# D-finiteness: Algorithms and Applications

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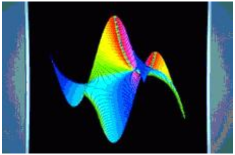

July 25, 2005

# I D-finiteness in One Variable

# Warm-up: Algebraic Numbers and Finite Dimension

$$x = \frac{\sin\frac{2\pi}{7}}{\sin^2\frac{3\pi}{7}} - \frac{\sin\frac{\pi}{7}}{\sin^2\frac{2\pi}{7}} + \frac{\sin\frac{3\pi}{7}}{\sin^2\frac{\pi}{7}} = 2\sqrt{7}.$$

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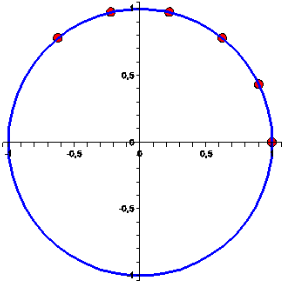
$\mathbb{Q}(\exp(i\pi/7))$ has dim 6 over $\mathbb{Q}$



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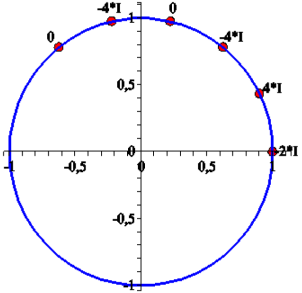
Coordinates of $x$




Tools: Euclidean division, (extended) Euclidean algorithm, linear algebra.

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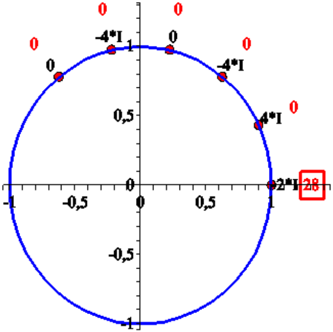


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# Warm-up: Algebraic Numbers and Finite Dimension

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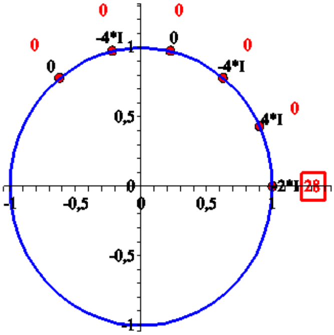



## Definition

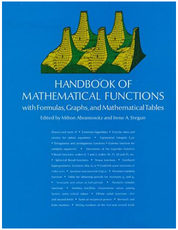
A number $x\in\mathbb{C}$ is *algebraic* when its powers generate a finite-dimensional vector space over $\mathbb{Q}$.

Tools: Euclidean division, (extended) Euclidean algorithm, linear algebra.

# D-finite Series & Sequences

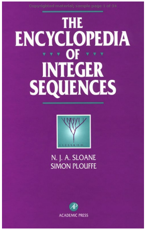
## Definition

A *series* $f(x) \in \mathbb{K}[[x]]$ is D-finite over $\mathbb{K}$ when its derivatives generate a finite-dimensional vector space over $\mathbb{K}(x)$. (LDE)
A *sequence* $u_n$ is D-finite over $\mathbb{K}$ when its shifts $(u_n, u_{n+1}, \dots)$ generate a finite-dimensional vector space over $\mathbb{K}(n)$. (LRE)




About 25% of Sloane's encyclopedia,
60% of Abramowitz & Stegun.

$\boxed{\text{eqn+ini. cond.=data structure}}$




Tools: right Euclidean division; right (extended) Euclidean algorithm; linear algebra; equivalence via generating series.
Implemented in `gfun` [SaZi94].

