

Flip-width: Cops and Robber on dense graphs

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Abstract

We define new graph parameters, called *flip-width*, that generalize treewidth, degeneracy, and generalized coloring numbers for sparse graphs, and clique-width and twin-width for dense graphs. The flip-width parameters are defined using variants of the Cops and Robber game, in which the robber has speed bounded by a fixed constant $r \in \mathbb{N} \cup \{\infty\}$, and the cops perform flips (or perturbations) of the considered graph. We then propose a new notion of tameness of a graph class, called *bounded flip-width*, which is a dense counterpart of classes of bounded expansion of Nešetřil and Ossona de Mendez, and includes classes of bounded twin-width of Bonnet, Kim, Thomassé, and Watrigant. This unifies Sparsity Theory and Twin-width Theory, for the first time providing a common language for studying the central notions of the two theories, such as weak coloring numbers and twin-width – corresponding to winning strategies of one player – or dense shallow minors, rich divisions, or well-linked sets, corresponding to winning strategies of the other player. To demonstrate the robustness of the introduced notions, we prove that boundedness of flip-width is preserved by first-order interpretations, or transductions, generalizing previous results concerning classes of bounded expansion and bounded twin-width. We also show that the considered notions are amenable to algorithms, by providing an algorithm approximating the flip-width of a given graph, which runs in slice-wise polynomial time (XP) in the size of the graph. Finally, we propose a more general notion of tameness, called *almost bounded flip-width*, which is a dense counterpart of nowhere dense classes. We conjecture, and provide evidence, that classes with almost bounded flip-width coincide with monadically dependent (or monadically NIP) classes, introduced by Shelah in model theory. We also provide evidence that classes of almost bounded flip-width characterise the hereditary graph classes for which the model-checking problem is fixed-parameter tractable, which is of central importance in structural and algorithmic graph theory.



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1 Introduction

A recent focus of algorithmic and structural graph theory, and of finite model theory, is to find graph parameters that extend the parameters used in the context of sparse graphs, to the dense setting. More generally, the goal is to extend the Sparsity theory of Nešetřil and Ossona de Mendez [NdM12] to dense graph classes. Two central parameters used in the sparse setting are treewidth and degeneracy; both have found numerous applications in algorithms and combinatorics. Whereas the dense analogue of treewidth – clique-width, or rank-width – is well-understood, there is not even a clear candidate for the dense analogue of degeneracy¹. Recall that a graph has degeneracy at most d if its vertices can be totally ordered so that every vertex has at most d neighbors before it in the order. Generalized coloring numbers are related parameters, specified by a radius r , which impose restrictions on neighborhoods of radius r , degeneracy being the case of radius $r = 1$. Sparsity theory is a very successful theory studying classes of sparse graphs in which degeneracy and generalized coloring numbers play a central role. The fundamental notions of this theory are two tameness conditions for graph classes: *bounded expansion* and *nowhere denseness*. A class of graphs has bounded expansion if and only if each generalized coloring number is bounded by a constant (depending on the radius), on all graphs in the class. Examples include every class with bounded maximum degree, the class of planar graphs, classes of bounded treewidth, and every class that excludes some graph as a minor or as a topological minor. The more general nowhere dense classes are characterized analogously, with the constant bound on the generalized coloring numbers of n -vertex graphs replaced with $O(n^\varepsilon)$, for any fixed $\varepsilon > 0$. In particular, nowhere dense classes are sparse – every n -vertex graph in such a class has $O(n^{1+\varepsilon})$ edges for any fixed $\varepsilon > 0$, and every nowhere dense class excludes some biclique $K_{t,t}$ as a subgraph. Classes with bounded expansion and nowhere dense classes can be characterized in many other ways, in terms of their remarkable combinatorial, algorithmic, and logical properties, yielding multiple applications in those areas.

One of the driving questions in this line of work, on the algorithmic side, is to characterize those graph classes for which the model-checking problem for first-order logic is fixed-parameter tractable: there is an algorithm that determines whether a given graph G from the class satisfies a given first-order sentence φ in time $f(\varphi) \cdot |G|^c$, for some constant c and function f that depend only on the class. Such a characterization is known in the special case of *monotone* graph classes, that is, graph classes that are closed under removing vertices and edges. In a landmark result, Grohe, Kreutzer and Siebertz [GKS17a] proved that a monotone graph class has fixed-parameter tractable model-checking if and only if it is nowhere dense (under a complexity-theoretic assumption, $\text{FPT} \neq \text{AW}[*]$).

There is an ongoing effort to lift Sparsity theory to *hereditary* graph classes, that is, classes that are closed under removing vertices. For example, the class of cliques is hereditary and combinatorially and logically very simple, but lies outside of the realm of Sparsity theory, as the notions studied in Sparsity theory – generalized coloring numbers, bounded expansion, nowhere denseness, etc. – are monotone under edge removals, and thus blow up entirely when considering any graph class with arbitrarily large cliques or balanced bicliques. Yet, there are many known hereditary graph classes that are well-behaved in a similar way to classes with bounded expansion and nowhere dense classes, but are not monotone, and are not sparse (e.g. contain arbitrarily large cliques). Those include for instance classes of bounded clique-width, the class of unit interval graphs, or proper hereditary classes of permutation graphs – which are all subsumed by the recently introduced classes of bounded twin-width [BKTW20] (see below) – as well as *structurally nowhere dense* classes [DMS] – classes of graphs that can be obtained from a nowhere dense graph class by redefining the edges using a fixed first-order formula $\varphi(x, y)$ – for instance, the edge-complements of graphs from a nowhere dense class, or the squares of graphs from a nowhere dense class.

The developments in Structural graph theory – where many results concerning graph

¹We are only aware of the recent notion of graph functionality [AAL21] as a possible candidate; see Sec. 5.5.

classes of bounded treewidth are extended to the setting of classes of bounded clique-width – serves as an inspiration in attempts of lifting the results of Sparsity theory from the sparse (monotone) setting to the dense (hereditary) setting. It is expected that the fundamental notions of Sparsity theory – bounded expansion and nowhere denseness – should extend to more general tameness conditions for graph classes that are possibly dense, similarly as treewidth extends to clique-width. However, currently, even the most fundamental questions remain unanswered: What is the dense analogue of degeneracy? Of generalized coloring numbers? The pursuit after such notions has been a driving factor, and a major open problem in the area (see *Related Work* below). To date, no such combinatorial notions, with compelling evidence of their utility, have been proposed.

The recent and already very successful Twin-width theory [BKTW20, BGK⁺21a, BGK⁺21b, BGOdM⁺22, BGdMT22, BKRT22, BGTT22, BCK⁺22, GPT22, GPPT22] provides a robust tameness condition for graph classes that are not necessarily sparse, classes of *bounded twin-width*. Those include many studied sparse graph classes, such as classes that exclude a fixed minor, as well as dense graph classes, such as unit interval graphs, proper hereditary classes of permutation graphs, or posets of bounded width. However, some very simple classes of bounded expansion, such as the class of subcubic graphs (with maximum degree three), have unbounded twin-width. Thus, the scopes of Twin-width theory and Sparsity theory are incomparable. This motivates the quest of finding a unified theory that encompasses both Sparsity theory and Twin-width theory, and provides a common framework for studying the fundamental notions of the two theories.

Both Sparsity theory and Twin-width theory have found multiple combinatorial and algorithmic applications, and the same is expected of a theory unifying the two. As a concrete application, and a motivation for our pursuit, the sought theory is expected ultimately to resolve one of the central open problems in structural and algorithmic graph theory – of characterizing those hereditary graph classes for which the model-checking problem is fixed-parameter tractable [Gro07, Sec. 8.2] [GKS17b, Sec. 9].

It is conjectured² (see [war16, GH0⁺20]) that first-order model-checking is fixed-parameter tractable on a hereditary graph class \mathcal{C} if and only if \mathcal{C} is *monadically dependent* (also called *monadically NIP*). This notion, formulated in logical terms, originates in model theory, and was introduced by Shelah [She86] (see also Braunfeld and Laskowski [BL21]) in his momentous classification program of logical theories. Intuitively, a hereditary class \mathcal{C} is monadically dependent if for any fixed first-order formula $\varphi(x, y)$, there is some graph H that cannot be represented in any graph $G \in \mathcal{C}$, using the formula $\varphi(x, y)$ to define the edges of H in G .

Conjecture 1.1. *Let \mathcal{C} be a hereditary class of graphs. Then the model-checking problem for first-order logic is fixed-parameter tractable on \mathcal{C} if and only if \mathcal{C} is monadically dependent.*

Monadically dependent graph classes include all the graph classes mentioned above, and are considered (see [AA14, BL21, NdMP⁺21, GPT22]) to be the dense counterpart of nowhere dense classes, as expressed e.g. by Conjecture 1.1. For instance, nowhere dense classes are exactly those monadically dependent classes that exclude some biclique as a subgraph. However, to date, no *combinatorial* characterization of monadically dependent graph classes – akin to the multiple characterizations of nowhere dense classes – is known. As a consequence, in general, monadically dependent classes are currently beyond the reach of algorithmic methods.

Contribution We propose a new family of graph parameters, called *flip-width* of radius r , for $r \in \mathbb{N} \cup \{\infty\}$, that are based on new *flipper games*. Those games are similar to the Cops and Robber game considered by Seymour and Thomas to characterize treewidth, in their classic

²The conjecture has been circulating in the community since around 2016. As far as we know, it has first been stated explicitly during the open problem session of [war16]. There, dependent (or NIP) classes were considered instead of monadically dependent classes, but those two notions coincide for hereditary classes, by a result of Braunfeld and Laskowski [BL22].

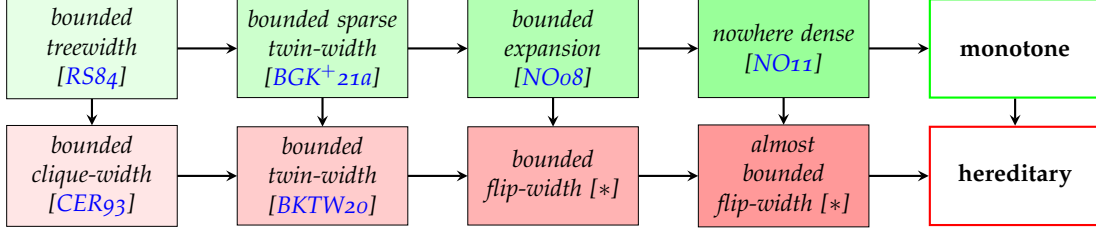


Figure 1: Properties of graph classes, and implications among them. Each property in the lower row, restricted to monotone graph classes, yields the property above it. The properties marked [*] are introduced in this paper. In this figure, *almost bounded flip-width* could be replaced by *monadically dependent* [She86]; we conjecture that those properties are identical.

paper [ST93]. Variants of our game can be used to characterize all the aforementioned parameters: treewidth, degeneracy, and generalized coloring numbers used in the context of sparse classes, as well as clique-width and twin-width in the context of dense graph classes. More importantly, they provide generalizations of degeneracy, and of generalized coloring numbers, to the setting of graphs that are not necessarily sparse, and offer a compelling dense counterpart of classes of bounded expansion – called classes of bounded *flip-width* – and of nowhere dense classes – called classes of *almost bounded flip-width*. Classes of bounded flip-width include classes of bounded expansion as well as classes of bounded twin-width, and provide a unified framework for understanding those fundamental notions (see Figure 1.) We conjecture, and confirm in many special cases, that classes of *almost bounded flip-width* coincide with monadically dependent classes. If true, this would give a combinatorial, quantitative characterization of monadic dependence, analogous to the characterization of nowhere dense classes in terms of generalized coloring numbers. Moreover, we verify that classes of almost bounded flip-width include most known examples of hereditary graph classes that are known to have a fixed-parameter tractable model-checking problem (we could not confirm this only for classes with structurally bounded local cliquewidth).

On a high level, the main contribution of this paper is to lay new foundations of a theory of structurally tractable graph classes, extending Sparsity theory to the dense setting, and unifying it with Twin-width theory. In this paper, we define the fundamental notions of our theory, and provide overwhelming evidence that they are the sought generalizations of the fundamental notions of Sparsity theory to the dense setting. This evidence is provided by multiple results, demonstrating that various central concepts of structural graph theory can be uniformly explained in terms of our notions. In the next section, we present an overview of our main results.

2 Overview

In this section, we give a high-level overview of the main results of this paper. These results are discussed systematically in greater detail, with many illuminating examples and proof sketches, in the following sections. Many of the statements recounted in this section are simplified versions of the more precise statements given in the next sections. The full statement of each result is referenced in parenthesis. All relevant notions are defined in Section 3.

Cop-width Our starting point is the – apparently new – observation that classes with bounded expansion can be characterized in terms of a variant of the Cops and Robber game considered by Seymour and Thomas [ST93]. Recall that in this game, k cops and a robber are occupying the vertices of a graph. In each round, some of the cops move to new positions in helicopters – that is, not necessarily along edges in the graph – whereas the robber may traverse any path in the graph which avoids the vertices occupied by the cops that remain on

ground. The minimal number k of cops needed to capture the robber is equal to one plus the treewidth of the graph.

In Section 4 we consider a variant of the Cops and Robber game, in which the robber runs at speed r , for some fixed $r \in \mathbb{N} \cup \{\infty\}$: they may traverse a path of length at most r that does not run through a cop. We call the parameter r the *radius* of the game, while the number of cops is called the *width* of the game. The *radius- r cop-width* of G , denoted $\text{copwidth}_r(G)$, is the least number k such that k cops win the game with radius r .

Thus, we obtain a family of graph parameters, one for each $r \in \mathbb{N} \cup \{\infty\}$. As we observe, variants of the above game characterize the graph parameters mentioned earlier: treewidth (for $r = \infty$), degeneracy (for $r = 1$), and generalized coloring numbers (for $1 \leq r < \infty$). Namely, the result of Seymour and Thomas can be phrased as follows.

Theorem 2.1 ([ST93]). *For every graph G ,*

$$\text{copwidth}_\infty(G) = \text{treewidth}(G) + 1.$$

On the other extreme, for radius $r = 1$, we have the following:

Theorem 2.2 (Thm. 4.4). *For every graph G ,*

$$\text{copwidth}_1(G) = \text{degeneracy}(G) + 1.$$

For higher radii, we prove a correspondence of the cop-width parameters with generalized coloring numbers. Recall that a graph class has bounded expansion if and only if each generalized coloring number is bounded by a constant on all graphs from the class.

Theorem 2.3 (Cor. 4.6). *A graph class \mathcal{C} has bounded expansion if and only if $\text{copwidth}_r(\mathcal{C}) < \infty$ for every fixed $r \in \mathbb{N}$.*

Here and later on, for a graph parameter f and graph class \mathcal{C} , by $f(\mathcal{C})$ we denote the supremum of $f(G)$, for $G \in \mathcal{C}$. In Theorem 2.3, we show that a witness to having degeneracy d – an ordering of the vertices such that every vertex has at most d vertices before it – yields a winning strategy for the Cops to win the game of width $d + 1$ and radius 1, in which the Cops move to the current position of the robber and all neighbors before it, thus forcing the robber to move upwards in the order. Conversely, a witness to having degeneracy larger than d – a set U of vertices such that every vertex in U has more than d neighbors in U – yields a winning strategy for the robber, allowing him to always remain in the set U . Similarly, to prove Theorem 2.3, we consider the Cops and Robber game for higher radii $1 \leq r < \infty$, and rely on central duality theorems of Sparsity theory, which describe combinatorial obstructions to having a small *weak coloring number*, in terms of a *dense shallow minor* [NO08]. We show that a witness to having a small weak coloring number yields a winning strategy for the Cops, while a dense shallow minor yields a winning strategy for the Robber in the Cops and Robber game of a fixed, finite radius. Those observations are analogues to the central duality result of Seymour and Thomas [ST93], which shows that a witness to having small treewidth yields a winning strategy for the Cops, while the dual obstruction to small treewidth, called a *haven* or *bramble*, yields a winning strategy for the Robber in the Cops and Robber game of infinite radius.

Parallel to Theorem 2.3, we obtain a characterization of nowhere dense classes:

Theorem 2.4 (Cor. 4.7). *A hereditary graph class \mathcal{C} is nowhere dense if and only if for every fixed radius $r \in \mathbb{N}$ and $\varepsilon > 0$, and for all n -vertex graphs $G \in \mathcal{C}$, we have that $\text{copwidth}_r(G) \leq O(n^\varepsilon)$.*

As is made apparent by the results above (and discussed in Section 4) the cop-width parameters capture a substantial part of the fundamental notions of Sparsity theory. However, being monotone under edge removals, they are only suited to the study of sparse graphs.

Flip-width In Section 5, we introduce a variant of the Cops and Robber game that is suited for dense graphs, dubbed the *flipper game*. This is similar to the recent development [GHN⁺12, GMM⁺23], where in some contexts, it is shown that vertex removals in sparse graphs correspond to *flips* in dense graphs, and a different variant of the flipper game is considered (see *Related work* below).

Flipping a pair of sets $X, Y \subseteq V(G)$ in a graph G results in the graph G' obtained from G by inverting the adjacency of every pair (x, y) of vertices with $x \in X$ and $y \in Y$: every such pair that is adjacent in G becomes non-adjacent in G' , and vice-versa. A k -flip of a graph G is obtained by partitioning $V(G)$ into k parts, and then performing flips between some pairs X, Y of parts of the partition (possibly with $X = Y$).

To motivate the flipper game, observe that in the Cops and Robber game of width k , we may think of the Cops as *isolating* at most k vertices, instead of placing the Cops on those vertices, where isolating a vertex amounts to (temporarily) removing all edges that are incident to it. Isolating a single vertex v in a graph G can be achieved by performing a 3-flip of G : partition the vertices into $\{v\}$, the set of neighbors of v , and the remaining vertices, and flip the first two sets. Similarly, a graph G obtained by isolating k vertices v_1, \dots, v_k is a $(k + 2^k)$ -flip of G , by taking the common refinement of the partitions used for isolating each vertex v_i separately. In the flipper game defined below, we will allow the Cops to perform flips, significantly extending their power comparing to the Cops and Robber game.

The *flipper game* of radius r and width k on a graph G proceeds as follows. Initially, the robber picks any vertex v_0 of G . In the i th round, the cops announce a k -flip G_i of the original graph G . The robber can move from their previous position v_{i-1} to a new position v_i , by traversing a path of length at most r in the *previous* k -flip G_{i-1} of G (where $G_0 = G$). The game is won by the cops if robbers new position v_i is isolated in the announced k -flip G_i .

The *flip-width* of radius r of a graph G , denoted $\text{fw}_r(G)$, is the smallest number k for which the cops have a winning strategy in the game described above. We say that a graph class \mathcal{C} has *bounded flip-width* if $\text{fw}_r(\mathcal{C}) < \infty$ for all $r \in \mathbb{N}$.

As we argue below, the flip-width parameters are the sought dense analogues of the generalized coloring numbers, while classes of bounded flip-width are the dense analogues of classes of bounded expansion. This is supported by vast evidence, which we detail below. Additionally, we show that variants of the flipper game can be used to uniformly characterize several among the most important graph parameters studied in structural graph theory: degeneracy, treewidth, and generalized coloring numbers, as well as clique-width and twin-width. For the first time, this provides a common perspective on parameters such as degeneracy and twin-width, or generalized coloring numbers and clique-width – all of which can be seen as representations of winning strategies in the flipper game. This common perspective also allows to view in a unified way various key combinatorial notions representing obstructions – namely *havens* or *brambles* studied in the context of treewidth [ST93], *well-linked sets* studied in the context of clique-width [iOs06], *dense shallow minors* studied in the context of bounded expansion classes [NO08], or *rich divisions* studied in the context of twin-width [BGdM⁺22]. Those notions lie at the core of duality theorems for the related parameters, and also have important algorithmic applications. As we will see, those notions can be seen as representations of winning strategies for the robber in the flipper game.

We now give a summary of our results concerning flip-width.

Relating flip-width to other notions We start with comparing classes of bounded flip-width with the fundamental notions of Sparsity theory and Twin-width theory.

First, regarding the sparse graph parameters – degeneracy, treewidth, and generalized coloring numbers – we show that those correspond to the parameters fw_1 , fw_∞ and fw_r , for $1 \leq r < \infty$, when considering *weakly sparse* graph classes. A graph class \mathcal{C} is *weakly sparse* if it excludes some fixed biclique as a subgraph.

Theorem 2.5. *Let \mathcal{C} be a graph class. Then:*

1. (Thm. 6.1) \mathcal{C} has bounded degeneracy if and only if \mathcal{C} is weakly sparse and $\text{fw}_1(\mathcal{C}) < \infty$,
2. (Cor. 6.2) \mathcal{C} has bounded treewidth if and only if \mathcal{C} is weakly sparse and $\text{fw}_\infty(\mathcal{C}) < \infty$,
3. (Thm. 6.3) \mathcal{C} has bounded expansion if and only if \mathcal{C} is weakly sparse and \mathcal{C} has bounded flip-width.

This shows that the flipper game of radius r is a sensible generalization of the Cops and Robber game of radius r discussed earlier, since we already know that the above notions (degeneracy, treewidth, bounded expansion) can be characterized in terms of the latter game. This fact only gives the rightwards implications in the above statements. To prove the converse implications, we rely on a refined characterization of the obstructions to small degeneracy/treewidth/expansion which are suitable in the setting of hereditary graph classes, as opposed to the obstructions in terms of forbidden subgraphs that were sufficient in the case of cop-width. Such obstructions can be exploited by the robber to win the flipper game, for $K_{t,t}$ -free graphs.

We then move to studying the flip-width parameters in graph classes that are not necessarily sparse. As a first case study, we verify that the flipper game with radius ∞ corresponds to clique-width, similarly as the Cops and Robber game corresponds to treewidth. Namely, we prove:

Theorem 2.6 (Thm. 5.17). *A class \mathcal{C} of graphs has bounded clique-width if and only if $\text{fw}_\infty(\mathcal{C}) < \infty$.*

This gives the first, to our knowledge, game characterization of classes of bounded clique-width, analogous to the classic characterization of treewidth by Seymour and Thomas. To prove the rightwards implication, we view a clique-width expression (used to construct a graph of bounded clique-width) as a description of a winning strategy for the cops in the radius- ∞ flipper game. For the converse implication, again we show that obstructions for bounded clique-width – called *well-linked sets* – yield winning strategies for the robber.

As clique-width is the dense analogue of treewidth, Theorem 2.6 is another indication that our flipper game is an adequate generalization of the Cops and Robber game for the study dense graph classes. In Section 7 we provide further evidence of this, by demonstrating that another crucial graph parameter studied in the context of dense graphs, twin-width, can be characterized in terms of flip-width. First, we show that classes of bounded flip-width include all classes of bounded twin-width. Those include, for example, the class of unit interval graphs, or every proper hereditary class of permutation graphs.

Theorem 2.7 (Thm. 7.1). *Every class of bounded twin-width has bounded flip-width.*

The proof of this result views *contraction sequences*, which are recursive decompositions of graphs of bounded twin-width, as strategies for the cop player in the flipper game.

By the results above, classes of bounded flip-width include all classes with bounded expansion as well as classes of bounded twin-width. As those two notions are incomparable, classes of bounded flip-width strictly extend each of them. In particular, the converse to Theorem 2.7 fails.

To characterize twin-width using flip-width, we naturally extend the flip-width parameters to arbitrary structures equipped with one or more binary relation, such as *ordered graphs* – graphs equipped with a total order. We prove that for classes of ordered graphs, bounded flip-width and bounded twin-width coincide:

Theorem 2.8 (Thm. 7.3). *A class of ordered graphs has bounded twin-width if and only if it has bounded flip-width.*

The rightwards implication follows from Theorem 2.7. To prove the converse, we view the combinatorial obstructions to small twin-width, called *rich divisions* [BGdM⁺22], as descriptions of winning strategies for the robber in the flipper game.

It is known [BGdM⁺22] that a class of graphs has bounded twin-width if and only if it can be obtained from a class of ordered graphs of bounded twin-width, by forgetting the

order. Thus, classes of bounded twin-width are exactly classes of ordered graphs of bounded flip-width, with the order forgotten.

Reassuringly, variants of our flipper game capture degeneracy, treewidth, bounded expansion, clique-width, and twin-width, all of which are of central importance in structural and algorithmic graph theory. Moreover, structural results can be employed to describe winning strategies for the cops, while the dual combinatorial obstructions can be used to obtain strategies for the robber.

Closure properties Another main contribution of this paper are results showing that classes of bounded flip-width enjoy good closure properties, in particular, closure under first-order interpretations (or transductions, see Theorem 8.2). More precisely, for a first-order formula $\varphi(x, y)$ and graph G , define the graph $\varphi(G)$ with vertices $V(G)$ and edges uv with $u \neq v$, such that $\varphi(u, v) \vee \varphi(v, u)$ holds in G . For a graph class \mathcal{C} , let $\varphi(\mathcal{C}) := \{\varphi(G) \mid G \in \mathcal{C}\}$. For example, if $\varphi(x, y)$ is the formula $\exists z. E(x, z) \wedge E(z, y)$ (where E denotes adjacency), then $\varphi(G)$ is the square of G . In Section 8, we prove the following.

Theorem 2.9 (Thm. 8.2). *Let $\varphi(x, y)$ be a first-order formula and \mathcal{C} be a graph class. If \mathcal{C} has bounded flip-width then $\varphi(\mathcal{C})$ has bounded flip-width.*

This generalizes a prior analogous result for classes of bounded twin-width [BKTW20] and a result concerning classes of bounded expansion. Also, this result provides further examples of classes of bounded flip-width, for instance, classes of *structurally bounded expansion* [GKN⁺20], that is, classes of the form $\varphi(\mathcal{C})$, where \mathcal{C} has bounded expansion and $\varphi(x, y)$ is a first-order formula. Moreover, Theorem 2.9 implies that classes of bounded flip-width are monadically dependent, a notion that conjecturally delimits the tractability frontier for the model checking problem (see Conjecture 1.1).

The proof of Theorem 2.9 uses a tool from finite model theory – namely, locality of first-order logic – that has so far been used mostly in the context of sparse graph classes, but more recently has also been successfully applied in the context of dense graph classes [BKTW20, BDG⁺22, DMS].

Approximation Determining the exact flip-width of radius r of a given graph G seems computationally difficult (the naive approach is exponential in the size of G). As our next contribution, in Section 9, we obtain a slicewise polynomial (XP) algorithm that approximates the flip-width of a given graph G , which runs in time polynomial in the size of G , when the flip-width is considered fixed.

Theorem 2.10 (Thm. 9.7). *There is a constant $C > 0$ and an algorithm that inputs an n -vertex graph G and numbers $r, k \in \mathbb{N}$, runs in time $n^{O(k)} \cdot 2^{O(2^k)}$, and either concludes that $\text{fw}_r(G) \leq 2^k$, or concludes that $\text{fw}_{5r}(G) \geq C \cdot k^{1/3}$.*

Note that no XP algorithm approximating twin-width is known. This suggests that flip-width might be easier to approximate than twin-width, even if it defines a more general notion.

To prove Theorem 2.10, we define a variant of the flipper game, called the *definable flipper game*, in which the partitions used by the Cops for defining flips are *definable*: they partition the vertex set according to their adjacency in a set of vertices of bounded size. The advantage of this game variant is that it has polynomially many (in terms of the graph size) possible configurations, as opposed to the flipper game, which has exponentially many configurations. Although not every k -flip of a graph G can be defined by using a partition definable using a vertex set of bounded size, we prove that nevertheless the graph parameters defined by the definable flipper game can still be bounded in terms of the flip-width parameters. The proof uses tools from Vapnik-Chervonenkis theory.

Almost bounded flip-width As our final contribution, in Section 10, we introduce and study classes of almost bounded flip-width, as a candidate dense counterpart of classes of nowhere dense classes. A graph class \mathcal{C} has *almost bounded flip-width* if for every $\varepsilon > 0$ and $r \in \mathbb{N}$, we have $\text{fw}_r(G) \leq O(n^\varepsilon)$ for every n -vertex graph G which is an induced subgraph of a graph in \mathcal{C} . Thus, every class of bounded flip-width has almost bounded flip-width, but the converse does not hold.

We provide substantial evidence that classes of almost bounded flip-width coincide with monadically dependent classes (cf. Conjecture 1.1), and also that the model checking problem is fixed-parameter tractable for those classes. This is corroborated by the following results, which examine this notion in special cases.

First, in the setting of weakly sparse classes, we prove:

Theorem 2.11 (Thm. 10.9). *Let \mathcal{C} be a weakly sparse graph class. The following conditions are equivalent:*

1. \mathcal{C} has almost bounded flip-width,
2. \mathcal{C} is nowhere dense,
3. \mathcal{C} is monadically dependent.

In the setting of classes of ordered graphs, we prove an analogous statement, with nowhere denseness replaced by bounded twin-width. Moreover, in this case, almost bounded flip-width collapses to bounded flip-width. This collapse is, ultimately, a consequence of the Marcus-Tardos theorem/Stanley-Wilf conjecture [MT04] from enumerative combinatorics.

Theorem 2.12 (Thm. 10.17). *Let \mathcal{C} be a class of ordered graphs. The following conditions are equivalent:*

1. \mathcal{C} has almost bounded flip-width,
2. \mathcal{C} has bounded flip-width,
3. \mathcal{C} has bounded twin-width,
4. \mathcal{C} is monadically dependent.

In particular, in the two settings considered in the theorems above, the model checking problem is fixed-parameter tractable, by the results of [GKS17b] and [BGOdM⁺22], respectively.

We now move to a more general setting than that of weakly sparse classes, namely of *stable* classes. A class \mathcal{C} is *edge-stable* if there is a number k such no graph $G \in \mathcal{C}$ contains vertices $a_1, \dots, a_k, b_1, \dots, b_k$ such that $E(a_i, b_j) \iff i < j$ for $i, j \in \{1, \dots, k\}$ (see Fig. 2). Edge-stable, monadically dependent graph classes coincide with *monadically stable* classes [BS85, NdMP⁺21], an important subfamily of monadically dependent classes. Examples include all *structurally nowhere dense classes*, that is, classes of the form $\varphi(\mathcal{C})$, for some first-order formula $\varphi(x, y)$ and nowhere dense class \mathcal{C} . It is conjectured [NdMP⁺21, Conjecture 6.1] that all monadically stable graph classes are structurally nowhere dense. By a recent result [DMS], the model-checking problem is fixed-parameter tractable for all structurally nowhere dense classes, thus significantly extending the result of [GKS17b] concerning model-checking on nowhere dense classes.

We prove the following.

Theorem 2.13 (Thm. 10.12). *Every structurally nowhere dense class has almost bounded flip-width.*

In the proof, we use the recent description of structurally nowhere dense classes [DGK⁺22a], which provides certain treelike decompositions for such classes, that can be used to produce a winning strategy for the cops. As a weak converse, we prove:

Theorem 2.14 (Thm. 10.15). *Every edge-stable class of almost bounded flip-width is monadically stable.*

The proof relies on a recent characterization of obstructions to monadic stability [GMM⁺23], which can be used to produce a winning strategy for the robber.

In particular, if all monadically stable classes are structurally nowhere dense as conjectured in [NdMP⁺21], the two results imply that among edge-stable graph classes, almost bounded flip-width coincides with monadic stability (and thus with monadic dependence). Furthermore, by the result of [DMS], the conjecture would imply fixed-parameter tractability for all edge-stable classes of almost bounded flip-width.

Those results provide substantial evidence indicating that classes of almost bounded flip-width form the dense counterpart of nowhere dense classes, that they coincide with monadically dependent classes, and that they admit fixed-parameter tractable model-checking. In fact almost all³ hereditary graph classes having fixed-parameter tractable model-checking that we are aware of [GKS17b, BKTW20, DMS, BCK⁺22], have almost bounded flip-width. Based on this, we believe that classes of almost bounded flip-width will play a key role in combinatorial and algorithmic approaches to the analysis of monadically dependent classes.

We supplement our results with a discussion, in Section 11, providing evidence for various stated conjectures, and outline a potential approach towards an algorithmic and combinatorial understanding of the classes introduced in this paper.

Related work Our flipper game is inspired by the paper [GMM⁺23]. There, another game based on performing flips is introduced, and is also called *flipper game*. We consider both games to be variants of a broader family of a flipper games, similarly as there are many variants of the Cops and Robber game. As far as we know, only two variants of the flipper game have been considered so far, but we anticipate that more variants might emerge. To distinguish the two variants, we can call the variant introduced in this paper the *open flipper game*, and the variant from [GMM⁺23] the *confined flipper game of bounded duration*. The latter game is an analogue of the *splitter game*, which characterizes nowhere dense classes [GKS17b]. Similarly, the flipper game from [GMM⁺23] characterizes monadically stable graph classes, a strict subset of monadically dependent classes. It differs from the open flipper game in two major ways. Firstly, the duration of the game is bounded, whereas in our variant, it is unbounded. Furthermore, in each round of the confined flipper game, the fugitive is confined to the radius- r ball around his present position in the current flip of the graph in all future rounds, not just the next round as in the open game. The confined flipper game of bounded duration is used in [GMM⁺23] to characterize monadically stable classes, which are incomparable with classes of bounded flip-width, and are likely to be strictly contained in classes of almost bounded flip-width (see Section 10).

There are multiple variations of the Cops and Robber game, and some of them are similar to the one considered above for sparse graphs. In some of those variants, the robber has bounded speed, as in our games. See [FKL12, FGK⁺10, AM15] and [FT08] for a survey. The cop-width parameters that we study for sparse graphs are closely related to the parameters studied in [RT08] and [LPPT20], which are defined in terms of a similar game, as we discuss in Section 4 and in Appendix A.1.

Some attempts at defining graph classes that are dense analogues of classes of bounded expansion were made in [KPS17, GKN⁺18, NORS21, NdMP⁺21, GPT22, BNdMS22b]. In [KPS17, NORS21, NdMP⁺21], the property of having low rankwidth covers is proposed as the dense analogue of bounded expansion. However, this notion does not include⁴ classes of bounded twin-width, and it is not known whether this notion is closed under transductions. It has been proved recently that those classes are monadically dependent [BNdMS22a]. In [GPT22, Section 4] and [BNdMS22b, Section 8], an attempt at formalizing an abstract notion of a dense analogue of classes with bounded expansion was made, but those papers don't propose any workable combinatorial definition.

³One exception are classes with structurally bounded local clique-width [BDG⁺22], which are tractable and monadically dependent, but we do not know whether they have almost bounded flip-width.

⁴An example was provided by Jakub Gajarský (private communication)

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Organization The organization of this paper is as follows. Results marked with (*) are proved in the appendix.

Section 3 introduces some standard notation, and recalls some basic notions and results, e.g. from Sparsity theory and Vapnik-Chervonenkis theory.

In Section 4 we introduce the cop-width parameters, and demonstrate they characterize degeneracy, treewidth, and generalized coloring numbers. Using those parameters, we characterize classes of bounded expansion and nowhere dense classes.

In Section 5 we introduce our two main notions: the flip-width parameters, and classes of bounded flip-width. We give many examples, and obtain some combinatorial properties. We show that radius- ∞ flip-width is equivalent to clique-width.

In Section 6 we study the behavior of the flip-width parameters in the case of $K_{t,t}$ -free graphs. We show that radius-one flip-width corresponds to degeneracy, and that for higher radii, the flip-width parameters correspond to the generalized coloring numbers. Consequently, a weakly sparse graph class has bounded flip-width if and only if it has bounded expansion.

In Section 7 we study the relationship between twin-width and flip-width. We prove that every class of bounded twin-width has bounded flip-width. We also prove that for classes of ordered graphs, bounded flip-width and bounded twin-width are equivalent.

In Section 8 we prove that classes of bounded flip-width are preserved by first-order transductions.

In Section 9 we introduce a definable variant of flip-width, and prove its equivalence with flip-width. As a consequence, we get a slicewise polynomial (XP) approximation algorithm for the flip-width parameters.

In Section 10 we introduce our third main notion, classes of almost bounded flip-width. We prove that they contain all structurally nowhere dense classes, and study their relationship with monadically stable and monadically dependent classes.

Finally, in Section 11, we discuss possible directions of further research, conjectures and questions.

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3 Preliminaries

We introduce basic notation in Section 3.1. In Section 3.2 we recount the fundamental notions of Sparsity theory. In Section 3.3 we recall the notion of Vapnik-Chervonenkis dimension of a set system, of a graph, and of a binary relation. In Section 3.4 we recall basic notions from logic (structures, formulas), and a result characterizing nowhere dense classes in terms of the VC-dimension of certain set systems.

3.1 Notation

$\mathbb{N} = \{0, \dots\}$ denotes the set of nonnegative integers. For two sets A and B their symmetric difference is denoted $A \triangle B := (A - B) \cup (B - A)$. We write $O(n)$ (resp. $\Omega(n)$) to denote a value that is bounded from above (resp. from below) by $c \cdot n + d$, for some reals c, d with $c > 0$. Sometimes we write $O_p(n)$ (resp. $\Omega_p(n)$), where p is a list of parameters, to indicate that the constants c and d above depend on the parameters p . For a function $f: \mathbb{N} \rightarrow \mathbb{R}$, we write $o(f(n))$ to denote a function $g: \mathbb{N} \rightarrow \mathbb{R}$ with $\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = 0$.

Graphs are finite, undirected and without self-loops. The set of vertices of a graph G is denoted $V(G)$, and the set of edges of G is denoted $E(G)$. An edge joining u and v is denoted uv . In particular, $uv = vu$ and $u \neq v$ for all $uv \in E(G)$. We write $|G|$ for the number of vertices of G . For a vertex v of a graph G the (open) *neighborhood* of v in G is $N_G(v) := \{u \mid uv \in E(G)\}$, denoted $N(v)$ if G is understood from the context. The set of vertices at distance at most r from v in G is denoted $B_G^r(v)$.

A graph H is a *subgraph* of G if H is obtained by removing vertices and/or edges from G , and is an *induced subgraph* of G if H is obtained by removing vertices from G , alongside with the edges incident to them. The subgraph of G induced by a set of vertices $X \subseteq V(G)$ is the graph $G[X]$ with vertices X and edges $uv \in E(G)$ with $u, v \in X$. For $X, Y \subseteq V(G)$, the bipartite graph *semi-induced* by X and Y in a graph G has parts X and Y and edges xy such that $x \in X, y \in Y$ and $xy \in E(G)$. Note that X and Y need not be disjoint in G ; in this case, $G[X, Y]$ contains two copies of every vertex in $X \cap Y$. Two sets X, Y are *complete* in G if $G[X, Y]$ is the complete bipartite graph, and *anti-complete* in G if $G[X, Y]$ has no edges, and *homogeneous* if they are either complete or anti-complete. K_t denotes the complete graph on t vertices, and $K_{s,t}$ denotes the complete bipartite graph with parts of sizes s and t .

