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Obstructions for bounded shrub-depth and rank-depth

O-joung Kwon [a,b,1,2], Rose McCarty [c], Sang-il Oum [b,d,1], Paul Wollan [e]

[a] *Department of Mathematics, Incheon National University, Incheon, South Korea*
[b] *Discrete Mathematics Group, Institute for Basic Science (IBS), Daejeon, South Korea*
[c] *Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada*
[d] *Department of Mathematical Sciences, KAIST, Daejeon, South Korea*
[e] *Department of Computer Science, University of Rome, "La Sapienza", Rome, Italy*

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ABSTRACT

Shrub-depth and rank-depth are dense analogues of the tree-depth of a graph. It is well known that a graph has large tree-depth if and only if it has a long path as a subgraph. We prove an analogous statement for shrub-depth and rank-depth, which was conjectured by Hliněný et al. (2016) [11]. Namely, we prove that a graph has large rank-depth if and only if it has a vertex-minor isomorphic to a long path. This implies that for every integer t, the class of graphs with no vertex-minor isomorphic to the path on t vertices has bounded shrub-depth.

E-mail addresses: ojoungkwon@gmail.com (O. Kwon), rose.mccarty@uwaterloo.ca (R. McCarty), sangil@ibs.re.kr (S. Oum), wollan@di.uniroma1.it (P. Wollan).

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1. Introduction

Nešetřil and Ossona de Mendez [17] introduced the *tree-depth* of a graph G, which is defined as the minimum height of a rooted forest whose closure contains the graph G as a subgraph. This concept has been proved to be very useful, in particular in the study of graph classes of bounded expansion [18]. Similar to the grid theorem for tree-width of Robertson and Seymour [24], it is known that a graph has large tree-depth if and only if it has a long path as a subgraph, see [17, Proposition 6.1]. For more information on tree-depth, the readers are referred to the surveys [20,17] by Nešetřil and Ossona de Mendez.

There have been attempts to define an analogous concept suitable for dense graphs. For tree-width, this line of research has resulted in width parameters such as clique-width [3] and rank-width [22]. In a conference paper published in 2012, Ganian, Hliněný, Nešetřil, Obdržálek, Ossona de Mendez, and Ramadurai [8] introduced the *shrub-depth* of a graph class, as an extension of tree-depth for dense graphs. Recently, DeVos, Kwon, and Oum [6] introduced the *rank-depth* of a graph as an alternative to shrub-depth and showed that shrub-depth and rank-depth are equivalent in the following sense.

Theorem 1.1 *(DeVos, Kwon, and Oum [6]). A class of graphs has bounded rank-depth if and only if it has bounded shrub-depth.*

Theorem 1.1 allows us to work exclusively with rank-depth going forward, and we omit the definition of shrub-depth. The definition of rank-depth is presented in Section 2.

One useful feature of rank-depth is that it does not increase under taking vertex-minors. In other words, if H is a vertex-minor of G, then the rank-depth of H is at most that of G. This allows us to consider obstructions for having small rank-depth in terms of vertex-minors. DeVos, Kwon, and Oum [6] showed that the rank-depth of the n-vertex path is larger than $\log n/\log(1+4\log n)$ for $n \geqslant 2$ and thus graphs having a long path as a vertex-minor have large rank-depth. Hliněný, Kwon, Obdržálek, and Ordyniak [11] conjectured that the converse is also true. Their original conjecture was stated in terms of shrub-depth but is equivalent by Theorem 1.1. We prove their conjecture as follows.

Theorem 1.2. *For every positive integer t, there exists an integer $N(t)$ such that every graph of rank-depth at least $N(t)$ contains a vertex-minor isomorphic to the path on t vertices.*

Courcelle and Oum [4] showed that there is a CMSO_1 transduction that maps a graph to its vertex-minors. Therefore, Theorem 1.2 implies that a class $\mathcal{G}$ of graphs has bounded rank-depth if and only if for every CMSO_1 transduction τ, there exists an integer t such that $P_t \notin \tau(\mathcal{G})$, which was conjectured by Ganian, Hliněný, Nešetřil, Obdržálek, and Ossona de Mendez [7].

If we apply the same proof for bipartite graphs, then we prove the following theorem on pivot-minors of graphs. Pivot-minors are more restricted in a sense that every pivot-minor of a graph is a vertex-minor but not every vertex-minor is a pivot-minor. This theorem allows us to deduce a corollary for binary matroids of large branch-depth.

Theorem 1.3. *For every positive integer t, there exists an integer $N(t)$ such that every bipartite graph of rank-depth at least $N(t)$ contains a pivot-minor isomorphic to P_t.*

The paper is organized as follows. In Section 2, we review vertex-minors and rank-depth and prove a few useful properties related to rank-depth. In Section 3, we present the proof of Theorem 1.2. In Section 4, we obtain Theorem 1.3 and discuss its consequence to binary matroids of large branch-depth. Finally, in Section 5 we conclude the paper by giving some remarks on linear χ-boundedness of graphs with no P_t vertex-minors.

