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# EXPONENTIAL POLYNOMIALS

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The exponential numbers considered in another paper[1] suggested the extensive generalizations to the polynomials of the present paper. The term "exponential polynomial" here refers to polynomials generated by operating on exponential functions exp $(f(x))$ by differentiation, or by expanding the exponential into a power series in $x$.

The polynomials $\xi$(§§1–3) are one of the many possible generalizations of Hermite's polynomials. From the $\xi$ are constructed functions $\zeta(t)$ which are orthogonal in the interval $-\infty \leqq t \leqq +\infty$. With respect to linear differential equations the Hermite polynomials are a sort of singular case of the $\xi$, see §2.

The polynomials $\phi$ generalize the $\xi$, and are a generalization of Appell polynomials; see §§4–6, 10. Closely connected with the $\phi$ are the polynomials $Y$ (§§7–9) which generalize the $\xi$ and (in a sense which will be clear from (7.16)), also the $\phi$. They are introduced in connection with a certain determinant in (6.5) on which the arithmetical properties of the $\phi$ depend. The $Y$ also give rise to a set of orthogonal functions. Neither $\phi_n$ nor $Y_n$ satisfies any differential equation whose order is independent of $n$. The number of terms in $\phi_n$ is the total number of partitions of $n$; the sum of all the coefficients is the positive integer $\epsilon_n$, and the $\epsilon$'s are one of the sequences whose congruence properties were discussed in the paper mentioned.

The polynomials $\psi$ (§§11, 12) are a generalization, in another direction, of Appell polynomials. The possible types of differential equations which, under suitable hypotheses, the $\psi$ can satisfy, are determined, and the degeneration in a special case to Appell's equations is noted.

All of the polynomials have interesting arithmetical properties, some of which are discussed.

## I. The polynomials $\xi$

1. Let $r$ denote a constant integer $> 0$, $n$ an arbitrary integer $\geqq 0$, and $x$, $t$ independent variables; write $D_t \equiv d/dt$. The polynomials $\xi_n \equiv \xi_n(x, t; r)$ are defined by $\xi_{-n-1} = 0$, and

$$(1.1) \qquad \xi_n \equiv \exp(-xt^r)D_t^n(\exp(xt^r)).$$

[1] To be published in 1934; see the abstract in the *Bulletin of the American Mathematical Society*, vol. 39, 1933, p. 667.

For $r = 2$, the $\xi_n$ are Hermite polynomials; for $r > 2$ the $\xi_n$ do not seem to have been discussed, although Hoppe[2] gave the equivalent of the explicit form (1.6) as an illustration of his method of obtaining higher derivatives without, however, investigating any of the properties of the polynomials.

Operation with $D_t$ on both sides of (1.1) gives

$$\xi_{n+1} = rxt^{r-1} + D_t\xi_n, \tag{1.2}$$

from which it follows by mathematical induction that

$$t^n\xi_n = \sum_{i=0}^{n} c_i(n)\ (xt^r)^{n-i}, \tag{1.3}$$

where $c_i(n)$ is a polynomial of degree $n$ in $r$ with integer coefficients, and

$$c_0(0) = 1, \qquad c_0(n+1) = rc_0(n), \tag{1.4}$$
$$c_j(n+1) = rc_j(n) - [n - r(n-j+1)]c_{j-1}(n) \qquad (0 < j < n+1).$$

The $c$'s can be calculated successively from (1.4); thus

$$c_0(n) = r^n, \qquad c_1(n) = \binom{n}{2}(r-1)r^{n-1}, \cdots;$$

but it is shorter to obtain the explicit forms otherwise, as in (1.6). For numerical checks we write down the following from $\xi_0 = 1$ and (1.2).

$$\begin{aligned} \xi_1 &= rxt^{r-1}, \\ \xi_2 &= r^2x^2t^{2r-2} + r(r-1)xt^{r-2}, \\ \xi_3 &= r^3x^3t^{3r-3} + 3r^2(r-1)x^2t^{2r-3} + r(r-1)(r-2)xt^{r-3}. \end{aligned}$$

From (1.3),

$$\xi_n = \sum_{i=0} c_i(n)\, x^{n-i}\, t^{(n-i)r-n}, \tag{1.5}$$

where the sum continues so long as $n - i \geqq 0$, $(n-i)r - n \geqq 0$. Hence, if $[y]$ denotes the greatest integer in $y$, the upper limit of the summation is $[n\,(r-1)/r]$.

To obtain the explicit form, multiply

$$D_t^n \exp(xt^r) = n! \sum_{m \geqq [n/r]}^{\infty} \frac{1}{m!}\binom{rm}{n} x^m t^{rm-n}$$

[2] R. Hoppe, *Journal für Mathematik*, vol. 33, 1846, pp. 76–79. I owe this reference to Professor Bateman, who has given an alternative expression for the $\xi_n$ involving Stirling polynomials, which follows readily from (1.6) here. Professor Bateman has also discussed the polynomials $\exp(-xt^r)D_t^n[x^{sr}\exp(xt^r)]$, where $s$ is an interger $> 0$.

by the power series for $\exp(-xt^r)$; thus

$$\xi_n = n! \sum_{h=0}^{\infty} \frac{x^h t^{rh-n}}{h!} \sum_{s=0}^{\leqq h-1} (-1)^s \binom{h}{s} \binom{r(h-s)}{n}.$$

The last binomial coefficient is zero unless $r(h - s) \geqq n$, and by (1.5) the degree in $x$ is $n$. Hence finally,

$$\xi_n = n! \sum_{h=a}^{n} \frac{x^h t^{rh-n}}{h!} \sum_{s=0}^{b} (-1)^s \binom{h}{s} \binom{r(h-s)}{n}, \tag{1.6}$$

$$a \equiv n - [n(r-1)/r], \qquad b = [(rh-n)/r].$$

It follows incidentally that

$$\sum_{s=0}^{b} (-1)^s \binom{h}{s} \binom{r(h-s)}{n} = 0, \qquad h > n,$$

which is a property of the Stirling polynomials.

The generating function for $\xi_n$ follows from (1.1) by Taylor's theorem,

$$\exp\,[x((h+t)^r - t^r)] = \exp\,(h\xi), \tag{1.7}$$

where the exponential on the right is $\exp\,(h\xi) \equiv \xi_0 + h\xi_1 + h^2\xi_2/2! + \cdots$, as usual in the symbolic or umbral calculus of Blissard.[3]

Operating on (1.7) with $D_h, D_t, D_x$ respectively, we get

$$\xi_{n+1} = n!\, rx \sum_{s=0}^{r-1} \binom{r-1}{s} t^s \frac{\xi_{n-r+s+1}}{(n-r+s+1)!},$$

$$D_t\xi_n = n!\, rx \sum_{s=0}^{r-2} \binom{r-1}{s} t^s \frac{\xi_{n-r+s+1}}{(n-r+s+1)!}, \qquad r > 1,$$

$$D_x\xi_n = n! \sum_{s=0}^{r-1} \binom{r}{s} t^s \frac{\xi_{n-r+s}}{(n-r+s)!},$$

in which (by the notation explained) $\xi_j \equiv \xi_j(x, t; r)$, and the sums either continue

