Deciding What is Good-for-MDPs

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- Abstract

Nondeterministic good-for-MDPs (GFM) automata are for MDP model checking and reinforcement learning what good-for-games automata are for reactive synthesis: a more compact alternative to deterministic automata that displays nondeterminism, but only so much that it can be resolved locally, such that a syntactic product can be analysed. GFM has recently been introduced as a property for reinforcement learning, where the simpler Büchi acceptance conditions it allows to use is key. However, while there are classic and novel techniques to obtain automata that are GFM, there has not been a decision procedure for checking whether or not an automaton is GFM. We show that GFM-ness is decidable and provide an EXPTIME decision procedure as well as a PSPACE-hardness proof.

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1 Introduction

Omega-automata [21, 13] are formal acceptors of ω -regular properties, which often result from translating a formula from temporal logics like LTL [15], as a specification for desired model properties in quantitative model checking and strategy synthesis [3], and reinforcement learning [20].

Especially for reinforcement learning, having a simple Büchi acceptance mechanism has proven to be a breakthrough [9], which led to the definition of the "good-for-MDPs" property in [10]. Just like for good-for-games automata in strategy synthesis for strategic games [11], there is a certain degree of nondeterminism allowed when using a nondeterministic automaton on the syntactic product with an MDP to learn how to control it, or to apply quantitative model checking. Moreover, the degree of freedom available to control MDPs is higher than the degree of freedom for controlling games. In particular, this always allows for using nondeterministic automata with a Büchi acceptance condition, both when using the classically used suitable limit deterministic automata [22, 7, 8, 18, 9] and for alternative GFM automata like the slim automata from [10].

This raises the question of whether or not an automaton is good-for-MDPs. While [10] has introduced the concept, there is not yet a decision procedure for checking the GFM-ness of an automaton, let alone for the complexity of this test.

We will start by showing that the problem of deciding GFM-ness is PSPACE-hard by a reduction from the NFA universality problem [19]. We then define the auxiliary concept of

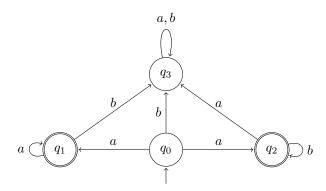


Figure 1 A nondeterministic Büchi word automaton over $\{a,b\}$. This NBW is complete and accepts the language $\{a^{\omega},ab^{\omega}\}$.

qualitative GFM, QGFM, which relaxes the requirements for GFM to qualitative properties, and develop an automata based EXPTIME decision procedure for QGFM. This decision procedure is constructive in that it can provide a counter-example for QGFM-ness when such a counter-example exists. We then use it to provide a decision procedure for GFM-ness that uses QGFM queries for all states of the candidate automaton. Finally, we show that the resulting criterion for GFM-ness is also a necessary criterion for QGFM-ness, which leads to a collapse of the two concepts. This entails that the EXPTIME decision procedure we developed to test QGFM-ness can be used to decide GFM-ness, while our PSPACE-hardness proofs extend to QGFM-ness.

2 Preliminaries

We write \mathbb{N} for the set of nonnegative integers. Let S be a finite set. We denote by $\operatorname{Distr}(S)$ the set of probability distributions on S. For a distribution $\mu \in \operatorname{Distr}(S)$ we write $\operatorname{support}(\mu) = \{s \in S \mid \mu(s) > 0\}$ for its support. The cardinal of S is denoted |S|. We use Σ to denote a finite alphabet. We denote by Σ^* the set of finite words over Σ and Σ^ω the set of ω -words over Σ . We use the standard notions of prefix and suffix of a word. By $w\alpha$ we denote the concatenation of a finite word w and an ω -word α . If $L \subseteq \Sigma^\omega$ and $w \in \Sigma^*$, the residual language (left quotient of L by w), denoted by $w^{-1}L$ is defined as $\{\alpha \in \Sigma^\omega \mid w\alpha \in L\}$.

2.1 Automata

A nondeterministic Büchi word automaton (NBW) is a tuple $\mathcal{A} = (\Sigma, Q, q_0, \delta, F)$, where Σ is a finite alphabet, Q is a finite set of states, $q_0 \in Q$ is the initial state, $\delta : Q \times \Sigma \to 2^Q$ is the transition function, and $F \subseteq Q$ is the set of final (or accepting) states. An NBW is complete if $\delta(q, \sigma) \neq \emptyset$ for all $q \in Q$ and $\sigma \in \Sigma$. Unless otherwise mentioned, we consider complete NBWs in this paper. A run r of \mathcal{A} on $w \in \Sigma^{\omega}$ is an ω -word $q_0, w_0, q_1, w_1, \ldots \in (Q \times \Sigma)^{\omega}$ such that $q_i \in \delta(q_{i-1}, w_{i-1})$ for all i > 0. An NBW \mathcal{A} accepts exactly those runs, in which at least one of the infinitely often occurring states is in F. A word in Σ^{ω} is accepted by the automaton if it has an accepting run, and the language of an automaton, denoted $\mathcal{L}(\mathcal{A})$, is the set of accepted words in Σ^{ω} . An example of an NBW is given in Figure 1.

Let $C \subset \mathbb{N}$ be a finite set of colours. A nondeterministic parity **word automaton** (NPW) is a tuple $P = (\Sigma, Q, q_0, \delta, \pi)$, where Σ, Q, q_0 and δ have the same definitions as for NBW, and $\pi: Q \to C$ is the priority (colouring) function that maps each state to a priority (colour).

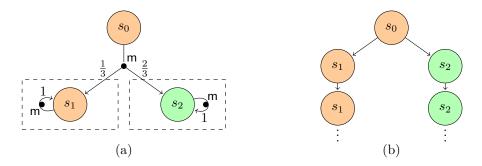


Figure 2 (a) An MDP with initial state s_0 . The set of labels is $\{a, b\}$ and the labelling function for the MDP is as follows: $\ell(s_0) = \ell(s_1) = a$, $\ell(s_2) = b$. The labels are indicated by different colours. Since each state has only one available action m, the MDP is actually an MC. There are two end-components in this MDP labelled with the two dashed boxes. (b) The tree that stems from unravelling of the MC with initial state s_0 on the left, while disregarding probabilities.

A run is accepting if and only if the highest priority (colour) occurring infinitely often in the infinite sequence is even. Similar to NBW, a word in Σ^{ω} is accepted by an NPW if it has an accepting run, and the language of the NPW P, denoted $\mathcal{L}(P)$, is the set of accepted words in Σ^{ω} . An NBW is a special case of an NPW where $\pi(q) = 2$ for $q \in F$ and $\pi(q) = 1$ otherwise with $C = \{1, 2\}$.

A nondeterministic word automaton is *deterministic* if the transition function δ maps each state and letter pair to a singleton set, a set consisting of a single state.

A nondeterministic automaton is called *good-for-games (GFG)* if it only relies on a limited form of nondeterminism: GFG automata can make their decision of how to resolve their nondeterministic choices on the history at any point of a run rather than using the knowledge of the complete word as a nondeterministic automaton normally would without changing their language. They can be characterised in many ways, including as automata that simulate deterministic automata. The NBW in Figure 1 is neither GFG nor good-for-MDPs (GFM) as shown later.

2.2 Markov Decision Processes (MDPs)

A (finite, state-labelled) Markov decision process (MDP) is a tuple $\langle S, \mathsf{Act}, \mathsf{P}, \Sigma, \ell \rangle$ consisting of a finite set S of states, a finite set Act of actions, a partial function $\mathsf{P}: S \times \mathsf{Act} \to \mathsf{Distr}(S)$ denoting the probabilistic transition and a labelling function $\ell: S \to \Sigma$. The set of available actions in a state s is $\mathsf{Act}(s) = \{\mathsf{m} \in \mathsf{Act} \mid \mathsf{P}(s,\mathsf{m}) \text{ is defined}\}$. An MDP is a (labelled) Markov chain (MC) if $|\mathsf{Act}(s)| = 1$ for all $s \in S$.

An infinite $run\ (path)$ of an MDP \mathcal{M} is a sequence $s_0 \mathbf{m}_1 \dots \in (S \times \mathsf{Act})^\omega$ such that $\mathsf{P}(s_i, \mathbf{m}_{i+1})$ is defined and $\mathsf{P}(s_i, \mathbf{m}_{i+1})(s_{i+1}) > 0$ for all $i \geq 0$. A finite run is a finite such sequence. Let $\Omega(\mathcal{M})$ (Paths (\mathcal{M})) denote the set of (finite) runs in \mathcal{M} and $\Omega(\mathcal{M})_s$ (Paths $(\mathcal{M})_s$) denote the set of (finite) runs in \mathcal{M} starting from s. Abusing the notation slightly, for an infinite run $r = s_0 \mathbf{m}_1 s_1 \mathbf{m}_2 \dots$ we write $\ell(r) = \ell(s_0)\ell(s_1) \dots \in \Sigma^\omega$.

A strategy for an MDP is a function μ : Paths $(\mathcal{M}) \to \text{Distr}(\mathsf{Act})$ that, given a finite run r, returns a probability distribution on all the available actions at the last state of r. A memoryless (positional) strategy for an MDP is a function $\mu: S \to \text{Distr}(\mathsf{Act})$ that, given a state s, returns a probability distribution on all the available actions at that state. The set of runs $\Omega(\mathcal{M})_s^{\mu}$ is a subset of $\Omega(\mathcal{M})_s$ that correspond to strategy μ and initial state s. Given a memoryless/finite-memory strategy μ for \mathcal{M} , an MC $(\mathcal{M})_{\mu}$ is induced [3, Section 10.6].

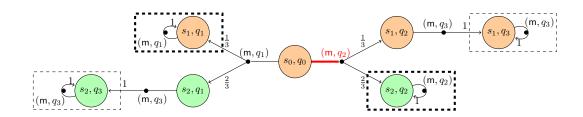


Figure 3 An example of a product MDP $\mathcal{M} \times \mathcal{N}$ with initial state (s_0, q_0) and $F^{\times} = \{(s_1, q_1), (s_2, q_1), (s_1, q_2), (s_2, q_2)\}$ where \mathcal{M} is the MDP (MC) in Figure 2(a) and \mathcal{N} is the NBW in Figure 1. The states $(s_0, q_0), (s_1, q_1), (s_1, q_2)$ and (s_1, q_3) are labelled with a while all the other states are labelled with b. Again, the four end-components of the MDP are labelled with dashed boxes; the upper left and lower right end-components are accepting (highlighted in thick dashed boxes).

A sub-MDP of \mathcal{M} is an MDP $\mathcal{M}' = \langle S', \mathsf{Act}', \mathsf{P}', \Sigma, \ell' \rangle$, where $S' \subseteq S$, $\mathsf{Act}' \subseteq \mathsf{Act}$ is such that $\mathsf{Act}'(s) \subseteq \mathsf{Act}(s)$ for every $s \in S'$, and P' and ℓ' are analogous to P and ℓ when restricted to S' and Act' . In particular, \mathcal{M}' is closed under probabilistic transitions, i.e. for all $s \in S'$ and $\mathsf{m} \in \mathsf{Act}'$ we have that $\mathsf{P}'(s,\mathsf{m})(s') > 0$ implies that $s' \in S'$. An end-component [1, 3] of an MDP \mathcal{M} is a sub-MDP \mathcal{M}' of \mathcal{M} such that its underlying graph is strongly connected and it has no outgoing transitions. An example MDP is presented in Figure 2(a).

A strategy μ and an initial state $s \in S$ induce a standard probability measure on sets of infinite runs, see, e.g., [3]. Such measurable sets of infinite runs are called events or objectives. We write \Pr_s^{μ} for the probability of an event $E \subseteq sS^{\omega}$ of runs starting from s.

▶ Theorem 1. (End-Component Properties [1, 3]). Once an end-component E of an MDP is entered, there is a strategy that visits every state-action combination in E with probability 1 and stays in E forever. Moreover, for every strategy the union of the end-components is visited with probability 1.

2.3 The Product of MDPs and Automata

Given an MDP $\mathcal{M} = \langle S, \mathsf{Act}, \mathsf{P}, \Sigma, \ell \rangle$ with initial state $s_0 \in S$ and an NBW $\mathcal{N} = \langle \Sigma, Q, \delta, q_0, F \rangle$, we want to compute an optimal strategy satisfying the objective that the run of \mathcal{M} is in the language of \mathcal{N} . We define the semantic satisfaction probability for \mathcal{N} and a strategy μ from state $s \in S$ as: $\mathrm{PSem}_{\mathcal{N}}^{\mathcal{M}}(s, \mu) = \mathrm{Pr}_s^{\mu} \{ r \in \Omega_s^{\mu} : \ell(r) \in \mathcal{L}(\mathcal{N}) \}$ and $\mathrm{PSem}_{\mathcal{N}}^{\mathcal{M}}(s) = \sup_{\mu} (\mathrm{PSem}_{\mathcal{N}}^{\mathcal{M}}(s, \mu))$. In the case that \mathcal{M} is an MC, we simply have $\mathrm{PSem}_{\mathcal{N}}^{\mathcal{M}}(s) = \mathrm{Pr}_s \{ r \in \Omega_s : \ell(r) \in \mathcal{L}(\mathcal{N}) \}$.

The product of \mathcal{M} and \mathcal{N} is an MDP $\mathcal{M} \times \mathcal{N} = \langle S \times Q, \mathsf{Act} \times Q, \mathsf{P}^\times, \Sigma, \ell^\times \rangle$ augmented with the initial state (s_0, q_0) and the Büchi acceptance condition $F^\times = \{(s, q) \in S \times Q \mid q \in F\}$. The labelling function ℓ^\times maps each state $(s, q) \in S \times Q$ to $\ell(s)$.

We define the partial function $\mathsf{P}^\times: (S \times Q) \times (\mathsf{Act} \times Q) \to \mathsf{Distr}(S \times Q)$ as follows: for all $(s,\mathsf{m}) \in \mathsf{support}(\mathsf{P}), \ s' \in S \ \mathrm{and} \ q,q' \in Q,$ we have $\mathsf{P}^\times \big((s,q), (\mathsf{m},q') \big) \big((s',q') \big) = \mathsf{P}(s,\mathsf{m})(s')$ for all $q' \in \delta(q,\ell(s))^1$.

We define the syntactic satisfaction probability for the product MDP and a strategy

When \mathcal{N} is complete, there always exists a state q' such that $q' \in \delta(q, \ell(s))$.

