

Skolem Meets Bateman-Horn

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Abstract. The Skolem Problem asks to determine whether a given integer linear recurrence sequence has a zero term. This problem arises across a wide range of topics in computer science, including loop termination, (weighted) automata theory, and the analysis of linear dynamical systems, amongst many others. Decidability of the Skolem Problem is notoriously open. The state of the art is a decision procedure for recurrences of order at most 4: an advance achieved some 40 years ago based on Baker’s theorem on linear forms in logarithms of algebraic numbers.

Recently, a new approach to the Skolem Problem was initiated in [7,8] via the notion of a Universal Skolem Set: a set \mathcal{S} of positive integers such that it is decidable whether a given non-degenerate linear recurrence sequence has a zero in \mathcal{S} . Clearly, proving decidability of the Skolem Problem is equivalent to showing that \mathbb{N} is a Universal Skolem Set. The main contribution of the present paper is to exhibit a Universal Skolem Set of positive density that moreover has density one subject to the Bateman-Horn conjecture in number theory. The latter is a central unifying hypothesis concerning the frequency of prime numbers among the values of systems of polynomials, and provides a far-reaching generalisation of many classical results and conjectures on the distribution of primes.

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1 Introduction

An (integer) linear recurrence sequence (LRS) $\langle u_n \rangle_{n=0}^\infty$ is a sequence of integers satisfying a recurrence of the form

$$u_{n+k} = a_1 u_{n+k-1} + \cdots + a_k u_n \quad (n \in \mathbb{N}), \quad (1)$$

where the coefficients a_1, \dots, a_k are integers. The celebrated theorem of Skolem, Mahler, and Lech [12,9,6] states that the set $\{n \in \mathbb{N} : u_n = 0\}$ of zero terms is the union of a finite set and finitely many arithmetic progressions. This result can be refined using the notion of *non-degeneracy* of an LRS. An LRS is non-degenerate if in its minimal recurrence no quotient of distinct characteristic roots is a root of unity. A given LRS can be effectively decomposed as the merge of finitely many non-degenerate sequences, some of which may be identically zero. The core of the Skolem-Mahler-Lech Theorem is the fact that a non-zero non-degenerate linear recurrence sequence has finitely many zero terms. Unfortunately, all known proofs are ineffective—it is not known how to compute the finite set of zeros of a given non-degenerate linear recurrence sequence; equivalently, it is not known how to decide whether an arbitrary given LRS has a zero.

The problem of deciding whether an LRS has a zero is known as the Skolem Problem. Decidability of this problem is known only for recurrences of order at most 4 [10,13]: an advance made some 40 years ago. Recently [3] gave a procedure to decide the Skolem Problem for the class of simple LRS (those with simple characteristic roots) of any order subject to two known conjectures about the exponential function. The present paper follows a different approach, via the notion of *Universal Skolem Set* [7]. This is an infinite set $\mathcal{S} \subseteq \mathbb{N}$ for which there is an effective procedure that, given a non-degenerate LRS $\langle u_n \rangle_{n=0}^\infty$, outputs the finite set $\{n \in \mathcal{S} : u_n = 0\}$. Evidently, establishing decidability of the Skolem Problem is equivalent to showing that \mathbb{N} is a Universal Skolem Set. Towards this objective, it is natural to ask whether there exists a Universal Skolem Set of density one. Studying this question leads in the present paper to new connections between the Skolem Problem and classical questions on the distribution of prime numbers.

The paper [7] exhibited a Universal Skolem Set of density zero. Subsequently [8] produced a set $\mathcal{S}_0 \subseteq \mathbb{N}$ of positive lower density and an effective procedure that, given a non-degenerate *simple* LRS $\langle u_n \rangle_{n=0}^\infty$, computes its set of zeros $\{n \in \mathcal{S}_0 : u_n = 0\}$. The present paper contains two significant advances over these two results. First, we exhibit a set $\mathcal{S} \subseteq \mathbb{N}$ of positive lower density, such that we can compute the set of zeros $\{n \in \mathcal{S} : u_n = 0\}$ for *any* non-degenerate LRS, not just the simple ones. In fact we give an explicit upper bound for the largest such zero. The second contribution is to show that \mathcal{S} has density one subject to the [Bateman-Horn conjecture](#) in number theory [2]. The latter is a [central unifying hypothesis concerning the frequency of prime numbers among the values of a system of polynomials](#); it generalises many classical results and conjectures on the distribution of primes, including [Hardy and Littlewood's twin primes conjecture](#).

A key ingredient of the present paper are deep results of Schlickewei and Schmidt [11] that yield explicit bounds on the number of solutions of certain polynomial-exponential Diophantine equations. Indeed, it is striking that while there is no known method to elicit the zero set of a given non-degenerate LRS, thanks to the above mentioned results [there are fully explicit upper bounds \(depending only on the order of the recurrence\) on the cardinality of its zero set](#). Such bounds do not suffice to solve the Skolem Problem, which would require effective bounds on the *magnitude* of the zeros of an LRS. The key idea of our approach is to leverage explicit upper bounds on the number of zeros of polynomial-exponential equations to obtain bounds on the magnitude of the zeros of LRS. Specifically, our Universal Skolem Set \mathcal{S} consists of positive integers n that admit sufficiently many representations of the form $n = Pq + a$, with P, q prime and q, a logarithmic in n . Given an LRS $\langle u_n \rangle_{n=0}^\infty$ we associate with the equation $u_n = 0$ a *companion equation* such that

each representation of n yields a solution of the companion equation. We then use upper bounds on the number of solutions of the companion equation to derive upper bounds on the magnitude of n . In general, we believe that such a transfer principle is a promising direction to make progress on Skolem's Problem.

A major difference between the present paper and [8] is that the latter used an existing bound of [11] on the number of solutions of a certain class of exponential Diophantine equations. To handle the case of non-simple LRS it appears that one cannot use existing results "off the shelf" and must instead adapt the techniques of [1,4,11] to our setting.

