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COMPLETENESS IN THE THEORY OF TYPES[1]

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The first order functional calculus was proved complete by Gödel[3] in 1930. Roughly speaking, this proof demonstrates that each formula of the calculus is a formal theorem which becomes a true sentence under every one of a certain intended class of interpretations of the formal system.

For the functional calculus of second order, in which predicate variables may be bound, a very different kind of result is known: no matter what (recursive) set of axioms are chosen, the system will contain a formula which is valid but not a formal theorem. This follows from results of Gödel[4] concerning systems containing a theory of natural numbers, because a finite categorical set of axioms for the positive integers can be formulated within a second order calculus to which a functional constant has been added.

By a valid formula of the second order calculus is meant one which expresses a true proposition whenever the individual variables are interpreted as ranging over an (arbitrary) domain of elements while the functional variables of degree n range over all sets of ordered n-tuples of individuals. Under this definition of validity, we must conclude from Gödel's results that the calculus is essentially incomplete.

It happens, however, that there is a wider class of models which furnish an interpretation for the symbolism of the calculus consistent with the usual axioms and formal rules of inference. Roughly, these models consist of an arbitrary domain of individuals, as before, but now an *arbitrary*[5] *class* of sets of ordered n-tuples of individuals as the range for functional variables of degree n. If we

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[1] The material in this paper is included in *The completeness of formal systems*, a Thesis presented to the faculty of Princeton University in candidacy for the degree of Doctor of Philosophy and accepted in October, 1947. The results were announced at the meeting of the Association for Symbolic Logic in December, 1947 (cf. this JOURNAL, vol. 13 (1948), p. 61).

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[3] Kurt Gödel, *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*, ***Monatshefte für Mathematik und Physik***, vol. 37 (1930), pp. 349–360.

[4] Kurt Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*, ***Monatshefte für Mathematik und Physik***, vol. 38 (1931), pp. 173–198.

[5] These classes cannot really be taken in an altogether arbitrary manner if every formula is to have an interpretation. For example, if the formula $F(x)$ is interpreted as meaning that x is in the class F, then $\sim F(x)$ means that x is in the complement of F; hence the range for functional variables such as F should be closed under complementation. Similarly, if G refers to a set of ordered pairs in some model, then the set of individuals x satisfying the formula $(\exists y)G(x, y)$ is a projection of the set G; hence, we require that the various domains be closed under projection. In short, each method of compounding formulas of the calculus has associated with it some operation on the domains of a model, with respect to which the domains must be closed. The statement of completeness can be given precisely and proved for models meeting these closure conditions.

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redefine the notion of valid formula to mean one which expresses a true proposition with respect to every one of *these* models, we can then prove that the usual axiom system for the second order calculus is complete: a formula is valid if and only if it is a formal theorem.[6]

A similar result holds for the calculi of higher order. In this paper, we will give the details for a system of order ω embodying a simple theory of (finite) types. We shall employ the rather elegant formulation of Church,[7] the details of which are summarized below:

Type symbols (to be used as subscripts):
1. o and ι are type symbols
2. If α, β are type symbols so is $(\alpha\beta)$.

Primitive symbols (where α may be any type symbol):
Variables: f_α, g_α, x_α, y_α, z_α, f'_α, g'_α, $\cdots$
Constants: $N_{(oo)}$, $A_{((oo)o)}$, $\Pi_{(o(o\alpha))}$, $\iota_{(\alpha(o\alpha))}$
Improper: λ, (,).

Well-formed formulas (wffs) and their *type*:
1. A variable or constant alone is a wff and has the type of its subscript.
2. If $A_{\alpha\beta}$ and B_β are wffs of type $(\alpha\beta)$ and β respectively, then $(A_{\alpha\beta}\ B_\beta)$ is a wff of type α.
3. If A_α is a wff of type α and a_β a variable of type β then $(\lambda a_\beta A_\alpha)$ is a wff of type $(\alpha\beta)$.

An occurrence of a variable a_β is *bound* if it is in a wff of the form $(\lambda a_\beta A_\alpha)$; otherwise the occurrence is *free*.

Letters A_α, B_α, C_α, will be used as syntactical variables for wffs of type α.

Abbreviations:

$(\sim A_o)$	for $(N_{(oo)}A_o)$
$(A_o \vee B_o)$	for $((A_{((oo)o)}A_o)B_o)$
(A_oB_o)	for $(\sim((\sim A_o) \vee (\sim B_o)))$
$(A_o \supset B_o)$	for $((\sim A_o) \vee B_o)$
$(a_\alpha)B_o$	for $(\Pi_{(o(o\alpha))}(\lambda a_\alpha B_o))$
$(\exists a_\alpha)B_o$	for $(\sim((a_\alpha)(\sim A_o)))$
$(\imath a_\alpha B_o)$	for $(\iota_{(\alpha(o\alpha))}(\lambda a_\alpha B_o))$
$Q_{((o\alpha)\alpha)}$	for $(\lambda x_\alpha(\lambda y_\alpha(f_{o\alpha})((f_{o\alpha}x_\alpha) \supset (f_{o\alpha}y_\alpha))))$
$(A_\alpha = B_\alpha)$	for $((Q_{((o\alpha)\alpha)}A_\alpha)(B_\alpha)$.

