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Dependence Logic with a Majority Quantifier

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Abstract We study the extension of dependence logic $\mathcal{D}$ by a majority quantifier M over finite structures. We show that the resulting logic is equi-expressive with the extension of second-order logic by second-order majority quantifiers of all arities. Our results imply that, from the point of view of descriptive complexity theory, $\mathcal{D}(\mathsf{M})$ captures the complexity class counting hierarchy. We also obtain characterizations of the individual levels of the counting hierarchy by fragments of $\mathcal{D}(\mathsf{M})$.

Keywords Dependence logic · Dependence atom · Team semantics · Logic · Dependence · Complexity theory · Expressivity · Second order logic · Counting hierarchy · Majority

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1 Introduction

We study the extension of dependence logic $\mathcal{D}$ by a majority quantifier M over finite structures. Dependence logic (Väänänen 2007) extends first-order logic by dependence atomic formulas

$$=(t_1, \ldots, t_n)$$

the intuitive meaning of which is that the value of the term t_n is completely determined by the values of $t_1, \ldots, t_{n-1}$. While in first-order logic the order of quantifiers solely determines the dependence relations between variables, in dependence logic more general dependencies between variables can be expressed. Historically dependence logic was preceded by partially ordered quantifiers (Henkin quantifiers) of Henkin (1961) and Independence-Friendly (IF) logic of Hintikka and Sandu (1989). It is known that both IF logic and dependence logic are equivalent to existential second-order logic ESO in expressive power. From the point of view of descriptive complexity theory, this means that dependence logic captures the class NP.

The framework of dependence logic has turned out be flexible to allow interesting generalizations. For example, the extensions of dependence logic in terms of so-called intuitionistic implication and linear implication were introduced in Abramsky and Väänänen (2009). In Yang (2013) it was shown that extending $\mathcal{D}$ by the intuitionistic implication makes the logic equivalent to full second-order logic SO.

Recently, new variants of the dependence atomic formulas have been introduced in Grädel and Väänänen (2013) and Galliani (2012). Also a modal version of dependence logic was introduced in Väänänen (2008) and has been studied in Lohmann and Vollmer (2013) and Sevenster (2009). In this paper we are concerned with introducing a new quantifier to dependence logic: the majority quantifier. Adding majority and, more generally, counting capabilities to logical formalisms or computational devices has deserved a lot of attention in theoretical computer science. Understanding the power of counting is an important problem both in logic and in computational complexity:

- The circuit class TC^0, the class of problems solvable by polynomial-size constant-depth circuits with majority gates, is at the current frontier for lower bound techniques [see, e.g., Vollmer (1999)]. We have strict separations of classes within TC^0, but above TC^0 we have essentially no lower bounds. By a diagonalization it follows that TC^0 is different from the second level of the exponential-time hierarchy, but a separation from a lower class seems to be far away. In particular, the question if TC^0 equals NC^1 (logarithmic-depth circuits with bounded fan-in gates) is considered the P-NP problem of circuit complexity. [Concerning uniform circuits, we know that uniform TC^0 is strictly included in the class PP of probabilistic polynomial time (Allender 1999).]
- The counting-hierarchy (the oracle hierarchy built upon PP) can be characterized using majority quantifiers in just the same way as by Wrathall's theorem existential and universal quantifiers characterize the polynomial hierarchy (Torán 1991).
- By Toda's theorem, one majority quantifier is as powerful as the whole polynomial hierarchy (Toda 1991).

Here we suggest a definition of a majority quantifier for dependence logic. The proposed semantics mimics that of the existential and universal quantifiers in $\mathcal{D}$. The present paper is devoted to a first study of the resulting logic, denoted by $\mathcal{D}(\mathsf{M})$. We examine some of its basic properties, prove strong normal forms (some of our technically most involved proofs are found here), and show in our main result, that dependence logic with the majority quantifier leads to a new descriptive complexity characterization of the counting hierarchy: $\mathcal{D}(\mathsf{M})$ captures CH. Fragments of the logic $\mathcal{D}(\mathsf{M})$, defined by restricting the number of nested uses of the majority quantifier, are shown to capture the individual levels of the counting hierarchy.

Engström (2012), Engström and Kontinen (2013) has also studied generalized quantifiers in dependence logic. He considered different conservative extensions of $\mathcal{D}$—informally this means that he extends $\mathcal{D}$ by generalized quantifiers in a first-order manner. From a descriptive complexity point of view, his logics do not lead out of NP, i.e., ESO, assuming the quantifier in question is ESO-definable (e.g., the majority quantifier). Our approach and results differ from that of Engström since we are in a sense extending dependence logic by a dependence majority quantifier, whose semantics is defined in close analogy with the semantics of $\exists$ and $\forall$ in dependence logic. The results of our paper show that our extension behaves like an extension of SO by second-order generalized quantifiers.

This article is organized as follows. In Sect. 2 we define dependence logic and discuss some basic results on it. Then we introduce a majority quantifier for the dependence logic setting and discuss the basic properties of $\mathcal{D}(\mathsf{M})$. In Sect. 2.3 we discuss the complexity class counting hierarchy and the second-order majority quantifiers Most^k that have been used to characterize it in Kontinen (2009). In Sect. 3, we introduce second-order majority quantifiers $\mathsf{Most}_\mathrm{f}^k$ ranging over functions and in Sect. 4 we show that, for sentences the logics $\mathrm{SO}(\mathsf{Most}_\mathrm{f})$ (the extension of second-order logic SO by $\mathsf{Most}_\mathrm{f}^k$ for $k \geq 1$) and $\mathcal{D}(\mathsf{M})$ are equivalent.

2 Preliminaries

In this section we first define dependence logic and discuss its basic properties. Then we define the counting hierarchy and the logic corresponding to it.

2.1 Dependence Logic

Dependence logic ($\mathcal{D}$) extends the syntax of first-order logic by new dependence atomic formulas. In this article we consider only formulas of $\mathcal{D}$ that are in negation normal form.

Definition 2.1 (Väänänen 2007) Let τ be a vocabulary. The τ-formulas of dependence logic ($\mathcal{D}[\tau]$) is defined by extending $\mathrm{FO}[\tau]$, defined in terms of $\vee$, $\wedge$, $\neg$, $\exists$ and $\forall$, by atomic dependence formulas

$$=\!(t_1, \ldots, t_n), \tag{1}$$

where $t_1, \ldots, t_n$ are terms.

The meaning of the formula (1) is that the value of the term t_n is functionally determined by the values of the terms $t_1, \ldots, t_{n-1}$. The formula $=()$ is interpreted as $\top$. The semantics of $\mathcal{D}$ will be formally presented shortly.

