

# Proof-relevant Category Theory in Agda

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## Abstract

The generality and pervasiveness of category theory in modern mathematics makes it a frequent and useful target of formalization. It is however quite challenging to formalize, for a variety of reasons. Agda currently (i.e. in 2020) does not have a standard, working formalization of category theory. We document our work on solving this dilemma. The formalization revealed a number of potential design choices, and we present, motivate and explain the ones we picked. In particular, we find that alternative definitions or alternative proofs from those found in standard textbooks can be advantageous, as well as “fit” Agda’s type theory more smoothly. Some definitions regarded as equivalent in standard textbooks turn out to make different “universe level” assumptions, with some being more polymorphic than others. We also pay close attention to engineering issues, in particular so that the library integrates well with Agda’s own standard library, as well as supporting as many of the modes of Agda as possible.

**2012 ACM Subject Classification** General and reference → General literature; General and reference

**Keywords and phrases** Agda, category theory, formal mathematics

## 1 Introduction

There have been many formalizations of category theory [8, 21] in many different proof assistants, over more than 25 years [30, 18, 33, 1, 5, 16, 25, 35, 26, 27, 28, etc.]. All of them embody *choices*; some were forced by the ambient logic of the host system, others were pragmatic decisions, some were philosophical stances, while finally others were simply design decision.

Category theory is often picked as a challenge, as it has shown itself to both be quite amenable to formalization and to involve many non-trivial decisions that can have drastic effects on the usability and effectiveness of the results [16]. With the rapid rise in the use of category theory as a tool in computer science, and with the advent of *applied* category theory, having a stable formalization in the *standard library* of one’s favourite proof assistant becomes necessary.

Our journey started as both authors were trying to keep the “old” category theory library for Agda [25] alive. Unfortunately, as Agda [4] evolved, some of the features used in that library were no longer well-supported, and eventually the library simply stopped working. As it became clear that simply continuing to patch that library, as had worked for a couple of releases, was no longer viable, a new version was in order.

This gave us the opportunity to revisit various design decisions of the earlier implementation — which we will document. We also wanted to preserve as many of the, frankly, rather brilliant design decisions as possible and while also using language features introduced in Agda 2.6.0.1. This new version is then partly a “port” of the previous one to current versions of Agda, but also heavily refactored, including some large changes in design.

Our principal theoretical contribution is to show that *proof-relevant category theory* works just as well as various other “flavours” of category theory. Our main engineering contribution

is a coherent set of design decisions for a widely reusable library of category theory, freely available<sup>1</sup>.

This paper is structured as follows. In Section 2, we discuss our global design choices. We discuss the rationale behind non-strictness, proof-relevance, hom-setoids, universe polymorphism, (not) requiring extra axioms and concepts as record types. In Section 3, we give examples on how proof-relevance drives us to find concepts in an alternative way. In Section 4, we discuss other details in our design decisions. In Section 5, we compare category theory libraries in other systems. Finally, we conclude in Section 6.

For reasons of space, we have to make some non-trivial assumptions of our readership, namely that they are familiar with:

1. category theory,
2. dependent type theory,
3. formalization, and
4. proof assistants (e.g. familiarity with Agda and a passing knowledge of other systems).

## 2 Design Choices

Choices arise from both the system and its logic, as well as from the domain itself.

### 2.1 Fitting with Agda

The previous formalization [25] was done in a much older Agda, with a seriously underdeveloped standard library. To better fit with *modern* Agda, we choose to:

1. use dependent types,
2. be constructive,
3. re-use as much of the standard library [14] as possible,
4. use its naming convention whenever meaningful,
5. use the `variable` generalization feature for levels and categories,
6. try to fit with as many modes of Agda as possible.

The first two requirements are “obvious”, as choosing otherwise would create a philosophical clash between the system and one of its libraries. The next two are just good software engineering, while the fifth is mere convenience. Note that re-using the standard library pushes us towards setoids (more on that later) as its formalization of algebra uses them extensively.

The last requirement is more subtle: we want to allow others to assume more axioms if they wish, and still be able to use our library. This means that we need to avoid using some features that are incompatible with certain axioms. For example, when added to Martin-Löf Type Theory (MLTT) [23], axiom K [29], equivalent to *Uniqueness of Identity Proofs* (UIP), creates a propositionally extensional type theory incompatible with *univalence* [31]. Thus there are options such as `--without-K` [11] to access the intensional type theory MLTT, and conversely `--with-K` to turn axiom K on. Separately, there is cubical type theory (`--cubical`) [32]. Intensional type theory is compatible with the latter two, and thus if we build our library using `--without-K`, it can be re-used. This further implies that we essentially have to avoid propositional equality as much as possible, as MLTT gives us very few tools to work with it. In addition, we turn on the `--safe` option to disallow the use of any features that might threaten logical consistency.

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<sup>1</sup> at <https://github.com/agda/agda-categories>

## 2.2 Which Category Theory?

Category theory is often presented as a single theory, there are in fact a wealth of flavours: set-theoretic, where a category has a single hom-set equipped with source and target maps; ETCS-style [20], where there are no objects at all; dependently-typed, where hom-“sets” are parametrized by two objects; proof-irrelevant, where the associativity and identity axioms are considered to be unique [5, 33, 16, 25]; setoid-based, where each category relies on a local notion of equivalence of hom-sets rather than relying on a *global* equality relation [35, 25]. There are also questions of being strict or weak, whether to do 1-categories,  $n$ -categories or even  $\infty$ -categories. What to choose?

