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Theories of Automata on ω-Tapes: A Simplified Approach

YAACOV CHOUEKA

Mathematics Department, Bar-Ilan University, Ramat Gan, Israel

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Using a combinatorial lemma on regular sets, and a technique of attaching a control unit to a parallel battery of finite automata, a simple and transparent development of McNaughton's theory of automata on ω-tapes is given. The lemma and the technique are then used to give an independent and equally simple development of Büchi's theory of nondeterministic automata on these tapes. Some variants of these models are also studied. Finally a third independent approach, modelled after a simplified version of Rabin's theory of automata on infinite trees, is developed.

1. INTRODUCTION AND MOTIVATION[1]

In the pioneering work of Büchi [1] and Elgot [5], the fundamental relationship between finite automata and second-order monadic calculi was discovered, and was used for a thorough study of the (weak) second-order monadic theory of the successor function (and especially for solving its decision problem). Since then, interest has been aroused in developing theories of finite automata which act on more general structures than finite tapes, such as, for example, finite trees or infinite, even transfinite, tapes. By developing such a theory we mean: defining some sort of device which "accepts" or "rejects" such a structure; studying the closure of the family of all sets which are defined by such devices under various set-theoretical operations, and, in particular, under union, intersection, complementation, projection and cylindrification; and finding an algorithm which effectively decides, for any given device of this sort, whether the set of structures defined by it is empty or not.

Experience has shown that for "natural" generalizations of finite automata, closure under union and cylindrification is quite direct; also intersection can be reduced as usual, by using DeMorgan's laws, to union and complementation. Thus the problem really focuses on the complementation and projection operations. Now if one chooses

[1] The introduction presupposes some knowledge of finite automata theory terminology and results; all the terms used in it are, nevertheless, defined in the paper (and, in particular, in Section 2). The reader who is not familiar with the "folklore" of this subject is invited to reread the introduction after reading the paper itself.

a deterministic approach, that is an approach in which the moves of the automaton at each given time t are uniquely determined by its present state and the present input (and, moreover, the "state" of the automaton when reaching the end of the structure is also uniquely determined), then closure under complementation becomes quite trivial. Closure under projection remains then the main fact to prove, and this is in general equivalent to showing that the nondeterministic devices are equivalent to (i.e., not more powerful than) the deterministic ones. On the other hand, in a nondeterministic approach, closure under projection is trivial, and complementation is the real problem.

The first step in building such theories was made by Büchi in [2]. Here he developed a theory of nondeterministic automata, call them B-automata, on tapes of length ω, and proved that the complement of a set definable by a B-automaton is also definable by a B-automaton. The proof made essential use of the Ramsey lemma; because of that, apparently, it was not suitable for generalization to other structures.

Somewhat later, and working in a totally different context, Muller [8] suggested another approach, essentially deterministic in nature; the theory developed there, however, contained an error. The idea was picked up by McNaughton, who, in the important paper [6], developed a correct version of Muller's approach. Three essential points characterize McNaughton's proof. First, new and powerful techniques for combining automata in parallel, and attaching a "control unit" to them, were introduced. As was shown in [10] and will be also exhibited in the present paper, these fruitful ideas can be successfully applied in a variety of different situations. Second, Büchi's result follows immediately from the principal theorem in [6], while the converse is not true; this shows that the deterministic theory developed in [6] is more powerful, in some sense, than that developed in [2]. Third, as remarked by Büchi in [3] (and, independently, about the same time, but in a somewhat different and weaker version, by the author; see [4]), McNaughton's ideas can be quite directly generalized to tapes of denumerable length, thus solving one of the problems raised in [2].

Unfortunately, McNaughton's original and ingenious proof is somewhat complicated, and even after following its numerous details, the proof as a whole is not easily grasped. Moreover, it contains some inaccuracies which, when properly remedied, make it even more complicated. In this paper we present, among other results, a new and simple version of the proof of the main theorem in [6], which, though essentially based on McNaughton's ideas, has nevertheless a very transparent structure, and can be explained in a few lines.

Let us call a set W of ω-tapes *finitely definable* if there exists an automaton, in the sense of [6], which defines it. As is clear from [6], the main point is to prove that if U and V are regular sets of finite tapes, then UV^{ω} is finitely definable. As will be shown, this result follows immediately from a "flag-construction" and a "basic lemma," which are, interestingly enough, totally independent of any model whatsoever of automata on infinite tapes. The basic lemma (see 5.2) states that for any regular set of tapes V, one can effectively find some regular $\tilde{V}$ such that $V^{\omega} = V^{*}(\lim \tilde{V})$, where $\lim \tilde{V}$ is the set of all ω-tapes which have an infinite number of initial sections in $\tilde{V}$. The lemma (which is perhaps interesting in itself) can be proved by using Ramsey's lemma; one can avoid, however, the use of Ramsey's lemma at a little extra cost (both variants are given in the paper). Now, using the flag-construction, which will be described in the next paragraph, one can show that if U is regular and W is finitely definable, then UW is also finitely definable. Since the limit of a regular set is (trivially) finitely definable, it follows that $UV^{\omega} = UV^{*}(\lim \tilde{V})$ is finitely definable too.[2]

As for the flag-construction, let us consider first automata working on finite tapes. The usual proof, in that case, of the closure of the family of regular sets under product (and star) operations is essentially nondeterministic, and only via the "subset construction" [9, Theorem 11] is a deterministic automaton provided by the proof. If we wish to avoid this construction, which cannot be generalized to infinite tapes, then a direct deterministic proof can be given along the following lines. Suppose U, V are regular sets of finite tapes defined by $\mathscr{A}$ and $\mathscr{B}$, respectively. Take $n + 2$ copies of $\mathscr{B}$ (where n is the number of states of $\mathscr{B}$) and connect them in parallel with $\mathscr{A}$. At the initial state, $\mathscr{A}$ is in its initial state, and all $\mathscr{B}$ copies are switched off. A control unit C switches on a "dormant" copy of $\mathscr{B}$ each time that $\mathscr{A}$ is in an acceptable state, and, at every time t, switches off all copies of $\mathscr{B}$ which are in the same state, except for the copy with the least index. The acceptable states of this configuration $\mathscr{C}$ are those in which at least one of $\mathscr{B}$ copies is in an acceptable state. It is quite clear that $\mathscr{C}$ defines UV. This construction, and the corresponding proof, carries over word for word to the infinite case too.

The basic lemma is given in Section 5; the flag-construction is described, together with some other techniques for combining tables (which work equally well for finite or infinite tapes) in Section 3. In Section 4, a deterministic theory of automata on finite tapes is developed. All of the results presented there are certainly well-known and already classical; there may be, none-the-less, some novelty in one or two points of the presentation. Thus, closure under projection (or the equivalence between deterministic and nondeterministic automata) is proved as a consequence of the Kleene synthesis and analysis theorem, together with the deterministic proof of the closure under product and star operations. Also the solution of the decision problem of the emptiness or finiteness of the set defined by an automaton $\mathscr{A}$, is formulated in a rather "unorthodox" algorithm, which is perhaps very uneconomical, but has the advantage of being easily stated, proved and generalized to the infinite case. All the presentation is aimed at showing the quite complete parallelism between the cases of finite and infinite tapes.

[2] The nucleus of this simplified proof already appears in the author's doctoral dissertation [4]. It has been reproduced, with some minor changes, by Rabin in his report [11]. The present version seems to be the simplest one.

The latter case is dealt with in Section 6, where three different models are introduced, and the deterministic theory of the Müller–McNaughton automata is developed. This theory is then used to prove the equivalence of the three approaches, and to analyze the structure of the sets defined by these models and some of their variants. In Section 7 we show how an independent nondeterministic theory of B-automata can be simply and directly derived from the flag-construction and the basic lemma, in much the same way as the deterministic theory was derived.

Finally we give in Section 8 a third approach, modelled after the methods of [10]. In this paper, Rabin developed a rather complex theory of automata on infinite trees, which used both Büchi's and McNaughton's results, together with new techniques of "well-founded mappings" and "simultaneous runs." Part of the difficulty of Rabin's original approach is inherent to the special method used by him to analyze the structure of the set of trees defined by a given automaton, while the other part is only technical, and has to do with the fact that we are dealing with binary trees, which have an infinite number of branches, instead of with "unary trees" (tapes) which have only a unique branch. In [4, Chaps. 7, 8] the author has shown how to develop a Rabin approach to a theory of automata on infinite tapes which is independent of both McNaughton's and Büchi's results, and in which ordinary transfinite induction replaces well-founded mappings, and flag sets replace simultaneous runs. The main point in [4] was to develop the theory in such a way that the cumbersome notation associated with trees could be avoided, while retaining all the essential points of the proof, so that it could be immediately generalized to infinite trees.[3] In contrast, full advantage is taken, in the proof given in Section 8, of the fact that we are dealing here with tapes rather than trees. In particular we use the remark, due to D. Cohen, that transfinite induction is not necessary in this case, since the sequence of sets $B_\alpha(s)$ (see [4, Lemma 7.21]) collapses after a finite number of steps.

The paper is written so as to be wholly self-contained; all needed terminology and definitions are presented in Section 2. Nevertheless, we feel that it can be fully "tasted" only by a reader who has tried to go through all the details of at least one of the papers [2, 6, or 10]. For such a reader, definition 3.9 and Lemma 5.2 will be quite sufficient for a quick proof of the principal result of [6], while Section 8 will provide an adequate simple version of some of the main ideas of [10].

2. Notation and Terminology: Sequences, Tapes and Tables

Notation. Union, intersection and complementation of sets will be denoted as usual by $\cup$, $\cap$, $\overline{}$, respectively. For a given set A, $P(A)$ is the power-set of A, and $\| A \|$ the number of elements of A. A is a *singleton* set if $\| A \| = 1$.

[3] The original proof in [4] contained an error which was pointed out to me by D. Cohen. A corrected version will appear elsewhere.

N is the set of nonnegative integers; ω is the first infinite ordinal; i, j, k, l, m, n are variables ranging over N. Sometimes n, m will be taken to range over $N \cup \{\omega\}$, in which case we state explicitly that $n, m \leqslant \omega$.

Sequences and Cartesian Products

Let A be a given set. A *sequence of length $n \leqslant \omega$ on A* is a function φ: $\{i: i < n\} \to A$; φ will be also written as $(\varphi(i))$, the explicit reference to the length of φ being generally omitted. For such a sequence φ we let $\varphi(*)$ be the last element of φ if $n < \omega$, and the set of all elements of A which appear infinitely often in φ, if $n = \omega$.

Given k sets $A_1, \ldots, A_k$ let $A = \prod_j A_j$ be their cartesian product. The *j-projection* of A $(1 \leqslant j \leqslant k)$ is the function $p_j : A \to A_j$ defined by $p_j(a_1, \ldots, a_n) = a_j$. p_j can be naturally extended to a function from $P(A)$ to $P(A_j)$ by letting $p_j(b) = \{p_j(a) : a \in b\}$, for every $b \in P(A)$. Similarly, if φ is a sequence of length $n \leqslant \omega$ on A, then its *j-projection* $p_j(\varphi)$ is the sequence of length n on A_j defined by $p_j(\varphi)(i) = p_j(\varphi(i))$. Conversely, if φ_j are sequences of length $n \leqslant \omega$ on A_j, for $1 \leqslant j \leqslant k$, then their *product* is the sequence $\varphi = \prod_j \varphi_j$ on A defined by $\varphi(i) = (\varphi_1(i), \ldots, \varphi_k(i))$ for every $i < n$.

Clearly $p_m(\prod_j \varphi_j) = \varphi_m$, and $\prod_j p_j(\varphi) = \varphi$.

The following fact, whose proof is elementary, is essential for all the subsequent development.

