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On modularity in infinitary term rewriting

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Abstract

We study modular properties in strongly convergent infinitary term rewriting. In particular, we show that:

- Confluence is not preserved across direct sum of a finite number of systems, even when these are non-collapsing.
- Confluence modulo equality of hypercollapsing subterms is not preserved across direct sum of a finite number of systems.
- Normalization is not preserved across direct sum of an infinite number of left-linear systems.
- Unique normalization with respect to reduction is not preserved across direct sum of a finite number of left-linear systems.

Together, these facts constitute a radical departure from the situation in finitary term rewriting. Positive results are:

- Confluence is preserved under the direct sum of an infinite number of left-linear systems iff at most one system contains a collapsing rule.
- Confluence is preserved under the direct sum of a finite number of non-collapsing systems if only terms of finite rank are considered.
- Top-termination is preserved under the direct sum of a finite number of left-linear systems.
- Normalization is preserved under the direct sum of a finite number of left-linear systems.

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All of the negative results above hold in the setting of weakly convergent rewriting as well, as do the positive results concerning modularity of top-termination and normalization for left-linear systems.

Keywords: Term rewriting; Modularity; Infinitary rewriting; Strong convergence; Church-rosser property; Confluence; Normalization

1. Introduction

1.1. Infinitary term rewriting

Term rewriting is the study of objects called *terms*, built from finite-arity functions symbols, constants and variables, and the stepwise, context-free transformation of these by *rules* [6,15,1,31]. As such, term rewriting is a fundamental tool in logic, and is one of the most popular and transparent operational models of functional programming languages.

Term rewriting has been extended in two major ways to model lazy programming languages such as HASKELL [10]. The first is the study of *graph rewriting* that provides the theoretical background closest to actual implementation [26,2], but many simple analyses from rewriting require elaborate refactoring to be used in graph rewriting, since both term *formation* (now requiring arcs), and term *rewriting* (no longer simple substitutions in contexts) is more complex in that setting.

The second way allows terms to be *infinite*. The rationale is that a lazy program "approximate" the infinite term, halting after only evaluating part of it; or, if the program does not halt, it approximates the term arbitrarily well. From a mathematical point-of-view, it is much cleaner to consider a single object that represents all possible finite behaviours, and their eventual limit, than to truncate the object, and computations involving it, at some finite level and consider all such truncations. Hence the study of infinite terms, and potentially infinite reductions.

The inception of this field of *infinitary term rewriting* was the POPL '89 paper by Dershowitz et al. who considered what has become known as *weakly* (or *Cauchy*) *convergent rewriting* [7,8]. As it turned out, weak convergence was not quite enough to ensure desirable properties (cf. [29]), and Kennaway et al. subsequently introduced *strongly convergent rewriting* [13,14,11], recovering—at least partially—many results from finitary rewriting. Strongly convergent rewriting has been extended to infinitary lambda calculus [3,12,4,28,5], and—coming full circle—has been used to study the semantics of finitary rewriting [18].

1.2. Modularity

The direct sum of two or more term rewriting systems over disjoint alphabets is the set of terms over the union of all the respective alphabets, equipped with the union of all rules, now applicable to the new, larger, set of terms. *Modularity* is the study of what properties are preserved across the direct sum.

As an example, consider the below system; one of many possible for computing Parallel Or:

$$R_0 \triangleq \left\{ \begin{array}{l} \mathbf{por}(\mathbf{t}, y) \longrightarrow \mathbf{t} \\ \mathbf{por}(x, \mathbf{t}) \longrightarrow \mathbf{t} \\ \mathbf{por}(\mathbf{f}, \mathbf{f}) \longrightarrow \mathbf{f} \end{array} \right\}$$

R_0 is weakly orthogonal, hence confluent by standard results in term rewriting.

Consider, next, the first-order version of **map** below:

$$R_1 \triangleq \left\{ \begin{array}{rcl} \mathbf{map}(\mathbf{p}(x), \mathbf{nil}) & \longrightarrow & \mathbf{nil} \\ \mathbf{map}(\mathbf{p}(x), \mathbf{cons}(y, z)) & \longrightarrow & \mathbf{cons}(\mathbf{p}(y), \mathbf{map}(\mathbf{p}(x), z)) \end{array} \right\}$$

R_1 is orthogonal, hence confluent by standard results. By Toyama's Theorem [33,16] confluence is modular, and the direct sum of the two above systems is thus confluent.

Reverting to the functional programming analogy, we may think of two different "modules," each consisting of a number of function definitions that may be combined into actual programs. If each of these provably has desirable properties, e.g., every program is Church-Rosser, or every program terminates, then it is of obvious interest whether this property is preserved when considering programs built from elements of each module. Furthermore, such reasoning allows for application of divide-and-conquer strategies when trying to ascertain whether some system has some property: split the system into subsystems more easily managed, and prove the property for each subsystem. If the property is modular, the original system will have the desired property as well.

Modularity in term rewriting is well-understood. Confluence [33,16], weak normalization [17], and unique normalization [20] are all modular. Termination [32], completeness [32,9], and unique normalization w.r.t. reduction [20] are not. Certain restrictions on the systems can recover modularity; termination is modular for non-duplicating, and for non-collapsing systems [22], and unique normalization w.r.t. reduction [19] and completeness [27] are both modular for left-linear systems.

In variations on term rewriting, modularity has been investigated in a variety of settings, e.g., conditional rewriting [21], systems with shared constructors [23], and composable systems [24]; a good overview can be found in [25].

In lazy languages, lists are potentially infinite objects, and it makes sense to ask whether confluence is modular for the two above systems in this context, where, for instance one may have terms such as

$$\mathbf{cons}(\mathbf{t}, \mathbf{cons}(\mathbf{por}(\mathbf{t}, \mathbf{f}), \mathbf{cons}(\mathbf{por}(\mathbf{t}, \mathbf{por}(\mathbf{t}, \mathbf{f})), \mathbf{cons}(\cdots))))$$

In infinitary rewriting, the system R_1 above is orthogonal and almost-non-collapsing, hence (transfinitely) confluent [13,11]; a moment's thought reveals that R_0 is (transfinitely) confluent as well, since it is almost non-collapsing and has the diamond property. However, it is not a priori clear whether the direct sum is (transfinitely) confluent.

1.3. This paper

In the present paper, we perform the first investigation of modularity issues in infinitary term rewriting. We work solely within the setting of *strongly convergent* rewriting, leaving modularity for weakly convergent rewriting open. However, all of our counterexamples apply to weakly convergent rewriting as well.

A preliminary version of the material of Sections 3, 4, 6, and 12 has previously appeared in the conference paper [30]; minor errors occurring in that paper have been corrected. The material on normalization in Sections 5, 7, 8, 9, and 10 is new.

The table of contents for the paper is as follows:

- Section 2 gives relevant definitions from the field of infinitary rewriting, paying particular attention to modularity in that setting.
- Section 3 contains a counterexample to the *modularity of confluence* that uses only non-collapsing systems.
- Section 4 gives necessary and sufficient conditions for the *modularity of confluence for left-linear systems.*
- Section 5 contains a counterexample to the modularity of *confluence modulo equality of hyper-collapsing subterms for left-linear systems.*
- Section 6 shows that *confluence is preserved across the direct sum of non-collapsing systems if only terms of finite rank are considered.*
- Section 7 contains a counterexample to *modularity of normalization for left-linear systems.*
- Section 8 shows that, for *left-linear systems*, *normalization is preserved across direct sum if only preserved terms (i.e., terms where no "modular collapses" can occur) are considered* and *top-termination is modular*.
- Section 9 shows that *normalization is modular when the direct sum is taken over a finite number of systems.*
- Section 10 contains a counterexample to the modularity of *unique normalization with respect to reduction for left-linear systems.*
- Section 11 informally outlines the difficulties inherent in extending our results to constructor-sharing unions.
- Section 12 summarizes the results of this paper that hold in *weakly* convergent rewriting, and discusses why the proofs of other properties fail to apply in that setting.
- Section 13 concludes and catalogues a number of open questions.

The reader is assumed to have a basic knowledge of ordinals and first-order, finitary term rewriting.

2. Preliminaries

We now introduce basic definitions from infinitary rewriting. All concepts until Section 2.4 are standard, cf. [11].

Throughout the paper, we work with signatures Σ consisting of finite-arity function symbols, presuppose a countably infinite set $\mathcal{X}$ of variables, and assume existence of a suitable "Hilbert-hotel" style renaming of the variables of any term if fresh variables are needed. The least infinite ordinal is denoted by ω, the least uncountable ordinal by Ω. We assume $\mathbb{N} = \{1, 2, \ldots\}$ and set $\mathbb{N}_0 = \omega = \{0, 1, 2, \ldots\}$.

Definition 1. The set of positions in a finite term s over Σ, denoted $Pos(s)$, is the subset of $\mathbb{N}^*$ defined in the usual way. The *strict* order on positions is denoted by $\prec$, equality of positions by $=$. If two distinct positions, u and v, are incomparable w.r.t. $\prec$, we say that they are *parallel*, written $u \parallel v$. The length of a position u is denoted by $|u|$ with the length of the empty position, ϵ, being 0. The set of finite terms $Ter\,(\Sigma)$ over Σ (and $\mathcal{X}$) is equipped with a metric $d : Ter\,(\Sigma) \times Ter\,(\Sigma) \longrightarrow [0\,;\,1]$ by letting $d(s,t) = 0$ if $s = t$ and otherwise $d(s,t) = 2^{-k}$ where k is the length of the shortest position at which s and t differ. The set of finite and infinite terms (henceforth just the set of *terms*) $Ter^{\infty}(\Sigma)$ over Σ is the metric completion of $(Ter\,(\Sigma), d)$. The notions of position and subterm carries over to $Ter^{\infty}(\Sigma)$ *mutatis mutandis*; if p is a position in a term s, we denote by $s|_p$ the subterm of s at p.

Definition 2. An *infinitary rewrite rule* is a pair $\mathbf{l} \longrightarrow \mathbf{r}$ where $\mathbf{l} \in Ter\,(\Sigma)$ and $\mathbf{r} \in Ter^{\infty}(\Sigma)$ such that $\mathbf{l}$ is not a variable and every variable of $\mathbf{r}$ also occurs in $\mathbf{l}$. An *infinitary term rewriting system* (abbreviated iTRS) is a pair (Σ, R) of a signature Σ and a set of infinitary rewrite rules R. The notions of one-hole context, rewrite step, left-linearity, collapsing rule, orthogonality, etc. carry over from the finitary setting *mutatis mutandis*. A one-hole context with the hole at position p is written $C[]_p$, where we occassionally suppress the p; we shall have occasion to write multi-hole contexts as $C[]_p$ if particular attention is directed at the hole at position p. If s rewrites to t in one step, we write $s \longrightarrow t$ as usual. The reflexive closure of $\longrightarrow$ is written as $\longrightarrow^{=}$, the reflexive, transitive closure as $\longrightarrow^{*}$.

Observe, in the above definition, that left-hand sides of rewrite rules are taken to be *finite*.

Definition 3. If $\mathcal{I}$ is a set of pairwise parallel positions in a term s over variable set $\mathcal{X} \cup \{[]\}$ such that the holes $[]$ occur exactly at the positions in $\mathcal{I}$, we write $s = C[]_{i\in\mathcal{I}}$. The substitution of a sequence $(s_i)_{i\in\mathcal{I}}$ of terms into $C[]_{i\in\mathcal{I}}$ is written $C[s_i]_{i\in\mathcal{I}}$. The replacement, in term s, of the subterm at position i (regardless of whether a hole is present at i) with a term t is written $s|_{i\mapsto t}$.

Observe that holes in a term can always be well-ordered by "lexicographically" going top-down and left-to-right in a term. As the holes can be well-ordered and there are at most a countable number of positions in a term, $\mathcal{I}$ will be order-isomorphic to $\mathbb{N}$, and one can think of the set $\mathcal{I}$ as indexing the holes.

Definition 4. A (transfinite) *reduction of length* α, where α is an ordinal, is a sequence of rewrite steps $(s_\beta \longrightarrow s_{\beta+1})_{\beta<\alpha}$. In the step $s_\beta \longrightarrow s_{\beta+1}$, let the redex be contracted at position u_β; then $|u_\beta|$ is called the *depth* of the redex and is denoted d_β. The reduction is called *weakly convergent* if, for every limit ordinal $\lambda \leq \alpha$, the distance $d(s_\beta, s_\lambda)$ tends to 0 as β approaches λ from below. It is called *strongly convergent* if, in addition, d_β tends to infinity as β approaches λ from below. We write $s \twoheadrightarrow t$ for a *strongly* convergent reduction of any length and say that t is a *reduct* of s.

It is immediate from the definition that the concatenation of any *finite* number weakly (strongly) convergent reductions is weakly (strongly) convergent.

2.1. Confluence

Definition 5. A *peak* in an iTRS is a triple of terms $t \twoheadleftarrow s \twoheadrightarrow t'$ where s, t and t' are terms such that $s \twoheadrightarrow t$ and $s \twoheadrightarrow t'$. A *valley* is a triple of terms $t \twoheadrightarrow s' \twoheadleftarrow t'$ where t, s' and t' are terms such that $t \twoheadrightarrow s'$ and $t' \twoheadrightarrow s'$; in this case, we say that t and t' are *joinable*, written $t \downarrow t'$. A term s is said to be *confluent* if, for each peak $t \twoheadleftarrow s \twoheadrightarrow t'$, there is a valley $t \twoheadrightarrow s' \twoheadleftarrow t'$. An iTRS is called *confluent* if all of its terms are confluent.

