Infinite-Duration Bidding Games

Guy Avni[1], Ventsislav Chonev[2], and Thomas A. Henzinger[1]

1 **IST Austria**
2 **Max Planck Institute for Software Systems (MPI-SWS)**

Abstract

Two-player games on graphs are widely studied in formal methods as they model the interaction between a system and its environment. The game is played by moving a token throughout a graph to produce an infinite path. There are several common modes to determine how the players move the token through the graph; e.g., in turn-based games the players alternate turns in moving the token. We study the *bidding* mode of moving the token, which, to the best of our knowledge, has never been studied in infinite-duration games. Both players have separate *budgets*, which sum up to 1. In each turn, a bidding takes place. Both players submit bids simultaneously, and a bid is legal if it does not exceed the available budget. The winner of the bidding pays his bid to the other player and moves the token. For reachability objectives, repeated bidding games have been studied and are called *Richman games* [32, 31]. There, a central question is the existence and computation of *threshold* budgets; namely, a value $t \in [0, 1]$ such that if Player 1's budget exceeds t, he can win the game, and if Player 2's budget exceeds $1 - t$, he can win the game. We focus on parity games and mean-payoff games. We show the existence of threshold budgets in these games, and reduce the problem of finding them to Richman games. We also determine the strategy-complexity of an optimal strategy. Our most interesting result shows that memoryless strategies suffice for mean-payoff bidding games.

1998 ACM Subject Classification J.4 Social and Behavioral Sciences, F.1.2 Modes of Computation

Keywords and phrases Bidding games, Richman games, Parity games, Mean-payoff games

1 Introduction

Two-player infinite-duration games on graphs are an important class of games as they model the interaction of a system and its environment. Questions about automatic synthesis of a reactive system from its specification [36] are reduced to finding a winning strategy for the "system" player in a two-player game. The game is played by placing a token on a vertex in the graph and allowing the players to move it throughout the graph, thus producing an infinite trace. The winner or value of the game is determined according to the trace. There are several common modes to determine how the players move the token (c.f., [4]) that are used to model different types of systems. The most well-studied mode is *turn-based*, where the vertices are partitioned between the players and the player who controls the vertex on which the token is placed, moves it. Other modes include *probabilistic* and *concurrent* moves.

We study a different mode of moving, which we refer to as *bidding*, and to the best of our knowledge, was never studied for infinite-duration games. Both players have *budgets*, where for convenience, we have $B_1 + B_2 = 1$. The players bid for the right to move the token; the players submit bids simultaneously, where a bid is legal if it does not exceed the available budget. Thus, a bid is a real number in $[0, B_i]$, for $i \in \{1, 2\}$. The player who bids higher pays the other player and decides where the token moves. Draws can occur and one needs to devise a mechanism for resolving them (e.g., giving advantage to Player 1), and our results do not depend on a specific mechanism.

Bidding arises in many settings and we list several examples below. The players in a two-player game often model concurrent processes. Bidding for moving can model an interaction with a scheduler. The process that wins the bidding gets scheduled and proceeds with its computation.

Thus, moving has a cost and processes are interested in moving only when it is critical. When and how much to bid can be seen as quantifying the resources that are needed for a system to achieve its objective, which is an interesting question. Other takes on this problem include reasoning about which input signals need to be read by the system at its different states [17, 2] as well as allowing the system to read chunks of input signals, thereby "delaying" the output signals, and quantify the necessary amount of delay to satisfy the system's objective [25, 24, 30]. Also, our bidding game can model *scrip systems* that use internal currencies for bidding in order to prevent "free riding" [28]. Such systems are successfully used in various settings as databases [39], group decision making [38], resource allocation, and peer-to-peer networks (see [26] and references therein). Finally, repeated bidding is a form of a sequential auction [33], which is used in many settings including online advertising.

Recall that the winner or value of the game is determined according to the outcome, which is an infinite trace. There are several well-studied objectives in games. The simplest objective is *reachability*, where Player 1 has a target vertex and a trace is winning for him iff it visits the target. Bidding reachability games are known as *Richman games* [32, 31], named after David Richman. Richman games are the first to study the bidding mode of moving. The central question that is studied on Richman games regards a *threshold budget*, which is a function $\text{THRESH} : V \to [0, 1]$ such that if Player 1's budget exceeds $\text{THRESH}(v)$ at a vertex v, then he has a strategy to win the game. On the other hand, if Player 2's budget exceeds $1 - \text{THRESH}(v)$, he can win the game (recall that the budgets add up to 1). In [32, 31], the authors show that threshold budgets exist, are unique, and that finding them is in NP. We slightly improve their result by showing that the problem is in NP and coNP.

We introduce and study infinite duration bidding games with richer qualitative objectives as well as quantitative objectives. *Parity games* are an important class of qualitative games as the problem of reactive synthesis from LTL specifications is reduced to a parity game. The vertices in a parity game are labeled by an index in $\{0, \dots, d\}$, for some $d \in \mathbb{N}$, and an infinite trace is winning for Player 1 iff the parity of the maximal index that is visited infinitely often is odd. The quantitative games we focus on are *mean-payoff* games. Here, the vertices are labeled by values in $\mathbb{Z}$. Consider an infinite trace π. The mean-payoff value of a prefix of length n of π is the sum of the values it traverses divided by n. The mean-payoff value of π is the $\liminf$ of the mean-payoff values of its prefixes, which can be thought of as the amount of money that Player 1 pays Player 2. Accordingly, we refer to the players in a mean-payoff game as *Maximizer* (Max, for short) and *Minimizer* (Min, for short). We stress the point that there are two "currencies" in the game: a "monopoly money" that is used to determine who moves the token and which the players do not care about once the game ends, and the values on the vertices, which is the value that Min and Max seek to minimize and maximize, respectively. We illustrate mean-payoff games with the following example.

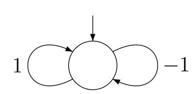

Figure 1 A bidding mean-payoff game where the weights are depicted on the edges.

▶ **Example 1.** Consider the mean-payoff bidding game that is depicted in Figure 1, where for convenience the values are placed on the edges and not on the vertices. We claim that Min has a strategy that guarantees a $\liminf$ value of at most 0. Without loss of generality, Min always chooses the (-1)-valued edge and Max always chooses the 1-valued edge. The strategy is a *tit-for-tat*-like strategy. Initially, Min bids 0. Assume Max wins a bidding with $b_1 > 0$. Min will try and *match* this win: he bids b_1 until he wins with it. Suppose Min matches Max's bid at turn n. His next bid is the smallest un-matched Max bid. We call a point in which all Max's wins are matched as a *match-point*. A key observation is that in a match-point, the sum of values is 0, thus infinitely many match-points imply that the $\liminf$ value is at most 0. The tit-for-tat strategy guarantees a slightly weaker property; if there are only finitely many match-points, then there is a bound on the number of un-matched loses of Min, which suffices to show that mean-payoff value is non-positive. Suppose there are finitely many match-points, and let n be the first turn after the last match-point in which Max bids positively (and wins the bid). Let $b > 0$ be Max's bid. We claim that for every $m > n$, the number of un-matched Min loses is at most $\lceil 1/b \rceil$. The proof follows from the fact that Min's bids after turn n are at least b. Thus, every un-matched win of Max reduces his budget by at least b. Since his budget is at most 1, there can be at most $\lceil 1/b \rceil$ such unmatched wins. ◀

We study the existence and computation of threshold budgets in parity and mean-payoff bidding games. Also, we determine the strategy complexity that is necessary for winning. Recall that a winning strategy in a game typically corresponds to an implementation of a system. A strategy that uses an unbounded memory, like the tit-for-tat strategy above, is not useful for implementing. Thus, our goal is to find strategies that use little or no memory, which are known as *memoryless* strategies.

We show that parity bidding games are linearly-reducible to Richman games allowing us to obtain all the positive results from these games; threshold budgets exist, are unique, and computing them is no harder than for Richman games, i.e., the problem is in NP and coNP. We find this result quite surprising since for most other modes of moving, parity games are considerably harder than reachability games. The crux of the proof considers bottom strongly-connected components (BSCCs, for short) in the arena, i.e., SCCs with no exiting edges. We show that the threshold budgets of the vertices of a BSCC are in $\{0, 1\}$, thus one of the players wins in all the vertices with every initial budget. If the vertex with highest parity in a BSCC is odd, then Player 1 wins, and otherwise Player 2 wins. We can thus construct a Richman game by setting the target of Player 1 to the BSCCs that are winning for him and the target of Player 2 to the ones that are winning for him. Moreover, we show that memoryless strategies are sufficient for winning in these games.

We proceed to study mean-payoff bidding games. We adapt the definition of threshold values; we say that $t \in [0, 1]$ is a threshold value for Min if with a budget that exceeds t, Min can guarantee a non-positive mean-payoff value. On the other hand, if Max's budget exceeds $1 - t$, he can guarantee a positive mean-payoff value. We show that threshold values exist and are unique in mean-payoff bidding games. The crux of the existence proof again considers the BSCCs of the game. We show that in a strongly-connected mean-payoff bidding game, the threshold budgets are in $\{0, 1\}$, thus either Min "wins" or Max "wins" the game. Moreover, the classification can be determined in NP and coNP, thus the complexity of solving bidding mean-payoff games coincides with Richman games. Our results for strongly-connected games develop the connection that was observed in [32, 31] between the threshold budget and the reachability probability in a probabilistic model on the same structure as the game. We show a connection between bidding mean-payoff games and *one-counter* 2.5-*player games* [11, 10] to prove the classification of BSCCs. In turn, these games are equivalent to discrete *quasi-birth-death processes* [21] and generalize *solvency games* [8], which can be thought of as a rewarded Markov decision process with a single vertex.

Our most technically challenging results concern the constructions of memoryless strategies for Min and Max. The challenging part of the construction is reasoning about strongly-connected bidding mean-payoff games. Consider a strongly-connected game in which Min can guarantee a non-positive mean-payoff value. The idea of our construction is to tie between changes in Min's budget with changes in the values; *investing* one unit of budget (with the appropriate normalization) implies a decrease of a unit of value, and on the other hand, an increase of a unit of value implies a *gain* of one unit of budget. Since the budgets are bounded by 1, the value cannot increase arbitrarily. Finding the right bids in a general SCC is not trivial, and we find our solution to be surprisingly elegant. The case where Max can guarantee a positive mean-payoff value, is more challenging and we show memoryless strategies in a fragment of strongly-connected games. The challenge stems from the fact that, unlike a memoryless strategy for Min, the normalization factor must decrease as the value increases so that Max does not exhaust his budget.