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[3] The account in Lucas' *Théorie des Nombres*, Chap. 13, may be supplemented by my own extensions in *Algebraic Arithmetic*, 1927. Umbral derivatives, as in

$$D_\xi \exp\,(h\xi) = \xi \exp\,(h\xi)$$

satisfy the formal laws of differentiation; all indicated operations upon umbrae are to be performed as in common algebra before exponents are lowered, and if $a, b$ are ordinaries such that $ab \neq 0$, and $\alpha, \beta$ umbrae, the value of

$$(a\alpha + b\beta)^0 \text{ is } a^0\alpha^0 b^0 \beta^0, = \alpha_0\beta_0.$$

The last avoids exceptions in the manipulations and results. The formal use of infinite processes in this connection is justified in the work cited.

to the upper limits shown, or terminate with the largest $s$ which makes the suffix of $\xi$ non-negative. These may be written

$$\xi_{n+1} = rx \sum_{s=0}^{r-1} (r - s - 1)! \binom{r-1}{s} \binom{n}{r-s-1} t^s \xi_{n-r+s+1}, \tag{1.8}$$

$$D_t \xi_n = rx \sum_{s=0}^{r-2} (r - s - 1)! \binom{r-1}{s} \binom{n}{r-s-1} t^s \xi_{n-r+s+1}, \qquad r > 1, \tag{1.9}$$

$$D_x \xi_n = \sum_{s=0}^{r-1} (r - s)! \binom{r}{s} \binom{n}{r-s} t^s \xi_{n-r+s}. \tag{1.10}$$

We may write (1.7) more symmetrically as

$$\exp [x(h^r - t^r)] = \exp [(h - t)\xi], \tag{1.11}$$

from which (1.6) follows by equating coefficients of $h^n$. We may take (1.11) as the definition of the polynomials $\xi_n \equiv \xi_n(x, t; r)$, since (1.8), (1.9), which follow from (1.7), and hence from (1.11), imply (1.2). Replacing $x$ by $x + y$ in (1.11) we have the addition theorem

$$\xi_n(x + y, t; r) = [\xi(x, t; r) + \xi(y, t; r)]^n, \tag{1.12}$$

and hence

$$\sum_{s=0}^{n} \binom{n}{s} \xi_{n-s} (x, t; r) \, \xi_s (-x, t; r) = \delta_{0n}, \tag{1.13}$$

where $\delta_{00} = 1$, $\delta_{0n} = 0$ $(n > 0)$. In (1.7) replace $h$, $t$ by $ch$, $ct$, where $c$ is independent of $h$. Then

$$\xi_n(c^r x, t; r) = c^n \xi_n(x, ct; r). \tag{1.14}$$

Substitution of $t^{-1}$ for $t$ in (1.7) gives

$$\xi_n(x, t^{-1}; r) = t^n \xi_n(xt^{-r}, 1; r). \tag{1.15}$$

Similarly we have

$$\xi_{sr}(x, 0; r) = \frac{(sr)!}{s!} x^s; \; \xi_n(x, 0; r) = 0, \quad n \not\equiv 0 \bmod r. \tag{1.16}$$

From (1.14),

$$\xi_n(c^r x, t; r) + a_1 \xi_{n-1}(c^r x, t; r) + \cdots + a_n \xi_0(c^r x, t; r) \equiv 0, \tag{1.17}$$

identically in $x$, $t$, where $c$ is any root of $c^n + a_1 c^{n-1} + \cdots + a_n = 0$. This is equivalent, by (1.6), to a set of identities in Stirling polynomials.

2. From (1.6), (1.8), (1.9) we readily find that

$$D_t \xi_n(x, t; 1) = 0,$$
$$D_t^2 \xi_n(x, t; 2) + 2xt D_t \xi_n(x, t; 2) - 2nx \xi_n(x, t; 2) = 0.$$

the second of which is Hermite's equation when $2x = -1$. Using (1.10) we get

$$xD_x\xi_n(x, t; 1) - n\xi_n(x, t; 1) = 0.$$

*It will now be shown that if $r > 2$, the polynomials $\xi_n(x, t; r) \equiv \xi_n$ satisfy no linear equation of constant order $m > 0$ of the form*

$$P_0 D_t^m \xi_n + P_1 D_t^{m-1}\xi_n + \cdots + P_m\xi_n = P_{m+1}, \tag{2.1}$$

*where the $P$'s are polynomials in $r, n, x, t$ alone* (the case where some or all $P$'s are constants included), *and if $r > 1$, the $\xi_n$ satisfy no equation of the form*

$$P_0 D_x^m \xi_n + P_1 D_x^{m-1}\xi_n + \cdots + P_m\xi_n = P_{m+1}. \tag{2.2}$$

Thus, with respect to the property (2.1), $x^n$ and the Hermite polynomials (with the parameter $x$) are unique among the $\xi_n(x, t; r)$, and $x^n$ is unique with respect to (2.2).

Let $r$ be $> 1$, $m > 0$. By repeated application of (1.9) it follows that $D_t^m\xi_n$, considered as a function of $\xi$'s, is linear in $\xi_{n-j}$ $(j = m, m+1, \cdots, m(r-1))$ alone, where account is taken of $\xi_{-s} = 0$, $s > 0$, say $D_t^m\xi_n = \lambda_m(\xi_{n-m}, \xi_{n-m-1}, \cdots, \xi_{n-m(r-1)})$, and that the coefficients of the $\xi$'s in $\lambda_m$ are polynomials in $r, n, x, t$ alone. Similarly, from (1.8),

$$D_t^0\xi_n \equiv \xi_n = \lambda_0(\xi_{n-1}, \xi_{n-2}, \cdots, \xi_{n-r}),$$

where $\lambda_0$ is a linear function of the same kind as $\lambda_m$. Hence

$$\begin{aligned} D_t^0\xi_n &= \lambda_0(\xi_{n-1}, \xi_{n-2}, \cdots, \xi_{n-r}), \\ D_t^1\xi_n &= \lambda_1(\xi_{n-1}, \xi_{n-2}, \cdots, \xi_{n-(r-1)}), \\ &\cdots \\ D_t^m\xi_n &= \lambda_m(\xi_{n-1}, \xi_{n-2}, \cdots, \xi_{n-m(r-1)}). \end{aligned} \tag{2.3}$$

Since $m, r$ are constant integers $> 0$, we can choose $n$ so that $n - m(r-1) \geqq 0$. If (2.1) holds, it must be possible to eliminate $\xi_{n-1}, \xi_{n-2}, \cdots, \xi_{n-m(r-1)}$ from the $(m+1)$ equations (2.3). But these $m(r-1)$ $\xi$'s are distinct. Hence, in order that the elimination be always possible, it is necessary that $m(r-1) = m$, or $m(r-2) = 0$. But $m > 0$; hence $r = 2$. Similarly for (2.2).