Definition 2.2 Let $\phi \in \mathcal{D}$. The set $\mathrm{Fr}(\phi)$ of free variables of a formula ϕ is defined as for first-order logic, except that we have the new case

$$\mathrm{Fr}(=(t_1, \dots, t_n)) = \mathrm{Var}(t_1) \cup \cdots \cup \mathrm{Var}(t_n),$$

where $\mathrm{Var}(t_i)$ is the set of variables occurring in term t_i. If $\mathrm{Fr}(\phi) = \emptyset$, we call ϕ a sentence.

The semantics of $\mathcal{D}$ is formulated using the concept of a *Team*. Let $\mathfrak{A}$ be a model with domain A. *Assignments* of $\mathfrak{A}$ are finite mappings from variables into A. The value of a term t in an assignment s is denoted by $t^{\mathfrak{A}}\langle s\rangle$. If s is an assignment, x a variable, and $a \in A$, then $s(a/x)$ denotes the assignment (with domain $\mathrm{dom}(s) \cup \{x\}$) that agrees with s everywhere except that it maps x to a.

Definition 2.3 Let A be a set and $\{x_1, \dots, x_k\}$ a finite (possibly empty) set of variables.

1. A *team* X of A with domain $\mathrm{dom}(X) = \{x_1, \dots, x_k\}$ (we call A the *co-domain* of X) is any set of assignments $s\colon \{x_1, \dots, x_k\} \to A$.
2. The relation $\mathrm{rel}(X) \subseteq A^k$ corresponding to X is defined as

$$\mathrm{rel}(X) = \{(s(x_1), \dots, s(x_k)) : s \in X\}.$$

   In general, the order of the values of variables in the tuples of $\mathrm{rel}(X)$ is the one inherited from the natural order of the indices of the variables.
3. For a function $F\colon X \to A$, we define

$$X(F/x) = \{s(F(s)/x) : s \in X\}$$
$$X(A/x) = \{s(a/x) : s \in X \text{ and } a \in A\}.$$

We will next define the semantics of dependence logic. Below, atomic formulas and their negations are called literals.

Definition 2.4 (Väänänen 2007) Let $\mathfrak{A}$ be a model and X a team of A. The satisfaction relation $\mathfrak{A} \models_X \phi$ is defined as follows:

1. If ϕ is a first-order literal, then $\mathfrak{A} \models_X \phi$ iff for all $s \in X$ we have $\mathfrak{A} \models_s \phi$.
2. $\mathfrak{A} \models_X =(t_1, \dots, t_n)$ iff for all $s, s' \in X$ such that $t_1^{\mathfrak{A}}\langle s\rangle = t_1^{\mathfrak{A}}\langle s'\rangle, \dots, t_{n-1}^{\mathfrak{A}}\langle s\rangle = t_{n-1}^{\mathfrak{A}}\langle s'\rangle$, we have $t_n^{\mathfrak{A}}\langle s\rangle = t_n^{\mathfrak{A}}\langle s'\rangle$.
3. $\mathfrak{A} \models_X \neg =(t_1, \dots, t_n)$ iff $X = \emptyset$.
4. $\mathfrak{A} \models_X \psi \wedge \theta$ iff $\mathfrak{A} \models_X \psi$ and $\mathfrak{A} \models_X \theta$.
5. $\mathfrak{A} \models_X \psi \vee \theta$ iff $X = Y \cup Z$ such that $\mathfrak{A} \models_Y \psi$ and $\mathfrak{A} \models_Z \theta$.
6. $\mathfrak{A} \models_X \exists x\psi$ iff $\mathfrak{A} \models_{X(F/x)} \psi$ for some $F\colon X \to A$.
7. $\mathfrak{A} \models_X \forall x\psi$ iff $\mathfrak{A} \models_{X(A/x)} \psi$.

Above, we assume that the domain of X contains the variables free in ϕ. Finally, a sentence ϕ is true in a model $\mathfrak{A}$ (in symbols: $\mathfrak{A} \models \phi$) if $\mathfrak{A} \models_{\{\emptyset\}} \phi$. Above, $A \models_s \phi$ denotes satisfaction in first-order logic.

Let us then recall some basic properties of dependence logic that will be needed later. The following lemma shows that the truth of a $\mathcal{D}$-formula depends only on the interpretations of variables occurring free in the formula. Below, for $V \subseteq \mathrm{dom}(X)$, $X \upharpoonright V$ is defined by

$$X \upharpoonright V := \{s \upharpoonright V \mid s \in X\}.$$

Lemma 2.5 (Väänänen 2007) *Suppose* $V \supseteq \mathrm{Fr}(\phi)$. *Then* $\mathfrak{A} \models_X \phi$ *if and only if* $\mathfrak{A} \models_{X \upharpoonright V} \phi$.

All formulas of dependence logic also satisfy the following strong monotonicity property called *downward closure*.

Proposition 2.6 (Väänänen 2007) *Let* ϕ *be a formula of dependence logic,* $\mathfrak{A}$ *a model, and* $Y \subseteq X$ *teams. Then* $\mathfrak{A} \models_X \phi$ *implies* $\mathfrak{A} \models_Y \phi$.

On the other hand, the expressive power of sentences of $\mathcal{D}$ coincides with that of existential second-order sentences:

Theorem 2.7 (Väänänen 2007) $\mathcal{D} \equiv \mathrm{ESO}$.

Finally, we note that dependence logic is a conservative extension of first-order logic.

Definition 2.8 A formula ϕ of $\mathcal{D}$ is called a first-order formula if it does not contain dependence atomic formulas as subformulas.

First-order formulas of dependence logic satisfy the so-called *flatness* property:

Theorem 2.9 (Väänänen 2007) *Let* ϕ *be a first-order formula of dependence logic. Then for all* $\mathfrak{A}$ *and* X*:*

$$\mathfrak{A} \models_X \phi \textit{ if and only if for all } s \in X \text{ we have } \mathfrak{A} \models_s \phi.$$

2.2 Dependence Logic with a Majority Quantifier

The main topic of the present paper is the study of a logic obtained from $\mathcal{D}$ by the introduction of a majority quantifier M. We denote this extended logic by $\mathcal{D}(\mathsf{M})$. We will use the following shorthand notation in the definition of the semantics of M: Assume $V \subseteq \mathrm{dom}(X)$ and F is a function $F\colon X \upharpoonright V \to A$. Then F determines a unique $F'\colon X \to A$ defined by $F'(s) = F(s \upharpoonright V)$. In the following, the mapping F' is also denoted by F. The semantics of the quantifier M is now defined by the following clause:

$$\mathfrak{A} \models_X \mathsf{M}x\, \phi \text{ iff for at least } \lceil |A|^{|X \upharpoonright V|}/2 \rceil \text{ many } F\colon X \upharpoonright V \to A \text{ we have } \mathfrak{A} \models_{X(F/x)} \phi,$$

where $V = \mathrm{Fr}(\phi) - \{x\}$.

