THE THIRD HOMOLOGY OF SL_2 OF REAL QUADRATICALLY CLOSED FIELDS.

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ABSTRACT. For a real closed field \mathbf{R} , we use the theory of the refined Bloch group to give a new short proof of the isomorphisms $H_3(SL_2(\mathbf{R}), \mathbb{Z}) \cong K_3^{\mathrm{ind}}(\mathbf{R})$ and $H_3(SL_n(\mathbf{R}), \mathbb{Z}) \cong H_3(SL_2(\mathbf{R}), \mathbb{Z}) \oplus K_3^M(\mathbf{R})^0$ for n > 3.

1. Introduction.

In the paper Homology of classical Lie groups made discrete III ([7]) the author, Chih-Han Sah, proves the injectivity of the map $H_3(SL_2(F), \mathbb{Z}) \to H_3(SL_3(F), \mathbb{Z})$ and $H_3(SL_3(F), \mathbb{Z}) \cong H_3(SL_2(F), \mathbb{Z}) \oplus K_3^M(F)^0$ for $F = \mathbb{R}$, \mathbb{H} or F algebraically closed ([7, Theorem 3.0] and [7, Proposition 2.15] respectively). He states, without giving details, that the result is also valid for any real closed field [7, Section 3].

In the present article we use the theory of the refined Bloch group ([2]) to give a new proof of this result for real closed fields \mathbf{R} , and we deduce the structure of $H_3(SL_n(\mathbf{R}), \mathbb{Z})$, $n \geq 3$, in terms of K-theory. In fact, all the results are valid for the larger class of real quadratically closed fields (See section 2 below).

1.1. **Some notation.** For a field F, we let F^{\times} denote the group of units of F. For $x \in F^{\times}$ we will let $\langle x \rangle \in F^{\times}/(F^{\times})^2$ denote the corresponding square class. Let R_F denote the integral group ring $\mathbb{Z}\left[F^{\times}/(F^{\times})^2\right]$ of the group $F^{\times}/(F^{\times})^2$. We will use the notation $\langle\langle x \rangle\rangle$ for the basis element, $\langle x \rangle - 1$, of the augmentation ideal \mathcal{I}_F of R_F .

2. Real quadratically closed fields.

Recall that an ordered field (\mathbf{R}, \geq) is real closed if it satisfies the following two properties:

- (1) Any positive element x > 0 in **R** has a square root in **R**; i.e. there exist $a \in \mathbf{R} \setminus \{0\}$ such that $x = a^2$.
- (2) Any odd degree polynomial with coefficients in \mathbf{R} has a root in \mathbf{R} .

In fact, our main results will apply to any ordered field satisfying property (1). For convenience, in this paper we will refer to such fields as real quadratically closed fields

Example 2.1. The field of (straightedge-and-compass) constructible real numbers is real quadratically closed but not real closed.

We will frequently use the following easily verified fact

Lemma 2.2. Let (\mathbf{R}, \geq) be an ordered field and $\mathbf{R}_{>0}$ denote the group of positive elements in \mathbf{R} under the multiplication of \mathbf{R} .

- (i) $\mu_{\mathbf{R}} = \mu_2$ and $\mathbf{R}^{\times} = \mu_2 \oplus \mathbf{R}_{>0}$.
- (ii) If **R** is real quadratically closed then $\mathbf{R}_{>0} = (\mathbf{R}_{>0})^2$

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Since $\mathbf{R}_{>0} = (\mathbf{R}_{>0})^2$, it follows that $\mathbf{R}^{\times}/(\mathbf{R}^{\times})^2 = \mu_2$ and hence $R_{\mathbf{R}} = \mathbb{Z}[\mu_2] = \mathbb{Z}[\langle -1 \rangle]$. Furthermore we will need the following observation:

Corollary 2.3. If **R** is real quadratically closed then $\bigwedge_{\mathbb{Z}}^2(\mathbf{R}^{\times}) \cong \bigwedge_{\mathbb{Z}}^2(\mathbf{R}_{>0})$ which is 2-divisible

3. Review of Bloch groups of fields.

In this section we will recall the definition and applications of the classical pre-Bloch group $\mathcal{P}(F)$ and the refined pre-Bloch group $\mathcal{RP}(F)$.

