ADVANCES IN MATHEMATICS **108**, 91–103 (1994)

A Finiteness Condition for Semigroups Generalizing a Theorem of Coudrain and Schützenberger*

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Let S be a semigroup. For $s, t \in S$ we set $s \leqslant_B t$ if $s \in \{t\} \cup tS^1t$; we say that S satisfies the condition $\min_B$ if and only if any strictly descending chain w.r.t. $\leqslant_B$ of elements of S has a finite length. The main result of the paper is the following theorem: *Let T be a semigroup satisfying* $\min_B$. *Let T' be a subsemigroup of T such that all subgroups of T are locally finite in T'. Then T' is locally finite.* This result is a noteworthy generalization of a theorem of Coudrain and Schützenberger. Moreover, as a corollary we obtain the theorem of McNaughton and Zalcstein which gives a positive answer to the Burnside problem for semigroups of matrices on a field. © 1994 Academic Press, Inc.

1. INTRODUCTION

A famous theorem of Coudrain and Shützenberger [2] states that if S is a finitely generated semigroup satisfying the minimal condition on principal bi-ideals and all subgroups of S are finite then S is finite. In 1970 Hotzel [4], solving a conjecture of Shützenberger, proved a similar finiteness condition with the only difference that S satisfies the minimal condition on principal right-ideals ($\min_R$), instead of principal bi-ideals. In a recent paper [8] we gave a generalization of Hotzel's theorem by requiring that only finitely generated subgroups, instead of all subgroups, be finite. This kind of generalization is important since it allows one to obtain

* This work was partially supported by the Italian Ministry of Universities and Scientific Research and in part by Esprit-Ebra project ASMICS under Contract 3166.

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0001-8708/94 $9.00

noteworthy finiteness conditions for finitely generated semigroups which can be brought back to finiteness conditions for finitely generated groups.

In this paper we prove a generalization of the theorem of Coudrain and Shützenberger which moves in the same direction. More precisely let S be a semigroup. For $s, t \in S$ we set $s \leqslant_B t$ if $s \in \{t\} \cup tS^1t$; we say that S satisfies the condition $\min_B$ if and only if any strictly descending chain w.r.t. $\leqslant_B$ of elements of S has a finite length. One can easily prove (cf. Proposition 3.3) that if S satisfies the minimal condition on principal bi-ideals then S satisfies $\min_B$.

Let T be a semigroup and T' a subsemigroup of T. We say that a subgroup G of T is *locally finite in* T' if any subgroup of G which is generated by a finite subset of T' is finite.

The main result of this paper is the following stronger version of the Coudrain and Shützenberger theorem (cf. Theorem 3.1). *Let T be a semigroup satisfying* $\min_B$. *Let T' be a subsemigroup of T such that all subgroups of T are locally finite in T'. Then T' is locally finite.* The theorem of Coudrain and Schützenberger is then derived when (i) $T' = T$, (ii) condition $\min_B$ is replaced by the stronger minimal condition on principal bi-ideals, (iii) the local finiteness of subgroups of T in T' is replaced by the finiteness of all subgroups.

In Section 4 we consider semigroups of matrices with elements in a field. A noteworthy application of our main theorem is a straightforward new proof of the famous theorem of McNaughton and Zalcstein [12]: *A torsion semigroup of matrices with elements in a field is finite.* We recall that G. Jacob [5] gave a different proof of this theorem; moreover, he proved that it is possible to decide whether a finitely generated semigroup of matrices with coefficients in a field is finite.

2. Notations and Preliminaries

In the sequel A denotes a finite set or *alphabet*, and A^+ (resp. A^*) the *free semigroup* (resp. *free monoid*) over A. The elements of A are called *letters* and those of A^* *words*. The identity element of A^* is denoted by Λ. For any word w, $|w|$ denotes its *length*. In the following we identify (up to an isomorphism) a finitely generated semigroup S with $A^+/\phi\phi^{-1}$, where A is a finite alphabet and $\phi: A^+ \to S$ is a surjective morphism.

