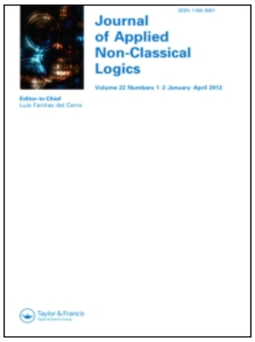

Journal of Applied Non-Classical Logics

ISSN: 1166-3081 (Print) 1958-5780 (Online) Journal homepage: https://www.tandfonline.com/loi/tncl20

Complexity of intuitionistic propositional logic and its fragments

Mikhail Rybakov

To cite this article: Mikhail Rybakov (2008) Complexity of intuitionistic propositional logic and its fragments, Journal of Applied Non-Classical Logics, 18:2-3, 267-292, DOI: 10.3166/jancl.18.267-292

To link to this article: https://doi.org/10.3166/jancl.18.267-292

Published online: 13 Apr 2012.

Complexity of intuitionistic propositional logic and its fragments

Mikhail Rybakov

Department of Mathematics
Tver State University
Zhelyabova Street, 33
Tver, 170000 (Russia)

m_rybakov@mail.ru

ABSTRACT. *In the paper we consider complexity of intuitionistic propositional logic and its natural fragments such as implicative fragment, finite-variable fragments, and some others. Most facts we mention here are known and obtained by logicians from different countries and in different time since 1920s; we present these results together to see the whole picture.*

KEYWORDS: *non-classical logic, intuitionistic logic, propositional logic, decision problem, complexity,* PSPACE-*completeness, complexity function.*

DOI:10.3166/JANCL.18.267–292

1. Introduction. Main notions

As a set of formulas, formally, intuitionistic propositional logic is a part of classical one, however, justifications—philosophical and mathematical—of intuitionistic logic are not connected directly with classical logic: these two logics were created to reflect different aspects of our reasonings, to solve different problems, and to answer to different questions. The main difference between these logics is in that the intended meaning of the intuitionistic logical connectives are defined via such notions as "proof" and "construction" (regarded as primary and understandable). Note that classical logic provide us only with true *in principle* reasonings: we regard a proposition as true in a classical theory if its negation contradicts the theory. Roughly speaking, we may understand classical logic as a logic of consistent reasonings, while intuitionistic one as a logic of constructive—or effective—reasonings.

Let's explain intuitionistic connectives in terms of proof and construction. Assume we know what a proof of an atomic proposition is; then

– to prove $A \wedge B$ we must prove both A and B;
– to prove $A \vee B$ we must prove at least one of A and B;
– to prove $A \rightarrow B$ we must present a construction which, for any proof of A, returns a proof of B;
– to prove $\neg A$ we must present a construction which, for any proof of A, returns a proof of $\perp$;
– $\perp$ have no proof.

A significate example—to fill the difference between classical and intuitionistic logics—is the law of the excluded middle $p \vee \neg p$: it is true from classical point of view but not from intuitionistic one (for example, because there are open mathematical problems; if p is such problem—for example, Goldbach's conjecture "every even number $n > 2$ can be represented as the sum of two prime numbers" or the statement "$\mathrm{P} = \mathrm{NP}$" or the statement "$\mathrm{NP} = \mathrm{PSPACE}$", etc.—then we have not yet both a proof of p and a proof of $\neg p$).

This interpretation—maybe with some inessential differences—was given by L. Brouwer, A. Kolmogorov, and A. Heyting. It is not provide us immediately with precisely defined semantics. However, it can be made more precise by several ways: for example, via Kleene realizability, Medvedev finite problem interpretation, or using extra—relative to classical language—provability operator; in particular, in the latter case we may understand intuitionistic logic as a certain fragment of modal logic **S4**: intuitionistic logic is embedded into **S4** by the Gödel translation (we are dealing with the Gödel translation in p. 277).

For practical use some another interpretation—given by E. Beth and S. Kripke—is more suitable; it expresses an epistemic feature of intuitionistic logic. To explain it, observe that from "classical" point of view, every proposition must be true or false. But for some propositions really we do not know whether they are true or false; nevertheless, it is quite possible that we can know about this in the future. So, to give another interpretation of intuitionistic logic, let's imagine that our knowledge is developing nondeterministically passing from one state of knowledge to another. At every state we know which propositions are true and which are not established yet; it is also reasonable to think that all established propositions will be kept at the future states. If, for some state x, we *know* that an atomic proposition p is true at x then it is natural to regard p as true at x and all subsequent states; as for compound propositions,

– $A \wedge B$ is true at a state x if both A and B are true at x;
– $A \vee B$ is true at a state x if at least on of A and B is true at x;
– $A \rightarrow B$ is true at x if, for any subsequent state y (in particular, x) such that A is true at y, we have B is true at y;
– $\neg A$ is true at x if, for any subsequent state y (in particular, x), we have that A is not true at y;
– $\perp$ is not true at x, for any state x.

This interpretation leads us to Kripke semantics for intuitionistic language. Let's give formal definitions.

We consider formulas constructed from propositional variables $p_0, p_1, p_2, p_3, \ldots$ and constant $\bot$ using $\wedge$, $\vee$, $\to$, and brackets. We define $\neg$, $\top$, and $\leftrightarrow$ as usual abbreviations: $\neg\varphi = \varphi \to \bot$, $\top = \neg\bot$, $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$. We will use standard conventions on representation of formulas: $\neg$ is stronger than $\wedge$ and $\vee$, which in their turn are stronger than $\to$ and $\leftrightarrow$.

DEFINITION. — *An intuitionistic Kripke frame is a pair $\mathfrak{F} = \langle W, R\rangle$ consisting of non-empty set W with partial order R on W,* i.e.*, such that for all $x, y, z \in W$,*

$$\begin{array}{ll} xRx & (\textit{reflexivity of } R), \\ xRy \wedge yRz \Rightarrow xRz & (\textit{transitivity of } R), \\ xRy \wedge yRx \Rightarrow x = y & (\textit{antisymmetry of } R). \end{array}$$

The elements of W are called the worlds of the frame $\mathfrak{F}$, R is called the accessibility relation. If xRx' holds, we say that x' is accessible from x.

DEFINITION. — *A valuation in an intuitionistic frame $\mathfrak{F} = \langle W, R\rangle$ is a function v mapping each propositional variable p into a subset $v(p)$ of W, so that for every $x \in v(p)$ and every $x' \in W$, xRx' implies $x' \in v(p)$.*

DEFINITION. — *An intuitionistic Kripke model is a pair $\mathfrak{M} = \langle \mathfrak{F}, v\rangle$, where $\mathfrak{F}$ is an intuitionistic Kripke frame and v is a valuation in $\mathfrak{F}$.*

DEFINITION. — *Let $\mathfrak{F} = \langle W, R\rangle$ be an intuitionistic Kripke frame, $\mathfrak{M} = \langle \mathfrak{F}, v\rangle$ an intuitionistic Kripke model, x a world in $\mathfrak{F}$. By induction we define the relation $(\mathfrak{M}, x) \models \varphi$:*

$$\begin{array}{lll} (\mathfrak{M}, x) \not\models \bot; & & \\ (\mathfrak{M}, x) \models p & \leftrightharpoons & x \in v(p), \textit{ where } p \textit{ is a propositional variable;} \\ (\mathfrak{M}, x) \models \psi' \wedge \psi'' & \leftrightharpoons & (\mathfrak{M}, x) \models \psi' \textit{ and } (\mathfrak{M}, x) \models \psi''; \\ (\mathfrak{M}, x) \models \psi' \vee \psi'' & \leftrightharpoons & (\mathfrak{M}, x) \models \psi' \textit{ or } (\mathfrak{M}, x) \models \psi''; \\ (\mathfrak{M}, x) \models \psi' \to \psi'' & \leftrightharpoons & \textit{for every } x' \in W \textit{ such that } xRx', \\ & & (\mathfrak{M}, x') \not\models \psi' \textit{ or } (\mathfrak{M}, x') \models \psi''. \end{array}$$

From the definition it follows that for every $x \in W$, $(\mathfrak{M}, x) \models \top$ and

$$\begin{array}{lll} (\mathfrak{M}, x) \models \neg\psi' & \Longleftrightarrow & \text{for every } x' \in W \text{ such that } xRx', \\ & & (\mathfrak{M}, x') \not\models \psi'; \\ (\mathfrak{M}, x) \models \psi' \leftrightarrow \psi'' & \Longleftrightarrow & \text{for every } x' \in W \text{ such that } xRx', \\ & & (\mathfrak{M}, x') \models \psi' \text{ if and only if } \\ & & (\mathfrak{M}, x') \models \psi''. \end{array}$$

If $(\mathfrak{M}, x) \models \varphi$ holds, we say that φ *is true at the world x in $\mathfrak{M}$*; otherwise that φ *is refuted at x*. We say that φ *is true in a model* $\mathfrak{M} = \langle \mathfrak{F}, v\rangle$ defined on a frame $\mathfrak{F} = \langle W, R\rangle$ if $(\mathfrak{M}, x) \models \varphi$, for every $x \in W$; this is denoted by $\mathfrak{M} \models \varphi$. We say that φ

is valid in a frame $\mathfrak{F} = \langle W, R \rangle$ if φ is true in every model based on $\mathfrak{F}$; this is denoted by $\mathfrak{F} \models \varphi$.

We define propositional intuitionistic logic **Int** as the set of all formulas that are valid in every intuitionistic frame.

Taking into account philosophical and mathematical motivation for introducing of intuitionistic logic, it is not surprising that there is a lot of investigations connected—directly or not—with this logic, its properties, and possibilities for use; we send the reader for details—in particular, for quite rich bibliography—to (Gabbay, 1981; Chagrov *et al.*, 1997). Here we stay on only one aspect of **Int**: its complexity in the sense of computational complexity of the decision problem for **Int** and some its natural fragments.

2. Complexity of intuitionistic logic: Kuznetsov's questions

Complexity issues for superintuitionistic[1] logics—and for non-classical logics as well—were first addressed by A. Kuznetsov, see (Kuznetsov, 1975). One of the main questions of A. Kuznetsov was about polynomial equivalency of intuitionistic propositional logic **Int** and classical propositional logic **Cl**. Due to V. Glivenko, it is known (Glivenko, 1929) that

$$\varphi \in \mathbf{Cl} \quad \Longleftrightarrow \quad \neg\neg\varphi \in \mathbf{Int},$$

for any formula φ; i. e., **Cl** is reducible to **Int** by a polynomial time algorithm (for Glivenko's theorem and some other embeddings of **Cl** into **Int** see also (Chagrov *et al.*, 1997), Section 2.7). Therefore, the Kuznetsov's question was only about polynomial reducibility of **Int** to **Cl**. Then, in 1979 he showed (Kuznetsov, 1979) that if **Int** has a polynomial model property then it is reducible to **Cl** by a polynomial time algorithm. We give briefly his argumentation.