Observe that confluence involves *transfinite* reductions. To distinguish between confluence for TRSs (which involves only finite reductions), we will usually refer to the latter as *finitary* confluence.

Recall that an iTRS is called *almost-non-collapsing* if it contains at most one collapsing rule $C[x] \longrightarrow x$ and there are no variables distinct from x in $C[x]$.

The following theorem is fundamental [13]:

Theorem 6. *An orthogonal iTRS is confluent iff it is almost-non-collapsing.*

A way to work around this restrictive result is contained in the following.

Definition 7. A term is said to be *hypercollapsing* if every reduct of the term can be reduced to a term with a collapsing redex at the root. Two terms, s and t, are said to be *equivalent* modulo hypercollapsing subterms, written $\equiv_{hc}$ if there is a context $C[]_{i \in \mathcal{I}}$ and hypercollapsing terms s_i, t_i, for all $i \in \mathcal{I}$ such that $s = C[s_i]_{i \in \mathcal{I}}$ and $t = C[t_i]_{i \in \mathcal{I}}$.

An iTRS R is said to be *confluent modulo* $\equiv_{hc}$ if, for all terms s, s', t, t' with $s \equiv_{hc} s'$, $s \twoheadrightarrow t$ and $s' \twoheadrightarrow t'$, there exist terms w and w' with $w \equiv_{hc} w'$, $t \twoheadrightarrow w$, and $t' \twoheadrightarrow w'$, i.e., the below diagram commutes:

$$\begin{tikzcd}
 & s \arrow[r, twoheadrightarrow] \arrow[dl, equal, "hc"] & t \arrow[d, dotted, twoheadrightarrow] \\
s' \arrow[d, twoheadrightarrow] & & w \arrow[dl, equal, "hc"'] \\
t' \arrow[r, dotted, twoheadrightarrow] & w' &
\end{tikzcd}$$

Another fundamental positive result in strongly convergent rewriting is that orthogonal systems are confluent *modulo identification of hypercollapsing subterms*:

Theorem 8. [13,11] *Orthogonal iTRSs are confluent modulo* $\equiv_{hc}$.

This theorem settles a large number of cases where all "meaningful" terms of an orthogonal system are confluent, but the entire system is not, as it is not almost-non-collapsing; An example is the system $\{\mathbf{por}(x, \mathbf{f}) \longrightarrow x, \mathbf{por}(\mathbf{f}, y) \longrightarrow y\}$ containing two distinct collapsing rules.

2.2. Normalization and top-termination

Definition 9. A term t is said to be a *normal form* if there is no term t' with $t \longrightarrow t'$. A term s is said to be *normalizing* if there exists a normal form t with $s \twoheadrightarrow t$. An iTRS is said to be *normalizing* if all of its terms are normalizing.

Definition 10. A term s is said to be *uniquely normalizing w.r.t. reduction* if, for each peak $t \twoheadleftarrow s \twoheadrightarrow t'$ with t and t' normal forms, we have $t = t'$. An iTRS is said to be *uniquely normalizing w.r.t. reduction*, (abbreviated $\mathrm{UN}^{\rightarrow}$), if all of its terms are uniquely normalizing w.r.t. reduction.

Termination has no direct analogue in infinitary rewriting; one proposed analogue, cf. [8], is *top-termination*:

Definition 11. An iTRS is said to be *top-terminating* if, for all terms s, there are no (not necessarily convergent) reductions of length ω starting from s having an infinite number of rewrite steps at the root.

It is easy to see that if a left-linear iTRS R is top-terminating, then all of its reductions are strongly convergent and it is normalizing.

2.3. Auxiliary lemmata

The following lemma is standard [11].

Lemma 12 (Compression). *If $s \twoheadrightarrow t$, all strongly convergent reductions from s to t are of length $\leq \Omega$. If the underlying iTRS is left-linear, there is a reduction from s to t of length $\leq \omega$.*

And the next lemma is straightforward.

Lemma 13 (Dovetailing). *If $(s_i)_{i \in \mathcal{I}}$ is a sequence of pairwise parallel subterms of some term $s = C[s_i]_{i \in \mathcal{I}}$ such that $s_i \twoheadrightarrow t_i$ for all $i \in \mathcal{I}$, then $s \twoheadrightarrow C[t_i]_{i \in \mathcal{I}}$.*

Proof. As function symbols have finite arities, there are at most a finite number of the s_i rooted at any given depth. For each such depth n, the finite number of strongly convergent reductions can be concatenated to yield a single strongly convergent reduction. Performing these reductions in a top-down fashion clearly yields a strongly convergent reduction. □

2.4. Modularity

We now define modularity for iTRSs.

Definition 14. Let $A \triangleq \{(\Sigma_k, R_k)\}_{k \in \mathcal{K}}$ be a set of iTRSs such that $\Sigma_i \cap \Sigma_j = \emptyset$ for all $i, j \in \mathcal{K}$ with $i \neq j$ (i.e., the iTRSs are *pairwise disjount*). The *direct sum* of the elements of A, denoted $\oplus A$ is the iTRS with signature $\bigcup_{k \in \mathcal{K}} \Sigma_k$ and rule set $\bigcup_{k \in \mathcal{K}} R_k$. If $A = \{(\Sigma_0, R_0), (\Sigma_1, R_1)\}$, we write $R_0 \oplus R_1$ for $\oplus A$. A term over $\bigcup_{k \in \mathcal{K}} \Sigma_k$ is called *monochrome* if it contains only function symbols from a single Σ_k, and *polychrome* otherwise.

Take note that A may be infinite in the above definition.

Definition 15. Let $\mathcal{P}$ be a predicate on the class of iTRSs (a "property" of iTRSs). $\mathcal{P}$ is said to be *modular* if, for all $A = \{(\Sigma_k, R_k)\}_{k \in \mathcal{K}}$, $\oplus A$ has property $\mathcal{P}$ iff each (Σ_k, R_k) has property $\mathcal{P}$.

$\mathcal{P}$ is said to be *finitely* modular if, for each $A = \{(\Sigma_k, R_k)\}_{k \in \mathcal{K}}$ with $|\mathcal{K}| < \infty$, $\oplus A$ has property $\mathcal{P}$ iff each (Σ_k, R_k) has property $\mathcal{P}$.

This paper will only concern properties where the "only if" part of the definition of modularity is trivially satisfied. Note that a modular property is automatically finitely modular, and, conversely, that any property that fails to be finitely modular fails to be modular.

A number of differences from finitary rewriting are immediately clear: a single term can contain function symbols from an infinite number of different systems, and a maximal path in the term starting from the root may encounter infinitely many "shifts" in signature:

Definition 16. The *rank* of a (mono- or polychrome) term s,denoted $\mathbf{rank}(s)$, is the maximal number of signature changes occurring in maximal paths from the root in s, if that number exists. Otherwise, we set $\mathbf{rank}(s) \triangleq \infty$.

Observe that $\mathbf{rank}(s) = \infty$ does not necessarily entail existence of a single path with an infinite number of signatures changes, since the existence of a set of paths with arbitrarily large *finite* numbers of signature changes will imply $\mathbf{rank}(s) = \infty$ as well.

We define principal and special subterms as in finitary rewriting:

Definition 17. Let s be a term over $\bigcup_{k \in \mathcal{K}} \Sigma_k$ where the Σ_k are pairwise disjoint. If $s = C[s_i]_{i \in \mathcal{I}}$ such that $C[]_{i \in \mathcal{I}}$ is a term over some Σ_m (with variable set $\mathcal{X} \cup \{[]\}$), and for all $i \in \mathcal{I}$ it is the case that $s_i \notin \mathcal{X} \cup \{[]\}$, and the root symbol of s_i is not in Σ_m, then we write $s = C[\![s_i]\!]_{i \in \mathcal{I}}$ and call $C[]_{i \in \mathcal{I}}$ the *cap* of s (denoted by $\mathbf{cap}(s)$), and the s_i the *principal subterms* of s.

The *multiset of special subterms of* s, denoted $S(s)$, is defined by:

$$S(s) \triangleq \begin{cases} \emptyset & \text{if } s \text{ is a variable} \\ \{s\} & \text{if } s \text{ is monochrome and not a variable} \\ \left(\bigcup_{k \in \mathcal{K}} S(s_i)\right) \uplus \{s\} & \text{if } s = C[\![s_i]\!]_{i \in \mathcal{I}} \end{cases}$$

The cap of a special subterm of s is called a *block* of s.

Observe that if s is monochrome then $s = \mathbf{cap}(s)$.

Definition 18. Let s be a polychrome term. A rewrite step $s \longrightarrow t$ is said to be *root-collapsing* if $s = C[\![s_i]\!]_{i \in \mathcal{I}}$ and $t = s_i$ for some $i \in \mathcal{I}$. A reduction is said to be root-collapsing if it contains a root-collapsing step. A (mono- or polychrome) term s is said to be *root-preserved* if there are no root-collapsing reductions starting from s. A rewrite step $s \longrightarrow t$ is said to be *m-collapsing* (short for "modular-collapsing") if there is some special subterm s' of s such that $s \longrightarrow t$ is the step $s = C[s'] \longrightarrow C[t']$ where $s' \longrightarrow t'$ is root-collapsing. A reduction is said to be m-collapsing if it contains an m-collapsing step. A (mono- or polychrome term) s is said to be *preserved* if there are no m-collapsing reductions starting from s.

We need to distinguish between redexes (and rewrite steps) in the cap and elsewhere:

Definition 19. A redex in a polychrome term s is said to be *outer* if it is in the cap of s. Otherwise, the redex is *inner*. A rewrite step $s \longrightarrow t$ is *outer* if the redex is outer, otherwise it is *inner*. An outer step is indicated by $\stackrel{o}{\longrightarrow}$, an inner step by $\stackrel{i}{\longrightarrow}$.

2.5. The modular descendant relation

A final auxiliary definition is that of the *descendant relation*. We will be tracking positions in blocks across reductions solely occurring in other blocks, and positions of variables of some block across reduction in the same block. Descendants of these are well-defined, since we are only considering *strongly* convergent reductions. The descendant relation below is identical to the standard one for iTRSs (see e.g. [11]), and is the obvious generalization of the corresponding relation for finitary, not necessarily orthogonal TRSs.

Definition 20 (*The descendant relation*). Let R be an iTRS, let s be a term of R, and let $s \twoheadrightarrow t$. The set of *descendants* of any position $u \in Pos(s)$ across $s \twoheadrightarrow t$, denoted $u/(s \twoheadrightarrow t)$, is defined by induction on the length, α, of $s \twoheadrightarrow t$:

- $\alpha = 0$. Then, $u/(s \twoheadrightarrow t) = \{u\}$.
- $\alpha = \beta + 1$. Let q be any position of s_β and assume that the redex r contracted in $s_\beta \longrightarrow s_{\beta+1}$ is at position v and is of the rule $\mathbf{l} \longrightarrow \mathbf{r}$. If $q \preceq v$, then $q/(s_\beta \longrightarrow s_{\beta+1}) = \{q\}$. If $v \prec q$, there are two subcases:
  - If $v \cdot p = q$ for some position p with $\mathbf{l}|_p \notin \mathcal{X}$, then $q/(s_\beta \longrightarrow s_{\beta+1}) \triangleq \emptyset$.
  - Otherwise, there is exactly one variable occurrence x in $\mathbf{l}$ at position p_x such that $v \cdot p_x \cdot p' = q$ for some position p'. Let $\{p_x^k : k \in \mathcal{K}\}$ be the set of positions of occurrences of x in $\mathbf{r}$. Then, $q/(s \twoheadrightarrow s_{\beta+1}) = \{v \cdot p_x^k \cdot p' : k \in \mathcal{K}\}$

  We then define $u/(s \longrightarrow s_{\beta+1})$ to be $\bigcup_{q \in u/(s \longrightarrow s_\beta)} (q/(s_\beta \longrightarrow s_{\beta+1}))$.
- $Lim(\alpha)$. Here, a position q of s_α is a descendant of a position u of s iff q is a descendant of u in s_β for all sufficiently large $\beta < \alpha$.

Note that strong convergence is essential for the limit case above to yield well-defined descendants. Abusing notation slightly, we will write of descendants of variable occurrences and of principal subterms, meaning "the position of a variable occurrence" and "position of the root symbol of a principal subterm."

The above definition allows us to track descendants of principal subterms:

Proposition 21. *Let $s = C[\![s_i]\!]_{i \in \mathcal{I}}$ be polychrome and preserved, and assume that $s \twoheadrightarrow D[\![t_j]\!]_{j \in \mathcal{J}}$. Then, for each $j \in \mathcal{J}$, there exists exactly one $i \in \mathcal{I}$ such that $j \in i/(s \twoheadrightarrow D[\![t_j]\!]_{j \in \mathcal{J}})$ and $s_i \twoheadrightarrow t_j$.*

Proof. By induction on the length, α, of $s \twoheadrightarrow D[\![t_j]\!]_{j \in \mathcal{J}}$.

