# On Extensions of Elementary Logic

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In this paper we present some results to the effect that certain combinations of theorems from the theory of models of elementary logic (EL) cannot be generalized to proper extensions of EL satisfying various conditions depending on the theorems in question. For example, we prove (Theorem 2) that Löwenheim's theorem together with the compactness theorem for denumerable sets of sentences cannot be extended to any generalized first order logic (defined below) which properly extends EL. Some of our results are improvements of earlier theorems of Mostowski [6] and Lindström [5].

We use the following notation and terminology. By a type we understand a sequence $t = \langle t_0, \ldots, t_{m-1} \rangle$ of positive integers; $Dt = m = \{0, \ldots, m-1\}$. A structure of type t, $\mathfrak{A} = \langle |\mathfrak{A}|, R_k^{\mathfrak{A}} \rangle_{k<m}$, consists of a non-empty set $|\mathfrak{A}|$ and relations $R_k^{\mathfrak{A}} \subseteq |\mathfrak{A}|^{t_k}$ for $k<m$; $t^{\mathfrak{A}} = t$. Structures are said to be similar if they are of the same type. By the cardinality of $\mathfrak{A}$ we understand that of $|\mathfrak{A}|$. $\mathfrak{B}$ is an expansion of $\mathfrak{A}$ if $Dt^{\mathfrak{A}} \leq Dt^{\mathfrak{B}}$ and $R_k^{\mathfrak{B}} = R_k^{\mathfrak{A}}$ for $k < Dt^{\mathfrak{A}}$. If $\pi$ is a permutation of $Dt^{\mathfrak{A}}$, then $\mathfrak{A}_\pi = \langle |\mathfrak{A}|, R_{\pi(k)}^{\mathfrak{A}} \rangle_{k<m}$, where $m = Dt^{\mathfrak{A}}$. If $t_k^{\mathfrak{A}} \geq 2$ for $k<m=Dt^{\mathfrak{A}}$ and $a \in |\mathfrak{A}|$, then $\mathfrak{A}^{(a)}$ is the structure $\langle |\mathfrak{A}|, S_k \rangle_{k<m}$ such that for any $k<m$ and any $b_0, \ldots, b_{n-2} \in |\mathfrak{A}|$, where $n = t_k^{\mathfrak{A}}$, $\langle b_0, \ldots, b_{n-2} \rangle \in S_k$ iff $\langle b_0, \ldots, b_{n-2}, a \rangle \in R_k^{\mathfrak{A}}$. K, M, N are classes of pairwise similar structures. K is said to be a free expansion of M if for some type t, $\mathfrak{A} \in K$ iff $\mathfrak{A}$ is of type t and $\mathfrak{A}$ is an expansion of a member of M. $K_\pi = \{\mathfrak{A}_\pi : \mathfrak{A} \in K\}$ and $K^+ = \{\mathfrak{A} : \mathfrak{A}^{(a)} \in K$ for every $a \in |\mathfrak{A}|\}$. $\overline{K}$ is the complement of K with respect to the class of structures similar to the members of K. K is said to characterize the class M of

structures of type t if for every $\mathfrak{A}$ of type t, $\mathfrak{A}$ is isomorphic to a member of M iff some expansion of $\mathfrak{A}$ is a member of K. K characterizes the structure $\mathfrak{B}$ if K characterizes $\{\mathfrak{B}\}$. We write $K \in F$ ($K \in F_\omega$) to mean that K is a class of structures of type $\langle 1 \rangle$ such that (K has no finite member and) if $\langle A, R \rangle \in K$, then R is finite and $\neq 0$ and for every n, there is a countable member $\langle B, S \rangle$ of K such that S is of power $n+1$.

Let $L = (\Sigma, T)$, where $\Sigma$ is an arbitrary non-empty set and T is a binary relation between members of $\Sigma$ on the one hand and structures on the other; $\Sigma_L = \Sigma$, $T_L = T$. Members of $\Sigma_L$ will be called L-sentences. If $\varphi$ is an L-sentence, then $\mathrm{Mod}_{t,L}(\varphi)$ is the class of structures of type t, $\mathfrak{A}$, such that $T_L(\varphi, \mathfrak{A})$. $K \in C_L$ means that there are $\varphi \in \Sigma_L$ and t such that $K = \mathrm{Mod}_{t,L}(\varphi)$. K ($\mathfrak{A}$) is said to be L-characterizable if some $M \in C_L$ characterizes K ($\mathfrak{A}$). We say that L is a generalized first order logic if L satisfies the following conditions:

(i) If $K \in C_L$, then K is closed under isomorphism.
(ii) If $K \in C_L$, the members of K are of type t, and $\pi$ is a permutation of Dt, then $K_\pi \in C_L$.
(iii) If $K \in C_L$ and M is a free expansion of K, then $M \in C_L$.
(iv) If $K \in C_L$, then $\overline{K} \in C_L$.
(v) If $K, M \in C_L$, then $K \cap M \in C_L$.

