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Type Theories, Normal Forms, and D_∞-Lambda-Models*

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A (non-standard) inverse-limit λ-model D_∞^* is constructed which has a non Hilbert–Post complete theory. Moreover, in D_∞^*, a simple semantic characterization of normalizable terms is given. These results are proved using the properties of a generalized type assignment system which yields a filter model (Barendregt, Coppo, and Dezani-Ciancaglini, 1983, *J. Symbolic. Logic* **48**, 931–940; Coppo, Dezani-Ciancaglini, Honsell, and Longo, 1983, pp. 241–262, "Logic Colloquium '82," North-Holland, Amsterdam), isomorphic to D_∞^*. The type assignment system is also proved complete with respect to an interpretation of types (in the term model of β-equality) based only on normalization properties. As an application a class of maximal monoids of normalizable terms is characterized.

INTRODUCTION

In Barendregt *et* al. (1983) a new λ-model (filter model) was introduced to study completeness properties of type assignment systems for terms of the pure λ-calculus. One interesting feature of this model (indeed, the reason why it was introduced) is that the objects of the domain can be interpreted as sets of formal types (Coppo *et al.*, 1980) of a type assignment system (which is a conservative extension of Curry's Functionality Theory) and that the interpretation of a term coincides with the set of all types that can be assigned to it. The domain of the filter model, in particular, is built by all the sets of types which satisfy a given closure condition (corresponding to type inclusion). By changing the closure conditions we can define a class of λ-models (*filter models*) whose basic properties have been investigated in Coppo *et al.* (1983b).

Filter models turn out to be a very rich class. In particular each inverse-limit space, built by the classical Scott's D_∞ construction (Scott, 1972) starting from a countably based lattice, is isomorphic to a filter model (Coppo *et al.*, 1984).

In this paper we study a filter model $\mathscr{F}^*$ induced by a type assignment system introduced by two of the authors to study normalization properties

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of terms (Coppo and Dezani, 1979a). $\mathscr{F}^*$ turns out to be isomorphic to an inverse limit space D_∞^* built starting from a three point lattice with a non standard initial projection. The formal theory of D_∞^* coincides with Morris' extensional theory (Morris, 1968) and, hence, is not Hilbert–Post complete. This proves that not all D_∞-models have a complete theory, solving (in negative sense) a conjecture stated in Böhm (1975, open problem No. 11.3). Moreover D_∞^* provides a simple semantic characterization of normalizable terms. In fact we show that a term is normalizable iff its interpretation (in a suitable environment) is contained in a specific open subset (with respect to the Scott topology) of D_∞^*.

The type assignment system on which the definition of $\mathscr{F}^*$ is based has also some interest in itself. Let us interpret the two basic types φ_* and $\varphi_{\top}$ as, respectively, the set of all terms which have a normal form and the set of all (normalizable) terms which preserve this property under application (i.e., all M such that NM is normalizable whenever N is normalizable). Then the system provides a complete characterization of the normalization properties represented by types built from φ_*, $\varphi_{\top}$, and ω (the universal type) by means of the "$\rightarrow$" type constructor. For example a term has type $\varphi_* \rightarrow \varphi_*$ (from a suitable basis) iff it yields a normalizable term whenever applied to a normalizable term. As an application of this result we give a simple characterization (in terms of types) of a class of monoids (under composition) of normalizable terms which are maximal in the sense that all their extensions contain at least one non-normalizable term. This was proved in a particular case in Dezani and Ermine (1982).

In this paper Section 1 contains an overview of generalized type assignment systems and the proof of some basic properties. In Section 2 filter models are introduced while in Section 3 the model D_∞^* is defined and its theory is characterized. Finally the properties of the type assignment system are studied in Section 4.

1. Type Assignment Systems

The following systems of type assignment for terms of the (pure) λ-calculus have been introduced in Coppo *et al.* (1983b), Barendregt *et al.* (1983) and further developed in Coppo *et al.* (1984). They extend the classical Curry's type assignment system (see, e.g., Curry *et al.*, 1958; Morris, 1968), by introducing a "universal" type "ω", a new operator "$\wedge$" (intersection) of type formation and an inclusion relation between types.

1.1. Definition. Let A be a set of atomic types (ranged over by $\varphi_0, \varphi_1, \ldots$) and ω a type $\notin A$. The *set of types* T_A is the minimal set such that:

(1) $A \cup \{\omega\} \subseteq T_A$

(2) $\alpha, \beta \in T_A \Rightarrow \alpha \to \beta \in T_A, \alpha \wedge \beta \in T_A$.

Let α, β, γ, δ range over T_A.

1.2. DEFINITION. (i) An *inclusion relation* is a transitive and reflexive relation $\leqslant \subseteq T_A \times T_A$ which satisfies:

(1) $\alpha \leqslant \omega$,

(2) $\omega \leqslant \omega \to \omega$,

(3) $\alpha \leqslant \alpha \wedge \alpha$,

(4) $\alpha \wedge \beta \leqslant \alpha$, $\alpha \wedge \beta \leqslant \beta$,

(5) $(\alpha \to \beta) \wedge (\alpha \to \gamma) \leqslant \alpha \to (\beta \wedge \gamma)$,

(6) $\alpha \leqslant \alpha'$, $\beta \leqslant \beta' \Rightarrow \alpha \wedge \beta \leqslant \alpha' \wedge \beta'$,

(7) $\alpha' \leqslant \alpha$, $\beta \leqslant \beta' \Rightarrow \alpha \to \beta \leqslant \alpha' \to \beta'$.

(ii) Let $\Sigma \subseteq T_A \times T_A$ be an arbitrary relation. Then the *type theory* generated by Σ, denoted $\leqslant_\Sigma$, is the minimal inclusion relation which contains Σ.

(iii) $\alpha \sim_\Sigma \beta$ iff $\alpha \leqslant_\Sigma \beta \leqslant_\Sigma \alpha$.

$\alpha \leqslant_\Sigma \beta$ can be viewed as saying that (the interpretation of) α is included in (the interpretation of) β. Observe that, for all Σ, $\omega \sim_\Sigma \omega \to \omega$. $\leqslant_\varnothing$ is the minimal type theory ($\varnothing$ denotes the empty set).

Types are assigned to terms by a set of rules which are a natural extension of the rules of Functionality Theory.

1.3. DEFINITION. (i) A *statement* is an expression αM, where M is a term (the *subject*) and α is a type (the *predicate*). A *basis* is a set of statements with only variables as subjects.

(ii) Let $\leqslant_\Sigma$ be a type theory. The *type assignment* induced by $\leqslant_\Sigma$ is defined by the natural deduction system

$$(\to I)\ \frac{\begin{matrix}[\alpha x]\\ \vdots \\ \beta M\end{matrix}}{\alpha \to \beta \lambda x.M}\,{}^{1} \qquad (\to E)\ \frac{\alpha \to \beta M \quad \alpha N}{\beta(MN)}$$

$$(\wedge I)\ \frac{\alpha M \quad \beta M}{\alpha \wedge \beta M} \qquad (\wedge E)\ \frac{\alpha \wedge \beta M}{\alpha M} \quad \frac{\alpha \wedge \beta M}{\beta M}.$$

$$(\leqslant_\Sigma)\ \frac{\alpha M \quad \alpha \leqslant_\Sigma \beta}{\beta M} \qquad (\omega)\ \frac{}{\omega M}$$

[1] If x is not free in assumptions on which βM depends other than αx.

(iii) $B \vdash^{\Sigma} \alpha M$ means that αM is derivable from the basis B in this system.

Note that all rules, except $(\leqslant_\Sigma)$, are independent of Σ.

Let M be a term and B a basis. Define $B \upharpoonright M = \{\alpha x \in B \mid x \in FV(M)\}$ and $FV(B) = \{x \mid \alpha x \in B\}$. We have immediately that $B \vdash^{\Sigma} \alpha M \Leftrightarrow B \upharpoonright M \vdash^{\Sigma} \alpha M$. We say that a type theory $\leqslant_\Sigma$ is *invariant* if $B \vdash^{\Sigma} \alpha M$ and $M =_\beta N$ imply $B \vdash^{\Sigma} \alpha N$.

We assume that the reader is familiar with the notion of λ-model (see Barendregt, 1984, Chap. 5 and, in particular, Sect. 5.3).

The following interpretation of types in λ-models was introduced by Scott in Böhm (1975, Open Problem II.4) and extended to "ω" and "$\wedge$" in Barendregt *et al.* (1983). This semantics of types is sometimes referred to as the *simple* semantics of types. Alternative notions of type semantics have been introduced in the literature, as F-semantics (Hindley, 1983) and quotient set semantics (Scott, 1976; Hindley, 1983).

1.4. Definition. (i) Let $\mathscr{M} = \langle D, \cdot, [\![\ ]\!] \rangle$ be a λ-model, A a set of atomic types and $\mathscr{V}: A \to \mathscr{P}(D)$. Then $\mathscr{V}$ extends to all $\alpha \in T_A$ as follows ($\mathscr{V}$ is a *type interpretation in* $\mathscr{M}$):

$$\mathscr{V}(\omega) = D$$
$$\mathscr{V}(\alpha \to \beta) = \{d \mid \forall e \in \mathscr{V}(\alpha) \quad d \cdot e \in \mathscr{V}(\beta)\}$$
$$\mathscr{V}(\alpha \wedge \beta) = \mathscr{V}(\alpha) \cap \mathscr{V}(\beta).$$

(ii) Let $\leqslant_\Sigma$ be a type theory. A type interpretation $\mathscr{V}$ *respects* $\leqslant_\Sigma$ if $\alpha \leqslant_\Sigma \beta \Rightarrow \mathscr{V}(\alpha) \subseteq \mathscr{V}(\beta)$.

