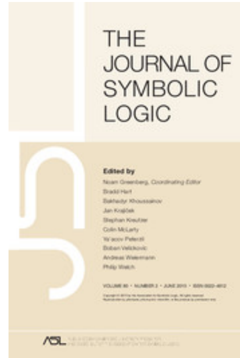

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THE FINITE MODEL PROPERTY FOR VARIOUS FRAGMENTS OF INTUITIONISTIC LINEAR LOGIC

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Abstract. Recently Lafont [6] showed the finite model property for the multiplicative additive fragment of linear logic (**MALL**) and for affine logic (**LLW**), i.e., linear logic with weakening. In this paper, we shall prove the finite model property for intuitionistic versions of those, i.e. intuitionistic **MALL** (which we call **IMALL**), and intuitionistic **LLW** (which we call **ILLW**). In addition, we shall show the finite model property for *contractive linear logic* (**LLC**), i.e., linear logic with contraction, and for its intuitionistic version (**ILLC**). The finite model property for related substructural logics also follow by our method. In particular, we shall show that the property holds for all of **FL** and $\mathbf{GL}^-$-systems except $\mathbf{FL_c}$ and $\mathbf{GL}_\mathbf{c}^-$ of Ono [11], that will settle the open problems stated in Ono [12].

§1. Introduction. In Lafont [6], it is shown that the finite model property holds for the multiplicative additive fragment of linear logic (**MALL**) and for affine logic (**LLW**), i.e., linear logic with weakening. In this paper we show the finite model property for some fragments of intuitionistic linear logic (**ILL**) and related systems, including the intuitionistic versions of **MALL** and **LLW**. Our study also includes classical and intuitionistic linear logics with contraction, which are not treated in [6].

In [6], the notion of *logical congruence* is introduced in the framework of (classical) phase semantics (Girard [3]) and it is observed that a quotient model induced by a logical congruence preserves satisfiability. Thus in order to show the finite model property, it is sufficient to find a proper logical congruence on a certain syntactical (canonical) model which induces a *finite* quotient model. To achieve this, Lafont [6] utilizes a syntactical (canonical) model construction of the first author (cf. [10]) for a phase-semantic "strong" completeness proof, from which the cut-elimination theorem is derivable. This model construction is quite useful in Lafont's proof, particularly in the case of **MALL** where the cut-elimination property is essential, and his refined canonical model construction thus implies the completeness, cut-elimination and finite model property theorems at once for all.

It is not very clear and should be carefully examined whether his method also applies to the intuitionistic cases, due to the "second-order" nature of the closure operation of intuitionistic phase semantics in contrast with the simple double-negation closure operation of classical phase semantics; for example, in intuitionistic phase semantics of Okada [10], given a subset X of an intuitionistic phase space domain, the *closure* X^C of X is defined to be the intersection of *all* facts that contain X.

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We adapt the notion of logical congruence in [6] to the *intuitionistic phase spaces* of Okada [10] in Section 2. Now the (intuitionistic) logical congruence must be defined with reference to all the facts (not only to one specific fact, as in the case of the classical logical congruence). To make the syntactical (canonical) models *finitely generated*, which is essential to prove the finite model property, we slightly modify the construction of syntactical models of Okada [10]. In this setting we are able to adapt Lafont's idea to the intuitionistic cases. We shall show the finite model property for the multiplicative additive fragment of intuitionistic linear logic (**IMALL**) in Section 3, and that for intuitionistic affine logic (**ILLW**) in Section 4. It turns out that a similar technique also applies to logics with contraction, which are not treated in [6]. In Section 5, we shall show the finite model property for intuitionistic *contractive* linear logic (**ILLC**), i.e., **ILL** with contraction, and the classical contractive linear logic (**LLC**). In particular, these results give us proofs of the decidability of **LLC** and **ILLC**.

There have been several attempts to show the finite model property for implicational fragments of some substructural logics, for example, for the implicational fragment of intuitionistic linear logic **ILL** (or equivalently the implicational fragment of $\mathbf{FL_e}$) by Buszkowski [2], for the implicational fragment of **ILLW** (or that of $\mathbf{FL_{ew}}$) by Meyer & Ono [8], and for the implicational fragment of **ILLC** (or that of $\mathbf{FL_{ec}}$) by Meyer [7] (cf. Meyer & Ono [8]). See Ono [12] for a short survey. In many cases, an extension with the additive conjunction & is not problematic. But it is not so easy to extend these results with other connectives (e.g., the multiplicative conjunction). Indeed, the question whether any of **FL**(*Full Lambek*)-systems of Ono [11] has the finite model property remains open according to Section 3 of Ono [12]. Our phase semantic framework avoids the difficulty arising in the existing frameworks, and provides the finite model property results for a wide range of substructural logics. Specifically we shall show, in Section 6, the finite model property for all of **FL** and $\mathbf{GL}^-$-systems except $\mathbf{FL_c}$ and $\mathbf{GL_c^-}$ of Ono [11].

All our results are summarized by Table 1 and Table 2 at the end of the last section (Section 6), and all necessary definitions on the syntax of intuitionistic substructural logics at Appendix, for the convenience of the reader.

§2. Intuitionistic phase spaces. We shall define *intuitionistic phase spaces*, following Okada [10], (cf. also Abrusci [1], Sambin [13], Troelstra [14] for slightly different formulations).