- $\alpha = 0$. Trivial.
- $\alpha = \beta + 1$. Consider the step $s_\beta \longrightarrow D[\![t_j]\!]_{j \in \mathcal{J}}$. It is clear by preservation and the definition of the descendant relation that each t_j is the descendant of exactly one principal subterm, s_k of s_β, that $s_k \longrightarrow t_j$ if the rewrite step $s_\beta \longrightarrow D[\![t_j]\!]_{j \in \mathcal{J}}$ occurs inside s_k, and that $t_j = s_k$ otherwise. The Induction Hypothesis now yields the desideratum.
- $Lim(\alpha)$. By strong convergence, there are no more rewrite steps at depths $\leq j$ from s_β onwards, for some $\beta < \alpha$. Hence, the principal subterm at position j is fixed at some position from s_β onwards. The Induction Hypothesis yields existence of a unique principal subterm s_i of s with a descendant, t'_j, at position j in s_β. Clearly, $t'_j \twoheadrightarrow t_j$. □

Strong convergence is a crucial assumption in the above proposition; see Section 12.

3. Confluence is not finitely modular

There is a trivial counterexample to the modularity of confluence based on the presence of two collapsing rules: If $R_0 = \{\mathbf{f}(x) \longrightarrow x\}$ and $R_1 = \{\mathbf{g}(x) \longrightarrow x\}$, then both R_0 and R_1 are confluent, but in $R_0 \oplus R_1$ there is the peak $\mathbf{f}^\omega \twoheadleftarrow \mathbf{f}(\mathbf{g}(\mathbf{f}(\mathbf{g}(\cdots)))) \twoheadrightarrow \mathbf{g}^\omega$, and $\mathbf{f}^\omega$ and $\mathbf{g}^\omega$ are obviously not joinable.

The above example is not new: exactly the same thing goes wrong when considering confluence of orthogonal systems [11]. However, orthogonal systems *are* confluent when they contain no collapsing rules; we could therefore, naïvely be led to believe that when restricting our attention to *non-collapsing* systems, confluence could be modular, even if the considered systems were not left-linear. We investigate this in the following, first noting a few facts about confluence in the presence of non-left-linear rules.

In infinitary rewriting, we may need "balancing" rules to make non-left-linear rules applicable. To see this, consider $S \triangleq \{\mathbf{f}(x,x) \longrightarrow \mathbf{a}\}$ which, by Newman's Lemma, is (finitarily) confluent as a TRS; when considering S as an iTRS, (transfinite) confluence is lost:

Example 22. Consider S. From the term $h \triangleq \mathbf{f}(h,h)$ we get the two reducts $k \triangleq \mathbf{f}(\mathbf{a},k)$ and $p \triangleq \mathbf{f}(p,\mathbf{a})$, both of which are normal forms, i.e., S is not confluent.

Suitably extending S with balancing rules yields a confluent iTRS. Indeed, consider the following right-ground system:

$$R_2 \triangleq \begin{cases} \mathbf{f}(x,x) \longrightarrow \mathbf{a}, \\ \mathbf{f}(\mathbf{a},x) \longrightarrow \mathbf{a}, \\ \mathbf{f}(x,\mathbf{a}) \longrightarrow \mathbf{a}, \\ \mathbf{f}(\mathbf{f}(x,y),z) \longrightarrow \mathbf{a}, \\ \mathbf{f}(x,\mathbf{f}(y,z)) \longrightarrow \mathbf{a} \end{cases}$$

We have:

Proposition 23. *R_2 is confluent.*

Proof. We claim that if $\mathbf{f}(s,s')$ is a term and if $\mathbf{f}(s,s') \twoheadrightarrow t$ has length at least 1, then $t \longrightarrow^= \mathbf{a}$

Clearly, $t \in \mathcal{X}$ is impossible and if $t = \mathbf{a}$, we are done. Otherwise, we may write $t = \mathbf{f}(w,w')$ and split on cases according to w and w':

(1) $w = \mathbf{a}$ or $w' = \mathbf{a}$ Here, $t \longrightarrow \mathbf{a}$ by an application of either the rule $\mathbf{f}(\mathbf{a},x) \longrightarrow \mathbf{a}$, or the rule $\mathbf{f}(x,\mathbf{a}) \longrightarrow \mathbf{a}$
(2) $w = \mathbf{f}(r,r')$ or $w' = \mathbf{f}(r,r')$. In this case, $t \longrightarrow \mathbf{a}$ by an application of either the rule $\mathbf{f}(\mathbf{f}(x,y),z) \longrightarrow \mathbf{a}$, or the rule $\mathbf{f}(x,\mathbf{f}(y,z)) \longrightarrow \mathbf{a}$
(3) $w = x$ and $w' = y$ for $x, y \in \mathcal{X}$. Since there are no collapsing rules, this is only possible if $s = x$ and $s' = y$. If $x \neq y$, $\mathbf{f}(x,y)$ is a normal form, contradicting the assumption that $\mathbf{f}(s,s') \twoheadrightarrow t$ has length at least 1. Thus, we must have $x = y$, i.e., $w = w'$ and the rule $\mathbf{f}(x,x) \longrightarrow \mathbf{a}$ yields $\mathbf{f}(w,w') \longrightarrow \mathbf{a}$, as desired. $\square$

Make a copy, R_3, of R_2, by performing the renaming $\{\mathbf{f} \mapsto \mathbf{g}, \mathbf{a} \mapsto \mathbf{b}\}$. Clearly, R_3 is confluent by Proposition 23. However, $R_2 \oplus R_3$ is not confluent:

Proposition 24. *The term* $s \triangleq \mathbf{f}(\mathbf{g}(s,s), \mathbf{g}(s,s))$ *is not confluent* (*in* $R_2 \oplus R_3$).

Proof. It is clear that $s \longrightarrow \mathbf{a}$ and that $\mathbf{g}(s,s) \longrightarrow \mathbf{b}$. There is a strongly convergent reduction starting from the "right" subterm $\mathbf{g}(s,s)$ with limit $s'' \triangleq \mathbf{g}(\mathbf{a}, \mathbf{f}(\mathbf{b}, s''))$. As the "left" subterm $\mathbf{g}(s,s)$ rewrites in one step to $\mathbf{b}$, s can in ω steps be rewritten to $s' \triangleq \mathbf{f}(\mathbf{b}, \mathbf{g}(\mathbf{a}, s'))$, which is a normal form. Thus, there is a peak $\mathbf{a} \twoheadleftarrow s \twoheadrightarrow s'$ for which no corresponding valley exists. □

Thus:

Theorem 25. *Confluence is not a finitely modular property of iTRSs.*

The counterexample constructed above crucially employs two facts:

(1) At least one of the considered systems has a rule that is not left-linear.
(2) The specific term considered does not have finite rank.

The next two sections of this paper will show that if restrictions are imposed on one of the two facts above, modularity of confluence can in some cases be recovered.

4. Necessary and sufficient conditions for the modularity of confluence for left-linear systems

In this section, we consider direct sums of confluent, left-linear, pairwise disjoint systems and derive necessary and sufficient conditions for the modularity of confluence. We begin by proving our results for non-collapsing iTRSs—by considering preserved terms—and subsequently extend them to sets, A, of iTRSs that are "essentially non-collapsing," i.e., contain at most one system with a collapsing rule.

4.1. Preserved terms

We first establish that inner and outer reductions commute; left-linearity is used crucially in the proof of the proposition.

Proposition 26 (*Outer and inner reductions commute*). *Let A be a set of left-linear, pairwise disjoint iTRSs and let s be a preserved term with a peak* $t \xleftarrow{\;i\;}\!\!\!\twoheadleftarrow$ …

$$t \overset{i}{\twoheadleftarrow} s \overset{o}{\twoheadrightarrow} t'.$$

Then there exists a valley such that the following diagram commutes.

$$\begin{tikzcd} s \arrow[r, twoheadrightarrow, "i"] \arrow[d, twoheadrightarrow, "o"'] & t \arrow[d, dotted, twoheadrightarrow, "o"'] \\ t' \arrow[r, dotted, twoheadrightarrow, "i"] & s' \end{tikzcd}$$

Proof. By left-linearity and preservation, redexes in the cap are unaffected by inner reductions. Thus, we may project the outer reduction over the inner, obtaining a strongly convergent reduction $t \twoheadrightarrow s'$ for some term s'; this reduction is clearly outer. Write $s = C[\![s_i]\!]_{i \in \mathcal{I}}$ and $t = C[\![t_i]\!]_{i \in \mathcal{I}}$. We then have $s_i \twoheadrightarrow t_i$ for all $i \in \mathcal{I}$. Write $t' = D[\![t'_j]\!]_{j\in\mathcal{J}}$. By Proposition 21, each j' is the descendant of exactly one $i \in \mathcal{I}$, and as the reduction is outer, we obtain $t'_j = s_i$ whenever $j \in i/(s \xleftarrow{o}\!\!\!\!\leftarrow t')$. Thus, we have strongly convergent reductions $t'_j \twoheadrightarrow t_i$ (which are inner, by preservation). These may be dovetailed to obtain an inner reduction $t' \twoheadrightarrow s''$, and by observing the behaviour of the descendants of $i \in \mathcal{I}$, we obtain $s'' = s'$, concluding the proof. □

Proposition 27 (*Postponement of inner reduction*). *Let A be a set of left-linear, pairwise disjoint iTRSs and let s be a preserved term with $s \twoheadrightarrow t$. Then there is a term t' such that the following diagram commutes.*

$$\begin{tikzcd} s \arrow[rr, two heads] \arrow[dr, two heads, dotted, "o"] & & t \\ & t' \arrow[ur, two heads, dotted, "i"'] & \end{tikzcd}$$

Proof. Consider any outer redex u at some position p in a term s and a reduction on the form $s \xrightarrow{u} s' \xrightarrow{u} s''$. By left-linearity and preservation, u cannot have been created by $s \xrightarrow{u} s'$, we must have $\mathbf{cap}(s) = \mathbf{cap}(s')$, and there is thus a redex, v in s such that v is at position p in s, and v is of the same rule as u. Contracting v in s (yielding a term s''') may copy or erase the principal subterms in which $s \xrightarrow{u} s'$ occurs, but we can simply perform the inner steps in the copies of the principal subterms in s''' and use Dovetailing to collect them into a single reduction $s''' \xrightarrow{u} s'$ (where we know that the reduction is inner as s was preserved and the cap thus does not collapse). We hence have a strongly convergent reduction $s \xrightarrow{v} s''' \twoheadrightarrow s''$. As v occurs at the same position as u, we can thus "pull back" all outer steps. As the initial reduction $s \twoheadrightarrow t$ was strongly convergent, so will the resulting reduction consisting of outer steps and there is thus a term t' and a strongly convergent outer reduction $s \xleftarrow{o}\!\!\!\!\leftarrow t'$ such that $\mathbf{cap}(t') = \mathbf{cap}(t)$, and such that the set of descendants of any position of a hole in $\mathbf{cap}(s)$ is identical under $s \twoheadrightarrow t$ and $s \xleftarrow{o}\!\!\!\!\leftarrow t'$. By Proposition 21, every principal subterm, t_j, of t is a descendant of some principal subterm s_i of s such that $s_i \twoheadrightarrow t_j$. As t_j was arbitrary, the same argument holds for all principal subterms of t, and an application of the Dovetailing lemma concludes the proof. □

Proposition 28. *Let A be a set of left-linear, pairwise disjoint, confluent iTRSs, and let s be a preserved term. Then all squares in the following diagram commute for any peak $t \twoheadleftarrow s \twoheadrightarrow t'$:*

$$\begin{tikzcd} s \arrow[r, two heads, "o"] \arrow[d, two heads, "o"'] & t_0 \arrow[r, two heads, "i"] \arrow[d, two heads, "o"'] & t \arrow[d, two heads, "o"'] \\ t'_0 \arrow[r, two heads, "o"] \arrow[d, two heads, "i"'] & s_1 \arrow[r, two heads, "i"] \arrow[d, two heads, "i"'] & t_1 \\ t' \arrow[r, two heads, "o"] & t'_1 & \end{tikzcd}$$

where all rewrite steps in the peak $t_1 \xleftarrow{i}\!\!\!\!\twoheadleftarrow s_1 \xrightarrow{u} t'_1$ take place at depths ≥ 1.