 μ^{\times} from a state (s,q) as: $\mathrm{PSyn}_{\mathcal{N}}^{\mathcal{M}}\big((s,q),\mu^{\times}\big) = \mathrm{Pr}_{s,q}^{\mu^{\times}}\{r\in\Omega_{s,q}^{\mu^{\times}}:\ell^{\times}(r)\in\mathcal{L}(\mathcal{N})\}^2$ and $\mathrm{PSyn}_{\mathcal{N}}^{\mathcal{M}}(s) = \sup_{\mu^{\times}}(\mathrm{PSyn}_{\mathcal{N}}^{\mathcal{M}}\big((s,q_0),\mu^{\times}\big))$. The set of actions is Act in the MDP \mathcal{M} while it is Act \times Q in the product MDP. This makes PSem and PSyn potentially different. In general, $\mathrm{PSyn}_{\mathcal{N}}^{\mathcal{M}}(s) \leq \mathrm{PSem}_{\mathcal{N}}^{\mathcal{M}}(s)$ for all $s \in S$, because accepting runs can only occur on accepted words. If \mathcal{N} is deterministic, $\mathrm{PSyn}_{\mathcal{N}}^{\mathcal{M}}(s) = \mathrm{PSem}_{\mathcal{N}}^{\mathcal{M}}(s)$ holds for all $s \in S$.

End-components and runs are defined for products just like for MDPs. A run of $\mathcal{M} \times \mathcal{N}$ is accepting if it satisfies the product's acceptance condition. An accepting end-component of $\mathcal{M} \times \mathcal{N}$ is an end-component which contains some states in F^{\times} .

An example of a product MDP is presented in Figure 3. It is the product of the MDP in Figure 2(a) and the NBW in Figure 1. Since $\ell(r)$ is in the language of the NBW for every run r of the MDP, we have $\operatorname{PSem}_{\mathcal{N}}^{\mathcal{M}}(s_0)=1$. However, the syntactic satisfaction probability $\operatorname{PSyn}_{\mathcal{N}}^{\mathcal{M}}(s_0)=\frac{2}{3}$ is witnessed by the memoryless strategy which chooses the action (m,q_2) at the initial state. We do not need to specify the strategy for the other states since there is only one available action for any remaining state. According to the following definition, the NBW in Figure 1 is not GFM as witnessed by the MDP in Figure 2(a).

▶ **Definition 2.** [10] An NBW \mathcal{N} is good-for-MDPs (GFM) if, for all finite MDPs \mathcal{M} with initial state s_0 , $\operatorname{PSyn}_{\mathcal{N}}^{\mathcal{M}}(s_0) = \operatorname{PSem}_{\mathcal{N}}^{\mathcal{M}}(s_0)$ holds.

3 PSPACE-Hardness

We show that the problem of checking whether or not a given NBW is GFM is PSPACE-hard. The reduction is from the NFA universality problem, which is known to be PSPACE-complete [19]. Given an NFA \mathcal{A} , the NFA universality problem asks whether \mathcal{A} accepts every string, that is, whether $\mathcal{L}(\mathcal{A}) = \Sigma^*$.

We first give an overview of how this reduction works. Given a complete NFA \mathcal{A} , we first construct an NBW \mathcal{A}_f (Definition 4) which can be shown to be GFM (Lemma 6). Using this NBW \mathcal{A}_f , we then construct another NBW fork(\mathcal{A}_f) (Definition 7). We complete the argument by showing in Lemma 8 that the NBW fork(\mathcal{A}_f) is GFM if, and only if, \mathcal{A} accepts the universal language.

We start with the small observation that 'for all finite MDPs' in Definition 2 can be replaced by 'for all finite MCs'.

▶ **Theorem 3.** An NBW \mathcal{N} is GFM iff, for all finite MCs \mathcal{M} with initial state s_0 , $\mathrm{PSyn}_{\mathcal{N}}^{\mathcal{M}}(s_0) = \mathrm{PSem}_{\mathcal{N}}^{\mathcal{M}}(s_0)$ holds.

Proof. 'if': This is the case because there is an optimal finite memory control for an MDP \mathcal{M} , e.g. by using a language equivalent DPW \mathcal{P} [14] and using its memory structure as finite memory. That is, we obtain an MC \mathcal{M}' by applying an optimal memoryless strategy for $\mathcal{M} \times \mathcal{P}$ [4]. Naturally, if \mathcal{N} satisfies the condition for \mathcal{M}' , then it also satisfies it for \mathcal{M} .

'only if': MCs are just special cases of MDPs.

Given a complete NFA \mathcal{A} , we construct an NBW \mathcal{A}_f by introducing a new letter \$ and a new state. As an example, given an NFA (DFA) \mathcal{B} in Figure 4(a), we obtain an NBW \mathcal{B}_f in Figure 4(b). It is easy to see that $\mathcal{L}(\mathcal{B}) = \Sigma^*$ where $\Sigma = \{a, b\}$.

² Let $\inf(r)$ be the set of states that appears infinite often in a run r. We also have $\operatorname{PSyn}^{\mathcal{M}}_{\mathcal{N}}((s,q),\mu^{\times}) = \operatorname{Pr}^{\mu^{\times}}_{s,q} \{ r \in \Omega^{\mu^{\times}}_{s,q} : \inf(r) \cap F^{\times} \neq \emptyset \}.$

- **Figure 4** (a) \mathcal{B} is a complete universal NFA. Let $\Sigma = \{a, b\}$. We have $\mathcal{L}(\mathcal{A}) = \Sigma^*$. (b) On the right is the corresponding complete NBW \mathcal{B}_f . The new final state f and the added transitions are highlighted in red. We have $\mathcal{L}(\mathcal{B}_f) = \{w_1 \$ w_2 \$ w_3 \$ \dots\}$ where $w_i \in \Sigma^*$.
- ▶ **Definition 4.** Given a complete NFA $\mathcal{A} = (\Sigma, Q, q_0, \delta, F)$, we define the NBW $\mathcal{A}_f = (\Sigma_\$, Q_f, q_0, \delta_f, \{f\})$ with $\Sigma_\$ = \Sigma \cup \{\$\}$ and $Q_f = Q \cup \{f\}$ for a fresh letter $\$ \notin \Sigma$ and a fresh state $f \notin Q$, and with $\delta_f(q, \sigma) = \delta(q, \sigma)$ for all $q \in Q$ and $\sigma \in \Sigma$, $\delta_f(q, \$) = \{f\}$ for all $q \in F$, $\delta_f(q, \$) = \{q_0\}$ for all $q \in Q \setminus F$, and $\delta_f(f, \sigma) = \delta_f(q_0, \sigma)$ for all $\sigma \in \Sigma_\$$.

The language of \mathcal{A}_f consists of all words of the form $w_1 \$ w_1' \$ w_2 \$ w_2' \$ w_3 \$ w_3' \$ \dots$ such that, for all $i \in \mathbb{N}$, $w_i \in \Sigma_{\* and $w_i' \in \mathcal{L}(\mathcal{A})$. This provides the following lemma.

▶ **Lemma 5.** Given two NFAs \mathcal{A} and \mathcal{B} , $\mathcal{L}(\mathcal{B}) \subseteq \mathcal{L}(\mathcal{A})$ if, and only if, $\mathcal{L}(\mathcal{B}_f) \subseteq \mathcal{L}(\mathcal{A}_f)$.

The following lemma simply states that the automaton A_f from the above construction is GFM. This lemma is technical and is key to prove Lemma 8, the main lemma, of this section.

▶ **Lemma 6.** For every complete NFA A, A_f is GFM.

Proof. Consider an arbitrary MC \mathcal{M} with initial state s_0 . We show that \mathcal{A}_f is good for \mathcal{M} , that is, $\operatorname{PSem}_{\mathcal{A}_f}^{\mathcal{M}}(s_0) = \operatorname{PSyn}_{\mathcal{A}_f}^{\mathcal{M}}(s_0)$. It suffices to show $\operatorname{PSyn}_{\mathcal{A}_f}^{\mathcal{M}}(s_0) \geq \operatorname{PSem}_{\mathcal{A}_f}^{\mathcal{M}}(s_0)$ since by definition the converse $\operatorname{PSem}_{\mathcal{A}_f}^{\mathcal{M}}(s_0) \geq \operatorname{PSyn}_{\mathcal{A}_f}^{\mathcal{M}}(s_0)$ always holds.

First, we construct a language equivalent deterministic Büchi automaton (DBW) \mathcal{D}_f by first determinising the NFA \mathcal{A} to a DFA \mathcal{D} by a standard subset construction and then obtain \mathcal{D}_f by Definition 4. Since $\mathcal{L}(\mathcal{A}_f) = \mathcal{L}(\mathcal{D}_f)$, we have that $\operatorname{PSem}_{\mathcal{D}_f}^{\mathcal{M}}(s_0) = \operatorname{PSem}_{\mathcal{D}_f}^{\mathcal{M}}(s_0)$. In addition, since \mathcal{D}_f is deterministic, we have $\operatorname{PSem}_{\mathcal{D}_f}^{\mathcal{M}}(s_0) = \operatorname{PSyn}_{\mathcal{D}_f}^{\mathcal{M}}(s_0)$.

It remains to show $\operatorname{PSyn}_{\mathcal{A}_f}^{\mathcal{M}}(s_0) \geq \operatorname{PSyn}_{\mathcal{D}_f}^{\mathcal{M}}(s_0)$. For that, it suffices to show that for an arbitrary accepting run r of $\mathcal{M} \times \mathcal{D}_f$, there is a strategy for $\mathcal{M} \times \mathcal{A}_f$ such that r' (the corresponding run in the product) is accepting in $\mathcal{M} \times \mathcal{A}_f$ where the projections of r and r' on \mathcal{M} are the same.

Consider an accepting run of $\mathcal{M} \times \mathcal{D}_f$. Before entering an accepting end-component of $\mathcal{M} \times \mathcal{D}_f$, any strategy to resolve the nondeterminism in $\mathcal{M} \times \mathcal{A}_f$ (thus \mathcal{A}_f) can be used. This will not block \mathcal{A}_f , as it is a complete automaton, and \mathcal{A}_f is essentially re-set whenever it reads a \$. Once an accepting end-component of $\mathcal{M} \times \mathcal{D}_f$ is entered, there must exist a word of the form \$w\$, where $w \in \mathcal{L}(\mathcal{D})$ (and thus $w \in \mathcal{L}(\mathcal{A})$), which is a possible initial sequence of letters produced from some state m of $\mathcal{M} \times \mathcal{D}_f$ in this end-component. We fix such a word \$w\$; such a state m of the end-component in $\mathcal{M} \times \mathcal{D}_f$ from which this word \$w\$ can be produced; and strategy of the NBW \mathcal{A}_f to follow the word w\$ from q_0 (and f) to the accepting state f. (Note that the first \$ always leads to q_0 or f.)

In an accepting end-component we can be in two modes: tracking, or not tracking. If we are not tracking and reach m, we start to $track \$ w\\$: we use A_f to reach an accepting state after reading \\$w\\$ (ignoring what happens in any other case) with the strategy we have fixed.

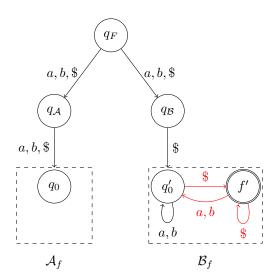
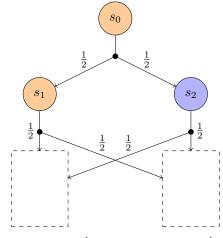


Figure 5 Given a complete NFA \mathcal{A} , an NBW \mathcal{A}_f and an NBW fork(\mathcal{A}_f) are constructed. In this example, $\Sigma = \{a, b\}$ and $\Sigma_{\$} = \{a, b, \$\}$. From the state $q_{\mathcal{A}}$ (resp. $q_{\mathcal{B}}$), the NBW fork(\mathcal{A}_f) transitions to the initial state of \mathcal{A}_f (resp. \mathcal{B}_f).



generate $w_a\cdot\$$ repeatedly with probability one generate $w_b\cdot\$$ repeatedly with probability one

Figure 6 An example MC in the proof of Lemma 8. Assume $\Sigma_{\$} = \{a, b, \$\}$. The labelling of the MC is as follows: $\ell(s_0) = \ell(s_1) = a$ and $\ell(s_2) = \$$.

Note that, after reading the first \$, we are in either q_0 or f, such that, when starting from m, it is always possible, with a fixed probability p > 0, to read \$w\$, and thus to accept. If we read an unexpected letter (where the 'expected' letter is always the next letter from \$w\$) or the end of the word \$w\$ is reached, we move to not tracking.

The automata choices when *not tracking* can be resolved arbitrarily.

Once in an accepting end-component of $\mathcal{M} \times \mathcal{D}_f$, tracking is almost surely started infinitely often, and it is thus almost surely successful infinitely often. Thus, we have $\operatorname{PSyn}_{\mathcal{D}_f}^{\mathcal{M}}(s_0) \geq \operatorname{PSyn}_{\mathcal{D}_f}^{\mathcal{M}}(s_0)$.

Let \mathcal{B} be an universal NFA in Figure 4(a) and $\mathcal{B}_f = (\Sigma_\$, Q'_f, q'_0, \delta'_f, \{f'\})$ be the NBW in Figure 4(b). Assume without loss of generality that the state space of \mathcal{A}_f , \mathcal{B}_f , and $\{q_F, q_A, q_B\}$ are pairwise disjoint. We now define the fork operation. An example of how to construct an NBW fork (\mathcal{A}_f) is shown in Figure 5.

- ▶ **Definition 7.** Given an NBW $A_f = (\Sigma_\$, Q_f, q_0, \delta_f, \{f\})$, we define the NBW fork $(A_f) = (\Sigma_\$, Q_F, q_F, \delta_F, \{f, f'\})$ with
- $Q_F = Q_f \cup Q'_{f'} \cup \{q_F, q_A, q_B\};$
- $\delta_F(q,\sigma) = \delta_f(q,\sigma) \text{ for all } q \in Q_f \text{ and } \sigma \in \Sigma_\$;$
- $\delta_F(q,\sigma) = \delta_{f'}(q,\sigma) \text{ for all } q \in Q'_{f'} \text{ and } \sigma \in \Sigma_{\$};$
- $\delta_F(q_F, \sigma) = \{q_A, q_B\} \text{ for all } \sigma \in \Sigma_{\$};$
- $\delta_F(q_{\mathcal{A}}, \sigma) = \{q_0\} \text{ for all } \sigma \in \Sigma_{\$};$
- $\delta_F(q_{\mathcal{B}},\$) = \{q_0'\}, \text{ and } \delta_F(q_{\mathcal{B}},\sigma) = \emptyset \text{ for all } \sigma \in \Sigma.$

Following from Lemma 5 and Lemma 6, we have:

▶ **Lemma 8.** The NBW fork(A_f) is GFM if, and only if, $\mathcal{L}(A) = \Sigma^*$.

Proof. We first observe that $\mathcal{L}(\mathsf{fork}(\mathcal{A}_f)) = \{\sigma\sigma'w \mid \sigma, \sigma' \in \Sigma_\$, \ w \in \mathcal{L}(\mathcal{A}_f)\} \cup \{\sigma\$w \mid \sigma \in \Sigma_\$, \ w \in \mathcal{L}(\mathcal{B}_f)\}.$

'if': When $\mathcal{L}(\mathcal{A}) = \Sigma^* = \mathcal{L}(\mathcal{B})$ holds, Lemma 5 provides $\{\sigma\sigma'w \mid \sigma, \sigma' \in \Sigma_{\$}, w \in \mathcal{L}(\mathcal{A}_f)\} \supset \{\sigma\$w \mid \sigma \in \Sigma_{\$}, w \in \mathcal{L}(\mathcal{B}_f)\}$, and therefore $\mathcal{L}(\mathsf{fork}(\mathcal{A}_f)) = \{\sigma\sigma'w \mid \sigma, \sigma' \in \Sigma_{\$}, w \in \mathcal{L}(\mathcal{A}_f)\}$.

