A new correctness criterion for MLL proof nets

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Abstract

In Girard's original presentation, proof structures of Linear Logic are hypergraphs whose hyperedges are labeled by logical rules and vertices represent the connections between these logical rules. Presentations of proof structures based on interaction nets have the same kind of graphical flavor. Other presentations of proof structures use terms instead of graphs or hypergraphs. The atomic ingredient of these terms are variables related by axiom links. However, the correctness criteria developed so far are adapted to the graphical presentations of proof structures and not to their term-based presentations. We propose a new correctness criterion for constant-free Multiplicative Linear Logic with Mix which applies to a coherence space structure that a term-based proof structure induces on the set of its variables in a straightforward way.

Categories and Subject Descriptors F.4.1 [*Mathematical Logic*]: Lambda calculus and related systems

Keywords Linear logic, proof nets, correctness criteria

1. Introduction

One of the major outcomes of the discovery of Linear Logic by Girard in the mid 1980's, see Girard (1987), was the introduction of *proof nets* which are a particularly elegant and canonical representation of proofs, identifying many derivations of the sequent calculus which are distinct for "bad reasons". These distinctions between derivations are due to the very sequential character of the sequent calculus: any derivation must have a last rule, but very often the choice of this last rule is arbitrary and several choices can be made without changing the "meaning" of the proof — for instance, without changing its denotational semantics in the relational model of Linear Logic, see Bucciarelli and Ehrhard (2001). When represented as proof nets, proofs which differ by such choices in the sequent calculus correspond to the same structures.

Proof nets are particular *proof structures*. Usually, proof structures are presented as graphical objects which can be of various kinds. Let us mention two examples.

- In Girard's original presentation, logical rules (*tensor* $\otimes$ and *par* $⅋$) as well as axioms and cuts are represented as *links* which are hyperedges connecting vertices. Each of these links has several *premise* vertices and *conclusion* vertices: an axiom link has no premises and two conclusions, a cut link has two premises and no conclusions, a tensor and a par link have two premises and one conclusion. The only constraint is that, in a proof structure, any premise of a link must be a conclusion of a link. The vertices which do not appear as a premise of a link are the conclusions of the proof structures.
- In Lafont's Interaction Nets, see Lafont (1990) — which allow to represent many other calculi as well — the logical rules tensor and par are represented as *cells*. Each of the cells of a proof net has two *auxiliary ports* (corresponding to the premises) and one main port (corresponding to the conclusion). A port can be free (and then it is a conclusion of the proof structure) or is connected to another port by a *wire* (which is simply a pair of distinct ports). The main difference wrt. Girard's proof structures is that interaction nets feature no cells for representing axioms and cuts: cuts are wires connecting main ports and axioms are wires connecting auxiliary ports.

There is also another way of presenting a proof structure, as a finite set of terms. These terms are built using variables, and function symbols which represent logical connectives. There is also a binary constructor for representing cuts, similar to the *parallel composition* operator of process algebras. The variables are used to represent the axiom links. Such formalisms have been introduced by Abramsky, and then by Fernandez and Mackie, see for instance Abramsky (1993); Fernández and Mackie (1999); Mackie and Sato (2008).

In all of these representation, there are very simple *typing rules* for proof structures with formulas of linear logic, and in that way one can associate a sequence of formulas Γ with the conclusions of a proof structure when it is typeable (this is always assumed to be the case).

Whatever be the choice of representation, the main feature of proof structures is that they are a calculus: there are (one or two, depending on the presentation) very simple reduction rules which implement the cut elimination of Linear Logic. It is here that the superiority of proof structures with respect to sequent calculus derivations is particularly dramatic. In the sequent calculus, cut elimination requires additional *commutation reduction rules* whose purpose is to transform an arbitrary cut into a cut where both formulas are introduced just above the cut (*key case* in the terminology of Girard et al. (1989)). The reduction rules of proof structures correspond exactly to these key cases, and commutative reductions are superfluous.

Given a proof of a sequence of formulas Γ in the sequent calculus, it is easy to turn it into a proof structure whose conclusions are labeled with Γ. A proof structure which can be obtained in that way is called a *proof net*. As we have seen this *unsequentialization* mapping is not injective since the sequent calculus imposes irrelevant distinctions between derivations. This mapping is also far from being surjective. The purpose of a *correctness criterion* is precisely

CSL-LICS 2014, July 14–18, 2014, Vienna, Austria.

http://dx.doi.org/10.1145/2603088.2603125

to characterize those proof structures which belong to the range of this mapping, *ie.* which can be sequentialized. A correctness criterion sorts out proof nets among general proof structures.

The most obvious correctness criterion is the definition itself of a proof net: a proof structure of conclusion Γ is correct if it can be sequentialized into a sequent calculus proof of Γ. Besides the fact that this "criterion" does not provide any new insight about proof nets, it is not suitable because it is difficult to prove directly — that is, without using another correctness criterion — that it is preserved under cut elimination. The most popular correctness criterion is the acyclicity criterion of Danos and Regnier (1989) — see also Bellin and van de Wiele (1995) — which is a simplification of the original long trip criterion of Girard (1987); it is easily proved to be preserved under cut reduction. Other criteria, based on the introduction of graph rewriting systems on proof structures have been introduced in Danos (1990); Guerrini (1999) and more recently in de Naurois and Mogbil (2011), with applications to the complexity of proof structure correctness.

These criteria are adapted to the graphical presentations of proof structures but are not very convenient for term-based presentations in the style *eg.* of Mackie and Sato (2008).

Content. We propose a correctness criterion adapted to this kind of term-based presentation of proof structures. Our criterion does not apply to the proof structure itself, but to a coherence space structure (that is, an antireflexive and symmetric binary relation) that the proof structure induces on its variables. The axiom links are implemented by the simple fact that variables come in pairs $x, \overline{x}$ of a variable and its co-variable (it is intended that $x \neq \overline{x}$ and that $\overline{\overline{x}} = x$): if a variable occurs in a proof structure, its co-variable must appear as well and this pair $x, \overline{x}$ represents an axiom link in the proof structure. This is a slight modification of the more standard approach where axiom links are represented by the variables themselves: each variable must appear exactly twice and the intended meaning is that there is an axiom link between the two occurrences.

Given a proof structure p, its set of variables X has a structure of coherence space defined as follows: x and y are related (written $x \frown y$) if x and y are distinct variables, and the highest common parent of x and y in p (seen as a forest[1]) is a $\otimes$ node. We call *cycle* a sequence $(x_1, \ldots, x_{2k})$ of pairwise distinct elements of X such that $x_i \frown x_{i+1}$ if i is odd and $\overline{x_i} = x_{i+1}$ if i is even (and $\overline{x_{2k}} = x_1$).

We prove that p is a proof net iff for any such cycle one can find a pair (i, j) of indices such that $1 \leq i, j \leq 2k$, $i + 2 \leq j$, $(i, j) \neq (1, 2k)$ and $x_i \frown x_j$. Such a pair (i, j) is called a *short-cut*, it is simply a $\frown$-edge between two non-adjacent vertices of the cycle.

To prove this result, we use a notion of *closed coherence space* which is a coherence space whose web (set of vertices) is equipped with an equivalence relation. A typical example of such a structure is of course the coherence space associated with a proof structure p as explained above: the equivalence classes are the sets $\{x, \overline{x}\}$ where x ranges in the set of variables occurring in p.

But we also use this concept in another and completely different way. With a proof structure p in which we assume that no outermost logical rule is a $⅋$ rule — this assumption is justified by the fact that the $⅋$ rule of Linear Logic is reversible — we associate a closed coherence space as follows. The web has one element for each premise of each outermost $\otimes$ rule. Two elements of the web are equivalent if they correspond to premises of the same $\otimes$ rule. They are related by the coherence relation if the corresponding trees contain variables which are related by an axiom link. Our correctness criterion allows to prove that this new closed coherence space has no cycles, and from this, we deduce that it can be "split", meaning that there is an equivalence class whose removal splits the coherence space in several connected components. In other words, there is an outermost $\otimes$ connective which can be introduced by a $\otimes$ rule of the sequent calculus. This is the key step in the proof of the Sequentialization Theorem.

Last we prove that this correctness criterion is preserved by cut elimination. For this purpose we describe first the effect of one step of cut reduction on the closed coherence space associated with a proof structure: this is a simple modification of its edges. Then we prove the result by examining the effect of such modifications on cycles. This proof is fairly simple and straightforward, and does not use the Sequentialization Theorem, showing that the correctness criterion is well-behaved.

Extensions. Our criterion deals only with the "acyclicity" aspect of correctness, this is why it applies to MLL extended with the MIX rule. Taking connectedness into account, and therefore rejecting the MIX rule, can be achieved by counting $⅋$, $\otimes$ and cuts in acyclic proof structures as in Guerrini (1999).

Adding exponential rules and boxes would not be a problem as far as correctness is concerned: our criterion will extend straightforwardly to this fragment of Linear Logic: it is precisely for this purpose that we consider $\otimes$ rules of an arbitrary arity, and not only binary tensor rules. Indeed, from the view point of correctness, an exponential promotion box with n auxiliary ports behaves like an n-ary tensor rule, and a contraction rule behaves like a $⅋$ rule.

Related works. The idea of restricting one's attention to the coherence space associated with a proof structure instead of considering the whole proof structure has been suggested to us by two discussions: a first one with Jean-Yves Girard in the mid-1990's and a second one, ten years after, with Séverine Maingaud. A similar idea is used by Dominic Hughes *eg.* in Hughes (2006).

In Retoré (1997) another correctness criterion for multiplicative proof nets based on coherence spaces is introduced: a proof structure is a proof net as soon as its relational semantics is a clique in the standard coherence space semantics of Girard (1987). There is no clear connection between our result and Rétoré's criterion since we use coherence spaces in a completely different way.

2. Syntax of proof structures

2.1 General constructions

2.1.1 MLL proof structures. Let $\mathcal{V}$ be an infinite and countable set of variables equipped with an involution $x \mapsto \overline{x}$ such that $x \neq \overline{x}$ for each $x \in \mathcal{V}$.

Let $u \subseteq \mathcal{V}$. An element $x \in u$ is *bound* in u if $\overline{x} \in u$. One says that u is *closed* if all the elements of u are bound in u. If x is not bound in u, one says that x is *free* in u.