In Barendregt *et al.* (1983) it has been proved that $\alpha \leqslant_\varnothing \beta$ iff for all $\mathscr{M}$ and for all $\mathscr{V}$ in $\mathscr{M}$, $\mathscr{V}(\alpha) \subseteq \mathscr{V}(\beta)$, i.e., all type interpretations respect $\leqslant_\varnothing$, and $\leqslant_\varnothing$ is complete with respect to type inclusion. (This was the original motivation in Barendregt *et al.*, 1983 for the introduction of $\leqslant_\varnothing$.)

1.5. Definition. (i) Let $\mathscr{M}$ be a λ-model, ξ an environment and $\mathscr{V}$ a type interpretation in $\mathscr{M}$,

$$\mathscr{M}, \xi, \mathscr{V} \models \alpha M \quad \text{iff} \quad [\![M]\!]_\xi \in \mathscr{V}(\alpha)$$
$$\mathscr{M}, \xi, \mathscr{V} \models B \quad \text{iff} \quad \forall \alpha x \in B\colon \mathscr{M}, \xi, \mathscr{V} \models \alpha x.$$

(ii)
$$\mathscr{M}, \mathscr{V}, B \models \alpha M \quad \text{iff} \quad \forall \xi\colon [\mathscr{M}, \xi, \mathscr{V} \models B \Rightarrow \mathscr{M}, \xi, \mathscr{V} \models \alpha M]$$
$$\mathscr{M}, B \models^{\Sigma} \alpha M \quad \text{iff} \quad \forall \mathscr{V} \text{ respecting } \leqslant_\Sigma\colon \mathscr{M}, \mathscr{V}, B \models \alpha M$$
$$B \models^{\Sigma} \alpha M \quad \text{iff} \quad \forall \mathscr{M}\colon \mathscr{M}, B \models^{\Sigma} \alpha M.$$

The soundness of $\vdash^{\Sigma}$ with respect to $\models^{\Sigma}$ follows easily.

1.6. THEOREM. $B \vdash^{\Sigma} \alpha M \Rightarrow B \models^{\Sigma} \alpha M$.

Proof. Easy induction on the deduction showing $B \vdash^{\Sigma} \alpha M$. For rule $(\leqslant_{\Sigma})$ notice that $\mathscr{V}$ must respect $\leqslant_{\Sigma}$. ∎

The completeness of $\vdash^{\varnothing}$ has been shown in Barendregt *et al.* (1983) using the filter model $\mathscr{F}$ (defined in Sect. 2) and in Hindley (1982) using the term model of β-equality. In Coppo *et al.* (1983b) the result of Barendregt *et al.* (1983) was generalized to all invariant type theories.

In the rest of this section we prove the completeness of type assignment for invariant type theories with respect to the term model of β-equality. The proof is given using the technique of Hindley (1982). This result is not needed in the rest of the paper, and can be skipped at a first reading (some technical lemmas will be used in the following).

The following properties have been proved in Barendregt *et al.* (1983) for the theory $\leqslant_{\varnothing}$ and can easily be extended to all $\leqslant_{\Sigma}$.

If B is a basis, let $B/x = \{\alpha y \mid \alpha y \in B \text{ and } y \not\equiv x\}$.

1.7. LEMMA. (i) *$B \vdash^{\Sigma} \alpha x$ iff $\exists \beta_1 x, ..., \beta_n x \in B$ $(n \geqslant 0)$ such that $\omega \wedge \beta_1 \wedge \cdots \wedge \beta_n \leqslant_{\Sigma} \alpha$.*

(ii) *$B \vdash^{\Sigma} \alpha(MN)$ implies $\exists \beta \in T_A$ such that $B \vdash^{\Sigma} \beta \to \alpha M$ and $B \vdash^{\Sigma} \beta N$.*

(iii) *$B/z \cup \{\beta z\} \vdash^{\Sigma} \alpha(Mz)$ and $z \notin FV(M) \Rightarrow B \vdash^{\Sigma} \beta \to \alpha M$.*

Proof. (i) By the rules (ω), $(\leqslant_{\Sigma})$, $(\wedge I)$ and $(\wedge E)$ (no other rule is applicable).

(ii) By induction on the derivation of $\alpha(MN)$. The only interesting case is when the last applied rule is $(\wedge I)$, i.e.,

$$\frac{\alpha_1(MN) \quad \alpha_2(MN)}{\alpha_1 \wedge \alpha_2(MN)}.$$

By the induction hypothesis there are β_1, β_2 such that $B \vdash^{\Sigma} \beta_i \to \alpha_i M$, $B \vdash^{\Sigma} \beta_i N$ for $i = 1, 2$. Then $B \vdash^{\Sigma} \beta_1 \wedge \beta_2 N$ and $B \vdash^{\Sigma} (\beta_1 \to \alpha_1) \wedge (\beta_2 \to \alpha_2) M$. It is easy to verify that $(\beta_1 \to \alpha_1) \wedge (\beta_2 \to \alpha_2) \leqslant_{\Sigma} (\beta_1 \wedge \beta_2) \to (\alpha_1 \wedge \alpha_2)$, so we can take $\beta \equiv \beta_1 \wedge \beta_2$.

(iii) By (ii) $\exists \gamma$: $B/z \cup \{\beta z\} \vdash^{\Sigma} \gamma \to \alpha M$ and $B/z \cup \{\beta z\} \vdash^{\Sigma} \gamma z$. (i) ensures that $\beta \leqslant_{\Sigma} \gamma$, so, by rule $(\leqslant_{\Sigma})$, $B/z \cup \{\beta z\} \vdash^{\Sigma} \beta \to \alpha M$. Then $B \vdash^{\Sigma} \beta \to \alpha M$, since $z \notin FV(M)$. ∎

Recall that the term model of β-equality $\mathcal{M}(\beta)=\langle \Lambda/\beta, \cdot, [\![\ ]\!]^{\mathcal{M}(\beta)}\rangle$ is defined by

$$\Lambda/\beta=\{[M] \mid M \text{ is a term}\} \text{ where } [M]=\{N \mid N=_\beta M\}$$

$$[M]\cdot[N]=[MN]$$

and

$$[\![M]\!]_\xi^{\mathcal{M}(\beta)}=[M[x_1:=M_1,..., x_n:=M_n]], \qquad \text{where} \quad \xi(x_i)=[M_i] \quad \text{and} \quad FV(M)=\{x_1,..., x_n\}.$$

As proved in Hindley (1982, 1983), given a term M we can extend any basis B to a basis B^+ which contains infinitely many statements $\alpha y_{\alpha,i}$ for all $\alpha \in T_A$ and $i \in \mathbf{N}$, where the variables $y_{\alpha,1}$ are all distinct and do not occur in B and M. We refer to Hindley (1982, 1983) for the technical details. Note that $B \vdash^\Sigma \alpha M$ iff $B^+ \vdash^\Sigma \alpha M$.

Let Var denote the set of term variables.

1.8 Definition. (i) $\mathcal{V}_B^\Sigma(\varphi)=\{[M] \mid B^+ \vdash^\Sigma \varphi M\}$.

(ii) ξ_0: Var $\rightarrow \Lambda/\beta$ is the environment defined by

$$\xi_0(x)=[x].$$

It is easy to verify that $[\![M]\!]_{\xi_0}^{\mathcal{M}(\beta)}=[M]$.

1.9. Lemma. *Let $\leqslant_\Sigma$ be an invariant type theory.*

(i) $\mathcal{V}_B^\Sigma(\alpha)=\{[M] \mid B^+ \vdash^\Sigma \alpha M\}$ *for all* $\alpha \in T_A$.

(ii) $\mathcal{V}_B^\Sigma$ *respects* $\leqslant_\Sigma$.

(iii) $\mathcal{M}(\beta), \xi_0, \mathcal{V}_B^\Sigma \models B$.

Proof. (i) By induction on α. If α is atomic, use Definition 1.8. If $\alpha \equiv \beta \wedge \gamma$, use the induction hypothesis and rule $(\wedge I)$. If $\alpha \equiv \beta \rightarrow \gamma$, we have

$$(\supseteq) \qquad B^+ \vdash^\Sigma \beta \rightarrow \gamma M \Rightarrow \forall N(B^+ \vdash^\Sigma \beta N \Rightarrow B^+ \vdash^\Sigma \gamma(MN))$$

by $(\rightarrow E)$

$$\Rightarrow \forall N([N] \in \mathcal{V}_B^\Sigma(\beta) \Rightarrow [MN] \in \mathcal{V}_B^\Sigma(\gamma))$$

by the induction hypothesis

$$\Rightarrow [M] \in \mathcal{V}_B^\Sigma(\beta \rightarrow \gamma).$$

$$(\subseteq) \qquad [M]\in\mathscr{V}_B^{\Sigma}(\beta\to\gamma)\Rightarrow(\forall N[N]\in\mathscr{V}_B^{\Sigma}(\beta)\Rightarrow[MN]\in\mathscr{V}_B^{\Sigma}(\gamma))$$

by definition

$$\Rightarrow \forall N(B^+\vdash^{\Sigma}\beta N\Rightarrow B^+\vdash^{\Sigma}\gamma(MN))$$

by the induction hypothesis

$$\Rightarrow B^+\vdash^{\Sigma}\gamma(Mz)$$

where $z\equiv y_{\beta,i}\notin FV(M)$

$$\Rightarrow B^+\vdash^{\Sigma}\beta\to\gamma M$$

by 1.7(iii) since $B^+=B^+/z\cup\{\beta z\}$.

(ii) Immediate from (i) and rule $(\leqslant_{\Sigma})$.