2. Preliminaries and basic lemmas

All graphs in this paper are simple, meaning that neither loops nor parallel edges are allowed. For two sets X and Y, we write $X \Delta Y$ for $(X \setminus Y) \cup (Y \setminus X)$.

Let G be a graph. We write $V(G)$ and $E(G)$ for the vertex set and the edge set of G, respectively. For a vertex v of G, we write $N_G(v)$ to denote the set of all neighbors of v in G. For a vertex v of G, let $G - v$ denote the graph obtained from G by removing v and all edges incident with v. For an edge e of G, let $G - e$ denote the graph obtained from G by removing e. For a vertex subset S of G, we write $G[S]$ for the subgraph of G induced by S. We write $\overline{G}$ for the *complement* of G; that is, u and v are adjacent in G if and only if they are not adjacent in $\overline{G}$.

We write $A(G)$ for the *adjacency matrix* of G over the binary field, that is, the $V(G) \times V(G)$ matrix over the binary field such that the (x, y)-entry is one if $x \neq y$ and x is adjacent to y in G, and zero otherwise. For an $X \times Y$ matrix M and $X' \subseteq X$, $Y' \subseteq Y$, we write $M[X', Y']$ for the $X' \times Y'$ submatrix of M.

Let P_n denote the path on n vertices, and let K_n denote the complete graph on n vertices. The *radius* of a tree is the minimum r such that there is a node having distance at most r from every node.

For two n-vertex graphs G and H with fixed orderings $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ on their respective vertex sets, let $G \boxslash H$ be the graph with vertex set $V(G) \cup V(H)$ such that $(G \boxslash H)[V(G)] = G$, $(G \boxslash H)[V(H)] = H$, and for all $i, j \in \{1, 2, \ldots, n\}$, $v_i w_j \in E(G \boxslash H)$ if and only if $i \geqslant j$. See Fig. 1 for an example. An induced subgraph isomorphic to $G \boxslash H$ for some G and H is called a *semi-induced half-graph* in [19].

2.1. Vertex-minors

For a vertex v of a graph G, *local complementation* at v is an operation which results in a new graph $G * v$ on $V(G)$ such that

$$E(G * v) = E(G) \Delta \{xy : x, y \in N_G(v), x \neq y\}.$$

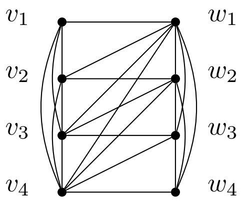

Fig. 1. The graph $K_4 \boxslash K_4$.

For an edge uv of a graph G, the operation of *pivoting* uv, denoted $G \wedge uv$, is defined as $G \wedge uv := G * u * v * u$. See Oum [21] for further background and properties of local complementation and pivoting. In particular, note that if G is bipartite, then so is $G \wedge uv$.

A graph H is *locally equivalent* to G if H can be obtained from G by a sequence of local complementations. A graph H is *pivot equivalent* to G if H can be obtained from G by a sequence of pivots. A graph H is a *vertex-minor* of G if H is an induced subgraph of a graph that is locally equivalent to G. Finally, a graph H is a *pivot-minor* of G if H is an induced subgraph of a graph that is pivot equivalent to G.

For a subset S of $V(G)$, let $\rho_G(S)$ be the rank of the $S \times (V(G) \setminus S)$ submatrix of $A(G)$. This function is called the *cut-rank* function of G. It is easy to show that the cut-rank function is invariant under taking local complementations, again see Oum [21]. Thus we have the following fact.

Lemma 2.1. *If H is a vertex-minor of G and $X \subseteq V(G)$, then*

$$\rho_H(X \cap V(H)) \leqslant \rho_G(X).$$

The following lemmas will be used to find a long path.

Lemma 2.2 *(Kim, Kwon, Oum, and Sivaraman [12, Lemma 5.6]). The graph $K_n \boxslash \overline{K_n}$ has a pivot-minor isomorphic to P_{n+1}.*

Lemma 2.3 *(Kwon and Oum [14, Lemma 2.8]). The graph $\overline{K_n} \boxslash \overline{K_n}$ has a pivot-minor isomorphic to P_{2n}.*

2.2. Rank-depth

We now review the notion of rank-depth, which was introduced by DeVos, Kwon, and Oum [6]. A *decomposition* of a graph G is a pair (T, σ) of a tree T and a bijection σ from $V(G)$ to the set of leaves of T. The *radius* of a decomposition (T, σ) is the radius of the tree T. For a non-leaf node $v \in V(T)$, the components of the graph $T - v$ give rise to a partition $\mathcal{P}_v$ of $V(G)$ by σ. The *width* of v is defined to be

$$\max_{\mathcal{P}' \subseteq \mathcal{P}_v} \rho_G \left(\bigcup_{X \in \mathcal{P}'} X \right).$$

The *width* of the decomposition (T, σ) is the maximum width of a non-leaf node of T. We say that a decomposition (T, σ) is a (k, r)-*decomposition* of G if the width is at most k and the radius is at most r. The *rank-depth* of a graph G is the minimum integer k such that G admits a (k, k)-decomposition. If $|V(G)| < 2$, then there is no decomposition and the rank-depth is zero. Note that every tree in a decomposition has radius at least one and therefore the rank-depth of a graph is at least one if $|V(G)| \geqslant 2$.