Proof trees are defined as follows, together with their associated set of variables ($\mathsf{V}(t)$ is the set of variables of the tree t):

- if $x \in \mathcal{V}$ then x is a tree and $\mathsf{V}(x) = \{x\}$;
- if $t_1, \ldots, t_n$ are trees with $\mathsf{V}(t_i) \cap \mathsf{V}(t_j) = \emptyset$ for $i \neq j$, then $t = t_1 \otimes \cdots \otimes t_n$ is a tree with $\mathsf{V}(t) = \mathsf{V}(t_1) \uplus \cdots \uplus \mathsf{V}(t_n)$. Similarly, $t = t_1 ⅋ \cdots ⅋ t_n$ is a tree with $\mathsf{V}(t) = \mathsf{V}(t_1) \uplus \cdots \uplus \mathsf{V}(t_n)$. We use the symbol $\uplus$ to denote unions when we want to stress that the sets are disjoint.

A *cut* is an expression $\langle t \mid t' \rangle$ where t and t' are trees such that $\mathsf{V}(t) \cap \mathsf{V}(t') = \emptyset$. We set $\mathsf{V}(c) = \mathsf{V}(t) \uplus \mathsf{V}(t')$. The cut construction is commutative: we make no difference between $\langle t \mid t' \rangle$ and $\langle t' \mid t \rangle$.

[1] We draw the forest with the leaves — which are the variables — located upwards.

A *proof structure* is a pair $p = (\overrightarrow{c}\ ;\ \overrightarrow{t})$ where $\overrightarrow{t}$ is a finite list of proof trees and $\overrightarrow{c}$ is a finite list of cuts, assuming that the sets of variables of these cuts and of these terms are pairwise disjoint.

Remark: The order of the elements of $\overrightarrow{c}$ does not matter; we could have used a set instead of a sequence (the cuts are pairwise distinct since their sets of variables are pairwise disjoint). In the sequel, we consider these sequences of cuts up to permutation.

Bound variables of $\mathsf{V}(p)$ can be renamed in the obvious way in p (rename simultaneously x and $\overline{x}$ avoiding clashes with other variables which occur in p) and proof structures are considered up to such renaming: this is *α-conversion*.

The simplest proof structure is of course $(\ ;\ )$. A less trivial closed proof structure is $(\langle x \,|\, \overline{x}\rangle\ ;\ )$ which is a loop.

2.1.2 MLL types.

Let $\mathcal{A}$ be a set of type atoms ranged over by $\alpha, \beta, \ldots$, together with an involution $\alpha \mapsto \overline{\alpha}$ such that $\overline{\alpha} \neq \alpha$ for all $\alpha \in \mathcal{A}$. Types are defined just as trees, apart from the fact that there is no disjointness assumption:

- if $\alpha \in \mathcal{A}$ then α is a type;
- if $k \geq 2$ and $A_1, \ldots, A_k$ are types then $A_1 \otimes \cdots \otimes A_k$ and $A_1 \parr \cdots \parr A_k$ are types.

The linear negation $A^\perp$ of a type A is given by the following inductive definition:

$$\alpha^\perp = \overline{\alpha}$$
$$(A_1 \otimes \cdots \otimes A_k)^\perp = A_1^\perp \parr \cdots \parr A_k^\perp$$
$$(A_1 \parr \cdots \parr A_k)^\perp = A_1^\perp \otimes \cdots \otimes A_k^\perp$$

so that $A^{\perp\perp} = A$ for any formula A.

2.1.3 Typing judgments.

A *typing context* is a finite partial function Φ (of domain $\mathsf{D}(\Phi)$) from $\mathcal{V}$ to formulas such that $\Phi(\overline{x}) = \Phi(x)^\perp$ whenever $x, \overline{x} \in \mathsf{D}(\Phi)$.

We first explain how to type proof trees. The corresponding typing judgments have shape $\Phi \vdash_0 t : A$ where Φ is a typing context, t is a proof tree and A is a formula. The rules are as follows:

$$\frac{}{\Phi, x : A \vdash_0 x : A}$$

$$\frac{\Phi \vdash_0 s_1 : A_1 \qquad \cdots \qquad \Phi \vdash_0 s_k : A_k}{\Phi \vdash_0 s_1 \,\square \cdots \square\, s_k : A_1 \,\square \cdots \square\, A_k}$$

where $\square \in \{\otimes, \parr\}$.

Given a cut $c = \langle s \,|\, s' \rangle$ and a typing context Φ, one writes $\Phi \vdash_0 c$ if there is a type A such that $\Phi \vdash_0 s : A$ and $\Phi \vdash_0 s' : A^\perp$.

Last, given a proof structure $p = (\vec{c}; \vec{s})$ with $\vec{s} = (s_1, \ldots, s_n)$ and $\vec{c} = (c_1, \ldots, c_k)$, a sequence $\Gamma = (A_1, \ldots, A_l)$ of formulas and a typing context Φ, one writes $\Phi \vdash_0 p : \Gamma$ if $l = n$, $\Phi \vdash_0 s_i : A_i$ for $1 \leq i \leq n$ and $\Phi \vdash_0 c_i$ for $1 \leq i \leq k$.

2.2 MLL proof nets

A *logical judgment* is an expression $\Phi \vdash p : \Gamma$ where Φ is a typing context, p is a proof structure and Γ is a list of formulas. The rules for deriving logical judgments are given in Figure 1. These rules correspond to the constant-free fragment of MLL sequent calculus, extended with the mix rule which allows to "glue together" unrelated proofs.

One checks easily that if $\Phi \vdash p : \Gamma$ then $\Phi \vdash_0 p : \Gamma$ and p is closed, but the converse is far from being true. Let p be a proof structure such that $\Phi \vdash_0 p : \Gamma$. One says that p is a *proof net* if $\Phi \vdash p : \Gamma$ holds. A derivation of $\Phi \vdash p : \Gamma$ is called a *sequentialization* of p.

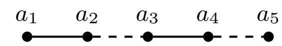

Figure 2. A path of length 5

Remark: These term-based proof nets (up to α-equivalence and permutation rule) are in bijective correspondence with the usual proof nets defined as (hyper)graphs.

3. Graphical considerations

We give in this sections a few definitions and properties about the *closed coherence spaces* that we'll use in our correctness criterion. This section can safely be skipped over and used "by need" whilst reading the remainder of the paper.

3.1 Coherence spaces

A *coherence space* is a structure $X = (|X|, \frown_X)$ where $|X|$ is a set and $\frown_X$ is a binary, symmetric and anti-reflexive relation on $|X|$. We write $a \smile_X b$ if $a \neq b$ and $a \frown_X b$ does not hold.

Let X and Y be coherence spaces such that $|X| \cap |Y| = \emptyset$. One defines the coherence space $X \oplus Y$ by $|X \oplus Y| = |X| \uplus |Y|$ and, for $a, b \in |X| \uplus |Y|$, one has $a \frown_{X \oplus Y} b$ if $a, b \in |X|$ and $a \frown_X b$, or if $a, b \in |Y|$ and $a \frown_Y b$.

One defines $X \,\&\, Y$ by $|X \,\&\, Y| = |X| \uplus |Y|$ and, for $a, b \in |X| \uplus |Y|$, one has $a \frown_{X \& Y} b$ if $a, b \in |X| \Rightarrow a \frown_X b$ and $a, b \in |Y| \Rightarrow a \frown_Y b$. So that $a \frown_{X \& Y} b$ if $a \in |X|$ and $b \in |Y|$.

3.2 Closed coherence spaces

A *closed coherence space* is a structure $X = (|X|, \frown_X, \tau_X)$ where $(|X|, \frown_X)$ is a coherence space and τ_X is an equivalence relation[2] on $|X|$. If $a \in |X|$, we denote as $(a)_X$ the equivalence class of a for the relation τ_X. We set

$$(a)_X^+ = (a)_X \setminus \{a\}\,.$$

A closed coherence space is *strict* if all the elements of $|X|/\tau_X$ have cardinality ≥ 2.

A subset U of $|X|$ is *closed* if $\forall a \in U\ (a)_X \subseteq U$.

Let X be a closed coherence space. We introduce a few useful notions.

The length of a sequence $\gamma = (a_1, \ldots, a_k)$ is $\mathsf{len}(\gamma) = k$.

3.2.1 Paths.

A *path* in X is a sequence $(a_1, \ldots, a_{2n-1})$ of odd length (we assume $n \geq 1$) of elements of $|X|$ such that

- the a_i's are pairwise distinct
- for each $i \in \{1, \ldots, n-1\}$, $a_{2i-1} \frown_X a_{2i}$ and $a_{2i+1} \in (a_{2i})_X^+$.

So a path strictly alternates coherence and equivalence edges between its vertices, starting with a coherence edge, see Figure 2. We use continuous lines to represent the coherence relation and dashed lines to represent the equivalence relation.

3.2.2 Loops.

A *loop* is a path $\gamma = (a_1, \ldots, a_{2n-1})$ such that

- $a_1 \frown_X a_{2n-1}$
- and $a_i \notin (a_1)_X$ for all $i = 2, \ldots, 2n-1$.

In γ, the vertex a_1 plays a particular role: it is strictly coherent with both a_2 and a_{2n-1}. We call a_1 the *anchor* of γ.

