Realization of Nonlinear Time-Delay Input–Output Equations

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***Abstract*—In this letter the problem of transforming a nonlinear multi-input multi-output time-delay control system, described by a set of higher order functional differential equations relating system inputs and outputs, into a set of first order functional differential equations, is studied. The globally linearized system equations are described in terms of polynomials in a delay operator and derivative operator. The Euclidean left division of polynomials is used to compute the basis of a certain free submodule of differential 1-forms. The integrability of the submodule is proved to be necessary and sufficient for the solvability of the problem.**

***Index Terms*—Delay systems, algebraic/geometric methods.**

I. Introduction

Time-delay is a very natural phenomenon of models in many applications, such as life sciences [17], [23], engineering [6], [18], economics [14], [20], biology [5], [19] etc (see [7], [15], [22] for models in different fields). A delay may have natural (e.g., in population dynamics it takes time for an individual to grow) or technical (e.g., computation time) meaning. In most cases, a time-delay is caused by the fact that nothing happens instantly, it takes time for information to 'travel'. Moreover, a delay allows to use more general feedback to achieve the required performance. The delay in such systems may be constant or state-dependent. To simplify the analysis, often one makes the assumption that the delays are commensurable, i.e., multiples of a minimal constant delay.

In this letter we consider nonlinear time-delay control systems. Such systems are not described by ordinary differential equations, but by functional differential equations. Typically control systems may be described by a set of higher order functional (ordinary) differential equations (the so-called input-output equations) or by a set of first order functional (ordinary) differential equations (in delay-free case called state equations). However, for time-delay systems, the state is infinite dimensional, and therefore it is not correct to call the vector, whose dynamics is described by the first order differential equations, a state vector.

The problem of transforming the input-output equations to a set of first order differential equations is called the realization problem. It is an important problem, since most of the analysis and control solutions are developed for systems described by a set of first order differential equations, while modeling often yields the input-output equations. The realization problem is solved for delay-free systems (see [1], [2], [27] and references therein), but only special cases are studied for nonlinear time-delay systems. The paper [13] solves the realization problem for single-input single-output (SISO) nonlinear time-delay systems. The special case of linear realization up to a nonlinear input-output injection term is studied in [8] for the SISO case. The problem has also been studied for linear time-delay systems [9], [11] and for the case of state-dependent delays [25].

The realization problem of nonlinear multi-input multi-output (MIMO) time-delay input-output equations is addressed in this letter. Differentiating the system equations allows to represent the system in terms of non-commutative polynomials in the derivative operator, while the coefficients of the polynomials are, in turn, from the ring of polynomials in the delay operator. We have found the polynomial formulas, allowing to compute the differentials of the 'state' coordinates, necessary for the realization. The formulas are based on Euclidean left division of polynomials. In general, the Euclidean division algorithm does not work for polynomials over rings, i.e., when the polynomial coefficients are the elements of the ring and not the field. The reason is that one has to divide the coefficients, the operation which is not defined in the ring. However, since we only need division by the polynomial indeterminate s (i.e., time-derivative operator), in this special case the coefficients of the quotient are still polynomials.

This letter uses the algebraic approach of differential 1-forms (see [4] and generalizations to time-delay systems [12], [21], [26]) and extends the results on realization from delay-free case [2] and SISO case [13] to MIMO time-delay systems. In all cases the solution depends on integrability of certain set of differential 1-forms. The integrability aspect is typical for nonlinear systems and as such studied separately in [12] and [13] for time-delay systems. In

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principal, the main novelty of this letter lies in a way to compute these 1-forms. Different methods are described for delay-free systems [1], [2]. However, the generalization of polynomial formulas from delay-free case [2] to time-delay systems is not trivial. As mentioned before, the Euclidean left division is not always possible in the time-delay case, while in the same setting for delay-free case, it can be always done. Also, the main algebraic objects one works with have different properties in the delay-free and time-delay cases. Compared to the results for SISO case in [13], a completely different method of computing the basis for a certain free submodule of 1-forms, which gives the solution of the realization problem, is presented. In [13] an algorithm was given to compute the basis of the aforementioned submodule. Unfortunately, this algorithm does not allow to find the rank of the submodule under consideration in the MIMO case which is necessary to prove the main result of this letter. Here the exact formulas, instead of an algorithm, for computing the basis 1-forms of the submodule, in terms of which the solution to the realization problem is given, are found. With these formulas, the rank of the corresponding free submodule is easy to find.

The results of this letter hold under the assumption of commensurable delays. This assumption has been made in many papers [8], [12], [13], [21], [26], where a similar algebraic approach is used. The reason is that it allows to define the delay operator and describe the globally linearized system equations as a system over certain polynomial ring, defined by the delay operator. In principle, this assumption can be weakened to let all the delays be only constant and not commensurable. In this case one has to define a set of delay operators and the corresponding polynomial ring will then depend on multiple variables. When the state-dependent delays are considered, then the delay operator also becomes state-dependent, and it is not clear to the authors, whether the algebraic approach used in this letter can be extended to such a case.

This letter is organized as follows. First, Section II describes the problem statement and the mathematical approach applied in this letter. Section III presents the main results of this letter. Formulas are derived to compute the basis of a certain submodule of 1-forms. Then, the submodule is used to solve the realization problem. Finally, some examples are presented to verify the results and conclusions are drawn.

The following notations are used throughout this letter. By $d\alpha$ the differential of a function α from the field of functions $\mathcal{K}$, defined below, is denoted. For time-derivative of ξ of order $k \in \mathbb{N}$, $k \geq 3$ the usual notation $\xi^{(k)}$ is used. The first and second time-derivative of ξ are denoted as $\dot{\xi}$ and $\ddot{\xi}$. Finally, $\xi^{[-k]}$ stands for $\xi(t-k)$ and $\xi_{[\rho]} := (\xi(t), \xi(t-1), \ldots, \xi(t-\rho))$.