An *intuitionistic phase space* (M, D) consists of a commutative monoid M and a set D (called the *facts*) of subsets of M that satisfies

(P1) D is closed under arbitrary $\bigcap$; in particular $M \in D$;
(P2) if $X \subseteq M$ and $Y \in D$, then $X \multimap Y \in D$,

where $\multimap$ is defined by $X \multimap Y = \{y | \forall x \in X xy \in Y\}$ for any $X, Y \subseteq M$. We also define XY as $\{xy | x \in X, y \in Y\}$ and X^C as $\bigcap\{Z \in D | X \subseteq Z\}$ (the smallest fact that includes X).

Then, we can define $\mathbf{1} = \{1\}^C$ (1 stands for the unit element of M), $\top = M$, $\mathbf{0} = \emptyset^C$, and for any facts X, Y,

- $X \otimes Y = (XY)^C$,
- $X \& Y = X \cap Y$,

- $X \oplus Y = (X \cup Y)^C$.

We shall often omit D and simply write M to denote an intuitionistic phase space (M, D).

Note that $(D, \&, \oplus, \mathbf{0}, \multimap, \otimes, \mathbf{1})$ forms an *IL-algebra* in the sense of Troelstra [14] (Associativity of $\otimes$ is nontrivial, but follows from the observation that $X^C Y^C \subseteq (XY)^C$ for any $X, Y \subseteq M$).

$D_0 \subseteq D$ is said to be a *base* for D if each $X \in D$ is of the form $X = \bigcap_{i \in \Lambda} Y_i$ where $Y_i \in D_0$. Thus, given D_0, the set D of facts for which D_0 is a base is uniquely obtained by taking arbitrary intersections of elements of D_0 as the facts. Clearly D is finite whenever D_0 is.

The following definition is analogous to that of *enriched* (classical) phase spaces in Lafont [6]. If M is an intuitionistic phase space, then $J(M) = \{x \in \mathbf{1} | x \in \{xx\}^C\}$ is a submonoid of M. An *enriched intuitionistic phase space* is an intuitionistic phase space M endowed with a submonoid K of $J(M)$ (not necessary to be a fact).

For any fact X of an enriched intuitionistic phase space, define

- $!X = (X \cap K)^C$.

See Girard [3, 4], Lafont [6] for the basic properties of the modality (exponential) !. From now on, we suppose all intuitionistic phase spaces be enriched whenever we are concerned with the fragments including the modality (exponential) !.

An *intuitionistic phase model* is given by an (enriched) intuitionistic phase space M and an *interpretation* which maps each atom α to a fact $\alpha^\bullet$ of M. Any formula A is interpreted by a fact $A^\bullet$ along the above definitions, and we say that A is *satisfied* in M if $1 \in A^\bullet$.

Now let us define the syntactical (canonical) model for completeness/cut-elimination. As we shall need finitely generated syntactical models in the subsequent argument, we modify the construction of Okada [10]. Fix a formula A and consider the system $\mathbf{ILL}[A]$ which is $\mathbf{ILL}$ restricted to the subformulas of A. Then the *syntactical model* $\mathbf{ILL}^\bullet[A]$ for $\mathbf{ILL}[A]$ is defined as follows;

- The underlying monoid of $\mathbf{ILL}^\bullet[A]$ is the free commutative monoid generated by all subformulas of A (hence finitely generated).
- For a sequent $\Gamma \vdash C$, let $[\![\Gamma \vdash C]\!]$ be $\{\Sigma \mid \Sigma, \Gamma \vdash C$ is *cut-free* provable in $\mathbf{ILL}[A]\}$, which we call the *outer value* of $\Gamma \vdash C$. Write $[\![C]\!]$ for $[\![\vdash C]\!]$.
- Let $D_0 = \{[\![\Gamma \vdash C]\!] \mid \Gamma \vdash C$ is a sequent in $\mathbf{ILL}[A]\}$ be the base for the facts D of $\mathbf{ILL}^\bullet[A]$. Namely, D consists of the elements of the form $\bigcap_{i \in \Lambda} [\![\Gamma_i \vdash C_i]\!]$.
- $\alpha^\bullet = [\![\alpha]\!]$ for any atom α in $\mathbf{ILL}[A]$.
- K is the submonoid of $\mathbf{ILL}^\bullet[A]$ generated by all formulas of the form $!B$.

One can show that $\mathbf{ILL}^\bullet[A]$ is indeed an intuitionistic phase model. Note that Okada [10] uses the notion of outer value of the form $[\![C]\!]$ for his completeness/cut-elimination proof, while we consider the form $[\![\Gamma \vdash C]\!]$. It is this change that allows us to prove that $\mathbf{ILL}^\bullet[A]$ satisfies (P2); observe that $X \multimap Y = \bigcap\{[\![\Delta, \Gamma \vdash C]\!] \mid \Delta \in X$ and $Y \subseteq [\![\Gamma \vdash C]\!]\} \in D$ whenever $Y \in D$.

The following completeness/cut-elimination argument is a modification of Okada [10].

LEMMA 1. *For any formula B in* $\mathbf{ILL}[A]$, *$B \in B^\bullet \subseteq [\![B]\!]$. In particular, any formula satisfied in* $\mathbf{ILL}^\bullet[A]$ *is provable in cut-free* $\mathbf{ILL}[A]$, *hence in cut-free* $\mathbf{ILL}$.

