A New Primitive for a Diffie-Hellman-like Key Exchange Protocol Based on Multivariate Ore Polynomials

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Abstract. In this paper we present a new primitive for a key exchange protocol based on multivariate non-commutative polynomial rings, analogous to the classic Diffie-Hellman method. Our technique extends the proposed scheme of Boucher et al. from 2010. Their method was broken by Dubois and Kammerer in 2011, who exploited the Euclidean domain structure of the chosen ring. However, our proposal is immune against such attacks, without losing the advantages of non-commutative polynomial rings as outlined by Boucher et al. Moreover, our extension is not restricted to any particular ring, but is designed to allow users to readily choose from a large class of rings when applying the protocol. Our primitive can also be applied to other cryptographic paradigms. In particular, we develop a three-pass protocol, a public key cryptosystem, a digital signature scheme and a zero-knowledge proof protocol.

Keywords: Cryptography, non-commutative, multivariate polynomial rings, skew-polynomials, Ore algebra

1 Introduction

In 2010, Boucher et al. [BGG+10] proposed a novel Diffie-Hellman-like key exchange protocol [DH76] based on skew-polynomial rings. An outline of their method can be given as follows (with adapted notation): Two communicating parties, Alice and Bob, publicly agree on an element L in a predetermined skew-polynomial ring, and on a subset $\mathcal{S}$ of commuting elements in this ring. Alice then chooses two private keys P_A, Q_A from $\mathcal{S}$ and sends Bob the product $P_A \cdot L \cdot Q_A$. Bob similarly chooses P_B, Q_B from $\mathcal{S}$ and sends Alice $P_B \cdot L \cdot Q_B$. Alice computes $P_A \cdot P_B \cdot L \cdot Q_B \cdot Q_A$, while Bob computes $P_B \cdot P_A \cdot L \cdot Q_A \cdot Q_B$. Since $P_A \cdot P_B = P_B \cdot P_A$, and $Q_A \cdot Q_B = Q_B \cdot Q_A$, Alice and Bob have computed the same final element, which can be used as a secret key, either directly or by hashing. Boucher et al. claimed that it would be intractable for an eavesdropper, Eve, to compute this secret key with knowledge only of L, $\mathcal{S}$, $P_A \cdot L \cdot Q_A$ and $P_B \cdot L \cdot Q_B$. They based their claim on the difficulty of the factorization problem in skew-polynomial rings, in particular the non-uniqueness of factorizations.

However, in 2011, Dubois and Kammerer exploited the fact that the concrete skew-polynomial ring chosen by Boucher et al. is a Euclidean domain to successfully attack their protocol [DK11]. Following their approach, an eavesdropper Eve chooses a random element $e \in \mathcal{S}$, and computes the greatest common right divisor of $P_A \cdot L \cdot Q_A \cdot e = P_A \cdot L \cdot e \cdot Q_A$ with $P_A \cdot L \cdot Q_A$, which is with high probability equal to Q_A. From this point on, Eve can easily recover the agreed upon key between Alice and Bob. Moreover, the authors also criticized the suggested brute-force method for Alice and Bob to generate commuting polynomials, as most of the commuting polynomials turn out to be central and thus the possible choices for private keys becomes fairly small. (An element of a ring is **central** if it commutes with all other elements of the ring.)

After Dubois and Kammerer's paper, interest in the application of non-commutative polynomial rings appears to have dwindled, since to the best of our knowledge no further publications considering non-commutative polynomial rings in cryptographic contexts have appeared.

It is our position that such rings can still be used as a foundation of a secure Diffie-Hellman-like protocol. The basic weakness in the scheme presented in [BGG$^+$10] lies in the choice of a univariate Ore extension as the underlying ring of the protocol, as these rings are Euclidean domains. However, the construction of Ore extensions can be iterated, and the resulting multivariate Ore polynomial rings will no longer be principal ideal domains (and therefore not Euclidean domains). This would preclude any attack of the type proposed by Dubois and Kammerer.

The contributions of this paper are the following:

- We present a method of constructing non-commutative algebras to use in the protocol as presented in [BGG$^+$10]. The creation of algebras in this fashion ensures that the Diffie-Hellman-like key exchange will not be subject to attacks as described in [DK11]. At the same time the desirable properties such as non-uniqueness of the factorization remain present, as well as key-generation in a feasible amount of time.
- For our choice of non-commutative algebras, no polynomial time factorization algorithm for their elements is known. For most of them, there is not even a general factorization technique, i.e. one that is applicable to any element, discovered yet and, as some of them do not even have the property of being Noetherian, factorization algorithms may not even exist.
- This paper addresses the critique given in [DK11] concerning the feasible construction of a set $\mathcal{S}$ of commuting, non-central elements, from which the two communicating parties choose their private keys. We show an efficient way to construct commuting polynomials, which is independent from the choice of the algebra.
- We have made an experimental implementation of the key-exchange protocol using our proposed rings, in order to examine the practical feasibility of our primitive. Furthermore, we have created challenge problems for the reader who wishes to examine the security of our schemes.
- Attacks based on the key-choice of A and B are studied for a concrete algebra and an overview of weak keys and their detection is presented.
- We also study the application of multivariate Ore polynomials to other cryptographic paradigms: a three-pass protocol, a public key cryptosystem, a digital signature scheme and a zero-knowledge proof protocol.

In the rest of this section, we will introduce some basic notations and definitions. Moreover, we present the main ring structures which will be used for the various cryptographic protocols. Furthermore, related work and the potential for these rings to be used in post-quantum cryptosystems will be discussed.

In section 2, we show how multivariate Ore polynomial rings can be applied as a primitive for a Diffie-Hellman-like key exchange protocol. We will argue its correctness, its efficiency and its security.

Section 3 describes an implementation of our proposal for a specific choice of the non-commutative ring. Experimental results are presented for different input-sizes.

In section 4, we discuss some known insecure keys for a particular ring, namely the n^{th} Weyl algebra. The described construction cannot always be generalized to other rings, albeit it provides a guideline for the study of insecure keys in rings that can be used in cryptographic protocols.

Before our conclusion in section 6, we present in section 5 applications of multivariate Ore polynomial rings to other cryptographic paradigms.

1.1 Basic Notations and Definitions

Throughout the whole paper, R denotes an arbitrary domain with identity. For practical reasons we furthermore assume for any introduced ring that it is computable, i.e. one can find a finite representation of its elements, and that all arithmetics can be done in polynomial time. Also a random choice of an element in the ring R is assumed to be possible with polynomial costs.

Let us address the basic construction principles of so-called Ore extensions [Ore33] of R. We follow the notions from [BGTV03], which we also recommend as a resource for a thorough introduction into the field of algorithmic non-commutative algebra.

Definition 1 ([BGTV03], Definition 3.1). *Let σ be a ring endomorphism of R. A* ***σ-derivation*** *of R is an additive endomorphism $\delta : R \to R$ with the property that $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for all $r, s \in R$. We call the pair (σ, δ) a* ***quasi-derivation****.*

Proposition 1 ([BGTV03], Proposition 3.3.). *Let (σ, δ) be a quasi-derivation on R. Then there exists a ring S with the following properties:*

1. *R is a subring of S;*
2. *there exists an element $\partial \in S$ such that S is freely generated as a left R-module by the positive powers $1, \partial, \partial^2, \ldots$ of ∂;*
3. *for every $r \in R$, we have $\partial r = \sigma(r)\partial + \delta(r)$.*

Definition 2 (cf. [BGTV03], Definition 3.4). *The ring S defined by the previous result and denoted by $R[\partial\,; \sigma, \delta]$ is usually referred to as an* ***Ore extension of*** *R.*

General Assumption: As we want the Ore extension to have at least the property of being a domain, we assume from now on that σ is always injective (compare [BGTV03], Proposition 3.10). In order to keep the costs of arithmetics in $R[\partial\,; \sigma, \delta]$ polynomial, we make two additional assumptions:

(i) There exist polynomial time algorithms to compute $\sigma(r)$ and $\delta(r)$ for any given $r \in R$.
(ii) Either σ is the identity map, or δ is the zero map.

While item (i) seems to be a natural assumption, item (ii) may seem highly restrictive. But these cases cover several algebras that are studied in practice, as we will point out in the examples below. The need for this condition comes from the result of the following lemma, which can be easily proven by induction on $n \in \mathbb{N}$.

Lemma 1. *Let $R[\partial\,; \sigma, \delta]$ be an Ore extension of R, and let r be an arbitrary element in R. Then we have the following identity for $n \in \mathbb{N}$:*

$$\partial^n r = \sigma^n(r)\partial^n + \left(\sum_{\theta \in S_n \bullet [\underbrace{\sigma, \ldots, \sigma}_{n-1\ times}, \delta]} \theta_1 \circ \ldots \circ \theta_n \circ r\right)\partial^{n-1} + \ldots$$

$$+ \left(\sum_{\theta \in S_n \bullet [\sigma, \delta, \ldots, \delta]} \theta_1 \circ \ldots \circ \theta_n \circ r\right)\partial + \delta^n(r),$$

where S_n denotes the permutation group on n elements and $\bullet$ the canonical action of the group on a list with n elements.

Without item (ii), when multiplying elements in S, we would have to compute up to 2^n images of an element $r \in R$ resulting from all different ways of applying n times functions chosen from the set $\{\sigma, \delta\}$. This is avoided by choosing one of the maps to be trivial, i.e. by our assumption (ii).

Example 1. For the Ore extensions considered in the paper [BGG+10], the authors assumed R to be a finite field, σ to be a power of the Frobenius automorphism on R and δ to be the zero map.

Example 2. The construction of a commutative polynomial ring over a given ring R can be viewed as an Ore extension by choosing σ to be the identity map and δ to be the zero map.

Remark 1. If we choose σ not to be an automorphism, then our constructed ring is not necessarily Noetherian, which makes the general factorization problem even harder to solve. An example of a non-Noetherian Ore extension is the following:
Let $\mathbb{K}$ be a field. Set $R := \mathbb{K}[y]$, the univariate polynomial ring over $\mathbb{K}$. Define $\sigma : R \to R, f(y) \mapsto f(y^2)$ and set δ to be the zero map. Then (σ, δ) is a quasi-derivation, and the ring $R[\partial\,; \sigma, \delta]$ is not Noetherian. A proof of this, and a more thorough discussion, can be found in [MR01], section 1.3.2.

The process of building an Ore extension can be iterated. The rings that we propose to use for a key exchange protocol are of the form

$$S := R[\partial_1; \sigma_1, \delta_1][\partial_2; \sigma_2, \delta_2] \dots [\partial_n; \sigma_n, \delta_n], \tag{1}$$

where $\mathbb{N} \ni n > 1$, R is a domain with identity element, and for all $i \in \{1, \dots, n\}$, either σ_i is the identity map, or δ_i is the zero map, according to our general assumptions. We will refer to these rings throughout the paper as **rings of type (1)**. Furthermore, the σ_i are injective, and there exists a subring $\tilde{R} \neq \{0\}$ of the center of R, such that for all $i \in \{1, \dots, n\}$ and $r \in \tilde{R} : \sigma_i(r) = r$ and $\delta_i(r) = 0$. We refer to $\tilde{R}$ as the **subring of constants**.