A *graph class* \mathcal{C} is a set of graphs. A class \mathcal{C} is *hereditary* if it is closed under taking induced subgraphs. The *hereditary closure* of a class \mathcal{C} is the class of all induced subgraphs of graphs from \mathcal{C} . \mathcal{C} is *weakly sparse* if there is some $t \in \mathbb{N}$ such that every $G \in \mathcal{C}$ excludes the biclique $K_{t,t}$ as a subgraph. A *graph parameter* is a function f assigning reals to graphs, which is invariant under graph isomorphism. For a graph class \mathcal{C} and graph parameter f , denote $f(\mathcal{C}) := \sup_{G \in \mathcal{C}} f(G)$, with $f(\mathcal{C}) = \infty$ if f is unbounded on \mathcal{C} . Say that f is *bounded in terms of* g if there is a function $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(G) \leq \alpha(g(G))$ for all graphs G . Two functions f and g are *functionally equivalent* if each of them is bounded in terms of the other.

A *k-colored graph* is a graph G equipped with a function assigning a color from $\{1, \dots, k\}$ to each vertex of G . A class $\hat{\mathcal{C}}$ of k -colored graphs is a *k-coloring* of a graph class \mathcal{C} , if for every colored graph in $\hat{\mathcal{C}}$, the underlying uncolored graph belongs to \mathcal{C} , and for every graph $G \in \mathcal{C}$, some k -coloring of G belongs to $\hat{\mathcal{C}}$.

3.2 Sparsity theory

We briefly recall the fundamental notions of Sparsity theory. See [NdM12] for more background.

A graph G is *d-degenerate* if there is a total order on the vertices of G such that every vertex has at most d neighbors before it in the order. The *degeneracy* of G is the least d such that G is d -degenerate.

An *exact r -subdivision* of a graph G is the graph obtained by replacing every edge of G by a path of length $r + 1$. If every edge is replaced by a path of length at most $r + 1$, the resulting graph is called a $\leq r$ -subdivision of G . For a graph G , let $\tilde{\nabla}_r(G)$ denote the maximum average degree, $2|E(H)|/|V(H)|$, of all graphs H whose $\leq r$ -subdivision is a subgraph of G (called *shallow topological minors of G at depth r*). Note that $\text{degeneracy}(G) \leq \tilde{\nabla}_0(G) \leq 2 \cdot \text{degeneracy}(G)$.

Definition 3.1. A graph class \mathcal{C} has bounded expansion if for every $r \geq 1$ we have $\tilde{\nabla}_r(\mathcal{C}) < \infty$.

Example 3.2. Classes of bounded expansion include many well-studied sparse classes: the class of planar graphs, every class of bounded maximum degree, classes of bounded tree-width, classes that exclude a fixed (topological) minor.

The weak coloring and admissibility numbers are two families of graph parameters that generalize the degeneracy number to higher radii $r \geq 1$, and are defined as follows. The *r -weak coloring number* of a graph G , denoted $\text{wcol}_r(G)$, is the smallest number k with the following property: There is a total order $<$ on the vertices of G such that for every vertex v , there are at most k vertices w with $w < v$ that are *weakly r -reachable* from v : there is a path of length at most r from v to w , in which w is the $<$ -smallest vertex. On the other hand, the *r -admissibility* of a graph G , denoted $\text{adm}_r(G)$, is the smallest number k with the following property: There is a total order $<$ on $V(G)$ such that for every $v \in V(G)$ one cannot find more than k paths of length at most r that start at v , end in some vertex $w < v$, and such that any two of the paths share only v as a common vertex.

Both parameters are of central importance in Sparsity theory. It is known that $\text{wcol}_r(G)$ is bounded in terms of $\text{adm}_r(G)$, and vice versa [Dvo13, Lemma 6]. More specifically, we have the following.

Fact 3.3. For all graphs G and $r \geq 1$ we have

$$\text{adm}_r(G) \leq \text{wcol}_r(G) \leq O(\text{adm}_r(G))^r. \quad (1)$$

The fundamental notion of Sparsity theory can be characterized using weak coloring numbers as follows:

Fact 3.4 ([Zhu09]). A class \mathcal{C} of graphs has bounded expansion if and only if $\text{wcol}_r(\mathcal{C}) < \infty$ for every $r \in \mathbb{N}$.

By (1), we could replace $\text{wcol}_r(\mathcal{C})$ by $\text{adm}_r(\mathcal{C})$ in this characterization.

We use the following inequalities.

Fact 3.5. For all graphs G and $r \geq 1$ we have

$$\tilde{\nabla}_{r-1}(G)/2 < \text{adm}_r(G) \leq 6(\tilde{\nabla}_{r-1}(G))^3. \quad (2)$$

Proof. The upper bound below is [GKR⁺18, Theorem 3.1]. We prove the lower bound. Suppose G contains an $\leq(r-1)$ -subdivision of a graph with average degree s . Then G contains an $\leq(r-1)$ -subdivision of a graph H with minimum degree larger than $s/2$. Let $U \subseteq V(G)$ consist of the principal vertices, corresponding to the vertices of H . Then for every vertex $u \in U$ there are more than $s/2$ paths of length r that start at u , end at vertices of U , and are vertex-disjoint apart from u . The set U witnesses that $\text{adm}_r(G) > s/2$. \square

We now move to nowhere dense classes.

Definition 3.6. A graph class \mathcal{C} is nowhere dense if for every $r \geq 1$ there is some $n \geq 1$ such that for all $G \in \mathcal{C}$, no $\leq r$ -subdivision of K_n is contained as a subgraph of $G \in \mathcal{C}$.

It is immediate that every class with bounded expansion is nowhere dense, and there exist nowhere dense classes which do not have bounded expansion.

One of the central results of Sparsity theory is the following characterization of nowhere dense classes (recall that $|G|$ is the number of vertices of G).

Fact 3.7. A hereditary graph class \mathcal{C} is nowhere dense if and only if for every $r \geq 1$ and $\varepsilon > 0$ there is a constant $n_{r,\varepsilon}$ such that $\text{wcol}_r(G) < |G|^\varepsilon$ for every $G \in \mathcal{C}$ with $|G| > n_{r,\varepsilon}$.

The condition in Fact 3.7 can be equivalently phrased as follows: for every $r \geq 1$ and $\varepsilon > 0$, $\text{wcol}_r(G) \leq O_{r,\varepsilon}(|G|^\varepsilon)$ for every $G \in \mathcal{C}$. This can be written more concisely as $\text{wcol}_r(G) \leq |G|^{o(1)}$.

This fundamental result opens the door for multiple algorithmic applications of nowhere denseness, thanks to the existence of efficient algorithms approximating weak coloring numbers. In particular, the model-checking result [GKS17b] relies on Fact 3.7.

3.3 Vapnik-Chervonenkis dimension

An important parameter measuring the complexity of graphs, and more generally, of set systems, is the *Vapnik-Chervonenkis dimension*, or *VC-dimension*.

A *set system* is a pair (X, \mathcal{F}) with $\mathcal{F} \subseteq 2^X$. Its *VC-dimension* is the maximal size of a subset $Y \subseteq X$ such that $\{Y \cap F \mid F \in \mathcal{F}\} = 2^Y$. We recall the fundamental Sauer-Shelah-Perles lemma [Sau72, She72].

Lemma 3.8 (Sauer-Shelah-Perles lemma). *Let (X, \mathcal{F}) be a set system of VC-dimension d . Then $|\mathcal{F}| \leq O(|X|^d)$.*

The VC-dimension of a graph G , denoted $\text{VCdim}(G)$, is defined as the VC-dimension of the set system $(V(G), \{N(v) \mid v \in V(G)\})$. More explicitly, $\text{VCdim}(G)$ is the maximal size of a subset $X \subseteq V(G)$ such that $\{N(v) \cap X \mid v \in V(G)\} = 2^X$.

For a binary relation $R \subseteq V \times W$, and elements $a \in V$ and $b \in W$, denote $\vec{R}(a) := \{w \in W \mid (a, w) \in R\}$ and $\vec{R}(b) := \{v \in V \mid (v, b) \in R\}$. The *VC-dimension* of R is the maximum of the VC-dimensions of the two set systems

$$(V, \{\vec{R}(a) \mid a \in V\}) \quad \text{and} \quad (W, \{\vec{R}(b) \mid b \in W\}).$$

3.4 Logic

In this paper, we only consider *binary* signatures, that is signatures Σ consisting of unary relation symbols, binary relation symbols, and unary function symbols. Fix a binary signature Σ . A Σ -structure B consists of a set of vertices $V(B)$, and is equipped with interpretations for each of the symbols in Σ , as unary relations, binary relations, and unary functions on $V(B)$, respectively. The number of vertices of B is denoted $|B|$. A graph G is seen as a structure over the signature Σ consisting of one binary relation E , which is interpreted as adjacency in G .

Terms are defined inductively, as either a variable symbol, or a function symbol applied to a term, e.g. x and $f(g(y))$ are terms if f, g are function symbols and x, y are a variable symbols. A *quantifier-free* formula over the signature Σ is a boolean combination of *atomic formulas* of the form $U(t(x))$, or $t(x) = t'(y)$, or $R(t(x), t'(y))$, where $U \in \Sigma$ is a unary relation, $R \in \Sigma$ is a binary relation, and $t(x)$ and $t'(y)$ are terms.

A *first-order formula* is built inductively: every atomic formula is a first-order formula, and if φ, ψ are first-order formulas and x is a variable, then so are $\varphi \vee \psi$, $\neg\varphi$, and $\exists x.\varphi$. The notation $\varphi \wedge \psi$, $\varphi \rightarrow \psi$, and $\forall x.\varphi$ is used as syntactic sugar. The *quantifier rank* of a formula φ is the maximal nesting of quantifiers in it.

We may write $\varphi(x_1, \dots, x_k)$ to indicate that the free-variables of φ are contained in $\{x_1, \dots, x_k\}$. For a formula $\varphi(x_1, \dots, x_k)$, structure A and elements a_1, \dots, a_k of A , we write $A \models \varphi(a_1, \dots, a_k)$ to denote that the valuation of the variables $x_1 \mapsto a_1, \dots, x_k \mapsto a_k$ satisfies the formula φ in A .

Say that a Σ -formula $\varphi(x, y)$ is *symmetric* if $G \models \varphi(u, v) \leftrightarrow \varphi(v, u)$ holds for every Σ -structure G and elements u, v of G .

The *Gaifman graph* of a Σ -structure B is the graph with vertices $V(B)$ and edges uv with $u \neq v$ such that $(u, v) \in R$ or $(v, u) \in R$ for some binary relation $R \in \Sigma$, or $f(u) = v$ or $f(v) = u$ for some unary function $f \in \Sigma$.

Dependence Let $\varphi(\bar{x}; \bar{y})$ be a first-order formula, whose set of free variables is partitioned into two disjoint sets \bar{x} and \bar{y} . For a structure H define the binary relation $R_H^\varphi \subseteq V(H)^{\bar{x}} \times V(H)^{\bar{y}}$ as

$$R_H^\varphi = \{(\bar{u}, \bar{v}) \in V(H)^{\bar{x}} \times V(H)^{\bar{y}} \mid H \models \varphi(\bar{u}; \bar{v})\}.$$

Say that a class \mathcal{C} of Σ -structures is *dependent*, or *NIP* [SS71, Adlo8] if for every first-order formula $\varphi(\bar{x}; \bar{y})$ there is some $k_\varphi \geq 1$ such that for every $H \in \mathcal{C}$ the binary relation R_H^φ has VC-dimension at most k_φ . This is equivalent to saying that for every first-order formula $\varphi(\bar{x}; \bar{y})$, there is some bipartite graph G_φ , such that for all $H \in \mathcal{C}$, the bipartite graph with parts $V(H)^{\bar{x}}$, $V(H)^{\bar{y}}$ and edges (\bar{u}, \bar{v}) such that $H \models \varphi(\bar{u}; \bar{v})$, does not contain G_φ as an induced subgraph.

The following fact is due to Podewski and Ziegler [PZ78] (see also [AA14] and [PST18]).

Fact 3.9. *Every nowhere dense graph class \mathcal{C} is dependent. Conversely, every monotone, dependent graph class \mathcal{C} is nowhere dense.*

Hereditary, dependent graph classes are much more general than nowhere dense classes. The study of those classes is the main motivation of this paper. A closely related notion, of a *monadically dependent* class, is discussed in Section 10. Monadically dependent classes are dependent, but in general, the converse implication does not hold. However, as shown by Braunfeld and Laskowski [BL22], for hereditary classes, the two notions coincide.

4 Cop-width

We start with defining and analyzing the Cops and Robber game with finite radius, and the related cop-width parameters. To define the game, we invoke the original description of the Cops and Robber game by Seymour and Thomas [ST93]: “The robber stands on a vertex of the graph, and can at any time run at great speed to any other vertex along a path of the graph. He is not permitted to run through a cop, however. There are k cops, each of whom at any time either stands on a vertex or is in a helicopter (that is, is temporarily removed from the game). The objective of the player controlling the movement of the cops is to land a cop via helicopters on the vertex occupied by the robber, and the robber’s objective is to elude capture. (The point of the helicopters is that cops are not constrained to move along paths of the graph – they move from vertex to vertex arbitrarily.) The robber can see the helicopter approaching its landing spot and may run to a new vertex before the helicopter actually lands.”

Seymour and Thomas proved that the least number of cops needed to catch a robber on a graph G is equal to one plus the treewidth of G . To this end, they proved a *min-max theorem*: either the cops have a winning strategy of a particularly simple, *monotone* form, which can be described by a tree decomposition of the graph, or otherwise, the robber has a winning strategy of a particularly simple form, called a *haven*.

In our variant of the game the robber runs at speed r , for some fixed $r \in \mathbb{N} \cup \{\infty\}$. That is, in each round, after the cops have taken off in their helicopters to their new positions, which are known to the robber, and before the helicopters have landed, the robber may traverse a path of length at most r that does not run through a cop that remains on the ground (he may also stay put). We call this game the *Cops and Robber game* with radius r and width k , if there are k cops, and the robber can run at speed r .

Definition 4.1. *The radius- r cop-width of G , denoted $\text{copwidth}_r(G)$, is the least number k such that the cops win the Cops and Robber game with radius r and width k .*

Note that there is a graph parameter called the *cop-number* of a graph [AF84], which is equal to the number of cops needed to catch the robber in a game where the cops and the robber move at speed one in each turn. It could seem that $\text{copwidth}_1(G)$ is upper bounded by the cop-number of G . However, there is a crucial difference with our notion: the cops do not announce their moves in advance. So for instance, every graph with a universal vertex (a vertex adjacent to all other vertices) has cop-number equal to one (such graphs are called *cop-win graphs*). On the other hand, $\text{copwidth}_1(G)$ can be arbitrarily large on such graphs. Indeed,

it is easy to see that for each $r \in \mathbb{N} \cup \{\infty\}$, the parameter $\text{copwidth}_r(G)$ is monotone with respect to the subgraph relation: if H is a subgraph of G then $\text{copwidth}_r(H) \leq \text{copwidth}_r(G)$.

Example 4.2. Let G be a graph with maximum degree d . Then $\text{copwidth}_r(G) < d^{r+1}$, for all $r \in \mathbb{N}$. To see this, consider the following strategy for the cops: if the robber is initially placed on a vertex v , direct the cops to all (less than d^{r+1}) vertices that are at distance at most r from v . Before the cops land at those locations, the robber traverses a path of length at most v to one of those vertices, and is caught by the landing cop.

Example 4.3. Let T be a rooted tree. Then G has tree-width at most one, so by the Seymour-Thomas result, $\text{copwidth}_\infty(G) \leq 2$. The strategy with two cops is as follows. In the first round, land one cop on the root of the tree, leaving the other cop in his helicopter. In each subsequent round, direct the cop that is further from the robber, to the child of the other cop's position which is closest to the robber's position. With this strategy, in round i the robber will be at distance at least i from the root, so they will be caught after at most as many rounds as the height of T .

We start with comparing the most fundamental parameter of Sparsity theory, degeneracy, with the cop-width parameter for radius 1.

Theorem 4.4. *For every graph G ,*

$$\text{copwidth}_1(G) = \text{degeneracy}(G) + 1.$$

The simple proof of this fact relies on the fundamental duality result concerning degeneracy: for every $d \in \mathbb{N}$, every graph G is either d -degenerate, or it has a subgraph H in which every vertex has degree larger than d . This duality theorem is nothing else than a min-max theorem for the radius-1 Cops and Robber game, quite analogous to the min-max theorem in the eponymous paper of Seymour and Thomas. Indeed, a d -degeneracy order can now be viewed as a compact representation of a winning strategy for the cops involving $d + 1$ cops, in the Cops and Robber game with radius 1: when robber is on a vertex v , place $d + 1$ cops on v and the neighbors of v before v . Then the robber needs to move rightwards in the order, and eventually loses. This proves $\text{copwidth}_1(G) \leq \text{degeneracy}(G) + 1$. Dually, a subgraph H of G whose all vertices have degree larger than d , can be seen as a haven for the robber: they can forever evade d -cops by always moving to an unoccupied vertex of H (or remaining in place). This proves $\text{copwidth}_1(G) \geq \text{degeneracy}(G) + 1$, thus proving Theorem 4.4.

Next we observe that for higher radii r , the parameter $\text{copwidth}_r(G)$ is closely related to the generalized coloring numbers: the weak coloring number $\text{wcol}_r(G)$ and the admissibility numbers $\text{adm}_r(G)$ (see Section 3.2). Recall (see Fact 3.3) that the two parameters are functionally equivalent, and that $\text{adm}_r(G) \geq \nabla_{r-1}(G)$.

We prove the following:

Theorem 4.5 (*). *For $r \in \mathbb{N}$,*

$$\text{adm}_r(G) + 1 \leq \text{copwidth}_r(G) \leq \text{wcol}_{2r}(G) + 1.$$

In particular, by Fact 3.4, this gives the first, arguably very natural, characterization of classes with bounded expansion, in terms of a game:

Corollary 4.6. *A graph class \mathcal{C} has bounded expansion if and only if for every $r \in \mathbb{N}$ we have that $\text{copwidth}_r(\mathcal{C}) < \infty$.*

Thus, all the classes \mathcal{C} from Example 3.2 satisfy $\text{copwidth}_r(\mathcal{C}) < \infty$, for all $r \in \mathbb{N}$. Similarly, by Fact 3.7 we get a new characterization of nowhere dense classes.

Corollary 4.7. *A hereditary graph class \mathcal{C} is nowhere dense if and only if for every $r \in \mathbb{N}$ and $\varepsilon > 0$ we have that $\text{copwidth}_r(G) \leq O_{r,\varepsilon}(|G|^\varepsilon)$ for $G \in \mathcal{C}$.*

The proof of Theorem 4.5, which we now give, is quite simple thanks to the theory of sparsity, but also sheds a new light on the fundamental notions of that theory. The total order

$<$ that appears in the definition of the weak coloring number $\text{wcol}_{2r}(G)$ can be viewed as a compact representation of a (very particular) winning strategy for the cops in the Cops and Robber game with radius r . Indeed, the following yields a winning strategy for the cops: if the robber is at a vertex v , then the cops are placed on v and the vertices $w < v$ that are $2r$ -weakly reachable from v . To see that this strategy is winning for the cops, consider the paths π_1, π_2, \dots in G , where π_i is the path of length at most r along which robber traversed from his i th position v_i to his $(i+1)$ -st position v_{i+1} . If m_i denotes the $<$ -minimal vertex of the path π_i , then we have that $m_{i+1} > m_i$. Otherwise, m_{i+1} is $2r$ -weakly reachable from v_i (as witnessed by the path $\pi_i; \pi_{i+1}$, truncated at m_{i+1}), so at the time the robber traversed the path π_{i+1} , the vertex m_{i+1} that was occupied by a cop, which is impossible. Therefore, we have $m_1 < m_2 < \dots$, so the cops win after at most $|G|$ rounds. This gives the upper bound in Theorem 4.5.

For the lower bound, we use the well-known (and straightforward) min-max characterization of admissibility: a graph G has r -admissibility number at least d if there is a set of vertices $X \subseteq V(G)$ such that for every vertex $v \in X$ there is a set of d paths of length at most r that start at v and end in some vertex of $X - v$, such that any two paths only share v as a common vertex.

A set $X \subseteq V(G)$ that witnesses that $\text{adm}_r(G) \geq d$ can be used as a haven for the robber, to elude d cops forever, similarly as in the case of degeneracy. If the robber is occupying a vertex v in X and the cops are moving to a set S of at most d new positions, then either $v \notin S$, or there is some path from v to a vertex in $X - S$ of length $\leq r$, and the robber moves along this path. This proves $\text{copwidth}_r(G) \geq d + 1$.

Therefore, the equivalence of the weak coloring numbers and the admissibility numbers can again be seen as a min-max theorem for the Cops and Robber game with radius r .

Note that Theorem 4.5 does not give an exact min-max theorem, as there is a gap between the upper and lower bounds. We can get a family of parameters based on another variant of the cops and robber game, which does admit an exact min-max theorem. In this game, in each round first the cops move to some k vertices of the graph, and then the robber moves via a path of length at most r , that does not run through a cop, and loses if no such path exists. See Appendix A.1 for more details. This family of parameters again characterizes classes with bounded expansion (see Corollary A.3), and does admit an exact min-max theorem (Lemma A.1) that exhibits a duality between total orders describing winning strategies for the cops, and havens for the robber. Those parameters essentially appear⁵ in the work [RT08, LPPT20]. However, the paper does not relate those notions with generalized coloring numbers, and with classes of bounded expansion. Curiously, the limit version of those parameters, for radius $r = \infty$, does not correspond to treewidth, but to a notion called ∞ -admissibility [Dvo12]. Classes with bounded ∞ -admissibility are characterized as clique-sums of graphs of almost bounded degree [Dvo12, Cor. 5 and Thm. 6]. Despite the appealing properties of the parameters based on this variant of the cops and robber game, they seem to be less suited for our purposes, of generalizing to dense graphs.

To summarize, our new cop-width parameters exactly characterize treewidth (for $r = \infty$), degeneracy (for $r = 1$), classes of bounded expansion, and nowhere dense classes. This captures an appreciable fragment of the theory of sparse graphs, while offering a new perspective on the fundamental graph parameters used for measuring sparsity, and the dualities between them. We have

$$\begin{aligned} \text{degeneracy}(G) + 1 = \text{copwidth}_1(G) &\leq \text{copwidth}_2(G) \leq \dots \\ &\leq \text{copwidth}_\infty(G) = \text{treewidth}(G) + 1, \end{aligned}$$

⁵The paper [RT08] considers a variant of the game in which the robber is *lazy*, that is, does not move unless a cop is placed at his location, whereas the [LPPT20] considers a variant where the cops occupy edges instead of vertices, and the robber never remains put.

so for a class \mathcal{C} of graphs, if any of those parameters is bounded by a constant, then \mathcal{C} has bounded degeneracy by Theorem 4.4. So those parameters are only well suited to the study of sparse graphs: every class \mathcal{C} for which either of those parameters is bounded, is sparse, in the sense of having a bound on the edge density $|E(G)|/|V(G)|$ for all graphs G in the class. Moreover, copwidth_r is monotone with respect to the subgraph relation: if H is a subgraph of G , then $\text{copwidth}_r(H) \leq \text{copwidth}_r(G)$.

5 Flip-width

To lift the Cops and Robber game to the setting of dense graphs, we enhance the power of the cops. Now, instead of placing cops on at most k vertices of the graph, which can be alternatively seen as removing at most k vertices, or isolating them, the player controlling the cops can perform *flips* on subsets of the graph G . For a fixed graph G , applying a *flip* between a pair of sets of vertices $A, B \subseteq V(G)$ results in the graph obtained from G by inverting the adjacency between any pair of vertices a, b with $a \in A$ and $b \in B$. For example, applying a flip between $V(G)$ and $V(G)$ in G results in the complement of G . And if v is a vertex of G , then applying a flip between $\{v\}$ and the neighborhood $N(v)$ of v has the same effect as *isolating* v , that is, removing all the edges adjacent to v in G . If G is a graph and \mathcal{P} is a partition of its vertex set, then call a graph G' a \mathcal{P} -*flip* of G if G' can be obtained from G by performing a sequence of flips between pairs of parts $A, B \in \mathcal{P}$ (possibly with $A = B$). Since flips are involutive and commute with each other, such a sequence of flips can be specified by a set of at most $\binom{|\mathcal{P}|+1}{2}$ unordered pairs of elements of \mathcal{P} . Finally, call G' a k -*flip* of G , if G' is a \mathcal{P} -flip of G , for some partition \mathcal{P} of $V(G)$ with $|\mathcal{P}| \leq k$.

Remark 5.1. There are many other, functionally equivalent, ways to measure the complexity of a k -flip (also called a *perturbation*) G' of G . For example, say that G' is a k -*sequential-flip* of G if G' is obtained from G by applying a sequence of flips between k pairs of arbitrary subsets of $V(G)$. If G' is a k -sequential-flip of G then G' is a 2^{2k} -flip of G , and conversely, if G' is a k -flip of G , then G' is a $\binom{k+1}{2}$ -sequential-flip of G . We could also require that flips are only applied to pairs of the form (A, A) ; this would lead to a functionally equivalent parameter, as flipping a pair (A, B) can be obtained by flipping three pairs: $(A \cup B, A \cup B)$, (A, A) and (B, B) . Other, functionally equivalent measures of the complexity of a flip G' of a graph G can be defined by considering the graph $G' \triangle G$ with vertices $V(G)$ and edges $E(G') \triangle E(G)$. Note that G' is a k -flip of G if and only if $G' \triangle G$ is a k -flip of the edgeless graph on $V(G)$. This is equivalent to $G \triangle G'$ having *neighborhood diversity* [Lam12] k . The rank of the adjacency matrix of $G \triangle G'$ over a fixed finite field, see [NiO20, DK06] leads to a further, functionally equivalent complexity measure of a flip.

Flipper game We now come to the central notions of this paper. The *flipper game* with radius $r \in \mathbb{N} \cup \{\infty\}$ and width $k \in \mathbb{N}$, $k \geq 1$, is played on a graph G . In each round i of the game, a k -flip G_i of G is declared by the cops, and the new position $v_i \in V(G)$ is selected by the robber, as follows. Initially, $G_0 = G$ and v_0 is a vertex of G chosen by the robber. In round $i > 0$, the cops announce a new k -flip G_i of G , that will be put into effect momentarily. The robber, knowing G_i , moves to a new vertex v_i by following a path of length at most r from v_{i-1} to v_i in the *previous* graph G_{i-1} . The game terminates when the robber is trapped, that is, when v_i is isolated in G_i .

Definition 5.2. Fix $r \in \mathbb{N} \cup \{\infty\}$. The radius- r flip-width of a graph G , denoted $\text{fw}_r(G)$, is the smallest number $k \in \mathbb{N}$ such that the cops have a winning strategy in the flipper game of radius r and width k on G .

Definition 5.3. A class \mathcal{C} of graphs has bounded flip-width if $\text{fw}_r(\mathcal{C}) < \infty$ for every $r \in \mathbb{N}$. More explicitly: for every radius $r \in \mathbb{N}$ there is some $c_r \in \mathbb{N}$ such that $\text{fw}_r(G) < c_r$ for all $G \in \mathcal{C}$.

Remark 5.4. The cop-width parameters considered in Section 4 are functionally equivalent (more precisely, each parameter can be bounded from above by a linear function of the other) to parameters defined by a variant of the flipper game, call it the *isolation game*, which is played as the flipper game, but each graph G_i announced by the cops is obtained from G by isolating at most k vertices in G (see Lemma A.4). The difference between the isolation game and the Cops and Robber game is that in the Cops and Robber game, the robber can move through a vertex from which a cop has just departed by a helicopter, while in the isolation game, they cannot.

We now argue that the flip-width parameters are bounded in terms of the corresponding cop-width parameters.

Lemma 5.5. *For every $r \in \mathbb{N} \cup \{\infty\}$ and graph G , we have*

$$\text{fw}_r(G) \leq \text{copwidth}_r(G) + 2^{\text{copwidth}_r(G)}. \quad (3)$$

Proof. The main observation is that isolating a set S of at most k vertices in G can be achieved by performing a $(k + 2^k)$ -flip: consider the partition \mathcal{P}_S that partitions S into singletons and $V(G) - S$ according to the neighborhood in S , and flip $\{s\}$ with every class of the partition that is complete to $\{s\}$. Note that $|\mathcal{P}_S| \leq k + 2^k$. Now, if the cops have a winning strategy in the Cops and Robber game of radius r and width k on a graph G , we can use this strategy in the flipper game of radius r and width $k + 2^k$, as follows: whenever the cops announce a new set S of positions of the cops in the Cops and Robber game, in the flipper game, the cops announce the graph G' with the vertices in S isolated, which is a $(k + 2^k)$ -flip of G . It is easy to verify that if the cops win in the Cops and Robber game, then, playing according to the above strategy, they also win in the flipper game. Inequality (3) follows. \square

The following gives an improved bound. Note that if $\text{copwidth}_1(G) \leq t$ then G excludes $K_{t,t}$ as a subgraph, by Theorem 4.4 and the fact that $K_{t,t}$ is not $(t - 1)$ -degenerate. From this, one can bound the size of the partition \mathcal{P}_S considered above by k^t (see Lemma B.3), and obtain:

Theorem 5.6. *Fix $r \in \mathbb{N} \cup \{\infty\}$. Let G be a graph and let t be the smallest number such that G excludes $K_{t,t}$ as a subgraph; in particular $t \leq \text{copwidth}_1(G) \leq \text{copwidth}_r(G)$. If $t \geq 3$ then*

$$\text{fw}_r(G) \leq \text{copwidth}_r(G)^t,$$

and if $t = 2$ then

$$\text{fw}_r(G) \leq O(\text{copwidth}_r(G)^t).$$

In particular, every class \mathcal{C} with bounded expansion has bounded flip-width.

Whereas the cop-width parameters are monotone with respect to the subgraph relation, the flip-width parameters are monotone with respect to the induced subgraph relation. This is expressed by the following, immediate lemma.

Lemma 5.7. *Fix $r \in \mathbb{N} \cup \{\infty\}$. If H is an induced subgraph of G , then $\text{fw}_r(H) \leq \text{fw}_r(G)$. In particular, a class \mathcal{C} has bounded flip-width if and only if its hereditary closure has bounded flip-width.*

5.1 Examples

We start by giving some example classes of bounded flip-width. By Theorem 5.6, every class of bounded expansion has bounded flip-width, see Example 3.2 for some specific classes. Unlike cop-width, flip-width is not limited to sparse graphs, and is geared towards the study of dense graphs.

Example 5.8. If \bar{G} is the complement of G and $r \in \mathbb{N} \cup \{\infty\}$, then $\text{fw}_r(\bar{G}) = \text{fw}_r(G)$, since any k -flip of G is also a k -flip of \bar{G} (just complement the set of flipped pairs). Therefore, if \mathcal{C} is a class with bounded expansion, then the class $\bar{\mathcal{C}} := \{\bar{G} \mid G \in \mathcal{C}\}$, has bounded flip-width, and if \mathcal{C} has bounded treewidth, then $\text{fw}_\infty(\bar{\mathcal{C}}) = \text{fw}_\infty(\mathcal{C}) < \infty$. In particular, as every edgeless graph G has $\text{fw}_\infty(G) = 1$, it follows that every clique G also has $\text{fw}_\infty(G) = 1$.

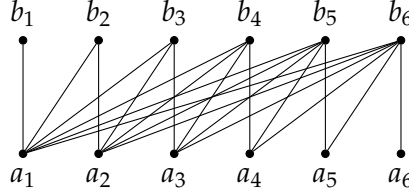


Figure 2: A half-graph of order 6.

Example 5.9. Consider the *half-graph* H_n of order n , as depicted in Figure 2. We show that $\text{fw}_\infty(H_n) \leq 4$. Observe that applying a flip between $\{a_1, \dots, a_{i-1}\}$ and $\{b_i, \dots, b_n\}$ breaks the half-graph into two connected components (each being a half-graph). Additionally flipping $\{a_i\}$ and $\{b_i, \dots, b_n\}$ makes a_i and b_i isolated. The strategy of the cops is to perform the above two flips in the i th round, thus pushing the robber rightwards in each round. To perform those flips, partition $V(H_n)$ into four parts: $\{1, \dots, a_{i-1}\}$, $\{a_i\}$, $\{b_i, \dots, b_n\}$, and the rest.

Example 5.10. The comparability graph G of a rooted tree T is the graph with vertices $V(T)$, where two vertices are adjacent if and only if one is an ancestor of the other in T . Generalizing half-graphs, those graphs also have $\text{fw}_\infty(G) \leq 4$. The strategy of the cops is, in round i , to consider the node u_i at depth i in the tree, that is the ancestor of the current position of the robber, and to isolate u_i , and remove all edges between the descendants of u_i and the ancestors of u_i . This can be done by partitioning $V(G)$ into four parts: $\{u_i\}$, the ancestors of u_i , the descendants of u_i , and the rest.

Example 5.11. Fix $r \in \mathbb{N} \cup \{\infty\}$. If G_1, \dots, G_m are graphs and G is their disjoint union, then the following inequality holds:

$$\text{fw}_r(G) \leq \max_{1 \leq i \leq m} (\text{fw}_r(G_i)) + 1.$$

The $+1$ comes from the fact that a partition \mathcal{P} of $V(G_i)$ into k parts induces a partition of $V(G)$ into $k+1$ parts, namely the k parts of \mathcal{P} , and $V(G) - V(G_i)$. Thus, if the robber is hiding in the graph G_i , the cops may translate a winning strategy on G_i of width $k := \text{fw}_r(G_i)$, into a winning strategy on G of width $k+1$.

It follows that if \mathcal{C} has bounded flip-width, then the class of disjoint unions of graphs from \mathcal{C} has bounded flip-width.

A *modular partition* of a graph G is a partition \mathcal{P} of $V(G)$ such that any two distinct parts are homogeneous in G . The quotient graph G/\mathcal{P} has as vertices the parts of \mathcal{P} , and as edges pairs of distinct parts that are complete in G . An extension of the idea in Example 5.11 yields the following.

Lemma 5.12 (*). *Let G be a graph and \mathcal{P} be its modular partition. Then*

$$\text{fw}_r(G) \leq \max \left(\text{fw}_r(G/\mathcal{P}), \max_{A \in \mathcal{P}} \text{fw}_r(G[A]) + 2 \right).$$

The strategy for the cops on G first follows the strategy on G/\mathcal{P} , where each k -flip of G/\mathcal{P} is lifted naturally to a k -flip of G . Once a part $A \in \mathcal{P}$ is isolated in the game on G/\mathcal{P} , the strategy on $G[A]$ is used. The $+2$ in the statement is due to the fact that every partition \mathcal{Q} of A into k parts induces a partition of G into $k+2$ parts: the k parts of \mathcal{Q} , the (common) set of neighbors of vertices in A outside of A , and the rest. See Appendix C for details. In particular, if G is the lexicographic product of two graphs H and K (obtained by blowing up each node of H to a copy of K) then $\text{fw}_r(G) \leq \max(\text{fw}_r(H), \text{fw}_r(K) + 2)$.

The *substitution closure* of a class of graphs \mathcal{C} is the smallest class \mathcal{C}^* containing \mathcal{C} such that if G is a graph with a modular partition \mathcal{P} into modules A satisfying $G[A] \in \mathcal{C}^*$, and $G/\mathcal{P} \in \mathcal{C}$, then $G \in \mathcal{C}^*$. Intuitively, a graph in \mathcal{C}^* can be obtained from a single vertex by repeatedly blowing up vertices to graphs from \mathcal{C} .

Using similar ideas as in Lemma 5.12, we prove Lemma 5.13.

Lemma 5.13 (*). *For every $r \in \mathbb{N} \cup \{\infty\}$ and graph class \mathcal{C} , we have*

$$\text{fw}_r(\mathcal{C}^*) \leq \text{fw}_r(\mathcal{C}) + 2.$$

In particular, if \mathcal{C} has bounded flip-width, then \mathcal{C}^ has bounded flip-width.*

Example 5.14. The substitution closure \mathcal{C}^* of the class \mathcal{C} of subcubic graphs (graphs with maximum degree 3) has bounded flip-width. This follows from Example 4.2, Theorem 5.6, and Lemma 5.13.

More examples are given in the following sections: they include classes of bounded clique-width (see Section 5.3), classes of bounded twin-width (see Section 7), and interpretations of classes of bounded expansion (see Section 8). We remark that the class from Example 5.14 is not among any of those classes (it has unbounded twin-width, and is not edge-stable; see Section 10).

5.2 Hideouts

We have already seen several examples of classes of bounded flip-width. To give examples of classes of unbounded flip-width, we need a tool for proving lower bounds. We now introduce the notion of a hideout, which is a set of vertices allowing the robber to evade the Cops indefinitely, thus allowing to prove lower bounds on $\text{fw}_r(G)$. In Section 5.4 we use this notion to prove some combinatorial properties of graphs with bounded flip-width. In particular, it will follow easily that every graph of radius-one flip-width at most k has a pair of vertices whose neighborhoods differ in at most $2k$ vertices.

Although we will use hideouts on several occasions to prove lower bounds on flip-width, we do not know whether the existence of hideouts is a necessary condition for having large flip-width. This is stated as Question 11.3 in Section 11.

Definition 5.15. *Fix $k, d \geq 1$ and $r \in \mathbb{N} \cup \{\infty\}$. A (r, k, d) -hideout in a graph G is a set of vertices $U \subseteq V(G)$ with $|U| > d$, satisfying the following property. For every k -flip G' of G ,*

$$|\{v \in U : |B_{G'}^r(v) \cap U| \leq d\}| \leq d, \quad (4)$$

that is, there are at most d vertices $v \in U$ such that there are at most d vertices $u \in U$ that are connected with v by a path of length at most r in G' .

We will show that in the flipper game with radius r and width k , a robber can hide infinitely in a (r, k, d) -hideout. Intuitively, when a k -flip G' of G is announced, the robber will want to avoid all vertices $v \in U$ with $|B_{G'}^r(v) \cap U| \leq d$. The condition in a hideout guarantees that there are at most d such vertices. This will allow the robber to always move to some vertex with $|B_{G'}^r(v) \cap U| > d$.

