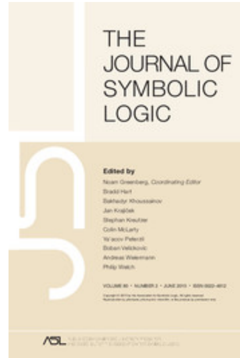

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STRONGLY MAJORIZABLE FUNCTIONALS OF FINITE TYPE: A MODEL FOR BARRECURSION CONTAINING DISCONTINUOUS FUNCTIONALS

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Abstract. In this paper a model for barrecursion is presented. It has as a novelty that it contains discontinuous functionals. The model is based on a concept called strong majorizability. This concept is a modification of Howard's majorizability notion; see [T, p. 456].

Contents.

§1. The typestructure of strongly majorizable functionals.

1.0. Types are 0 and with σ and τ also $(\sigma)\tau$. We will define an extensional typestructure over $\mathbf{N}$. Elements of a typestructure are called functionals, and we will use capital letters to denote them. Functionals of type 0 are natural numbers and functionals of type $(\sigma)\tau$ are mappings from the functionals of type σ to the functionals of type τ. The result of applying a functional F of type $(\sigma)\tau$ to a functional G of type σ will be denoted by FG. In expressions like FGH association is assumed to be to the left and the functionals F, G and H are assumed to have appropriate types. Whenever we think it convenient to stress the fact that a functional is of type 0 (i.e. a natural number), we will denote it by a lower case letter.

1.1. The typestructure $\mathfrak{M} = \bigcup M_\sigma$ of strongly majorizable functionals is defined by simultaneous inductive definition of sets M_σ and relations s-maj$_\sigma \subseteq M_\sigma \times M_\sigma$:

$$M_0 = \mathbf{N}, \qquad n \text{ s-maj}_0\, m \quad \text{iff} \quad n \geq m,$$

and F^* s-maj$_{(\sigma)\tau}$ F (in words: F^* is a majorant of F) iff

$$F^*, F \in M_\tau^{M_\sigma} \wedge \forall G^*, G \in M_\sigma$$
$$[G^* \text{ s-maj}_\sigma\, G \to (\underline{F^*G^* \text{ s-maj}_\tau\, F^*G} \wedge F^*G^* \text{ s-maj}_\tau\, FG)];$$

also,

$$M_{(\sigma)\tau} = \{F \in M_\tau^{M_\sigma} \mid \exists F^* \in M_\tau^{M_\sigma}\, F^* \text{ s-maj}_{(\sigma)\tau}\, F\}.$$

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We have s-maj$_{(\sigma)\tau} \subseteq M_{(\sigma)\tau} \times M_{(\sigma)\tau}$ by the following

LEMMA. F^* s-maj$_{(\sigma)\tau}$ $F \rightarrow F^*$ s-maj$_{(\sigma)\tau}$ $F^* \rightarrow F^* \in M_{(\sigma)\tau}$.

PROOF. Observe the underlined part of the definition of s-maj$_{(\sigma)\tau}$. □

1.2. The definition in the previous subsection has been chosen from various possibilities. Let us motivate our choice.

Firstly we modified Howard's majorizability notion (see [T, p. 456]) by adding the underlined part in the definition of s-maj$_{(\sigma)\tau}$. Consequently we obtain Lemma 1.1 and keep, so to say, all majorants within the model $\mathfrak{M}$. This is crucial, since the majorant of the barrecursor is barrecursive itself and has to be well defined.

Secondly, a concept of majorizability can be defined in any typestructure $\mathfrak{T} = \bigcup T_\sigma$, by replacing $M_\tau^{M_\sigma}$ by $T_{(\sigma)\tau}$ and M_σ by T_σ in the definitions of the previous subsection. In fact the result of the present paper has been preceded by the observation of the author that in any typestructure $\mathfrak{T}$, provided that it is a model for barrecursion, the barrecursive functionals are majorizable in $\mathfrak{T}$. Here we did not choose this possibility, simply because any known model for barrecursion contained only continuous functionals and our aim is to show that this is not always the case. Finally we could have chosen the typestructure of strongly majorizable functionals in the full typestructure $\mathfrak{T} = \bigcup T_\sigma$ defined by $T_0 = \mathbf{N}$ and $T_{(\sigma)\tau} = T_\tau^{T_\sigma}$. This yields a model for barrecursion (which the full typestructure is not), but not an extensional one. For, functionals that only differ on arguments that are not strongly majorizable are not identified. Applying to this intensional typestructure Zucker's construction $(\ )^E$ (see [T, 2.4.5]), yields an extensional typestructure which can be proved isomorphic with the one described in §1.1.