3. In this section we show that if

$$\zeta_n \equiv \zeta_n(x, t; 2r) \equiv e^{-xt^{2r}/2}\,\xi_n(-x, t; 2r), \qquad x > 0, \tag{3.1}$$

then the $\zeta_n$ form an orthogonal set in the interval $-\infty \leqq t \leqq +\infty$,

$$\int_{-\infty}^{\infty} \zeta_m\zeta_n\, dt = 0, \qquad m \neq n. \tag{3.2}$$

For the moment let $x$, the integer $r > 0$, and $f(t) \equiv f$ be unrestricted beyond the conditions

$$[e^{-xt^r}D_t^s f(t)]_{t=\pm\infty} = 0 \ (s = 0, \cdots, n-1). \tag{3.3}$$

Then

$$[(e^{-xt^r}D_t^s f)\xi_n(-x, t; r)]_{t=\pm\infty} = 0,$$

and hence, by repeated partial integration of the left of (3.4),

$$\int_{-\infty}^{\infty} f(t)\ (D_t^n e^{-xt^r})\ dt = (-1)^n \int_{-\infty}^{\infty} e^{-xt^r}(D_t^n f)\ dt. \tag{3.4}$$

Thus if we choose $f(t) \equiv P(t)$, where $P$ is a polynomial of degree $< n$ in $t$, we have

$$\int_{-\infty}^{\infty} e^{-xt^r}P(t)\xi_n(-x, t; r)\ dt = 0. \tag{3.5}$$

The conditions (3.3) with $f \equiv P$ are satisfied when $x > 0$ and $r$ is an even integer $> 0$. Hence (3.2) follows from (3.5) when we take $m < n$.

## II. The Polynomials $\phi$, $Y$

4. The $\phi_n$ are a generalization of the $\xi_n$; see (4.51). Let $\alpha_0, \alpha_1, \cdots, \alpha_n, \cdots$ be an infinite sequence of independent variables. The case when some or all of the $\alpha$ are absolute constants is included; in particular, $\alpha_n$ may be zero for all $n > m$, where $m$ is a constant integer. The polynomials

$$\phi_n \equiv \phi_n(\alpha) \equiv \phi_n(\alpha_1, \cdots, \alpha_n) \qquad (n = 0, 1, \cdots)$$

are defined by

$$\phi_0 = 1, \qquad \phi_{-n-1} = 0, \qquad \phi^{n+1} = \alpha(\phi + \alpha)^n, \qquad n \geqq 0. \tag{4.1}$$

In ordinary notation the last is

$$\phi_{n+1} = \sum_{s=0}^{n} \binom{n}{s} \alpha_{s+1}\phi_{n-s}. \tag{4.2}$$

For verifications of formulas we give the first 6 $\phi$'s

$$\begin{aligned} &\phi_0 = 1, \qquad \phi_1 = \alpha_1, \qquad \phi_2 = \alpha_1^2 + \alpha_2, \qquad \phi_3 = \alpha_1^3 + 3\alpha_1\alpha_2 + \alpha_3, \\ &\phi_4 = \alpha_1^4 + 6\alpha_1^2\alpha_2 + 4\alpha_1\alpha_3 + 3\alpha_2^2 + \alpha_4, \\ &\phi_5 = \alpha_1^5 + 10\alpha_1^3\alpha_2 + 10\alpha_1^2\alpha_3 + 15\alpha_1\alpha_2^2 + 5\alpha_1\alpha_4 + 10\alpha_2\alpha_3 + \alpha_5. \end{aligned}$$

Let $t$ be a parameter. Then, writing

$$f(t) \equiv \exp(t\alpha) \equiv \alpha_0 + \alpha_1 t + \alpha_2 t^2/2\,! + \cdots,$$

we have the generating identity of the $\phi$,

$$\exp[f(t) - \alpha_0] = \exp(t\phi) \tag{4.3}$$

For, operation with $D_t$ on (4.3) gives

$$\alpha \exp[(\alpha + \phi)t] = \phi \exp(t\phi),$$

and hence (4.1) by comparing coefficients of $t^n$. Conversely, (4.1) implies (4.3). By (4.1) the coefficients in $\phi_n$ are positive integers, and it follows by mathematical induction from (4.1) that

$$(4.4) \qquad \phi_n = \alpha_1^n + \alpha_n + P(\alpha_1, \cdots, \alpha_{n-1}), \qquad n > 1,$$

where $P(\alpha_1, \cdots, \alpha_{n-1})$ is a polynomial in $\alpha_1, \cdots, \alpha_{n-1}$ alone and is of degree $< n$.

From (4.3),

$$(4.41) \qquad \exp\,(t\alpha_1 + t^2\alpha_2/2! + \cdots) = \phi_0 + t\phi_1 + t^2\phi_2/2! + \cdots,$$

and hence we have the explicit form

$$(4.5) \qquad \phi_n = n! \sum \frac{\alpha_1^{s_1}\alpha_2^{s_2} \cdots \alpha_n^{s_n}}{(1!)^{s_1}(2!)^{s_2} \cdots (n!)^{s_n}.\ \ s_1!\, s_2! \cdots s_n!} \ (n > 0),$$

where the sum refers to all integers $s_1, \cdots, s_n \geqq 0$ such that $n = s_1 + 2s_2 + \cdots + ns_n$. Hence *the number* $\pi_n$ *of terms in* $\phi_n$ *is equal to the total number of partitions of* $n$. Thus the complexity increases rapidly with $n$; for example, $\pi_6 = 11$, $\pi_{22} = 1002$.

Comparing (1.7), (4.41), we see that for the particular sequence $\alpha$ indicated in (4.51) we have

$$(4.51) \quad \phi_n(\alpha) = \xi_n(x, t; r), \ \alpha_j = j! \binom{r}{j} x t^{r-j} (j = 1, \cdots, r); \ \alpha_n = 0, \quad n > r.$$

Another interesting sequence of positive integers connected with $\phi$ are the $\epsilon$, defined by

$$(4.6) \qquad \epsilon_n \equiv \phi_n(1, \cdots, 1), \qquad n > 0; \qquad \epsilon_0 = 1;$$

$\epsilon_n$ is the sum of the coefficients of $\phi_n$. The recurrence for the $\epsilon_n$ is, from (4.1) with all $\alpha$'s replaced by 1,

$$(4.7) \qquad \epsilon_{n+1} = (\epsilon + 1)^n, \qquad \epsilon_0 = 1,$$

and it is readily shown[4] that

$$\epsilon_n = \left(\frac{\Delta}{1!} + \frac{\Delta^2}{2!} + \cdots + \frac{\Delta^n}{n!}\right) 0^n,$$

the notation being that of finite differences, so that

$$\epsilon_n = \sum_{s=1}^{n} \frac{1}{(s-1)!} \left[ \sum_{r=0}^{s-1} (-1)^r \binom{s-1}{r} (s - r)^{n-1} \right], \qquad n > 0.$$

[4] By Herschel's theorem in finite differences. The $\epsilon$ were given as an example in the paper mentioned in the introduction; the table up to $\epsilon_{20}$ here was computed by J. L. Bell by machine from Cayley's table of $\Delta^n 0^m/n!$ (Collected Papers, vol. 11, pp. 145–6.) The congruences for the 0's incidentally check Cayley's table. For the combinatorial meaning of the $\epsilon$ see Whitworth, Choice and Chance, p. 95.

