square-free. By [Theorem 2.6](#), there are two cases to consider: first, when  $d \not\equiv 1 \pmod{4}$  and second, when  $d \equiv 1 \pmod{4}$ .

We concentrate on the changes to the proof of [Proposition 3.1](#) when  $d \not\equiv 1 \pmod{4}$ . Like before, we can use the recurrence formula to write  $\Gamma(a + b\sqrt{d}) = \theta\Gamma(b\sqrt{d})$  where  $\theta \in \mathbb{N}$ . Thus all that remains is to evaluate products  $\Gamma(b\sqrt{d})\Gamma(-b\sqrt{d})$  of conjugate elements. By the reflection formula, we have

$$\Gamma(b\sqrt{d})\Gamma(-b\sqrt{d}) = -\frac{\pi}{b\sqrt{d} \sin(b\pi\sqrt{d})} = \frac{2\pi}{b\sqrt{-d}(e^{\pi b\sqrt{-d}} - e^{-\pi b\sqrt{-d}})}.$$

The important update in (4) is the product  $\prod_m (\sinh(b_m\pi))^{\varepsilon_m}$ . In our new setting, the product takes the form  $\prod_m (e^{\pi b\sqrt{-d}} - e^{-\pi b\sqrt{-d}})^{\varepsilon_m}$ . Observe that  $e^{\pi\sqrt{-d}}$  is transcendental (once again by Gelfond–Schneider) and that for each  $-d \in \mathbb{N}$  the numbers  $\pi$  and  $e^{\pi\sqrt{-d}}$  are algebraically independent over  $\mathbb{Q}$  [[15](#), Corollary 6]. The rest of the proof in this case follows as before.

In the second case where  $d \equiv 1 \pmod{4}$  we must additionally deal with contributions of the form  $\Gamma(b/2 + b\sqrt{d}/2)\Gamma(b/2 - b\sqrt{d}/2)$ . This setting introduces cases where  $2 \nmid b$ , which we resolve by repeated application of the recurrence formula and the cosine variant of Euler’s reflection formula. Indeed, we have

$$\Gamma(1/2 + b\sqrt{d}/2)\Gamma(1/2 - b\sqrt{d}/2) = \frac{\pi}{\cos(\pi b\sqrt{d}/2)} = \frac{2\pi}{e^{\pi b\sqrt{-d}/2} + e^{-\pi b\sqrt{-d}/2}},$$

and so we can construct an updated version of the product in (4). This update and analogous arguments for the transcendental properties of  $e^{\pi\sqrt{-d}/2}$  let us conclude decidability in this case too.  $\square$

#### 4. DECIDABILITY SUBJECT TO SCHANUEL'S CONJECTURE

We now generalise the decidability results [Proposition 3.1](#) and [Theorem 3.5](#). In this section we consider parameters from a wider class. Here the parameters will possess 2-fold symmetries such that the real part of the centre of rotation of each symmetry is an integer (or half-integer). The main result in this section is [Theorem 4.5](#). Before we state and prove [Theorem 4.5](#), we first introduce some preliminary results and notation in [Subsection 4.1](#).

It is worth noting that our assumption on the parameters in this section is invariant under a shift  $n \mapsto n + 1$  to the indexing. Thus we can assume, without loss of generality, that none of the parameters in our investigation is a negative integer.

**4.1. Algebraic numbers with rational real part.** Let  $C$  be the class of minimal polynomials associate with algebraic numbers that have rational real parts as discussed in [6]. Trivially, a linear polynomial with rational coefficients is always a member of  $C$  since the single root of the polynomial is rational. It is straightforward to see that an irreducible quadratic polynomial  $x^2 + a_1x + a_0 \in \mathbb{Q}[x]$  is in  $C$  if and only if  $a_1^2 - 4a_0 < 0$ .

**Example 4.1.** The irreducible polynomial  $f(x) = x^8 - 8x^7 + 28x^6 - 56x^5 + 74x^4 - 72x^3 + 84x^2 - 88x + 41$  has four roots with rational real part 1, namely  $1 \pm i \pm \sqrt{1 - \sqrt{2}}$ . The other four roots  $1 \pm i \pm \sqrt{1 + \sqrt{2}}$  of  $f$  do not have rational real part. Note that  $f(x + 1) = x^8 + 4x^4 + 32x^2 + 4$  is an even polynomial.

The authors of [6] achieve a complete classification of the class  $C$  with the following result.

