International Journal of Mathematics
Vol. 21, No. 11 (2010) 1439–1459

DOI: 10.1142/S0129167X10006574

PUISEUX POWER SERIES SOLUTIONS FOR SYSTEMS OF EQUATIONS

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Received 19 April 2009
Revised 24 November 2009

We give an algorithm to compute term-by-term multivariate Puiseux series expansions of series arising as local parametrizations of zeroes of systems of algebraic equations at singular points. The algorithm is an extension of Newton's method for plane algebraic curves, replacing the Newton polygon by the tropical variety of the ideal generated by the system. As a corollary we deduce a property of tropical varieties of quasi-ordinary singularities.

Keywords: Puiseux series; Newton polygon; singularity; tropical variety; quasi-ordinary projection.

Mathematics Subject Classification 2000: 14J17, 52B20; 14B05, 14Q15, 13P99

1. Introduction

Newton described an algorithm to compute term-by-term the series arising as y-roots of algebraic equations $f(x, y) = 0$ [25]. The main tool used in the algorithm is a geometrical object called the Newton polygon. The roots found belong to a field of power series called Puiseux series [29].

The extension of Newton–Puiseux's algorithm for equations of the form $f(x_1, \ldots, x_N, y) = 0$ is due to McDonald [23]. As can be expected, the Newton polygon is extended by the Newton polyhedron.

An extension for systems of equations of the form $\{f_1(x, y_1, \ldots, y_M) = \cdots = f_r(x, y_1, \ldots, y_M) = 0\}$ is described in [22] using tropism and in [19] using tropical geometry.

McDonald gives an extension to systems of equations

$$\{f_1(x_1,\ldots,x_N,y_1,\ldots,y_M)=\cdots=f_r(x_1,\ldots,x_N,y_1,\ldots,y_M)=0\}$$

using the Minkowski sum of the Newton polyhedra. However, this algorithm works only for "general" polynomials [24].

In this note we extend Newton's method to any dimension and codimension. The Newton polyhedron of a polynomial is replaced by its normal fan. The tropical variety comes in naturally as the intersection of normal fans. We prove that, in an algebraically closed field of characteristic zero, the given algorithm always works.

The natural field embedding the algebraic closure of polynomials in one variable is the field of Puiseux series. When it comes to several variables there is a family of fields to choose from. Each field is determined by the choice of a vector $\omega \in \mathbb{R}^N$ of rationally independent coordinates. The need to choose ω already appeared when working with a hypersurface [16, 23]. The introduction of the family of fields is done in [7].

We start the article recalling the main statements on which the Newton–Puiseux method for algebraic plane curves relies (Sec. 2) and extending these statements to the general case (Sec. 3).

In the complex case, a series of positive order, obtained by the Newton–Puiseux method for algebraic plane curves, represents a local parametrization of the curve around the origin. In Sec. 4 we explain how an M-tuple of series arising as a solution to the general Newton–Puiseux statement also represents a local parametrization, recalling results form [3].

In Sec. 5 we recall the definition given in [7] of the family of fields of ω-positive Puiseux series and their natural valuation. Then, in Sec. 6 we reformulate the question using the fields introduced and show how it becomes a lot simpler.

Then we work in the ring of polynomials with coefficients ω-positive Puiseux series (Secs. 7–9).

The Newton–Puiseux algorithm is based on the fact that the first term of a y-root is the y-root of the equation restricted to an edge of the Newton polygon. The analogous of this fact is expressed in terms of initial ideals. In Secs. 7 and 8, weighted orders and initial ideals are defined. In Sec. 9 we prove that initial parts of zeroes are zeroes of weighted initial ideals.

Then we consider ideals in the ring of polynomials $\mathbb{K}[x^*,y] := \mathbb{K}[x_1,{x_1}^{-1},\ldots,x_N,{x_N}^{-1},y_1,\ldots,y_M]$ (Sec. 10) and characterize initial ideals with zeroes in a given torus. This is done in terms of the tropical variety of the ideal (Sec. 11).

Sections 12 to 16 are devoted to explaining the algorithm. In the last section we show the theoretical implications of the extension of Newton–Puiseux algorithm by giving a property of the tropical variety associated to a quasi-ordinary singularity.

2. Newton–Puiseux's Method

Given an algebraic plane curve $\mathcal{C} := \{f(x,y) = 0\}$, the Newton–Puiseux method constructs all the fractional power series $y(x)$ such that $f(x, y(x)) = 0$. These series turn out to be Puiseux series.

Newton–Puiseux's method is based on two points:

Given a polynomial $f(x, y) \in \mathbb{K}[x, y]$,

(1) cx^μ is the first term of a Puiseux series $y(x) = cx^\mu + \cdots$ with the property $f(x, y(x)) = 0$ if and only if
- $\frac{-1}{\mu}$ is the slope of some edge L of the Newton polygon of f.
- cx^μ is a solution of the characteristic equation associated to L.

(2) If we iterate the method: take $c_i x^{\mu_i}$ to be a solution of the characteristic equation associated to the edge of slope $\frac{-1}{\mu_i}$ of $f_i := f_{i-1}(x, y + c_{i-1}x^{\mu_{i-1}})$ with $\mu_i > \mu_{i-1}$. We do get a Puiseux series $\sum_{i=0}^{\infty} c_i x^{\mu_i}$ with the property $f(x, y(x)) = 0$.

In this paper we prove the extension of these points: Point 1 is extended in Sec. 11, Theorem 11.2 and, then, Point 2 in Sec. 16, Theorem 16.1.

Point 1 is necessary to assure that the sequences in Point 2 always exist. But Point 1 does not imply that any sequence constructed in such a way leads to a solution. Both results have led to a deep understanding of algebraic plane curves.

3. The General Statement

Take an N-dimensional algebraic variety $V \subset \mathbb{K}^{N+M}$. There is no hope to find $k \in \mathbb{N}$ and an M-tuple of series $y_1, \dots, y_M$ in $\mathbb{K}[[x_1^k, \dots, x_M^k]]$ such that the substitution $x_j \mapsto y_j(x_1, \dots, x_N)$ makes f identically zero for all f vanishing on V. (Parametrizations covering a whole neighborhood of a singularity do not exist in general.)

McDonald's great idea was to look for series with exponents in cones. Introducing rings of series with exponents in cones served to prove Newton–Puiseux's statement for the hypersurface case [23] and has been the inspiration of lots of other results (both in algebraic geometry [16, 31] and differential equations [2, 5]).

In order to give a general statement for all dimension and codimension, we need to recall some definitions of convex geometry:

A *convex rational polyhedral cone* is a subset of $\mathbb{R}^N$ of the form

$$\sigma = \{\lambda_1 v_1 + \cdots + \lambda_r v_r \mid \lambda_i \in \mathbb{R}, \lambda_i \geq 0\},$$

where $v_1, \dots, v_r \in \mathbb{Q}^N$ are vectors.

A cone is said to be *strongly convex* if it contains no nontrivial linear subspaces.

A *fractional power series* φ in N variables is expressed as

$$\varphi = \sum_{\alpha \in \mathbb{Q}^N} c_\alpha x^\alpha, \quad c_\alpha \in \mathbb{K}, \quad x^\alpha := x_1^{\alpha_1} \cdots x_N^{\alpha_N}.$$

The *set of exponents* of φ is the set

$$\mathcal{E}(\varphi) := \{\alpha \in \mathbb{Q}^N \mid c_\alpha \neq 0\}.$$

A fractional power series φ is a *Puiseux series* when its set of exponents is contained in a lattice. That is, there exists $K \in \mathbb{N}$ such that $\mathcal{E}(\varphi) \subset \frac{1}{K}\mathbb{Z}^N$.

Let $\sigma \subset \mathbb{R}^N$ be a strongly convex cone. We say that a Puiseux series φ has *exponents in a translate of* σ when there exists $\gamma \in \mathbb{Q}^N$ such that $\mathcal{E}(x^\gamma \varphi) \subset \sigma$.

It is easy to see that the set of Puiseux series with exponents in the translates of a strongly convex cone σ is a ring. (But, when $N > 1$, it is not a field).

Given a nonzero vector $\omega \in \mathbb{R}^N$, we say that a cone σ is ω*-positive* when for all $v \in \sigma$ we have $v \cdot \omega \geq 0$. If ω has rationally independent coordinates, an ω-positive rational cone is always strongly convex.

Denote by $\mathsf{V}(\mathcal{I})$ the set of common zeroes of the ideal $\mathcal{I}$. Extending Newton–Puiseux's statement for an algebraic variety of any dimension and codimension is equivalent to answering the following question:

Question 3.1. Given an ideal $\mathcal{I} \subset \mathbb{K}[x_1, \ldots, x_{N+M}]$ such that the projection

$$\begin{array}{rccc} \pi: & \mathsf{V}(\mathcal{I}) & \to & \mathbb{K}^N \\ & (x_1, \ldots, x_{N+M}) & \mapsto & (x_1, \ldots, x_N) \end{array} \tag{3.1}$$

is dominant and of generic finite fiber.