As the $\epsilon$ are among the simplest $\phi$ for constant values of the variables, they are valuable for checking the general $\phi$ congruences. Accordingly we list the first 20, which will be sufficient.

| $n$ | $\epsilon_n$ | $n$ | $\epsilon_n$ |
|---|---|---|---|
| 1 | 1 | 20 | 51724158235372 |
| 2 | 2 | 19 | 5832742205057 |
| 3 | 5 | 18 | 682076806159 |
| 4 | 15 | 17 | 82864869804 |
| 5 | 52 | 16 | 10480142147 |
| 6 | 203 | 15 | 1382958545 |
| 7 | 877 | 14 | 190899322 |
| 8 | 4140 | 13 | 27644437 |
| 9 | 21147 | 12 | 4213597 |
| 10 | 115975 | 11 | 678570 |

In passing, the connection of the $\epsilon$ with certain other polynomials $\eta$, $G$, the second of which were introduced by Steffensen[5] in connection with Makeham's formula, may be noted.

The polynomials $\eta(x)$ are defined by

$$\eta_0(x) = 1, \qquad \eta_s(x) \equiv \sum_{n=1}^{s} x^n \frac{\Delta^n 0^s}{s!} \, (s > 0),$$

so that $\eta(1) = \epsilon$. The generating identity is

$$\exp\,[x(e^t - 1)] = \exp\,[t\eta(x)];$$

whence, as a definition of $G$,

$$\exp[zt - x(e^t - 1)] \equiv \exp\,[tG(x, z)] = \exp\,[t(z + \eta(-x))].$$

Thus $G_n(x, z)$ is the Appell polynomial $(z + \eta(-x))^n$ in $z$;

$$\eta(x) = G(-x, 0); \qquad \epsilon = G(-1, 0);$$
$$G_n(x, z) = \phi_n(\beta), \qquad \beta_1 = z - x, \qquad \beta_s = -x(s > 0).$$

If in (4.3) $t$ be replaced by $ct$, where $c$ is constant, we see that

$$\phi_n(c\alpha_1, c^2\alpha_2, \cdots, c^n\alpha_n) = c^n\phi_n(\alpha_1, \alpha_2, \cdots, \alpha_n), \tag{4.8}$$

so that $\phi_n$ is isobaric in $\alpha_1, \cdots, \alpha_n$ of weight $n$. From (4.3) we get immediately the addition theorem

$$\phi_n(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots, \alpha_n + \beta_n) = [\phi(\alpha) + \phi(\beta)]^n. \tag{4.9}$$

[5] J. F. Steffensen, *Some recent researches in the theory of statistics and actuarial science*, Cambridge Press, 1930, p. 24.

5. Write $D_{\alpha_j} \equiv D_j$. Then, from (4.3), we have at once

$$(5.1) \qquad D_0\phi_n = 0, \qquad D_j\phi_n = \binom{n}{j}\phi_{n-j} \; (j > 0).$$

Thus $\phi_n(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is an Appell polynomial in $\alpha_1$, and hence the $\phi_n(\alpha_1, \cdots, \alpha_n)$ may be considered as a generalization of the Appell polynomials.

6. The derivatives of the $\phi_n$ are of importance in the arithmetical properties. By repetitions of (5.1) we get

$$(6.1) \qquad D_j^r\phi_n = \frac{n!}{(j!)^r\,(n-rj)!}\phi_{n-rj},\ j > 0, \qquad 0 < r \leqq [n/j],$$

$$D_j^r\phi_n = 0 \text{ if } r > [n/j].$$

From the way in which (6.1) is obtained it follows that the coefficient of $\phi_{n-rj}$ is a positive integer which is divisible by $\binom{n}{j}$. By (6.1) the degree of $\phi_n$ in $\alpha_j$ is $[n/j]$ $(j = 1, \cdots, n)$.

When, as in the general definition, the $\alpha$ are independent variables, a congruence involving them (or polynomials $\phi(\alpha)$) is interpreted to mean that the coefficients of the several power products of $\alpha$'s on both sides of the congruence are congruent with respect to the modulus. In particular, if all the coefficients are integers, the congruences hold for all sets of integer values of the $\alpha$'s.

*As always henceforth, let $p$ denote a rational prime.* Then (5.1), (6.1) and the elementary congruence properties of binomial coefficients imply

$$(6.2) \qquad D_j^r\phi_p \equiv 0 \bmod p, \qquad 0 < r, \qquad 0 < j < p.$$

Hence, by (4.4), for the same $r, j$,

$$(6.3) \qquad D_j^r P(\alpha_1, \cdots, \alpha_{p-1}) \equiv 0 \bmod p,$$

where $P$ is of degree $< p$. Consider a particular term of $P$, say

$$k\alpha_1^{a_1} \cdots \alpha_j^{a_j} \cdots \alpha_{p-1}^{a_{p-1}};$$

$k$ is an integer $> 0$. Since the $\alpha$'s are independent variables, it follows from (6.3) that

$$ka_j(a_j - 1) \cdots (a_j - r + 1) \equiv 0 \bmod p,$$

which holds for all integers $r > 0$. Take $r = 1$. Then $ka_j \equiv 0 \bmod p$, and therefore, since $p$ is prime, at least one of $k \equiv 0 \bmod p$, $a_j \equiv 0 \bmod p$ holds. But $a_j \equiv 0 \bmod p$ is impossible, since the degree of $P$ is $< p$. Hence $k \equiv 0 \bmod p$. Or, since $a_j, a_j - 1, \cdots, a_j - r + 1$ are all $< p$, and $p$ is prime, it follows at once that $k \equiv 0 \bmod p$. This proves

$$(6.4) \qquad \phi_p \equiv \alpha_1^p + \alpha_p \bmod p.$$

Again, from (4.1) we have

$$\phi^{p+r+1} = \alpha(\phi + \alpha)^{p+r},$$

and hence, by a general theorem established in a previous paper on anharmonic polynomials,[6]

$$\phi^{p+r+1} \equiv \alpha(\phi^p + \alpha^p)\,(\phi + \alpha)^r \bmod p, \qquad 0 \leqq r < p,$$

which gives

$$\phi_{p+r+1} - \sum_{s=0}^{r} \binom{r}{s} \alpha_{s+1}\phi_{p+r+s} - \alpha^{p+1}(\phi + \alpha)^r \equiv 0 \bmod p.$$

In the last take $r$ successively equal to $r - 1$, $r - 2$, $\cdots$, 0, and solve the resulting system of congruences (the system is obviously solvable) for $\phi_{p+r+1}$. Then

$$(6.5) \qquad (-1)^{r+1}\phi_{p+r+1}$$

is congruent mod $p$ to

$$\begin{vmatrix}
\alpha^{p+1}(\phi+\alpha)^r & \alpha_1 & \binom{r}{1}\alpha_2 & \binom{r}{2}\alpha_3 & \binom{r}{3}\alpha_4 & \cdots & \binom{r}{r}\alpha_{r+1} \\
\alpha^{p+1}(\phi+\alpha)^{r-1} & -1 & \alpha_1 & \binom{r-1}{1}\alpha_2 & \binom{r-1}{2}\alpha_3 & \cdots & \binom{r-1}{r-1}\alpha_r \\
\alpha^{p+1}(\phi+\alpha)^{r-2} & 0 & -1 & \alpha_1 & \binom{r-2}{1}\alpha_2 & \cdots & \binom{r-2}{r-2}\alpha_{r-1} \\
\alpha^{p+1}(\phi+\alpha)^{r-3} & 0 & 0 & -1 & \alpha_1 & \cdots & \binom{r-3}{r-3}\alpha_{r-2} \\
\alpha^{p+1}(\phi+\alpha)^{r-4} & 0 & 0 & 0 & -1 & \cdots & \binom{r-4}{r-4}\alpha_{r-3} \\
 & \cdots & & \cdots & \cdots & & \\
\alpha^{p+1}(\phi+\alpha)^0 & 0 & 0 & 0 & 0 & \cdots & \alpha_1 \\
\alpha_1{}^p + \alpha_p & 0 & 0 & 0 & 0 & \cdots & -1
\end{vmatrix}$$