The condition $n > 1$ gives us the property that our ring is neither a left nor a right principal ideal domain and therefore there exists no notion of a left- or right greatest common divisor. Thus, our construction of a Diffie-Hellman-like key exchange protocol would not be vulnerable to the methods introduced in [DK11].

The condition that $\tilde{R}$, whose elements are not subject to any non-commutative relation, exists, is needed later to construct commutative subrings in rings of type (1).

There will be two kinds of rings of type (1) that will serve as model examples throughout the paper:

Definition 3. *The so-called* $\boldsymbol{n^{\mathrm{th}}}$ ***Weyl algebra*** $\boldsymbol{A_n}$ *is an Ore extension of type (1). For this, let $\tilde{\mathbb{K}}$ be an arbitrary field, and $\mathbb{K} := \tilde{\mathbb{K}}(x_1, \dots, x_n)$. Define for all $i \in \{1, \dots, n\}$ the σ_i to be identity maps, and define δ_i to be the partial derivation with respect to x_i. Thus, $\partial_i x_i = \sigma_i(x_i)\partial_i + \delta_i(x_i) = x_i\partial_i + x_i' = x_i\partial_i + 1$. But ∂_i commutes with all other x_j, where $j \neq i$. Also, ∂_i, ∂_j always commute, as do x_i, x_j. Finally $\tilde{\mathbb{K}}$ is the subring of constants of S.*

Example 3. The rings used in [BGG+10], namely $\mathbb{F}_q[\partial\,; \sigma]$ with $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$, are single Ore extensions of a finite field $\mathbb{F}_q$, where q is some positive power of a prime number p. One can iterate the extension of this ring, and create $\mathbb{F}_q[\partial_1; \sigma_1] \dots [\partial_n; \sigma_n]$, where $\sigma_1, \dots, \sigma_n \in \mathrm{Aut}(\mathbb{F}_q)$. (Thus, if $c \in \mathbb{F}_q$, then $\partial_i c = \sigma_i(c)\partial_i$. But ∂_i, ∂_j commute, for all i, j.)

The factorization properties of a ring S that has the form (1) are different from commutative multivariate polynomial rings. In particular, S is not a unique factorization domain in the classical sense, i.e. factors are not just unique up to permutation and multiplication by units. Factors in S are unique up to the following notion of similarity.

Definition 4 (cf. [BGTV03], Definition 4.9). *Let R be a domain. Two elements $r, s \in R$ are said to be **similar**, if R/Rr and R/Rs are isomorphic as left R-modules.*

In general, given an element $p \in S$, which has two different complete factorizations

$$p = p_1 \cdots p_m = \tilde{p}_1 \cdots \tilde{p}_k,$$

where $m, k \in \mathbb{N}$, there exists for every $i \in \{1, \ldots, m\}$ at least one $j \in \{1, \ldots, k\}$ such that p_i is similar to $\tilde{p}_j$ (cf. [Jac43], [Ore33]). This means, that the position of similar elements in different factorizations is not fixed.

Example 4. Let us state a simple example in A_2, that can be found in [Lan02]:

$$h := (\partial_1 + 1)^2(\partial_1 + x_1\partial_2) \in A_2.$$

Besides the given factorization in the definition of h, we have the following decomposition into irreducible elements:

$$h = (x_1\partial_1\partial_2 + \partial_1^2 + x_1\partial_2 + \partial_1 + 2\partial_2)(\partial_1 + 1).$$

The corresponding decision problem, of deciding whether two given polynomials are similar, is not known to be possible in polynomial time to the best of our knowledge, although attempts have been made [CVHL10].

One possible attack is to factor elements in non-commutative polynomial rings. However, besides the fact that factoring is currently intractable in this setup, the non-uniqueness of the factorization adds another difficulty for an attack based on factorization. As the next example illustrates, one might end up with infinitely many factorizations, from which one has to choose the correct one.

Example 5. Let $\mathbb{K}$ in Definition 3 be of characteristic zero and consider $\partial_1^2 \in A_n$. Besides factoring into $\partial_1 \cdot \partial_1$, it also factors for all $c \in \mathbb{K}$ into

$$\left(\partial_1 + \frac{1}{x_1 + c}\right) \cdot \left(\partial_1 - \frac{1}{x_1 + c}\right).$$

Remark 2. In Algorithm 1, we are exclusively using the multiplicative structure of the ring S. Only the generation of random elements causes us to apply addition. It is to be emphasized, that $(S, \cdot)$, i.e. the elements in S equipped with the multiplication operator, do not form a group but a monoid. This is due to the fact that the variables introduced by the Ore extension (i.e. $\partial_1, \ldots, \partial_n$) do not have a multiplicative inverse.

1.2 Potential as a Post-Quantum Cryptosystem

Here we will try to give some plausible grounds for our conjecture, that the factorization problem for our rings cannot be solved in polynomial time, even with quantum algorithms.

This stems from the observation that factors of an Ore polynomial p can be very large compared with p itself. Indeed, in terms of bit length representations, the size of the factors can be exponential in the size of p. For example, consider the Chebyshev differential operator

$$L := \left(1 - x_1^2\right) \partial_1^2 - x_1 \partial_1 + n^2, \tag{2}$$

as an element in A_1, where n is a real constant. When n is a positive integer, one can show that L has two possible factorizations. Furthermore, when n is prime one can show that these factors will contain $\Omega(n)$ non-zero terms in expanded representation. Thus their bit size grows at least as fast as n, while the size of L itself grows only as $\log n$. Consequently, the sizes of the factors are exponentially larger than the size of L. If the reader wishes to try some experimentation in MAPLE, we provide a code snippet in appendix A.1.

Now, for the decision problem, "Is L factorable?", the obvious certificate for verification of a "yes" answer would be an actual pair of factors of L. But as we can see from this example, the size of such a certificate may not be polynomial in the input size of L. Furthermore, this problem is already occurring for the simplest possible case: a second-order operator in a univariate Ore ring. Our proposal works with much higher-order operators over many variables, so the relative size of such a certificate of factors will not improve, and may even become worse.

Of course, this does not prove that a polynomial-sized certificate could not exist. But we do not know of any, and hence we suspect that this problem may not even belong to the class NP. As there is some thought that NP-complete problems would not have polynomial time quantum algorithms (see e.g. [Bro01], page 297), we are therefore led to conjecture that our factorization problem would not have any such algorithm, either.

Note, though, that the above example was over a field of characteristic zero. We actually prefer to work over finite fields, to reduce expression swell in the computations. For such fields, we do not know of any examples where the bit-size of the factors is exponentially larger than the input-polynomial. However, even for univariate differential polynomials, there are no known polynomial time algorithms for factorization or deciding irreducibility, for either classical or quantum computers. For Noetherian rings over finite fields, the hardness of factorization is less clear, though there are still no known polynomial time algorithms for the multivariate case. But even if one exists for Noetherian rings over finite fields, one could instead choose a ring having a non-Noetherian extension. As mentioned above, we are skeptical that there is any polynomial time algorithm for this case, using a classical computer. Furthermore, unless there is some property of the non-Noetherian ring that a quantum algorithm can take advantage of, we conjecture the same is true for quantum computers.

1.3 Related Work

There exist a polynomial time algorithms that factor univariate Ore-polynomials over finite fields, namely [Gie98, GZ03]. This includes the skew-polynomials as used in [BGG+10]. Boucher et al. argued that even if an attacker can find a factorization using this algorithm, then it might not be the right one to discover the key A and B agreed upon. This can be true for certain choices of polynomials, but there is more theory needed to prove that there is a certain lower bound on the number of different factorizations.

For certain single Ore extensions of a univariate commutative polynomial ring or function field there are several algorithms and even implementations available. This is due to the fact that those extensions are algebraic generalizations of operator algebras. The most prominent publications that deal with factoring in the first Weyl algebra are [vH97b, vH96, vH97a, vHY10, GS04]; the algorithms of the first four papers are implemented in the computer algebra system Maple [MGH+08], and that of the fifth paper in ALLTYPES [Sch09]. For factoring elements in the first Weyl algebra with polynomial coefficients, there is an implementation [HL13] in the computer algebra system Singular [DGPS12]. The implementation also extends to the shift algebra and classes of polynomials in the so-called first $\underline{q}$-Weyl algebra. Theoretical results for those operator algebras are shown in [Tsa94] and [Tsa96], which extend the papers [Loe03] and [Loe06].

The factorization problem for general multivariate Ore algebras has not, as of yet, been as well investigated.

A thorough theoretic overview of the factorization problem in Ore domains is presented in [BGTV03].

Recently, the techniques from [HL13] were extended to factor elements in the n^{th} Weyl algebra, the n^{th} shift algebra and classes of polynomials in the n^{th} $\underline{q}$-Weyl algebra [GHL14]. However, the algorithm uses Gröbner-bases [Buc97], and therefore does not run in polynomial time [MM82].

From an algebraic point of view, and dealing only with strictly polynomial non-commutative algebras, Melenk and Apel [MA94] developed a package for the computer algebra system REDUCE. That package provides tools to deal with certain non-commutative polynomial algebras and also contains a factorization algorithm for the supported algebras.

Beals and Kartashova [BK05] consider the problem of factoring polynomials in the second Weyl algebra, where they are able to deduce parametric factors. Research in a similar direction was done by Shemyakova in [She07, She09, She10].

Another key exchange protocol based on non-commutative rings is presented in [CNT12]. The ring chosen in this publication is the ring of endomorphisms of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$, which is also not a principal ideal domain and therefore not subject to an attack as shown in [DK11]. It should be noted that the authors used the same technique as in this paper for constructing commuting elements.

Using a different non-commutative ring, which is a $\mathbb{Z}$-module, Cao et al. have presented a similar key-exchange protocol in [CDW07]. The authors also use the same idea to construct commuting elements, but their work furthermore considers some other non-abelian groups. We are not aware of any known attack on this system.

An approach for using non-abelian groups to generate a public key cryptosystem was developed by Ko et al. in [KLC+00] for the special case of braid groups [Art47]. The authors used the conjugacy problem of groups as their hard problem. However, Jun and Cheon presented in [CJ03] a polynomial time algorithm for exactly their setup (but not for the conjugacy problem in general). This attack exploits the Lawrence-Krammer representation of braid groups [Kra02], which is a linear representation of the braid group.

Concerning the task of finding commuting polynomials in the first Weyl algebra, a very thorough study is presented in [BC23], which also demonstrates the hardness to find all commuting polynomials in the ring of ordinary linear differential operators.

2 The Key Exchange Protocol

2.1 Description of the Protocol

We refer to our communicating parties as Alice (abbreviated A) and Bob (abbreviated B). Alice and Bob wish to agree on a common secret key using a Diffie-Hellmann-like cryptosystem.

