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Envy-Free Division of Land

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Abstract. Classic cake-cutting algorithms enable people with different preferences to divide among them a heterogeneous resource ("cake") such that the resulting division is fair according to each agent's individual preferences. However, these algorithms either ignore the geometry of the resource altogether or assume it is one-dimensional. In practice, it is often required to divide multidimensional resources, such as land estates or advertisement spaces in print or electronic media. In such cases, the geometric shape of the allotted piece is of crucial importance. For example, when building houses or designing advertisements, in order to be useful, the allotments should be squares or rectangles with bounded aspect ratio. We, thus, introduce the problem of *fair land division*—fair division of a multidimensional resource wherein the allocated piece must have a prespecified geometric shape. We present constructive division algorithms that satisfy the two most prominent fairness criteria, namely *envy-freeness* and *proportionality*. In settings in which proportionality cannot be achieved because of the geometric constraints, our algorithms provide a *partially proportional* division, guaranteeing that the fraction allocated to each agent be at least a certain positive constant. We prove that, in many natural settings, the envy-freeness requirement is compatible with the best attainable partial-proportionality.

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Keywords: fairness • land division • cake cutting • envy free • two-dimensional • cutting and packing

1. Introduction

Fair division is an active field of research with various applications. A frequently mentioned application is division of land (e.g., Berliant and Raa [10], Berliant et al. [11], Legut et al. [31], Chambers [18], Dall'Aglio and Maccheroni [21], Hüsseinov [25], Nicolò et al. [37]). The basic setting considers a heterogeneous good, such as a land estate, to be divided among several agents. The agents may have different preferences over the possible pieces of the good, for example, one agent prefers the forests and the other prefers the seashore. The goal is to divide the good among the agents in a way deemed "fair." The common fairness criterion in economics is *envy-freeness*, which means that no agent prefers getting a piece allotted to another agent.

Envy-freeness on its own is trivially satisfied by the empty allocation. The task becomes more interesting when envy-freeness is combined with an *efficiency* criterion. The most common such criterion is Pareto efficiency. Indeed, Weller [54] has proved that, when the agents' preferences are represented by nonatomic measures over the good, there always exists a Pareto-efficient and envy-free allocation. However, Weller's [54] allocation gives no guarantees about the *geometric shape* of the allotted pieces. A "piece" in his allocation might even contain an infinite number of disconnected bits. So Weller's [54] positive result is valid only when agents' preferences ignore the geometry of their allotted pieces. Although such preferences make sense when dividing a pudding or an ice cream, they are less sensible when dividing land.

Many authors have noted the importance of imposing some geometric constraints on the pieces. The most common constraint is *connectivity*—the good is assumed to be the one-dimensional interval $[0,1]$, and the allotted pieces are subintervals (e.g., Stromquist [49], Su [51], Nicolò and Yu [36], Azrieli and Shmaya [3]). This is usually justified by the argument that higher dimensional settings can always be projected onto one dimension, and hence, fairness in one dimension implies fairness in higher dimensions. However, projecting back from the one dimension, the resulting two-dimensional plots are thin rectangular slivers of little use in most practical applications; it is hard to build a house on a $10 \times 1{,}000$ meter plot even though its area is a full hectare, and a thin 0.1-inch-wide advertisement space would ill serve most advertisers regardless of its height.

This paper studies the fair division of a multidimensional resource with geometric constraints on the pieces. We call this problem *fair land division* to differentiate it from the one-dimensional problem often called *fair cake cutting* (Steinhaus [48], Dubins and Spanier [23], Brams and Taylor [13], Robertson and Webb [40], Procaccia [38]). A remarkable feature of the land-division setting is that Pareto-efficient, envy-free allocations might not exist even when there are two agents; see Figure 1 for a simple example.

Thus, to get an envy-free allocation among agents with geometric preferences, we replace Pareto efficiency with a different efficiency criterion. A natural candidate is *proportionality*—every agent should receive at least $1/n$ of the total resource value. This was the first fairness criterion studied in the context of cake cutting (Steinhaus [48]), and it is still very common in the cake-cutting literature (Robertson and Webb [40], Procaccia [38]). With geometric preferences, a proportional division might not exist; see Figure 1 again for an example. Hence, we relax the proportionality requirement and consider *partial proportionality*. Partial proportionality means that each agent receives a piece worth at least a fraction p of the total value, where p is a positive constant in $[0, 1]$. Obviously, we would like p to be as large as possible. In a previous paper (Segal-Halevi et al. [47]), we showed that partial proportionality can be attained in various geometric settings. For example, in the setting of Figure 1 (square land and two agents who want square pieces), each agent can be guaranteed at least a fraction $1/4$ of the total value, and this is the largest fraction that can be guaranteed. However, these results did not consider envy. The present paper (which can be read independently of the previous one) studies the following question:

> When each agent wants a plot of land with a given geometric shape, what is the largest fraction of the total value that can be guaranteed to every agent in an envy-free allocation?

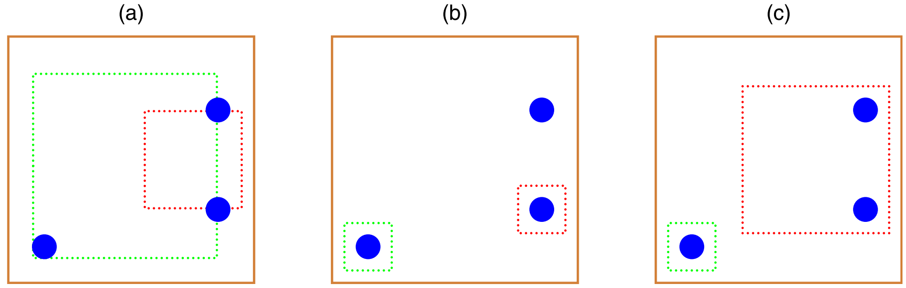
The following example shows that existing cake-cutting algorithms are insufficient for answering this question.

Example 1. You and a partner are going to divide a square land estate. It is 100 by 100 square meters, and its western side is adjacent to the sea. Your desire is to build a house near the seashore. You decide to use the classic algorithm for envy-free cake cutting: "you cut, I choose." You let your partner divide the land into two plots, knowing that you have the right to choose the plot that is more valuable according to your personal preferences. Your partner makes a cut parallel to the shoreline at a distance of only one meter from the sea.[1] Which of the two plots would you choose? The western plot contains a lot of seashore, but it is so narrow that it has no room for building anything. On the other hand, the eastern plot is large but does not contain any shore land. Whichever plot you choose, the division is not proportional for you because your utility is far less than half the utility of the original land estate.

Of course the land *could* be cut in a more sensible way (e.g., by a line perpendicular to the sea), but existing cake-cutting algorithms say nothing about how exactly to cut in each situation in order to guarantee that the division is fair with respect to the geometric preferences. Although the cut-and-choose algorithm still yields an envy-free division, it does not satisfy partial proportionality because it does not ensure any positive utility to agents who want square pieces.

This paper presents fair division algorithms that ensure both envy-freeness and partial proportionality. Our algorithms focus on agents who want *fat pieces*—pieces with a bounded length/width ratio, such as squares. The rationale is that a fat shape is more convenient to work with, build on, cultivate, and so forth.

Figure 1. (Color online) Impossibility of a Pareto-efficient envy-free land division. A square land estate has to be divided between two people. The land estate is mostly barren except for three water pools (discs). The agents have the same preferences: each agent wants a square land plot with as much water as possible. The squares must not overlap. Hence, (a) it is impossible to give both agents more than one third of the water. Hence, (b) an envy-free division must give each agent at most one third of the water, but (c) such a division cannot be Pareto-efficient because it is dominated by a division that gives one agent one third and the other two thirds of the water.



1.1. Fatness

We use the following formal definition of fatness, which is adapted from the computational geometry literature, for example, Agarwal et al. [1] and Katz [29]:

Definition 1. Let $R \geq 1$ be a real number. A d-dimensional piece is called *R-fat* if it contains a d-dimensional cube B^- and is contained in a parallel d-dimensional cube B^+ such that the ratio between the side lengths of the cubes is at most R: $\text{len}(B^+)/\text{len}(B^-) \leq R$.

A two-dimensional cube is a square. So, for example, the only two-dimensional, 1-fat shape is a square (it is also 2-fat, 3-fat, etc.). An $L \times 1$ rectangle is L-fat; a right-angled isosceles triangle is 2-fat (but not 1-fat), and a circle is $\sqrt{2}$-fat (see Figure 2).

Note that the fatness requirement is inherently multidimensional and cannot be reduced to a one-dimensional requirement. Hence, it cannot be satisfied by methods developed for a one-dimensional cake.[2]

1.2. Results

We prove that envy-freeness and partial proportionality are *compatible* in progressively more general geometric settings. Our proofs are constructive: In every geometric setting (geometric shape of the land and preferred shape of the pieces), we present an algorithm that divides the land with the following guarantees:

- Envy-freeness: Every agent weakly prefers their allotted piece over the piece given to any other agent.
- Partial proportionality: Every agent receives a piece worth for them at least a fraction p of the total land value, where p is a positive constant that depends on the geometric requirements.

In the following theorems, the partial proportionality guarantee p is given in parentheses.

The first theorem handles division between two agents.

Theorem 1. *There is an algorithm for finding an envy-free and partially proportional allocation of land between two agents in the following cases:*

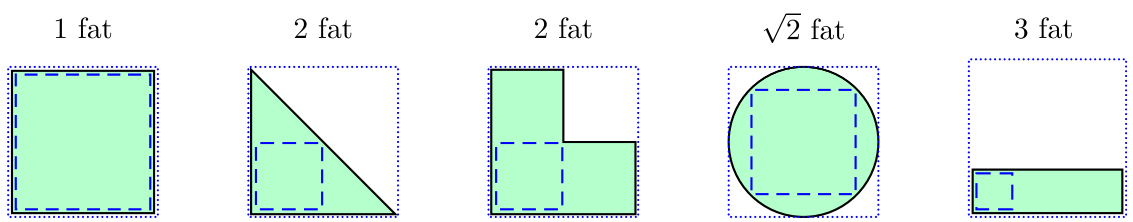
a. *The land is square and the usable pieces are squares ($p \geq 1/4$).*
b. *The land is an R-fat rectangle and the usable pieces are R-fat rectangles with $R \geq 2$ ($p \geq 1/3$).*
c. *The land is an arbitrary R-fat object and the usable pieces are 2R-fat, where $R \geq 1$ ($p \geq 1/2$).*

1.2.1. Value–Shape Trade-off. Theorem 1 illustrates a multiple-way trade-off between value and shape. Consider two agents who want to divide a square land estate with no envy. They have the following options:

- By projecting a one-dimensional division obtained by any classic cake-cutting algorithm, they can achieve a proportional allocation (a value of at least 1/2) with rectangular pieces but with no bound on the aspect ratio; the pieces might be arbitrarily thin.
- By (a), they can achieve an allocation with square pieces but only partial proportionality; the proportionality might be as low as 1/4.
- By (b), they can achieve a proportionality of 1/3 with 2-fat rectangles, which is a compromise between the previous two options.
- By (c), they can achieve an allocation that is *both* proportional *and* with 2-fat pieces, but the pieces might be nonrectangular.

The proportionality constants in Theorem 1 are tight in the following sense: It may be impossible to find an allocation with a larger proportionality even if envy is allowed. This means that envy-freeness is compatible with the largest possible proportionality; we do not have to compromise on proportionality to prevent envy.

Figure 2. (Color online) Fatness of several two-dimensional geometric shapes. The dashed square is the largest contained cube; the dotted square is the smallest containing parallel cube. The shape is R-fat if the ratio of the side lengths of these squares is at most R.



Moreover, whenever the pieces should be R-fat with $R < 2$, it might be impossible to guarantee more than 1/4 proportionality, and whenever the pieces should be R-fat with any finite R, it might be impossible to guarantee more than 1/3 proportionality. This implies that 2-fat rectangles are a good practical compromise between fatness and fairness: If we require fatter pieces ($R < 2$), then the proportionality guarantee drops from 1/3 to 1/4, and if we allow thinner pieces ($R > 2$), the proportionality remains 1/3 for all $R < \infty$.

The second theorem handles division among any number of agents.

Theorem 2. *There exists an envy-free and partially proportional allocation of land among n agents in the following cases:*
a. *The land is square and the usable pieces are squares* ($p > 1/(4n^2)$).
b. *The land is an R-fat rectangle and the pieces are R-fat rectangles, where* $R \geq 1$ ($p > 1/(4n^2)$).
c. *The land is a d-dimensional R-fat object and the pieces are* $\lceil n^{1/d} \rceil R$*-fat,*[3] *where* $d \geq 2$ *and* $R \geq 1$ ($p \geq 1/n$).

1.2.2. Value–Shape Trade-off. Parts (a) and (c) are duals in the following sense:

• Part (a) guarantees an envy-free division with perfect pieces (squares) but compromises on the proportionality level.

• Part (c) guarantees an envy-free division with perfect proportionality ($1/n$) but compromises on the fatness of the pieces.

The "magnitude" of the first compromise is $4n$ because the proportionality drops from $1/n$ to $1/(4n^2)$. We do not know if this magnitude is tight: We know that it is possible to attain a division with square pieces and a proportionality of $1/O(n)$, which is not necessarily envy-free (Segal-Halevi et al. [47]), but we do not know if a proportionality of $1/O(n)$ is compatible with envy-freeness.

The magnitude of the second compromise is $\lceil n^{1/d} \rceil$. This magnitude is asymptotically tight. We prove that, in order to guarantee a proportional division of an R-fat land with or without envy, we must allow the pieces to be $\Omega(n^{1/d})R$-fat.

1.3. Challenges and Solutions

The main challenge in land division is that utility functions depending on geometric shape are not *additive*. For example, consider an agent who wants to build a square house the utility of which is determined by its area. The utility of this agent from a 20×20 plot is 400, but if this plot is divided to two 20×10 plots, the utility from each plot is 100 and the sum of utilities is only 200. Most existing algorithms for proportional cake cutting assume that the valuations are additive, so they are not applicable in our case. Although there are some previous works on cake cutting with nonadditive utilities, they too cannot handle geometric constraints:

• Berliant et al. [11] and Maccheroni and Marinacci [33] focus on subadditive, or concave, utility functions, in which the sum of the utilities of the parts is *more* than the utility of the whole. These utility functions are inapplicable in our scenario because, as illustrated in the previous paragraph, utility functions that consider geometry are not necessarily subadditive; the sum of the utilities of the parts might be less than the utility of the whole.

• Dall'Aglio and Maccheroni [21] do not explicitly require subadditivity, but they require *preference for concentration*: If an agent is indifferent between two pieces Z_1 and Z_2, then the agent prefers 100% of Z_1 to 50% of Z_1 plus 50% of Z_2. This axiom may be incompatible with geometric constraints: The agent in this example is indifferent between the two 20×10 rectangles but prefers 50% of their union (the 20×20 square) to 100% of a single rectangle.[4]

• Sagara and Vlach [42] and Hüsseinov and Sagara [26] consider general nonadditive utility functions but provide only nonconstructive existence proofs.

• Su [51], Caragiannis et al. [17], and Mirchandani [35] provide practical division algorithms for nonadditive utilities, but they crucially assume that the cake is a one-dimensional interval and cannot handle two-dimensional constraints.

• Berliant and Dunz [9] study the division of a multidimensional good with geometric constraints. Their results are mostly negative: When general value measures are combined with geometric preferences, a competitive equilibrium might not exist. This is in contrast to the situation without geometric constraints in which a competitive equilibrium always exists (Weller [54], Segal-Halevi and Sziklai [47]).

