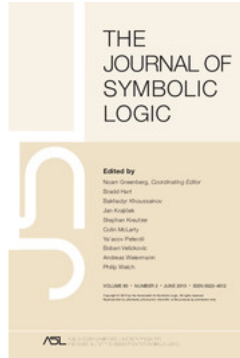

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Piergiorgio Odifreddi

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FORCING AND REDUCIBILITIES

PIERGIORGIO ODIFREDDI[1]

Dedicated to Carl Jockusch, Jr.

Introduction. We see far away, Newton said, if we stand on giants' shoulders. We take him seriously here and moreover (as appropriate to recursion-theorists) we will jump from one giant to another, since this paper is mostly an exegesis of two fundamental works: Feferman's *Some applications of the notions of forcing and generic sets* [4] and Sacks' *Forcing with perfect closed sets* [19]. We hope the reader is not afraid of heights: our exercises are risky ones, since the two giants are in turn on the shoulders of others! Feferman [4] rests on the basic works of Cohen [2], who introduced forcing with finite conditions in the context of set theory; Sacks [19] relies on Spector [24], who realized—in recursion theory—the necessity of more powerful approximations than the finite ones.

To minimize the risk we will try to keep technicalities to a minimum, choosing to give priority to the methodology of forcing. We do not suppose any previous knowledge of forcing in the reader, but we do require some acquaintance with recursion theory. After all, our interest lies in the applications of the forcing method to the study of various recursion-theoretic notions of degrees. The farther we go, the deeper we plunge into recursion theory.

In Part I only very basic notions and results are used, like the definitions of the arithmetical hierarchy and of the jump operator and their relationships. In Part II we need Kleene's system of notations $\mathcal{O}$ and some structural properties of Δ_1^1 and Π_1^1 sets, as well as the elementary theory of hyperdegrees. In Part III some notions on admissible sets and the constructible universe will help. However we explicitly state along the way the results we need and define the basic notions, in the hope that the paper can be read with even a partial, sketchy knowledge of, say, Rogers' or Shoenfield's books [17], [22].

In our forthcoming book [15] the reader will find a treatment of classical recursion theory from (its) Middle Age to the present day, with all the details he or she may regret were left out of this paper. Since the present work is going to appear in three separate parts, we include here its table of contents.

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PART I. FORCING IN ARITHMETIC

§1. Feferman forcing in arithmetic. The approach we follow in this section is standard and goes back to Cohen [2]: he had the idea that truth can be approximated by finite information. The adaptation of his work to the context of arithmetic was made by Feferman [4]. Although the technique remains basically the same in the transfer from set theory to arithmetic, many complications vanish and forcing in arithmetic is a transparent and easy way to approach forcing.

Let $\mathscr{L}$ be a language for first-order arithmetic with equality such that the complexity of formulas of $\mathscr{L}$ reflects soundly the arithmetical hierarchy. For example we can have as atomic formulas the graphs of sum and product or (if we want to avoid the use of the solution to Hilbert's tenth problem in the proof of the expressibility of Σ_1^0 formulas as existential formulas of $\mathscr{L}$) even all the primitive recursive predicates. We also suppose that $\mathscr{L}$ contains a constant $\bar{n}$ for each natural number n. Finally, $\mathscr{L}$ has a set constant X and the membership relation $\in$.

Given a set $A \subseteq \omega$ and a sentence φ of $\mathscr{L}$, we define inductively on the construction of φ the *truth* of φ in the structure of arithmetic augmented by the constant A (written $A \models \varphi$) as follows:

If φ is atomic and does not contain X, $A \models \varphi$ iff φ is true in arithmetic.

If $\varphi \equiv \bar{n} \in X$, $A \models \varphi$ iff $n \in A$.

If $\varphi \equiv \sim\psi$, $A \models \varphi$ iff not $A \models \psi$.

If $\varphi \equiv \psi_0 \vee \psi_1$, $A \models \varphi$ iff $A \models \psi_0$ or $A \models \psi_1$.

If $\varphi \equiv \exists x\psi(x)$, $A \models \varphi$ iff for some n, $A \models \psi(\bar{n})$.

Of course from the definition it also follows:

If $\varphi \equiv \psi_0 \wedge \psi_1$, $A \models \varphi$ iff $A \models \psi_0$ and $A \models \psi_1$.

If $\varphi \equiv \forall x\psi(x)$, $A \models \varphi$ iff for all n, $A \models \psi(\bar{n})$.

As we have already announced, the basic idea of Cohen was to try to approximate truth by using finite approximations, i.e. instead of all of A we only use strings $\sigma, \tau \ldots$ of 0's and 1's. The intent is that if σ tells us that φ holds, then $A \models \varphi$ for all $A \supseteq \sigma$ (we will usually identify a set and its characteristic function). The whole idea makes sense because (by definition) we only use finite pieces of A to determine truth for atomic formulas.

DEFINITION. Let φ be a sentence of $\mathscr{L}$. Then $\sigma \Vdash \varphi$ (*σ forces φ*) is defined by induction on φ as follows:

If φ is atomic and does not contain X, $\sigma \Vdash \varphi$ iff φ is true in arithmetic.

If $\varphi \equiv \bar{n} \in X$, $\sigma \Vdash \varphi$ iff $\sigma(n) = 1$.

If $\varphi \equiv \sim\psi$, $\sigma \Vdash \varphi$ iff $(\forall \tau \supseteq \sigma)(\tau \nVdash \psi)$.

If $\varphi \equiv \psi_0 \vee \psi_1$, $\sigma \Vdash \varphi$ iff $\sigma \Vdash \psi_0$ or $\sigma \Vdash \psi_1$.

If $\varphi \equiv \exists x\psi(x)$, $\sigma \Vdash \varphi$ iff for some n, $\sigma \Vdash \psi(\bar{n})$.

The definition of forcing is hence similar to that of truth, except for the negation case. The reason we do not let $\sigma \Vdash \sim\psi$ iff $\sigma \nVdash \psi$ is that σ only gives an approximation to truth, and the fact that $\sigma \nVdash \psi$ could simply mean that σ does not contain enough information to decide ψ, but some bigger string could decide it. If this does not happen, i.e. if no extension of σ forces ψ, then we say that σ forces $\sim\psi$.

Se let $\sigma \Vdash \bar{n} \in X$ if $\sigma(n) = 1$ by definition. Note that similarly we have $\sigma \Vdash \bar{n} \notin X$ iff $\sigma(n) = 0$: in fact if $\sigma(n) = 0$ then $\sigma \nVdash \bar{n} \in X$ and similarly if $\tau \supseteq \sigma$ then $\tau \nVdash \bar{n} \in X$, so $\sigma \Vdash \bar{n} \notin X$; and if $\sigma \Vdash \bar{n} \notin X$ then $\sigma(n)$ must be defined (otherwise σ can be extended to a τ s.t. $\tau(n) = 1$, and $\tau \Vdash \bar{n} \in X$) and cannot be 1, so $\sigma(n) = 0$.

Note that, because of the way we decided to choose to force negation, forcing does not behave with respect to conjunction and universal quantifier the same way truth does. We have instead:

If $\varphi \equiv \psi_0 \wedge \psi_1$, from $\varphi \equiv \sim(\sim \psi_0 \vee \sim \psi_1)$ we have

$$\begin{aligned}\sigma \Vdash \varphi &\text{ iff } (\forall \tau \supseteq \sigma) \text{ not } (\sigma \Vdash \sim\psi_0 \text{ or } \sigma \Vdash \sim\psi_1)\\ &\text{ iff } (\forall \tau \supseteq \sigma)(\sigma \nVdash \sim\psi_0 \text{ and } \sigma \nVdash \sim\psi_1)\\ &\text{ iff } (\forall \tau \supseteq \sigma)(\exists \tau_1, \tau_2 \supseteq \tau)(\tau_1 \Vdash \psi_0 \text{ and } \tau_2 \Vdash \psi_1).\end{aligned}$$

If $\varphi \equiv \forall x\psi(x)$, from $\varphi \equiv \sim\exists x \sim\psi(x)$ we have

$$\sigma \Vdash \varphi \text{ iff } (\forall n)(\forall \tau \supseteq \sigma)(\exists \tau_1 \supseteq \tau)(\tau_1 \Vdash \psi(\bar{n})).$$

$$\sigma \Vdash \sim\ \sim\varphi \text{ iff } (\forall \tau \supseteq \sigma)(\exists \tau_1 \supseteq \tau)(\tau_1 \Vdash \varphi).$$

In particular, if we define $\sigma \Vdash^{w} \varphi$ (*σ weakly forces φ*) iff $\sigma \Vdash \sim\ \sim\varphi$ then:

$\sigma \Vdash \forall x\varphi(x)$ iff $(\forall n)(\sigma \Vdash^{w} \varphi(\bar{n}))$,

$\sigma \Vdash \psi_0 \wedge \psi_1$ iff $\sigma \Vdash^{w} \psi_0$ and $\sigma \Vdash^{w} \psi_1$.

Note that nothing is magic above. We could have let by definition $\sigma \Vdash \forall x\varphi(x)$ iff $(\forall n)(\sigma \Vdash \varphi(\bar{n}))$ and $\sigma \Vdash \psi_0 \wedge \psi_1$ iff $\sigma \Vdash \psi_0$ and $\sigma \Vdash \psi_1$. Then the connectives $\sim$, $\vee$, $\wedge$ and the quantifiers $\exists$, $\forall$ are regarded as independent and what would fail would be, e.g., $\sigma \Vdash \exists x\varphi(x)$ iff $\sigma \Vdash \sim \forall x \sim \varphi(x)$, which now holds by definition. The choice of the definition of $\Vdash$ is usually made with an eye to the applications: we will introduce different definitions in the following sections.

PROPOSITION 1.1. (a) *Monotonicity*: *if* $\sigma \Vdash \varphi$ *and* $\tau \supseteq \sigma$, *then* $\tau \Vdash \varphi$.

(b) *Consistency*: *not* ($\sigma \Vdash \varphi$ *and* $\sigma \Vdash \sim\varphi$).

(c) *Quasi-completeness*: $(\forall\sigma)(\exists\tau \supseteq \sigma)(\tau \Vdash \varphi$ *or* $\tau \Vdash \sim\varphi)$.

PROOF. (a) By induction on φ, using for negation the fact that every extension of τ is an extension of σ.

(b) By definition of forcing for negation.

(c) Given σ, either for some $\tau \supseteq \sigma$, $\tau \Vdash \varphi$ or (by definition) $\sigma \Vdash \sim\varphi$. □

PROPOSITION 1.2. *Relationships between forcing and weak forcing.*

(a) *If* $\sigma \Vdash \varphi$ *then* $\sigma \Vdash^{w} \varphi$, *but not conversely.*

(b) $\sigma \Vdash \sim\varphi$ *iff* $\sigma \Vdash^{w} \sim\varphi$.

PROOF. (a) The implication follows from monotonicity. For a counterexample to the opposite implication, let, e.g., $\varphi \equiv \bar{n} \in X \vee \bar{n} \notin X$ for a fixed n, and σ be not defined on n. Then $\sigma \nVdash \varphi$ (by definition of forcing on disjunction and atomic formulas) but $\sigma \Vdash^{w} \varphi$ (since every $\tau \supseteq \sigma$ is either defined on n or can be extended to a string defined on n).