It is worth noting that the above definition of the quantifier M differs slightly from the one used in the previous version Durand et al. (2011) of this article. The majority quantifier of Durand et al. (2011) states that a majority of the functions $F: X \to A$ satisfy a formula whereas in this article we count only functions $F: X \upharpoonright V \to A$, where $V = \mathrm{Fr}(\phi) - \{x\}$. The drawback of the previous definition is the failure of the locality property (Proposition 2.12) (Durand et al. 2011) which holds under the definition of M adopted in this article. A consequence of referring to the free variables of a formula in the definition of M is that the semantics of $\mathcal{D}(\mathsf{M})$ becomes sensitive to "dummy" variables: the formulas $P(x)$ and $P(x) \wedge y = y$ are equivalent over models and teams with domain $\{x, y\}$ but it is easy to check that the sentences $\forall y \mathsf{M} x\, P(x)$ and $\forall y \mathsf{M} x\, (P(x) \wedge y = y)$ are not logically equivalent.

Analogously to $\mathcal{D}$ the logic $\mathcal{D}(\mathsf{M})$ has the so-called empty team property:

Proposition 2.10 *For all models $\mathfrak{A}$ and formulas ϕ of $\mathcal{D}(\mathsf{M})$, it holds that $\mathfrak{A} \models_{\emptyset} \phi$.*

Proof The claim is proved using induction on ϕ. □

We also observe that $\mathcal{D}(\mathsf{M})$ satisfies the downward closure property (compare to Proposition 2.6).

Proposition 2.11 *Let ϕ be a formula of $\mathcal{D}(\mathsf{M})$, $\mathfrak{A}$ a model, and $Y \subseteq X$ teams. Then $\mathfrak{A} \models_X \phi$ implies $\mathfrak{A} \models_Y \phi$.*

Proof The claim is proved using induction on ϕ. We consider the case where ϕ is $\mathsf{M}x\, \psi$. The other cases are proved exactly as for dependence logic [see Proposition 3.10 in Väänänen (2007)]. Let $\mathfrak{A}$, X and Y be as above and suppose that $|A| = n$ and $|X \upharpoonright V| = m$, where $V = \mathrm{Fr}(\psi) - \{x\}$. Since we restrict attention to finite structures, we may without loss of generality assume that $|Y| = |X| - 1$. Now obviously $|Y \upharpoonright V| \geq |X \upharpoonright V| - 1$. In the trivial case, where $X \upharpoonright V = Y \upharpoonright V$, the claim follows easily using the induction assumption for ψ.

Suppose then that $|Y \upharpoonright V| = |X \upharpoonright V| - 1$, and let $F: X \upharpoonright V \to A$. By the induction hypothesis we have that if $\mathfrak{A} \models_{X(F/x)} \psi$ holds, then it also holds that $\mathfrak{A} \models_{Y(G/x)} \psi$, where G is the reduct of F to the domain $Y \upharpoonright V$. Note that in the worst case at most n different functions F give rise to the same reduct G. Therefore, since $\mathfrak{A} \models_X \mathsf{M}x\, \psi$, the number of functions $G: Y \upharpoonright V \to A$ satisfying $\mathfrak{A} \models_{Y(G/x)} \psi$ is at least $n^m/2n = n^{m-1}/2 = |A|^{|Y \upharpoonright V|}/2$ and hence $\mathfrak{A} \models_Y \phi$. □

Proposition 2.12 (Locality of $\mathcal{D}(\mathsf{M})$) *Let $\mathfrak{A}$ be a structure, X a team and ϕ be a $\mathcal{D}(\mathsf{M})$ formula. Then it holds that*

$$\mathfrak{A} \models_X \phi \textit{ if and only if } \mathfrak{A} \models_{X \upharpoonright \mathrm{Fr}(\phi)} \phi.$$

Proof We proof the claim via induction on ϕ. We consider the case $\phi = \mathsf{M}x\, \psi$ only since other cases are analogous to the proof of Lemma 2.5 (Väänänen (2007)). Without loss of generality we may assume that $x \in \mathrm{Fr}(\psi)$. Let us first assume $\mathfrak{A} \models_X \phi$. Let $V = \mathrm{Fr}(\phi)$. Then $\mathfrak{A} \models_{X(F/x)} \psi$ holds for at least $|A|^{|X \upharpoonright V|}/2$ many functions

$$F: X \upharpoonright V \to A.$$

By the induction hypothesis, for each such F, it also holds that $\mathfrak{A} \models_{X(F/x)\upharpoonright V\cup\{x\}} \psi$. It is straightforward to check that

$$X(F/x) \upharpoonright V \cup \{x\} = (X \upharpoonright V)(F/x),$$

hence it follows that $\mathfrak{A} \models_{(X\upharpoonright V)(F/x)} \psi$ holds for at least $|A|^{|X\upharpoonright V|}/2$ many functions $F : X \upharpoonright V \to A$, and thus $\mathfrak{A} \models_{X\upharpoonright V} \phi$; note that here we use the trivial fact that

$$(X \upharpoonright V) \upharpoonright V = X \upharpoonright V.$$

The converse implication is proved by reversing the steps above. □

2.3 Second-order Majority Quantifiers and the Counting Hierarchy

In this section we define the counting hierarchy and the relevant generalized quantifiers.

Definition 2.13 Let $k \geq 1$. We define the k-ary second-order generalized quantifier Most^k binding a k-ary relation symbol X in a formula ϕ. Assume $\mathfrak{A}$ is a structure with domain A such that $|A| = n$. Then the semantics of this quantifier is defined as follows:

$$\mathfrak{A} \models \mathsf{Most}^k X\phi(X) \iff \left|\left\{B \subseteq A^k \mid \mathfrak{A} \models \phi(B)\right\}\right| \geq 2^{n^k}/2.$$

We will also make use of the so-called k-ary second-order *Rescher quantifier*, defined as follows:

$$\begin{aligned}&\mathfrak{A} \models \mathsf{R}^k X, Y(\phi(X), \psi(Y)) \iff \\ &\left|\left\{B \subseteq A^k \mid \mathfrak{A} \models \phi(B)\right\}\right| \geq \left|\left\{B \subseteq A^k \mid \mathfrak{A} \models \psi(B)\right\}\right|.\end{aligned}$$

It is quite easy to see that the Most^k-quantifier can be defined in terms of the quantifier R^k. In Kontinen (2009) it was shown that the k-ary Rescher quantifier R^k can be defined in first order logic with Most^{k+1}, and, for $k \geq 2$, already with Most^k. It is worth noting that in Kontinen (2009) the quantifiers Most^k and R^k are interpreted as strict majority and strict inequality, respectively. All the results of Kontinen (2009) that we use also hold under the "non-strict" interpretation adopted in this article.

The counting hierarchy (CH) is the analogue of the polynomial hierarchy, defined as the oracle hierarchy using as building block probabilistic polynomial time (the class PP) instead of NP:

1. $\mathrm{C_0P} = \mathrm{P}$,
2. $\mathrm{C}_{k+1}\mathrm{P} = \mathrm{PP}^{\mathrm{C}_k\mathrm{P}}$,
3. $\mathrm{CH} = \bigcup_{k\in\mathbb{N}} \mathrm{C}_k\mathrm{P}$.

