Notes on Equivalence and Minimization of Weighted Automata

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Abstract

This set of notes re-proves known results on weighted automata (over a field, also known as multiplicity automata). The text offers a unified view on theorems and proofs that have appeared in the literature over decades and were written in different styles and contexts. None of the results reported here are claimed to be new.

The content centres around fundamentals of equivalence and minimization, with an emphasis on algorithmic aspects.

The presentation is minimalistic. No attempt has been made to motivate the material. Weighted automata are viewed from a linear-algebra angle. As a consequence, the proofs, which are meant to be succinct, but complete and almost self-contained, rely mainly on elementary linear algebra.

1 Preliminaries

Let \mathbb{K} be a field. When speaking about algorithms and computational complexity, we will implicitly take as \mathbb{K} the field \mathbb{Q} of rational numbers (where we assume that rational numbers are encoded as quotients of integers encoded in binary). For a finite alphabet Σ we call a map $s: \Sigma^* \to \mathbb{K}$ a *series*.

An automaton $\mathcal{A} = (n, \Sigma, M, \alpha, \eta)$ consists of a natural number n (to which we refer as the number of states), a finite alphabet Σ , a map M: $\Sigma \to \mathbb{K}^{n \times n}$, an initial (row) vector $\alpha \in \mathbb{K}^n$, and a final (column) vector $\eta \in \mathbb{K}^n$. Extend M to a monoid homomorphism $M: \Sigma^* \to \mathbb{K}^{n \times n}$ by setting $M(a_1 \cdots a_k) := M(a_1) \cdots M(a_k)$ and $M(\varepsilon) := I_n$, where ε is the empty word and $I_n \in \{0,1\}^{n \times n}$ the $n \times n$ identity matrix. The semantics of an automaton \mathcal{A} is the series $[\![\mathcal{A}]\!]: \Sigma^* \to \mathbb{K}$ with $[\![\mathcal{A}]\!](w) = \alpha M(w)\eta$. Automata $\mathcal{A}_1, \mathcal{A}_2$ over the same alphabet Σ are said to be equivalent if $[\![\mathcal{A}_1]\!] = [\![\mathcal{A}_2]\!]$. An automaton \mathcal{A} is minimal if there is no equivalent automaton \mathcal{A}' with fewer states. If n = 0, it is natural to put $[\![\mathcal{A}]\!](w) = 0$ for all $w \in \Sigma^*$.

We have the following closure properties:

Proposition 1.1. Let $A_i = (n_i, \Sigma, M_i, \alpha_i, \eta_i)$ for $i \in \{1, 2\}$ be automata. One can compute in logarithmic space (hence, in polynomial time) automata A_+, A_-, A_\otimes with $[\![A_+]\!](w) = [\![A_1]\!](w) + [\![A_2]\!](w)$ and $[\![A_-]\!](w) = [\![A_1]\!](w)$

 $[\![\mathcal{A}_1]\!](w) - [\![\mathcal{A}_2]\!](w)$ and $[\![\mathcal{A}_{\otimes}]\!](w) = [\![\mathcal{A}_1]\!](w) \cdot [\![\mathcal{A}_2]\!](w)$ for all $w \in \Sigma^*$. One can compute $\mathcal{A}_+, \mathcal{A}_-$ with $O(|\Sigma|(n_1 + n_2)^2)$ arithmetic operations. One can compute \mathcal{A}_{\otimes} with $O(|\Sigma|n_1^2n_2^2)$ arithmetic operations.

Proof. It is straightforward to check that the automaton $\mathcal{A}_{+}=(n_1+n_2,\Sigma,M_+,(\alpha_1,\alpha_2),\eta_+)$ with

$$M_{+}(a) = \begin{pmatrix} M_{1}(a) & 0_{n_{1},n_{2}} \\ 0_{n_{2},n_{1}} & M_{2}(a) \end{pmatrix} \quad \text{for all } a \in \Sigma \quad \text{and} \quad \eta_{+} = \begin{pmatrix} \eta_{1} \\ \eta_{2} \end{pmatrix}$$

is the desired automaton, where $0_{m,n}$ stands for the $m \times n$ zero matrix.

The automaton \mathcal{A}_{-} can be constructed similarly to \mathcal{A}_{+} , but (α_{1}, α_{2}) is replaced with $(\alpha_{1}, -\alpha_{2})$.