(iii) $\alpha x\in B\Rightarrow B^+\vdash^{\Sigma}\alpha x$

$$\Rightarrow [x]\in\mathscr{V}_B^{\Sigma}(\alpha) \qquad \text{from (i)}$$

$$\Rightarrow [\![x]\!]_{\xi_0}^{\mathscr{M}(\beta)}\in\mathscr{V}_B^{\Sigma}(\alpha) \qquad \text{by definition of } \xi_0. \quad ∎$$

1.10. THEOREM (Completeness). *Let $\leqslant_{\Sigma}$ be an invariant type theory.*

(i) $\mathscr{M}(\beta),\ \xi_0,\ \mathscr{V}_B^{\Sigma}\models\alpha M\Rightarrow B\vdash^{\Sigma}\alpha M.$

(ii) $B\vdash^{\Sigma}\alpha M\Leftrightarrow B\models^{\Sigma}\alpha M.$

Proof. (i)

$$\begin{aligned}\mathscr{M}(\beta),\ \xi_0,\ \mathscr{V}_B^{\Sigma}\models\alpha M &\Rightarrow [\![M]\!]_{\xi_0}^{\mathscr{M}(\beta)}\in\mathscr{V}_B^{\Sigma}(\alpha)\\ &\Rightarrow [M]\in\mathscr{V}_B^{\Sigma}(\alpha)\\ &\Rightarrow B^+\vdash^{\Sigma}\alpha M \qquad \text{by 1.9(i)}\\ &\Rightarrow B\vdash^{\Sigma}\alpha M.\end{aligned}$$

(ii) $(\Rightarrow)$ By 1.6.

$(\Leftarrow)$ Immediate from (i) and 1.9(ii). ∎

Remarks. (i) As suggested by one of the referees, one could extend Definitions 1.4 and 1.5 to structures $\langle D,\cdot,[\![\ ]\!]\rangle$ that do not satisfy β and ask about a general completeness theorem over such weaker structures. We conjecture that a result of this kind could be proved by using $\langle\mathscr{F}^{\Sigma},\cdot,[\![\ ]\!]^{\Sigma}\rangle$ (as defined in Sect. 2).

(ii) Our completeness theorem extends easily to quotient set semantics (in fact completeness with respect to the simple semantics implies completeness with respect to the quotient set semantics), but it does not extend, in general, to F-semantics (Dezani and Margaria, 1984).

2. Filter Models

Filter models have been introduced in Barendregt *et al.* (1983) and have been studied to some extent in Coppo *et al.* (1983b).

Given a type theory $\leqslant_\Sigma$ over T_A, we define the corresponding filter domain by taking the subsets of T_A closed under $\wedge$ and $\leqslant_\Sigma$. We refer to Coppo *et al.* (1983b) for more discussions and proofs.

2.1. Definition. Let $\leqslant_\Sigma$ be a type theory over T_A.

(i) An *abstract filter d* with respect to $\leqslant_\Sigma$ is a non empty subset of T_A such that:

(1) $\alpha, \beta \in d \Rightarrow \alpha \wedge \beta \in d$,

(2) $\alpha \in d$ and $\alpha \leqslant_\Sigma \beta \Rightarrow \beta \in d$.

(ii) If $X \subseteq T_A$, ${}^\Sigma\!\uparrow X$ is the minimal abstract filter which contains X. If $a \in T_A$, then we use the abbreviation ${}^\Sigma\!\uparrow a$ for ${}^\Sigma\!\uparrow \{a\}$.

(iii) $\mathscr{F}^\Sigma = \{d \mid d$ is an abstract filter with respect to $\leqslant_\Sigma\}$ is the *filter domain* of the type theory $\leqslant_\Sigma$.

Filter domains can be seen as a particular case of Scott's Information Systems (Scott, 1982) over T_A. In fact, we can interpret "$\wedge$" as the constructor of finite subsets of T_A, whose elements either are atomic types or are built by the "$\rightarrow$" operator. Our inclusion relation corresponds to the entailment relation. More precisely, $\beta_1 \wedge \cdots \wedge \beta_n \leqslant_\Sigma \alpha$ iff $\{\beta_1,..., \beta_n\} \vdash \alpha$ in the corresponding information system (the translation is explicitly given in Coppo *et al.*, 1983a).

Let us recall that an applicative structure is a pair $\langle D, \cdot\rangle$ where "$\cdot$": $D \times D \rightarrow D$. Filter domains can be turned into applicative structures in the following way.

2.2. Definition. "$\cdot$": $\mathscr{F}^\Sigma \times \mathscr{F}^\Sigma \rightarrow \mathscr{F}^\Sigma$ is defined by $d \cdot e = \{\beta \mid \exists \alpha \in e\ \alpha \rightarrow \beta \in d\}$.

It can be easily checked that application is well defined, i.e., that $d \cdot e \in \mathscr{F}^\Sigma$ whenever d, $e \in \mathscr{F}^\Sigma$.

In Coppo *et al.* (1983b) it has been proved that filter domains are complete ω-algebraic lattices (with respect to set inclusion) in which ${}^\Sigma\!\uparrow \omega$ is the least element and $\{{}^\Sigma\!\uparrow \alpha \mid \alpha \in T_A\}$ is the set of the finite elements (recall that an element d of a complete lattice D is finite iff, for every directed subset $X \subseteq D$, $d \sqsubseteq \bigsqcup X$ implies that $\exists e \in X$ such that $d \sqsubseteq e$). Moreover "$\cdot$" is continuous in both arguments (with respect to the Scott topology; Barendregt, 1984, 1.2).

An applicative structure $\langle D, \cdot \rangle$ is *extensional* iff $d_1 \cdot d = d_2 \cdot d$ for all $d \in D$ implies $d_1 = d_2$. There is a simple characterization of the type theories whose filter domains determine extensional applicative structures. The result is from Coppo *et al.* (1983b, Theorem 2.16).

2.3. THEOREM. *Let* $\leqslant_\Sigma$ *be a type theory over* T_A. $\langle \mathscr{F}^\Sigma, \cdot \rangle$ *is an extensional applicative structure iff for all* $\varphi \in A$, $\varphi \sim_\Sigma (\alpha_1 \to \beta_1) \wedge \cdots \wedge (\alpha_n \to \beta_n)$ *for some* $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n \in T_A$.

We can now introduce the notion of interpretation of terms of the λ-calculus in filter domains. If D is a domain, let $[D \to D]$ denote the set of continuous functions from D to D (in the Scott topology; Barendregt, 1984, 1.2).

2.4. DEFINITION. (i) Let $G^\Sigma : [\mathscr{F}^\Sigma \to \mathscr{F}^\Sigma] \to \mathscr{F}^\Sigma$ be defined as $G^\Sigma(f) = {}^\Sigma\!\uparrow \{\alpha \to \beta \mid \beta \in f({}^\Sigma\!\uparrow \alpha)\}$.

(ii) Let $\xi : \mathrm{Var} \to \mathscr{F}^\Sigma$ and M a term. The interpretation of M in $\mathscr{F}^\Sigma$ via ξ (denoted $[\![M]\!]^\Sigma_\xi$) is defined by:

(1) $[\![x]\!]^\Sigma_\xi = \xi(x)$,

(2) $[\![MN]\!]^\Sigma_\xi = [\![M]\!]^\Sigma_\xi \cdot [\![N]\!]^\Sigma_\xi$,

(3) $[\![\lambda x.M]\!]^\Sigma_\xi = G^\Sigma(\lambda v.[\![M]\!]^\Sigma_{\xi[v/x]})$.

(iii) $\mathscr{F}^\Sigma$ is a *filter model* iff $\langle \mathscr{F}^\Sigma, \cdot, [\![\ ]\!]^\Sigma \rangle$ is a λ-model.

The crucial point in Definition 2.4(ii) is, obviously, (3). In fact, we have that $\langle \mathscr{F}^\Sigma, \cdot, [\![\ ]\!]^\Sigma \rangle$ is a λ-model iff $[\![\lambda x.M]\!]^\Sigma_\xi \cdot d = [\![M]\!]^\Sigma_{\xi[d/x]}$. This follows from Definitions 5.3.1–5.3.2 of Barendregt (1984), since we can easily show (Coppo *et al.*, 1983b) that rule (ξ) of λ-calculus (weak extensionality) is always satisfied in filter domains. As an immediate consequence we have that if $\langle \mathscr{F}^\Sigma, \cdot, [\![\ ]\!]^\Sigma \rangle$ is a λ-algebra (in the sense of Barendregt and Koymans, 1980; Barendregt, 1984, 5.2; Meyer, 1981) then it is also a λ-model (note that there are λ-algebras which are not λ-models, see Barendregt and Koymans, 1980).

Not all type theories yield filter models. We give a syntactic characterization of them in Theorem 2.6.

Given a type theory $\leqslant_\Sigma$, the interpretation of a term in $\langle \mathscr{F}^\Sigma, \cdot, [\![\ ]\!]^\Sigma \rangle$ coincides with the set of types that can be assigned to it, as stated in the next theorem, proved in Coppo *et al.* (1983b, Theorem 4.7). If $\xi : \mathrm{Var} \to \mathscr{F}^\Sigma$, B_ξ is the basis defined as $B_\xi = \{\alpha x \mid \alpha \in \xi(x)\}$.

2.5. THEOREM. $[\![M]\!]^{\Sigma}_{\xi} = \{\alpha \mid B_{\xi} \vdash^{\Sigma} \alpha M\}$.

Theorem 2.5 is very useful since it gives a constructive definition of the interpretation of M in $\mathscr{F}^{\Sigma}$. It has been used in Barendregt *et al.* (1983) to prove the completeness of the type assignment induced by $\leqslant_{\varnothing}$.

Through Theorem 2.5 we can also give a characterization of the type theories that yield λ-models. This is also a result of Coppo *et al.* (1983b).