Standard textbooks often define a category as follows:

► **Definition 1.** A category  $\mathcal{C}$  consists of the following data:

1. a collection of objects,  $\mathcal{C}_0$ ,
2. a collection of morphisms,  $\mathcal{C}_1$ , mapping from one object  $A$  to another  $B$ , denoted as  $A \Rightarrow B$ ,
3. for each object  $A$ , we have an identity morphism  $1_A : A \Rightarrow A$ , and
4. morphism composition  $\circ$  composing two morphisms  $f : B \Rightarrow C$  and  $g : A \Rightarrow B$  into another morphism  $f \circ g : A \Rightarrow C$ .

These must satisfy the following axioms:

1. identity: for any morphism  $f : A \Rightarrow B$ , we have  $f \circ 1_A = f = 1_B \circ f$ , and
2. associativity: for any three morphisms  $f, g$  and  $h$ , we have  $(f \circ g) \circ h = f \circ (g \circ h)$ .

Embedded in the above definition are a variety of decisions, and we will use these as a running example to explain our design decisions.

### 2.2.1 Collections and Strictness

The first item to notice is the use of *collection* rather than (say) *set* or *type*. Textbooks tend to do this to side-step “size” issues, and then define various kinds of categories depending on whether each of the collections (objects, all morphisms, all morphisms given a pair of objects) is “small”, i.e. a set. This matters because a number of constructions in category theory produce large results. But even these textbooks tend to implicitly assume that collections are somewhat still set-like, in that *equality* is taken for granted, i.e. that it always makes sense to ask whether two items from a collection are equal. Not only that it always makes sense, but that the underlying meta-theory will always answer such queries in finite time<sup>2</sup>.

The *Principle of Isomorphism* [22] already tells us that we should not assume that we have any relation on objects other than the one given by categorical principles (isomorphism); a related *Principle of Equivalence* [6] can be stated formally in the context of homotopy type theory (HoTT) [31]. That we normally do not have, and should not assume, such a relation have motivated some to create the concept of a *strict category*, where we have given ourselves the ability to compare objects for equality. Classically, sets have equality defined as a total relation, so that this comes “for free”. In other words, given two elements  $x, y$  of a set  $S$ , in set theory it always makes sense to “ask” the question  $x = y$ , and this has a *boolean* answer. This is one reason why it took a while for the Principle of Equivalence to emerge as meaningful. As such global extensionality is hard to mechanize (impossible without assuming axioms

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<sup>2</sup> That we should not ask whether two objects are equal is an issue well described at the *Principle of Equivalence* page of the nLab. <https://ncatlab.org/nlab/show/principle+of+equivalence>

that are non-constructive), it is simplest to forgo having an equality relation on objects at all.

Thus it makes sense to refine *collections of objects* to *types*, with no further assumptions or requirements. We do know that in MLTT types are well modeled by  $\infty$ -groupoids [17, 34] — so wouldn't this higher structure be a problem? No! This is because we never look at it, i.e. we never look at the identity type (or their identity types) of objects.

The collection of morphisms is trickier, and splits into two:

1. Is there a single collection of morphisms?
2. What about equality of morphisms?

The first item will be treated here, the second in the next subsection.

If we try to put all the morphisms of certain categories together in a single collection, size issues arise, but there is also another issue: if we consider composition as a function of pairs of morphisms, then this function is partial. Luckily, our dependent type theory allows one to side-step both issues at the same time: rather than a single collection of morphisms, we have a (dependently-typed) family of morphisms, one for each pair of objects. In category theory, one rarely considers the “complete collection” of all morphisms. This solves the composition problem too, as we can only compose morphisms that have the right type, leading us to the following (partial) definition:

```
record Category : Set where
  field
    Obj : Set
     $\Rightarrow$  : (A B : Obj) → Set
     $\circ$  :  $\forall \{A B C\} \rightarrow B \Rightarrow C \rightarrow A \Rightarrow B \rightarrow A \Rightarrow C$ 
```

## 2.2.2 Proof-relevant Setoids

In the definition of category (Definition 1), a notion is introduced quite implicitly: equality. The axioms use equality, blithely assuming that the meta-theory defines it. In MLTT, contrarily, which equality we use matters. Usually, there are three options: local equality (setoids), propositional equality in intensional type theory ( $\equiv$ ), and equality in **hSets**, in the sense of HoTT or cubical type theory [12].

Propositional equality does not work very well in MLTT without further axioms which are necessary to be able to compare functions (such as univalence or function extensionality). Many categories have functions as morphisms. In the third case, assuming that morphisms in a category are in fact **hSets** seems to be a throwback to set-centric foundations. Assuming that morphisms are set-truncated does, however, seem to work well in HoTT, as it gives that equality proofs are unique (UIP). However, this cannot be done inside MLTT without assuming an extra axiom, which we do not want to do.

Earlier formalizations of category theory in type theory already used setoids [3, 18, 25, 35], which associate an equivalence relation to each type. This generalizes “hom-sets” to proof-relevant “hom-setoids”, i.e. the definition of category gets augmented as follows:

```
 $\approx$  :  $\forall \{A B\} \rightarrow (f g : A \Rightarrow B) \rightarrow \text{Set}$ 
equiv :  $\forall \{A B\} \rightarrow \text{IsEquivalence } (\approx \{A\} \{B\})$ 
```

**IsEquivalence** is a predicate that express that  $\approx$  is an equivalence relation. Furthermore, composition must respect this equivalence relation, which we can express as<sup>3</sup>:

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<sup>3</sup> We use variable generalization to leave implicit variables out and let Agda infer them.

$\circ\text{-resp-}\approx : f \approx h \rightarrow g \approx i \rightarrow f \circ g \approx h \circ i$

Note that  $\approx$  can be specialized to  $\equiv$  to work in the other settings.

Proof-relevance is a significant difference between this library and the previous one [25], which relied heavily on irrelevant arguments [2]. In particular, all of the proof obligations (for example left and right identities, and associativity in the case of a category) were marked irrelevant, making these proofs “unique” by fiat. Thus two categories that differed only in their proofs were automatically regarded as (definitionally) equal. Ignoring the details of proofs is convenient — but unfortunately irrelevant arguments are not part of MLTT. Worse yet, they are not a stable, well-maintained feature in Agda.