which holds for $0 \leqq r < p$. The binomial coefficient $\binom{0}{0}$ in this is by convention 1. For example, taking $r = 0, 1, 2$ we get

$$\begin{aligned}
(6.6)\quad \phi_{p+1} &\equiv \alpha_1{}^{p+1} + \alpha_1\alpha_p + \alpha_{p+1} \bmod p, \\
\phi_{p+2} &\equiv (\alpha_1^2 + \alpha_2)\alpha_p + 2\alpha_1\alpha_{p+1} + \alpha_{p+2} + \alpha_1{}^{p+2} + \alpha_2\alpha_1{}^p \bmod p, \\
\phi_{p+3} &\equiv 3(\alpha_1^2 + \alpha_2)\alpha_{p+1} + 3\alpha_1\alpha_{p+2} + \alpha_{p+3} + (\alpha_1^3 + 3\alpha_1\alpha_2 + \alpha_3)\alpha_p \\
&\qquad + \alpha_1{}^{p+3} + 3\alpha_2\alpha_1{}^{p+1} + \alpha_3\alpha_1{}^p \bmod p, \qquad p > 2,
\end{aligned}$$

and these may be verified for the $\epsilon$ by setting each $\alpha = 1$. With (6.4) we have thus, mod $p$,

$$\epsilon_p \equiv 2, \qquad \epsilon_{p+1} \equiv 3, \qquad \epsilon_{p+2} \equiv 7; \qquad \epsilon_{p+3} \equiv 20 \ (p > 2).$$

The condition $p > 2$ in the last happens to be superfluous for the $\epsilon$, but this does not follow from the proof of the general congruence. These congruences are checked by the values in the table in §4.

[6] *Trans. American Math. Soc.*, vol. 34, 1922, p. 109.

The congruence (6.5) can be considerably generalized by applying theorems proved in a previous paper[7] on residues of certain types of binomial coefficients. For example, it has shown that

$$\binom{2p}{h} \equiv 2\binom{p}{h} \bmod p^2, \qquad 0 < h < p;$$
$$\binom{2p}{h} \equiv 2\binom{p}{h-p} \bmod p^2, \qquad p < h < 2p.$$

Hence, proceeding as above, we get

$$(6.6)\quad \phi_{2p+1} \equiv \alpha_{2p+1} + \binom{2p}{p}\alpha_{p+1}\phi_p + \alpha_1\phi_{2p} + 2\sum_{h=1}^{p-1}\binom{p}{h}(\alpha_{h+1}\phi_{2p-h} + \alpha_{p+h+1}\phi_{p-h}) \bmod p^2.$$

Similar results may be found for moduli $p^a$, $p^a q^b$, $p^a q^b r^c$, $\cdots$, where $p, q, r, \cdots$ are distinct primes, and $a, b, c, \cdots$ are any integers $> 0$, by means of the theorems cited.

Having derived any congruence of the type (6.5) for a particular $r$, we can obtain from it congruences for $r - s(s > 0)$ by differentiating the given congruence and applying (6.1). For this it is more convenient to rewrite (6.1) as

$$D_j^s\phi_n = \binom{n}{j}\binom{n-j}{j}\binom{n-2j}{j}\cdots\binom{n-(s-1)j}{j}\phi_{n-sj}.$$

For, (6.5) is equivalent to an identity in the $\alpha$ of the form

$$\phi_{p+r+1} \equiv P(\alpha_{p+r+1}, \alpha_{p+r}, \cdots, \alpha_p, \alpha_1, \alpha_2, \cdots, \alpha_{r+1}) + pQ(\alpha_1, \alpha_2, \cdots, \alpha_{p+r+1}),$$

where $P$, $Q$ are polynomials, with integer coefficients; $P$ is $(-1)^{r+1}$ times the determinant in (6.5). Hence

$$(6.7)\qquad D_j^s\phi_{p+r+1} \equiv D_j^s P \bmod p.$$

The coefficients on both sides of (6.7) are integers; the left of (6.7) is $\phi_{p+r+1-sj}$ times the integer

$$\binom{p+r+1}{j}\binom{p+r+1-j}{j}\binom{p+r+1-2j}{j}\cdots\binom{p+r+1-(s-1)j}{j}.$$

If the left of (6.7) is not to vanish identically, $s$, $j$ must be chosen such that $p + r + 1 - sj \geqq 0$; the maximum value of $j$ is $p + r + 1$. Subject to these

[7] *Journal of the London Math. Soc.*, vol. 5, 1930, pp. 253-258.

restrictions the above product of binomial coefficients is reduced modulo $p$ by means of

$$\binom{m}{n} \equiv \binom{m_1}{n_1}\binom{a}{b} \bmod p,$$

$$m > 0,\ n > 0,\ m = m_1p + a, \qquad n = n_1p + b,$$

$$0 \leqq a, \qquad b < p, \qquad \binom{a}{b} = 0 \text{ if } b > a.$$

The result of this reduction is an integer, say $c$. Finally then, from (6.7) we have

(6.8) $$c\phi_{p+r+1-sj} \equiv D_j^s P \bmod p,$$
$$j > 0, \qquad p + r + 1 \geqq sj, \qquad s \geqq 0.$$

7. The polynomials $Y_n$ next considered are a generalization of the $\xi_n$. Write (see (4.3))

(7.1) $$y \equiv e^{2t} - \alpha_0 = \alpha_1 t + \alpha_2 t^2/2! + \cdots + \alpha_n t^n/n! + \cdots,$$

and define the $Y_n$ by

(7.2) $$e^{-y} D_t^n e^y \equiv Y_n, \qquad n = 0, 1, \cdots; \qquad Y_0 = 1.$$

Obviously, with $Y_s \equiv D_t^s y$ we have

(7.3) $$Y_n \equiv Y_n(y_1, \cdots, y_n),$$

and if $\alpha_r = r!\,x$, $\alpha_n = 0$, $n \neq r$, then $Y_n = \xi_n(x, t; r)$.
Operating on (7.2) with $D_t$ we get

(7.4) $$Y_{n+1} = (y_1 + D_t)Y_n,$$

which corresponds to (1.2). For checks on subsequent formulas we write down from (7.4)

$$Y_1 = y_1, \qquad Y_2 = y_1^2 + y_2, \qquad Y_3 = y_1^3 + 3y_1y_2 + y_3,$$
$$Y_4 = y_1^4 + 6y_1^2y_2 + 4y_1y_2 + 3y_2^2 + y_4,$$

which may be compared with the corresponding $\phi_n$ in §4. In fact we have generally

(7.5) $$Y_n = \phi_n(y_1, \cdots, y_n).$$

To prove (7.5) we observe from (4.2), (6.1) that

(7.6) $$\phi_{n+1} = \left(\alpha_1 + \sum_{s=1}^{n} \alpha_{s+1}\frac{\partial}{\partial\alpha_s}\right)\phi_n.$$