Let \otimes denote the Kronecker product. Define $\mathcal{A}_{\otimes} = (n_1 n_2, \Sigma, M_{\otimes}, (\alpha_1 \otimes \alpha_2), (\eta_1 \otimes \eta_2))$, where $M_{\otimes}(a) = M_1(a) \otimes M_2(a)$ for all $a \in \Sigma$. Using the mixed-product property of \otimes (i.e., $(AB) \otimes (CD) = (A \otimes C)(B \otimes D)$), we have for all $a_1 \cdots a_k \in \Sigma^*$:

For a set V of vectors we use the notation $\langle v \mid v \in V \rangle$ to denote the vector space spanned by V. For an automaton \mathcal{A} , define its forward space as the (row) vector space $\langle \alpha M(w) \mid w \in \Sigma^* \rangle$. Similarly, the backward space of \mathcal{A} is the (column) vector space $\langle M(w) \eta \mid w \in \Sigma^* \rangle$.

Let $s: \Sigma^* \to \mathbb{K}$. The *Hankel matrix* of s is the (infinite) matrix $H \in \mathbb{K}^{\Sigma^* \times \Sigma^*}$ with H[x,y] = s(xy) for all $x,y \in \Sigma^*$. Define $\operatorname{rank}(s) := \operatorname{rank}(H)$.

2 Equivalence Checking

First we discuss how to efficiently compute a basis of the forward space $\mathcal{F} := \langle \alpha M(w) \mid w \in \Sigma^* \rangle$ of an automaton $\mathcal{A} = (n, \Sigma, M, \alpha, \eta)$. It is a matter of basic linear algebra to check that \mathcal{F} is the smallest vector space that contains α and is closed under post-multiplication of M(a) (i.e., $\mathcal{F}M(a) \subseteq \mathcal{F}$) for all $a \in \Sigma$. Hence Algorithm 1 computes a basis of \mathcal{F} .

Algorithm 1 actually computes a set W of words such that $\{\alpha M(w) \mid w \in W\}$ is a basis of the forward space \mathcal{F} . These words will be of interest, e.g., to compute a word w that "witnesses" the inequivalence of two automata. Since \mathcal{F} is a subspace of \mathbb{K}^n , its dimension, say \overrightarrow{n} , is at most n. It follows that $|W| = \overrightarrow{n} \leq n$ and $|w| \leq \overrightarrow{n} - 1$ holds for all $w \in W$.

We want to make Algorithm 1 efficient. First, in addition to the words w we save the vectors $\alpha M(w)$ to avoid unnecessary vector-matrix computations. Second, we use a worklist, implemented as a queue, to keep track of

Algorithm 1: Computing a basis of the forward space of an automaton $\mathcal{A} = (n, \Sigma, M, \alpha, \eta)$.

```
1 if \alpha=0 then
2 | return \emptyset
3 W:=\{\varepsilon\}
4 while \exists w\in W\ \exists\ a\in\Sigma: \alpha M(wa)\not\in\langle\alpha M(w)\mid w\in W\rangle do
5 | W:=W\cup\{wa\}
6 return \{\alpha M(w)\mid w\in W\}
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Algorithm 2: Computing $\{(w_1, v_1), \dots, (w_{\overrightarrow{n}}, v_{\overrightarrow{n}})\} \subseteq \Sigma^{\leq \overrightarrow{n}-1} \times \mathbb{K}^n$ such that $\{v_1, \dots, v_{\overrightarrow{n}}\}$ is a basis of the forward space of an automaton $\mathcal{A} = (n, \Sigma, M, \alpha, \eta)$