The use of proof-relevant hom-setoids makes our definition of category more general than most textbook definitions. In particular, the hom-setoids do not necessarily behave like sets, in contrast to textbooks and other libraries based on HoTT [5, 16]. There, hom-sets are truly sets by requiring an additional axiom that forces the contractibility of equality proofs. Of course, such proofs are never encountered in textbooks either!

### 2.2.3 Explicit Universe Level

In Agda, users are exposed to the issue of explicit handling of universe levels (i.e. of type `Level`). Some find it cumbersome, but we have found it to be quite useful. To help with reuse, we choose to make our definitions universe-polymorphic by parameterizing them by `Levels`. For example, a `Category` is refined as follows:

```
record Category (o l e : Level) : Set (suc (o ⊔ l ⊔ e)) where
  field
    Obj : Set o
    _⇒_ : (A B : Obj) → Set l
    _≈_ : ∀ {A B} → (f g : A ⇒ B) → Set e
  -- other fields omitted
```

Since the definition of `Category` contains three `Sets` representing objects, morphisms and the equivalence relations respectively, the definition must be indexed by three `Levels` and thus live at least one level above their supremum.

One significant advantage of such a level-parametric definition is that it allows us to easily formalize concepts such as the category of categories and the category of setoids directly. We do not have to duplicate definitions, nor do we have to sprinkle various size constraints about (such as a category being “locally small”) to avoid set-theoretic troubles.

With explicit `Levels`, we do observe new phenomena. In set-based category theory, one might be tempted to talk about the (large) category of all sets or all setoids. In Agda, we can only talk about the category of all `Setoids` with particular `Levels`:

```
Setoids : ∀ c l → Category (suc (c ⊔ l)) (c ⊔ l) (c ⊔ l)
Setoids c l = record
  { Obj      = Setoid c l -- ... other fields omitted.
  }
```

Here `c` and `l` are the `Levels` of the carrier and the equivalence of a `Setoid c l`, respectively. We can clearly see the ensuing size issue. The definition *must* be indexed by `Levels`, as there is no term *in* the type theory in which all `Setoids` (for example) exist. The set of types `Set l` is somewhat analogous to a *Grothendieck universe*, as it is closed under similar operations,

but not unrestricted unions, where one must then move to a larger universe. `Set (suc l)` is indeed sometimes called a Russell-style universe.

However explicit `Levels` combined with non-cumulative universes lead to other issues. In a type theory with cumulative universes, a type in one universe automatically inhabits all larger universes. In Agda, one must explicitly lift terms to larger levels, which adds a certain amount of “noise” to some code. For example, consider two categories of `Setoids`, `Setoids 0 1` and `Setoids 1 1`, differing only in the first index. With cumulative universes, this would allow `Setoids 0 1` to be embedded in `Setoids 1 1` by a straightforward functor. In Agda, explicit calls to `lift` must be inserted.

## 2.2.4 Duality

In category theory, duality is omnipresent. However, in type theory and in formalized mathematics, subtleties arise. Some are due to proof relevance, while others are usability issues, which we discuss here.

### 2.2.4.1 Additional Axioms for Duality

In category theory, there is a very precise sense in which, if a theorem holds, then its dual statement also holds. Thus, in theory, we obtain two theorems by proving one. This is the *Principle of Duality* [8], which we would like to exploit.

But first, we need to make sure that the most basic duality, that of forming the opposite category, should be involutive. We can easily prove that the double-opposite of a category  $C$  is equivalent to  $C$ . This equivalence is true *definitionally* with proof-irrelevant definitions. Can we recover this here as well? Yes – we can follow [16] and require two (symmetric) proofs of associativity of composition in the definition of a `Category`:

```
assoc      : (h ∘ g) ∘ f ≈ h ∘ (g ∘ f)
sym-assoc : h ∘ (g ∘ f) ≈ (h ∘ g) ∘ f
```

Specifically, with `sym-assoc`, we can define opposite category as follows:

```
op : Category → ℓ → e
op = record
  { assoc      = sym-assoc
  ; sym-assoc = assoc -- other fields omitted
  }
```

Otherwise, without `sym-assoc`, we would have to use the symmetry of `_≈_`:

```
assoc = sym assoc
```

But now, applying duality twice gives `sym (sym assoc)` for the associativity proof, which is propositionally but not definitionally equal to `assoc`. This makes the properties of an opposite category less useful than ones of the original one.

Another axiom which is convenient to add is

```
identity2 : id ∘ id ≈ id
```

This axiom can be proved by taking `f` as `id` in either the left identity or right identity axiom:

```
identityl : id ∘ f ≈ f
identityr : f ∘ id ≈ f
```

We add this additional axiom for the following reasons:

1. When proving  $\text{id} \circ \text{id} \approx \text{id}$ , we need to choose between  $\text{identity}^l$  and  $\text{identity}^r$ , while there is no particular reason to prefer one to another. Adding this axiom neutralizes the need to make this choice.
2. In the implementation, we sometimes rely on constant functors, which ignore the domain categories and constantly return fixed objects in the codomain categories and their identity morphisms. Since the domain categories are completely ignored, these functors are intuitively “the same” as their duals.  $\text{identity}^2$  allows constant functors to be definitionally equal to their duals even with proof-relevance.

#### 2.2.4.2 Independent Definitions of Dual Concepts

In other libraries [30, 33, 5, 16], it is typical to define one concept and use duality to obtain the opposite one. For example, we could define the initial object of  $\mathcal{C}$ , `Initial C` as usual, and then define the terminal object by taking the opposite, as follows:

```
Terminal : ∀ {o l e} (C : Category o l e) → Set _
Terminal C = Initial (Category.op C)
```

However, we do not take this approach. Instead, we always define concepts explicitly in terms of data and axioms and define conversions between duals in modules of the form `*.Duality`. While this might appear redundant, there are also advantages:

1. when constructing or using the concepts, the names of the fields are more familiar;
2. redundancy helps maintain the Principle of Duality.