Observe that, if $(a_1, \ldots, a_{2n-1})$ is a loop, then

$$(a_1, a_{2n-1}, a_{2n-2}, \ldots, a_2)$$

[2] We could call it the *tensor-axiom relation* as this corresponds to its two usages in this paper.

$$\frac{}{\Phi, x : A, \overline{x} : A^{\perp} \vdash (\,;\, x, \overline{x}) : A, A^{\perp}}\ \text{axiom} \qquad \frac{\Phi \vdash (\vec{c};\, t_1, \ldots, t_n) : A_1, \ldots, A_n}{\Phi \vdash (\vec{c};\, t_{\sigma(1)}, \ldots, t_{\sigma(n)}) : A_{\sigma(1)}, \ldots, A_{\sigma(n)}}\ \text{permutation rule}, \sigma \in \mathfrak{S}_n$$

$$\frac{\Phi \vdash (\overrightarrow{c}\,;\, \overrightarrow{s}, s) : \Gamma, A \qquad \Phi \vdash (\overrightarrow{d}\,;\, \overrightarrow{t}, t) : \Delta, A^{\perp}}{\Phi \vdash (\overrightarrow{c}, \overrightarrow{d}, \langle s \,|\, t \rangle\,;\, \overrightarrow{s}, \overrightarrow{t}) : \Gamma, \Delta}\ \text{cut rule} \qquad \frac{\Phi \vdash (\overrightarrow{c}\,;\, \overrightarrow{t}, s_1, \ldots, s_k) : \Gamma, A_1, \ldots, A_k}{\Phi \vdash (\overrightarrow{c}\,;\, \overrightarrow{t}, s_1 ⅋ \cdots ⅋ s_k) : \Gamma, A_1 ⅋ \cdots ⅋ A_k}\ ⅋\text{-rule}$$

$$\frac{\Phi \vdash (\overrightarrow{c_1}\,;\, \overrightarrow{s_1}, s_1) : \Gamma_1, A_1 \qquad \cdots \qquad \Phi \vdash (\overrightarrow{c_k}\,;\, \overrightarrow{s_k}, s_k) : \Gamma_k, A_k}{\Phi \vdash (\overrightarrow{c_1}, \ldots, \overrightarrow{c_k}\,;\, \overrightarrow{s_1}, \ldots, \overrightarrow{s_k}, s_1 \otimes \cdots \otimes s_k) : \Gamma_1, \ldots, \Gamma_k, A_1 \otimes \cdots \otimes A_k}\ \otimes\text{-rule}$$

$$\frac{\Phi \vdash (\overrightarrow{c_1}\,;\, \overrightarrow{s_1}) : \Gamma_1 \qquad \cdots \qquad \Phi \vdash (\overrightarrow{c_k}\,;\, \overrightarrow{s_k}) : \Gamma_k}{\Phi \vdash (\overrightarrow{c_1}, \ldots, \overrightarrow{c_k}\,;\, \overrightarrow{s_1}, \ldots, \overrightarrow{s_k}) : \Gamma_1, \ldots, \Gamma_k}\ \text{mix rule}$$

Figure 1. The MLL logical rules

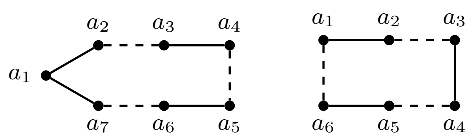

Figure 3. A loop of length 7 and a cycle of length 6

is also a loop (the same loop traveled in the opposite direction). See Figure 3 for an example.

Let X be a closed coherence space and let $\gamma = (a_1, \ldots, a_{2n-1})$ be a loop in X. One defines a new closed coherence space X_γ by "contracting the loop γ" (up to τ_X) as follows.

The set $|X_\gamma|$ is $|X| \setminus \bigcup_{i=2}^{2n-1} (a_i)_X$; observe that $a_1 \in |X_\gamma|$ by definition of a loop (the anchor of the loop is not removed).

The coherence relation $\frown_{X_\gamma}$ is the least symmetric and antireflexive relation on $|X_\gamma|$ such that

- for all $c, d \in |X_\gamma|$, if $c \frown_X d$ then $c \frown_{X_\gamma} d$;
- if $d \in |X_\gamma| \setminus \{a_1\}$ and there exists $c \in \bigcup_{i=2}^{2n-1} (a_i)_X$ such that $c \frown_X d$, then $a_1 \frown_{X_\gamma} d$.

Last τ_{X_γ} is the restriction of the equivalence relation τ_X to $|X_\gamma|$. Since $\bigcup_{i=2}^{2n-1} (a_i)_X$ is a closed subset of $|X|$ which does not contain a_1 (that set is closed with respect to τ_X, of course), we have

$$|X_\gamma|/\tau_{X_\gamma} = (|X|/\tau_X) \cap \mathcal{P}(|X_\gamma|)\,. \tag{1}$$

3.2.3 Cycles. A *cycle* is a sequence $(a_1, \ldots, a_{2n})$ (with $n \geq 1$) of pairwise distinct elements of $|X|$ such that $(a_1, \ldots, a_{2n-1})$ is a path and such that $a_{2n-1} \frown_X a_{2n}$ and $a_1 \in (a_{2n})^+_X$.

Observe that, for any $i \in \{1, \ldots, 2n\}$, if i is odd then $(a_i, a_{i+1}, \ldots, a_{2n}, a_1, \ldots, a_{i-1})$ is a cycle and if i is even then $(a_i, a_{i-1}, \ldots, a_2, a_1, a_{2n}, \ldots, a_{i+1})$ is also a cycle. They are actually the same cycle but traveled in a different way.

3.2.4 Splitting nodes. An element α of $|X|/\tau_X$ is a *splitting node* if there is a family $(U_a)_{a \in \alpha}$ of subsets of $|X|$ which are

- closed,
- pairwise disjoint,

such that $\cup_{a \in \alpha} U_a = |X| \setminus \alpha$ and, moreover, for all $a, a' \in \alpha$:

$$a \neq a' \Rightarrow (\forall b \in U_a \cup \{a\},\, b' \in U_{a'} \cup \{a'\} \quad b \smile_X b')\,.$$

A splitting node α allow to decompose the closed coherence space in completely independent components $(U_a \cup \{a\})_{a \in \alpha}$ intuitively corresponding to proofs of the premises of a $\otimes$ rule represented by α, see Paragraph 4.1.2.

Lemma 1 *Let X be a closed coherence space and let γ be a loop in X.*

- *Any splitting node of X_γ is a splitting node of X.*
- *If X_γ has a cycle then X has a cycle.*

Proof. Let $\gamma = (a_1, \ldots, a_{2n-1})$ (with $n \geq 1$) be a loop of X.

$\triangleright$ Let first α be a splitting node of X_γ and let $(U_a)_{a \in \alpha}$ be a family of closed and pairwise disjoint subsets of $|X_\gamma|$ such that

- $\bigcup_{a \in \alpha} U_a = |X_\gamma| \setminus \alpha$
- and for any $a, a' \in \alpha$ with $a \neq a'$ and any $b \in U_a \cup \{a\}$ and $b' \in U_{a'} \cup \{a'\}$, one has $b \smile_{X_\gamma} b'$.

Remember that $\alpha \in |X|/\tau_X$. We prove that α is a splitting node of X.

Since $|X_\gamma|$ is the disjoint union of the sets $(U_a \cup \{a\})_{a \in \alpha}$, there is an unique $a_0 \in \alpha$ such that $a_1 \in U_{a_0} \cup \{a_0\}$. Let $(V_a)_{a \in \alpha}$ be the family of subsets of $|X|$ such that $V_a = U_a$ if $a \neq a_0$ and

$$V_{a_0} = U_{a_0} \cup \bigcup_{i=2}^{2n-1} (a_i)_X\,.$$

Then the V_a's are closed (relative to τ_X), pairwise disjoint and clearly satisfy $\bigcup_{a \in \alpha} V_a = |X| \setminus \alpha$.

Let $a, a' \in \alpha$ with $a \neq a'$. Let $b \in V_a \cup \{a\}$ and $b' \in V_{a'} \cup \{a'\}$, we prove that $b \smile_X b'$ considering several cases.

If $a \neq a_0$ and $a' \neq a_0$ the assertion directly follows from our hypothesis that α is a splitting node of X_γ. So assume that $a = a_0$.

- If $b \notin \bigcup_{i=2}^{2n-1} (a_i)_X$ then $b \in |X_\gamma|$, hence $b \in U_{a_0} \cup \{a_0\}$ and therefore $b \smile_{X_\gamma} b'$, that is $b \smile_X b'$.
- If $b \in \bigcup_{i=2}^{2n-1} (a_i)_X$ and if $b \frown_X b'$, then we have $a_1 \frown_{X_\gamma} b'$ by definition of $\frown_{X_\gamma}$, which is impossible since $a_1 \in U_{a_0} \cup \{a_0\}$ and $b' \in U_{a'} \cup \{a'\}$. Therefore $b \smile_X b'$ (we know that $b \neq b'$ because $a \neq a'$).

$\triangleright$ We prove now the second statement of the Lemma. Let $\delta = (b_1, \ldots, b_{2k})$ (with $k \geq 1$) be a cycle in X_γ and let us build a cycle in X. If $b_j \neq a_1$ for all $j = 1, \ldots, 2k$, then δ is already a cycle in X, so assume that $b_j = a_1$ for some j. Up to reindexing the elements of δ, we can assume without loss of generality that $j = 1$, that is $b_1 = a_1$.

Observe that

$$\forall j \in \{2, \ldots, 2k\} \quad b_j \notin \{a_1, \ldots, a_{2n-1}\}\,.$$

We have $a_1 = b_1 \frown_{X_\gamma} b_2$. If $a_1 \frown_X b_2$, then δ is a cycle in X and we are done, so assume that this not the case. By definition of

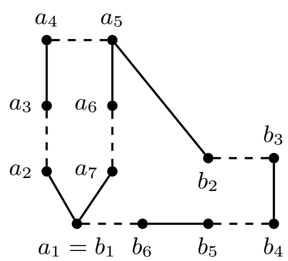

Figure 4. A configuration in the proof of Lemma 1

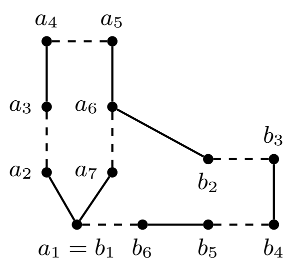

Figure 5. A configuration in the proof of Lemma 1

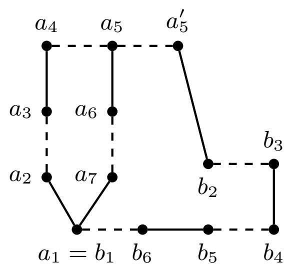

Figure 6. A configuration in the proof of Lemma 1

$\frown_{X_\gamma}$, there exists $i \in \{2, \dots, 2n-1\}$ and $a_i' \in (a_i)_X$ such that $a_i' \frown_X b_2$.

Assume first that $a_i' = a_i$. If $i \in 2\mathbb{N}+1$ then

$$\delta' = (a_1, \dots, a_i', b_2, \dots, b_{2k})$$

is a cycle in X. This case is illustrated in Figure 4. And if $i \in 2\mathbb{N}$ then

$$\delta' = (a_1, a_{2n-1}, a_{2n-2}, \dots, a_{i+1}, a_i', b_2, \dots, b_{2k})$$

is a cycle in X. This case is illustrated in Figure 5.

Assume last that $a_i' \neq a_i$. If $i \in 2\mathbb{N}+1$ then

$$\delta' = (a_1, a_{2n-1}, a_{2n-2}, \dots, a_{i+1}, a_i, a_i', b_2, \dots, b_{2k})$$

is a cycle in X and if $i \in 2\mathbb{N}$ then

$$\delta' = (a_1, \dots, a_i, a_i', b_2, \dots, b_{2k})$$

is a cycle in X. These two cases are illustrated in Figures 6 and 7.

