Bounds for D-finite Closure Properties

Manuel Kauers

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A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called D-finite if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

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- Many sequences which have no name and no closed form.

A formal power series $f(x) \in K[[x]]$ is called D-finite if there exist polynomials $p_0, \dots, p_r$, not all zero, such that

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Examples: $Dx = xD + 1$, $Sn = (n+1)S$.

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If there is an action $A \times F \to F$ of A on a $C[x]$-module F (a "function space"), then $f \in F$ is called D-finite (w.r.t. this action), if there exists $L \in A \setminus \{0\}$ with $L \cdot f = 0$.

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Examples:

- Differential operators: $\sigma = \mathrm{id}$, $\delta = \frac{d}{dx}$, $F = C[[x]]$
- Recurrence operators: $\sigma(x) = x + 1$, $\delta = 0$, $F = C^{\mathbb{N}}$

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Example:

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

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Example:

$$\underbrace{\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\ \text{of order 1}}} \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \underbrace{\frac{1}{1-x}\Big(2 - \underbrace{\underbrace{P_n(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}} - \underbrace{P_{n+1}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}}_{\text{recurrence of order 4}}\Big)}_{\text{recurrence of order 3}}}_{\text{recurrence of order 7}} = 0$$

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(Compare $\dim_{\mathbb{Q}} \mathbb{Q}[\sqrt{2}] = \dim_{\mathbb{Q}} \mathbb{Q}[x]/\langle x^2 - 2 \rangle = 2$.)

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\le\ &\dim_{C(x)}\big(C(x)[\partial] \cdot f\big) + \dim_{C(x)}\big(C(x)[\partial] \cdot g\big) \\
\le\ &r + s.
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Any $r+s+1$ elements of $C(x)[\partial]\cdot(f+g)$ are $C(x)$-linearly dependent.

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A similar reasoning shows that the product fg is D-finite.

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$$p_2(n)\big(a_{n+2} + b_{n+2}\big) + p_1(n)\big(a_{n+1} + b_{n+1}\big) + p_0(n)\big(a_n + b_n\big) = 0$$

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$$\begin{aligned} & p_2(n)\big(4a_n + (n{+}2)(n{+}1)b_n\big) \\ + \, & p_1(n)\big(2a_n + (n{+}1)b_n\big) \\ + \, & p_0(n)\big(a_n + b_n\big) = 0 \end{aligned}$$

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$$\begin{aligned}&\big(p_2(n)\ 4 + p_1(n)\ 2 + p_0(n)\big)a_n\\ +&\big(p_2(n)\ (n{+}2)(n{+}1) + p_1(n)\ (n{+}1) + p_0(n)\big)b_n = 0\end{aligned}$$

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More columns than rows $\Rightarrow$ must have nontrivial solution.

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- $b_n = n! \qquad b_{n+1} - (n+1)b_n = 0 \qquad$ (order 1)

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$$\begin{pmatrix} 4 & 2 & 1 \\ (n+2)(n+1) & n+1 & 1 \end{pmatrix} \begin{pmatrix} p_2(n) \\ p_1(n) \\ p_0(n) \end{pmatrix} = 0$$

More columns than rows $\Rightarrow$ must have nontrivial solution.

$$\big(p_2(n), p_1(n), p_0(n)\big) = \big(n-1, -n^2-3n+2, 2n(n+1)\big).$$

Example $f_n = 2^n + n!$

- $a_n = 2^n \qquad a_{n+1} - 2a_n = 0$ (order 1)
- $b_n = n! \qquad b_{n+1} - (n+1)b_n = 0$ (order 1)

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$$(n-1)f_{n+2} - (n^2 + 3n - 2)f_{n+1} + 2n(n+1)f_n = 0.$$

- D-finite
  - Solutions of linear recurrences / differential equations with polynomial coefficients.
- Closure properties
  - If f and g are D-finite, then also $f + g$ and fg are D-finite (and some others, too).
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  - Want to know the sizes of the resulting recurrences / differential equations in advance.