The main idea is similar to the key exchange protocol presented in [BGG+10]. The main differences are that (i) the ring S and (ii) the commuting subsets are not fixed, but agreed upon as part of the key-exchange protocol. It is summarized by the following algorithm.

Algorithm 1 DH-like protocol with rings of type (1)

1: A and B publicly agree on a ring S of type (1), a security parameter $\nu \in \mathbb{N}$ representing the size of the elements to be picked from S in terms of total degree and coefficients, a non-central element $L \in S$, and two multiplicatively closed, commutative subsets of $\mathcal{C}_l, \mathcal{C}_r \subset S$, whose elements do not commute with L.
2: A chooses a tuple $(P_A, Q_A) \in \mathcal{C}_l \times \mathcal{C}_r$.
3: B chooses a tuple $(P_B, Q_B) \in \mathcal{C}_l \times \mathcal{C}_r$.
4: A sends the product $A_{\text{part}} := P_A \cdot L \cdot Q_A$ to B.
5: B sends the product $B_{\text{part}} := P_B \cdot L \cdot Q_B$ to A.
6: A computes $P_A \cdot B_{\text{part}} \cdot Q_A$.
7: B computes $P_B \cdot A_{\text{part}} \cdot Q_B$.
8: $P_A \cdot P_B \cdot L \cdot Q_B \cdot Q_A = P_B \cdot P_A \cdot L \cdot Q_A \cdot Q_B$ is the shared secret key of A and B.

Proof (Correctness of Algorithm 1). As $P_A, P_B \in \mathcal{C}_l$ and $Q_A, Q_B \in \mathcal{C}_r$, we have the identity in step 8. Therefore, by the end of the key exchange, both A and B are in possession of the same secret key.

Before discussing the complexity and security of the proposed scheme, we deal with the feasibility of constructing the sets $\mathcal{C}_l, \mathcal{C}_r$ in Algorithm 1. We propose the following technique, which is applicable independent of the choice of S. Let $P, Q \in S$ be chosen, such that they do not commute with L. Define

$$C_l := \left\{ f(P) \mid f = \sum_{i=0}^{m} f_i X^i \in \tilde{R}[X], m \in \mathbb{N}, f_0 \neq 0 \right\},$$
$$C_r := \left\{ f(Q) \mid f = \sum_{i=0}^{m} f_i X^i \in \tilde{R}[X], m \in \mathbb{N}, f_0 \neq 0 \right\}, \tag{3}$$

where $\tilde{R}$ is the subring of constants of S, and $\tilde{R}[X]$ is the univariate commutative polynomial ring over $\tilde{R}$. For an element $f \in \tilde{R}[X]$, we let $f(P)$ denote the substitution of X in the terms of f by P, and similarly $f(Q)$ denotes the substitution of X by Q. By this construction, all the elements in $\mathcal{C}_l$ commute, as do the elements in $\mathcal{C}_r$. The choice of the coefficient f_0 in both sets to be non-zero is motivated by the following fact: If f_0 is allowed to be zero, Eve could find that out by simply trying to divide the resulting polynomial by P on the left (resp. by Q on the right). Moreover, Eve could iterate this process for increasing indices, until an f_i for $i \in \{0, \dots, m\}$ is reached, which is not equal to zero. This could lead to a decrease of the amount of coefficients Eve has to figure out for certain choices of keys. By the additional condition of having $f_0 \neq 0$, Eve cannot retrieve any further information in the described way.

Using this technique, the first steps of Algorithm 1 can be altered in the following way. In step 1, A and B agree upon two elements $P, Q \in S$, which represent C_l and C_r, respectively, as in (3). Then each one of them chooses two random polynomials f, g in $\tilde{R}[X]$, and obtains the tuple of secret keys in steps 2 and 3 by computing $f(P)$ and $g(Q)$. (Note that it could happen that one, or even both, of $f(P)$, $g(Q)$ commutes with L, even though neither P nor Q do so. This appears to be unlikely in practice, but in any case, it is straightforward to deal with this possibility. A (resp. B) simply checks if the chosen $f_A(P)$ or $g_A(Q)$ (resp. $f_B(P)$ or $g_B(Q)$) commutes with L. If a commutation with L should occur, say for A, A just chooses a new polynomial for $f_A(X)$ or $g_A(X)$. As P and Q are chosen to not commute with L, A will quickly find polynomials $f_A(P)$ and $g_A(Q)$ that do not commute with L.)

Example 6. Let S be the third Weyl algebra A_3 over the finite field $\mathbb{F}_{71}$, upon which A and B agree. Let

$$\begin{aligned} L &:= 3x_2^2 - 5\partial_2^2 - x_2\partial_3 - x_3 - \partial_2, \\ P &:= -5x_3^2 - 2x_1\partial_3 + 34, \quad \text{and} \\ Q &:= x_2^2 + x_1x_3 - \partial_3^2 + \partial_3, \end{aligned}$$

where L is the public polynomial as required in Algorithm 1, and P, Q, such that they define the sets $\mathcal{C}_l$ and $\mathcal{C}_r$ as in (3).

Suppose A chooses polynomials $f_A(X) = 48X^2 + 22X + 27$, $g_A(X) = 58X^2 + 5X + 52$, while B chooses $f_B(X) = 3X^2 + X + 31$, $g_B(X) = 24X^2 + 4X + 11$. Then the tuples are $(P_A, Q_A) = (48P^2 + 22P + 27,\ 58Q^2 + 5Q + 52)$, and $(P_B, Q_B) = (3P^2 + P + 31,\ 24Q^2 + 4Q + 11)$.

As described in the protocol, A subsequently sends the product $A_{\text{part}} := P_A \cdot L \cdot Q_A$ to B, while B sends $B_{\text{part}} := P_B \cdot L \cdot Q_B$ to A, and their secret key is $P_A \cdot P_B \cdot L \cdot Q_B \cdot Q_A = P_B \cdot P_A \cdot L \cdot Q_A \cdot Q_B$. (For brevity, the final expanded product is not shown here.)

Remark 3. For practical purposes, the degree of L should be chosen to be of a sufficiently large degree in order to perturb the product $Q_B \cdot Q_A$ well enough before it is multiplied to $P_A \cdot P_B$. An examination of the best choices for the degree of L is a subject of future work that includes practical applications of our primitive for a Diffie-Hellman-like key exchange protocol.

Remark 4. As we will see in section 4, there are known insecure choices of keys for certain rings S. Obviously, in a practical implementation, one has to check for these and avoid them.

2.2 Complexity of the Protocol

Of course, as our definition of the rings we consider in Algorithm 1 – namely rings of type (1) – is chosen to be as general as possible, a complexity discussion is highly dependent on the choice of the specific algebra. In practice, we envision that a certain finite subset of those algebras (such as, for example, the Weyl algebras, or iterated extensions of the rings used in [BGG$^+$10]) will be studied for practical applications. Our complexity discussion here focusses rather on the general setup than on concrete examples.

As we generally assume, all arithmetics in R, and therefore also in its subring of constants $\tilde{R}$, can be computed in polynomial time. We suppose the same holds for the application of σ_i and δ_i, for $i \in \{1, \ldots, n\}$, to the elements of R, and that the time needed to choose a random element in R is polynomial in the desired bit length of this random element. Thus, the choice of a random element in S is just a finitely iterated application of the choice of coefficients, which lie in R. Let $\omega_i(k)$ denote the cost of applying σ_i (or δ_i, depending on which one of them is non-trivial) to an element $f \in R$ of bit-length $k \in \mathbb{N}$. For two elements $f, g \in R$ of bit-sizes $k_1, k_2 \in \mathbb{N}$, we denote the cost of multiplying them in R by $\theta(k_1, k_2)$, and the cost of adding them by $\rho(k_1, k_2)$.

For the key exchange protocol the main cost that we need to address is the cost of multiplying two polynomials in S. For a monomial $m := \partial_1^{e_1} \cdots \partial_n^{e_n}$, where $e \in \mathbb{N}_0^n$, one can generalize Lemma 1 to the multivariate case and find that multiplying m and f, where f has bit-size k, costs $O(\prod_{i=1}^n e_i \cdot \omega_i(k))$ bit-operations. For general polynomials in S, we obtain therefore the following property:

Lemma 2. *Let n be the number of Ore extensions as in (1). For two polynomials $h_1, h_2 \in S$, let $d \in \mathbb{N}_0$ be the maximal degree among the ∂_i that appears in h_1 and h_2, and let $k_1, k_2 \in \mathbb{N}$ be the maximal bit-length among the coefficients of h_1 and h_2, respectively. For notational convenience, we define $\zeta := \prod_{i=1}^n \omega_i(k_2)$. Then the cost of computing the product $h_1 \cdot h_2$ is in*

$$O\left(d^{2n} \cdot \zeta \cdot \theta(k_1, \zeta) \cdot \rho(\theta(k_1, \zeta), \theta(k_1, \zeta))\right).$$

Proof. We have at most d^n terms in h_1. When we multiply h_1 and h_2, we have to regard each term separately, and compute the non-commutative relations. This results in the d^{2n} different computations of size ζ. Then, for every one of those results, we need to apply a multiplication in R with the coefficients of h_1. In the end, the results of those multiplications have to be added together appropriately, which results in the above complexity.

This lemma shows that multiplying two elements in S has polynomial time complexity in the size of the elements, since the value of n is fixed for a chosen S.

Remark 5. The cost in Lemma 2 assumes the worst case, where each Ore extension of R has a non-trivial δ_i. If for one of the extensions, δ_i is equal to the zero map, then the worst case complexity in this variable is lower, as the term-wise multiplication does not result each time in a sum of different terms in ∂_i. One can see here, that in general, when the cost of the protocol is crucial for a resource-limited practical implementation, one should prefer Ore extensions where δ is the zero map, i.e. skew-polynomial rings.

2.3 Security Analysis

2.3.1 The Attacker's Problem The security of our scheme relies on the difficulty of the following problem, which is similar to the computational Diffie Hellman problem (CDH) [Mau94].

Given a ring S, a security parameter ν, two sets $\mathcal{C}_l, \mathcal{C}_r$ of multiplicatively closed, commutative subsets of S, whose elements do not commute with a certain given $L \in S$. Furthermore, let the products $P_A \cdot L \cdot Q_A$ and $P_B \cdot L \cdot Q_B$ for some $(P_A, Q_A), (P_B, Q_B) \in \mathcal{C}_l \times \mathcal{C}_r$ also be known.

Difficult Problem (Ore Diffie Hellman (ODH)): Compute $P_B \cdot P_A \cdot L \cdot Q_A \cdot Q_B$ $(= P_A \cdot P_B \cdot L \cdot Q_B \cdot Q_A)$ with the given information.