1.3. From now on we will omit the type in the denotation of the relations s-maj$_\sigma$. From now on we let F^* s-maj $F_1, \ldots, F_n$ be an abbreviation of F^* s-maj $F_1 \wedge \cdots \wedge F^*$ s-maj F_n.

LEMMA. *For all* $\sigma = (\sigma_1)\cdots(\sigma_k)\tau$ *we have* F^* s-maj F *iff*

$$\forall G_1^*, G_1, \ldots, G_k^*, G_k \left[\bigwedge_{i=1}^{k} G_i^* \text{ s-maj } G_i \rightarrow (F^* G_1^* \cdots G_k^* \text{ s-maj } F^* G_1^* \cdots G_{k-1}^* G_k, \ldots, F^* G_1 \cdots G_k, F G_1 \cdots G_k) \right].$$

PROOF. By induction on k.

$k = 0$. □

$k + 1$. This follows from the fact that $F^* G_1^* \cdots G_k^*$ s-maj $F^* G_1^* \cdots G_i^* G_{i+1} \cdots G_k$ iff

$$\forall G_{k+1}^*, G_{k+1} [G_{k+1}^* \text{ s-maj } G_{k+1} \rightarrow (F^* G_1^* \cdots G_k^* G_{k+1}^* \text{ s-maj } F^* G_1^* \cdots G_k^* G_{k+1}, F^* G_1^* \cdots G_i^* G_{i+1} \cdots G_{k+1})]. \quad \square$$

1.4. Inductively we define $\max^\sigma$ by:

$$\max_{i \leq n}{}^0(m_i) = \max(m_0, \ldots, m_n); \qquad \max_{i \leq n}{}^{(\sigma)\tau}(X_i) = \lambda Y. \max_{i \leq n}{}^\tau(X_i Y).$$

For $F \in M_\sigma^{M_0}$ we define $F^+ = \lambda n . \max_{i \leq n}^\sigma(Fi)$.

From now on we will omit the type in the denotation $\max^\sigma$.

1.5. LEMMA. *Suppose* $F, G \in M_\sigma^{M_0}$ *are such that* Fn s-maj Gn *for all* $n \in \mathbf{N}$. *Then* F^+ s-maj G^+, G.

PROOF. Let $\sigma = (\sigma_1)\cdots(\sigma_k)0$, $n \geq m$ and H_1^* s-maj $H_1, \ldots, H_k^*$ s-maj H_k. For all $0 \leq j \leq k$ let

$$p_j = F^+ n H_1^* \cdots H_j^* H_{j+1} \cdots H_k = \max_{i \leq n} (FiH_1^* \cdots H_j^* H_{j+1} \cdots H_k),$$

$$q = F^+ m H_1 \cdots H_k = \max_{i \leq m} (FiH_1 \cdots H_k), \qquad r = G^+ m H_1 \cdots H_k = \max_{i \leq m} (GiH_1 \cdots H_k),$$

and $s = GmH_1 \cdots H_k$.

By $\forall n \in \mathbf{N}$ Fn s-maj Gn and Lemma 1.3 it is easily seen that $p_k \geq p_j, q, r, s$ for all $0 \leq j < k$. Hence again by Lemma 1.3 it follows that F^+ s-maj G^+, G. $\square$

1.6. The class of primitive recursive functionals is the smallest class of functionals which is closed under application and contains 0, $S+ = \lambda n.n + 1$, $K = \lambda XY.X$, $S = \lambda XYZ.XZ(YZ)$ and

$$R = \lambda nMN. \begin{cases} M & \text{if } n = 0, \\ N(R(n-1)MN)(n-1) & \text{else} \end{cases}$$

(S, K and R of all appropriate types).