It is worthwhile expanding on the second point: like with `sym-assoc`, we want duality to be a definitional involution for a number of concepts. We were able to identify a number of concepts which require additional axioms to achieve this goal, which we detail next.

#### 2.2.4.3 Duality-Completeness of Axioms

The involution of duality turns out to be a very general design principle. We sometimes can obtain it for free, e.g. `Functor` and `Adjoint`. Other times we need to supply a symmetric version of an axiom. For example, `Category`, `NaturalTransformation`, `DinaturalTransformation` and `Monad` all need the addition of extra axioms. As a rule of thumb, if a conversion to the dual concept requires any equational reasoning, even completely trivial (such as the use of `sym` for `assoc`), then we need to add that equation as an axiom. In other words, our axioms should either be self-dual, or come in dual pairs (quite reminiscent of work on reversible computation [10] where the same property is desirable; amusingly, while it is the second author who has worked in reversible computation, it is the first author who discovered the application to category theory). We have taken care to check that our definitions follow this principle.

### 2.3 Encodings

Another important design decision is how to encode each definition. Generally, two different styles are used: records [35, 16] or nested  $\Sigma$  types [5, 33]. In the latter style, developers typically need to write a certain amount of boilerplate accessor code. In Agda it is much more natural to use record definitions:

1. It aligns very well with the design principle of the standard library,
2. Records allow various *syntactic sugar*, as well as having good IDE (via Emacs) support,

3. Most importantly records also behave as modules. That is, we can export symbols to the current context from a record when it is unambiguous to do so.

The record module feature enables some structural benefits as well. Consider the following definition of a `Monad` over a category:

```
record Monad {o ℓ e} (C : Category o ℓ e) : Set (o ⊔ ℓ ⊔ e) where
  field
    F : Endofunctor C
    η : NaturalTransformation idF F
    μ : NaturalTransformation (F ∘ F) F
  -- ... axioms are omitted
```

We often need to refer to components of the `Functor F` or the `NaturalTransformations`  $\eta$  or  $\mu$  when working with a `Monad`. By adding the following module definitions inside the `Monad` record, we can use dot accessors to access deeper fields.

```
module F = Functor F
module η = NaturalTransformation η
module μ = NaturalTransformation μ
```

For example, if we have two `Monads` `M` and `N` in scope, we can declare `module M = Monad M` and `module N = Monad N`, and get the following convenient nested dot accessors:

```
M.F.F0 -- the mapping of objects of F of M
N.F.F1 -- the mapping of morphisms of F of N
M.μ.η X -- the component of the NaturalTransformation μ of M at object X
N.η.commute f -- the naturality square of the NaturalTransformation η of N
               -- at morphism f
```

The original syntax involves many applications, so the module syntax is significantly more convenient:

```
Functor.F0 (Monad.F M)
Functor.F1 (Monad.F N)
NaturalTransformation.η (Monad.μ M) X
NaturalTransformation.commute (Monad.η N) f
```

We use this style throughout the code base. Compared to spending effort on renaming symbols when opening modules, we find that this style is more elegant and more readable.

### 3 Choosing Definitions

While implementing the library, we noticed several times that “standard” definitions needed to be adjusted, for technical reasons. Certain direct translations of concepts from classical category theory are not well-typed! Proof-relevance also forces us to pay close attention to the axioms embedded in each concept, to obtain more definitional equalities, rather than relying on extensional behavior for “sameness”. The resulting formalization is more robust, and it also eases type checking.

Various categorical concepts are well-known to have multiple, equivalent definitions. We have found that, although classically equivalent, some turn out to be technically superior for our formalization. We are sometimes even forced to introduce new ones. Here we discuss the choices we made when defining concepts related to closed monoidal categories in detail, focusing on the underlying rationale.



### 3.1 Adjoint Functors

Adjoint functors are frequently regarded as one of the most fundamental concept in category theory, and play a critical part in the definition of closed monoidal categories. The following two definitions of adjoint functors are equivalent in classical category theory.

► **Definition 2.** *Functors  $F : \mathcal{C} \Rightarrow \mathcal{D}$  and  $G : \mathcal{D} \Rightarrow \mathcal{C}$  are adjoint,  $F \dashv G$ , if there is a natural isomorphism  $\text{Hom}(FX, Y) \simeq \text{Hom}(X, GY)$  in  $\mathcal{X}$  and  $\mathcal{Y}$ .*

► **Definition 3.** *Functors  $F : \mathcal{C} \Rightarrow \mathcal{D}$  and  $G : \mathcal{D} \Rightarrow \mathcal{C}$  are adjoint,  $F \dashv G$ , if there exist two natural transformation, unit  $\eta : 1_{\mathcal{C}} \Rightarrow GF$  and counit  $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$ , so that the triangle identities below hold:*

1.  $\epsilon F \circ F \eta = 1_F$
2.  $G \epsilon \circ \eta G = 1_G$

These two definitions are classically equivalent. Definition 2 is typically very easy to use in classical category theory, as it is about *hom-sets*, and so partly set-theoretic in its formulation. However, this definition is not very natural in Agda, especially in the presence of non-cumulative universes and level-polymorphic morphisms (Section 2.2.3), so that the morphisms of  $\mathcal{C}$  and  $\mathcal{D}$  do not always live in the same universe level. Thus  $\text{Hom}(FX, Y) \simeq \text{Hom}(X, GY)$  is not well-typed as is. Instead,  $\text{Hom}(FX, Y)$  and  $\text{Hom}(X, GY)$  need to be precomposed by lifting functors, which lift both hom-setoids to the universe at their supremum level. One might think that this technicality is classically not present – but that is because many textbooks make the blanket assumption that all their categories are locally small. This “technical noise” added here is related to lifting this restriction, but set theory has no means to express size (as in set, proper class, superclass, etc) polymorphism. However, such coercions are neither intuitive nor easy to work with.