When envy-free division protocols are applied to agents with nonadditive utility functions, the division is still envy-free, but the utility per agent might be arbitrarily small. This is true for cut and choose (as shown in Example 1), and it is also true for all other algorithms for envy-free division that we are aware of (Stromquist [49], Brams and Taylor [12], Reijnierse and Potters [39], Su [51], Barbanel and Brams [4], Manabe and Okamoto [34], Cohler et al. [20], Deng et al. [22], Chen et al. [19], Kurokawa et al. [30], Aziz and Mackenzie [2], Segal-Halevi et al. [46]).

Our way to cope with this challenge is to explicitly handle the geometric constraints in the algorithms. The main tool we use is the *geometric knife function*.

Moving-knife algorithms have been used for envy-free cake cutting since its earliest years (Dubins and Spanier [23], Stromquist [49], Brams et al. [15], Saberi and Wang [41]). For example, consider the following simple algorithm for envy-free division among two agents. A referee moves a knife slowly over the cake from left to right. Whenever an agent feels that the piece to the left of the knife is worth for the agent exactly half the total cake value, the agent shouts, "Stop!" Then, the cake is cut at the current knife location, the shouter receives the piece to its left and the nonshouter receives the piece to its right.

In this paper, we formalize the notion of a knife and add geometric constraints ensuring that the final pieces have a sufficiently high utility for agents who care about geometric shape.

1.4. Other Related Work

The prominent model in the cake-cutting literature assumes that the cake is an interval. Several authors diverge from the interval model by assuming a circular cake (e.g., Thomson [52], Brams et al. [14], Barbanel et al. [5]), but they still work in one dimension, so the pieces are one-dimensional arcs corresponding to thin wedge-like slivers.

The importance of the multidimensional geometric shape of the plots was noted by several authors.

Hill [24], Beck [6], Webb [53], and Berliant et al. [11] study the problem of dividing a disputed territory between several bordering countries with the constraint that each country should get a piece that is adjacent to its border.

Ichiishi and Idzik [27], Dall'Aglio and Maccheroni [21], and Segal-Halevi [43] require the plots to be convex shapes, such as multidimensional simplices or rectangles. However, the allocated pieces can be arbitrarily thin; their methods cannot handle requirements that are inherently two-dimensional, such as squareness.

Iyer and Huhns [28] describe an algorithm for giving each agent a rectangular plot with an aspect ratio determined by the agent. However, their algorithm is not guaranteed to succeed. If even a single rectangle of Alice intersects two rectangles of George (for example), then the algorithm fails, and no agent gets any piece.

In a related paper (Segal-Halevi et al. [47]), we considered the problem of partially proportional division when the pieces must be squares or fat rectangles. We presented an algorithm for dividing a square among n agents such that each agent receives a square piece with a value of at least $1/(4n-4)$. When all agents have the same value function, the proportionality improves to $1/(2n)$. We also proved that the upper bound in both cases is $1/(2n)$. The algorithms in the present paper use very different techniques, in order to satisfy envy-freeness in addition to partial proportionality. Their downside is that their proportionality guarantee (in the case of square pieces) is only $1/(4n^2)$. Additionally, in the present paper, we handle general fat objects rather than just squares and rectangles.

1.5. Paper Layout

The formal definitions and model are provided in Section 2. Section 3 introduces the core geometric concepts and techniques. These geometric techniques are then applied in the construction of the envy-free division algorithms for two agents (Section 4) and n agents (Section 5). Some directions for future work are presented in Section 6. The appendix contains some of the more technical proofs related to continuity of knife functions.

The preprint version of the paper (https://arxiv.org/abs/1609.03938) contains some more appendices with additional peripheral results: an alternative model in which the pieces should be convex in addition to being fat, an alternative model in which the proportionality of an allocation is measured relative to the maximum attainable utility rather than the total land value, and some negative results—upper bounds on attainable partial proportionality in various cases.

2. Model

2.1. Land and Pieces

The resource to be divided is called a *land estate* or just *land* for short. It is denoted by C. It is assumed to be a Borel subset of a Euclidean space $\mathbb{R}^d$. In most of the paper, $d = 2$. Pieces are Borel subsets of $\mathbb{R}^d$. Pieces of C are Borel subsets of C.

There is a family S of pieces that are considered usable. An S-piece is an element of S.

2.2. Agents and Utilities

There are $n \geq 1$ *agents*. Each agent $i \in \{1, \ldots, n\}$ has a value density function v_i, which is an integrable, nonnegative, and bounded function on C. It represents the quality of each land spot in the eyes of the agent. It may depend upon factors such as the fertility of soil, the probability of finding oil, the presence of trees, etc.

The *value* of a piece Z to agent i is denoted by $V_i(Z)$, and it is the integral of the value density:

$$V_i(Z) = \int_{z \in Z} v_i(z) dz.$$

We assume that, for all agents i, $V_i(\mathbb{R}^d) < \infty$. Hence, each V_i is a finite measure which is absolutely continuous with respect to the Lebesgue measure. In particular, the boundary of a piece has a value of zero to all agents.

In the standard cake-cutting model (Weller [54], Chambers [18], LiCalzi and Nicolò [32], Chen et al. [19]), the *utility function* of an agent is identical to their value measure. The present paper diverges from this model by considering agents whose utility functions depend on both value and geometric shape. We assume that an agent can derive utility only from an S-piece; when the agent's allotted land plot is not an S-piece, the agent selects the most valuable S-piece contained therein and utilizes it. For each agent i, we define the *S-value function*, which assigns to each piece $Z \subseteq \mathbb{R}^d$ the value of the most valuable usable piece contained therein:

$$V_i^S(Z) = \sup_{Y \in S, Y \subseteq Z} V_i(Y).$$

We assume that the utility of agent i is equal to the agent's S-value function V_i^S. In general, V_i^S is not a measure because it is not additive (it is not even subadditive). Hence, cake-cutting algorithms that require additivity are not applicable. Note that the two most common cake-cutting models are special cases of our model:

• The model in which each agent may receive an arbitrary Borel subset (Weller [54]) is a special case in which S is the set of all pieces.

• The model in which each agent must receive a connected piece (Stromquist [49]) is a special case in which C is an interval and S is the set of intervals.

2.3. Allocations and Fairness

An *allocation* of C is a vector of n pieces of C, $X = (X_1, \ldots, X_n)$, such that the X_i are pairwise disjoint,[5] and their union is a subset of C. We express the latter two facts succinctly using the "disjoint union" operator, $\sqcup$:

$$X_1 \sqcup \cdots \sqcup X_n \subseteq C.$$

We assume free disposal; some of C may remain unallocated. This assumption is natural in division of land: It is common to leave some land unallocated to make it available for public use.

An allocation is called envy-free if the utility of an agent from the agent's allotment is at least as large as the agent's utility from every piece allocated to another agent:

$$\forall i, j \in \{1, \ldots, n\}: \; V_i^S(X_i) \geq V_i^S(X_j).$$

We call an allocation *p-proportional* for some $p \in [0, 1]$ if the utility of each agent from the agent's allotment is at least a fraction p of the agent's total land value:

$$\forall i \in \{1, \ldots, n\}: \; V_i^S(X_i) \geq p \cdot V_i(C).$$

We call an allocation *p-relative proportional* if the utility of each agent from the agent's allotment is at least a fraction p of the agent's largest attainable *utility*:

$$\forall i \in \{1, \ldots, n\}: \; V_i^S(X_i) \geq p \cdot V_i^S(C).$$

Because $V_i(C) \leq V_i^S(C)$, every p-proportional allocation is also p-relative proportional. The present paper focuses on p-proportionality, so all positive results are valid for p-relative proportionality as well. Moreover, whenever C itself is an element of S, $V_i(C) = V_i^S(C)$, so p-proportionality and p-relative proportionality are equivalent. This is the case in all settings mentioned in our main Theorems 1 and 2; therefore, they are valid and tight by both criteria.

However, when $C \notin S$, it may be possible to attain relative proportionality that is higher than the maximum attainable absolute proportionality. Exploiting this possibility requires new techniques. Some results in this direction are presented in the preprint version (https://arxiv.org/abs/1609.03938).

2.4. Fairness Guarantees

Given the geometric shape of C and the family S, we would like to know what proportionality can be ensured for any combination of agents with and without the additional requirement of envy-freeness. Formally,

Definition 2. Let C be a land estate, S a family of usable shapes, and $n \geq 1$ an integer.

a. The *proportionality guarantee* for C, S, and n, denoted $\textsc{Prop}(C, S, n)$, is the largest fraction $p \in [0, 1]$ such that, for every n value measures $(V_1, \ldots, V_n)$, a p-proportional allocation exists.

b. The *envy-free proportionality guarantee* of C, S, and n, denoted $\textsc{PropEF}(C, S, n)$, is the largest fraction $p \in [0, 1]$ such that, for every n value measures $(V_1, \ldots, V_n)$, an envy-free and p-proportional allocation exists.

For example, classic cake cutting results can be presented as

$$\forall C : \textsc{Prop}(C, \mathit{AllPieces}, n) = \textsc{PropEF}(C, \mathit{AllPieces}, n) = 1/n$$
$$\textsc{Prop}(\mathit{Interval}, \mathit{Intervals}, n) = \textsc{PropEF}(\mathit{Interval}, \mathit{Intervals}, n) = 1/n,$$

and our results for two agents can be stated as

$$\textsc{Prop}(\mathit{Square}, \mathit{Squares}, 2) = \textsc{PropEF}(\mathit{Square}, \mathit{Squares}, 2) = 1/4$$

$$\begin{aligned}\forall R \geq 2 :\ & \textsc{Prop}(R \textit{ fat rectangle}, R \textit{ fat rectangles}, 2)\\ &= \textsc{PropEF}(R \textit{ fat rectangle}, R \textit{ fat rectangles}, 2) = 1/3\end{aligned}$$

$$\begin{aligned}\forall R \geq 1 :\ & \textsc{Prop}(R \textit{ fat object}, R \textit{ fat objects}, 2)\\ &= \textsc{PropEF}(R \textit{ fat object}, 2R \textit{ fat objects}, 2) = 1/2.\end{aligned}$$

2.5. Strategy Considerations

In the present paper, we ignore strategic considerations and assume that all agents act according to their true value functions. In fact, even without geometric constraints, it is impossible to build a general division protocol that is both fair and strategy-proof (Brânzei and Miltersen [16]). Strategy-proof algorithms exist only in very special cases, for example, when all agents have piecewise uniform valuations (Chen et al. [19]; Bei et al. [7], [8]).

However, the guarantees of our algorithms are valid for any *single* agent who acts according to the agent's own value function. For example, the algorithm of Theorem 2(c) ensures that every agent acting according to the agent's true value function receives a piece with a utility of at least $1/n$ and at least as good as the other pieces regardless of what the other agents do.

3. Geometric Preliminaries

3.1. Covers and Choosers

The key feature of our geometric setting is that the utility of each agent i is given by V_i^S rather than V_i. Therefore, we would like to bound the ratio between V_i^S and V_i. The key concept we use is a *cover*.

Definition 3.

a. A cover of a piece $Z \subseteq \mathbb{R}^d$ is a set of pieces of Z whose union equals Z:

$$Z_1 \cup \cdots \cup Z_m = Z.$$

b. An *S-cover* of Z is a cover in which all pieces are elements of S.

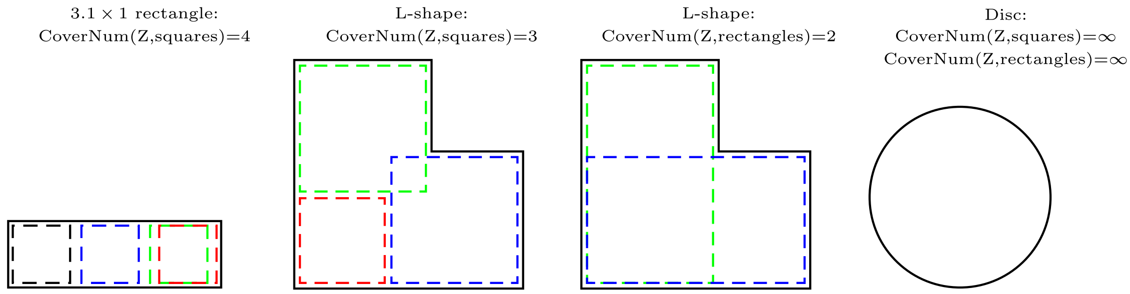
c. The *S-cover number* of Z, $\textsc{CoverNum}(Z, S)$, is the smallest size of an S-cover of Z.

Some examples of cover numbers are shown in Figure 3.

Consider a piece Z with an S-cover of size m. For any value-measure V, the sum of values of the m S-pieces is at least $V(Z)$. Therefore, the most valuable S-piece in the cover has a value of at least $V(Z)/m$:

$$\max_{i \in \{1,\ldots,m\}} V(Z_i) \geq V(Z)/m. \tag{1}$$

Figure 3. (Color online) Cover numbers of several geometric shapes. In the first three figures, the dashed squares/rectangles denote minimal covers. In the fourth figure, there is no finite cover, so the cover number is defined as ∞.



Using this simple fact, we now prove that, in any cover of Z (not necessarily an S-cover), any agent can choose a piece with a utility of at least $1/M$ the total value of Z, where M is the sum of cover numbers of the covering pieces:

Lemma 1 (Chooser Lemma)**.** *Let Z be a piece covered by m pieces $Z_1 \cup \cdots \cup Z_m = Z$, and let $M = \sum_{i=1}^{m} \textsc{CoverNum}(Z_i, C)$. Then, for every value measure V,*

$$\max_{i \in \{1,\ldots,m\}} V^S(Z_i) \geq V(Z)/M.$$

Proof. For every $i \in \{1, \ldots, m\}$, denote $m_i := \textsc{CoverNum}(Z_i, S)$. Suppose that we replace each Z_i in the cover of Z by m_i S-pieces in a minimal S-cover of Z_i, for example, $Z_{i,1}, \ldots, Z_{i,m_i}$. We now have an S-cover of Z with M S-pieces. By (1), at least one of these M pieces, say $Z_{i,j}$, has a value of at least $V(Z)/M$. By definition of V^S, $V^S(Z_i) \geq V(Z_{i,j}) \geq V(Z)/M$.

As a simple corollary of the chooser lemma, we get the following:

Lemma 2 (Allocation Lemma)**.** *Given a land C and some integer $m \geq n$, let $(C_1, \ldots, C_m)$ be a cover of C with a total cover number of M, that is, $M = \sum_{i=1}^{m} \textsc{CoverNum}(Z_i, C)$. Suppose each agent $i \in \{1, \ldots, n\}$ chooses a piece C_i that gives them a highest utility among the m pieces, and the n choices are pairwise disjoint. Then, the resulting allocation is envy-free and $(1/M)$-proportional.*

3.2. Knife Functions

Moving knives have been used in fair division procedures ever since the seminal paper of Dubins and Spanier [23]. We generalize the concept of a moving knife to handle geometric shape constraints.

Definition 4. A *knife function* on a land C is a function K_C from the real interval $[0, 1]$ to Borel subsets of C with the following continuity property: For every $\epsilon > 0$, there is a $\delta > 0$ such that $|t' - t| < \delta$ implies $\textsc{Volume}[K_C(t') \ominus K_C(t)] < \epsilon$.[6]

If $K_C(0) = C_{\text{start}}$ and $K_C(1) = C_{\text{end}}$, we say that K_C is a knife function from C_{start} to C_{end}.

Some examples of knife functions are shown in Figure 4. In Appendix A.1, we show a general construction of an increasing knife function from C_{start} to C_{end}—the *growing-ball function*.

The *complement* of K_C is denoted $\overline{K_C}$ and defined by $\overline{K_C}(t) = C \setminus K_C(t)\ \forall t \in [0, 1]$. If K_C is a knife function, then $\overline{K_C}$ is a knife function too. If K_C is increasing, then $\overline{K_C}(t)$ is decreasing and vice versa.