II. Preliminaries

A. Control Systems

In this letter we consider two types of system descriptions. First of them is described by nonlinear time-delay input-output (i/o) equations in the form

$$y_i^{(n_i)} = \Phi_i(y_{j,[\rho]}, \ldots, y_{j,[\rho]}^{(n_{ij})}, u_{\tau,[\rho]}, \ldots, u_{\tau,[\rho]}^{(r_{i\tau})}), \tag{1}$$

where $i, j = 1, \ldots, \sigma$, $\tau = 1, \ldots, m$, y_i are the outputs of the system, u_τ are the inputs of the system, the functions Φ_i are assumed to be real analytic in their variables and the indices in (1) satisfy the conditions

$$\begin{aligned} &n_1 \leq n_2 \leq \cdots \leq n_\sigma, n_{ij} < n_j \quad r_{i\tau} < n_i \\ &n_{ij} < n_i, \quad j \leq i \\ &n_{ij} \leq n_i, \quad j > i. \end{aligned} \tag{2}$$

Additionally, let $\sum_{i=1}^{\sigma} n_i = n$ and $r := \max_{i,\tau}\{r_{i\tau}\}$. For delay-free systems, it has been proved, that causal i/o systems can be taken, at least locally, to the form (1), where indices satisfy (2) [24]. The same has not yet been proved for time-delay systems. However, to keep the analogy with delay-free case and keeping in mind that the subclass (1), (2) is very general, we make the assumption, that the indices in (1) satisfy (2). This is also necessary to simplify the mathematical formalism below.

Second, the time-delay system is described by a set of first order functional differential equations

$$\begin{aligned} \dot{x} &= f(x_{[q]}, u_{[q]}) \\ y &= h(x_{[q]}), \end{aligned} \tag{3}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}^\sigma$ is the output of the system. Similarly as in [28], we define the observability property of system (3) as follows.

Definition 1: The system (3) is said to be causally observable if x_i, $i = 1, \ldots, n$, can be written in terms of system outputs, inputs, their derivatives in current and past time instances.

Note that the word 'causally' in Definition 1 means that one does not need future time-instances of system outputs, inputs and their derivatives, to express the variables x_i, $i = 1, \ldots, n$. For delay-free systems the causal observability is related to the observability notion, defined in [4], i.e., to the notion, defined through the rank of the observable space.

Problem statement: Given the system (1), find, whenever possible, the variables $x = \varphi(y_{[\rho]}, u_{[\rho]}, \dot{y}_{[\rho]}, \dot{u}_{[\rho]}, \ldots)$, such that the equations (1) are transformed into form (3), which is causally observable.

B. Mathematical Formalism

In what follows we relay on the algebraic approach from [12] and [13]. The approach is here described for systems of the form (1), similar approach for systems (3) is developed in [12], [21], and [26].

Denote by $\mathcal{K}$ the field of meromorphic functions depending on finite number of independent variables from the set $\mathcal{C} = \{y_i^{(l)}(t-\beta), u_j^{(k)}(t-\beta); \beta, k \in \mathbb{N}; i = 1, \ldots, \sigma; j = 1, \ldots, m; l = 0, \ldots, n_i - 1\}$. The delay operator δ is defined on $\mathcal{K}$ by the rule $\delta\varphi(\xi(t-\tau)) = \varphi(\xi(t-\tau-1))$, where $\varphi \in \mathcal{K}$ and $\xi(t-\tau) \in \mathcal{C}$.

Define the vector space of 1-forms as $\mathcal{E} = \text{span}_{\mathcal{K}}\{\text{d}\varphi | \varphi \in \mathcal{K}\}$. The operator δ is extended to $\mathcal{E}$ as $\delta(\sum_j a_j \text{d}\xi_j) = \sum_j \delta(a_j)\text{d}\delta(\xi_j)$, where $a_j \in \mathcal{K}$ and $\xi_j \in \mathcal{C}$. Using the delay operator δ, a non-commutative polynomial ring $\mathcal{K}[\vartheta]$ can be constructed. The addition in $\mathcal{K}[\vartheta]$ is defined as usual, but for multiplication the following rule is used: $\vartheta\varphi = \delta(\varphi)\vartheta$ for $\varphi \in \mathcal{K}$. Now, the 1-forms may be alternatively viewed as elements of the module $\mathcal{M} = \text{span}_{\mathcal{K}[\vartheta]}\{\text{d}\varphi | \varphi \in \mathcal{K}\}$. Unlike a vector space, not every module has a basis. The modules, that do have a basis, are called free modules. Since $\mathcal{K}[\vartheta]$ satisfies the left Ore condition [26], any two bases of a free module have the same number of elements, which is called the *rank* of the free module.

Definition 2 [26]: The closure of a free submodule $\mathcal{F}$ of $\mathcal{M}$, denoted by $cl_{\mathcal{K}[\vartheta]}\mathcal{F}$, is defined as $cl_{\mathcal{K}[\vartheta]}\mathcal{F} = \{\omega \in \mathcal{M} \mid \exists p(\vartheta) \in \mathcal{K}[\vartheta] \wedge p(\vartheta)\omega \in \mathcal{F}\}$.

By definition, the closure of the free submodule $\mathcal{F}$ is the largest free submodule, containing $\mathcal{F}$, and having the same rank as $\mathcal{F}$. If the closure of the submodule $\mathcal{F}$ is equal to itself, then $\mathcal{F}$ is said to be *closed*.