Let φ be an intuitionistic formula, $p_1, \dots, p_n$ be all its variables, and m be a positive integer; we will think about m as a number of worlds in a frame refuting φ. With every pair $\langle i, j \rangle$ of numbers in $\{1, \dots, m\}$, we associate a propositional variable r_{ij}, whose intended meaning is "the world number j is accessible from the world number i". Let $\mathrm{sub}\,\varphi$ denotes the set of all subformulas of φ; with every $i \in \{1, \dots, m\}$ and every $\psi \in \mathrm{sub}\,\varphi$ we associate a propositional variable p_i^{ψ}, whose intended meaning is "ψ is true at the world number i". Using these variables we construct a formula describing the condition "φ is valid in all Kripke frames with m worlds".

1. I. e., extensions of **Int** closed under Modus Ponens and Substitution.

Let $Rel(\varphi, m)$ be a formula describing that the relation is a partial order; we define $Rel(\varphi, m)$ as a conjunction of the following formulas:

$$
\begin{array}{ll}
r_{ii}, & \text{where } i \in \{1, \ldots, m\}; \\
r_{ij} \to \neg r_{ji}, & \text{where } i, j \in \{1, \ldots, m\} \text{ and } i \neq j; \\
r_{ij} \wedge r_{jk} \to r_{ik}, & \text{where } i, j, k \in \{1, \ldots, m\};
\end{array}
$$

Let $Truth(\varphi, m)$ describes properties of the truth-relation in intuitionistic Kripke models; we define $Truth(\varphi, m)$ as a conjunction of the following formulas:

$$
\begin{array}{ll}
p_i^{p_k} \wedge r_{ij} \to p_j^{p_k}, & \text{where } i, j \in \{1, \ldots, m\},\, k \in \{1, \ldots, n\}; \\
p_i^{\bot} \leftrightarrow \bot, & \text{where } i \in \{1, \ldots, m\}; \\
p_i^{\psi \wedge \chi} \leftrightarrow p_i^{\psi} \wedge p_i^{\chi}, & \text{where } i \in \{1, \ldots, m\},\, \psi \wedge \chi \in \mathrm{sub}\, \varphi; \\
p_i^{\psi \vee \chi} \leftrightarrow p_i^{\psi} \vee p_i^{\chi}, & \text{where } i \in \{1, \ldots, m\},\, \psi \wedge \chi \in \mathrm{sub}\, \varphi; \\
p_i^{\psi \to \chi} \leftrightarrow \bigwedge\limits_{j=1}^{m} (r_{ij} \wedge p_j^{\psi} \to p_j^{\chi}), & \text{where } i \in \{1, \ldots, m\},\, \psi \to \chi \in \mathrm{sub}\, \varphi.
\end{array}
$$

Let also $Val(\varphi, m)$ describes the condition that φ is true in any model defined on any of the frames; we put

$$
Val(\varphi, m) \quad = \quad \bigwedge_{i=1}^{m} p_i^{\varphi}.
$$

Now we are ready to say that φ is valid in every frame of m worlds. To do this we define the following formula:

$$
\varphi^*(m) \quad = \quad Rel(\varphi, m) \wedge Truth(\varphi, m) \to Val(\varphi, m).
$$

We will use the following denotations: for a (finite) set X we denote by $|X|$ the number of its elements, for a formula ψ we denote by $|\psi|$ the length of ψ.

PROPOSITION 1. — *For every formula φ and for every integer $m > 0$,*

$$
\varphi^*(m) \in \mathbf{Cl} \quad \Longleftrightarrow \quad \begin{array}{l} \textit{for every intuitionistic frame } \mathfrak{F} = \langle W, R \rangle \textit{ such that} \\ |W| = m, \textit{ we have } \mathfrak{F} \models \varphi. \end{array}
$$

PROOF. — Left to the reader. ■

PROPOSITION 2. — *Formula $\varphi^*(m)$ is computable on $\langle \varphi, m \rangle$ in time polynomial on $|\varphi| + m$.*

PROOF. — Left to the reader. ■

Let's show what happens if **Int** has polynomial model property. In this case there is a polynomial q such that, for every formula $\varphi \notin \mathbf{Int}$, there exists a frame $\mathfrak{F} = \langle W, R \rangle$ and a model $\mathfrak{M} = \langle \mathfrak{F}, v \rangle$ such that

– $|W| \leqslant q(|\varphi|)$;
– $(\mathfrak{M}, x) \not\models \varphi$, for some $x \in W$.

Then, by Proposition 1,

$$\varphi \in \mathbf{Int} \quad \Longleftrightarrow \quad \varphi^*(q(|\varphi|)) \in \mathbf{Cl},$$

and, by Proposition 2, **Int** must be reducible to **Cl** by a polynomial time algorithm.

REMARK 3. — In fact, in (Kuznetsov, 1979) A. Kuznetsov showed that if a Kripke complete propositional logic L has a polynomial model property relative to some "good" semantics then L is reducible to **Cl** by a polynomial time algorithm. The word "good" means that it is possible to describe Kripke semantics for L by a formula like $Rel(\varphi, m)$. In particular, one may see that it is the case for such modal logics as **K5** and **S5**. An example of using of this method the reader may find in (Chagrov *et al.*, 1997), see Theorem 18.3 and Example 18.1 before. □

Apparently, A. Kuznetsov presumed that **Int** has a polynomial model property. In any case, one of the problems put by A. Kuznetsov was the polynomial model property of **Int** and some its extensions, see (Kuznetsov, 1979). Now we know the answer: **Int** has exponential model property.

3. Exponential model property for Int

The fact that **Int** has an exponential model property was proved by M. Zakharyaschev (Popov *et al.*, 1979; Popov *et al.*, 1980) at the same time as A. Kuznetsov put corresponded problem in (Kuznetsov, 1979). In his investigations M. Zakharyaschev used the following function f_L defined for a logic L.

DEFINITION. — *The complexity function for a logic (a set of formulas) L is defined in the following way:*

$$f_L(n) \quad = \quad \max_{\substack{|\varphi| \leqslant n \\ \varphi \notin L}} \; \min_{\substack{\mathfrak{F} \models L \\ \mathfrak{F} \not\models \varphi}} |\mathfrak{F}|,$$

where $|\mathfrak{F}|$ denotes the number of worlds in the frame $\mathfrak{F}$.

Then polynomial model property and exponential model property of a logic L can be defined via the complexity function for L as follows.

DEFINITION. — *Polynomial model property for a (Kripke complete) logic L means that f_L is bounded by a polynomial function, i. e., that there is a polynomial q such that $f_L(n) \leqslant q(n)$, for all n; exponential model property for L means that f_L is exponential, i. e.,*

$$2^{k \cdot n^m} \leqslant f_L(n) \leqslant 2^{r \cdot n^s},$$

for some strictly positive real numbers k, m, r, s.

Note, the upper exponential bound for $f_{\mathbf{Int}}$ follows from "standard" constructions such as Hintikka systems or filtration of canonical model for **Int**, see (Chagrov *et al.*, 1997): if $\varphi \notin \mathbf{Int}$ then φ is refuted in a frame with at most $2^{|\varphi|}$ worlds. So, the question is about the lower bound. To obtain the exponential lower bound for countermodels we will use formulas from (Chagrov *et al.*, 1997); let for any integer $n \geqslant 1$,

$$\begin{array}{rcl}\alpha_n & = & (\neg p_n \to q_n) \vee (p_n \to q_n);\\ \beta_n & = & \bigwedge\limits_{i=1}^{n-1} (\alpha_{i+1} \to q_i) \to \alpha_1,\end{array}$$

where p_j's and q_j's are pairwise different variables and the empty conjunction (in the premise of β_1) is $\top$.

Observe, the countermodel for β_n has to contain a binary tree of depth $n+1$. Indeed, let for a model $\mathfrak{M}$ and a world x in it, $(\mathfrak{M}, x) \not\models \beta_n$. Then there is a world y accessible from x such that

$$(\mathfrak{M}, y) \models \bigwedge_{i=1}^{n-1} (\alpha_{i+1} \to q_i), \quad (\mathfrak{M}, y) \not\models (\neg p_1 \to q_1) \vee (p_1 \to q_1).$$

The former condition gives us that for any z accessible from y,

$$(\mathfrak{M}, z) \models \bigwedge_{i=1}^{n-1} (\alpha_{i+1} \to q_i).$$

The latest one means that there are y' and y'' accessible from y such that

$$(\mathfrak{M}, y') \models \neg p_1, \ (\mathfrak{M}, y') \not\models q_1, \ (\mathfrak{M}, y'') \models p_1, \ (\mathfrak{M}, y'') \not\models q_1.$$

Since $(\mathfrak{M}, y') \models \neg p_1$ and $(\mathfrak{M}, y'') \models p_1$, we conclude that y' and y'' are distinct; moreover, there is no y''' accessible from both y' and y'' because at every world accessible from y', p_1 must be true and at every one accessible from y'', $\neg p_1$ must be true.

Then, let $(\mathfrak{M}, z) \not\models q_i$ and $(\mathfrak{M}, z) \models \alpha_{i+1} \to q_i$. In this case $(\mathfrak{M}, z) \not\models \alpha_{i+1}$, and hence there are z' and z'' accessible from z such that

$$(\mathfrak{M}, z') \models \neg p_{i+1}, \ (\mathfrak{M}, z') \not\models q_{i+1}, \ (\mathfrak{M}, z'') \models p_{i+1}, \ (\mathfrak{M}, z'') \not\models q_{i+1}.$$

Since $(\mathfrak{M}, z') \models \neg p_i$ and $(\mathfrak{M}, z'') \models p_i$, we conclude that $z' \neq z''$ and there is no z''' accessible from both z' and z''.

As a result we get that $\mathfrak{M}$ contains a binary tree of depth $n+1$. It remains to observe that β_n is indeed refuted in some model (in particular, in a binary tree of depth $n+1$; as an example, the counermodel for β_2 is depicted on Fig 1) and that the length of β_n grows linearly on n; so, we have

THEOREM 4 (M. ZAKHARYASCHEV). — *The complexity function for* $\mathbf{Int}$ *is exponential.*

REMARK 5. — The argumentation above works for many logics; so, using formulas like β_n, one may prove that the lower bound of the complexity function for any logic in intervals $[\mathbf{K}, \mathbf{GL}]$ and $[\mathbf{K}, \mathbf{Grz}]$—in particular, for **T**, **K4**, **S4**—is exponential. □

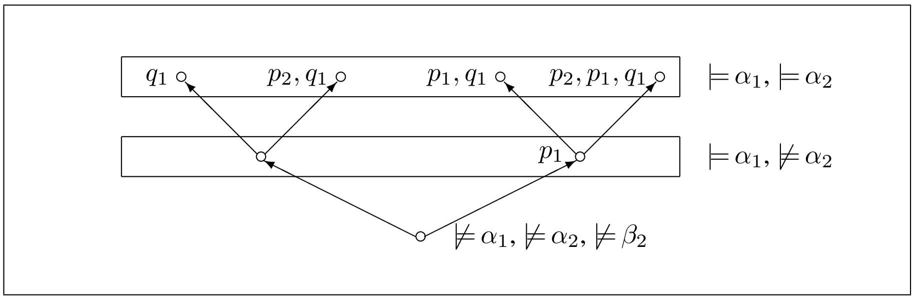

Figure 1. *Countermodel for* β_2

In view of Theorem 4, **Int** has not polynomial model property. But this does not mean that **Int** is not reducible to **Cl** by a polynomial time algorithm: we may imagine that there is another way—not based on describing of countermodels for formulas—to construct polynomial reduction of **Int** to **Cl**. So, the Kuznetsov's question is still without an answer. Moreover, this question is connected with open problem "NP = PSPACE?" which is known as one of the basic problems in Complexity Theory. Let's see some details.

