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ABSTRACT

In this paper, the notion of bisimulation relation for linear input-state-output systems is extended to general linear differential-algebraic (DAE) systems. Geometric control theory is used to derive a linear-algebraic characterisation of bisimulation relations, and an algorithm for computing the maximal bisimulation relation between two linear DAE systems. The general definition is specialised to the case where the matrix pencil $sE - A$ is regular. Furthermore, by developing a one-sided version of bisimulation, characterisations of simulation and abstraction are obtained.

1. Introduction

A fundamental concept in the broad area of systems theory, concurrent processes, and dynamical systems, is the notion of *equivalence*. In general, there are different ways to describe systems (or, processes); each with their own advantages and possibly disadvantages. This call for systematic ways to convert one representation into another, and for means to determine which system representations are 'equal'. It also involves the notion of *minimal* system representation.

Furthermore, in systems theory and the theory of concurrent processes, the emphasis is on determining which systems are *externally equivalent*; we only want to distinguish between systems if the distinction can be detected by an external system interacting with these systems. This is crucial in any modular approach to the control and design of complex systems.

Classical notions developed in systems and control theory for external equivalence are *transfer matrix equality* and *state space equivalence*. Within computer science the basic notion has been called *bisimulation relation* (Clarke, Grumberg, & Peled, 1999). An extension of the notion of bisimulation to continuous dynamical systems has been explored before in a series of innovative papers by Pappas and co-authors (Pappas, 2003; Tabuada & Pappas, 2004). More recently, motivated by the rise of hybrid and cyber-physical systems, a reapproachment of these notions stemming from different backgrounds has been initiated. In particular, it has been shown how for linear systems a notion of bisimulation relation can be developed mimicking the notion of bisimulation relation for transition systems, and directly extending classical notions of transfer matrix equality and state space equivalence (van der Schaft, 2004a). An important aspect of this approach in developing bisimulation theory for continuous linear systems is that the conditions for existence of a bisimulation relation are formulated directly in terms of the differential equation description, instead of the corresponding dynamical behaviour (the solution set of the differential equations). This has dramatic consequences for the complexity of bisimulation computations, which reduce to linear-algebraic computations on the matrices specifying the linear system descriptions, very much in the spirit of linear geometric control theory (Basile & Marro, 1992; Wonham, 1974). For extensions to nonlinear systems exploiting corresponding nonlinear geometric theory we refer to van der Schaft, (2004a).

The present paper continues on these developments by extending the notion of bisimulation relation to general linear *differential-algebraic* (DAE) systems involving disturbances (capturing non-determinism). This is well motivated since complex system descriptions usually arise from interconnection of system components, and generally lead to descriptions involving both differential equations and algebraic equations. Indeed, *network modelling* almost invariably leads to DAE systems. The aim of this paper is to determine linear-algebraic conditions for the existence of a bisimulation relation, directly in terms of the differential-algebraic equations instead of computing the solution trajectories. The extension with respect to van der Schaft, (2004a) (where the linear-algebraic

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conditions were derived in case of ordinary differential equation models) is non-trivial because of the following two reasons. First, since bisimulation is an equivalence between system trajectories we need to characterise the set of solution trajectories of DAE systems, involving the notion of the consistent set of initial conditions. This is fundamentally different from the scenario considered in van der Schaft,(2004a) where the solutions of the differential equations exist for arbitrary initial states. In fact, in this paper we use geometric control theory, see in particular (Trentelman, Stoorvogel, & Hautus, 2001), in order to explicitly describe the set of consistent states and the set of state trajectories. This appears to be a new contribution to the literature on DAE or descriptor systems (Armentano, 1986; Bernhard, 1982; Berger & Reis, 2013; Campbell, 1980; Dai, 1989; Karcanicas & Hayton, 1982; Lewis, 1986; Trenn, 2013). Second, the notion of bisimulation between state trajectories needs to be characterised in terms of the differential-algebraic equations, containing the conditions previously obtained in van der Schaft,(2004a) as a special case.

As in previous work on bisimulation theory for input-state-output systems (van der Schaft, 2004b), we explicitly allow for the possibility of ' *non-determinism*' in the sense that the state may evolve according to different time-trajectories for the same values of the external variables. This 'non-determinism' may be explicitly modelled by the presence of *internal 'disturbances'* or implicitly by non-uniqueness of the solutions of differential-algebraic equations. Non-determinism may be an intrinsic feature of the system representation (as due e.g. to non-uniqueness of variables in the internal subsystem interconnections), but may also arise by *abstraction* of the system to a lower dimensional system representation. By itself, the notion of abstraction can be covered by a one-way version of bisimulation, called *simulation*, as will be discussed in Section 5.

As a simple motivating example for the developments in this paper let us consider two DAE systems (for simplicity without inputs) given by

$$
\Sigma_1 : \begin{aligned} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} d_1, \\ y_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x_1, \end{aligned} \tag{1}
$$

$$
\Sigma_2 : \begin{aligned} \dot{x}_2 = x_2 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d_2, \\ y_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} x_2. \end{aligned}
$$

What is the relation between Σ_1 and Σ_2? Are the systems Σ_1 and Σ_2 equivalent? At the end of Section 3.1 we will provide an answer exemplifying some of the results that have been obtained.

The structure of this paper is as follows. In Section 2, we provide the theory concerning DAE systems which will be used in the sequel. These DAE systems are given in *descriptor system* format $E\dot{x} = Ax + Bu + Gd, y = Cx$, with u, y being the external variables (inputs and outputs), d the disturbances modelling internal non-determinism, and x the (not necessarily minimal) state. In Section 3, we give the definition of bisimulation relation for DAE systems, and a full linear-algebraic characterisation of them, together with a geometric algorithm to compute the maximal bisimulation relation between two linear systems. In Section 4, we study the implication of adding the condition of *regularity* to the matrix pencil $sE - A$, and show how in this case bisimilarity reduces to equality of transfer matrices. Finally, simulation relations and the accompanying notion of abstraction are discussed in Section 5.

2. Preliminaries on linear DAE systems

In this paper, we consider the following general class of linear DAE systems:

$$
\Sigma : \begin{aligned} E\dot{x} = Ax + Bu + Gd, \ x \in \mathcal{X}, u \in \mathcal{U}, d \in \mathcal{D} \\ y = Cx, \qquad\qquad\quad y \in \mathcal{Y}, \end{aligned} \tag{2}
$$

where $E, A \in \mathbb{R}^{q \times n}$ and $B \in \mathbb{R}^{q \times m}$, $G \in \mathbb{R}^{q \times s}, C \in \mathbb{R}^{p \times n}$; $\mathcal{X}, \mathcal{U}, \mathcal{D}$ and $\mathcal{Y}$ are finite dimensional linear spaces, of dimension, respectively, n, m, s, p . Here, x denotes the state of the system (possibly constrained by linear equations), u the input, y the output and d the 'disturbance' acting on the system. Furthermore, q denotes the total number of (differential and algebraic) equations describing the dynamics of the system. The allowed time-functions $x : \mathbb{R}^+ \to \mathcal{X}, u : \mathbb{R}^+ \to \mathcal{U}, y : \mathbb{R}^+ \to \mathcal{Y}, d : \mathbb{R}^+ \to \mathcal{D}$, with $\mathbb{R}^+ = [0, \infty)$, will be denoted by $\mathfrak{X}, \mathfrak{U}, \mathfrak{Y}, \mathfrak{D}$. The exact choice of function classes is for purposes of this paper not really important, as long as the state trajectories $x(\cdot)$ are at least continuous. For convenience, we will take $\mathfrak{U}, \mathfrak{D}$ to be the class of piecewise-continuous and $\mathfrak{X}, \mathfrak{Y}$ the class of continuous and piecewise-differentiable functions on $\mathbb{R}^+$. We will denote these functions by $x(\cdot), u(\cdot), y(\cdot), d(\cdot)$, and if no confusion can arise simply by x, u, y, d . We will primarily regard d as an internal generator of 'non-determinism': multiple state trajectories may occur for the same initial condition $x(0)$ and input function $u(\cdot)$. This, for example, occurs by *abstracting* a deterministic system; see the developments in Section 5.