(b) One direction follows from (a). For the other, given σ and any $\tau \supseteq \sigma$, from $\sigma \Vdash^{w} \sim\varphi$ we have $\tau_1 \Vdash \sim\varphi$ for some $\tau_1 \supseteq \tau$. To have $\sigma \Vdash \sim\varphi$ we want $\tau \nVdash \varphi$, but if $\tau \Vdash \varphi$ then $\tau_1 \Vdash \varphi$ by monotonicity, contradicting consistency. □

We now come back to the whole set A.

DEFINITION. $A \Vdash \varphi$ (*A forces* φ) if for some $\sigma \subseteq A$, $\sigma \Vdash \varphi$.

We now have two ways to check if a sentence φ holds for A: one using A entirely ($A \models \varphi$) and one using only finite parts of A ($A \Vdash \varphi$). We would like these two ways to coincide, but we cannot expect them to always be the same. Suppose for example that φ is $(\forall x)(x \in X)$: then for $A = \omega$ we have $A \models \varphi$, but for no σ we have $\sigma \Vdash \varphi$, since even if σ gives value 1 to all x's for which it is defined, we can always extend it with some τ which is 0 on some arguments. But suppose that A is such that for every sentence φ we can decide if φ holds for A *or not* only using finite parts of A. Then by induction on φ we can certainly prove that $A \models \varphi$ iff $A \Vdash \varphi$.

DEFINITION. (a) A is *n-generic* if for all sentences $\varphi \in \Sigma_n^0$ either $A \Vdash \varphi$ or $A \Vdash \sim\varphi$ (Hinman [7]).

(b) A is *ω-generic* or generic for arithmetic if the same happens for all arithmetical sentences, i.e. if A is n-generic for all n (Feferman [4]).

Since every set is 0-generic, we will tacitly assume $n \geq 1$ in the following. Also, note that if A is n-generic and $\varphi \in \Pi_n^0$ then $A \Vdash \varphi$ or $A \Vdash \sim\varphi$ ($\varphi \equiv \sim\psi$ where $\psi \in \Sigma_n^0$, so $A \Vdash \psi$ or $A \Vdash \sim\psi$: in the latter case $A \Vdash \varphi$, in the first $A \Vdash \sim\sim\psi$ as well, hence $A \Vdash \sim\varphi$).

THEOREM 1.3. *Forcing* = *Truth* (*global version*).

(a) *A is n-generic iff for any sentence* $\varphi \in \Sigma_n^0 \cup \Pi_n^0$, $A \models \varphi$ *iff* $A \Vdash \varphi$.

(b) *Similarly for A ω-generic and* φ *arithmetical.*

PROOF. (a) Let A be n-generic. By induction on φ: the only case in which forcing is not defined as truth is negation. Let $\varphi \equiv \sim\psi$. Then

$$\begin{aligned} A \models \sim\psi \text{ iff } & A \not\models \psi \text{ (by definition of truth)} \\ \text{iff } & A \nVdash \psi \text{ (by induction hypothesis)} \\ \text{iff } & A \Vdash \sim\psi \text{ (by genericity and consistency).} \end{aligned}$$

To prove the converse, let $\varphi \in \Sigma_n^0$ be a sentence. Since $A \models \varphi$ or $A \models \sim \varphi$, by hypothesis $A \Vdash \varphi$ or $A \Vdash \sim\varphi$. Hence A is n-generic.

(b) from (a).

PROPOSITION 1.4. *Forcing = Truth (local version).*

(a) *For any σ and any sentence $\varphi \in \Sigma_n^0 \cup \Pi_n^0$, $\sigma \Vdash^{w} \varphi$ iff for every n-generic set $A \supseteq \sigma$, $A \models \varphi$.*

(b) *Similarly for φ arithmetical and ω-generic sets.*

PROOF. (a) Let $\sigma \Vdash^{w} \varphi$ and A be n-generic: if $A \supseteq \sigma$ then $A \Vdash \sim\sim\varphi$ and by genericity $A \models \sim\sim\varphi$, so $A \models \varphi$. Suppose now $\sigma \nVdash^{w} \varphi$: then for some $\tau \supseteq \sigma$, every extension of τ does not force φ, so $\tau \Vdash \sim\varphi$. Let $A \supseteq \tau$ be n-generic (see 1.6 for such an A): $A \supseteq \sigma$ and $A \Vdash \sim\varphi$, so $A \models \sim\varphi$ and $A \not\models \varphi$.

(b) from (a).

Because of the way genericity is defined (every property of a certain arithmetical complexity is decided by a finite amount of information) any two generic sets code the same infinitary information expressible within that complexity. So generic sets all look alike. Part (a) of 1.4 makes this fact precise. The following fact gives a property of the same flavour. Given a set A, let A_i be the ith component of A, defined by $x \in A_i \Leftrightarrow \langle x, i\rangle \in A$ (where $\langle x, i\rangle$ is a fixed recursive one-one onto function). If A is n-generic (ω-generic) so are all the A_i's. Also the complement of a generic set is generic and any set which differs finitely from a generic set is generic.

Of course, since truth and forcing are related as just shown, we may expect to get definability results for forcing similar to those known for truth. In the next proposition we identify strings and formulas with their codes.

Recall that $\mathbf{0}^{(n)}$ is the highest Turing-degree containing $\Sigma_n^0 \cup \Pi_n^0$ sets or, equivalently, the degree of the truth-predicate for Σ_n^0 sets. Similarly, $\mathbf{0}^{(\omega)}$ is the degree of the truth-predicate for arithmetic. Formally, define by induction $0^{(n)}$ as: $0^{(0)} = \varnothing$, $0^{(n+1)}$ = jump of $0^{(n)}$. Moreover, let $\langle x, n\rangle \in 0^{(\omega)} \Leftrightarrow x \in 0^{(n)}$. Then $\mathbf{0}^{(n)}$ is the degree of $0^{(n)}$, and $\mathbf{0}^{(\omega)}$ is the degree of $0^{(\omega)}$. See [17, Chapter 13].

THEOREM 1.5. *Definability of forcing.*

(a) *$\{(\sigma, \varphi)$: φ is a Σ_n^0 sentence and $\sigma \Vdash \varphi\}$ is Σ_n^0, hence recursive in $0^{(n)}$. Similarly for Π_n^0.*

(b) *$\{(\sigma, \varphi)$: φ is an arithmetical sentence and $\sigma \Vdash \varphi\}$ is recursive in $0^{(\omega)}$.*

(c) *For any fixed arithmetical sentence φ, $\{\sigma: \sigma \Vdash \varphi\}$ is arithmetic.*

PROOF. (a) By induction on φ. First note that to check if φ is an arithmetical sentence, and in case it is to see if it is Σ_n^0 or Π_n^0 for a given n, is a recursive procedure. It is also recursive to decide whether $\sigma \Vdash \varphi$ or not, for φ atomic. The crucial cases are:

$\varphi \equiv \exists x\, \psi(x)$: $\sigma \Vdash \varphi$ iff $(\exists n)(\sigma \Vdash \psi(\bar{n}))$,

$\varphi \equiv \sim\psi$: $\sigma \Vdash \varphi$ iff $(\forall\tau)(\tau \supseteq \sigma \Rightarrow \tau \nVdash \psi)$.

They speak for themselves.

(b) By part (a), there is a uniform procedure to reduce forcing for Σ_n^0 sentences to $0^{(n)}$. By uniformity, forcing for arithmetical sentences is recursive in $0^{(\omega)}$.

(c) By (a), since if φ is arithmetical then $\varphi \in \Sigma_n^0$ for some n. □

The results above are actually the best possible. E.g. part (b) cannot be improved to read: $\{(\sigma, \varphi)$: σ is an arithmetical sentence $\wedge\, \sigma \Vdash \varphi\}$ is arithmetical, since otherwise by the proof of the next result we could get an arithmetical ω-generic set. This observation is the obvious analogue of Tarski's theorem to the effect that arithmetical truth is not arithmetical.

A final observation with respect to the relationship between forcing and truth is in order: *if A is not ω-generic then there is a sentence ψ such that $A \Vdash \psi$ but $A \models \sim \psi$.* In fact, if A is not ω-generic then for some φ, $A \nVdash \varphi$ and $A \nVdash \sim\varphi$. Let $\psi \equiv \sim\sim(\exists\sigma \subseteq X)(\sigma \Vdash \varphi \vee \sim\varphi)$: ψ is arithmetical by 1.5, and by definition $A \models \sim \psi$. Moreover $(\exists\sigma \subseteq X)(\sigma \Vdash \varphi \vee \sim\varphi)$ is true of every ω-generic set, hence (by 1.4) every string weakly forces it, i.e. every string forces ψ. In particular $A \Vdash \psi$. Hence forcing does not imply truth, and this suggests that the use of the word "forcing" is slightly misleading. Forcing is an open approximation to truth and agrees with it on a comeager subset of ${}^{\omega}2$ in its usual topology, but it is not an approximation from inside or from outside.

THEOREM 1.6. GENERIC EXISTENCE THEOREM. (a) *There are comeager many ω-generic sets. (In fact the family of ω-generic sets is comeager in the subspace of ${}^{\omega}2$ determined by any given string.)*

(b) *n-generic sets are not Σ_n^0 or Π_n^0, but there are n-generic sets recursive in $0^{(n)}$.*

(c) *ω-generic sets are not arithmetical, but there are ω-generic sets recursive in $0^{(\omega)}$.*

PROOF. (a) Since A is ω-generic iff A is n-generic for every n, and the intersection of countably many comeager sets is comeager, we prove that for all n there are comeager many n-generic sets. But A is n-generic iff for all sentences $\varphi \in \Sigma_n^0$, $A \Vdash \varphi$ or $A \Vdash \sim\varphi$. Hence

$$n\text{-generic sets} = \bigcap_{\varphi \in \Sigma_n^0} \{A: A \Vdash \varphi \text{ or } A \Vdash \sim\varphi\}.$$

So it is enough to note that for fixed φ the set $\{A: A \Vdash \varphi \text{ or } A \Vdash \sim\varphi\}$ is open dense. Indeed it is open because, for example, $A \Vdash \varphi$ iff $(\exists\sigma \subseteq A)(\sigma \Vdash \varphi)$, and it is dense by quasi-completeness. A similar proof works above any given string.

(b) To find an n-generic set recursive in 0^n, we build it by stages. We have a recursive enumeration of the Σ_n^0 sentences. Suppose that at stage s we have already defined an initial segment τ of our n-generic set, and that φ is the first Σ_n^0 sentence we have not yet considered. We search for a proper extension $\sigma \supseteq \tau$ such that $\sigma \Vdash \varphi$ or $\sigma \Vdash \sim\varphi$: we know it exists by quasi-completeness, and by definability of forcing the procedure is recursive in $0^{(n)}$.

