COMPLETENESS OF CALCULII FOR AXIOMATICALLY DEFINED CLASSES OF ALGEBRAS

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1. Introduction

Garrett Birkhoff has shown [1, p. 440] that the identities of an equationally defined class of algebras are provable from the equations defining the class, using a very simple calculus all of whose proofs involve no linguistic expressions other than equations. This paper will present analogues of Birkhoff's result for the following two classes of algebras: classes all of whose defining conditions can be given by equations and equation implications, and classes all of whose defining conditions can be given by equations and expressions of the form $E_1 \wedge \cdots \wedge E_n \rightarrow E$, $n \geqslant 1$, where $E_1, \ldots, E_n$, E are equations. The first of these results answers a question posed by G. Birkhoff in [2, pp. 323–324]. Axioms of the forms considered comprise almost all axiom systems used in algebra. Section 6 at the end of this paper gives an algebraic characterization of axiomatically defined classes of algebras which are definable by axioms of the latter form.

In Section 2 a formal system for equation implications is described and the precise statement of our first result (Theorem 1) is given. The proof of Theorem 1 is sketched in Section 4, after key lemmas are presented in Section 3. Our second result (Theorem 2) is stated and proved in Section 5. We have deleted those details of the proof of Theorem 2 which are either straightforward or identical to steps in the proof of Theorem 1.

The method of L. Henkin [4] will be used to prove our theorems. As will be seen, the principal difficulty is to avoid the natural use of 'long' formulas (implications of non-atomic formulas) in a Henkin type completeness argument.

This paper is in part a revised version of results first announced in [9]. Donald Loveland has noted that two of the rules of inference present in [9] are redundant. It has also been pointed out that A. Robinson in [8] has proved completeness of a calculus for a class of languages whose syntax is similar to ours. Also, the author wishes to express his gratitude to Professors Hugo Ribeiro and George Grätzer for suggesting this problem to the author.

2. A calculus of equation implications

We shall consider a class of languages called *equation implication* languages. For each of these the primitive symbols consist of (1) a denumerably infinite set **VR** of

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variables, (2) an arbitrary (possibly empty) set **CN** of individual constants, (3) a non-empty set **FN** of function letters, (4) a binary relation symbol, $=$, (5) the implication sign, $\rightarrow$, and (6) grouping symbols, (,). With every member of **FN** is associated some finite rank. The terms of an equation implication language are to be defined inductively in the usual way. The only atomic formulas are the equations, formed by applying $=$ to the terms. The set **FL** of formulas is defined to be the set of all atomic formulas and expressions $E_1 \rightarrow E_2$, where E_1 and E_2 are atomic formulas. The expressions $E_1 \rightarrow E_2$ are the equation implications.

If $\mathscr{L}$ is an equation implication language, let $\mathscr{A}(\mathscr{L})$ denote the corresponding class of algebras.

THEOREM 1. *Let $\mathscr{L}$ be an equation implication language. Suppose* $\mathrm{S} \subseteq \mathbf{FL}$, $\mathrm{A} \in \mathbf{FL}$, *and for all algebras $\mathfrak{A}$ of $\mathscr{A}(\mathscr{L})$, if each formula of* S *is valid in $\mathfrak{A}$, then A is valid in $\mathfrak{A}$ (if $\models_{\mathfrak{A}} \mathrm{S}$, then $\models_{\mathfrak{A}} A$). Then a proof of A from* S *exists (in symbols,* $\mathrm{S} \vdash_{\mathscr{L}} A$*) using the following axioms and deduction rules. The set of axioms is the set of instances of the following three schemas, where E is any equation and p and q are any terms:*

(α1) $E \rightarrow E$;

(α2) $p = p$;

(α3) $p = q \rightarrow q = p$.

If E_1, E_2, and E_3 are equations; $p, q, r, p_1, q_1, \ldots, p_n, q_n$, are terms; $B \in$ **FL**; *and $f \in$* **FN** *has rank n; then the following are rules of inference:*

(ϱ1) *From E_2 to infer $E_1 \rightarrow E_2$;*

(ϱ2) *From $E_1 \rightarrow E_2$ and $E_2 \rightarrow E_3$ to infer $E_1 \rightarrow E_3$;*

(ϱ3) *From E_1 and $E_1 \rightarrow E_2$ to infer E_2;*

(ϱ4) *From B to infer the result of replacing all occurrences of a variable z in B by p;*

(ϱ5) *From $E_1 \rightarrow p = q$ and $E_1 \rightarrow q = r$ to infer $E_1 \rightarrow p = r$;*

(ϱ6) *From $E_1 \rightarrow p_1 = q_1, \ldots, E_1 \rightarrow p_n = q_n$ to infer $E_1 \rightarrow f(p_1, \ldots, p_n) = f(q_1, \ldots, q_n)$.* [1]

3. The lemmas

An easy induction argument yields the following lemma:

LEMMA 1. *Let $A \in$* **FL**. *Let $x_1, \ldots, x_n$ be the collection of variables occurring in A, $c_1, \ldots, c_n$ a collection of individual constants not belonging to* **CN**, *$\mathscr{L}'$ the extension of $\mathscr{L}$ defined by enlarging* **CN** *to include $c_1, \ldots, c_n$, and A' the result of replacing each occurrence of x_i in A by c_i. Then* $\mathrm{S} \vdash_{\mathscr{L}'} A'$ *implies* $\mathrm{S} \vdash_{\mathscr{L}} A$.