Let $ky_1^{a_1} \cdots y_n^{a_n}$ be a particular term of $Y_n$. Then, since $D_t y_s = y_{s+1}$, we have

$$D_t(ky_1^{a_1} \cdots y_n^{a_n}) = \left(y_2 \frac{\partial}{\partial y_1} + \cdots + y_{n+1} \frac{\partial}{\partial y_n}\right) (ky_1^{a_1} \cdots y_n^{a_n}),$$

and therefore by (7.4),

$$Y_{n+1} = \left(y_1 + \sum_{s=1}^{n} y_{s+1} \frac{\partial}{\partial y_s}\right) Y_n. \tag{7.7}$$

These functions are connected with the determinant in (6.5). For, by the definition of $y_s$ we have

$$y_s = \alpha^s e^{\alpha t} = \sum_{n=0}^{\infty} \alpha_{n+s} t^n/n! \ (s > 0),$$

and if we define $\alpha^{(s)}$ by the MacLaurin expansion of $Y_s$ in powers of $t$ thus,

$$Y_s = \exp\,(t\alpha^{(s)}) \equiv \sum_{n=0}^{\infty} \alpha_n^{(s)} t^n/n!, \tag{7.8}$$

we see from (7.1), (4.3), (7.8) that

$$Y_s e^y = \phi^s e^{t\phi} = \exp\,[t(\alpha^{(s)} + \phi)],$$

and hence

$$\phi_{n+s} = (\phi + \alpha^{(s)})^n. \tag{7.9}$$

In (7.9) take $n = p$, $s = r + 1$, as in (6.5). Then, referring to (6.4), we have

$$(-1)^{r+1}\phi_{p+r+1} \equiv \alpha_1^p + \alpha_p + \alpha_p^{(r+1)} \bmod p. \tag{7.10}$$

The symbolic recurrence for $Y$,

$$Y^{n+1} = y(Y + y)^n, \qquad Y_0 = 1, \qquad y^n \equiv y_n, \tag{7.11}$$

follows from (7.6), (7.7) compared with (4.1). From the last we deduce the recurrence for the $\alpha^{(s)}$. For,

$$Y_{n+1} = \sum_{s=0}^{n} \binom{n}{s} Y_{n-s} y_{s+1}, \ Y_{n+s} = \exp(t\alpha^{(n-s)}), \qquad y_{s+1} = \alpha^{s+1} e^{t\alpha};$$

hence, on substituting in the first for $Y_{n-s}$, $y_{s+1}$ and comparing coefficients of $t^m$, we have

$$\alpha_m^{(n+1)} = \sum_{s=0}^{n} \binom{n}{s} \sum_{r=0}^{m} \binom{m}{r} \alpha_{m-r+s+1} \alpha_r^{(n-s)}, \tag{7.12}$$

which may be written

$$\alpha_m^{(n+1)} = \sum_{s=0}^{n} \binom{n}{s} \alpha^{s+1}(\alpha + \alpha^{(n-s)})^m. \tag{7.13}$$

In the congruences for the $\phi$ we are interested only in the residues mod $p$ of the $\alpha^{(s)}$, as in (7.10). From (7.13) we have

$$\alpha_m^{(p+1)} \equiv \alpha(\alpha + \alpha^{(p)})^m + \alpha_{m+p+1} \bmod p, \tag{7.14}$$

or, in non-symbolic form,

$$\alpha_m^{(p+1)} \equiv \sum_{s=0}^{m} \binom{m}{s} \alpha_{m-s+1}\alpha_s{}^{(p)} + \alpha_{m+p+1} \bmod p. \tag{7.15}$$

Considering $Y_n$ as a function $Y_n(t)$ of $t$ we have

$$e^{t\phi} = e^y = \sum_{n=0}^{\infty} \frac{t^n}{n!} [D_t^n(e^y)]_{t=0};$$

hence

$$\phi_n \equiv \phi_n(\alpha) = Y_n(0) = \alpha_0^{(n)}; \; Y_n(0) \equiv [Y_n(t)]_{t=0}. \tag{7.16}$$

8. By the remark after (7.3) and §2 it follows that *the $Y_n$ satisfy no differential equation of the type*

$$R_0 D_t^m Y_n + R_1 D_t^{m-1} Y_n + \cdots + R_m = R_{m+1}$$

*of constant order $m$, where the $R$'s are polynomials in $\alpha$'s and $t$.* Otherwise we should have a contradiction with §2 for suitably chosen $\alpha$'s.

9. For properly restricted $\alpha$, the $Y_n$ give rise to an orthogonal set in $-\infty \leqq t \leqq +\infty$. The restrictions are $\alpha_{2n} > 0$, $\alpha_{2n-1} = 0 (n = 1, 2, \cdots)$. For these $\alpha$, write $Y_n \equiv Y_n^1$, and let $y \equiv y(t)$, for the same $\alpha$, have a radius of convergence $> 0$. Write $Z_n \equiv e^{-y/2}\, Y_n^1$. Then, as in §3, it may be shown that

$$\int_{-\infty}^{\infty} Z_m Z_n \, dt = 0, \qquad m \neq n.$$

10. Returning to the $\phi$, we shall give some miscellaneous properties which follow readily from what has been given. Let $0_s$ denote the sequence $0, \cdots, 0$ of $s$ zeros. Then

$$k!\, \phi_{km}\left(0_{m-1}, \frac{m!}{1!}\alpha_1, 0_{m-1}, \frac{(2m)!}{2!}\alpha_2, 0_{m-1}, \cdots, 0_{m-1}, \frac{(km)!}{k!}\alpha_k\right) \tag{10.1}$$
$$= (km)!\, \phi_k(\alpha_1, \cdots, \alpha_k),$$

where $0_0$ indicates that the symbol $0_s$, $s = 0$, is to be suppressed. This follows from ($y$ as in §7)

$$t^h \exp y(t^m) = \sum_{n=0}^{\infty} \phi_n(\alpha_1, \cdots, \alpha_n) t^{mn+h}/n!\,.$$

The integrals are found from (5.1). Let $\phi_n^{(j)}$ denote the result of putting $\alpha_j = 0$ in $\phi_n$. Then

$$\phi_n^{(j)} = \sum_{s=0} \frac{(-1)^s}{s!} \alpha_j^s D_j^s \phi_n, \tag{10.2}$$

as we see on multiplying exp $y$ by exp $(-\alpha_j t^j/j\,!)$. From (5.1),

$$\binom{n+j}{j} \int_0^{\alpha_j} \phi_n \, d\alpha_j = \phi_{n+j} - \phi_{n+j}^{(j)},$$

which may be written, by (10.2), (6.1),

$$\int_0^{\alpha_j} \phi_n \, d\alpha_j = n! \sum_{s=0}^{[n/j]} \frac{(-1)^s}{(j!)^s (s+1)!\,(n-sj)!} \alpha_j^{s+1} \phi_{n-sj}. \tag{10.3}$$

The differences of the $\phi$ are found by applying (6.1) to the following formula from finite differences,

$$\Delta_j^r = D_j^r + \frac{\Delta^r 0^{r+1}}{(r+1)!} D_j^{r+2} + \frac{\Delta^r 0^{r+2}}{(r+2)!} D_j^{r+2} + \cdots,$$

where $\Delta_j^r$ indicates the $r$th difference with respect to $\alpha_j$. Thus

$$\Delta_j^r \phi_n = \sum_{s=0}^{[(n-rj)/s]} \frac{\Delta^r 0^{r+s}}{(r+s)!} \frac{n!}{(j!\,)^{r+s}(n-(r+s)j)!} \phi_{n-(r+s)j}; \tag{10.4}$$

the coefficients are positive integers.