Further bidding games Variants of bidding games where studied in the past. Already in [31] several variants are studied including a *poorman* version in which the winner of the bidding pays the bank, thus the amount of money in the game decreases as the game proceeds. Motivated by recreational games, e.g., bidding chess, *discrete bidding games* are studied in [20], where the money is divided into chips, so a bid cannot be arbitrarily small as in the bidding games we study. In *all-pay* bidding games [34], the players all pay their bids to the bank. Non-zero-sum two-player games were recently studied in [27]. They consider a bidding game on a directed acyclic graph. Moving the token throughout the graph is done by means of bidding. The game ends once the token reaches a sink, and each sink is labeled with a pair of payoffs for the two players that do not necessarily sum up to 0. They show existence of *subgame perfect equilibrium* for every initial budget and a polynomial algorithm to compute it.

Due to lack of space, some of the proofs appear in the appendix.

2 Preliminaries

An *arena* is a pair $\langle G, \alpha \rangle$, where G is a directed graph and α is an objective. A game is played on an arena as follows. A token is placed on a vertex in the arena and the players move it throughout the graph. The *outcome* is an infinite path π. The winner or value is determined according to π and α as we elaborate below. There are several common modes in which the players move the token. In *turn-based games* the vertices are partitioned between the players and the player who controls the vertex on which the token is placed, moves it. Another mode is *probabilistic* choices, where the game can be thought of as a *Markov chain*, thus the edges are labeled with probabilities, and the edge on which the token proceeds is chosen randomly. A combination of these two modes is called *2.5-player* games, where the vertices are partitioned into three sets: Player 1 vertices, Player 2 vertices, and probabilistic vertices. Finally, in *concurrent* games, each player has a possible (typically finite) set of actions he can choose from in a vertex. The players select an action simultaneously, and the choice of actions dictates to which vertex the token moves.

We study a different mode of moving, which we call *bidding*. Both players have budgets, where for convenience, we have $B_1 + B_2 = 1$. In each turn, a bidding takes place to determine who moves the token. Both players submit bids simultaneously, where a bid is a real number in $[0, B_i]$, for $i \in \{1, 2\}$. The player who bids higher pays the other player and decides where the token moves. Note that the sum of budgets always remains 1. While draws can occur, in the questions we study we try avoid the issue of draws.

A *strategy* prescribes to a player which *action* to take in a game, given a finite *history* of the game, where we define these two notions below. In 2.5-player games, histories are paths and actions are vertices. Thus, a strategy for Player i, for $i \in \{1, 2\}$, takes a finite path that ends in a Player i vertex, and prescribes to which vertex the token moves to next. In bidding games, histories and strategies are more complicated as they maintain the information about the bids and winners of the bids. A history is a sequence of the form $v_0, \langle v_1, b_1, i_1 \rangle, \langle v_2, b_2, i_2 \rangle, \ldots, \langle v_k, b_k, i_k \rangle \in V \cdot (V \times [0, 1] \times \{1, 2\})^*$, where, for $j \geq 1$, in the j-th round, the token is placed on vertex v_{j-1}, the winning bid is b_j, and the winner is Player i_j, and Player i_j moves the token to vertex v_j. An action for a player is $\langle b, v \rangle \in ([0, 1] \times V)$, where b is the bid and v is the vertex to move to upon winning. An initial vertex v_0 and strategies f_1 and f_2 for Players 1 and 2, respectively, determine a unique *outcome* π for the game, denoted $out(v_0, f_1, f_2)$, which is an infinite sequence in $V \cdot (V \times [0, 1] \times \{1, 2\})^\omega$. We sometimes abuse notation and refer to $out(v_0, f_1, f_2)$ as a finite prefix of the infinite outcome. We drop v_0 when it is clear from the context. We define the outcome inductively. The first element of the outcome is v_0. Suppose $\pi_1, \ldots, \pi_j$ is defined. The players bids are given by $\langle b_1, v_1 \rangle = f_1(\pi_1, \ldots, \pi_j)$ and $\langle b_2, v_2 \rangle = f_2(\pi_1, \ldots, \pi_j)$. If $b_1 > b_2$, then $\pi_{j+1} = \langle v_1, b_1, 1 \rangle$, and dually when $b_1 < b_2$, we have $\pi_{j+1} = \langle v_2, b_2, 2 \rangle$. We assume there is some tie-breaking mechanism that determines who the winner is when $b_1 = b_2$, and our results are not affected by what the tie-breaking mechanism is. Consider a finite outcome π. The *payment* of Player 1 in π, denoted $\mathcal{B}_1(\pi)$, is $\sum_{1 \leq j \leq |\pi|} (-1)^{3-i_j} b_j$, and Player 2's payment, denoted $\mathcal{B}_2(\pi)$ is defined similarly. For $i \in \{1, 2\}$, consider an initial budget $B_i^{init} \in [0, 1]$ for Player i. A strategy f is *legal* for Player i with respect to B_i^{init} if for every $v_0 \in V$ and strategy g for the other player, Player i's bid in a finite outcome $\pi = out(v_0, f, g)$ does not surpass his budget. Thus, for $\langle b, v \rangle = f(\pi)$, we have $b \leq B_i^{init} - \mathcal{B}_i(\pi)$.

Richman games and threshold budgets The simplest qualitative objective is reachability: Player 1 has a target vertex v_R and an infinite outcome is winning for him if it visits v_R. Reachability bidding games are known as Richman games [32, 31]. In Richman games both players have a target, which we denote by v_R and v_S. The game ends once one of the targets is reached. Note that this definition is slightly different from standard reachability games since there, Player 2 has no target and his goal is to keep the game from v_R. As we show below, one of the players in a bidding reachability game can draw the game to a BSCC, thus v_S can be thought of as the vertices with no path to v_R.

The central question that is studied on bidding games regards a *threshold budget*. A threshold budget is a function $\text{THRESH} : V \to [0, 1]$ such that if Player 1's budget exceeds $\text{THRESH}(v)$ at a vertex v, then he has a strategy to win the game. On the other hand, if Player 2's budget exceeds $1 - \text{THRESH}(v)$, he can win the game. We sometimes use $\text{THRESH}_1(v)$ to refer to $\text{THRESH}(v)$ and $\text{THRESH}_2(v)$ to refer to $1 - \text{THRESH}(v)$. We formalize the problem of finding threshold budgets as a decision problem. We define the THRESH-BUDG problem, which takes as input a bidding game $\mathcal{G}$, a vertex v, and a value $t \in [0, 1]$, and the goal is to decide whether $\text{THRESH}(v) = t$.

Threshold values are shown to exist in [32] as well as how to compute them. We review their results. Consider a Richman game $\mathcal{G} = \langle V, E, v_R, v_S \rangle$. We define the *Richman function* as follows. We first define $R(v, i)$, for $i \in \mathbb{N} \cup \{0\}$, where the intuition is that if Player 1's budget exceeds $R(v, i)$, he can win in at most i steps. We define $R(v_R, 0) = 0$ and $R(v, 0) = 1$ for every other vertex $v \in V$. Indeed, Player 1 can win in 0 steps from v_R no matter what his initial budget is, and even if he has all the budget, he cannot win in 0 steps from anywhere else. Consider $i \in \mathbb{N}$ and $v \in V$. We denote by $adj(v) \subseteq V$, the adjacent vertices to v, so $u \in adj(v)$ iff $E(v, u)$. Let v^+ be the vertex that maximizes the expression $\max_{u \in adj(v)} R(u, i-1)$, and let v^- be the vertex that minimizes the expression $\min_{u \in adj(v)} R(u, i-1)$. We define $R(v, i) = \frac{1}{2}\big(R(v^+, i-1) + R(v^-, i-1)\big)$. We define $R(v) = \lim_{i \to \infty} R(v, i)$. The following theorem shows that $R(v)$ equals $\text{THRESH}(v)$, and throughout the paper we use them interchangeably. We give the proof of the theorem for completeness.

▶ **Theorem 2.** *[32] For every $v \in V$, we have $\text{THRESH}(v) = R(v)$, thus if Player 1's budget at v exceeds $R(v)$, he can win from v, and if Player 2's budget exceeds $1 - R(v)$, he can win from v.*

Proof. We prove for Player 1 and the proof for Player 2 is dual. Let $t \in \mathbb{N}$ be an index such that $B_1^{init} > R(t, v)$. We prove by induction on t that Player 1 wins in at most t steps. The base case is easy. For the inductive step, assume Player 1 has a budget of $R(v, i) + \epsilon$. He bids $b_1 = \frac{1}{2}\big(R(v^+, i-1) - R(v^-, i-1)\big)$. If he wins the bidding, he proceeds to v^- with a budget of $R(v^-, i-1) + \epsilon$. If he loses, then Player 2's bid exceeds b_1 and the worst he can do is move to v^+. But then Player 1's budget is at least $R(v^+, i-1) + \epsilon$. By the induction hypothesis, Player 1 wins in at most $i - 1$ steps from both positions. ◀

Finding threshold budgets The authors in [31] study the complexity of threshold-budget problem and show that is in NP. They guess, for each vertex v its neighbors v^- and v^+, and devise a linear program with the constraints $R(v) = \frac{1}{2}\big(R(v^-) + R(v^+)\big)$ and, for every neighbor v' of v, we have $R(v^-) \leq R(v') \leq R(v^+)$. The program has a solution iff the guess is correct. They leave open the problem of determining the exact complexity of finding the threshold budgets, and they explicitly state that it is not known whether the problem is in P or NP-hard.

We improve on their result by showing that THRESH-BUDG is in NP and coNP. Our reduction uses an important observation that is made in [32], which will be useful later on. They connect between threshold budgets and reachability probabilities in Markov chains.

▶ Observation 3. Consider a Richman game $\mathcal{G} = \langle V, E, v_R, v_S \rangle$. Let $M(\mathcal{G})$ be a Markov chain in which for each vertex $v \in V$, the probability of the edges $\langle v, v^+ \rangle$ and $\langle v, v^- \rangle$ is $\frac{1}{2}$ and the other outgoing edges from v have probability 0. Then, since $R(v) = \frac{1}{2}\big(R(v^+) + R(v^-)\big)$, in $M(\mathcal{G})$, the probability of reaching v_R from v is $\textsc{Thresh}(v)$.