2 Background

2.1 Number fields

Let \mathbb{K} be a finite Galois extension of \mathbb{Q} . The ring of algebraic integers in \mathbb{K} is denoted $\mathcal{O}_{\mathbb{K}}$. We denote by $\text{Gal}(\mathbb{K}/\mathbb{Q})$ the group of automorphisms of \mathbb{K} . The *norm* of $\alpha \in \mathbb{K}$ is defined by

$$N_{\mathbb{K}/\mathbb{Q}}(\alpha) = \prod_{\sigma \in \text{Gal}(\mathbb{K}/\mathbb{Q})} \sigma(\alpha).$$

The *norm* $N_{\mathbb{K}/\mathbb{Q}}(\alpha)$ is rational for all $\alpha \in \mathbb{K}$ and $N_{\mathbb{K}/\mathbb{Q}}(\alpha)$ is an integer if $\alpha \in \mathcal{O}_{\mathbb{K}}$. Clearly we have $|N(\alpha)| < M^{d_{\mathbb{K}}}$, where $d_{\mathbb{K}}$ is the degree of \mathbb{K} and

$$M := \max_{\sigma \in \text{Gal}(\mathbb{K}/\mathbb{Q})} |\sigma(\alpha)|$$

is the *house* of α . Furthermore, given a rational prime $p \in \mathbb{Z}$ and a prime ideal factor \mathfrak{p} of p in $\mathcal{O}_{\mathbb{K}}$, we have $p \mid N_{\mathbb{K}/\mathbb{Q}}(\alpha)$ for all $\alpha \in \mathfrak{p}$.

We say that $\alpha, \beta \in \mathbb{K}$ are *multiplicatively dependent* if there exist integers k, ℓ , not both zero, such that $\alpha^k = \beta^\ell$. Observe that if $\alpha \in \mathbb{K}$ is not a root of unity then given $\sigma \in \text{Gal}(\mathbb{K}/\mathbb{Q})$, every multiplicative relation $\alpha^k = \sigma(\alpha)^\ell$ is such that $k = \pm\ell$. Indeed, repeatedly applying σ to this relation we deduce that $\alpha^{k^d} = (\sigma^d(\alpha))^{\ell^d}$ for all $d \geq 1$. In particular, choosing d to be the order of σ we get that $\alpha^{k^d} = \alpha^{\ell^d}$ and hence $k = \pm\ell$.

2.2 Polynomial-exponential equations

Let \mathbb{K} be a number field of degree d and consider the equation

$$\sum_{i=1}^s P_i(\mathbf{x}) \alpha_i^{\mathbf{x}} = 0, \tag{2}$$

in variables $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$, where $P_1, \dots, P_s \in \mathbb{K}[\mathbf{x}]$, and $\alpha_i^{\mathbf{x}} = \alpha_{i1}^{x_1} \cdots \alpha_{in}^{x_n}$ with $\alpha_{ij} \in \mathbb{K}^\times$ for all i, j . We say that Equation (2) is *non-degenerate* if no proper sub-sum vanishes. Schlickewei and Schmidt [11, Theorem 1] have proved the following upper bound on the number of non-degenerate solutions:

Theorem 1. *Let δ_i be the total degree of polynomial P_i for $i \in \{1, \dots, s\}$. Put $A = \sum_{i=1}^s \binom{n+\delta_i}{n}$ and $B = \max(n, A)$. Suppose that there is no non-zero $\mathbf{x} \in \mathbb{Z}^n$ such that $\alpha_i^{\mathbf{x}} = \alpha_j^{\mathbf{x}}$ for all $i, j \in \{1, \dots, s\}$. Then Equation (2) has at most $2^{35B^3} d^{6B^2}$ non-degenerate solutions.*

2.3 Distribution of primes

Consider the linear forms $f_1(t) := a_1t + b_1$ and $f_2(x) = a_2t + b_2$ for integers a_1, a_2, b_1, b_2 , with $a_1, a_2 > 0$. The following result [5, Chapter 2.6, Theorem 2.3] gives an upper bound on the number of times that f_1 and f_2 are simultaneously prime. Here φ denotes Euler's totient function and we use the Vinogradov notation $f \ll g$ for $f \in O(g)$.

Theorem 2. *Suppose that $D := |a_1a_2(a_1b_2 - a_2b_1)|$ is non-zero. Then*

$$\#\{x \leq X : f_1(x), f_2(x) \text{ both prime}\} \ll \frac{X}{(\log X)^2} \frac{D}{\varphi(D)},$$

where the implied constant is independent of f_1 and f_2 .

We say that $f := f_1f_2 \in \mathbb{Z}[x]$ is *admissible* if it does not vanish identically modulo any prime. Note that f_1 and f_2 are simultaneously prime only finitely many times if f is not admissible. For a prime p , let $\omega_f(p)$ denote the number of $x \in \mathbb{F}_p$ such that $f(x) = 0$. The following instance of the Bateman-Horn conjecture provides a much stronger statement than Theorem 2, with matching upper and lower bounds.

Conjecture 3 (Bateman-Horn Conjecture). Let f_1, f_2 be a pair of linear forms such that $f = f_1f_2$ is admissible. Then

$$\#\{x \leq X : f_1(x), f_2(x) \text{ both prime}\} \sim C \frac{X}{(\log X)^2}, \quad \text{where } C := \prod_{p \text{ prime}} \frac{p(p - \omega_f(p))}{(p - 1)^2}.$$

In particular, the infinite product above converges.

In general, the Bateman-Horn conjecture concerns the set of positive integers on which a family f_1, \dots, f_k of polynomials is simultaneously prime. Here we have the case that $k = 2$ and the f_i have degree one.

3 A Universal Skolem Set

For a positive real number $x > 0$, denote by $\log x$ the natural logarithm of x . For a positive integer $k \geq 1$, we inductively define the iterated logarithm function $\log_k x$ as follows: $\log_1 x := \log x$, and for $k \geq 2$ we set $\log_k x := \max\{1, \log_{k-1}(\log x)\}$. Thus, for x sufficiently large, $\log_k x$ is the k -fold iterate of \log applied to x . We omit the subscript when $k = 1$.

