

What is an EUTxO blockchain?

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We condense the theory of blockchains down to a simple and compact set of four type equations (Idealised EUTxO), and to an algebraic characterisation (abstract chunk systems), and exhibit an adjoint pair of functors between them. This gives a novel account of the essential mathematical structures underlying blockchain technology, such as Bitcoin.

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1. INTRODUCTION

Blockchain is a young field — young enough that no consensus has yet developed as to its underlying mathematical structures. There are many blockchain implementations, but what (mathematically) are they implementations of?

Two major blockchain architectures exist: *UTxO-based* blockchains (like Bitcoin), and *accounts-based* blockchains (like Ethereum). We consider UTxO-style blockchains in this paper, and specifically the *Extended* UTxO-style model [6], which as the name suggests extends the UTxO structure (how, is described in Remark 3.1.5) of Bitcoin, which is still the canonical blockchain application.

So our question becomes: what, mathematically speaking, is an EUTxO blockchain?

In the literature, Figure 3 of [6] exhibits Extended UTxO as an inductive datatype, designed with implementation and formal verification in mind. Most blockchains exist only in code, so to have an inductive specification to work from in a published academic paper is a luxury for which we can be grateful.

However, this does not answer our question.

It would be like answering “What are numbers?” with the inductive definition $\mathbb{N} = 1 + \mathbb{N}$: this an important structure (and to be fair, it yields an important inductive principle) but this does not tell us that \mathbb{N} is a ring; or about primes and the fundamental theorem of arithmetic; or that \mathbb{N} is embedded in \mathbb{Q} and \mathbb{R} ; or even about binary representations. In short: Figure 3 of [6] gives us the raw data structure of one particular blockchain implementation, which is certainly important, but this is not a *mathematics of blockchains*.

As we shall see, there is more to be said here.

1.1. Map of the paper

A map of this paper, and our answer, is as follows:

- (1) In Section 3 we present *Idealised EUTxO* (Definition 3.1.1), which is four type equations (Figure 1).
This captures the essence of [6, Figure 3], but far more succinctly — four lines vs. one full page.¹
So: EUTxO is a solution to the IEUTxO equations in Figure 1.
- (2) The approach to blockchain in this paper is novel — we concentrate not on *blockchains* but on *blockchain segments*, which we call *chunks* (Definition 3.2.1).
Chunks have many properties that blockchains do not have: if you cut a blockchain into pieces you get chunks, not blockchains, and chunks have much more structure, e.g. they form a partially-

¹This is not a criticism of the original inductive definition.

ordered partial monoid (Theorem 3.2.8) which communicate across *channels* (much like the π -calculus [12]).

A blockchain is the special case of a chunk with no active input channels (Definition 3.2.20).

So: EUTxO is a system of chunks (a partially-ordered partial monoid with channels).

- (3) IEUTxO models form a category (Definition 3.4.1).

So now EUTxO is the category of partially-ordered partial monoid solutions to the IEUTxO models, and arrows between them.

- (4) Our answers are still quite concrete, in the sense that objects are solutions to type equations. To go further, we use algebra.

We introduce *abstract chunk systems* (Definition 4.5.1), which are oriented atomic monoids of chunks (Definitions 4.4.1, 4.2.5, and 4.2.1). These too form a category (Definition 4.5.2), with objects and arrows.

Thus, we extract relevant properties of what makes solutions to the IEUTxO equations Figure 1 *interesting*, as explicit and testable algebraic properties (see the discussion in Subsection 8.3).

Several design choices exist in this space: we discuss some of them in Remarks 6.4.4, 7.1.5, and 7.1.6.

So now EUTxO is a bundle of abstract algebraic, testable properties, which exists in a clean design space which could be explored in future work.

- (5) Finally, we pull this all together by constructing functors between the categories of IEUTxO models and of abstract chunk systems (Definitions 5.1.1 and 6.2.1), and we exhibit an adjunction between them (Theorem 7.3.4).

So finally, EUTxO becomes an adjoint pair of categories — one of concrete solutions to some type equations, and the other of abstract algebras — related by adjoint functors mapping between them.

There are many definitions and results in this paper, and this leads to a broader point of it: that the mathematics we observe is *possible*. There is a mathematics of blockchain here which we have not seen commented on before.

This may help make blockchain more accessible and interesting to a mathematical audience, and improve communication — since there is no more effective language for handling complexity than mathematics. Furthermore, as we argue in Subsections 4.4.2 and 8.3, our analysis of blockchain structure in this paper is not just of interest to mathematicians — it may also be of practical interest to programmers, by suggesting ways to structure and transform code, and establishing properties for unit tests, property-based testing, and formal verification of correctness.

More exposition and discussion is in the body of the paper and in Section 8, including discussions of future work.²

2. SOME BACKGROUND

We need to set up some basic machinery. The reader is welcome to skip or skim this Section, and refer back to it as required.

2.1. Basic data structures

DEFINITION 2.1.1. (1) Fix a countably infinite set $\mathbb{A} = \{a, b, c, p, \dots\}$ of **atoms**.

(2) A **permutation** is a bijection on \mathbb{A} ; write $\pi, \pi' \in \text{Perm}$ for permutations.

REMARK 2.1.2. Following the ideas in [8; 10] atoms will be the atoms of ZFA of Zermelo-Fraenkel set theory with Atoms³ — this is a fancy way to saying that \mathbb{A} will be a type of atomic identifiers.

We will use atoms to locate inputs and outputs on a blockchain. More on this in Subsection 2.2.

²Tip: search the pdf for ‘future work’.

³Also called *urelemente* or *urelements* in the set-theoretic literature.

NOTATION 2.1.3. (1) Write $\mathbb{N} = \{0, 1, 2, \dots\}$.

- (2) If X is a set, write $\text{fin}(X)$ for the finite powerset of X and $\text{fin}_!(X)$ for the **pointed finite powerset**.
In symbols:

$$\text{fin}(X) = \{(X, x) \in \text{fin}(X) \times X \mid x \in X.$$

- (3) If X and Y are sets, we use a convenient shorthand in Figure 1 by writing

$$(\text{fin}(X), Y)_! \text{ as shorthand for } (\text{fin}(X)_!, Y).$$

That is, we take $(-)_!$ to act on a pair functorially, on the first component. We do this in Figure 1 when we write $\text{Transaction}_!$.

- (4) If X is a set then write $[X]$ for the set of (possibly empty) finite lists of elements from X . We write \cdot for list concatenation, so $l \cdot l'$ is l followed by l' .
More generally, we will write \cdot for any monoid composition; list concatenation is one instance. It will always be clear what is intended.
- (5) If X is a set then order $l, l' \in [X]$ by the **sublist inclusion** relation, where $l \leq l'$ when l can be obtained from l' by deleting (but not rearranging) some of its elements.
- (6) If X is a set and $x \in X$ then we may call the one-element list $[x] \in [X]$ a **singleton**.
- (7) If $V : X \rightarrow \text{Bool}$ and $x \in X$ then we may write $V(x)$ or $V x$ for $V x = \text{True}$.

2.2. The permutation action

REMARK 2.2.1. We spend this Subsection introducing permutations and their action on elements. We will need this most visibly in two places:

- (1) To state the key Definition 4.3.4.
- (2) To prove Lemma 5.2.1, and thus Proposition 5.2.2.

Because we assume atoms and are working in a ZFA universe, everything has a standard permutation action. We describe it in Definition 2.2.2. Programmers can think of the permutation action as a *generic* definition in the ZFA universe (given below in this Remark), which is sufficiently generic that it exists for all the datatypes considered in this paper. By this perspective, Definition 2.2.2 specifies how this generic action interacts with the specific type-formers of interest for this paper.

For reference we write out the ZFA generic definition, which is by ϵ -induction on the sets universe:

$$\begin{array}{ll} \pi \cdot a = \pi(a) & a \in \mathbb{A} \\ \pi \cdot X = \{\pi \cdot x \mid x \in X\} & X \text{ a set.} \end{array}$$

More information on this sets inductive definition is in [8; 7]. Definition 2.2.2 is then a concrete instance for the datatypes of interest in this paper:

DEFINITION 2.2.2. Permutations π act concretely as follows:

- (1) If $\pi \in \text{Perm}$ and $a \in \mathbb{A}$ then π acts on a as a function:

$$\pi \cdot a = \pi(a).$$

- (2) If $\pi \in \text{Perm}$ and X is any set then π acts **pointwise** on X as follows:

$$\pi \cdot X = \{\pi \cdot x \mid x \in X\}.$$

Note as a corollary of this that $x \in X \iff \pi \cdot x \in \pi \cdot X$.

- (3) If $\pi \in \text{Perm}$ and (x_1, \dots, x_n) is a tuple then π acts **pointwise** on (x_1, \dots, x_n) as follows:

$$\pi \cdot (x_1, \dots, x_n) = (\pi \cdot x_1, \dots, \pi \cdot x_n).$$

Note therefore that $\pi \cdot ((x_1, \dots, x_n)!!i) = \pi \cdot x_i$, where $1 \leq i \leq n$ and $!!$ indicates lookup.

- (4) If $\pi \in \text{Perm}$ and (X, x) is a pointed set (Notation 2.1.3(2)) then π acts **pointwise** on (X, x) as follows:

$$\pi \cdot (X, x) = (\pi \cdot X, \pi \cdot x).$$

(This is indeed just a special case of the previous case, for tuples.)

$$\begin{aligned}
\text{Input} &= \mathbb{A} \times \alpha \\
\text{Output} &= \mathbb{A} \times \beta \times \text{Validator} \\
\text{Transaction} &\subseteq \text{fin}(\text{Input}) \times \text{fin}(\text{Output}) \\
\text{Validator} &\subseteq \text{pow}(\beta \times \text{Transaction}_!)
\end{aligned}$$

Fig. 1. Type equations of Idealised EUTxO

- (5) If $\pi \in \text{Perm}$ and f is a function then π has the **conjugation action** on f as follows:

$$(\pi \cdot f)(x) = \pi \cdot (f(\pi^{-1} \cdot x)).$$

Note therefore that $\pi \cdot (f(x)) = (\pi \cdot f)(\pi \cdot x)$.

- (6) In particular, π acts as the above on the inputs, outputs, sets of inputs, sets of outputs, transactions, and validators from Figure 1.
- (7) If $\pi \in \text{Perm}$ and R is a relation, then π acts **pointwise** such that

$$\pi \cdot R = \{(\pi \cdot x, \pi \cdot y) \mid (x, y) \in R\}$$

so that

$$x \pi \cdot R y \iff \pi^{-1} \cdot x R \pi^{-1} \cdot y.$$

We use Definition 2.2.3 in Definition 4.3.4, but it is a useful concept so we include it here:

DEFINITION 2.2.3. If $a \in \mathbb{A}$ then write $\text{fix}(a)$ for the set of permutations $\pi \in \text{Perm}$ such that $\pi(a) = a$. In symbols:

$$\text{fix}(a) = \{\pi \in \text{Perm} \mid \pi(a) = a\}.$$

DEFINITION 2.2.4. Call an element **equivariant** when $\pi \cdot x = x$ for every $\pi \in \text{Perm}$. Concretely:

- (1) \mathbb{A} is equivariant, and no individual atom $a \in \mathbb{A}$ is equivariant.
- (2) A set X is equivariant precisely when $\forall \pi \in \text{Perm}. (x \in X \iff \pi \cdot x \in X)$. In words:
A set is equivariant precisely when it is closed in the orbits of its elements under the permutation action.
- (3) (x_1, \dots, x_n) is equivariant precisely when x_i is equivariant for every $1 \leq i \leq n$.
- (4) A pointed set (X, x) is equivariant precisely when X and x are equivariant.
- (5) A function f is equivariant when $\pi \cdot (f(x)) = f(\pi \cdot x)$ for every x and every π .
- (6) A relation R is equivariant when $x R y$ if and only if $\pi \cdot x R \pi \cdot y$ for every x and every π .

3. IDEALISED EUTxO: IEUTxO

3.1. IEUTxO equations, solutions, and models

DEFINITION 3.1.1. Let **idealised EUTxO** be the type equations in Figure 1.

It might be helpful to note that diagrams of Definition 3.1.2 will appear later in Example 3.3.1:

DEFINITION 3.1.2. Let a **solution** to the IEUTxO type equations in Figure 1 be a tuple

$$\mathbb{T} = (\alpha, \beta, \text{Transaction}, \text{Validator}, \nu : \text{Validator} \hookrightarrow \text{pow}(\beta \times \text{Transaction}))$$

where:

- (1) $\alpha, \beta, \text{Transaction}$, and Validator are equivariant sets (Definition 2.2.4(2)).
- (2) Transaction is an equivariant subset of $(\text{fin}(\mathbb{A} \times \alpha) \times \text{fin}(\mathbb{A} \times \beta \times \text{Validator})) \setminus \{(\emptyset, \emptyset)\}$
- (3) ν is an equivariant injective function (Definition 2.2.4(5)) from Validator to $\text{pow}(\beta \times \text{Transaction}_!)$.

NOTATION 3.1.3. We may elide equivariance conditions Definition 3.1.2 henceforth. Any such type-like definition will be equivariant — i.e. closed under taking orbits of the permutation action — unless stated otherwise.

REMARK 3.1.4. Equivariance comes from the underlying ZFA universe. Notation 3.1.3 can be viewed as an assertion that the definition exists in the category of equivariant ZFA sets and equivariant functions between them (or if the reader prefers: sets with a permutation action, and equivariant functions between them).

This paper will not be heavy on sets or categorical foundations: we use just enough so that readers from various backgrounds get a hook on the ideas that speaks to them, and so it is always clear what is meant and how it could be made fully formal.

Note that just because a set is equivariant does not mean all its elements must be; for instance, \mathbb{A} is equivariant (and consists of a single permutation orbit), but none of its elements $a \in \mathbb{A}$ are equivariant.

REMARK 3.1.5. In Figure 1 note from Notation 2.1.3(2) that $\text{Transaction}_! = \text{fin}_!(\text{Input}) \times \text{fin}(\text{Output})$.

Then the UTxO model — which is what Bitcoin is based on — is the special case of EUTxO where validators just examine the point in the pointed input of $\text{Transaction}_!$. So intuitively, in the UTxO model a validator sees the input that points to it, but does not see the context of the transaction in which that input occurs.

So we can convert Figure 1 into an *Idealised UTxO* model by changing the line for validators to:

$$\text{Validator} \subseteq \text{pow}(\beta \times \text{Input}).$$

REMARK 3.1.6. The sets subtraction in Definition 3.1.2(2) disallows the empty transaction, having no inputs or outputs.

We also note that Definition 3.1.2(2) uses a subset inclusion, whereas Definition 3.1.2(3) uses an injection ν . Why?

First, in practice we would expect $\nu(v)$ to be a *computable* subset of $\beta \times \text{Transaction}_!$, since we have implementations in mind (though nothing in the mathematics to follow will depend on this).

Also, sets are well-founded, so a pure subset inclusion solution to Figure 1, for both clauses, would be difficult; we use ν to break the downward chain of sets inclusions.⁴

Just as we may write

$$\mathbb{N} \subseteq \mathbb{Q} \subseteq \mathbb{R},$$

eliding (or neglecting to consider) that their realisations may differ (finite cardinals vs. equivalence classes of pairs vs. Dedekind cuts), so — since ν is an injection — we may abuse notation and treat Validator as a literal subset of $\text{pow}(\beta \times \text{Transaction}_!)$, where convenient.

This is standard, provided we clearly state what is intended and are confident that we could unroll the injections if required. See

— Definition 3.1.7 for an example of the elided presentation, and

— Definition 6.2.1 for an instance where we unpack the definition in full detail.

DEFINITION 3.1.7. An **IEUTxO model** \mathbb{T} is a solution (Definition 3.1.2) to the type equations in Figure 1. Thus (modulo Remark 3.1.6) \mathbb{T} is a tuple

$$\mathbb{T} = (\alpha_{\mathbb{T}}, \beta_{\mathbb{T}}, \text{Transaction}_{\mathbb{T}}, \text{Validator}_{\mathbb{T}})$$

of sets that solves the equations in Figure 1. We may drop the \mathbb{T} subscripts where these are clear from the context.

NOTATION 3.1.8. (1) If $tx \in \text{Transaction}$ appears in $txs \in [\text{Transaction}]$ then write $tx \in txs$.

(2) If $tx, tx' \in txs$ and tx appears before tx' in txs , then call tx **earlier** than tx' and tx' **later** than tx (in txs).

⁴A universe of non-wellfounded sets [1] would also solve this. That might be overkill for this paper, but this might become relevant if the theorems of this paper are implemented in a universe with more direct support for non-wellfounded objects than ZFA provides.

$$\begin{array}{ll}
i = (p, d) \in \text{Input} & \text{pos}(i) = \{p\} \\
o = (p, V) \in \text{Output} & \text{pos}(o) = \{p\} \\
tx = (I, O) \in \text{Transaction} & \text{pos}(tx) = \bigcup \{\text{pos}(x) \mid x \in I \cup O\} \\
ctx = ((I, i), O) \in \text{Transaction}_! & \text{pos}(ctx) = \text{pos}(I, O) \\
txs \in [\text{Transaction}] & \text{pos}(txs) = \bigcup \{\text{pos}(tx) \mid tx \in txs\}
\end{array}$$

Fig. 2. Positions of

- (3) If $tx = (I, O) \in \text{Transaction}$ and $o \in \text{Output}$, say o **appears in** tx and write $o \in tx$ when $o \in O$; similarly for an input $i \in \text{Input}$.
 We may silently extend this notation to larger data structures, writing e.g. $i \in txs$ when $i \in tx \in txs$ for some tx .
- (4) If $tx = (I, O) \in \text{Transaction}$ and $i \in I$ then write $tx@i \in \text{Transaction}_!$ for the input-in-context $((I, i), O)$ obtained by pointing I at $i \in I$ (Notation 2.1.3).

