Journal of Pure and Applied Algebra 31 (1984) 1-4
North-Holland

PUISEUX'S THEOREM REVISITED

P.M. COHN
Dept. of Mathematics, Bedford College, Regents Park, London NW1 4NS, England

Communicated by F. Van Oystaeyen
Received December 1982

In the older accounts of algebraic function theory a central place is occupied by Puiseux's theorem, which allows one to express the branches of an algebraic function locally by fractional power series. The usual proof in this form is a laborious computation using Newton's polygon [2]. But there are also more algebraic proofs which take cognizance of the fact that the result just asserts that the field of fractional power series over the complex numbers is algebraically closed [3]. Our purpose here is to give another proof of Puiseux's theorem, expressed in matrix form. Such a form is possibly more transparent; it is also the natural one to consider when one tries to extend the theorem to skew fields, although we shall not do so here. The proof uses a matrix analogue of Hensel's lemma and another result on the normal form of matrices over fields of fractional power series (Lemma 2) which may be of independent interest.

Let k be a (commutative) field and $k((x))$ the field of formal Laurent series in x over k. We can embed $k((x))$ in $k((x^{1/n}))$ in a natural way, for any $n \geq 1$, and the fields $k((x^{1/n}))$ so obtained from a direct system whose limit will be denoted by $k\{x\}$. Now Puiseux's theorem states that $C\{x\}$ is algebraically closed. In fact most proofs apply when C is replaced by any algebraically closed field of characteristic 0. We shall prove the theorem by showing that every matrix over $k\{x\}$ has an eigenvalue. Given any polynomial f, its companion matrix has f as characteristic polynomial, and it follows that any eigenvalue of the matrix is a zero of the polynomial. Conversely, the usual form of the theorem provides a root of the characteristic equation and so an eigenvalue of the matrix. This shows that the two formulations are equivalent.

Lemma 1 ('Hensel's lemma'). *Let K be a complete valued field (with a principal valuation). Denote by V the ring of integers in K, by $\mathfrak{p}$ its maximal ideal and by $x \mapsto \bar{x}$ the natural mapping to the residue class field $k = V/\mathfrak{p}$. Given a partitioned matrix*

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

over V, assume that $\bar{A}_2=0$ and that the characteristic polynomials of $\bar{A}_1, \bar{A}_4$ are coprime, or more generally, there exists $f \in k[x]$ such that $f(\bar{A}_1)$ is a unit and $f(\bar{A}_4)=0$, then A is similar over K to a matrix of the form

$$\begin{pmatrix} A_1' & 0 \\ A_3 & A_4' \end{pmatrix}$$

i.e. A_3 is unchanged and $A_2=0$.

Proof. We have

$$\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}=\begin{pmatrix} A_1-XA_3 & A_2+A_1X-XA_4-XA_3X \\ A_3 & A_4+A_3X \end{pmatrix}. \tag{1}$$

Let f be the characteristic polynomial of A_4, then $f(A_4)=0$, hence $\bar{f}(\bar{A}_4)=0$ and $\bar{f}(\bar{A}_1)$ is a unit in k, by hypothesis, hence $f(A_1)$ is a unit in K, and by Lemma 2.3 of [1] the equation

$$M=XA_4-A_1X \tag{2}$$

has a unique solution X for any M. To prove the lemma we have to solve

$$A_2=XA_4-A_1X+XA_3X. \tag{3}$$

Let us define matrices X_r recursively by $X_0=0$ and for $r>0$,

$$X_rA_4-A_1X_r=A_2-X_{r-1}A_3X_{r-1}.$$

Since $\bar{A}_2=0$, it follows that $\bar{X}_1=0$. Moreover, we have

$$\begin{aligned}(X_{r+1}-X_r)A_4-A_1(X_{r+1}-X_r)&=X_rA_3X_r-X_{r-1}A_3X_{r-1}\\ &=(X_r-X_{r-1})A_3X_r+X_{r-1}A_3(X_r-X_{r-1}).\end{aligned}$$

Let v be the normalized valuation on K and π a uniformizer, then if $v(X_r-X_{r-1})=t$, we can divide the right-hand side by π^t.

Since $v(X_r)\geq 1$, the right-hand side is congruent to 0, so by the uniqueness of the solution of (2), $(X_r-X_{r+1})\pi^{-t}\equiv 0$, hence $v(X_r-X_{r+1})\geq t+1$. It follows that X_r converges to a solution of (3) and with this solution in (1) we get $A_2=0$, while A_3 remains unchanged.