3.1. Classical Bloch Group $\mathcal{B}(F)$. For a field F, with at least 4 elements, the *pre-Bloch group* or *Scissors congruence group*, $\mathcal{P}(F)$, is the group generated by the elements [x], with $F^{\times} \setminus \{1\}$, subject to the relations

$$R_{x,y}: 0 = [x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)], \quad x \neq y.$$

Let $S_{\mathbb{Z}}^2(F^{\times})$ denote the group

$$\frac{F^{\times} \otimes_{\mathbb{Z}} F^{\times}}{\langle x \otimes y + y \otimes x \mid x, y \in F^{\times} \rangle}$$

and denote by $x \circ y$ the image of $x \otimes y$ in $S^2_{\mathbb{Z}}(F^{\times})$. The map

$$\lambda: \mathcal{P}(F) \to S_{\mathbb{Z}}^2(F^{\times}), \quad [x] \mapsto (1-x) \circ x$$

is well-defined, and the Bloch group of F, $\mathcal{B}(F) \subset \mathcal{P}(F)$, is defined to be the kernel of λ .

The Bloch group $\mathcal{B}(F)$ of a general field F is of interest because of the following result of Suslin on K_3^{ind} :

Theorem 3.1. [9, Theorem 5.2] Let F be an infinite field, then there is a short exact sequence

$$0 \longrightarrow \operatorname{Tor}_{1}^{\mathbb{Z}}(\widetilde{\mu_{F}}, \widetilde{\mu_{F}}) \longrightarrow K_{3}^{\operatorname{ind}}(F) \longrightarrow \mathcal{B}(F) \longrightarrow 0$$

where $\operatorname{Tor}_{1}^{\mathbb{Z}}(\widetilde{\mu_{F}}, \mu_{F})$ is the unique nontrivial extension of $\operatorname{Tor}_{1}^{\mathbb{Z}}(\mu_{F}, \mu_{F})$ by $\mathbb{Z}/2$ when $\operatorname{Char}(F) \neq 2$ (and $\operatorname{Tor}_{1}^{\mathbb{Z}}(\widetilde{\mu_{F}}, \mu_{F}) = \operatorname{Tor}_{1}^{\mathbb{Z}}(\mu_{F}, \mu_{F})$ if $\operatorname{Char}(F) = 2$).

3.2. The refined Bloch Group $\mathcal{RB}(F)$. The refined pre-Bloch group $\mathcal{RP}(F)$, of a field F which has at least 4 elements, is the R_F -module with generators [x], $x \in F^{\times}$ subject to the relations [1] = 0 and

$$S_{x,y}: 0 = [x] - [y] + \langle x \rangle [y/x] - \langle x^{-1} - 1 \rangle [(1 - x^{-1})/(1 - y^{-1})] + \langle 1 - x \rangle [(1 - x)/(1 - y)], \quad x, y \neq 1.$$

From the definitions, it follows that $\mathcal{P}(F) = (\mathcal{RP}(F))_{F^{\times}}$. Let $\Lambda = (\lambda_1, \lambda_2)$ be the R_F -module homomorphism

$$\mathcal{RP}(F) \to \mathcal{I}_F^2 \oplus S^2_{\mathbb{Z}}(F^{\times})$$

where $\lambda_1: \mathcal{RP}(F) \to \mathcal{I}_F^2$ is the map $[x] \mapsto \langle \langle 1-x \rangle \rangle \langle \langle x \rangle \rangle$ and λ_2 is the composite

$$\mathcal{RP}(F) \longrightarrow \mathcal{P}(F) \xrightarrow{\lambda} S_{\mathbb{Z}}^{2}(F^{\times})$$
,

where $S_{\mathbb{Z}}^2(F^{\times})$ has the trivial R_F -module structure.

