Zeitschr. f. math. Logik und Grundlagen d. Math.
Bd. 21, S. 401–404 (1975)

SENTENCES WITH FINITE MODELS

by C. J. Ash in Monash University Clayton, Victoria (Australia)

It is well known, see for example [2], p. 156, that, in a language with only unary predicate letters, any sentence which has a model has a finite model. We delineate those languages in which all sentences have this property and also those in which all universal sentences have this property. Clearly whenever every member of a class of sentences has a finite model or no model at all there is a decision procedure for the satisfiability of each of the sentences.

§ 0. We deal with sentences in a first order predicate calculus whose non-logical symbols are those of a language L. The equality symbol is treated as a non-logical constant and its interpretation in all models is the identity relation. For further amplification see [2]. A sentence of L is said to have *property* (F) if either it has no model or it has a finite model.

§ 1. The first two theorems yield all the languages in which every sentence has property (F).

Theorem 1. *In a language with only unary predicate letters, constant letters and equality, every sentence has property* (F).

Proof. We can clearly treat constant letters as additional predicate letters. Now suppose that φ is a sentence involving only unary predicate letters $P_0, P_1, \ldots, P_{n-1}$ and equality, and that φ has a denumerable model $\mathfrak{A}$.

For each function $\pi\colon n \to 2$, $i < n$, let $Q_i^\pi(v)$ be the formula $P_i(v)$ if $\pi(i) = 0$, $\neg P_i(v)$ if $\pi(i) = 1$, and let $P^\pi(v)$ be $Q_1^\pi(v) \mathbin{\&} Q_2^\pi(v) \mathbin{\&} \cdots \mathbin{\&} Q_{n-1}^\pi(v)$. In general let $(\exists^{>k} v)\, \psi(v)$ be the formula $\forall v_1, \ldots, \forall v_k\, \exists v(\psi(v) \mathbin{\&} v \neq v_1 \mathbin{\&} \cdots \mathbin{\&} v \neq v_k)$ and $(\exists^k v)\, \psi(v)$ be $(\exists^{>k-1} v)\, \psi(v) \mathbin{\&} \neg(\exists^{>k} v)\, \psi(v)$ for $k \geqq 1$ and $\neg(\exists v)\, \psi(v)$ for $k = 0$.

Now for each $\pi\colon n \to 2$, if $\{a \in |\mathfrak{A}| : \mathfrak{A} \models P^\pi[a]\}$ is finite with k elements, let T_π be $\{(\exists^k v)\, P^\pi(v)\}$, otherwise let T_π be $\{(\exists^{>k} v)\, P^\pi(v) : k = 0, 1, \ldots\}$. Let $T = \bigcup_{\pi: n \to 2} T_\pi$. Now since $\mathfrak{A}$ is infinite, so is some T_π; thus T has no finite models. Clearly any denumerable model for T is isomorphic to $\mathfrak{A}$, so T is $\aleph_0$-categorical. Thus by Vaught's test, [2] or [4], T is complete. $\mathfrak{A} \models \varphi$ and so $T \vdash \varphi$. Thus $T' \vdash \varphi$ for some finite subset T' of T. But any such T' clearly has a finite model which will be a model for φ.

Theorem 2. *In a language with only unary predicate letters, unary function letters and constant letters, but without equality, every sentence has property* (F).

Proof. We can treat constant letters as additional function letters. Now suppose φ is a sentence involving only unary predicate letters and unary function letters $f_0, \ldots, f_{q-1}$ which has a model $\langle A, R_0, \ldots, R_{p-1}, g_0, \ldots, g_{q-1}\rangle$. Let n be the greatest length of terms involved in φ. We define equivalence relations $\sim_m$ on A, for $0 \leqq m \leqq n$, inductively: $a \sim_0 b$ if, for each $k < q$, $R_k(a)$ if and only if $R_k(b)$. $a \sim_{m+1} b$ if $a \sim_m b$ and, for each $k < p$, $g_k(a) \sim_m g_k(b)$. Thus each $\sim_m$ has finitely many equivalence classes. Choose one element from each $\sim_n$ equivalence class. For $a \in A$ let $\bar{a}$ denote the element chosen from $a/\sim_n$. Let $\bar{A} = \{\bar{a}: a \in A\}$. Define $\bar{g}_k$ on $\bar{A}$ for $k < q$ by $\bar{g}_k(\bar{a}) = g_k(a)$. Let $\bar{\mathfrak{A}}$ be $\langle \bar{A}, R_0 \upharpoonright \bar{A}, \ldots, R_{p-1} \upharpoonright \bar{A}, \bar{g}_0, \ldots, \bar{g}_{q-1}\rangle$. Observe that if g_k is constant, so is $\bar{g}_k$.