From now on we assume that L and L′ are generalized first order logics. L is a strong generalized first order logic if L satisfies the following condition:

(vi) If $K \in C_L$, then $K^+$ is L-characterizable.

Let t be any type. By a t-formula (t-sentence) we understand an elementary formula (sentence) with no non-logical symbols other than the predicates $P_k$ for $k < Dt$, where $P_k$ is $t_k$-ary. Let $\Gamma$ be the union of the sets of t-sentences for arbitrary t and let $T_0$ be the relation such that $T_0(\varphi, \mathfrak{A})$ iff $\varphi \in \Gamma$ and $\varphi$ holds in $\mathfrak{A}$. Set $EL = (\Gamma, T_0)$. Thus $K \in C_{EL}$ iff K is elementarily definable. Next let $Q_0, \ldots, Q_{k-1}$ be arbitrary (generalized) quantifiers and let Q be the set of the corresponding quantifier symbols. (For definitions of the relevant notions pertaining to logic with generalized quantifiers see [5]. We assume here that Q-formulas may contain the ordinary quantifier symbols and sentential connectives.) Let $\Delta$ be the set of Q-sentences and let $T_1$ be the relation such that $T_1(\varphi, \mathfrak{A})$ iff $\varphi \in \Delta$ and $\varphi$ holds in $\mathfrak{A}$. Set $L(Q) = (\Delta, T_1)$. Clearly EL and L(Q) are strong generalized first order logics. If $\varphi$ is a Q-formula with no free variables other than $v_0, \ldots, v_{m-1}$ and $x \in |\mathfrak{A}|^m$, then $\mathfrak{A} \models \varphi[x]$ means that x satisfies $\varphi$ in $\mathfrak{A}$.

L′ is an extension of L, in symbols $L \subseteq L'$, if for every L-sentence $\varphi$ and every type t, there is an L′-sentence $\psi$ such that $\mathrm{Mod}_{t,L}(\varphi) = \mathrm{Mod}_{t,L'}(\psi)$. L is equivalent to L′, $L \equiv L'$, if $L \subseteq L'$ and $L' \subseteq L$. $L \subseteq_{\inf} L'$ means that for every L-sentence $\varphi$ and every type t, there is an L′-sentence $\psi$ such that $\mathrm{Mod}_{t,L}(\varphi)$ and $\mathrm{Mod}_{t,L'}(\psi)$ have the same infinite members. $L \equiv_{\inf} L'$ iff $L \subseteq_{\inf} L'$ and $L' \subseteq_{\inf} L$.

Consider now the following conditions on L:

(I) If $K_n \in C_L$ for every n and $\bigcap_{n<\omega} K_n = 0$, then $\bigcap_{n\leq m} K_n = 0$ for some m.

(II) If $K \in C_L$ and K has an infinite member, then K has a denumerable member.

(III) If $K \in C_L$ and K has a denumerable member, then K has an uncountable member.

As is well-known, EL satisfies these conditions and certain much stronger conditions as well. In what follows we shall obtain some results in the converse direction. These results are simple consequences of the following

THEOREM 1. If L satisfies (II), $EL \subseteq_{\inf} L$, and $L \not\subseteq_{\inf} EL$, then some member of $F_\omega$ is L-characterizable.