Definition 3 [12]: A set of 1-forms $\{\omega_1, \ldots, \omega_k\}$, independent over $\mathcal{K}[\vartheta]$, is said to be strongly (weakly) integrable if there exist k independent functions $\{\varphi_1, \ldots, \varphi_k\}$, such that

$$\text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\} = \text{span}_{\mathcal{K}[\vartheta]}\{\text{d}\varphi_1, \ldots, \text{d}\varphi_k\}$$
$$(\text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\} \subseteq \text{span}_{\mathcal{K}[\vartheta]}\{\text{d}\varphi_1, \ldots, \text{d}\varphi_k\}).$$

If the set of 1-forms $\{\omega_1, \ldots, \omega_k\}$ is strongly (respectively weakly) integrable, then the submodule $\text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$ is said to be strongly (respectively weakly) integrable. For more information about integrability of submodules and conditions to check whether a submodule is strongly or weakly integrable, see [12], [13]. Note also that in the case a submodule $\mathcal{A}$ is closed, the definitions of strong and weak integrability coincide, in which case we only speak about integrability of $\mathcal{A}$.

We also consider the polynomial ring $\mathcal{K}[\vartheta][s]$ whose elements are polynomials in the form

$$a(s) = \sum_{i=0}^{n} a_i(\vartheta)s^i = \sum_{i=0}^{n}\left(\sum_{j=0}^{k} a_{ij}\vartheta^j\right)s^i, \quad a_{ij} \in \mathcal{K}, \tag{4}$$

where s stands for the time-derivative operator. The coefficients of $a(s) \in \mathcal{K}[\vartheta][s]$ are polynomials $a_i(\vartheta) \in \mathcal{K}[\vartheta]$.

The solution to the realization problem below requires dividing polynomials $a(s) \in \mathcal{K}[\vartheta][s]$ by the polynomial s, i.e., finding the left quotient $\gamma(s)$ and left remainder $r(s)$, such that

$$a(s) = s\gamma(s) + r(s),$$

where $\gamma(s) \in \mathcal{K}[\vartheta][s]$ and $r(s) = r(\vartheta) \in \mathcal{K}[\vartheta]$.

Polynomials $\gamma(s)$ and $r(s)$ can be found, in principle, by left Euclidean division algorithm, given in [3, p. 11]. The following recursive formula allows to find left quotient $\gamma(s)$ and left remainder $r(s)$ for polynomials $a(s)$, given by (4), and s. To initialize the computations, we denote $n = \deg_s a(s)$, $a^n(s) := a(s)$ and then compute recursively for $i = n-1, \ldots, 0$ the polynomials

$$\gamma^i(s) := a_{i+1}^{i+1}(\vartheta)s^i, \quad a^i(s) := a^{i+1}(s) - s\gamma^i(s), \tag{5}$$

where $a_{i+1}^{i+1}(\vartheta) \in \mathcal{K}[\vartheta]$ is the coefficient of s^{i+1} in $a^{i+1}(s)$. The left quotient $\gamma(s) := \gamma^{n-1}(s) + \cdots + \gamma^1(s) + \gamma^0(s)$ and left remainder $r(s) := a^0(s)$. Due to the construction,

$$\deg_s \gamma(s) = n - 1, \quad \deg_s r(s) = 0.$$

Example 1: Consider the polynomial $a(s) = u\vartheta s^3 + 3s + y =: a^3(s)$ and find $\gamma(s)$, $r(s)$ such that $a(s) = s\gamma(s) + r(s)$. By (5),

$$\begin{aligned} \gamma^2(s) &= u\vartheta s^2, & a^2(s) &= -\dot{u}\vartheta s^2 + 3s + y,\\ \gamma^1(s) &= -\dot{u}\vartheta s, & a^1(s) &= (\ddot{u}\vartheta + 3)s + y,\\ \gamma^0(s) &= \ddot{u}\vartheta + 3, & a^0(s) &= u^{(3)}\vartheta + y. \end{aligned}$$

Thus $\gamma(s) = \gamma^2(s) + \gamma^1(s) + \gamma^0(s) = u\vartheta s^2 - \dot{u}\vartheta s + \ddot{u}\vartheta + 3$, $r(s) = a^0(s) = -u^{(3)}\vartheta + y$ and $\deg_s r(s) = 0$.

III. Realization Problem

In this section the realization problem, i.e., transforming equations (1) into the strongly observable form (3), is addressed. The solution is found in terms of a sequence of left-submodules $\mathcal{H}_1 \supset \mathcal{H}_2 \supset \ldots$ of $\mathcal{M}$, computed for system (1) as follows:

$$\begin{aligned} \mathcal{H}_1 &= \text{span}_{\mathcal{K}[\vartheta]}\{\text{d}y_i, \ldots, \text{d}y_i^{(n_i-1)}, \text{d}u_j, \ldots, \text{d}u_j^{(r)}; \\ &\qquad i = 1, \ldots, \sigma; j = 1, \ldots, m\} \end{aligned} \tag{6a}$$
$$\mathcal{H}_k = \text{span}_{\mathcal{K}[\vartheta]}\{\omega \in \mathcal{H}_{k-1} \mid \dot{\omega} \in \mathcal{H}_{k-1}\}. \tag{6b}$$

It has been proved in [26] for systems of the form (3) that sequence (6) converges to a submodule, denoted by $\mathcal{H}_\infty$, and all the submodules $\mathcal{H}_k$ are closed. Similar results can be proved for subspaces (6) defined for i/o systems (1).