Given $\omega \in \mathbb{R}^N$ of rationally independent coordinates. Can one always find an ω-positive rational cone σ and an M-tuple $\phi_1, \ldots, \phi_M$ of Puiseux series with exponents in some translate of σ such that

$$f(x_1, \ldots, x_N, \phi_1(x_1, \ldots, x_N), \ldots, \phi_M(x_1, \ldots, x_N)) = 0.$$

for any $f \in \mathcal{I}$?

If the projection is not dominant the problem has no solution. If the generic fiber is not finite an output will not be a parametrization.

To emphasize the roll of the projection, the indeterminates will be denoted by $x_1, \ldots, x_N, y_1, \ldots, y_M$. We will work with an ideal $\mathcal{I} \subset \mathbb{K}[x, y] := \mathbb{K}[x_1, \ldots, x_N, y_1, \ldots, y_M]$.

With this notation, the set of common zeroes of $\mathcal{I}$ is given by

$$\mathsf{V}(\mathcal{I}) = \{(x, y) \in \mathbb{K}^{N+M} \mid f(x, y) = 0, \forall f \in \mathcal{I}\}.$$

Definition 3.2. We will say that an ideal $\mathcal{I} \subset \mathbb{K}[x, y]$ is N*-admissible* when the Projection (3.1) is dominant and of finite generic fiber.

We will say that an algebraic variety $V \subset \mathbb{K}^{N+M}$ is N-admissible when its defining ideal is N-admissible.

Given an N-admissible ideal $\mathcal{I} \subset \mathbb{K}[x, y]$ and a vector $\omega \in \mathbb{R}^N$ of rationally independent coordinates, an M-tuple $\phi_1, \ldots, \phi_M$ solving Question 3.1 will be called an *ω-solution for $\mathcal{I}$*.

4. The Local Parametrizations Defined by the Series

Let $(\mathcal{C}, (0,0))$ be a complex plane algebraic curve singularity

$$(0,0) \in \mathcal{C} := \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\}$$

where f is a polynomial with complex coefficients.

Each output of the Newton–Puiseux method $y(x) = c_0 x^{\mu_0} + \cdots$ with $\mu_0 > 0$ is a convergent series in a neighborhood of 0. This series corresponds to a multi-valued mapping defined in a neighborhood of the origin $0 \in U \subset \mathbb{C}$

$$\begin{aligned} \varphi : U &\to \mathcal{C} \\ x &\mapsto (x, y(x)) \end{aligned}$$

that is compatible with the projection

$$\begin{aligned} \pi : \mathcal{C} &\to \mathbb{C} \\ (x,y) &\mapsto x, \end{aligned}$$

that is, $\pi \circ \varphi$ is the identity on U.

When $\mathcal{C}$ is analytically irreducible at $(0,0)$, the image $\varphi(U)$ is a neighborhood of the curve at $(0,0)$. The series φ contains all the topological and analytical information of $(\mathcal{C}, (0,0))$, and there are different ways to recover it (see for example [10, 33]).

If $\omega \in \mathbb{R}^N_{>0}$ has rationally independent positive coordinates, then the first orthant is ω-positive and we may suppose that the series of an output of the extended Newton–Puiseux method has exponents in a cone σ that contains the first orthant.

Let σ be a strongly convex cone that contains the first orthant. In [3] it is shown that (when it is not empty) the domain of convergence of a series with exponents in a strongly convex cone σ contains an open set W that has the origin as an accumulation point. Moreover, by the results of [3, Prop. 3.4], the intersection of a finite number of such domains is nonempty.

Let V be an N-admissible complex algebraic variety embedded in $\mathbb{C}^{N+M}$ and let $\omega \in \mathbb{R}^N_{>0}$ be of rationally independent coordinates. Each M-tuple of series $(y_1(\underline{x}), \ldots, y_M(\underline{x}))$ found solving Question 3.1 corresponds to a multi-valued function defined on an open set $W \subset \mathbb{C}^N$ that has the origin as accumulation point

$$\begin{aligned} \varphi : W &\to V \\ \underline{x} &\mapsto (\underline{x}, y_1(\underline{x}), \ldots, y_M(\underline{x})). \end{aligned}$$

The image $\varphi(W)$ contains an open set (a wedge) of V.

When

$$\omega \cdot \alpha > 0 \quad \text{for all } \alpha \in \bigcup_{j=1,\ldots,M} \mathcal{E}(y_j) \tag{4.1}$$

(when for each j, y_j does not have constant term and its set of exponents is contained in an ω-positive cone with apex at the origin) the open set has the origin an accumulation point.

Since analytic continuation is unique, when the origin is an analytically irreducible singularity, this parametrization contains all the topological and analytic information of the singularity.

5. The Field of ω-Positive Puiseux Series

In all that follows ω will be a vector in $\mathbb{R}^N$ of rationally independent coordinates. We will work with an algebraically closed field $\mathbb{K}$ of characteristic zero.

Given an N-admissible ideal we are looking for solutions in the ring of Puiseux series with exponents in some translate of an ω-positive cone σ. The cone σ may be different for different ideals. It is only natural to work with the infinite union of all these rings.

We say that a Puiseux series φ is *ω-positive* when there exists $\gamma \in \mathbb{Q}^N$ and an ω-positive cone σ such that $\mathcal{E}(x^\gamma \varphi) \subset \sigma$. The set of ω-positive Puiseux series was introduced in [7] where it was proved that it is an algebraically closed field. This field is called the *field of ω-positive Puiseux series* and will be denoted by S_ω.

The vector ω induces a total order on $\mathbb{Q}^N$

$$\alpha \leq \alpha' \Leftrightarrow \omega \cdot \alpha \leq \omega \cdot \alpha'.$$

This gives a natural way to choose the first term of a series in S_ω. This is the order we will use to compute the ω-solutions term-by-term. More precisely, the *order* of an element $\phi = \sum_\alpha c_\alpha x^\alpha$ in S_ω is

$$\mathrm{val}\,(\phi) := \min_{\alpha \in \mathcal{E}(f)} \omega \cdot \alpha$$

and its *first term* is

$$\mathrm{in}\,(\phi) := c_\alpha x^\alpha \quad \text{where } \omega \cdot \alpha = \mathrm{val}\,(\phi).$$

Set $\mathrm{val}\,(0) \colon = \infty$ and $\mathrm{in}\,(0) = 0$.

Remark 5.1. For $\phi, \phi' \in S_\omega$

(1) $\mathrm{val}\,(\phi + \phi') \geq \min\{\mathrm{val}\,(\phi), \mathrm{val}\,(\phi')\}$.
(2) $\mathrm{val}\,(\phi + \phi') \neq \min\{\mathrm{val}\,(\phi), \mathrm{val}\,(\phi')\}$ if and only if $\mathrm{val}\,(\phi) = \mathrm{val}\,(\phi')$ and $\mathrm{in}\,(\phi) + \mathrm{in}\,(\phi') = 0$.
(3) $\mathrm{val}\,(\phi \cdot \phi') = \mathrm{val}\,(\phi) + \mathrm{val}\,(\phi')$. Moreover $\mathrm{in}\,(\phi \cdot \phi') = \mathrm{in}\,(\phi) \cdot \mathrm{in}\,(\phi')$.
(4) $\mathrm{in}\,(\mathrm{in}\,(\phi)) = \mathrm{in}\,(\phi)$.
(5) $\mathrm{val}\,(\phi(x_1^r, \ldots, x_N^r)) = r\mathrm{val}\,(\phi(x_1, \ldots, x_N))$ for any $r \in \mathbb{Q}$.

A map from a ring into the reals with Properties (1) and (3) of the above remark is called a *valuation*.

The *first M-tuple* of an element $\varphi = (\varphi_1, \ldots, \varphi_M) \in S_\omega^M$ is the M-tuple of monomials

$$\mathrm{in}\,(\varphi) = (\mathrm{in}\,(\varphi_1), \ldots, \mathrm{in}\,(\varphi_M))$$

and the *order* of φ is the M-tuple of orders

$$\mathrm{val}\,(\varphi) = (\mathrm{val}\,(\varphi_1), \ldots, \mathrm{val}\,(\varphi_M)).$$

Remark 5.2. With the language introduced, Eq. (4.1) is equivalent to $\mathrm{val}\,(y) \in {\mathbb{R}_{>0}}^M$.

6. The Extended Ideal

Given an ideal $\mathcal{I} \subset \mathbb{K}[x, y]$, let $\mathcal{I}^* \subset \mathbb{K}[x^*, y]$ be the extension of $\mathcal{I}$ to $\mathbb{K}[x^*, y]$ via the natural inclusion.

We have

$$\mathbf{V}(\mathcal{I}^* \cap \mathbb{K}[x, y]) = \overline{\mathbf{V}(\mathcal{I}) \backslash \{x_1 \cdots x_N = 0\}}.$$

In regard to our question, it is then equivalent to work with ideals in $\mathbb{K}[x, y]$ or in $\mathbb{K}[x^*, y]$. For technical reasons we will start with ideals in $\mathbb{K}[x^*, y]$.

Definition 6.1. And ideal $\mathcal{I} \subset \mathbb{K}[x^*, y]$ is said to be *N-admissible* if the ideal $\mathcal{I} \cap \mathbb{K}[x, y] \subset \mathbb{K}[x, y]$ is N-admissible.