DEFINITION 3.1.9. Suppose \mathbb{T} is an IEUTxO model. We define **positions of**, written pos , as in Figure 2.

REMARK 3.1.10. Intuitively, $\text{pos}(x)$ collects the positions mentioned explicitly on the inputs or outputs of a structure. Note that validators may also act depending on positions of their inputs, but this information is not detected by pos . For instance, consider a (arguably odd, but imaginable) output o having the form

$$o = (b, 0, \lambda i. \text{pos}(i) = \{a\}).$$

So this output is at position b , $\beta = \mathbb{N}$ and o carries data 0, and o has a validator that validates an input precisely when it is at position a . Then $\text{pos}(o) = \{b\}$.⁵

LEMMA 3.1.11. Suppose \mathbb{T} is an IEUTxO model and $txs \in [\text{Transaction}]$. Then

$$\text{pos}(txs) = \emptyset \quad \text{implies} \quad txs = [].$$

Proof. Recall from Definition 3.1.2(2) that the empty transaction is disallowed. Now examining Figure 2 we see that the only way txs can mention no positions *at all*, is by having no transactions and so being empty. \square

DEFINITION 3.1.12. (1) Suppose $i \in \text{Input}$ and $o \in \text{Output}$. Then say that i **points to** o and write $i \mapsto o$ when they share a position:

$$i \mapsto o \quad \text{when} \quad \text{pos}(i) = \text{pos}(o).$$

(The use of ‘point’ here is unrelated to the ‘pointed sets’ from Notation 2.1.3.)

(2) Recall the notation $tx@i$ from Notation 3.1.8(4). Suppose that:

$$\begin{array}{l}
i = (p, k) \in \text{Input} \\
i \in tx \in \text{Transaction} \quad \text{and} \\
o = (p, V) \in \text{Output}.
\end{array}$$

Then write

$$\text{validates}(o, tx@i) \quad \text{when} \quad tx@i \in V$$

and say that the output o **validates** the input-in-context $tx@i$.

⁵Note to experts in nominal techniques: so the support of validators, if any, is not counted in pos . See also the discussion in Subsection 8.4.

3.2. Chunks and blockchains

3.2.1. Chunks

DEFINITION 3.2.1. Suppose $\mathbb{T} = (\alpha, \beta, \text{Transaction}, \text{Validator})$ is an IEUTxO model. Call a transaction-list $txs \in [\text{Transactions}]$ a **chunk** when:

- (1) Distinct outputs appearing in txs have distinct positions.
- (2) Distinct inputs appearing in txs have distinct positions.
It may be that the position of an input equals that of some output, or in symbols: $i \mapsto o$.
- (3) Every input i appearing in some transaction tx in txs — so $i \in tx \in txs$ — points to *at most one* output $o \in tx' \in txs$. Write this output $txs(i)$ if it exists, so $o \in txs(i) \in txs$.
- (4) For every $i \in tx \in txs$, if $txs(i)$ is defined then it must be strictly earlier than tx in txs .
- (5) For every $i \in tx \in txs$, if an output $txs(i)$ is defined then $\text{validates}(txs(i), tx@i)$.

NOTATION 3.2.2. (1) Write

$$\text{Chunk}_{\mathbb{T}} \subseteq [\text{Transaction}_{\mathbb{T}}] \quad \text{for the set of chunks over } \mathbb{T}.$$

We may drop the \mathbb{T} subscripts; the meaning will always be clear.

- (2) We may also call a transaction-list **valid**, when it is a chunk. That is: chunks are precisely the *valid* transaction-lists.

REMARK 3.2.3. (1) We can summarise Definition 3.2.1 as follows:

A list of transactions is a *chunk* when all input positions are distinct, all output positions are distinct, and all inputs point to at most one earlier validating output.

- (2) The way we have formulated the structure of chunks in Definitions 3.2.1 and 3.1.7 reminds this author of the π -calculus, where positions correspond to π -calculus channel names (and outputs are outputs and inputs are inputs).

When we have this intuition in mind, we may occasionally call positions **channels**, as in the *blocked channels* of Subsection 5.5. See also the discussion in Remark 4.3.8.

An important special case of Notation 3.2.2 is when the chunk is a singleton list, i.e. it contains just one transaction:

LEMMA 3.2.4. Suppose \mathbb{T} is an IEUTxO model and suppose $tx \in \text{Transaction}$. Then

$$[tx] \in \text{Chunk} \quad \text{if and only if} \quad \text{input}(tx) \cap \text{output}(tx) = \emptyset.$$

Proof. We consider the conditions in Definition 3.2.1 and see that condition 4 forces the inputs and outputs of the transaction to be disjoint, and then none of the other conditions are relevant. \square

3.2.2. Algebraic and closure properties of chunks. Lemma 3.2.5 expresses that a list of transactions is a valid chunk if and only if every sublist of it of length at most three, is a valid chunk. In this sense, (in)validity is a *local* phenomenon:

LEMMA 3.2.5. Suppose $\mathbb{T} = (\alpha_{\mathbb{T}}, \beta_{\mathbb{T}}, \text{Transaction}_{\mathbb{T}}, \text{Validator}_{\mathbb{T}})$ is an IEUTxO model, and suppose $[tx_1, \dots, tx_n] \in [\text{Transaction}]$. Then the following conditions are equivalent:

- $[tx_i, tx_j, tx_k] \in \text{Chunk}$ for every $1 \leq i \leq j \leq k \leq n$ ⁶
- $[tx_1, \dots, tx_n] \in \text{Chunk}$

Proof. We note of the well-formedness conditions on chunks from Definition 3.2.1 that they all concern relationships involving at most three transactions. \square

COROLLARY 3.2.6 (Validity is down-closed). Suppose we have an IEUTxO model $\mathbb{T} = (\alpha, \beta, \text{Transaction}, \text{Validator})$ and $l, l' \in [\text{Transaction}]$.

⁶This condition holds trivially if $n = 0$, i.e. for the empty list.

Recall from Notation 2.1.3 that we order lists by sublist inclusion, so $l' \leq l$ when l' is a sublist of l . Then

$$ch \in \text{Chunk} \wedge l' \in [\text{Transaction}] \wedge l' \leq ch \text{ implies } l' \in \text{Chunk}.$$

In words: every sublist of a chunk, is itself a chunk.

Proof. From Lemma 3.2.5. □

We can wrap up Corollary 3.2.6 in a nice mathematical package:

DEFINITION 3.2.7. Suppose (X, \leq, \cdot, e) has the following structure:

- (1) X is a set.
- (2) (X, \leq) is a partial order, for which e is a bottom element.
- (3) \cdot is a **partial monoid** action on X , meaning that $(x \cdot y) \cdot z$ exists if and only if $x \cdot (y \cdot z)$ exists, and if both exist then they are equal.⁷
- (4) \cdot is **down-closed**, meaning that if $x' \leq x$ and $x \cdot y$ exists, then so does $x' \cdot y$, and similarly for $y \cdot x$ and $y \cdot x'$.
- (5) \cdot is **monotone** where defined, meaning that if $x' \leq x$ then $x' \cdot y \leq x \cdot y$ (provided $x \cdot y$ exists), and similarly for $y \cdot x$ and $y \cdot x'$.

In this case, call (X, \leq, \cdot, e) a **partially-ordered partial monoid**.

THEOREM 3.2.8. Suppose \mathbb{T} is an IEUTxO model (Definition 3.1.7). Then its set of chunks $\text{Chunk}_{\mathbb{T}}$ (Definition 3.2.1) forms a partially-ordered partial monoid (Definition 3.2.7), where \leq is sublist inclusion, \cdot is list concatenation, and the unit element is \square the empty set.

Proof. By facts of lists, and Corollary 3.2.6. □

3.2.3. Some observations on observational equivalence

REMARK 3.2.9. Lemmas 3.2.10 and 3.2.13 apply to IEUTxO models and essentially give criteria for observational equivalence when positions are disjoint.

We find them echoed in the theory of abstract chunk systems as Definitions 4.4.1(5) and 4.4.1(4), and we need them for Proposition 4.4.8.

LEMMA 3.2.10. Suppose $\mathbb{T} = (\alpha, \beta, \text{Transaction}, \text{Validator})$ is an IEUTxO model and $ch, ch' \in \text{Chunk}_{\mathbb{T}}$. Then

$$\text{pos}(ch) \cap \text{pos}(ch') = \emptyset \text{ implies } ch \cdot ch' \in \text{Chunk}_{\mathbb{T}}.$$

Proof. By routine checking of possibilities, using the fact that if $\text{pos}(ch) \cap \text{pos}(ch') = \emptyset$ (Definition 3.1.9) then they have no positions in common, so no output in one can be called on to validate an input in the other. □

DEFINITION 3.2.11. Suppose $ch, ch' \in \text{Chunk}_{\mathbb{T}}$. Then call ch and ch' **commuting** when for every $ch'' \in \text{Chunk}_{\mathbb{T}}$,

$$\begin{aligned} ch \cdot ch' \cdot ch'' \in \text{Chunk}_{\mathbb{T}} &\iff ch' \cdot ch \cdot ch'' \in \text{Chunk}_{\mathbb{T}} \quad \text{and} \\ ch'' \cdot ch \cdot ch' \in \text{Chunk}_{\mathbb{T}} &\iff ch'' \cdot ch' \cdot ch \in \text{Chunk}_{\mathbb{T}}. \end{aligned}$$

REMARK 3.2.12. Definition 3.2.11 is clearly a notion of observational equivalence between $ch \cdot ch'$ and $ch' \cdot ch$ where the observable is ‘forms a valid chunk with’. This observable does not depend on internal structure, so we will develop it further once we have abstract chunk systems; see Definition 4.3.13.

For now, Definition 3.2.11 gives us just enough of the background theory of observational equivalence, to state and prove Lemma 3.2.13, Proposition 3.2.18, and Theorem 3.3.2.

⁷A slightly weaker possibility is that $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z)$ need not exist or not exist together, but if they both exist, then they are equal. This is the natural notion if we bind names of spent output-input pairs. More discussion in Subsection 8.2.

LEMMA 3.2.13. Suppose $\mathbb{T} = (\alpha, \beta, \text{Transaction}, \text{Validator})$ is an IEUTxO model. Then:

- (1) If $tx, tx', tx'' \in \text{Transaction}$ and $\text{pos}(tx) \cap \text{pos}(tx') = \emptyset$ (Definition 3.1.9) then the following all hold:

$$\begin{aligned} [tx, tx'] \in \text{Chunk} &\iff [tx', tx] \in \text{Chunk} \iff [tx], [tx'] \in \text{Chunk} \\ [tx, tx', tx''] \in \text{Chunk} &\iff [tx', tx, tx''] \in \text{Chunk} \\ [tx'', tx, tx'] \in \text{Chunk} &\iff [tx'', tx', tx] \in \text{Chunk} \end{aligned}$$

- (2) As a corollary, if $ch, ch' \in \text{Chunk}$ and $\text{pos}(ch) \cap \text{pos}(ch') = \emptyset$ then ch and ch' are commuting (Definition 3.2.11).

Proof. (1) By routine checking of possibilities, using the fact that if $\text{pos}(tx) \cap \text{pos}(tx') = \emptyset$ (Definition 3.1.9) then they have no positions in common, so no output in one can be called upon to validate an input in the other.

- (2) It is a fact that if $tx \in l$ then $\text{pos}(tx) \subseteq \text{pos}(l)$ and similarly for $tx' \in l'$. The corollary now follows by a routine argument from part 1 of this result and Lemma 3.2.5. \square

3.2.4. UTxOs and UTxIs

DEFINITION 3.2.14. Suppose \mathbb{T} is an IEUTxO model and $txs \in [\text{Transaction}]$.

- (1) If $i \in tx \in txs$ and $txs(i)$ is not defined⁸ then call the unique atom $a \in \text{pos}(i) \subseteq \mathbb{A}$ an **unspent transaction input**, or **UTxI**. Write

$$utri(txs) \subseteq_{fin} \mathbb{A} \quad \text{for the UTxIs of } txs.$$

- (2) If $o \in tx \in txs$ and $o \neq txs(i)$ for all later $i \in tx \in txs$ ⁹ then call the unique atom $a \in \text{pos}(o) \subseteq \mathbb{A}$ an **unspent transaction output**, or **UTxO**. Write

$$utxo(txs) \subseteq_{fin} \mathbb{A} \quad \text{for the set of UTxOs of } txs.$$

- (3) If $o \in tx \in txs$ and $o = txs(i)$ for some later $i \in tx \in txs$, then call the atom $a \in \text{pos}(o) \subseteq \mathbb{A}$ a **spent transaction**, or **STx**. Write

$$stx(txs) \subseteq_{fin} \mathbb{A} \quad \text{for the set of STxs of } txs.$$

REMARK 3.2.15. Some comments on the interpretation of $utri$ and $utxo$ and stx from Definition 3.2.14:

$utri(txs)$, $utxo(txs)$, and $stx(txs)$ are all finite sets of atoms, but we interpret them somewhat differently:

- (1) Intuitively, an atom $a \in utxo(txs)$ identifies an output $o \in tx \in txs$ with position a . So $utxo(txs)$ is a finite set of names of outputs in txs .
- (2) Intuitively, an atom $a \in utri(txs)$ identifies an input-in-context $tx@i$, for $i \in tx \in txs$ with $\text{pos}(i) = a$.

We say this because the validator of an output takes as argument an input-in-context $tx@i \in \text{Transaction}$. So $utri(txs)$ is a finite set of names for inputs-in-contexts.

- (3) Intuitively, an atom $a \in stx(txs)$ identifies a pair of an output and the input-in-context that spends it. Thus a could be thought of as this pair, or a could be thought of as an edge in a graph that joins a node representing the output, to a node representing the input.

So $stx(txs)$ is a finite set of internal names of already-spent communications between outputs and inputs-in-context within txs .

Lemma 3.2.16 uses Definition 3.2.14 to note some simple properties of Definition 3.2.1.

LEMMA 3.2.16. Suppose \mathbb{T} is an IEUTxO model and $ch, ch' \in \text{Chunk}_{\mathbb{T}}$. Then:

⁸...so i does not point to an earlier validating output in txs ...

⁹...so o does not validate some later input in txs ...

- (1) $utxi(ch) \cap utxo(ch) = \emptyset$
- (2) If $ch \cdot ch' \in \text{Chunk}_{\mathbb{T}}$ then $pos(ch) \cap pos(ch') \subseteq utxo(ch) \cap utxi(ch')$.
- (3) $\emptyset = utxi(ch) \cap stx(ch)$
 $\emptyset = utxo(ch) \cap stx(ch)$
 $pos(ch) = utxi(ch) \uplus utxo(ch) \uplus stx(ch) \quad \uplus \text{ is disjoint union}$

Proof. (1) An input cannot point to a later output, because of Definition 3.2.1(4), and if it points to an earlier output then by construction in Definition 3.2.14 this position no longer labels a UTxO or UTxI. Furthermore a position can be used at most once in an input-output pair, by Definition 3.2.1(3).

(2) From Definition 3.2.1(4), as for the previous case.

(3) All facts of Definition 3.2.14 and Figure 2. □

REMARK 3.2.17. Proposition 3.2.18 can be viewed as a considerably stronger version of Lemma 3.2.10. It is an important result because it relates three apparently different observables:

- (1) A statically observable property, that ch and ch' mention disjoint sets of positions.
- (2) A locally observable property, that ch and ch' compose in both directions.
- (3) An abstract global observable, that $ch \cdot ch'$ and $ch' \cdot ch$ can be commuted in any larger chunk.

Compare also with Proposition 4.4.8, which is a similar result but for the very differently-constructed abstract chunk systems.

PROPOSITION 3.2.18. Suppose $\mathbb{T} = (\alpha, \beta, \text{Transaction}, \text{Validator})$ is an IEUTxO model and $ch, ch' \in \text{Chunk}_{\mathbb{T}}$. Then the following are equivalent:

- (1) $pos(ch) \cap pos(ch') = \emptyset$
- (2) $ch \cdot ch' \in \text{Chunk}_{\mathbb{T}} \wedge ch' \cdot ch \in \text{Chunk}_{\mathbb{T}}$.
- (3) ch and ch' are commuting.

Proof. The top-to-bottom implication is Lemma 3.2.10.

For the bottom-to-top implication, suppose that $ch \cdot ch', ch' \cdot ch \in \text{Chunk}_{\mathbb{T}}$. From Lemma 3.2.16(2) we have

$$pos(ch) \cap pos(ch') \subseteq (utxo(ch) \cap utxi(ch')) \cap (utxo(ch') \cap utxi(ch)).$$

We can rearrange this:

$$pos(ch) \cap pos(ch') \subseteq (utxo(ch) \cap utxi(ch)) \cap (utxo(ch') \cap utxi(ch')).$$

Now we use Lemma 3.2.16(1).