# Example: Mehler's Identity for Hermite Polynomials

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

1. Definition of Hermite polynomials (D-finite over $\mathbb{Q}(x)$): recurrence of order 2
2. Product by linear algebra: $H_{n+k}(x)H_{n+k}(y)/(n+k)!, k \in \mathbb{N}$ generated over $\mathbb{Q}(x,n)$ by

$$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

$\longrightarrow$ recurrence of order at most 4;

3. Translation into a differential equation

# I. Definition

> $R_1 := \{H(n+2) = (-2n-2)H(n) + 2H(n+1)x, H(0)=1, H(1)=2x\}$ :

> $R_2 := subs(H = H_2, x = y, R_1);$

$$R_2 := \{H_2(0)=1, H_2(n+2) = (-2n-2)H_2(n) + 2H_2(n+1)y, H_2(1) = 2y\}$$

# II. Product

> $R_3 := gfun:\text{-}poltorec(H(n)\cdot H_2(n)\cdot v(n), [R_1, R_2, \{v(n+1)\cdot(n+1) = v(n), v(1)=1\}], [H(n), H_2(n), v(n)], c(n));$

$$R_3 := \Big\{c(0)=1, c(1)=4xy, c(2)=8x^2y^2+2-4y^2-4x^2, c(3)=\frac{32}{3}x^3y^3+24xy-16xy^3-16x^3y, (16n+16)c(n)-16xy\,c(n+1)+(-8n-20+8y^2+8x^2)c(n+2)-4x\,c(n+3)y+(n+4)c(n+4)\Big\}$$

# III. Differential Equation

> $gfun:\text{-}rectodiffeq(R_3, c(n), f(u));$

$$\left\{(16u^3-16u^2yx-4u+8uy^2+8ux^2-4xy)f(u)+(16u^4-8u^2+1)\left(\frac{d}{du}f(u)\right), f(0)=1\right\}$$

> $dsolve(\%, f(u));$

$$f(u) = \frac{I\,e^{\left(\frac{-4xyu+y^2+x^2}{(2u-1)(2u+1)}\right)}}{e^{(-y^2-x^2)}\sqrt{2u+1}\sqrt{2u-1}}$$

>

# Euclidean Division & Finite Dimension

**Theorem (XIXth century)**

*D-finite series and sequences over $\mathbb{K}$ form $\mathbb{K}$-algebras.*

**Proof.**

Linear algebra □

**Corollary**

*D-finite series are closed under Hadamard (termwise) product, Laplace transform, Borel transform (ogf$\leftrightarrow$egf).*

# Euclidean Division & Finite Dimension

**Theorem (Tannery 1874)**

*D-finite series composed with algebraic power series are D-finite.*

**Proof.**

$$P(x,y)=0 \text{ and } AP+BP_y=1 \Rightarrow y'=-\frac{P_x}{P_y}=-BP_x \bmod P$$

$$\Rightarrow y^{(k)} \in \underbrace{\bigoplus_{i<\deg_y P} \mathbb{K}(x)y^i}_{\text{finite dim}}.$$

$(f\circ y)^{(p)}$ linear combination of $(f^{(j)}\circ y)y^k$.

$\square$

# Example: Airy Ai at Infinity

$$\mathrm{Ai}(z) = \frac{\sqrt{z}e^{-\xi}}{2\pi}\int_{-\infty}^{\infty} e^{-\xi[(u-1)(4u^2+4u+1)]}\,dv, \quad \xi = \frac{2}{3}z^{3/2}, u = \sqrt{1+\frac{v^2}{3}}$$

$$\sim \frac{1}{2}\pi^{-1/2}z^{-1/4}e^{-\xi}\sum_{n=0}^{\infty}(-1)^n\xi^{-n}\frac{\Gamma(3n+\frac{1}{2})}{54^n n!\Gamma(n+\frac{1}{2})}.$$

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## Computation:

1. algebraic change of variables $t^2 = (u-1)(4u^2+4u+1)$;

$$\to \int_{-\infty}^{\infty} e^{-\xi t^2} f(t)\,dt, \quad f(t) = \frac{dv}{dt},$$

2. recurrence satisfied by the coefficients of $f$ (generating series);
3. termwise integration (Hadamard product).