4. PSPACE-completeness of Int

We will use complexity classes; let's recall basic definitions (for details see, for example, (Papadimitriou, 1995)). We will consider only algorithmic recognition problems, i. e., problems that can be formulated as questions "$x \in X$?", for some set X; for example, the decision problem for a logic L looks as "$\varphi \in L$?". The complexity classes P, NP, and PSPACE are the classes of problems that can be solved respectively by polynomial time deterministic algorithms, by polynomial time nondeterministic algorithms, and by polynomial space deterministic[2] algorithms.

Let C be NP or PSPACE. A problem "$x \in X$?" is called *C-hard* if any problem "$y \in Y$?" in C is polynomially reducible to "$x \in X$?", i. e., if there exists a polynomial time function (algorithm) f such that for every y,

$$y \in Y \quad \Longleftrightarrow \quad f(y) \in X.$$

A problem "$x \in X$?" is called *C-complete* if it belongs to C and is C-hard.

We recall that $\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE}$ but it is not known yet whether $\mathrm{P} = \mathrm{NP}$, $\mathrm{NP} = \mathrm{PSPACE}$, and even $\mathrm{P} = \mathrm{PSPACE}$.

2. The class PSPACE can also be defined as the class of problems solvable by nondeterministic polynomial space algorithms; the equivalence between the definitions follows from (Savitch, 1970).

It is known that the satisfiability problem for **Cl** is NP-complete (Cook, 1971). R. Statman proved (Statman, 1979) that the decision problem for **Int** is PSPACE-complete; therefore, **Int** is polynomially reducible to **Cl** if and only if $\mathrm{NP} = \mathrm{PSPACE}$.

Let's consider Statman's argumentation; below we will use a slight modification of formulas in (Chagrov *et al.*, 1997) but the general idea of the construction is the same as in (Statman, 1979).

To show that the decision problem for **Int** is PSPACE-hard, it suffices to reduce some PSPACE-hard problem to it. As such problem we take the problem **SAT-QBF**, see (Stockmeyer, 1987): *given a quantified Boolean formula φ, to check whether φ is satisfiable*. Without a loss of generality we may (and will) assume that any quantified Boolean formula φ is in the form $Q_1 p_1 \dots Q_n p_n\, \varphi'$, where $Q_1, \dots, Q_n \in \{\forall, \exists\}$ and φ' is a quantifier-free formula in conjunctive normal form in variables $p_1, \dots, p_n$.

Let $\varphi = Q_1 p_1 \dots Q_n p_n\, \varphi'$ be a quantified Boolean formula, where φ' is in conjunctive normal form, i. e.,

$$\varphi' \quad = \quad \bigwedge_{t=1}^{m} \Big(\bigvee_{s=1}^{k_t} p_{i_{st}} \vee \bigvee_{s=k_t+1}^{j_t} \neg p_{i_{st}} \Big);$$

here we assume i_{st} be always in $\{1, \dots, n\}$.

To simulate the variable p_i, where $i \in \{1, \dots, n\}$, we will use two formulas on propositional variables q_i and q_i':

$$\delta_i = q_i \to q_i', \quad \overline{\delta}_i = q_i' \to q_i.$$

The intended meaning of these formulas is as follows: if δ_i is refuted then p_i is true and if $\overline{\delta}_i$ is refuted then p_i is false.

For technical purposes we will use also four extra formulas: $\delta_{n+1} = q_{n+1} \to q_{n+1}'$, $\overline{\delta}_{n+1} = q_{n+1}' \to q_{n+1}$, and $\delta_0 = \overline{\delta}_0 = q_0$.

To simulate quantifiers, we define two formulas, where r_1, r_2, r_3, r_4, r_5 are pairwise different propositional variables:

$$A(r_1, r_2, r_3, r_4, r_5) = (r_1 \wedge r_2 \to r_3) \wedge (r_1 \wedge r_2 \to r_4) \wedge (r_1 \to r_5) \wedge (r_2 \to r_5) \to r_5,$$
$$E(r_1, r_2, r_3, r_4, r_5) = (r_1 \wedge r_2 \to r_3) \wedge (r_1 \wedge r_2 \to r_4) \wedge (r_1 \wedge r_2 \to r_5) \to r_5.$$

We briefly comment these formulas. Let $\delta_{i-1}^* \in \{\delta_{i-1}, \overline{\delta}_{i-1}\}$. Let x be a world of a Kripke model $\mathfrak{M}$ such that $(\mathfrak{M}, x) \not\models \delta_{i-1}^*$, i. e., a truth-value of p_{i-1} is already defined. If $(\mathfrak{M}, x) \not\models A(\delta_i, \overline{\delta}_i, \delta_{i+1}, \overline{\delta}_{i+1}, \delta_{i-1}^*)$ then, accordingly to the third conjunct $\delta_i \to \delta_{i-1}^*$ and the fourth conjunct $\overline{\delta}_i \to \delta_{i-1}^*$ of the premise, we have $(\mathfrak{M}, x) \not\models \delta_i$ and $(\mathfrak{M}, x) \not\models \overline{\delta}_i$. Therefore, there are (different) worlds x' and x'' accessible from x such that

$$(\mathfrak{M}, x') \models q_i, \;\; (\mathfrak{M}, x') \not\models q_i' \;\; \text{(note, in particular, } (\mathfrak{M}, x') \not\models \delta_i),$$
$$(\mathfrak{M}, x'') \models q_i', \;\; (\mathfrak{M}, x'') \not\models q_i \;\; \text{(note, in particular, } (\mathfrak{M}, x'') \not\models \overline{\delta}_i),$$

i. e., we have an "explanation" of the quantifier $\forall p_i$.

Similarly, if $(\mathfrak{M}, x) \not\models \delta^*_{i-1}$ and $(\mathfrak{M}, x) \not\models E(\delta_i, \overline{\delta}_i, \delta_{i+1}, \overline{\delta}_{i+1}, \delta^*_{i-1})$ then we have an "explanation" of the quantifier $\exists p_i$: accordingly to the third conjunct $\delta_i \wedge \overline{\delta}_i \to \delta^*_{i-1}$ of the premise, there is a world x' accessible from x such that

$$\begin{array}{ll} \text{either} & (\mathfrak{M}, x') \models q_i \text{ and } (\mathfrak{M}, x') \not\models q'_i \\ \text{or} & (\mathfrak{M}, x') \models q'_i \text{ and } (\mathfrak{M}, x') \not\models q_i \end{array}$$

(note that in particular, we have $(\mathfrak{M}, x') \not\models \delta_i$ or $(\mathfrak{M}, x') \not\models \overline{\delta}_i$).

As for the first and second conjuncts of the premises—which are $\delta_i \wedge \overline{\delta}_i \to \delta_{i+1}$ and $\delta_i \wedge \overline{\delta}_i \to \overline{\delta}_{i+1}$ in the both cases above,—they ensure that if a truth-value of p_{i+1} is already defined (i. e., one of δ_{i+1} and $\overline{\delta}_{i+1}$ is refuted) then a truth-value of p_i is defined too (because one of δ_i and $\overline{\delta}_i$ is refuted), i. e., we go throw quantifiers $Q_1p_1, \dots, Q_np_n$ accordingly to their order in φ.

Note also that if one of worlds y and z is accessible from the other then it is impossible that simultaneously $(\mathfrak{M}, y) \not\models \delta_i$ and $(\mathfrak{M}, z) \not\models \overline{\delta}_i$, i. e., the truth-value of p_i cannot be changed at consequent worlds.

Now we are ready to write intuitionistic formulas describing the condition that φ is a satisfiable quantified Boolean formula. For better understanding of the formulas below, try to "read" any implication $\alpha \to \beta$ as "if β is refuted (at a world) then α is refuted (at the same world)".

With any quantifier Q_ip_i in the formula φ we associate the formula F_i defined as follows: if $Q_ip_i = \forall p_i$ then

$$F_i = (A(\delta_i, \overline{\delta}_i, \delta_{i+1}, \overline{\delta}_{i+1}, \delta_{i-1}) \to \delta_{i-1}) \wedge (A(\delta_i, \overline{\delta}_i, \delta_{i+1}, \overline{\delta}_{i+1}, \overline{\delta}_{i-1}) \to \overline{\delta}_{i-1})$$

and if $Q_ip_i = \exists p_i$ then

$$F_i = (E(\delta_i, \overline{\delta}_i, \delta_{i+1}, \overline{\delta}_{i+1}, \delta_{i-1}) \to \delta_{i-1}) \wedge (E(\delta_i, \overline{\delta}_i, \delta_{i+1}, \overline{\delta}_{i+1}, \overline{\delta}_{i-1}) \to \overline{\delta}_{i-1}).$$

Let $Q = F_1 \wedge \dots \wedge F_n$.

Thus, we have a tool to describe quantifiers of φ; now let's consider the formula φ'. It is in conjunctive normal form; let $\psi_1, \dots, \psi_m$ be its conjuncts. With any conjunct ψ_t of φ' we associate the formula

$$\psi^*_t \quad = \quad \bigwedge_{s=1}^{k_t} \delta_{i_{st}} \wedge \bigwedge_{s=k_t+1}^{j_t} \overline{\delta}_{i_{st}} \to \delta_n \wedge \overline{\delta}_n.$$

Note that if ψ^*_t is true at some world at which $\delta_n \wedge \overline{\delta}_n$ is refuted, then we obtain some truth-values for the variables $p_1, \dots, p_n$ (corresponding to the intended meaning of δ_i and $\overline{\delta}_i$) such that ψ_t is true.

Let $\psi^* = \psi^*_1 \wedge \dots \wedge \psi^*_m$.

DEFINITION. — *Let $\varphi = Q_1p_1 \dots Q_np_n\, \varphi'$ be a quantified Boolean formula, where $Q_1, \dots, Q_n \in \{\forall, \exists\}$ and φ' is a quantifier-free formula in conjunctive normal form* *in variables $p_1, \dots, p_n$. Let Q and ψ^* be intuitionistic formulas defined on φ as it described above. We define φ^* in the following way: $\varphi^* = \psi^* \to (Q \to q_0)$.*

Note that if φ^* is refuted in an intuitionistic Kripke model $\mathfrak{M}$ then there is a world x in $\mathfrak{M}$ such that $(\mathfrak{M}, x) \not\models q_0$, $(\mathfrak{M}, x) \models Q$, $(\mathfrak{M}, x) \models \psi^*$. From the first and the second conditions we get the "explanation"—step by step—of quantifiers of φ, the third condition then gives us that φ' is true on all tuples of truth-values of $p_1, \dots, p_n$ checked during the "explanation".