THEOREM. *$\mathfrak{M}$ contains all primitive recursive functionals.*

PROOF. Since $\mathfrak{M}$ is closed under application it suffices to show that $0, S+, K, S$ and R are strongly majorizable. We have 0 s-maj 0 and $S+$ s-maj $S+$ trivially, K s-maj K and S s-maj S by Lemma 1.3, and (see below) $\forall n \in \mathbf{N}$ Rn s-maj Rn by induction and Lemma 1.3. Hence R^+ s-maj R by Lemma 1.5. $\square$

Proof of $\forall n \in \mathbf{N}$ Rn s-maj Rn. Let M^* s-maj M, N^* s-maj N.

$n = 0$. $R0M^*N^* = M^*$ s-maj M^*, M and $M^* = R0M^*N$, $M = R0MN$. Hence $R0$ s-maj $R0$ by Lemma 1.3. $\square$

$n + 1$. Suppose Rn s-maj Rn; then we have RnM^*N^* s-maj RnM^*N, $RnMN$. So we have $R(n+1)M^*N^* = N^*(RnM^*N^*)n$ s-maj $N(RnM^*N)n$, $N(RnMN)n$. Since $R(n+1)M^*N = N(RnM^*N)n$ and $R(n+1)MN = N(RnMN)n$, it follows by Lemma 1.3 that $R(n+1)$ s-maj $R(n+1)$. $\square$

§2. Barinduction and barrecursion.

2.0. Finite sequences of functionals from M_σ are denoted by $\langle C_0, \ldots, C_{n-1} \rangle$. Let $M_\sigma^{\smile}$ be the set of all these sequences. We will use α to denote a variable over $M_\sigma^{\smile}$. All other Greek letters shall thus remain available to denote types. For all $\alpha \in M_\sigma^{\smile}$, $\alpha = \langle C_0, \ldots, C_{n-1} \rangle$, we define $\mathrm{lh}\,\alpha = n$ and $\alpha * D = \langle C_0, \ldots, C_{n-1}, D \rangle$ for all $D \in M_\sigma$. For all $F \in M_{(0)\sigma}$ and $n \in \mathbf{N}$ we define $\bar{F}n = \langle F0, \ldots, F(n-1) \rangle$ and $\alpha \prec F$ iff $\alpha = \bar{F}\,\mathrm{lh}\,\alpha$.

2.1. PROPOSITION. *Classical barinduction holds for $\mathfrak{M}$. More precisely, let $P(\cdot)$ be an unary predicate over $M_\sigma^{\smile}$. If*

$$\forall C \in M_{(0)\sigma} \exists n_0 \in \mathbf{N} \forall n \geq n_0 \, P(\bar{C}n), \tag{1}$$

and

$$\forall \alpha \in M_\sigma^{\smile} [(\forall D \in M_\sigma \, P(\alpha * D)) \rightarrow P(\alpha)], \tag{2}$$

then $\forall \alpha \in M_\sigma^{\smile} \, P(\alpha)$.

PROOF (reductio ad absurdum). Suppose (1) and (2) and not $P(\alpha)$ for some $\alpha = \langle C_0, \ldots, C_{n-1} \rangle \in M_\sigma^{\smile}$. By repeated application of (2) we obtain an infinite sequence $C_0, \dots, C_{n-1}, C_n, C_{n+1}, \dots$ in M_σ such that not $P(\langle C_0, \dots, C_{k-1}\rangle)$ for all $k \geq n$. By the axiom of choice we have $F = \lambda m.C_m \in M_\sigma^{M_0}$. We will show $F \in M_{(0)\sigma}$ and obtain a contradiction by (1). $\square$

From $\forall m \in \mathbf{N}\, C_m \in M_\sigma$ it follows that for all $m \in \mathbf{N}$ there exists $C_m^* \in M_\sigma$ such that C_m^* s-maj C_m. By Lemma 1.5 we conclude that $(\lambda m.C_m^*)^+$ s-maj F. Hence $F \in M_{(0)\sigma}$. $\square$

REMARK. In fact we proved $M_\sigma^{M_0} = M_{(0)\sigma}$ in the last lines.

2.2. For all $F \in M_{(0)\sigma}$ and $n, m \in \mathbf{N}$ we define

$$[F]_n m = \begin{cases} Fm & \text{if } m \leq n, \\ Fn & \text{else.} \end{cases}$$

If F^* s-maj F, then F^* s-maj $[F]_n$ for all n, since for $j \geq i$ we have $j \geq \min(n, i)$ and so F^*j s-maj $F \min(n, i) = [F]_n i$.