The refined Bloch group of F is the module

$$\mathcal{RB}(F) := \operatorname{Ker}(\Lambda : \mathcal{RP}(F) \to \mathcal{I}_F^2 \oplus S_{\mathbb{Z}}^2(F^{\times}).$$

Furthermore, the refined scissor congruence group of F is the R_F -module

$$\mathcal{RP}_1(F) := \operatorname{Ker}(\lambda_1 : \mathcal{RP}(F) \to \mathcal{I}_F^2).$$

Thus
$$\mathcal{RB}(F) = \operatorname{Ker}(\lambda_2 : \mathcal{RP}_1(F) \to S_{\mathbb{Z}}^2(F^{\times})).$$

Now for any infinite field F, Suslin shows that there is a natural homomorphism $H_3(SL(F), \mathbb{Z}) \to K_3^{\text{ind}}(F)$, where $SL(F) = \bigcup SL_n(F)$ and $K_3^{\text{ind}}(F)$ is the indecomposable K_3 of F. It follows that there are induced

maps $H_3(SL_n(F), \mathbb{Z}) \to K_3^{\text{ind}}(F)$ for all n. Hutchinson and Tao show that $H_3(SL_2(F), \mathbb{Z}) \to K_3^{\text{ind}}(F)$ is surjective for any infinite field [3, Lemma 5.1].

The refined Bloch group is of interest because of the following result on the third homology of SL_2 over a general field F:

Theorem 3.2. [2, Theorem 4.3] There is a complex of R_F -modules

$$0 \longrightarrow \operatorname{Tor}_{1}^{\mathbb{Z}}(\mu_{F}, \mu_{F}) \longrightarrow H_{3}(SL_{2}(F), \mathbb{Z}) \longrightarrow \mathcal{RB}(F) \longrightarrow 0,$$

which is exact except at the middle term where the homology is annihilated by 4.

Furthermore we have that the following diagram is commutative:

$$0 \longrightarrow \operatorname{Tor}_{1}^{\mathbb{Z}}(\mu_{F}, \mu_{F}) \longrightarrow H_{3}(SL_{2}(F), \mathbb{Z}) \longrightarrow \mathcal{RB}(F) \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow \operatorname{Tor}_{1}^{\mathbb{Z}}(\widetilde{\mu_{F}, \mu_{F}}) \longrightarrow K_{3}^{\operatorname{ind}}(F) \longrightarrow \mathcal{B}(F) \longrightarrow 0$$

Remark 3.3. We will examine the sequence of Theorem 3.2 below in more detail in the case of real quadratically closed fields.

3.3. The elements $\psi_i(x)$ in $\mathcal{RP}(F)$. For $x \in F^{\times}$, we define the following elements of $\mathcal{RP}(F)$

$$\psi_1(x) := [x] + \langle -1 \rangle [x^{-1}] \quad \text{ and } \quad \psi_2(x) := \left\{ \begin{array}{ll} \langle x^{-1} - 1 \rangle [x] + \langle 1 - x \rangle [x^{-1}], & x \neq 1; \\ 0, & x = 1. \end{array} \right..$$

From the definitions of the the elements $\psi_i(x)$, we get that $\langle -1 \rangle \psi_i(-1) = \psi_i(-1)$ for $i \in \{1,2\}$. We review some of the algebraic properties of the elements $\psi_i(x)$:

Proposition 3.4. [4, Lemma 3.1, Proposition 3.2] Let F be a field. For $i \in \{1, 2\}$, we have:

- (1) $\psi_i(xy) = \langle x \rangle \psi_i(y) + \psi_i(x)$, for all x, y.
- (2) $\langle \langle x \rangle \rangle \psi_i(y) = \langle \langle y \rangle \rangle \psi_i(x)$, for all x, y.
- (3) $2\psi_i(-1) = 0$.
- (4) $\psi_i(x^2) = \langle \langle x \rangle \rangle \psi_i(-1)$, for all x. (5) $2\psi_i(x^2) = 0$ for all x and if $-1 \in (A^{\times})^2$, then $\psi_i(x^2) = 0$ for all x.

3.4. The constant C_F . In the section 3.2 of [4], it is shown that the elements

$$C(x) = [x] + \langle -1 \rangle [1 - x] + \langle \langle 1 - x \rangle \rangle \psi_1(x) \in \mathcal{RP}(F)$$

are constant for a field with at least 4 elements (Lemma 3.5); i.e. C(x) = C(y) for all $x, y \in F^{\times}$. Therefore we have the following definition.

