International Journal of Bifurcation and Chaos, Vol. 8, No. 7 (1998) 1479–1490

ASYMPTOTIC INDEPENDENCE AND UNIFORM DISTRIBUTION OF QUANTIZATION ERRORS FOR SPATIALLY DISCRETIZED DYNAMICAL SYSTEMS

P. DIAMOND* and I. VLADIMIROV†
Department of Mathematics, The University of Queensland, Brisbane, QLD 4072, Australia

Received November 14, 1997; Revised March 1, 1998

Computer simulation of dynamical systems involves a state space which is the finite set of computer arithmetic. Restricting state values to this grid produces roundoff effects which can be studied by replacing the original system with a spatially discretized dynamical system. Study of the deviation of the discretized trajectories from those of the original system reduces to that of appropriately defined quantization errors. As the grid is refined, the asymptotic behavior of these quantization errors follows probabilistic laws. These results are applied to discretized polynomial mappings of the unit interval.

1. Introduction

Finiteness of machine arithmetic results in roundoff errors when continuous dynamical systems are simulated. This often results in dramatic differences between the computed phase portrait of the resulting spatially discretized system and that of the original system [Blank, 1997; Diamond *et al.*, 1995a, 1995b; Kuznetsov & Kloeden, 1997]. In fixed point arithmetic, the evolution of the simulation can be modeled by dynamical systems on uniform grids. These discretized systems are generated by composing the transition operator of the original smooth system with a computer discretization procedure [Diamond *et al.*, 1995a, 1995b, 1996a, 1996b, 1998a, 1998b; Diamond & Pokrovskii, 1996].

Two important features often arise in these simulation studies. First, the discretized system can evolve quite differently from the true, underlying system, and this aberrant behavior persists even with refinement of machine accuracy [Diamond & Pokrovskii, 1996]. Moreover, as the grid is refined, deviations between the discretized and original systems behaves in a markedly irregular manner that can best be described in probabilistic terms [Diamond *et al.*, 1995a, 1996a, 1996b, 1998a, 1998b].

In this paper it is shown that under increasing refinement, these deviations can be asymptotically expressed linearly in terms of appropriate quantization errors and the empirical joint distribution of these is weakly convergent to a direct product of uniform distributions on cubes. This reflects a type of asymptotic statistical independence of the quantization errors and generalizes the result for quantized linear systems [Vladimirov, 1996] to the nonlinear case. A key condition here is that a certain resonance set, pertaining to the original dynamical system, has zero Lebesgue measure.

*E-mail: pmd@maths.uq.oz.au.
†E-mail: igv@maths.uq.oz.au
Permanent address: Institute for Information Transmission Problems, 19 Bolshoi Karetny Lane, GSP–4, 101447 Moscow, Russia.

These results on the asymptotic distribution of quantization errors can be interpreted as a form of low level universal law governing the distortion produced by discretization of smooth nonlinear dynamical systems. The principal techniques that are used below involve weak convergence of probability measures [Billingsley, 1968] and skew product of measure preserving automorphisms of finite-dimensional tori [Cornfeld *et al.*, 1982; Martin & England, 1984].

The paper is organized as follows. In Sec. 2, the class of spatially discretized dynamical systems is specified and a mapping is introduced which describes the deviation between positive semitrajectories of the discretized and original system. In Sec. 3, the quantization errors are defined, and the deviation is expressed in terms of these. In Sec. 4, the main theorem about the asymptotical independence and uniform distribution of the quantization errors is formulated. Section 5 applies these results to one-dimensional dynamical systems generated by rounded off piecewise polynomial mappings of the unit interval, including the logistic mapping. In Secs. 6–9, the complete proof of the main theorem is given, along with preliminary lemmas.

The machinery and results of the present paper should be applicable to quantitative analysis of aliasing structures and other artifacts connected with computer visualization of smooth objects [Izmailov & Pokrovskii, 1992; Izmailov *et al.*, 1996], as well as to rigorous justification of the heuristic and *ad hoc* models for computational collapse in chaotic systems [Diamond *et al.*, 1995, 1996a, 1996b, 1998a, 1998b; Grebogi *et al.*, 1988].

2. Preliminaries

Let $f : X \to X$ be a continuously differentiable mapping on an open subset X of $\mathbb{R}^n$. This mapping will be interpreted as the transition operator of an autonomous dynamical system with phase space X.

Let $\mathbb{Z}^n$ denote the n-dimensional integer lattice in $\mathbb{R}^n$ and define the roundoff operator $R : \mathbb{R}^n \to \mathbb{Z}^n$ which maps a vector to the nearest node of $\mathbb{Z}^n$,

$$R((u_k)_{1\le k\le n}) = (\lfloor u_k + 1/2 \rfloor)_{1\le k\le n}$$

where $\lfloor \cdot \rfloor$ denotes the integer part of a number. Clearly, R commutes with the additive group of translations of the lattice $\mathbb{Z}^n$,

$$R(u+z) = R(u) + z \quad \text{for all } u \in \mathbb{R}^n,\ z \in \mathbb{Z}^n\,, \tag{1}$$

and the full preimage of the zero vector with respect to the operator is the half open cube

$$V = \{u \in \mathbb{R}^n : R(u) = 0\} = [-1/2,\, 1/2)^n\,. \tag{2}$$

For arbitrary $\varepsilon > 0$ define the ε-*grid*

$$X_\varepsilon = X \bigcap (\varepsilon\mathbb{Z}^n)\,, \tag{3}$$

and the corresponding ε-*discretization* $f_\varepsilon : X_\varepsilon \to \varepsilon\mathbb{Z}^n$ of the transition operator f, defined as the superposition

$$f_\varepsilon = R_\varepsilon \circ f\,, \tag{4}$$

where the mapping $R_\varepsilon : \mathbb{R}^n \to \varepsilon\mathbb{Z}^n$ is given by

$$R_\varepsilon = \varepsilon R \circ \varepsilon^{-1}\,. \tag{5}$$

Note that the ε-grid (3) is, in general, not invariant with respect to the mapping f_ε. However, for any positive integer N and compact subset $K \subset X$, the N first images of $K \bigcap X_\varepsilon$ lie in X_ε for all sufficiently small $\varepsilon > 0$,

$$f_\varepsilon^k(K \textstyle\bigcap X_\varepsilon) \subset X_\varepsilon\,, \quad 1 \le k \le N\,.$$

In this sense, for any given mapping f, the corresponding discretized mapping f_ε may be interpreted as the transition operator of a spatially discrete autonomous dynamical system with phase space X_ε. Such a system can be regarded as a model for the computer implementation of the original dynamical system in fixed-point machine arithmetic.

The basic aim of this paper is to study the asymptotic behavior of the deviation of positive semitrajectories of the system f_ε from those of the original system f, as $\varepsilon \to +0$. To formalize this, for any $\varepsilon > 0$ and positive integer N define the *normalized deviation mapping* $d_{\varepsilon,N} : X_\varepsilon \to \mathbb{R}^n$

$$d_{\varepsilon,N}(x) = (f^N(x) - f_\varepsilon^N(x))/\varepsilon\,, \quad x \in X_\varepsilon\,. \tag{6}$$

It will be shown that the asymptotics of this mapping reduce to that of appropriately defined quantization errors.