So, the following propositions hold; we leave their detailed proofs to the reader.

PROPOSITION 6. — *For every quantified Boolean formula φ of the form as above, the following equivalence holds:*

$$\varphi \in \textbf{SAT-QBF} \iff \varphi^* \notin \mathbf{Int}.$$

PROPOSITION 7. — *The formula φ^* is computable on φ by a polynomial time algorithm.*

As a corollary we get that the decision problem for **Int** is PSPACE-hard. To prove that this problem is in PSPACE it is sufficient to observe that **Int** is embedded into **S4** which is known to be PSPACE-complete due to R. Ladner (Ladner, 1977). Indeed, let T be the Gödel translation defined as follows:

$$\begin{array}{lcl} \mathsf{T}(\bot) & = & \Box\bot; \\ \mathsf{T}(p) & = & \Box p, \text{ where } p \text{ is a propositional variable;} \\ \mathsf{T}(\varphi \wedge \psi) & = & \mathsf{T}(\varphi) \wedge \mathsf{T}(\psi); \\ \mathsf{T}(\varphi \vee \psi) & = & \mathsf{T}(\varphi) \vee \mathsf{T}(\psi); \\ \mathsf{T}(\varphi \to \psi) & = & \Box(\mathsf{T}(\varphi) \to \mathsf{T}(\psi)). \end{array}$$

It is well known (Chagrov *et al.*, 1997) that for every formula φ,

$$\varphi \in \mathbf{Int} \iff \mathsf{T}(\varphi) \in \mathbf{S4},$$

and hence, the decision problem for **Int** is in PSPACE. Thus, the following theorem holds.

THEOREM 8 (R. STATMAN). — *The decision problem for* **Int** *is* PSPACE*-complete.*

This result means that to answer to Kuznetsov's question about polynomial equivalence of **Int** and **Cl**, in fact, we must prove or disprove the equality "NP = PSPACE", more exactly,

$$\begin{array}{l} \text{the problem "}\varphi \in \mathbf{Int}\text{?" is polynomially} \\ \text{reducible to the problem "}\varphi \in \mathbf{Cl}\text{?"} \end{array} \iff \text{NP} = \text{PSPACE}.$$

Note that the problem "NP = PSPACE?" seems to be, in a certain sense, unassailable. Nevertheless, we give some remarks.

5. On ideas of proofs concerned complexity of Int

Note that in the proofs of that **Int** is PSPACE-hard and that the lower bound of the complexity function for **Int** is exponential, in fact, the same idea was used: both M. Zakhariaschev and R. Statman described binary trees by intuitionistic formulas. The same idea was used by other authors to get similar results for other logics, see (Ladner, 1977; Halpern *et al.*, 1992; Zakharyaschev *et al.*, 2001). Moreover, one may prove PSPACE-hardness of **Int** using a slight modification of Zakharyaschev's formulas and prove that $f_{\mathbf{Int}}$ has exponential lower bound using Statman's formulas constructed for quantified Boolean formulas of the form $\forall p_1 \ldots \forall p_n \top$. It is interesting that the purposes were different—to evaluate the bounds for the complexity function for a logic L and to prove PSPACE-hardness of L—but the proofs are similar. What it may mean?

Let's make a note about the difference of the purposes. When we prove PSPACE-completeness of a certain logic L, we usually do not make any attempt to solve the problem "$\mathrm{NP} = \mathrm{PSPACE}$?" or to answer to the Kuznetsov's question; we show only that the problem is equivalent—in the sense of polynomial reducibility—to some others. But is it possible that we are lucky with the answer if we study of the complexity function for L? As it is shown above, if a Kripke complete logic L has a "good" Kripke semantics and it has polynomial model property, then L is polynomially reducible to **Cl**; so, if it is the case for some PSPACE-complete logic, then it implies that, first, NP = PSPACE and hence, second, **Int** is polynomially reducible to **Cl**. Therefore, studying the complexity function, we are trying to obtain the answer.

But, if we think that $\mathrm{NP} \neq \mathrm{PSPACE}$, then we must presuppose also that Kripke complete PSPACE-hard logics have not polynomial model property. Another hypothesis is that if a Kripke complete logic has exponential model property then it is PSPACE-hard.

PROBLEM 9. — Is there a Kripke complete PSPACE-hard propositional logic with polynomial model property? □

PROBLEM 10. — Is there a Kripke complete propositional logic L such that L has exponential model property and the satisfiability problem for L is in NP? □

Let's return to complexity of **Int**. The logic is quite expressive and complex but what happens if we restrict the language? The question is quite natural because, first, for many purposes we need not the whole language and, second, there are examples when fragments of logics are simpler than the logics itself, see for example (Halpern, 1995). We start with restrictions on connectives.

6. Restrictions on connectives: PSPACE-hardness of the implicative fragment

Let L be a logic, C be a set of connectives; by LC we will denote the fragment of L consisting of all formulas with connectives in C. Observe that $\mathbf{Int}\{\wedge, \vee, \bot\}$ is trivial: $\mathbf{Int}\{\wedge,\vee,\bot\}\subseteq\mathbf{Cl}\{\wedge,\vee,\bot\}=\varnothing$. Thus, to obtain PSPACE-hardness of **Int**, we need implication. Note that implication is quite expressive even if it used only to define negation. For example, we have the following

PROPOSITION 11. — *The satisfiability problem for* $\mathbf{Int}\{\wedge,\neg\}$ *is* NP-*complete.*

PROOF. — For the proof, it suffices to show that $\mathbf{Int}\{\wedge,\neg\}=\mathbf{Cl}\{\wedge,\neg\}$; the last fact easily follows from Glivenko's theorem (see (Chagrov *et al.*, 1997), Corollary 2.51).

Since $\mathbf{Int}\subset\mathbf{Cl}$, we have $\mathbf{Int}\{\wedge,\neg\}\subseteq\mathbf{Cl}\{\wedge,\neg\}$.

Let's prove that $\mathbf{Cl}\{\wedge,\neg\}\subseteq\mathbf{Int}\{\wedge,\neg\}$. Let $\varphi\in\mathbf{Cl}\{\wedge,\neg\}$. Then φ must have the form $\varphi_1\wedge\ldots\wedge\varphi_m$, where each φ_i is either a variable or has the form $\neg\psi_i$. The case $\varphi_i=p_k$ is impossible (because if we define p_k to be false, φ will be false), therefore $\varphi_i=\neg\psi_i$, for every $i\in\{1,\ldots,m\}$. Since $\varphi\in\mathbf{Cl}$ and $\varphi=\neg\psi_1\wedge\ldots\wedge\neg\psi_m$, we have $\neg\psi_i\in\mathbf{Cl}$, for every $i\in\{1,\ldots,m\}$. Then, by Glivenko's theorem, for every formula ψ,

$$\psi\in\mathbf{Cl}\quad\Longleftrightarrow\quad\neg\neg\psi\in\mathbf{Int},$$

and hence, $\neg\neg\neg\psi_1\in\mathbf{Int},\ldots,\neg\neg\neg\psi_m\in\mathbf{Int}$. Note that $\neg\neg\neg p\to\neg p\in\mathbf{Int}$ and **Int** is closed under Modus Ponens. Therefore, $\neg\psi_1\in\mathbf{Int},\ldots,\neg\psi_m\in\mathbf{Int}$. Then, note that $p\to(q\to p\wedge q)\in\mathbf{Int}$; hence, $\neg\psi_1\wedge\ldots\wedge\neg\psi_m\in\mathbf{Int}$, i. e., $\varphi\in\mathbf{Int}\{\wedge,\neg\}$. ∎

Indeed in the proof in (Statman, 1979) we need conjunction and implication only; therefore, in fact R.Statman proved that $\mathbf{Int}\{\wedge,\to\}$ is PSPACE-complete. But, as it was observed by A. Chagrov (Chagrov, 1985), one may eliminate conjunction in that proof. This fact is also mentioned in passing in (Chagrov *et al.*, 1997), moreover, corresponding formulas in (Chagrov *et al.*, 1997) are specially defined so that one can easily eliminate conjunction from them and get PSPACE-hardness for $\mathbf{Int}\{\to\}$. Above we used a slight modification of those formulas; let's show how to eliminate conjunction from them.

First, we eliminate one conjunction from the consequent of every ψ_t^*: we replace ψ_t^* with the formula

$$\tilde{\psi}_t^*\quad=\quad[\bigwedge_{s=1}^{k_t}\delta_{i_{st}}\wedge\bigwedge_{s=i_t+1}^{j_t}\overline{\delta}_{st}\to\delta_n]\wedge[\bigwedge_{s=1}^{k_t}\delta_{i_{st}}\wedge\bigwedge_{s=i_t+1}^{j_t}\overline{\delta}_{st}\to\overline{\delta}_n].$$

Since $(p\to q\wedge r)\leftrightarrow[(p\to q)\wedge(p\to r)]\in\mathbf{Int}$, we have $\tilde{\psi}_t^*\leftrightarrow\psi_t^*\in\mathbf{Int}$. Let $\tilde{\varphi}^*$ be the result of replacing of every ψ_t^* with $\tilde{\psi}_t^*$ in φ^*; then $\tilde{\varphi}^*$ is computable on φ^* in polynomial time and

$$\tilde{\varphi}^*\in\mathbf{Int}\quad\Longleftrightarrow\quad\varphi^*\in\mathbf{Int}.$$

Second, we eliminate conjunction in $\tilde{\varphi}^*$. To do this, it suffices to observe that $(p\wedge q\to r)\leftrightarrow(p\to(q\to r))\in\mathbf{Int}$ and that every conjunction in $\tilde{\varphi}^*$ occurs in a premise of some implication. Therefore, the following fact holds.

THEOREM 12 (A. CHAGROV). — *The decision problem for the implicative fragment of* **Int** *is* PSPACE-*complete. The complexity function for the implicative fragment of* **Int** *is exponential.*

Note that the same holds for many other logics, see (Chagrov, 1985), in particular, for all logics in intervals $[\mathbf{BPL}, \mathbf{FPL}]$ and $[\mathbf{BPL}, \mathbf{KC}]$, where **BPL** and **FPL** are Visser basic and formal propositional logics (Visser, 1981), **KC** is the logic of the weak law of the excluded middle, i. e., $\mathbf{KC} = \mathbf{Int} + \neg p \vee \neg\neg p$. Note also that for modal logics it means that they may have PSPACE-hard fragment with strong implication[3] only (Chagrov, 1985).

Thus, we may say that **Int** is so complex due to implication. But let's consider another else restriction: the number of variables in the language. The picture will be more interesting.