Given an ideal $\mathcal{I} \subset \mathbb{K}[x^*, y]$, let $\mathcal{I}^{\mathrm{e}} \subset S_\omega[y]$ be the extension of $\mathcal{I}$ via the natural inclusion

$$\mathbb{K}[x^*, y] = \mathbb{K}[x^*][y] \hookrightarrow S_\omega[y].$$

When $\mathcal{I}$ is an N-admissible ideal, $\mathrm{V}(\mathcal{I}^{\mathrm{e}})$ is a discrete subset of S_ω^M. By definition, $\phi \in \mathrm{V}(\mathcal{I}^{\mathrm{e}})$, if and only if ϕ is an ω-solution for $\mathcal{I}$.

Question 3.1 may be reformulated as follows:

Question 6.2 (Reformulation of Question 3.1). Given an N-admissible ideal $\mathcal{I} \subset \mathbb{K}[x^*, y]$, and a vector $\omega \in \mathbb{R}^N$ of rationally independent coordinates, find the (discrete) set of zeroes of in S_ω^M of the extended ideal $\mathcal{I}^{\mathrm{e}} \subset S_\omega[y]$.

A polynomial $f \in \mathbb{K}[x^*, y]$ may be considered a polynomial in $N + M$ variables with coefficients in $\mathbb{K}$, or a polynomial in M variables with coefficients in $\mathbb{K}[x^*] \subset S_\omega$. To cope with this fact we will use a slightly different notation:

* val and in refer to the field S_ω (Sec. 5)
* VAL_η, IN_η and $\mathcal{I}\mathrm{N}_\eta$ refer to the ring $S_\omega[y]$ (Secs. 7 and 8)
* $\mathsf{val}_{\omega,\eta}$, $\mathsf{In}_{\omega,\eta}$ and $\mathcal{I}\mathsf{n}_{\omega,\eta}$ refer to the ring $\mathbb{K}[x^*, y]$ (Sec. 10)

Given an ideal $\mathcal{I} \subset \mathbb{K}[x, y]$ the notation $\mathsf{V}(\mathcal{I})$ will stand for the set of common zeroes of $\mathcal{I}$ in $\mathbb{K}^{N+M}$. Given an ideal $\mathfrak{I} \subset S_\omega[y]$ the set of common zeroes of $\mathfrak{I}$ in S_ω^M will be denoted by $\mathrm{V}(\mathfrak{I})$.

7. Weighted Orders and Initial Parts in $S_\omega[y]$

The classical definition of weighted order and initial part considers as weights only vectors in $\mathbb{R}^M$. For technical reasons we need to extend the classical definition to weights in $(\mathbb{R} \cup \{\infty\})^M$.

A polynomial in M variables with coefficients in S_ω is written in the form

$$f = \sum_{\beta \in E \subset (\mathbb{Z}_{\geq 0})^M} \phi_\beta y^\beta, \quad \phi_\beta \in S_\omega, \quad y^\beta := {y_1}^{\beta_1} \cdots {y_M}^{\beta_M}$$

where E is a finite set.

Set $\infty \cdot a = \infty$ for $a \in \mathbb{R}^*$ and $\infty \cdot 0 = 0$. A vector $\eta \in (\mathbb{R} \cup \{\infty\})^M$ induces a (not necessarily total) order on the terms of f. The *η-order of f as an element of $S_\omega[y]$* is

$$\textsc{val}_\eta(f) := \min_{\phi_\beta \neq 0} (\mathrm{val}\, \phi_\beta + \eta \cdot \beta)$$

and, if $\textsc{val}_\eta f < \infty$, the *$\eta$-initial part of f as an element of $S_\omega[y]$* is

$$\textsc{In}_\eta(f) := \sum_{\mathrm{val}\, \phi_\beta + \eta \cdot \beta = \textsc{val}_\eta(f)} (\mathrm{in}\, \phi_\beta) y^\beta.$$

Example 7.1. Consider a binomial of the form $y^\beta - \phi$ we have

$$\textsc{val}_\eta(y^\beta - \phi) = \begin{cases} \eta \cdot \beta & \text{if } \eta \cdot \beta \leq \mathrm{val}\,(\phi) \\ \mathrm{val}\,(\phi) & \text{if } \mathrm{val}\,(\phi) \leq \eta \cdot \beta \end{cases}$$

and

$$\textsc{In}_\eta(y^\beta - \phi) = \begin{cases} y^\beta & \text{if } \eta \cdot \beta < \mathrm{val}\,(\phi), \\ y^\beta - \mathrm{in}\,(\phi) & \text{if } \eta \cdot \beta = \mathrm{val}\,(\phi), \\ \mathrm{in}\,(\phi) & \text{if } \mathrm{val}\,(\phi) < \eta \cdot \beta. \end{cases}$$

Lemma 7.2. *If $\varphi \in S_\omega^M$ is a zero of $f \in S_\omega[y]$, then $\mathrm{in}\,(\varphi)$ is a zero of $\textsc{In}_{\mathrm{val}\,\varphi}(f)$.*

Proof. Set $\eta := \mathrm{val}\,(\varphi)$. For $\phi \in S_\omega$ and $\beta \in \mathbb{Z}^M_{\geq 0}$ the following equality holds:

$$\mathrm{val}\,(\phi\varphi^\beta) \overset{4.1,\,3}{=} \mathrm{val}\,(\phi) + \eta \cdot \beta = \textsc{val}_\eta \phi y^\beta. \tag{7.1}$$

Suppose that $\varphi \in S_\omega^M$ is a zero of $f = \sum_\beta \phi_\beta y^\beta$, we have

$$\begin{aligned} \sum_\beta \phi_\beta \varphi^\beta = 0 &\overset{(6.1)+4.1,\,2}{\Longrightarrow} \sum_{\mathrm{val}\,(\phi_\beta\varphi^\beta) = \textsc{val}_\eta(f)} \mathrm{in}\, \left(\phi_\beta \varphi^\beta\right) = 0 \\ &\overset{4.1,\,3}{\Longrightarrow} \sum_{\mathrm{val}\,(\phi_\beta) + \eta\cdot\beta = \textsc{val}_\eta(f)} \mathrm{in}\, \phi_\beta (\mathrm{in}\, \varphi)^\beta = 0 \\ &\overset{\text{By definition}}{\Longrightarrow} \textsc{In}_\eta(f)(\mathrm{in}\,\varphi) = 0. \end{aligned}$$

□

For any $f \in S_\omega[y]$ and $\eta \in (\mathbb{R} \cup \{\infty\})^M$ we have $\mathrm{I{\scriptsize N}}_\eta(f) \in \mathbb{K}(x^{\frac{1}{K}})[y]$.

An element of the form cx^α with $c \in \mathbb{K}$ will be called a monomial.

Lemma 7.3. *Given $f \in S_\omega[y]$, let $\mathfrak{m}(x) \in \mathbb{K}(x^{\frac{1}{K}})^M$ be an M-tuple of monomials. Set $\eta := \operatorname{val}\mathfrak{m}$. We have $\mathrm{I{\scriptsize N}}_\eta(f) \in \mathbb{K}(x^{\frac{1}{K}})[y]$ and*

$$\mathrm{I{\scriptsize N}}_\eta(f(x, \mathfrak{m}(x))) = 0 \Leftrightarrow \mathrm{I{\scriptsize N}}_\eta(f(\underline{1}, \mathfrak{m}(\underline{1}))) = 0.$$

An M-tuple of monomials $\mathfrak{m} \in \mathbb{K}(x^{\frac{1}{K}})^M$ with $\operatorname{val}\mathfrak{m} = \eta$ is a zero of $\mathrm{I{\scriptsize N}}_\eta(f)$ as an element of $\mathbb{K}(x^{\frac{1}{K}})[y]$ if and only if $\mathfrak{m}(\underline{1})$ is a zero of $\mathrm{I{\scriptsize N}}_\eta(f(\underline{1}, y))$.

Proof. If $\operatorname{val}(\mathfrak{m}) = \eta$ then $\operatorname{val} x^\alpha m^\beta = \omega \cdot \alpha + \eta \cdot \beta$. Since ω has rationally independent coordinates, $x^\alpha \mathfrak{m}^\beta = ax^\gamma$ where $a = \mathfrak{m}(\underline{1})^\beta \in \mathbb{K}$ and γ is the unique vector in $\mathbb{Q}^N$ such that $w \cdot \gamma = \omega \cdot \alpha + \eta \cdot \beta$.

Now write

$$\mathrm{I{\scriptsize N}}_\eta(f) = \sum_{\omega\cdot\alpha+\eta\cdot\beta=\mathrm{VAL}_\eta(f)} a_{\alpha,\beta} x^\alpha y^\beta$$

we have $\sum_{\omega\cdot\alpha+\eta\cdot\beta=\mathrm{VAL}_\eta(f)} a_{\alpha,\beta} x^\alpha \mathfrak{m}^\beta = 0$ if and only if

$$\sum_{\omega\cdot\alpha+\eta\cdot\beta=\mathrm{VAL}_\eta(f)} a_{\alpha,\beta} \frac{x^\alpha \mathfrak{m}^\beta}{x^\gamma} = 0 \Leftrightarrow \sum_{\omega\cdot\alpha+\eta\cdot\beta=\mathrm{VAL}_\eta(f)} a_{\alpha,\beta} \mathfrak{m}(\underline{1})^\beta = 0.$$

□

8. Initial Ideals in $S_\omega[y]$

For an M-tuple $\eta \in (\mathbb{R} \cup \{\infty\})^M$ we will denote by $\Lambda(\eta)$ the set of subindices

$$\Lambda(\eta) := \{i \in \{1, \ldots, M\} \mid \eta_i \neq \infty\}.$$

Remark 8.1. $\mathrm{VAL}_\eta(f) = \infty$ if and only if f is in the ideal generated by $\{y_i \mid i \in \Lambda(\eta)^{\mathrm{C}}\}$.