[1] The completeness property expressed in this theorem is sometimes called *strong* completeness, to distinguish it from the special case that **S** is the empty set of formulas.

The class of terms of an equation implication language is denoted by **T**. A term is defined to be closed if it contains no occurrence of variables. The class of closed terms is denoted by $\bar{\mathbf{T}}$. A formula of $\mathscr{L}$ is closed if every term occurring in it is a closed term.

A deduction of a formula A from a set S of formulas is to be one in tree form; see for example [5, end §24]. In a use of the rule $\varrho 3$, E_1 is called the major premise, and $E_1 \to E_2$ is called the minor premise.

LEMMA 2. *If $p, q, r, p_1, \ldots, p_n, q_1, \ldots, q_n$ are terms and $f \in$ FN has rank n, then the following are derived rules of inference.*

($\varrho 7$) *From $p = q$ and $q = r$ to infer $p = r$;*

($\varrho 8$) *From $p_1 = q_1, \ldots, p_n = q_n$ to infer $f(p_1, \ldots, p_n) = f(q_1, \ldots, q_n)$.*

Proof.

$$\dfrac{\dfrac{\dfrac{p = q}{p = q \to p = q}\,\varrho 1 \qquad \dfrac{q = r}{p = q \to q = r}\,\varrho 1}{p = q \to p = r}\,\varrho 5}{p = r}\,\varrho 3$$

$$\dfrac{p_1 = q_1 \qquad \dfrac{\dfrac{p_1 = q_1, \ldots, p_n = q_n}{p_1 = q_1 \to p_1 = q_1, \ldots, p_1 = q_1 \to p_n = q_n}\,\varrho 1}{p_1 = q_1 \to f(p_1, \ldots, p_n) = f(q_1, \ldots, q_n)}\,\varrho 6}{f(p_1, \ldots, p_n) = f(q_1, \ldots, q_n).}\,\varrho 3$$

It is important to note that we will use rules $\varrho 7$ and $\varrho 8$ freely. Thus, we will prove completeness for a system with rules of inference $\varrho 1$–$\varrho 8$. The proof of our theorem then follows immediately from Lemma 2. The purpose of this indirect procedure is that the addition of ($\varrho 7$) and ($\varrho 8$) enables us to write deductions in a *standard* form, according to the following definition and lemma.

DEFINITION 1. A deduction from a set S of formulas of an equation implication language is *standard* if the minor premise of each application of rule $\varrho 3$ is an axiom, a formula of S, or the result of applying rule $\varrho 4$ one or more times to a formula of S.

LEMMA 3. *If $S \vdash_{\mathscr{L}} B$, then there exists a standard deduction W of B from* S.

Proof. Let W' be any deduction of B from S. We proceed in two steps.

I. By successive applications of the following instruction (whose purpose is to 'push-up' the uses of rule $\varrho 4$), W' may be altered to a deduction W'' of B from S with the property that the premise of each application of ($\varrho 4$) is an axiom, a formula of S, or the result of applying rule $\varrho 4$ one or more times to a formula of S.

Instruction I. In a deduction replace an occurrence of the form

$$\cfrac{\cfrac{W_1, \ldots, W_{n_i}}{Y}\,\varrho i}{Y(p)}\,\varrho 4$$

where $i = 1, 2, 3, 5, 6, 7, 8$; n_i is the numbeı of premises of (ϱi); W_j is a deduction from S of a formula X_j, for $j = 1, \ldots, n_i$ and Y is an immediate consequence of (ϱi) applied to $X_1, \ldots, X_{n_i}$ by

$$\cfrac{\cfrac{W_1}{X_1(p)}\,\rho 4, \ldots, \cfrac{W_{n_i}}{X_{n_i}(p)}\,\varrho 4}{Y(p)}\,\varrho i.$$

Since proofs are finite, and the result of applying ($\varrho 4$) to an axiom is again an axiom, successive applications of this instruction indeed yield a deduction W'' of B from S with the property described above.

Instruction II. Observing that in any deduction, an expression $X \to Y$ is either an axiom, a member of S, or an immediate consequence of one of the rules $\varrho 1$, $\varrho 2$, $\varrho 4$, $\varrho 5$, or $\varrho 6$, we see now that by successive applications of the following four instructions, W'' may be changed to a standard deduction W.

Instruction IIa. In a deduction replace an occurrence of the form

$$\cfrac{W_0 \qquad \cfrac{W_1}{X \to Y}\,\rho 1}{Y}\,\varrho 3,$$

where W_0 is a deduction from S of X, and W_1 is a deduction from S of Y, by W_1.

