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A SHARP BOUND FOR SOLUTIONS OF LINEAR DIOPHANTINE EQUATIONS

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ABSTRACT. Let $Ax = b$ be an $m \times n$ system of linear equations with rank m and integer coefficients. Denote by Y the maximum of the absolute values of the $m \times m$ minors of the augmented matrix (A, b). It is proved that if the system has an integral solution, then it has an integral solution $x = (x_i)$ with $\max |x_i| \leq Y$. The bound is sharp.

I. INTRODUCTION

The existence of small integral solutions to systems of linear equations with integral coefficients has been discussed previously in [1, 2,3,4,5,6,7,8, 11]. Two types of problems have been considered.

In the first type the system is assumed to have a nonzero integer solution and the existence of a small solution is proved. A typical result of this type is the classical Siegel's Lemma [7] for homogeneous systems which has been used extensively in the theory of transcendental numbers. This result was generalized in [1] where the existence of a small integral basis for systems of linear homogeneous equations is proved.

In the second type of problems the system is assumed to have a nontrivial nonnegative integral solution and the existence of a small solution with these properties is proved. More work has been devoted recently to this type because of its implications for the complexity of integer programming [11]. In [3] the conjecture was made that for the second type of problems a nonnegative integral solutions exists with components bounded by the $p \times p$ minors of the augmented matrix, where p is the rank of the matrix. This conjecture was proved in several special cases and weaker results were proved in the general case in [4, 5]; however, it is still open in the general case.

In [6] the corresponding conjecture for the first type problem is discussed and proved under various additional conditions. In particular it is proved for

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an $m \times n$ system of rank m when $n - m \leq 8$. The object of this paper is to prove this latter conjecture, namely:

If $Ax = b$ is an $m \times n$ system of linear equations of rank m with integer coefficients and if the system has a nonzero integer solution, then it has an integral solution $x = (x_i)$ with $0 < \max |x_i| \leq Y$, where Y is the maximum of the absolute values of the $m \times m$ minors of (A, b).

This bound is sharp as we can see in the case $A = (A' \,|\, 0)$ and A' is a unimodular matrix, or if (1) A is an $m \times (m+1)$ matrix with the property that the gcd of all the $m \times m$ minors of A is 1, and (2) $b = 0$. Such an A can be obtained, for example, by taking m rows of an $(m+1) \times (m+1)$ unimodular matrix.

2. The main result

Let $Ax = b$ be a matrix equation of the form

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1,n+1} \\ \vdots \\ a_{m,n+1} \end{bmatrix} \tag{1}$$

where each a_{ij} is an integer. Assume that $n > m$, that the rows of A are linearly independent, and that (1) has a solution $y = (y_i)$, where each y_i is an integer.

The main result of this paper is the following:

Theorem. *If $Ax = b$ has a solution in integers, it has such a solution within the bound Y.*

Proof. Since A has full row rank, we may assume, without loss of generality, that the first m columns of A are linearly independent. Accordingly, partition A as (B, N), where B is $m \times m$ and nonsingular, and N is $m \times (n - m)$. Similarly, partition x as $(x_B^T, x_N^T)^T$, where $x_B^T = (x_1, x_2, \ldots, x_m)$ and $x_N^T = (x_{m+1}, \ldots, x_n)$. Let δ be the determinant of B.

The system (1) can be expanded as

$$Bx_B + Nx_N = b \tag{2}$$

and the general solution to (2) in real numbers is given by

$$x_B = B^{-1}(b - Nx_N), \qquad x_N \text{ arbitrary.} \tag{3}$$

From (3), it follows that finding integer solutions to (1) is equivalent to finding integer solutions x_N to

$$B^{-1}b \equiv B^{-1}Nx_N \pmod 1. \tag{4}$$

Since (1) is assumed to have a solution in integers, it follows that (4) also has a solution. Gomory [10] has shown that if (4) has an integer solution, then it has a nonnegative integer solution with

$$x_{m+1} + x_{m+2} + \cdots + x_n \leq |\delta| - 1. \tag{5}$$

(See also Theorem 5 on p. 275 of [9].)

Let $\bar{x}_N$ be such a solution to (4), and substitute $\bar{x}_N$ into (3) to compute $\bar{x}_B$. Then $\bar{x} = (\bar{x}_B^T, \bar{x}_N^T)^T$ is an integer solution to (1). The proof will be completed when we demonstrate that each component of $\bar{x}$ has absolute value at most Y.

For $i = m+1, m+2 \dots, n$ it follows immediately from (5) that $|\bar{x}_i| \le Y$. For $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n-m$ let δ_{ij} be the determinant of the matrix obtained by replacing the ith column of B with the jth column if N (i.e., by the $(j+m)$th column of A), and let δ_{i0} be the determinant of the matrix obtained by replacing the ith column of B with b. It now follows from Cramer's rule and (3) that

$$\begin{aligned}
|\bar{x}_i| &= |\delta_{i0} - \delta_{i1}\bar{x}_{m+1} - \delta_{i2}\bar{x}_{m+2} - \cdots - \delta_{i,n-m}\bar{x}_n|/|\delta| \\
&\le (|\delta_{i0}| + |\delta_{i1}|\bar{x}_{m+1} + |\delta_{i2}|\bar{x}_{m+2} + \cdots + |\delta_{i,n-m}|\bar{x}_n)/|\delta| \\
&\le Y(1 + \bar{x}_{m+1} + \bar{x}_{m+2} + \cdots + \bar{x}_n)/|\delta| \\
&\le Y(1 + (|\delta| - 1))/|\delta| \quad (\text{by}(5)) \\
&\le Y.
\end{aligned}$$

Hence all components of $\bar{x}$ are bounded in absolute value by Y, completing the proof of the theorem.

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