In writing wffs and subscripts, we shall practise the omission of parentheses and their supplantation by dots on occasion, the principal rules of restoration

[6] A demonstration of this type of completeness can be carried out along the lines of the author's recent paper, *The completeness of the first order functional calculus*, this JOURNAL, vol. 14 (1949), pp. 159–166.

[7] Alonzo Church, *A formulation of the simple theory of types*, this JOURNAL, vol. 5 (1940), pp. 56–68.

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being first that the formula shall be well-formed; secondly, that association is to the left; and thirdly, that a dot is to be replaced by a left parenthesis having its mate as far to the right as possible. (For a detailed statement of usage, refer to Church.[7])

Axioms and axiom schemata:

1. $(x_o \vee x_o) \supset x_o$
2. $x_o \supset (x_o \vee y_o)$
3. $(x_o \vee y_o) \supset (y_o \vee x_o)$
4. $(x_o \supset y_o) \supset . (z_o \vee x_o) \supset (z_o \vee y_o)$

5^α. $\Pi_{o(o\alpha)} f_{o\alpha} \supset f_{o\alpha} x_\alpha$

6^α. $(x_\alpha)(y_o \vee f_{o\alpha} x_\alpha) \supset . y_o \vee \Pi_{o(o\alpha)} f_{o\alpha}$

10. $x_o \equiv y_o \supset x_o = y_o \qquad (x_\beta)(f_{\alpha\beta} x_\beta = g_{\alpha\beta} x_\beta) \supset f_{\alpha\beta} = g_{\alpha\beta}$

11^α. $f_{o\alpha} x_\alpha \supset f_{o\alpha}(\iota_{\alpha(o\alpha)} f_{o\alpha})$

Rules of Inference:

I. To replace any part A_α of a formula by the result of substituting a_β for b_β throughout A_α, provided that b_β is not a free variable of A_α and a_β does not occur in A_α.

II. To replace any part $(\lambda a_\gamma A_\beta) B_\gamma$ of a wff by the result of substituting B_γ for a_γ throughout A_β, provided that the bound variables of A_β are distinct both from a_γ and the free variables of B_γ.

III. To infer A_o from B_o if B_o may be inferred from A_o by a single application of Rule II.

IV. From $A_{o\alpha} a_\alpha$ to infer $A_{o\alpha} B_\alpha$ if the variable a_α is not free in $A_{o\alpha}$.

V. From $A_o \supset B_o$ and A_o to infer B_o.

VI. From $A_{o\alpha} a_\alpha$ to infer $\Pi_{o(o\alpha)} A_{o\alpha}$ provided that the variable a_α is not free in $A_{o\alpha}$.

A finite sequence of wffs each of which is an axiom or obtained from preceding elements of the sequence by a single application of one of the rules I–VI is called a *formal proof*. If A is an element of some formal proof, we write $\vdash A$ and say that A is a *formal theorem*.

This completes our description of the formal system. In order to discuss the question of its completeness, we must now give a precise account of the manner in which this formalism is to be *interpreted*.

By a *standard model*, we mean a family of domains, one for each type-symbol, as follows: D_ι is an arbitrary set of elements called *individuals*, D_o is the set consisting of two truth values, T and F, and $D_{\alpha\beta}$ is the set of all functions defined over D_β with values in D_α.

By an *assignment* with respect to a standard model $\{D_\alpha\}$, we mean a mapping ϕ of the variables of the formal system into the domains of the model such that for a variable a_α of type α as argument, the value $\phi(a_\alpha)$ of ϕ is an element of D_α.

We shall associate with each assignment ϕ a mapping V_ϕ of all the formulas of the formal system such that $V_\phi(A_\alpha)$ is an element of D_α for each wff A_α of type

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α. We shall define the values $V_\phi(A_\alpha)$ simultaneously for all ϕ by induction on the length of the wff A_α :

(i) If A_α is a variable, set $V_\phi(A_\alpha) = \phi(A_\alpha)$. Let $V_\phi(N_{oo})$ be the function whose values are given by the table

x	$V_\phi(N_{oo})(x)$
T	F
F	T

Let $V_\phi(A_{ooo})$ be the function whose value for arguments T, F are the functions given by the tables 1, 2 respectively.

1.

x	$V_\phi(A_{ooo})(\mathrm{T})(x)$
T	T
F	T

2.

x	$V_\phi(A_{ooo})(\mathrm{F})(x)$
T	T
F	F

Let $V_\phi(\Pi_{o(o\alpha)})$ be the function which has the value T for just the single argument which is the function mapping D_α into the constant value T. Let $V_\phi(\iota_{\alpha(o\alpha)})$ be some fixed function whose value for any argument f of $D_{o\alpha}$ is one of the elements of D_α mapped into T by f (if there is such an element).