Lemma 5.16. *Fix $k \geq 1$ and $r \in \mathbb{N} \cup \{\infty\}$. If a graph G has a (r, k, d) -hideout U for some $d \geq 1$, then $\text{fw}_r(G) > k$.*

Proof. Let $U \subseteq V(G)$ be a (r, k, d) -hideout. We describe a strategy for the robber in the flipper game on G with radius r and width k , which allows him to elude the cops indefinitely. The strategy is as follows: when the cops announce a k -flip G' of G , the robber moves to some vertex $v \in U$ such that $|B_{G'}^r(v) \cap U| > d$. In the first move, pick any $v \in U$ with $|B_G^r(v) \cap U| > d$. Such a vertex exists, by (4) applied to $G' = G$, since $|U| > d$.

We show it is always possible to make a move as described in the strategy. Suppose at some point in the game, the current position v of the robber is such that

$$|B_P^r(v) \cap U| > d. \quad (5)$$

where P is the previous k -flip of G announced by the cops (in the first round, $P = G$), and that the cops now announce the next k -flip N of G . Since U is a (r, k, d) -hideout, the set $X \subseteq U$

of vertices $w \in U$ such that $|B_N^r(w) \cap U| \leq d$ satisfies $|X| \leq d$. By (5), $B_P^r(v)$ contains at least one vertex $v' \in U - X$. The robber moves from v to v' along a path of length at most r in P . As $v' \in U - X$, the invariant is maintained. Therefore, playing according to this strategy, the robber can elude the cops indefinitely, so $\text{fw}_r(G) > k$. \square

5.3 Flip-width with infinite radius

As a simple case study, we first analyze the parameter fw_∞ . Recall that its sparse analogue, copwidth_∞ , corresponds to treewidth. We show that fw_∞ is functionally equivalent to the clique-width and rank-width parameters. Those parameters extend treewidth to the setting of dense graphs, and are recalled later below. To the best of our knowledge, our result is the first characterization of graph classes of bounded clique-width, in terms of a game⁶, analogous to the game characterization of treewidth.

Theorem 5.17 (*). *For every graph G , we have*

$$\text{rankwidth}(G) \leq 3 \text{fw}_\infty(G) + 1 \leq O(2^{\text{rankwidth}(G)}).$$

In particular, a graph class \mathcal{C} has bounded rank-width if and only if $\text{fw}_\infty(\mathcal{C}) < \infty$.

The *rank-width* parameter is functionally equivalent to clique-width, and is often more convenient to work with. We will only use rank-width. Rank-width and clique-width are functionally equivalent, as expressed below [iOSo6, Prop. 6.3]:

$$\text{rankwidth}(G) \leq \text{cliquewidth}(G) < 2^{\text{rankwidth}(G)+1}.$$

A graph G has rank-width at most k if there is a tree T whose leaves are the vertices of G , and inner nodes have degree at most 3, such that for every edge e of the tree, the bipartition $A \uplus B$ of the leaves of T into the leaves on either side of e , has *cut-rank* at most k . The cut-rank of a bipartition $A \uplus B$ of the vertex set of a graph G , denoted $\text{rk}_G(A, B)$, is defined as the rank, over the two-element field, of the 0, 1-matrix with rows A , columns B , where the entry at row $a \in A$ and column $b \in B$ is 1 if $ab \in E(G)$ and 0 otherwise.

The upper bound in Theorem 5.17 is a generalization of the bound in Example 5.10, where it is shown that comparability graphs of trees have $\text{fw}_\infty(G) \leq 4$. We briefly sketch the argument now.

Note that a 0, 1-matrix of rank at most k over the two-element field has at most 2^k distinct rows and at most 2^k distinct columns. It follows that if G is a graph and $V(G) = A \uplus B$ is a bipartition of its vertex sets with cut-rank $\text{rk}_G(A, B) \leq k$, then A and B can be partitioned as $A = A_1 \uplus \dots \uplus A_p$ and $B = B_1 \uplus \dots \uplus B_q$ with $p, q \leq 2^k$, so that A_i and B_j are complete in G . This implies that there is a 2^{k+1} -flip G' of G which has no edges with one endpoint in A and one endpoint in B .

Therefore, if G is a graph of rank-width k , then there is a subcubic tree T with leaves $V(G)$ such that for every edge e of T , if $V(G) = A \uplus B$ is the bi-partition induced by e (into the leaves on either side of e), then there is a 2^{k+1} -flip G' of G which has no edges with one endpoint in A and one endpoint in B . Moreover, for every inner node v of T (of degree at most three), there is a $O(2^k)$ -flip G' of G such that for any two vertices $a, b \in V(G)$, if a and b are in the same connected component of G' , then a and b are connected in T by a path that avoids v . This flip can be used by the cops in their winning strategy, to restrain the robber to the leaves of smaller and smaller subtrees of T , similarly as in Example 5.10. See Appendix D for details.

The lower bound relies⁷ on the following result describing certain obstructions that can be found in classes of unbounded clique-width.

⁶The radius- ∞ flipper game arose in a private discussion in 2018 with Michał Pilipczuk.

⁷We thank Rose McCarty for providing the following proof.

A set U of vertices of G is *well-linked* if for every bipartition $A \uplus B$ of $V(G)$, the cut-rank of $A \uplus B$ satisfies $\text{rk}_G(A, B) \leq \min(|A \cap U|, |B \cap U|)$. Oum and Seymour [iOSo6, Theorem 5.2] prove the following:

Fact 5.18 ([iOSo6]). *Every graph of rank-width greater than k contains well-linked set of size k .*

Using this, the lower bound in Theorem 5.17 follows from the next lemma, which is proved in Appendix D.

Lemma 5.19 (*). *Fix a graph G and number $k \in \mathbb{N}$. Every well-linked set U with $|U| > 3k$ is a (∞, k, k) -hideout.*

This implies the lower bound in Theorem 5.17 as follows. Suppose $\text{rankwidth}(G) > 3k + 1$ for some $k \in \mathbb{N}$. By Fact 5.18, G contains a well-linked set U of size $3k + 1$. By Lemma 5.19, U is a (∞, k, k) -hideout. By Lemma 5.16, $\text{fw}_\infty(G) > k$. By contrapositive, this shows that if $\text{fw}_\infty(G) = k$ then $\text{rankwidth}(G) \leq 3k + 1 = 3\text{fw}_\infty(G) + 1$.

5.4 Radius-one flip-width

We move to the study of finite radii, which are our main focus, starting with the first parameter, fw_1 . We have seen that its sparse analogue, copwidth_1 , corresponds precisely to degeneracy (plus one), which is a very well-understood parameter, with many good algorithmic and combinatorial properties. The parameter fw_1 enjoys many useful combinatorial properties, relating it to near-twins, and to the VC-dimension.

Near-twins We prove a first combinatorial property of graphs with small $\text{fw}_1(G)$, namely that such graphs have near-twins. This has several consequences. Say that two vertices u, v of a graph G are δ -near-twins if $|N(u) \Delta N(v)| \leq \delta$, where Δ denotes the symmetric difference. We show that every graph G with $\text{fw}_1(G) \leq k$ has a pair of $2k$ -near-twins. More generally, we prove:

Lemma 5.20. *Let $b, k \in \mathbb{N}$ and let G be a graph with $\text{fw}_1(G) \leq k$. Then G contains a set of at least $b + 1$ vertices which are mutual $2bk$ -near-twins.*

Proof. Assume that G has no set containing $b + 1$ mutual $2bk$ -near-twins. We prove that $V(G)$ is a $(1, k, bk)$ -hideout in G , which implies that $\text{fw}_1(G) > k$ by Lemma 5.16.

Let G' be a k -flip of G and let B be the set of vertices of degree at most bk in G' . We show that $|B| \leq bk$, proving that $V(G)$ is a $(1, k, bk)$ -hideout in G .

Suppose that $|B| > bk$. Let \mathcal{P} be a partition with $|\mathcal{P}| \leq k$ such that G' is a \mathcal{P} -flip of G . As $|B| > bk$, there is a set $B_0 \subseteq B$ with $|B_0| > b$, such that B_0 is contained in one part of \mathcal{P} . Any two vertices of B_0 are $2bk$ -near-twins in G' , as they both have degree at most bk in G' . Since B_0 is contained in a single part of \mathcal{P} , it follows that any two vertices of B_0 are $2bk$ -near-twins in G , too. But $|B_0| > b$, so this contradicts the assumption. Hence, $|B| \leq bk$. \square

A bipartite variant of Lemma 5.20, with a very similar proof, is as follows. Recall that if $X, Y \subseteq V(G)$ are two sets of vertices of a graph G then $G[X, Y]$ denotes the bipartite graph semi-induced by X and Y in G .

Lemma 5.21 (*). *Let $b, k \in \mathbb{N}$ and let G be a graph with $\text{fw}_1(G) \leq k$. Then for every two sets $X, Y \subseteq V(G)$ there is a subset of X or a subset of Y consisting of $b + 1$ mutual $2bk$ -near-twins in $G[X, Y]$.*

Setting $b := 1$ in Lemma 5.20 we get:

Corollary 5.22. *Let G be a graph with $\text{fw}_1(G) \leq k$. Then G has a pair of $2k$ -near-twins.*

We can now verify that there exist graphs G with arbitrarily large $\text{fw}_1(G)$. It is well known that there exist graphs of arbitrarily large girth and minimum degree.

Corollary 5.23. *A graph G with girth larger than 4 and minimum degree larger than k has $\text{fw}_1(G) > k$. Therefore, there exist graphs G with arbitrarily large $\text{fw}_1(G)$.*

Proof. Any two distinct vertices u and v have at most one common neighbor, so $|N(u) \triangle N(v)| > 2k$, so G has no pair of $2k$ -near-twins, hence $\text{fw}_1(G) > k$ by Corollary 5.22. \square

VC dimension We now show that the VC-dimension of a graph G is bounded in terms of $\text{fw}_1(G)$. Set systems and graphs of bounded VC-dimension have many useful properties, and we use one of them later for studying classes of bounded flip-width. We also consider a related parameter, called *2VC-dimension* [BT15], and denoted $2\text{VCdim}(G)$. This is the maximal size of a set $X \subseteq V(G)$ such that for every two distinct $a, b \in X$ there is a vertex $c \in V(G)$ with $N_G(c) \cap X = \{a, b\}$. Clearly, $\text{VCdim}(G) \leq 2\text{VCdim}(G)$. We prove the following.

Theorem 5.24 (*). *For every graph G we have*

$$\text{VCdim}(G) \leq 8\text{fw}_1(G), \quad (6)$$

$$2\text{VCdim}(G) \leq 8\text{fw}_2(G) + 2 \quad (7)$$

Note that 2VCdim cannot be bounded in terms of fw_1 , only in terms of fw_2 , as witnessed by 1-subdivided cliques, which are 2-degenerate, and hence (by Theorem 5.6 and Theorem 4.4), have bounded fw_1 , and clearly have unbounded fw_2 . Furthermore, graphs of girth larger than 4 have VC-dimension at most two, but have arbitrarily large fw_1 by Corollary 5.23. Hence, fw_1 is not bounded in terms of $\text{VCdim}(G)$.

Inequality (6) in Theorem 5.24 follows from Lemma 5.21 (for $b = 1$) and the following.

Lemma 5.25 (*). *Let G be a graph with $\text{VCdim}(G) \geq 2^m$, for some m . Then there are two sets X, Y such that the bipartite graph $G[X, Y]$ contains no pair of $(2^{m-1} - 1)$ -near-twins in either of the parts X, Y .*

In the proof, the sets X and Y are two copies of the m -dimensional vector space over the two-element field, with edges connecting vectors with a nonzero dot product. See Appendix E.1.

We prove inequality (7) in Corollary 6.8 later.

From Lemma 5.25 and Lemma 5.21 (for $b = 1$) we conclude the following.

Corollary 5.26. *Let G be a graph with $\text{VCdim}(G) \geq d$. Then G contains an induced subgraph H with $O(d)$ vertices and with $\text{fw}_1(H) \geq d/8$.*

Corollary 5.27. *If \mathcal{C} is a hereditary class of graphs such that $\text{fw}_1(G) \leq o(|G|)$ for $G \in \mathcal{C}$, then $\text{VCdim}(\mathcal{C}) < \infty$.*

Proof. If $\text{VCdim}(\mathcal{C}) = \infty$ then for every d there is a graph $G \in \mathcal{C}$ with $\text{VCdim}(G) \geq d$, and by Corollary 5.26 there is $H \in \mathcal{C}$ with $O(d)$ vertices and $\text{fw}_1(H) = \Omega(d)$. Since this holds for all $d \in \mathbb{N}$, it cannot be that $\text{fw}_1(G) \leq o(|G|)$ for all $G \in \mathcal{C}$. \square

5.5 Relationship of radius-one flip-width with other graph parameters

In this section, we compare radius-one flip-width with existing graph parameters. First, we mention that in Section 6.1 we show that for graphs G that exclude a fixed biclique $K_{t,t}$ as a subgraph, we have that $\text{fw}_1(G)$ is functionally equivalent to the degeneracy of G . However, unlike degeneracy, fw_1 is bounded for some graphs that contain arbitrarily large cliques or bicliques. Thus, $\text{fw}_1(G)$ may be thought of as a dense extension of degeneracy. Two other graph parameters – *symmetric difference* and *functionality* – which could serve a similar role, are discussed below.

For a graph G , define the *symmetric difference* of G , denoted $\text{sd}(G)$, as the least number $k \in \mathbb{N}$, such that every induced subgraph H of G contains a pair u, v of k -near-twins. Classes \mathcal{C} with $\text{sd}(\mathcal{C}) < \infty$ are called *classes with bounded symmetric difference*. For example, if G has

degeneracy k , then $\text{sd}(G) \leq 2k$. In particular, every graph class with bounded degeneracy has bounded symmetric difference.

Since fw_1 is monotone under taking induced subgraphs, Corollary 5.22 implies that $\text{sd}(G) \leq 2\text{fw}_1(G)$ for every graph G . Hence, every graph class \mathcal{C} with $\text{fw}_1(\mathcal{C}) < \infty$ has bounded symmetric difference. We do not know whether the converse implication holds.

A more general graph parameter, called *graph functionality* and denoted $\text{fun}(G)$, is defined in [AAL21]. The following is a rephrasing of the original definition.

Say that a vertex $v \in V(G)$ is a *function* of a set $S \subseteq V(G)$ of vertices, with $s \notin S$, if for every $w \in V(G) - (S \cup \{v\})$, the adjacency of w and v in G depends only on $N(w) \cap S$. For instance, if v is a k -near-twin of u , then v is a function of $\{u\} \cup (N(v) \triangle N(u)) - \{v\}$. Define $\text{fun}(G)$ as the least number k such that every induced subgraph H of G has a vertex v that is a function of at most k other vertices of H .

Clearly, $\text{fun}(G) \leq \text{sd}(G) + 1$ holds for every graph G , so classes of bounded symmetric difference have bounded functionality. The converse implication is false, as witnessed by the class of permutation graphs [AAL21, Sec. 2.4], which have bounded functionality and unbounded symmetric difference.

Classes of bounded functionality have bounded VC-dimension, as shown in [AAL21, Thm. 8]. However, no effective bound is known (see [AAL21, Open problem 4]).

It follows from [ACLZ15, Thm. 2] that every hereditary graph class of bounded functionality is at most *factorial* – contains at most $2^{O(n \log n)}$ labelled graphs on n vertices. In fact, every class that is at most factorial has bounded VC-dimension (see [AAL21, Sec. 3]).

Figure 3 summarizes the relationships among the discussed properties of graph classes.

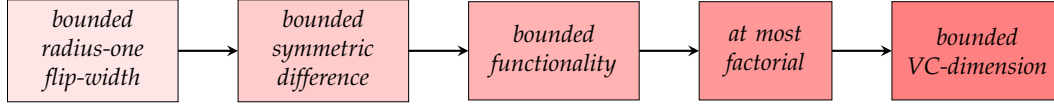


Figure 3: Properties of graph classes and implications among them.

6 Flip-width in weakly sparse classes

We have seen in Theorem 5.6 radius-one flip-width is upper bounded in terms of degeneracy, and radius- r flip-width is upper bounded in terms of generalized coloring numbers. In this section, we provide bounds in the other direction, in weakly sparse classes. It follows that for weakly sparse classes, having bounded flip-width is equivalent to having bounded expansion.

6.1 Radius-one flip-width and degeneracy

As fw_1 is bounded in terms of copwidth_1 , which is equivalent to degeneracy by Theorem 4.4, it follows that every class of bounded degeneracy has bounded fw_1 . Clearly, every class of degeneracy bounded by t is weakly sparse, as it excludes $K_{t+1,t+1}$ as a subgraph. We show that for weakly sparse classes, bounded degeneracy is equivalent to having bounded fw_1 .

Theorem 6.1. *If G is a graph that avoids $K_{t,t}$ as a subgraph, then*

$$\text{degeneracy}(G)/(2t^2) < \text{fw}_1(G) \leq (\text{degeneracy}(G) + 1)^t. \quad (8)$$

As a consequence, if \mathcal{C} is a weakly sparse class of graphs then $\text{fw}_1(\mathcal{C}) < \infty$ if and only if \mathcal{C} has bounded degeneracy.

Proof. The second inequality is by Theorem 5.6 and Theorem 4.4. To prove the first inequality, we show that $\text{degeneracy}(G) < 2kt^2$, where $k := \text{fw}_1(G)$.

Recall that G is d -degenerate if and only if every induced subgraph of G contains a vertex of degree at most d . Therefore, to prove $\text{degeneracy}(G) < 2kt^2$ it is enough to show that G contains some vertex of degree less than $2kt^2$ (since the same holds for every induced subgraph H of G , as $\text{fw}_1(H) \leq \text{fw}_1(G) \leq k$).

Setting $b := t - 1$ in Lemma 5.20 we get that G contains a set U of t mutual $2k(t - 1)$ -near-twins. Pick any $v \in U$. Then all $u \in U$ are $2k(t - 1)$ -near-twins of v , so $|N(v) \triangle N(u)| \leq 2k(t - 1)$ for all $u \in U$.

We show that $|N(v)| < 2k(t - 1)^2 + 2t$. Suppose that $|N(v)| \geq 2k(t - 1)^2 + 2t$. Then the set $W := \bigcap_{u \in U} N(u)$ has at least $2t$ elements, and so $W - U$ has at least t elements. As every $u \in U$ is adjacent to every $w \in W$ and U and W are disjoint, we have a copy of $K_{t,t}$ as a subgraph of G , a contradiction. Hence, $|N(v)| < 2k(t - 1)^2 + 2t < 2kt^2$. \square

6.2 Radius- r flip-width and r -admissibility

We have seen that every weakly sparse class \mathcal{C} has bounded degeneracy if and only if $\text{fw}_1(\mathcal{C}) < \infty$. It is known that a weakly sparse class \mathcal{C} has bounded clique-width if and only if \mathcal{C} has bounded tree-width, as clique-width and treewidth are functionally equivalent in weakly sparse classes [GW00]. This proves the following.

Corollary 6.2. *A graph class \mathcal{C} has bounded treewidth if and only if \mathcal{C} is weakly sparse and $\text{fw}_\infty(\mathcal{C}) < \infty$.*

A general theme in structural graph theory is that a dense graph parameter is often functionally equivalent to its sparse counterpart in weakly sparse classes. We now show that for weakly sparse classes, bounded flip-width is indeed equivalent to bounded expansion:

Theorem 6.3. *A class \mathcal{C} has bounded expansion if and only if \mathcal{C} is weakly sparse and has bounded flip-width.*

We prove a more precise result, from which Theorem 6.3 follows.

Theorem 6.4. *For every $r \geq 1$ and graph G we have*

$$\tilde{\nabla}_{r-1}(G) \leq O(r \cdot \text{fw}_r(G) \cdot \text{degeneracy}(G))^{36}.$$

We first show how Theorem 6.4 implies Theorem 6.3.

Proof of Theorem 6.3. For the forward direction, assume \mathcal{C} has bounded expansion. Then it has bounded flip-width by Theorem 5.6. Also, as remarked before Theorem 5.6, every class with bounded expansion is weakly sparse.

Conversely, suppose \mathcal{C} is weakly sparse and has bounded flip-width. In particular, $\text{fw}_1(\mathcal{C}) < \infty$, and by Theorem 6.1, \mathcal{C} has degeneracy bounded by some constant d . By Theorem 6.4, for all $r \geq 1$ we have that $\tilde{\nabla}_{r-1}(\mathcal{C}) \leq O(r \cdot \text{fw}_r(\mathcal{C}) \cdot d)^{36} < \infty$. Hence, \mathcal{C} has bounded expansion. \square

In the remainder of Section 6.2 we prove Theorem 6.4. Our proof relies on a result of Dvořák [Dvo18], extending a result of Kühn and Osthus [KO04], which we now recall.

An *exact r -subdivision* of a graph G is the graph obtained by replacing every edge of G by a path of length $r + 1$. If every edge is replaced by a path of length at most $r + 1$, the resulting graph is an $\leq r$ -subdivision of G . For a graph G , let $\tilde{\nabla}_r^e(G)$ denote the maximum average degree of all graphs H whose exact r -subdivision is an induced subgraph of G .

The following is [Dvo18, Lemma 9].

Lemma 6.5. *For every $r, k, d \geq 1$ there is a number $s = s(r, k, d) \leq O(rdk)^{12}$ such that for every graph G , if $\tilde{\nabla}_0(G) \leq d$ and $\tilde{\nabla}_r^e(G) < k$, then $\tilde{\nabla}_r(G) < \tilde{\nabla}_{r-1}(G) + s$.*

An easy induction on r yields the following.

Corollary 6.6. *For every $r, d \geq 1$ and d -degenerate graph G ,*

$$\tilde{\nabla}_r(G) \leq O(dr \cdot \sum_{i=1}^r \tilde{\nabla}_i^e(G))^{12}.$$

It is well known that every graph of average degree at least d contains a subgraph with minimum degree at least $d/2$. The following proposition yields Theorem 6.4, using Corollary 6.6 and Fact 3.5.

Proposition 6.7 (*). *Fix $r \geq 2, k \geq 1$. Let G be the exact $(r-1)$ -subdivision of some graph H with minimum degree at least $2rk$. Then $\text{fw}_r(G) > k$.*

Proposition 6.7 is proved in Appendix F, by showing that the vertices of G that correspond to the vertices of H (those of degree larger than 2) form a (r, k, k) -hideout in G . We now show how this proves Theorem 6.4.

Proof of Theorem 6.4. Denote $k = \text{fw}_r(G)$. Since $\text{fw}_r(G)$ is monotone in r , by Proposition 6.7, for all $t < r$, the graph G does not contain an induced t -subdivision of a graph of minimum degree at least $2rk$. Hence, $\tilde{\nabla}_t^e(G) \leq 4rk$ for all $t < r$.

If G is d -degenerate then by Corollary 6.6 we have:

$$\tilde{\nabla}_{r-1}(G) \leq O\left(dr \sum_{1 \leq t < r} \tilde{\nabla}_t^e(G)\right)^{12} \leq O(dr^3k)^{12} \leq O(drk)^{36},$$

as required. \square

Proposition 6.7 easily implies the following corollary (see Appendix B.3 for a proof), yielding inequality (7) in Theorem 5.24.

Corollary 6.8 (*). *If G is the exact 1-subdivision of an n -clique, then $\text{fw}_2(G) > (n-1)/4$. Furthermore, for every graph G , $2\text{VCdim}(G) \leq 8\text{fw}_2(G) + 2$.*

Hence, weakly sparse classes of bounded flip-width, being exactly the classes with bounded expansion, are by now very well-understood and characterized in multiple ways. For instance, the model checking problem for first-order logic is fixed-parameter tractable for such classes, by the result of Dvořák, Král, and Thomas [DKT13].

7 Flip-width of ordered graphs and twin-width

Let us return to dense graph classes, which are our main focus. As follows from Theorem 5.17, radius- ∞ flip-width is functionally equivalent to clique-width, so classes of bounded clique-width are examples of dense graph classes of bounded flip-width. In this section we show that classes of bounded twin-width have bounded flip-width, but the converse does not hold. To characterize twin-width in terms of flip-width, we study flip-width of graphs equipped with a total order. We start with recalling the definition of twin-width.

An *uncontraction sequence* of a graph G is a sequence $\mathcal{P}_1, \dots, \mathcal{P}_n$ of partitions of $V(G)$ that starts with the partition \mathcal{P}_1 with one part, ends with the partition of $V(G)$ into singletons, and such that every partition \mathcal{P}_{i+1} , for $i < n$, is obtained from the previous partition \mathcal{P}_i by splitting one of the parts into two. The *red graph* of a partition \mathcal{P} is the graph whose nodes are the parts of \mathcal{P} , with *red edges* connecting two distinct nodes $A, B \in \mathcal{P}$ if A and B are not homogeneous in G . A graph has twin-width at most d if it has an uncontraction sequence such that at every time i , the red graph of the partition \mathcal{P}_i has maximum degree at most d .

First, we prove that $\text{fw}_r(G)$ is bounded in terms of r and the twin-width $\text{tw}(G)$ of G .

Theorem 7.1 (*). *Fix $r \in \mathbb{N}$. For every graph G of twin-width d we have:*

$$\text{fw}_r(G) \leq 2^d \cdot d^{O(r)}.$$

In particular, every class of bounded twin-width has bounded flip-width.

In the proof, the cops use an uncontraction sequence in order to vanquish the robber by constraining him, in round i to some ball of radius r in the red-graph of the partition \mathcal{P}_i . In round $i = n$, as every ball of radius r in the red graph of \mathcal{P}_n comprises a single vertex of G , the robber is trapped. We use the bounds from [BFLP23] to optimize the bounds. See Appendix G for details.

Therefore, bounded twin-width implies bounded flip-width, but the converse does not hold: the class of subcubic graphs has bounded expansion, and hence bounded flip-width, but does not have bounded twin-width [BGK⁺21a]. So can twin-width be exactly characterized in terms of flip-width?

The twin-width parameter is defined not only for graphs, but also for structures equipped with one or more binary relations. As argued in [BGdM⁺22], twin-width may – and perhaps even should – be seen as a parameter of *ordered* graphs, rather than graphs. An ordered graph $G = (V, E, <)$ is equipped with a (symmetric, irreflexive) edge relation E and a total order relation $<$. Every graph can be equipped with some total order without increasing the twin-width, so we may assume the order is present (also such an order can be easily computed from an uncontraction sequence, but the problem of finding it efficiently given the graph G only, remains open).

Similarly as twin-width, the notion of flip-width extends to structures with several binary relations, as follows. Fix a signature Σ consisting of several binary relations. For a Σ -structure A and partition \mathcal{P} of $V(A)$, a \mathcal{P} -flip of A is specified by a Σ -structure F with domain \mathcal{P} . Applying the flip F results in a Σ -structure A' with domain $V(A)$ and relations

$$R_{A'} := R_A \triangle \bigcup_{(P,Q) \in R_F} P \times Q,$$

for each binary relation symbol $R \in \Sigma$. A k -flip of A is any structure A' obtained in this way, for some partition \mathcal{P} with $|\mathcal{P}| \leq k$. Using this notion of a k -flip, the flipper game and the flip-width parameters $\text{fw}_r(A)$ are defined just as for graphs (see Appendix B.4 for details), where in each round, the robber can traverse at most r edges of the Gaifman graph of the previously announced k -flip. Theorem 7.1 holds also in the case when G is a binary structure, rather than a graph, with the same proof. In the case of ordered graphs, it is convenient to work with the following variant of flip-width which takes into account that one of the binary relations is a total order. Fix an ordered graph $G = (V, E, <)$. A k -cut-flip of G is a triple $G' = (V, E', S)$, where (V, E') is a (usual) graph that is a k -flip of the graph (V, E) , and $S \subseteq V$ is a set with $|S| \leq k$. Intuitively, the robber will be able to instantaneously move between any two points that are not separated by an element of S . Let \sim_S denote the equivalence relation on S such that $u \sim_S v$ if and only if $u = v$ or $u < v$ and there is no $s \in S$ with $u \leq s \leq v$. The *weighted graph* associated to $G' = (V, E', S)$ is the graph with vertices V and edges uv such that $uv \in E'$ or $u \sim_S v$, where each edge uv with $u \sim_S v$ has weight 0, and the remaining edges have weight 1. Fix a radius $r \in \mathbb{N} \cup \{\infty\}$, a width parameter k , and an ordered graph $G = (V, E, <)$. In the *ordered flipper game* with radius r and width k on an ordered graph $G = (V, E, <)$, in round i , the cops announce a k -cut-flip $G_i = (V, E', S)$ of the ordered graph $G = (V, E, <)$, and the robber moves from his previous position v_{i-1} to a new position v_i by following a path of total weight at most r in the weighted graph associated with the previous k -cut-flip G_{i-1} (in round $i = 1$, as the cops announce G_1 , the robber picks $v_1 \in V$ arbitrarily). The cops win if v_i is isolated in the weighted graph associated to G_i , that is, there is no $w \neq v$ with $vw \in E'$ or $v \sim_S w$. The *radius- r ordered flip-width* of an ordered graph $G = (V, E, <)$, denoted $\text{fw}_r^<(G)$, is the smallest number k such that the cops have a winning strategy in the ordered flipper game with radius r and width k on G .

For an ordered graph $G = (V, E, <)$, the parameter $\text{fw}_r^<(G)$ relates to the flip-width of G , treated as a binary structure, as follows (see Appendix G.2).

Lemma 7.2 (*). Fix $r \in \mathbb{N} \cup \{\infty\}$ and an ordered graph $G = (V, E, <)$. Then

$$\sqrt{\text{fw}_r(G) + 1} \leq \text{fw}_r^<(G) + 1 \leq \text{fw}_{3r+2}(G) + 1.$$

We now prove the main result of Section 7, which says that for ordered graphs $G = (V, E, <)$, the parameters $\text{tw}(G)$ and $\text{fw}_1^<(G)$ are functionally equivalent. In other words, a class of ordered graphs has bounded twin-width if and only if it has bounded radius-one ordered flip-width.

Theorem 7.3. *The following conditions are equivalent for a class \mathcal{C} of ordered graphs:*

1. \mathcal{C} has bounded twin-width,
2. \mathcal{C} has bounded flip-width, that is, $\text{fw}_r(\mathcal{C}) < \infty$ for every $r \in \mathbb{N}$,
3. \mathcal{C} has bounded radius-five flip-width (as a class of binary structures), that is, $\text{fw}_5(\mathcal{C}) < \infty$,
4. \mathcal{C} has bounded ordered flip-width, that is, $\text{fw}_r^<(\mathcal{C}) < \infty$ for every $r \in \mathbb{N}$,
5. \mathcal{C} has bounded radius-one ordered flip-width, that is, $\text{fw}_1^<(\mathcal{C}) < \infty$.

So for ordered graphs, among the parameters $\text{fw}_r^<$, only the parameters $\text{fw}_\infty^<$ (characterizing bounded clique-width) and $\text{fw}_1^<$ (characterizing bounded twin-width) are relevant, since it follows that $\text{fw}_r^<(G)$ is bounded in terms of $\text{fw}_1^<(G)$ and r , for every $r \in \mathbb{N}$ and ordered graph G .

We obtain the following characterization of twin-width of usual, unordered graphs, in terms of flip-width. Recall that every graph G can be equipped with some total order without increasing the twin-width, and conversely, forgetting a total order of an ordered graph does not increase the twin-width.

Corollary 7.4. *A class \mathcal{C} of graphs has bounded twin-width if and only if every graph in \mathcal{C} can be equipped with a total order, so that the resulting class of ordered graphs has bounded flip-width.*

Proof of Theorem 7.3. The implication $1 \rightarrow 2$ in Theorem 7.1 is by Theorem 7.3 (stated for binary structures); the implications $2 \rightarrow 3$ and $4 \rightarrow 5$ are immediate; the implications $2 \rightarrow 4$ and $3 \rightarrow 5$ are by Lemma 7.2. It remains to prove the implication $5 \rightarrow 1$. This is done in Lemma 7.5 below, which proves that for ordered graphs, twin-width is bounded in terms of $\text{fw}_1^<$. \square

We show that ordered graphs of large twin-width have large $\text{fw}_1^<$. To this end, we use a core result of [BGODM⁺22], which states that an ordered graph has large twin-width if and only if it contains a k -rich division, for a large number k . A k -rich division of an ordered graph G is a pair of partitions \mathcal{L}, \mathcal{R} of $V(G)$, whose parts are intervals with respect to the order, such that for every interval $A \in \mathcal{L}$ and k intervals $B_1, \dots, B_k \in \mathcal{R}$, there are at least k vertices in A with pairwise distinct neighborhoods in $V(G) - (B_1 \cup \dots \cup B_k)$, and symmetrically, for every interval $B \in \mathcal{R}$ and k intervals $A_1, \dots, A_k \in \mathcal{L}$, there are at least k vertices in B with pairwise distinct neighborhoods in $V(G) - (A_1 \cup \dots \cup A_k)$. It is shown in [BGODM⁺22, Theorem 21] that if G has no k -rich division, then $\text{tw}(G) \leq 2^{O(k^2)}$. We now show that a $(k+1)$ -rich division can be employed by the robber to evade the cops in the ordered flipper game of radius 1 and width k . The following lemma immediately yields the implication $4 \rightarrow 1$ in Theorem 7.3, and finishes its proof.

Lemma 7.5. *Let G be an ordered graph with $\text{fw}_1^<(G) \leq k$. Then G does not have a $(k+1)$ -rich division \mathcal{L}, \mathcal{R} . In particular, $\text{tw}(G) \leq 2^{O(k^2)}$.*

Proof. Suppose G has a $(k+1)$ -rich division \mathcal{L}, \mathcal{R} . We show that $\text{fw}_1^<(G) > k$, by describing a winning strategy for the robber in the ordered flipper game with radius one and width k .

The strategy is as follows: in round i , when the cops announce a new k -cut-flip $G_i = (V, E_i, S_i)$ of G , the robber always moves to any reachable vertex in one of the parts of \mathcal{L} (in even-numbered rounds) or of \mathcal{R} (in odd-numbered rounds) that does not contain any element of S_i . We show that the robber can always reach such a vertex v_i from his previous position v_{i-1} , by following a path of weight 1 in the previous k -cut-flip G_{i-1} of G .

By inductive assumption, suppose that v_{i-1} belongs to a part A of \mathcal{L} or \mathcal{R} that does not contain any element of S_{i-1} . Suppose, by symmetry, that A is a part of \mathcal{L} . Since $|S_i| \leq k$, there are at most k parts of \mathcal{R} that contain some element of S_i ; denote those parts by B_1, \dots, B_k .

By the condition of a $(k+1)$ -rich division there are $k+1$ vertices in A with pairwise distinct neighborhoods in $B := V(G) - (B_1 \cup \dots \cup B_k)$. Then in the previous k -cut-flip G_{i-1} of G , there is an edge joining some vertex $a \in A$ with some vertex $b \in B$. Indeed, let \mathcal{P} denote the partition of $V(G)$ underlying the flip G_{i-1} of G , with $|\mathcal{P}| \leq k$. By the pigeonhole principle, two vertices a_1, a_2 of A with distinct neighborhoods in B , ie. $N_G(a_1) \cap B \neq N_G(a_2) \cap B$ belong to the same part of \mathcal{P} . Therefore also $N_{G_{i-1}}(a_1) \cap B \neq N_{G_{i-1}}(a_2) \cap B$, so one of those sets must be nonempty.

Thus, there is $a \in A$ and $b \in B$ such that $ab \in E(G_{i-1})$. The robber thus moves to b , by the path $v_{i-1} - a' - b$ of weight 1 in G_{i-1} , maintaining the invariant. \square

Reassuming, the duality result of [BGOdM⁺22] proving the equivalence of unbounded twin-width and having k -rich divisions for all k , can be seen as a min-max theorem for the flipper game of radius 1, for ordered graphs. Moreover, we now see that degeneracy and twin-width are two flip sides of the same coin: $\text{fw}_1(G)$ corresponds to the degeneracy of G for weakly sparse graphs (see Theorem 4.4 and Theorem 6.3), and $\text{fw}_1^<(G)$ corresponds to the twin-width of G for ordered graphs (see Theorem 7.3). Similarly, classes of bounded flip-width coincide with classes of bounded expansion in the weakly sparse case, and with classes of bounded twin-width in the ordered case. Both of those cases is by now well-understood from an algorithmic, combinatorial, and logical perspective. In particular, the model checking problem for first-order logic is fixed-parameter tractable in each of the two special cases.

8 Closure under transductions

As we have seen, classes of bounded flip-width include all classes of bounded expansion and all classes of bounded twin-width, and characterize those notions in the weakly sparse and totally ordered settings. We argue that classes of bounded flip-width enjoy many good closure properties. For instance, if two classes \mathcal{C} and \mathcal{D} have bounded flip-width, then their union $\mathcal{C} \cup \mathcal{D}$ also has bounded flip-width. Other such properties include: closure under disjoint unions (see Example 5.11), and closure under substitution (see Lemma 5.13).

An entire family of closure properties is provided by the notion of first-order *interpretations* or *transductions*. As a very special instance, we saw in Example 5.8 that if \mathcal{C} has bounded flip-width, then the class of edge-complements of graphs from \mathcal{C} also has bounded flip-width. What about, say, the class of squares of graphs from \mathcal{C} ? (The square of a graph G has vertices $V(G)$ and edges uv such that u and v have a common neighbor in G .) We show that this class also has bounded flip-width, and a similar result holds for every operation that can be defined by a first-order formula, as we now describe. We phrase our result in greater generality for colored graphs. This result has multiple corollaries, and generalizes previous results.

8.1 Preservation of flip-width under transductions

We start with defining interpretations and transductions, and then state the main result of this section.

Colored graphs Recall that a c -colored graph is a graph together with an assignment of colors from $\{1, \dots, c\}$ to its vertices. For a c -colored graph G , its radius- r flip-width $\text{fw}_r(G)$, for $r \in \mathbb{N} \cup \{\infty\}$, is defined as the same parameter for the underlying uncolored graph. Say that a class \mathcal{C} of c -colored graphs has *bounded flip-width* if the underlying class of uncolored graphs has bounded flip-width. A c -colored graph is seen as a structure over the signature consisting of the binary relation $E(x, y)$ denoting adjacency, as well as unary predicates $C_1(x), \dots, C_c(x)$ denoting the respective colors.

Interpretations The following notion is a special case of a (simple, domain-preserving) first-order interpretation. Let G be a c -colored graph and $\varphi(x, y)$ be a first-order formula in

the signature of c -colored graphs. Define the graph $\varphi(G)$ with vertices $V(G)$ and edges uv such that $u \neq v$ and $\varphi(u, v) \vee \varphi(v, u)$ holds in G . For a class \mathcal{C} of c -colored graphs, denote $\varphi(\mathcal{C}) := \{\varphi(G) \mid G \in \mathcal{C}\}$. The class $\varphi(\mathcal{C})$ is called an *interpretation* of \mathcal{C} , via φ . For example, for the formula $\varphi(x, y) = \neg E(x, y)$ and a graph G , the graph $\varphi(G)$ is the complement \bar{G} of G . And for the formula $\varphi(x, y) = \exists z.[E(x, z) \wedge E(z, y)]$, the graph $\varphi(G)$ is the square of G .