The final part is direct from Lemma 3.2.13(2). □

We conclude with Lemma 3.2.19, which we will need later in Lemma 5.2.1:

LEMMA 3.2.19. Suppose $\mathbb{T} \in \text{IEUTxO}$ is an IEUTxO model and $\pi \in \text{Perm}$ is a permutation of atoms and $txs \in [\text{Transaction}_{\mathbb{T}}]$. Then:

$$\begin{aligned} f(\pi \cdot txs) &= \pi \cdot f(txs) & \text{for } f \in \{utxi, utxo, stx, pos\} \\ &= \{\pi(a) \mid a \in f(txs)\} \end{aligned}$$

In the terminology of Definition 2.2.4: $utxi$, $utxo$, stx , and pos are all equivariant.

Proof. Direct from Figure 2 and Definitions 3.2.14 and 2.2.2. □

3.2.5. Blockchains. With the machinery we have already constructed, it is quick and easy to define blockchains:

DEFINITION 3.2.20. A **blockchain** is a chunk $ch \in \text{Chunk}$ such that $utxi(ch) = \emptyset$.

LEMMA 3.2.21. *We note two alternative characterisation of blockchains (Definition 3.2.20). A chunk is a blockchain when ...*

- (1) ...the ‘at most one’ in Definition 3.2.1(3) is strengthened to ‘precisely one’.
- (2) ...the function $i \mapsto txs(i)$ (Definition 3.2.1(3)) is a total function on the inputs in txs (so that every input points to precisely one output in an earlier transaction).

REMARK 3.2.22. We step back to reflect on Definition 3.2.20. This is supposed to be a paper about blockchains; why did it take us this long to get to them? Because they are a special case of something better and more pertinent.

A blockchain is just a left-closed chunk. There is nothing wrong with this definition, but mathematically, chunks seem more interesting:

- A sublist of a blockchain is a chunk, not a blockchain.
- A composition of blockchains is not naturally another blockchain; whereas a composition of chunks is naturally a chunk (provided the composition is valid).
- If we cut a blockchain into n pieces then we get one *blockchain* (the initial segment) ... and $n - 1$ *chunks*.
- Chunks can in any case be viewed as a natural generalisation of blockchains, to allow UTxIs as well as UTxOs.

Definitions and results like Definition 3.2.7, Theorem 3.2.8, and Lemma 3.2.13 inhabit a universe of chunks, not blockchains.

Even in implementation, where we care about real blockchains on real systems, a lot of development work goes into allowing users in practice to download only partial histories of the blockchain rather than having to download and store a complete record — the motivation here is practical, not mathematical — and in the terminology of this paper, we would say that for efficiency we may prefer to work with chunks where possible, because they can be partial and so can be more lightweight.

So the focus of this paper is on chunks: they generalise blockchains, have better mathematical structure; and chunks are in any case where we arrive even if we *start off* asserting (de)compositional properties of blockchains; and finally — though this is not rigorously explored in this paper, but we would suggest that — chunks are also where we arrive when we consider space-efficient blockchain implementations.

Finally, we mention that blockchains have a right monoid action given by concatenating chunks. Thus, by analogy here with rings and modules, we could imagine for future work a mathematics of blockchains generalising Definition 3.2.20 such that a ‘blockchain set’ is just any set with a suitable chunk action.

3.3. Some diagrams, and an application

We conclude with some diagrams, and Theorem 3.3.2 which is an application of our machinery so far:

EXAMPLE 3.3.1. Example transactions, blockchains, and chunks are illustrated in Figures 3, 4, 5, 6, and 7.¹⁰

We leave it as an exercise to the reader to verify that: $\mathcal{B}, \mathcal{B}', [tx_1, tx_2]$ and $[tx_1, tx_3]$ are blockchains; and $[tx]$, $[tx_3, tx_4]$ and $[tx_2, tx_4]$ are chunks but not blockchains. Also, e.g. $[tx_2, tx_1]$ is neither blockchain nor chunk, though it is a list of transactions, because the b -input of tx_2 points to the later b -output of tx_1 .

¹⁰These diagrams are imported from [5], with my coauthor’s agreement.

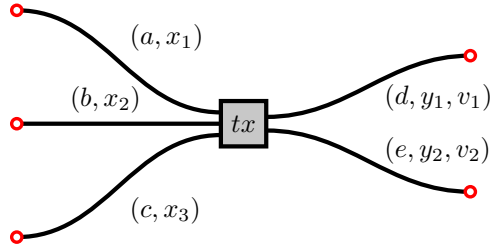


Fig. 3. A transaction tx with three inputs and two outputs, positions a, b, c, d, e , input data x_i , output data y_j , and validators v_j

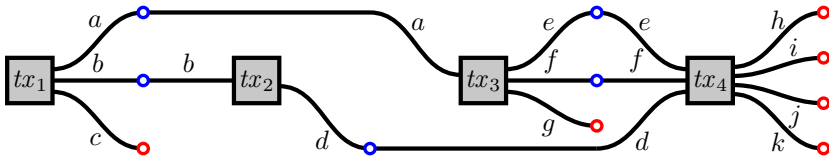


Fig. 4. A blockchain $\mathcal{B} = [tx_1, tx_2, tx_3, tx_4]$

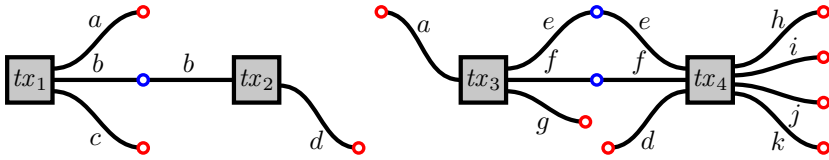


Fig. 5. B chopped up as a blockchain $[tx_1, tx_2]$ and a chunk $[tx_3, tx_4]$

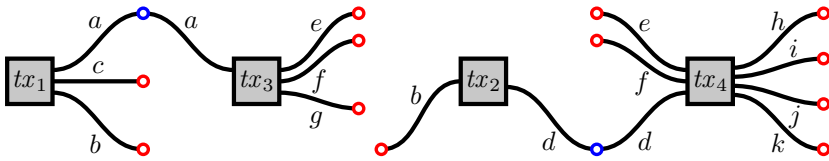


Fig. 6. B chopped up as a blockchain $[tx_1, tx_3]$ and a chunk $[tx_2, tx_4]$

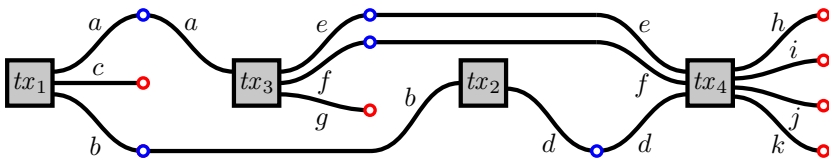


Fig. 7. The blockchain $\mathcal{B}' = [tx_1, tx_3, tx_2, tx_4]$

THEOREM 3.3.2. Suppose $\mathbb{T} \in \text{IEUTxO}$ and $y, x, x' \in \text{Chunk}_{\mathbb{T}}$. Suppose further that

$$y \cdot x \cdot x' \in \text{Chunk}_{\mathbb{T}} \quad \text{and} \quad \text{utxi}(y \cdot x') = \text{utxi}(y \cdot x \cdot x').$$

Then we have that:

- (1) $y \cdot x' \cdot x \in \text{Chunk}_{\mathbb{T}}$ and
- (2) x and x' are commuting (Definition 3.2.11).

Proof. We know by Corollary 3.2.6 (because $y \cdot x \cdot x'$ is a chunk) that $x \cdot x'$ is a chunk, so $\text{pos}(x) \cap \text{pos}(x') \subseteq \text{utxo}(x) \cap \text{utxi}(x')$.

We also know that $\text{utxi}(y \cdot x') = \text{utxi}(y \cdot x \cdot x')$ and it follows from Definition 3.2.14 that ($\text{utxo}(y) \cap \text{utxi}(x) = \emptyset$ and) $\text{utxo}(x) \cap \text{utxi}(x') = \emptyset$.¹¹

Therefore $\text{pos}(x) \cap \text{pos}(x') = \emptyset$. By Proposition 3.2.18, x and x' are commuting, and it follows (since $y \cdot x' \cdot x \in \text{Chunk}_{\mathbb{T}}$) that $y \cdot x \cdot x' \in \text{Chunk}_{\mathbb{T}}$. \square

REMARK 3.3.3. Let us recall Definition 3.2.20 and consider for concreteness the special case where $\text{utxi}(y \cdot x') = \text{utxi}(y \cdot x \cdot x') = \emptyset$, so that $y \cdot x'$ and $y \cdot x \cdot x'$ are blockchains and not just chunks.

Then Theorem 3.3.2 models a situation where someone submits a chunk x to attach to a blockchain y , but since this is a distributed system, somebody else gets in first and attaches x' .

Then Theorem 3.3.2 tells us that x and x' are commuting, and what makes this Theorem interesting is that it is an important correctness property from the point of view of the user who created x . It *does not matter* that x' got in first — $y \cdot x \cdot x'$ and $y \cdot x' \cdot x$ are both chunks, they are equivalent up to observable behaviour, and x' and x cannot affect each other's inputs and outputs.

3.4. The category IEUTxO of IEUTxO models

We can organise our IEUTxO models into a category:

DEFINITION 3.4.1. Let IEUTxO be a category such that:

- (1) Objects \mathbb{S}, \mathbb{T} are IEUTxO models (Definition 3.1.2).
- (2) An arrow $f : \mathbb{S} \rightarrow \mathbb{T}$ is a map

$$f : \text{Transaction}_{\mathbb{S}} \rightarrow \text{Chunk}_{\mathbb{T}}$$

such that if $tx, tx', tx'' \in \text{Transaction}_{\mathbb{S}}$ then

- (a) $[tx, tx'] \in \text{Chunk}_{\mathbb{S}}$ implies $f(tx) \cdot f(tx') \in \text{Chunk}_{\mathbb{T}}$, and
- (b) $[tx, tx', tx''] \in \text{Chunk}_{\mathbb{S}}$ implies $f(tx) \cdot f(tx') \cdot f(tx'') \in \text{Chunk}_{\mathbb{T}}$.

Above, \cdot denotes monoid composition, which on chunks is list concatenation; see Notation 2.1.3(4).

- (3) The identity arrow maps tx to $[tx]$.
- (4) Composition of arrows is pointwise, meaning that if

$$\begin{array}{ll} f : \mathbb{S} \rightarrow \mathbb{S}' & tx \in \text{Transaction}_{\mathbb{S}} \\ f' : \mathbb{S}' \rightarrow \mathbb{S}'' & f(tx) = [tx'_1, \dots, tx'_n] \end{array}$$

then $f' \circ f : \mathbb{S} \rightarrow \mathbb{S}''$ is such that

$$tx \in \text{Transaction}_{\mathbb{S}} \mapsto f'(f(tx)) \in \text{Chunk}_{\mathbb{S}''}.$$

We prove this mapping is indeed an arrow — thus, it maps to chunks — in Corollary 3.4.4.

LEMMA 3.4.2. An arrow $f : \mathbb{S} \rightarrow \mathbb{T}$ (Definition 3.4.1(2)) induces a mapping of chunks $\text{Chunk}_{\mathbb{S}} \rightarrow \text{Chunk}_{\mathbb{T}}$, by acting on the individual transactions and composing the results:

$$f([tx_1, \dots, tx_n]) = f(tx_1) \cdot \dots \cdot f(tx_n).$$

Proof. The nontrivial part is to check that

¹¹We really use here the fact that names can only be used to link an output to an input *once*.

- if $ch = [tx_1, \dots, tx_n]$ is a valid chunk in $\text{Chunk}_{\mathbb{S}}$,
- then $f(tx_1) \cdot \dots \cdot f(tx_n)$ is a valid chunk in $\text{Chunk}_{\mathbb{T}}$.

This follows by combining condition 2 of Definition 3.4.1 with Lemma 3.2.5 and Corollary 3.2.6. \square

COROLLARY 3.4.3. *Condition 2 of Definition 3.4.1 is equivalent to the following condition:*

$$[tx_1, \dots, tx_n] \in \text{Chunk}_{\mathbb{S}} \text{ implies } f(tx_1) \cdot \dots \cdot f(tx_n) \in \text{Chunk}_{\mathbb{T}}.$$

Proof. Condition 2 of Definition 3.4.1 is just a special case of the abstract condition above. The reverse implication is Lemma 3.4.2. \square

COROLLARY 3.4.4. *Composition of arrows as given in Definition 3.4.1(4) is well-defined; that is, the composition $f'f$ really is a map from transactions to chunks.*

Proof. Continuing the notation of Definition 3.4.1(4), by assumption f maps $tx \in \text{Transaction}_{\mathbb{S}}$ to some $f(tx) \in \text{Chunk}_{\mathbb{S}'}$, and then by Lemma 3.4.2 the action of f' maps $f(tx)$ to a valid chunk $f'(f(tx)) \in \text{Chunk}_{\mathbb{S}''}$. \square

REMARK 3.4.5. We briefly discuss the design decisions embedded in Definition 3.4.1:

- (1) Corollary 3.4.3 is arguably more readable than conditions 2a and 2b of Definition 3.4.1, but note how this comes at the cost of an additional universally quantified parameter n . It is a matter of taste which version we take as primitive: the one in the Definition has fewest parameters and is easiest to check.
- (2) We could relax the condition to allow f to be a partial map.
This would exhibit IEUTxO as a subcategory of a larger category with the same objects but more arrows, and in particular it would allow chunks in \mathbb{S} to cease to be valid when mapped to \mathbb{T} — we would still insist that f be a *partial* monoid homomorphism on chunks, where everything is defined.
We did not choose this design for this paper, but it might be useful for future work; e.g. following an intuition that \mathbb{S} is a liberal universe of chunks, and f maps it to a stricter universe \mathbb{T} in which additional restrictions are appended to validators. Thus, chunks in the liberal world might cease to be valid in the stricter universe.
- (3) We could also restrict f further so that $f : \text{Transaction}_{\mathbb{S}} \rightarrow \text{Transaction}_{\mathbb{T}}$.
This would yield fewer arrows, and we prefer to allow the flexibility of mapping a single transaction in \mathbb{S} to a chunk of transactions in \mathbb{T} ; following an intuition that \mathbb{S} is a coarse-grained representation which f maps into a finely-grained representation where something that was considered a single transaction is now a chunk.

4. ABSTRACT CHUNK SYSTEMS: ACS

4.1. Basic definitions

We recall some basic definitions:

DEFINITION 4.1.1. Suppose X is a set and $\leq \subseteq X^2$ is a relation on X . Call (X, \leq) a **well-ordering** when:

- (1) \leq is a partial order (reflexive, transitive, anti-symmetric), and
- (2) \leq is well-founded (every descending chain is eventually stationary).¹²

As per Notation 3.1.3, X and \leq are also assumed equivariant.

EXAMPLE 4.1.2. This should be familiar, but we give examples:

- (\mathbb{Z}, \leq) is not well-founded.

¹² Alternative and equivalent definition: every *strictly* descending chain is finite.

- $(pow(\mathbb{A}), \subseteq)$ and $(fin(\mathbb{A}), \subseteq)$ are well-founded.
- $[\mathbb{N}]$ (lists of numbers) with sublist inclusion is well-founded.

DEFINITION 4.1.3. Suppose (X, e, f, \leq) is a partial order with an equivariant least element e and an equivariant greatest element f . Call $x \in X$ **atomic** when

- (1) $e \leq x \leq f$ and
- (2) for every $x' \in X$ if $x' \leq x$ then either $x' = e$ or $x' = x$.

Write $atomic(X)$ for the set of atomic elements of X (see also Definition 4.2.5).

If we call $x \in X$ **proper** when it is neither e nor f (following the standard terminology of *proper subset*), then an atomic element is a “least proper element”.

REMARK 4.1.4. The set of atomic elements $atomic(X)$ is not to be confused with the set of atoms \mathbb{A} from Definition 2.1.1. This name collision is just a coincidence.

Lemma 4.1.5 will be useful later:

LEMMA 4.1.5. Suppose $\mathbb{T} \in \text{IEUTxO}$ and consider $\text{Chunk}_{\mathbb{T}}$ (valid lists of transactions; see Definition 3.2.1) as a partial order under sublist inclusion \leq .

Then the atomic elements in $(\text{Chunk}_{\mathbb{T}}, \leq)$ are precisely the singleton chunks (Notation 3.2.2).

Proof. Using Corollary 3.2.6. □

4.2. Monoid of chunks

DEFINITION 4.2.1. Assume we have equivariant data (X, e, f, \leq, \cdot) where:

- X is a set.
- $e, f \in X$ are called **unit** and **fail** respectively.
- $\leq \subseteq X^2$ is a relation.
- $\cdot : X^2 \rightarrow X$ is a **composition**.

Call (X, e, f, \leq, \cdot) a **monoid of chunks** when:

- (1) $e \cdot x = x = x \cdot e$.
- (2) $f \cdot x = f = x \cdot f$.
- (3) \leq is a well-ordering for which the unit e is a bottom element and the paradoxical element f is a top element.
- (4) Composition \cdot is **associative**, and **monotone** in both components, meaning that

$$\begin{array}{lcl} x' \leq x & \text{implies} & x' \cdot y \leq x \cdot y \quad \text{and} \\ y \leq y' & \text{implies} & x \cdot y \leq x \cdot y'. \end{array}$$

- (5) Composition is **increasing** in the sense that

$$x \leq x \cdot y \quad \text{and} \quad y \leq x \cdot y.$$

- (6) If $x_1, \dots, x_n \in X$ and $x_1 \cdot \dots \cdot x_n = f$, then there must exist $1 \leq i \leq j \leq k \leq n$ such that $x_i \cdot x_j \cdot x_k = f$.

