Automata with Semilinear Constraints

Ph.D. defense

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Université de Montréal

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Ex: $L = \{a, ba, abb, babb, \cdots\}$

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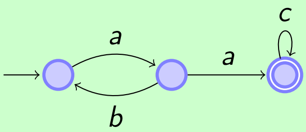
Example



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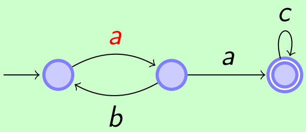

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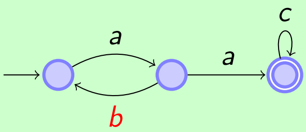

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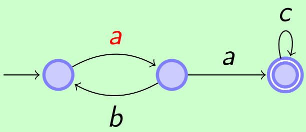

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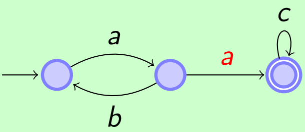

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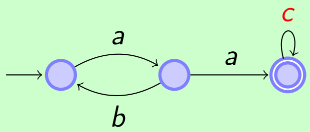

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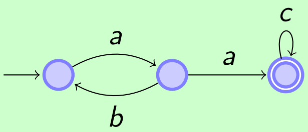

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This thesis: add counting mechanisms to automata, continuing a line of work started by [Minsky, 1961]

Outline

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How are context-free and regular similar?

The fundamental result of [Parikh, 1966]

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Parikh's theorem [1966]

L CFL or REG $\Rightarrow \mathrm{Pkh}(L)$ semilinear
C semilinear $\Rightarrow \exists L \in \mathrm{REG}\ \mathrm{Pkh}(L) = C$

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- Generalized: over commutative rings [Kudlek, 2007]; as *semipolynomial* sets [Karianto et al., 2006], ...

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- We locate them within other frameworks

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Constraining the Parikh image of words

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- In hindsight: the *automaton* should take better advantage of counting

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Semilinear sets: our counters

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Constrained Automata

Affine Constrained Automata

Conclusion

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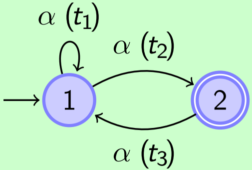

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The language of accepting runs

Automaton $A = (Q, \Sigma, \delta, q_0, F)$, $\delta \subseteq Q \times \Sigma \times Q$
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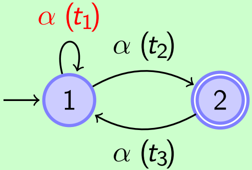

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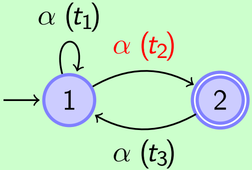

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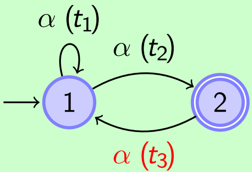

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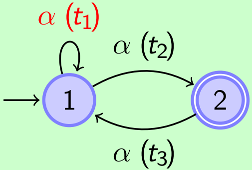

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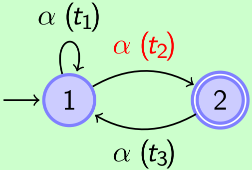

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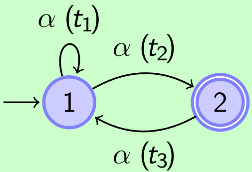

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- Semilinear constraint on Pkh(a run)

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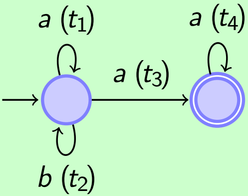

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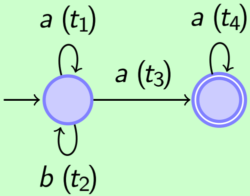

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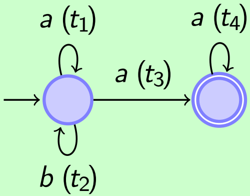

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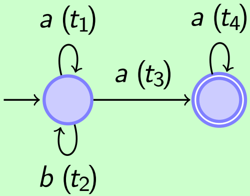

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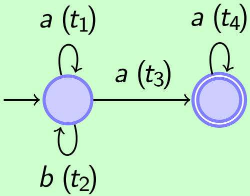

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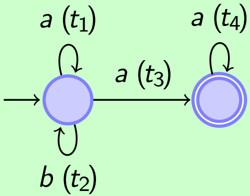