The *consistent subset* $\mathcal{V}^*$ for a system Σ is given as the maximal subspace $\mathcal{V} \subset \mathbb{R}^n$ satisfying

$$
\begin{aligned} &(i) \ \ A\mathcal{V} \subset E\mathcal{V} + \mathcal{G} \\ &(ii) \ \text{im} \, B \subset E\mathcal{V} + \mathcal{G} \end{aligned} \tag{3}
$$

where $\mathcal{G} = \operatorname{im} G$, or is *empty* in case there does not exist any subspace $\mathcal{V}$ satisfying Equation (3). It follows that $\mathcal{V}^*$ equals the set of all initial conditions x_0 for which for every piecewise-continuous input function $u(\cdot)$ there exist a piecewise-continuous function $d(\cdot)$ and a continuous and piecewise-differentiable solution trajectory $x(\cdot)$ of Σ with $x(0) = x_0$.

Remark 2.1: The definition of consistent subset $\mathcal{V}^*$ as given above *extends* the standard definition given in the literature on linear DAE and descriptor systems (see e.g. Berger & Reis, 2013). In fact, the above definition reduces to the definition in Berger and Reis,(2013) for the case $B = 0$ when additionally *renaming* the disturbance d by u. (Thus in the standard definition the consistent subset is the set of initial conditions for which there exists an *input* function u and a corresponding solution of the DAE with $d = 0$.) This extended definition of consistent subset, as well as the change in terminology between u and d, is directly motivated by the notion of bisimulation where we wish to consider solutions of the system for *arbitrary* external input functions $u(\cdot)$; see also the definition of bisimulation for labelled transition systems (Clarke et al., 1999). Note that for $B = 0$ or void the zero subspace $\mathcal{V} = \{0\}$ always satisfies Equation (3), and thus $\mathcal{V}^*$ is a subspace. However for $B \neq 0$ there may *not* exist any subspace $\mathcal{V}$ satisfying Equation (3) in which case the consistent subset is *empty* (and thus strictly speaking not a subspace). In the latter case, such a system has empty input–output behaviour from a bisimulation point of view.

Remark 2.2: Note that we *can* accommodate for additional restrictions on the allowed values of the input functions u, depending on the initial state, by making use of the following standard construction, incorporating u into an extended state vector. Rewrite system (2) as

$$\Sigma_e : \quad \begin{aligned} [E \quad 0] \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} &= [A \quad B] \begin{bmatrix} x \\ u \end{bmatrix} + Gd \\ y &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \end{aligned} \quad (4)$$

Denote by $x_e = \begin{bmatrix} x \\ u \end{bmatrix}$ the *extended* state vector, and define $E_e := [E\ 0]$, $A_e := [A\ B]$. Then the consistent *subspace* $\mathcal{V}_e^*$ of system (4) is given by the maximal subspace $\mathcal{V}_e \subset \mathcal{X} \times \mathcal{U}$ satisfying

$$A_e \mathcal{V}_e \subset E_e \mathcal{V}_e + \mathcal{G} \quad (5)$$

It can be easily seen that $\mathcal{V}^* \subset \pi_x(\mathcal{V}_e^*)$, where π_x is the canonical projection of $\mathcal{X} \times \mathcal{U}$ on $\mathcal{X}$. The case $\mathcal{V}^* \subsetneq \pi_x(\mathcal{V}_e^*)$ corresponds to the presence of initial conditions

which are consistent only for input functions taking value in a strict *subspace* of $\mathcal{U}$.

In order to analyse the solutions of the linear DAE (2), an important observation is that we can always *eliminate* the disturbances d. Indeed, given Equation (2) we can construct matrices $G^\perp$, $G^\dagger$ and an $q \times q$ matrix P such that

$$G^\perp G = 0,\ G^\dagger G = I_s,\ P = \begin{bmatrix} G^\perp \\ G^\dagger \end{bmatrix},\ \operatorname{rank}(P) = q \quad (6)$$

($G^\perp$ is a left annihilator of G of maximal rank, and $G^\dagger$ is a left inverse of G.) By pre-multiplying both sides of Equation (2) by the invertible matrix P it follows (Karcanicas & Hayton, 1982) that system (2) is equivalent to

$$\begin{aligned} G^\perp E\dot{x} &= G^\perp Ax + G^\perp Bu \\ d &= G^\dagger (E\dot{x} - Ax - Bu) \\ y &= Cx \end{aligned} \quad (7)$$

Hence, the disturbance d is specified by the second line of Equation (7), and the solutions $u(\cdot)$, $x(\cdot)$ are determined by the first line of Equation (7) not involving d. We thus conclude that for the theoretical study of the state trajectories $x(\cdot)$ corresponding to input functions $u(\cdot)$ we can always, without loss of generality, restrict attention to linear DAE systems of the form:

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad (8)$$

On the other hand, for computational purposes it is usually not desirable to eliminate d, since this will often complicate the computations and result in loss of insight into the model.

The next important observation is that for theoretical analysis any linear DAE system (8) can be assumed to be in the following *special form*, again without loss of generality. Take invertible matrices $S \in \mathbb{R}^{q \times q}$ and $T \in \mathbb{R}^{n \times n}$ such that

$$SET = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad (9)$$

where the dimension n_a of the identity block I is equal to the rank of E. Split the transformed state vector $T^{-1}x$ correspondingly as $T^{-1}x = \begin{bmatrix} x^a \\ x^b \end{bmatrix}$, with $\dim x^a = n_a$, $\dim x^b = n_b$, $n_a + n_b = n$. It follows that by pre-multiplying the linear DAE (8) by S it transforms into an equivalent system (in the new state vector $T^{-1}x$) of the

form:

$$\begin{bmatrix} \dot{x}^a \\ 0 \end{bmatrix} = \begin{bmatrix} A^{aa} & A^{ab} \\ A^{ba} & A^{bb} \end{bmatrix} \begin{bmatrix} x^a \\ x^b \end{bmatrix} + \begin{bmatrix} B^a \\ B^b \end{bmatrix} u$$
$$y = \begin{bmatrix} C^a & C^b \end{bmatrix} \begin{bmatrix} x^a \\ x^b \end{bmatrix} \tag{10}$$

One of the advantages of the special form (10) is that the consistent subset $\mathcal{V}^*$ can be explicitly characterised using geometric control theory.

Proposition 2.1: *The set $\mathcal{V}^*$ of consistent states of Equation* (10) *is non-empty if and only if $B^b = 0$ and $\operatorname{im} B^a \subset \mathcal{W}(A^{aa}, A^{ab}, A^{ba})$, where $\mathcal{W}(A^{aa}, A^{ab}, A^{ba})$ denotes the maximal controlled invariant subspace of the auxiliary system*

$$\dot{x}^a = A^{aa}x^a + A^{ab}v$$
$$w = A^{ba}x^a \tag{11}$$

with state x^a, input v, and output w. Furthermore, in case $\mathcal{V}^*$ is non-empty it is given by the subspace

$$\mathcal{V}^* = \left\{ \begin{bmatrix} x^a \\ x^b \end{bmatrix} \mid x^a \in \mathcal{W}, x^b = Fx^a + z, \right.$$
$$\left. z \in \ker A^{bb} \cap (A^{ab})^{-1}\mathcal{W}(A^{aa}, A^{ab}, A^{ba}) \right\} \tag{12}$$

where $(A^{ab})^{-1}$ denotes set-theoretic inverse, and where the matrix F is a friend of $\mathcal{W}(A^{aa}, A^{ab}, A^{ba})$, i.e.

$$(A^{aa} + A^{ab}F)\mathcal{W}(A^{aa}, A^{ab}, A^{ba}) \subset \mathcal{W}(A^{aa}, A^{ab}, A^{ba}) \tag{13}$$

Proof: The first claim follows from the fact that the subset $\mathcal{V}^*$ of consistent states for Equation (8) is non-empty if and only if, see Equation (3), $\operatorname{im} B \subset E\mathcal{V}^*$. The characterisation of $\mathcal{V}^*$ given in Equation (12) follows from the characterisation of the maximal controlled invariant subspace of a linear system with feedthrough term as given, e.g. in Trentelman et al., (2001, Theorem 7.11). ∎

Remark 2.3: The characterisation of the consistent subspace $\mathcal{V}^*$ given in Equation (12), although being a direct consequence of geometric control theory, seems relatively unknown within the literature on DAE systems.