3. Quantization Errors

Note that the mapping $d_{\varepsilon,1}$ takes its values in the cube V. With this in mind, define for any $\varepsilon > 0$ and $k \in \mathbb{N}$, the kth *quantization error* $E_{\varepsilon,k} : X_\varepsilon \to V$ as the superposition

$$E_{\varepsilon,k} = d_{\varepsilon,1} \circ f_\varepsilon^{k-1} \tag{7}$$

of the $(k-1)$th iterate of the discretized f_ε and $d_{\varepsilon,1}$.

The first result expresses the deviations between the discretized and original systems in terms of the quantization errors. To state the lemma, some notations are required.

Define the mapping D_ε which, for any sufficiently small $\varepsilon > 0$, maps (x, y), $x \in X$, $y \in \mathbb{R}^n$ to

$$D_\varepsilon(x, y) = (f(x) - f(x - \varepsilon y))/\varepsilon \in \mathbb{R}^n\,, \qquad (8)$$

and define a sequence of mappings $G_{\varepsilon,N} : X \times V^N \to \mathbb{R}^n$ by the recursion

$$\begin{aligned} &G_{\varepsilon,N+1}(x, y_1, \ldots, y_{N+1}) \\ &= D_\varepsilon(f^N(x), G_{\varepsilon,N}(x, y_1, \ldots, y_N)) + y_{N+1} \end{aligned} \qquad (9)$$

for any $x \in X$, $y_1, \ldots, y_{N+1} \in V$ and $N \in \mathbb{N}$, with initial condition

$$G_{\varepsilon,1}(x, y) = y\,. \qquad (10)$$

Lemma 1. *The normalized deviation* (*6*) *can be expressed in terms of the quantization errors* (*7*) *as*

$$d_{\varepsilon,N}(x) = G_{\varepsilon,N}(x, \mathcal{E}_{\varepsilon,N}(x))\,, \qquad (11)$$

where the mapping $\mathcal{E}_{\varepsilon,N} : X_\varepsilon \to V^N$ *is defined by*

$$\mathcal{E}_{\varepsilon,N}(x) = \begin{bmatrix} E_{\varepsilon,1}(x) \\ \vdots \\ E_{\varepsilon,N}(x) \end{bmatrix}. \qquad (12)$$

Proof. The proof proceeds by induction on $N \in \mathbb{N}$. The formula holds for $N = 1$ by (10), since the first quantization error $E_{\varepsilon,1}$ is just $d_{\varepsilon,1}$. Assuming (11) holds for some $N \in \mathbb{N}$, rewrite the subsequent mapping (6) as

$$\begin{aligned} d_{\varepsilon,N+1}(x) &= (f^{N+1}(x) - f_\varepsilon(f_\varepsilon^N(x)))/\varepsilon \\ &= (f^{N+1}(x) - f(f_\varepsilon^N(x)) + \varepsilon d_{\varepsilon,1}(f_\varepsilon^N(x)))/\varepsilon \\ &= (f(f^N(x)) - f(f^N(x) - \varepsilon d_{\varepsilon,N}(x)))/\varepsilon \\ &\quad + E_{\varepsilon,N+1}(x) \\ &= D_\varepsilon(f^N(x), G_{\varepsilon,N}(x, \mathcal{E}_{\varepsilon,N}(x))) \\ &\quad + E_{\varepsilon,N+1}(x)\,, \end{aligned} \qquad (13)$$

using (7) and (8). From (9), (12) and the right hand side of (13) it follows that

$$d_{\varepsilon,N+1}(x) = G_{\varepsilon,N+1}(x, \mathcal{E}_{\varepsilon,N+1}(x))\,,$$

and the proof is complete. ∎

The expression (11) can be linearized asymptotically as $\varepsilon \to +0$. Indeed, by the continuous differentiability of f,

$$\lim_{\varepsilon \to 0} D_\varepsilon(x, y) = f'(x)y \qquad (14)$$

holds uniformly in (x, y) over any compact subset of $X \times \mathbb{R}^n$. Here, $f'(x)$ is the Jacobian matrix of the mapping f at x. Hence, from (9) and (10), one can easily obtain by induction on $N \in \mathbb{N}$ that the following holds uniformly on $K \times V^N$ for any compact set $K \subset X$:

$$\begin{aligned} &\lim_{\varepsilon \to 0} G_{\varepsilon,N}(x, y_1, \ldots, y_N) \\ &\qquad = \sum_{k=1}^{N} (f^{N-k})'(f^k(x)) y_k \\ &\qquad = G_N(x, y_1, \ldots, y_N)\,. \end{aligned} \qquad (15)$$

Clearly, the mappings $G_N : X \times V^N \to \mathbb{R}^n$ satisfy the recursion

$$\begin{aligned} &G_{N+1}(x, y_1, \ldots, y_{N+1}) \\ &\quad = f'(f^N(x)) G_N(x, y_1, \ldots, y_N) + y_{N+1} \end{aligned}$$

for any $x \in X$, $y_1, \ldots, y_{N+1} \in V$ and $N \in \mathbb{N}$, with the initial condition $G_1(x, y) = y$.

4. Asymptotic Distribution of Quantization Errors

Recall that the vectors $x_1, \ldots, x_p$ are said to be *rationally dependent* if there exist rational real numbers $\lambda_1, \ldots, \lambda_p$, not all zero, such that

$$\lambda_1 x_1 + \cdots + \lambda_p x_p = 0\,.$$

If the rows of a $m \times n$ matrix A are rationally dependent, say that A *resonates*. Define the mapping $F_N : X \to X^N$ by

$$F_N(x) = \begin{bmatrix} f(x) \\ \vdots \\ f^N(x) \end{bmatrix}. \qquad (16)$$

Define the *resonance set* $\mathcal{R}(f)$ of the original dynamical system generated by f as

$$\mathcal{R}(f) = \left\{ x \in X : \begin{bmatrix} I_n \\ F_N{}'(x) \end{bmatrix} \right. \text{ resonates for some } N \in \mathbb{N} \left. \vphantom{\begin{bmatrix} I_n \\ F_N{}'(x) \end{bmatrix}} \right\}, \qquad (17)$$

where I_n is the identity matrix of order n.

By a Jordan measurable set $A \subset \mathbb{R}^n$ is meant a bounded measurable set, the boundary of which has zero n-dimensional Lebesgue measure.

Theorem 1. *Let the resonance set $\mathcal{R}(f)$ have zero n-dimensional Lebesgue measure,*

$$\operatorname{mes}_n \mathcal{R}(f) = 0 \,.$$

Then for any $N \in \mathbb{N}$, any Jordan measurable set $A \subset X$ and any bounded function $\varphi : A \times V^N \to \mathbb{R}$ which is continuous $\operatorname{mes}_{Nn+n}$*-almost everywhere,*

$$\lim_{\varepsilon \to +0} \left(\varepsilon^n \sum_{x \in A \bigcap X_\varepsilon} \varphi(x, \mathcal{E}_{\varepsilon,N}(x)) \right) = \int_{A \times V^N} \varphi(z) dz \,.$$

The proof will be given in Secs. 6–9. Note an important consequence of the theorem. Since

$$\lim_{\varepsilon \to +0} (\varepsilon^n \#(A \textstyle\bigcap X_\varepsilon)) = \operatorname{mes}_n A \,, \qquad (18)$$

for any Jordan measurable set $A \subset X$ ($\#(\,\cdot\,)$ is the cardinality of a finite set), from Theorem 1 it follows that for any $N \in \mathbb{N}$ and any Jordan measurable sets $A \subset X$ and $B_1, \ldots, B_N \subset V$, the following holds:

$$\lim_{\varepsilon \to +0} \frac{\#\{x \in A \bigcap X_\varepsilon : E_{\varepsilon,k}(x) \in B_k \text{ for all } 1 \le k \le N\}}{\#(A \bigcap X_\varepsilon)} = \prod_{k=1}^{N} \operatorname{mes}_n B_k \,.$$

It is precisely this property of the quantization errors that can be interpreted as their asymptotic independence and uniform distribution on the cube V.

