Journal of Symbolic Computation 47 (2012) 16–31

Contents lists available at SciVerse ScienceDirect

Journal of Symbolic Computation

journal homepage: www.elsevier.com/locate/jsc

Deciding polynomial-transcendental problems

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ARTICLE INFO

Article history:
Received 31 January 2009
Accepted 31 May 2011
Available online 16 September 2011

Keywords:
Decision procedure
Exponential polynomials

ABSTRACT

This paper presents a decision procedure for a certain class of sentences of first order logic involving integral polynomials and a certain specific analytic transcendental function trans(x) in which the variables range over the real numbers. The list of transcendental functions to which our decision method directly applies includes exp(x), the exponential function with respect to base e, ln(x), the natural logarithm of x, and arctan(x), the inverse tangent function. The inputs to the decision procedure are prenex sentences in which only the outermost quantified variable can occur in the transcendental function. In the case trans(x) $=$ exp(x), the decision procedure has been implemented in the computer logic system REDLOG. It is shown how to transform a sentence involving a transcendental function from a much wider collection of functions (such as hyperbolic and Gaussian functions, and trigonometric functions on a certain bounded interval) into a sentence to which our decision method directly applies. Closely related work is reported by Anai and Weispfenning (2000), Collins (1998), Maignan (1998), Richardson (1991), Strzebonski (in press) and Weispfenning (2000).

1. Introduction

Tarski in 1948 (Tarski, 1951, 1998) published a proof that the first order theory of the real numbers is decidable: indeed he exhibited a decision method for this theory (which he had discovered

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0747-7171/$ – see front matter
doi:10.1016/j.jsc.2011.08.004

in 1930). In his monograph, Tarski briefly considered an extended system in which one introduces a unary function symbol for exponentiation with respect to a fixed base. He remarked that the decision problem for such a system, which was still an open problem in 1948, is of great theoretical and practical interest. Over the following years many efforts were made to resolve this decision problem. The problem was conditionally solved in the positive sense by Macintyre and Wilkie (1996). Their solution relies upon the plausible yet unproven Schanuel's conjecture in transcendental number theory (Lang, 1966; Ax, 1971; Baker, 1975). Their solution also is an indirect one, using powerful model-theoretic machinery; so it is not well suited for implementation.

A number of papers have addressed, without recourse to Schanuel's conjecture and with practical implementability in mind, decision problems for fragments of the full first order theory of the reals with a specific analytic transcendental function. For example, Anai and Weispfenning (2000) and Weispfenning (2000) show how to decide a certain kind of linear-transcendental problem using a relatively elementary and explicit approach well suited for implementation. Examples of related work are provided by a line of research initiated by Richardson (1991). This line of work addresses the computation of one and two dimensional polynomial-exponential systems using a variant of Sturm theory (Maignan, 1998, 2001). Further examples of related work, in which algorithms are proposed (though not implemented) and relevant complexity bounds derived, can be found; for instance, see Pericleous and Vorobjov (2001).

The present paper is in the spirit of work of Anai and Weispfenning (2000) and Weispfenning (2000) in that its method addresses a certain fragment of the first order theory of the reals with a certain specific analytic transcendental function $\mathrm{trans}(x)$, relies on no unproven conjecture and, in the case $\mathrm{trans}(x) = \exp(x)$, the exponential function with respect to base e, has been implemented in the computer logic system REDLOG (Dolzmann and Sturm, 2004). Our decision method is also directly applicable in the cases $\mathrm{trans}(x) = \ln(x)$, the natural logarithm of x, and $\mathrm{trans}(x) = \arctan(x)$, the inverse tangent function. In particular an unconditional and implementable decision method is presented for prenex sentences (that is, prenex formulae containing no free variables) having bound variables $x_1, x_2, \ldots, x_n$, in which only x_1 occurs as argument of the transcendental function trans(x). This non-trivial fragment of the extended system of real algebra is thus decided both in theory and, so far in the case of the exponential function, in practice. It is further shown how to transform a sentence involving a transcendental function from a much wider collection of functions (such as hyperbolic and Gaussian functions, and trigonometric functions on a certain bounded interval) into a sentence to which our decision method directly applies. Our method and the REDLOG module embodying it thus provide an extension of computer algebra tools for real algebra into real analysis. Further extension to problems with several exponential variables, for example, appears difficult while Schanuel's conjecture remains unresolved.

The present paper is an extended version of Achatz et al. (2008). A different kind of generalisation of Achatz et al. (2008) is found in Strzebonski (in press).

Our decision method is based upon an algorithm for isolating the real zeros of a certain kind of generalised integral polynomial in $\mathrm{trans}(x)$ $f(x, \mathrm{trans}(x))$ (where $f(x, y)$ is a given polynomial in y whose coefficients are elements of the ring of fractions of $\mathbb{Z}[x]$ with respect to powers of a specific integral polynomial $d(x)$). Our recursive real root isolation algorithm uses pseudodifferentiation and Rolle's theorem in the spirit of Collins and Loos (1976), and also relies upon a classical result of Lindemann (Shidlovskii, 1989) (see Section 2). The root isolation method of Maignan (1998), in contrast, is applicable to exponential polynomials only and is based upon the construction of local Sturm sequences. We have reason to believe that, for the exponential case, our root isolation method is more efficient than that of Maignan (1998) (see Section 3).

The remainder of the paper is organised as follows. Section 2 presents the formal framework and recalls the most essential background material for reading the paper. In Section 3 we outline an algorithm which decides *univariate* polynomial-transcendental problems. This involves a careful study of real zero isolation for generalised integral transcendental polynomials. Section 4 presents a decision procedure for more general polynomial-transcendental problems which uses the method of Section 3 as a subalgorithm. Section 5 reports on the REDLOG implementation of the decision procedure of Section 4 in the case $\mathrm{trans}(x) = \exp(x)$, and discusses some examples. Section 6 discusses extensions and refinements of the work reported in the previous sections.

2. Formal framework and background material

Recall that a complex number is *transcendental* if it is not algebraic. F. Lindemann proved the following important result (Shidlovskii, 1989) concerning the transcendence of the values of the complex exponential function:

Theorem 1 (*Lindemann*)**.** *If z is a nonzero algebraic number, then e^z is transcendental.*

The following concept was introduced by Anai and Weispfenning (2000):

Definition 2. A real or complex valued function f defined in some open domain of $\mathbb{R}$ or $\mathbb{C}$, respectively, is called *strongly transcendental* (with exceptional point ξ) if for all numbers x in the domain of f excluding ξ not both x and $f(x)$ are algebraic.

Using this concept we can restate Lindemann's theorem: the complex function e^z is strongly transcendental with exceptional point 0.

As immediate consequences of Lindemann's theorem we see that the real exponential function $\exp(x)$ and the real natural logarithm function $\ln(x)$ $(x > 0)$ are strongly transcendental with exceptional points 0 and 1, respectively. Moreover, as pointed out by Anai and Weispfenning (2000), it can easily be deduced from Lindemann that the real trigonometric functions $\sin(x)$ and $\cos(x)$ are strongly transcendental with exceptional point 0; and the inverse functions $\arcsin(x)$ and $\arccos(x)$ are strongly transcendental with exceptional points 0 and 1, respectively. Also, $\tan(x)$ and $\arctan(x)$ are strongly transcendental with exceptional point 0.

For the remainder of this section, and indeed the rest of this paper, we shall let $\mathrm{trans}(x)$ denote a specific real valued function defined and analytic on some nonempty open interval I of the real line, which is strongly transcendental with exceptional point $\xi \in I$. For simplicity we shall assume that both ξ and $\mathrm{trans}(\xi)$ are integers.

We shall be concerned with a certain extension of the first order theory of the real numbers. This extension, which we shall denote by $\mathcal{T}_{\mathrm{trans}}$, is a certain class of true sentences for the structure $\mathbb{R}_{\mathrm{trans}} = \langle \mathbb{R}, +, -, \cdot, 0, 1, =, <, \mathrm{trans}\rangle$. Sentences of $\mathcal{T}_{\mathrm{trans}}$ are expressed in a language $\mathcal{L}_{\mathrm{trans}}$ which is an extension of the well known language of Tarski algebra (Tarski, 1951, 1998; Collins, 1998). In $\mathcal{L}_{\mathrm{trans}}$ the variables are $x_1, x_2, \ldots$, the constant symbols are $0, 1$, the binary function symbols are $+$, $-$ and $\cdot$, the unary function symbol is trans, to be applied *only* to the variable x_1, and the binary relation symbols are $=$, $<$, etc. *Terms* of this language are integral polynomials in the variables $x_1, x_2, \ldots$ and another variable y, where every occurrence of y is replaced by $\mathrm{trans}(x_1)$. (Terms involving x_1 only are particularly important. We call such univariate terms *integral polynomials in* $\mathrm{trans}(x_1)$.) By an *atomic formula* we mean an equation, inequation or inequality of the form $\tau = 0$, $\tau \neq 0$, $\tau < 0$, etc. where τ is a term. *Formulae* are constructed from atomic formulae using Boolean connectives and quantifiers. A *sentence* is a formula without free variables, usually expressed in prenex form:

$$(Q_1x_1)(Q_2x_2)\ldots(Q_nx_n)\psi(x_1, x_2, \ldots, x_n)$$

where ψ is a quantifier-free formula, and the (Q_ix_i) are quantifiers. The following are examples of sentences:

(1) $(\forall x_1)(\exists x_2)[(x_1 + x_2) + x_2^2 y \neq x_1 y^2 \vee x_1 - x_2 = 0]$, where $y = \mathrm{trans}(x_1)$.
(2) $(\exists x_1)[y - x_1 - 1 = 0 \wedge x_1 > 0]$, where $y = \mathrm{trans}(x_1)$.