Let $\mathfrak{I}$ be an ideal of $S_\omega[y]$ and $\eta \in (\mathbb{R} \cup \infty)^M$. The *$\eta$-initial part of* $\mathfrak{I}$ is the ideal of $S_\omega[y]$ generated by the η-initial parts of its elements:

$$\mathcal{I}\mathrm{N}_\eta \mathfrak{I} = \left\langle \{\mathrm{I{\scriptsize N}}_\eta f \mid f \in \mathfrak{I}\} \cup \{y_i\}_{i\in\Lambda(\eta)^{\mathrm{C}}} \right\rangle.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be ideals. We have

$$\mathcal{I}\mathrm{N}_\eta(\mathcal{A} \cap \mathcal{B}) \subset \mathcal{I}\mathrm{N}_\eta\mathcal{A} \cap \mathcal{I}\mathrm{N}_\eta\mathcal{B} \tag{8.1}$$

and

$$\mathcal{A} \subset \mathcal{B} \Rightarrow \mathcal{I}\mathrm{N}_\eta\mathcal{A} \subset \mathcal{I}\mathrm{N}_\eta\mathcal{B}. \tag{8.2}$$

Since $\mathcal{A} \cdot \mathcal{B} \subset \mathcal{A} \cap \mathcal{B}$ then

$$\mathcal{I}\mathrm{N}_\eta(\mathcal{A} \cdot \mathcal{B}) \subset \mathcal{I}\mathrm{N}_\eta(\mathcal{A} \cap \mathcal{B}) \tag{8.3}$$

and, since $\mathrm{I{\scriptsize N}}_\eta(a \cdot b) = \mathrm{I{\scriptsize N}}_\eta a \cdot \mathrm{I{\scriptsize N}}_\eta b$ then

$$\mathcal{I}\mathrm{N}_\eta\mathcal{A} \cdot \mathcal{I}\mathrm{N}_\eta\mathcal{B} \subset \mathcal{I}\mathrm{N}_\eta(\mathcal{A} \cdot \mathcal{B}). \tag{8.4}$$

Let A be an arbitrary set. For an M-tuple $y \in A^M$ and a subset $\Lambda \subset \{1, \ldots, M\}$ we will use the following notation:

$$y_\Lambda := (y_i)_{i \in \Lambda}. \tag{8.5}$$

Given two subsets $B \subset A$ and $C \subset A$ the set $B^\Lambda \times C^{\Lambda^C}$ is defined to be:

$$B^\Lambda \times C^{\Lambda^C} := \left\{ y \in A^M \mid y_\Lambda \in B^{\#\Lambda} \text{ and } y_{\Lambda^C} \in C^{\#\Lambda^C} \right\}.$$

We will use the notation $\mathbf{T}_\eta$ for the $\#\Lambda(\eta)$-dimensional torus

$$\mathbf{T}_\eta := (S_\omega^*)^{\Lambda(\eta)} \times \{0\}^{\Lambda(\eta)^C}.$$

Remark 8.2. $\mathrm{V}(\mathcal{I}\mathrm{N}_\eta \mathfrak{I}) \subset \overline{\mathbf{T}_\eta}$.

Example 8.3. For a point $\varphi = (\varphi_1, \ldots, \varphi_M) \in S_\omega^M$, denote by $\mathcal{J}_\varphi$ be the maximal ideal

$$\mathcal{J}_\varphi = \langle y_1 - \varphi_1, \ldots, y_M - \varphi_M \rangle \subset S_\omega[y].$$

Given $\eta \in (\mathbb{R} \cup \{\infty\})^M$ we have

$$\begin{cases} \mathcal{I}\mathrm{N}_\eta \mathcal{J}_\varphi = S_\omega[y] & \text{if } \mathrm{val}\,(\varphi_i) < \eta_i \quad \text{for some } i \in \{1, \ldots, M\}, \\ y_i \in \mathcal{I}\mathrm{N}_\eta \mathcal{J}_\varphi & \text{if } \mathrm{val}\,(\varphi_i) > \eta_i, \\ \mathcal{I}\mathrm{N}_\eta \mathcal{J}_\varphi = \mathcal{J}_{\mathrm{in}\,(\varphi)} & \text{if } \mathrm{val}\,(\varphi) = \eta. \end{cases}$$

The first two points and the inclusion $\mathcal{I}\mathrm{N}_\eta \mathcal{J}_\varphi \supset \mathcal{J}_{\mathrm{in}\,(\varphi)}$ in the third are direct consequence of Example 7.1. The inclusion $\mathcal{I}\mathrm{N}_\eta \mathcal{J}_\varphi \subset \mathcal{J}_{\mathrm{in}\,(\varphi)}$ in the third point is equivalent to $\mathrm{in}\,(\varphi) \in \mathrm{V}\,(\mathcal{I}\mathrm{N}_\eta \mathcal{J}_\varphi)$ which follows from Lemma 7.2. And then

$$\mathbf{T}_\eta \cap \mathrm{V}\,(\mathcal{I}\mathrm{N}_\eta \mathcal{J}_\varphi) = \begin{cases} \emptyset & \text{if } \mathrm{val}\,(\varphi) \neq \eta, \\ \mathrm{in}\,(\varphi) & \text{if } \mathrm{val}\,(\varphi) = \eta. \end{cases} \tag{8.6}$$

9. Zeroes of the Initial Ideal in $S_\omega[y]$

Now we are ready to characterize the first terms of the zeroes of the ideal $\mathfrak{I} \subset S_\omega[y]$. The following is the key proposition to extend Point 1 of Newton–Puiseux's method.

Proposition 9.1. *Let $\mathfrak{I} \subset S_\omega[y]$ be an ideal with a finite number of zeroes and let η be an M-tuple in $(\mathbb{R} \cup \{\infty\})^M$.*

An element $\phi \in \mathbf{T}_\eta$ is a zero of the ideal $\mathcal{I}\mathrm{N}_\eta \mathfrak{I}$ if and only if $\mathrm{val}\,(\phi) = \eta$ and there exists $\varphi \in \mathrm{V}(\mathfrak{I})$ such that $\mathrm{in}\,(\varphi) = \phi$.

Proof. Given $\varphi = (\varphi_1, \ldots, \varphi_M) \in S_\omega^M$ consider the ideal

$$\mathcal{J}_\varphi = \langle y_1 - \varphi_1, \ldots, y_M - \varphi_M \rangle \subset S_\omega[y].$$

Set $H := \mathrm{V}(\mathfrak{I})$. By hypothesis H is a finite subset of S_ω^M. By the *Nullstellensatz* there exists $k \in \mathbb{N}$ such that

$$\left(\bigcap_{\varphi \in H} \mathcal{J}_\varphi \right)^k \subset \mathfrak{I} \subset \bigcap_{\varphi \in H} \mathcal{J}_\varphi.$$

By (8.2) and (8.4) we have

$$\left(\mathcal{I}\mathrm{N}_\eta \bigcap_{\varphi\in H} \mathcal{J}_\varphi\right)^k \subset \mathcal{I}\mathrm{N}_\eta \mathfrak{I} \subset \mathcal{I}\mathrm{N}_\eta \bigcap_{\varphi\in H} \mathcal{J}_\varphi. \tag{9.1}$$

On the other hand

$$\prod_{\varphi\in H} \mathcal{I}\mathrm{N}_\eta \mathcal{J}_\varphi \overset{(8.4)+(8.3)}{\subset} \mathcal{I}\mathrm{N}_\eta \bigcap_{\varphi\in H} \mathcal{J}_\varphi \overset{(8.1)}{\subset} \bigcap_{\varphi\in H} \mathcal{I}\mathrm{N}_\eta \mathcal{J}_\varphi. \tag{9.2}$$

The zeroes of the right-hand and left-hand side of Eq. (9.2) coincide. Therefore

$$\mathrm{V}\left(\mathcal{I}\mathrm{N}_\eta \bigcap_{\varphi\in H} \mathcal{J}_\varphi\right) \overset{(9.2)}{=} \mathrm{V}\left(\bigcap_{\varphi\in H} \mathcal{I}\mathrm{N}_\eta \mathcal{J}_\varphi\right) = \bigcup_{\varphi\in H} \mathrm{V}\left(\mathcal{I}\mathrm{N}_\eta \mathcal{J}_\varphi\right) \tag{9.3}$$

and then, by (8.6),

$$\mathbf{T}_\eta \cap \mathrm{V}\left(\mathcal{I}\mathrm{N}_\eta \bigcap_{\varphi\in H} \mathcal{J}_\varphi\right) = \{\mathrm{in}\,\varphi \mid \varphi\in H, \mathrm{val}\,(\varphi)=\eta\}. \tag{9.4}$$

The conclusion follows directly from (9.1) and (9.4). □

Corollary 9.2. *Let $\mathfrak{I}\subset S_\omega[y]$ be an ideal with a finite number of zeroes and let η be an M-tuple in $(\mathbb{R}\cup\{\infty\})^M$.*

The zeroes of the ideal $\mathcal{I}\mathrm{N}_\eta\mathfrak{I}$ in $\mathbf{T}_\eta$ are M-tuples of monomials of order η.