Definition 3, on the other hand, has no such problem. Both natural transformations and triangle identities involve no explicit universe level management. For this reason, we choose Definition 3 as our primary definition of adjoint functors and have Definition 2 as a theorem. While both are logically equivalent, the added polymorphism of the unit-counit definition makes it more suitable when working in type theory.

### 3.2 Closed Monoidal Category

Intuitively, a closed monoidal category is a category possessing both a closed and a monoidal structure, in a compatible way. In the literature, we can find various definitions of a closed monoidal category:

1. (a monoidal category with an added closed structure): given a monoidal category (with bifunctor  $\otimes$ ), there is also a family of functors  $[X, -]$  for each object  $X$ , such that  $- \otimes X \dashv [X, -]$ . The closed bifunctor  $[-, -]$  is then induced uniquely up to natural isomorphism.
2. (a closed category with an added monoidal structure): given a closed category with bifunctor  $[-, -]$ , it is additionally equipped with a family of functors  $- \otimes X$  for each object  $X$ , such that  $- \otimes X \dashv [X, -]$ . The monoidal bifunctor  $\otimes$  is then induced uniquely up to natural isomorphism.
3. (via a natural isomorphism of hom-sets): given a category, for each object  $X$ , there are two families of functors  $- \otimes X$  and  $[X, -]$ , such that the isomorphism  $\text{Hom}(Y \otimes X, Z) \simeq \text{Hom}(Y, [X, Z])$  is natural in  $X, Y$  and  $Z$ . Both bifunctors  $\otimes$  and  $[-, -]$  are then induced uniquely up to natural isomorphism.

Note that the third definition above is not biased towards either the closed or monoidal structure. All three can be shown equivalent (classically). But in the proof-relevant setting, problems arise. One that all three share is that they all induce at least one bifunctor from a *family* of functors. For example, in the first definition, the closed bifunctor  $[-, -]$  is the result of a theorem; two different instances of  $[-, -]$  (which might potentially differ in their proofs) can only be related by a natural isomorphism, which is often too weak. In other words, we want *both* bifunctors  $\otimes$  and  $[-, -]$  to be part of the definition. None of the three definitions above satisfy this requirement. We thus arrive at the following definition, which is the one we use:

► **Definition 4.** *A closed monoidal category is a monoidal category with two bifunctors  $\otimes$  and  $[-, -]$ , so that*

1.  $- \otimes X \dashv [X, -]$  for each object  $X$ , and
2. for a morphism  $f : X \Rightarrow Y$ , the induced natural transformations  $\alpha_f : - \otimes X \Rightarrow - \otimes Y$  and  $\beta_f : [Y, -] \Rightarrow [X, -]$  form a mate (or a conjugate in the sense of [21]) for the two pairs of adjunctions,  $- \otimes X \dashv [X, -]$  and  $- \otimes Y \dashv [Y, -]$ , formed by previous constraint.

This definition is *better*, in the sense that it is 1) unbiased, 2) incremental (it simply adds more constraints on both bifunctors). Further note that both bifunctors are given as part of the data, rather than derived, which allows us to consistently refer to both uniquely. The following theorem strengthens our confidence:

► **Theorem 5.** *A closed monoidal category according to Definition 4 is a closed category.*

In addition, the closed bifunctor  $[-, -]$  from the closed category in this theorem is precisely the same one given in Definition 4. This allows closed monoidal categories to inherit all properties of closed categories as they are talking about the same  $[-, -]$ .

A potential downside of this definition is that it depends on *mates* which are not present in previous definitions. Though this seems to add complexity, we argue that the benefit is worth the effort. We now discuss mates in order to justify that this new definition is equalent to the previous three.

### 3.3 Mate

Mates between adjunctions are typically defined by two natural isomorphisms between hom-sets as follows:

► **Definition 6.** *For functors  $F, F' : \mathcal{C} \Rightarrow \mathcal{D}$  and  $G, G' : \mathcal{D} \Rightarrow \mathcal{C}$ , two natural transformations  $\alpha : F \Rightarrow F'$  and  $\beta : G' \Rightarrow G$  form a mate for two pairs of adjunctions  $F \dashv G$  and  $F' \dashv G'$ , if the following diagram commutes:*

$$\begin{array}{ccc} \text{Hom}(F'X, Y) & \xrightarrow{\sim} & \text{Hom}(X, G'Y) \\ \text{Hom}(\alpha_X, Y) \downarrow & & \downarrow \text{Hom}(X, \beta_Y) \\ \text{Hom}(FX, Y) & \xrightarrow{\sim} & \text{Hom}(X, GY) \end{array}$$

This definition is not very convenient because it is defined via hom-set(oid)s. The situation described in Sections 2.2.3 and 3.1 recurs, and the two natural isomorphisms need to be composed by lifting functors in order to be well-typed. As before, there is another definition which does not depend on hom-sets.

► **Definition 7.** For functors  $F, F' : \mathcal{C} \Rightarrow \mathcal{D}$  and  $G, G' : \mathcal{D} \Rightarrow \mathcal{C}$ , two natural transformation  $\alpha : F \Rightarrow F'$  and  $\beta : G' \Rightarrow G$  form a mate for two pairs of adjunctions  $(\eta, \epsilon) : F \dashv G$  and  $(\eta', \epsilon') : F' \dashv G'$ , if the following two diagrams commute:

$$\begin{array}{ccc} 1_{\mathcal{C}} & \xrightarrow{\eta} & GF \\ \eta' \downarrow & & \downarrow G\alpha \\ G'F' & \xrightarrow{\beta F'} & GF' \end{array} \quad \begin{array}{ccc} FG' & \xrightarrow{\alpha G'} & F'G' \\ F\beta \downarrow & & \downarrow \epsilon' \\ FG & \xrightarrow{\epsilon} & 1_{\mathcal{D}} \end{array}$$

Both definitions are equivalent [21], but Definition 7 is simpler to work with in our setting.