One way to solve this problem would be to recover one of the elements P_A, Q_A, P_B or Q_B. This can be done via factoring $P_A \cdot L \cdot Q_A$ or $P_B \cdot L \cdot Q_B$ which appears, as mentioned in the introduction, to be hard. Furthermore, even if one is able to factor an intercepted product, the factorization may not be the correct one due to the non-uniqueness of the factorization in Ore extensions.

Another attack for the potential eavesdropper is to guess the degrees of (P_A, Q_A) (or (P_B, Q_B)) and to create an ansatz with the coefficients as unknowns, to form a system of multivariate polynomial equations to solve. This type of attack and its infeasibility was discussed already in [BGG+10], Section 5.2., and the argumentation of the authors translates analogously to our setup. Finally, another attempt, which seems natural, is to generalize the attack of Dubois and Kammerer to the multivariate setup. We will discuss such a possible generalization for certain rings of type (1) and show that it is impractical in the following subsection.

We are not aware of any other way to obtain the common key of A and B while eavesdropping on their communication channel in Algorithm 1 other than trying to recover the correct factorization from the exchanged products of the form $P \cdot L \cdot Q$.

Remark 6. Concerning the attack where Eve forms an ansatz and tries to solve multivariate polynomial systems of equations: In fact, each element in our system has total degree at most two. There exist attempts to improve the Gröbner computations for these kinds of systems [CKPS00, KS99], but the assumptions are quite restrictive. Besides the assumption that the given ideal must be zero-dimensional (which is only guaranteed in the case when the subring of constants is finite), there are certain relations between the number of generators and variables necessary to apply these improvements. We are not aware of any further progress on the techniques presented in [CKPS00, KS99] since 2000, which have fewer restrictions on the system to be solved.

Remark 7. Note that there is a corresponding decision problem related to ODH: Given a candidate for the final secret key, determine if this key is consistent with the public information exchanged by Alice and Bob. To the best of our knowledge, this is also currently intractable.

2.3.2 Generalization of the attack by Dubois and Kammerer In this subsection, we assume that our ring S is Noetherian, and that there exists a notion of a left or right Gröbner basis. Alice and Bob have applied Algorithm 1 and their communication channel has been eavesdropped by Eve. Now Eve knows about the chosen ring S, the commuting subsets $\mathcal{C}_l, \mathcal{C}_r$ and the exchanged products $P_A \cdot L \cdot Q_A$ and $P_B \cdot L \cdot Q_B$ for some (P_A, Q_A), $(P_B, Q_B) \in \mathcal{C}_l \times \mathcal{C}_r$. Let us assume without loss of generality that Eve wants to compute Q_A.

Eve does not have a way to compute greatest common right divisors, but she can utilize Gröbner basis theory. For this, she picks a finite family $\{E_i\}_{i=1}^m$, $m \in \mathbb{N}$, of elements in C_r. After that, she computes the set $G := \{P_A \cdot L \cdot Q_A \cdot E_i \mid i \in \{1, \ldots, n\}\}$.

All elements in G (along with $P_A \cdot L \cdot Q_A$) have Q_A as a right divisor in common, since E_i commutes with Q_A for all $i \in \{1, \ldots, m\}$. This means, the left ideal I in S generated by the elements in $G \cup \{P_A \cdot L \cdot Q_A\}$ lies in – or is even equal to – the left ideal generated by Q_A. Hence, a Gröbner basis computation of I might reveal Q_A. If not, a set of polynomials of possibly smaller degree than the ones given in G that have Q_A as a right divisor will be the result of such a computation.

Besides having no guarantee that Eve obtains Q_A from the computations described above, the computation of a Gröbner basis is an exponential space hard problem [MM82]. We tried to attack our protocol using this idea. We chose the second Weyl algebra as a possible ring, as there is a notion of a Gröbner basis and there are implementations available. It turned out that our computer ran out of memory after days of computation on several examples where L, Q_A, Q_B, P_A and P_B each exceed a total degree of ten. For practical choices, of course, one must choose degrees which are higher (dependent on the choice of the ring S). Hence, we consider our proposal secure from this generalization of the attack by Dubois and Kammerer.

2.3.3 Recommended Key Lengths The question of recommended key lengths has to be discussed for each ring of type (1) separately. With lengths, one means in the context of this paper the degree of the chosen public polynomials L, P and Q in the ∂_i for $i \in \{1, \ldots, n\}$ and the size of their respective coefficients in R. We cannot state a general recommendation for key-lengths that lead to secure keys for arbitrary choices of S. For the Weyl algebra, where some implementations of factoring algorithms are available, we could observe through experiments that generic choices of P and Q in $\mathcal{C}_l$ and $\mathcal{C}_r$, respectively, each of total degree 20, lead to products $P \cdot L \cdot Q$ which cannot be factored after a feasible amount of time. If one chooses our approach (3) to find commuting elements, the choice of the degree of the polynomials in $\tilde{R}[X]$ is the critical part, and the polynomials P and Q – as they are publicly known – can be chosen to be of small degree for performance's sake.

In general, for efficiency, we recommend choosing $n = 2$ for the ring S, as it already ensures that S is not a principal ideal domain and keeps multiplication costs low.

For the case where our underlying ring is a finite field, we are able to present in Table 1 a more detailed cost estimate on the hardness to attack our protocol by using brute-force. There, we assume that $R = \mathbb{F}_q$, where $q = p^k$ for a prime p and $k > 2$. For efficiency, as outlined above, we pick $n = 2$ and further $k = 3$. Then we define S as being $R[\partial_1; \sigma_1][\partial_2; \sigma_2]$, where σ_1, σ_2 are different powers of the Frobenius automorphism on $\mathbb{F}_q$. We assume that the polynomials are stored in dense representation in memory. The two commuting subsets $\mathcal{C}_l, \mathcal{C}_r$ are defined as in (3). We will measure the time in computation steps. We assume that any arithmetic operation on $\mathbb{F}_q$, as well as the application of σ_1 resp. σ_2, takes one step. Then, the cost formula as presented in Lemma 2 will be in the worst case $d^4 \cdot 2$, as addition and multiplication are assumed to take one computation step, and $\zeta \leq 3$ (due to the automorphism group of R having order 3). The security parameter is given as a tuple (d_L, d_{PQ}, ν), where d_L is the total degree of L, d_{PQ} is the total degree of each of P and Q, and ν is the maximal degree of the polynomials in $\mathbb{F}_p[X]$ chosen to compute each of P_A, P_B, Q_A and Q_B. To simplify the analysis, we assume for our estimates that the degree in each ∂_1 and ∂_2 will be half of the total degree for L, P and Q. As for the cost of Alice resp. Bob to compute the messages they are sending, and to compute the final key, we used the following formulas to make a prudent estimation:

- **Computing all powers of P and Q:** The cost $c(\nu)$ to compute all these powers up to a certain exponent ν can be estimated by the following recursive formula:

$$c(1) = 0, \text{ (P or Q are given, no need to compute)}$$
$$c(j+1) = \frac{(j \cdot d_{PQ})^4}{8} + c(j).$$

  As a closed formula, we can we can write it as

$$c(\nu) = \sum_{j=0}^{\nu} \frac{(j \cdot d_{PQ})^4}{8} = \frac{d_{PQ}^4}{8} \cdot \sum_{j=0}^{\nu} j^4 = \frac{d_{PQ}^4}{8} \cdot \left(5 \cdot (\nu+1)^5 - \frac{1}{2} \cdot (\nu+1)^4 + \frac{1}{3} \cdot (\nu+1)^3 - \frac{1}{30} \cdot \nu - \frac{1}{30}\right).$$

- **Generating private polynomials:** Both Alice and Bob have to compute P_A and Q_A resp. P_B and Q_B. In order to do so, each power of P and Q has to be computed and multiplied by an element in $\mathbb{F}_p$, which results in

$$2 \cdot \sum_{j=0}^{\nu} \frac{j^2 \cdot d_{PQ}^2}{4} = \sum_{j=0}^{\nu} \frac{j^2 \cdot d_{PQ}^2}{2} = \frac{d_{PQ}^2}{2} \cdot 3 \cdot \left((\nu+1)^3 - \frac{1}{2} \cdot (\nu+1)^2 + \frac{1}{6} \cdot \nu + \frac{1}{6}\right)$$

  operations. Adding all these together adds another $2 \cdot \frac{\nu^2 \cdot d_{PQ}^2}{4}$ operations for Alice resp. Bob.

- **Computing initial message:** We assume that we have the private polynomials for A and B already computed, and their respective degree is $d_{PQ} \cdot \nu$. Then, in order to compute the initial message, we need $\frac{(d_{PQ}\cdot\nu)^4}{8}$ steps to compute $P_A \cdot L$, assuming that the degree of L is smaller. Afterwards, to obtain $P_A \cdot L \cdot Q_A$, we have to do $\frac{(d_{PQ}\cdot\nu+\frac{d_L}{2})^4}{8}$ additional steps.
- **Computing the shared secret key:** The shared secret takes then $\frac{(2\cdot d_{PQ}\cdot\nu+d_L/2)^4}{8}+\frac{(3\cdot d_{PQ}\cdot\nu+d_L/2)^4}{8}$ steps to compute by directly applying the cost estimate for multiplication.

The worst case for the size in bits of the shared key in the end can be estimated by adding the degrees of the computed L, P_A, Q_A, Q_B and P_B together. This results in the formula

$$\left(\frac{d_L + 8\cdot\nu\cdot d_{PQ}}{2}\right)^2 \cdot \lceil \log(p) \rceil,$$

where we assume that the partial degree in ∂_1 and ∂_2 is about half of the total degree of the final polynomial. In practice, one would probably prefer to use a sparse representation, which would on average lead to smaller final key sizes.

As for the cost for an attacker to do a brute-force attack, i.e. trying to determine the shared secret of Alice and Bob, we assume that an attacker would try all possibilities for one of the polynomials P_A, P_B, Q_A or Q_B and check, for each possibility, if the computed polynomial divides one of the messages between Alice and Bob. Hence, for every possibility, Eve must solve a linear system of equations of size d_m^2, where d_m is the maximal total degree of one of the messages (usually $2\cdot\nu\cdot d_{PQ}+d_L$). I.e. there arise $(\frac{d_m}{2})^{2\omega}$ additional computation steps for each possibility, where ω is the matrix multiplication constant (currently $\omega \approx 2.373$). Initially, the attacker has to also compute all powers of P resp. Q, and then the additions, as listed above. The following table lists our computed costs for different security parameters.