FACT 2.1. *Let $A_1, \ldots, A_k$ be finite nonempty sets, $A = \prod_j A_j$ their product, and φ a sequence of length $n \leqslant \omega$ on A. Then for every $1 \leqslant j \leqslant k$, $p_j(\varphi(*)) = p_j(\varphi)(*)$.*

Proof. The assertion is trivial for $n < \omega$; so assume $n = \omega$. That $p_j(\varphi(*)) \subseteq p_j(\varphi)(*)$ is immediate. Take now any $a' \in p_j(\varphi)(*)$; a' appears infinitely often in $p_j(\varphi)$, so it appears there, infinitely often, as a j-projection of elements $\varphi(i)$ of φ. All the elements of φ are however from A which is finite. There is then some $a \in A$ such that $p_j(a) = a'$ and a appears infinitely often in $(\varphi(i))$, which shows that $a \in \varphi(*)$ and $a' \in p_j(\varphi(*))$.

Q.E.D.

Tapes

Let Σ be a given finite nonempty set which will be called the *alphabet*. *A tape X of length $n \leqslant \omega$ on Σ* is a sequence of length n on Σ; $X(i)$ will be also denoted by x_i, and the tape X by (x_i). $|X|$ is the length of X. Tapes will be denoted by $X, Y, Z, \ldots$; Λ is the unique tape of length zero. We also put $\Sigma^* = \{X : |X| < \omega\}$, $\Sigma^+ = \Sigma^* - \{\Lambda\}$, $\Sigma^\omega = \{X : |X| = \omega\}$ and $\Sigma(\omega) = \{X : |X| \leqslant \omega\} = \Sigma^* \cup \Sigma^\omega$.

For $X \in \Sigma^*$ and $Y \in \Sigma(\omega)$, where $|X| = n$, we define the *product* of X and Y as the tape $Z = XY$ obtained by concatenating X and Y, i.e., Z is given by $Z(i) = X(i)$ for $0 \leqslant i < |X|$, $Z(n + i) = Y(i)$ for $0 \leqslant i < |Y|$ $(X\Lambda = \Lambda X = X)$. For a given X and $0 \leqslant i \leqslant j \leqslant |X|$ we let $X(i, j)$ be the restriction of X to the set $\{k: i \leqslant k < j\}$. In particular, $X(i, i) = \Lambda$, $X(0, |X|) = X$ and for $i \leqslant j \leqslant k$,

$X(i, j)\, X(j, k) = X(i, k)$. Y is a *section* of X if $Y = X(0, i)$ for some $0 \leqslant i \leqslant |X|$, i.e., if $X = YZ$ for some $Z \in \Sigma(\omega)$. This induces a partial ordering on $\Sigma(\omega)$ which will be denoted by $\leqslant$.

Sets of tapes will be denoted by $U, V, W, \ldots$; V is a *trivial* set if $V = \varnothing$ or $V = \{\Lambda\}$. For $U \subseteq \Sigma^*$ and $V \subseteq \Sigma(\omega)$, the *product* of U and V is the set $UV = \{XY : X \in U, Y \in V\}$. For $U \subseteq \Sigma^*$, we also put $U^0 = \{\Lambda\}$, $U^{n+1} = U^n U$, $U^* = \bigcup_0^\omega U^n$ and $U^+ = \bigcup_1^\omega U^n$; U^* is the *star* of U ($\varnothing^* = \{\Lambda\}$).

Given two alphabets Σ and Σ', we call a function $f\colon \Sigma \to \Sigma'$ a *projection* from Σ to Σ'. Such a function can be naturally extended to a function $f\colon \Sigma(\omega) \to \Sigma'(\omega)$ by letting $f(X) = (f(x_i))$ for $X = (x_i)$ in $\Sigma(\omega)$. Also for $V \subseteq \Sigma(\omega)$ we put $f(V) = \{f(X) : X \in V\}$. Conversely, if $V' \subseteq \Sigma'(\omega)$, then its *cylindrification* $f^{-1}(V')$ is the set $V = \{X : f(X) \in V'\}$. Naturally $f(f^{-1}(V')) = V'$; in general, however, we have only $f^{-1}(f(V)) \supseteq V$.

Tables

A *table* T on Σ is a triple $T = \langle S, M, S^* \rangle$ where S is a finite set (the set of *states*), M is a function from $S \times \Sigma$ to $P(S)$ (the *transition* function) and $S^* \subseteq S$ is the set of *initial states*. Given a tape $X \in \Sigma(\omega)$ and a state $s \in S$, we define an *s-run* of T on X as any sequence $\varphi = (s_i)$ of length $1 + |X|$ of states, which satisfy $s_0 \in S^*$, $s_{i+1} \in M(s_i, x_i)$ for every $i < |X|$. The set of all s-runs of T on X will be denoted by $R_s(T, X)$. We put $R(T, X) = \bigcup_{s \in S^*} R_s(T, X)$; elements of $R(T, X)$ will be called just *runs* of T on X.

A table $T = \langle S, M, S^* \rangle$ is *deterministic* if S^* and $M(s, \sigma)$ are singleton sets (for any $s \in S$, $\sigma \in \Sigma$). For such a table, and for $s \in S$ and $X \in \Sigma(\omega)$, it is obvious that $R_s(T, X)$ is a singleton set. The function M can be naturally extended in this case to a function from $S \times \Sigma(\omega)$ to $S \cup P(S)$ by letting $M(s, X) = \varphi(*)$ where φ is the unique element of $R(T, X)$. By definition, $M(s, \Lambda) = s$, and it is easy to verify that for $s \in S$, $X \in \Sigma^*$ and $Y \in \Sigma(\omega)$, $M(s, XY) = M(M(s, X), Y)$. By adjoining to a given table T some mechanism for "accepting" tapes X (the acceptance's test being based in general on $\varphi(*)$, for $\varphi \in R(T, X)$) one gets various models of finite automata working on finite or infinite tapes. Before going into the details of such models, however, we want to gather in the next section a number of techniques and results on tables and runs which will be valid for any such model.

3. Operations on Tables

Definition 3.1. Given k tables $T_j = \langle S_j, M_j, S_j^* \rangle$ on Σ (where we assume $S_j \cap S_{j'} = \varnothing$ for $j \neq j'$) we define their *union* $\bigcup_j T_j$ as the table $T = \langle S, M, S^* \rangle$ where $S = \bigcup_j S_j$, $S^* = \bigcup_j S_j^*$ and $M = \bigcup_j M_j$, i.e. $M(s, \sigma) = M_j(s, \sigma)$ for every $s \in S_j$ and $\sigma \in \Sigma$ $(1 \leqslant j \leqslant k)$.

PROPERTY 3.2. *If $T = \bigcup_j T_j$ then for any $X \in \Sigma(\omega)$, $R(T, X) = \bigcup_j R(T_j, X)$.*

DEFINITION 3.3. Let f be a given projection from Σ to Σ'. If $T = \langle S, M, S^* \rangle$ is a table on Σ then its *projection* $f(T)$ is the table $T' = \langle S, M', S^* \rangle$ on Σ' where $M'(s, \sigma') = \{M(s, \sigma) : f(\sigma) = \sigma'\}$.

On the other hand if $T' = \langle S, M', S^* \rangle$ is a [deterministic] table on Σ', then its *cylindrification* $f^{-1}(T')$ is the [deterministic] table $T = \langle S, M, S^* \rangle$ on Σ where $M(s, \sigma) = M'(s, f(\sigma))$.

PROPERTY 3.4. *Let f be a projection from Σ to Σ'.*

(a) *If T is a table on Σ and $X' \in \Sigma'(\omega)$ then*

$$R(f(T), X') = \bigcup \{R(T, X) : f(X) = X'\}.$$

(b) *If T' is a table on Σ' and $X \in \Sigma(\omega)$ then $R(f^{-1}(T'), X) = R(T', f(X))$.*

DEFINITION 3.5. Let $T = \langle S, M, S^* \rangle$ be a nondeterministic table on Σ. The *deterministic image* of T is the deterministic table $T' = \langle S', M', s'^* \rangle$ on $\Sigma' = \Sigma \times S$, defined as follows. $S' = S \cup \{0, 1\}$ (where 0, 1 are two new states), $s'^* = 0$, $M'(s_1, (\sigma, s_2)) = s_2$ if $s_2 \in M(s_1, \sigma)$, $M'(0, (\sigma, s)) = s$ if $s \in \bigcup_{s' \in S^*} M(s', \sigma)$, and $M'(s', (\sigma, s'')) = 1$ in all other cases.

Given a tape $X \in \Sigma(\omega)$ and a sequence $\varphi \in S(\omega)$ of length $1 + |X|$, we let $X' = (X, \varphi)$ be the tape in $\Sigma'(\omega)$ defined by $X'(i) = (X(i), \varphi(i + 1))$ for $0 \leqslant i < |X|$.

PROPERTY 3.6. *Let $T = \langle S, M, S^* \rangle$ be a nondeterministic table on Σ, and $T' = \langle S', M', s'^* \rangle$ its deterministic image on $\Sigma' = \Sigma \times S$. Then for any $X \in \Sigma(\omega)$:*

$$R(T, X) = \{\varphi : \varphi \in S(\omega), |\varphi| = 1 + |X|, M'(s'^*, (X, \varphi)) = \varphi(*)\}.$$

DEFINITION 3.7. Given k [deterministic] tables $T_j = \langle S_j, M_j, S_j^* \rangle$ on Σ, we define their *product* $\prod_j T_j$ as the [deterministic] table $T = \langle S, M, S^* \rangle$ given by: $S = \prod_j S_j$, $S^* = \prod_j S_j^*$ and $M = \prod_j M_j$ is defined "by coordinates", that is $M((s_1, \ldots, s_k), \sigma) = \prod_j M_j(s_j, \sigma)$.

PROPERTY 3.8. *If $T = \prod_j T_j$ and $X \in \Sigma(\omega)$ then $R(T, X) = \prod_j R(T_j, X)$ (More explicitly: $\varphi \in R(T, X)$ if and only if $\varphi = \prod_j \varphi_j$ where $\varphi_j \in R(T_j, X)$ for every $1 \leqslant j \leqslant k$).*

The verification of properties 3.2, 3.4, and 3.6 is straightforward. Property 3.8 follows immediately from 2.1.

Informally speaking, when we form the product of some given tables we are in fact connecting them in parallel, letting the input tape enter them simultaneously, and allowing each table to act on this input, independently of the others. It will be useful to somewhat enrich this construction by attaching to this parallel battery of tables a "control unit" which may switch On or Off some of the components, according to simple given criteria. This led us to the "flag-construction," which we first informally describe.

Anticipating somewhat the notions to be developed in the next section, suppose we are given a deterministic "finite automaton" $\mathscr{A}$, i.e., a table $T = \langle S, M, s^* \rangle$ together with a set $F \subseteq S$ of "final" states; a tape $X \in \Sigma^*$ is accepted by $\mathscr{A}$ if $M(s^*, X) \in F$. Suppose we are also given some table T' on Σ with n states. We take $n + 2$ copies of T', number them 1 to $n + 2$, connect them in parallel with $\mathscr{A}$, and add to this battery a "control unit" C. Each of T' copies may be either *dormant*, in which case it is insensitive to the input and remains so until it is switched On by C, or *active* in which case it acts according to the table T', and remains so until it is switched Off and made dormant by C.

The configuration works as follows. In the initial state, $\mathscr{A}$ is in its initial state, and all T' copies are dormant. At each time t, C checks for all T' copies which are in the same state, and switches them Off, except for the one with the least index. It then checks $\mathscr{A}$ state to see whether it is a final state; if so, it switches On one of the dormant copies (of T').

Suppose that at some time t, n of T' copies are active and in different states, and that, further, C switches On then one of the dormant copies. At $t + 1$, there will be only $t + 1$ active copies, so that one copy is still available for switching On by C if necessary. Moreover, one (at least) of the active copies at $t + 1$ is then made dormant; thus we are assured that C will never have to "simultaneously" switch Off and then On some active copy.

Finally we remark that if T' is deterministic, then this configuration is deterministic too.

The formalization of this structure, and the way it works, should be now quite obvious; we give it in the following definitions.