Remark 2.4: Usually, the maximal controlled invariant subspace is denoted by $\mathcal{V}^*(A^{aa}, A^{ab}, A^{ba})$ (see e.g. Trentelman, Stoorvogel, & Hautus, 2001). However, in order to distinguish it from the consistent subset $\mathcal{V}^*$ we have chosen the notation $\mathcal{W}(A^{aa}, A^{ab}, A^{ba})$. In the rest of the paper we will abbreviate this, if no confusion is possible, to $\mathcal{W}$.

Based on Proposition 2.1 we derive the following fundamental statement regarding solutions of linear DAE systems.

Theorem 2.1: *Consider the linear DAE system* (8), *with $\operatorname{im} B \subset E\mathcal{V}^*$. Then for all $u(\cdot) \in \mathfrak{U}$ continuous at $t = 0$ and for all $x_0 \in \mathcal{V}^*$ and $f \in \mathcal{V}^*$ satisfying*

$$Ef = Ax_0 + Bu(0) \tag{14}$$

there exists a continuous and piecewise-differentiable solution $x(\cdot)$ of Equation (8) *satisfying*

$$x(0) = x_0, \quad \dot{x}(0) = f. \tag{15}$$

Conversely, for all $u(\cdot) \in \mathfrak{U}$ every continuous and piecewise-differentiable solution $x(\cdot)$ of Equation (8) *which is differentiable at $t = 0$ defines by Equation* (15) *$x_0, f \in \mathcal{V}^*$ satisfying Equation* (14).

Proof: The last statement is trivial. Indeed, if $x(\cdot)$ is a differentiable solution of $E\dot{x} = Ax + Bu$ then $x(t) \in \mathcal{V}^*$ for all t, and thus $x(0) \in \mathcal{V}^*$ and by linearity $\dot{x}(0) \in \mathcal{V}^*$. Furthermore, $E\dot{x}(0) = Ax(0) + Bu(0)$.

For the first claim, take $u(\cdot) \in \mathfrak{U}$ and consider any $x_0, f \in \mathcal{V}^*$ satisfying Equation (14). As noted above we can assume that the system is in the form (10). Then by Equation (12)

$$x_0 = \begin{bmatrix} x_0^a \\ x_0^b \end{bmatrix}, \quad x_0^a \in \mathcal{W}, x_0^b = Fx_0^a + z_0,$$
$$z_0 \in \ker A^{bb} \cap (A^{ab})^{-1}\mathcal{W}$$
$$f = \begin{bmatrix} f^a \\ f^b \end{bmatrix}, \quad f^a \in \mathcal{W}, f^b = Ff^a + z_f, \tag{16}$$
$$z_f \in \ker A^{bb} \cap (A^{ab})^{-1}\mathcal{W}$$

Then consider the unique solution $x^a(\cdot)$ of

$$\dot{x}^a = A^{aa}x^a + A^{ab}(Fx^a + z) + B^a u, \quad x^a(0) = x_0^a \tag{17}$$

where the constant vector z is chosen such that

$$A^{aa}x_0^a + A^{ab}(Fx_0^a + z) + B^a u(0) = f^a. \tag{18}$$

Furthermore, define the time-function

$$x^b(t) = Fx^a(t) + z_0 + tz_f \tag{19}$$

Then by construction

$$x(0) = \begin{bmatrix} x^a(0) \\ x^b(0) \end{bmatrix} = \begin{bmatrix} x_0^a \\ Fx_0^a + z_0 \end{bmatrix} = x_0 \tag{20}$$

while

$$\begin{bmatrix} \dot{x}^a(0) \\ \dot{x}^b(0) \end{bmatrix} = \begin{bmatrix} A^{aa}x_0^a + A^{ab}(Fx_0^a + z) + B^a u(0) \\ F\dot{x}^a(0) + z_f \end{bmatrix}$$
$$= \begin{bmatrix} f^a \\ Ff^a + z_f \end{bmatrix} = \begin{bmatrix} f^a \\ f^b \end{bmatrix}.$$

∎

By recalling the equivalence between systems with disturbances (2) with systems without disturbances (8) we obtain the following corollary.

Corollary 2.1: *Consider the linear DAE system* (2)*, with* $\mathrm{im}\, B \subset EV^* + \mathcal{G}$*. Then for all* $u(\cdot) \in \mathfrak{U}, d(\cdot) \in \mathfrak{D}$*, continuous at* $t = 0$ *, and for all* $x_0 \in \mathcal{V}^*$ *and* $f \in \mathcal{V}^*$ *satisfying*

$$Ef = Ax_0 + Bu(0) + Gd(0) \qquad (21)$$

there exists a continuous and piecewise-differentiable solution $x(\cdot)$ *of Equation* (2) *satisfying*

$$x(0) = x_0, \quad \dot{x}(0) = f. \qquad (22)$$

Conversely, for all $u(\cdot) \in \mathfrak{U}, d(\cdot) \in \mathfrak{D}$ *every continuous and piecewise-differentiable solution* $x(\cdot)$ *of Equation* (2) *which is differentiable at* $t = 0$ *defines by Equation* (22) $x_0, f \in \mathcal{V}^*$ *satisfying Equation* (21).

3. Bisimulation relations for linear DAE systems

Now, let us consider two systems of the form (2)

$$\Sigma_i : \begin{array}{l} E_i \dot{x}_i = A_i x_i + B_i u_i + G_i d_i, \ x_i \in \mathcal{X}_i, u_i \in \mathcal{U}, d_i \in \mathcal{D}_i \\ y_i = C_i x_i, \qquad\qquad y_i \in \mathcal{Y}, \quad i = 1, 2. \end{array}$$
$$(23)$$

where $E_i, A_i \in \mathbb{R}^{q_i \times n_i}$ and $B_i \in \mathbb{R}^{q_i \times m}, G_i \in \mathbb{R}^{q_i \times s_i}, C_i \in \mathbb{R}^{p \times n_i}$ for $i = 1, 2$, with $\mathcal{X}_i, \mathcal{D}_i, i = 1, 2$, the state space and disturbance spaces, and $\mathcal{U}, \mathcal{Y}$ the common input and output spaces. The fundamental definition of bisimulation relation is given as follows.

Definition 3.1: A subspace

$$\mathcal{R} \subset \mathcal{X}_1 \times \mathcal{X}_2,$$

with $\pi_i(\mathcal{R}) \subset \mathcal{V}_i^*$, where $\pi_i : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_i$ denote the canonical projections for $i = 1, 2$, is a *bisimulation relation* between two systems Σ_1 and Σ_1 with consistent subsets $\mathcal{V}_i^*, i = 1, 2$, if and only if for all pairs of initial conditions $(x_1, x_2) \in \mathcal{R}$ and any joint input function $u_1(\cdot) = u_2(\cdot) = u(\cdot) \in \mathfrak{U}$ the following properties hold:

(1) For every disturbance function $d_1(\cdot) \in \mathfrak{D}_1$ for which there exists a solution $x_1(\cdot)$ of Σ_1 (with $x_1(0) = x_1$), there exists a disturbance function

$d_2(\cdot) \in \mathfrak{D}_2$ such that the resulting solution trajectory $x_2(\cdot)$ of Σ_2 (with $x_2(0) = x_2$) satisfies

$$(x_1(t), x_2(t)) \in \mathcal{R}, \ t \geq 0, \qquad (24)$$

and conversely for every disturbance function $d_2(\cdot)$ for which there exists a solution $x_2(\cdot)$ of Σ_2 (with $x_2(0) = x_2$), there exists a disturbance function $d_1(\cdot)$ such that the resulting solution trajectory $x_1(\cdot)$ of Σ_1 (with $x_1(0) = x_1$) satisfies (24).