Now suppose $t(v)$ is a term of length $m \leqq n$, and $h, \bar{h}$ are its interpretations in $\mathfrak{A}$, $\bar{\mathfrak{A}}$. Then, by induction on m, for each $a \in A$, $h(a) \sim_{n-m} \bar{h}(\bar{a})$. If $m = 0$ this follows since $a \sim_n \bar{a}$. Otherwise $t(v)$ is $f_k(t_1(v))$, h is $g_k \circ h_1$ and $\bar{h}$ is $\bar{g}_k \circ \bar{h}_1$ where, by induction hypothesis, $h_1(a) \sim_{n-m+1} \bar{h}_1(\bar{a})$. Then $h(a) = g_k(h_1(a)) \sim_{n-m} g_k(\bar{h}_1(\bar{a}))$, but $g_k(\bar{h}_1(\bar{a})) \sim_n \sim_n \overline{g_k(\bar{h}_1(\bar{a}))} = \bar{g}_k(\bar{h}_1(\bar{a})) = \bar{h}(\bar{a})$.

It then follows by induction that for each formula $\psi(v_1, \ldots, v_r)$ involving only terms of length $\leqq n$ and for all $a_1, \ldots, a_r \in A$, $\mathfrak{A} \models \psi[a_1, \ldots, a_r]$ if and only if $\bar{\mathfrak{A}} \models \psi[\bar{a}_1, \ldots, \bar{a}_r]$. In particular, $\bar{\mathfrak{A}} \models \varphi$, so φ has a finite model.

§ 2. To see that theorems 1 and 2 give all languages in which every sentence has property (F), we indicate counter-examples by considering the languages $L_1, \ldots, L_4$ with the following symbols:

L_1: One binary predicate letter.

L_2: One unary function letter and equality.

L_3: One binary function letter and one unary predicate letter.

L_4: One binary function letter and equality.

Examples can be given of sentences in each of these languages which do not have property (F). In the case of L_3 and L_4 these may be taken to be universal sentences. The sentences are those which ensure that in any model with interpretations R_1, R_2, g_1, g_2, for the unary, binary, predicate, function letters respectively:

L_1: R_2 is an irreflexive partial ordering without a maximal element.

L_2: g_1 is one-one but not onto.

L_3: The relation $R_1(g_2(x, y))$ is an irreflexive partial ordering, $<$, and $x < g_2(x, x)$.

L_4: g_2 is a one-one pairing function.

See Appendix I.

§ 3. That universal sentences cannot be given as counter-examples for L_1 or L_2 follows from the following remark and theorem.

Remark 1. In a language with no function letters every universal sentence has property (F).

Theorem 3. *In a language with only unary predicate letters, one unary function letter and equality, every universal sentence has property* (F).

Proof. Let φ be a universal sentence in such a language, L, involving only unary predicate letters $P_0, \ldots, P_{n-1}$, unary function letter f and equality. Assume that φ has a model. Consider in this model the submodel, $\mathfrak{B}$, generated by a single element. If $\mathfrak{B}$ is finite nothing remains to be proved.

Assuming $\mathfrak{B} = \langle B, R_0, \ldots, R_{n-1}, g\rangle$ is infinite, a linear ordering can be introduced on $|\mathfrak{B}|$ to give a structure $\mathfrak{A}_0 = \langle B, \leqq, R_0, \ldots, R_{n-1}\rangle$ in which $\langle B, \leqq\rangle$ is a linear ordering of order type ω whose successor function is g. Conversely, any discrete linear ordering without last element with n unary predicates, $\mathfrak{A}$, gives rise to an interpretation $\mathfrak{A}'$ of L by interpreting f as the successor function. Clearly there is a formula ψ in the appropriate language for which $\mathfrak{A} \models \psi$ if and only if $\mathfrak{A}' \models \varphi$.

We may now utilise known results about the class, M, of linear orderings with n unary predicates. Let ψ have m quantifiers when expressed in prenex normal form. We assume the definition of the equivalence relation $\equiv_m$ on M which partitions M into finitely many equivalence classes and has the property that if $\mathfrak{A}, \mathfrak{B} \in M$, $\mathfrak{A} \equiv_m \mathfrak{B}$ and $\mathfrak{A} \models \psi$ then $\mathfrak{B} \models \psi$ (see e.g. [1]). By $\mathfrak{A} + \mathfrak{B}$ we mean the result of canonically placing a copy of $\mathfrak{B}$ after $\mathfrak{A}$. If $\boldsymbol{K} = \langle K, \leqq \rangle$ is a linear ordering, $\mathfrak{A}_k \in M$ then by $\sum_{k \in K} \mathfrak{A}_k$ we mean the result of combining copies of the $\mathfrak{A}_k$ according to the ordering of $\boldsymbol{K}$. If all the $\mathfrak{A}_k$ are $\mathfrak{A}$ and $\boldsymbol{K}$ has order type α, we write this as $\mathfrak{A} \cdot \alpha$. As in [1] we then have that if $\mathfrak{A}, \mathfrak{A}_k, \mathfrak{B}, \mathfrak{B}_k \in M$, $\mathfrak{A} \equiv_m \mathfrak{B}$, $\mathfrak{A}_k \equiv_m \mathfrak{B}_k$ then $\mathfrak{A} + \sum_{k \in K} \mathfrak{A}_k \equiv_m \mathfrak{B} + \sum_{k \in K} \mathfrak{B}_k$.