10. Initial Ideals in $\mathbb{K}[x^*, y]$

A polynomial $f\in\mathbb{K}[x^*,y]=\mathbb{K}[x^*][y]$ is an expression of the form:

$$\sum_{(\alpha,\beta)\in(\mathbb{Z}^N\times\mathbb{Z}_{\geq 0})^M} a_{(\alpha,\beta)} x^\alpha y^\beta \quad a_{(\alpha,\beta)}\in\mathbb{K}.$$

The η-order of $f\in\mathbb{K}[x^*][y]$ as an element of $S_\omega[y]$ is called the (ω,η)-*order* of f. That is

$$\mathsf{val}_{\omega,\eta}(f) := \min_{a_{(\alpha,\beta)}\neq 0} \omega\cdot\alpha + \eta\cdot\beta.$$

And the η-initial part of f as an element of $S_\omega[y]$ is called the (ω,η)-*initial part* f. That is: if $\mathsf{val}_{\omega,\eta} f < \infty$, then

$$\mathsf{In}_{\omega,\eta}(f) := \sum_{\omega\cdot\alpha+\eta\cdot\beta=\mathsf{val}_{\omega,\eta}(f)} a_{(\alpha,\beta)} x^\alpha y^\beta$$

and, if $\mathsf{val}_{\omega,\eta}(f)=\infty$, $\mathsf{In}_{\omega,\eta}(f)=0$.

Given an ideal $\mathcal{I}\subset\mathbb{K}[x^*][y]$ the (ω,η)-*initial ideal of* $\mathcal{I}$ is the ideal

$$\mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I} := \left\langle \{\mathsf{In}_{\omega,\eta}(f) \mid f\in\mathcal{I}\} \cup \{y_i\}_{i\in\Lambda(\eta)^{\mathrm{C}}} \right\rangle \subset \mathbb{K}[x^*][y].$$

Given an ideal $\mathcal{I} \subset \mathbb{K}[x^*][y]$ let $\mathcal{I}^{\mathrm{e}}$ denote the extension of $\mathcal{I}$ to $S_\omega[y]$.

Proposition 10.1. *Given $\eta \in (\mathbb{R} \cup \{\infty\})^M$ and an ideal $\mathcal{I} \subset \mathbb{K}[x^*, y]$ we have that*

$$(\mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I})^{\mathrm{e}} = \mathcal{I}\mathrm{N}_\eta\mathcal{I}^{\mathrm{e}}.$$

Proof. The inclusion $(\mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I})^{\mathrm{e}} \subset \mathcal{I}\mathrm{N}_\eta\mathcal{I}^{\mathrm{e}}$ is straightforward.

Now, $h \in \{\mathrm{I{\small N}}_\eta f \mid f \in \mathcal{I}^{\mathrm{e}}\}$ if and only if $h = \mathrm{I{\small N}}_\eta(\sum_{i=1}^r g_i P_i)$ where $g_i \in S_\omega[y]$ and $P_i \in \mathcal{I}$. Let $\Lambda = \{i \mid \mathrm{VAL}_\eta(g_i P_i) = \min_{j=1,\dots,r} \mathrm{VAL}_\eta(g_j P_j)\}$. If $\sum_{i\in\Lambda} \mathrm{I{\small N}}_\eta(g_i P_i) = 0$ then $h = \mathrm{I{\small N}}_\eta(\sum_{i=1}^r g_i')P_i$, where $g_i' = g_i - \mathrm{I{\small N}}_\eta(g_i)$ for $i \in \Lambda$ and $g_i' = g_i$ otherwise. Then we can suppose that $\sum_{i\in\Lambda} \mathrm{I{\small N}}_\eta(g_i P_i) \neq 0$. Then $h = \sum_{i\in\Lambda} \mathrm{I{\small N}}_\eta(g_i P_i) = \sum_{i\in\Lambda} \mathrm{I{\small N}}_\eta(g_i)\mathrm{I{\small N}}_\eta(P_i)$ is an element of $(\mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I})^{\mathrm{e}}$. □

We will be using the following technical result:

Lemma 10.2. *Given $\eta \in (\mathbb{R} \cup \{\infty\})^M$ and an ideal $\mathcal{I} \subset \mathbb{K}[x^*][y]$ we have that*

$$\mathcal{I}\mathsf{n}_{\omega,\eta}(\mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}) = \mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}.$$

Proof. It is enough to see that for any $g \in \mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}$ there exists $f \in \mathcal{I}$ such that $\mathcal{I}\mathsf{n}_{\omega,\eta} g = \mathcal{I}\mathsf{n}_{\omega,\eta}(f)$:

Given $p = \sum_{i=1}^d a_i x^{\alpha_i} y^{\beta_i} \in \mathbb{K}[x^*][y]$ and $h \in \mathcal{I}$, we have

$$p\mathcal{I}\mathsf{n}_{\omega,\eta}(h) = \sum_{i=1}^d a_i x^{\alpha_i} y^{\beta_i} \mathcal{I}\mathsf{n}_{\omega,\eta}(h) = \sum_{i=1}^d \mathcal{I}\mathsf{n}_{\omega,\eta}(a_i x^{\alpha_i} y^{\beta_i} h).$$

Then the product $p\mathcal{I}\mathsf{n}_{\omega,\eta}(h)$ is a sum of (ω,η)-initial parts of elements of $\mathcal{I}$.

Therefore, $g \in \mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}$ if and only if there exists $f_1, \dots, f_r \in \mathcal{I}$, such that $g = \sum_{i\in\{1,\dots,r\}} \mathcal{I}\mathsf{n}_{\omega,\eta}(f_i)$. The f_i's may be chosen such that $\sum_{i\in\Lambda} \mathcal{I}\mathsf{n}_{\omega,\eta}(f_i) \neq 0$ for all nonempty $\Lambda \subset \{1,\dots,r\}$. Let $m = \min_{i\in\{1,\dots,r\}} \mathsf{val}_{\omega,\eta}(f_i)$. Since $\sum_{\mathsf{val}_{\omega,\eta}(f_i)=m} \mathcal{I}\mathsf{n}_{\omega,\eta}(f_i) \neq 0$ then $\mathcal{I}\mathsf{n}_{\omega,\eta}(g) = \sum_{\mathsf{val}_{\omega,\eta}(f_i)=m} \mathcal{I}\mathsf{n}_{\omega,\eta}(f_i)$, and then $f := \sum_{\mathsf{val}_{\omega,\eta}(f_i)=m} f_i$ has the property we were looking for. □

Proposition 10.3. *Let $\omega \in \mathbb{R}^N$ be of rationally independent coordinates, let η be an M-tuple in $(\mathbb{R} \cup \{\infty\})^M$ and let $\mathcal{I} \subset \mathbb{K}[x^*, y]$ be an N-admissible ideal.*

An element $\phi \in \mathbf{T}_\eta$ is an ω-solution for the ideal $\mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}$ if and only if $\mathrm{val}(\phi) = \eta$ *and there exists $\varphi \in S_\omega^M$, an ω-solution for $\mathcal{I}$, such that* $\mathrm{in}(\varphi) = \phi$.

Proof. This is a direct consequence of Propositions 10.1 and 9.1. □

11. The Tropical Variety

The tropical variety of a polynomial $f \in \mathbb{K}[x^*, y]$ is the $(N + M - 1)$-skeleton of the normal fan of its Newton polyhedron. The tropical variety of an ideal $\mathcal{I} \subset \mathbb{K}[x^*, y]$ is the intersection of the tropical varieties of the elements of $\mathcal{I}$.

More precisely, the *tropical variety* of $\mathcal{I}$ is the set

$$\tau(\mathcal{I}) := \{(\omega, \eta) \in \mathbb{R}^N \times (\mathbb{R} \cup \{\infty\})^M \mid \mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I} \cap \mathbb{K}[x^*, y_{\Lambda(\eta)}] \\ \text{does not have a monomial}\}.$$

Tropical varieties have become an important tool for solving problems in algebraic geometry (see for example [14, 18, 30]). In [9, 17] algorithms to compute tropical varieties are described.

Proposition 11.1. *Let $\mathcal{I}$ be an ideal of $\mathbb{K}[x^*, y]$. Given $\eta \in (\mathbb{R} \cup \{\infty\})^M$ the ideal $\mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}$ has an ω-solution in $\mathbf{T}_\eta$ if and only if (ω, η) is in the tropical variety of $\mathcal{I}$.*

Proof. Suppose that $\varphi \in \mathbf{T}_\eta$ is an ω-solution of $\mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}$ and that $cx^\alpha y^\beta \in \mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I} \cap \mathbb{K}[x^*, y_{\Lambda(\eta)}]$. We have $x^\alpha \varphi^\beta = 0$ and then, $\varphi_i = 0$ for some $i \in \Lambda(\eta)$ which gives a contradiction.