Definition 3.5. Let F be a field with at least 4 elements. We will denote by C_F the common value of the expression C(x) for $x \in F \setminus \{0,1\}$; i.e.

$$C_F := [x] + \langle -1 \rangle [1-x] + \langle \langle 1-x \rangle \rangle \psi_1(x)$$
 in $\mathcal{RP}(F)$.

One can deduce some properties of the element C_F

Proposition 3.6. [4, Corollary 3.7] Let F be a field. Then

- (1) $\langle -1 \rangle C_F = C_F$.
- (2) For any field $F, C_F \in \mathcal{RB}(F)$; i.e. $\Lambda(C_F) = 0$.

Observe that the image of C_F in $\mathcal{P}(F)$ is [x] + [1-x] for $x \neq 0,1$. We also denote this element of $\mathcal{B}(F) \subset \mathcal{P}(F)$ by C_F .

4. The Refined Bloch Group of a real quadratically closed field

In this section we show that $\langle -1 \rangle$ acts trivially on $\mathcal{RP}(\mathbf{R})$ where **R** is a real quadratically closed field; i.e. $R_{\mathbf{R}}$ acts trivially on $\mathcal{RP}(\mathbf{R})$.

Lemma 4.1. Let $x \in \mathbb{R}$, if x > 0 then $\psi_i(x) = 0$ for $i \in \{1, 2\}$ in $\mathcal{RP}(\mathbb{R})$.

Proof. If x > 0, then $x = a^2$ for some $a \in \mathbf{R}$ and hence $\psi_i(x) = \langle \langle a \rangle \rangle \psi_i(-1)$ by Proposition 3.4 (4). Replacing a by -a if necessary, we can suppose a>0. Thus a is a square so that $\langle\langle a\rangle\rangle=0$ and hence $\psi_i(x) = 0.$

Proposition 4.2. Let $x \in \mathbb{R}$, if x > 0 then the action of $\langle -1 \rangle$ on [x] is trivial i.e. $\langle -1 \rangle[x] = [x]$.

Proof. Since x > 0, we have that $x = a^2$ for some $a \in \mathbb{R} \setminus \{0\}$. Therefore

$$\langle x^{-1} - 1 \rangle = \langle 1 - x \rangle \langle x \rangle = \langle 1 - x \rangle \langle a^2 \rangle = \langle 1 - x \rangle.$$

By Lemma 4.1, we have that

$$0 = \psi_2(x) = \langle x^{-1} - 1 \rangle [x] + \langle 1 - x \rangle [x^{-1}] = \langle 1 - x \rangle ([x] + [x^{-1}]).$$

Multiplying the equation $(1-x)([x]+[x^{-1}])=0$ by (1-x), we get that $[x]+[x^{-1}]=0$, whence we get that $[x] = -[x^{-1}]$. Similarly applying Lemma 4.1 to $\psi_1(x)$, we get that $[x] = -\langle -1 \rangle [x^{-1}]$. By combining these last two equalities, we get our desired result.

Proposition 4.3. If x < 0 then $C_{\mathbf{R}} = [x] + [1 - x]$ in $\mathcal{RP}(\mathbf{R})$.

Proof. If x < 0 then 1 - x > 0. Hence $\langle (1 - x) \rangle \psi_1(x) = 0$ and by Proposition 4.2, we get that $\langle -1 \rangle [1-x] = [1-x]$. It follows that $C_{\mathbf{R}} = [x] + [1-x]$ in $\mathcal{RP}(\mathbf{R})$.

Proposition 4.4. Let $x \in \mathbb{R}$, if x < 0 then the action of $\langle -1 \rangle$ on [x] is trivial i.e. $\langle -1 \rangle [x] = [x]$.