(2)

$$C_1 x_1 = C_2 x_2, \ \text{for all } (x_1, x_2) \in \mathcal{R}. \qquad (25)$$

Using the geometric notion of a *controlled invariant subspace* (Basile & Marro, 1992; Wonham, 1974), a linear-algebraic characterisation of a bisimulation relation $\mathcal{R}$ is given in the following proposition and subsequent theorem.

Proposition 3.1: *Consider two systems* Σ_i *as in Equation*(23)*, with consistent subsets* $\mathcal{V}_i^*, i = 1, 2$*. A subspace* $\mathcal{R} \subset \mathcal{X}_1 \times \mathcal{X}_2$ *satisfying* $\pi_i(\mathcal{R}) \subset \mathcal{V}_i^*, i = 1, 2$*, is a bisimulation relation between* Σ_1 *and* Σ_2 *if and only if for all* $(x_1, x_2) \in \mathcal{R}$ *and for all* $u \in \mathcal{U}$ *the following properties hold*:

(1) *For every* $d_1 \in \mathcal{D}_1$ *for which there exists* $f_1 \in \mathcal{V}_1^*$ *such that* $E_1 f_1 = A_1 x_1 + B_1 u + G_1 d_1$ *, there exists* $d_2 \in \mathcal{D}_2$ *for which there exists* $f_2 \in \mathcal{V}_2^*$ *such that* $E_2 f_2 = A_2 x_2 + B_2 u + G_2 d_2$ *while*

$$(f_1, f_2) \in \mathcal{R}, \qquad (26)$$

and conversely for every $d_2 \in \mathcal{D}_2$ *for which there exists* $f_2 \in \mathcal{V}_2^*$ *such that* $E_2 f_2 = A_2 x_2 + B_2 u + G_2 d_2$ *, there exists* $d_1 \in \mathcal{D}_1$ *for which there exists* $f_1 \in \mathcal{V}_1^*$ *such that* $E_1 f_1 = A_1 x_1 + B_1 u + G_1 d_1$ *while Equation*(26) *holds*.

(2)

$$C_1 x_1 = C_2 x_2. \quad \text{for all } (x_1, x_2) \in \mathcal{R} \qquad (27)$$

Proof: Properties (2) of Definition 3.1 and Proposition 3.1, cf. (25) and (27), are equal, so we only need to prove equivalence of Properties (1) of Definition 3.1 and Proposition 3.1.

In order to do this we will utilise the fact (as explained above) that the DAEs $E_i \dot{x}_i = A_i x_i + B_i u_i + G_i d_i, i = 1, 2$, can be transformed, see Equation(7), to DAEs of the form $E_i \dot{x}_i = A_i x_i + B_i u_i, i = 1, 2$, not containing disturbances. Hence, it is sufficient to prove equivalence of Properties

(1) of Definition 3.1 and Proposition 3.1 for systems Σ_1 and Σ_2 of the form (8). For clarity we will restate Property (1) in this simplified case briefly as follows:

Property (1) of Definition 3.1: For every solution $x_1(\cdot)$ of Σ_1 with $x_1(0) = x_1$ there exists a solution $x_2(\cdot)$ of Σ_2 with $x_2(0) = x_2$ such that Equation (24) holds, and conversely.

Property (1) of Proposition 3.1: For every $f_1 \in \mathcal{V}_1^*$ such that $E_1 f_1 = A_1 x_1 + B_1 u$ there exists $f_2 \in \mathcal{V}_2^*$ such that $E_2 f_2 = A_2 x_2 + B_2 u$ such that Equation (26) holds, and conversely.

'Only if part'. Take $u(\cdot) \in \mathfrak{U}$ and $(x_1, x_2) \in \mathcal{R}$, and let $f_1 \in \mathcal{V}_1^*$ be such that $E_1 f_1 = A_1 x_1 + B_1 u(0)$. According to Theorem 2.1, there exists a solution $x_1(\cdot)$ of Σ_1 such that $x_1(0) = x_1$ and $\dot{x}_1(0) = f_1$. Then, based on Property (1) of Definition 3.1, there exists a solution $x_2(\cdot)$ of Σ_2 with $x_2(0) = x_2$ such that Equation (24) holds. By differentiating $x_2(t)$ with respect to t and denoting $f_2 := \dot{x}_2(0)$, we obtain Equation (26). The same argument holds for the case where the indices 1 and 2 are interchanged.

'If part'. Let $(x_1, x_2) \in \mathcal{R}$, $u(\cdot) \in \mathfrak{U}$. Consider any solution $x_1(\cdot)$ of Σ_1 corresponding to $x_1(0) = x_1$. Transform systems Σ_1 and Σ_2 into the form (10). This means that $x_1(\cdot) = \begin{bmatrix} x_1^a(\cdot) \\ x_1^b(\cdot) \end{bmatrix}$, $t \geq 0$, is a solution to

$$\Sigma_1 : \begin{array}{l} \dot{x}_1^a(t) = (A_1^{aa} + A_1^{ab} F_1)x_1^a(t) + A_1^{ab} z_1(t) + B_1^a u(t), \\ \qquad x_1^a(t) \in \mathcal{W}_1 \\ x_1^b(t) = F_1 x_1^a(t) + z_1(t), \\ \qquad z_1(t) \in \ker A_1^{bb} \cap (A_1^{ab})^{-1}\mathcal{W}_1, \ t \geq 0 \end{array}$$
(28)

Equivalently, $x_1^a(\cdot)$, $t \geq 0$, is a solution to

$$\begin{array}{l} \dot{x}_1^a(t) = (A_1^{aa} + A_1^{ab} F_1)x_1^a(t) + A_1^{ab} z_1(t) + B_1^a u(t), \\ \qquad x_1^a(t) \in \mathcal{W}_1 \\ \dot{z}_1(t) = e_1(t), \quad z_1(t) \in \ker A_1^{bb} \cap (A_1^{ab})^{-1}\mathcal{W}_1, \end{array}$$
(29)

where $e_1(\cdot)$ is a disturbance function, while additionally $x_1^b(t) = F_1 x_1^a(t) + z_1(t)$, $t \geq 0$.

Similarly, the solutions $x_2(\cdot) = \begin{bmatrix} x_2^a(\cdot) \\ x_2^b(\cdot) \end{bmatrix}$, $t \geq 0$, of Σ_2 are generated as solutions $x_2^a(\cdot)$ of

$$\begin{array}{l} \dot{x}_2^a(t) = (A_2^{aa} + A_1^{ab} F_2)x_1^a(t) + A_2^{ab} z_1(t) + B_2^a u(t), \\ \qquad x_2^a(t) \in \mathcal{W}_2 \\ \dot{z}_2(t) = e_2(t), \quad z_2(t) \in \ker A_2^{bb} \cap (A_2^{ab})^{-1}\mathcal{W}_2, \end{array}$$
(30)

where $e_2(\cdot)$ is a disturbance function, while additionally $x_2^b(t) = F_2 x_2^a(t) + z_2(t)$, $t \geq 0$.

Now, the systems (29) and (30) with state vectors $\begin{bmatrix} x_1^a(t) \\ z_1(t) \end{bmatrix}$, respectively $\begin{bmatrix} x_2^a(t) \\ z_2(t) \end{bmatrix}$ are *ordinary* (no algebraic constraints) linear systems with disturbances e_1 and e_2 , to which the bisimulation theory of van der Schaft (2004a) for ordinary linear systems applies. In particular, given the solution $x_1^a(\cdot), z_1(\cdot)$, and corresponding 'disturbance' $e_1(\cdot)$ by Proposition 2.9 in van der Schaft (2004a), Property (1) in Proposition 3.1 implies that there exists a disturbance $e_2(\cdot)$ with $e_2(t) = e_2(x_1^a(t), z_1(t), x_2^a(t), z_2(t), e_1(t))$ such that the combined dynamics of (x_1^a, z_1) and (x_2^a, z_2) remain in $\mathcal{R}$. This implies Property (1) in Definition 3.1.