LEMMA. *For all $Y \in M_{((0)\sigma)0}$ and $F \in M_{(0)\sigma}$ there exists $n_0 \in \mathbf{N}$ such that $Y[F]_n < n$ for all $n \geq n_0$.*

PROOF. Let Y^* s-maj Y and F^* s-maj F; then we have F^* s-maj $[F]_n$, so $Y^*F^* \geq Y[F]_n$ for all n. Hence for $n_0 = 1 + Y^*F^*$ we have $Y[F]_n \leq Y^*F^* < n_0 \leq n$ for all $n \geq n_0$. $\square$

2.3. Consider the following definition scheme, called *barrecursion*:

$$BYGHFn = \begin{cases} GFn & \text{if } Y[F]_n < n, \\ H(\lambda X.BYGH[F *_{n+1} X]_{n+1}(n+1))Fn & \text{else,} \end{cases}$$

where

$$(F *_{n+1} X)m = \begin{cases} X & \text{if } m = n+1, \\ Fm & \text{else.} \end{cases}$$

Our scheme is unusual, mainly because we use $[\cdot]_\cdot$ instead of $[\cdot]_\cdot^0$, where

$$[F]_n^0 m = \begin{cases} Fm & \text{if } m < n, \\ 0^\sigma & \text{else.} \end{cases}$$

Functionals $0^\sigma \in M_\sigma$ are inductively defined by $0^0 = 0$ and $0^{(\sigma)\tau} = \lambda X.0^\tau$. In §3 we will show how to define Spector's barrecursor primitive recursively in ours. The reason for our unusual definition of B is that later on we need the following

LEMMA. *If F^* s-maj F and $n \geq m$, then $[F^*]_n$ s-maj $[F]_m$.*

PROOF. Let F^* s-maj F, $n \geq m$ and $j \geq i$; then we have $\min(n,j) \geq \min(n, i)$, $\min(m, i)$. So

$$[F^*]_n j = F^* \min(n,j) \text{ s-maj } F^* \min(n, i), F \min(m, i)$$

and so $[F^*]_n j$ s-maj $[F^*]_n i$, $[F]_m i$. Hence $[F^*]_n$ s-maj $[F]_m$ by the definition of s-maj. $\square$

This lemma does not hold for $[\cdot]_\cdot^0$.

2.4. THEOREM. *The above scheme of barrecursion yields a well defined strongly majorizable functional B. In short: $B \in \mathfrak{M}$.*

PROOF. See 2.5–2.11 below. $\square$

COROLLARY. *Since $\mathfrak{M}$ is closed under application and contains all primitive recursive functionals as well as functionals B of all appropriate types, we can summarize:*

$\mathfrak{M}$ is a model for barrecursion.

It is clear that $\mathfrak{M}$ contains discontinuous functions; we only mention

$$Y \in M_{((0)0)0} \quad \text{with } YF = \begin{cases} 1 & \text{if } \forall n \in \mathbf{N}\, Fn = 0, \\ 0 & \text{else.} \end{cases}$$

2.5. Define $H_G = \lambda ZFn.\max(GFn, H(\lambda X.Z\max(X, F(n+1)))Fn)$. Then we have

$$\begin{aligned} &H_G(\lambda X.BYGH_G[F *_{n+1} X]_{n+1}(n+1))Fn \\ &\quad = \max(GFn, H(\lambda X.BYGH_G[F *_{n+1} \max(X, F(n+1))]_{n+1}(n+1))Fn). \end{aligned}$$

We will prove Theorem 2.4 by showing that

$$B^* = \lambda YGHF.(BYGH_GF)^+ \text{ s-maj } B.$$

First we have to derive some properties of the functionals max.

2.6. LEMMA. *If* $X, Y \in M_\sigma$, X^* s-maj X *and* Y^* s-maj Y, *then*

$$\max(X^*, Y^*) \text{ s-maj } \max(X, Y), X, Y.$$

PROOF. Straightforward by induction on σ. □

ALTERNATIVE PROOF. Define $F, G \in M_\sigma^{M_0}$ by

$$Fn = \begin{cases} X^* & \text{if } n = 0, \\ Y^* & \text{else,} \end{cases} \qquad Gn = \begin{cases} X & \text{if } n = 0, \\ Y & \text{else.} \end{cases}$$

Then we have $\forall n \in \mathbf{N}\, Fn$ s-maj Gn, so F^+ s-maj G^+, G by Lemma 1.5. Hence F^+1 s-maj $G^+1, G0, G1$. □

REMARK. Conversely, we could have proved Lemma 1.5 by iterating Lemma 2.6.