We reduce THRESH-BUDG to the problem of "solving" a *simple stochastic game* (SSG, for short) [19]. An SSG has two players; one tries to minimize the probability that the target is reached, and the second player tries to minimize it. It is well-known that the game has a *value*, which is the probability of reaching the target when both players play optimally. The problem of finding the value of an SSG is known to be in NP $\cap$ coNP. The SSG we construct can be seen as a turn-based game in which the player whose turn it is to move is chosen uniformly at random. The details of the proof can be found in Appendix A.

▶ **Theorem 4.** *THRESH-BUDG for Richman games is in NP $\cap$ coNP.*

We stress the fact that the strategies in SSGs are very different from bidding games. As mentioned above, there, the strategies only prescribe which vertex to move the token to, whereas in bidding games, a strategy also prescribes what the next bid should be. So, a solution of a Richman game by reducing it to an SSG is existential in nature and does not give insight on the bids a player uses in his winning strategy. We will return to this point later on.

Objectives We study zero-sum games. The qualitative games we focus on are parity games. A parity game is a triple $\langle V, E, p \rangle$, where $p : V \to \{0, \ldots, d\}$ is a parity function that assigns to each vertex a *parity index*. An infinite outcome is winning for Player 1 iff the maximal index that is visited infinitely often is odd. The quantitative games we focus on are mean-payoff games. A mean-payoff game is $\langle V, E, w \rangle$, where $w : V \to \mathbb{Z}$ is a weight function on the vertices. We often refer to the sum of weights in a path as its *energy*. Consider an infinite outcome $\pi = v_0, \langle v_1, b_1, i_1 \rangle, \ldots$. For $n \geq 0$, we use π^n to refer to the prefix of length n of π. The energy of π^n, denoted $E(\pi^n)$, is $\sum_{0 \leq i \leq n-1} w(v_i)$. We define the mean-payoff value of π to be $\liminf_{n \to \infty} \frac{E(\pi^n)}{n}$. The value of π can be thought of as the amount of money Player 1 pays Player 2. Note that the mean-payoff values do not affect the budgets of the players. That is, the game has two currencies: a "monopoly money" that is used to determine who moves the token and which the players do not care about once the game ends, and the mean-payoff value that is determined according to the weights of the vertices, which is the value that Min and Max seek to minimize and maximize, respectively. Consider a finite outcome π. We use $\mathcal{B}_m(\pi)$ and $\mathcal{B}_M(\pi)$ to denote the sum of payments of Min and Max in the bids. Throughout the paper we use m and M to refer to Min and Max, respectively.

Strategy complexity Recall that a winning strategy in a two-player game often corresponds to a system implementation. Thus, we often search for strategies that use limited or no memory. That is, we ask whether a player can win even with a *memoryless* strategy, which is a strategy in which the action depends only on the position of the game and not on the history. For example, in turn-based games, for $i \in \{1,2\}$, a memoryless strategy for Player i prescribes, for each vertex $v \in V_i$, a successor vertex u. It is well known that memoryless strategies are sufficient for winning in a wide variety of games, including turn-based parity games and turn-based mean-payoff games. In Richman games, the threshold budgets tell us who the winner of the game is. But, they do not give insight on how the game is won game, namely what are the bids the winning player bids in order to win. Particularly, when the threshold budgets are 0 as we shall see in Lemmas 5 and 11.

We extend the definition of memoryless strategies to bidding games, though the right definition is not immediate. One can define a memoryless strategy as a function from vertex and budget to action (i.e., bid and vertex) similar to the definition in other games. However, this definition does not preserve the philosophy of implementation with no additional memory. Indeed, recall the proof of Theorem 2. One can define a strategy that, given a vertex $v \in V$ and a budget B, bids according to $R_t(v)$, where t is the minimal index such that $R_t(v) < B$. Clearly, the memory that is needed to implement such a strategy is infinite.

To overcome this issue, we use a different definition. We define a memoryless strategy in a vertex $v \in V$ with initial budget $B \in [0,1]$ as a pair $\langle u, f_v^B \rangle$, where $u \in adj(v)$ is the vertex to proceed to upon winning and $f_v^B : [0,1] \to [0,1]$ is a function that takes the current budget and, in mean-payoff games, also the energy, and returns a bid. We require that f_v^B is *simple*, namely a polynomial or a selection between a constant number of polynomials. For simplicity, we assume a memoryless strategy is generated for an initial vertex with an initial budget, thus there can be different strategies depending where the game starts and with what budget. Also, we call a concatenation of memoryless strategies, a memoryless strategy.

3 Parity Bidding Games

We study threshold budgets in bidding parity games. We first study strongly-connected parity games and show a classification for them; either Player 1 wins with any initial budget or Player 2 wins with every initial budget.

▶ **Lemma 5.** *Consider a strongly-connected parity game $\mathcal{G} = \langle V, E, p \rangle$. There exists $\tau \in \{0,1\}$ such that for every $v \in V$, we have $R(v) = \tau$. Moreover, we have $\tau = 0$ iff the maximal parity index of a vertex in V is odd.*

Proof. The proof relies on a claim that in a Richman game in which only Player 1's target is reachable, Player 1 wins with every initial budget. The claim clearly implies the lemma as we view a strongly-connected bidding parity game as a Richman game in which Player 1 tries to force the game to the vertex with the highest parity index, and similarly for Player 2. Intuitively, the claim follows from the fact that the threshold budget of a vertex $v \in V$ is some average between $\text{THRESH}(v_R)$ and $\text{THRESH}(v_S)$, and the average depends on the distances of v to the two targets. When only Player 1's target is reachable, we have $\text{THRESH}(v) = 0$. The details of the proof can be found in Appendix B. ◀

Consider a bidding parity game $\mathcal{G} = \langle V, E, p \rangle$. Let R and S be union of vertices in the BSCCs that are winning for Player 1 and Player 2, respectively. Let $\mathcal{G}'$ be the Richman game that is obtained from $\mathcal{G}$ by setting the target of Player 1 to be the vertices in R and the target of Player 2 to be the vertices in S. The following lemma follows from Lemma 5.

▶ **Lemma 6.** *For every $v \in V$, we have that $\text{THRESH}(v)$ in $\mathcal{G}$ coincides with $\text{THRESH}(v)$ in $\mathcal{G}'$.*

Lemma 6 allows us to obtain the positive results of Richman games in parity bidding games.

▶ **Theorem 7.** *The threshold budgets in parity bidding games exist, are unique, and THRESH-BUDG is in NP $\cap$ coNP.*

We continue to study the strategy complexity in parity games.

▶ **Theorem 8.** *Consider a parity game $\mathcal{G} = \langle V, E, p\rangle$ and a vertex $v \in V$. For $i \in \{1, 2\}$, Player i has a memoryless strategy that is winning if $B_i^{init} > \text{THRESH}_i(v)$.*

Proof. We show a memoryless strategy for Richman games. In order to obtain a memoryless strategy for parity games we proceed as the above; we first find a memoryless strategy in the Richman game in which the winning BSCCs are the targets for Player 1 and the losing BSCCs are the targets for Player 2, and then find a memoryless strategy in the internal Richman game by letting Player 1 draw the game to the vertex with maximal parity index. Consider a Richman game $\mathcal{G} = \langle V, E, v_R, v_S\rangle$, a vertex $v_0 \in V$, and an initial budget $B_1^{init} \in [0, 1]$ for Player 1 with $B > R(v_0)$. Let $\epsilon = (B - \frac{R(v_0^+)-R(v_0^-)}{2})$. Since $R(v_0) = \frac{R(v_0^+)+R(v_0^-)}{2}$, we have $\epsilon > 0$. We trim $\mathcal{G}$ by keeping only edges of the form $\langle v, v^-\rangle$, for every $v \in V$. Note that every vertex v that has $R(v) < 1$ has a path to v_R. Let $dist(v)$ be the distance from v_R in the trimmed graph. For every vertex $v \in V$, there is a constant in $[0, 1]$ that is sufficient for winning $dist(v)$ times in a row. If Player 1's budget exceeds this constant, he bids accordingly and draws the game to v_R. Otherwise, Player 1 bids $\frac{R(v^+)+R(v^-)}{2} + \epsilon \cdot 2^{-dist(v)}$. Note that if Player 1 wins for $dist(v)$ times, he wins the game. Otherwise, he loses a bid and gains at least $\epsilon \cdot 2^{-|V|}$, thus eventually he will be able to win $|V|$ times in a row. ◀

4 Mean-Payoff Bidding Games

We proceed to study mean-payoff games. We adjust the definition of threshold budgets to the quantitative setting.

▶ **Definition 9.** Consider a mean-payoff bidding game $\mathcal{G} = \langle V, E, w\rangle$. The threshold budget in a vertex $v \in V$, denoted $\text{THRESH}(v)$, is a value $t \in [0, 1]$ such that

1. If Min's budget exceeds t at v, then he can guarantee a non-positive mean-payoff value, and
2. if Max's budget exceeds $1 - t$, then he can guarantee a strictly positive value.

4.1 Solving Bidding Mean-Payoff Games

In this section we solve the problem of finding threshold values in bidding mean-payoff games. Our solution relies on work on probabilistic models, namely *one-counter simple stochastic games* [11, 10], and it is existential in nature. Namely, knowing what the threshold budget is in v does not give much insight on how Min guarantees a non-negative value even if he has sufficient budget, and similarly for Max. Constructing concrete memoryless strategies for the two players is much more challenging and we show constructions in the following sections.

Recall that in bidding parity games, we showed a classification for strongly-connected games; namely, the threshold budgets in all vertices are in $\{0, 1\}$, thus either Player 1 wins with every initial budget or Player 2 wins with every initial budget. We show a similar classification for strongly-connected bidding mean-payoff games: the threshold budgets in all vertices of a strongly-connected bidding mean-payoff game are in $\{0, 1\}$, thus in a strongly-connected bidding mean-payoff game, for every initial energy and every initial budget, either Min can guarantee a non-positive mean-payoff value or Max can guarantee a positive mean-payoff value. The classification uses a generalization of the Richman function to weighted graphs. Consider a strongly-connected bidding mean-payoff game $\mathcal{G} = \langle V, E, w\rangle$ and a vertex $u \in V$. We construct a graph $\mathcal{G}^u = \langle V^u, E^u, w^u\rangle$ by making two copies u_s and u_t of u, where u_s has no incoming edges and u_t has no outgoing edges. Thus, a path from u_s to u_t in $\mathcal{G}^u$ corresponds to a loop in $\mathcal{G}$. Recall that we denote by $w(v)$ the weight of the vertex v.