# I. Algebraic change of variables

$eq_u := u^2 - \left(1 + \frac{v^2}{3}\right):$

$eq_t := t^2 - (u-1)\cdot\left(4\cdot u^2 + 4\cdot u + 1\right):$

$res := resultant\left(eq_u, eq_t, u\right);$

$$t^4 + 2\,t^2 - 3\,v^2 - \frac{8}{3}\,v^4 - \frac{16}{27}\,v^6$$

$gfun :- algeqtodiffeq\left(res, v(t), \left\{v(0)=0, D(v)(0)=sqrt\left(\frac{2}{3}\right)\right\}\right);$

$$\left\{-4\,v(t) + 9\,t\left(\frac{\mathrm{d}}{\mathrm{d}t}v(t)\right) + \left(9\,t^2+18\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}v(t)\right), v(0)=0, (\mathrm{D}(v))(0)=\frac{1}{3}\sqrt{6}\right\}$$

# II. Recurrence satisfied by the coefficients of f

$gfun :- poltodiffeq(diff(v(t),t), [\%], [v(t)], f(t));$

$$\left\{5\,f(t) + 27\,t\left(\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right) + \left(9\,t^2+18\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)\right), f(0)=\frac{1}{3}\sqrt{6}, (\mathrm{D}(f))(0)=0\right\}$$

$R_f := gfun :- diffeqtorec(\%, f(t), c(n));$

$$\left\{\left(5+18\,n+9\,n^2\right)c(n) + \left(18\,n^2+54\,n+36\right)c(n+2), c(0)=\frac{1}{3}\sqrt{6}, c(1)=0\right\}$$

# III. Hadamard product

$assume(\xi > 0);\ s := Int(\exp(-\xi * t\hat{}2) * t\hat{}n,\ t = -\infty\ ..\ \infty):\ s = student[intparts](s, \exp(-\xi * t\hat{}2));$

$$\int_{-\infty}^{\infty} e^{(-\xi{\sim} t^2)}\, t^n \,dt = -\int_{-\infty}^{\infty} -\frac{2\,\xi{\sim}\, t\, e^{(-\xi{\sim} t^2)}\, t^{(n+1)}}{n+1}\,dt$$

$R_i := \left\{ c(n) = \frac{2\cdot\xi}{(n+1)} \cdot c(n+2),\ c(0) = value(eval(s, n=0)),\ c(1) = value(eval(s, n=1)) \right\};$

$$\left\{ c(n) = \frac{2\,\xi{\sim}\, c(n+2)}{n+1},\ c(0) = \frac{\sqrt{\pi}}{\sqrt{\xi{\sim}}},\ c(1) = 0 \right\}$$

```
> FinalRec:=gfun:-`rec*rec`(R[i],R[f],c(n));
```

$$FinalRec := \left\{ \left(5 + 18\,n + 9\,n^2\right) c(n) + \left(36\,\xi{\sim}\, n + 72\,\xi{\sim}\right) c(n+2),\ c(1) = 0,\ c(0) = \frac{1}{3}\,\frac{\sqrt{\pi}\,\sqrt{6}}{\sqrt{\xi{\sim}}} \right\}$$

$Sol := rsolve(FinalRec, c(n));$

$$\begin{cases} \frac{1}{3}\,\frac{(-1)^{\left(\frac{1}{2}n\right)}\, 2^{\left(-1-\frac{1}{2}n\right)}\, \Gamma\left(\frac{1}{2}n+\frac{5}{6}\right)\Gamma\left(\frac{1}{2}n+\frac{1}{6}\right)\xi{\sim}^{\left(-\frac{1}{2}n\right)}\sqrt{6}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}n+1\right)\sqrt{\xi{\sim}}} & n::even \\ 0 & n::odd \end{cases}$$

# Analytic Properties

$$\mathcal{L}y = a_0(x)y^{(k)} + \cdots + a_k(x)y = 0$$

1. Singular points: roots $\rho$ of $a_0$;
2. Indicial polynomial: $\mathcal{L}(x-\rho)^\sigma \sim P(\sigma)(x-\rho)^{\sigma+m}$ [Fuchs1868]
3. Basis of formal solutions [Fabry1885]
   - $\deg P = k$: *regular* singular point.

$$\Psi_i(z) = (z-\rho)^{\sigma_i} \sum_{j=0}^{d_i} \log^j(z-\rho) \underbrace{\Phi_{i,j}(z-\rho),}_{\text{convergent p. s.}} \qquad P(\sigma_i) = 0.$$

   - $\deg P < k$: *irregular* singular point

$$y_i(t) = \exp\big(\underbrace{P_i(1/t)}_{\text{polynomial}}\big) \underbrace{\Psi_i(t)}_{\text{as above}}, \qquad \underbrace{t^{\mu_i}}_{\mu_i \in \mathbb{N}^*} = (z-\rho).$$