By Lemma 2.1, it is easy to see the following.

Lemma 2.4 *(DeVos, Kwon, and Oum [6]). If H is a vertex-minor of G, then the rank-depth of H is at most the rank-depth of G.*

The next two lemmas will serve as a base case for induction in the proof of Theorem 1.2.

Lemma 2.5. *Let G be a graph of rank-depth m. Then G has a connected component of rank-depth at least $m - 1$.*

Proof. If $m < 2$, then it is trivial, as the one-vertex graph has rank-depth zero. Thus, we may assume that $m \geqslant 2$.

Suppose for contradiction that every connected component of G has rank-depth at most $m - 2$. Let $C_1, C_2, \ldots, C_t$ be the connected components of G. For each $i \in \{1, 2, \ldots, t\}$,

- if C_i contains at least two vertices, then we take an $(m - 2, m - 2)$-decomposition (T_i, σ_i) where r_i is a node of T_i having distance at most $m - 2$ to every node of T_i, and
- if C_i consists of one vertex, then let T_i be the one-node graph on $\{r_i\}$ and let $\sigma_i : V(C_i) \to \{r_i\}$ be the uniquely possible function.

We obtain a new decomposition (T, σ) of G by taking the disjoint union of T_i's and adding a new node r and adding edges rr_i for all $i \in \{1, 2, \ldots, t\}$. For every vertex v of G, define $\sigma(v) = \sigma_i(v)$ if v is a vertex of C_i. Then (T, σ) has depth at most $m - 1$ and width at most $m - 2$. This contradicts the assumption that G has rank-depth m.

We conclude that G has a connected component of rank-depth at least $m - 1$. $\square$

The following lemma can be proven similarly to Lemma 2.5. For a graph G of rank-depth m and a non-empty vertex set A, it is easy to check that $G - A$ has rank-depth at least $m - |A|$, and by Lemma 2.5, $G - A$ has a connected component of rank-depth at least $m - |A| - 1$. But, by a direct argument, we can guarantee that there is a connected component of $G - A$ of rank-depth at least $m - |A|$. We include the full proof for completeness.

Lemma 2.6. *Let G be a graph of rank-depth m and A be a non-empty proper subset of $V(G)$. Then $G - A$ has a connected component of rank-depth at least $m - |A|$.*

Proof. If $|A| \geqslant m$, then any connected component has rank-depth at least zero. Thus, we may assume that $|A| < m$. This implies that $m \geqslant 2$ as A is non-empty.

Suppose for contradiction that every connected component of $G - A$ has rank-depth at most $m - |A| - 1$. Let $C_1, C_2, \ldots, C_t$ be the connected components of $G - A$. For each $i \in \{1, 2, \ldots, t\}$,

- if C_i contains at least two vertices, then we take an $(m - |A| - 1, m - |A| - 1)$-decomposition (T_i, σ_i) where r_i is a node of T_i having distance at most $m - |A| - 1$ to every node of T_i, and
- if C_i consists of one vertex, we set T_i to be the one-node graph on $\{r_i\}$ and let $\sigma_i : V(C_i) \to \{r_i\}$ be the uniquely possible function.

We obtain a new decomposition (T, σ) of G by taking the disjoint union of T_i's and adding a new node r and adding edges rr_i for all $i \in \{1, 2, \ldots, t\}$, and additionally appending $|A|$ leaves to r and assigning each vertex of A to a distinct leaf with the map σ. For every vertex v of $G - A$, define $\sigma(v) = \sigma_i(v)$ if v is a vertex of C_i. Then (T, σ) has depth at most $m - |A|$ and width at most $m - 1$. Because $|A| \geqslant 1$, this contradicts the assumption that G has rank-depth m.

We conclude that $G - A$ has a connected component of rank-depth at least $m - |A|$. $\square$

Lemma 2.7. *Let m and d be positive integers. Let G be a graph with a vertex partition (A, B) such that connected components of $G[A]$ and $G[B]$ have rank-depth at most m and $\rho_G(A) \leqslant d$. Then G has rank-depth at most $m + d + 1$.*

Proof. Let $C_1, \ldots, C_p$ be the connected components of $G[A]$, and $D_1, \ldots, D_q$ be the connected components of $G[B]$. For each $i \in \{1, 2, \ldots, p\}$,

- if C_i contains at least two vertices, then we take an (m, m)-decomposition (T_i, σ_i) where r_i is a node of T_i having distance at most m to every node of T_i, and
- if C_i consists of one vertex, then set T_i as the one-node graph on $\{r_i\}$ and $\sigma_i : V(C_i) \to \{r_i\}$ as the uniquely possible function.