Instruction IIb. In a deduction replace an occurrence of the form

$$\cfrac{W_0 \qquad \cfrac{W_1 \qquad\qquad W_2}{X \to Y}\,\varrho 2}{Y}\,\varrho 3,$$

where W_0 is a deduction from S of X, W_1 is a deduction from S of a formula $X \to Y_1$, and W_2 is a deduction from S of $Y_1 \to Y$, by

$$\cfrac{\cfrac{W_0 \qquad W_1}{Y_1}\,\varrho 3 \qquad W_2}{Y}\,\varrho 3 .$$

Instruction IIc. In a deduction replace an occurrence of the form

$$\cfrac{W_0 \qquad \cfrac{W_1 \qquad W_2}{X \to Y}\,\varrho 5}{Y}\,\varrho 3 ,$$

where W_0 is a deduction from S of X, W_1 is a deduction from S of a formula $X \to Y_1$, and W_2 is a deduction from S of a formula $X \to Y_2$, by

$$\cfrac{\cfrac{W_0 \qquad W_1}{Y_1}\,\varrho 3 \qquad \cfrac{W_0 \qquad W_2}{Y_2}\,\varrho 3}{Y}\,\varrho 7 .$$

Instruction IId. In a deduction replace an occurrence of the form

$$\cfrac{W_0 \qquad \cfrac{W_1, \ldots, W_n}{X \to f(p_1, \ldots, p_n) = f(q_1, \ldots, q_n)}\,\varrho 6}{f(p_1, \ldots, p_n) = f(q_1, \ldots, q_n)}\,\varrho 3 ,$$

where W_0 is a deduction from S of X; and W_i is a deduction from S of $X \to p_i = q_i$, for $i = 1, \ldots, n$; by

$$\cfrac{\cfrac{W_0 \qquad W_1}{p_1 = q_1}\,\rho 3, \ldots, \cfrac{W_0 \qquad W_n}{p_n = q_n}\,\varrho 3}{f(p_1, \ldots, p_n) = f(q_1, \ldots, q_n)}\,\varrho 8 .$$

LEMMA 4. (Restricted Deduction Theorem) *If E_1 and E_2 are equations, E_1 is closed, and* $S \cup \{E_1\} \vdash_{\mathscr{S}} E_2$, *then* $S \vdash_{\mathscr{S}} E_1 \to E_2$.

Proof. If E_2 is E_1, then $S \vdash_{\mathscr{S}} E_1 \to E_2$, by ($\alpha$1). If $E_2 \in S$ or E_2 is an axiom, then $S \vdash_{\mathscr{S}} E_2$. Thus $S \vdash_{\mathscr{S}} E_1 \to E_2$, by ($\varrho$1).

Otherwise, let W be a standard proof of E_2 from $\mathrm{S} \cup \{E_1\}$. For each equation X occurring above E_2 in W, hence such that $\mathrm{S} \cup \{E_1\} \vdash_{\mathscr{S}} X$, we may assume as an induction hypothesis that $\mathrm{S} \vdash_{\mathscr{S}} E_1 \rightarrow X$. Since E_2 is an equation, it is an immediate consequence of an application of (ϱ3), (ϱ4), (ϱ7), or (ϱ8). We consider each of these cases.

Case 1. E_2 is an immediate consequence of an application of (ϱ3). Then the last step in the deduction is of the form

$$\frac{X \qquad X \rightarrow E_2}{E_2}.$$

Since W is standard, $X \rightarrow E_2$ does not depend on E_1. Thus $\mathrm{S} \vdash_{\mathscr{S}} X \rightarrow E_2$. By induction hypothesis $\mathrm{S} \vdash_{\mathscr{S}} E_1 \rightarrow X$. Thus, by ($\varrho$2), $\mathrm{S} \vdash_{\mathscr{S}} E_1 \rightarrow E_2$.

Case 2. E_2 is an immediate consequence of an application of (ϱ4). Then $\mathrm{S} \vdash_{\mathscr{S}} E_1 \rightarrow$ $\rightarrow E_2$ follows by the induction hypothesis and (ϱ4), since E_1 is closed.

Case 3. E_2 is an immediate consequence of an application of (ϱ7). Then $\mathrm{S} \vdash_{\mathscr{S}} E_1 \rightarrow$ $\rightarrow E_2$ follows by the induction hypothesis and (ϱ5).

Case 4. E_2 is an immediate consequence of an application of (ϱ8). This case is treated as in case 3.

Hence, $\mathrm{S} \vdash_{\mathscr{S}} E_1 \rightarrow E_2$ in all possible cases.

LEMMA 5. *Let E be an equation. If there exist closed equations E_1 and E_2 so that* $\mathrm{S} \cup \{E_1\} \vdash_{\mathscr{S}} E$ *and* $\mathrm{S} \cup \{E_1 \rightarrow E_2\} \vdash_{\mathscr{S}} E$, *then* $\mathrm{S} \vdash_{\mathscr{S}} E$.

Proof. If the proof of E from $\mathrm{S} \cup \{E_1 \rightarrow E_2\}$ does not depend on $E_1 \rightarrow E_2$, then $\mathrm{S} \vdash_{\mathscr{S}} E$. Suppose the proof of E from $\mathrm{S} \cup \{E_1 \rightarrow E_2\}$ does not depend on $E_1 \rightarrow E_2$, and, by Lemma 3, let W be a standard deduction. Since here the proof of an equation depends on the equation implication $E_1 \rightarrow E_2$, there is a step in the deduction of the form

$$\frac{X \qquad X \rightarrow Y}{Y}\,\varrho 3$$

where one of the premises depends on $E_1 \rightarrow E_2$. Since proofs are finite, we may assume that neither X nor any formula in the proof above X is an immediate consequence of (ϱ3). Thus X does not depend on $E_1 \rightarrow E_2$. Since W is standard and $E_1 \rightarrow E_2$ is closed, it follows that $X \rightarrow Y$ is $E_1 \rightarrow E_2$. So $\mathrm{S} \vdash_{\mathscr{S}} E_1$, since E_1 is X and $\mathrm{S} \vdash_{\mathscr{S}} X$. By Lemma 4, $\mathrm{S} \vdash_{\mathscr{S}} E_1 \rightarrow E$, and, by ($\varrho$3), $\mathrm{S} \vdash_{\mathscr{S}} E$.