Transductions Say that a graph class \mathcal{C} *transduces* a graph class \mathcal{D} , or that \mathcal{D} is a *transduction* of \mathcal{C} , if there is some $c \geq 1$ and class $\hat{\mathcal{C}}$ of c -colored graphs which is a c -coloring of \mathcal{C} , and some first-order formula $\varphi(x, y)$ in the signature of c -colored graphs, such that every graph in \mathcal{D} is an induced subgraph of some graph in $\varphi(\hat{\mathcal{C}})$ (that is, \mathcal{D} is contained in the hereditary closure of $\varphi(\hat{\mathcal{C}})$).

Example 8.1. Let \mathcal{C} be the class of all half-graphs (see Figure 2), where the half-graph H_n of order n has vertices a_1, \dots, a_n and b_1, \dots, b_n , and edges $a_i b_j$ for $1 \leq i < j \leq n$. We show that \mathcal{C} transduces the class \mathcal{D} consisting of disjoint unions of cliques.

We use two colors. Let $\hat{\mathcal{C}}$ be the class of all 2-colored half-graphs. Consider the formula $\varphi(x, y)$ expressing that there is no vertex of color 2 which is adjacent to one of x, y , and not the other:

$$\varphi(x, y) \equiv \forall z. C_2(z) \rightarrow (E(x, z) \leftrightarrow E(y, z)).$$

We argue that the hereditary closure of $\varphi(\hat{\mathcal{C}})$ contains \mathcal{D} , implying that \mathcal{D} transduces in \mathcal{C} .

Let $F \in \mathcal{D}$ be a disjoint union of cliques. Let $1, \dots, n$ be the vertices of F , for some $n \geq 0$, and assume that every connected component of F consists of consecutive vertices in the usual order $1 < \dots < n$. Consider the half-graph H_n with vertices a_1, \dots, a_n and b_1, \dots, b_n , and color a vertex b_i with color 2 if i is the largest element of its connected component in F , and with color 1 otherwise. Now, for all $1 \leq i, j \leq n$, $H_n \models \varphi(a_i, a_j)$ if and only if i and j are adjacent in F . Hence, $\varphi(H_n)[\{a_1, \dots, a_n\}]$ is isomorphic to F , and therefore F is an induced subgraph of some graph in $\varphi(\hat{\mathcal{C}})$.

Transductions and flip-width We prove the following theorem.

Theorem 8.2 (*). Fix $q \geq 0$. There is a computable function $T_q: \mathbb{N} \rightarrow \mathbb{N}$ with the following property. Fix numbers $r, c \geq 1$ and a first-order formula $\varphi(x, y)$ of quantifier rank q in the signature of c -colored graphs. Set $r' := 2^q \cdot r$. Then for every c -colored graph G we have

$$\text{fw}_r(\varphi(G)) \leq T_q(\text{fw}_{r'}(G) \cdot c). \quad (9)$$

In particular, if \mathcal{C} has bounded flip-width, then $\varphi(\mathcal{C})$ has bounded flip-width. Moreover, $T_0(k) = k$ for every k .

Before discussing the proof of Theorem 8.2, we discuss its consequences.

The following is an immediate consequence of Theorem 8.2, and the fact that radius- r flip-width is monotone with respect to induced subgraphs.

Corollary 8.3. If a class \mathcal{C} has bounded flip-width and transduces a class \mathcal{D} , then \mathcal{D} has bounded flip-width.

Since the class of all graphs has unbounded flip-width by Corollary 5.23, we get the following.

Corollary 8.4. If \mathcal{C} is a class of bounded flip-width, then \mathcal{C} does not transduce the class of all graphs.

In the language of model theory, Corollary 8.4 says that classes of bounded flip-width are *monadically dependent*, see Section 10. In the phrasing of [GPT22, BNdMS22b], Corollary 8.3 says that classes of bounded flip-width form a *transduction ideal*, whose weakly sparse part, by Theorem 6.3, consists exactly of classes of bounded expansion.

A graph class has *structurally bounded expansion* [GKN⁺20] if it is a transduction of a class with bounded expansion. Corollary 8.3 immediately implies that those classes have bounded flip-width.

Corollary 8.5. *Every class of structurally bounded expansion has bounded flip-width.*

We remark that an extension of Theorem 8.2 and of Corollary 8.3 also hold, with the same proof (but a slightly different bound in (9)), for input structures over a binary signature Σ , rather than graphs G , and where instead of one formula $\varphi(x, y)$ defining a new edge relation of the output graph, we have a tuple of formulas $\bar{\varphi} = (\varphi_1(x, y), \dots, \varphi_k(x, y))$ defining k binary relations of the output structure $\bar{\varphi}(G)$ with domain $V(G)$ and i th relation R_i , for $i = 1, \dots, k$, interpreted as

$$R_i := \{(a, b) \mid a, b \in V(G), G \models \varphi_i(a, b)\}.$$

We then obtain that if \mathcal{C} is a class of Σ -structures of bounded flip-width (see Appendix B.4), then the class $\bar{\varphi}(\mathcal{C}) := \{\bar{\varphi}(G) \mid G \in \mathcal{C}\}$ also has bounded flip-width.

This implies (using Corollary 7.4) the result of [BKTW20], that transductions preserve classes of bounded twin-width.

Corollary 8.6 ([BKTW20]). *If \mathcal{C} has bounded twin-width and $\varphi(x, y)$ is a first-order formula, then $\varphi(\mathcal{C})$ has bounded twin-width.*

Indeed, let \mathcal{C} be a class of bounded twin-width and $\varphi(x, y)$ a symmetric formula in the signature of graphs. By Corollary 7.4 there is a class $\hat{\mathcal{C}}$ of ordered graphs of bounded flip-width such that \mathcal{C} is obtained from $\hat{\mathcal{C}}$ by forgetting the order. Let $\bar{\varphi}$ be the pair consisting of the formula $\varphi(x, y)$ and the formula $x < y$ defining the order. Then $\bar{\varphi}(\hat{\mathcal{C}})$ is a class of ordered graphs which has bounded flip-width, since $\hat{\mathcal{C}}$ has bounded flip-width. Moreover, the graph class $\varphi(\mathcal{C})$ is obtained from the class of ordered graphs $\bar{\varphi}(\hat{\mathcal{C}})$ by forgetting the order. Hence, $\varphi(\mathcal{C})$ has bounded twin-width, by Corollary 7.4.

Together with Theorem 6.3, Corollary 8.3 implies the following consequence of [GKN⁺18].

Corollary 8.7. *Every weakly sparse class with structurally bounded expansion has bounded expansion.*

The proof of Theorem 8.2, which is sketched below, relies on locality of first-order logic, a central notion for analysing first-order formulas on sparse graphs [See96], which, in some form, also plays a key role in understanding first-formulas on classes of bounded twin-width [BKTW20, GPPT22]. The function $T_q(k)$, although computable, is astronomical:

$$T_q(k) := \underbrace{2^{2^{\cdot^{\cdot^{\cdot^2^m}}}}}_{\text{height } q}$$

where m is the number of distinct k -colored graphs with vertex set $\{1, \dots, q+1\}$. However, for $q = 0$ we have $T_0(k) = k$, so the bound (9) becomes $\text{fw}_r(\varphi(G)) \leq c \cdot \text{fw}_r(G)$ in the case when $\varphi(x, y)$ is a quantifier-free formula and G is a c -colored graph with underlying graph G_0 .

CMSO transductions and radius- ∞ flip-width For the case of radius $r = \infty$, we prove in that classes \mathcal{C} with $\text{fw}_\infty(\mathcal{C}) < \infty$ are preserved under transductions expressed in the more powerful logic called CMSO. We remark that this result is known for classes of bounded clique-width, so our result follows from Theorem 5.17, which states that \mathcal{C} has bounded clique-width if and only if $\text{fw}_\infty(\mathcal{C}) < \infty$. However, we provide a separate proof here, as it is very analogous to the proof of Theorem 8.2, and nicely illustrates the parallels between the bounded flip-width case, and the limit case of bounded radius- ∞ flip-width. We also show how the closure under CMSO transductions can be used to obtain an alternative proof of Theorem 5.17. Let us first define the logic, and state the result.

Counting Monadic Second Order Logic (CMSO) is the extension of first-order logic, where apart from first-order quantifiers $\exists x, \forall x$, that range over vertices x of a graph, we have second-order quantifiers $\exists X, \forall X$, that range over sets X of vertices of the graph. We additionally have the atomic predicate $x \in X$ that allows to check whether a given vertex belongs to a given

set, and the divisibility predicates $\text{div}_k(X)$, for $k \geq 1$, where $\text{div}_k(X)$ holds for a given set of vertices X if and only if $|X|$ is divisible by k . The usual constructs of first-order logic (boolean connectives and relation symbols, such as adjacency in a graph) are also included.

CMSO is able to express non-local properties. For instance, the formula

$$\varphi(x, y) = \neg \exists X. [(x \in X) \wedge (y \notin X) \wedge \forall z. \forall t. [(z \in X) \wedge E(z, t) \rightarrow (t \in X)]]$$

expresses that x and y lie in the same connected component.

Still, CMSO enjoys property similar to locality of first-order logic, called compositionality (or a Feferman-Vaught-Mostowski type result), which is an analogue of locality for the radius $r = \infty$. Using this instead of locality, with the same proof as in Theorem 8.2, we get the following.

Theorem 8.8 (*). *Let \mathcal{C} be a class of c -colored graphs of bounded ∞ -flip-width and let $\varphi(x, y)$ be a formula of CMSO. Then $\varphi(\mathcal{C})$ has bounded ∞ -flip-width.*

Say that a class \mathcal{C} CMSO-transduces a class \mathcal{D} , or that \mathcal{D} is a CMSO-transduction of \mathcal{C} , if there is a c -coloring $\hat{\mathcal{C}}$ of \mathcal{C} , for some $c \geq 1$, and some CMSO formula $\varphi(x, y)$ in the signature of c -colored graphs, such that every graph in \mathcal{D} is an induced subgraph of some graph in $\varphi(\hat{\mathcal{C}})$ (that is, \mathcal{D} is contained in the hereditary closure of $\varphi(\hat{\mathcal{C}})$).

Corollary 8.9. *If \mathcal{C} is a class with $\text{fw}_\infty(\mathcal{C}) < \infty$ that CMSO-transduces a class \mathcal{D} , then $\text{fw}_\infty(\mathcal{D}) < \infty$.*

We now show that, together with a result of Courcelle and Oum, Corollary 8.9 yields an alternative proof of the backwards implication in Theorem 5.17, that classes of bounded radius- ∞ flip-width have bounded clique-width.

Courcelle and Oum [CiO07, Corollary 7.5] proved the following result⁸.

Theorem 8.10. *Every graph class of unbounded clique-width CMSO-transduces the class of all graphs.*

In particular, as the class of all graphs has unbounded radius-one flip-width by Corollary 5.23, it follows from Corollary 8.9 and Theorem 8.10 that every class \mathcal{C} of unbounded clique-width has $\text{fw}_\infty(\mathcal{C}) = \infty$, proving the backward implication in Theorem 5.17.

8.2 Transferring strategies

Before giving the details about the proof of Theorem 8.2, we define a tool that will be also useful in other contexts. The following notion allows to transfer a winning strategy of the cops from a graph G to a graph H . Let G and H be two graphs with $V(H) \subseteq V(G)$. Fix $r_G, r_H \in \mathbb{N} \cup \{\infty\}$ and $k, \ell \in \mathbb{N}$. Suppose furthermore that the following are given:

- a mapping F that maps each k -flip G' of G to an ℓ -flip $H' = F(G')$ of H ,
- a strategy of the cops in the game on the graph G with radius r_G and width k .

This induces the following strategy of the cops in the flipper game of radius r_H and width ℓ on the graph H ; call this game the H -game. Consider a play of this game the H -game. Simultaneously, initiate the G -game on G , with radius r_G and width k , in which we will copy robber's moves from the H -game. Whenever the cops announce a k -flip G' of G in the G -game, then in the H -game the cops announce the ℓ -flip $H' = F(G')$ of H . Whenever the robber moves to a vertex v in the H -game, we also move the robber to v in the H -game. Note that this move might not be a valid move in the H -game. In this case, the cops continue the G -game by playing arbitrarily (e.g. always announcing G as the next flip) until the end of the game. We say the resulting strategy of the cops (in the G -game) is *transferred* from the considered strategy in the H -game, according to the mapping F .

The following lemma gives a condition which implies that if the original strategy on G is winning, then the transferred strategy on H is winning.

⁸In fact, they proved an equivalent statement, but with 'all square grids' instead of 'all graphs'. However, it is easy to see that the class of all square grids CMSO-transduces the class of all graphs, and transductions can be composed.

Lemma 8.11. Fix $r \in \mathbb{N} \cup \{\infty\}$ and $k, \ell, s \geq 1$. Let H, G be two graphs with $V(H) \subseteq V(G)$. Suppose that for all $k \geq 1$ and every k -flip G' of G there is some ℓ -flip $H' = F(G')$ of H , such that

$$\text{dist}_{G'}(u, v) \leq s \quad \text{for all } uv \in E(H'). \quad (10)$$

Then transferring a winning strategy of the cops from the flipper game on G with radius rs and width k , according to the mapping F , results in a winning strategy of the cops in the flipper game on H with radius r and width ℓ . In particular,

$$\text{fw}_{rs}(G) \leq k \quad \text{implies} \quad \text{fw}_r(H) \leq \ell.$$

In the case $r = \infty$, the conclusion reads “ $\text{fw}_\infty(G) \leq k$ implies $\text{fw}_\infty(H) \leq \ell$ ”. Moreover we can then replace $\leq s$ by $< \infty$ in (10), since for $s := |V(G)|$ we have that $\text{dist}_{G'}(u, v) \leq s \iff \text{dist}_{G'}(u, v) < \infty$. We state this explicitly below.

Corollary 8.12. Fix $k, \ell \geq 1$. Let H, G be two graphs with $V(H) \subseteq V(G)$. Suppose that for all $k \geq 1$ and every k -flip G' of G there is some ℓ -flip $H' = F(G')$ of H , such that

$$\text{dist}_{G'}(u, v) < \infty \quad \text{for all } uv \in E(H'). \quad (11)$$

Then

$$\text{fw}_\infty(G) \leq k \quad \text{implies} \quad \text{fw}_\infty(H) \leq \ell.$$

Proof of Lemma 8.11. Consider a play in the flipper game on H of radius r and width ℓ , according to the strategy transferred from G , as described above. Suppose that, in some round, G' is the announced flip in the G -game and $H' = F(G')$ is the announced flip in the H -game, and that v is the new vertex chosen by the robber in the H -game. By (10), the following holds:

$$B_{H'}^r(v) \subseteq B_{G'}^{rs}(v).$$

In the case $r = \infty$, each side of the inclusion should be interpreted as the connected component of v in the appropriate graph.

In particular, in the next round, every valid move of robber in the H -game will also be a valid move of robber in the G -game (this is trivially satisfied in the first round), and if robber is trapped in the G -game, that is, $|B_{G'}^{rs}(v)| = 1$, then also $|B_{H'}^r(v)| = 1$, so he is also trapped in the H -game. Thus, this describes a winning strategy for the cops in the flipper game on H with radius r and width ℓ . In particular, $\text{fw}_r(H) \leq \ell$. \square

8.3 Proof of Theorem 8.2

We now sketch the proof of Theorem 8.2. The details are presented in Appendix H, along with the proof of Theorem 8.8.

For simplicity, we assume the case $c = 1$, that is, when the considered graphs G have no colors. The general case proceeds analogously.

Our aim is to transfer a winning strategy of the cops from G to $H = \varphi(G)$ (with appropriate radii), by applying Lemma 8.11. So for every flip G' of G we need to produce a flip $\varphi(G)'$ of $\varphi(G)$ such that adjacent vertices in $\varphi(G)'$ are not too far in G' . We first show how to achieve this in the case when $G' = G$, using locality, a well known tool from finite model theory, which we now recall.

Locality Fix a number $s \in \mathbb{N}$. Say that a formula $\varphi(x, y)$ is s -local, if for any graph G there is a labelling of the vertices of G using a bounded number of labels (depending only on φ , and not on G) such that for any two vertices u, v of G with $\text{dist}_G(u, v) > s$, whether or not $G \models \varphi(u, v)$ depends only on the label of u and the label of v . It is well-known (and follows for instance from Gaifman’s locality theorem) that every formula $\varphi(x, y)$ of first-order logic is s -local for some radius s depending only on the quantifier rank q of φ . Namely, one can take $s := 2^q$. The label assigned to a vertex v of G as above, is essentially the set of formulas $\alpha(x)$ of quantifier rank at most q , such that $\alpha(v)$ holds in G . The number of such formulas, up to equivalence, is finite, and is bounded by $T_q(1)$, where T_q is the function described above.

Flipping $\varphi(G)$ We now show how to obtain a flip $\varphi(G)'$ of $\varphi(G)$ such that vertices that are adjacent in $\varphi(G)'$ are not too far in G . Let \mathcal{P} be the partition of $V(G) = V(\varphi(G))$ such that two vertices of G are in the same part if they get the same label. In particular, $|\mathcal{P}| \leq T_q(1)$. Now, in the graph $\varphi(G)$ flip a pair of parts A, B of \mathcal{P} if and only if there is a pair of vertices $u \in A$ and $v \in B$, such that $\text{dist}_G(u, v) > s$ and $G \models \varphi(u, v)$, equivalently, $uv \in E(\varphi(G))$. The statement above implies that whether or not we flip A and B , does not depend on the choice of $u \in A$ and $v \in B$ such that $\text{dist}_G(u, v) > s$. This yields a \mathcal{P} -flip of $\varphi(G)$, which we denote $\varphi(G)'$. Then the following holds for all $u, v \in V(G)$:

$$uv \in E(\varphi(G)') \quad \text{implies} \quad \text{dist}_G(u, v) \leq s.$$

We now generalize the above reasoning to the case when G' is a k -flip of G . Again, the goal is to construct a flip $\varphi(G)'$ of $\varphi(G)$ such that vertices that are adjacent in $\varphi(G)'$ are not too far in G' .

We treat G' as a k -colored graph, by adding colors that mark parts of the partition that is used to produce the k -flip G' of G . The key observation is that we can write a formula $\varphi'(x, y)$ that makes use of those colors, and such that

$$G' \models \varphi'(u, v) \quad \Longleftrightarrow \quad G \models \varphi(u, v) \quad \text{for all } u, v \in V(G).$$

This is because we can write a formula $\varepsilon(x, y)$ such that $G' \models \varepsilon(u, v)$ if and only if $G \models E(u, v)$ (the formula $\varepsilon(x, y)$ checks the colors of x and y , whether x and y are adjacent in G' , and inverts the flip). The formula $\varphi'(u, v)$ is obtained by replacing each atomic formula $E(z, t)$ with $\varepsilon(z, t)$. In particular, φ' has the same quantifier rank as φ , and is therefore also s -local. Applying the same argumentation as above, but this time to the formula φ' and the k -colored graph G' , we obtain a \mathcal{P} -flip $\varphi(G)'$ such that the following holds for all $u, v \in V(G)$:

$$uv \in E(\varphi(G)') \quad \text{implies} \quad \text{dist}_{G'}(u, v) \leq s.$$

Moreover, the size of \mathcal{P} can be bounded by a number ℓ depending on the formula φ and the number k . Since this holds for every k -flip G' of G , we can now apply Lemma 8.11 and conclude that $\text{fw}_r(\varphi(G)) \leq \ell$.

9 Definable flip-width

Determining the flip-width of radius r of a given graph G seems computationally difficult. The space of all configurations in the flipper game of radius r and width k (consisting of a k -flip of G and robber's position) has size exponential in $|G|$, and the naive algorithm for determining whether $\text{fw}_r(G) \leq k$, which explores the space of all configurations, therefore runs in time exponential in $|G|$. We expect that for possible algorithmic applications, an algorithm which approximates $\text{fw}_r(G)$, instead of computing it exactly, should suffice. This is what happens in the case of weak coloring numbers, and of twin-width, where such an approximation algorithm is not known in general, but is known in various special cases.

In this section, we introduce the *definable flipper game* of radius r and width k , in which the moves allowed for the cops are parameterized by tuples of vertices of the graph, rather than by partitions of the vertex set. Among other things, this reduces (comparing to the flipper game) the number of configurations, and the computational complexity of determining the definable flip-width of a given graph. The main result of this section is Theorem 9.4 that says that the definable flip-width at radius r can be bounded in terms of the flip-width at radius $5r$. This allows to obtain, in Theorem 9.7, an algorithm for approximating flip-width, which runs in time $|G|^{O(k)} \cdot O_k(1)$, where k is the flip-width. We use tools related to VC-dimension to accomplish this.

Atomic types and definable flips Let S be a set of vertices of a graph G . Consider the partition \mathcal{P} of $V(G)$ such that two vertices $u, v \in V(G)$ are in the same part if $N(u) \cap S = N(v) \cap S$. The equivalence classes of the partition \mathcal{P} are called S -types. In particular, $|\mathcal{P}| \leq 2^{|S|}$.

Say that a graph G' is an S -definable flip of G if G' is a \mathcal{P} -flip of G , where \mathcal{P} is the partition of $V(G)$ into S -types. Say that G' is a k -definable flip of G , if G' is an S -definable flip of G for some $S \subseteq V(G)$ with $|S| \leq k$.

Note that a k -definable flip of G is a 2^k -flip of G . However, there is no function f such every k -flip of a graph G is a $f(k)$ -definable flip of G . For instance, the graph G_n obtained from an n -clique K_n by adding n isolated vertices, is a 2-flip of K_{2n} , but is not an $(n-1)$ -definable flip of K_{2n} .

Definable flipper game Fix $r \in \mathbb{N} \cup \{\infty\}$. The *definable flipper game of radius r and width k* is defined in the same way as the flipper game of radius r , but now in each round the cops are allowed to announce a k -definable flip G' of G , rather than a k -flip of G .

Definition 9.1. Fix $r \in \mathbb{N} \cup \{\infty\}$. The *radius- r definable flip-width of a graph G* , denoted $\text{dfw}_r(G)$, is the smallest number k such that the cops have a winning strategy in the definable flipper game of radius r and width k on G .

As every k -definable flip of G is a 2^k -flip of G , it follows that for every $r \in \mathbb{N} \cup \{\infty\}$ and graph G we have:

$$\text{fw}_r(G) \leq 2^{\text{dfw}_r(G)}. \quad (12)$$

One advantage of the definable version of the flipper game is that it has far fewer configurations than the original flipper game. As there are only $O(|G|^{k+1} \cdot 2^{4^k})$ configurations in the definable flipper game of width k , we get that it can be decided in time $|G|^{O(k)} \cdot 2^{O(2^k)}$ whether a given graph G has $\text{dfw}_r(G) \leq k$ (see Appendix B.5 for a proof).

Lemma 9.2 (*). There is an algorithm that, given a graph G and numbers $k \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{\infty\}$, determines whether $\text{dfw}_r(G) \leq k$ in time $n^{O(k)} \cdot 2^{O(2^k)}$.

The following lemma is an immediate consequence of the Sauer-Shelah-Perles lemma (Lemma 3.8).

Lemma 9.3 (*). Fix $r \in \mathbb{N} \cup \{\infty\}$. For every graph G we have:

$$\text{fw}_r(G) \leq O(\text{dfw}_r(G)^{\text{VCdim}(G)}). \quad (13)$$

The following is the main result of Section 9. It says that the definable flip-width can be bounded in terms of the flip-width, at the cost of increasing the radius. Below, $5 \cdot \infty$ is interpreted as ∞ .

Theorem 9.4. Fix $r \in \mathbb{N} \cup \{\infty\}$. For every graph G we have:

$$\text{dfw}_r(G) \leq O(\text{fw}_{5r}(G)^3). \quad (14)$$

We prove Theorem 9.4 below. We first observe some consequences. The bounds (12) and (14) give the following.

Corollary 9.5. The following conditions are equivalent for a graph class \mathcal{C} :

1. \mathcal{C} has bounded flip-width, that is, $\text{fw}_r(\mathcal{C}) < \infty$ for all $r \geq 1$,
2. $\text{dfw}_r(\mathcal{C}) < \infty$ for all $r \geq 1$.

Similarly, for the case $r = \infty$, using Theorem 5.17 we get the following characterization of classes of bounded clique-width in terms of the definable flipper game with radius ∞ :

Corollary 9.6. The following conditions are equivalent for a graph class \mathcal{C} :

1. \mathcal{C} has bounded clique-width,
2. $\text{dfw}_\infty(\mathcal{C}) < \infty$.

Finally, we get an algorithm for approximating the flip-width of a given graph. The algorithm is an approximation algorithm: unlike the algorithm in Lemma 9.2, it does not allow to exactly determine whether the radius- r flip-width of a given graph G is smaller than a given number k . Rather, it recognises one of two, non-exclusive, cases: whether $\text{fw}_r(G)$ is small comparing to k , and whether $\text{fw}_{5r}(G)$ is large comparing to k . Note that there is a gap in the radii, r and $5r$. The running time of the algorithm is $O_k(1) \cdot n^{O(k)}$, which is called an XP algorithm (parameterized by k) in the language of parameterized complexity.

Theorem 9.7. *There is a constant $C > 0$ and an algorithm that inputs a graph G and numbers $r, k \in \mathbb{N}$, runs in time $n^{O(k)} \cdot 2^{O(2^k)}$, and either concludes that $\text{fw}_r(G) \leq 2^k$, or concludes that $\text{fw}_{5r}(G) \geq C \cdot k^{1/3}$.*

Proof. The algorithm tests whether $\text{dfw}_r(G) \leq k$ in time $n^{O(k)} \cdot 2^{O(2^k)}$, using Lemma 9.2. If $\text{dfw}_r(G) \leq k$ it concludes that $\text{fw}_r(G) \leq 2^k$, by (12). If $\text{dfw}_r(G) > k$, it concludes that $\text{fw}_{5r}(G) \geq Ck^{1/3}$ by (14), where $C > 0$ is some fixed constant. \square

Proof of Theorem 9.4 We now turn to the proof of Theorem 9.4. We use a result concerning graphs of small VC-dimension. Recall that $\text{VCdim}(G) \leq O(\text{fw}_1(G))$ by Theorem 5.24. The following result from [BDG⁺22], relies on the (p, q) -theorem of Alon-Kleitman-Matoušek [Mat04] (see Appendix E.2 and Appendix E.3).

Lemma 9.8 (*). *Fix $k, d \in \mathbb{N}$. Let V be a set equipped with:*

- *a binary relation $E \subseteq V \times V$ of VC-dimension at most d ,*
- *a pseudometric $\text{dist}: V \times V \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ (that is, a function satisfying the triangle inequality),*
- *and a partition \mathcal{P} of size at most k ,*

such that $E(u, v)$ depends only on the \mathcal{P} -class of u and the \mathcal{P} -class of v whenever $\text{dist}(u, v) > 1$. Then there is a set $S \subseteq V$ of size $O(dk^2)$, such that $E(u, v)$ depends only on the S -types of u and of v , whenever $\text{dist}(u, v) > 5$.

We reformulate Lemma 9.8 in terms of flips, as follows.

Corollary 9.9. *Let G be a graph and $d = \text{VCdim}(G)$, and let G' be a k -flip of G . Then there is a $O(dk^2)$ -definable flip H' of G such that*

$$\text{dist}_{G'}(u, v) \leq 5 \quad \text{for all } u, v \in E(H') \quad (15)$$

Proof. Apply Lemma 9.8 to $V = V(G)$, $E = E(G)$, $\text{dist}: V \times V \rightarrow \mathbb{N} \cup \{\infty\}$ denoting the shortest path metric in G' , and the partition \mathcal{P} with $|\mathcal{P}| \leq k$ such that G' is a \mathcal{P} -flip of G . Note that for any two vertices $u, v \in V$, whether or not $uv \in E(G)$ holds, can be determined basing only on the \mathcal{P} -class of u , the \mathcal{P} -class of v , and on the information whether $uv \in E(G')$ holds. In particular, $uv \in E(G)$ depends only on the \mathcal{P} -class of u and the \mathcal{P} -class of v , for all $u, v \in V$ that are not adjacent in G' , equivalently, with $\text{dist}(u, v) > 1$. Hence, the assumption of Lemma 9.8 is satisfied.

Let S with $|S| \leq O(dk^2)$ be as in the conclusion of the lemma, so that whether or not $uv \in E(G)$, depends only on the S -type of u and the S -type of v , for all $u, v \in V$ with $\text{dist}(u, v) > 5$.

Let \mathcal{P}_S denote the partition of $V(G)$ into S -types. Let H' be the \mathcal{P}_S -flip of G that flips between two S -types A and B if and only if there are some $u \in A, v \in B$ with $\text{dist}(u, v) > 5$ and $uv \in E(G)$. The conclusion follows. \square

Theorem 9.4 easily follows from Corollary 9.9.

Proof of Theorem 9.4. Let G be a graph and let $k = \text{fw}_{5r}(G)$. In particular, by Theorem 5.24, we have that $\text{VCdim}(G) \leq O(k)$. By Corollary 9.9 applied to $H := G$, the assumptions of

Lemma 8.11 are satisfied, where the ℓ -flip H' of $H = G$, for $\ell = 2^{O(k^3)}$ is a $O(k^3)$ -definable flip, as provided by Corollary 9.9. By transferring the winning strategy of the cops in the flipper game of radius $5r$ and width k on G , according to this mapping $G' \mapsto H'$, we get a strategy for the cops in flipper game of radius r on G , which uses only $O(dk^2)$ -definable flips. By Lemma 8.11, this yields a winning strategy of the cops that will use only $O(k^3)$ -definable flips. Hence, $\text{dfw}_r(G) \leq O(k^3)$. \square

10 Almost bounded flip-width

Recall from Fact 3.7 that a hereditary graph class \mathcal{C} is nowhere dense if and only if for every $r \geq 1$ and $G \in \mathcal{C}$ we have $\text{wcol}_r(G) = |G|^{o(1)}$. Inspired by this characterization, we extend the notion of bounded flip-width as follows.

Definition 10.1. *A graph class \mathcal{C} has almost bounded flip-width if for every $r \geq 1$ and real $\varepsilon > 0$ we have $\text{fw}_r(G) \leq O_{\varepsilon,r}(|G|^\varepsilon)$ for every graph G in the hereditary closure of \mathcal{C} .*

Note that we consider all graphs G from the hereditary closure of \mathcal{C} . Otherwise, the class consisting of every graph G with $2^{|G|}$ isolated vertices added to it, would have almost bounded flip-width, while according to the above definition, it does not. Indeed, we have the following lemma, which is an immediate consequence of Lemma 5.26.

Lemma 10.2. *Every graph class with almost bounded flip-width has bounded VC-dimension.*

Clearly, every class with bounded flip-width has almost bounded flip-width. As we conjecture (see Conjecture 10.7), classes of almost bounded flip-width coincide with monadically dependent classes (see definition below), analogously to the characterization of nowhere dense classes in Fact 3.7.

In this section, we provide some evidence towards this conjecture. In Theorem 10.9, we prove that a weakly sparse class has almost bounded flip-width if and only if it is nowhere dense, if and only if it is monadically dependent. In Theorem 10.12 we prove that structurally nowhere dense classes have almost bounded flip-width. In Theorem 10.15 we prove that edge-stable classes of almost bounded flip-width are monadically dependent. In Theorem 10.17 we prove that classes of ordered graphs of almost bounded flip-width coincide with classes of bounded twin-width, and with classes of bounded flip-width. We start with recalling the discussed notions.

10.1 Monadic dependence and monadic stability

The following notion, due to Shelah [She86] (see also [BL21]), originates in model theory.

Definition 10.3. *A graph class \mathcal{C} is monadically dependent (or monadically NIP) if and only if it does not transduce the class of all graphs.*

Monadically dependent classes have recently attracted attention in areas of structural and algorithmic graph theory [AA14, NdMP⁺21, BGOdM⁺22, GPT22], as it is conjectured (see Conjecture 1.1) that monadically dependent classes are precisely those for which model checking first-order logic is fixed-parameter tractable.

Monadically dependent classes include all nowhere dense classes, and in fact, among weakly sparse classes, they provide an exact characterization:

Fact 10.4 (Consequence of [Dvo18]+[AA14]). *Let \mathcal{C} be a weakly sparse graph class. Then \mathcal{C} is nowhere dense if and only if \mathcal{C} is monadically dependent.*

Monadically dependent classes also include all classes of bounded flip-width, by Corollary 8.4.

An important subfamily of monadically dependent classes consists of monadically stable classes. A graph class is *monadically stable* if it does not transduce the class of all half-graphs

(see Fig. 2). A graph class is *edge-stable* if it excludes some half-graph as a semi-induced bipartite graph. More precisely, there is $k \geq 1$ such that there do not exist $G \in \mathcal{C}$ and vertices $a_1, \dots, a_k, b_1, \dots, b_k$ of G such that $a_i b_j \in E(G) \iff i < j$ for all $i, j \in \{1, \dots, k\}$.

The following result is proved in [NdMP⁺21, Theorem 1.3], see also [BL22, Theorem 3.20].

Fact 10.5. *Let \mathcal{C} be an edge-stable graph class. Then \mathcal{C} is monadically stable if and only if it is monadically dependent.*

Monadically stable classes include all nowhere dense classes [AA14], as well as transductions of nowhere dense classes, called *structurally nowhere dense* classes. It is not known whether all monadically stable classes are structurally nowhere dense (this has been conjectured in [NdMP⁺21, Conjecture 6.1]). The class of half-graphs is clearly not monadically stable, and has bounded (linear) clique-width. Hence, monadically stable classes are incomparable with classes of bounded clique-width. They are also incomparable with classes of bounded flip-width, as witnessed by the class of half-graphs on one side, and any nowhere dense class which does not have bounded expansion on the other side.

Monadic dependence can be defined not only for graph classes, but for arbitrary classes of structures, e.g. classes of ordered graphs. A class of ordered graphs is monadically dependent if it does not transduce the class of all graphs, where now the transduction may involve the edge relation symbol, as well as the total order $<$ (and the color predicates). The following result is proved in [BGdM⁺22].

Fact 10.6. *Let \mathcal{C} be a class of ordered graphs. Then \mathcal{C} is monadically dependent if and only if \mathcal{C} has bounded twin-width.*

In this section, we study classes of almost bounded flip-width in the three settings discussed above: of weakly sparse classes, of edge-stable classes, and of classes of ordered graphs. We show that in the first and last settings, those classes coincide with monadically dependent classes, and that in the edge-stable case they also coincide, assuming all monadically stable classes are structurally nowhere dense.

We conjecture that classes of almost bounded flip-width coincide exactly with the monadically dependent classes.

Conjecture 10.7. *A graph class has almost bounded flip-width if and only if it is monadically dependent.*

Currently, we are unable to prove neither implication in this conjecture. However, in the rest of Section 10, we provide evidence towards this conjecture, by confirming it in restricted settings.

Another conjecture that is supported by the evidence presented below, predicts a collapse result: that a bound $o(n^{1/2})$ on the flip-width parameters implies a bound $o(n^\varepsilon)$, for every fixed $\varepsilon > 0$.

Conjecture 10.8. *The following conditions are equivalent for a hereditary graph class \mathcal{C} :*

1. *\mathcal{C} has almost bounded flip-width, that is, for every fixed $r \in \mathbb{N}$ and $\varepsilon > 0$ we have that $\text{fw}_r(G) = o(|G|^\varepsilon)$ holds for all $G \in \mathcal{C}$,*
2. *for every fixed $r \in \mathbb{N}$ we have that $\text{fw}_r(G) = o(|G|^{1/2})$ holds for all $G \in \mathcal{C}$.*

A similar collapse occurs in the sparse case [NdM12].

10.2 Weakly sparse classes of almost bounded flip-width

Analogously to Theorem 6.3, which characterizes classes with bounded expansion as exactly the weakly sparse classes of bounded flip-width, we get a characterization of nowhere dense classes in terms of almost bounded flip-width.

Theorem 10.9. *Let \mathcal{C} be a weakly sparse graph class. Then the following conditions are equivalent:*

1. \mathcal{C} is nowhere dense,
2. \mathcal{C} has almost bounded flip-width,
3. \mathcal{C} is monadically dependent.

The equivalence of the first and last condition is by Fact 10.4, so we only prove the equivalence of the first two.

Proof. We first show that every nowhere dense class has almost bounded flip-width. Every nowhere dense class \mathcal{C} is weakly sparse, so excludes some $K_{t,t}$ as a subgraph. By Theorem 5.6, Theorem 4.5, and the forward implication in Fact 3.7, we have that $\text{fw}_r(G) \leq O_{r,\varepsilon}(n^{t\varepsilon})$ for every n -vertex graph $G \in \mathcal{C}$. Since this holds for every $\varepsilon > 0$ and t is fixed, the conclusion follows by rescaling ε .

Conversely, suppose that \mathcal{C} has almost bounded flip-width, and excludes $K_{t,t}$ as a subgraph. Without loss of generality, we may assume that \mathcal{C} is hereditary. We have $\text{fw}_1(G) \leq O_\varepsilon(n^\varepsilon)$ for every n -vertex graph $G \in \mathcal{C}$, and by Theorem 6.1, $\text{degeneracy}(G) \leq O_\varepsilon(n^\varepsilon \cdot 2t^2) \leq O_\varepsilon(n^\varepsilon)$. By Theorem 6.4, for every $r \geq 1$ and $\varepsilon > 0$ we have

$$\tilde{\text{v}}_{r-1}(G) \leq (r \cdot \text{fw}_r(G) \cdot \text{degeneracy}(G))^{O(1)} \leq (r \cdot O_{r,\varepsilon}(|G|^\varepsilon) \cdot O_\varepsilon(|G|^\varepsilon))^{O(1)} \leq O_{r,\varepsilon}(|G|^{O(\varepsilon)}),$$

for all $G \in \mathcal{C}$. By Facts 3.3 and 3.5 we have $\text{wcol}_r(G) \leq O_{r,\varepsilon}(|G|^{O(\varepsilon)})$ for every $r \geq 1$ and graph $G \in \mathcal{C}$. Therefore, \mathcal{C} is nowhere dense, by the backwards implication in Fact 3.7. \square

As there exist nowhere dense classes of unbounded degeneracy, we get the following.

