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On modal extensions of Product fuzzy logic

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Abstract

In this article, we study modal extensions of Product fuzzy logic with both algebraic semantics and relational semantics based on Kripke structures with crisp accessibility relations, when the underlying product fuzzy logic is expanded with truth-constants, the Δ operator and with two infinitary inference rules. We provide completeness results for both kinds of semantics. Finally, we also consider a generalization of possibilistic logic evaluated over product algebras.

Keywords: Product fuzzy logic, modal fuzzy logics, strong standard completeness, rational truth-constants, infinitary rule.

1 Introduction

The main purpose of this article is to study some modal extensions (with generalized $\Box$ and $\Diamond$ modalities) of the Product fuzzy logic [28], one of the distinguished systems of mathematical fuzzy logic [11]. Modal extensions of main systems of mathematical fuzzy logic have appeared in the literature, either as purely technical contributions or as systems with practical motivations in the sense of Hájek in [26]: the underlying fuzzy logics appear as a suitable framework to formalize reasoning under vagueness, and the modalities address the logical formalization of intensional notions such as belief, uncertainty, knowledge, preference, time, etc. For instance, [14–16] present results concerning different modal extensions of Gödel fuzzy logic, [2] is a systematic study of modal logics over finite residuated lattices and [30] deals with modal extensions of both infinitely and finitely-valued Łukasiewicz logics. On the other hand, for instance, finitely-valued modal Łukasiewicz logics have been considered in [3] to provide formal grounds for fuzzy autoepistemic logic and certain logics of belief.

However, the study of modal extensions over the Product fuzzy logic, with Kripke style semantics based on either crisp or fuzzy accessibility relations, has remained open so far. We consider its study as a crucial step towards a better understanding of the modal extensions of Hájek's Basic fuzzy logic BL and their extensions. Here, we present some results that partially fill this gap, namely by providing an axiomatization of the minimal local and global modal logics that arise from Kripke models with worlds evaluating formulas over the canonical standard product algebra and with a crisp accessibility relation. We also explore an algebraic semantics for these logics and how it relates to the relational Kripke semantics.

It is worth noticing that the underlying propositional logic we use is not the plain Product logic but its expansion with rational constants, the Baaz–Monteiro Δ operator and two additional infinitary inference rules, that make the resulting logic strongly standard complete (i.e. with respect to the semantics defined by the standard product algebra on the real unit interval $[0,1]$). We have two reasons to do this, one for the addition of truth constants and another for the addition of the Δ operator and the two infinitary rules. First, truth constants play a fundamental role in our proof of completeness of the proposed axiomatization: whereas in the standard Gödel algebra, one can always differentiate two different elements by an endomorphism (sending the greatest to 1 and the

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other to some value less than 1) and in Łukasiewicz logic there are terms that can play this role (see e.g. [35]), in Product logic we have to resort to truth constants to be able to discriminate among different elements in the standard algebra. Second, for technical reasons, we need to have strong completeness of the underlying propositional logic (this is also the case for the modal extensions of Gödel and Łukasiewicz logics). In our case of Product logic with rational constants, strong completeness can be achieved by expanding the logic with the Δ operator and adding two infinitary rules.

The article is structured as follows. To make the paper self contained as much as possible, Section 2 gathers some preliminaries about Product fuzzy logic as well as some basic notions on abstract algebraic logic. Section 3 presents some results concerning strong completeness of an infinitary[1] Product logic with rational truth constants and the Monteiro–Baaz Δ operator, extending and providing alternative proofs to some related contributions in [9, 10] and [33]. The main result is the strong standard completeness of our proposed axiomatization (Theorem 3.10). In Section 4, we expand the language with modal operators $\Box$ and $\Diamond$, and we present two axiomatizations. The main results from this section are Theorems 4.11 and 4.12, which prove the strong completeness of these axiomatizations with respect to the local and global logical consequences relations induced by Kripke structures with crisp accessibility relations.

Section 5 focuses on the algebraic semantics of these logics, defining the class of modal product algebras, studying its classification within the Leibniz hierarchy and presenting completeness theorems and other characterization results. The most remarkable results in this respect are the completeness of the global logic with respect to its corresponding class of algebras (Theorem 5.4), the non-algebraizability of the local modal logic (Lemma 5.7), as it could be expected but nevertheless non-trivial because of the infinitary feature of our logics, and the subsequent characterization of its reduced models (Theorem 5.8). Moreover, we prove in Theorem 5.9 that both the local and the global logics are not complete with respect to classes of linearly ordered modal algebras. This section also provides details on the deep relationship between the Kripke semantics and the algebraic semantics for these modal logics. First, this allows us to present an algebraic completeness result for the local modal logic resorting to the order preserving logics paradigm (Theorem 5.13). We also define mappings between modal algebras and Kripke models, which allow us to obtain interesting results concerning the behaviour of the logic and to study how the deductions over Kripke models and modal algebras are related (for instance, Theorems 5.19 and 5.20). On this matter, we leave as an open question the study of the possible duality between these two classes of structures. Section 6 continues the investigation of product modal logics with several further related results: some interdefinability issues between the modal operators (Theorem 6.2); the definition of several axiomatic extensions with some of the well-known axioms (D, T, 4, 5) and their completeness with respect to corresponding classes of product Kripke models (Theorem 6.5); and, finally, a possibilistic extension of the fuzzy modal product logic and its axiomatization. We end up in Section 7 with some final conclusions and some interesting open problems. For the sake of readability, the proofs of some theorems have been moved to an annex, they are rather technical proofs and not essential for the comprehension of the article.

1.1 Notation coventions

The main conventions about notation that we will use throughout this article are as follows. Classes of algebras will be denoted as $\mathbb{C}, \mathbb{D}, \ldots$; algebras as $\boldsymbol{A}, \boldsymbol{B}, \ldots$; sets (and so, algebra universes) as $A, B, \ldots$

[1]In the sense of having infinitary inference rules.

and elements of a set (algebra) as $a, b, \ldots$ In particular, for a logic $\mathcal{L}$, $\boldsymbol{Fm}_{\mathcal{L}}$ will denote the algebra of formulas built from a denumerable set of propositional variables on the language of the logic. If not specified otherwise, $x, y, \ldots$ will denote propositional variables, and $\varphi, \psi, \ldots$ formulas (over the corresponding language). Kripke models will be denoted by $\mathfrak{M}, \mathfrak{G}, \ldots$ Finally, given two sets A, B we will write A^B to denote the set of mappings from B to A, $f, g, \ldots$ to name the elements of this set, and for $f \in A^B$, we will also write $[b \mapsto f(b)]$ to denote it.

2 Preliminaries

In this section, we provide some background definitions and existing results that are needed in order to understand the article without continuously resorting to external sources. First, we overview Product fuzzy logic, which will be the starting point of our study. Second, we provide a refresher of several notions and results from abstract algebraic logic, which will be used in Sections 3 and 5.

2.1 Product fuzzy logic

Product fuzzy logic Π was introduced by Hájek *et al.* in [28] with the aim of axiomatizing a $[0,1]$-valued propositional logic based on the conjunction interpreted as product of reals together with the corresponding residuated implication (Goguen implication) and the corresponding negation (Gödel negation). The logic is defined over the language $\mathfrak{L} = \{\&, \to, \overline{0}\}$. Other usual connectives can be defined from these: $\overline{1} := \varphi \to \varphi$, $\neg\varphi := \varphi \to \overline{0}$, $\varphi \wedge \psi := \varphi \& (\varphi \to \psi)$ and $\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$. Several simpler but equivalent axiomatizations of Π have been proposed since then, for instance Hájek in [26], Cintula in [8] and Montagna *et al.* in [34]. Here, we consider the latter:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2) $\varphi \& \psi \to \varphi$
(A3) $\varphi \& \psi \to \psi \& \varphi$
(A4) $\varphi \& (\varphi \to \psi) \to (\psi \& (\psi \to \varphi))$
(A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$
(A5b) $((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(Π) $\neg\varphi \vee ((\varphi \to (\varphi \& \psi)) \to \psi)$

The only inference rule of Π is *Modus Ponens* (*MP*).

It is well known that $(A1)$ – $(A6)$ together with (MP) define Hájek's Basic Fuzzy Logic *BL* [26], whose extensions with suitable axioms provide, respectively, axiomatizations of Gödel-Dummet, Łukasiewicz, and Product logics, the logics of the three basic continuous t-norms: the minimum, Łukasiewicz and product t-norms. The finitary notion of proof in Π is the usual one, and for an arbitrary set of formulas $\Gamma \cup \{\varphi\}$ in the language $\mathfrak{L}$, we shall write $\Gamma \vdash_{\Pi} \varphi$ to denote that there exists a finite $\Gamma_0 \subseteq \Gamma$ and a proof of φ from Γ_0.

The algebraic companion of Π is the variety of *product algebras*, i.e. algebras of the form $\boldsymbol{A} = \langle A, \odot, \Rightarrow, \wedge, \vee, 0, 1\rangle$ which are commutative, integral and bounded residuated lattices satisfying the following conditions:

- $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$,
- $\neg\neg x \odot (x \odot z \Rightarrow y \odot z) \leqslant x \Rightarrow y$,
- $x \wedge \neg x = 0$,
- $x \odot (x \Rightarrow y) = x \wedge y$

where $\neg x := x \Rightarrow 0$. For the sake of a simpler notation, we will write $\langle A, \odot, \Rightarrow, 0, 1\rangle$ to actually denote the product algebra $\langle A, \odot, \Rightarrow, \wedge, \vee, 0, 1\rangle$, with the $\wedge$ and $\vee$ operations defined from $\odot$ and $\Rightarrow$ as in Π.

An evaluation of formulas h on a product algebra $\boldsymbol{A}$ is a mapping from the set of formulas into A satisfying $h(\overline{0}) = 0$, $h(\varphi \& \psi) = h(\varphi) \odot h(\psi)$ and $h(\varphi \to \psi) = h(\varphi) \Rightarrow h(\psi)$.

As proved in [28], Product logic enjoys finite strong completeness with respect to the *standard product algebra* $[0,1]_\Pi = \langle [0,1], \cdot, \Rightarrow_\Pi, 0, 1\rangle$ (where $x \cdot y$ is the usual product on $[0,1]$ and $x \Rightarrow_\Pi y$ equals 1 if $x \leqslant y$ and y/x otherwise). That is to say, for any finite set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm$, $\Gamma \vdash_\Pi \varphi$ holds if and only if for any evaluation of formulas h into $[0,1]_\Pi$, if $h([\Gamma]) \subseteq \{1\}$ then $h(\varphi) = 1$. Unless specified otherwise, we will write $\Gamma \models_{\boldsymbol{A}} \varphi$ for an arbitrary algebra $\boldsymbol{A}$ whenever for any evaluation of formulas h into $\boldsymbol{A}$, if $h([\Gamma]) \subseteq \{1\}$ then $h(\varphi) = 1$.

Extending [28], in [36] the authors studied the expansion of Product logic over a language expanded with countably many truth constants, one constant $\overline{c}$ for each rational $c \in [0,1]$. They also included in their study the expansion of the language with the Monteiro–Baaz unary operator Δ, whose interpretation in a product chain is the function defined by $\Delta(1) = 1$, $\Delta(x) = 0$ for all $x < 1$. They propose an axiomatization that extends Π with book-keeping axioms that model the behaviour of the rational constant symbols and some new axioms and an inference rule regarding the Δ operation. We will denote by Π^c_Δ the axiomatic system resulting from adding to Π the following axioms and rule:

(A$_\Delta$1) $\Delta\varphi \vee \neg\Delta\varphi$
(A$_\Delta$2) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$
(A$_\Delta$3) $\Delta\varphi \to \varphi$
(A$_\Delta$4) $\Delta\varphi \to \Delta\Delta\varphi$
(A$_\Delta$5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

(A$_c$1) $\overline{c} \& \overline{d} \leftrightarrow \overline{c \cdot d}$
(A$_c$2) $(\overline{c} \to \overline{d}) \leftrightarrow \overline{c \Rightarrow_\Pi d}$
(A$_c$3) $\neg\Delta\overline{c}$, for any $c < 1$;

(**G**$_\Delta$) $\dfrac{\varphi}{\Delta\varphi}$

Again, the notion of proof in Π^c_Δ is the usual finitary one. The expansion of the standard product algebra with rational constants interpreted by their value and with the Δ operator will be called **canonical standard product algebra**. Given that this will be the only product algebra referred to in the following sections, there is no place for confusion, and for the sake of readability we will abuse notation and denote the canonical standard product algebra by $[0,1]_\Pi$.

Π^c_Δ was proved in [36] to be finitely strong complete with respect to the canonical standard product algebra (i.e. $[0,1]_\Pi$ generates the variety of product algebras with rational constants and Δ), that is to say, for any finite set of formulas $\Gamma \cup \{\varphi\}$ in the language $\mathfrak{L} \cup \{\overline{c}\}_{c \in (0,1)_\mathbb{Q}} \cup \{\Delta\}$, it holds that $\Gamma \vdash_{\Pi^c_\Delta} \varphi$ if and only if $\Gamma \models_{[0,1]_\Pi} \varphi$.

2.2 *Abstract algebraic logic*

The study of the algebraization of a given logic and, in particular, its classification within the Leibniz hierarchy determines the extent to which that logic may be studied using known algebraic methods and techniques. Over the last 30 years, a powerful arsenal of results on the different levels of the Leibniz hierarchy have been obtained, and they can be used to obtain an exhaustive characterization of a logic once it has been assigned to a level of this hierarchy.

We will recall here some basic notions that will allow us to classify the logics defined in Section 4 into the Leibniz hierarchy and to study their algebraic semantics. Most of the definitions and results recalled here are folklore; for a systematic exposition of them, see for instance [4], [5], [19] and [25].

Recall that, given a class of algebras $\mathbb{K}$ and an algebra $\boldsymbol{A}$, a congruence θ of $\boldsymbol{A}$ is a **$\mathbb{K}$-congruence**, or a **congruence of $\boldsymbol{A}$ relative to** $\mathbb{K}$, when $\boldsymbol{A}/\theta \in \mathbb{K}$. The class of congruences of $\boldsymbol{A}$ relative to $\mathbb{K}$ is denoted by $Co_{\mathbb{K}}\boldsymbol{A}$. If $\mathbb{K}$ is closed under subdirect products, the class of congruences of any algebra relative to $\mathbb{K}$ forms a complete lattice.

In this work, we will restrict ourselves to the case when $\mathbb{K}$ is a **generalized quasi-variety**, i.e. a class of algebras axiomatized with generalized quasi-equations (quasi-equations with a possibly infinite set of premises written with countably many variables). It is known that these are, in particular, $\mathbb{ISP}_S$ classes (closed under isomorphism, subalgebras and subdirect products), so it will make sense to talk about the lattice of congruences relative to $\mathbb{K}$.

Definition 2.1
Let $\langle \boldsymbol{A}, D\rangle$ be a logical matrix, i.e. $\boldsymbol{A}$ is an algebra and $D \subseteq A$. The **Leibniz congruence of** $\langle \boldsymbol{A}, D\rangle$, denoted $\boldsymbol{\Omega}^{\boldsymbol{A}}D$, is the largest congruence of $\boldsymbol{A}$ compatible with D (i.e. if $a \in D$ and $a \equiv b(\boldsymbol{\Omega}^{\boldsymbol{A}}D)$ then $b \in D$).

Given two algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same type, we will denote by $Hom(\boldsymbol{A},\boldsymbol{B})$ the set of homomorphisms from $\boldsymbol{A}$ to $\boldsymbol{B}$. Then, for a logic $\mathcal{L}$ and an algebra $\boldsymbol{A}$,[2] a set $F \subseteq A$ is called a **filter** of $\mathcal{L}$ (or a **$\mathcal{L}$-filter**) on $\boldsymbol{A}$ whenever for any $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ (i.e. formulas in the language $\mathcal{L}$) and $h \in Hom(\boldsymbol{Fm}_{\mathcal{L}}, \boldsymbol{A})$, if $\Gamma \vdash_{\mathcal{L}} \varphi$ and $h[\Gamma] \subseteq F$, then $h(\varphi) \in F$. The set of all $\mathcal{L}$-filters on the algebra $\boldsymbol{A}$ is denoted by $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ and the class of **models** of $\mathcal{L}$, $Mod\mathcal{L}$, will consist of pairs $\langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A}$ is an algebra of the type of $\mathcal{L}$ and $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$.

It is a general fact that any logic $\mathcal{L}$ is strongly complete with respect to its class $Mod^*\mathcal{L}$ of reduced models, that is, those models $\langle \boldsymbol{A}, F\rangle$ of $\mathcal{L}$ for which $\boldsymbol{\Omega}^{\boldsymbol{A}}F = \mathrm{Id}_{\boldsymbol{A}}$. We will denote by $Alg^*\mathcal{L}$ the class of algebras $\boldsymbol{A}$ for which there exists $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ such that $\langle \boldsymbol{A}, F\rangle \in Mod^*\mathcal{L}$.

Lemma 2.2
Let $\mathcal{L}$ be a logic and $\boldsymbol{\Delta}(x,y)$ a set of formulas with only x and y as variables. The following conditions are equivalent:

(1) $\boldsymbol{\Delta}(x,y)$ satisfies the following conditions:

- (R) $\vdash_{\mathcal{L}} \boldsymbol{\Delta}(x,x)$
- (MP) $x, \boldsymbol{\Delta}(x,y) \vdash_{\mathcal{L}} y$
- (Re) $\bigcup_{i<n} \boldsymbol{\Delta}(x_i,y_i) \vdash_{\mathcal{L}} \boldsymbol{\Delta}(\lambda(x_0 \ldots x_{n-1}), \lambda(y_0 \ldots y_{n-1}))$ for each λ in the language of $\mathcal{L}$, with $n = ar\lambda$.

(2) For all $\langle \boldsymbol{A}, F\rangle \in Mod\mathcal{L}$, $a \equiv b(\boldsymbol{\Omega}^{\boldsymbol{A}}F)$ if and only if $\boldsymbol{\Delta}^{\boldsymbol{A}}(a,b) \subseteq F$.

A logic $\mathcal{L}$ for which there exists a set of formulas $\boldsymbol{\Delta}(x,y)$ that satisfies any of the two previous equivalent properties is said to be **equivalential with congruence formulas $\boldsymbol{\Delta}(x,y)$**. [3]

Lemma 2.3
Let $\mathcal{L}$ be an equivalential logic. Then for every algebra $\boldsymbol{A}$, the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is monotonic over the $\mathcal{L}$-filters on $\boldsymbol{A}$.

[2] We will always assume that the type of the algebras is the one of the logic.

[3] Do not confuse this $\boldsymbol{\Delta}$ binary function that provides a set of equations with the $\triangle$ Baaz–Monterio unary operator of the logic.

Definition 2.4

A logic $\mathcal{L}$ is **algebraizable** when there is a class $\mathbb{K}$ of algebras, a set of formulas $\boldsymbol{\Delta}(x,y)$ in at most two variables (**congruence formulas**), and a set of equations $\mathbf{E}(x)$ in at most one variable (**defining equations**) such that for all $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ the following conditions hold:

- $\Gamma \vdash_{\mathcal{L}} \varphi$ if and only if $\mathbf{E}(\Gamma) \models_{\mathbb{K}} \mathbf{E}(\varphi)$;
- $x \approx y \models_{\mathbb{K}} \mathbf{E}(\boldsymbol{\Delta}(x,y))$ and $\mathbf{E}(\boldsymbol{\Delta}(x,y)) \models_{\mathbb{K}} x \approx y$.

where for an arbitrary set of formulas X, we write $\mathbf{E}(X)$ to denote the set of equations $\bigcup_{x\in X}\mathbf{E}(x)$, and for $E1 \cup E2$ a set of equations in the language of $\mathcal{L}$ (with at most $\aleph_0$ variables), $E1 \models_{\mathbb{K}} E2$ if and only if for all $\boldsymbol{A} \in \mathbb{K}$ and for all $h \in Hom(\boldsymbol{Fm}_{\mathcal{L}}, \boldsymbol{A})$, if $h(\varepsilon) = h(\delta)$ for every $\varepsilon \approx \delta \in E1$, then $h(\alpha) = h(\beta)$ for all $\alpha \approx \beta \in E2$.

In such a case, $\mathbb{K}$ is called an **algebraic semantics** of $\mathcal{L}$. The greatest algebraic semantics $\mathbb{K}$ of $\mathcal{L}$ (ordered by inclusion) is called **equivalent algebraic semantics** and it coincides with the quasi-variety generated by any algebraic semantics of $\mathcal{L}$.

We will finish this brief reminder on abstract algebraic logic by stating a very important result that deeply relates the filters of an algebraizable logic and the congruences relative to its class of algebras.

Theorem 2.5 (Isomorphism Theorem)

Let $\mathcal{L}$ be a logic and let $\mathbb{K}$ be a generalized quasi-variety. $\mathcal{L}$ is algebraizable with equivalent algebraic semantics $\mathbb{K}$ if and only if, for any algebra $\boldsymbol{A}$, the Leibniz operator is an isomorphism between the lattices $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ and $Co_{\mathbb{K}}\boldsymbol{A}$ that commutes with endomorphisms.

As a last remark, relying on this isomorphism, if $\mathcal{L}$ is an algebraizable logic with algebraic semantics $\mathbb{K}$ we will simplify the notation by writing $\boldsymbol{A}/F$ instead of $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F$, whenever $\boldsymbol{A} \in \mathbb{K}$ and F is an $\mathcal{L}$-filter on $\boldsymbol{A}$.

3 Infinitary Product logic with rational constants and $\boldsymbol{\Delta}$

In the previous section, we have remarked the fact that both Π and Π^c_Δ enjoy finite strong standard (canonical) completeness. However, for reasons that were commented in the introduction and that will become clearer in Section 4, we are interested in defining a logic that is complete with respect to arbitrary logical consequences (in the sense that infinite sets of premises are considered too) over the canonical standard product algebra. A simple counterexample shows that neither Π nor Π^c_Δ are strongly standard complete.