For the proof of Theorem 1 we require the following two lemmas the first of which is due to R. Fraissé [4] and A. Ehrenfeucht [2] and the second to R. Fraissé [3]. To state these lemmas we need the following additional notation and terminology. If $x = \langle x_0, \ldots, x_{m-1} \rangle$, then $x^\frown a = \langle x_0, \ldots, x_{m-1}, a \rangle$. By an I-sequence of length m+1 for $\langle \mathfrak{A}, \mathfrak{B} \rangle$, where $\mathfrak{A}$ and $\mathfrak{B}$ are of type t, we understand a sequence $\langle I_k \rangle_{k \leq m}$ of relations such that

(1) $I_k \subseteq |\mathfrak{A}|^k \times |\mathfrak{B}|^k$ for $k \leq m$,
(2) $\langle\rangle\, I_0 \,\langle\rangle$,
(3) if $k < m$ and $xI_k y$, then for every $a \in |\mathfrak{A}|$($b \in |\mathfrak{B}|$), there is a $b \in |\mathfrak{B}|$ ($a \in |\mathfrak{A}|$) such that $x^\frown a I_{k+1} y^\frown b$,
(4) if $xI_m y$, then for any atomic t-formula $\varphi$ with no variables othen than $v_0, \ldots, v_{m-1}$, $\mathfrak{A} \models \varphi[x]$ iff $\mathfrak{B} \models \varphi[y]$.

$\langle I_k \rangle_{k<\omega}$ is an I-sequence of length $\omega$ for $\langle \mathfrak{A}, \mathfrak{B} \rangle$ if for every m, $\langle I_k \rangle_{k \leq m}$ is an I-sequence of length $m+1$ for $\langle \mathfrak{A}, \mathfrak{B} \rangle$.

LEMMA I. Let $\varkappa$ be any cardinal. If K does not have the same members of power $\varkappa$ as any member of $C_{EL}$, then for every m, there are structures $\mathfrak{A}$, $\mathfrak{B}$ of power $\varkappa$ such that $\mathfrak{A} \in K$, $\mathfrak{B} \in \overline{K}$ and an I-sequence of length $m+1$ for $\langle \mathfrak{A}, \mathfrak{B} \rangle$.

PROOF. Suppose the members of K are of type t. For $n \leq m$ we define the notions of an (m, n)-condition and of a complete (m, n)-condition as follows. An (m, m)-condition is an atomic t-formula with no variables other than $v_0, \ldots, v_{m-1}$. Let $\varphi^{(1)}$ be $\varphi$ or $\neg\varphi$ according as $i=0$ or $i=1$. Next let $\varphi_0, \ldots, \varphi_k$ be all (m, n)-conditions. Then for any $i_0, \ldots, i_k$, $\varphi_0^{(i_0)} \wedge \ldots \wedge \varphi_k^{(i_k)}$ is a complete (m, n)-condition. Finally, if $n > 0$ and $\varphi$ is a complete (m, n)-condition, then $\exists v_{n-1}\varphi$ is an (m, n − 1)-condition. No other formulas are (complete) (m, n)-conditions. Note that the free variables of an (m, n)-condition are among $v_0, \ldots, v_{n-1}$.

By hypothesis, the class of members of K of power $\varkappa$ does not coincide with the class of models of power $\varkappa$ of any disjunction of complete (m, 0)-conditions. It follows that there are $\mathfrak{A}$ and $\mathfrak{B}$ of power $\varkappa$ such that $\mathfrak{A} \in K$, $\mathfrak{B} \in \overline{K}$, and the same (m, 0)-conditions hold in $\mathfrak{A}$ and $\mathfrak{B}$. Now for $k \leq m$, define the relation $I_k$ thus: $xI_k y$ iff $x \in |\mathfrak{A}|^k$, $y \in |\mathfrak{B}|^k$, and for every (m, k)-condition $\varphi$, $\mathfrak{A} \models \varphi[x]$ iff $\mathfrak{B} \models \varphi[y]$. Obviously, the sequence $\langle I_k \rangle_{k \leq m}$ satisfies conditions (1), (2), and (4). To show that it also satisfies (3) suppose $k < m$, $xI_k y$, and $a \in |\mathfrak{A}|$. Let $\varphi$ be the complete (m, k + 1)-condition such that $\mathfrak{A} \models \varphi[x^\frown a]$. Then $\psi = \exists v_k \varphi$ is an (m, k)-condition and $\mathfrak{A} \models \psi[x]$. Hence $\mathfrak{B} \models \psi[y]$, whence there is a $b \in |\mathfrak{B}|$ such that $\mathfrak{B} \models \varphi[y^\frown b]$. Clearly $x^\frown a I_{k+1} y^\frown b$. This proves one half of (3). The proof of the other half is the same. Thus

$\langle I_k \rangle_{k \leq m}$ is an I-sequence of length $m+1$ for $\langle \mathfrak{A}, \mathfrak{B} \rangle$ and Lemma 1 is proved.