Corollary 10.10. *There is a class that has almost bounded flip-width, but does not have bounded flip-width.*

By taking the substitution closure (see Section 5.1) of the class from the corollary above, we obtain a class which has almost bounded flip-width (by Lemma 5.13), but is not nowhere dense (not even edge-stable), and has unbounded flip-width.

10.3 Structurally nowhere dense classes

We are unable to determine whether classes of almost bounded flip-width are closed under transductions. Observe that the bound $\text{fw}_r(\varphi(G)) \leq T_q(\text{fw}_{r'}(G))$ in Theorem 8.2 is not polynomial in $\text{fw}_{r'}(G)$. It is, however, linear in the case when φ is a quantifier-free formula $\varphi(x, y)$, so we get the following.

Corollary 10.11. *Let \mathcal{C} be a class of k -colored graphs of almost bounded flip-width, and let $\varphi(x, y)$ be a quantifier-free formula. Then the class $\varphi(\mathcal{C})$ has almost bounded flip-width.*

Even though we do not know whether classes of almost bounded flip-width are closed under transductions, we confirm that structurally nowhere dense classes (transductions of nowhere dense classes) have almost bounded flip-width.

Theorem 10.12. *Every structurally nowhere dense class has almost bounded flip-width.*

To prove Theorem 10.12, we use the main result of [DGK⁺22a], which essentially implies that for every structurally nowhere dense class \mathcal{C} there is an *almost nowhere dense* class \mathcal{B} of structures equipped with functions, and a quantifier-free formula $\varphi(x, y)$ involving function symbols, such that $\mathcal{C} \subseteq \varphi(\mathcal{B})$. This is made precise below.

Say that a class \mathcal{C} of graphs is *almost nowhere dense* if for every $r \geq 1$ and $\varepsilon > 0$ we have $\text{wcol}_r(G) \leq O_{r,\varepsilon}(|G|^\varepsilon)$, for all $G \in \mathcal{C}$. Crucially, \mathcal{C} does not need to be hereditary, otherwise this notion would coincide with nowhere denseness by Fact 3.7.

Fix a signature Σ consisting of unary relation symbols, binary relation symbols, and unary function symbols. The VC-dimension of a Σ -structure B , denoted $\text{VCdim}(B)$, is the maximum of the VC-dimensions of the binary relations of B (see Section 3.3). Here, the functions of B are ignored.

The following result, apart from the ‘moreover’ part, is a straightforward consequence of [DGK⁺22a, Theorem 3]. It says that every structurally nowhere dense class \mathcal{C} interprets in an almost nowhere dense class \mathcal{B} of structures, via a quantifier-free interpretation using a unary function symbol. Additionally, every $G \in \mathcal{C}$ interprets in some $B \in \mathcal{B}$ with $|B| \leq O(|G|)$. Moreover, the binary relations of the binary structures in \mathcal{B} have bounded VC-dimension, which will be important in the next lemma, for controlling the bounds on the flip-width of G .

Theorem 10.13 (*). *Let \mathcal{C} be a structurally nowhere dense graph class. There is a signature Σ consisting of unary and binary relation symbols and one function symbol, a class \mathcal{B} of Σ -structures which is almost nowhere dense, and a quantifier-free symmetric formula $\varphi(x, y)$ with the following property. For every graph $G \in \mathcal{C}$ there is some $B \in \mathcal{B}$ with $|B| \leq O(|G|)$, such that G is an induced subgraph of $\varphi(B)$. Moreover, $\text{VCdim}(\mathcal{B}) < \infty$.*

The ‘moreover’ part is shown by analysing the construction from [DGK⁺22a] (see [DGK⁺22b] for the full version), and observing that the constructed quasi-bushes interpret (via a k -dimensional interpretation, for some fixed d) in graphs from a nowhere dense classes, and thus have bounded VC-dimension by the results of [AA14]. The following lemma is proved by extending the ideas used in the proof of Theorem 8.2.

Lemma 10.14 (*). *Let Σ be a signature consisting of unary and binary relation symbols, and unary function symbols. Fix $k, r \geq 0$, and a symmetric quantifier-free Σ -formula $\varphi(x, y)$. There are numbers $p \leq O_\varphi(k)$ and $r' \leq O_\varphi(r)$ such that the following holds. Let B be a Σ -structure with $\text{VCdim}(B) \leq k$ and G_B be its Gaifman graph. Then*

$$\text{fw}_r(\varphi(B)) \leq O(\text{copwidth}_{r'}(G_B))^p.$$

The key insight is that a bound k on the VC-dimension implies that if the cops in the Cops and Robber game on G occupy a set S of vertices of a graph G , then, by the Sauer-Shelah-Perles lemma (Lemma 3.8), the partition of $V(G)$ into S -types has size $O(|S|^k)$, and this partition is used by the cop in the flipper game.

The two statements above are proved in Appendix I. Theorem 10.12 follows, as we now show.

Proof of Theorem 10.12. Let \mathcal{C} be a structurally nowhere dense class. Without loss of generality, \mathcal{C} is hereditary. Let \mathcal{B} and $\varphi(x, y)$ be as in Theorem 10.13, and $k \in \mathbb{N}$ be such that $\text{VCdim}(B) < k$ for $B \in \mathcal{B}$.

Let $G \in \mathcal{C}$ and $B \in \mathcal{B}$ be such that G is an induced subgraph of $\varphi(B)$ and $|B| \leq O(|G|)$, and let G_B be the Gaifman graph of B . Let p and r' be as in Lemma 10.14. Fix $\varepsilon > 0$. Then we have:

$$\text{fw}_r(G) \leq \text{fw}_r(\varphi(B)) \leq O(\text{copwidth}_{r'}(G_B))^p \leq O_{r', \varepsilon}(|B|^{\varepsilon p}) \leq O_{r', \varepsilon}(|G|^{\varepsilon p}).$$

Since r' depends only on r and φ , and p is a constant, and $\varepsilon > 0$ is arbitrary, this proves that \mathcal{C} has almost bounded flip-width. \square

10.4 Edge-stable classes of almost bounded flip-width

Conjecture 10.7 predicts that a graph class has almost bounded flip-width if and only if it is monadically dependent. Currently, we are able to prove neither of the two implications. However, we prove the forward implication under the assumption that \mathcal{C} is edge-stable (that is, excludes some half-graph as a semi-induced bipartite graph).

Theorem 10.15. *Let \mathcal{C} be an edge-stable, hereditary graph class such that for all $r \in \mathbb{N}$,*

$$\text{fw}_r(G) = o(|G|^{1/2}) \quad \text{for } G \in \mathcal{C}.$$

Then \mathcal{C} is monadically stable. In particular, every edge-stable, hereditary graph class of almost bounded flip-width is monadically stable.

As far as we know, every monadically stable class might be structurally nowhere dense. This is conjectured in [NdMP⁺21, Conjecture 6.1]. If this were true, then Theorem 10.15 and Theorem 10.12 would imply that among edge-stable graph classes, almost bounded flip-width coincides with monadically stable class (and thus with monadically dependent classes, by Fact 10.5). Moreover, this would imply a collapse result, confirming Conjecture 10.8.

To prove Theorem 10.15, we use the following result of [GMM⁺23, Thm. 1.4].

Fact 10.16 ([GMM⁺23]). *Let \mathcal{C} be a hereditary, edge-stable class of graphs. If \mathcal{C} is not monadically stable then there are $r, k \geq 1$, a k -coloring $\hat{\mathcal{C}}$ of \mathcal{C} and a quantifier-free formula $\varphi(x, y)$ such that $\varphi(\hat{\mathcal{C}})$ contains the exact r -subdivision of every graph.*

Proof of Theorem 10.15. Let \mathcal{C} be a hereditary, edge-stable graph class, such that for every fixed $r \in \mathbb{N}$, we have that $\text{fw}_r(G) = o(|G|^{1/2})$ holds for all $G \in \mathcal{C}$. We prove that \mathcal{C} is monadically stable. Suppose otherwise. Let $r, k, \hat{\mathcal{C}}$ and $\varphi(x, y)$ be as in Fact 10.16.

Pick a number $n \geq 1$. Let $K_n^{(r)}$ denote the exact r -subdivision of the clique K_n . Then $\text{fw}_{r+1}(K_n^{(r)}) \geq \Omega_r(n)$ by Proposition 6.7. By Fact 10.16, there is some $G_n \in \hat{\mathcal{C}}$ such that $\varphi(G_n)$ is isomorphic to $K_n^{(r)}$. In particular, $|V(G_n)| = |V(K_n^{(r)})| \leq O(rn^2)$.

As φ is quantifier-free and involves k colors, by Theorem 8.2, we have $\text{fw}_{r+1}(\varphi(G_n)) \leq k \cdot \text{fw}_{r+1}(G_n)$, and altogether:

$$\Omega_r(n) \leq \text{fw}_{r+1}(K_n^{(r)}) \leq \text{fw}_{r+1}(\varphi(G_n)) \leq k \cdot \text{fw}_{r+1}(G_n) = o((n^2)^{1/2}) = o(n),$$

which is impossible. Hence, \mathcal{C} is monadically stable. \square

10.5 Classes of ordered graphs of almost bounded flip-width

Recall that flip-width is defined for arbitrary binary relational structures (see Appendix B.4), and there is a variant of flip-width tailored to ordered graphs, defined in Section 7. It follows from Lemma 7.2, that a class \mathcal{C} of ordered graphs has almost bounded flip-width if and only if for every r , we have $\text{fw}_r^<(\mathcal{C}) < \infty$.

In this section, we prove that for hereditary classes of ordered graphs, almost bounded flip-width coincides with bounded twin-width, as well as with bounded flip-width (by Theorem 7.3). Moreover, in the setting of ordered graphs, a very strong form of the collapse predicted by Conjecture 10.8, holds: from $o(n^{1/2})$ all the way down to $O(1)$ (as opposed to just $n^{o(1)}$, as predicted by the conjecture). This total collapse is, ultimately, a consequence of the Marcus-Tardos theorem/Stanley-Wilf conjecture.

Theorem 10.17. *The following conditions are equivalent for a hereditary class \mathcal{C} of ordered graphs:*

1. \mathcal{C} is monadically dependent,
2. \mathcal{C} has bounded twin-width,
3. \mathcal{C} has bounded flip-width,
4. \mathcal{C} has almost bounded flip-width,
5. $\text{fw}_1^<(G) = o(|G|^{1/2})$ for all $G \in \mathcal{C}$.

The equivalence $1 \leftrightarrow 2$ is proved in [BGDM⁺22]. The equivalence $2 \leftrightarrow 3$ is by Theorem 7.3. The implications $3 \rightarrow 4$ and $4 \rightarrow 5$ are immediate. We prove the implication $5 \rightarrow 2$. For this, we use the following consequence of the main result of [BGDM⁺22], which itself is a manifestation of the Marcus-Tardos theorem/Stanley-Wilf conjecture [MT04].

Denote $[n] := \{1, \dots, n\}$. Fix a symbol $s \in \{=, \neq, \leq_l, \geq_l, \leq_r, \geq_r\}$ and let $G = (V, E, <)$ be an ordered graph and $A, B \subseteq V(G)$. Say that A and B form a s -pattern of order n^2 if $|A| = |B| = n^2$, and the following condition holds (see Fig. 4). Let $\alpha: [n]^2 \rightarrow A$ and $\beta: [n]^2 \rightarrow B$ be order-preserving bijections, where $[n]^2$ is ordered by the lexicographic order $<_{\text{lex}}$, while A and B are ordered by $<$. Then for $(i, j), (i', j') \in [n]^2$, we have that $\alpha(i, j)\beta(i', j') \in E$ if and only if the following condition holds:

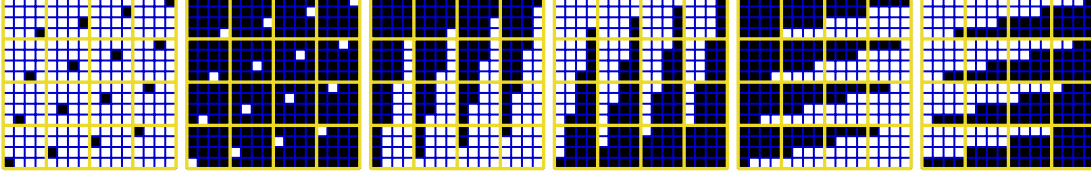


Figure 4: An s -pattern A, B of order n^2 for $n = 4$ and $s \in \{=, \neq, \leq_l, \geq_l, \leq_r, \geq_r\}$, represented as incidence matrices of $G[A, B]$. The origin is at the lower-left corner and black entries denote incidence.

- $(i, j) = (j', i')$, if s is $=$,
- $(i, j) \neq (j', i')$, if s is \neq ,
- $(j, i) \leq_{\text{lex}} (i', j')$, if s is \leq_l ,
- $(j, i) \geq_{\text{lex}} (i', j')$, if s is \geq_l ,
- $(i, j) \leq_{\text{lex}} (j', i')$, if s is \leq_r ,
- $(i, j) \geq_{\text{lex}} (j', i')$, if s is \geq_r .

Fact 10.18 ([BGdM⁺22]). *Let \mathcal{C} be a class of ordered graphs of unbounded twin-width. Then there is $s \in \{=, \neq, \leq_l, \geq_l, \leq_r, \geq_r\}$ such that for every $n \geq 1$, there is some graph $G \in \mathcal{C}$ and sets A and B that form an s -pattern of order n^2 .*

Proof. Follows from [BGdM⁺21, Lemma 40], by considering the (s, σ) -matching (in their notation), where σ is the permutation of $[n]^2$ (ordered lexicographically) such that $\sigma((i, j)) = (j, i)$ for $(i, j) \in [n]^2$. \square

Recall the definition of an n -rich division in an ordered graph G , and that the existence of such a division implies $\text{fw}_1^<(G) \geq n$ (see Lemma 7.5 and definition before it).

Lemma 10.19. *Let n be even, and G be an ordered graph containing an s -pattern of order n^2 . Then G has an $n/2$ -rich division. In particular, $\text{fw}_1^<(G) \geq n/2$.*

Proof. Let A and B form an s -pattern of order n^2 . Let $\mathcal{L} = \{I_1, \dots, I_n\}$ be the partition of $V(G)$ into n intervals, with respect to the order of G restricted to A , each containing exactly n elements of A . Similarly, let $\mathcal{R} = \{J_1, \dots, J_n\}$ be the partition of $V(G)$ into n intervals, each containing exactly n elements of B . We show that \mathcal{L} and \mathcal{R} form an $n/2$ -rich division of G .

Let I_i be an interval of \mathcal{L} and pick a family $\mathcal{R}' \subseteq \mathcal{R}$ of $n/2$ intervals of \mathcal{R} . Let $\{j_1, \dots, j_{n/2}\}$ be the indices of the $n/2$ intervals J_j in \mathcal{R}' and let J be their union. We show that there are $n/2$ vertices in $A \cap I_i$ with pairwise distinct neighborhoods in $B \cap J$. This (by symmetry) implies that \mathcal{L} and \mathcal{R} form an $n/2$ -rich division of G .

Let α and β be as in the definition of an s -pattern of order n^2 . Then, for $p, q \in [n/2]$ we have that $\alpha(i, j_p) \in A \cap I_i$ and $\beta(j_q, i) \in B \cap J$. Furthermore, $\alpha(i, j_p)\beta(j_q, i) \in E(G)$ if and only if:

- $p = q$, if s is $=$,
- $p \neq q$, if s is \neq ,
- $p \leq q$, if s is \leq_l or \leq_r ,
- $p \geq q$, if s is \geq_l or \geq_r .

In any case, the vertices $\alpha(i, j_1), \dots, \alpha(i, j_{n/2}) \in A \cap I_i$ have distinct neighborhoods in $B \cap J$. This (and a symmetric argument, exchanging the roles of \mathcal{L} and \mathcal{R}) demonstrates that \mathcal{L} and \mathcal{R} form an $n/2$ -rich division. By Lemma 7.5, we have that $\text{fw}_1^<(G) \geq n/2$. \square

Proof of Theorem 10.17. As argued, it remains to prove the implication $5 \rightarrow 2$. Let \mathcal{C} be a hereditary class of ordered graphs with unbounded twin-width. We prove that there are infinitely many ordered graphs $G \in \mathcal{C}$ with $\text{fw}_1^<(G) \geq \Omega(|G|)^{1/2}$, which contradicts condition 5.

Let $s \in \{=, \neq, \leq_l, \geq_l, \leq_r, \geq_r\}$ be as in Fact 10.18.

Fix an even $n \geq 2$. By Fact 10.18 there is an ordered graph $G_n \in \mathcal{C}$ and sets $A, B \subseteq V(G_n)$ of size n^2 , that form an s -pattern of order n^2 in G . As \mathcal{C} is hereditary, we may assume that $V(G_n) = A \cup B$, so in particular $|G_n| \leq 2n^2$. By Lemma 10.19, $\text{fw}_1^<(G_n) \geq n/2$. Altogether, we have that

$$\text{fw}_1^<(G_n) \geq \frac{1}{2}(|G_n|/2)^{1/2} = \Omega(|G_n|^{1/2}).$$

Since $G_n \in \mathcal{C}$ for all even $n \geq 2$, it cannot be that $\text{fw}_1^<(G) \leq o(|G|)^{1/2}$ holds for all $G \in \mathcal{C}$. \square

Summary In Section 10 we have defined graph classes of almost bounded flip-width. We conjecture that they coincide with monadically dependent graph classes. We provide the following evidence towards this conjecture. We have shown that, when restricted to weakly sparse graph classes, almost bounded flip-width coincides with nowhere denseness, and with monadic dependence. And when restricted to edge-stable graph classes, almost bounded flip-width generalizes structurally nowhere denseness, and is generalized by monadic stability. As it is conjectured that structurally nowhere dense classes coincide with monadically stable classes, this would imply that for edge-stable graph classes, almost bounded flip-width coincides with monadic stability, and with monadic dependence. We have shown that for classes of ordered graphs, almost bounded flip-width coincides with bounded flip-width, and with bounded twin-width, and therefore, with monadic dependence.

In all the special cases studied above – of weakly sparse graph classes, of edge-stable graph classes, and of classes of ordered graphs – the model-checking problem is known to be fixed-parameter tractable [GKS17b, DMS, BGOdM⁺22] (in the edge-stable case, this assumes that monadic stability coincides with structural nowhere denseness). We therefore obtain the following corollary.

Corollary 10.20. *Let \mathcal{C} be a class of almost bounded flip-width that is either weakly sparse, or is a class of ordered graphs, or is structurally nowhere dense. Then the model checking problem for first-order logic is fpt on \mathcal{C} .*

We remark that, up to best of our knowledge, all hereditary graph classes for which the model-checking problem is currently known to be fixed-parameter tractable – apart from one family of examples – have almost bounded flip-width. Those includes the listed cases above, as well as some classes of bounded twin-width [BCK⁺22, GT22]. However, we do not know whether classes of *structurally bounded local clique-width* [BDG⁺22] have almost bounded flip-width (they are monadically dependent, and have tractable model-checking).

11 Discussion

In this section, we discuss some conjectured relationships between various notions defined in this paper, and other known notions. We speculate on possible routes towards proving some conjectures.

11.1 Obstructions to small flip-width

The results of this paper strongly suggest that the flip-width parameters are the sought dense analogues of generalized coloring numbers, that classes of bounded flip-width are the sought dense analogues of classes with bounded expansion, and almost bounded flip-width is a dense counterpart of nowhere dense classes. What is currently missing to complete this picture, is an analogue of the key result of Sparsity theory, which is a min-max theorem that relates, on one hand, explicit descriptions of winning strategies of the robber, and on the other hand, explicit descriptions of winning strategies for the cops in the Cops and Robber game (see Section 4). In the sparse case, the former are obstructions to having small r -admissibility numbers, or (by Fact 3.5), \leq_r -subdivisions of graphs with large minimum degree. Finding

an analogous notion in the dense case seems to be a major challenge. It seems plausible that finding such notions is related to the question of efficiently approximating the flip-width parameters. This goal can be formalized as follows.

Goal 11.1. *Obtain an fpt approximation algorithm for radius- r flip-width: an algorithm running in time $f(r, k) \cdot n^c$, for some constant c and function $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, which given an n -vertex graph G and numbers r, k , either concludes that $\text{fw}_r(G) > k$, or that $\text{fw}_r(G) < f(r, k)$.*

A first step in this direction is achieved by Theorem 9.7, which achieves an XP approximation algorithm, rather than an fpt algorithm.

A related, but less concrete goal is the following.

Goal 11.2. *Describe explicit forbidden weak obstructions at depth r and density ℓ , such that there are functions $f, g: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that for all $r, \ell \geq 1$, every graph G with $\text{fw}_r(G) \geq f(r, \ell)$ contains such an obstruction of density ℓ as an induced subgraph, and conversely, no graph G which contains such an obstruction of density $g(r, \ell)$ as an induced subgraph satisfies $\text{fw}_r(G) \geq \ell$.*

One attempt at formalizing what we mean by “explicit”, is by requiring that the following holds: for every fixed r there is a formula $\varphi_r(x, y)$ (possibly involving colors) and a function $f_r: \mathbb{N} \rightarrow \mathbb{N}$ such that if G is a forbidden weak obstruction at depth r and density at least $f_r(\ell)$ then there is some k -coloring of G such that $\varphi_r(G)$ contains an r -subdivision of a graph with minimum degree at least ℓ as an induced subgraph.

The following question suggests a path towards achieving Goal 11.2.

Question 11.3. *Fix $r \geq 1$. Is it the case that $\text{fw}_r(\mathcal{C}) < \infty$ if and only if there is a $k \in \mathbb{N}$ such that no graph $G \in \mathcal{C}$ contains a (r, k, k) -hideout (see Definition 5.15)?*

Hideouts are not induced subgraphs, as posited in Goal 11.2, but this could serve as a starting point.

The following notions are defined in [GPT21]. A *transduction ideal* is a property of graph classes that is preserved under first-order transductions: if a class \mathcal{C} has the considered property, and \mathcal{D} transduces in \mathcal{C} , then \mathcal{D} has the property, too. The *dense analogue of bounded expansion* is the largest transduction ideal such that all weakly sparse classes that belong to it have bounded expansion.

By Corollary 8.3, classes of bounded flip-width form a transduction ideal, and by Theorem 6.3, they are contained in the dense analogue of bounded expansion. We conjecture that bounded flip-width is exactly the dense analogue of bounded expansion, according to the above definition. This is equivalently stated as follows.

Conjecture 11.4. *Let \mathcal{C} be a class that does not have bounded flip-width. Then \mathcal{C} transduces a weakly sparse class \mathcal{D} which does not have bounded expansion.*

A road towards proving Conjecture 11.4 is through Goal 11.2. Indeed, if \mathcal{C} contains obstructions at a fixed depth $r \geq 1$ and of arbitrarily large density ℓ , then according to the requirement given below Goal 11.2, \mathcal{C} transduces a class \mathcal{D} which contains r -subdivisions of arbitrarily dense graphs, and therefore \mathcal{D} does not have bounded expansion.

11.2 Model checking

Recall that Conjecture 1.1 predicts that for a hereditary class \mathcal{C} , first-order model-checking is fixed-parameter tractable (fpt) on \mathcal{C} if and only if \mathcal{C} is monadically dependent. Both implications are open. We now discuss approaching the backward implication, giving the upper bound: that model-checking is fpt on every monadically dependent graph class.

The question whether the model checking problem is fpt on every class \mathcal{C} of graphs that has bounded twin-width, remains open. (The result of [BKTW20] says that this is the case, if the input graph G is given together with its contraction sequence. In some special cases [BGOdM⁺22, BCK⁺22, GT22], such a contraction sequence can be efficiently computed.) As

bounded twin-width implies bounded flip-width, we do not know whether the model checking problem is fpt on every class \mathcal{C} of graphs that has bounded flip-width. However, we believe that approximating the radius- r flip-width of a given graph G , might be easier than approximating its twin-width. This is indicated for example by Theorem 9.7, which gives an XP approximation algorithm for flip-width, while no such algorithm for twin-width is known. Still, an analogue of the model-checking result of [BKTW20] is missing, and is posed as the following goal.

Goal 11.5. *Devise an efficient representation of winning strategies for the cops such that for a fixed formula φ there is a number r , such that given an efficient representation of a winning strategy for the cops in the radius- r flipper game on a given graph G , allows to efficiently check whether φ holds in G .*

In particular, Goal 11.5 combined with Goal 11.1 could allow to solve the model-checking problem on classes of bounded flip-width (and also on classes of bounded twin-width, overcoming the lack of an fpt approximation algorithm for twin-width).

An extension of Goal 11.5 to monadically dependent classes, with the hope of confirming Conjecture 1.1, could lead through Conjecture 10.7, which characterizes the hereditary monadically dependent classes as those with almost bounded flip-width. This, however, remains very speculative, and most likely requiring further insights, on top of the ones needed to achieve Goal 11.5 and to solve Conjecture 10.7.

11.3 Stable classes of bounded flip-width

It is conjectured [GPT21, Conjecture 6] that the edge-stable classes in the dense analogue of bounded expansion have structurally bounded expansion, i.e., are transductions of classes with bounded expansion. As classes of bounded flip-width are contained in the dense analogue of bounded expansion, this would imply the following.

Conjecture 11.6. *Every edge-stable class with bounded flip-width has structurally bounded expansion.*

Conjecture 11.6 would generalize the main result of [GPT22, GPT21], which states that every edge-stable class of bounded twin-width has structurally bounded expansion. The conjunction of Conjecture 11.4 and Conjecture 11.6 implies the following duality statement for edge-stable graph classes.

Conjecture 11.7. *For every edge-stable graph class \mathcal{C} , either \mathcal{C} is a transduction of a class with bounded expansion, or \mathcal{C} transduces a weakly sparse class with unbounded expansion.*

Conjecture 11.4 and Conjecture 11.6 imply Conjecture 11.7: If \mathcal{C} is not monadically stable, then \mathcal{C} is not monadically dependent (by Fact 10.5), and hence transduces every weakly sparse class of graphs. If \mathcal{C} is monadically stable and has unbounded flip-width, then \mathcal{C} transduces a weakly sparse class that does not have bounded expansion by Conjecture 11.4. Finally, if \mathcal{C} is monadically stable and has bounded flip-width, then \mathcal{C} transduces in a class of bounded expansion, by Conjecture 11.6.

11.4 Restrictions of the flipper game

Restricted classes of bounded flip-width can be defined by imposing additional constraints on the flipper game. We consider the following restrictions⁹ on the strategy of the cops in the flipper game. We say that the strategy of the cops is:

- blind** if their move does not depend on the current position of the robber (but may depend on the graph and on the round number);
- positional** if their move is only based on the current position of the robber (and not on the past moves, nor the round number);

⁹I thank Rose McCarty and Pierre Ohlmann for suggesting the positional and bounded depth variants.

of bounded depth if there is a bound ℓ such that the cops win within ℓ rounds;

branching-blind if in each round move, the cops propose a partition \mathcal{P} of the vertex set, and bases his move only on the part of \mathcal{P} which is occupied by the robber, and furthermore, ensure that the robber will remain in that part until the end of the game.

Each of those variants of the game defines a variant of the flip-width parameter, and the related classes. Thus, a class \mathcal{C} has bounded X flip-width, for $X \in \{\text{blind}, \text{branching blind}, \text{positional}\}$, if for every radius r , there is a k such that the cops win the flipper game of radius r and width k on every graph $G \in \mathcal{C}$, using a strategy with property X . Similarly, \mathcal{C} has *bounded flip-depth* if for every $r \in \mathbb{N}$ there are k, ℓ such that for every $G \in \mathcal{C}$, the cops have a strategy in the flipper game of radius r and width k , that wins in at most ℓ rounds.

We may also consider the limit case of the above games, for $r = \infty$, thus defining classes with *blind ∞ -flip-width*, *positional ∞ -flip-width*, *∞ -flip-depth*, *branching-blind ∞ -flip-width*, analogously as above.

It turns out that these notions relate to notions and conjectures that have been studied earlier, as we discuss below. We start with the following observation.

Proposition 11.8. *All the above properties are preserved under first-order transductions, and under CMSO transductions for the ∞ variants.*

Proposition 11.8 follows by observing that in the proof of Theorem 8.2 and Theorem 8.8, the strategy on $\varphi(G)$ is obtained by transferring a winning strategy from G (see Section 8.2). And transferring a strategy with one of the listed properties, results in a strategy with the same property.

Infinite radius In the limit case of radius ∞ , the picture is quite well-understood, thanks to the following.

Theorem 11.9. *Let \mathcal{C} be a class of graphs. Then:*

1. \mathcal{C} has branching-blind ∞ -flip-width if and only if \mathcal{C} has bounded clique-width;
2. \mathcal{C} has blind ∞ -flip-width if and only if \mathcal{C} has bounded linear clique-width;
3. \mathcal{C} has bounded positional ∞ -flip-width, if and only if \mathcal{C} has bounded ∞ -flip-depth, if and only if \mathcal{C} has bounded shrubdepth.

A class \mathcal{C} has *bounded shrubdepth* [GHN⁺12] if and only if it transduces in a class of trees of depth bounded by a constant.

Sketch of proof. (1). For the forward implication, observe that the strategy presented in the proof of Theorem 5.17 (see Appendix D) is branching blind. For the backward implication, note that if \mathcal{C} has unbounded clique-width then by Theorem 5.17 it has unbounded flip-width, so in particular, it has unbounded branching-blind flip-width.

(2). For the forward implication, observe that the strategy presented in Example 5.9, for half-graphs, is blind. Therefore, the class of half-graphs has bounded blind flip-width. It is well-known that every class \mathcal{C} of bounded linear clique-width is a CMSO transduction of the class of half-graphs (or of the class of finite total orders). By Proposition 11.8, \mathcal{C} has bounded blind flip-width.

Conversely, if a class \mathcal{C} has unbounded linear clique-width, then it CMSO transduces the class of all trees (this follows from [HJMW20, DT17] and from [CiO07]). It is not difficult to verify that the class of trees does not have bounded blind ∞ -flip-width (this is a special case of Proposition 11.12 below). Again by Proposition 11.8, this implies that \mathcal{C} has unbounded blind ∞ -flip-width.

(3). First observe that in a tree of depth at most d , the cops have a positional winning strategy in the radius- ∞ flipper game, that wins in at most d rounds (this is essentially the same as the strategy in Example 4.3). Hence, every class \mathcal{D} of trees of depth bounded by a constant has bounded positional ∞ -flip-width, and bounded ∞ -flip-depth. By Proposition 11.8, the same holds for every class that transduces in \mathcal{D} , hence for all classes of bounded shrubdepth.

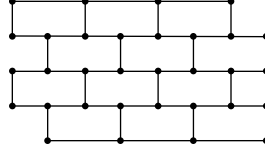


Figure 5: A wall graph

Conversely, if a class \mathcal{C} has unbounded shrubdepth, then it CMSO-transduces the class of all paths, by [KMOW21]. It is not difficult to verify that the class of all paths does not have bounded positional ∞ -flip-width, and does not have bounded ∞ -flip-depth. Proposition 11.8 implies that \mathcal{C} has unbounded positional ∞ -flip-width, and unbounded ∞ -flip-depth. \square

Blind case We now move the case of classes of bounded blind flip-depth. We start with some examples.

Example 11.10. Classes of bounded blind flip-width include classes of bounded pathwidth, and more generally, classes of bounded linear rank-width (or linear clique-width), by Theorem 11.9.

Example 11.11. On the other hand, the strategies described in Example 4.3 and in Example 5.10, for trees and for comparability graphs of trees, respectively, are not blind. One can show that those classes do not have bounded blind flip-width. A more general statement is given below.

By a *subdivision* of a graph G we mean any graph obtained from G by replacing each edge by a path of arbitrary length. We omit the proof of the following proposition.

Proposition 11.12. *Let \mathcal{C} be a class that contains a subdivision of every cubic tree. Then \mathcal{C} does not have bounded blind flip-width.*

[GPT21, Conjecture 3] predicts that every class of unbounded linear clique-width transduces some class that contains a subdivision of every cubic tree. This would imply the following, possibly weaker conjecture.

Conjecture 11.13. *A class \mathcal{C} has bounded blind flip-width if and only if \mathcal{C} has bounded linear clique-width.*

As bounded linear clique-width is equivalent to bounded blind ∞ -flip-width by Theorem 11.9, Conjecture 11.13 states that bounded blind flip-width collapses to bounded blind ∞ -flip-width.

Branching-blind case We now discuss the branching-blind case. By Theorem 11.9, every class of bounded clique-width has bounded branching-blind flip-width.

A wall is a graph as depicted in Figure 5. We omit the proof of the following proposition.

Proposition 11.14. *Let \mathcal{C} be a class that contains a subdivision of every wall. Then \mathcal{C} does not have bounded branching blind flip-width.*

[GPT21, Conjecture 4] predicts that every class of unbounded clique-width transduces some class that contains a subdivision of every wall. This would imply the following, weaker conjecture.

Conjecture 11.15. *A class \mathcal{C} has bounded branching-blind flip-width if and only if \mathcal{C} has bounded clique-width.*

As bounded clique-width is equivalent to bounded branching-blind ∞ -flip-width by Theorem 11.9, Conjecture 11.15 states that bounded branching-blind flip-width collapses to bounded branching-blind ∞ -flip-width.

Positional case Let us now look at some examples related to bounded positional flip-width.

Example 11.16. Every class with bounded expansion has bounded positional flip-width. This follows from the proof of the upper bound in Theorem 4.5, as the cops' strategy in the radius- r Cops and Robber game is to occupy (or isolate) the vertices that are $2r$ -weakly reachable from the vertex that is currently occupied by the robber. Clearly, this is a positional strategy.

By Proposition 11.8 we get the following.

Corollary 11.17. *Every class with structurally bounded expansion has bounded positional flip-width.*

On the other hand, we have:

Example 11.18. Half-graphs have unbounded positional flip-width.

It follows that every class \mathcal{C} with bounded positional flip-width is edge-stable. Since \mathcal{C} is also monadically dependent by Corollary 8.4, and every monadically dependent, edge-stable class is monadically stable by Fact 10.5, we get the following.

Corollary 11.19. *Every class with bounded positional flip-width is monadically stable.*

Conjecture 11.6 would therefore imply the following characterization of classes with bounded positional flip-width.

Conjecture 11.20. *A class \mathcal{C} has bounded positional flip-width if and only if \mathcal{C} has structurally bounded expansion.*

Bounded flip-depth Finally, we look at classes of bounded flip-depth.

Example 11.21. Classes of bounded degree have bounded flip-depth, as seen in Example 4.2. Also, classes of bounded treedepth, or more generally, classes of bounded shrubdepth, have bounded flip-depth.

A class of examples combining the above is provided by the following notion. Fix $d, h \geq 1$. A *hybrid tree* of depth h and degree d is a graph G that can be obtained from a rooted tree T of depth h by:

- first adding some edges connecting siblings in T , in such a way that every vertex is adjacent in G to at most d of its siblings in T ,
- afterwards, taking a subgraph of the resulting graph.

Proposition 11.22. *Fix $d, h \geq 1$. The class $\mathcal{H}_{d,h}$ of hybrid trees of depth h and degree d has bounded flip-depth.*

The following is a consequence of Simon's factorisation theorem [Sim90]. We omit the details.

Theorem 11.23. *Every class \mathcal{C} of bounded pathwidth, and more generally, every stable class of bounded linear clique-width, transduces in $\mathcal{H}_{2,h}$, for some fixed h .*

Corollary 11.24. *Every class of bounded pathwidth, and more generally, every stable class of bounded linear clique-width has bounded flip-depth.*

We now pose two conjectures characterizing classes of bounded flip-depth in complementary ways.

Define the *tree-rank* of a graph class \mathcal{C} as the largest number $d \in \mathbb{N}$ such that \mathcal{C} transduces the class of all forests of depth d , or ∞ if the largest such number does not exist. Note that classes of rank $d \in \mathbb{N}$ are in particular monadically dependent, in fact, monadically stable, since the class of half-graphs transduces every class of forests of bounded depth.

Next, the following conjecture relates the above notion to bounded flip-depth.

Conjecture 11.25. *A class \mathcal{C} has bounded flip-depth if and only if \mathcal{C} has rank $< d$, for some $d \in \mathbb{N}$.*

We finish with the following conjecture, relating classes of bounded flip-width with the complexity of the model-checking problem for first-order logic.

Say that a graph class \mathcal{C} has *non-uniform elementary-fpt* first-order model-checking if there are numbers $h, c \geq 1$ such that for every first-order formula φ of length k there is an algorithm which determines if a given n -vertex graph $G \in \mathcal{C}$ satisfies φ in time at most

$$\underbrace{2^{2^{\dots 2^k}}}_{\text{height } h} \cdot n^c.$$

In the *uniform* variant, there is a single algorithm, which inputs φ and $G \in \mathcal{C}$, and has the above running time.

It is known [DGKS07] that every class of bounded degree has uniform elementary-fpt model-checking. Furthermore, the class of trees does not have uniform elementary-fpt model-checking, unless $\text{AW}[*] = \text{FPT}$, by a result of Frick and Grohe [FG04] (see also [DGKS07]), and the class of colored linear orders does not have uniform elementary-fpt model-checking unless $\text{P} = \text{NP}$, also by Frick and Grohe.

Together with Michał Pilipczuk, we pose the following.

Conjecture 11.26. *A class \mathcal{C} of graphs has non-uniform elementary-fpt first-order model-checking if and only if \mathcal{C} does not transduce the class of forests of depth d , for some $d \geq 1$.*

Together with Conjecture 11.25, this would characterize classes with non-uniform elementary-fpt first-order model-checking precisely as those with bounded flip-depth.

Appendices

Appendix A Variants of the cop-width parameter

A.1 A variant without announced moves

Fix parameters $k, r \in \mathbb{N}$, and consider the variant of the Cops and Robber game in which there are k cops, and a robber with speed r , and in each round, first the cops pick a set A of k chosen vertices of the graph, and then the robber may either stay in his last position v , if $v \notin A$, or moves to any vertex u via a path $v = v_0, \dots, v_i = u$ of length $0 \leq i \leq r$ such that $v_1, \dots, v_i, u \notin A$. If they cannot do so, the cops win the game, and if the robber can evade the cops forever, then the robber wins. Denote the smallest number k for which the cops have a winning strategy on a graph G by $\text{copwidth}'_r(G)$.

Those parameters essentially appear in the work [RT08, LPPT20]. The paper [RT08] considers a variant of the game in which the robber is *lazy*, that is, does not move unless a cop is placed at his location, whereas the [LPPT20] considers a variant where the cops occupy edges instead of vertices, and the robber never remains put. Analogues of the next notion and lemma also appear in those papers.

Call a set U of vertices of a graph G a (k, r) -hideout if for every $v \in U$ and set $A \subseteq V(G) - \{v\}$ with $|A| < k$, there is some path from v to $U - \{v\}$ of length at most r in $G - A$.