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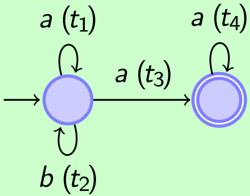

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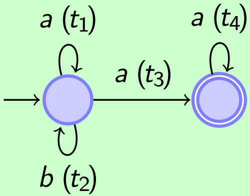

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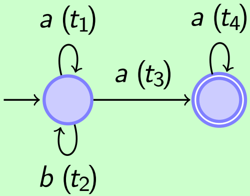

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Pkh(traced run): $(0, 0, 0, 0)$

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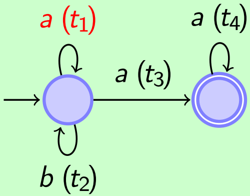

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Pkh(traced run): (1, 0, 0, 0)
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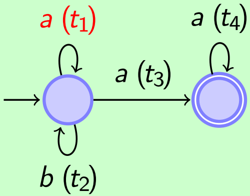

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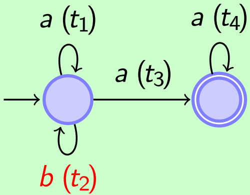

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Pkh(traced run): (2, 1, 0, 0)
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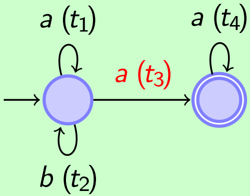

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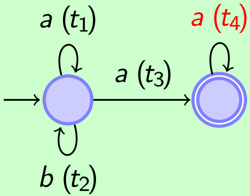

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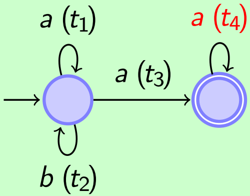

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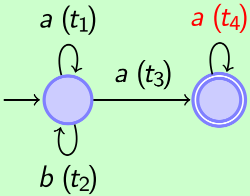

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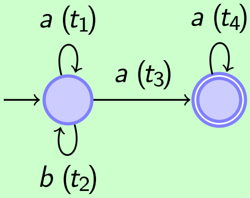

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Pkh(traced run): $(2, 1, 1, 3) \in C$

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Constrained automata formally

Definition (Constr. Automata [Klaedtke and Rueß, 2003])

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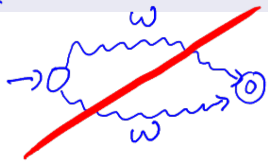

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$\to \forall$ positions $u_i = v_i$

$\exists$ position $u_i \neq v_i$

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We continue the study of closure:

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- Use of expressiveness lemmata reminiscent of pumping

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We also contribute (un)decidability properties:

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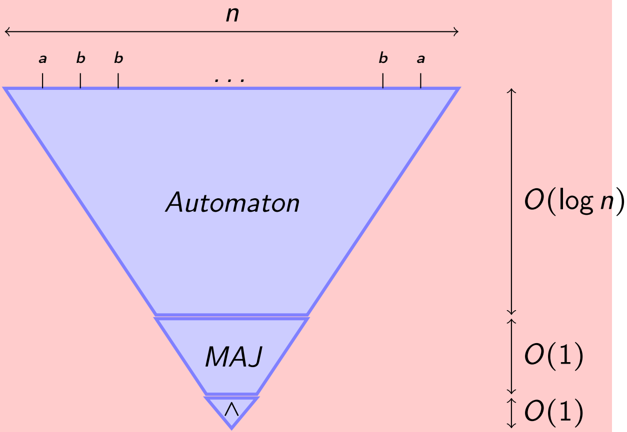

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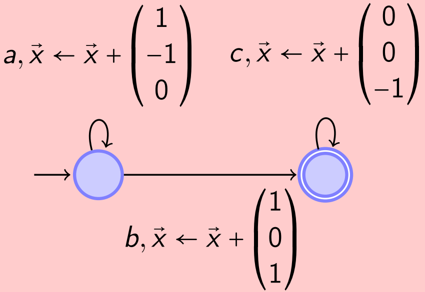

Constraint: Bool($x_i > 0$)

Outline

Affine constrained automata

Adding linear transformations to CA transitions

Definition (Affine Constrained Automata (ACA))

- *Affine constrained automaton* given by:
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Related to *linear systems with Presburger guards* [Finkel and Leroux, 2002]

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How much more powerful are ACA?

CA variants are restrictions of ACA:

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  - Connection with probabilistic languages, Markov chains, ... [Paz, 1971]

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Further work:

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Thank you!

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