Let the alphabet A be totally ordered by the relation $<$. We can totally order A^+ by the relation $<_a$, called *alphabetic order*, defined as follows: For all $u, v \in A^+$

$$u < v \text{ if and only if } |u| < |v| \text{ or if } |u| = |v| \text{ then } u <_{\text{lex}} v,$$

where $<_{\text{lex}}$ denotes the *lexicographic order*. From the definition it follows that $<_a$ is a well order.

Let $s \in S$. In the set $s\phi^{-1}$ there is a unique minimal element with respect to the alphabetic order, usually called the *canonical representative* of s. Let $s, t \in S$; we say that s is a *factor* of t if $t \in S^1 s S^1$. If $t \in sS^1$ (resp. $t \in S^1 s$) then s is called a *prefix* (resp. *suffix*) of t. For any $s \in S$ we denote by $F(s)$ the set of factors of s. For any subset X of S, $F(X) = \bigcup_{x \in X} F(x)$. One says that X is *closed by factors* if $F(X) = X$. The following lemma holds. We omit the proof since it is straightforward (cf. [10]).

LEMMA 2.1. *Let S be a finitely generated semigroup and T any subset of S closed by factors. Then the set C_T of the canonical representatives of T is closed by factors.*

We denote by $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$, $\mathscr{J}$ the Green's relation in a semigroup S. We say that a semigroup S satisfies $\min_R$ (resp. $\min_L$, $\min_J$) if any strictly descending chain of principal right-ideals (resp. left-ideals, two-sided ideals) is finite.

An infinite word w (from left to right) over A is any map $w: N \to A$. For each $n \geqslant 0$ we set $w_n = w(n)$. A word $u \in A^+$ is a *finite factor* of w if there exist integers i, j, $0 \leqslant i \leqslant j$, such that $u = w_i \cdots w_j$; the sequence $w[i, j] = w_i \cdots w_j$ is also called an *occurrence* of u in w. We denote by $F(w)$ the set of all finite factors of w. The set of all infinite words over A is denoted by A^{∞}.

DEFINITION 1. An infinite word $t: N \to A$ is uniformly recurrent if there exists a map $k: A^* \to N$ with the property that any word $u \in F(t)$ is a factor of all words $w \in F(t)$ whose length $|w| \geqslant k(u)$. We call k the uniform recurrence function of t.

The relevance of uniformly recurrent infinite words is due to the following (cf. [3, 9]).

LEMMA 2.2. *Let $L \subseteq A^*$ be an infinite language. There exists an infinite word $w: N \to A$ such that*

(i) *w is uniformly recurrent,*

(ii) *$F(w) \subseteq F(L)$.*

3. MINIMAL AND FINITENESS CONDITIONS

Let S be a semigroup, We consider some quasi-order relations in S. We recall that a *quasi-order* $\leqslant$ in S is a reflexive and transitive relation in S. The meet $\leqslant \cap \leqslant^{-1}$ is an equivalence relation $\equiv$ and the quotient of S by

$\equiv$ is a partially ordered set. An element $s \in X \subseteq S$ is minimal (resp. maximal) in X w.r.t. $\leqslant$ if, for every $x \in X$, $x \leqslant s$ (resp. $s \leqslant x$) implies $x \equiv s$. For $s, t \in S$ if $s \leqslant t$ and $s \not\equiv t$ then we set $s < t$. The order $<$ is strict. Natural quasi-orders are associated to the Green relations $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{J}$ as follows. For $s, t \in S$

$$s \leqslant_R t \Leftrightarrow sS^1 \subseteq tS^1$$

$$s \leqslant_L t \Leftrightarrow S^1 s \subseteq S^1 t$$

$$s \leqslant_J t \Leftrightarrow S^1 s S^1 \subseteq S^1 t S^1.$$

One easily verifies the the equivalences $\equiv_R$, $\equiv_L$, and $\equiv_J$ coincide with the relations $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{J}$, respectively. We now quasi-order S by a further relation $\leqslant_B$, which plays an important role in the sequel. More precisely, following Clifford and Preston [1], we recall that a *bi-ideal* B of S is a subsemigroup of S such that $BSB \subseteq B$. If C is any non-empty subset of S, then the bi-ideal $C \cup CS^1C$ is the bi-ideal generated by C, i.e., the smallest bi-ideal of S containing C. For any $s \in S$, we denote by $B(s)$ the bi-ideal generated by s, i.e.,