Example 3.1

Let p,q be propositional variables and let $\Gamma = \{p \to q^n \mid n \in \mathbb{N}\}$, where q^n is the abbreviation of $q \& \overset{n}{\ldots} \& q$. For any evaluation v on $[0,1]_\Pi$, if $v(p \to q^n) = 1$ for all $n \in \mathbb{N}$, then either $v(p) = 0$ or $v(q) = 1$, which is equivalent to $v(\neg p \vee q) = 1$, i.e. $\Gamma \models_{[0,1]_\Pi} \neg p \vee q$. However, it is clear that for any finite subset Γ_0 of Γ, $\Gamma_0 \not\models_{[0,1]_\Pi} \neg p \vee q$. Hence $\models_{[0,1]_\Pi}$ is infinitary, while $\vdash_\Pi$ is finitary.

In [33], Montagna deals with the problem of syntactically coping with the infinitary character of the semantical consequence relation of the logics based on continuous t-norms, and obtains a general result for the family of BL logics. He defines a unary operator $*$, called *storage operator*, over any BL algebra, with $*x$ being the maximum idempotent element smaller or equal to x, and studies the semilinear expansion of any BL logic with an storage operator,[4] with its corresponding

[4] In [33], these logics correspond to representable BL-algebras with storage.

axiom schemata, and the infinitary rule

$$\frac{\chi \vee (\varphi \to \psi^k) \text{ for all } k \in \omega}{\chi \vee (\varphi \to *\psi)},$$

where, as usual, ψ^k denotes $\psi \& \overset{k}{\ldots} \& \psi$. This expansion is proved to be strongly complete (for infinite theories) with respect to the corresponding class of (expanded) standard BL-chains. In particular, it can be seen that for Product logic, the $*$ operator coincides with the Monteiro–Baaz operator Δ in $[0,1]$.

On the other hand, when working with logics extended with rational constants, some results have been obtained, in the frame of rational Pavelka-like logics, by Cintula in his PhD dissertation [9] and in a very recent paper [10]. In particular, in the case of the Product logic with the Δ operator, he proves that the logic Π^∞, defined from Π^c_Δ with two additional infinitary rules that manage the behaviour of the constants in the discontinuity points in $[0,1]$ of Δ (the point $x=1$) and $\Rightarrow_\Pi$ (the point $(0,0)$), is Pavelka complete, That is to say, for an arbitrary set of formulas $\Gamma \cup \{\varphi\}$, the truth degree of φ over Γ, defined as $\|\varphi\|_\Gamma = \inf\{e(\varphi) \colon\ e \in Hom(\boldsymbol{Fm}_{\boldsymbol{\Pi}^\infty}, [0,1]_\Pi) \text{ such that } e([\Gamma]) \subseteq \{1\}\}$, coincides with the provability degree of φ over Γ, defined as $|\varphi|_\Gamma = \sup\{c \mid \Gamma \vdash_{\Pi^\infty} \overline{c} \to \varphi\}$.

Definition 3.2
$\boldsymbol{\Pi}^{\boldsymbol{\infty}}$ is the infinitary logic obtained by extending Π^c_Δ with the following two infinitary rules:

$$(\mathrm{R}_\Delta)\ \frac{\overline{c} \to \varphi,\ \text{for all } c \in (0,1)_{\mathbb{Q}}}{\varphi} \qquad (\mathrm{R}_\to)\ \frac{\varphi \to \overline{c},\ \text{for all } c \in (0,1)_{\mathbb{Q}}}{\neg\varphi};$$

The notion of infinitary deduction for these kinds of logics is worth to be recalled (cf. [9, Definition 2.1.10]).

Definition 3.3
Let $\Gamma \cup \{\varphi\} \subseteq Fm_{\Pi^\infty}$. A **proof** of φ from Γ in Π^∞ is a well-founded tree (with possibly infinite width) labelled by formulae in such a way that

- The root is labelled by φ, and the leaves are axioms of Π^∞ or elements from Γ.
- For each intermediate node ψ with Σ_ψ being its successors in the tree, there is a rule $\Sigma_\psi \vdash \psi$ in Π^∞.

As usual, we write $\Gamma \vdash_{\Pi^\infty} \varphi$ whenever there exists a proof of φ from Γ in Π^∞.

Note that the Δ-deduction theorem holds for Π^∞: $\Gamma \cup \{\varphi\} \vdash_{\Pi^\infty} \psi$ iff $\Gamma \vdash_{\Pi^\infty} \Delta\varphi \to \psi$, see e.g. [37]. Actually in [37] some preliminary results about the logic Π^∞ have been presented.

It is clear that, enjoying Pavelka style completeness and having the infinitary rule (R_Δ), Π^∞ also enjoys strong completeness with respect to $[0,1]_\Pi$. However, it is also possible to follow a different path, namely an algebraic one, to prove the strong standard completeness of Π^∞. That is, first to characterize and study the algebraic companion of this logic and prove strong completeness with respect to linearly ordered algebras from this class, and then, gaining inspiration from ideas in [33], to prove strong completeness as well with respect to the canonical standard product algebra. The advantage of this latter path, which we will explore in the rest of this section, is to obtain a deeper insight into the logic through its algebraic counterpart, which will be of use in future sections.

Facing the algebraic study of Π^∞, it is immediate to check that it is a Rasiowa-implicative logic, and thus algebraizable in the sense of Blok and Pigozzi [5]. Its algebraic companion coincides with the class $\mathbb{P}^{\infty}$ of Π^{∞}-algebras $\boldsymbol{A}=\langle A,\odot,\Rightarrow,\Delta,\{c^{\boldsymbol{A}}\}_{c\in[0,1]_{\mathbb{Q}}}\rangle$ where:

- $\langle A,\odot,\Rightarrow,\Delta,0\rangle$ is a $\mathbb{P}_{\Delta}$-algebra, i.e. a product algebra with Δ operator.
- The rational constants $\{c^{\boldsymbol{A}}\}$ form a subalgebra isomorphic to $[0,1]_{\mathbb{Q}}$ (as $\mathbb{P}_{\Delta}$-algebras) such that for each $c,d\in[0,1]_{\mathbb{Q}}$ and $x\in A$ the following equations and generalised quasi-equations hold:
  - $c^{\boldsymbol{A}}\odot d^{\boldsymbol{A}}=(c\cdot d)^{\boldsymbol{A}}$;
  - $c^{\boldsymbol{A}}\Rightarrow d^{\boldsymbol{A}}=\min\{1,(d/c)^{\boldsymbol{A}}\}$;
  - $\Delta c^{\boldsymbol{A}}=0$ for $c<1$;
  - If $x\geqslant c^{\boldsymbol{A}}$ for all $c\in(0,1)_{\mathbb{Q}}$ then $x=1$; $\quad(\mathcal{Q}_{\Delta})$
  - If $x\leqslant c^{\boldsymbol{A}}$ for all $c\in(0,1)_{\mathbb{Q}}$ then $x=0$. $\quad(\mathcal{Q}_{\rightarrow})$

Since Π^{∞} is implicative, it enjoys the following form of completeness: For any $\Gamma\cup\{\varphi\}\subseteq Fm_{\Pi^{\infty}}$,

$$\Gamma\vdash_{\Pi^{\infty}}\varphi \text{ iff } h([\Gamma])\subseteq\{1\} \text{ implies } h(\varphi)=1, \tag{1}$$
$$\text{for all } h\in Hom(\boldsymbol{Fm_{\Pi^{\infty}}},\boldsymbol{A}) \text{ and all } \boldsymbol{A}\in\mathbb{P}^{\infty}.$$

The class $\mathbb{P}^{\infty}$ is a *generalised quasi-variety*, and it can be proven to be not a quasi-variety, since it is not closed under ultraproducts.

Example 3.4
Let U be the set of cofinite subsets of ω, and consider the ultraproduct $\prod_{n\in\omega}[0,1]_{\Pi}/U$, i.e. for $a,b\in\prod_{n\in\omega}[0,1]_{\Pi}$, $a\sim_U b$ iff $\{n\in\omega\colon\ a[n]\neq b[n]\}$ is a finite set. Let $p\in\prod_{n\in\omega}[0,1]_{\Pi}$ be the element defined by $p[n]=\frac{2^n-1}{2^n}$ for each $n\in\omega$. It is a strictly increasing series, and given that $\lim_{n\to\infty}\frac{2^n-1}{2^n}=1$, for any $c\in(0,1)_{\mathbb{Q}}$ there is some $s\in\omega$ such that $\frac{2^s-1}{2^s}\geqslant c$. Denote by c^{ω} the element such that $c^{\omega}[n]=c$ for each $n\in\omega$. Then $(c^{\omega}\to p)[i]=1$ for all $i\geqslant s$ and thus, by definition, $c^{\omega}\to p\sim 1$ for all $c\in(0,1)_{\mathbb{Q}}$. Since the constants in this ultraproduct are by definition the equivalence classes of c^{ω} for each $c\in(0,1)_{\mathbb{Q}}$, it holds that $c^{\omega}\to p=1$ for each $c\in(0,1)_{\mathbb{Q}}$. However, $p[n]<1$ for each $n\in\omega$, and thus it is not true that $p=1$ in $\prod_{n\in\omega}[0,1]_{\Pi}/U$, getting that $\prod_{n\in\omega}[0,1]_{\Pi}/U\notin\mathbb{P}^{\infty}$.

Completeness of Π^{∞} with respect to linearly ordered algebras of $\mathbb{P}^{\infty}$ can be deduced from the fact that for any set of formulas $\Gamma\cup\{\alpha\}$ such that $\Gamma\nvdash_{\Pi^{\infty}}\alpha$, Γ can be extended to a linear Π^{∞}-theory Γ' (i.e. a theory where for any pair of formulas φ,ψ, either $\varphi\to\psi\in\Gamma'$ or $\psi\to\varphi\in\Gamma'$) for which still it holds that $\Gamma'\nvdash_{\Pi^{\infty}}\alpha$. This result can be found implicit within the proof of Lemma 3.4.25 in [9], and we detail the proof in the Appendix for the interested reader.

Lemma 3.5
Let $T\cup\{\alpha\}$ be a set of formulas such that $T\nvdash_{\Pi^{\infty}}\alpha$. Then there is a prime theory T' such that $T\subseteq T'$ and $T'\nvdash_{\Pi^{\infty}}\alpha$.

Strong completeness of Π^{∞} with respect to the linearly ordered Π^{∞}-algebras is now a direct corollary from the previous lemma.

Corollary 3.6
For any set of formulas $\Gamma\cup\{\varphi\}$, the following are equivalent:

(1) $\Gamma\vdash_{\Pi^{\infty}}\varphi$;
(2) $\Gamma\models_{C\mathbb{P}^{\infty}}\varphi$, with $C\mathbb{P}^{\infty}$ denoting the class of chains in $\mathbb{P}^{\infty}$.

PROOF. Soundness is easy to check. Concerning the other direction, if $\Gamma \not\vdash_{\Pi^\infty} \varphi$, from the previous lemma it follows that there is Γ' prime theory such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \not\vdash_{\Pi^\infty} \varphi$. Let $\boldsymbol{A} = \boldsymbol{Fm}/\boldsymbol{\Omega}\Gamma'$,[5]. It is clear that $\boldsymbol{A}$ is linearly ordered since Γ' is prime, and taking the homomoprhism $h: \boldsymbol{Fm} \to \boldsymbol{A}$ that sends each formula to its congruence class, $h[\Gamma] \subseteq h[\Gamma'] = \{\overline{1}^{\boldsymbol{A}}\}$ and $h(\varphi) < \overline{1}^{\boldsymbol{A}}$, which concludes the proof. ∎

From this point, in order to prove strong standard completeness we gain inspiration in the work of Montagna [33] and extend some of his results in order to show that any chain from $\mathbb{P}^\infty$ can be completely embedded into the canonical standard algebra. We will use the notion of **Archimedean property** in product algebras, i.e. for all $0 < a, b < 1$ there is a positive integer n such that $b^n < a$ (where b^n denotes the iteration of b with itself under the $\odot$ operation n times). Indeed, Montagna realizes the importance of this property in the standard *BL* algebras, and his infinitary rule translates it into syntactical terms. In the particular case of product algebras he provides a very neat characterization.

LEMMA 3.7 ([33, Lemma 10])
Let $\boldsymbol{A}$ be a linearly ordered product algebra. $\boldsymbol{A}$ is Archimedean if and only if for any $a, b \in A$ such that $a \leqslant b^n$ for all n, then $a \leqslant \Delta b$.

When considering a language extended not only with Δ but also with rational constants, we can check that, whenever the generalized quasiequations $\mathcal{Q}_\Delta$ and $\mathcal{Q}_\to$ hold, the previous characterization of the Archimedean property applies.

LEMMA 3.8
Any linearly ordered $\boldsymbol{A} \in \mathbb{P}^\infty$ is Archimedean.

PROOF. Let $a, b \in \boldsymbol{A}$ with $a \leqslant b^n$ for all positive integer n. We can prove by cases that $a \leqslant \Delta b$.

- If there is $c \in (0,1)_{\mathbb{Q}}$ such that $b \leqslant c^{\boldsymbol{A}}$, then $a \leqslant b^n \leqslant (c^{\boldsymbol{A}})^n$ for all n. Moreover $[0,1]_{\mathbb{Q}}$ is Archimedean, and so, by axiom (A_c1) $a \leqslant d^{\boldsymbol{A}}$ for any $d \in (0,1)_{\mathbb{Q}}$. Then, by $\mathcal{Q}_\to$, $a = 0$, and trivially $a \leqslant \Delta b$.
- If there is no $c \in (0,1)_{\mathbb{Q}}$ such that $b \leqslant c^{\boldsymbol{A}}$, since $\boldsymbol{A}$ is linearly ordered we have that $c^{\boldsymbol{A}} \leqslant b$ for all $c \in (0,1)_{\mathbb{Q}}$. Then, by $\mathcal{Q}_\Delta$, $b = 1$ and trivially $a \leqslant \Delta b = 1$.

∎

From this result, one can build a complete embedding from any chain in $\mathbb{P}^\infty$ into $[0,1]_\Pi$ in a natural way. It is only necessary to ensure that the constants are sent to their corresponding values in $[0,1]$.

LEMMA 3.9
Let $\boldsymbol{A}$ be a countable linearly ordered Π^∞-algebra. Then $\boldsymbol{A}$ can be embedded in the canonical standard product algebra $[0,1]_\Pi$ by a complete embedding.

PROOF. The details of the proof, being rather technical and not necessary for the comprehension of the general result, can be found in the Appendix. ∎

We can finally state the completeness result as a corollary from the previous lemmas.

THEOREM 3.10 (Strong Standard Completeness)
For any $\Gamma \cup \{\varphi\} \subseteq Fm_{\Pi^\infty}$, the following are equivalent:

(1) $\Gamma \vdash_{\Pi^\infty} \varphi$;

[5] Recall that $\boldsymbol{\Omega}\Gamma'$ is the congruence in $\boldsymbol{Fm}$ such that $\varphi \sim \psi \Leftrightarrow \Gamma' \vdash_{\Pi^\infty} \varphi \leftrightarrow \psi$

(2) $\Gamma \models_{\mathbb{P}^\infty} \varphi$;
(3) $\Gamma \models_{C\mathbb{P}^\infty} \varphi$, where $C\mathbb{P}^\infty$ denotes the class of chains in $\mathbb{P}^\infty$;
(4) $\Gamma \models_{[0,1]_\Pi} \varphi$.

Proof. (1)$\Rightarrow$(4) is the soundness of the axioms and rules, and it is a matter of routine checking. On the other hand, (4)$\Rightarrow$(3) follows directly from Lemma 3.9, while (3)$\Rightarrow$(2) is the strong completeness of Π^∞ with respect to the linearly ordered Π^∞-algebras, and this is Corollary 3.6. Finally, (2)$\Rightarrow$(1) is the completeness result stated in Condition (1), that is a consequence of Π^∞ being a Rasiowa-implicative logic. ■

A remarkable fact that arises from this strong completeness result is the very natural behaviour of the constants over any Π^∞-algebra.

Lemma 3.11
Let $\boldsymbol{A} \in \mathbb{P}^\infty$ and $a, b \in \boldsymbol{A}$. If $a < b$ then there is $c \in (0,1)_{\mathbb{Q}}$ such that $c^{\boldsymbol{A}} \not\leqslant a$ and $b \not\leqslant c^{\boldsymbol{A}}$.

Proof. Using the fact that the rational numbers are dense in the real interval $[0,1]$ it is clear that for any $x, y \in [0,1]$ we have that if $x \leqslant y$ and $(c^{\boldsymbol{A}} \to x) \vee (y \to c^{\boldsymbol{A}}) = 1$ for each $c \in (0,1)_{\mathbb{Q}}$ then $y \leqslant x$ (i.e. $y = x$). This can be written as, for any $\varphi, \psi \in Fm$,

$$\{\varphi \to \psi\} \cup \{(\overline{c} \to \varphi) \vee (\psi \to \overline{c})\}_{c \in (0,1)_{\mathbb{Q}}} \models_{[0,1]_\Pi} \psi \to \varphi.$$

By the strong completeness, it follows that

$$\{\varphi \to \psi\} \cup \{(\overline{c} \to \varphi) \vee (\psi \to \overline{c})\}_{c \in (0,1)_{\mathbb{Q}}} \vdash_{\Pi^\infty} \psi \to \varphi,$$

and thus, by completeness with respect to the whole class of algebras, we have that, in an arbitrary $\boldsymbol{A}$, if $a < b \in \boldsymbol{A}$ there must exist $c_0 \in (0,1)_{\mathbb{Q}}$ for which $(c_0^{\boldsymbol{A}} \to a) \vee (b \to c_0^{\boldsymbol{A}}) < 1$ (otherwise, $b = a$). Then, in particular, $c_0^{\boldsymbol{A}} \to a < 1$ and $b \to c_0^{\boldsymbol{A}} < 1$, which means, by the definition of the order in $\boldsymbol{A}$, that $c_0^{\boldsymbol{A}} \not\leqslant a$ and $b \not\leqslant c_0^{\boldsymbol{A}}$. ■

Corollary 3.12
If $\boldsymbol{A} \in \mathbb{P}^\infty$ is linearly ordered, then for each $a < b \in \boldsymbol{A}$ there exists $c \in (0,1)_{\mathbb{Q}}$ such that $a < c^{\boldsymbol{A}} < b$.

4 Minimal local and global modal logics over Π^∞

In this section, we will focus on the study of the local and global modal logics associated with the class of product algebras with the Δ operator and rational constants. It is important to remark that we will consider the modal logics as closure operators, not just as a closed set of formulas (theorems), studying this way a richer problem whose solutions have implicit the other possible definition of the logic. The main result presented is the definition of two axiomatic systems that are (respectively) strongly complete with respect to the local and global modal logics that arise from a class of crisp Kripke models evaluated over product algebras, and in particular, from those evaluated over the canonical standard product algebra.

We will begin by defining the modal logics in semantical terms. First, we let the **modal language** be the expansion of the propositional language for product logic with rational constants and the Δ operator with two unary operators $\Box$ and $\Diamond$. From now on, unless specified otherwise ***Fm*** will denote the the algebra of formulas built from a countable set of variables $\mathcal{V}$ and rational constants $\{\overline{c}\}_{c \in [0,1]_{\mathbb{Q}}}$ using the connectives of the modal language.

The notion of **Kripke frame** is the usual one: a pair $\langle W,R\rangle$ where W is a non-empty set of worlds and $R\subseteq W\times W$ is an accessibility relation.[6] For the sake of a simpler notation, we will write Rvw instead of $\langle v,w\rangle\in R$.

Definition 4.1
Let $\mathbf{A}\in\mathbb{P}^{\infty}$. An **A-Kripke model** $\mathfrak{M}=\langle W,R,e\rangle$ is a Kripke frame $\langle W,R\rangle$ endowed with an evaluation of propositional variables on $\mathbf{A}$ for each world, $e\colon W\times\mathcal{V}\to\mathbf{A}$. The map e is extended to formulas interpreting the connectives by their corresponding operations in $\mathbf{A}$, i.e. fulfilling $e(v,\varphi\&\psi)=e(v,\varphi)\odot e(v,\psi)$, $e(v,\varphi\to\psi)=e(v,\varphi)\Rightarrow e(v,\psi)$, $e(v,\Delta\varphi)=\Delta e(v,\varphi)$ and $e(v,\overline{c})=c^{\mathbf{A}}$, and the modal operators by the following expressions:

$$e(v,\Box\varphi)=\inf\{e(w,\varphi)\colon Rvw\}\qquad e(v,\Diamond\varphi)=\sup\{e(w,\varphi)\colon Rvw\}$$

whenever these values are defined, and are left undefined otherwise.

A model $\mathfrak{M}$ is called **safe** whenever the evaluation of $\Box\varphi$ and $\Diamond\varphi$ is defined over any world for any $\varphi\in Fm$. For the sake of simplicity, from now on we will only consider safe models, and we will denote by PK the class of safe models over the algebras in $\mathbb{P}^{\infty}$, and its elements will be called **Product Kripke models**.

Over the class of models PK, different notions of truth arise. Given $\varphi\in Fm$, $\mathfrak{M}\in\mathsf{PK}$ and $w\in W$, we write $(\mathfrak{M},w)\models\varphi$ whenever $e(w,\varphi)=1$, and $\mathfrak{M}\models\varphi$ whenever $(\mathfrak{M},w)\models\varphi$ for all $w\in W$. Then, as usual, two notions of logical consequence can be defined, a local and a global one. For an arbitrary set of formulas $\Gamma\cup\{\varphi\}\subseteq Fm$, we respectively define:

- $\Gamma\models^{l}_{\mathsf{PK}}\varphi$ if for any $\mathfrak{M}\in\mathsf{PK}$ and $w\in W$, $(\mathfrak{M},w)\models\Gamma$ implies that $(\mathfrak{M},w)\models\varphi$.
- $\Gamma\models^{g}_{\mathsf{PK}}\varphi$ if for any $\mathfrak{M}\in\mathsf{PK}$, $\mathfrak{M}\models\Gamma$ implies that $\mathfrak{M}\models\varphi$.