The same argument holds for the case where the indices 1 and 2 are interchanged. ∎

The next step in the linear-algebraic characterisation of bisimulation relations for linear DAE systems is provided in the following theorem.

Theorem 3.1: *A subspace $\mathcal{R} \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a bisimulation relation between Σ_1 and Σ_2 satisfying $\pi_i(\mathcal{R}) \subset \mathcal{V}_i^*$, $i = 1, 2$, if and only if*

(a) $\mathcal{R} + \begin{bmatrix} E_1^{-1}(\operatorname{im} G_1) \cap \mathcal{V}_1^* \\ 0 \end{bmatrix}$

$\qquad = \mathcal{R} + \begin{bmatrix} 0 \\ E_2^{-1}(\operatorname{im} G_2) \cap \mathcal{V}_2^* \end{bmatrix}$,

(b) $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \mathcal{R} \subset \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \mathcal{R} + \operatorname{im} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}$,   (31)

(c) $\operatorname{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \mathcal{R} + \operatorname{im} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}$,

(d) $\mathcal{R} \subset \ker [C_1 : -C_2]$.

Proof: *'If part'*. Condition (27) of Proposition 3.1 follows trivially from condition (31 d). From Equation (31 b,c) it follows that for every $(x_1, x_2) \in \mathcal{R}$ and $u \in \mathcal{U}$ there exist $(f_1, f_2) \in \mathcal{R}$, and $d_1 \in \mathcal{D}_1, d_2 \in \mathcal{D}_2$, such that

$$\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$
$$+ \begin{bmatrix} G_1 \\ 0 \end{bmatrix} d_1 + \begin{bmatrix} 0 \\ G_2 \end{bmatrix} d_2.$$
(32)

This implies $\pi_i(\mathcal{R}) \subset \mathcal{V}_i^*$, $i = 1, 2$.

Now let $(x_1, x_2) \in \mathcal{R}$ and $u \in \mathcal{U}$. Then as above, by Equation (31 b,c), there exist $(f_1, f_2) \in \mathcal{R}$, and $d_1 \in \mathcal{D}_1, d_2 \in \mathcal{D}_2$ such that Equation (32) holds. Now consider any $f_1' \in \mathcal{V}_1^*$ and $d_1' \in \mathcal{D}_1$ such that $E_1 f_1' = A_1 x_1 + B_1 u + G_1 d_1'$. Then $f_1' = f_1 + v_1$ for some $v_1 \in E_1^{-1}(\operatorname{im} G_1) \cap \mathcal{V}_1^*$. Hence by Equation (31 a) there exist $v_2 \in E_2^{-1}(\operatorname{im} G_2) \cap \mathcal{V}_2^*$ and $(f_1'', f_2'') \in \mathcal{R}$ such that

$$\begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1'' \\ f_2'' \end{bmatrix} - \begin{bmatrix} 0 \\ v_2 \end{bmatrix}$$

with $E_2 v_2 = G_2 d_2''$ for some $d_2'' \in \mathcal{D}_2$. Therefore,

$$\begin{bmatrix} f_1' \\ f_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} f_1'' \\ f_2'' \end{bmatrix} - \begin{bmatrix} 0 \\ v_2 \end{bmatrix}$$
$$= \begin{bmatrix} f_1' \\ f_2' \end{bmatrix} - \begin{bmatrix} 0 \\ v_2 \end{bmatrix},$$

with $f_2' := f_2 + f_2''$. Clearly $(f_1', f_2') \in \mathcal{R}$. It follows that

$$E_2 f_2' = E_2 f_2 + E_2 v_2 = A_2 x_2 + B_2 u + G_2 d_2',$$

with $d_2' := d_2 + d_2''$. Similarly, for every $f_2' \in \mathcal{V}_2^*$ and $d_2' \in \mathcal{D}_2$ such that $E_2 f_2' = A_2 x_2 + B_2 u + G_2 d_2'$ there exist $f_1' \in \mathcal{V}_1^*$ with $(f_1', f_2') \in \mathcal{R}$, while $E_1 f_1' = A_1 x_1 + B_1 u + G_1 d_1'$ for some $d_1' := d_1 + d_1''$. Hence, we have shown Property (1) of Proposition 3.1.

   'Only if part'. Property (2) of Proposition 3.1 is trivially equivalent with Equation (31 d). Since $\pi_i(\mathcal{R}) \subset \mathcal{V}_i^*$ for $i = 1, 2$ we have

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \mathcal{R} \subset \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \mathcal{R} + \text{im} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \quad (33)$$

and

$$\text{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \mathcal{R} + \text{im} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}. \quad (34)$$

Furthermore, since Property (1) of Proposition 3.1 holds, by taking $(x_1, x_2) = (0, 0)$ and $u = 0$, then for every d_1 for which there exists $f_1 \in \mathcal{V}_1^*$ such that $E_1 f_1 = G_1 d_1$, there exists d_2 and $f_2 \in \mathcal{V}_2^*$ such that $E_2 f_2 = G_2 d_2$, while $(f_1, f_2) \in \mathcal{R}$. Hence

$$\begin{bmatrix} f_1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \begin{bmatrix} 0 \\ f_2 \end{bmatrix} \in \mathcal{R} + \begin{bmatrix} 0 \\ E_2^{-1}(\text{im } G_2) \cap \mathcal{V}_2^* \end{bmatrix}, \quad (35)$$

and thus

$$\begin{bmatrix} E_1^{-1}(\text{im } G_1) \cap \mathcal{V}_1^* \\ 0 \end{bmatrix} \subset \mathcal{R} + \begin{bmatrix} 0 \\ E_2^{-1}(\text{im } G_2) \cap \mathcal{V}_2^* \end{bmatrix}. \quad (36)$$

Similarly, one obtains

$$\begin{bmatrix} 0 \\ E_2^{-1}(\text{im } G_2) \cap \mathcal{V}_2^* \end{bmatrix} \subset \mathcal{R} + \begin{bmatrix} E_1^{-1}(\text{im } G_1) \cap \mathcal{V}_1^* \\ 0 \end{bmatrix}. \quad (37)$$

Combining Equations (36) and (37) implies condition (31a). ∎

Remark 3.1: In the special case E_i, $i = 1, 2$, equal to the identity matrix, it follows that $\mathcal{V}_i^* = \mathcal{X}_i$, $i = 1, 2$, and

Equation (31) reduces to

$$(a) \ \mathcal{R} + \begin{bmatrix} \text{im } G_1 \\ 0 \end{bmatrix} = \mathcal{R} + \begin{bmatrix} 0 \\ \text{im } G_2 \end{bmatrix} =: \mathcal{R}_e,$$

$$(b) \ \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \mathcal{R} \subset \mathcal{R} + \text{im} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix},$$

$$(c) \ \text{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset \mathcal{R} + \text{im} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix},$$

$$(d) \ \mathcal{R} \subset \ker [C_1 \vdots - C_2].$$

(38)

Hence in this case Theorem 3.1 reduces to van der Schaft (2004a, Theorem 2.10).