The counting hierarchy was first defined by Wagner (1986) but the above equivalent formulation is due to Torán (1991). The class PP consists of languages L for which there is a polynomial time-bounded nondeterministic Turing machine N such that, for all inputs x, $x \in L$ iff more than half of the computations of N on input x accept.

In Kontinen (2009) it was shown that the extension FO(Most) of FO by the quantifiers Most^k, for $k \in \mathbb{N}$, describes exactly the problems in the counting hierarchy. The proof therein used the fact that the second-order existential quantifier can be simulated by Most^k and first-order logic.

Theorem 2.14 FO(Most) $\equiv$ SO(Most) $\equiv$ CH.

By the above remark we see that in the previous theorem the Most quantifiers can be replaced by Rescher quantifiers.

3 Majority over Functions

For our main result that compares second-order logic and dependence logic with majority-quantifiers, it turns out to be helpful to consider a version of the Most-quantifier that ranges over functions instead of relations.

Definition 3.1 Let $k \geq 1$. We define the k-ary second-order generalized quantifier $\mathsf{Most}_\mathrm{f}^k$ binding a k-ary function symbol g in a formula ϕ. Assume $\mathfrak{A}$ is a structure with domain A such that $|A| = n$. Then

$$\mathfrak{A} \models \mathsf{Most}_\mathrm{f}^k g\, \phi(g) \iff \left|\left\{f : A^k \to A \mid \mathfrak{A} \models \phi(f)\right\}\right| \geq n^{n^k}/2.$$

We denote by SO(Most_f) the extension of SO by the quantifiers $\mathsf{Most}_\mathrm{f}^k$ for all $k \geq 1$. The following elementary properties of SO(Most_f) will be useful.

Proposition 3.2 *The following equivalences hold:*

1. $(\phi \vee \mathsf{Most}_\mathrm{f}^k g\, \psi) \equiv \mathsf{Most}_\mathrm{f}^k g\, (\phi \vee \psi)$, *if g does not appear free in ϕ,*
2. $(\phi \wedge \mathsf{Most}_\mathrm{f}^k g\, \psi) \equiv \mathsf{Most}_\mathrm{f}^k g\, (\phi \wedge \psi)$, *if g does not appear free in ϕ.*

The equivalences of Proposition 3.2 obviously hold also for the relational majority quantifiers Most^k.

The next proposition states the intuitively obvious fact that the extensions of SO by the quantifiers Most^k or alternatively by $\mathsf{Most}_\mathrm{f}^k$, for $k \in \mathbb{N}$, are equal in expressive power.

Proposition 3.3 SO(Most) $\equiv$ SO(Most_f).

Proof We prove the claim by an argument analogous to Theorem 3.4 in Kontinen (2009). We will show how to express the quantifier $\mathsf{Most}_\mathrm{f}^k$ in the logic SO(Most) implying SO(Most_f) $\leq$ SO(Most). The converse inclusion is proved analogously.

Let us consider a formula of the form $\mathsf{Most}_\mathrm{f}^k g\, \phi(g) \in$ SO(Most_f). Let $\mathfrak{A}$ be a structure. We may assume that $\mathfrak{A}$ is ordered (we can existentially quantify an order) and hence there is an FO-formula $\delta(\overline{x}, \overline{y})$ defining the lexicographic ordering of the set A^{k+1}. We can construct a formula $\chi(X, Y)$ which, for $A_1, A_2 \subseteq A^{k+1}$, defines the lexicographic ordering ($A_1 \leq_l A_2$) of $k+1$-ary relations induced by $\delta(\overline{x}, \overline{y})$.

It is now fairly straightforward to express $\mathsf{Most}_\mathrm{f}^k g\, \phi(g)$ in the logic SO(Most). Let

$$G = \{B \subseteq A^{k+1} \mid B \text{ is graph of some } g : A^k \to A \text{ and } \mathfrak{A} \models \phi(g)\},$$

$$G^c = \{B \subseteq A^{k+1} \mid B \text{ is graph of some } g\colon A^k \to A \text{ and } \mathfrak{A} \not\models \phi(g)\}.$$

It now suffices to express $|G| \geq |G^c|$ in the logic SO(Most). For a $D \subseteq A^{k+1}$, define the set IS(D) (the "initial segment" determined by D) by

$$\mathrm{IS}(D) = \{D' \subseteq A^{k+1} | \ D' \notin G \cup G^c \text{ and } D' \leq_l D\}.$$

The condition $|G| \geq |G^c|$ can be now expressed by

$$\forall D\big(|G^c \cup \mathrm{IS}(D)| \geq 2^{n^{k+1}}/2 \Rightarrow |G \cup \mathrm{IS}(D)| \geq 2^{n^{k+1}}/2\big).$$

It is straightforward to express this in the logic SO(Most). □

The following lemma will be needed in the proof of the next proposition.

Lemma 3.4 *Let $k \geq 1$ and let g be a k-ary function variable symbol. There exists an* ESO *sentence χ of vocabulary $\{\leq\}$ such that for all structures $\mathfrak{A} = (A, \leq^{\mathfrak{A}})$, where $A = \{0, 1, \ldots, n-1\}$, $\leq^{\mathfrak{A}}$ is the natural order on A, and $n > 2$, χ is satisfied by exactly $\lceil n^{n^k}/2 \rceil - 2^{n^k - 1}$ many k-ary functions g none of which is a characteristic function of some k-ary relation, i.e., $g(\overline{a}) \notin \{0, 1\}$ for some $\overline{a} \in A^k$.*

Proof Let us first consider the case that n is even. Let $\varphi(U)$, where U is a unary relation symbol, be the sentence

$$\forall x(U(0) \wedge (U(x) \leftrightarrow \neg U(x+1))). \tag{2}$$

Note that, for a U satisfying $\varphi(U)$, there is a natural bijection between functions g such that $g(\overline{0}) \in U$ and functions h satisfying $h(\overline{0}) \notin U$, namely, if $f\colon A \to A$ is such that $f(a) = a + 1$ if a is even and $f(a) = a - 1$ otherwise, then

$$F\colon g \mapsto f \circ g,$$

is such a bijection of k-ary functions of A.

Define now $\chi'(g)$ as follows

$$\chi'(g) := \exists U(\varphi(U) \wedge U(g(\overline{0})) \wedge \exists \overline{x}\, g(\overline{x}) \notin \{0, 1\}).$$

Note that the last conjunct eliminates functions that correspond to a characteristic function of some k-ary relation. Now, over structures with domain of even cardinality, the sentence $\chi'(g)$ satisfies the claim of the lemma.