Observe indeed that in all cases δ' are repetition-free sequences because the points of δ belong to $|X_\gamma|$. $\square$

We can now easily prove the main graph-theoretical property which will allow us to establish our new Sequentialization Theorem.

Proposition 2 *Let X be a finite closed coherence space. If X has no cycle, then X has a splitting node.*

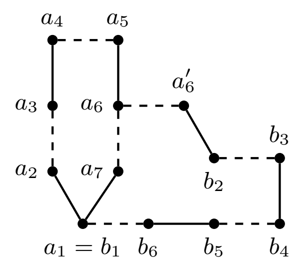

Figure 7. A configuration in the proof of Lemma 1

Proof. By induction on the cardinality of $|X|$.

Assume that X has no cycle. Let $\gamma = (a_1, \dots, a_{2n-1})$ be a path of maximal length in X. Observe that

$$\forall i \in \{2, \dots, 2n-1\} \quad a_i \notin (a_1)_X \tag{2}$$

because X has no cycle. Indeed, assume that $a_i \in (a_1)_X$. In each of the two cases $i \in 2\mathbb{N}+1$ and $i \in 2\mathbb{N}$, using transitivity of τ_X in the first case, one builds a cycle in X.

If, for all $a \in (a_1)_X^+$ and all $b \in |X| \setminus \{a\}$ one has $b \smile_X a$, then $\alpha = (a_1)_X$ is a splitting node of X (for $a \in \alpha$, set $U_a = \emptyset$ it $a \neq a_1$ and $U_{a_1} = |X| \setminus \alpha$ then for all $b \in U_a \cup \{a\}$ and $b' \in U_{a'} \cup \{a'\}$ one has $b \smile_X b'$ for all $a, a' \in \alpha$ with $a \neq a'$; indeed, one of the two points a and a' is distinct from a_1, say for instance that $a \neq a_1$, then $b = a$ and hence $b \smile_X b'$ by our assumption).

Assume now that there exists $a \in (a_1)_X^+$ and $b \in |X|$ such that $b \frown_X a$. By (2) we cannot have $a \in \{a_1, \dots, a_{2n-1}\}$ and hence, since γ is a path of maximal length, there must exist $i \in \{1, \dots, 2n-1\}$ such that $b = a_i$ (otherwise, $(b, a, a_1, \dots, a_{2n-1})$ is a longer path). This index i is unique because the elements of γ are pairwise distinct. If $i \in 2\mathbb{N}+1$ then $(a_1, \dots, a_i, a)$ is a cycle and this is impossible since we have assumed that X has no cycle. So $i \in 2\mathbb{N}$ (and actually $i \geq 2$) and $\delta = (a_i = b, a, a_1, \dots, a_{i-1}) = (d_1, \dots, d_{i+1})$ is a loop of length ≥ 3 (the fact that $d_j \notin (d_1)_X$ for all $j = 2, \dots, i+1$ results again from the acyclicity of X).

Since X has no cycle, X_δ has no cycle either, by Lemma 1. Since $|X_\delta|$ has strictly less elements than $|X|$, it follows by inductive hypothesis that X_δ has a splitting node. By Lemma 1 again, X has a splitting node as contended. $\square$

3.2.5 Short-cuts.

Given a closed coherence space X and a cycle $\gamma = (a_1, \dots, a_{2n})$ with $n \geq 1$ in X (see Section 3.2), we call *short-cut of* γ any pair (i, j) such that $i, j \in \{1, \dots, 2n\}$ and

- $(i, j) \neq (1, 2n)$
- $i + 2 \leq j$
- and $a_i \frown_X a_j$.

A cycle $\gamma = (a_1, \dots, a_{2n})$ can have two kinds of short-cuts.

- The short-cuts (i, j) where i and j have not the same parity (one is odd and the other is even) are called *reducible*.
- The short-cuts (i, j) where i and j have the same parity are called *irreducible*.

These configurations are illustrated in Figure 8.

This terminology is justified by the following observation.

Lemma 3 *Let $\gamma = (a_1, \dots, a_{2n})$ be a cycle and let (i, j) be a reducible short-cut of γ. If $i \in 2\mathbb{N}+1$ and $j \in 2\mathbb{N}$ then $\gamma' = (a_i, a_j, a_{j+1}, \dots, a_{2n}, a_1, \dots, a_{i-1})$ is a cycle, and if $i \in 2\mathbb{N}$ and $j \in 2\mathbb{N}+1$ then $\gamma' = (a_i, a_j, a_{j-1}, \dots, a_{i+1})$ is a cycle.*

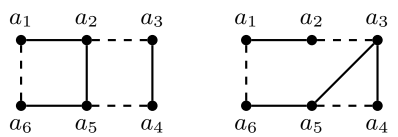

Figure 8. Reducible and irreducible short-cuts $(2, 5)$ and $(3, 5)$ in a cycle of length 6

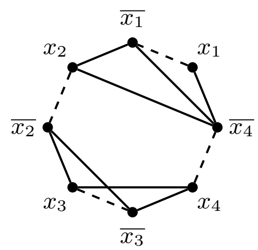

Figure 9. The closed coherence space $\mathsf{G}(p)$ associated with the proof net $p = (\,;\, (\overline{x_3} \parr x_3) \otimes (\overline{x_2} \parr x_4), \overline{x_4} \otimes (x_1 \parr (\overline{x_1} \otimes x_2)))$

The proof is straightforward. We call γ' the cycle *induced by* (i, j) and we introduce a notation for this cycle: $\gamma|_{i,j} = \gamma'$.

Lemma 4 *Let $\gamma = (a_1, \ldots, a_{2n})$ be a cycle and let (i, j) be a reducible short-cut of γ. We have*

$$\mathsf{len}(\gamma|_{i,j}) = \begin{cases} 2n - j + i + 1 & \text{if } i \in 2\mathbb{N} + 1 \\ j - i + 1 & \text{if } i \in 2\mathbb{N} \end{cases}$$

Moreover, any short-cut of $\gamma|_{i,j}$ is also a short-cut of γ. It is irreducible in $\gamma|_{i,j}$ iff it is irreducible in γ.

The proof is a simple verification.

Lemma 5 *If any cycle of X has a short-cut, then any cycle of X has an irreducible short-cut.*

Proof. Assume that any cycle of X has a short-cut. By induction on γ, we prove that any cycle γ has an irreducible short-cut. Let γ be a cycle. Let (i, j) be a short-cut of γ. If (i, j) is irreducible, we are done. If not, then by Lemma 3 there is a shorter cycle γ' as described in the statement of that Lemma. By inductive hypothesis, γ' has an irreducible short-cut which is easily seen to be an irreducible short-cut of γ. □

4. Sequentialization

4.1 Closed coherence space associated with a proof structure

Given a tree t, we define a coherence space $\mathsf{G}(t)$ such that $|\mathsf{G}(t)| = \mathsf{V}(t)$ as follows: $\mathsf{G}(x)$ is the unique coherence space whose web is $\{x\}$.

$$\mathsf{G}(s_1 \otimes \cdots \otimes s_k) = \mathsf{G}(s_1) \mathbin{\&} \cdots \mathbin{\&} \mathsf{G}(s_k)$$
$$\mathsf{G}(s_1 \parr \cdots \parr s_k) = \mathsf{G}(s_1) \oplus \cdots \oplus \mathsf{G}(s_k)\,.$$

Given a cut $c = \langle s \mid s' \rangle$ one defines $\mathsf{G}(c) = \mathsf{G}(s) \mathbin{\&} \mathsf{G}(s')$.

Given a closed proof structure $p = (c_1, \ldots, c_k\,;\, s_1, \ldots, s_n)$, we set

$$\mathsf{G}(p) = \mathsf{G}(s_1) \oplus \cdots \oplus \mathsf{G}(s_n) \oplus \mathsf{G}(c_1) \oplus \cdots \oplus \mathsf{G}(c_k)$$

and we equip this coherence space with the equivalence relation on $|\mathsf{G}(p)|$ defined by: $x \;\tau_{\mathsf{G}(p)}\; y$ if $x = y$ or $x = \overline{y}$. Then $\mathsf{G}(p)$ is a closed coherence space and $|\mathsf{G}(p)| = \mathsf{V}(p)$. Figure 9 provides an example of this construction. Observe that the closed coherence space has a cycle of length 8, but that this cycle has 4 short-cuts, and that 3 of them are irreducible. We leave it to the reader to find a sequentialization of this proof net.

Remark: The coherence spaces $X = \mathsf{G}(p)$ produced in that way are *serial-parallel*, see Brandstädt et al. (1999). These coherence spaces are characterized by the following property: $|X|$ is finite and, given four pairwise distinct elements a_1, a_2, a_3, a_4 of $|X|$ such that $a_i \frown_X a_{i+1}$ for $i = 1, 2, 3$, one has necessarily $a_1 \frown_X a_3$, $a_2 \frown_X a_4$ or $a_1 \frown_X a_4$.

The starting point of this work was the idea of representing formulas by means of such spaces, as suggested by Girard in the 1990's during private discussions. It was not clear to us however that a correctness criterion might be formulated directly in this setting, without coming back to the ordinary representations of formulas as trees.

4.1.1 Correctness of proof nets.

We establish that any proof net satisfies our correctness criterion.

Proposition 6 *Assume that $\Phi \vdash p : \Gamma$. In the closed coherence space $\mathsf{G}(p)$, any cycle has a short-cut.*

Remark: Since a cycle of length 2 cannot have a short-cut, it follows that if $\Phi \vdash p : \Gamma$ then $\mathsf{G}(p)$ has no cycle of length 2.

Proof. By induction on a derivation of $\Phi \vdash p : \Gamma$. If the derivation consists of one axiom then $p = (\,;\, x, \overline{x})$ and $|\mathsf{G}(p)| = \{x, \overline{x}\}$ and one has $x \smile_{\mathsf{G}(p)} \overline{x}$ so that $\mathsf{G}(p)$ has no cycle.

The case where the derivation ends with a $\parr$-rule is trivial since the closed coherence space associated with the conclusion coincides with the closed coherence space associated with the premise.