(ii) If A_α has the form $B_{\alpha\beta}C_\beta$ define $V_\phi(B_{\alpha\beta}C_\beta)$ to be the value of the function $V_\phi(B_{\alpha\beta})$ for the argument $V_\phi(C_\beta)$.

(iii) Suppose A_α has the form $(\lambda a_\beta B_\gamma)$. We define $V_\phi(\lambda a_\beta B_\gamma)$ to be the function whose value for the argument x of D_β is $V_\psi(B_\gamma)$, where ψ is the assignment which has the same values as ϕ for all variables except a_β , while $\psi(a_\beta)$ is x.

We can now define a wff A_o to be *valid in the standard sense* if $V_\phi(A_o)$ is T for every assignment ϕ with respect to every standard model $\{D_\alpha\}$.[8] Because the theory of recursive arithmetic can be developed within our formal system as shown by Church,[7] it follows by Gödel's methods[4] that we can construct a particular wff A_o which is valid in the standard sense, but not a formal theorem.

We can, however, interpret our formalism with respect to other than the standard models. By a *frame*, we mean a family of domains, one for each type symbol, as follows: D_ι is an arbitrary set of individuals, D_o is the set of two truth values, T and F, and $D_{\alpha\beta}$ is some class of functions defined over D_β with values in D_α .

Given such a frame, we may consider assignments ϕ mapping variables of the formal system into its domains, and attempt to define the functions V_ϕ exactly as for standard models. For an arbitrary frame, however, it may well happen that one of the functions described in items (i), (ii), or (iii) as the value of some $V_\phi(A_\alpha)$ is not an element of any of the domains.

A frame such that for every assignment ϕ and wff A_α of type α, the value $V_\phi(A_\alpha)$ given by rules (i), (ii), and (iii) is an element of D_α , is called a *general model*. Since this definition is impredicative, it is not immediately clear that any non-standard models exist. However, they do exist (indeed, there are general models for which every domain D_α is denumerable), and we shall give a method

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of constructing every general model without resorting to impredicative processes.

Now we define a *valid formula in the general sense* as a formula A_o such that $V_\phi(A_o)$ is T for every assignment ϕ with respect to any general model. We shall prove a completeness theorem for the formal system by showing that A_0 is valid in the general sense if and only if $\vdash A$.

By a *closed* well-formed formula (cwff), we mean one in which no occurrence of any variable is free. If Λ is a set of cwffs such that, when added to the axioms 1–6^α, 10^α, 11^α, a formal proof can be obtained for some wff A_0, we write $\Lambda \vdash A_0$. If $\Lambda \vdash A_0$ for every wff A_0, we say that Λ is *inconsistent*, otherwise *consistent*.

THEOREM 1. *If* Λ *is any consistent set of* cwffs, *there is a general model* (*in which each domain* D_α *is denumerable*) *with respect to which* Λ *is satisfiable.*[8]

We shall make use of the following derived results about the formal calculus which we quote without proof:

VII. The deduction theorem holds: If Λ, $A_o \vdash B_o$, then $\Lambda \vdash A_o \supset B_o$ where Λ is any set of cwffs, A_o is any cwff, and B_o is any wff. (A proof is given in Church.[7])

12. $\vdash A_o \supset . \sim A_o \supset B_o$
13. $\vdash A_o \supset B_o \supset . \sim A_o \supset B_o \supset . B_o$
14. $\vdash A_\alpha = A_\alpha$
15. $\vdash A_\alpha = B_\alpha \supset B_\alpha = A_\alpha$
16. $\vdash A_\alpha = B_\alpha \supset . B_\alpha = C_\alpha \supset . A_\alpha = C_\alpha$
17. $\vdash A_o \supset . (A_o = B_o) \supset B_o$
18. $\vdash \sim A_o \supset . (A_o = B_o) \supset \sim B_o$
19. $\vdash A_o \supset . B_o \supset . A_o = B_o$
20. $\vdash \sim A_o \supset . \sim B_o \supset . A_o = B_o$
21. $\vdash A_{\alpha\beta} = A'_{\alpha\beta} \supset . B_\beta = B'_\beta \supset . A_{\alpha\beta}B_\beta = A'_{\alpha\beta}B'_\beta$
22. $\vdash A_{\alpha\beta}((\iota x_\beta) \sim (A_{\alpha\beta}x_\beta = A'_{\alpha\beta}x_\beta)) = A'_{\alpha\beta}((\iota x_\beta) \sim (A_{\alpha\beta}x_\beta = A'_{\alpha\beta}x_\beta)) \supset . A_{\alpha\beta} = A'_{\alpha\beta}$
23. $\vdash A_o \supset \sim\sim A_o$
24. $\vdash C_o \supset . C_o \vee A_o$
25. $\vdash \sim C_o \supset . A_o \supset . C_o \vee A_o$
26. $\vdash \sim C_o \supset . \sim A_o \supset . \sim(C_o \vee A_o)$
27. $\vdash \Pi_{o(o\alpha)}A_{o\alpha} \supset A_{o\alpha}C_\alpha$
28. $\vdash A_{o\alpha}((\iota x_\alpha) \sim (A_o x_\alpha)) \supset \Pi_{o(o\alpha)}A_{o\alpha}$
29. $\vdash A_{o\alpha}C_\alpha \supset A_{o\alpha}(\iota_{\alpha(o\alpha)}C_\alpha)$
30. $\vdash (\sim B_o \supset B_o) \supset B_o$
31. $\vdash (x_\alpha)A_o \supset A_o$