PROOF. By induction on the complexity of B. See Okada [10]. ⊣

The above lemma as well as the usual soundness proof argument implies;

THEOREM 1 (Soundness, Completeness and Cut-Elimination). *For any formula A, the following statements are equivalent*:

(1) *A is provable in* **ILL**;
(2) *A is satisfied in all intuitionistic phase models*;
(3) *A is satisfied in* $\mathbf{ILL}^{\bullet}[A]$;
(4) *A is provable in cut-free* **ILL**.

The notion of *logical congruence* for classical phase models introduced by Lafont [6] is modified to that for intuitionistic phase models as follows; a *logical congruence* on an (enriched) intuitionistic phase model (M, D, K) is a congruence $\sim$ such that any fact is closed for $\sim$. A logical congruence $\sim$ induces a quotient model $M/\sim$. K is not necessarily closed for $\sim$, but it raises no problem; one can always replace K by its *closure* $\overline{K} = \{x \in M | x \sim y$ for some $y \in K\}$ and obtain $(X \cap K)^C = (X \cap \overline{K})^C$ for any fact X. Let us write $\pi : M \to M/\sim$ to denote the canonical map, and $A^{\bullet}_{\sim}$ to denote the interpretation of A in the quotient model $M/\sim$. Then as in Lafont [6], the following lemma holds.

LEMMA 2. *For any formula A, one has $\pi^{-1}(A^{\bullet}_{\sim}) = A^{\bullet}$. In particular, $M/\sim$ satisfies the same formulas as M.*

We see that there is a logical congruence $\sim$ inducing a finite quotient model $M/\sim$ if and only if the set D of facts is finite; the only-if part is due to the observation that each fact $X \in D$ can be written as $\bigcup\{[x]_{\sim} | x \in X\}$ where $[x]_{\sim}$ denotes the equivalence class of x, and that there are only finitely many equivalence classes by the assumption. To show the reverse, take the following $\equiv$;

$$x \equiv y \text{ iff } (x \in X \Leftrightarrow y \in X \text{ for every } X \in D) \text{ (or equivalently, } \{x\}^C = \{y\}^C).$$

Then $x \equiv y$ implies $xz \equiv yz$ because $\{xz\}^C = (\{x\}^C\{z\}^C)^C = (\{y\}^C\{z\}^C)^C = \{yz\}^C$. Hence it is obvious that $\equiv$ is a logical congruence (indeed the coarsest one) and $M/\equiv$ is finite. Lafont's strategy to show the finite model property is to find for each fragment in question a proper logical congruence that induces a finite monoid structure on a certain syntactical model, whereas ours consists in finding just a syntactical model in which D is finite, so that the above $\equiv$ induces a finite intuitionistic phase model.

§3. Finite model property for intuitionistic multiplicative-additive linear logic. First we shall show the finite model property for **IMALL**, the multiplicative additive fragment of intuitionistic linear logic (see Appendix for the **IMALL** syntax). The argument is analogous to that for **MALL** in Lafont [6]. Let $\sqsubseteq$ be the smallest preordering on **IMALL** sequents satisfying the following;

- If $\dfrac{S_1, \dots, S_n}{S}$ $(n \geq 1)$ is an instance of an inference rule of *cut-free* **IMALL**, then $S_i \sqsubseteq S$ for each $1 \leq i \leq n$.
- $\Gamma \vdash C \sqsubseteq B, \Gamma \vdash C$.

It is clear from the inference rules of *cut-free* **IMALL** that for any **IMALL** formula A, there are only finitely many sequents S such that $S \sqsubseteq \vdash A$. Let **IMALL**$\langle A \rangle$ be the fragment **IMALL**$[A]$ extended with the set $\langle A \rangle = \{S \mid S \not\sqsubseteq \vdash A\}$ of axioms, and let

IMALL$^\bullet\langle A\rangle$ be the syntactical model obtained from **IMALL**$\langle A\rangle$ in the same way as **ILL**$^\bullet[A]$ from **ILL**$[A]$ in the previous section. Since $\{S|S \sqsubseteq\vdash A\}$ is finite and $[\![S]\!]$ is the whole set for $S \not\sqsubseteq \vdash A$, the base $D_0 = \{[\![S]\!] \mid S$ is a sequent in **IMALL**$\langle A\rangle\}$ is obviously finite. Hence D is also finite and $\equiv$ induces the finite model **IMALL**$^\bullet\langle A\rangle/\equiv$.

THEOREM 2 (Finite Model Property for **IMALL**). *For any formula A in* **IMALL**, *the following statements are equivalent*:

(1) *A is provable in* **IMALL**;
(2) *A is satisfied in all finite intuitionistic phase models*;
(3) *A is satisfied in* **IMALL**$^\bullet\langle A\rangle/\equiv$;
(4) *A is satisfied in* **IMALL**$^\bullet\langle A\rangle$;
(5) *A is provable in cut-free* **IMALL**$\langle A\rangle$.

PROOF. (1) implies (2) by the soundness argument, and (2) implies (3) trivially. (3) implies (4) by Lemma 2, and (4) implies (5) by the analogue of Lemma 1 for **IMALL**$\langle A\rangle$. (5) implies (1) because no cut-free proof of A can contain any of the axioms in $\langle A\rangle$. ⊣

The equivalence between (1) and (2) in the above theorem expresses that **IMALL** has the *finite model property* in the usual sense.

§4. Finite model property for intuitionistic affine logic.

