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Zero-Sum Differential Games Involving Hybrid Controls[1,2]

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Abstract. We study a zero-sum differential game with hybrid controls in which both players are allowed to use continuous as well as discrete controls. Discrete controls act on the system at a given set interface. The state of the system is changed discontinuously when the trajectory hits predefined sets, an autonomous jump set A or a controlled jump set C, where one controller can choose to jump or not. At each jump, the trajectory can move to a different Euclidean space. One player uses all the three types of controls, namely, continuous controls, autonomous jumps, and controlled jumps; the other player uses continuous controls and autonomous jumps. We prove the continuity of the associated lower and upper value functions V^- and V^+. Using the dynamic programming principle satisfied by V^- and V^+, we derive lower and upper quasivariational inequalities satisfied in the viscosity sense. We characterize the lower and upper value functions as the unique viscosity solutions of the corresponding quasivariational inequalities. Lastly, we state an Isaacs like condition for the game to have a value.

Key Words. Dynamic programming principle, viscosity solutions, quasivariational inequalities, hybrid control, differential games.

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1. Introduction

We study here a zero sum differential game involving hybrid controls. The main motivation comes from the hybrid control systems arising in many engineering problems like constrained robotic systems, automated highway systems, and flight control systems; see Ref. 1. Branicky, Borkar, and Mitter (Ref. 1) have presented a model of a hybrid control system where continuous controls and discrete controls act on the system at a given set interface. The state of the system is changed discontinuously when the trajectory hits predefined sets, namely, an autonomous jump set A or a controlled jump set C, where the controller can choose to jump or not; at each jump, the trajectory can move to a different Euclidean space. The authors of Ref. 1 prove the right continuity of the value function. By using the dynamic programming principle, they have arrived at a quasivariational inequalities satisfied by the value functions in the viscosity sense. In Ref. 2, Bensoussan and Menaldi study a similar system and prove that the value function is continuous. They prove the uniqueness for a special case when the autonomous jump set is empty. For this hybrid control problem, we have shown in Ref. 3 that the value function is Hölder continuous under a transversality condition. We have proved the uniqueness of the value function by a method very different from that in Ref. 2 and for a more general case, namely the autonomous jump set being nonempty.

In this work, we extend the model in Ref. 1 to the game theoretic set up. We allow player P1 to use all the three types of controls, namely continuous controls, autonomous jumps, and controlled jumps; we allow the other player P2 to use continuous controls and autonomous jumps. As in Ref. 4, we use the Elliot-Kalton approach to define the strategies and the lower and upper value functions. The viscosity solution techniques applied to differential game problems using the Elliot-Kalton strategies and the dynamic programming principle give good existence and uniqueness results for the Hamilton-Jacobi-Isaacs equations satisfied by the value functions of the game problem; see chapter 6 of Ref. 4 and the references therein. Games where both players use continuous controls (Refs. 4–5), when both players use switching strategies (Ref. 6), and a problem where one player uses impulse controls and the other use continuous control and/or switching controls (Ref. 7) are particular examples, to name a few.

In this paper, our aim is to prove the local Hölder continuity of the lower and upper value functions, to derive the corresponding lower and upper quasivariational inequalities (QVIs) satisfied by them in the viscosity sense, and to characterize them as the unique viscosity solutions of these QVIs. Finally, we give an Isaacs like condition for the upper and lower value to coincide and thus for the game to have a value.

2. Notations and Assumptions

The trajectory $X(t)$ of the hybrid system evolves according to the following equation:

$$X(t) = f_i(X(t), u_1(t), u_2(t)), \quad X(0) = x, \tag{1}$$

where $x \in \Omega_i$, Ω_i is the closure of a connected open subset of $\mathbb{R}^{d_i}, d_i \in Z_+$. On hitting the autonomous jump set A_i, the trajectory jumps to the destination set $D_j, D_j \subseteq \Omega_j$ possibly, according to the given transition map g depending on discrete controls from the discrete control sets V_1 for P1 and V_2 for P2, where V_1, V_2 are compact metric spaces. Then, trajectory continues its evolution under f_j till it hits again A_j or C_j. On hitting C_j, player P1 can either choose to jump or choose not to jump. If it chooses to jump, then the trajectory is moved to a new point in say D_k. The state space Ω is actually $\Omega := \cup \Omega_i$, with $\Omega_i \in \mathbb{R}^{d_i}$ and $d_i, i \in Z_+$. The predefined sets A, C, D are unions of sets in $\mathbb{R}^{d_i}$,

$$A = \bigcup_{i=1}^{\infty} A_i, \;\; C = \bigcup_{i=1}^{\infty} C_i, \;\; D = \bigcup_{i=1}^{\infty} D_i, \;\text{ with } A_i, C_i, D_i \subseteq \Omega_i \subseteq \mathbb{R}^{d_i}.$$

Also, $f : \Omega \times U_1 \times U_2 \to \Omega$, with the understanding that $f = f_i(X(t), u_1(t), u_2(t))$, whenever $X(t) \in \Omega_i$ and u_j is in the continuous control space $\mathcal{U}_j = \{u_j : [0, \infty) \to U_j | u_j \text{ measurable}\}$, where U_j are compact metric spaces, j = 1, 2. Similarly, the transition map $g : A \times V_1 \times V_2 \to D$ is actually $g(x, v_1, v_2) = g_i(x, v_1, v_2)$, whenever $x \in A_i$.

Thus, the trajectory of the hybrid problem gives rise to a sequence of hitting times of A, which we denote by σ_i, and a sequence of hitting times of C, where P1 chooses to jump, denoted by ξ_i. We denote $X(\sigma_i, u_1(\cdot), u_2(\cdot))$ by x_i, $g(x_i, v_1, v_2)$ by x_i', and the destination of $X(\xi_i, u_1(\cdot), u_2(\cdot))$ by $X(\xi_i)'$. When it is clear from the context, we use $X_x(t)$ instead of $(X_x(t, u_1(\cdot), u_2(\cdot)))$.

We give the inductive limit topology on Ω, namely $x_n \in \Omega$ converges to $x \in \Omega$ if, for some N large and $\forall n \geq N, x, x_n \in \Omega_i$, for some i, and $\| x_n - x \|_{\mathbb{R}^{d_i}} < \varepsilon$.

The following are our basic assumptions.

- (A1) A_i, C_i, D_i are closed, with nonempty interior; $\partial A_i, \partial C_i$ are C^1; D_i are uniformly bounded; $\partial A_i \supseteq \partial \Omega_i, \forall i$.
- (A2) The transition map g is bounded, uniformly Lipschitz continuous, with Lipschitz constant G.

(A3) The vector field f is Lipschitz continuous with Lipschitz constant L in the state variable x and uniformly continuous in the variables u_1 and u_2. Moreover,

$$|f(x,u_1,u_2)|\le F, \quad \forall x\in\Omega \text{ and } \forall(u_1,u_2)\in U_1\times U_2. \tag{2}$$

Furthermore, ∂A_i is compact for all i; for some $\xi_0>0$, the following transversality condition holds:

$$f(x_0,u_1,u_2)\cdot\eta(x_0)\le-\xi_0, \quad \forall x_0\in\partial A_i,\ \forall(u_1,u_2)\in U_1\times U_2, \tag{3}$$

where $\eta(x_0)$ is the unit outward normal to ∂A_i at x_0.

(A4) Let d be the distance between two sets A_i and C_i defined by $d(A_i,C_i)=\inf_{x\in A_i,y\in C_i} d(x,y)$. We assume that $d(A_i,C_i)>\beta>0, \forall i$, and that $\inf_i d(A_i,D_i)\geqslant\beta>0$. Note that this rules out infinitely many autonomous jumps in a finite time.

(A5) The sets U_1,U_2 and V_1,V_2 are compact.

The total discounted cost functional is given by

$$\begin{aligned} &J(x,u_1(\cdot),u_2(\cdot),v_1,v_2,\xi_i,X(\xi_i)')\\ &\quad=\int_0^\infty k(X_x(t),u_1(t),u_2(t))\exp(-\lambda t)dt\\ &\qquad+\sum_{i=1}^\infty c_a(X(\sigma_i),v_1,v_2)\exp(-\lambda\sigma_i)\\ &\qquad+\sum_{i=1}^\infty c_c(X(\xi_i),X(\xi_i)')\exp(-\lambda\xi_i), \end{aligned} \tag{4}$$

where λ is the discount factor, $k:\Omega\times U_1\times U_2\to\mathbb{R}_+$ is the running cost, $c_a:A\times V_1\times V_2\to\mathbb{R}_+$ is the autonomous jump cost, and $c_c: C\times D\to\mathbb{R}_+$ is the controlled jump cost.