Similarly, we define (F_j, μ_j) for each D_j where f_j is a node of F_j having distance at most m to every node of F_j.

Now, we obtain a new decomposition (T, σ) of G as follows. Let T be the tree obtained by taking the disjoint union of all of T_i's and F_j's, adding new vertices x and y, an edge xy, edges xr_i for all $i \in \{1, \dots, p\}$, and edges yf_j for all $j \in \{1, \dots, q\}$. Define $\sigma(v) = \sigma_i(v)$ if v is a vertex of C_i, and $\sigma(v) = \mu_j(v)$ if v is a vertex of D_j. Then (T, σ) has depth at most $m+2$ and width at most $m+d$. Because $d \geqslant 1$, G has rank-depth at most $\max\{m+2, m+d\} \leqslant m+d+1$. $\square$

2.3. Rank-width

We now review the definition of rank-width. A *rank-decomposition* of a graph G is a pair (T, L) of a tree T whose vertices each have degree either one or three, and a bijection L from $V(G)$ to the set of leaves of T. The *width* of an edge e of T is the cut-rank in G of the set of all leaves assigned to one of the components of $T - e$. The *width* of the rank-decomposition (T, L) is the maximum width of an edge of T. Finally, the *rank-width* of G is the minimum width over all rank-decompositions of G. Graphs with at most one vertex do not admit rank-decompositions and we define their rank-width to be zero.

3. The proof

We write $R(n; k)$ to denote the minimum number N such that every coloring of the edges of K_N with k colors induces a monochromatic complete subgraph on n vertices. The classical theorem of Ramsey [23] implies that $R(n; k)$ exists.

The following lemma is well known. We include its proof for the sake of completeness.

Lemma 3.1. *Let G be a graph of rank-width at most q and let $M \subseteq V(G)$. If $|M| \geqslant 3k+1$ for a positive integer k, then there is a vertex partition (X, Y) of G such that $\rho_H(X) \leqslant q$ and $\min(|M \cap X|, |M \cap Y|) > k$.*

Proof. Suppose that there is no such vertex partition. Let (T, L) be a rank-decomposition of width at most q. For each edge uv of T, let us orient e towards v if the component of $T - e$ containing u has at most k vertices in $L(M)$. By the assumption, every edge is oriented. Since T is acyclic, there is a node w of T such that all edges of T incident with w are oriented towards w. But this implies that $|M| \leqslant 3k$, a contradiction. $\square$

For a path P with an endpoint x and a graph H and a non-empty subset of vertices $S \subseteq V(H)$, we denote by $(P, x) + (H, S)$ the graph obtained from the disjoint union of P and H by adding all edges between x and S. We now prove our main proposition; Theorem 1.2 will follow quickly after.

Proposition 3.2. *For all positive integers a, b, t, q, there exists an integer $f(a, b, t, q)$ such that every graph of rank-width at most q and rank-depth at least $f(a, b, t, q)$ has a vertex-minor isomorphic to either P_t or $(P_a, x) + (H, S)$ where x is an endpoint of P_a, H is a connected graph of rank-depth at least b, and S is a non-empty subset of $V(H)$.*

Proof. For all positive integers b, t, q, we set

$$f(1,b,t,q) := b+2,$$

and for $a \geqslant 2$, we set

$$\begin{aligned}
u &:= \max(3\cdot(2^q-1)+1, t-1),\\
r &:= R(u+1; 2^{a-1}),\\
g_i &:= \begin{cases} b+q+2 & \text{if } i=r,\\ f(a-1, g_{i+1}, t, q) & \text{if } i\in\{0,1,2,\dots,r-1\},\end{cases}\\
f(a,b,t,q) &:= g_0.
\end{aligned}$$

We prove the proposition by induction on a. Let G be a graph whose rank-depth is at least $f(a,b,t,q)$ and rank-width is at most q. If $a=1$, then it has a component G' of rank-depth at least $b+1$ by Lemma 2.5. Let $v\in V(G')$. By Lemma 2.6, $G'-v$ has a connected component H of rank-depth at least b. So, $(G'[\{v\}], v) + (H, N_G(v)\cap V(H))$ is the second outcome.

Thus, we may assume that $a \geqslant 2$. Suppose that G has no vertex-minor isomorphic to P_t. We claim that G contains the second outcome.

Let $H_0 := G$. Observe that H_0 has rank-depth at least $f(a,b,t,q) = g_0$.

For $i\in\{1,2,\dots,r\}$, we recursively find tuples (A_i, x_i, H_i, S_i) from H_{i-1} such that

- A_i is isomorphic to P_{a-1} and x_i is an endpoint of A_i,
- H_i is a connected graph of rank-depth at least g_i,
- S_i is a non-empty subset of $V(H_i)$, and
- $(A_i, x_i) + (H_i, S_i)$ is a vertex-minor of H_{i-1}.