3.3. Continuity of Utility Covered by Knife Functions

The value covered by a knife function always changes continuously with time. Formally, we prove in Appendix A.2 the following:

Lemma 3. *If K_C is a knife function and V is an absolutely continuous measure, then $V \circ K_C$ is a uniformly continuous real function.*

However, in our setting, the agents care not about V but about V^S. In general, the function $V^S \circ K_C$ might be discontinuous. For example, let K_C be the knife function in Figure 4(f), V the area measure, and S the family of rectangles. Then, the function $V^S \circ K_C$ (the largest area of a rectangle covered by the knife) is discontinuous; it is less than $1/2$ when $t < 1/2$ and jumps to one when $t \geq 1/2$. To handle this issue, we define two different properties of knife functions.

3.3.1. *S*-Continuity. S-continuity means (informally) that all S-pieces in $K_C(t)$ and $\overline{K_C}(t)$ grow or shrink continuously; no S-piece with a positive area is created or destroyed abruptly. See Appendix A.3 for a formal definition. In Figure 4, all knives except for knife f are square continuous and rectangle continuous.[7] In Appendix A.3, we prove the following:

Lemma 4. *If K_C is an S-continuous knife function and V is an absolutely continuous measure, then both $V^S \circ K_C$ and $V^S \circ \overline{K_C}$ are uniformly continuous real functions.*

In Appendix A.4, we show a general construction of an increasing S-continuous knife function on C—the *sweeping-plane function* (the knives in Figure 4(a, c, and e) are special cases of this function).

3.3.2. *S*-Smoothness. S-smoothness means (informally) that $K_C(t)$ is a finite union of S-pieces that grow continuously and $\overline{K_C}(t)$ is a finite union of S-pieces that shrink continuously. To define it formally, we need some preliminary definitions:

- The *union* of two knife functions, K_1 and K_2, is a knife function $K_1 \cup K_2$ defined by $(K_1 \cup K_2)(t) = K_1(t) \cup K_2(t)\ \forall t \in [0,1]$.
- A knife function K is an S-knife function if it is into S, that is, $K(t) \in S\ \forall t \in [0,1]$.
- An S-cover of a knife function K_C is a set of S-knife functions whose union equals K_C.

Definition 5. A knife function K_C is called *S-smooth* if
- K_C has a finite S-cover $K^1 \cup \cdots \cup K^m$ for some integer $m \geq 1$.
- $\overline{K_C}$ has a finite S-cover $\overline{K}^1 \cup \cdots \cup \overline{K}^{m'}$ for some integer $m' \geq 1$.
- For every $j \in \{1,\ldots,m\}$, $K^j(1) = K_C(1)$. That is, the knives covering K_C coincide at $t = 1$.[8]

If K_C is S-smooth, then we define the *cover number* of K_C, $\text{KCoverNum}(K_C, S)$, as the smallest sum $m + m'$ of integers that satisfy the preceding definition.

Lemma 4 is not necessarily true for S-smooth knife functions; $V^S \circ K_C$ and $V^S \circ \overline{K_C}$ are not necessarily continuous. Therefore, we define the following alternative functions (where V is any value measure and the K^j are the S-knife functions covering K_C):

$$V^{K_C}(t) := \max_{j=1}^{m} V\big(K^j(t)\big) \qquad V^{\overline{K_C}}(t) := \max_{j=1}^{m'} V\big(\overline{K}^j(t)\big). \tag{2}$$

Note that, for all t, $V^{K_C}(t) \leq V^S(K_C(t))$ because V^{K_C} considers a most valuable S-piece from a finite set of S-pieces contained in $K_C(t)$, and V^S considers a supremum over *all* S-pieces contained in $K_C(t)$. Similarly, $V^{\overline{K_C}}(t) \leq V^S(\overline{K_C}(t))$.

Lemma 5. *If K_C is an S-smooth knife function and V is an absolutely continuous measure, then both V^{K_C} and $V^{\overline{K_C}}$ are uniformly continuous real functions.*

Proof. Each of these functions is a maximum over functions that are uniformly continuous (by Lemma 3). The maximum of uniformly continuous functions is uniformly continuous. □

3.3.2.1. Examples (See Figure 4).

a. Both $K_C(t)$ and its complement are rectangles, and their areas are continuous functions of t. Therefore, K_C is rectangle smooth with $m = m' = 1$, and its rectangle cover number is two. In contrast, $K_C(t)$ is not a union of a fixed finite number of squares. Therefore, K_C is not square smooth.

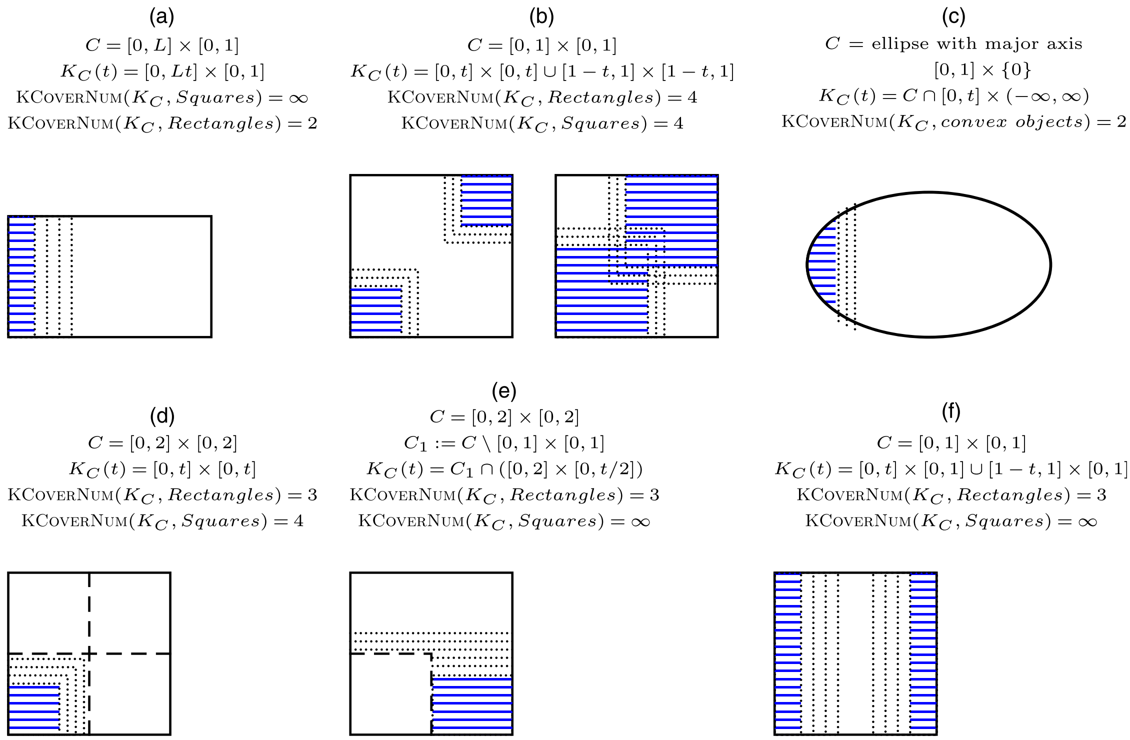
b. $K_C(t)$ can be covered by two square knife functions: $K^1(t) = [0,t] \times [0,t]$ and $K^2(t) = [1-t,1] \times [1-t,1]$. Its complement too can be covered by two square knife functions: $\overline{K}^1(t) = [0,1-t] \times [t,1]$ and $\overline{K}^2(t) = [t,1] \times [0,1-t]$. Therefore, K_C is square smooth with $\text{KCoverNum}(K_C, \mathit{Squares}) = 2 + 2 = 4$. K_C is also rectangle smooth with the same cover number.

c. C is an arbitrary convex object, and S is the family of convex objects. Both $K_C(t)$ and its complement are convex, so K_C is S-smooth with a cover number of two.

d. This is a knife function from $\emptyset$ to $[0,1] \times [0,1]$ = the bottom-left quarter of C. Here, $K_C(t)$ is a square that grows continuously. Its complement $\overline{K_C}(t)$ is an L-shape, similar to the L-shapes in Figure 3, which can be covered by a union of three squares that shrink continuously. Hence, K_C is square smooth with $m = 1$ and $m' = 3$, and its square cover number is four.

e. This is a knife function from $\emptyset$ to $C \setminus [0,1] \times [0,1]$; it sweeps over this L-shape continuously from bottom to top. It is square continuous (see proof in Appendix A.4) but not square smooth.

Figure 4. (Color online) Several knife functions. The area filled with horizontal lines marks $K_C(t)$ in a certain intermediate time $t \in (0,1)$. Dotted lines mark future knife locations.



f. $K_C(t)$ is a union of two rectangles that grow continuously, and its complement is a rectangle that shrinks continuously. It is rectangle smooth with a cover number of three but not rectangle continuous (see proof in Appendix A.5).

Examples (e) and (f) show that S-continuity and S-smoothness are independent properties that do not imply each other.

4. Envy-Free Division for Two Agents

In this section, we use the tools developed in Section 3 to build complete land division algorithms.

4.1. Dividing Squares

Our first algorithm uses a single S-smooth knife function and generalizes the classic cut-and-choose protocol. It is presented in Algorithm 1. The following lemma proves that it is correct.

Algorithm 1 (Smooth Knife Algorithm)

INPUT:

a. C_{end}—an S-piece contained in C such that, for each agent $i \in \{1, 2\}$,

$$V_i^S(C_{\text{end}}) \geq V_i^S(C \setminus C_{\text{end}}).$$

b. K_C—an S-smooth knife function from $\emptyset$ to C_{end} such that, for some $M \geq 2$,

$$\text{KCoverNum}(K_C, S) = M$$

and the corresponding covers by S-knife functions as in Definition 5, where $m + m' = M$:

$$K^1 \cup \cdots \cup K^m = K_C \qquad \overline{K}^1 \cup \cdots \cup \overline{K}^{m'} = \overline{K_C}.$$

OUTPUT:

An envy-free and $(1/M)$-proportional allocation of C between the two agents.

ALGORITHM:

1. Agent 1 selects a time $t^* \in [0, 1]$ such that (where $V_1^{K_C}$ and $V_1^{\overline{K_C}}$ are as defined in (2)):

$$V_1^{K_C}(t^*) = V_1^{\overline{K_C}}(t^*).$$

2. Agent 2 selects one of the following two options:
2.1. Agent 2 picks an S-piece $K^j(t^*)$ that maximizes V_2 for $j \in \{1,\ldots,m\}$, and agent 1 picks an S-piece $\overline{K}^{j'}(t^*)$ that maximizes V_1 for $j' \in \{1,\ldots,m'\}$.
2.2. Agent 2 picks an S-piece $\overline{K}^{j'}(t^*)$ that maximizes V_2 for $j' \in \{1,\ldots,m'\}$, and agent 1 picks an S-piece $K^j(t^*)$ that maximizes V_1 for $j \in \{1,\ldots,m\}$.

Lemma 6 (Correctness of Smooth Knife Algorithm). *Let C_{end} be a piece of C such that, for each agent $i \in \{1,2\}$, $V_i^S(C_{end}) \geq V_i^S(C \setminus C_{end})$. Let K_C be an S-smooth knife function from $\emptyset$ to C_{end} such that* KCoverNum$(K_C, S) = M$. *Then, Algorithm 1 produces an envy-free and $(1/M)$-proportional allocation between the two agents.*

Proof. By Lemma 5, for all $i \in \{1,2\}$, both $V_i^{K_C}(t)$ and $V_i^{\overline{K_C}}(t)$ are continuous functions of t. At $t=0$, $V_i^{K_C}(t) \leq V_i^{\overline{K_C}}(t)$,[9] and at $t=1$, $V_i^{K_C}(t) \geq V_i^{\overline{K_C}}(t)$,[10] so by the intermediate value theorem, for some $t^* \in [0,1]$, the two functions are equal. Therefore, agent 1 can indeed find a $t^* \in [0,1]$ such that $V_1^{K_C}(t^*) = V_1^{\overline{K_C}}(t^*)$ as required in step 1.

Now, in step 2, each agent i receives an S-piece that maximizes V_i from a covering of size $m + m' = M$. Hence, by the Allocation Lemma, the allocation is envy-free and $1/M$-proportional. □

Based on this lemma, we can now prove our first subtheorem:

Theorem 1(a). PropEF(*Square*, *Squares*, 2) $\geq 1/4$.

Proof. Given a square land C, apply the single knife algorithm with $C_{\text{end}} = C$ and K_C the knife function shown in Figure 4(b). As explained in Section 3.2, this K_C is square smooth and KCoverNum$(K_C, \textit{Squares}) = 4$. Because $C_{\text{end}} = C$, $C \setminus C_{\text{end}} = \emptyset$, so obviously, $\forall i \in \{1,2\} : V_i^S(C_{\text{end}}) \geq V_i^S(C \setminus C_{\text{end}})$. Therefore, by Lemma 6, the resulting division is envy-free and $(1/4)$-proportional. □

This lower bound is tight; it is not possible to guarantee both agents a larger utility even if envy is allowed. See Segal-Halevi et al. [47].

4.2. Dividing Cubes and Archipelagos

In many cases, it is difficult to find a single S-smooth knife function that covers the entire land. To handle such cases, we first present a subroutine (Algorithm 2) and then a division algorithm that uses this subroutine (Algorithm 3).

We prove the correctness of the subroutine and then the correctness of the full algorithm.

Algorithm 2 (Continuous Knife Subroutine)
INPUT:
a. C_{end}—a piece of C such that, for each agent $i \in \{1,2\}$,

$$V_i^S(C_{\text{end}}) \geq V_i^S(C \setminus C_{\text{end}}).$$

b. K_C—an increasing S-continuous knife function from $\emptyset$ to C_{end}.
OUTPUT:
An envy-free allocation of C in which every agent i gets a utility of at least $V_i^S(C \setminus C_{\text{end}})$.
ALGORITHM:
1. Agent 1 selects a time $t^* \in [0,1]$ such that

$$V_1^S(K_C(t^*)) = V_1^S\big(\overline{K_C}(t^*)\big).$$

2. Agent 2 picks either $K_C(t^*)$ or $\overline{K_C}(t^*)$; agent 1 receives the remaining piece.

Lemma 7 (Correctness of Continuous Knife Subroutine). *Let C_{end} be a piece of C such that, for each agent $i \in \{1,2\}$, $V_i^S(C_{end}) \geq V_i^S(C \setminus C_{end})$. Let K_C be an increasing S-continuous knife function from $\emptyset$ to C_{end}. Then, Algorithm 2 produces an envy-free allocation of C between the two agents such that the utility of agent i is at least $V_i^S(C \setminus C_{end})$.*

Proof. By Lemma 4, for all $i \in \{1,2\}$, both $V_i^S K_C(t))$ and $V_i^S(\overline{K_C}(t))$ are continuous functions.

At $t = 0$, the first function is weakly smaller because $V_i^S(K_C(0)) = V_i^S(\emptyset) = 0$.

At $t = 1$, the first function is weakly larger because $V_i^S(K_C(1)) = V_i^S(C_{\text{end}}) \geq V_i^S(C \setminus C_{\text{end}}) = V_i^S(\overline{K_C}(1))$ by assumption.

Therefore, by the intermediate value theorem, agent 1 can indeed pick a $t^* \in [0,1]$ such that $V_1^S(K_C(t^*)) = V_1^S(\overline{K_C}(t^*))$ as required in step 1.

In step 2, agent 1 is indifferent between the two pieces, and agent 2 picks a best piece, so the allocation is envy-free.