2.7. LEMMA. *If* $F \in M_{(0)\sigma}$, $X \in M_\sigma$, F^* s-maj F, X^* s-maj X *and* $n \in \mathbf{N}$, *then*

$$\begin{aligned} [F^* *_{n+1} \max(X^*, F^*(n+1))]_{n+1} \text{ s-maj } &[F^* *_{n+1} \max(X, F^*(n+1))]_{n+1}, \\ &[F *_{n+1} \max(X, F(n+1))]_{n+1}, \\ &[F *_{n+1} X]_{n+1}. \end{aligned}$$

PROOF. Let X^* s-maj X and F^* s-maj F; then $F^*(n+1)$ s-maj F^*k, Fk for all $n \in \mathbf{N}$ and $k \leq n+1$. So by applying Lemma 2.6 several times we obtain

$$\max(X^*, F^*(n+1)) \text{ s-maj } \max(X, F^*(n+1)), \max(X, F(n+1)), X, F^*k, Fk.$$

Let $j \geq i$; then we have $n + 1 \geq j' = \min(n+1, j) \geq \min(n+1, i) = i'$. It follows that

$$\begin{aligned} (F^* *_{n+1} \max(X^*, F^*(n+1)))j' \text{ s-maj } &(F^* *_{n+1} \max(X^*, F^*(n+1)))i', \\ &(F^* *_{n+1} \max(X, F^*(n+1)))i', \\ &(F *_{n+1} \max(X, F(n+1)))i', \\ &(F *_{n+1} X)i'. \end{aligned}$$

Since $\forall C \in M_{(0)\sigma} \forall a \in \mathbf{N}\, [C]_{n+1}\, a = C\min(n+1, a)$, the conclusion of the lemma is obtained by applying several times the definition of s-maj. □

2.8. We continue the proof of Theorem 2.4.

Let Y^* s-maj Y, G^* s-maj G and H^* s-maj H. Abbreviate $BY^*G^*H^*_{G^*}$ by B_0, $BY^*G^*H_{G^*}$ by B_1, BY^*GH_G by B_2, $BYGH_G$ by B_3 and $BYGH$ by B_4. By the

Lemmas 1.3 and 1.5, it suffices for

$$B^* = \lambda YGHF.(BYGH_G F)^+ \text{ s-maj } B$$

to show that for all F^* s-maj F and $n \in \mathbf{N}$ we have $Q(F^*, F, n)$, where $Q(F^*, F, n)$ is the proposition

$$B_0 F^* n \text{ s-maj } B_0 Fn,\ B_1 Fn,\ B_2 Fn,\ B_3 Fn,\ B_4 Fn.$$

This follows from $\forall\alpha \in M_\sigma^{\smile} P(\alpha)$, where $P(\alpha)$ is the proposition

$$\operatorname{lh}\alpha > 0 \to \forall F^*, F[(F^* \text{ s-maj } F \wedge \alpha \prec F^*) \to Q(F^*, F, \operatorname{lh}\alpha - 1)].$$

(The occurrences of $\operatorname{lh}\alpha > 0$ and $\operatorname{lh}\alpha - 1$ in the definition of $P(x)$ might seem rather ad hoc to the reader, but they are consequences of the definition of $[\cdot]$. In this connection, note that we do not have $[C]_{n+1} = [F^*]_{n+1}$ in 2.9 below, but $[C]_n = [F^*]_n$.)

We will prove $\forall\alpha \in M_\sigma^{\smile} P(\alpha)$ by barinduction. This means that we have to show (1) and (2) from Proposition 2.1.