**Theorem 4.2.** *Let  $F$  be a polynomial of degree at least three. Then  $F \in C$  if and only if  $F(x) = G((x - \rho)^2)$  for some  $\rho \in \mathbb{Q}$  and monic irreducible  $G \in \mathbb{Q}[X]$  that has a negative real root. In this case,  $F$  has a root with a rational real part  $\rho$ .*

**Corollary 4.3.** *Let  $F$  be a polynomial in  $C$  of degree at least three. The roots of  $F$  that have rational real part have the same real part  $\rho$ . Further, we have  $F(\rho - w) = F(\rho + w)$ .*

We partition  $C$  as follows. Let  $C_\rho$  be the subclass of minimal polynomials of algebraic numbers that have rational real part  $\rho$ . It follows from [Theorem 4.2](#) that  $C = \bigsqcup_{\rho \in \mathbb{Q}} C_\rho$ .

We now evaluate gamma products whose inputs are given by polynomials in  $\bigsqcup_{n \in \mathbb{Z}} C_n \cup C_{1/2+n}$ . It is clear from [Theorem 4.2](#) that if  $p \in C$  and  $\deg(p) \geq 3$ , then  $\deg(p)$  is even. Further, by [Corollary 4.3](#), roots with irrational real parts come in pairs.

**Lemma 4.4.** *Suppose that  $p \in C$  and  $\deg(p) = 2d > 3$ . Let  $\alpha_1, \bar{\alpha}_1, \dots, \alpha_m, \bar{\alpha}_m$  be the roots of  $p$  with rational real part  $\rho$  and let  $\alpha_{m+1}, \alpha_{m+1} - 2w_{m+1}, \dots, \alpha_d, \alpha_d - 2w_d$  be the roots of  $p$  with irrational real parts. Here for  $k \in \{1, \dots, m\}$  we write  $b_k := \text{Im}(\alpha_k)$  and for each  $k \in \{m + 1, \dots, d\}$  write  $w_k := \alpha_k - \rho$ . If*

$p \in \bigsqcup_{n \in \mathbb{Z}} C_n \cup C_{1/2+n}$ , then (up to multiplication by an algebraic integer) the gamma product  $\prod_{\alpha : p(\alpha)=0} \Gamma(\alpha)$  is given by

$$\begin{cases} \prod_{k=1}^m \frac{2\pi e^{\pi b_k}}{b_k(e^{2\pi b_k} - 1)} \cdot \prod_{k=m+1}^d \frac{2\pi i e^{\pi w_k i}}{w_k(1 - e^{2\pi w_k i})} & \text{if } \rho \text{ is an integer, or} \\ \prod_{k=1}^m \frac{2\pi e^{\pi b_k}}{e^{\pi b_k} + 1} \cdot \prod_{k=m+1}^d \frac{2\pi e^{\pi w_k i}}{e^{2\pi w_k i} + 1} & \text{if } \rho \text{ is a half-integer.} \end{cases}$$

*Proof.* Taken in combination, the results of [Theorem 4.2](#), [Corollary 4.3](#), and [Lemma 2.1](#) give the desired result.  $\square$

**4.2. Main result.** Consider the set of polynomials  $\bigsqcup_{n \in \mathbb{Z}} C_n \cup C_{1/2+n}$ . This set contains all the minimal polynomials of algebraic integers of degree at most two. The octic  $f$  in [Example 4.1](#) is also a member of this set. We shall now generalise the results in [Proposition 3.1](#) and [Theorem 3.5](#) by considering the class of hypergeometric sequences whose parameters are drawn from the above set of polynomials.

**Theorem 4.5.** Let  $\mathfrak{C}$  be the class of hypergeometric sequences whose parameters have minimal polynomials in  $\bigsqcup_{n \in \mathbb{Z}} C_n \cup C_{1/2+n}$ . If Schanuel's conjecture is true, then we can decide the membership and threshold problems for sequences in  $\mathfrak{C}$ .

*Proof.* A hypergeometric sequence  $\langle u_n \rangle_n$  is in class  $\mathfrak{C}$  if and only if the numerator and denominator of the shift quotient  $r(n) = cP(n)/Q(n)$  are given by products of irreducible factors each of which is a polynomial in  $\bigsqcup_{n \in \mathbb{Z}} C_n \cup C_{1/2+n}$ . As noted previously, we can assume that  $r$  is harmonious and so  $c = 1$  without loss of generality,  $p_1, \dots, p_k$  and  $q_1, \dots, q_{k'}$  be the irreducible factors of  $P$  and  $Q$ , respectively.

