Defining Fairness

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Abstract. We propose a definition for the class of all fairness properties of a given system. We provide independent characterizations in terms of topology, language theory and game theory. All popular notions of fairness from the literature satisfy our definition. Moreover our class is closed under union and countable intersection, and it is, in a sense, the maximal class having this property. On the way, we characterize a class of liveness properties, called *constructive liveness*, which is interesting by itself because it is also closed under union and countable intersection. Furthermore, we characterize some subclasses of liveness that are closed under arbitrary intersection.

1 Introduction

The distinction of *safety* and *liveness* properties, first proposed by Lamport [10] and later formalized by Lamport [11] and Alpern and Schneider [2], is now well-established in the specification, analysis, and verification of *reactive systems* [6]. The main reasons for the success of these concepts is their natural and convincing intuition, their stringent mathematical formalization, and the fact that every property can be expressed as the conjunction of a safety and a liveness property (see [7] for a survey). In particular, it turned out that safety properties are the closed sets and liveness properties are the dense sets in the natural topology of runs [2].

The distinction of safety and liveness is also reflected in the operational model of a reactive system: Some sort of state machine or transition system defines the set of all possible runs of the system, which is a safety property. In order to guarantee something to happen at all and to guarantee that some particular choices will eventually be made, there is an additional liveness property. That liveness property is usually called the *fairness assumption* of the reactive system.

Fairness usually means that a particular choice is taken sufficiently often provided that it is sufficiently often possible [3]. Depending on the interpretation of 'choice', 'sufficiently often', and 'possible', many different fairness notions arise (cf. e.g. [13,4,8]).

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In contrast to safety and liveness properties, there is no satisfactory characterization of fairness. Apt, Francez, and Katz [3] gave some criteria that must be met by fairness. Following Lamport [12], we think that their most important criterion is that a fairness assumption must be *machine closed*[1] with respect to (w.r.t.) the safety property defined by the underlying transition system. This, basically, means that fairness is imposed in such a way to the transition system that the system 'cannot paint itself into a corner' [3]; i. e. , whatever the system does, it is possible to continue in such a way that the fairness assumption is met. However, machine closedness does not exclude some properties that, we think, should not be considered to be fairness properties. For example, consider the two properties:

P_1: Transition t is always eventually taken if it is always eventually enabled.
P_2: Transition t is eventually henceforth never taken.

While P_1 (called *strong fairness* w.r.t. t) is a typical fairness assumption that enforces a transition to be taken sufficiently often, P_2 rather prevents a particular choice (transition t) from being taken sufficiently often. P_2 is therefore not a fairness property from our point of view. However, both properties are machine closed with respect to any safety property[2].

Another issue is that fairness should be closed under intersection, i.e., the intersection of finitely many, or better: countably many, fairness assumptions should be a fairness assumption. This is because fairness assumptions are usually imposed stepwise and componentwise, e.g., with respect to a particular process or with respect to a particular transition. The fairness assumption for the system is then the intersection of all fairness assumptions for its components.

Machine-closure is not sufficient to guarantee closure under intersection: The intersection of P_1 and P_2 is the empty set in some systems, and the empty set is not machine closed w.r.t. any nonempty safety property. Kwiatkowska [9] proposes a definition of fairness[3] that is closed under countable intersection. However, many popular fairness notions, such as strong fairness are not covered by her definition.

We propose a definition of fairness that refines machine-closure and excludes properties like P_2 that prohibits a choice to be taken sufficiently often. Roughly, a fairness property w.r.t. a system is a property that can be realized by a scheduler that always eventually gets control over the system. We show that fairness is then closed under union and countable intersection and that popular fairness notions satisfy our definition. We give independent characterizations in terms of game theory, language theory, and topology. It turns out that fairness as we define it coincides with the *co-meager sets* of the natural topology of runs, a subclass of

[1] *Machine closedness* was originally called *feasibility.* The term *machine closedness* was introduced in [1].

[2] P_2 also meets the other criteria of Apt, Francez, and Katz [3].

[3] Kwiatkowska [9] works on the domain of Mazurkiewicz traces. She defines a fairness property for a system to be a G_δ set of maximal traces that is machine closed w.r.t. the safety property of the system.

the dense sets. Co-meager sets are ‘large’, which in our context means that they, besides of possibly enforcing some choices, also leave enough choices.

2 Preliminaries

Runs. A *run* is a nonempty finite or infinite sequence over some fixed countable set Σ of *states*. Σ^+, Σ^ω, and $\Sigma^\infty = \Sigma^+ \cup \Sigma^\omega$ denote the set of all *finite runs*, *infinite runs*, and of all *runs* respectively. We will use the symbols α, β for denoting finite runs, and x, y for arbitrary runs. The length of a run x is denoted by $|x|$ ($= \omega$ if x is infinite). Concatenation of sequences is denoted by juxtaposition; $\sqsubseteq$ denotes the usual (reflexive) *prefix order* on sequences. Two runs x and x' are *compatible*, if $x \sqsubseteq x'$ or $x' \sqsubseteq x$. By $x\!\uparrow = \{y \mid x \sqsubseteq y\}$ and $x\!\downarrow = \{y \mid y \sqsubseteq x\}$ we denote the set of all *extensions* and *prefixes* of a run x respectively. The least upper bound of a sequence $(\alpha_i)_{i=0,1,\ldots}$ of finite runs where $\alpha_i \sqsubseteq \alpha_{i+1}$ is denoted by $\sup_i \alpha_i$. For a run $x = s_0, s_1, \ldots$ and a position i where $0 \le i < |x|$ of x, x_i denotes the i-th prefix $s_0, \ldots, s_i$ of x.