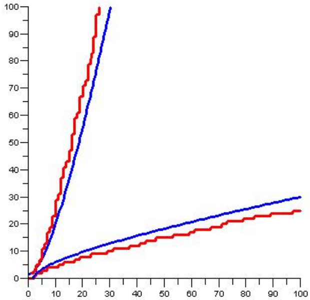
Algorithms for everything [Tournier87,vanHoeij97]
Incorporated in ESF: http://algo.inria.fr/esf [MeSa03]

# Not Everything is D-finite

## Analytic behaviour

- $\tan z$, Lambert $W$ are not D-finite;
- Bell numbers are not D-finite ($\sum B_n z^n/n! = \exp(e^z - 1)$);
- sequences $\log n$, $n^\alpha$ ($\alpha \notin \mathbb{N}$), $p_n$ not D-finite [FlGeSa05];

$$\pi(x) \sim \mathrm{Li}(x) + R(x) \Rightarrow p_n - nH_n \sim n \log\log n \Rightarrow \text{g.f.} \sim \frac{\log\log(1-z)}{(1-z)^2}$$



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### Algebraic behaviour (mostly Galois theory)

- $f$ and $1/f$ D-finite iff $f'/f$ algebraic [HaSi85];
- $f_n$ and $1/f_n$ D-finite iff $f_n$ interlacing of hypergeometric sequences (= rec. of order 1) [vdPSi97];
- $f$ and $\exp \int f$ D-finite iff $f$ algebraic;
- $g$ algebraic of genus $\geq 1$. $f$ and $g \circ f$ D-finite iff $f$ is algebraic. [Singer86]

# Fast Algorithms & Applications

1. Power series expansion: $O(n)$ arithmetic ops (no product)
2. $n$th coefficient: $\tilde{O}(\sqrt{n})$ arithmetic ops (baby steps/giant steps)
3. $n$th coefficient over $\mathbb{Q}$: $\tilde{O}(n)$ binary ops (binary splitting)
4. evaluation at an algebraic point.

Examples:

- hypergeometric formula for $1/\pi$ [ChCh87]
- Sigsam Challenges'97, Problem 4.
- Rational solutions of LDEs and LREs [BoClSa05].

[Hakmem,Brent76,ChCh86,vdH00,BoGaSc04]

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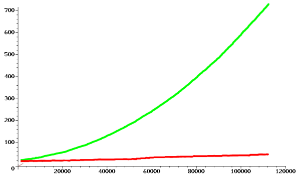
Examples:

- hypergeometric formula for $1/\pi$ [ChCh87]

$$\frac{1}{\pi} = \frac{1}{53360\sqrt{640320}} \sum_{n\geq 0} \frac{(-1)^n (6n)!(13591409 + 545140134n)}{n!^3 (3n)!(8 \cdot 100100025 \cdot 327843840)^n};$$




Used in Maple & Mathematica. Recurrence of order 1.

- Sigsam Challenges'97, Problem 4.

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Examples:

- hypergeometric formula for $1/\pi$ [ChCh87]
- Sigsam Challenges'97, Problem 4. Coefficient of $x^{3000}$ in

$$(x+1)^{2000}(x^2+x+1)^{1000}(x^4+x^3+x^2+x+1)^{500}$$

(first compute a rec. of order 7, then use binary splitting).

- Rational solutions of LDEs and LREs [BoClSa05].

[Hakmem,Brent76,ChCh86,vdH00,BoGaSc04]

# II D-finiteness in Several Variables

# Ore Polynomials & Ore Algebras

- Skew polynomial ring: $\mathbb{A}[\partial;\sigma,\delta]$, $\mathbb{A}$ integral domain and commutation $\partial P=\sigma(P)\partial+\delta(P)$, $P\in\mathbb{A}$
(ex. $\partial_x P(x)=P(x)\partial_x+P'(x)$, $S_nP(n)=P(n+1)S_n$).
Technical conditions on $\sigma,\delta$ to make product associative.
- Ore algebra: $\mathbb{A}[\partial_1;\sigma_1,\delta_1]\cdots[\partial_n;\sigma_n,\delta_n]$, $\sigma,\delta$ s. t. $\partial_i\partial_j=\partial_j\partial_i$.
Aim [ChSa98]: manipulate (solutions of) systems of mixed linear ($q$-)differential or ($q$-)difference operators.