LEMMA 2. If $\mathfrak{A}$ and $\mathfrak{B}$ are denumerable and there is an I-sequence of length $\omega$ for $\langle \mathfrak{A}, \mathfrak{B} \rangle$, then $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

PROOF. Let $|\mathfrak{A}| = \{a_n\colon n \in \omega\}$, $|\mathfrak{B}| = \{b_n\colon n \in \omega\}$, and let $\langle I_k \rangle_{k<\omega}$ be an I-sequence for $\langle \mathfrak{A}, \mathfrak{B} \rangle$. We define sequences $\langle c_n \rangle_{n<\omega}$ and $\langle d_n \rangle_{n<\omega}$ such that for every n,

(5) $c_{2n} = a_n$,
(6) $d_{2n+1} = b_n$,
(7) $\langle c_0, \ldots, c_{n-1} \rangle \, I_n \langle d_0, \ldots, d_{n-1} \rangle$

as follows. Suppose $c_n$ and $d_n$ have been defined for $n<k$. If k is even, $k=2r$, set $c_k = a_r$. By (3), there is a least index s such that $\langle c_0, \ldots, c_k \rangle \, I_{k+1} \langle d_0, \ldots, d_{k-1}, b_s \rangle$. Set $d_k = b_s$. If, on the other hand, k is odd, $k=2r+1$, set $d_k = b_r$. Again by (3), and (2) if $r=0$, there is a least index s such that $\langle c_0, \ldots, c_{k-1}, a_s \rangle \, I_{k+1} \langle d_0, \ldots, d_k \rangle$. Set $c_k = a_s$.

Now let $f = \{\langle c_n, d_n \rangle\colon n \in \omega\}$. Then, by (4)—(7), f is an isomorphism on $\mathfrak{A}$ onto $\mathfrak{B}$.

PROOF OF THEOREM I. Let $K_0 \in C_L$ be such that there is no $M \in C_{EL}$ such that $K_0$ and M have the same infinite members. Then

(8) there is no $M \in C_{EL}$ such that $K_0$ and M have the same denumerable members.

Indeed, Suppose $M \in C_{EL}$. There is then a class $M_0 \in C_L$ such that M and $M_0$ have the same infinite members. Let $M_1 = (\overline{K}_0 \cap M_0) \cup \cup (K_0 \cap \overline{M}_0)$. By (iv) and (v), $M_1 \in C_L$. Clearly $M_1$ has an infinite member. Hence, by (II), $M_1$ has a denumerable member and so (8) follows.

Suppose, for simplicity, that the members of $K_0$ are of type $\langle 2 \rangle$. Let $K_1$ be the class of structures $\langle A, R_k \rangle_{k<7}$ of type $t = \langle 1, 2, 2, 2, 2, 3, 3 \rangle$ such that

(9) $R_0$ is non-empty,
(10) $R_3$ is a one-one function on A into a proper subset of A,

(11) $R_4$ is a linear ordering of $R_0$ such that $R_0$ has an $R_4$-first member and every member of $R_0$ which has an $R_4$-successor has an immediate $R_4$-successor,

(12) for every $a \in A$, the relation $f_a = \{\langle x, y\rangle : \langle a, x, y\rangle \in R_5\}$ is a function on $R_0$ into A,

(13) if x is the $R_4$-first member of $R_0$, then there are a, b such that $\langle x, a, b\rangle \in R_6$,

(14) if $\langle x, a, b\rangle \in R_6$, $x \in R_0$, y is the immediate $R_4$-successor of x, and z is any member of A, then there are c, d, u such that $\langle y, c, d\rangle \in R_6$, $f_c(y) = z$, $f_d(y) = u$, and for every $v \in R_0$, if $v \neq y$, then $f_c(v) = f_a(v)$ and $f_d(v) = f_b(v)$,

(15) (Like (14) except that a and b are interchanged.),

(16) if $\langle x, a, b\rangle \in R_6$ and y, z are $R_4$-predecessors of x, then $\langle f_a(y), f_a(z)\rangle \in R_1$ iff $\langle f_b(y), f_b(z)\rangle \in R_2$.

It is easily seen that $K_1 \in C_{EL}$ and that it has no finite members, whence $K_1 \in C_L$. Next, it follows at once from (ii)—(v) that there is a class $K_2 \in C_L$ of structures of type t such that $\mathfrak{A} \in K_2$ iff $\langle |\mathfrak{A}|, R_1^{\mathfrak{A}}\rangle \in K_0$ and $\langle |\mathfrak{A}|, R_2^{\mathfrak{A}}\rangle \in \overline{K}_0$. Now set $K = K_1 \cap K_2$. Then, by (v), $K \in C_L$. We propose to show that K characterizes a member of $F_\omega$.