A. Computation of $\mathcal{H}_k$

By applying the operator d to equation (1) we obtain for $i = 1, \ldots, \sigma$

$$\begin{aligned} &\text{d}y_i^{(n_i)} - \sum_{j=1}^{\sigma}\sum_{\ell=0}^{n_{ij}}\sum_{\lambda=0}^{\rho} \frac{\partial\Phi_i}{\partial y_j^{(\ell)}(t-\lambda)}\text{d}y_j^{(\ell)}(t-\lambda)\\ &\quad - \sum_{\tau=1}^{m}\sum_{\iota=0}^{r_{i\tau}}\sum_{\beta=0}^{\rho} \frac{\partial\Phi_i}{\partial u_\tau^{(\iota)}(t-\beta)}\text{d}u_\tau^{(\iota)}(t-\beta) = 0, \end{aligned} \tag{7}$$

called the *globally linearized* i/o equation. The equation (7) can be represented in terms of non-commutative polynomials from the ring $\mathcal{K}[\vartheta][s]$ by rewriting (7) as

$$\sum_{j=1}^{\sigma} p_{i,j}(\vartheta, s)\text{d}y_j + \sum_{\tau=1}^{m} q_{i,\tau}(\vartheta, s)\text{d}u_\tau = 0, \tag{8}$$

where $i = 1, \ldots, \sigma$ and

$$\begin{aligned} p_{i,i} &= s^{n_i} - \sum_{\ell=0}^{n_{ii}}\sum_{\lambda=0}^{\rho} \frac{\partial\Phi_i}{\partial y_i^{(\ell)}(t-\lambda)}\vartheta^\lambda s^\ell\\ p_{i,j} &= -\sum_{\ell=0}^{n_{ij}}\sum_{\lambda=0}^{\rho} \frac{\partial\Phi_i}{\partial y_j^{(\ell)}(t-\lambda)}\vartheta^\lambda s^\ell \qquad j \neq i\\ q_{i,\tau} &= -\sum_{\iota=0}^{r_{i\tau}}\sum_{\beta=0}^{\rho} \frac{\partial\Phi_i}{\partial u_\tau^{(\iota)}(t-\beta)}\vartheta^\beta s^\iota. \end{aligned} \tag{9}$$

Introduce the 1-forms, based on which the bases of the submodules $\mathcal{H}_k$, $k = 1, \dots, r+2$, are computed for system (1). Let

$$\omega_{i,\ell} := \sum_{j=1}^{\sigma} p_{i,j}^{\ell}(\vartheta, s)\mathrm{d}y_j + \sum_{\tau=1}^{m} q_{i,\tau}^{\ell}(\vartheta, s)\mathrm{d}u_\tau, \tag{10}$$

where $i = 1, \dots, \sigma$, $\ell = 1, \dots, n_i$, $p_{i,j}^{\ell}$ and $q_{i,\tau}^{\ell}$ are polynomials, which can be recursively calculated from the equalities

$$\begin{aligned} p_{i,j}^{\ell-1} &= s p_{i,j}^{\ell} + \kappa_{i,j}^{\ell}, \quad \deg_s \kappa_{i,j}^{\ell} = 0, \\ q_{i,\tau}^{\ell-1} &= s q_{i,\tau}^{\ell} + \alpha_{i,\tau}^{\ell}, \quad \deg_s \alpha_{i,\tau}^{\ell} = 0 \end{aligned} \tag{11}$$

with the initial polynomials $p_{i,j}^{0} := p_{i,j}$, $q_{i,\tau}^{0} := q_{i,\tau}$, given by (9).

Theorem 1: For the i/o model (1) the submodules $\mathcal{H}_k$, $k = 1, \dots, r+2$, satisfy rank $\mathcal{H}_k = n + (r+2-k)m$ and can be calculated as

$$\begin{aligned} \mathcal{H}_k &= \mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_{i,1}, \dots, \omega_{i,n_i}, \mathrm{d}u_j, \dots, \mathrm{d}u_j^{(r-k+1)}\} \\ \mathcal{H}_{r+2} &= \mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_{i,1}, \dots, \omega_{i,n_i}\}, \end{aligned} \tag{12}$$

where $k = 1, \dots, r+1$, $i = 1, \dots, \sigma$, $j = 1, \dots, m$ and the 1-forms $\omega_{i,1}, \dots, \omega_{i,n_i}$ are defined by (10)-(11).

Proof: First, we prove that (12) is true. Then, rank $\mathcal{H}_k = n + (r+2-k)m$ is proved by showing that the 1-forms $\omega_{i,1}, \dots, \omega_{i,n_i}$, $i = 1, \dots, \sigma$, are independent over the polynomial ring $\mathcal{K}[\vartheta]$.

To begin with, note that the 1-forms $\omega_{i,1}, \dots, \omega_{i,n_i}$, $i = 1, \dots, \sigma$, are defined such that the submodules $\tilde{\mathcal{H}}_k := \mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_{i,1}, \dots, \omega_{i,n_i}, \mathrm{d}u_j, \dots, \mathrm{d}u_j^{(r-k+1)}\}$, $k = 1, \dots, r+1$, and $\tilde{\mathcal{H}}_{r+2} := \mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_{i,1}, \dots, \omega_{i,n_i}\}$ are closed. The submodules $\tilde{\mathcal{H}}_k$ are closed if there does not exist a polynomial $\gamma(\vartheta) \in \mathcal{K}[\vartheta]$, $\deg_\vartheta \gamma(\vartheta) > 0$ and a 1-form $\omega \in \mathcal{M}$, such that $\gamma(\vartheta)\omega \in \tilde{\mathcal{H}}_k$. For submodules $\tilde{\mathcal{H}}_k$ such polynomial and 1-form do not exist, because the polynomials $p_{i,i}^{\ell}$ are monic and thus there does not exist any left factors of higher degree than 0 for $p_{i,i}^{\ell}$.

