Discounting the Future in Systems Theory*

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Abstract. Discounting the future means that the value, today, of a unit payoff is 1 if the payoff occurs today, a if it occurs tomorrow, a^2 if it occurs the day after tomorrow, and so on, for some real-valued discount factor $0 < a < 1$. Discounting (or inflation) is a key paradigm in economics and has been studied in Markov decision processes as well as game theory. We submit that discounting also has a natural place in systems engineering: for nonterminating systems, a potential bug in the far-away future is less troubling than a potential bug today. We therefore develop a systems theory with discounting. Our theory includes several basic elements: discounted versions of system properties that correspond to the ω-regular properties, fixpoint-based algorithms for checking discounted properties, and a quantitative notion of bisimilarity for capturing the difference between two states with respect to discounted properties. We present the theory in a general form that applies to probabilistic systems as well as multicomponent systems (games), but it readily specializes to classical transition systems. We show that discounting, besides its natural practical appeal, has also several mathematical benefits. First, the resulting theory is robust, in that small perturbations of a system can cause only small changes in the properties of the system. Second, the theory is computational, in that the values of discounted properties, as well as the discounted bisimilarity distance between states, can be computed to any desired degree of precision.

1 Introduction

In systems theory, one models systems and analyzes their properties. Nonterminating discrete-time models, such as transition systems and games, are important in many computer science applications, and the ω-regular properties offer an accomplished theory for their analysis. The theory is expressive from a practical point of view [22,27], computational (algorithmic) [5,28], and abstract (language-independent) [21,34]. In its general setting, the theory considers games with ω-regular winning conditions [17,28], provides fixpoint-based algorithms for their solution [13,15], and property-preserving equivalence relations

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between structures [4,24]. From a systems engineering point of view, however, the theory has a significant drawback: it is too exact [1]. Since the ω-regular properties generalize finite behavior by considering behavior at infinity, they can distinguish behavior differences that occur arbitrarily late. This exactness becomes even more pronounced for probabilistic models [6,29,33], whose behaviors are specified using numerical quantities, because the theory can distinguish arbitrarily small perturbations of a system.

We propose an alternative formalism that is (in a certain sense) as expressive as the ω-regular properties, and yet achieves continuity in the Cantor topology by sacrificing exactness. In other words, we introduce an *approximate* theory of nonterminating discrete-time systems. The approximation is in two directions. First, instead of giving boolean answers to logical questions, we consider the value of a property to be a real number in the interval [0,1] [19]. Second, we generalize, as in [11,12,18], the classical notions of state equivalences to pseudometrics on states. Both are achieved by defining a *discounted* version of the classical theory. Discounting is inspired by similar ideas in Markov decision processes, economics, and game theory [16,31], and captures the natural engineering intuition that the far-away future is not as important as the near future. Consider, for example, the safety property that no unsafe state is visited. In the classical theory, this property is either true or false. In the discounted theory, its value is 1 if no unsafe state is visited ever, and $1 - a^k$, for some discount factor $0 < a < 1$, if no unsafe state is visited for k steps: the longer the system stays in safe states, the greater the value of the property. Our theory is *robust*, in that small perturbations of a system imply small differences in the numerical values of properties, and *computational*, in that numerical approximation schemes are available which converge geometrically to property values from both directions.

The key insight of this work is that discounting is most naturally and fundamentally applied not to properties, nor to state equivalences, but to the μ-calculus [20]. We introduce the *discounted μ-calculus*, a *quantitative* fixpoint calculus: rather than computing with sets of states, as the traditional μ-calculus does, we compute with functions that assign to each state a value between 0 and 1. A quantitative μ-calculus was introduced in [9] to compute the values of probabilistic ω-regular games by iterating a quantitative version of the predecessor (pre) operator. The discounted μ-calculus is obtained from the calculus of [9] by discounting the pre operator through multiplication with a discount factor $a < 1$. In the classical setting, there is a connection between (linear-time) ω-regular properties, (branching-time) μ-calculus, and games. By discounting the μ-calculus while maintaining this connection, we obtain a notion of discounted ω-regular properties, as well as algorithms for solving games with discounted ω-regular objectives. In the classical setting, the connection is as follows. The solution of a game with an ω-regular winning condition can be written as a μ-calculus formula [13,14]. The fixpoint formula defines the property: when evaluated on linear traces, it holds exactly on the initial states of the traces that satisfy the property. We extend this correspondence to the discounted setting by considering discounted versions of the μ-calculus formula: the discounted fixpoint formula, evaluated on linear traces, defines a discounted version of the original ω-regular property. At the same time, we show that the discounted formula, when evaluated on a game structure, computes the value of the game whose payoff is given by the discounted ω-regular property.

We develop our theory on the system model of concurrent probabilistic game structures [9,16]. These structures generalize several standard models of computation, including nondeterministic transition systems, Markov decision processes [10], and deterministic two-player games [2,32]. The use of discounting gives our theory two main features: *computationality* and *robustness*. Computationality is due to the fact that discount factors strictly less than 1 ensure the geometric convergence of each fixpoint computation by successive approximation (Picard iteration). This enables us to compute every fixpoint value to any desired degree of precision. Moreover, discounting entails the uniqueness of fixpoints. Together, the monotonicity of the μ-calculus operators, the geometric convergence of Picard iteration, and the uniqueness of fixpoints mean that we can iteratively compute geometrically converging lower and upper bounds for the value of every discounted μ-calculus formula. The existence of such approximation schemes is in sharp contrast to the situation for the undiscounted μ-calculus, where least and greatest fixpoints generally differ, where each (least or greatest) fixpoint can be approximated in one direction only (from below, or from above), and where in the quantitative case, no rate of convergence is known.

In the classical setting, the μ-calculus characterizes bisimilarity: two states are bisimilar iff they satisfy the same μ-calculus formulas. To extend this connection to the discounted setting, we define a quantitative, discounted notion of bisimilarity, which assigns a real-valued *distance* in the interval [0,1] to every pair of states: the distance between two states is 1 if they satisfy different propositions, and otherwise it is coinductively computed from discounted distances between successor states. We show that in the discounted setting, the bisimilarity distance between two states is equal to the supremum, over all μ-calculus formulas, of the difference between the values of a formula at the two states. This is in fact the characterization of discounted bisimilarity from [11,12] extended to games. However, while in [11,12] the above characterization is taken to be the definition of discounted bisimilarity, in our case it is a theorem that can be proved from the coinductive definition. The theorem demonstrates the *robustness* of the theory: small perturbations in the numerical values of transition probabilities, as well as (small or large) perturbations that come far in the future, correspond to small bisimilarity distance, and hence to small differences in the numerical values of discounted properties. The numerical computation of discounted bisimilarity by successive approximation enjoys the same properties as the numerical evaluation of discounted μ-calculus formulas; in particular, geometrically-converging approximation schemes are available for computing both lower and upper bounds.