Lemma A.1. Fix numbers $k, r \in \mathbb{N}$ and a graph G . The following conditions are equivalent:

1. $\text{copwidth}'_r(G) \leq k$,
2. G has no (k, r) -hideout,
3. there is a total order on $V(G)$ such that for every $v \in V$ there is some set $A \subseteq V(G) - \{v\}$ with $|A| < k$ such that there is no path from v to any vertex $w < v$ in $G - A$.

Proof. (1 \rightarrow 2) We show that if U is a (k, r) -hideout in G , then there is a winning strategy for the robber in the game of radius r and width k corresponding to the $\text{copwidth}'_r$ parameter. As a first move, the robber picks an arbitrary vertex $v \in U$, and we show that they may always remain in the set U . In each round, when the cops place the cops on a set A of at most k vertices, then robber stays put if his current position v is not in A , and otherwise, as $|A - \{v\}| < k$, the robber moves to any vertex $u \in U - A$ that is connected by a path of length at most r from v in $G - A$.

(2 \rightarrow 3) Suppose that G has no (k, r) -hideout. Start with $U = V(G)$ and \bar{w} being the empty sequence. As long as U is nonempty, pick any vertex $u \in U$ such that there is some set $A \subseteq V(G) - \{v\}$ with $|A| < k$ such that there is no path from v to any vertex $w \in U$ in $G - A$. Such a vertex exists, since U is not a (k, s) -hideout. Remove u from U and prepend it to \bar{w} , and repeat. Once U becomes empty, the sequence \bar{w} gives a total order on $V(G)$ satisfying the required condition.

(3 \rightarrow 1) We show how to turn a total order as in condition (3) into a winning strategy for the cops in the Cops and Robber game of radius r and width w . When the robber is occupying a vertex v , the cops pick any set $A \subseteq V(G) - \{v\}$ with $|A| < k$ such that there is no path from v to any vertex $w < v$, and place the cops on all the vertices of $A \cup \{v\}$. Then the robber needs to move right in the order, so eventually they will lose. \square

We now relate the parameter $\text{copwidth}'_r$ to generalized coloring numbers. The r -strong coloring number of a graph G , denoted $\text{scol}_r(G)$ is the smallest number k with the following property. There is a total order $<$ on $V(G)$ such that every for vertex v , there are at most k vertices $w < v$ that can be reached from v by a path of length at most r . Clearly, $\text{adm}_r(G) \leq \text{scol}_r(G) \leq \text{wcol}_r(G)$.

Lemma A.2. For every $r \in \mathbb{N}$ and graph G the following inequalities hold:

$$\text{adm}_r(G) + 1 \leq \text{copwidth}'_r(G) \leq \text{scol}_r(G) + 1.$$

Proof. We prove the first inequality, by showing that $\text{adm}_r(G) \geq k - 1$ implies $\text{copwidth}'_r(G) > k$. Indeed, by $\text{adm}_r(G) \geq k - 1$ there is a set $U \subseteq V(G)$ such that for every $v \in U$ there are k

paths of length at most r from v to $U - v$, which are vertex-disjoint apart from v . In particular, no set $A \subseteq V(G) - \{v\}$ with $|A| < k$ hits all of those k paths. Hence, U is a (k, r) -hideout, and $\text{copwidth}'_r(G) > k$ by Lemma A.1.

For the second inequality, suppose $\text{scol}_r(G) \leq k$ and let $<$ be a total order witnessing it. Then condition 3 of Lemma A.1 holds. \square

Also note that $\text{copwidth}'_r(G) \leq \text{copwidth}_r(G)$, since a winning strategy in the game corresponding to copwidth_r is also a winning strategy in the game corresponding to $\text{copwidth}'_r$.

By Lemma A.2, we have the following.

Corollary A.3. *The following conditions are equivalent for a graph class \mathcal{C} :*

1. \mathcal{C} has bounded expansion,
2. $\text{copwidth}_r(\mathcal{C}) < \infty$, for every $r \in \mathbb{N}$,
3. $\text{copwidth}'_r(\mathcal{C}) < \infty$, for every $r \in \mathbb{N}$.

A.2 Isolation game

Consider the following variant of the flipper game. The *isolation-width* game with radius $r \in \mathbb{N} \cup \{\infty\}$ and width $k \in \mathbb{N}$, $k \geq 1$, is played on a graph G . In round i of the game we have a set $S_i \subseteq V(G)$ with $|S_i| \leq k$, which are the new positions of the cops declared by the cops, and the current position $v_i \in V(G)$ of the robber. Initially, $S_0 = \emptyset$ and v_0 is a vertex of G chosen by the robber. In round $i > 0$, the cops announce a set $S_i \subseteq V(G)$ of next positions of the cops with $|S_i| \leq k$, that will be put into effect momentarily. The robber, knowing S_i , moves to a new vertex v_i by following a path of length at most r from v_{i-1} to v_i that avoids the previous cop positions S_{i-1} . The game terminates when $v_i \in S_i$. Write $\text{iw}_r(G)$ for the smallest number k such that the cops have a winning strategy in the isolation game with radius r and width k .

Lemma A.4.

$$\text{iw}_r(G) \leq \text{copwidth}_r(G) \leq 2 \text{iw}_r(G).$$

Proof sketch. A winning strategy in the Cops and Robber game of radius r and width k can be translated into a winning strategy in the isolation game of radius r and width k : when in the Cops and Robber game the cops are directed to their new set of positions X , in the isolation game the cops define X as their next positions. In particular, $|X| \leq k$, and a response of the robber in the isolation game is a valid response in the Cops and Robber game.

Conversely, a winning strategy in the isolation game of radius r and width k can be translated into a winning strategy in the Cops and Robber game of radius r and width $2k$: when in the isolation game the cops declare the new set X of cops positions and Y is the previous set of positions, in the Cops and Robber game the cops define $X \cup Y$ as the next positions of the cops. In particular, $|X \cup Y| \leq 2k$, and a response of the robber in the Cops and Robber game is a valid response in the isolation game. \square

Appendix B Flip-width

B.1 Bipartite variants

Let G be a bipartite graph. We define the parameter $\text{bfw}_r(G)$ analogously to $\text{fw}_r(G)$, but the flips played by the cops are now *bipartite* flips: flips between two subsets of opposite parts of G . We only consider partitions of $V(G)$ that refine the bipartition of $V(G)$, and measure its size by the maximum, over the two parts of G , of the number of parts of \mathcal{P} that are contained in it. By definition, we have the following.

Lemma B.1. *Let G be a bipartite graph and $r \in \mathbb{N} \cup \infty$. Then $\text{fw}_r(G) \leq 2 \text{bfw}_r(G)$.*

Let G be a graph and $X, Y \subseteq V(G)$, and let $G[X, Y]$ be the bipartite graph semi-induced by X and Y in G , with parts X and Y and edges xy such that $x \in X, y \in Y, xy \in E(G)$.

Lemma B.2. *Let G be a graph and $X, Y \subseteq V(G)$, Then $\text{bfw}_r(G[X, Y]) \leq \text{fw}_r(G)$.*

Proof. We convert a winning strategy for the cops on G to a winning strategy on $H := G[X, Y]$. If the cops play a k -partite flip G' of G , and \mathcal{P} is the corresponding partition of $V(G)$, then consider the bi-partition \mathcal{Q} of $X \uplus Y$ $\mathcal{Q} := \{P \cap X, P \cap Y \mid P \in \mathcal{P}\}$, and the \mathcal{Q} -flip H' of H , in which two parts of $R, S \in \mathcal{Q}$ are flipped if and only if the two unique parts of $A, B \in \mathcal{P}$ such that $A \cap X = R$ and $B \cap Y = S$, are flipped in the \mathcal{P} -flip producing G' from G . The key property of this construction is that any path in H' , starting at some vertex u and ending at a vertex v , determines a path in G' of the same length, starting at (a copy of) u and ending at (a copy of) v . In particular, if the robber moves to a vertex v' along a path of length at most r in the previous flip of H , then this induces a path of length at most r in the previous flip of G . Hence, if the cops win in G , then they also win in H . \square

B.2 Excluding a $K_{t,t}$

Lemma B.3. *Fix $t > 1$ and a graph G that excludes $K_{t,t}$ as a subgraph. Let $S \subseteq V(G)$ and let \mathcal{P}_S be the partition of $V(G)$ that partitions S into singletons and vertices in $v \in V(G) - S$ according to $N(v) \cap S$. Then $|\mathcal{P}_S| \leq |S|^t$ if $t \geq 3$, and $|\mathcal{P}_S| \leq O(|S|^t)$ if $t \geq 2$.*

Proof. Fix a set $S \subseteq V(G)$ and let $s = |S|$.

To every vertex $v \in V(G)$ with $|N(v) \cap S| \geq t$ assign any set $A \subseteq N(v) \cap S$ with $|A| = t$. Then every set $A \subseteq S$ with $|A| = t$ is assigned to at most $t - 1$ vertices v , since otherwise we have a $K_{t,t}$ as a subgraph of G . It follows that $\{N(v) \cap S \mid v \in V(G), |N(v) \cap S| \geq t\}$ has at most $(t - 1) \cdot \binom{s}{t}$ elements. On the other hand, $\{N(v) \cap S \mid v \in V(G), |N(v) \cap S| < t\}$ has at most s^{t-1} elements. Altogether, $\{N(v) \cap S \mid v \in V(G)\} \uplus S$, which is in bijection with the partition \mathcal{P}_S of $V(G)$, has at most s^t elements for $t \geq 3$ and $O(s^t)$ elements for $t = 2$. \square

B.3 2VC-dimension

Corollary (6.8). *If G is the exact 1-subdivision of an n -clique, then $\text{fw}_2(G) > (n - 1)/4$. Furthermore, for every graph G , $2\text{VCdim}(G) \leq 8\text{fw}_2(G) + 2$.*

Proof. The first part follows immediately from Proposition 6.7, for $r = 2$ and $k = (n - 1)/4$. We prove the second part. Let $V = V(G)$, and consider the bipartite graph $G[V, V]$. Suppose $2\text{VCdim}(G) \geq k$. Then $G[V, V]$ contains the 1-subdivision of K_k as an induced subgraph. By the first part of Corollary 6.8, we have that $\text{fw}_2(G[V, V]) > (k - 1)/4$. By Lemma B.1, we have $\text{bfw}_r(G[V, V]) \geq \text{fw}_r(G[V, V])/2 > (k - 1)/8$, and by Lemma B.2, we have $\text{fw}_r(G) \geq \text{bfw}_r(G[V, V]) > (k - 1)/8$. \square

B.4 Flip-width of binary structures

We extend the definition of flip-width to structures equipped with one or more binary relations. To this end, we extend the notion of flips as follows. Let $R \subseteq V \times V$ be a binary relation on a set V , and let (A, B) be a pair of subsets of V . The relation $R' \subseteq V \times V$ obtained from R by *flipping* the pair (A, B) (now the order of the pair matters) is defined as $R' := R \Delta (A \times B)$, where Δ is the symmetric difference. We will apply such flips in the context of binary relational structures, as defined below.

Fix a binary relational signature Σ , that is, a signature consisting of unary and binary relation symbols only (see Section 3.4). Let B be a Σ -structure and \mathcal{P} be a partition of $V(G)$. A \mathcal{P} -flip is an operation which is specified by a Σ -structure F with vertex set \mathcal{P} . Applying this operation to B results in the Σ -structure B' with $V(B') = V(B)$ and relations

$$R_{B'} := R_B \Delta \bigcup_{(P, Q) \in R_F} P \times Q,$$

for each binary relation symbol $R \in \Sigma$. The unary relation symbols are interpreted in B' in the same way as in B . By slight abuse of language, we sometimes call the structure B' a \mathcal{P} -flip of B .

For $k \geq 1$, a k -flip of B is a Σ -structure B' which is a \mathcal{P} -flip of B , for some partition \mathcal{P} of $V(B)$ with $|\mathcal{P}| \leq k$.

Fix $r \in \mathbb{N} \cup \{\infty\}$. We now define the *flipper game* of radius r and width k on a Σ -structure B similarly as in the case of graphs, with the following differences: in each round, the cops announce a k -flip B' of B , whereas the robber moves along a path of length at most r in the Gaifman graph of the k -flip of B that was announced in the previous round (and B in the first round). The radius- r flip-width of B , denoted $\text{fw}_r(B)$ is the smallest k such that the cops win the flipper game of radius r and width k on B .

Graphs are viewed as structures over the signature Σ consisting of a single binary relation symbol E , interpreted in a given graph G as the (symmetric, irreflexive) adjacency relation. Note that in principle, applying a \mathcal{P} -flip F to a graph G can result in a Σ -structure G' in which the binary relation is no longer symmetric. This is the case when the binary relation of F is not symmetric. However, for the cops, it never pays off to apply such flips, since the robber moves in the Gaifman graph of the resulting structure, so the directions of the edges are of no relevance to the robber. Hence, $\text{fw}_r(G)$ is the same when G is regarded as a graph, or as a binary structure.

Example B.4. Let $B = (V, <)$ be a totally ordered set, viewed as a structure over the signature $\Sigma = \{<\}$. Let \mathcal{P} be a partition of V into k sets that are intervals with respect to $<$, and let I_1, \dots, I_k denote those intervals in increasing order. Consider the \mathcal{P} -flip of B specified by the Σ -structure F with vertices \mathcal{P} , with $<_F = \{(I_i, I_j) \mid 1 \leq i < j \leq k\}$. Applying the flip F results in the Σ -structure $B' = (V, <')$, where $a <' b$ if and only if $a < b$ and a and b belong to the same part of \mathcal{P} .

Now consider the \mathcal{P} -flip of B specified by the Σ -structure F' with vertices \mathcal{P} , with $<_{F'} = \{(I_i, I_j) \mid 1 \leq i \leq j \leq k\}$ (so we also flip pairs (I_i, I_i)). Applying the flip F' results in the Σ -structure $B' = (V, <')$, where $a <' b$ if and only if $a \geq b$ and a and b belong to the same part of \mathcal{P} (so in each part, the order is reversed and becomes reflexive).

Example B.5. The radius- ∞ flip-width of a totally ordered set $B = (\{1, \dots, n\}, <)$ is at most three. The strategy is similar as in Example 5.9, but now in the i th round, the cops apply the 3-flip F of B as in the previous example, for the partition \mathcal{P} of V into the intervals $\{1, \dots, i\}, \{i\}, \{i+1, \dots, n\}$, removing all relations between distinct intervals.

B.5 Definable flip-width

We prove Lemma 9.2.

Lemma (9.2). *There is an algorithm that, given a graph G and numbers $k \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{\infty\}$, determines whether $\text{dfw}_r(G) \leq k$ in time $n^{O(k)} \cdot 2^{O(2^k)}$.*

Proof. Fix k, r , and a graph G . A *configuration* in the definable flipper game with radius r consists of:

- a set $S \subseteq V(G)$ of size at most k , specifying the partition \mathcal{P} played by the cops,
- a \mathcal{P} -flip of G ,
- the current position of the robber.

The set of all configurations has size $O(n^{k+1} \cdot 2^{4^k})$, and the winner of the game can be computed using a fixpoint computation running in time polynomial in $n^{k+1} \cdot 2^{4^k}$. \square

Lemma (9.3). *Fix $r \in \mathbb{N} \cup \{\infty\}$. For every graph G we have:*

$$\text{fw}_r(G) \leq O(\text{dfw}_r(G)^{\text{VCdim}(G)}). \quad (16)$$

Proof. Let G be a graph and let $d = \text{VCdim}(G)$ and let $k = \text{dfw}_r(G)$. Consider a set $S \subseteq V(G)$ with $|S| \leq k$. Then the set system $(S, \{N(v) \cap S \mid v \in V(G)\})$ has VC-dimension at most d . By the Sauer-Shelah-Perles lemma, we have that $\{N(v) \cap S \mid v \in V(G)\}$ has $O(|S|^d) = O(k^d)$ elements.

It follows that every k -definable flip of G is a $O(k^d)$ -flip of G . Since $\text{dfw}_r(G) \leq k$ it follows that $\text{fw}_r(G) = O(k^d)$. \square

Appendix C Modular partition and substitution closure

A set of vertices $X \subseteq V(G)$ in a graph G is a *module* if all vertices in X have the same neighbors outside of X . Note that in a modular partition, all the parts of the partition are modules.

Lemma (5.12). *Let G be a graph and \mathcal{P} be its modular partition. Then*

$$\text{fw}_r(G) \leq \max \left(\text{fw}_r(G/\mathcal{P}), \max_{A \in \mathcal{P}} \text{fw}_r(G[A]) + 2 \right).$$

Proof sketch for Lemma 5.12. The strategy of the cops is as follows. First, use a strategy of width $\ell = \text{fw}_r(G/\mathcal{P})$ on G/\mathcal{P} . Each ℓ -flip G' of G/\mathcal{P} induces an ℓ -flip \widehat{G} of G . Playing according to this strategy, the robber eventually reaches a vertex $A \in V(G/\mathcal{P}) = \mathcal{P}$ that is isolated in the current flip G' of G/\mathcal{P} . The corresponding play in G leads to a ℓ -flip \widehat{G} of G such that the current vertex v of the robber is in A , and there are no edges joining A and $V(G) - A$ in \widehat{G} .

Since A is a module in G , the graph G_A obtained from G by removing all edges with one endpoint in A and one endpoint in $V(G) - A$, can be obtained from G by flipping A and $N(A) - A$. The cops now announce the graph G_A , and the robber is still confined to A .

Now, we use a winning strategy of the cops in the graph $G[A]$, of width $k = \text{fw}_r(G[A])$. Whenever in the game on $G[A]$ the cops announce a k -flip G' of $G[A]$, in the game on G the cops announce the graph \widehat{G} such that $\widehat{G}[A] = G'[A]$, $\widehat{G}[V(G) - A] = G[V(G) - A]$, and there are no edges joining A and $V(G) - A$ in \widehat{G} . The graph \widehat{G} is a $(k+2)$ -flip of G , where the partition partitions A into k parts, according to the partition of $G[A]$ used in the k -flip G' of $G[A]$, and partitions $V(G) - A$ into two parts, $N(A) - A$ and $V(G) - (N(A) \cup A)$. It follows that playing according to this strategy, the cops will win, once the cops win in the game on $G[A]$. \square

Lemma (5.13). *Fix $r \in \mathbb{N} \cup \{\infty\}$. For every $r \in \mathbb{N} \cup \{\infty\}$ and graph class \mathcal{C} , we have*

$$\text{fw}_r(\mathcal{C}^*) \leq \text{fw}_r(\mathcal{C}) + 2.$$

In particular, if \mathcal{C} has bounded flip-width, then \mathcal{C}^ has bounded flip-width.*

The idea is to use the winning strategy on the class \mathcal{C} , and confine the robber to $L(w)$, for deeper and deeper nodes of w , similarly as in the strategy given Example 4.3 for the Cops and Robber game on trees.

Proof sketch for Lemma 5.13. Suppose $\text{fw}_r(\mathcal{C}) \leq k$. Let $G \in \mathcal{C}^*$. Note that G has a modular partition \mathcal{P} such that $G/\mathcal{P} \in \mathcal{C}^*$ and each part A of \mathcal{P} induces a graph $G[A] \in \mathcal{C}^*$, or is a singleton. By repeatedly applying the argument as in the proof of Lemma 5.12, we conclude that $\text{fw}_r(G) \leq k+2$. \square

Appendix D Flip-width with infinite radius

The following yields the upper bound in Theorem 5.17.

Lemma D.1. *For every graph G with $\text{rankwidth}(G) = k$,*

$$\text{fw}_\infty(G) \leq O(2^k).$$

Proof. We prove that every class \mathcal{C} of bounded clique-width has bounded flip-width at radius- ∞ . The other implication was shown in Section 5.3.

Fix a number k and let \mathcal{C} be a class of graphs of rank-width at most k . Then for every $G \in \mathcal{C}$ there is a rooted binary tree T with leaves $V(G)$, such that for every node w of T the set L_w of leaves that are descendants of w induces at most 2^k distinct neighborhoods over its complement $V(G) - L_w$, and conversely, $V(G) - L_w$ induces at most 2^k distinct neighborhoods in L_w .

We present a strategy for the cops, for radius $r = \infty$. The cops keep track of a node w_i of T , maintaining the invariant that in round i , w_i is at distance exactly i from the root, and $A_i \subseteq L_{w_i}$, where A_i denotes the set of possible next moves of the robber.

Initially, w_0 is the root, and $A_0 = V(G)$, so the invariant is satisfied.

We now describe the cops strategy that maintains the invariant. Suppose we are in round i . If w_i is a leaf then the cops win, since $A_i \subseteq \{w_i\}$ by the invariant. Otherwise, partition $V(G)$ into three parts:

- B_0 – the leaves of T below the left child of w_{i-1} ,
- B_1 – the leaves of T below the right child of w_{i-1} ,
- B_2 – the remaining leaves of T .

For $j = 0, 1, 2$, let \mathcal{Q}_j be the partition of the set B_j into equivalence classes of the relation of having equal neighborhoods in $V(G) - B_j$. Then $|\mathcal{Q}_j| \leq O(2^k)$, for $j = 0, 1, 2$. Define the partition $\mathcal{P}_i := \mathcal{Q}_0 \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$. Then there is a \mathcal{P}_i -flip G_i of G such that for any edge $uv \in E(G_i)$, the vertices u and v belong to the same part of the partition $\{B_0, B_1, B_2\}$. The cops play the partition \mathcal{P}_i and the flip G_i .

Suppose the robber responds by moving to a vertex $c_i \in A_{i-1}$. Since A_{i-1} consists of descendants of w_{i-1} by the invariant, it follows that either $c_i \in B_0$ or $c_i \in B_1$. Therefore, the set A_i , which is defined as the connected component of c_i in G_i , is either contained in B_0 , or in B_1 . In the first case, define w_i as the left child of w_{i-1} , and in the latter case, as the right child of w_{i-1} . Thus, the invariant is maintained.

Since T is a finite tree, after at most $|T|$ rounds w_i is a leaf, and then cops win. \square

We now prove Lemma 5.19, completing the proof of the lower bound in Theorem 5.17. First, a lemma.

Lemma D.2. *Let G be a graph and $A \uplus B$ a bipartition of $V(G)$ with $\text{rk}_G(A, B) > k$. Then for every k -flip G' there is some edge $ab \in E(G')$ with $a \in A$ and $b \in B$.*

Proof. Let G' be a k -flip of G . Denote by H the graph with vertices $V(G)$ and edges $E(G) \triangle E(G')$. Since G' is a k -flip of G , it follows that H is a k -flip of the edgeless graph, and thus $\text{rk}_H(A, B) \leq k$. Since $E(G) = E(G') \triangle E(H)$, we have that

$$\text{rk}_G(A, B) \leq \text{rk}_{G'}(A, B) + \text{rk}_H(A, B), \quad (17)$$

as the rank of the sum of two matrices is at most the sum of the ranks. If there are no edges $ab \in E(G')$ with $a \in A, b \in B$, then we have $\text{rk}_{G'}(A, B) = 0$, and therefore $\text{rk}_G(A, B) \leq k$ by (17). \square

Lemma (5.19). *Fix a graph G and number $k \in \mathbb{N}$. Every well-linked set U with $|U| > 3k$ is a (∞, k, k) -hideout.*

Proof. Let G' be a k -flip of G . For a vertex $v \in U$, let $U(v) \subseteq U$ denote the set of vertices $w \in U$ that are reachable from v by a path in G' (including v). We show that there are at most k vertices $v \in U$ with $|U(v)| \leq k$, proving that U is a (∞, k, k) -hideout.

Suppose otherwise: that there are $k+1$ such distinct vertices $v_1, \dots, v_{k+1} \in U$. Let i be the smallest number such that $|U(v_1) \cup \dots \cup U(v_i)| > k$. Then $2 \leq i \leq k+1$. Let $U' = U(v_1) \cup \dots \cup U(v_i)$. Then $k < |U'| \leq 2k$, where the second inequality holds because $|U(v_i)| \leq k$ and

$|U(v_1) \cup \dots \cup U(v_{i-1})| \leq k$ by minimality of i . Since $|U| > 3k$, we have that $|U - U'| > k$ and $|U'| > k$. Let $A \subseteq V(G)$ be the union of the connected components of G' that intersect U' , and let $B = V(G) - A$. Then there is no edge in G' with one endpoint in A and one endpoint in B . As $A \cap U = U'$ and $B \cap U = U - U'$ have both size greater than k , we have that $\text{rk}_G(A, B) > k$ as U is well-linked. By Lemma D.2, there is an edge of G' with one endpoint in A and one endpoint in B , a contradiction. \square

Appendix E VC-dimension

E.1 Bounding VC-dimension in terms of radius-one flip-width

By a similar argument as in Lemma 5.20, we prove the following (see Appendix B.1 for the definition of bipartite flip-width, bfw).

Lemma (5.21). *Let $b, k \in \mathbb{N}$ and let G be a bipartite graph with $\text{bfw}_1(G) \leq k$. Then G contains more than b mutual $2bk$ -near-twins contained in a single part of G .*

To prove Theorem 5.24, we show that every bipartite graph of sufficiently large VC-dimension contains an induced subgraph with no k -near-twins. First we construct such bipartite graphs. For a number $m \in \mathbb{N}$ let \mathbb{Z}_2^m denote the m -dimensional vector space over the two-element field \mathbb{Z}_2 , and for $v, w \in \mathbb{Z}_2^m$ denote $v \cdot w = \sum_{i=1}^m v_i \cdot w_i \pmod{2}$.

Lemma E.1. *Fix a number $m \in \mathbb{N}$ and consider the bipartite graph H whose parts are two copies of \mathbb{Z}_2^m , with edges vw such that $v \cdot w \neq 0$. Then H has no $(2^{m-1} - 1)$ -near-twins in either part.*

Proof. Denote the two copies of \mathbb{Z}_2^m with V and V^* , and write v^* for the copy of a vector $v \in V$ in V^* . We claim that for all $u, v \in V$ we have $|N_H(u) \triangle N_H(v)| = 2^{m-1}$. Indeed,

$$N_H(u) \triangle N_H(v) = \{w^* \mid w \in V, w \cdot u \neq w \cdot v\} = \{w^* \mid w \in V, w \cdot (u - v) \neq 0\}.$$

Since $\{w \in V \mid w \cdot (u - v) \neq 0\}$ has 2^{m-1} elements, so does $N_H(u) \triangle N_H(v)$.

By a dual argument, $|N_H(u^*) \triangle N_H(v^*)| = 2^{m-1}$ for all $u^*, v^* \in V^*$. Therefore, H_m has no pair of $(2^{m-1} - 1)$ -near-twins in either part. \square

Lemma (5.25). *Let G be a graph with $\text{VCdim}(G) \geq 2^m$, for some m . Then there are two sets X, Y such that the bipartite graph $G[X, Y]$ contains no pair of $(2^{m-1} - 1)$ -near-twins in either of the parts X, Y .*

Proof. Let X be a subset of $V(G)$ of size 2^m that is shattered by $\{N(v) \mid v \in V(G)\}$.

Arbitrarily identify the elements of X with the elements of \mathbb{Z}_2^m . Denote $v^\perp := \{u \in X \mid u \cdot v = 0\} \subseteq X$. Since X is shattered in G , for every $v \in X$ there is a vertex $v^* \in V(G)$ such that $N(v^*) \cap X = v^\perp$. Denote $Y := \{v^* \mid v \in X\}$. The function $v \mapsto v^*$ is then a bijection between X and Y , and for $v, w \in X$, v is adjacent to w^* in G if and only if $v \cdot w \neq 0$. Therefore, $G[X, Y]$ is isomorphic to the graph from Lemma E.1, and the conclusion follows. \square

Proof of Theorem 5.24. We show that if $\text{VCdim}(G) \geq 2^m$, then $\text{fw}_1(G) > 2^{m-2}$. Assume $\text{VCdim}(G) \geq 2^m$, and let $G[X, Y]$ be as in Lemma 5.25. By Lemma 5.21 (setting $b = 1$), this implies that $\text{bfw}_1(H) > 2^{m-2}$. By Lemma B.2, $\text{fw}_1(G) > 2^{m-2}$. \square

E.2 Duality result

We reprove a duality result which is a corollary of the (p, q) -theorem of Alon-Kleitman-Matoušek [Mat04, Theorem 4]. We present a self-contained proof, to analyze the bounds. Our presentation is based on a proof by Simon [Sim15, Cor 6.3; Sec. 6.3].

Say that a binary relation $E \subseteq A \times B$ has a *duality of order k* , where $k \in \mathbb{N}$, if at least one of two cases holds:

- there is a set $B' \subseteq B$, $|B'| \leq k$, such that for all $a \in A$ there is some $b \in B'$ with $(a, b) \in E$,
or
- there is a set $A' \subseteq A$, $|A'| \leq k$, such that for all $b \in B$ there is some $a \in A'$ with $(a, b) \notin E$.

Recall that the VC-dimension of a binary relation E is the maximum of the VC-dimension of two set systems:

$$(B, \{\vec{E}(a) \mid a \in A\}) \quad \text{and} \quad (A, \{\tilde{E}(b) \mid b \in B\}).$$

Here's the duality result.

Theorem E.2. *Let $E \subseteq A \times B$ be a binary relation, with A, B finite. Then E has a duality of order $O(d)$, where $d = \text{VCdim}(E)$.*

The result follows from a combination of two classic results: the Vapnik-Chervonenkis theorem, and von Neumann's minimax theorem. We first state a corollary of von Neumann's minimax theorem (also of Farkas' lemma, the Hahn-Banach theorem, or of the strong duality theorem for linear programming).

Theorem E.3 (Minimax Theorem). *Let $E \subseteq A \times B$ be a binary relation with A, B finite, and let $\alpha \in \mathbb{R}$. Then exactly one of the following two cases holds:*

1. *there is some probability distribution ν on B such that $\nu(\vec{E}(a)) \geq \alpha$ holds for all $a \in A$,*
2. *there is some probability distribution μ on A such that $\mu(\tilde{E}(b)) < \alpha$ holds for all $b \in B$.*

To formulate the VC-theorem, we introduce the following notions.

For a multiset S of elements of a set V and a set $F \subseteq V$, denote by $\text{Av}_S(F)$ the proportion of elements of S (counted with multiplicities) that belong to F , that is

$$\text{Av}_S(F) = \frac{|S \cap F|}{|S|}.$$

Let (V, \mathcal{F}) be a set system with V finite, μ a probability distribution on V , and $\varepsilon > 0$ a real. We say that a set S is an ε -approximation of μ on \mathcal{F} , if for every $F \in \mathcal{F}$ we have

$$|\text{Av}_S(F) - \mu(F)| \leq \varepsilon.$$

Theorem E.4 (VC-theorem). *Let (V, \mathcal{F}) be a set system with V finite, and let d be its VC-dimension. For every $\varepsilon > 0$ and probability distribution μ on V , there exists an ε -approximation S of μ on \mathcal{F} with*

$$|S| \leq O(d) \cdot \frac{1}{\varepsilon^2} \log \left(\frac{1}{\varepsilon} \right).$$

Proof of Theorem E.2. Set $\alpha := \frac{1}{2}$ and $\varepsilon := \frac{1}{3}$; the point is that $0 < \alpha \pm \varepsilon < 1$. Fix $d \in \mathbb{N}$, and let k be the number from the VC-theorem, with $k \leq O(d) \frac{1}{\varepsilon^2} \log(\frac{1}{\varepsilon}) = O(d)$. By the Minimax Theorem applied to $\alpha = \frac{1}{2}$, one of the two cases below holds.

Case 1: *There is some probability distribution ν on B such that $\nu(\vec{E}(a)) \geq \frac{1}{2}$ holds for all $a \in A$. By the VC-theorem applied to $\varepsilon = \frac{1}{3}$, there is a multiset $B' \subseteq B$ with $|B'| \leq k$ such that*

$$|\text{Av}_{B'}(\vec{E}(a)) - \nu(\vec{E}(a))| \leq \frac{1}{3} \quad \text{for all } a \in A.$$

Pick $a \in A$. Since $\nu(\vec{E}(a)) \geq \frac{1}{2}$, we have that $\text{Av}_{B'}(\vec{E}(a)) > 0$, so $B' \cap \vec{E}(a)$ is nonempty. Therefore, the first condition in the definition of a duality is satisfied.

Case 2: *There is some probability distribution μ on A such that $\mu(\tilde{E}(b)) < \frac{1}{2}$ holds for all $b \in B$. By the VC-theorem again, there is a multiset $A' \subseteq A$ with $|A'| \leq k$ such that*

$$|\text{Av}_{A'}(\tilde{E}(b)) - \mu(\tilde{E}(b))| \leq \frac{1}{3} \quad \text{for all } b \in B.$$

Pick $b \in B$. Since $\mu(\tilde{E}(b)) < \frac{1}{2}$, we have that $\text{Av}_{A'}(\tilde{E}(b)) < 1$, so A' is not contained in $\tilde{E}(b)$. Therefore, the second condition in the definition of a duality is satisfied. \square

E.3 Definability result

The following result is proved in [BDG⁺22, Thm. 3.5], up to a slight variation of the assumptions (see proof below).

Lemma (9.8). *Fix $k, d \in \mathbb{N}$. Let V be a set equipped with:*

- *a binary relation $E \subseteq V \times V$ of VC-dimension at most d ,*
- *a pseudometric $\text{dist}: V \times V \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ (that is, a function satisfying the triangle inequality),*
- *and a partition \mathcal{P} of size at most k ,*

such that $E(u, v)$ depends only on the \mathcal{P} -class of u and the \mathcal{P} -class of v whenever $\text{dist}(u, v) > 1$. Then there is a set $S \subseteq V$ of size $O(dk^2)$, such that $E(u, v)$ depends only on the S -types of u and of v , whenever $\text{dist}(u, v) > 5$.

Proof. [BDG⁺22, Thm. 3.5] prove the same statement, except that the assumption $\text{VCdim}(E) \leq d$ is replaced with: *for every $A, B \subseteq V$, the relation $E \cap (A \times B)$ has a duality of order d .* By Theorem E.2, this implies the formulation above. \square

Appendix F Exact subdivisions

We prove Proposition 6.7, which is repeated below.

Proposition (6.7). *Fix $r \geq 2, k \geq 1$. Let G be the exact $(r-1)$ -subdivision of some graph H with minimum degree at least $2rk$. Then $\text{fw}_r(G) > k$.*

We first prove two lemmas.

Lemma F.1. *Fix $k, \ell, m \geq 1$. Let H be a bipartite graph with bipartition (L, R) , in which every vertex in L has degree at least ℓ , and any two distinct vertices in L have at most m common neighbors. Let H' be a k -flip of H . Then there is a set $X \subseteq L$ with $|X| \geq |L| - k$, and a bijection $\pi: X \rightarrow X$, such that every $v \in X$ is adjacent in H' to at least $\lceil \frac{\ell-m}{2} \rceil$ vertices in $N_H(\pi(v))$.*

Proof. Let \mathcal{P} be the partition of $V(G)$ with $|\mathcal{P}| \leq k$ such that H' is a \mathcal{P} -flip of H .

First consider the case when all the vertices of L are in one part A of \mathcal{P} . Let W be the union of the parts B of \mathcal{P} that are not flipped with A in the flip that produces H' from H . Let X_1 consist of those vertices $v \in L$ such that $|N_H(v) \cap W| \geq \frac{\ell-m}{2}$, and let X_2 consist of the remaining vertices in L .

Every vertex in X_1 is adjacent in H' to at least $\frac{\ell-m}{2}$ vertices in $N_H(v)$ (namely, to $N_H(v) \cap W$), so we can set $\pi(v) = v$ for all $v \in X_1$.

If v and v' are two distinct vertices in X_2 , then $N_{H'}(v) \supseteq N_H(v') \cap W$ and $|N_H(v') \cap W| \geq \frac{\ell-m}{2}$. If $|X_2| \leq 1$ then set $X := X_1 = L - X_2$, and let $\pi: X \rightarrow X$ be the identity on X . If $|X_2| > 2$, set $X := L = X_1 \cup X_2$, and let $\pi: X \rightarrow X$ be a permutation that maps every vertex in X_1 to itself, and acts as a cyclic permutation on the vertices in X_2 . In any case, $|X| \geq |L| - 1$, and every $v \in X$ is adjacent in H' to at least $\lceil \frac{\ell-m}{2} \rceil$ neighbors of $\pi(v)$.

In the general case, partition L as $L = L_1 \uplus \dots \uplus L_s$, for some $s \leq k$, following the partition \mathcal{P} restricted to L . For each $1 \leq i \leq s$, let $H_i = H[L_i, R]$, and $H'_i = H'[L_i, R]$; then H'_i is a k -flip of H_i , and they fall into the special case considered above. Hence, for each $1 \leq i \leq s$ there are set $X_i \subseteq L_i$ with $|L_i| \geq |X_i| - 1$ and a bijection $\pi_i: X_i \rightarrow X_i$. Set $X = X_1 \uplus \dots \uplus X_s$, and let $\pi: X \rightarrow X$ be such that $\pi(x) = \pi_i(x)$ for $x \in X_i$. Then X and π satisfy the required condition. \square

Applying Lemma F.1 in the case $\ell = 1$ and $m = 0$, we get the following.

Corollary F.2. *Fix $k \geq 1$. Let M be a matching between two sets L and R , and let M' be a k -flip of M . Then M' contains, as a subgraph, a matching between all but k vertices of L , and a set of vertices of R of the same size.*

For $r, \ell \geq 1$, let $G_{r,\ell}$ denote the union of ℓ paths, each of length r (and with $r + 1$ vertices), and in each path, call one of the vertices of degree one a *source*, and the other one a *target*. By an easy induction on $r \geq 1$, Corollary F.2 gives the following.

Lemma F.3. *Fix $k, r, \ell \geq 1$. If G' is a k -flip of $G_{r,\ell}$, then at least $\ell - rk$ target vertices of $G_{r,\ell}$ are joined by a path of length r in G' with some source vertex.*

Proof of Proposition 6.7. Fix $r \geq 2$ and let G be an exact $(r - 1)$ -subdivision of some graph H with minimum degree at least $2rk$. We aim to prove that $\text{fw}_r(G) > k$. Let $P \subseteq V(G)$ denote the set of *principal vertices* of G , that is, vertices of degree larger than two; those vertices correspond to the vertices of H . We show that P forms a (r, k, k) -hideout in G . By Lemma 5.16, this implies that $\text{fw}_r(G) > k$.