Proof. Use Proposition 27 twice to construct the leftmost and uppermost sides of the diagram. Since s is preserved and all systems are confluent and left-linear, outer reduction is confluent, whence we obtain commutativity of the upper-left square. Two applications of Proposition 26 furnish commutativity of the two remaining squares. All rewrite steps in the peak $t_1' \twoheadleftarrow s_1 \twoheadrightarrow t_1$ are inner, and so, by preservation of s, take place at depth ≥ 1. $\square$

We can now prove confluence for preserved terms:

Lemma 29. *Let $A = \{(\Sigma_k, R_k)\}_{k \in \mathcal{K}}$ be a set of left-linear, confluent, pairwise disjoint iTRSs. Then every preserved term over $\bigcup_{k \in \mathcal{K}} \Sigma_k$ is confluent (in $\oplus A$).*

Proof. Let $t \twoheadleftarrow s \twoheadrightarrow t'$ be a peak of $\oplus A$ with s preserved. By Proposition 28, we can construct a diagram as in that proposition. Consider the peak $t_1 \twoheadleftarrow s_1 \twoheadrightarrow t_1'$ and observe that $\mathbf{cap}(s_1) = \mathbf{cap}(t_1) = \mathbf{cap}(t_1')$, since s was preserved. Write $s_1 = C[\![s_i']\!]_{i \in \mathcal{I}}$; then the inner reductions in $t_1 \twoheadleftarrow s_1 \twoheadrightarrow t_1'$ occur in the s_i. Applying the proposition repeatedly (see Fig. 1) to the inner reductions in the s_i—using the Dovetailing Lemma to arrange reductions in parallel subterms—yields strongly convergent reductions $t \twoheadrightarrow s' \twoheadleftarrow t'$ for some term s', as all redex contractions at the "kth application" of Proposition 28 take place at depth $\geq k$. $\square$

An immediate consequence of Lemma 29 is:

Corollary 30. *Confluence is modular for left-linear, non-collapsing, iTRSs.*

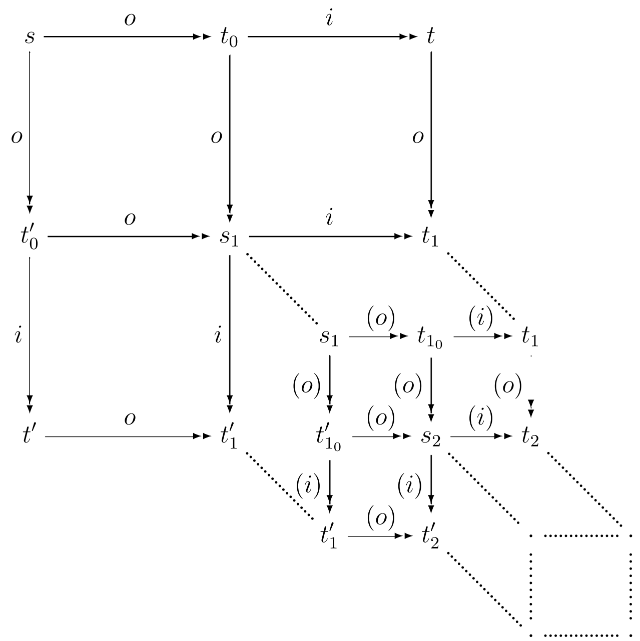

Fig. 1. Repeated application of Proposition 28 in the proof of Lemma 29.

4.2. Essentially non-collapsing sets of iTRSs

We now give a simple condition on sets of iTRSs that will turn out to be necessary and sufficient for the modularity of confluence of left-linear systems.

Definition 31. A set, A, of pairwise disjoint iTRSs is said to be *essentially non-collapsing* if at most one iTRS in A contains a collapsing rule. If there exists an $R \in A$ such that R is the unique iTRS among the elements of A that contains a collapsing rule, we call R the *collapsing colour* of A.

The definition is similar to the notion of *almost-non-collapsing* iTRS well-known from the study of orthogonal iTRSs [11]; observe however, that the term is used here as a property of a *set* of iTRSs, not the individual iTRSs, and that collapsing rules $\mathbf{l} \longrightarrow x$ may contain variables distinct from x in the left-hand-side, unlike the case with almost-non-collapsing systems.

A few auxiliary results need to be established:

Proposition 32. *Let R be a left-linear iTRS. If, for some variable x, $s \twoheadrightarrow x$, then $s \longrightarrow^k x$ for some $k \in \omega$.*

Proof. By the Compression Lemma, we may assume that $s \longrightarrow^{\leq\omega} x$. The fact that $s \longrightarrow^{\leq\omega} x$ is *convergent* now furnishes the desideratum. $\square$

Lemma 33. *If A is a set of left-linear, pairwise disjoint, confluent iTRSs such that $\oplus A$ is confluent, then A is essentially non-collapsing.*

Proof. By contradiction. Assume that $\oplus A$ were confluent, and that A were not essentially non-collapsing. Then there would be at least two iTRSs, (Σ_0, R_0) and (Σ_1, R_1) each containing at least one collapsing rule that we may write as $C_0[x] \longrightarrow x$ and $C_1[x] \longrightarrow x$, respectively (note that the left-hand sides may contain variables different from x). The infinite term $s \triangleq C_0[C_1[s]]$ has the two reducts $C_0[C_0[\cdots]]$ and $C_1[C_1[\cdots]]$; these are terms over disjoint signatures and are hence joinable only if both terms can be reduced to some variable y. By Proposition 32, this can only happen if $C_i[C_i[\cdots]] \longrightarrow^* y$ for $i \in \{0, 1\}$. By finiteness of this reduction and finiteness of the left-hand sides of rules, there exists an n such that a finite stack, $C_i[\cdots[C_i[x]]]$, of n copies of $C_i[x]$, rewrites to y. But, clearly, $C_i[\cdots[C_i[x]]] \longrightarrow^* x$, whence confluence of the underlying iTRSs yields $x = y$. But by left-linearity, we may assume that there are no copies of x in the infinite term $C_i[C_i[\cdots]]$, whence $C_0[C_0[\cdots]]$ and $C_1[C_1[\cdots]]$ can have no common join, contradicting confluence of $\oplus A$. $\square$

Thus, essential non-collapsingness is a necessary condition for modularity of confluence. To see that it is also sufficient, we proceed as follows:

Definition 34. Let A be an essentially non-collapsing set of left-linear, confluent, pairwise disjoint iTRSs. Let s be a term and write $s = C[\![s_i]\!]_{i \in \mathcal{I}}$ (if s is monochrome we have $\mathcal{I} = \emptyset$). Choose fresh, distinct variables $(x_i)_{i \in \mathcal{I}}$; we then define the term $\tilde{s}$ as follows:

$$\tilde{s} \triangleq \begin{cases} C[\tilde{s}_i]_{i \in \mathcal{I}} & \text{if } C[x_i]_{i \in \mathcal{I}} \text{ does not collapse to any } x_i \\ \tilde{s}_m & \text{if } C[x_i]_{i \in \mathcal{I}} \twoheadrightarrow x_m \text{ for some } m \in \mathcal{I} \end{cases}$$

That is, $\tilde{s}$ is the term obtained from s by collapsing all collapsing blocks of the collapsing colour in a top-down fashion, and leaving all other blocks untouched. Observe that by confluence of the elements of A, each block can collapse in at most one way, whence $\tilde{s}$ is well-defined.

Proposition 35. *$\tilde{s}$ is preserved, $s \twoheadrightarrow \tilde{s}$, and for any position u in s, we have $|u/(s \twoheadrightarrow \tilde{s})| \leq 1$.*

Proof. If it exists, the collapsing colour is unique. Thus, when we construct $\tilde{s}$, the "nth" unfolding of the definition of $\tilde{s}$ will have all of the rewrite steps in its collapsing reduction occur at depths $\geq n$, whence the entire reduction will be strongly convergent. Preservation of $\tilde{s}$ follows from the fact that the collapsing colour, if it exists, is unique, and the considered systems are left-linear (hence, no redexes giving rise to collapses can be created by reductions deeper in the term). Since a block collapses to a single position, a position in s is either erased by $s \twoheadrightarrow \tilde{s}$, or has exactly one descendant. $\square$

We need to "project" any strongly convergent reduction $s \twoheadrightarrow t$ to a strongly convergent reduction $\tilde{s} \twoheadrightarrow \tilde{t}$. There is the slight snag that strong convergence requires redex contractions to occur at eventually increasing depths, but the construction of $\tilde{s}$ and $\tilde{t}$ entails that the depth of positions may *decrease*. The following proposition is a first step around this, showing that if a block in s collapses along a reduction of length ω, then any descendant of it in t collapses as well.

Proposition 36. *Let A be an essentially non-collapsing set of left-linear, confluent iTRSs, and let $s \longrightarrow^{\omega} t$ be strongly convergent. Assume that q is the position of a block $C[x_i]_{i \in \mathcal{I}}$ in s such that there is a $p \in \mathcal{I}$ with $C[x_i]_{i \in \mathcal{I}} \twoheadrightarrow x_p$.*

For any $q' \in q/(s \twoheadrightarrow t)$, write $t|_{q'} = D[\![t_j]\!]_{j \in \mathcal{J}}$; then there is an $r \in (q \cdot p)/(s \twoheadrightarrow t)$ such that $r = q' \cdot p'$ for some p' with the property that $p' \in \mathcal{J}$ and $D[x_j]_{j \in \mathcal{J}} \twoheadrightarrow x_{p'}$.

Proof. Note first that since there is a collapsing block, there is exactly one (not zero) collapsing colour(s).

Either there is a block of the collapsing colour at q', or there is not. If there is no such block, then the descendant of the block at q in s that ends up at q' has collapsed in the course of $s \longrightarrow^{\omega} t$ and thus $q' \in (q \cdot p)/(s \twoheadrightarrow t)$. We may thus choose $p' = \epsilon$, $r = q'$ and $D[]_{p'} = []$. (in which case $[x]_{\epsilon} \twoheadrightarrow x$ trivially by an empty reduction).

If there *is* a block of the collapsing colour at q', proceed as follows: Assume there were no r satisfying the proposition. We may, by strong convergence, consider $\beta < \omega$ sufficiently large that all rewrite steps in $s_{\beta} \longrightarrow^{\omega} t$ occur at depths strictly greater than $|q'|$. Consider the block at position q' in any term along $s_{\beta} \longrightarrow^{\omega} t$. As $s \longrightarrow^{*} s_{\beta}$ and all the systems are left-linear, this block is clearly collapsing by confluence of the collapsing colour and the fact that only one colour is collapsing. We may thus write $s_{\beta}|_{q'} = D'[\![s_{j'}]\!]_{j' \in \mathcal{J}'}$ such that there is a $k \in (q \cdot p)/(s \longrightarrow^{*} s_{\beta})$ with $k = q' \cdot k'$ for some $k' \in \mathcal{J}'$ satisfying $D'[x_{j'}]_{j' \in \mathcal{J}'} \twoheadrightarrow x_{k'}$. As all rewrite steps from β onwards occur at depths greater than $|q'|$, we have $D'[\![s_{j'}]\!]_{j' \in \mathcal{J}'} \twoheadrightarrow t|_{q'} = D[\![t_j]\!]_{j \in \mathcal{J}}$. Thus, by left-linearity and the fact that only one colour collapses, there is a reduction $D'[x_{j'}]_{j' \in \mathcal{J}'} \twoheadrightarrow D[x_j]_{j \in \mathcal{J}}$ such that

$$(q' \cdot j')/(s_{\beta} \twoheadrightarrow t) = q' \cdot (j'/(D'[x_{j'}]_{j' \in \mathcal{J}'} \twoheadrightarrow D[x_j]_{j \in \mathcal{J}}))$$

for all $j' \in \mathcal{J}'$. By confluence of the collapsing colour, there is some p' such that $p' \in \mathcal{J}$ and $p' \in k'/(D'[x_{j'}]_{j' \in \mathcal{J}'} \twoheadrightarrow D[x_j]_{j \in \mathcal{J}})$ (as otherwise $x_{k'} \twoheadleftarrow D'[x_{j'}]_{j' \in \mathcal{J}'} \twoheadrightarrow D[x_j]_{j \in \mathcal{J}}$ would not have a corresponding valley), and $D[x_j]_{j \in \mathcal{J}} \twoheadrightarrow x_{p'}$. Clearly, $r = q' \cdot p'$ is a descendant of $q' \cdot k'$ across $s_\beta \longrightarrow^{\omega} t$, and hence a descendant of $q \cdot p$ across $s \longrightarrow^{\omega} t$, concluding the proof. □

We can now prove the desired projection property:

Proposition 37. *Let A be an essentially non-collapsing set of left-linear, confluent, pairwise disjoint iTRSs, and let $s \twoheadrightarrow t$. Then $\tilde{s} \twoheadrightarrow \tilde{t}$.*

Proof. By the compression lemma, we may assume that $s \longrightarrow^{\leq \omega} t$, and proceed by induction on the length, α, of the reduction:

- $\alpha = 0$. Trivial.
- $\alpha = \beta + 1$. Let the redex contracted in $s_\beta \longrightarrow s_{\beta+1}$ be at position u. If $u/(s_\beta \twoheadrightarrow \tilde{s}_\beta) = \emptyset$, then u is inside some collapsing block of s_β and $s_{\beta+1}$, whence $\tilde{s}_\beta = \tilde{s}_{\beta+1}$, and we are done. Otherwise, the construction of $\tilde{\cdot}$ entails that there is exactly one element $v \in u/(s_\beta \twoheadrightarrow \tilde{s}_\beta)$. This v is not inside any collapsing blocks of s_β, and we then clearly have $\tilde{s}_\beta \longrightarrow \tilde{s}_{\beta+1}$ by contraction of the redex at position v.
- $\alpha = \omega$. By the induction hypothesis, we can construct a reduction $\tilde{s} \longrightarrow \tilde{s}_1 \longrightarrow \tilde{s}_2 \longrightarrow \cdots$. Assume that this reduction were not strongly convergent. Then there is a least depth, $m \in \mathbb{N}_0$ such that an infinite number of rewrite steps take place at depth m. Let $\beta < \omega$ be so large that all rewrite steps in $s_\beta \twoheadrightarrow t$ take place at depths greater than m. As there is at most one collapsing system in A, the absence of strong convergence can only be caused by the presence of at least one collapsing block in s that has been collapsed by $\tilde{\cdot}$ and that did *not* collapse in $s \twoheadrightarrow t$. As $s \twoheadrightarrow t$ is strongly convergent, some descendant of such a block must be at position w in s_β with $|w| = m$, and hence have precisely one descendant, occurring at position w in t.
  We must ensure that this descendant is itself a collapsing block; observe that this is not clear a priori, as an infinite number of rewrite steps could potentially erase the subterm to which the block collapsed by "pushing it out" of the term. To see that this cannot happen, observe that the block was collapsing in s, whence Proposition 36 yields that its descendant in t collapses to one of its principal subterms (having root at some position q in t with $w \preceq q$).
  By strong convergence of $s_\beta \twoheadrightarrow t$, q is fixed after $s_{\beta'}$ for some $\beta' \geq \beta$, since all rewrite steps take place at depths $> |q|$ after some point. But by the construction in the successor step of the induction, the steps in $\tilde{s} \longrightarrow \tilde{s}_1 \longrightarrow \tilde{s}_2 \longrightarrow \cdots$ are exactly those of $s \twoheadrightarrow t$ that had "descendants" across the $\tilde{\cdot}$-operation. Thus, there would be an infinite number of rewrite steps at position q in $s_{\beta'} \twoheadrightarrow t$, contradicting strong convergence of that reduction. □

Confluence is a crucial assumption in the proposition, to wit the following example.