5. Application: Discretized Interval Mappings

To illustrate some of the ideas above, let $n = 1$ and consider the mapping $f : X \to X$ of the unit interval $X = [0, 1]$

$$f(x) = 1 - |2x - 1|^a \qquad (19)$$

where $a \ge 2$ is an integer. In previous sections, X was an open set in $\mathbb{R}^n$. However, conventionally, interval mappings are considered on a closed interval, usually $[0, 1]$ or $[-1, 1]$. Obviously, for any $\nu \in \mathbb{N}$, the $1/\nu$-grid

$$X_{1/\nu} = \{0,\, 1/\nu,\, \ldots,\, (\nu - 1)/\nu,\, 1\}$$

is invariant under the $1/\nu$-discretization $f_{1/\nu}$ of (19),

$$f_{1/\nu}(X_{1/\nu}) \subset X_{1/\nu} \,.$$

Since f is a polynomial spline for odd values of a and is a polynomial of degree a for even a, it is easily seen that the resonance set $\mathcal{R}(f)$ is countable. Indeed, for each $N \in \mathbb{N}$, let

$$\Delta_{N,1}, \ldots,\, \Delta_{N,2^N} \qquad (20)$$

be a collection of 2^N intervals recursively defined by

$$\Delta_{N+1,2k-1} = [0,\, 1/2) \textstyle\bigcap f^{-1}(\Delta_{N,k}) \,,$$
$$\Delta_{N+1,2k} = [0,\, 1] \setminus \Delta_{N+1,2k-1}$$

for all $1 \le k \le 2^N$, with the initial condition

$$\Delta_{1,1} = [0,\, 1/2) \,, \quad \Delta_{1,2} = [1/2,\, 1] \,.$$

It is straightforward that, for every $N \in \mathbb{N}$, the intervals (20) form a partition of $[0, 1]$, with the first derivative $(f^k)'$ of the kth iterate of f a polynomial of degree $a^k - 1$, $1 \le k \le N$ on each of these intervals. Hence, for any set of real numbers $c_0, \ldots, c_N$, not all zero,

$$\#\left\{ x \in X : \sum_{k=0}^{N} c_k (f^k)'(x) = 0 \right\} \le 2^N (a^N - 1) \,.$$

So the resonance set $\mathcal{R}(f)$, consisting of $x \in X$ such that the $N + 1$ numbers 1, $f'(x), \ldots, (f^N)'(x)$ are rationally dependent for some $N \in \mathbb{N}$, is countable and so

$$\operatorname{mes}_1 \mathcal{R}(f) = 0 \,.$$

Consequently, Theorem 1 applies to any mapping of the type (19). In particular, for any such mapping the empirical variance of the normalized deviation (6) converges as

$$s^2_{\nu,N} = \frac{1}{\nu+1} \sum_{x \in X_{1/\nu}} (d_{1/\nu,N}(x))^2$$

$$\longrightarrow \int_0^1 (\sigma_N(x))^2 dx \quad \text{as } \nu \to +\infty\,, \qquad (21)$$

where

$$\sigma_N(x) = \sqrt{\frac{1}{12} \int_0^1 \sum_{k=1}^{N} ((f^{N-k})'(f^k(x)))^2 dx} \qquad (22)$$

denotes the root mean square value for the asymptotic distribution of the deviation $d_{1/\nu,N}$. Note that $\sigma_N(x)$ is majorized by

$$\mu_N(x) = \frac{1}{2} \sum_{k=1}^{N} |(f^{N-k})'(f^k(x))| \qquad (23)$$

which is the asymptotic local upper bound for the deviation (6) in the sense that

$$\limsup_{\nu \to +\infty} |d_{1/\nu,N}(x)| \le \mu_N(x)\,.$$

We compare some numerical experiments with the theoretically predicted asymptotic results for the logistic mapping

$$f(x) = 4x(1-x)\,. \qquad (24)$$

The first two quantization errors $E_{0.0001,1}(x)$ and $E_{0.0001,2}(x)$ for the 0.0001-discretization of the mapping (24), with x running over the grid $X_{0.0001}$, are graphed against x in Figs. 1 and 2. These figures visually confirm that the empirical marginal distributions of the quantization errors is close to the uniform distribution on the interval $[-1/2,\, 1/2)$. In Fig. 2, the "gap" in the cloud of points is due to the absorbing center [Diamond & Pokrovskii, 1996] and asymptotically narrows as $\nu \to \infty$ as $O(\nu^{-1/2})$. Figures 3 and 4 show the empirical joint distribution over $[-1/2,\, 1/2)^2$ and again apparently confirm the closeness to the uniform distribution.

In Figs. 5 and 6, the normalized deviations $d_{0.0001,2}(x)$ and $d_{0.0001,3}(x)$ are graphed against $x \in X_{0.0001}$ along with the corresponding root mean square values (22) (solid line) and upper bounds

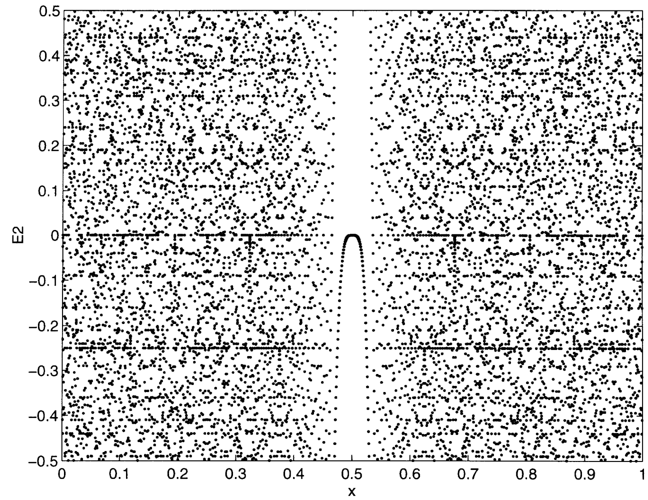

Fig. 2. The second quantization error for the 0.0001-discretized logistic mapping.

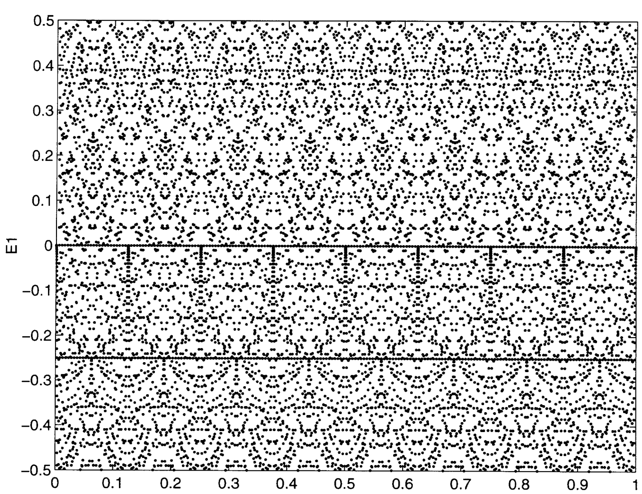

Fig. 1. The first quantization error for the 0.0001-discretized logistic mapping.