2 Systems: Concurrent Game Structures

For a countable set U, a *probability distribution* on U is a function $p\colon U \mapsto [0,1]$ such that $\sum_{u \in U} p(u) = 1$. We write $\mathcal{D}(U)$ for the set of probability distributions on U. A two-player (*concurrent*) *game structure* [2,7] $\mathcal{G} = \langle Q, M, \Gamma_1, \Gamma_2, \delta \rangle$ consists of the following components:

- A finite set Q of states.
- A finite set M of moves.
- Two move assignments $\Gamma_1, \Gamma_2\colon Q \mapsto 2^M \setminus \emptyset$. For $i \in \{1,2\}$, the assignment Γ_i associates with each state $s \in Q$ the nonempty set $\Gamma_i(s) \subseteq M$ of moves available to player i at state s.
- A probabilistic transition function $\delta\colon Q \times M^2 \mapsto \mathcal{D}(Q)$. For a state $s \in Q$ and moves $\gamma_1 \in \Gamma_1(s)$ and $\gamma_2 \in \Gamma_2(s)$, the function δ provides a probability distribution of successor states. We write $\delta(t \mid s, \gamma_1, \gamma_2)$ for the probability $\delta(s, \gamma_1, \gamma_2)(t)$ that the successor state is $t \in Q$.

At every state $s \in Q$, player 1 chooses a move $\gamma_1 \in \Gamma_1(s)$, and simultaneously and independently player 2 chooses a move $\gamma_2 \in \Gamma_2(s)$. The game then proceeds to the successor state $t \in Q$ with probability $\delta(t \mid s, \gamma_1, \gamma_2)$. The outcome of the game is a path. A *path* of $\mathcal{G}$ is an infinite sequence $s_0, s_1, s_2, \ldots$ of states in $s_k \in Q$ such that for all $k \geq 0$, there are moves $\gamma_1^k \in \Gamma_1(s_k)$ and $\gamma_2^k \in \Gamma_2(s_k)$ with $\delta(s_{k+1} \mid s_k, \gamma_1^k, \gamma_2^k) > 0$. We write Σ for the set of all paths.

The following are special cases of concurrent game structures. The structure $\mathcal{G}$ is *deterministic* if for all states $s \in Q$ and moves $\gamma_1 \in \Gamma_1(s)$, $\gamma_2 \in \Gamma_2(s)$, there is a state $t \in Q$ with $\delta(t \mid s, \gamma_1, \gamma_2) = 1$; in this case, with abuse of notation we write $\delta(s, \gamma_1, \gamma_2) = t$. The structure $\mathcal{G}$ is *turn-based* if at every state at most one player can choose among multiple moves; that is, for all states $s \in Q$, there exists at most one $i \in \{1,2\}$ with $|\Gamma_i(s)| > 1$. The turn-based deterministic game structures coincide with the games of [32]. The structure $\mathcal{G}$ is *one-player* if at every state only player 1 can choose among multiple moves; that is, $|\Gamma_2(s)| = 1$ for all states $s \in Q$. The one-player game structures coincide with Markov decision processes (MDPs) [10]. The one-player deterministic game structures coincide with transition systems: in every state, each available move of player 1 determines a possible successor state.

A *strategy* for player $i \in \{1,2\}$ is a function $\pi_i\colon Q^+ \mapsto \mathcal{D}(M)$ that associates with every nonempty finite sequence $\sigma \in Q^+$ of states, representing the history of the game, a probability distribution $\pi_i(\sigma)$, which is used to select the next move of player i. Thus, the choice of the next move can be history-dependent and randomized. We require that the strategy π_i can prescribe only moves that are available to player i; that is, for all sequences $\sigma \in Q^*$ and states $s \in Q$, if $\pi_i(\sigma s)(\gamma) > 0$, then $\gamma \in \Gamma_i(s)$. We write Π_i for the set of strategies for player i. The strategy π_i is *deterministic* if for all sequences $\sigma \in Q^+$, there exists a move $\gamma \in M$ such that $\pi(\sigma)(\gamma) = 1$. Thus, deterministic strategies are functions from Q^+ to M. The strategy π_i is *memoryless* if for all sequences $\sigma, \sigma' \in Q^*$ and states $s \in Q$, we have $\pi(\sigma s) = \pi(\sigma' s)$. Thus, the moves chosen by memoryless strategies depend only on the current state and not on the history of the game.

Given a starting state $s \in Q$ and two strategies π_1 and π_2 for the two players, the game is reduced to an ordinary stochastic process, denoted $\mathcal{G}_s^{\pi_1,\pi_2}$, which defines a probability distribution on the set Σ of paths. An *event* of $\mathcal{G}_s^{\pi_1,\pi_2}$ is a measurable set $A \subseteq \Sigma$ of paths. For an event $A \subseteq \Sigma$, we write $\Pr_s^{\pi_1,\pi_2}(A)$ for the probability that the outcome of the game belongs to A when the game starts from s and the players use the strategies π_1 and π_2. A *payoff function* v: $\Sigma \mapsto [0,1]$ is a measurable function that associates with every path a real in the interval $[0,1]$. Payoff functions define the rewards of the two players for each outcome of the game. For a payoff function v, we write $\mathrm{E}_s^{\pi_1,\pi_2}\{v\}$ for the expected value of v on the outcome when the game starts from s and the strategies π_1 and π_2 are used. If v defines the reward for player 1, then the (*player 1*) *value* of the game is a function that maps every state $s \in Q$ to the maximal expected reward $\sup_{\pi_1 \in \Pi_1} \inf_{\pi_2 \in \Pi_2} \mathrm{E}_s^{\pi_1,\pi_2}\{v\}$ that player 1 can achieve no matter which strategy player 2 chooses.

3 Algorithms: Discounted Fixpoint Expressions

Quantitative region algebra. The classical μ-calculus specifies algorithms for iterating boolean and predecessor (pre) operators on regions, where a region is a set of states. In our case a region is a function from states to reals. This notion of quantitative region admits the analysis both of probabilistic transitions and of real-valued discount factors. Consider a concurrent game structure $\mathcal{G} = \langle Q, M, \Gamma_1, \Gamma_2, \delta \rangle$. A (*quantitative*) *region* of $\mathcal{G}$ is a function f: $Q \mapsto [0,1]$ that maps every state to a real in the interval $[0,1]$. For example, for a given payoff function, the value of a game on the structure $\mathcal{G}$ is a quantitative region. We write $\mathcal{F}$ for the set of quantitative regions. By $\mathbf{0}$ and $\mathbf{1}$ we denote the constant functions in $\mathcal{F}$ that map all states in Q to 0 and 1, respectively. Given two regions $f, g \in \mathcal{F}$, define $f \leq g$ if $f(s) \leq g(s)$ for all states $s \in Q$, and define the regions $f \wedge g$ and $f \vee g$ by $(f \wedge g)(s) = \min\{f(s), g(s)\}$ and $(f \vee g)(s) = \max\{f(s), g(s)\}$, for all states $s \in Q$. The region $\mathbf{1} - f$ is defined by $(\mathbf{1} - f)(s) = 1 - f(s)$ for all $s \in Q$; this has the role of complementation. Given a set $T \subseteq Q$ of states, with abuse of notation we denote by T also the indicator function of T, defined by $T(s) = 1$ if $s \in Q$, and $T(s) = 0$ otherwise. Let $\mathcal{F}_{\mathbb{B}} \subseteq \mathcal{F}$ be the set of indicator functions (also called *boolean regions*). Note that in $\mathcal{F}_{\mathbb{B}}$, the operators $\wedge$, $\vee$, and $\leq$ correspond respectively to intersection, union, and set inclusion.