Temporal Properties. A *temporal property* (*property* for short) is a set $E \subseteq \Sigma^\infty$; E is *finitary* if $E \subseteq \Sigma^+$, and *infinitary* if $E \subseteq \Sigma^\omega$. Sometimes (e.g. [2,14]), a temporal property is defined to be a subset of Σ^ω. That results in the underlying topology having nicer properties[4]. However, that needs finite runs to be ruled out a priori or to be mimicked by infinite runs, e.g., by repeating the last state infinitely often. Moreover, including finite runs gives rise to a more natural generalization to other domains such as non-sequential runs (cf. Sect. 6). We say that some run x *satisfies* a property E if $x \in E$, otherwise we say that x *violates* E. A property S is a *safety property* if for any run x violating S, there exists a finite prefix α of x that violates S and each extension of a run violating S violates S as well, i.e.:

$$\forall x \notin S : \exists \alpha \sqsubseteq x : \alpha\!\uparrow \cap\, S = \varnothing.$$

Safety properties are exactly those sets S that are *downward-closed* and *complete*, where the former means $x \in S$ and $y \sqsubseteq x$ implies $y \in S$ and the latter $\alpha_i \in S$ for $i \in \mathbb{N}$ with $\alpha_i \sqsubseteq \alpha_{i+1}$ implies $\sup_i \alpha_i \in S$. A property E is *live in* a finite run α if there exists a run $x \in E$ such that $\alpha \sqsubseteq x$. A property E is *live* (or a *liveness property*) if E is live in every $\alpha \in \Sigma^+$. Let S be a safety property. E is *live w.r.t. S* (or (S, E) is *machine closed*) if $E \cap S$ is live in every $\alpha \in \Sigma^+ \cap S$.

Σ^+ and Σ^ω are simple examples of liveness properties. The empty set $\varnothing = \Sigma^\omega \cap \Sigma^+$ is not a liveness property, which shows that liveness properties are not closed under finite intersection. It is easy to see that for a liveness property E, every property $E' \supseteq E$ is also a liveness property. No property except Σ^∞ is a safety and a liveness property.

[4] The natural topology on Σ^ω is metrizable while the natural topology on Σ^∞ does not satisfy the separation axiom T_1.

Basic Notions from General Topology. A *topology* on a nonempty set Ω is a family $\mathscr{T} \subseteq 2^{\Omega}$ that is closed under union and finite intersection such that $\Omega, \varnothing \in \mathscr{T}$. The elements of $\mathscr{T}$ are called *open sets*. A family $\mathscr{B} \subseteq \mathscr{T}$ is a *base* for $\mathscr{T}$ if every open set $G \in \mathscr{T}$ is the union of members of $\mathscr{B}$. The complement of an open set is called a *closed set*. The *closure* of a set $X \subseteq \Omega$, denoted by $\overline{X}$, is the smallest closed set that contains X. A set X is closed if and only if $X = \overline{X}$. A set X is *dense* if $\overline{X} = \Omega$. A G_δ *set* is a set that is the intersection of countably many open sets. Let $X \subseteq \Omega$ be a nonempty set. The family $\mathscr{T}_X = \{G \cap X \mid G \in \mathscr{T}\}$ is a topology, called the *relativization* of $\mathscr{T}$ to X.

Scott Topology. The *Scott topology* on Σ^∞ is the family of sets G such that

$$\forall x \in G : \exists \alpha \sqsubseteq x : \alpha\!\uparrow \subseteq G.$$

The family $\{\alpha\!\uparrow \mid \alpha \in \Sigma^+\}$ is a basis for the Scott topology. Note that open sets are generated by finitary properties Q by $G = Q\!\uparrow = \bigcup_{\alpha \in Q} \alpha\!\uparrow$, i.e., there is an exact correspondence between open sets and finitary properties. Open sets can therefore be interpreted as *observations* that can be recognized in finite time.

It is easy to see that safety properties are exactly the closed sets and that liveness properties are exactly the dense sets of the Scott topology. It follows that each property E is the intersection of a safety property and a liveness property [2], viz. $E = \overline{E} \cap \mathrm{lex}(E)$ where $\overline{E}$ is the smallest safety property that contains E, and $\mathrm{lex}(E)$ is the *liveness extension* of E, defined by

$$\mathrm{lex}(E) = E \cup \neg\overline{E} = E \cup \bigcup_{\alpha\uparrow \cap E = \varnothing} \alpha\!\uparrow.$$