Proof. Since 1-x>0, by Proposition 4.2, we get that $\langle -1\rangle[1-x]=[1-x]$. Therefore from this latter result, Proposition 4.3 and Proposition 3.6 (1), we obtain that:

$$0 = C_{\mathbf{R}} - \langle -1 \rangle C_{\mathbf{R}} = [x] + [1 - x] - \langle -1 \rangle [x] - \langle -1 \rangle [1 - x] = [x] - \langle -1 \rangle [x].$$

It follows that $\langle -1 \rangle [x] = [x]$.

Corollary 4.5. For any real quadratically closed field **R**. The natural map $\mathcal{RP}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is an isomorphism.

Proof. $\mathcal{RP}(\mathbf{R}) = \mathcal{RP}(\mathbf{R})_{\mathbf{R}^{\times}} \cong \mathcal{P}(\mathbf{R})$ by Proposition 4.2 and Proposition 4.4

Corollary 4.6. For any real quadratically closed field \mathbf{R} , the natural map $\mathcal{RP}_1(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is an isomorphism.

Proof. Note that the map $\lambda_1 : \mathcal{RP}(\mathbf{R}) \to \mathcal{I}^2_{\mathbf{R}}$ given by $[x] \mapsto \langle \langle 1-x \rangle \rangle \langle \langle x \rangle \rangle$ is the zero map for all $x \in \mathbf{R}^{\times}$ since either x > 0 or 1 - x > 0. Hence $\mathcal{RP}_1(\mathbf{R}) = \mathcal{RP}(\mathbf{R})$.

Corollary 4.7. For any real quadratically closed field \mathbf{R} , the natural map $\mathcal{RB}(\mathbf{R}) \to \mathcal{B}(\mathbf{R})$ is an isomorphism.

Proof. $\mathcal{RB}(\mathbf{R}) = \operatorname{Ker}(\lambda_2 : \mathcal{RP}_1(\mathbf{R}) \to S_{\mathbb{Z}}^2(\mathbf{R}^{\times})) = \operatorname{Ker}(\lambda : \mathcal{P}(\mathbf{R}) \to S_{\mathbb{Z}}^2(\mathbf{R}^{\times})) = \mathcal{B}(\mathbf{R})$ by Corollary 4.6.

5. The third homology of $SL_2(\mathbf{R})$

We consider the spectral sequence used in section 4 of [2] to relate $H_3(SL_2(F), \mathbb{Z})$ to $\mathcal{RB}(F)$.

First we recall two standard subgroups of $SL_2(F)$.

$$T:=\left\{\left(\begin{array}{cc}a&0\\0&a^{-1}\end{array}\right)\;|a\in F^{\times}\right\}\quad B:=\left\{\left(\begin{array}{cc}a&b\\0&a^{-1}\end{array}\right)\;|a\in F^{\times}\;,b\in F\right\}.$$

We will now examine this spectral sequence in the case of real quadratically closed fields. We will need the following lemma:

Lemma 5.1. [2, Lemma 4.1] If F is an infinite field, then the inclusion $T \to B$ induces homology isomorphisms

$$H_k(T,\mathbb{Z}) \cong H_k(B,\mathbb{Z})$$

for all k > 0.

5.1. The Spectral Sequence. Let F be a field, and $G = SL_2(F)$ act (on the left) on $\mathbb{P}^1(F)$ by fractional linear transformations. Let X_n be the set of ordered (n+1)-tuples (x_0, \ldots, x_n) of distinct points of $\mathbb{P}^1(F)$. X_n is a left G-set by diagonal action. Let $L_n = \mathbb{Z}[X_n]$ and let $d_n : L_n \to L_{n-1}$ given by

$$d_n(x_0, \dots, x_n) = \sum_{i=0}^n (-1)^i(x_0, \dots, \widehat{x}_i, \dots, x_n).$$

The natural augmentation $\varepsilon: L_0 \to \mathbb{Z}$ gives a weak equivalence $L_{\bullet} \to \mathbb{Z}$ where \mathbb{Z} is a complex concentrated in degree 0. It follows that there is a spectral sequence

$$E_{p,q}^1 = H_p(G, L_q) \Rightarrow H_{p+q}(G, \mathbb{Z}).$$

In fact $E_{0,q}^1 = H_0(G, L_q) = (L_q)_G$.