▶ **Definition 10.** Consider a strongly-connected bidding mean-payoff game $\mathcal{G} = \langle V, E, w \rangle$, a vertex $u \in V$. The weighted Richman function $W : V \to \mathbb{Q}$ is first defined on G^u. We define $W(u_t) = 0$ and for every $v \in (S \setminus \{u_t\})$, we define $W(v) = \frac{1}{2}\big(W(v^+) + W(v^-)\big) + w(v)$. In order to define W on $\mathcal{G}$, we define $W(u)$ to be $W(u_s)$ in $\mathcal{G}^u$.

We use the connection with probabilistic models as in Observation 3 in order to show that W is well defined. We view $\mathcal{G}^u$ as a rewarded Markov chain, in which, for $v \in V$, the outgoing edges from v with positive probability probabilities are $\langle v, v^+ \rangle$ and $\langle v, v^- \rangle$, and their probability is $1/2$. The function W coincides with the expected reward of a run that starts and returns to u, which in turn is well-defined since the probability of returning to u is 1. The classification of strongly-connected bidding mean-payoff games is the following.

▶ **Lemma 11.** *Consider a strongly-connected bidding mean-payoff game $\mathcal{G} = \langle V, E, w \rangle$. There is $\tau \in \{0, 1\}$ such that for every $v \in V$, we have $\text{THRESH}(v) = \tau$. Moreover, we have $\tau = 0$ iff there exists $u \in V$ with $W(u) \leq 0$.*

Similarly to the connection we show in Theorem 4 between Richman values and reachability probabilities in a simple-stochastic game, we prove Lemma 11 by connecting the threshold value in bidding mean-payoff games to the probability that a counter in a one-counter simple-stochastic games reaches value 0. We then use results from [11, 10] on this model to prove the lemma. The proof can be found in Appendix C. Lemmas 12 and 13 below, which are also helpful in the following sections, show how to connect the mean-payoff value with the objective of reaching energy 0 or maintaining non-negative energy.

▶ **Lemma 12.** *Consider a strongly-connected bidding mean-payoff game $\mathcal{G}$ and a vertex u in $\mathcal{G}$. Suppose that for every initial budget and initial energy, Min has a strategy f_m and there is a constant $N \in \mathbb{N}$ such that for every Max strategy f_M, a finite outcome $\pi = out(u, f_m, f_M)$ either reaches energy 0 or the energy is bounded by N throughout π. Then, Min can guarantee a non-positive mean-payoff value in $\mathcal{G}$.*

Proof. Suppose Min has a strategy f_m as the above, and we describe a Min strategy f'_m that guarantees a non-positive mean-payoff value. Min plays according to f_m until an energy of 0 is reached. He bids 0 until the energy increases, then he forces the game back to u, which is possible due to Lemma 5, and plays again according to f_m. Since the mean-payoff value of an infinite outcome is the $\liminf$ of the mean-payoff values of its finite prefixes, reaching 0 energy infinitely often implies a non-positive mean-payoff value. On the other hand, consider an infinite outcome $\pi_1 \cdot \pi_2 \in out(u, f'_m, f_M)$, where f_M is some Max strategy and π_1 is the prefix after which 0 energy is never reached. Then, the energy level in π_2 is bounded by N. Thus, the mean-payoff value of π is 0, and we are done. ◀

We turn to the complementary result.

▶ **Lemma 13.** *Consider a strongly-connected bidding mean-payoff game $\mathcal{G} = \langle V, E, w \rangle$ with a vertex $u \in V$ having $W(u) > 0$. If for every initial budget $B_M^{init} \in [0, 1]$ for Max there exists an energy level $n \in \mathbb{N}$ such that Max can guarantee a non-negative energy level in $\mathcal{G}$, then Max can guarantee a positive mean-payoff value in $\mathcal{G}$.*

Proof. Consider an initial budget B_M^{init} for Max. Let $\epsilon = W(v) \cdot \big(2 \sum_{v \in V} cont(v)\big)^{-1}$ and let $\mathcal{G}^\epsilon$ be the game obtained from $\mathcal{G}$ by decreasing all the weights by ϵ. Then, the weighted Richman value of u in $\mathcal{G}^\epsilon$ is $W(u)/2$ and in particular, it is positive. Thus, there exists $n \in \mathbb{N}$ such that Max can keep the energy level in $\mathcal{G}^\epsilon$ non-negative with an initial budget of B_M^{init}, if the initial energy level is n. Max plays in $\mathcal{G}$ according to his strategy in $\mathcal{G}^\epsilon$. Thus, he guarantees that for every finite outcome

π in $\mathcal{G}$, the energy is at least $|\pi|\epsilon - n$. Since n is constant, the mean-payoff value of an infinite outcome is at least ϵ. ◀

The proof of the following theorem can be found in Appendix D. Deciding the classification in Lemma 11 can be done in NP and coNP by guessing the neighbors the vertices and using linear programming, similarly to Richman games. Then, we reduce bidding mean-payoff games to Richman games in a similar way to the proof of Lemma 6 for parity games.

▶ **Theorem 14.** *Threshold budgets exist in bidding mean-payoff games, they are unique, and THRESH-BUDG for bidding mean-payoff games is in NP $\cap$ coNP.*

4.2 A Memoryless Optimal Strategy for Min

We turn to the more challenging task of finding memoryless strategies for the players, and in this section we focus on constructing a strategy for Min. Theorem 8 and Lemma 11 allow us to focus on strongly-connected bidding mean-payoff games. Consider a strongly-connected bidding mean-payoff game $\mathcal{G} = \langle V, E, w \rangle$ that has a vertex $u \in V$ with $W(u) \leq 0$. We construct a Min memoryless strategy that guarantees that for every initial energy and every initial budget, either the energy level reaches 0 or it is bounded. By Lemma 12, this suffices for Min to guarantee a non-positive mean-payoff value in $\mathcal{G}$.

The idea behind our construction is to tie between changes in the energy level and changes of the budget. That is, in order to decrease the energy by one unit, Min needs to *invest* at most one unit of budget (with an appropriate normalization), and when Max increases the energy by one unit, Min's *gain* is at least one unit of budget. Our solution builds on an alternative solution to the two-loop game in Figure 1. This solution is inspired by a similar solution in [31].

▶ **Example 15.** Consider the bidding mean-payoff game that is depicted in Figure 1. We show a Min strategy that guarantees a non-positive mean-payoff value. Consider an initial Min budget of $B_m^{init} \in [0,1]$ and an initial energy level of $k_I \in \mathbb{N}$. Let $N \in \mathbb{N}$ be such that $B_m^{init} > \frac{k}{N}$. Min bids $\frac{1}{N}$ and takes the (-1)-weighted edge upon winning. Intuitively, Min invests $\frac{1}{N}$ for every decrease of unit of energy and, since by losing a bidding he gains at least $\frac{1}{N}$, this is also the amount he gains when the energy increases. Formally, it is not hard to show that the following invariant is maintained: if the energy level reaches $k \in \mathbb{N}$, Min's budget is at least $\frac{k}{N}$. Note that the invariant implies that either an energy level of 0 is reached infinitely often, or the energy is bounded by N. Indeed, in order to cross an energy of N, Max would need to invest a budget of more than 1. Lemma 12 implies that the mean-payoff value is non-positive, and we are done. ◀

Extending this result to general strongly connected games is not immediate. Consider a strongly-connected game $\mathcal{G} = \langle V, E, w \rangle$ and a vertex $u \in V$. We would like to maintain the invariant that upon reaching u with energy k, the budget of Min exceeds k/N, for a carefully chosen N. The game in the simple example above has two favorable properties that general SCCs do not necessarily have. First, unlike the game in the example, there can be infinite paths that avoid u, thus Min might need to invest budget in drawing the game back to u. Moreover, different paths from u to itself may have different energy levels, so bidding a uniform value (like the $\frac{1}{N}$ above) is not possible. The solution to these problems is surprisingly elegant and uses the weighted Richman function in Definition 10.

Consider an initial budget of $B_m^{init} \in [0,1]$ for Min and an initial energy $k_I \in \mathbb{N}$. We describe Min's strategy f_m. At a vertex $v \in V$, Min's bid is $\frac{W(v^+)-W(v^-)}{2} \cdot \frac{1}{N}$ and he proceeds to v^- upon winning, where we choose $N \in \mathbb{N}$ in the following. Let w_M be the maximal absolute weighted Richman value in $\mathcal{G}$, thus $w_M = \max_{v \in V} |W(v)|$. Let b_M be the maximal absolute "bid", thus $b_M = \max_{v \in V} |\frac{W(v^+)-W(v^-)}{2}|$. We choose $N \in \mathbb{N}$ such that $B_m^{init} > \frac{k_I + b_M + w_M}{N}$.

In the following lemmas we prove that f_m guarantees that an outcome either reaches energy level 0 or that the energy is bounded, as well as showing that f_m is legal, i.e., that Min always bids less than his budget. The following lemma is the crux of the construction as it connects the weighted Richman function with the change in energy and in budget. Recall that for a finite outcome π the accumulated energy in π is $E(\pi)$ and the payments of Min throughout π is $\mathcal{B}_m(\pi)$.