Security Tuple	Computation Costs for Alice and Bob			Final Key Size in KB	Brute-Force Cost
	Secret Parameter	Initial Message	Shared Secret		
$(30,5,10)$	$1.247955E+08$	$3.012579E+06$	$1.145127E+08$	46	$2.066009E+16$
$(30,5,15)$	$8.144450E+08$	$1.215633E+07$	$5.073701E+08$	97	$9.616857E+18$
$(30,5,20)$	$3.176336E+09$	$3.436258E+07$	$1.497794E+09$	169	$2.237607E+21$
$(30,5,25)$	$9.248193E+09$	$7.853758E+07$	$3.508245E+09$	260	$3.467317E+23$
$(30,5,30)$	$2.229704E+10$	$1.559313E+08$	$7.074856E+09$	370	$4.110897E+25$
$(50,5,35)$	$4.711215E+10$	$3.172363E+08$	$1.391021E+10$	514	$5.144021E+27$
$(50,5,40)$	$9.029806E+10$	$5.203613E+08$	$2.315166E+10$	665	$4.258507E+29$
$(50,5,45)$	$1.605675E+11$	$8.086426E+08$	$3.637583E+10$	836	$3.176783E+31$
$(50,5,50)$	$2.690343E+11$	$1.203174E+09$	$5.458994E+10$	1027	$2.179949E+33$

Table 1. Computation costs (given as number of primitive computation steps) for Alice and Bob to perform Algorithm 1 with $R = \mathbb{F}_q$, $p = 2$ and $S = R[\partial_1;\sigma_1][\partial_2;\sigma_2]$ and costs for Eve to perform a brute-force attack.

Remark 8. We tried to factor the exchanged products $P_A \cdot L \cdot Q_A$ and $P_A \cdot L \cdot Q_B$ from the small Example 6 in section 2.1 using Singular and REDUCE, and it turned out that both were not able to provide us with one factorization after 48 hours of computation on an iMac with 2.8Ghz (4 cores) and 8GB RAM available. This means that even for rather small choices of keys, the recovery of P and Q via factoring appears already to be hard using available tools. Of course, for this small key-choice, a brute-force ansatz attack (as described above in Remark 6) would succeed fairly quickly. We also tried 150 examples with different degrees for P, Q, L and the respective polynomials in C_l and C_r. In particular, we let the degree of L range from 5 to degree 50, the degree of P and Q respectively between 5 and 10, and the degrees of the elements in C_l and C_r — which are created with the help of P and Q – are having degrees ranging between 25 and 50. We gave each factorization process a time limit of 4 hours to be finished. None of the polynomials has been factored within that time-frame. The examples can be downloaded from the following website: `https://cs.uwaterloo.ca/~aheinle/software_projects.html`.

2.3.4 Attacks On Similar Systems Here, we discuss why known attacks on protocols similar to Algorithm 1 do not apply to our contexts.

As emphasized before, the attack developed by Dubois and Kammerer on the protocol by Boucher et al. is prevented by choosing rings that are not principal ideal domains. Thus, there is no general algorithmic way to compute greatest common right divisors.

When applying the rings S of type (1) to exchange keys, one does in fact not utilize the whole ring structure, but only the multiplicative monoid structure. Therefore it appears to be reasonable to consider also attacks developed for protocols based on non-abelian groups (albeit they contain more structure than just monoids, the latter being the correct description of our setup). The most famous protocol is given by Ko et al., as discussed in the section on related work. The attack developed by Jun and Cheon exploits the fact that braid groups are linear. However, there is currently no linear representation known for our rings of type (1) (though it would be an interesting subject of future research), so there is at present no analogous attack on protocols based on our primitive. Furthermore, even if a linear representation for our rings were discovered, it is not clear whether Jun and Cheon's attack could be extended to our case, as the authors make use of invertible elements in their algorithm (which our structures, only being monoids, do not possess).

3 Implementation and Experiments

We developed an experimental implementation of the key exchange protocol as presented in Algorithm 1 in the programming language C[1]. We decided to develop such a low-level implementation after we found that commodity computer algebra systems appear to be too slow to make experiments with reasonably large elements. This may be due to the fact that their implemented algorithms are designed to be generally applicable to several classes of rings and therefore come with a large amount of computational overhead. Our goal is to examine key-lengths and the time it takes for computing the secret keys. It is to be emphasized that our code leaves considerable room for improvement.

For the implementation, we chose our ring S to have the form as described in Example 3. In particular, our ring for the coefficients R is set to $\mathbb{F}_{125}$, and we fixed $n := 2$. Internally, we view $\mathbb{F}_{125}$ isomorphically as $\mathbb{F}_5(\alpha) := \mathbb{F}_5[x]/\langle x^3 + 3x + 3\rangle$. Our non-commutative polynomial ring S is

[1] One can download the implementation at `https://github.com/ioah86/diffieHellmanNonCommutative`

$R[\partial_1;\sigma_1][\partial_2;\sigma_2]$, where

$$\sigma_1 : \mathbb{F}_5(\alpha) \to \mathbb{F}_5(\alpha),\ a_0 + a_1\alpha + a_2\alpha^2 \mapsto a_0 + a_1 + a_2 + 3a_2\alpha + (3a_1 + 4a_2)\alpha^2$$
$$\sigma_2 : \mathbb{F}_5(\alpha) \to \mathbb{F}_5(\alpha),\ a_0 + a_1\alpha + a_2\alpha^2 \mapsto a_0 + 4a_1 + 3a_2 + (4a_1 + 2a_2)\alpha + 2a_1\alpha^2.$$

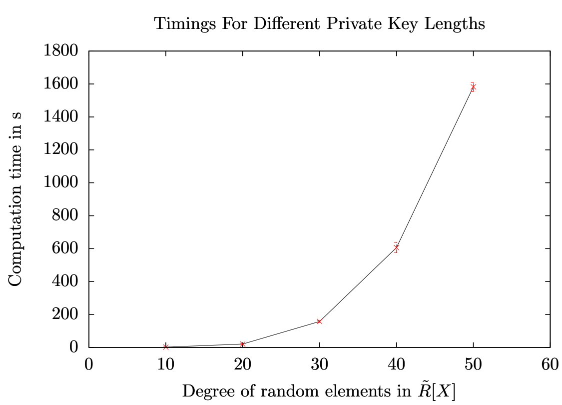
The ring of constants is therefore $\tilde{R} := \mathbb{F}_5 \subset R$. These two automorphisms are given by different powers of the Frobenius automorphism, and they are the only two distinct non-trivial automorphisms on $\mathbb{F}_5(\alpha)$ (cf. [Gar86, Theorem 12.4]).

Note, that the multiplication of two elements f and g in this ring requires $O(n^4)$ integer multiplications, where $n = \max\{\deg(f), \deg(g)\}$.

Following the notation as in Algorithm 1, our implementation generates random polynomials L, P and Q in S. Our element L is chosen to have total degree 50, and P, Q each have total degree 5. Afterwards, it generates four polynomials in $\tilde{R}[X]$ to obtain $(P_A, Q_A), (P_B, Q_B)$ in the fashion of (3).

Subsequently, the program computes the products $P_A \cdot L \cdot Q_A$, $P_B \cdot L \cdot Q_B$ and the secret key $P_A \cdot P_B \cdot L \cdot Q_B \cdot Q_A = P_B \cdot P_A \cdot L \cdot Q_A \cdot Q_B$. Naturally, some of those computations would be performed in parallel when the protocol is applied, but we did not incorporate parallelism into our experimental setup. At runtime, all computed values are printed out to the user.

We experimented with different degrees for the polynomials in $\tilde{R}[X]$ to generate the private keys, namely 10, 20, 30, 40 and 50. This leads to respectively 20, 40, 60, 80 and 100 indeterminates for Eve to solve for if she eavesdrops the channel between Alice and Bob and tries to attack the protocol using an ansatz by viewing the coefficients as unknown parameters. Even if she decides to attack the protocol using brute-force, she has to go through $5^{10}, 5^{20}, 5^{30}, 5^{40}$ and 5^{50} possibilities respectively (note here, that for a brute-force attack, Eve only needs to extract a right or left hand factor of the products $P_A \cdot L \cdot Q_A$ and $P_B \cdot L \cdot Q_B$ that Bob and Alice exchange). The file sizes and the timings for the experiments are illustrated in Figure 1.



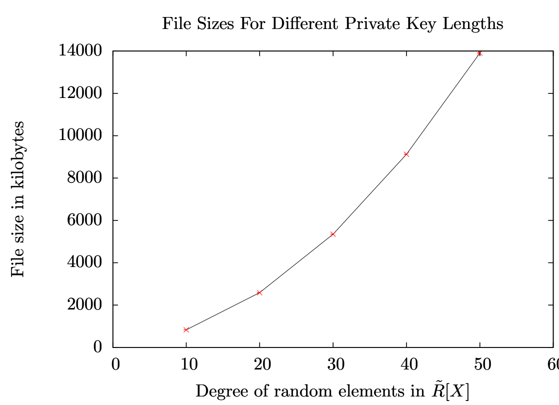


Fig. 1. Timings and file sizes for different degrees of elements in $\tilde{R}[X]$

Note, that the file sizes are not indicative of the actual bit-size of the keys, as the files we produced are made to be human-readable. Allowing for this fact, the bit-sizes of our keys are comparable to those found necessary for secure implementations of the McEliece cryptosystem [McE78, BLP08], which is a well-studied post-quantum encryption scheme.

In our experimental setup, we can see that one can generate a reasonably secure key (degree 30 for the elements in $\tilde{R}[X]$) in less than five minutes at the current stage of the implementation. For larger degrees, we believe that machine-optimized code would decrease the computation time significantly. An interesting question is whether arithmetics in our class of non-commutative rings can be implemented in a smart way on a quantum computer.

3.1 Challenge Problems

For readers who would like to try to attack the keys generated by this particular implementation, we have generated a set of challenge problems. They can be found, with description, on the website of one of the authors (`https://cs.uwaterloo.ca/~aheinle/miscellaneous.html#challenges`). There are also challenges included for the three-pass protocol as described in section 5.1.

4 Insecure Keys

In this section, we will present an insecure key-choice for the Weyl algebras. The construction of those insecure keys, which is related to finding commutative subrings, can be applied to other algebras of type (1).

4.1 Insecure Keys for the Weyl Algebras

4.1.1 Graded Polynomials. Based on the paper [GHL14], there is a large subset of the polynomial n^{th} Weyl algebra, where the factorization problem of their elements can be reduced to factoring in a commutative, multivariate polynomial ring.

In particular, there exists a non-trivial $\mathbb{Z}^n$-grading on the polynomial n^{th} Weyl algebra, where the z^{th} **graded part**, for $z \in \mathbb{Z}^n$, can be characterized in the following way:

$$A_n^{(z)} := \left\{ \sum_{\substack{e,w \in \mathbb{N}_0^n \\ w-e=z}} r_{ew} \cdot x_1^{e_1} \cdots x_n^{e_n} \cdot \partial_1^{w_1} \cdots \partial_n^{w_n} \quad | \quad r_{ew} \in \tilde{\mathbb{K}} \right\}.$$

We call an element h in the polynomial n^{th} Weyl algebra **graded**, if $h \in A_n^{(z)}$ for some $z \in \mathbb{Z}^n$. These graded polynomials are exactly the ones for which the factorization problem can be reduced to commutative factorization as mentioned above.