Proposition 5.2. Let F be a field

- (1) $SL_2(F)$ acts transitively on X_1 . The stabilizer of (∞) is the subgroup B.
- (2) $SL_2(F)$ acts transitively on X_2 . The stabilizer of $(\infty,0)$ is the subgroup T.

Corollary 5.3. Let F be an infinite field, then

- (1) $E_{p,0}^1 \cong H_p(G, \mathbb{Z}[G/B]) \cong H_p(B, \mathbb{Z}) \cong H_p(T, \mathbb{Z}).$ (2) $E_{p,0}^1 \cong H_p(G, \mathbb{Z}[G/T]) \cong H_p(T, \mathbb{Z}).$

Proposition 5.4. [2, Section 2.3] Let F be an infinite field. Let

$$\phi(x,y,z) := \begin{cases} (z-x)(x-y)(z-y)^{-1}, & x,y,z \neq \infty; \\ (y-z)^{-1}, & x = \infty; \\ z-x, & y = \infty; \\ x-y, & z = \infty. \end{cases}$$

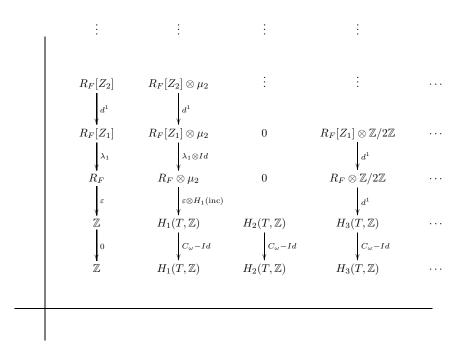
Let $Z_0 = \emptyset$ and for $n \ge 1$, let Z_n denote the set of ordered n-tuples $[z_1, \ldots, z_n]$ of distinct points of $F^{\times} \setminus \{1\}$. Then for $n \ge 2$, there is a isomorphism of R_F -modules

$$E_{0,n+1}^{1} = (L_{n+1})_{G} \longleftrightarrow R_{F}[Z_{n-2}]$$

$$(x_{0},\ldots,x_{n}) \longmapsto \langle \phi(x_{0},x_{1},x_{2}) \rangle \left[\frac{\phi(x_{0},x_{1},x_{3})}{\phi(x_{0},x_{1},x_{2})},\cdots,\frac{\phi(x_{0},x_{1},x_{n})}{\phi(x_{0},x_{1},x_{2})} \right]$$

$$(0,\infty,1,z_{1},\ldots,z_{n-2}) \longleftarrow [z_{1},\ldots,z_{n-2}]$$

Lemma 5.5. [2, Section 4.4] Let F be an infinite field, the E^1 -page of the spectral sequence $E^1_{p,q} = H_n(G, L_{\bullet}) \Rightarrow$ $H_n(G,\mathbb{Z})$ has the form



where C_{ω} denotes the map induced in homology by conjugation $\omega := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\varepsilon : R_F \to \mathbb{Z}$ is the augmentation map.

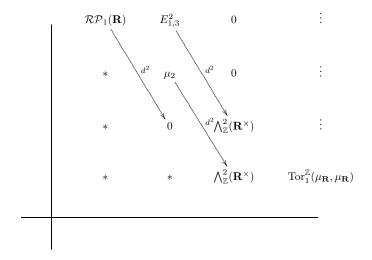
5.2. Third homology of SL_2 of a real quadratically closed field. Since $\lambda_1 = 0$, then $E_{1,2}^2 = \text{Ker}(\varepsilon) \otimes$ $\mu_2 = \mathcal{I}_{\mathbf{R}} \otimes \mu_2 \cong \mu_2 \text{ (since } \mathcal{I}_{\mathbf{R}} = \mathbb{Z}\langle\langle -1 \rangle\rangle \cong \mathbb{Z}).$

Now $H_1(T,\mathbb{Z}) = T \cong \mathbf{R}^{\times}$ and the map C_{ω} induces $x \mapsto x^{-1}$, so $d^1 = C_{\omega} - \mathrm{Id} : E^1_{1,1} = H_1(T,\mathbb{Z}) \cong \mathbf{R}^{\times} \to \mathbf{R}^{\times} \cong H_1(T,\mathbb{Z}) = E^1_{1,0}$ is given by $x \mapsto x^{-2}$. Therefore $E^2_{1,0} = \mathbf{R}^{\times}/(\mathbf{R}^{\times})^2 = \mu_2$. Also $\mathrm{Ker}(C_{\omega} - \mathrm{Id}) = \mu_2 = \mathrm{Im}(\varepsilon \otimes H_1(\mathrm{inc}))$. It follows that $E^2_{1,1} = 0$.