For the negative part, we prove that *if A is n-generic it does not have any infinite Σ_n^0 subset* (since an n-generic set is infinite, this proves that $A \notin \Sigma_n^0$). Let $B \in \Sigma_n^0$ be infinite. If $B \subseteq A$ then if $\varphi \equiv (\forall x)(x \in B \Rightarrow x \in X)$, φ is a Π_n^0 formula true of A, so by forcing $=$ truth $A \Vdash \varphi$ and for some $\sigma \subseteq A$, $\sigma \Vdash \varphi$. Let τ be a string extending σ and giving value 0 to some element $x \in B$ not in the domain of σ (B is infinite, so τ exists). Then if C is n-generic extending τ we get a contradiction, because $C \supseteq \tau$ (so in particular $x \notin C$) and $C \supseteq \sigma$ (so $C \Vdash \varphi$, and by forcing $=$ truth $C \models \varphi$ and $B \subseteq C$, hence $x \in C$).

Similarly we can prove that if A is n-generic then $A \notin \Pi_n^0$, by proving that A does not have any infinite Π_n^0 superset.

(c) The positive part is proved similarly to (b), the negative part actually follows from (b). □

We did the proof in some detail because it is very typical of the kind of arguments used over and over when dealing with generic sets. Other typical examples are:

PROPOSITION 1.7. *There are hyperarithmetical sets not implicitly definable in arithmetic* (*Feferman* [4]).

PROOF. By 1.6(c) we have ω-generic sets recursive in $\mathbf{0}^{(\omega)}$, hence hyperarithmetical. Suppose an ω-generic set A is implicitly definable in arithmetic: then for some arithmetical φ we would have $X = A$ iff $A \models \varphi$ iff $A \Vdash \varphi$. So for some $\sigma \subseteq A$, $\sigma \Vdash \varphi$. Contradiction, because then $A = B$ for any ω-generic $B \supseteq \sigma$. □

PROPOSITION 1.8. *The set of arithmetical sets is not arithmetical* (*Addison* [1]).

PROOF. Suppose otherwise. Then some arithmetical φ would satisfy X arithmetical $\Leftrightarrow X \models \varphi$ for all X. Fix n such that $\varphi \in \Sigma_n^0$ and let A be n-generic and arithmetical (e.g. let A be recursive in $0^{(n)}$). Then $A \Vdash \varphi$ and for some $\sigma \subseteq A$, $\sigma \Vdash \varphi$. It follows that every n-generic set extending σ is arithmetical, a contradiction because there are uncountably many such sets (by 1.6(a)) but only countably many arithmetical sets. □

The same kind of argument gives: the set of Δ_n^0 sets is not Σ_n^0. An elaboration of the proof gives the best possible result: the set of Δ_n^0 sets is not Π_{n+2}^0 (see Hinman [7]).

We conclude this section with some alternative characterizations of genericity. Sometimes in the literature the definition of 1-genericity is stated in a different form, suggested by the proof of Friedberg's completeness criterion for the jump operator (see 3.2):

> A is *1-generic* iff for every e and x, $\{e\}^A(x)$ is defined or $\{e\}^A(x)$ is strongly undefined, i.e. $(\exists\sigma \subseteq A)(\forall\tau \supseteq \sigma)(\{e\}^\tau(x)$ is undefined);

i.e. not only is it true that if a recursive computation from A converges then a finite amount of information on A will tell us (this holds for every A): the same is true also when the computation diverges. The next criterion tells us that this definition is equivalent to the one we gave (any Σ_1^0 set of strings determines the same subset of ${}^\omega 2$ as one of $\{\sigma : \{e\}^\sigma(x)$ converges$\}$ for some e, x):

PROPOSITION 1.9. (a) *A is n-generic iff for any Σ_n^0 set of strings S, there is a $\sigma \subseteq A$ such that either $\sigma \in S$ or $(\forall\tau \supseteq \sigma)(\tau \notin S)$.*

(b) *Similarly for A ω-generic and S arithmetical* (*Posner* [16] *for* $n = 1$, *Jockusch* [8]).

PROOF. (a) Let A be n-generic and $S \in \Sigma_n^0$. Let φ be $(\exists\tau \subseteq X)(\tau \in S)$: $\varphi \in \Sigma_n^0$ so by n-genericity, $A \Vdash \varphi$ or $A \Vdash \sim\varphi$, and for some $\sigma \subseteq A$ either $\sigma \Vdash \varphi$ or $\sigma \Vdash \sim\varphi$.

If $\sigma \Vdash \varphi$ then by forcing = truth, $A \models \varphi$ and for some $\tau \subseteq A$, $\tau \in S$.

If $\sigma \Vdash \sim\varphi$ then suppose for some $\tau \supseteq \sigma$, $\tau \in S$. Let $B \supseteq \tau$ be n-generic. $B \Vdash \sim\varphi$ since $B \supseteq \sigma$. But since $B \supseteq \tau$ and $\tau \in S$, also $B \models \varphi$ and $B \Vdash \varphi$, contradiction.

For the converse, let $\varphi \in \Sigma_n^0$ and $S = \{\sigma : \sigma \Vdash \varphi\}$. $S \in \Sigma_n^0$ by 1.5(a), so let $\sigma \subseteq A$ be as in the hypothesis. If $\sigma \in S$ then $A \Vdash \varphi$; if for every $\tau \supseteq \sigma$, $\tau \notin S$ then by definition $A \Vdash \sim\varphi$. Hence A is n-generic.

(b) from (a). □

The next characterization ties up the development of forcing used here with the one adopted in Shoenfield [23]. Note that (the characteristic function of) a set can be thought of as the union of a set of compatible strings. We now see which condition on a maximal set of compatible strings guarantees the genericity of the set determined by the union. Let us call a set of strings D dense if every string has an extension in D.

PROPOSITION 1.10. *A is ω-generic iff $\{\sigma\colon \sigma \subseteq A\}$ meets every dense arithmetical set of strings.*

PROOF. If A is ω-generic, let D be arithmetical and dense. By 1.9, for some $\sigma \subseteq A$ either $\sigma \in D$ (hence D is met), or $(\forall \tau \supseteq \sigma)(\tau \notin D)$: this last is ruled out by density. For the converse, given φ consider $D = \{\sigma;\ \sigma \Vdash \varphi \text{ or } \sigma \Vdash {\sim}\varphi\}$. D is arithmetical by definability of forcing (1.5) and dense by quasi-completeness. So D is met and for some $\sigma \subseteq A$, $\sigma \Vdash \varphi$ or $\sigma \Vdash {\sim}\varphi$. $\square$

Note that we do not have a similar characterization for n-genericity. What is true is that every n-generic set meets every dense Σ^0_n set of strings, but for the converse it suffices to meet either every Δ^0_{n+1} dense set of strings (to use the same proof as in 1.10) or every Σ^0_n set of strings dense over A (to use 1.9).

Observe that every Σ^0_{n+1} set of strings determines the same open subset of ${}^{\omega}2$ as some Π^0_n set of strings. (If $S \in \Sigma^0_{n+1}$, say $\sigma \in S$ iff $(\exists u)R(u, \sigma)$ with $R \in \Pi^0_n$, let T be the set of strings τ such that $R(u, \sigma)$ holds for some $\sigma \subseteq \tau$ and $u < |\tau|$. Then T is a Π^0_n set of strings which defines the same subset of ${}^{\omega}2$ as S.) It follows that a set A meets every dense Π^0_n set of strings iff A meets every dense Δ^0_{n+1} set of strings iff A meets every dense Σ^0_{n+1} set of strings. This condition on A, called *weak $(n + 1)$-genericity*, is intermediate in strength between n-genericity and $(n + 1)$-genericity but is not equivalent to either. In particular, no Σ^0_{n+1} or Π^0_{n+1} set can be weakly $(n + 1)$-generic and thus, by Theorem 1.6(b), for each $n \geq 1$, there are n-generic sets which fail to be weakly $(n + 1)$-generic. Stuard Kurtz (still unpublished) has recently shown that a Turing degree contains a weakly 1-generic set iff it contains a hyperimmune set.

The last two results proved that the notion of genericity is independent of the notion of forcing. This is made explicit in Shoenfield [23], and allows an alternative approach to the matter discussed here: we could *define* genericity as in 1.9 (or 1.10) and weak-forcing as in 1.4. The generic existence theorem is then proved by meeting all dense sets of strings considered in the definition. To prove however that weak forcing = truth (note that using weak or strong forcing here does not matter, since the truth of a sentence or of its double negation is equivalent) we have to reintroduce forcing to mimic truth, and *prove* that a generic set forces a sentence or its negation and that $\sigma \Vdash^{w} \varphi$ iff $\sigma \Vdash {\sim}{\sim}\varphi$. The two approaches are clearly equivalent. However note that the complexity of weak forcing in the arithmetical hierarchy is higher than that of forcing.

We finish the section by introducing two strengthenings of the notion of genericity. The first one is simply a relativization to a given set B: we add to the language $\mathscr{L}$ a set constant that has to be interpreted as B. We then get the notion of A being ω-generic over B, and all the theorems of this section go through, simply relativized to B. In particular,

A is ω-generic over B iff $\{\sigma\colon \sigma \subseteq A\}$ meets every dense set of strings arithmetical in B.

A second generalization consists of considering a language $\mathscr{L}'$ similar to $\mathscr{L}$, where however instead of having only one set constant X for the generic set, we have two set constants X_1 and X_2 for two generic sets. This leads to the notion of a pair (A, B) being ω-generic, and again the theorems of this section go through, in particular:

(A, B) is ω-generic iff $\{(\sigma, \tau): \sigma \subseteq A \wedge \tau \subseteq B\}$ meets every dense arithmetical set of pairs of strings.

(The notion of density for sets of pairs of strings is the natural one, relative to the partial ordering $(\sigma, \tau) \subseteq (\sigma', \tau')$ iff $\sigma \subseteq \sigma'$ and $\tau \subseteq \tau'$.) This second notion of forcing is usually referred to as *product forcing*. The next proposition connects the various notions introduced so far:

PROPOSITION 1.11. *(A, B) is ω-generic iff A is ω-generic and B is ω-generic over A iff B is ω-generic and A is ω-generic over B.*

PROOF. Let (A, B) be ω-generic. To prove that A is ω-generic, given D dense arithmetical set of strings consider $D' = \{(\sigma, \tau): \sigma \in D\}$. D' is still dense arithmetical, hence for some $\sigma \subseteq A$ and $\tau \subseteq B$, $(\sigma, \tau) \in D'$ and hence $\sigma \in D$. So D is met and A is ω-generic.

To prove that B is ω-generic over A, let D be dense and arithmetical in A. Then "$\tau \in D$" can be expressed in the forcing language relative to A (we will abuse language and avoid the distinction). Let $\sigma_0 \subseteq A$ be such that $\sigma_0 \Vdash$ "D is dense", and let $D' = \{(\sigma, \tau): \sigma_0 \subseteq \sigma \wedge \sigma \Vdash \tau \in D\}$. Given $(\sigma', \tau') \supseteq (\sigma_0, \emptyset)$ some $\sigma \supseteq \sigma'$ forces some $\tau \supseteq \tau'$ to belong to D. So D' is dense above $(\sigma_0, \emptyset)$ and arithmetical: for some $\sigma \subseteq A$ and $\tau \subseteq B$, $(\sigma, \tau) \in D'$ and $\sigma \Vdash \tau \in D$, so $A \Vdash \tau \in D$ and by forcing $=$ truth $\tau \in D$. So D is met and B is ω-generic over A.