Fix a positive integer parameter X . We define disjoint intervals

$$A(X) := \left[\log_2 X, \sqrt{\log X} \right] \quad \text{and} \quad B(X) := \left[\frac{\log X}{\sqrt{\log_3 X}}, \frac{2 \log X}{\sqrt{\log_3 X}} \right].$$

We further define a *representation* of an integer $n \in [X, 2X]$ to be a triple (q, P, a) such that $q \in A(X)$, $a \in B(X)$, P and q are prime, and $n = Pq + a$. We say that two representations $n = Pq + a$ and $n = P'q' + a'$ are *correlated* if

$$q \neq q', \quad a \neq a' \quad \text{and} \quad |(a + \eta q) - (a' + \eta q')| < \sqrt{\log X}$$

for some $\eta \in \{\pm 1\}$.

We denote by $r(n)$ the number of representations of n . Finally we put

$$\mathcal{S}(X) := \{n \in [X, 2X] : r(n) > \log_4 X \text{ and no two representations of } n \text{ are correlated}\}$$

and we define

$$\mathcal{S} := \bigcup_{k \geq 10} \mathcal{S}(2^k).$$

The following result shows that \mathcal{S} is a Universal Skolem Set and furthermore gives an explicit upper bound on the largest element of \mathcal{S} that is a zero of a given non-degenerate LRS.

Theorem 4. *Let $\mathbf{u} = \langle u_n \rangle_{n=0}^\infty$ be a non-degenerate LRS of order $k \geq 2$ given by*

$$u_{n+k} = a_1 u_{n+k-1} + \cdots + a_k u_n$$

for $n \geq 1$, with given initial terms u_1, \dots, u_k not all zero. If $u_n = 0$ and $n \in \mathcal{S}$, then

$$n < \max\{\exp_3(A^2), \exp_5(10^{10}k^6)\}, \quad \text{where } A := \max\{10, |u_i|, |a_i| : 1 \leq i \leq k\}.$$

The rest of this section is devoted to the proof of Theorem 4, which comprises four main steps.

Step 1: Rescaling. We rescale \mathbf{u} so that all the coefficients of the polynomials in its closed form representation are algebraic integers. To this end, let

$$\Psi(X) := X^k - a_1 X^{k-1} - \cdots - a_k = \prod_{i=1}^s (X - \alpha_i)^{\sigma_i}$$

be the characteristic polynomial of \mathbf{u} and let $\mathbb{K} := \mathbb{Q}(\alpha_1, \dots, \alpha_s)$ be the splitting field of Ψ , which has degree at most $k!$ over \mathbb{Q} . If $|\alpha_i| > 1$, then

$$|\alpha_i| = \left| a_1 + \frac{a_2}{\alpha_i} + \cdots + \frac{a_k}{\alpha_i^{k-1}} \right| < kA,$$

for A as in the statement of Theorem 4. Writing $\rho := \max |\alpha_i|_{i=1}^s$, we have $\rho < kA$.

The sequence \mathbf{u} admits the closed-form solution $u_n = \sum_{i=1}^s Q_i(n) \alpha_i^n$, where the coefficients of the polynomials $Q_i(x)$ are computed from the initial values u_1, \dots, u_k by solving a system of linear equations. By Cramer's rule, each of the coefficients of $Q_i(x)$ is the quotient of an algebraic integer by the determinant

$$\Delta := \begin{vmatrix} 1 & \cdots & 0 & 1 & \cdots & 0 & 1 & \cdots \\ \alpha_1 & \cdots & \alpha_1 & \alpha_2 & \cdots & \alpha_{s-1} & \alpha_s & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_1^{k-1} & \cdots & (k-1)\alpha_1^{k-2} & \alpha_2^{k-1} & \cdots & (k-1)\alpha_{s-1}^{k-2} & \alpha_s^{k-1} & \cdots \end{vmatrix}.$$

The length of each column vector above is at most

$$\sqrt{k(k-1)^{2(k-1)} \rho^{2k}} < k^k (kA)^k = k^{2k} A^k.$$

Thus, by the Hadamard inequality, $\Delta^2 < (k^{2k^2} A^{k^2})^2 = (k^2 A)^{2k^2}$.

Solving with Cramer's rule for the coefficients of $Q_i(x)$ gives, via the Hadamard inequality again, that they are bounded by $kA|\Delta|$. Thus, replacing \mathbf{u} by $\Delta \mathbf{u}$, we have that

$$Q_i(x) := \sum_{j=0}^{\sigma_i-1} c_{i,j} X^j, \quad \text{where } |c_{i,j}| \leq (k^2 A)^{2k^2+1} \quad \text{and} \quad c_{i,j} \in \mathcal{O}_{\mathbb{K}}. \quad (3)$$

Step 2: Reduction modulo P . Fix $n \in \mathcal{S}(X)$ such that $u_n = 0$ and consider a representation $n = qP + a$. Let \mathfrak{p} be a prime ideal factor of P in $\mathcal{O}_{\mathbb{K}}$ and let $\sigma \in \text{Gal}(\mathbb{K}/\mathbb{Q})$ be the Frobenius automorphism corresponding to \mathfrak{p} , such that $\sigma(\alpha) \equiv \alpha^P \pmod{\mathfrak{p}}$ for all $\alpha \in \mathcal{O}_{\mathbb{K}}$. From $u_n = 0$ and $n = qP + a$ we have

$$\sum_{i=1}^s Q_i(a) \alpha_i^a \sigma(\alpha_i)^q \equiv 0 \pmod{\mathfrak{p}}. \quad (4)$$

Recall that $n \in \mathcal{S}(X)$ and hence $n \leq 2X$. In view of desired upper bound on n we may freely assume that $X > \exp(10k^2 \log k)$, which gives $a \geq \log X / \sqrt{\log_3 X} > 4k^2 + 3$. It follows that $a^k < k^a$. Noting also that $q \leq a$, the absolute value of the left-hand side of (4) is at most

$$\begin{aligned} (k^2 A)^{2k^2+1} (ka^k \rho^{2a}) &\leq (k^2 A)^{2k^2+1} (ka^k (kA)^{2a}) \\ &< (k^2 A)^{2k^2+1} (k(k^a)(kA)^{2a}) \\ &\leq (kA)^{4k^2+3} (kA)^{4a} \\ &= (kA)^{4k^2+3+4a} < (kA)^{5a}. \end{aligned}$$