4. Proof of theorem 1

To show that if A is not provable from S (in symbols, $\mathrm{S} \nvdash_{\mathscr{S}} A$), then there exists an algebra $\mathfrak{U}$ so that $\vDash_{\mathfrak{U}} \mathrm{S}$ and $\nvDash_{\mathfrak{U}} A$, we first observe that it suffices to consider the case that A is a closed equation. Firstly, suppose A is an equation, say E_1, and S $\nvdash_{\mathscr{L}} E_1$. Then, by Lemma 1, S $\nvdash_{\mathscr{L}'} E_1'$, where E_1' is a closed equation. If there exists an algebra $\mathfrak{U}$ so that $\models_{\mathfrak{U}}$ S and not $\models_{\mathfrak{U}} E_1'$, then by definition we also have $\nvDash E_1$. Secondly, suppose A is of the form $E_1 \rightarrow E_2$, and S $\nvdash_{\mathscr{L}} E_1 \rightarrow E_2$. By Lemma 1, S $\nvdash_{\mathscr{L}'} E_1' \rightarrow E_2'$. By Lemma 4, $S \cup \{E_1'\} \nvdash_{\mathscr{L}'} E_2'$. E_2' is closed. If there exists an algebra $\mathfrak{U}$ so that $\models_{\mathfrak{U}}$ S, $\models_{\mathfrak{U}} E_1'$, and $\nvDash_{\mathfrak{U}} E_2'$, then by definition we also have $\nvDash_{\mathfrak{U}} E_1 \rightarrow E_2$. Thus in both cases a reduction is possible.

Now, suppose A is a closed equation, E, and S $\nvdash_{\mathscr{L}} E$. By Zorn's lemma applied to the set of extensions of S from which E is not derivable in $\mathscr{L}$, there is a maximal extension S_1 of S satisfying $S_1 \nvdash_{\mathscr{L}} E$.

Let γ be the following interpretation of $\mathscr{L}$ with domain $\bar{\mathbf{T}}$, defined in Section 3. The interpretation of an individual constant c shall be c itself. If $f \in$ FN, the associated operation f^* in γ shall be defined so that $f^*(p_1, \ldots, p_n)$ is $f(p_1, \ldots, p_n)$, where $p_1, \ldots, p_n \in \bar{\mathbf{T}}$. γ shall contain a binary relation $\approx$, defined so that $p \approx q$ if and only if $(p = q) \in S_1$, for $p, q \in \bar{\mathbf{T}}$.

We show that a closed formula is valid in γ if and only if it belongs to S_1. This is true, by definition, for closed equations. Suppose E_1 and E_2 are closed equations so that $(E_1 \rightarrow E_2) \in S_1$. By rule $\varrho 3$, either $E_1 \notin S_1$ or $E_2 \in S_1$. Hence, either $\nvDash_\gamma E_1$ or $\models_\gamma E_2$. In either case, $\models_\gamma E_1 \rightarrow E_2$. Conversely, suppose $\models_\gamma E_1 \rightarrow E_2$. Either $\nvDash_\gamma E_1$ or $\models_\gamma E_2$. Hence, either $E_1 \notin S_1$ or $E_2 \in S_1$. If $E_1 \notin S_1$, then $S_1 \cup \{E_1\} \vdash_{\mathscr{L}} E$, since S_1 is maximal. So $S_1 \cup \{E_1 \rightarrow E_2\} \vdash_{\mathscr{L}} E$ would imply $S_1 \vdash_{\mathscr{L}} E$, by Lemma 5. Thus $(E_1 \rightarrow E_2) \in S_1$. If $E_2 \in S_1$, then $(E_1 \rightarrow E_2) \in S_1$, by rule $\varrho 1$. Therefore, $\models_\gamma E_1 \rightarrow E_2$ if and only if $(E_1 \rightarrow E_2) \in S_1$.

In particular, $\nvDash_\gamma E$.

Let B be an arbitrary formula of S. If $\mathbf{f}$ is any function from VR into $\bar{\mathbf{T}}$, let $B(\mathbf{f})$ denote the result of replacing each variable z occurring in B by $\mathbf{f}(z)$ at all of its occurrences. By rule $\varrho 4$, $B(\mathbf{f}) \in S_1$, for each function $\mathbf{f}$. Hence $\models_\gamma B(\mathbf{f})$, for each function $\mathbf{f}$, since $B(\mathbf{f})$ is closed. Thus, we see that $B \in$ S implies $\models_\gamma B$, that is, $\models_\gamma$ S.

Finally, we notice that $\approx$ is a congruence relation on γ, by (α2), (α3), (ϱ7), and (ϱ8). Therefore, $\gamma/\approx$ is an algebra in which each formula of S is valid and A is not valid. To complete the proof of Theorem 1, take $\mathfrak{U}$ to be $\gamma/\approx$.