```
1 if \alpha = 0 then
 _{\mathbf{2}} return \emptyset
 \mathbf{3} \ \dot{P} := \{(\varepsilon, \alpha)\}
 4 Q := [(\varepsilon, \alpha)]
 5 repeat
         (w,v) := dequeue(Q)
        forall a \in \Sigma do
             w' := wa
             v' := vM(a)
             if v' \notin \langle u \mid (x, u) \in P \rangle then
10
                 P := P \cup \{(w', v')\}
11
               Q := enqueue(Q, (w', v'))
13 until isEmpty(Q)
14 return P
```

which vectors are new in the basis of \mathcal{F} computed so far. These refinements result in Algorithm 2.

Line 10 of Algorithm 2 requires a check for linear independence. Using Gaussian elimination, such a check can be carried out with $O(n^3)$ arithmetic operations. To make this more efficient, one can keep a basis of the vector space $\langle u \mid (x,u) \in P \rangle$ in echelon form. With such a basis at hand, the check for linear independence amounts to performing one iteration of Gaussian elimination, which takes $O(n^2)$ operations, and checking if the resulting vector is non-zero. If it is indeed non-zero, it can be added to the basis, thus preserving its echelon form.¹

Since $\overrightarrow{n} \leq n$, it follows that line 10 is executed $O(n|\Sigma|)$ times. Hence we have:

Proposition 2.1. Let $\mathcal{A} = (n, \Sigma, M, \alpha, \eta)$ be an automaton. One can compute in polynomial time (with $O(|\Sigma|n^3)$ arithmetic operations) a set $\{(w_1, v_1), \ldots, (w_{\overrightarrow{n}}, v_{\overrightarrow{n}})\} \subseteq \Sigma^{\leq \overrightarrow{n}-1} \times \mathbb{K}^n \text{ such that } \{v_1, \ldots, v_{\overrightarrow{n}}\} \text{ is a basis of } \mathcal{F} \text{ and } v_i = \alpha M(w_i) \text{ holds for all } 1 \leq i \leq \overrightarrow{n}.$

An automaton \mathcal{A} is called zero if $[\![\mathcal{A}]\!](w) = 0$ for all $w \in \Sigma^*$. We show:

Proposition 2.2. Let $A = (n, \Sigma, M, \alpha, \eta)$ be an automaton. One can check in polynomial time (with $O(|\Sigma|n^3)$ arithmetic operations) whether A is zero, and if it is not, output $w \in \Sigma^*$ with $|w| \le n - 1$ such that $|A|(w) \ne 0$.

Proof. Automaton $\mathcal{A} = (n, \Sigma, M, \alpha, \eta)$ is zero if and only if its forward space $\mathcal{F} := \langle \alpha M(w) \mid w \in \Sigma^* \rangle$ is orthogonal to η , i.e., $v\eta = 0$ for all $v \in \mathcal{F}$. Let $S \subseteq \mathbb{K}^n$ (with $|S| \leq n$) be a basis of \mathcal{F} . Then \mathcal{A} is zero if and only if $v\eta = 0$ holds for all $v \in S$. But by Proposition 2.1 one can compute such S. Similarly, one can compute, if it exists, the "counterexample" w.

Theorem 2.3. Let $A_i = (n_i, \Sigma, M_i, \alpha_i, \eta_i)$ for $i \in \{1, 2\}$ be automata. One can check in polynomial time (with $O(|\Sigma|(n_1 + n_2)^3)$ arithmetic operations) whether A_1, A_2 are equivalent, and if they are not, output $w \in \Sigma^*$ with $|w| \leq n_1 + n_2 - 1$ such that $[A_1](w) \neq [A_2](w)$.

Proof. Compute automaton \mathcal{A}_{-} from Proposition 1.1. Then the theorem follows from Proposition 2.2.

Bibliographic Remarks

Equivalence checking goes back to the seminal paper by Schützenberger from 1961 [21]. A polynomial-time algorithm could be derived from there but was not made explicit. The books by Paz [20] and Eilenberg [11] from 1971

¹For improved numerical stability of the computation, instead of using a basis in echelon form, one may keep an orthonormal basis, against which the new vector is orthogonalized using one iteration $(O(n^2)$ operations) of the modified Gram-Schmidt process.

and 1974, respectively, describe an exponential-time algorithm based on the fact that shortest "counterexamples" have length at most $n_1 + n_2 - 1$. An $O(|\Sigma|(n_1+n_2)^4)$ (in terms of arithmetic operations) algorithm was explicitly provided in 1992 by Tzeng [22]. Improvements to $O(|\Sigma|(n_1+n_2)^3)$ were then (re-)discovered, e.g., in [9, 16, 3]. These improvements are all based on the idea described before Proposition 2.1. The abstract of the 2002 paper [1] indicates that this improvement was already known to some. Incidentally, a different algorithm, also cubic in n, was proposed in [1].