Notice that the local deduction is strictly weaker than the global one, since $\Gamma\models^{l}_{\mathsf{PK}}\varphi$ implies $\Gamma\models^{g}_{\mathsf{PK}}\varphi$, but not conversely. It is also remarkable that the sets of theorems (i.e. deductions from the empty set) under the local and the global deductions coincide.

The main goal is to provide a complete axiomatization for these logics. We will focus on the proof of completeness of the axiomatization of the local consequence, since the results needed in the case of the global consequence can be easily obtained from the ones of the local one. The proposed axiomatization for the local consequence $\models^{l}_{\mathsf{PK}}$ is the following.

Definition 4.2
$\mathbf{K}\Pi^{\infty}$ is the axiomatic system on the modal language given as the extension of Π^{∞} with the following axioms and rules:

(K) $\Box(\varphi\to\psi)\to(\Box\varphi\to\Box\psi)$
($\mathrm{A}_{\Box}1$) $\Box(\overline{c}\to\varphi)\leftrightarrow(\overline{c}\to\Box\varphi)$
($\mathrm{A}_{\Diamond}1$) $\Box(\varphi\to\overline{c})\leftrightarrow(\Diamond\varphi\to\overline{c})$
($\mathrm{A}_{\Box}2$) $\Delta\Box\varphi\to\Box\Delta\varphi$
($\mathrm{N}_{\Box}$) From $\vdash\varphi$ derive $\vdash\Box\varphi$.

[6] It is not in the scope of this article to deal with fuzzy accesibility relations of the form $R\colon W\times W\to\mathbf{A}$ for some algebra $\mathbf{A}\in\mathbb{P}^{\infty}$.

As usual, for an arbitrary set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm$, we will write $\Gamma \vdash_{\mathbf{K}\Pi^\infty} \varphi$ whenever there is a proof (in the sense of Definition 3.3) of φ from elements in Γ (clearly, we are assuming the deductions are closed under uniform substitution).[7]

The axiomatic system corresponding to the global consequence $\models^g_{\mathsf{PK}}$ will be denoted by $\mathbf{GK}\Pi^\infty$. Its definition is as follows.

Definition 4.3
$\mathbf{GK}\Pi^\infty$ is the axiomatic system resulting from $\mathbf{K}\Pi^\infty$ by just replacing the local necessitation rule ($\mathrm{N}_\Box$) by the more general rule:

($N^g_\Box$) From φ derive $\Box\varphi$

The corresponding notion of proof will be denoted by $\vdash_{\mathbf{GK}\Pi^\infty}$.

Note that the rule ($N^g_\Box$) is to be applied to any formula, not only to theorems as it happens with the rule ($\mathrm{N}_\Box$).

As it happened at the semantical level, $\vdash_{\mathbf{K}\Pi^\infty}$ is weaker than $\vdash_{\mathbf{GK}\Pi^\infty}$. It is also clear that the theorems of both logics coincide, and we will refer to this set by $Th_{\mathbf{K}\Pi^\infty}$.

The soundness of the previous axiomatizations with respect to their semantic counterparts is easy to check and we omit the proof. To show that $\mathbf{K}\Pi^\infty$ indeed axiomatizes the local consequence $\models^l_{\mathsf{PK}}$ we have to prove that, for any set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm$, if $\Gamma \nvdash_{\mathbf{K}\Pi^\infty} \varphi$ there exist a model $\mathfrak{M} = \langle W, R, e\rangle \in \mathsf{PK}$ and a world $w \in W$ for which $(\mathfrak{M}, w) \models \Gamma$ but $(\mathfrak{M}, w) \not\models \varphi$. This will be done through the usual canonical model construction. But first, we will present some results about $\mathbf{K}\Pi^\infty$ that will be useful for this task.

Since the only modal deduction rule of $\mathbf{K}\Pi^\infty$ applies just to theorems, it is clear that the Δ-Deduction Theorem of Π^∞ (see Section 2) keeps holding, i.e.

$$\Gamma \cup \{\alpha\} \vdash_{\mathbf{K}\Pi^\infty} \varphi \text{ iff } \Gamma \vdash_{\mathbf{K}\Pi^\infty} \Delta\alpha \to \varphi.$$

However, the fact that $\mathbf{K}\Pi^\infty$ has infinitary rules imposes a detour in the completeness proof. Indeed, if the logic were finitary, the Deduction Theorem would allow to treat the logic as just a set of theorems, and then the proof of correctness of the canonical model could be more directly done simply using the rule ($\mathrm{N}_\Box$). In this more general context, it is necessary to show that the deductions in $\mathbf{K}\Pi^\infty$ are closed under the $\Box$ operator.

Lemma 4.4
Let $\Gamma \cup \{\varphi\} \subseteq Fm$. $\Gamma \vdash_{\mathbf{K}\Pi^\infty} \varphi$ implies that $\Box\Gamma \vdash_{\mathbf{K}\Pi^\infty} \Box\varphi$, where $\Box\Gamma$ stands for $\{\Box\gamma : \gamma \in \Gamma\}$.

Proof. Applying induction on the last step of the proof of φ from Γ, we consider the possible last step in that proof:

- If φ is a theorem, then by ($\mathrm{N}_\Box$), $\Box\varphi$ is a theorem too.
- If φ was obtained by (*MP*), then there exists ψ such that $\Gamma \vdash_{\mathbf{K}\Pi^\infty} \psi$ and $\Gamma \vdash_{\mathbf{K}\Pi^\infty} \psi \to \varphi$. Applying the induction hypothesis, axiom (K) and the (*MP*) rule, it follows that $\Box\Gamma \vdash_{\mathbf{K}\Pi^\infty} \Box\varphi$.
- If φ is $\Delta\psi$ and it was obtained by (G_Δ), then $\Gamma \vdash_{\mathbf{K}\Pi^\infty} \psi$. Applying the induction hypothesis, the rule (G_Δ) and axiom ($\mathrm{A}_\Box 2$), we get that $\Box\Gamma \vdash_{\mathbf{K}\Pi^\infty} \Box\Delta\psi$.

[7]Note we could restrict ourselves to study the finitary modal logic ($\Gamma \vdash^f_{\mathbf{K}\Pi^\infty} \varphi$ whenever there is a proof, also in the sense of Definition 3.3, of φ from *a finite subset* of Γ), but as we will see, no extra assumptions are needed to work directly with the infinitary case (since the infinitary inference is already treated at the non-modal level).

- If φ was obtained by the rule (R_Δ), then $\Gamma \vdash_{\mathbf{K}\Pi^\infty} \overline{c} \to \varphi$ for all $c \in (0,1)_{\mathbb{Q}}$. By the induction hypothesis we get $\Box\Gamma \vdash_{\mathbf{K}\Pi^\infty} \Box(\overline{c} \to \varphi)$ for all $c \in (0,1)_{\mathbb{Q}}$, and thus, by axiom (A$_\Box$1) it follows that $\Box\Gamma \vdash_{\mathbf{K}\Pi^\infty} \overline{c} \to \Box\varphi$ for all $c \in (0,1)_{\mathbb{Q}}$. By applying again (R_Δ) it then follows that $\Box\Gamma \vdash_{\mathbf{K}\Pi^\infty} \Box\varphi$.
- If φ is $\neg\psi$ and it was obtained by the rule $(R_\to)$, then $\Gamma \vdash_{\mathbf{K}\Pi^\infty} \psi \to \overline{c}$ for all $c \in (0,1)_{\mathbb{Q}}$. As before, by applying the induction hypothesis, (A$_\Diamond$1), $(R_\to)$ and again (A$_\Diamond$1), we conclude that $\Box\Gamma \vdash_{\mathbf{K}\Pi^\infty} \Box\neg\psi$.

∎

One of the usual methods to prove completeness in modal logics is by defining a suitable translation of modal formulas into non-modal ones over an extended set of variables, and then translating proofs in the modal logic into proofs from an extended set of premises (possibly infinite) in the underlying non-modal logic. Actually, this is the reason behind the necessity of having strong completeness at the non-modal level. In our case, we can still proceed in this way thanks to the fact that Π^∞ is strongly complete and any use of (N$_\Box$) rule in a proof in $\mathbf{K}\Pi^\infty$ produces a theorem of the logic.

Formally, we denote by $\mathcal{V}^\star$ the extended set of variables $\mathcal{V} \cup \{\varphi_\Box, \varphi_\Diamond \mid \varphi \in Fm\}$, and by $Fm^\star$ the set of formulas in the language of Π^c_Δ (*without* modal operators) over the extended set of variables $\mathcal{V}^\star$.

Definition 4.5
We inductively define a one-to-one translation to non-modal formulas $\star\colon Fm \to Fm^\star$ by

$$\begin{aligned}
&\overline{c}^\star := \overline{c}, \quad x^\star := x \text{ for } x \in \mathcal{V} && (\Delta\varphi)^\star := \Delta\varphi^\star\\
&(\varphi \,\&\, \psi)^\star := \varphi^\star \,\&\, \psi^\star && (\Box\varphi)^\star := \varphi_\Box\\
&(\varphi \to \psi)^\star := \varphi^\star \to \psi^\star && (\Diamond\varphi)^\star := \varphi_\Diamond.
\end{aligned}$$

Note that the modal axioms are simply translated to the following schemes:

$$\begin{aligned}
&(\mathrm{K})^\star : (\varphi \to \psi)_\Box \to (\varphi_\Box \to \psi_\Box) && (\mathrm{A}_\Diamond 1)^\star : (\varphi \to \overline{c})_\Box \leftrightarrow (\varphi_\Diamond \to \overline{c})\\
&(\mathrm{A}_\Box 1)^\star : (\overline{c} \to \varphi)_\Box \leftrightarrow (\overline{c} \to \varphi_\Box) && (\mathrm{A}_\Box 2)^\star : \Delta\varphi_\Box \leftrightarrow (\Delta\varphi)_\Box
\end{aligned}$$

This translation can be used to obtain an equivalence between deductions in the local modal logic and deductions in the non-modal product logic. This is immediate since all the propositional rules are shared and all the formulas deduced in $\mathbf{K}\Pi^\infty$ by the (N$_\Box$) rule belong to $Th_{\mathbf{K}\Pi^\infty}$.

Lemma 4.6
For any set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm$, $\Gamma \vdash_{\mathbf{K}\Pi^\infty} \varphi$ if and only if $\Gamma^\star \cup (Th_{\mathbf{K}\Pi^\infty})^\star \vdash_{\Pi^\infty} \varphi^\star$.

To prove strong standard completeness of $\mathbf{K}\Pi^\infty$ with respect to the local deduction $\models^l_{\mathsf{PK}}$ we will define a canonical $[0,1]_\Pi$-model with the property that for any set of modal formulas $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \nvdash_{\mathbf{K}\Pi^\infty} \varphi$, there is a world in the model which assigns the value 1 to all $\gamma \in \Gamma$ but a value strictly smaller than 1 to φ.

Definition 4.7
The **canonical model** is the $[0,1]_\Pi$-model $\mathfrak{M}_\mathfrak{c} = \langle W_\mathfrak{c}, R_\mathfrak{c}, e_\mathfrak{c}\rangle$ where

- $W_\mathfrak{c} := \{h \in Hom(\mathbf{Fm}^\star, [0,1]_\Pi)\colon h([(Th_{\mathbf{K}\Pi^\infty})^\star]) \subseteq \{1\}\}$;
- $R_\mathfrak{c} vw$ if for any $\psi \in Fm$ such that $v(\psi_\Box) = 1$ it holds that $w(\psi^\star) = 1$;
- $e_\mathfrak{c}(w,p) := w(p)$, for every $p \in \mathcal{V}$.

One may wonder about the above definition of $R_{\mathfrak{c}}$ as it is only depends on the $\Box$ modality and on formulas evaluated to 1. However as next lemmas show, this suffices to prove the truth lemma for this canonical model.

Lemma 4.8
$R_{\mathfrak{c}}vw$ if and only if for any $\psi \in Fm$, both inequalities hold:

- $v(\psi_{\Box}) \leqslant w(\psi^{\star})$;
- $v(\psi_{\Diamond}) \geqslant w(\psi^{\star})$.

Proof. To prove the non-direct case, assume $v(\psi_{\Box}) > w(\psi^{\star})$ for some $\psi \in Fm$. Then there is $c \in (0,1)_{\mathbb{Q}}$ such that $w(\psi^{\star}) < c < v(\psi_{\Box})$, and so, $1 = c \Rightarrow_{\Pi} v(\psi_{\Box}) = v(\overline{c} \to \psi_{\Box})$. Axiom $(\mathrm{A}_{\Box}1)^{\star}$ leads to $1 = v((\overline{c} \to \psi)_{\Box})$. However, $w((\overline{c} \to \psi)^{\star}) = c \Rightarrow_{\Pi} w(\psi^{\star}) < 1$, and so it does not hold that $R_{\mathfrak{c}}vw$. The other proof, showing that $v(\psi_{\Diamond}) < w(\psi^{\star})$ leads to the fact that v is not related with w, is analogous using $(\mathrm{A}_{\Diamond}1)^{\star}$. ∎

In the way of proving the *Truth Lemma* for the canonical model, we first show a crucial result.

Lemma 4.9
Let $v \in W_{\mathfrak{c}}$ and $\varphi \in Fm$ such that $w(\varphi^{\star}) = 1$ for all $w \in W_{\mathfrak{c}}$ such that $R_{\mathfrak{c}}vw$. Then it holds that $v(\varphi_{\Box}) = 1$.

Proof. Let $w \in W_{\mathfrak{c}}$. Then, by definition, $R_{\mathfrak{c}}vw$ if and only if $w([(Th_{\mathbf{K}\Pi^{\infty}})^{\star}]) \subseteq \{1\}$ and $w(\psi^{\star}) = 1$ for all $\psi \in Fm$ such that $v(\psi_{\Box}) = 1$. In other words, for any $w \in Hom(\mathbf{Fm}^{*}, [0,1]_{\Pi})$, $R_{\mathfrak{c}}vw$ if and only if $w([(Th_{\mathbf{K}\Pi^{\infty}} \cup T)^{\star}]) \subseteq \{1\}$, where $T = \{\psi \in Fm \colon v(\psi_{\Box}) = 1\}$.

Therefore, the hypothesis of the lemma amounts to assume $(Th_{\mathbf{K}\Pi^{\infty}} \cup T)^{\star} \models_{[0,1]_{\Pi}} \varphi^{\star}$. By strong standard completeness of Π^{∞}, it follows that $(Th_{\mathbf{K}\Pi^{\infty}} \cup T)^{\star} \vdash_{\Pi^{\infty}} \varphi^{\star}$, and then, by Lemma 4.6, $T \vdash_{\mathbf{K}\Pi^{\infty}} \varphi$ as well.

Now Lemma 4.4 can be applied, obtaining $\Box T \vdash_{\mathbf{K}\Pi^{\infty}} \Box\varphi$. By the same reasoning in the opposite sense (Lemma 4.6 and then applying strong standard completeness), it follows that $(Th_{\mathbf{K}\Pi^{\infty}} \cup \Box T)^{\star} \models_{[0,1]_{\Pi}} \varphi_{\Box}$. But then, given that $v \in W_{\mathfrak{c}}$, it follows by definition that $v([(Th_{\mathbf{K}\Pi^{\infty}})^{\star}]) \subseteq \{1\}$. On the other hand, for each $\psi \in T$ it holds that $v(\psi_{\Box}) = 1$, hence $v([(\Box T)^{\star}]) \subseteq \{1\}$. Therefore, since the whole set of premises is evaluated to 1, it also holds that $v(\varphi_{\Box}) = 1$, which concludes the proof. ∎

Using the density of the rational numbers within the reals, we get the Truth Lemma as a consequence of the previous result.

Lemma 4.10 (Truth Lemma)
For any $\varphi \in Fm$ and any $v \in W_{\mathfrak{c}}$ it holds that

$$e(v,\varphi) = v(\varphi^{\star}).$$

Proof. This can be proven by induction on the structure of the formulas, and the only cases that are worth to be detailed are the modal ones.

To show that $v(\varphi_{\Box}) = \inf\{w(\varphi^{\star}) : R_{\mathfrak{c}}vw\}$, first notice that by Lemma 4.8, $v(\varphi_{\Box}) \leqslant w(\varphi^{\star})$ for any w such that $R_{\mathfrak{c}}vw$, so $v(\varphi_{\Box}) \leqslant \inf\{w(\varphi^{\star}) : R_{\mathfrak{c}}vw\}$. To prove the converse inequality, by way of a contradiction, assume that $v(\varphi_{\Box}) < \inf\{w(\varphi^{\star}) \colon R_{\mathfrak{c}}vw\}$. Then there is $c \in (0,1)_{\mathbb{Q}}$ such that $v(\varphi_{\Box}) < c < w(\varphi^{\star})$ for any w such that $R_{\mathfrak{c}}vw$. This implies that $w((\overline{c} \to \varphi)^{\star}) = 1$ for any w such that $R_{\mathfrak{c}}vw$, and by Lemma 4.9 we have that $v((\overline{c} \to \varphi)_{\Box}) = 1$ as well. However, by axiom $(\mathrm{A}_{\Box}1)^{\star}$, $v((\overline{c} \to \varphi)_{\Box}) = v(\overline{c} \to \varphi_{\Box})$, and hence $c \to v(\varphi_{\Box}) = 1$ too, which contradicts the fact that $v(\varphi_{\Box}) < c$.

To show that $v(\varphi_\Diamond)=\sup\{w(\varphi^\star):R_\mathfrak{c}vw\}$, the proof is analogous using axiom $(A_\Diamond 1)^\star$ instead of $(A_\Box 1)^\star$, when needed. ■

Now, completeness of $\mathbf{K}\Pi^\infty$ with respect to the local semantics is a quite direct consequence.

THEOREM 4.11 (Completeness of $\mathbf{K}\Pi^\infty$)
For any set of modal formulas $\Gamma\cup\{\varphi\}$ the following are equivalent:

(1) $\Gamma\vdash_{\mathbf{K}\Pi^\infty}\varphi$;
(2) $\Gamma\models^{l}_{[0,1]_\Pi\mathsf{K}}\varphi$, where $[0,1]_\Pi\mathsf{K}$ denotes the class of $[0,1]_\Pi$-Kripke models;[8]
(3) $\Gamma\models^{l}_{\mathcal{C}\mathsf{PK}}\varphi$, where $\mathcal{C}\mathsf{PK}$ denotes the class of safe $\mathbf{A}$-Kripke models such that $\mathbf{A}\in\mathbb{P}^\infty$ is a chain;
(4) $\Gamma\models^{l}_{\mathsf{PK}}\varphi$.

PROOF. Observe that (4) implies (3), and that (3) implies (2). Soundness is simple to check in general for the $\models^{l}_{\mathsf{PK}}$ case, so (1) implies (4). Therefore it remains to prove (2) implies (1), using the canonical model, and the rest follows directly.

Assume $\Gamma\nvdash_{\mathbf{K}\Pi^\infty}\varphi$. By Lemma 4.6, this happens if and only if $\Gamma^\star\cup(Th_{\mathbf{K}\Pi^\infty})^\star\nvdash_{\Pi^\infty}\varphi^\star$. By strong completeness of Π^∞, there is an homomorphism v from $Fm^\star$ into the canonical standard product algebra $[0,1]_\Pi$ such that $v([(Th_{\mathbf{K}\Pi^\infty})^\star])\subseteq\{1\}$ and $v([\Gamma^\star])\subseteq\{1\}$ but $v(\varphi^\star)<1$. Hence, $v\in W_\mathfrak{c}$ and by the Truth Lemma 4.10, $e(v,[\Gamma])=v([\Gamma^\star])\subseteq\{1\}$ and $e(v,\varphi)=v(\varphi^\star)<1$, so $\mathfrak{M}_\mathfrak{c}\models_v\Gamma$ and $\mathfrak{M}_\mathfrak{c}\not\models_v\varphi$. ■

THEOREM 4.12 (Completeness $\mathbf{GK}\Pi^\infty$)
For any set of (modal) formulas $\Gamma\cup\{\varphi\}$ the following are equivalent:

(1) $\Gamma\vdash_{\mathbf{GK}\Pi^\infty}\varphi$;
(2) $\Gamma\models^{g}_{[0,1]_\Pi\mathsf{K}}\varphi$;
(3) $\Gamma\models^{g}_{\mathcal{C}\mathsf{PK}}\varphi$;
(4) $\Gamma\models^{g}_{\mathsf{PK}}\varphi$.

PROOF. (Sketch) The proof of completeness of the logic $\mathbf{GK}\Pi^\infty$ with respect to the global consequence relation $\models^{g}_{\mathsf{PK}}$ can be proved in a similar way to the local one. The main difference is defining a slightly different canonical model ${\mathfrak{M}_\mathfrak{c}}^\Gamma$ for each set of formulas $\Gamma\cup\{\varphi\}\subseteq Fm$ such that $\Gamma\nvdash_{\mathbf{GK}\Pi^\infty}\varphi$. For each of such sets, the worlds of ${\mathfrak{M}_\mathfrak{c}}^\Gamma$ are defined as the homomorphisms in $Hom(\mathbf{Fm}^\star,[0,1]_\Pi)$ that satisfy the whole (but fixed) set of formulas deducible from Γ, $[C_{\mathbf{GK}\Pi^\infty}\Gamma]^\star=\{\chi^\star\in Fm^\star:\Gamma\vdash_{\mathbf{GK}\Pi^\infty}\chi\}$. To prove that the model so defined enjoys the Truth Lemma, the simplest way is to resort to the results concerning the local logic, already proven. In particular, we can prove the result corresponding to Lemma 4.9 in this new context using *the same methods* (and with that we mean resorting to the local logic $\mathbf{K}\Pi^\infty$ in the proof). Finally, the result analogous to Lemma 4.6 that is now needed turns to be no more than an obvious observation: $\Gamma\nvdash_{\mathbf{GK}\Pi^\infty}\varphi$ if and only if $[C_{\mathbf{GK}\Pi^\infty}\Gamma]^\star\nvdash_{\Pi^\infty}\varphi^\star$. This points out to an homomorphism h within the standard product algebra, $[0,1]_\Pi$, such that $h([C_{\mathbf{GK}\Pi^\infty}\Gamma]^\star)\subseteq\{1\}$ and $h(\varphi^\star)<1$. This concludes the proof, since h is a world of the canonical model ${\mathfrak{M}_\mathfrak{c}}^\Gamma$ (which is a *global model* for Γ by definition). ■

[8] Observe $[0,1]_\Pi$ is a complete algebra, and so all the models defined over it are safe.