DEFINITION 3.9. Given a deterministic finite automaton $\mathscr{A} = \langle S, M, s^*, F \rangle$ on Σ and a [deterministic] table $T' = \langle S', M', S' \rangle$ with n states on Σ, we define the *flag-table of $\mathscr{A}$ relative to T'*, to be denoted by $\mathrm{fl}(\mathscr{A}, T')$, as the [deterministic] table $T'' = \langle R, P, r^* \rangle$ constructed as follows.

Add to S' a new *dormant* state 0 and extend M' on it by $M'(0, \sigma) = \{0\}$ for every $\sigma \in \Sigma$. Let $G = (S' \cup \{0\})^k - S'^k$, where $k = n + 2$, and put $R = S \times G$, $r^* = (s^*, 0, \ldots, 0)$. States r of R will be written as $r = (s, g)$ where $g' = (g_1, \ldots, g_k)$; for $1 \leqslant j \leqslant k$, $g_j = p_j(r)$ is *the state of copy j*, and $s = p_0(r)$ is the state of the *driving automaton* $\mathscr{A}$.

For $s \notin F$, we put $P((s, g), \sigma) = \{(s', g') \colon s' = M(s, \sigma), g_j' \in M'(g_j'', \sigma), \text{ for } 1 \leqslant j \leqslant k\}$, where $g_j'' = g_j$, unless there is some $m < j$ such that $g_j = g_m$, in which case $g_j'' = 0$.

For $s \in F$, $P((s, g), \sigma)$ is as before, except that we now require $g'_{j_0} \in \bigcup_{s \in S^*} M'(s, \sigma)$, where j_0 is the least index j for which $g_j = 0$ (by definition, there is always such an index).

We now introduce some suggestive terminology.

Let $X \in \Sigma(\omega)$ and $\varphi'' \in R(T'', X)$. Given some $i < | X |$, let $\varphi''(i) = (s, g)$, $\varphi''(i + 1) = (s', g')$. We say that copy j is *switched On* at i if $g_j = 0, g_j' \neq 0$; *switched Off* if $g_j \neq 0, g_j' = 0$; *dormant* if $g_j = g_j' = 0$; *active* if $g_j \neq 0, g_j' \neq 0$. If copy j is switched Off at i and j' is the index of the unique copy which is active at i and is in the same state as copy j, then we say that j has been switched Off *because* of j'.

Suppose now that some copy j_0 has been switched On at i_0. Its *representative* at $i > i_0$, relative to i_0, to be denoted by $\operatorname{rep}(j_0, i_0, i)$ is defined as follows: $\operatorname{rep}(j_0, i_0, i_0 + 1) = j_0$, and $\operatorname{rep}(j_0, i_0, i + 1) = \operatorname{rep}(j_0, i_0, i)$, unless this right-hand copy has been switched Off at i because of j', in which case $\operatorname{rep}(j_0, i_0, i + 1) = j'$. Since $\operatorname{rep}(j_0, i_0, i)$ is a nonincreasing sequence of numbers bounded by j_0, there is some $j^* \leqslant j_0$ such that, ultimately $\operatorname{rep}(j_0, i_0, i)$ is constant and equal to j^*. We call j^* the *ultimate representative of j_0 relative to i_0*, and denoted it by $\operatorname{urep}(j_0, i_0)$. Finally we define *the virtual run of j_0 relative to i_0* (and, of course, relative to φ'') as the sequence $\varphi' = \nu(j_0, i_0, \varphi'')$ defined by $\varphi'(0) = s'$ where s' is any state for which $p_{j_0}(\varphi''(i_0 + 1)) \in M(s', x_{i_0})$, and, for $i > 0$, $\varphi'(i) = p_{j'}(\varphi''(i + i_0))$, where $j' = \operatorname{rep}(j_0, i_0, i)$. It is clear that $\varphi' \in R(T', X(i_0, | X |))$, and $\varphi'(*) = p_{j^*}(\varphi''(*))$, where $j^* = \operatorname{urep}(j_0, i_0)$.

Conversely, suppose $X = YZ$ where $Y = X(0, i)$ is accepted by $\mathscr{A}$, and let $\varphi' \in R(T', Z)$. There is then some $\varphi'' \in R(T'', X)$ and some $1 \leqslant j_0 \leqslant k$ such that $\varphi' = \nu(j_0, i_0, \varphi'')$. (In fact, φ'' may be arbitrarily defined, as a run, up to and including i_0; if j_0 is the copy switched On at i_0, then φ'' need by defined only for copy j_0 and at $i > i_0$ where j_0 is active; but then it can be defined according to φ').

We have thus proved the following.

PROPERTY 3.10. *Let $\mathscr{A}$ be a deterministic finite automaton, T' a table, $T'' = \mathrm{fl}(\mathscr{A}, T')$ and $X \in \Sigma(\omega)$. If $\varphi'' \in R(T'', X)$, copy j_0 is switched On at i_0, $Y = X(0, i_0)$ and $Z = X(i_0, | X |)$, then Y is accepted by $\mathscr{A}$, $\varphi' = \nu(j_0, i_0, \varphi'') \in R(T', Z)$, and $\varphi'(*) = p_{j^*}(\varphi''(*))$. On the other hand if $X = YZ$ where $Y = X(0, i_0)$ is accepted by $\mathscr{A}$, then for every $\varphi' \in R(T', Z)$ there is some $\varphi'' \in R(T'', X)$ and some $1 \leqslant j_0 \leqslant k$ such that $\varphi' = \nu(j_0, i_0, \varphi'')$ and $\varphi'(*) = p_{j^*}(\varphi''(*))$. (In both cases, $j^* = \operatorname{urep}(j_0, i_0)$).*

4. Finite Automata on Finite Tapes

DEFINITION 4.1. *A finite [deterministic] automaton on Σ* is a pair $\mathscr{A} = \langle T, F \rangle$ where $T = \langle S, M, S^* \rangle$ is a [deterministic] table, and $F \subseteq S$ is a set of specified *final* states. $X \in \Sigma^*$ is *accepted* by $\mathscr{A}$, in symbols: $X \in \mathrm{T}(\mathscr{A})$, if there is some $\varphi \in R(T, X)$ such that $\varphi(*) \in F$. $\mathrm{T}(\mathscr{A})$ is the set *defined* by $\mathscr{A}$. The collection of sets of tapes on Σ

defined by [deterministic] finite automata will be denoted by $\mathbf{F}(\Sigma)$ [$\mathbf{DF}(\Sigma)$, respectively].

THEOREM 4.2. *$\mathbf{DF}(\Sigma)$ is closed under complementation, union, intersection, product and star operations; it also contains all cylindrifications of sets in $\mathbf{DF}(\Sigma')$ for any alphabet Σ'.*

Proof. (All automata considered in the proof will be assumed to be deterministic).

If $V = \mathrm{T}(\mathscr{A})$ where $\mathscr{A} = \langle T, F\rangle$ then $\bar{V} = \mathrm{T}(\mathscr{A}')$ where $\mathscr{A}' = \langle T, \bar{F}\rangle$. If $V_j = \mathrm{T}(\mathscr{A}_j)$ for $1 \leqslant j \leqslant k$, then $\bigcup_j V_j$ is defined by $\mathscr{A} = \langle T, F\rangle$ where $T = \prod_j T_j$, and $f \in F$ if and only if $p_j(f) \in F_j$ for some $1 \leqslant j \leqslant k$ (use 2.1 and 3.8). If $V_j \in \mathbf{DF}(\Sigma)$ and $W = \bigcap_j V_j$ then $\overline{W} = \bigcup_j \overline{V_j}$ so that $W \in \mathbf{DF}(\Sigma)$. Also if $V' \in \mathbf{DF}(\Sigma')$ is defined by $\mathscr{A}' = \langle T', F'\rangle$, and $f: \Sigma \to \Sigma'$ is a projection then (by 3.4) $f^{-1}(V')$ is defined by $\mathscr{A} = \langle f^{-1}(T'), F'\rangle$.

Suppose that $U = \mathrm{T}(\mathscr{A})$, $V = \mathrm{T}(\mathscr{B})$, where $\mathscr{B} = \langle T', F'\rangle$ has n states, and assume first that the initial state of T' is not in F' (that is, $\Lambda \notin V$). Take $\mathscr{C} = \langle T'', G'\rangle$ where $T'' = \mathrm{fl}(\mathscr{A}, T')$ and $g' \in G'$ if and only if $p_j(g') \in F'$ for some $1 \leqslant j \leqslant n+2$. It follows directly from 3.10 that $\mathrm{T}(\mathscr{C}) = UV$. (One has only to remark that since T' is deterministic, T'' is deterministic too). If now $\Lambda \in V$ then $UV = \mathrm{T}(\mathscr{C}) \cup U$ (where $\mathscr{C}$ is as before), so that, again, $UV \in \mathbf{DF}(\Sigma)$.

Turning finally to the star operation, let $V = T(\mathscr{A})$, $\mathscr{A} = \langle T, F\rangle$ where $T = \langle S, M, s^*\rangle$ and $\| S \| = n$. We need only a slight variation of the flag-construction of 3.9, as follows.

Take $k = n + 2$ copies of T (without any "driving automaton") and let the control unit switch On a dormant copy every time that *any* one of T copies is in a final state of $\mathscr{A}$. In the initial state, the first copy is in the initial state of T; all other copies are dormant. Let T'' be the corresponding table, and put $\mathscr{C} = \langle T'', G'\rangle$ where $g' \in G'$ if and only if $p_j(g') \in F$ for some $1 \leqslant j \leqslant k$. One can easily show using 3.10, that for any $X \neq \Lambda$, $X \in \mathrm{T}(\mathscr{C})$ if and only if $X = X_1 \cdots X_m$ where $X_i \in V$, $X_i \neq \Lambda$. In fact, suppose $X \in \mathrm{T}(\mathscr{C})$ and let j_0 be the copy which is in a final state at the end of X. If j_0 was lastly switched On at i_0, then $X(i_0, |X|) \in V$. Some copy, say j_1, is in a final state at i_0; if j_1 was switched On at $i_1 < i_0$, then $X(i_1, i_0) \in V$. After a finite number of steps, we arrive at a splitting of X as a finite product of nonempty tapes from V. On the other hand if $X = X_1 \cdots X_m$ where $X_i \in V$, then one easily sees, by induction on r, that some copy is in a final state at the end of $X_1 \cdots X_r$, for every $1 \leqslant r \leqslant m$, so that in particular, $X \in \mathrm{T}(\mathscr{C})$.

Taking now the automaton $\mathscr{B}$ which defines $\mathrm{T}(\mathscr{C}) \cup \{\Lambda\}$, we get that $\mathrm{T}(\mathscr{B}) = V^*$.

Q.E.D.

DEFINITION 4.3. A set $V \subseteq \Sigma^*$ is *regular*, in symbols: $V \in \mathbf{R}(\Sigma)$, if it can be obtained from the empty set and the singleton sets $\{\sigma\}$ (for $\sigma \in \Sigma$) by a finite number of union, product and star operations.

THEOREM 4.4. $\mathbf{R}(\Sigma) = \mathbf{DF}(\Sigma)$ (*Moreover, the union operation, in* $\mathbf{R}(\Sigma)$, *can be restricted so that it applies on disjoint sets only*).

Proof[4]. Since $\varnothing$ and $\{\sigma\}$ are certainly in $\mathbf{DF}(\Sigma)$, and this collection is closed under union, product and star operations, we immediately get that $\mathbf{R}(\Sigma) \subseteq \mathbf{DF}(\Sigma)$. Suppose now that $V = \mathrm{T}(\mathscr{A})$ where $\mathscr{A} = \langle S, M, s^*, F\rangle$, $S = \{s_1, ..., s_n\}$ and $s^* = s_1$. For $1 \leqslant i, j \leqslant n$ and $0 \leqslant k \leqslant n$, we define the sets V_{ij}^k as follows.

(a) $V_{ij}^0 = \{X : M(s_i, X) = s_j, |X| \leqslant 1\}$.

(b) $V_{ij}^{k+1} = V_{ij}^k \cup V_{i\,k+1}^{k}(V_{k+1\,k+1}^{k})^* V_{k+1\,j}^k$.