2.9. Ad(1) $\forall C \in M_{(0)\sigma} \exists n_0 \in \mathbf{N}\, \forall n \geq n_0\, P(\bar{C}n)$.

PROOF. Let $C \in M_{(0)\sigma}$; then by Lemma 2.2 (since $Y^* \in \mathfrak{M}$) there exists $n_0 \in \mathbf{N}$ such that $Y^*[C]_n < n$ for all $n \geq n_0$. We will prove $\forall n \geq n_0\, P(\bar{C}(n+1))$, which suffices for (1). Suppose $n \geq n_0$, F^* s-maj F and $\langle C0, \ldots, Cn\rangle \prec F^*$; then we have $[C]_n = [F^*]_n$ by the definition of $[\cdot]$., and $[F^*]_n$ s-maj $[F]_n$ by Lemma 2.3. It follows that

$$n > Y^*[C]_n = Y^*[F^*]_n \geq Y^*[F]_n, Y[F]_n.$$

Hence $B_0 F^* n = G^* F^* n$, $B_0 Fn = B_1 Fn = G^* Fn$ and $B_2 Fn = B_3 Fn = B_4 Fn = GFn$. The reader can easily verify this by inspecting the write-out in 2.10. It follows that $B_0 F^* n$ s-maj $B_0 Fn$, $B_1 Fn$, $B_2 Fn$, $B_3 Fn$, $B_4 Fn$. Hence $Q(F^*, F, n)$, so we have $P(\bar{C}(n+1))$. $\square$

2.10. Recalling the definitions and abbreviations from 2.5 and 2.8, we have

$$B_0 F^* n = \begin{cases} G^* F^* n & \text{if } Y^*[F^*]_n < n, \\ \max(G^* F^* n, H^*(\lambda X.B_0[F^* *_{n+1} \max(X, F^*(n+1))]_{n+1}(n+1))F^* n) & \text{else,} \end{cases}$$

$$B_0 Fn = \begin{cases} G^* Fn & \text{if } Y^*[F]_n < n, \\ \max(G^* Fn, H^*(\lambda X.B_0[F *_{n+1} \max(X, F(n+1))]_{n+1}(n+1))Fn) & \text{else,} \end{cases}$$

$$B_1 Fn = \begin{cases} G^* Fn & \text{if } Y^*[F]_n < n, \\ \max(G^* Fn, H(\lambda X.B_1[F *_{n+1} \max(X, F(n+1))]_{n+1}(n+1))Fn) & \text{else,} \end{cases}$$

$$B_2 Fn = \begin{cases} GFn & \text{if } Y^*[F]_n < n, \\ \max(GFn, H(\lambda X.B_2[F *_{n+1} \max(X, F(n+1))]_{n+1}(n+1))Fn) & \text{else,} \end{cases}$$

$$B_3 Fn = \begin{cases} GFn & \text{if } Y[F]_n < n, \\ \max(GFn, H(\lambda X.B_3[F *_{n+1} \max(X, F(n+1))]_{n+1}(n+1))Fn) & \text{else,} \end{cases}$$

$$B_4 Fn = \begin{cases} GFn & \text{if } Y[F]_n < n, \\ H(\lambda X.B_4[F *_{n+1} X]_{n+1}(n+1))Fn & \text{else.} \end{cases}$$

2.11. Ad(2) $\forall\alpha \in M_\sigma^{\smile}[(\forall D \in M_\sigma P(\alpha * D)) \to P(\alpha)]$.

PROOF. Let $\alpha \in M_\sigma^<$, and suppose $\forall D \in M_\sigma P(\alpha * D)$ and $\text{lh}\,\alpha > 0$. Let $n = \text{lh}\,\alpha - 1$ and F^* s-maj F with $\alpha \prec F^*$.

Claim.

$$\begin{aligned}\lambda X.B_0[F^* *_{n+1} \max(X, F^*(n+1))]_{n+1}(n+1) \text{ s-maj } \\ &\lambda X.B_0[F *_{n+1} \max(X, F(n+1))]_{n+1}(n+1),\\ &\lambda X.B_1[F *_{n+1} \max(X, F(n+1))]_{n+1}(n+1),\\ &\lambda X.B_2[F *_{n+1} \max(X, F(n+1))]_{n+1}(n+1),\\ &\lambda X.B_3[F *_{n+1} \max(X, F(n+1))]_{n+1}(n+1),\\ &\lambda X.B_4[F *_{n+1} X]_{n+1}(n+1).\end{aligned}$$

Proof of claim. Let X^* s-maj X; then by Lemma 2.7 we have

$$\begin{aligned}[F^* *_{n+1} \max(X^*, F^*(n+1))]_{n+1} \text{ s-maj } &[F^* *_{n+1} \max(X, F^*(n+1))]_{n+1},\\ &[F *_{n+1} \max(X, F(n+1))]_{n+1},\\ &[F *_{n+1} X]_{n+1}.\end{aligned}$$

Now the claim follows from the barinduction hypothesis $\forall D \in M_\sigma P(\alpha * D)$ and the definition of s-maj (applied 5 times).