Now, there are two possible scenarios for weak key choices of A and B. Let L be the public key, and P_A, Q_A and P_B, Q_B be the private keys of A and B respectively, i.e. the final key of A and B is $P_A \cdot P_B \cdot L \cdot Q_B \cdot Q_A$. The first scenario is that all of the keys that A and B use are graded. Then a possible eavesdropper E can recover the private keys by factoring $P_A \cdot L \cdot Q_A$ and $P_B \cdot L \cdot Q_B$, applying techniques presented by [GHL14]. The second scenario is that one of the private keys of P_A, P_B, Q_A and Q_B is graded. Without loss of generality, let P_A be graded. Then E can recover P_A by factoring every graded summand of $P_A \cdot L \cdot Q_A$, and therefore E can also recover Q_A and the security is broken.

Fortunately, A and B can check their keys for being graded in polynomial time. In particular, for an element h the polynomial n^{th} Weyl algebra, one has to check if for every $i \in \{1, \ldots, n\}$ the difference of the exponents of x_i and ∂_i is the same in every term of h. This is the case if and only if h is graded.

Remark 9. One can argue that the Weyl algebras as we define them assume that the x_i are units, and therefore the attack as described here is not possible once we choose for our keys coefficients that have in the denominator some nontrivial polynomial in the x_i. But there is a possibility to lift factorizations into the polynomial Weyl algebra, which is described in [GHL14]. Thus one has to lift the keys and double check for them being graded or not. This check obviously still requires only polynomial time.

Generally, as we can see here, one should study the concrete ring of type (1) chosen for Algorithm 1, in order to determine in which cases factorization can be reduced to an easy problem in commutative algebra, and avoid those cases for the key-choice. The construction of the insecure keys in the case of the Weyl algebra as presented here gives an idea how to find commutative subrings and how to avoid them.

5 Enhancements

In what follows, we show that the use of multivariate Ore polynomials is not limited to the Diffie-Hellman-like protocol discussed and analyzed in the previous sections, but can also be utilized to develop other cryptographic applications.

It should be noted that two of the protocols described below, namely the digital signature scheme and the zero-knowledge-proof protocol, do not require the use of commuting subsets, which reduces the necessary amount of public information to be exchanged between Alice and Bob.

5.1 The Three-pass Protocol and private L

As we saw in Algorithm 1, Alice and Bob must make inter alia the following information public at the start: $\mathcal{C}_l$, $\mathcal{C}_r$, and L. If we can find a way to make any of these objects private between Alice and Bob, the security of the system may be increased.

One possible method to make L private is based on a well-known puzzle, known variously as the Locks and Boxes, the Knight and the Princess, and perhaps other names. In this puzzle, Alice wishes to send Bob an item in a securely locked box. Both have a supply of locks and keys to the locks, which they could use to lock the box. However, neither Alice nor Bob have keys to the other party's locks. So if Alice sends the item to Bob in a locked box, he will be unable to open it. Furthermore, it does no good to send an unopened lock or a key to the other party to use, as it would simply be stolen, or the key copied, negating any security.

The solution is for Alice to first send Bob the item in a box, sealed with one of her own locks. Bob cannot open Alice's lock, but he can add one of his own locks to the box (so it has now has two locks on it) and return it to Alice. Alice then removes her lock, and sends the box a second time to Bob. Finally, Bob removes his lock, and opens the box.

We use this idea in the following protocol, for Alice to send a secret choice of L to Bob.

Algorithm 2 Three-pass exchange protocol with rings of type (1)

1: A and B publicly agree on a ring S of type (1), and two multiplicatively closed, commutative subsets $\mathcal{C}_l, \mathcal{C}_r \subset S$.
2: A chooses a secret $L \in S$, which is not central in $\mathcal{C}_l$ and $\mathcal{C}_r$, that she wants to share with B.
3: A picks random polynomials $P_A \in \mathcal{C}_l$ and $Q_A \in \mathcal{C}_r$, which form her private tuple (P_A, Q_A). If coincidentally either P_A or Q_A commute with L, A must choose a different pair P_A, Q_A.
4: A computes the product $P_A \cdot L \cdot Q_A$, and sends it to B.
5: B picks random polynomials $P_B \in \mathcal{C}_l$ and $Q_B \in \mathcal{C}_r$, which form his private tuple (P_B, Q_B).
6: B computes the intermediate product $P_{\text{int}} = P_B \cdot P_A \cdot L \cdot Q_A \cdot Q_B (= P_A \cdot P_B \cdot L \cdot Q_B \cdot Q_A)$ and sends it to A.
7: A divides P_{int} on the right by Q_A, and on the left by P_A. A sends the result, $P_B \cdot L \cdot Q_B$, to B.
8: B divides $P_B \cdot L \cdot Q_B$ on the right by Q_B, and on the left by P_B, to recover the secret L.

Remark 10. With L secretly agreed upon, Alice and Bob may now use Algorithm 1 to agree upon a secret key, with only the commuting elements from $\mathcal{C}_l$ and $\mathcal{C}_r$ being public. Note that they may choose a new set of tuples $(P_A,\ Q_A)$, $(P_B,\ Q_B)$, and indeed, may even publicly agree on a new choice of $\mathcal{C}_l$ and $\mathcal{C}_r$ for Algorithm 1. In this way, the information exchanged during Algorithm 2 cannot help the eavesdropper to know the $(P_A,\ Q_A)$ and $(P_B,\ Q_B)$ used in Algorithm 1.

Remark 11. Naturally, Algorithm 2 can be used as a key exchange protocol by itself, where L is the key being exchanged. Deciding which approach would be better in a given situation would require further investigation. However, there is an advantage in defending against the attack where Eve forms an ansatz and tries to solve non-linear systems of equations: Eve would deal with tertiary systems of equations (instead of quadratic ones) in this case, as the coefficients of L are also unknown.

Remark 12. Regarding the requirement, at various points in Algorithm 2, that elements must not commute: If the elements did commute, we actually do not know of any attacks that would take advantage of this property. So these requirements may be unnecessary. However, in general terms, commuting algebraic objects are often easier to analyze than non-commuting ones, and might be easier to attack. Prudence therefore suggests that we choose non-commuting elements as much as possible, allowing commutativity only where it is needed by the protocol in order to function properly.

The security assumption here differs slightly from the one for Algorithm 1. Here, Eve is given $\mathcal{C}_l$, $\mathcal{C}_r$, $P_A \cdot L \cdot Q_A$, $P_B \cdot P_A \cdot L \cdot Q_A \cdot Q_B$ and $P_B \cdot L \cdot Q_B$.

Difficult Problem (Ore Three Pass Protocol (OTPP)): Determine L with the given information.

As is the case for the Diffie-Hellman-like protocol, the ability to feasibly compute all factorizations of elements in S would of course allow Eve to determine Alice's secret. But this is a hard problem with currently known methods, as mentioned in the introduction. We are not aware of any other feasible way to obtain L.

Remark 13. Similarly to ODH, there is also a corresponding decision problem related to OTPP: Given a candidate for the final secret L, determine if L is consistent with the public information exchanged by Alice and Bob. To the best of our knowledge, this is also currently intractable.

5.2 ElGamal-like Encryption and Signature Schemes

In 1984, ElGamal showed that a cryptosystem and digital signature scheme were possible using exponentiation in finite fields [ElG85]. Here, we will show that such schemes are also possible using non-commutative polynomial rings. There will necessarily be some differences between our schemes and those of ElGamal, since finite fields are commutative and all (non-zero) elements are invertible, whereas our structures are non-commutative and the elements are non-invertible. Nevertheless, approaches very similar to those of ElGamal can be developed.

5.2.1 An ElGamal-like Encryption Scheme Suppose Bob wishes to send a secret message to Alice. An inconvenience of the Diffie-Hellman-like protocols is that Bob must wait for Alice to respond before the key is decided. Only then can Bob encrypt and send his message.

It would be desirable to have a scheme whereby Bob, or anyone else, can send an encrypted message to Alice whenever they wish, without the need to wait for Alice to respond. This is possible, as we now show, but it requires Alice to precompute and publish some information. It should be noted that the following encryption scheme is only intended to provide a basic level of security (i.e. an eavesdropper who intercepts the ciphertext is not able to compute the corresponding plaintext). Further enhancements to the scheme are required for stronger notions of security (e.g. indistinguishability against adaptive chosen-ciphertext attacks).

Preparation: Alice chooses $L \in S$, and two multiplicatively closed, commutative subsets $\mathcal{C}_l, \mathcal{C}_r \subset S$. Then she picks random polynomials $P_A \in \mathcal{C}_l$ and $Q_A \in \mathcal{C}_r$, which form her private tuple (P_A, Q_A). If coincidentally either P_A or Q_A commute with L, Alice must choose a different pair P_A, Q_A. Alice publishes L, $\mathcal{C}_l$, $\mathcal{C}_r$, and $P_{\text{Alice}} := P_A \cdot L \cdot Q_A$. The tuple $(P_A,\ Q_A)$ is kept secret.

Encryption: Let $m \in S$ be a message which Bob wishes to encrypt before sending to Alice. (We assume that a plaintext message has already been mapped to Ore polynomial form, so $m \in S$.)

Bob picks random polynomials $P_B \in \mathcal{C}_l$ and $Q_B \in \mathcal{C}_r$, which form his private tuple (P_B, Q_B). Again, Bob ensures that P_B, Q_B and L are pairwise non-commutative.

Bob computes $P_{\text{Bob}} := P_B \cdot L \cdot Q_B$ and $P_{\text{final}} := P_B \cdot P_{\text{Alice}} \cdot Q_B$, and encrypts the message as $m_e := m \cdot P_{\text{final}}$. Bob then sends the pair (m_e, P_{Bob}) to Alice.

Decryption: Given (m_e, P_{Bob}), Alice first computes $P_{\text{final}} = P_A \cdot P_{\text{Bob}} \cdot Q_A$. She then divides $m_e(= m \cdot P_{\text{final}})$ on the right by P_{final} to recover m.

Correctness: The correctness depends on Alice computing the same P_{final} as Bob. But this will be true because P_A, P_B commute, as do Q_A, Q_B, so $P_B \cdot P_{\text{Alice}} \cdot Q_B = P_A \cdot P_{\text{Bob}} \cdot Q_A$.

Security: Comparing this scheme with Algorithm 1, it is clear that P_{Alice} is the product sent from Alice to Bob, P_{Bob} the product from Bob to Alice, and P_{final} is the shared final key that is agreed upon. In addition to knowing these exchanged values, Eve will also know the encrypted message m_e. So the security is essentially the same as that of Algorithm 1, except for an additional assumption, that the attacker will find it difficult to factor $m_e := m \cdot P_{\text{final}}$.

Remark 14. As ElGamal observes, the multiplication $m_e := m \cdot P_{\text{final}}$ in the encryption step could be replaced by any invertible operation, e.g., by addition. Indeed, setting $m_e := m + P_{\text{final}}$ gives essentially the encryption scheme proposed by Boucher et al. [BGG+10]. This has the advantage of being easier to compute, and giving an encrypted m_e of smaller size than one that uses multiplication. We would caution, however, that for such a scheme to be secure, both m and P_{final} should probably be dense polynomials. If one polynomial is dense but the other sparse, for example, much of the structure of the dense polynomial would be visible in m_e, and could aid an attacker.