▶ **Lemma 16.** *Consider a Max strategy f_M, and let $\pi = out(f_m, f_M)$ be a finite outcome that starts in a vertex v and ends in v'. Then, we have $W(v) - W(v') \geq E(\pi) + N \cdot \mathcal{B}_m(\pi)$.*

Proof. We prove by induction on the length of π. In the base case $v = v'$, thus $E(\pi) = \mathcal{B}_m(\pi) = 0$ and the claim is trivial. For the induction step, let b be the winning bid in the first round and let π' be the suffix of π after the first bidding. We distinguish between two cases. In the first case, Min wins the bidding, pays $b = \frac{W(v^+)-W(v^-)}{2} \cdot \frac{1}{N}$, and proceeds to v^-. Thus, we have $E(\pi) + N \cdot \mathcal{B}_m(\pi) = w(v) + E(\pi') + N\big(b + \mathcal{B}_m(\pi')\big)$. By the induction hypothesis, we have $E(\pi') + N \cdot \mathcal{B}_m(\pi') \leq W(v^-) - W(v')$, thus $E(\pi) + N \cdot \mathcal{B}_m(\pi) \leq w(v) + \frac{W(v^+)-W(v^-)}{2} + W(v^-) - W(v') = W(v) - W(v')$, and we are done. For the second case, suppose Max wins the bidding. Min's gain is $-b < -\frac{W(v^+)-W(v^-)}{2} \cdot \frac{1}{N}$, and Max proceeds to v'' having $W(v'') \leq W(v^+)$. Similar to the previous case, we have $E(\pi) + N \cdot \mathcal{B}_m(\pi) = w(v) + E(\pi') + N\big(-b + B_m(\pi')\big) \leq w(v) - \frac{W(v^-)-W(v^+)}{2} + W(v^+) - W(v') = W(v) - W(v')$, and we are done. ◀

The following corollary of Lemma 16 explains why we refer to our technique as "tying energy and budget". Its proof follows from the fact that $W(u_s) \leq 0$ and $W(u_t) = 0$.

▶ **Corollary 17.** *Consider a Max strategy f_M, and let $\pi = out(f_m, f_M)$ be a finite outcome from u to u. Then, we have $-N \cdot \mathcal{B}_m(\pi) \leq E(\pi)$.*

The following lemma formalizes the intuition above by means of an invariant that is maintained throughout the outcome. Recall that the game starts from a vertex $u \in V$ with $W(u) \leq 0$, the initial energy is $k_I \in \mathbb{N}$, Min's initial budget is $B_m^{init} \in [0,1]$, and N is such that $B_m^{init} > \frac{k_I + b_M + w_M}{N}$.

▶ **Lemma 18.** *Consider a Max strategy f_M, and let $\pi = out(f_m, f_M)$ be a finite outcome. Then, when the energy level reaches k, Min's budget is at least $\frac{k+b_M}{N}$.*

Proof. The invariant clearly holds initially. Consider a partition $\pi = \pi_1 \cdot \pi_2$, where π_1 is a maximal prefix of π that ends in u and π_2 starts in u and ends in a vertex $v \in V$. The energy level at the end of π is $k = k_I + E(\pi)$. Recall that $\mathcal{B}_m(\pi)$ is the sum of Min's payments in π, thus his budget at the end of π is $B_m^{init} - \big(\mathcal{B}_m(\pi_1) + \mathcal{B}_m(\pi_2)\big)$. By Corollary 17, we have $-\mathcal{B}_m(\pi_1) \geq \frac{1}{N} E(\pi_1)$ and by Lemma 16, we have $-\mathcal{B}_m(\pi_2) \geq \frac{1}{N}\big(E(\pi_2) - W(u) + W(v)\big) \geq \frac{1}{N}\big(E(\pi_2) - 0 - w_M\big)$. Combining with $B_m^{init} \geq \frac{k_I + b_M + w_M}{N}$, we have that the new budget is at least $\frac{k_I + b_M + w_M}{N} + \frac{E(\pi_1)}{N} + \frac{E(\pi_2) - w_M}{N} = \frac{k+b_M}{N}$, and we are done. ◀

Lemma 18 implies that Min always has sufficient budget to bid according to f_m, thus the strategy is legal. Moreover, since Min's budget cannot exceed 1, Lemma 18 implies that if the energy does not reach 0, then it is bounded by $N - b_M$. Thus, Lemma 12 implies that Min has a memoryless strategy that guarantees a non-positive mean-payoff value in a strongly-connected bidding mean-payoff game having a vertex u with $W(u) \leq 0$. Combining with the memoryless strategy in parity games, we have the following.

▶ **Theorem 19.** *Consider a bidding mean-payoff game $\mathcal{G} = \langle V, E, w \rangle$ and a vertex $v \in V$. If Min's initial budget exceeds* $\text{THRESH}(v)$*, he has a memoryless strategy that guarantees a non-positive mean-payoff value.*

4.3 A Memoryless Optimal Strategy for Max

This section is devoted to the complementary result of the previous section; finding a memoryless strategy for Max that guarantees a positive mean-payoff value in a strongly-connected bidding mean-payoff game $\mathcal{G}$ with a vertex u that has $W(u) > 0$. We show such a strategy for a fragment of the general case, which we refer to as *recurrent SCCs*, and we leave the problem open for general SCCs. We say that an SCC $G = \langle V, E\rangle$ is a recurrent if there is a vertex $u \in V$ such that every cycle in G includes u. We refer to u as the *root* of G.

Intuitively, the construction has two ingredients. First, we develop the idea of tying energy and budget. We construct a Max strategy f_M that guarantees the following: if he invests a unit of budget (with an appropriate normalization), then the energy increases by at least one unit, and when the energy decreases by one unit, Max's gain is at least $z > 1$ units of budget, where z arises from the game graph. The second ingredient concerns the normalization factor. Recall that in the previous section it was a constant $\frac{1}{N}$. Here on the other hand, it cannot be constant. Indeed, if the normalization does not decrease as the energy increases, Max's budget will eventually run out, which is problematic since with a budget of 1, Min can guarantee reaching energy level 0, no matter how high the energy is. We split $\mathbb{N}$ into *energy blocks* of size M, for a carefully chosen $M \in \mathbb{N}$. The normalization factor of the bids depends on the block in which the energy is in, and we refer to it as the *currency* of the block. The currency of the n-th block is z^{-n}. Note that the currency of the $(n-1)$-th block is higher by a factor of z from the currency of the n-th block. This is where the first ingredient comes in: investing in the n-th block is done in the currency of the n-th block, whereas gaining is done in the currency of the $(n-1)$-th block. We switch between the currencies only at the root u of $\mathcal{G}$, and this is possible since $\mathcal{G}$ is a recurrent SCC. The mismatch between gaining and investing is handy when switching between currencies as we cannot guarantee that when we reach u the energy is exactly in the boundary of an energy block.

We formalize this intuition. We start by finding an alternative definition for the weighted Richman function. Recall that in order to define W, we constructed a new graph $\mathcal{G}^u$ by splitting u into u_s and u_t. We define the *contribution* of a vertex $v \in V$ to $W(u_s)$, denoted $cont(v)$, as follows. We have $cont(u_s) = 1$. For a vertex $v \in V$, we define $pre(v) = \{v' \in V : v = v'^{-} \text{ or } v = v'^{+}\}$. For $v \in V$, we define $cont(v) = \sum_{v' \in pre(v)} \frac{1}{2} \cdot cont(v')$. The proof of the following lemma is by induction on the size of the graph and can be found in Appendix E.

▶ **Lemma 20.** *We have* $W(u) = \sum_{v \in V} \big(cont(v) \cdot w(v)\big)$.

Let $z = \big(\sum_{v:w(v)\geq 0} cont(v) \cdot w(v)\big) \cdot \big(\sum_{v:w(v)<0} cont(v) \cdot |w(v)|\big)^{-1}$. Since $W(u) > 0$, we have $z > 1$. Let $\mathcal{G}^z$ be the game that is obtained from $\mathcal{G}$ by multiplying the negative-weighted vertices by z, thus $\mathcal{G}^z = \langle V, E, w^z\rangle$, where $w^z(v) = w(v)$ if $w(v) \geq 0$ and otherwise $w^z(v) = z \cdot w(v)$. We denote by W^z the weighted threshold budgets in $\mathcal{G}^z$. The following lemma follows immediately from Lemma 20.

▶ **Lemma 21.** *We have* $W^z(u) = 0$.

We define the partition into energy blocks. Let $cycles(u)$ be the set of simple cycles from u to itself and $w_M = \max_{\pi \in cycles(u)} |E(\pi)|$. We choose M such that $M \geq (b_M + 3w_M)/(1 - z^{-1})$, where b_M is the maximal bid as in the previous section. We partition $\mathbb{N}$ into blocks of size M. For $n \geq 1$, we refer to the n-th block as M_n, and we have $M_n = \{M(n-1), M(n-1)+1, \ldots, Mn-1\}$. We use $\beta_n^{\downarrow}$ and $\beta_n^{\uparrow}$ to mark the upper and lower boundaries of M_n, respectively. Consider a finite outcome π that ends in u and let $visit_u(\pi)$ be the set of indices in which π visits u. Let $k_I \in \mathbb{N}$ be an initial energy. We say that π *visits* M_n if $k_I + E(\pi) \in M_n$. We say that π *stays in* M_n starting from an index $1 \leq i \leq |\pi|$ if for all $j \in visit_u(\pi)$ such that $j \geq i$, we have $k_I + E(\pi_1, \ldots, \pi_j) \in M_n$.

We are ready to describe Max's strategy f_M. Consider an initial Max budget $B_M^{init} \in [0,1]$. Let $k_I = M(m-1)$ be an initial energy level such that $w_M \cdot z^{-(m-1)} + \sum_{n=m}^{\infty} z^{-n} M < B_M^{init}$ and $\sum_{n=1}^{m} M \cdot z^{-n} > 1$. Suppose the game reaches a vertex v and the energy in the last visit to u was in M_n, for $n \geq 1$. Then, Max bids $z^{-n} \cdot \frac{1}{2}\big(W^z(v^+) - W^z(v^-)\big)$ and proceeds to v^+ upon winning. Note that currency changes occur only in u. Recall that for an outcome π, the sum of payments of Max is $\mathcal{B}_M(\pi)$ and let $E^z(\pi)$ be the change in energy in $\mathcal{G}^z$. The proof of Lemma 16 can easily be adjusted to this setting.

▶ **Lemma 22.** *Consider a Min strategy f_m, and let $\pi = out(f_m, f_M)$ be a finite outcome that starts in v, ends in v', and stays within a block M_n, for $n \geq 1$. We have $W^z(v) - W^z(v') \leq E^z(\pi) - z^n \cdot \mathcal{B}_M(\pi)$. In particular, for $\pi \in cycles(u)$, we have $E^z(\pi) \leq z^n \cdot \mathcal{B}_M(\pi)$.*

We relate between the changes in energy in the two structures. The proof of the following lemma can be found in Appendix F.