We remark that if $i = j = k + 1$ then, denoting V_{ij}^k by V', we get that $V_{ij}^{k+1} = V' \cup V'V'^*V' = V' \cup V'^* = V'^*$, so that for this case we just put $V_{ij}^{k+1} = (V_{ij}^k)^*$. In the other cases, the two components of the union in (b) are clearly disjoint. By induction on k, one immediately verifies that V_{ij}^k is the set of all tapes X such that $M(s_i, X) = s_j$ and $M(s_i, Y) \in S_k = \{s_1, ..., s_k\}$ for every $\Lambda < Y < X$ ($S_0 = \varnothing$). Thus $V = \bigcup_{s_i \in F} V_{1i}^n$, where the sets V_{1i}^n are pairwise disjoint. Since, by induction on k, $V_{ij}^k \in \mathbf{R}(\Sigma)$ for every $1 \leqslant i, j \leqslant n$, we get in particular that $V \in \mathbf{R}(\Sigma)$.
Q.E.D.

THEOREM 4.5. *If* $V \in \mathbf{R}(\Sigma)$ *and* $f: \Sigma \to \Sigma'$, *then* $f(V) \in \mathbf{R}(\Sigma')$.

Proof. This follows immediately from the preceeding theorem if we note that $f(\varnothing) = \varnothing$, $f(\{\sigma\}) = \{f(\sigma)\}$, and f is preserved under union, product, and star operations; that is,

$$f(U \cup V) = f(U) \cup f(V), \qquad f(UV) = f(U)f(V) \qquad \text{and} \qquad f(V^*) = f(V)^*.$$
Q.E.D.

Thus we have here a proof of the closure under projection of the class of all regular sets (on all alphabets), which does not mention nondeterministic automata or the "subset construction." The equivalence of deterministic and nondeterministic finite automata is now a corollary of 4.5:

COROLLARY 4.6. $\mathbf{DF}(\Sigma) = \mathbf{F}(\Sigma)$.

Proof. By definition $\mathbf{DF}(\Sigma) \subseteq \mathbf{F}(\Sigma)$. If now $V \in \mathbf{F}(\Sigma)$ is defined by the nondeterministic automaton $\mathscr{A} = \langle T, F\rangle$ where $T = \langle S, M, S^*\rangle$, let $\mathscr{A}' = \langle T', F\rangle$ be the deterministic automaton on $\Sigma \times S$, in which T' is the deterministic image of T. If $V' = \mathrm{T}(\mathscr{A}')$, and $f: \Sigma \times S \to \Sigma$ is defined by $f(\sigma, s) = \sigma$, then it is immediate by 3.6 that $f(V') = V$, so that $V \in \mathbf{DF}(\Sigma)$. Q.E.D.

[4] Following [7, Theorem 2.1].

Alternatively, letting $S = \{s_1, ..., s_n\}$, we can define for $1 \leqslant i,j \leqslant n$ and $0 \leqslant k \leqslant n$, the sets V_{ij}^k as in 4.4 (except for an obvious change in V_{ij}^0) and conclude that V_{ij}^k is the set of all tapes X for which there is a run φ such that $\varphi(0) = s_i$, $\varphi(|X|) = s_j$, and $\varphi(m) \in S_k$ for $0 < m < |X|$. Thus $V = \bigcup_{s_i \in S^* \, s_j \in F} V_{ij}^n$ is in $\mathbf{R}(\Sigma)$, so that it is in $\mathbf{DF}(\Sigma)$. Q.E.D.

THEOREM 4.7. *Given a finite automaton $\mathscr{A}$, one can effectively decide whether the set $V = \mathrm{T}(\mathscr{A})$ is empty, trivial, singleton, finite, or infinite.*

Proof. Given $\mathscr{A}$, we have seen in 4.4, how to effectively construct $V = \mathrm{T}(\mathscr{A})$ from $\varnothing$ and $\{\sigma\}$ by a finite number of union, product and star operations.

Now, by definition, these questions are decidable for $\varnothing$ and $\{\sigma\}$. Suppose they are decidable for V_1 and V_2, and take $W = V_1 \cup V_2$, where we can assume without loss of generality that $V_1 \cap V_2 = \varnothing$.

W is empty, trivial or finite if and only if both V_1 and V_2 are empty, trivial, or finite respectively; it is singleton if and only if one of them is singleton and the other empty; and it is infinite in all other cases. Similarly, $W = V_1 V_2$ is empty if and only if V_1 or V_2 are empty; it is finite [singleton] if and only if both V_1 and V_2 are finite [singleton] or one of them is empty, and infinite otherwise; and it is trivial if and only if one of them is empty or both are trivial. Finally V^* is never empty; it is trivial [singleton] if and only if V is trivial; and it is infinite in all other cases. Q.E.D.

DEFINITION 4.8. The *kernel* of a set $V \subseteq \Sigma^*$, to be denoted by $k(V)$, is the set of all minimal elements of V. V is *minimal* if $V = k(V)$.

Since $k(k(V)) = k(V)$, $k(V)$ is always minimal. Also, if V is minimal then $\Lambda \in V$ if and only if $V = \{\Lambda\}$. Finally we notice that $k(V) = V - V\Sigma^+$, so that if V is regular, then $k(V)$ is regular too.

THEOREM 4.9. *$V \in \mathbf{R}(\Sigma)$ if and only if it is a finite union of sets of the form $V_1 V_2{}^*$ where V_1, $V_2 \subseteq \Sigma^*$ are regular and minimal sets.*

Proof. Let $V = \mathrm{T}(\mathscr{A})$, $\mathscr{A} = \langle S, M, s^*, F \rangle$ and for $s, s' \in S$, let $V_{ss'}$ be the set defined by $\langle S, M, s, s' \rangle$. Put $V'_{ss'} = k(V_{ss'})$. Then $V = \bigcup_{s' \in F} V'_{s^*s'} V'^{*}_{s's'}$. Q.E.D.

5. THE BASIC LEMMA

DEFINITION 5.1. For a given set $V \subseteq \Sigma^*$ we define the *limit* of V, in symbols: $\lim V$, to be the set of all tapes $X \in \Sigma^\omega$ which have an infinite number of sections in V, and V^ω to be the set of all tapes $X \in \Sigma^\omega$ which are an infinite concatenation of nonempty tapes in V.

Thus, $X \in \lim V$ if and only if there is an increasing infinite sequence of numbers (i_j) such that $X(0, i_j) \in V$; $X \in V^\omega$ if and only if there is such a sequence (i_j) for which $X(i_j, i_{j+1}) \in V$ and $i_0 = 0$.

As we shall see later, it is simpler, from an automata point of view, to deal with $\lim V$ than with V^ω; so we seek some set-theoretical relation between these two sets. Certainly one cannot say, in general, that $V^\omega = \lim V$ or $V^\omega = \lim V^*$: the set $V = \{1\}\{0\}^*$ is a counterexample to both assertions, since 10^ω is not in V^ω, although it is in $\lim V$ and in $\lim V^*$.

The desired relationship is given, for regular sets V, by the following lemma, which lies at the heart of McNaughton's theory, but is, however, of a rather combinational nature, and does not relate directly to any model of automata on infinite tapes.

LEMMA 5.2. *For every regular $V \subseteq \Sigma^*$ one can effectively find some regular $\tilde{V} \subseteq \Sigma^*$ such that $V^\omega = V^*(\lim \tilde{V})$.*

Proof. If V is trivial, then $\lim V$ and V^ω are both empty; so we may take $\tilde{V} = \varnothing$. Assume then that V is a nontrivial regular set, and let V^* be defined by the deterministic automaton $\mathscr{A} = \langle S, M, s^*, F\rangle$, where $S = \{s_1, \ldots, s_n\}$, $s^* = s_1$.

Define $\tilde{V}$ to be the set of all nonempty tapes X for which there is a corresponding tape Y, $\Lambda < Y < X$, such that, for $X = YZ$, the following holds.

$$\text{(a) } Y \in V^*. \qquad \text{(b) } M(s^*, X) = M(s^*, Z). \qquad \text{(c) For all } \Lambda < Z' < Z,$$
$$M(s^*, Z') \neq M(s^*, YZ').$$

We claim that $\tilde{V}$ is regular, and $V^\omega = V^*(\lim \tilde{V})$. The first claim can be proved directly by constructing a (nondeterministic) automaton defining $\tilde{V}$. We prefer however to show this as follows. For $1 \leqslant i, j \leqslant n$ put $\mathscr{A}_{ij} = \langle S, M, s_i, s_j\rangle$, $V_{ij} = \mathrm{T}(\mathscr{A}_{ij})$, $W_{ij} = V_{ij} \cap V_{1j} \cap \Sigma^+$, $W'_{ij} = W_{ij} \cap \bigcap_{k=1}^n \overline{W_{ik}\Sigma^+}$ and $V_i = (V_{1i} \cap V^*)(\bigcup_{j=1}^n W'_{ij})$. One easily checks, by direct verification of the definition, that $\tilde{V} = \bigcup_i V_i$. (We note that this construction gives the same "wave set" $\tilde{V}'$ for all sets V' for which $V'^* = V^*$; this is indeed as it should be, since $V'^\omega = (V'^*)^\omega$).

Turning now to the second claim, we first show that $\lim \tilde{V} \subseteq V^\omega$, thus showing also that $V^*(\lim \tilde{V}) \subseteq V^\omega$.

Let $X \in \Sigma^\omega$ be any tape which has an infinite number of initial sections $X_j \in \tilde{V}$ and, for each j, let Y_j be the corresponding initial section of X_j satisfying conditions (a)–(c) above. If $j \neq j'$, say $j < j'$, then $Y_j \neq Y_{j'}$, otherwise (c) would not be satisfied for the pair $(Y_{j'}, X_{j'})$. Thus, by possibly dropping some of the X_j, we may assume that $Y_1 < X_1 < \cdots < Y_j < X_j < \cdots$.

Defining Z_j' by $Y_{j+1} = X_j Z_j'$, and Z_j by $X_j = Y_j Z_j$, we get that

$$X = Y_1(Z_1 Z_1') \cdots (Z_j Z_j') \cdots.$$

By definition, $Y_1 \in V^*$; also $M(s^*, Y_{j+1}) = M(s^*, X_jZ_j') = M(M(s^*, X_j), Z_j') = M(M(s^*, Z_j), Z_j') = M(s^*, Z_jZ_j')$. Since $Y_{j+1} \in V^*$, we get that $Z_jZ_j' \in V^*$ too, thus proving that $X \in V^\omega$.

Suppose now $X \in V^\omega$, i.e., $X(i_j, i_{j+1}) \in V$ for some infinite sequence (i_j), where $i_0 = 0$. (In particular, $M(s^*, X(i_j, i_{j+1})) \in F$). For each $s \in F$, let

$$Q(s) = \{\{j, j'\}: j < j', M(s^*, X(i_j, i_{j'})) = s\}.$$

This induces a partition of the set of all pairs of natural numbers into m disjoint classes (where $\|F\| = m$) so that by Ramsey lemma,[5] there is some infinite set N' and some $s' \in F$ such that for all $j, j' \in N'$, $\{j, j'\} \in Q(s')$. Let $N' = (j_k)$ and put $Y = X(0, i_{j_0})$, $Z = X(i_{j_0}, \omega)$. Clearly $Y \in V^*$, we prove that $Z \in \lim \tilde{V}$. For each $k \geqslant 1$ let r_k be the least number which satisfy $i_{j_k} < r_k \leqslant i_{j_{k+1}}$ and $M(s^*, X(i_{j_k}, r_k)) = M(s^*, X(i_{j_0}, r_k))$. Put $X_k = X(i_{j_0}, r_k)$, $Y_k = X(i_{j_0}, i_{j_k})$, $Z_k = X(i_{j_k}, r_k)$. Then $X_k = Y_kZ_k$, $Y_k \in V^*$, $M(s^*, X_k) = M(s^*, Z_k)$ and for all

$$\Lambda < Z' < Z_k, \qquad M(s^*, Z') \neq M(s^*, Y_kZ'),$$

that is $X_k \in \tilde{V}$. Q.E.D.