5 Algebraic semantics for the product modal logics

In this section we will study the algebraic semantics of the modal systems $\mathbf{K}\Pi^{\infty}$ and $\mathbf{GK}\Pi^{\infty}$. We will begin by classifying $\mathbf{K}\Pi^{\infty}$ and $\mathbf{GK}\Pi^{\infty}$ into the Leibniz Hierarchy and obtain their algebraic classes, and then proceed to study the classes of matrices over these algebras corresponding to each logic, providing completeness results more specific than the general ones (of each logic with respect to its reduced models). We will finish by studying the relation between the Kripke and the algebraic semantics.

5.1 Algebraic semantics for $\mathbf{K}\Pi^{\infty}$ *and* $\mathbf{GK}\Pi^{\infty}$

For the algebraic counterpart of $\mathbf{K}\Pi^{\infty}$ and $\mathbf{GK}\Pi^{\infty}$ it is natural to consider the class of algebras obtained expanding product algebras in $\mathbb{P}^{\infty}$ with unary modal operations $\Box$ and $\Diamond$ satisfying equations related to the axioms of $\mathbf{K}\Pi^{\infty}$.

Definition 5.1
A $\mathbf{K}\Pi^{\infty}$-agebra, or **modal product algebra** with (rational) constants and Δ, is an algebra $\boldsymbol{A} = \langle A, \odot, \rightarrow, \Delta, \Box, \Diamond, \{c^{\boldsymbol{A}}\}_{c\in[0,1]_{\mathbb{Q}}}\rangle$ such that $\langle A, \odot, \rightarrow, \Delta, \{c^{\boldsymbol{A}}\}_{c\in[0,1]_{\mathbb{Q}}}\rangle \in \mathbb{P}^{\infty}$, and $\Box$ and $\Diamond$ are two unary operations for which the following equations hold:

($\mathrm{E_K}$) $\Box(x\rightarrow y)\rightarrow(\Box x\rightarrow \Box y)\approx 1^{\boldsymbol{A}}$;
($\mathrm{E_{A_\Box 1}}$) $\Box(c^{\boldsymbol{A}}\rightarrow x)\approx c^{\boldsymbol{A}}\rightarrow \Box x$, for every $c\in[0,1]_{\mathbb{Q}}$;
($\mathrm{E_{A_\Diamond 1}}$) $\Box(x\rightarrow c^{\boldsymbol{A}})\approx \Diamond x\rightarrow c^{\boldsymbol{A}}$, for every $c\in[0,1]_{\mathbb{Q}}$;
($\mathrm{E_{A_\Box 2}}$) $\Box\Delta x\rightarrow \Delta\Box x\approx 1^{\boldsymbol{A}}$;
($\mathrm{E_{N_\Box}}$) $\Box 1^{\boldsymbol{A}}\approx 1^{\boldsymbol{A}}$.

We will denote by $\mathbb{KP}^{\infty}$ the class of $\mathbf{K}\Pi^{\infty}$-algebras, and for each $\boldsymbol{A}\in\mathbb{KP}^{\infty}$, $\overline{\boldsymbol{A}}$ will denote its non-modal reduct.

A **modal evaluation** e over a modal product algebra $\boldsymbol{A}$ is an homomorphism from the algebra of modal formulae (with Δ and rational constants) into $\boldsymbol{A}$.

From the general definition of a filter, we have that the deductive filters of the (non-modal) logic Π^{∞} over a product algebra $\boldsymbol{A}\in\mathbb{P}^{\infty}$ are given by subsets $F\subseteq A$ such that

- $1^{\boldsymbol{A}}\in F$,
- If $x\in F$ and $x\rightarrow y\in F$ then $y\in F$,
- If $x\in F$ then $\Delta x\in F$,
- If $c^{\boldsymbol{A}}\rightarrow x\in F$ for all $c\in(0,1)_{\mathbb{Q}}$ then $x\in F$,
- If $x\rightarrow c^{\boldsymbol{A}}\in F$ for all $c\in(0,1)_{\mathbb{Q}}$ then $\neg x\in F$.

The filters of $\mathbf{K}\Pi^{\infty}$ are those of Π^{∞} that are also closed under $N_\Box$ (i.e. $\Box 1^{\boldsymbol{A}}\in F$), and the filters of $\mathbf{GK}\Pi^{\infty}$ will be those of of Π^{∞} closed under $N^g_\Box$ (i.e. if $x\in F$ then $\Box x\in F$).

In order to simplify the notation in the characterization of the filters of the modal logics, we shall introduce a natural concept. We say that a subset X of an algebra $\boldsymbol{A}\in\mathbb{KP}^{\infty}$ is **open** whenever for any $x\in X$, it holds that $\Box x\in X$ as well. Then, we can more clearly characterize the filters of $\mathbf{K}\Pi^{\infty}$ and $\mathbf{GK}\Pi^{\infty}$ over a modal product algebra $\boldsymbol{A}\in\mathbb{KP}^{\infty}$.

Lemma 5.2
Given $\boldsymbol{A}\in\mathbb{KP}^{\infty}$, and $F\subseteq A$

- F is a filter of $\mathbf{K}\Pi^{\infty}$ over $\boldsymbol{A}$ if and only if $F\in\mathcal{F}i_{\Pi^{\infty}}\overline{\boldsymbol{A}}$

- F is a filter of $\mathbf{GK}\Pi^\infty$ over $\boldsymbol{A}$ if and only if $F \in \mathcal{F}i_{\Pi^\infty}\overline{\boldsymbol{A}}$ and F is an open set.

It is routine to prove that $\{1^{\boldsymbol{A}}\}$ is a filter of both of $\mathbf{K}\Pi^\infty$ and of $\mathbf{GK}\Pi^\infty$ logics over all $\boldsymbol{A} \in \mathbb{KP}^\infty$.
It turns out that the classification of the global modal logic $\mathbf{GK}\Pi^\infty$ in the Leibniz Hierachy is now quite immediate, just by checking the definition.

LEMMA 5.3
$\mathbf{GK}\Pi^\infty$ is algebraizable with algebraic semantics $\mathbb{KP}^\infty$, with equivalence formulas $\boldsymbol{\Delta}(x,y) := \{x \leftrightarrow y\}$ and with defining equations $E(x) = \{x \approx 1\}$.

From this it follows that $Alg^*\mathbf{GK}\Pi^\infty = \mathbb{KP}^\infty$ and thus, its reduced models are of the form $\langle \boldsymbol{A}, \{1^{\boldsymbol{A}}\}\rangle$ for $\boldsymbol{A} \in \mathbb{KP}^\infty$. That is to say, we have the following result.

THEOREM 5.4 (Algebraic Completeness of $\mathbf{GK}\Pi^\infty$)
For any set of modal formulas $\Gamma \cup \{\varphi\}$

$$\Gamma \vdash_{\mathbf{GK}\Pi^\infty} \varphi \text{ iff } \Gamma \models_{\mathbb{KP}^\infty} \varphi.$$

This fact, together with the Isomorphism Theorem 2.5, ensures that for every algebra $\boldsymbol{A}$, the mappings

$$\boldsymbol{\rho}^+ : \mathcal{F}i_{\mathbf{GK}\Pi^\infty}\boldsymbol{A} \longleftrightarrow Co_{\mathbb{KP}^\infty}\boldsymbol{A} : \boldsymbol{\tau}^+,$$

defined respectively as

$$\boldsymbol{\rho}^+(F) = \boldsymbol{\Omega}^{\boldsymbol{A}} F \qquad \text{and} \qquad \boldsymbol{\tau}^+(\theta) = 1/\theta$$

for every $F \in \mathcal{F}i_{\mathbf{GK}\Pi^\infty}\boldsymbol{A}$ and $\theta \in Co_{\mathbb{KP}^\infty}\boldsymbol{A}$, are complete lattice isomorphisms with one being the inverse of the other (see for instance [5]). We will make use of this isomorphism in the study of the local modal logic, which is not as direct as in the case of $\mathbf{GK}\Pi^\infty$.

Concerning the local modal logic $\mathbf{K}\Pi^\infty$, we begin by proving a weaker property than being algebraizable.

LEMMA 5.5
$\mathbf{K}\Pi^\infty$ is equivalential with congruence formulas $\boldsymbol{\Delta}(x,y) \equiv \{\Box^n(x \leftrightarrow y)\}_{n\in\omega}$, where $\Box^n\varphi = \Box\Box^{n-1}\varphi$ for $n > 0$ and $\Box^0\varphi = \varphi$.

PROOF. It suffices to see that $\boldsymbol{\Delta}(x,y)$ meets the fisrt condition of Lemma 2.2 for $\mathbf{K}\Pi^\infty$. It is routine to show this by using the already proven completeness of $\mathbf{K}\Pi^\infty$ with respect to $\models^l_{\mathsf{PK}}$. ■

This characterization is important for checking that the class of modal product algebras $\mathbb{KP}^\infty$ is indeed the algebraic counterpart of $\mathbf{K}\Pi^\infty$.

PROPOSITION 5.6
$Alg^*\mathbf{K}\Pi^\infty = \mathbb{KP}^\infty$.

PROOF. Let $\boldsymbol{A} \in \mathbb{KP}^\infty$. Taking into account that $\{1^{\boldsymbol{A}}\}$ is a filter of $\mathbf{K}\Pi^\infty$ on $\boldsymbol{A}$, by (2) of Lemma 2.2, $\langle a,b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}\{1^{\boldsymbol{A}}\}$ iff $\boldsymbol{\Delta}(a,b) \subseteq \{1^{\boldsymbol{A}}\}$. In particular, this implies that $a \leftrightarrow b = 1^{\boldsymbol{A}}$, and so, $a = b$. Therefore, $\boldsymbol{\Omega}^{\boldsymbol{A}}\{1^{\boldsymbol{A}}\} = \mathrm{Id}_{\boldsymbol{A}}$, and thus, $\langle \boldsymbol{A}, \{1^{\boldsymbol{A}}\}\rangle \in Mod^*\mathbf{K}\Pi^\infty$.

For the converse inclusion, we know that $\boldsymbol{A} \in Alg^*\mathbf{K}\Pi^\infty$ if and only if there exists $F \in \mathcal{F}i_{\mathbf{K}\Pi^\infty}\boldsymbol{A}$ for which $\boldsymbol{\Omega}^{\boldsymbol{A}}F = \mathrm{Id}_{\boldsymbol{A}}$. To see that $\boldsymbol{A} \in \mathbb{KP}^\infty$ if $\boldsymbol{A} \in Alg^*\mathbf{K}\Pi^\infty$ we have to check that the equations and quasiequations that define $\mathbb{KP}^\infty$ hold in $\boldsymbol{A}$ as well.

Each equation $\alpha_i \approx \beta_i$ corresponding to one of the identities that must hold in an algebra belonging to $\mathbb{KP}^\infty$ comes from an associated theorem $\alpha_i \leftrightarrow \beta_i$ of $\mathbf{K}\Pi^\infty$. By the ($\mathrm{N}_\Box$) rule, it follows that

$\Box^n(\alpha_i \leftrightarrow \beta_i)$ is also a theorem of the logic for each $n\in\omega$, and so, for any $h\in Hom(\mathbf{Fm},\boldsymbol{A})$ it holds that $\boldsymbol{\Delta}(h(\alpha_i),h(\beta_i))\subseteq F$, with $\boldsymbol{\Delta}$ being the set of congruence formulas of $\mathbf{K}\Pi^\infty$ defined in the previous lemma. By (2) of Lemma 2.2, it follows that $\langle h(\alpha_i),h(\beta_i)\rangle \in \Omega^{\boldsymbol{A}}F$ for all $h\in Hom(\mathbf{Fm},\boldsymbol{A})$, and given that by assumption $\Omega^{\boldsymbol{A}}F=\mathrm{Id}_{\boldsymbol{A}}$, it must hold that $h(\alpha_i)=h(\beta_i)$ for all $h\in Hom(\mathbf{Fm},\boldsymbol{A})$ (so $\boldsymbol{A}\models\alpha_i\approx\beta_i$).

For the generalized quasiequations ($\mathcal{Q}_\Delta$) and ($\mathcal{Q}_\rightarrow$), the proof is similar using that the filters are closed under the correspondent infinitary rules. We will detail the proof for ($\mathcal{Q}_\Delta$), the case of ($\mathcal{Q}_\rightarrow$) being analogous.

First observe that R_Δ can be written as $\{(\overline{c}\to\varphi)\leftrightarrow\overline{1}\}_{c\in(0,1)_{\mathbb{Q}}}\vdash_{\mathbf{K}\Pi^\infty}\varphi\leftrightarrow\overline{1}$. By Theorem 4.4, it follows that for any $n\in\omega$, $\{\Box^n((\overline{c}\to\varphi)\leftrightarrow\overline{1})\}_{c\in(0,1)_{\mathbb{Q}}}\vdash_{\mathbf{K}\Pi^\infty}\Box^n(\varphi\leftrightarrow\overline{1})$. By definition, filters of $\mathbf{K}\Pi^\infty$ are closed under deductions, so for any $a\in A$ and $n\in\omega$,

$$\text{if } \{\Box^n((c^{\boldsymbol{A}}\to a)\leftrightarrow 1^{\boldsymbol{A}})\}_{c\in(0,1)_{\mathbb{Q}}}\subseteq F \text{ then } \Box^n(a\leftrightarrow 1^{\boldsymbol{A}})\in F. \tag{2}$$

To see that $\mathcal{Q}_\Delta$ holds in $\boldsymbol{A}$, let $a\in A$ be such that $c^{\boldsymbol{A}}\to a=1^{\boldsymbol{A}}$ for all $c\in(0,1)_{\mathbb{Q}}$. Since $\boldsymbol{\Omega}^{\boldsymbol{A}}F=\mathrm{Id}_{\boldsymbol{A}}$, $\langle c^{\boldsymbol{A}}\to a, 1^{\boldsymbol{A}}\rangle\in\boldsymbol{\Omega}^{\boldsymbol{A}}F$ for all $c\in(0,1)_{\mathbb{Q}}$. Given that $\mathbf{K}\Pi^\infty$ is equivalential, this happens if and only if $\{\Box^n((c^{\boldsymbol{A}}\to a)\leftrightarrow 1^{\boldsymbol{A}})\}_{c\in(0,1)_{\mathbb{Q}},n\in\omega}\subseteq F$. From (2) it follows that $\{\Box^n(a\leftrightarrow 1^{\boldsymbol{A}})\}_{n\in\omega}\subseteq F$, i.e. $\boldsymbol{\Delta}^{\boldsymbol{A}}(a,1^{\boldsymbol{A}})\subseteq F$. Following the same reasoning as before this is equivalent to $\langle a,1^{\boldsymbol{A}}\rangle\in\boldsymbol{\Omega}^{\boldsymbol{A}}F$, and so, to $a=1^{\boldsymbol{A}}$ in $\boldsymbol{A}$, which concludes the proof. ∎

We are now able to conclude that, as in the case for the local classical modal logic, $\mathbf{K}\Pi^\infty$ is not algebraizable. However, to prove this, we need a different approach than the one used in classical modal logic, where it is enough to define a 4-element Boolean algebra with modalities that is simple but for which there exist more than two filters of the local modal logic. In our case, there do not exist finite $\mathbf{K}\Pi^\infty$-algebras, so this ad-hoc construction is more complicated and it is in fact easier to prove it in a more general way. Moreover, since classical logic is not an extension of Π^∞ (because of the rational constants and the Δ operator), we cannot use the fact that classical local modal logic is not algebraizable to straightforwardly derive that $\mathbf{K}\Pi^\infty$ is not algebraizable either.

Lemma 5.7
$\mathbf{K}\Pi^\infty$ is not algebraizable.

Proof. For the sake of a simpler notation, in this proof we let $\mathcal{C}nX=\{\theta\in Fm\colon X\vdash_{\mathbf{K}\Pi^\infty}\theta\}$, and we will simply denote by $\boldsymbol{\Omega}$ the Leibniz operator $\boldsymbol{\Omega}^{\mathbf{Fm}}$ over the Lindenbaum-Tarski algebra $\mathbf{Fm}$ of formulas of $\mathbf{K}\Pi^\infty$ (i.e. the set of formulas modulo $\mathbf{K}\Pi^\infty$-equivalence).

It is clear that for $x,y\in Fm$ such that $x\neq y$, we have $\{x,y\}\nvdash_{\mathbf{K}\Pi^\infty}\Box(x\leftrightarrow y)$, and so, in particular, $\boldsymbol{\Delta}(x,y)\nsubseteq\mathcal{C}n\{x,y\}$. From the definition of equivalential logic and Lemma 5.5 it follows that $\langle x,y\rangle\notin\boldsymbol{\Omega}\mathcal{C}n\{x,y\}$, which in particular proves that $\mathcal{C}n\{x,y\}/\boldsymbol{\Omega}\mathcal{C}n\{x,y\}$ has at least two different elements: $x/\boldsymbol{\Omega}\mathcal{C}n\{x,y\}$ and $y/\boldsymbol{\Omega}\mathcal{C}n\{x,y\}$.

For simplicity, let $\boldsymbol{A}$ denote $\mathbf{Fm}/\boldsymbol{\Omega}\mathcal{C}n\{x,y\}$. Since $\mathcal{C}n\{x,y\}\in\mathcal{F}i_{\mathbf{K}\Pi^\infty}\mathbf{Fm}$ it follows that $\boldsymbol{\Omega}^{\boldsymbol{A}}(\mathcal{C}n\{x,y\}/\boldsymbol{\Omega}\mathcal{C}n\{x,y\})=\mathrm{Id}_{\boldsymbol{A}}$, and so

$$\langle\boldsymbol{A},\mathcal{C}n\{x,y\}/\boldsymbol{\Omega}\mathcal{C}n\{x,y\}\rangle\in Mod^*\mathbf{K}\Pi^\infty,$$

which implies that $\boldsymbol{A}\in Alg^*\mathbf{K}\Pi^\infty=\mathbb{KP}^\infty$. We know that $\{\overline{1}\}$ is a filter of $\mathbf{K}\Pi^\infty$ over $\boldsymbol{A}$, so $\{\overline{1}/\boldsymbol{\Omega}\mathcal{C}n\{x,y\}\}\in\mathcal{F}i_{\mathbf{K}\Pi^\infty}\boldsymbol{A}$. Now, by Lemma 2.3, we have that

$$\boldsymbol{\Omega}^{\boldsymbol{A}}\{\overline{1}/\boldsymbol{\Omega}\mathcal{C}n\{x,y\}\}\subseteq\boldsymbol{\Omega}^{\boldsymbol{A}}(\mathcal{C}n\{x,y\}/\boldsymbol{\Omega}\mathcal{C}n\{x,y\})=\mathrm{Id}_{\boldsymbol{A}},$$

and so, $\boldsymbol{\Omega}^{\boldsymbol{A}}\{\overline{1}/\boldsymbol{\Omega}\mathcal{C}n\{x,y\}\} = \mathrm{Id}_{\boldsymbol{A}}$ as well. Then, both

$$\langle \boldsymbol{A}, \mathcal{C}n\{x,y\}/\boldsymbol{\Omega}\mathcal{C}n\{x,y\}\rangle \quad \text{and} \quad \langle \boldsymbol{A}, \{\overline{1}/\boldsymbol{\Omega}\mathcal{C}n\{x,y\}\}\rangle$$

are reduced models of $\mathbf{K}\Pi^{\infty}$. Since their algebraic counterparts coincide, but they are different models (given that $\mathcal{C}n\{x,y\}/\boldsymbol{\Omega}\mathcal{C}n\{x,y\}$ has at least two elements, and $\{\overline{1}/\boldsymbol{\Omega}\mathcal{C}n\{x,y\}\}$ is a singleton), we have that $\boldsymbol{\Omega}$ is not injective, and thus, it is not an isomorphism, as the isomorphism Theorem 2.5 requires. ■

Recall that $\mathbf{K}\Pi^{\infty}$ is strongly complete with respect to its reduced models. Inspired by [32], where the analogous problem for the classical modal logic is studied, we can give a more specific characterization of this class of models, and thus, obtain a more concrete completeness result for $\mathbf{K}\Pi^{\infty}$ as an immediate corollary.

THEOREM 5.8
$\langle \boldsymbol{A}, F\rangle$ is a reduced model of $\mathbf{K}\Pi^{\infty}$ if and only if $\boldsymbol{A} \in \mathbb{KP}^{\infty}$, $F \in \mathcal{F}i_{\mathbf{K}\Pi^{\infty}}\boldsymbol{A}$ and $\{1^{\boldsymbol{A}}\}$ is the only open filter included in F.

PROOF. As for the left to right direction, let $\langle \boldsymbol{A}, F\rangle \in Mod^{*}\mathbf{K}\Pi^{\infty}$. From Lemma 5.6 it follows that $\boldsymbol{A} \in \mathbb{KP}^{\infty}$ and it also holds that $F \in \mathcal{F}i_{\Pi^{\infty}}(\overline{\boldsymbol{A}})$. By way of contradiction, suppose there is an open $\mathbf{K}\Pi^{\infty}$-filter G on $\boldsymbol{A}$ such that $\{1\} \subsetneq G \subseteq F$. Since both $\{1\}$ and G are open filters, they are also deductive $\mathbf{GK}\Pi^{\infty}$-filters on $\boldsymbol{A}$. Given that $\mathbf{GK}\Pi^{\infty}$ is algebraizable, $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is a lattice isomorphism between $\mathcal{F}i_{\mathbf{GK}\Pi^{\infty}}\boldsymbol{A}$ and $Co_{\mathbb{KP}^{\infty}}\boldsymbol{A}$, and, in particular, $\boldsymbol{\Omega}^{\boldsymbol{A}}: \mathcal{F}i_{\mathbf{GK}\Pi^{\infty}}\boldsymbol{A} \to Co_{\mathbb{KP}^{\infty}}\boldsymbol{A}$ is injective. Then, given that $\{1\} \neq G$, $\boldsymbol{\Omega}^{\boldsymbol{A}}\{1\} \neq \boldsymbol{\Omega}^{\boldsymbol{A}}G$. It is clear that $\boldsymbol{\Omega}^{\boldsymbol{A}}\{1\} = \mathrm{Id}_{\boldsymbol{A}}$ and so, $\boldsymbol{\Omega}^{\boldsymbol{A}}G \neq \mathrm{Id}_{\boldsymbol{A}}$. $\mathbf{K}\Pi^{\infty}$ is equivalential so $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is monotone over the $\mathbf{K}\Pi^{\infty}$-filters on $\boldsymbol{A}$. Given that both F and G are $\mathbf{K}\Pi^{\infty}$-filters on $\boldsymbol{A}$ it follows that $\mathrm{Id}_{\boldsymbol{A}} \neq \boldsymbol{\Omega}^{\boldsymbol{A}}G \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}F$, so $\boldsymbol{\Omega}^{\boldsymbol{A}}F \neq \mathrm{Id}_{\boldsymbol{A}}$ which contradicts the assumption that $\langle \boldsymbol{A}, F\rangle \in Mod^{*}\mathbf{K}\Pi^{\infty}$.