REMARK 4.2.2. A few comments on Definition 4.2.1:

- (1) $x \cdot y$ is not necessarily a least upper bound for $\{x, y\}$.
Take $X = \{1, 2\}$ and $x = [1]$ and $y = [2]$ in Example 4.2.6(4) (finite lists with a top element). Then $x \cdot y$ and $y \cdot x$ are distinct and incomparable, so both are upper bounds for $\{x, y\}$ but $x \cdot y \not\leq y \cdot x$ and $y \cdot x \not\leq x \cdot y$.
- (2) We see that condition 6 of Definition 4.2.1 closely resembles Lemma 3.2.5, and indeed the condition is inspired by that very Lemma. We will use this in Proposition 5.1.3.

NOTATION 4.2.3. As is standard, we may write X for both a monoid of chunks and its carrier set. See for instance the first line of Definition 4.2.4.

DEFINITION 4.2.4. Suppose $X = (X, e, f, \leq, \cdot)$ is a monoid of chunks.

- (1) If $x \in X$ and $[x_1, \dots, x_n] \in [atomic(X)]$ is a finite list of atomic elements¹³ in X and

$$x = x_1 \cdot \dots \cdot x_n \quad \text{then say that } x \text{ **factorises** as } [x_1, \dots, x_n].$$
- (2) Say that X is **generated by its atomic elements** when every $x \in X \setminus \{f\}$ has a (possibly non-unique) factorisation into atomic elements.

DEFINITION 4.2.5. (1) Call a monoid of chunks $X = (X, e, f, \leq, \cdot)$ **atomic** when:

- (a) X is generated as a monoid by its atomic elements (Definition 4.2.4).
- (b) There exists a **factorisation function** $factor : X \setminus \{f\} \rightarrow [atomic(X)]$ such that for every $x, y \in X$
 - i. $factor(x)$ factorises x (Definition 4.2.4) and
 - ii. $factor(x \cdot y) = factor(x) \cdot factor(y)$ (the right-hand \cdot denotes list concatenation; the left-hand \cdot is the monoid action in X).

In words we say that X is atomic when there is a homomorphism of partially-ordered monoids from $X \setminus \{f\}$ to the space of possible factorisations of its elements. The relevance of this condition is discussed in Remark 6.4.4.

- (2) Call X **perfectly atomic** when it is atomic and furthermore:
 - (a) factorisations into atom elements are unique and
 - (b) if $x \leq y < f$ and $x = x_1 \cdot \dots \cdot x_m$ and $y = y_1 \cdot \dots \cdot y_n$ then $[x_1, \dots, x_m] \leq [y_1, \dots, y_n]$ (sublist inclusion).

The relevance of this condition is discussed in Remark 7.1.5.

EXAMPLE 4.2.6. Suppose X is an equivariant set. Then:

- (1) $pow(X)$ forms a monoid of chunks, where $e = \emptyset$ and $f = X$, and \leq is subset inclusion, and composition \cdot is sets union. It is atomic if and only if X is finite (recall: factorisations must be finite).
We obtain a factorisation function by choosing any order on X , and listing elements of any $X' \subseteq X$ in order.
- (2) $pow(X)$ forms a monoid of chunks, where:
 - $e = \emptyset$ and $f = X$.
 - \leq is subset inclusion.
 - $x \cdot y = x \cup y$ if $x \cap y = \emptyset$, and $x \cdot y = f$ otherwise.
 It is atomic if and only if X is finite.
- (3) $fin(X) \cup \{X\}$ (finite sets of atoms, with a top element) forms an atomic monoid of chunks, using either of the two definitions above for $pow(X)$.
- (4) Finite lists with a top element $[X]^\top$ — meaning finite lists of elements from X , plus one extra ‘top’ element \top — form a perfectly atomic monoid of chunks as follows:
 - $e = []$ and $f = \top$.
 - \leq is sublist inclusion (Notation 2.1.3(5)) and $l \leq f$ for every finite list l .
 - Composition \cdot is list concatenation on lists, and $x \cdot f = f = f \cdot x$ for any x (list, or f).
- (5) Finite lists with a top element $[X]^\top$ form an atomic (but not perfectly atomic) monoid of chunks as above, except that \cdot is defined as follows:
 - $[] \cdot l' = l \cdot [] = l$ for any list.
 - $f \cdot x = f = x \cdot f$ for any x .

¹³Functional programmers, who may be used to distinguishing between types (which are primitive) and sets (which inhabit powerset types), may perceive this definition as subtly broken, since it appears to apply a type-former $[\dots]$, to a set $atoms(X)$. This is just a cultural ‘thing’ and is not an issue with the maths as set up in this paper.

We are working in ZFA; the carrier set X is a set and so is $atomic(X)$ (and both are equivariant); the list set-former $[\dots]$ is a set-former, not a type-former. Thus, $[atomic(X)]$ is well-defined by Notation 2.1.3(4) as the set of finite lists of atomic elements from X .

- If l and l' are non-empty lists, then $l \cdot l'$ is l_{init} concatenated with l'_{tail} , where l_{init} is everything except for the last element of l , and l'_{tail} is everything except for the first element of l' .
- (6) If \mathbb{T} is an IEUTxO model then \mathbb{T} gives rise to a perfectly atomic monoid of chunks in a sense made formal and proved in Proposition 5.1.3.

REMARK 4.2.7. It might seem counterintuitive to make failure f a *top* element in Definition 4.2.1, especially if we are used to seeing domain models where ‘failure’ is intuitively ‘non-termination’ and features as \perp a bottom element.

We have a concrete reason for this: our canonical IEUTxO models are based on lists ordered by sublist inclusion, so bottom is already occupied by the empty list $[]$ which plays the role of e (see Definition 5.1.1).

But also we have abstract justifications: if we think of X a chunk system as a many-valued logic (in which truth-values are chunks or blockchains and \leq reflects how they accumulate transactions over time), then to exhibit a \top is to *fail* to exhibit a concrete witness. Or (thinking perhaps of callCC) we can think of f as a ‘final’ or ‘escape’ element.

4.3. Left- and right-behaviour

4.3.1. Left- and right-behaviour

DEFINITION 4.3.1. Suppose $X = (X, e, f, \leq, \cdot)$ is a monoid of chunks. Then we have natural **left-** and **right-behaviour** functions:

$$\begin{aligned} leftB : X &\rightarrow pow(X) & rightB : X &\rightarrow pow(X) \\ leftB : x &\mapsto \{y \in X \mid y \cdot x < f\} & rightB : x &\mapsto \{y \in X \mid x \cdot y < f\} \end{aligned}$$

REMARK 4.3.2. If we think of f as a failure element, and we think of $x \cdot y$ as being a composition of which we can observe whether it fails or succeeds,¹⁴ then

- $rightB$ maps $x \in X$ to its right-observational behaviour, and
- $leftB$ maps $x \in X$ to its left-observational behaviour.

LEMMA 4.3.3. Suppose X is a monoid of chunks. Then we have:

- (1) $leftB(e) = rightB(e) = X \setminus \{f\}$.
- (2) $leftB(f) = rightB(f) = \emptyset$.

Proof. A fact of Definition 4.2.1(1&2). □

DEFINITION 4.3.4. Suppose X is a monoid of chunks. Define $posi(f_X) = \emptyset$ and for $x \in X \setminus \{f_X\}$ write

$$a \in posi(x) \quad \text{when} \quad \forall \pi \in fix(a). \pi \cdot x \notin leftB(x) \cup rightB(x).$$

($fix(a)$ from Definition 2.2.3.)

Call the set

$$posi(x) \text{ the } \mathbf{positions} \text{ of } x.$$

REMARK 4.3.5. In words, $posi(x)$ from Definition 4.3.4 is those atoms in x , such that there is no permutation fixing a such that $\pi \cdot x$ can be successfully combined (left or right) with x . What is the intuition here?

The name $posi$ reminds us of pos from Definition 3.1.9, though the definitions are quite different. They are indeed related; in fact, they are equal in a sense made formal in Proposition 5.2.2.

We do not have all the machinery in place yet, so it may be helpful to point forwards here and observe that conditions 3 and 4 of Definition 4.4.1 can be read as a way to make name-clash into an observable.

¹⁴Much as in the untyped λ -calculus a traditional observable is non-termination; here the observable is ‘is equal to f ’.

So intuitively, Definition 4.3.4 — once combined with the notion of an oriented monoid from Definition 4.4.1) — can use permutations to observe name-clash: it measures the live communication channels $a \in \mathbb{A}$ in an element x by forcing name-clashes between a -channels with π -renamed variants $\pi \cdot x$ for $\pi \in \text{fix}(a)$. More details will follow.

DEFINITION 4.3.6. Suppose $X = (X, e, f, \leq, \cdot)$ is a monoid of chunks, and suppose $x \in X$ and $a \in \mathbb{A}$.

(1) If

$$a \in \text{posi}(x) \quad \text{and} \quad \exists y \in \text{left}B(x). a \in \text{posi}(y),$$

then say that a **points left** in x .

Write $\text{left}(x) \subseteq \mathbb{A}$ for the set of atoms that point left in x .

(2) If

$$a \in \text{posi}(x) \quad \text{and} \quad \exists y \in \text{right}B(x). a \in \text{posi}(y),$$

then say that a **points right** in x .

Write $\text{right}(x) \subseteq \mathbb{A}$ for the set of atoms that point right in x .

(3) If a points neither left nor right in a and yet $a \in \text{posi}(x)$, so that

$$a \in \text{posi}(x) \quad \text{and} \quad \forall y \in \text{left}B(x) \cup \text{right}B(x). a \notin \text{posi}(y),$$

then say that a **points up** in x .

Write $\text{up}(x) \subseteq \mathbb{A}$ for the set of atoms that point up in x .

Lemma 4.3.7 expresses, intuitively that atoms that point ‘up’ in a transaction cannot engage in successful (non-failing) combination; they are ‘stuck interfaces’:

LEMMA 4.3.7. Suppose X is a monoid of chunks and $x, y \in X$ and $a \in \text{up}(x)$. Then

$$a \in \text{posi}(y) \quad \text{implies} \quad x \cdot y = y \cdot x = f.$$

Proof. Direct from Definition 4.3.6(3). □

REMARK 4.3.8. The reader who sees similarities between the *left*, *right*, and *up* of Definition 4.3.6, and the *utxi*, *utxo*, and *stx* of Definition 3.2.14 is right: see Propositions 5.2.2, Lemma 5.3.1, and Proposition 5.5.4.

Parallels can also be made with the π -calculus [12] and the λ -calculus [2], which may be helpful:

(1) In the λ -calculus, a standard observable is non-termination.

Here (as noted in Remark 4.3.2) we are doing something similar, except instead of failure to terminate (\perp) we observe failure to compose (f), and we consider combination both to the left and to the right.

(2) The π -calculus has notions of communications across channels, and as noted in Remark 3.2.3(2) we see a resemblance with communication of an input and output on a position. However there are differences, including:

(a) *Validation* is not primitive in the π -calculus but it is a core precept here.

(b) Communication in the plain π -calculus (without considering dialects) is non-deterministic — one channel name can be invoked by multiple inputs and outputs — whereas here a key assumption is that every channel name (i.e. position) must have one input and one output — and if not, the chunk collapses to a failure error-state f (cf. the conditions in Definitions 3.2.1 and 3.2.20).

(c) Name-restriction in the π -calculus is not automatic but instead is managed by an explicit restriction term-former. In contrast here a communicating channel (an output-input pair) automatically closes when used once. We say ‘closed’ and not ‘bound’ because the name remains visible in *up* (see also *stx* in the IEUTxO models); it is just that no further communication may occur along it. We discuss garbage-collecting names in chunks in Subsection 8.2.

A simple lemma will be helpful:

LEMMA 4.3.9. Suppose X is a monoid of chunks and $x \in X$. Then:

$$\begin{aligned} up(x) &= posi(x) \setminus (left(x) \cup right(x)) \\ \emptyset &= left(x) \cap up(x) \\ \emptyset &= right(x) \cap up(x) \\ posi(x) &= left(x) \cup right(x) \cup up(x) \end{aligned}$$

Proof. This just rephrases clause 3 of Definition 4.3.6. \square

REMARK 4.3.10. Lemmas 4.3.9 and 3.2.16(3) are similar but note that the status of the underlying datatypes is somewhat different:

- A chunk $ch \in \text{Chunk}$ of an IEUTxO model is full of internal structure, and operations on it are defined in terms of that structure, whereas
- an element $x \in X$ in a monoid of chunks is an abstract entity and we assume nothing about its internal structure.

Thus, a similarity between them has significance: it is a sanity check on our model and indicates that something rather abstract (monoids of chunks) is accurately following the behaviour of something more concrete (IEUTxO models).

REMARK 4.3.11. Lemma 4.3.9 expresses that every position in some $x \in X$ (Definition 4.3.4) must point in a direction in $\{left, right, up\}$, and it cannot point both left and up, or both right and up.

Note that Definition 4.3.6 admits a possibility that an atom could point both left and right; this *cannot* happen in the IEUTxO models (see Lemma 3.2.16(1)). We will exclude this when we introduce the notion of an oriented monoid of chunks; see Corollary 4.4.5.

LEMMA 4.3.12. (1) $left(e) = right(e) = up(e) = \emptyset$.

(2) $left(f) = right(f) = up(f) = \emptyset$.

(3) As a corollary using Lemma 4.3.9, $posi(e) = posi(f) = \emptyset$.

Proof. We check the behaviour of e and f as specified in Definition 4.2.1 against the definitions of $left$, $right$, and up in Definition 4.3.6 and see that this is true. \square

4.3.2. Observational equivalence

DEFINITION 4.3.13. Suppose X is a monoid of chunks.

(1) Call x and x' in X **observationally equivalent** and write

$$x \sim x' \quad \text{when} \quad (leftB(x), rightB(x)) = (leftB(x'), rightB(x')).$$

(2) Say that x and y **commute (up to observational equivalence)** when

$$x \cdot y \sim y \cdot x.$$

We start with a simple but useful sanity check:

LEMMA 4.3.14. Suppose X is a monoid of chunks and $x, y \in X$. Then if x and y commute then

$$x \cdot y < f \iff y \cdot x < f \quad \text{and} \quad x \cdot y = f \iff y \cdot x = f.$$

Proof. We unpack Definitions 4.3.13(1&2) and 4.3.1 and conclude that

$$x \cdot y \cdot e < f \iff y \cdot x \cdot e < f.$$

The result follows, since e is the unit for \cdot . \square

LEMMA 4.3.15. Suppose X is a monoid of chunks. Then if $x \sim x'$ (Definition 4.3.13) then

$$left(x) = left(x') \quad \text{and} \quad right(x) = right(x') \quad \text{and} \quad up(x) = up(x').$$

Proof. A fact of Definitions 4.3.13(1) and 4.3.6. \square

4.4. Oriented monoids

4.4.1. Definition and properties

DEFINITION 4.4.1. Suppose $X = (X, e, f, \leq, \cdot)$ is a monoid of chunks.

Call X **oriented** when for all $x, y \in X$:

- (1) $\text{posi}(x) \subseteq_{\text{fin}} \mathbb{A}$.
- (2) If $\text{posi}(x) = \emptyset$ then $x = e$ or $x = f$.
- (3) If $\text{left}(x) \cap \text{right}(y) \neq \emptyset$ then $x \cdot y = f$.
- (4) If $\text{posi}(x) \cap \text{posi}(y) = \emptyset$ then x and y commute up to observational equivalence (Definition 4.3.13(2)).
- (5) If $\text{posi}(x) \cap \text{posi}(y) = \emptyset$ and $f \notin \{x, y\}$ then $x \cdot y < f$.

REMARK 4.4.2. We discuss the conditions of Definition 4.4.1 in turn:

- (1) An element $x \in X$ can only be accessible on finitely many channel interfaces.
- (2) The only elements without any interface (meaning atoms that point left right or up) are the unit element (\leq -bottom) and the failure element (\leq -top). Compare with the IEUTxO property Lemma 3.1.11.
We use this in Lemmas 4.4.6 and 6.3.1.
- (3) Interfaces always try to connect, but can only *successfully* connect if the directions of their interfaces match up; if not, the whole combination fails.
We use this in Lemma 4.4.6, which is required for Proposition 4.4.8.
- (4) This condition echoes Lemma 3.2.13(2). We use it in Proposition 4.4.8.
- (5) Elements with no channels in common, cannot fail to compose.

We will show later that the IEUTxO models from Definition 3.1.7 are models of Definition 4.4.1 in a suitable sense; see Proposition 5.3.3.