▶ **Lemma 23.** *Consider an outcome $\pi \in cycles(u)$. Then, $E(\pi) \geq E^z(\pi)$ and $E(\pi) \geq zE^z(\pi)$.*

A corollary of Lemmas 22 and 23 is the following. Recall that $\mathcal{B}_M(\pi)$ is the amount that Max pays, thus it is negative when Max gains budget. Intuitively, the corollary states that if the energy increases in M_n, then Max invests in the currency of M_n, and if the energy decreases, he gains in the currency of M_{n-1}.

▶ **Corollary 24.** *Consider a Min strategy f_m, and let $\pi = out(f_m, f_M)$ be a finite outcome such that $\pi \in cycles(u)$. Then, we have $E(\pi) \geq z^n \cdot \mathcal{B}_M(\pi)$ and $zE(\pi) \geq z^n \cdot \mathcal{B}_M(\pi)$.*

Consider a Min strategy f_m, and let $\pi = \pi_1 \cdot \pi_2 \cdot \ldots \cdot \pi_k = out(f_m, f_M)$ be a finite outcome that is partitioned such that for every $i \geq 1$, the energy during the sub-sequence π_i stays within some energy block. For $1 \leq i \leq i-1$, let $\pi^i = \pi_1 \cdot \ldots \cdot \pi_i$ and we use $e^i = k_I + E(\pi^i)$ to refer to the energy at the end of π^i. Note that a sequence π^i marks a change between energy blocks, which in turn imply a change in currency. Suppose π_i stays in M_n. There can be two options; either the energy decreases in π_i, thus the energy before it e^{i-1} is in M_{n+1} and the energy after it e^i is in M_n, or it increases, thus $e^{i-1} \in M_{n-1}$ and $e^i \in M_n$. We then call π^i *decreasing* and *increasing*, respectively. Recall that $w_M = \max_{\pi \in cycles(u)} E(\pi)$ and that b_M is the maximal bid. The definition of w_M and the fact that $\mathcal{G}$ is recurrent imply that upon entering M_n, the energy is within w_M of the boundary. Thus, in the case that π^i is decreasing, the energy at the end of π^i is $e^i \geq \beta_n^{\uparrow} - w_M$ and in the case it is increasing, we have $e^i \leq \beta_n^{\downarrow} + w_M$. Let $\ell_0 = 0$, and for $i \geq 1$, let $\ell_i = (\beta_{n+1}^{\downarrow} - w_M) - e^i$ in the first case and $\ell_i = (\beta_n^{\downarrow} + w_M) - e^i$ in the second case. Note that $\ell_i \in \{0, \ldots, 2w_M\}$.

▶ **Lemma 25.** *For every $i \geq 0$, suppose π^i ends in M_n. The budget of Max at the end of outcome π^i is at least $(w_M + b_M + \ell_i) \cdot z^{-(n^i - 1)} + \sum_{j=n^i}^{\infty} Mz^{-j}$, where $n^i = n+1$ if π^i is decreasing and $n^i = n$ if π^i is increasing.*

Proof. The proof is by induction. The base case follows from our choice of initial energy. For $i \geq 1$, assume the claim holds for π^{i-1} and we prove for π^i. There are four cases depending on the energy level e^{i-1} at the end of π^{i-1}. The first three cases are when $e^{i-1} \in M_{n+1}$ near $\beta_{n+1}^{\downarrow}$, thus π^i is decreasing, or when $e^{i-1} \in M_{n-1}$, thus π^i is increasing. We prove the first of these case and the others are similar. Suppose $e^{i-1} \in M_{n+1}$, thus π_i decreases into M_n and e^i is near $\beta_n^{\uparrow}$. Thus, we have $\ell_{i-1} = (\beta_{n+1}^{\downarrow} + w_M) - e^{i-1}$ and $\ell_i = (\beta_{n+1}^{\downarrow} + w_M) - e^i$. since we decrease in blocks, we have $\ell_{i-1} < \ell_i$ and $E(\pi_i) = \ell_{i-1} - \ell_i$. By Corollary 24, we have $z^{n+1} \cdot \mathcal{B}_M(\pi_i) \geq z \cdot (\ell_{i-1} - \ell_i)$, thus the gain in budget in π_i is at least $(\ell_i - \ell_{i-1}) z^{-n}$. The induction hypothesis states that Max's budget in π^{i-1} is at least $(w_M + b_M + \ell_{i-1}) \cdot z^{-n} + \sum_{j=n}^{\infty} Mz^{-j}$, thus his budget after π^i is at least $(w_M + b_M + \ell_i) \cdot z^{-n} + \sum_{j=n}^{\infty} Mz^{-j}$, and we are done. The final case is when π^i is decreasing and crosses M_{n+1} from top to bottom. That is, the energy at π^{i-1} is in M_{n+1} and $e^{i-1} \geq \beta^{\uparrow}_{n+1} - w_M$ and $e^i \leq \beta^{\downarrow}_{n+1} = \beta^{\uparrow}_n$. The decrease in energy is $E(\pi_i) = (2w_M - \ell_{i-1}) + (M - 2w_M) + \ell_i$, thus by Corollary 24, the increase in budget is $E(\pi_i) \cdot z^{n-1}$. We chose M such that $(M - 2w_M) \cdot z^{-(n-1)} \geq (w_M + b_M) \cdot z^{-(n-1)} + M \cdot z^{-n}$. The claim follows from combining with the induction hypothesis, and we are done. ◀

It is not hard to show that Lemma 25 implies that f_M is legal. Moreover, combining it with our choice of the initial energy, we get that the energy never reaches 0 as otherwise Min invests a budget of more than 1. Lemma 13 implies that Max guarantees a positive mean-payoff value in a strongly-connected game, and combining with the memoryless strategy in parity games of Theorem 8, we have the following.

▶ **Theorem 26.** *Consider a bidding mean-payoff game $\mathcal{G} = \langle V, E, w \rangle$ such that all its BSCCs are recurrent. For a vertex $v \in V$, if Max has an initial budget that is greater than $1 - \text{THRESH}(v)$, he has a memoryless strategy that guarantees a positive mean-payoff value.*

5 Discussion and Future Directions

We introduce and study infinite-duration bidding games in which the players bid for the right to move the token. The players have budgets and in each turn, a bidding takes place, where the players simultaneously submit bids under the restriction that a bid must not exceed a player's budget. The winner pays the other player and moves the token. The questions we ask on these games are the existence and computation of threshold budgets as well as constructing memoryless strategies. We showed reductions from parity and mean-payoff bidding games to Richman games. The crux of the reduction was to show a classification for strongly-connected games; one of the players "wins" in an SCC for every initial budget. We showed memoryless strategies for both players in bidding parity games and in bidding mean-payoff games.

This work belongs to a line of works that transfer concepts and ideas between the areas of formal methods and algorithmic game theory (AGT, for short). Richman games originated in the game theory community in the 90s and recently gained interest by the AGT community [27]. We combine them with the study of infinite-duration games, which is well-studied in the formal methods community. Prior to this work, a series of works focused on applying concepts and ideas from formal methods to *resource-allocation games* [7, 5, 6], which constitutes a well-studied class of games in AGT. More to the formal methods side, there are many works on games that share similar concepts to these that are studied in AGT. For example, logics for reasoning about multi-agent systems [3, 16, 35], studies of equilibria in games related to synthesis and repair problems [15, 22, 1, 12], and studies of infinite-duration non-zero-sum games [18, 13, 14, 9].

There are several problems we left open as well as plenty of future research directions. We list a handful of them below. We showed that the complexity of THRESH-BUDG is in NP and coNP. We leave open the problem of determining its exact complexity. We conjecture that it is reducible from solving simple stochastic games, which will show that it is as hard as several other problems whose exact complexity is unknown. In this work we focused on parity and mean-payoff games. *Energy games* are games that are played on a weighted graph, where one of the players tries to reach negative energy and the second player tries to prevent it. Note that unlike parity and mean-payoff, the energy objective is not prefix independent, so we cannot trivially reduce an energy bidding game to a Richman game by classifying the BSCCs of the graph to winning and losing. Still we can show that threshold budgets exist in energy games. The complexity of THRESH-BUDG in energy games is interesting and is tied with recent work on optimizing the probability of reaching a destination in a weighted MDP [23, 37]. For acyclic energy bidding games, the problem is PP-hard using a result in [23], and for a single vertex with two self loops having two integer weights (similar to Figure 1), we can show that the problem is in P by viewing the game as an instance of a general gambler's ruin problem and using the direct formula of [29]. For general games the problem is open.

Acknowledgments

We thank Petr Novotný for helpful discussions and pointers.

References

1 S. Almagor, G. Avni, and O. Kupferman. Repairing multi-player games. In *Proc. 26th CONCUR*, pages 325–339, 2015.

2 S. Almagor, D. Kuperberg, and O. Kupferman. The sensing cost of monitoring and synthesis. In *Proc. 35th FSTTCS*, pages 380–393, 2015.

3 R. Alur, T. A. Henzinger, and O. Kupferman. Alternating-time temporal logic. *J. ACM*, 49(5):672–713, 2002.

4 K.R. Apt and E. Grädel. *Lectures in Game Theory for Computer Scientists*. Cambridge University Press, 2011.

5 G. Avni, T. A. Henzinger, and O. Kupferman. Dynamic resource allocation games. In *Proc. 9th SAGT*, pages 153–166, 2016.

6 G. Avni, O. Kupferman, and T. Tamir. Congestion games with multisets of resources and applications in synthesis. In *Proc. 35th FSTTCS*, pages 365–379, 2015.

7 G. Avni, O. Kupferman, and T. Tamir. Network-formation games with regular objectives. *Inf. Comput.*, 251:165–178, 2016.

8 N. Berger, N. Kapur, L. J. Schulman, and V. V. Vazirani. Solvency games. In *Proc. 28th FSTTCS*, pages 61–72, 2008.

9 P. Bouyer, R. Brenguier, N. Markey, and M. Ummels. Nash equilibria in concurrent games with büchi objectives. In *Proc. 31st FSTTCS*, pages 375–386, 2011.

10 T. Brázdil, V. Brozek, K. Etessami, and A. Kucera. Approximating the termination value of one-counter mdps and stochastic games. In *Proc. 38th ICALP*, pages 332–343, 2011.