- Main property: the leading term of a product is (up to a cst) the product of leading terms.

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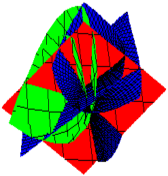
  Technical conditions on $\sigma,\delta$ to make product associative.
- Ore algebra: $\mathbb{A}[\partial_1;\sigma_1,\delta_1]\cdots[\partial_n;\sigma_n,\delta_n]$, $\sigma,\delta$ s. t. $\partial_i\partial_j=\partial_j\partial_i$.
- Main property: the leading term of a product is (up to a cst) the product of leading terms.
- Consequences:
  1. Univariate: Right Euclidean division and extended Euclidean algorithm [Ore 33];
  2. Multivariate: Buchberger's algorithm for Gröbner bases works in Ore algebras [Kredel93].

# 0-dimensionality & D-finiteness



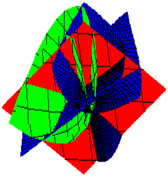
| Polynomial algebra | Ore algebra |
|---|---|
| 0-dimensional ideal | D-finite left ideal |
| $\Updownarrow$ | $\Updownarrow$ def |

quotient is a finite dimensional vector space

Exs: Orthogonal polynomials, hypergeometric series, their $q$-analogues,...

$$\boxed{\text{system+ini. cond.=data structure}}$$

# 0-dimensionality & D-finiteness



| Polynomial algebra | Ore algebra |
|---|---|
| 0-dimensional ideal | D-finite left ideal |
| $\Updownarrow$ | $\Updownarrow$ def |
| quotient is a finite dimensional vector space | |
| $\Downarrow$ | $\Downarrow$ |
| polynomial expressions are algebraic | polynomials and $\partial$'s are D-finite |

Tools: linear algebra, Gröbner bases. Implemented in `Mgfun` [Chyzak98]

Exs: Orthogonal polynomials, hypergeometric series, their $q$-analogues,. . .

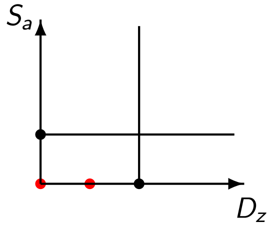
$\boxed{\text{system+ini. cond.=data structure}}$

# Example: Contiguity of Hypergeometric Series

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\underbrace{\frac{(a)_n(b)_n}{(c)_n n!}}_{u_{a,n}}z^n,\qquad (x)_n:=x(x+1)\cdots(x+n-1).$$

$$\frac{u_{a,n+1}}{u_{a,n}}=\frac{(a+n)(b+n)}{(c+n)(n+1)}\rightarrow\ z(1-z)F''+(c-(a+b+1)z)F'-abF=0,$$

$$\frac{u_{a+1,n}}{u_{a,n}}=\frac{n}{a}+1\rightarrow S_aF:=F(a+1,b;c;z)=\frac{z}{a}F'+F.$$



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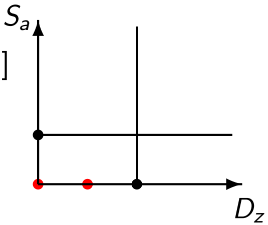


dim=2 $\Rightarrow S_a^2F, S_aF, F$ linearly dependent [Gauss1812]

Also:

- $S_a^{-1}$ in terms of Id, $D_z$;
- relation between any three polynomials in $S_a, S_b, S_c$;
  - generalizes to any ${}_pF_q$ and multivariate case [Takayama89].

# Creative Telescoping [Zeilberger 90]

$$F_n = \sum_k u_{n,k} = ?$$

**IF** one knows $A(n, S_n)$ and $B(n, k, S_n, S_k)$ such that

$$(A(n, S_n) + \Delta_k B(n, k, S_n, S_k)) \cdot u_{n,k} = 0,$$

then the sum "telescopes", leading to $A(n, S_n) \cdot F_n = 0$.