The remaining part of the proof is by mathematical induction. First, we show that formula (12) holds for $k = 1$. Taking $k = 1$ in (12), yields a subspace, which we denote by

$$\tilde{\mathcal{H}}_1 := \mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \dots, \omega_n, \mathrm{d}u, \dots, \mathrm{d}u^{(r)}\}. \tag{13}$$

We show that $\tilde{\mathcal{H}}_1 = \mathcal{H}_1$, i.e., that the basis 1-forms of (13) belong to $\mathcal{H}_1$, defined by (6a), and vice versa. By (10), the 1-forms

$$\omega_{i,\ell} := \sum_{j=1}^{\sigma}\sum_{\mu=0}^{n_i-\ell} p_{i,j,\mu}^{\ell}(\vartheta)\mathrm{d}y_j^{(\mu)} + \sum_{\tau=1}^{m}\sum_{\pi=0}^{r_{i\tau}-\ell} q_{i,\tau,\pi}^{\ell}(\vartheta)\mathrm{d}u_\tau^{(\pi)},$$

where $i = 1, \dots, \sigma$, $\ell = 1, \dots, n_i$, $p_{i,j,\mu}^{\ell}, q_{i,\tau,\pi}^{\ell} \in \mathcal{K}[\vartheta]$. This holds due to the fact that $\deg_s p_{i,j}^{\ell} \le n_i - \ell$, $\deg_s q_{i,\tau}^{\ell} \le r_{i\tau} - \ell$ for $\ell = 0, \dots, r_{i\tau}$ and polynomials $q_{r_{i\tau}+1} = \dots = q_n = 0$. Thus, the 1-forms $\omega_{i,1}, \dots, \omega_{i,n_i}$ are represented as the linear combination of basis 1-forms of (6a). The remaining 1-forms $\mathrm{d}u_j, \dots, \mathrm{d}u_j^{(r)}$, $j = 1, \dots, m$, are explicitly written in the basis of (6a). Therefore, $\tilde{\mathcal{H}}_1 \subset \mathcal{H}_1$, but since $\tilde{\mathcal{H}}_1$ and $\mathcal{H}_1$ are both closed, then $\tilde{\mathcal{H}}_1 = \mathcal{H}_1$.

Assume next that formula (12) holds for k and we prove it to be valid for $k+1$. The proof is based on the definition of the submodules $\mathcal{H}_k$. Since the submudules in (12) are closed, all we have to prove is that the basis elements of

$$\tilde{\mathcal{H}}_{k+1} = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_{i,1}, \dots, \omega_{i,n_i}, \mathrm{d}u_j, \dots, \mathrm{d}u_j^{(r-k)}\}, \tag{14}$$

where $i = 1, \dots, \sigma$, $j = 1, \dots, m$, calculated according to formula (12), satisfy condition (6b). We have to prove that the derivatives of the basis 1-forms of (14) belong to $\mathcal{H}_k$. By (10), we have

$$\dot{\omega}_{i,\ell} := s\sum_{j=1}^{\sigma} p_{i,j}^{\ell}(\vartheta, s)\mathrm{d}y_j + s\sum_{\tau=1}^{m} q_{i,\tau}^{\ell}(\vartheta, s)\mathrm{d}u_\tau,$$

that yields, using the relations (11):

$$\begin{aligned} \dot{\omega}_{i,\ell} = &\sum_{j=1}^{\sigma}[p_{i,j}^{\ell-1}(\vartheta, s) - \kappa_{i,j}^{\ell}(\vartheta)]\mathrm{d}y_j \\ &+ \sum_{\tau=1}^{m}[q_{i,\tau}^{\ell-1}(\vartheta, s) - \alpha_{i,\tau}^{\ell}(\vartheta)]\mathrm{d}u_\tau. \end{aligned} \tag{15}$$

After reordering the terms in (15) we get

$$\dot{\omega}_{i,\ell} = \omega_{i,\ell-1} - \sum_{j=1}^{\sigma}\kappa_{i,j}^{\ell}(\vartheta)\mathrm{d}y_j - \sum_{\tau=1}^{m}\alpha_{i,\tau}^{\ell}(\vartheta)\mathrm{d}u_\tau, \tag{16}$$

where $\omega_{i,0} = 0$ due to the polynomial system description (8). Since $\omega_{i,\ell-1} \in \mathcal{H}_k$ for $i = 1, \dots, \sigma$, $\ell = 1, \dots, n_i$, then it remains to show that $\mathrm{d}y_j \in \mathcal{H}_k$ and $\mathrm{d}u_\tau \in \mathcal{H}_k$. The latter is true by the assumption that the formula (12) is true for $\mathcal{H}_k$. Next, it is shown that $\mathrm{d}y_j \in \mathcal{H}_k$. Note that

$$\omega_{i,n_i} = \sum_{j=1}^{\sigma} p_{i,j}^{n_i}(\vartheta)\mathrm{d}y_j \in \mathcal{H}_k,$$

because $q_{i,\tau}^{n_i} = 0$ and $\deg_s p_{i,j}^{n_i}(\vartheta, s) = 0$. Since submodules $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_{i,n_i}\}$ and $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}y_i\}$ are both closed, then they are equal and thus, $\mathrm{d}y_i \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_{i,n_i}\} \subset \mathcal{H}_k$.

Finally, we observe that the derivatives of the rest of the basis 1-forms in (14) are $\mathrm{d}\dot{u}_j, \dots, \mathrm{d}u_j^{(r-k+1)}$, $j = 1, \dots, m$, which are also in $\mathcal{H}_k$. Thus, we have proved that $\mathcal{H}_{k+1}$, computed according to (12), agrees with the definition (6b).