3 Minimization

Let $\mathcal{A}=(n,\Sigma,M,\alpha,\eta)$ be an automaton. Let $F\in\mathbb{K}^{\overrightarrow{n}\times n}$ with $\overrightarrow{n}\leq n$ be a matrix whose rows form a basis of the forward space \mathcal{F} . Similarly, let $B\in\mathbb{K}^{n\times n}$ with $\overleftarrow{n}\leq n$ be a matrix whose columns form a basis of the backward space \mathcal{B} . Since $\mathcal{F}M(a)\subseteq\mathcal{F}$ and $M(a)\mathcal{B}\subseteq\mathcal{B}$ for all $a\in\Sigma$, there exist maps $\overrightarrow{M}:\Sigma\to\mathbb{K}^{\overrightarrow{n}\times \overrightarrow{n}}$ and $\overleftarrow{M}:\Sigma\to\mathbb{K}^{\overleftarrow{n}\times \overleftarrow{n}}$ such that

$$FM(a) = \overrightarrow{M}(a)F$$
 and $M(a)B = B\overleftarrow{M}(a)$ for all $a \in \Sigma$.

These maps \overrightarrow{M} , \overleftarrow{M} are unique, as F, B have full rank. The above equalities extend inductively to words:

$$FM(w) = \overrightarrow{M}(w)F$$
 and $M(w)B = B\overleftarrow{M}(w)$ for all $w \in \Sigma^*$ (1)

Let $\overrightarrow{\alpha} \in \mathbb{K}^{\overrightarrow{n}}$ be the unique row vector with $\overrightarrow{\alpha}F = \alpha$, and $\overleftarrow{\eta} \in \mathbb{K}^{\overleftarrow{n}}$ be the unique column vector with $B\overleftarrow{\eta} = \eta$. Call $\overrightarrow{\mathcal{A}} := (\overrightarrow{n}, \Sigma, \overrightarrow{M}, \overrightarrow{\alpha}, F\eta)$ the forward conjugate of \mathcal{A} with base F, and $\overleftarrow{\mathcal{A}} := (\overleftarrow{n}, \Sigma, \overleftarrow{M}, \alpha B, \overleftarrow{\eta})$ the backward conjugate of \mathcal{A} with base B.

Proposition 3.1. Let \mathcal{A} be an automaton. Then $[\![\mathcal{A}]\!] = [\![\overrightarrow{\mathcal{A}}]\!] = [\![\overrightarrow{\mathcal{A}}]\!]$.

Proof. By symmetry, it suffices to show the first equality. Indeed, we have for all $w \in \Sigma^*$:

$$[\![\overrightarrow{\mathcal{A}}]\!](w) = \overrightarrow{\alpha} \overrightarrow{M}(w) F \eta$$

$$= \overrightarrow{\alpha} F M(w) \eta \qquad \text{by (1)}$$

$$= \alpha M(w) \eta \qquad \text{definition of } \overrightarrow{\alpha}$$

$$= [\![\mathcal{A}]\!](w) \qquad \square$$

Proposition 3.2. Let $A = (n, \Sigma, M, \alpha, \eta)$ be an automaton. One can compute in polynomial time (with $O(|\Sigma|n^3)$ arithmetic operations)

- a matrix F whose rows form a basis of the forward space of A, and
- the forward conjugate of A with base F.

The statement holds analogously for backward conjugates.

Proof. By Proposition 2.1 one can compute a basis of the forward space, and hence F, in the required time. Having computed F it is straightforward to compute \overrightarrow{A} in the required time. The same holds analogously for \overleftarrow{A} . \square

Proposition 3.3. Let A be an automaton. Then rank([A]) $\leq n$.

Proof. Consider the matrices $\widehat{F}: \mathbb{K}^{\Sigma^* \times n}$ and $\widehat{B}: \mathbb{K}^{n \times \Sigma^*}$ with $\widehat{F}[w,\cdot] = \alpha M(w)$ and $\widehat{B}[\cdot,w] = M(w)\eta$ for all $w \in \Sigma^*$. Note that $\operatorname{rank}(\widehat{F}) \leq n$ (and similarly $\operatorname{rank}(\widehat{B}) \leq n$). Let $x,y \in \Sigma^*$. Then $(\widehat{F}\widehat{B})[x,y] = \alpha M(x)M(y)\eta = [\![\mathcal{A}]\!](xy)$, so $\widehat{F}\widehat{B}$ is the Hankel matrix of $[\![\mathcal{A}]\!]$. Hence $\operatorname{rank}([\![\mathcal{A}]\!]) = \operatorname{rank}(\widehat{F}\widehat{B}) \leq \operatorname{rank}(\widehat{F}) \leq n$.

Call an automaton with n states forward-minimal (resp., backward-minimal) if its forward (resp., backward) space has dimension n.

Proposition 3.4. A forward conjugate is forward-minimal. A backward conjugate is backward-minimal.