Temporal Operators. Manna and Pnueli [14] define four operators that construct temporal properties from finitary properties. While they consider only sets $E \subseteq \Sigma^\omega$ as temporal properties, we generalize their operators here to our setting in a natural way. Let Q be a finitary property. Define $\mathrm{A}(Q) = \{x \mid \forall i < |x| : x_i \in Q\}$, $\mathrm{E}(Q) = \{x \mid \exists i : x_i \in Q\}$, $\mathrm{R}(Q) = \{x \mid \forall i < |x| : \exists j \geq i : x_j \in Q\}$, and $\mathrm{P}(Q) = \{x \mid \exists i : \forall j : i \leq j < |x| : x_j \in Q\}$. Properties of the form $\mathrm{A}(Q)$ are exactly the safety properties. Properties of the form $\mathrm{E}(Q)$, $\mathrm{R}(Q)$, and $\mathrm{P}(Q)$ are called *guarantee*, *recurrence*, and *persistence properties* respectively. It is easy to see that guarantee properties are exactly the open sets (where Q is the corresponding observation). We have $\neg\mathrm{A}(Q) = \mathrm{E}(\neg Q)$, $\neg\mathrm{E}(Q) = \mathrm{A}(\neg Q)$, $\neg\mathrm{R}(Q) = \mathrm{P}(\neg Q)$, and $\neg\mathrm{P}(Q) = \mathrm{R}(\neg Q)$ where $\neg\cdot$ denotes the complement w.r.t. the appropriate universe. Since $\mathrm{A}(Q) = \mathrm{R}(\mathrm{A}(Q) \cap \Sigma^+)$ and $\mathrm{E}(Q) = \mathrm{R}(\mathrm{E}(Q) \cap \Sigma^+)$, we have that each safety property and each guarantee property is a recurrence property. Similarly, each safety and each guarantee property is also a persistence property. We will use a simple linear-time temporal logic with the modalities $\Diamond$ and $\Box$ to be interpreted on finite and infinite runs in their usual meaning. The properties $\Box\,\varphi$, $\Diamond\,\varphi$, $\Box\Diamond\,\varphi$, and $\Diamond\Box\,\varphi$ are simple examples of safety, guarantee, recurrence, and persistence properties respectively where φ denotes any state property.

3 Constructive Liveness

Fairness is always defined with respect to a particular system, where a system can be seen as a safety property. In this section, we define fairness with respect to the system where every transition is possible at any time, i.e., with respect to the safety property Σ^∞. The generalization to arbitrary safety properties will be a simple step, which we take in Sect. 4. Fairness properties with respect to Σ^∞ are special liveness properties, which we call *constructive liveness*. Constructive liveness is interesting by itself because it is closed under union and countable intersection. We give three independent characterizations of constructive liveness: a game-theoretic, a language-theoretic, and a topological characterization. Additional proofs can be found in the full version of this paper [17].

3.1 A Game-Theoretic View

Fairness may enforce that a particular choice is taken sufficiently often while it must not prevent any other choice from being taken sufficiently often. This can be formalized by thinking of a party, which we will call the *scheduler*, that enforces a choice to be taken sufficiently often while it cannot prevent other choices from being taken by another party, called the *opponent*. Fairness properties (or here: constructive liveness properties) are those properties that can be realized by the scheduler regardless of the behavior of the opponent. In detail: We view runs now as the result of an infinite interaction between the *scheduler* and the *opponent*. The opponent starts by performing a nonempty sequence α_0. The scheduler then appends a finite, possibly empty, sequence of states yielding a finite run α_1 such that $\alpha_0 \sqsubseteq \alpha_1$. Now it is the turn of the opponent again, which also appends a finite, possibly empty, sequence and so on. The result of this interaction is the run $x = \sup_i \alpha_i$. A liveness property is *constructive* if, regardless of what the opponent does, the scheduler can guarantee that a run is obtained that satisfies the property. This game is similar to the *Banach-Mazur game* (see [5], cf. Sect. 3.4).

Definition 1. *A* play *on Σ is an infinite sequence of finite runs $(\alpha_i)_{i\in\mathbb{N}}$, such that $\alpha_i \sqsubseteq \alpha_{i+1}$. Given a play $(\alpha_i)_{i\in\mathbb{N}}$, we say that the scheduler* wins *the play for the game with* target *$E \subseteq \Sigma^\infty$ if $\sup_i \alpha_i \in E$. Otherwise the opponent wins. A* strategy *(for the scheduler) is a mapping*[5] *$f : \Sigma^+ \to \Sigma^+$ such that $\alpha \sqsubseteq f(\alpha)$ for all $\alpha \in \Sigma^+$. A strategy f is* progressive *if $f(\alpha) \neq \alpha$ for all $\alpha \in \Sigma^+$. A play $(\alpha_i)_{i\in\mathbb{N}}$ is f-*compliant *if for every i, $f(\alpha_{2i}) = \alpha_{2i+1}$. A run x is f-*compliant *if it is the result of an f-compliant play $(\alpha_i)_{i\in\mathbb{N}}$, i.e., $x = \sup_i \alpha_i$. The set of all f-compliant runs is denoted by R_f. A strategy f is* winning *for E if $R_f \subseteq E$.*

Note that a finite run α is f-compliant if and only if $f(\alpha) = \alpha$. Therefore, all f-compliant runs are infinite if f is progressive. We could indeed restrict to consider progressive strategies, as the following result shows.

[5] Considering strategies that depend on the full history of the play does not increase their power in the game considered here.

Lemma 1. *There exists a winning strategy for E if and only if there exists a progressive winning strategy for $E \cap \Sigma^\omega$.*

Proof. Let f be a winning strategy for E. Let $\beta \in \Sigma^+$ be any finite run and define $f'(\alpha) = f(\alpha)\beta$. Then $R_{f'} \subseteq R_f \subseteq E$. Moreover $R_{f'} \subseteq \Sigma^\omega$ and hence f' is also winning for $E \cap \Sigma^\omega$. The converse is trivial. □

However, we will use non-progressive strategies to neatly characterize some interesting subclasses of liveness.

It is easy to see that, for every strategy f, the property R_f is a liveness property. Therefore, if a target E has a winning strategy, it is a liveness property. This justifies the following definition.