Next, it will be shown that the 1-forms $\omega_{i,1}, \dots, \omega_{i,n_i}$, $i = 1, \dots, \sigma$, are linearly independent over the polynomial ring $\mathcal{K}[\vartheta]$. Note that, by construction, the polynomials $p_{i,i}^{l}$, $i = 1, \dots, \sigma$, $l = 1, \dots, n_i$, are monic and $p_{i,i}^{l} = s^{n_i-l+1} + \tilde{p}_{i,i}^{l}$, where $\deg \tilde{p}_{i,i}^{l} < n_i - l + 1$. Therefore, since $\deg p_{i,i}^{l} > \deg p_{i,j}^{l}$, $i \neq j$, then $\omega_{i,l} = \mathrm{d}y_i^{(n_i-l)} + \tilde{\omega}_{i,l}$, where $\tilde{\omega}_{i,l}$ does not depend on $\mathrm{d}y_i^{(n_i-l)}$. Because the 1-forms $\mathrm{d}y_i^{(n_i-l)}$, $i = 1, \dots, \sigma$, $l = 1, \dots, n_i$, are linearly independent over the ring $\mathcal{K}[\vartheta]$, then so are $\omega_{i,l}$. Since the 1-forms $\omega_{i,l}$ are also independent from the 1-forms $\mathrm{d}u_j^{(k)}$, $j = 1, \dots, m$, $k \ge 0$, then from the formulas (12) it becomes clear that rank $\mathcal{H}_k = n + (r+2-k)m$. ■

Remark 1: The proof of Theorem 1 is based on the similar result for delay-free systems, see [2]. A new proof is necessary for time-delay systems, because of the different properties of modules, compared to vector spaces, which were used in delay-free case. In particular, while equality of vector spaces is shown by proving that the ranks of vector spaces are equal and bases elements of one of the vector spaces belong to the other, the same cannot be said for modules, in general. However, it is true for closed modules. Therefore, as a first step, it is proved, that the modules (12) are closed. The remaining proof follows the one in [2], although with some additional explanations, such as detailed proof that $\mathrm{d}y_j \in \mathcal{H}_k$ for $j = 1, \ldots, \sigma$, $k = 1, \ldots, r+2$ and that the 1-forms $\omega_{i,l}$ are independent.

B. Solution to Realization Problem

The following theorem is a generalization both of delay-free [2] and time-delay SISO case [13].

Theorem 2: The system (1) can be transformed into the form (3), which is causally observable, if and only if the submodule $\mathcal{H}_{r+2}$ is integrable.

Proof: Sufficiency. Assume that $\mathcal{H}_{r+2}$ is integrable. By Theorem 1, rank $\mathcal{H}_{r+2} = n$ and thus, let $\mathcal{H}_{r+2} = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_n\}$. Define $x_i = \varphi_i$, $i = 1, \ldots, n$. Now, since

$$\begin{aligned} \mathrm{d}\dot{x}_i = \mathrm{d}\varphi_i^{(1)} &\in \mathcal{H}_{r+1} \\ &= \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n, \mathrm{d}u_1, \ldots, \mathrm{d}u_m\}, \end{aligned}$$

then $\dot{x} = f(x_{[q]}, u_{[q]})$ for some function f and some $q \in \mathbb{N}$. In the proof of Theorem 1 it was shown that $\mathrm{d}y_i \in \mathcal{H}_{r+2}$, which means, that there exists h, such that $y = h(x_{[q]})$. Obviously, the obtained system form $\dot{x} = f(x_{[q]}, u_{[q]})$, $y = h(x_{[q]})$ is causally observable, since x_i, $i = 1, \ldots, n$, are selected such that $\mathrm{d}x_i = \mathrm{d}\varphi_i(\cdot) \in \mathcal{M}$.

Necessity: Causal observability of (3) yields, after eliminating the x_i coordinates (see [10]), that $\mathrm{d}x_i = \mathrm{d}\varphi_i(\cdot) \in \mathcal{H}_1$ for $i = 1 \ldots, n$. Because of Theorem 1

$$\mathcal{H}_1 = \mathcal{H}_{r+2} \oplus \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}u, \ldots, \mathrm{d}u^{(r)}\}.$$

Since $\mathrm{d}x_i \notin \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}u, \ldots, \mathrm{d}u^{(r)}\}$, one has that $\mathrm{d}x_i \in \mathcal{H}_{r+2}$ for $i = 1, \ldots, n$. Finally, since rank $\mathcal{H}_{r+2} = n$ and $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\}$ is closed submodule contained in $\mathcal{H}_{r+2}$, one has $\mathcal{H}_{r+2} = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\}$, i.e., $\mathcal{H}_{r+2}$ is integrable. ■

Remark 2: Consider the special case when $r = 0$. From the definition of submodules $\mathcal{H}_k$, one has that

$$\mathcal{H}_2 = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}y_i, \ldots, \mathrm{d}y_i^{(n_i-1)}, \mathrm{d}u_j, \ldots, \mathrm{d}u_j^{(r-1)}\},$$

where $i = 1, \ldots, \sigma$, $j = 1, \ldots, m$ and which, for $r = 0$, becomes

$$\mathcal{H}_2 = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}y_i, \ldots, \mathrm{d}y_i^{(n_i-1)}; i = 1, \ldots, \sigma\}. \tag{17}$$

Thus, for $r = 0$ in (1), the realization problem is always solvable, since $\mathcal{H}_{r+2} = \mathcal{H}_2$ is, by definition, integrable.