Let $i\in\{1,2,\dots,r\}$ and assume that H_{i-1} is a given graph of rank-depth at least g_{i-1}. Then by the induction hypothesis, H_{i-1} has a vertex-minor $(A_i, x_i) + (H_i, S_i)$ where A_i is isomorphic to P_{a-1}, x_i is an endpoint of A_i, H_i is a connected graph of rank-depth at least g_i, and S_i is a non-empty subset of $V(H_i)$. By the choice of functions $g_0, g_1, \dots, g_r$, we can obtain the tuples for all $i\in\{1,2,\dots,r\}$.

Observe that for $i<j$, no vertex in $V(A_i)\setminus\{x_i\}$ has a neighbor in H_i, and therefore, the sequence of local complementations to obtain $(A_j, x_j) + (H_j, S_j)$ from H_{j-1} does not change previous paths $A_1, \dots, A_{j-1}$, but may change the edges between x_1, x_2, …, x_{j-1}.

By definition, H_r is connected and has rank-depth at least g_r. Let G_1 be the graph obtained from G by following the sequence of local complementations to obtain $(A_1, x_1) + (H_1, S_1), \dots, (A_r, x_r) + (H_r, S_r)$. See Fig. 2 for a depiction. Note that $G_1[V(H_i)\cup\{x_i\}]$ is connected for each i, as H_i is connected, S_i is non-empty, and we apply local complementations only inside H_i to obtain later H_j's.

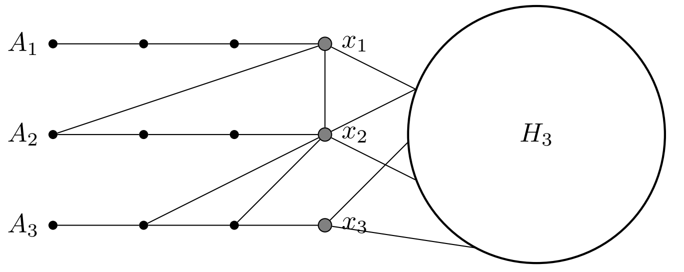

Fig. 2. The graph $G_1[A_1 \cup \cdots \cup A_r \cup V(H_r)]$, when $r = 3$ and $a = 5$.

If for some $i \in \{1, 2, \ldots, r\}$, $N_{G_1}(x_i) \cap V(H_r) = \emptyset$, then by taking a shortest path from x_i to $V(H_r)$ in the graph $G_1[V(H_i) \cup \{x_i\}]$, we can directly obtain the second outcome. So, we may assume that each of $\{x_1, x_2, \ldots, x_r\}$ has a neighbor in $V(H_r)$.

Note that for $i < j$, only the endpoint x_i in A_i can have a neighbor in A_j in G_1, and therefore, there are 2^{a-1} possible ways of having edges between A_i and A_j in G_1. Since $r = R(u + 1; 2^{a-1})$, by applying the theorem of Ramsey, we deduce that there exists a subset $W \subseteq \{1, 2, \ldots, r\}$ of size $u + 1$ such that for all $i < j$ with $i, j \in W$, $\{\ell :$ the ℓ-th vertex of A_j is adjacent to x_i in $G_1\}$ are identical.

If x_i has a neighbor in $V(A_j - x_j)$ in G_1 for some $i < j$ with $i, j \in W$, then G_1 has $\overline{K_u} \boxslash \overline{K_u}$ or $\overline{K_u} \boxslash K_u$ as an induced subgraph. Since $u \geqslant t - 1$, by Lemmas 2.2 and 2.3, G_1 contains a pivot-minor isomorphic to P_t, contradicting the assumption. So, for all $i < j$ with $i, j \in W$, x_i has no neighbors in $V(A_j - x_j)$.

Note that $\{x_i : i \in W\}$ is an independent set or a clique in G_1. If it is an independent set, then for some $i' \in W$, we set

- $G_2 := G_1$ and $W' := W \setminus \{i'\}$.

If $\{x_i : i \in W\}$ is a clique, then we choose a vertex $x_{i'}$ for some $i' \in W$ and locally complement at $x_{i'}$. Then $\{x_i : i \in W \setminus \{i'\}\}$ becomes an independent set. We set

- $G_2 := G_1 * x_{i'}$ and $W' := W \setminus \{i'\}$.

Let $M := \{x_i : i \in W'\}$ and $H := G_2[V(H_r) \cup M \cup \{x_{i'}\}]$.

By definition, H is locally equivalent to the graph $G_1[V(H_r) \cup M \cup \{x_{i'}\}]$. Thus, as the latter is connected, H is also connected. Similarly, since $H_r = G_1[V(H_r)]$ has rank-depth at least g_r, H has rank-depth at least g_r. Also, note that H has rank-width at most q and M is an independent set of size $u \geqslant 3 \cdot (2^q - 1) + 1$ in H. Thus, by Lemma 3.1, H admits a vertex partition (X, Y) such that $|M \cap X| > 2^q - 1$, $|M \cap Y| > 2^q - 1$, and $\rho_H(X) \leqslant q$.