2.6. THEOREM. *$\langle \mathscr{F}^{\Sigma}, \cdot, [\![\;]\!]^{\Sigma} \rangle$ is a λ-model iff*

$$[B \vdash^{\Sigma} \alpha \to \beta \; \lambda x.M \Rightarrow B/x \cup \{\alpha x\} \vdash^{\Sigma} \beta M].$$

Since, in a λ-model, all β-convertible terms have the same value (in a given environment), we have easily the following:

2.7. COROLLARY. (i) *$\langle \mathscr{F}^{\Sigma}, \cdot, [\![\;]\!]^{\Sigma} \rangle$ is a λ-model iff the type theory $\leqslant_{\Sigma}$ is invariant.*

(ii) *Types are invariant under β-conversion of terms iff they are so under β-reduction.*

Proof. (i) (Only if) Immediate by 2.5.

(If) By 2.6 we must prove that $B \vdash^{\Sigma} \alpha \to \beta \; \lambda x.M \Rightarrow B/x \cup \{\alpha x\} \vdash^{\Sigma} \beta M$. If we assume $z \notin FV(M)$ we have

$$\begin{aligned} B \vdash^{\Sigma} \alpha \to \beta \; \lambda x.M &\Rightarrow B/z \cup \{\alpha z\} \vdash^{\Sigma} \beta((\lambda x.M)\, z) \\ &\Rightarrow B/z \cup \{\alpha z\} \vdash^{\Sigma} \beta M[x := z] \\ &\Rightarrow B/x \cup \{\alpha x\} \vdash^{\Sigma} \beta M. \end{aligned}$$

(ii) It is easy to verify that types are invariant under β-expansion. Let $\beta_1 N, \ldots, \beta_n N$ be all the statements whose subject is N in a proof of $B \vdash^{\Sigma} \alpha M[x := N]$. Then we can build a proof of $B \vdash^{\Sigma} (\beta_1 \wedge \cdots \wedge \beta_n) \to \alpha \; \lambda x.M$, a proof of $B \vdash^{\Sigma} \beta_1 \wedge \cdots \wedge \beta_n N$ and conclude (using $(\to E)$) $B \vdash^{\Sigma} \alpha((\lambda x.M)\, N)$. ∎

An interesting class of type theories induces filter models which are isomorphic to λ-models defined by the classical Scott's inverse limit construction (Scott, 1972).

We assume the reader has some familiarity with the construction of Scott (1972). An inverse limit space D which satisfies $D = [D \to D]$ (up to isomorphism) is defined starting from a continuous lattice D_0 and a projection (i, j) of $D_1 = [D_0 \to D_0]$ on D_0. Let us recall that if D, D' are two continuous lattices, a projection of D' on D is a pair of continuous maps $i: D \to D'$ and $j: D' \to D$ such that $i \circ j \sqsubseteq \mathrm{id}_{D'}$ and $j \circ i = \mathrm{id}_D$ (where id_E represents the identity function on the domain E).

The choice of D_0 and (i, j) determines for each $n \geqslant 0$ a projection (i_n, j_n) of $D_{n+1} = [D_n \to D_n]$ on D_n. The inverse limit $D_\infty = \varprojlim D_n$ of this chain is, in its turn, a continuous lattice. Moreover, each D_n can be embedded in D_∞. In the following discussion we identify the elements of D_n with their projections in D_∞. As proved in Scott (1972), D_∞ satisfies $D_\infty = [D_\infty \to D_\infty]$ (up to isomorphism) and, then, yields an extensional λ-model (Barendregt, 1984, 18.2).

Given a lattice D and $a, b \in D$, let $f_{a,b}$ be the (step) function defined by $f_{a,b}(x) = \mathit{if}\ a \sqsubseteq x\ \mathit{then}\ b\ \mathit{else}\ \bot$. In the standard construction of D_∞, the initial projections (i_s, j_s) are defined by $i_s(a) = f_{\bot,a}$ and $j_s(f) = f(\bot)$, while the choice of D_0 is somewhat arbitrary. In fact it is well known (Barendregt, 1984, 19.2) that, in this case, the local structure of the model is independent of D_0.

The connections between (extensional) filter domains and inverse limit spaces have been investigated in Coppo *et al.* (1983b) and Coppo *et al.* (1983a, 1984). The general result of Coppo *et al.* (1984) is that each inverse limit space D_∞, built from a countably based algebraic D_0, is isomorphic both as lattice and as applicative structure to a (constructively defined) filter domain.

2.8. DEFINITION. Let D_0 be a countably based algebraic complete lattice (with respect to a partial order $\sqsubseteq$) and (i, j) a projection of $[D_0 \to D_0]$ on D_0. Define A as the set of atomic types indexed by the finite elements of D_0 different from $\bot$ (i.e., $A = \{\varphi_a \mid a \neq \bot$ and a is a finite element of $D_0\}$) and $\leqslant_+$ as the type theory over T_A generated by

$$\varphi_a \leqslant_+ \varphi_b \qquad \text{if} \quad b \sqsubseteq a \qquad (b \neq \bot),$$

$$\varphi_a \sim_+ (\varphi_{a_1} \to \varphi_{b_1}) \wedge \cdots \wedge (\varphi_{a_n} \to \varphi_{b_n}) \qquad \text{if} \quad i(a) = \bigsqcup_{1 \leqslant j \leqslant n} f_{a_j, b_j} \quad (a \neq \bot)$$

where we assume $\varphi_\bot \equiv \omega$.

2.9. THEOREM. *Let D_0, (i, j), A, and $\leqslant_+$ be defined as in 2.8, D_∞ be the inverse limit space built starting from D_0 using (i, j), and $\mathscr{F}^+$ be the filter domain of the type theory $\leqslant_+$. Then D_∞ and $\mathscr{F}^+$ are isomorphic both as lattices and as applicative structures.*

For example, the standard D_∞ of Scott (1982) built starting from a two point lattice $D_0 = \{\bot, \top\}$ is isomorphic to $\mathscr{F}^{\Sigma_0}$ where $A_0 = \{\varphi_\top\}$ and $\Sigma_0 \subseteq T_{A_0} \times T_{A_0}$ is the type theory generated by $\varphi_\top \sim_{\Sigma_0} \omega \to \varphi_\top$.

Theorem 2.9 is proved in Coppo *et al.* (1984) using the (slightly different) formalism of Scott's information systems (Scott, 1982) (in the same paper, however, the connections between the two formalisms are clarified). But the idea behind this construction can be easily shown. Note that in this isomorphism, ${}^{+}\!\uparrow \varphi_a$ corresponds to (the projection in D_∞ of) $a \in D_0$, ${}^{+}\!\uparrow (\varphi_a \to \varphi_b)$ corresponds to (the projection in D_∞ of) $f_{a,b} \in [D_0 \to D_0]$ and if ${}^{+}\!\uparrow \alpha$ corresponds to d, ${}^{+}\!\uparrow \beta$ corresponds to e, then ${}^{+}\!\uparrow \alpha \wedge \beta$ corresponds to $d \sqcup e$. Therefore, since $\bigsqcup_{1 \leqslant j \leqslant n} f_{aj,bj}$ is projected on a, we force φ_a to be equivalent to $(\varphi_{a_1} \to \varphi_{b_1}) \wedge \cdots \wedge (\varphi_{a_n} \to \varphi_{b_n})$, thus obtaining

$$ {}^{+}\!\uparrow \varphi_a = {}^{+}\!\uparrow ((\varphi_{a_1} \to \varphi_{b_1}) \wedge \cdots \wedge (\varphi_{a_n} \to \varphi_{b_n})). $$

Recall that if $\langle D, \cdot \rangle$ is an extensional applicative structure, there is (if any) a unique possible choice of the interpretation $[\![\ ]\!]$ of a term (Hindley and Longo, 1980; Meyer, 1981). Therefore the isomorphism of $\langle D_\infty, \cdot \rangle$ and $\langle \mathscr{F}^+, \cdot \rangle$ as applicative structures determines the isomorphism of the corresponding λ-models (Barendregt, 1983).

3. The Model D_∞^*

In this section we define an inverse limit space D_∞^* which has a non Hilbert–Post complete theory. Moreover D_∞^* provides a simple semantic characterization of normalizable terms (see Sect. 4).

The definition of D_∞^* is given through the corresponding type theory $\leqslant_*$, which has been suggested by a type assignment system introduced in Coppo and Dezani (1979a) for studying termination property of terms, and further developed in Sallé (1978) and Coppo *et al.* (1979b).

3.1. Definition. Let $A^* = \{\varphi_*, \varphi_\top\}$. $\leqslant_* \subseteq T_{A^*} \times T_{A^*}$ is the minimal inclusion relation such that:

(1) $\varphi_\top \leqslant_* \varphi_*$,

(2) $\varphi_* \sim_* \varphi_\top \to \varphi_*$,

(3) $\varphi_\top \sim_* \varphi_* \to \varphi_\top$.

In the theory $\leqslant_*$ type φ_* is intended to represent the class of normalizable terms while type $\varphi_\top$ is intended to represent the class of terms M such that NM is normalizable whenever N is normalizable. So the property represented by $\varphi_\top$ is stronger than the property represented by φ_* (point 1 of Definition 3.1) and a normalizable term yields a normalizable term whenever applied to a term having type $\varphi_\top$ (point 2 of Definition 3.1). Analogously the application of a term having type $\varphi_\top$ to a term having type φ_* yields a term having type $\varphi_\top$ (Definition 3.1(3)). These properties of the type assignment $\vdash^*$ are proved in Section 4.

Theorem 2.9 suggests the following definition of an inverse limit space D_∞^* isomorphic to the filter domain $\mathscr{F}^*$ generated by $\leqslant_*$.