# Creative Telescoping [Zeilberger 90]

$$I(x) = \int_\Omega u(x,y)\,dy = ?$$

**IF** one knows $A(x,\partial_x)$ and $B(x,y,\partial_x,\partial_y)$ such that

$$(A(x,\partial_x) + \partial_y B(x,y,\partial_x,\partial_y)) \cdot u(x,y) = 0,$$

then the integral "telescopes", leading to $A(x,\partial_x)\cdot I(x) = 0.$

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$$(A(x,\partial_x) + \partial_y B(x,y,\partial_x,\partial_y)) \cdot u(x,y) = 0,$$

then the integral "telescopes", leading to $A(x,\partial_x)\cdot I(x) = 0.$

> *Then I come along and try differentating under the integral sign, and often it worked. So I got a great reputation for doing integrals.*

Richard P. Feynman 1985

Creative telescoping="differentiation" under integral+"integration" by parts

# Creative Telescoping [Zeilberger 90]

$$I(x) = \int_\Omega u(x,y)\,dy = ?$$

**IF** one knows $A(x,\partial_x)$ and $B(x,y,\partial_x,\partial_y)$ such that

$$(A(x,\partial_x) + \partial_y B(x,y,\partial_x,\partial_y)) \cdot u(x,y) = 0,$$

then the integral "telescopes", leading to $A(x,\partial_x)\cdot I(x) = 0$.

Creative telescoping="differentiation" under integral+"integration" by parts

- General case: Find annihilators of

$$I(x_1,\dots,x_{n-1}) = \partial_n^{-1}\big|_\Omega f(x_1,\dots,x_n)$$

knowing generators of $\operatorname{Ann}_f$ in
$\mathbb{O}_n = \mathbb{K}(x_1,\dots,x_n)[\partial_1;\sigma_1,\delta_1]\cdots[\partial_n;\sigma_n,\delta_n]$;

- Crucial step: compute $(\mathbb{O}_n \operatorname{Ann}_f + \partial_n\mathbb{O}_n)\cap\mathbb{O}_{n-1}$.

# Example: $\zeta(3)$ is Irrational [Apéry78]

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad b_n = a_n \sum_{k=1}^{n} \frac{1}{k^3} + \sum_{k=1}^{n} \sum_{m=1}^{k} \frac{(-1)^{m+1} \binom{n}{k}^2 \binom{n+k}{k}^2}{2m^3 \binom{n}{m} \binom{n+m}{m}}.$$

1. $b_n/a_n \to \zeta(3)$, $n \to \infty$; $d_n^3 b_n \in \mathbb{Z}$, where $d_n = \operatorname{lcm}(1, \dots, n)$;
2. By creative telescoping, both $a_n$ and $b_n$ satisfy

$$(n+1)^3 u_{n+1} = (34n^3 + 51n^2 + 27n + 5)u_n - n^3 u_{n-1}, \quad n \geq 1;$$

3. $0 < \zeta(3) - \dfrac{b_n}{a_n} = \displaystyle\sum_{k \geq n+1} \frac{b_k}{a_k} - \frac{b_{k-1}}{a_{k-1}}$: $b_k a_{k-1} - b_{k-1} a_k = \dfrac{6}{k^3}$;
4. $\lambda a_n + \mu b_n \approx \alpha_\pm^n$, with $\alpha_\pm^2 = 34\alpha_\pm - 1$;
5. Conclusion: $0 < \underbrace{a_n d_n^3}_{\in \mathbb{N}} \zeta(3) - \underbrace{d_n^3 b_n}_{\in \mathbb{N}} \approx \alpha_-^n e^{3n} \to 0.$

Algolib can be downloaded from http://algo.inria.fr/libraries.