Moreover, because K_C is increasing, $\overline{K_C}$ is decreasing, so $\overline{K_C}(t^*)$ contains $\overline{K_C}(1) = C \setminus C_{\text{end}}$. Each agent receives either $\overline{K_C}(t^*)$ or a piece with a weakly larger utility. Therefore, the utility of each agent i is at least $V_1^S(C \setminus C_{\text{end}})$. □

Algorithm 3 (Single Partition Algorithm)
INPUT:
a. A partition of C into m pieces with a total cover number of at most M (for some integers $M \geq m \geq 2$):

$$\bigsqcup_{j=1}^{m} C_j = C \qquad\qquad \sum_{j=1}^{m} \textsc{CoverNum}(C_j, S) \leq M.$$

b. For every $j \in \{1, \ldots, m\}$, an S-smooth knife function K_{C_j} from $\emptyset$ to C_j such that

$$\forall j \in \{1, \ldots, m\} : \textsc{KCoverNum}\Big(K_{C_j}, S\Big) \leq M.$$

c. For every $j \in \{1, \ldots, m\}$, an increasing S-continuous knife function $K^{\overline{C_j}}$ from $\emptyset$ to $\overline{C_j} := C \setminus C_j$.
OUTPUT:
An envy-free and $(1/M)$-proportional allocation of C between the two agents.
ALGORITHM:
1. From the input partition (a), each agent chooses the part C_j that gives them maximum utility. If the choices are different, then each agent receives their choice, and we are done.
2. If both agents chose the same part C_j, then ask each agent to choose either C_j or its complement $\overline{C_j}$. If the choices are different, then each agent receives their choice, and we are done.

If the choices are identical, then there are two cases:
3.1. Both agents chose C_j. Apply Algorithm 1 with $C_{\text{end}} = C_j$ and the S-smooth knife function K_{C_j} of input (b).
3.2. Both agents chose $\overline{C_j}$. Apply Algorithm 2 with $C_{\text{end}} = \overline{C_j}$ and the S-continuous knife function $K^{\overline{C_j}}$ of input (c).

Lemma 8 (Correctness of Single Partition Algorithm). *If there exist a partition and knife functions satisfying the input requirements of Algorithm 3, then this algorithm produces an envy-free and $(1/M)$-proportional allocation of C between the two agents.*

Proof. The algorithm may end in step 1, 2, 3.1, or 3.2. We prove that, in each of these cases, the resulting allocation is envy-free and $(1/M)$-proportional.

In steps 1 and 2, if the choices are different, then by the Allocation Lemma (Lemma 2) and the condition on input (a), each agent i receives an envy-free share with a utility of at least $V_i(C)/M$.

In step 3.1, we know that both agents prefer C_j over its complement $\overline{C_j}$. Therefore, $C_{\text{end}} = C_j$ satisfies requirement (a) of Algorithm 1. The knife function K_{C_j} is S-smooth with a cover number of at most M, so it satisfies requirement (b). Hence, by Lemma 6, Algorithm 1 gives to each agent i an envy-free share with a utility of at least $V_i(C)/M$.

In step 3.2, we know that both agents prefer $\overline{C_j}$ over its complement C_j. Therefore, $C_{\text{end}} = \overline{C_j}$ satisfies requirement (a) of Algorithm 2. The knife function $K^{\overline{C_j}}$ is increasing and S-continuous, so it satisfies requirement (b). Hence, by Lemma 7, Algorithm 2 gives to each agent i an envy-free share with a utility of at least $V_i^S(C \setminus \overline{C_j}) = V_i^S(C_j)$. Now, in step 1, both agents chose C_j from a partition with a total cover number of at most M. Therefore, by the chooser lemma (Lemma 1), $V_i^S(C_j) \geq V_i(C)/M$. □

Several applications of Algorithm 3 are presented below.

a. Multidimensional cubes: $\textsc{PropEF}(d\ dimensional\ cube, d\ dimensional\ cubes, 2) \geq 1/2^d$.

Proof. C can be partitioned into 2^d subcubes of equal side length. The total cube cover number of this partition is 2^d, satisfying input condition (a). For each subcube C_j, there is an S-smooth knife function analogous to Figure 4(d)—a cube growing from the corner toward the center of C. The complement of the cube can always be covered by a union of $2^d - 1$ cubes (possibly overlapping). Therefore, this knife function is S-smooth with a cover number of 2^d, satisfying input condition (b). For each complement $\overline{C_j}$, the sweeping-plane knife function on $\overline{C_j}$ (see appendix A.4 and Figure 4(e)) is increasing and S-continuous, satisfying condition (c). □

b. Rectangle archipelagos: Let C be an archipelago that is a union of m disjoint rectangular islands. Then, $\textsc{PropEF}(C, \textit{Rectangles}, 2) \geq \frac{1}{m+1}$.

Proof. The total rectangle cover number of the partition of C into m rectangles is obviously $m < m+1$, satisfying condition (a). For each part C_j, define a knife function K_{C_j} based on a line sweeping from one side of the rectangle to the other side, similar to Figure 4(a). $K_{C_j}(t)$ is always a rectangle. Its complement is a union of m rectangles: the shrinking rectangle $C_j \setminus K_{C_j}(t)$ and the remaining $m-1$ fixed rectangular islands. Hence, K_{C_j} is rectangle smooth, and its cover number is $1+1+m-1 = m+1$, satisfying condition (b) (see Figure 5). For the complements, a sweeping-line knife function (as in Appendix A.4) satisfies (c).

It is easy to prove that these bounds are tight. See the preprint version for details.

4.3. Dividing Fat Rectangles

To prove Theorem 1(b), we add a partition step as shown in Algorithm 4. Note that steps 1, 2, and 3.2 of that algorithm are the same as in Algorithm 3; they are repeated for completeness. Step 3.1 is refined.

Algorithm 4 (Multiple Partition Algorithm)
INPUT:
a. A partition of C into m pieces with a total cover number of at most M

$$\bigsqcup_{j=1}^{m} C_j = C \qquad \sum_{j=1}^{m} \text{CoverNum}(C_j, S) \le M.$$

b. For every part C_j, a partition such that, if C_j is replaced with its partition, then the total cover number of the resulting partition of C is at most M; that is, for every j, there exist $C_j^1, \ldots, C_j^{m_j}$ with

$$\bigsqcup_{k=1}^{m_j} C_j^k = C_j \qquad \sum_{j' \neq j} \text{CoverNum}(C_{j'}, S) + \sum_{k=1}^{m_j} \text{CoverNum}\left(C_j^k, S\right) \le M.$$

c. For every $j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, m_j\}$, an S-smooth knife function $K_{C_j^k}$ from $\emptyset$ to C_j^k with a cover number of at most M

$$\forall j: \text{ KCoverNum}\left(K_{C_j^k}, S\right) \le M.$$

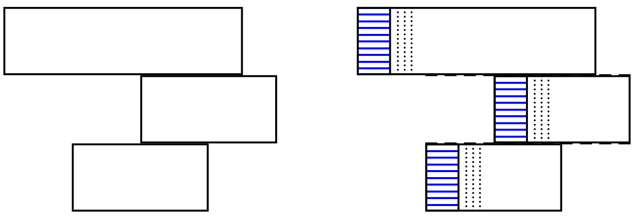
d. For every $j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, m_j\}$, an increasing S-continuous knife function from $\emptyset$ to C_j and to $\overline{C_j}$ and to $\overline{C_j^k}$.
OUTPUT:
An envy-free and $(1/M)$-proportional allocation of C between the two agents.
ALGORITHM:
1. From the input partition (a), each agent chooses the part C_j that gives the agent maximum utility. If the choices are different, then each agent receives their choice, and we are done.
2. If both agents chose the same part C_j, then ask each agent to choose either C_j or its complement $\overline{C_j}$. If the choices are different, then each agent receives their choice, and we are done.

If the choices are identical, then there are two cases, denoted by (3.1) and (3.2):
3.1. Both agents chose C_j. Refine the partition of C by replacing C_j with its subpartition

$$\left(\bigsqcup_{j' \neq j} C_{j'}\right) \sqcup \left(\bigsqcup_{k=1}^{m_j} C_j^k\right) = C.$$

Let each agent choose a best part from this refined partition.
- (3.11) If the choices are different, then each agent receives their choice, and we are done.

Figure 5. (Color online) Knife functions for archipelagos of rectangles. Left: A land estate made of a union of three disjoint rectangles. Right: Three knife functions, each having a cover number of four, proving that PropEF(C, *rectangles*, 2) $\ge 1/4$.



- (3.12) If both agents chose the same part from the main partition, for example, $C_{j'}$ for some $j' \neq j$, then apply Algorithm 2 with $C_{\text{end}} = C_j$ (where C_j is the piece chosen by both agents in step (3.1)) and the S-continuous knife function from $\emptyset$ to C_j of input (d).
- (3.13) If both agents chose the same part from the subpartition, for example, C_j^k for some k, then ask each agent to choose either C_j^k or $\overline{C_j^k}$ (where $\overline{C_j^k} := C \setminus C_j^k$). If the choices are different, then each agent receives their choice, and we are done.

If the choices are identical, then there are two subcases:

- (3.141) Both agents chose C_j^k. Apply Algorithm 1 with $C_{\text{end}} = C_j^k$ and the S-smooth knife function of input (c).
- (3.142) Both agents chose $\overline{C_j^k}$. Apply Algorithm 2 with $C_{\text{end}} = \overline{C_j^k}$ and the S-continuous knife function from $\emptyset$ to $\overline{C_j^k}$ of input (d).

3.2. Both agents chose $\overline{C_j}$. Apply Algorithm 2 with $C_{\text{end}} = \overline{C_j}$ and the S-continuous knife function from $\emptyset$ to $\overline{C_j}$ of input (d).

Lemma 9 (Correctness of Multiple Partition Algorithm). *If there exist partitions and knife functions satisfying the input requirements of Algorithm 4, then this algorithm produces an envy-free and* $(1/M)$*-proportional allocation of* C *between the two agents.*

Proof. Steps 1, 2, and 3.2 are the same as in Algorithm 3, and their correctness proof is the same too. It remains to prove that, if the algorithm ends in one of the substeps of 3.1, then the resulting allocation is envy-free and $(1/M)$-proportional.

In step 3.11, if the choices are different, then by the Allocation Lemma and the condition on input (b), each agent i receives an envy-free share with a utility of at least $V_i(C)/M$.

In step 3.12, we know that both agents prefer C_j to its complement $\overline{C_j}$ (from the choice of step 3.1). Therefore, $C_{\text{end}} = C_j$ and the S-continuous knife function from input (d) satisfy the input requirements of Algorithm 2. The algorithm gives each agent an envy-free share with utility at least $V_i^S(\overline{C_j})$. This $\overline{C_j}$ contains all other parts of the main partition, including $C_{j'}$. The fact that both agents chose $C_{j'}$ in the refined partition proves, by the chooser lemma, that $V_i^S(C_{j'}) \geq V_i(C)/M$. Hence, also $V_i^S(\overline{C_j}) \geq V_i(C)/M$.

In step 3.13, if the choices are different, then by the Allocation Lemma and the condition on input (b), each agent i receives an envy-free share worth at least $V_i(C)/M$.

In step 3.141, we know that both agents prefer C_j^k to its complement $\overline{C_j^k}$. Therefore, $C_{\text{end}} = C_j^k$ and the S-smooth knife function from input (c) satisfy the input requirements of Algorithm 1. The cover number of this knife function is at most M. Hence, Algorithm 1 gives to each agent i an envy-free share with a utility of at least $V_i(C)/M$.

In step 3.142, we know that both agents prefer $\overline{C_j^k}$ to its complement C_j^k. Therefore, $C_{\text{end}} = \overline{C_j^k}$ and the S-continuous knife function from input (d) satisfy the input requirements of Algorithm 2. Each agent receives an envy-free share with utility at least $V_i^S(C \setminus \overline{C_j^k}) = V_i^S(C_j^k)$. Now, in step 3.13, both agents chose C_j^k from a partition with a total cover number of at most M. Therefore, by the chooser lemma, $V_i^S(C_j^k) \geq V_i(C)/M$. □

Based on this lemma, we can now prove the second part of our first theorem:

Theorem 1(b). *For every* $R \geq 2$,

$$\textsc{PropEF}(R\ \textit{fat rectangle},\ R\ \textit{fat rectangles},\ 2) \geq 1/3.$$

Proof. Let C be an R-fat rectangle. We divide C using Algorithm 4 with $M = 3$ in the following way (see Figure 6).

The main partition (input (a)) is a partition of C into two halves in the middle of its longer side. Because $R \geq 2$, the two halves are R-fat too so the total cover number of the partition is $1 + 1 < 3$.

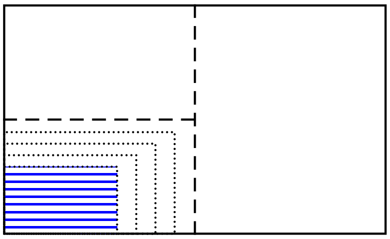
The subpartitions (input (b)) are again partitions of each half to two quarters in the middle of its longer side. When a part is replaced by its subpartition, the resulting partition contains three R-fat rectangles, so its total cover number is three.

The S-smooth knife functions on the quarters (input (c)) are rectangles growing from the corner toward the center as in Figure 6. K_C is a single R-fat rectangle, and its complement can always be covered by two R-fat rectangles, so the cover number of these knife-functions is at most three.

The increasing S-continuous knife functions (input (d)) are sweeping-lines; see Appendix A.4 and Figure 4(e). All input conditions are satisfied, so the resulting division is envy-free and 1/3-proportional. □

The fraction 1/3 is tight; see the preprint version for details.

Figure 6. (Color online) A 2-fat rectangle is partitioned into two 2-fat rectangles. One of them is further partitioned into two smaller 2-fat rectangles. On the bottom-left one, there is a knife-function with a cover number (relative to the family of 2-fat rectangles) of at most 3. The partitions and knife-functions are used in the proof of Theorem 1(b).



Algorithm 4 can be further refined by adding more subpartition steps. For example, by adding a third subpartition step, we can prove that, if C is an archipelago of m disjoint R-fat rectangles (with $R \geq 2$), then

$$\textsc{PropEF}(C, R\text{-fat rectangles}, 2) \geq \frac{1}{m+2},$$

and this bound is tight. We omit the proof details as the proof is analogous to examples (b) and (c) after Lemma 8.

4.4. Dividing Fat Lands of Arbitrary Shape

Our most general result involves land estates that are arbitrary Borel sets. The result is proved for any number of dimensions; Figure 7 illustrates the proof for $d = 2$ dimensions.

Theorem 1(c). *For every $R \geq 1$, if C is R-fat and S is the family of $2R$-fat pieces, then*

$$\textsc{PropEF}(C, S, 2) = \textsc{Prop}(C, S, 2) = 1/2.$$

Proof. The proof uses Algorithm 3 (the single partition algorithm). We show a partition of C into two parts and a knife function on each part.

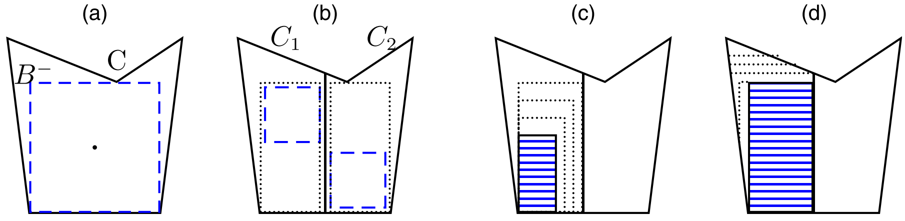
Scale, rotate, and translate C such that the largest cube contained in C is $B^- = [-1, 1]^d$; see Figure 7(a). By definition of fatness (see Section 1.1), C is now contained in a cube B^+ of side length at most $2R$. Using the hyperplane $x = 0$, bisect the cube B^- into two boxes $B_1 = [-1, 0] \times [-1, 1]^{d-1}$ and $B_2 = [0, 1] \times [-1, 1]^{d-1}$. This hyperplane also bisects C into two parts, C_1 and C_2; see Figure 7(b). Every C_j contains B_j, which contains a cube with a side length of one. Every C_j is, of course, still contained in B^+, which is cube with a side length of $2R$. Hence, every C_j is $2R$-fat. Hence, the total cover number of the partition $C = C_1 \sqcup C_2$ with respect to the family of $2R$-fat objects, is two, satisfying input condition (a).