We can strengthen Definition 4.4.1(2) to a logical equivalence:

LEMMA 4.4.3. Suppose X is an oriented monoid of chunks and $x \in X$. Then

$$\text{posi}(x) = \emptyset \quad \text{if and only if} \quad x \in \{e, f\}.$$

Proof. The right-to-left implication is direct from Definition 4.4.1(2). The left-to-right implication is Lemma 4.3.12. \square

Lemma 4.4.4 is a nice way to repackage Definition 4.4.1(3) in a slightly more accessible wrapper. In its form it resembles Lemma 3.2.16(2), and we use it for Corollary 4.4.5:

LEMMA 4.4.4. Suppose X is an oriented monoid of chunks and suppose $x, y \in X$. Then

$$x \cdot y < f \quad \text{implies} \quad \text{posi}(x) \cap \text{posi}(y) \subseteq \text{right}(x) \cap \text{left}(y).$$

Proof. We consider the possibilities, using Lemma 4.3.9:

- Suppose $a \in \text{left}(x) \cap \text{posi}(y)$. From Definition 4.4.1(3) $a \notin \text{right}(x)$, and by Lemma 4.3.9 $a \in \text{posi}(x)$. It follows from Definition 4.3.6(2) that $x \cdot y = f$.
- Suppose $a \in \text{right}(y) \cap \text{posi}(x)$. From Definition 4.4.1(3) $a \notin \text{left}(y)$, and by Lemma 4.3.9 $a \in \text{posi}(y)$. It follows from Definition 4.3.6(1) that $x \cdot y = f$.
- Other cases are from Lemma 4.3.7 (or by direct reasoning from Definition 4.3.6(3)).

\square

Corollary 4.4.5 is a slightly magical result, in the sense that it is perhaps not immediately obvious that it should follow from our definitions so far. In its form, if not its proof, it clearly resembles Lemma 3.2.16(1). We need it for Lemma 6.2.3, that atomic elements in X generate valid singleton chunks under a mapping to IEUTxO models F :

COROLLARY 4.4.5. *Suppose X is an oriented monoid of chunks and $x \in \mathsf{X}$. Then:*

$$\text{left}(x) \cap \text{right}(x) = \emptyset.$$

Proof. Suppose $a \in \text{left}(x)$; we will show that $a \in \text{right}(x)$ is impossible. Consider some $y \in \mathsf{X}$ with $a \in \text{posi}(y)$, so that by Lemma 4.3.9 $a \in \text{left}(y) \cup \text{right}(y) \cup \text{up}(y)$. Then:

- If $a \in \text{left}(y)$ then $a \in \text{left}(x) \cap \text{left}(y)$ and by Lemma 4.4.4 $x \cdot y = \mathsf{f}$.
- If $a \in \text{right}(y)$ then $a \in \text{left}(x) \cap \text{right}(y)$ and by Lemma 4.4.4 $x \cdot y = \mathsf{f}$.
- If $a \in \text{up}(y)$ then $a \in \text{left}(x) \cap \text{up}(y)$ and by Lemma 4.4.4 $x \cdot y = \mathsf{f}$.

Thus $a \in \text{posi}(y)$ implies $x \cdot y = \mathsf{f}$ and so $y \notin \text{right}B(x)$. It follows from Definition 4.3.6(2) that $a \notin \text{right}(x)$ as required. \square

We use Lemma 4.4.6 for Proposition 4.4.8:

LEMMA 4.4.6. *Suppose $\mathsf{X} = (\mathsf{X}, \mathsf{e}, \mathsf{f}, \leq, \cdot)$ is an oriented monoid of chunks. Then at least one of the following must hold:*

$$x \cdot y = \mathsf{f} \quad y \cdot x = \mathsf{f} \quad \text{posi}(x) \cap \text{posi}(y) = \emptyset$$

Proof. If x or y are equal to e or f then $\text{posi}(x) \cap \text{posi}(y) = \emptyset$ is immediate from Lemma 4.4.3.

So suppose $x, y \notin \{\mathsf{e}, \mathsf{f}\}$, from which it follows by Lemma 4.4.3 (or direct from Definition 4.4.1(2)) that $\text{posi}(x) \neq \emptyset$ and $\text{posi}(y) \neq \emptyset$. Suppose we have some $a \in \text{posi}(x) \cap \text{posi}(y)$. We reason by cases using our assumption that X is oriented (Definition 4.4.1):

- If $a \in \text{left}(x) \cap \text{right}(y)$ then $x \cdot y = \mathsf{f}$ by Definition 4.4.1(3).
- If $a \in \text{right}(x) \cap \text{left}(y)$ then $y \cdot x = \mathsf{f}$ by Definition 4.4.1(3).
- If $a \in \text{up}(x)$ or $a \in \text{up}(y)$ then $x \cdot y = y \cdot x = \mathsf{f}$ by Lemma 4.3.7.
- Other cases are no harder.

\square

REMARK 4.4.7. Proposition 4.4.8 is a partial converse to Definition 4.4.1(4) (compare also with Proposition 3.2.18, which is the same result but for a different structure, and with a very different proof). It is significant because it equates

- a static property of their positions, with
- a local property of being combinable in either order, with
- a global operational property of being commutative up to observation.

Commutativity is of particular interest in the context of blockchains, because they are by design intended to be distributed, so that we cannot in general know or assume in what order transactions get appended.

PROPOSITION 4.4.8. *Suppose $\mathsf{X} = (\mathsf{X}, \mathsf{e}, \mathsf{f}, \leq, \cdot)$ is an oriented monoid of chunks, and $x, y \in \mathsf{X}$. Suppose further that $x \cdot y < \mathsf{f}$ or $y \cdot x < \mathsf{f}$ (not necessarily both). Then the following conditions are equivalent:*

- (1) $\text{posi}(x) \cap \text{posi}(y) = \emptyset$.
- (2) $x \cdot y < \mathsf{f} \wedge y \cdot x < \mathsf{f}$.
- (3) x and y commute up to observational equivalence (Definition 4.3.13(2)).

Proof. First, note that since $x \cdot y < \mathsf{f}$ or $y \cdot x < \mathsf{f}$, it must from Definition 4.2.1(2) be the case that $x < \mathsf{f}$ and $y < \mathsf{f}$.

If $x \cdot y < \mathsf{f} \wedge y \cdot x < \mathsf{f}$ then $\text{posi}(x) \cap \text{posi}(y) = \emptyset$ by Lemma 4.4.6. Conversely if $\text{posi}(x) \cap \text{posi}(y) = \emptyset$ then (since $x < \mathsf{f}$ and $y < \mathsf{f}$) by Definition 4.4.1(5) $x \cdot y < \mathsf{f}$ and $y \cdot x < \mathsf{f}$.

If $\text{posi}(x) \cap \text{posi}(y) = \emptyset$ then x and y commute by Definition 4.4.1(4). Conversely, if $\text{posi}(x) \cap \text{posi}(y) \neq \emptyset$ then by Lemma 4.4.6 (since $x \cdot y < \mathsf{f}$ or $y \cdot x < \mathsf{f}$) we have that $y \cdot x = \mathsf{f}$ or $x \cdot y = \mathsf{f}$

respectively, and in particular we have that $x \cdot y \neq y \cdot x$. Using Lemma 4.3.14 we conclude that x and y do not commute. \square

REMARK 4.4.9 (Further properties). There is design freedom to Definition 4.4.1, and we mention this briefly for future work. One plausible condition is:

$$up(x \cdot y) \subseteq up(x) \cup up(y) \cup (right(x) \cap left(y)).$$

Intuitively, this states that combining x and y can *only* bind positions that point right in x and point left in y . An even stricter variant would be to insist on equality provided that $x \cdot y \neq f$.

4.4.2. A brief discussion. We might ask:

Why bother with monoids of chunks? Why not just work with IEUTxOs?

The answer is that we need both: IEUTxOs are the motivating concrete model, and monoids of chunks are their abstraction.

Of course, the IEUTxO equations from Figure 1 themselves are an abstraction and generalisation of a concrete inductive definition in [6, Figure 3], so in overview this paper has the following hierarchy of models, given in increasing order of generality —

- The inductive EUTxO structures from [6].
- IEUTxO models.
- Abstract chunk systems (ACS).

— though there is also more going on, because we also give a representation map from ACS back down to IEUTxO: see Theorem 7.3.4.

Programmers can think of APIs, which abstract away from internal structure of a concrete implementation; this makes programs more modular, and easier to test and document.

Modern programming languages make it easier to program efficiently using abstract denotations, instantiating only when needed — one might call this *just-in-time instantiation*. So even if we have just one concrete model and want one implementation, a good denotational theory is still relevant to producing producing working code, because:

- It may help structure the mathematics and the code.
- An abstraction can be read as a library of testable properties, against which implementations can be checked. If an implementation fails an axiom — it's wrong.

So a better question is this:

What are the essential properties of IEUTxO that make them interesting? How are those properties layered, and nested; and how can they be compactly represented?

Definitions 4.2.1 and 4.4.1 are one set of answers to these questions. And then, a further question is this:

What other models, aside from IEUTxO, exist of these axioms?

This paper contains one answer (developing others is future work): in Definitions 6.2.1 and 6.4.1 and Proposition 6.4.3 we give a functor taking *any* oriented monoid of chunks to IEUTxOs. We also map functorially in the other direction in Definitions 5.1.1 and 5.4.1 and Theorem 5.4.4.

This is a *representation result*, culminating in an adjoint functor result (Theorem 7.3.4).

4.5. The category ACS of abstract chunk systems

DEFINITION 4.5.1. An **abstract chunk system** (ACS) is an oriented atomic monoid of chunks (Definitions 4.4.1, 4.2.5, and 4.2.1).

DEFINITION 4.5.2. Define ACS the **category of abstract chunk systems** by:

- (1) Objects are abstract chunk systems (Definition 4.5.1).

- (2) Arrows $g : X \rightarrow Y$ are sets functions from X to Y such that:
- (a) $g(e_X) = e_Y$ and $g(f_X) = f_Y$
 - (b) $x \leq y < f_X$ implies $g(x) \leq g(y) < f_Y$
 - (c) $g(x) \cdot g(y) = g(x \cdot y)$

Composition of arrows is composition of functions, and the identity arrow is the identity function.

REMARK 4.5.3. (1) We do not insist in Definition 4.5.2 that $g(x)$ must be atomic if x is. This corresponds to our choice in Definition 3.4.1(2) to let f map from transactions to chunks, and not from transactions to transactions.

- (2) We do insist in Definition 4.5.2(2b) that if x is not the failure element f_X in X then $g(x)$ is also not the failure element f_Y in Y . This corresponds to our choice in Definition 3.4.1(2) to make f a total function from transactions to chunks, rather than a partial one (cf. discussion in Remark 3.4.5(2)).

We could relax this condition by allowing g to map $x < f_X$ to f_Y . There would be nothing wrong with this and it would just exhibit ACS as embedded in a larger category with the same objects but more arrows.

Lemma 4.5.4 just repackages Definition 4.2.5 for objects in ACS:

LEMMA 4.5.4. Suppose $X, Y \in \text{ACS}$ and $x \in X$ and $g : X \rightarrow Y \in \text{ACS}$. Then:

- If $x \neq f$ then there exists some finite (possibly empty, possibly non-unique) list of atomic elements $x_1, \dots, x_n \in \text{atomic}(X)$ such that $x = x_1 \cdot \dots \cdot x_n$.
- If $x \neq f$ (and in particular by Definition 4.1.3(1) if x is atomic) then there exists some finite (possibly empty, possibly non-unique) list of atomic elements $y_1, \dots, y_n \in \text{atomic}(Y)$ such that $g(x) = y_1 \cdot \dots \cdot y_n$.

Proof. Immediate since by Definition 4.5.1 X is atomic (Definition 4.2.5). □

PROPOSITION 4.5.5. (1) An arrow $g : X \rightarrow Y \in \text{ACS}$ (Definition 4.5.2(2)) is uniquely determined by its action on $\text{atomic}(X)$.

- (2) As a corollary, if $g, g' : X \rightarrow Y$ are two arrows, then to check the equality of arrows $g = g'$ it suffices to check that $g(x) = g'(x)$ for every $x \in \text{atomic}(X)$.

Proof. (1) Consider some $x \in X$. If $x \in \{f, e\}$ then the action of f is determined by Definition 4.5.2(2a).

Otherwise, using Lemma 4.5.4 write $x = x_1 \cdot \dots \cdot x_n$ for atomic $x_1, \dots, x_n \in \text{atomic}(X)$. We then have from Definition 4.5.2(2c) that

$$f(x_1) \cdot \dots \cdot g(x_n) = g(x).$$

Thus $g(x)$ is determined by the values of $g(x_1), \dots, g(x_n)$.

- (2) By a routine argument from Definition 4.2.5 (since X is atomic) and Definition 4.5.2(2c). □

5. THE FUNCTOR $F : \text{IEUTXO} \rightarrow \text{ACS}$

5.1. Action on objects

DEFINITION 5.1.1. Suppose $\mathbb{T} = (\alpha, \beta, \text{Transaction}) \in \text{IEUTxO}$ and recall $\text{Chunk}_{\mathbb{T}}$ from Notation 3.2.2. Then we define an abstract chunk system

$$F(\mathbb{T}) = (\text{Chunk}_{\mathbb{T}} \cup \{\text{fail}\}, [], \text{fail}, \leq, \cdot) \in \text{ACS}$$

as follows:

- The underlying set is $\text{Chunk}_{\mathbb{T}} \cup \{\text{fail}\}$ — so if we write “ $x \in F(\mathbb{T})$ ” this means “either $x = \text{fail}$ or $x \in \text{Chunk}_{\mathbb{T}}$ ”.
- $e = []$ and $f = \text{fail}$.

- \leq is sublist inclusion (Notation 2.1.3(5)) on chunks that are lists of transactions, with f as a top element, so $ch \leq f$ always.
- $x \cdot f = f = f \cdot x$, and if x and y are both chunks then \cdot is validated concatenation of chunks, defaulting to the failure element f if validation fails.

REMARK 5.1.2. So if $x, y \in \text{Chunk}_{\mathbb{T}}$ but $x \cdot y \in \text{Transaction}_{\mathbb{T}} \setminus \text{Chunk}_{\mathbb{T}}$ in \mathbb{T} , then $x \cdot y = \text{fail}$ in $F(\mathbb{T})$. We will always be clear whether \cdot means ‘concatenate as lists’ or ‘compose in $F(\mathbb{T})$ ’.

$F(\mathbb{T})$ resembles \mathbb{T} , except with an explicit failure element *fail* added, which makes monoid composition total, since not every pair of chunks composes to form a list of transactions that is a chunk as per the criteria in Definition 3.2.1, so any combinations of chunks that it not itself a chunk, can be set to *fail*.

We still need to prove that $F(\mathbb{T})$ is an abstract chunk system (Definition 4.5.1) in the category ACS (Definition 4.5.2(1)) thus:

- $F(\mathbb{T})$ is a monoid of chunks (Definition 4.2.1),
- $F(\mathbb{T})$ is atomic (Definition 4.2.5) — in fact it is perfectly atomic (Definition 4.2.5(2)) — and
- $F(\mathbb{T})$ is oriented (Definition 4.4.1).

We do this next, culminating with Theorem 5.3.4.

Note that this gives us a dual view of a chunk:

- as a chunk in the IEUTxO universe, and
- as an abstract element in the ACS universe.

To what extent do these two views correspond, and how can we make this formal? We prove Proposition 5.2.2, which verifies that the positions in a chunk as an element in the IEUTxO universe, coincide with its positions as an element in the ACS universe. This is refined in further results; notably Lemma 5.3.1 and its sharper corollary Proposition 5.5.4.

PROPOSITION 5.1.3. (1) $F(\mathbb{T})$ from Definition 5.1.1 is a perfectly atomic (Definition 4.2.5(2)) monoid of chunks (Definition 4.2.1).

Unpacking Definition 4.2.5(2), any $x \in F(\mathbb{T}) \setminus \{f\}$ can be uniquely decomposed as a (possibly empty) finite list of atomic elements

$$x = x_1 \cdot \dots \cdot x_n,$$

and if $x \leq y < f_{F(\mathbb{T})}$ then the factorisation of x is a sublist of the factorisation of y .

- (2) Furthermore, the atomic elements $x_i \in \text{atomic}(F(\mathbb{T}))$ are singleton lists of the form $[tx(tx_i)]$ for $tx \in \text{Transaction}_{\mathbb{T}}$.

Proof. Most of the properties in Definition 4.2.1 are facts of sublist inclusion and concatenation. Condition 6 of Definition 4.2.1 is Lemma 3.2.5.

Recall that being a chunk is down-closed by Corollary 3.2.6, so in a perfectly atomic monoid of chunks, every chunk is above its atomic chunks. It is just a fact of lists, sublist inclusion, and the construction in Definition 5.1.1 that $F(\mathbb{T})$ is perfectly atomic with atomic elements singleton chunks of the form $[tx(tx)]$ for $tx \in \text{Transaction}_{\mathbb{T}}$. \square

5.2. Relation between the partial monoid $\text{Chunk}_{\mathbb{T}}$ and the monoid of chunks $F(\mathbb{T})$

LEMMA 5.2.1. Suppose $T \in \text{IEUTxO}$ and $ch \in \text{Chunk}_{\mathbb{T}}$ and $a \in \text{pos}(ch)$. Suppose further that $\pi \in \text{Perm}$ is a permutation and $\pi(a) = a$, and write $ch' = \pi \cdot ch$ (Definition 2.2.2).