3.2. DEFINITION. (i) Let D^* be the three element complete lattice $\{\bot, *, \top\}$ where $\bot \sqsubseteq * \sqsubseteq \top$ and (i_*, j_*) be the projection of $[D^* \to D^*]$ on D^* such that

$$i_*(\bot) = f_{\bot,\bot}$$
$$i_*(*) = f_{\top,*}$$
$$i_*(\top) = f_{*,\top}.$$

(ii) D^*_∞ is the inverse limit space built from D^* and (i_*, j_*).

(iii) $\langle D^*_\infty, \cdot, [\![\ ]\!]\rangle$ is the λ-model determined by D^*_∞ (where "$\cdot$" is defined following Scott, 1972).

Observe that (i_*, j_*) is well defined by Proposition 3.10 of Scott (1972) since given i_* there is a unique j_* such that (i_*, j_*) is a projection of $[D^* \to D^*]$ on D^* (namely $j_*(f) = \bigsqcup\{x \mid i_*(x) \sqsubseteq f\}$ where $f \in [D^* \to D^*]$ and $x \in D^*$). As an immediate consequence of 2.9 we have

3.3. THEOREM. *$\langle D^*_\infty, \cdot\rangle$ and $\langle \mathscr{F}^*, \cdot\rangle$ are isomorphic (and, hence, $\langle \mathscr{F}^*, \cdot, [\![\ ]\!]^*\rangle$ is a λ-model).*

From Theorem 3.3 we obtain the following corollary, which will be useful later.

3.4. COROLLARY. (i) $M =_{\beta\eta} N \Rightarrow [B \vdash^* \alpha M \Leftrightarrow B \vdash^* \alpha N]$.

(ii) $B \vdash^* \alpha \to \beta \lambda x.M \Rightarrow B/x \cup \{\alpha x\} \vdash^* \beta M$.

(iii) $\alpha \to \beta \leqslant_* \gamma \to \delta$, *where* $\delta \not\sim_* \omega \Rightarrow \gamma \leqslant_* \alpha$ *and* $\beta \leqslant_* \delta$.

Proof. (i) By 2.7(i) and the fact that $\mathscr{F}^*$ is extensional.

(i) By 2.6.

(iii) By Theorem 2.13 of Coppo *et al.* (1983b) since all continuous functions (and, hence, all step functions) are representable in $\mathscr{F}^*$. ∎

In the rest of this section we characterize the formal theory of $\mathscr{F}^*$. Obviously all results hold also for D^*_∞.

Recall that the *theory of a λ-model* $\mathscr{M} = \langle D, \cdot, [\![\ ]\!]\rangle$ is the set $\mathrm{Th}(\mathscr{M}) = \{M = N \mid [\![M]\!]_\xi = [\![N]\!]_\xi$ for all environments $\xi\}$ (cf. e.g., Barendregt, 1984, 19). A theory Th is *Hilbert–Post complete* iff for every equation $M = N$ either $M = N \in \mathrm{Th}$ or $\mathrm{Th} \cup \{M = N\}$ is inconsistent.

It is well known (Hyland, 1976; Barendregt, 1984, 19.2) that the theory of each inverse limit space D_∞ built from the standard projections (i_s, j_s) is complete independently of the initial lattice D_0. More precisely, we have $\mathrm{Th}(D_\infty) = \mathscr{H}^*$ where $\mathscr{H}^*$ is the (unique) completion of the theory $\mathscr{H}$ obtained by equating all unsolvable terms. In the case of D^*_∞ we prove

$\mathscr{H} \subsetneq \mathrm{Th}(D_\infty^*) \subsetneq \mathscr{H}^*$. In particular $\mathrm{Th}(D_\infty^*)$ coincides with Morris' extensional theory (Morris, 1968) called $\mathscr{I}_{NF}$ in Barendregt (1984, 16.4).

We need the standard definitions of λ–Ω–calculus and Ω-reduction rules. Following Barendregt (1984, 14.3) the set of λ–Ω-terms is obtained by adding a constant Ω to the formation rules of terms. In the λ–Ω-calculus we have the following Ω-reduction rules (besides rules α and β),

$$\Omega M \to \Omega$$

$$\lambda x.\Omega \to \Omega$$

for all M and x.

The notion of $\beta\eta\Omega$-conversion $=_{\beta\eta\Omega}$ is introduced in the usual way (Wadsworth, 1976). A λ–Ω-term A is a *$\beta\eta\Omega$-normal form* ($\beta\eta\Omega$-n.f.) iff A cannot be reduced using β, η and Ω-reduction rules.

Let P be a λ–Ω-term and A and $\beta\eta\Omega$-n.f., A is an *approximant* of P ($A \precapprox_\eta P$) iff $\exists P' =_{\beta\eta} P$ such that A matches P' except at occurrences of Ω in A. Last, define $\mathscr{A}_\eta(P) = \{A \mid A \precapprox_\eta P\}$. It is well known that the set of $\beta\eta\Omega$-n.f.s is a coherent c.p.o. with respect to $\precapprox_\eta$ and that, for all P, $\mathscr{A}_\eta(P)$ is directed. The type assignment of Definition 1.3 can be extended to λ–Ω-terms without modifications.

3.5. THEOREM. *$B \vdash^* \alpha M \Leftrightarrow \exists A \in \mathscr{A}_\eta(M)$ such that $B \vdash^* \alpha A$.*

Proof. ($\Leftarrow$) By definition, there is $M' =_{\beta\eta} M$ such that A matches M' except at occurrences of Ω in A. Therefore we obtain a deduction of $B \vdash^* \alpha M'$ using rule (ω) to assign type ω to the terms which are replaced by Ω in A. Last, we obtain $B \vdash^* \alpha M$ by 3.4(i).

($\Rightarrow$) The rather technical proof is given in Appendix A. ∎

3.6. COROLLARY. $M =_{\beta\eta\Omega} N \Rightarrow [B \vdash^* \alpha M \Leftrightarrow B \vdash^* \alpha N]$.

Proof. $B \vdash^* \alpha M \Rightarrow \exists A \precapprox_\eta M\ B \vdash^* \alpha A \Rightarrow B \vdash^* \alpha N$ since $A \precapprox_\eta N$. ∎

Let us extend $[\![\ ]\!]^*$ to λ–Ω-terms by assuming $[\![\Omega]\!]_\xi^* = {}^*\!\uparrow \omega$.

3.7 THEOREM (Approximation theorem). $[\![M]\!]_\xi^* = \bigsqcup \{[\![A]\!]_\xi^* \mid A \in \mathscr{A}_\eta(M)\}$.

Proof. Immediate from 3.5 and 2.5. ∎

By 3.7 we have that $\mathscr{F}^*$ is a continuous λ-model in the sense of Barendregt (1984, 19.3). Continuous λ-models have many interesting properties. The most important one is that the fixed point combinator **Y** is interpreted as the least fixed point operator.

Let $M \sqsubseteq^* N$ iff $\mathscr{F}^* \models M \subseteq N$ (i.e., if, for all ξ, $[\![M]\!]^*_\xi \subseteq [\![N]\!]^*_\xi$) and $M = {}^*N$ iff $\mathscr{F}^* \models M = N$.

The definition of context $C[\ ]$ is standard (see, e.g., Barendregt, 1984, 2.1.18).

3.8 Lemma. (i) *$M \sqsubseteq^* N$ and $B \vdash^* \alpha C[M]$ imply $B \vdash^* \alpha C[N]$.*

(ii) $B \vdash^* \alpha\Omega \Leftrightarrow \alpha \in {}^*\!\uparrow \omega$.

(iii) $\varphi_* \to \varphi_* \notin {}^*\!\uparrow \omega$.

Proof. (i) Note that by 2.5 $M \sqsubseteq^* N$ iff for all basis B': $\{\beta \mid B' \vdash^* \beta M\} \subseteq \{\beta \mid B' \vdash^* \beta N\}$. Therefore, given a deduction of $B \vdash^* \alpha C[M]$, we obtain a deduction of $B \vdash^* \alpha C[N]$ simply by replacing each subdeduction of βM by a deduction of βN.

(ii) $(\Rightarrow)$ By induction on deductions (note that we can apply only rules (ω), $(\wedge I)$, $(\wedge E)$, and $(\leqslant_*)$).

$(\Leftarrow)$ Trivial.

(iii) Define $\mathbf{\Omega} \subseteq T_{A^*}$ inductively by $\omega \in \mathbf{\Omega}$; $\alpha \in \mathbf{\Omega} \Rightarrow \beta \to \alpha \in \mathbf{\Omega}$; $\alpha, \beta \in \mathbf{\Omega} \Rightarrow \alpha \wedge \beta \in \mathbf{\Omega}$. By induction on the definition of $\leqslant_*$ one can show $\alpha \in \mathbf{\Omega}$, $\alpha \leqslant_* \beta \Leftrightarrow \beta \in \mathbf{\Omega}$. It follows that $\alpha \in \mathbf{\Omega} \Leftrightarrow \alpha \sim_* \omega$. Clearly $\varphi_* \to \varphi_* \notin \mathbf{\Omega}$. ∎

Let us recall the following properties of $\lesssim_\eta$ which are proved in Wadsworth (1976).

3.9. Lemma. *$A \lesssim_\eta M$ and $A' \lesssim_\eta C[A]$ imply $A' \lesssim_\eta C[M]$ for all contexts $C[\ ]$.*

(ii) *If $A \not\lesssim_\eta N$ then there is a context $C[\ ]$ such that $C[A] =_{\beta\eta\Omega} \mathbf{I} \equiv \lambda x.x$ and $C[N] =_{\beta\eta\Omega} \Omega$.*

3.10. Theorem (Characterization of $\sqsubseteq^*$).

(i) $M \sqsubseteq^* N \Leftrightarrow \mathscr{A}_\eta(M) \subseteq \mathscr{A}_\eta(N)$.