> $libname := "/Users/salvy/lib/maple/Algolib", libname:$

> $a := binomial(n,k)^2 \cdot binomial(n+k,k)^2;$

$$a := \mathrm{binomial}(n,k)^2\,\mathrm{binomial}(n+k,k)^2$$

> $Mgfun[creative\_telescoping](a, n::shift, k::shift);$

$$\left[\left(-n^3-3\,n^2-3\,n-1\right)\_f(n,k)+\left(34\,n^3+153\,n^2+231\,n+117\right)\_f(n+1,k)+\left(-n^3-6\,n^2-12\,n-8\right)\_f(n+2,k), -\frac{4\,k^4\left(4\,n^2+12\,n+8+3\,k-2\,k^2\right)(2\,n+3)\_f(n,k)}{4+12\,n-12\,k-4\,n\,k^3+13\,n^2+13\,k^2+k^4-26\,n\,k+n^4+6\,n^3-6\,k^3+6\,n^2\,k^2-18\,n^2\,k+18\,n\,k^2-4\,n^3\,k}\right]$$

*Neither Cohen nor I had been able to prove [this] in the intervening two months.* [Van der Poorten]

# Applications of Creative Telescoping

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3 \quad \text{[Strehl92]}$$

$$\int_0^{+\infty} xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx=-\frac{\ln(1-a^4)}{2\pi a^2} \quad \text{[GlMo94]}$$

$$\frac{1}{2\pi i}\oint\frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy=\frac{H_n(x)}{\lfloor n/2\rfloor!} \quad \text{[Doetsch30]}$$

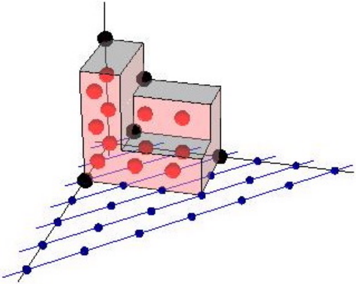
$$\sum_{k=0}^{n}\frac{q^{k^2}}{(q;q)_k(q;q)_{n-k}}=\sum_{k=-n}^{n}\frac{(-1)^kq^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}} \quad \text{[Andrews74]}$$

$$\sum_{j=0}^{n}\sum_{i=0}^{n-j}\frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i(q;q)_j}=\sum_{k=-n}^{n}\frac{(-1)^kq^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}} \quad \text{[Paule85].}$$

# (Partial) Algorithms for Creative Telescoping

Aim: $\mathcal{I} = (\mathbb{O}_n \operatorname{Ann}_f + \partial_n \mathbb{O}_n) \cap \mathbb{O}_{n-1}$

- By Gröbner bases, eliminate $x_n$ and set $\partial_n$ to 0 [ChSa98]
  $\to (\mathbb{O}_n \operatorname{Ann}_f \cap \mathbb{O}_{n-1}[\partial_n] + \partial_n \mathbb{O}_{n-1}) \cap \mathbb{O}_{n-1} \subset \mathcal{I}$
- Differential case: algorithms from $\mathcal{D}$-module theory [SaStTa00,Tsai00], Gröbner bases with negative weights.
- Shift case, $n = 2$, dim 1 ($=$ hypergeometric): [Zeilberger91]
  For increasing $k$, search for $a_i$ and $B$
  $$\mathbb{O}_{n-1} \ni \sum_{i=0}^{k} a_i \partial_{n-1}^{i} f = \partial_n B f$$
  Termination [Abramov03].
- Arbitrary $n$ and $\mathbb{O}_n$: [Chyzak00]
  $$\mathbb{O}_{n-1} \ni \sum_{\lambda} a_\lambda \partial^\lambda = \partial_n B \bmod \operatorname{Ann}_f$$
  $B$ is given by rational solutions of a linear system in $\sigma_n, \partial_n$.



# III D-finiteness in Infinitely Many Variables

# $k$-uniform Young Tableaux

| | | | | | |
|---|---|---|---|---|---|
| 4 | 4 | | | | |
| 3 | 3 | 5 | | | |
| 2 | 2 | 3 | 4 | | |
| 1 | 1 | 1 | 2 | 5 | 5 |

Question: Asymptotic number of semi-standard Young tableaux filled with $k$ 1's, $k$ 2's,..., $k$ n's?

## Result [ChMiSa05]

$$\frac{1}{\sqrt{2}}\left(\frac{e^{k-2}}{2\pi}\right)^{k/4} n!^{k/2-1}\left(\frac{k^{k/2}}{k!}\right)^{n}\frac{\exp\sqrt{kn}}{n^{k/4}},\qquad n\to\infty.$$

Method:

Combinatorics $\rightarrow$ Symmetric functions $\rightarrow$ LDE $\rightarrow$ Asymptotics.

