On Computation of Gröbner Bases for Linear Difference Systems

Vladimir P. Gerdt

Laboratory of Information Technologies
Joint Institute for Nuclear Research
141980, Dubna
Russia

ACAT 2005, May 23, DESY, Zeuthen, Germany

Contents

Contents

Generation of Difference Schemes

Consider PDEs in the conservation law form

$$\frac{\partial \mathbf{v}}{\partial x} + \frac{\partial}{\partial y}\mathbf{F}(\mathbf{v}) = 0 \Longleftrightarrow \oint_{\Gamma} -\mathbf{F}(\mathbf{v})dx + \mathbf{v}dy = 0\,.$$

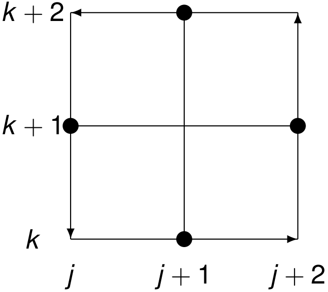
Γ is arbitrary closed contour, $\mathbf{v}$ is a $m-$vector function in unknown $n-$vector function $\mathbf{u}$ and its partial derivatives $\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_{xx}, \mathbf{u}_{xy}, \mathbf{u}_{yy}, \ldots$
F is a function that maps R^m into R^m.

To do discretization we set

$$\mathbf{u}(x,y) = \mathbf{u}(x_j, y_k) \equiv \mathbf{u}_{j\,k},\ \mathbf{u}_x(x,y) = \mathbf{u}_x(x_j,y_k) \equiv (\mathbf{u}_x)_{j\,k}, \ldots$$

Generation of Difference Schemes

Choose the integration contour and add the integral relations, e.g.,



$$\int_{x_j}^{x_{j+2}} \mathbf{u}_x dx = \mathbf{u}(x_{j+2}, y) - \mathbf{u}(x_j, y), \quad \int_{y_k}^{y_{k+2}} \mathbf{u}_y dy = \mathbf{u}(x, y_{k+2}) - \mathbf{u}(x, y_k), \ldots .$$

Generation of Difference Schemes

Using a numerical integration method, e.g. the midpoint one, with

$$x_{j+1} - x_j = y_{k+1} - y_k = \triangle h$$

we rewrite the equations and the relations as

$$\begin{gathered}
-(\mathbf{F}(\mathbf{v})_{j+1\,k} - \mathbf{F}(\mathbf{v})_{j+1\,k+2}) + (\mathbf{v}_{j+2\,k+1} - \mathbf{v}_{j\,k+1}) = 0, \\
(\mathbf{u}_x)_{j+1\,k} \cdot 2\triangle h = \mathbf{u}_{j+2\,k} - \mathbf{u}_{j\,k}, \\
(\mathbf{u}_y)_{j\,k+1} \cdot 2\triangle h = \mathbf{u}_{j\,k+2} - \mathbf{u}_{j\,k}, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{gathered}$$

A difference scheme for $\mathbf{u}$ is obtained (Mozzhilkin, Blinkov'01) by elimination of all partial derivatives $\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_{xx}, \ldots$ from the above system. The elimination can be achieved by constructing a Gröbner basis (GB), if it exists (finite). For linear PDEs GB always exists.

Contents

Reduction of Feynman Integrals

Consider scalar $L-$loop integral with n internal lines

$$\mathcal{I}_\nu := \int d^d k_1 \cdots d^d k_L \, \frac{1}{\prod_{i=1}^n P_i^{\nu_i}} \, .$$

P_i are propagators and $\nu = \{\nu_1, \nu_2, \ldots, \nu_n\} \in \mathbb{Z}^n$ is multi-index.

$\mathcal{I}_\nu$ satisfies recurrence relations (RR) derived from the integration by part method (Chetyrkin, Tkachov'81).