Now partition pairs of elements a, b of $|\mathfrak{A}_0|$ with $a < b$ according to the $\equiv_m$ class of $\mathfrak{A}_0(a, b) = \mathfrak{A}_0 \restriction \{c \in |\mathfrak{A}_0| : a < c \leqq b\}$. By RAMSEY's theorem [3] there is an infinite sequence $a_0 < a_1 < \ldots$ of elements of $|\mathfrak{A}_0|$ such that all $\mathfrak{A}_0(a_k, a_{k+1})$ are m-equivalent. Thus $\mathfrak{A}_0 \cong \mathfrak{C}_0 + \sum_{k \in N} \mathfrak{A}_0(a_k, a_{k+1}) \equiv_m \mathfrak{C}_0 + \mathfrak{C} \cdot \omega$ where $\mathfrak{C} = \mathfrak{A}_0(a_0, a_1)$ and so is finite. Consequently $(\mathfrak{C}_0 + \mathfrak{C} \cdot \omega) \models \psi$ and so $(\mathfrak{C}_0 + \mathfrak{C} \cdot \omega)' \models \varphi$. Since φ is universal, clearly $(\mathfrak{C} \cdot (\omega^* + \omega))' \models \varphi$.

It remains to give axioms for the theory of $(\mathfrak{C} \cdot (\omega^* + \omega))'$. Let $\mathfrak{C}$ have r elements $c_0, \ldots, c_{r-1}$, let $P^n(v)$ be the formulae of theorem 1 and let $\theta_i(v)$ be the unique $P^n(v)$ for which $\mathfrak{C} \models P^n[c_i]$ for $i < r$. Let T be the theory in L whose axioms are:

$$\exists v(\theta_0(v) \ \& \ \theta_1(f(v)) \ \& \cdots \& \ \theta_{r-1}(f^{(r-1)}(v))),$$

$$\forall v(\mathop{\&}_{i=0}^{n-1} (P_i(v) \leftrightarrow P_i(f^{(r)}(v))),$$

$$\forall v(f^{(s)}(v) \neq v) \quad (s = 1, 2, \ldots).$$

Then T has no finite model, $(\mathfrak{C} \cdot (\omega^* + \omega))' \models T$ and one can easily see that any two models of T of power $\varkappa > \aleph_0$ are isomorphic. Thus T is $\varkappa$-categorical and so complete by VAUGHT's test, showing that $T \vdash \varphi$. Consequently $T' \vdash \varphi$ for some finite subset T' of T. But a finite model for any such T' can easily be constructed; this will then be a model for φ.

§ 4. Again we indicate examples of universal sentences without property (F) in those languages not included in those of Remark 1 and theorems 1, 2, 3. We already have counter-examples for L_3 and L_4 of 2. In addition we need to consider the languages with the following symbols:

L_5: One binary predicate letter, one unary function letter.

L_6: One unary function letter, one constant letter and equality.

L_7: Two unary function letters and equality.

The sentences are those which ensure that in any model if R, g, h, a are the interpretations of the binary predicate letters, two unary function letters and constant letter respectively:

L_5: R is an irreflexive partial ordering and $R(x, g(x))$.

L_6: g is one-one and a is not in the range of g.

L_7: g is one-one and the ranges of g, h are disjoint.

See Appendix II.

Thus these results are the best possible along these lines.

Appendices

In determining for which languages certain sentences have property (F) we need only consider languages with finitely many symbols. The *type* of such a language is the number of predicate letters and function letters of each number of arguments and of constant letters and equality symbols. We say L is included in L' if L' has a sublanguage of the same type as L.

Appendix I. It is clear that counter-examples for languages with >2-ary predicate letters or function letters can be given similarly. Thus if a language L is not included in those of theorems 1, 2 then either it has a 2-ary predicate letter as in L_1 or it has only unary predicate letters. In this case it has a function letter, and if this is $\geqq 2$-ary then we have counter-examples as in L_3 or L_4. Otherwise L contains only unary predicate letters, a unary function letter, constants and therefore equality and so includes L_2.

Appendix II. If a language L is not included in those of Remark 1 or theorems 1, 2 or 3 then it has a function letter. Either it has a $\geqq 2$-ary function letter as in L_3 or L_4, or it has only unary function letters. In this case either L has a $\geqq 2$-ary predicate letter as in L_5 or it has only unary predicate letters, and so must have equality. L must then contain either two unary function letters as in L_7 or one unary function letter and a constant letter as in L_6.

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(Eingegangen am 8. Februar 1974)