As for the other direction, let $F \in \mathcal{F}i_{\mathbf{K}\Pi^{\infty}}\boldsymbol{A}$ and suppose that $\boldsymbol{\Omega}^{\boldsymbol{A}}F \neq \mathrm{Id}_{\boldsymbol{A}}$. We know that $\boldsymbol{\Omega}^{\boldsymbol{A}}: \mathcal{F}i_{\mathbf{GK}\Pi^{\infty}}\boldsymbol{A} \to Co_{\mathbb{KP}^{\infty}}\boldsymbol{A}$ is an isomorphism, and also that $\boldsymbol{\Omega}^{\boldsymbol{A}}F \in Co_{\mathbb{KP}^{\infty}}\boldsymbol{A}$. Then, there is $G \in \mathcal{F}i_{\mathbf{GK}\Pi^{\infty}}\boldsymbol{A}$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}G = \boldsymbol{\Omega}^{\boldsymbol{A}}F$, and so, $\boldsymbol{\Omega}^{\boldsymbol{A}}G \neq \mathrm{Id}_{\boldsymbol{A}}$. For any $a \in G$, $\langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}G$ and so $\langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F$ as well. Since $1 \in F$ and $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is congruent with F (by definition), $a \in F$, and so $G \subseteq F$, which concludes the proof. ■

Finally, the existence and behaviour of the constants in the algebras in $\mathbb{KP}^{\infty}$ leads to a very clear characterization of all the linearly ordered modal product algebras.

THEOREM 5.9
If an algebra $\boldsymbol{A} = \langle A, \odot, \to, \Delta, \Box, \Diamond, \{c^{\boldsymbol{A}}\}_{c\in[0,1]_{\mathbb{Q}}}\rangle \in \mathbb{KP}^{\infty}$ is linearly ordered then one of the following conditions holds:

- $\Box = \Diamond = Id_{\boldsymbol{A}}$ (the identity function in $\boldsymbol{A}$);
- $\Box = 1^{\boldsymbol{A}}$ and $\Diamond = 0^{\boldsymbol{A}}$ (the constant functions of value 1 and 0 respectively).

PROOF. First we can easily see that for any $a \in A$ both $a \leqslant \Box a$ and $\Diamond a \leqslant a$ hold. Indeed, if $\Box a < a$, from Corollary 3.12, it would follow that there is $c \in [0,1]_{\mathbb{Q}}$ such that $\Box a < c^{\boldsymbol{A}} < a$. In that case, it is clear that $\Box(c^{\boldsymbol{A}} \to a) = 1$, but at the same time, $1 > c^{\boldsymbol{A}} \to \Box a = \Box(c^{\boldsymbol{A}} \to a)$, which is a contradiction. For $\Diamond$, the proof is analogous.

Now, we will prove by cases that if the interpretation of $\Box$ or $\Diamond$ is not the identity function, then $\Box = \overline{1}$ and $\Diamond = \overline{0}$. First, suppose that $\Box$ is not the identity function. Since for all $a \in A$ it holds that $a \leqslant \Box a$, then there must exist $b \in A$ such that $b < \Box b$ ($\mathbf{A}$ is a chain by assumption). By Corollary 3.12 there exists $c \in [0,1]_{\mathbb{Q}}$ such that $b < c^{\boldsymbol{A}} < \Box b$, from where it follows that $1 = c^{\boldsymbol{A}} \to \Box b = \Box(c^{\boldsymbol{A}} \to b) =$

$\Delta\Box(c^{\boldsymbol{A}} \to b)$. Given that Δ and $\Box$ commute in $\boldsymbol{A}$, it follows that $\Box\Delta(c^{\boldsymbol{A}} \to b) = 1$. Now note that $(c^{\boldsymbol{A}} \to b) < 1$ by assumption, and since $\boldsymbol{A}$ is a product chain, $\Delta(c^{\boldsymbol{A}} \to b) = 0$. Then $1 = \Box\Delta(c^{\boldsymbol{A}} \to b) = \Box 0$, and using that $\Box$ is an increasing function[9] we can conclude that $\Box = \overline{1}$ (since 0 is the minimum element of $\boldsymbol{A}$). Since $\Box\neg x = \neg\Diamond x$, it is immediate that $\Diamond = \overline{0}$.

On the other hand, suppose that for some $b \in A$, $\Diamond b < b$, and so $\Diamond b < c^{\boldsymbol{A}} < b$ for some $c \in [0,1]_{\mathbb{Q}}$. It follows that $1 = \Diamond b \to c^{\boldsymbol{A}} = \Box(b \to c^{\boldsymbol{A}})$, and an analogous reasoning as before leads to $\Box 0 = 1$. From there, again as above, one can prove that $\Box = \overline{1}$ and $\Diamond = \overline{0}$, and this concludes the proof. ∎

With this result, it is clear that neither $\mathbf{K}\Pi^{\infty}$ nor $\mathbf{GK}\Pi^{\infty}$ are complete with respect to $\mathbb{KP}^{\infty}$-chains. Indeed, from the previous lemma, $\Box\overline{0} \vee (\varphi \leftrightarrow \Box\varphi)$ is valid in all linearly ordered modal product algebras, but clearly it is not a theorem of $\mathbf{K}\Pi^{\infty}$.

5.2 Complex algebras and Canonical models

Since we have developed two semantics for our modal logics, namely Kripke and algebraic semantics, it is natural to study their relationship. Complex algebras are a family of algebras that arise from Kripke frames through a dual construction, and were originally defined in the context of classical modal logic (see for instance [17]). The construction method can be easily generalized to the case of many-valued logics, taking into account the fact that the algebra over which the Kripke model evaluates formulas takes an explicit role in the construction of the dual algebra from the model.

In our case, we will describe a way of translating the Kripke semantics into the algebraic one by associating to each product Kripke model a $\mathbf{K}\Pi^{\infty}$-algebra and an evaluation over it.

Remember that for two sets A, B, A^B denotes the set of maps from B to A, and that for simplicity we will sometimes denote by $[b \mapsto f(b)]$ the element $f \in A^B$ that sends $b \in B$ to $f(b) \in A$.

Definition 5.10

Let $\boldsymbol{A} \in \mathbb{P}^{\infty}$ and $\mathfrak{M} = \langle W, R, e\rangle$ be a (safe) $\boldsymbol{A}$-Kripke model. The **complex algebra associated to** $\mathfrak{M}$ is the partial algebra

$$\mathbb{A}lg(\mathfrak{M}) = \langle A^W, \odot, \to, \Delta, \Box, \Diamond, \{c^{\mathbb{A}lg(\mathfrak{M})}\}_{c\in[0,1]_{\mathbb{Q}}}\rangle,$$

where for every $f, g \in A^W$ the non-modal operations $\odot, \to, \Delta$ and $\{c^{\mathbb{A}lg(\mathfrak{M})}\}_{c\in[0,1]_{\mathbb{Q}}}$ are defined component-wise from the ones of $\mathbf{A}$ and the modal operations are given by

$$\Box f := [v \mapsto \inf\{f(w) : w \in W, Rvw\}] \qquad \Diamond f := [v \mapsto \sup\{f(w) : w \in W, Rvw\}].$$

The associated evaluation over $\mathbb{A}lg(\mathfrak{M})$ is $e_{\mathfrak{M}} : Fm \to A^W$ with

$$e_{\mathfrak{M}}(\varphi) = [v \mapsto e(v, \varphi)].$$

Notice that, although the algebra may be partial in the sense that $\Box f$ and $\Diamond f$ may be not defined for all $f \in A^W$, the evaluation $e_{\mathfrak{M}}(\varphi)$ is defined over every formula φ because we assume the model $\mathfrak{M}$ to be safe.

It is routine to see that for any product Kripke model $\mathfrak{M}$, its complex algebra $\mathbb{A}lg(\mathfrak{M})$ is a $\mathbf{K}\Pi^{\infty}$-algebra. Also, it is easy to check that the evaluation associated with $\mathfrak{M}$ is indeed a modal evaluation

[9] It follows from (E_K) that for any $x, y \in A$, if $x \leqslant y$ then $\Box x \leqslant \Box y$.

over $\mathbb{A}lg(\mathfrak{M})$ (i.e. $e_{\mathfrak{M}} \in Hom(\mathbf{Fm}, \mathbb{A}lg(\mathfrak{M}))$) and that it coincides with the homomorphism that sends each propositional variable x to $[v \mapsto e(v,x)]$.

The previous construction provides us with several concrete modal product algebras, since the class of complex algebras is included in $\mathbb{KP}^{\infty}$. Moreover, using the completeness of $\mathbf{K}\Pi^{\infty}$ with respect to $\models^{l}_{\mathsf{PK}}$, and in particular with respect to the canonical Kripke model $\mathfrak{M}_{\mathfrak{c}}$, we will obtain another interesting algebraic completeness result for this logic.

The intuition behind the algebraic completeness result that can be achieved following this approach is as follows. For $\Gamma \cup \{\varphi\} \subseteq Fm$ it holds that $\Gamma \vdash_{\mathbf{K}\Pi^{\infty}} \varphi$ if and only if for all $w \in W_c$ for which $w([\Gamma]) \subseteq \{1\}$ it holds $w(\varphi) = 1$ as well. This condition is easily expressible within the associated complex algebra $\mathbb{A}lg(\mathfrak{M}_{\mathfrak{c}})$: it means that for all $w \in W_c$ for which $e_{\mathfrak{M}_{\mathfrak{c}}}([\Gamma])(w) \subseteq \{1\}$ then $e_{\mathfrak{M}_{\mathfrak{c}}}(\varphi)(w) = 1$. Given that the Δ operator is defined on the complex algebras component-wise it is clear that, for any $w \in W_c$ and for any formula ψ, either $e_{\mathfrak{M}_{\mathfrak{c}}}(\Delta\psi)(w) = 1$ or $e_{\mathfrak{M}_{\mathfrak{c}}}(\Delta\psi)(w) = 0$. Then, the previous condition can be rewritten as $\inf e_{\mathfrak{M}_{\mathfrak{c}}}([\Delta\Gamma]) \leqslant e_{\mathfrak{M}_{\mathfrak{c}}}(\varphi)$.[10]

Given a product algebra $\boldsymbol{A}$, we will say that $b \in \boldsymbol{A}$ is a **Boolean element** whenever there exists an element $b' \in \boldsymbol{A}$, called **complement of** b, such that $b \vee b' = 1$ and $b \wedge b' = 0$. We will denote the set of Boolean elements of $\boldsymbol{A}$ by $\mathfrak{B}_{\boldsymbol{A}}$.

Remark 5.11
Given $\boldsymbol{A} \in \mathbb{P}^{\infty}$, observe that $\Delta a \in \mathfrak{B}_{\boldsymbol{A}}$ for any $a \in \boldsymbol{A}$ and that for any $b \in \mathfrak{B}_{\boldsymbol{A}}$ it holds that $b \odot x = b \wedge x$ for any $x \in \boldsymbol{A}$.

The first statement is direct by definition. As for the second, we have $b \wedge x = b \odot (b \to x)$ by definition. We know from [28, (12) of Lemma 2]) that for any $z \in \boldsymbol{A}$ it holds that $(b \to x) \leqslant b \vee z \to x \vee z$. Letting then z be the complement of b, we have that $b \wedge x \leqslant b \odot (x \vee b')$ (since $b \vee b' = 1$), and thus, by commutativity of $\odot$ and $\vee$ in BL (see for instance [26]), $b \wedge x \leqslant (b \odot x) \vee (b \odot b') = b \odot x$ (since $b \wedge b' = 0$).

Now, with the aim at generalizing the above idea of logic preserving the Boolean degrees of truth to the whole class of $\mathbf{K}\Pi^{\infty}$-algebras, we will refer to the usual definition of a (many-valued) logic preserving degrees of truth (cf. [1, Definition 2.1]).

Definition 5.12
Let $\Gamma \cup \{\varphi\} \subseteq Fm$, $\mathbb{K} \cup \{\boldsymbol{A}\} \subseteq \mathbb{KP}^{\infty}$ and $h \in Hom(\mathbf{Fm}, \boldsymbol{A})$. Then we define:

- $\Gamma \models^{\leqslant}_{\boldsymbol{A},h} \varphi$ if for any $a \in A$, if $a \leqslant h(\gamma)$ for all $\gamma \in \Gamma$ then $a \leqslant h(\varphi)$.
- $\Gamma \models^{\leqslant}_{\boldsymbol{A}} \varphi$ if $\Gamma \models^{\leqslant}_{\boldsymbol{A},h} \varphi$ for any $h \in Hom(\mathbf{Fm}, \boldsymbol{A})$.
- $\Gamma \models^{\leqslant}_{\mathbb{K}} \varphi$ if $\Gamma \models^{\leqslant}_{\mathbf{B}} \varphi$ for each $\mathbf{B} \in \mathbb{K}$.

In particular, if $\mathbf{A}$ is a complete algebra, the first definition is equivalent to say that $\inf\{h(\Gamma)\} \leqslant h(\varphi)$.

Now, a second algebraic completeness result for $\mathbf{K}\Pi^{\infty}$ can be obtained just checking the soundness of $\mathbf{K}\Pi^{\infty}$ with respect to the $\models^{\leqslant}_{\mathbb{KP}^{\infty}}$ logic defined above in which just the preservation for Boolean elements is required.

Theorem 5.13 (Algebraic completeness of $\mathbf{K}\Pi^{\infty}$)
For any set of modal formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash_{\mathbf{K}\Pi^{\infty}} \varphi \text{ iff } \Delta\Gamma \models^{\leqslant}_{\mathbb{KP}^{\infty}} \varphi,$$

[10] $\leqslant$ is defined component-wise, and thus for each $w \in W_c$, either $e_{\mathfrak{M}_{\mathfrak{c}}}([\Delta\Gamma])(w) \subseteq 1$, in which case by assumption $e_{\mathfrak{M}_{\mathfrak{c}}}(\varphi)(w) = 1$, or there is $\gamma \in \Gamma$ with $e_{\mathfrak{M}_{\mathfrak{c}}}(\Delta\gamma)(w) = 0$, and thus $\inf e_{\mathfrak{M}_{\mathfrak{c}}}([\Delta\Gamma]) = 0$.

where $\Delta\Gamma$ stands for the set $\{\Delta\gamma : \gamma \in \Gamma\}$.[11]

PROOF. Completeness is direct because $\mathbb{A}lg(\mathfrak{M}_{\mathfrak{c}}) \in \mathbb{KP}^{\infty}$. If $\Gamma \nvdash_{\mathbf{K}\Pi^{\infty}} \varphi$, by completeness with respect to $\mathfrak{M}_{\mathfrak{c}}$, there exists a world $v \in W_c$ such that $v([\Gamma]) \subseteq \{1\}$ and $v(\varphi) < 1$. Then we have that $e_{\mathfrak{M}_{\mathfrak{c}}}([\Gamma])(v) = 1$ (and so, that $e_{\mathfrak{M}_{\mathfrak{c}}}([\Delta\Gamma])(v) = 1$) but $e_{\mathfrak{M}_{\mathfrak{c}}}(\varphi)(v) < 1$. But taking into account how the order is defined in $\mathbb{A}lg(\mathfrak{M}_{\mathfrak{c}})$, this means that $e_{\mathfrak{M}_{\mathfrak{c}}}([\Delta\Gamma]) \not\leqslant e_{\mathfrak{M}_{\mathfrak{c}}}(\varphi)$.

To prove soundness, let $\boldsymbol{A} \in \mathbb{KP}^{\infty}$ and $h \in Hom(\mathbf{Fm}, \boldsymbol{A})$. Since all axioms of $\mathbf{K}\Pi^{\infty}$ are evaluated to 1 under any homomorphism, we just need to check the soundness of the deduction rules of $\mathbf{K}\Pi^{\infty}$ in $\models^{\leqslant}_{\mathbb{KP}^{\infty}}$:

- (MP) $h(\Delta\psi) \wedge h(\Delta(\psi \to \varphi)) = h(\Delta\psi) \odot h(\Delta(\psi \to \varphi)) \leqslant h(\Delta\psi) \odot (h(\Delta\psi) \to h(\Delta\varphi)) \leqslant h(\Delta\varphi) \leqslant h(\varphi)$, by the Remark 5.11 and definitions and axioms of product logic.
- (G_{Δ}) It is direct by definition.
- $(\mathrm{R}_{\Delta}/\mathrm{R}_{\to})$ Let $a \in \boldsymbol{A}$ such that $a \leqslant h(\Delta(\overline{c} \to \varphi))$ for all $c \in (0,1)_{\mathbb{Q}}$. Then, $a \leqslant h(\overline{c} \to \varphi) = c^{\boldsymbol{A}} \to h(\varphi)$ for all $c \in (0,1)_{\mathbb{Q}}$. By residuation, $c^{\boldsymbol{A}} \leqslant a \to h(\varphi)$ for all $c \in (0,1)_{\mathbb{Q}}$, so by $(\mathcal{Q}_{\Delta})$ it follows that $a \to h(\varphi) = 1$, i.e. that $a \leqslant h(\varphi)$. The $R_{\to}$ case can be proved analogously using $(\mathcal{Q}_{\to})$.
- $(N_{\Box})$ This rule is only applicable over theorems, so assume that for any modal product algebra, and for any evaluation e, $e(\psi) = 1$. Then, by definition of modal product algebra, in particular for any modal morphism h, we have $h(\Box\psi) = \Box h(\psi) = \Box 1 = 1$, so $\models_{\mathbb{KP}^{\infty}} \Box\psi$, and thus, $\models^{\leqslant}_{\mathbb{KP}^{\infty}} \Box\psi$ as well.

■

It is natural to close this section by showing how to build a product Kripke model from a $\mathbf{K}\Pi^{\infty}$-algebra $\boldsymbol{A}$ and a modal evaluation over it, in an inverse way to the complex algebra construction, and studying the relation between these two constructions.[12] A suitable choice for the set of worlds of a Kripke model associated with a $\mathbf{K}\Pi^{\infty}$-algebra $\boldsymbol{A}$ are the set of homomorphisms from $\overline{\boldsymbol{A}}$ into $[0,1]_{\Pi}$.

DEFINITION 5.14
Let $\boldsymbol{A} \in \mathbb{KP}^{\infty}$, $e \in Hom(\mathbf{Fm}, \boldsymbol{A})$ and $\boldsymbol{A^e} = e(\boldsymbol{Fm})$ (the subalgebra of $\boldsymbol{A}$ given by the image of e). The **canonical $\langle \boldsymbol{A}, e\rangle$-Kripke model** is the model

$$\mathfrak{Mod}(\boldsymbol{A}, e) = \langle W_{\boldsymbol{A}}, R_{\boldsymbol{A}}, e_{\boldsymbol{A}} \rangle,$$

where:

- $W_{\boldsymbol{A}} = Hom(\overline{\boldsymbol{A^e}}, [0,1]_{\Pi})$;
- $R_{\boldsymbol{A}} vw$ if and only if $w(a) = 1$ for all $a \in A^e$ such that $v(\Box a) = 1$;
- $e_{\boldsymbol{A}}(v, x) = v(e(x))$.

Our aim is to prove that this construction is inverse to that of the complex algebras from product Kripke models, in the sense that applying these constructions one after the other over a structure of the corresponding type produces a new structure N of the same type in which the original one O can be 'embedded' (in the sense that the consequence relation induced by N is a subset of the one induced by O). That is to say, for an arbitrary set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \models_N \varphi \text{ implies that } \Gamma \models_O \varphi.$$

[11] Do not forget that, in general, modal product algebras are not linearly ordered.

[12] It is remarkable that this construction leads to a more abstract proof of completeness of the logic $\mathbf{K}\Pi^{\infty}$ with respect to its correspondent canonical models. It turns out that the canonical model of the Lindenbaum-Tarski formula algebra of the correspondent logic is the canonical model of the logic.

In order to do this we need first to see that the evaluation $e_{\boldsymbol{A}}$ of the above canonical Kripke model $\mathfrak{Mod}(\boldsymbol{A},e)$ verifies the Truth Lemma, i.e. $e_{\boldsymbol{A}}(v,\varphi)=v(e(\varphi))$ for any $\varphi\in Fm$ and any $e\in Hom(\mathbf{Fm},\boldsymbol{A})$.

We will prove this by checking that for any $a\in A'$ (i.e. any element of A that is the image of some formula), both $v(\Box a)=\inf\{w(a)\colon R_{\boldsymbol{A}}vw\}$ and $v(\Diamond a)=\sup\{w(a)\colon R_{\boldsymbol{A}}vw\}$ hold true.