Proof. By symmetry, it suffices to prove the statement about forward conjugates. Let $\overrightarrow{A} = (\overrightarrow{n}, \Sigma, \overrightarrow{M}, \overrightarrow{\alpha}, F\eta)$ be the forward conjugate of $A = (n, \Sigma, M, \alpha, \eta)$ with base $F \in \mathbb{K}^{\overrightarrow{n} \times n}$. We have:

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\dim \langle \overrightarrow{\alpha} \overrightarrow{M}(w) \mid w \in \Sigma^* \rangle
= \dim \langle \overrightarrow{\alpha} \overrightarrow{M}(w) F \mid w \in \Sigma^* \rangle \quad \text{the rows of } F \text{ are linearly independent}
= \dim \langle \overrightarrow{\alpha} FM(w) \mid w \in \Sigma^* \rangle \quad \text{by (1)}
= \dim \langle \alpha M(w) \mid w \in \Sigma^* \rangle \quad \text{definition of } \overrightarrow{\alpha}
= \dim \mathcal{F} \quad \text{definition of } \mathcal{F}
= \overrightarrow{n} \quad \text{definition of } \overrightarrow{n} \quad \square
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Proposition 3.5. A backward conjugate of a forward-minimal automaton is minimal. A forward conjugate of a backward-minimal automaton is minimal.

Proof. By symmetry, it suffices to show the first statement. Let $\mathcal{A} = (n, \Sigma, M, \alpha, \eta)$ be forward-minimal. Let $B \in \mathbb{K}^{n \times \overline{n}}$ be a matrix whose columns form a basis of the backward space of \mathcal{A} . By Proposition 3.3 it suffices to show that $\overline{n} = \operatorname{rank}(H)$, where H is the Hankel matrix of $[\![\mathcal{A}]\!]$. Let \widehat{F} and \widehat{B} be the matrices from the proof of Proposition 3.3. Since \mathcal{A} is

forward-minimal, the columns of \widehat{F} are linearly independent. We have:

 $\begin{array}{ll} \overleftarrow{n} \\ = \operatorname{rank}(B) & \operatorname{definition \ of} \ B \\ = \operatorname{rank}(\widehat{F}B) & \operatorname{the \ columns \ of} \ \widehat{F} \ \operatorname{are \ linearly \ independent} \\ = \operatorname{dim} \langle \widehat{F}M(w)\eta \mid w \in \Sigma^* \rangle & \operatorname{definition \ of} \ B \\ = \operatorname{rank}(\widehat{F}\widehat{B}) & \operatorname{definition \ of} \ \widehat{B} \\ = \operatorname{rank}(H) & \operatorname{proof \ of \ Proposition \ 3.3} \end{array} \ \Box$

Theorem 3.6. Let \mathcal{A} be an automaton. Let \mathcal{A}' be a backward conjugate of a forward conjugate of \mathcal{A} (or a forward conjugate of a backward conjugate of \mathcal{A}). Then \mathcal{A}' is minimal and equivalent to \mathcal{A} . It can be computed in polynomial time (with $O(|\Sigma|n^3)$) arithmetic operations).

Proof. Minimality follows from Propositions 3.4 and 3.5. Equivalence follows from Proposition 3.1. Polynomial-time computability follows by invoking Proposition 3.2 twice.

Let $A_i = (n, \Sigma, M_i, \alpha_i, \eta_i)$ for $i \in \{1, 2\}$ be minimal, where A_2 is the forward conjugate of A_1 with some base $Q \in \mathbb{K}^{n \times n}$. By minimality and Proposition 3.1, matrix Q is invertible. Since

$$\alpha_2 Q = \alpha_1, \quad \eta_2 = Q \eta_1, \quad Q M_1(a) = M_2(a) Q \quad \text{for all } a \in \Sigma,$$

automaton A_1 is the backward conjugate of A_2 with base Q. Since

$$\alpha_2 = \alpha_1 Q^{-1}, \quad Q^{-1} \eta_2 = \eta_1, \quad M_1(a) Q^{-1} = Q^{-1} M_2(a) \quad \text{for all } a \in \Sigma,$$

automaton \mathcal{A}_1 is the forward conjugate of \mathcal{A}_2 with base Q^{-1} , and \mathcal{A}_2 is the backward conjugate of \mathcal{A}_1 with base Q^{-1} .

This motivates the following definition. Call minimal automata A_1, A_2 conjugate if one is a forward (equivalently, backward) conjugate of the other.

Theorem 3.7. Two minimal automata are conjugate if and only if they are equivalent.

Proof. The forward direction follows from Proposition 3.1.