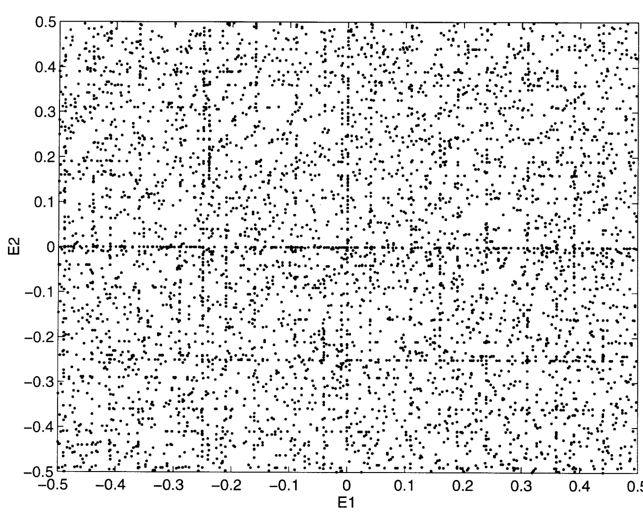

Fig. 3. The second versus first quantization errors for the 0.0001-discretized logistic mapping.

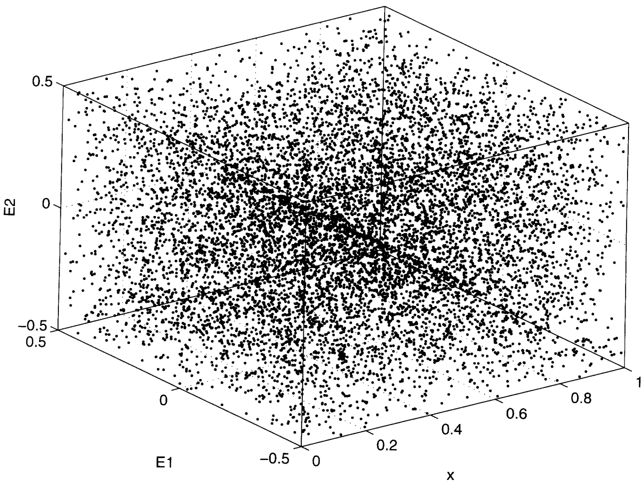

Fig. 4. The first and second quantization errors for the 0.0001-discretized logistic mapping.

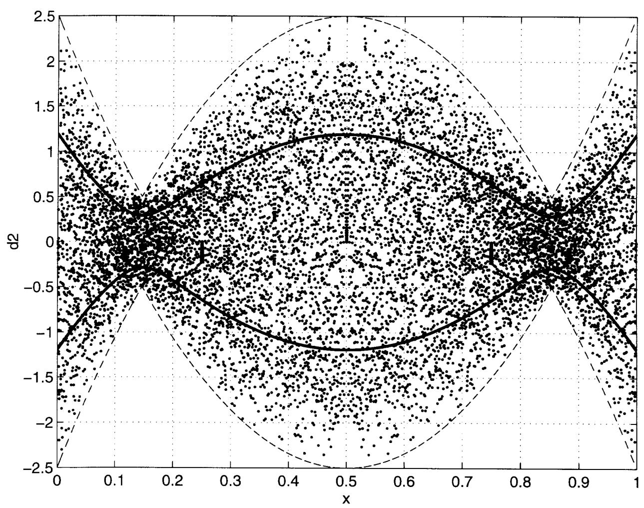

Fig. 5. The second normalized deviation for the 0.0001-discretized logistic mapping.

(23) (dashed line). The three first empirical variances (21) and their theoretically predicted limits, along with the relative discrepancies between theory and observation, are given by the following table:

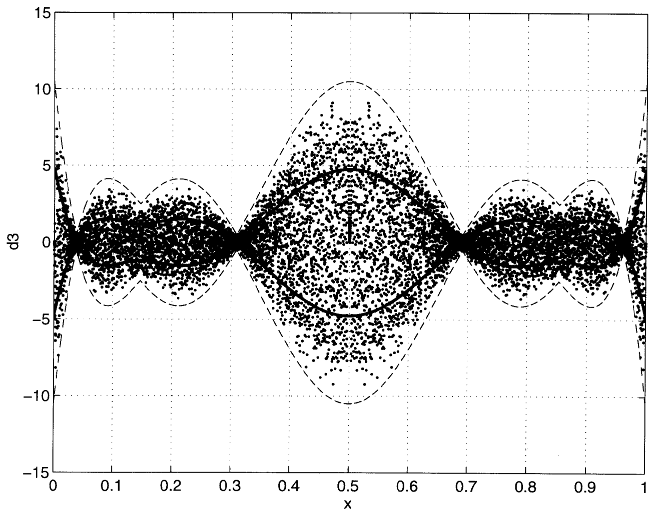

Fig. 6. The third normalized deviation for the 0.0001-discretized logistic mapping.

N	$s^2_{10000,N}$	$\int_0^1(\sigma_N(x))^2dx$	Rel. Gap, %
1	0.0812	$1/12 = 0.0833$	2.6033
2	0.6903	$127/180 = 0.7056$	2.1670
3	5.2373	$108831/20020 = 5.4361$	3.6579

This gives strong experimental evidence of the validity of Theorem 1 when applied to the logistic mapping and to similar dynamical systems.

6. Conditionally Periodic Representation of Quantization Errors

In similar vein to the roundoff operator $R(u)$, define a mapping $H : \mathbb{R}^n \to V$

$$\begin{aligned} H(u) &= u - R(u) \\ &= (\{\!\{u_k + 1/2\}\!\} - 1/2)_{1\le k\le n}\,, \\ &\qquad u = (u_k)_{1\le k\le n}\,, \end{aligned} \tag{25}$$

where $\{\!\{v\}\!\} = v - \lfloor v\rfloor$ denotes the fractional part of v. Define the mappings $H_{\varepsilon,N} : X \times \mathbb{R}^{Nn} \to V^N$ by

$$H_{\varepsilon,N+1}(x,\, y_1, \ldots,\, y_{N+1}) = \begin{bmatrix} H_{\varepsilon,N}(x,\, y_1, \ldots,\, y_N) \\ H(y_{N+1} - D_\varepsilon(f^N(x),\, G_{\varepsilon,N}(x,\, H_{\varepsilon,N}(x,\, y_1, \ldots,\, y_N)))) \end{bmatrix}, \tag{26}$$

where $x \in X$, $y_1, \ldots, y_{N+1} \in \mathbb{R}^n$ and $N \in \mathbb{N}$, with initial condition

$$H_{\varepsilon,1}(x,\, y) = H(y)\,. \tag{27}$$

Lemma 2. *For any $N \in \mathbb{N}$, the vector of quantization errors (12) can be represented in the form*

$$\mathcal{E}_{\varepsilon,N}(x) = H_{\varepsilon,N}(x,\, \varepsilon^{-1}F_N(x))\,. \tag{28}$$

where we have used the notation (16).