Finally, let A be ω-generic and B ω-generic over A. Let D be a dense arithmetical set of pairs of strings. Consider $D' = \{\tau: (\exists \sigma \subseteq A)((\sigma, \tau) \in D)\}$: it is arithmetical in A. To prove that it is dense, fix τ'. Then $D'' = \{\sigma: (\exists \tau \supseteq \tau')((\sigma, \tau) \in D)\}$ is dense and arithmetical, and by genericity of A there is $\sigma \subseteq A$ in it. Hence D' is dense and by genericity over A of B, there is $\tau \subseteq B$ in it. Hence D is met and (A, B) is ω-generic. $\square$

§2. Sacks forcing in arithmetic. In §1 we introduced forcing as a way to approximate truth by finite information. For many of the applications we have in mind, however, finite approximations are too weak. Sacks [19] introduced a more powerful way, using a special kind of infinite approximations: closed perfect trees. We view *closed perfect trees* as nonempty sets of strings P such that each string in P has two incompatible extensions in P. The name comes from the fact that each such set determines a set of reals (or subsets of ω) which is closed and perfect (i.e. with no isolated branch) in the usual topology of ${}^{\omega}2$ (with the basic open sets determined by finite strings). We will write $A \in P$ to indicate that A is a branch of the tree determined by P.

As a first attempt to define the forcing relation we decide to use arithmetical perfect closed sets $P, Q, R, \ldots$ and to consider the behaviour of every branch of such trees. The natural definition of $P \Vdash \varphi$ for φ an arithmetical sentence would be, by induction on φ:

if φ is atomic and does not contain X, $P \Vdash \varphi$ iff φ is true;
if $\varphi \equiv \bar{n} \in X$, $P \Vdash \varphi$ iff $(\forall A \in P)(\bar{n} \in A)$ iff $(\forall \sigma \in P)(\sigma(n)$ defined $\Rightarrow \sigma(n) = 1)$;
if $\varphi \equiv \sim \psi$, $P \Vdash \varphi$ iff $(\forall Q \subseteq P)(Q \nVdash \psi)$;
if $\varphi \equiv \psi_0 \vee \psi_1$, $P \Vdash \varphi$ iff $P \Vdash \psi_0$ or $P \Vdash \psi_1$;
if $\varphi \equiv \exists x \psi(x)$, $P \Vdash \varphi$ iff for some n, $P \Vdash \psi(\bar{n})$.

This is in perfect analogy with the definition of §1: the forcing relation mimics the truth relation, except for the negation case. And a condition forces $\sim\psi$ if no stronger condition forces ψ (we consider Q stronger than P if $Q \subseteq P$, i.e. if it determines fewer sets). But there is one problem: since the set of arithmetical perfect closed sets is not arithmetical and in the definition of $P \Vdash \sim\psi$ we quantify over it, even for very simple formulas (e.g. negations of atomic formulas) the forcing relation is apparently not arithmetic. We hence retreat from our first attempt and brutally define:

DEFINITION. Let φ be a sentence of $\mathscr{L}$ and P be an arithmetical perfect closed set. Then $P \Vdash \varphi$ iff $(\forall A \in P)(A \models \varphi)$.

We can still think of the forcing relation as an approximation to truth. But we immediately run into trouble when we try to prove the analogue of 1.1: monotonicity and consistency are trivial, but quasi-completeness is no longer immediate. To prove it, we need a way to control truth along the branches of our forcing conditions, and the first thought is to use the approach of §1: finite forcing!

DEFNITION. If P is a perfect closed set and φ is an arithmetical sentence of $\mathscr{L}$, $\sigma \Vdash^P \varphi$ (*local forcing on P*) is the relativization of $\sigma \Vdash \varphi$ to P.

What the definition means is that we repeat the definition of $\sigma \Vdash \varphi$, but we only use conditions in P. So $\sigma \Vdash^P \varphi$ is different from $\sigma \in P \wedge \sigma \Vdash \varphi$. E.g. $\sigma \Vdash^P \sim\varphi$ iff $\sigma \in P \wedge (\forall \tau \supseteq \sigma)(\tau \in P \Rightarrow \tau \nVdash^P \varphi)$. In other words, we replace the original full tree of strings on which we worked in §1 with the homeomorphic one P. If P is in particular arithmetical, the homeomorphism is arithmetical as well and hence we have for example:

PROPOSITION 2.1. *Definability of local forcing. If P is arithmetical then*:

(a) *for any fixed n*, $\{(\sigma, \varphi)$: *φ is a Σ_n^0 sentence and* $\sigma \Vdash^P \varphi\}$ *is arithmetical*;

(b) $\{(\sigma, \varphi)$: *φ is an arithmetical sentence and* $\sigma \Vdash^P \varphi\}$ *is recursive in* $0^{(\omega)}$.

PROOF. (a) By relativization of 1.5(a), the relation is recursive in $P^{(n)}$. Since P is arithmetical, $P^{(n)}$ is arithmetical.

(b) Similar, since $P^{(\omega)}$ is recursive in $0^{(\omega)}$ when P is arithmetical. $\square$

We will freely use in the future properties of local forcing on P that are analogous to properties of the forcing relation studied in §1, e.g. the fact that for generic sets, forcing = truth. Note, however, that a generic set relative to local forcing on P is a branch of P.

PROPOSITION 2.2. (a) *Monotonicity*: *if* $P \Vdash \varphi$ *and* $Q \subseteq P$, *then* $Q \Vdash \varphi$.

(b) *Consistency*: *not* ($P \Vdash \varphi$ *and* $P \Vdash \sim\varphi$).

(c) *Quasi-completeness*: $(\forall P)(\exists Q \subseteq P)(Q \Vdash \varphi$ *or* $Q \Vdash \sim\varphi)$.

PROOF. The first two assertions are trivial. For the third, fix P and φ. By local forcing we know that for some $\sigma \in P, \sigma \Vdash^P \varphi$ or $\sigma \Vdash^P \sim \varphi$, but we cannot, however, simply take as Q the full subtree of P above σ (i.e. all the strings in P extending σ). Suppose e.g. $\sigma \Vdash^P \varphi$: then every branch of P extending σ does force φ on P, but we would like it to make φ true, and this happens only for generic branches. But this also gives us the start for the solution: if $\varphi \in \Sigma_n^0$, then we build Q as a perfect subtree of P of n-generic sets. By definability (2.1) Q can be made arithmetical, and if e.g. $\sigma \Vdash^P \varphi$ then $A \supseteq \sigma$ for every branch A of Q, hence $A \Vdash^P \varphi$ and $A \models \varphi$. $\square$

We now turn to the set A.

DEFINITION. (a) $A \Vdash \varphi$ (*A forces* φ) if for some arithmetical perfect closed set P, $A \in P$ and $P \Vdash \varphi$.

(b) A is *Sacks ω-generic* or *Sacks generic for arithmetic* if for every arithmetical sentence φ, $A \Vdash \varphi$ or $A \Vdash \sim\varphi$.

By repeating the proof of 1.3 we immediately get:

PROPOSITION 2.3. *Forcing = Truth. A is Sacks ω-generic iff for every arithmetical sentence φ, $A \models \varphi$ iff $A \Vdash \varphi$.*

The quasi-completeness property of forcing allows us to prove the existence of generic sets:

PROPOSITION 2.4. GENERIC EXISTENCE THEOREM (SACKS [19]). (a) *There is a perfect tree of Sacks ω-generic sets (contained in any given condition).*

(b) *Sacks ω-generic sets are not arithmetical, but there are Sacks ω-generic sets recursive in* $0^{(\omega)}$.

PROOF. (a) Fix an arithmetical enumeration $\{\varphi_n\}_{n\in\omega}$ of the arithmetical sentences. We define for each string σ an arithmetical perfect closed set T_σ: the intersection of the T_σ's will be the desired tree of Sacks ω-generic sets. Let T_0 be a fixed arithmetical perfect tree (say, a given condition). Given T_σ, choose two incompatible strings τ_1, τ_2 in it. Let $T_{\sigma * i}$ $(i = 0, 1)$ be an arithmetical perfect tree contained in the full subtree of T_σ above τ_i and forcing φ_n or $\sim\varphi_n$, where $n + 1 =$ length of σ. Such a $T_{\sigma * i}$ exists by quasi-completeness.

(b) By checking the proof of 2.2(c) the proof of part (a) can easily be improved to give: the tree of Sacks ω-generic sets is built recursively in $0^{(\omega)}$. By taking a branch of it recursive in it (say, the leftmost branch) we get a Sacks ω-generic set recursive in $0^{(\omega)}$. The fact that Sacks ω-generic sets are not arithmetical can be proved by showing that they are not even implicitly definable in arithmetic. Otherwise, for some arithmetical φ, $X = A$ iff $A \Vdash \varphi$. Then for some P, $A \in P$ and $P \Vdash \varphi$. But we can build a whole subtree of Sacks ω-generic sets contained in P, and all its branches should be equal to A, contradiction. □

Although the proof of 2.4(b) tells us that a Sacks ω-generic set is not implicitly definable in arithmetic, as was the case for ω-generic sets (1.7), we cannot expect the same properties of ω-generic sets to hold for Sacks ω-generic sets as well. The reason is that perfect conditions are much stronger than finite ones, and they can code infinitary information. For example a Sacks ω-generic set can contain an infinite arithmetic subset. Actually, for any coinfinite arithmetical set, the set of strings giving value 1 to elements of it and arbitrary otherwise is a perfect closed arithmetical set and hence contains a whole tree of Sacks ω-generic sets, each one containing the given arithmetical set as subset. In §4 we will see a striking difference between the arithmetical degrees of ω-generic sets and of Sacks ω-generic sets. We see now another type of difference. Let $\varphi_n \equiv \bar{n} \in X \vee \bar{n} \notin X$: we already know that every string weakly forces every φ_n (see proof of 1.2(a)), but no finite string can force them all together, since to force φ_n it must be defined on n. This does not happen with Sacks forcing.

First let $P \Vdash^{w} \varphi$ iff $(\forall Q \subseteq P)(\exists R \subseteq Q)(R \Vdash \varphi)$. We do not define $P \Vdash^{w} \varphi$ directly as $P \Vdash \sim\sim\varphi$ since this is equivalent to $P \Vdash \varphi$, because forcing has been defined as truth on all branches of P. The present definition of weak forcing is, however, in accordance with that of §1.

PROPOSITION 2.5. FUSION LEMMA. *If $\{\varphi_n\}_{n\in\omega}$ is an arithmetical set of arithmetical sentences of bounded complexity and* $(\forall n)(P \Vdash^{w} \varphi_n)$, *then* $(\exists Q \subseteq P)(\forall n)(Q \Vdash \varphi_n)$.