Let n be any natural number $\neq 0$. By (8) and Lemma 1, there are relations $R_1, R_2 \subseteq \omega^2$ such that $\mathfrak{A} = \langle \omega, R_1\rangle \in K_0$, $\mathfrak{B} = \langle \omega, R_2\rangle \in \overline{K}_0$, and an I-sequence $\langle I_k\rangle_{k<n}$ of length n for $\langle \mathfrak{A}, \mathfrak{B}\rangle$. Let $R_0 = n$ and let $R_4$ be the $<$-relation restricted to $R_0$. Let g be a one-one function on $\omega$ onto the set of functions on $R_0$ into $\omega$. We write $g_x$ for $g(x)$. Next let $R_5 = \{\langle m, x, y\rangle : x \in R_0 \text{ and } g_m(x) = y\}$. Let $R_6 = \{\langle m, x, y\rangle : m \in R_0 \text{ and } \langle g_x(0), \ldots, g_x(m-1)\rangle\ I_m\ \langle g_y(0), \ldots, g_y(m-1)\rangle\}$. Finally, let $R_3$ be a one-one function on $\omega$ into a proper subset of $\omega$. The verification that $\langle \omega, R_k\rangle_{k<7}$ is a member of K presents no difficulties.

Suppose now, for reductio ad absurdum, that there is a structure $\mathfrak{A} \in K$ such that $R_0^{\mathfrak{A}}$ is infinite. Let $M_1$ be the free expansion of K whose members are of type $t' = t^\frown 2$. Next let $M_2$ be the class of structures $\mathfrak{B}$ of type $t'$ such that $R_7^{\mathfrak{B}}$ is a one-one function on $R_0^{\mathfrak{B}}$ into a proper subset of $R_0^{\mathfrak{B}}$. Clearly $M_0 = M_1 \cap M_2 \in C_L$ and $M_0$ has an infinite member. Hence, by (II), $M_0$ has a denumerable member $\mathfrak{C}$. Clearly $R_0^{\mathfrak{C}}$ is infinite. Hence, by (11), for every n, there is a member $c_n$ of $R_0^{\mathfrak{C}}$ which has exactly n $R_4^{\mathfrak{C}}$-predecessors. For $a \in |\mathfrak{C}|$ let $f_a = \{\langle x, y\rangle : \langle a, x, y\rangle \in R_5^{\mathfrak{C}}\}$. Now, for each n, define the relation $I_n$ as follows: $\langle a_0, \ldots, a_{n-1}\rangle I_n \langle b_0, \ldots, b_{n-1}\rangle$ iff there are $a, b \in |\mathfrak{C}|$ such that $\langle c_n, a, b\rangle \in R_6^{\mathfrak{C}}$ and $f_a(c_k) = a_k$ and $f_b(c_k) = b_k$ for $k < n$. It is then easily checked, using (12)—(16), that $\langle I_n\rangle_{n<\omega}$ is an I-sequence of length $\omega$ for $\langle \mathfrak{C}_1, \mathfrak{C}_2\rangle$, where $\mathfrak{C}_m = \langle |\mathfrak{C}|, R_m^{\mathfrak{C}}\rangle$, $m = 1, 2$. Hence, by Lemma 2, $\mathfrak{C}_1$ is isomorphic to $\mathfrak{C}_2$. But $\mathfrak{C}_1 \in K_0$ and $\mathfrak{C}_2 \in \overline{K}_0$. Hence, by (i), $\mathfrak{C}_1$ is not isomorphic to $\mathfrak{C}_2$. A contradiction from which it follows that if $\mathfrak{A} \in K$, then $R_0^{\mathfrak{A}}$ is finite. Since, finally, in view of (9) and (10), if $\mathfrak{A} \in K$, then $\mathfrak{A}$ is infinite and $R_0^{\mathfrak{A}} \neq 0$, this concludes our proof that K characterizes a member of $F_\omega$.

COROLLARY 1. If L satisfies (II), $EL \subseteq L$, and $L \nsubseteq EL$, then some member of F is L-characterizable.