Proof. The proof proceeds by induction on N. To begin,

$$E_{\varepsilon,1}(x) = d_{\varepsilon,1}(x) = H(\varepsilon^{-1}f(x))\,,$$

which easily follows from (4), (6) and (25). Now suppose that the representation (28) holds for some $N \in \mathbb{N}$. Rewrite the subsequent quantization error (7) as follows

$$\begin{aligned} E_{\varepsilon,N+1}(x) &= d_{\varepsilon,1}(f_\varepsilon^N(x)) = H(\varepsilon^{-1}f(f^N(x) - \varepsilon d_{\varepsilon,N}(x))) \\ &= H(\varepsilon^{-1}f^{N+1}(x) - D_\varepsilon(f^N(x),\, d_{\varepsilon,N}(x))) \\ &= H(\varepsilon^{-1}f^{N+1}(x) - D_\varepsilon(f^N(x),\, G_{\varepsilon,N}(x,\, \mathcal{E}_{\varepsilon,N}(x)))) \\ &= H(\varepsilon^{-1}f^{N+1}(x) - D_\varepsilon(f^N(x),\, G_{\varepsilon,N}(x,\, H_{\varepsilon,N}(x,\, \varepsilon^{-1}F_N(x)))))\,, \end{aligned} \tag{29}$$

where Lemma 1 has been used. From (26), (16), and the right hand side of (29) it easily follows that

$$\begin{aligned} \mathcal{E}_{\varepsilon,N+1}(x) &= \begin{bmatrix} \mathcal{E}_{\varepsilon,N}(x) \\ E_{\varepsilon,N+1}(x) \end{bmatrix} \\ &= \begin{bmatrix} H_{\varepsilon,N}(x,\, \varepsilon^{-1}F_N(x)) \\ H(\varepsilon^{-1}f^{N+1}(x) - D_\varepsilon(f^N(x),\, G_{\varepsilon,N}(x,\, H_{\varepsilon,N}(x,\, \varepsilon^{-1}F_N(x))))) \end{bmatrix} \\ &= H_{\varepsilon,N+1}(x,\, \varepsilon^{-1}F_{N+1}(x))\,. \end{aligned}$$

The lemma is proved. ∎

The next lemma shows that (28) is, in a sense, a *conditionally periodic* representation for the quantization errors, and gives further properties of the mappings $H_{\varepsilon,N}$.

Lemma 3. *For any fixed $x \in X$, each of the mappings $H_{\varepsilon,N}(x,\cdot) : \mathbb{R}^{Nn} \to V^N$ is periodic in its Nn variables with unit period,*

$$H_{\varepsilon,N}(x,\, y+z) = H_{\varepsilon,N}(x,\, y) \quad \text{for all } y \in \mathbb{R}^{Nn},\ z \in \mathbb{Z}^{Nn}\,.$$

Moreover, the restriction $H_{\varepsilon,N}(x,\cdot)|_{V^N}$ is a mes$_{Nn}$*-preserving bijection of the set V^N onto itself. That is, for any Borel subset $B \subset V^N$,*

$$\begin{aligned} \text{mes}_{Nn}\{y \in V^N : H_{\varepsilon,N}(x,\, y) \in B\} &= \text{mes}_{Nn} H_{\varepsilon,N}(x,\, B) \\ &= \text{mes}_{Nn} B\,. \end{aligned}$$

Proof. Proof proceeds by induction on N. Since the mapping $H_{\varepsilon,1}$ coincides with the mapping H given by (25), begin from the unit periodicity of H in each of its n variables, and the property that H is the identity bijection of the cube V onto itself that implies the measure preserving property. Suppose the assertions of the lemma hold for some $N \in \mathbb{N}$. Consider the mapping $H_{\varepsilon,N+1}$ given by (26), and for any $x \in X$ and $y \in \mathbb{R}^{Nn}$, define $\Phi_{\varepsilon,N}(x,\, y,\, \cdot) : \mathbb{R}^n \to V$ described by the second subvector on the right hand side of (26):

$$\begin{aligned} &\Phi_{\varepsilon,N}(x,\, y,\, v) \\ &\quad = H(v - D_\varepsilon(f^N(x),\, G_{\varepsilon,N}(x,\, H_{\varepsilon,N}(x,\, y)))) \,. \end{aligned} \tag{30}$$

Clearly, for any fixed $x \in X$ and $y \in \mathbb{R}^{Nn}$, $\Phi_{\varepsilon,N}(x,\, y,\, \cdot) : \mathbb{R}^n \to V$ is unit periodic in its n variables and is a mes$_n$-preserving bijection of the cube V. This implies the unit periodicity of the mapping $H_{\varepsilon,N+1}(x,\, \cdot) : \mathbb{R}^{(N+1)n} \to V^{N+1}$ in its $(N+1)n$ variables. That the restriction $H_{\varepsilon,N+1}(x,\, \cdot)|_{V^{N+1}}$ is a mes$_{(N+1)n}$-preserving bijection of the set V^{N+1} onto itself follows, since the mapping $H_{\varepsilon,N+1}(x,\, \cdot)$ is the skew product [Cornfeld *et al.*, 1982; Martin & England, 1984] of the mes$_{Nn}$-preserving bijection $H_{\varepsilon,N}(x,\cdot)$ of the set V^N with the family $\{\Phi_{\varepsilon,N}(x, y, \cdot) : y \in V^N\}$ of mes_n-preserving bijections of the cube V for any fixed $x \in X$. This completes the inductive step and the lemma is proved. ■

Denote the n-dimensional torus endowed with the appropriate topology and the uniform distribution by $\tilde{V}$. There is a natural bijection between the cube V and $\tilde{V}$. With each of the mappings $H_{\varepsilon,N}$ associate a mapping $\tilde{H}_{\varepsilon,N} : X \times \tilde{V}^N \to \tilde{V}^N$ by the following commutative diagram:

$$\begin{array}{ccc} V^N & \xrightarrow{H_{\varepsilon,N}(x,\cdot)} & V^N \\ \updownarrow & & \updownarrow \\ \tilde{V}^N & \xrightarrow[\tilde{H}_{\varepsilon,N}(x,\cdot)]{} & \tilde{V}^N \end{array}$$

Now Lemma 3 can be restated in terms of the mapping $\tilde{H}_{\varepsilon,N}$ being a conditionally measure preserving automorphism of the Nn-dimensional torus $\tilde{V}^N$.

7. Convergence to Conditionally Piecewise Linear Automorphisms of Tori

Define mappings $H_N : X \times \mathbb{R}^{Nn} \to V^N$ by

$$H_{N+1}(x, y_1, \ldots, y_{N+1}) = \begin{bmatrix} H_N(x, y_1, \ldots, y_N) \\ H(y_{N+1} - f'(f^N(x))G_N(x, H_N(x, y_1, \ldots, y_N))) \end{bmatrix}, \tag{31}$$

$x \in X$, $y_1, \ldots, y_{N+1} \in \mathbb{R}^n$, $N \in \mathbb{N}$, with the initial condition

$$H_1(x, y) = H(y). \tag{32}$$

As in the proof of Lemma 3, it is easy to show inductively that for any given $x \in X$, $H_N(x,\cdot) : \mathbb{R}^{Nn} \to V^N$ is unit periodic in its Nn variables and the restriction $H_N(x,\cdot)|_{V^N}$ of this mapping is a mes_{Nn}-preserving piecewise linear bijection of the cube V^N onto itself.

The following lemma clarifies the role of the mappings H_N in the asymptotic behavior of the mappings $H_{\varepsilon,N}$ as $\varepsilon \to +0$.