11 T. Brázdil, V. Brozek, K. Etessami, A. Kucera, and D. Wojtczak. One-counter markov decision processes. In *Proc. 21st SODA*, pages 863–874, 2010.

12 R. Brenguier, L. Clemente, P. Hunter, G. A. Pérez, M. Randour, J.-F. Raskin, O. Sankur, and M. Sassolas. Non-zero sum games for reactive synthesis. In *Proc. 10th LATA*, pages 3–23, 2016.

13 T. Brihaye, V. Bruyère, J. De Pril, and H. Gimbert. On subgame perfection in quantitative reachability games. *Logical Methods in Computer Science*, 9(1), 2012.

14 V. Bruyère, N. Meunier, and J.-F. Raskin. Secure equilibria in weighted games. In *Proc. CSL-LICS*, pages 26:1–26:26, 2014.

15 K. Chatterjee, T. A. Henzinger, and M. Jurdzinski. Games with secure equilibria. *Theor. Comput. Sci.*, 365(1-2):67–82, 2006.

16 K. Chatterjee, T. A. Henzinger, and N. Piterman. Strategy logic. *Inf. Comput.*, 208(6):677–693, 2010.

17 K. Chatterjee, R. Majumdar, and T. A. Henzinger. Controller synthesis with budget constraints. In *Proc. 11th HSCC*, pages 72–86, 2008.

18 K. Chatterjee, R. Majumdar, and M. Jurdzinski. On nash equilibria in stochastic games. In *Proc. 13th CSL*, pages 26–40, 2004.

19 A. Condon. On algorithms for simple stochastic games. In *Proc. DIMACS*, pages 51–72, 1990.

20 M. Develin and S. Payne. Discrete bidding games. *The Electronic Journal of Combinatorics*, 17(1):R85, 2010.

21 K. Etessami, D. Wojtczak, and M. Yannakakis. Quasi-birth-death processes, tree-like qbds, probabilistic 1-counter automata, and pushdown systems. *Perform. Eval.*, 67(9):837–857, 2010.

22 D. Fisman, O. Kupferman, and Y. Lustig. Rational synthesis. In *Proc. 16th TACAS*, pages 190–204, 2010.

23 C. Haase and S. Kiefer. The odds of staying on budget. In *Proc. 42nd ICALP*, pages 234–246, 2015.

24 M. Holtmann, L. Kaiser, and W. Thomas. Degrees of lookahead in regular infinite games. *Logical Methods in Computer Science*, 8(3), 2012.

25 F. A. Hosch and L. H. Landweber. Finite delay solutions for sequential conditions. In *ICALP*, pages 45–60, 1972.

26 K. Johnson, D. Simchi-Levi, and P. Sun. Analyzing scrip systems. *Operations Research*, 62(3):524–534, 2014.

27 G. Kalai, R. Meir, and M. Tennenholtz. Bidding games and efficient allocations. In *Proc. 16th EC*, pages 113–130, 2015.

28 I. A. Kash, E. J. Friedman, and J. Y. Halpern. Optimizing scrip systems: crashes, altruists, hoarders, sybils and collusion. *Distributed Computing*, 25(5):335–357, 2012.

29 G. Katriel. Gambler's ruin probability – a general formula. *Statistics and Probability Letters*, 83:2205–2210, 2013.

30 F. Klein and M. Zimmermann. How much lookahead is needed to win infinite games? *Logical Methods in Computer Science*, 12(3), 2016.

31 A. J. Lazarus, D. E. Loeb, J. G. Propp, W. R. Stromquist, and D. H. Ullman. Combinatorial games under auction play. *Games and Economic Behavior*, 27(2):229–264, 1999.

32 A. J. Lazarus, D. E. Loeb, J. G. Propp, and D. Ullman. Richman games. 29:439–449, 1996.

33 R. Paes Leme, V. Syrgkanis, and É. Tardos. Sequential auctions and externalities. In *Proc. 23rd SODA*, pages 869–886, 2012.

34 M. Menz, J. Wang, and J. Xie. Discrete all-pay bidding games. *CoRR*, abs/1504.02799, 2015.

35 F. Mogavero, A. Murano, and M. Y. Vardi. Reasoning about strategies. In *Proc. 30th FSTTCS*, pages 133–144, 2010.

36 A. Pnueli and R. Rosner. On the synthesis of a reactive module. In *Proc. 16th POPL*, pages 179–190, 1989.

37 M. Randour, J.-F. Raskin, and O. Sankur. Variations on the stochastic shortest path problem. In *Proc. 16th VMCAI*, pages 1–18, 2015.

38 D. M. Reeves, B. M. Soule, and T. Kasturi. Yootopia! *SIGecom Exchanges*, 6(2):1–26, 2007.

39 M. Stonebraker, P. M. Aoki, W. Litwin, A. Pfeffer, A. Sah, J. Sidell, C. Staelin, and A. Yu. Mariposa: A wide-area distributed database system. *VLDB J.*, 5(1):48–63, 1996.

A Proof of Theorem 4

We reduce the problem of finding the Richman value to the problem of finding the *value* in a 2.5-player reachability game, also called *simple stochastic games* [19] (SSGs, for short), which is known to be in NP and coNP. Let v_R be Player 1's target vertex. Consider an SSG $\mathcal{S}$ and two strategies f and g for the two players. We can construct a Markov chain from $\mathcal{S}$ using f and g, and compute the probability that v_R is reached when the two players follow f and g, which we denote by $\Pr_{f,g}[\text{Reach}(v_R)]$. The value of a vertex v, denoted $val(v)$, is $\sup_f \inf_g \Pr_{f,g}[\text{Reach}(v_R)]$. It is known that $val(v)$ is achieved by memoryless strategies for both players.

Consider a Richman game $\mathcal{G} = \langle V, E, v_R, v_S \rangle$ and a vertex $v \in V$. We construct an SSG $\mathcal{S}_\mathcal{G}$ by splitting every vertex $v \in V$, into a probabilistic vertex v_c, a Player 1 vertex v_1, and a Player 2 vertex v_2. There are edges $\langle v_c, v_1 \rangle$ and $\langle v_c, v_2 \rangle$ with probability $1/2$ each, which correspond to choosing randomly the player that moves next. From v_i, for $i \in \{1, 2\}$, there are edges to a vertex u_c iff $\langle v, u \rangle$ is an edge in $\mathcal{G}$. We claim that $val(v_c) = R(v)$. We show that $val(v_c) \geq R(v)$, and the other direction is dual. Consider a vertex $v \in V$. Let f be the memoryless Player 1 strategy in $\mathcal{S}_\mathcal{G}$ that proceeds from v_1 to v_c^-. Consider a memoryless strategy g for Player 2. We have $val(v) \geq \Pr_{f,g}[\text{Reach}(v_R)]$. For a vertex $u \in V$ in the Richman game, let R_g be a function that satisfies $R_g(v_R) = 0$, $R_g(v_S) = 1$, and for every other vertex $u \in V$, we have $R_g(u) = \frac{1}{2}\big(R_g(u^-) + R_g(g(u))\big)$. It is not hard to see that $1 - R_g(v) = \Pr_{f,g}[\text{Reach}(v_R)]$ and $R_g(v) \leq R(v)$, thus the claim follows, and we are done.

B Proof of Lemma 5

We prove the claim for an odd maximal parity index, and the proof is similar for even maximal parity index. We claim that from every vertex $v \in S$, if Player 1 has a positive budget, he can force the game to reach the vertex $u \in S$ with maximal parity index. Moreover, he can force the game to reach u with positive budget. A special vertex in S is u. Trivially, Player 1 can reach u, but we show that he can also reach u with positive budget in a non-trivial path, i.e., a loop.

We construct a Richman game on a graph $\langle V|_S \cup \{t\}, E' \rangle$, where t is a new vertex, and we redirect edges that have u as a target to the vertex t, thus for every $e = \langle v, w \rangle \in E|_S$, if $w \neq u$, then $e \in E'$ and otherwise $\langle v, t \rangle \in E'$. Player 1's objective is t and Player 2 has no objective (alternatively, his objective cannot be reached). We claim that $R(v) = 0$, for every $v \in S$. Assume towards contradiction that this is not the case. Let v be a vertex with maximal $R(v)$ in S, and we denote $\gamma = R(v)$. In particular, we have $\gamma > 0$. Since S is strongly connected, there is a path π from v to u. Let w be the last vertex on π before t. Since $R(w^-) = R(t) = 0$ and $R(w^+) \leq \gamma$, we have $R(w) < \gamma$. Let w' be the vertex before w in π. We have $R(w'^+) \leq \gamma$ and $R(w'^-) \leq R(w) < \gamma$, thus $R(w') < \gamma$. Continuing inductively we get $R(v) < \gamma$, which is a contradiction.

Consider a vertex $v \in S$ and assume Player 1's budget is $\epsilon_1 > 0$. We describe a winning strategy for Player 1. By the above, $R(v) = 0$. Thus, there is a number $i_1 \in \mathbb{N}$ such that $R(v, i_1) < \epsilon_1$. By Theorem 2, Player 1 has a strategy that forces the game to u. Furthermore, the winning strategy that is described there guarantees that u is reached with a positive budget. Let that budget be $\epsilon_2 > 0$. Again, there is an index i_2 such that $R(u, i_2) < \epsilon_2$. We continue ad infinitum thereby guaranteeing that u is visited infinitely often. Since u has maximal parity index in S and it is odd, Player 1 wins the game.

C Proof of Lemma 11

Consider a strongly-connected bidding mean-payoff game $\mathcal{G} = \langle V, E, w \rangle$. We construct a 2.5-player game $\mathcal{S}(\mathcal{G})$ similar to the proof of Theorem 4. We split every vertex $v \in V$ into three vertices; a probability vertex v_c, a Min vertex v_m, and a Max vertex v_M. We add edges with probability $1/2$ from v_c to v_m and v_M, and edges with probability 1 from v_m and v_M to every v'_c such that $\langle v, v' \rangle \in E$. The difference between the constructions is that now, the game has weights and we define the weight of v_c to be $w(v)$. The game we construct is referred to as *one-counter SSG* in [11]. There, the authors consider the objective of *termination*, i.e., reaching an energy level of 0. Formally, for a vertex $v \in V$ and an energy level $i \in \mathbb{N}$, let $Term(v, i)$ be the set of infinite paths that start at v and have a decrease of at least i units of energy. It is shown in [11] that if there exists a vertex $u \in V$ with $W(u) \geq 0$ as well as a cycle from u to itself with negative sum of weights, then Min has a memoryless strategy f that has $\Pr_{f,g}[Term(v, i)] = 1$, for every Max strategy g. On the other hand, by combining the results of [11] with [10], we have that if $W(u) > 0$, for all $u \in V$, then there exists a memoryless Max strategy g such that $\lim_{n \to \infty} \Pr_{f,g}[Term(v, n)] = 0$, for every Min strategy f.