7. One-variable formulas and Nishimura lattice

Note that for real-life applications of a logic—or its language with semantics of some kind—one does usually need only finitely many propositional variables. Furthermore, we can, without a loss of generality, assume that all we need is a fixed, finite set of variables. Therefore, decision problems that are of genuine practical importance are the ones for so restricted fragments of logics, rather than the ones for logics whose language contains an infinity of propositional variables.

Let's denote by $L(n)$ the n-variable fragment of a logic (a set of formulas) L, i. e., the set of formulas of L built up from the propositional constant $\perp$ as well as propositional variables $p_1, \ldots, p_n$ with the help of the connectives of the language of L. In particular, $L(0)$ stands for the variable-free fragment of L.

The reader may check that $\mathbf{Int}(0) = \mathbf{Cl}(0)$, and hence, $\mathbf{Int}(0)$ is decidable in polynomial time. Moreover, it follows from (Nishimura, 1960) that $\mathbf{Int}(1)$ is decidable in polynomial time too. Let's see, why.

For our purposes we define the following formulas in one variable, called the *Nishimura formulas*: let p be a variable and let

$$nf_\omega = \top, \quad nf_0 = \perp, \quad nf_1 = p, \quad nf_2 = \neg p,$$

and let for any $n \geqslant 0$,

$$nf_{2n+3} = nf_{2n+1} \vee nf_{2n+2}, \quad nf_{2n+4} = nf_{2n+3} \to nf_{2n+1}.$$

These formulas are connected with a model $\mathfrak{N}$ called *Rieger–Nishimura lattice*. To define it we consider the frame $\mathfrak{F} = \langle \mathbb{N}^+, R \rangle$, where R is the reflexive and transitive closure of the following relation:

$$\{\langle 3, 1\rangle\} \ \cup \ \{\langle 2n+3, 2n\rangle, \langle 2n+3, 2n+1\rangle : n \geqslant 1\} \ \cup$$
$$\{\langle 2n+4, 2n+1\rangle, \langle 2n+4, 2n+2\rangle : n \geqslant 0\}.$$

3. Strong implication $\Rightarrow$ can be defined as follows: $(\varphi \Rightarrow \psi) = \Box(\varphi \to \psi)$.

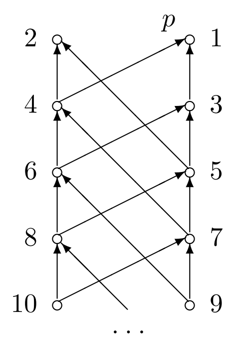

Figure 2. *Model* $\mathfrak{N}$

Let $\mathfrak{N} = \langle \mathfrak{F}, v \rangle$, where $v(p) = \{1\}$; model $\mathfrak{N}$ is depicted on Fig. 2.

Let $v(nf_m) = \{k \in \mathbb{N}^+ : (\mathfrak{N}, k) \models nf_m\}$. Let also $R(m) = \{k \in \mathbb{N}^+ : mRk\}$.

PROPOSITION 13. — *For any integer $n \geqslant 1$, the following equalities hold:*

$$\begin{array}{lllll} v(nf_{2n}) & = & R(n+1) & = & \{1, \ldots, n-2\} \cup \{n\}, \\ v(nf_{2n+1}) & = & R(n) \cup R(n+1) & = & \{1, \ldots, n+1\}. \end{array}$$

PROOF. — Induction on n. ■

REMARK 14. — As for another three Nishimura formulas, we have $v(nf_0) = \varnothing$, $v(nf_1) = \{1\}$, $v(nf_\omega) = \mathbb{N}^+$. □

PROPOSITION 15. — *Every one-variable formula is equivalent in* **Int** *to some Nishimura formula.*

PROOF. — Observe, it is sufficient to show that the set of all Nishimura formulas is closed (modulo equivalence in **Int**) under $\wedge$, $\vee$, and $\rightarrow$, i. e., that any conjunction, disjunction, and implication of Nishimura formulas is equivalent in **Int** to some Nishimura formula: indeed, p and $\perp$ are Nishimura formulas and any one-variable formula is constructed from them with $\wedge$, $\vee$, and $\rightarrow$; so, it remains to use only equivalent replacement theorem (if $\alpha \leftrightarrow \beta \in \mathbf{Int}$, then if we replace some occurrences of α in a formula ψ with ones of β, then the resulting formula will be equivalent in **Int** to ψ).

We give only an idea of the proof here; the details we leave to the reader. First of all, check that $nf_k \rightarrow nf_{2n+1} \in \mathbf{Int}$, for any n and k such that $0 \leqslant k \leqslant 2n+1$; it meant that these formulas are equivalent in **Int** to nf_ω. Then, check that for all $i, j \in \mathbb{N}$ such that $\langle i, j \rangle \neq \langle 2n+1, 2n+2 \rangle$ and $\langle i, j \rangle \neq \langle 2n+2, 2n+4 \rangle$, we have $nf_i \rightarrow nf_j \in \mathbf{Int}$ too. Now to complete the proof it remains to consider only finitely many cases; see for details (Gabbay, 1981), Chapter 6, Theorem 7. ■

So, we have many ways to define a set of formulas like Nishimura formulas; but Nishimura formulas are "economic": the reader may check that if φ is an one-variable formula and $v(\varphi) = v(nf_m)$ then $|nf_m| \leqslant |\varphi|$. Note that by Proposition 13, the set of Nishimura formulas has polynomial model property, therefore as a corollary from the Nishimura construction we get the following well known fact.

THEOREM 16. — *The complexity function for* $\mathbf{Int}(1)$ *is polynomial;* $\mathbf{Int}(1)$ *is decidable in polynomial time.*

REMARK 17. — We note that Nishimura construction provides us with other corollaries; for example, we obtain that there are exactly countably many extensions of **Int** with one-variable extra axiom. For details concerned the Nishimura formulas see also (Gabbay, 1981; Chagrov *et al.*, 1997). $\square$

So, $\mathbf{Int}$ is quite hard—it has exponential model property and PSPACE-complete decision problem—but $\mathbf{Int}(0)$ and $\mathbf{Int}(1)$ are "simple": they have polynomial (more exactly, linear) model property and they are decidable in polynomial time. A natural desire in a view of these results is to extend them to $\mathbf{Int}(n)$, for $n \geqslant 2$. But is it possible?

8. Two-variable formulas

The question about complexity of n-variable fragments of **Int** for $n \geqslant 2$ was raised by A. Chagrov on several conferences about twenty years ago; later corresponding problem was published in (Chagrov *et al.*, 1997), see Problem 18.4. The Chagrov's hypothesis was in that $\mathbf{Int}(n)$ has polynomial model property, for any $n \in \mathbb{N}$. Note that in the case the hypothesis is true, we might get—using the Kuznetsov's construction described in Section 2—that $\mathbf{Int}(n)$ is polynomially reducible to **Cl**. But indeed $\mathbf{Int}(2)$ is PSPACE-complete (Rybakov, 2006); let's see the argumentation.[4]

To prove that $\mathbf{Int}(2)$ is PSPACE-hard, we construct a polynomial reduction of $\mathbf{Int}\{\to\}$ to $\mathbf{Int}(2)$. Let p and q be different propositional variables. We define the following formulas constructed from p and q:

$$
\begin{array}{lllllllll}
D_1 & = & p \to q; \quad D_2 \;=\; q \to p; & & A_1^1 & = & A_1^0 \wedge A_2^0 \to B_1^0 \vee B_2^0;\\
D_3 & = & D_1 \wedge D_2 \to p \vee q; & & A_2^1 & = & A_1^0 \wedge B_1^0 \to A_2^0 \vee B_2^0;\\
A_1^0 & = & D_2 \to D_1 \vee D_3; & & A_3^1 & = & A_1^0 \wedge B_2^0 \to A_2^0 \vee B_1^0;\\
A_2^0 & = & D_3 \to D_1 \vee D_2; & & B_1^1 & = & A_2^0 \wedge B_1^0 \to A_1^0 \vee B_2^0;\\
B_1^0 & = & D_1 \to D_2 \vee D_3; & & B_2^1 & = & A_2^0 \wedge B_2^0 \to A_1^0 \vee B_1^0;\\
B_2^0 & = & A_1^0 \wedge A_2^0 \wedge B_1^0 \to D_1 \vee D_2 \vee D_3; & & B_3^1 & = & B_1^0 \wedge B_2^0 \to A_1^0 \vee A_2^0.
\end{array}
$$

4. The proof in (Rybakov, 2006) contains a confusing definition of used linear order $\prec$. Here we correct the definition thanks to the referee of the article.

To define other formulas we need, let's fix the following strict linear order $\prec$ on the set $\{(i,j) : i,j \geqslant 2\}$:

$$\begin{array}{rcl}(i,j) \prec (i',j') & \Longleftrightarrow & \max\{i,j\} < \max\{i',j'\}, \\ & & \text{or } \max\{i,j\} = \max\{i',j'\} = i = i' \text{ and } j > j', \\ & & \text{or } \max\{i,j\} = \max\{i',j'\} = j = j' \text{ and } i < i', \\ & & \text{or } \max\{i,j\} = \max\{i',j'\} = i' = j \text{ and } (i,j) \neq (i',j').\end{array}$$

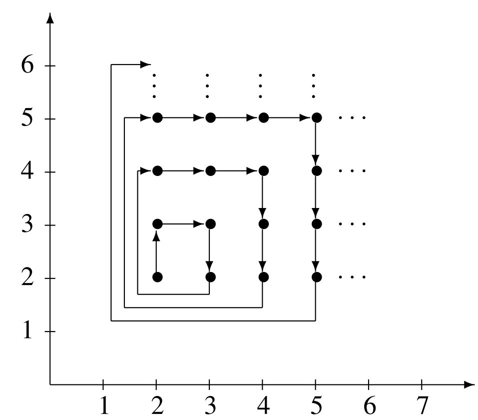

Figure 3. *Order g on* $(\mathbb{N} \setminus \{0,1\})^2$

Let g be the function enumerating the pairs of numbers accordingly with the order $\prec$, i. e., $g(1) = (2,2)$, $g(2) = (2,3)$, $g(3) = (3,3)$, $g(4) = (3,2)$, etc., see Fig. 3. Suppose the formulas $A_1^k, \dots, A_{N_k}^k, B_1^k, \dots, B_{N_k}^k$ are already defined (here and below N_k denotes the number of formulas A_i^k and also the number of formulas B_i^k). For any $i,j \in \{2,\dots,N_k\}$ and s such that $(i,j) = g(s)$, we put

$$A_s^{k+1} \;=\; A_1^k \to B_1^k \vee A_i^k \vee B_j^k; \qquad B_s^{k+1} \;=\; B_1^k \to A_1^k \vee A_i^k \vee B_j^k.$$

We point out some properties of A_i^k, B_i^k, and N_k which will be important for us below. Let

$$l = \max_{1 \leqslant i \leqslant 2} \{|A_i^0| + |B_i^0|\}.$$

LEMMA 18. — *For any $k \geqslant 0$ and any $i \in \{1,\dots,N_k\}$, the inequalities $|A_i^k| < 5^k \cdot l$ and $|B_i^k| < 5^k \cdot l$ hold.*