Lemma 4. *For any $N \in \mathbb{N}$, the set L_N of continuity points of the mapping H_N on $X \times V^N$ has full $(N+1)n$-dimensional Lebesgue measure, and*

$$\lim_{\varepsilon \to +0, w \to u} H_{\varepsilon,N}(w) = H_N(u) \quad \text{for all } u \in L_N. \tag{33}$$

Proof. The proof proceeds by induction on N. First, from (32) the continuity set of the mapping H_1 is

$$L_1 = X \times V.$$

Now let the statement of the lemma hold for some $N \in \mathbb{N}$. Consider the continuity set L_{N+1} of the mapping H_{N+1}. It is straightforward to show that this set can be represented as

$$L_{N+1} = \{(u, v) \in X \times V^{N+1} : u \in L_N \text{ and } v \in V \setminus \Psi_N(u)\}, \tag{34}$$

where $\Psi_N : X \times V^N \to 2^V$ is a set-valued mapping which assigns to $(x, y) \in X \times V^N$ the discontinuity set with respect to $v \in V$ of the mapping

$$\Phi_N(x, y, v) = H(v - f'(f^N(x))G_N(x, H_N(x, y))). \tag{35}$$

Now, consider the structure of the set $\Psi_N(x, y)$. By definition (25), the discontinuity set of the periodic and piecewise linear mapping H is described by the set $\Gamma + \mathbb{Z}^n$, where the set

$$\Gamma = \left\{ u = (u_k)_{1 \le k \le n} \in \mathbb{R}^n : \prod_{k=1}^{n} (u_k + 1/2) = 0 \right\}$$

is the union of n mutually perpendicular $(n-1)$-dimensional hyperplanes containing the vector $(-1/2, \ldots, -1/2)$. Hence, $\mathrm{mes}_n \Gamma = 0$. Therefore, the set $\Psi_N(x, y)$ of discontinuity with respect to $v \in V$ of the mapping (35), for a given point $(x, y) \in X \times V^N$ can be written as

$$\Psi_N(x, y) = (f'(f^N(x))G_N(x, H_N(x, y)) + \Gamma + Z^n) \bigcap V.$$

This is the intersection of the shifted set Γ with the cube V. Hence, the set $\Psi_N(u)$ has zero n-dimensional Lebesgue measure for every $(x, y) \in X \times V^N$. Therefore, the set L_{N+1} given by (34) is of

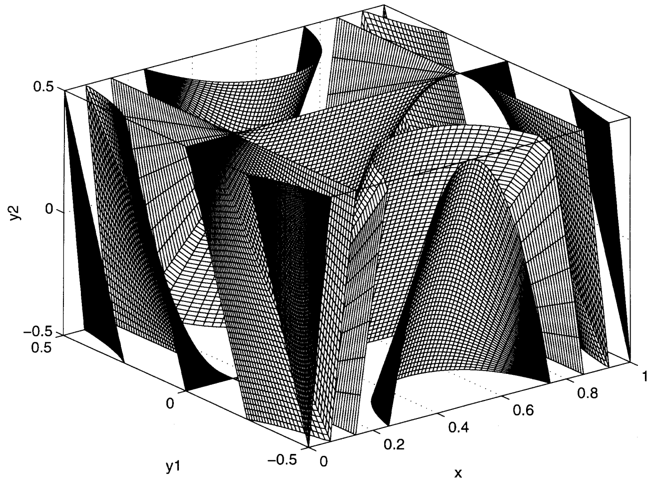

Fig. 7. The discontinuity set of $H_2(x, y_1, y_2)$ for the logistic mapping.

full $(N+2)n$-dimensional Lebesgue measure. Furthermore, using the uniform limits (14) and (15), it is easy to show that the mapping $\Phi_{\varepsilon,N}$ converges to the mapping (35) on the continuity set

$$\lim_{\varepsilon\to+0,w\to u,z\to v} \Phi_{\varepsilon,N}(w,\, z) = \Phi_N(u,\, v)$$

$$\text{for all } u \in L_N,\ v \in V \setminus \Psi_N(u)\,.$$

This gives the assertion for the set L_{N+1}, completing the inductive step and the lemma is proved. ■

Some insight into the complicated structure of the discontinuity sets $(X \times V^N)\backslash L_N$ of H_N is provided by Fig. 7 which shows the set for the one-dimensional dynamical system generated by the logistic mapping for $N = 2$.

8. Asymptotically Uniform Distribution of Toral Projections

For any $\omega \in \mathbb{Z}^m$, define the elementary trigonometric polynomial $T_\omega : \mathbb{R}^m \to \mathbb{C}$ given by

$$T_\omega(u) = \exp(2\pi i \omega^T u)\,. \tag{36}$$

Lemma 5. *Let $F : X \to \mathbb{R}^m$ be a continuously differentiable mapping on an open set $X \subset \mathbb{R}^n$ satisfying the following condition*:

($*$) *the rows of the matrix $\begin{bmatrix} I_n \\ F'(x)\end{bmatrix} \in \mathbb{R}^{(n+m)\times n}$ are rationally independent for almost all $x \in X$ with respect to n-dimensional Lebesgue measure.*

Then for any Jordan measurable set $A \subset X$, the following limit relation holds:

$$\lim_{\varepsilon\to+0}\left(\varepsilon^n \sum_{x\in A\bigcap X_\varepsilon} T_\omega(\varepsilon^{-1}F(x))\right) = \begin{cases} \operatorname{mes}_n A & \textit{for} \quad \omega = 0 \\ 0 & \textit{for} \quad \omega \in \mathbb{Z}^m \setminus \{0\}\end{cases}. \tag{37}$$

Proof. The validity of (37) for $\omega = 0$ immediately follows from the identity $T_0 \equiv 1$. For any $\alpha \in \mathbb{N}$, consider the discrete set

$$Q_\alpha = \{x = (x_k)_{1\le k\le n} \in \mathbb{Z}^n : |x_k| \le \alpha \quad \text{for all } 1 \le k \le n\}$$

consisting of $(2\alpha+1)^n$ points of the lattice $\mathbb{Z}^n$. By appropriate translations of this cube, obtain a partition of the lattice $\mathbb{Z}^n$

$$\{x + Q_\alpha : x \in (2\alpha+1)\mathbb{Z}^n\}\,.$$

It is straightforward to see that for any fixed $\alpha \in \mathbb{N}$ and any Jordan measurable set $A \subset X$,

$$\#((A \textstyle\bigcap X_\varepsilon)\triangle((A \bigcap X_{(2\alpha+1)\varepsilon}) + \varepsilon Q_\alpha)) = o(\varepsilon^{-n}) \quad \text{as } \varepsilon \to +0\,. \tag{38}$$

Let $F : X \to \mathbb{R}^m$ be a continuously differentiable mapping satisfying the condition ($*$) of Lemma 5. For any given $\omega \in \mathbb{Z}^m \setminus \{0\}$ and any $\varepsilon > 0$, define the function $\varphi_\varepsilon : X \to \mathbb{C}$ by

$$\varphi_\varepsilon = T_\omega \circ \varepsilon^{-1} \circ F\,.$$

Since $|\varphi_\varepsilon| = 1$, from (38) it follows that for any $\alpha \in \mathbb{N}$

$$\limsup_{\varepsilon\to+0}\left|\varepsilon^n \sum_{x\in A\bigcap X_\varepsilon} \varphi_\varepsilon(x)\right| \le \limsup_{\varepsilon\to+0}\left(((2\alpha+1)\varepsilon)^n \sum_{x\in A\bigcap X_{(2\alpha+1)\varepsilon}} \psi_{\alpha,\varepsilon}(x)\right), \tag{39}$$

where $\psi_{\alpha,\varepsilon} : X \to \mathbb{R}_+$ is a non-negative-valued function given by

$$\psi_{\alpha,\varepsilon}(x) = (2\alpha+1)^{-n}\left|\sum_{y\in Q_\alpha}\varphi_\varepsilon(x+\varepsilon y)\right| . \qquad (40)$$