[8] In addition to the notion of validity, the mappings V_ϕ may be used to define the concept of the *denotation* of a wff A_α containing no free occurrence of any variable. We first show (by induction) that if ϕ and ψ are two assignments which have the same value for every variable with a free occurrence in the wff B_α, then $V_\phi(B_\alpha) = V_\psi(B_\alpha)$. Then the denotation of A_α is simply $V_\phi(A_\alpha)$ for any ϕ. We also define the notion of satisfiability. If Γ is a set of wffs and ϕ an assignment with respect some model $\{D_\alpha\}$ such that $V_\phi(A_o)$ is T for every A_o in Γ, then we say that Γ is *satisfiable with respect to the model* $\{D_\alpha\}$. If Γ is satisfiable with respect to some model, we say simply that it is *satisfiable*.

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The first step in our proof of Theorem 1 is to construct a maximal consistent set Γ of cwffs such that Γ contains Λ, where by maximal is meant that if A_o is any cwff not in Γ then the enlarged set $\{\Gamma, A_o\}$ is inconsistent. Such a set Γ may be obtained in many ways. If we enumerate all of the cwffs in some standard order, we may test them one at a time, adding them to Λ and previously added formulas whenever this does not result in an inconsistent set. The union of this increasing sequence of sets is then easily seen to be maximal consistent.

Γ has certain simple properties which we shall use. If A_o is any cwff, it is clear that we cannot have both $\Gamma \vdash A_o$ and $\Gamma \vdash \sim A_o$ for then by 12 and V, we would obtain $\Gamma \vdash B_o$ for any B_o, contrary to the consistency of Γ. On the other hand, at least one of the cwffs A_o, $\sim A_o$ must be in Γ. For otherwise, using the maximal property of Γ we would have $\Gamma, A_o \vdash B_o$ and $\Gamma, \sim A_o \vdash B_o$ for any B_o. By VII, it then follows that $\Gamma \vdash A_o \supset B_o$ and $\Gamma \vdash \sim A_o \supset B_o$, whence by 13 and V $\Gamma \vdash B_o$ contrary to the consistency of Γ.

Two cwffs A_α, B_α of type α will be called *equivalent* if $\Gamma \vdash A_\alpha = B_\alpha$. Using 14, 16, and V, we easily see that this is a genuine congruence relation so that the set of all cwffs of type α is partitioned into disjoint equivalent classes $[A_\alpha]$, $[B_\alpha]$, $\cdots$ such that $[A_\alpha]$ and $[B_\alpha]$ are equal if and only if A_α is equivalent to B_α.

We now define by induction on α a frame of domains $\{D_\alpha\}$, and simultaneously a one-one mapping Φ of equivalence classes onto the domains D_α such that $\Phi([A_\alpha])$ is in D_α.

D_o is the set of two truth values, T and F, and for any cwff A_o of type o $\Phi([A_o])$ is T or F according as A_o or $\sim A_o$ is in Γ. We must show that Φ is a function of equivalence classes and does not really depend on the particular representative A_o chosen. But by 17 and V, we see that if $\Gamma \vdash A_o$ and B_o is equivalent to A_o (i.e., $\Gamma \vdash A_o = B_o$), then $\Gamma \vdash B_o$; and similarly if $\Gamma \vdash \sim A_o$ and B_o is equivalent to A_o, then $\Gamma \vdash \sim B_o$ by 18. To see that Φ is one-one, we use 19 to show that if $\Phi([A_o])$ and $\Phi([B_o])$ are both T (i.e., $\Gamma \vdash A_o$ and $\Gamma \vdash B_o$), then $\Gamma \vdash A_o = B_o$ so that $[A_o]$ is $[B_o]$. Similarly 20 shows that $[A_o]$ is $[B_o]$ in case $\Phi([A_o])$ and $\Phi([B_o])$ are both F.

D_ι is simply the set of equivalence classes $[A_\iota]$ of all cwffs of type ι. And $\Phi([A_\iota])$ is $[A_\iota]$ so that Φ is certainly one-one.