(ii) $M =^* N \Leftrightarrow \mathscr{A}_\eta(M) = \mathscr{A}_\eta(N)$.

Proof. (i)$(\Leftarrow)$ Immediate from 3.7.

$(\Rightarrow)$ Assume that $\mathscr{A}_\eta(M) \not\subseteq \mathscr{A}_\eta(N)$ and $M \sqsubseteq^* N$.

$$\begin{aligned}
&\mathscr{A}_\eta(M) \not\subseteq \mathscr{A}_\eta(N) \\
&\quad \Rightarrow \exists A \lesssim_\eta M\ A \not\lesssim_\eta N \\
&\quad \Rightarrow \exists A \lesssim_\eta M\ \exists C[\ ]\ C[A] =_{\beta\eta\Omega} \mathbf{I} \text{ and } C[N] =_{\beta\eta\Omega} \Omega \qquad \text{by 3.9(ii).}
\end{aligned}$$

Now we have $\vdash^* \varphi_* \to \varphi_* C[M]$ by 3.5 since $\mathbf{I} \precsim_\eta C[M]$ by 3.9(i) and $\vdash^* \varphi_* \to \varphi_* \mathbf{I}$. But, then, $M \sqsubseteq^* N$ implies $\vdash^* \varphi_* \to \varphi_* C[N]$ by 3.8(i) which is a contradiction by 3.8(ii) and (iii) since $C[N] =_{\beta\eta\Omega} \Omega$.

(ii) Immediate from (i). ∎

As proved in Hyland (1975) the theory $\{M = N \mid \mathscr{A}_\eta(M) = \mathscr{A}_\eta(N)\}$ is exactly $\mathscr{I}_{NF}$ of Barendregt (1984, 16.4), where it is proved that $\mathscr{H} \subsetneq \mathscr{I}_{\mathrm{NF}} \subsetneq \mathscr{H}^*$. Then we have $\mathscr{H} \subsetneq \mathrm{Th}(\mathscr{F}^*) \subsetneq \mathscr{H}^*$. This result extends immediately to D_∞^* by the isomorphism proved in 3.3.

3.11. THEOREM. $\mathrm{Th}(D_\infty^*) = \mathscr{I}_{NF}$ *is not Hilbert–Post complete.*

4. D_∞^* AND NORMALIZATION PROPERTIES OF TERMS

In this section we use the results of Sections 2 and 3 to prove some properties of normalizable terms. The first one is that the terms having a normal form can be characterized in $\mathscr{F}^*$ (D_∞^*) as the terms whose value, in a suitable environment, is greater than $*\uparrow\varphi_*$ (the projection in D_∞^* of $*$). This is, to the author's knowledge, the first purely semantic characterization of the normalizable terms.

Moreover we prove the completeness of $\vdash^*$ for all types without occurrences of "$\wedge$" with respect to a type interpretation $\mathscr{V}^*$ in $\mathscr{M}(\beta)$ which is based on normalization properties of terms. More precisely $\mathscr{V}^*(\varphi_*)$ is the set of all the normalizable terms (modulo $=_\beta$) and $\mathscr{V}^*(\varphi_\top)$ is the set of all terms M such that, for all normalizable N, NM is normalizable (modulo $=_\beta$). This justifies also the properties of types φ_*, $\varphi_\top$ introduced informally in Section 3. Another result is the characterization of a class of maximal monoids (with respect to the operation of composition) of normalizable terms.

Let $B_\top = \{\varphi_\top x \mid x \in \mathrm{Var}\}$.

4.1. THEOREM. (i) *M has a head normal form iff there exists a basis B and a type $\alpha \not\sim_* \omega$ such that $B \vdash^* \alpha M$.*

(ii) *M has a normal form iff $B_\top \vdash^* \varphi_* M$.*

Proof. (i) ($\Leftarrow$) By Theorem 3.5, $B \vdash^* \alpha M \Rightarrow \exists A \precsim_\eta M$: $B \vdash^* \alpha A$. $B \vdash^* \alpha A$, with $\alpha \not\sim_* \omega$ implies $A \not\equiv \Omega$ by 3.8(ii), i.e., M has a head normal form.

($\Rightarrow$) Straightforward.

(ii) ($\Leftarrow$) $B_\top \vdash^* \varphi_* M \Rightarrow \exists A \precsim_\eta M$: $B_\top \vdash^* \varphi_* A$. We prove that Ω does not occur in A by induction on A.

$A \equiv \Omega$. Impossible by (i).

$A \equiv x$. Trivial.

$$A \equiv \lambda x.A'.\ B_\top \vdash\!\!-^* \varphi_* \lambda x.A' \Rightarrow B_\top \vdash\!\!-^* \varphi_\top \to \varphi_* \lambda x.A'$$

$$\Rightarrow B_\top / x \cup \{\varphi_\top x\} \vdash\!\!-^* \varphi_* A' \qquad \text{by 3.4(ii)}$$

$$\Rightarrow B_\top \vdash\!\!-^* \varphi_* A'.$$

$A \equiv xA_1 \cdots A_n$. $B_\top \vdash\!\!-^* \varphi_* A \Rightarrow B_\top \vdash\!\!-^* \varphi_* A_i$ $(1 \leqslant i \leqslant n)$ by Lemma A.3(i) given in Appendix A.

($\Rightarrow$) Induction on the normal form of M. ∎

A semantic characterization of the terms having a normal form follows immediately from 4.1.

Let $\xi_\top : \mathrm{Var} \to \mathscr{F}^*$ be defined by $\xi_\top(x) = {}^*\!\uparrow\varphi_\top$.

4.2. COROLLARY. *M has a normal form iff* ${}^*\!\uparrow\varphi_* \subseteq [\![M]\!]^*_{\xi_\top}$.

Proof. By 2.5, ${}^*\!\uparrow\varphi_* \subseteq [\![M]\!]^*_{\xi_\top}$ iff $B_{\xi_\top} \vdash\!\!-^* \varphi_* M$. Now observe that $B_\top \subseteq B_{\xi_\top}$ and $\alpha x \in B_{\xi_\top}$ implies $B_\top \vdash\!\!-^* \alpha x$. So we complete the proof using 4.1(ii). ∎

To obtain the statement of Theorem 4.2 for D^*_∞ we must only replace ${}^*\!\uparrow\varphi_*$, ${}^*\!\uparrow\varphi_\top$ by (the projections in D^*_∞ of) $*$, $\top$, and $\subseteq$ by $\sqsubseteq$. Observe that $\{x \mid * \sqsubseteq x\}$ is an open set in the Scott topology and that, for closed terms, the interpretation is independent of the environment.

We can also give a characterization of the terms having type $\varphi_\top$ (via $B_\top$).

4.3. THEOREM. *The following four conditions are equivalent.*

(i) $B_\top \vdash\!\!-^* \varphi_\top M$.

(ii) ${}^*\!\uparrow\varphi_\top = [\![M]\!]^*_{\xi_\top}$.

(iii) *For all normalizable N, NM is normalizable.*

(iv) *For all $n \geqslant 0$ and $N_1, \ldots, N_n$ normalizable, $MN_1 \cdots N_n$ is normalizable.*

Proof. (i)$\Leftrightarrow$(ii) Immediate by 2.5.

(i)$\Rightarrow$(iii) Immediate by $\varphi_* \sim_* \varphi_\top \to \varphi_*$ and 4.1(ii).

(iii)$\Rightarrow$(i) The rather technical proof is given in Appendix B.

(iii)$\Rightarrow$(iv) Take $N \equiv \lambda z.z\, N_1 \cdots N_n$.

(iv)$\Rightarrow$(iii) By induction on the normal form of N. ∎

Theorem 4.3(iii) justifies the axiom $\varphi_* \sim_* \varphi_\top \to \varphi_*$.

More generally we can give a complete characterization in terms of normalization properties of the terms having a type without occurrences of "$\wedge$." To this aim we introduce a new interpretation $\mathscr{V}^*$ of types in $\mathscr{M}(\beta)$. We prove the completeness of $\vdash^*$ (restricted to arrow types) with respect to this interpretation.

4.4. Definition. (i) The set $T_\rightarrow$ of *arrow types* is defined by

(1) $\omega, \varphi_*, \varphi_\top \in T_\rightarrow$,

(2) $\alpha, \beta \in T_\rightarrow \Rightarrow \alpha \rightarrow \beta \in T_\rightarrow$.

(ii) An *arrow basis* is a basis in which all predicates are arrow types and no two statements have the same subject.

(iii) $\mathscr{V}^*: T_{A^*} \rightarrow \mathscr{P}(\Lambda/\beta)$ is defined by

(1) $\mathscr{V}^*(\varphi_*) = \{[M] \mid M \text{ is normalizable}\}$,

(2) $\mathscr{V}^*(\varphi_\top) = \{[M] \mid \text{for all normalizable } N, NM \text{ is normalizable}\}$.

So, for instance, $\mathscr{V}^*(\varphi_* \rightarrow \varphi_*)$ is the set of all terms which yield a normalizable term whenever applied to a normalizable term.

Note that $\mathscr{V}^*$ is different from the interpretation $\mathscr{V}_B^{\Sigma *}$ introduced in Section 1. For instance, $\mathscr{V}^*(\varphi_*) \subsetneq \mathscr{V}_{B_\top}^{\Sigma *}(\varphi_*)$: in fact $B_\top^+ \vdash^* \varphi_* N$ does not imply $B_\top \vdash^* \varphi_* N$ (N could contain a variable $y_{\alpha,i}$ which does not occur in $B_\top$).

Let $B \uparrow M = \{\alpha x \mid \alpha x \in B, \alpha \not\equiv \varphi_T \text{ and } x \in FV(M)\} \cup \{\omega x \mid x \in FV(M) \text{ and } x \notin FV(B)\}$.