Using the continuous differentiability of the mapping F, it is easy to show that

$$\begin{aligned}\lim_{\varepsilon\to+0}\psi_{\alpha,\varepsilon}(x) &= (2\alpha+1)^{-n}\left|\sum_{y\in Q_\alpha}T_\omega(F'(x)y)\right| \\ &= \xi_\alpha((F'(x))^T\omega) \qquad (41)\end{aligned}$$

for any $\alpha\in\mathbb{N}$ and uniformly in x lying in any given compact subset of X. Here, the function $\xi_\alpha:\mathbb{R}^n\to[0,\,1]$ is defined by

$$\xi_\alpha((u_k)_{1\le k\le n}) = \prod_{k=1}^n \eta_\alpha(u_k) \qquad (42)$$

where, in turn,

$$\eta_\alpha(v) = \begin{cases} 1 & \text{for} \quad v\in\mathbb{Z} \\ \left|\dfrac{\sin(\pi(2\alpha+1)v)}{(2\alpha+1)\sin(\pi v)}\right| & \text{for} \quad v\in\mathbb{R}\setminus\mathbb{Z}\end{cases} . \qquad (43)$$

Since the convergence (41) is uniform, the upper limit of the expression on the right hand side of (39) coincides with its limit and the inequality can thus be rewritten as

$$\limsup_{\varepsilon\to+0}\left|\varepsilon^n\sum_{x\in A\bigcap X_\varepsilon}\varphi_\varepsilon(x)\right| \le \int_A \xi_\alpha((F'(x))^T\omega)dx . \qquad (44)$$

From (42) and (43) it follows that

$$\lim_{\alpha\to+\infty}\xi_\alpha(u) = \begin{cases}1 & \text{for} \quad u\in\mathbb{Z}^n \\ 0 & \text{for} \quad u\in\mathbb{R}^n\setminus\mathbb{Z}^n\end{cases} ,$$

so, applying the Lebesgue Dominated Convergence Theorem to the integral on the right of (44), obtain

$$\begin{aligned}&\limsup_{\varepsilon\to+0}\left|\varepsilon^n\sum_{x\in A\bigcap X_\varepsilon}\varphi_\varepsilon(x)\right| \\ &\qquad \le \operatorname{mes}_n\{x\in A : (F'(x))^T\omega\in\mathbb{Z}^n\} .\end{aligned}$$

Since $\omega\in\mathbb{Z}^m\setminus\{0\}$, then, by the assumption $(*)$, the right hand side of the last inequality is zero. Hence,

$$\lim_{\varepsilon\to+0}\left(\varepsilon^n\sum_{x\in A\bigcap X_\varepsilon}\varphi_\varepsilon(x)\right) = 0 . \quad \blacksquare$$

Lemma 6. *Let F be a mapping satisfying the conditions of Lemma 5, and let $\varphi:\mathbb{R}^m\to\mathbb{C}$ be a continuous function, unit periodic in its m variables. Then for any Jordan measurable set $A\subset X$,*

$$\begin{aligned}&\lim_{\varepsilon\to+0}\left(\varepsilon^n\sum_{x\in A\bigcap X_\varepsilon}\varphi(\varepsilon^{-1}F(x))\right) \\ &\qquad = \operatorname{mes}_n A\int_{[0,1]^m}\varphi(u)du . \qquad (45)\end{aligned}$$

Proof. Let $\varphi:\mathbb{R}^m\to\mathbb{C}$ be a trigonometric polynomial,

$$\varphi = \sum_{\omega\in\Omega}c_\omega T_\omega \qquad (46)$$

where c_ω are complex coefficients, $T_\omega(u) = \exp(2\pi i\omega^T u)$ and Ω is a finite subset of $\mathbb{Z}^m$ containing the zero vector. Clearly

$$c_0 = \int_{[0,1]^m}\varphi(u)du . \qquad (47)$$

On the other hand, application of Lemma 5 easily gives

$$\lim_{\varepsilon\to+0}\left(\varepsilon^n\sum_{x\in A\bigcap X_\varepsilon}\varphi(\varepsilon^{-1}F(x))\right) = c_0\operatorname{mes}_n A . \qquad (48)$$

Comparing (47) and (48) shows the validity of (45) for any trigonometric polynomial φ satisfying (46). The proof of the lemma now follows because the set of all such polynomials φ is dense in the Banach space of continuous functions $g:\mathbb{R}^m\to\mathbb{C}$, unit periodic in all m variables, endowed with the uniform metric. $\blacksquare$

Lemma 7. *Let F be a mapping satisfying the conditions of Lemma 5. Then, for any compact set $K\subset X$ and any continuous function $\varphi: K\times\mathbb{R}^m\to\mathbb{R}$ such that $\varphi(x,\,y)$ is unit periodic in the m*

coordinates of $y \in \mathbb{R}^m$ for any fixed $x \in X$,

$$\lim_{\varepsilon\to+0}\left(\varepsilon^n \sum_{x\in K\bigcap X_\varepsilon} \varphi(x,\, \varepsilon^{-1}F(x))\right) = \int_{K\times[0,1]^m} \varphi(z)dz\,. \tag{49}$$

Proof. Let A_k, $k = 1, \ldots, N$, be Jordan measurable sets forming a partition of K, and let $\psi_k : \mathbb{R}^m \to \mathbb{R}$ be continuous functions, unit periodic in their m variables. Define the function $\psi : K \times \mathbb{R}^m \to \mathbb{R}$ by

$$\psi(x,\, y) = \sum_{k=1}^{N} \mathcal{I}_{A_k}(x)\psi_k(y)\,, \tag{50}$$

where $\mathcal{I}_{\mathrm{M}}$ stands for the indicator function of a set M. From Lemma 6, it is easily seen that

$$\lim_{\varepsilon\to+0}\left(\varepsilon^n \sum_{x\in K\bigcap X_\varepsilon} \varphi(x,\, \varepsilon^{-1}F(x))\right) = \sum_{k=1}^{N} \mathrm{mes}_{\,n} A_k \int_{[0,1]^m} \psi_k(y)dy\,.$$

Now, the limit relation (49) follows from the observation that for any $\delta > 0$ there exists a function ψ of form (50) such that

$$\sup_{z\in K\times[0,1]^m} |\varphi(z) - \psi(z)| < \delta\,. \quad \blacksquare$$

Given a mapping $F : X \to \mathbb{R}^m$, for each $\varepsilon > 0$ introduce a countably additive measure μ_ε defined on the Borel subsets B of $X \times [-1/2,\, 1/2)^m$ by

$$\mu_\varepsilon(B) = \varepsilon^n \#\{x \in X_\varepsilon : (x,\, P_m(\varepsilon^{-1}F(x))) \in B\}\,, \tag{51}$$

where the mapping $P_m : \mathbb{R}^m \to [-1/2,\, 1/2)^m$ is

$$P_m((u_k)_{1\le k\le m}) = (\{\!\{u_k + 1/2\}\!\} - 1/2)_{1\le k\le m} \tag{52}$$

which can be regarded as a projection onto the m-dimensional torus. Note that P_n is the mapping H given by (25).