Suppose then that n is odd. Let $c = n - 1$, i.e., a definable constant from the linear order. Define $\varphi'(U)$ as follows:

$$\forall x(U(0) \wedge \neg U(c) \wedge (x < c - 1 \to (U(x) \leftrightarrow \neg U(x+1))).$$

Note that the formula $\varphi'(U)$ induces a partition of the domain of a structure into three parts: $\{c\}$, $U = \{0 \leq a \leq n-2 \mid a \text{ even}\}$, and $\{0 \leq a \leq n-2 \mid a \text{ odd}\}$ of which the two last ones have equal size. Analogously to the case "n even" above, we use this partition to define a formula that accepts exactly half of the k-ary functions which are not characteristic functions of k-ary relations. This can be achieved by sorting functions g according to whether the smallest element $\overline{a} \in A^k$ (in the lexicographic order) such that $g(\overline{a}) \neq c$ satisfies $g(\overline{a}) \in U$ or not. Now the following formula $\psi(g, U)$ accepts, in addition to the constant function $g(\overline{x}) = c$, exactly those functions g for which the smallest such tuple $\overline{a}$ is in U:

$$\psi(g, U) := (\exists \overline{x}\, U(g(\overline{x})) \wedge \forall \overline{y} < \overline{x}\, g(\overline{y}) = c) \vee (\forall \overline{x}\, g(\overline{x}) = c).$$

Define now $\chi''(g)$ as follows:

$$\chi''(g) = \exists U(\varphi'(U) \wedge \psi(g, U) \wedge \exists \overline{x}\, g(\overline{x}) \notin \{0, 1\}).$$

It is straightforward to check that over structures with domain of odd cardinality, the sentence $\chi''(g)$ satisfies the claim of the lemma. This can be explained algorithmically as follows. For each fixed $\overline{a} \in A^k$, the family of functions $g\colon A^k \to A$ such that $g(\overline{a}) \neq c$ and $g(\overline{b}) = c$ for all $\overline{b} < \overline{a}$ is divided into two parts of equal size by checking whether $g(\overline{a}) \in U$ or not (an analogous argument as in the case "n even" shows that the parts have equal size). Note that at the end ($\overline{a} = \overline{c}$) only the constant function $g(\overline{x}) = c$, for all $\overline{x} < \overline{n}$ remain. It is put explicitly into the "good" side by the second disjunct of formula $\psi(g, U)$. Therefore, $\chi''(g)$ is satisfied by half of the k-ary functions which are not characteristic functions of k-ary relations hence half of the number:

$$n^{n^k} - 2^{n^k} \text{ that is } \lceil n^{n^k}/2 \rceil - 2^{n^k - 1}.$$

The formula $\chi(g)$ satisfying the claim of the lemma is now defined as

$$\chi(g) := |A| \geq 3 \wedge ((\chi'(g) \wedge \theta_{even}) \vee (\chi''(g) \wedge \theta_{odd})),$$

where θ_{even} (respectively θ_{odd}) is a ESO-sentence expressing that $|A|$ is even (respectively odd). □

The next proposition gives a useful normal form for sentences of the logic $\mathrm{SO}(\mathsf{Most}_\mathrm{f})$.

Proposition 3.5 *Every sentence of* $\mathrm{SO}(\mathsf{Most}_\mathrm{f})$ *is equivalent to a sentence of the form*

$$\exists \overline{h}^1 \mathsf{Most}_\mathrm{f}^k g_1 \cdots \mathsf{Most}_\mathrm{f}^k g_l\, \exists \overline{h}^2 \theta,$$

where the function symbols in $\overline{h}^1$*, and* g_i *for* $1 \leq i \leq l$*, are* k*-ary* ($k \geq 3$)*, and* θ *is a universal first-order sentence.*

Proof Note that by Proposition 3.3 it suffices to show that every sentence of the logic $\mathrm{SO}(\mathsf{Most})$ can be transformed to this form. The result in Kontinen (2009) shows [as pointed out in Lemma 10.5 in Kontinen and Niemistö (2011)] that, in the presence of built-in relations $\{<, +, \times\}$, sentences of $\mathrm{SO}(\mathsf{Most})$ can be assumed to have the form

$$\mathsf{Most}^{i_1} Y_1 \cdots \mathsf{Most}^{i_l} Y_l\, \psi, \tag{3}$$

where ψ is first-order. Furthermore, when l in (3) is fixed, we get a fragment of $\mathrm{SO}(\mathsf{Most})$ characterizing the lth level of CH, i.e., the class $\mathrm{C}_l\mathrm{P}$.

We will next show how to transform any sentence of the form (3) to the required form. The first step is to quantify out the built-in relations $\{<, +, \times\}$ to get a sentence of the form

$$\exists X_< \exists X_+ \exists X_\times \mathsf{Most}^{i_1} Y_1 \cdots \mathsf{Most}^{i_l} Y_l\, \psi^*. \tag{4}$$

The relations $X_<$, X_+, and $X_\times$ can be axiomatized as part of ψ^* (compare to case 2 of Proposition 3.2). Then we modify the sentence (4) to change the arities of all the quantified relations to some big enough k. We need only to replace all occurrences, say $Y_i(t_1, \ldots, t_{i_j})$, of the quantified relation symbols in ψ^* by $Y_i(t_1, \ldots, t_{i_j}, 0, \ldots, 0)$. (Note that the needed constant 0 can be defined using the linear order.) Increasing the arity of the second-order existential quantifiers in (4) is clearly unproblematic. For the majority quantifiers Most^{i_j}, we note that for any structure $\mathfrak{A}$ of cardinality n and $B \subseteq A^v$, the number of k-ary relations $D \subseteq A^k$ such that

$$\{\overline{a} \in A^v \mid (\overline{a}, 0, \ldots, 0) \in D\} = B \tag{5}$$

is $2^{n^k - n^v}$, which is independent of B. Furthermore, obviously the truth of ψ^* with respect to a tuple of k-ary relations $D_1, \ldots, D_{l+3}$ only depends on whether $\psi^*(B_1, \ldots, B_{l+3})$ holds, where B_i is the restriction of D_i defined analogously to (5). This fact allows us to increase also the arity of the majority quantifiers without changing the meaning of the sentence (4).

Let us then show how to transform the relational quantifiers in (4) into function quantifiers. We claim that it is possible to replace $\psi^*(X_<, X_+, X_\times, Y_1, \ldots, Y_l)$ by a formula of the form

$$\theta(\overline{g}) \vee (\forall \overline{x} \left(\bigwedge_{1 \le i \le l} g_i(\overline{x}) \in \{0, 1\}) \wedge \right.$$
$$\left. \psi'(g_</X_<, g_+/X_+, g_\times/X_\times, g_1/Y_1, \ldots, g_l/Y_l) \right), \tag{6}$$

where $\overline{g} = (g_<, g_+, g_\times, g_1, \ldots, g_l)$, the new function symbols are all k-ary and ψ' is obtained from ψ^* by substituting subformulas $Z(t_1, \ldots, t_k)$ by the corresponding

$$g_{(.)}(1_1, \ldots, t_k) = 1,$$

where $Z \in \{Y_1, \ldots, Y_l, X_<, X_+, X_\times\}$. The formula $\theta(\overline{g})$ is a ESO-formula that accepts certain dummy functions in order to shift the border of acceptance from $(2^{|\mathfrak{A}|^k})/2$ (half of k-ary relations) to $|\mathfrak{A}|^{|\mathfrak{A}|^k}/2$ (half of k-ary functions). The logical form of θ is

$$\chi(g_1) \vee \chi(g_2) \vee \cdots \vee \chi(g_l),$$

where $\chi(g)$ is defined in Lemma 3.4. Note that we repeatedly use case 1 of Proposition 3.2 to gather all the formulas $\chi(g_i)$ into θ which is placed after the block of all majority quantifiers.

