Equilibrium in Muller games with preference ordering

Soumya Paul and Sunil Simon

The Institute of Mathematical Sciences, Chennai, India

{soumya,sunils}@imsc.res.in

Game model

Let N be a finite set of players.

An infinite multiplayer game arena $\mathcal{G}$ consists of:

- A directed graph (W, E) where,
  - W is the set of game positions.
  - $E \subseteq W \times W$ specifies the move relation.
- An initial game position w_0.
- A partition W^i assigning each game position a player.

Play in the arena

A play is an infinite path in $\mathcal{G}$.

Intuitively, a token is placed in the initial game position and then moved through the graph. Whenever the token reaches a game position $w \in W^i$, player i moves the token to a position v such that $(w, v) \in E$.

Strategies

A strategy for a player i is a map $\mu^i : W^* W^i \to W$.
We denote the set of all strategies of player i by $\Omega^i(\mathcal{G})$.

Interesting subclasses of strategies:

- Bounded memory strategies: Depend only on bounded information about the sequence of game positions observed in the past.
- Positional (memoryless) strategies: Depends only on the current game position. That is, $\mu^i : W^i \to W$.

A strategy profile is a tuple of strategies $\mu = (\mu^i)_{i \in N}$, one for each player. A strategy profile μ determines a unique play in the arena denoted ρ_μ.

Objectives

Most common: Zero sum objectives (for two player games). Winning condition of one of the player (say player 1) is specified as a set $\Phi_1 \subseteq W^\omega$. The winning set of player 2 is $\Phi_2 = W^\omega \setminus \Phi_1$.

Definition: A two player zero sum game is then specified by the pair $G = (\mathcal{G}, \Phi_1)$.

Question: Is the game determined? That is, does one of the players always have a winning strategy?

Theorem [Martin 75]: Every two player zero sum Borel game is determined.

Objectives

For algorithmic analysis, objectives need to be finitely representable conditions. Of particular interest in the context of computer science are omega regular objectives.

Definition: A two player zero sum Muller game is specified by the pair $(\mathcal{G}, \mathcal{F})$ where

- $\mathcal{G}$ is a game arena with the set of game positions W.
- $\mathcal{F} \subseteq 2^W$.

A play ρ in $\mathcal{G}$ is said to be winning for player 1 if $Inf(\rho) \in \mathcal{F}$.

Theorem [Büchi and Landweber 69]: For Muller games played on finite graphs, the winner can be determined and the winning strategy can be effectively synthesised in finite memory strategies.

From zero sum to non-zero sum

Multiplayer games with binary objectives: $G = (\mathcal{G}, \{\mathcal{F}_i\}_{i \in N})$.

- Each player is associated with a Muller condition $\mathcal{F}_i \subseteq 2^W$.

Note: Objectives of players are allowed to overlap.

Solution concept: Nash equilibrium

- A strategy profile μ is a Nash equilibrium if for all players i and all strategies ν^i of player i we have:
  - if $\rho_{(\mu^{-i}, \nu^i)}$ is winning for i then ρ_μ is winning for i.

Games with binary objectives

Question: Does Nash equilibrium always exist?

Theorem [Chatterjee-Jurdziński-Majumdar 04]: Every game with regular winning condition has a Nash equilibrium.

Theorem [Ummels 05]: Every game with regular winning condition has a sub-game perfect equilibrium.

Non-zero sum objectives

- From a game theoretic perspective, it is natural to look at games where players have preference orderings over plays.
- If we restrict our attention to classifying regular plays this can be captured by associating with each player a preference ordering over the Muller table.

Generalised Muller games

Definition: A generalised Muller game is a tuple $(\mathcal{G}, \mathcal{F}, \{\sqsubseteq^i\}_{i \in N})$ where:

- $\mathcal{G}$ is a game arena with set of game positions W.
- $\sqsubseteq^i \subseteq \mathcal{F} \times \mathcal{F}$.

For simplicity we assume that $\mathcal{F} = 2^W$.

The relation $\sqsubseteq^i$ induces a preference ordering on plays in $\mathcal{G}$ as follows:

- $\rho \preceq^i \rho'$ iff $Inf(\rho) \sqsubseteq^i Inf(\rho')$.

Generalised Muller games

Generalised Muller games define multiplayer non-zero sum infinite games.

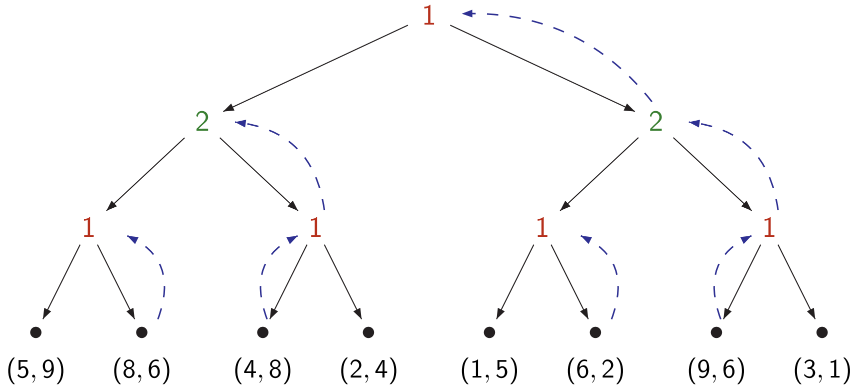
Questions:

- Does Nash equilibrium always exists?
- Is it possible to synthesize an equilibrium profile (if it exists)?

Our approach

- Borrow ideas from equilibrium computation in finite extensive form games.
- In finite extensive form games the backward induction procedure [Zermello 1913] synthesises an equilibrium profile.

Backward induction



Backward induction

Not quite straight forward.