Let $R = N_G(P) = \bigcup_{v \in P} N_G(v)$ denote the set of neighbors of the principal vertices in G . Note that R and P are disjoint, as $r \geq 2$. Then $G[P, R]$ is a bipartite graph in which every vertex in P has at least $2rk$ neighbors in R , and any two vertices in P have at most one common neighbor in R (in fact, no common neighbors if $r \geq 2$). For each $v \in P$, there is an induced subgraph G_v of G that is isomorphic to $G_{r-1,\ell}$, for some $\ell \geq 2rk$ with the source vertices equal to $N_G(v)$, and where each target vertex is a principal vertex.

Let G' be a k -flip of G . Call a principal vertex $v \in P$ *good* if there is some $w \in P$ such that v is adjacent in G' to at least rk elements of $N_G(w)$, and call v *bad* otherwise. We show that (1) there are at most k bad vertices, and (2) every good vertex has more than k principal vertices in its r -neighborhood in G' . It follows that the principal vertices form a (r, k, k) -hideout in G .

(1) Apply Lemma F.1 to the bipartite graph $G[P, R]$ and to $G'[P, R]$, which is a k -flip of F . Since every vertex in P has degree at least $2rk$, and any two distinct vertices in L have at most one common neighbor, and $\lceil (2rk - 1)/2 \rceil = rk$, there are at most k bad principal vertices by Lemma F.1.

(2) Let $v \in P$ be a good principal vertex, and let $w \in P$ be such that some set A of rk vertices of $N_G(w)$ are adjacent to v in G' . Consider the subgraph K of G_w , consisting of vertex-disjoint paths of length $r - 1$ in G joining the vertices of A with rk principal vertices. In particular, K is isomorphic to $G_{r-1,rk}$. Let $K' = G'[V(K)]$; then K' is a k -flip of K . By Lemma F.3, at least $rk - (r - 1)k = k$ principal vertices are joined by a path of length $r - 1$ in K' with some vertex in A . Those k principal vertices are therefore at distance at most r in G' from v . \square

Appendix G Bounded twin-width

G.1 Bounding flip-width in terms of twin-width

We prove:

Theorem (7.1). *Fix $r \in \mathbb{N}$. For every graph G of twin-width d we have:*

$$\text{fw}_r(G) = 2^d \cdot d^{O(r)}.$$

In particular, every class of bounded twin-width has bounded flip-width.

We use the following result, proved in [BFLP23].

Theorem G.1. *Let G be a graph of twin-width $d \geq 1$, and let $A \subseteq V(G)$. Then we have:*

$$|\{N(v) \cap A \mid v \in V(G)\}| \leq 2^{d+O(\log d)} \cdot |A|.$$

This improves a previous result [BKR⁺22, Prz22], which gave a doubly-exponential dependency on d .

Proof. Fix an uncontraction sequence $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ of G of red-degree d . For $v \in V(G)$ and $1 \leq i \leq n$ let $B_r^i(v) \subseteq \mathcal{P}_i$ denote the ball of radius r in the red graph of \mathcal{P}_i , around the part of \mathcal{P}_i containing v . In particular, $|B_r^i(v)| < d^{r+1} = O_{r,d}(1)$.

We describe a strategy of the cops that guarantees that the following invariant holds after round i , for $i = 1, 2, \dots, n$:

$$A_i \subseteq \bigcup B_r^i(c_i), \quad (18)$$

where c_i is robber's position in round i , and $A_i \subseteq V(G)$ is the set of possible positions the robber can move to in round $i + 1$ of the game, that is, A_i is the ball of radius r around c_i in the graph G_i announced by the cops in round i , with $G_1 = G$ and c_1 being the initial vertex of the robber.

Note that for $i = n$, the inclusion (18) implies that $|A_n| = 1$ as in \mathcal{P}_n there are no red edges and each part is a singleton. Therefore, the cops win the game after n rounds according to this strategy. We need to show how the cops can maintain the invariant (18), by playing k -flips of G for some k bounded in terms of r and d .

Before describing the strategy, we make an observation that will be useful in the inductive reasoning. It gives a description of a ball in the red graph of \mathcal{P}_{i-1} , in terms of the red graph of \mathcal{P}_i . Below, for $\mathcal{F} \subseteq \mathcal{P}_i$, the set $B_r^i(\mathcal{F}) \subseteq \mathcal{P}_i$ denotes the set of parts of \mathcal{P}_i that are at distance at most r from some part in \mathcal{F} in the red graph of \mathcal{P}_i .

Claim G.1. *Let $1 < i \leq n$ and let $v \in V(G)$. Then there is a set $\mathcal{F} \subseteq \mathcal{P}_i$ with $|\mathcal{F}| \leq d + 3$, such that $B_r^{i-1}(v) \subseteq B_r^i(\mathcal{F})$.*

Proof. The family \mathcal{F} consists of:

- the part of \mathcal{P}_i that contains v ,
- the parts A, B of \mathcal{P}_i such that $A \cup B$ is a part of \mathcal{P}_{i-1} ,
- the parts in \mathcal{P}_i that are not homogeneous towards $A \cup B$ in G .

It can be easily verified that \mathcal{F} satisfies the statement of the claim. \square

We now describe the cops' strategy. In the first round, we have $G_1 = G$, and the invariant (18) is trivially satisfied since \mathcal{P}_1 has just one part, and that part contains $A_1 = B_G^r(c_1)$, regardless of the robber's choice of c_1 .

Suppose that the invariant (18) is satisfied after round $i - 1$, for some $1 < i \leq n$, so that $A_{i-1} \subseteq \bigcup B_r^{i-1}(c_{i-1})$. We describe how the cops should play to maintain invariant (18) after round i . Apply Claim G.1 to $v = c_{i-1}$, obtaining a family $\mathcal{F} \subseteq \mathcal{P}_i$ with $|\mathcal{F}| \leq d + 3$ such that $B_r^{i-1}(c_{i-1}) \subseteq B_r^i(\mathcal{F})$. In particular, $A_{i-1} \subseteq \bigcup B_r^i(\mathcal{F})$.

Let $R = V(G) - \bigcup B_{2r}^i(\mathcal{F})$, and let \mathcal{R} be the partition of R according to the equivalence relation of having the same neighborhood in the set $\bigcup B_{2r-1}^i(\mathcal{F})$. Note that we are simultaneously considering balls around \mathcal{F} with radii $r, 2r - 1$, and $2r$.

Claim G.2. *We have $|\mathcal{R}| = 2^d \cdot d^{O(r)}$, and every part $P \in \mathcal{R}$ is homogeneous towards all parts in $B_{2r-1}^i(\mathcal{F})$.*

Proof. Pick a set X which contains one representative of each part in $B_{2r-1}^i(\mathcal{F})$. Then $|X| \leq d^{O(r)}$. Note that every part $P \in \mathcal{P}_i - B_{2r}^i(\mathcal{F})$ is homogeneous towards every part $Q \in B_{2r-1}^i(\mathcal{F})$. Therefore, the neighborhood of a vertex $v \in R$ in $\bigcup B_{2r-1}^i(\mathcal{F})$ is completely determined by $N(v) \cap X$. By Theorem G.1, there are at most

$$2^{d+O(\log d)} |X| \leq 2^d \cdot d^{O(r)}$$

different neighborhoods of vertices $v \in R$ in X , so $|\mathcal{R}| \leq 2^d \cdot d^{O(r)}$. \square

Let $\mathcal{P}'_i = B_{2r}^i(\mathcal{F}) \cup \mathcal{R}$. Then \mathcal{P}'_i is a partition of $V(G)$. We have $|B_{2r}^i(\mathcal{F})| \leq |\mathcal{F}| \cdot d^{2r+1} \leq O(d^{2r+2})$, and it follows from Claim G.2 that $|\mathcal{P}'_i| \leq 2^d \cdot d^{O(r)}$.

The cops play the \mathcal{P}'_i -flip G_i of G obtained by flipping any pair P, Q of distinct parts of \mathcal{P}'_i such that the pair P, Q is complete in G .

Now, the robber makes his move, and picks a vertex $c_i \in A_{i-1}$, and we set $A_i := B_r^{G_i}(c_i)$. We now prove that the invariant (18) holds.

Claim G.3. $B_r^{G_i}(c_i) \subseteq \bigcup B_r^i(c_i)$.

Proof. Note that $c_i \in A_{i-1} \subseteq \bigcup B_r^i(\mathcal{F})$.

Let $v \in B_r^{G_i}(c_i)$ be a vertex at distance k from c_i in G_i , for some $0 \leq k \leq r$. We show by induction on k that if A is the part of \mathcal{P}_i that contains v , then $A \in B_k^i(c_i)$. For $k = r$, this immediately yields the claim.

For $k = 0$ the statement holds trivially, so suppose that $k > 0$ and the statement holds for $k - 1$. Let w be a vertex that is a neighbor of v and is at distance $k - 1$ from c_i in G_i , and let B be the part of \mathcal{P}_i that contains w . By inductive assumption, $B \in B_{k-1}^i(c_i)$, and since $c_i \in \bigcup B_r^i(\mathcal{F})$, it follows that $B \in B_{r+k}^i(\mathcal{F})$. As $k < r$, we have that $B \in B_{2r-1}^i(\mathcal{F})$.

Since A and B are connected by an edge in G_i , it must be the case that A and B are not homogeneously connected in G . As $B \in B_{2r-1}^i(\mathcal{F})$, it cannot be that $A \in \mathcal{R}$ (since all parts in \mathcal{R} are homogeneous towards all parts in $B_{2r-1}^i(\mathcal{F})$ by Claim G.2), so $A \in B_{2r}^i(\mathcal{F})$. As A and B are not homogeneously connected in G , A is a neighbor of B in the red graph of \mathcal{P}_i . As $B \in B_{k-1}^i(c_i)$, it follows that $A \in B_k^i(c_i)$, finishing the inductive step. \square

The invariant (18) now follows from Claim G.3. In particular, if the cops continue playing this way, at the end of round n we have that $|A_n| = 1$, so the cops win. \square

G.2 Ordered flip-width

We prove Lemma 7.2, repeated below.

Lemma (7.2). Fix $r \in \mathbb{N} \cup \{\infty\}$ and an ordered graph $G = (V, E, <)$. Then

$$\sqrt{\text{fw}_r(G) + 1} \leq \text{fw}_r^<(G) + 1 \leq \text{fw}_{3r+2}(G) + 1.$$

We first study the effects of applying flips to a set equipped with a total order.

Lemma G.2. Let $L = (V, <)$ be a total order and let L' be a k -flip of L (as a binary structure). Then there is a set $S \subseteq V$ with $|S| \leq k$ such that any two vertices of $V - S$ with no vertex of S between them are at distance at most 2 in the Gaifman graph of L' .

Proof. Let \mathcal{P} be a partition of V with $|\mathcal{P}| \leq k$ such that L' is a \mathcal{P} -flip of L . Note that the parts of \mathcal{P} need not be convex in the order $<$; this is the main challenge here. Let $S = \{\max(A) \mid A \in \mathcal{P}\}$ be the set of $<$ -maximal elements of each part of \mathcal{P} . We claim that S satisfies the required condition.

Observe first that each part $A \in \mathcal{P}$ forms a clique in the Gaifman graph of L' . Indeed, the relation $<'$ of L' restricted to A either coincides with $<$, if (A, A) is not flipped in the \mathcal{P} -flip producing L' from L , or coincides with \geq , if (A, A) is flipped. In any case, $<'$ is a total relation on A .

Let $a, b \in V - S$ be vertices belonging to different parts A, B of \mathcal{P} , with no element of S between a and b . We show that a and b are either adjacent, or have a common neighbor in the Gaifman graph of L' . By symmetry, suppose that $a < b$. If the pair (A, B) is not flipped in the \mathcal{P} -flip producing L' from L , then $a <' b$ in L' , so a and b are adjacent in the Gaifman graph of L' .

Suppose that the pair (A, B) is flipped. Let $m_a := \max A$ and $m_b := \max B$. Clearly, $b < m_b$ and $a < m_a$. As m_a is not between a and b , we have that $b < m_a$. Since (A, B) is flipped and $m_a \not<' b$ in L , it follows that $m_a <' b$ in L' . Since a and m_a are in the same part of A , they are adjacent in the Gaifman graph of L' . Therefore, m_a is a common neighbor of a and of b in the Gaifman graph of L' . \square

Corollary G.3. Let $G = (V, E, <)$ be an ordered graph, and let $G' = (V, E', <')$ be a k -flip of G , in the sense of binary structures. Then there is a k -cut-flip $G'' = (V, E', S)$ of G , such that for all $u, v \in V$, if u and v are connected by a path π of total weight at most r in G'' , then u and v are within distance at most $3r + 2$ in the Gaifman graph of G' .

Proof. Let $L = (V, <)$ be the total order underlying G , and let $G' = (V, E', <')$ be a k -flip of G . Then $L' := (V, <')$ is a k -flip of L . Apply Lemma G.2, yielding a set S . Let $G'' := (V, E', S)$. Then G'' is a k -cut-flip of G . We check that it satisfies the condition.

Every path of total weight 0 can be replaced by a path of length ≤ 2 in the Gaifman graph of G' , by Lemma G.2. A path of total weight r decomposes into at most $r + 1$ paths of total weight 0 and at most r edges of weight 1. By replacing the paths of total weight 0 as above, we get a path of length at most $3r + 2$. \square

Proof of Lemma 7.2. Fix $G = (V, E, <)$ and r .

To prove the second inequality, we show that if $\text{fw}_{3r+2}(G) \leq k$ (in the sense of binary structures), then $\text{fw}_r^<(G) \leq k$ (in the sense of ordered graphs). This is done by transferring the strategy (cf. Section 8.2), using Corollary G.3.

More precisely, suppose $\text{fw}_{3r+2}(G) \leq k$, so the robber wins the flipper game on G , as a binary structure, of radius $3r + 2$ and width k . The cops copy their winning strategy when playing the radius- r ordered flipper game on G , as follows: Whenever the cops announce a k -flip G' of G in the flipper game, then in the ordered flipper game, the cops announce the k -cut-flip $G'' = (V, E', S)$ of G , as given by Corollary G.3. By an argument analogous to Lemma 8.11, Corollary G.3 shows that this way, the cops win the ordered flipper game on G as an ordered graph, so $\text{fw}_r^<(G) \leq k$.

For the first inequality, we show that if (V, E', S) is a k -cut-flip of G , then there is a $(k^2 + 2k)$ -flip $G' = (V, E', <')$ of G such for all distinct $u, v \in V$, if there is no $s \in S$ with $u \leq s \leq v$ then $u <' v$ or $v <' u$. Namely, consider the partition \mathcal{P} of V with $|\mathcal{P}| \leq k$ such that (V, E') is a \mathcal{P} -flip of (V, E) , let \mathcal{Q} be the partition of V that partitions S into singletons and $V - S$ into maximal $<$ -intervals that are disjoint with S , and let \mathcal{R} be the common refinement of \mathcal{Q} and \mathcal{R} . Then $|\mathcal{Q}| \leq |\mathcal{P}| \cdot (|S| + 1) + |S| \leq k(k + 1) + k = k^2 + 2k$. Define $<'$ so that $u <' v$ if and only if u and v are in the same part of \mathcal{Q} and $u < v$. Now, $G' = (V, E', <')$ is a \mathcal{Q} -flip of G , and has the desired properties.

It follows that a winning strategy for the cops in the ordered flipper game on G with radius r and width k can be transferred into a winning strategy for the cops in the flipper game on G with radius r and width $k^2 + 2k = (k + 1)^2 - 1$, by replacing each k -cut-flip played by the cops by the $(k^2 + 2k)$ -flip G' as above. Hence, $\text{fw}_r(G) \leq (\text{fw}_r^<(G) + 1)^2 - 1$, proving the first inequality. \square

Appendix H Closure under transductions

In this section, we show that bounded flip-width is preserved by first-order transductions, and that bounded ∞ -flip-width is preserved by CMSO transductions.

We work with c -colored graphs G . We consider formulas $\varphi(x, y)$ of first-order logic, or CMSO. The same arguments work to other logics with suitable locality properties.

Theorem (8.2). *There is a computable function $T_q : \mathbb{N} \rightarrow \mathbb{N}$ with the following property. Fix a radius $r \geq 1$ and a first-order formula $\varphi(x, y)$ of quantifier rank q in the signature of c -colored graphs, for some $c \geq 0$. Set $r' := 2^q \cdot r$. Then for every c -colored graph G we have*

$$\text{fw}_r(\varphi(G)) \leq T_q(\text{fw}_{r'}(G) \cdot c).$$

In particular, if \mathcal{C} has bounded flip-width, then $\varphi(\mathcal{C})$ has bounded flip-width.

Say that a formula $\varphi(x, y)$ is r -local, where $r \in \mathbb{N}$, if the following condition holds: there is a finite set T_φ of local types such that for every colored graph G each vertex v of G can be labelled by an element $\text{ltp}(v) \in T_\varphi$ in such a way that for any pair of vertices (a, b) with distance larger than r in G , whether or not $\varphi(a, b)$ holds in G depends only on $\text{ltp}(a)$ and $\text{ltp}(b)$. More precisely, there is a binary relation $\Phi \subseteq T_\varphi \times T_\varphi$ (which may depend on G) such that for all vertices a, b with $\text{dist}(a, b) > r$,

$$G \models \varphi(a, b) \iff (\text{ltp}(a), \text{ltp}(b)) \in \Phi.$$

We say that $\varphi(x, y)$ is ∞ -local if the above condition holds, where instead of $\text{dist}(a, b) > r$ we require that a and b are in different connected components of the graph.

Fact H.1. Fix $c \geq 0$ and consider the signature of c -colored graphs. Every formula $\varphi(x, y)$ of first-order logic is 2^q -local, where q is the quantifier rank of φ . The number $|T_\varphi|$ of local types is at most

$$T_q(k) := \underbrace{2^{2^{\cdot^{\cdot^{\cdot^2^m}}}}}_{\text{height } q}$$

where m is the number of c -colored graphs with vertex set $\{1, \dots, q+1\}$.

Fact H.2. In the setting of the previous fact, every formula $\varphi(x, y)$ of CMSO is ∞ -local. The number $|T_\varphi|$ is again bounded by a number $T'_q(k)$ that is non-elementary in q .

Theorem 8.2 follows easily from the next lemma, using Lemma 8.11.

Lemma H.3. Fix $k, c \geq 1$, a first-order formula $\varphi(x, y)$ of quantifier rank q , and let $s = 2^q$ and $\ell = T_q(k \cdot c)$, where T_q is the function from Fact H.1. Let G be a c -colored graph. For every k -flip G' of the uncolored graph underlying G there is an ℓ -flip $\varphi(G)'$ of $\varphi(G)$ such that:

$$\text{dist}_{G'}(u, v) \leq s \quad \text{for all } uv \in E(\varphi(G)'). \quad (19)$$

Proof. Let $\mathcal{P} = \{A_1, \dots, A_s\}$ be the partition of $V(G)$ with $s \leq k$, such that G' is a \mathcal{P} -flip of G . Color the vertices of G' using $k \cdot c$ colors $[k] \times [c]$ so that a vertex v has color (i, j) if and only if $v \in A_i$ and v has color j in G . Below, we treat G' as a relational structure equipped with the edge relation of G' , and $k \cdot c$ unary predicates marking the colors of G' . In particular, for $i = 1, \dots, s$ we can write a quantifier-free formula $A_i(x)$ such that $A_i(v)$ holds in G' if and only if $v \in A_i$.

We now write a formula $\psi(x, y)$ such that for all $a, b \in V(G)$ we have

$$G' \models \psi(a, b) \iff G \models \varphi(a, b). \quad (20)$$

The formula $\psi(x, y)$ is obtained from $\varphi(x, y)$ by replacing each atom $E(z, t)$ by the quantifier-free formula

$$\varepsilon(z, t) := E(z, t) \triangle \alpha(z, t) := (E(z, t) \wedge \neg \alpha(z, t)) \vee (\alpha(z, t) \wedge \neg (E(z, t))),$$

where $\alpha(z, t)$ is the disjunction of formulas $A_i(z) \wedge A_j(t)$, for all pairs $i, j \in [k]$ such that the parts A_i and A_j are flipped in the \mathcal{P} -flip G' of G . In particular, $\psi(x, y)$ is a formula of quantifier rank q over the signature of graphs colored with $k \cdot c$ colors.

Hence, by Fact H.1, there is a labelling $\text{ltp}: V(G) \rightarrow T_\psi$, for some set of local types T_ψ with $|T_\psi| \leq T_q(k \cdot c) = \ell$, and a binary relation $\Phi \subseteq T_\psi \times T_\psi$, such that

$$G' \models \psi(a, b) \iff (\text{ltp}(a), \text{ltp}(b)) \in \Phi$$

for all pairs $a, b \in V(G)$ with distance larger than $s = 2^q$ in G' . With (20) this implies that

$$G \models \varphi(a, b) \iff (\text{ltp}(a), \text{ltp}(b)) \in \Phi, \quad (21)$$

for all pairs $a, b \in V(G)$ with distance larger than s in G' .

Let \mathcal{Q} be the partition of $V(G)$ defined by ltp , with $\mathcal{Q} = \{\text{ltp}^{-1}(p) \mid p \in T_\psi\} - \{\emptyset\}$. In particular, $|\mathcal{Q}| \leq \ell$. Construct the \mathcal{Q} -flip $\varphi(G)'$ of $\varphi(G)$ by flipping two parts $\text{ltp}^{-1}(p), \text{ltp}^{-1}(q)$ of \mathcal{Q} whenever $(p, q) \in \Phi$ or $(q, p) \in \Phi$. In particular, if a and b are adjacent in $\varphi(G)'$, then it must be the case that $\text{dist}_{G'}(a, b) \leq s$ by (21). \square

Proof of Theorem 8.2. Fix a formula $\varphi(x, y)$ of quantifier rank q , and a c -colored graph H . Let $k = \text{fw}_{sr}(G_0)$, where H_0 is the uncolored graph underlying H . Set $s := 2^q$ and $\ell := T_q(k \cdot c)$. Denote $G := \varphi(H)$. Then Lemma H.3 says that the assumptions of Lemma 8.11 hold. Therefore, $\text{fw}_r(\varphi(H)) \leq \ell = T_q(k \cdot c)$, as required. \square

We now consider the case of CMSO-transductions.

Theorem (8.8). *Let \mathcal{C} be a class of bounded ∞ -flip-width and let $\varphi(x, y)$ be a formula of CMSO. Then $\varphi(\mathcal{C})$ has bounded ∞ -flip-width.*

The proof of Theorem 8.8 is the same as the proof of Theorem 8.2, with the difference that the use of Lemma H.3 is replaced by Lemma H.4 below.

Lemma H.4. *Fix $k \geq 1$, a CMSO formula $\varphi(x, y)$ of quantifier rank q , and let $\ell = T'_q(k)$, where $T'_q(k)$ is the function from Fact H.2. For every k -flip G' of a graph G there is an ℓ -flip $\varphi(G)'$ of $\varphi(G)$ such that*

$$\text{dist}_{G'}(u, v) < \infty \quad \text{for } uv \in E(\varphi(G)'). \quad (22)$$

In turn, the proof of Lemma H.4 is the same as the proof of Lemma H.3, with the difference that the use of Fact H.1 is replaced by the use of Fact H.2.

Appendix I Structurally nowhere dense classes have almost bounded flip-width

In this Appendix, we prove Theorem 10.13 and Lemma 10.14. Theorem 10.13 is proved in Section I.1. Lemma 10.14 is proved in Section I.2. Theorem 10.12 is an immediate consequence of those two results, as argued in Section 10.

I.1 Proof of Theorem 10.13

In Section I.1 we prove Theorem 10.13, restated below.

Theorem (10.13). *Let \mathcal{C} be a structurally nowhere dense graph class. There is a signature Σ consisting of unary and binary relation symbols and a unary function symbol, a class \mathcal{B} of Σ -structures such that the class of Gaifman graphs of the structures in \mathcal{B} is almost nowhere dense, and a symmetric quantifier-free formula $\varphi(x, y)$, such that every $G \in \mathcal{C}$ is an induced subgraph of $\varphi(B)$ for some $B \in \mathcal{B}$ with $|B| = O(|G|)$. Moreover, $\text{VCdim}(B) < \infty$.*

The proof of Theorem I.1 is based on the proof of [DGK⁺22a, Theorem 3] (see also [DGK⁺22b]), stated below.

A quasi-bush B is a rooted tree T equipped with:

- a set D of directed edges from the leaves of T to inner nodes of T , called *pointers*; every leaf has a pointer to the root of T ,
- a labelling function $\lambda: \text{Leaves}(T) \rightarrow \Lambda$, where Λ is a finite set of labels,
- a labelling function $\lambda^D: D \rightarrow 2^\Lambda$.

A quasi-bush B defines a directed graph $G(B)$ whose vertices are the leaves of T and directed edges (u, v) such that u, v are distinct leaves of T , and the closest ancestor w of u such that $(v, w) \in D$ satisfies $\lambda(u) \in \lambda^D((v, w))$. In particular, for $G(B)$ to be equal to an undirected graph, we require that the directed edge relation of $G(B)$ is symmetric.

Say that a class \mathcal{B} of quasi-bushes is almost nowhere dense if the class of underlying graphs (where we keep the edges of the tree T and turn the pointers in D into undirected edges) form an almost nowhere dense graph class.

Theorem I.1. [DGK⁺22a, Theorem 3] *Let \mathcal{D} be a structurally nowhere dense graph class. Then there are $d, \ell \in \mathbb{N}$, and an almost nowhere dense class \mathcal{B} of quasi-bushes, each of depth at most d and using at most ℓ labels, such that for every $G \in \mathcal{D}$ there is some quasi-bush $B \in \mathcal{B}$ with $G(B) = G$.*

We now prove Theorem 10.13.

Proof of Theorem 10.13. Fix d, ℓ as in Theorem I.1. Every quasi-bush $B \in \mathcal{B}$ may be viewed as a Σ -structure, over a fixed signature Σ consisting of:

- a unary function, interpreted in B as the parent function of the tree, and mapping the root to itself,
- ℓ unary relation symbols, interpreted in B as the labels of the leaves of T according to the function $\lambda: \text{Leaves}(T) \rightarrow \Lambda$, where $|\Lambda| \leq \ell$,
- 2^ℓ binary relation symbols D_M , for $M \subseteq \Lambda$, where $D_M(u, v)$ holds for a leaf u and inner node v and if and only if $\lambda^D((u, v)) = M$.

It is straightforward to construct a quantifier-free formula $\gamma_0(x, y)$ such that for every $B \in \mathcal{B}$ and leaves u, v of B we have $B \models \gamma_0(u, v)$ if and only if the lowest ancestor w of u such that $(v, w) \in D$ satisfies $\lambda(u) \in \lambda^D((v, w))$. Let $\gamma(x, y)$ be the symmetric formula $\gamma_0(x, y) \vee \gamma_0(y, x)$. Then for every quasi-bush $B \in \mathcal{B}$ such that $G(B)$ is an undirected graph G , we have that G is the subgraph of $\gamma(B)$ induced by the leaves of B .

By Theorem I.1, for every $G \in \mathcal{D}$ there is a quasi-bush $B \in \mathcal{B}$ such that $G(B) = G$, and hence, G is the subgraph of $\gamma(B)$ induced by the leaves of B . Since there is a tree T with leaves $V(G)$, depth at most d , and $V(T) = V(B)$, it follows that $|B| \leq d \cdot |G| = O(|G|)$. The class \mathcal{B} is almost nowhere dense. This proves the statement of Theorem 10.13, apart from the ‘moreover’ part.

It remains to argue that the class \mathcal{B} , viewed as Σ -structures as described above, has bounded VC-dimension. Therefore, we need to show that each of the binary relations D_M , for $M \subseteq \Lambda$, has VC-dimension bounded by a constant independent of $B \in \mathcal{B}$ and of $M \subseteq \Lambda$. To do this, we inspect how the set of pointers D and labeling function λ^D are defined in the construction in [DGK⁺22b]. The key property of the construction, from which the bounds on the VC-dimension follows, is encapsulated in the claim below.

Let \mathcal{B} be as constructed in the proof of [DGK⁺22b, Theorem 3].

Claim I.1. *There is a nowhere dense graph class \mathcal{C} and numbers $s, q \geq 1$ with the following property. For every quasi-bush $B \in \mathcal{B}$ there is a graph $G \in \mathcal{C}$ with $\text{Leaves}(B) = V(G)$, a function $\beta: V(B) \rightarrow V(G)^s$, and for each $M \in 2^\Lambda$ a formula $\psi_M(x_0, x_1, \dots, x_s)$ of quantifier rank at most q , such that*

$$(u, w) \in D_M \iff G \models \psi_M(u, \beta(w)) \quad \text{for all } u, w \in V(B). \quad (23)$$

We first show how Claim I.1 implies the bound on the VC-dimension of \mathcal{B} .

Let Δ be the set of all formulas $\psi(x_0, \dots, x_s)$ of quantifier rank q , where q and s are as in Claim I.1. Since Δ is finite (up to equivalence), it follows from Fact 3.9 that there is a bound k depending only on q, s and the class \mathcal{C} such that for all $G \in \mathcal{C}$ and every formula $\psi \in \Delta$, the binary relation $R_G^\psi \subseteq V(G) \times V(G)^s$ (as considered in Fact 3.9) has VC-dimension at most k . In particular, $R_G^{\psi_M}$ has VC-dimension at most k , for all $G \in \mathcal{C}$ and ψ_M as in Claim I.1. Then (23) is restated as follows:

$$(u, w) \in D_M \iff (u, \beta(w)) \in R_G^{\psi_M} \quad \text{for all } u, w \in V(B).$$

It follows that D_M has VC-dimension bounded by the VC-dimension of the binary relation $R_G^{\psi_M}$, so at most $k = O_{q,s}(1)$. Since q and s are independent of $B \in \mathcal{B}$, the ‘moreover’ part of Theorem 10.13 follows.

It therefore remains to analyse the construction from [DGK⁺22b], and argue that Claim I.1 holds.

First, Theorem I.1 is proved (in [DGK⁺22b]) in the special case when $\mathcal{D} = \varphi(\mathcal{C})$ for some nowhere dense class \mathcal{C} of colored graphs and first-order formula $\varphi(x, y)$ involving color predicates. This is done in [DGK⁺22b, Theorem 28]. In general, in Theorem I.1, \mathcal{D} is contained in the hereditary closure of $\varphi(\mathcal{C})$, rather than in $\varphi(\mathcal{C})$ itself.

The general case is reduced to the special case at the end of Section 5 in [DGK⁺22b], as follows. It is shown that there is a coloring $\hat{\mathcal{C}}$ of \mathcal{C} (that is, a class of k -colored graphs for some

$k \geq 1$, where each underlying graph belongs to \mathcal{C}), a formula $\widehat{\varphi}(x, y)$, and constants $c, d > 0$, such that for every $G \in \mathcal{D}$ there is some colored graph $\widehat{H} \in \widehat{\mathcal{C}}$ such that $\widehat{\varphi}(\widehat{H})$ contains G as a subgraph induced by some set $A \subseteq V(\widehat{H})$, and moreover $|\widehat{H}| \leq c|G|^d$. A quasi-bush B is constructed for the graph $\widehat{\varphi}(\widehat{H})$ using the special case of Theorem I.1 applied to the class $\widehat{\varphi}(\widehat{\mathcal{C}})$. To get a quasi-bush B' for $G = \widehat{\varphi}(\widehat{H})[A]$, the nodes of B that have no descendants in $A \subseteq \text{Leaves}(B)$ are removed. It is then argued that the class of quasi-bushes B' obtained in this way, for each $G \in \mathcal{D}$, satisfies the conditions of Theorem I.1. For us here, it only matters that B' is a quasi-bush obtained from restricting the quasi-bush B as obtained in the special case of Theorem I.1. It is immediate that the VC-dimension of B' is bounded by the VC-dimension of B , so it is enough to argue that the quasi-bushes constructed in the special case of Theorem I.1 have bounded VC-dimension.

We may therefore focus on analyzing the proof of the special case of Theorem I.1 (stated as Theorem 28 in [DGK⁺22b]) where it is assumed that $\mathcal{D} = \varphi(\mathcal{C})$ for some nowhere dense class \mathcal{C} of colored graphs and first-order formula $\varphi(x, y)$. We argue that Claim I.1 holds in this case.

Types Before analysing the construction, we recall the following notion. Fix the signature Σ consisting of the edge relation symbol E and unary predicates corresponding to the colors of the graphs in \mathcal{C} . Fix $q, m \geq 0$. For a colored graph H and m -tuple $\bar{v} \in V(H)^m$, the *quantifier rank q type* of \bar{v} , denoted $\text{tp}_H^q(\bar{v})$, is the of all first-order formulas $\varphi(x_1, \dots, x_m)$ over the signature Σ of quantifier rank q such that $H \models \varphi(\bar{v})$. Let Γ_q^m denote the set of all possible quantifier rank q types of m tuples:

$$\Gamma_q^m := \{\text{tp}_H^q(\bar{v}) \mid H \text{ - colored graph, } \bar{v} \in V(H)^m\}.$$

The following fact is well known, and follows from the observation that up to equivalence, there are finitely many formulas $\varphi(x_1, \dots, x_m)$ of quantifier rank q .

Fact I.2. *The set Γ_q^m is finite. For every type $\tau \in \Gamma_q^m$ there is a first-order formula, denoted $\tau(x_1, \dots, x_m)$, such that for every colored graph H and tuple $\bar{v} \in V(H)^m$, we have:*

$$\text{tp}_H^q(\bar{v}) = \tau \iff H \models \tau(\bar{v}).$$

Analysis of the proof We now go through the proof of Theorem 28 in [DGK⁺22b], that is, Theorem I.1 in the case where $\mathcal{D} = \varphi(\mathcal{C})$ for some class \mathcal{C} of colored graphs and first-order formula $\varphi(x, y)$. We argue that Claim I.1 holds.

For each graph $G \in \mathcal{C}$, a quasi-bush B with $G(B) = \varphi(G)$ is constructed as follows. First, an r -separator quasi-bush T for G is constructed, for some number r depending on the quantifier rank of $\varphi(x, y)$. An r -separator quasi-bush is a tree T with leaves $V(G)$, equipped with:

- a set $D \subseteq V(T) \times V(T)$ of *pointers*, where each pointer (u, w) points from some leaf u of T to some inner node w of T (and each leaf points to the root), and
- a function α mapping each inner node v of T to a set $\alpha(v) \subseteq V(G)$ with the following property. For every two leaves u, v of T , and node w which is the lowest ancestor of v such that $(v, w) \in D$, the set $\alpha(w)$ is an r -separator between u and v in G , that is, every path from u to v of length at most r in G passes through $\alpha(w)$.

Crucially (see Lemma 36 and second item in Lemma 33 in [DGK⁺22b]), the size of the set $\alpha(w)$ is bounded by a constant m (depending only on \mathcal{C} and φ). Below, the set $\alpha(w)$ is treated as a tuple of length at most m , by enumerating its elements according to any fixed order on $V(G)$.

Next, an r -separator quasi-bush T is converted into a quasi-bush B , by assigning a label $\lambda(v) \in \Lambda$ (where Λ is some finite set) to each leaf v of T , and a label $\lambda^D((u, w)) \in 2^\Lambda$ to each pointer $(u, w) \in D$. For $M \in 2^\Lambda$, let D_M denote the set of pointers $(u, w) \in D$ with $\lambda^D((u, w)) = M$.

The following statement is immediate from the construction (Proof of Theorem 28 in [DGK⁺22b]): There is a number q (depending only on φ and \mathcal{C}), such that the label $\lambda^D((u, w))$ of a pointer $(u, w) \in D$ depends only on $\text{tp}_G^q(u\alpha(w))$, where $\alpha(w)$ is viewed as a tuple. Hence, for every label $M \in 2^\Lambda$, whether or not a pointer $(u, w) \in D$ belongs to D_M , depends only on $\text{tp}_G^q(u\alpha(w))$. That means that for each $M \in 2^\Lambda$ there is a set $\Phi_M \subseteq \Gamma_q^{m+1}$ such that for all $(u, w) \in D$ we have

$$(u, w) \in D_M \iff \text{tp}_G^q(u\alpha(w)) \in \Phi_M.$$

Let $\psi_M^0(x, \bar{y})$ denote the disjunction of all formulas $\tau(x, \bar{y})$ representing the types $\tau \in \Phi_M$ (as described in Fact I.2). We conclude that the following claim holds.

Claim I.2. *There is a number q depending only on \mathcal{C} and φ such that the following holds. For every $M \in 2^\Lambda$ there is a formula $\psi_M^0(x, \bar{y})$ of quantifier rank q such that for every pointer $(u, w) \in D$ we have*

$$(u, w) \in D_M \iff G \models \psi_M^0(u, \alpha(w)).$$

We also need to argue that the set D can be defined by a first-order formula, as made precise below.

Claim I.3. *There is a number t depending only on φ and \mathcal{C} , a function $\delta: V(T) \rightarrow V(G)^t$ and a first-order formula $\psi_D(x_0, \dots, x_t)$, such that the following holds for all $u, w \in V(B)$:*

$$(u, w) \in D \iff G \models \psi_D(u, \delta(w)). \quad (24)$$

First, we show how Claim I.2 and Claim I.3 imply Claim I.1. From the two claims it follows that for each $M \in 2^\Lambda$ and pair $(u, w) \in V(T) \times V(T)$ we have that

$$(u, w) \in D_M \iff G \models \psi_D(u, \delta(w)) \wedge \psi_M^0(u, \alpha(w)).$$

For a node w of T , let $\beta(w)$ be the concatenation of the tuples $\alpha(w)$ and $\delta(w)$. For $M \in 2^\Lambda$ define

$$\psi_M(x, \bar{y}, \bar{z}) := \psi_D(x, \bar{z}) \wedge \psi_M^0(x, \bar{y}).$$

Then for each pair $(u, w) \in V(T) \times V(T)$ we have that

$$(u, w) \in D_M \iff G \models \psi_M(u, \beta(w)).$$

This proves Claim I.1, assuming Claim I.2 and Claim I.3.

Claim I.3 is argued below, by analysing the construction the r -separator quasi-bush for G . Definition 32 of [DGK⁺22b] associates to each vertex v of G and number $k \geq 0$ two sets of vertices of G , denoted $M_r^k[v]$ and $S_r^k[v]$. Those sets are treated as tuples according to some fixed enumeration of $V(G)$. It is shown (see Lemma 36 of [DGK⁺22b]) that those sets have size bounded by some constant d . According to Definition 37 of [DGK⁺22b], the nodes w of the r -separator quasi-bush T are sets of the form $M_r^k[v]$, for all $v \in V(G)$ and all $k \leq d$. And the pointers D of T are defined so that $(u, M_r^k[v]) \in D$, for $v \in V(G)$ and $k \geq 1$, if and only if $S_r^{k-1}[v]$ does not r -separate u and v in G .