An operator F: $\mathcal{F} \mapsto \mathcal{F}$ is *monotonic* if for all regions $f, g \in \mathcal{F}$, if $f \leq g$, then $F(f) \leq F(g)$. The operator F is *Lipschitz continuous* if for all regions $f, g \in \mathcal{F}$, we have $|F(f) - F(g)| \leq |f - g|$, where $|\cdot|$ is the L_∞ norm. Note that Lipschitz continuity implies continuity: for all infinite increasing sequences $f_1 \leq f_2 \leq \cdots$ of regions in $\mathcal{F}$, we have $\lim_{n\to\infty} F(f_n) = F(\lim_{n\to\infty} f_n)$. The operator F is *contractive* if there exists a constant $0 < c < 1$ such that for all regions $f, g \in \mathcal{F}$, we have $|F(f) - F(g)| \leq c \cdot |f - g|$. For $i \in \{1, 2\}$, we consider so-called *pre operators* Pre_i: $\mathcal{F} \mapsto \mathcal{F}$ with the following properties: (1) Pre_1 and Pre_2 are monotonic and Lipschitz continuous, and (2) for all regions $f \in \mathcal{F}$, we have $\mathrm{Pre}_1(f) = \mathbf{1} - \mathrm{Pre}_2(\mathbf{1} - f)$; that is, the operators Pre_1 and Pre_2 are dual. The following pre operators have natural interpretations on (subclasses of) concurrent game structures. The *quantitative pre operator* [9] $\mathrm{Qpre}_1\colon \mathcal{F} \mapsto \mathcal{F}$ is defined for every quantitative region $f \in \mathcal{F}$ and state $s \in Q$ by

$$\mathrm{Qpre}_1(f)(s) \;=\; \sup_{\pi_1 \in \Pi_1} \inf_{\pi_2 \in \Pi_2} \mathrm{E}_s^{\pi_1,\pi_2}\{v_{\bigcirc f}\},$$

where $v_{\bigcirc f}\colon \Sigma \mapsto [0,1]$ is the payoff function that maps every path $s_0, s_1, \ldots$ in Σ to the value $f(s_1)$ of f at the second state of the path. In words, $\mathrm{Qpre}_1(f)(s)$ is the maximal expectation for the value of f that player 1 can achieve in a successor state of s. The value $\mathrm{Qpre}_1(f)(s)$ can be computed by solving a matrix game:

$$\mathrm{Qpre}_1(f)(s) \;=\; \mathrm{val}_1\Big[\sum_{t \in Q} f(t) \cdot \delta(t \mid s, \gamma_1, \gamma_2)\Big]_{\gamma_1 \in \Gamma_1(s), \gamma_2 \in \Gamma_2(s)}$$

where $\mathrm{val}_1[\cdot]$ denotes the player 1 value (i.e., maximal expected reward for player 1) of a matrix game. The minmax theorem guarantees that this matrix game has optimal strategies for both players [35]. The matrix game can be solved by linear programming [9,26]. The player 2 operator Qpre_2 is defined symmetrically. The minmax theorem permits the exchange of the sup and inf in the definition, and thus ensures the duality of the two pre operators.

By specializing the quantitative pre operators Qpre_i to turn-based deterministic game structures, we obtain the *controllable pre operators* [2] Cpre_i: $\mathcal{F}_{\mathbb{B}} \mapsto \mathcal{F}_{\mathbb{B}}$, which are closed on boolean regions. In particular, for every boolean region $f \in \mathcal{F}_{\mathbb{B}}$ and state $s \in Q$, $\mathrm{Cpre}_1(f)(s) = 1$ iff $\exists \gamma_1 \in \Gamma_1(s).\, \forall \gamma_2 \in \Gamma_2(s).\, f(\delta(s, \gamma_1, \gamma_2)) = 1$. In words, for a set $T \subseteq Q$ of states, $\mathrm{Cpre}_1(T)$ is the set of states from which player 1 can ensure that the next state lies in T. For one-player game structures, this characterization further simplifies to $\mathrm{Epre}_1(f)(s) = 1$ iff $\exists \gamma_1 \in \Gamma_1(s).\, f(\delta(s, \gamma_1, \cdot)) = 1$. This is the traditional definition of the existential pre operator on a transition system: for a set $T \subseteq Q$ of states, $\mathrm{Epre}_1(T)$ is the set of predecessor states.

Discounted μ-calculus. We define a fixpoint calculus that permits the iteration of pre operators. The calculus is discounted, in that every occurrence of a pre operator is multiplied by a *discount factor* from [0,1]. If the discount factor of a pre operator is less than 1, this has the effect that each additional application of the operator in a fixpoint iteration carries less weight. We use a fixed set Θ of *propositions*; every proposition $T \in \Theta$ denotes a boolean region $[\![T]\!] \in \mathcal{F}_{\mathbb{B}}$. For a state $s \in Q$ with $[\![T]\!](s) = 1$, we write $s \models T$ and say that s is a *T-state.* The formulas of the *discounted μ-calculus* are generated by the grammar

$$\begin{aligned}\phi \;::=\;& T \mid \neg T \mid x \mid \phi \vee \phi \mid \phi \wedge \phi \mid \alpha \cdot \mathrm{pre}_1(\phi) \mid \alpha \cdot \mathrm{pre}_2(\phi) \mid \\ & (1-\alpha) + \alpha \cdot \mathrm{pre}_1(\phi) \mid (1-\alpha) + \alpha \cdot \mathrm{pre}_2(\phi) \mid \mu x.\, \phi \mid \nu x.\, \phi\end{aligned}$$

for propositions $T \in \Theta$, variables x from some fixed set X, and parameters α from some fixed set Λ. The syntax defines formulas in positive normal form. The definition of negation in the calculus, which is given below, makes it clear that we need two discounted pre modalities, $\alpha \cdot \mathrm{pre}_i(\cdot)$ and $(1-\alpha) + \alpha \cdot \mathrm{pre}_i(\cdot)$, for each player $i \in \{1,2\}$. A formula ϕ is *closed* if every variable x in ϕ occurs in the scope of a least-fixpoint quantifier μx or greatest-fixpoint quantifier νx.