As A_f is GFM by Lemma 6, this provides the GFM strategy "move first to q_A , then to q_0 , and henceforth follow the GFM strategy of A_f for fork (A_f) ". Thus, fork (A_f) is GFM in this case

'only if': Assume $\mathcal{L}(\mathcal{A}) \neq \Sigma^* = \mathcal{L}(\mathcal{B})$, that is, $\mathcal{L}(\mathcal{A}) \subset \mathcal{L}(\mathcal{B})$. There must exist words $w_a \in \mathcal{L}(\mathcal{A})$ and $w_b \in \mathcal{L}(\mathcal{B}) \setminus \mathcal{L}(\mathcal{A})$. We now construct an MC which witnesses that fork (\mathcal{A}_f) is not GFM.

The MC emits an a at the first step and then either an a or a \$ with a chance of $\frac{1}{2}$ at the second step. An example is provided in Figure 6.

After these two letters, it then moves to one of two cycles (independent of the first two chosen letters) with equal chance of $\frac{1}{2}$; one of these cycles repeats a word w_a .\$ infinitely often, while the other repeats a word w_b .\$ infinitely often, where $w_a \in \mathcal{L}(\mathcal{A})$.

It is easy to see that the semantic chance of acceptance is $\frac{3}{4}$ – failing if, and only if, the second letter is a and the word w_b \$ is subsequently repeated infinitely often – whereas the syntactic chance of satisfaction is $\frac{1}{2}$: when the automaton first moves to q_A , it accepts if, and only if, the word w_a \$ is later repeated infinitely often, which happens with a chance of $\frac{1}{2}$; when the automaton first moves to q_B , it will reject when \$ is not the second letter, which happens with a chance of $\frac{1}{2}$.

It follows from Lemma 8 that the NFA universality problem is polynomial-time reducible to the problem of whether or not a given NBW is GFM.

▶ Theorem 9. The problem of whether or not a given NBW is GFM is PSPACE-hard.

Using the same construction of Definition 4, we can show that the problem of minimising a GFM NBW is PSPACE-hard. The reduction is from a problem which is similar to the NFA universality problem.

▶ Theorem 10. Given a GFM NBW and a bound k, the problem whether there is a language equivalent GFM NBW with at most k states is PSPACE-hard. It is PSPACE-hard even for (fixed) k = 2.

Proof. Using the construction of Definition 4, PSPACE-hardness follows from a reduction from the problem whether a nonempty complete NFA accepts all nonempty words. The latter problem is PSPACE-complete, following the PSPACE-completeness of the universality problem of (general) NFAs [19].

The reduction works because, for a nonempty complete NFA A the following hold:

- (a) A GFM NBW equivalent to \mathcal{A}_f must have at least 2 states, one final and one nonfinal. This is because it needs a final state (as some word is accepted) as well as a nonfinal one (words that contain finitely many \$s are rejected).
- (b) For a 2-state minimal GFM NBW equivalent to \mathcal{A}_f , there cannot be a word $w \in \Sigma^+$ that goes from the final state back to it as this would produce an accepting run with finitely many \$s (as there is some accepting run). Therefore, when starting from the final state, any finite word can only go to the nonfinal state and stay there or block. But blocking is no option, as there is an accepted continuation to an infinite word. Thus, all letters in Σ lead from either state to the nonfinal state (only).

In order for a word starting with a letter in Σ to be accepted, there must therefore be a \$ transition from the nonfinal to the final state.

These two points imply that for a nonempty complete NFA \mathcal{A} such that there is a 2-state GFM NBW equivalent to \mathcal{A}_f iff \mathcal{A} accepts all nonempty words.

4 Decision Procedure for Qualitative GFM

In this section, we first define the notion of qualitative GFM (QGFM) and then provide an EXPTIME procedure that decides QGFM-ness.

The definition of QGFM is similar to GFM but we only need to consider MCs with which the semantic chance of success is one:

▶ **Definition 11.** An NBW \mathcal{N} is qualitative good-for-MDPs (QGFM) if, for all finite MDPs \mathcal{M} with initial state s_0 and $\mathrm{PSem}_{\mathcal{N}}^{\mathcal{M}}(s_0) = 1$, $\mathrm{PSyn}_{\mathcal{N}}^{\mathcal{M}}(s_0) = 1$ holds.

Similar to Theorem 3, we can also replace 'MDPs' by 'MCs' in the definition of QGFM:

▶ **Theorem 12.** An NBW \mathcal{N} is QGFM iff, for all finite MCs \mathcal{M} with initial state s_0 and $\operatorname{PSem}_{\mathcal{N}}^{\mathcal{M}}(s_0) = 1$, $\operatorname{PSyn}_{\mathcal{N}}^{\mathcal{M}}(s_0) = 1$ holds.

To decide QGFM-ness, we make use of the well known fact that qualitative acceptance, such as $\operatorname{PSem}_{\mathcal{N}}^{\mathcal{M}}(s_0) = 1$, does not depend on the probabilities for an MC \mathcal{M} . This can, for example, be seen by considering the syntactic product $\operatorname{PSyn}_{\mathcal{D}}^{\mathcal{M}}(s_0) = 1$ with a deterministic parity automaton \mathcal{D} (for a deterministic automaton, $\operatorname{PSem}_{\mathcal{D}}^{\mathcal{M}}(s_0) = \operatorname{PSyn}_{\mathcal{D}}^{\mathcal{M}}(s_0)$ trivially holds), where changing the probabilities does not change the end-components of the product MC $\mathcal{M} \times \mathcal{D}$, and the acceptance of these end-components is solely determined by the highest colour of the states (or transitions) occurring in it, and thus also independent of the probabilities: the probability is 1 if, and only if, an accepting end-component can be reached almost surely, which is also independent of the probabilities. As a result, we can search for the (regular) tree that stems from the unravelling of an MC, while disregarding probabilities. See Figure 2(b) for an example of such a tree.

This observation has been used in the synthesis of probabilistic systems before [16]. The set of directions (of a tree) Υ could then, for example, be chosen to be the set of states of the unravelled finite MC; this would not normally be a full tree.

In the following, we show an exponential-time algorithm to decide whether a given NBW is QGFM or not. This procedure involves transformations of tree automata with different acceptance conditions. Because this is quite technical, we only provide an outline in the main paper. More notations (Section A.1) and details of the constructions (Section A.2) are provided in the appendix.

For a given candidate NBW C, we first construct a language equivalent NBW G that we know to be GFM, such as the slim automaton from [10] or a suitable limit deterministic automaton [22, 7, 8, 18, 9]. For all known constructions, G can be exponentially larger than G. We use the slim automata from [10]; they have $O(3^{|Q|})$ states and transitions.

We then construct a number of tree automata as outlined in Figure 7. In a first construction, we discuss in the appendix how to build, for an NBW \mathcal{N} , a (symmetric) alternating Büchi tree automaton (ABS) $\mathcal{T}_{\mathcal{N}}$ that accepts (the unravelling of) an MC (without probabilities, as discussed above) \mathcal{M} if, and only if, the syntactic product of \mathcal{N} and \mathcal{M} almost surely accepts. This construction is used twice: once to produce $\mathcal{T}_{\mathcal{G}}$ for the GFM automaton \mathcal{G} we have constructed, and once to produce $\mathcal{T}_{\mathcal{C}}$ for our candidate automaton \mathcal{C} . Since \mathcal{G} is QGFM, $\mathcal{T}_{\mathcal{G}}$ accepts all the MCs \mathcal{M} that almost surely produce a run in $\mathcal{L}(\mathcal{G})$ (which is the same as $\mathcal{L}(\mathcal{C})$), that is, PSem $_{\mathcal{G}}^{\mathcal{M}}(s_0) = \text{PSem}_{\mathcal{C}}^{\mathcal{M}}(s_0) = 1$.

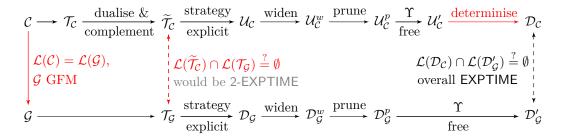


Figure 7 Flowchart of the algorithms. The first algorithm is in 2-EXPTIME, which is to check the nonemptiness of the intersection of $\widetilde{\mathcal{T}}_{\mathcal{C}}$ and $\mathcal{T}_{\mathcal{G}}$. The second algorithm is in EXPTIME, which is to check the nonemptiness of the intersection of $\mathcal{D}_{\mathcal{C}}$ and $\mathcal{D}'_{\mathcal{G}}$. The steps that have exponential blow-up are highlighted in red.

Therefore, to check whether or not our candidate NBW \mathcal{C} is QGFM, we can test language equivalence of $\mathcal{T}_{\mathcal{C}}$ and $\mathcal{T}_{\mathcal{G}}$, e.g. by first complementing $\mathcal{T}_{\mathcal{C}}$ to $\widetilde{\mathcal{T}}_{\mathcal{C}}$ and then checking whether or not $\mathcal{L}(\widetilde{\mathcal{T}}_{\mathcal{C}}) \cap \mathcal{L}(\mathcal{T}_{\mathcal{G}}) = \emptyset$ holds: the MCs in the intersection of the languages witness that \mathcal{C} is not QGFM. Thus, \mathcal{C} is QGFM if, and only if, these languages do not intersect, that is, $\mathcal{L}(\widetilde{\mathcal{T}}_{\mathcal{C}}) \cap \mathcal{L}(\mathcal{T}_{\mathcal{G}}) = \emptyset$. This construction leads to a 2-EXPTIME procedure for deciding QGFM: we get the size of the larger automaton (\mathcal{G}) and the complexity of the smaller automaton \mathcal{C} . The purpose of the following delicate construction is to contain the exponential cost to the syntactic material of the smaller automaton, while still obtaining the required level of entanglement between the structures and retaining the size advantage from the GFM property of \mathcal{G} .

Starting from $\widetilde{\mathcal{T}}_{\mathcal{C}}$, we make a few transformations by mainly controlling the number of directions the alternating tree automaton needs to consider and the set of decisions player accept³ has to make. This restricts the scope in such a way that the resulting intersection might shrink, but cannot become empty⁴.

We rein in the number of directions in two steps: in a first step, we *increase* the number of directions by widening the run tree with one more direction than the size of the state space of the candidate automaton \mathcal{C} . The larger amount of directions allows us to concurrently untangle the decisions of player *accept* within and between $\widetilde{\mathcal{T}}_{\mathcal{C}}$ and $\mathcal{T}_{\mathcal{G}}$, which intuitively creates one distinguished direction for each state q of $\widetilde{\mathcal{T}}_{\mathcal{C}}$ used by player *accept*, and one (different) distinguished direction for $\mathcal{T}_{\mathcal{G}}$. In a second step, we only keep these directions, resulting in an automaton with a *fixed* branching degree (just one bigger than the size of the state space of \mathcal{C}), which is easy to analyse with standard techniques.

The standard techniques mean to first make the remaining choices of player accept in $\mathcal{T}_{\mathcal{C}}$ explicit, which turns it into a universal co-Büchi automaton ($\mathcal{U}_{\mathcal{C}}$). The automaton is then simplified to the universal co-Büchi automaton $\mathcal{U}'_{\mathcal{C}}$ which can easily be determinised to a deterministic parity automaton $\mathcal{D}_{\mathcal{C}}$.

For $\mathcal{T}_{\mathcal{G}}$, a sequence of similar transformations are made; however, as we do not need to complement here, the automaton obtained from making the decisions explicit is already deterministic, which saves the exponential blow-up obtained in the determinisation of a universal automaton (determinising $\mathcal{U}'_{\mathcal{C}}$ to $\mathcal{D}_{\mathcal{C}}$).

³ The acceptance of a tree by a tree automaton can be viewed as the outcome of a game played by player accept and player reject. We refer to the appendix for details.

⁴ It can be empty to start with, of course, and will stay empty in that case. But if the intersection is not empty, then these operations will leave something in the language.

Therefore, $\mathcal{D}_{\mathcal{C}}$ and $\mathcal{D}'_{\mathcal{G}}$ can both be constructed from \mathcal{C} in time exponential in \mathcal{C} , and checking their intersection for emptiness can be done in time exponential in \mathcal{C} , too. With that, we obtain the membership in EXPTIME for QGFM-ness:

▶ **Theorem 13.** The problem of whether or not a given NBW is QGFM is in EXPTIME.

5 Membership in EXPTIME for GFM

In this section, we start out with showing a sufficient (Lemma 14) and necessary (Lemma 15) criterion for a candidate NBW to be GFM in Section 5.1.

We show in Section 5.2 that this criterion is also sufficient and necessary for QGFM-ness. This implies that GFM-ness and QGFM-ness collapse, so that the EXPTIME decision procedure from Section 4 can be used to decide GFM-ness, and the PSPACE hardness from Section 3 extends to QGFM.

5.1 Key Criterion for GFM-ness

In order to establish a necessary and sufficient criterion for GFM-ness, we construct two safety automata⁵ S and T.

Given a candidate NBW \mathcal{C} , we define some notions for the states and transitions. We say a state q of \mathcal{C} is productive if $\mathcal{L}(\mathcal{C}_q) \neq \emptyset$ where \mathcal{C}_q is the automaton obtained from \mathcal{C} by making q the initial state. A state q of the NBW \mathcal{C} is called QGFM if the automaton \mathcal{C}_q is QGFM. A transition (q, σ, r) is called residual if $\mathcal{L}(\mathcal{C}_r) = \sigma^{-1}\mathcal{L}(\mathcal{C}_q)$ [12, 2]. In general, $\mathcal{L}(\mathcal{C}_r) \subseteq \sigma^{-1}\mathcal{L}(\mathcal{C}_q)$ holds. See Figure 1 for an example of non-residual transitions. Selecting either of the two transitions from q_0 will lose language: when selecting the transition to the left, the word $a \cdot b^{\omega}$ is no longer accepted. Likewise, when selecting the transition to the right, the word a^{ω} is no longer accepted. Thus, this automaton cannot make the decision to choose the left or the right transition, and neither (q_0, a, q_1) nor (q_0, a, q_2) is a residual transition.

Now we are ready to define S and T. In the NBW S, we include the states from the candidate NBW C that are productive and QGFM at the same time. We only include the residual transitions (in C) between those states. In the NBW T, we include only the productive states of C and the transitions between them. We then make both S and T safety automata by marking all states final. We first show that the criterion, L(S) = L(T) and S is GFG⁶, is sufficient for C to be GFM. Similar to the proof of Lemma 6, to show the NBW C is GFM, we show there exists a strategy for C such that the syntactic and semantic chance of winning are the same for any MC.