4.5. Lemma. *Let α be an arrow type, B an arrow basis and $B \uparrow M = \{\alpha_i x_i \mid 0 \leqslant i \leqslant n\}$ $(n \geqslant 0)$. If $B \not\vdash^* \alpha M$ then there exist $N_1,..., N_n$ such that $B_\top \vdash^* \alpha_i N_i$ $(0 \leqslant i \leqslant n)$ and $B_\top \not\vdash^* \alpha M[x_1 := N_1,..., x_n := N_n]$.*

The proof is given in Appendix C.

4.6. Theorem. (i) *Let α be an arrow type. Then $B_\top \vdash^* \alpha M \Leftrightarrow [M] \in \mathscr{V}^*(\alpha)$.*

(ii) (*Relative completeness*). *Let α be an arrow type and B an arrow basis. Then $B \vdash^* \alpha M \Leftrightarrow \mathscr{M}(\beta), \mathscr{V}^*, B \models \alpha M$.*

Proof. (i) By induction on α. If α is ω the proof is trivial. If α is φ_* or $\varphi_\top$ the proof follows from 4.1(ii) and 4.3. Let $\alpha \equiv \beta \rightarrow \gamma$.

($\Rightarrow$) Let $[N] \in \mathscr{V}^*(\beta)$. By the induction hypothesis $B_\top \vdash^* \beta N$. So $B_\top \vdash^* \gamma(MN)$ which implies, by the induction hypothesis, $[MN] \in \mathscr{V}^*(\gamma)$.

($\Leftarrow$) Assume $B_\top \not\vdash^* \beta \rightarrow \gamma M$. Then $B_\top / z \cup \{\beta z\} \not\vdash^* \gamma(Mz)$, where $z \notin FV(M)$, by 1.7(iii). By 4.5, then, $\exists N$ such that $B_\top \vdash^* \beta N$ (i.e.,

$[N] \in \mathscr{V}^*(\beta)$ by the induction hypothesis) and $B_\top \not\vdash^* \gamma(MN)$. Then by the induction hypothesis $[MN] \notin \mathscr{V}^*(\gamma)$ which implies $[M] \notin \mathscr{V}^*(\alpha)$.

(ii) ($\Rightarrow$) By soundness.

($\Leftarrow$) Assume $B \not\vdash^* \alpha M$. Let $B \upharpoonright M = \{\alpha_i x_i \mid 0 \leqslant i \leqslant n\}$. By Lemma 4.5, $\exists N_1, \ldots, N_n$ such that $B_\top \vdash^* \alpha_i N_i$ (i.e., $N_i \in \mathscr{V}^*(\alpha_i)$ by (i)) and $B_\top \not\vdash^* \alpha M[x_1 := N_1, \ldots, x_n := N_n]$ which implies by (i) $[M[x_1 := N_1, \ldots, x_n := N_n]] \notin \mathscr{V}^*(\alpha)$, i.e., by definition, $\mathscr{M}(\beta), \mathscr{V}^*, B \not\models \alpha M$. ∎

We can easily obtain a result analogous to Corollary 4.3 for all arrow types.

4.7. COROLLARY. *If α is an arrow type, the following three conditions are equivalent:*

(i) $B_\top \vdash^* \alpha M$.

(ii) $*\uparrow\alpha \subseteq [\![M]\!]^*_{\xi_\top}$.

(iii) *For all N such that $B_\top \vdash^* \alpha \to \varphi_* N$, NM is normalizable.*

Proof. (i) $\Leftrightarrow$ (ii) Immediate by 2.5.

(i) $\Rightarrow$ (iii) Immediate by 4.1(ii).

(iii) $\Rightarrow$ (i) By cases on α; $\alpha \sim_* \omega$ trivial.

$$\alpha \equiv \alpha_1 \to \cdots \to \alpha_n \to \varphi_*,$$

$$\begin{aligned} B_\top \not\vdash^* \alpha M \Rightarrow\ & B_\top / x_1 \ldots / x_n \cup \{\alpha_i x_i \mid 1 \leqslant i \leqslant n\} \not\vdash^* \varphi_*(M x_1 \cdots x_n), \\ & \text{where } x_i \notin FV(M) \quad \text{for} \quad 1 \leqslant i \leqslant n \quad \text{by} \quad 1.7\text{(iii)} \\ \Rightarrow\ & \exists N_1, \ldots, N_n \quad \text{such that } B_\top \vdash^* \alpha_i N_i \ (1 \leqslant i \leqslant n) \\ & \text{and } B_\top \not\vdash^* \varphi_*(M N_1 \cdots N_n) \quad \text{by} \quad 4.5 \end{aligned}$$

(note that when $\alpha_i \equiv \varphi_\top$ we can choose $N_i \equiv x_i$).

Then if we choose $N \equiv \lambda z.z N_1 \cdots N_n$ (where $z \notin FV(N_1 \cdots N_n)$) we have that $B_\top \vdash^* \alpha \to \varphi_* N$ and NM is not normalizable by 4.1(ii).

$$\alpha \equiv \alpha_1 \to \cdots \to \alpha_n \to \varphi_\top,$$

$$\begin{aligned} B_\top \not\vdash^* \alpha M \Rightarrow\ & \exists N_1, \ldots, N_n \\ & \text{such that } B_\top \vdash^* \alpha_i N_i \ (1 \leqslant i \leqslant n) \\ & \text{and} \quad B_\top \not\vdash^* \varphi_\top(M N_1 \cdots N_n) \quad \text{as in previous case} \\ \Rightarrow\ & \exists N_{n+1}, \ldots, N_m \\ & \text{such that } B_\top \vdash^* \varphi_* N_i \ (n+1 \leqslant i \leqslant m) \\ & \text{and} \quad M N_1 \cdots N_m \quad \text{is not normalizable by 4.3.} \end{aligned}$$

Then if we choose $N \equiv \lambda z.zN_1 \cdots N_m$ (where $z \notin FV(N_1 \cdots N_m)$) we have $B_\top \vdash^* \alpha \to \varphi_* N$ and NM not normalizable. ∎

Theorem 4.6 does not hold for arbitrary types in T_{A^*}. In fact, if $\alpha_0 \equiv \omega \to \varphi_* \wedge \varphi_* \to \varphi_\top$, we have $\mathcal{M}(\beta)$, $\mathcal{V}^*$, $\{\alpha_0 x\} \models \omega \to \varphi_\top x$ while $\{\alpha_0 x\} \not\vdash^* \omega \to \varphi_\top x$ since $\alpha_0 \not\leqslant_* \omega \to \varphi_\top$. To prove $\mathcal{M}(\beta)$, $\mathcal{V}^*$, $\{\alpha_0 x\} \models \omega \to \varphi_\top x$ take an arbitrary $[M] \in \mathcal{V}^*(\omega \to \varphi_* \wedge \varphi_* \to \varphi_\top)$. $[M] \in \mathcal{V}^*(\omega \to \varphi_*)$ means that for all N, MN is normalizable, and this is possible only if M reduces to a term $\lambda x.M'$, where x does not occur in M'. $[M] \in \mathcal{V}^*(\varphi_* \to \varphi_\top)$ implies $B_\top \vdash^* \varphi_* \to \varphi_\top M$ by 4.6(i), i.e., $B_\top \vdash^* \varphi_* \to \varphi_\top \lambda x.M'$ and, since x does not occur in M', $B_\top / x \vdash^* \varphi_\top M'$. So we conclude $B_\top \vdash^* \omega \to \varphi_\top \lambda x.M'$ (i.e., by 4.6(i) $[M] \in \mathcal{V}^*(\omega \to \varphi_\top)$).

In the last part of this section we will show an application of previous results to the characterization of a class of maximal monoids of normalizable terms.

Church (1937) proved that the whole set of terms forms a semigroup with respect to the composition operator $\mathbf{B} \equiv \lambda xyz.x(yz)$. This is not true for the set of normalizable terms since the composition of two arbitrary normalizable terms can have no normal form. In Coppo and Dezani (1979a) a sufficient condition is given to characterize sets of normalizable terms which form semigroups with respect to $\mathbf{B}$. In this section we show that there is an infinite number of maximal monoids of normalizable terms, in particular we define a maximal monoid for any arrow type of the shape $\alpha \to \alpha$ such that $\varphi_\top \leqslant_* \alpha \leqslant_* \varphi_*$. Let $M \circ N$ be an infix form for $\mathbf{B}MN$.

4.8. Definition. (i) A *semigroup* is any set of terms closed under composition.

(ii) A semigroup $\mathcal{S}$ is a *monoid* if the combinator $\mathbf{I} \equiv \lambda x.x$ belongs to $\mathcal{S}$.

(iii) $\mathcal{S}^\alpha_B = \{M \mid B \vdash^* \alpha M\}$.

(iv) A type α is *proper* iff α is an arrow type and $\varphi_\top \leqslant_* \alpha \leqslant_* \varphi_*$.

$\mathcal{S}^\alpha$ is short for $\mathcal{S}^\alpha_{B_\top}$.

Observe that, for all $\beta \leqslant_* \alpha$, $\mathcal{S}^{\alpha \to \beta}_B$ is a semigroup (in fact, in this case, $\vdash^* (\alpha \to \beta) \to (\alpha \to \beta) \to \alpha \to \beta \mathbf{B}$) and that $\mathcal{S}^{\alpha \to \alpha}_B$ is a monoid. Moreover, if $\alpha \to \beta \leqslant_* \varphi_*$ ($\alpha \to \alpha \leqslant_* \varphi_*$), by Theorem 4.1(ii) $\mathcal{S}^{\alpha \to \beta}$ ($\mathcal{S}^{\alpha \to \alpha}$) is a semigroup (monoid) of normalizable terms. Note that $\alpha \to \alpha \leqslant_* \varphi_*$ iff $\varphi_\top \leqslant_* \alpha \leqslant_* \varphi_*$ iff ω does not occur in α.