The proof is split into two parts: a reduction to an equality testing problem and an application of Schanuel's conjecture.

*Reduction to an equality testing problem.* From [Proposition 2.5](#), we know that decidability of the membership and threshold problems restricted to class  $\mathfrak{C}$  reduces to the problem of determining whether equality holds in

$$(5) \quad \frac{\prod_{j=1}^k \prod_{\alpha : p_j(\alpha)=0} \Gamma(\alpha)}{\prod_{j=1}^{k'} \prod_{\beta : q_j(\beta)=0} \Gamma(\beta)} = t$$

for a given rational  $t$ . Let us consider how we evaluate the gamma product  $\prod_{\alpha : p_j(\alpha)=0} \Gamma(\alpha)$  in accordance with the degree  $d_j$  of the associated polynomial factor  $p_j$  (evaluating the gamma product  $\prod_{j=1}^{k'} \prod_{\beta : q_j(\beta)=0} \Gamma(\beta)$  is similar). If  $d_j = 1$ , then  $p_j$  is a linear polynomial with a single root. Thus the aforementioned term is of the form  $\Gamma(n) = (n-1)!$  or  $\Gamma(n+1/2) = \frac{(2n)!}{4^n n!} \sqrt{\pi}$ . If  $d_j = 2$ , then the roots of the quadratic polynomial  $p_j$  are  $a \pm bi$  where  $a$  is either an integer or a half-integer by assumption. This contribution is discussed in [Corollary 2.2](#). We now turn to the case where  $d_j \geq 3$ . The contributions in this third case are summarised in [Lemma 4.4](#).

For each root  $\alpha$  of either  $P$  or  $Q$ , let  $b$  be either i) the imaginary part of  $\alpha$  if  $\text{Re}(\alpha)$  is an integer or a half-integer, or ii)  $b := \rho_\alpha - \alpha$  if  $\text{Re}(\alpha)$  is

irrational (here  $\rho_\alpha$  is the rational real part of the minimal polynomial of  $\alpha$  as in [Corollary 4.3](#)). Consider the set of such numbers  $\{b_1, \dots, b_M\}$ . We denote by  $S' := \{s'_1, \dots, s'_m\}$  a maximal  $\mathbb{Q}$ -linearly independent subset of  $\{b_1, \dots, b_M\}$  with an additional condition that  $\sqrt{\pi} \cup \pi S'$  is also  $\mathbb{Q}$ -linearly independent (here  $\pi S' := \{\pi s'_1, \dots, \pi s'_m\}$ ). Then for each  $k$ , write  $b_k$  as a  $\mathbb{Q}$ -linear sum of elements in  $\{s'_1, \dots, s'_m\}$  so that

$$b_k = \frac{x_{k1}}{y_{k1}} s'_1 + \dots + \frac{x_{km}}{y_{km}} s'_m.$$

We define  $s_j := s'_j / \text{lcm}(y_{1j}, y_{2j}, \dots, y_{mj})$  for each  $j \in \{1, \dots, m\}$ . Now we can write each  $b_k$  as a  $\mathbb{Z}$ -linear sum of elements in the normalised set  $S := \{s_1, \dots, s_m\}$ .

For a given problem instance  $(\langle u_n \rangle_n, t)$ , we piece together the above contributions and consider the product on the left-hand side of (5) in its entirety. Our approach is similar to the equality testing procedure for the product (4) in [Proposition 3.1](#). By clearing denominators, we deduce that testing the equality in (5) reduces to testing whether a certain non-trivial polynomial with rational coefficients vanishes at a given point. More specifically, we want to test whether a given polynomial  $\mathcal{P} \in \mathbb{Q}[X_1, \dots, X_{4m+4}]$  satisfies

$$\mathcal{P}(\sqrt{\pi}, \pi i, \pi S, \pi Si, e^{\sqrt{\pi}}, e^{\pi i}, e^{\pi S}, e^{\pi Si}) = 0.$$

Here  $\pi Si := \{\pi s_1 i, \dots, \pi s_m i\}$ ,  $e^{\pi S} := \{e^{\pi s_1}, \dots, e^{\pi s_m}\}$ , and likewise for  $e^{\pi Si}$ . Further, we need only consider a polynomial in  $4m + 4$  variables as the elements  $\{b_1, \dots, b_M\}$  for our problem instance are given by  $\mathbb{Q}$ -linear combinations of the elements of  $S \cup Si$ .