Intuitionistic affine logic (**ILLW**) is **ILL** with weakening (cf. Appendix of this paper). Meyer & Ono [8] showed the finite model property for the implicational fragment of **ILLW**, called **BCK**. They also showed that the property holds for the fragment extended with the additive conjunction, but left open the extension with other connectives (see also Ono [12]). Here we shall settle this open problem by showing that the property holds for full **ILLW**, that is the intuitionistic version of Lafont [6]'s result for (*classical*) *affine logic* (**LLW**). Some notions and lemmas needed for the next section are also included below.

Let M be a commutative monoid. For $x, y \in M$, we say that $x \leq y$ if $y = xz$ for some $z \in M$. An *ideal* of M is an $X \subseteq M$ such that $XM \subseteq X$, or equivalently, $XM = X$. The complement of an ideal is called a *coideal*. Alternatively, one may say that X is an ideal if $x \in X$ and $x \leq y$ imply $y \in X$, and that X is a coideal if $x \notin X$ and $x \leq y$ imply $y \notin X$. Given $Y \subseteq M$, YM is always an ideal, which is denoted by $Id_M(Y)$. The complement of $Id_M(Y)$ is denoted by $Co_M(Y)$. We drop the subscript M when they are obvious from the context. Remark that $Id_M(X) = \{y \in M | \exists x \in X\ x \leq y\}$ and $Co_M(X) = \{y \in M | \forall x \in X\ x \not\leq y\}$.

If M is a free commutative monoid generated by $\{p_1, \dots, p_k\}$, then each $x \in M$ can be represented as $p_1^{x_1} \cdots p_k^{x_k}$ for some nonnegative integers $x_1, \dots, x_k$. We define $x \dot{-} y$ to be $p_1^{x_1 \dot{-} y_1} \cdots p_k^{x_k \dot{-} y_k}$ where $x_i \dot{-} y_i$ is $x_i - y_i$ if $x_i \geq y_i$ and is 0 otherwise. For $X \subseteq M$ and $y \in M$, $X \dot{-} y$ denotes the set $\{x \dot{-} y | x \in X\}$.

LEMMA 3. *Let M be a finitely generated free commutative monoid.*

(1) *For any ideal X of M, $xy \in X$ iff $x \in X \dot{-} y$. Hence we have $\{y\} \multimap X = X \dot{-} y$ when X is an ideal.*
(2) *For any $X \subseteq M$, $\{y\} \multimap Id(X) = Id(X \dot{-} y)$.*
(3) *For any $X \subseteq M$, $\{y\} \multimap Co(X) = Co(X \dot{-} y)$.*

PROOF. Here we only prove (3) since (1) and (2) are more or less immediate.

First we show that $x \leq y$ iff $x \dot{-} z \leq y \dot{-} z$, whenever $z \leq y$. The only-if-part is obvious. The if-part holds since $x \leq (x \dot{-} z)z \leq (y \dot{-} z)z = y$ (the last equation holds because $z \leq y$).

Now, $x \in \{y\} \multimap Co(X)$ iff $xy \in Co(X)$ iff $\forall z \in X \;\; z \not\leq xy$ iff $\forall z \in X \;\; z \dot{-} y \not\leq xy \dot{-} y = x$, which means $x \in Co(X \dot{-} y)$. ⊣

We need, as in Lafont [6], the following lemma which was also crucial in the first proof of the decidability of **LLW** by Kopylov [5].

LEMMA 4. *Let M be a finitely generated free commutative monoid, and X be an ideal of it. Then $X = Id(\{x_1, \ldots, x_n\})$ for some $\{x_1, \ldots, x_n\} \subseteq M$.*

PROOF. See Lafont [6]. ⊣

An *intuitionistic affine phase space* is an intuitionistic phase space in which each fact is an ideal. Given a formula A, the system **ILLW**$[A]$ and the syntactical model **ILLW**$^\bullet[A]$ are defined as before, and we can show that **ILLW**$^\bullet[A]$ is an intuitionistic affine phase space.

LEMMA 5. *The base D_0 for the facts of* **ILLW**$^\bullet[A]$ *is finite.*

PROOF. There are only finitely many outer values of the form $[\![B]\!]$ in D_0. By Lemma 4, $[\![B]\!] = Id(\{\Delta_1, \ldots, \Delta_n\})$ for some $\{\Delta_1, \ldots, \Delta_n\}$. For a given Σ, $[\![\Sigma \vdash B]\!] = \{\Sigma\} \multimap [\![B]\!] = Id(\{\Delta_1 \dot{-} \Sigma, \ldots, \Delta_n \dot{-} \Sigma\})$ by Lemma 3(2). But there are only finitely many Π's such that $\Pi \leq \Delta_i$. Hence for a given B, the set $\{[\![\Sigma \vdash B]\!] \mid \Sigma$ is a sequence of formulas in **ILLW**$[A]\}$ is finite. Therefore

$$D_0 = \{[\![S]\!] \mid S \text{ is a sequent in } \mathbf{ILLW}[A]\}$$

is finite. ⊣

Thus we obtain;

THEOREM 3 (Finite Model Property for **ILLW**). *For any formula A, the following statements are equivalent*:

(1) *A is provable in* **ILLW**;
(2) *A is satisfied by all finite intuitionistic phase models*;
(3) *A is satisfied by* **ILLW**$^\bullet[A]/\equiv$;
(4) *A is satisfied by* **ILLW**$^\bullet[A]$;
(5) *A is provable in cut-free* **ILLW**.