PROOF. If $L \nsubseteq_{\inf} EL$, then the conclusion follows from Theorem 1. Suppose then $L \subseteq_{\inf} EL$. There are then classes K, M such that $K \in C_L$, $K \notin C_{EL}$, $M \in C_{EL}$, and K and M have the same infinite members. It follows that for every m, there is an $n > m$ such that K and M does not have the same members of power n. Indeed, otherwise, as is easily seen, we would have $K \in C_{EL}$. Suppose the members of K are of type $t = \langle t_0, \ldots, t_{m-1}\rangle$. Set $t' = \langle 1, t_0, \ldots, t_{m-1}\rangle$. Let N be the class of structures such that $\mathfrak{A} \in N$ iff $\mathfrak{A}$ is of type $t'$, $R_0^{\mathfrak{A}} \neq 0$ and $\langle |\mathfrak{A}|, R_{k+1}^{\mathfrak{A}}\rangle_{k<Dt} \in (\overline{K} \cap M) \cup (K \cap \overline{M})$. Then, clearly, $N \in C_L$ and N characterizes a member of F.

COROLLARY 2. If L is strong, L satisfies (II), $EL \subseteq_{\inf} L$, and $L \nsubseteq_{\inf} EL$, then $\langle \omega, \leq\rangle$ is L-characterizable.

PROOF. By Theorem 1, there is a class $K \in C_L$ such that K characterizes a member of $F_\omega$. Since L is strong, it follows that there is a class $M \in C_L$ which characterizes $K^+$. Let N be the class of structures of the same type as the members of M, $\mathfrak{A}$, such that $R_0^{\mathfrak{A}}$ is a reflexive linear ordering of $|\mathfrak{A}|$ such that every member of $|\mathfrak{A}|$ has an $R_0^{\mathfrak{A}}$-successor. Then $M \cap N \in C_L$ and characterizes $\langle \omega, \leq\rangle$.

From Corollary 1 we can now easily derive the following

THEOREM 2. If L satisfies (I) and (II) and $EL \subseteq L$, then $L \equiv EL$.

PROOF. Suppose L satisfies (II), $EL \subseteq L$, and $L \not\subseteq EL$. Then, by Corollary 1, there is a class $K_0 \in C_L$ which characterizes a member of F. Let t be the type of the members of $K_0$. For $n > 0$ let $K_n$ be the class of structures $\mathfrak{A}$ of type t such that $R_0^{\mathfrak{A}}$ has at least n members. Then clearly $K_n \in C_L$ for every n, $\bigcap_{n<\omega} K_n = 0$, and $\bigcap_{n\leq m} K_n \neq 0$ for every m. Thus L does not satisfy (I).

If K characterizes $\langle \omega, \leq \rangle$, then, obviously, K has a denumerable member but no uncountable member. Hence, in view of Corollary 2, we have the following

THEOREM 3. If L is strong, L satisfies (II) and (III), and $EL \subseteq_{inf} L$, then $L \equiv_{inf} EL$.

Theorems 2 and 3 are improvements of Theorems 5.1 and 5.6 [5]. The present proofs are, however, almost the same as those given in [5].

In order to be able to apply the concepts of recursive function theory we now assume that the members of $\Sigma_L$ and $\Sigma_{L'}$ are finite configurations of symbols from certain given finite sets or, equivalently, natural numbers. We write $L \subseteq_{eff} L'$ to mean that there is an effective method whereby, given any type t and any L-sentence $\varphi$, an L'-sentence $\psi$ can be found such that $Mod_{t,L}(\varphi) = Mod_{t,L'}(\psi)$. $L \equiv_{eff} L'$ iff $L \subseteq_{eff} L'$ and $L' \subseteq_{eff} L$. L is n.c.-effective if (a) there is an effective method by means of which for any type t and any L-sentence $\varphi$, an L-sentence $\psi$ can be found such that $Mod_{t,L}(\psi) = \overline{Mod_{t,L}(\varphi)}$ and (b) there is an effective method by means of which for any type t and any L-sentences $\varphi$, $\psi$, an L-sentence $\theta$ can be found such that $Mod_{t,L}(\theta) = Mod_{t,L}(\varphi) \cap Mod_{t,L}(\psi)$. Clearly $EL \subseteq_{eff} L(Q)$ and EL and L(Q) are both n.c.-effective. Set $V_L = \{\langle \varphi, t \rangle$: t is any type, $\varphi \in \Sigma_L$, and $T_L(\varphi, \mathfrak{A})$ for every $\mathfrak{A}$ of type t$\}$. L is said to be axiomatizable if $V_L$ is recursively enumerable. As is well-known, EL is axiomatizable. In the converse direction we have the following

THEOREM 4. If L is n.c.-effective and axiomatizable, L satisfies (II), and $EL \subseteq_{eff} L$, then $L \equiv_{eff} EL$.