*Applying Schanuel's conjecture.* By assumption, the elements of the set  $\{\sqrt{\pi}, \pi i, \pi S, \pi Si\}$  are  $\mathbb{Q}$ -linearly independent. Let us apply Schanuel's conjecture to the larger set

$$\{\sqrt{\pi}, \pi i, \pi S, \pi Si, e^{\sqrt{\pi}}, e^{\pi i}, e^{\pi S}, e^{\pi Si}\}$$

with cardinality  $4m + 4$ . If Schanuel's conjecture is true, then this larger set has an algebraically independent subset of size at least  $2m + 2$ . By construction, this algebraically independent subset is necessarily

$$\{\sqrt{\pi}, e^{\sqrt{\pi}}, e^{\pi S}, e^{\pi Si}\}$$

since the  $2m + 2$  elements of  $\{\sqrt{\pi}, \pi i, \pi S, \pi Si\}$  are pairwise algebraically dependent and  $e^{\pi i} = -1$ .

We now rewrite the preceding displayed equality in terms of the (obvious) polynomial  $\hat{\mathcal{P}}$  that absorbs the algebraically dependent inputs  $S \cup Si$  into the coefficients. That is to say, we employ a polynomial  $\hat{\mathcal{P}} \in \mathbb{L}(X_1, \dots, X_{2m+2})$  where  $\mathbb{L}$  is the Galois closure of the number field  $\mathbb{Q}(S, Si)$  and evaluate  $\hat{\mathcal{P}}$  on our algebraically independent subset. It follows that the above equality holds if and only if

$$\hat{\mathcal{P}}(\sqrt{\pi}, e^{\sqrt{\pi}}, e^{\pi S}, e^{\pi Si}) = 0.$$

By [Lemma 2.7](#), the above holds if and only if there exists a polynomial  $Q \in \mathbb{Q}[X_1, \dots, X_{2m+2}]$  such that  $Q(\sqrt{\pi}, e^{\sqrt{\pi}}, e^{\pi S}, e^{\pi Si}) = 0$ . If Schanuel's conjecture is true, then this equality cannot hold. This assertion follows



from our preceding work. Indeed, if Schanuel's conjecture is true, then the elements of the set  $\{\sqrt{\pi}, e^{\sqrt{\pi}}, e^{\pi S}, e^{\pi Si}\}$  are algebraically independent over  $\mathbb{Q}$ , from which we deduce that the equality in (5) cannot hold. The desired result follows.  $\square$

**Remark 4.6.** Recall [Remark 3.3](#) where we extended our class of problem instances to include setups with  $u_0, t \in \mathbb{L}(\pi, e^\pi)$ . Under our assumption in [Theorem 4.5](#) that Schanuel's conjecture is true we can include a broader range of setups. Notice, for instance, that the algebraic independence of  $\pi$  and  $e$  is currently unknown; however, if Schanuel's conjecture is true, then it follows that  $\pi$  and  $e$  are algebraically independent. Thus, subject to the truth of Schanuel's conjecture, we can extend our results in [Proposition 3.1](#), [Theorem 3.5](#), and [Theorem 4.5](#) to instances where  $u_0, t \in \mathbb{L}(\pi, e)$ . In fact, the truth of Schanuel's conjecture implies the algebraic independence of the numbers  $e, e^\pi, e^e, e^i, \pi, \pi^\pi, \pi^e, \pi^i, 2^\pi, 2^e, 2^i, \log \pi, \log 2, \log 3, \log \log 2, (\log 2)^{\log 3}, 2^{\sqrt{2}}$ , and many more (cf. [\[20, Conjecture S7\]](#)). It follows that we can extend our decidability results to instances where  $u_0, t \in \mathbb{L}(x_1, \dots, x_m)$  with  $x_1, \dots, x_m$  drawn from the preceding list.