Assume that the derivation ends with a $\otimes$-rule. More precisely assume that $p = (\overrightarrow{c_1}, \ldots, \overrightarrow{c_k}\,;\, \overrightarrow{s_1}, \ldots, \overrightarrow{s_k}, s_1 \otimes \cdots \otimes s_k)$, $\Gamma = (\Gamma_1, \ldots, \Gamma_k, A_1 \otimes \cdots \otimes A_k)$ and the derivation ends with

$$\frac{\Phi \vdash p_1 : \Gamma_1, A_1 \qquad \cdots \qquad \Phi \vdash p_k : \Gamma_k, A_k}{\Phi \vdash p : \Gamma}\ \otimes\text{-rule}$$

where $p_i = (\overrightarrow{c_i}\,;\, \overrightarrow{s_i}, s_i)$ for $i = 1, \ldots, k$. Observe that

$$|\mathsf{G}(p)| = |\mathsf{G}(p_1)| \uplus \cdots \uplus |\mathsf{G}(p_k)|$$

and that, given $z_1, z_2 \in |\mathsf{G}(p)|$, one has $z_1 \frown_{\mathsf{G}(p)} z_2$ only in the following situations:

- $z_1, z_2 \in |\mathsf{G}(p_i)|$ and $z_1 \frown_{\mathsf{G}(p_i)} z_2$ for some $i \in \{1, \ldots, k\}$
- there is $i, j \in \{1, \ldots, k\}$ with $i \neq j$ and $z_1 \in \mathsf{V}(s_i)$, $z_2 \in \mathsf{V}(s_j)$.

Let $\gamma = (z_1, \ldots, z_{2n})$ be a cycle in the closed coherence space $\mathsf{G}(p)$.

Assume first that $\{z_1, \ldots, z_{2n}\} \subseteq \mathsf{V}(p_i)$ for some i. Then γ is a cycle in $\mathsf{G}(p_i)$ and hence must have a short-cut, by inductive hypothesis, hence γ has a short-cut in $\mathsf{G}(p)$.

So assume that none of these inclusions holds. Without loss of generality (up to reindexing the trees $s_1, \ldots, s_k$) we can assume that $z_1 \in |\mathsf{G}(p_1)|$. Let i be the least index such that $z_i \notin |\mathsf{G}(p_1)|$. Again, up to reindexing, we can assume that $z_i \in |\mathsf{G}(p_2)|$. We have $i \geq 2$ and we cannot have $z_{i-1} = \overline{z_i}$ because $|\mathsf{G}(p_1)|$ and $|\mathsf{G}(p_2)|$ are closed and disjoint. Therefore $i \in 2\mathbb{N}$, $z_{i-1} \in \mathsf{V}(s_1)$ and $z_i \in \mathsf{V}(s_2)$ because we know that $z_{i-1} \frown_{\mathsf{G}(p)} z_i$. Hence $z_{i+1} = \overline{z_i} \in \mathsf{V}(p_2)$ since this set is closed.

Remember that, by definition of a cycle we have $z_{2n} = \overline{z_1}$ and hence $z_{2n} \in \mathsf{V}(p_1)$ since $z_1 \in \mathsf{V}(p_1)$ and this set is closed. Hence there is a $j \in \{i + 2, \ldots, 2n\}$ such that $z_j \in \mathsf{V}(p_1)$ (we have seen that $j = 2n$ has this property). Choose $j \geq i+2$ minimal with this property. We have $j \in 2\mathbb{N}$ because $\mathsf{V}(p_1)$ is closed and j is minimal such that $z_j \in \mathsf{V}(p_1)$. So $z_j \frown_{\mathsf{G}(p)} z_{j-1}$ by definition of a cycle and hence $z_j \in \mathsf{V}(s_1)$ (and $z_{j-1} \in \mathsf{V}(s_l)$ for some $l \in \{2,\dots,k\}$, because $z_{j-1} \notin \mathsf{V}(p_1)$). Then (i,j) is a short-cut of γ since $z_i \in \mathsf{V}(s_2)$ and $z_j \in \mathsf{V}(s_1)$, $i \neq 1$ and $j \geq i+2$. Observe that $(i-1, j-1)$ is another short-cut.

The case where the derivation ends with a cut rule is identical to the case of a $\otimes$-rule.

The case where the derivation ends with a mix rule with premises $\Phi \vdash p_i : \Gamma_i$ (for $i = 1,\dots,k$) is trivial because, if $\mathsf{G}(p)$ contains a cycle, then this cycle must be contained in $\mathsf{G}(p_i)$ for some i. □

4.1.2 Sequentialization.

We want now to prove a converse statement.

Proposition 7 *Let p be a closed proof structure and assume that $\Phi \vdash_0 p : \Gamma$. If, in the closed coherence space $\mathsf{G}(p)$, any cycle has an irreducible short-cut, then $\Phi \vdash p : \Gamma$.*

A natural strategy for proving this would be to show that our criterion implies the acyclicity criterion of Danos and Regnier (1989). We have not been able to do so yet and the connection between the two criteria has still to be investigated.

The general idea of the proof is to associate with p another closed coherence space X whose vertices are the premises of the outermost $\otimes$ rules of p and whose edges witness the presence of axiom links. Using the criterion, we show that this closed coherence space has no cycles in the sense of Paragraph 3.2.3. This closed coherence space has a splitting node by Proposition 2 and this provides the last tensor rule of a sequentialization of p.

Proof. Let $p = (\vec{c};\, t_1,\dots,t_n)$ be a closed proof structure such that $\Phi \vdash_0 p : \Gamma$ (with $\Gamma = (A_1,\dots,A_n)$), and assume that all cycles of $\mathsf{G}(p)$ have a short-cut. The proof is by induction on the number of $\otimes$-constructions occurring in p.

Let $i \in \{1,\dots,n\}$. If $t_i = t_{i,1} \parr \cdots \parr t_{i,k}$ (with $k \geq 2$) then $A_i = A_{i,1} \parr \cdots \parr A_{i,k}$. Let

$$p' = (\vec{c};\, t_1,\dots,t_{i-1},t_{i,1},\dots,t_{i,k},t_{i+1},\dots,t_n)$$
$$\Gamma' = (A_1,\dots,A_{i-1},A_{i,1},\cdots,A_{i,k},A_{i+1},\dots,A_n)\,.$$

Then $\mathsf{G}(p') = \mathsf{G}(p)$ and $\Phi \vdash p' : \Gamma' \Rightarrow \Phi \vdash p : \Gamma$ (by applying a $\parr$-rule). Iterating this reduction, we can assume that, for each i, t_i is either an element of $\mathcal{V}$ (and then A_i can be any formula) or is of shape $t_i = t_{i,1} \otimes \cdots \otimes t_{i,k_i}$ (and then $A_i = A_{i,1} \otimes \cdots \otimes A_{i,k_i}$) with $k_i \geq 2$.

Since the cut rule and the $\otimes$-rule are handled in the same way, we also assume for simplifying notations that the list $\vec{c}$ is empty. Let I be the set of all $i \in \{1,\dots,n\}$ such that $t_i = t_{i,1} \otimes \cdots \otimes t_{i,k_i}$ with $k_i \geq 2$. Therefore, saying that $i \in \{1,\dots,n\} \setminus I$ simply means that t_i is a variable, that we always denote as y_i.

To summarize our notations,

$$\forall i \in \{1,\dots,n\} \quad t_i = \begin{cases} t_{i,1} \otimes \cdots \otimes t_{i,k_i} & \text{if } i \in I \\ y_i \in \mathcal{V} & \text{if } i \notin I \end{cases}$$

We define now another closed coherence space X as follows.

- $|X| = \{(i,\lambda) \mid i \in I \text{ and } 1 \leq \lambda \leq k_i\} \subseteq \mathbb{N} \times \mathbb{N}$
- $(i,\lambda) \frown_X (i',\lambda')$ if $(i,\lambda) \neq (i',\lambda')$ and there exists $x \in \mathcal{V}$ such that $x \in \mathsf{V}(t_{i,\lambda})$ and $\overline{x} \in \mathsf{V}(t_{i',\lambda'})$.
- $(i,\lambda)\ \tau_X\ (i',\lambda')$ if $i = i'$.

We prove that X has no cycle.

Towards a contradiction, assume that

$$\delta = ((i_1,\lambda_1),\dots,(i_{2l},\lambda_{2l}))$$

is a cycle in X (with $l \geq 1$). Assume moreover that this cycle is of minimal length.

Saying that δ is a cycle means that the (i_j,λ_j)'s are pairwise distinct, that $(i_1,\lambda_1) \frown_X (i_2,\lambda_2)$, $i_2 = i_3$ and $\lambda_2 \neq \lambda_3$, $(i_3,\lambda_3) \frown_X (i_4,\lambda_4)$,…,$(i_{2l-1},\lambda_{2l-1}) \frown_X (i_{2l},\lambda_{2l})$, $i_{2l} = i_1$ and $\lambda_{2l} \neq \lambda_1$.

For each $j \in \{1,\dots,2l\}$, by definition of $\frown_X$, we can find $x_j \in \mathsf{V}(t_{i_j,\lambda_j})$ such that $x_{j+1} = \overline{x_j}$ if $j \in 2\mathbb{N}+1$.

If $j \in 2\mathbb{N}$ we have $x_j \frown_{\mathsf{G}(p)} x_{j+1}$ since $i_j = i_{j+1}$ and $\lambda_j \neq \lambda_{j+1}$, and because of the definition of the relation $\frown_{\mathsf{G}(p)}$. This holds also in the case where $j = 2n$, replacing then $j+1$ with 1 (we actually work modulo $2l$).

Therefore $\gamma = (x_2,\dots,x_{2l},x_1) = (z_1,\dots,z_{2l})$ is a cycle in $\mathsf{G}(p)$; indeed, the x_j's are pairwise distinct because the sets $\mathsf{V}(t_{i_j,\lambda_j})$ are pairwise disjoint since the (i_j,λ_j) are pairwise distinct.

Hence γ has an irreducible short-cut (h',j'): we have $1 \leq h',j' \leq 2l$, $h'+2 \leq j'$, $(h',j') \neq (1,2l)$ and $z_{h'} \frown_{\mathsf{G}(p)} z_{j'}$, and h' and j' have the same parity. In other words, we have two possibilities:

(1) either we can find $h,j \in \{1,\dots,2l\}$ having the same parity and such that $2 \leq h,j \leq 2l$ with $h+2 \leq j$ and $x_h \frown_{\mathsf{G}(p)} x_j$

(2) or we can find $h \in \{3,\dots,2l-1\} \cap (2\mathbb{N}+1)$ such that $x_1 \frown_{\mathsf{G}(p)} x_h$; in that case we set $j = 1$.