3.1 Computing the maximal bisimulation relation

The *maximal* bisimulation relation between two DAE systems, denoted $\mathcal{R}^{\text{max}}$, can be computed, whenever it exists, in the following way, similarly to the well-known algorithm (Basile & Marro, 1992; Wonham, 1974) from geometric control theory to compute the *maximal controlled invariant subspace*. For notational convenience define

$$E^\times := \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \quad A^\times := \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

$$C^\times := [C_1 \vdots - C_2], \quad \bar{G}^\times := \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix},$$

$$\mathcal{G}_1^\times := \begin{bmatrix} E_1^{-1}(\text{im } G_1) \cap \mathcal{V}_1^* \\ 0 \end{bmatrix},$$

$$\mathcal{G}_2^\times := \begin{bmatrix} 0 \\ E_2^{-1}(\text{im } G_2) \cap \mathcal{V}_2^* \end{bmatrix}. \quad (39)$$

Algorithm 3.1: Given two systems Σ_1 and Σ_2. Define the following sequence $\mathcal{R}^j$, $j = 0, 1, 2, \ldots$, of subsets of $\mathcal{X}_1 \times \mathcal{X}_2$

$$\mathcal{R}^0 = \mathcal{X}_1 \times \mathcal{X}_2,$$
$$\mathcal{R}^1 = \{z \in \mathcal{R}^0 \mid z \in \ker C^\times, \mathcal{R}^1 + \mathcal{G}_1^\times = \mathcal{R}^1 + \mathcal{G}_2^\times\},$$
$$\mathcal{R}^2 = \{z \in \mathcal{R}^1 \mid A^\times z \subset E^\times \mathcal{R}^1 + \text{im } \bar{G}^\times,$$
$$\mathcal{R}^2 + \mathcal{G}_1^\times = \mathcal{R}^2 + \mathcal{G}_2^\times\},$$
$$\vdots$$
$$\mathcal{R}^j = \{z \in \mathcal{R}^{j-1} \mid A^\times z + \subset E^\times \mathcal{R}^{j-1} + \text{im } \bar{G}^\times,$$
$$\mathcal{R}^j + \mathcal{G}_1^\times = \mathcal{R}^j + \mathcal{G}_2^\times\}.$$

(40)

Proposition 3.2: *The sequence $\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^j, \ldots$ satisfies the following properties.*

(1) $\mathcal{R}^j$, $j \neq 0$, *is a linear space or empty. Furthermore,* $\mathcal{R}^0 \supset \mathcal{R}^1 \supset \mathcal{R}^2 \supset \cdots \supset \mathcal{R}^j \supset \mathcal{R}^{j+1} \supset \cdots$.

(2) *There exists a finite k such that $\mathcal{R}^k = \mathcal{R}^{k+1} =: \mathcal{R}^*$, and then $\mathcal{R}^j = \mathcal{R}^*$ for all $j \neq k$.*

(3) $\mathcal{R}^*$ *is either empty or equals the maximal subspace of* $\mathcal{X}_1 \times \mathcal{X}_2$ *satisfying the properties*

$$(i) \quad \mathcal{R}^* + \begin{bmatrix} E_1^{-1}(\operatorname{im} G_1) \cap \mathcal{V}_1^* \\ 0 \end{bmatrix}$$

$$= \mathcal{R}^* + \begin{bmatrix} 0 \\ E_2^{-1}(\operatorname{im} G_2) \cap \mathcal{V}_2^* \end{bmatrix},$$

$$(ii) \quad \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \mathcal{R}^* \subset \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \mathcal{R}^* \qquad (41)$$

$$+ \operatorname{im} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix},$$

$$(iii) \quad \mathcal{R}^* \subset \ker \begin{bmatrix} C_1 \vdots - C_2 \end{bmatrix}.$$

Proof: Analogous to the proof of van der Schaft (2004a, Theorem 3.4). ∎

If $\mathcal{R}^*$ as obtained from Algorithm 3.1 is non-empty and satisfies condition (31 c) in Theorem 3.1, then it follows that $\mathcal{R}^*$ is the *maximal bisimulation relation* $\mathcal{R}^{\max}$ between Σ_1 and Σ_2, while if $\mathcal{R}^*$ is empty or does not satisfy condition (31 c) in Theorem 3.1 then there does not exist any bisimulation relation between Σ_1 and Σ_2.

Furthermore, two systems are called *bisimilar* if there exists a bisimulation relation relating *all* states. This is formalised in the following definition and corollary.

Definition 3.2: Two systems Σ_1 and Σ_2 as in Equation (23) are *bisimilar*, denoted $\Sigma_1 \sim \Sigma_2$, if there exists a bisimulation relation $\mathcal{R} \subset \mathcal{X}_1 \times \mathcal{X}_2$ with the property that

$$\pi_1(\mathcal{R}) = \mathcal{V}_1^*, \ \pi_2(\mathcal{R}) = \mathcal{V}_2^*, \qquad (42)$$

where $\mathcal{V}_i^*$ is the consistent subset of Σ_i, $i = 1, 2$.

Corollary 3.1: Σ_1 *and* Σ_2 *are bisimilar if and only if* $\mathcal{R}^*$ *is non-empty and satisfies condition* (31 c) *in Theorem* 3.1 *and equation* (42).

Bisimilarity is implying the *equality of external behavior*. Consider two systems Σ_i, $i = 1, 2$, as in Equation (23), with external behavior $\mathcal{B}_i$ defined as

$$\mathcal{B}_i := \{(u_i(\cdot), y_i(\cdot)) \mid \exists x_i(\cdot),$$
$$d_i(\cdot) \text{ such that (23) is satisfied}\}. \qquad (43)$$

Analogously to van der Schaft (2004a) we have the following result.

Proposition 3.3: *Let* Σ_i, $i = 1, 2$, *be bisimilar. Then their external behaviors* $\mathcal{B}_i$ *are equal.*

However, due to the possible non-determinism introduced by the matrices G and E in Equation (2), two systems of the form (2) may have the same external behavior

while not being bisimilar. This is already illustrated in van der Schaft (2004a) for the case $E = I$.

Example 3.1: Recall the example given in the Introduction, cf. (1). The maximal bisimulation relation between Σ_1 and Σ_2 can be computed as the one-dimensional subspace $\mathcal{R}$ given by

$$\mathcal{R} = \operatorname{span} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T. \qquad (44)$$

Since $\mathcal{V}_1^* = \operatorname{span} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ every trajectory of Σ_1 is simulated by a trajectory of Σ_2. However, since $\mathcal{V}_2^* = \mathbb{R}^2$ the two systems are not bisimilar.

3.2 Bisimulation relation for deterministic case

In this section, we specialise the results to DAE systems *without* disturbances d. Consider two systems of the form

$$\Sigma_i : \begin{array}{ll} E_i \dot{x}_i = A_i x_i + B_i u_i, & x_i \in \mathcal{X}_i, u_i \in \mathcal{U}, \\ y_i = C_i x_i, & y_i \in \mathcal{Y}, i = 1, 2, \end{array} \qquad (45)$$

where $E_i, A_i \in \mathbb{R}^{q_i \times n_i}$ and $B_i \in \mathbb{R}^{q_i \times m}$, $C_i \in \mathbb{R}^{p \times n_i}$ for $i = 1, 2$. Theorem 3.1 can be specialised as follows.

Corollary 3.2: *A subspace* $\mathcal{R} \subset \mathcal{X}_1 \times \mathcal{X}_2$ *is a bisimulation relation between* Σ_1 *and* Σ_2 *given by Equation* (45), *satisfying* $\pi_i(\mathcal{R}) \subset \mathcal{V}_i^*$, $i = 1, 2$, *if and only if*

$$(a) \ \mathcal{R} + \begin{bmatrix} \ker E_1 \cap \mathcal{V}_1^* \\ 0 \end{bmatrix} = \mathcal{R} + \begin{bmatrix} 0 \\ \ker E_2 \cap \mathcal{V}_2^* \end{bmatrix},$$

$$(b) \ \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \mathcal{R} \subset \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \mathcal{R},$$

$$(c) \ \operatorname{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \mathcal{R}, \qquad (46)$$

$$(d) \ \mathcal{R} \subset \ker \begin{bmatrix} C_1 \vdots - C_2 \end{bmatrix}.$$

Corollary 3.2 can be applied to the following situation considered in van der Schaft (2004a). Consider two linear systems given by

$$\Sigma_i : \begin{array}{l} \dot{x}_i = A_i x_i + B_i u_i + G_i d_i, \\ y_i = C_i x_i. \end{array} \qquad (47)$$