$$B(s) = sS^1 s \cup \{s\}.$$

DEFINITION 2. Let S be a semigroup. For $s, t \in S$ we set $s \leqslant_B t$ if and only if $B(s) \subseteq B(t)$.

The relation $\leqslant_B$ is a quasi-order since it is reflexive and transitive; moreover one easily verifies that $s \leqslant_B t$ if and only if $s = t$ or $s \in tS^1 t$. Let $\equiv_B$ be the equivalence relation

$$\equiv_B = \leqslant_B \cap (\leqslant_B)^{-1}.$$

One then has

$$s \equiv_B t \quad \Leftrightarrow \quad s = t \qquad \text{or} \qquad \exists u, v \in S^1 \text{ such that } s = tut \text{ and } t = svs.$$

DEFINITION 3. A semigroup S satisfies $\min_B$ if and only if any strictly descending chain with respect to $\leqslant_B$ has a finite length. S satisfies $\min_B^*$ if $s \leqslant_B t$ and $s \mathscr{J} t \Rightarrow s \equiv_B t$.

PROPOSITION 3.1. *If a semigroup S satisfies* $\min_B$, *then S satisfies* $\min_B^*$.

Proof. Let I be a $\mathscr{J}$-class of S. I contains at least one element which is minimal in I with respect to $\leqslant_B$. In fact, otherwise, there would exist an infinite strictly descending chain with respect to $\leqslant_B$ of elements of I, which is absurd. Now we prove that any element of I is minimal with respect to $\leqslant_B$. Let a be a minimal element of I and be $b \mathscr{J} a$. One has

$$b = xay, \qquad x, y \in S^1.$$

Let $c \leqslant_B b$. We prove that $c \equiv_B b$. Since $c \leqslant_B b$, if $c \neq b$ then

$$c = b\lambda b = xay\lambda xay.$$

Let $d = ay\lambda xa$. One has $d \mathscr{J} a$. In fact a is a factor of d and d is a factor of c so that

$$S^1aS^1 \supseteq S^1\,dS^1 \supseteq S^1cS^1.$$

Moreover,

$$S^1cS^1 = S^1bS^1 = S^1aS^1,$$

hence $S^1aS^1 = S^1\,dS^1$.

Since $d = ay\lambda xa$, one has $d \leqslant_B a$ and, by the minimality of a, it follows that $d \equiv_B a$. If $d = a$ then $c = x\,dy = xay = b$. Let us then suppose $d \neq a$. One has that there exists $\mu \in S^1$, such that

$$a = d\mu d = ay\lambda xa\mu ay\lambda xa.$$

By substitution

$$\begin{aligned} a &= ay\lambda x(ay\lambda xa\mu ay\lambda xa)\,\mu(ay\lambda xa\mu ay\lambda xa)\,y\lambda xa \\ &= (ay\lambda xay)\,\lambda xa\mu ay\lambda xa\mu ay\lambda xa\mu ay\lambda(xay\lambda xa) \\ &= (ay\lambda xay)\,\mu'(xay\lambda xa), \end{aligned}$$

where $\mu' = \lambda xa\mu ay\lambda xa\mu ay\lambda xa\mu ay\lambda$. Hence

$$b = xay = (xay\lambda xay)\,\mu'(xay\lambda xay) = c\mu'c.$$

Thus $b \leqslant_B c \Rightarrow b \equiv_B c$. ∎

PROPOSITION 3.2. *If a semigroup S is periodic, then S satisfies* $\min_B^*$.