Since $x_h \frown_{\mathsf{G}(p)} x_j$, we must have $i_h = i_j$ (and $\lambda_h \neq \lambda_j$ since the elements of the cycle δ are pairwise distinct). We consider now various cases.

Assume first that we are in case (1).

- If $h,j \in 2\mathbb{N}$ then we know that $i_{h+1} = i_h$ and we must have $\lambda_{h+1} \neq \lambda_j$ because $i_j = i_h$ and $j \neq h+1$ (and the elements of δ are pairwise distinct). Therefore, the sequence $((i_{h+1},\lambda_{h+1}),\dots,(i_j,\lambda_j))$ is a cycle whose length is less than the length of δ, contradicting our assumption that δ is a cycle of minimal length.
- If $h,j \in 2\mathbb{N}+1$ then we have $i_j = i_{j-1}$ and hence $i_{j-1} = i_h$, and we have $j-1 \neq h$. Hence the sequence $((i_h,\lambda_h),\dots,(i_{j-1},\lambda_{j-1}))$ is a cycle whose length is less than the length of δ, contradiction.

Assume now that we are in case (2). Remember that $i_{2l} = i_1 = i_h$. Since $h \in 2\mathbb{N}+1$ the sequence $((i_h,\lambda_h),\dots,(i_{2l},\lambda_{2l}))$ is a cycle whose length is less than the length of δ, contradiction.

Therefore, by Proposition 2, X has a splitting node which is an element of $|X|/\tau_X$. It is a set of shape $\{i\} \times \{1,\dots,\lambda_{k_i}\}$ for one element i of $\{1,\dots,n\}$. Without loss of generality we can assume that $i = 1$. We can therefore find a family $(U_\lambda)_{\lambda=1}^{k_1}$ of sets

$$U_\lambda \subseteq |X| \setminus (\{1\} \times \{1,\dots,k_1\})$$

which are closed, disjoint, such that

$$U_1 \uplus \cdots \uplus U_{k_1} = |X| \setminus (\{1\} \times \{1,\dots,k_1\})$$

and such that for all $\lambda,\lambda' \in \{1,\dots,k_1\}$ such that $\lambda \neq \lambda'$ and all $(i,\mu) \in U_\lambda \cup \{(1,\lambda)\}$ and $(i',\mu') \in U_{\lambda'} \cup \{(1,\lambda')\}$, one has $(i,\mu) \smile_X (i',\mu')$.

For each $\lambda = 1,\dots,k_1$, we define three subsets of $\{2,\dots,n\}$.

- U^1_λ is the set of all $j \in I \cap \{2,\dots,n\}$ such that $(j,\mu) \in U_\lambda$ for each $\mu \in \{1,\dots,k_j\}$ (since U_λ is closed, this is equivalent to saying that $(j,\mu) \in U_\lambda$ for some $\mu \in \{1,\dots,k_j\}$).

- U^2_λ is the set of all $j \in \{2,\dots,n\} \setminus I$ such that $\overline{y_j} \in \mathsf{V}(t)$ for some $t \in \{t_{1,\lambda}\} \cup \{t_i \mid i \in U^1_\lambda\}$ (remember that $t_j = y_j$ since $j \notin I$).
- W is the set of all $j \in \{2,\dots,n\} \setminus I$ such that there exists $j' \in \{2,\dots,n\} \setminus I$ with $y_{j'} = \overline{y_j}$ (we have $t_j = y_j$ and $t_{j'} = y_{j'}$ since $j, j' \notin I$).

The sets W and U^θ_λ for $\lambda \in \{1,\dots,k_1\}$ and $\theta \in \{1,2\}$ are pairwise disjoint, as we check now.

- Let $\lambda, \lambda' \in \{1,\dots,k_1\}$ with $\lambda \neq \lambda'$ and assume that $j \in U^1_\lambda \cap U^1_{\lambda'}$. This means that there are $\mu, \mu' \in \{1,\dots,k_j\}$ such that $(j,\mu) \in U_\lambda$ and $(j,\mu') \in U_{\lambda'}$. Since U_λ is closed and since $(j,\mu)\ \tau_X\ (j,\mu')$, we have therefore $(j',\mu') \in U_\lambda \cap U_{\lambda'}$ and this is impossible since the U_λ's are pairwise disjoint.
- Let $\lambda, \lambda' \in \{1,\dots,k_1\}$. We have $U^1_\lambda \subseteq I$ and $U^2_{\lambda'} \cap I = \emptyset$ and hence $U^1_\lambda \cap U^2_{\lambda'} = \emptyset$.
- Let $\lambda, \lambda' \in \{1,\dots,k_1\}$ with $\lambda \neq \lambda'$ and assume that $j \in U^2_\lambda \cap U^2_{\lambda'}$. Then we have $\overline{y_j} \in \mathsf{V}(t) \cap \mathsf{V}(t')$ for some $t \in \{t_{1,\lambda}\} \cup \{t_i \mid i \in U^1_\lambda\}$ and $t' \in \{t_{1,\lambda'}\} \cup \{t_i \mid i \in U^1_{\lambda'}\}$. But this is impossible because $t \neq t'$ and hence $\mathsf{V}(t) \cap \mathsf{V}(t') = \emptyset$.
- $U^1_\lambda \cap W = \emptyset$ because $U^1_\lambda \subseteq I$ and $W \cap I = \emptyset$.
- Let $\lambda \in \{1,\dots,k_1\}$ and let $j \in U^2_\lambda \cap W$. This means that $\overline{y_j} \in \mathsf{V}(t)$ for some $t \in \{t_{1,\lambda}\} \cup \{t_i \mid i \in U^1_\lambda\}$, and also that $\overline{y_j} = t_{j'}$ for some $j' \in \{2,\dots,n\} \setminus I$. This is impossible because we must have $\mathsf{V}(t) \cap \mathsf{V}(t_{j'}) = \emptyset$.

A simple inspection shows that (one has just to check the right to left inclusions)

$$\bigcup_{\lambda=1}^{k_1} U^1_\lambda = I \cap \{2,\dots,n\} \quad \text{and} \quad W \cup \bigcup_{\lambda=1}^{k_1} U^2_\lambda = \{2,\dots,n\} \setminus I\,.$$

For each $\lambda = 1,\dots,k_1$, we define a proof structure

$$q_\lambda = (\,;\ (u_{l,\lambda})_{l=1}^{h(\lambda)})$$

where $u_{1,\lambda} = t_{1,\lambda}$ and $(u_{l,\lambda})_{l=2}^{h(\lambda)}$ is an enumeration of $(t_i)_{i \in U^1_\lambda \cup U^2_\lambda}$. So we have a bijection $\varphi_\lambda : \{2,\dots,h(\lambda)\} \to U^1_\lambda \cup U^2_\lambda$ such that

$$\forall l \in \{2,\dots,h(\lambda)\} \quad u_{l,\lambda} = t_{\varphi_\lambda(l)}\,.$$

Let $\Delta_\lambda = (D_{l,\lambda})_{l=1}^{h(\lambda)}$ where $D_{1,\lambda} = A_{1,\lambda}$ and $D_{l,\lambda} = A_{\varphi_\lambda(l)}$ for $l = 2,\dots,h(\lambda)$.

Then we have $\Phi \vdash_0 q_\lambda : \Delta_\lambda$.

We have also

$$\bigcup_{\lambda=1}^{k_1} \mathsf{V}(q_\lambda) \cup \{y_j \mid j \in W\} = \mathsf{V}(p)\,.$$

Let $\lambda \in \{1,\dots,k_1\}$, we claim that $\mathsf{V}(q_\lambda)$ is closed. Let $y \in \mathsf{V}(q_\lambda)$, there are several possibilities.

- Assume first that $y \in \mathsf{V}(t_{1,\lambda})$. Let $\lambda' \in \{1,\dots,k_1\} \setminus \{\lambda\}$. We have $\overline{y} \notin \mathsf{V}(t_{1,\lambda'})$ because $(1,\lambda) \smile_X (1,\lambda')$ (otherwise X has a cycle of length 2). Given $j \in U^1_{\lambda'}$ we have $\overline{y} \notin \mathsf{V}(t_j)$ because $(1,\lambda) \smile_X (j,\lambda')$ by definition of the U_μ's. We cannot have $j \in U^2_{\lambda'}$ such that $y = y_j$ since otherwise there would exist $j' \in U^1_{\lambda'}$ such that $y = \overline{\overline{y}} \in \mathsf{V}(t_{j'})$ which is impossible since $t_{j'} \neq t_{1,\lambda}$ and hence $\mathsf{V}(t_{j'}) \cap \mathsf{V}(t_{1,\lambda}) = \emptyset$. We cannot have $\overline{y} \in \{y_j \mid j \in W\}$ because this latter set is closed so that we should also have $y \in \{y_j \mid j \in W\}$. Therefore $\overline{y} \in \mathsf{V}(t_j)$ for some $j \in U^1_\lambda \cup U^2_\lambda$ because $\mathsf{V}(p)$ is closed, and hence $\overline{y} \in \mathsf{V}(q_\lambda)$.
- Assume next that $y \in \mathsf{V}(t_j)$ for some $j \in U^1_\lambda$. Let $\lambda' \in \{1,\dots,k_1\} \setminus \{\lambda\}$. Then, given $t \in \{t_{1,\lambda'}\} \cup \{t_k \mid k \in U^1_{\lambda'}\}$, one has $\overline{y} \notin \mathsf{V}(t)$ because $(j,\lambda) \smile_X (1,\lambda')$ and $(j,\lambda) \smile_X (k,\lambda')$ for each $k \in U_{1,\lambda'}$. As before we cannot have $\overline{y} \in \{y_j \mid j \in U^2_{\lambda'} \cup W\}$ and therefore $\overline{y} \in \mathsf{V}(q_\lambda)$.
- Assume last that $y = y_j$ with $j \in U^2_\lambda$. Then we have $\overline{y} \in \mathsf{V}(t)$ for some $t \in \{t_{1,\lambda}\} \cup \{t_i \mid i \in U^1_\lambda\}$ and hence $\overline{y} \in \mathsf{V}(q_\lambda)$.