Suppose that the left-hand side of (4) is non-zero. Then it is a non-zero algebraic integer of degree at most $k!$, all of whose conjugates have absolute value at most $(kA)^{5a}$, and which is divisible by \mathfrak{p} . This implies that P divides an integer of size at most $(kA)^{5ak!}$. Since $P \geq \frac{X-a}{q} \geq \frac{X-\log X}{\sqrt{\log X}} > \sqrt{X}$ for $X > X_0 := 100$, and $a \leq 2 \log X / \sqrt{\log_3 X}$, taking logs we have

$$5k! \log(kA) \frac{2 \log X}{\sqrt{\log_3 X}} > \frac{\log X}{2}.$$

It follows that $\sqrt{\log_3 X} < 20k! \log(kA)$ and so $X < \exp_3((20k! \log(kA))^2)$. But this last inequality yields the desired upper bound on $n \leq 2X$ since either

- $\frac{A}{\log A} > 40k!$, in which case $X < \exp_3(A^2)$, or
- $\frac{A}{\log A} < 40k!$, and so $A < 80k! \log(40k!)$ and $X < \max\{\exp_4(14), \exp_4(10k \log k)\}$.

Step 3: Companion equation. In Step 2 we have proved the desired upper bound on n under the assumption that the left-hand side of (4) is non-zero for some representation of n . Now suppose, on the contrary, that the left-hand side of (4) is zero for each of the $r(n) > \log_4 X$ representations of n . Of these representations, at least $(\log_4 X)/k!$ have the same Frobenius automorphism σ . For this choice of σ we have that the *companion equation* (the equation analog of the congruence (4))

$$\sum_{i=1}^s Q_i(a) \alpha_i^a \sigma(\alpha_i)^q = 0 \quad (5)$$

has least $(\log_4 X)/k!$ solutions in integer variables q, a . The remainder of the proof is dedicated to deriving an upper bound on the number of solutions of (5) that arise from representations of n . From this we obtain the desired upper bound on X .

Every solution of (5) has a non-degenerate vanishing sub-sum and we focus on bounding the number of such sub-sums. The following claim, proven in Section A.1, considers sub-sums that involve only terms from a single summand $Q_i(a) \alpha_i^a \sigma(\alpha_i)^q$ of (5).

Claim 5 *Suppose $R_i(a) \alpha_i^a \sigma(\alpha_i)^q = 0$, where R_i a sub-polynomial of Q_i for some $i \in \{1, \dots, s\}$. Then $X < \max\{\exp_4(14), \exp_3 A, \exp_3(4k \log k)\}$.*

Since the upper bound on X in Claim 5 entails the desired bound on n , it remains to bound the total number of non-degenerate solutions of each of the at most 2^k sub-equations of the form

$$\sum_{i \in I} R_i(a) \sigma(\alpha_i)^q \alpha_i^a = 0, \quad (6)$$

of (5), where $I \subseteq \{1, \dots, s\}$ contains at least two elements, and where $R_i(x)$ is a sub-polynomial of $Q_i(x)$ for all $i \in I$. For this task, a key structure is the group \mathcal{P} of $\mathbf{z} = (z_1, z_2) \in \mathbb{Z}^2$ such that

$$\sigma(\alpha_i)^{z_1} \alpha_i^{z_2} = \sigma(\alpha_j)^{z_1} \alpha_j^{z_2} \quad \text{for all } i, j \in I.$$

For $\mathbf{z} = (z_1, z_2) \in \mathcal{P}$ we have $\sigma(\alpha_i/\alpha_j)^{z_1} = (\alpha_j/\alpha_i)^{z_2}$ for all $i, j \in I$. As shown in Section 2, since α_i/α_j is not a root of unity, this entails $z_1 = z_2$ or $z_1 = -z_2$. There are thus three possibilities for \mathcal{P} : either $\mathcal{P} = \{\mathbf{0}\}$, \mathcal{P} is parallel to $(1, 1)$, or \mathcal{P} is parallel to $(1, -1)$.

The simplest case is that $\mathcal{P} = \{\mathbf{0}\}$. Here, Theorem 1 shows that if we put

$$A := \sum_{i \in I} \binom{2 + \sigma_i - 1}{2} \quad \text{and} \quad B = \max(2, A)$$

then the number of solutions (a, q) of (6) is at most $2^{35B^3} (k!)^{6B^2}$. But

$$A \leq \sum_{i \in I} \binom{\sigma_i + 1}{2} = \sum_{i \in I} \frac{\sigma_i(\sigma_i + 1)}{2} \leq \sum_{i \in I} \sigma_i^2 \leq k^2,$$

so the number of solutions of (6) is at most $2^{35k^6} (k!)^{6k^4}$. After multiplying the above bound by 2^k to account for the number of non-degenerate sub-sums, the resulting quantity is greater than the number $(\log_4 X)/k!$ of solutions of the companion equation. In other words,

$$\log_4 X < 2^k k! 2^{35k^6} (k!)^{6k^4},$$

from which we obtain

$$X < \max\{\exp_5(10^{10}), \exp_5(25k^6)\}. \quad (7)$$

Step 4: The hard case. We are left with the case $\mathcal{P} \neq \{\mathbf{0}\}$, where we cannot apply Theorem 1. In this case \mathcal{P} is either a subgroup of $\{(z, z) : z \in \mathbb{Z}\}$ or a subgroup of $\{(z, -z) : z \in \mathbb{Z}\}$. It follows that either $\sigma(\alpha_i)\alpha_i$ is constant for all $i \in I$ or $\sigma(\alpha_i)/\alpha_i$ is constant for all $i \in I$. Cancelling the common value of $(\sigma(\alpha_i)\alpha_i)^q$ or $(\sigma(\alpha_i)/\alpha_i)^q$ in (6) we have

$$\sum_{i \in I} R_i(a) \alpha_i^{a+\eta q} = 0 \quad \text{for some } \eta \in \{\pm 1\}. \quad (8)$$

Similar to Step 3, we will obtain the desired upper bound on n by giving an upper bound on the number of solutions of (8) and hence of the number of representations of n . In lieu of Theorem 1 we use a bespoke argument that uses ideas of [1,4,11], but which is greatly simplified by exploiting the assumption that no two representations of n are correlated.