Remark 15. The message encryption step (i.e. computing $m_e = m \cdot P_{\text{final}}$) is not required to be performed inside S. In fact, P_{final} is the key with which Bob encrypts the message m, and the algorithm that is used for this step can be replaced by any known and well-studied private key system, such as AES [DR02]. In this case, the security of our protocol reduces to the security of our Algorithm 1, along with the security of the chosen private key system.

5.2.2 An ElGamal-like Digital Signature Scheme

Suppose Alice wishes to prove to Bob that a message she is sending him did, in fact, come from her. For his part, Bob may want this proof, both as a guard against forgers, and also to prevent Alice from denying at a later time that the message had come from her. The digital signature scheme shown below is intended to accomplish this. (Note that a message m need not be encrypted to be signed; Alice may also use her signature to show that she publicly approves of a cleartext message.) As in the encryption scheme in section 5.2.1, this requires Alice to precompute and publish information at some location, e.g., a secure webpage. This must be securely associated with her, both to prevent forgers from altering the information, and to prevent her from trying to repudiate any messages bearing her digital signature. Again, as in section 5.2.1, our proposed signature scheme is only intended to provide a basic level of security (i.e. Eve cannot forge Alice's signature on an arbitrary message of Eve's choice). Further enhancements are needed for stronger notions of security.

Preparation: Alice chooses $a_1, L, a_2 \in S$, pairwise non-commutative. Alice publishes L and $P_{\text{Alice}} := a_1 \cdot L \cdot a_2$. The tuple (a_1, a_2) is kept secret.

Signature Creation: Let $m \in S$ be a message which Alice wishes to sign. (m may be either encrypted or unencrypted.)

Alice chooses two new random polynomials $k_1, k_2 \in S$. (Alice checks that L, k_1, k_2 are pairwise non-commutative.) Alice computes $\gamma := k_1 \cdot L \cdot k_2$, $\epsilon_1 := k_1 \cdot L \cdot a_2$, and $\epsilon_2 := a_1 \cdot L \cdot k_2$. The tuple (k_1, k_2) is kept secret.

Now Alice chooses $q_1 \in S$ and computes $r_1 := m - \gamma a_1 - q_1 k_1 \in S$, so that:

$$m - \gamma a_1 = q_1 k_1 + r_1. \tag{4}$$

Analogously, Alice also chooses $q_2 \in S$ and computes $r_2 := m - a_2 \gamma - k_2 q_2 \in S$, so that:

$$m - a_2 \gamma = k_2 q_2 + r_2. \tag{5}$$

Alice sends Bob the signed message m as the 8-tuple $(m, \gamma, q_1, r_1, q_2, r_2, \epsilon_1, \epsilon_2)$. Note that k_1, k_2, a_1, a_2 are all kept secret by Alice.

Signature Verification: Given $(m, \gamma, q_1, r_1, q_2, r_2, \epsilon_1, \epsilon_2)$, (and the public L and P_{Alice}), Bob computes:

$$\begin{aligned} \text{sig}_{\text{left}} &:= (m - r_1) \cdot L \cdot (m - r_2)\,, \\ \text{sig}_{\text{right}} &:= q_1 \gamma q_2 + q_1 \epsilon_1 \gamma + \gamma \epsilon_2 q_2 + \gamma P_{\text{Alice}} \gamma. \end{aligned} \tag{6}$$

If $\text{sig}_{\text{left}} = \text{sig}_{\text{right}}$, then the signature is accepted as valid. Otherwise, the signature is rejected.

Correctness: We have

$$\begin{aligned}
\mathrm{sig}_{\mathrm{left}} &= (m - r_1) \cdot L \cdot (m - r_2) \\
&= (q_1 k_1 + \gamma a_1) \cdot L \cdot (k_2 q_2 + a_2 \gamma)\,, \quad \text{using} \quad (4),\ (5) \\
&= q_1 k_1 L k_2 q_2 + q_1 k_1 L a_2 \gamma + \gamma a_1 L k_2 q_2 + \gamma a_1 L a_2 \gamma \\
&= q_1 \gamma q_2 + q_1 \epsilon_1 \gamma + \gamma \epsilon_2 q_2 + \gamma P_{\mathrm{Alice}} \gamma \\
&= \mathrm{sig}_{\mathrm{right}},
\end{aligned}$$

using $k_1 L k_2 = \gamma$, $k_1 L a_2 = \epsilon_1$, $a_1 L k_2 = \epsilon_2$, and $a_1 L a_2 = P_{\mathrm{Alice}}$.

Security: To forge a signature for a given message m, Eve must find values of γ, q_1, r_1, q_2, r_2, ϵ_1, ϵ_2 which result in $\mathrm{sig}_{\mathrm{left}} = \mathrm{sig}_{\mathrm{right}}$. We cannot see any easy way to do this. She can, of course, choose her own values of k_1, k_2, and compute a corresponding $\gamma = k_1 L k_2$. This is, indeed, a plausible value for γ, since Alice could have chosen these values for k_1, k_2 herself. But Eve does not have the values of a_1 and a_2. If she guesses them, as say $\tilde{a}_1$, $\tilde{a}_2$, then eventually she will fail to form P_{Alice} correctly when the product $\tilde{a}_1 L \tilde{a}_2$ is formed in the correctness proof above, and will not have $\mathrm{sig}_{\mathrm{left}} = \mathrm{sig}_{\mathrm{right}}$.

Furthermore, given a legitimate signature from Alice, it appears doubtful that an attacker can recover a_1 and a_2. For example, from (4), we have $m - r_1 = q_1 k_1 + \gamma a_1$. Here, the left side is known, as are q_1 and γ on the right. But we know no way of determining k_1 and a_1, other than computing the syzygy-module $M \in S^3$ of q_1, γ, and $m - r_1$ and considering the subset $\{\ell = [\ell_1, \ell_2, \ell_3] \in M \mid \ell_3 = -1\} \subseteq M$. As this is in general an infinite set, this does not yield a practical way of recovering k_1 and a_1.

It will be noted, however, that computing a signature may result in very large expressions. Thus, $(m - r_1) \cdot L \cdot (m - r_2)$ is likely to be much larger than the message m itself. However, in practice, one would only create a signature on a certain hash-sum of the message m, which has fixed length for all possible messages.

5.3 A Zero Knowledge Proof Protocol Using Multivariate Ore Polynomials

As usual, let $S := R[\partial_1; \sigma_1, \delta_1][\partial_2; \sigma_2, \delta_2] \ldots [\partial_n; \sigma_n, \delta_n]$, for some fixed $n \geq 2$. Let $L \in S$. Suppose Alice knows a factorization (possibly partial) of L into two nontrivial factors ℓ_1 and ℓ_2. That is, $L = \ell_1 \cdot \ell_2$, where $L, \ell_1, \ell_2 \in S$. (By nontrivial, we mean that for at least one i, where $1 \leq i \leq n$, we have $\deg_{\partial_i}(\ell_1) \geq 1$, and similarly for ℓ_2. Furthermore, the factorization may only be partial, i.e., ℓ_1 or ℓ_2 may themselves factor over S. But we will not be concerned here with this possibility.) Note that as we are assuming that factorization of multivariate Ore polynomials is, in general, computationally infeasible, Alice may well have accomplished her feat by first choosing ℓ_1 and ℓ_2, then creating a suitable L by computing $L = \ell_1 \cdot \ell_2$, and then finally publishing L. But the particular method she used to produce an L and its factors is not important to the protocol.

Bob also knows L, but does not know ℓ_1 or ℓ_2. Alice wishes to convince Bob, beyond any reasonable doubt, that she knows a factorization of L. But she does not wish to tell Bob her factors ℓ_1 or ℓ_2, or to give Bob enough information that would allow him to compute the factors within any feasible time. We have, therefore, a situation which calls for a zero knowledge proof protocol.

Alice can do this using the following protocol, which is repeated as many times as Bob desires, until he is convinced that Alice does, indeed, possess a factorization of L:

Step 1: Alice chooses two polynomials $p_1, p_2 \in S$, and forms the product $\pi := p_1 \cdot L \cdot p_2$. She sends π to Bob. She also tells Bob the degrees of p_1 and p_2 in each of the ∂_i. That is, for each i, where

$1 \leq i \leq n$, she sends Bob $\deg_{\partial_i}(p_1)$ and $\deg_{\partial_i}(p_2)$. Apart from this information, however, Alice keeps p_1 and p_2 private, unless Bob specifically asks for them in Step 2.

Note that $\pi = p_1 \cdot L \cdot p_2 = p_1 \cdot \ell_1\ell_2 \cdot p_2 = \pi_1 \cdot \pi_2$, where we define $\pi_1 := p_1\ell_1$, and $\pi_2 := \ell_2 p_2$. Thus, two different partial factorizations of π are $\pi = p_1 \cdot L \cdot p_2$ and $\pi = \pi_1 \cdot \pi_2$. Essentially, in Step 2., Bob may ask Alice for one, and only one, of these two factorizations.

Step 2: Having received π and the degree information for p_1 and p_2 from Alice, Bob asks Alice for exactly one of the following: Either (i) p_1 and p_2, or (ii) π_1 and π_2.

Step 2a: If Bob asked for p_1 and p_2, he checks that $\pi = p_1 \cdot L \cdot p_2$. He also checks that p_1 and p_2 satisfy the degree bounds sent earlier by Alice in Step 1.

Step 2b: If Bob asked for π_1 and π_2, he checks that $\pi = \pi_1 \cdot \pi_2$. He also checks that π_1 and π_2 satisfy the following degree conditions: For each i, where $1 \leq i \leq n$, he requires that $\deg_{\partial_i}(\pi_1) \geq \deg_{\partial_i}(p_1)$. Furthermore, for at least one j, $1 \leq j \leq n$, the inequality must be strict, i.e., $\deg_{\partial_j}(\pi_1) > \deg_{\partial_j}(p_1)$. Likewise, π_2 must satisfy $\deg_{\partial_i}(\pi_2) \geq \deg_{\partial_i}(p_2)$ for all i, where $1 \leq i \leq n$, and for at least one k, where $1 \leq k \leq n$, must satisfy $\deg_{\partial_k}(\pi_2) > \deg_{\partial_k}(p_2)$.

If any of these checks fail, then the protocol terminates, and Bob rejects Alice's claim to know a nontrivial factorization of L. If the checks hold true, then Alice has passed this cycle of the protocol. Steps 1. through 2b. constitute one cycle of the protocol. For each cycle, Alice chooses a new pair of polynomials p_1, p_2 in Step 1., never using the same polynomial more than once. The cycle is repeated until either one of the checks fail, or until Bob is convinced, beyond a reasonable doubt, that Alice does indeed posses a nontrivial factorization of L.