Example 38. Consider $R_0 \triangleq \{\mathbf{f}(x) \longrightarrow x, \mathbf{f}(x) \longrightarrow \mathbf{f}(\mathbf{f}(x))\}$ and $R_1 \triangleq \{\mathbf{a} \longrightarrow \mathbf{b}, \mathbf{b} \longrightarrow \mathbf{a}\}$, and observe that R_0 is not confluent, witnessed by the peak $\mathbf{f}^\omega \twoheadleftarrow \mathbf{f}(x) \longrightarrow x$.
We have:

$$\mathbf{f}(\underline{\mathbf{a}}) \longrightarrow \underline{\mathbf{f}}(\mathbf{b}) \longrightarrow \mathbf{f}(\mathbf{f}(\underline{\mathbf{b}})) \longrightarrow \mathbf{f}(\underline{\mathbf{f}}(\mathbf{a})) \longrightarrow \mathbf{f}(\mathbf{f}(\mathbf{f}(\underline{\mathbf{a}}))) \longrightarrow \cdots \mathbf{f}^\omega$$

where the reduction is strongly convergent since, for any $k \in \mathbb{N}$, all steps after the $(2k+1)$th occur at depths $\geq k$.

However, constructing $\widetilde{\mathbf{f}(\mathbf{a})}$ by collapsing the block at the root, and subsequently projecting the above reduction, yields a non-convergent reduction

$$\underline{\mathbf{f}}(\mathbf{a}) \longrightarrow \underline{\mathbf{a}} \longrightarrow \underline{\mathbf{b}} \longrightarrow \underline{\mathbf{a}} \longrightarrow \underline{\mathbf{b}} \longrightarrow \cdots$$

Proposition 39. *Let A be an essentially non-collapsing set of left-linear, confluent iTRSs. If $\tilde{s} \twoheadrightarrow t$, then $s \twoheadrightarrow t$.*

Proof. By Proposition 35, we have $s \twoheadrightarrow \tilde{s} \twoheadrightarrow t$; concatenation of a finite number of strongly convergent reductions yields a strongly convergent reduction. $\square$

We are now in position to prove the first positive result of the paper:

Theorem 40. *Let A be a set of confluent, left-linear, pairwise disjoint iTRS. Then, $\oplus A$ is confluent iff A is essentially non-collapsing.*

Proof. If $\oplus A$ is confluent, it follows from Lemma 33 that A must be essentially non-collapsing. Conversely, if A is essentially non-collapsing, let $t \twoheadleftarrow s \twoheadrightarrow t'$ be a peak of $\oplus A$. By Proposition 37, there exists a peak $\tilde{t} \twoheadleftarrow \tilde{s} \twoheadrightarrow \tilde{t}'$. Lemma 29 now implies existence of a term s' and strongly convergent reductions $\tilde{t} \longrightarrow s'$ and $\tilde{t}' \longrightarrow s'$, and an application of Proposition 39 concludes the proof (see Fig. 2). $\square$

5. Confluence modulo equality of hypercollapsing subterms

Essential non-collapsingness looks suspiciously like almost-non-collapsingness, and it is a standard result that *orthogonal* iTRSs are (1) confluent iff they are almost-non-collapsing and (2) are confluent modulo $\equiv_{hc}$ [11]. In light of the previous section, it is therefore natural to ask whether confluence modulo $\equiv_{hc}$ is preserved under direct sum. Unfortunately, this turns out not to be the case, as we shall presently show.

Define $\Sigma_0 \triangleq \{\mathbf{f}/1, \mathbf{m}/1\} R_0 \triangleq \{\mathbf{m}(x) \longrightarrow x, \mathbf{f}(\mathbf{m}(x)) \longrightarrow \mathbf{f}(\mathbf{f}(x)), \mathbf{f}(\mathbf{f}(x)) \longrightarrow \mathbf{f}(x)\}$.

$$\begin{tikzcd}
s \arrow[rr, two heads] \arrow[dr, dotted, two heads] \arrow[dd, two heads] & & t \arrow[dr, dotted, two heads] \arrow[dddr, dotted, two heads] & \\
& \tilde{s} \arrow[rr, dotted, two heads] \arrow[dd, dotted, two heads] & & \tilde{t} \arrow[dd, dotted, two heads] \\
t' \arrow[dr, dotted, two heads] \arrow[drrr, dotted, two heads] & & & \\
& \tilde{t}' \arrow[rr, dotted, two heads] & & s'
\end{tikzcd}$$

Fig. 2. The proof of Theorem 40.

Proposition 41. *R_0 is terminating and (finitarily) confluent as a TRS.*

Proof. To see that any finite term is terminating, observe that the lexicographic order on finite terms s defined by

$$(\text{no. of function symbols in } s) \times (\text{no. of occurrences of } \mathbf{m} \text{ in } s)$$

is strictly decreasing across rewrite steps. Also, there are three critical pairs in R_0, corresponding to the peaks $\mathbf{f}(\mathbf{f}(x)) \longleftarrow \mathbf{f}(\mathbf{f}(\mathbf{f}(x))) \longrightarrow \mathbf{f}(\mathbf{f}(x))$, $\mathbf{f}(\mathbf{f}(\mathbf{f}(x))) \longleftarrow \mathbf{f}(\mathbf{f}(\mathbf{m}(x))) \longrightarrow \mathbf{f}(\mathbf{m}(x))$, and $\mathbf{f}(x) \longleftarrow \mathbf{f}(\mathbf{m}(x)) \longrightarrow \mathbf{f}(\mathbf{f}(x))$. In the first of these, the critical pair is trivially joinable. In the second case, we have $\mathbf{f}(\mathbf{f}(\mathbf{f}(x))) \longrightarrow \mathbf{f}(x)^{+} \longleftarrow \mathbf{f}(\mathbf{f}(\mathbf{m}(x)))$, and in the third case $\mathbf{f}(x) \longleftarrow \mathbf{f}(\mathbf{f}(x))$, i.e., all critical pairs are joinable, and Newman's Lemma thus ensures that any finite term is (finitarily) confluent, as desired. □

Proposition 42. *Let s be an infinite term distinct from $\mathbf{m}^{\omega}$. Then $s \twoheadrightarrow \mathbf{f}^{\omega}$.*

Proof. If s is distinct from $\mathbf{m}^{\omega}$, it is on one of the three forms:

(1) $C[\mathbf{f}^{\omega}]$,
(2) $C[\mathbf{f}(\mathbf{m}^{\omega})]$, or
(3) $\mathbf{f}^{n_0}(\mathbf{m}^{n_1}(\mathbf{f}^{n_2}(\mathbf{m}^{n_3}(\cdots))))$.

In the first case, we can remove all occurrences of $\mathbf{m}$ in the finite term $C[x]$ in a finite number of steps. Since $\mathbf{f}$ is the only other function symbol in Σ_0, the resulting term will be $\mathbf{f}^{\omega}$. In the second case, we clearly have $\mathbf{f}(\mathbf{m}^{\omega}) \twoheadrightarrow \mathbf{f}^{\omega}$, i.e., $C[\mathbf{f}(\mathbf{m}^{\omega})] \twoheadrightarrow C[\mathbf{f}^{\omega}]$, and the case reduces to the previous one. In the third case, we may remove all the occurrences of $\mathbf{m}$ by first using n_1 steps to remove the uppermost n_1 occurrences, then using n_3 steps to collapse the next n_3 occurrences, and so on. Since there are occurrences of $\mathbf{f}$ interspersed between the blocks of occurrences of $\mathbf{m}$, this results in a strongly convergent reduction to $\mathbf{f}^{\omega}$, as desired.

Lemma 43. *R_0 is confluent, and confluent modulo $\equiv_{hc}$ as an iTRS.*

Proof. Consider any term s. If s is finite, then Proposition 41 yields that there are no infinite reductions starting from s, and confluence hence follows from finitary confluence, also obtainable from Proposition 41.

If s is infinite, the fact that both symbols in Σ_0 are unary entails that all reducts of s are infinite as well. If $s = \mathbf{m}^{\omega}$, then any reduct of s is $\mathbf{m}^{\omega}$, and we trivially have confluence. If s is *not* $\mathbf{m}^{\omega}$, the two facts that (1) no right-hand side in R_0 contains an $\mathbf{m}$, and (2) that no rule other than $\mathbf{m}(x) \longrightarrow x$ is collapsing, imply that no reduct of s can be $\mathbf{m}^{\omega}$. Hence, any two reducts of s must reduce to $\mathbf{f}^{\omega}$, by Proposition 42, ensuring confluence.

As there is exactly one hypercollapsing term in (Σ_0, R_0), namely $\mathbf{m}^{\omega}$, confluence modulo $\equiv_{hc}$ efollows from confluence. □

Consider, now, the system R_1 consisting of the single collapsing rule $\mathbf{a}(x) \longrightarrow x$. This system is obviously orthogonal and almost-non-collapsing, hence confluent and confluent modulo $\equiv_{hc}$. However:

Proposition 44. *$R_0 \oplus R_1$ is not confluent modulo $\equiv_{hc}$.*

Proof. Consider the term $s \triangleq \mathbf{m}(\mathbf{a}(s))$. Then $\mathbf{f}(s) \twoheadrightarrow \mathbf{f}(\mathbf{m}^\omega) \twoheadrightarrow \mathbf{f}^\omega$, but also $\mathbf{f}(s) \twoheadrightarrow \mathbf{f}(\mathbf{a}^\omega)$, where $\mathbf{f}(\mathbf{a}^\omega)$ contains the hypercollapsing subterm $\mathbf{a}^\omega$. The two terms $\mathbf{f}^\omega$ and $\mathbf{f}(\mathbf{a}^\omega)$ reduce only to themselves, but $\mathbf{f}^\omega$ contains no hypercollapsing subterms, whence $\neg(\mathbf{f}^\omega \equiv_{hc} \mathbf{f}(\mathbf{a}^\omega))$. □

Thus:

Theorem 45. *Confluence modulo $\equiv_{hc}$ is not a finitely modular property of left-linear iTRSs.*

6. Preservation of confluence for non-collapsing systems of finite rank

In this section, we show that when only terms of *finite rank* are considered, confluence is modular for non-collapsing, not necessarily left-linear, systems. The methods employed are akin to Toyama's original proof of (finitary) confluence of TRSs [33] and the initial part of the later, more elegant proof found in [16]. The parts of these two papers that deal with collapsing rules do not appear to be applicable when working with strongly convergent reductions, since they concatenate reductions with no common constraints on the *depth* of the rewrite steps.

We first note that, for confluent terms, joinability is preserved across reduction:

Proposition 46. *If the confluent terms s and s' are joinable, and $s \twoheadrightarrow t$, respectively $s' \twoheadrightarrow t'$, then t and t' are joinable.*

Proof. A simple diagram chase:

$$\begin{tikzcd}
 & s \arrow[dl, ->>] \arrow[dr, ->>] & & s' \arrow[dl, ->>] \arrow[dr, ->>] & \\
t \arrow[dr, ->>, dotted] & & \cdot \arrow[dl, ->>, dotted] \arrow[dr, ->>, dotted] & & t' \arrow[dl, ->>, dotted] \\
 & \cdot \arrow[dr, ->>, dotted] & & \cdot \arrow[dl, ->>, dotted] & \\
 & & \cdot & &
\end{tikzcd}$$

We need a way to ensure that non-left-linear rules remain applicable when we replace principal subterms by fresh variables; as two principal subterms can become equal after a number of rewrite steps—thus possibly creating a new redex of a non-left-linear rule—we are forced to consider *joinability* of principal subterms instead of mere equality. To this end, we introduce the standard way [16] of replacing subterms by variables in a non-linear way according to joinability:

Definition 47. For sequences $(s_k)_{k\in\mathcal{K}}$ and $(t_k)_{k\in\mathcal{K}}$ of terms, we write $(s_k)_{k\in\mathcal{K}} \propto (t_k)_{k\in\mathcal{K}}$ when $t_{k'} = t_{k''}$ iff $s_{k'} \downarrow s_{k''}$ for all $k', k'' \in \mathcal{K}$.

Observe that $(s_k)_{k\in\mathcal{K}} \propto (t_k)_{k\in\mathcal{K}}$ ensures that any *finite* number of s_i that are mapped to some t_j have a single, common, join. On the other, if an *infinite* number of s_i are mapped to some t_j, they may potentially *fail* to have a single, common join (even though each pair of them are joinable). As left-hand sides of rules are finite, we shall only have occasion to join a finite number of subterms.