Following the Elliot-Kalton approach as in Refs. 4–5, we define the set of all nonanticipating or Elliot-Kalton strategies for P1 and denote it by Γ,

$$\Gamma=\{\alpha:U_2\times V_2\to U_1\times V_1\times[0,\infty)\times D\},$$

where the α are such that

$$\begin{aligned} &(u_2(s),v_2)=(u_2'(s),v_2'), \qquad \text{for } s\le t_0\\ &\quad\Rightarrow\alpha(u_2(s),v_2)=\alpha(u_2'(s),v_2'), \ \text{ for } s\le t_0. \end{aligned}$$

Similarly, we can define the set of nonanticipating strategies β for P2, denoted by Δ. The lower value function V^- and upper value function V^+ are defined as follows:

$$V^-(x) = \inf_{\alpha\in\Gamma} \sup_{u_2(\cdot),v_2} J(x, \alpha(u_2, v_2), u_2(\cdot), v_2),$$

$$V^+(x) = \sup_{\beta\in\Delta} \inf_{(u_1(\cdot),v_1,\xi_i,X(\xi_i)')} J(x, u_1(\cdot), v_1, \xi_i, X(\xi_i)', \beta(u_1, v_1, \xi_i, X(\xi_i)')(\cdot)).$$

We assume the following conditions on the cost functionals:

(C1) k is uniformly continuous in x and is bounded by k_0. Also, k is Lipschitz continuous in the x variable with Lipschitz constant k_1.

(C2) c_a and c_c are uniformly continuous and bounded below by $C' > 0$. Moreover c_a is bounded above by C_0 and is Lipschitz continuous with Lipschitz constant C_1.

In Section 3, we show that the lower and upper value functions are bounded and locally Hölder continuous. In Section 4, we use the dynamic programming principle to derive a PDE and the Hamilton-Jacobi-Isaacs quasivariational inequalities (QVI) satisfied by V^- and V^+ in the viscosity sense. In Section 5, we characterize the value functions as the unique viscosity solutions of the respective QVI and state a condition for the value to exist.

3. Continuity of the Value Function

Let the trajectory given by the solution of (1) and starting from the point x be denoted by $X_x(t, u_1(\cdot), u_2)(\cdot)$. Since $x \in \Omega$, in particular it belongs to some $\Omega_1 \subseteq \mathbb{R}^{d_1}$. Then, from Assumption A3 and the theory of ordinary differential equations, we have

$$|X_x(t, u_1(\cdot), u_2(\cdot)) - X_z(t, u_1(\cdot), u_2(\cdot))| \le \exp(Lt)|x - z|, \tag{5}$$

$$|X_x(t, u_1(\cdot), u_2(\cdot)) - X_x(\bar{t}, u_1(\cdot), u_2(\cdot)) \le F|t - \bar{t}|. \tag{6}$$

Define the first hitting time of the trajectory starting from x and evolving with the fixed controls $u_1(\cdot), u_2(\cdot)$ by

$$t(x, u_1(\cdot), u_2(\cdot)) = \inf\ \{t > 0 | X_x(t, u_1(\cdot), u_2(\cdot)) \in A\}.$$

The next proposition deals with the continuity of the first hitting time in the topology of $\mathbb{R}^{d_i}$ if $x \in \Omega_i$. Note that this is equivalent to proving its continuity in Ω with respect to the inductive limit topology.

Proposition 3.1. Assume (A1)–(A5). Then, the first hitting time t is locally Lipschitz continuous with respect to the initial point; i.e., there exists a $\delta_1 > 0$ depending on f, the distance function from A_i, and ξ_0 such that, $\forall\, y, \bar{y}$ in $B(x, \delta_1)$, a δ_1-neighborhood of x in Ω, we have

$$|t(y, u_1(\cdot), u_2(\cdot)) - t(\bar{y}, u_1(\cdot), u_2(\cdot))| < \tilde{C}|y - \bar{y}|, \quad \text{where } \tilde{C} \text{ depends on } \xi_0.$$

Proof. Step 1. Estimate for Points Near ∂A_i. For simplicity of notation, we drop the suffix i from now onward remembering that the distances are in $\mathbb{R}^{d_i}$. First, we show that there exists $\delta > 0$ and $\tilde{C} > 0$ such that

$$t(x, u_1(\cdot), u_2(\cdot)) < \tilde{C} d(x), \quad \forall x \in B(A, \delta) \backslash \mathring{A},$$

where $B(A, \delta)$ is δ-neighborhood of A and $d(x)$ is a signed distance of x from ∂A, given by

$$d(x) = \begin{cases} -d(x, \partial A), & \text{if } x \in \mathring{A}, \\ 0, & \text{if } x \in \partial A, \\ d(x, \partial A), & \text{if } x \in \bar{A}^c. \end{cases}$$

It is possible to choose $R > 0$ such that, in a small neighborhood of ∂A, say $B(\partial A, R)$, the above signed distance function d is C^1, thanks to Assumption (A1). We denote the derivative of d by ∇d. For all $(\hat{u}_1, \hat{u}_2)$ in $U_1 \times U_2$, we can choose $r_0 < R$ such that

$$f(x, \hat{u}_1, \hat{u}_2) \cdot \nabla d(x) < -\xi_0, \quad \forall x \in B(x_0, r_0). \tag{7}$$

Observe that we can choose r_0 independent of x_0 by using the compactness of ∂A. We fix the controls $u_1(\cdot)$ and $u_2(\cdot)$; from now on, we denote them by u_1, u_2 respectively. For the trajectory starting from $x \in B(x_0, r_0)$, evolving with the fixed controls u_1, u_2, we have

$$\begin{aligned} d(X(s)) - d(x) = & \int_0^s \nabla d(x) \cdot f(x, u_1, u_2) d\tau \\ & + \int_0^s (\nabla d(X(\tau)) - \nabla d(x)) \cdot f(X(\tau), u_1, u_2) d\tau \\ & + \int_0^s \nabla d(x) \cdot (f(X(\tau), u_1, u_2) - f(x, u_1, u_2)) d\tau. \end{aligned}$$

Let c be the bound on ∇d on $B(\partial A, r_0)$. If s is small, so that $X(\tau)$ is in the r_0-neighborhood of ∂A, then ∇d is continuous. So is f. Also, using (3) and (2), we have

$$d(X(s)) - d(x) \le -\xi_0 s + o(F_s) + o(cLs) < -\xi_0 s/2,$$

for $0 < s < \bar{s}$, where $\bar{s}$ is dependent on only the modulus of continuity of f and ∇d and is independent of x. Choose

$$\delta = \min\{r_0, \bar{s}\xi_0/2\}.$$

If x is in a δ-ball around x_0, then $d(x) < \bar{s}/\xi_0 2$; choosing $s_x = 2d(x)/\xi_0$ will imply $s_x < \bar{s}$ and hence $d(X(s_x)) < 0$. Thus, by our definition of d, $X(s_x) \in A$, which implies

$$t(x, u_1, u_2) < s_x = 2d(x)/\xi_0.$$

Then, for $\tilde{C} = 2/\xi_0$, we have

$$t(x, u_1, u_2) < \tilde{C} d(x), \quad \forall x \in B(x_0, \delta)\backslash \mathring{A}.$$

Step 2. Estimate for Any Two Points in Ω. Using the estimate (6), we have

$$|X_{\bar{x}}(t, u_1, u_2) - X_x(t, u_1, u_2)| \leq |\bar{x} - x| \exp(Lt). \tag{8}$$

Let us denote $t(x, u_1, u_2)$ by τ. Then, $X_x(\tau) \in \partial A$. Define $\delta_1 = \delta \exp(-L\tau)$, where δ is as in Step 1. Let us choose x such that $|x - \bar{x}| < \delta_1$. Then,

$$|X_{\bar{x}}(\tau, u_1, u_2) - X_x(\tau, u_1, u_2)| < \delta.$$

Hence,

$$X_{\bar{x}}(\tau, u_1, u_2) \in B(X_x(\tau), \delta)\backslash \mathring{A}.$$

Therefore, by Step 1,

$$t(X_{\bar{x}}(\tau, u_1, u_2), u_1, u_2) < \tilde{C} d(X_{\bar{x}}(\tau, u_1, u_2), u_1, u_2), \tag{9a}$$

$$\begin{aligned} t(\bar{x}, u_1, u_2) &\leq \tau + \inf\{t | X_{X_{\bar{x}(\tau, u_1, u_2)}}(t, u_1, u_2) \in \partial A\} \\ &\leq \tau + t(X_{\bar{x}}(\tau, u_1, u_2), u_1, u_2). \end{aligned} \tag{9b}$$

Using (9), we have

$$t(\bar{x}, u_1, u_2) - \tau \leq t(X_{\bar{x}}(\tau, u_1, u_2), u_1, u_2) \leq \tilde{C}|x - \bar{x}| \exp(L\tau). \tag{10}$$

Interchanging the roles of x and $\bar{x}$, we see that

$$\tau - t(\bar{x}, u_1, u_2) \leq \tilde{C}|x - \bar{x}| \exp(Lt(\bar{x}, u_1, u_2)).$$

Thus, we have

$$|t(x, u_1, u_2) - t(\bar{x}, u_1, u_2)| \leq \tilde{C}|x - \bar{x}| \exp(L\{(\bar{x}, u_1, u_2) \vee t(x, u_1, u_2)\}). \tag{11}$$

$\square$

Theorem 3.1. Continuity of the Lower and Upper Value Functions. Assume (A1)–(A5) and (C1)–(C2). Then, the lower and upper value functions V^- and V^+ of the hybrid game problem are bounded and locally Hölder continuous.