§5. Finite model property for intuitionistic and classical contractive linear logic. *Intuitionistic contractive linear logic* (**ILLC**) is **ILL** with contraction (cf. Appendix for the syntax of **ILLC**). Meyer [7] (cf. Meyer & Ono [8]) showed the finite model property for the implicational fragment of **ILLC**, called **BCIW**, and that for its extension with the additive conjunction, but left open the question of the finite model property for richer fragments. In this section we shall settle this question in the form of **ILLC**. The finite model property for classical contractive linear logic (**LLC**), i.e., **LL** with contraction, can be obtained similarly, as we shall briefly explain it at the end of this section. These results will show that **ILLC** and **LLC** are decidable.

Let M be a commutative monoid and L be a subset of M. $X \subseteq L$ is said to be a *coideal relative to L* if $x \notin X$ implies $y \notin X$ for any $x, y \in L$ such that $x \leq y$.

LEMMA 6. *Let M be a finitely generated free commutative monoid, L be an ideal of M, and X be a coideal relative to L. Then $X = Co(\{x_1, \dots, x_n\}) \cap L$ for some $\{x_1, \dots, x_n\} \subseteq M$.*

PROOF. Observe that $X \cup M \backslash L$ is a coideal of M (here $M \backslash L$ means the difference of M and L). Hence, by Lemma 4, $X \cup M \backslash L = Co(\{x_1, \dots, x_n\})$ for some $\{x_1, \dots, x_n\} \subseteq M$. Therefore the lemma follows because $X = (X \cup M \backslash L) \cap L$. ⊣

An *intuitionistic contractive phase space* is an intuitionistic phase space M in which $x \in \{xx\}^C$ holds for any $x \in M$ (or equivalently, $xx \in X$ implies $x \in X$ for any fact X). Let us fix a formula A and consider the system **ILLC**$[A]$. The syntactical model **ILLC**$^\bullet[A]$ are defined as in former sections, and is easily shown to be an intuitionistic contractive phase model.

If every formula occurring in Γ also occurs in Δ and *vice versa*, we say that Γ and Δ are *cognate*. This cognation relation induces a family of equivalence classes on **ILLC**$^\bullet[A]$, which we call the *cognation classes*. Clearly there are only finitely many cognation classes, say, $L_1, \dots, L_k$ on **ILLC**$^\bullet[A]$.

For a given cognation class L_i, let M_i be the smallest submonoid of **ILLC**$^\bullet[A]$ that includes L_i (i.e., the free commutative monoid generated by the formulas occurring in some $\Sigma \in L_i$). Remark that L_i is an ideal of M_i.

Let $[\![\Sigma \vdash B]\!]_{Li}$ be $\{\Delta | \Delta, \Sigma \in L_i$ and $\Delta, \Sigma \vdash B$ is *cut-free* provable in **ILLC**$[A]\}$. The smallest multiset that is cognate to Σ is denoted by $|\Sigma|$ (namely, every formula in Σ occurs exactly once in $|\Sigma|$).

LEMMA 7. *The base D_0 for the facts of* **ILLC**$^\bullet[A]$ *is finite.*

PROOF. First note the following;

(1) Each outer value $[\![\Sigma \vdash B]\!]$ is of the form $\bigcup_{i=1,\dots,k} [\![\Sigma \vdash B]\!]_{L_i}$.
(2) $[\![\vdash B]\!]_{L_i}$ is a coideal relative to L_i.
(3) For a cognation class L_i and $\Sigma \in M_i$, $L_i \dot{-} |\Sigma| = L_i \dot{-} \Sigma$.
(4) If $\Sigma \in M_i$ and $X \subseteq M_i$, $\{\Sigma\} \multimap (Co_{M_i}(X) \cap L_i) = Co_{M_i}(X \dot{-} \Sigma) \cap (L_i \dot{-} |\Sigma|)$.

(1) is obvious. (2) is due to the contraction rule. (3) is shown using Lemma 3(1); $\Gamma \in L_i \dot{-} \Sigma$ iff $\Sigma, \Gamma \in L_i$ iff $|\Sigma|, \Gamma \in L_i$ iff $\Gamma \in L_i \dot{-} |\Sigma|$. (4) follows from (1) and (3) of Lemma 3 and (3) above, together with the fact that $X \multimap (Y \cap Z) = X \multimap Y \cap X \multimap Z$.

To prove the lemma, note that there are only finitely many outer values of the form $[\![B]\!]$ in D_0. Consider a cognation class L_i. Then by Lemma 6 and (2) above, $[\![B]\!]_{L_i} = Co_{M_i}(\{\Delta_1, \dots, \Delta_n\}) \cap L_i$ for some $\{\Delta_1, \dots, \Delta_n\} \subseteq M_i$. Moreover, for a given Σ, $[\![\Sigma \vdash B]\!]_{L_i} = \{\Sigma\} \multimap [\![B]\!]_{L_i}$ is empty if $\Sigma \notin M_i$ and is of the form $Co_{M_i}(\{\Delta_1 \dot{-} \Sigma, \dots, \Delta_n \dot{-} \Sigma\}) \cap (L_i \dot{-} |\Sigma|)$ if $\Sigma \in M_i$ (by (4) above). The latter means that $[\![\Sigma \vdash B]\!]_{L_i}$ is completely determined by $\{\Delta_1 \dot{-} \Sigma, \dots, \Delta_n \dot{-} \Sigma\}$ and $|\Sigma|$. There are only finitely many Π's such that $\Pi \leq \Delta_i$ and only finitely many different $|\Sigma|$'s. These observations show that $[\![\Sigma \vdash B]\!]_{L_i}$ can be only finitely many different values when Σ ranging over the multisets of formulas in **ILLC**$[A]$. Moreover the number of cognation classes is finite, hence there are finitely many outer values of the form $[\![\Sigma \vdash B]\!]$. Therefore $D_0 = \{[\![S]\!] | S$ is a sequent in **ILLC**$[A]\}$ is finite. ⊣