Proof. Let $s \leqslant_B t$ and $s \mathscr{J} t$. If $s = t$ the result is trivially true; let us then suppose $s \neq t$. One then has $s = txt$, $s = \alpha t\beta$ and $t = \gamma s\delta$, with x, α, β, γ, $\delta \in S^1$. Let us now introduce the sequence

$$\begin{aligned} f_0 &= s \\ f_1 &= f_0\,\delta x\gamma f_0 \\ f_2 &= f_1\,\delta^2 x\gamma^2 f_1 \\ &\vdots \\ f_n &= f_{n-1}\,\delta^n x\gamma^n f_{n-1} \\ &\vdots \end{aligned}$$

It is easy to prove by induction that for any $n \geqslant 0$ one has $s = \gamma^n f_n \delta^n$. Indeed $f_0 = s$ and

$$\begin{aligned}\gamma^n f_n \delta^n &= \gamma^n f_{n-1} \delta^n x \gamma^n f_{n-1} \delta^n \\ &= \gamma\gamma^{n-1} f_{n-1} \delta^{n-1} \delta x \gamma\gamma^{n-1} f_{n-1} \delta^{n-1} \delta \\ &= \gamma s \delta x \gamma s \delta = txt = s.\end{aligned}$$

On the other hand for any $n \geqslant 1$ one has also

$$t = \gamma s \delta = \gamma\gamma^n f_n \delta^n \delta = \gamma^{n+1} f_n \delta^{n+1}.$$

Since S is periodic, there exist two integers h, k, $h < k$, such that $\gamma^h = \gamma^k$ and $\delta^h = \delta^k$. Thus

$$t = \gamma^k f_{k-1} \delta^k = \gamma^h f_{k-1} \delta^h.$$

Now by construction one has $f_{k-1} = f_h \delta^h \xi \gamma^h f_h$ for a suitable $\xi \in S^1$; so that $t = \gamma^h f_h \delta^h \xi \gamma^h f_h \delta^h$. Now $s = \gamma^h f_h \delta^h$ and then $t = s\xi s$, which implies $s \equiv_B t$. ∎

We now give the following definition (cf. [2]).

Definition 4. A semigroup S satisfies the minimal condition on principal bi-ideals if any strictly descending chain

$$s_1 S^1 s_1 \supset s_2 S^1 s_2 \supset \cdots \supset s_n S^1 s_n \cdots$$

has a finite length.

Proposition 3.3. *If a semigroup S satisfies the minimal condition on principal bi-ideals, then S satisfies* $\min_B$.

Proof. Suppose that there exists an infinite strictly descending chain w.r.t. $\leqslant_B$ of elements of S

$$s_1 >_B s_2 >_B \cdots s_n >_B s_{n+1} >_B \cdots.$$

One has $s_{n+1} = s_n x_n s_n$, $x_n \in S^1$, $n > 0$, and

$$s_{n+1} S^1 s_{n+1} = s_n x_n s_n S^1 s_n x_n s_n \subseteq s_n S^1 s_n.$$

From the minimal condition on principal bi-ideals there exists m such that for $n \geqslant m$ one has $s_n S^1 s_n = s_m S^1 s_m$. Thus

$$s_{n+1} = s_n x_n s_n = s_{n+2}\, y s_{n+2},\ y \in S^1,$$

which implies $s_{n+1} \leqslant_B s_{n+2}$. Since $s_{n+2} \leqslant_B s_{n+1}$, it follows that $s_{n+1} \equiv_B s_{n+2}$, which is a contradiction. ∎

PROPOSITION 3.4. *If a semigroup S satisfies* $\min_R$ *and* $\min_L$, *then S satisfies* $\min_B$.