In order to ensure that a sentence of the form given above is meaningful in case the domain I of trans is a proper subinterval of $\mathbb{R}$ and the term $\mathrm{trans}(x_1)$ occurs at least once in the sentence, we shall assume in such a case that the bound variable x_1 ranges over I instead of $\mathbb{R}$. Consider the case $\mathrm{trans}(x) = \ln(x)$, for which $I = (0, \infty)$, for instance. For example (1) above, we assume that x_1 ranges over I: thus, this sentence is understood to mean $(\forall x_1 \in I)(\exists x_2 \in \mathbb{R})[\ \ldots\ ]$. For example (2) above, we similarly assume that x_1 ranges over I (though such clarification is not really necessary in this case because "$x_1 > 0$" is a conjunct of the formula).

In Sections 3 and 4 we shall present a decision method for $\mathcal{T}_{\mathrm{trans}}$. The most essential component of this method is a univariate decision method, which is described in Section 3. This univariate method is based on real root isolation of integral polynomials in $\mathrm{trans}(x_1)$ using pseudodifferentiation (a process akin to differentiation) and recursion. However, in order to provide an appropriate setting for our real root isolation algorithm we shall need to introduce a slight extension of the ring of integral polynomials in $\mathrm{trans}(x_1)$. For the remainder of this section, and the next one, we shall denote x_1 by x, for simplicity.

We suppose that $\mathrm{trans}'(x) = (a(x) + b(x)\mathrm{trans}(x))/d(x)$, for some $a(x), b(x), d(x) \in \mathbb{Z}[x]$, with $d(\alpha) \neq 0$ for all $\alpha \in I$. (Note that the derivatives of the analytic functions $\exp(x)$, $\ln(x)$ and $\arctan(x)$ are of this form.) Denote by $\mathbb{Z}[x]_d$ the ring of fractions of $\mathbb{Z}[x]$ with respect to powers of $d = d(x)$; that is

$$\mathbb{Z}[x]_d = \{c(x)/d(x)^k \mid c(x) \in \mathbb{Z}[x],\ k \geq 0\}.$$

We shall be especially interested in polynomials in y with coefficients in $\mathbb{Z}[x]_d$, and we shall denote the ring of all such polynomials by R_d: that is $R_d = \mathbb{Z}[x]_d[y]$. Now any $f(x, y) \in R_d$ can be expressed uniquely in the form $f(x, y) = p(x, y)/d(x)^k$, where $p(x, y) \in \mathbb{Z}[x, y]$ and $k \geq 0$ is least possible. We call $p(x, y)$ the *integral polynomial associated to* $f(x, y)$, and write $p = \mathrm{ipol}f$. For given $f(x, y) \in R_d$ we put $f^*(x) = f(x, \mathrm{trans}(x))$, for all $x \in I$: then f^* is defined and analytic on I. We put $R_d^* = \{f^*(x) \mid f \in R_d\}$. The following basic result states that the rings R_d and R_d^* are isomorphic.

Proposition 3. *The mapping $\Phi : R_d \to R_d^*$ defined by $\Phi(f) = f^*$ is an isomorphism of rings.*

Proof. The injectivity of Φ is the only non-trivial fact. Take any $f \in R_d$, with $f \neq 0$, and put $p = \mathrm{ipol}f$. Choose a rational number $\alpha \in I$, with $\alpha \neq \xi$, so that $p(\alpha, y) \neq 0$. We claim that $p^*(\alpha) \neq 0$. For if this is not the case then $\beta = \mathrm{trans}(\alpha)$ is a root of the nonzero polynomial $p(\alpha, y)$ whose coefficients are rational numbers. Hence β is algebraic. By the strong transcendence of trans, this implies that $\alpha = \xi$, which contradicts the choice of α. This proves the claim. We have shown that the real valued function $p^*(x)$ is not identically zero. Therefore the same is true for $f^*(x)$. Hence the mapping Φ is injective. □

In view of the above proposition, any element r of R_d^* has a unique representation as $r = f^*(x)$, with $f \in R_d$. Take any $f \in R_d$. Concerning the derivative $(f^*)'$ of f^* we can observe the following. By the chain rule we have

$$(f^*)'(x) = f_x^*(x) + f_y^*(x)\mathrm{trans}'(x)$$

where f_x and f_y denote the partial derivatives of f with respect to x and y, respectively. Since $f(x, y) = p(x, y)/d(x)^k$, for some $p(x, y) \in \mathbb{Z}[x, y]$ and $k \geq 0$, clearly f_x and f_y belong to R_d. Also, $\mathrm{trans}'(x) \in R_d^*$, by assumption. Hence $(f^*)'(x) \in R_d^*$. We have shown that R_d^* is closed under differentiation.

We now consider pseudodifferentiation, for which we describe an appropriate setting. We shall assume that there is a certain distinguished subset S (which we will sometimes denote by S_{trans}) of R_d, closed under multiplication, for which $S^* = \Phi(S)$ is equipped with functions $\mathrm{pdeg} : S^* - \{0\} \to \mathbb{N} \times \mathbb{N}$ (a *pseudodegree* function) and $\mathrm{pder} : S^* - \{0\} \to S^*$ (a *pseudoderivative* function). We shall assume that $(S^*, \mathrm{pdeg}, \mathrm{pder})$ enjoys some special properties in relation to $\mathrm{trans}(x)$, which we present as three axiom groups:

- PDS1. Firstly, we suppose that for each $f \in R_d$, there is a distinguished element $\hat{f} \in S$ such that f and $\hat{f}$ have the same associated integral polynomial, up to a factor of a power of $d(x)$ (more precisely, $\mathrm{ipol}f = \mathrm{ipol}\hat{f} \times d(x)^k$, for some $k \geq 0$). Furthermore we assume that the mapping $f \to \hat{f}$ is multiplicative, and for each $f \in S$ we have $f = \widehat{\mathrm{ipol}f}$.
- PDS2. Secondly, we suppose that for all nonzero $f^*, g^* \in S^*$, $\mathrm{pdeg}f^* = (0, 0)$ if and only if f is an integer constant; $\mathrm{pdeg}f^* \leq \mathrm{pdeg}f^*g^*$; $\mathrm{pdeg}(\mathrm{pder}f^*) < \mathrm{pdeg}f^*$, provided $\mathrm{pder}f^* \neq 0$; and $\mathrm{pder}f^*$ and $(f^*)'$ (the true derivative of f^*) have the same set of real zeros, multiplicities taken into account. (Note that we have used the lexicographic order $\leq$ on $\mathbb{N}^2$; see Section 3 for the definition and some discussion of this concept.)
- PDS3. Thirdly, we assume that, for each $f(x, y) \in S$ for which $\mathrm{ipol}f$ is irreducible of positive degree in y, the resultant with respect to y of $\mathrm{ipol}f$ and $\mathrm{ipol}g$, denoted by $\mathrm{res}_y(\mathrm{ipol}f, \mathrm{ipol}g)$, is a nonzero polynomial in x, where $g^* = \mathrm{pder}f^*$.

Definition 4. We will call such a set S^* (or S) together with the associated functions pdeg and pder, satisfying the above axioms, a *pseudoderivation system for* $\mathrm{trans}(x)$. Individual elements of S^* will sometimes be referred to as *generalised integral polynomials in* $\mathrm{trans}(x)$.

We illustrate these concepts by exhibiting specific pseudoderivation systems for the two strongly transcendental functions $\exp(x)$ and $\arctan(x)$. Firstly, in case $\mathrm{trans}(x) = \exp(x)$, we have $\mathrm{trans}'(x) = \mathrm{trans}(x)$, so $d(x) = 1$. We have $R_d = \mathbb{Z}[x, y]$ and we put $S_{\exp} = \mathbb{Z}[x, y]$. For each $f \in R_d$ we put $\hat{f} = f$. Let $f^*(x)$ be a nonzero element of $S^*_{\exp}$. For the pseudodegree of $f^*(x)$ we set $\mathrm{pdeg} f^* = (m, n)$, where m is the degree in y of $f(x, y)$, and n is the degree in x of $f(x, 0)$ if $f(x, 0) \neq 0$, with $n = 0$ otherwise. For the pseudoderivative of $f^*(x)$ we set

$$\mathrm{pder} f^*(x) = \begin{cases} (f^*)'(x) & \text{if } n > 0 \\ (f^*)'(x)/\exp(x) & \text{otherwise.} \end{cases}$$

That $S_{\exp}$, together with pdeg and pder, is a pseudoderivation system for $\exp(x)$ is not difficult to verify. For the record, we prove that it satisfies the third axiom PDS3:

Proposition 5. *Let $f(x, y) \in S_{\exp}$ and suppose that f is irreducible of positive degree in y. Let $g^* = \mathrm{pder} f^*$. Then* $\mathrm{res}_y(f, g) \neq 0$.