Definition 2. *A property E is called* constructive liveness property *if there exists a winning strategy for E.*

Corollary 1. *A property E is a constructive liveness property if and only if $E \cap \Sigma^\omega$ is a constructive liveness property.*

Corollary 1 is a direct consequence of Lemma 1. We now get:

Proposition 1. *The family of constructive liveness properties is closed under union and countable intersection.*

Proof. Closure under union is trivial. Let E_i be a constructive liveness property and f_i a progressive winning strategy for E_i for each $i \in \mathbb{N}$. Define for $\alpha \in \Sigma^+$ with $|\alpha| = k$: $f(\alpha) = f_k(f_{k-1}(\ldots f_0(\alpha)\ldots))$. It is straight-forward to check that f is a winning strategy for $\bigcap_{i\in\mathbb{N}} E_i$. □

Σ^ω is a constructive liveness property, while Σ^+ is a liveness property but not constructive because the opponent can enforce the outcome of the play to be infinite. Similarly for any run x, the property $\{\alpha x \mid \alpha \in \Sigma^+\}$ is a liveness property but not constructive. The property $\Box\Diamond\varphi$ is a constructive liveness property while $\Diamond\Box\varphi$ is a liveness property but not constructive—for any non-trivial state property φ. More examples for constructive liveness properties are $\Box(\varphi \Rightarrow \Diamond\psi)$, $\Diamond\Box\varphi \Rightarrow \Box\Diamond\psi$, and $\Box\Diamond\varphi \Rightarrow \Box\Diamond\psi$.

Call a run *periodic* if it is of the form $\alpha\beta^\omega$ for $\alpha, \beta \in \Sigma^+$ and *aperiodic* otherwise. The set of aperiodic runs is a constructive liveness property while the set of periodic runs is a liveness property but not constructive; f defined by $f(\alpha) = \alpha s^k r$ where $k = |\alpha|, s, r \in \Sigma, s \neq r$ is a winning strategy for aperiodic runs.

3.2 A Language-Theoretic View

In this section, we study what guarantee, recurrence, and persistence properties are constructive liveness properties. (Recall that a safety property is a liveness property only if it equals Σ^∞.) Furthermore, we derive an independent characterization of constructive liveness that is based on recurrence properties.

Proposition 2. *Let Q be a finitary property.*

1. $\mathrm{E}(Q)$ *is a liveness property if and only if Q is a* pseudo-liveness property*, that is, for each $\alpha \in \Sigma^+$ exists an $x \in Q$ that is compatible with α.*
2. $\mathrm{R}(Q)$ *is a liveness property if and only if Q is a liveness property.*
3. $\mathrm{P}(Q)$ *is a liveness property if and only if Q is a liveness property.*

It is easy to check that each live guarantee as well as each live recurrence property is constructive. More precisely, live recurrence properties correspond to the runs complying with *idempotent* strategies, i.e., strategies f that satisfy $f(f(\alpha)) = f(\alpha)$ for all $\alpha \in \Sigma^+$. Live guarantee properties correspond to the runs complying with *stable* strategies, i.e., strategies f that satisfy $f(\alpha) \sqsubseteq \beta \Rightarrow f(\beta) = \beta$ for all $\alpha, \beta \in \Sigma^+$. Each stable strategy is idempotent.

Proposition 3. *We have:*

1. $\{\mathrm{E}(Q) \mid Q$ *is a pseudo-liveness property*$\} = \{R_f \mid f$ *is a stable strategy*$\}$ *and*
2. $\{\mathrm{R}(Q) \mid Q$ *is a liveness property*$\} = \{R_f \mid f$ *is an idempotent strategy*$\}$.

It follows from Prop. 3 that each property that contains a live guarantee property or a live recurrence property is a constructive liveness property. We show now that live persistence properties are in general not constructive.

Proposition 4. *A live persistence property is constructive if and only if it contains a live guarantee property.*

We have shown that each property that contains a live recurrence property is a constructive liveness property. The converse does not hold. However, we can give a characterization of constructive liveness in terms of recurrence properties if we restrict ourselves to the infinitary subset of a recurrence property in the spirit of Corollary 1. Define $\mathrm{R}^\omega(Q) = \mathrm{R}(Q) \cap \Sigma^\omega$, i.e., $\mathrm{R}^\omega(Q)$ consists of all runs that have infinitely many prefixes in Q. A property is called *infinitary recurrence property* if it is of the form $\mathrm{R}^\omega(Q)$. Infinitary recurrence properties are closed under countable intersection (and finite union) [14]. In contrast, recurrence properties are not closed under finite intersection.

Proposition 5. *The family of live infinitary recurrence properties is closed under finite union and countable intersection.*

We obtain the following characterization of constructive liveness.

Proposition 6. *A property is a constructive liveness property if and only if it contains a live infinitary recurrence property.*

Proof. The claim is part of the more general Thm. 1 below. □

The property $\Diamond \varphi$ is a live guarantee property, $\Box \Diamond \varphi$ is a live recurrence property and hence, $\Box \Diamond \varphi \cap \Sigma^\omega$ ('infinitely often φ') is a live infinitary recurrence property.

3.3 A Topological View

In this section, we characterize constructive liveness in terms of dense G_δ sets. As we have stated, a dense open set is a live guarantee property. Such a property has a nice intuition: It requires one finite observation to be made. Often it is natural to require countably many finite observations to be made. This corresponds to the intersection of countably many dense open sets, which is again dense in our topology:

Proposition 7. *A property E is a dense G_δ set if and only if it is the intersection of countably many dense open sets.*

Topological spaces that satisfy Prop. 7 are called *Baire spaces*.

Corollary 2. *The family of dense G_δ sets is closed under finite union and countable intersection.*

Each G_δ set E is, like any open set, *upward-closed*, i.e., $x \in E$ and $x \sqsubseteq y$ implies $y \in E$. Recurrence properties are therefore not G_δ sets in general. However, they are related as follows.