Remark 1. It is easy to see that the following three statements are equivalent. There exists an $A \in$ FL such that S $\nvdash_{\mathscr{L}} A$. For all variables x and y, S $\nvdash_{\mathscr{L}} x = y$. There exist distinct variables x and y such that S $\nvdash_{\mathscr{L}} x = y$. If any of these three statements is taken as a definition of consistency, then our result implies the following corollary: every consistent set of equations and equation implications is valid in an algebra having at least two elements.

Remark 2. Consider a class of languages having the same primitive symbols and formation rules as the class of equation implication languages, except that the set **FL** of formulas is to be the smallest set containing all the atomic formulas and closed under the operation of forming $A \to B$ from A and B. Then we can prove a strong completeness theorem for this class without making the kind of analysis that appears in Lemma 3 above. Indeed, it is enough to have as propositional axiom schemas $A \to (B \to A)$, $(A \to B) \to ((A \to (B \to C)) \to (A \to C)$, and $(A \to C) \to (((A \to B) \to C) \to C)$ (these schemas appear in [3, p. 43]); the obvious axioms for equality; and *modus ponens* and substitution (rule $\varrho 4$) as the only rules of inference. The restricted Deduction Theorem for this system is known to follow, and the proof of the analogue of Lemma 5 for this system follows immediately from the third propositional axiom schema and two uses of the restricted Deduction Theorem. Then an argument almost identical to the one presented in Section 4 above will furnish the proof of strong completeness.

5. A calculus of equation conjunction implications

We consider now a class of languages called *equation conjunction implication* (ECI) languages. The set of primitive symbols contains, in addition to the primitive symbols of an equation implication language, the conjunction sign, $\wedge$. Terms and equations are defined in the usual way, and, as before, the only atomic formulas are the equations. A *conjunctive formula* is defined inductively so that (1) an equation is a conjunctive formula and, (2) if A and B are conjunctive formulas, then so is $A \wedge B$. The set **FL** of formulas is defined to be the set of all equations and expressions $A \to E$, where A is a conjunctive formula and E is an equation. Note that, except for equations, conjunctive formulas are not formulas. If A is a conjunctive formula, write $A(E_1, \ldots, E_n)$ for A, where A consists precisely of occurrences of the equations $E_1, \ldots, E_n$ (in any order and with possible repetitions).

Since ECI languages have greater expressibility than equation implication languages, transitivity and substitivity of equality can now be written as axioms rules $\alpha 3$ and $\alpha 4$ below). On the other hand, new axioms and rules are required, principally the conjunction rule, $\varrho 5$.

For each ECI language $\mathscr{L}$, $\mathscr{A}(\mathscr{L})$ denotes the corresponding class of algebras.

THEOREM 2. *Let $\mathscr{L}$ be an ECI language. Suppose* $\mathrm{S} \subseteq \mathbf{FL}$, $A \in \mathbf{FL}$, *and for all* $\mathfrak{U}$ *of* $\mathscr{A}(\mathscr{L})$, *if* $\models_{\mathfrak{U}} \mathrm{S}$, *then* $\models_{\mathfrak{U}} A$. *Then a proof of A from* S *exists using the following axioms and deduction rules. The set of axioms is the set of instances of the following four schemas, where E_1 and E_2 are equations; $p, q, p_1, q_1, \ldots, p_n, q_n$ are terms; and* $f \in \mathbf{FN}$ *has rank n, $n \geqslant 1$.*

($\alpha 1$) $E_1 \wedge E_2 \to E_2$;

($\alpha 2$) $p = p$;

($\alpha 3$) $p = q \wedge q = r \to r = p$

($\alpha 4$) $p_1 = q_1 \wedge \cdots \wedge p_n = q_n \to f(p_1, \ldots, p_n) = f(q_1, \ldots, q_n)$.

If $E_1, E_2, E_3, E, F_1, \ldots, F_n, G_1, \ldots, G_m, n \geqslant 1, m \geqslant 1$, are equations; p is a term; $B(F_1, \ldots, F_n)$ and $C(F_1, \ldots, F_n)$ are conjunctive formulas; and $D \in \mathbf{FL}$, then the following are rules of inference.

($\varrho 1$) *From E_2 to infer $E_1 \to E_2$;*

($\varrho 2$) *From E_1 and $E_1 \to E_2$ to infer E_2;*

($\varrho 3$) *From $F_1 \wedge \cdots \wedge F_n \to E_1$ and $E_1 \to E$ to infer $F_1 \wedge \cdots \wedge F_n \to E$;*

($\varrho 4$) *From $F_1 \wedge \cdots \wedge F_n \to E_1$ and $E_1 \wedge G_1 \wedge \cdots \wedge G_m \to E$ to infer $F_1 \wedge \cdots \wedge F_n \wedge G_1 \wedge \wedge \cdots \wedge G_m \to E$;*

($\varrho 5$) *From $B(F_1, \ldots, F_n) \to E$ to infer $C(F_1, \ldots, F_n) \to E$;*

($\varrho 6$) *From D to infer the result of replacing all occurrences of a variable z in D by p.*

It may be observed that ($\varrho 5$) is equivalent to the following three rules:

From $E_1 \wedge E_1 \to E$ to infer $E_1 \to E$;

From $F_1 \wedge F_1 \wedge F_2 \wedge F_3 \wedge \cdots \wedge F_n \to E$ to infer $F_1 \wedge F_2 \wedge \cdots \wedge F_n \to E$; and,

From $E_1 \wedge \cdots \wedge E_n \to E$ to infer $E_{\pi(1)} \wedge \cdots \wedge E_{\pi(n)} \to E$, where π is a permutation of $\{1, \ldots, n\}$.