Now suppose that D_α and D_β have been defined, as well as the value of Φ for all equivalence classes of formulas of type α and of type β, and that every element of D_α, or D_β is the value of Φ for some $[A_\alpha]$, or $[B_\beta]$ respectively. Define $\Phi([A_{\alpha\beta}])$ to be the function whose value, for the element $\Phi([B_\beta])$ of D_β, is $\Phi([A_{\alpha\beta}B_\beta])$. This definition is justified by the fact that if $A'_{\alpha\beta}$ and B'_β are equivalent to $A_{\alpha\beta}$ and B_β respectively, then $A'_{\alpha\beta}B'_\beta$ is equivalent to $A_{\alpha\beta}B_\beta$, as one sees by 21. To see that Φ is one-one, suppose that $\Phi([A_{\alpha\beta}])$ and $\Phi([A'_{\alpha\beta}])$ have the same value for every $\Phi([B_\beta])$ of D_β. Hence $\Phi([A_{\alpha\beta}B_\beta]) = \Phi([A'_{\alpha\beta}B_\beta])$ and so, by the induction hypothesis that Φ is one-one for equivalence classes of formulas of type α, $A_{\alpha\beta}B_\beta$ is equivalent to $A'_{\alpha\beta}B_\beta$ for each cwff B_β. In particular, if we take B_β to be $(\imath x_\beta) \sim(A_{\alpha\beta}x_\beta = A'_{\alpha\beta}x_\beta)$, we see by 22 that $A_{\alpha\beta}$ and $A'_{\alpha\beta}$ are equivalent so that $[A_{\alpha\beta}] = [A'_{\alpha\beta}]$. The one-one function Φ having been thus completely defined, we define $D_{\alpha\beta}$ to be the set of values $\Phi([A_{\alpha\beta}])$ for all cwffs $A_{\alpha\beta}$.

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Now let ϕ be any assignment mapping each variable x_α into some element $\Phi([A_\alpha])$ of D_α, where A_α is a cwff. Given any wff B_β, let B_β^ϕ be a cwff obtained from B_β by replacing all free occurrences in B_β of any variable x_α by some cwff A_α such that $\phi(x_\alpha) = \Phi([A_\alpha])$.

LEMMA. *For every ϕ and B_β we have $V_\phi(B_\beta) = \Phi([B_\beta^\phi])$.*

The proof is by induction on the length of B_β.

(i) If B_β is a variable and $\phi(B_\beta)$ is the element $\Phi([A_\beta])$ of D_β, then by definition B_β^ϕ is some cwff A_β' equivalent to A_β and $V_\phi(B_\beta) = \phi(B_\beta) = \Phi([A_\beta]) = \Phi([A_\beta']) = \Phi([B_\beta^\phi])$.

Suppose B_β is N_{oo}, whence B_β^ϕ is N_{oo}. If $\Phi([A_o])$ is T, then by definition $\Gamma \vdash A_o$ whence by 23 $\Gamma \vdash \sim N_{oo}A_o$ so that $\Phi([N_{oo}A_o])$ is F. That is, $\Phi([N_{oo}])$ maps T into F. Conversely, if $\Phi([A_o])$ is F, then by definition $\Gamma \vdash N_{oo}A_o$ so that $\Phi([N_{oo}A_o])$ is T; i.e., $\Phi([N_{oo}])$ maps F into T. Hence $V_\phi(B_\beta) = \Phi([B_\beta^\phi])$ in this case.

Suppose B_β is A_{ooo}, whence B_β^ϕ is A_{ooo}. If $\Phi([C_o])$ is T, then by definition $\Gamma \vdash C_o$ whence by 24 $\Gamma \vdash A_{ooo}C_oA_o$ for any A_o so that $\Phi([A_{ooo}C_oA_o])$ is T no matter whether $\Phi[(A_o)]$ is T or F. Similarly, using 25 and 26, we see that $\Phi([A_{ooo}C_oA_o])$ is T, or F, if $\Phi([C_o])$ is F, and $\Phi([A_o])$ is T, or F respectively. Comparing this with the definition of $V_\phi(A_{ooo})$, we see that the lemma holds in this case also.

Suppose B_β is $\Pi_{o(o\alpha)}$, whence B_β^ϕ is $\Pi_{o(o\alpha)}$. If the value of $\Phi([\Pi_{o(o\alpha)}])$ for the argument $\Phi([A_{o\alpha}])$ is T, then $\Gamma \vdash \Pi_{o(o\alpha)}A_{o\alpha}$ whence by 27 $\Gamma \vdash A_{o\alpha}C_\alpha$ for every cwff C_α so that $\Phi([A_{o\alpha}])$ maps every element of D_α into T. On the other hand, if $\Phi([A_{o\alpha}])$ maps every $\Phi([C_\alpha])$ into T, then we have, taking the particular case where C_α is $(\imath x_\alpha)\sim(A_{o\alpha}x_\alpha)$, $\Gamma \vdash A_{o\alpha}((\imath x_\alpha)\sim(A_{o\alpha}x_\alpha))$ whence by 28 $\Gamma \vdash \Pi_{o(o\alpha)}A_{o\alpha}$. That is $\Phi([\Pi_{o(o\alpha)}])$ maps $\Phi([A_{o\alpha}])$ into T. The lemma holds in this case.