One can easily avoid the use of Ramsey lemma, at a little extra cost, by using the following lemma (see [6, Lemma 1]):

LEMMA. *Let $T = \langle S, M, s^* \rangle$ be a deterministic table on Σ, and $X \in \Sigma^\omega$. For $i, j \in N$ let $Q(i, j) = \{k: k > i, j$ and $M(s^*, X(i, k)) = M(s^*, X(j, k))\}$. Define $i \equiv j$ if $Q(i, j) \neq \varnothing$. Then $\equiv$ is an equivalence relation and the number of equivalence classes is at most $\| S \| = n$.*

Proof. Since $i + 1 \in Q(i, i)$ and $Q(i, j) = Q(j, i)$, the relation is clearly reflexive and symmetric. Also, if $Q(i, j) \neq \varnothing$ and k_0 is the least element in $Q(i, j)$, then $Q(i, j) = [k_0, \omega)$; which shows that if $Q(i, j) \neq \varnothing$, $Q(j, k) \neq \varnothing$ then $Q(i, k) \supseteq Q(i, j) \cap Q(j, k) \neq \varnothing$, so that $\equiv$ is also transitive. Finally if $k_1 < k_2 < \cdots < k_{n+1}$ is any sequence of $n + 1$ numbers, and $m = k_{n+1} + 1$, then since $\{M(s^*, X(k_j, m)$: $1 \leqslant j \leqslant n + 1\}$ contains at most n different elements, we immediately get that at least two of these numbers are equivalent, so that the number of equivalence classes of $\equiv$ is at most n. Q.E.D.

Given now $X \in V^\omega$, where $X(m_j, m_{j+1}) \in V$, there is an infinite subsequence of (m_j), denoted by (i_j), all of whose elements are equivalent, and certainly $X(i_j, i_{j+1}) \in V^*$. Putting $Y = X(0, i_1)$, $Z = X(i_1, \omega)$ we get that $Y \in V^*$, and by essentially the same arguments as before, $Z \in \lim \tilde{V}$. Q.E.D.

Remark 1. If $\Sigma = \{\sigma\}$ is a one-letter alphabet, then the theorem is trivially true for *any* set $V \subseteq \Sigma^*$: Just take (for nontrivial V) $\tilde{V} = \Sigma^*$.

[5] The use of Ramsey Lemma in this context was suggested by A. Lichtman.

Remark 2. Call a set $V \subseteq \Sigma^*$ *closed under quotients* if whenever $X = YZ \in V$ and $Y \in V$, then $Z \in V$. It is easy to check that Lemma 5.2 is true for any set V such that V (or V^*) is closed under quotients (for, in that case $V^\omega = \lim V^*$). In particular, the lemma is true for all minimal sets, since a minimal set is closed under quotients. Incidentally this shows that the lemma may hold for nonregular sets too, since there are such sets which are minimal ($V = \{0^n 1^n\}$). The exact characterization of the collection of all sets which satisfy the assertion of the lemma is an interesting question in itself, which will not be dealt with here.

6. Automata on Infinite Tapes: Definitions, and Development of the Deterministic Approach

We begin this section by defining three different variants of automata which act on infinite tapes, to be called here M-automata, R-automata, and B-automata after their originators: Muller [8] (also McNaughton [6]), Rabin [10], and Buchi [2], respectively. The definition will be expressed in a unified formalism, which will exhibit their basic analogous nature.

The bulk of this section will be then devoted to the development of a deterministic theory of M-automata. This theory will closely parallel the theory of finite automata on finite tapes as developed in Section 4, and using the basic Lemma 5.2, it will be seen to be quite simple and transparent. This theory will then be used to prove the basic equivalence of the three variants mentioned above, and to study the structure of the sets defined by them.

Two different independent nondeterministic approaches to the theory of finite automata on infinite tapes will be given in the next sections.

Definition 6.1. Let $T = \langle S, M, S^* \rangle$ be a given table on Σ. An *R-automaton* on Σ is a pair $\mathscr{A} = \langle T, \Omega \rangle$ where Ω is a two-place relation on $P(S)$ (i.e., $\Omega \subseteq P(S) \times P(S)$). An *M-automaton* on Σ is a pair $\mathscr{A} = \langle T, Q \rangle$ where Q is a one-place relation on $P(S)$ (i.e., $Q \subseteq P(S)$). A *B-automaton* on Σ is a pair $\mathscr{A} = \langle T, q \rangle$ where q is a zero-place relation on $P(S)$ (i.e., $q \in P(S)$).

An infinite tape $X \in \Sigma^\omega$ is *accepted* by such an automaton if there is a run φ of T on X such that: $\varphi(*) \cap H \neq \varnothing$, $\varphi(*) \cap L = \varnothing$ for some $(H, L) \in \Omega$, in the first case; $\varphi(*) \in Q$ in the second case; and $\varphi(*) \cap q \neq \varnothing$ in the third case.

The set of tapes accepted by $\mathscr{A}$ will be denoted, in all three cases, by $\mathrm{T}(\mathscr{A})$, and this is the set *defined* by $\mathscr{A}$. The collections of sets of infinite tapes on Σ defined by the various models will be denoted by the corresponding bold-type letters: $\mathbf{IR}(\Sigma)$, $\mathbf{IM}(\Sigma)$ and $\mathbf{IB}(\Sigma)$ respectively. The collections of such sets defined by the corresponding *deterministic* automata will be denoted by $\mathbf{DIR}(\Sigma)$, $\mathbf{DIM}(\Sigma)$, and $\mathbf{DIB}(\Sigma)$ respectively.

Remark. A B-automaton $\mathscr{A}$ is in fact also a finite automaton which can act and accepts finite tapes too, so that the notation $T(\mathscr{A})$ is somewhat ambiguous in this case. Since, however, it will be always clear from the context whether we are dealing with finite or infinite tapes, we retain this notation in this and the following sections.

DEFINITION 6.2. A set $W \subseteq \Sigma^\omega$ is an *ω-regular set* if it is a finite union of sets of the form UV^ω where $U, V \in \mathbf{R}(\Sigma)$. The collection of all ω-regular sets will be denoted by $\mathbf{R}_\omega(\Sigma)$.

From now on we concentrate on deterministic M-automata.

THEOREM 6.3. *$\mathbf{DIM}(\Sigma)$ is closed under complementation, union and intersection and contains all cylindrifications of sets in $\mathbf{DIM}(\Sigma')$ for any alphabet Σ'.*

Proof. Exactly as in the finite case (Theorem 4.2).

LEMMA 6.4. *If $V \in \mathbf{R}(\Sigma)$, then* $\lim V \in \mathbf{DIM}(\Sigma)$.

Proof. If V is defined by $\mathscr{A} = \langle T, F \rangle$ then $\lim V$ is defined by $\mathscr{A}' = \langle T, Q \rangle$ where $F' \in Q$ if and only if $F' \cap F \neq \varnothing$. Q.E.D.

LEMMA 6.5. *If $U \in \mathbf{R}(\Sigma)$ and $V \in \mathbf{DIM}(\Sigma)$ then $UV \in \mathbf{DIM}(\Sigma)$.*

Proof. As for the finite case (Theorem 4.2), by using the flag-construction (Theorem 3.10) and fact 2.1.

THEOREM 6.6. *If $U, V \in \mathbf{R}(\Sigma)$ then $UV^\omega \in \mathbf{DIM}(\Sigma)$.*

Proof. By the basic lemma 5.2, we can find some $\tilde{V} \in \mathbf{R}(\Sigma)$ such that $V^\omega = V^*(\lim \tilde{V})$. But then

$$UV^\omega = U(V^*(\lim \tilde{V})) = (UV^*)(\lim \tilde{V}), \qquad \text{where} \quad UV^* \in \mathbf{R}(\Sigma).$$

By Lemma 6.4, $\lim \tilde{V} \in \mathbf{DIM}(\Sigma)$ so that by Lemma 6.5 $UV^\omega \in \mathbf{DIM}(\Sigma)$. Q.E.D.

To fully appreciate how troublesome a direct proof of Theorem 6.6 can be, the reader is invited to reread the concluding remarks of the original proof of the same theorem in [6]. After showing that every ω-regular set is defined by some machine with green and red lights (which is in fact an M-automaton) an example is given to show that "in general both red and green lights are necessary to represent ω-events." A second remark is then made as follows. "For the second example let $\Sigma_i = \{0, 1, \ldots, i\}$ and consider the regular ω-event

$$E = \Sigma_4{}^*[\Sigma_1{}^*\, 0\, \Sigma_1{}^*\, 1 \cup \Sigma_3{}^*\, 0\, \Sigma_3{}^*\, 1\, \Sigma_3{}^*\, 2\, \Sigma_3{}^*\, 3]^\omega.$$

A machine with two sets of lights (each one consisting of a green and red light, Y. C.) suffices for this ω-event $\cdots$ No machine for this ω-event has just one green light and one red light $\cdots$. This example shows it is not the case that all ω-events of the form

$\alpha\beta^\omega$ can be represented by machines with one set of lights, which would have been a natural conjecture to make."

THEOREM 6.7. $\mathbf{R}_\omega(\Sigma) = \mathbf{DIM}(\Sigma)$.

Proof. If $W \in \mathbf{R}_\omega(\Sigma)$, then by Theorem 6.6 and the closure of $\mathbf{DIM}(\Sigma)$ under unions, we get that $W \in \mathbf{DIM}(\Sigma)$. Let now $W = \mathrm{T}(\mathscr{A})$ where $\mathscr{A} = \langle S, M, s^*, Q\rangle$ and, because of the closure of $\mathbf{R}_\omega(\Sigma)$ under unions, we may assume that $Q = \{F\}$ is a singleton set. Put $S = \{s_1, \ldots, s_n\}$, $F = \{s_1, \ldots, s_k\}$ and let $S' = S \cup \{s_{n+1}\}$, and $M' = S' \times \Sigma \to S'$ be defined by $M'(s_i, \sigma) = M(s_i, \sigma)$ for $i \leqslant k$, $M'(s_i, \sigma) = s_{n+1}$ for $i > k$. Let U be the set defined by $\langle S, M, s^*, s_1\rangle$ and for $1 \leqslant i, j \leqslant n$ let V_{ij} be the set defined by $\langle S', M', s_i, s_j\rangle$, and $V = V_{12}V_{23} \cdots V_{k-1k}V_{k1}$. It is clear that $U, V \in \mathbf{R}(\Sigma)$ and $W = UV^\omega$. Q.E.D.

COROLLARY 6.8. If $W \in \mathbf{DIM}(\Sigma)$, $f : \Sigma \to \Sigma'$ is a projection, then $f(W) \in \mathbf{DIM}(\Sigma')$.

Proof. For any $U, V \subseteq \Sigma^*$, $f(UV^\omega) = f(U)f(V)^\omega$; the corollary is now immediate by Theorems 6.3, 6.6, 6.7. Q.E.D.

COROLLARY 6.9. $\mathbf{DIM}(\Sigma) = \mathbf{IM}(\Sigma)$.

Proof. As in the finite case (Corollary 4.6) one can show that if $W \subseteq \Sigma^\omega$ is accepted by the nondeterministic M-automaton $\mathscr{A} = \langle T, Q\rangle$ on Σ, then $W = f(W')$ where $W' \subseteq \Sigma'^\omega(\Sigma' = \Sigma \times S)$ is the set defined by $\mathscr{A}' = \langle T', Q\rangle$, T' is the deterministic image of T, and $f = \Sigma' \to \Sigma$ is defined by $f(\sigma, s) = \sigma$.

Alternatively one can show, by a slight variation of the proof in Theorem 6.7, that $W \in \mathbf{R}_\omega(\Sigma)$, so that $W \in \mathbf{DIM}(\Sigma)$ too. Q.E.D.

COROLLARY 6.10. *$W \in \mathbf{R}_\omega(\Sigma)$ if and only if it is a finite union of sets of the form $U(\lim V)$ where $U, V \in \mathbf{R}(\Sigma)$.*

The following theorem is the counterpart of Theorem 4.9 in the finite case.