After a proper shift of indices $\mu = \nu - \lambda$, $\lambda \in \mathbb{Z}^n_{\geq 0}$, RR can be written in the form

$$f_j := \sum_\alpha b^j_\alpha \, \theta^\alpha \circ \mathcal{I}_\mu = 0 \, , \qquad j = 1, \ldots, p \, .$$

Reduction of Feynman Integrals

$\theta^{\alpha} = \theta_1^{\alpha_1} \cdots \theta_n^{\alpha_n}$, $\alpha = \{\alpha_1, \ldots, \alpha_n\} \in \mathbb{Z}^n_{\geq 0}$. θ_i denotes the right-shift operator for the i-th index, i.e.,

$$\theta_i \circ \mathcal{I}_{\mu} = \mathcal{I}_{\mu_1,\ldots,\mu_i+1,\ldots,\mu_n} .$$

Coefficients b^j_α are polynomials in indices $\{\nu_1, \ldots, \nu_n\}$ and physical parameters: masses, scalar products of external momenta, space-time dimension d.

Converting difference polynomials f_j into the Gröbner basis form allows (Gerdt'04):

- Define basic (master) integrals as those independent modulo RR.
- Reduce an integral $\mathcal{I}_{\bar{\nu}}$ with shifted indices $\nu \longrightarrow \bar{\nu}$ to the basic integrals by using the standard Gröbner reductions.

Contents

Difference Algebra

Let $\{y^1, \ldots, y^m\}$ be the set of *indeterminates* such, for example, as functions of $n-$variables $\{x_1, \ldots, x_n\}$ and $\theta_1, \ldots, \theta_n$ be the set of mutually commuting *difference operators (differences)*, e.g.,

$$\theta_i \circ y^j = y^j(x_1, \ldots, x_i + 1, \ldots, x_n).$$

A *difference ring R with differences $\theta_1, \ldots, \theta_n$* is a commutative ring R such that $\forall f, g \in R,\ 1 \leq i, j \leq n$

$$\theta_i\theta_j = \theta_j\theta_i,\ \theta_i \circ (f + g) = \theta_i \circ f + \theta_i \circ g,\ \theta_i \circ (f\,g) = (\theta_i \circ f)(\theta_i \circ g)$$

Similarly one defines a *difference field*.

Difference Algebra

Let $\mathbb{K}$ be a difference field. Denote by $\mathbb{R} := \mathbb{K}\{y^1, \ldots, y^m\}$ the difference ring of polynomials over $\mathbb{K}$ in variables

$$\{\ \theta^\mu \circ y^k \ \mid \mu \in \mathbb{Z}^n_{\geq 0},\ k = 1, \ldots, m\ \}\,.$$

Denote by $\mathbb{R}_L$ the set of linear polynomials in $\mathbb{R}$ and use the notations

$$\Theta = \{\ \theta^\mu \ \mid\ \mu \in \mathbb{Z}^n_{\geq 0}\ \},\ \deg_i(\theta^\mu \circ y^k) = \mu_i,\ \deg(\theta^\mu \circ y^k) = |\mu| = \sum_{i=1}^{n} \mu_i\,.$$

A *difference ideal* I in $\mathbb{R}$ is an ideal $I \in \mathbb{R}$ close under the action of any operator from Θ. If $F := \{f_1, \ldots, f_k\} \subset \mathbb{R}$ is a finite set, then the smallest difference ideal containing F denoted by $\mathrm{Id}(F)$. If $F \subset \mathbb{R}_L$, then $\mathrm{Id}(F)$ is *linear difference ideal*.

Contents

Difference Algebra

A total ordering $\prec$ over the set of $\theta_\mu y^j$ is a *ranking* if it satisfies

1. $\theta_i \theta^\mu \circ y^j \succ \theta^\mu \circ y^j$
2. $\theta^\mu y^j \succ \theta^\nu \circ y^k \quad \Longleftrightarrow \quad \theta_i \theta^\mu \circ y^j \succ \theta_i \theta^\nu \circ y^k \quad \forall\, i, j, k, \mu, \nu.$

If $\mu \succ \nu \Longrightarrow \theta_\mu \circ y^j \succ \theta_\nu \circ y^k$ the ranking is *orderly*.
If $i \succ j \Longrightarrow \theta_\mu \circ y^j \succ \theta_\nu \circ y^k$ the ranking is *elimination*.