The argument is by induction on the number of summands $|I| \leq k$. To get started, we write $|I| = \ell$, relabel the roots so that $I = \{1, \dots, \ell\}$, and restate Equation (8) as follows:

$$\sum_{i=1}^{\ell} R_i(a) \alpha_i^{a+\eta q} = 0.$$

We dehomogenize the above equation by dividing through by the first summand, yielding

$$1 = \sum_{i=2}^{\ell} (-R_i(a)/R_1(a))(\alpha_i/\alpha_1)^{a+\eta q}. \quad (9)$$

Our goal is to apply [1, Theorem 6.1] in order to find a homogeneous linear relation among the summands on the right-hand side of (9). This will yield an equation similar to (8) but with strictly fewer summands. To this end, let Γ be the rank-one multiplicative subgroup of $(\mathbb{C}^\times)^{\ell-1}$ generated by $\gamma := (\alpha_i/\alpha_1 : i = 2, \dots, \ell)$. Then Equation (9) can be written $\mathbf{x}^\top \mathbf{y} = 1$, where $\mathbf{x} = \gamma^{a+\eta q}$ and $\mathbf{y} = (-R_i(a)/R_1(a) : i = 2, \dots, \ell)$. Denote by h the absolute logarithmic Weil height (see [11, Section 7] for the definition and relevant properties of h) and define $\varepsilon := (8k)^{-6k^3}$. Then we have the following claim, which is proven in Section A.2.

Claim 6 *If $X > 2 \max\{\exp_3(A^2), \exp_5(10^{10}k^6)\}$ then $h(\mathbf{y}) \leq (1 + h(\mathbf{x}))\varepsilon$.*

The inequality $X \leq 2 \max\{\exp_3(A^2), \exp_5(10^{10}k^6)\}$ implies the bound on n that we are ultimately trying to prove, and thus we may apply Claim 6 to deduce that $h(\mathbf{y}) \leq (1 + h(\mathbf{x}))\varepsilon$. This height inequality allows us to apply [1, Theorem 6.1] to conclude that there is a collection of at most $(8k)^{6k^3(k+1)}$ vectors $\mathbf{A} = (A_2, \dots, A_\ell) \in \overline{\mathbb{Q}}^{\ell-1}$ such that each solution of (9) satisfies

$$\sum_{i=2}^{\ell} A_i R_i(a) \alpha_i^{a+\eta q} = 0$$

for one of these vectors \mathbf{A} . We will use such linear relations to proceed by induction.

Fix a vector $\mathbf{A} = (A_2, \dots, A_\ell)$ among the $(8k)^{6k^3(k+1)}$ possibilities and consider the equation

$$\sum_{i=2}^{\ell} A_i R_i(a) \alpha_i^{a+\eta q} = 0.$$

Assume that at least three of the A_i 's are nonzero and relabel so that the non-zero A_i 's have indices $i = 2, 3, \dots, \ell'$, where $\ell' \leq \ell$. We dehomogenize the above relation to get

$$1 = \sum_{i=3}^{\ell'} (-A_i/A_2)(R_i(a)/R_2(a))(\alpha_i/\alpha_2)^{a+\eta q}.$$

We take now $\Gamma \subseteq (\mathbb{C}^\times)^{\ell'-2}$ to be the rank-two multiplicative subgroup generated by $(\alpha_3/\alpha_2, \dots, \alpha_{\ell'}/\alpha_2)$ and $((-A_3/A_2), \dots, (-A_{\ell'}/A_2))$. The above equation is again of the form $\mathbf{x}^\top \mathbf{y} = 1$, where now

$$\mathbf{x} = ((-A_3/A_2)(\alpha_3/\alpha_2)^{a+\eta q}, \dots, (-A_{\ell'}/A_2)(\alpha_{\ell'}/\alpha_2)^{a+\eta q}),$$

and $\mathbf{y} = (R_3(a)/R_2(a), \dots, R_{\ell'}(a)/R_2(a))$.

To continue the induction we need to establish again the height bound

$$h(\mathbf{y}) \leq (1 + h(\mathbf{x}))\varepsilon, \quad (10)$$

where ε is as in Claim 6. The challenge is that the components of \mathbf{A} (arising from the application of [1, Theorem 6.1]) are not known. But in the case at hand this is easy thanks to the following lemma, which is proved in Section A.3:

Claim 7 *There is at most one value of $a + \eta q$ such that (10) fails for the corresponding \mathbf{x} , provided $X > \max\{\exp_4(10), \exp_3(4k \log k)\}$.*

By Claim 7, for large X , there is at most one representation for which the corresponding vector \mathbf{x} fails to satisfy (10). This allows us to continue the induction. In summary, at step one, the group Γ had rank 1 and the application of [1, Theorem 6.1] lead to a homogeneous equation in at most $k - 1$ unknowns whose coefficients had unknown heights. At most one solution of the equation failed to satisfy the height bound (10) for the induction step, for which we had now a group Γ of rank 2 yielding an equation in at most $k - 2$ of the unknowns. At each step, when the rank of Γ was r then the number of equations was at most $2^k \cdot (8k)^{6k^3(k+r)}$ and at each step there was at most one solution violating the height bound (10). So, if we have at least

$$\sum_{j=1}^{k-2} 2^k \prod_{i=1}^j (8k)^{6k^3(k+i)} < k 2^{k(k-2)} (8k)^{6k^3(k-2)+6k^3(k-1)(k-2)/2} < (8k)^{3k^5}$$

solutions of the original Equation (8), then we arrive at a two-dimensional equation that has at least two solutions. That is, we have

$$A_i R_i(a) \alpha_i^{a+\eta q} + A_j R_j(a) \alpha_j^{a+\eta q} = 0,$$

for some $i \neq j$ and some A_i, A_j nonzero, $\eta \in \{\pm 1\}$, and the same with (q, a) replaced by (q', a') . Hence,

$$\left(\frac{R_i(a)}{R_i(a')} \right) \left(\frac{R_j(a')}{R_j(a)} \right) = \left(\frac{\alpha_j}{\alpha_i} \right)^{(a+\eta q) - (a'+\eta q')}.$$