IV. Examples

Example 2: Consider the i/o equations

$$\begin{aligned} \ddot{y}_1 &= \dot{y}_2^{[-1]} + \dot{u}_2^{[-1]} + u_1 y_2 \\ \ddot{y}_2 &= -u_1^{[-2]} \dot{y}_1^{[-2]} \end{aligned} \tag{18}$$

and their polynomial representation (8),

$$\begin{aligned} p_{1,1}\mathrm{d}y_1 + p_{1,2}\mathrm{d}y_2 + q_{1,1}\mathrm{d}u_1 + q_{1,2}\mathrm{d}u_2 &= 0 \\ p_{2,1}\mathrm{d}y_1 + p_{2,2}\mathrm{d}y_2 + q_{2,1}\mathrm{d}u_1 + q_{2,2}\mathrm{d}u_2 &= 0, \end{aligned}$$

where

$$\begin{aligned} p_{1,1} &= s^2 & q_{1,1} &= -y_2 \\ p_{1,2} &= -\vartheta s - u_1 & q_{1,2} &= -\vartheta s \\ p_{2,1} &= u_1^{[-2]}\vartheta^2 s & q_{2,1} &= \dot{y}_1^{[-2]}\vartheta^2 \\ p_{2,2} &= s^2 & q_{2,2} &= 0. \end{aligned}$$

After denoting $p_{i,j}^0 := p_{i,j}$, $q_{i,\tau}^0 := q_{i,\tau}$ one can compute by (11) recursively the left quotients

$$\begin{aligned} p_{1,1}^1 &= s2 & p_{1,1}^2 &= 1 \\ p_{1,2}^1 &= -\vartheta 2 & p_{1,2}^2 &= 0 \\ p_{2,1}^1 &= u_1^{[-2]}\vartheta^2 2 & p_{2,1}^2 &= 0 \\ p_{2,2}^1 &= s2 & p_{2,2}^2 &= 1 \\ q_{1,2}^1 &= -\vartheta 2 & q_{1,2}^2 &= 0 \end{aligned}$$
$$q_{1,1}^1 = q_{1,1}^2 = q_{2,1}^1 = q_{2,1}^2 = q_{2,2}^1 = q_{2,2}^2 = 0$$

and by (10) the 1-forms

$$\begin{aligned} \omega_{1,1} &= \mathrm{d}(\dot{y}_1 - y_2^{[-1]} - u_2^{[-1]}) \\ \omega_{1,2} &= \mathrm{d}y_1 \\ \omega_{2,1} &= \mathrm{d}\dot{y}_2 + u_1^{[-2]}\mathrm{d}y_1^{[-2]} \\ \omega_{2,2} &= \mathrm{d}y_2, \end{aligned}$$

which, by Theorem 1, form a basis for submodule $\mathcal{H}_{r+2} = \mathcal{H}_3$. One can see that $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_{1,1}, \omega_{1,2}, \omega_{2,1}, \omega_{2,2}\} = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_{1,1}, \omega_{1,2}, \omega_{2,1} - u_1^{[-2]}\omega_{1,2}^{[-2]}, \omega_{2,2}\} = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}y_1, \mathrm{d}y_2, \mathrm{d}(\dot{y}_1 - y_2^{[-1]} - u_2^{[-1]}), \mathrm{d}\dot{y}_2\}$. Clearly, the latter is integrable and thus, by Theorem 2, the system (18) can be transformed into the causally observable form (3). According to the proof of Theorem 2 the x coordinates can be defined as follows:

$$\begin{aligned} x_1 &:= y_1 \\ x_2 &:= \dot{y}_1 - y_2^{[-1]} - u_2^{[-1]} \\ x_3 &:= y_2 \\ x_4 &:= \dot{y}_2. \end{aligned}$$

In these coordinates the system (18) takes the form

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3^{[-1]} + u_2^{[-1]} \\ \dot{x}_2 &= x_3 u_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -u_1^{[-2]}(x_2^{[-2]} + x_3^{[-3]} + u_2^{[-3]}) \\ y_1 &= x_1 \\ y_2 &= x_3. \end{aligned}$$

Example 3: Consider the situation where one car is following the other without overtaking. We assume that the position y_1 of the first car is controlled by u, while the position y_2 of the second car depends on the positions and speeds of both cars.

Taking into consideration the reaction time T of the driver of the second car, the system can be modeled as [15]

$$\begin{aligned} \ddot{y}_1 &= u \\ \ddot{y}_2 &= \frac{\alpha[\dot{y}_1^{[-T]} - \dot{y}_2^{[-T]}]}{m[y_1^{[-T]} - y_2^{[-T]}]}, \end{aligned} \tag{19}$$

where m is the mass of each car and α is the sensitivity coefficient of the driver. Since in (19) $r = 0$, then by Remark 2 the realization problem is solvable for (19) and the x coordinates can be chosen as $x_1 = y_1$, $x_2 = \dot{y}_1$, $x_3 = y_2$ and $x_4 = \dot{y}_2$. This yields the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{\alpha[x_2^{[-T]} - x_4^{[-T]}]}{m[x_1^{[-T]} - x_3^{[-T]}]}. \end{aligned}$$

V. Conclusion

This letter addressed the realization problem for nonlinear MIMO time-delay systems. A two-step solution was provided. First, formulas for computation of a basis of a certain submodule of differential 1-forms were derived. Second, it was proved that the realization problem is solvable if and only if the computed basis elements were integrable. For time-delay systems, one does not work with nice algebraic objects, like vector spaces and fields anymore. Instead, modules and rings are used, whose properties are different. Nevertheless, we have proven that similar results hold for time-delay systems as for the delay-free case.

For nonlinear systems, minimal realization can be defined in two ways, either by requiring that the dimension of x is minimal or that the system of first order equations (3) is both accessible and observable [16]. Observability of (3) is guaranteed by the solution given in this letter. Accessibility of (3) can be characterized through the submodule $\mathcal{H}_\infty$ [12]. When the maximal integrable submodule $\hat{\mathcal{H}}_\infty$ of $\mathcal{H}_\infty$ is $\hat{\mathcal{H}}_\infty = \{0\}$, then the realization is also accessible, otherwise it is non-accessible. In the delay-free case it is known that if one starts from the irreducible set of i/o equations, then the realization will be accessible. We conjecture that this holds also in the case of time-delay systems. However, the goal of this letter was not to find the minimal realization; this problem is left for future studies. Note that extension of system reduction from delay-free case to time-delay systems is far from trivial, since the standard approach relies on finding the largest left common factor of two polynomials in the description of globally linearized i/o equations. The latter can be found with the help of the Euclidean division algorithm which, as said before, does not work for polynomials, defined over rings.