A variable valuation $\mathcal{E}\colon X \mapsto \mathcal{F}$ is a function that maps every variable $x \in X$ to a quantitative region in $\mathcal{F}$. We write $\mathcal{E}[x \mapsto f]$ for the function that agrees with $\mathcal{E}$ on all variables, except that x is mapped to f. A formula may contain several different discount factors. A parameter valuation $\mathcal{P}\colon \Lambda \mapsto [0,1]$ is a function that maps every parameter $\alpha \in \Lambda$ to a real-valued discount factor in the interval $[0,1]$. Given a real $r \in [0,1]$, the parameter valuation $\mathcal{P}$ is *r-bounded* if $\mathcal{P}(\alpha) \le r$ for all parameters $\alpha \in \Lambda$. An *interpretation* is a pair that consists of a variable valuation and a parameter valuation. Given an interpretation $(\mathcal{E}, \mathcal{P})$, every formula ϕ of the discounted μ-calculus defines a quantitative region $[\![\phi]\!]^{\mathcal{G}}_{\mathcal{E},\mathcal{P}} \in \mathcal{F}$ (the superscript $\mathcal{G}$ is omitted if the game structure is clear from the context):

$$\begin{aligned}
[\![T]\!]_{\mathcal{E},\mathcal{P}} &= [\![T]\!] \\
[\![\neg T]\!]_{\mathcal{E},\mathcal{P}} &= \mathbf{1} - [\![T]\!] \\
[\![x]\!]_{\mathcal{E},\mathcal{P}} &= \mathcal{E}(x) \\
[\![\alpha \cdot \mathrm{pre}_i(\phi)]\!]_{\mathcal{E},\mathcal{P}} &= \mathcal{P}(\alpha) \cdot \mathrm{Qpre}_i([\![\phi]\!]_{\mathcal{E},\mathcal{P}}) \\
[\![(1-\alpha) + \alpha \cdot \mathrm{pre}_i(\phi)]\!]_{\mathcal{E},\mathcal{P}} &= (1-\mathcal{P}(\alpha)) + \mathcal{P}(\alpha) \cdot \mathrm{Qpre}_i([\![\phi]\!]_{\mathcal{E},\mathcal{P}})
\end{aligned}$$

$$\begin{aligned}
[\![\phi_1 \left\{\begin{smallmatrix}\vee\\ \wedge\end{smallmatrix}\right\} \phi_2]\!]_{\mathcal{E},\mathcal{P}} &= [\![\phi_1]\!]_{\mathcal{E},\mathcal{P}} \left\{\begin{smallmatrix}\vee\\ \wedge\end{smallmatrix}\right\} [\![\phi_2]\!]_{\mathcal{E},\mathcal{P}} \\
[\![\left\{\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right\} x.\, \phi]\!]_{\mathcal{E},\mathcal{P}} &= \left\{\begin{smallmatrix}\inf\\ \sup\end{smallmatrix}\right\} \{f \in \mathcal{F} \mid f = [\![\phi]\!]_{\mathcal{E}[x \mapsto f],\mathcal{P}}\}
\end{aligned}$$

The existence of the required fixpoints is guaranteed by the monotonicity and continuity of all operators. The region $[\![\phi]\!]_{\mathcal{E},\mathcal{P}}$ is in general not boolean even if the game structure is turn-based deterministic, because the discount factors introduce real numbers. The discounted μ-calculus is closed under negation: if we define the negation of a formula ϕ inductively using $\neg(\alpha \cdot \mathrm{pre}_1(\phi')) = (1-\alpha) + \alpha \cdot \mathrm{pre}_2(\neg\phi')$ and $\neg((1-\alpha) + \alpha \cdot \mathrm{pre}_1(\phi')) = \alpha \cdot \mathrm{pre}_2(\neg\phi')$, then $[\![\neg\phi]\!]_{\mathcal{E},\mathcal{P}} = \mathbf{1} - [\![\phi]\!]_{\mathcal{E},\mathcal{P}}$. This generalizes the duality $\mathbf{1} - \mathrm{Qpre}_1(f) = \mathrm{Qpre}_2(\mathbf{1} - f)$ of the undiscounted pre operators.

A parameter valuation $\mathcal{P}$ is *contractive* if $\mathcal{P}$ maps every parameter to a discount factor strictly less than 1. A fixpoint quantifier μx or νx occurs *syntactically contractive* in a formula ϕ if a pre modality occurs on every syntactic path from the quantifier to a quantified occurrence of the variable x. For example, in the formula $\mu x.\,(T \vee \alpha \cdot \mathrm{pre}_i(x))$ the fixpoint quantifier occurs syntactically contractive; in the formula $(1-\alpha) + \alpha \cdot \mathrm{pre}_i(\mu x.\,(T \vee x))$ it does not. Under a contractive parameter valuation, every syntactically contractive occurrence of a fixpoint quantifier defines a contractive operator on the values of the free variables that are in the scope of the quantifier. Hence, by the Banach fixpoint theorem, the fixpoint is unique. In such cases, since there are unique fixpoints, we need not distinguish between μ and ν quantifiers, and we use a single (self-dual) fixpoint quantifier λ. Fixpoints can be computed by Picard iteration: $[\![\mu x.\, \phi]\!]_{\mathcal{E},\mathcal{P}} = \lim_{k\to\infty} f_k$ where $f_0 = \mathbf{0}$, and $f_{k+1} = [\![\phi]\!]_{\mathcal{E}[x\mapsto f_k],\mathcal{P}}$ for all $k \geq 0$; and $[\![\nu x.\, \phi]\!]_{\mathcal{E},\mathcal{P}} = \lim_{k\to\infty} f_k$ where $f_0 = \mathbf{1}$, and f_{k+1} is defined as in the μ case. If the fixpoint is unique, then both sequences converge to the same region $[\![\lambda x.\, \phi]\!]_{\mathcal{E},\mathcal{P}}$, one from below, and the other from above.

Approximating the undiscounted semantics. If $\mathcal{P}(\alpha) = 1$, then both discounted pre modalities $\alpha \cdot \mathrm{pre}_i(\cdot)$ and $(1-\alpha) + \alpha \cdot \mathrm{pre}_i(\cdot)$ collapse, and are interpreted as the quantitative pre operator $\mathrm{Qpre}_i(\cdot)$, for $i \in \{1, 2\}$. In this case, we may omit the parameter α from formulas, writing instead the undiscounted modality $\mathrm{pre}_i(\cdot)$. The *undiscounted semantics* of a formula ϕ is the quantitative region $[\![\phi]\!]_{\mathcal{E},\mathit{1}}$ obtained from the parameter valuation $\mathit{1}$ that maps every parameter in Λ to 1. The undiscounted semantics coincides with the quantitative μ-calculus of [9,23]. In the case of turn-based deterministic game structures, it coincides with the alternating-time μ-calculus of [2], and in the case of transition systems, with the classical μ-calculus of [19]. The following theorem justifies discounting as an approximation theory: the undiscounted semantics of a formula can be obtained as the limit of the discounted semantics as all discount factors tend to 1.[1]

Theorem 1. *Let $\phi(x)$ be a formula of the discounted μ-calculus with a free variable x and parameter α, which always occur in the context $\alpha \cdot \mathrm{pre}_i(x)$, for $i \in \{1, 2\}$. Then*

$$\lim_{a\to 1} [\![\lambda x.\, \phi(\alpha \cdot \mathrm{pre}_i(x))]\!]_{\mathcal{E},\mathcal{P}[\alpha\mapsto a]} \;=\; [\![\mu x.\, \phi(\mathrm{pre}_i(x))]\!]_{\mathcal{E},\mathcal{P}}.$$