In the closed coherence space $\mathsf{G}(q_\lambda)$, each cycle has a short-cuts since this property holds in $\mathsf{G}(p)$. By inductive hypothesis, since q_λ has strictly less $\otimes$ connectives than p, we have therefore $\Phi \vdash q_\lambda : A_{1,\lambda}, \Delta_\lambda$ for each $\lambda \in \{1,\dots,k_1\}$. Applying a $\otimes$-rule we get

$$\Phi \vdash (\,;\ t_1, \overrightarrow{u}) : A_1, \Delta_1, \dots, \Delta_{k_1}$$

where $\overrightarrow{u} = ((u_{l,1})_{l=2}^{h(1)}, \dots, (u_{l,k_1})_{l=2}^{h(k_1)})$

For each $j \in W$ we must have $\Phi(y_j) = A_j$ and $\Phi(\overline{y_j}) = A_j^\perp$ because $\Phi \vdash_0 p : \Gamma$ and hence we have the axiom $\Phi \vdash (\,;\ y_j, \overline{y_j}) : A_j, A_j^\perp$. Therefore, applying several times the $\mathsf{mix\ rule}$ we get $\Phi \vdash (\,;\ \overrightarrow{y}) : \overrightarrow{A}$ with $\overrightarrow{y} = (y_{j_1}, \dots, y_{j_{2m}})$ where $j_1, \dots, j_{2m}$ is a repetition-free enumeration of W and $\overrightarrow{A} = (A_{j_1}, \dots, A_{j_{2m}})$. Applying once more the $\mathsf{mix\ rule}$ we get a proof of

$$\Phi \vdash (\,;\ t_1, \overrightarrow{u}, \overrightarrow{y}) : A_1, \Delta_1, \dots, \Delta_{k_1}, \overrightarrow{A}$$

So (up to permutations), we have obtained a proof of $\Phi \vdash p : \Gamma$ and this ends the proof of the proposition. □

Theorem 8 (sequentialization) *Let p be a closed proof structure and assume that $\Phi \vdash_0 p : \Gamma$. The following conditions are equivalent.*

(1) $\Phi \vdash p : \Gamma$;
(2) any cycle of the closed coherence space $\mathsf{G}(p)$ has an irreducible short-cut;
(3) any cycle of the closed coherence space $\mathsf{G}(p)$ has a short-cut.

Proof. (1)⇒(3) by Proposition 6. (3)⇒(2) by Lemma 5. (2)⇒(1) by Proposition 7. □

5. Preservation under cut elimination

5.1 Cut elimination

We define a rewriting system on closed proof structures which consists of two rules. The $\otimes/\parr$ reduction rule is

$$(\overrightarrow{c}, \langle s_1 \otimes \cdots \otimes s_n \mid s'_1 \parr \cdots \parr s'_n \rangle\,;\ \overrightarrow{s}) \rightsquigarrow (\overrightarrow{c}, \langle s_1 \mid s'_1 \rangle, \dots, \langle s_n \mid s'_n \rangle\,;\ \overrightarrow{s})$$

and the axiom reduction rule is

$$(\overrightarrow{c}, \langle t \mid x \rangle\,;\ \overrightarrow{s}) \rightsquigarrow (\overrightarrow{c}\,;\ \overrightarrow{s})\,[t/\overline{x}] \quad \text{if } \overline{x} \notin \mathsf{V}(t)$$

where substitution of a tree for a variable is defined in the obvious way. This rule applies only under the proviso that $\overline{x} \notin \mathsf{V}(t)$. For instance, the "loop" $(\langle x \mid \overline{x} \rangle\,;\ )$ is a normal closed proof structure.

Observe that, if p is a proof structure and if $p \rightsquigarrow p'$ then p' is a proof structure, meaning that each variable occurs at most once in p'.

There is an apparent critical pair in the second reduction rule, in the case where t is a variable y which cannot belong to $\{x, \overline{x}\}$. We have

$$(\overrightarrow{c}, \langle y \mid x \rangle\,;\ \overrightarrow{s}) \rightsquigarrow (\overrightarrow{c}\,;\ \overrightarrow{s})\,[y/\overline{x}]$$

and

$$(\overrightarrow{c}, \langle y \mid x \rangle\,;\ \overrightarrow{s}) \rightsquigarrow (\overrightarrow{c}\,;\ \overrightarrow{s})\,[x/\overline{y}]\,.$$

The resulting proof structures $(\overrightarrow{c}\,;\ \overrightarrow{s})\,[y/\overline{x}]$ and $(\overrightarrow{c}\,;\ \overrightarrow{s})\,[x/\overline{y}]$ are easily seen to be α-equivalent. So all critical pairs are trivial and this rewriting system is confluent.

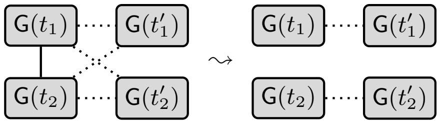

Figure 10. $\otimes/\parr$ reduction in the case $n = 2$

Lemma 9 *Let p be a proof structure and assume that $p \rightsquigarrow p'$. If p is closed then p' is closed.*

Proof. If $p \rightsquigarrow p'$ by the $\otimes/\parr$ reduction rule, the proof is straightforward. If the reduction results from an application of the axiom reduction rule, let us use the notations of the definition above of that reduction rule. Since $p = (\overrightarrow{c}, \langle t \mid x \rangle\,;\, \overrightarrow{s})$ is a closed proof structure, x does not occur in t, $\overrightarrow{c}$ and $\overrightarrow{s}$. Hence, by the proviso, $\overline{x}$ does not occur in t and occurs exactly once in $(\overrightarrow{c}\,;\,\overrightarrow{s})$. Therefore $p' = (\overrightarrow{c}\,;\,\overrightarrow{s})\,[t/\overline{x}]$ is a closed proof structure. □

Lemma 10 *Let p be a proof structures and assume that $p \rightsquigarrow p'$. If $\Phi \vdash_0 p : \Gamma$ then $\Phi \vdash_0 p' : \Gamma$.*

Coming back to the definitions of Section 2.1, the proof is straightforward (one needs an obvious Substitution Lemma for trees to deal with the axiom reduction rule).

Proposition 11 *If $\Phi \vdash p : \Gamma$ and $p = (\overrightarrow{c}\,;\,\overrightarrow{s})$ where the list $\overrightarrow{c}$ is not empty, then there exists p' such that $p \rightsquigarrow p'$.*

Proof. We know that $\Phi \vdash_0 p : \Gamma$. We also know that $\overrightarrow{c} = (\overrightarrow{d}, \langle s \mid s' \rangle)$ since the sequence $\overrightarrow{c}$ is not empty.

- If none of s and s' is a variable, then, without loss of generality, we have $s = t_1 \otimes \cdots \otimes t_n$ and $s' = t'_1 \parr \cdots \parr t'_n$. This is due to the fact that we know that $\Phi \vdash_0 s : A$ and $\Phi \vdash_0 s' : A^\perp$ for some type A.
- Assume that $s' = x \in \mathcal{V}$. We cannot have $\overline{x} \in \mathsf{V}(s)$ since otherwise $(x, \overline{x})$ is a cycle of length 2 in $\mathsf{G}(p)$, which is impossible by Proposition 6, since such a cycle cannot have a short-cut. Therefore, the second reduction rule applies. □

So if $\Phi \vdash p : \Gamma$ and $p = (\overrightarrow{c}\,;\,\overrightarrow{s})$ then p is normal iff $\overrightarrow{c}$ is empty. The reduction relation is strongly normalizing since the size of proof structures decreases along reduction steps.

5.2 Evolution of the coherence space of a proof structure during cut reduction

Let p and p' be proof structures and assume that $p \rightsquigarrow p'$. We have $p = (\overrightarrow{c}\,;\,\overrightarrow{s})$, $\overrightarrow{c} = (\overrightarrow{d}, \langle t \mid t' \rangle)$ and one of the two reduction rules applies to the cut $\langle t \mid t' \rangle$.

Lemma 12 *If $t = t_1 \otimes \cdots \otimes t_n$, $t' = t'_1 \parr \cdots \parr t'_n$ and $p' = (\overrightarrow{d}, \langle t_1 \mid t'_1 \rangle, \ldots, \langle t_n \mid t'_n \rangle\,;\,\overrightarrow{s})$, then $\mathsf{V}(p') = \mathsf{V}(p)$ and, given $x, y \in \mathcal{V}(p')$, one has $x \frown_{\mathsf{G}(p')} y$ if and only if*

- *$x \frown_{\mathsf{G}(p)} y$,*
- *and, if $x, y \in \bigcup_{i=1}^n (\mathsf{V}(t_i) \cup \mathsf{V}(t'_i))$, then $x, y \in \mathsf{V}(t_i) \cup \mathsf{V}(t'_i)$ for some $i \in \{1, \ldots, n\}$.*

This reduction is illustrated in Figure 10 where we use dotted lines to represent coherence edges corresponding to cuts. Grey boxes represent subgraphs and an edge between two such boxes means that all vertices of the first boxes are connected through the corresponding relation to all vertices of the second box.

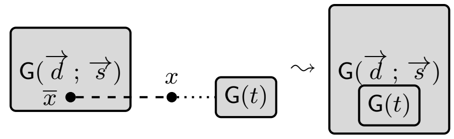

Figure 11. Axiom reduction

Lemma 13 *If $t' = x$, $\overline{x} \notin \mathsf{V}(t)$ and $p' = (\overrightarrow{d}\,;\,\overrightarrow{s})\,[t/\overline{x}]$, then $\mathsf{V}(p') = \mathsf{V}(p) \setminus \{x, \overline{x}\}$ and, given $y, z \in \mathcal{V}(p')$, one has $y \frown_{\mathsf{G}(p')} z$ if and only if*

- *$y \frown_{\mathsf{G}(p)} z$,*
- *or $y \in \mathsf{V}(t)$, $z \notin \mathsf{V}(t)$ and $z \frown_{\mathsf{G}(p)} \overline{x}$,*
- *or $z \in \mathsf{V}(t)$, $y \notin \mathsf{V}(t)$ and $y \frown_{\mathsf{G}(p)} \overline{x}$.*

The corresponding reduction is pictured in Figure 11.

The proofs of these lemmas are straightforward.

5.3 Preservation of correctness by cut reduction

The main statement of this section is that our correctness criterion is preserved under cut elimination.