THEOREM 6.11. *$W \in \mathbf{R}_\omega(\Sigma)$ if and only if it is a finite union of pairwise disjoint sets of the form $UV^*V'^\omega$, where $U, V, V' \subseteq \Sigma^*$ are regular minimal sets, and $V' \subseteq V$.*

Proof. If $W \in \mathbf{R}_\omega(\Sigma)$, then $W = \mathrm{T}(\mathscr{A})$ for some deterministic M-automaton $\mathscr{A} = \langle S, M, s^*, Q\rangle$ where we may assume without loss of generality that $Q = \{F\}$ is a singleton set. Put $S = \{s_1, \ldots, s_n\}$, $F = \{s_1, \ldots, s_k\}$, $s^* = s_{i_1}$, and for $1 \leqslant i, j \leqslant n$ let V_{ij} be the set defined by $\langle S, M, s_i, s_j\rangle$. Put $U = k(V_{i_1 1})$, $U' = V_{12}V_{23} \cdots V_{k-1k}V_{k1}$, $V = k(U')$, and $V' = V - \bigcup_{j=k+1}^{n} V_{1j}V_{j1}$. It is clear that U, V, V' are regular minimal sets, $V' \subseteq V$, and $W = UV^*V'^\omega$. Q.E.D.

COROLLARY 6.12. *$W \in \mathbf{R}_\omega(\Sigma)$ if and only if it is a finite union of pairwise disjoint sets of the form UV^ω where $U, V \in \mathbf{R}(\Sigma)$, and V is minimal.*

Remark 1. An independent combinatorial proof of Corollary 6.12 can give a new proof of Theorem 6.6, which is independent of Lemma 5.2. In fact, since for minimal V, $V^\omega = \lim V^*$ we get immediately by Lemma 6.4 that $V^\omega \in \mathbf{DIM}(\Sigma)$, so that by Lemma 6.5, $W = UV^\omega \in \mathbf{DIM}(\Sigma)$ too.

Remark 2. The more natural counterpart of 4.9, namely the assertion that every ω-regular set is a finite union of sets of the form UV^ω where U, V are regular minimal sets, is not true. We shall prove below that the collection of such sets is exactly $\mathbf{DIB}(\Sigma)$ (Theorem 6.21) and this collection is properly contained in $\mathbf{R}_\omega(\Sigma)$ (Remark 6.18).

THEOREM 6.13. *Given an M-automaton* $\mathscr{A}$ *one can effectively decide whether* $W = \mathrm{T}(\mathscr{A})$ *is empty, singleton, finite or infinite.*

Proof. Assume first that $W = \mathrm{T}(\mathscr{A})$ can be effectively written as $W = UV^*V'^\omega$ where U, V, V' are regular minimal sets and $V' \subseteq V$. Clearly W is empty if and only if U is empty or V' is trivial. So assume U nonempty and V' nontrivial. If $X \neq Y$ are both in U, then for any $Z \in \Sigma^\omega$, $XZ \neq YZ$. Also if $Z \in V$, $Y \in V'$, and $Y \neq Z$, then for any $j, k > 0$, $Z^jY^\omega \neq Z^jZ^kY^\omega$. From which we conclude that W is finite [singleton] if and only if U is finite [singleton] and V is singleton.

Now every set W defined by an M-automaton $\mathscr{A}$ can be effectively written as a finite union of pairwise disjoint sets W_i of the form dealt with above. Such a union is empty [finite] if and only if all its components are empty [finite]; it is a singleton set if and only if exactly one of the components is singleton, the others empty; and it is infinite if and only if at least one of the components is infinite. Q.E.D.

For R-automata we have the following.

THEOREM 6.14. $\mathbf{DIR}(\Sigma) = \mathbf{DIM}(\Sigma)$, $\mathbf{IR}(\Sigma) = \mathbf{IM}(\Sigma)$.

Proof. Let $\mathscr{A} = \langle S, M, S^*, \Omega\rangle$ be a [deterministic] R-automaton and define the [deterministic] M-automaton $\mathscr{B} = \langle S, M, S^*, Q\rangle$ by $F \in Q$ if and only if $F \cap H \neq \varnothing$, $F \cap L = \varnothing$ for some $(H, L) \in \Omega$; clearly $\mathrm{T}(\mathscr{A}) = \mathrm{T}(\mathscr{B})$. Thus $\mathbf{DIR}(\Sigma) \subseteq \mathbf{DIM}(\Sigma)$, $\mathbf{IR}(\Sigma) \subseteq \mathbf{IM}(\Sigma)$.

Suppose now $\mathscr{A} = \langle S, M, S^*, Q\rangle$ is a [deterministic] M-automaton, where $Q = \{S_1, \ldots, S_k\}$ and define the [deterministic] R-automaton $\mathscr{B} = \langle S', M', S'^*, \Omega\rangle$ as follows: $S' = S \times \prod_1^k P(S_i)$, $S'^* = \{(s, \varnothing, \ldots, \varnothing) : s \in S^*\}$, $\Omega = \{(H_i, L_i) : 1 \leqslant i \leqslant k\}$ where $H_i = \{s' : p_i(s') = S_i\}$ and $L_i = \{s' : p_0(s') \notin S_i\}$. Finally $s'' \in M'(s', \sigma)$ if and only if $p_0(s'') \in M(p_0(s'), \sigma)$ and $p_i(s'') = \varnothing$ if $p_i(s') = S_i$, $p_i(s'') = (p_i(s') \cup \{p_0(s')\}) \cap S_i$ if $p_i(s') \neq S_i$. Again it is easy to verify that $\mathrm{T}(\mathscr{A}) = \mathrm{T}(\mathscr{B})$. This completes the proof of the theorem. Q.E.D.

COROLLARY 6.15. $\mathbf{DIR}(\Sigma) = \mathbf{IR}(\Sigma)$.

We turn now to B-automata.

THEOREM 6.16. $\mathbf{IB}(\Sigma) = \mathbf{R}_\omega(\Sigma)$.

Proof. If $W = \mathrm{T}(\mathscr{A})$ where $\mathscr{A} = \langle S, M, S^*, q\rangle$ is a B-automaton, let $V_{ss'}$ be the set defined by the finite automaton $\langle S, M, s, s'\rangle$. Clearly $W = \bigcup\{V_{ss'}V_{s's'}^{\omega} : s \in S^*, s' \in q\}$, so that $W \in \mathbf{R}_\omega(\Sigma)$. Let now $W = UV^\omega$ where $U, V \in \mathbf{R}(\Sigma)$ are defined by $\mathscr{A} = \langle S, M, s^*, F\rangle$ and $\mathscr{B} = \langle S', M', s'^*, F'\rangle$ respectively. Define the (nondeterministic) B-automaton $\mathscr{C} = \langle R, P, R^*, q\rangle$ as follows.

$R = S \cup (S' \times \{0, 1\})$, $R^* = \{s^*\}$ (and if $s^* \in F$, $R^* = \{s^*, s'^*\}$), $q = \{(s'^*, 1)\}$ and P is given by:

$$P(s, \sigma) = \{M(s, \sigma)\} \quad \text{if} \quad M(s, \sigma) \notin F;$$
$$P(s, \sigma) = \{M(s, \sigma), (s'^*, 1)\} \quad \text{if} \quad M(s, \sigma) \in F;$$
$$P((s', \epsilon), \sigma) = \{(M'(s', \sigma), 0)\} \quad \text{if} \quad M'(s', \sigma) \notin F';$$
$$P((s', \epsilon), \sigma) = \{(M'(s', \sigma), 0), (s'^*, 1)\} \quad \text{if} \quad M'(s', \sigma) \in F'. \qquad (\epsilon = 0, 1).$$

It is easily seen that $\mathrm{T}(\mathscr{C}) = UV^\omega$.

Since $\mathbf{IB}(\Sigma)$ is closed under unions (if $W_j \in \mathbf{IB}(\Sigma)$ is defined by $\mathscr{A}_j = \langle T_j, q_j\rangle$ then $\bigcup_j W_j$ is defined by $\langle \bigcup_j T_j, \bigcup_j q_j\rangle$), the theorem is proved. Q.E.D.

COROLLARY 6.17. $\mathbf{IB}(\Sigma)$ *is closed under complementation.*

Remark 6.18. It is well-known [2] that $\mathbf{DIB}(\Sigma) \neq \mathbf{R}_\omega(\Sigma)$ $(\Sigma^*\{0\}^\omega \notin \mathbf{DIB}(\Sigma))$. Thus $\mathbf{DIB}(\Sigma) \neq \mathbf{IB}(\Sigma)$. The following three theorems give a characterization of $\mathbf{DIB}(\Sigma)$.

THEOREM 6.19. $W \in \mathbf{DIB}(\Sigma)$ *if and only if* $W = \lim V$ *where* $V \in \mathbf{R}(\Sigma)$.

Proof. If $\mathscr{A} = \langle T, F\rangle$ is a deterministic finite automaton, $V \subseteq \Sigma^*$ is the set of finite tapes accepted by $\mathscr{A}$, and $W \subseteq \Sigma^\omega$ is the set of infinite tapes accepted by $\mathscr{A}$ (when viewed as a B-automaton) then $W = \lim V$. The assertion of the theorem now follows immediately. Q.E.D.

COROLLARY 6.20. $\mathbf{R}_\omega(\Sigma)$ $(=\mathbf{IB}(\Sigma))$ *is the closure of* $\mathbf{DIB}(\Sigma)$ *under finite unions and left product by regular sets.*

THEOREM 6.21. $W \in \mathbf{DIB}(\Sigma)$ *if and only if it is a finite union of sets of the form* UV^ω *where* U, V *are regular minimal sets.*

Proof. If $W \subseteq \Sigma^\omega$ is defined by the deterministic B-automaton $\langle S, M, s^*, q\rangle$ let $V_{ss'} \subseteq \Sigma^*$ be the set defined by $\langle S, M, s, s'\rangle$, and put $V_{ss'} = k(V_{ss'})$. Then

$$W = \bigcup_{s' \in q} V'_{s^*s'}V'^{\omega}_{s's'}.$$

On the other hand suppose $U, V \subseteq \Sigma^*$ are regular minimal sets defined by $\mathscr{A} = \langle S, M, s^*, F\rangle$ and $\mathscr{B} = \langle S', M', s'^*, F'\rangle$ respectively. $W = UV^\omega$ is then defined by the deterministic B-automaton $\mathscr{C} = \langle S \cup S', s^*, R, F'\rangle$ where $R(s, \sigma) = M(s, \sigma)$ if $s \in S \cap \bar{F}$, $R(s, \sigma) = M'(s'^*, \sigma)$ if $s \in S \cap F$, $R(s', \sigma) = M'(s', \sigma)$ if $s' \in S' \cap \bar{F}'$ and $R(s', \sigma) = M'(s'^*, \sigma)$ if $s' \in S' \cap F'$. Since $\mathbf{DIB}(\Sigma)$ is closed under unions (the proof is immediate by the product-table technique), the theorem is proved. Q.E.D.

The equivalence between nondeterministic M-automata and B-automata shown above, can be used to show that the parallelism between M-automata and R-automata, suggested by Theorem 6.14, is not complete.

THEOREM 6.22. *There are ω-regular sets which are not accepted by any M-automaton $\langle T, Q\rangle$ in which Q is a singleton set, or contains only singleton sets. On the other hand, every ω-regular set is accepted by some R-automaton $\langle T, \Omega\rangle$ in which Ω is a singleton set (or, alternatively, Ω contains only pairs of singleton sets).*

Proof. Let $\Sigma = \{0, 1\}$ and take $W = \{0^\omega, 1^\omega, (01)^\omega\}$. Suppose W is defined by $\mathscr{A} = \langle T, Q\rangle$ where $Q = \{F\}$ is a singleton set. There are then some $m, n_1, n_2 > 0$ and some $s \in F$ such that $M(s^*, 0^m) = s$, $M(s, 0^{n_1}) = M(s, 1^{n_2}) = s$, and $\{M(s, 0^j\colon 0 \leqslant j \leqslant n_1\} = \{M(s, 1^j)\colon 0 \leqslant j \leqslant n_2\} = F$. But then the tape $X = 0^m(0^{n_1}1^{n_2})^\omega$ is also accepted by $\mathscr{A}$, a contradiction. Suppose now that Q contains only singleton sets; there is then some $s' \in S$ and some $m \geqslant 0$ such that $\{s'\} \in Q$, $M(s^*, (01)^m) = s'$, $M(s', 0) = M(s', 1) = s'$. But then every tape $X = (01)^m Y$ where $Y \in \Sigma^\omega$ is also acceptable by $\mathscr{A}$, again a contradiction.