▶ **Lemma 14.** If $\mathcal{L}(S) = \mathcal{L}(T)$ and S is GFG, then the candidate NBW C is GFM.

Proof. As \mathcal{T} contains all states and transitions from \mathcal{S} , $\mathcal{L}(\mathcal{S}) \subseteq \mathcal{L}(\mathcal{T})$ always holds. We assume that $\mathcal{L}(\mathcal{S}) \supseteq \mathcal{L}(\mathcal{T})$ holds and \mathcal{S} is GFG.

By Theorem 3, to show \mathcal{C} is GFM, it suffices to show that \mathcal{C} is good for an arbitrary MC \mathcal{M} with initial state s_0 . We first determinise \mathcal{C} to a DPW \mathcal{D} [14]. Since \mathcal{D} is deterministic and $\mathcal{L}(\mathcal{D}) = \mathcal{L}(\mathcal{C})$, we have $\operatorname{PSyn}_{\mathcal{D}}^{\mathcal{M}}(s_0) = \operatorname{PSem}_{\mathcal{D}}^{\mathcal{M}}(s_0) = \operatorname{PSem}_{\mathcal{C}}^{\mathcal{M}}(s_0)$. Since $\operatorname{PSem}_{\mathcal{C}}^{\mathcal{M}}(s_0) \geq \operatorname{PSem}_{\mathcal{C}}^{\mathcal{M}}(s_0)$

⁵ A safety automaton is one where all states are final. These automata can be viewed as NFAs where convenient.

⁶ GFG as a general property is tricky, but S is a safety automaton, and GFG safety automata are, for example, determinisable by pruning, and the property whether or not a safety automaton is GFG can be checked in polynomial time [6].

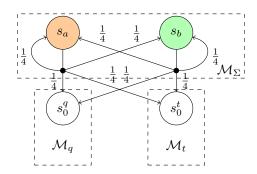


Figure 8 An example MC in the proof of Lemma 15. In this example, $\Sigma = \{a, b\}$. Also, the state q of the candidate NBW \mathcal{C} is the only state that is not QGFM and the transition t of the NBW \mathcal{C} is the only non-residual transition. We have $\ell(s_a) = a$ and $\ell(s_b) = b$. The states s_a and s_b and the transitions between them form \mathcal{M}_{Σ} .

 $\operatorname{PSyn}_{\mathcal{C}}^{\mathcal{M}}(s_0)$ always holds, we establish the equivalence of syntactic and semantic chance of winning for $\mathcal{M} \times \mathcal{C}$ by proving $\operatorname{PSyn}_{\mathcal{C}}^{\mathcal{M}}(s_0) \geq \operatorname{PSyn}_{\mathcal{D}}^{\mathcal{M}}(s_0) = \operatorname{PSem}_{\mathcal{C}}^{\mathcal{M}}(s_0)$.

For that, we show for an arbitrary accepting run r of $\mathcal{M} \times \mathcal{D}$, there is a strategy for $\mathcal{M} \times \mathcal{C}$ such that r' (the corresponding run in the product) is accepting in $\mathcal{M} \times \mathcal{C}$ where the projections of r and r' on \mathcal{M} are the same.

We define the strategy for $\mathcal{M} \times \mathcal{C}$ depending on whether an accepting end-component of $\mathcal{M} \times \mathcal{D}$ has been entered. Since r is accepting, it must enter an accepting end-component of $\mathcal{M} \times \mathcal{D}$ eventually. Let the run r be $(s_0, q_0^{\mathcal{D}})(s_1, q_1^{\mathcal{D}}) \cdots$ and it enters the accepting end-component on reaching the state $(s_n, q_n^{\mathcal{D}})$. Before r enters an accepting end-component of $\mathcal{M} \times \mathcal{D}$, \mathcal{C} follows the GFG strategy for \mathcal{S} to stay within the states that are productive and QGFM. Upon reaching an accepting end-component of $\mathcal{M} \times \mathcal{D}$, the run r is in state $(s_n, q_n^{\mathcal{D}})$, assume the run for $\mathcal{M} \times \mathcal{C}$ is in state $(s_n, q_n^{\mathcal{C}})$ at this point. We then use the QGFM strategy of $\mathcal{C}_{q_n^{\mathcal{C}}}$ from here since $q_n^{\mathcal{C}}$ is QGFM.

We briefly explain why this strategy for $\mathcal{M} \times \mathcal{C}$ would lead to $\operatorname{PSyn}_{\mathcal{C}}^{\mathcal{M}}(s_0) \geq \operatorname{PSyn}_{\mathcal{D}}^{\mathcal{M}}(s_0)$. Since $(s_n, q_n^{\mathcal{D}})$ is in the accepting end-component of $\mathcal{M} \times \mathcal{D}$, we have $\operatorname{PSem}_{\mathcal{D}_{q_n^{\mathcal{D}}}}^{\mathcal{M}}(s_n) = \operatorname{PSyn}_{\mathcal{D}_{q_n^{\mathcal{D}}}}^{\mathcal{M}}(s_n) = 1$ by Theorem 1. We show $\operatorname{PSem}_{\mathcal{C}_{q_n^{\mathcal{C}}}}^{\mathcal{M}}(s_n) = 1$ so that $\operatorname{PSyn}_{\mathcal{C}_{q_n^{\mathcal{C}}}}^{\mathcal{M}}(s_n) = \operatorname{PSem}_{\mathcal{C}_{q_n^{\mathcal{C}}}}^{\mathcal{M}}(s_n) = 1$ as $q_n^{\mathcal{C}}$ is QGFM. For that, it suffices to show $\mathcal{L}(\mathcal{C}_{q_n^{\mathcal{C}}}) = \mathcal{L}(\mathcal{D}_{q_n^{\mathcal{D}}}) = w^{-1}\mathcal{L}(\mathcal{C})$ where $w = \ell(s_0s_1 \cdots s_{n-1})$. This can be proved by induction in the appendix over the length of the prefix of words from $\mathcal{L}(\mathcal{S})$.

In order to show that this requirement is also necessary, we build an MC witnessing that \mathcal{C} is not GFM in case the criterion is not satisfied. An example MC is given in Figure 8. We produce the MC by parts. It has a central part denoted by \mathcal{M}_{Σ} . The state space of \mathcal{M}_{Σ} is S_{Σ} where $S_{\Sigma} = \{s_{\sigma} \mid \sigma \in \Sigma\}$. Each state s_{σ} is labelled with σ and there is a transition between every state pair.

For every state q that is not QGFM, we construct an MC $\mathcal{M}_q = \langle S_q, P_q, \Sigma, \ell_q \rangle$ from Section 4 witnessing that \mathcal{C}_q is not QGFM, that is, from a designated initial state s_0^q , $\mathrm{PSyn}_{\mathcal{C}_q}^{\mathcal{M}_q}(s_0^q) \neq \mathrm{PSem}_{\mathcal{C}_q}^{\mathcal{M}_q}(s_0^q) = 1$, and for every non-residual transition $t = (q, \sigma, r)$ that is not in \mathcal{S} due to $\mathcal{L}(\mathcal{C}_r) \neq \sigma^{-1}\mathcal{L}(\mathcal{C}_q)$, we construct an MC $\mathcal{M}_t = \langle S_t, P_t, \Sigma, \ell_t \rangle$ such that, from an initial state s_0^t , there is only one ultimately periodic word w_t produced, such that $w_t \in \sigma^{-1}\mathcal{L}(\mathcal{C}_q) \setminus \mathcal{L}(\mathcal{C}_r)$.

Finally, we produce an MC \mathcal{M} , whose states are the disjoint union of the MCs \mathcal{M}_{Σ} , \mathcal{M}_q and \mathcal{M}_t from above. The labelling and transitions within the MCs \mathcal{M}_q and \mathcal{M}_t are preserved

while, from the states in S_{Σ} , \mathcal{M} also transitions to all initial states of the individual \mathcal{M}_q and \mathcal{M}_t from above. It remains to specify the probabilities for the transitions from S_{Σ} : any state in S_{Σ} transitions to its successors uniformly at random.

▶ **Lemma 15.** If $\mathcal{L}(S) \neq \mathcal{L}(T)$ or S is not GFG, the candidate NBW C is not GFM.

Proof. Assume $\mathcal{L}(S) \neq \mathcal{L}(T)$. There must exist a word $w = \sigma_0, \sigma_1, \ldots \in \mathcal{L}(T) \setminus \mathcal{L}(S)$.

Let us use \mathcal{M} with initial state s_{σ_0} as the MC which witnesses that \mathcal{C} is not GFM. We first build the product MDP $\mathcal{M} \times \mathcal{C}$. There is a non-zero chance that, no matter how the choices of \mathcal{C} (thus, the product MDP $\mathcal{M} \times \mathcal{C}$) are resolved, a state sequence $(s_{\sigma_0}, q_0), (s_{\sigma_1}, q_1), \ldots, (s_{\sigma_k}, q_k)$ with $k \geq 0$ is seen, and \mathcal{C} selects a successor q such that (q_k, σ_k, q) is not a transition in \mathcal{S} .

For the case that this is because C_q is not QGFM, we observe that there is a non-zero chance that the product MDP moves to (s_0^q, q) , such that $\operatorname{PSyn}_{\mathcal{C}}^{\mathcal{M}}(s_{\sigma_0}) < \operatorname{PSem}_{\mathcal{C}}^{\mathcal{M}}(s_{\sigma_0})$ follows.

For the other case that this is because the transition $t = (q_k, \sigma_k, q)$ is non-residual, that is, $\mathcal{L}(\mathcal{C}_q) \neq \sigma_k^{-1} \mathcal{L}(\mathcal{C}_{q_k})$, we observe that there is a non-zero chance that the product MDP moves to (s_0^t, q) , such that $\operatorname{PSyn}_{\mathcal{C}}^{\mathcal{M}}(s_{\sigma_0}) < \operatorname{PSem}_{\mathcal{C}}^{\mathcal{M}}(s_{\sigma_0})$ follows.

For the case that S is not GFG, no matter how the nondeterminism of C is resolved, there must be a shortest word $w = \sigma_0, \ldots, \sigma_k$ $(k \ge 0)$ such that w is a prefix of a word in $\mathcal{L}(S)$, but the selected control leaves S. For this word, we can argue in the same way as above.

Lemma 14 and Lemma 15 suggest that GFM-ness of a NBW can be decided in EXPTIME by checking whether the criterion holds or not. However, as shown in the next section that QGFM = GFM, we can apply the EXPTIME procedure from Section 5 to check QGFM-ness, and thus, GFM-ness.

$5.2 \quad QGFM = GFM$

To show that QGFM = GFM, we show that the same criterion from the previous section is also sufficient and necessary for QGFM. By definition, if an NBW is GFM then it is QGFM. Thus, the sufficiency of the criterion follows from Lemma 14. We are left to show the necessity of the criterion. To do that, we build an MC \mathcal{M}' witnessing that \mathcal{C} is not QGFM in case the criterion of Lemma 15 is not satisfied. We sketch in Figure 9 the construction of \mathcal{M}' .

The principle difference between the MC \mathcal{M}' constructed in this section and \mathcal{M} from the previous section is that the new MC \mathcal{M}' needs to satisfy that $\operatorname{PSem}_{\mathcal{C}}^{\mathcal{M}'}(s_0) = 1$ (s_0 is the initial state of \mathcal{M}'), while still forcing the candidate NBW \mathcal{C} to make decisions that lose probability of success, leading to $\operatorname{PSyn}_{\mathcal{C}}^{\mathcal{M}'}(s_0) < 1$. This makes the construction of \mathcal{M}' more complex, but establishes that qualitative and full GFM are equivalent properties.

The MC will also be constructed by parts and it has a central part. It will also have the MCs \mathcal{M}_q for each non QGFM state q and \mathcal{M}_t for each non-residual transition t from the previous section. We now describe the three potential problems of \mathcal{M} of the previous section and the possible remedies.

The first potential problem is in the central part as it might contain prefixes that cannot be extended to words in $\mathcal{L}(\mathcal{C})$. Such prefixes should be excluded. This can be addressed by building a cross product with a deterministic safety automaton that recognises all the prefixes of $\mathcal{L}(\mathcal{C})$ (the safety hull of the language of \mathcal{C}) and removing the states that have no outgoing transitions in the product.

The second problem is caused by the transitions to all MCs \mathcal{M}_q and \mathcal{M}_t from every state in the central part. This can, however, be avoided by including another safety automaton in

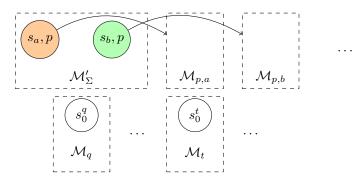


Figure 9 An illustration of the MC in the proof of Lemma 16. In this example, we have $\Sigma = \{a, b\}$. The new central part \mathcal{M}'_{Σ} is obtained by removing the states that have no outgoing transitions in the cross product of \mathcal{M}_{Σ} and \mathcal{D}' . For the states (s_a, p) and (s_b, p) of \mathcal{M}'_{Σ} , we have $\ell((s_a, p)) = a$ and $\ell((s_b, p)) = b$. For each state (s_σ, p) of the central part, we construct an MC $\mathcal{M}_{p,\sigma}$ using \mathcal{D}' . There is a transition from each state (s_σ, p) of \mathcal{M}'_{Σ} to the initial state of $\mathcal{M}_{p,\sigma}$. The MCs \mathcal{M}_q and \mathcal{M}_t are as before. Whether there is a transition from a state from \mathcal{M}'_{Σ} to the MCs \mathcal{M}_q and \mathcal{M}_t is determined by the overestimation provided by \mathcal{D}' .

the product that tracks, which of these transitions *could* be used by our candidate NBW C at the moment, and only using those transitions. Note that this retains all transitions used in the proof to establish a difference between $\operatorname{PSem}_{\mathcal{C}}^{\mathcal{M}'}(s_0)$ and $\operatorname{PSyn}_{\mathcal{C}}^{\mathcal{M}'}(s_0)$ while guaranteeing that the semantic probability of winning after progressing to \mathcal{M}_q or \mathcal{M}_t is 1.

Removing the transitions to some of the \mathcal{M}_q and \mathcal{M}_t can potentially create a third problem, namely that no transitions to \mathcal{M}_q and \mathcal{M}_t are left so that there is no way leaving the central part of the MC. To address this problem, we can create a new MC for each state in the central part so that a word starting from the initial state of the MC can always be extended to an accepting word in $\mathcal{L}(\mathcal{C})$ by transitioning to this new MC. How such MCs can be constructed is detailed later.

Zooming in on the construction, the MC \mathcal{M} should satisfy that all the finite runs that start from an initial state s_0 , before transitioning to \mathcal{M}_q or \mathcal{M}_t , can be extended to a word in the language of \mathcal{C} are retained. The language of all such initial sequences is a safety language, and it is easy to construct an automaton that (1) recognises this safety language and (2) retains the knowledge of how to complete each word in the language of \mathcal{C} . To create this automaton, we first determinise \mathcal{C} to a deterministic parity word automaton \mathcal{D} [14]. We then remove all non-productive states from \mathcal{D} and mark all states final, yielding the safety automaton \mathcal{D}' .