For every $j \in \{1, 2\}$, define the following knife function K_{C_j} on C_j; see Figure 7(c and d):

- For $t \in [0, \frac{1}{2}]$, $K_{C_j}(t) = (B_j)^{2t}$; that is, the box B_j dilated by a factor of $2t$. Hence, $K_{C_j}(0) = \emptyset$ and $K_{C_j}(\frac{1}{2}) = B_j$.
- For $t \in [\frac{1}{2}, 1]$, $K_{C_j}(t)$ is any knife function from B_j to C_j, for example, the growing ball function of Appendix A.1.

$K_{C_j}(t)$ is always $2R$-fat because, in $[0, \frac{1}{2}]$, it is a scaled-down version of the box B_j (which is 2-fat), and in $[\frac{1}{2}, 1]$, it contains B_j and is contained in the cube B^+. Also $\overline{K_{C_j}(t)}$ is $2R$-fat becasue it contains the box B_{3-j} and is

Figure 7. (Color online) Dividing a general R-fat land estate between two people.



a. The R-fat land C and its largest contained square B^- (the smallest containing square B^+ is not shown).
b. The parts C_1 and C_2 (solid), the two rectangles B_1 and B_2 (dotted) and their largest contained squares (dashed).
c. The knife function on C_1 in $t \in [0, \frac{1}{2}]$.
d. The knife function on C_1 in $t \in [\frac{1}{2}, 1]$.

contained in B^+. Hence, K_{C_j} is an S-smooth knife function with a cover number of $1 + 1 = 2$, satisfying input condition (b).

For the complements, we can use, for example, the sweeping-plane knife function of Appendix A.4, satisfying input condition (c).

By Lemma 8, Algorithm 3 finds an envy-free and (1/2)-proportional division. □

Theorem 1(c) implies that we can satisfy the two main fairness requirements, proportionality and envy-freeness, while keeping the allocated pieces sufficiently fat. The fatness guarantee means that each allotted piece (a) contains a sufficiently *large* square and (b) is contained in a sufficiently *small* square. In the context of land division, these guarantees can be interpreted as follows: (a) each land plot has sufficient room for building a large house having a convenient shape (square), and (b) the parts of the land that are valuable to the agent are close together because they are bounded in a sufficiently small square.

Finally, we note that a different technique leads to a version of Theorem 1(c), which guarantees that the pieces are not only $2R$-fat but also *convex* (if the original land is convex); hence, an agent can walk in a straight line from the agent's square house to the agent's valuable spots without having to enter or circumvent the neighbor's fields. See the preprint version for details.

4.5. Between Envy-Freeness and Proportionality

For all lands C and families of usable pieces S studied in this section, we proved that there exists a positive constant p such that $\text{PropEF}(C, S, 2) \geq p$ and $\text{Prop}(C, S, 2) \leq p$. Because $\text{PropEF}(C, S, 2) \leq \text{PropEF}(C, S, 2)$ always, we get that, for all settings studied here,

$$\text{PropEF}(C, S, 2) = \text{Prop}(C, S, 2).$$

In other words, in these cases, envy-freeness is compatible with the best possible proportionality guarantee. It is an open question whether this equality holds for *every* combination of lands C and families S.

What *can* we say about the relation between proportionality and envy-freeness for arbitrary C and S? In addition to the trivial upper bound $\text{PropEF}(C, S, 2) \leq \text{Prop}(C, S, 2)$, we have the following lower bound:

Lemma 10. *For every C and S,*

$$\text{PropEF}(C, S, 2) \geq p_S \cdot \text{Prop}(C, S, 2),$$

where $p_S := \inf_{Z \in S} \text{PropEF}(Z, S, 2)$.

Proof. Let $p_C = \text{Prop}(C, S, 2)$. The following meta-algorithm yields an envy-free allocation of C in which the utility of each agent i is at least $p_S \cdot p_C \cdot V_i(C)$.

By the definition of $\text{Prop}(C, S, 2)$, there exists an allocation of C, say $X = (X_1, X_2)$, with a partial proportionality of at least p_C; that is, for each agent i, $V_i^S(X_i) \geq p_C \cdot V_i(C)$. This means that, for each i, the piece X_i contains an S-piece Z_i with $V_i(Z_i) \geq p_C \cdot V_i(C)$.

Ask each agent whether the agent prefers Z_1 or Z_2 and proceed accordingly.

a. If each agent i prefers Z_i, then the allocation (Z_1, Z_2) is envy-free. Both the value and the utility of each agent i are at least $p_C \cdot V_i(C)$, which is at least $p_S \cdot p_C \cdot V_i(C)$ (because $p_S \leq 1$).

b. If each agent i prefers Z_{3-i}, then the allocation (Z_2, Z_1) is envy-free. Both the value and the utility of each agent i are now even more than $p_C \cdot V_i(C)$.

c. The remaining case is that both agents prefer the same piece, say Z_2. So, for each agent i, $V_i(Z_2) \geq p_C \cdot V_i(C)$. By the assumptions of the lemma, because $Z_2 \in S$, $\text{PropEF}(Z_2, S, 2) \geq p_S$. Therefore, there exists an envy-free allocation of Z_2 in which the utility of each agent i is at least $p_S \cdot V_i(Z_2) \geq p_S \cdot p_C \cdot V_i(C)$. □

So, by previous results, we have the following bounds for *every* C:

- $\text{Prop}(C, \textit{Squares}, 2) \geq \text{PropEF}(C, \textit{Squares}, 2) \geq \frac{1}{4}\text{Prop}(C, \textit{Squares}, 2)$.
- $\text{Prop}(C, R \textit{ fat rects}, 2) \geq \text{PropEF}(C, R \textit{ fat rects}, 2) \geq \frac{1}{3}\text{Prop}(C, R \textit{ fat rects}, 2)$ (for $R \geq 2$).
- $\text{Prop}(C, \textit{Rectangles}, 2) \geq \text{PropEF}(C, \textit{Rectangles}, 2) \geq \frac{1}{2}\text{Prop}(C, \textit{Rectangles}, 2)$.

5. Envy-Free Division for Many Agents

5.1. The One-Dimensional Algorithm

Existence of envy-free allocations in one dimension was first proved by Stromquist [49]. Later, Su [51] presented an algorithm, attributed to Simmons, for generating an infinite sequence of allocations that converges to an envy-free allocation. In this section, we generalize the Simmons–Su algorithm to handle geometric constraints. We first briefly describe the one-dimensional algorithm.

C is a one-dimensional interval $[0,1]$, and S is the family of intervals. A partition of C to n intervals can be described by a vector of length n whose coordinates are the lengths of the intervals. The sum of all lengths in a partition is one, so the set of all partitions is equivalent to Δ^{n-1}—the standard $(n-1)$-dimensional simplex in $\mathbb{R}^n$. The algorithm proceeds as follows (see Figure 8):

1. Preparation. Triangulate the simplex of partitions to a collection of $(n-1)$-dimensional subsimplices. Assign each vertex of the triangulation to one of the n agents, such that, in each subsimplex, all n agents are represented. Su [51] shows that there always exists such a triangulation.

2. Evaluation. For each vertex v of the triangulation, ask its assigned agent, "If C is partitioned as prescribed by the coordinates of v, which piece would you prefer?" The answer is an integer $j \in \{1,\ldots,n\}$; label that vertex with j.

3. Selection. The labeling created in step 2 satisfies *Sperner's boundary condition*: in every face of the simplex, every vertex is labeled with one of the labels on the end points of that face because the other labels correspond to empty pieces (see Figure 8(b), in which the three vertices of the large triangle are labeled by 1–3). By Sperner's lemma, there exists a *fully labeled subsimplex*—a subsimplex in which all vertices are labeled differently.

4. Refinement. Steps 1–3 can be repeated again and again, each time with a finer triangulation. This yields an infinite sequence of fully labeled simplices. By compactness of the simplex, there is a subsequence that converges to a single point. By the continuity of the agents' valuations, this point corresponds to a partition in which each of the n agents prefers a different piece. By definition, this partition is envy-free.

Note that the algorithm is infinite; the envy-free partition is found only at the limit of an infinite sequence. In fact, Stromquist [50] proved that, when $n \geq 3$, an envy-free partition among n agents with connected pieces cannot be found by a finite algorithm. Therefore, Simmons' infinite algorithm is the best that can be hoped for.

5.2. Knife Tuples

Both Stromquist's existence proof and the Simmons–Su algorithm do not work directly on C; they work on the unit simplex, each point of which represents a partition of C. Therefore, we can extend these results to multiple dimensions if we find an appropriate way to map each point of the unit simplex to a partition of a multidimensional land.

Our main tool is a *knife tuple*—a generalization of the knife function defined in Definition 4.

Definition 6. An *n-knife tuple on C* is a vector of n functions $(K_1,\ldots,K_n)$ from Δ^{n-1} to pieces of C such that

- To every $t \in \Delta^{n-1}$, the tuple assigns a partition of C so that $K_1(t) \sqcup \cdots \sqcup K_n(t) = C$.
- For every $\ell \in \{1,\ldots,n\}$ and every $t \in \Delta^{n-1}$ such that $t_\ell = 0$, $K_\ell(t) = \emptyset$.
- [Continuity]. For every $\epsilon > 0$, there is a $\delta > 0$ such that $|t' - t| < \delta$ implies

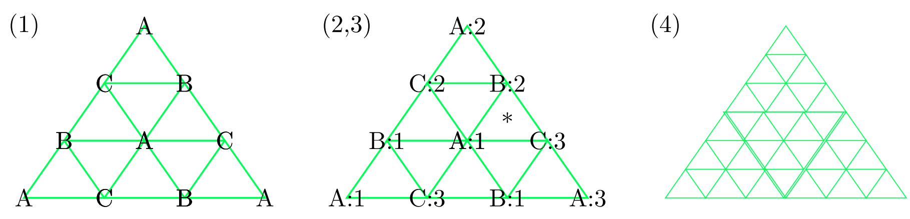
$\forall \ell \in \{1,\ldots,n\}$: $\textsc{Volume}[K_\ell(t') \ominus K_\ell(t)] < \epsilon$, where $|t - t'|$ denotes the distance $\sum_{\ell=1}^{n} |t_\ell - t'_\ell|$.

The definitions of S-continuity and S-smoothness can be easily generalized from knife functions to knife tuples. For the present paper, it is sufficient to generalize S-smoothness (Definition 5):

Definition 7. A knife tuple $(K_1,\ldots,K_n)$ is S-smooth if, for every $\ell \in \{1,\ldots,n\}$, the function K_ℓ has a finite S-cover $K_\ell^1 \cup \cdots \cup K_\ell^{m_\ell}$ for some integer $m_\ell \geq 1$.

If a knife tuple K is S-smooth, then we define its cover number, $\textsc{KCoverNum}(K,S)$, as the smallest sum $\sum_{\ell=1}^{n} m_\ell$ of integers that satisfy the definition.

Figure 8. (Color online) An illustration of the Simmons–Su algorithm for $n = 3$ agents, A, B, and C.



1. A triangulation of the simplex of partitions in which each vertex is assigned to an agent.
2–3. Each vertex is labeled with the index of the piece preferred by its assigned agent. The fully labeled triangle is starred.
4. The process is repeated with a finer triangulation of the original simplex.

Given an S-smooth knife tuple $(K_1, \ldots, K_n)$ and an absolutely continuous measure V, define the function V^{K_ℓ} (analogously to (2)) as the maximum value of an S-piece in the given cover of $K_\ell(t)$:

$$\forall \ell \in \{1, \ldots, m\}: \qquad V^{K_\ell}(t) := \max_{j=1}^{m_\ell} V\left(K_\ell^j(t)\right).$$

Lemma 5 can be generalized as follows:

Lemma 11. *If $(K_1, \ldots, K_n)$ is an S-smooth knife tuple and V is an absolutely continuous measure, then, for every $\ell \in \{1, \ldots, n\}$, V^{K_ℓ} is a uniformly continuous real function.*

The proof is the same: The maximum of uniformly continuous functions is uniformly continuous.

5.3. Constructing Knife Tuples

Knife tuples can be constructed from knife functions.

Lemma 12. *Let K be a knife function from $\emptyset$ to C, and let K_1, K_2 be the following functions from Δ^1 to pieces of C:*

$$K_1(t_1, t_2) := K(t_1)$$
$$K_2(t_1, t_2) := C \setminus K(1 - t_2).$$

Then, (K_1, K_2) is a 2-knife tuple on C.

Moreover, if the knife function K_C is S-smooth and its cover number is M, then the knife tuple (K_1, K_2) is S-smooth, and its cover number is M too.

Proof. We prove that (K_1, K_2) satisfies the conditions of Definition 6.

• For every $t \in \Delta^1$, $1 - t_2 = t_1$. Hence, $K_2(t) = C \setminus K(1 - t_2) = C \setminus K(t_1) = C \setminus K_1(t)$. Hence, $K_1(t), K_2(t)$ is indeed a partition of C.

• If $t_1 = 0$, then $K_1(t) = K(0) = \emptyset$; if $t_2 = 0$, then $K_2(t) = C \setminus K(1) = C \setminus C = \emptyset$.

• For every $\epsilon > 0$, because K is a knife function, there exists a $\delta > 0$ such that $|t_1 - t_1'| < \delta$ implies $\textsc{Volume}[K(t_1) \ominus K(t_1')] < \epsilon$. For every $t, t' \in \Delta^1$, $|t - t'| = |t_1 - t_1'| + |t_2 - t_2'| = |t_1 - t_1'| + |(1 - t_1) - (1 - t_1')| = 2|t_1 - t_1'|$. If $|t - t'| < \delta$, then $|t_1 - t_1'| < \delta/2 < \delta$, so $\textsc{Volume}[K_1(t) \ominus K_1(t')] < \epsilon$ and $\textsc{Volume}[K_2(t) \ominus K_2(t')] < \epsilon$.

Now, if K is S-smooth, then K can be covered by m_1 S-knife functions, and $\overline{K}$ can be covered by m_2 S-knife functions (for some integers $m_1, m_2 \geq 1$), so K_1 and K_2 can be covered by the same S-knife functions, so (K_1, K_2) is S-smooth with the same cover number. □

For example, from the knife function in Figure 4(b)—a growing pair of squares—we get the following two-knife tuple:

$$\begin{aligned} K_1(t) &= [0, t_1] \times [0, t_1] \cup [1 - t_1, 1] \times [1 - t_1, 1] \\ K_2(t) &= [0, t_2] \times [1 - t_2, 1] \cup [1 - t_2, 1] \times [0, t_2]. \end{aligned} \tag{3}$$

It is square smooth with a square cover number of four.

Longer knife tuples can be constructed by splitting existing knife tuples. Let $(K_1, \ldots, K_n)$ be an n-knife tuple on C. Suppose that, for some $\ell \in \{1, \ldots, n\}$, for every $t \in \Delta^{n-1}$ for which $t_\ell > 0$, we have a knife function K^t from $\emptyset$ to $K_\ell(t)$.