In the proof of Theorem 4 we use the following result due to Trakhtenbrot [7]. A t-sentence is said to be finitely valid if it holds in all finite structures of type t.

LEMMA 3. There is a type t such that the set of finitely valid t-sentences is not recursively enumerable.

PROOF OF THEOREM 4. We first prove that $L \subseteq EL$. Suppose not. Then, by Corollary 1, there is an L-sentence $\theta$ and a type $t' = \langle t'_0, \ldots, t'_{m'-1} \rangle$ such that $\mathrm{Mod}_{t',L}(\theta)$ characterizes a member of F. Let $t = \langle t_0, \ldots, t_{m-1} \rangle$ be as in Lemma 3, set $t^+ = \langle t'_0, \ldots, t'_{m'-1}, t_0, \ldots, t_{m-1} \rangle$, and let $\theta^+$ be an L-sentence such that $\mathrm{Mod}_{t^+,L}(\theta^+)$ is a free expansion of $\mathrm{Mod}_{t',L}(\theta)$. Now let $\varphi$ be any t-sentence. For every r, replace $P_r$ everywhere in $\varphi$ by $P_{m'+r}$ and then relativize all quantifier expressions to $P_0$, i.e. replace $\exists v_k \psi$ by $\exists v_k(P_0 v_k \wedge \psi)$ and $\forall v_k \psi$ by $\forall v_k(P_0 v_k \rightarrow \psi)$. Let $\varphi_0$ be the sentence thus obtained. Since $EL \subseteq_{\mathrm{eff}} L$, we can now effectively find an L-sentence $\psi$ such that $\mathrm{Mod}_{t^+,L}(\psi) = \mathrm{Mod}_{t^+,EL}(\varphi_0)$. Next, since L is n.c.-effective, we can find an L-sentence $\eta$ such that

$$\mathrm{Mod}_{t^+,L}(\eta) = \overline{\mathrm{Mod}_{t^+,L}(\theta^+)} \cup \mathrm{Mod}_{t^+,L}(\psi).$$

Clearly, $\langle \eta, t^+ \rangle \in V_L$ iff $\varphi$ is finitely valid. Since $\eta$ was found effectively from $\varphi$ and since $V_L$ is recursively enumerable, we may now conclude that the set of finitely valid t-sentences is recursively enumerable. But this contradicts Lemma 3. Thus the assumption that $L \not\subseteq EL$ was false and so $L \subseteq EL$.

That $L \subseteq_{\mathrm{eff}} EL$ can now be shown as follows. Let $\varphi$ be any L-sentence and t any type. Given t we can obviously find an effective enumeration $\psi_0, \psi_1, \psi_2, \ldots$ of all t-sentences. Next, for each n, an L-sentence $\eta_n$ can be found such that $\mathrm{Mod}_{t,L}(\eta_n) = \mathrm{Mod}_{t,EL}(\psi_n)$. Finally, we can find L-sentences $\xi_n$ such that for every n,

$$\mathrm{Mod}_{t,L}(\xi_n) = (\mathrm{Mod}_{t,L}(\varphi) \cap \mathrm{Mod}_{t,L}(\eta_n)) \cup \cup (\overline{\mathrm{Mod}_{t,L}(\varphi)} \cap \overline{\mathrm{Mod}_{t,L}(\eta_n)}).$$

Clearly $\langle \xi_n, t \rangle \in V_L$ iff $\mathrm{Mod}_{t,L}(\varphi) = \mathrm{Mod}_{t,EL}(\psi_n)$. But, since $L \subseteq EL$, there is an n for which this equation holds. Hence, since $V_L$ is recursively enumerable, we can find an n such that $\langle \xi_n, t \rangle \in$

$\in V_L$. Thus, given a type t and an L-sentence $\varphi$ a t-sentence $\psi_n$ can be found effectively such that $\mathrm{Mod}_{t,EL}(\psi_n) = \mathrm{Mod}_{t,L}(\varphi)$; in other words, $L \subseteq_{\mathrm{eff}} EL$, as was to be proved.

Theorem 4 is a generalization of a theorem of Mostowski (Theorem 4 [6]).