To prove the claim we transform the ESO-formula (6) into Skolem normal form to get a sentence of the form

$$\exists g_< \exists g_+ \exists g_\times \mathsf{Most}_{\mathrm{f}}^k g_1 \cdots \mathsf{Most}_{\mathrm{f}}^k g_l \, \exists \overline{h} \psi', \tag{7}$$

where ψ' is a universal FO-sentence and $\overline{h}$ is a tuple of function symbols. Note that we used Lemma 3.4 in the construction of the formula (7), and hence equivalence with the formula (3) holds for structures of cardinality at least three. Again using Proposition 3.2, the formulas (3) and (7) can be made equivalent over all finite structures by modifying the subformula $\exists \overline{h} \psi'$ suitably (Skolem normal form is retained). □

4 SO(Most) ≡ $\mathcal{D}$(M)

In this section we show that the logics SO($\mathsf{Most}_{\mathrm{f}}$) (and thus, by the previous section, SO(Most)) and $\mathcal{D}$(M) are equivalent with respect to sentences.

We will first show a compositional translation mapping formulas of $\mathcal{D}$(M) into sentences of SO($\mathsf{Most}_{\mathrm{f}}$). This translation is analogous to the translation from $\mathcal{D}$ into ESO of Väänänen (2007).

Lemma 4.1 *Let τ be a vocabulary. For every $\mathcal{D}(\mathsf{M})[\tau]$-formula ϕ there is a $\tau \cup \{S\}$-sentence ϕ^* of* SO($\mathsf{Most}_{\mathrm{f}}$)*, where S has arity $|\mathrm{Fr}(\phi)|$, such that for all models $\mathfrak{A}$ and teams X with $\mathrm{dom}(X) = \mathrm{Fr}(\phi)$ it holds that*

$$\mathfrak{A} \models_X \phi \iff (\mathfrak{A}, \mathrm{rel}(X)) \models \phi^*.$$

Proof We will prove the claim using induction on the structure of $\mathcal{D}$(M)-formulas. We consider only the case, where ϕ is of the form

$$\phi := \mathsf{M} y_n \gamma(y_1, \ldots, y_n). \tag{8}$$

The other cases are defined exactly as in the analogous translation from $\mathcal{D}$ into ESO in Väänänen (2007). Assume then that γ is a formula for which we have already a translation into an SO($\mathsf{Most}_{\mathrm{f}}$)[$\tau \cup S$] sentence $\gamma^*(S)$. Recall that the quantifiers R^k can be uniformly defined in the logic SO(Most), hence by the results of the previous section, also in SO($\mathsf{Most}_{\mathrm{f}}$). Therefore, we may use the quantifiers R^k in the translation below. We claim that ϕ can be translated as follows:

$$\phi^*(S) := \mathsf{R}^n \, Y, Z(\theta_1(Y), \theta_2(Z)) \tag{9}$$

where

$$\begin{aligned}\theta_1(Y) &:= \gamma^*(Y/S) \wedge \forall y_1 \ldots \forall y_{n-1} \exists^{=1} y_n Y(\overline{y}) \wedge \\ &\quad\ \forall y_1 \ldots \forall y_{n-1} (\exists y_n Y(\overline{y}) \leftrightarrow S(y_1, \ldots, y_{n-1})) \\ \theta_2(Z) &:= \neg\gamma^*(Z/S) \wedge \forall y_1 \ldots \forall y_{n-1} \exists^{=1} y_n Z(\overline{y}) \wedge \\ &\quad\ \forall y_1 \ldots \forall y_{n-1} (\exists y_n Z(\overline{y}) \leftrightarrow S(y_1, \ldots, y_{n-1})).\end{aligned}$$

The following equivalence is now obvious for all $\mathfrak{A}$ and X:

$$\mathfrak{A} \models_X \phi \Leftrightarrow (\mathfrak{A}, \mathrm{rel}(X)) \models \phi^*(S).$$

□

Next we will show that, for sentences, Lemma 4.1 can be reversed.

Lemma 4.2 *Let τ be a vocabulary and $\phi \in \mathrm{SO}(\mathsf{Most}_\mathrm{f})[\tau]$. Then there is a sentence $\psi \in \mathcal{D}(\mathsf{M})[\tau]$ such that for all models $\mathfrak{A}$:*

$$\mathfrak{A} \models \phi \iff \mathfrak{A} \models \psi.$$

Proof By Proposition 3.5 we may assume that ϕ is of the form:

$$\exists \overline{h}^1 \mathsf{Most}^k_\mathrm{f} g_1 \cdots \mathsf{Most}^k_\mathrm{f} g_n \, \exists \overline{h}^2 \forall x_1 \cdots \forall x_m \psi, \tag{10}$$

where the function symbols in $\overline{h}^1$ and $g_1, \ldots, g_n$ are k-ary, and ψ is quantifier free. Before translating this sentence into $\mathcal{D}(\mathsf{M})$, we will first apply certain reductions to it. First of all, we make sure that the functions g_i have only occurrences of the form $g_i(x_1, \ldots, x_k)$ in ψ. We can achieve this by existentially quantifying new names f_i for these symbols and passing on to the sentence

$$\exists \overline{h}^1 \mathsf{Most}^k_\mathrm{f} g_1 \cdots \mathsf{Most}^k_\mathrm{f} g_n \, \exists \overline{h}^2 \exists f_1 \cdots \exists f_n \forall x_1 \cdots \forall x_m \left(\bigwedge_{1 \le j \le n} g_j(x_1, \ldots, x_k) = f_j(x_1, \ldots, x_k) \wedge \psi^* \right), \tag{11}$$

where ψ^* is obtained from ψ by replacing all occurrences of g_i by f_i for $1 \le j \le n$. Analogously, we may also assume that the functions h in $\overline{h}^1$ have only occurrences $h(x_1, \ldots, x_k)$ in ψ. Here m can always be made at least k.

The next step is to transform the quantifier-free part ψ^* to satisfy the condition that for each of the function symbols h in $\overline{h}^2$ (also f_i) there is a unique tuple $\overline{x}$ of pairwise distinct variables such that all occurrences of it in ψ^* are of the form $h(\overline{x})$ ($f_i(\overline{x})$). In order to achieve this, we might have to introduce new existentially quantified functions and also universal first-order quantifiers (see Theorem 6.15 in Väänänen (2007)), but the quantifier structure of the sentence (10) does not change.