The following proposition shows that strongly convergent reduction in a polychrome term gives rise to strongly convergent reduction in the cap of the term.

Proposition 48. *Let A be a set of non-collapsing, pairwise disjoint, confluent, iTRSs, let $s = C[\![s_i]\!]_{i\in\mathcal{I}}$, and assume that $s \twoheadrightarrow t$ with $t = C'[\![t_j]\!]_{j\in\mathcal{J}}$. Let $(x_i)_{i\in\mathcal{I}}$ be a sequence of variables such that $(s_i)_{i\in\mathcal{I}} \propto (x_i)_{i\in\mathcal{I}}$.*

Then $C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow C'[y_j]_{j\in\mathcal{J}}$, and for any $i \in \mathcal{I}$ and any $j \in \mathcal{J}$, we have $j \in i/(C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow C'[y_j]_{j\in\mathcal{J}})$ iff $j \in i/(s \twoheadrightarrow t)$.

Proof. By induction on the length, α, of $s \twoheadrightarrow t$.

- $\alpha = 0$. Straightforward.
- $\alpha = \beta + 1$. Write $s_\beta = D[\![t'_k]\!]_{k\in\mathcal{K}}$; by the induction hypothesis we may assume that there exists a strongly convergent reduction $C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow D[z_k]_{k\in\mathcal{K}}$ such that $k \in i/(C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow D[z_k]_{k\in\mathcal{K}})$ iff $k \in i/(s \twoheadrightarrow s_\beta)$, for all $k \in \mathcal{K}$, $i \in \mathcal{I}$. Consider the single rewrite step $s_\beta \longrightarrow s_{\beta+1}$. Assume that the redex contracted in the step is at position u. If the redex is not outer, or the rule is left-linear, the desideratum follows immediately. Assume, then, that the redex is outer and that the rule employed is not left-linear. By the Induction Hypothesis and Proposition 46, we gather that $(t'_k)_{k\in\mathcal{K}} \propto (z_k)_{k\in\mathcal{K}}$, and the rule is applicable at position u in $D[z_k]_{k\in\mathcal{K}}$. The demand on the descendants is clearly fulfilled.
- $Lim(\alpha)$. Observe that the rewrite steps constructed in the successor step correspond exactly to, and occur at exactly the same positions as, the outer steps in $s \twoheadrightarrow t$. Thus, if $C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow C'[y_j]_{j\in\mathcal{J}}$ were not strongly convergent, neither would $s \twoheadrightarrow t$ be. It is clear by the definition of the descendant relation that the demand on the descendants is fulfilled.

We need a "converse" to Proposition 48, i.e., that a reduction in the cap of the term (with variables inserted suitably to mimic joinability of principal subterms), gives rise to a reduction in the original term. To achieve this, we may have to perform "balancing" steps in the principal subterms to make non-left-linear rules applicable, as evident in the successor case of the proof of the following proposition.

Proposition 49. *Let A be a set of non-collapsing, pairwise disjoint iTRSs, let $s = C[\![s_i]\!]_{i\in\mathcal{I}}$ such that the s_i are all confluent, and choose variables $(x_i)_{i\in\mathcal{I}}$ such that $(s_i)_{i\in\mathcal{I}} \propto (x_i)_{i\in\mathcal{I}}$. If $C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow C'[z_k]_{k\in\mathcal{K}}$, then we have $C[\![s_i]\!]_{i\in\mathcal{I}} \twoheadrightarrow C'[\![t_k]\!]_{k\in\mathcal{K}}$ such that $k \in i/(C[\![s_i]\!]_{i\in\mathcal{I}} \twoheadrightarrow C'[\![t_k]\!]_{k\in\mathcal{K}})$ iff $k \in i/(C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow C'[z_k]_{k\in\mathcal{K}})$.*

Proof. By induction on the length, α, of $C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow C'[z_k]_{k\in\mathcal{K}}$.

- $\alpha = 0$. Straightforward
- $\alpha = \beta + 1$. We have $C[x_i]_{i\in\mathcal{I}} \longrightarrow^{\beta} D[z_l]_{l\in\mathcal{L}}$, and, by the Induction Hypothesis, we get $C[s_i]_{i\in\mathcal{I}} \twoheadrightarrow D[r_l]_{l\in\mathcal{L}}$ for suitable $(r_l)_{l\in\mathcal{L}}$ such that $l \in i/(C\llbracket s_i\rrbracket_{i\in\mathcal{I}} \twoheadrightarrow D\llbracket r_l\rrbracket_{l\in\mathcal{L}})$ iff $l \in i/(C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow D[z_l]_{l\in\mathcal{L}})$. Assume that the redex contracted in $D[z_l]_{l\in\mathcal{L}} \longrightarrow C'[z_k]_{k\in\mathcal{K}}$ is at position u and of the rule $\mathbf{l} \longrightarrow \mathbf{r}$. If the rule is left-linear, the desideratum follows immediately. If the rule is not left-linear, applicability of the rule in $D[z_l]_{l\in\mathcal{L}}$, $(s_i)_{i\in\mathcal{I}} \propto (x_i)_{i\in\mathcal{I}}$, and the Induction Hypothesis, furnish that if $z_j = z_{j'}$, then r_j and $r_{j'}$ are joinable. Since $\mathbf{l}$ is a finite term, only a finite number of principal subterms need to be reduced to a common term in order for the rule to be applicable at position u in $D[r_l]_{l\in\mathcal{L}}$. Thus, by Proposition 46, we have $D[r_l]_{l\in\mathcal{L}} \twoheadrightarrow D[r'_l]_{l\in\mathcal{L}}$ (with all steps performed at depth $\geq |u|$) such that $\mathbf{l} \longrightarrow \mathbf{r}$ is applicable at position u in $D[r'_l]_{l\in\mathcal{L}}$. The demand on the descendant relation is clearly satisfied.
- $Lim(\alpha)$. There are two kinds of rewrite steps constructed in the successor case: "authentic" steps corresponding to, and the same depth as, the steps in $s \twoheadrightarrow C'[z_k]_{k\in\mathcal{K}}$, and "balancing" steps performed to make non-left-linear rules applicable in the successor case above. The balancing steps are all performed at a depth greater than that of the authentic step in $C[x_i]_{i\in\mathcal{I}} \twoheadrightarrow C'[z_k]_{k\in\mathcal{K}}$ that prompted them. Hence, the resulting reduction is strongly convergent, and the demand on the descendant relation is clearly satisfied. $\square$

We can now tie the previous two propositions together:

Lemma 50. *Let A be a set of non-collapsing iTRSs and let $s = C\llbracket s_i\rrbracket_{i\in\mathcal{I}}$. Assume that outer reduction and the s_i are confluent for all $i \in \mathcal{I}$, and let $D\llbracket t_l\rrbracket_{l\in\mathcal{L}} \twoheadleftarrow s \twoheadrightarrow D'\llbracket t_{l'}\rrbracket_{l'\in\mathcal{L}'}$ be a peak. Then there exists a valley $t \twoheadrightarrow s' \twoheadleftarrow t'$.*

Proof. Apply Propositions 48 and 49 twice:

$$
\begin{tikzcd}
C\llbracket s_i\rrbracket_{i\in\mathcal{I}} \arrow[rrr, twoheadrightarrow] \arrow[dr, dotted, no head] \arrow[ddd, twoheadrightarrow] & & & D\llbracket t_l\rrbracket_{l\in\mathcal{L}} \arrow[dl, dotted, no head] \arrow[dd, dotted, twoheadrightarrow] \\
& C[x_i]_{i\in\mathcal{I}} \arrow[r, twoheadrightarrow] \arrow[r, phantom, shift left=6ex, "\text{Prop. 48}"] \arrow[d, twoheadrightarrow] \arrow[dl, phantom, "\text{Prop. 48}"] & D[y_l]_{l\in\mathcal{L}} \arrow[d, dotted, twoheadrightarrow] \arrow[dr, phantom, "\text{Prop. 49}"] & \\
& D'[y'_{l'}]_{l'\in\mathcal{L}'} \arrow[r, dotted, twoheadrightarrow] \arrow[dl, dotted, no head] & C'[z_k]_{k\in\mathcal{K}} \arrow[r, dotted, no head] \arrow[d, dotted, no head] \arrow[dl, phantom, "\text{Prop. 49}"] \arrow[dr, phantom, "r_k \downarrow r'_k"] & C'\llbracket r_k\rrbracket_{k\in\mathcal{K}} \arrow[d, dotted, twoheadrightarrow] \\
D'\llbracket t_{l'}\rrbracket_{l'\in\mathcal{L}'} \arrow[rr, dotted, twoheadrightarrow] & & C'\llbracket r'_k\rrbracket_{k\in\mathcal{K}} \arrow[r, dotted, twoheadrightarrow] & C'\llbracket s'_k\rrbracket_{k\in\mathcal{K}}
\end{tikzcd}
$$

In the lower right rectangle, observe that the demand on the descendant relations in Propositions 48 and 49 ensures that r_k and r'_k are descendants of the same s_i for all $k \in \mathcal{K}$. Since the s_i were confluent, Proposition 46 yields that $r_k \downarrow r'_k$ for all $k \in \mathcal{K}$. An application of the Dovetailing lemma shows that performing the $|\mathcal{K}|$ reductions needed to obtain $C'[\![s'_k]\!]$ from $C'[r_k]$ (respectively, $C'[r'_k]$) can be done in a strongly convergent fashion. □

We can now prove the second positive result of this paper:

Theorem 51. *Let A be a set of non-collapsing, confluent iTRSs. Then every polychrome term with finite rank is confluent (in $\oplus A$).*

Proof. By induction on $\mathbf{rank}(s)$. If $\mathbf{rank}(s) = 0$, the result follows immediately, since monochrome terms were assumed to be confluent. If $\mathbf{rank}(s) > 0$, note that outer reduction is confluent by assumption, as are all principal subterms of s, since they have rank strictly less than $\mathbf{rank}(s)$. The result follows by an application of Lemma 50. □

7. Normalization is not modular

We now show that normalization is not preserved across the direct sum of an infinite number of systems by giving a counterexample.

Definition 52. Let, for each $n \in \mathbb{N}$, the iTRS (Σ_n, R_n) be defined by setting $\Sigma_n \triangleq \{\mathbf{f}_n/1, \mathbf{c}_n/0\}$, and setting $R_n \triangleq \{\mathbf{f}_n(\mathbf{f}_n(x)) \longrightarrow \mathbf{c}_n, \mathbf{f}_n(x) \longrightarrow x\}$, where $\mathbf{f}_m \neq \mathbf{f}_n$ and $\mathbf{c}_m \neq \mathbf{c}_n$ for $m \neq n$.

Proposition 53. *For each $n \in \mathbb{N}$, (Σ_n, R_n) is normalizing.*

Proof. Every term over Σ_n is either a variable, $\mathbf{c}_n$, $\mathbf{f}_n(x)$, $\mathbf{f}_n(\mathbf{c}_n)$, or $\mathbf{f}_n(\mathbf{f}_n(s))$ for some term s. Variables and $\mathbf{c}_n$ are normal forms; $\mathbf{f}_n(x)$, $\mathbf{f}_n(\mathbf{c}_n)$ and $\mathbf{f}_n(\mathbf{f}_n(s))$ all reduce to normal forms in one step. □

Theorem 54. *Normalization is not modular, not even for left-linear systems.*

Proof. Consider the systems (Σ_n, R_n), all of which are normalizing by Proposition 53 and the direct sum $\oplus_{n \in \mathbb{N}} R_n$. Ponder the polychrome term $\mathbf{f}_1(\mathbf{f}_2(\mathbf{f}_3(\cdots)))$. It is a trivial induction over ordinals to show that any reduct of this term by a weakly convergent reduction will be infinite and will contain at most one function symbol from each Σ_n. Since the reduct is infinite, all function symbols in question will be from $\{\mathbf{f}_n \mid n \in \mathbb{N}\}$. In particular there will be some $\mathbf{f}_n$ at the root of the term so that we can use the rule $\mathbf{f}_n(x) \longrightarrow x$ at the root, whence the reduct cannot be a normal form. □

8. Normalization for preserved terms and top-termination

If we consider only left-linear systems, it is straightforward to see that restricting to *preserved* terms recovers modularity of normalization.

Lemma 55. *Let A be a set of left-linear, normalizing iTRSs. Then, any preserved term in $\oplus A$ is normalizing.*

Proof. By left-linearity and preservation, no rewrite step in any block can create a redex in any other block. Hence, we may normalize in a top-down fashion. If strong convergence is desired, we may collecting the top-down reductions in a single reduction by using the Dovetailing Lemma repeatedly. □

This simple lemma will prove to be of use in the next section where we will prove that normalization is *finitely* modular for left-linear iTRSs.

The argument used in the proof of the lemma carries over to top-terminating systems as well; we have:

Proposition 56. *A left-linear system containing a collapsing rule is not top-terminating.*

Proof. If $C[x] \longrightarrow x$ is a collapsing rule (possibly with more variables than x in the left-hand side), the term $s \triangleq C[s]$ will have reductions of any (finite or infinite) length weakly converging to itself in which all steps occur at the root. □

Thus, the third positive result of the paper:

Theorem 57. *Top-termination is modular for left-linear systems.*

Proof. By Proposition 56, all terms in the direct sum of any set of top-terminating systems will be preserved. Hence, by left-linearity, no rewrite step in any block can create a redex in any other block, and top-termination of the summands ensures top-termination of each term in the direct sum. □

Remark 58. The original definition of top-termination occurs in the work of Dershowitz et al. [18] on weakly convergent rewriting. There, the emphasis is on reductions starting from finite terms, and ordinary (finitary) termination of a system R thus entails that R is top-terminating.