From here, it is straightforward to see that our definition of closed monoidal category is equivalent to the previous ones. We need to show Definition 4 is equivalent to requiring  $\text{Hom}(Y \otimes X, Z) \simeq \text{Hom}(Y, [X, Z])$  is natural in  $X, Y$ , and  $Z$ . Since we require  $- \otimes X \dashv [X, -]$  for any object  $X$ , this requirement is equivalent to a natural isomorphism in  $Y$  and  $Z$ . Moreover, the naturality of  $X$  is ensured by the mate condition, due to Definition 6.

### 3.4 Morphism Equality over Natural Isomorphism

Our experience with closed monoidal categories can be generalized into a guideline. We find that in general, characterization in morphism equality (such as triangle identities in Definition 3) is better than one in natural isomorphism (such as the natural isomorphism between hom-sets in Definition 2). The latter can be proved as a theorem.

We observe that natural isomorphisms tend to be more difficult to type-check, for a variety of reasons. For example, in the definition of monoidal categories, classically the associativity of  $\otimes$  requires the natural isomorphism  $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$  to be natural in all  $X, Y$  and  $Z$ . We cannot express this natural isomorphism literally:  $(X \otimes Y) \otimes Z$  has type  $\text{Functor } ((\mathcal{C} \times \mathcal{C}) \times \mathcal{C}) \mathcal{C}$ , while  $X \otimes (Y \otimes Z)$  has type  $\text{Functor } (\mathcal{C} \times (\mathcal{C} \times \mathcal{C})) \mathcal{C}$ . As the domain categories are not definitionally equal, there cannot be a natural isomorphism between them. The natural isomorphism must involve an associator functor from  $(\mathcal{C} \times \mathcal{C}) \times \mathcal{C}$  to  $\mathcal{C} \times (\mathcal{C} \times \mathcal{C})$  or vice versa. This is not mere pedantry: we know that “one level up”, this is an unavoidable issue. In other words, some issues that show up as type-checking problems of 1-category theory are actually previews of 2-categorical subtleties “peeking through”, that can be ignored in informal paper-math. Our definition characterizes this associativity condition in low level using the following two steps:

1. require an isomorphism between  $(X \otimes Y) \otimes Z$  and  $X \otimes (Y \otimes Z)$ , and
2. two naturality squares to complement the missing axioms so that the isomorphisms are natural.

This leads to definitions that are easier to use, and the required natural isomorphism becomes a theorem.

### 3.5 Finite Categories

Category theorists have developed terminology to talk about the cardinality (size) of components of a category. In Section 2.2.3, we use universe levels to make size issue explicit. For small categories, since we know both objects and morphisms “fit” in sets, we can use more set-theoretic language. Among these, “finiteness” is of particular importance, especially in its guise as enabling **enumeration**.

However when we attempt to define finite categories, a problem arises: MLTT does not give us primitives to count the elements of a type. For example [30] and [36] both implement finiteness as a predication requiring an isomorphism between a type and finite natural

numbers. We could also do this, but that approach has the drawback of (implicitly) putting a canonical order on elements, which is undesirable<sup>4</sup>. It also forces a notion of equivalence on objects, which does not always exist for any `Set`. We do not want finiteness to force us into strictness. We instead base our definition on adjoint equivalence:

► **Definition 8.** *Two categories  $\mathcal{C}$  and  $\mathcal{D}$  are adjoint equivalent if there are two functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  so that they form a pair of adjoint functors  $F \dashv G$  and their unit and counit natural transformations are isomorphisms.*

Then a finite category can be defined as follows:

► **Definition 9.** *A category  $\mathcal{C}$  is finite, if it is adjoint equivalent to a finite diagram.*

► **Definition 10.** *Given a function  $|a, b| : \mathbb{N}$  for  $a, b : \mathbf{Fin}\ x$ , and a compatible notion of composition of  $|a, b|$  functions, a finite diagram is a category with*

1.  *$\mathbf{Fin}\ x$  as objects, and*
2.  *$\mathbf{Fin}\ |a, b|$  as morphisms*

Here `Fin n` is the Agda type representing the discrete finite set of natural numbers  $[0, \dots, n - 1]$ .

For example, as adjoint equivalence respects equivalence, a contractible Groupoid is always finite. Note that this is still somewhat problematic: coming up with such an adjoint equivalence can be fiendishly difficult and, in some cases, may require the Axiom of Choice.

Nevertheless, the above definition lets us prove:

► **Theorem 11.** *A category with all finite products and equalizers has all finite limits.*

The proof is constructive, i.e. an algorithm that builds a finite limit from products and equalizers given any finite diagram. This seems to justify the adequacy of finite diagrams.

We can also verify that a finite category as per Definition 9 can serve as an index category for a finite limit in the general case. We have proven:

► **Theorem 12.** *Limits respect adjoint equivalence, i.e. if  $\mathcal{J}$  is adjoint equivalent to  $\mathcal{J}'$  with  $F : \mathcal{J}' \rightarrow \mathcal{J}$ , then for a functor  $L : \mathcal{J} \rightarrow \mathcal{C}$ ,  $\varprojlim L = \varprojlim (L \circ F)$ .*

Thus Definition 9 is an adequate definition of finite categories. Definition 9 does not involve equivalence of objects. How much the choice of adjoint equivalence reveals about the inner structure of a category still remains to be investigated.