Proposition 8. *E is a G_δ set if and only if $E = \mathrm{E}(Q) \cup \mathrm{R}^\omega(Q')$ for some finitary Q and Q'.*

In particular, each infinitary recurrence property is a G_δ set. Furthermore, $\mathrm{lex}(E)$ is a dense open set if E is open and a dense G_δ set if E is a G_δ set. We define now two more classes of liveness properties.

Definition 3. *A property is an* open-liveness property *if it contains a dense open set. A property is a* G_δ-liveness property *if it contains a dense* G_δ *set.*

An open-liveness property is a property satisfying

$$\forall \alpha \in \Sigma^+ : \exists \beta : \alpha \sqsubseteq \beta \wedge \beta\!\uparrow \subseteq E.$$

Examples of open-liveness properties are $\Diamond \varphi \Rightarrow \Diamond \psi$ and $\Box \Diamond \varphi \Rightarrow \Diamond \psi$ if ψ is nonempty. E is a dense open set if and only if $E = \mathrm{lex}(\mathrm{E}(Q))$ for some finitary Q. Due to Prop. 7, a property is a G_δ-liveness property if and only if it is the intersection of countably many open-liveness properties.

G_δ-liveness properties have another topological characterization—they are the *co-meager* sets of our topology: In a topological space, we say that a set is *nowhere dense* if its closure does not contain any nonempty open set. A set is *meager*, if it is the countable union of nowhere dense sets. The complement of a meager set is called *co-meager* (or *residual*).

Proposition 9 (E.g. [15], page 41). *In a Baire space, a set is co-meager if and only if it contains a dense* G_δ *set.*

Co-meagerness is a topological notion of 'largeness' of a set. The class of co-meager sets shares many properties with the class of sets of measure 1 [15], which are the 'large' sets in probability theory. We prove now that our three views on constructive liveness coincide.

Theorem 1. *Let E be a temporal property. The following statements are equivalent:*

1. *E is a constructive liveness property.*
2. *E contains a live infinitary recurrence property.*
3. *E is a G_δ-liveness property.*

Proof.

1. *$\Rightarrow$ 2.* Let f be a progressive winning strategy for E. It is easy to check that $R_f = \mathrm{R}(f(\Sigma^+)) \cap \Sigma^\omega$.
2. *$\Rightarrow$ 3.* Each infinitary recurrence property is a G_δ set due to Prop. 8.
3. *$\Rightarrow$ 1.* Each open-liveness property is constructive (Prop. 3.1). Since constructive liveness is closed under countable intersection, each G_δ-liveness property is constructive as well. □

Fig. 1.a shows the relationships between the subfamilies of constructive liveness.

3.4 A Maximality Result

We would like to argue that constructive liveness is the most permissive definition that suits our purposes, i.e., it is in some sense maximal among all the subclasses of liveness that are closed under countable intersection. We are not able to prove such a result. However we are able to prove that it is maximal if we restrict to *determinate* sets.

Definition 4. *A* counter strategy *(for the opponent) is a pair $g = (\alpha, f)$ of a finite run $\alpha \in \Sigma^+$ and a strategy f; g is* progressive *if f is progressive. A play $(\alpha_i)_{i\in\mathbb{N}}$ is g-*compliant *if $\alpha_0 = \alpha$ and for every i, $f(\alpha_{2i+1}) = \alpha_{2i+2}$. A run x is said to be g-compliant if it is the result of a g-compliant play $(\alpha_i)_{i\in\mathbb{N}}$, i.e., $x = \sup_i \alpha_i$. The set of all g-compliant runs is denoted by R_g. A counter strategy g is* winning *for target E if $R_g \subseteq \neg E$. We say that E is* determinate *if it has either a winning strategy or a winning counter strategy.*

Determinate sets have been studied in the classical theory of Banach-Mazur games. In the standard definition of a Banach-Mazur game, both players must play progressively and strategies may also depend on the full history of the previous play. However, in our setting, both definitions characterize the same class of sets (see [5]). In particular

Proposition 10. *E is determinate if and only if $E \cap \Sigma^\omega$ is determinate in the Banach-Mazur game.*

Using the axiom of choice, it is possible to show the existence of indeterminate sets. Nevertheless, the class of determinate sets is quite general. For instance, every *Borel set* of the natural topology[6] on Σ^ω is determinate in the Banach-Mazur game (see [5]), where the family of *Borel sets* of a topology is the smallest family of sets that contains the open sets and is closed under countable union and complementation. It easily follows from Prop. 10 that each Borel set of the Scott topology is determinate. In particular, this means that the class of determinate sets contains all properties that can be expressed by Büchi automata and hence all properties that can be expressed by common linear-time temporal logics.

We show now the maximality of constructive liveness within the determinate sets. Note that each constructive liveness property is determinate.

Theorem 2. *The family of constructive liveness properties is the largest family of determinate liveness properties that contains all dense* G_δ *sets and is closed under finite intersection.*

Proof. Consider a determinate set E that is not a constructive liveness property. E must therefore have a winning counter strategy $g = (\alpha_0, f)$. Note that R_f is a constructive liveness property. We claim that $R_f \cap E$ is not dense: Consider the finite run α_0. Since g is a winning counter strategy, any extension of α_0 into R_f is in $\neg E$. Therefore, there is no extension of α_0 into $R_f \cap E$ and hence $R_f \cap E$ is not dense. Since R_f is constructive, it contains a dense G_δ set L. It follows that $L \cap E$ is not a liveness property. Hence no non-constructive determinate liveness property can be added to the dense G_δ sets without losing closedness under finite intersection. □

Note that we have a complete proof strategy for showing that a determinate set is a constructive liveness property or not: Either display a winning strategy for the scheduler or a winning counter strategy for the opponent.