Given a ranking $\succ$, every linear polynomial $f \in \mathbb{R}_L \setminus \{0\}$ has the *leading term* $a\,\theta \circ y^j$, $\theta \in \Theta$; $\mathrm{lc}(f) := a \in \mathbb{K} \setminus \{0\}$ is the *leading coefficient* and $\mathrm{lm}(f) := \theta \circ y^j$ is the *leading monomial*.
In $\mathbb{R}_L$ a ranking is a *monomial order*. If $F \in \mathbb{R}_L$, $\mathrm{lm}(F)$ is the set of the leading monomials and $\mathrm{lm}_j(F)$ is its subset with indeterminate y^j.
Thus,

$$\mathrm{lm}(F) = \cup_{j=1}^{m} \mathrm{lm}_j(F)\,.$$

Contents

Gröbner Bases

Given nonzero linear difference ideal $I = \mathrm{Id}(G)$ and term order $\succ$, its generating set $G = \{g_1, \ldots, g_s\} \subset \mathbb{R}_L$ is a *Gröbner basis* (GB) (Buchberger, Winkler'98, Mikhalev et al'99) of I if

$$\forall f \in I \cap \mathbb{R}_L \setminus \{0\}\ \exists g \in G, \theta \in \Theta\ :\ \mathrm{lm}(f) = \theta \circ \mathrm{lm}(g)\,.$$

It follows that $f \in I$ *is reducible modulo G*

$$f \underset{g}{\longrightarrow} f' := f - \mathrm{lc}(f)\,\theta \circ (g/lc(g)), \quad f' \in I, \ldots, \quad f \underset{G}{\longrightarrow} 0\,.$$

Similarly, a polynomial $h \in \mathbb{R}_L$, whose terms are reducible (if any) modulo set $F \in \mathbb{R}_L$, can be reduced to an irreducible polynomial $\bar{h}$, which is said to be in the *normal form modulo F* ($\bar{h} = NF(h, F)$).

Gröbner Bases

In our algorithmic construction of GB we shall use a restricted set of reductions called *Janet-like* (Gerdt, Blinkov'05) and defined as follows.

For a finite set $F \in \mathbb{R}_L$ and order $\succ$, partition every $\mathrm{lm}_k(F)$ groups labeled by $d_0, \ldots, d_i \in \mathbb{Z}_{\geq 0}$, $(0 \leq i \leq n)$, $([0]_k = \mathrm{lm}_k(F))$

$$[d_0, \ldots, d_i]_k := \{u \in \mathrm{lm}_k(F) \mid d_0 = 0, d_1 = \deg_1(u), ..., d_i = \deg_i(u)\}.$$

Define $h_i(u, \mathrm{lm}_k(F)) := \max\{\deg_i(v) \mid u, v \in [d_0, ..., d_{i-1}]_k\} - \deg_i(u)$.
If $h_i(u, \mathrm{lm}_k(F)) > 0$, then $\theta_i^{s_i}$ where

$$s_i := \min\{\deg_i(v) - \deg_i(u) \mid u, v \in [d_0, ..., d_{i-1}]_k, \deg_i(v) > \deg_i(u)\}$$

is called a *difference power* for u.

Contents

Gröbner Bases

Denote the set of difference powers for $u \in \mathrm{lm}_k(F)$ by $DP(u, \mathrm{lm}(F))$ and define the following subset of Θ

$$\mathcal{J}(u, \mathrm{lm}(F)) := \{\theta \in \Theta \mid \forall \vartheta_i^{s_i} \in DP(u, \mathrm{lm}(F)) \ : \ \deg_i(\theta \circ u) < s_i\}.$$

A GB of $I = \mathrm{Id}(G)$ is called Janet-like (Gerdt, Blinkov'05) if

$$\forall f \in I \cap \mathbb{R}_L \setminus \{0\} \ \exists g \in G, \theta \in \mathcal{J}(\mathrm{lm}(g), \mathrm{lm}(G)) : \ \mathrm{lm}(f) = \theta \circ \mathrm{lm}(g)\,.$$

This implies $\mathcal{J}-$reductions and $\mathcal{J}-$normal form: $NF_{\mathcal{J}}(f, F)$.