In order to connect the threshold budget in $\mathcal{G}$ with the termination probability in $\mathcal{S}(\mathcal{G})$, we "unwind" both structures into a graph with infinite-many vertices in which we keep track of the accumulated energy. We refer to the unwinding of $\mathcal{G}$ as $\mathcal{G}^\infty$. Intuitively, reaching a vertex $\langle v, i\rangle \in (V \times \mathbb{N})$ in $\mathcal{G}^\infty$ corresponds to reaching v with energy i in $\mathcal{G}$. The goal for Min is to reach energy level 0, thus $\mathcal{G}^\infty$ is a Richman game with infinite many vertices. The unwinding of $\mathcal{S}(\mathcal{G})$ coincides with $\mathcal{G}^\infty$. Since $\mathcal{G}^\infty$ is locally finite, by [31], it has threshold budgets that satisfy the same properties as in finite graphs. When $W(u) \geq 0$, we show that $R(\langle v, i\rangle) = 0$, for every $v \in V$ and $i \in \mathbb{N}$, thus Min can guarantee reaching an energy level of 0 from v with initial energy i, for every initial budget, thus by Lemma 12 he guarantees a non-negative mean-payoff value. On the other hand, if $W(u) > 0$, we can show that for every $v \in V$, we have $\lim_{i\to\infty} R(\langle v, i\rangle) = 1$, thus for an initial Max budget B_M, there is an initial energy level n such that $1 - R(\langle v, n\rangle) < B_M$. So, by Lemma 13, Max guarantees a positive mean-payoff value.

We show how to connect the Richman value of $\langle v, i\rangle$ in $\mathcal{G}^\infty$ with the termination probability in $\mathcal{S}(\mathcal{G})$. We prove for the case that there exists $u \in V$ with $W(u) \geq 0$, and the other case is similar. First note that if there are no negative-weight cycles in $\mathcal{G}$, then there are also no positive-weight cycles, thus all outcomes have a mean-payoff value of 0. Suppose there is at least one negative-weight cycle in $\mathcal{G}$. We construct from $\mathcal{G}$ an (unweighted) Richman game with infinite many vertices by keeping track of the accumulated energy throughout a play and setting Min's goal to reach energy level 0. We have $\mathcal{G}^\infty = \langle V^\infty, E^\infty\rangle$, where $V^\infty = V \times \mathbb{N} \cup \{v_{win}\}$ and we describe E^∞ below. Intuitively, reaching a vertex $\langle v, i\rangle \in V^\infty$ in $\mathcal{G}^\infty$ corresponds to reaching v with energy i in $\mathcal{G}$. For convenience we assume the weights are on the edges rather than the vertices. Accordingly, for $i, j \geq 0$, we have $\langle\langle v, i\rangle, \langle u, j\rangle\rangle \in E^\infty$ iff $e = \langle v, u\rangle \in E$ and $w(e) = j - i$, and $\langle\langle v, i\rangle, v_{win}\rangle \in E^\infty$ iff there is a vertex $u \in V$ with $e = \langle v, u\rangle \in E$ and $i + w(e) < 0$. Note that $\mathcal{G}^\infty$ is locally finite. It is shown in [31] that threshold budgets exist for such infinite-graph games and that for every vertex $v \in V^\infty$, there are $v^+, v^- \in V^\infty$ with $R(v) = \frac{1}{2}\big(R(v^-) + R(v^+)\big)$.

Recall that a strategy for a player in a bidding game consists of two components; a bid and a vertex to move to upon winning. We fix the second component for Min: we call f^m a Min strategy that, at a vertex $v \in V$, proceeds to $v^- \in V$ upon winning, where recall that v^- is the neighbor of v with the minimal weighted-Richman value. We define a Richman value R_{f^m} for $\mathcal{G}^\infty$, where Min proceeds according to f^m. Using the same proof of [31], we can show that R_{f^m} is defined and for every $\langle v, i\rangle \in V^\infty$, there is $\langle v', i'\rangle, \langle v^-, i''\rangle \in V^\infty$ such that $R_{f^m}(\langle v, i\rangle) = \frac{1}{2}\big(R_{f^m}(\langle v', i'\rangle + R_{f^m}(\langle v^-, i''\rangle))\big)$. Note that $R(\langle v, i\rangle) \leq R_{f^m}(\langle v, i\rangle)$.

Let g^M be a Max strategy in $\mathcal{S}(\mathcal{G})$ that proceeds from v_M with energy $i \in \mathbb{N}$ to $v'_c \in V$ such that $R_{f^m}(\langle v, i\rangle) = \frac{1}{2}\big(R_{f^m}(\langle v', i'\rangle + R_{f^m}(\langle v^-, i''\rangle))\big)$. Note that f^m can be seen as a Min strategy in $\mathcal{S}(\mathcal{G})$. Moreover, note that $\Pr_{f^m, g^M}[Term(v_p, i)] = 1 - R_{f^m}(\langle v, i\rangle)$. Indeed, for $v \in V$, we have $\Pr_{f^m, g^M}[Term(v_c, 0)] = 1$ and $\Pr_{f^m, g^M}[Term(v_c, i)] = \frac{1}{2}\Big(\Pr_{f^m, g^M}[Term(v^-, i')] + \Pr_{f^m, g^M}[Term(g^M(v), i'')]\Big)$, where i' and i'' are the appropriate energy levels. Since memoryless strategies suffice in one-counter SSGs, there is a Max memoryless strategy f^M for which $\Pr_{f^m, g^M}[Term(v_c, i)] \leq \Pr_{f^m, f^M}[Term(v_c, i)]$.

Assume towards contradiction that $\Pr_{f^m, f^M}[Term(v_c, i)] < 1$. Let W_{f^M} be a function on V that is defined similarly to W. We split u into two vertices u_s and u_t, define $W_{f^M}(u_t) = 0$, and, for $v \in V$ we have $W_{f^M}(v) = \frac{1}{2}\big(W_{f^M}(v^-) + W_{f^M}(f^M(v))\big)$, where $v^- \in V$ is the neighbor of v that has the minimal value according to W_{f^M}. Clearly, we have $W_{f^M}(v) \leq W(v)$. Since $W(u) \leq 0$, we have $W_{f^M}(u) \leq 0$, thus it follows from [11] that $\Pr_{f^m, f^M}[Term(v_c, i)] = 1$, and we are done.

D Proof of Theorem 14

Consider a bidding mean-payoff game $\mathcal{G} = \langle V, E, w\rangle$. Lemma 11 induces a reduction from bidding mean-payoff games to Richman games: we classify the BSCCs of $\mathcal{G}$, relate Min with Player 1 and Max with Player 2, and set the BSCCs with threshold budget 0 as Player 1's target and these with threshold budget 1 as Player 2's target. For every $v \in V$ that is not in a BSCC, we have $\text{THRESH}(v) = R(v)$. Indeed, if Min's budget exceeds $R(v)$ in v, he can draw the game to a BSCC from which he can guarantee a non-negative mean-payoff value, and the claim for a budget below $R(v)$ is dual. Thus, we only need to show how to determine whether a BSCC S of $\mathcal{G}$ has a vertex $u \in S$ with $W(u) \leq 0$. This can easily be done in NP and coNP. For a vertex $u \in S$ we guess, for every $v \in S$ two neighbors v^+ and v^-, and add the appropriate linear constraints: we have $W(u_t) = 0$, for every neighbor v' of v we have $W(v^-) \leq W(v') \leq W(v^+)$ and $W(v) = \frac{1}{2}\big(W(v^-) + W(v^+)\big) + w(v)$. Such a system can be solved in polynomial time, thus we are done.

E Proof of Lemma 20

The proof is by induction on the size of the graph. For the base case, we consider a graph with only one vertex u, and indeed, since $cont(u) = 1$, we have $W(u) = cont(u) \cdot w(u) = w(u)$. For the induction hypothesis, consider the subgraphs $\mathcal{G}^- = \langle V^-, E^-, w\rangle$ and $\mathcal{G}^+ = \langle V^+, E^+, w\rangle$ of $\mathcal{G}$ whose roots are u^- and u^+, respectively. For $v \in V^-$ we denote by $cont^-(v)$ the contribution of v to $W(u^-)$ in $\mathcal{G}^-$, and similarly for $\mathcal{G}^+$. By the induction hypothesis, we have $W(u^-) = \sum_{v\in V^-} cont^-(v)$ and $W(u^+) = \sum_{v\in V^+} cont^+(v)$. Note that $cont^-(u^-) = cont^+(u^+) = 1$ whereas in $\mathcal{G}$, we have $cont(u) = 1$, thus $cont(u^-) = cont(u^+) = 1/2$. Accordingly, for every $v \in V^-$, we have $cont^-(v) = 1/2 \cdot cont(v)$, and similarly for $v \in V^+$. Thus, $W(u) = \frac{1}{2}\big(W(u^-) + W(u^+)\big) = \sum_{v\in V^-} 1/2 \cdot cont^-(v) + \sum_{v\in V^+} 1/2 \cdot cont^+(v) = \sum_{v\in V} cont(v)$, and we are done.

F Proof of Lemma 23

Let $E^{\geq 0}(\pi)$ and $E^{<0}(\pi)$ be the sum of non-negative weights and negative weights in π, respectively. We have $E(\pi) = E^{\geq 0}(\pi) + E^{<0}(\pi)$ and $E^z(\pi) = E^{\geq 0}(\pi) + zE^{<0}(\pi)$. The inequality $E(\pi) \geq E^z(\pi)$ is immediate. For the second inequality, we multiply the first equality by z and subtract it from the first to get $E^z(\pi) - zE(\pi) = E^{\geq 0}(\pi) - zE^{\geq 0}(\pi) \leq 0$, and we are done.