Combined with previous results the last shows that, in an obvious sense, *the set of all $\phi$ is closed under operations of differentiation, integration, and finite differencing.*

Any polynomial (or power series, if convergent) in $\alpha_1, \cdots, \alpha_r$ can be expressed as a polynomial (or power series) in suitably chosen $\phi$'s. For we have

$$\alpha_s^n = \frac{(-1)^{sn}(s!)^n n!}{(sn)!} \phi_{sn}^{(s)}(k_s\alpha_1 - \phi, \alpha_2, \alpha_3, \cdots, \alpha_{sn}), \tag{10.5}$$

where $k_1 = 1$, $k_s = 0$, $s > 1$. The proof is from the identity exp $y$ = exp $(t\phi)$, which gives an exponential expression for exp $(\alpha_r t^r/r!)$. By the formula preceding (10.3), the last gives the curious result

$$\alpha_s^n = \frac{(-1)^{sn}(s!)^n n!}{(sn)!} \left[ \phi_{sn} - \binom{sn}{s} \int_0^{\alpha_s} \phi_{sn-s}(k_s\alpha_1 - \phi, \alpha_2, \cdots, \alpha_{sn}) d\alpha_s \right].$$

When $y$(as in §7) satisfies a differential equation of the type

$$(D_t^m + A_1 D_t^{m-1} + \cdots + A_{m-1}D_t + A_m)y = A_{m+1},$$

or, what is the same,

$$y_m + A_1 y_{m-1} + \cdots + A_{m-1}y_1 + A_m y = A_{m+1}, \tag{10.6}$$

where the $A$'s are functions of $t$, some or all of which may be constants, possessing derivatives of all orders, the $\phi_n$ will be connected by relations of the form

$$R(\phi_1, \cdots, \phi_{m+s+1}) = 0, \qquad s \geqq m + 2, \tag{10.7}$$

where $R$ is a polynomial of degree $m + 2$. The $\phi$, $Y$ are connected by (7.16). Hence we find a relation for the $Y$ from which (10.7) follows by setting $t = 0$. From (7.11) we have

$$Y_{m+n+1} = y_1 Y_{m+n} + \binom{m+n}{1} y_2 Y_{m+n-1} + \cdots + y_{m+n+1}Y_0, \tag{10.8}$$

and $Y_0 = 1$; from (10.6), by successive operations with $D_t$ and reductions after each by means of (10.6), $y_{m+s}$ is expressed in a linear function of $y, y_1, \cdots, y_{m-1}$. Thus (10.8) is equivalent to

$$Y_{m+n+1} = \lambda_0(n)y + \lambda_1(n)y_1 + \cdots + \lambda_{m-1}(n)y_{m-1}, \tag{10.9}$$

where the $\lambda$'s do not contain $y, y_1, \cdots, y_{m-1}$, and are linear functions of $Y_1, Y_2, \cdots, Y_{m+n}$. Give to $n$ in (10.9) $m + 2$ distinct integer values $> 0$, of which the greatest is $s$, and eliminate $y, y_1, \cdots, y_{m-1}$. Setting $t = 0$ in the result gives (10.7).

It is obvious that $\phi_n$, in the general case of unrestricted $\alpha$ or $y$, can satisfy no differential or difference equation of constant finite order, for all $n$, whose coefficients involve only a finite number of the $\alpha$, since $\phi_n \equiv \phi_n(\alpha_1, \cdots, \alpha_n)$.

The general discussion has included the case when certain of the $\alpha$'s are replaced by zeros; in particular (10.7) holds when $y \equiv y(t)$ is a polynomial of degree $m$.

## III. The Polynomials x.

11. The Appell polynomial in $x$ (ordinary variable) to the *base* $\beta$ (umbra) of rank $n$ is $(x + \beta)^n (n = 0, 1, \cdots)$, namely $\sum_{s=o}^{n} \binom{n}{s} \beta_s x^{n-s}$, and this polynomial is generated by $\exp(hx) \times \exp(h\beta)$, where $h$ is a parameter, since

$$\exp(hx)\exp(h\beta) = \exp[h(x + \beta)].$$

The obvious property $D_x(x + \beta)^n = n(x + \beta)^{n-1}$ is that which makes possible the discussion of the differential equation satisfied by $(x + \beta)^n$ when it is given that $\exp(h\beta)$ satisfies a differential equation of prescribed type (with respect to $h$).

In line with the polynomials $\xi$, $\phi$, we now generalize the Appell polynomials to the polynomials $\psi$, defined by

$$\psi_n \equiv \psi_n(x, \beta; r), \psi_{-n-1} = 0, n = 0, 1, \cdots ;\quad \exp(h^r x) \times \exp(h\beta) \equiv \exp(h\psi), \tag{11.1}$$

where $r$ is a constant integer $> 0$. Hence $\psi_n(x, \beta, 1)$ is the general Appell polynomial of rank $n$ to the base $\beta$. Obviously if $r > 1$, and $\beta$ is an arbitrary sequence, $D_x\psi_n \neq n\psi_{n-1}$, so that the differential equation satisfied by $\psi_n$ when the equation satisfied by $\exp(h\beta)$ is given, cannot be found by the same process as for Appell's polynomials. We shall see in fact that the equation for $\exp(h\beta)$ must be of a certain restricted type if the equation for $\psi_n$ is to be of constant order (independent of $n$). The restriction degenerates to no restriction when $r = 1$.

From (11.1), by $D_t$, $D_x^s$ we have

$$\psi_{m+1} = \beta\psi^m + \frac{m!\,rx}{(m - r + 1)!}\psi_{m-r+1}\ (m \geqq 0), \qquad \psi_0 = \beta_0; \tag{11.2}$$

$$D_x^s\psi_m = \frac{m!}{(m - rs)!}\psi_{m-rs}\ (0 \leqq s \leqq [m/r]), \tag{11.3}$$

$$D_x^s\psi_m = 0\ (s > [m/r]),$$

and by $D_\beta^s$,

$$D_\beta^s\psi_m = m(m - 1) \cdots (m - s + 1)\psi_{m-s}. \tag{11.4}$$

Directly from (11.1) the explicit form is

$$\psi_m = m! \sum_{s=0}^{[m/r]} \frac{\beta_{m-sr}x^s}{(m - sr)!\, s!}; \tag{11.5}$$

the coefficient of $\beta_{m-sr}x^s$ is a non-negative integer, and there is the addition theorem

$$\psi_m(x + y, \beta; r) = [\psi(x, \beta; r) + \psi(y, \beta; r)]^m. \tag{11.6}$$

Writing (11.5) in the form

$$\psi_m = \sum_{s=0}^{[m/r]} (sr - s)!\binom{sr}{r}\binom{m}{sr}\beta_{m-sr}x^s,$$

we see that ($p$ prime)

$$\psi_p(x, \beta; r) \equiv \beta_p \bmod p, \qquad r \neq p. \tag{11.7}$$

The addition theorem may be written

$$\psi_m(x + y, \beta; r) = \psi_m(\psi(x, \beta; r), y; r), \tag{11.8}$$

and $x$, $y$ are interchangeable on the right. This follows by inspection from (11.1).