THEOREM 4 (Finite Model Property for **ILLC**). **ILLC** *has the finite model property.*

PROOF. Similar to Theorem 3. ⊣

The above method can be adapted to **LLC** almost straightforwardly. An intuitionistic phase space (M, D) is called a *classical phase space* if D consists of all X's such that $X = X^{\perp\perp}$ for a fixed set $\perp \subseteq M$ (here $X^{\perp}$ means $X \multimap \perp$). We refer to Girard [3, 4] for the details of classical phase semantics. It is easy to see that X^C coincides with $X^{\perp\perp}$ in a classical phase space. Hence, $x \equiv y$ iff $\{x\}^{\perp\perp} = \{y\}^{\perp\perp}$ iff $\{x\}^{\perp} = \{y\}^{\perp}$.

We can construct the classical syntactical model $\mathbf{LLC}^{\bullet}[A]$ for $\mathbf{LLC}[A]$ (cf. Lafont [6]), which is actually an classical contractive phase space, and define the cognation classes $L_1, \dots, L_k$ on it. Now we claim that there are finitely many facts of the form $\{\Sigma\}^{\perp}$ in $\mathbf{LLC}^{\bullet}[A]$. To show this, define $\{\Sigma\}^{\perp}_{L_i}$ as $\{\Delta | \Delta, \Sigma \in L_i$ and $\Delta, \Sigma \in \perp\}$. Then $\perp = \{1\}^{\perp}$ can be written as $\bigcup_{i=1,\dots,k}\{1\}^{\perp}_{L_i}$ and $\{1\}^{\perp}_{L_i}$ is a coideal relative to L_i. If $\{1\}^{\perp}_{L_i}$ is determined by $\{\Delta_1, \dots, \Delta_n\}$, then $\{\Sigma\}^{\perp}_{L_i}$ is determined by $\{\Delta_1 \dot{-} \Sigma, \dots, \Delta_k \dot{-} \Sigma\}$ and $|\Sigma|$ if $\Sigma \in M_i$ (and is empty otherwise). Hence just in the same way as the intuitionistic case, we can show that our claim holds. Therefore $\equiv$ induces a finite quotient model $\mathbf{LLC}^{\bullet}[A]/\equiv$. Hence we obtain;

THEOREM 5 (Finite Model Property for **LLC**). **LLC** *has the finite model property.*

COROLLARY 1. *Both* **LLC** *and* **ILLC** *are decidable.*

§6. Related systems of substructural logics — FL-family and GL⁻-family. So far we have shown the finite model property (FMP, for short) for **IMALL**, **ILLW**, **ILLC** and **LLC**, meanwhile Lafont [6] proved that for **MALL** and **LLW**. On the other hand, **IMELL** (hence **ILL**) fails to have FMP by just the same counterexample which Lafont came up with for **MELL**: the formula

$$(!a \otimes !(a \otimes b) \otimes !(a \otimes b \multimap \mathbf{1})) \multimap b$$

is satisfied in all finite intuitionistic phase models, but not provable.

To consider related systems of substructural logics, it is convenient to follow Ono [11]'s systematic notation (cf. also Appendix of this paper). In his notation, **FL** (*Full Lambek calculus*) means **IMALL** without exchange, and the subscripts **e, w, c** express exchange, weakening and contraction, respectively; for example $\mathbf{FL_{ew}}$ means **IMALL** with weakening. Corresponding to **FL**-family, he also introduced a family of classical substructural logics, called $\mathbf{GL}^-$-family. In his notation, $\mathbf{GL}^-_{\mathbf{X}}$ denotes, roughly, the classical counterpart of $\mathbf{FL_X}$. According to Ono [12], FMP for any of **FL**-systems and $\mathbf{GL}^-$-systems seems to be open. (As for the syntax of $\mathbf{GL}^-$-family, we refer to Ono [11].)

Some of **FL**-systems have already been shown to have FMP in former sections; $\mathbf{FL_e}$ is **IMALL**, and $\mathbf{FL_{ew}}$ and $\mathbf{FL_{ec}}$ are conservative subsystems of **ILLW** and **ILLC**. On the other hand, $\mathbf{FL_{ewc}}$ is just the intuitionistic logic, which is well-known to have FMP (cf. Troelstra & van Dalen [15]). $\mathbf{FL_{wc}}$ is equivalent to $\mathbf{FL_{ewc}}$ because in $\mathbf{FL_{wc}}$ the exchange rule is admissible.

To show FMP for **FL** and $\mathbf{FL_W}$, we consider *noncommutative intuitionistic phase spaces*, in which $Y \circ\!- X$ is defined as $\{y | \forall x \in X yx \in Y\}$ and, together with (P1) and (P2), the following (P3) holds;

(P3) If $X \subseteq M$ and $Y \in D$, then $Y \circ\!- X \in D$.

A *noncommutative intuitionistic affine phase space* is a noncommutative intuitionistic phase space in which $XM \subseteq X$ and $MX \subseteq X$ hold for any fact X.