By applying the lemma twice (or rather, once as it stands and once in transposed form) we obtain the following

Corollary. *If A is as in Lemma 1 and $\bar{A}_1, \bar{A}_4$ have coprime characteristic polynomials and $\bar{A}_2=\bar{A}_3=0$, then A is similar to a matrix for which $A_2=A_3=0$.*

We now apply Lemma 1 to obtain a normal form for matrices over a field of formal Laurent series. At this stage we need not restrict the characteristic nor assume algebraic closure.

Lemma 2. *Let k be any field and put $V=k[\![x]\!]$, $K=k((x))$. If C is an $n\times n$ matrix over V, non-singular over K, then it is similar to a diagonal sum of terms*

$$y^r(C_0+C_1y+\cdots),\quad \text{where } y=x^{1/\nu},\tag{4}$$

and the C_i have entries in k, while C_0 is non-singular.

Proof. Since C has entries in V, $\det C=x^m\gamma_0+\cdots$, where $\gamma_0\in k$, $\gamma_0\neq 0$ and $m\geq 0$. We shall use double induction: on n, and for given n, on m. For $n=1$ there is nothing to prove, so let $n>1$.

We can write C in the form

$$C=x^s(C_0+C_1x+\cdots),$$

and here we may suppose that $s=0$, since otherwise we can diminish m by dividing by x^s. If C_0 is non-singular, we have the desired conclusion, otherwise we can transform C_0 over k to the form

$$C_0=\begin{pmatrix} B & 0\\ 0 & N\end{pmatrix},$$

where B is non-singular and N is nilpotent. Thus we have

$$C=\begin{pmatrix} B+A_1x & A_2x\\ A_3x & N+A_4x\end{pmatrix},$$

where the A_i have entries in V. By the Corollary to Lemma 1 we can transform this matrix so that $A_2=A_3=0$, to obtain

$$C=\begin{pmatrix} B+A_1x & 0\\ 0 & N+A_4x\end{pmatrix}.$$

If both diagonal terms are present, we can apply induction on n to treat them separately. Since C_0 is singular, the second diagonal term is definitely present, and this only leaves the case $C=N+Ax$, where N is nilpotent; without loss of generality we may take N to be in normal form (0 or 1 in the diagonal above the main diagonal, 0's elsewhere). Let us replace x by y^n and transform by $\mathrm{diag}(1,y,y^2,\ldots,y^{n-1})$; then C becomes $C_1=Ny+A_1y$, where A has entries in V, and $\det C=\det C_1=x\cdot\det(N+A_1)$. Thus $N+A_1$ has a lower value for m and we can use induction to complete the proof.

We now come to the main result.

Theorem (Puiseux). *Let k be an algebraically closed field of characteristic 0 or greater than n. Then any $n\times n$ matrix A over the fractional power series field $k\{x\}$ has an eigenvalue.*

Proof. On multiplying A by a suitable power of x (and replacing x by x^ν if necessary) we may assume that the entries of A are in $k[x]$. By induction on n we may take A to be indecomposable over $k\{x\}$. Further, we may replace A by $A-\alpha$, where $\alpha=(\operatorname{tr} A)/n$; this leaves the problem unaffected, but we now have a matrix of trace 0. If A is singular, 0 is an eigenvalue; otherwise A is similar to

$$y^r(A_0+A_1y+\cdots),$$

where A_i has entries in k, and A_0 is in triangular form, and is non-singular, by Lemma 2. If A_0 has more than one eigenvalue we can use Lemma 1 to reduce A and then use induction on n. This leaves the case where A_0 has only one eigenvalue. Since clearly $\operatorname{tr} A_0=0$, this single eigenvalue must be 0, so A_0 is then singular, but this contradicts the fact that A_0 is non-singular, and this proves the result.

It seems likely that the result holds even in finite characteristic, although the method used here cannot be applied as it stands; its use would require a completion of $k\{x\}$, but then Lemma 1 is no longer available.

References

[1] P.M. Cohn, The similarity reduction of matrices over a skew field, Math. Z. 132 (1973) 151–163.
[2] S. Lefschetz, Algebraic Geometry (Oxford University Press, Oxford, 1953).
[3] B.L. van der Waerden, Einführung in die algebraische Geometrie, (Springer, Berlin, 1939; Dover, New York, 1945).