Suppose B_β is $\iota_{\alpha(o\alpha)}$, whence B_β^ϕ is $\iota_{\alpha(o\alpha)}$. Let $A_{o\alpha}$ be a cwff such that $\Phi([A_{o\alpha}])$ maps some $\Phi([C_\alpha])$ into T so that $\Gamma \vdash A_{o\alpha}C_\alpha$. Then by 29 $\Gamma \vdash A_{o\alpha}(\iota_{\alpha(o\alpha)}A_{o\alpha})$ so that the value of $\Phi([\iota_{\alpha(o\alpha)}])$ for the argument $\Phi([A_{o\alpha}])$ is mapped into T by the latter. Therefore, we may take $\Phi([\iota_{\alpha(o\alpha)}])$ to be $V_\phi(\iota_{\alpha(o\alpha)})$.

(ii) Suppose that B_β has the form $B_{\beta\gamma}C_\gamma$. We assume (induction hypothesis) that we have already shown $\Phi([B_{\beta\gamma}^\phi]) = V_\phi(B_{\beta\gamma})$ and $\Phi([C_\gamma^\phi]) = V_\phi(C_\gamma)$.

Now $V_\phi(B_{\beta\gamma}C_\gamma)$ is the value of $V_\phi(B_{\beta\gamma})$ for the argument $V_\phi(C_\gamma)$, or the value of $\Phi([B_{\beta\gamma}^\phi])$ for the argument $\Phi([C_\gamma^\phi])$, which is $\Phi([B_{\beta\gamma}^\phi C_\gamma^\phi])$. But $(B_{\beta\gamma}C_\gamma)^\phi$ is simply $B_{\beta\gamma}^\phi C_\gamma^\phi$. Hence $V_\phi(B_{\beta\gamma}C_\gamma) = \Phi([(B_{\beta\gamma}C_\gamma)^\phi])$.

(iii) Suppose that B_β has the form $\lambda a_\gamma C_\alpha$ and our induction hypothesis is that $\Phi([C_\alpha^\phi]) = V_\phi(C_\alpha)$ for every assignment ϕ. Let $\Phi([A_\gamma])$ be any element of D_γ. Then the value of $\Phi([(\lambda a_\gamma C_\alpha)^\phi])$ for the argument $\Phi([A_\gamma])$ is by definition $\Phi([(\lambda a_\gamma C_\alpha)^\phi A_\gamma])$.

But by applying II to the right member of the instance $\vdash (\lambda a_\gamma C_\alpha)^\phi A_\gamma = (\lambda a_\gamma C_\alpha)^\phi A_\gamma$ of 14, we find $\vdash (\lambda a_\gamma C_\alpha)^\phi A_\gamma = C_\alpha^\psi$, where ψ is the assignment which has the same value as ϕ for every argument except the variable a_γ and $\psi(a_\gamma)$ is $\Phi([A_\gamma])$. That is, $[(\lambda a_\gamma C_\alpha)^\phi A_\gamma] = [C_\alpha^\psi]$ so that the value of $\Phi([(\lambda a_\gamma C_\alpha)^\phi])$ for the argument $\Phi([A_\gamma])$ is $\Phi([C_\alpha^\psi])$—or $V_\psi(C_\alpha)$ by induction hypothesis. Since for every argument $\Phi([(\lambda a_\gamma C_\alpha)^\phi])$ and $V_\phi(\lambda a_\gamma C_\alpha)$ have the same value, they must be equal.

This concludes the proof of our lemma.

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Theorem 1 now follows directly from the lemma. In the first place, the frame of domains $\{D_\alpha\}$ is a general model since $V_\phi(B_\beta)$ is an element of D_β for every wff B_β and assignment ϕ. Because the elements of any D_α are in one-one correspondence with equivalence classes of wffs each domain is denumerable. Since for every cwff $A_o^\phi = A_o$, ϕ being an arbitrary assignment, since therefore for every cwff A_o of Γ we have $\Phi([A_o]) = \mathrm{T}$, and since Λ is a subset of Γ, it follows that $V_\phi(A_o)$ is T for any element A_o of Λ; i.e., Λ is satisfiable with respect to the model $\{D_\alpha\}$.

THEOREM 2. *For any* wff A_o, *we have* $\vdash A_o$ *if and only if* A_o *is valid in the general sense.*

From the definition of validity, we easily see that A_o is valid if and only if the cwff $(x_{\alpha_1}) \cdots (x_{\alpha_n})A_o$ is valid, where $x_{\alpha_1}, \cdots, x_{\alpha_n}$ are the variables with free occurrences in A_o; and hence A_o is valid if and only if $V_\phi(\sim(x_{\alpha_1}) \cdots (x_{\alpha_n})A_o)$ is F for every assignment ϕ with respect to every general model $\{D_\alpha\}$. By Theorem 1, this condition implies that the set Δ whose only element is the cwff $\sim(x_{\alpha_1}) \cdots (x_{\alpha_n})A_o$ is inconsistent and hence, in particular, $\sim(x_{\alpha_1}) \cdots (x_{\alpha_n})A_o \vdash (x_{\alpha_1}) \cdots (x_{\alpha_n})A_o$. Now applying VII, 30, and 31 (several times), we see that if A_o is valid, then $\vdash A_o$. The converse can be verified directly by checking the validity of the axioms and noticing that the rules of inference operating on valid formulas lead only to valid formulas.