Since $|(a + \eta q) - (a' + \eta q')| > \sqrt{\log X}$, taking heights we get

$$8k \log_2 X > \sqrt{\log X} h(\alpha_i/\alpha_j) > \frac{\sqrt{\log X}}{4k^4} \left(\frac{\log_2(k^2)}{\log k^2} \right)^3,$$

and this gives an upper bound on X that is much smaller than some of the ones encountered before. Hence, if X is larger than any of the previous bounds then we have

$$\frac{\log_4 X}{k!} < (8k)^{3k^5}, \quad \text{hence} \quad X < \exp_5(13k^5 \log k).$$

This last upper bound is smaller than (7), which is the largest of all upper bounds on X in the proof. Modifying the coefficient 25 of k^6 to 10^{10} to absorb the first term of the max into the second term in (7), we get the desired bound on n . This finishes the proof of Theorem 4.

4 The Density of \mathcal{S}

In this section we show that \mathcal{S} has density one subject to the Bateman-Horn conjecture. Recall from Section 3 that we exclude from \mathcal{S} all $n \in \mathbb{N}$ that have two correlated representations. In Section 4.1 we show that the set of numbers thus excluded has density zero. Then, in Section 4.2, we show that set of $n \in [X, 2X]$ that have more than $\log_4 X$ representations has density one. We conclude that \mathcal{S} itself has density one.

In this section the indices p, q, P, P' in summations and products run over positive primes.

4.1 Counting correlated representations

We will need the following simple result. See [8, Proposition 6] for a proof.

Proposition 8. $\sum_{q \in A(X)} \frac{1}{q} \sim \log_3 X$

We now have:

Lemma 9. *The set of $n \in [X, 2X]$ with two correlated representations $n = Pq + a = P'q' + a'$, i.e., such that*

$$q \neq q', a \neq a' \quad \text{and} \quad |(a + \eta q) - (a' + \eta q')| < \sqrt{\log X}$$

for some $\eta \in \{\pm 1\}$, is of cardinality $O(X/(\log X)^{1/3})$.

Proof. We first fix $q \neq q' \in A(X)$ and $a \neq a' \in B(X)$ and count the number of pairs of primes P and P' such that

$$qP + a = q'P' + a' \in [X, 2X]. \quad (11)$$

The general solution of the above equation in nonnegative integers P and P' can be written in the form $P = P_0 + q't$ and $P' = P'_0 + qt$, where t is a nonnegative integer parameter and P_0, P'_0 is the minimal solution (simultaneously, in both coordinates) among positive integers. The condition that $qP + a \leq 2X$ implies that $P \leq 2X/q$ and hence that $t \leq \frac{2X}{qq'}$. Using the assumption $a \neq a'$, we can apply Theorem 2 to deduce that the number of $t \leq \frac{2X}{qq'}$ such that $P_0 + q't$ and $P'_0 + qt$ are both prime is

$$\ll \frac{X}{qq'(\log X)^2} \left(\frac{|a - a'|}{\varphi(|a - a'|)} \right) \ll \frac{X \log_3 X}{qq'(\log X)^2},$$

where we have used the inequality $m/\varphi(m) \ll \log \log m$ in the case $m = |a - a'| \leq \log X$.

We next sum up the number of solutions of (11) over the different choices of $q \neq q' \in A(X)$ and $a \neq a' \in B(X)$ such that $|(a + \eta q) - (a' + \eta q')| < \sqrt{\log X}$. Note here that since $q, q' \leq \sqrt{\log X}$, the condition $|(a + \eta q) - (a' + \eta q')| < \sqrt{\log X}$ implies that $|a - a'| < 2\sqrt{\log X}$ and hence a' is determined in at most $2\sqrt{\log X}$ different ways by the choice of a . Since $a \in B(X)$, there at most $\frac{2 \log X}{\sqrt{\log_3 X}}$ choices of a . We thus get a count of

$$\frac{X(\log_3 X)}{(\log X)^2} \left(\sum_{q \leq \sqrt{\log X}} \frac{1}{q} \right)^2 \left(\frac{\log X}{\sqrt{\log_3 X}} \right) \sqrt{\log X} \ll \frac{X(\log_3 X)^{2.5}}{\sqrt{\log X}},$$

where we use the inequality $\sum_{q \in A(X)} \frac{1}{q} \ll \log_3 X$ from Proposition 8. This is a count on the number of sextuples (q, q', a, a', P, P') subject to the above conditions, so the number of n 's arising as $Pq + a$ from such a sextuple is also $O(X/(\log X)^{1/3})$. \square

4.2 Counting all representations

Let n and X be positive integers with $n \in [X, 2X]$. Recall that $r(n)$ denotes the number of representations of $n = qP + a$ with q, P prime, $q \in A(X)$ and $a \in B(X)$. By the prime number theorem,

we have that

$$\begin{aligned}
\sum_{n \in [X, 2X]} r(n) &= \sum_{\substack{q \in A(X) \\ a \in B(X)}} \sum_{\substack{\frac{X-a}{q} \leq P \leq \frac{2X-a}{q}}} 1 \\
&= (1 + o(1)) \sum_{\substack{q \in A(X) \\ a \in B(X)}} \frac{X}{q \log X} \\
&= (1 + o(1)) X \sqrt{\log_3 X}.
\end{aligned}$$

If we can show that

$$\sum_{n \in [X, 2X]} r(n)^2 = (1 + o(1)) X (\sqrt{\log_3 X})^2, \quad (12)$$

then it follows that

$$\sum_{n \in [X, 2X]} (r(n) - \sqrt{\log_3 X})^2 = o(X (\sqrt{\log_3 X})^2),$$

and so $r(n) = (1 + o(1)) \sqrt{\log_3 X}$ for $(1 + o(1))X$ integers $n \in [X, 2X]$. In combination with Lemma 9 we conclude that $\#S(X) = (1 + o(1))X$ and hence that \mathcal{S} has density one.