For a node $w = M_r^k[v]$ of T , define $\delta(w)$ as the concatenation of the following tuples:

- v (where v is arbitrarily chosen so that $M_r^k[v] = w$),
- $S_r^{k-1}[v]$, padded to a tuple of length d .

Let $\psi_D(x, y, \bar{z})$ with $|\bar{z}| = d$ be a first-order formula expressing “ \bar{z} does not r -separate x and y .” Then, by definition of D , we have that for every pair $u, w \in V(T)$ (24) holds. This proves Claim I.3. \square

I.2 Quantifier-free interpretations with function symbols

In this section, we prove Lemma 10.14, which is repeated below.

Lemma (10.14). *Let Σ be a signature consisting of unary and binary relation symbols, and unary function symbols. Fix $k, r \geq 0$, and a symmetric quantifier-free Σ -formula $\varphi(x, y)$. There are numbers $p = O_\varphi(k)$ and $r' = O_\varphi(r)$ such that the following holds. Let B be a Σ -structure of VC-dimension at most k and G_B be its Gaifman graph. Then*

$$\text{fw}_r(\varphi(B)) \leq O(\text{copwidth}_{r'}(G_B))^p.$$

In Section I.2, fix a signature Σ consisting of unary relation symbols, binary relation symbols, and unary function symbols. All considered formulas are over this signature, and are quantifier-free.

The *depth* of a term $t(x)$ is the nesting of function symbols occurring in t , where the term x has depth 0, $f(x)$ has depth 1, etc. The depth of a quantifier-free formula is the maximal depth of a term occurring in it. Note that there are $O_d(1)$ terms and atomic formulas of depth d .

We first prove the following lemma. For a set $S \subseteq V(G)$ and two vertices $u, v \in V(B)$, let $\text{dist}_S(u, v)$ denote the distance between u and v in the subgraph of the Gaifman graph of B obtained by isolating S , that is, removing the edges incident to vertices in S .

Lemma I.3. *Fix $k, d \geq 0$ and a quantifier-free Σ -formula $\varphi(x, y)$ of depth at most d . Then there is a number $m = O_d(k)$ such that for every Σ -structure B of VC-dimension at most k and set $S \subseteq V(B)$ there is a set T of labels with $|T| \leq O_\varphi(|S|^m)$, a binary relation $\Phi \subseteq T \times T$, and a function $\lambda: V(B) \rightarrow T$, such that for all vertices $u, v \in V(G)$ with $\text{dist}_S(u, v) > 2d + 1$ we have*

$$B \models \varphi(u, v) \iff (\lambda(u), \lambda(v)) \in \Phi.$$

Proof. Fix $d \geq 0$. For a vertex $v \in V(B)$ and set of vertices $S \subseteq V(G)$, define the *atomic S -type of depth d of v* , denoted $\text{atp}^d(v/S)$, as the set of all pairs consisting of an atomic formula $\alpha(x, y)$ of depth at most d and an element $s \in S$, such that $\alpha(v, s)$ holds in B . For a set $S \subseteq V(G)$, define

$$T^d(S) := \{\text{atp}^d(v/S) \mid v \in V(G)\}.$$

Claim I.4. *Fix $d, k \geq 0$. There is a number $m = O_d(k)$ such that for every structure B with $k = \text{VCdim}(B)$ and set $S \subseteq V(B)$ we have*

$$|T^d(S)| = O(|S|^m).$$

Proof. We first prove the claim in the case $d = 0$. We have that $\text{atp}^0(v/S)$ is determined by the following data:

- the set of atomic formulas $\alpha(x)$ of depth 0 such that $\alpha(v)$ holds in G ,
- the sets $R(v; S)$ and $R(S; v)$, for each binary relation symbol $R \in \Sigma$,
- the set of elements $s \in S$ such that $s = v$; this set is either empty, or a singleton.

There are $O(1)$ formulas of depth 0, and for each binary relation symbol $R \in \Sigma$, we have

$$|\{R(v; S) \mid v \in V(B)\}| \leq O(|S|^k)$$

and

$$|\{R(S; v) \mid v \in V(B)\}| \leq O(|S|^k)$$

by the Sauer-Shelah-Perles lemma and the assumption that $\text{VCdim}(B) \leq k$. We get that $|T^d(S)| \leq O(|S|^{2k|\Sigma|+1}) = |S|^{O(k)}$, since we consider Σ as fixed.

We now consider the case $d > 0$. Let $S^{(d)}$ denote the set of vertices that can be obtained in B by applying a term $t(x)$ of depth at most d to a vertex $s \in S$:

$$S^{(d)} := \{t(s) \mid s \in S, t(x) \text{ is a term of depth } \leq d\}.$$

Then $|S^{(d)}| \leq O_d(|S|)$, as there are $O_d(1)$ terms of depth at most d .

For a vertex $v \in V(B)$, $\text{atp}^d(v/S)$ is uniquely determined by the tuple

$$(\text{atp}^0(t(v)/S^{(d)}))_{t(x)}$$

where $t(x)$ ranges over all terms of depth at most d . As there are $O_d(1)$ such terms $t(v)$, and $|S^{(d)}| \leq O_d(|S|)$ the conclusion follows from the case $d = 0$ considered earlier. \square

Claim I.5. Fix $d \geq 0$, and let $\varphi(x, y)$ be a quantifier-free formula of depth at most d . Fix a Σ -structure B and a set $S \subseteq V(B)$. For all vertices $u, v \in V(B)$ with $\text{dist}_S(u, v) > 2d + 1$, whether or not $\varphi(u, v)$ holds in B , depends only on $\text{atp}^d(u/S)$ and $\text{atp}^d(v/S)$. More precisely, there is a binary relation $\Phi \subseteq T^d(S) \times T^d(S)$ such that for all vertices $u, v \in V(B)$ with $\text{dist}_S(u, v) > 2d + 1$ we have

$$B \models \varphi(u, v) \iff (\text{atp}^d(u/S), \text{atp}^d(v/S)) \in \Phi.$$

Proof. It is enough to consider the case when $\varphi(x, y)$ is an atomic formula, since if the statement holds for two formulas $\varphi(x, y)$ and $\psi(x, y)$ of nesting depth at most d , then it also holds for $\neg\varphi(x, y)$ and for $\varphi(x, y) \vee \psi(x, y)$.

Thus assume that $\varphi(x, y)$ is of the form

$$\varphi(x, y) \equiv R(t(x), t'(y)),$$

where R is either a binary relation symbol occurring in Σ , or is the equality relation, and $t(x)$ and $t'(y)$ are two terms of depth at most d .

Fix two vertices $u, v \in V(B)$ with $\text{dist}_B(u, v) > 2d + 1$. We show how to determine whether $\varphi(u, v)$ holds in B , from the information contained in $\text{atp}^d(u/S)$ and $\text{atp}^d(v/S)$.

Suppose first that there is a subterm $t_0(x)$ of $t(x)$ such that $t_0(u) \in S$. Note that whether this is the case can be determined from $\text{tp}^d(u/S)$.

Let $s = t_0(u) \in S$, and let $t_1(z)$ be a term such that $t_1(t_0(x)) = t(x)$. In particular, $t(u) = t_1(s)$. Then

$$B \models \varphi(u, v) \iff B \models R(t_1(s), t'(v)).$$

Since $R(t_1(s), t'(v))$ is an atomic formula of depth at most d , whether or not $R(t_1(s), t'(v))$ holds in B is determined by $\text{atp}^d(v/S)$. Hence, in this case, whether or not $B \models \varphi(u, v)$, is determined by $\text{atp}^d(v/S)$.

Similarly, if there is a subterm $t'_0(y)$ of $t'(y)$ such that $t'_0(v) \in S$, then whether or not $B \models \varphi(u, v)$, is determined by $\text{atp}^d(u/S)$. Moreover, whether this case holds can be determined from $\text{tp}^d(u/S)$.

We show that if neither of the two cases holds, then $B \models \neg\varphi(u, v)$. First, note that $\text{dist}_S(u, t(u)) \leq d$, as witnessed by the path formed by $u, f_1(u), f_2(f_1(u)), \dots, t(u)$, where $t(x) = f_d(\dots(f_1(x))\dots)$. Similarly, $\text{dist}_S(v, t'(v)) \leq d$. Since $\text{dist}_S(u, v) > 2d + 1$, by the triangle inequality we have $\text{dist}_S(t(u), t'(v)) > 1$. As $t(u), t'(v) \notin S$, it follows that $t(u)$ and $t'(v)$ are non-adjacent in the Gaifman graph of G . We conclude that $B \models \neg R(t(u), t'(v))$, equivalently, $B \models \neg\varphi(u, v)$.

The claim follows. \square

Lemma I.3 follows immediately from Claim I.4 and Claim I.5, by taking $T = T^d(S)$ and $\lambda(v) = \text{atp}^d(v/S)$. \square

From Lemma I.3 we get the following.

Corollary I.4. Fix $d, k \geq 0$ and a symmetric quantifier-free Σ -formula $\varphi(x, y)$ of depth at most d . There is a number $m = O_d(k)$ with the following property. For every Σ -structure B of VC-dimension at most k and set $S \subseteq V(B)$ there is a $O_\varphi(|S|^m)$ -flip $\varphi(B)'$ of $\varphi(B)$ such that for every vertex $v \in V(B)$ we have

$$\text{dist}_S(u, v) \leq 2d + 1 \quad \text{for } u, v \in E(\varphi(B)'), \quad (25)$$

where $\text{dist}_S(\cdot, \cdot)$ denotes the distance in the Gaifman graph of B with the vertices in S isolated.

Proof. Let $\lambda: V(B) \rightarrow T$ be the labelling from Lemma I.3. Let \mathcal{P} be the partition of $V(B)$ into parts $\lambda^{-1}(a)$, for $a \in T$. Then $|\mathcal{P}| \leq |T| = O_\varphi(|S|^m)$ for some $m = O_d(k)$. Define $\varphi(B)'$ as the \mathcal{P} -flip of $\varphi(B)$, obtained by flipping two parts P, Q of \mathcal{P} if and only if there are $u \in P, v \in Q$ such that $\text{dist}_S(u, v) > 2d + 1$ and $B \models \varphi(u, v)$. By construction, if u and v are adjacent in $\varphi(B)'$, then $\text{dist}_S(u, v) \leq 2d + 1$. The conclusion follows. \square

Lemma 10.14 now follows along the same lines as Theorem 8.2.

Proof of Lemma 10.14. Let m be as in Corollary I.4. We fix a winning strategy of the cops in the Cops and Robber game of radius $r' := r(2d + 1)$ and width $\ell := \text{copwidth}_{r'}(G)$, and transfer this strategy to the flipper game of radius r and width $O(\ell^m)$ on $\varphi(B)$, so that whenever the cops announce a new set $S \subseteq V(B)$ of vertices in the Cops and Robber game, then in the flipper game the cops announce the $O(|S|^m)$ -flip $\varphi(B)'$ of $\varphi(B)$, as obtained by Corollary I.4. It follows from (25) and Lemma 8.11 that this yields a winning strategy in the flipper game. Hence, $\text{fw}_r(\varphi(B)) = O(\ell^m)$. \square

References

- [AA14] Hans Adler and Isolde Adler. Interpreting nowhere dense graph classes as a classical notion of model theory. *Eur. J. Comb.*, 36:322–330, 2014. doi:[10.1016/j.ejc.2013.06.048](https://doi.org/10.1016/j.ejc.2013.06.048).
- [AAL21] Bogdan Alecu, Aistis Atminas, and Vadim Lozin. Graph functionality. *Journal of Combinatorial Theory, Series B*, 147:139–158, 2021. URL: <https://www.sciencedirect.com/science/article/pii/S0095895620301052>, doi:<https://doi.org/10.1016/j.jctb.2020.11.002>.
- [ACLZ15] A. Atminas, A. Collins, V. Lozin, and V. Zamaraev. Implicit representations and factorial properties of graphs. *Discrete Mathematics*, 338(2):164–179, 2015. URL: <https://www.sciencedirect.com/science/article/pii/S0012365X14003690>, doi:<https://doi.org/10.1016/j.disc.2014.09.008>.
- [Adlo8] Hans Adler. An introduction to theories without the independence property. available online, 2008.
- [AF84] M. Aigner and M. Fromme. A game of cops and robbers. *Discrete Applied Mathematics*, 8(1):1–12, 1984. URL: <https://www.sciencedirect.com/science/article/pii/0166218X84900738>, doi:[https://doi.org/10.1016/0166-218X\(84\)90073-8](https://doi.org/10.1016/0166-218X(84)90073-8).
- [AM15] Noga Alon and Abbas Mehrabian. Chasing a fast robber on planar graphs and random graphs. *Journal of Graph Theory*, 78(2):81–96, 2015. URL: <https://onlinelibrary.wiley.com/doi/abs/10.1002/jgt.21791>, arXiv:<https://onlinelibrary.wiley.com/doi/pdf/10.1002/jgt.21791>, doi:<https://doi.org/10.1002/jgt.21791>.
- [BCK⁺22] Édouard Bonnet, Dibyayan Chakraborty, Eun Jung Kim, Noleen Köhler, Raul Lopes, and Stéphan Thomassé. Twin-width VIII: delineation and win-wins. In Holger Dell and Jesper Nederlof, editors, *17th International Symposium on Parameterized and Exact Computation, IPEC 2022, September 7-9, 2022, Potsdam, Germany*, volume 249 of *LIPICs*, pages 9:1–9:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. doi:[10.4230/LIPICs.IPEC.2022.9](https://doi.org/10.4230/LIPICs.IPEC.2022.9).
- [BDG⁺22] Édouard Bonnet, Jan Dreier, Jakub Gajarský, Stephan Kreutzer, Nikolas Mählmann, Pierre Simon, and Szymon Toruńczyk. Model Checking on Interpretations of Classes of Bounded Local Cliquewidth. In *37th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2022*. ACM, 2022. URL: <https://hal.archives-ouvertes.fr/hal-03751012>.
- [BFLP23] Édouard Bonnet, Florent Foucaud, Tuomo Lehtilä, and Aline Parreau. Neighbourhood complexity of graphs of bounded twin-width, 2023. URL: <https://arxiv.org/abs/2301.04217>, doi:[10.48550/ARXIV.2301.04217](https://doi.org/10.48550/ARXIV.2301.04217).
- [BGdM⁺21] Édouard Bonnet, Ugo Giocanti, Patrice Ossona de Mendez, Pierre Simon, Stéphan Thomassé, and Szymon Toruńczyk. Twin-width iv: ordered graphs and matrices, 2021. URL: <https://arxiv.org/abs/2102.03117>, doi:[10.48550/ARXIV.2102.03117](https://doi.org/10.48550/ARXIV.2102.03117).
- [BGdMT22] Édouard Bonnet, Ugo Giocanti, Patrice Ossona de Mendez, and Stéphan Thomassé. Twin-width V: linear minors, modular counting, and matrix multiplication. *arXiv*, abs/2209.12023, 2022. arXiv:[2209.12023](https://arxiv.org/abs/2209.12023), doi:[10.48550/arXiv.2209.12023](https://doi.org/10.48550/arXiv.2209.12023).
- [BGK⁺21a] Édouard Bonnet, Colin Geniet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width II: small classes. In Dániel Marx, editor, *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms, SODA 2021, Virtual Conference, January 10 - 13, 2021*, pages 1977–1996. SIAM, 2021. doi:[10.1137/1.9781611976465.118](https://doi.org/10.1137/1.9781611976465.118).

- [BGK⁺21b] Édouard Bonnet, Colin Geniet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width III: Max Independent Set, Min Dominating Set, and Coloring. In Nikhil Bansal, Emanuela Merelli, and James Worrell, editors, *48th International Colloquium on Automata, Languages, and Programming (ICALP 2021)*, volume 198 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 35:1–35:20, Dagstuhl, Germany, 2021. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. URL: <https://drops.dagstuhl.de/opus/volltexte/2021/14104>, doi:10.4230/LIPIcs.ICALP.2021.35.
- [BGdM⁺22] Édouard Bonnet, Ugo Giocanti, Patrice Ossona de Mendez, Pierre Simon, Stéphan Thomassé, and Szymon Toruńczyk. Twin-width iv: Ordered graphs and matrices. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2022*, pages 924–937, New York, NY, USA, 2022. Association for Computing Machinery. doi:10.1145/3519935.3520037.
- [BGT22] Édouard Bonnet, Colin Geniet, Romain Tessera, and Stéphan Thomassé. Twin-width VII: groups. *arXiv*, abs/2204.12330, 2022. [arXiv:2204.12330](https://arxiv.org/abs/2204.12330), doi:10.48550/arXiv.2204.12330.
- [BKR⁺22] Édouard Bonnet, Eun Jung Kim, Amadeus Reinald, Stéphan Thomassé, and Rémi Watrigant. Twin-width and polynomial kernels. *Algorithmica*, 84(11):3300–3337, 2022. doi:10.1007/s00453-022-00965-5.
- [BKRT22] Édouard Bonnet, Eun Jung Kim, Amadeus Reinald, and Stéphan Thomassé. Twin-width VI: the lens of contraction sequences. In Joseph (Seffi) Naor and Niv Buchbinder, editors, *Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022, Virtual Conference / Alexandria, VA, USA, January 9 - 12, 2022*, pages 1036–1056. SIAM, 2022. doi:10.1137/1.9781611977073.45.
- [BKTW20] Édouard Bonnet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width I: tractable FO model checking. In Sandy Irani, editor, *61st IEEE Annual Symposium on Foundations of Computer Science, FOCS 2020, Durham, NC, USA, November 16-19, 2020*, pages 601–612. IEEE, 2020. doi:10.1109/FOCS46700.2020.00062.
- [BL21] Samuel Braunfeld and Michael C. Laskowski. Characterizations of monadic nip. *Transactions of the American Mathematical Society, Series B*, 2021.
- [BL22] Samuel Braunfeld and Michael C Laskowski. Existential characterizations of monadic NIP. *arXiv preprint arXiv:2209.05120*, 2022.
- [BNdMS22a] Samuel Braunfeld, Jaroslav Nešetřil, Patrice Ossona de Mendez, and Sebastian Siebertz. Decomposition horizons: from graph sparsity to model-theoretic dividing lines, 2022. URL: <https://arxiv.org/abs/2209.11229>, doi:10.48550/ARXIV.2209.11229.
- [BNdMS22b] Samuel Braunfeld, Jaroslav Nesetril, Patrice Ossona de Mendez, and Sebastian Siebertz. On the first-order transduction quasiorder of hereditary classes of graphs. *arXiv*, abs/2208.14412, 2022. [arXiv:2208.14412](https://arxiv.org/abs/2208.14412), doi:10.48550/arXiv.2208.14412.
- [BS85] J.T. Baldwin and S. Shelah. Second-order quantifiers and the complexity of theories. *Notre Dame Journal of Formal Logic*, 26(3):229–303, 1985.
- [BT15] Nicolas Bousquet and Stéphan Thomassé. Vc-dimension and erdős-pósa property. *Discrete Mathematics*, 338(12):2302–2317, 2015. doi:<https://doi.org/10.1016/j.disc.2015.05.026>.
- [CER93] Bruno Courcelle, Joost Engelfriet, and Grzegorz Rozenberg. Handle-rewriting hypergraph grammars. *Journal of Computer and System Sciences*, 46(2):218–270, 1993. URL: <https://www.sciencedirect.com/science/article/pii/002200009390004G>, doi:[https://doi.org/10.1016/0022-0000\(93\)90004-G](https://doi.org/10.1016/0022-0000(93)90004-G).

- [CiO07] Bruno Courcelle and Sang-il Oum. Vertex-minors, monadic second-order logic, and a conjecture by seese. *Journal of Combinatorial Theory, Series B*, 97(1):91–126, 2007. URL: <https://www.sciencedirect.com/science/article/pii/S0095895606000463>, doi:<https://doi.org/10.1016/j.jctb.2006.04.003>.
- [DGK⁺22a] Jan Dreier, Jakub Gajarský, Sandra Kiefer, Michal Pilipczuk, and Szymon Torunczyk. Treelike decompositions for transductions of sparse graphs. In Christel Baier and Dana Fisman, editors, *LICS '22: 37th Annual ACM/IEEE Symposium on Logic in Computer Science, Haifa, Israel, August 2 - 5, 2022*, pages 31:1–31:14. ACM, 2022. doi:[10.1145/3531130.3533349](https://doi.org/10.1145/3531130.3533349).
- [DGK⁺22b] Jan Dreier, Jakub Gajarský, Sandra Kiefer, Michal Pilipczuk, and Szymon Torunczyk. Treelike decompositions for transductions of sparse graphs. *arXiv*, abs/2201.11082, 2022. URL: <https://arxiv.org/abs/2201.11082>, arXiv:2201.11082.
- [DGKS07] Anuj Dawar, Martin Grohe, Stephan Kreutzer, and Nicole Schweikardt. Model theory makes formulas large. In Lars Arge, Christian Cachin, Tomasz Jurdziński, and Andrzej Tarlecki, editors, *Automata, Languages and Programming*, pages 913–924, Berlin, Heidelberg, 2007. Springer Berlin Heidelberg.
- [DKo6] Guoli Ding and Andrei Kotlov. On minimal rank over finite fields. *ELA. The Electronic Journal of Linear Algebra [electronic only]*, 15:210–214, 2006. URL: <http://dml.mathdoc.fr/item/05279950>.
- [DKT13] Zdeněk Dvořák, Daniel Král, and Robin Thomas. Testing first-order properties for subclasses of sparse graphs. *J. ACM*, 60(5):36:1–36:24, 2013. doi:[10.1145/2499483](https://doi.org/10.1145/2499483).
- [DMS] Jan Dreier, Nikolas Mählmann, and Sebastian Siebertz. First-order model checking on structurally sparse graph classes. *STOC 2023* (accepted). URL: <https://arxiv.org/abs/2102.03117>.
- [DT17] Thanh N. Dang and Robin Thomas. Minors of two-connected graphs of large path-width, 2017.
- [Dvo12] Zdeněk Dvořák. A stronger structure theorem for excluded topological minors. *ArXiv*, abs/1209.0129, 2012.
- [Dvo13] Zdeněk Dvořák. Constant-factor approximation of the domination number in sparse graphs. *European Journal of Combinatorics*, 34(5):833–840, 2013. URL: <https://www.sciencedirect.com/science/article/pii/S0195669812002028>, doi:<https://doi.org/10.1016/j.ejc.2012.12.004>.
- [Dvo18] Zdeněk Dvořák. Induced subdivisions and bounded expansion. *Eur. J. Comb.*, 69(C):143–148, mar 2018. doi:[10.1016/j.ejc.2017.10.004](https://doi.org/10.1016/j.ejc.2017.10.004).
- [FGo4] Markus Frick and Martin Grohe. The complexity of first-order and monadic second-order logic revisited. *Annals of Pure and Applied Logic*, 130(1):3–31, 2004. Papers presented at the 2002 IEEE Symposium on Logic in Computer Science (LICS). URL: <https://www.sciencedirect.com/science/article/pii/S0168007204000612>, doi:<https://doi.org/10.1016/j.apal.2004.01.007>.
- [FGK⁺10] Fedor V. Fomin, Petr A. Golovach, Jan Kratochvíl, Nicolas Nisse, and Karol Suchan. Pursuing a fast robber on a graph. *Theoretical Computer Science*, 411(7):1167–1181, 2010. URL: <https://www.sciencedirect.com/science/article/pii/S0304397509008342>, doi:<https://doi.org/10.1016/j.tcs.2009.12.010>.
- [FKL12] Alan Frieze, Michael Krivelevich, and Po-Shen Loh. Variations on cops and robbers. *Journal of Graph Theory*, 69(4):383–402, 2012. URL: <https://onlinelibrary.wiley.com/doi/abs/10.1002/jgt.20591>, arXiv:<https://onlinelibrary.wiley.com/doi/pdf/10.1002/jgt.20591>, doi:<https://doi.org/10.1002/jgt.20591>.

- [FTo8] Fedor V. Fomin and Dimitrios M. Thilikos. An annotated bibliography on guaranteed graph searching. *Theoretical Computer Science*, 399(3):236–245, 2008. Graph Searching. URL: <https://www.sciencedirect.com/science/article/pii/S0304397508001606>, doi:<https://doi.org/10.1016/j.tcs.2008.02.040>.
- [GHN⁺12] Robert Ganian, Petr Hliněný, Jaroslav Nesetril, Jan Obdržálek, Patrice Ossona de Mendez, and Reshma Ramadurai. When trees grow low: Shrubs and fast MSO₁. In Branislav Rován, Vladimiro Sassone, and Peter Widmayer, editors, *Mathematical Foundations of Computer Science 2012 - 37th International Symposium, MFCS 2012, Bratislava, Slovakia, August 27–31, 2012. Proceedings*, volume 7464 of *Lecture Notes in Computer Science*, pages 419–430. Springer, 2012. doi:[10.1007/978-3-642-32589-2_38](https://doi.org/10.1007/978-3-642-32589-2_38).
- [GHO⁺20] Jakub Gajarský, Petr Hliněný, Jan Obdržálek, Daniel Lokshtanov, and M. S. Ramanujan. A new perspective on model checking of dense graph classes. *ACM Trans. Comput. Logic*, 21(4), jul 2020. doi:[10.1145/3383206](https://doi.org/10.1145/3383206).
- [GKN⁺18] Jakub Gajarský, Stephan Kreutzer, Jaroslav Nesetril, Patrice Ossona de Mendez, Michal Pilipczuk, Sebastian Siebertz, and Szymon Torunczyk. First-order interpretations of bounded expansion classes. In Ioannis Chatzigiannakis, Christos Kaklamanis, Dániel Marx, and Donald Sannella, editors, *45th International Colloquium on Automata, Languages, and Programming, ICALP 2018, July 9–13, 2018, Prague, Czech Republic*, volume 107 of *LIPIcs*, pages 126:1–126:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018. doi:[10.4230/LIPIcs.ICALP.2018.126](https://doi.org/10.4230/LIPIcs.ICALP.2018.126).
- [GKN⁺20] Jakub Gajarský, Stephan Kreutzer, Jaroslav Nesetril, Patrice Ossona de Mendez, Michal Pilipczuk, Sebastian Siebertz, and Szymon Torunczyk. First-order interpretations of bounded expansion classes. *ACM Trans. Comput. Log.*, 21(4):29:1–29:41, 2020. doi:[10.1145/3382093](https://doi.org/10.1145/3382093).
- [GKR⁺18] Martin Grohe, Stephan Kreutzer, Roman Rabinovich, Sebastian Siebertz, and Konstantinos Stavropoulos. Coloring and covering nowhere dense graphs. *SIAM Journal on Discrete Mathematics*, 32(4):2467–2481, 2018. arXiv:<https://arxiv.org/abs/1801.08000>, doi:[10.1137/18M1168753](https://doi.org/10.1137/18M1168753).
- [GKS17a] Martin Grohe, Stephan Kreutzer, and Sebastian Siebertz. Deciding first-order properties of nowhere dense graphs. *J. ACM*, 64(3):17:1–17:32, 2017. doi:[10.1145/3051095](https://doi.org/10.1145/3051095).
- [GKS17b] Martin Grohe, Stephan Kreutzer, and Sebastian Siebertz. Deciding first-order properties of nowhere dense graphs. *J. ACM*, 64(3):17:1–17:32, 2017. doi:[10.1145/3051095](https://doi.org/10.1145/3051095).
- [GMM⁺23] Jakub Gajarský, Nikolas Mählmann, Rose McCarty, Pierre Ohlmann, Michal Pilipczuk, Wojciech Przybyszewski, Sebastian Siebertz, Marek Sokołowski, and Szymon Toruńczyk. Flipper games for monadically stable graph classes. *arXiv*, abs/2301.13735, 2023. URL: <https://arxiv.org/abs/2301.13735>, doi:[10.48550/ARXIV.2301.13735](https://doi.org/10.48550/ARXIV.2301.13735).
- [GPPT22] Jakub Gajarský, Michal Pilipczuk, Wojciech Przybyszewski, and Szymon Torunczyk. Twin-width and types. In Mikolaj Bojanczyk, Emanuela Merelli, and David P. Woodruff, editors, *49th International Colloquium on Automata, Languages, and Programming, ICALP 2022, July 4–8, 2022, Paris, France*, volume 229 of *LIPIcs*, pages 123:1–123:21. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. doi:[10.4230/LIPIcs.ICALP.2022.123](https://doi.org/10.4230/LIPIcs.ICALP.2022.123).
- [GPT21] Jakub Gajarský, Michal Pilipczuk, and Szymon Toruńczyk. Stable graphs of bounded twin-width, 2021. URL: <https://arxiv.org/abs/2107.03711>, doi:[10.48550/ARXIV.2107.03711](https://doi.org/10.48550/ARXIV.2107.03711).

- [GPT22] Jakub Gajarský, Michał Pilipczuk, and Szymon Toruńczyk. Stable graphs of bounded twin-width. In *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS '22*, New York, NY, USA, 2022. Association for Computing Machinery. doi:10.1145/3531130.3533356.
- [Gro07] Martin Grohe. Logic, graphs, and algorithms. *Electron. Colloquium Comput. Complex.*, TR07-091, 2007. URL: <https://eccc.weizmann.ac.il/eccc-reports/2007/TR07-091/index.html>, arXiv:TR07-091.
- [GT22] Colin Geniet and Stéphan Thomassé. First order logic and twin-width in tournaments and dense oriented graphs, 2022. arXiv:2207.07683.
- [GW00] Frank Gurski and Egon Wanke. The tree-width of clique-width bounded graphs without kn,n. In Ulrik Brandes and Dorothea Wagner, editors, *Graph-Theoretic Concepts in Computer Science*, pages 196–205, Berlin, Heidelberg, 2000. Springer Berlin Heidelberg.
- [HJMW20] Tony Huynh, Gwenaél Joret, Piotr Micek, and David Wood. Seymour’s conjecture on 2-connected graphs of large pathwidth. *Combinatorica*, 40:839–868, 2020. doi:10.1007/s00493-020-3941-3.
- [iOSo6] Sang il Oum and Paul Seymour. Approximating clique-width and branch-width. *Journal of Combinatorial Theory, Series B*, 96(4):514–528, 2006. URL: <https://www.sciencedirect.com/science/article/pii/S0095895605001528>, doi: <https://doi.org/10.1016/j.jctb.2005.10.006>.
- [KMOW21] O-joung Kwon, Rose McCarty, Sang-il Oum, and Paul Wollan. Obstructions for bounded shrub-depth and rank-depth. *Journal of Combinatorial Theory, Series B*, 149:76–91, 2021. URL: <https://www.sciencedirect.com/science/article/pii/S0095895621000058>, doi: <https://doi.org/10.1016/j.jctb.2021.01.005>.
- [KOo4] Daniela Kühn and Deryk Osthus. Induced subdivisions in ks,s-free graphs of large average degree. *Combinatorica*, 24(2):287–304, 2004. doi:10.1007/s00493-004-0017-8.
- [KPS17] O-joung Kwon, Michał Pilipczuk, and Sebastian Siebertz. On low rank-width colorings. In Hans L. Bodlaender and Gerhard J. Woeginger, editors, *Graph-Theoretic Concepts in Computer Science*, pages 372–385, Cham, 2017. Springer International Publishing.
- [Lam12] Michael Lampis. Algorithmic meta-theorems for restrictions of treewidth. *Algorithmica*, 64(1):19–37, 2012. doi:10.1007/s00453-011-9554-x.
- [LPPT20] Stratis Limnios, Christophe Paul, Joanny Perret, and Dimitrios M. Thilikos. Edge degeneracy: Algorithmic and structural results. *Theoretical Computer Science*, 839:164–175, 2020. URL: <https://www.sciencedirect.com/science/article/pii/S030439752030342X>, doi: <https://doi.org/10.1016/j.tcs.2020.06.006>.
- [Mato4] Jiří Matoušek. Bounded VC-dimension implies a fractional Helly theorem. *Discrete Comput. Geom.*, 31(2):251–255, 2004.
- [MT04] Adam Marcus and Gábor Tardos. Excluded permutation matrices and the stanley–wilf conjecture. *Journal of Combinatorial Theory, Series A*, 107(1):153–160, 2004. URL: <https://www.sciencedirect.com/science/article/pii/S0097316504000512>, doi: <https://doi.org/10.1016/j.jcta.2004.04.002>.
- [NdM12] Jaroslav Nešetřil and Patrice Ossona de Mendez. *Sparsity - Graphs, Structures, and Algorithms*, volume 28 of *Algorithms and combinatorics*. Springer, 2012. doi:10.1007/978-3-642-27875-4.
- [NdMP⁺21] Jaroslav Nešetřil, Patrice Ossona de Mendez, Michał Pilipczuk, Roman Rabinovich, and Sebastian Siebertz. Rankwidth meets stability. In *Proceedings of the*

- Thirty-Second Annual ACM-SIAM Symposium on Discrete Algorithms, SODA '21*, pages 2014–2033, USA, 2021. Society for Industrial and Applied Mathematics.
- [NiO20] Huy-Tung Nguyen and Sang il Oum. The average cut-rank of graphs. *European Journal of Combinatorics*, 90:103183, 2020. URL: <https://www.sciencedirect.com/science/article/pii/S0195669820301049>, doi:<https://doi.org/10.1016/j.ejc.2020.103183>.
- [NO08] Jaroslav Nešetřil and Patrice Ossona de Mendez. Grad and classes with bounded expansion i. decompositions. *European Journal of Combinatorics*, 29(3):760–776, 2008. URL: <https://www.sciencedirect.com/science/article/pii/S019566980700056X>, doi:<https://doi.org/10.1016/j.ejc.2006.07.013>.
- [NO11] Jaroslav Nešetřil and Patrice Ossona de Mendez. On nowhere dense graphs. *Eur. J. Comb.*, 32(4):600–617, 2011. doi:[10.1016/j.ejc.2011.01.006](https://doi.org/10.1016/j.ejc.2011.01.006).
- [NORS21] Jaroslav Nešetřil, Patrice Ossona de Mendez, Roman Rabinovich, and Sebastian Siebertz. Classes of graphs with low complexity: The case of classes with bounded linear rankwidth. *European Journal of Combinatorics*, 91:103223, 2021. Colorings and structural graph theory in context (a tribute to Xuding Zhu). URL: <https://www.sciencedirect.com/science/article/pii/S019566982030144X>, doi:<https://doi.org/10.1016/j.ejc.2020.103223>.
- [Prz22] Wojciech Przybyszewski. Vc-density and abstract cell decomposition for edge relation in graphs of bounded twin-width. *arXiv*, abs/2202.04006, 2022. URL: <https://arxiv.org/abs/2202.04006>, arXiv:2202.04006.
- [PST18] Michał Pilipczuk, Sebastian Siebertz, and Szymon Toruńczyk. On the number of types in sparse graphs. In Anuj Dawar and Erich Grädel, editors, *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 09–12, 2018*, pages 799–808. ACM, 2018. doi:[10.1145/3209108.3209178](https://doi.org/10.1145/3209108.3209178).
- [PZ78] Klaus-Peter Podewski and Martin Ziegler. Stable graphs. *Fundamenta Mathematicae*, 100(2):101–107, 1978.
- [RS84] Neil Robertson and P.D Seymour. Graph minors. iii. planar tree-width. *Journal of Combinatorial Theory, Series B*, 36(1):49–64, 1984. URL: <https://www.sciencedirect.com/science/article/pii/0095895684900133>, doi:[https://doi.org/10.1016/0095-8956\(84\)90013-3](https://doi.org/10.1016/0095-8956(84)90013-3).
- [RT08] David Richerby and Dimitrios M. Thilikos. Searching for a visible, lazy fugitive. In Hajo Broersma, Thomas Erlebach, Tom Friedetzky, and Daniël Paulusma, editors, *Graph-Theoretic Concepts in Computer Science, 34th International Workshop, WG 2008, Durham, UK, June 30 - July 2, 2008. Revised Papers*, volume 5344 of *Lecture Notes in Computer Science*, pages 348–359, 2008. doi:[10.1007/978-3-540-92248-3_31](https://doi.org/10.1007/978-3-540-92248-3_31).
- [Sau72] N Sauer. On the density of families of sets. *Journal of Combinatorial Theory, Series A*, 13(1):145–147, 1972. URL: <https://www.sciencedirect.com/science/article/pii/0097316572900192>, doi:[https://doi.org/10.1016/0097-3165\(72\)90019-2](https://doi.org/10.1016/0097-3165(72)90019-2).
- [See96] Detlef Seese. Linear time computable problems and first-order descriptions. *Mathematical Structures in Computer Science*, 6(6):505–526, 1996. doi:[10.1017/S0960129500070079](https://doi.org/10.1017/S0960129500070079).
- [She72] Saharon Shelah. A combinatorial problem; stability and order for models and theories in infinitary languages. *Pacific Journal of Mathematics*, 41(1):247 – 261, 1972. URL: <https://doi.org/10.2307/2371110>, doi:[pjm/1102968432](https://doi.org/10.2307/2371110).
- [She86] Saharon Shelah. *Monadic logic: Hanf Numbers*, pages 203–223. Springer Berlin Heidelberg, Berlin, Heidelberg, 1986. doi:[10.1007/BFb0098511](https://doi.org/10.1007/BFb0098511).

- [Sim90] Imre Simon. Factorization forests of finite height. *Theoretical Computer Science*, 72(1):65–94, 1990. URL: <https://www.sciencedirect.com/science/article/pii/030439759090047L>, doi:[https://doi.org/10.1016/0304-3975\(90\)90047-L](https://doi.org/10.1016/0304-3975(90)90047-L).
- [Sim15] Pierre Simon. *A Guide to NIP Theories*. Lecture Notes in Logic. Cambridge University Press, 2015. doi:[10.1017/CB09781107415133](https://doi.org/10.1017/CB09781107415133).
- [SS71] Saharon Shelah and Saharon Shelah. Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory. *Annals of Mathematical Logic*, 3:271–362, 1971.
- [ST93] P.D. Seymour and R. Thomas. Graph searching and a min-max theorem for tree-width. *Journal of Combinatorial Theory, Series B*, 58(1):22–33, 1993. URL: <https://www.sciencedirect.com/science/article/pii/S0095895683710270>, doi:<https://doi.org/10.1006/jctb.1993.1027>.
- [war16] Algorithms, logic and structure workshop in warwick, open problem session. https://warwick.ac.uk/fac/sci/math/people/staff/daniel_kral/alglogstr/openproblems.pdf, 2016. [Online; accessed 23-Jan-2023].
- [Zhu09] Xuding Zhu. Colouring graphs with bounded generalized colouring number. *Discrete Mathematics*, 309(18):5562–5568, 2009.