We must be careful about the outer values when defining the syntactical model; we have to consider the outer values of the form $[\![\Gamma\_\Delta \vdash C]\!] = \{\Pi \mid \Gamma, \Pi, \Delta \vdash C$ is *cut-free* provable$\}$ instead of our original form $[\![\Gamma \vdash C]\!]$. The rest of the proof is almost the same as that for **IMALL**. Thus we have;

COROLLARY 2. *Each of the* **FL**-*systems except* $\mathbf{FL_C}$ *has the finite model property.*

REMARK. There are some minor differences between our formulation of **FL**-systems and the original one of Ono [11].

1. Ono [11] includes constant $\perp$ within **FL** syntax and allows a sequent whose right hand side is empty and accordingly the right weakening rule, while we do not include $\perp$ within **FL** and do not allow such a sequent. We can easily modify our definition of phase spaces to include $\perp$ and the whole argument of this section works with this modification.

2. Ono [11] considers the contraction rule of the form;

$$\frac{\Gamma_1, A, A, \Gamma_2 \vdash C}{\Gamma_1, A, \Gamma_2 \vdash C},$$

whereas we consider the contraction rule of the following generalized form;

$$\frac{\Gamma_1, \Sigma, \Sigma, \Gamma_2 \vdash C}{\Gamma_1, \Sigma, \Gamma_2 \vdash C}.$$

These are of course equivalent in the presence of the exchange rule, but also equivalent without the exchange rule (in $\mathbf{FL_C}$ and $\mathbf{FL_{WC}}$) as far as provability is concerned; the latter generalized form of the contraction rule is derivable from the former using $\otimes l$, $\otimes r$ and *cut*. (However, these are different with respect to provability when one deals with the implicational fragments of $\mathbf{FL_C}$ and $\mathbf{FL_{WC}}$, see Ono [12].) The latter form is, in our opinion, more appropriate since it makes the phase semantics more natural and allows us to have the cut-elimination property for $\mathbf{FL_{WC}}$ and $\mathbf{FL_C}$.

The decidability and FMP for $\mathbf{FL_C}$ remain open, whichever form of the contraction rule mentioned above is adopted, (even for its implicational fragment).

Most of $\mathbf{GL}^-$-systems have been shown to have FMP in Lafont [6] and our present paper, namely, $\mathbf{GL}^-$, $\mathbf{GL}^-_{\mathbf{e}}$ and $\mathbf{GL}^-_{\mathbf{ew}}$ by Lafont and $\mathbf{GL}^-_{\mathbf{ec}}$ in this paper. $\mathbf{GL}^-_{\mathbf{wc}}$=$\mathbf{GL}^-_{\mathbf{ewc}}$ is just the classical logic, so has FMP trivially. FMP for $\mathbf{GL}^-_{\mathbf{w}}$ is easily shown in Lafont's framework. The decidability and FMP for $\mathbf{GL}^-_{\mathbf{c}}$ are still unknown.

COROLLARY 3. *Each of the* $\mathbf{GL}^-$-*systems except* $\mathbf{GL}^-_{\mathbf{c}}$ *has the finite model property.*

Finally we summarize FMP results shown so far by Table 1 and Table 2 below.

Table 1: Finite Model Property Results on $\mathbf{GL}^-$ and FL-Families.

with	$\mathbf{GL}^-$-family		FL-family	
$\emptyset$	Yes	(= noncommutative **MALL**), Lafont [6]	Yes	this paper
e	Yes	(= **MALL**), Lafont [6]	Yes	(= **IMALL**), this paper
w	Yes	this paper	Yes	this paper
c	??	open (open also for the decidability)	??	open (open also for the decidability)
ew	Yes	($\subseteq$ **LLW**), Lafont [6]	Yes	($\subseteq$ **ILLW**), this paper
ec	Yes	($\subseteq$ **LLC**), this paper	Yes	($\subseteq$ **ILLC**), this paper
wc=ewc	Yes	(= classical logic), trivial	Yes	(= intuitionistic logic), folklore

Table 2: Finite Model Property Results on Classical and Intuitionistic Linear Logics.

classical linear logics			intuitionistic linear logics		
MALL	Yes	Lafont [6]	**IMALL**	Yes	this paper
MELL	No	Lafont [6]	**IMELL**	No	essentially by Lafont [6]
LL	No	cf. Lafont [6]	**ILL**	No	cf. Lafont [6]
LLW	Yes	Lafont [6]	**ILLW**	Yes	this paper
LLC	Yes	this paper	**ILLC**	Yes	this paper

In Table 1, FMP result for $\mathbf{GL}^-_{\mathbf{x}}$ is represented on column labelled by $\mathbf{GL}^-$-family and row labelled by **x**, where **x** stands for any combination of **e**, **w** and **c**. Likewise FMP result for $\mathbf{FL}_{\mathbf{x}}$ is represented on column labelled by **FL**-family and row labelled by **x**. For example, "Yes" on column $\mathbf{GL}^-$-family and on row **ew** means "FMP holds for $\mathbf{GL}^-_{\mathbf{ew}}$". The attached comment "($\subseteq$ **LLW**), Lafont [6]" means that the result follows from the fact that $\mathbf{GL}^-_{\mathbf{ew}}$ is a conservative subsystem of **LLW**, which was shown to have FMP by Lafont [6]. We identify $\mathbf{GL}^-_{\mathbf{wc}}$ with $\mathbf{GL}^-_{\mathbf{ewc}}$, and $\mathbf{FL}_{\mathbf{wc}}$ with $\mathbf{FL}_{\mathbf{ewc}}$ according to the former remark.