PROOF. Suppose $(\forall n)(\varphi_n \in \Sigma^0_m)$ for a fixed m (the φ_n's are of bounded complexity). Build an arithmetical subtree Q of P of m-generic sets by using local forcing on P. Suppose now that for some n, $Q \nVdash \varphi_n$. Then for some $A \in Q$, $A \nvDash \varphi_n$ and hence $A \vDash \sim\varphi_n$. But A is m-generic over P, so there is $\sigma \subseteq A$, $\sigma \in P$ and $\sigma \Vdash^{P} \sim\varphi_n$. By the construction of 2.2(c) (i.e. again by local forcing on P) we get a condition $R \subseteq P$, $R \Vdash \sim\varphi_n$. This is a contradiction, because $P \Vdash^{w} \varphi_n$. □

The fusion lemma is actually the characteristic property of Sacks forcing. For example from it we can derive the quasi-completeness property by induction on the complexity of φ. The crucial case is $\varphi \equiv \exists x\psi(x)$. By induction hypothesis $(\forall x)(\forall P)(\exists Q \subseteq P)(Q \Vdash \psi(\bar{x})$ or $Q \Vdash \sim\psi(\bar{x}))$. Fix P: if for some x and $Q \subseteq P$, $Q \Vdash \psi(\bar{x})$ then we are done, since $Q \Vdash \varphi$. Otherwise we have $(\forall x)(\forall Q \subseteq P)$ $(Q \nVdash \psi(\bar{x}))$ so by induction hypothesis, $(\forall x)(\forall Q \subseteq P)(\exists R \subseteq Q)(R \Vdash \sim\psi(\bar{x}))$, i.e. $(\forall x)(R \Vdash^{w} \sim\psi(\bar{x}))$. By the fusion lemma for some $Q \subseteq R$, $(\forall x)(Q \Vdash \sim\psi(\bar{x}))$ and hence $Q \Vdash \sim\varphi$ by the definition of forcing.

This proof of quasi-completeness is not simpler than the one we gave in 2.2(c) (after all, already the proof of the fusion lemma is more complicated than that), but in §§7, 10 we will need appropriate forms of the fusion lemma for other purposes, and we will derive quasi-completeness from it.

A kind of forcing related to the Sacks forcing studied above is *pointed-forcing*. Again this was introduced by Sacks [19], and the forcing conditions are now arithmetical perfect trees with one additional feature.

DEFINITION. A closed perfect set is called *recursively pointed* if it is recursive in each of its branches.

The basic properties of recursively pointed trees are collected in the following.

PROPOSITION 2.6. *Let P be a recursively pointed tree. Then*:

(a) *If $Q \subseteq P$ and $Q \leq_T P$ then Q is recursively pointed and $Q \equiv_T P$.*

(b) *The degrees of branches of P are exactly the degrees above that of P.*

(c) *If $P \leq_T A$ then there is $Q \subseteq P$ recursively pointed and such that $Q \equiv_T A$* (*Sacks* [19]).

PROOF. (a) If $A \in Q \subseteq P$ then $Q \leq_T P \leq_T A$ so Q is recursively pointed; if A is the leftmost branch of Q then $P \leq_T A$ (by pointedness of P) and $A \leq_T Q$, so $P \leq_T Q$.

(b) If A is a branch of P, $P \leq_T A$ by pointedness. Let A be such that $P \leq_T A$. To find a branch B of P with the same degree as A, consider P as a function of the same degree from strings to strings defined by induction as: $P(\emptyset) = \emptyset$; $P(\sigma * 0)$, $P(\sigma * 1)$ = the smallest incompatible extensions of $P(\sigma)$ on P. If length $\sigma = n$ we say $P(\sigma)$ is a node of P of level n. Let $B = \bigcup_{\sigma\subseteq A} P(\sigma)$ be the branch of P determined by A. Then:

$B \leq_T P \oplus A$ by definition and $P \leq_T A$ by hypothesis, so $B \leq_T A$.

$A \leq_T P \oplus B$ and by pointedness $P \leq_T B$, so $A \leq_T B$.

(c) Define $Q \subseteq P$ by induction on σ. Given $Q(\sigma)$ (in P) we look at the next level of P above $Q(\sigma)$, and go right or left depending on whether the length of σ is in A or not. Since we also want Q to be perfect, we add then the two next nodes of P.

$Q \leq_T P \oplus A$ and, since $P \leq_T A$, $Q \leq_T A$. Q is pointed since given any $B \in Q$, by the uniformity used in the construction of Q we can actually recover it from $B \oplus P$; by pointedness $P \leq_T B$, so $Q \leq_T B$. Finally $A \leq_T Q$ because from any path B (of Q) and P itself we can recover A. But $P \leq_T B$, hence $A \leq_T B$. Choose a path $B \leq_T Q$ (e.g. the leftmost path) to have $A \leq_T Q$. □

It is instructive to see what happens if we require our forcing conditions to be not only arithmetical perfect trees, but also recursively pointed. The natural definition would be: A is *pointedly generic* if for every arithmetical sentence φ there is P arithmetical recursively pointed such that $A \in P$ and $P \Vdash \varphi$ or $P \Vdash \sim\varphi$. Sacks [19] claims that pointedly generic sets exist. There is, however, an obvious difficulty in extending the proof of 2.4 to this effect: we do not have the quasi-completeness property for recursively pointed trees. We prove now that in fact pointedly generic sets do not exist. Some of the computations we carry on will be useful in the future. Let A be pointedly generic.

(a) $(\forall n)\,(0^{(n)} \leq_T A)$.

Let φ be $0^{(n)} \leq_T X$: φ is an arithmetical sentence, so $A \Vdash \varphi$ or $A \Vdash \sim\varphi$. If $A \Vdash \varphi$ then by forcing = truth we are finished. Suppose $A \Vdash \sim\varphi$: for some arithmetical recursively pointed P, $A \in P$ and $P \Vdash \sim\varphi$. Hence, by definition of forcing, $(\forall B \in P)\,(0^{(n)} \not\leq_T B)$. Let m be big enough so that both $0^{(n)}$ and P are recursive in $0^{(m)}$. Then $P \leq_T 0^{(m)}$ and by 2.6(c) there is $Q \subseteq P$ recursively pointed and of degree $\mathbf{0}^{(m)}$. By pointedness, for any branch $B \in Q$, $0^{(m)} \leq_T B$. Contradiction, because $B \in P$ and $0^{(n)} \not\leq_T B$.

(b) $0^{(\omega)} \leq_T A^{(2)}$.

We use the following fact from Enderton and Putnam [3]: *for any set B, if $(\forall n)(0^{(n)} \leq_T B$ then $0^{(\omega)} \leq_T B^{(2)}$.* To get the conclusion it is enough to prove that $0^{(n)} \leq_T B^{(2)}$ uniformly in n. This comes from the observation that since $0^{(n+1)} = (0^{(n)})'$ then for some e,

$$(\forall x)(x \in 0^{(n+1)} \Leftrightarrow \{e\}^{0^{(n)}}(x)\downarrow)$$

and if we express $0^{(n)}$ recursively in B (by hypothesis), then we can obtain such an e recursively in $B^{(2)}$ (the part in brackets is recursive in B' and the quanitfier is responsible for one more jump).

(c) $0^{(\omega)}$ is arithmetical.

By above $0^{(\omega)}$ is arithmetical in A, hence for some arithmetical formula φ, $A \models \varphi$ and $(\forall B)(B \models \varphi \Rightarrow 0^{(\omega)}$ is arithmetical in $B)$. By forcing = truth $A \Vdash \varphi$ so for some arithmetical perfect tree, $P \Vdash \varphi$. By definition of forcing, $(\forall B \in P)(B \models \varphi)$. Let B be an arithmetical branch of P (say, the leftmost branch of it): $0^{(\omega)}$ is arithmetical in B, hence arithmetical.

Of course (c) is a contradiction that proves what we wanted. Pointed genericity was only used in part (a), hence the same proof gives: *for no Sacks ω-generic A is $(\forall n)(0^{(n)} \leq_T A)$.*

We note that what we lacked to get pointedly generic sets was the analogue of quasi-completeness. The next lemma gives a special case of it and it is very useful in applications.

PROPOSITION 2.7. *Given P and a Σ^0_2 sentence φ, there is $Q \subseteq P$ such that $Q \leq_T P$ and $Q \Vdash \varphi$ or $Q \Vdash \sim\varphi$ (Spector [24]).*

PROOF. Let $\varphi \equiv \exists x \forall y \psi(x, y)$. For some σ, $\sigma \Vdash^P$ or $\sigma \Vdash^P \sim\varphi$.

If $\sigma \Vdash^P \varphi$ then for some x, $\sigma \Vdash^P \forall y \psi(\bar{x}, y)$. By definition of forcing,

$$(\forall y)(\forall \tau \in P)(\exists \tau' \in P)[\tau \supseteq \sigma \Rightarrow \tau' \supseteq \tau \wedge \tau' \Vdash^P \psi(\bar{x}, \bar{y})].$$

Build, recursively in P, a subtree Q all of whose branches force $\psi(\bar{x}, \bar{y})$ for all y. E.g. take τ_0, τ_1 incompatible and $\tau_i' \supseteq \tau_i$ such that $\tau_i' \Vdash^P \psi(\bar{x}, \bar{0})$, etc. Then remember that for quantifier-free formulas forcing = truth by definition.

If $\sigma \Vdash^P \sim\varphi$, similarly, since $\sigma \Vdash^P \forall x \exists y \sim \psi(xy)$ and

$$(\forall x)(\forall \tau \in P)(\exists y)(\exists \tau' \in P)[\tau \supseteq \sigma \Rightarrow \tau' \supseteq \tau \wedge \tau' \Vdash^P \sim\psi(\bar{x}, y)]. \quad \square$$

Certainly the lemma cannot be improved to higher levels of the arithmetical hierarchy because we will see in the next section that it can be used straightforwardly to get an A such that $(\forall n)(0^{(n)} \leq_T A)$ and $A^{(2)} \leq_T 0^{(\omega)}$ and we certainly cannot get such an A with $A^{(3)} \leq_T 0^{(\omega)}$ by the Enderton-Putnam computation above. So it appears that, as long as we keep the definition of forcing with trees as truth for all branches, the best possible notion of pointed genericity is the following.

DEFINITION. A is *pointedly 2-generic* if for every Σ_2^0 sentence φ there is an arithmetical recursively pointed P such that $A \in P$ and $P \Vdash \varphi$ or $P \Vdash \sim\varphi$.

PROPOSITION 2.8. POINTEDLY GENERIC EXISTENCE THEOREM. (a) *There is a perfect tree of pointedly 2-generic sets (contained in any given condition).*

(b) *Pointedly 2-generic sets are not recursive in $0'$ but there are pointedly 2-generic sets recursive in $0^{(2)}$.*

PROOF. (a) Same proof as 2.4(a), using 2.7 and the observation 2.6(a). In particular, if we start with a given recursively pointed tree, all the conditions used in the construction will be recursively pointed trees of the same degree as the initial condition.

(b) In the construction of part (a) we can start with a recursive tree (say the full binary tree). Then all the trees are recursive, and the only nonconstructive part of 2.7 is to decide, given $\varphi \in \Sigma_2^0$ and P, if $\sigma \Vdash^P \varphi$ or $\sigma \Vdash^P \sim \varphi$. Since P is recursive, this is a Σ_2^0 question, hence recursive in $0^{(2)}$. We thus get a tree recursive in $0^{(2)}$, and we can get a branch recursive in it as usual.