First, observe that a generalization of Lemma 4.8 easily follows from the fact that each $w\in W_{\boldsymbol{A}}$ is a homomorphism from $\overline{\boldsymbol{A^e}}$ into $[0,1]_{\Pi}$. Indeed, for any $a,b\in\boldsymbol{A^e}$ and any $v,w\in W_{\boldsymbol{A}}$, if $v(a)\nleqslant w(b)$ then $w(b)<v(a)$, and thus, there is $c\in[0,1]_{\mathbb{Q}}$ such that $w(b)<c<v(a)$, i.e. $v(c^{\boldsymbol{A}}\to a)=1$ and $w(c^{\boldsymbol{A}}\to b)<1$. Using the equations ($\mathrm{E_{A_\Box}1}$) and ($\mathrm{E_{A_\Diamond}1}$) we get

- $v(\Box a)\leqslant w(a)$ for all $v,w\in W_{\boldsymbol{A}}$ with $R_{\boldsymbol{A}}vw$;
- $v(\Diamond a)\geqslant w(a)$ for all $v,w\in W_{\boldsymbol{A}}$ with $R_{\boldsymbol{A}}vw$.

As for the converse inequalities, the corresponding version of Lemma 4.9 can be proven.

Lemma 5.15
Let $\boldsymbol{A}\in\mathbb{KP}^{\infty}$, $e\in Hom(\mathbf{Fm},\boldsymbol{A})$. Further, let $v\in W_{\boldsymbol{A}}$ and $\varphi\in Fm$. If $w(e(\varphi))=1$ for all $w\in W_{\boldsymbol{A}}$ such that $R_{\boldsymbol{A}}vw$, then $v(e(\Box\varphi))=1$.

Proof. First, it is easy to see that for any $w\in W_{\boldsymbol{A}}$ the following hold:

- $e^{-1}w^{-1}(1)=\{\varphi\in Fm\colon w(e(\varphi))=1\}$ is a prime Π^{∞}-theory and
- $\Box^{-1}e^{-1}w^{-1}(1)=\{\varphi\in Fm\colon w(e(\Box\varphi))=1\}$ is a Π^{∞}-theory. This is easy to check using that w is an homomorphism from $\overline{\boldsymbol{A}}$ into $[0,1]_{\Pi}$, that e is a (modal) homomorphism from $\boldsymbol{Fm}$ into $\boldsymbol{A}$ and that the equations and generalised quasi-equations arising from the axiomatization of $\mathbf{K}\Pi^{\infty}$ hold in $\boldsymbol{A}$. In order to check that any axiom α belongs to $\Box^{-1}e^{-1}w^{-1}(1)$ and that $\Box^{-1}e^{-1}w^{-1}(1)$ is closed under MP, ($\mathbf{G}_{\Delta}$), R_{Δ} and $\mathrm{R}_{\to}$, it is just necessary to use, respectively, ($\mathrm{E_K}$), ($\mathrm{E_K}$), ($\mathrm{E_{A_\Box}2}$), ($\mathrm{E_{A_\Diamond}1}$) and ($\mathrm{E_{A_\Box}1}$) from the definition of the class of $\mathbb{KP}^{\infty}$ algebras.

On the other hand, we can prove the following claim:

Claim: The set of prime theories that contain $\Box^{-1}e^{-1}v^{-1}(1)$ coincides with the set $\{e^{-1}w^{-1}(1)\colon w\in W_{\boldsymbol{A}}$ and $R_{\boldsymbol{A}}vw\}$.

For each $w\in W_{\boldsymbol{A}}$ such that $R_{\boldsymbol{A}}vw$, the fact that $e^{-1}w^{-1}(1)$ is a prime theory that contains $\Box^{-1}e^{-1}v^{-1}(1)$ is immediate by definition of $R_{\boldsymbol{A}}vw$. We want to check that each prime theory containing $\Box^{-1}e^{-1}v^{-1}(1)$ in fact coincides with $e^{-1}w^{-1}(1)$ for some $w\in W_{\boldsymbol{A}}$ related with v. Define for each prime theory T containing $\Box^{-1}e^{-1}v^{-1}(1)$ the homomorphism $h_T\in Hom(\overline{\boldsymbol{A^e}},[0,1]_{\Pi})$ by letting $h_T=\rho_T\circ\Pi_T$, where:

- $\Pi_T:\overline{\boldsymbol{A^e}}\to\overline{\boldsymbol{A^e}}/\boldsymbol{\Omega}^{\overline{\boldsymbol{A^e}}}e(T)$ is the projection over the quotient with respect to the Leibniz congruence of the image of the theory, (that is to say, two elements a,b of A' are identified whenever there are $\varphi_1,\varphi_2\in Fm$ with $T\vdash_{\Pi^{\infty}}\varphi_1\leftrightarrow\varphi_2$ and such that $e(\varphi_1)=a$ and $e(\varphi_2)=b$).
- ρ_T is the embedding from $\overline{\boldsymbol{A^e}}/\boldsymbol{\Omega}^{\overline{\boldsymbol{A^e}}}e(T)$ into $[0,1]_{\Pi}$ built in the proof of Lemma 3.9 (since T is prime, $\overline{\boldsymbol{A^e}}/\boldsymbol{\Omega}^{\overline{\boldsymbol{A^e}}}e(T)$ is a Π^{∞}-chain).

It is clear that $T=e^{-1}h_T^{-1}(1)$. To check that $R_{\boldsymbol{A}}vh_T$, take for any $a\in A^e$ such that $v(\Box a)=1$, $\psi\in Fm$ with $e(\psi)=a$. By definition $\psi\in\Box^{-1}e^{-1}v^{-1}(1)$, and so, $\psi\in T$. Then $h_T(a)=h_T(e(\psi))=1$.

Assume now that $v(e(\Box\varphi)) < 1$. This means that $\varphi \notin \Box^{-1}e^{-1}v^{-1}(1)$. By Lemma 3.5, there exists a prime theory T that extends $\Box^{-1}e^{-1}v^{-1}(1)$ with $\varphi \notin T$. Using the previous claim, there is $w \in W_{\mathbf{A}}$ such that $\varphi \notin e^{-1}w^{-1}(1)$, i.e. $w(e(\varphi)) < 1$, which contradicts the assumptions of the Lemma. ■

From here it is straighforward to prove the next lemma, following the same reasoning as in the proof of Lemma 4.10, and from which it follows the previously mentioned more general Truth Lemma, that refers to the evaluation $e_{\mathbf{A}}$ of the canonical Kripke model $\mathfrak{Mod}(\boldsymbol{A}, e)$ for a $\mathbf{K}\Pi^{\infty}$-algebra $\mathbf{A}$ and a modal $\mathbf{A}$-evaluation e.

Lemma 5.16
Let $\boldsymbol{A}$ be a countable $\mathbf{K}\Pi^{\infty}$-algebra. For any $a \in A$ and $v \in W_{\mathbf{A}}$ the following hold.

- $v(\Box a) = \inf\{w(a) : R_{\mathbf{A}}vw\}$;
- $v(\Diamond a) = \sup\{w(a) : R_{\mathbf{A}}vw\}$.

Corollary 5.17 (Truth Lemma II)
Let $\boldsymbol{A}$ be a $\mathbf{K}\Pi^{\infty}$-algebra. For any $e \in Hom(\mathbf{Fm}, \boldsymbol{A})$, any formula φ and any $v \in W_{\mathbf{A}}$, it holds that $e_{\mathbf{A}}(v, \varphi) = v(e(\varphi))$.

It is now possible to study how the compositions of the $\mathfrak{Mod}()$ and the $\mathbb{A}lg()$ functions behave. For the sake of a lighter notation, in what follows, given a product Kripke model $\mathfrak{M}$, we will write $\mathfrak{M}'$ to denote the model $\mathfrak{Mod}(\mathbb{A}lg(\mathfrak{M}), e_{\mathfrak{M}})$ (its associated canonical model). Analogously, for a given $\boldsymbol{A} \in \mathbb{KP}^{\infty}$ and any modal $\boldsymbol{A}$-evaluation e, we will write $\boldsymbol{A}'$ instead of $\mathbb{A}lg(\mathfrak{Mod}(\boldsymbol{A}, e))$ (the complex algebra associated with $\boldsymbol{A}$).

A general approach towards the relation of the deductions at a model and a its associated canonical model, and similarly concerning algebras, is considering arbitrary algebras from the class.

Concerning the model relations, in the case when this algebras are linearly ordered, it is possible to obtain an embedding-like result, and prove that deductions are maintained at both sides. We first check the following technical lemma, which provides a simpler approach to the problem. For simplicity, given a product algebra $\boldsymbol{A}$ and an arbitrary set W, we will denote by $\boldsymbol{A}^W$ the product algebra of universe A^W and with its operations defined component-wise.

Lemma 5.18
Let $\boldsymbol{A} \in \mathbb{P}^{\infty}$ be linearly ordered, W an arbitrary denumerable set and F a linear Π^{∞}-filter on $\boldsymbol{A}^W$. Then there exists $I \subseteq W$ such that $F = \{f \in A^W : f(w) = 1 \text{ for all } w \in I\}$.

Proof. Let $I := \bigcap_{f \in F}\{w \in W : f(w) = 1\}$, and $B = \{g \in A^W : g(w) = 1 \text{ for all } w \in I\}$. To see that $F \subseteq B$, pick any $f \in F$, and notice that for any $v \in W$ such that $f(v) < 1$, $v \notin I$. Then, it is clear that $f \in B$. On the other hand, to prove that $B \subseteq F$, let $b \in B$. It is straightforward to see that, for any $x \in A^W$, either $\Delta x \in F$ or $\neg\Delta x \in F$, so in particular, either $\Delta b \in F$ or $\neg\Delta b \in F$. Towards a contradiction, suppose that $\neg\Delta b \in F$. Then, given that for all $v \in I$, $b(v) = 1$, it holds that $(\neg\Delta b)(v) = 0$, so by definition of I in terms of the elements in F, $v \notin I$, which is a contradiction. Then, $\Delta b \in F$, and since the equality $\Delta x \to x = 1$ holds in $\mathbb{P}^{\infty}$, it implies that $b \in F$ as well, concluding the proof. ■

Theorem 5.19
Let $\boldsymbol{A} \in \mathbb{P}^{\infty}$ be linearly ordered, $\mathfrak{M} = \langle W, R, e\rangle$ be an $\boldsymbol{A}$-Kripke model and $\Gamma \cup \{\varphi\} \subseteq Fm$. Then

$$\Gamma \models^{l}_{\mathfrak{M}} \varphi \text{ iff } \Gamma \models^{l}_{\mathfrak{M}'} \varphi.$$

Proof. We begin by proving the right to left direction. Suppose there is $w \in W$ such that $e(w, \Gamma) \subseteq \{1\}$ and $e(w, \varphi) < 1$. Consider then the world $h \in W' = Hom(\overline{\mathbb{A}lg(\mathfrak{M})^e}, [0,1]_{\Pi})$ given by $h = \rho_A \circ \pi_w$, where ρ_A is the embedding from $\boldsymbol{A}$ into $[0,1]_\Pi$ defined in Theorem 3.9 and π_w denotes the projection over the wth component (i.e. $\pi_w([v \mapsto f_v]) = f_w$). Then, using the previous Truth Lemma we get that $e'(h,\Gamma) = h(e_{\mathfrak{M}}(\Gamma)) = \rho_A e(w,\Gamma) \subseteq \{1\}$ and $e'(h,\varphi) = h(e_{\mathfrak{M}}(\varphi)) = \rho_A e(w,\varphi) < 1$.

To prove the other direction, suppose there is $h \in Hom(\overline{\mathbb{A}lg(\mathfrak{M})^e}, [0,1]_\Pi)$ such that $e'(h,\Gamma) \subseteq \{1\}$ and $e'(h,\varphi) < 1$. By Lemma 5.16, and letting F be the linear filter given by the set $h^{-1}(1)$, we know that $[v \mapsto e(v,\gamma)] \in F$ for all $\gamma \in \Gamma$ and $[v \mapsto e(v,\varphi)] \notin F$. By the previous lemma, there exists $I \subseteq W$ such that $F = \{f \in A^W : f(w) = 1 \text{ for all } w \in I\}$. Then, $[v \mapsto e(v,\gamma)] \in F$ for all $\gamma \in \Gamma$ is equivalent to say that $e(w,\gamma) = 1$ for all $w \in I$ and all $\gamma \in \Gamma$. On the other hand, we have that $[v \mapsto e(v,\varphi)] \notin F$ implies that there is some $w \in I$ such that $e(w,\varphi) < 1$, so we have that $\Gamma \not\models^l_{\mathfrak{M}} \varphi$. ■

On the other hand, we can also study the relation between a modal algebra $\boldsymbol{A}$ and its associated complex algebra $\boldsymbol{A}'$.

THEOREM 5.20
Let $\boldsymbol{A}$ be a countable $\mathbf{K}\Pi^\infty$-algebra and e be a modal evaluation in $\boldsymbol{A}$. Then $\boldsymbol{A}$ can be embedded into $\boldsymbol{A}'$.

PROOF. Note that the universe of the algebra $\mathbb{A}lg(\mathfrak{Mod}(\boldsymbol{A},e))$ is the set $[0,1]^{Hom(\overline{\boldsymbol{A}},[0,1]_\Pi)}$. Let ρ be the function that maps each element $a \in A$ to the function from $Hom(\overline{\boldsymbol{A}},[0,1]_\Pi)$ to $[0,1]$ that sends each h to $h(a)$, i.e. $\rho(a) = [h \mapsto h(a)]$ for each $h \in Hom(\overline{\boldsymbol{A}},[0,1]_\Pi)$. From the Truth Lemma 5.17 it follows that ρ is a modal homomorphism from $\boldsymbol{A}$ to $\mathbb{A}lg(\mathfrak{Mod}(\boldsymbol{A},e))$. It is straightforward to see that it is injective. ■

As a corollary, it is easy to see that, for any modal algebra $\boldsymbol{A}$ and any set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm$, we have:

$$\Delta\Gamma \not\models^{\leqslant}_{\boldsymbol{A}} \varphi \text{ implies that } \Delta\Gamma \not\models^{\leqslant}_{\boldsymbol{A}'} \varphi.$$

6 Some properties of $\mathbf{K}\Pi^\infty$

In this section we will further study the logic $\mathbf{K}\Pi^\infty$ in different aspects, presenting a miscellanea of results. We begin by focusing on some interdefinability issues between the modal operators of the logic. Then we consider axiomatic extensions of $\mathbf{K}\Pi^\infty$ with some usual modal axioms and their corresponding Kripke models. We finish by studying a possibilistic product logic, its embedding into a S5-type extension of $\mathbf{K}\Pi^\infty$, and its expansion with an involutive negation.

6.1 Interdefinability of $\Box$ and $\Diamond$

As we remarked in Section 2, a very interesting consequence of having rational constants in the language of $\mathbf{K}\Pi^\infty$ is obtaining a very strong relation between the $\Diamond$ and the $\Box$ operators. Even though they are not completely interdefinable, as it happens when the logic has an involutive negation, it is possible to use the constants to define a negation that will partially behave in an involutive way. This will allows us to show a partial interdefinability between the modal operators.

For any $c \in [0,1]_{\mathbb{Q}}$, let us consider a new unary connective $\neg_c$ defined as $\neg_c\varphi := \varphi \to \overline{c}$. These connectives are a sort of generalized negation, in fact $\neg_{\mathbf{0}} = \neg$, and e.g. they were already used by Jenei in [31] to study the question of determining the product t-norm (and others) by some level sets of its graph. Clearly, whenever either $e(\varphi) \geqslant c > 0$ or $e(\varphi) \in \{0,1\}$ and $c = 0$, $\neg_c$ behaves in an involutive way over φ. That is to say, for each $c \in (0,1]$, $\neg_c$ is an involutive operator in the range

$[c,1]$ of the $[0,1]_\Pi$ algebra (while, as it was already known, $\neg_0 = \neg$ is only involutive over $\{0,1\}$, and $\neg_1$ is trivially involutive only over 1). By completeness of Π^∞, this amounts to express that for all $c>0$, the following formula is a theorem of Π^∞ (and so, of $\mathbf{K}\Pi^\infty$ too):

$$\Delta(\overline{c}\to\varphi)\to(\neg_c\neg_c\varphi\leftrightarrow\varphi).$$

It is easy to see (semantically) that, for any constant, and in particular, for $c=0$, $\Box\neg_c\varphi\leftrightarrow\neg_c\Diamond\varphi$ is a theorem of $\mathbf{K}\Pi^\infty$. Then, whenever $\neg_c$ behaves in an involutive way over $\Diamond\varphi$, it is clear that it is possible to (locally) express $\Diamond$ in terms of the $\Box$ operator, and viceversa. This is shown in next lemma.

Lemma 6.1
Let φ be a modal formula, and v an arbitrary world of the canonical model. Then there exist $c,d\in[0,1]_\mathbb{Q}$ such that

(1) $v(\Diamond\varphi)=v(\neg_c\Box\neg_c\varphi)$
(2) if $v(\Box\varphi)>0$ then $v(\Box\varphi)=v(\neg_d\Diamond\neg_d\varphi)$

Proof. (1) We can split the proof in two cases. First, if $v(\Diamond\varphi)=0$, let $c=0$ and observe that $v(\neg\Box\neg\varphi)=\neg\neg v(\Diamond\varphi)=\neg\neg 0=0$. If $v(\Diamond\varphi)>0$, then pick c such that $0<c<v(\Diamond\varphi)$. Then, $v(\neg_c\Box\neg_c\varphi)=(v(\Diamond\varphi)\to c)\to c=c/(c/v(\Diamond\varphi))=v(\Diamond\varphi)$.

(2) $v(\Box\varphi)>0$ implies that there exists $0<d<v(\Box\varphi)$, and by definition, $d<w(\varphi)$ for all $w\in W_\mathfrak{c}$ such that $R_\mathfrak{c}vw$. Then for all the worlds w as before it holds that $w(\neg_d\neg_d\varphi)=w(\varphi)$. Using axiom ($A_\Diamond 1$) we immediately get $v(\neg_d\Diamond\neg_d\varphi)=v(\Box\neg_d\neg_d\varphi)=v(\Box\varphi)$.

■

It is interesting to notice that the second statement in the above lemma is just a reformulation of the quasi-witnessed model property for predicate Product logic, adapted to the particular case of the Product modal logic (see for instance [6] for more details at this respect).

A more general statement can be given by adding a precondition ensuring the involutive behaviour of the negations $\neg_c$.

Theorem 6.2
The following are theorems of $\mathbf{K}\Pi^\infty$ for any constant $c\in(0,1)_\mathbb{Q}$.

(1) $\neg\Delta\Box\neg_c\varphi\to(\Diamond\varphi\leftrightarrow\neg_c\Box\neg_c\varphi)$;
(2) $\neg\Diamond\Delta\neg_c\varphi\to(\Box\varphi\leftrightarrow\neg_c\Diamond\neg_c\varphi)$;

Proof. (1) For $v\in W_\mathfrak{c}$, $v(\neg\Delta\Box\neg\varphi)=1$ whenever $v(\Delta\Box\neg_c\varphi)=0$, and $v(\neg\Delta\Box\neg_c\varphi)=0$ otherwise, in which case the formula is trivially true. $v(\Delta\Box\neg_c\varphi)=0$ if and only if $v(\Box\neg_c\varphi)<1$ and from axiom ($A_\Diamond 1$) it follows that $v(\neg_c\Diamond\varphi)<1$. Using that in this case, $\neg_c$ is involutive we conclude that $v(\neg_c\Box\neg_c\varphi)=v(\neg_c\neg_c\Diamond\varphi)=v(\Diamond\varphi)$.

(2) Similarly, for $v\in W_\mathfrak{c}$, $v(\neg\Diamond\Delta\neg_c\varphi)=1$ whenever $v(\Diamond\Delta\neg_c\varphi)=0$, and it equals 0 in any other case. $v(\Diamond\Delta\neg_c\varphi)=0$ if and only if $w(\Delta\neg_c\varphi)=0$ for all $w\in W_\mathfrak{c}$ such that $R_\mathfrak{c}vw$, and so, if and only if $w(\neg_c\varphi)<1$ for any $w\in W_\mathfrak{c}$ such that $R_\mathfrak{c}vw$. Then, it holds that $w(\neg_c\neg_c\varphi)=w(\varphi)$ for any $w\in W_\mathfrak{c}$ such that $R_\mathfrak{c}vw$. Thus $v(\neg_c\Diamond\neg_c\varphi)=v(\Box(\neg_c\neg_c\varphi))=v(\Box\varphi)$.

■

6.2 *Some usual extensions of the minimal modal logic* $\mathbf{K}\Pi^{\infty}$

As expected, we can check there is a correspondence between extensions of $\mathbf{K}\Pi^{\infty}$ with common modal axioms and their corresponding classes of product Kripke models.

First, we recall the main properties that are generally studied over accessibility relations, which allow the use of the adequate class of structures to treat a certain problem.

DEFINITION 6.3
Let R be a binary crisp relation on W. We say that R is

- **Serial** iff for any $v \in W$ there is $w \in W$ such that Rvw ($\mathcal{D}$);
- **Reflexive** iff Rvv for every $v \in W$ ($\mathcal{T}$);
- **Symmetric** iff Rvw implies that Rwv for all $v, w \in W$ ($\mathcal{B}$);
- **Euclidean** iff for any $v, w, z \in X$ such that Rvw and Rvz it holds that Rwz (5);
- **Transitive** iff for any $v, w, z \in W$ such that Rvw and Rwz it holds that Rvz (4).

We will say that a model enjoys a set of properties whenever its accessibility relation does so. For each $p \in \{\mathcal{D}, \mathcal{T}, \mathcal{B}, 5, 4\}$, we will write PK^p to denote the class of product Kripke models that enjoy the property p. Then, for any subset of properties $\mathcal{A} \subseteq \{\mathcal{D}, \mathcal{T}, \mathcal{B}, 5, 4\}$, we define $\mathsf{PK}^{\mathcal{A}} = \bigcap_{p \in \mathcal{A}} \mathsf{PK}^p$.

DEFINITION 6.4
We let the main **structural axioms** be the following set of modal formulas:

(D) $\Box\varphi \to \Diamond\varphi$;
(T) $\Box\varphi \to \varphi$;
(B) $\Diamond\Box\varphi \to \varphi$;
(5) $\Diamond\Box\varphi \to \Box\varphi$;
(4) $\Box\varphi \to \Box\Box\varphi$.