Proof. Suppose, by contradiction, that there exists an infinite strictly descending chain

$$f_1 >_B f_2 >_B \cdots f_n >_B f_{n+1} >_B \cdots$$

with $f_i \in S$, $i > 0$. For any $n > 1$ one then has

$$f_{n+1} = f_n x_n f_n, \; x_n \in S^1. \tag{1}$$

This implies that $f_n S^1 \supseteq f_{n+1} S^1$ and $S^1 f_n \supseteq S^1 f_{n+1}$. Therefore, by $\min_R$ and $\min_L$ it follows that there exists an integer m such that for any $n \geqslant m$ one has $f_n \mathscr{R} f_{n+1}$ and $f_n \mathscr{L} f_{n+1}$; thus $f_n = u f_{n+1}$ and $f_n = f_{n+1} v$, with $u, v \in S^1$. One has

$$f_n S^1 f_n = f_{n+1} v S^1 u f_{n+1} \subseteq f_{n+1} S^1 f_{n+1}.$$

From (1),

$$f_n S^1 f_n \supseteq f_{n+1} S^1 f_{n+1},$$

so that for any $n \geqslant m$

$$f_n S^1 f_n = f_{n+1} S^1 f_{n+1}.$$

Hence $f_{n+1} = f_n x_n f_n = f_{n+2} \lambda f_{n+2}$, with $\lambda \in S^1$. Since $f_{n+2} = f_{n+1} x_{n+1} f_{n+1}$, one has $f_{n+1} \equiv_B f_{n+2}$. This is a contradiction. ∎

LEMMA 3.1. *Let S be a semigroup and $s, t \in S$. If $s \equiv_B t$, then $s \mathscr{H} t$.*

Proof. Let $s \leqslant_B t$. If $s = t$ the result is trivially true; let us then suppose $s \neq t$. Then there exists $x \in S^1$ such that $s = txt$. This implies $sS^1 = txtS^1 \subseteq tS^1$ and $S^1 s = S^1 txt \subseteq S^1 t$. Hence $s \equiv_B t \Rightarrow s \mathscr{R} t$ and $s \mathscr{L} t$; thus $s \equiv_B t \Rightarrow s \mathscr{H} t$. ∎

LEMMA 3.2. *If a semigroup S satisfies* $\min_J$ *and* $\min_B^*$, *then it satisfies* $\min_B$.

Proof. Indeed, suppose by contradiction that there exists in S an infinite strictly descending chain:

$$s_1 >_B s_2 >_B \cdots >_B s_n >_B \cdots.$$

Since $s_n >_B s_{n+1}$ implies $s_n \geqslant_J s_{n+1}$, $n \geqslant 1$, then by $\min_J$ there exists an integer m such that $s_n \mathscr{J} s_m$, for all $n \geqslant m$. Using $\min_B^*$ it follows that $s_n \equiv_B s_m$, for $n \geqslant m$. ∎

The following theorem gives a finiteness condition for semigroups which is a strong generalization of the Coudrain and Schützenberger theorem [2].

THEOREM 3.1. *Let T be a semigroup satisfying* $\min_B$. *Let T' be a subsemigroup of T such that the subgroups of T are locally finite in T'. Then T' is locally finite.*

Proof. Let S be a finitely generated subsemigroup of T'; we want to prove that S is finite. Since S is finitely generated, there exists a finite subset $X \subseteq S$ such that $X^+ = S$. Then consider the canonical epimorphism $\phi: A^+ \to S$, where A is a finite alphabet with the same cardinality of X such that $\phi(A) = X$ and A^+ is the free semigroup over A. Let C denote the set of canonical representatives of S. By Lemma 2.1, C is closed by factors so that by Lemma 2.2. there exists a uniformly recurrent infinite word $m \in A^\infty$ such that $F(m) \subseteq C$ and then $\phi(F(m)) \subseteq S$. The word m is irreducible, i.e., for any $w \in F(m)$ there is no word $u \in A^+$, such that $|u| < |w|$ and $\phi(u) = \phi(w)$.

We prove now that the following fact holds:

Fact 1. There exists $K > 0$ such that if $\lambda w \mu \in F(m)$ and $|w| \geqslant K$, then $\phi(w)\, \mathscr{R} \phi(w\mu)$ and $\phi(w)\, \mathscr{L} \phi(\lambda w)$.