The negative part comes from the observation that any $B \leq_T 0'$ has a Π_2^0 implicit definition (since in the expression $(\forall x)(x \in X \Leftrightarrow x \in B)$ we may use the fact that $B \in \Delta_2^0$) and if a 2-generic set satisfies it, then a whole tree satisfies it, contradiction. $\square$

The notions of this section can be easily adapted to the case of product forcing.

§3. **Arithmetical sets and Turing degrees.** In [8] Jockusch studies the (uniquely determined up to elementary equivalence) structure of degrees below the degree of an ω-generic set and proves that it is not a lattice, embeds all countable partial orderings, does not have minimal elements and is not dense. We are interested here instead in the possible applications of the concept of genericity to the study of the structure of degrees itself.

PROPOSITION 3.1. (a) *If A is n-generic, $A^{(n)} \equiv_T A \oplus 0^{(n)}$.*

(b) *If A is ω-generic, $A^{(\omega)} \equiv_T A \oplus 0^{(\omega)}$.*

PROOF. (a) Let A be n-generic and $\varphi(x) \equiv x \in X^{(n)}$. Then $\varphi \in \Sigma_n^0$ and by forcing = truth

$$x \in A^{(n)} \Leftrightarrow A \Vdash \varphi(\bar{x}) \Leftrightarrow (\exists \sigma \subseteq A)(\sigma \Vdash \varphi(\bar{x})).$$

By definability, the expression $\sigma \Vdash \varphi(\bar{x})$ is Σ_n^0, so the right-hand side is r.e. in $A \oplus 0^{(n)}$. Similarly $x \notin A^{(n)} \Leftrightarrow A \Vdash \sim\varphi(\bar{x})$ is so, hence by Post's theorem $A^{(n)} \leq_T A \oplus 0^{(n)}$. The dual reduction always holds.

(b) Since the above proof is uniform in n, for A ω-generic we get $A^{(\omega)} \leq_T A \oplus 0^{(\omega)}$. Again the dual reduction always holds. □

It follows at once that no 1-generic (hence no n-generic or ω-generic) set can be of degree above $\mathbf{0}'$.

We derive now the completeness criterion for jumps. It might be objected that for $n = 1$ this was obtained by Friedberg [6] and that for $n > 1$ the result follows by successive relativizations of it. But Friedberg's proof is really the place where the notion of 1-genericity was introduced (long before the invention of forcing in set theory!). Moreover, we may define a relative notion of 1-genericity, e.g. as

> A is *1-generic over* B if for every set S of strings r.e. in B, there is a $\sigma \subseteq A$ such that either $\sigma \in S$ or $(\forall^\tau \supseteq \sigma)(\tau \notin S)$.

Then A is $(n + 1)$-generic iff A is 1-generic over $0^{(n)}$. Hence the suggestion of relativizing Friedberg's result is simply the suggestion of using n-genericity. We follow the usual convention of saying that a degree has a certain property if it contains a set with that property.

PROPOSITION 3.2. (a) *If* $\boldsymbol{a} \geq \mathbf{0}^{(n)}$ *then for some n-generic* $\boldsymbol{b}$, $\boldsymbol{b}^{(n)} = \boldsymbol{b} \cup \mathbf{0}^{(n)} = \boldsymbol{a}$.

(b) *If* $\boldsymbol{a} \geq \mathbf{0}^{(\omega)}$ *then for some ω-generic* $\boldsymbol{b}$, $\boldsymbol{b}^{(\omega)} = \boldsymbol{b} \cup \mathbf{0}^{(\omega)} = \boldsymbol{a}$ *(Friedberg* [6] *for* $n = 1$, *Selman* [20] *for* $n > 1$, *MacIntyre* [11] *for* ω).

PROOF. (b) Given $A \in \boldsymbol{a}$ we define B ω-generic such that $B \oplus 0^{(\omega)} \equiv_T A$. The result will follow from 3.1(b). Let φ_n be a recursive enumeration of the arithmetical sentences. At stage $n + 1$ suppose we already have σ_n ($\sigma_0 = \emptyset$). First define σ_n' as the smallest proper extension of σ_n which forces either φ_n or $\sim\varphi_n$. Then let $\sigma_{n+1} = \sigma_n' * \langle A(n)\rangle$. Let $B = \bigcup \sigma_n \cdot B \oplus 0^{(\omega)} \leq_T A$ because $0^{(\omega)} \leq_T A$ by hypothesis, the construction of σ_n' from σ_n is arithmetical (hence recursive in $0^{(\omega)}$) and the construction of σ_{n+1} from σ_n' is recursive in A.

$A \leq_T B \oplus 0^{(\omega)}$: to know if $x \in A$, do the construction of B until σ_{x+1} is determined. The step from σ_n to σ_n' is recursive in $0^{(\omega)}$, the one from σ_n' to σ_{n+1} is recursive in B (ask what $B(y)$ is for y = length of σ_n').

(a) Similar, using n-genericity. □

The previous proposition proves that the n-jump has range $\{\boldsymbol{c}: \boldsymbol{c} \geq \mathbf{0}^{(n)}\}$ and the ω-jump has range $\{\boldsymbol{c}: \boldsymbol{c} \geq \mathbf{0}^{(\omega)}\}$. The next proposition proves that these jumps are never one-one on their range.

PROPOSITION 3.3. (a) *If* $\boldsymbol{a} \geq \mathbf{0}^{(n)}$ *then for some n-generic* $\boldsymbol{b}, \boldsymbol{c}$, $\boldsymbol{b}^{(n)} = \boldsymbol{c}^{(n)} = \boldsymbol{b} \cup \boldsymbol{c} = \boldsymbol{a}$ *(Spector* [24] *for* $n = 1$, *Selman* [20] *for* $n > 1$).

(b) *If* $\boldsymbol{a} \geq \mathbf{0}^{(\omega)}$ *then for some ω-generic* $\boldsymbol{b}, \boldsymbol{c}$, $\boldsymbol{b}^{(\omega)} = \boldsymbol{c}^{(\omega)} = \boldsymbol{b} \cup \boldsymbol{c} = \boldsymbol{a}$ *(Selman* [20] *for* $\boldsymbol{a} = \mathbf{0}^{(\omega)}$).

PROOF. (b) We use the same procedure as above to get B, C ω-generic of the required ω-jump. We also want to be able to recover A from both B and C. At stage $n + 1$ suppose we have σ_n, τ_n of the same length and defined on an initial segment of the integers. Then let:

σ_n^1 a proper extension of σ_n forcing φ_n or $\sim\varphi_n$,

τ_n^1 an extension of τ_n agreeing with σ_n^1 on the new elements of it,

τ_n^2 a proper extension of τ_n^1 forcing φ_n or $\sim\varphi_n$,

σ_n^2 an extension of σ_n^1 agreeing with τ_n^2 on the new elements of it,

$\sigma_{n+1} = \sigma_n^2 * \langle A(n)\rangle$,

$\tau_{n+1} = \tau_n^2 * \langle 1 - A(n)\rangle$.

Let $B = \bigcup \sigma_n$, $C = \bigcup \tau_n$. The procedure is recursive in $0^{(\omega)}$ and A, so as above $B^{(\omega)} \equiv_T C^{(\omega)} \equiv_T A$. Moreover the procedure is recursive in A because $0^{(\omega)} \leq_T A$, so $B \oplus C \leq_T A$. Finally $A \leq_T B \oplus C$: to know $A(n)$ look at the value of B on the nth place where B and C differ.

(a) similar. □

The similarities among the theorems above for all n's should not be taken for granted: Nerode and Shore [14] have proved that if $\mathscr{D}_n$ is the theory of degrees with the usual ordering and the operation of n-jump, then $\mathscr{D}_n$ and $\mathscr{D}_m$ are not elementarily equivalent if $n \neq m$. If $\mathscr{D}_\omega$ is defined similarly using the ω-jump, $\mathscr{D}_n$ and $\mathscr{D}_\omega$ are also not elementarily equivalent.

Our second application of genericity deals with incomparability questions. Given A let $x \in A_i \Leftrightarrow \langle x, i\rangle \in A$ and $A_i^* = \bigcup_{j\neq i} A_j$. The A_i's are called the components of A. We say that the A_i's are Σ_n^0-independent if for each i, A_i is not Σ_n^0 in A_i^*. Similarly for arithmetical independence.

PROPOSITION 3.4. (a) *If A is n-generic then the components of A are Σ_n^0-independent.*

(b) *If A is ω-generic then the components of A are arithmetically independent* (*Feferman* [4] *for part* (b)).

PROOF. (a) Similar to the fact that no n-generic set is Σ_n^0. (b) from (a), or similar to the fact that no ω-generic is implicitly definable in arithmetic. Actually the components of A are strongly independent in the sense that no component is arithmetical in the infinite join of the other components. □

PROPOSITION 3.5. *There are Σ_n^0-incomparable Δ_{n+1}^0 sets* (*Kleene and Post* [10] *for* $n = 1$, *Selman* [20] *for* $n > 1$).

PROOF. Take an n-generic set recursive in $0^{(n)}$ and apply 3.4(a). We sketch, however, a direct proof as a warm-up for the next theorem. We build A and B n-generic and recursive in $0^{(n)}$ and diagonalize against the possible Σ_n^0 reductions, using the fact that, by n-genericity, if $\varphi \in \Sigma_n^0$ is a reduction of A to B then $x \in A \Leftrightarrow (\exists\sigma \subseteq B)(\sigma \Vdash \varphi(\bar{x}))$, hence to spoil it is enough to have x for which this fails. At stage $s + 1$ we will have to insure that the sth Σ_n^0 sentence or its negation is forced by both A and B, and that the sth Σ_n^0 reduction is spoiled (both ways). □

The general pattern of the proofs above has been to mix forcing with the original proof of the result for the first level of the arithmetical hierarchy. In the next result we do the same, and the proof mixes the two sophisticated tools of priority and forcing.

PROPOSITION 3.6. *There are Δ_n^0-incomparable Σ_n^0 sets* (*Friedberg* [5] *and Muchnik* [12] *for* $n = 1$, *Hinman* [7] *for* $n > 1$).

PROOF. We find A and B Σ_{n+1}^0 and Δ_{n+1}^0-incomparable. We cannot expect to make them $n + 1$-generic because then they could not be Σ_{n+1}^0. What we can do is to make them n-generic, but first we have to understand why this is enough to control all Δ^0_{n+1}-reductions (and not only Σ^0_n-reductions) by finite strings. To say that A is Δ^0_{n+1} in B means that the graph of the characteristic function of A (we call it c_A) is so. The advantage of working with c_A is that we can then only consider Σ^0_{n+1} reductions, since they are automatically Δ^0_{n+1}:

$$c_A(x) \neq y \Leftrightarrow (\exists z)(c_A(x) = z \wedge z \neq y).$$

And this does not conflict with the fact that A has to be Σ^0_{n+1} (hence Σ^n_{0+1} in B). We give separately the strategies to handle single requirements. Each requirement will be taken in consideration infinitely many times and there will be an assignment of priorities in the usual way. Let φ_e and ψ_e be, respectively enumerations of the Σ^0_n sentences and of the Π^0_n formulas with three free variables.