The two automata \mathcal{D}' and \mathcal{D} can be used to address all the problems we have identified. To address the first problem, we build a cross product MC of \mathcal{M}_{Σ} (the central part of \mathcal{M} in last section) and \mathcal{D}' . We then remove all the states in the product MC without any outgoing transitions and make the resulting MC the new central part denoted by \mathcal{M}'_{Σ} . Every state in \mathcal{M}'_{Σ} is of the form (s_{σ}, p) where $s_{\sigma} \in S_{\Sigma}$ is from \mathcal{M}_{Σ} and $p \in \mathcal{D}'$.

The states of the deterministic automaton \mathcal{D} (and thus those of \mathcal{D}') also provide information about the possible states of \mathcal{C} that could be after the prefix we have seen so far. To address the second problem, we use this information to overestimate whether \mathcal{C} could be in

⁷ From every state p in \mathcal{D}' , we can construct an extension to an accepted word by picking an accepted lasso path through \mathcal{D} that starts from p. Note that \mathcal{D}' is not complete, but every state has some successor.

some state q, or use a transition t, which in turn is good enough for deciding whether or not to transition to the initial states of \mathcal{M}_q resp. \mathcal{M}_t from every state of the new central part.

To address the third problem, we build, for every state p of \mathcal{D}' and every letter $\sigma \in \Sigma$ such that σ can be extended to an accepted word from state p, an MC $\mathcal{M}_{p,\sigma}$ that produces a single ω -regular word (sometimes referred to as lasso word) $w_{p,\sigma}$ with probability 1. The word $\sigma \cdot w_{p,\sigma}$ will be accepted from state p (or: by \mathcal{D}_p). From every state (s_{σ}, p) of the central part, there is a transition to the initial state of $\mathcal{M}_{p,\sigma}$.

The final MC transitions uniformly at random, from a state (s_{σ}, p) in \mathcal{M}'_{Σ} , to one of its successor states, which comprise the initial state of $\mathcal{M}_{p,\sigma}$ and the initial states of some of the individual MCs \mathcal{M}_q and \mathcal{M}_t .

▶ **Lemma 16.** If $\mathcal{L}(S) \neq \mathcal{L}(T)$ or S is not GFG, the candidate NBW C is not QGFM.

Proof. The proof of the difference in the probability of winning in case $\mathcal{L}(\mathcal{S}) \neq \mathcal{L}(\mathcal{T})$ or in case \mathcal{S} is not GFG are the same as in Lemma 15.

We additionally have to show that $\operatorname{PSem}_{\mathcal{C}}^{\mathcal{M}'}(s_0) = 1$. But this is easily provided by the construction: when we move on to some $\mathcal{M}_{p,\sigma}$, \mathcal{M}_q , or \mathcal{M}_t , we have sure, almost sure, and sure satisfaction, respectively, of the property by construction, while staying for ever in the central part of the new MC happens with probability 0.

By definition, if a candidate NBW \mathcal{C} is GFM, it is QGFM. Together with Lemma 14 and Lemma 16, we have that $\mathcal{L}(\mathcal{S}) = \mathcal{L}(\mathcal{T})$ and \mathcal{S} is GFG iff the candidate NBW \mathcal{C} is QGFM. Thus, we have

- ▶ **Theorem 17.** The candidate NBW C is GFM if, and only if, C is QGFM.
 - By Theorem 13 and Theorem 17, we have:
- ▶ Corollary 18. The problem of whether or not a given NBW is GFM is in EXPTIME.

Likewise, by Theorem 9, Theorem 10, and Theorem 17, we have:

▶ Corollary 19. The problem of whether or not a given NBW is QGFM is PSPACE-hard. Given a QGFM NBW and a bound k, the problem whether there is a language equivalent QGFM NBW with at most k states is PSPACE-hard. It is PSPACE-hard even for (fixed) k=2.

6 Discussion

We have started out with introducing the prima facie simpler auxiliary concept of *qualitative* GFM-ness.

We have then established that deciding GFM-ness is PSPACE-hard by a reduction from the NFA universality problem and developed an algorithm for checking *qualitative* GFM-ness in EXPTIME.

We then closed with first characterising GFM-ness with a heavy use of QGFM-ness tests, only to find that this characterisation also proves to be a necessary requirement for QGFM-ness, which led to a collapse of the qualitative and full GFM-ness. The hardness results for GFM-ness therefore carry over to QGFM-ness, while the decision procedure for QGFM-ness proves to be a decision procedure for GFM-ness by itself.

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A Detailed Construction of Section 4

A.1 Tree Automata

Trees. Consider a finite set $\Upsilon = \{v_1, \dots, v_n\}$ of directions. An Υ -tree is a prefixed closed set $T \subseteq \Upsilon^*$. A run or path r of a tree is a set $r \subseteq T$ such that $\epsilon \in T$ and, for every $w \in r$, there exists a unique $v \in \Upsilon$ such that $w \cdot v \in r$. For a node $w \in T$, $\operatorname{succ}(w) \subseteq \Upsilon$ denotes exactly those directions v, such that $w \cdot v \in T$. A tree is called *closed* if it is non-empty (i.e. if it contains the empty word ε) and every node of the tree has at least one successor.

Given a finite alphabet Σ , a Σ -labelled Υ -tree is a pair $\langle T, V \rangle$ where T is a closed Υ -tree and $V: T \to \Sigma$ maps each node of T to a letter in Σ .

An Υ -tree T and a labelled Υ -tree $\langle T, V \rangle$ are called *full* if $T = \Upsilon^*$.

Tree automata. For the infinite tree automata in this paper, we are using transition-based acceptance conditions. An alternating parity tree automaton is a tuple $\mathcal{A} = (\Sigma, Q, q_0, \delta, \alpha)$, where Q denotes a finite set of states, $q_0 \in Q$ denotes an initial state, $\delta: Q \times \Sigma \to \mathbb{B}^+(\Upsilon \times Q \times C)$ denotes⁸, for a finite set $C \subset \mathbb{N}$ of colours, a transition function and the acceptance condition α is a transition-based parity condition. An alternating automaton runs on Σ -labelled Υ -trees. The acceptance mechanism is defined in terms of run trees.

A run of an alternating automaton \mathcal{A} on a Σ -labelled Υ -tree $\langle T, V \rangle$ is a tree $\langle T_r, r \rangle$ in which each node is labelled with an element of $T \times Q$. Unlike T, in which each node has at most $|\Upsilon|$ children, the tree T_r may have nodes with many children and may also have leaves (nodes with no children). Thus, $T_r \subset \mathbb{N}^*$ and a path in T_r may be either finite, in which case it ends in a leaf, or infinite. Formally, $\langle T_r, r \rangle$ is the $(T \times Q)$ -labelled tree such that:

- $\epsilon \in T_r \text{ and } r(\epsilon) = (\epsilon, q_0);$
- Consider a node $w_r \in T_r$ with $r(w_r) = (w, q)$ and $\theta = \delta(q, V(w))$. Then there is a (possibly empty) set $S \subseteq \Upsilon \times Q \times C$ such that S satisfies θ , and for all $(c, q', i) \in S$, the following hold: if $c \in \mathsf{succ}(w)$, then there is a $j \in \mathbb{N}$ such that $w_r \cdot j \in T_r$, $r(w_r \cdot j) = (w \cdot c, q')$ and the transition in T_r between w_r and $w_r \cdot j$ is assigned colour i.

An infinite path of the run tree fulfils the parity condition, if the highest colour of the transitions (edges) appearing infinitely often on the path is even. A run tree is accepting if all infinite paths fulfil the parity condition. A tree T is accepted by an automaton \mathcal{A} if there is an accepting run tree of \mathcal{A} over T. We denote by $\mathcal{L}(\mathcal{A})$ the language of the automaton \mathcal{A} ; i.e., the set of all labelled trees that \mathcal{A} accepts.

The acceptance of a tree can also be viewed as the outcome of a game, where player accept chooses, for a pair $(q, \sigma) \in Q \times \Sigma$, a set of atoms satisfying $\delta(q, \sigma)$, and player reject chooses one of these atoms, which is executed. The tree is accepted iff player accept has a strategy enforcing a path that fulfils the parity condition. One of the players has a memoryless winning strategy, that is, a strategy where the moves only depend on the state of the automaton and the node of the tree, and, for player reject, on the choice of player accept in the same move.

A nondeterministic tree automaton is a special alternating tree automaton, where the image of δ consists only of such formulas that, when rewritten in disjunctive normal form, contains exactly one element of $\{v\} \times Q \times C$ for every $v \in \Upsilon$ in every disjunct. A nondeterministic tree automaton is deterministic if the image of δ consists only of such formulas

⁸ $\mathbb{B}^+(\Upsilon \times Q \times C)$ are positive Boolean formulas: formulas built from elements in $\Upsilon \times Q \times C$ using \wedge and \vee . For a set $S \subseteq \Upsilon \times Q \times C$ and a formula $\theta \in \mathbb{B}^+(\Upsilon \times Q \times C)$, we say that S satisfies θ iff assigning **true** to elements in S and assigning **false** to elements not in S makes θ true. For technical convenience, we use *complete* automata, i.e. \mathbb{B}^+ does not contain *true* or *false*, so that our run trees are closed.

that, when rewritten in disjunctive normal form, contain only one disjunct. An automaton is universal when the image of δ consists only of formulas that can be rewritten as conjunctions.

An automaton is called a Büchi automaton if $C = \{1, 2\}$ and a co-Büchi automaton if $C = \{0, 1\}.$

Symmetric Automata. Symmetric automata are alternating automata that abstract from the concrete directions, replacing them by operators $\Omega = \{\Box, \Diamond, \boxtimes_i\}$, where \Box and \Diamond are standard operators that mean 'send to all successors (\Box) ' and 'send to some successor (\diamondsuit) ', respectively.

A symmetric (alternating) parity tree automaton is a tuple $\mathcal{S} = (\Sigma, Q, q_0, \delta, \alpha)$ that differs from ordinary alternating automata only in the transition function. Here, $\delta: Q \times \Sigma \to \mathbb{R}$ $\mathbb{B}^+(\Omega \times Q \times C)$, where, for a node w in a labelled tree $\langle T, V \rangle$,

$$(\boxtimes_j^v,q,i)$$
 can (subsequently) be replaced by $(v,q,j) \land \bigwedge_{v \neq c \in \mathsf{succ}(w)} (c,q,i)$

An alternating parity tree automaton is nearly symmetric if there is \boxtimes_i^v in the transition function, that is, $\delta: Q \times \Sigma \to \mathbb{B}^+ ((\Omega \cup \{\boxtimes_i^v \mid j \in C \land v \in \Upsilon\}) \times Q \times C)$.

While the \boxtimes_i and \boxtimes_i^v operators are less standard, they have a simple semantics: (\boxtimes_i, q, i) sends q in every direction (much like \square), but uses different colours: i in all directions but one, and j into one (arbitrary) direction; \bigotimes_{i}^{v} is similar, but fixes the direction to v into which q is sent with a different colour.

So, while nondeterministic tree automata send exactly one copy to each successor, symmetric automata can send several copies to the same successor. On the other hand, symmetric automata cannot distinguish between left and right and can send copies to successor nodes only in either a universal or an existential manner.

In this paper, we use common three-letter abbreviations to distinguish types of automata. The first (D, N, U, A) tells the automaton is deterministic, nondeterministic, universal or alternating if it does not belong to any of the previous types. The second denotes the acceptance condition (B for Büchi, C for co-Büchi, P for parity if it is neither Büchi nor co-Büchi). The third letter (W, T, S, or N) says that the automaton is an ω -word automaton (W) or runs on infinite trees (T, S or N); S is only used when the automaton is symmetric, N is only used when the automaton is nearly symmetric, while T is used when it is neither symmetric nor nearly symmetric. For example, an NBW is a nondeterministic Büchi ω -word automaton, and a DCT is a deterministic co-Büchi tree automaton.

Detailed Constructions A.2

An MC and an initial state induces a (state-labelled) tree by unravelling and disregarding the probabilities. See Figure 2 for an example. We say an MC with an initial state is accepted by a tree automaton if the tree induced by the MC and the initial state is accepted by that tree automaton.

From Word Automata to Tree Automata. From the candidate NBW $\mathcal{C}=$ $\langle \Sigma, Q, \delta, q_0, F \rangle$, we build a symmetric alternating Büchi tree automaton (ABS) $\mathcal{T}_{\mathcal{C}}$ $\langle \Sigma, Q, \delta_{\mathcal{T}}, q_0, \alpha \rangle \text{ where } \delta_{\mathcal{T}}(q, \sigma) = \bigvee_{q' \in F \cap \delta(q, \sigma)} (\square, q', 2) \vee \bigvee_{q' \in (Q \setminus F) \cap \delta(q, \sigma)} (\square, q', 2) \wedge (\lozenge, q', 1)$ for $(q,\sigma) \in \mathsf{support}(\delta)$ and the accepting condition α is a Büchi one (and thus a parity condition with two colours), which specifies that a run is accepting if, and only if, the highest colour that occurs infinitely often is even. In fact, all other accepting conditions in this section are parity ones. A run tree is accepted by the ABS $\mathcal{T}_{\mathcal{C}}$ if colour 2 is seen infinitely often on all infinite paths. Similar constructions can be found in [16, Section 3].

▶ **Lemma 20.** The ABS $\mathcal{T}_{\mathcal{C}}$ accepts an MC \mathcal{M} with initial state s_0 if, and only if, the syntactic product of \mathcal{M} and \mathcal{C} almost surely accepts, that is, $\operatorname{PSyn}_{\mathcal{C}}^{\mathcal{M}}(s_0) = 1$.

Proof. Consider an arbitrary MC \mathcal{M} with initial state s_0 . Let Υ be the set of states in the MC \mathcal{M} . Let $\langle T, V \rangle$ be the Σ -labelled Υ -tree induced by \mathcal{M} with s_0 .

'if:' We have that $\operatorname{PSyn}_{\mathcal{C}}^{\mathcal{M}}(s_0) = 1$. As Büchi MDPs have positional optimal strategies [1], there is a positional strategy μ^{\times} for the product MDP, such that $\operatorname{PSyn}_{\mathcal{C}}^{\mathcal{M}}(s_0) = \operatorname{PSyn}_{\mathcal{C}}^{\mathcal{M}}((s_0, q_0), \mu^{\times})) = 1$ holds.

For every reachable state in the Markov chain $(\mathcal{M} \times \mathcal{C})_{\mu^{\times}}$, there is a (not necessarily unique) shortest path to the next final state, and for each state (s,q) of $(\mathcal{M} \times \mathcal{C})_{\mu^{\times}}$, we pick one such direction $d_{(s,q)} \in \Upsilon$ of \mathcal{M} (note that the direction in \mathcal{C} is determined by μ^{\times}).