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A typical element of $\mathbb{Z}[x][\partial]$ looks like this:

$$
\begin{aligned}
&(20x^6 + 36x^5 + 14x^4 + 8x^3 - 7x^2 - 43x - 46)\partial^4 \\
&\quad - (24x^6 + 38x^5 - 29x^4 - 18x^3 + 14x^2 + 19x - 43)\partial^3 \\
&\quad + (5x^6 + 36x^5 + 50x^4 + 16x^3 - 36x^2 - 41x + 43)\partial^2 \\
&\quad - (19x^6 - x^5 - 40x^4 - 11x^3 + 9x^2 + 25x + 27)\partial \\
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order

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&\quad + (5x^6 + 36x^5 + 50x^4 + 16x^3 - 36x^2 - 41x + 43)\partial^2 \\
&\quad - (19x^6 - x^5 - 40x^4 - 11x^3 + 9x^2 + 25x + 27)\partial \\
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degree

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&\quad - (24x^6 + 38x^5 - 29x^4 - 18x^3 + 14x^2 + 19x - 43)\partial^3 \\
&\quad + (5x^6 + 36x^5 + \overset{\text{exp(height)}}{\boxed{50}}x^4 + 16x^3 - 36x^2 - 41x + 43)\partial^2 \\
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degree

Want: bounds on the order, degree, and height of operators obtained from executing closure properties.

Theorem A. If $L_1, \dots, L_n$ are operators of degree $\leq d$ and height $\leq h$ for $f_1, \dots, f_n$, then there is an operator L for $f_1 + f_2 + \cdots + f_n$ with

$$\begin{aligned}
\operatorname{ord}(L) &\leq r := \sum_{k=1}^{n} \operatorname{ord}(L_k) \\
\deg(L) &\leq (n(r+1) - r)d \\
\operatorname{height}(L) &\leq \operatorname{height}(r) + \operatorname{height}((n(r+1) - r - 1)!) \\
&\quad + (n(r+1) - r)(\operatorname{height}(d) + c^{(r)}(d, h))
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$c^{(r)}(d, h)$:≈ bound on $\operatorname{height}(\sigma^r(p))$ and $\operatorname{height}(\delta^r(p))$ for all $p \in C[x]$ with $\deg(p) \leq d$ and $\operatorname{height}(p) \leq h$

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from degrees and heights of the matrix entries

Theorem B. If $L_1, \dots, L_n$ are operators for $f_1, \dots, f_n$, then there is an operator L for $f_1 + f_2 + \cdots + f_n$ with

$$\operatorname{ord}(L) = r \quad \text{and} \quad \deg(L) = d$$

for every r, d with

$$r \geq \sum_{k=1}^{n} \operatorname{ord}(L_k) \text{ and } d \geq \frac{(r+1)\sum_{k=1}^{n} \deg(L_k) - \sum_{k=1}^{n} \operatorname{ord}(L_k)\deg(L_k)}{r+1-\sum_{k=1}^{n} \operatorname{ord}(L_k)}.$$

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We now consider operators L for functions f of the form

$$(3x+8)f_1(x)^3 f_2'(x) + (1+x)f_1(x)f_1''(x)f_2(x) + (2x+5)f_1'(x)^2 f_2''(x).$$

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Let $f = P(f_1, \partial f_1, \ldots, f_2, \ldots)$ be such an expression, and let D_i be the total degree of P with respect to the variables corresponding to $f_i, \partial f_i, \ldots, \partial^{\operatorname{ord}(L_i)-1} f_i$.

Theorem C. There exists an operator L for f with

$$\operatorname{ord}(L) \le m := \prod_{i=1}^{n} \binom{D_i + \operatorname{ord}(L_i) - 1}{D_i}$$

$$\deg(L) \le m \deg(P) + m^2 \sum_{i=1}^{n} D_i \deg(L_i)$$

$$\begin{aligned}\operatorname{height}(L) \le{}& \operatorname{height}(m!) + m\, c^{(m)}(\deg(P), \operatorname{height}(P)) \\ &+ (m-1)\operatorname{height}\Big(\deg(P) + m \sum_{i=1}^{n} D_i \deg(L_i)\Big) \\ &+ m^2 \sum_{i=1}^{n} \Big(\operatorname{height}(4) D_i + \operatorname{height}(D_i{+}1) + D_i \operatorname{height}(\operatorname{ord}(L_i){+}m) \\ &\qquad + \operatorname{height}(\deg(L_i)) + c^{(m)}(\deg(L_i), \operatorname{height}(L_i))\Big).\end{aligned}$$

Theorem D. Furthermore, there exist operators L for f with

$$\mathrm{ord}(L) = r \quad \text{and} \quad \deg(L) = d$$

for every r, d with

$$r \geq m := \prod_{i=1}^{n} \binom{D_i + \mathrm{ord}(L_i) - 1}{D_i} \quad \text{and}$$

$$d \geq \frac{r\,m \sum_{i=1}^{n} D_i \deg(L_i) + m \deg(P)}{r + 1 - m}$$

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