It is a structural fact that $ch \cdot ch'$, $ch' \cdot ch \in [\text{Transaction}_{\mathbb{T}}]$ since a concatenation of two lists is a list. However:

$$ch' \cdot ch \notin \text{Chunk}_{\mathbb{T}} \quad \text{and} \quad ch \cdot ch' \notin \text{Chunk}_{\mathbb{T}}.$$

Proof. By assumption $a \in \text{pos}(ch)$, so using Lemmas 3.2.16(3) and 3.2.19 (since $\pi(a) = a$) precisely one of

$$\begin{aligned} & a \in \text{utxi}(ch) \cap \text{utxi}(ch'), \\ & a \in \text{utxo}(ch) \cap \text{utxo}(ch'), \\ \text{or } & a \in \text{stx}(ch) \cap \text{stx}(ch') \end{aligned}$$

must hold. In each of these three cases the result follows from Lemma 3.2.16(2). \square

PROPOSITION 5.2.2. *Suppose \mathbb{T} is an IEUTxO model and $x \in \text{Chunk}_{\mathbb{T}}$, and*

— *recall $\text{pos}(x)$ from Definition 3.1.9, and*

— *noting from Proposition 5.1.3 that x can also be viewed as an element $x \in F(\mathbb{T})$, recall also $\text{posi}(x)$ from Definition 4.3.4.*

Then

$$\text{pos}(x) = \text{posi}(x).$$

Proof. By a routine calculation from Definitions 3.2.1 and 4.3.4 using Lemma 5.2.1. \square

5.3. $F(\mathbb{T})$ is oriented, so $F(\mathbb{T}) \in \text{ACS}$

LEMMA 5.3.1. *Suppose $\mathbb{T} \in \text{IEUTxO}$. By Proposition 5.1.3 $F(\mathbb{T}) = \text{Chunk}_{\mathbb{T}} \cup \{\text{fail}\}$ is a monoid of chunks, so it has notions of left, right, and up from Definition 4.3.6.*

*Then for $x \in \text{Chunk}_{\mathbb{T}}$ we have:*¹⁵

$$\begin{aligned} \text{left}(x) &\subseteq \text{utxi}(x) \\ \text{right}(x) &\subseteq \text{utxo}(x) \\ \text{stx}(x) &\subseteq \text{up}(x) \end{aligned}$$

Proof. (1) If $a \in \text{left}(x)$ then by Lemma 4.3.9 $a \in \text{posi}(x)$ so by Proposition 5.2.2 also $a \in \text{pos}(x)$.

By Definition 4.3.6 there exists $y \in \text{Chunk}_{\mathbb{T}}$ such that $a \in \text{posi}(y)$, so by Proposition 5.2.2 also $a \in \text{pos}(y)$, and $y \cdot x \in \text{Chunk}_{\mathbb{T}}$. It follows from Lemma 3.2.16(2) that $(a \in \text{utxo}(y))$ and $a \in \text{utxi}(x)$ as required.

(2) If $a \in \text{right}(x)$ then $a \in \text{pos}(x)$ and by Definition 4.3.6 there exists $y \in \text{Chunk}_{\mathbb{T}}$ such that $a \in \text{pos}(y)$ and $x \cdot y \in \text{Chunk}_{\mathbb{T}}$, so by Lemma 3.2.16(2) $(a \in \text{utxi}(y))$ and $a \in \text{utxo}(x)$ as required.

(3) The reasoning to prove $\text{stx}(x) \subseteq \text{up}(x)$ is no harder. \square

REMARK 5.3.2. Lemma 5.3.1 is interesting as much for what it is *not*, namely it is not the equality $\text{left} = \text{utxi}$ and $\text{right} = \text{utxi}$ and $\text{up} = \text{stx}$ that one might initially expect. Why?

(1) Consider a chunk $ch \in \text{Chunk}_{\mathbb{T}}$ with an IEUTxO output located at a but with an empty validator (one which validates no inputs). Then $a \in \text{utxo}(ch)$ in \mathbb{T} , but $a \in \text{up}(ch)$ in $F(\mathbb{T})$.

(2) Similarly consider a chunk ch with an input located at a but such that no validator will validate it — just because an input exists, does not mean a validator must exist to accept it. Then $a \in \text{utxi}(ch)$ in \mathbb{T} , but $a \in \text{up}(ch)$ in $F(\mathbb{T})$.

We return to this with the notion of a *blocked channel*, in Subsection 5.5.

PROPOSITION 5.3.3. *$F(\mathbb{T})$ from Definition 5.1.1 is oriented (Definition 4.4.1).*

Proof. We check each condition of Definition 4.4.1 in turn:

¹⁵...so we are looking at a ‘real chunk’ here, for which utxi , utxo , and stx are defined, and excluding our extra failure element, for which they are not defined ...

- (1) We check that $\text{posi}(x) \subseteq_{fin} \mathbb{A}$.

It is a structural fact of Definition 3.1.9 that $\text{pos}(x) \subseteq_{fin} \mathbb{A}$. We use Proposition 5.2.2.

- (2) We check that $\text{posi}(x) = \emptyset$ implies $x = e_{F(\mathbb{T})}$ or $x = f_{F(\mathbb{T})}$.

An element $x \in F(\mathbb{T})$ is either a chunk or the failure element. If x is a chunk and $\text{posi}(x) = \emptyset$ then by Proposition 5.2.2 $\text{posi}(x) = \emptyset$ and by Lemma 3.1.11 $x = []$, which by Definition 5.1.1 is $e_{F(\mathbb{T})}$. If $x = \text{fail}$ then there is nothing to prove, since $\text{fail} = f_{F(\mathbb{T})}$.

- (3) We check that $\text{left}(x) \cap \text{right}(y) \neq \emptyset$ implies $x \cdot y = f$.

If x or y are fail then $x \cdot y = f$.

So suppose that x and y are chunks, that is, suppose $x, y \in \text{Chunk}_{\mathbb{T}}$. By Lemma 5.3.1 $\text{utxi}(x) \cap \text{utxo}(y) \neq \emptyset$. We use Lemma 3.2.16(2).

- (4) We check that if $\text{posi}(x) \cap \text{posi}(y) = \emptyset$ then x and y commute (Definition 4.3.13(2)).

From Proposition 5.2.2 and Lemma 3.2.13(2).

- (5) We check that if $\text{posi}(x) \cap \text{posi}(y) = \emptyset$ and $f \notin \{x, y\}$ then $x \cdot y < f_{F(\mathbb{T})}$.

Using Proposition 5.2.2, this just rephrases Lemma 3.2.10 in the language of a monoid of chunks.

□

THEOREM 5.3.4. $F(\mathbb{T})$ (Definition 5.1.1) is an abstract chunk system in ACS (Definition 4.5.1). In symbols:

$$F(\mathbb{T}) \in \text{ACS}.$$

Proof. From Propositions 5.1.3 and 5.3.3.

□

5.4. Action of F on arrows

DEFINITION 5.4.1. Suppose $f : \mathbb{S} \rightarrow \mathbb{T} \in \text{IEUTxO}$ is an arrow (Definition 3.4.1(2)). We define an arrow

$$F(f) : F(\mathbb{S}) \rightarrow F(\mathbb{T}) \in \text{ACS}$$

by

$$\begin{aligned} F(f)([tx_1, \dots, tx_n]) &= f(tx_1) \cdot \dots \cdot f(tx_n) \\ F(f)(\text{fail}_{\mathbb{S}}) &= \text{fail}_{\mathbb{T}}. \end{aligned} \tag{1}$$

LEMMA 5.4.2. $F(f)$ from Definition 5.4.1 does indeed map from $F(\mathbb{S})$ to $F(\mathbb{T})$.

Proof. We need to check that validity is preserved, meaning that if $[tx_1, \dots, tx_n]$ is a chunk then so is $f(tx_1) \cdot \dots \cdot f(tx_n)$. This is Lemma 3.4.2. □

PROPOSITION 5.4.3. Continuing Definition 5.4.1, we have that

$$f : \mathbb{S} \rightarrow \mathbb{T} \in \text{IEUTxO} \quad \text{implies} \quad F(f) : F(\mathbb{S}) \rightarrow F(\mathbb{T}) \in \text{IEUTxO}.$$

Furthermore, $F(f' f) = F(f') F(f)$ and $F(\text{id}_{\mathbb{S}}) = \text{id}_{F(\mathbb{S})}$.

Proof. We check the properties in Definition 4.5.2(2) in turn:

- (1) We check that $F(f)(e_{F(\mathbb{S})}) = e_{F(\mathbb{T})}$ and $F(f)(f_{F(\mathbb{S})}) = f_{F(\mathbb{T})}$.

This is just the fact that $F(f)([]) = []$ and $F(f)(\text{fail}_{\mathbb{S}}) = \text{fail}_{\mathbb{T}}$.

- (2) We check that $x \leq y < \text{fail}_{F(\mathbb{S})}$ implies $F(f)(x) \leq F(f)(y) < f_{F(\mathbb{T})}$.

It is a fact of the construction in Definition 5.4.1 that $x \leq y$ (x is a sublist of y) implies $F(f)(x) \leq F(f)(y)$.

- (3) We check that $F(f)(x) \cdot F(f)(y) = F(f)(x \cdot y)$.

A fact of the first clause of equation (1) in Definition 5.4.1.

We can check $F(f' f) = F(f') F(f)$ and $F(\text{id}_{\mathbb{S}}) = \text{id}_{F(\mathbb{S})}$ by routine calculations which we elide.

□

THEOREM 5.4.4. *The map F , with the action on IEUTxO models from Definition 5.1.1, and with the action on arrows from Definition 5.4.1, is a functor*

$$F : \text{IEUTxO} \rightarrow \text{ACS}.$$

Proof. This is Theorem 5.3.4 and Proposition 5.4.3. □

5.5. Blocked channels

Recall from Remark 3.2.3 that we can think of positions as communication channels in the π -calculus sense. We conclude this Section by taking a little time to refine the subset inclusions from Lemma 5.3.1. To do so, we need to consider the possibility of a channel which is (intuitively) *blocked*, in the sense that no successful validation can occur across it:

DEFINITION 5.5.1. Suppose that \mathbb{T} is an IEUTxO model and $ch \in \text{Chunk}_{\mathbb{T}}$ and $a \in \mathbb{A}$.

- (1) Suppose that
 - $a \in \text{utxi}(ch)$ and
 - for every $ch' \in \text{Chunk}_{\mathbb{T}}$ with $a \in \text{utxo}(ch')$, $ch' \cdot ch$ is not a chunk.
 Then call a a **blocked utxi** in ch . Write $\text{blockedUtxi}(ch)$ for the blocked utxis of ch .
- (2) Similarly define $\text{blockedUtxo}(ch)$ the **blocked utxos** of ch to be those $a \in \mathbb{A}$ such that
 - $a \in \text{utxo}(ch)$ and
 - for every $ch' \in \text{Chunk}_{\mathbb{T}}$ with $a \in \text{utxi}(ch')$, $ch' \cdot ch$ is not a chunk.

REMARK 5.5.2. So a blocked UTxI or UTxO in a chunk is an input or output that exists, but which fails if you try to interact with it. This could happen for an output whose validator is the empty set (it fails on any input), or for an input such that no validator in \mathbb{T} exists to validate it (see Remark 5.3.2).

LEMMA 5.5.3. *Suppose \mathbb{T} is an IEUTxO model and $x, y \in \text{Chunk}_{\mathbb{T}}$ and $x \cdot y \in \text{Chunk}_{\mathbb{T}}$. Then*

$$\text{utxo}(x) \cap \text{utxi}(y) \subseteq \text{right}(x) \cap \text{left}(y).$$

Proof. Suppose $a \in \text{utxo}(x) \cap \text{utxi}(y)$. In particular then by Lemma 3.2.16(3) $a \in \text{pos}(x) \cap \text{pos}(y)$ so by Proposition 5.2.2 also $a \in \text{posi}(x) \cap \text{posi}(y)$.

$F(\mathbb{T})$ is a monoid of chunks by Proposition 5.1.3, and since $x \cdot y \in \text{Chunk}_{\mathbb{T}}$ it follows that $x \cdot y <_{F(\mathbb{T})}$. It follows from Definition 4.3.1 that $y \in \text{right}B(x)$ and $x \in \text{left}B(y)$.

The result now follows by Definition 4.3.6. □

PROPOSITION 5.5.4. *Suppose $\mathbb{T} \in \text{IEUTxO}$ and $x \in F(\mathbb{T}) \setminus \{f\}$ (that is, $x \in \text{Chunk}_{\mathbb{T}}$). Then:*

$$\begin{aligned} \text{left}(x) &= \text{utxi}(x) \setminus \text{blockedUtxi}(x) \\ \text{right}(x) &= \text{utxo}(x) \setminus \text{blockedUtxo}(x) \\ \text{up}(x) &= \text{stx}(x) \cup \text{blockedUtxi}(x) \cup \text{blockedUtxo}(x) \end{aligned}$$

Proof. We know by Lemma 5.3.1 that $\text{left}(x) \subseteq \text{utxi}(x)$ and $\text{right}(x) \subseteq \text{utxo}(x)$. Now suppose $a \in \text{utxi}(x)$ and $a \notin \text{blockedUtxi}(x)$; unpacking Definition 5.5.1 this means that there exists a $y \in \text{Chunk}_{\mathbb{T}}$ such that $a \in \text{utxo}(y)$ and $y \cdot x \in \text{Chunk}_{\mathbb{T}}$. By Lemma 5.5.3 it follows that ($a \in \text{right}(y)$ and) $a \in \text{left}(x)$.

The case of $\text{right}(x)$ is similar, and the case of $\text{up}(x)$ follows from the previous two cases and Lemma 5.3.1. □

6. THE REPRESENTATION FUNCTOR $G : \text{ACS} \rightarrow \text{IEUTxO}$

6.1. A brief discussion: why represent?

In Subsection 4.4.2 we observed a hierarchy of models, from concrete EUTxO structures to IEUTxO models to abstract chunk systems.

The mapping from IEUTxO to ACS is the functor $F : \text{IEUTxO} \rightarrow \text{ACS}$ from Section 5. We will now exhibit a *representation functor* $G : \text{ACS} \rightarrow \text{IEUTxO}$.¹⁶

This is interesting for two reasons: one general, and one specific. We consider each in turn.

Representation maps like G are generally interesting because:

- (1) A representation map tells us that an abstraction reasonably represents the concrete models. There is nothing the abstract model can do that is so crazy that it could not be engineered back down to a concrete structure. This may involve some ugly concrete fiddling, emulation, and choices — as one might expect going from an abstract to a concrete object — but it can be done, and seeing how, can be helpful for understanding both worlds.
- (2) Sometimes, theorems are better proved in the concrete world than the abstract world. This can be particularly useful to prove negative properties, that something *cannot* happen in the abstract world, because it would correspond to something that would be impossible in the concrete world. A well-known example is that every Boolean Algebra can be represented as a powerset, and thus every finite Boolean Algebra has cardinality a power of two. Thus, to prove that some abstract structure does *not* admit any Boolean Algebra structure, it suffices if its carrier set is finite and has cardinality that is *not* a power of two.¹⁷

Now to understand the relevance of G specifically for this paper, consider the following question:

In what sense is Definition 4.5.1 a good abstraction of Definition 3.1.2?

Design decisions are embedded in the conditions of Definition 4.5.1, and some of these were not trivial to make, having more than one plausible outcome. Why did we choose as we did? How do these choices interact? In what sense were they appropriate?

To answer these questions, F is not necessarily the greatest help on its own. To illustrate why, consider that we can obtain a general ‘theory of blockchains’ merely by insisting that an ‘abstract chunk system’ is a set. We impose no further structure: *et voilà*: instant generality!¹⁸ But this is rather useless: it tells us very little, e.g. F would just be the forgetful functor, mapping an IEUTxO model to its underlying set. We could also map just to monoids, if we add the failure element, and again an F would exist, but this would be only slightly less uninformative.

So where is the sweet spot, and why? As we observed, merely exhibiting an F -style functor does not help: we need to get an algebraic measure of what it is about Figure 1 that gives it its essential nature.

We get a formally meaningful measure of an appropriate level of abstraction by locating one at which we can build a sensible representation functor G *going back*, and seeing how conditions in Definition 4.5.1 interact with its construction — and, we can observe how tweaking them can affect, or even break, these constructions. A discussion of such tweaks, and their effects, is in Remarks 6.4.4, 7.1.5, and 7.1.6.

To be clear, other design choices are interesting, and indeed we consider some in the Remarks cited above. G and the choices we have made in building it are not direct value judgements: they are a way to measure and explore the fine structure of a large, abstract, and rather interesting design space.

¹⁶Note that G consists of an action on objects, and an action on arrows, and we can usefully have the former without the latter. See Remark 6.4.4.

¹⁷We do not exhibit any such application of our representation result in this paper; we are just making the general observation. Still, it is possible that in future work our representation functor could be put to such use.

¹⁸This really happened. An author lifted an algebra from one of my papers, deleted crucial structure, and claimed superior generality. When the paper went to me to referee, I observed that deleting this structure also deleted all the interesting theorems. This was not necessarily fatal; but what other theorems or properties were there to replace them? None were forthcoming.

6.2. Action on objects

Recall from Definition 3.1.2 the notion of an IEUTxO model, and the accompanying discussion in Remark 3.1.6 about the status of the injection ν . Continuing that Remark, in Definition 6.2.1 we must be explicit about ν :

DEFINITION 6.2.1. Suppose $(X, e, f, \leq, \cdot) \in \text{ACS}$. We define an IEUTxO model $G(X)$

$$G(X) = (\alpha, \beta, \text{Transaction}, \text{Validator}, \nu : \text{Validator} \hookrightarrow \text{pow}(\beta \times \text{Transaction}_!))$$

as follows:

(1) We take $\alpha = \beta = \text{atomic}(X)$ (Definition 4.1.3).