(a) To satisfy $c_A(x) = y \not\Leftrightarrow B \models \exists z\psi_e(x, y, z)$ for some x, y (in the case it has not yet been satisfied and not injured afterwards) note that in the end

$$\begin{aligned} B \models \exists z\psi_e(\bar{x}, \bar{y}, z) &\Leftrightarrow (\exists z)(B \models \psi_e(x, y, z)) \\ &\Leftrightarrow (\exists z)(\exists \sigma \subseteq B)(\sigma \Vdash \psi_e(\bar{x}, \bar{y}, \bar{z})) \end{aligned}$$

because B will be n-generic and $\psi_e \in \Pi^0_n$. Pick a fresh witness x and at stage $s + 1$ (if the requirement is considered) see if for some $\sigma \subseteq B_s$ (with code $\leq s$) and some $z \leq s$, $\sigma \Vdash \psi_e(\bar{x}, \bar{0}, \bar{z})$.

If this never happens, $B \not\models \exists z\psi_e(\bar{x}, \bar{0}, z)$ and x is never put into A, so $c_A(x) = 0$.

If this happens at stage $s + 1$, then put x into A (so that $c_A(x) = 1$) and freeze all numbers z such that $\sigma(z) = 0$: if they are never put in B after this stage, then $\sigma \subseteq B$ and $B \models \exists z\psi_e(\bar{x}, \bar{0}, z)$.

(b) To satisfy $A \Vdash \varphi_e$ or $A \Vdash \sim\varphi_e$ (again if this has not yet been satisfied and not injured) at stage $s + 1$, we search for a string σ such that $\sigma \Vdash \varphi_e$ or $\sigma \Vdash \sim\varphi_e$ and such that σ gives:

value 1 to the elements of A_s,

value 0 to the elements that we do not want to put in A,

either because they have been frozen when a condition has been satisfied, or because they are witnesses (like x in (a)) and have not yet been put into A (both cases only relative to conditions with higher priority).

The elements such that $\sigma(z) = 1$ go into A; those for which $\sigma(z) = 0$ are frozen out of A.

Of course both the strategies have to be duplicated, by interchanging the roles of A and B. □

PROPOSITION 3.7. (a) *If A is not Σ^0_n, there are comeager many sets Σ^0_n-incomparable with it.*

(b) *If A is not arithmetical, there are comeager many sets arithmetically incomparable with it (Hinman [7]).*

PROOF. (a) Since the intersection of two comeager sets is comeager and $\{B\colon B$ is Σ^0_n in $A\}$ is countable (hence meager), it is enough to prove that $\{B\colon A$ is not Σ^0_n in $B\}$ is comeager. For this we build a set B n-generic and such that A is not Σ^0_n in B, by stages. If φ_s and ψ_s are enumerations of the Σ^0_n sentences and formulas with one free variable, and at stage $s + 1$ we have σ_s, then:

To have B n-generic let $\tau \supseteq \sigma_s$ be such that $\tau \Vdash \varphi_s$ or $\tau \Vdash \sim\varphi_s$.

To spoil $(\forall x)[x \in A \Leftrightarrow (\exists\sigma \subseteq B)(\sigma \Vdash \psi_s(\bar{x}))]$ consider the set C so defined: $x \in C$ iff $(\exists\sigma \supseteq \tau)(\sigma \Vdash \psi_s(\bar{x}))$. Clearly C is a Σ_n^0 set, so $C \neq A$. If $A - C$ is nonempty, we do nat have anything to do. Otherwise there is $x \in C - A$: then take $\sigma \supseteq \tau$ such that $\sigma \Vdash \psi_s(\bar{x})$ and define $\sigma_{s+1} = \sigma$, so that $x \notin A$ and $B \Vdash \psi_s(\bar{x})$.

(b) follows from (a), because the intersection of countably many comeager sets is comeager (arithmetically incomparable means Σ_n^0-incomparable for all n's). $\square$

We turn now to applications of the method of trees of §2. We first recast 2.7 in its original form. Given an arithmetical formula φ, we indicate by A_φ the set defined by φ from A: $x \in A_\varphi \Leftrightarrow A \models \varphi(x)$.

PROPOSITION 3.8. *Given P and φ recursive, there is $Q \subseteq P$ such that $Q \leq_T P$ and one of the following holds*:

(a) $(\forall A \in Q)(A_\varphi \leq_T Q)$,

(b) $(\forall A \in Q)(A \leq_T A_\varphi \oplus Q)$ (*Spector* [24]).

PROOF. Consider the Σ_2^0 sentence ψ asserting there is $\sigma \in P$ such that for every $\tau_1, \tau_2 \supseteq \sigma$ with $\tau_1, \tau_2 \in P$ and every x it is never the case that $\tau_1 \models \varphi(x)$ and $\tau_2 \models \sim\varphi(x)$. If ψ is true pick such a σ and build Q as the subtree of P consisting of all the strings of P extending σ (the full subtree above σ). Then for every $A \in Q$ the set A_φ is independent of A, since to check if $x \in A_\varphi$ it is enough to find $\tau \in Q$ such that $\tau \models \varphi(x)$. This proves that $(\forall A \in Q)(A_\varphi \leq_T Q)$.

If ψ is false then build Q with the following property: for every $\sigma \in Q$, if τ_1, τ_2 are the two immediate extensions of it, then for some x, $\tau_1 \models \varphi(x)$ and $\tau_2 \models \sim\varphi(x)$. Given then $A \in Q$, we want to prove that (b) holds. Suppose we know that $\sigma \subseteq A$ and let τ_1, τ_2, x be as above: we only have to decide which of τ_1, τ_2 is contained in A. This is the τ_i such that $\tau_i \models \varphi(x) \Leftrightarrow A \models \varphi(x) \Leftrightarrow x \in A_\varphi$. $\square$

Recall that a degree $\boldsymbol{a}$ is a *minimal upper bound* for a set of degrees $\mathscr{A}$ if $(\forall \boldsymbol{c} \in \mathscr{A})(\boldsymbol{c} < \boldsymbol{a})$ and there is no $\boldsymbol{b} < \boldsymbol{a}$ such that $(\forall \boldsymbol{c} \in a)(\boldsymbol{c} < \boldsymbol{b})$. If $\mathscr{A} = \{\boldsymbol{0}\}$ then a minimal upper bound for $\mathscr{A}$ is called a *minimal degree*.

PROPOSITION 3.9. *Every countable set of degrees has uncountably many minimal upper bounds* (*Sacks* [18]).

PROOF. Let $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots\}$ be an enumeration of the given set of degrees, and consider the ascending chain of degrees so defined: $\boldsymbol{b}_0 = \boldsymbol{a}_0$, $\boldsymbol{b}_{n+1} = \boldsymbol{b}_n \cup \boldsymbol{a}_{n+1}$. We do an argument in the style of 2.4, insuring that if σ has length n, then T_σ is recursively pointed of degree $\boldsymbol{b}_n$ (this takes care of the condition $(\forall n)(\boldsymbol{a}_n \leq \boldsymbol{a})$ where $\boldsymbol{a}$ is the degree of any branch of the final tree). Suppose T_σ is given: choose τ_1, τ_2 incomparable on it. First take the full subtree of T_σ above τ_i; then choose a recursively pointed subtree of it of degree $\boldsymbol{b}_{n+1}$ (by 2.6(c)), then apply 3.8 to get $T_{\sigma * i}$ with the property there (we still get a recursively pointed tree of degree $\boldsymbol{b}_{n+1}$ by 2.6(a)) where φ is the nth recursive formula. The result follows by noting that the tree we built has uncountably many branches, and only countably many of them may be recursive in some $\boldsymbol{a}_n$. $\square$

PROPOSITION 3.10. *There are uncountably many minimal degrees* (*Spector* [24], *Lacombe*).

PROOF. Apply 3.9 to the set $\{\boldsymbol{0}\}$.

PROPOSITION 3.11. *No infinite ascending sequence of degrees has a least upper bound* (*Spector* [24]).

PROOF. By 3.9 if the least upper bound of a set of degrees exists, it must be the join of a finite subset. □

PROPOSITION 3.12. *The degrees are not a lattice* (*Kleene and Post* [10]).

PROOF. Two minimal upper bounds of an ascending chain have no greatest lower bound, since this should be an upper bound of the chain and be below both of them. □

A nonempty set of degrees I is called an *ideal* if it is closed downward (if $\boldsymbol{a} \leq \boldsymbol{b}$ and $\boldsymbol{b} \in I$ then $\boldsymbol{a} \in I$) and under join (if $\boldsymbol{a}, \boldsymbol{b} \in I$ then $\boldsymbol{a} \cup \boldsymbol{b} \in I$). An exact pair for I is a pair $(\boldsymbol{a}, \boldsymbol{b})$ such that $I = \{\boldsymbol{c}: \boldsymbol{c} \leq \boldsymbol{a} \text{ and } \boldsymbol{c} \leq \boldsymbol{b}\}$.

PROPOSITION 3.13. *Every countable ideal has an exact pair* (*Spector* [24]).

PROOF. Let $\{\boldsymbol{a}_n\}_{n\in\omega}$ be an enumeration of the ideal and $\{\boldsymbol{b}_n\}_{n\in\omega}$ be as in 3.9. It is enough to build by product forcing a pair (A, B) generic with respect to Π^0_2 sentences, by using recursively pointed trees of degree $\boldsymbol{b}_n$ at stage n. Suppose in fact $C \leq_T A, B$: then $C = \{e\}^A = \{i\}^B$ for some e, i. Since $\{e\}^{X_1} = \{i\}^{X_2}$ is a Π^0_2 sentence true of (A, B), it is forced by some (P_1, P_2) such that $A \in P_1, B \in P_2$. Then for every branch $D \in P_1$ $\{e\}^D = \{i\}^B = C$ (by definition of forcing), hence to know C it is enough to compute $\{e\}^D$ for any branch of P_1. In particular $C \leq_T P_1$ and the degree of P_1 is in the ideal by hypothesis, hence so is C. □

Note that this is the best possible result because it is not always true that for any ideal I there is $\boldsymbol{a}$ such that $I = \{\boldsymbol{c}: \boldsymbol{c} \leq \boldsymbol{a}\}$ or $I = \{\boldsymbol{c}: \boldsymbol{c} < \boldsymbol{a}\}$. As a counterexample to both cases, take I as the ideal of the degrees of arithmetical sets: the first case fails because I is closed under jump; the second fails because any $\boldsymbol{a}$ with the property is above $\boldsymbol{0}'$, and by 3.3(a) there are $\boldsymbol{b}, \boldsymbol{c}$ such that $\boldsymbol{b} \cup \boldsymbol{c} = \boldsymbol{a}$: if $\boldsymbol{b}, \boldsymbol{c}$ were both in I then $\boldsymbol{a}$ should be, contradiction.

The next result is a degree-theoretic characterization of $\boldsymbol{0}^{(\omega)}$. Although naturally defined as an upper bound of the $\boldsymbol{0}^{(n)}$'s, the degree $\boldsymbol{0}^{(\omega)}$ is not the least upper bound of them for the simple reason that no ascending sequence has one (3.11). We may however generalize the notion and say that $\boldsymbol{a}$ is the *n-least upper bound* of $\mathscr{A}$ if it is the least element of $\{\boldsymbol{c}^{(n)} : (\forall \boldsymbol{b} \in \mathscr{A})(\boldsymbol{b} \leq \boldsymbol{c})\}$. In particular 0-l.u.b. means l.u.b.