**Remark 4.7.** We note that the equality test

$$Q(\sqrt{\pi}, e^{\sqrt{\pi}}, e^{\pi S}, e^{\pi Si}) = 0$$

can be realised as a proposition in the first order theory of the reals with exponentiation. Macintyre and Wilkie [\[12\]](#) established decidability of said theory subject to the truth of Schanuel's conjecture. As noted in previous works, careful inspection of Macintyre and Wilkie's algorithm reveals that correctness is independent of the truth of Schanuel's conjecture. Indeed, Schanuel's conjecture is only used to prove termination. Thus if we apply Macintyre and Wilkie's algorithm to determine whether the equality  $Q(\sqrt{\pi}, e^{\sqrt{\pi}}, e^{\pi S}, e^{\pi Si}) = 0$  holds and find the procedure terminates, then the output is certainly correct.

We note that Macintyre and Wilkie's algorithm terminates unless the inputs constitute a counterexample to Schanuel's conjecture. Thus, the process underlying the proof of [Theorem 4.5](#) presents an interesting prospect in the sense described by Richardson in [\[22\]](#) (see also [\[21\]](#)) "A failure of the [process] to terminate would be even more interesting than [its] success."

**4.3. Unnested radical and cyclotomic parameters.** We draw the reader's attention to classes of hypergeometric sequences in  $\mathfrak{C}$  (those that are amenable to the transcendental techniques laid out above). We focus here on parameters associated with unnested radicals and cyclotomic polynomials (both of which lie beyond the reach of [\[17\]](#)); that is to say, when the shift quotient of a hypergeometric sequence has an irreducible factor of the form  $n^m - a$  or  $\Phi_m(n)$ . Such sequences appear elsewhere in the literature. We direct the interested reader to [\[19, pp. 753–757\]](#) and [\[7\]](#) (and to further references in the latter).

**Remark 4.8.** We adopt the notation  $\sqrt[m]{a}$  for the principal  $m$ th root of  $a$  and  $\omega_m := e^{2\pi i/m}$ .

- (1) Note that the polynomial  $n^m - a \in \mathbb{Z}[n]$  has roots  $\sqrt[m]{a}\omega_m^j$  for  $j \in \{0, \dots, m-1\}$ . For  $m$  even,  $n^m - a$  is both even and its irreducible factors all lie in  $C_0$ .
- (2) Let us discuss cyclotomic polynomials  $\Phi_m(n)$  whose roots are the primitive  $m$ th roots of unity. Under the assumption that  $m$  is a multiple of four, it is clear that  $\Phi_m(n) \in C_0$ . This assumption permits us to pair together the gamma terms, indexed by  $j$  and  $j + m/2$ , for evaluation. As an illustration to what happens without this additional assumption, observe that  $\omega_{18}, \omega_{18}^5, \omega_{18}^7, \omega_{18}^{11}, \omega_{18}^{13}$ , and  $\omega_{18}^{17}$  are the 18th primitive roots of unity and cannot be paired so.

The next corollary is an immediate consequence of applying [Theorem 4.5](#) to sequences with parameters drawn from the rational integers and the quadratic integers. That is to say, sequences whose parameters are integers drawn from any number of quadratic fields. We can equivalently ask that the polynomial coefficients of the hypergeometric sequences are monic and all the irreducible factors of  $p$  and  $q$  (as in (1)) are either linear or quadratic.

**Corollary 4.9.** *The Hypergeometric Membership and Threshold Problems with parameters drawn from the rational integers and quadratic integers are both decidable subject to Schanuel's conjecture.*

**4.4. Limitations of the technique.** In [Remark 4.10](#), we show the limitation of our technique: we cannot even handle parameters drawn from a particularly well-behaved class of quartic fields, the so called biquadratic fields.

**Remark 4.10.** Our approach already fails at biquadratic fields (such as  $\mathbb{Q}(\sqrt{5}, \sqrt{13})$  and  $\mathbb{Q}(\sqrt{21}, \sqrt{33})$ ) because the rings of integers associated with these fields contain elements that are not amenable to our approach. Take, for example,  $\theta = (5 + 3\sqrt{5} + \sqrt{13} + 3\sqrt{65})/4 \in \mathbb{Q}(\sqrt{5}, \sqrt{13})$  that satisfies  $\theta^4 - 5\theta^3 - 71\theta^2 + 120\theta + 1044 = 0$ . As a second example, take  $\tilde{\theta} = (1 + \sqrt{21} + \sqrt{33} - \sqrt{77})/4 \in \mathbb{Q}(\sqrt{21}, \sqrt{33})$  that satisfies  $\tilde{\theta}^4 - \tilde{\theta}^3 - 16\tilde{\theta}^2 + 37\tilde{\theta} - 17 = 0$ . Both of these examples are taken from [29].