Proof. Let $r(x) = \mathrm{res}_y(f, g)$ and suppose $r(x) = 0$. By Theorem 2 of Collins (1971) (extended slightly) f and g have a common factor of positive degree in y. Since f is irreducible it follows that f is a divisor of g. Let (m, n) be the pseudodegree of $f^*(x)$. We claim that $n = 0$. The claim is proved by contradiction as follows. Suppose that $n > 0$. Then $g^*(x) = (f^*)'(x)$, by definition. Since f divides g, we have $f(x, 0) \mid g(x, 0)$. But $g(x, 0)$ is the ordinary derivative of $f(x, 0)$. Therefore $g(x, 0) = 0$. So $f(x, 0)$ is constant, contradicting $n > 0$. The claim is proved. So $g^*(x) = (f^*)'(x)/\exp(x)$, hence $g(x, y) = (\partial f/\partial x + y\partial f/\partial y)/y$. Therefore $\deg_y g < \deg_y f$, hence $g = 0$, hence $(f^*)'$ is identically zero. By the mean value theorem, it follows that $f^*(x)$ is identically constant, contradicting the assumptions about f. We have shown that $r(x) \neq 0$, as required. $\square$

Secondly, consider the case $\mathrm{trans}(x) = \arctan(x)$. We have $\mathrm{trans}'(x) = 1/(1+x^2)$, so $d(x) = 1+x^2$. We have $R_d = \mathbb{Z}[x]_d[y]$ and we put

$$S_{\arctan} = \{f(x, y) = f_0(x) + f_1(x)y + \cdots + f_m(x)y^m \in R_d \mid f_m(x) \in \mathbb{Z}[x],\ f_m(x) \notin (1 + x^2)\}.$$

Then for each $f \in R_d$ there is a unique $\hat{f} \in S_{\arctan}$ such that f and $\hat{f}$ have the same associated integral polynomial, up to a factor of a power of $d(x)$. Moreover the mapping $f \to \hat{f}$ is multiplicative, and for each $f \in S_{arctan}$ we have $f = \widehat{\mathrm{ipol} f}$. Now let $f^*(x)$ be a nonzero element of $S^*_{\arctan}$. For the pseudodegree of $f^*(x)$ we set $\mathrm{pdeg} f^* = (m, n)$, where m is the degree in y of f, and n is the degree in x of $f_m(x)$, the leading coefficient of f. For the pseudoderivative of $f^*(x)$ we set

$$\mathrm{pder} f^*(x) = \widehat{(f^*)'}(x).$$

That $S_{\arctan}$, together with pdeg and pder, is a pseudoderivation system for $\arctan(x)$ is not difficult to verify. (A proof that it satisfies the third axiom could be obtained by suitably adjusting the proof of Proposition 5.)

As a simple example, let $f(x, y) = x + y \in S_{\arctan}$, with $\mathrm{pdeg} f^* = (1, 0)$. Then $(f^*)'(x) = 1 + 1/(1 + x^2) = (2 + x^2)/(1 + x^2) \notin S^*_{\arctan}$. So

$$\mathrm{pder} f^*(x) = \widehat{(f^*)'}(x) = 2 + x^2 \in S^*_{\arctan}$$

and $(0, 2) = \mathrm{pdeg}(\mathrm{pder} f^*) < \mathrm{pdeg} f^*$.

It is straightforward to exhibit a pseudoderivation system for the strongly transcendental function $\ln(x)$ defined for $x > 0$, and we leave it to the reader to supply the details if so desired.

We can now prove two important theorems and a corollary concerning the exceptionality of a nonsimple root of a given $f^*(x) \in S^*$, and of a common root of given $f^*(x), g^*(x) \in S^*$, where S is any pseudoderivation system. For definiteness we summarise all our assumptions. We let $\mathrm{trans}(x)$ denote a specific real valued function defined and analytic on I, strongly transcendental with exceptional point ξ. We suppose that $\mathrm{trans}'(x) = (a(x) + b(x)\mathrm{trans}(x))/d(x)$, for some $a(x), b(x), d(x) \in \mathbb{Z}[x]$, with $d(\alpha) \neq 0$ for all $\alpha \in I$. We let $S \subset R_d$ be a pseudoderivation system for $\mathrm{trans}(x)$, equipped with pseudodegree and pseudoderivative functions pdeg and pder, respectively.

Theorem 6. *Let $f(x,y) \in S$ and suppose that $p(x,y) = \text{ipol}f(x,y)$ is an irreducible element of $\mathbb{Z}[x,y]$. Then the only possible non-simple real zero of $f^*(x)$ is the exceptional point ξ of* $\text{trans}(x)$.

Proof. $f^*(x)$ has no non-simple zeros if p has degree 0 in y. So suppose that p has positive degree in y. Let $g^*(x) = \text{pder}\, f^*(x)$, and put $q(x,y) = \text{ipol}g(x,y)$. Let α be a non-simple real zero of $f^*(x)$. Then $(f^*)'(\alpha) = 0$. Therefore $g^*(\alpha) = 0$ by PDS2. Hence, with $\beta = \text{trans}(\alpha)$, we have

$$p(\alpha,\beta) = q(\alpha,\beta) = 0.$$

Therefore α is a root of $r(x) = \text{res}_y(p,q)$, which is a nonzero polynomial by PDS3. Hence α is algebraic. Now β is a root of $p(\alpha,y)$, which is a nonzero polynomial by the assumed irreducibility of $p(x,y)$. Hence β is also algebraic. By the strong transcendence of trans, α must be ξ. □

Theorem 7. *Let $f(x,y)$ and $g(x,y)$ be nonzero elements of S and suppose that $p(x,y) = \text{ipol}f(x,y)$ and $q(x,y) = \text{ipol}g(x,y)$ are relatively prime elements of $\mathbb{Z}[x,y]$. Then the only possible common real zero of $f^*(x)$ and $g^*(x)$ is the exceptional point ξ of* $\text{trans}(x)$.

Proof. $f^*(x)$ and $g^*(x)$ have no common zeros if both p and q have degree 0 in y. So suppose that at least one of p and q has positive degree in y. Let α be a common zero of $f^*(x)$ and $g^*(x)$. Then, with $\beta = \text{trans}(\alpha)$, we have

$$p(\alpha,\beta) = q(\alpha,\beta) = 0.$$

Therefore α is a root of the resultant $res_y(p,q)$, which is a nonzero polynomial since p and q are assumed relatively prime. Hence β is also algebraic, since β is a root of the polynomials $p(\alpha,y)$ and $q(\alpha,y)$, at least one of which is nonzero by the relative primality of p and q. By the strong transcendence of trans α must be the exceptional point ξ of trans. □

Corollary 8. *Let $f(x,y) \in S$ and suppose that $p(x,y) = \text{ipol}f(x,y)$ is a squarefree element of $\mathbb{Z}[x,y]$. Then the only possible non-simple real zero of $f^*(x)$ is the exceptional point ξ of* $\text{trans}(x)$.

Proof. Write $p(x,y) = p_1(x,y)p_2(x,y)\cdots p_t(x,y)$, where the $p_i(x,y)$ are pairwise relatively prime, irreducible elements of $\mathbb{Z}[x,y]$. We have $p^*(x) = p_1^*(x)p_2^*(x)\cdots p_t^*(x)$. Let α be a non-simple real zero of $f^*(x)$. Then α is a nonsimple zero of $p^*(x)$. Suppose first that $p_i^*(\alpha) = p_j^*(\alpha)$, for some i and j, with $1 \le i < j \le t$. Then $(\hat{p}_i)^*(\alpha) = (\hat{p}_j)^*(\alpha)$. Hence, by Theorem 7, $\alpha = \xi$. Suppose on the other hand that α is a zero of at most one $p_i^*(x)$. Then α must be a nonsimple zero of $p_i^*(x)$, hence of $(\hat{p}_i)^*(x)$. Hence, by Theorem 6, $\alpha = \xi$. □

3. Deciding univariate polynomial-transcendental problems

The goal of this section is to present a decision method for those sentences of the theory $\mathcal{T}_{\text{trans}}$ defined in Section 2 which involve only the variable x_1. Indeed, we shall describe an algorithm *DUPTP* that decides univariate polynomial-transcendental problems. In this section we shall continue to denote the variable x_1 by x, for simplicity. We will also continue to adopt the key assumptions stated in the preamble to Theorem 6. Moreover we shall further assume that a method is available which, given an element $f^*(x)$ of S^*, computes a bounded interval $[\gamma_1,\gamma_2] \subset I$, with binary rational endpoints, such that every real zero of $f^*(x)$ is contained in (γ_1,γ_2). In this section we shall first consider such methods of determining bounds for the real zeros of generalised integral polynomials in $\text{trans}(x)$ for the cases $\text{trans}(x) = \exp(x)$, $\text{trans}(x) = \arctan(x)$ and $\text{trans}(x) = \ln(x)$. Then we present a key subalgorithm *ISOL* that isolates the real zeros of a generalised integral polynomial in $\text{trans}(x)$. Finally, a full description of the algorithm *DUPTP* is given.