## 4 Further Design Decisions

While the previous section detailed decisions that lie in the intersection of category theory and formalization in type theory, here we document issues closer to software engineering.

### 4.1 Module Structure

The previous library favoured a flat module structure, we use a deeper hierarchy, and thus fewer top-level modules. We use the following principles as a guide:

1. Important concepts have their top level modules. `Object`, `Morphism`, `Diagram`, `Functor`, `NaturalTransformation`, `Kan`, `Monad`, `Yoneda` and `Adjoint` belong to this category.

---

<sup>4</sup> Propositional truncation could be used, if we had it, to get around this problem.

2. Different flavours of category theory: `Category`, `Enriched`, `Bicategory` and `Minus2-Category` contain the definitions and properties of categories, enriched categories, bicategories and -2-categories, respectively. `Pseudofunctor` contains the instances of pseudofunctors.

Submodules also follow conventions so that definitions and properties are easier to locate.

1. `*.Instance` contains instances of some concept. For example, the category of all setoids is defined in `Category.Instance`. Generally, only instances that are re-used in the library itself (making them “special”) are defined.
2. `*.Construction` contains instances induced from some input. The difference with instances is that `*.Construction` takes parameters beyond just `Levels`. For example, the Kleisli category of a monad is defined in `Category.Construction`.
3. `*.Properties` contains properties of the corresponding concepts.
4. `*.Duality` contains conversions to dual concepts (see Section 2.2.4).

This module structure was inspired by a recent restructuring of Agda’s standard library along similar lines, which we believe helps users find what they need faster.

## 4.2 Predicates versus Structures

Like [16] before us, we are faced with the problem of how to organize a concept. There are two typical choices: as predicates and as structures. A predicate expresses an “is-a” relation, while a structure expresses a “has-a” relation. For example, we define monoidal categories as a predicate:

```
record Monoidal {o l e} (C : Category o l e) : Set (o ⊔ l ⊔ e) where
  -- omit the fields
```

It asserts that `C` is a monoidal category. Alternatively, we could encapsulate `C` inside the record, yielding a structure:

```
record Monoidal' o l e : Set (suc (o ⊔ l ⊔ e)) where
  field
    C : Category o l e -- omit other fields
```

We choose one style by judging whether a concept is closer to a predicate or a structure. In the case of monoidal categories, we find the predicate style is more suitable. In the case of `Groupoid`, for example, we find that a `Groupoid` is closer to a structure, which leads us to the current definition:

```
record Groupoid (o l e : Level) : Set (suc (o ⊔ l ⊔ e)) where
  field
    category    : Category o l e
    isGroupoid  : IsGroupoid category
```

where `IsGroupoid` is the corresponding predicate version of the definition.

This is related to (un)bundling of definitions, which is discussed in [7] along with tools for moving between the two equivalent styles of definitions.

## 5 Related Work

Table 1 gives a list of formalized libraries of category theory. For each we specify the proof assistant, the foundation, lines of code and whether it uses hom-setoids and is proof-relevant. In Tables 2 and 3, we compare a list of features implement by these libraries.

libraries	proof assistants	foundation	hom-setoids	proof-relevant	LoC <sup>†</sup>
Ours	Agda 2.6	MLTT	✓	✓	16945
[25]	Agda 2.5.2	MLTT + K	✓	✗	11770
[30]	Coq 8.5	CIC	✗	✗	14711
[35]	Coq 8.10.2	CIC	✓	✓	22695
[18]	Coq 8.11.0	CIC	✓	✓	7879
[33, 5]	Coq 8.11.0	HoTT	✗	✗	95460
[16]	Hoq 8.10 <sup>††</sup>	HoTT with HIT	✗	✗	10601
[1]	Lean	CIC	✗	✗	6270
[26, 27, 28]	Isabelle	HOL	✗	✗	84464

<sup>†</sup> The lines of code are counted by `cloc` of Al Danial and code in Isabelle is counted by `wc`, because `cloc` does not recognize Isabelle. The lines of code might include documentation text. Only folders directly related to category theory are counted.

<sup>††</sup> Hoq is a modified version of Coq which implements a part of HoTT.

■ **Table 1** Tools and key characteristics of various libraries

We have completely ported all definitions and theorems from [25], except those requiring UIP or axiom K. We also extend it with many new definitions and new theorems, as shown in Tables 2 and 3 (roughly 50% more material). Moreover, since we turn on the `--safe` flag, we do not have `postulates` in our code base. This helps us to avoid inheriting a postulated unsound axiom [2], which would, for example, let us incorrectly mix relevance and irrelevance, including “recovering” a relevant value from an irrelevant one.

From Table 1, we can see that much effort has been spent in Coq (or its Hoq dialect) on category theory. The reason for the multiple efforts can be seen when comparing the versions, and foundations used. Some believe that Coq’s tactics and hint databases provide a significant boost in the productivity of formalizations. We suspect that this may be somewhat illusory, as the explicit equational proofs in  $n$ -category theory (which can be automated via tactics) tend to turn up as data in  $n + 1$ -category theory, and then no longer avoidable. [35] stands out by its use of other Coq mechanisms, such as type classes, rather than record or  $\Sigma$  types, for structuring of the development.

Like us, [35, 18] use hom-setoids and proof-relevance. Unfortunately, [35] has not been described in a paper, so we do not know what lessons the author learned from their experience. [18] was a smaller scale but pioneering effort that taught us the basics of formalizing category theory in MLTT, but not the kinds of design decisions we faced here.