The expressions for both  $\theta$  and  $\tilde{\theta}$  in terms of radicals demonstrate that their respective irreducible polynomials are not elements of  $\bigsqcup_{n \in \mathbb{Z}} C_n \cup C_{1/2+n}$ . Thus a sequence whose shift quotient contains such factors lies outside of  $\mathfrak{C}$  (and thus beyond the method presented here).

## 5. CONCLUSION

Our transcendence approach extends the state of the art for the membership and threshold problems for hypergeometric sequences in a novel direction. Major obstacles to extending this method to a larger class of hypergeometric sequences are the functional properties of the gamma function. Indeed, the reflection and recurrence properties of the gamma function are indispensable for our approach. Without further functional properties or the inclusion of techniques from other mathematical disciplines, we are particularly limited to polynomial factors whose root sets have both twofold symmetries and centres of rotation at a point  $\rho$  in the complex plane with  $\text{Re}(\rho)$  either an integer or a half-integer.

It is not obvious that even if new functional properties were to be discovered that they would be of use when it comes to linear factors in the shift quotient. In fact, we cannot evaluate the infinite product associated with the shift quotient (2) because almost nothing is known about  $\Gamma(1/9)$ . Indeed, we know very little about the transcendental properties of the gamma function. On the one hand,  $\Gamma(1/2) = \sqrt{\pi}$ . On the other hand, we do not know how to express  $\Gamma$  evaluated at general rational values. Indeed, for  $s \in \{1/6, 1/4, 1/3, 2/3, 3/4\}$  and  $n \in \mathbb{N}$ , it is known that  $\Gamma(n + s)$  is a transcendental number and algebraically independent of  $\pi$  (cf. [27]). However, transcendence of the gamma function at other rational points is not known.

We close by noting that such obstacles are observed elsewhere in the literature; in particular, in relation to evaluating infinite products and gamma product identities (e.g., the product identity in (3)).

## REFERENCES

- [1] J.-P. Allouche. “Paperfolding infinite products and the gamma function”. In: *Journal of Number Theory* 148 (Mar. 2015), pp. 95–111. doi: [10.1016/j.jnt.2014.09.012](#).
- [2] G. E. Andrews, R. Askey, and R. Roy. *Special Functions*. Vol. 71. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1999. doi: [10.1017/CB09781107325937](#).
- [3] A. Baker. *Transcendental Number Theory*. Cambridge Mathematical Library. Cambridge University Press, 1975. doi: [10.1017/CB09780511565977](#).
- [4] J. M. Borwein, D. H. Bailey, and R. Girgensohn. *Experimentation in Mathematics*. A K Peters/CRC Press, Apr. 2004. doi: [10.1201/9781439864197](#).
- [5] M. Chamberland and A. Straub. “On gamma quotients and infinite products”. In: *Advances in Applied Mathematics* 51.5 (2013), pp. 546–562. doi: [10.1016/j.aam.2013.07.003](#).
- [6] K. Dilcher, R. Noble, and C. Smyth. “Minimal polynomials of algebraic numbers with rational parameters”. In: *Acta Arithmetica* 148.3 (2011), pp. 281–308. doi: [10.4064/aa148-3-5](#).
- [7] K. Dilcher and C. Vignat. “Infinite products involving Dirichlet characters and cyclotomic polynomials”. In: *Advances in Applied Mathematics* 100 (Sept. 2018), pp. 43–70. doi: [10.1016/j.aam.2018.05.003](#).
- [8] G. Kenison, O. Klurman, E. Lefauchaux, F. Luca, P. Moree, J. Ouaknine, A. Whitley, and J. Worrell. “On Inequality Decision Problems for Low-Order Holonomic Sequences”. Submitted. 2023.
- [9] G. Kenison, K. Nosan, M. Shirmohammadi, and J. Worrell. *The Membership Problem for Hypergeometric Sequences with Quadratic Parameters*. 2023. arXiv: [2303.09204 \[cs.LG\]](#).
- [10] M. Kontsevich and D. Zagier. “Periods”. In: *Mathematics unlimited—2001 and beyond*. Springer, Berlin, 2001, pp. 771–808.
- [11] S. Lang. *Introduction to transcendental numbers*. Addison-Wesley series in mathematics. Addison-Wesley Pub. Co., 1966.
- [12] A. Macintyre and A. J. Wilkie. “On the decidability of the real exponential field”. In: *Kreiseliana*. A K Peters, Wellesley, MA, 1996, pp. 441–467.
- [13] G. Martin. *A product of Gamma function values at fractions with the same denominator*. arXiv: 0907.4384. Dec. 2009.
- [14] M. Mignotte, T. Shorey, and R. Tijdeman. “The distance between terms of an algebraic recurrence sequence”. In: *Journal für die Reine und Angewandte Mathematik* (1984), pp. 63–76.
- [15] Y. V. Nesterenko. “Modular functions and transcendence questions”. In: *Sbornik: Mathematics* 187.9 (Oct. 1996), pp. 1319–1348. doi: [10.1070/sm1996v187n09abeh000158](#).
- [16] A. Nijenhuis. “Short Gamma Products with Simple Values”. In: *The American Mathematical Monthly* 117.8 (2010), p. 733. doi: [10.4169/000298910x515802](#).
- [17] K. Nosan, A. Pouly, M. Shirmohammadi, and J. Worrell. “The Membership Problem for Hypergeometric Sequences with Rational Parameters”. In: *Proceedings of the 2022 International Symposium on Symbolic and Algebraic Computation*. ISSAC ’22.