PROOF. — Induction on k. ■

LEMMA 19. — *For any $k \geqslant 6$, the inequality $5^k \cdot l < N_k$ holds.*

PROOF. — Straightforward. Use the facts that $N_1 = 3$, $N_{k+2} = (N_{k+1} - 1)^2$, and $l < 1000$. ■

Now we construct a Kripke model refuting all these formulas such that for every formula A_i^k and B_i^k, there exists a unique maximal world at which this formula is refuted (i. e., a world x such that A_i^k is refuted in some y if and only if x is accessible from y; and the same for B_i^k). We put

$$W \quad = \quad \{c_0, d_1, d_2, d_3\} \cup \{a_i^k, b_i^k : k \geqslant 0,\ 1 \leqslant i \leqslant N_k\}.$$

We call a_i^k and b_j^k *worlds of level* k. To define the accessibility relation R on W, let

$$\begin{aligned} R_0 \quad &= \quad \{\langle d_1, c_0\rangle, \langle d_2, c_0\rangle, \langle d_3, c_0\rangle, \langle a_1^0, d_1\rangle, \langle a_2^0, d_1\rangle, \langle b_2^0, d_1\rangle, \\ &\qquad \langle a_2^0, d_2\rangle, \langle b_1^0, d_2\rangle, \langle b_2^0, d_2\rangle, \langle a_1^0, d_3\rangle, \langle b_1^0, d_3\rangle, \langle b_2^0, d_3\rangle\}; \\ R_1 \quad &= \quad \{\langle a_1^1, b_1^0\rangle, \langle a_1^1, b_2^0\rangle, \langle a_2^1, a_2^0\rangle, \langle a_2^1, b_2^0\rangle, \langle a_3^1, a_2^0\rangle, \langle a_3^1, b_1^0\rangle\} \cup \\ &\qquad \{\langle b_1^1, a_1^0\rangle, \langle b_1^1, b_2^0\rangle, \langle b_2^1, a_1^0\rangle, \langle b_2^1, b_1^0\rangle, \langle b_3^1, a_1^0\rangle, \langle b_3^1, a_2^0\rangle\}, \end{aligned}$$

and let for every $k \geqslant 1$,

$$\begin{aligned} R_{k+1} \quad = \quad &\{\langle a_m^{k+1}, b_1^k\rangle, \langle a_m^{k+1}, a_i^k\rangle, \langle a_m^{k+1}, b_j^k\rangle : A_m^{k+1} = A_1^k \to B_1^k \vee A_i^k \vee B_j^k\} \cup \\ &\{\langle b_m^{k+1}, a_1^k\rangle, \langle b_m^{k+1}, a_i^k\rangle, \langle b_m^{k+1}, b_j^k\rangle : B_m^{k+1} = B_1^k \to A_1^k \vee A_i^k \vee B_j^k\}. \end{aligned}$$

Then we put

$$R_\omega \quad = \quad \bigcup_{k=0}^{\infty} R_k$$

and take as R the reflexive and transitive closure of the relation R_ω. Let $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{M} = \langle \mathfrak{F}, v\rangle$, where v is defined as follows: $v(p) = \{c_0, d_1\}$, $v(q) = \{c_0, d_2\}$. The model $\mathfrak{M}$ is depicted on Fig. 4.

LEMMA 20. — *Let x be a world in $\mathfrak{M}$. Then*

$$\begin{aligned} (\mathfrak{M}, x) \not\models A_m^k \quad &\Longleftrightarrow \quad xRa_m^k; \\ (\mathfrak{M}, x) \not\models B_m^k \quad &\Longleftrightarrow \quad xRb_m^k. \end{aligned}$$

PROOF. — Induction on k. ■

Let φ be an implicative formula in variables $p_1, \ldots, p_n$. We denote by k the least integer such that $|\varphi| < 5^k \cdot l$. Note that $N_{k+6} > 5^{k+6} \cdot l > 5^6 \cdot |\varphi| > |\varphi| > n$ (for the first inequality see Lemma 19), and so the following formulas are well-defined for every $i \in \{1, \ldots, n\}$:

$$\alpha_i \ = \ A_i^{k+6} \vee B_i^{k+6}.$$

Let φ_α be the result of substituting the formulas $\alpha_1, \ldots, \alpha_n$ respectively for the variables $p_1, \ldots, p_n$ in φ.

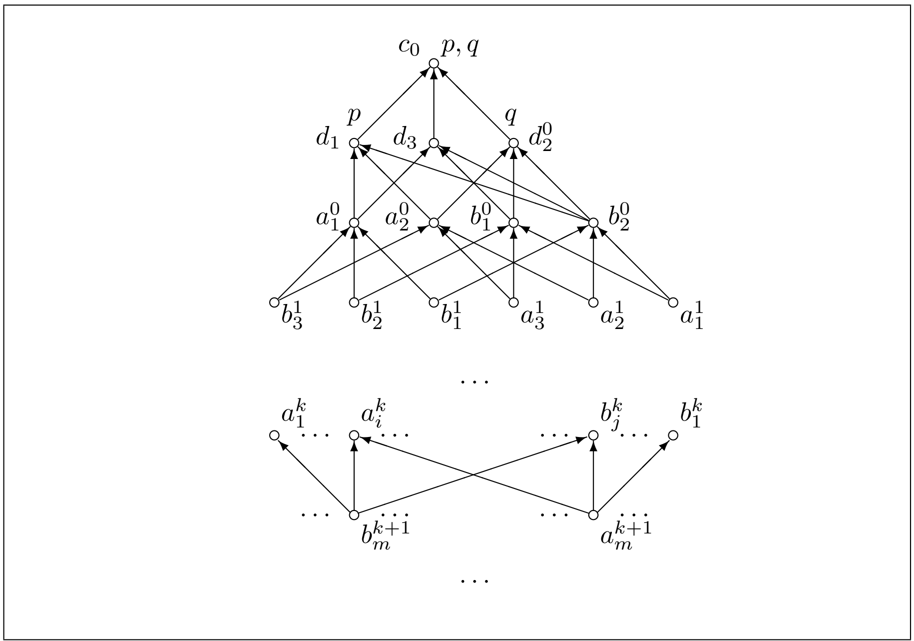

Figure 4. *Model* $\mathfrak{M}$

LEMMA 21. — *For any* $i \in \{1, \ldots, n\}$*, the formula* α_i *is computable in polynomial time on* $|\varphi|$.

PROOF. — It suffices to show that $|\alpha_i| \leqslant c \cdot |\varphi|$, for some positive number c.

Since k is the least integer satisfying the condition $|\varphi| < 5^k \cdot l$, we have $5^{k-1} \cdot l \leqslant |\varphi|$, and hence $5^{k+6} \cdot l \leqslant 5^7 \cdot |\varphi|$. By Lemma 18, $|A_i^{k+6}| < 5^{k+6} \cdot l$ and $|B_i^{k+6}| < 5^{k+6} \cdot l$, wherefore $|\alpha_i| < 2 \cdot 5^{k+6} \cdot l \leqslant 2 \cdot 5^7 \cdot |\varphi|$.

Other details we leave to the reader. ■

LEMMA 22. — *There exists a polynomial time algorithm that for any implicative formula* φ*, computes* φ_α.

PROOF. — We must find k, then compute the formulas $\alpha_1, \ldots, \alpha_n$, and then replace each p_i in φ with α_i; it remains to observe that all these steps take a polynomial time on $|\varphi|$. ■

Thus, to get PSPACE-hardness of $\mathbf{Int}(2)$ it suffices to prove

LEMMA 23. — *For every implicative formula* φ,

$$\varphi \in \mathbf{Int}\{\rightarrow\} \iff \varphi_\alpha \in \mathbf{Int}(2).$$

PROOF. — If $\varphi \in \mathbf{Int}\{\rightarrow\}$, then $\varphi_\alpha \in \mathbf{Int}(2)$ since $\mathbf{Int}$ is closed under Substitution.

Let φ be an implicative formula such that $\varphi \notin \mathbf{Int}$. Then there exists an intuitionistic model $\mathfrak{M}_\varphi = \langle \mathfrak{F}_\varphi, v_\varphi \rangle$ defined on a frame $\mathfrak{F}_\varphi = \langle W_\varphi, R_\varphi \rangle$ such that $(\mathfrak{M}_\varphi, x_0) \not\models \varphi$, for some $x_0 \in W_\varphi$.

We construct an intuitionistic model refuting φ_α. The general idea how to construct the model is in the following: for any world x in $\mathfrak{M}_\varphi$ such that $(\mathfrak{M}_\varphi, x) \not\models p_i$ we make accessible the worlds a_i^{k+6} and b_i^{k+6} of $\mathfrak{M}$ from x; this will ensure that in the new model the formula α_i will be refuted at x. Then, we must make sure that φ_α will be refuted at x_0.

Without any loss of generality we may (and will) assume that $W \cap W_\varphi$ is empty. We put $W^* = W \cup W_\varphi$. Let

$$\begin{aligned} R' \quad = \quad & \{\langle x, a_i^{k+6}\rangle, \langle x, b_i^{k+6}\rangle : x \in W_\varphi,\ (\mathfrak{M}_\varphi, x) \not\models p_i,\ 1 \leqslant i \leqslant n\} \cup \\ & \{\langle x, a_{n+1}^{k+6}\rangle, \langle x, b_{n+1}^{k+6}\rangle : x \in W_\varphi\}. \end{aligned}$$

We define the relation R^* as the reflexive and transitive closure of $R \cup R_\varphi \cup R'$ and put $\mathfrak{F}^* = \langle W^*, R^* \rangle$. We define the model $\mathfrak{M}^* = \langle \mathfrak{F}^*, v^* \rangle$ on $\mathfrak{F}^*$ putting $v^*(p) = \{c_0, d_1\}$, $v^*(q) = \{c_0, d_2\}$.

For every $\psi \in \operatorname{sub}\varphi$, let ψ_α be the result of substituting the formulas $\alpha_1, \dots, \alpha_n$ respectively for $p_1, \dots, p_n$ in ψ. By induction on the construction of ψ, let's show that for every $x \in W_\varphi$,

$$(\mathfrak{M}^*, x) \models \psi_\alpha \iff (\mathfrak{M}_\varphi, x) \models \psi. \tag{$*$}$$

Let $\psi = p_m$. If $(\mathfrak{M}_\varphi, x) \not\models p_m$, then the worlds a_m^{k+6} and b_m^{k+6} in $\mathfrak{M}^*$ are accessible from x. But $(\mathfrak{M}, a_m^{k+6}) \not\models A_m^{k+6}$, $(\mathfrak{M}, b_m^{k+6}) \not\models B_m^{k+6}$. It is clear that in this case $(\mathfrak{M}^*, a_m^{k+6}) \not\models A_m^{k+6}$ and $(\mathfrak{M}^*, b_m^{k+6}) \not\models B_m^{k+6}$, hence $(\mathfrak{M}^*, x) \not\models \alpha_m$, i.e., $(\mathfrak{M}^*, x) \not\models \psi_\alpha$.