Indeed, consider the set $\phi(F(m))$. Since T satisfies $\min_B$, there exists an element $s_0 \in \phi(F(m))$ which is minimal with respect to the relation $\leqslant_B$. Let $x \in F(m)$ be such that $\phi(x) = s_0$. We take $K = \delta(x)$, where δ is the uniform recurrence function of m.

Let $\lambda w \mu \in F(m)$ with $\lambda, \mu \in A^*$ and $|w| \geqslant K$. Since m is uniformly recurrent, there exist $z', z \in A^*$ such that $w = z'xz$. Moreover, there exists $u \in A^*$ such that

$$\lambda z'xz\mu u\lambda z'xz\mu \in F(m).$$

This implies

$$xz\mu u\lambda z'x \in F(m).$$

Since

$$\phi(xz\mu u\lambda z'x) = s_0 \phi(z\mu u\lambda z')\, s_0 \leqslant_B s_0,$$

one has, by the minimality of s_0 with respect to $\leqslant_B$, that

$$s_0 \phi(z\mu u\lambda z')\, s_0 \equiv_B s_0.$$

By Lemma 3.1 one easily derives

$$\phi(x)\, \mathscr{R} \phi(xz)\, \mathscr{R} \phi(xz\mu)$$

and

$$\phi(x) \mathscr{L} \phi(z'x) \mathscr{L} \phi(\lambda z'x).$$

Since $\mathscr{R}$ is left-invariant and $\mathscr{L}$ is right-invariant, it follows that

$$\phi(w) = \phi(z'xz) \mathscr{R} \phi(w\mu)$$

and

$$\phi(w) = \phi(z'xz) \mathscr{L} \phi(\lambda w),$$

which concludes the proof of Fact 1.

Since m is uniformly recurrent, we can factorize m as

$$m = w\lambda_0 w\lambda_1 w\lambda_2 \cdots w\lambda_n \cdots,$$

where $|w| = K$ and for all $i \geq 0$, $|\lambda_i| \leq \delta(w) = D$, where δ is the uniform recurrence function of m. We can then consider the alphabet $Y = \{w\lambda_i \mid i \geq 0\}$; Y is trivially finite; one can rewrite m on the alphabet Y as an infinite word $s \in Y^\infty$,

$$s = y_0 y_1 y_2 \cdots y_n \cdots,$$

where $y_i = w\lambda_i \in Y$ for $i \geq 0$. Moreover, we can suppose that s is uniformly recurrent as an element of Y^∞. Indeed, otherwise, by Lemma 2.2 there exists a uniformly recurrent word $t \in Y^\infty$ such that $F(t) \subseteq F(s)$, so that we can identify s with t. Then we can write

$$m = y_0 \mu_0 y_0 \mu_1 y_0 \cdots y_0 \mu_n y_0 \cdots,$$

where $y_0 = w\lambda_0 = w\lambda$ (we set $\lambda = \lambda_0$) and

$$\begin{aligned} \mu_0 &= w\lambda_1 \cdots w\lambda_{j_1} \\ \mu_1 &= w\lambda_{j_1+1} \cdots w\lambda_{j_2} \\ &\vdots \\ \mu_k &= w\lambda_{j_k+1} \cdots w\lambda_{j_{k+1}} \\ &\vdots \end{aligned}$$

In view of the uniform recurrence of s, for any $k \geq 0$ one has $j_{k+1} - j_k < M$ for a suitable M. We prove now that for any $k \geq 0$

$$\phi(w\lambda) \mathscr{H} \phi(\mu_k w\lambda) \mathscr{H} \phi(w\lambda\mu_k w\lambda). \tag{2}$$

Indeed, since $w\lambda\mu_k w\lambda \in F(m)$, by Fact 1 one has

$$\phi(w) \mathscr{R} \phi(w\lambda) \mathscr{R} \phi(w\lambda\mu_k w\lambda) \tag{3}$$

and

$$\phi(w\lambda)\,\mathscr{L}\phi(\mu_k\,w\lambda)\,\mathscr{L}\phi(w\lambda\mu_k w\lambda). \tag{4}$$

Moreover, $\mu_k \in wA^*$ so that, since $\mu_k w\lambda \in F(m)$, one has by Fact 1

$$\phi(w)\,\mathscr{R}\phi(\mu_k w\lambda). \tag{5}$$

Hence from Eqs. (3), (4), and (5) one derives Eq. (2).