From F^* s-maj F it follows by Lemma 2.3 that $[F^*]_n$ s-maj $[F]_n$, so we have $Y^*[F^*]_n \geq Y^*[F]_n, Y[F]_n$. By the claim, Lemma 2.6 and the write-out from 2.10, it follows that B_0F^*n s-maj B_0Fn, B_1Fn, B_2Fn, B_3Fn, B_4Fn. Hence $Q(F^*, F, n)$, so we have $P(\alpha)$. □

This completes the proof of Theorem 2.4.

§3. Spector's barrecursor.

3.1. Spector's scheme of barrecursion, introduced in [S], can be formulated as follows (see 2.3 for the definition of $[\cdot]^0_\cdot$):

$$B^0YGHFn = \begin{cases} G[F]^0_n n & \text{if } Y[F]^0_n < n,\\ H(\lambda X.B^0YGH[F *_n X]^0_{n+1}(n+1))[F]^0_n n & \text{else.}\end{cases}$$

We will prove that B^0 can be defined primitive recursively in B.

3.2. Let $+^\sigma$, $-^\sigma$ and Σ^σ be inductively defined in the same way as $\max^\sigma$ in 1.4. From now on we will omit the type in the denotations of $+^\sigma$, $-^\sigma$ and Σ^σ. For $F \in M_{(0)\sigma}$ we define

$$F^\oplus = \lambda n.\sum_{i<n} Fi \quad (\text{with } F^\oplus 0 = 0^\sigma) \quad \text{and} \quad F^\ominus = \lambda n.F(n+1) - F(n).$$

Then we have $F^{\oplus\ominus} = F$ and $[F^\oplus]^\ominus_n = [F]^0_n$ for all $n \in \mathbf{N}$.

3.3. Let $Y^0, G^0, H^0 \in \mathfrak{M}$ be given. We define

(a) $Y = \lambda C.Y^0C^\ominus$, then $Y[F^\oplus]_n = Y^0[F]^0_n$;

(b) $G = \lambda Cn.G^0[C^\ominus]^0_n n$, then $GF^\oplus n = G^0[F]^0_n n$;

(c) $H = \lambda ACn.H^0(\lambda D.A(D + Cn))[C^\ominus]^0_n n$.

From now on we abbreviate $BYGH$ by B and $B^0Y^0G^0H^0$ by B^0. We have

$$\begin{aligned}&H(\lambda X.B[F^\oplus *_{n+1} X]_{n+1}(n+1))F^\oplus n\\ &= H^0(\lambda D.B[F^\oplus *_{n+1} (D + F^\oplus n)]_{n+1}(n+1))[F^{\oplus\ominus}]^0_n n\\ &= H^0(\lambda D.B[F *_n D]^{0\oplus}_{n+1}(n+1))[F]^0_n n.\end{aligned}$$

The last equality follows from $F^{\oplus\ominus} = F$ and the fact that

$$[F^{\oplus} *_{n+1} (D + F^{\oplus} n)]_{n+1} i = [F *_n D]_{n+1}^{0\oplus} i$$

for all $i \in \mathbf{N}$. Since Y, G and H are primitive recursive in Y^0, G^0 and H^0, it follows that $Y, G, H \in \mathfrak{M}$.

3.4. THEOREM. *B^0 can be defined primitive recursively in B.*

PROOF. Let $P(\alpha)$ be the proposition

$$\operatorname{lh}\alpha > 0 \to \forall F \in M_{(0)\sigma} [\alpha \prec F^{\oplus} \to BF^{\oplus}(\operatorname{lh}\alpha - 1) = B^0 F(\operatorname{lh}\alpha - 1)].$$

By barinduction we will prove $\forall \alpha \in M_\sigma^\smile P(\alpha)$, which implies

$$\forall F \in M_{(0)\sigma} \forall n \in \mathbf{N}\, B^0 F n = BF^{\oplus} n.$$

From this the theorem immediately follows, so it remains to show (1) and (2) from Proposition 2.1.

Ad(1) $\forall C \in M_{(0)\sigma} \exists n_0 \in \mathbf{N}\, \forall n \geq n_0\, P(\bar{C}n)$.