Since H has rank-depth at least $g_r = b + q + 2$ and $\rho_H(X) \leqslant q$, by Lemma 2.7, $H[X]$ or $H[Y]$ has a connected component of rank-depth at least $b + 1$. Without loss of generality, we assume that $H[X]$ has a connected component Q of rank-depth at least $b + 1$.

Now, if $M \cap Y$ has a vertex x_i that has no neighbor in Q, then by taking a shortest path from x_i to Q in H, along with A_i, we can find the second outcome.

Thus, we may assume that in H, all vertices in $M \cap Y$ have a neighbor in Q. Since $\rho_H(X) \leqslant q$, there are at most $2^q - 1$ distinct non-zero rows in the matrix $A(H)[M \cap Y, V(Q)]$. As $|M \cap Y| \geqslant 2^q$, by the pigeon-hole principle, H has two vertices x_{i_1} and x_{i_2} in $M \cap Y$ for some $i_1, i_2 \in W'$ that have the same neighborhood in Q.

First assume that x_{i_1} has exactly one neighbor in Q, say w. As Q has rank-depth at least $b+1$, $Q - w$ has a connected component Q' having rank-depth at least b by Lemma 2.6. Then

$$(G_2[V(A_{i_1}) \cup \{w\}], w) + (Q', N_{G_2}(w) \cap V(Q'))$$

is the required second outcome. So, we may assume that x_{i_1} has at least two neighbors in Q. Let w be a neighbor of x_{i_1} in Q.

Since x_{i_1} and x_{i_2} have the same neighborhood in Q and they are not adjacent, if we pivot $x_{i_2}w$, then the edges between x_{i_1} and $N_H(x_{i_1}) \cap V(Q)$ are removed and x_{i_2} becomes the unique neighbor of x_{i_1} in $V(Q) \cup \{x_{i_2}\}$. Note that $G_2[V(Q) \cup \{x_{i_2}\}]$ is connected, and thus $(G_2 \wedge x_{i_2}w)[V(Q) \cup \{x_{i_2}\}]$ is also connected. As Q has rank-depth at least $b+1$, $(G_2 \wedge x_{i_2}w)[V(Q) \cup \{x_{i_2}\}] - x_{i_2}$ has a connected component Q' that has rank-depth at least b. Then

$$((G_2 \wedge x_{i_2}w)[V(A_{i_1}) \cup \{x_{i_2}\}], x_{i_2}) + (Q', N_{G_2 \wedge x_{i_2}w}(x_{i_2}) \cap V(Q'))$$

is the second outcome. This proves the proposition. □

Proposition 3.2 implies the following result.

Theorem 3.3. *For all positive integers t and q, there exists an integer $F(t,q)$ such that every graph of rank-width at most q and rank-depth at least $F(t,q)$ contains a vertex-minor isomorphic to P_t.*

Proof. We take $F(t,q) := f(t-1, 1, t, q)$ where f is the function in Proposition 3.2. □

A *circle graph* is the intersection graph of chords on a circle. It is easy to see that P_t is a circle graph. We can derive Theorem 1.2 by taking $q = \beta(P_t)$ from the following recent theorem.

Theorem 3.4 *(Geelen, Kwon, McCarty, and Wollan [10]). For every circle graph H, there exists an integer $\beta(H)$ such that every graph of rank-width more than $\beta(H)$ contains a vertex-minor isomorphic to H.*

Theorem 1.2. *For every positive integer t, there exists an integer $N(t)$ such that every graph of rank-depth at least $N(t)$ contains a vertex-minor isomorphic to P_t.*

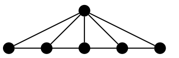

Fig. 3. The fan graph F_5.

Proof. We take $N(t) := F(t, \beta(P_t))$ where β is the function given in Theorem 3.4 and F is the function from Theorem 3.3. If a graph has rank-width more than $\beta(P_t)$, then by Theorem 3.4, it contains a vertex-minor isomorphic to P_t. So, we may assume that a graph has rank-width at most $\beta(P_t)$. Then by Theorem 3.3, it contains a vertex-minor isomorphic to P_t. $\square$

4. Pivot-minors

We can prove a stronger result on bipartite graphs, by slightly modifying the proof of Proposition 3.2. Suppose that a given graph G is bipartite in the proof of Proposition 3.2. The only place that we have to apply local complementation instead of pivoting is when the set $\{x_i : i \in W\}$ is a clique, and we want to change it into an independent set. But if G is bipartite, then the obtained set $\{x_i : i \in W\}$ has no triangle, and so it is an independent set since $|W| \geqslant 3$. Therefore, we can proceed only with pivoting. For bipartite graphs, we can use the following theorem due to Oum [21], obtained as a consequence of the grid theorem for binary matroids [9].

Theorem 4.1 *(Oum [21]). For every bipartite circle graph H, there exists an integer $\gamma(H)$ such that every bipartite graph of rank-width more than $\gamma(H)$ contains a pivot-minor isomorphic to H.*

Thus we deduce the following theorem for bipartite graphs.