Termination is not modular for TRSs [32], but there is a priori no conflict with Theorem 57, as top-termination does not imply termination, even when only finite terms are considered (to wit the system $\{\mathbf{f}(x) \longrightarrow \mathbf{g}(\mathbf{f}(x))\}$ which is top-terminating, but not terminating for finite terms).

However, there is a significant difference in the systems that are top-terminating, according to whether only finite terms are considered. If only finite terms are considered, then the system $\{\mathbf{f}(x) \longrightarrow x\}$ is top-terminating, whereas it is *not* top-terminating in our sense, as $\mathbf{f}^{\omega}$ is not. Intuitively, the ability to construct reductions starting from infinite terms makes it harder to be top-terminating, as collapsing rules give rise to terms from which reductions with infinitely many redex contractions at the root will exist.

Example 59. Toyama's counterexample from [32] to the modularity of termination is also a counterexample to the modularity of top-termination if only finite terms are considered: Let $R_0 \triangleq \{\mathbf{f}(\mathbf{0},\mathbf{1},x) \longrightarrow \mathbf{f}(x,x,x)\}$ and $R_1 \triangleq \{\mathbf{g}(x,y) \longrightarrow x, \mathbf{g}(x,y) \longrightarrow y\}$. Then each system is terminating, hence top-terminating when only finite terms are considered, but in $R_0 \oplus R_1$, there is a cyclic reduction starting from $\mathbf{f}(\mathbf{g}(\mathbf{0},\mathbf{1}),\mathbf{g}(\mathbf{0},\mathbf{1}),\mathbf{g}(\mathbf{0},\mathbf{1}))$, showing that the the direct sum is not top-terminating, even when only finite terms are considered.

Note that R_1 is not top-terminating, as evidenced by the term $s \triangleq \mathbf{g}(x,s)$ which reduces to itself in one step.

A way around these troubles is given in the following:

Definition 60. An iTRS is said to be *finite-top-terminating* if, for each *finite* term s, there are no reductions of length ω starting from s having an infinite number of rewrite steps at the root.

Proposition 61. *Finite-top-termination is not finitely modular, not even for left-linear systems.*

Proof. Toyama's counterexample to modularity of termination provides a counterexample (see Example 59). $\square$

Theorem 62. *Finite-top-termination is modular for non-collapsing, left-linear systems.*

Proof. By induction on the rank of a term. If $\mathbf{rank}(s) = 0$, the result follows by assumption. Let $\mathbf{rank}(s) > 0$, and assume, for purposes of contradiction, that there is an $n \in \mathbb{N}_0$ and a reduction $s_0 \longrightarrow s_1 \longrightarrow \cdots$ contracting an infinite number of steps at depth n. We may wlog. assume that $n = 0$. By left-linearity and preservation, there is a reduction in $\mathbf{cap}(s_0)$ contracting an infinite number of steps at depth 0. This contradicts the assumption that each of the considered systems was finite-top-terminating. $\square$

9. Finite modularity of normalization for left-linear systems

If we consider only left-linear systems, we may proceed with a top-down approach to normalization. The trick employed in this section is to consider finitely (thus, wlog., two) many systems, and in that situation to leave the cap of the term untouched and m-collapse as many principal subterms as possible. When this happens, a principal subterm will leave at least one function symbol of the same colour as the cap, thus "enlarging" the cap at that point. Taking limits, we will obtain a term in which no principal subterms root-collapse. This procedure can then be (co-)iterated in a top-down fashion until a term is produced in which all principal subterms are preserved. This term is easily seen to be normalizing.

To show that normalization is finitely modular, it clearly suffices to show it for two arbitrary systems.

Proposition 63. *Let $s = C[\![s_i]\!]_{i \in \mathcal{I}}$ be a term over $\Sigma_0 \cup \Sigma_1$, and consider a specific principal subterm s_k. Assume that s_k root-collapses, say by a reduction $s_k \twoheadrightarrow t_k$. Then, if $s_{k'}$ is a principal subterm of $C[\![s_i]\!]_{i \in \mathcal{I}}|_{k \mapsto t_k}$, we either have $k \parallel k'$ or $k \prec k'$.*

Proof. If s_k root-collapses to t_k, then the cap of t_k is of the same colour as the cap of s. The principal subterms of $C[\![s_i]\!]_{i \in \mathcal{I}}|_{k \mapsto t_k}$ are of two types: (1) the principal subterms of s, except for s_k, and (2) "new" principal subterms that are all at positions $k \cdot j$ where $t_k = D[\![s'_j]\!]_{j \in \mathcal{J}}$. Observe that $|j| = 0$ entails that $t_k \in \mathcal{X}$, which is impossible, since we assumed that s_k root-collapsed. $\square$

Lemma 64. *Let $s_0 = C[\![t_i]\!]_{i \in \mathcal{I}}$ be a term over $\Sigma_0 \cup \Sigma_1$. Then $s_0 \twoheadrightarrow s_\omega$ where s_ω is a term, every principal subterm of which is root-preserved.*

Proof. Consider all principal subterms t_i of s_0 such that t_i has a root-collapsing reduction, and let $\mathcal{K}$ be the set of positions at which these principal subterms occur. By the Dovetailing Lemma, all these reductions can be collected in a strongly convergent reduction $s_0 \twoheadrightarrow s_1$ (in which all rewrite steps occur at depths ≥ 1). Write $s_1 = C'[\![s'_j]\!]_{j \in \mathcal{J}}$. By Proposition 63, if any s'_j is root-collapsing, then there exists $k \in \mathcal{K}$ such that $k \prec j$. Thus, if any principal subterm of s_1 is root-collapsing, it occurs at a strictly greater depth than any root-collapsing subterm in s_0. We can obviously repeat the above construction, obtaining a reduction $s_0 \twoheadrightarrow s_1 \twoheadrightarrow s_2 \twoheadrightarrow \cdots$ in which all steps in $s_n \twoheadrightarrow s_{n+1}$ occur at depth $\geq n$, whence the reduction strongly converges to some limit s_ω. Since any principal subterm of s_ω is rooted at some finite depth n, no principal subterm of s_ω can root-collapse, as it would have been collapsed in the reduction $s_n \twoheadrightarrow s_{n+1}$ at the latest. □

Corollary 65. Let s be a term over $\Sigma_0 \cup \Sigma_1$. Then s has a root-preserved reduct.

Proof. By Lemma 64, $s \twoheadrightarrow t'$ where all principal subterms of t' are root-preserved. If t' is root-preserved, we are done. If t' is not root-preserved, there is a root-collapsing reduction $t' \twoheadrightarrow t''$ where t'' is a descendant of a principal subterm of t', in which case t'' is root-preserved. Hence, we have $s \twoheadrightarrow t' \twoheadrightarrow t''$, concluding the proof. □

Lemma 66. *Let s^0 be a term over $\Sigma_0 \cup \Sigma_1$. Then s^0 has a preserved reduct s^ω_ω.*

Proof. Corollary 65 ensures that s^0 has a root-preserved reduct s^0_ω. Write $s^0_\omega = C[\![s'_j]\!]_{j \in \mathcal{J}}$. By repeated application of Corollary 65 and the Dovetailing Lemma, there is a strongly convergent reduction $s^0_\omega \twoheadrightarrow s^1_\omega$ where s^1_ω and all of its principal subterms are root-preserved, and where all rewrite steps in $s^0_\omega \twoheadrightarrow s^1_\omega$ occur at depths ≥ 1. We can repeat this construction by applying it to the principal subterms of s^1_ω, thus obtaining a reduction $s^1_\omega \twoheadrightarrow s^2_\omega \twoheadrightarrow \cdots$ in which, for all $n \in \mathbb{N}$, every special subterm in s^n_ω at depth $\leq N$ is root-preserved, and in which all rewrite steps in $s^n_\omega \twoheadrightarrow s^{n+1}_\omega$ occur at depths $\geq n$. Hence $s^1_\omega \twoheadrightarrow s^2_\omega \twoheadrightarrow \cdots$ converges strongly to some limit s^ω_ω which must be preserved. □

We then have the fourth positive result of the paper:

Theorem 67. *Normalization is finitely modular for left-linear iTRSs.*

Proof. Let (Σ_0, R_0) and (Σ_1, R_1) be normalizing and left-linear, and let s be any term over $\Sigma_0 \cup \Sigma_1$. By Lemma 66, s has a preserved reduct s^ω_ω, and the desideratum follows from Lemma 55. □

On a negative note, the above proof is not constructive in the sense that the presence of "strategies" for obtaining normal forms in each of the summands yields a "strategy" for normalization of the terms over $\Sigma_0 \cup \Sigma_1$ analogous to the way that normalizing strategies can be combined in TRSs [17]. The reason is that we are asking the obviously undecidable question, for each principal subterm, whether the subterm—which may be infinite—root-collapses.

10. Unique normalization w.r.t. reduction is not finitely modular for left-linear iTRSs

We now show that unique normalization w.r.t. reduction is not finitely modular for left-linear systems by providing yet another explicit counterexample.

Definition 68. Define the iTRS (Σ_0, R_0) by $\Sigma_0 \triangleq \{\mathbf{f}/1, \mathbf{g}/1, \mathbf{m}/1\}$ and $R_0 \triangleq \{\mathbf{f}(x) \longrightarrow \mathbf{f}(x), \mathbf{m}(x) \longrightarrow x, \mathbf{m}(x) \longrightarrow \mathbf{f}(x), \mathbf{f}(\mathbf{f}(x)) \longrightarrow \mathbf{g}(\mathbf{f}(x))\}$.

Define the iTRS (Σ_1, R_1) by the renaming $\{\mathbf{f} \mapsto \mathbf{a}, \mathbf{g} \mapsto \mathbf{b}, \mathbf{m} \mapsto \mathbf{n}\}$ and letting R_1 be the rules of R_0 *mutatis mutandis.*

Proposition 69. (Σ_0, R_0) *and* (Σ_1, R_1) *are* $UN^{\rightarrow}$.

Proof. It suffices to prove the result for (Σ_0, R_0). Observe that the normal forms of (Σ_0, R_0) are the terms of the form x, $\mathbf{g}^k(x)$ (for $k \in \mathbb{N}$), and $\mathbf{g}^\omega$, where x is a variable. Observe also that, in all rules but $\mathbf{m}(x) \longrightarrow x$, the depth of the single variable in the left-hand side is the same as in the right-hand side. Since all rules but $\mathbf{m}(x) \longrightarrow x$ have an occurrence of $\mathbf{f}$ in the right-hand side, these facts entail that no convergent reduction can remove all $\mathbf{f}$ s from a finite term. Since we have the rule $\mathbf{f}(x) \longrightarrow \mathbf{f}(x)$, this implies that no finite term containing an $\mathbf{f}$ can be reduced to normal form.

Let s be a term and assume that $t' \twoheadleftarrow s \twoheadrightarrow t$ where t and t' are normal forms. We split on cases according to t:

- $t = x$. Since all function symbols are unary, we must have $s \longrightarrow^* x$ (since otherwise x could not appear in s). Also, since all rules but $\mathbf{m}(x) \longrightarrow x$ are depth-preserving, we must have $s = \mathbf{m}^j(x)$ for some $j \in \mathbb{N}_0$. The only rule applicable to $\mathbf{m}^j(x)$ other than $\mathbf{m}(x) \longrightarrow x$ is $\mathbf{m}(x) \longrightarrow \mathbf{f}(x)$ which results in a finite term containing exactly one $\mathbf{f}$. By the above comments, this term cannot have a normal form. Hence, $t' = t$.
- $t = \mathbf{g}^k(x)$ for some $k \in \mathbb{N}$. As in the previous case, we must have $s \longrightarrow^* t$. Since function symbols are unary and no rule erases a variable, s must thus be a finite term. By the above comments, s cannot contain any occurrences of $\mathbf{f}$, and must thus be a finite interleaving of $\mathbf{m}$s and $\mathbf{g}$s. As no reduction of s to normal form can contain a (finite) term with an occurrence of $\mathbf{f}$, the reduction $s \twoheadrightarrow t'$ can only consist of steps using the rule $\mathbf{m}(x) \longrightarrow x$, whence $t' = t$.
- $t = \mathbf{g}^\omega$. As no rule increases the depth of a position, s cannot be finite, hence must be infinite. As the only rule that can decrease the length of a position is $\mathbf{m}(x) \longrightarrow x$, any reduct of s must be infinite as well. The only infinite normal form in R_0 is $\mathbf{g}^\omega$, and we thus have $t' = t$. □

Proposition 70. $R_0 \oplus R_1$ *is not* $UN^{\rightarrow}$.