On the other hand if $W \in \mathbf{R}_\omega(\Sigma)$ then W is defined by some B-automaton $\langle S, M, S^*, q\rangle$; but then W is also defined by the R-automaton $\langle S, M, S^*, \Omega\rangle$ where $\Omega = \{(q, \varnothing)\}$ is a singleton set. (Alternatively we can take $\Omega = \{(\{s\}, \{0\})\colon s \in q\}$ where 0 is a new state, so that Ω contains only pairs of singleton sets). Q.E.D.

7. AN INDEPENDENT NONDETERMINISTIC APPROACH

In this section we develop a theory of B-automata which is independent of the theory of M-automata, but uses the basic Lemma 5.2. In the next section, a third approach will be given, which does not use Lemma 5.2, and is patterned after Rabin's method for dealing with automata on infinite trees [10].

Both these approaches require a variant of the flag-construction of Section 3, which is now given:

DEFINITION 7.1. Let $U \subseteq \Sigma^*$ and $W \subseteq \Sigma^\omega$. We denote by $U \mid W$ the set of all tapes $X \in \Sigma^\omega$ such that if $X = YZ$ and $Y \in U$ then $Z \in W$.

The usefulness of this concept lies in the fact that for U, W as before, we have $\overline{UW} = U \mid \overline{W}$.

THEOREM 7.2. *If $U \in \mathbf{R}(\Sigma)$ and $W \in \mathbf{IB}(\Sigma)$ then $U \mid W \in \mathbf{IB}(\Sigma)$.*

Proof. Let U be defined by the deterministic finite automaton $\mathscr{A} = \langle S, M, s^*, F\rangle$ and W by the B-automaton $\mathscr{B} = \langle T', q'\rangle$ where $T' = \langle S', M', S'^*\rangle$, $\| S' \| = n$ and $q' \in P(S')$. Let $k = n + 2$, $G = (S' \cup \{0\})^k - S'^k$, $R = S \times G$ and $T'' = \langle R, P, r^*\rangle$ be the flag set of $\mathscr{A}$ relative to T' as defined in 3.9. We use a slight variation of T'', as follows. $T_1 = \langle R \times P(K), P_1, (r^*, \varnothing)\rangle$ where $K = \{1,..., k\}$, $P_1(r, K'), \sigma) = \{P(r, \sigma)\} \times \{K''\}$ and $K'' = \varnothing$ if $K' = K$, otherwise $K'' = K' \cup \{j : p_j(r) = 0$ or $p_j(r) \in q'\}$. Put $\mathscr{C} = \langle T'', H\rangle$ where $H = \{(r, K): r \in R\}$. We claim that $\mathrm{T}(\mathscr{C}) = U \mid W$.

Indeed let $X \in \mathrm{T}(\mathscr{C})$ and $\varphi'' \in R(T'', X)$ be such that $(r_0, K) \in \varphi''(*)$, for some $r_0 \in R$. Suppose $X = YZ$ where $Y = X(0, i_0) \in U$. Let j_0 be the copy switched On at i_0, $j^* = \mathrm{urep}(j_0, i_0)$ and $\varphi' \in R(T', Z)$ be the virtual run of T' on $Z = X(i_0, \omega)$. If j^* was switched On at i^*, then for $i > i^* p_{j^*}(\varphi''(i)) = \varphi'(i - i_0)$. Since $j^* \in K$ and $(r_0, K) \in \varphi''(*)$, there is an infinite number of i such that $p_{j^*}(\varphi''(i)) \in q'$, which shows that $\varphi'(*) \cap q' \neq \varnothing$. Q.E.D.

Conversely, suppose $X \in U \mid V$. We define $\varphi'' \in R(T'', X)$ as follows: if copy j is active (or made dormant) at i, and was last activated at $i_0 < i$, then put $p_j(\varphi''(i)) = \varphi'(i - i_0)$ where $\varphi' \in R(T', X(i_0, \omega))$ is such that $\varphi'(*) \cap q' \neq \varnothing$. All we have to show is that every copy j which is ultimately in a switched On mode, passes through some state of q' an infinite number of times. This is however obvious by the construction of φ''. Q.E.D.

LEMMA 7.3. *$\mathbf{IB}(\Sigma)$ is closed under intersections.*

Proof. Let W_j be defined by $\mathscr{A}_j = \langle S_j, M_j, S_j^*, q_j\rangle$ for $1 \leqslant j \leqslant k$, and let $K = \{1,..., k\}$, $\mathscr{A} = \langle S \times P(K), M, S^*, q\rangle$ where $S = \prod_j S_j$, $S^* = \prod_j S_j^* \times \{\varnothing\}$, $q = \{(s, K): s \in S\}$ and $M((s, K'), \sigma)$ is the set of all (s'', K'') for which $p_j(s'') \in M_j(p_j(s), \sigma)$ (for $1 \leqslant j \leqslant k$) and $K'' = \varnothing$ if $K' = K$, $K'' = K' \cup \{j : p_j(s) \in q_j\}$ otherwise. It is clear that $\mathrm{T}(\mathscr{A}) = \bigcap_j W_j$. Q.E.D.

LEMMA 7.4. *If $V \in \mathbf{R}(\Sigma)$ then $\overline{\lim V} \in \mathbf{IB}(\Sigma)$.*

Proof. If V is defined by the deterministic finite automaton $\mathscr{A} = \langle S, M, s^*, F\rangle$ then $\overline{\lim V}$ is defined by the B-automaton $\mathscr{A}' = \langle S', M', s^*, q'\rangle$ where $S' = S \cup [S \times \{0\}]$, $q' = S \times \{0\}$, $M'(s, \sigma) = \{M(s, \sigma), (M(s, \sigma), 0)\}$, $M'((s, 0), \sigma) = (\{M(s, \sigma)\} \cap \bar{F}) \times \{0\}$.

THEOREM 7.5. *$\mathbf{IB}(\Sigma)$ is closed under complementation.*

Proof. Suppose first $W = UV^\omega$ where $U, V \in \mathbf{R}(\Sigma)$. By Lemma 5.2 $W = (UV^*)$

$(\lim \vec{V})$ where $\vec{V} \in \mathbf{R}(\Sigma)$. Thus $\overline{W} = \overline{(UV^*)(\lim \vec{V})} = (UV^*) \mid \overline{(\lim \vec{V})}$, where $UV^* \in \mathbf{R}(\Sigma)$, $\overline{\lim \vec{V}} \in \mathbf{IB}(\Sigma)$ and thus by 7.2, $\overline{W} \in \mathbf{IB}(\Sigma)$. If now $W \in \mathbf{IB}(\Sigma)$, then by 6.16, W is a finite union of sets of the form UV^ω so that by the preceeding result and the closure of $\mathbf{IB}(\Sigma)$ under intersection we get that $\overline{W} \in \mathbf{IB}(\Sigma)$. Q.E.D.

8. A Nondeterministic Rabin Approach

We now give a third proof for the closure of $\mathbf{IB}(\Sigma)$ under complementation. The only results needed for the proof are the following.

(a) $\mathbf{IB}(\Sigma)$ is closed under unions and intersections, and contains all projections and cylindrifications of sets in $\mathbf{IB}(\Sigma')$ (for any Σ'), and all limits of sets in $\mathbf{R}(\Sigma)$. (All these assertions are quite trivial; see also the end of the proof of Theorems 6.16 and 6.19; Lemma 7.3).

(b) If $V \in \mathbf{R}(\Sigma)$ and $W \in \mathbf{IB}(\Sigma)$ then $V \mid W \in \mathbf{IB}(\Sigma)$. This has been proved, using the flag-construction, in Theorem 7.2.

From now on we assume that we are given a B-automaton $\mathscr{A} = \langle S, M, S^*, q\rangle$ on Σ, and $\| S \| = n$. For $s \in S$ we denote by T_s the table: $\langle S, M, s\rangle$ and we define the sets $A(s) \subseteq \Sigma^\omega$ and $B_k(s) \subseteq \Sigma^\omega$ (for $k \geqslant 0$) as follows.

$$A(s) = \{X \colon \varphi \in R(T_s, X) \to \varphi(*) \cap q = \varnothing\},$$

$$B_0(s) = \{X \colon \varphi \in R(T_s, X) \to [\varphi] \cap q = \varnothing\}.$$

$$B_{k+1}(s) = \{X \colon \varphi \in R(T_s, X) \to ([\varphi] \cap q = \varnothing \vee (\exists i > 0)(X(i, \omega) \in B_k(\varphi(i))))\}$$

where $[\varphi] = \{\varphi(i) \colon i > 0\}$.

THEOREM 8.1. *$B_n(s) = A(s)$, for every $s \in S$.*

Proof. We first show by induction on k that for every $s \in S$, $B_k(s) \subseteq A(s)$. For $k = 0$ this is trivial. Assume $B_k(s) \subseteq A(s)$ for every $s \in S$ and let $X \in B_{k+1}(s')$ for some $s' \in S$.

Given $\varphi \in R(T_{s'}, X)$, if $[\varphi] \cap q = \varnothing$, we are finished; otherwise there is some $i > 0$ such that $X(i, \omega) \in B_k(\varphi(i))$ so that $X(i, \omega) \in A_k(\varphi(i))$, and certainly $\varphi(*) \cap q = \varnothing$. Thus $X \in A(s')$. Q.E.D.

We now show that $B_n(s) = B_{n+1}(s)$ for all $s \in S$. It is obvious from the very definition of the sets $B_k(s)$ that $B_k(s) \subseteq B_{k+1}(s)$, for all $s \in S$ and $k \geqslant 0$, and if $B_k(s) = B_{k+1}(s)$ then $B_k(s) = B_j(s)$ for all $j \geqslant k$. For a fixed $X \in \Sigma^\omega$ let $S_k(X) = \{s \colon X \in B_k(s)\}$. By the preceeding remarks $S_k(X) \subseteq S_{k+1}(X)$ and if $S_k(X) = S_{k+1}(X)$, then $S_k(X) = S_j(X)$ for all $j \geqslant k$. Thus, if for some $0 \leqslant j \leqslant n$ $S_j(X) = S_{j+1}(X)$, then $S_n(X) = S_{n+1}(X)$; otherwise, since $\| S_0(X)\| \geqslant 0$, we get that $\| S_j(X)\| \geqslant j$, so that, again $S_n(X) = S_{n+1}(X)$. Thus, in all cases $S_n(X) = S_{n+1}(X)$ and this holds for every $X \in \Sigma^\omega$, from which we conclude that $B_n(s) = B_{n+1}(s)$ for all $s \in S$.

Finally we show that $B_n(s) \supseteq A(s)$.