(2) We take ($\text{right}B$ is from Definition 4.3.1)

$$\text{Validator} = \{\text{right}B(x) \mid x \in \text{atomic}(X)\}.$$

(3) For each atomic $x \in \text{atomic}(X)$ we admit a transaction $\text{tx}(x) \in \text{Transaction}$ such that:

$$\begin{aligned} \text{input}(\text{tx}(x)) &= \{(a, x) \mid a \in \text{left}(x)\} \\ \text{output}(\text{tx}(x)) &= \{(b, x, \text{right}B(x)) \mid b \in \text{right}(x) \cup \text{up}(x)\} \end{aligned}$$

(4) We define $\nu : \text{Validator} \hookrightarrow \text{pow}(\beta \times \text{Transaction}_!)$ to map $X = \text{right}B(x) \in \text{Validator}$ as follows:

$$\nu(X) = \{(x'', \text{tx}(x')@i) \mid x'', x' \in X, i \in \text{input}(\text{tx}(x'))\}.$$

REMARK 6.2.2. It may be helpful to rewrite ν from Definition 6.2.1(4) in a more functional notation, where $X \in \text{Validator} = \{\text{right}B(x) \mid x \in \text{atomic}(X)\}$ and we use underscore for wildcards and identify powersets with functions to $\{\top, \perp\}$:

$$\nu(X) = \lambda(-, \text{tx}(x')@i). (X \ x' \wedge \text{input}(\text{tx}(x')) \ i)$$

Note also of the construction that x' can also be recovered just from the pointed input i above, with no need to look at the transaction, so this validator is actually a UTxO validator in the sense of Remark 3.1.5.¹⁹

Several things about Definition 6.2.1 need checked. We start with Lemma 6.2.3:

LEMMA 6.2.3. (1) If $X \in \text{ACS}$ and $x \in \text{atomic}(X)$ then $\text{tx}(x)$ has the right type to be a transaction as per Figure 1.

(2) If $X \in \text{ACS}$ and $x \in \text{atomic}(X)$ then $[\text{tx}(x)]$ is a chunk.

(3) As a corollary, atomic elements in $F(X)$ are precisely the singleton chunks of $\text{tx}(x)$, where x ranges over atomic elements of X — or more concisely in symbols:

$$\text{atomic}(F(X)) = \{[\text{tx}(x)] \mid x \in \text{atomic}(X)\}.$$

Proof. (1) From Definitions 4.4.1(1) and 6.2.1(3) and Lemma 4.3.9, $\text{tx}(x)$ has finitely many inputs and outputs; so as per Figure 1 it is indeed a pair of a *finite* set of inputs and a *finite* set of outputs.

(2) By Lemma 3.2.4, to show $[\text{tx}]$ is a valid chunk it would suffice to show that $\text{input}(\text{tx}(x)) \cap \text{output}(\text{tx}(x)) = \emptyset$. This follows from Lemma 4.3.9 and Corollary 4.4.5.

(3) By construction and Lemma 4.1.5, noting that in lists ordered by subset inclusion, atomic elements are singleton lists. □

REMARK 6.2.4. It remains to prove that ν is well-defined and (as required by Definition 3.1.2(3)) injective, and that $G(X)$ is indeed an IEUTxO model. See Corollaries 6.3.3 and 6.3.4.

¹⁹So interestingly, GF is actually a functor from IEUTxO models to UTxO models, and $\eta_{\mathbb{T}}$ from Definition 7.2.2 is also a natural mapping from an IEUTxO model to a corresponding UTxO model. We never use this, but it seems worth pointing out.

6.3. ν is injective

LEMMA 6.3.1. Suppose $X \in \text{ACS}$ (Definition 4.5.2). Then

$$x \in \text{atomic}(X) \text{ implies } \text{posi}(x) \neq \emptyset.$$

Proof. Suppose x is atomic. By Definition 4.1.3(1) $x \notin \{e, f\}$. We use Lemma 4.4.3. \square

LEMMA 6.3.2. Suppose $X \in \text{ACS}$. Then the assignment

$$x \in \text{atomic}(X) \longmapsto \text{tx}(x) \in \text{Transaction}_{G(X)}$$

from Definition 6.2.1(3) is injective.

Proof. By Lemma 6.3.1 (since x is atomic) $\text{posi}(x) \neq \emptyset$, so by Lemma 4.3.9 at least one of $\text{left}(x)$ or $\text{right}(x)$ or $\text{up}(x)$ must be nonempty.

If $\text{left}(x)$ is nonempty then $\text{tx}(x)$ has an input and we can read x off the data in that input.

Otherwise $\text{tx}(x)$ has an output and we can read x off the data in that output. \square

As promised in Remark 6.2.4, we prove:

COROLLARY 6.3.3. The map

$$\nu : \text{Validator} \hookrightarrow \text{pow}(\beta \times \text{Transaction}_1)$$

from Definition 6.2.1(4) is well-defined and injective.

Proof. By Lemma 6.3.2 we can deduce y from $\text{tx}(y)$. The result follows. \square

COROLLARY 6.3.4. If X is an abstract chunk system (Definition 4.5.1) then $G(X)$ is an IEUTxO model (Definition 3.1.2).

Proof. We just need to check the conditions of Definition 3.1.2; the only nontrivial part is that ν is an injection, and that is Corollary 6.3.3. \square

6.4. Action on arrows

DEFINITION 6.4.1. Suppose $g : X \rightarrow Y \in \text{ACS}$ and recall from Definition 3.4.1(2) that an arrow

$$G(g) = G(X) \rightarrow G(Y) \in \text{IEUTxO}$$

should be a mapping from $\text{Transaction}_{G(X)}$ to $\text{Chunk}_{G(Y)}$. Recall *factor* from Definition 4.2.5(1b), which factorises non-failure elements into atomic constituents, and recall from Definition 4.5.2(2b) that g maps non-failure elements to non-failure elements.

Then define $G(g)$ by

$$G(g) : \text{tx}(x) \longmapsto \text{tx}(y_1) \cdot \dots \cdot \text{tx}(y_n) \in \text{Chunk}_{G(Y)} \\ \text{where } \text{factor}(g(x)) = [y_1, \dots, y_n] \in [\text{atomic}(Y)].$$

LEMMA 6.4.2. $G(g)$ from Definition 6.4.1 is well-defined.

Proof. We must check that $\text{tx}(y_1) \cdot \dots \cdot \text{tx}(y_n)$ is a chunk; this follows by Lemma 3.4.2. \square

PROPOSITION 6.4.3. The map G , with the action on abstract chunk systems from Definition 6.2.1, and with the action on arrows from Definition 6.4.1, is a functor

$$G : \text{ACS} \rightarrow \text{IEUTxO}.$$

Proof. Given the results above, the only remaining thing to check is that if $g : X \rightarrow Y$ and $g' : Y \rightarrow Z$ then $G(g'g) = G(g')G(g)$. This follows by a routine argument from the definitions, using Definition 4.2.5(1b)ii). \square

REMARK 6.4.4. The significance of condition 1b of Definition 4.2.5 is not that elements can be factored into atomic elements — this follows already from condition 1a — but that a factorisation can be *selected*, as a monoid homomorphism.

We use this condition in just one place: to define the action of G on arrows in Definition 6.4.1 and prove it functorial in Proposition 6.4.3.

It would be legitimate to remove condition 1b of Definition 4.2.5.

This would exhibit our category ACS as a subcategory of a larger category which would include more objects — ‘even more abstract’ abstract chunk systems — at a cost of no longer being able to functorially map this larger space back down to IEUTxO. Intuitively, this larger space behaves more like a space of all possible denotations, rather than the space of IEUTxO-representable ones.

Interestingly, the action on objects from Definition 6.2.1 would still be well-defined even without Definition 4.2.5(1b), so that we can still represent our ‘even more abstract’ abstract chunk systems concretely in IEUTxO models — this just would not correspond to a functor.

7. AN ADJUNCTION BETWEEN $F : \text{IEUTxO} \rightarrow \text{ACS}$ AND $G : \text{ACS} \rightarrow \text{IEUTxO}$

7.1. The map $\epsilon_X : FG(X) \rightarrow X$

REMARK 7.1.1. Suppose $X \in \text{ACS}$; we wish to define an arrow $\epsilon_X : FG(X) \rightarrow X \in \text{ACS}$. Unpacking Definitions 6.2.1 and 5.1.1, we see that an $x \in FG(X)$ has one of the following forms:

- $x = \text{fail}_{FG(X)}$ for $\text{fail}_{FG(X)}$ the *failure* element added by F to $\text{Chunk}_{G(X)}$ in Definition 5.1.1.
- $x = [\text{tx}(x_1), \dots, \text{tx}(x_n)]$ for some unique $[x_1, \dots, x_n] \in [\text{atomic}(X)]$.

We also know from Lemma 6.3.2 that $\text{tx} : \text{atomic}(X) \rightarrow \text{Transaction}(G(X))$ is injective, and it follows that we can recover each x_i from the unique corresponding $[\text{tx}(x_i)]$ above.

DEFINITION 7.1.2. Let $\epsilon_X : FG(X) \rightarrow X$ be determined by:

- (1) $\epsilon_X([\]) = e_X$ (this would be a special case of the next clause, for $n = 0$)
- (2) $\epsilon_X([\text{tx}(x_1), \dots, \text{tx}(x_n)]) = x_1 \cdot \dots \cdot x_n$ for $n \geq 1$ and $x_1, \dots, x_n \in \text{atomic}(X)$
- (3) $\epsilon_X(f_{FG(X)}) = f_X$

LEMMA 7.1.3. *Definition 7.1.2 is well-defined and determines an arrow in ACS.*

Proof. Well-definedness follows as per Remark 7.1.1 from Lemma 6.3.2, since we can recover each x_i from its $[\text{tx}(x_i)]$. It remains to check the arrow conditions from Definition 4.5.2(2):

- (1) We check that $\epsilon_X(e_{FG(X)}) = e_X$ and $\epsilon(f_{FG(X)}) = f_X$.

A fact of Definition 7.1.2.

- (2) We check that $x \leq y < f_{FG(X)}$ implies $\epsilon_X(x) \leq \epsilon_X(y) < f_X$.

By Proposition 5.1.3 $FG(X)$ is perfectly atomic, and if $x \leq y < f_{FG(X)}$ then x and y factorise uniquely into a composition of lists of singleton chunks, which by construction in Definition 6.2.1(3) have the form $\text{tx}(x_i)$ and $\text{tx}(y_j)$, such that the factorisation of x is a sublist of the factorisation of y .

The result follows by a routine calculation from Definition 7.1.2.

- (3) $\epsilon_X(x) \cdot \epsilon_X(y) = \epsilon_X(x \cdot y)$

A fact of Definition 7.1.2, again using the fact that by Proposition 5.1.3 $FG(X)$ is perfectly atomic.

□

PROPOSITION 7.1.4. $\epsilon_X : FG(X) \rightarrow X$ is a surjection on underlying sets.

Proof. Suppose we are given $x \in X$; we want to exhibit an element in $FG(X)$ that maps to it under ϵ_X .

If $x = f_X$ then by construction in Definition 5.1.1 $fail_{FG(X)} = f_{FG(X)}$ and also by construction $\epsilon_X(fail_{FG(X)}) = f_X$ so we are done.

Otherwise by Proposition 5.1.3 (or just by Definition 4.2.5) we can write

$$x = x_1 \cdot \dots \cdot x_n$$

for some atomic $x_1, \dots, x_n \in atomic(X)$, and looking at Definition 7.1.2 we immediately have that

$$\epsilon_X([tx(x_1), \dots, tx(x_n)]) = x_1 \cdot \dots \cdot x_n = x.$$

□

In a sense Remark 7.1.5 continues Remark 6.4.4, in that they both discuss tweaks to the design of abstract chunk systems:

REMARK 7.1.5. It can be shown that if we were to strengthen Definition 4.5.1 so that an abstract chunk system is a *perfectly atomic* oriented monoid of chunks, then

- surjectivity of ϵ_X in Proposition 7.1.4 would strengthen to bijectivity (Definition 4.2.5(2a) gives the injectivity, but does not ensure that every \leq -relationship is preserved), and
- the adjunction in Theorem 7.3.4 would become an equivalence of categories (Definition 4.2.5(2b) ensures that $FG(X)$ has all the \leq -structure of X).

G and F would still have non-trivial work to do, as we see e.g. from Propositions 5.5.4 and 5.2.2. Still: there is a sense in which the subcategory of *perfectly atomic* abstract chunk systems, and all arrows between them, is the ‘properly IEUTxO-like’ abstract chunk systems.

REMARK 7.1.6. By Proposition 7.1.4 ϵ_X surjects $FG(X)$ onto X as sets, but is it not necessarily surjective on the \leq structure, meaning that $\epsilon_X(ch) \leq_X \epsilon_X(ch')$ does not imply $ch \leq_{FG(X)} ch'$.

We can have this if we add a third condition in Definition 4.2.5, after condition 1(b)ii, that: if $x \leq y < f$ then $factor(x) \leq factor(y)$ (the right-hand \leq denotes sublist inclusion; the left-hand \leq is the partial order on X).

7.2. The map $\eta_{\mathbb{T}} : \mathbb{T} \rightarrow GF(\mathbb{T})$

REMARK 7.2.1. Suppose $\mathbb{T} = (\alpha, \beta, \text{Transaction}, \text{Validator}) \in \text{IEUTxO}$. We wish to define an arrow

$$\eta_{\mathbb{T}} : \mathbb{T} \rightarrow GF(\mathbb{T}).$$

We can make some observations:

- By construction in Definition 5.1.1, $F(\mathbb{T})$ is isomorphic as a partial ordering to \mathbb{T} , with the addition of the *fail* top element.
As noted in Proposition 5.1.3 it follows that the atomic elements of $F(\mathbb{T})$ correspond precisely with the singleton chunks in $\text{Chunk}_{\mathbb{T}}$, and thus with transactions in $\text{Transaction}_{\mathbb{T}}$.
The chunks in $F(\mathbb{T})$ are then determined by combining the singleton chunks, subject to the well-formedness conditions of Definition 3.2.1 and the locality properties noted in Lemma 3.2.5.
- By construction in Definition 6.2.1 the atomic elements of $GF(\mathbb{T})$ are isomorphic to $atomic(F(\mathbb{T}))$, and by Lemma 6.2.3(3) also to $\text{Transaction}_{\mathbb{T}}$.

DEFINITION 7.2.2. Let $\eta_{\mathbb{T}} : \mathbb{T} \rightarrow GF(\mathbb{T})$ be determined by mapping $tx \in \text{Transaction}_{\mathbb{T}}$ to $[tx(tx)] \in \text{Chunk}_{GF(\mathbb{T})}$. Thus using Lemma 3.4.2 we have:

$$\eta_{\mathbb{T}}([tx_1, \dots, tx_n]) = [tx(tx_1), \dots, tx(tx_n)].$$

LEMMA 7.2.3. If

- $x = [tx_1, \dots, tx_n] \in \text{Chunk}_{\mathbb{T}}$ then
- $\eta_{\mathbb{T}}(x) = [tx(tx_1), \dots, tx(tx_n)] \in \text{Chunk}_{GF(\mathbb{T})}$.

As a corollary, Definition 7.2.2 does indeed map chunks to chunks.

Proof. By a routine check on Definition 6.2.1(3), using Proposition 5.5.4. \square

LEMMA 7.2.4. Recall the notions of $utxi$, $utxo$, and stx from Definition 3.2.14 and the notion of $blockedUtxi$ from Definition 5.5.1. Recall η from Definition 7.2.2 and suppose $x \in X \in ACS$. Then:

- (1) $utxi(\eta_{\mathbb{T}}(x)) = utxi(x) \setminus blockedUtxi(x)$
- (2) $utxo(\eta_{\mathbb{T}}(x)) = utxo(x) \cup blockedUtxi(x)$
- (3) $stx(\eta_{\mathbb{T}}(x)) = stx(x)$

Proof. By routine calculations using Proposition 5.5.4. \square

PROPOSITION 7.2.5. The action $\eta_{\mathbb{T}} : \text{Chunk}_{\mathbb{T}} \rightarrow \text{Chunk}_{GF(\mathbb{T})}$ is an isomorphism on underlying sets.

Proof. Using Lemma 6.3.2 and the fact that from Definitions 5.1.1 and 6.2.1, any element of $GF(\mathbb{T})$ has the form $[tx(tx_1), \dots, tx(tx_n)]$ for some transactions $tx_1, \dots, tx_n \in \text{Transaction}_{\mathbb{T}}$. \square

7.3. F is left adjoint to G

REMARK 7.3.1. Naturality of ϵ and η is Propositions 7.3.2 and 7.3.3. The proofs are by diagram-chasing. We do need to be a little careful because of the choice made in the *factor* function (Definition 4.2.5(1b)), which propagates to the action of G on arrows (Definition 6.4.1). In the event, the diagram-chasing is all standard and it works fine.

PROPOSITION 7.3.2. ϵ is a natural transformation from FG to 1_{ACS} .