PROOF. Let $C \in M_{(0)\sigma}$. By Lemma 2.3 there exists $n_0 \in \mathbf{N}$ such that $Y[C]_n < n$ for all $n \geq n_0$. We will prove $P(\bar{C}(n + 1))$ for all $n \geq n_0$, which suffices for (1). Let $n \geq n_0$ and $F \in M_{(0)\sigma}$ be such that $\langle C0, \ldots, Cn \rangle \prec F^{\oplus}$. Then we have $[C]_n = [F^{\oplus}]_n$, so $n > Y[C]_n = Y[F^{\oplus}]_n$. By 3.3(a) we have $Y[F^{\oplus}]_n = Y^0[F]_n^0$, so by 3.3(b) and the definition of B and B^0 it follows that $BF^{\oplus}n = GF^{\oplus}n = G^0[F]_n^0 n = B^0 Fn$ (see the write-out below). Hence $P(\bar{C}(n + 1))$. This proves (1).

Ad(2) $\forall \alpha \in M_\sigma^\smile [(\forall D \in M_\sigma P(\alpha * D)) \to P(\alpha)]$.

PROOF. Let $\alpha \in M_\sigma^\smile$ and suppose $\forall D \in M_\sigma P(\alpha * D)$ and $\operatorname{lh}\alpha > 0$. Let $n = \operatorname{lh}\alpha - 1$ and $F \in M_{(0)\sigma}$ with $\alpha \prec F^{\oplus}$. By the barinduction hypothesis $\forall D \in M_\sigma P(\alpha * D)$ it follows that

$$\lambda D.B[F *_n D]_{n+1}^{0\oplus}(n + 1) = \lambda D.B^0[F *_n D]_{n+1}^0(n + 1).$$

Hence by 3.3(a), (b), (c) and the definitions of B and B^0:

$$BF^{\oplus}n = \begin{cases} GF^{\oplus}n & \text{if } Y[F^{\oplus}]_n < n, \\ H(\lambda X.B[F^{\oplus} *_{n+1} X]_{n+1}(n + 1))F^{\oplus}n & \text{else,} \end{cases}$$

$$B^0 Fn = \begin{cases} G^0[F]_n^0 n & \text{if } Y^0[F]_n^0 < n, \\ H^0(\lambda X.B^0[F *_n X]_{n+1}^0(n + 1))[F]_n^0 n & \text{else,} \end{cases}$$

we conclude $BF^{\oplus}n = B^0 Fn$. Hence $P(\alpha)$. This proves (2).

This completes the proof of Theorem 3.4. □

3.5. In the last subsections we gave in detail the construction of B^0 from B. This yields the validity for B^0 of the conclusions from §2. For equivalence of both definition schemes we need, of course, also the converse, the construction of B from B^0. We refrain from giving the details of this construction. The subsections above could be considered as a typical example, but for the converse there are some technical complications. These are mainly caused by the fact that $[F]_n^0$ contains less information than F, so that the construction of B from B^0 requires (inevitably?) a B^0 of a higher type than $\mathbf{B}$.

3.6. We conclude this paper by the following

APPLICATION. The Dialectica interpretation of the extensionality axiom for functionals of type ((0)0)0 cannot be satisfied by a barrecursive functional. This strengthens Howard's result from [T, p. 454]). His argument, based on the fact that majorizable functionals are hereditarily bounded, goes through for strongly majorizable functionals.

Postscript. Recent investigations have shown the following facts:

1. The concepts of strong majorizability and majorizability, as well as the concept of compactness (see [T, 2.8.6]), coincide with respect to the full typestructure. The proof uses the comprehension axiom

$$[\forall G \in \sigma\, \exists!\, X \in \tau\, A(G, X)] \rightarrow \exists F \in (\sigma)\tau\, \forall G \in \sigma\, A(G, FG),$$

where $\exists!$ means "there exists a unique". The notion of compactness is also due to Howard, who proved that the compact functionals are a model of barrecursion of type 0. Although Howard left open the general case, he made an observation that can be seen as a predecessor of our Lemma 2.2.

2. As stated in 1.2, concepts of majorizability can be defined in any typestructure. If we do so in HEO (see [T, 2.4.11]), the typestructure of the hereditarily effective operations, we do not obtain a model of barrecursion.

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