For $A \subseteq \{(\mathrm{D}), (\mathrm{T}), (\mathrm{B}), (5), (4)\}$ we will denote by $\mathbf{K}\Pi^{\infty A}$ the logic obtained by extending $\mathbf{K}\Pi^{\infty}$ with the set of axioms in A. It is obvious there is a 1-1 relation between subsets of structural axioms A and subsets of properties $\mathcal{A}$, relating each axiom with its correspondent property over the Kripke models. It is not hard to prove that indeed this relation preserves the completeness of the extensions with respect to the classes of models that enjoy the corresponding properties.

THEOREM 6.5
Let $A \subseteq \{(\mathrm{D}), (\mathrm{T}), (\mathrm{B}), (5), (4)\}$ and $\Gamma \cup \{\varphi\} \subseteq Fm$. Then $\Gamma \vdash_{\mathbf{K}\Pi^{\infty A}} \varphi$ if and only if $\Gamma \models^{l}_{\mathsf{PK}^{\mathcal{A}}} \varphi$.

PROOF. It is enough to define the canonical model $\mathfrak{M}_c^A$ of $\mathbf{K}\Pi^{\infty A}$ by letting its universe be $W_{\mathfrak{c}}^{A} := \{w \in W_{\mathfrak{c}} : w([Th_{\mathbf{K}\Pi^{\infty A}}]) \subseteq \{1\}\}$. The completeness of $\mathbf{K}\Pi^{\infty A}$ with respect to this canonical model is direct, by an analogous reasoning as that of the one done for $\mathbf{K}\Pi^{\infty}$. To see that the extension is complete with respect to the class of models that enjoy a certain set of properties, it is enough to check that for each axiom in A, the canonical model $\mathfrak{M}_c^A$ enjoys the corresponding property. We will detail two cases, the rest can be done similarly.

(1) Assume $\mathrm{D} \in A$. Suppose towards a contradiction that there is $v \in W_{\mathfrak{c}}^{A}$, for which there is no $w \in W_{\mathfrak{c}}^{A}$ such that $R_{\mathfrak{c}}^{A} vw$. Then, by definition it holds that $v(\Box 0) = 1$ and $v(\Diamond 0) = 0$, but this contradicts axiom (D), so $\mathfrak{M}_c^A$ is serial.

(2) Assume $\mathrm{T} \in A$. Suppose towards a contradiction that there is $v \in W_{\mathfrak{c}}^{A}$ such that it does not hold that $R_{\mathfrak{c}}^{A} vv$, i.e. by definition of the canonical relation, there is $\theta \in Fm$ such that $v(\Box\theta) = 1$ but $v(\theta) < 1$. This contradicts axiom (T), so $\mathfrak{M}_{c}^{A}$ is reflexive.

■

6.3 *Possibilistic product logic*

It is not in the scope of this article to study product modal logics defined over *non-crisp* product Kripke Frames, i.e. Kripke frames $\langle W, R\rangle$ where the accessibility relation R generalizes to a fuzzy relation valued in an arbitrary product algebra $\boldsymbol{A}$, i.e. relations of the form $R\colon W \times W \to \boldsymbol{A}$. The gap between the *crisp* product modal logics that have been studied in this article and these non-crisp modal logics is quite big, since there is a lack of general tools or approaches in the literature for this subject, with the only exceptions of modal logics over Gödel fuzzy logic ([15, 16]) and over logics with truth-values in a finite residuated lattice [2]. However, there is a particular kind of non-crisp modal logic over product algebras that can be studied with the tools developed in this article, namely logics with *possibilistic* semantics.

Possibilistic logic (see e.g. [21, 22]) is a well-known uncertainty logic for reasoning with graded beliefs on classical propositions by means of necessity and possiblity measures. Possibilistic logic deals with weighted formulas (φ, r), where φ is a classical proposition and $r \in [0,1]$ is a weight, interpreted as a lower bound for the necessity degree of φ. The semantics of these degrees is defined in terms of possibility distributions $\pi: \Omega \to [0,1]$ on the set Ω of classical interpretations of a given propositional language. A possibility distribution π on Ω ranks interpretations according to its plausibility level: $\pi(w) = 0$ means that w is rejected, $\pi(w) = 1$ means that w is fully plausible, while $\pi(w) < \pi(w')$ means that w' is more plausible than w. A possibility distribution $\pi: \Omega \to [0,1]$ induces a pair of dual possibility and necessity measures on propositions, defined respectively as:

$$\Pi(\varphi) := \sup\{\pi(w) \mid w \in \Omega, w(\varphi) = 1\}$$
$$\mathsf{N}(\varphi) := \inf\{1 - \pi(w) \mid w \in \Omega, w(\varphi) = 0\}\ .$$

They are dual in the sense that $\Pi(\varphi) = 1 - N(\neg\varphi)$ for every proposition φ. From a logical point of view, possibilistic logic can be seen as a sort of graded extension of the non-nested fragment of the well-known modal logic of belief KD45.

When we go beyond the classical framework of Boolean algebras of events to generalized algebras of many-valued events, one has to come up with appropriate extensions of the notion of necessity and possibility measures for many-valued events, as explored in [20]. A natural generalization is to consider Ω as the set of propositional interpretations of some many-valued calculi defined by a t-norm $\odot$ and its residuum $\Rightarrow$, in particular when $\odot$ is the product t-norm. Then, a possibility distribution $\pi: \Omega \to [0,1]$ induces the following generalized possibility and necessity measures over many-valued propositions:

$$\Pi(\varphi) := \sup\{\pi(w) \odot w(\varphi) \mid w \in \Omega\}$$
$$\mathsf{N}(\varphi) := \inf\{\pi(w) \Rightarrow w(\varphi) \mid w \in \Omega\}\ .$$

Actually, these definitions agree with the ones commonly used in many-valued modal logics with Kripke semantics based on frames (W, R) with R being a $[0,1]$-valued binary relations $R: W \times W \to [0,1]$ (see e.g. [2]), in the particular case where the many-valued accessibility relations R are of the form $R(w, w') = \pi(w')$, for some possibility distributions $\pi: W \to [0,1]$. In the frame of this article, we generalize this possibilistic semantics by replacing the unit real interval $[0,1]$ by an arbitrary product algebra.

The set of formulas over the language of the product logic Π^{∞} extended with two unary operators, N and Π, will be denoted by Fm_{Pos_Π} and we will refer to them as *possibilistic formulas*.

DEFINITION 6.6

Let $\boldsymbol{A}$ be a Π^{∞}-algebra. An $\boldsymbol{A}$**-possibilistic model** is a structure $\langle W,\pi,e\rangle$ such that

- W is a non-empty set of worlds;
- $\pi: W\to \boldsymbol{A}$ is an $\boldsymbol{A}$-valued possibility distribution;
- $e: W\times\mathcal{V}\to\boldsymbol{A}$ is an evaluation of variables in each world. It extends to all the formulas by interpreting the propositional connectives by the corresponding operations of $\boldsymbol{A}$, and the possibilistic operations by

$$e(v,\Pi\varphi)=\sup_{w\in W}(\pi(w)\odot e(w,\varphi)\} \qquad e(v,\mathsf{N}\varphi)=\inf_{w\in W}(\pi(w)\Rightarrow e(w,\varphi)\}.$$

Again, if these latter two values exist for any formula in any world, the model is called *safe*. The class of safe Possibilistic models over product algebras will be denoted by Pos_Π. For $\Gamma\cup\{\varphi\}\subseteq Fm_{\mathsf{Pos}_\Pi}$, we will write $\Gamma\models_{\mathsf{Pos}_\Pi}\varphi$ whenever for any $\mathfrak{M}\in\mathsf{Pos}_\Pi$ and any $w\in W$, $e(w,[\Gamma])\subseteq\{1\}$ implies that $e(w,\varphi)=1$.

Notation: For the sake of uniformity with the notation used in existing literature, from now on we will denote $\mathsf{PK}^{\{T,5\}}$ by $\mathsf{PS5}$ (i.e. the set of reflexive and euclidean safe product models), and $\mathbf{K}\Pi^{\infty\{\mathrm{T},5\}}$ by $\mathbf{S5}_\Pi$, the logic which is strongly complete with respect to that class of models.

In [29], it is shown how a possibilistic modal logic over a finitely-valued Łukasiewicz logic can be embedded in a $S5$-like extension (with axioms (K), (T) and (5)) over a language extended with a new propositional variable, playing the special role of the possibility distribution. We follow the same idea here.

We let $\star$ be the translation from Fm_{Pos_Π} (built from a set $\mathcal{V}$ of propositional variables) to the set of modal formulas[13] built from $\mathcal{V}\cup\{p\}$, with $\{p\}$ a new fresh variable not in $\mathcal{V}$, defined as:

$$\begin{array}{ll}
\overline{c}^\star := \overline{c}, \quad x^\star := x \text{ for } x\in\mathcal{V} & (\Delta\varphi)^\star := \Delta\varphi^\star\\
(\varphi\&\psi)^\star := \varphi^\star\&\psi^\star & (\mathsf{N}\varphi)^\star := \Box(p\to\varphi^\star)\\
(\varphi\to\psi)^\star := \varphi^\star\to\psi^\star & (\Pi\varphi)^\star := \Diamond(p\&\varphi^\star).
\end{array}$$

THEOREM 6.7

For any set of possibilistic formulas $\Gamma\cup\{\varphi\}\subseteq Fm_{\mathsf{Pos}_\Pi}$ it holds that

$$\Gamma\models_{\mathsf{Pos}_\Pi}\varphi \text{ iff } \Gamma^\star\vdash_{\mathbf{S5}_\Pi}\varphi^\star.$$

PROOF. First, given $\mathfrak{M}=\langle W,\pi,e\rangle\in\mathsf{Pos}_\Pi$, a model $\mathfrak{M}'=\langle W',R',e'\rangle\in\mathsf{PS5}$ can be defined by taking $W'=W$, $R'=W\times W$, $e'(v,x)=e(v,x)$ for all $x\in\mathcal{V}$ and $e'(v,p)=\pi(v)$. By the definition of the evaluation of the modal formulas in a possibilistic model, and the behaviour of the translation $\star$ over them, it is immediate to see, by induction on the length of the formula, that the evaluations coincide, i.e. $e(v,\psi)=e'(v,\psi^\star)$ for any $\psi\in Fm_{\mathsf{Pos}_\Pi}$. Take for instance the possibilistic formula $\mathsf{N}\chi$. By definition,

[13] With usual $\Box$ and $\Diamond$ operators.

$e(v,\mathsf{N}\chi) = \inf_{w\in W}(\pi(w) \Rightarrow e(w,\chi)\} = \inf_{w\in W}\{e'(w,p) \Rightarrow e(w,\chi)\}$. By the induction hypothesis, this is equal to $\inf_{w\in W}\{e'(w,p) \Rightarrow e'(w,\chi^\star)\}$, and thus, to $\inf_{w\in W}\{e'(w,p) \Rightarrow \chi^\star)\} = e'(v,\Box(p\to\chi^\star)) = e'(v,(\mathsf{N}\chi)^\star)$.

Conversely, given a model $\mathfrak{M}\in\mathsf{PS5}$, define $\mathfrak{M}'\in\mathsf{Pos}_\Pi$ by letting $W' = W$, $\pi(v) = e(v,p)$ and $e'(v,x) = e(v,x)$ for all $x\in\mathcal{V}$ and $v\in W$. Then it is also easy to check, again by induction, that $e(v,\psi^\star) = e'(v,\psi)$ for all $\psi\in Fm_{\mathsf{Pos}_\Pi}$ and $v\in W$. Similarly as before, consider the case where $\psi = \mathsf{N}\chi$, i.e. $\psi^\star = \Box(p\to\chi^\star)$. Then, by definition, $e(v,\Box(p\to\chi^\star) = \inf_{w\in W}\{e(w,p\to\chi^\star)\} = \inf_{w\in W}\{e(w,p) \Rightarrow e(w,\chi^\star)\} = \inf_{w\in W}\{\pi(w) \Rightarrow e(w,\chi^\star)\}$. By the induction hypothesis, this is equal to $\inf_{w\in W}\{\pi(w) \Rightarrow e'(w,\chi)\}$, which is equal, by definition, to $e'(v,\mathsf{N}\chi)$. ■

At this point, one can wonder if this logic indeed behaves, as expected, as a logic for reasoning about graded possibilities and necessities. The main discrepancy is the behaviour of the necessity operator over negated formulas. Indeed, it is straightforward to check that, in any possibilistic model $\mathfrak{M} = (W,\pi,e)\in\mathsf{Pos}_\Pi$, we have that for any $v\in W$:

$$e(v,\mathsf{N}\neg\varphi) = \begin{cases} 1, & \text{if, for all } w\in W, \text{ either } \pi(w)=0 \text{ or } e(w,\varphi)=0 \\ 0, & \text{otherwise .} \end{cases}$$

This means that N actually behaves as a Boolean operator over the negated events, which is rather unintuitive since, in contrast to classical Possibilistic logic, it means that uncertainty (in terms of necessity measures) on negated formulas (even if they are Boolean themselves) cannot be graded: a negated formula can only be fully believed or fully disbelieved. Therefore, it makes full sense to consider a language where it is possible to build more expressive formulas to reason about negated events. A natural solution is to expand the language with a new negation connective $\sim$ and endowing it with a behaviour of an involutive negation (that within the standard product algebra is simply given by $\sim a = 1-a$).

At a propositional level, there are several works about the extension of product logic with an involutive negation $\sim$ (see e.g. [13],[12], [23]), and one of the first remarks is that the Baaz–Monteiro operator Δ is in fact definable from the two negations: $\Delta\varphi = \neg{\sim}\varphi$. Moreover, with the involutive negation it is not necessary to add the rule R_Δ, since it turns to be derivable.

So we define $\Pi^\infty_\sim$ to be the following axiomatic system for a product logic with rational constants and an involutive negation:

- axioms of Π^c, i.e. Product logic axioms and rules plus the following ones:

(A_c1) $\overline{c}\,\&\,\overline{d} \leftrightarrow \overline{c\cdot d}$
(A_c2) $(\overline{c}\to\overline{d}) \leftrightarrow \overline{c \Rightarrow_\Pi d}$
$(N_{\neg\sim})$ $\dfrac{\varphi}{\Delta\varphi}$
$(\mathrm{R}_\to)$ $\dfrac{\{\varphi\to\overline{c}\}_{c\in(0,1)_\mathbb{Q}}}{\neg\varphi}$

- $\neg\varphi \to {\sim}\varphi$
- ${\sim}{\sim}\varphi \to \varphi$
- $\Delta(\varphi\to\psi) \to ({\sim}\psi \to {\sim}\varphi)$
- ${\sim}\overline{c} \leftrightarrow \overline{1-c}$

It was remarked in [9] that this logic coincides with the so-called Łukasiewicz Product Logic with rational truth constants $\text{Ł}\Pi\frac{1}{2}$ [24]. This is so because the definition of the $\sim$ operator over the rational constants, which are dense in $[0,1]$, uniquely determines that in the whole real unit interval $[0,1]$ the involutive negation $\sim$ has to be exactly the standard Łukasiewicz negation $1-x$. This makes all the other Łukasiewicz connectives (in particular the Łukasiewicz strong conjunction $\&_{Ł}$ and implication $\to_{Ł}$), as well as all rational constants besides $\overline{\frac{1}{2}}$, to be definable in $\Pi^{\infty}_{\sim}$.

The axiomatization of a modal logic (with modal operators $\Box$ and $\Diamond$), strongly complete with respect to the class of (crisp) Kripke models over product algebras with an involutive negation, is naturally obtained by extending $\Pi^{\infty}_{\sim}$ with the expected axioms (K), $(\mathrm{A}_{\Box}1)$, $(\mathrm{A}_{\Diamond}1)$ and $(\mathrm{A}_{\Box}2)$. We will denote this logic by $\mathbf{K}\Pi^{\infty}_{\sim}$. It is easy to check that it is complete with respect to the class of Kripke structures evaluated over product algebras with rational constants and with an involutive negation. Note that the evaluation of a modal formula of the form $\mathsf{N}{\sim}\varphi$ in a possibilistic model $\mathfrak{M}=(W,\pi,e)$ over the canonical standard product algebra (expanded with the involutive negation $1-x$) is now as follows:

$$e(v,\mathsf{N}{\sim}\varphi)=\inf_{w\in W}(\pi(x)\Rightarrow_{\Pi} 1-e(w,\varphi)),$$

which is no longer $\{0,1\}$-valued. Moreover, one could consider as well another necessity operator N', definable in $\mathbf{K}\Pi^{\infty}_{\sim}$, resulting from dualizing the possibility operator Π with the involutive negation $\sim$, namely one can define $\mathsf{N}'\varphi := {\sim}\Pi{\sim}\varphi$. The semantics of this operator (in the above model over $[0,1]$) is given by the following expression:

$$e(v,\mathsf{N}'\varphi)=\inf_{w\in W}(1-\pi(x)+\pi(x)\cdot e(w,\varphi)).$$

The semantics of the pair of operators Π and N' are indeed proper extensions of the original possibility and necessity measures for classical propositions used in Dubois and Prade's Possibilistic logic, while the one for N is not, as shown above.

Following an analogous approach to the one used for proving completeness of $\mathbf{K}\Pi^{\infty}$ by means of a canonical model construction (Theorem 4.11), we can check completeness of $\mathbf{K}\Pi^{\infty}_{\sim}$ as well. Indeed, the canonical model of $\mathbf{K}\Pi^{\infty}$ can be defined in the natural way: $\mathfrak{M}^{\sim}_{\mathfrak{c}}=\langle W^{\sim}_{c}, R^{\sim}_{\mathfrak{c}}, e^{\sim}_{\mathfrak{c}}\rangle$ where $W^{\sim}_{c} := \{h\in Hom((Fm_{\mathbf{K}\Pi^{\infty}_{\sim}})^{*},[0,1]_{\Pi_{\sim}}) : h((Th_{\mathbf{K}_{\Pi_{\sim}}})^{*})\subseteq\{1\}\}$, $R^{\sim}_{\mathfrak{c}}vw$ iff for any $\psi\in Fm_{\mathbf{K}\Pi^{\infty}_{\sim}}$, $v(\theta_{\Box})=1$ implies $w(\theta^{*})=1$, and $e^{\sim}_{\mathfrak{c}}(w,x):=w(x)$ for every $x\in\mathcal{V}$.

Notice that, since $\mathbf{K}\Pi^{\infty}_{\sim}$ extends Łukasiewicz logic, it follows that the modal operators $\Box$ and $\Diamond$ are now fully interdefinable, i.e. $\Diamond\varphi$ is equivalent to ${\sim}\Box{\sim}\varphi$, and $\Box\varphi$ equivalent to ${\sim}\Diamond{\sim}\varphi$.

It is now possible to follow the same reasoning as the one done for Theorem 6.7 to prove a completeness result for the possibilistic (local) modal logic $\mathsf{Pos}_{\Pi_{\sim}}$ by embedding it into the logic $\mathbf{S5}_{\Pi_{\sim}}$ (the extension of $\mathbf{S5}_{\Pi}$ with an involutive negation in the natural way). It is only necessary to observe that, as expected, $({\sim}\varphi)^{\star}={\sim}\varphi^{\star}$.

THEOREM 6.8
For any set of possibilistic formulas $\Gamma\cup\{\varphi\}\subseteq Fm_{\mathsf{Pos}_{\Pi_{\sim}}}$

$$\Gamma\models_{\mathsf{Pos}_{\Pi_{\sim}}}\varphi \text{ iff } \Gamma^{\star}\vdash_{\mathbf{S5}_{\Pi_{\sim}}}\varphi^{\star}.$$

As a final observation, we show how the translations into the language of $\mathbf{S5}_{\Pi_{\sim}}$ of the $\sim$-dual pairs of operators $(\mathsf{N},{\sim}\mathsf{N}{\sim})$ and $({\sim}\Pi{\sim},\Pi)$ look like

LEMMA 6.9
The following are theorems in $\mathbf{S5}_{\Pi_{\sim}}$:

(1) $(\Pi'\varphi)^{\star}\leftrightarrow\Diamond(\neg\neg p\wedge(p\to p\&_{Ł}\varphi))$;

(2) $(\mathsf{N}'\varphi)^\star \leftrightarrow \Box(p \to_{\text{Ł}} p\&\varphi)$.

where $\Pi'\varphi$ is an abbreviation for $\sim\mathsf{N}\sim\varphi$ (recall that $\mathsf{N}'\varphi$ stands for $\sim\Pi\sim\varphi$).

PROOF. (1) By completeness, we have that on any Kripke $[0,1]_{\Pi_\sim}$-model $\mathfrak{M} = (W,\pi,e)$ with a reflexive and euclidean accessibility relation R, for any $w \in W$, we have $e(w,(\sim\mathsf{N}\sim\varphi)^\star) = 1 - e(w,\Box(p \to \sim\varphi^\star)) = 1 - \inf\{e(v,p \to \sim\varphi^\star) : v \in W\} = \sup\{1 - e(v,p \to \sim\varphi^\star) : v \in W\}$. We can prove by cases (depending on the value of p in the world) that for each $v \in W$, $1 - e(v,p \to \sim\varphi^\star) = e(v,\neg\neg p \wedge (p \to p\&_{\text{Ł}}\varphi))$.

- For $v \in W$ such that $e(v,p) = 0$, it holds that $1 - e(v,p \to \sim\varphi^\star) = 0$, which is equal to $e(v,\neg\neg p)$.
- For $v \in W$ such that $e(v,p) > 0$, note that $e(v,\neg\neg p) = 1$, so it does not affect the value of the conjunction. Therefore it remains to be proved that $1 - e(v,p \to \sim\varphi^\star) = e(v,p \to p\&_{\text{Ł}}\varphi^\star)$. We consider two cases:
  (*) If $e(v,p) \leqslant e(v,\sim\varphi^\star)$ then $1 - e(v,p \to \sim\varphi^\star) = 0$. Note that $e(v,p) \leqslant e(v,\sim\varphi^\star)$ whenever $e(v,p) - 1 + e(v,\varphi^\star) \leqslant 0$, or equivalently $e(v,p\&_{\text{Ł}}\varphi^\star) = 0$. Given that $e(v,p) > 0$, we have that $e(v,p \to p\&_{\text{Ł}}\varphi^\star) = 0$ as well.
  (*) If $e(v,p) > e(v,\sim\varphi^\star)$, by the definition of the product implication on $[0,1]$ we can directly write $1 - e(v,p \to \sim\varphi^\star) = \frac{e(v,p) - 1 + e(v,\varphi^\star)}{e(v,p)} = \frac{e(v,p\&_{\text{Ł}}\varphi^\star)}{e(v,p)}$ and, since $e(v,p\&_{\text{Ł}}\chi) \leqslant e(v,p)$ for any formula χ, this equals to $e(v,p \to p\&_{\text{Ł}}\varphi^\star)$.