We will now assume that the sentence (10) has the properties discussed above:

1. The function symbols $h \in \overline{h}^1$ and g_i have only occurrences of the form $h(x_1, \ldots, x_k)$ and $g_i(x_1, \ldots, x_k)$ in ψ, respectively.
2. For each h in $\overline{h}^2$ (f_i, for $1 \leq i \leq n$) there is a unique tuple $\overline{x}$ of pairwise distinct variables such that all occurrences of h in ψ^* are of the form $h(\overline{x})$ ($f_i(\overline{x})$).

We will now show how the sentence (10) can be translated into $\mathcal{D}(\mathsf{M})$. For the sake of bookkeeping, we assume that $\overline{h}^1 = h_1 \ldots h_p$, $\overline{h}^2 = h_{p+1} \ldots h_r$, and that h_i appears in ψ only as $h_i(\overline{x}^i)$. We claim now that the following sentence of $\mathcal{D}(\mathsf{M})$ is a correct translation for (10):

$$\begin{aligned}&\forall x_1 \cdots \forall x_k \exists y_1 \cdots \exists y_p \mathsf{M} z_1 \cdots \mathsf{M} z_n \\ &\quad \forall x_{k+1} \cdots \forall x_m \exists y_{p+1} \cdots \exists y_r \left(\bigwedge_{p+1 \leq j \leq r} =\!(\overline{x}^i, y_i) \wedge \theta \right),\end{aligned} \tag{12}$$

where θ is obtained from ψ by replacing all occurrences of the term $g_i(x_1, \ldots, x_k)$ by the variable z_i and, similarly, each occurrence of $h_i(\overline{x}^i)$ by y_i. Note that, by the definition of the team semantics, the values of the variables z_i are determined by the values of the universally quantified variables $x_1, \ldots, x_k$, hence dependence atoms $=\!(\overline{x}, z_i)$ are not needed in the translation.

Let us then show that the sentence ϕ (see (10)) and sentence (12) are logically equivalent. Let $\mathfrak{A}$ be a structure and let $\mathbf{h}_1, \ldots, \mathbf{h}_r$ and $\mathbf{g}_1, \ldots, \mathbf{g}_n$ interpret the corresponding function symbols. We will show that the following holds:

$$(\mathfrak{A}, \overline{\mathbf{h}}, \overline{\mathbf{g}}) \models_X \psi \Leftrightarrow \mathfrak{A} \models_{X^*} \theta, \tag{13}$$

where $X = \{\emptyset\}(A/x_1) \cdots (A/x_m)$ and

$$\begin{aligned} X^* = \{\emptyset\} \quad & (A/x_1) \cdots \\ & (A/x_k)(H_1/y_1) \cdots \\ & (H_p/y_p)(G_1/z_1) \cdots \\ & (G_n/z_n)(A/x_{k+1}) \cdots \\ & (A/x_m)(H_{p+1}/y_1) \cdots \\ & (H_r/y_r), \end{aligned}$$

where the supplement functions H_i and G_i are defined using the functions $\mathbf{h}_i$ and $\mathbf{g}_i$ as follows:

$$\begin{aligned} H_i(s) &= \mathbf{h}_i(s(x_1), \ldots, s(x_k)) \text{ for } 1 \leq i \leq p, \\ H_i(s) &= \mathbf{h}_i(s(\overline{x}^i)) \text{ for } p+1 \leq i \leq r, \\ G_i(s) &= \mathbf{g}_i(s(x_1), \ldots, s(x_k)) \text{ for } 1 \leq i \leq n, \end{aligned}$$

and where $s(\overline{x}^i)$ is the tuple obtained by pointwise application of s. The claim in (13) is now proved using induction on the structure of the quantifier-free formula ψ. Note that ψ is a first-order formula of dependence logic; hence, by Theorem 2.9, (13) holds iff the equivalence holds for each $s \in X$ (equivalently $s \in X^*$ since the values of the universally quantified variables functionally determine the values of all the other variables) individually. We can now show, using induction on the construction of ψ, that for all $s \in X^*$ it holds that

$$\mathfrak{A} \models_s \theta \iff (\mathfrak{A}, \overline{\mathbf{h}}, \overline{\mathbf{g}}) \models_{s'} \psi, \tag{14}$$

where $s' = s \upharpoonright \{x_1, \ldots, x_m\}$. The key to this result is the fact that, for every s, the interpretation of the variables z_i and y_i agree with the interpretation of the terms $h_i(\overline{x}^i)$ and $g(x_1, \ldots, x_k)$, respectively.

Finally, we note that there is a one-to-one correspondence between all possible interpretations $\mathbf{h}_1, \ldots, \mathbf{h}_r$ and $\mathbf{g}_1, \ldots, \mathbf{g}_n$ for the function symbols and teams X^* satisfying the dependence atomic formulas in (12). Therefore, sentence ϕ (see (10)) and sentence (12) are logically equivalent. □

The just given two lemmas immediately yield the following extension of Theorem 2.14, giving a descriptive complexity characterization of the counting hierarchy in terms of dependence logic.

Corollary 4.3 FO(Most) $\equiv$ SO(Most) $\equiv \mathcal{D}(\mathsf{M}) \equiv$ CH.

Making use of a well-known result by Toda and Watanabe Toda and Watanabe (1992) we conclude that certain fragments of $\mathcal{D}(\mathsf{M})$ characterize the classes that form the counting hierarchy.

Corollary 4.4 *In the presence of built-in relations* $\{<, +, \times\}$*, the class* $\mathrm{C}_k\mathrm{P}$ *is captured by the fragment of* $\mathcal{D}(\mathsf{M})$ *that consists only of sentences of the form*

$$\forall x_1 \cdots \forall x_l \mathsf{M} y_1 \cdots \mathsf{M} y_k \phi, \tag{15}$$

where $k \geq 2$, $l \geq 0$ *and* ϕ *is an arbitrary* $\mathcal{D}$*-formula.*

Proof The class $\mathrm{C}_k\mathrm{P}$ is captured by second-order formulas involving sequences of k nested Most-quantifiers (Kontinen and Niemistö 2011) (Lemma 10.5). By Lemma 4.2, such formulas can be transformed into the shape of (15).

Conversely, given a formula of the form (15), it follows from our results and the fact that $\mathcal{D}$ captures NP that it defines a language in the complexity class $\mathrm{C}_k\mathrm{P}^{\mathrm{NP}}$. From Toda and Watanabe's result that every function in #PH Turing-reduces to a function in #P, it follows that $\mathrm{PP}^{\mathrm{PH}} \subseteq \mathrm{P}^{\mathrm{PP}}$. Hence we conclude

$$\mathrm{C}_k\mathrm{P}^{\mathrm{NP}} \subseteq \mathrm{C}_k\mathrm{P}^{\mathrm{PH}} \subseteq \mathrm{C}_{k-1}\mathrm{P}^{\mathrm{P}^{\mathrm{PP}}} \subseteq \mathrm{C}_k\mathrm{P}.$$

□

5 Conclusion and Open Questions

We have seen that extending dependence logic by a majority quantifier increases the expressive power of dependence logic considerably. One particular consequence of our result is that $\mathcal{D}(\mathsf{M})$ is closed under classical negation on the level of sentences. Note further that, for open formulas, this does not hold because of the downward closure property of formulas.