3.1. Real zero bounds

Our real zero isolation algorithm *ISOL* will require the determination of a real zero bound for generalised integral polynomials in $\text{trans}(x)$. In this subsection we show how this can be achieved for the cases $\text{trans}(x) = \exp(x)$, $\text{trans}(x) = \arctan(x)$ and $\text{trans}(x) = \ln(x)$. We begin with the first case.

Theorem 9. *Let $p(x, y) = \sum_{i=0}^{n} p_i(x)y^i$ with $p_i(x) \in \mathbb{Z}[x]$ and $p_n(x) \neq 0$. Then an upper bound C for the real zeros of $p^*(x)$ can be obtained with the following procedure:*

(1) *Find $C_1 > 0$ such that for all $x > C_1$, $|p_n(x)| \geq 1$.*
(2) *Find $C_2 > 0$ and $k \in \mathbb{N}$ such that for all i in the range $0 \leq i < n$ and for all $x > C_2$, $|p_i(x)| \leq \frac{x^k}{n}$.*
(3) *Find $C_3 > 0$ such that for all $x > C_3$, $x^k < \exp(\frac{x}{2})$.*
(4) *Set $C \leftarrow \max\{C_1, C_2, C_3\}$.*

Proof. Let $x > C$. Then we can derive the inequality

$$\left|\sum_{i=0}^{n-1} p_i(x) \exp(x)^i\right| < |p_n(x)|e^{nx}$$

by applying (1)–(3). Therefore, for $x > C$, we have $p^*(x) \neq 0$. Thus C is an upper bound for the real zeros of $p^*(x)$. Complete details concerning the determination of the numbers C_1, C_2, C_3 are provided by Achatz (2006). □

A lower bound for the real zeros of $p^*(x)$ can be obtained by applying an analogous procedure to the exponential polynomial $g^*(y) = e^{ny}p^*(-y)$. The details can be found in Achatz (2006). Maignan (1998) provides an alternative method for finding a real zero bound for exponential polynomials.

Let us consider next the case $\mathrm{trans}(x) = \arctan(x)$. Let $p(x, y)$ be a nonzero integral polynomial. We can determine a bound for the absolute values of the zeros of $p^*(x) = p(x, \arctan(x))$ as follows. To prepare, write $p(x, y)$ in the form

$$p(x, y) = \sum_{j=0}^{m} q_j(y)x^j$$

with $q_j(y) \in \mathbb{Z}[y]$ and $q_m(y) \neq 0$. Suppose first that $q_m = q_m(y)$ is constant, for simplicity. For $0 \leq j \leq m - 1$, compute a bound $B_j > 0$ for $|q_j(\arctan(x))|$ on the whole real line, using the fact that $|\arctan(x)| < \pi/2$ for all $x \in \mathbb{R}$. Then apply Cauchy's classical root bound to obtain:

$$|z| < 1 + \max(|B_0|, \ldots, |B_{m-1}|)/|q_m|,$$

for every root z of $p^*(x)$. The remaining case, in which $q_m(y)$ is not constant, can be handled with a slight modification of the above straightforward construction.

Finally we consider the case $\mathrm{trans}(x) = \ln(x)$. Let $p(x, y)$ be a nonzero integral polynomial. Put $q(x, y) = p(y, x)$. Determine an upper bound $C > 0$ for the real roots of $q(x, \exp(x))$ using the construction outlined above. Then put $B = \exp(C)$ to obtain an upper bound on the real roots of $p(x, \ln(x))$. (The reason this works is as follows. Take any positive root w of $p(x, \ln(x))$. Put $z = \ln(w)$. Then $w = \exp(z)$. Hence $q(z, \exp(z)) = p(\exp(z), z) = 0$. Therefore $z < C$, from which we obtain $w < B$.) A (positive) lower bound on the real roots of $p(x, \ln(x))$ can be obtained using a similar process.

3.2. *Isolating real zeros of generalised integral polynomials in* trans(x)

We shall describe an algorithm to isolate the real zeros of a nonzero generalised integral polynomial $f^*(x) = f(x, \mathrm{trans}(x))$. This algorithm is based on pseudodifferentiation and recursion on the pseudodegree $\mathrm{pdeg}\, f^*$ of $f^*(x) \neq 0$. Related algorithms for polynomial real root isolation are described by Collins and Loos (1982) and Johnson (1998). We shall use the lexicographic order $\leq$ on $\mathbb{N}^2$ defined, for example, in Exercise 4.61 of Becker et al. (1998): $(k, l) \leq (m, n)$ means that $k < m$ or ($k = m$ and $l < n$) or ($k = m$ and $l = n$). Theorem 4.62 of Becker et al. (1998) implies that this linear (indeed admissible) order on $\mathbb{N}^2$ is a *well-order*, that is, every non-empty subset of $\mathbb{N}^2$ has a least element. Hence the principle of *noetherian induction* can be used to prove a claim of the kind, "$P(m, n)$ is true for all $(m, n) \in \mathbb{N}^2$", as explained by Becker et al. (1998). Our real zero isolation algorithm uses recursion on the pseudodegree, and we will demonstrate its validity using noetherian induction.

Our algorithm uses the concept of a *modulus of continuity (moc)* (Bishop, 1967; Bishop and Bridges, 1985) for a real valued function $f(x)$ on a nonempty compact interval $[a, b]$ of the real line. A positive real valued function δ defined on the set of all positive real numbers is called a *moc* for $f(x)$ on $[a, b]$ if for all $\epsilon > 0$ and for all $x, y \in [a, b]$ $|x - y| \leq \delta(\epsilon)$ implies $|f(x) - f(y)| \leq \epsilon$. By definition $f(x)$ is uniformly continuous on $[a, b]$ if and only if there is a moc for $f(x)$ on $[a, b]$. Moreover by a standard theorem of analysis the assertion of the preceding sentence remains valid if one omits the modifier 'uniformly'. The following proposition provides an explicit linear moc for a continuously differentiable function.

Proposition 10. *Let $f(x)$ be real valued and continuously differentiable on $[a, b]$. Let M be a positive number with $M \geq \max_{x\in[a,b]} |f'(x)|$. Then a linear moc δ for $f(x)$ on $[a, b]$ can be obtained by putting $\delta(\epsilon) = \epsilon/M$.*

Proof. Let $\epsilon > 0$, let $x, y \in [a, b]$, and suppose that $|x - y| \leq \delta(\epsilon)$. By the mean value theorem, $f(x) - f(y) = f'(c)(x - y)$, for some c between x and y. Therefore

$$|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y| \leq M\delta(\epsilon) = M\epsilon/M = \epsilon.$$

This completes the proof. □

We define some terminology. An *isolation list* for a real-valued function $f(x)$ defined on the nonempty open interval $I = (a, b)$ is a list $L = (I_1, I_2, \ldots, I_k)$, such that

(a) k is the number of distinct real zeros of f;
(b) each $I_j = (a_j, b_j)$, where a_j and b_j are binary rational numbers;
(c) each I_j contains a unique zero of f; and
(d) $a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_k < b_k \leq b$.

Recall that we assume that $S \subset R_d$ is a pseudoderivation system for trans(x) whose domain is I. We can now describe our algorithm for isolating the real zeros of a given nonzero element $f^*(x)$ of S^* for which ipol$f$ is squarefree. The basic idea of the algorithm in case pdeg $f^* > (0, 0)$ is: firstly, isolate the real zeros of pder f^* recursively; secondly, refine these isolating intervals I_j so as to ensure they contain no zeros of f^* (with the possible exception of ξ), by Corollary 8; thirdly, identify those complementary intervals J_i which contain a root of f^*: the collection of such complementary intervals comprises an isolation list for $f^*(x)$, by Rolle's theorem. Now we describe the steps in detail. The algorithm commands are interspersed with comments enclosed in square brackets []. Braces {} are used to group commands.

Algorithm 1.

$L \leftarrow ISOL(f)$

Input: $f(x, y) \in S$, such that $f \neq 0$ and ipol$f$ is squarefree.
Output: L, an isolation list for $f^*(x) = f(x, \text{trans}(x))$.