Proof. We prove the theorem for only the lower value function as the proof for the upper value function is completely analogous. First, we show that the lower value function is bounded. Consider the total discounted cost functional J given by (4). Assuming that player P2 chooses not to do any controlled jumps, by our assumptions (C1) and (C2), we have

$$\begin{aligned}|J(x, u_1(\cdot), u_2(\cdot), v_1, v_2, \xi_i, X(\xi_i)')| &\leq k_0 \int_0^\infty \exp(-\lambda t)dt + \sum_{i=1}^{+\infty} C_0 \exp(-\lambda\sigma_i)\\ &\leq k_0/\lambda + C_0 \sum_{i=1}^{\infty} \exp(-\lambda\sigma_i).\end{aligned}$$

From (A4), recalling that $\beta > d(A_i, D_i)$, we have

$$\sigma_{i+1} \geq \sigma_i + \beta/\sup|f| \geq \sigma_i + \beta/F.$$

Hence,

$$\begin{aligned}&\sum_{i=1}^{\infty} \exp(-\lambda\sigma_1) \leq \exp(-\lambda\sigma_i)(1/1 - \exp(-\lambda\beta/F)\\ &\Rightarrow |J(x, u_1(\cdot), u_2(\cdot), v_1, v_2, \xi_i, X(\xi_i)')|\\ &\quad \leq k_0/\lambda + C_0 \exp(-\lambda\sigma_1)(1/1 - \exp(-\lambda\beta/F)). \qquad (12)\end{aligned}$$

Since this is true for all the controls, we can conclude that $V^-(x)$ is bounded.

We now show that V^- is locally Hölder continuous. Let $x, z \in \Omega$. We denote by Π_i the projection on the ith variable of any strategy $\alpha \in \Gamma$. Given $\varepsilon > 0$, there exists $\alpha_0 \in \Gamma$ depending on ε such that

$$\begin{aligned}V^-(z) &\geq \sup_{u_2, v_2} J(z, \alpha_0(u_2, v_2)(\cdot), u_2(\cdot), v_2) - \varepsilon\\ &\geq \int_0^\infty k(X_z(t), (\Pi_1 \circ \alpha_0)(u_2, v_2)(t), u_2(t)) \exp(-\lambda t)dt\\ &\quad + \sum_{i=1}^{\infty} c_a(X_z(\Sigma_i), (\Pi_2 \circ \alpha_0)(u_2, v_2), v_2) \exp(-\lambda\Sigma_i) - \varepsilon.\end{aligned}$$

By the definition of $V^-(x)$, for every α and hence in particular for α_0,

$$V^-(x) \le \sup_{u_2, v_2} J(x, \alpha_0(u_2, v_2)(\cdot), u_2(\cdot), v_2),$$

and there exists $\bar{u_2}, \bar{v_2}$ such that

$$V^-(x) \le J(x, \alpha_0(\bar{u_2}, \bar{v_2})(\cdot), \bar{u_2}(\cdot), \bar{v_2}(\cdot)) + \varepsilon.$$

Thus,

$$\begin{aligned} V^-(x) - V^-(z) &\le J(x, \alpha_0(\bar{u_2}, \bar{v_2})(\cdot), \bar{u_2}(\cdot), \bar{v_2}) \\ &\quad - J(z, \alpha_0(\bar{u_2}, \bar{v_2})(\cdot), \bar{u_2}(\cdot), \bar{v_2}) + 2\varepsilon \\ &\le \int_0^\infty I\,dt + \sum_{i=1}^{\infty} S + 2\varepsilon, \end{aligned}$$

where

$$\begin{aligned} I = {} & |k(X_x(t), (\Pi_1 \circ \alpha_0)(\bar{u_2}, \bar{v_2})(t), \bar{u_2}(t)) \\ & - k(X_z(t), (\Pi_1 \circ \alpha_0)(\bar{u_2}, \bar{v_2})(t), \bar{u_2}(t))| \exp(-\lambda t), \end{aligned}$$

$$\begin{aligned} S = {} & |c_a(X_x(\sigma_i), (\Pi_2 \circ \alpha_0)(\sigma_i), \bar{v_2}) \\ & - c_a(X_z(\Sigma_i), (\Pi_2 \circ \alpha_0)(\Sigma_i), \bar{v_2})| \exp(-\lambda(\sigma_i \vee \Sigma_i)). \end{aligned}$$

Here, we have denoted by $\{\Sigma_i\}$ the hitting times of A for the trajectory starting from z and by $\{\sigma_i\}$ the hitting times of A for the trajectory starting from x evolving with the controls $\alpha_0(\bar{u_2}, \bar{v_2}), \bar{u_2}(\cdot), \bar{v_2}$. We split the integral and summation as follows:

$$V^-(x) - V^-(z) \le \int_0^T I\,dt + \sum_{i=1}^{n} S + \int_T^\infty I\,dt + \sum_{i=n+1}^{\infty} S + 2\varepsilon, \tag{13}$$

where both n and T are to be chosen suitably large so that the tail end of the integral and the summation become small and $\sigma_n < T < \sigma_{n+1}$. By using the bound k_0 on k given by (C1), we get

$$\int_T^\infty I\,dt \le 2k_0/\lambda \exp(-\lambda T); \tag{14}$$

by using the bound C_0 on c_a given by (C2) and doing calculations on lines similar to (12), we get the estimate

$$\sum_{i=n+1}^{\infty} S \le 2C_0(\exp(-\lambda\beta/F))^n (1/\{1 - \exp(-\lambda\beta/F)\}). \tag{15}$$

Without loss of generality, let $\sigma_1 < \Sigma_1$. We will show that, for T large chosen suitably, there exists $\bar{\delta} > 0$ such that, if $|x - z| < \bar{\delta}$, then the sequence of σ_i and Σ_i is such that

$$0 < \sigma_1 \le \Sigma_1 < \sigma_2 \le \cdots \le \sigma_n \le \Sigma_n, \tag{16}$$

where n is the number of hitting times of A in the interval $[0, T]$. Assuming the above claim, we further split the integral as follows:

$$\int_0^T I dt \le \int_0^{\sigma_1} I dt + \int_{\sigma_1}^{\Sigma_1} I dt + \cdots + \int_{\sigma_n}^{\Sigma_n} I dt. \tag{17}$$

In order to evaluate these integrals, we need to find some estimates on the successive hitting times of the trajectories starting from x and z. We state these estimates in following lemmas which will be proved after the proof of the theorem is complete. For simplicity, from here onward, we suppress the fixed control variables $\bar{u}_2(\cdot)$, $\bar{v}_2$, and $\alpha_0(\bar{u}_2, \bar{v}_2)(\cdot)$ for X and k.

Lemma 3.1. Let σ_1 and Σ_1 be as in the theorem above and let

$$x_1 = X_x(\sigma_1), \quad z_1 = X_z(\Sigma_1), \quad x_1, z_1 \in \partial A.$$

If $|x - z| < \delta_1$, where δ_1 is as in Proposition 3.1, then there exists $\tilde{C}$, a constant depending on ξ_0, such that

$$|\sigma_1 - \Sigma_1| \le \tilde{C} \exp(L(\Sigma_1 \vee \sigma_1))|x - z|,$$
$$|x_1 - z_1| \le (1 + F\tilde{C}) \exp(L(\Sigma_1 \vee \sigma_1))|x - z|.$$

Let the jump destination points of x_1 and z_1 be

$$x_1' = g(x_1, (\Pi_2 \circ \alpha_0)(\sigma_1), \bar{v}_2), \; z_1' = g(z_1, (\Pi_2 \circ \alpha_0)(\Sigma_1), \bar{v}_2), \text{ in } \Omega_2 \subseteq \mathbb{R}^{d_2},$$

and let the evolution of the trajectories take place in Ω_2 till the next hitting time. Let σ_i and Σ_i be the ith hitting times of the trajectories starting from x and z, respectively. We denote by x_i, z_i and x_i', z_i' the following:

$$x_i = X_{x_{i-1}}(\sigma_i - \sigma_{i-1}), \qquad x_i' = g(x_i, (\Pi_2 \circ \alpha_0)(\sigma_i), \bar{v}_2),$$
$$z_i = X_{z_{i-1}}(\Sigma_i - \Sigma_{i-1}), \qquad z_i' = g(z_i, (\Pi_2 \circ \alpha_0)(\Sigma_i), \bar{v}_2).$$

Lemma 3.2. Assume (16) and let us denote $F\tilde{C} + G(1 + F\tilde{C})$ by μ. Then,

$$|\sigma_i - \Sigma_i| \le \tilde{C} \exp(L\Sigma_i)\mu^{i-1}|x - z|,$$
$$|x_i - z_i| \le \exp(L\Sigma_i)(F\tilde{C} + 1)\mu^{i-1}|x - z|,$$

whenever $|x - z| < \delta_i$, where $\delta_i := \min\{\delta_1, \delta_2, \ldots, \delta_1 \exp(-L\Sigma_i)/\mu^{i-1}\}$.