Furthermore, if x and α always occur in the context $(1-\alpha) + \alpha \cdot \mathrm{pre}_i(x)$, then

$$\lim_{a\to 1} [\![\lambda x.\, \phi((1-\alpha) + \alpha \cdot \mathrm{pre}_i(x))]\!]_{\mathcal{E},\mathcal{P}[\alpha\mapsto a]} \;=\; [\![\nu x.\, \phi(\mathrm{pre}_i(x))]\!]_{\mathcal{E},\mathcal{P}}.$$

Note that the assumption of the theorem ensures that the fixpoint quantifiers on x occur syntactically contractive on the discounted left-hand side, and therefore define unique fixpoints. Depending on the form of the discounted pre modality, the unique discounted fixpoints approximate either the least or the greatest undiscounted fixpoint. This implies that, in general, limits of discount factors are not interchangeable. Consider the formula

$$\varphi \;=\; \lambda y.\, \lambda x.\, ((\neg T \wedge \beta \cdot \mathrm{pre}_1(x)) \vee (T \wedge ((1-\alpha) + \alpha \cdot \mathrm{pre}_1(y)))).$$

Then $\lim_{\alpha\to 1} \lim_{\beta\to 1} \varphi$ is equivalent to $\nu y.\, \mu x.\, ((\neg T \wedge \mathrm{pre}_1(x)) \vee (T \wedge \mathrm{pre}_1(y)))$, which characterizes, in the turn-based deterministic case, the player 1 winning

[1] It may be noted that Picard iteration itself offers an approximation theory for fixpoint calculi: the longer the iteration sequence, the closer the approximation of the fixpoint. This approximation scheme, however, is neither syntactically robust nor compositional, because it is not closed under the unrolling of fixpoint quantifiers. By contrast, for every discounted μ-calculus formula $\kappa x.\, \phi(x)$, where $\kappa \in \{\mu, \nu\}$, we have $[\![\kappa x.\, \phi(x)]\!]_{\mathcal{E},\mathcal{P}} = [\![\phi(\kappa x.\, \phi(x))]\!]_{\mathcal{E},\mathcal{P}}$.

states of a Büchi game (infinitely many T-states must be visited). The inner (β) limit ensures that a T-state will be visited; the outer (α) limit ensures that this remains always the case. On the other hand, $\lim_{\beta \to 1} \lim_{\alpha \to 1} \varphi$ is equivalent to $\mu x. \nu y. ((\neg T \wedge \mathrm{pre}_1(x)) \vee (T \wedge \mathrm{pre}_1(y)))$, which characterizes the player 1 winning states of a coBüchi game (eventually only T-states must be visited). This is because the inner (α) limit ensures that only T-states are visited, and the outer (β) limit ensures that this will happen.

4 Properties: Discounted ω-Regular Winning Conditions

In the classical setting, the ω-regular languages can be used to specify system properties (or winning conditions of games), while the μ-calculus provides algorithms for verifying the properties (or computing the winning states). In our discounted approach, the discounted μ-calculus provides the algorithms; what, then, are the properties? We establish a connection between the semantics of a discounted fixpoint expression over a concurrent game structure, and the semantics of the same expression over the paths of the structure. This provides a trace semantics for the discounted μ-calculus, thus giving rise to a notion of "discounted ω-regular properties."

Reachability and safety conditions. A *discounted reachability game* consists of a concurrent game structure $\mathcal{G}$ (with state space Q) together with a winning condition $\diamond_a T$, where $T \in \Theta$ is a proposition and $a \in [0,1]$ is a discount factor. Starting from a state $s \in Q$, player 1 has the objective to reach a T-state as quickly as possible, while player 2 tries to prevent this. The reward for player 1 is a^k if a T-state is visited for the first time after k moves, and 0 if no T-state is ever visited. Formally, we define the payoff function $v^a_{\diamond T} \colon \Sigma \mapsto [0,1]$ on paths by $v^a_{\diamond T}(s_0, s_1, \ldots) = a^k$ for $k = \min\{i \mid s_i \models T\}$, and $v^a_{\diamond T}(s_0, s_1, \ldots) = 0$ if $s_k \not\models T$ for all $k \geq 0$. Then, for every state $s \in Q$, the *value* of the discounted reachability game at s is $(\langle\!\langle 1 \rangle\!\rangle \diamond_a T)(s) = \sup_{\pi_1 \in \Pi_1} \inf_{\pi_2 \in \Pi_2} \mathrm{E}_s^{\pi_1, \pi_2}\{v^a_{\diamond T}\}$. This defines a discounted stochastic game [31]. For $a = 1$, the value can be computed as a least fixpoint; for $a < 1$, as the unique fixpoint

$$\langle\!\langle 1 \rangle\!\rangle \diamond_a T \;=\; [\![\lambda x. (T \vee \alpha \cdot \mathrm{pre}_1(x))]\!]_{\cdot, [\alpha \mapsto a]}.$$

Picard iteration yields $\langle\!\langle 1 \rangle\!\rangle \diamond_a T = \lim_{k \to \infty} f_k$ where $f_0 = \mathbf{0}$, and $f_{k+1} = (T \vee a \cdot \mathrm{Qpre}_1(f_k))$ for all $k \geq 0$. This gives an approximation scheme from below to solve the discounted reachability game. The sequence converges geometrically in $a < 1$; more precisely, $(\langle\!\langle 1 \rangle\!\rangle \diamond_a T)(s) - f_k(s) \leq a^k$ for all states $s \in Q$ and all $k \geq 0$. This permits the approximation of the value of the game for any desired precision. Furthermore, as the fixpoint is unique, an approximation scheme from above, which starts with $f_0 = \mathbf{1}$, also converges geometrically. For turn-based deterministic game structures, the value of the discounted reachability game $\langle\!\langle 1 \rangle\!\rangle \diamond_a T$ at state s is a^k, where k is the length of the shortest path that player 1 can enforce to reach a T-state, if such a path exists (in the case of one-player structures, k is the length of the shortest path from s to a T-state). For general game structures and $a = 1$, the value $\langle\langle 1\rangle\rangle \diamond_1 T$ at s is the maximal probability with which player 1 can achieve to reach a T-state [9]. A strategy π_1 for player 1 is *optimal* (resp., ϵ-*optimal* for $\epsilon > 0$) for the reachability condition $\diamond_a T$ if for all states $s \in Q$, we have $\inf_{\pi_2 \in \Pi_2} \mathrm{E}_s^{\pi_1,\pi_2}\{v^a_{\diamond T}\} = (\langle\langle 1\rangle\rangle \diamond_a T)(s)$ (resp., $\inf_{\pi_2 \in \Pi_2} \mathrm{E}_s^{\pi_1,\pi_2}\{v^a_{\diamond T}\} \geq (\langle\langle 1\rangle\rangle \diamond_a T)(s) - \epsilon$). While undiscounted ($a = 1$) reachability games admit only ϵ-optimal strategies [16], discounted reachability games have optimal memoryless strategies for both players [16,31].