Proof. Consider some arrow $g : X \rightarrow Y \in ACS$. We must check a commuting square in ACS that

$$\epsilon_Y FG(g) = g \epsilon_{FG(X)}.$$

Using Proposition 4.5.5(2) it suffices to check for each $x' \in \text{atomic}(FG(X))$ that

$$\epsilon_Y (FG(g)(x')) = g(\epsilon_{FG(X)} x').$$

From Lemma 6.2.3(3), $x' = [tx(x)]$ for $x \in \text{atomic}(X)$.

Write $g(x) = y_1 \dots y_n$ where $\text{factor}(g(x)) = [y_1, \dots, y_n] \in [\text{atomic}(Y)]$ (Definition 4.2.5(1b)), so that

$$(FG(g))([tx(x)]) = [tx(y_1), \dots, tx(y_n)] \in FG(Y).$$

Then

$$\epsilon_Y (FG(g)([tx(x)])) = \epsilon_Y ([tx(y_1), \dots, tx(y_n)]) = y_1 \dots y_n = y$$

and

$$g(\epsilon_X([tx(x)])) = g(x) = y$$

as required. \square

Consider $f : \mathbb{S} \rightarrow \mathbb{T}$ and $F(f) : F(\mathbb{S}) \rightarrow F(\mathbb{T})$ and some $[tx] \in \text{Chunk}_{\mathbb{S}}$ and $f(tx) = [tx_1, \dots, tx_n] \in \text{Chunk}_{\mathbb{T}}$. Then $\eta_{\mathbb{S}}([tx_1, \dots, tx_n]) = [tx(tx_1), \dots, tx(tx_n)]$ and

$$F(f)([tx])([tx_1, \dots, tx_n])$$

PROPOSITION 7.3.3. η is a natural transformation from $1_{\text{IEUT} \times \text{O}}$ to GF .

Proof. Consider some arrow $f : \mathbb{S} \rightarrow \mathbb{T} \in \text{IEUT} \times \text{O}$. We must check a commuting square in $\text{IEUT} \times \text{O}$ that

$$\eta_{\mathbb{T}} f = GF(f) \eta_{\mathbb{S}}.$$

Using Lemma 3.4.2 it suffices to check for each $tx \in \text{Transaction}_{\mathbb{S}}$ that

$$\eta_{\mathbb{T}}(f([tx])) = GF(f)(\eta_{\mathbb{S}}([tx])).$$

Suppose $f(tx) = [tx_1, \dots, tx_n] \in \text{Chunk}_{\mathbb{T}}$. Then

$$\eta_{\mathbb{T}}(f([tx])) = \eta_{\mathbb{T}}([tx_1, \dots, tx_n]) = [\text{tx}(tx_1), \dots, \text{tx}(tx_n)]$$

and

$$GF(f)(\eta_{\mathbb{S}}([tx])) = GF(f)([\text{tx}(tx)]) = [\text{tx}(tx_1), \dots, \text{tx}(tx_n)]$$

as required. \square

THEOREM 7.3.4. *The functors*

$$F : \text{IEUTxO} \rightarrow \text{ACS} \quad \text{and} \quad G : \text{ACS} \rightarrow \text{IEUTxO}$$

from Definitions 5.1.1 and 5.4.1 (for F) and from Definitions 6.2.1 and 6.4.1 (for G) form an adjoint pair. In symbols:

$$F \dashv G : \text{IEUTxO} \rightarrow \text{ACS}$$

Proof. The natural transformations are $\epsilon : FG \rightarrow 1$ and $\eta : 1 \rightarrow GF$ from Definitions 7.1.2 and 7.2.2, which are natural by Propositions 7.3.2 and 7.3.3. \square

8. CONCLUSIONS

We have presented the EUTxO blockchain model in a novel and compact form and derived from it an algebra-style theory of blockchains as partially-ordered partial monoids with channel name communication. This builds on previous work [5].

We hope this paper will make two contributions:

- (1) its specific definitions and theorems, but also,
- (2) an *idea* that blockchain structures can be subjected to this kind of analysis.

Or to put it another way: we illustrate that an algebraic theory of blockchains is possible, and what it might look like.

The reader can apply these ideas to their favourite blockchain architecture, and if this were widespread practice then this might help make the field accessible to an even broader audience, ease technical comparisons between systems, add clarity to a fast-changing field — and as we have argued, it might suggest structures and tests for practical programs, as is already reflected in a recent work [5; 9].

We now reflect on the design decisions made along the way and suggest possibilities for future work:

8.1. Observational equivalence

We touched on notions of observational equivalence in Subsections 3.2.3 and 4.3.2.

The theory of EUTxO observational equivalence is a little weaker than we might like, in the sense that not as many things get identified as one might first anticipate.

This is because a validator gets access to the whole transaction from which an input emanates — see the line for Validator in Figure 1. (This is specific to EUTxO; UTxO is more local, see Remark 3.1.5.)

This makes it hard to factor out internal structure. For instance, even if all but one of the inputs and outputs of a transaction have been spent, so that it has just one dangling input — then that entire transaction is still observable at the final dangling input. Recall Proposition 3.2.18, and consider $tx, tx' \in \text{Transaction}$ such that $\text{pos}(tx) \cap \text{pos}(tx') = \emptyset$, so that tx and tx' are commuting. We might reasonably wish to identify $tx \cdot tx'$ and $tx' \cdot tx$ with a composite transaction which we might write $tx \cup tx'$, but we cannot do this because a validator could see the difference.

Thus, currently the EUTxO framework can observe the difference between

— one large transaction, and

— a composite chain (i.e. a chunk; see Definition 3.2.1) of smaller ones

— even if they are ‘morally’ the same. This is not good for developing compositional theories of observational equivalence.

We propose it might be helpful if the EUTxO model allowed us to limit access to the channels in a transaction, such that an input can limit validators to access only certain inputs and outputs in a transaction, e.g. by nominating positions that are ‘examinable’ along that input.²⁰ The pure UTxO model (Remark 3.1.5) would correspond to the special case when an input nominates only itself for validators to access.

Then, we would in suitable circumstances be able to switch between a single large transaction, and a composite chunk with the same inputs and outputs.

Another observable of a transaction is the names of its spent transactions, and this brings us on to garbage-collection:

8.2. Garbage-collection

Our model has no garbage-collection of spent positions — where by *spent positions* we mean the *stx*-atoms from Definition 3.2.14; see also the *up*-atoms from Definition 4.3.6 (a connection is precisely stated in Proposition 5.5.4).

There is nothing wrong with this; the EUTxO presentation in [6] does not garbage-collect either (i.e. positions on the blockchain can be occupied at most once, and can never be un-occupied).

However, since we name our positions, it might be nice to consider garbage-collecting (i.e. locally binding) the names of spent positions, by removing or α -converting them in some way.

We do this in the implementation in [9].

However, the maths indicates that this does come at a certain price. For instance, consider a very simple model of finite lists of atoms in which the first atom is an ‘input’ and the last atom is an ‘output’, and we garbage-collect by removing matching atoms, like so:

$$[a, b] \cdot [b, c] = [a, c] \quad [a, c] \cdot [c, b] = [a, b].$$

Thus,

$$([a, b] \cdot [b, c]) \cdot [c, b] = [a, b].$$

Now if we bracket the other way, then $[b, c] \cdot [c, b]$ is ill-defined, and therefore so is $[a, b] \cdot ([b, c] \cdot [c, b])$. We cannot have $[b, b]$ because this would violate the condition that an input must point to an earlier (not a later) output (Definition 3.2.1(4)) — note that we consider relaxing this condition in item 2 of Subsection 9).

The partiality is no issue — chunks are already a partial monoid (Theorem 3.2.8). But, our monoid of garbage-collecting chunks would not be *associative*. It would still be *nearly* associative, meaning that $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ where both sides are defined. However this is a weaker property that would be messier to work with, and for now we do not need it to make our case, so we defer this to future work.

In this paper we do not garbage-collect. We leave deciding whether we should, and if so how, for future work.

8.3. Tests

A few more words on the equations in Figure 1 (IEUTxO type equations) vs. those in Definition 4.5.2 (its algebraic counterpart of abstract chunk systems).

We want the equations in Figure 1 to be short and sweet, so that the system looks simple and solutions are easy to build and manipulate.

²⁰So a validator would be passed the *restriction* of a transaction obtained by including only nominated inputs and outputs in that transaction, and withholding the rest.

The design parameters in Definition 4.5.2 are somewhat different: we do not mind if there are plenty of algebraic properties, because this means that we have captured as many interesting properties as possible. The adjunction in Theorem 7.3.4 gives a formal sense in which the two correspond.

(This leaves it for future work to see how these conditions could be relaxed, as discussed e.g. in Remarks 6.4.4, 7.1.5, and 7.1.6.)

This has mathematical interest, but not only that.

As noted in Subsection 4.4.2, modern programming languages support efficient programming on abstract denotations, thus delaying instantiating to specific instances until truly necessary. Also, they allow us to express and test against properties — and equality properties in particular can be helpful for optimising transformations.

So an algebraic theory can be relevant to producing concrete working code, because:

- (1) it can help structure code; and
- (2) an axiom can be read both as a testable property and as a program transformation; so that
- (3) the more axioms we have, the more transformations and tests are available, and the more scope we have to transform, structure, and test our programs.

8.4. Connections with nominal techniques

This paper borrows ideas from *nominal techniques* [10] and in particular it follows the ideas on Equivariant ZFA from [8] in handling the atoms which we use to name positions. We use atoms to name positions in IEUTxO models and in abstract chunk systems (ACS).

Partly, this is just using nominal-style names and permutations as a standard vocabulary. We can think of this application of nominal ideas as reaching for a familiar API, typeclass, or algebra, and there is nothing wrong with that.

The reader can find this implemented in [9], where the IEUTxO equations in Figure 1 are combined with this author’s nominal datatypes package to create first a Haskell typeclass, and then a working implementation of chunks and blockchains, following the IEUTxO model of this paper.

But in parts of this paper, something deeper is also taking place. For example:

- (1) The notion of abstract chunk system — which makes up the more abstract half of this paper — depends on that of an oriented monoids of chunks from Definition 4.4.1, which depends on the notion of *posi* from Definition 4.3.4, which depends on the permutation action. *posi* is a nameful definition, in the nominal sense, and it is not clear how would even go about expressing it, and thus the notion of ACS, were it not for our nominal use of names.
- (2) In [9] we go somewhat further than in this paper in developing IEUTxO models, in that we permit α -conversion to garbage-collect spent conditions as discussed in Subsection 8.2. This too is a fully nominal definition, which uses the nominal model of binding in specific ways.

REMARK 8.4.1 (No support assumed). Experts on nominal techniques should note that no finite support conditions are imposed; for example, there is nothing to insist that a validator $\nu(v) \subseteq \beta \times \text{Transaction}_!$ should be a finitely-supported subset. This paper is in ZFA, not Fraenkel-Mostowski set theory.

We could construct a finitely-supported account of the theory in this paper in FM, just by imposing finite support conditions appropriately — but it would cost us complexity, and since we do not seem to need this, we do not do it. We hope we have struck a good balance in the mathematics between being rigorous in our treatment of names, and not drowning the reader in detail.

Note that finiteness conditions on sets of names are still important: notably in Definition 4.4.1(1); and name-management is key to some nontrivial results, notably Corollary 4.4.5. So there is a ‘finite support’ flavour to the maths, but not as a direct translation of the Fraenkel-Mostowski notion of finite support.

REMARK 8.4.2 (Disjointness conditions). Continuing the previous remark, ‘freshness-flavoured’ set disjointness conditions like $\text{input}(tx) \cap \text{output}(tx) = \emptyset$ in Lemma 3.2.4, and $\text{pos}(ch) \cap \text{pos}(ch') =$

\emptyset in Lemma 3.2.10, and $\text{posi}(x) \cap \text{posi}(y) = \emptyset$ in Definition 4.4.1 and elsewhere, are quite important in this paper.

We could have imported a nominal notation and written these $\#$, as in $ch\#ch'$ or $x\#y$. This would not be wrong, but it might mislead because, as discussed above, we do not assume nominal notions of support and freshness. Thus, $\text{pos}(ch)$ and $\text{posi}(x)$ are *not* necessarily equal to the support of ch and x respectively, and if e.g. x and y were in an abstract chunk system that happened also to be finitely-supported (which is a very plausible scenario), it would not be guaranteed that $x\#y$ would coincide with $\text{posi}(x) \cap \text{posi}(y) = \emptyset$.

We could insist that $\text{supp}(x)$ must exist and coincide with $\text{posi}(x)$, of course — but that would be an additional restriction.

8.5. Concrete formalisation

Can we implement all or parts of this paper in a theorem-prover, and use that to verify properties of a blockchain system?

This paper has a lot of moving parts, and it is not all or nothing: a user can import whichever components (IEUTxO? ACS?) they wish — though we also hope that the overall mathematical vision could provide useful guidance.

How the ideas in this paper can be brought to bear in a formal verification cannot have a clear-cut answer, because it depends on interactions between the maths, what we want to verify, the resources available,²¹ and what facilities are offered by a particular theorem-proving environment.

A reasonable (but naive) implementation of the EUTxO inductive definition in [6] would suggest modelling positions by numbers which are essentially de Bruijn indices. The maths in this paper suggests against this:

- We want to talk about chunks. Indices only make sense in a blockchain which has an initial genesis block from which we start to index.
- We want to rearrange transactions and chunks, e.g. to talk about how they commute, as in (for example) Definition 4.3.13. With indices, a transaction in a different place is a *different transaction*, and potentially subject to different validation if a validator were to directly inspect its indices.
- Because we use names, we never have to worry about reindexing functions, as can be an issue with the de Bruijn indexed approach.

This is borne out by the practice: the developers of the EUTxO implementation underlying [6] have explored theorem-provers and they do not reference transactions by position (even though the mathematical description in the literature makes it look like they do, at least to the uninformed reader).

What they actually do is maintain an explicit naming context. This is ongoing research, but the interested reader can find a brief description in [11].

The difficulty with this hands-on context-based approach is that we end up having to curate our context of names, using explicit context-weakening and context-combining operations. These get limited, extremely tedious to formulate and prove, and clutter the proofs of properties.²²

It is standard and known that maintaining contexts of ‘known names’ can get painful. Indeed, this is one of the issues that nominal techniques were developed to alleviate.

In the context of this paper, we would advise looking at how the implementation in [9] manages names and binding using permutations and the Nom binding context (which is the nominal construct that closely corresponds to a local context of known names).

So it works in Haskell — but determining the extent to which this can be translated to a *theorem-prover* remains to be seen. We could imitate the nominal datatypes package; there is a nominal pack-

²¹Very important: a single person working alone will make different tradeoffs than a large team, and a short-term project will make different tradeoffs than a longer-term project that can e.g. afford to invest in building basic tooling.

²²*Properties* often get called *meta-theorems* in this field. So wherever we say ‘(algebraic) property’, a reader with a theorem-proving background can say ‘meta-theorem’, and they will not go too far wrong.

age in Isabelle [13]; and this author has written some recommendations on implementing nominal techniques in theorem-proving environments [8].²³ Exploring this is future work.

9. FUTURE WORK

We discussed future work above and in the body of the paper. We conclude with some further observations:

- (1) The authors of [6] map their concrete models to *Constraint Emitting Machines*, which are a novel variant of Mealy machines. It is future work to see whether the idealised EUTxO solutions from Definition 3.1.2 admit corresponding descriptions. For the interested reader, we can note that a body of work on nominal automata does exist [4; 3].
- (2) In condition 4 of Definition 3.2.1 we restrict inputs to point to strictly earlier outputs. This restriction makes operational sense for a blockchain, and it is unavoidable and required for well-definedness in [6] because of its inductive construction. However, there is no mathematical necessity to retain it here; it would be perfectly valid and possible to contemplate a generalisation of Definition 3.2.1 which permits loops from a transaction to itself, or even forward pointers from inputs to later outputs (i.e. ‘feedback loops’). Mathematically and structurally, loops are perfectly admissible in the framework of this paper. Loops from a transaction to itself are particularly interesting, because this would remove the need for a *genesis block* — which has no inputs, and therefore exists *sui generis* in that it is not subject to the action of any validation from other blocks. So, we could insist that all transactions have at least one input, but that input may loop to an output on the same transaction. This might seem like a minor difference, but it is not, because *we* control the set of validators (the injection ν in Definition 3.1.2), so we can enforce checks of good behaviour — even of the first transaction, which would e.g. be forced to validate itself via a loop — by controlling the set of validators.
- (3) We have considered EUTxO blockchains in this paper. It would be natural to attempt a similar analysis for an accounts-based blockchain architecture such as Ethereum. A start on this is the *Idealised Ethereum* equations in [5, Figure 4]; developing this further is future work.
- (4) As is often the case, in practice there are desirable features of real systems that would break our model.

For instance, it can be useful to make transactions time-sensitive using *slot ranges*, also called *validity intervals*; a transaction can only be accepted into a block whose slot is inside the transaction’s slot range. Clearly, this could compromise results having to do with chunks and commutations, because these are by design ‘pure’ notions, with no notion of time.

It is future work to see to what extent a theory of chunks might be compatible with explicit time dependence constraints, but in any case, it is common to find a pure sublanguage with a nice theory embedded in a more expressive and larger language with less good behaviour, as e.g. pure SML is embedded in SML with global variables.

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²³This last paper essentially says: make sure that permuting names in properties is a pushbutton operation. Once you have that, the rest should follow; and if you do not then there may be trouble.

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