Using these lemmas, now we estimate the integral in (17). This involves two types of integrals,

$$\int_{\sigma_i}^{\Sigma_i} I dt \quad \text{and} \quad \int_{\Sigma_i}^{\sigma_{i+1}} I dt.$$

If

$$|x - z| < \delta_n, \qquad \text{where } \delta_n = \min\{\delta_1, \delta_2, \ldots, \delta_1 \exp(-L\Sigma_n)/\mu^{n-1}\},$$

we can estimate the above integrals using Lemma 3.1 and Lemma 3.2. We use the bound on k to evaluate the first integral,

$$\int_{\sigma_i}^{\Sigma_i} I dt \leq (2k_0/\lambda)[\exp(-\lambda\sigma_i) - \exp(-\lambda\Sigma_i)] \leq 2k_0|\sigma_i - \Sigma_i|.$$

Using Lemma 3.2, we have

$$\int_{\sigma_i}^{\Sigma_i} I dt \leq 2k_0\tilde{C}\mu^{i-1}\exp(L\Sigma_i)|x - z|. \tag{18}$$

To evaluate the second integral, we use the Lipschitz continuity of k. Hence,

$$\int_{\Sigma_i}^{\sigma_{i+1}} I dt \leq k_1 \int_{\Sigma_i}^{\sigma_{i+1}} |X_{x_i'}(t - \sigma_i) - X_{z_i'}(t - \Sigma_i)| \exp(-\lambda t) dt. \tag{19}$$

By the semigroup property and (6),

$$\begin{aligned}|X_{x_i'}(t - \sigma_i) - X_{z_i'}(t - \Sigma_i)| &= |X_{X_{x_i'}(\Sigma_i - \sigma_i)}(t - \Sigma_i) - X_{z_i'}(t - \Sigma_i)| \\ &\leq \exp(L(t - \Sigma_i))|X_{x_i'}(\Sigma_i - \sigma_i) - z_i'|.\end{aligned}$$

By similar calculations, we get

$$|X_{x_i'}(\Sigma_i - \sigma_i) - z_i'| \leq \mu^i \exp(L\Sigma_i)|x - z|. \tag{20}$$

Substituting the above estimates in (19), we get

$$\int_{\Sigma_i}^{\sigma_{i+1}} I dt \leq k_1 \exp(-L\Sigma_i)\mu^i \exp(L\Sigma_i)|x - z| \int_{\Sigma_i}^{\sigma_{i+1}} \exp((L - \lambda)t) dt.$$

For $L \neq \lambda$,

$$\int_{\Sigma_i}^{\sigma_{i+1}} I dt \leq k_1\mu^i|x - z|\{\exp((L - \lambda)T - 1)\}/L - \lambda, \tag{21}$$

and for $L=\lambda$,

$$\int_{\Sigma_i}^{\sigma_{i+1}} I\,dt \le k_1\mu^i|x-z||\sigma_{i+1}-\Sigma_i| \le k_1\mu^i\ \ |x-z|\,2T. \tag{22}$$

For $L\neq\lambda$, by using (18), (21), we have

$$\int_0^T I\,dt \le \sum_{i=1}^n 2k_0\tilde{C}\mu^{i-1}\exp(LT)|x-z| \\ +\sum_{i=1}^n (k_1\mu^i/L-\lambda)\{\exp((L-\lambda)T-1)\}|x-z|;$$

hence, for $L\neq\lambda$,

$$\int_0^T I\,dt \le [\mu^{n-1}/\mu-1]|x-z|\Big[2k_0\tilde{C}+k_1\exp((L-\lambda)T-1)/L-\lambda\Big]. \tag{23}$$

For $L=\lambda$, using (18) and (22), we have

$$\int_0^T I\,dt \le \sum_{i=1}^n 2k_0\tilde{C}\mu^{i-1}|x-z| + \sum_{i=1}^n k_1T\mu^i|x-z|;$$

thus, for $L=\lambda$,

$$\int_0^T I\,dt \le [\mu^n-1/\mu-1]|x-z|\Big[2k_0\tilde{C}+2k_1T\Big]. \tag{24}$$

Furthermore, we need to estimate the summation term, arising from discrete cost, in the estimate (13). By using (C2) and Lemma 3.2, we get

$$\sum_{i=1}^n S \le \sum_{i=1}^n 2C_1|x_i-z_i|\exp(-\lambda(\sigma_i\vee\Sigma_i)) \\ \le 2C_1\sum_{i=1}^n (F\tilde{C}+1)\exp(LT)\mu^{i-1}|x-z|.$$

Hence,

$$\sum_{i=1}^n S \le 2C_1(F\tilde{C}+1)\exp(LT)|x-z|(\mu^{n-1}-1/\mu-1). \tag{25}$$

Since μ is a constant, without loss of generality we can assume that

$$\mu^n/\mu-1 < 2\mu^n.$$

Also, observe that

$$\sigma_i - \sigma_{i+1} \geq \beta/F$$

implies that

$$T \geq \sigma_{n+1} - \sigma_1 \geq n\beta/F;$$

hence,

$$n < TF/\beta. \tag{26}$$

Using (23), (25), (26), (14), (15) in (13), for $L \neq \lambda$ we have

$$\begin{aligned} V^-(x) - V^-(z) \leq\ & 4k_0\tilde{C}e^{LT}\mu^{TF/\beta}|x-z| \\ & +2k_1\mu^{TF/\beta}\exp((L-\lambda)T-1)/L-\lambda|x-z| \\ & +2k_0/\lambda\exp-\lambda T + 2C_1\exp(LT)\mu^{TF/\beta}|x-z| \\ & +2C_0(\exp(-\lambda\beta/F))^{TF/\beta}1/1-\exp(-\lambda\beta/F)+2\varepsilon. \end{aligned} \tag{27}$$

Now, we further restrict

$$|x-z| < (\delta_1)^{1/1-\theta}, \quad \text{for some } \theta \text{ such that } 0<\theta<1,$$

where δ_1 is as in Proposition 3.1. Then, choosing T such that

$$\mu^{TF/\beta}\exp(LT) = |x-z|^{-\theta}$$

or

$$T = -\theta\log|x-z|/\lambda + F\log\mu/\beta, \tag{28}$$

this together with the choice of $|x-z|$ implies

$$\begin{aligned} \delta_n &= \delta_1/\exp(L\Sigma_n)\mu^{n-1} \\ &> \delta_1/\exp(LT)\mu^{TF/\beta} \\ &= \delta_1|x-z|^\theta \\ &> |x-z|. \end{aligned}$$

Thus,

$$|x-z| < \delta_n$$

and hence, the estimate (27) holds true for our choice of T. Then, substituting the value of T in this estimate, for $L \neq \lambda$ we get

$$
\begin{aligned}
V^-(x) - V^-(z) \leq 4k_0\tilde{C}|x-z|^{1-\theta} &+ (k_1/L - \lambda)|x-z|^{1-\theta} + C_1|x-z|^{1-\theta} \\
&+ (2k_0/\lambda)|x-z|^{\lambda\theta/(F\log\mu/\beta)+L} \\
&+ 2C_0|x-z|^{\lambda\theta/(F\log\mu/\beta)+L} + 2\varepsilon.
\end{aligned}
$$

Thus, V^- is Hölder continuous in a $\delta_1^{1/1-\theta}$ ball around x, with Hölder constant θ_1,

$$\theta_1 = \min\{1-\theta, \lambda\theta/\{(F\log\mu/\beta) + L\}\}, \quad 0 < \theta < 1.$$

Similar calculations for $L = \lambda$ yield that V^- is locally Hölder continuous in the $\delta_1^{1/1-\theta}$ ball around x with Hölder constant θ_1 for all θ_1 such that

$$\theta_1 < \min\{1-\theta, L\theta/\{(F\log\mu/\beta) + L\}\}, \quad 0 < \theta < 1.$$

This proves the local Hölder continuity of V^-.