12. We now examine the possible differential equations of order independent of $n$ satisfied by $\psi_n$ when $\exp(h\beta)$ satisfies a linear differential equation (with respect to $h$) with polynomial coefficients in $h$. For this we introduce the intermediary functions (not polynomials)

$$\psi_n^{(s)} \equiv \psi_n^{(s)}(x, \beta; r), \qquad \psi_{-n-1}^{(s)} \equiv 0 \ (n = 0, 1, \cdots ; s > 0),$$

defined by

$$\exp(xh^r)D_h^s \exp(h\beta) \equiv \exp(h\psi^{(s)}). \tag{12.1}$$

Take $D_h^s$ of (11.1) and refer to (1.1). Then

$$\psi^s \exp(h\psi) = \sum_{j=0}^{s} \binom{s}{j} \exp(h\psi^{(s-j)}),$$

$$\xi_j \equiv \xi_j(x, h; r), \ \psi_n^{(s-j)} \equiv \psi_n^{(s-j)}(x, \beta; r), \ \psi_n \equiv \psi_n(x, \beta; r).$$

Hence, by (1.5),

$$\exp(h\psi^{(s)}) = \psi^s \exp(h\psi) - \sum_{j=0}^{s} \binom{s}{j} \exp(h\psi^{(s-j)}) \sum_{i=0} c_i(j)x^{j-i}h^{(j-i)r-j}, \tag{12.2}$$

from which $\psi_m^{(s)}$ can be calculated. It will be sufficient to show that

$$\psi_m^{(s)} = \sum_{a=0}^{s} A_a^{(s)} D_x^a \psi_{m+s}, \qquad A_0^{(s)} = 1, \tag{12.3}$$

where the point to be noted is that only derivatives of the *single* $\psi$ function $\psi_{m+s}$ appear on the right. The precise form of the $A$'s is immaterial; actually we shall prove that

$$A_0^{(s+1)} = A_0^{(s)} - \frac{rx}{m+1} D_x A_0^{(s)}, \qquad A_{s+1}^{(s+1)} = -\frac{rx}{m+1} A_s^{(s)}, \tag{12.4}$$

$$A_a^{(s+1)} = A_a^{(s)} - \frac{rx}{m+1}(D_x A_a^{(s)} + A_{a-1}^{(s)}) \qquad (a = 1, \cdots, s),$$

whence the $A$'s can be found explicitly if desired. The proof will be from (12.3) by mathematical induction on $s$.

Taking $D_h$ of (12.1), and comparing coefficients of $h^m$ in the result, we get

$$r!\binom{m}{r-1} x\psi_{m-r+1}^{(s)} + \psi_m^{(s+1)} = \psi_{m+1}^{(s)}; \tag{12.5}$$

and $D_x$ of (12.1) gives

$$D_x \psi_m^{(s)} = r!\binom{m}{r} \psi_{m-r}^{(s)}. \tag{12.6}$$

From (12.6), (12.5) we have

$$\frac{rx}{m+1} D_x \psi_{m+1}^{(s)} + \psi_m^{(s+1)} = \psi_{m+1}^{(s)}. \tag{12.7}$$

In (12.3) replace $m$ by $m + 1$ and apply (12.7); then

$$\psi_{m+1}^{(s+1)} + \frac{rx}{m+1} D_x \psi_{m+1}^{(s)} = \sum_{a=0}^{s} A_a^{(s)} D_x \psi_{m+1+s}. \tag{12.8}$$

Now (merely to simplify the writing) define

$$P_a^{(s)} \equiv A_a^{(s)} - \frac{rx}{m+1} D_x A_a^{(s)}, \quad -Q_a^{(s)} \equiv \frac{rx}{m+1} A_a^{(s)}.$$

To the second term of (12.8) apply (12.3) with $m$ replaced by $m + 1$ in (12.3). Then

$$\psi_m^{(s+1)} = \sum_{a=0}^{s} [P_a^{(s)} D_x^a \psi_{m+1+s} + Q_a^{(s)} D_x^{a+1} \psi_{m+1+s}], \tag{12.9}$$

and therefore, with the $A^{(s+1)}$ defined in (12.4) we have completed the induction on $s$ (the case $s = 1$ obviously is true).

From the foregoing we can construct the required differential equations. Let $c$ be independent of $t$, and let $k$ be an arbitrary non-negative integer. Then, from (12.1), (12.3),

$$ch^k \exp(xt^r) D_h^s (\exp(h\beta)) = c \sum_{n=0}^{\infty} \frac{h^{n+k}}{n!} \sum_{a=0}^{s} A_a^{(s)} D_x^a \psi_{n+s}, \tag{12.10}$$

the second member of which is equal to

$$c \sum_{m=k}^{\infty} \frac{h^m}{(m-k)!} \sum_{a=0}^{s} A_a^{(s)} D_x^a \psi_{m+s-k}.$$

For our purpose, as stated at the beginning of this section, it is now required to choose $s$, $k$ in the last so that all derivatives $D_x^a$ are taken with respect to the *same* function $\psi$. From (11.3) we have

$$D_x^a \psi_{m+s-k} = (ra)! \binom{m+s-k}{ra} \psi_{m+s-k-ra}; \tag{12.11}$$

the restriction on $a$ need not be included, since $\psi_{-j} = 0$, $j > 0$. Hence we choose $s - k \equiv q \bmod r$; say $s - k = q - ur$. Application of (11.3) to (12.11) now gives

$$D_x^a \psi_{m+s-k} = K D_x^{a+u} \psi_{m+q},$$

$$K \equiv (ra)! \binom{m+q-ur}{ra} / (r(a+u))! \binom{m+q}{r(a+u)}; \tag{12.12}$$

and therefore, from (12.10),

$$\exp (xt^r) \; [ch^k D_h^s \; (\exp \; (h\beta))] = c \sum_{m=k}^{\infty} \frac{h^m}{(m-k)!} \sum_{a=0}^{s} A_a^{(s)} K D_x^{a+u} \psi_{m+q},$$

where $s = q - k - ur$.

It follows that *if* $\exp\ (h\beta)$ *satisfies the differential equation*

$$\sum_{c,k,u} ch^k D_h^{q-k-ur} \; (\exp \; (h\beta)) = 0,$$

*then the polynomials* $\psi_{m+q}$ *satisfy the differential equation*

$$\sum_{c,k,u} \frac{c}{(m-k)!} \sum_{a=0}^{q-k-ur} A_a^{(q-k-ur)} K D_x^{a+u} \psi_{m+q} = 0,$$

*whose order is independent of* $m$*, where the constants* $K$ *are as in* (12.12), *and the* $A$*'s are given by* (12.4).

The derivatives in the given equation for $\exp\ (h\beta)$ are of orders in arithmetical progression with common difference $r$; when the $\psi$'s are Appell polynomials ($r = 1$), the implied restriction disappears.

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