(1) [Basis.] Set $(m, n) \leftarrow$ pdeg $f^*(x)$. If $(m, n) = (0, 0)$ then {Set $L \leftarrow$ the empty list. Return}.
(2) [Recursion.] Set $g^*(x) \leftarrow$ pder $f^*(x)$. Set $q(x, y) \rightarrow$ ipol $g(x, y)$. Set $s(x, y) \leftarrow$ gsfd $q(x, y)$. ['gsfd' denotes 'greatest squarefree divisor'.] Set $L' \leftarrow ISOL(\hat{s})$. [Our validity proof below will show that pdeg $\hat{s}^*(x) \leq$ pdeg $g^*(x) <$ pdeg $f^*(x)$. By noetherian induction hypothesis, L' is an isolation list for $\hat{s}^*(x)$, hence for $g^*(x)$, hence also for $(f^*)'(x)$.]
(3) [Bound for real zeros.] Compute a bounded interval $[\gamma_1, \gamma_2] \subset I$, with binary rational endpoints, such that every real zero of $f^*(x)$ is contained in (γ_1, γ_2).
(4) [Prepare for induction step.] Let $I = (a, b)$ and let $L' = (I_1, I_2, \ldots, I_k)$, with $I_j = (a_j, b_j)$. Let $\alpha'_0 = a$ and $\alpha'_{k+1} = b$, and for $1 \leq j \leq k$, let α'_j denote the unique zero of $(f^*)'(x)$ in I_j. [Observe that, by Corollary 8, $\hat{s}^*(a_j)\hat{s}^*(b_j) < 0$, for every j in the range $1 \leq j \leq k$, unless $\alpha'_j = \xi$.] By expanding (γ_1, γ_2) as necessary ensure that $\gamma_1 \leq a_1$ and $b_k \leq \gamma_2$. Put $b_0 \leftarrow \gamma_1$ and $a_{k+1} \leftarrow \gamma_2$. For $i = 0, 1, \ldots, k$, put $J_i \leftarrow (b_i, a_{i+1})$. [In case $b_i = a_{i+1}$ J_i is empty. By Rolle's theorem, each interval $[\alpha'_i, \alpha'_{i+1}]$ contains at most one zero of $f^*(x)$. Hence each complementary interval $J_i = (b_i, a_{i+1})$ contains at most one such zero. Moreover, by Corollary 8, $f^*(x)$ has no non-simple zeros, with the possible exception of ξ. Hence neither α'_i nor α'_{i+1} is a zero of $f^*(x)$, unless $\alpha'_i = \xi$ or $\alpha'_{i+1} = \xi$. The next step will ensure that, after suitable refinement of the I_js, no $[a_j, b_j]$ contains a zero of $f^*(x)$, unless $\alpha'_j = \xi$.] Set $L \leftarrow$ the empty list.
(5) [Interval refinement.] For $j = 1, 2, \ldots, k$ {If $\alpha'_j = \xi$ and $f^*(\xi) = 0$ then insert I_j into L else {Compute bound M for $|(f^*)'(x)|$ on the (initial) interval $[a_j, b_j]$. Repeatedly bisect $I_j = (a_j, b_j)$,

always retaining the subinterval of I_j which contains α'_j, that is, always maintaining the invariance of the relation $\hat{s}^*(a_j)\hat{s}^*(b_j) < 0$. But if $\hat{s}^*(m_j) = 0$, with $m_j = (a_j+b_j)/2$, then retain the subinterval $(a_j+(b_j-a_j)/4, a_j+3(b_j-a_j)/4)$ centred at $m_j = \alpha'_j$. Terminate the bisection process when $b_j-a_j \le |f^*(a_j)|/M$ and $b_j-a_j \le |f^*(b_j)|/M$. } } [See discussion which follows the algorithm description for clarification of the operations of this step. For each j the repeated bisection process must terminate because the values of the a_j and the b_j approach α'_j, for which $f^*(\alpha'_j) \ne 0$, by Corollary 8. Our validity proof below will show that upon termination $[a_j, b_j]$ contains no zero of $f^*(x)$.]

(6) [Completion of induction step.] For $i = 0, 1, \ldots, k$ {If $f^*(b_i)f^*(a_{i+1}) < 0$ then insert J_i into L }. [By step (5), $(\alpha'_i, \alpha'_{i+1})$ contains a zero of $f^*(x)$ if and only if J_i does, which occurs if and only if $f^*(b_i)f^*(a_{i+1}) < 0$.] Return.

We clarify some of the operations used in steps (5) and (6). For each value of j step (5) requires the computation of a bound M for $|(f^*)'(x)|$ on the initial interval $[a_j, b_j]$. Such a bound could be obtained using the triangle inequality and appropriate estimates for the component terms of $|(f^*)'(x)|$ on $[a_j, b_j]$. The reader will note that a number of comparison tests are to be performed by steps (5) and (6). The first such comparison, namely $\alpha'_j = \xi$ (step (5)), is equivalent to $\hat{s}^*(\xi) = 0 \wedge \xi \in I_j$. Since ξ and trans(ξ) are assumed to be integers, testing whether or not $\hat{s}^*(\xi) = 0 \wedge \xi \in I_j$ can be done using integer and rational number arithmetic. Determining $f^*(\xi) = 0$ (step (5)) can be done similarly. A method to evaluate the sign of a generalised integral polynomial $h^*(x)$ at a binary rational number r, in case $h^*(r) \ne 0$, could be obtained by iterating sufficiently often a standard numerical procedure for computing an interval $[u, w]$ of specified length $\epsilon > 0$ guaranteed to contain the value $v = h^*(r)$. Such a method could be used to evaluate the conditions $\hat{s}^*(a_j)\hat{s}^*(b_j) < 0$ (step (5)) and $f^*(b_i)f^*(a_{i+1}) < 0$ (step (6)). For a method to determine the equality $\hat{s}^*(m_j) = 0$ (step (5)) see Section 3.3 below. The termination test (step (5)) can be performed using a combination of such methods.

For the record, we prove the validity of our algorithm.

Theorem 11. *For all $(m, n) \in \mathbb{N}^2$, the following statement $P(m, n)$ is true: for every valid input $f(x, y) \in S$, with* pdeg $f^* = (m, n)$*, ISOL returns an isolation list L for $f^*(x) = f(x, \text{trans}(x))$.*

Proof. The proof of this theorem is by noetherian induction on (m, n). We first address the induction base, in which we aim to prove that $P(0, 0)$ is true. Let $f(x, y) \in S$ be a valid input with pdeg $f^* = (0, 0)$. By PDS2, f is an integer constant. Therefore, since $f \ne 0$ by the input assumption (precondition), $f^*(x)$ has no real zeros. So the empty list returned by *ISOL* is indeed the isolation list for $f^*(x)$. Next we address the induction step. Let $(m, n) > (0, 0)$. Suppose that $P(k, l)$ is true for every $(k, l) < (m, n)$ (the induction hypothesis). We must prove that $P(m, n)$ is true. Let $f(x, y) \in S$ be a valid input with pdeg $f^* = (m, n)$. Since $(m, n) > (0, 0)$, step (2) and subsequent steps are performed; and by PDS2, f is not an integer constant. Therefore, by Proposition 3, f^* is not identically constant. Hence, by the mean value theorem, $(f^*)'$ is not identically zero. By PDS2, $g^* =$ pder f^* is not identically zero, so $g \ne 0$, hence $\hat{s} \ne 0$. By PDS1, ipol $\hat{s} \mid s$, hence ipol $\hat{s}$ is squarefree. So $\hat{s}$ is a valid input to *ISOL*. Now $q(x, y) = s(x, y)t(x, y)$ for some $t(x, y) \in \mathbb{Z}[x, y]$. Therefore $\hat{q}(x, y) = \hat{s}(x, y)\hat{t}(x, y)$, by PDS1. Now $g(x, y) = \hat{q}(x, y)$, by PDS1. Therefore $g^*(x) = \hat{s}^*(x)\hat{t}^*(x)$, by multiplicativity of Φ. Therefore pdeg $\hat{s}^*(x) \le$ pdeg $g^*(x) <$ pdeg $f^*(x)$, by PDS2. Therefore, by induction hypothesis, L' is an isolation list for $\hat{s}^*(x)$, hence for $g^*(x)$, hence also for $(f^*)'(x)$. As noted in a comment in step (4), by Corollary 8, $\hat{s}^*(a_j)\hat{s}^*(b_j) < 0$, for every j in the range $1 \le j \le k$, unless $\alpha'_j = \xi$. By Rolle's theorem, each interval $[\alpha'_i, \alpha'_{i+1}]$ contains at most one zero of $f^*(x)$. Hence each complementary interval $J_i = (b_i, a_{i+1})$ contains at most one such zero. Moreover, by Corollary 8, $f^*(x)$ has no non-simple zeros, with the possible exception of ξ. Hence neither α'_i nor α'_{i+1} is a zero of $f^*(x)$, unless $\alpha'_i = \xi$ or $\alpha'_{i+1} = \xi$. The aim of step (5) is to ensure that, after suitable refinement of the I_js, no $[a_j, b_j]$ contains a zero of $f^*(x)$, unless $\alpha'_j = \xi$. We show that the interval refinement process of step (5) achieves this aim. Now for each j the repeated interval bisection process of step (5) terminates because the values of the a_j and the b_j approach α'_j, for which $f^*(\alpha'_j) \ne 0$, by Corollary 8. Consider the situation upon termination. Let $\epsilon = |f^*(b_j)|$, and put $\delta(\epsilon) = \epsilon/M$, as in Proposition 10. Since $|b_j - \alpha'_j| < b_j - a_j \le \delta(\epsilon)$, it follows by Proposition 10 that $|f^*(b_j) - f^*(\alpha'_j)| \le \epsilon$. Therefore $|f^*(b_j) - f^*(\alpha'_j)| < |f^*(b_j)| + |f^*(\alpha'_j)|$, since $f^*(\alpha'_j) \ne 0$. Hence $f^*(b_j)$ and $f^*(\alpha'_j)$ have the same nonzero sign, by the triangle inequality (with particular attention to the conditions under which strict inequality occurs). Hence $[\alpha_j', b_j]$ contains no zero of $f^*(x)$. We could similarly show that $[a_j, \alpha_j']$ contains no zero of $f^*(x)$. Combining these two conclusions we see that $[a_j, b_j]$ contains no zero of $f^*(x)$. So step (5) achieves its aim. Step (6) identifies which of the complementary intervals J_i contains a zero of $f^*(x)$, and retains all such intervals in L, which therefore comprises an isolation list for $f^*(x)$. $\square$

It would be natural and reasonable to ask whether or not the theoretical computing time of *ISOL* could be estimated in terms of suitable parameters relating to the length of the input f. We think that a complete analysis of this kind would be challenging, and probably best attempted for each specific transcendental function of interest. We offer here a modest start: an estimate for the total number of recursive calls made by *ISOL* applied to f in the case $\mathrm{trans}(x) = \exp(x)$.