Theorem 2. *Let F be a mapping satisfying the conditions of Lemma 5. Then, for any compact set $K \subset X$, the restriction of the measure (51) onto the set $K \times [-1/2,\, 1/2)^m$ converges weakly to the corresponding restriction of the measure $\mathrm{mes}_{\,n+m}$ as $\varepsilon \to +0$.*

Proof. From the definition of the measure μ_ε, for any function $\varphi : K \times \mathbb{R}^m \to \mathbb{R}$ such that $\varphi(x,\, y)$ is unit periodic in the m coordinates of $y \in \mathbb{R}^m$ for any fixed $x \in X$,

$$\varepsilon^n \sum_{x\in K\bigcap X_\varepsilon} \varphi(x,\, \varepsilon^{-1}F(x)) = \int_{K\times[-1/2,1/2)^m} \varphi(z)\mu_\varepsilon(dz)\,.$$

Hence, by Lemma 7, if φ is continuous,

$$\lim_{\varepsilon\to+0} \int_{K\times[-1/2,1/2)^m} \varphi(z)\mu_\varepsilon(dz) = \int_{K\times[0,1)^m} \varphi(z)dz = \int_{K\times[-1/2,1/2)^m} \varphi(z)dz\,. \tag{53}$$

Since φ is otherwise arbitrary, (53) immediately implies the theorem. $\blacksquare$

9. Proof of the Main Theorem

For every $N \in \mathbb{N}$ and $\varepsilon > 0$, define a countably additive measure $\lambda_{\varepsilon,N}$ on the Borel sets $B \subset X \times V^N$ by

$$\lambda_{\varepsilon,N}(B) = \varepsilon^n \#\{x \in X_\varepsilon : (x,\, \mathcal{E}_{\varepsilon,N}(x)) \in B\}\,. \tag{54}$$

Clearly, it suffices to show that the restriction of the measure $\lambda_{\varepsilon,N}$ to the set $K \times V^N$ converges weakly to the corresponding restriction of $\mathrm{mes}_{\,(N+1)n}$ for any compact set $K \subset X$. To do this, define another countably additive measure $\mu_{\varepsilon,N}$ on the Borel sets $B \subset X \times V^N$ by

$$\mu_{\varepsilon,N}(B) = \varepsilon^n \#\{x \in X_\varepsilon : (x,\, P_{Nn}(\varepsilon^{-1}F_N(x))) \in B\}\,, \tag{55}$$

recalling the notations (16) and (52). Using the conditionally periodic representation of the quantization errors of Lemmas 2 and 3,

$$\lambda_{\varepsilon,N} = \mu_{\varepsilon,N} \circ S_{\varepsilon,N}^{-1}\,, \tag{56}$$

where the mapping $S_{\varepsilon,N} : X \times V^N \to X \times V^N$ is given by

$$S_{\varepsilon,N}(x,\, y) = (x,\, H_{\varepsilon,N}(x,\, y))$$

where $H_{\varepsilon,N}$ is as in (26), (27). From Lemma 4,

$$\lim_{\varepsilon\to+0,w\to u} S_{\varepsilon,N}(w) = S_N(u)$$

$$\text{for mes}_{(N+1)n}\text{-almost all } u \in X \times V^N\,, \quad (57)$$

where $S_N : X \times V^N \to X \times V^N$ is

$$S_N(x,\, y) = (x,\, H_N(x,\, y))\,. \quad (58)$$

Note that the mapping S_N is a $\text{mes}_{(N+1)n}$-preserving bijection of the set $X \times V^N$ onto itself. Now since by assumption the resonance set (17) of the transition operator f has zero n-dimensional Lebesgue measure, the mapping F_N satisfies the condition $(*)$ of Lemma 5. Hence, by Theorem 2 the restriction of the measure $\mu_{\varepsilon,N}$ to $K \times V^N$ given by (55) weakly converges to the corresponding restriction of $\text{mes}_{(N+1)n}$ for any compact set $K \subset X$,

$$\mu_{\varepsilon,N} \Longrightarrow \text{mes}_{(N+1)n} \quad \text{as } \varepsilon \to +0\,. \quad (59)$$

From (56), by [Billingsley, 1968, Theorem 5.5], the relations (57) and (59) imply the weak convergence of the measure (54),

$$\lambda_{\varepsilon,N} \Longrightarrow \text{mes}_{(N+1)n} \circ S_N^{-1} \quad \text{as } \varepsilon \to +0\,.$$

The proof of the theorem now follows from the $\text{mes}_{(N+1)n}$-preserving property of the mapping (58). ■

Acknowledgments

The authors thank V. Kozyakin and A. Pokrovskii for helpful discussions which significantly improved the first version of the paper. The work is supported by the Australian Research Council Grant A 4970 2246.

References

Billingsley, P. [1968] *Convergence of Probability Measures* (John Wiley & Sons, NY).

Blank, M. [1997] *Discreteness and Continuity in Problems of Chaotic Dynamics* (American Mathematical Society, Providence, Rhode Island).

Cornfeld, I. P., Fomin, S. V. & Sinai, Ya. G. [1982] *Ergodic Theory* (Springer-Verlag, NY).

Diamond, P., Kloeden, P., Pokrovskii, A. & Vladimirov, A. [1995a] "Collapsing effects in numerical simulation of a class of chaotic dynamical systems and random mappings with a single attracting centre," *Physica* **D86**, 559–571.

Diamond, P., Kloeden, P. & Pokrovskii, A. [1995b] "Interval stochastic matrices, a combinatorial lemma and the computation of invariant measures," *J. Dyn. Diff. Eqns.* **7**, 341–364.

Diamond, P. & Pokrovskii, A. [1996] "Statistical laws for computational collapse of discretized chaotic mappings," *Int. J. Bifurcation and Chaos* **6**, 2389–2399.

Diamond, P., Kloeden, P., Klemm, A. & Pokrovskii, A. [1996a] "Basin of attraction of measurable systems and their discretizations," *J. Stat. Phys.* **84**, 713–733.

Diamond, P., Kloeden, P., Pokrovskii, A. & Suzuki, M. [1996b] "Statistical properties of discretizations of a class of chaotic dynamical systems," *Comput. Math. Appl.* **31**, 83–95.

Diamond, P., Kloeden, P., Kozyakin, V. & Pokrovskii, A. [1998a] "Monotone dynamical systems under spatial discretization," *Proc. Amer. Math. Soc.* **126**, 2169–2174.

Diamond, P., Kloeden, P., Kozyakin, V. & Pokrovskii, A. [1998b] "Boundedness and dissipativity of truncated rotations on uniform planar lattices," *Math. Nachr.* **191**, 59–81.

Grebogi, E., Ott, E. & Yorke, J. A. [1988] "Roundoff–induced periodicity and the correlation dimension of chaotic attractors," *Phys. Rev.* **A34**, 3688–3692.

Izmailov, R. & Pokrovskii, A. [1992] "Asymptotic analysis of aliasing structures," *J. Appl. Math. Stochastic Anal.* **5**, 193–204.

Izmailov, R., Pokrovskii, A. & Vladimirov, A. [1996] "Visualization of polynomials," *Comput. Graph.* **20**, 95–105.

Kuznetsov, N. & Kloeden, P. [1997] "The problem of information stability in computer studies of continuous systems," *Math. Comput. Sim.* **43**, 143–158.

Martin, N. & England, J. [1984] *Mathematical Theory of Entropy* (Cambridge University Press, London).

Vladimirov, I. [1996] "Quantized linear systems on integer lattices: Frequency-based approach," Parts I, II, Center for Applied Dynamical Systems and Environmental Modeling, CADSEM Reports 96-032, 96-033, Geelong, Vic. Australia.