Suppose $X \notin B_n(s)$; then $X \notin B_{n+1}(s)$ and so it satisfies the condition:

$$C(s){:}\ (\exists\varphi \in R(T_s, X))([\varphi] \cap q \neq \varnothing \wedge (\forall i > 0)(X(i, \omega) \notin B_n(\varphi(i)))).$$

Let $\varphi_0 \in R(T_s, X)$ be a run which satisfies the clause of $C(s)$; there is some $i_1 > 0$ such that $\varphi_0(i_1) \in q$, so that $Y_1 = X(i_1, \omega) \notin B_n(\varphi_0(i_1))$. Thus Y_1 satisfies the condition $C(\varphi_0(i_1))$, so there is some $\varphi_1 \in R(T_{\varphi_0(i_1)}, Y_1)$ and some $i_2 > 0$ such that $\varphi_1(i_2) \in q$ and $Y_2 = Y_1(i_2, \omega) \notin B_n(\varphi_1(i_2))$. Y_2 satisfies $C(\varphi_1(i_2))$, so we can continue as before, defining thus, by induction, a sequence of numbers i_{j+1} $(i_0 = 0)$, of tapes $Y_{j+1} = Y_j(i_{j+1}, \omega)$ $(Y_0 = X)$ and of runs $\varphi_{j+1} \in R(T_{\varphi_j(i_{j+1})}, Y_{j+1})$ such that $\varphi_j(i_{j+1}) \in q$. Let then $\psi = \varphi_0(0, i_1)\, \varphi_1(0, i_2) \cdots \varphi_j(0, i_{j+1}) \cdots$; it is obvious that $\psi \in R(T_s, X)$, and $\psi(*) \cap q \neq \varnothing$, so that $X \notin A(s)$. Q.E.D.

Remark. The observation that the sequence $B_k(s)$ "collapses" (for infinite tapes, in contrast to the case of infinite trees) when $k = n$ is due to Daniel Cohen, and I am indebted to him for pointing it out to me while commenting on [4]. This "collapse" turns out to be very convenient, since it will allow us to prove that $A(s) \in \mathbf{IB}(\Sigma)$ by showing (by induction on k) that $B_k(s) \in \mathbf{IB}(\Sigma)$. Nevertheless, it is important to note that this fact is not really necessary for the proof. Thus one can define the sets $B_\alpha(s)$, for every ordinal $\alpha \geqslant 0$, by the condition:

$$\varphi \in R(T_s, X) \Rightarrow \Big([\varphi] \cap q = \varnothing \vee (\exists i > 0)\Big(X(i, \omega) \in \bigcup_{\delta<\alpha} B_\delta(\varphi(i))\Big)\Big).$$

Letting then μ be the least ordinal such that $B_{\mu+1}(s) = B_\mu(s)$, one can show, exactly as before, that $B_\mu(s) = A(s)$. In this case, however, the proof that $B_\mu(s) \in \mathbf{IB}(\Sigma)$ is more lengthy, and requires an additional lemma, which is a rather complicated version of the flag-construction of definition 3.9.

For $B_0(s)$ we have:

LEMMA 8.2. $B_0(s) \in \mathbf{IB}(\Sigma)$.

Proof. Let $B_0'(s) \subseteq \Sigma^*$ be the set of all finite tapes which satisfy the condition that defines $B_0(s)$. Certainly $B_0'(s) \in \mathbf{R}(\Sigma)$ and since, obviously, $B_0(s) = \lim B_0'(s)$, we get that $B_0(s) \in \mathbf{IB}(\Sigma)$. Q.E.D.

For the induction step we need the following additional notation[6]: Let $\bar{\bar{\Sigma}} = \Sigma \times P(S)$; the projection functions $p_0 : \bar{\bar{\Sigma}} \to \Sigma$ and $p_1 : \bar{\bar{\Sigma}} \to P(S)$ are defined as usual. For each $s \in S$ the table T_s can be naturally extended to a table on $\bar{\bar{\Sigma}}$, by

[6] Because of typographical reasons, we use a double over-bar on certain letters, instead of underlining them. No confusion should arise, since we don't use double complementation in this section.

defining $M(s, \bar{\bar{\sigma}}) = M(s, p_0(\bar{\bar{\sigma}}))$ for every $s \in S$ and $\bar{\bar{\sigma}} \in \bar{\bar{\Sigma}}$. It is then clear that for any $\bar{\bar{X}} \in \bar{\bar{\Sigma}}(\omega)$, $R(T_s, \bar{\bar{X}}) = R(T_s, p_0(\bar{\bar{X}}))$.

LEMMA 8.3. *Let $\bar{C}(s)$ be the set of all infinite tapes $\bar{X} \in \bar{\Sigma}^\omega$ which satisfy the following condition.*

$$\varphi \in R(T_s, \bar{X}) \Rightarrow (\exists i)(\varphi(i) \in p_1(\bar{x}_i)).$$

Then $\bar{C}(s) \in \mathbf{IB}(\bar{\Sigma})$.

Proof. Let $\bar{C}'(s)$ be the set of all finite tapes $\bar{X} \in \bar{\Sigma}^*$ which satisfy the condition of the lemma. Certainly $\bar{C}'(s) \in \mathbf{R}(\bar{\Sigma})$. We shall show that $\bar{C}(s) = \bar{C}'(s)\,\bar{\Sigma}^\omega = \lim(\bar{C}'(s)\,\bar{\Sigma}^*)$, thus proving the assertion. It is obvious that $\bar{C}'(s)\,\bar{\Sigma}^\omega \subseteq \bar{C}(s)$. Assume now that $\bar{X} \notin \bar{C}'(s)\,\bar{\Sigma}^\omega$. For each $m > 0$ there is a run $\varphi_m \in R(T_s, \bar{X}(0, m))$ such that for all i, $\varphi_m(i) \notin p_1(\bar{x}_i)$. Using the familiar diagonal argument we define a run $\varphi \in R(T_s, \bar{X})$ as follows: $\varphi(0) = s$; assuming that we have already defined $\varphi(j)$ for every $j \leqslant k$ in such a way that there is an infinite set N_k such that $\varphi(0, k+1) = \varphi_m(0, k+1)$ for all $m \in N_k$, let $\varphi(k+1)$ be any state s' for which the set $\{m: m \in N_k, \varphi_m(k+1) = s'\}$ is infinite. Clearly $\varphi \in R(T_s, \bar{X})$ and for all i, $\varphi(i) \notin p_1(\bar{x}_i)$, i.e., $\bar{X} \notin \bar{C}(s)$. Q.E.D.

LEMMA 8.4. $B_k(s) \in \mathbf{IB}(\Sigma)$.

Proof. The proof is by induction on k. The case $k = 0$ has been dealt with in Lemma 8.2. Assume the lemma proved for k and let s be any fixed state in S. We prove that $B_{k+1}(s) \in \mathbf{IB}(\Sigma)$.

For $s' \in S$ we define the following sets of tapes:

$$\bar{C}(s') = \{\bar{X} \in \bar{\bar{\Sigma}}^\omega: \varphi \in R(T_{s'}, \bar{X}) \Rightarrow (\exists i)(\varphi(i) \in p_1(\bar{X}_i))\}$$

$$\bar{\bar{D}}(s') = \{\bar{\bar{X}} \in \bar{\bar{\Sigma}}^\omega: p_0(\bar{\bar{X}}) \in B_k(s') \vee s' \notin p_1(\bar{\bar{x}}_0)\}$$

$$\bar{\bar{F}}(s, s') = \{\bar{\bar{X}} \in \bar{\bar{\Sigma}}^+: (\exists \varphi \in R(T_s, \bar{\bar{X}}))(\varphi(*) = s')\}$$

$$\bar{G}(s, s') = \{\bar{X} \in \bar{\Sigma}^+: (\exists \varphi \in R(T_s, \bar{X}))(\varphi(*) = s' \wedge (\forall i)(0 < i < |X| \Rightarrow \varphi(i) \notin p_1(\bar{x}_i))\}$$

$\bar{\bar{H}}(s) = \bigcap_{s' \in S} [\bar{\bar{F}}(s, s') \mid \bar{\bar{D}}(s')]$, $\bar{L}(s) = \bigcap_{s' \in q} [\bar{G}(s, s') \mid \bar{C}(s')]$ and, finally, $\bar{\bar{E}}(s) = \bar{\bar{H}}(s) \cap \bar{L}(s)$.

We claim that $B_{k+1}(s) = p_0(\bar{\bar{E}}(s))$. Since $\bar{C}(s') \in \mathbf{IB}(\bar{\Sigma})$ by Lemma 8.3, $\bar{\bar{D}}(s') \in \mathbf{IB}(\bar{\bar{\Sigma}})$ by the induction hypothesis, and $\bar{\bar{F}}(s, s')$ and $\bar{G}(s, s')$ are certainly regular, this will prove that $B_{k+1}(s) \in \mathbf{IB}(\Sigma)$. Assume first $X = p_0(\bar{\bar{X}})$, where $\bar{\bar{X}} \in \bar{\bar{E}}(s)$, and let $\varphi \in R(T_s, X) = R(T_s, \bar{\bar{X}})$. If $[\varphi] \cap q = \varnothing$, we are finished. Let then $i_1 > 0$ be such that $\varphi(i_1) = s' \in q$. If $\bar{X}(0, i_1) \in \bar{G}(s, s')$ then $\bar{X}(i_1, \omega) \in \bar{C}(s')$ so that there is some $i_2 \geqslant i_1 > 0$ such that $\varphi(i_2) = s'' \in p_1(\bar{x}_{i_2})$. If $\bar{X}(0, i_1) \notin \bar{G}(s, s')$, then such an i_2 certainly exists. In either case $\bar{\bar{X}}(0, i_2) \in \bar{\bar{F}}(s, s'')$ so that $\bar{\bar{X}}(i_2, \omega) \in \bar{\bar{D}}(s'')$ and since $s'' \in p_1(\bar{x}_{i_2})$, we conclude that $p_0(\bar{\bar{X}}(i_2, \omega)) = X(i_2, \omega) \in B_k(\varphi(i_2))$. Q.E.D.

Suppose now $X \in B_{k+1}(s)$ and define $\overline{X} = (\bar{x}_i)$ by the conditions: $p_0(\bar{x}_i) = x_i$, and for every $s' \in S$, $s' \in p_1(\bar{x}_i)$ if and only if $X(i, \omega) \in B_k(s')$. Certainly $X = p_0(\overline{X})$; we show that $\overline{X} \in \overline{E}(s)$. If $\overline{X}(0, i) \in \overline{F}(s, s')$ for some $i > 0$, then either $s' \notin p_1(\bar{x}_i)$, in which case $\overline{X}(i, \omega) \in \overline{D}(s')$, or $s' \in p_1(\bar{x}_i)$, so that, by definition, $X(i, \omega) \in B_k(s')$, and again $\overline{X}(i, \omega) \in \overline{D}(s')$. This proves that $\overline{X} \in \overline{H}(s)$. Suppose now $\overline{X}(0, i_1) \in \overline{G}(s, s')$ for some $i_1 > 0$ and $s' \in q$, we show that $\overline{Y} = \overline{X}(i_1, \omega) \in \overline{C}(s')$. Let $\psi \in R(T_{s'}, \overline{Y})$, and $\varphi \in R(T_s, \overline{X}(0, i_1))$ be such that $\varphi(*) = s'$, and for all $0 < i < i_1$, $\varphi(i) \notin p_1(\bar{x}_i)$. The run $\varphi' = \varphi(0, i_1)\, \psi(0, \omega)$ of T_s on X satisfy $[\varphi'] \cap q \neq \varnothing$, so (since $X \in B_{k+1}(s)$) there is some $i_2 > 0$ such that $X(i_2, \omega) \in B_k(\varphi'(i_2))$, that is $\varphi'(i_2) \in p_1(\bar{x}_{i_2})$. Since, however, $\varphi(i) = \varphi'(i)$ for $i < i_1$, we necessarily have that $i_2 \geqslant i_1$, so that $m = i_2 - i_1 \geqslant 0$, and $\psi(m) \in p_1(\bar{y}_m)$; thus $\overline{Y} \in \overline{C}(s')$. Q.E.D.

THEOREM 8.5. $\mathbf{IB}(\Sigma)$ *is closed under complementation.*

Proof. Let $W \subseteq \Sigma^\omega$ be defined by the B-automaton $\mathscr{A} = \langle S, M, S^*, q \rangle$. Retaining all the previous notation, we have that $\overline{W} = \bigcap_{s \in S^*} A(s)$. Since $A(s) = B_n(s)$ and $B_n(s) \in \mathbf{IB}(\Sigma)$, we get immediately that $\overline{W} \in \mathbf{IB}(\Sigma)$. Q.E.D.

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