4 Defining Fairness

We consider now an arbitrary system, represented by a safety property S. We are interested in properties $E \subseteq S$ of the system under consideration. These properties are equipped with the Scott topology relativized to S. Liveness of a property F w.r.t. S is exactly density of $F \cap S$ in the Scott topology relative to S. We now define fairness properties in S analogously to constructive liveness. All notions and theorems from Sect. 3 easily carry over to the relativized case.

Definition 5. *Let S be a safety property, F a temporal property, and let $S^\top = \{x \in S \mid x{\uparrow} \cap S = \{x\}\}$ denote the set maximal runs w.r.t. S. A strategy f is* closed in S *if $\alpha \in S \Rightarrow f(\alpha) \in S$ for all $\alpha \in \Sigma^+$; f is* progressive in S *if $f(\alpha) = \alpha \Rightarrow \alpha \in S^\top$; f is a* winning strategy for F in S *if f is closed in S and $R_f \cap S \subseteq F$. F is a* fairness property *for S if there is a winning strategy for*

[6] This means the Cantor topology on Σ^ω, which coincides with the Scott topology on Σ^∞ relativized to Σ^ω.

F in S. A fairness notion *is a mapping that maps each safety property S to a fairness property for S.*

Clearly, each fairness property for S is live w.r.t. S, moreover:

Theorem 3. *The family of fairness properties for S is closed under union and countable intersection.*

Theorem 4. *The following statements are equivalent:*

1. *F is a fairness property for S.*
2. *There exists a finitary Q such that $F' = \mathrm{R}(Q) \cap S^\top \subseteq F$ and F' is live w.r.t. S.*
3. *There exists a G_δ set E such that $F' = E \cap S \subseteq F$ and F' is live w.r.t. S.*

Note that statement 3 is equivalent with F being co-meager in the Scott topology relativized to S.

Theorem 5. *The family of all fairness properties w.r.t. S is the largest family of live determinate properties w.r.t. S that contains the live G_δ sets w.r.t. S and that is closed under finite intersection.*

Note that E being a liveness property does not imply that E is live w.r.t. S, nor does the converse hold. However for the converse case we have: If E is live w.r.t. S then $E \cup \neg S$ is a liveness property that is live w.r.t. S. Therefore, it is neither necessary nor wrong to think of fairness properties as liveness properties, i.e., we would not gain or lose anything if we additionally required that a fairness property for S has to be a liveness property.

4.1 Examination of Popular Fairness Notions

We show now that our definition of fairness covers popular fairness notions in the literature. To check this, one can use the following proposition. Define $\mathrm{lex}_S(E)$ to be the *liveness extension of E relative to S* by

$$\mathrm{lex}_S(E) = E \cup \bigcup_{\alpha \in S, \alpha\uparrow \cap E \cap S = \varnothing} \alpha\uparrow .$$

Proposition 11. *If E is a constructive liveness property, then (any superset of)* $\mathrm{lex}_S(E)$ *is a fairness property for S.*

Define a *transition* to be a relation $t \subseteq \Sigma \times \Sigma$ over states. Let S be a safety property and $x = s_0, s_1, \ldots \in S$. Transition t is *enabled* in S at position i of x if there exists a state s such that $x_i s \in S$ and $(s_i, s) \in t$; t is *taken* at position i if $(s_i, s_{i+1}) \in t$. The following examples of fairness notions can be checked by using Prop. 11, but it is also easy to define a winning strategy in each case. The following list cannot be exhaustive due to lack of space. We also omit here the references to the papers where the fairness notions were introduced. Those can be found in the full version of this paper [17].

1. Maximality w.r.t. a transition t defined as $\Box(\text{enabled}_S(t) \Rightarrow \exists t' : \Diamond\, \text{taken}(t'))$ is a fairness notion.
2. Weak and strong fairness w.r.t. a transition t defined as $\Diamond\Box\, \text{enabled}_S(t) \Rightarrow \Box\Diamond\, \text{taken}(t)$ and $\Box\Diamond\, \text{enabled}_S(t) \Rightarrow \Box\Diamond\, \text{taken}(t)$ respectively are fairness notions. Weak and strong fairness w.r.t. words is similar.
3. Let $\varphi \subseteq \Sigma$ be a state property. Say that φ is *enabled* in S at a position i of a run x if there is a state $s \in \varphi$ such that $x_i s \in S$. State fairness w.r.t. φ defined as $\Box\Diamond\, \text{enabled}_S(\varphi) \Rightarrow \Box\Diamond\, \varphi$ is a fairness notion.
4. Extreme fairness w.r.t. a transition t and a state property φ defined as $\Box\Diamond(\varphi \wedge \text{enabled}_S(t)) \Rightarrow \Box\Diamond(\varphi \wedge \text{taken}(t))$ is a fairness notion. The notion of α-fairness is similar.
5. Say that a transition t is k-*enabled* in S at position i of x if there is a finite sequence α with $|\alpha| \leq k$ such that $x_i\alpha \in S$ and $x_i\alpha$ enables t; k-fairness w.r.t. t defined as $\Box\Diamond\, \text{enabled}_S(k,t) \Rightarrow \Box\Diamond\, \text{taken}(t)$ is a fairness notion.
6. Say that a transition t is ∞-*enabled* in S at position i of x if there exists a k such that t is k-enabled at i; ∞-fairness w.r.t. t (called *hyperfairness* in [12]) defined as $\Box\Diamond\, \text{enabled}_S(\infty,t) \Rightarrow \Box\Diamond\, \text{taken}(t)$ equals $\text{lex}_S(\Box\Diamond\, \text{taken}(t))$ and is therefore a fairness notion. Note that ∞-fairness is not the intersection of all k-fairness for $k \in \mathbb{N}$.
7. Unconditional fairness, defined as $\Box\Diamond\, \text{taken}(t)$ is not a fairness notion because it is not live w.r.t. all S.
8. Say that a transition t is k-*taken* at position i of a run x if t is taken at a position $j \leq i + k$ in x. The property $\Box\, \text{enabled}_S(t) \Rightarrow \text{taken}(k,t)$ is in general not a fairness property for S since it is a safety property and only live w.r.t. S if it coincides with S.
9. Let $y \in \Sigma^\omega$. Say that y is *enabled* in S at a position i of a run x if $x_i y \in S$; it is *taken* at i if $x = x_i y$. The property $\Box\Diamond\, \text{enabled}_S(y) \Rightarrow \Diamond\, \text{taken}(y)$ is live w.r.t. S but not a fairness property in general. Similarly, $\text{lex}_S(\Diamond\Box\, \varphi)$ is not a fairness property in general.
10. Finitary fairness w.r.t. a transition t, which is defined as $\bigcup_k \Box(\text{enabled}_S(t) \Rightarrow \text{taken}(k,t))$ is live w.r.t. S but not a fairness property. A winning counter strategy is defined by $f(\alpha) = \alpha s^k$ where $k = |\alpha|, s \in \Sigma$. Note that finitary fairness w.r.t. t is in conflict with the intersection of countably many strong fairness requirements (w.r.t. transitions $t_i, i \in \mathbb{N}$).