We create a new tuple by replacing the index ℓ with two indices $\ell 1$ and $\ell 2$ and replacing the function K_ℓ with two complementary functions $K'_{\ell 1}$ and $K'_{\ell 2}$ split by the knife function K^t. This gives a new vector of $n + 1$ functions $(K'_1, \ldots, K'_{\ell 1}, K'_{\ell 2}, \ldots, K'_n)$ from Δ^n to partitions of C:

$$K'^{\ell 1}(t_1, \ldots, t_{\ell 1}, t_{\ell 2}, \ldots, t_n) := \begin{cases} K^{t_1, \ldots, t_{\ell 1} + t_{\ell 2}, \ldots, t_n}\left(\frac{t_{\ell 1}}{t_{\ell 1} + t_{\ell 2}}\right) & [t_{\ell 1} + t_{\ell 2} > 0] \\ \emptyset & [t_{\ell 1} = t_{\ell 2} = 0] \end{cases}$$

$$K'^{\ell 2}(t_1, \ldots, t_{\ell 1}, t_{\ell 2}, \ldots, t_n) := \begin{cases} K_\ell(t_1, \ldots, t_{\ell 1} + t_{\ell 2}, \ldots, t_n) \setminus \\ \qquad K^{t_1, \ldots, t_{\ell 1} + t_{\ell 2}, \ldots, t_n}\left(\frac{t_{\ell 1}}{t_{\ell 1} + t_{\ell 2}}\right) & [t_{\ell 1} + t_{\ell 2} > 0] \\ \emptyset & [t_{\ell 1} = t_{\ell 2} = 0] \end{cases}$$

$$\forall j \neq \ell: \quad K'^j(t_1, \ldots, t_{\ell 1}, t_{\ell 2}, \ldots, t_n) := K_j(t_1, \ldots, t_{\ell 1} + t_{\ell 2}, \ldots, t_n).$$

It is easy to see that, to every $t \in \Delta^n$, the new tuple indeed assigns a partition of C and that, whenever one of the new $n+1$ time variables is zero, the corresponding part of the knife tuple returns an empty set. The continuity or smoothness of the new knife tuple does not follow automatically from the construction; it has to be verified separately.

As an example, we apply this construction with $n = 2$, the 2-knife tuple (K_1, K_2) of (3), and $\ell = 2$.

For every $(t_1, t_2) \in \Delta^1$, we have to define a knife function K^{t_1,t_2} from $\emptyset$ to $K_2(t_1, t_2)$. Recall that $K_2(t_1, t_2)$ is a union of two squares. For each such square, we create a square pair knife function analogous to Figure 4(b)—two squares growing from opposite corners. We define K^{t_1,t_2} as the union of these two square pairs; see Figure 9. This construction yields the following three functions:

$$
\begin{aligned}
K_1'(t_1,t_2,t_3) &= K_1(t_1, t_2+t_3) = [0,t_1]\times[0,t_1] \cup [1-t_1,1]\times[1-t_1,1]\\
K_2'(t_1,t_2,t_3) &= [0,t_2]\times[1-t_2-t_3, 1-t_3] \cup [t_3, t_2+t_3]\times[1-t_2,1]\\
&\quad \cup [1-t_2-t_3, 1-t_3]\times[0,t_2] \cup [1-t_2,1]\times[t_3,t_2+t_3]\\
K_3'(t_1,t_2,t_3) &= [0,t_3]\times[1-t_3,1] \cup [t_2,t_2+t_3]\times[1-t_2-t_3,1-t_2]\\
&\quad \cup [1-t_2-t_3,1-t_2]\times[t_2,t_2+t_3] \cup [1-t_3,1]\times[0,t_3].
\end{aligned}
$$

It can be verified that (K_1', K_2', K_3') is indeed a 3-knife tuple: For every $t \in \Delta^2$, it returns a partition of C; whenever $t_\ell = 0$, the corresponding K_ℓ' is empty; and for every $\ell \in \{1,2,3\}$, the function $K_\ell(t)$ satisfies the continuity requirement of Definition 6 (because it is a union of squares whose boundaries are continuous functions of t).

Moreover, K_1' is covered by two square knife functions, and each of K_2', K_3' is covered by four square knife functions, so (K_1', K_2', K_3') is square smooth, and its cover number is $2+4+4 = 10$.

In exactly the same manner, we can replace K_1'—the growing square pair—with two growing square quadruplets. This yields a new 4-knife tuple that is square smooth with a cover number of $4+4+ 4+4 = 16$.

5.4. Land Division Using Knife Tuples

Using knife tuples, Lemma 6 can be generalized as follows:

Lemma 13. *Let C be a land and S a family of pieces. If there is an S-smooth n-knife tuple on C with a cover number of at most M, then there exists an envy-free and 1/M-proportional division of C among the n agents.*

Proof. The proof generalizes the infinite algorithm of Simmons–Su (Section 5.1).[11]

1. The preparation step is entirely the same: Triangulate the standard simplex Δ^{n-1} and assign each triangulation vertex to an agent such that, in each subsimplex, all agents are represented.

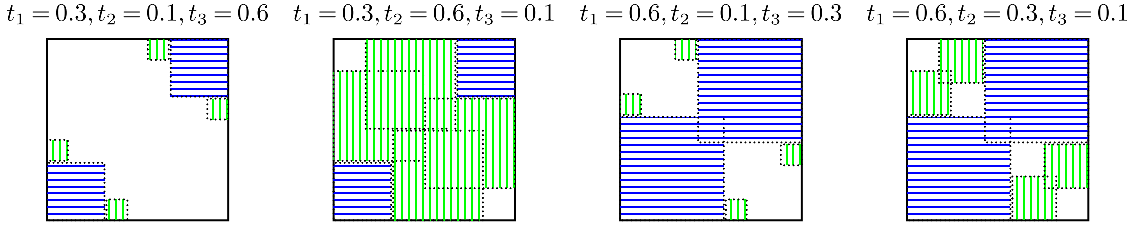
2. The evaluation step is different: for each vertex $t = (t_1, \ldots, t_n)$ of the triangulation, a partition of C is defined by the given n-knife tuple: $K_1(t), \ldots, K_n(t)$. Each part $K_\ell(t)$ in this partition is covered by m_ℓ S-pieces. Ask the owner of vertex t (e.g., agent i) to calculate, for each $\ell \in \{1, \ldots, n\}$, the value of $V_i^{K_\ell}(t)$, that is, the most valuable S-piece from the set of m_ℓ S-pieces covering $K_\ell(t)$. Then, find the maximum of these n maxima:

$$\arg\max_{\ell \in \{1,\ldots,n\}} V_j^{K_\ell}(t),$$

and label the vertex t with the result.

3. By the definition of a knife tuple, whenever $t_\ell = 0$, $K_\ell(t) = \emptyset$, so for every agent i, $V_i^{K_\ell}(t) = 0$. Hence, any agent asked to label vertex t will never label it with ℓ. Hence, the resulting labeling satisfies Sperner's labeling condition, so in the selection step, a fully labeled subsimplex exists.

Figure 9. (Color online) A 3-knife tuple on a square cake. Four partitions induced by the three-knife tuple (K_1', K_2', K_3') in different points (t_1, t_2, t_3) of the unit simplex. $K_1'(\cdot)$ is filled with horizontal lines, $K_2'(\cdot)$ is filled with vertical lines and $K_3'(\cdot)$ is blank.



4. By repeating steps 1–3 infinitely many times with finer and finer triangulations, we get a sequence of smaller and smaller fully labeled simplices. This sequence has a subsequence that converges to a single point t^*. Because the knife tuple is S-smooth, by Lemma 11, all agents' utilities are continuous functions of t. Therefore, in the partition corresponding to the limit point, $K_1(t^*),\dots,K_n(t^*)$, each agent is allocated a different S-piece with a maximum value.

The cover number of the knife tuple is at most M. Therefore, by the Allocation Lemma (Lemma 2), this allocation is envy-free and $1/M$-proportional. □

5.5. Dividing Squares and Rectangles

We apply Lemma 13 to prove our second theorem.

Theorem 2(a). *For every $n \geq 1$,*

$$\text{PropEF}(Square, n, Squares) \geq \frac{1}{2^{2\lceil \log_2 n\rceil}} > \frac{1}{4n^2}.$$

Proof. We first consider the case in which n is a power of two. We construct an n-knife tuple $(K_1,\dots,K_n)$, in which, for every $t \in \Delta^{n-1}$ and for every $\ell \in \{1,\dots,n\}$, $K_\ell(t)$ is a union of at most n squares. Hence, the cover number of the n-knife tuple $(K_1,\dots,K_n)$ is at most $n \cdot n = n^2$.

The construction is recursive. The base is $n = 2$. Take the knife function in Figure 4(b)—a union of two corner squares growing toward the center. By Lemma 12, this knife function defines a 2-knife tuple, which we denote by (K_1, K_2). For each t_1 and t_2, $K_1(t_1,t_2)$ and $K_2(t_1,t_2)$ are square pairs—unions of two squares.

Consider next the case $n = 4$. In every square pair in the 2-knife tuple, define a knife function as shown in Figure 9—a union of four squares growing from opposite corners toward the center. This yields a 4-knife tuple (K'_1, K'_2, K'_3, K'_4). For each $\ell \in \{1,2,3,4\}$ and for each $t \in \Delta^3$, $K'_\ell(t)$ is a union of four squares.

After k steps, we have a 2^k-knife tuple in which each component is a union of 2^k squares. We split each component using a knife function made of a union of 2^k square pairs—a square-pair for each of the 2^k squares in the cover.[12] This gives a new, 2^{k+1}-knife tuple in which each component is a union of 2^{k+1} squares. After $\log_2 n$ steps, we get an n-knife tuple. It is square smooth because each element of the tuple can be covered by n square knife functions—n squares changing continuously with t. Its square cover number is, therefore, at most n^2.

Applying Lemma 13 with this knife tuple implies that there exists an envy-free and $(1/n^2)$-proportional division of C among the n agents.

Now, suppose n is not a power of two. Define $n' = 2^{\lceil \log_2 n\rceil}$ =the smallest power of two larger than n. Add $n' - n$ dummy agents and apply the proof of the first case. There exists an envy-free and $(1/n'^2)$-proportional division of C among the n' agents. Because free disposal is assumed, the pieces allocated to the $n' - n$ dummy agents can be discarded. We remain with an envy-free division with a proportionality of at least $1/2^{2\lceil \log_2 n\rceil} > 1/4n^2$. □

The second part of the second theorem is a simple corollary of the first part:

Theorem 2(b). *If C is an R-fat rectangle and S the family of R-fat rectangles, then*

$$\text{PropEF}(C, n, S) \geq \frac{1}{2^{2\lceil \log_2 n\rceil}} > \frac{1}{4n^2}.$$

Proof. Rescale the axes such that C becomes a square. By Theorem 2(a), there exists an allocation in which each agent i receives a piece that contains a square Z_i with a value of at least $V_i(C)/2^{2\lceil \log_2 n\rceil}$. Rescale the axes back. Now, each Z_i is an R-fat rectangle. □

We do not know if the $1/(4n^2)$ lower bound is asymptotically tight. The best upper bound currently known is $\text{Prop}(Square, squares, n) \leq 1/(2n)$, and there is an algorithm for non–envy-free division that proves $\text{Prop}(Square, squares, n) \geq 1/(4n-4)$ (Segal-Halevi et al. [47]). We do not know if it is possible to attain an envy-free division with a proportionality of $1/O(n)$.

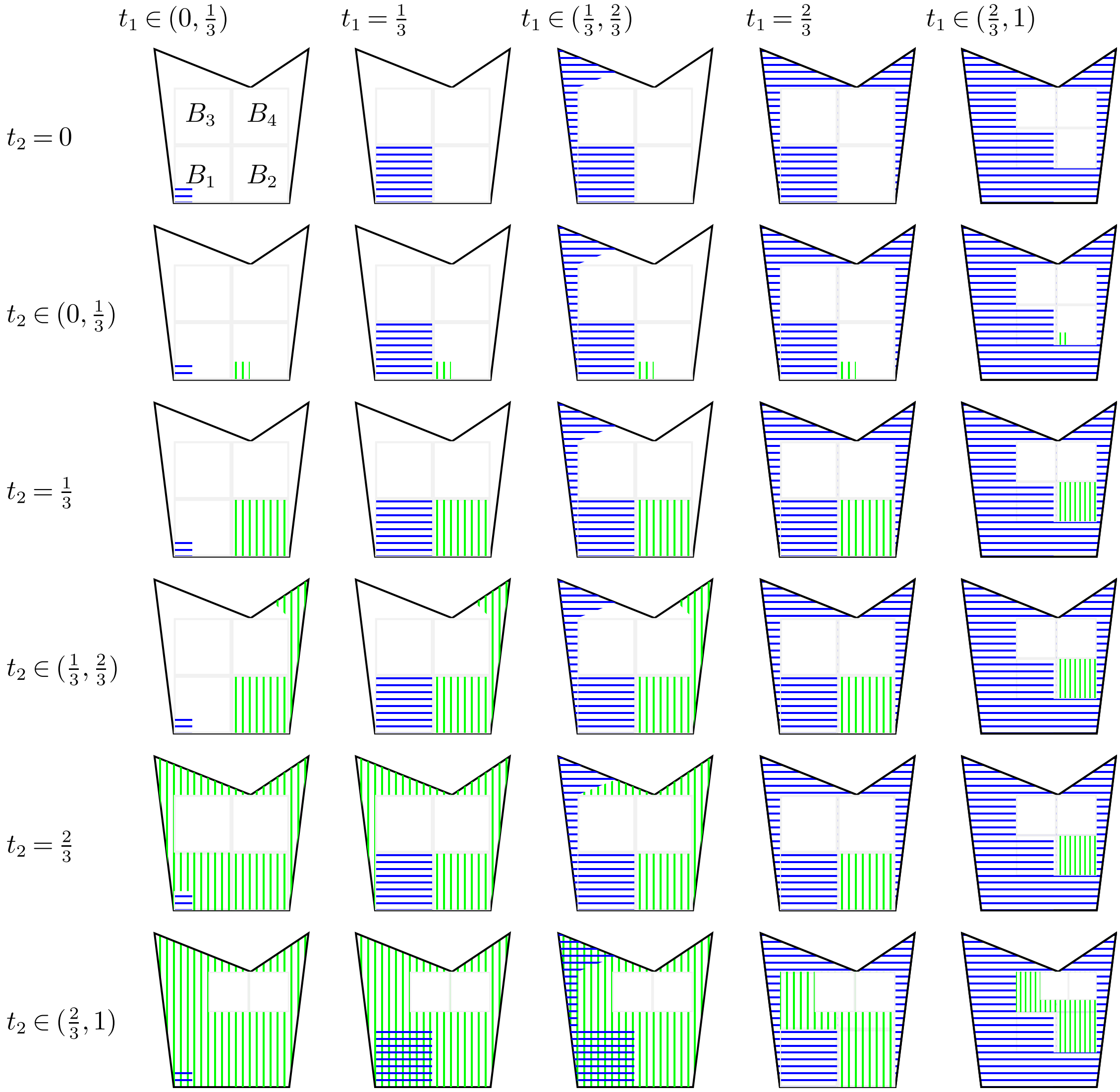
5.6. Dividing Fat Objects of Arbitrary Shape

In this section, we show that it is possible to attain an envy-free and proportional division for every n in return for a compromise on the fatness of the pieces.

Theorem 2(c). *Let C be a d-dimensional R-fat land and $n \geq 2$ an integer. Let S be the family of mR-fat pieces, where m is the smallest integer such that $n \leq m^d$ (i.e. $m = \lceil n^{1/d}\rceil$). Then,*

$$\text{PropEF}(C, S, n) = 1/n.$$

Figure 10. (Color online) Dividing a general R-fat cake for $n = 3$ people. K_1 is filled with horizontal lines, K_2 is filled with vertical lines, and K_3 is white. Note that each of these three pieces is $2R$-fat, where R is the fatness of the original cake.



Proof. The proof is illustrated in Figure 10 for the case of $d = 2$ dimensions. Let C be an R-fat d-dimensional land. By definition of fatness, it contains a cube B^- of side length x, and it is contained in a parallel cube B^+ of side length $R \cdot x$ for some $x > 0$.