THEOREM 3. *A set* Γ *of* cwffs *is satisfiable with respect to some model of denumerable domains* D_α *if and only if every finite subset* Λ *of* Γ *is satisfiable.*

By Theorem 1, if Γ is not satisfiable with respect to some model of denumerable domains, then Γ is inconsistent so that, in particular, $\Gamma \vdash (x_o)x_o$. Since the formal proof of $(x_o)x_o$ contains only a finite number of formulas, there must be some finite subset $\Lambda = \{A_1, \cdots, A_n\}$ of Γ such that $A_1, \cdots, A_n \vdash (x_o)x_o$, whence by repeated applications of VII, $\vdash A_1 \supset . \cdots \supset . A_n \supset (x_o)x_o$. But then by Theorem 2, the cwff $A_1 \supset . \cdots . A_n \supset (x_o)x_o$ is valid so that we must have some $V_\phi(A_i) = \mathrm{F}$, $i = 1, \cdots, n$, for any ϕ with respect to any model; i.e., Λ is not satisfiable. Thus, if every finite subset Λ of Γ is satisfiable, then Γ is satisfiable with respect to a model of denumerable domains. The converse is immediate.

If Γ is satisfiable, then so are its finite subsets, and hence Γ is satisfiable with respect to some model of denumerable domains. This may be taken as a generalization of the Skolem-Löwenheim theorem for the first order functional calculus.

Analogues of Theorems 1, 2, and 3 can be proved for various formal systems which differ in one respect or another from the system which we have here considered in detail. In the first place, we may add an arbitrary set of *constants* S_α as new primitive symbols. In case the set of constants is infinite, we must replace the condition of denumerability, in the statement of Theorems 1 and 3, by the condition that the domains of the model will have a cardinality not greater than that of the set of constants. The proofs for such systems are exactly like the ones given here.

In the second place, the symbols $\iota_{\alpha(o\alpha)}$ and the axioms of choice (11^α) may be

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dropped. In this case, we have to complicate the proof by first performing a construction which involves forming a sequence of formal systems built up from the given one by adjoining certain constants u_α^{ij}, $i, j = 1, 2, \cdots$, and providing suitable axioms for them. The details can be obtained by consulting the paper mentioned in footnote 6.

The axioms of extensionality (10^α) can be dropped if we are willing to admit models whose domains contain functions which are regarded as distinct even though they have the same value for every argument.

Finally, the functional abstraction of the present system may either be replaced by set-abstraction or dropped altogether. In the latter case, the constants $\Pi_{o(o\alpha)}$ must be replaced by a primitive notion of quantifiers.

Theorem 3 can be applied to throw light on formalized systems of number theory.

The concepts of elementary number theory may be introduced into the pure functional calculus of order ω by definition, a form particularly suited to the present formulation being given in Church.[7] Under this approach, the natural numbers are identified with certain functions. Alternatively we may choose to identify the natural numbers with the individuals making up the domain of type ι. In such a system, it is convenient to construct an applied calculus by introducing the constants 0_ι and $S_{\iota\iota}$ and adding the following formal equivalents of Peano's postulates:

P1. $(x_\iota) . \sim S_{\iota\iota}x_\iota = 0_\iota$

P2. $(x_\iota)(y_\iota) . S_{\iota\iota}x_\iota = S_{\iota\iota}y_\iota \supset x_\iota = y_\iota$

P3. $(f_{o\iota}) . f_{o\iota}0_\iota \supset . (x_\iota)[f_{o\iota}x_\iota \supset f_{o\iota}(S_{\iota\iota}x_\iota)] \supset (x_\iota)f_{o\iota}x_\iota$.

The Peano axioms are generally thought to characterize the number-sequence fully in the sense that they form a categorical axiom set any two models for which are isomorphic. As Skolem[9] points out, however, this condition obtains only if "set"—as it appears in the axiom of complete induction (our P3)—is interpreted with its standard meaning. Since, however, the scope ("all sets of individuals") of the quantifier $(f_{o\iota})$ may vary from one general model to another,[10] it follows that we may expect non-standard models for the Peano axioms.

This argument may be somewhat clearer if we consider in detail the usual proof of the categoricity of Peano's postulates. One easily shows that any model for the axioms must *contain* a sequence of the order-type of the natural numbers by considering the individuals 0_ι, $S_{\iota\iota}0_\iota$, $S_{\iota\iota}(S_{\iota\iota}0_\iota)$, $\cdots$ and using P1 and P2 to show them distinct and without other predecessors. Then the proof continues as follows.

Suppose that the domain of individuals contained elements other than those of this sequence (which we may as well identify with the natural numbers themselves). Then consider the class of individuals consisting of just the natural

[9] Thoralf Skolem, *Über einige Grundlagenfragen der Mathematik*, ***Skrifter utgitt av Det Norske Videnskaps-Akademi***, I, no. 4 (1929), 49 pp.

[10] Here we are identifying a set X of elements of D_ι with the function (element of $D_{o\iota}$) which maps every element of X into T and every other element of D_ι into F.