In Kontinen and Väänänen (2009) it was shown that the open formulas of $\mathcal{D}$ correspond to the downwards monotone properties of NP [see Kontinen and Väänänen (2009) for the exact formulation]. We conjecture that the open formulas of $\mathcal{D}(\mathsf{M})$ correspond in an analogous manner to the downwards monotone properties of CH.

The majority quantifier is only one particular example of so-called *generalized quantifiers* (or, *Lindström quantifiers*), introduced in Lindström (1966) and studied extensively in the context of descriptive complexity theory [see surveys Väänänen (1999) and Ebbinghaus and Flum (1999)]. In Burtschick and Vollmer (1998), second-order Lindström quantifiers were introduced and some results concerning their expressive power were obtained [see also Kontinen and Szymanik (2014), Kontinen (2010), Kontinen (2006), Andersson (2002)]. We consider it an interesting study to enrich in a similar way dependence logic by further generalized quantifiers [see the study of Ebbing of the parity quantifier (Ebbing 2014)] and relate the obtained logics to those studied in Burtschick and Vollmer (1998).

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References

Abramsky, S., & Väänänen, J. (2009). From IF to BI. *Synthese*, *167*(2), 207–230.

Allender, E. (1999). The permanent requires large uniform threshold circuits. *Chicago Journal of Theoretical Computer Science*, *7*, 19. (electronic).

Andersson, A. (2002). On second-order generalized quantifiers and finite structures. *Annals of Pure and Applied Logic*, *115*(1–3), 1–32.

Burtschick, H.-J., & Vollmer, H. (1998). Lindström quantifiers and leaf language definability. *International Journal of Foundations of Computer Science*, *9*(3), 277–294.

Durand, A., Ebbing, J., Kontinen, J., & Vollmer, H. (2011). Dependence logic with a majority quantifier. In S. Chakraborty & A. Kumar (Eds.), *IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2011)* (Vol. 13, pp. 252–263)., Leibniz International Proceedings in Informatics (LIPIcs) Dagstuhl, Germany: Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik.

Ebbing, J. (2014). *Complexity and expressivity of dependence logic extensions*. PhD thesis, Leibniz Universität Hannover.

Ebbinghaus, H.-D., & Flum, J. (1999). *Finite model theory perspectives in mathematical logic* (2nd ed.). Heidelberg: Springer.

Engström, F. (2012). Generalized quantifiers in dependence logic. *Journal of Logic, Language and Information*, *21*(3), 299–324.

Engström, F., & Kontinen, J. (2013). Characterizing quantifier extensions of dependence logic. *The Journal of Symbolic Logic*, *78*(1), 307–316.

Galliani, P. (2012). Inclusion and exclusion dependencies in team semantics—On some logics of imperfect information. *Annals of Pure and Applied Logic*, *163*(1), 68–84.

Grädel, E., & Väänänen, J. A. (2013). Dependence and independence. *Studia Logica*, *101*(2), 399–410.

Henkin, L. (1961). Some remarks on infinitely long formulas. In *Infinitistic Methods (Proceedings of the Symposium of Foundations of Mathematics, Warsaw, 1959)* (pp. 167–183). Pergamon, Oxford.

Hintikka, J., & Sandu, G. (1989). Informational independence as a semantical phenomenon. In *Logic, methodology and philosophy of science, VIII (Moscow, 1987), Vol 126 of Studies in Logic and the Foundations of Mathematics* (pp. 571–589). North-Holland, Amsterdam.

Kontinen, J. (2006). The hierarchy theorem for second order generalized quantifiers. *The Journal of Symbolic Logic*, *71*(1), 188–202.

Kontinen, J. (2009). A logical characterization of the counting hierarchy. *ACM Transactions on Computational Logic (TOCL)*, *10*(1), 7.

Kontinen, J. (2010). Definability of second order generalized quantifiers. *Archive for Mathematical Logic*, *49*(3), 379–398.

Kontinen, J., & Niemistö, H. (2011). Extensions of MSO and the monadic counting hierarchy. *Information and Computation*, *209*(1), 1–19.

Kontinen, J., & Szymanik, J. (2014). A characterization of definability of second-order generalized quantifiers with applications to non-definability. *Journal of Computer and System Sciences*, *80*(6), 1152–1162.

Kontinen, J., & Väänänen, J. (2009). On definability in dependence logic. *Journal of Logic, Language and Information*, *18*(3), 317–332.

Lindström, P. (1966). First order predicate logic with generalized quantifiers. *Theoria*, *32*, 186–195.

Lohmann, P., & Vollmer, H. (2013). Complexity results for modal dependence logic. *Studia Logica*, *101*(2), 343–366.

Sevenster, M. (2009). Model-theoretic and computational properties of modal dependence logic. *Journal of Logic and Computation*, *19*(6), 1157–1173.

Toda, S. (1991). PP is as hard as the polynomial-time hierarchy. *SIAM Journal on Computing*, *20*(5), 865–877.

Toda, S., & Watanabe, O. (1992). Polynomial time 1-turing reductions from #PH to #P. *Theoretical Computer Science*, *100*(1), 205–221.

Torán, J. (1991). Complexity classes defined by counting quantifiers. *Journal of the ACM*, *38*(3), 753–774.

Väänänen, J. (1999). Generalized quantifiers, an introduction. In *Generalized quantifiers and computation (Aix-en-Provence, 1997), Vol 1754 of Lecture Notes in Computer Science* (pp. 1–17). Berlin: Springer.

Väänänen, J. (2007). *Dependence logic: A new approach to independence friendly logic, volume 70 of London mathematical society student texts*. Cambridge: Cambridge University Press.

Väänänen, J. (2008). Modal dependence logic. In K. Apt & R. van Rooij (Eds.), *New perspectives on games and interaction, volume 5 of texts in logic and games* (pp. 237–254). Amsterdam: Amsterdam University Press.

Vollmer, H. (1999). Introduction to circuit complexity—-A uniform approach. In W. Brauer, G. Rozenberg & A. Salomaa (Eds.), *Texts in Theoretical Computer Science—An EATCS Series*. Berlin, Heidelberg: Springer.

Wagner, K. (1986). The complexity of combinatorial problems with succint input representation. *Acta Informatica*, *23*, 325–356.

Yang, F. (2013). Expressing second-order sentences in intuitionistic dependence logic. *Studia Logica*, *101*(2), 323–342.