Now we have that C_{ω} induces the identity in $H_2(T,\mathbb{Z}) = \bigwedge_{\mathbb{Z}}^2(\mathbf{R}^{\times})$, thus $d^1 = C_{\omega} - \mathrm{Id} : E_{2,1}^1 = H_2(T,\mathbb{Z}) \cong \mathbf{R}^{\times} \wedge \mathbf{R}^{\times} \to \mathbf{R}^{\times} \wedge \mathbf{R}^{\times} \cong H_2(T,\mathbb{Z}) = E_{2,0}^1$ is the zero map. Hence $E_{2,1}^2 = E_{2,0}^2 = \bigwedge_{\mathbb{Z}}^2(\mathbf{R}^{\times})$.

By the arguments of [2, Section 4.5], we have that $E_{3,0}^2 = \operatorname{Tor}_1^{\mathbb{Z}}(\mu_{\mathbf{R}}, \mu_{\mathbf{R}})$.

Therefore the relevant part of the E^2 -page has the form



5.3. The calculation of $H_3(SL_2(\mathbf{R}), \mathbb{Z})$. Now, $E_{1,2}^{\infty}$ is a subquotient of $E_{1,2}^2 = \mathcal{I}_{\mathbf{R}} \otimes \mu_2 \cong \mu_2$. So $E_{1,2}^{\infty} = \{1\}$ or μ_2 .

In [2, Section 4.7], it is shown that $E_{0,3}^{\infty} = \mathcal{RB}(F)$ and $E_{0,3}^{\infty} = \operatorname{Tor}_{1}^{\mathbb{Z}}(\mu_{F}, \mu_{F})$ for any infinite field F. Thus $E_{0,3}^{\infty} = \mathcal{B}(\mathbf{R})$ and $E_{0,3}^{\infty} = \operatorname{Tor}_{1}^{\mathbb{Z}}(\mu_{\mathbf{R}}, \mu_{\mathbf{R}})$ respectively.

By Corollary 2.3, we have that $E_{2,1}^2 = \bigwedge_{\mathbb{Z}}^2(\mathbf{R}^{\times}) = 2 \cdot \bigwedge_{\mathbb{Z}}^2(\mathbf{R}^{\times}) = 2 \cdot E_{2,1}^2$. Now in the section 4.7 of [2], it was proved that $2 \cdot E_{2,1}^{\infty} = 0$. Since $E_{2,1}^2$ maps onto $E_{2,1}^{\infty}$, it follows that $E_{2,1}^{\infty} = 2 \cdot E_{2,1}^{\infty} = 0$.

Now let $K := \text{Ker}(H_3(SL_2(\mathbf{R}), \mathbb{Z}) \to \mathcal{B}(\mathbf{R}))$. Thus the convergence of the spectral sequence gives us short exact sequences

$$0 \longrightarrow K \longrightarrow H_3(SL_2(\mathbf{R}), \mathbb{Z}) \longrightarrow \mathcal{B}(\mathbf{R}) \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Tor}_{1}^{\mathbb{Z}}(\mu_{\mathbf{R}}, \mu_{\mathbf{R}}) \longrightarrow K \longrightarrow E_{1,2}^{\infty} \longrightarrow 0$$
.

In particular, it follows that |K| = 2 or 4.

6. The indecomposable K_3 of a real quadratically closed field

Theorem 6.1. Let **R** be a real quadratically closed field, then the natural map $H_3(SL_2(F), \mathbb{Z}) \to K_3^{\text{ind}}(F)$ is an isomorphism.