Let $\mathbb{K}(x)$ denote the field of fractions of $\mathbb{K}[x]$ and let $\widetilde{\mathcal{I}}$ be the extension of $\mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}$ to $\mathbb{K}(x)[y]$ via the natural inclusion $\mathbb{K}[x, y] = \mathbb{K}[x][y] \subset \mathbb{K}(x)[y]$.

Since S_ω contains the algebraic closure of $\mathbb{K}(x)$, the zeroes of $\mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}$ are the algebraic zeroes of $\tilde{\mathcal{I}}$. Suppose that $\mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}$ does not have zero in $\mathbf{T}_\eta$ then, by Remark 8.2

$$\mathrm{V}\left(\mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}\right) \subset \overline{\mathbf{T}_\eta} \backslash \mathbf{T}_\eta.$$

Let v be the only element of $\{1\}^{\Lambda(\eta)} \times \{0\}^{\Lambda(\eta)^{\mathrm{C}}}$. The monomial y^v vanishes in all the algebraic zeroes of $\tilde{\mathcal{I}}$. By the *Nullstellensatz*, there exists $k \in \mathbb{N}$ such that y^{kv} belongs to $\tilde{\mathcal{I}}$.

Now y^{kv} belongs to $\tilde{\mathcal{I}}$ if and only if there exists $h_1, \ldots, h_r \in \mathbb{K}[x] \backslash \{0\}$ and $f_1, \ldots, f_r \in \mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}$ such that

$$y^{kv} = \sum_{i=1}^{r} \frac{1}{h_i} f_i \Rightarrow \left(\prod_{i=1}^{r} h_i(x)\right) y^{kv} = \sum_{i=1}^{r} \left(\prod_{\substack{j=1 \\ i \neq j}}^{r} h_j\right) f_i \in \mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}.$$

Then, by Lemma 10.2, $\mathsf{In}_{\omega,\eta}((\prod_{i=1}^r h_i(x)) y^{kv}) \in \mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}$ and $\mathsf{In}_{\omega,\eta}((\prod_{i=1}^r h_i(x)) y^{kv}) = \mathrm{in}\,(\prod_{i=1}^r h_i(x)) y^{kv}$ is a monomial, and the result is proved. □

As a direct consequence of of Propositions 10.3 and 11.1 we have the extension Point 1 of Newton–Puiseux's method.

Theorem 11.2. *Let $\mathcal{I} \subset \mathbb{K}[x^*, y]$ an N-admissible ideal and let $\omega \in \mathbb{R}^N$ be of rationally independent coordinates.*

$\phi = (c_1 x^{\alpha^{(1)}}, \ldots, c_M x^{\alpha^{(M)}})$ is the first term of an ω-solution of $\mathcal{I}$ if and only if

- *$(\omega, \mathrm{val}\,\phi)$ is in the tropical variety of $\mathcal{I}$.*
- *ϕ is an ω-solution of the ideal $\mathcal{I}\mathsf{n}_{\omega,\mathrm{val}\,\phi}\mathcal{I}$.*

These statements recall Kapranov's theorem. Kapranov's theorem was proved for hypersurfaces in [12] and the first published proof for an arbitrary ideal may be found in [11]. There are several constructive proofs [20, 27] in the literature. An other proof of Proposition 9.1 could probably be done by using Proposition 10.1, showing that $(\omega, \eta) \in \mathcal{T}(\mathcal{I})$ if and only if $\eta \in \mathcal{T}(\mathcal{I}^e)$, and checking each step of one of the constructive proofs.

12. ω-Set

At this stage we need to introduce some more notation: Given an $M \times N$ matrix

$$\Gamma = \begin{pmatrix} \Gamma_{1,1} & \ldots & \Gamma_{1,N} \\ \vdots & & \vdots \\ \Gamma_{M,1} & \ldots & \Gamma_{M,N} \end{pmatrix}.$$

The ith row will be denoted by $\Gamma_{i,*} := (\Gamma_{i,1}, \ldots, \Gamma_{i,N})$ and

$$x^{\Gamma} := \begin{pmatrix} x^{\Gamma_{1,*}} \\ \vdots \\ x^{\Gamma_{M,*}} \end{pmatrix}.$$

In particular, if $I \in \mathcal{M}_{N \times N}$ is the identity, then $x^{\frac{1}{k}I} = (x_1{}^{\frac{1}{k}}, \ldots, x_N{}^{\frac{1}{k}})$.

An M-tuple of monomials $\mathfrak{m} \in \mathbb{K}(x^{\frac{1}{K}I})^M$ can be written as an entrywise product

$$\mathfrak{m} = x^{\Gamma} c = \begin{pmatrix} c_1 x^{\Gamma_{1,*}} \\ \vdots \\ c_M x^{\Gamma_{M,*}} \end{pmatrix}.$$

Given an M-tuple of monomials $\mathfrak{m} \in \mathbb{K}(x^{\frac{1}{K}I})^M$ the *defining data of* $\mathfrak{m}$ is the three-tuple

$$D(\mathfrak{m}) = \{\operatorname{val} \mathfrak{m}, \Gamma, \mathfrak{m}(\underline{1})\}$$

where $\Gamma \in \mathcal{M}_{M \times N}(\mathbb{Q} \cup \{\infty\})$ is the unique matrix such that $\omega \cdot \Gamma^T = \operatorname{val} \mathbf{m}$ and $\Gamma_{i,*} = \underline{\infty}$ for all $i \in \Lambda(\operatorname{val} \mathbf{m})^{\mathrm{C}}$.

Example 12.1. If $\omega = (1, \sqrt{2})$ and

$$\mathfrak{m} = \begin{pmatrix} 3x_1{}^3 \\ 7x_1{}^2 x_2 \\ 0 \end{pmatrix}$$

then

$$D(\mathfrak{m}) = \left\{ (3, 2+\sqrt{2}, \infty), \begin{pmatrix} 3 & 0 \\ 2 & 1 \\ \infty & \infty \end{pmatrix}, (3, 7, 0) \right\}.$$

Definition 12.2. An *ω-set* is a three-tuple $\{\eta, \Gamma, c\}$ where

$$\eta \in (\mathbb{R} \cup \{\infty\})^M, \quad \Gamma \in \mathcal{M}_{M\times N}(\mathbb{Q} \cup \{\infty\}), \quad c \in \mathbb{K}^M \tag{12.1}$$

and

- $\omega \cdot \Gamma^T = \eta$,
- $\Gamma_{i,*} = \underline{\infty}$ for all $i \in \Lambda(\eta)^{\mathrm{C}}$,
- $c \in {\mathbb{K}^*}^{\Lambda(\eta)} \times \{0\}^{\Lambda(\eta)^{\mathrm{C}}}$.

Given an ω-set $D = \{\eta, \Gamma, c\}$ the *M-tuple defined by D* is the M-tuple of monomials

$$\mathfrak{M}_D := x^{\Gamma} c.$$

We have

$$\mathfrak{M}_{\{\eta,\Gamma,c\}}(x^{rI}) = \mathfrak{M}_{\{r\eta,r\Gamma,c\}}(x).$$

Remark 12.3. $\mathfrak{m} = \mathfrak{M}_{D(\mathfrak{m})}$ and $D(\mathfrak{M}_D) = D$.

13. Starting ω-Set for $\mathcal{I}$

Given an N-admissible ideal $\mathcal{I} \subset \mathbb{K}[x^*, y]$. *A starting ω-set for $\mathcal{I}$* is an ω-set $D = \{\eta, \Gamma, c\}$ such that:

- the vector (ω, η) is in the tropical variety of $\mathcal{I}$,
- c is a zero of the system $\{f(\underline{1}, y) = 0 \mid f \in \mathrm{In}_{\omega,\eta}\mathcal{I}\}$.

Example 13.1. Let $\mathcal{I} = \langle x_1 + y_1 - y_2 + y_1y_2 + y_3, x_2 - y_1 + y_2 + 2y_1y_2, y_3\rangle$. For $\omega = (1, \sqrt{2})$ there are two possible starting ω-sets

$$D1 = \left\{(1,1,\infty), \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ \infty & \infty \end{pmatrix}, (1,1,0)\right\}$$

and

$$D2 = \left\{(0,0,\infty), \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \infty & \infty \end{pmatrix}, \left(\frac{1}{3}, \frac{1}{5}, 0\right)\right\}.$$

and

$$\mathfrak{M}_{D1}(x) = \begin{pmatrix} x_1 \\ x_1 \\ 0 \end{pmatrix}, \quad \mathfrak{M}_{D1}(x^{\frac{1}{3}I}) = \begin{pmatrix} {x_1}^{\frac{1}{3}} \\ {x_1}^{\frac{1}{3}} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{M}_{D2}(x) = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{5} \\ 0 \end{pmatrix}.$$

Proposition 13.2. *The ω-set $D = \{\eta, \Gamma, c\}$ is a starting ω-set for $\mathcal{I}$ if and only if $\mathfrak{M}_D$ is an ω-solution of $\mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}$.*

Moreover all the ω-solutions of $\mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}$ in $\mathbf{T}_\eta$ are of the form $\mathfrak{M}_D$ where $D = \{\eta, \Gamma, c\}$ is a starting ω-set for $\mathcal{I}$.