Now suppose that $(\mathfrak{M}^*, x) \not\models \alpha_m$, for some world $x \in W_\varphi$; let $g(m) = (i, j)$. Then there exist worlds x' and x'' in $\mathfrak{M}^*$ such that xR^*x', xR^*x'', and

$$\begin{array}{lll} (\mathfrak{M}^*, x') \models A_1^{k+5}, & (\mathfrak{M}^*, x') \not\models B_1^{k+5}, & (\mathfrak{M}^*, x') \not\models A_i^{k+5} \vee B_j^{k+5}; \\ (\mathfrak{M}^*, x'') \not\models A_1^{k+5}, & (\mathfrak{M}^*, x'') \models B_1^{k+5}, & (\mathfrak{M}^*, x'') \not\models A_i^{k+5} \vee B_j^{k+5}. \end{array}$$

Note that $x', x'' \notin W_\varphi$. Indeed, for any $y \in W_\varphi$, we have $yR^*a_{n+1}^{k+6}$ and $yR^*b_{n+1}^{k+6}$. Since $(\mathfrak{M}^*, a_{n+1}^{k+6}) \not\models B_1^{k+5}$ and $(\mathfrak{M}^*, b_{n+1}^{k+6}) \not\models A_1^{k+5}$, we obtain $(\mathfrak{M}^*, y) \not\models A_1^{k+5}$ and $(\mathfrak{M}^*, y) \not\models B_1^{k+5}$. But $(\mathfrak{M}^*, x') \models A_1^{k+5}$ and $(\mathfrak{M}^*, x'') \models B_1^{k+5}$, therefore it is clear that $x', x'' \notin W_\varphi$.

Since x' and x'' are R^*-accessible from some world in W_φ, we see that their level is not greater than $k+6$. On the other hand, $(\mathfrak{M}^*, x') \not\models A_m^{k+6}$ and $(\mathfrak{M}^*, x'') \not\models B_m^{k+6}$, therefore the level of x' and x'' is not less than $k+6$. So, x' and x'' are worlds of level $k+6$. But the unique world of level $k+6$ refuting A_m^{k+6} is a_m^{k+6} and the unique world of level $k+6$ refuting B_m^{k+6} is b_m^{k+6}, hence $x' = a_m^{k+6}$ and $x'' = b_m^{k+6}$.

Thus, $xR^* a_m^{k+6}$. This means that there exists a world $y \in W_\varphi$ such that $xR_\varphi y$ and $yR' a_m^{k+6}$. But if $yR' a_m^{k+6}$ holds, then $yR' b_m^{k+6}$ must hold, which is possible only if $(\mathfrak{M}_\varphi, y) \not\models p_m$. Since $xR_\varphi y$, we obtain $(\mathfrak{M}_\varphi, x) \not\models p_m$.

Let $\psi = \psi' \to \psi''$, $\psi \in \mathrm{sub}\,\varphi$, and for every $x \in W_\varphi$,

$$\begin{array}{lcl}(\mathfrak{M}^*, x) \models \psi'_\alpha & \iff & (\mathfrak{M}_\varphi, x) \models \psi'; \\ (\mathfrak{M}^*, x) \models \psi''_\alpha & \iff & (\mathfrak{M}_\varphi, x) \models \psi''.\end{array}$$

If $(\mathfrak{M}_\varphi, x) \not\models \psi$, then there is a world x' in $\mathfrak{M}_\varphi$ such that xRx', $(\mathfrak{M}_\varphi, x') \models \psi'$, and $(\mathfrak{M}_\varphi, x') \not\models \psi''$; but then $(\mathfrak{M}^*, x') \models \psi'_\alpha$ and $(\mathfrak{M}^*, x') \not\models \psi''_\alpha$, hence $(\mathfrak{M}^*, x) \not\models \psi_\alpha$.

Suppose $(\mathfrak{M}^*, x) \not\models \psi_\alpha$, for some $x \in W_\varphi$. Then there is $x' \in W^*$ such that xR^*x', $(\mathfrak{M}^*, x') \models \psi'_\alpha$, and $(\mathfrak{M}^*, x') \not\models \psi''_\alpha$. Note that the formulas $\alpha_1, \ldots, \alpha_n$ are true in every world $y \in W$ of level not greater than $k+6$. Hence in all such worlds the formulas constructed from $\alpha_1, \ldots, \alpha_n$ using implication only, must be true; in particular, ψ''_α must be true. Since $(\mathfrak{M}^*, x') \not\models \psi''_\alpha$ we see that x' cannot be a world in W and hence $x' \in W_\varphi$. By applying the induction hypothesis, we obtain $(\mathfrak{M}_\varphi, x') \models \psi'$ and $(\mathfrak{M}_\varphi, x') \not\models \psi''$, hence $(\mathfrak{M}_\varphi, x) \not\models \psi$.

Thus, $(*)$ is proved. Since $(\mathfrak{M}_\varphi, x_0) \not\models \varphi$, by $(*)$ we have $(\mathfrak{M}^*, x_0) \not\models \varphi_\alpha$, and therefore $\varphi_\alpha \notin \mathbf{Int}$. ■

As a result we immediately obtain

THEOREM 24. — *The decision problem for the two-variable fragment of* **Int** *is* PSPACE-*complete.*

Now let's turn to the complexity function for the two-variable fragments of **Int**. Recall that if this function is polynomial then we might prove, using the Kuznetsov's argumentation, that $\mathbf{Int}(2)$ is polynomially reducible to **Cl**. But it follows from Theorem 24 that

$$\begin{array}{l}\text{the problem “}\varphi \in \mathbf{Int}(2)\text{?” is polynomi-} \\ \text{ally reducible to the problem “}\varphi \in \mathbf{Cl}\text{?”}\end{array} \iff \mathrm{NP} = \mathrm{PSPACE}.$$

Since the equality $\mathrm{NP} = \mathrm{PSPACE}$ seems to be doubtful, we may expect that the complexity function for $\mathbf{Int}(2)$ is not polynomial, and it is indeed the case: using implicative version of the formulas in Section 4 constructed for quantified Boolean formulas of the form $\forall p_1 \ldots \forall p_n \top$ and the construction from the proof of Theorem 24, one may prove the following assertion.

COROLLARY 25. — *The complexity function for the two-variable fragment of* **Int** *is exponential.*

9. Restrictions on both connectives and variables

To prove PSPACE-completeness of $\mathbf{Int}(2)$ we used $\to$, $\wedge$, and $\vee$. But one may eliminate conjunction from the formulas in the proof: it suffices to observe that $(p\wedge q\to r)\leftrightarrow(p\to(q\to r))\in\mathbf{Int}$. So, we have

COROLLARY 26. — *The decision problem for the two-variable fragment of* **Int** *with disjunction and implication only is* PSPACE-*complete; the complexity function for this fragment is exponential.*

What we can say about complexity of other fragments? The answer follows from Diego's theorem, see (Diego, 1966; Chagrov *et al.*, 1997). Let's recall it. Let $\mathcal{L}$ be a set of formulas; for every formula $\varphi\in\mathcal{L}$, we define the equivalence class $[\varphi]_{\mathcal{L}}=\{\psi\in\mathcal{L}:\psi\leftrightarrow\varphi\in\mathbf{Int}\}$.

THEOREM 27 (DIEGO). — *Let n be a non-negative integer, $\mathcal{L}_n$ be the set of formulas built up from the variables $p_1,\dots,p_n$ with the help of $\wedge$, $\to$, and $\bot$. Then there are formulas $\varphi_1,\dots,\varphi_m\in\mathcal{L}_n$ such that $\mathcal{L}_n=[\varphi_1]_{\mathcal{L}_n}\cup\ldots\cup[\varphi_m]_{\mathcal{L}_n}$.*

So, any disjunction-free formula in n variables is equivalent in **Int** to a disjunction-free formula of some fixed finite set of formulas in n variables. It means that n-variable fragment of $\mathbf{Int}\{\wedge,\to,\bot\}$ is locally tabular,[5] and hence, it is decidable in polynomial time. Therefore, if $\mathrm{P}\neq\mathrm{PSPACE}$, to prove PSPACE-hardness of $\mathbf{Int}(n)$ we need implication and disjunction.

REMARK 28. — Observe that to prove PSPACE-hardness of $\mathbf{Int}(2)$ disjunction was used without restrictions on its nesting. Let $\mathrm{nd}(\varphi)$ be the number of disjunction nesting in φ, i. e.,

$$\begin{aligned}&\mathrm{nd}(p_i)=\mathrm{nd}(\bot)=0;\\&\mathrm{nd}(\psi'\wedge\psi'')=\mathrm{nd}(\psi'\to\psi'')=\max\{\mathrm{nd}(\psi'),\mathrm{nd}(\psi'')\};\\&\mathrm{nd}(\psi'\vee\psi'')=\max\{\mathrm{nd}(\psi'),\mathrm{nd}(\psi'')\}+1.\end{aligned}$$

From Diego's theorem we immediately obtain that the set $\{\varphi\in\mathbf{Int}(n):\mathrm{nd}(\varphi)\leqslant m\}$ is decidable in polynomial time, for every non-negative integers n and m. □

10. Some corollaries and questions

Note that for most results mentioned above we need only formulas without $\bot$; such formulas are called *positive*. Therefore, we can apply the same argumentation for many superintuitionistic logics. Let's consider the logic of the weak law of the excluded middle $\mathbf{KC}=\mathbf{Int}+\neg p\vee\neg\neg p$. This logic is Kripke complete: it is characterized by the class of rooted strongly directed[6] (or convergent) Kripke frames,

5. A logic is called *locally tabular* if for every integer $n\geqslant 0$, it contains only a finite number of pairwise nonequivalent formulas built up from variables $p_1,\dots,p_n$.

6. A rooted Kripke frame is called *strongly directed* if for any two worlds in it, there exists a world accessible from both of them.

see (Chagrov *et al.*, 1997). It is known also (Jankov, 1968) that **KC** is a biggest superintuitionistic logic with the same positive fragment as **Int**, i. e., for every superintuitionistic logic L,

$$L\{\wedge,\vee,\rightarrow\} = \mathbf{Int}\{\wedge,\vee,\rightarrow\} \quad \Longleftrightarrow \quad L \subseteq \mathbf{KC}.$$

In particular, we have

– the positive and implicative fragments of every logic L in the interval $[\mathbf{Int},\mathbf{KC}]$ are PSPACE-complete;

– the two-variable fragment of $L\{\rightarrow,\vee\}$ is PSPACE-complete, for every logic L in the interval $[\mathbf{Int},\mathbf{KC}]$;

– the one-variable fragment of every logic L in the interval $[\mathbf{Int},\mathbf{KC}]$ is decidable in polynomial time;

– the n-variable fragment of $L\{\wedge,\rightarrow,\bot\}$ is decidable in polynomial time, for every logic L in the interval $[\mathbf{Int},\mathbf{KC}]$ and every non-negative integer n.