Theorem 1.3. *For every positive integer t, there exists an integer $N(t)$ such that every bipartite graph of rank-depth at least $N(t)$ contains a pivot-minor isomorphic to P_t.*

Theorem 1.3 allows us to obtain the following corollary for binary matroids, solving a special case of a conjecture of DeVos, Kwon, and Oum [6] on general matroids. We need a few terms to state the corollary. The branch-depth of a matroid is defined analogously to the definition of the rank-depth obtained by replacing the cut-rank function with the matroid connectivity function [6]. Let F_t be the fan graph, that is the union of P_t with one vertex adjacent to all vertices of P_t, see Fig. 3. As usual, $M(F_t)$ denotes the cycle matroid of F_t.

Corollary 4.2. *For every positive integer t, there exists an integer $N(t)$ such that every binary matroid of branch-depth at least $N(t)$ contains a minor isomorphic to $M(F_t)$.*

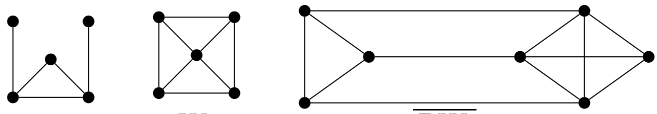

Fig. 4. The graphs bull, W_4, and $\overline{BW_3}$.

Proof. It is known [2,21] that the connectivity function of a binary matroid is equal to the cut-rank function of a corresponding bipartite graph, called a *fundamental graph*. Furthermore for two binary matroids M and N, if N is connected, and a fundamental graph of N is a pivot-minor of a fundamental graph of M, then either N or N^* is a minor of M, see Oum [21, Corollary 3.6]. Since $(M(F_t))^*$ has a minor isomorphic to $M(F_{t-1})$, we deduce the corollary from Theorem 1.3, because the path graph P_{2t-1} is a fundamental graph of $M(F_t)$. $\square$

We show that the class $\{K_n \boxslash K_n : n \geqslant 1\}$ has unbounded rank-depth, while for every positive integer n, $K_n \boxslash K_n$ has no pivot-minor isomorphic to P_5. It implies that contrary to Theorem 1.3, the class of graphs having no P_n pivot-minor has unbounded rank-depth for each $n \geqslant 5$.

Kwon and Oum [16, Lemma 6.5] showed that for every integer $n \geqslant 2$, $K_n \boxslash K_n$ contains a vertex-minor isomorphic to P_{2n-2}. Thus, $\{K_n \boxslash K_n : n \geqslant 1\}$ has unbounded rank-depth.

Now, we show that for $n \geqslant 1$, $K_n \boxslash K_n$ has no pivot-minor isomorphic to P_5. We prove a stronger statement that $K_n \boxslash K_n$ has no pivot-minor isomorphic to $K_{1,3}$. Dabrowski et al. [5] characterized the class of graphs having no pivot-minor isomorphic to $K_{1,3}$ in terms of forbidden induced subgraphs. See Fig. 4 for bull, W_4, and $\overline{BW_3}$.

Theorem 4.3 *(Dabrowski et al. [5]). A graph has a pivot-minor isomorphic to $K_{1,3}$ if and only if it has an induced subgraph isomorphic to one of $K_{1,3}$, P_5,* bull*, W_4, and $\overline{BW_3}$.*

Lemma 4.4. *For $n \geqslant 1$, $K_n \boxslash K_n$ has no induced subgraph isomorphic to one of $K_{1,3}$, P_5,* bull*, W_4, and $\overline{BW_3}$.*

Proof. As the maximum size of an independent set in $K_n \boxslash K_n$ is 2, $K_n \boxslash K_n$ has no induced subgraph isomorphic to one of $K_{1,3}$, P_5, and bull.

Also $K_n \boxslash K_n$ has no induced cycle of length 4 because such a cycle should contain two vertices in each K_n but the edges between two K_n's have no induced matching of size 2. Therefore, it has no induced subgraph isomorphic to W_4 or $\overline{BW_3}$. $\square$

By Theorem 4.3 and Lemma 4.4, $K_n \boxslash K_n$ has no pivot-minor isomorphic to $K_{1,3}$, and to P_5. Thus, for all $n \geqslant 5$, the class of graphs having no P_n pivot-minor includes $\{K_n \boxslash K_n : n \geqslant 1\}$, which has unbounded rank-depth. It may be interesting to see whether every graph with sufficiently large rank-depth contains either P_n or $K_n \boxslash K_n$ as a pivot-minor. We leave it as an open question.

Question 1. Does there exist a function f such that for every n, every graph with rank-depth at least $f(n)$ contains a pivot-minor isomorphic to P_n or $K_n \boxslash K_n$?

5. Concluding remarks

5.1. Linear χ-boundedness

We define linear rank-width. For an ordering $(v_1, v_2, \ldots, v_n)$ of the vertex set of a graph G, its *width* is defined as the maximum of $\rho_G(\{v_1, \ldots, v_i\})$ for all $i \in \{1, 2, \ldots, n-1\}$, and the *linear rank-width* of G is defined as the minimum width of all orderings of G. If $|V(G)| < 2$, then the linear rank-width of G is defined as 0.