The length of every path that follows these directions to the next final state is at most the number of states of $\mathcal{M} \times \mathcal{C}$.

Thus, we choose the direction $d_{s,q}$ when the \Diamond is resolved in a state s of the (unravelled) MC \mathcal{M} and $\mathcal{T}_{\mathcal{C}}$ is in state q, there is no consecutive section in any path of the run tree longer than this without a colour 2. Thus, the dominating colour of every path for this strategy is 2, and (the unravelling of) \mathcal{M} is accepted by $\mathcal{T}_{\mathcal{C}}$.

'only if': We consider the set of states of (s,q) such that the automaton can be in a state q and in a node that ends in a state s of the Markov chain, and denote by $Q_{s,q} \subseteq \delta(q, \ell(s))$ the set of successor states selected at any such state reachable in a given accepting run.

For every run, this defines an MDP $\mathcal{M}' \subseteq \mathcal{M} \times \mathcal{C}$, and as the run we use is accepting, a final state in \mathcal{M}' is reachable from every state in \mathcal{M}' . (To see this, assume for contradiction that \mathcal{M}' contains a state, from which no final state is reachable. Then we can henceforth follow the path from any such state defined by the \Diamond choices, this path has henceforth only colour 1, and is therefore rejecting. This contradicts the assumption that our run is accepting.)

Therefore, there is a shortest path, and we can fix, for every state (s,q) a successor in $q_{(s,q)} \in Q_{s,q}$ such that a minimal path from (s,q) takes this decision. For the resulting positional strategy μ , the minimal distance to the next final state for \mathcal{M}'_{μ} is the same as for \mathcal{M}' , and thus finite for every reachable state. Thus, all reachable end-components in \mathcal{M}'_{μ} —and thus all reachable end-components in $(\mathcal{M} \times \mathcal{C})_{\mu}$ —contain a final state. We therefore have $\operatorname{PSyn}^{\mathcal{M}}_{\mathcal{C}}((s_0, q_0), \mu)) = 1$.

Next, we dualise $\mathcal{T}_{\mathcal{C}}$ into a symmetric alternating co-Büchi tree automaton (ACS), which is a complement of the ABS $\mathcal{T}_{\mathcal{C}}$. In other words, this automaton accepts an MC \mathcal{M} if, and only if, the syntactical product of \mathcal{C} and \mathcal{M} does not almost surely accept. Dualising the ABS $\mathcal{T}_{\mathcal{C}}$ consists of switching the roles of player accept and player reject, complementing the transition function and decreasing the colours by one. Thus, we define the ACS $\widetilde{\mathcal{T}}_{\mathcal{C}} = \langle \Sigma, Q, \delta_{\widetilde{\mathcal{T}}}, q_0, \widetilde{\alpha} \rangle$ where $\delta_{\widetilde{\mathcal{T}}}(q, \sigma) = \bigwedge_{q' \in F \cap \delta(q, \sigma)} (\lozenge, q', 1) \wedge \bigwedge_{q' \in (Q \setminus F) \cap \delta(q, \sigma)} (\lozenge, q', 1) \vee (\square, q', 0)$ for $(q, \sigma) \in \text{support}(\delta)$. Since 0 and 1 are the only colours, $\widetilde{\mathcal{T}}_{\mathcal{C}}$ is a co-Büchi automaton, and a run tree is accepted when colour 1 is seen only finitely often on all infinite paths.

▶ **Lemma 21.** The ACS $\mathcal{T}_{\mathcal{C}}$ accepts the complement of $\mathcal{L}(\mathcal{T}_{\mathcal{C}})$.

Proof. To prove $\mathcal{L}(\mathcal{T}'_{\mathcal{G}}) \subseteq \mathcal{L}(\mathcal{T}_{\mathcal{G}})$, we show that any tree rejected by $\mathcal{T}_{\mathcal{G}}$ is rejected by $\mathcal{T}'_{\mathcal{G}}$, that is, for a tree t, if player reject in $\mathcal{T}_{\mathcal{G}}$ has a winning strategy, player reject in $\mathcal{T}'_{\mathcal{G}}$ also has a winning strategy. It is trivially true as player reject in $\mathcal{T}'_{\mathcal{G}}$ can just use the winning strategy by player reject in $\mathcal{T}_{\mathcal{G}}$, since the set of choices of player reject in $\mathcal{T}'_{\mathcal{G}}$ is a superset of that in $\mathcal{T}_{\mathcal{G}}$.

Assume that player reject in $\mathcal{T}'_{\mathcal{G}}$ has a memoryless winning strategy over an input tree. We modify this winning strategy such that player reject sends a copy of the automaton in state q with colour 1 regardless of which colour is chosen in the original winning strategy, whenever player reject has the choice of sending with either colour 1 or 2. We argue that this modified strategy is also a winning strategy for player reject. The run tree induced by the modified strategy remains the same with possibly more colour 1's on some edges of the run tree. For any tree rejected by $\mathcal{T}'_{\mathcal{G}}$, player reject in $\mathcal{T}_{\mathcal{G}}$ can use this modified winning strategy to win the rejection game. It follows that $\mathcal{L}(\mathcal{T}_{\mathcal{G}}) \subseteq \mathcal{L}(\mathcal{T}'_{\mathcal{G}})$.

From an NBW $\mathcal{G} = \langle \Sigma, Q_{\mathcal{G}}, \delta_{\mathcal{G}}, q_0^{\mathcal{G}}, F_{\mathcal{G}} \rangle$ that is known to be GFM and language equivalent to \mathcal{C} ($\mathcal{L}(\mathcal{G}) = \mathcal{L}(\mathcal{C})$), similar to \mathcal{C} , we can construct an ABS $\mathcal{T}'_{\mathcal{G}}$. Instead, we construct a symmetric nondeterministic Büchi tree automaton (NBS) $\mathcal{T}_{\mathcal{G}} = \langle \Sigma, Q_{\mathcal{G}}, \delta_{\mathcal{T}_{\mathcal{G}}}, q_0^{\mathcal{G}}, \alpha_{\mathcal{G}} \rangle$ where $\delta_{\mathcal{T}_{\mathcal{G}}}(q,\sigma) = \bigvee_{q' \in F_{\mathcal{G}} \cap \delta_{\mathcal{G}}(q,\sigma)} (\Box, q', 2) \vee \bigvee_{q' \in (Q_{\mathcal{G}} \setminus F_{\mathcal{G}}) \cap \delta_{\mathcal{G}}(q,\sigma)} (\boxtimes_1, q', 2)$ for all $(q,\sigma) \in \text{support}(\delta_{\mathcal{G}})$. The automaton $\mathcal{T}_{\mathcal{G}}$ differs from $\mathcal{T}'_{\mathcal{G}}$ only in the transition functions for the successor states that are non-accepting: in $\mathcal{T}_{\mathcal{G}}$, a copy of the automaton in a non-accepting state q' is sent to some successor with colour 1 and all the other successors with colour 2; in $\mathcal{T}'_{\mathcal{G}}$, a non-accepting state q' is sent to some successor with both colour 1 and 2. The following lemma claims that these two tree automata are language equivalent:

▶ Lemma 22. $\mathcal{L}(\mathcal{T}_{\mathcal{G}}) = \mathcal{L}(\mathcal{T}'_{\mathcal{G}})$.

By Lemma 20 and Lemma 22, we have:

▶ **Lemma 23.** The NBS $\mathcal{T}_{\mathcal{G}}$ accepts an MC \mathcal{M} with initial state s_0 if, and only if, the syntactic product of \mathcal{G} and \mathcal{M} almost surely accepts, that is, $\operatorname{PSyn}_{\mathcal{G}}^{\mathcal{M}}(s_0) = 1$.

Strategy of the ACS $\widetilde{\mathcal{T}}_{\mathcal{C}}$. Define a function $\delta': Q \to \{\Box\} \cup \Upsilon$ such that a state $q \in Q$ is mapped to an $v \in \Upsilon$ if $q \in F$ and is mapped to either \Box or an $v \in \Upsilon$ if $q \notin F$. Let $\Delta_{\mathcal{C}}$ be the set of such functions. Intuitively, a function δ' can be considered as the strategy of player accept at a tree node. Since an input tree is accepted by a parity automaton iff player accept has a memoryless winning strategy on the tree, we can define the strategy of $\widetilde{\mathcal{T}}_{\mathcal{C}}$ for a Σ -labelled Υ -tree $\langle T, V \rangle$ as an $\Delta_{\mathcal{C}}$ -labelled tree $\langle T, f_{\mathcal{C}} \rangle$ where $f_{\mathcal{C}}: T \to \Delta_{\mathcal{C}}$. A strategy $\langle T, f_{\mathcal{C}} \rangle$ induces a single run $\langle T_r, r \rangle$ of $\widetilde{\mathcal{T}}_{\mathcal{C}}$ on $\langle T, V \rangle$. The strategy is winning if the induced run is accepting. Whenever the run $\langle T_r, r \rangle$ is in state q as it reads a node $w \in T$, it proceeds according to $f_{\mathcal{C}}(w)$. Formally, $\langle T_r, r \rangle$ is the $\langle T \times Q \rangle$ -labelled tree such that:

- $\epsilon \in T_r \text{ and } r(\epsilon) = (\epsilon, q_0);$
- Consider a node $w_r \in T_r$ with $r(w_r) = (w, q)$. Let $\sigma = V(w)$. For all $q' \in \delta(q, \sigma)$ and $c = f_{\mathcal{C}}(w)(q')$ we have the following two cases:
 - 1. if $c = \square$, then for each successor node w' of w, there is a $j \in \mathbb{N}$ such that $w_r \cdot j \in T_r$, $r(w_r \cdot j) = (w', q')$, the transition in T_r between w_r and $w_r \cdot j$ is assigned colour 0;
 - 2. if $c \in \Upsilon$ and $w \cdot c \in \mathsf{succ}(w)$, then there is a $j \in \mathbb{N}$ such that $w_r \cdot j \in T_r$, $r(w_r \cdot j) = (w \cdot c, q')$ and the transition in T_r between w_r and $w_r \cdot j$ is assigned colour 1.
- ▶ **Lemma 24.** The ACS $\widetilde{\mathcal{T}}_{\mathcal{C}}$ accepts a Σ -labelled Υ -tree $\langle T, V \rangle$ if, and only if, there exists a winning strategy $\langle T, f_{\mathcal{C}} \rangle$ for the tree.

Strategy of the NBS $\mathcal{T}_{\mathcal{G}}$. Let $\Delta_{\mathcal{G}} \subseteq (\{\Box\} \cup \Upsilon) \times Q_{\mathcal{G}}$ such that $(\Box, q) \in \Delta_{\mathcal{G}}$ if $q \in F_{\mathcal{G}}$ and $(v, q) \in \Delta_{\mathcal{G}}$ for some $v \in \Upsilon$ if $q \notin F_{\mathcal{G}}$. An item in the set $\Delta_{\mathcal{G}}$ can be considered as the strategy of player *accept* at a tree node in the acceptance game of $\mathcal{T}_{\mathcal{G}}$. Intuitively, by choosing both, the atoms that are going to be satisfied and the directions in which the existential requirements are going to be satisfied, an item in $\Delta_{\mathcal{G}}$ resolves all the nondeterminism in $\delta_{\mathcal{T}_{\mathcal{G}}}$.

A tree is accepted if, and only if, player accept has a memoryless winning strategy. Thus, we can define player accept's strategy of $\mathcal{T}_{\mathcal{G}}$ on a Σ -labelled Υ -tree $\langle T, V \rangle$ as a $\Delta_{\mathcal{G}}$ -labelled tree $\langle T, f_{\mathcal{G}} \rangle$ with $f_{\mathcal{G}} : T \to \Delta_{\mathcal{G}}$. A strategy $\langle T, f_{\mathcal{G}} \rangle$ induces a single run $\langle T_r, r \rangle$ and it is winning if, and only if, the induced run is accepting.

Whenever the run $\langle T_r, r \rangle$ is in state q as it reads a node $w \in T$, it proceeds according to $f_{\mathcal{G}}(w)$. Formally, $\langle T_r, r \rangle$ is the $(T \times Q_{\mathcal{G}})$ -labelled tree such that:

- $\epsilon \in T_r \text{ and } r(\epsilon) = (\epsilon, q_0^{\mathcal{G}});$
- Consider a node $w_r \in T_r$ with $r(w_r) = (w, q)$. Let $\sigma = V(w)$ and $(c, q') = f_{\mathcal{G}}(w)$. We have the following two cases:
 - 1. if $c = \square$, then $q' \in F_{\mathcal{G}}$ and, for each successor node w' of w, there is a $j \in \mathbb{N}$ s.t. $w_r \cdot j \in T_r$ and $r(w_r \cdot j) = (w', q')$, and the transition in T_r between w_r and $w_r \cdot j$ is assigned colour 2;
 - 2. if $c \in \Upsilon$ and $w \cdot c \in \mathsf{succ}(w)$, then $q' \notin F_{\mathcal{G}}$ and there is a $j \in \mathbb{N}$ such that $w_r \cdot j \in T_r$, $r(w_r \cdot j) = (w \cdot c, q')$, and the transition in T_r between w_r and $w_r \cdot j$ is assigned colour 1. Also, for all successor node w' of w with $w' \neq w \cdot c$, there is a $j \in \mathbb{N}$ s.t. $w_r \cdot j \in T_r$, $r(w_r \cdot j) = (w', q')$, and the transition in T_r between w_r and $w_r \cdot j$ is assigned the colour 2.
- ▶ Lemma 25. The NBS $\mathcal{T}_{\mathcal{G}}$ accepts a Σ -labelled Υ -tree $\langle T, V \rangle$ if, and only if, there exists a winning strategy $\langle T, f_{\mathcal{G}} \rangle$ for the tree.

Making Strategies Explicit. Annotating input trees with functions from $\Delta_{\mathcal{C}}$ and $\Delta_{\mathcal{G}}$ enables us to transform the ACS $\widetilde{\mathcal{T}}_{\mathcal{C}}$ to a universal co-Büchi tree automaton (UCT) and the NBS $\mathcal{T}_{\mathcal{G}}$ to a nearly symmetric deterministic Büchi tree automaton (DBN).

We first define a UCT that runs on $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled Υ -trees $\mathcal{U}_{\mathcal{C}} = \langle \Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}} \rangle$. For all $q \in Q$ and $(\sigma, \delta', g) \in \Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}}$, define $\delta_{\mathcal{U}_{\mathcal{C}}}(q, \langle \sigma, \delta', g \rangle) = \bigwedge_{q' \in \delta(q, \sigma) \wedge \delta'(q') = \square} (\square, q', 0) \wedge \bigwedge_{q' \in \delta(q, \sigma) \wedge \upsilon = \delta'(q') \in \Upsilon} (\upsilon, q', 1)$. Then, we have:

▶ **Lemma 26.** The UCT $\mathcal{U}_{\mathcal{C}}$ accepts a tree $\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle$ if, and only if, the ACS $\widetilde{\mathcal{T}}_{\mathcal{C}}$ accepts $\langle T, V \rangle$ with $\langle T, f_{\mathcal{C}} \rangle$ as the winning strategy.