Towards the backward direction, let $A_i = (n, \Sigma, M_i, \alpha_i, \eta_i)$ for $i \in \{1, 2\}$ be minimal equivalent automata. For $i \in \{1, 2\}$, consider the matrices $\widehat{F}_i : \mathbb{K}^{\Sigma^* \times n}$ and $\widehat{B}_i : \mathbb{K}^{n \times \Sigma^*}$ with $\widehat{F}_i[w, \cdot] = \alpha_i M_i(w)$ and $\widehat{B}_i[\cdot, w] = M_i(w) \eta_i$ for all $w \in \Sigma^*$. It follows from minimality and Proposition 3.1 that \widehat{F}_i and \widehat{B}_i have full rank n. Thus, there is an invertible matrix $Q \in \mathbb{K}^{n \times n}$ with $\widehat{F}_1 = \widehat{F}_2 Q$. We show that A_2 is the forward conjugate of A_1 with base Q.

We have $\alpha_1 = \widehat{F}_1[\varepsilon, \cdot] = \widehat{F}_2[\varepsilon, \cdot]Q = \alpha_2 Q$. Letting H denote the Hankel matrix of $[A_1] = [A_2]$, we have $\widehat{F}_2 Q \widehat{B}_1 = \widehat{F}_1 \widehat{B}_1 = H = \widehat{F}_2 \widehat{B}_2$. Since \widehat{F}_2

has full rank, it follows that $Q\widehat{B}_1 = \widehat{B}_2$. In particular, $Q\eta_1 = Q\widehat{B}_1[\cdot, \varepsilon] = \widehat{B}_2[\cdot, \varepsilon] = \eta_2$.

For any $a \in \Sigma$ let $H_a \in \mathbb{K}^{\Sigma^* \times \Sigma^*}$ be the matrix with $H_a[x, y] = [A_i](xay)$ for all $x, y \in \Sigma^*$. We have $\widehat{F}_i M_i(a) \widehat{B}_i = H_a$ for all $a \in \Sigma$. Thus, for all $a \in \Sigma$ we have:

$$\widehat{F}_{2}QM_{1}(a)\widehat{B}_{1} \ = \ \widehat{F}_{1}M_{1}(a)\widehat{B}_{1} \ = \ H_{a} \ = \ \widehat{F}_{2}M_{2}(a)\widehat{B}_{2} \ = \ \widehat{F}_{2}M_{2}(a)Q\widehat{B}_{1}$$

Since \widehat{F}_2 and \widehat{B}_1 have full rank, we have $QM_1(a) = M_2(a)Q$ for all $a \in \Sigma$. \square

Bibliographic Remarks

Minimization is closely related to equivalence and also goes back to [21]. The book [4, Chapter II] describes a minimization procedure. An $O(|\Sigma|n^4)$ minimization algorithm (for a related probabilistic model) was suggested in [13]. The algorithm given in this note reminds of Brzozowski's algorithm for minimizing DFAs [6]. The succinct formulation in this note is essentially from [18]. Further generalizations of Brzozowski's algorithm are discussed in [5].

Theorem 3.7 also goes back to [21]. See also [12] and [4, Chapter II].

4 The Hankel Automaton

Let $s: \Sigma^* \to \mathbb{K}$ be a series of rank n, with Hankel matrix H. Call a set $C = \{c_1, \ldots, c_n\} \subseteq \Sigma^*$ complete if the columns of $H[\cdot, C]$ form a basis of the column space of H.

Note that for any $G \subseteq \Sigma^*$ and $w \in \Sigma^*$ we have H[Gw, C] = H[G, wC], where $Gw := \{gw \mid g \in G\}$ and $wC := \{wc \mid c \in C\}$.

Let $C = \{c_1, \ldots, c_n\} \subseteq \Sigma^*$ be complete. Then, for any $w \in \Sigma^*$ there is a unique column vector $\eta_w \in \mathbb{K}^n$ with $H[\cdot, w] = H[\cdot, C]\eta_w$, and for all $a \in \Sigma$ a unique matrix $\overline{M}(a) \in \mathbb{K}^{n \times n}$ with $H[\cdot, C]\overline{M}(a) = H[\cdot, aC]$. We define the Hankel automaton for s, C as $\overline{\mathcal{A}} = (n, \Sigma, \overline{M}, H[\varepsilon, C], \eta_{\varepsilon})$.

Proposition 4.1. Let $\overline{A} = (n, \Sigma, \overline{M}, H[\varepsilon, C], \eta_{\varepsilon})$ be the Hankel automaton for s, C. Then for all $w \in \Sigma^*$ we have $H[\cdot, C]\overline{M}(w) = H[\cdot, wC]$. Hence, if $G = \{g_1, \ldots, g_n\} \subseteq \Sigma^*$ is such that H[G, C] has full rank, we have $\overline{M}(w) = H[G, C]^{-1}H[G, wC]$ for all $w \in \Sigma^*$.

Proof. We proceed by induction on the length of w. The induction base $(w = \varepsilon)$ is trivial. For the step, let $w \in \Sigma^*$ and $a \in \Sigma$. We have:

$$H[\cdot,C]\overline{M}(wa) = H[\cdot,wC]\overline{M}(a)$$
 induction hypothesis
$$= H[\cdot w,C]\overline{M}(a)$$

$$= H[\cdot w,aC]$$
 definition of $\overline{M}(a)$
$$= H[\cdot,waC]$$

Proposition 4.2. Let $\overline{A} = (n, \Sigma, \overline{M}, H[\varepsilon, C], \eta_{\varepsilon})$ be the Hankel automaton for s, C. Then $\|\overline{A}\| = s$ and \overline{A} is minimal.

Proof. We have for all $w \in \Sigma^*$:

Minimality follows from Proposition 3.3.