(2) Similarly, we have that $e(w,(\sim\Pi\sim\varphi)^\star) = 1 - e(w,\Diamond(p\&\sim\varphi^\star)) = 1 - \sup\{e(v,p\&\sim\varphi^\star) : v \in W\} = \inf\{1 - e(v,p\&\sim\varphi^\star) : v \in W\} = \inf\{1 - (e(v,p)\cdot(1 - e(v,\varphi^\star))) : v \in W\} = \inf\{1 - e(v,p) + e(v,p)\cdot e(v,\varphi^\star) : v \in W\} = \inf\{e(v,p) \to_{\text{Ł}} e(v,p\&\varphi^\star)\}$ (since $e(v,p) \geqslant e(v,p\&\varphi^\star)$ in any case). ■

7 Concluding remarks

In this article, we have mainly been concerned with the study and axiomatization of modal extensions of the Product fuzzy logic with semantics given by Kripke frames with crisp accessibility relations and evaluations over the canonical standard product algebra. To do this, we have first developed an infinitary product logic Π^∞ with rational constants and with Monteiro–Baaz's Δ operator (both from axiomatic and semantical points of view), with the objective of providing a solid basis to build modal extension, that is the main aim of this article. Then, we have provided axiomatizations for the minimal local and global modal logics extending Π^∞ and proved completeness with respect to different classes of Kripke frames evaluated over product algebras. In order to deepen the analysis of these logics, we have also developed their algebraic semantics and studied its relationship with the Kripkean semantics. Finally, we have also considered some usual axiomatic extensions of modal logics in our particular framework.

The need of resorting to the use of rational constants together with two infinitary inference rules to handle them, as well the Monteiro–Baaz Δ operator, has been motivated by our aim to get complete modal axiomatizations with respect to corresponding classes of Product Kripke models. We admit this approach may be felt as not very elegant, but we have not been able to find a better alternative. Note that also in [2], an intensive use of constants in the language is made for the analysis of modal logics over finite MTL-algebras.

There are a number of interesting open problems that we plan to address in future works. In particular, one natural question is the study of a modal extension of product fuzzy logic arising from

Kripke models where the accessibility relation is fuzzy as well, as it is the case e.g. in [2, 15, 16]. In such a case, axiom K is no longer sound, and this may make the problem more complex. Another pending question is to study decidability issues of the logics defined in this article. Note that in [27] the arithmetical complexity of a modal extension of product logic (without truth-constants and Δ operator) with an S5 semantics was also left open. There are however, a few available results in the literature in similar settings: in [7] the authors show that checking whether a formula is positive-satisfiable or 1-valid in a modal product logic (without truth-constants and Δ) is decidable. However, the techniques used in that paper cannot be applied here because of the presence of truth-constants and the Δ operator. Also, in [14] it is shown that Gödel modal logics are decidable by defining an alternative new semantics for which the finite model property holds.

On the other hand, one can realise that many of the results obtained in this article do not rely on characteristic properties of Product logic itself but on weaker assumptions. With this in mind, it seems that the step to understand modal extensions of weaker fuzzy logics (for instance, classes of MTL logics) could be not too far. This problem is part of ongoing work.

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A Appendix

A.1 [Proof of Lemma 3.5]

We will first present some technical results that will simplify the proof.

LEMMA A.1 (c.f. [33, Lemma 6])
Π^∞ is closed under

$$(\mathrm{R}_\Delta{}'):\ \frac{\chi\vee(\overline{c}\to\varphi),\ \text{for all } c<1}{\chi\vee\varphi}$$

PROOF. Suppose $T\vdash_{\Pi^\infty}\chi\vee(\overline{c}\to\varphi)$ for all c. By $(D.1)$, $T\vdash_{\Pi^\infty}\overline{c}\to(\varphi\vee\chi)$ for all c. Then, by rule R_Δ, $T\vdash_{\Pi^\infty}\varphi\vee\chi$. ■

The following is a direct application of the previous lemma and the axioms and rules of Δ.

COROLLARY A.2 (c.f. [33, Lemma 7])
If for all c, if $T\vdash_{\Pi^\infty}\Delta(\varphi\to\overline{c})\to\alpha$ and $T\vdash_{\Pi^\infty}\Delta\varphi\to\alpha$ then $T\vdash_{\Pi^\infty}\alpha$.

Moreover, the following are quite simple but useful observations.

LEMMA A.3
The following conditions hold for any consistent theory T and $c\in(0,1)_{\mathbb{Q}}$:

(1) If $T\vdash_{\Pi^\infty}\varphi\to\overline{c}$, then, for any consistent theory T' extending T, $T'\nvdash_{\Pi^\infty}\overline{c'}\to\varphi$ for some c'.
(2) If $T\vdash_{\Pi^\infty}\overline{c}\to\varphi$, then, for any consistent theory T' extending T, $T'\nvdash_{\Pi^\infty}\varphi\to\overline{c\cdot c}$.

Now we are ready to prove that is possible to extend a theory to another one that is complete and closed under the rules (R_Δ) and $(\mathrm{R}_\to)$. In the following, for $T\subset Fm$ we let $C_{\Pi^\infty}(T):=\{\varphi:\ \Gamma\vdash_{\Pi^\infty}\varphi\}$ be the closure of Γ under Π^∞.

LEMMA 3.5
Let $T\cup\{\alpha\}$ be a set of formulas such that $T\nvdash_{\Pi^\infty}\alpha$. Then there is a prime theory T' such that $T\subseteq T'$ and $T'\nvdash_{\Pi^\infty}\alpha$.

PROOF. As Montagna noticed, it is interesting to observe that by the presence of infinitary rules, many standard constructions do not work. For instance, one might be tempted to use Zorn's lemma to obtain a maximal theory closer under (R_Δ) and ($R_\to$) T' extending Γ such that $\varphi \notin T'$. But in this case Zorn's lemma does not apply, as the union of a chain of ($R_\Delta, R_\to$)-theories may fail to be a theory closer under R_Δ and $R_\to$: for instance, let $\Gamma_n = K_{\Pi^\infty}(\{\overline{\frac{1}{n}} \to \psi\})$. Then, $\Gamma_1 \subseteq \ldots \subseteq \Gamma_n \ldots$, and every Γ_n is a theory closer under R_Δ, but their union T' is not, as $\psi \notin T'$. Thus we will proceed in another way.

Let $\langle \varphi_n, \psi_n \rangle$ be an enumeration of all the possible couples of formulas from Fm. We can define a series of theories T_n such that:

(1) $T_0 = C_{\Pi^\infty}(T) \subseteq T_1 \subseteq \ldots \subseteq T_n \subseteq \ldots$;
(2) For each n, $\alpha \notin T_n$.
(3) For each n, either $\varphi_n \to \psi_n \in T_{2n+1}$ or $\psi_n \to \varphi_n \in T_{2n+1}$. (linearity)
(4) For each n, if φ_n or ψ_n is a constant symbol $\overline{r_n}$ (and in that case, we name χ_n the other formula):

- If $\overline{r_n} \to \chi_n \in T_{2n+1}$, then either $\chi_n \to \overline{c} \in T_{2n+2}$ for some $1 > c > 0$ or $\chi_n \in T_{2n+2}$. (closure under (R_Δ)).
- If $\overline{r_n} \to \chi_n \notin T_{2n+1}$, then either $\overline{c} \to \chi_n \in T_{2n+2}$ for some $c \in (0, r_n]_{\mathbb{Q}}$ or $\neg \chi_n \in T_{2n+2}$. (closure under $R_\to$).

Indeed, we define them as follows:

Step 0: $T_0 := C_{\Pi^\infty}(T)$.

Step $2n+1$: If $\alpha \notin C_{\Pi^\infty}(T_{2n} \cup \{\varphi_n \to \psi_n\})$, put $T_{2n+1} := C_{\Pi^\infty}(T_{2n} \cup \{\varphi_n \to \psi_n\})$. Otherwise, put $T_{2n+1} := C_{\Pi^\infty}(T_{2n} \cup \{\psi_n \to \varphi_n\})$. Notice that in any case, $\alpha \notin T_{2n+1}$. Otherwise, $T_{2n} \vdash_{\Pi^\infty} \Delta(\varphi_n \to \psi_n) \to \alpha$ and $T_{2n} \vdash_{\Pi^\infty} \Delta(\psi_n \to \varphi_n) \to \alpha$ (by the deduction theorem). Then, $T_{2n} \vdash_{\Pi^\infty} \alpha$, which is a contradiction.

Step $2n+2$: If both φ_n, ψ_n are different from constant symbols, let $T_{2n+2} := T_{2n+1}$; if at least one of $\{\varphi_n, \psi_n\}$ is a truth constant, let $\overline{r_n}$ be that constant and χ_n be the other formula. Then, we have two cases:

(1) If $\overline{r_n} \to \chi_n \in T_{2n+1}$ then we have two subcases:

- If $\alpha \notin C_{\Pi^\infty}(T_{2n+1} \cup \{\chi_n \to \overline{c}\})$ for some c. Then let $T_{2n+2} := C_{\Pi^\infty}(T_{2n+1} \cup \{\chi_n \to \overline{c}\})$.
- If $\alpha \in C_{\Pi^\infty}(T_{2n+1} \cup \{\chi_n \to \overline{c}\})$ for any $c \in (0,1)_{\mathbb{Q}}$, by the Deduction theorem we have that $T_{2n+1} \vdash_{\Pi^\infty} \Delta(\chi_n \to \overline{c}) \to \alpha$ for each c. By Corollary A.2, and since $T_{2n+1} \nvdash_{\Pi^\infty} \alpha$, we know that $T_{2n+1} \nvdash_{\Pi^\infty} \Delta \chi_n \to \alpha$. Then, put $T_{2n+2} := K_R(T_{2n+1, R_\to} \cup \{\chi_n\})$.

(2) If $\overline{r_n} \to \chi_n \notin T_{2n+1}$ (i.e. $\chi_n \to \overline{r_n} \in T_{2n+1}$) then we have two subcases again:

- If $\alpha \notin C_{\Pi^\infty}(T_{2n+1} \cup \{\overline{c} \to \chi_n\})$ for some $c \in (0, r_n]_{\mathbb{Q}}$, let $T_{2n+2} := C_{\Pi^\infty}(T_{2n+1} \cup \{\overline{c} \to \chi_n\})$.
- If $\alpha \in C_{\Pi^\infty}(T_{2n+1} \cup \{\overline{c} \to \chi_n\})$ for any $c \in (0,1)_{\mathbb{Q}}$, then (by the Deduction Theorem) we have that $T_{2n+1} \vdash_{\Pi^\infty} \Delta(\overline{c} \to \chi_n) \to \alpha$ for each c. It follows by properties of Π that $T_{2n+1} \vdash_{\Pi^\infty} \Delta(\chi_n \to \overline{c}) \vee \alpha$ for each c. And so, by G_Δ, $T_{2n+1} \vdash_{\Pi^\infty} \Delta(\chi_n \to \overline{c}) \vee \Delta\alpha$ for each c. This implies that $T_{2n+1} \vdash_{\Pi^\infty} (\neg\Delta\alpha \wedge \chi_n) \to \overline{c})$ for each c, and so, by rule ($R_\to$), we know that $T_{2n+1} \vdash_{\Pi^\infty} \neg(\neg\Delta\alpha \wedge \chi_n)$. By the DeMorgan Laws, and using G_Δ and axiom ($\mathcal{A}_\Pi^\Delta 3$) we have that $T_{2n+1} \vdash_{\Pi^\infty} \alpha \vee \Delta\neg\chi_n$. Then, $T_{2n+1} \nvdash_{\Pi^\infty} \Delta\neg\chi_n \to \alpha$ (otherwise, $T_{2n+1} \vdash_{\Pi^\infty} \alpha$, which is a contradiction).
  Then, let $T_{2n+2} := C_{\Pi^\infty}(T_{2n+1} \cup \{\neg\chi_n\})$.

It is clear that the sequence $\{T_n\}_{n\in\omega}$ satisfies conditions (1), (2), (3) and (4). Then, let $T^+ := \bigcup_{n\in\omega} T_n$. Clearly, T^+ is a complete theory extending T and $\alpha \notin T^+$. We just need to check it is closed under R_Δ and $\mathrm{R}_\rightarrow$.

On the one hand suppose that $\overline{r} \to \varphi \in T^+$ for some $0 < r < 1$. Then, by our construction, there must be n such that $\overline{r} \equiv \overline{r_n}$ and $\varphi \equiv \chi_n$ (this implication has been added at step $2n+1$). Then, by construction, we have two options: either $\varphi \to \overline{c} \in T_{2n+2}$ for some c or $\varphi \in T_{2n+2}$. In the first case, by Lemma A.3(1), it does not hold that T^+ contains $\overline{d} \to \varphi$ for any constant d (so (R_Δ) will not be applied over this formula). On the second case, we have that $\varphi \in T^+$, so it is closed under (R_Δ) for this formula.

On the other hand, suppose that $\varphi \to \overline{r} \in T^+$, $0 < r < 1$. Then, by our construction, there must be n such that $\overline{r} \equiv \overline{r_n}$ and $\varphi \equiv \chi_n$ (this implication has been added at step $2n+1$). Then, by construction, we have two options: either $\overline{c_0} \to \varphi \in T_{2n+2}$ for some $c_0 \in (0, r]_{\mathbb{Q}}$ or $\neg\varphi \in T_{2n+2}$. In the first case, by Lemma A.3(2), it does not hold that T^+ contains $\varphi \to \overline{c}$ for any $c \in (0,1)_{\mathbb{Q}}$ (so $\mathrm{R}_\rightarrow$ will not be applied over this pair of formula-constant). On the second case, we have that $\neg\varphi \in T^+$ so it is closed under ($\mathrm{R}_\rightarrow$) for this formula. ■

A.2 [Proof of Lemma 3.9]

Lemma 3.9

Let $\boldsymbol{A}$ be a linearly ordered Π^∞-algebra. Then $\boldsymbol{A}$ can be embedded in the canonical standard product algebra $[0,1]_\Pi$ by a unique complete embedding.

Proof. In [18] it is shown there exist two functors $\mathfrak{G} : PL_c \leftrightarrow LG : \mathfrak{B}$ that induce a natural equivalence between the full subcategory of product algebras satisfying a certain condition and the category of the lattice-ordered Abelian groups. For the particular case of linearly ordered product algebras (that satisfy the mentioned condition) and linearly ordered Abelian groups (which by cardinality reasons are isomorphic to the additive group of the real numbers, $\mathbb{R}_+ = \langle \mathbb{R}, +, -, 0\rangle$), this equivalence is as follows. Given a linearly ordered Π-algebra $\boldsymbol{A} = \langle A, \odot, \to, 0^{\boldsymbol{A}}, 1^{\boldsymbol{A}}\rangle$, $\mathfrak{G}(\boldsymbol{A})$ is defined as the linearly ordered Abelian group whose universe is $\{a_+ : a \in A \setminus \{\bot, 1\}\} \cup \{a_- : a \in A \setminus \{\bot, 1\}\} \cup \{0\}$, where $\mathbf{0}$ is the neutral element (that coincides with $1^{\boldsymbol{A}}_+$ and $1^{\boldsymbol{A}}_-$), the order is given by $a_- <_{\mathfrak{G}(A)} b_- <_{\mathfrak{G}(A)} 0 <_{\mathfrak{G}(A)} b_+ <_{\mathfrak{G}(A)} a_+$ for $0 <_{\boldsymbol{A}} a <_{\boldsymbol{A}} b \in A$, and the group operations $+$ and $-$ (addition and inverse resp.) are defined by:

$$
\begin{aligned}
a_+ + b_+ &= (a \odot b)_+ \\
a_- + b_- &= (a \odot b)_- \\
a_+ + b_- &= \begin{cases} \mathbf{0} & \text{if } a =_{\boldsymbol{A}} b \\ (b \to a)_- & \text{if } a <_{\boldsymbol{A}} b \\ (a \to b)_+ & \text{if } b <_{\boldsymbol{A}} a. \end{cases} \\
-a_+ &= a_- \\
-a_- &= a_+
\end{aligned}
$$

Conversely, given a linearly ordered Abelian group $\boldsymbol{G} = \langle G, +, -, \leqslant, 0\rangle$, we define $\mathfrak{B}(G)$ as the product chain having as universe the set $\{g \in G : g \leqslant 0\} \cup \{-\infty\}$, the order $\leqslant_{\mathfrak{B}(G)}$ being the extension of the order of G, $\leqslant$, by letting $-\infty \leqslant_{\mathfrak{B}(G)} g$ for any $g \in G$, and with operations defined as follows:

$$a \odot b = \begin{cases} a+b & \text{if } a,b \in G \\ -\infty & \text{otherwise} \end{cases} \qquad a \to b = \begin{cases} 0 & \text{if } a \leqslant_{\mathfrak{B}(G)} b \\ -\infty & \text{if } a >_{\mathfrak{B}(G)} b = -\infty \\ b-a & \text{otherwise .} \end{cases}$$

Now, using the previous functors and results from [33], for each rational $c \in (0,1)_{\mathbb{Q}}$ we can construct a complete embedding σ_c of each Π^∞-chain $\boldsymbol{A}$ into $\mathfrak{B}(\mathbb{R}_+)$ suitably extended with rational constants (depending on c).[14] Indeed, pick an arbitrary element $c \in (0,1)_{\mathbb{Q}}$ and let $\sigma'_c\colon \mathfrak{G}(\boldsymbol{A}) \to \mathbb{R}_+$ be given by $\sigma'_c(0) = 0$, $\sigma'_c(x_+) = \sup\{\frac{n}{m}\colon (x_+)^m \geqslant (c_+^{\boldsymbol{A}})^n\}$ and $\sigma'_c(x_-) = -\sigma'_c(x_+)$, for $x \in A, 1 > x > 0$. In [33, Prop. 3] it is proven that this is a complete embedding. Define then the embedding $\sigma_c\colon \boldsymbol{A} \to \mathfrak{B}(\mathbb{R}_+)$ by $\sigma_c(0) = -\infty$, $\sigma_c(1) = \sigma'_c(0) = 0$ and $\sigma_c(x) = \sigma'_c(x_-)$ for all $0 < x < 1$, and observe that it is a complete embedding too. This is immediate since σ'_c is already complete, the only case worth to be checked is whether $\sigma_c(\bigwedge_{i\in I} x_i) = \bigwedge_{i\in I} \sigma_c(x_i)$ when $\{x_i\}_{i\in I} \subseteq A$ is such that $x_i > 0$ for all $i \in I$ and $\bigwedge_{i\in I} x_i = 0$. By definition, $\sigma_c(\bigwedge_{i\in I} x_i) = -\infty$. On the other hand, if we suppose towards a contradiction that $\bigwedge_{i\in I} \sigma_c(x_i) > -\infty$, by definition we have that there is $z \in \mathbb{R}_+$ such that $z = \bigwedge_{i\in I} \sigma'_c((x_i)_-)$. It is easy to see that, by the definition of σ'_c, there exists $x_- \in \mathfrak{G}(\boldsymbol{A})$ such that $\sigma'_c(x_-) \leqslant z$, and thus we get that $0 < x \leqslant x_i$ for all $i \in I$, and so, a contradiction (since $\bigwedge_{i\in I} x_i = 0$).

On the other hand, by the definition of the ordering in $\mathfrak{G}(\boldsymbol{A})$, it follows that $(x_+)^m \geqslant (c_+^{\boldsymbol{A}})^n$ if and only if $x^m \leqslant (c^{\boldsymbol{A}})^n$. Now, using that the rationals are dense in the reals and that $c < 1$, it is straightforward to check that for any $d \in [0,1]_{\mathbb{Q}}$, $\log_c d = \sup\{r \in \mathbb{Q} : d \leqslant c^r\}$, and thus, by axiom ($\mathrm{A_c}1$), we get that $\sigma_c(d^{\boldsymbol{A}}) = -\log_c d$.

Now it is a natural step to combine each one of the previously defined complete embeddings with their reciprocal one from $\mathfrak{B}(\mathbb{R}_+)$ into $[0,1]_\Pi$ (the corresponding exponential function with base c). Let $E_c\colon \mathfrak{B}(\mathbb{R}_+) \to [0,1]_\Pi$ be defined by

$$E_c(x) := \begin{cases} c^{-x} & \text{if } x \in \mathbb{R}^- \\ 0 & \text{if } x = 0. \end{cases}$$

Observe that the function E_c is clearly a complete embedding from $\mathfrak{B}(\mathbb{R})$ into $[0,1]_\Pi$ (it is a continuous, monotone and increasing function since $0 < c < 1$ and its domain is $\mathbb{R}^- \cup \{0\}$), and $E_c(-1) = c$.

We can now proceed to prove that the composition $E_c \circ \sigma_c$ is a complete embedding from $\boldsymbol{A}$ into $[0,1]_\Pi$. Since both functions are complete embeddings, the only thing that is left to prove is that for any constant $d^{\boldsymbol{A}}$ we have that $E_c \circ \phi_c(d^{\boldsymbol{A}}) = d$. But we know that $\sigma_c(d^{\boldsymbol{A}}) = -log_c d$, so it is direct that $E_c(\phi_c(d^{\boldsymbol{A}})) = c^{--log_c d} = d$.

Finally, it is straightforward to see that this embedding is unique. ■

[14] In the rest of the proof, we simplify the notation by dropping the subindices in the different order symbols since there is no danger of confusion.