To conclude that \mathcal{S} has density one it remains to prove (12). Establishing this is out of the reach of unconditional techniques, but it follows quite quickly from standard conjectures. In particular, we show that the Bateman-Horn conjecture implies that for any given $a \neq a' \in B(X)$ and $q \neq q' \in A(X)$, if $\gcd(a - a', qq') = 1$ and $2|(a - a')$ then the number of pairs of primes P, P' such that

$$qP + a = q'P' + a' \in [X, 2X] \quad (13)$$

is given by

$$(C' + o(1)) \frac{X}{qq'(\log X)^2} g(|a - a'|) \quad (14)$$

where

$$C' := 2 \prod_{p > 2} \left(\frac{p(p-2)}{(p-1)^2} \right) \approx 1.32 \quad \text{and} \quad g(m) := \prod_{\substack{p|m \\ p > 2}} \frac{p-1}{p-2}.$$

Indeed, as explained in the proof of Lemma 9, there exist two linear forms $f_1(t) := P_0 + q't$ and $f_2(t) := P'_0 + qt$ such that the number of solutions of Equation (13) in primes P and P' is equal to the number of values t with $\frac{X-o(X)}{qq'} \leq t \leq \frac{2X}{qq'}$ for which $f_1(t)$ and $f_2(t)$ are both prime. The assumptions $\gcd(a - a', qq') = 1$ and $2|(a - a')$ guarantee that $f := f_1 f_2$ is admissible (specifically, that f does not vanish identically modulo 2, q , or q'). If one of the previous two assumptions fails then there are no solutions of (13) in primes P and P' . We apply Conjecture 3 to obtain the estimate (14). Note here that the constant C in Conjecture 3 becomes $C'g(|a - a'|)$ in (14).

In summary, we see that

$$\begin{aligned}
\sum_{n \in [X, 2X]} r(n)^2 &= \sum_{\substack{a, a' \in B(X) \\ q, q' \in A(X)}} \sum_{\substack{P, P' \\ qP + a = q'P' + a' \in [X, 2X]}} 1 \\
&= \sum_{\substack{a \neq a' \in B(X) \\ q \neq q' \in A(X) \\ 2|(a - a') \\ \gcd(a - a', qq') = 1}} (C' + o(1)) \frac{X}{qq'(\log X)^2} g(|a - a'|) + O(X \sqrt{\log_3 X}).
\end{aligned}$$

(Here the $O(X\sqrt{\log_3 X})$ term comes from bounding the contribution when $q = q'$ or $a = a'$). Applying Proposition 8 twice to evaluate the inner summation over $q \neq q' \in A(X)$ we get

$$(C' + o(1)) \frac{X(\log_3 X)^2}{(\log X)^2} \sum_{\substack{a \neq a' \in B(X) \\ 2|(a-a') \\ \gcd(a-a', qq')=1}} g(|a - a'|) + O(X\sqrt{\log_3 X})$$

Since g is a multiplicative function with $g(p) = 1 + O(1/p)$ standard estimates show that

$$\begin{aligned} \sum_{\substack{a \neq a' \in B(X) \\ 2|(a-a')}} g(|a - a'|) &= \frac{1 + o(1)}{2} \left(\frac{\log X}{\sqrt{\log_3 X}} \right)^2 \prod_{p>2} \left(1 + \frac{g(p) - 1}{p} \right) \\ &= \frac{1 + o(1)}{C'} \left(\frac{\log X}{\sqrt{\log_3 X}} \right)^2. \end{aligned}$$

Putting this all together, we have

$$\sum_{n \in [X, 2X]} r(n)^2 = (1 + o(1)) X (\sqrt{\log_3 X})^2.$$

This is what we wanted to prove, and we conclude that \mathcal{S} has density one subject to the Bateman-Horn conjecture. We note also that \mathcal{S} has positive density unconditionally. Indeed, \mathcal{S} arises by taking a set that was shown to have positive density in [8, Section 3] and removing the set of all natural numbers with two correlated representations, which has density zero by Lemma 9.

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A Deferred Proofs

A.1 Proof of Claim 5

Claim 5 Suppose $R_i(a)\alpha_i^a\sigma(\alpha_i)^q = 0$, where R_i a sub-polynomial of Q_i for some $i \in \{1, \dots, s\}$. Then $X < \max\{\exp_4(14), \exp_3 A, \exp_3(4k \log k)\}$.

Proof. Suppose that $R_i(a)\alpha_i^a\sigma(\alpha_i)^q = 0$. Write

$$R_i(x) = b_{i_0}x^{i_0} + b_{i_1}x^{i_1} + \dots + b_{i_t}x^{i_t},$$

where $t \geq 1$, $0 \leq i_0 < i_1 < \dots < i_t \leq \sigma_i - 1$ and b_{i_0}, \dots, b_{i_t} are nonzero algebraic integers. Simplifying across by x^{i_0} , we may assume that $x = a$ is a root of

$$b_{i_0} + b_{i_1}x^{i_1-i_0} + \dots + b_{i_t}x^{i_t-i_0}.$$

But then a divides the norm of b_{i_0} , a nonzero integer of size at most $(k^2 A)^{(2k^2+1)k!}$. Since $a \in B(X)$, this implies that

$$\frac{\log X}{\sqrt{\log_3 X}} < (kA)^{4(k+2)!},$$

and so

$$\log X < 2(kA)^{4(k+2)!} \log((kA)^{2(k+2)!}) < (kA)^{8(k+2)!} < \exp(8(k+2)^{k+2} \log(kA)).$$

This implies that

$$X < \exp_3((k+2) \log(k+2) + \log(8 \log(kA)))$$

and this in turn yields the upper bound stated in the claim. That is, either

$$A > 10k \log k, \quad \text{and then the above gives } X < \max\{\exp_4(14), \exp_3 A\}$$

or

$$A < 10k \log k, \quad \text{in which case } X < \max\{\exp_4(14), \exp_3(4k \log k)\}.$$

□

A.2 Proof of Claim 6

Claim 6 If $X > 2 \max\{\exp_3(A^2), \exp_5(10^{10}k^6)\}$ then $h(\mathbf{y}) \leq (1 + h(\mathbf{x}))\varepsilon$.