Proof. Consider (Σ_1, R_1) and (Σ_2, R_2), both of which are $\mathrm{UN}^{\rightarrow}$ by Proposition 69. The term $s \triangleq \mathbf{m}(\mathbf{n}(s))$ reduces to both $\mathbf{m}^\omega$ and $\mathbf{n}^\omega$ by strongly convergent reductions. But $\mathbf{m}^\omega \twoheadrightarrow \mathbf{g}^\omega$, and $\mathbf{n}^\omega \twoheadrightarrow \mathbf{b}^\omega$, and both of the terms $\mathbf{g}^\omega$ and $\mathbf{b}^\omega$ are normal forms. □

Hence:

Theorem 71. $UN^{\rightarrow}$ *is not finitely modular for left-linear iTRSs.*

The counterexample crucially employs the presence of a collapsing rule. Before Marchiori's proof of the modularity of $\mathrm{UN}^{\rightarrow}$ for left-linear TRSs, Middeldorp showed a weaker result [20]: that $\mathrm{UN}^{\rightarrow}$ is modular for left-linear, *non-collapsing* TRSs. We can adapt his proof by, roughly, replacing induction on the rank of a term by coinduction:

Theorem 72. $UN^{\rightarrow}$ *is modular for left-linear, non-collapsing iTRSs.*

Proof. Let s be a polychrome term and assume that $s \twoheadrightarrow t$, respectively $s \twoheadrightarrow t'$ with t and t' normal forms. As the considered systems are non-collapsing, s is a preserved term, and, writing $s = C[\![s_i]\!]_{i \in \mathcal{I}}$, left-linearity and Proposition 27 thus yield

$$C[\![s_i]\!]_{i \in \mathcal{I}} \overset{o}{\twoheadleftarrow} D[\![s_j]\!]_{j \in \mathcal{J}} \xrightarrow{u} D[\![t_j]\!]_{j \in \mathcal{J}} = t$$

respectively,

$$C[\![s_i]\!]_{i\in\mathcal{I}} \overset{o}{\twoheadleftarrow} D'[\![s'_{j'}]\!]_{j'\in\mathcal{J}'} \xrightarrow{u} D'[\![t'_{j'}]\!]_{j'\in\mathcal{J}'} = t'$$

As the considered systems are non-collapsing and both t and t' are normal forms, both D and D' must be normal forms of the cap $C[x_i]_{i\in\mathcal{I}}$ which, by hypothesis, was $\mathrm{UN}^{\rightarrow}$. We hence have $\mathcal{J} = \mathcal{J}'$, $D[]_{j\in\mathcal{J}} = D'[]_{j'\in\mathcal{J}'}$, and $s_j = s'_j$ for all $j \in \mathcal{J}$. Again due to non-collapsingness of the considered systems, both contexts D and D' are non-empty and both reductions $D[\![s_j]\!]_{j\in\mathcal{J}} \xrightarrow{u} D[\![t_j]\!]_{j\in\mathcal{J}} = t$ and $D[\![s_j]\!]_{j\in\mathcal{J}} \xrightarrow{u} D[\![t'_j]\!]_{j\in\mathcal{J}} = t'$ thus occur at depth ≥ 1. Repeating the above arguments coinductively on each term s_j in the inner reduction thus yields reductions at successively greater depths, and an application of the Dovetailing Lemma hence allows us, for each $d \in \mathbb{N}$ to construct reductions $s \twoheadrightarrow t_d \twoheadrightarrow t$, respectively, $s \twoheadrightarrow t'_d \twoheadrightarrow t'$ where t_d and t'_d are identical up to depth d and the reductions $t_d \twoheadrightarrow t$, respectively, $t'_d \twoheadrightarrow t'$ contract only steps below depth d, and the result follows. □

11. Constructor-sharing unions

Modularity assumes disjointness of the considered systems; however, modularity for certain classes systems over non-disjoint alphabets exist in finitary rewriting (i.e., *constructor-sharing* unions rather than disjoint unions are considered)—see e.g. [25] for an overview.

In this section, we informally describe the technical difficulties occurring in infinitary rewriting when tackling modularity of confluence for the most innocuous of constructor-sharing unions, namely the case of *mutually orthogonal systems*.

In both first- and higher-order finitary rewriting, mutual orthogonality (and the more lax *mutual weak orthogonality*) is sufficient for left-linear, confluent systems to be confluent under direct sum [34]. The techniques of [13,11] for proving confluence in orthogonal infinitary rewriting use reasoning about residuals and the depths of redexes contracted in valleys as their linchpin. However, this does not readily generalize to the modular case where each system is merely required to be confluent (i.e., not necessarily orthogonal). It is easy to see that a "modular" Parallel Moves Lemma holds for strongly convergent reductions in left-linear systems:

$$\begin{tikzcd}
\cdot \arrow[r, two heads, "\mathcal{R}\setminus\{R\}"] \arrow[d, "R"'] & \cdot \arrow[d, dotted, two heads, "R"] \\
\cdot \arrow[r, dotted, two heads, "\mathcal{R}\setminus\{R\}"'] & \cdot
\end{tikzcd}$$

where the rightmost vertical reduction consists of contraction of parallel redexes.

As the systems of $\mathcal{R}$ are merely required to be confluent, a peak $t \overset{r}{\longleftarrow} s \twoheadrightarrow t'$ in R where the redex r is at depth k and all rewrite steps in $s \twoheadrightarrow t'$ are assumed to occur at depths $k' > k$ is only guaranteed to yield a term s' and R-reductions $t \twoheadrightarrow s'$ and $t' \twoheadrightarrow s'$ with all rewrite steps below k. Thus, there appears to be no constructive way to relate the depth of u and the depth of redexes contracted in $t' \twoheadrightarrow s'$, something which, for proof-technical purposes, is crucial in standard techniques in infinitary rewriting are to be used.

When considering reductions in $\oplus\mathcal{R}$, we will hence be in the following situation (assuming all rewrite steps in the upper horizontal R-reduction take place at depth $\geq k$):

$$\begin{tikzcd}
s \arrow[r, "R", "*"'] \arrow[d, "R"', "u"] & \cdot \arrow[r, "\mathcal{R}\setminus\{R\}", "+"'] \arrow[d, dotted, "R"', "\mathrm{dep} \geq k"] & \cdot \arrow[r, "R", "+"'] \arrow[d, dotted, "R"', "\mathrm{dep} \geq k"] & \cdot \arrow[d, dotted, "R"', "\mathrm{dep} \geq k"] \cdots\cdots \\
\cdot \arrow[r, dotted, twoheadrightarrow, "R", "\mathrm{dep} \geq k"'] & \cdot \arrow[r, dotted, twoheadrightarrow, "\mathcal{R}\setminus\{R\}", "\mathrm{dep} \geq k"'] & \cdot \arrow[r, dotted, twoheadrightarrow, "R", "\mathrm{dep} \geq k"'] & \cdot \cdots\cdots
\end{tikzcd}$$

Thus, the fact that R is confluent appears to yield no useful information about the depths of the reductions when trying to find suitable valleys for R-peaks, and it is not clear whether the bottom horizontal reduction above is even convergent.

In light of these phenomena, it appears that any proof of confluence of the direct sum of mutually orthogonal confluent systems must employ quite novel tools and proof methods.

12. Weakly convergent rewriting

While stated for strongly convergent systems, none of the counterexamples in this paper crucially employ *strong* convergence. In addition, the proofs concerning normalization, while carefully ensuring that strong convergence is possible, do not crucially *use* that fact, and are hence applicable in the setting of weak convergence.

The reader can satisfy herself that the proofs of the following results from this paper hold for weakly convergent rewriting, as they neither assume, nor employ the mechanics of *strong* convergence:

- Confluence is not (finitely) modular (Theorem 25).
  None of the auxiliary results (Propositions 23 and 24) use strong convergence.
- Normalization is not modular, not even for left-linear system (Theorem 54).
  The proof of the ancillary Proposition 53 does not use strong convergence.
- Top-termination is modular (Theorem 57).
  The auxiliary results of Lemma 55 and Proposition 56 do not employ strong convergence, unless this is desired.
- Normalization is finitely modular for left-linear systems (Theorem 67).
  The auxiliary results of Lemma 64 and Proposition 63 mention strong convergence explicitly, but clearly strong convergence is only necessary if one wishes strongly convergent normalizing reductions in the direct sum; as the considered terms are all preserved, the top-down reduction argument goes through even when the normalizing reductions in each separate system are only weakly convergent.
- Unique Normalization w.r.t. reduction is not finitely modular (Theorem 71).

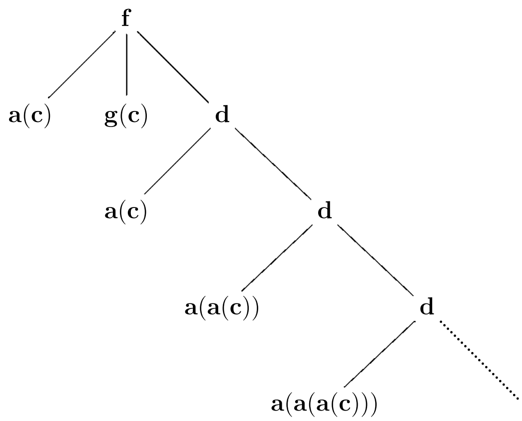

Fig. 3. A troublesome term in weakly convergent rewriting (Example 73).

The (proofs of the) other main results of this paper employ the ability to work with a well-defined notion of descendant, something that is notoriously hard to pin down in weakly convergent rewriting [11,29]. We illustrate the impact of this fact on the study of modularity by providing an explicit counterexample to Proposition 21 in the form a weakly convergent reduction $s \longrightarrow^{\alpha} t$ such that, for a certain principal subterm t_j of t, there is *no* principal subterm, s_i, of s satisfying $s_i \longrightarrow\!\!\!\!\!\twoheadrightarrow t_j$.

Example 73. Let $R_0 \triangleq \{\mathbf{a}(x) \longrightarrow \mathbf{b}(x)\}$ and let R_1 by the system consisting of the following infinite set of rules:

$$\mathbf{f}(x, \mathbf{g}^k(\mathbf{c}), \mathbf{d}(y, z)) \longrightarrow \mathbf{f}(y, \mathbf{g}^{k+1}(\mathbf{c}), z) \text{ for } k \in \omega$$

Clearly, the two systems are disjoint, and both are orthogonal. Let s be coinductively defined by $s \triangleq \mathbf{d}\;(\mathbf{a}^{\omega}, s)$ and ponder the term:

$$\mathbf{f}(\mathbf{a}(\mathbf{c}), \mathbf{g}(\mathbf{c}), \mathbf{d}\;(\mathbf{a}(\mathbf{c}), \mathbf{d}(\mathbf{a}(\mathbf{a}(\mathbf{c})), \mathbf{d}(\mathbf{a}(\mathbf{a}(\mathbf{a}(\mathbf{c}))), \mathbf{d}(\cdots)))))$$

(see Fig. 3), from which there is a weakly convergent reduction having limit $\mathbf{f}(\mathbf{a}^{\omega}, \mathbf{g}^{\omega}, s)$ (contract redexes at the root repeatedly). But there is no principal subterm, s_i of the starting term such that a weakly convergent reduction $s_i \longrightarrow^{\beta} \mathbf{a}^{\omega}$ exists for any ordinal β.

13. Conclusion and open problems

We have studied modular and non-modular properties in strongly convergent infinitary rewriting, and found that few of the properties known to be modular in the finitary setting proved recoverable in the infinitary setting. A number of positive results were proven as well, notably a necessary and sufficient condition for the modularity of confluence for left-linear systems. A brief summary of results can be found in the table below:

	Modular in finitary rewriting	Finitely modular	Modular	Modular with finite rank
WN	Yes	Yes	No	Yes
WCR	Yes	?	?	?
CR	Yes	No	No	? (yes, if non-collapsing)
NF	No	?	?	?
UN	Yes	?	?	?
$\text{UN}^{\rightarrow}$	No (yes if left-linear)	No (even if left-linear)	No (even if left-linear)	No (even if left-linear)
TT	No	? (yes, if left-linear)	? (yes, if left-linear)	? (yes, if left-linear)
FTT	No	No	No	No

where "TT" and "FTT" abbreviate "top-termination" and "finite-top-termination," respectively.

A number of pertinent open questions remain:

(1) Are there counterexamples to finite modularity of normalization and top-termination for non-left-linear systems?
(2) Is the property of having unique normal forms modular? For left-linear systems? Observe that conversion of normal forms in the infinitary case still consists of a *finite* number "back-and-forth" concatenations of strongly convergent reductions (see [11]).
(3) The counterexample to confluence modulo equality of hypercollapsing subterms crucially employs the spurious term $\mathbf{f}^{\omega}$ which, though not hypercollapsing, reduces only to itself. Perhaps some notion of confluence modulo a "suitable" class of meaningless terms, vis-à-vis [14], is modular.
(4) In finitary rewriting, *mutual* orthogonality allows the sum of two different systems to retain confluence, even if the systems share function symbols (are constructor-sharing). Does a result similar to Theorem 40 hold in this case?
In addition, many interesting modularity results hold for constructor-sharing unions of TRSs in the finitary case (see e.g [25] for an overview). What, if any, of these results can be transferred to the infinitary setting?
(5) What positive results, if any, hold in weakly convergent rewriting?
(6) What results are modular when initial reductions must start from finite terms (corresponding to the "modular for finite terms" column in the table above)? Note that not all reductions have to start from finite terms: For confluence, in every peak $t \twoheadleftarrow s \twoheadrightarrow t'$, s must be finite, but in a corresponding valley $t \twoheadrightarrow s' \twoheadleftarrow t'$, either t, t' or both may be infinite, as may s'.

It does not appear to be possible to produce (easy) answers to these problems using the methods of this paper.

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