There are a lot of quite important and interesting logics between **Int** and **KC**; let's say some words about only two of them: Kreisel–Putnam logic **KP** and Medvedev logic **ML**. Kreisel–Putnam logic is obtained from **Int** by adding the Kreisel–Putnam axiom:

$$\mathbf{KP} \quad = \quad \mathbf{Int} + (\neg p \rightarrow q \vee r) \rightarrow (\neg p \rightarrow q) \vee (\neg p \rightarrow r).$$

Medvedev logic **ML** can be defined semantically as follows. Let n be a strictly positive integer and let W_n be the set of all non-empty subsets of a certain n-element set. Let xR_ny mean $y \subseteq x$, for every $x, y \in W_n$. The pair $\mathfrak{F}_n = \langle W_n, R_n\rangle$ is clearly a Kripke frame. Then

$$\mathbf{ML} \quad = \quad \{\varphi : \text{for all } n > 0,\ \mathfrak{F}_n \models \varphi\}.$$

Because **ML** has the same positive fragment as **Int** (Szatkowski, 1981), from the result of V. Jankov mentioned above it follows that $\mathbf{ML} \subseteq \mathbf{KC}$, and hence the interval $[\mathbf{KP},\mathbf{ML}]$ is a subinterval of $[\mathbf{Int},\mathbf{KC}]$; in particular, both **KP** and **ML** are PSPACE-hard. In general, the decision problem for **KP** and **ML** is rather complex. To see this, let's consider the complexity function for these logics. As it is known (see (Chagrov *et al.*, 1997), Exercise 2.10), finite rooted frames for any logic $L \in [\mathbf{KP},\mathbf{ML}]$ have the following property: for every subset of the set of all final worlds in such frame, there is a world in the frame from which all worlds in the subset are accessible and all other final worlds are not accessible. Therefore, using the formulas β_n defined in Section 3, one may get that for any logic $L \in [\mathbf{KP},\mathbf{ML}]$,

$$f_L(n) \geqslant 2^{2^{k\cdot n}},$$

for some $k > 0$; see (Chagrov *et al.*, 1997), Theorem 18.7. It is also known that **KP** is decidable (Gabbay, 1970) but the question if the decision problem for **KP** is in PSPACE, is still open. The decidability of **ML** is a famous open problem.

We put some else questions for **KP** and **ML**.

PROBLEM 29. — Let n be an integer, $n \geqslant 2$. Is $\mathbf{ML}(n)$ decidable? If the answer is "yes" then

– to which complexity class $\mathbf{ML}(n)$ belongs, in particular, is it PSPACE-complete?
– is the complexity functions for $\mathbf{ML}(n)$ exponential?

$\Box$

The same questions are actual for **KP** too; but we must take into account that **KP** is decidable.

PROBLEM 30. — Let n be an integer, $n \geqslant 2$. To which complexity class $\mathbf{KP}(n)$ belongs, in particular, is it PSPACE-complete? Is the complexity functions for $\mathbf{KP}(n)$ exponential? $\Box$

Presently, we see an interest to various modal logics; note, however, that many of them are extensions—modulo the Gödel translation—of **Int**. Therefore, most of ideas working for **Int** can be applied to modal logics too; moreover, some ideas are applied simpler than in "intuitionistic" case; see, for example, (Ladner, 1977; Halpern *et al.*, 1992; Chagrov *et al.*, 1997; Zakharyaschev *et al.*, 2001) for PSPACE-complete logics, (Fisher *et al.*, 1979; Spaan, 1993; Hemaspaandra, 1996) for EXPTIME-complete logics, (Spaan, 1993; Halpern, 1995; Chagrov *et al.*, 2003) for fragments of modal logics. Here we mention only several logics for which their complexity was proved recently; namely, let

$$
\begin{array}{lcl}
\mathbf{LM}_1 & = & \mathbf{K4} \oplus \Diamond p \wedge \Diamond q \to \Diamond(\Diamond p \wedge \Diamond q) \oplus \Diamond\top; \\
\mathbf{LM}_2 & = & \mathbf{LM}_1 \oplus \Diamond\Box p \to \Box\Diamond p; \\
\mathbf{LM}_3 & = & \mathbf{K4} \oplus \Diamond p \wedge \Diamond q \to \Diamond(\Diamond p \wedge \Diamond q) \oplus \Diamond\top \to \Diamond\Box\bot,
\end{array}
$$

where $\oplus$ means closure under Modus Ponens, Necessary Rule (Gödel Rule), and Substitution. These logics are PSPACE-hard and even have PSPACE-hard fragments in the language with strong implication only (to prove this it suffices to take the results of the Gödel translation for the formulas considered in Section 4 and modified as it described before Theorem 12). These logics belong to PSPACE, and hence, PSPACE-complete (Shapirovskiy, 2004; Shapirovskiy, 2005). Using the construction in Section 8, we obtain that $\mathbf{LM}_1(2)$, $\mathbf{LM}_2(2)$, and $\mathbf{LM}_3(2)$ are PSPACE-complete too. But we cannot obtain complexity results for $\mathbf{LM}_1(1)$, $\mathbf{LM}_2(1)$, and $\mathbf{LM}_3(1)$ from complexity of one-variable fragments of **Int** and its extensions directly. Nevertheless, it seems to be probable the following conjecture.

CONJECTURE 31. — The one-variable fragments of $\mathbf{LM}_1$, $\mathbf{LM}_2$, and $\mathbf{LM}_3$ are PSPACE-complete. $\Box$

Acknowledgements

I am very grateful to the anonymous referee, whose notes and advises substantially helped to improve the article. The work on this paper was supported by RFBR, Projects 06–06–80380, 06–01–72555, and 07–06–00318.

11. References

Chagrov A., "On the Complexity of Propositional Logics", *Complexity Problems of Mathematical Logic*, Kalinin State University, Kalinin, pp. 80–90, 1985. (In Russian).

Chagrov A., Rybakov M., "How Many Variables One Needs to Prove PSPACE-Hardness of Modal Logics?", *Advances in Modal Logic*, vol. 4, pp. 71–81, 2003.

Chagrov A., Zakharyaschev M., *Modal Logic*, Oxford University Press, 1997.

Cook S., "The Complexity of Theorem-Proving Procedures", *Proceedings of the Third Annual ACM Symposium on the Theory of Computation*, pp. 151–158, 1971.

Diego A., *Sur les algèbres de Hilbert*, Gauthier–Villars, Paris, 1966.

Fisher M., Ladner R., "Propositional Dynamic Logic of Regular Programs", *J. Comput. Syst. Sci.*, vol. 18, num. 2, pp. 194–211, 1979.

Gabbay D., "The Decidability of the Kreisel–Putnam System", *The Journal of Symbolic Logic*, vol. 35, num. 3, pp. 431–437, 1970.

Gabbay D., *Semantical Investigations in Heyting's Intuitionistic Logic*, D. Reidel Publishing Company, Dordrecht, Holland, 1981.

Glivenko V., "Sur quelques points de la logique de M. Brouwer", *Bulletin de la Classe des Sciences de l'Académie Royale de Belgique*, vol. 15, pp. 183–188, 1929.

Halpern J., "The Effect of Bounding the Number of Primitive Propositions and the Depth of Nesting on the Complexity of Modal Logic", *Artificial Intelligence*, vol. 75, num. 2, pp. 361–372, 1995.

Halpern J., Moses Y., "A Guide to Completeness and Complexity for Modal Logics of Knowledge and Belief", *Artificial Intelligence*, vol. 54, num. 3, pp. 319–379, 1992.

Hemaspaandra E., "The Price of Universality", *Notre Dame Journal of Formal Logic*, vol. 37, num. 2, pp. 174–203, 1996.

Jankov V., "On The Calculus of the Weak Law of Excluded Middle", *Mathematics of the USSR, Izvestia*, vol. 2, num. 5, pp. 1044–1051, 1968. (In Russian).

Kuznetsov A., "On Superintuitionistic Logics", *Proceedings of the International Congress of Mathematicians*, Vancouver, pp. 243–249, 1975.

Kuznetsov A., "On Facilities for Detection of Non-Derivability and Non-Expressibility", *Logical Derivation*, Nauka, Moscow, pp. 5–33, 1979. (In Russian).

Ladner R., "The Computational Complexity of Provability in Systems of Modal Logic", *SIAM Journal on Computing*, vol. 6, pp. 467–480, 1977.

Nishimura I., "On formulas of the one variable in intuitionistic propositional calculus", *The Journal of Symbolic Logic*, vol. 25, pp. 327–331, 1960.

Papadimitriou C., *Computational Complexity*, Addison–Wesley Publishing Company, 1995.

Popov S., Zakharyaschev M., "On the Complexity of Kripke Countermodels in Intuitionistic Propositional Calculus", *Proceedings of the 2nd Soviet–Finland Logic Colloquium*, pp. 32–36, 1979. (In Russian).

Popov S., Zakharyaschev M., On the Cardinality of Countermodels of Intuitionistic Calculus, Technical Report num. 45, Institute of Applied Mathematics, Russian Academy of Science, 1980. (In Russian).

Rybakov M., "Complexity of Intuitionistic and Visser's Basic and Formal Logic in Finitely Many Variables", *Advances in Modal Logic*, vol. 6, pp. 393–411, 2006.

Savitch W., "Relationships Between Nondeterministic and Deterministic Tape Complexity", *Journal of Computer and System Science*, vol. 4, pp. 177–192, 1970.

Shapirovskiy I., "PSPACE-Decision Procedure for Some Transitive Modal Logics", *Proc. of Advances in Modal Logics*, King's College Publications, pp. 331–343, 2004.

Shapirovskiy I., "On PSPACE-Decidability in Transitive Modal Logics", *Advances in Modal Logics*, vol. 5, pp. 269–287, 2005.

Spaan E., Complexity of Modal Logics, PhD thesis, University of Amsterdam, Department of Mathematics and Computer Science, 1993.

Statman R., "Intuitionistic Propositional Logic is Polynomial-Space Complete", *Theoret. Comput. Sci.*, vol. 9, num. 1, pp. 67–72, 1979.

Stockmeyer L., "Classifying the Computational Complexity of Problems", *The Journal of Symbolic Logic*, vol. 52, num. 1, pp. 1–43, 1987.

Szatkowski M., "On Fragments of Medvedev's Logic", *Studia Logica*, vol. 40, num. 1, pp. 39–54, 1981.

Visser A., "A Propositional Logic with Explicit Fixed Points", *Studia Logica*, vol. 40, pp. 155–175, 1981.

Zakharyaschev M., Wolter F., Chagrov A., "Advanced Modal Logic", *in* D. Gabbay, F. Guenthner (eds), *Handbook of Philosophical Logic*, 2nd edn, vol. 3, Kluwer Academic Publishers, pp. 83–266, 2001.