By multiplying both sides of the first equation of (47) by an annihilating matrix $G_i^{\perp}$ of maximal rank one obtains the equivalent system representation without disturbances

$$G_i^{\perp} \dot{x}_i = G_i^{\perp} A_i x_i + G_i^{\perp} B_i u_i,$$
$$y_i = C_i x_i, \qquad (48)$$

which is of the general form (45); however, satisfying the special property $\mathcal{V}_i^* = \mathcal{X}_i$. This implies that $\mathcal{R}$ is a bisimulation relation between Σ_1 and Σ_2 given by Equation (47) *if and only if* it is a bisimulation relation between Σ_1 and Σ_2 given by Equation (48), as can be seen as follows. As already noted in Remark 2.6 a bisimulation relation between Σ_1 and Σ_2 as in Equation (47) is a subspace $\mathcal{R} \subset \mathcal{X}_1 \times \mathcal{X}_2$ satisfying Equation (38). Now let $\mathcal{R}$ satisfy Equation (38). We will show that it will satisfy Equation (46) for systems (48). First, since $\mathcal{V}_i = \mathcal{X}_i$ and $\ker E_i = \ker G_i^\perp = \operatorname{im} G_i$ we see that Equation (46 a) is satisfied. Furthermore, by pre-multiplying both sides of Equation (38 b,c) with

$$\begin{bmatrix} G_1^\perp & 0 \\ 0 & G_2^\perp \end{bmatrix}, \qquad (49)$$

we obtain

$$\begin{bmatrix} G_1^\perp A_1 & 0 \\ 0 & G_1^\perp A_2 \end{bmatrix} \mathcal{R} \subset \begin{bmatrix} G_1^\perp & 0 \\ 0 & G_2^\perp \end{bmatrix} \mathcal{R},$$

$$\operatorname{im} \begin{bmatrix} G_1^\perp B_1 \\ G_2^\perp B_2 \end{bmatrix} \subset \begin{bmatrix} G_1^\perp & 0 \\ 0 & G_2^\perp \end{bmatrix} \mathcal{R}, \qquad (50)$$

showing satisfaction of Equation (46 b,c). Conversely, let $\mathcal{R}$ be a bisimulation relation between Σ_1 and Σ_2 given by Equation (48), having consistent subsets $\mathcal{V}_i^* = \mathcal{X}_i$, $i = 1, 2$. Then according to Equation (46) it is satisfying

$$(a) \ \mathcal{R} + \begin{bmatrix} \ker G_1^\perp \\ 0 \end{bmatrix} = \mathcal{R} + \begin{bmatrix} 0 \\ \ker G_2^\perp \end{bmatrix},$$

$$(b) \ \begin{bmatrix} G_1^\perp A_1 & 0 \\ 0 & G_1^\perp A_2 \end{bmatrix} \subset \begin{bmatrix} G_1^\perp & 0 \\ 0 & G_2^\perp \end{bmatrix} \mathcal{R},$$

$$(c) \ \operatorname{im} \begin{bmatrix} G_1^\perp B_1 \\ G_2^\perp B_2 \end{bmatrix} \subset \begin{bmatrix} G_1^\perp & 0 \\ 0 & G_2^\perp \end{bmatrix} \mathcal{R}, \qquad (51)$$

$$(d) \ \mathcal{R} \subset \ker \begin{bmatrix} C_1 \vdots -C_2 \end{bmatrix}.$$

Using again $\operatorname{im} G_i = \ker G_i^\perp$ it immediately follows that $\mathcal{R}$ is satisfying Equation (38), and thus is a bisimulation relation between the systems (47).

4. Bisimulation relations for regular DAE systems

In this section, we will specialise the notion of bisimulation relation for general DAE systems of the form (2) to *regular* DAE systems. Regularity is usually defined for DAE systems *without disturbances*

$$\Sigma : \begin{array}{l} E\dot{x} = Ax + Bu, \ x \in \mathcal{X}, \ u \in \mathcal{U} \\ y = Cx, \qquad y \in \mathcal{Y}, \end{array} \qquad (52)$$

Hence, the consistent subset $\mathcal{V}^*$ is either empty or equal to the maximal subspace $\mathcal{V} \subset \mathcal{X}$ satisfying $A\mathcal{V} \subset E\mathcal{V}$, $\operatorname{im} B \subset E\mathcal{V}$.

Definition 4.1: The matrix pencil $sE - A$ is called *regular* if the polynomial $\det(sE - A)$ in $s \in \mathbb{C}$ is *not* identically zero. The corresponding DAE system (52) is called regular whenever the pencil $sE - A$ is regular.

Define additionally $\mathcal{V}_0^*$ as the maximal subspace $\mathcal{V} \subset \mathcal{X}$ satisfying $A\mathcal{V} \subset E\mathcal{V}$. (Note that *if* there exists a subspace $\mathcal{V}$ satisfying $A\mathcal{V} \subset E\mathcal{V}$, $\operatorname{im} B \subset E\mathcal{V}$ then $\mathcal{V}_0^* = \mathcal{V}^*$.) Then (Armentano, 1986)

Theorem 4.1: *Consider Equation* (52). *The following statements are equivalent:*

(1) *$sE - A$ is a regular pencil,*
(2) *$\mathcal{V}_0^* \cap \ker E = 0$.*

Regularity thus means uniqueness of solutions from any initial condition in the consistent subset $\mathcal{V}^*$ of Equation (52). We immediately obtain the following consequence of Corollary 3.2.

Corollary 4.1: *A subspace $\mathcal{R} \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a bisimulation relation between Σ_1 and Σ_2 satisfying $\pi_i(\mathcal{R}) \subset \mathcal{V}_i^*$, $i = 1, 2$, if and only if*

$$(a) \ \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \mathcal{R} \subset \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \mathcal{R},$$

$$(b) \ \operatorname{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \mathcal{R}, \qquad (53)$$

$$(c) \ \mathcal{R} \subset \ker \begin{bmatrix} C_1 \vdots -C_2 \end{bmatrix}.$$

In the regular case, the existence of a bisimulation relation can be characterised in terms of *transfer matrices*.

Theorem 4.2: *Let $\mathcal{R}$ be a bisimulation relation between regular systems Σ_1 and Σ_2 given in Equation* (45)*, then their transfer matrices $G_i(s) := C_i(sE_i - A_i)^{-1}B_i$ for $i = 1, 2$ are equal.*

Proof: Let $\mathcal{R}$ be a bisimulation relation between Σ_1 and Σ_2 thus it is satisfying Equation (53). According to Equations (53 a) and (53 b), for $(x_1, x_2) \in \mathcal{R}$ and $u \in \mathcal{U}$, there exist $(\dot{x}_1, \dot{x}_2) \in \mathcal{R}$ such that

$$\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u. \qquad (54)$$

Taking the *Laplace* transform of Equation (54), we have

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} (sE_1 - A_1)^{-1}B_1 \\ (sE_2 - A_2)^{-1}B_2 \end{bmatrix}. \qquad (55)$$

Since Equation (53 c) holds and taking *Laplace* transform, we have

$$C_1(sE_1 - A_1)^{-1}B_1 = C_2(sE_2 - A_2)^{-1}B_2. \qquad (56)$$

∎

The converse statement holds provided the matrices E_i are *invertible*.

Theorem 4.3: *Assume E_i, $i = 1, 2$, is invertible. Then there exists a bisimulation relation $\mathcal{R}$ between Σ_1 and Σ_2 if and only if their transfer matrices $G_i(s) := C_i(sE_i - A_i)^{-1}B_i$ for $i = 1, 2$ are equal.*

Proof: Let $G_1(s) = G_2(s)$. Then

$$\mathcal{R} := \text{im} \begin{bmatrix} E_1^{-1}B_1 & E_1^{-1}A_1E_1^{-1}B_1 & (E_1^{-1}A_1)^2E_1^{-1}B_1 & \cdots \\ E_2^{-1}B_2 & E_2^{-1}A_2E_2^{-1}B_2 & (E_2^{-1}A_2)^2E_2^{-1}B_2 & \cdots \end{bmatrix} \qquad (57)$$

satisfies Equation (53).

∎

The following example shows that Theorem 4.3 does not hold if E_i is not invertible.

Example 4.1: Consider two systems, given by

$$\Sigma_1 : \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1,$$
$$y_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} x_1,$$

$$\Sigma_2 : \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \dot{x}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_2,$$
$$y_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} x_2.$$

Systems Σ_1 and Σ_2 are regular and their transfer matrices are equal. However, there does not exist any bisimulation relation $\mathcal{R}$ satisfying Equation (53), since in fact the consistent subsets for both system are empty.