Discussion and Security of the Protocol: For each cycle of the protocol, Bob can randomly choose whether to ask Alice for p_1, p_2, or for π_1, π_2. Suppose this is repeated many times, with the answers always satisfying the checks. Then Bob should eventually be convinced, beyond a reasonable doubt, that for any π offered by Alice in Step 1., she is always able to factor π both as p_1Lp_2 and as $\pi_1\pi_2$, with the degree conditions also satisfied.

The two questions we must address are: (i) should Bob believe that Alice knows some nontrivial factorization $\ell_1\ell_2$ of L?; and (ii) do Alice's answers to Bob's queries give Bob a practical method to determine a nontrivial factorization of L?

(i) Alice can factor L:

First, consider the requirement that Alice must give degree conditions on p_1, p_2 in Step 1., before Bob says whether he wishes to know p_1, p_2, or π_1, π_2 in Step 2. If we did not impose this condition on Alice, she could trick Bob into believing that she had a factorization of L as follows. First, instead of a p_1, she chooses $p_1', p_1'', p_2 \in S$, and computes $\pi := p_1' \cdot p_1'' \cdot L \cdot p_2$, sending only π to Bob. Now, if Bob asks for p_1, p_2, she sends him $p_1 := p_1'p_1''$, and p_2. He then computes $p_1 \cdot L \cdot p_2$, and finds this product to equal π, as expected. On the other hand, if he asks for π_1, π_2, Alice sends him $\pi_1 := p_1'$, $\pi_2 := p_1''Lp_2$. Again, Bob will find that $\pi_1 \cdot \pi_2$ yields π, as expected. So Alice appears to have passed the test, even though she needed no knowledge of any factorization of L.

Forcing Alice to give the degree conditions in Step 1., before Bob announces his choice in Step 2., prevents Alice from using this strategy to deceive Bob. Furthermore, the degree conditions on π_1 and π_2, if Bob should ask for these two polynomials, prevents Alice from simply setting either $\pi_1 := p_1$, $\pi_2 := Lp_2$, or $\pi_1 := p_1L$, $\pi_2 := p_2$, neither of which would have required any knowledge of the factors of L. Informally, the degree conditions force both π_1 and π_2 to partially "overlap" L in the product π.

Now, the above might appear to imply that π_1 and π_2 must have the forms $\pi_1 = p_1\ell_1$, $\pi_2 = \ell_2 p_2$. However, factorization in S is unique only up to similarity (cf. Section 1.1, Definition 4). Consequently, it is possible that Alice has found p_1, p_2, π_1, π_2, such that $p_1 L p_2 = \pi_1\pi_2$ (which she sets equal to π), and yet p_1 is not a left divisor of π_1, nor is p_2 a right divisor of π_2. With such a choice of polynomials, Alice would be able to satisfy the demands of the protocol, without necessarily having any knowledge of the factors of L. However, finding such a set of polynomials is, as far as we know, computationally impractical with currently known methods. The only reasonable conclusion for Bob, therefore, is that $\pi_1 = p_1\ell_1$, $\pi_2 = \ell_2 p_2$.

However, if Alice knows both π_1 and p_1, she can simply perform exact division of π_1 on the left by p_1 to obtain ℓ_1. Similarly, (exactly) dividing π_2 on the right by p_2 will yield ℓ_2. Hence Bob must conclude that Alice does, indeed, know some nontrivial factorization $\ell_1\ell_2$ of L.

(ii) Bob cannot factor L:

Again, let us first consider the effect that knowing the degrees of p_1 and p_2 will have on Bob's attempts to find a factorization of L. Certainly, this knowledge makes it easier for him to set up an ansatz, e.g., of the form $\pi = p_1\ell_1\ell_2 p_2$. However, even if Bob did not know these degree conditions, the number of possibilities he would have to consider would increase only by a factor that is polynomial in n and the maximum degree in any ∂_i of π. That is, if Bob could find some factors of L in polynomial time by some algorithm which makes use of the given degree conditions, he could also find the factors in polynomial time without knowing these conditions. Thus, these degree conditions do not, in themselves, compromise the security of the protocol.

Now, Bob obtains two sorts of information from Alice. The first is p_1 and p_2 such that $\pi = p_1 L p_2$. Clearly, this does not help Bob to factor L, since he could just as well have chosen his own p_1 and p_2, computed the product $p_1 L p_2$ to form his own π, and done so as many times as he wished. Such queries, therefore, do not lead to any exploitable weakness.

The second type of query gives Bob π_1 and π_2 such that $\pi = \pi_1\pi_2$. Let us first consider π_1 (the situation for π_2 is similar): Bob knows that $\pi_1 = p_1\ell_1$, though of course p_1 and ℓ_1 are unknown to him. Bob also knows L, though again its factors ℓ_1, ℓ_2 are also unknown to him. Hence one type of attack would be to use the fact that ℓ_1 is a right divisor of π_1, and also a left divisor of L. It is interesting to note, however, that even in (non-commutative) Euclidean domains, there is currently no known practical algorithm to find such a left and right simultaneous divisor. Our domain S is not even Euclidean, so the situation is even worse for an attacker. The only general, though impractical, approach involves forming an ansatz, and solving the resulting quadratic system of multivariate polynomial equations. So this attack fails.

Another approach Bob might try is to ask many queries of this sort, to generate a growing set of polynomials $\pi = p_1\ell_1$, $\pi' = p_1'\ell_1$, $\pi'' = p_1''\ell_1$, et cetera. Now, ℓ_1 is a common right divisor of all these polynomials, and as the set grows in size, it will very likely be the gcrd of the set. Hence, if S were a Euclidean domain, Bob could use the Euclidean algorithm to find ℓ_1 with high probability. However, S is not a Euclidean domain. The only hope for a potential attacker could be – similar to the attack described in section 2.3.2 – a computation of a left Gröbner basis of the ideal generated by the πs, if the notion of a left Gröbner basis exists for the chosen ring S (which does not, if S is chosen non-Noetherian). Then, it is possible that the basis consists of only one element, namely ℓ_1. We made experiments in the polynomial second Weyl algebra, choosing very small degrees (total degree less or equal to five) for ℓ_1. With a sufficiently large number of these πs, the computed Gröbner basis had indeed the desired structure. However, as the computation of a Gröbner basis is exponential space hard [MM82], this is not a feasible approach in general. Even with slightly larger choices for these degrees, the Gröbner basis computation in case of the second Weyl algebra fails to terminate within a reasonable time. Other than this, we are not aware of any known practical method to find such a common divisor for S.

Again, a general, though impractical, approach involves forming an ansatz, and solving the resulting quadratic system of multivariate polynomial equations.

Similar remarks apply to Bob's attempts to determine ℓ_2.

We conclude, therefore, that it will be impractical for Bob to determine a nontrivial factorization of L by using Alice's answers to his queries.

6 Conclusion

The key exchange primitive as presented in [BGG+10] has been altered to be immune against the attack presented in [DK11], and extended. The new version presented in this paper continues to have the positive properties discussed by Boucher et al. in their paper. The security of our proposal is related to the hardness of factoring in non-commutative rings and the non-uniqueness of the factorization. A class of insecure key choices that would reduce the problem to commutative factorization was outlined. Moreover, we provide the freedom to choose rings that are not Noetherian, where a general factorization algorithm might not even exist.

An implementation for a specific ring is provided and we look forward to feedback from users. Furthermore, based on this implementation, we have published some challenge problems as described in section 3.1. We encourage the reader to attack our proposed schemes via these challenges.

We also mention here that related protocols can be developed using this primitive. We presented four such enhancements in section 5: a three-pass protocol, an ElGamal-like encryption scheme, and an ElGamal-like digital signature scheme, and a zero-knowledge proof protocol.

Of course the security and practicality of our protocols need to be examined further for particular choices of the non-commutative ring S, which is described in as general a way as possible in this article. All of them can be broken if an algorithm to find a specific factorization in a feasible amount of time is available. However, researchers have been interested in factoring Ore polynomials for decades, and this problem is in general perceived as very difficult. As Ore polynomial rings are in general abstractions of operator algebras, any success of breaking our protocols would lead to a further understanding of the underlying operators. Thus any successful attempt to break our protocols, even if it is just for one special choice of S, would benefit several scientific communities.

Moreover, we note that some of our proposed schemes bear a strong resemblance to others that have been known, studied, and withstood general attacks for years, or even decades. For example, our digital signature scheme can be viewed as an analogue of the ElGamal scheme, adapted for non-commutative rings. But to the best of our knowledge, no proof of security, showing that breaking the ElGamal signature scheme is equivalent to solving the discrete logarithm problem, has yet been found. Instead, we have the accumulated experience of many cryptographers, who have so far found no general method of attack that does not involve computing a discrete logarithm. This appears to be the basis for accepting the scheme as secure. Now, given the similarity of our proposal to that of ElGamal, one can plausibly suggest that ours will also be secure, unless – again – some practical method of factorization is found.

Interesting questions for future research are: For which choices of a ring S of type (1) can one construct an effective attack for the proposed schemes (possibly using quantum computers)? Note, that the rings we chose for our examples and for our implementations are among the simplest ones

(Noetherian, bivariate, over finite fields) that appear to be immune to known attacks. And furthermore, can one improve the computation of the arithmetics in those rings using a quantum computer?

We also hope for better implementations in the future for arithmetics in Ore polynomials, since the existing ones in commodity computer algebra systems appear to be slow on large examples. This fact forced us to write our own experimental implementation to evaluate the feasibility of our proposals.

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Appendix

A Code-Examples

A.1 Factorization of Chebyshev Differential Operators using MAPLE

A few simple MAPLE commands to set up and find right hand factors of the Chebyshev differential operator (2) of section 1.2, with parameter n. In the code, $\mathbf{D} = \partial_1$, $\mathbf{x} = x_1$.

```
> with(DEtools):

> _Envdiffopdomain:=[D,x]:

> n := 11;
                                  n := 11

> L := (1-x^2)*D^2 - x*D + n^2;
                                  2   2
                       L := (1 - x ) D  - x D + 121


> v := DFactorLCLM(L);
                2        4         6          8         10
         1 - 60 x  + 560 x  - 1792 x  + 2304 x  - 1024 x
v := [D - -----------------------------------------------,
                 3         5         7         9   1024  11
          x - 20 x  + 112 x  - 256 x  + 256 x  - ---- x
                                                  11
```

```
                                             3           5           7           9
           x           -120 x + 2240 x  - 10752 x  + 18432 x  - 10240 x
D - --------------- - --------------------------------------------------]
    (x - 1) (x + 1)             2         4          6          8          10
                       1 - 60 x  + 560 x  - 1792 x  + 2304 x  - 1024 x
```

Here, the two components of $\mathbf{v}$ are the two possible right hand factors of L. By rerunning the commands with other (positive integer) values of n, the growth in the factors can be observed. Other classical operators from the 1800's, such as those of Hermite, Legendre, and Laguerre, have similar behaviour.