Compared to other developments in Coq, [16, 33] are special: they build category theory in HoTT. [33] focuses more on fundamental constructions. It does not use any feature beyond the primitive type constructors like  $\Sigma$  and  $\Pi$ . By contrast, [16] experiments with the use of various HoTT ideas, and therefore is more permissive. It uses extended features like records and higher inductive types (HIT). Working in HoTT has some advantages. First, if one understands hom-sets to be literally classical sets, rendered as `hSets` in HoTT, this is straightforward. In HoTT this also implies that `hSets` have unique identity proofs, which make their category theory proof-irrelevant (or truncated), which is closer to the set-based understanding of classical category theory. Second, HoTT has a very natural way of expressing universal properties. Using Martin-Löf type theories, e.g. ours, [25, 35, 18, 30], universal properties are usually stated in two parts: a universal part returning a morphism and a uniqueness part equating morphisms from the universal part. In HoTT, this can be expressed compactly as constructing a contractible morphism.

features	Ours	[25]	[30]	[35]	[18]	[33, 5]	[16]	[1]	[26, 27, 28]
basic structures:									
initial / terminal	✓	✓	✓	✓	✓	✓	✓	✓	✓
product / coproduct	✓	✓	✓	✓	✓	✓	✓	✓	✓
limit / colimit	✓	✓	✓	✓	✓	✓	✓	✓	✓
end / coend	✓	✓		✓		✓			
exponential	✓	✓	✓		✓	✓			
categorical structures:									
product / coproduct <sup>†</sup>	✓	✓	✓	✓	✗	✗	✗	✓	✗
comma category	✓	✓	✓	✓	✓	✓	✓	✓	
cartesian category	✓	✓	✓	✓	✓				
closed category	✓								
CCC	✓	✓	✓	✓	✓				
LCCC	✓		✓						
biCCC	✓			✓					
rig category	✓	✓				✓			
topos	✓		✓						
Grothendieck topos						✓			
Eilenberg Moore	✓			✓					
Kleisli	✓			✓		✓		✓	
monoidal category	✓			✓		✓		✓	✓
Kelly's coherence [19]	✓								✓
closed monoidal category	✓								
closed monoidal categories are closed categories	✓								
braided monoidal category	✓	✓		✓		✓			
symmetric monoidal category	✓	✓		✓					
traced monoidal category	✓	✓							
instances:									
<b>Cat</b>	✓	✓	✓	✓	✓	✓	✓	✓	
<b>Set</b>	✓	✓	✓	✓	✓	✓	✓		✓
simplicial set	✓					✓			
functor	✓	✓	✓	✓	✓	✓	✓	✓	✓
(co)limit functor	✓		✓			✓	✓		✓
Hom functor	✓	✓	✓	✓	✓	✓	✓	✓	✓
Hom functors preserve limits	✓		✓						
T-algebra	✓	✓	✓	✓		✓			
Lambek's lemma	✓	✓				✓			
natural transformation	✓	✓	✓	✓	✓	✓	✓	✓	✓
dinatural transformation	✓	✓		✓					
enriched category	✓	✓		✓		✓			
2-category	✓	✓				✓			✓
bicategory	✓	✓		✓		✓			✓
pseudofunctor	✓	✓				✓	✓		✓

<sup>†</sup>: ✗ indicates that these libraries only implement product categories.

■ **Table 2** Feature comparison (part 1)

features	Ours	[25]	[30]	[35]	[18]	[33, 5]	[16]	[1]	[26, 27, 28]
Yoneda lemma	✓	✓	✓	✓	✓	✓	✓	✓	✓
Grothendieck construction	✓	✓					✓	✗ ††	
adjoint functors	✓	✓	✓	✓	✓	✓	✓	✓	✓
adjoint composition	✓	✓	✓				✓	✓	✓
Right(left) adjoints preserve (co)limits	✓		✓			✓		✓	✓
Adjoint functors induce monads	✓			✓		✓			
(Co)limit functors are left(right) adjoint to diagonal functor†	✓		✓	✗		✗	✓		✓
mate (conjugate)	✓					✓			
adjoint functor theorem			✓		✓				
Kan extension	✓	✓	✓	✓	✓	✓	✓		
(Co)limit is kan	✓		✓	✓		✓			
Kan extensions are preserved by adjoint functors			✓		✓				
presheaves	✓	✓	✓	✓		✓			
complete / cocomplete			✓						
cartesian closed			✓						
topos			✓						
Rezk completion						✓			

† ✗ indicates that these libraries only show a special case of the theorem.

†† ✗ indicates that [1] only implements the category of elements.

■ **Table 3** Feature comparison (part 2)

The mathematical library of Lean [15], mathlib [1], also implements some category theory. Since Lean has proof-irrelevance built in and mathlib uses propositional equality directly, category theory in this library is very close to the classical one.

There has also been some development of category theory developed for Nuprl [13], Idris [9] and Isabelle [24]. Due to space limitation, we are not able to fully survey all of them. We refer interested readers to [16] and the Coq discourse forum<sup>5</sup> for a more thorough list of formalizations of category theory.

## 6 Conclusion and Future Work

We implemented proof-relevant category theory in Agda, successfully. The concepts covered, and the theorems proved, are quite broad. We did not find any real barrier to doing so — strictness and hom-sets are not necessary features of modern category theory. We did find that some definitions work better for us than others, which we have explained in detail. Comparing with other libraries, we find that ours covers quite similar grounds, and often

<sup>5</sup> <https://coq.discourse.group/t/survey-of-category-theory-in-coq/371/4>.



more.

We are still actively developping this library — many theorems of classical category theory remain to be done; both bicategory theory and enriched category theory are also being built up. Some work has been done on “negative thinking” ( $-2$ -categories, etc) and should be extended. Both double categories and higher categories are still awaiting, along with multicategories, PROPs, operads and polycategories. We also intend to move parts of this library to the standard library.

We also need to look at performance. The current library takes a lot of memory and time to typecheck. This presents an opportunity to investigate the underlying reasons, and either fix our development or Agda, if possible.

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