Let $f \in S$ be nonzero. Put $f_0^* = f^*$ and $f_i^* = \mathrm{pder}\, f_{i-1}^*$, for $i = 1, 2, \ldots$, provided $\mathrm{pdeg}\, f_{i-1}^* \neq (0, 0)$. Then for some $N \geq 1$ we have $\mathrm{pdeg}\, f_{N-1}^* = (0, 0)$, by PDS2. We call $(f_0^*, f_1^*, \ldots, f_{N-1}^*)$ the *pseudoderivative sequence* for f^* (or f), and call N its *length*. (Note that algorithm *ISOL* applied to f computes the pseudoderivative sequence for f, provided that each f_{i-1} is squarefree.) Write $f(x, y) = a_0(x) + a_1(x)y + \cdots + a_m(x)y^m$, with the $a_i(x) \in \mathbb{Z}[x]$ and $a_m(x) \neq 0$. For each i, $0 \leq i \leq m$, put $n_i = \deg a_i(x)$ if $a_i(x) \neq 0$ and put $n_i = 0$ otherwise. We call the quantity $m + \sum_{i=0}^m n_i$ the *adjusted coefficient degree sum* of f, denoted by $ACDS(f)$.

Proposition 12. *Let $f \in S_{\exp} = \mathbb{Z}[x, y]$ be nonzero and let N be the length of its pseudoderivative sequence. Then $N = ACDS(f) + 1$.*

Proof. The proof is by induction on N. First consider the case $N = 1$. In this case $ACDS(f) = 0$, so the desired equation is true. Next suppose $N > 1$. Suppose that for any nonzero $g \in S_{\exp}$ whose pseudoderivative sequence has length $N - 1$ we have $N - 1 = ACDS(g) + 1$ (induction hypothesis). Put $g^* = \mathrm{pder}\, f^*$. By induction hypothesis $N - 1 = ACDS(g) + 1$. Write $f(x, y) = a_0(x) + a_1(x)y + \cdots + a_m(x)y^m$, with the $a_i(x) \in \mathbb{Z}[x]$ and $a_m(x) \neq 0$. Consider first the case in which $\deg a_0(x) > 0$. In this case

$$g^*(x) = a_0'(x) + (a_1'(x) + a_1(x)) \exp(x) + \cdots + (a_m'(x) + m a_m(x)) \exp^m(x).$$

Hence $ACDS(g) = ACDS(f) - 1$. Next consider the case in which $\deg a_0(x) \leq 0$. In this case

$$g^*(x) = (a_1'(x) + a_1(x)) + (a_2'(x) + 2a_2(x)) \exp(x) + \cdots + (a_m'(x) + m a_m(x)) \exp^{m-1}(x).$$

Hence $ACDS(g) = ACDS(f) - 1$. In both cases we have $N - 1 = ACDS(f)$, completing the proof. $\square$

Corollary 13. *Let $f \in S_{\exp} = \mathbb{Z}[x, y]$ be nonzero and suppose that the integral polynomial of each element of its pseudoderivative sequence is squarefree. Then the total number of recursive calls made when ISOL(f) is executed is $ACDS(f)$.*

Corollary 14. *Let $f \in S_{\exp} = \mathbb{Z}[x, y]$ be nonzero and suppose that the integral polynomial of each element of its pseudoderivative sequence is squarefree. Let $d = \max(\deg_x f, \deg_y f)$. Then the total number of recursive calls made when ISOL(f) is executed is $O(d^2)$.*

For ease of exposition and proof we have kept our description of *ISOL* conceptually simple. In practice there are elementary improvements which could be made to enhance the efficiency of the method. For example, we could insert an initial algorithm step (step 0, say) which finds the content $c(x)$ of $p(x, y) = \mathrm{ipol}\, f(x, y)$ with respect to y and computes an isolation list L_0 for $c(x)$ using any highly efficient real root isolation algorithm for $\mathbb{Z}[x]$ (Collins and Loos, 1982; Johnson, 1998). After setting $p(x, y) \leftarrow p(x, y)/c(x)$ we proceed with steps (1)–(6) (in which step (5) could be simplified slightly, by Proposition 15 below). We could then append a final algorithm step (step (7), say) which refines the isolating intervals in $L \leftarrow L_0 \cup L$ into an isolation list for the original $f^*(x)$.

Some comparison of *ISOL*, in the case $\mathrm{trans}(x) = \exp(x)$, with the root isolation method of Maignan (1998) is warranted. Actually Maignan (1998) is chiefly concerned with counting the number of real roots of a given exponential polynomial $f^*(x)$ in some interval (a, b). The method is based on construction of local Sturm sequences. It requires computation of the real roots of two auxiliary univariate polynomials whose degrees are high relative to $\deg_x f$ and $\deg_y f$. It also involves evaluation of the signs of a list of exponential polynomials at rational points. So, although no systematic empirical comparison of the two root isolation algorithms has yet been undertaken, we think that our method is more efficient than that of Maignan (1998).

We would also like to mention at this point comparatively recent related work of Strzebonski (2008, 2009). Both of these papers are concerned with the isolation of the real roots of certain classes of analytic functions of a single variable. The former paper treats exp–log functions, while the latter considers a more general function class, that of tame elementary functions. The root isolation algorithm of the former paper makes use of semi Fourier sequences, while that of the latter paper uses a notion of 'false derivative' (Richardson, 1991, 1992). Both algorithms require the ability to determine the sign of an elementary function at an elementary root constant, for which the only method presently available relies on the unproven Schanuel's conjecture (Lang, 1966; Ax, 1971; Baker, 1975) for its termination.

3.3. Sample points

Let $f^*(x)$ be a generalised integral polynomial in trans(x). We make no assumption about the squarefreeness of ipol$f$, so we cannot directly apply algorithm *ISOL* to isolate the zeros of $f^*(x)$. Let us nevertheless consider the zeros of $f^*(x)$, which determine a decomposition of the nonempty open interval I (the domain of trans), which we term an f^**-invariant* decomposition. Assuming that $f^*(x)$ has n zeros, then the decomposition consists of n 0-cells (the zeros) and $n+1$ 1-cells (the open intervals between the zeros). A *sample point* for a cell is an exact representation of a particular algebraic or non-algebraic (i.e. transcendental) number belonging to that cell. Suppose that we could somehow obtain an isolation list for $f^*(x)$. (A method for doing so, which uses the algorithm *ISOL*, will be described in the next subsection.) Then as sample points for the 1-cells we use appropriately chosen rational endpoints from the isolating intervals obtained. If a 0-cell α is an algebraic number we use a standard representation for α by its minimal polynomial and an isolating interval, as described by Loos (1982). For a 0-cell α which is not an algebraic number we represent α by the irreducible factor $p(x, y)$ of ipol$f(x, y)$ for which $p^*(\alpha) = 0$ and an isolating interval for α.

We now describe how to determine the sign of a given generalised integral polynomial $g^*(x)$ at a given sample point $\alpha \neq \xi$ of such an f^*-invariant decomposition of I. Such sign determination will be an important component of the algorithm *DUPTP*, described in the next subsection.

Suppose first that α is algebraic. For determining whether or not $g^*(\alpha) = 0$ the following proposition is relevant.

Proposition 15. *Let $h(x, y) \in \mathbb{Z}[x, y]$ be primitive and of positive degree in y. Let $\alpha \neq \xi$ be an algebraic number. Then $h^*(\alpha) \neq 0$.*

Proof. Suppose that $h^*(\alpha) = 0$. Let $\beta = \text{trans}(\alpha)$. Then $h(\alpha, \beta) = h^*(\alpha) = 0$. But $h(\alpha, y) \neq 0$, by the primitivity of $h(x, y)$. Therefore β is algebraic, contradicting Lindemann. $\square$

Let $q = \text{ipol}g$. We find the content $c(x)$ and the primitive part $h(x, y)$ of $q(x, y)$. Then $g^*(\alpha) = 0$ if and only if $c(\alpha) = 0$, by the above proposition. Suppose now that $g^*(\alpha) \neq 0$. Let J be an isolating interval for α. Using a method analogous to that of step (5) of *ISOL*, one refines J about α so that, after refinement, J contains no zero of $g^*(x)$. Then, by evaluating g^* at the left or right endpoint of the refined J, we can determine the sign of $g^*(\alpha)$.