Now, we want to justify our claim in (16); i.e., if $\sigma_1 < \Sigma_1$, we can choose $|x-z|$ small enough such that (16) holds. If we restrict $|x-z|$ such that

$$|x-z| \leq \min\left\{\beta/(4F\tilde{C}), (\beta/(4\tilde{C}F))^{1/1-\theta}\right\},$$

then by Lemma 3.2

$$|\Sigma_i - \sigma_i| \leq \tilde{C}\exp(LT)\mu^{TF/\beta}|x-z|.$$

By our choice of T,

$$|\Sigma_i - \sigma_i| \leq \tilde{C}|x-z|^{1-\theta} \leq \beta/(4F) < 1/2|\sigma_i - \sigma_{i+1}|.$$

This, together with the assumption $\sigma_1 < \Sigma_1$, implies

$$\sigma_i < \Sigma_i < \sigma_{i+1}, \quad \forall i.$$

So, our claim is justified. This completes the proof of the continuity of V^-. □

Proof of Lemma 3.1. Note here that, be Proposition 3.1, we have the estimate on $|\sigma_1 - \Sigma_1|$ given by (11),

$$|\sigma_1 - \Sigma_1| < \tilde{C} \exp(L(\Sigma_1 \vee \sigma_1)) \; |x - z|. \tag{29}$$

Using this, we estimate $|x_1 - z_1|$. By assumption, $\sigma_1 < \Sigma_1$. Thus,

$$\begin{aligned} |x_1 - z_1| &= |X_x(\sigma_1) - X_z(\Sigma_1)| \\ &\le |X_x(\sigma_1) - X_z(\sigma_1)| + |X_z(\sigma_1) - X_z(\Sigma_1)|. \end{aligned}$$

Using (6), we get

$$|X_x(\sigma_1) - X_z(\sigma_1)| < \exp(L\sigma_1)|x - z|,$$

while (6) and (29) lead to

$$|X_z(\sigma_1) - X_z(\Sigma_1)| \le F|\sigma_1 - \Sigma_1| \le F\tilde{C} \exp(L\Sigma_1)|x - z|.$$

Combining these estimates, we have

$$|x_1 - z_1| \le \exp(L\Sigma_1)|x - z|(1 + F\tilde{C}), \quad \text{for } z \in B(x, \delta_1). \qquad \square$$

Proof of Lemma 3.2. Without loss of generality, we assume that $x_i', z_i' \in \Omega_{i+1} \subseteq \mathbb{R}^{d_{i+1}}$. We apply Proposition 3.1 and the above lemma recursively to find estimates on successive hitting times and points to conclude the estimate in the lemma. $\square$

4. Quasivariational Inequality

The following theorem deals with the dynamic programming principle (DPP) satisfied by V^-. The DPP for V^+ is analogous.

Theorem 4.1. Dynamic Programming Principle DPP$^-$ for the Lower Value Function V^-. For any $T > 0$,

$$\begin{aligned} V^-(x) = \inf_{\alpha \in \Gamma} \sup_{u_2(\cdot), v_2} \Bigg\{ &\int_0^T k(X_x(t), \Pi_1 \circ \alpha(u_2, v_2)(t), u_2(t)) \exp(-\lambda t) dt \\ &+ \sum_{\sigma_i < T} S_a(\alpha) + \sum_{\Pi_3 \circ \alpha(u_2, v_2) < T} S_c(\alpha) \\ &+ \exp(-\lambda T) V^-(X_x(T, \Pi_1 \circ \alpha(u_2, v_2), u_2)) \Bigg\}, \end{aligned}$$

where for fixed controls $u_2(\cdot), v_2$,

$$S_a(\alpha) = \exp(-\lambda\sigma_i)c_a(X(\sigma_i), \Pi_2 \circ \alpha(u_2, v_2)(\sigma_i), v_2),$$

$$S_c(\alpha) = \exp(-\lambda\Pi_3 \circ \alpha(u_2, v_2))c_c(X(\Pi_3 \circ \alpha(u_2, v_2)), \Pi_4 \circ \alpha(u_2, v_2)).$$

If σ_1 is the first hitting time of A, then the DPPA$^-$ for V^- is given by

$$V^-(x) = \inf_{\alpha\in\Gamma} \sup_{u_2(\cdot), v_2} \left\{ \int_0^{\sigma_1} k(X_x(t, \Pi_1 \circ \alpha(u_2, v_2)(t), u_2(t)) \exp(-\lambda t)dt + \exp(-\lambda\sigma_1)M^- V^-(X_x(\sigma_1, \Pi_1 \circ \alpha(u_2, v_2), u_2) \right\},$$

where

$$M^-\phi(x) = \inf_{v_1} \sup_{v_2}\{\phi(g(x, v_1, v_2)) + c_a(x, v_1, v_2)\}.$$

If ξ_1 is the first hitting time of C, then the DPPC$^-$ is given by

$$V^-(x) = \inf_{\alpha\in\Gamma} \sup_{u_2(\cdot), v_2} \left\{ \int_0^{t_1} k(X_x(t, \Pi_1 \circ \alpha(u_2, v_2)(t), u_2(t)) \exp(-\lambda t)dt + \exp(-\lambda t_1)NV^-(X_x(\xi_1, \Pi_1 \circ \alpha(u_2, v_2), u_2)) \right\},$$

where

$$N\phi(x) = \inf_{x'\in D}\{\phi(x') + c_c(x, x')\}.$$

Proof. We prove the general dynamic programming principle for any $T > 0$ given by DPP$^-$. The proofs of DPPA$^-$ and DPPC$^-$ follow on similar lines. Let us denote RHS of DPP$^-$ by $w(x)$. Fix $\varepsilon > 0$. Note that, by the definition of V^-, for any z there exists $\alpha_z \in \Gamma$ such that, for all $u_2(\cdot), v_2$,

$$V^-(z) \geq J(z, \alpha_z(u_2, v_2)(\cdot), u_2(\cdot), v_2) - \varepsilon. \tag{30}$$

We first prove that

$$V^-(x) \leq w(x).$$

Choose $\bar{\alpha} \in \Gamma$ such that

$$w(x) \geq \sup_{u_2(\cdot), v_2} \left\{ \int_0^T k(X_x(t), \Pi_1 \circ \bar{\alpha}(u_2, v_2)(t), u_2(t)) \exp(-\lambda t) dt \right. \\ + \sum_{\sigma_i < T} S_a(\bar{\alpha}) + \sum_{\xi_i < T} S_c(\bar{\alpha}) \\ \left. + \exp(-\lambda T) V^-(X_x(T), \bar{\alpha}(u_2, v_2), u_2, v_2)) \right\} - \varepsilon. \quad (31)$$

Now, define $\delta \in \Gamma$ by

$$\delta(u_2, v_2)(s) = \bar{\alpha}(u_2, v_2)(s), \qquad \text{if } s \leq T,$$
$$\delta(u_2, v_2)(s) = \alpha_z(u_2, v_2)(s - T), \quad \text{if } s > T,$$

with

$$z = X_x(T, \bar{\alpha}(u_2, v_2), u_2, v_2)$$

and α_z as chosen in (30),

$$X_x(s + T, \delta(u_2, v_2), u_2, v_2) = X_z(s, \alpha_z(u_2, v_2)(s + T), u_2(s + T), v_2).$$

So, by a change of variables, $\tau = s + T$, we have

$$J(z, \alpha_z(u_2, v_2)(\cdot + T), u_2(\cdot + T), v_2) \\ = \left\{ \int_T^\infty k(X_x(\tau), \Pi_1 \circ \delta(u_2, v_2)(\tau), u_2(\tau)) \exp(-\lambda(T - \tau)) d\tau \right. \\ \left. + \sum_{\sigma_i > T} S_a(\delta) + \sum_{\xi_i > T} S_c(\delta) \right\}.$$

Substituting the above in (30), we get

$$V^-(z) \geq \left\{ \int_T^\infty k(X_x(\tau), \Pi_1 \circ \delta(u_2, v_2)(\tau), u_2(\tau)) \exp[-\lambda(T - \tau)] d\tau \right. \\ \left. + \sum_{\sigma_i > T} S_a(\delta) + \sum_{\xi_i > T} S_c(\delta) \right\} - \varepsilon.$$