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numbers. Since it contains zero (0_ι) and is closed under the successor function ($S_{\iota\iota}$), we infer from the axiom of complete induction (P3) that it contains all individuals, contrary to the hypothesis that some individuals were not numbers.

By examining this proof, we see that we can conclude only that if a general model satisfies Peano's axioms and at the same time possesses a domain of individuals not isomorphic to the natural numbers, then the domain $D_{o\iota}$ of sets of individuals *cannot* contain the set consisting of just those individuals which are numbers.

Although Skolem indicates that the meaning of "natural number" is relative to the variable meaning of "set" he does not give any example of a non-standard number system satisfying all of Peano's axioms. In two later papers,[11] however, he proves that it is impossible to characterize the natural number sequence by any denumerable system of axioms formulated within the first order functional calculus (to which may be added any set of functional constants denoting numerical functions and relations), the individual variables ranging over the "numbers" themselves. Skolem makes ingenious use of a theorem on sequences of functions (which he had previously proved) to construct, for each set of axioms for the number sequence (of the type described above) a set of numerical functions which satisfy the axioms, but have a different order type than the natural numbers. This result, for axiom systems which do not involve class variables, cannot be regarded as being at all paradoxical since the claim had never been made that such systems were categorical.

By appealing to Theorem 3, however, it becomes a simple matter to construct a model containing a non-standard number system which will satisfy all of the Peano postulates as well as any preassigned set of further axioms (which may include constants for special functions as well as constants and variables of higher type). We have only to adjoin a new primitive constant u_ι and add to the given set of axioms the infinite list of formulas $u_\iota \neq 0_\iota$, $u_\iota \neq S_{\iota\iota}0_\iota$, $u_\iota \neq S_{\iota\iota}(S_{\iota\iota}0_\iota)$, $\cdots$. Since any finite subset of the enlarged system of formulas is clearly satisfiable, it follows from theorem that some denumerable model satisfies the full set of formulas, and such a model has the properties sought. By adding a non-denumerable number of primitive constants v_ι^ξ together with all formulas $v_\iota^{\xi_1} \neq v_\iota^{\xi_2}$ for $\xi_1 \neq \xi_2$, we may even build models for which the Peano axioms are valid and which contain a number system having any given cardinal.[12]

These same remarks suffice to show more generally that no mathematical axiom system can be genuinely categorical (determine its models to within isomorphism) unless it constrains its domain of elements to have some definite

[11] Thoralf Skolem, *Über die Unmöglichkeit einer vollständigen Charakterisierung der Zahlenreihe mittels eines endlichen Axiomensystems*, ***Norsk matematisk forenings skrifter***—series 2 no. 10 (1933), pp. 73–82. And *Über die Nicht-charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen*, ***Fundamenta mathematicae***, vol. 23 (1934), pp. 150–161.

[12] A similar result for formulations of arithmetic within the first order functional calculus was established by A. Malcev, *Untersuchungen aus dem Gebiete der mathematischen Logik*, ***Recueil mathématique***, n.s. vol. 1 (1936), pp. 323–336. Malcev's method of proof bears a certain resemblance to the method used above. I am indebted to Professor Church for bringing this paper to my attention. (Added February 14, 1950.)

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finite cardinal number—provided that the logical notions of set and function are axiomatized along with the specific mathematical notions.

The existence of non-standard models satisfying axiom-systems for number theory throws new light on the phenomenon of ω-inconsistency, first investigated by Tarski and Gödel. A formal system is ω-inconsistent if for some formula $A_{o\iota}$ the formulas $A_{o\iota}0_\iota$, $A_{o\iota}(S_{\iota\iota}0_\iota)$, $A_{o\iota}(S_{\iota\iota}(S_{\iota\iota}0_\iota))$, $\cdots$, $\sim(x_\iota)A_{o\iota}x_\iota$ are all provable. Tarski, and later Gödel, showed the existence of consistent systems which were ω-inconsistent. We can now see that such systems can and must be interpreted as referring to a non-standard number system whose elements include the natural numbers as a proper subset.

It is generally recognized that all theorems of number theory now in the literature can be formalized and proved within the functional calculus of order ω with axioms P1–P3 added. (In fact, much weaker systems suffice.) On the one hand, it follows from Theorem 1 that these theorems can be re-interpreted as true assertions about a great variety of number-systems other than the natural numbers. On the other hand, it follows from the results of Gödel[4] that there are true theorems about the natural numbers which cannot be proved by extant methods (consistency assumed).

Now Gödel's proof furnishes certain special formulas which are shown to be true but unprovable, but there is no general method indicated for establishing that a given theorem cannot be proved from given axioms. From Theorem 1, we see that such a method is supplied by the procedure of constructing non-standard models for number theory in which "set" and "function" are reinterpreted. It, therefore, becomes of practical interest to number-theorists to study the structure of such models.

A detailed investigation of these numerical structures is beyond the scope of the present paper. As an example, however, we quote one simple result: Every non-standard denumerable model for the Peano axioms has the order type $\omega + (\omega^* + \omega)\eta$, where η is the type of the rationals.

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