Appendix: Syntax of intuitionistic linear logic and related substructural logics. Roman capitals $A, B, \ldots$ stand for formulas. The constants and the connectives which we deal with in this paper are classified into three groups;

- Multiplicatives: $\mathbf{1}$, $A \otimes B$, $A \multimap B$, $B \circ\!\!- A$;
- Additives: $\top$, $\mathbf{0}$, $A \,\&\, B$, $A \oplus B$;
- Modality (Exponential): $!A$.

Greek capitals $\Gamma_1, \Gamma_2, \Delta, \ldots$ stand for finite sequences of formulas. We list various inference rules below, and define various systems related to intuitionistic linear logic on Table 3.

Identity and Cut:

$$\frac{}{A \vdash A}\ \mathit{Identity} \qquad \frac{\Gamma \vdash A \quad \Delta_1, A, \Delta_2 \vdash C}{\Delta_1, \Gamma, \Delta_2 \vdash C}\ \mathit{Cut}.$$

Multiplicatives:

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash C}{\Gamma_1, A \otimes B, \Gamma_2 \vdash C}\ \otimes l \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes r \qquad \frac{\Gamma_1, \Gamma_2 \vdash C}{\Gamma_1, \mathbf{1}, \Gamma_2 \vdash C}\ \mathbf{1}l$$

$$\frac{}{\vdash \mathbf{1}}\ \mathbf{1}r \qquad \frac{\Gamma \vdash A \quad \Delta_1, B, \Delta_2 \vdash C}{\Delta_1, \Gamma, A \multimap B, \Delta_2 \vdash C}\ \multimap l \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap B}\ \multimap r$$

$$\frac{\Gamma \vdash A \quad \Delta_1, B, \Delta_2 \vdash C}{\Delta_1, B \circ\!\!- A, \Gamma, \Delta_2 \vdash C}\ \circ\!\!- l \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash B \circ\!\!- A}\ \circ\!\!- r.$$

Additives:

$$\frac{\Gamma_1, A, \Gamma_2 \vdash C \quad \Gamma_1, B, \Gamma_2 \vdash C}{\Gamma_1, A \oplus B, \Gamma_2 \vdash C}\ \oplus l \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\ \oplus r_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\ \oplus r_2$$

$$\frac{}{\Gamma_1, \mathbf{0}, \Gamma_2 \vdash C}\ \mathbf{0}l \qquad \frac{\Gamma_1, A, \Gamma_2 \vdash C}{\Gamma_1, A \,\&\, B, \Gamma_2 \vdash C}\ \&l_1 \qquad \frac{\Gamma_1, B, \Gamma_2 \vdash C}{\Gamma_1, A \,\&\, B, \Gamma_2 \vdash C}\ \&l_2$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\ \&r \qquad \frac{}{\Gamma \vdash \top}\ \top r.$$

Modality (Exponential):

$$\frac{\Gamma_1, A, \Gamma_2 \vdash C}{\Gamma_1, !A, \Gamma_2 \vdash C}\ !D \qquad \frac{\Gamma_1, !A, !A, \Gamma_2 \vdash C}{\Gamma_1, !A, \Gamma_2 \vdash C}\ !C \qquad \frac{\Gamma_1, \Gamma_2 \vdash C}{\Gamma_1, !A, \Gamma_2 \vdash C}\ !W \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ !r.$$

Here $!\Gamma$ stands for a sequence of the form $!A_1, \dots, !A_n$ $(n \geq 0)$.

By *exchange*, denoted by **e**, we mean the convention that the left-hand side of a sequent be considered as a *multiset*, hence that we identify a sequence of formulas with any of its permutations.

By *weakening*, denoted by **w**, we mean the following rule;

$$\frac{\Gamma_1, \Gamma_2 \vdash C}{\Gamma_1, A, \Gamma_2 \vdash C}\ \mathbf{w}.$$

By *contraction*, denoted by **c**, we mean the following rule;

$$\frac{\Gamma_1, \Sigma, \Sigma, \Gamma_2 \vdash C}{\Gamma_1, \Sigma, \Gamma_2 \vdash C}\ \mathbf{c}.$$

Every system considered in this paper has *Identity* and *Cut* in common. Then a particular system is obtained as a combination of some of the above rules (with corresponding constants and connectives in its language), as follows;

Table 3: Definition of logical systems used in this paper.

Intuitionistic Linear Logic	**ILL**	Multiplicatives + Additives + Modality + **e**
Multiplicative Additive Fragment of **ILL**	**IMALL**	Multiplicatives + Additives + **e**
Multiplicative Exponential Fragment of **ILL**	**IMELL**	Multiplicatives + Modality + **e**
Intuitionistic Affine Logic	**ILLW**	**ILL** + **w**
Intuitionistic Contractive Linear Logic	**ILLC**	**ILL** + **c**
Full Lambek Calculus	**FL**	Multiplicatives + Additives
FL with structural rules **x**	$\mathbf{FL_x}$	**FL** + **x**

where the first column represents the names of the systems and the second represents their abbreviations. **x** ranges over combinations of **e**, **w**, **c**; for example, $\mathbf{FL_{ec}}$ means **FL** + **e** + **c** (equivalently, **IMALL** + **c**, or **ILLC** without Modality). In systems with **e**, such as **ILL**, **IMALL**, etc., $A \multimap B$ and $B \circ\!\!- A$ are equivalent, hence we usually dispense with formulas of the form $B \circ\!\!- A$.

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