Substituting back in (31), we have

$$\begin{aligned} w(x) &\geq \sup_{u_2(\cdot),v_2} \left\{ \int_0^\infty k(X_x(t), \Pi_1 \circ \delta(u_2, v_2)(t), u_2(t)) \exp(-\lambda t) dt \right. \\ &\qquad \left. + \sum_{i=1}^\infty S_a(\delta) + \sum_{i=1}^\infty S_c(\delta) \right\} - 2\varepsilon \\ &\geq J(x, \delta(u_2, v_2)(\cdot), u_2(\cdot), v_2) - 2\varepsilon. \end{aligned}$$

Hence,

$$w(x) \geq V^-(x) - 2\varepsilon.$$

Since ε is arbitrary,

$$w(x) \geq V^-(x).$$

To prove the reverse inequality, i.e.,

$$w(x) \leq V^-(x),$$

we choose $\bar{u}_2(\cdot), \bar{v}_2$ such that

$$\begin{aligned} w(x) &\leq \int_0^T k(X_x(t), \Pi_1 \circ \alpha_x(\bar{u}_2, \bar{v}_2)(t), \bar{u}_2(t)) \exp(-\lambda t) dt \\ &\quad + \sum_{\sigma_i < T} S_a(\alpha_x) + \sum_{\xi_i < T} S_c(\alpha_x) \\ &\quad + \exp(-\lambda T) V^-(X_x(T, \alpha_x(\bar{u}_2, \bar{v}_2), \bar{u}_2, \bar{v}_2)) + \varepsilon, \end{aligned} \tag{32}$$

where α_x is as in (30) and S_a and S_c are now with respect to the fixed controls $\bar{u}_2, \bar{v}_2$. Now, for each $u_2(\cdot), v_2$, define $\tilde{u}_2, \tilde{v}_2$ by

$$(\tilde{u}_2(s), \tilde{v}_2) = (\bar{u}_2(s), \bar{v}_2), \qquad \text{if } s \leq T, \tag{33a}$$

$$(\tilde{u}_2(s), \tilde{v}_2) = (u_2(s - T), v_2), \quad \text{if } s > T. \tag{33b}$$

Then, define $\hat{\alpha} \in \Gamma$ and z by

$$\hat{\alpha}(u_2(s), v_2) = \alpha_x(\tilde{u}_2, \tilde{v}_2), \tag{34}$$

$$z = X_x(T, \alpha_x(\bar{u}_2, \bar{v}_2), \bar{u}_2, \bar{v}_2), \tag{35}$$

and choose $u_2(\cdot), v_2$ such that

$$V^-(z) \leq J(z, \hat{\alpha}(u_2, v_2)(\cdot), u_2(\cdot), v_2) + \varepsilon. \tag{36}$$

So, by using (32), (35), (36), we get

$$w(x) \le 2\varepsilon + \int_0^T k(X_x(t), \Pi_1 \circ \alpha_x(\bar{u}_2, \bar{v}_2)(t), \bar{u}_2(t)) \exp(-\lambda t) d\tau + \sum_{\sigma_i > T} S_a(\alpha_x) + \sum_{\xi_i > T} S_c(\alpha_x). \tag{37}$$

Observe that (33)–(35) imply

$$X_x(\tau, \alpha_x(\tilde{u}_2, \tilde{v}_2), \tilde{u}_2, \tilde{v}_2) = X_z(\tau - T, \hat{\alpha}(u_2, v_2), u_2, v_2), \quad \text{if } \tau > T.$$

So, doing the change of variables $\tau = s + T$, we have

$$\begin{aligned} &J(z, \hat{\alpha}(u_2, v_2)(\cdot), u_2(\cdot), v_2) \\ &= \int_T^\infty k(X_x(\tau, \alpha_x(\tilde{u}_2, \tilde{v}_2)(\tau), \tilde{u}_2(\tau), \tilde{v}_2) \exp(-\lambda(T - \tau)) d\tau \\ &\quad + \sum_{\sigma_i > T} S_a(\alpha_x) + \sum_{\xi_i > T} S_c(\alpha_x). \end{aligned} \tag{38}$$

Using (33)–(35) in (38) implies

$$w(x) \le J(x, \alpha_x(\tilde{u}_2, \tilde{v}_2)(\cdot), \tilde{u}_2(\cdot), \tilde{v}_2) + 2\varepsilon.$$

Combining this and (30), we have

$$w(x) \le V^-(x) + 3\varepsilon,$$

which proves the reverse inequality. □

Theorem 4.2. Quasivariational Inequality. Under Assumptions (A1)–(A5) and (C1), (C2), the value function V^- satisfies the following lower quasivariational inequality in the viscosity sense:

$$(\mathrm{QVI}^-) \quad V^-(x) = \begin{cases} M^- V^-(x), & \forall x \in A, \\ \max\{N V^-(x), -H^-(x, \nabla V^-(x))\}, & \forall x \in C, \\ -H^-(x, \nabla V^-(x)), & \forall x \in \Omega \backslash (A \cup C), \end{cases}$$

where

$$H^-(x, p) = \inf_{u_1} \sup_{u_2} \{-k(x, u_1, u_2) - f(x, u_1, u_2) \cdot p\}/\lambda.$$

Proof. Let $x \in A$. Then, the first hitting time of the trajectory is $\sigma_1 = 0$. Hence, by DPPA$^-$, we get

$$V^-(x) = M^- V^-(x).$$

Now, we consider the case $x \in \Omega \backslash A \cup C$. Let

$$r = \min\{d(x, \partial A), d(x, \partial C)\}.$$

Choose $R < r$. Then, in the ball $B(x, R)$, no impulses are applied. Now, V^- is continuous at x; let $\phi \in C^1(\Omega)$ be such that $V^- - \phi$ has a local maximum at x and $V^-(x) = \phi(x)$. Then, to show that V^- is a subsolution, we must prove that

$$\phi(x) + H^-(x, \nabla\phi(x)) \leq 0.$$

Should this fail, there would exist some $\nu > 0$ such that

$$\phi(x) + H^-(x, \nabla\phi(x)) \geq \nu > 0. \tag{39}$$

Then, a lemma similar to the one in Ref. 4 (Chapter 8, Lemma 1.11) will imply that, for sufficiently small $t > 0$, there exists $\alpha^* \in \Gamma$ such that, for all $u_2 \in U_2$,

$$\int_0^t \{k(X_x(s)) + f(X_x(s)) \cdot \nabla\phi(X_x(s)) - \lambda\phi(x)\} \exp(-\lambda s) ds \leq -\nu t/4, \tag{40}$$

where

$$X_x(s) = (X_x(s, \alpha^*(u_2, v_2)(s), u_2(s)), \Pi_1 \circ \alpha^*(u_2, v_2)(s), u_2(s)).$$

Choose τ small enough such that $\tau < t$ and $X_x(\tau) \in B(x, R)$. By our choice of R and τ, we see that τ is less than the first hitting time, that is, $\tau < \sigma_1$ and $\tau < \xi_1$. Since $V^- - \phi$ has a maximum at x and $V^-(x) = \phi(x)$,

$$\exp(-\lambda\tau)\phi(X_x(\tau)) - \phi(x) \geq \exp(-\lambda\tau)V^-(X_x(\tau)) - V^-(x). \tag{41}$$

Substituting (41) in (40), we get

$$\inf_{\alpha \in \Gamma} \sup_{u_2(\cdot), v_2} \left\{ \int_0^\tau k(X_x(s)) + \exp(-\alpha\tau)V^-(X_x(\tau, \alpha^*(u_2, v_2), u_2) \right\} - V^-(x)$$
$$\leq -\nu\tau/4 < 0.$$

This contradicts DPP$^-$. Hence, V^- is a viscosity subsolution of HJB equation.

To show that V^- is a viscosity supersolution, let $V^- - \phi$ have a local minimum at x and let

$$V^-(x) = \phi(x).$$

Assume by contradiction that

$$\phi(x) + H^-(x, \nabla\phi(x)) = -\nu < 0.$$

By definition of H^-, there exists $u_2^* \in U_2$ such that

$$\phi(x) - f(x, u_1, u_2^*).\nabla\phi(x) - k(x, u_1, u_2^*) \leq -\nu,$$

for all $u_1 \in U_1$. So, for τ small enough, that is $\tau < t$ and for any $\alpha \in \Gamma$,

$$\phi(x_x(s)) - f(X_x(s)) \cdot \nabla\phi(X_x(s)) - k(X_x(s)) \leq -\nu/2,$$

where

$$X_x(s) = (X_x(s), \Pi_1 \circ \alpha(u_2^*, v_2)(s), u_2^*(s)).$$

For $0 \leq s \leq \tau$, multiplying by $\exp(-\lambda s)$ and integrating from 0 to τ, for τ small, we have

$$\phi(x) - \exp(-\lambda\tau)\phi(X_x(\tau)) - \int_0^\tau k(X_x(s))\exp(-\lambda s)ds \leq -\nu t/4. \tag{42}$$

From

$$\phi(x) - \exp(-\lambda\tau)\phi(X_x(\tau)) \geq V^-(x) - \exp(-\lambda\tau)V^-(X_x(\tau)),$$

we obtain

$$\exp(-\lambda\tau)V^-(X_x(\tau)) + \int_0^\tau k(X_x(s))\exp(-\lambda s)ds \geq \nu t/2 + V^-(x).$$

Thus,

$$\inf_{\alpha\in\Gamma} \sup_{u_2(\cdot), v_2} \left\{ \int_0^\tau k(X_x(s))\exp(-\lambda s)ds + \exp(-\lambda\tau)V^-(X_x(\tau)) \right\} > V^-(x),$$

which is a contradiction to DPP$^-$ and this completes the proof that V^- is a viscosity supersolution of HJI equation.