PROPOSITION 3.14. $\boldsymbol{0}^{(\omega)}$ *is the* 2*-least upper bound of* $\{\boldsymbol{0}^{(n)}\}_{n\in\omega}$ (*Enderton and Putnam* [3], *Sacks* [19]).

PROOF. We want to find A such that $(\forall n)(0^{(n)} \leq_T A)$ and $A^{(2)} \equiv_T 0^{(\omega)}$. By the Enderton-Putnam computation of §2, the condition $0^{(\omega)} \leq_T A^{(2)}$ follows automatically from $(\forall n)(0^{(n)} \leq_T A)$. It is then enough to build A by using at stage n a recursively pointed tree P_n of degree $\boldsymbol{0}^{(n)}$ and forcing the Σ^0_2 sentence $\varphi \equiv \bar{n} \in X^{(2)}$ or its negation (2.7). To know which case applies only requires two more jumps, hence the question $n \in A^{(2)}$ is recursive in $0^{(n+2)}$ uniformly in n, so $A^{(2)} \leq_T 0^{(\omega)}$. □

We might as well use in the proof above Proposition 3.8 at every stage, and we would then actually get that $\boldsymbol{0}^{(\omega)}$ is the least element of $\{\boldsymbol{b}^{(2)} : \boldsymbol{b}$ is a minimal upper bound of $\{\boldsymbol{0}^{(n)}\}_{n\in\omega}\}$.

We can also build two sets at in the proof above, having incomparable jumps: the condition of having incomparable jumps can be handled by Σ^0_2 sentences. This proves that $\{\boldsymbol{0}^{(n)}\}_{n\in\omega}$ *does not have a* 1*-least upper bound* (Sacks [19]). Jockusch and Simpson [9] note that no ascending sequence of degrees has 1-l.u.b., unless the jumps of the degrees in the sequence are eventually constant.

§4. Arithmetical degrees. We turn now our attention to the structure of *arithmetical degrees*. We consider the relation $A \leq_a B$ iff A is arithmetical in B, and note that it is transitive. Then $A \leq_a B$ and $B \leq_a A$ is an equivalence relation, whose equivalence classes we call arithmetical degrees. The structure of arithmetical degrees admits a natural partial order $\leq$ induced by $\leq_a$, and a natural jump operation induced by the ω-jump. To see that the latter is well defined, note that if A and B are arithmetically equivalent then $A^{(\omega)} \equiv_T B^{(\omega)}$: e.g. $A \leq_a B$, hence for some m, $A \leq_T B^{(m)}$ and $A^{(n)} \leq_T B^{(n+m)}$ uniformly in n, so $A^{(\omega)} \leq_T B^{(\omega)}$. Actually $A^{(\omega)}$ and $B^{(\omega)}$ are recursively isomorphic (see [17, p. 258]). We use for arithmetical degrees the same notations as for Turing degrees: $\boldsymbol{a}$ is a degree, $\mathbf{0}$ is the smallest degree (containing exactly the arithmetical sets), $\boldsymbol{a}'$ is the degree of the ω-jump of any set in $\boldsymbol{a}$, $\boldsymbol{a} \cup \boldsymbol{b}$ is the least upper bound of $\boldsymbol{a}$ and $\boldsymbol{b}$ (it always exists, and it is the degree of $A \oplus B$ for any $A \in \boldsymbol{a}$, $B \in \boldsymbol{b}$), $\boldsymbol{a} \cap \boldsymbol{b}$ is the g.l.b. of $\boldsymbol{a}$ and $\boldsymbol{b}$ (when it exists).

PROPOSITION 4.1. *Every countable partial ordering is embeddable in the arithmetical degrees below* $\mathbf{0}'$ (*Feferman* [4]).

PROOF. Sacks [18] proves that the corresponding result for r.e. degrees follows from the existence of a strongly independent simultaneously r.e. sequence of r.e. degrees. By 3.4(b) there is such a strongly independent sequence uniformly recursive in any ω-generic set. By 1.6(c) there is such a sequence recursive in $0^{(\omega)}$. □

PROPOSITION 4.2. *If* $\boldsymbol{a} \geq \mathbf{0}'$ *there is a* $\boldsymbol{b}$ *such that* $\boldsymbol{b}' = \boldsymbol{b} \cup \mathbf{0}' = \boldsymbol{a}$ (*MacIntyre* [11]).

PROOF. Let $A \in \boldsymbol{a}$. Then $0^{(\omega)} \leq_a A$ by assumption, hence $0^{(\omega)} \leq_T A^{(n)}$ for some n. From 3.2(b) we get B such that $B^{(\omega)} \equiv_T B \oplus 0^{(\omega)} \equiv_T A^{(n)}$, hence $B^{(\omega)} \equiv_a B \oplus 0^{(\omega)} \equiv_a A$. □

Similarly, from 3.3(b) we get that the arithmetical jump is never one-one on its range.

Open problem. Determine the range of the arithmetical jump restricted to the degrees below $\mathbf{0}'$.

We are unable to make a conjecture: for Turing degrees the range consists of the degrees between $\mathbf{0}'$ and $\mathbf{0}^{(2)}$ which are r.e. in $\mathbf{0}'$ (Shoenfield [21]), for hyperdegrees instead every degree strictly below $\mathbf{0}'$ has hyperjump $\mathbf{0}'$ (see §8). Since the arithmetical degrees below $\mathbf{0}'$ we build are the degrees of ω-generic sets, by 3.1(b) their jump is always $\mathbf{0}'$.

PROPOSITION 4.3. *There are degrees* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *strictly between* $\mathbf{0}$ *and* $\mathbf{0}'$, *such that* $\boldsymbol{a} \cap \boldsymbol{b} = \mathbf{0}$ *and* $\boldsymbol{a} \cup \boldsymbol{b} = \mathbf{0}'$.

PROOF. By using the methods of 3.3(b) it is easy to build an ω-generic pair (A, B) such that $A \oplus B \equiv_T 0^{(\omega)}$. Since both A and B are ω-generic, it is then enough to prove that they form a minimal pair. Let $C \leq_a A, B$: we want to prove that C is arithmetic. There are two arithmetic formulas φ_1, φ_2 defining C from A and B, respectively. Let ψ be the arithmetical statement expressing the fact that the two sets defined by φ_1 from X_1 and by φ_2 from X_2 are equal. Then $(A, B) \models \psi$ and for some $(\sigma_1, \sigma_2) \subseteq (A, B)$, $(\sigma_1, \sigma_2) \Vdash \psi$. Then for any set $D \supseteq \sigma_1$ such that (D, B) is

n-generic (if $\psi \in \Sigma_n^0$) the set defined by φ_1 from D is equal to the set defined by φ_2 from B, hence to C. Take D arithmetical as above: C is then arithmetical. $\square$

PROPOSITION 4.4. *For any* $\boldsymbol{a} > \mathbf{0}$ *there is a degree* $\boldsymbol{b}$ *incomparable with it. If, moreover,* $\mathbf{0} < \boldsymbol{a} < \mathbf{0}'$, *then* $\boldsymbol{b}$ *can be chosen such that* $\mathbf{0} < \boldsymbol{b} < \mathbf{0}'$.

PROOF. The first assertion comes from 3.7(b) (a comeager set is not empty). The second follows from 4.3 (any degree strictly between $\mathbf{0}$ and $\mathbf{0}'$ must be incomparable with one of $\boldsymbol{a}$, $\boldsymbol{b}$). $\square$

PROPOSITION 4.5. *The degrees below* $\mathbf{0}'$ *are not a lattice.*

PROOF. Let A be ω-generic recursive in $\mathbf{0}^{(\omega)}$. Consider its components A_i and let $\boldsymbol{a}_i$ be the degree of $\bigoplus_{1\le j\le i} A_i$. By the independence property (3.4.(b)) the $\boldsymbol{a}_i$'s are strictly increasing. Let $A^* = \bigoplus_{i\neq 0} A_i$. Consider then B_i agreeing with A_i except by having the string coded by the ith element of A_0 as initial segment (of its characteristic function). Then the B_i's generate the same chain $\{\boldsymbol{a}_i\}_{i\in\omega}$. Let $B^* = \bigoplus_{i\neq 0} B_i$. Since everything has been obtained by A, by forcing, any set arithmetic in both A^*, B^* is already arithmetical in some $\boldsymbol{a}_i$, hence A^* and B^* do not have a g.l.b. $\square$

PROPOSITION 4.6. *Every countable set of arithmetical degrees has uncountably many minimal upper bounds.*

PROOF. Similar to 3.9, using arithmetically pointed trees (defined in a straightforward way and having the analogue of the properties in 2.6). The relevant fact (obtained from 2.2(c)) is that given any tree P and any arithmetical formula φ, if we call A_φ the set determined by φ from A ($x \in A_\varphi \Leftrightarrow A \models \varphi(x)$) then there is $Q \subseteq P$, $Q \le_a P$ such that one of the following holds:

(a) $(\forall A \in Q)(A_\varphi \le_a Q)$,

(b) $(\forall A \in Q)(A \le_a A_\varphi \oplus Q)$.

The arithmetical pointedness takes care of the upper bound requirement, and the fact above of the minimality. $\square$

The proposition has the usual consequences. In particular:

PROPOSITION 4.7. *No infinite ascending sequence of arithmetical degrees has a least upper bound.*

PROPOSITION 4.8. *There is a minimal arithmetical degree below* $\mathbf{0}'$ (*Sacks* [19]).

PROOF. As in 4.6, using arithmetical trees. $\square$

PROPOSITION 4.9. *Every countable ideal has an exact pair* (*Nerode and Shore* [13]).

PROOF. As in 3.13, using product forcing with arithmetically pointed trees, or as a direct corollary of 4.6. $\square$

Nerode and Shore [13] have proved that the elementary theory of the arithmetical degrees is recursively equivalent to second order arithmetic. In particular it is undecidable and has the same degree of unsolvability of the theory of Turing degrees. We do not know of any elementary difference between the two theories.

AFTERWORD. The notion of finite forcing was obviously instrumental in the proofs of the results in §§3, 4: it allowed one to reduce arithmetical computations from a set (requiring in theory infinite parts of the set itself) to finite strings. Having the machinery at hand, to mimic the proof of the original results for Turing degrees was routine. Perhaps the only somewhat delicate point occurred in the proof of

3.6, where injuries could occur not only for the sake of making the two sets incomparable (as in the original proof for r.e. sets) but also for the sake of making them sufficiently generic.

A different situation occurred instead for results invoving the notion of tree. After reading the proofs in §§3, 4 the reader is entitled to ask why the notion of Sacks-genericity was introduced, since it did not help at all. Sacks [19] does claim that a Sacks ω-generic set has minimal arithmetical degree, but we have been unable to verify this. Nevertheless, the work done in §2 was not unuseful: it is an introduction to forcing with trees in a particularly simple situation, and in the next parts of this work we will be in need of this kind of forcing as well. However, we now descend from the giants' shoulders for a little rest.

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ISTITUTO MATEMATICO
UNIVERSITÀ DI TORINO
TORINO, ITALY 10100

UNIVERSITY OF CALIFORNIA
LOS ANGELES, CALIFORNIA 90024