Theorem 4.3. Let $s: \Sigma^* \to \mathbb{K}$ be a series and $n \in \mathbb{N}$. Then $\operatorname{rank}(s) \leq n$ if and only if there is an automaton A with [A] = s that has at most n states.

Proof. Follows from Propositions 3.3 and 4.2. \square

The following proposition uses some notions from Section 3.

Proposition 4.4. Let $\mathcal{A} = (n, \Sigma, M, \alpha, \eta)$ be forward-minimal. Let $C = \{c_1, \ldots, c_r\} \subseteq \Sigma^*$ be such that the columns of the matrix $B := (M(c_1)\eta, \ldots, M(c_r)\eta) \in \mathbb{K}^{n \times r}$ form a basis of the backward space of \mathcal{A} . Then C is complete, and the backward conjugate of \mathcal{A} with base B is the Hankel automaton for $[\![\mathcal{A}]\!]$, C.

Proof. Let H be the Hankel matrix of $[\![A]\!]$. Let $\widehat{F}: \mathbb{K}^{\Sigma^* \times n}$ and $\widehat{B}: \mathbb{K}^{n \times \Sigma^*}$ be the matrices with $\widehat{F}[w,\cdot] = \alpha M(w)$ and $\widehat{B}[\cdot,w] = M(w)\eta$ for all $w \in \Sigma^*$. We have $\widehat{F}\widehat{B} = H$ and $\widehat{B}[\cdot,C] = B$, hence $\widehat{F}B = H[\cdot,C]$. Since the column spaces of \widehat{B} and B are equal, it follows that the column spaces of H and $H[\cdot,C]$ are equal. Since A is forward-minimal, \widehat{F} has full rank. Thus, $F = \operatorname{rank}(B) = \operatorname{rank}(\widehat{F}B) = \operatorname{rank}(H[\cdot,C])$, so the columns of $H[\cdot,C]$ are linearly independent. Hence, C is complete.

Let $\overline{\mathcal{A}} = (r, \Sigma, \overline{M}, H[\varepsilon, C], \eta_{\varepsilon})$ be the Hankel automaton for $\llbracket \mathcal{A} \rrbracket, C$. For all $x, w \in \Sigma^*$ and all $c \in C$ we have $\widehat{F}[x, \cdot]M(w)B[\cdot, c] = \alpha M(x)M(w)M(c)\eta = \llbracket \mathcal{A} \rrbracket(xwc) = H[x, wc]$. Thus, we have for all $w \in \Sigma^*$:

$$\widehat{F}M(w)B = H[\cdot, wC]$$

= $H[\cdot, C]\overline{M}(w)$ Proposition 4.1
= $\widehat{F}B\overline{M}(w)$

Since \widehat{F} has full rank, it follows that $M(w)B = B\overline{M}(w)$ for all $w \in \Sigma^*$. Similarly, we have $\widehat{F}B\eta_{\varepsilon} = H[\cdot, C]\eta_{\varepsilon} = H[\cdot, \varepsilon] = \widehat{F}\eta$, and since \widehat{F} has full rank, it follows that $B\eta_{\varepsilon} = \eta$. Finally, we have $\alpha B = \widehat{F}[\varepsilon, \cdot]B = H[\varepsilon, C]$. We conclude that \overline{A} is the backward conjugate of A with base B. **Theorem 4.5.** Let $A = (n, \Sigma, M, \alpha, \eta)$ be an automaton. One can compute in polynomial time (with $O(|\Sigma|n^3)$ arithmetic operations) a complete (for $[\![A]\!]$) set $C = \{c_1, \ldots, c_r\} \subseteq \Sigma^{\leq r-1}$ with $r \leq n$, and the Hankel automaton for $[\![A]\!]$, C.

Proof. Using Propositions 3.1, 3.2 and 3.4, first compute in the required time a forward-minimal automaton $\overrightarrow{A} = (\overrightarrow{n}, \Sigma, \overrightarrow{M}, \overrightarrow{\alpha}, \overrightarrow{\eta})$ with $[\![\overrightarrow{A}]\!] = [\![A]\!]$. By the backward analogue of Proposition 2.1 one can compute in the required time a set $C = \{c_1, \ldots, c_r\} \subseteq \Sigma^{\leq r-1}$ and the matrix $B = (\overrightarrow{M}(c_1)\overrightarrow{\eta}, \ldots, \overrightarrow{M}(c_r)\overrightarrow{\eta}) \in \mathbb{K}^{\overrightarrow{n}\times r}$ such that the columns of B form a basis of the backward space of \overrightarrow{A} . Let A' be the backward conjugate of \overrightarrow{A} with base B. By Proposition 3.2, it can be computed in the required time. By Proposition 4.4, A' is the Hankel automaton for $[\![A]\!]$, C.

Bibliographic Remarks

The material in this section, at least up to Theorem 4.3, is similar to [7, Section 2] and [12]. See also [19, Theorem 5.3] and [2, Section 2] for related treatments.

5 Computations in NC