Now, consider the case $x \in C$. If $V^-(x) = NV^-(x)$, there is nothing to prove. Suppose that $V^-(x) < NV^-(x)$. Then, we show that, whenever there exists a $r > 0$ and a ball $B(x, r)$ around x such that it is not optimal to apply any impulses on $B(x, r)$, then we can do the analysis in this ball to conclude as in the case of $x \in \Omega\backslash A \cup C$. We claim that there exists $\varepsilon > 0$ such that

$$V^-(x) = \inf_{\alpha} \sup_{u_2(\cdot), v_2} \left\{ \int_0^{t_1} k(X_x(t), \Pi_1 \circ \alpha(u_2, v_2)(t), u_2(t)) \exp(-\lambda t) dt \right.$$
$$\left. + N V^-(X_x(t_1)) | t_1 > \varepsilon \right\},$$

where t_1 is first time when the controller chooses to jump. If not, then $\varepsilon = 0$, which implies $\xi_1 = 0$, which by DPPC implies

$$V^-(x) = N V^-(x),$$

which is a contradiction to

$$V^-(x) < N V^-(x).$$

Hence, $\varepsilon > 0$. Choose

$$r < \min\{d(x, X_x(\varepsilon)), d(A, C)\}.$$

Then, in the ball $B(x, r)$, no impulses are applied. So, we can do the analysis in this ball around x and can conclude as in the earlier case. This proves the quasivariational inequality for the case $x \in C$. □

For the upper value function, define $M^+\phi(x)$ and $H^+(x, p)$ by

$$M^+\phi(x) = \sup_{v_2} \inf_{v_1} \{\phi(g(x, v_1, v_2)) + c_a(x, v_1, v_2)\},$$
$$H^+(x, p) = \sup_{u_2} \inf_{u_1} \{-k(x, u_1, u_2) - f(x, u_1, u_2) \cdot p\}/\lambda.$$

The upper value function V^+ satisfies the following quasivariational inequality:

$$(\mathrm{QVI}^+) \quad V^+(x) = \begin{cases} M^+ V^+(x), & \forall x \in A, \\ \max\{N V^+(x), -H^+(x, \nabla V^+(x))\}, & \forall x \in C, \\ -H^+(x, \nabla V^+(x)), & \forall x \in \Omega\backslash(A \cup C). \end{cases}$$

5. Existence of Value

Our aim in this section is to characterize the lower and upper value functions V^- and V^+ as the unique viscosity solutions of the corresponding quasivariational inequalities. Further, we give a condition for equality to hold between V^- and V^+ and thus the game to have the value.

Theorem 5.1. Uniqueness. Assume (A1)–(A5) and (C1), (C2). Let $w_1, w_2 \in BC(\Omega)$ be two viscosity solutions of the quasivariational inequality QVI$^-$. Then, $w_1 = w_2$.

Proof. The idea of the proof is to show that

$$w_1(x) \leq w_2(x), \quad \forall x \in \Omega.$$

For this, we define the following auxiliary function Φ on $\bigcup_{i=1}^{\infty}(\Omega_i \times \Omega_i)$:

$$\Phi(x, y) = \gamma w_1(x) - w_2(y) - (1/2\epsilon)|x - y|^2 - \kappa(\langle x\rangle^m + \langle y\rangle^m), \tag{43}$$

where

$$m \in (0, 1), \quad \langle x\rangle^m = (1 + |x|^2)^{m/2}, \quad \gamma \in (0, 1),$$

ε and κ are positive parameters to be chosen suitably later on. Let

$$\sup_i \sup_{\Omega_i \times \Omega_i} \Phi(x, y) > 0.$$

If the above supremum is not attained at some finite point (x_0, y_0) in some fixed Ω_i, we can work with an approximate supremum of Φ over $\Omega \times \Omega$; that is, for give $n > 0$, we can find $(x_n, y_n) \in \Omega_n$ such that

$$\sup_{\Omega \times \Omega} \Phi(x, y) < \Phi(x_n, y_n) + 1/n.$$

The proof in this case is very similar to the one given below; however, for simplicity, we give the proof in the case when the supremum of Φ is attained at (x_0, y_0) in some fixed Ω_i.

Since x_0 and y_0 can lie in different sets in Ω_i, $w_1(x_0)$ and $w_2(y_0)$ satisfy different equations from QVI$^-$. We list below the different cases which arise:

Case 1. $(x_0, y_0) \in A \times C$ or $C \times A$.

Case 2. $(x_0, y_0) \in \Omega\backslash(A \cup C) \times \Omega\backslash(A \cup C)$.

Case 3. $x_0, y_0 \notin A$ and one of x_0 or $y_0 \in C$.

Case 4. $x_0, y_0 \notin C$ and one of x_0 or $y_0 \in A$.

We claim that, under a suitable choice of ε, γ and κ, only Case 2 occurs and no other case can occur. In Case 2, both w_1 and w_2 satisfy the HJI equation and we can complete the proof by slightly modifying the usual comparison method. First, we list some estimates in the following lemma which can be proved by standard techniques using the facts that w_1, w_2 are bounded, continuous, and (x_0, y_0) is maximum point of Φ.

Lemma 5.1. Let Φ be as in (43) and let (x_0, y_0) be such that $\Phi(x_0, y_0) = \sup \Phi$. Let $\hat{C}$ be a generic constant. Then:

(i) $\left(|x_0 - y_0|^2/\epsilon\right) \leq \hat{C}$, for $\hat{C}$ independent of κ and ε.

(ii) $\sqrt{\kappa}\ |x_0|, \sqrt{\kappa}\ |y_0| \leq \hat{C}$, for some $\hat{C}$ independent of κ and ε.

(iii) $\left(|x_0 - y_0|^2/\epsilon\right) \leq \omega\left(\sqrt{\hat{C}\epsilon}\right)$, where ω is the local modulus of continuity of w_1 and w_2 in the ball of radius $\tilde{C}/\sqrt{\kappa}$.

Now, we consider the different cases listed earlier.

Case 1. $(x_0, y_0) \in A \times C$ or $C \times A$. Without loss of generality, let $(x_0, y_0) \in A \times C$. Note that $d(A, C) > \beta \Rightarrow |x_0 - y_0| > \beta$. On the other hand, by Lemma 5.1 (i), $|x_0 - y_0| < \sqrt{\hat{C}\epsilon}$. So, choosing ϵ such that $\sqrt{\hat{C}\epsilon} < \beta/2$, we have $|x_0 - y_0| < \beta/2$, which is a contradiction. Hence, Case 1 does not occur for small ε.

Case 2. Let $(x_0, y_0) \in \Omega\backslash(A \cup C) \times \Omega\backslash(A \cup C)$. Then, using the definition of viscosity subsolution and supersolution for w_1 and w_2, respectively, we have

$$\begin{aligned} r w_1(x_0) - w_2(y_0) &\leq H^-(y_0, (x_0 - y_0)/\varepsilon - \kappa m \langle y_0\rangle^{m-2} y_0) \\ &\quad - r H^-(x_0, (1/\gamma)((x_0 - y_0)/\varepsilon + \kappa m \langle x_0\rangle^{m-2} x_0)) \\ &\leq (L)(|x_0 - y_0|^2/\varepsilon) + (L/\gamma)\kappa m |\langle x_0\rangle^{m-2} x_0||x_0 - y_0| \\ &\quad + k_1|x_0 - y_0| + (1 - r)k \\ &\quad + F\kappa m |x_0\langle x_0\rangle^{m-2} + y_0\langle y_0\rangle^{m-2}|. \end{aligned}$$

Here, in the second inequality, we have used the fact that, under our Assumptions (A1)–(A5), the Hamiltonian H^- satisfies the structural condition

$$|H^-(x, p) - H^-(y, p) \leq F|p - q| + L|q||x - y| + k_1(|x - y|), \quad \forall x, p.$$