Let us now set $t_k = \mu_k w\lambda$; from (2) one has, for all $k \geqslant 0$,

$$\phi(w\lambda t_k)\,\mathscr{H}\phi(w\lambda)\,\mathscr{H}\phi(t_k).$$

If H is the $\mathscr{H}$-class of $\phi(w\lambda)$, then $\phi(w\lambda t_k)$, $\phi(w\lambda)$, $\phi(t_k) \in H$, so that $H^2 \cap H \neq \varnothing$ and H is a group (by a theorem of Green). Hence m can be rewritten as

$$m = w\lambda t_0 t_1 \cdots t_k \cdots,$$

where $\phi(t_j) \in H$, $j \geqslant 0$, and $|t_j| \leqslant (M+1)(D+K)$. Thus all the factors of m of the kind $t_r \cdots t_s$ are such that $\phi(t_r \cdots t_s)$ belong to a subgroup G of H generated by a finite subset of T'. By hypothesis G is finite so that there exist two factors $t_0 \cdots t_i$ and $t_0 \cdots t_j$, $i < j$, such that $\phi(t_0 \cdots t_i) = \phi(t_0 \cdots t_j)$; this is absurd since m is irreducible. ∎

We remark that a different proof of Theorem 3.1, based on a suitable J-depth decomposition on the semigroup S, was sketched by us in [11].

If in the preceding theorem we identify T' with T, then one obtains, by Proposition 3.3, the following:

COROLLARY 3.1. *Let T be a semigroup satisfying the minimal condition on principal bi-ideals. If the subgroups of T are locally finite, then T is locally finite.*

We note that Corollary 3.1 is a remarkable generalization of the theroem of Coudrain and Schützenberger [2], since in the latter theorem one supposes that *all subgroups* of T are finite.

COROLLARY 3.2. *Let T be a semigroup satisfying* $\min_B$. *If T' is a periodic subsemigroup whose subgroups are locally finite, then T' is locally finite.*

Proof. Let G be a subgroup of T generated by a finite subset X of T'. In view of the periodicity of T' one has $G \subseteq T'$; indeed in this case the inverse of any element $x \in X$ is still an element of T'. Since the subgroups of T' are locally finite then G is finite. Hence from Theorem 3.1 the result follows. ∎

Let us remark that if we drop the hypothesis that T' is periodic, Corollary 3.2 does not, in general, hold true. For instance, take T equal to the group of integers and T' equal to the subsemigroup of positive integers.

COROLLARY 3.3. *Let T be a periodic semigroup satisfying* $\min_J$. *If T' is a subsemigroup of T whose subgroups are locally finite, then T' is locally finite.*

Proof. From Proposition 3.2, T satisfies $\min_B^*$, so that by Lemma 3.2, T satisfies $\min_B$. Since T' is periodic, by Corollary 3.2 the result follows. ∎

4. AN APPLICATION TO THE BURNSIDE PROBLEM FOR SEMIGROUPS OF MATRICES

In this section we give a new proof of the McNaughton and Zalcstein theorem [12] on the local finiteness of periodic semigroups of matrices of a finite dimension on a field. In particular, we show that semigroups of all $n \times n$ matrices over a field satisfy some interesting minimal conditions. Using these conditions one can reduce the local finiteness of a subsemigroup of matrices to that of one of its subgroups.

Throughout this section F denotes a field and $\mathcal{M}_n(F)$ the semigroup of $n \times n$ squares matrices over F. We identify, up to an isomorphism, $\mathcal{M}_n(F)$ with the semigroup $\mathrm{End}_n(V, F)$ of endomorphisms of a vectorial space V of dimension n over the field F. Let us recall that for $f \in \mathrm{End}_n(V, F)$, $\mathrm{Im}(f) = \{v \in V \mid wf = v \textit{ for some } w \in V\}$, $\mathrm{Ker}(f) = \{v \in V \mid vf = 0\}$ and $\mathrm{rank}(f) = \dim(\mathrm{Im}(f)) = \dim(V/\mathrm{Ker}(f))$. The following theorem (cf. [7]) gives a characterization of Green's relations in $\mathrm{End}_n(V, F)$.