Graphs of bounded rank-depth have bounded linear rank-width, which was already known through the notions of shrub-depth and linear clique-width [7]. Kwon and Oum [15] proved it directly as follows.

Proposition 5.1 *(Kwon and Oum [15]). Every graph of rank-depth k has linear rank-width at most k^2.*

We write $\chi(G)$ to denote the chromatic number of G and $\omega(G)$ to denote the maximum size of a clique of G. A class $\mathcal{C}$ of graphs is *χ-bounded* if there is a function f such that $\chi(H) \leqslant f(\omega(H))$ for all induced subgraphs H of a graph in $\mathcal{C}$. In addition, if f can be taken as a polynomial function, then $\mathcal{C}$ is *polynomially χ-bounded*. If f can be taken as a linear function, then $\mathcal{C}$ is *linearly χ-bounded*.

Proposition 5.2 *(Nešetřil, Ossona de Mendez, Rabinovich, and Siebertz [19]). For every positive integer r, there exists an integer $c(r)$ such that for every graph G of linear rank-width at most r,*

$$\chi(G) \leqslant c(r)\,\omega(G).$$

By combining Proposition 5.1 and Proposition 5.2, we can prove the following, which answers a previous question by Kim, Kwon, Oum, and Sivaraman [12].

Theorem 5.3. *For every positive integer t, the class of graphs with no vertex-minor isomorphic to P_t is linearly χ-bounded.*

We remark that there is an alternative way to prove Theorem 5.3 without using linear rank-width. First, DeVos, Kwon, and Oum [6, Lemma 4.10] showed that if a graph has rank-depth k, then it has an (a, k)-shrubbery where

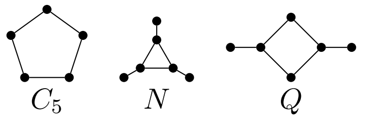

Fig. 5. Obstructions for being a vertex-minor of a path.

$$a = (1 + o(1))2^{(2^{2k+1}(2^{2k+2}-1)+1)k/2}.$$

(Please see [6] for the definition of an (a, k)-shrubbery.) Lemma 2.16 of Nešetřil, Ossona de Mendez, Rabinovich, and Siebertz [19] states that every class of bounded shrub-depth can be partitioned into bounded number of vertex-disjoint induced subgraphs, each of which is a cograph. Its (short and easy) proof shows that a graph with an (a, k)-shrubbery can be partitioned into at most a vertex-disjoint induced subgraphs, each of which is a cograph. Since cographs are perfect, we deduce that if G has rank-depth at most k, then $\chi(G) \leqslant \omega(G)(1 + o(1))2^{(2^{2k+1}(2^{2k+2}-1)+1)k/2}$.

5.2. When does the class of $\mathcal{H}$-vertex-minor-free graphs have bounded rank-depth?

For a set $\mathcal{H}$ of graphs, we say that G is *$\mathcal{H}$-minor-free* if G has no minor isomorphic to a graph in $\mathcal{H}$, and G is *$\mathcal{H}$-vertex-minor-free* if G has no vertex-minor isomorphic to a graph in $\mathcal{H}$. Robertson and Seymour [24] showed that $\mathcal{H}$-minor-free graphs have bounded tree-width if and only if $\mathcal{H}$ contains a planar graph. As an analogue, Geelen, Kwon, McCarty, and Wollan [10] showed that $\mathcal{H}$-vertex-minor-free graphs have bounded rank-width if and only if $\mathcal{H}$ contains a circle graph. Interestingly, Theorem 1.2 allows us to characterize the classes $\mathcal{H}$ such that $\mathcal{H}$-vertex-minor-free graphs have bounded rank-depth. This is due to the following theorem; the equivalence between (a) and (b) was shown by Kwon and Oum [13] and the equivalence between (a) and (c) was shown by Adler, Farley, and Proskurowski [1].

Theorem 5.4 *(Kwon and Oum [13]; Adler, Farley, and Proskurowski [1]). Let H be a graph. The following are equivalent.*

(a) H has linear rank-width at most one.
(b) H is a vertex-minor of a path.
(c) H has no vertex-minor isomorphic to C_5, N, or Q in Fig. 5.

We define linear rank-width in the next subsection. Here, we only need the fact that linear rank-width does not increase when we take vertex-minors and that paths have linear rank-width 1 and arbitrary large rank-depth to deduce the following corollary from Theorems 5.4 and 1.2.

Corollary 5.5. *Let $\mathcal{H}$ be a set of graphs. Then the following are equivalent.*

(a) The class of $\mathcal{H}$-vertex-minor-free graphs has bounded rank-depth.
(b) $\mathcal{H}$ contains a graph of linear rank-width at most one.
(c) $\mathcal{H}$ contains a graph with no vertex-minor isomorphic to C_5, N, or Q.

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