Suppose on the other hand that α is not algebraic. Recall that α is represented by the irreducible factor $p(x, y)$ of ipol$f(x, y)$ for which $p^*(\alpha) = 0$ and an isolating interval say J for α. Clearly $g^*(\alpha) = 0$ if p is a factor of q. So suppose that p is not a factor of q. Then, by Theorem 7, $g^*(\alpha) \neq 0$. Using a method analogous to that of step (5) of *ISOL*, one refines J about α so that, after refinement, J contains no zero of $g^*(x)$. Then, by evaluating g^* at the left or right endpoint of the refined J, we can determine the sign of $g^*(\alpha)$. We shall call the sign determination algorithm just described *SIGNEVAL*.

3.4. The algorithm DUPTP (to decide univariate polynomial-transcendental problems)

We present our decision algorithm for univariate polynomial-transcendental problems. Our algorithm is in the spirit of quantifier elimination (QE) by cylindrical algebraic decomposition (CAD) (Arnon et al., 1998; Collins, 1998) for univariate problems.

Algorithm 2.

$v \leftarrow DUPTP(\varphi)$

Input: A prenex sentence φ in $\mathcal{L}_{\text{trans}}$ involving only x.
Output: The truth value v of φ over the domain I of trans.

(1) [Extraction.] The input φ is of the following form

$$(Qx)\psi(x)$$

where ψ is a quantifier-free formula of $\mathcal{L}_{\text{trans}}$ and (Qx) is a quantifier for which x is assumed to range over I. Extract the list $P := (p_1, p_2, \ldots, p_n)$ of those polynomials $p_i(x, y) \in \mathbb{Z}[x, y]$ for which $p_i^*(x)$ occurs in $\psi(x)$.

(2) [Contents and primitive parts.] Compute the set $cont_y(P)$ of contents w.r.t. y of the elements of P and the set $pp_y(P)$ of primitive parts w.r.t. y of elements of P of positive degree in y.

(3) [Squarefree bases.] Compute squarefree bases K and Q of $cont_y(P)$ and $pp_y(P)$, respectively.

(4) [Root isolation.] Apply algorithm *ISOL* to each generalised integral polynomial $\hat{q}(x, y)$, with q in Q, and to each polynomial $c(x)$ in K, individually.

(5) [Isolation list for product.] By their relative primality, for any pair of distinct elements p and q of $K \cup Q$, $\hat{p}^*$ and $\hat{q}^*$ have a common real zero only at ξ by Theorem 7. Refine the isolating intervals for the zeros of all the $p^*(x) \in K \cup Q^*$ to obtain an isolation list for the product $f^*(x)$ of all $p^*(x)$.

(6) [Sample points.] Use the isolation list for $f^*(x)$ to construct sample points for all of the cells of the decomposition of I determined by the zeros of $f^*(x)$, as described in the previous subsection.

(7) [Evaluation.] Use the sample points to decide the original function $(Qx)\psi(x)$. [This can be done as follows. For each i express $p_i(x, y)$ as a product of elements of $K \cup Q$. By evaluating the signs of the generalised integral polynomials associated with the factors of each p_i at each sample point using algorithm *SIGNEVAL* described in the previous subsection, the sign of each p_i^* at each sample point can be determined. Then the truth value of the original formula φ over I can be decided.]

The most essential observation for the validity of the above method is that the formula $\psi(x)$ is truth-invariant in each cell of the decomposition of I determined by the zeros of $f^*(x)$. It is well known that QE by CAD for *one variable* problems is polynomial time (Collins, 1998). By its similarity with QE by CAD, we would therefore expect *DUPTP* to be polynomial time, provided that its subalgorithms *ISOL* and *SIGNEVAL* are polynomial time (as we think they are).

4. Deciding multivariate polynomial-transcendental problems

In Section 3 we described an algorithm *DUPTP* that decides univariate polynomial-transcendental problems. In this section we outline an extension to this procedure, i.e. an algorithm that decides polynomial-transcendental problems in general.

Our decision algorithm *DPTP* accepts as input a prenex sentence φ in $\mathcal{L}_{\text{trans}}$, with bound variables say $x_1, \ldots, x_n$. It produces as output the truth value v of the input φ over the real numbers (where x_1 is assumed to range over I).

The algorithm *DPTP* has two basic phases:

(1) [First phase.] Now the input prenex sentence φ has the form

$$(Q_1x_1)(Q_2x_2)\ldots(Q_nx_n)\psi(x_1, x_2, \ldots, x_n)$$

where ψ is a quantifier-free formula of $\mathcal{L}_{\text{trans}}$ and the (Q_ix_i) are quantifiers [with (Q_1x_1) understood to mean $(Q_1x_1 \in I)$]. Recall that ψ involves polynomials in the x_i and trans(x_1). We apply a quantifier-elimination algorithm for elementary real algebra such as QE by CAD (Arnon et al., 1998; Collins, 1998) to the following formula

$$\varphi' = (Q_2x_2)(Q_3x_3)\ldots(Q_nx_n)\psi(x_1, x_2, \ldots, x_n)$$

obtained by removing (Q_1x_1) from φ. [As the variable x_1 is not quantified in φ', the quantifier elimination algorithm can proceed without any precautions concerning the transcendental function.] The output is a quantifier-free formula $\psi_1(x_1)$ which is equivalent to φ'.

(2) [Second phase.] We combine the quantifier (Q_1x_1) of the input sentence φ with the output of the QE algorithm $\psi_1(x_1)$, thus obtaining a univariate polynomial-transcendental decision problem instance:

$$\varphi'' = (Q_1x_1)\psi_1(x_1)$$

[with (Q_1x_1) understood to mean $(Q_1x_1 \in I)$]. We apply algorithm *DUPTP* from Section 3 to this problem instance φ'', obtaining the truth value v of φ'' over I. Finally we return v as the truth value of the input sentence φ over the real numbers (with $x_1 \in I$).

The validity of algorithm *DPTP* is clear. Its computing time – like that of QE by CAD (Collins, 1998) – is likely to be doubly exponential in n.

We briefly mentioned related work of Strzebonski (in press) in Section 1. Here we offer some comparison between our algorithm *DPTP* and algorithm *CT1D* of Strzebonski (in press). Compared with *DPTP*, *CT1D* admits a wider class of functions transcendental in the first variable and produces a description of a full multidimensional cylindrical analytic decomposition representing the solution set of a given system. However *CT1D*, which in effect makes use of the real root isolation algorithms of Strzebonski (2008, 2009), relies on the unproven Schanuel's conjecture. On the other hand, *DPTP* uses exact methods based on Lindemann's theorem for representing and computing with the transcendental numbers which arise, methods which guarantee the safety and infallibility of *DPTP* both in principle and practice.

5. Implementation and examples

In the case $\mathrm{trans}(x) = \exp(x)$ the univariate decision procedure *DUPTP* described in Section 3 was implemented by Achatz (2006) in the REDLOG package of the computer algebra system REDUCE (Hearn, 2004) under the guidance of the second author. Based on the package for quantifier elimination by cylindrical algebraic decomposition (QE by CAD) in REDLOG (Seidl, 2006) this module was extended to an implementation of the more general decision procedure *DPTP* described in Section 4, again in case $\mathrm{trans}(x) = \exp(x)$.

The functions implemented use the logical REDLOG context OFSF for the ordered field of real numbers regarded as a structure for the first-order language of ordered rings (Dolzmann and Sturm, 2004). In order to avoid a cumbersome extension of this context for the handling of polynomial-exponential problems the following conventions were used: the variable x_1 is reserved as independent variable of the exponential function; and the value $\exp(x_1)$ of this function is represented by a new free variable y. Sentences φ of the formal language $\mathcal{L}_{\exp}$ are entered into the decision program *DPTP* in prenex form

$$(Q_1x_1)(Q_2x_2)\ldots(Q_nx_n)\psi(x_1, y, x_2, \ldots, x_n),$$

where the (Q_ix_i) are quantifiers. Semantically the variables x_i range over the real numbers, the variable y is treated as the value $\exp(x_1)$, and the truth value of φ is evaluated accordingly in the ordered field of real numbers with exponentiation.

In REDLOG the user has the following commands to start, and to obtain verbose output of, the decision procedure: `rldpep`, `rldpepverbose`, and `rldpepiverbose`. The second command turns on the verbose output option. The third command provides an even more detailed trace of the procedure's workings.