THEOREM 4.1. *Let $f, g \in \mathrm{End}_n(V, F)$. One has*

(i) $f \mathcal{J} g$ *if and only if* $\mathrm{rank}(f) = \mathrm{rank}(g)$,

(ii) $f \mathcal{L} g$ *if and only if* $\mathrm{Im}(f) = \mathrm{Im}(g)$,

(iii) $f \mathcal{R} g$ *if and only if* $\mathrm{Ker}(f) = \mathrm{Ker}(g)$.

PROPOSITION 4.1. *The semigroup* $\mathrm{End}_n(V, F)$ *satisfies* $\min_L$ *and* $\min_R$.

Proof. We prove that $\mathrm{End}_n(V, F)$ satisfies $\min_L$. Let us denote $\mathrm{End}_n(V, F)$ by S. Suppose, by contradiction, that there exists an infinite strictly descending chain of principal left ideals

$$S^1 f_1 \supset S^1 f_2 \supset \cdots \supset S^1 f_i \supset \cdots.$$

For any $i \geq 1$ one has $f_{i+1} = xf_i$, $x \in S^1$, and so $\text{rank}(f_{i+1}) \leq \text{rank}(f_i) \leq n$. Thus there exists an integer m such that for any $i \geq m$ one has

$$\text{rank}(f_i) = \text{rank}(f_m).$$

On the other hand for any $i \geq m$ one has also

$$\text{Im}(f_m) \supseteq \text{Im}(f_i)$$

and, since $\dim(\text{Im}(f_i)) = \dim(\text{Im}(f_m))$, one obtains $\text{Im}(f_m) = \text{Im}(f_i)$. Thus by Theorem 4.1, one derives for any $i \geq m$, $f_i \mathscr{L} f_m$ and this is a contradiction.

Let us prove now that $\text{End}_n(V, F)$ satisfies $\min_R$. Suppose by contradiction that there exists an infinite strictly descending chain of principal right-ideals

$$f_1 S^1 \supset f_2 S^1 \supset \cdots \supset f_i S^1 \supset \cdots.$$

As before, for any $i \geq 1$ one has $f_{i+1} = f_i x$, $x \in S^1$, and so $\text{rank}(f_{i+1}) \leq \text{rank}(f_i)$. Thus there exists an integer m such that for any $i \geq m$ one has $\text{rank}(f_i) = \text{rank}(f_m)$. Moreover for any $i \geq m$ one has $\text{Ker}(f_m) \subseteq \text{Ker}(f_i)$ and, since $\dim(V/\text{Ker}(f_i)) = \dim(V/\text{Ker}(f_m))$, one derives $\text{Ker}(f_m) = \text{Ker}(f_i)$. Thus by Theorem 4.1, for any $i \geq m$, $f_i \mathscr{R} f_m$ and this is absurd. ∎

COROLLARY 4.1 (McNaughton and Zalcstein). *Let S be a finitely generated subsemigroup of $\mathscr{M}_n(F)$. If S is periodic, then S is finite.*

Proof. Let S be a finitely generated and periodic subsemigroup of $\mathscr{M}_n(F)$. Since $\mathscr{M}_n(F)$ is isomorphic to $\text{End}_n(V, F)$, it follows by Proposition 4.1 that $\mathscr{M}_n(F)$ satisfies $\min_R$ and $\min_L$ and then, by Proposition 3.4, also the condition $\min_B$. It is well known that any finitely generated and periodic subgroup of $\mathscr{M}_n(F)$ is finite (cf. [6]). Then all finitely generated subgroups of S are finite. From Corollary 3.2 it follows that S is finite. ∎

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