Proof. From the previous remarks and the previous section, we have the following commutative diagram:

$$0 \longrightarrow K \longrightarrow H_3(SL_2(\mathbf{R}), \mathbb{Z}) \longrightarrow \mathcal{B}(\mathbf{R}) \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad$$

The exactness of the bottom row is the main theorem in Suslin (Theorem 3.1 above). The middle vertical arrow is a surjection by [3, Lemma 5.1]. Thus the map $K \to \operatorname{Tor}_1^{\mathbb{Z}}(\widetilde{\mu_{\mathbf{R}},\mu_{\mathbf{R}}})$ is surjective, and since $|K| \leq 4$, it follows that it is an isomorphism and the result follows.

Remark 6.2. From the proof of Corollary 6.1, it follows that the term $E_{1,2}^{\infty}$ of the spectral sequence of above is equal to $\mathcal{I}_{\mathbf{R}} \otimes \mu_2 \cong \mu_2$.

Now, let us recall the definition of the Milnor K-theory.

For a field F, let $T(F^{\times}) = \bigoplus_{n \geq 0} (F^{\times})^{\otimes n}$, where $(F^{\times})^{\otimes 0} = \mathbb{Z}$, be the tensor algebra over the \mathbb{Z} -module F^{\times} .

Let J be the two-sided homogeneous ideal in $T(F^{\times})$ generated by elements $a \otimes (1-a)$ with $a \in F^{\times} \setminus \{1\}$.

Definition 6.3. The Milnor K-theory of a field F is defined to be the graded ring $K_{\bullet}^{M}(F) = T(F^{\times})/J$.

The image of an element $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ in $K_n^M(F)$ is denoted by $\{a_1, \ldots, a_n\}$.

It follows that for any real quadratically closed field **R** that $K_n^M(\mathbf{R}) = \langle \{-1, \dots, -1 \rangle \oplus K_n^M(\mathbf{R})^0 \text{ where } K_n^M(\mathbf{R})^0 := \langle \{x_1, \dots, x_n\} | x_i > 0 \, \forall i \rangle \text{ is 2-divisible.}$

Note that, it follows that $K_n^M(\mathbf{R})^0 = 2 \cdot K_n^M(\mathbf{R})$ when **R** is real quadratically closed.

Proposition 6.4. For any real quadratically closed field \mathbf{R} and any $n \geq 3$, the inclusion $SL_2(\mathbf{R}) \to SL_n(\mathbf{R})$, induces an injection $H_3(SL_2(\mathbf{R}), \mathbb{Z}) \to H_3(SL_n(\mathbf{R}), \mathbb{Z})$ and $H_3(SL_n(\mathbf{R}), \mathbb{Z}) \cong H_3(SL_2(\mathbf{R}), \mathbb{Z}) \oplus K_n^M(\mathbf{R})^0$.

Proof. By definition, the map $H_3(SL_2(\mathbf{R}), \mathbb{Z}) \to K_3^{\text{ind}}(\mathbf{R})$ factors through $H_3(SL_3(\mathbf{R}), \mathbb{Z})$ and hence $H_3(SL_2(\mathbf{R}), \mathbb{Z}) \to H_3(SL_3(\mathbf{R}), \mathbb{Z})$ is injective.

Furthermore $H_3(SL_3(\mathbf{R}), \mathbb{Z}) \cong H_3(SL_n(\mathbf{R}), \mathbb{Z}) \ \forall n \geq 3 \ \text{by } [3, \text{ Theorem 4.7}].$

For any infinite field F the sequence

$$H_3(SL_2(F), \mathbb{Z}) \longrightarrow H_3(SL_3(F), \mathbb{Z}) \longrightarrow 2 \cdot K_3^M(F) \longrightarrow 0$$

is exact by [3, Theorem 4.7]. Thus, for real quadratically closed fields, we have an exact sequence

$$0 \longrightarrow H_3(SL_2(\mathbf{R}), \mathbb{Z}) \longrightarrow H_3(SL_3(\mathbf{R}), \mathbb{Z}) \longrightarrow K_3^M(\mathbf{R})^0 \longrightarrow 0.$$

The above sequence is split, since we have a commutative triangle

where the diagonal arrow is an isomorphism.

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