Proof. We make a case distinction on the value of $h(a)$. First, assume that $h(a) \geq 3k^2 \log(k^2 A)$. Then for some $i \in \{2, \dots, \ell\}$ we have

$$\begin{aligned} h(\mathbf{y}) &\leq h(R_i(a)) + h(R_1(a)) \leq 2kh(a) + 2k \log 2 + 2k \log((k^2 A)^{2k^2+1}) \\ &< 2kh(a) + 2k(2k^2 + 2) \log(k^2 A) < 2kh(a) + 6k^3 \log(k^2 A) < 4kh(a). \end{aligned}$$

We deduce that

$$h(\mathbf{y}) < 4kh(a) < 4k \log a < 4k \log_2 X.$$

Next we give a lower bound on $h(\mathbf{x})$. Let α be an element of maximum height in $\{\alpha_i/\alpha_j : 1 \leq i < j \leq s\}$. Then

$$h(\mathbf{x}) \geq (a - q)h(\alpha) \geq \left(\frac{\log X}{2\sqrt{\log_3 X}} \right) h(\alpha) \quad \text{for } X > 55,$$

since $\log X/(2\sqrt{\log_3 X}) > \sqrt{\log X}$ for $X > 55$. If at least one of the quotients α_i/α_j is not an algebraic integer then $h(\alpha)$ is at least $(\log 2)/k^2$. Suppose instead that all the quotients are algebraic integers. Since α is not a root of unity (by non-degeneracy of \mathbf{u}) it follows from Voutier's effective version [14] of Dobrowolski's result that

$$h(\alpha) \geq \frac{1}{4k^4} \left(\frac{\log \log(k^2)}{\log(k^2)} \right)^3. \quad (15)$$

Combining the upper bound on $h(\mathbf{y})$ and lower bound on $h(\mathbf{x})$, we see that for the desired height inequality $h(\mathbf{y}) \leq (1 + h(\mathbf{x}))\varepsilon$ it suffices that

$$4k \log_2 X < \frac{1}{(8k)^{6k^3}} \left(1 + \frac{\log X}{8k^4 \sqrt{\log_3 X}} \left(\frac{\log \log(k^2)}{\log(k^2)} \right)^3 \right).$$

But this inequality holds under the condition $X > \max\{\exp_4(10), \exp_2(7k^3 \log k)\}$, which is implied by the lower bound on X in the hypothesis of the current claim.

It remains to consider the case that $h(a) < 3k^2 \log(k^2 A)$. Here, since $a \geq \frac{\log X}{\sqrt{\log_3 X}}$, we have

$$\log X < 3k^2 \log(k^2 A) \sqrt{\log_3 X}.$$

This yields

$$\log X < 6k^3 \log(k^2 A) \log(3k^2 (\log k^2 A)) < 6k^3 (\log(3k^2 A))^2,$$

and so

$$X < \exp(6k^3 \log((3k^2 A))^2).$$

If $A > k \log k$, then $X < \max\{\exp_2(100), \exp(A^4)\}$ and if $A \leq k \log k$, then $X < \max\{\exp_2(100), \exp(k^4)\}$. But these upper bounds on X contradict the lower bound on X in the hypothesis of the claim and so the assumption $h(a) < 3k^2 \log(k^2 A)$ leads to a contradiction, i.e., the second case of the proof is vacuous. \square

A.3 Proof of Claim 7

Claim 7 *There is at most one value of $a + \eta q$ such that (10) fails for the corresponding \mathbf{x} , provided $X > \max\{\exp_4(10), \exp_3(4k \log k)\}$.*

Proof. Suppose that (10) fails for both $(q, a) \neq (q', a')$. For the vectors \mathbf{x}, \mathbf{y} and \mathbf{x}', \mathbf{y}' respectively corresponding to (q, a) and (q', a') we have $h(\mathbf{y}) > (1 + \varepsilon)h(\mathbf{x})$ and $h(\mathbf{y}') > (1 + \varepsilon)h(\mathbf{x}')$. Adding these two inequalities we get

$$\begin{aligned} 8k \log_2 X &> 4k \log a + 4k \log a' > h(\mathbf{y}) + h(\mathbf{y}') > (2 + h(\mathbf{x}) + h(\mathbf{x}'))\varepsilon \\ &> (2 + h(\mathbf{x}/\mathbf{x}'))\varepsilon. \end{aligned} \quad (16)$$

For the right-most inequality above see equation (7.6) in [4]. But in \mathbf{x}/\mathbf{x}' , the unknown vector \mathbf{A} is gone and

$$h(\mathbf{x}/\mathbf{x}') = h((\alpha_3/\alpha_2)^{(a+\eta q)-(a'+\eta q')}, \dots, (\alpha_{\ell'}/\alpha_2)^{(a+\eta q)-(a'+\eta q')}).$$

In particular, by (15) and the fact that $|(a + \eta q) - (a' + \eta q')| > \sqrt{\log X}$, we have

$$\begin{aligned} h(\mathbf{x}/\mathbf{x}') &\geq |(a + \eta q) - (a' + \eta q')| \min\{h(\alpha_i/\alpha_j) : i \neq j \in \{2, 3, \dots, \ell'\}\} \\ &\geq \sqrt{\log X} \left(\frac{1}{4k^4} \left(\frac{\log \log k^2}{\log k^2} \right)^3 \right). \end{aligned}$$

So the estimate (16) leads to

$$8k^2 \exp((4k)^{3k}) \log_2 X > 2 + \sqrt{\log X} \left(\frac{1}{4k^2} \left(\frac{\log_2(k^2)}{\log k^2} \right)^3 \right),$$

which gives $X < \max\{\exp_4(10), \exp_3(4k \log k)\}$. □