# D-finite Symmetric Series

Algebra of symmetric functions: $\Lambda := \mathbb{K}[[p_1, p_2, \dots]]$

| | |
|---|---|
| power $p_k$ | $p_3 = x_1^3 + x_2^3 + x_3^3 + \cdots$ |
| homogeneous $h_k$ | $h_3 = x_1^3 + x_2^3 + \cdots + x_1^2 x_2 + \cdots + x_1 x_2 x_3 + \cdots$ |
| monomial $m_\lambda$ | $m_{(3,2,1)} = x_1^3 x_2^2 x_3 + x_2^3 x_1^2 x_3 + \cdots$ |

## Definition

$F \in \Lambda[[t]]$ D-finite if for any $n$, $F(p_1, \dots, p_n, 0, \dots; t)$ D-finite.

## Theorem (Gessel 90)

- *Closed under* $+$, $\times$, $\partial/\partial p_i$, *algebraic substitution.*

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$$\begin{array}{rr} \text{power } p_k & p_3 = x_1^3 + x_2^3 + x_3^3 + \cdots \\ \text{homogeneous } h_k & h_3 = x_1^3 + x_2^3 + \cdots + x_1^2 x_2 + \cdots + x_1 x_2 x_3 + \cdots \\ \text{monomial } m_\lambda & m_{(3,2,1)} = x_1^3 x_2^2 x_3 + x_2^3 x_1^2 x_3 + \cdots \end{array}$$

## Definition

$F \in \Lambda[[t]]$ D-finite if for any $n$, $F(p_1, \dots, p_n, 0, \dots; t)$ D-finite.

## Theorem (Gessel 90)

- *Closed under $+$, $\times$, $\partial/\partial p_i$, algebraic substitution.*
- *[Technical conds] closed under plethysm and scalar product.*

Scalar product: $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$, where $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$

Adjoint: $\langle \phi F, G \rangle = \langle F, \phi^\perp G \rangle$, with $p_k^\perp = k \frac{\partial}{\partial p_k}$, $\left(\frac{\partial}{\partial p_k}\right)^\perp = \frac{p_k}{k}$.

# Algorithm

$$\Bigg\langle \underbrace{\exp\left(\sum_{n=1}^{\infty}\frac{p_n^2}{2n}+\sum_{n\text{ odd}}^{\infty}\frac{p_n}{n}\right)}_{\text{All semi-standard Young tableaux}}, \underbrace{\sum_{n\geq 0} h_k^n t^n}_{\text{Extract coeffs corresponding to } k\text{-uniform}} \Bigg\rangle$$

$$= \left\langle \exp\left(\sum_{n=1}^{k}\frac{p_n^2}{2n}+\sum_{n\text{ odd}}^{k}\frac{p_n}{n}\right), \sum_{n\geq 0} h_k^n t^n \right\rangle =: \langle F, G\rangle$$

Wanted: $(\underbrace{\mathrm{Ann}_F^{\perp}}_{\text{right ideal}} + \underbrace{\mathrm{Ann}_G}_{\text{left ideal}}) \cap \mathbb{K}[t,\partial_t]$.

## Algorithm [ChMiSa05]

For increasing $D$, from GB's of $\mathrm{Ann}_F$ and $\mathrm{Ann}_G$, compute bases of $\mathrm{Ann}_F^{\perp}$ and $\mathrm{Ann}_G$ up to degree $D$ as vector spaces.
Stop when Gaussian elimination yields an element of $\mathbb{K}[t,\partial_t]$.

Termination: granted by holonomy.

# IV Conclusion

# Future Work

## Efficiency

- Faster Gröbner bases;
- Other elimination techniques (adapt geometric resolution [GiHe93,GiLeSa01] to Ore algebras);
- Structured Padé-Hermite approximants.

## Understand non-minimality

- Remove apparent singularities by Ore closure , a generalization of Weyl closure [Tsai00], and of [AbBavH05] ([ChDuLeMaMiSa05] in progress);
- Exploit symmetry (extend [Paule94]).

## Easy-to-use Implementations

- Improve `gfun` and `Mgfun`. Make the ESF interactive.