References

[1] J. Belikov, Ü. Kotta, and M. Tõnso, "Adjoint polynomial formulas for nonlinear state-space realization," *IEEE Trans. Autom. Control*, vol. 59, no. 1, pp. 256–261, Jan. 2014.

[2] J. Belikov, Ü. Kotta, and M. Tõnso, "Realization of nonlinear MIMO system on homogeneous time scales," *Eur. J. Control*, vol. 23, pp. 48–54, May 2015.

[3] M. Bronstein and M. Petkovšek, "An introduction to pseudo-linear algebra," *Theor. Comput. Sci.*, vol. 157, no. 1, pp. 3–33, 1996.

[4] G. Conte, C. H. Moog, and A. M. Perdon, *Algebraic Methods for Nonlinear Control Systems. Theory and Applications*. London, U.K.: Springer, 2007.

[5] K. L. Cook, "Stability of delay differential equations with applications in biology and medicine," in *Mathematics in Biology and Medicine*, E. Grosso, V. Capasso, and S. L. Paveri-Fontana, Eds. Heidelberg, Germany: Springer, 1985.

[6] L. Dugard and E. I. Verriest, Eds., *Stability and Control of Time-Delay Systems*. Berlin, Germany: Springer-Verlag, 1998.

[7] T. Erneux, *Applied Delay Differential Equations*. New York, NY, USA: Springer, 2009.

[8] E. Garcia-Ramirez, C. H. Moog, C. Califano, and L. A. Márquez-Martínez, "Linearisation via input-output injection of time delay systems," *Int. J. Control*, vol. 89, no. 6, pp. 1125–1136, 2016.

[9] H. Glusing-Luersen, "Realization of behaviors given by delay-differential equations," in *Proc. Eur. Control Conf.*, Brussels, Belgium, 1997, pp. 1171–1176.

[10] M. Halas and M. Anguelova, "When retarded nonlinear time-delay systems admit an input–output representation of neutral type," *Automatica*, vol. 49, no. 2, pp. 561–567, 2013.

[11] T. Kaczorek, "Realization problem for a class of positive continuous-time systems with delays," *Int. J. Appl. Math. Comput. Sci.*, vol. 15, no. 4, pp. 447–453, 2005.

[12] A. Kaldmäe, C. Califano, and C. H. Moog, "Integrability for nonlinear time-delay systems," *IEEE Trans. Autom. Control*, vol. 61, no. 7, pp. 1912–1917, Jul. 2016.

[13] A. Kaldmäe and Ü. Kotta, "Realization of time-delay systems," *Automatica*, vol. 90, pp. 317–320, Apr. 2018.

[14] A. A. Keller, "Generalized delay differential equations to economic dynamics and control," in *Proc. Amer. Conf. Appl. Math.*, Cambridge, MA, USA, 2010, pp. 278–286.

[15] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*. Dordrecht, The Netherlands: Springer, 1999.

[16] Ü. Kotta, C. H. Moog, and M. Tõnso, "Minimal realizations of nonlinear systems," *Automatica*, 2018.

[17] Y. Kuang, *Delay Differential Equations With Applications in Population Dynamics*. Boston, MA, USA: Academic, 1993.

[18] Y. N. Kyrychko and S. J. Hogan, "On the use of delay equations in engineering applications," *J. Vib. Control*, vol. 16, nos. 7–8, pp. 943–960, 2010.

[19] N. MacDonald, *Time Lags in Biological Models*. Berlin, Germany: Springer-Verlag, 1978.

[20] A. Matsumoto and F. Szidarovszky, "Delay differential nonlinear economic models," in *Nonlinear Dynamics in Economics, Finance and Social Sciences*, C. Chiarella, G. Bischi, and L. Gardini, Eds. Heidelberg, Germany: Springer, 2010.

[21] C. H. Moog, R. Castro-Linares, M. Velasco-Villa, and L. A. Márquez-Martínez, "The disturbance decoupling problem for time-delay nonlinear systems," *IEEE Trans. Autom. Control*, vol. 45, no. 2, pp. 305–309, Feb. 2000.

[22] S. I. Niculescu, *Delay Effects on Stability*. London, U.K.: Springer, 2001.

[23] H. Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*. New York, NY, USA: Springer, 2011.

[24] A. J. van der Schaft, "Transformations of nonlinear systems under external equivalence," in *New Trends in Nonlinear Control Theory*, J. Descusse, M. Fliess, A. Isidori, and D. Leborgne, Eds. Berlin, Germany: Springer-Verlag, 1989, pp. 33–43.

[25] E. I. Verriest, "State space realization for systems with state dependent delay," in *Proc. 11th IFAC Workshop Time Delay Syst.*, Grenoble, France, 2013, pp. 451–456.

[26] X. Xia, L. A. Márquez-Martínez, P. Zagalak, and C. H. Moog, "Analysis of nonlinear time-delay systems using modules over non-commutative rings," *Automatica*, vol. 38, no. 9, pp. 1549–1555, 2002.

[27] J. Zhang, C. H. Moog, and X. Xia, "Realization of multivariable nonlinear systems via the approaches of differential forms and differential algebra," *Kybernetika*, vol. 46, no. 5, pp. 799–830, 2010.

[28] G. Zheng and J. P. Richard, "Identifiability and observability of nonlinear time-delay systems with unknown inputs," in *Recent Results on Nonlinear Delay Control Systems*. Cham, Switzerland: Springer, 2016, pp. 385–403.