- Villeneuve-d'Ascq, France: Association for Computing Machinery, 2022, pp. 381–389. doi: [10.1145/3476446.3535504](https://doi.org/10.1145/3476446.3535504).
- [18] W. Paulsen. “Gamma triads”. en. In: *The Ramanujan Journal* 50.1 (Oct. 2019), pp. 123–133. doi: [10.1007/s11139-018-0114-8](https://doi.org/10.1007/s11139-018-0114-8).
  - [19] A. Prudnikov, Y. Brychkov, and O. Marichev. *Integrals and Series*. Vol. 1: Elementary Functions. Gordon & Breach, 1986.
  - [20] P. Ribenboim. *My numbers, my friends*. Popular lectures on number theory. Springer-Verlag, New York, 2000, pp. xii+375.
  - [21] D. Richardson. “How to Recognize Zero”. In: *Journal of Symbolic Computation* 24.6 (1997), pp. 627–645. doi: <https://doi.org/10.1006/jsco.1997.0157>.
  - [22] D. Richardson. “The Elementary Constant Problem”. In: *Papers from the International Symposium on Symbolic and Algebraic Computation*. ISSAC '92. Berkeley, California, USA: Association for Computing Machinery, 1992, pp. 108–116. doi: [10.1145/143242.143284](https://doi.org/10.1145/143242.143284).
  - [23] J.ándor and L. Tóth. “A remark on the gamma function.” und. In: *Elemente der Mathematik* 44.3 (1989), pp. 73–76.
  - [24] I. Stewart and D. Tall. *Algebraic number theory and Fermat’s last theorem*. Fourth. CRC Press, Boca Raton, FL, 2016, pp. xix+322.
  - [25] N. Vereshchagin. “Occurrence of zero in a linear recursive sequence”. In: *Mathematical notes of the Academy of Sciences of the USSR* 38.2 (Aug. 1985), pp. 609–615.
  - [26] R. Vidunas. “Expressions for values of the gamma function”. In: *Kyushu Journal of Mathematics* 59.2 (2005), pp. 267–283. doi: [10.2206/kyushujm.59.267](https://doi.org/10.2206/kyushujm.59.267).
  - [27] M. Waldschmidt. “Transcendence of periods: the state of the art”. In: *Pure Appl. Math. Q.* 2.2, Special Issue: In honor of John H. Coates. Part 2 (2006), pp. 435–463. doi: [10.4310/PAMQ.2006.v2.n2.a3](https://doi.org/10.4310/PAMQ.2006.v2.n2.a3).
  - [28] E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis*. en. Cambridge University Press, Sept. 1996. doi: [10.1017/CBO9780511608759](https://doi.org/10.1017/CBO9780511608759).
  - [29] K. S. Williams. “Integers of Biquadratic Fields”. In: *Canadian Mathematical Bulletin* 13.4 (1970), pp. 519–526. doi: [10.4153/CMB-1970-094-8](https://doi.org/10.4153/CMB-1970-094-8).

GEORGE KENISON, INSTITUTE OF LOGIC AND COMPUTATION, TU WIEN, VIENNA, AUSTRIA  
 Email address: [george.kenison@tuwien.ac.at](mailto:george.kenison@tuwien.ac.at)