Next, we define a DBN that runs on $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled Υ -trees $\mathcal{D}_{\mathcal{G}} = \langle \Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}} \rangle$ where $\delta_{\mathcal{D}_{\mathcal{G}}}(q, \langle \sigma, \delta', (c, q') \rangle) = \begin{cases} (\Box, q', 2) & \text{if } c = \Box \\ (\boxtimes_1^c, q', 2) & \text{if } c \in \Upsilon \end{cases}$ for all $q \in Q_{\mathcal{G}}$ and $(\sigma, \delta', (c, q')) \in \Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}}$. Then, we have:

▶ **Lemma 27.** The DBN $\mathcal{D}_{\mathcal{G}}$ accepts a tree $\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle$ if, and only if, the NBS $\mathcal{T}_{\mathcal{G}}$ accepts $\langle T, V \rangle$ with $\langle T, f_{\mathcal{G}} \rangle$ as the winning strategy.

Tree Widening. The operator hide_Y maps a tree node $w \in (X \times Y)^*$ to tree node $w' \in X^*$ such that w' is obtained by replacing each $\langle x,y \rangle$ in w by x. The operator wide_Y maps a Σ -labelled Υ -tree $\langle T,V \rangle$ to a Σ -labelled $(\Upsilon \times Y)$ -tree $\langle T',V' \rangle$ such that, for every node $w \in T'$, we have $V'(w) = V(\mathsf{hide}_Y(w))$. Let $Q_{\dot{\approx}} = Q \cup \{\dot{\approx}\}$. We widen the input trees with $Q_{\dot{\approx}}$. We then define a UCT and a DBN that accept the widened trees.

Let a UCT which runs on $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled $(\Upsilon \times Q_{\dot{\cong}})$ -trees $\mathcal{U}_{\mathcal{C}}^{w} = \langle \Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}}, Q, \delta_{\mathcal{U}_{\mathcal{C}}^{w}}, q_{0}, \alpha_{\mathcal{U}_{\mathcal{C}}^{w}} \rangle$. For all $q \in Q$ and $(\sigma, \delta', g) \in \Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}}$, define $\delta_{\mathcal{U}_{\mathcal{C}}^{w}}(q, \langle \sigma, \delta', g \rangle) = \bigwedge_{q' \in \delta(q, \sigma) \wedge \delta'(q') = \square} (\square, q', 0) \wedge \bigwedge_{q' \in \delta(q, \sigma) \wedge \upsilon = \delta'(q') \in \Upsilon} ((\upsilon, q'), q', 1)$ for all $q \in Q$ and $(\sigma, \delta', g) \in \Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}}$.

▶ Lemma 28. The UCT $\mathcal{U}_{\mathcal{C}}$ accepts a $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled tree $\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle$ if, and only if, the UCT $\mathcal{U}_{\mathcal{C}}^w$ accepts a $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled tree wide $_{Q_{\pm}}(\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle)$.

Proof. Consider the run trees of $\mathcal{U}_{\mathcal{C}}$ and $\mathcal{U}_{\mathcal{C}}^{w}$ on an arbitrary input tree $\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle$ and the corresponding widened tree wide $_{Q_{\pm}}(\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle)$, respectively. For every tree path in the run tree of $\mathcal{U}_{\mathcal{C}}$, there is the same tree path in the run tree of $\mathcal{U}_{\mathcal{C}}^{w}$. In other words, the run tree of $\mathcal{U}_{\mathcal{C}}$ is a subtree of that of $\mathcal{U}_{\mathcal{C}}^{w}$. Furthermore, consider the paths that are in the run tree of $\mathcal{U}_{\mathcal{C}}^{w}$ but are not in the run tree of $\mathcal{U}_{\mathcal{C}}$. For those paths, there are more colour 0's, which will not affect the acceptance of the run tree of the UCT $\mathcal{U}_{\mathcal{C}}^{w}$. Thus, the run tree of $\mathcal{U}_{\mathcal{C}}$ is accepting if, and only if, the run tree of $\mathcal{U}_{\mathcal{C}}^{w}$ is accepting.

Define a DBN that runs on $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled $(\Upsilon \times Q_{\hat{\pi}})$ -trees $\mathcal{D}_{\mathcal{G}}^w = \langle \Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}} \rangle$ where $\delta_{\mathcal{D}_{\mathcal{G}}^w}(q, \langle \sigma, \delta', (c, q') \rangle)$ is $(\Box, q', 2)$ if $c = \Box$ and $(\boxtimes_1^{(c, \hat{\pi})}, q', 2)$ if $c \in \Upsilon$ for all $q \in Q_{\mathcal{G}}$, $(\sigma, \delta', (c, q')) \in \Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}}$. Similar to Lemma 28, we have:

▶ **Lemma 29.** The DBN $\mathcal{D}_{\mathcal{G}}$ accepts a $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled tree $\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle$ if, and only if, the DBN $\mathcal{D}_{\mathcal{G}}^{w}$ accepts a $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled tree wide $_{Q_{\varphi}}(\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle)$.

Proof. The proof is similar to that of Lemma 28. Consider the run trees of $\mathcal{D}_{\mathcal{G}}$ and $\mathcal{D}_{\mathcal{G}}^{w}$ on an arbitrary input tree $\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle$ and the corresponding widened tree wide $_{\mathcal{Q}_{\pm}}(\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle)$, respectively. For every tree path in the run tree of $\mathcal{D}_{\mathcal{G}}$, there is the same tree path in the run tree of $\mathcal{D}_{\mathcal{G}}^{w}$. In other words, the run tree of $\mathcal{D}_{\mathcal{G}}^{w}$ is a subtree of that of $\mathcal{D}_{\mathcal{G}}^{w}$. Furthermore, consider the paths that are in the run tree of $\mathcal{D}_{\mathcal{G}}^{w}$ but are not in the run tree of $\mathcal{D}_{\mathcal{G}}$. For those paths, there are more colour 2's, which will not affect the acceptance of the run tree of the DBN $\mathcal{D}_{\mathcal{G}}^{w}$. Thus, the run tree of $\mathcal{D}_{\mathcal{G}}$ is accepting if, and only if, the run tree of $\mathcal{D}_{\mathcal{G}}^{w}$ is accepting.

Tree Pruning. Given a widened tree wide_{Q_{\pm}} $(\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle)$, the operator prune keeps, for all $w \in \mathsf{wide}_{Q_{\pm}}(\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle)$ and all $q \in Q_{\pm}$, exactly one direction (v_q, q) from $\mathsf{succ}(w)$, such that, for all $q \in Q_{\pm}$, the following holds:

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if q = \not \cong and f_{\mathcal{G}}(\mathsf{hide}_{Q_{\mathfrak{D}}}(w)) = (v, q') \in \Upsilon \times Q_{\mathcal{G}}, then v_q = v; and
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• if $q \in Q$ and $f_{\mathcal{C}}(\mathsf{hide}_{Q_{\hat{\pi}}}(w))(q) = v \in \Upsilon$ then $v_q = v$.

If $q \in Q$ and $f_{\mathcal{C}}(\mathsf{hide}_{Q_{\pm}}(w))(q) = \square$ or if $q = \; \stackrel{\sim}{\approx} \; \text{and} \; f_{\mathcal{G}}(\mathsf{hide}_{Q_{\pm}}(w)) \in \{\square\} \times Q_{\mathcal{G}}, \; \text{then} \; v_q \; \text{can} \; \text{be chosen arbitrarily from } \mathsf{succ}(\mathsf{hide}_{Q_{\pm}}(w)).$

Next, all nodes w in the pruned tree are renamed to $\mathsf{hide}_{\Upsilon}(w)$. Pruning the tree by operator prune gives us full $Q_{\mathring{\pi}}$ -trees, and thus trees with fixed branching degree $|Q_{\mathring{\pi}}|$.

Let us define a UCT that runs on full $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled $Q_{\stackrel{\circ}{\alpha}}$ -trees $\mathcal{U}_{\mathcal{C}}^p = \langle \Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}}, Q, \delta_{\mathcal{U}_{\mathcal{C}}^p}, q_0, \alpha_{\mathcal{U}_{\mathcal{C}}^p} \rangle$, where $\delta_{\mathcal{U}_{\mathcal{C}}^p}(q, \langle \sigma, \delta', g \rangle) = \bigwedge_{q' \in \delta(q, \sigma) \wedge \delta'(q') = \square} (\square, q', 0) \wedge \bigwedge_{q' \in \delta(q, \sigma) \wedge \delta'(q') \in \Upsilon} (q', q', 1)$ for all $q \in Q$ and $(\sigma, \delta', g) \in \Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}}$.

Consider the run trees of $\mathcal{U}_{\mathcal{C}}^{w}$ and $\mathcal{U}_{\mathcal{C}}^{p}$ on an arbitrary widened tree wide_{Q_{\pm}} $(\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle)$ and the corresponding pruned tree prune $(\text{wide}_{Q_{\pm}}(\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle))$, respectively. The run tree of $\mathcal{U}_{\mathcal{C}}^{p}$ is a subtree of that of $\mathcal{U}_{\mathcal{C}}^{w}$:

▶ Lemma 30. If the UCT $\mathcal{U}_{\mathcal{C}}^{w}$ accepts a $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled tree wide $_{Q_{\hat{\pi}}}(\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle)$, then the UCT $\mathcal{U}_{\mathcal{C}}^{p}$ accepts the $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled tree prune $\Big(\text{wide}_{Q_{\hat{\pi}}}\big(\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle\big)\Big)$.

Let us define a DBN that runs on full $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled $Q_{\dot{\alpha}}$ -trees $\mathcal{D}_{\mathcal{G}}^{p} = \langle \Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}} \times \Delta_{$

▶ Lemma 31. If the DBN $\mathcal{D}_{\mathcal{G}}^{w}$ accepts a $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled tree wide_{Q_{\pm}} $(\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle)$, then the DBN $\mathcal{D}_{\mathcal{G}}^{p}$ accepts the full $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled tree prune (wide_{Q_{\pm}} $(\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle)$).

Proof. Consider the run trees of $\mathcal{D}_{\mathcal{G}}^{w}$ and $\mathcal{D}_{\mathcal{G}}^{p}$ on an arbitrary widened tree wide $_{Q_{\mathcal{L}}}\left(\langle T,(V,f_{\mathcal{C}},f_{\mathcal{G}})\rangle\right)$ and the corresponding pruned tree $\mathsf{prune}\left(\mathsf{wide}_{Q_{\mathcal{L}}}\left(\langle T,(V,f_{\mathcal{C}},f_{\mathcal{G}})\rangle\right)\right)$, respectively. It follows from the fact that the run tree of $\mathcal{D}_{\mathcal{G}}^{p}$ is a subtree of the run tree of $\mathcal{D}_{\mathcal{G}}^{w}$.

Note that a tree $\mathsf{prune}\Big(\mathsf{wide}_{Q_{\hat{\pi}}}\big(\langle T,(V,f_{\mathcal{C}},f_{\mathcal{G}})\rangle\big)\Big)$ being accepted by $\mathcal{U}^p_{\mathcal{C}}$ does not imply that the tree $\langle T,V\rangle$ is accepted by $\widetilde{\mathcal{T}}_{\mathcal{C}}$. This is because some nodes in the tree T may disappear after pruning. For the same reason, that a tree $\mathsf{prune}\Big(\mathsf{wide}_{Q_{\hat{\pi}}}\big(\langle T,(V,f_{\mathcal{C}},f_{\mathcal{G}})\rangle\big)\Big)$ is accepted by $\mathcal{D}^p_{\mathcal{G}}$ does not imply that the tree $\langle T,V\rangle$ is accepted by $\mathcal{T}_{\mathcal{G}}$.

Strategy Simplification – removing the direction Υ . Noting that the Υ part of the strategies has flown into the transformations of the tree, but are no longer used by the automata, we can remove the Υ part from the strategies: Let $\Delta'_{\mathcal{C}}$ be the set of functions $Q \to \{\Box, \Diamond\}$ and $\Delta'_{\mathcal{G}} \subseteq Q_{\mathcal{G}}$. For $\Delta'_{\mathcal{G}}$, we remove $\{\Box\} \cup \Upsilon$ from $\Delta_{\mathcal{G}}$, as it is clear from the construction of the DBT $\mathcal{D}^p_{\mathcal{G}}$ and the definition of strategy $\Delta_{\mathcal{G}}$ that we have a $(\Box, q, 2)$ transition when $q \in F_{\mathcal{G}}$ is chosen and a $(\boxtimes_1^{\uparrow \alpha}, q, 2)$ transition when $q \notin F_{\mathcal{G}}$ is chosen.

We first define the UCT $\mathcal{U}'_{\mathcal{C}} = \langle \Sigma \times \Delta'_{\mathcal{C}} \times \Delta'_{\mathcal{G}}, Q, \delta_{\mathcal{U}'_{\mathcal{C}}}, q_0, \alpha_{\mathcal{U}'_{\mathcal{C}}} \rangle$ that runs on full $(\Sigma \times \Delta'_{\mathcal{C}} \times \Delta'_{\mathcal{G}})$ -labelled $Q_{\stackrel{\leftarrow}{\approx}}$ -trees. Let $\delta_{\mathcal{U}'_{\mathcal{C}}}(q, \langle \sigma, \delta', g \rangle) = \bigwedge_{q' \in \delta(q,\sigma) \wedge \delta'(q') = \square}(\square, q', 0) \wedge \bigwedge_{q' \in \delta(q,\sigma) \wedge \delta'(q') = \lozenge}(q', q', 1)$ for all $q \in Q$ and $(\sigma, \delta', g) \in \Sigma \times \Delta'_{\mathcal{C}} \times \Delta'_{\mathcal{G}}$. We next define the DBN that runs on full $(\Sigma \times \Delta'_{\mathcal{C}} \times \Delta'_{\mathcal{G}})$ -labelled $Q_{\stackrel{\leftarrow}{\approx}}$ -trees $\mathcal{D}'_{\mathcal{G}} = \langle \Sigma \times \Delta'_{\mathcal{C}} \times \Delta'_{\mathcal{G}}, Q_{\mathcal{G}}, \delta_{\mathcal{D}'_{\mathcal{G}}}, q_0^{\mathcal{G}}, \alpha_{\mathcal{D}'_{\mathcal{G}}} \rangle$ where $\delta_{\mathcal{D}'_{\mathcal{G}}}(q, \langle \sigma, \delta', q' \rangle)$ is $(\square, q', 2)$ if $q' \in F_{\mathcal{G}}$ and $(\boxtimes_1^{\stackrel{\leftarrow}{\approx}}, q', 2)$ if $q' \notin F_{\mathcal{G}}$ for all $q \in Q_{\mathcal{G}}$ and $(\sigma, \delta', q') \in \Sigma \times \Delta'_{\mathcal{C}} \times \Delta'_{\mathcal{G}}$.