Partition the cube B^- to a grid of m^d subcubes, $B_1, \ldots, B_{m^d}$, each of side length $\frac{x}{m}$. For every $i \in \{1, \ldots, n-1\}$, denote by $B_{>i}$ the union of the $m^d - i$ squares with indices larger than i; that is,

$$B_{>i} := \bigcup_{j>i} B_j.$$

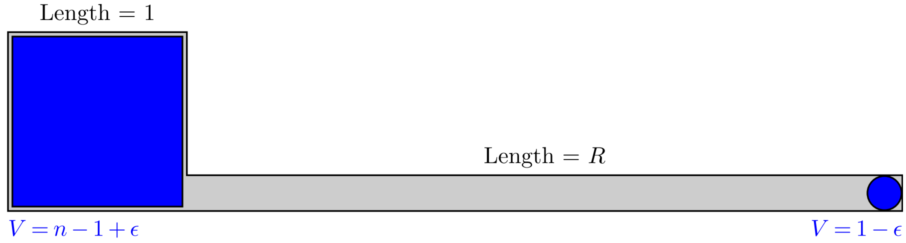
Denote by $\overline{B^-}$ the land outside the enclosed cube; that is,

$$\overline{B^-} := C \setminus B^-.$$

Define the following knife function K from $\emptyset$ to C:

- For $t \in [0, \frac{1}{3}]$, $K(t) = (B_1)^{3t}$; that is, the cube B_1 dilated by a factor of $3t$. Hence, $K(0) = \emptyset$ and $K(\frac{1}{3}) = B_1$.
- For $t \in [\frac{1}{3}, \frac{2}{3}]$, $K(t)$ is any knife function from B_1 to $C \setminus B_{>1}$, such as the growing-ball function of Appendix A.1.

Figure 11. (Color online) A fat land estate in which every proportional division must use slim pieces. See Lemma 14.



• For $t \in [\frac{2}{3}, 1]$, $K(t)$ is $C \setminus [(B_{>1})^{3(1-t)}]$; that is, the land not yet covered by the knife is $B_{>1}$ dilated by a factor proportional to the remaining time. Hence, $K(1) = C$.

By Lemma 12, K induces a 2-knife tuple (K_1, K_2). For every t_1, t_2 with $t_1 + t_2 = 1$, $K_1(t_1, t_2)$ is mR-fat:

• When $t_1 \in [0, \frac{1}{3}]$, K_1 it is a cube, which is 1-fat.

• When $t_1 \in [\frac{1}{3}, 1]$, K_1 contains the cube B_1, whose side length is x/m, and is contained in the cube B^+, whose side length is $x \cdot R$.

$K_2(t_1, t_2)$ is mR-fat too:

• When $t_1 \in [0, \frac{2}{3}]$, K_2 contains, for example, the cube B_n, whose side length is x/m, and is contained in the larger cube B^+, whose side length is $x \cdot R$.

• When $t_1 \in [\frac{2}{3}, 1]$, K_2 contains a dilated B_n, and it is contained in a dilated B^-; because they are dilated by the same factor, the ratio between their side lengths remains m, so $K_2(t)$ is m-fat.

Therefore, (K_1, K_2) is an S-smooth 2-knife tuple with a cover number of two.

For every t_1, t_2 with $t_1 + t_2 = 1$, we now define a knife function from $\emptyset$ to $K_2(t_1, t_2)$. K^{t_1,t_2} is analogous to K but uses the subcube B_2. This is possible because

• When $t_1 \in [0, \frac{2}{3}]$, K_2 contains the cube B_2 itself.

• When $t_1 \in (\frac{2}{3}, 1]$, K_2 contains a dilated B_2, which is contained in a dilated B^-.

The function K^{t_1,t_2} is defined as follows:

• For $t \in [0, \frac{1}{3}]$, $K^{t_1,t_2}(t) = (B_2)^{3t}$.

• For $t \in [\frac{1}{3}, \frac{2}{3}]$, $K^{t_1,t_2}(t)$ is any knife function from B_2 to $K_2(t_1, t_2) \setminus B_{>2}$ (e.g., the growing-ball function of Appendix A.1).

• For $t \in [\frac{2}{3}, 1]$, $K^{t_1,t_2}(t)$ is $K_2(t_1, t_2) \setminus [(B_{>2})^{3(1-t)}]$.

This process induces a 3-knife tuple (K'_1, K'_2, K'_3); see Figure 10.

To define an n-knife tuple, proceed in a similar way for the pieces $B_3, \ldots, B_n$. All components in the knife tuple are mR-fat. Therefore, the knife tuple is S-smooth with a cover number of n. By Lemma 13, there is an envy-free and $1/n$-proportional division of C with mR-fat pieces. □

Figure 10 shows an example of the construction for $d = 2$ dimensions and $n = 3$ agents. Here $m = \lceil\sqrt{3}\rceil = 2$, so each agent receives an envy-free $2R$-fat land plot with a utility of at least 1/3.

Theorem 2(c) implies that we can guarantee the highest possible level of proportionality ($1/n$) by compromising on the fatness of the pieces, allowing the pieces to be thinner than the original land by a factor of $\lceil n^{1/d}\rceil$. This factor is asymptotically optimal even when envy is allowed:

Lemma 14. *For every $R \geq 1$, there is an $(R+1)$-fat land C for which, for every $m' \leq (n-1)^{1/d}$,*

$$\textsc{PropEF}(C,\ n,\ m'R\ \textit{fat objects}) \leq \textsc{Prop}(C,\ m'R\ \textit{fat objects},\ n) < 1/n.$$

Proof. Let δ, ϵ be small positive constants. Let C be a land with the following two components:

• The left component is a cube with all sides of length one.

• The right component is a box with one side of length R and the other sides of length δ.

See Figure 11 for an illustration for $d = 2$. C is contained in a cube of side length $R + 1$, and it contains a cube of side length one, so it is $(R+1)$-fat.

C represents a desert with the following water sources:

• The left cube contains $n - 1 + \epsilon$ water units.

• A small disc at the end of the right box contains $1 - \epsilon$ water units.

C has to be divided among n agents whose value measures are equal to the amount of water. To get a proportional division, each agent must receive exactly one unit of water. This means that at least one piece, for example, X_i, must overlap both the right pool and the left pool.

The smallest cube containing X_i has a side length of at least R. For the largest cube contained in X_i, there are two options:

- If the largest contained cube is in the left side, then its side length must be at most $\left(\frac{1}{n-1+\epsilon}\right)^{1/d}$ because it must contain at most one unit of water.
- If the largest contained cube is in the right side, then its side length must be at most δ.

If δ is sufficiently small (in particular, $\delta < \left(\frac{1}{n-1}\right)^{1/d}$), then the piece X_i is not $m'R$-fat for every $m' \leq (n-1)^{1/d}$. This means that, if all pieces must be $m'R$-fat, a proportional division is impossible. □

6. Conclusion and Future Work

Fair division algorithms hold a great promise for resolving material disputes between people. But to realize this promise, these algorithms must consider practical requirements, such as the geometry of the pieces. The present paper contributed to this objective by presenting several algorithms for envy-free division considering geometric constraints. For two agents, the algorithms have the best possible partial proportionality guarantees in various geometric settings. For n agents, the algorithms guarantee a positive partial proportionality, and it is an open question whether this guarantee can be improved.

The tools developed in this paper are generic and can work for various geometric shapes. In fact, these tools reduce the envy-free division problem to a geometric problem: the problem of finding appropriate knife functions. Some topics not covered in the present paper are the following:

- Utility functions that take into account both the value contained in the best usable piece and the total value of the piece, for example, $U(Z) = w \cdot V^S(Z) + (1-w) \cdot V(Z)$, where w is some constant.
- Absolute size constraints on the usable pieces instead of the relative fatness constraints studied here; for example, let S be the family of all rectangles with length and width of at least 10 meters.
- Personal geometric preferences—letting each agent i specify a different family S_i of pieces.

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Appendix A. Knife Functions: Existence and Continuity

This appendix fills in some technical details related to Section 3.2.

A.1. Existence of Increasing Knife Functions

Given two bounded Borel subsets of $\mathbb{R}^d$, C_{start} and C_{end}, does there always exist an increasing knife function K from C_{start} to C_{end}?

Because K should be increasing, a necessary condition is that $C_{\text{start}} \subset C_{\text{end}}$. We show that this condition is also sufficient.[13] We denote by $D(r)$ the open d-dimensional ball of radius r around the origin.

Definition A.1. Let C_{start} and C_{end} be two bounded Borel subsets of $\mathbb{R}^d$ with $C_{\text{start}} \subsetneqq C_{\text{end}}$. Let r_{end} be a radius of a ball that contains C_{end} (it exists because C_{end} is bounded). Let $D^*(t) := D(t \cdot r_{\text{end}})$. The growing ball from C_{start} to C_{end} is the following function from $[0, 1]$ to Borel subsets of C_{end}:

$$K(t) := [C_{\text{start}} \cup D^*(t)] \cap C_{\text{end}}.$$

Lemma A.1. *If C_{start} and C_{end} are two bounded Borel subsets of $\mathbb{R}^d$ with $C_{start} \subsetneq C_{end}$, then the growing ball from C_{start} to C_{end} is an increasing knife function from C_{start} to C_{end}.*

Proof. Clearly, $K(0) = C_{\text{start}}$, $K(1) = C_{\text{end}}$, and K is increasing. It remains to show that K satisfies the continuity property of a knife function. For every two times t, t',

$$K(t) \ominus K(t') = [D^*(t) \ominus D^*(t')] \cap [C_{\text{end}} \setminus C_{\text{start}}].$$

So

$$\textsc{Volume}[K(t) \ominus K(t')] \le \textsc{Volume}[D^*(t) \ominus D^*(t')] = |\textsc{Volume}[D^*(t)] - \textsc{Volume}[D^*(t')]|.$$

Because $D^*(t)$ is a ball whose radius is a uniformly continuous function of t, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|t - t'| < \delta$ implies that the right-hand side is smaller than ϵ. □

Remark A.1. The growing ball function is not necessarily a "nice" knife function. For example, it may return disconnected pieces. It is useful mainly as a proof of existence.

A.2. Continuity of Value Covered by Knife

Lemma A.2. *If K is a knife function and V is an absolutely continuous measure, then $V \circ K$ is a uniformly continuous real function.*

Proof. We have to prove that, for every $\epsilon' > 0$, there exists $\delta > 0$ such that $|t - t'| < \delta$ implies $|V(K(t)) - V(K(t'))| < \epsilon'$.

Indeed, for every $\epsilon' > 0$, by the absolute continuity of V, there is an $\epsilon > 0$ such that $\textsc{Volume}[Z] < \epsilon$ implies $V(Z) < \epsilon'$. Given ϵ, by the definition of a knife function, there is a $\delta > 0$ such that $|t' - t| < \delta$ implies $\textsc{Volume}[K(t') \ominus K(t)] < \epsilon$, which implies $V(K(t') \ominus K(t)) < \epsilon'$.

By the additivity of V, for every two Borel sets A, B, $V(A) - V(B) \le V(A \setminus B)$. Therefore,

$$\begin{aligned}
|V(K(t')) - V(K(t))| &= \max(V(K(t')) - V(K(t)), V(K(t)) - V(K(t'))) \\
&\le \max(V(K(t') \setminus K(t)), V(K(t) \setminus K(t'))) \\
&\le V(K(t') \setminus K(t)) + V(K(t) \setminus K(t')) \\
&= V((K(t') \setminus K(t)) \cup (K(t) \setminus K(t'))) \\
&= V(K(t') \ominus K(t)) \\
&< \epsilon'.
\end{aligned}$$

So $V \circ K$ is a uniformly continuous real function.

A.3. *S*-Continuity of Knife Functions

In Section 3.3, we informally defined a knife function as S-continuous if "all S-pieces in $K_C(t)$ and $\overline{K_C}(t)$ grow or shrink continuously; no S-piece with a positive area is created or destroyed abruptly." We define this property formally here.

Definition A.2. A piece Z is called an ϵ-predecessor of a piece Z' if $Z \subseteq Z'$ and $\textsc{Volume}[Z' \setminus Z] < \epsilon$.

Definition A.3. Let S be a family of pieces. A knife function $K(t)$ is called S-continuous if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all t and t' having $|t' - t| < \delta$,

a. Every S-piece $Z_{t'} \subseteq K(t')$ has an ϵ-predecessor S-piece $Z_t \subseteq K(t)$.
b. Every S-piece $Z_{t'} \subseteq \overline{K}(t')$ has an ϵ-predecessor S-piece $Z_t \subseteq \overline{K}(t)$.

We now prove that S-continuity implies continuity of utility.

Lemma A.3. *If K_C is an S-continuous knife function and V is an absolutely continuous measure, then both $V^S \circ K_C$ and $V^S \circ \overline{K_C}$ are uniformly continuous real functions.*

Proof. Given $\epsilon' > 0$, we show the existence of $\delta > 0$ such that, for every t, t', if $|t' - t| < \delta$, then $|V^S(K(t')) - V^S(K(t))| < \epsilon'$.

Given ϵ', by the absolute continuity of V, there is an $\epsilon > 0$ such that

$$\textsc{Volume}[Z] < \epsilon \qquad \Longrightarrow \qquad V(Z) < \epsilon'. \tag{A.1}$$

Given ϵ, by the S-continuity of K, there is a $\delta > 0$ such that, if $|t' - t| < \delta$, then every S-piece $Z_{t'} \subseteq K(t')$ has an ϵ-predecessor S-piece $Z_t \subseteq K(t)$. This means that $Z_t \subseteq Z_{t'}$ and

$$\textsc{Volume}[Z_{t'} \setminus Z_t] < \epsilon,$$

which, by (A.1), implies

$$V(Z_{t'} \setminus Z_t) < \epsilon',$$

which, by additivity of V, implies

$$V(Z_t) > V(Z_{t'}) - \epsilon'.$$

The latter inequality is true for every S-piece $Z_{t'} \subseteq K(t')$, so it is also true for the supremum:

$$V(Z_t) > \sup_{Z_{t'} \in S, Z_{t'} \subseteq K(t')} V(Z_{t'}) - \epsilon'.$$

By definition, the S-value is the supremum, so

$$V(Z_t) > V^S(K(t')) - \epsilon'.$$

Because Z_t is an S-piece in $K(t)$, by definition of V^S, $V^S(K(t) \geq V(Z_t)$. Therefore,

$$V^S(K(t)) > V^S(K(t')) - \epsilon'.$$

By symmetric arguments (replacing the roles of t and t'), $V^S(K(t')) > V^S(K(t)) - \epsilon'$. Hence, $|V^S(K(t')) - V^S(K(t))| < \epsilon'$ as we wanted to prove.

An analogous proof applies to the function $V^S \circ \overline{K}$. □

A.4. Existence of S-Continuous Knife Functions

We show how S-continuous knife functions can be constructed. We denote by $H(r)$ the open half space of $\mathbb{R}^d$ defined by $x < r$.

Definition A.4. Let C be a bounded d-dimensional polytope in $\mathbb{R}^d$. Let $x_0 < x_1$ be real numbers such that C lies entirely to the right of the hyperplane $x = x_0$ and to the left of $x = x_1$ (they exist because C is bounded). Define $H^*(t) = H(x_0 + t \cdot (x_1 - x_0))$.