5 A Complete Lattice of Liveness Properties

The family of fairness properties for a given S and in particular the family of constructive liveness properties is not closed under arbitrary intersection. In particular, there is not a strongest fairness property in general.

Proposition 12. *Constructive liveness is not closed under arbitrary intersection.*

Proof. The property $\neg\{x\}$ is a constructive liveness property for each run x. $\bigcap_{x\in\Sigma^\infty} \neg\{x\} = \varnothing$ is not a liveness property. $\Box$

In this section, we identify a subclass of constructive liveness that is closed under arbitrary union and intersection, i.e., it forms a complete lattice. Therefore it possesses a strongest and a weakest property. We develop the theory here for constructive liveness. Analogous results can be obtained for fairness w.r.t. a given safety property. We start with the definition of two families of liveness properties which have been mentioned by Alpern and Schneider [2], where *absolute liveness* was introduced earlier by Sistla [16].

Definition 6. *A temporal property E is a* uniform liveness property *if there exists an x such that $\alpha x \in E$ for all $\alpha \in \Sigma^+$. E is an* absolute liveness property *if $E \neq \varnothing$ and $x \in E \Rightarrow \alpha x \in E$ for all $\alpha \in \Sigma^+$.*

Each absolute liveness property is a uniform liveness property and each uniform liveness property is a liveness property. Moreover:

Proposition 13. *A property is a uniform liveness property if and only if it contains an absolute liveness property.*

Both properties, $\Box \Diamond \varphi$ and $\Diamond \Box \varphi$ are absolute and hence uniform liveness properties. Absolute and uniform liveness properties are closed under union but not under finite intersection.

Definition 7. *Let E be a temporal property.*

1. *E is an* open-uniform liveness property *if there exists a finite run β such that $\alpha\beta\uparrow \subseteq E$ for all $\alpha \in \Sigma^+$.*
2. *E is a* G_δ-uniform liveness property *if it is the intersection of countably many open-uniform liveness properties.*
3. *E is a* G_δ-absolute liveness property *if it is the intersection of countably many absolute and open sets.*

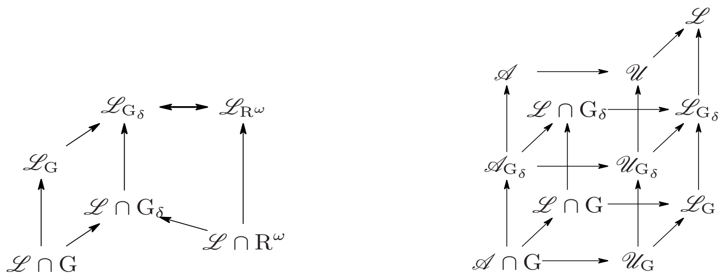
Proposition 14.

1. *Each open-uniform liveness property is a uniform open-liveness property.*
2. *Each G_δ-uniform liveness property is a uniform G_δ-liveness property.*
3. *Each G_δ-absolute liveness property is an absolute G_δ set.*

An example of an absolute open set is $\Diamond \varphi$. The properties $\Box \Diamond \varphi$ and $\Box(\varphi \Rightarrow \Diamond \psi)$ are G_δ-uniform liveness properties; $\Diamond \Box \varphi$ is uniform but not G_δ-uniform. The converse of Prop. 14.1 and 2 does not hold. Consider, for example, the property $E = \bigcap_{k \in \mathbb{N}} (s^k \Rightarrow \Diamond r^k)$ for $s, r \in \Sigma$. E is a uniform open-liveness property, the witness for uniformity being the infinite sequence r^ω. However, it is not an open-uniform liveness property.

Proposition 15.