Lemma 14 *Let p be a proof structure and assume that $p \rightsquigarrow p'$. If any cycle of $\mathsf{G}(p)$ has an irreducible short-cut, then any cycle of $\mathsf{G}(p')$ has a short-cut.*

Proof. We have $p = (\overrightarrow{c}\,;\,\overrightarrow{s})$, $\overrightarrow{c} = (\overrightarrow{d}, \langle t \mid t' \rangle)$ and one of the two reduction rules applies to the cut $\langle t \mid t' \rangle$. Therefore we consider two cases.

▷ Assume first that $t = t_1 \otimes \cdots \otimes t_n$ and $t' = t'_1 \parr \cdots \parr t'_n$ and that $p' = (\overrightarrow{d}, \langle t_1 \mid t'_1 \rangle, \ldots, \langle t_n \mid t'_n \rangle\,;\,\overrightarrow{s})$. Let $\gamma = (x_1, \ldots, x_{2k})$ be a cycle in $\mathsf{G}(p')$. Towards a contradiction, we assume that γ has a no short-cuts in p'. We refer to Paragraph 3.2.5 for definitions and notations about short-cuts.

Since γ is also a cycle in $\mathsf{G}(p)$ by Lemma 12, it has irreducible short-cuts (i, j) in $\mathsf{G}(p)$ (so $1 \leq i \leq j \leq 2k$, $i + 2 \leq j$, $(i, j) \neq (1, 2k)$, i and j have the same parity and $x_i \frown_{\mathsf{G}(p)} x_j$).

Let (i, j) be one of these short-cuts.

We assume first that $i, j \in 2\mathbb{N} + 1$.

Since $x_i \smile_{\mathsf{G}(p')} x_j$ and $x_i \frown_{\mathsf{G}(p)} x_j$, by Lemma 12 there must exist $h, l \in \{1, \ldots, n\}$ such that $h \neq l$, $x_i \in \mathsf{V}(t_h) \cup \mathsf{V}(t'_h)$, $x_j \in \mathsf{V}(t_l) \cup \mathsf{V}(t'_l)$, and $x_i \in \mathsf{V}(t_h)$ or $x_j \in \mathsf{V}(t_l)$. Since γ is a cycle in $\mathsf{G}(p')$ we have $x_{i+1} \frown_{\mathsf{G}(p')} x_i$ and $x_{j+1} \frown_{\mathsf{G}(p')} x_j$ and hence $x_{i+1} \in \mathsf{V}(t_h) \cup \mathsf{V}(t'_h)$ and $x_{j+1} \in \mathsf{V}(t_l) \cup \mathsf{V}(t'_l)$ by definition of p'. It follows that $x_i \frown_{\mathsf{G}(p)} x_{j+1}$ or $x_{i+1} \frown_{\mathsf{G}(p)} x_j$. In the first case, we cannot have $(i, j + 1) = (1, 2k)$ and in the second one we cannot have $(i + 1) + 1 = j$ since otherwise we would have cycles of length 2 in $\mathsf{G}(p)$. So, in the first case $(i, j + 1)$ is a reducible short-cut of γ and in the second case $(i + 1, j)$ is a reducible short-cut of γ. Let (i_0, j_0) be the element of $C = \{(i, j + 1), (i + 1, j)\}$ which is a short-cut and is such that $\mathsf{len}(\gamma|_{i_0,j_0})$ is minimal (among the $\mathsf{len}(\gamma|_{i_1,j_1})$ for the elements $(i_1, j_1) \in C$ which are short-cuts; if both are short-cuts and if $\mathsf{len}(\gamma|_{i,j+1}) = \mathsf{len}(\gamma|_{i+1,j})$, pick one of them randomly). Let $\gamma(i, j) = \gamma|_{i_0,j_0}$.

If $i, j \in 2\mathbb{N}$, we define $\gamma(i, j)$ in the same way (replace $i + 1$ with $i - 1$ and $j + 1$ with $j - 1$).

We choose now an irreducible short-cut (i, j) of γ in such a way that $\mathsf{len}(\gamma(i, j))$ be minimal. Then $\gamma(i, j)$ is a cycle in $\mathsf{G}(p)$ and therefore must have an irreducible short-cut (i', j'). But (i', j') is an irreducible short-cut of γ and satisfies $\mathsf{len}(\gamma(i', j')) < \mathsf{len}(\gamma(i, j))$ by Lemma 4, contradiction.

$\triangleright$ Assume now that $t' = x$, $\overline{x} \notin \mathsf{V}(t)$ and $p' = (\overrightarrow{d}\ ;\ \overrightarrow{s})\ [t/\overline{x}]$. Let $\gamma = (x_1, \ldots, x_{2k})$ be a cycle in $\mathsf{G}(p')$. Let I be the set of all $i \in \{1, \ldots, 2k\} \cap (2\mathbb{N} + 1)$ such that one of the two following conditions hold.

- $x_i \in \mathsf{V}(t)$, $x_{i+1} \notin \mathsf{V}(t)$ and $\overline{x} \frown_{\mathsf{G}(p)} x_{i+1}$.
- $x_{i+1} \in \mathsf{V}(t)$, $x_i \notin \mathsf{V}(t)$ and $\overline{x} \frown_{\mathsf{G}(p)} x_i$.

If I is empty then γ is a cycle of $\mathsf{G}(p)$ by Lemma 13, hence γ has a short-cut in $\mathsf{G}(p)$ which is also a short-cut in $\mathsf{G}(p')$. So assume that I is not empty, let l be the least element of I and h be its largest element. We consider 4 possibilities.

- $x_l \in \mathsf{V}(t)$ and $x_h \notin \mathsf{V}(t)$, so we have $x_l \frown_{\mathsf{G}(p')} x_h$ and hence (l, h) is a short-cut for γ in $\mathsf{G}(p')$ (remember that $l, h \in 2\mathbb{N}+1$) and this ends the proof in that case.
- Similarly, if $x_l \notin \mathsf{V}(t)$ and $x_h \in \mathsf{V}(t)$, then (l, h) is a short-cut for γ in $\mathsf{G}(p')$ and this ends the proof in that case.
- $x_l, x_h \in \mathsf{V}(t)$. So we have $x_{h+1} \notin \mathsf{V}(t)$ and $\overline{x} \frown_{\mathsf{G}(p)} x_{h+1}$ because $h \in I$. It follows that $x_l \frown_{\mathsf{G}(p')} x_{h+1}$ and hence that $(x_1, \ldots, x_l, x_{h+1}, \ldots, x_{2k})$ is a cycle in $\mathsf{G}(p')$.
- $x_l \notin \mathsf{V}(t)$ and $x_h \notin \mathsf{V}(t)$, then $x_{h+1} \in \mathsf{V}(t)$ and $\overline{x} \frown_{\mathsf{G}(p)} x_l$ because $h \in I$ and hence $x_l \frown_{\mathsf{G}(p')} x_{h+1}$ and therefore $(x_1, \ldots, x_l, x_{h+1}, \ldots, x_{2k})$ is a cycle in $\mathsf{G}(p')$.

So we can assume that I has exactly one element, say $I = \{1\}$, $x_1 \in \mathsf{V}(t)$ and $x_2 \notin \mathsf{V}(t)$, up to reindexing γ (if $x_1 \notin \mathsf{V}(t)$ and $x_2 \in \mathsf{V}(t)$, replace γ by the cycle $(x_2, x_1, x_{2k}, x_{2k-1}, \ldots, x_3)$). It is clear then that

$$\delta = (x_1, x, \overline{x}, x_2, \ldots, x_{2k}) = (y_1, \ldots, y_{2k+2})$$

is a cycle in $\mathsf{G}(p)$, hence δ has an irreducible short-cut (h, l) with $h, l \in \{1, \ldots, 2k+2\}$ (remember that saying that the short-cut is irreducible means that h and l have the same parity). If none of h and l belongs to $\{2, 3\}$ (remember that $y_2 = x$ and $y_3 = \overline{x}$) then (h, l) is a short-cut of γ and this ends the proof. So assume that $h \in \{2, 3\}$ or $l \in \{2, 3\}$. There are three cases.

- $h = 2$ and $l \in \{4, \ldots, 2k+2\} \cap 2\mathbb{N}$, so $y_h = x$. We have $x \frown_{\mathsf{G}(p)} y_l$ and hence $y_l \in \mathsf{V}(t)$. Therefore $l \neq 4$ since $y_4 = x_2$ and we know that $x_2 \notin \mathsf{V}(t)$. It follows that $(h, l-2)$ is a short-cut in γ and we are done.
- $h = 1$ and $l = 3$, so that $y_1 = x_1 \frown_{\mathsf{G}(p)} \overline{x} = y_3$. This is impossible because $x_1 \in \mathsf{V}(t)$, $\overline{x} \notin \mathsf{V}(t)$ and $\overline{x} \neq x$.
- $h = 3$ and $l \in \{5, \ldots, 2k+1\} \cap (2\mathbb{N}+1)$. Then we have $y_3 = \overline{x} \frown_{\mathsf{G}(p)} x_{l-2} = y_l$ and hence $x_1 \frown_{\mathsf{G}(p')} x_{l-2}$ by Lemma 13 because $x_1 \in \mathsf{V}(t)$. It follows that $(1, l-2)$ is a short-cut for γ in $\mathsf{G}(p')$ and this ends the proof. $\square$

Theorem 15 *Let p and p' be proof structures and assume that $p \rightsquigarrow p'$. If any cycle of $\mathsf{G}(p)$ has a short-cut, then any cycle of $\mathsf{G}(p')$ has a short-cut.*

Proof. Apply Lemmas 14 and 5. $\square$

6. Conclusion

We have presented a new correctness criterion for multiplicative proof nets with the MIX rule. The main feature of this criterion is that it does not involve transformations of proof structures, like criteria based on switchings or on graph rewriting.

This work suggests the possibility of replacing proof structures by closed coherence spaces in the spirit of the approach developed in Hughes (2006), and this idea is reinforced by the fact that cut reduction can essentially be expressed as a rewriting relation on these closed coherence spaces, as shown in Section 5.2.

Other natural questions remain to be addressed such as the computational complexity of the criterion.

Acknowledgments

I am very grateful to the referees of this paper for their very careful reading and for their extremely valuable suggestions.

This work has been partly funded by the French-Chinese project ANR-11-IS02-0002 *Locali*.

Addendum of June 2014: after the final version of this paper has been completed, a few weeks before the LICS-CSL conference in Vienna, we learned that this correctness criterion has already been published in Retoré (1999, 2003). We rediscovered it independently in 2013 and we are convinced that it is worth being further studied.

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