The dual of reachability is safety. A *discounted safety game* consists of a concurrent game structure $\mathcal{G}$ together with a winning condition $\Box_a T$, where $T \in \Theta$ and $a \in [0, 1]$. Starting from a state $s \in Q$, player 1 has the objective to stay within the set of T-states for as long as possible. The payoff function $v^a_{\Box T}$: $\Sigma \mapsto [0, 1]$ is defined by $v^a_{\Box T}(s_0, s_1, \dots) = 1 - a^k$ for $k = \min\{i \mid s_i \not\models T\}$, and $v^a_{\Box T}(s_0, s_1, \dots) = 1$ if $s_k \models T$ for all $k \geq 0$. For every state $s \in Q$, the *value* of the discounted safety game at s is $(\langle\langle 1\rangle\rangle \Box_a T)(s) = \sup_{\pi_1 \in \Pi_1} \inf_{\pi_2 \in \Pi_2} \mathrm{E}_s^{\pi_1,\pi_2}\{v^a_{\Box T}\}$. For $a = 1$, the value can be computed as a greatest fixpoint; for $a < 1$, as the unique fixpoint

$$\langle\langle 1\rangle\rangle \Box_a T \;=\; [\![\lambda x.\, (T \wedge ((1 - \alpha) + \alpha \cdot \mathrm{pre}_1(x))]\!]_{\cdot,[\alpha \mapsto a]}.$$

For $a < 1$, the Picard iteration $\langle\langle 1\rangle\rangle \Box_a T = \lim_{k\to\infty} f_k$ where $f_0 = \mathbf{0}$, and $f_{k+1} = (T \wedge a \cdot \mathrm{Qpre}_1(f_k))$ for all $k \geq 0$, converges geometrically from below, and with $f_0 = \mathbf{1}$, it converges geometrically from above. For turn-based deterministic game structures and $a < 1$, the value $\langle\langle 1\rangle\rangle \Box_a T$ at state s is $1 - a^k$, where k is the length of the longest path that player 1 can enforce to stay in T-states. For general game structures and $a = 1$, it is the maximal probability with which player 1 can achieve to stay in T-states forever [9].

In summary, the mathematical appeal of discounting reachability and safety, in addition to the practical appeal of emphasis on the near future, is threefold: (1) geometric convergence from both below and above (no theorems on the rate of convergence are known for $a = 1$); (2) the existence of optimal memoryless strategies (only ϵ-optimal strategies may exist for undiscounted reachability games); (3) the continuous approximation property (Theorem 1), which shows that for $a \to 1$, the values of discounted reachability and safety games converge to the values of the corresponding undiscounted games.

Trace semantics of fixpoint expressions. Reachability and safety properties are simple, and offer a natural discounted interpretation. For more general ω-regular properties, however, there are often multiple candidates for a discounted interpretation, as there are multiple algorithms for evaluating the property. Consider, for example, Büchi games. An undiscounted Büchi game consists of a concurrent game structure $\mathcal{G}$ together with a winning condition $\Box\diamond T$, where $T \in \Theta$ specifies a set of Büchi states, which player 1 tries to visit infinitely often. The value of the game at a state s, denoted $(\langle\langle 1\rangle\rangle \Box\diamond T)(s)$, is the maximal probability with which player 1 can enforce that a T-state is visited infinitely often. The value of an undiscounted Büchi game can be characterized as [9]

$$\langle\!\langle 1 \rangle\!\rangle \Box \Diamond T \;=\; \nu y.\, \mu x.\, ((\neg T \wedge \mathrm{pre}_1(x)) \vee (T \wedge \mathrm{pre}_1(y))).$$

This fixpoint expression suggests several alternative ways of discounting the Büchi game. For example, one may require that the distances between the infinitely many visits to T-states are as small as possible, obtaining $\nu y.\, \lambda x.\, ((\neg T \wedge \alpha \cdot \mathrm{pre}_1(x)) \vee (T \wedge \mathrm{pre}_1(y)))$. Alternatively, one may require that the number of visits to T-states is as large as possible, but arbitrarily spaced, obtaining $\lambda y.\, \mu x.\, ((\neg T \wedge \mathrm{pre}_1(x)) \vee (T \wedge ((1-\beta) + \beta \cdot \mathrm{pre}_1(y))))$. More generally, we can use both discount factors α and β, as in $\lambda y.\, \lambda x.\, ((\neg T \wedge \alpha \cdot \mathrm{pre}_1(x)) \vee (T \wedge ((1-\beta) + \beta \cdot \mathrm{pre}_1(y))))$, and study the effect of various relationships, such as $\alpha < \beta$, $\alpha = \beta$, and $\alpha > \beta$. All these discounted interpretations of Büchi games have two key properties: (1) the value of the game can be computed by algorithms that converge geometrically; and (2) if all discount factors tend to one, then the value of the discounted game tends to the value of the undiscounted game. So instead of defining a discounted Büchi (or more general ω-regular) winning condition, chosen arbitrarily from the alternatives, we take a discounted μ-calculus formula itself as specification of the game and show that, under each interpretation, the formula naturally induces a discounted property of paths.

We first define the semantics of a formula on a path. Consider a concurrent game structure $\mathcal{G}$. Every path $\sigma = s_0, s_1, \ldots$ of $\mathcal{G}$ induces an infinite-state[2] game structure in a natural way: the set of states is $\{(k, s_k) \mid k \geq 0\}$, and at each state (k, s_k), both players have exactly one move available, whose combination takes the game deterministically to the successor state $(k+1, s_{k+1})$, for all $k \geq 0$. With abuse of notation, we write σ also for the game structure that is induced by the path σ. For this structure and $i \in \{1,2\}$, $\mathrm{Qpre}_i(\{(k+1, s_{k+1})\})$ is the function that maps (k, s_k) to 1 and all other states to 0. For a closed discounted μ-calculus formula ϕ and parameter valuation $\mathcal{P}$, we define the *trace semantics* of ϕ under $\mathcal{P}$ to be the payoff function $[\phi]_{\mathcal{P}} \colon \Sigma \mapsto [0,1]$ that maps every path $\sigma \in \Sigma$ to the value $[\![\phi]\!]^{\sigma}_{\cdot,\mathcal{P}}(s_0)$, where s_0 is the first state of the path σ (the superscript σ indicates that the formula is evaluated over the game structure induced by σ).

The Cantor metric d_C is defined on the set Σ of paths by $d_C(\sigma_1, \sigma_2) = \frac{1}{2^k}$, where k is the length of the maximal prefix that is common to the two paths σ_1 and σ_2. The following theorem shows that for discount factors strictly less than 1, the trace semantics of every discounted μ-calculus formula is a continuous function from this metric space to the interval $[0,1]$. This is in contrast to undiscounted ω-regular properties, which can distinguish between paths that are arbitrarily close.