Proof. That $\mathfrak{M}_D$ is an ω-solution of $\mathcal{I}\mathsf{n}_{\omega,\eta}\mathcal{I}$ when D is a starting ω-set is a direct consequence of Lemma 7.3. The other implication is consequence of Proposition 11.1 and Lemma 7.3.

The last sentence follows from Corollary 9.2. □

14. The Ideal $\mathcal{I}_D$

Given a matrix $\Gamma \in \mathcal{M}_{M\times N}(\mathbb{Q} \cup \{\infty\})$ the minimum common multiple of the denominators of its entries will be denoted by $\mathbf{d}\Gamma$. That is

$$\mathbf{d}\Gamma := \min\{k \in \mathbb{N} \mid \Gamma \in \mathcal{M}_{N\times M}(\mathbb{Z} \cup \{\infty\})\}.$$

Given an ω-set $D = \{\eta, \Gamma, c\}$, we will denote by $\mathcal{I}_D$ the ideal in $\mathbb{K}[x^*, y]$ given by

$$\mathcal{I}_D := \left\langle \{f(x^{\mathbf{d}\Gamma I}, y + \mathfrak{M}_D(x^{\mathbf{d}\Gamma I})) \mid f \in \mathcal{I}\} \right\rangle \subset \mathbb{K}[x^*, y].$$

Remark 14.1. A series $\phi \in S_\omega^M$ is an ω-solution of $\mathcal{I}$ if and only if the series $\tilde{\phi} := \phi(x^{\mathbf{d}\Gamma I}) - \mathfrak{M}_D(x^{\mathbf{d}\Gamma I})$ is an ω-solution of $\mathcal{I}_D$.

Proposition 14.2. *Let $D = \{\eta, \Gamma, c\}$ be a starting ω-set for an ideal $\mathcal{I}$. There exists $\tilde{\eta} \in (\mathbb{R} \cup \{\infty\})^M$ such that $(\omega, \tilde{\eta}) \in \tau(\mathcal{I}_D)$ and $\tilde{\eta}_{\Lambda(\tilde{\eta})} > \mathbf{d}\Gamma\eta_{\Lambda(\tilde{\eta})}$ coordinatewise.*

Proof. By Propositions 9.1 and 13.2, $\mathfrak{M}_D$ is the first term of at least one ω-solution of $\mathcal{I}$. Say

$$\phi = \mathfrak{M}_D + \tilde{\phi} \in \mathrm{V}(\mathcal{I}), \quad \tilde{\phi} = \begin{pmatrix} \tilde{\phi}_1 \\ \vdots \\ \tilde{\phi}_M \end{pmatrix} \in S_\omega^M,$$

with $\mathrm{val}\,(\tilde{\phi}_i) > \omega \cdot \Gamma_{i,*} = \eta_i$ when $\tilde{\phi}_i \neq 0$.

Set

$$\tilde{\eta} := \mathbf{d}\Gamma \mathrm{val}\,(\tilde{\phi}) \overset{\text{Remarks } 5.1 \text{ and } 5.5}{=} \mathrm{val}\,(\tilde{\phi})(x^{\mathbf{d}\Gamma I})$$

then $\tilde{\eta}_i > \mathbf{d}\Gamma\eta_i$ for all $i \in \Lambda(\tilde{\eta})$.

By Remark 14.1 $\tilde{\phi}(x^{\mathbf{d}\Gamma I})$ is an ω-solution of $\mathcal{I}_D$. Then, by Theorem 9.1, in $\tilde{\phi}$ is an ω-solution of $\mathcal{I}\mathsf{n}_{\omega,\tilde{\eta}}\mathcal{I}_D$. Finally, by Proposition 11.1, $(\omega, \tilde{\eta})$ is in the tropical variety of $\mathcal{I}_D$. □

Proposition 14.3. *Let $D = \{\eta, \Gamma, c\}$ be a starting ω-set for an ideal $\mathcal{I}$. There exists a starting ω-set $D' = \{\eta', \Gamma', c'\}$ for $\mathcal{I}_D$ such that ${\eta'}_{\Lambda(\eta')} > \mathbf{d}\Gamma\eta_{\Lambda(\eta')}$ coordinatewise.*

Proof. By Proposition 14.2, there exists $\eta' \in (\mathbb{R} \cup \{\infty\})^M$ such that ${\eta'}_{\Lambda(\eta')} > \mathbf{d}\Gamma\eta_{\Lambda(\eta')}$ coordinatewise and $(\omega, \eta') \in \tau(\mathcal{I}_D)$. By Proposition 11.1, the ideal $\mathcal{I}\mathsf{n}_{\omega,\eta'}\mathcal{I}_D$ has an ω-solution ϕ in $\mathbf{T}_{\eta'}$. By Proposition 13.2, $\phi = \mathfrak{M}_{D'}$ where $D' = \{\eta', \Gamma', c'\}$ is a starting ω-set for $\mathcal{I}_D$. $\square$

15. ω-Sequences

Given an M-tuple $\phi \in S_\omega^M$ define inductively $\{\phi^{(i)}\}_{i=0}^\infty$, and $\{D^{(i)}\}_{i=0}^\infty$ by:

- For $i = 0$,
  - — $\phi^{(0)} := \phi$
  - — $D^{(0)}$ is the defining data of in ϕ.
- For $i > 0$,
  - — $\phi^{(i)} := \phi^{(i-1)}(x^{\mathbf{d}\Gamma^{(i-1)}I}) - \mathfrak{M}_{D^{(i-1)}}(x^{\mathbf{d}\Gamma^{(i-1)}I})$
  - — $D^{(i)}$ is the defining data of the M-tuple of monomials in $\phi^{(i)}$ ($D^{(i)} := D(\mathrm{in}\,\phi^{(i)})$).

The sequence above

$$\mathbf{seq}(\phi) := \{D^{(i)}\}_{i=0}^\infty$$

will be called *the defining data sequence for* ϕ.

Remark 15.1. For any $\phi \in S_\omega^M$. If $\mathbf{seq}(\phi) := \{\eta^{(i)}, \Gamma^{(i)}, c^{(i)}\}_{i=0}^\infty$ then ${\eta^{(i)}}_{\Lambda(\eta^{(i)})} > \mathbf{d}\Gamma^{(i-1)}{\eta^{(i-1)}}_{\Lambda(\eta^{(i)})}$ coordinatewise.

Given a sequence $S = \{D^{(i)}\}_{i=0\cdots K} = \{\eta^{(i)}, \Gamma^{(i)}, c^{(i)}\}_{i=0\cdots K}$, with $K \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Set

- $\mathcal{I}^{(0)} = \mathcal{I}$,
- $\mathcal{I}^{(i)} = \mathcal{I}^{(i-1)}_{D^{(i-i)}}$ for $i \in \{1, \ldots, K\}$.

S is called an *ω-sequence for* $\mathcal{I}$ if and only if for $i \in \{0, \ldots, K\}$

- $D^{(i)} = \{\eta^{(i)}, \Gamma^{(i)}, c^{(i)}\}$ is a starting ω-set for $\mathcal{I}^{(i)}$,
- ${\eta^{(i)}}_{\Lambda(\eta^{(i)})} > \mathbf{d}\Gamma^{(i-1)}{\eta^{i-1}}_{\Lambda(\eta^{(i)})}$ coordinatewise.

As a corollary to Proposition 14.3 we have:

Corollary 15.2. *Let $\{D^{(i)}\}_{i=0\cdots K}$ be an ω-sequence for $\mathcal{I}$. For any $K' \in \{K+1, \ldots, \infty\}$ there exists a sequence $\{D^{(i)}\}_{i=K+1\cdots K'}$ such that $\{D^{(i)}\}_{i=0\cdots K'}$ is an ω-sequence for $\mathcal{I}$.*

Proposition 15.3. *If ϕ is an ω-solution of $\mathcal{I}$ then* $\mathbf{seq}(\phi)$ *is an ω-sequence for $\mathcal{I}$.*

Proof. This is a direct consequence of Remarks 15.1 and 14.1. $\square$

16. The Solutions

Given an ω-sequence $S = \{D^{(i)}\}_{i=0\cdots K}$ for $\mathcal{I}$, with $D^{(i)} = \{\eta^{(i)}, \Gamma^{(i)}, c^{(i)}\}$ set

$$r^{(0)} := 1 \quad \text{and} \quad r^{(i)} := \frac{1}{\prod_{j=0}^{i-1} \mathbf{d}\Gamma^{(j)}} \quad \text{for } i > 0. \tag{16.1}$$

The *series defined by* S is the series

$$\mathbf{ser}(S) := \sum_{i=0}^{K} \mathfrak{M}_{D^{(i)}}(x^{r^{(i)}I}).$$

The following theorem is the extension of Point 2 of Newton–Puiseux's method:

Theorem 16.1. *If $S = \{D^{(i)}\}_{i=0}^{\infty}$ is an ω-sequence for $\mathcal{I}$ then* $\mathbf{ser}(S)$ *is an ω-solution of $\mathcal{I}$.*

Proof. Let $S = \{D^{(i)}\}_{i=1}^{\infty}$ be an ω-sequence for $\mathcal{I}$. Where $D^{(i)} = \{\eta^{(i)}, \Gamma^{(i)}, c^{(i)}\}$. Let $\{\mathcal{I}^{(i)}\}_{i=0}^{\infty}$ be defined by $\mathcal{I}^{(0)} := \mathcal{I}$ and $\mathcal{I}^{(i)} := \mathcal{I}^{(i-1)}_{D^{(i-1)}}$. We have that $D^{(i)}$ is a starting ω-set for $\mathcal{I}^{(i)}$.