Given a $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled Q_{\Leftrightarrow} -tree $\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle$, the operator relabel assigns all nodes $w \in T$ with old label $(\sigma, \delta', (c, q'))$ a new label (σ, δ'', q') , where $\delta''(q)$ is \square if $\delta'(q) = \square$ and \Diamond if $\delta'(q) \in \Upsilon$. Relabelling gives us $(\Sigma \times \Delta'_{\mathcal{C}} \times \Delta'_{\mathcal{G}})$ -labelled trees.

▶ Lemma 32. The UCT $\mathcal{U}_{\mathcal{C}}^{p}$ accepts a full $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled Q_{\Leftrightarrow} -tree $\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle$ if, and only if, the UCT $\mathcal{U}_{\mathcal{C}}'$ accepts a full $(\Sigma \times \Delta_{\mathcal{C}}' \times \Delta_{\mathcal{G}}')$ -labelled Q_{\Leftrightarrow} -tree relabel $(\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle)$. Similarly, the DBN $\mathcal{D}_{\mathcal{G}}^{p}$ accepts a full $(\Sigma \times \Delta_{\mathcal{C}} \times \Delta_{\mathcal{G}})$ -labelled Q_{\Leftrightarrow} -tree $\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle$ if, and only if, the DBN $\mathcal{D}_{\mathcal{G}}'$ accepts a full $(\Sigma \times \Delta_{\mathcal{C}}' \times \Delta_{\mathcal{G}}')$ -labelled Q_{\Leftrightarrow} -tree relabel $(\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}}) \rangle)$.

Language Intersection Emptiness Check. Let a Σ -labelled Υ -tree $\langle T, V \rangle$ be accepted by both the ACS $\widetilde{\mathcal{T}}_{\mathcal{C}}$ and the NBS $\mathcal{T}_{\mathcal{G}}$. Let $\langle T, f_{\mathcal{C}} \rangle$ be the winning strategy of $\widetilde{\mathcal{T}}_{\mathcal{C}}$ and $\langle T, f_{\mathcal{G}} \rangle$ be the winning strategy of $\mathcal{T}_{\mathcal{G}}$ over this tree. From the previous lemmas in Appendix A.2, we have that if a tree $\langle T, V \rangle$ is accepted by both $\widetilde{\mathcal{T}}_{\mathcal{C}}$ and $\mathcal{T}_{\mathcal{G}}$, then the tree $\langle T', V' \rangle = \text{prune}\Big(\text{wide}_{Q_{\hat{\pi}}}\big(\langle T, (V, f_{\mathcal{C}}, f_{\mathcal{G}})\rangle\big)\Big)$ is accepted by both $\mathcal{U}_{\mathcal{C}}^p$ and $\mathcal{D}_{\mathcal{G}}^p$, and the tree relabel $(\langle T', V' \rangle)$ is accepted by both $\mathcal{U}_{\mathcal{C}}'$ and $\mathcal{D}_{\mathcal{G}}'$:

▶ Corollary 33. We have that $\mathcal{L}(\widetilde{\mathcal{T}}_{\mathcal{C}}) \cap \mathcal{L}(\mathcal{T}_{\mathcal{G}}) \neq \emptyset$ implies $\mathcal{L}(\mathcal{U}'_{\mathcal{C}}) \cap \mathcal{L}(\mathcal{D}'_{\mathcal{G}}) \neq \emptyset$.

As argued before, the other directions of Lemma 30 and Lemma 31 do not necessarily hold. The other direction of Corollary 33, however, holds, because a tree in the intersection of $\mathcal{L}(\mathcal{U}'_{\mathcal{C}})$ and $\mathcal{L}(\mathcal{D}'_{\mathcal{G}})$ is accepted by both $\widetilde{\mathcal{T}}_{\mathcal{C}}$ and $\mathcal{T}_{\mathcal{G}}$ with $\Upsilon = Q_{\stackrel{\sim}{\approx}}$.

▶ Proposition 34. For any full $(\Sigma \times \Delta'_{\mathcal{C}} \times \Delta'_{\mathcal{G}})$ -labelled $Q_{\stackrel{\sim}{\pi}}$ -tree $\langle T, (V, f'_{\mathcal{C}}, f'_{\mathcal{G}}) \rangle$ in $\mathcal{L}(\mathcal{U}'_{\mathcal{C}})$ and $\mathcal{L}(\mathcal{D}'_{\mathcal{G}})$, respectively, we have that the full Σ -labelled $Q_{\stackrel{\sim}{\pi}}$ -tree $\langle T, V \rangle$ is accepted by $\widetilde{\mathcal{T}}_{\mathcal{C}}$ and $\mathcal{T}_{\mathcal{G}}$, respectively.

Proof. If $\langle T, (V, f'_{\mathcal{C}}, f'_{\mathcal{G}}) \rangle$ is accepted by $\mathcal{U}'_{\mathcal{C}}$ then, due to the construction of $\mathcal{U}'_{\mathcal{C}}, \langle T, V \rangle$ is accepted by $\widetilde{\mathcal{T}}_{\mathcal{C}}$ with $\langle T, f_{\mathcal{C}} \rangle$ as the winning strategy, where $f_{\mathcal{C}}(w)(q) = \begin{cases} \Box & \text{if } f'_{\mathcal{C}}(w)(q) = \Box \\ q & \text{if } f'_{\mathcal{C}}(w)(q) = \Diamond \end{cases}$

for all $w \in T$ and $q \in Q$. This is because the run tree of $\widetilde{\mathcal{T}}_{\mathcal{C}}$ over $\langle T, V \rangle$ with this strategy is the same as the run tree of $\mathcal{U}'_{\mathcal{C}}$ over $\langle T, (V, f'_{\mathcal{C}}, f'_{\mathcal{G}}) \rangle$, and $\widetilde{\mathcal{T}}_{\mathcal{C}}$ and $\mathcal{U}'_{\mathcal{C}}$ have the same acceptance condition.

Likewise, if $\langle T, (V, f'_{\mathcal{C}}, f'_{\mathcal{G}}) \rangle$ is accepted by $\mathcal{D}'_{\mathcal{G}}$, then $\langle T, V \rangle$ is accepted by $\mathcal{T}_{\mathcal{G}}$ with $\langle T, f_{\mathcal{G}} \rangle$ as the winning strategy, where $f_{\mathcal{G}}(w) = \begin{cases} (\Box, q) & \text{if } f'_{\mathcal{G}}(w) = q \in F_{\mathcal{G}} \\ (\Leftrightarrow, q) & \text{if } f'_{\mathcal{G}}(w) = q \notin F_{\mathcal{G}} \end{cases}$ for all $w \in T$. This is because the run tree of $\mathcal{T}_{\mathcal{G}}$ over $\langle T, V \rangle$ with this strategy is the same as the run tree of $\mathcal{D}'_{\mathcal{G}}$ over $\langle T, (V, f'_{\mathcal{C}}, f'_{\mathcal{G}}) \rangle$, and $\mathcal{T}_{\mathcal{G}}$ and $\mathcal{D}'_{\mathcal{G}}$ have the same acceptance condition. This concludes the proof.

In the next step, we determinise $\mathcal{U}'_{\mathcal{C}}$ and obtain a deterministic parity tree automaton (DPT) $\mathcal{D}_{\mathcal{C}}$ such that $\mathcal{L}(\mathcal{U}'_{\mathcal{C}}) = \mathcal{L}(\mathcal{D}_{\mathcal{C}})$. For universal automata, this is a standard transformation that works on the word level, as the UCT can be viewed as individual UCW running along every individual branch of the tree. It is therefore a standard operation in synthesis to determinise these automata by dualising them to nondeterministic word automata, determinising them to deterministic parity automata [14], and dualising the resulting DPW by increasing all colours by 1, which defines a DPT $\mathcal{D}_{\mathcal{C}}$, which is language equivalent to $\mathcal{U}'_{\mathcal{C}}$.

To check whether the candidate NBW \mathcal{C} is QGFM, it now suffices to check the emptiness of $\mathcal{L}(\mathcal{D}_{\mathcal{C}}) \cap \mathcal{L}(\mathcal{D}'_{\mathcal{G}})$. To this end, we construct a deterministic tree automaton \mathcal{D} that recognises the language $\mathcal{L}(\mathcal{D}) = \mathcal{L}(\mathcal{D}_{\mathcal{C}}) \cap \mathcal{L}(\mathcal{D}'_{\mathcal{G}})$.

The DPT \mathcal{D} is constructed as follows: the new states are triples $(q_{\mathcal{C}}, q_{\mathcal{G}}, i)$, where $q_{\mathcal{C}}$ is a state of $\mathcal{D}_{\mathcal{C}}$, and updated according to the rules of $\mathcal{D}_{\mathcal{C}}$, $q_{\mathcal{G}}$ is a state from $\mathcal{D}'_{\mathcal{G}}$, and updated according to the rules of $\mathcal{D}'_{\mathcal{G}}$, and i serves as memory for the highest colour that occurs in a run of $\mathcal{D}_{\mathcal{C}}$ between two accepting transitions (those with colour 2) of $\mathcal{D}'_{\mathcal{G}}$.

That is, when we are in a state $(q_{\mathcal{C}}, q_{\mathcal{G}}, i)$, $\sigma \in \Sigma'$ (where Σ' is the alphabet over which \mathcal{D} , $\mathcal{D}_{\mathcal{C}}$, $\mathcal{D}'_{\mathcal{G}}$ are interpreted), and $v \in Q_{\hat{\pi}}$, then $(v, (q'_{\mathcal{C}}, q'_{\mathcal{G}}, i'), j)$ is a conjunct of $\delta_{\mathcal{D}}((q_{\mathcal{C}}, q_{\mathcal{G}}, i), \sigma)$ if, and only if, (1) $\delta_{\mathcal{D}_{\mathcal{C}}}(q_{\mathcal{C}}, \sigma)$ has a conjunct $(v, q'_{\mathcal{C}}, i_{\mathcal{C}})$, (2) $\delta_{\mathcal{D}'_{\mathcal{G}}}(r, \sigma)$ has a conjunct $(v, q'_{\mathcal{G}}, i_{\mathcal{G}})$, and (3) either $i_{\mathcal{G}} = 1$, $i' = \max\{i, i_{\mathcal{C}}\}$, and j = 1, or $i_{\mathcal{G}} = 2$, $i' = i_{\mathcal{C}}$, and j = i.

This way, if there are only finitely many accepting transitions on a path in the run tree of $\mathcal{D}'_{\mathcal{G}}$, then the dominating colour of the same path in the run tree of \mathcal{D} is 1, and if there are infinitely many accepting transitions on a path in the run tree of $\mathcal{D}'_{\mathcal{G}}$, then the dominating colour of the same path in the run tree of \mathcal{D} is the dominating colour of the same path in the run tree of $\mathcal{D}_{\mathcal{C}}$.

▶ **Theorem 35** (QGFM is in EXPTIME). For an NBW \mathcal{C} with n states over an alphabet Σ , we can decide in time polynomial in $\max\{|\Sigma|, n!\}$ whether or not \mathcal{C} is QGFM and, if \mathcal{C} is not QGFM, construct a Markov chain \mathcal{M} with initial state s_0 and $\operatorname{PSyn}_{\mathcal{C}}^{\mathcal{M}}(s_0) \neq \operatorname{PSem}_{\mathcal{C}}^{\mathcal{M}}(s_0) = 1$.

Proof. The UCT $\mathcal{U}'_{\mathcal{C}}$ has |Q| states. From $\mathcal{U}'_{\mathcal{C}}$, we can construct a language equivalent DPT $\mathcal{D}_{\mathcal{C}}$ with $O(|Q|!^2)$ states and O(|Q|) colours [14, 17]. The GFM automaton of our choice \mathcal{G} has $|Q_{\mathcal{G}}| = 3^{|Q|}$ states. The DBN $\mathcal{D}'_{\mathcal{G}}$ has $|Q_{\mathcal{G}}| = 3^{|Q|}$ states. By the construction of the DPT \mathcal{D} described above, \mathcal{D} has $O(|Q|!^23^{|Q|}|Q|)$ states and O(|Q|) colours.

We note that solving the language non-emptiness of \mathcal{D} reduces to solving a parity game, where player accept and player reject take turns in selecting the letter on the node of the tree, and choosing the direction to follow.

This game has $O(|\Sigma \times \Delta'_{\mathcal{C}} \times \Delta'_{\mathcal{G}} \times Q_{\mathcal{D}}|)$ states (where $Q_{\mathcal{D}}$ is the set of states of \mathcal{D}) and O(|Q|) colours. Using one of the quasi polynomial time algorithms for solving parity games [5], this can be solved in time polynomial in $\max\{|\Sigma|, |Q|!\}$.

As this is the most expensive step in the construction, we get the overall cost and result from the procedure in Figure 7.

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B Proofs of Section 5

▶ **Lemma 14.** If $\mathcal{L}(S) = \mathcal{L}(T)$ and S is GFG, then the candidate NBW C is GFM.

Proof. Continuing the proof in the main section, we now prove by induction over the length of the prefix of words from $\mathcal{L}(\mathcal{S})$ that, after reading a prefix $w \in \Sigma^*$, \mathcal{S} is in a state q of \mathcal{C} , which satisfies $\mathcal{L}(\mathcal{C}_q) = w^{-1}\mathcal{L}(\mathcal{C})$.

Induction Basis $(w = \varepsilon)$: It trivially holds that $\mathcal{L}(\mathcal{C}) = \mathcal{L}(\mathcal{C}_{q_0}) = \varepsilon^{-1}\mathcal{L}(\mathcal{C})$, and q_0 is productive and QGFM if $\mathcal{L}(\mathcal{S}) \neq \emptyset$. (Otherwise $\mathcal{L}(\mathcal{S}) = \emptyset$ and there is nothing to show.)

Induction Step $(w \mapsto w \cdot \sigma)$: If $w \cdot \sigma$ is not a prefix of a word in $\mathcal{L}(\mathcal{S})$, there is nothing to show. If $w \cdot \sigma$ is a prefix of a word in $\mathcal{L}(\mathcal{S})$, then obviously w is a prefix of the same word in $\mathcal{L}(\mathcal{S})$, and \mathcal{S} was in a state q, which is QGFM and with $\mathcal{L}(\mathcal{C}_q) = w^{-1}\mathcal{L}(\mathcal{C})$. As \mathcal{S} , with the transition function denoted by $\delta_{\mathcal{S}}$, is following a GFG strategy, we have that the GFG strategy will select a state $r \in \delta_{\mathcal{S}}(q,\sigma) \neq \emptyset$. By construction, we have $\mathcal{L}(\mathcal{C}_r) = \sigma^{-1}\mathcal{L}(\mathcal{C}_q)$, which — together with $\mathcal{L}(\mathcal{C}_q) = w^{-1}\mathcal{L}(\mathcal{C})$, implies $\mathcal{L}(\mathcal{C}_r) = (w \cdot \sigma)^{-1}\mathcal{L}(\mathcal{C})$.

This concludes the inductive proof.