Theorem 2. *Let ϕ be a closed discounted μ-calculus formula, and let $\mathcal{P}$ be a contractive parameter valuation. For every $\epsilon > 0$, there is a $\delta > 0$ such that for all paths $\sigma_1, \sigma_2 \in \Sigma$ with $d_C(\sigma_1, \sigma_2) < \delta$, we have $|[\phi]_{\mathcal{P}}(\sigma_1) - [\phi]_{\mathcal{P}}(\sigma_2)| < \epsilon$.*

A formula ϕ of the discounted μ-calculus is *player-1 strongly guarded* [8] if (1) ϕ is closed and consists of a string of fixpoint quantifiers followed by a

[2] The infiniteness is harmless, because we do not compute in this structure.

quantifier-free part, (2) ϕ contains no occurrences of pre_2, and (3) every conjunction in ϕ has at least one constant argument; that is, every conjunctive subformula of ϕ has the form $T \wedge \phi'$, where T is a boolean combination of propositions. In the classical μ-calculus, all ω-regular winning conditions of turn-based deterministic games can be expressed by strongly guarded (e.g., Rabin chain) formulas [13]. For player-1 strongly guarded formulas ϕ the following theorem gives the correspondence between the semantics of ϕ on structures and the semantics of ϕ on paths: the value of ϕ at a state s under parameter valuation $\mathcal{P}$ is the value of the game with start state s and payoff function $[\phi]_{\mathcal{P}}$.

Theorem 3. *Let $\mathcal{G}$ be a concurrent game structure, let ϕ be a player-1 strongly guarded formula of the discounted μ-calculus, and let $\mathcal{P}$ be a parameter valuation. For every state s of $\mathcal{G}$, we have $[\![\phi]\!]^{\mathcal{G}}_{\cdot,\mathcal{P}}(s) \;=\; \sup_{\pi_1 \in \Pi_1} \inf_{\pi_2 \in \Pi_2} \mathrm{E}_s^{\pi_1,\pi_2}\{[\phi]_{\mathcal{P}}\}$.*

Rabin chain conditions. An undiscounted Rabin chain game [13,25] consists of a concurrent game structure $\mathcal{G}$ together with a winning condition $\bigvee_{i=0}^{n-1}(\Box\Diamond T_{2i} \wedge \neg\Box\Diamond T_{2i+1})$, where $n > 0$ and the T_j's are propositions with $\emptyset \subseteq [\![T_{2n}]\!] \subseteq [\![T_{2n-1}]\!] \subseteq \cdots \subseteq [\![T_0]\!] = Q$. A more intuitive characterization of this winning condition can be obtained by defining, for all $0 \le j \le 2n-1$, the set $C_j \subseteq Q$ of states of *color* j by $C_j = [\![T_j]\!] \setminus [\![T_{j+1}]\!]$. For a path $\sigma \in \Sigma$, let $MaxCol(\sigma)$ be the maximal j such that a state in C_j occurs infinitely often in σ. The winning condition for player 1 is that $MaxCol(\sigma)$ is even. The ability to solve games with Rabin chain conditions suffices for solving games with arbitrary ω-regular winning conditions, because every ω-regular property can be translated into a deterministic Rabin chain automaton [25,32].

As in the Büchi case, there are many ways to discount a Rabin chain game, so we use the corresponding fixpoint expression to explore various tradeoffs. Accordingly, for discount factors $a_0, \ldots, a_{2n-1} < 1$, we define the value of an $(a_0, \ldots, a_{2n-1})$-*discounted Rabin chain game* by

$$R(a_0, \ldots, a_{2n-1}) \;=\; [\![\lambda x_{2n-1} \ldots \lambda x_0 . \bigvee_{0 \le j < 2n} (C_j \wedge \mathrm{Rpre}(x_j))]\!]_{\cdot,\{\alpha_j \mapsto a_j \mid 0 \le j < 2n\}},$$

where $\mathrm{Rpre}(x_j) = \alpha_j \cdot \mathrm{pre}_1(x_j)$ if j is odd, and $\mathrm{Rpre}(x_j) = (1-\alpha_j) + \alpha_j \cdot \mathrm{pre}_1(x_j)$ if j is even. Note that the fixpoint expression is player-1 strongly guarded. The value $R(a_0, \ldots, a_{2n-1})$ of the discounted Rabin chain game can be approximated monotonically by Picard iteration from below and above. Moreover, if the j-th fixpoint is computed for k_j steps, we can bound the cumulative error of the process. Let ε_j be the error in the value of the j-th fixpoint; then $\varepsilon_0 \le a_0^{k_0}$, and $\varepsilon_j \le \frac{\varepsilon_{j-1}}{1-a_j} + a_j^{k_j}$ for all $1 \le j \le 2n-1$.

Theorem 4. *For a vector $(k_0, \ldots, k_{2n-1})$ of integers, let $R^{\perp}_{k_0,\ldots,k_{2n-1}}$ be the region obtained by approximating from below the j-th fixpoint of the discounted μ-calculus formula for $R(a_0, \ldots, a_{2n-1})$ for k_j iterations, and let $R^{\top}_{k_0,\ldots,k_{2n-1}}$ be the region obtained by approximating from above. If*

$a_0, \ldots, a_{2n-1} < 1$, *then for each state* s, *we have* $R(a_0, \ldots, a_{2n-1})(s) - \varepsilon_{2n-1} \leq R^{\perp}_{k_0,\ldots,k_{2n-1}}(s) \leq R^{\top}_{k_0,\ldots,k_{2n-1}}(s) \leq R(a_0, \ldots, a_{2n-1})(s) + \varepsilon_{2n-1}$. *Moreover, if* R *is the value of the corresponding undiscounted Rabin chain game, then* $\lim_{a_{2n-1}\to 1} \ldots \lim_{a_0 \to 1} R(a_0, \ldots, a_{2n-1}) = R$.

5 State Equivalences: Discounted Bisimilarity

Consider a concurrent game structure $\mathcal{G} = \langle Q, M, \Gamma_1, \Gamma_2, \delta \rangle$. A *distance function* d: $Q^2 \mapsto [0,1]$ is a pseudo-metric on the states with the range $[0,1]$. Distance functions provide a quantitative generalization for equivalence relations on states: distance 0 means "equivalent" in the boolean sense, and distance 1 means "different" in the boolean sense. For two distance functions d_1 and d_2, we write $d_1 \leq d_2$ if $d_1(s,t) \leq d_2(s,t)$ for all states $s, t \in Q$. Given a discount factor $a \in [0,1]$, we define the functor F_a mapping distance functions to distance functions: for every distance function d and all states $s, t \in Q$, we define $F_a(d)(s,t) = 1$ if there is a proposition $T \in \Theta$ such that $[\![T]\!](s) \neq [\![T]\!](t)$, and