Algorithmic characterization of Janet-like GB:

$$\forall g \in G \ \forall \vartheta \in DP(\mathrm{lm}(g), \mathrm{lm}(G)) : \ NF_{\mathcal{J}}(\vartheta \circ g, G) = 0\,.$$

They are similar to (but more compact than) involutive Janet bases.

Contents

Algorithm: Janet-like Gröbner Basis($F \subset \mathbb{R}_L, \succ$)

choose $f \in F$ with the lowest $\mathrm{lm}(f)$ w.r.t. $\succ$
$G := \{f\};\ \ Q := F \setminus G$
do
  $h := 0$
  while $Q \neq \emptyset$ and $h = 0$ **do**
    choose $p \in Q$ with the lowest $\mathrm{lm}(p)$ w.r.t. $\succ$
    $Q := Q \setminus \{p\};\ \ h :=$ **Normal Form**$(p, G, \prec)$
  od
  if $h \neq 0$ **then**
    for all $\{g \in G \mid \mathrm{lm}(g) = \theta^{\mu}(\mathrm{lm}(h)), |\mu| > 0\}$ **do**
      $Q := Q \cup \{g\};\ \ G := G \setminus \{g\}$
    od
    $G := G \cup \{h\}$
    $Q := Q \cup \{\, \theta^{\beta} \circ g \mid g \in G,\ \beta \in DP(\mathrm{lm}(g), \mathrm{lm}(G))\,\}$
  fi
od while $Q \neq \emptyset$
return G

Subalgorithm

Algorithm: Normal Form$(p, G, \prec)$

1: $h := p$
2: **while** $h \neq 0$ **and** h has a monomial u with coefficient $b \in \mathbb{K}$ $\mathcal{J}-$reducible modulo G **do**
3: **take** $g \in G$ s.t. $u = \theta^\gamma(\mathrm{lm}(g))$ with $\gamma \in \mathcal{J}(\mathrm{lm}(g), \mathrm{lm}(G))$
4: $h := h/b - \theta^\gamma \circ (g/\,\mathrm{lc}(g))$
5: **od**
6: **return** h

Algorithm **Janet-like Gröbner Basis** implemented (in an improved form) in Maple (Gerdt, Robertz'05) is an extension of the polynomial algorithm (Gerdt, Blinkov'05) to difference ideals.

Contents

FDS for Laplace Equation

Consider the Laplace equation $u_{xx} + u_{yy} = 0$ and rewrite it as the conservation law

$$\oint_{\Gamma} -u_y dx + u_x dy = 0\,.$$

Add the integral relations

$$\int_{x_j}^{x_{j+2}} u_x dx = u(x_{j+2}, y) - u(x_j, y), \quad \int_{y_k}^{y_{k+2}} u_y dy = u(x, y_{k+2}) - u(x, y_k)\,.$$

Thus, we obtain 3 integral relations for 3 unknown functions

$$u(x,y),\ u_x(x,y),\ u_y(x,y)\,.$$

FDS for Laplace Equation

Choose midpoint integration method for above rectangular contour.

This yields the discrete system

$$\begin{cases} -((u_y)_{j+1\,k} - (u_y)_{j+1\,k+2}) + ((u_x)_{j+2\,k+1} - (u_y)_{j\,k+1}) = 0, \\ (u_x)_{j+1\,k} \cdot 2\triangle h = u_{j+2\,k} - u_{j\,k}, \\ (u_y)_{j\,k+1} \cdot 2\triangle h = u_{j\,k+2} - u_{j\,k}. \end{cases}$$

Its difference form is

$$\begin{cases} (\theta_x\theta_y^2 - \theta_x) \circ u_y + (\theta_x^2\theta_y - \theta_y) \circ u_x = 0\,, \\ 2\triangle h\,\theta_x \circ u_x - (\theta_x^2 - 1) \circ u = 0\,, \\ 2\triangle h\,\theta_y \circ u_y - (\theta_y^2 - 1) \circ u = 0\,. \end{cases}$$