By Proposition 11.1 for each $i \in \mathbb{N}$ there exists $\phi^{(i)} \in S_\omega^M$ such that $\operatorname{val} \phi^{(i)} = \eta^{(i)}$ and $\phi^{(i)} \in \mathrm{V}(\mathcal{I}^{(i)})$. Set $\{r^{(i)}\}_{i=0}^{\infty}$ as in (16.1) and $\phi^{(\tilde{i})} := \mathbf{ser}(\{D^{(i)}\}_{j=0}^{i-1}) + \phi^{(i)}(x^{r^{(i)}I})$. For each $i \in \mathbb{N}$, by Remark 14.1, $\phi^{(\tilde{i})} \in \mathrm{V}(\mathcal{I}) \subset S_\omega^M$. Since there are only a finite number of zeroes there exists $K \in \mathbb{N}$ such that $\phi^{(\tilde{i})} = \phi^{(\tilde{K})}$ for all $i > K$. Then

$$\mathbf{ser}(S) = \phi^{(\tilde{K})} \in \mathrm{V}(\mathcal{I}) \subset S_\omega^M. \qquad \square$$

Theorem 16.1 together with Proposition 15.3 gives:

Corollary 16.2 (Answer to Question 3.1). *Let $\mathcal{I} \subset \mathbb{K}[x^*, y]$ be an N-admissible ideal and let $\omega \in \mathbb{R}^N$ be of rationally independent coordinates. The M-tuple of series defined by an ω-sequence for $\mathcal{I}$ is an element of S_ω^M. An M-tuple of series $\phi \in S_\omega^M$ is an ω-solution of $\mathcal{I}$ if and only if ϕ is an M-tuple of series defined by an ω-sequence for $\mathcal{I}$.*

17. The Tropical Variety of a Quasi-Ordinary Singularity

Let $(V, \underline{0})$ be a singular N-dimensional germ of algebraic variety. $(V, \underline{0})$ is said to be *quasi-ordinary* when it admits a projection $\pi : (V, \underline{0}) \to (\mathbb{K}^N, \underline{0})$ whose discriminant is contained in the coordinate hyperplanes. Such a projection is called a *quasi-ordinary projection*.

Quasi-ordinary singularities admit analytic local parametrizations. This was shown for hypersurfaces by Abhyankar [1] and extended to arbitrary codimension in [3]. Quasi-ordinary singularities have been the object of study of many research papers [8, 15, 28, 32].

Corollary 17.1. *Let V be an N-dimensional algebraic variety embedded in $\mathbb{C}^{N+M}$ with a quasi-ordinary analytically irreducible singularity at the origin. Let*

$$\begin{array}{rccc} \pi: & V & \to & \mathbb{C}^N \\ & (x_1,\dots,x_{N+M}) & \mapsto & (x_1,\dots,x_N) \end{array}$$

be a quasi-ordinary projection.

Let $\mathcal{I} \subset \mathbb{K}[x_1,\dots,x_{N+M}]$ be the defining ideal of V. Then for any $\omega \in \mathbb{R}^N_{>0}$ of rationally independent coordinates there exists a unique *$e \in \mathbb{R}^M_{>0}$ such that (ω, e) is in the tropical variety $\mathcal{T}(\mathcal{I})$.*

Proof. Since π is quasi-ordinary there exists k and an M-tuple of analytic series in N variables $\varphi_1,\dots,\varphi_M$ such that $(t_1^k,\dots,t_N^k,\varphi_1(\underline{t}),\dots,\varphi_M(\underline{t}))$ are parametric equations of $\mathsf{V}(\mathcal{I})$ about the origin [3].

For any $\omega \in \mathbb{R}^N_{>0}$ of rationally independent coordinates, the first orthant is ω-positive and then $(\varphi_1,\dots,\varphi_M)$ is an element of S^M_ω. Set $\eta_\omega := \operatorname{val}(\varphi_1,\dots,\varphi_M)$. The M-tuple of Puiseux series (with positive exponents) $y := (\varphi_1({x_1}^{\frac{1}{k}},\dots,{x_N}^{\frac{1}{k}}),\dots,\varphi_M({x_1}^{\frac{1}{k}},\dots,{x_N}^{\frac{1}{k}}))$ is an ω-solution of $\mathcal{I}$ with $\operatorname{val}(y) = \frac{1}{k}\eta_\omega$ By Theorem 11.2 $(\omega, \frac{1}{k}\eta_\omega)$ is of $\mathcal{I}$.

Now we will show that $(\omega, e) \in \mathcal{T}(\mathcal{I})$ implies $e = \frac{1}{k}\eta_\omega$. Set Φ to be the map.

$$\begin{array}{rccc} \Phi: & U & \to & \mathbb{C}^{N+M} \\ & (t_1,\dots,t_N) & \mapsto & (t_1^k,\dots,t_N^k,\varphi_1(\underline{t}),\dots,\varphi_M(\underline{t})). \end{array}$$

Since the singularity is analytically irreducible this parametrization covers a neighborhood of the singularity at the origin. Let $U \subset \mathbb{C}^N$ be the common domain of convergence of $\varphi_1,\dots,\varphi_M$. U contains a neighborhood of the origin. We have $\Phi(U) = \pi^{-1}(U) \cap V$.

Let $y_1,\dots,y_M$ be Puiseux series with exponents in some ω-positive rational cone σ (with $\operatorname{val}\underline{y} = e \in {\mathbb{R}_{>0}}^M$) such that $f(x_1,\dots,x_N,y_1(\underline{x}),\dots,y_M(\underline{x})) = 0$ for all $f \in \mathcal{I}$.

Take k' such that $\psi_j(\underline{t}) := y_j(t_1^{k'},\dots,t_N^{k'})$ is a series with integer exponents for all $j = 1,\dots,M$ (this may be done since they are Puiseux series).

There exists an open set $W \subset \mathbb{C}^N$ where all the series ψ_j converge, that has the origin as accumulation point. Set Ψ to be the map:

$$\begin{array}{rccc} \Psi: & W & \to & \mathbb{C}^{N+M} \\ & (t_1,\dots,t_N) & \mapsto & (t_1^{k'},\dots,t_N^{k'},\psi_1(\underline{t}),\dots,\psi_M(\underline{t})). \end{array}$$

Since U is a neighborhood of the origin, $W' := U \cap W \neq \emptyset$. Ramifying again, if necessary, we may suppose that $k = k'$. Take $\underline{t} \in W'$. There exists $\underline{t}' \in U$ such that $\Psi(\underline{t}) = \Phi(\underline{t}')$. Then $\pi \circ \Psi(\underline{t}) = \pi \circ \Phi(\underline{t}')$, and then $(t_1^k,\dots,t_N^k) = (t_1'^k,\dots,t_N'^k)$.

There exists an N-tuple of k-roots of the unity $\xi_1,\dots,\xi_N$ such that $(t_1,\dots,t_N) = (\xi_1 t_1',\dots,\xi_N t_N')$. By continuity this N-tuple is the same for all $\underline{t} \in W'$.

We have $\Psi(t_1,\dots,t_N) = \Phi(\xi_1 t_1,\dots,\xi_N t_N)$ on W'. Since W' contains an open set, we have the equality of the series $\psi_j(t_1,\dots,t_N) = \phi_j(\xi_1 t_1,\dots,\xi_N t_N)$.

Then $(y_1, \ldots, y_M) = (\varphi_1(\xi_1 x_1{}^{\frac{1}{k}}, \ldots, \xi_N x_N{}^{\frac{1}{k}}), \ldots, \varphi_M(\xi_1 x_1{}^{\frac{1}{k}}, \ldots, \xi_N x_N{}^{\frac{1}{k}}))$ for some N-tuple of roots of unity $(\xi_1, \ldots, \xi_N)$ and then

$$\mathrm{val}\,(y_1, \ldots, y_M) = \frac{1}{k}\mathrm{val}\,(\varphi_1, \ldots, \varphi_M).$$

The conclusion follows from Theorem 11.2. □

18. Closing Remarks

In the literature there are many results relating the Newton polyhedron of a hypersurface and invariants of its singularities (see for example [21]). To extend this type of theorems to arbitrary codimension the usual approach has been to work with the Newton polyhedra of a system of generators (see for example [26]). The results presented here suggest that using the notion of tropical variety better results may be obtained.

Both Newton–Puiseux's and McDonald's algorithm have been extended for an ordinary differential equation [13] and a partial differential equations [5, 6] respectively. The algorithm presented here can definitely be extended to systems of partial differential equations; a first step in this direction can be found in [4].

Acknowledgments

The first and the third author were partially supported by CONACyT 55084 and UNAM: PAPIIT IN 105806, IN 102307 and IN 117110.

During the preparation of this work the third author was profiting from the post-doctoral fellowship CONACyT 37035.

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