$$F_a(d)(s,t) \;=\; a \cdot \max \left\{ \begin{array}{l} \sup\limits_{\xi_1 \in \mathcal{D}_1(s)} \inf\limits_{\hat{\xi}_1 \in \mathcal{D}_1(t)} \sup\limits_{\hat{\xi}_2 \in \mathcal{D}_2(t)} \inf\limits_{\xi_2 \in \mathcal{D}_2(s)} \mathrm{E}^{s':\xi_1 \xi_2, t':\hat{\xi}_1, \hat{\xi}_2}_{s,t} \{d_a(s',t')\}, \\ \sup\limits_{\hat{\xi}_1 \in \mathcal{D}_1(t)} \inf\limits_{\xi_1 \in \mathcal{D}_1(s)} \sup\limits_{\xi_2 \in \mathcal{D}_2(s)} \inf\limits_{\hat{\xi}_2 \in \mathcal{D}_2(t)} \mathrm{E}^{s':\xi_1 \xi_2, t':\hat{\xi}_1, \hat{\xi}_2}_{s,t} \{d_a(s',t')\} \end{array} \right\}$$

otherwise. In the above formula, $\mathcal{D}_i(u) = \mathcal{D}(\Gamma_i(u))$ is the set of probability distributions over the moves of player $i \in \{1,2\}$ at the state $u \in Q$. By $\mathrm{E}^{s':\xi_1 \xi_2, t':\hat{\xi}_1, \hat{\xi}_2}_{s,t}\{d(s',t')\}$ we denote the expected value $d(s',t')$ of the distance function d when the state s' results from playing the distributions of moves ξ_1 and ξ_2 from s, and t' results from playing $\hat{\xi}_1$ and $\hat{\xi}_2$ from t. Formally,

$$\mathrm{E}^{s':\xi_1 \xi_2, t':\hat{\xi}_1, \hat{\xi}_2}_{s,t}\{d(s',t')\} = \sum_{s',t' \in Q} \sum_{\gamma_1 \in \Gamma_1(s)} \sum_{\gamma_2 \in \Gamma_2(s)} \sum_{\hat{\gamma}_1 \in \Gamma_1(t)} \sum_{\hat{\gamma}_2 \in \Gamma_2(t)} d(s',t') \cdot \delta(s' \mid s, \gamma_1, \gamma_2) \cdot \delta(t' \mid t, \hat{\gamma}_1, \hat{\gamma}_2) \cdot \xi_1(\gamma_1) \cdot \xi_2(\gamma_2) \cdot \hat{\xi}_1(\hat{\gamma}_1) \cdot \hat{\xi}_2(\hat{\gamma}_2).$$

The fixpoints of the functor F_a are called *a-discounted (game) bisimulations.* The least fixpoint of F_a is called *a-discounted (game) bisimilarity*, and denoted $B^{\mathcal{G}}_a$ (the superscript is omitted if the game structure $\mathcal{G}$ is clear from the context).[3] If $a < 1$, then F_a has a unique fixpoint; in this case, there is a unique a-discounted bisimulation, namely, B_a. If $a = 1$, instead of 1-discounted, we say *undiscounted.*

On MDPs (one-player game structures), for $a < 1$, discounted game bisimulation coincides with the discounted bisimulation of [12], and undiscounted game bisimulation coincides with the probabilistic bisimulation of [30]. On transition systems (one-player deterministic game structures), undiscounted game bisimulation coincides with classical bisimulation [24]. However, undiscounted game bisimulation is not equivalent to the alternating bisimulation of [3], which has been defined for deterministic game structures. By the minimax theorem [35],

[3] Bisimilarity is usually considered a greatest fixpoint, but in our setup, the distance function that considers all states to be equivalent in the boolean sense is the least distance function.

we can exchange the two middle sup and inf operators in the definition of F_a; that is, the roles of players 1 and 2 can be exchanged. Hence, there is only one version of (un)discounted game bisimulation, while there are distinct player 1 and player 2 alternating bisimulations. Alternating bisimulation corresponds to the case where the sets $\mathcal{D}_i(u)$, for $i \in \{1,2\}$ and $u \in Q$, consist only of deterministic distributions, where each player must choose a specific move (indeed, the minimax theorem does not hold if the players are forced to use deterministic distributions). In the case of turn-based deterministic game structures, the two definitions collapse, but for concurrent game structures the sup-inf interpretation of winning is strictly weaker than the deterministic interpretation [7].

The a-discounted bisimilarity B_a can be computed using Picard iteration: starting from $d_a^{(0)}$, with $d_a^{(0)}(s,t) = 0$ for all states $s, t \in Q$, let $d_a^{(k+1)} = F_a(d_a^{(k)})$ for all $k \geq 0$. If $a < 1$, then we may start from any distance function $d_a^{(0)}$ (because the fixpoint is unique) and the convergence is geometric with rate a. The theorem below relates discounted and undiscounted game bisimulation.

Theorem 5. *On every concurrent game structure,* $\lim_{a \to 1} B_a = B_1$*. Moreover, for two states s and t, we have $B_1(s,t) = 0$ iff $B_a(s,t) = 0$ for any and all discount factors $a > 0$, and $B_1(s,t) = 1$ iff $B_a(s,t) = 1$ for any and all $a > 0$.*

Our main theorem on discounted game bisimulation states that for all states, closeness in discounted game bisimilarity corresponds to closeness in the value of discounted μ-calculus formulas. In other words, a small perturbation of a system can only cause a small change of its properties.

Theorem 6. *Consider two states s and t of a concurrent game structure, and a discount factor $a < 1$. For all closed discounted μ-calculus formulas ϕ and a-bounded parameter valuations $\mathcal{P}$, we have $|[\![\phi]\!]_{\cdot,\mathcal{P}}(s) - [\![\phi]\!]_{\cdot,\mathcal{P}}(t)| \leq B_a(s,t)$. Also,* $\sup_\phi |[\![\phi]\!]_{\cdot,\mathcal{P}}(s) - [\![\phi]\!]_{\cdot,\mathcal{P}}(t)| = B_a(s,t)$.

Let ϵ be a nonnegative real. A game structure $\mathcal{G}' = \langle Q, M, \Gamma_1, \Gamma_2, \delta' \rangle$ is an *ϵ-perturbation* of the game structure $\mathcal{G} = \langle Q, M, \Gamma_1, \Gamma_2, \delta \rangle$ if for all states $s \in Q$, all sets $X \subseteq Q$, and all moves $\gamma_1 \in \Gamma_1(s)$ and $\gamma_2 \in \Gamma_2(s)$, we have $|\sum_{t \in X} \delta(t \mid s, \gamma_1, \gamma_2) - \sum_{t \in X} \delta'(t \mid s, \gamma_1, \gamma_2)| \leq \epsilon$. We write $B_a^{\mathcal{G} \uplus \mathcal{G}'}$ for the a-discounted bisimililarity on the disjoint union of the game structures $\mathcal{G}$ and $\mathcal{G}'$. The following theorem, which generalizes a result of [11] from one-player structures to games, shows that discounted bisimilarity is robust under perturbations.

Theorem 7. *Let $\mathcal{G}'$ be an ϵ-perturbation of a concurrent game structure $\mathcal{G}$, and let $a < 1$ be a discount factor. For every state s of $\mathcal{G}$ and corresponding state s' of $\mathcal{G}'$, we have $B_a^{\mathcal{G} \uplus \mathcal{G}'}(s,s') \leq K \cdot \epsilon$, where* $K = \sup_{k \geq 0} \{k \cdot a^k\}$.

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