Now, by using the estimates in Lemma 5.1 and the fact that $|\langle x_0\rangle^m x_0|$ remains bounded for $m \in (0, 1)$, we have that

$$r w_1(x_0) - w_2(y_0) \leq O(\sqrt{\varepsilon}) + \omega\left(\sqrt{\hat{C}\varepsilon}\right) + (1 - r)k + O(\kappa).$$

Also, for given $x \in \Omega$,

$$\Phi(x,x) \leq \Phi(x_0, y_0),$$

so

$$\gamma w_1(x) - w_2(x) \leq \gamma w_1(x_0) - w_2(y_0) + 2\kappa \langle x \rangle^m.$$

Hence, for x fixed,

$$\gamma w_1(x) - w_2(x) \leq O(\sqrt{\varepsilon}) + O(\kappa) + (1-r)k.$$

Now, first sending ε to 0 and then κ to 0 and γ to 1, we get that

$$w_1(x) \leq w_2(x), \quad \forall x \in \Omega.$$

Case 3. $x_0, y_0 \notin A$ and on of $x_0, y_0 \in C$. Without loss of generality, let $y_0 \in C$. Since $x_0 \notin A$ and the fact that w_1 is a subsolution of QVI$^-$ implies $w_1(x_0) + H^-(x_0, \nabla w_1(x_0)) \leq 0$, we have

$$y_0 \in C \Rightarrow \max\{w_2(y_0) + H^-(y_0, \nabla w_2(y_0)), w_2(y_0) - N w_2(y_0)\} = 0.$$

Also, w_2 is a solution of QVI$^-$, in particular it is a supersolution. Hence, either $w_2 + H^- \geq 0$ or $w_2 - N w_2 \geq 0$ at y_0. If $w_2(y_0) + H^-(y_0, \nabla w_2(y_0)) \geq 0$, we can proceed as in Case 2. Else, assume $w_2(y_0) - N w_2(y_0) \geq 0$. Since w_2 is also a subsolution,

$$w_2(x) \leq N w_2(x), \quad \forall x \in C;$$

therefore,

$$w_2(y_0) = N w_2(y_0) = \inf_{y' \in \mathrm{D}} w_2(y') + c_c(y_0, y').$$

If the infimum in $N w_2(y_0)$ is not attained, we deal with an approximate minimum at the expense of an extra term $1/n$. For given $n > 0$, we can choose $y_0' \in D_j$ such that

$$N w_2(y_0) > w_2(y_0') + c_c(y_0, y_0') - 1/n.$$

If $y_0' D_i$ and it belongs to some other D_j, we will have to work with an approximate supermum of Φ as indicated earlier. In that case, by similar calculations as below, for a suitable choice of γ we get a contradiction. But for simplicity of calculations, we work with a $y_0' \in D_i \subseteq \Omega_i$, where the infimum is attained in $N w_2(y_0)$.

Let R be the bound on D, fix κ such that $\kappa < \min\{1, C'/2R\}$, where C' is as in (C2).

Then, using the fact that

$$w_1(y_0) \leq N w_1(y_0) \leq w_1(y_0') + c_c(y_0, y_0'),$$

we have

$$\begin{aligned}
\Phi(x_0, y_0) &= \gamma w_1(x_0) - w_2(y_0') - c_c(y_0, y_0') - 1/\varepsilon \left(|x_0 - y_0|^2\right) \\
&\quad -\kappa \left(\langle x_0\rangle^m + \langle y_0\rangle^m\right) \\
&\leq \gamma w_1(x_0) + \gamma w_1(y_0') - \gamma w_1(y_0) - w_2(y_0') - (1-\gamma)c_c(y_0, y_0') \\
&\quad -1/\varepsilon(|x_0 - y_0|^2) - \kappa(\langle x_0\rangle^m + \langle y_0\rangle^m) \\
&\leq \Phi(y_0', y_0') + 2\kappa\langle y_0'\rangle^m + \gamma w_1(x_0) - \gamma w_1(y_0) \\
&\quad -1/\varepsilon(|x_0 - y_0|^2) - \kappa(\langle x_0\rangle^m + \langle y_0\rangle^m) - (1-\gamma)C' \\
&\leq \Phi(y_0', y_0') + 2\kappa R + \gamma\omega(\hat{C}\sqrt{\varepsilon}) - (1-\gamma)C'.
\end{aligned}$$

By choosing γ such that

$$\gamma < (C' - 2\kappa R)/[C' + \omega(\hat{C}\sqrt{\varepsilon})],$$

we can make

$$\Phi(x_0, y_0) < \Phi(y_0', y_0').$$

This is a contradiction as

$$\sup \Phi = \Phi(x_0, y_0).$$

Hence, the case $w_2(y_0) - N w_2(y_0) \geq 0$ cannot occur for this choice of γ depending on ε and κ and we can proceed as in Case 2 and conclude that $w_1 \leq w_2$.

Case 4. $x_0, y_0 \notin C$ and one of $x_0, y_0 \in A$. Without loss of generality, let $y_0 \in A$. $x_0 \notin C$ and the fact that w_1 is a subsolution of QVI$^-$ imply

$$w_1(x_0) + H^-(x_0, \nabla w_1(x_0)) \leq 0.$$

Since

$$y_0 \in A \Rightarrow w_2(y_0) - M^- w_2(y_0) = 0,$$

therefore

$$w_2(y_0) = M^- w_2(y_0) = \inf_{v_1} \sup_{v_2} \{w_2(g(y_0, v_1, v_2)) + c_a(y_0, v_1, v_2)\}.$$

By the compactness of V_1 and V_2 above, inf sup will be attained at some v_1 and v_2. Observe that the point $g(y_0, v_1, v_2) \in D$ and let R be the bound on D. We fix κ such that

$$\kappa < \min\{1, C'/2R\}.$$

Now, doing a calculation similar to the one in Case 3, we get

$$\begin{aligned}\Phi(x_0, y_0) \leq \Phi(&g(y_0, v_1, v_2), g(y_0, v_1, v_2)) \\ &+2\kappa R + \gamma w_1(x_0) - \gamma w_1(y_0) - (1-\gamma)C'.\end{aligned}$$

By choosing

$$\gamma < (C' - 2\kappa R)/[2C' + \omega(\hat{C}\sqrt{\varepsilon})],$$

we can make

$$\Phi(x_0, y_0) < \Phi(g(y_0, v_1, v_2), g(y_0, v_1, v_2)).$$

This is a contradiction. Hence, the case $w_2(y_0) - M^- w_2(y_0) \geq 0$ cannot occur and we can proceed as in Case 2 and conclude that $w_1 \leq w_2$ for this case also.

Thus,

$$w_1 \leq w_2, \quad \forall x \in \Omega.$$

By interchanging the roles of w_1 and w_2, we get the other way inequality; hence, $w_1 = w_2$ and we have the uniqueness. □

Similarly, one can show that V^+ is the unique viscosity solution of the upper quasivariational inequality QVI$^+$.

Corollary 5.1. Existence of Value. Assume (A1)–(A5) and (C1), (C2). If H^-, H^+, M^-, M^+ satisfy

$$H^-(x, p) = H^+(x, p), \qquad \forall x, p, \tag{44}$$

$$M^- V^-(x) = M^+ V^-(x), \quad \forall x \in A, \tag{45}$$

then $V^- = V^+$; hence, the game has a value.

Proof. By the assumptions (44), (45), V^- and V^+ are solutions of the QVI$^-$. By the above uniqueness theorem, V^- is the unique viscosity solution of QVI^-; hence,

$$V^-(x) = V^+(x), \quad \forall x \in \Omega.$$

Thus, the game has the value under the conditions (44), (45). □

Remark 5.1. We remark that, when f and k are linear in all of their variables, g and c_a are linear in v_1, v_2, the total discounted cost functional J turns out to be linear in u_1, u_2, v_1, v_2. For example, we can take $\Omega \subseteq \mathbb{R}^d$, U_i compact $\mathbb{R}^d$, $f(x, u_1, u_2) = \left[a_1^T u_1\right] x + \left[a_2^T u_2\right] x$, where $a_i \in \mathbb{R}^d$, constants, and a_i^T denotes its transpose. Note that

$$k(x, u_1, u_2) = k^1 \langle u_1, x\rangle + k^2 \langle u_2, x\rangle, \quad k^i \in \mathbb{R},$$
$$g = v_1 g^1(x) + v_2 g^2(x), c_a = \psi^1 \langle b_1, v_1\rangle + \psi^2 \langle b_2, v_2\rangle,$$

where b_1, b_2 are constants and g^1, g^2, ψ^1, ψ^2 are real-valued functions. Then, by the Von Neumann minimax theorem (see Ref. 8, Section 2.13, Theorem 2G), $V^- = V^+$, and $H^-(x, p) = H^+(x, p)$, $\forall x, p$, we can conclude also that $M^- V^- = M^+ V^-$; thus, the conditions (44), (45) holds.

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