- Backward induction is designed to work on finite trees.
- Tree unfolding of $\mathcal{G}$ results is an infinite tree.

To do: Construct a finite tree structure which preserves the equilibrium behaviour of players.

Core issue

For each play identify the Muller set that the play settles down to without actually performing the infinite tree unfolding.

Latest appearance record (LAR)

[Büchi 83]: Combine the latest appearance record (a permutation of the states) along with a hit position.

Objective: Keep a record of states in the order of their "last visit".

Example: Let $W = \{w_1, w_2, w_3\}$.

w_1	w_3	w_2	w_1	w_2	w_1	w_2	$\dots$
$w_3 w_2 \sharp w_1$	$\sharp w_2 w_1 w_3$	$\sharp w_1 w_3 w_2$	$\sharp w_3 w_2 w_1$	$w_3 \sharp w_1 w_2$	$w_3 \sharp w_2 w_1$	$w_3 \sharp w_1 w_2$	$\dots$

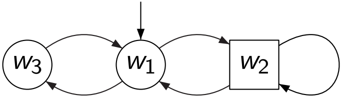
- Eventually only states w_1 and w_2 occur at positions of the vector after the hit position.
- Infinitely often the set $\{w_1, w_2\}$ occur after the hit position.

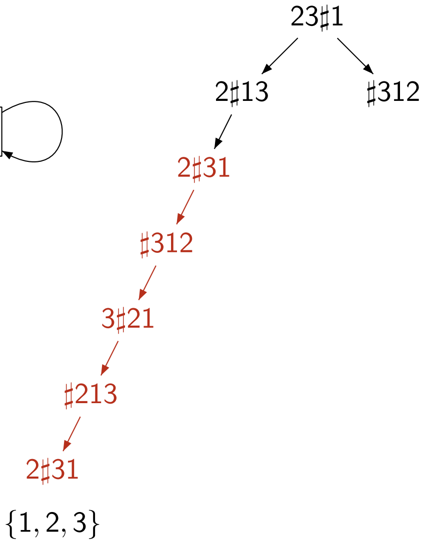
The LAR tree

For an arena $\mathcal{G}$ the LAR tree $T_{\mathbb{LAR}}(\mathcal{G})$ is built by taking the tree unfolding of $\mathcal{G}$ where:

- Paths consists of sequences of LAR vectors.
- Each path terminates when an LAR vector on the path repeats, which denotes a connected component in the game arena.
- We label the leaf nodes of the tree with the connected component that the path denotes.

The LAR tree


w_3
w_1
w_2


$23\sharp1$
$2\sharp13$
$\sharp312$
$2\sharp31$
$\sharp312$
$3\sharp21$
$\sharp213$
$2\sharp31$
$\{1, 2, 3\}$

The LAR tree

LAR tree as a finite extensive form game: Consider the partition of game positions where $S^i = \{x\sharp yw \mid w \in W^i\}$.

Let $\Omega^i(T_{\mathbb{LAR}}(\mathcal{G}))$ denote the set of all strategies of player i in the LAR tree $T_{\mathbb{LAR}}(\mathcal{G})$.

Translation functions

- $\mathfrak{f}^i : \Omega^i(T_{\mathbb{LAR}}(\mathcal{G})) \rightarrow \Omega^i(\mathcal{G})$ - translates strategies on the LAR tree to bounded memory strategies in the arena $\mathcal{G}$. The memory required for translation is $LAR(\mathcal{G})$.
- $\mathfrak{g}^i : \Omega^i(\mathcal{G}) \rightarrow \Omega^i(T_{\mathbb{LAR}}(\mathcal{G}))$ - translates "LAR implementable" strategies in $\mathcal{G}$ to strategies in $T_{\mathbb{LAR}}(\mathcal{G})$.

Purpose of the LAR tree

Claim 1: For any strategy profile ν in $\Omega^i(T_{\mathbb{LAR}}(\mathcal{G}))$ if $lab(\varrho_\nu) = F$ then $Inf(\rho_\mu) = F$ where $\mu = \mathfrak{f}(\nu)$.

Claim 2: For any strategy profile μ in $\mathcal{G}$ which is LAR implementable, if $Inf(\rho_\mu) = F$ then $lab(\varrho_\nu) = F$ where $\nu = \mathfrak{g}(\mu)$.

Backward induction

Procedure

- Initially, all interior nodes of $T_{\mathbb{LAR}}(\mathcal{G})$ are unlabelled.
- Repeat the following steps till the labelling function is defined on the root node.
  - Choose any node $x \sharp y$ which is not labelled and all of whose successors are labelled.
  - if $x \sharp y \in S^i$ then let $x_1 \sharp y_1$ be a successor node such that $lab(x_2 \sharp y_2) \sqsubseteq^i lab(x_1 \sharp y_1)$ for all other successor nodes $x_2 \sharp y_2$ of $x \sharp y$.

    Let $lab(x \sharp y) = lab(x_1 \sharp y_1)$ and $\nu^i(x \sharp y) = x_1 \sharp y_1$.

Equilibrium

Lemma: If ν is a Nash equilibrium profile in $T_{\mathbb{LAR}}(\mathcal{G})$ then μ is a Nash equilibrium profile in the game $G = (\mathcal{G}, \{\sqsubseteq^i\}_{i \in N})$ where $\mu = \mathfrak{f}(\nu)$.

Theorem: Nash equilibrium always exists for generalised Muller games and it is possible to synthesize an equilibrium profile.

Conclusion

- Suggested a possible model for non-zero sum games on graphs in terms of generalised Muller games.
- Showed that the standard backward induction algorithm can be adapted to work in this setting.
- Study the rationality assumptions which justify equilibrium behaviour in infinite games.