Also we have the following lemma.

LEMMA 6. *If $E, E_1, \ldots, E_n$, $n \geqslant 1$, are equations, and $A(E_1, \ldots, E_n)$ and B are conjunctive formulas, then the following are derived rules of inference.*

($\varrho 7$) *From $E_1, \ldots, E_n$ and $A(E_1, \ldots, E_n) \to E$ to infer E;*

($\varrho 8$) *From $E_1, \ldots, E_n$ and $A(E_1, \ldots, E_n) \wedge B \to E$ to infer $B \to E$;*

($\varrho 9$) *From $E_1 \to E$ to infer $A(E_1, \ldots, E_n) \to E$.*

($\varrho 10$) *From E to infer $A(E_1, \ldots, E_n) \to E$.*

Proof. To prove ($\varrho 7$), we are given $E_1, \ldots, E_n$ and $A(E_1, \ldots, E_n) \to E$. Suppose $A(E_1, \ldots, E_n)$ is $E_{i_1} \wedge \cdots \wedge E_{i_m}$.

$$\dfrac{\dfrac{E_{i_2}}{E_{i_3} \to E_{i_2}}\,\varrho 1 \qquad \dfrac{\dfrac{\dfrac{E_{i_1}}{E_{i_2} \to E_{i_1}}\,\varrho 1 \qquad E_{i_1} \wedge \cdots \wedge E_{i_m} \to E}{E_{i_2} \wedge E_{i_2} \wedge \cdots \wedge E_{i_m} \to E}\,\varrho 4}{E_{i_2} \wedge \cdots \wedge E_{i_m} \to E}\,\varrho 5}{\vdots}\,\varrho 4$$

$$\dfrac{E_{i_m} \qquad \dfrac{\vdots}{E_{i_m} \to E}\,\varrho 5}{E}\,\varrho 2.$$

The proof of ($\varrho 8$) is identical. In this case we start with $E_{i_1} \wedge \cdots \wedge E_{i_m} \wedge B \to E$ and our result is $B \to E$.

To prove ($\varrho 9$),

$$\cfrac{\cfrac{\cfrac{\cfrac{E_2 \wedge E_1 \to E_1\,(\alpha 1) \qquad E_1 \to E}{E_2 \wedge E_1 \to E}\,(\varrho 3) \qquad E_3 \wedge E_2 \to E_2\,(\alpha 1)}{E_3 \wedge E_2 \wedge E_1 \to E \qquad E_4 \wedge E_3 \to E_3\,(\alpha_1)}\,(\varrho 4)}{\vdots}\,(\varrho 4)}{\cfrac{E_n \wedge \cdots \wedge E_1 \to E}{A(E_1, \ldots, E_n) \to E.}\,\varrho 5}\,(\varrho 4)$$

($\varrho 10$) follows from ($\varrho 1$) and ($\varrho 9$).

It is clear that ($\varrho 7$) implies ($\varrho 2$). It is important to observe now that we do not prove completeness directly for the system given in Theorem 2. Instead, we prove completeness for a system with rules of inference ($\varrho 1$), ($\varrho 3$), ($\varrho 4$), ($\varrho 5$), ($\varrho 6$), and ($\varrho 7$). The proof of Theorem 2 then follows. The purpose of this indirect procedure is, as in the proof of Theorem 1, that the system with ($\varrho 7$) rather than ($\varrho 2$) enables us to write deductions in a standard form. In a use of ($\varrho 7$), $A(E_1, \ldots, E_n) \to E$ is called the minor premise.

DEFINITION 2. A deduction from a set S of formulas of an ECI language is *standard* if the minor premise of each application of rule $\varrho 7$ is an axiom, a formula of S, or the result of applying rule $\varrho 6$ one or more times to a formula of S.

LEMMA 7. *If $\mathscr{L}$ is an ECI language and* S $\vdash_{\mathscr{L}} B$, *then there is a standard deduction W of B from* S.

Proof. Let W' be any deduction of B from S (using rules $\varrho 1$, and $\varrho 3$–$\varrho 7$). As in the proof of Lemma 3, we proceed in two steps. The first step is identical to Instruction I in the proof of Lemma 3. By this instruction W' is altered to a deduction W'' of B from S with the property that the premise of each application of ($\varrho 6$) is an axiom, a formula of S, or the result of applying rule $\varrho 6$ one or more times to a formula of S.

In any deduction, a formula $A(X_1, \ldots, X_n) \to E$ can be an immediate consequence of rules $\varrho 1$, $\varrho 3$, $\varrho 4$, $\varrho 5$, or $\varrho 6$. The purpose of the second step is to 'push up' applications of ($\varrho 7$) above all applications of ($\varrho 1$), ($\varrho 3$), ($\varrho 4$), and ($\varrho 5$), thereby yielding a standard deduction W. The replacements needed are straightforward. ($\varrho 1$) above ($\varrho 7$) is simply deleted; ($\varrho 3$) above ($\varrho 7$) is replaced by two successive applications of ($\varrho 7$); ($\varrho 4$) above ($\varrho 7$) is also replaced by two uses of ($\varrho 7$); and ($\varrho 5$) above ($\varrho 7$) is replaced by one use of ($\varrho 7$).