1. *A property is an open-uniform liveness property if and only if it contains an absolute and open set.*
2. *A property is a G_δ-uniform liveness property if and only if it contains a G_δ-absolute liveness property.*



(a) Constructive liveness (b) Absolute and uniform liveness

Fig. 1. Relationships between various subclasses of liveness: An arrow denotes inclusion. $\mathscr{L}$, G, and R^ω denote the family of liveness properties, open sets, and infinitary recurrence properties respectively. $\mathscr{L}_{\mathscr{F}}$ denotes the family of sets that contain a dense set from the family $\mathscr{F}$. By $\mathscr{A}$, $\mathscr{U}$, $\mathscr{A}_{\mathrm{G}_\delta}$, $\mathscr{U}_{\mathrm{G}}$, and $\mathscr{U}_{\mathrm{G}_\delta}$ we denote the absolute, uniform, G_δ-absolute, open-uniform, and G_δ-uniform liveness properties respectively.

Proposition 16. *The family of* G_δ*-absolute liveness properties is closed under arbitrary intersection.*

Proof. Consider the property

$$\hat{E} = \{x \mid \forall \alpha \exists \beta : \beta\alpha \sqsubseteq x\} = \bigcap_{\alpha \in \Sigma^+} E(\alpha) \text{ where } E(\alpha) = \{x \mid \exists \beta : \beta\alpha \sqsubseteq x\}.$$

$\hat{E}$ is a G_δ-absolute liveness property because each $E(\alpha)$ is an absolute open set. Furthermore, each G_δ-absolute liveness property E contains $\hat{E}$: Let $E = \bigcap_{i \in \mathbb{N}} G_i$ where G_i is an absolute open set. Let $x \in \hat{E}$. Consider a G_i and a $y \in G_i$. Since G_i is absolute and open, there exists $\beta \sqsubseteq y$ such that $\alpha\beta\!\uparrow \subseteq G_i$ for all $\alpha \in \Sigma^+$. Since $x \in \hat{E}$, there is an α' such that $\alpha'\beta \sqsubseteq x$. Hence $x \in G_i$. □

Proposition 17. *The family of* G_δ*-uniform liveness properties is closed under arbitrary union and intersection.*

Proof. Closedness under union is trivial. Closedness under intersection follows from Props. 15.2 and 16. □

Fig. 1.b shows the inclusion of the defined families.

6 Conclusion

For this presentation, we have restricted ourselves to sequential runs. But our definitions and results can be generalized to non-sequential runs. In topological terms, the results can be generalized to any Baire space and, in particular, to the Scott topology of ω-algebraic domains. Since the configurations of an event structure form an ω-algebraic domain, our results immediately carry over to event structures. However, the game-theoretic point of view could allow us to refine fairness in a non-sequential setting. The details remain to be worked out.

Apt, Francez, and Katz [3] proposed that fairness should be machine-closed w.r.t. the safety property of the system. We refined this to exclude some properties that should not be called fairness properties from our point of view. We did not consider their other two criteria: *equivalence robustness* and *liveness enhancement*. Equivalence robustness is an issue when concurrency plays an important role in the modeling of the reactive system. That issue is then best dealt with in the domain of non-sequential runs. Since our results carry over to these domains, equivalence robustness is orthogonal to our definition of fairness. Liveness enhancement refers to the view that every system is equipped with the basic assumption of maximality with respect to every transition. Liveness enhancement means that fairness should be strictly stronger than this basic assumption—at least with respect to some safety property. Liveness enhancement is also orthogonal to our definition and can be additionally used when relevant.

References

1. M. Abadi and L. Lamport. The existence of refinement mappings. *Theoretical Computer Science*, 82:253–284, 1991.
2. B. Alpern and F. B. Schneider. Defining liveness. *IPL*, 21:181–185, Oct. 1985.
3. K. R. Apt, N. Francez, and S. Katz. Appraising fairness in languages for distributed programming. *Distributed Computing*, 2:226–241, 1988.
4. N. Francez. *Fairness*. Springer, 1986.
5. E. Grädel. Positional determinacy of infinite games. In *STACS*, pp. 4–18, 2004.
6. D. Harel and A. Pnueli. On the development of reactive systems. In K. Apt (ed.), *Logics and Models of Concurrent Systems*, pp. 477–498. Springer-Verlag, 1985.
7. E. Kindler. Safety and liveness properties: A survey. *EATCS Bulletin* 53, 1994.
8. M. Z. Kwiatkowska. Survey of fairness notions. *Information and Software Technology*, 31(7):371–386, 1989.
9. M. Z. Kwiatkowska. On topological characterization of behavioural properties. *Topology and Category Theory in Computer Science*, Oxford Univ. Press, 1991.
10. L. Lamport. Proving the correctness of multiprocess programs. *IEEE Transactions on Software Engineering*, SE-3(2):125–143, Mar. 1977.
11. L. Lamport. Formal foundation for specification and verification. *Distributed Systems: Methods and Tools for Specification*, *LNCS* 190. Springer-Verlag, 1985.
12. L. Lamport. Fairness and hyperfairness. *Distr. Computing*, 13(4):239–245, 2000.
13. D. Lehmann, A. Pnueli, and J. Stavi. Impartiality, justice, and fairness: The ethics of concurrent termination. In ICALP, *LNCS* 115, pp. 264–277. Springer, 1981.
14. Z. Manna and A. Pnueli. A hierarchy of temporal properties. In *9th PODC*, pp. 377–408. ACM, 1990.
15. J. C. Oxtoby. *Measure and Category. A Survey of the Analogies between Topological and Measure Spaces*. Springer-Verlag, 1971.
16. A. P. Sistla. On characterization of safety and liveness properties in temporal logic. In *4th PODC*, pp. 39–48. ACM, 1985.
17. H. Völzer, D. Varacca, and E. Kindler. Defining Fairness. SIIM Technical Report SIIM-TR-A-05-18, Universität zu Lübeck, 2005.