5. Simulation relations and abstractions

In this section, we will define a one-sided version of the notion of bisimulation relation and bisimilarity.

Definition 5.1: A subspace

$$\mathcal{S} \subset \mathcal{X}_1 \times \mathcal{X}_2, \qquad (58)$$

with $\pi_i(\mathcal{S}) \subset \mathcal{V}_i^*$, for $i = 1, 2$, is a *simulation relation* of Σ_1 by Σ_2 with consistent subsets $\mathcal{V}_i^*$, $i = 1, 2$ if and only if for all pairs of initial conditions $(x_1, x_2) \in \mathcal{S}$ and any joint input function $u_1(\cdot) = u_2(\cdot) = u(\cdot) \in \mathfrak{U}$ the following properties hold:

(1) for every disturbance function $d_1(\cdot) \in \mathfrak{D}_1$ for which there exists a solution $x_1(\cdot)$ of Σ_1 (with

$x_1(0) = x_1)$, there exists a disturbance function $d_2(\cdot) \in \mathfrak{D}_2$ such that the resulting solution trajectory $x_2(\cdot)$ of Σ_2 (with $x_2(0) = x_2$) satisfies for all $t \geq 0$

$$(x_1(t), x_2(t)) \in \mathcal{S}, \qquad (59)$$

(2)

$$C_1x_1 = C_2x_2, \quad \text{for all } (x_1, x_2) \in \mathcal{S}. \qquad (60)$$

Σ_1 is *simulated* by Σ_2 if the simulation relation $\mathcal{S}$ satisfies $\pi_1(\mathcal{S}) = \mathcal{V}_1^*$.

The one-sided version of Theorem 3.1 is given as follows.

Proposition 5.1: *A subspace $\mathcal{S} \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 satisfying $\pi_i(\mathcal{S}) \subset \mathcal{V}_i^*$, for $i = 1, 2$ if and only if*

$$(a) \ \mathcal{S} + \begin{bmatrix} E_1^{-1}(\text{im } G_1) \cap \mathcal{V}_1^* \\ 0 \end{bmatrix} \subset \mathcal{S}$$
$$+ \begin{bmatrix} 0 \\ E_2^{-1}(\text{im } G_2) \cap \mathcal{V}_2^* \end{bmatrix},$$
$$(b) \ \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \mathcal{S} \subset \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \mathcal{S} + \text{im} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}, \qquad (61)$$
$$(c) \ \text{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \mathcal{S} + \text{im} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix},$$
$$(d) \ \mathcal{S} \subset \ker \begin{bmatrix} C_1 \vdots - C_2 \end{bmatrix}.$$

The maximal simulation relation $\mathcal{S}^{\max}$ can be computed by the following simplified version of Algorithm 3.1.

Algorithm 5.1: Given two dynamical systems Σ_1 and Σ_2. Define the following sequence $\mathcal{S}^j$, $j = 0, 1, 2, \ldots$, of subsets of $\mathcal{X}_1 \times \mathcal{X}_2$

$$\mathcal{S}^0 = \mathcal{X}_1 \times \mathcal{X}_2,$$
$$\mathcal{S}^1 = \{z \in \mathcal{S}^0 | z \in \ker C^\times, \mathcal{S}^1 + \mathcal{G}_1^\times \subset \mathcal{S}^1 + \mathcal{G}_2^\times\}$$
$$\mathcal{S}^2 = \{z \in \mathcal{S}^1 | A^\times z + \subset E^\times \mathcal{S}^1 + \text{im } \bar{G}^\times,$$
$$\mathcal{S}^2 + \mathcal{G}_1^\times \subset \mathcal{S}^2 + \mathcal{G}_2^\times\},$$
$$\vdots$$
$$\mathcal{S}^j = \{z \in \mathcal{S}^{j-1} | A^\times z + \subset E^\times \mathcal{S}^{j-1} + \text{im } \bar{G}^\times,$$
$$\mathcal{S}^j + \mathcal{G}_1^\times \subset \mathcal{S}^j + \mathcal{G}_2^\times\}. \qquad (62)$$

Recall the definition of the inverse relation $\mathcal{T}^{-1} := \{(x_a, x_b) \mid (x_b, x_a) \in \mathcal{T}\}$. We have the following facts.

Proposition 5.2: *Let $\mathcal{S} \subset \mathcal{X}_1 \times \mathcal{X}_2$ be a simulation relation of Σ_1 by Σ_2 and let $\mathcal{T} \subset \mathcal{X}_2 \times \mathcal{X}_1$ be a simulation relation of Σ_2 by Σ_1. Then $\mathcal{R} := \mathcal{S} + \mathcal{T}^{-1}$ is a bisimulation relation between Σ_1 and Σ_2.*

Proof: Let $\mathcal{S}$ satisfy Equation (61) and let $\mathcal{T}$ satisfy Equation (61) with index 1 replaced by 2. Define $\mathcal{R} = \mathcal{S} + \mathcal{T}^{-1}$, then we have properties (31 a). Similarly, $\mathcal{R}$ satisfies (31 b,c,d). ∎

Proposition 5.3: *Suppose there exists a simulation of Σ_1 by Σ_2, and a simulation of Σ_2 by Σ_1. Let $\mathcal{S}^{max} \subset \mathcal{X}_1 \times \mathcal{X}_2$ denote the maximal simulation relation of Σ_1 by Σ_2, and $\mathcal{T}^{max} \subset \mathcal{X}_2 \times \mathcal{X}_1$ the maximal simulation relation of Σ_2 by Σ_1. Then $\mathcal{S}^{max} = (\mathcal{T}^{max})^{-1} = \mathcal{R}^{max}$, with $\mathcal{R}^{max}$ the maximal bisimulation relation.*

Proof: Analogous to the proof of van der Schaft (2004a, Proposition 5.4). ∎

Simulation relations appear naturally in the context of *abstractions* (see e.g. Pappas, 2003). Consider the DAE system

$$\Sigma : \begin{array}{l} E\dot{x} = Ax + Bu + Gd, \ x \in \mathcal{X}, u \in \mathcal{U}, d \in \mathcal{D}, \\ y = Cx, \qquad\qquad y \in \mathcal{Y}, \end{array} \quad (63)$$

together with a surjective linear map $H : \mathcal{X} \to \mathcal{Z}$, $\mathcal{Z}$ being another linear space, satisfying $\ker H \subset \ker C$. This implies that there exists a unique linear map $\bar{C} : \mathcal{Z} \to \mathcal{Y}$ such that

$$C = \bar{C}H. \quad (64)$$

Then define the following dynamical system on $\mathcal{Z}$

$$\Sigma : \begin{array}{l} \bar{E}\dot{z} = \bar{A}z + \bar{B}u + \bar{G}d, \ z \in \mathcal{Z}, u \in \mathcal{U}, d \in \mathcal{D}, \\ y = \bar{C}z, \qquad\qquad y \in \mathcal{Y} \end{array} \quad (65)$$

where H^+ denotes the Moore–Penrose pseudo-inverse of H, $\bar{E} := EH^+$, $\bar{A} := AH^+$, $\bar{B} := B$, and

$$\bar{G} := [G \vdots E(\ker H) \vdots A(\ker H)],$$

is an *abstraction* of Σ in the sense that we factor out the part of the state variables $x \in \mathcal{X}$ corresponding to $\ker H$. Since $H^+z = x + \ker H$, it can be easily proved that $\mathcal{S} := \{(x, z) \mid z = Hx\}$ is a simulation relation of Σ by $\bar{\Sigma}$.

6. Conclusions

In this paper we have defined and studied by methods from geometric control theory the notion of bisimulation relation for general linear DAE systems, including the special case of DAE systems with regular matrix pencil. Also the one-sided notion of simulation relation

related to abstraction has been provided. Avenues for further research include the use of bisimulation relations for model reduction, the consideration of switched DAE systems, as well as the generalisation to nonlinear DAE systems.

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