The sweeping-plane function on C is the following function from $[0, 1]$ to Borel subsets of C:

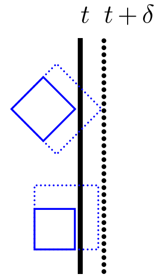
$$K(t) = C \cap H^*(t).$$

Lemma A.4. *Let C be a bounded d-dimensional object in $\mathbb{R}^d$. Let S be the family of d-dimensional cubes. The sweeping-plane function on C is an S-continuous increasing knife function from $\emptyset$ to C.*

Proof. Clearly, $K(0) = \emptyset$, $K(1) = C$, and K is increasing. The continuity of K can be proved exactly as in *Lemma A.1*. It remains to prove that K is S-continuous. To simplify the proof, we scale and translate C such that it is contained in the unit cube $[0, 1]^d$. In this case, we can take $x_0 = 0$ and $x_1 = 1$ so that $H^*(t) \equiv H(t)$.

The proof of S-continuity is based on the following geometric fact: For every cube $Z_{t'}$ contained in the half space $H(t + \delta)$, there exists a cube $Z_t \subseteq Z_{t'}$ contained in the half space $H(t)$ such that the side length of Z_t is smaller than that of $Z_{t'}$ by at most δ (it is smaller by exactly δ when $Z_{t'}$ is adjacent to the rightmost side of $H(t + \delta)$ and parallel to the axes; see Figure A.1 for an illustration of the two-dimensional case). Suppose $Z_{t'}$ is also contained in C. Because C is contained in the unit cube, the side length of $Z_{t'}$ is at most one. Therefore, the volume of Z_t is smaller than that of $Z_{t'}$ by at most $1 - (1 - \delta)^d \leq d \cdot \delta$.

Figure A.1. (Color online) Square continuity of the knife function defined in Lemma A.4. The solid line describes the knife location at time t; the dotted line describes its location at time $t + \delta$. The dotted squares are squares contained in $H(t + \delta)$; the solid squares are their predecessors in $H(t)$. At the bottom, the side length of the solid square Z_t is smaller than the dotted square $Z_{t+\delta}$ by exactly δ. At the top, the side length of the solid square Z_t is smaller than the dotted square $Z_{t+\delta}$ by less than δ.



Consider now the definition of S-continuity. For every $\epsilon > 0$, take $\delta := \epsilon/d$, let $t' = t + \delta$, and let $Z_{t'}$ be an S-piece contained in $K(t')$. By definition of K, $Z_{t'}$ is contained in both C and $H(t')$. By the geometric fact, $Z_{t'}$ has an ϵ-predecessor Z_t that is contained in $H(t)$. Because $Z_t \subseteq Z_{t'}$, it is also contained in C. Hence, it is contained in $K(t)$.

The S-continuity of $\overline{K}$ can be proved analogously. □

Using similar arguments, it is possible to prove that the sweeping-plane function on C is S-continuous also when S is the family of d-dimensional boxes or fat boxes.

A.5. Existence of Non-S-Continuous Knife Functions

We show how to prove that a knife function is *not* S-continuous.

Lemma A.5. *Let* $C := [0,1] \times [0,1]$ *and* K_C *be the following knife function on* C *(Figure 4(f)):*

$$K_C(t) = [0,t] \times [0,1] \cup [1-t, 1] \times [0,1].$$

Then, K_C *is not square continuous.*

Proof. Intuitively, a square of side length one is created at time $t = 0.5$ when the two components of $K_C(t)$ meet. Formally, let $\epsilon = 0.75$, and let's prove that there does not exist any δ satisfying the requirements of square continuity.

For every $\delta > 0$, select $t = 0.5 - \frac{\delta}{3}$ and $t' = 0.5 + \frac{\delta}{3}$. Then, $K_C(t')$ contains the square $Z' = [0,1] \times [0,1]$, but all squares $Z \subseteq K_C(t)$ have a side length of less than 0.5; hence, $\text{Volume}[Z' \setminus Z] > 0.75 = \epsilon$. □

Endnotes

[1] The reason why your partner decided to cut this way is irrelevant because a fair division algorithm is expected to ensure that the division is fair for every agent playing by the rules regardless of what the other agents do.

[2] In contrast, the simpler requirement that the pieces be rectangles with an arbitrary length/width ratio can easily be reduced to a one-dimensional requirement that the pieces are connected intervals. Such reduction is also possible for the requirements that the pieces be simplices (Ichiishi and Idzik [27]) or polytopes (Dall'Aglio and Maccheroni [21]) with an unbounded aspect ratio.

[3] $\lceil x \rceil$ denotes the *ceiling* of x—the smallest integer that is larger than x.

[4] We are grateful to Marco Dall'Aglio for his help in clarifying this issue.

[5] Throughout the paper, when we talk about "disjoint pieces," we allow the pieces to intersect in their boundary. We can ignore the question of which agent receives the boundary because the value of the boundary is zero for all agents.

[6] $\text{Volume}[Z]$ denotes the Lebesgue measure of a piece Z in $\mathbb{R}^d$. The symbol $\ominus$ denotes the symmetric set difference: For two sets A and B, $A \ominus B := (A \setminus B) \cup (B \setminus A)$.

[7] This is true for knife (c) too even though the pieces are not rectangular because no rectangle of positive area is created abruptly while the knife moves.

[8] The latter condition comes to ensure that there are no jumps in the S-value at $t = 1$. It implies that $K_C(1) \in S$.

[9] Because $K_C(0) = \emptyset$, for all $j \in \{1, \ldots, m\}$, $K^j(0) = \emptyset$, so $V_i^{K_C}(0) = V_i(\emptyset) = 0$.

[10] Because, by Definition 5, $\forall j \in \{1, \ldots, m\}$, $K^j(1) = K_C(1) = C_{\text{end}}$, so $V_i(K^j(1)) = V_i^S(K^j(1)) = V_i^S(C_{\text{end}})$, and so also $V_i^{K_C}(1) = V_i^S(C_{\text{end}})$. On the other hand, $\forall j \in \{1, \ldots, m\}$, $\overline{K_C}^j(1) \subseteq \overline{K}_C(1) = C \setminus C_{\text{end}}$. Because $\overline{K_C}^j(1)$ is an S-piece, $V_i(\overline{K_C}^j(1)) \le V_i^S(C \setminus C_{\text{end}})$. Because this is true for all $j \in \{1, \ldots, m\}$, also $V_i^{\overline{K_C}}(1) \le V_i^S(C \setminus C_{\text{end}})$. Now, by assumption, $V_i^S(C_{\text{end}}) \ge V_i^S(C \setminus C_{\text{end}})$.

[11] When $n = 3$, the three knives algorithm of Stromquist [49] can be used instead of Simmons' algorithm. See the conference version (Segal-Halevi et al. [45]) for details.

[12] In each square pair, the two squares should grow from opposite corners of the square in the cover. It does not matter which pair of opposite corners is used.

[13] Based on an answer by Christopher Fish here: http://math.stackexchange.com/a/1015267/29780.

References

[1] Agarwal PK, Katz MJ, Sharir M (1995) Computing depth orders for fat objects and related problems. *Comput. Geometry* 5(4):187–206.
[2] Aziz H, Mackenzie S (2016) A discrete and bounded envy-free cake cutting protocol for any number of agents. *2016 IEEE 57th Annual Sympos. Foundations Comput. Sci.* (IEEE, Piscataway, NJ), 416–427.
[3] Azrieli Y, Shmaya E (2014) Rental harmony with roommates. *J. Econom. Theory* 153:128–137.
[4] Barbanel JB, Brams SJ (2004) Cake division with minimal cuts: Envy-free procedures for three persons, four persons, and beyond. *Math. Soc. Sci.* 48(3):251–269.
[5] Barbanel JB, Brams SJ, Stromquist W (2009) Cutting a pie is not a piece of cake. *Amer. Math. Monthly* 116(6):496–514.
[6] Beck A (1987) Constructing a fair border. *Amer. Math. Monthly* 94(2):157–162.
[7] Bei X, Huzhang G, Suksompong W (2018) Truthful fair division without free disposal. *Proc. 27th Internat. Joint Conf. Artificial Intelligence* (AAAI Press, Menlo Park, CA), 63–69.
[8] Bei X, Chen N, Huzhang G, Tao B, Wu J (2017) Cake cutting: Envy and truth. *Proc. 26th Internat. Joint Conf. Artificial Intelligence* (AAAI Press, Menlo Park, CA), 3625–3631.
[9] Berliant M, Dunz K (2004) A foundation of location theory: Existence of equilibrium, the welfare theorems, and core. *J. Math. Econom.* 40(5): 593–618.

[10] Berliant M, Raa T (1988) A foundation of location theory: Consumer preferences and demand. *J. Econom. Theory* 44(2):336–353.
[11] Berliant M, Thomson W, Dunz K (1992) On the fair division of a heterogeneous commodity. *J. Math. Econom.* 21(3):201–216.
[12] Brams SJ, Taylor AD (1995) An envy-free cake division protocol. *Amer. Math. Monthly* 102(1):9–18.
[13] Brams SJ, Taylor AD (1996) *Fair Division: From Cake Cutting to Dispute Resolution* (Cambridge University Press, Cambridge, UK).
[14] Brams SJ, Jones M, Klamler C (2008) Proportional pie-cutting. *Internat. J. Game Theory* 36(3–4):353–367.
[15] Brams SJ, Taylor AD, Zwicker WS (1997) A moving-knife solution to the four-person envy-free cake division problem. *Proc. Amer. Math. Soc.* 125(2):547–554.
[16] Brânzei S, Miltersen PB (2015) A dictatorship theorem for cake cutting. *Proc. 24th Internat. Conf. Artificial Intelligence (IJCAI'15)* (AAAI Press, Menlo Park, CA), 482–488.
[17] Caragiannis I, Lai JK, Procaccia AD (2011) Toward more expressive cake cutting. *Proc. 22nd Internat. Joint Conf. Artificial Intelligence (IJCAI'11)* (AAAI Press), 127–132.
[18] Chambers CP (2005) Allocation rules for land division. *J. Econom. Theory* 121(2):236–258.
[19] Chen Y, Lai JK, Parkes DC, Procaccia AD (2013) Truth, justice, and cake cutting. *Games Econom. Behav.* 77(1):284–297.
[20] Cohler YJ, Lai JK, Parkes DC, Procaccia AD (2011) Optimal envy-free cake cutting. *Proc. 25th AAAI Conf. Artificial Intelligence* (AAAI Press, Menlo Park, CA), 626–631.
[21] Dall'Aglio M, Maccheroni F (2009) Disputed lands. *Games Econom. Behav.* 66(1):57–77.
[22] Deng X, Qi Q, Saberi A (2012) Algorithmic solutions for envy-free cake cutting. *Oper. Res.* 60(6):1461–1476.
[23] Dubins LE, Spanier EH (1961) How to cut a cake fairly. *Amer. Math. Monthly* 68(1):1–17.
[24] Hill TP (1983) Determining a fair border. *Amer. Math. Monthly* 90(7):438–442.
[25] Hüsseinov F (2011) A theory of a heterogeneous divisible commodity exchange economy. *J. Math. Econom.* 47(1):54–59.
[26] Hüsseinov F, Sagara N (2013) Existence of efficient envy-free allocations of a heterogeneous divisible commodity with nonadditive utilities. *Soc. Welfare* 41(4):923–940.
[27] Ichiishi T, Idzik A (1999) Equitable allocation of divisible goods. *J. Math. Econom.* 32(4):389–400.
[28] Iyer K, Huhns MN (2009) A procedure for the allocation of two-dimensional resources in a multiagent system. *Internat. J. Cooperative Inform. Systems* 18(3–4):381–422.
[29] Katz MJ (1997) 3-D vertical ray shooting and 2-D point enclosure, range searching, and arc shooting amidst convex fat objects. *Comput. Geometry* 8(6):299–316.
[30] Kurokawa D, Lai JK, Procaccia AD (2013) How to cut a cake before the party ends. *Proc. 27th AAAI Conf. Artificial Intelligence* (AAAI Press, Menlo Park, CA), 555–561.
[31] Legut J, Potters JAM, Tijs SH (1994) Economies with Land—A game theoretical approach. *Games Econom. Behav.* 6(3):416–430.
[32] LiCalzi M, Nicolò A (2009) Efficient egalitarian equivalent allocations over a single good. *Econom. Theory* 40(1):27–45.
[33] Maccheroni F, Marinacci M (2003) How to cut a pizza fairly: Fair division with decreasing marginal evaluations. *Soc. Choice Welfare* 20(3): 457–465.
[34] Manabe Y, Okamoto T (2010) Meta-envy-free cake-cutting protocols. Hliněný P, Kučera A, eds. *Mathematical Foundations of Computer Science 2010*, Lecture Notes in Computer Science, vol. 6281 (Springer, Berlin, Heidelberg), 501–512.
[35] Mirchandani RS (2013) Superadditivity and subadditivity in fair division. *J. Math. Res.* 5(3):78–91.
[36] Nicolò A, Yu Y (2008) Strategic divide and choose. *Games Econom. Behav.* 64(1):268–289.
[37] Nicolò A, Perea y Monsuwe A, Roberti P (2012) Equal opportunity equivalence in land division. *SERIES J. Spanish Econom. Assoc.* 3(1–2): 133–142.
[38] Procaccia AD (2015) Cake cutting algorithms. Brandt F, Conitzer V, Endriss U, Lang J, Procaccia AD, eds. *Handbook of Computational Social Choice* (Cambridge University Press, New York), 261–283.
[39] Reijnierse JH, Potters JAM (1998) On finding an envy-free Pareto-optimal division. *Math. Programming* 83(1–3):291–311.
[40] Robertson JM, Webb WA (1998) *Cake-Cutting Algorithms: Be Fair if You Can*, 1st ed. (A K Peters/CRC Press, Natick, MA).
[41] Saberi A, Wang Y (2009) Cutting a cake for five people. Goldberg AV, Zhou Y, eds. *Algorithmic Aspects in Information and Management*, Lecture Notes in Computer Science, vol. 5564 (Springer, Berlin, Heidelberg), 292–300.
[42] Sagara N, Vlach M (2009) Representation of preference relations on sigma-algebras of nonatomic measure spaces: Convexity and continuity. *Fuzzy Sets Systems* 160(5):624–634.
[43] Segal-Halevi E (2018) Redividing the cake. *Proc. 27th Internat. Joint Conf. Artificial Intelligence* (AAAI Press, Menlo Park, CA), 498–504.
[44] Segal-Halevi E, Sziklai BR (2018) Monotonicity and competitive equilibrium in cake-cutting. *Econom. Theory* 68(2):363–401.
[45] Segal-Halevi E, Hassidim A, Aumann Y (2015) Envy-free cake-cutting in two dimensions. *Proc. 29th AAAI Conf. Artificial Intelligence* (AAAI Press, Menlo Park, CA), 1021–1028.
[46] Segal-Halevi E, Hassidim A, Aumann Y (2016) Waste makes haste: Bounded time algorithms for envy-free cake cutting with free disposal. *ACM Trans. Algorithms (TALG)* 13(1):12:1–12:32.
[47] Segal-Halevi E, Nitzan S, Hassidim A, Aumann Y (2017) Fair and square: Cake-cutting in two dimensions. *J. Math. Econom.* 70:1–28.
[48] Steinhaus H (1948) The problem of fair division. *Econometrica* 16(1):101–104.
[49] Stromquist W (1980) How to cut a cake fairly. *Amer. Math. Monthly* 87(8):640–644.
[50] Stromquist W (2008) Envy-free cake divisions cannot be found by finite protocols. *Electr. J. Combinatorics* 15(1):#R11.
[51] Su FE (1999) Rental harmony: Sperner's lemma in fair division. *Amer. Math. Monthly* 106(10):930–942.
[52] Thomson W (2007) Children crying at birthday parties. Why? *Econom. Theory* 31(3):501–521.
[53] Webb WA (1990) A combinatorial algorithm to establish a fair border. *Eur. J. Combinatorics* 11(3):301–304.
[54] Weller D (1985) Fair division of a measurable space. *J. Math. Econom.* 14(1):5–17.