We show that some of the mentioned polynomial-time computations can even be carried out in the complexity class NC, which comprises those languages having L-uniform Boolean circuits of polylogarithmic depth and polynomial size, or, equivalently, those problems solvable in polylogarithmic time on parallel random-access machines with polynomially many processors. We have $NL \subseteq NC \subseteq P$.

Lemma 5.1. Let $A \in \mathbb{K}^{m \times n}$. The row spaces of A and $A^T A$ are equal.

Proof. It is clear that the row space of A^TA is included in the row space of A. For the converse, it suffices to show that the null space of A^TA is included in the null space of A. Let $x \in \mathbb{K}^n$ with $A^TAx = 0_n$, where 0_n denotes the zero vector. Then $(Ax)^T(Ax) = x^TA^TAx = x^T0_n = 0$, and hence $Ax = 0_m$.

In the following we assume $\mathbb{K} = \mathbb{Q}$.

Proposition 5.2 (cf. Proposition 2.1). Let $\mathcal{A} = (n, \Sigma, M, \alpha, \eta)$ be an automaton. One can compute in NC a basis of the forward space $\mathcal{F} := \langle \alpha M(w) \mid w \in \Sigma^* \rangle$.

Proof. Let $F \in \mathbb{K}^{\Sigma^{\leq n-1} \times n}$ be the matrix with $F[w,\cdot] = \alpha M(w)$ for $w \in \Sigma^{\leq n-1}$. We have shown in Section 2 that the rows of F span \mathcal{F} . By Lemma 5.1, the rows of $E := F^T F \in \mathbb{K}^{n \times n}$ also span \mathcal{F} .

Let $e_i \in \{0,1\}^n$ denote the *i*th coordinate column vector, and let \otimes denote the Kronecker product. Using the mixed-product property of \otimes (i.e., $(AB) \otimes (CD) = (A \otimes C)(B \otimes D)$), we have:

$$\begin{split} E[i,j] &= F^T[i,\cdot] \ F[\cdot,j] \\ &= \sum_{w \in \Sigma^{\leq n-1}} (\alpha M(w))[i] \ (\alpha M(w))[j] \\ &= \sum_{w \in \Sigma^{\leq n-1}} (\alpha M(w)e_i) \otimes (\alpha M(w)e_j) \\ &= \sum_{w \in \Sigma^{\leq n-1}} (\alpha \otimes \alpha)(M(w) \otimes M(w))(e_i \otimes e_j) \\ &= (\alpha \otimes \alpha) \left(\sum_{w \in \Sigma^{\leq n-1}} M(w) \otimes M(w) \right) (e_i \otimes e_j) \\ &= (\alpha \otimes \alpha) \left(\sum_{k=0}^{n-1} \sum_{w \in \Sigma^k} M(w) \otimes M(w) \right) (e_i \otimes e_j) \\ &= (\alpha \otimes \alpha) \left(\sum_{k=0}^{n-1} \sum_{a_1 \cdots a_k \in \Sigma^k} (M(a_1) \otimes M(a_1)) \cdots (M(a_k) \otimes M(a_k)) \right) \\ &= (\alpha \otimes \alpha) \left(\sum_{k=0}^{n-1} \left(\sum_{a \in \Sigma} M(a) \otimes M(a) \right)^k \right) (e_i \otimes e_j) \end{split}$$

Since Kronecker products, sums and matrix powers [8] can be computed in NC, one can compute E in NC. We include the ith row of E in the desired basis of \mathcal{F} if and only if $\operatorname{rank}(E[\{1,\ldots,i\},\cdot]) > \operatorname{rank}(E[\{1,\ldots,i-1\},\cdot])$. This can be done in NC, as the rank of a matrix can be determined in NC [14].

Proposition 5.3 (cf. Proposition 2.2). Let $A = (n, \Sigma, M, \alpha, \eta)$ be an automaton. One can check in NC whether A is zero.

Proof. Analogous to the proof of Proposition 2.2. \square

Theorem 5.4 (cf. Theorem 2.3). Let $A_i = (n_i, \Sigma, M_i, \alpha_i, \eta_i)$ for $i \in \{1, 2\}$ be automata. One can check in NC whether A_1, A_2 are equivalent.

Proof. Analogous to the proof of Theorem 2.3.

Proposition 5.5 (cf. Proposition 3.2). Let $A = (n, \Sigma, M, \alpha, \eta)$ be an automaton. One can compute in NC:

• a matrix F whose rows form a basis of the forward space of A,

• the forward conjugate of A with base F.

The statement holds analogously for backward conjugates.

Proof. The first item follows from Proposition 5.2. The second item follows from the fact that linear systems of equations can be solved in NC [10]. \Box

Theorem 5.6 (cf. Theorem 3.6). Given an automaton, one can compute in NC a minimal equivalent automaton.

Proof. Analogous to the proof of Theorem 3.6.

Bibliographic Remarks

Theorem 5.4 about equivalence checking was proved by Tzeng [23]. Theorem 5.6 about minimization was obtained in [15, Section 4.2]. It is not known whether "counterexample" words for equivalence can be computed in NC. They can be computed in randomized NC [17].

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