We conclude this paper by discussing the possibility of extending Beth's theorem on definability together with (II) to proper extensions of EL of the form L(Q). Let $\varphi$ be an L(Q)-sentence which contains the q-ary predicate $P_s$. $\varphi$ is said to define $P_s$ implicitly if for any two structures $\mathfrak{A}$ and $\mathfrak{B}$, if $t^{\mathfrak{A}} = t^{\mathfrak{B}}$, $\mathfrak{A} \models \varphi$, $\mathfrak{B} \models \varphi$, $|\mathfrak{A}| = |\mathfrak{B}|$, and $R_k^{\mathfrak{A}} = R_k^{\mathfrak{B}}$ for $k < Dt^{\mathfrak{A}}$ and $k \neq s$, then $R_s^{\mathfrak{A}} = R_s^{\mathfrak{B}}$. $\varphi$ is said to L(Q)-define $P_s$ explicitly if there is a Q-formula $\psi$ which contains no predicates other than those occurring in $\varphi$, does not contain $P_s$, contains no free variables othen than $v_0, \ldots, v_{q-1}$, and is such that

$$\mathfrak{A} \models \varphi \rightarrow \forall v_0 \ldots v_{q-1}(P_s v_0 \ldots v_{q-1} \leftrightarrow \psi),$$

for every structure $\mathfrak{A}$ of suitable type. We say that L(Q) has Beth's property if for every L(Q)-sentence $\varphi$ and every predicate $P_s$, if $\varphi$ defines $P_s$ implicitly, then $\varphi$ L(Q)-defines $P_s$ explicitly. As is well-known, EL (Q empty) has Beth's property. In the converse direction we have the following

THEOREM 5. If L(Q) has Beth's property and satisfies (II), then $L(Q) \equiv_{\mathrm{inf}} EL$.

PROOF IN OUTLINE. Let $L = L(Q)$. As noted above, L is a strong generalized first order logic. Suppose L satisfies (II) and $L \not\subseteq_{\mathrm{inf}} EL$. Then, by Corollary 2, there is an L-sentence $\theta$ and a type t such that $\mathrm{Mod}_{t,L}(\theta)$ characterizes $\langle \omega, \leq \rangle$. Suppose now we have defined a Gödel numbering of the Q-formulas satisfying the usual effectiveness conditions. Then, by a suitable modification of the construction used in the proof of Theorem 2.1[1] and using the sentence $\theta$, we can find a type t′ and an L-sentence $\varphi$ such that $\mathrm{Mod}_{t',L}(\varphi)$ is non-empty and if $\mathfrak{A} \in \mathrm{Mod}_{t',L}(\varphi)$, then there is a structure $\mathfrak{B}$ isomorphic to $\mathfrak{A}$ such that $|\mathfrak{B}| = \omega$, $R_0^{\mathfrak{B}} = \{\langle m, n \rangle :$ $: m \leq n\}$, $R_1^{\mathfrak{B}} = \{\langle k, m, n \rangle : k + m = n\}$, $R_2^{\mathfrak{B}} = \{k, m, n\rangle : k \cdot m = n\}$, and $R_3^{\mathfrak{B}} = \{\langle m, n \rangle$ : n is the Gödel number of a formula $\psi$ such that $\psi$ does not contain $P_3$ and for some r, $\psi$ contains no free variables other than $v_0, \ldots, v_{r-1}$, $m = 2^{m_0} \cdot \ldots \cdot p_{r-1}^{m_{r-1}}$, where $p_{k-1}$ is the $k$th prime, and $\mathfrak{B} \models \psi[m_0, \ldots, m_{r-1}]\}$. Clearly $\varphi$ defines $P_3$ implicitly. Suppose now L has Beth's property and let $\mathfrak{B}$ be as described. There is then a Q-formula $\eta$ not containing $P_3$ such that

$$\mathfrak{B} \models \forall v_0 v_1 (P_3 v_0 v_1 \leftrightarrow \eta).$$

It follows that there is a Q-formula $\xi$ with no free variable other than $v_0$ and not containing $P_3$ such that for every n, $\mathfrak{B} \models \xi[n]$ iff $\langle 2^n, n \rangle \in R_3^{\mathfrak{B}}$. Let p be the Gödel number of $\neg\xi$. Then $\mathfrak{B} \models \xi[p]$ iff $\langle 2^p, p \rangle \in R_3^{\mathfrak{B}}$ iff $\mathfrak{B} \models \neg\xi[p]$; a contradiction. Thus L does not have Beth's property and our proof is finished.

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