LEMMA 8. *If $E_1, \ldots, E_n$ are closed equations, $n \geqslant 1$, and* S $\cup \{E_1, \ldots, E_n\} \vdash_{\mathscr{L}} E$, *then* S $\vdash_{\mathscr{L}} E_1 \wedge \cdots \wedge E_n \to E$.

Proof. If $E \in \mathrm{S}$ or E is an axiom, then $S \vdash_{\mathscr{S}} E$. Thus $\mathrm{S} \vdash_{\mathscr{S}} E_1 \wedge \cdots \wedge E_n \rightarrow E$, by the derived rule $\varrho 10$. If E is one of E_i, $i=1,\ldots,10$. Then $\mathrm{S} \vdash_{\mathscr{S}} E_i \rightarrow E$. ($E \rightarrow E$ is a theorem by ($\alpha 1$) and ($\varrho 5$). Thus $\mathrm{S} \vdash_{\mathscr{S}} E_1 \wedge \cdots \wedge E_n \rightarrow E$, by ($\varrho 10$) again.

Otherwise, let W be a standard proof of E from $\mathrm{S} \cup \{E_1,\ldots,E_n\}$. For each equation X occurring above E in W we assume as induction hypothesis that $\mathrm{S} \vdash_{\mathscr{S}} E_1 \wedge \cdots \wedge E_n \rightarrow X$. E is an immediate consequence of ($\varrho 6$) or ($\varrho 7$). If E is an immediate consequence of ($\varrho 6$), then $\mathrm{S} \vdash_{\mathscr{S}} E$ or E is one of the E_i, $i=1,\ldots,n$, since $E_1,\ldots,E_n$ are closed and W is standard. Thus, by ($\varrho 10$), as in the previous paragraph, $\mathrm{S} \vdash_{\mathscr{S}} E_1 \wedge \cdots \wedge E_n \rightarrow E$.

Suppose E is a consequence of ($\varrho 7$). Then the last step in the deduction is of the form

$$\frac{X_1,\ldots,X_n \qquad X_1 \wedge \cdots \wedge X_n \rightarrow E}{E}.$$

Since W is standard, $X_1 \wedge \cdots \wedge X_n \rightarrow E$ does not depend on $E_1,\ldots,E_n$. Thus, $\mathrm{S} \vdash_{\mathscr{S}} X_1 \wedge \cdots \wedge X_n \rightarrow E$. Then,

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{S}{E_1 \wedge \cdots \wedge E_n \rightarrow X_1 \qquad X_1 \wedge \cdots \wedge X_n \rightarrow E}\ \varrho 4}{E_1 \wedge \cdots \wedge E_n \wedge X_2 \wedge \cdots \wedge X_n \rightarrow E}\ \rho 5}{E_1 \wedge \cdots \wedge E_n \rightarrow X_2,\ X_2 \wedge E_1 \wedge \cdots \wedge E_n \wedge X_3 \wedge \cdots \wedge X_n \rightarrow E}\ \varrho 4}{E_1 \wedge \cdots \wedge E_n \wedge E_1 \wedge \cdots \wedge E_n \wedge X_3 \wedge \cdots \wedge X_n \rightarrow E}\ \varrho 5}{\cfrac{E_1 \wedge \cdots \wedge E_n \wedge X_3 \wedge \cdots \wedge X_n \rightarrow E}{\cfrac{\vdots}{E_1 \wedge \cdots \wedge E_n \rightarrow E}}}$$

LEMMA 9. *If there exist closed equations E_1 and E_2 so that* $\mathrm{S} \cup \{E_1\} \vdash_{\mathscr{S}} E$ *and* $\mathrm{S} \cup \{E_1 \rightarrow E_2\} \vdash_{\mathscr{S}} E$, *then* $\mathrm{S} \vdash_{\mathscr{S}} E$.

Proof. If the proof of E from $S \cup \{E_1 \rightarrow E_2\}$ does not depend on $E_1 \rightarrow E_2$, then $S \vdash_{\mathscr{S}} E$. Otherwise, there is a step in the deduction of the form

$$\frac{X_1,\ldots,X_n \qquad A(X_1,\ldots,X_n) \rightarrow Y}{Y},$$

where one of the premises depends on $E_1 \rightarrow E_2$. Since proofs are finite we may assume that neither X nor any formula in the proof above X is an immediate consequence of ($\varrho 7$). Thus $X_1,\ldots,X_n$ do not depend on $E_1 \rightarrow E_2$. Since proofs are standard and $E_1 \rightarrow E_2$ is closed $A(X_1,\ldots,X_n) \rightarrow Y$ is $E_1 \rightarrow E_2$. (In particular, the $X_1,\ldots,X_n$ are identical.) $\mathrm{S} \vdash_{\mathscr{S}} E_1$ and, by Lemma 8, $\mathrm{S} \vdash E_1 \rightarrow E_2$. Thus, $\mathrm{S} \vdash_{\mathscr{S}} E$.