FDS for Laplace Equation

Computation of GB (in this case Janet-like GB is the reduced GB) for elimination order with $u_x \succ u_y \succ u$ and $\theta_x \succ \theta_y$ gives

$$
\begin{cases}
\theta_x \circ u_x - \frac{1}{2\triangle h}\,(\theta_x^2 - 1) \circ u = 0\,, \\
\theta_y \circ u_x + \theta_x \circ u_y - \frac{1}{2\triangle h}\,(\theta_x\theta_y((\theta_x^2 - 1) + (\theta_y^2 - 1))) \circ u = 0\,, \\
\theta_x^2 \circ u_y - \frac{1}{2\triangle h}\,(\theta_x^2\theta_y((\theta_x^2 - 1) + (\theta_y^2 - 1)) - \theta_y(\theta_x^2 - 1)) \circ u = 0\,, \\
\theta_y \circ u_y - \frac{1}{2\triangle h}\,(\theta_y^2 - 1) \circ u = 0\,, \\
\frac{1}{2\triangle h}\,(\theta_x^4\theta_y^2 + \theta_x^2\theta_y^4 - 4\theta_x^2\theta_y^2 + \theta_x^2 + \theta_y^2) \circ u = 0\,.
\end{cases}
$$

FDS for Laplace Equation

The last equation gives the difference scheme written in double nodes

$$\frac{u_{j+2\,k} - 2u_{j\,k} + u_{j-2\,k}}{4\triangle h^2} + \frac{u_{j\,k+2} - 2u_{j\,k} + u_{j\,k-2}}{4\triangle h^2} = 0.$$

Similarly, the trapezoidal rule for the relation integrals generates the same difference scheme but written in ordinary nodes

$$\frac{u_{j+1\,k} - 2u_{j\,k} + u_{j-1\,k}}{\triangle h^2} + \frac{u_{j\,k+1} - 2u_{j\,k} + u_{j\,k-1}}{\triangle h^2} = 0.$$

Conclusions

- GB are the most universal algorithmic tool for linear difference systems.
- In particular, they can be applied to generate differences schemes for linear PDEs and to reduce multiloop Feynman integrals.
- There is an efficient algorithm for construction of GB for linear difference ideals. The algorithm is based on the concept of Janet-like reductions.
- Janet-like GB are similar to (but more compact than) involutive Janet bases, and the reduced GB can be easily extracted from the Janet-like GB without any extra computational costs.
- The first implementation in Maple is already available.
- Computer experiments and open software for constructing polynomial Janet and Janet-like bases presented on the Web site `http://invo.jinr.ru`.

References I

B.Buchberger, F.Winkler (eds.)
Gröbner Bases and Applications.
Cambridge University Press, 1998.

A.V. Mikhalev, A.B. Levin, E.V. Pankratiev, M.V. Kondratieva
Differential and Difference Dimension Polynomials. Mathematics and Its Applications.
Kluwer, Dordrecht, 1999.

V.V.Mozzhilkin, Yu.A.Blinkov
Methods of Constructing Difference Schemes in Gas Dynamics (in Russian).
Transactions of Saratov University, 1(2):145–156, 2001.

References II

K.G.Chetyrkin, F.V.Tkachev
Integration by Parts: The Algorithm to Calculate $\beta-$Fuctions in 4 Loops.
Nuclear Physics B, 192:159–204, 1981.

V.P.Gerdt
Gröbner Bases in Perturbative Calculations.
Nuclear Physics B (Proc. Suppl.) , 135:232-237, 2004.

V.P.Gerdt, Yu.A.Blinkov
Janet-like Monomial Division. Janet-like Gröbner Bases.
CASC 2005 , submitted.

V.P.Gerdt, D.Robertz
A Maple Package for Computing Gröbner bases for Linear Recurrence Relations.
ACAT 2005.