As Achatz (2006) reports, the implementation of *DPTP* was tested on several examples with up to 3 quantifiers and produced correct results with running times in the range of 0.5–12 s on a Pentium 4 (2 GHz, 128 heap size). Here are a few of the examples used for testing purposes. In each example the correct truth value (true) was obtained in less than a second on the Pentium 4.

$$(\exists x_1)(\exists x_2)[y - x_2^2 = 0 \wedge x_1 - x_2 = 0], \quad \text{where } y = \exp(x_1)$$
$$(\forall x_1)[(1 - x_1) \cdot y \leq 1 \vee x_1 \geq 1], \quad \text{where } y = \exp(x_1)$$
$$(\exists x_1)(\exists x_2)(\exists x_3)[2x_1 - x_2 + x_3 + y^2 = 0 \wedge 3x_2 - x_3 = 0 \wedge$$
$$2x_1 + x_2 + 3x_3 + y = 0], \quad \text{where } y = \exp(x_1).$$

Sentences involving polynomials in $x_1, x_2, \ldots, x_n$ and a hyperbolic function such as $\cosh(x_1)$ can be massaged into a form for which the program *DPTP* (with $\text{trans}(x) = \exp(x)$) can be applied. For example suppose we want to decide

$$(\forall x_1)[x_1 > 7 \Rightarrow \cosh(x_1) > p(x_1)]$$

for some integral polynomial $p(x_1)$. Using the defining relation

$$\cosh(x_1) = \frac{\exp(x_1) + \exp(-x_1)}{2}$$

and noting that $\exp(x_1)$ is positive on the real line, we see that the sentence of interest is equivalent to the sentence

$$(\forall x_1)[x_1 > 7 \Rightarrow \exp(x_1)^2 + 1 > 2p(x_1)\exp(x_1)]$$

of $\mathcal{L}_{\text{exp}}$. As a specific example we set $p(x_1) = x_1^3 - 4x_1$. The program *DPTP* found that the above sentence is true within 17 s.

Sentences involving polynomials in x and a Gaussian function such as $\exp(-x^2)$ can also be massaged into sentences of $\mathcal{L}_{\text{exp}}$. For example suppose we want to decide

$$(\exists x)[\exp(-x^2) = p(x)]$$

for some integral polynomial $p(x)$. We introduce a new variable say z to represent $-x^2$ and thus obtain an equivalent sentence:

$$(\exists z)(\exists x)[\exp(z) = p(x) \wedge z + x^2 = 0].$$

Replacing z by x_1 and x by x_2 we obtain an equivalent sentence of $\mathcal{L}_{\text{exp}}$. In a similar manner sentences involving polynomials in x and a more general exponential function of the form $\exp(\pi(x))$, where $\pi(x)$ is a rational polynomial, can be converted into sentences of $\mathcal{L}_{\text{exp}}$.

6. Extensions and refinements

Sections 2 and 3 described in detail our decision method for univariate problems expressed in the language $\mathcal{L}_{\text{trans}}$, where $\text{trans}(x)$ is a specific analytic function on an interval I, strongly transcendental with exceptional point ξ, satisfying the assumptions and axioms stated. It was remarked that the three special functions $\exp(x)$, $\ln(x)$ and $\arctan(x)$ satisfy these assumptions and axioms, and hence the decision method applies to these functions.

The last two examples reported in the previous section suggest that our decision method could be adapted to a wider collection of special functions. Here we offer a brief systematic account of such extension. Let us consider three simple ways in which a new transcendental function can be defined using the given function $\text{trans}(x)$. For each one, we indicate how to transform a sentence involving the new function into an equivalent one in $\mathcal{L}_{\text{trans}}$, decidable by our method.

(1) We could *substitute the argument by a rational polynomial* say $\pi(x)$ to obtain $\text{trans}^*(x) = \text{trans}(\pi(x))$. Let $I^* = \pi^{-1}(I)$. Provided that I^* is a nonempty open interval, and the equation $\pi(x) = \xi$ has exactly one real root $\eta \in I^*$, $\text{trans}^*(x)$ is analytic on I^*, and strongly transcendental with exceptional point η. The Gaussian function $\exp(-x^2)$ considered in the last section is of this kind. A sentence of $\mathcal{L}_{\text{trans}^*}$ could be transformed into an equivalent one of $\mathcal{L}_{\text{trans}}$ by introducing a new variable say z to represent $\pi(x)$, in the manner illustrated above for the Gaussian function.

(2) We could *form a rational function in* $\text{trans}(x)$: that is, we could put

$$\text{trans}^*(x) = p(x, \text{trans}(x))/q(x, \text{trans}(x)),$$

where $p(x, y), q(x, y) \in \mathbb{Z}[x, y]$. Provided that $q(x, \text{trans}(x)) > 0$ for all $x \in I$, $\text{trans}^*(x)$ is analytic on I, and strongly transcendental with exceptional point ξ. The hyperbolic functions $\cosh(x)$, $\sinh(x)$ and $\tanh(x)$ are of this kind (with $\text{trans}(x) = \exp(x)$). An atomic formula of $\mathcal{L}_{\text{trans}^*}$ could be transformed into an equivalent one of $\mathcal{L}_{\text{trans}}$ by clearing the denominators, which are powers of the positive definite $q(x, \text{trans}(x))$. By treating in this way all the atomic formulae occurring in a given sentence of $\mathcal{L}_{\text{trans}^*}$, we transform the sentence into an equivalent one of $\mathcal{L}_{\text{trans}}$. (An illustrative example of this transformation was provided for $\cosh(x)$ in the previous section.)

(3) We could *define the inverse of* $\text{trans}(x)$ under suitable conditions. If $\text{trans}(x)$ is strictly monotonic on I with range J, then its inverse function $\text{trans}^{-1}(y)$ is a strictly monotonic analytic function

from J to I. Furthermore $\text{trans}^{-1}(y)$ is strongly transcendental with exceptional point $\text{trans}(\xi)$. A given sentence

$$(Qy \in J)\psi(y, \text{trans}^{-1}(y))$$

in $\text{trans}^{-1}(y)$ is transformed into an equivalent one

$$(Qx \in I)\psi(\text{trans}(x), x)$$

in $\text{trans}(x)$. Important instances of this approach are the transformation of a sentence of $\mathcal{L}_{\ln}$ into an equivalent one of $\mathcal{L}_{\exp}$; and the transformation of a sentence of $\mathcal{L}_{\tan}$ (with tan defined on $(-\pi/2, \pi/2)$) into an equivalent one of $\mathcal{L}_{\arctan}$. The former is of academic interest since $\ln(x)$ already satisfies the assumptions and axioms of Section 2. But the latter transformation opens the door to deciding problems involving the *circular* trigonometric functions on a bounded interval, as we will see below, and is therefore significant.

Still more transcendental functions can be defined from the given $\text{trans}(x)$ using a combination of the above techniques. Perhaps the most important examples of such a combined approach are the well known representations of $\sin(x)$ and $\cos(x)$ over $(-\pi, \pi)$ as rational functions of $\tan(x/2)$ (which we could consider as obtained from a combination of the techniques of (1) and (2) above). Since a sentence of $\mathcal{L}_{\tan}$ over $(-\pi/2, \pi/2)$ can be transformed into one of $\mathcal{L}_{\arctan}$ (as noted above under item (3)), we see that a sentence involving $\sin(x)$ and $\cos(x)$ over $(-\pi, \pi)$ can be successively transformed into one of $\mathcal{L}_{\arctan}$, hence decided by our method.

We mention a couple of refinements which we could make to the methods of this paper. A refinement of algorithm *DPTP* on efficiency grounds is contemplated. It would probably be desirable to try to develop a decision method for $\mathcal{T}_{\text{trans}}$ which is analogous to decision by cylindrical algebraic decomposition (CAD). In order to decide a sentence φ in the variables $x_1, x_2, \ldots, x_n$ we would first extract the set A of polynomials occurring in φ. Second, by analogy with the CAD method, we would compute a description of a cylindrical *analytic* decomposition D of the space $\mathbb{R}^n$ compatible with the zeros of the polynomials in A. Third, we would use the description of D to decide the truth or falsity of φ by analogy with QE by CAD. Some progress has already been made in this direction (McCallum and Weispfenning, 2006). An alternative to the transformation technique of item (1) above would be beneficial. We will try to extend our decision procedure for $\mathcal{T}_{\text{trans}}$ to decide sentences involving polynomials in x and the more general transcendental function $\text{trans}(\pi(x))$, with $\pi(x) \in \mathbb{Q}[x]$, *without* introducing a new variable. This would undoubtedly be more efficient than using the transformation described in item (1) above. Some progress has already been made towards this goal by McCallum and Weispfenning (2006).

Further work contemplated includes investigating ways to make our root isolation algorithm *ISOL* more efficient, trying to make further progress on analysing the theoretical computing time of *ISOL* and *SIGNEVAL*, carrying out implementation of our decision method for $\mathcal{T}_{\arctan}$, and conducting further experimentation with the implementations.

Extension of our decision method to the full first order theory of the reals with exponentiation, for example, would of course be highly desirable. However this is probably difficult to achieve while Schanuel's conjecture (Ax, 1971; Baker, 1975; Lang, 1966) remains unresolved.

Acknowledgements

We are grateful to Melanie Achatz for kindly granting us permission to report on the implementation and testing of *DUPTP* and *DPTP*, originally described in her thesis (Achatz, 2006). We thank the referee who provided valuable suggestions for the improvement of the manuscript.

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