We have now proved the needed lemmas and are ready to complete the proof of Theorem 2. We show that if S $\nvdash_{\mathscr{L}} A$, then there exists an algebra $\mathfrak{U}$ so that $\models_{\mathfrak{U}}$ S and $\not\models_{\mathfrak{U}} A$. First, observing that Lemma 1 holds for ECI languages as well as equation implication languages, we reduce to the case that A is a closed equation, E. This reduction is immediate if A is an equation. Suppose A is $E_1 \wedge \cdots \wedge E_n \rightarrow F$, $n \geqslant 1$. By Lemma 1, S $\nvdash_{\mathscr{L}'} E_1' \wedge \cdots \wedge E_n' \rightarrow E'$. By Lemma 7, S $\cup \{E_1', \ldots, E_n'\} \nvdash_{\mathscr{L}'} E'$. If there exists $\mathfrak{U}$ so that $\models_{\mathfrak{U}}$ S $\cup \{E_1', \ldots, E_n'\}$ and $\not\models_{\mathfrak{U}} E'$, then $\models_{\mathfrak{U}}$ S and $\not\models_{\mathfrak{U}} E_1 \wedge \cdots \wedge E_n \rightarrow E$.

Then, given a closed equation E so that S $\nvdash_{\mathscr{L}} E$, apply Zorn's lemma to obtain a maximal extension S_1 of S satisfying $S_1 \nvdash_{\mathscr{L}} E$. Let γ be the usual interpretation of $\mathscr{L}$ with domain $\overline{\mathrm{T}}$ so that $p \approx q$ if and only if $p = q \in S_1$, for $p, q \in \overline{\mathrm{T}}$.

A closed equation holds in γ if and only if it belongs to S_1. (In particular, $\not\models_{\gamma} E$.) We show this for all closed formulas. Suppose $E_1 \wedge \cdots \wedge E_n \rightarrow F$ belongs to S_1, where $E_1, \ldots, E_n$, F are closed equations. By rule $\varrho 7$, $E_i \notin S_1$, for some i, or $F \in S_1$. Thus, either $\not\models_{\gamma} E_i$ or $\models_{\gamma} F$; i.e., $\models E_1 \wedge \cdots \wedge E_n \rightarrow F$.

Suppose $\models_{\gamma} E_1 \wedge \cdots \wedge E_n \rightarrow F$. Then, either $\not\models_{\gamma} E_i$, for some i, or $\models_{\gamma} F$. Consider the case that $\not\models_{\gamma} E_i$. Then $E_i \notin S_1$. So $S_1 \cup \{E_i\} \vdash_{\mathscr{L}} E$. By Lemma 9, if $S_1 \cup \{E_i \rightarrow F\} \vdash_{\mathscr{L}} E$, then $S_1 \vdash_{\mathscr{L}} E$. Thus, $S_1 \cup \{E_i \rightarrow F\} \nvdash_{\mathscr{L}} E$. It follows by ($\varrho 9$), since S_1 is maximal, that $E_1 \wedge \cdots \wedge E_n \rightarrow F$ belongs to S_1. In the case that $\models_{\gamma} F$, $F \in S_1$, so it follows by rule $\varrho 10$ that $E_1 \wedge \cdots \wedge E_n \rightarrow F$.

Using rule $\varrho 6$, an argument identical to that in the proof of Theorem 1 shows that $B \in$ S implies $\models_{\gamma} B$. That is $\models_{\gamma} S$. By ($\alpha 2$), ($\alpha 3$), and ($\alpha 4$), $\approx$ is a congruence relation on γ. Let $\mathfrak{U}$ be $\gamma/\approx$. Then $\models_{\mathfrak{U}} S$ and $\not\models_{\mathfrak{U}} A$.

6. ECI definable classes

Call a formula of an ECI language an ECI-formula. We conclude this paper now with a characterization of those classes of algebras which are definable by sets of ECI-formulas. Our theorems follow directly from well-known results in the literature, and we believe they are essentially known. A class of algebras K is *axiomatic* if for some set S of first-order formulas, K is the class of all algebras in which every formula of S holds. A *basic Horn formula* is any first-order formula of the form $\theta_1 \wedge \cdots \wedge \theta_n$, where each θ_i is an atomic formula or the negation of an atomic formula, and at most one of the θ_i is atomic.

THEOREM 3. *If K is an axiomatic class of algebras, then K is closed under the formation of subalgebras and direct products if and only if K is definable by a set of basic-Horn formulas.*

Proof. By the results of J. Los and A. Tarski [10], if K is closed under the formation of subalgebras then K is a universal class. By the results of J. C. C. McKinsey [6] and W. Peremans [7], a universal class which is closed under the formation of direct products is definable by a set of basic Horn formulas. The proof in the other direction is straightforward and well known.

THEOREM 4. *A class K of algebras is definable by a set of ECI-formulas if and only if K is axiomatic, K contains the one element algebra, and K is closed under the formation of subalgebras and direct products.*

Proof. The theorem follows from Theorem 3 and the following two observations made in [6]. First, if A is a basic Horn formula and one the disjuncts is an atomic formula, then A is equivalent to an ECI-formula. Second, no disjunction of inequalities holds in the one element algebra.

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Remark (added in proof). The proofs of Theorems 1 and 2 can be simplified so that Lemmas 5 and 9 can be deleted. Given $S \subseteq \mathbf{FL}$, closed equation E for which $S \nvdash_{\mathscr{E}} E$, and γ, as in the proofs above, it can be shown directly for all closed formulas A, that $S \vdash_{\mathscr{E}} A$ implies $\models_\gamma A$. In particular, a maximal extension of S is not needed.