We say that a monoid $\mathcal{M}$ of normalizable terms is *maximal* if $\forall N \notin \mathcal{M}$ there exist M_1 and M_2 in $\mathcal{M}$ such that $M_1 \circ N \circ M_2$ does not have a normal form. Our result is that for all proper α $\mathcal{S}^{\alpha \to \alpha}$ is maximal. This was proved in Dezani and Ermine (1982) only in the particular case $\alpha \sim_* \varphi_*$.

4.9. THEOREM. *If* α *is proper,* $\mathscr{S}^{\alpha\to\alpha}$ *is a maximal monoid of normalizable terms.*

Proof. Since $\alpha\to\alpha \leqslant_* \varphi_*$, $\mathscr{S}^{\alpha\to\alpha}$ is a monoid of normalizable terms. Assume $B_\top \not\vdash^* \alpha\to\alpha N$,

$$B_\top \not\vdash^* \alpha\to\alpha N \Rightarrow B_\top/x\cup\{\alpha x\} \not\vdash^* \alpha(Nx)$$

$$\text{where } x\notin FV(N) \quad \text{by} \quad 1.7(\text{iii})$$

$$\Rightarrow \exists P\; B_\top \vdash^* \alpha P \quad \text{and} \quad B_\top \not\vdash^* \alpha(NP)$$

$$\text{by} \quad 4.5 \text{ (if } \alpha\equiv\varphi_\top \text{ we can choose } P\equiv x\text{)},$$

$$B_\top \not\vdash^* \alpha(NP) \Rightarrow \exists Q\; B_\top \vdash^* \alpha\to\varphi_* Q \quad \text{and}$$

$$Q(NP) \text{ has no normal form by 4.7.}$$

Then a possible choice is $M_1\equiv\lambda y.z(Qy)$ where $y\notin FV(Q)$ and $M_2\equiv\lambda t.P$ where $t\notin FV(P)$, since $B_\top \vdash^* \alpha\to\alpha M_1$, $B_\top \vdash^* \alpha\to\alpha M_2$ and $M_1\circ N\circ M_2 = \lambda x.z(Q(NP))$ has no normal form. ∎

Note that not all maximal monoids of normalizable terms are of the shape $\mathscr{S}^{\alpha\to\alpha}$, not even if we assume α to be an arbitrary type of T_A^* (such that $B_\top \vdash^* \alpha\to\alpha M$ implies that M is normalizable). For example, consider a maximal extension $\mathscr{M}_0$ of the monoid generated by the set of terms $\{\mathbf{I}, \mathbf{\Delta}, \mathbf{O}, \mathbf{K}\}$ where $\mathbf{\Delta}\equiv\lambda x.xx$, $\mathbf{O}\equiv\lambda xy.y$, and $\mathbf{K}\equiv\lambda xy.x$. We prove that there is no type α such that $\alpha \not\leqslant_* \omega$ and $\alpha\to\alpha$ can be deduced for all elements of this monoid. Suppose that such an α exists. Let $\alpha \sim_* \bigwedge_{1\leqslant i\leqslant m}(\alpha_i^{(1)}\to\cdots\to\alpha_i^{(n)}\to\psi_i)$ where $\psi_i\in\{\varphi_*, \varphi_\top\}$ (this can be assumed by 2.3 and $\beta\to\omega \sim_* \omega$ for all β). Observe that $\alpha \not\leqslant_* \varphi_*$ and $\alpha \not\leqslant_* \varphi_\top$ since $\not\vdash^* \varphi_*\to\varphi_*\mathbf{\Delta}$ and $\not\vdash^* \varphi_\top\to\varphi_\top\mathbf{O}$. From $\vdash^* \alpha\to\alpha\mathbf{K}^n$ (where $\mathbf{K}^n = \underbrace{\mathbf{K}\circ\cdots\circ\mathbf{K}}_{n\text{ times}} = \lambda xy_1\cdots y_n.x$ belongs to $\mathscr{M}_0$) we have $\alpha \leqslant_* \psi_i$ $(1\leqslant i\leqslant m)$ by 1.7(i), i.e., $\alpha\leqslant_* \bigwedge_{1\leqslant i\leqslant m}\psi_i$. From $\vdash^* \alpha\to\alpha\mathbf{\Delta}$ we have $\alpha \leqslant_* \alpha\to\alpha$, i.e.,

$$\bigwedge_{1\leqslant i\leqslant m} \alpha_i^{(1)}\to\cdots\to\alpha_i^{(n)}\to\psi_i \leqslant_* \underbrace{\alpha\to\cdots\to\alpha}_{n\text{ times}}\to\alpha$$

which implies, by 2.13 of Coppo *et al.* (1983b), $\bigwedge_{1\leqslant i\leqslant m}\psi_i \leqslant_* \alpha$. Then $\bigwedge_{1\leqslant i\leqslant m}\psi_i \leqslant_* \alpha \leqslant_* \bigwedge_{1\leqslant i\leqslant m}\psi_i$, i.e., either $\alpha \sim_* \varphi_*$ or $\alpha \sim_* \varphi_\top$ which is a contradiction.

APPENDIX A: Proof of Theorem 3.5($\Rightarrow$)

Theorem 3.5 is proved using a variant of Tait's notion of computability (Tait, 1967; Stenlund, 1972). The key point of this proof is the definition, by induction on types, of a predicate Comp(B, α, M). We prove (by induction on types) that Comp(B, α, M) implies $\exists A \precsim_\eta M$ such that $B \vdash^* \alpha A$ (this latter property being denoted App(B, α, M)) and (by induction on derivations) that $B \vdash^* \alpha M$ implies Comp(B, α, M).

Let $\vec{\mathbf{M}}$ denote a sequence $M_1, ..., M_n$ $(n \geqslant 0)$ of terms. $N\vec{\mathbf{M}}$ stands for $NM_1 \cdots M_n$. If $P(\ )$ is a predicate, $P(\vec{\mathbf{M}})$ stands for $P(M_1)$ *and* $\cdots$ *and* $P(M_n)$.

A.1. Definition. (i) App(B, α, M) iff $\exists A \precsim_\eta M$ such that $B \vdash^* \alpha A$.

(ii) Comp(B, α, M) is defined by induction on α:

(1) Comp(B, ω, M) is true,

(2) Comp(B, φ_*, M) iff App(B, φ_*, M),

(3) Comp($B, \varphi_\top, M$) iff App($B, \varphi_\top, M$) and [App($B', \varphi_*, \vec{\mathbf{N}}$) $\Rightarrow$ App($B \cup B', \varphi_\top, M\vec{\mathbf{N}}$)],

(4) Comp($B, \alpha \to \beta, M$) if Comp(B', α, N) $\Rightarrow$ Comp($B \cup B', \beta, MN$),

(5) Comp($B, \alpha \wedge \beta, M$) if Comp($B, \alpha, M$) *and* Comp($B, \beta, M$).

Observe that Comp, as well as App, is invariant under $\beta\eta\Omega$-conversion of terms, i.e., if $M =_{\beta\eta\Omega} N$ then Comp(B, α, M) iff Comp(B, α, N) (and the same holds for App).

A.2. Lemma. (i) App($B, \alpha, x\vec{\mathbf{M}}$) $\Rightarrow$ Comp($B, \alpha, x\vec{\mathbf{M}}$).

(ii) Comp(B, α, M) $\Rightarrow$ App(B, α, M).

Proof. We prove (i) and (ii) simultaneously by induction on α. If $\alpha \equiv \varphi_*, \varphi_\top$ or $\alpha \equiv \beta \wedge \gamma$ the proof is easy. Let $\alpha \equiv \beta \to \gamma$. We prove (i) first. Note that Comp(B', β, N) implies App(B', β, N) by the induction hypothesis (ii). Therefore by rule ($\to E$) we have App($B \cup B', \gamma, x\vec{\mathbf{M}}N$) which implies Comp($B \cup B', \gamma, x\vec{\mathbf{M}}N$) by the induction hypothesis (i). We conclude Comp($B, \beta \to \gamma, x\vec{\mathbf{M}}$).

(ii) Take $z \notin FV(M) \cup FV(B)$ (as remarked in Sect. 1, we can always find such a z). We have App($\{\beta z\}, \beta, z$) which implies, by the induction hypothesis (i), Comp($\{\beta x\}, \beta, z$).

$$\text{Comp}(B, \beta \to \gamma, M) \text{ and } \text{Comp}(\{\beta z\}, \beta, z) \Rightarrow \text{Comp}(B \cup \{\beta z\}, \gamma, Mz)$$

$$\Rightarrow \text{App}(B \cup \{\beta z\}, \gamma, Mz) \qquad \text{by the induction hypothesis (ii)}$$

$$\Rightarrow \exists A \precsim_\eta Mz \qquad \text{such that } B \cup \{\beta z\} \vdash^* \gamma A.$$

Last, observe that $A \precsim_\eta Mz \Rightarrow \lambda z.A \precsim_\eta M$ and, by rule $(\to I)$, $B \cup \{\beta z\} \vdash^* \gamma A \Rightarrow B \vdash^* \beta \to \gamma \lambda z.A$. ∎

A.3. LEMMA. Let $z \notin FV(B)$.

(i) $B \cup \{\varphi_\top z\} \vdash^* \alpha z\vec{\mathbf{M}}$ and $\alpha \not\sim_* \omega$ imply $\varphi_\top \leqslant_* \alpha$, $B \cup \{\varphi_\top z\} \vdash^* \varphi_* M_i$ for $1 \leqslant i \leqslant n$ and $B \cup \{\varphi_\top z\} \vdash^* \varphi_\top z\vec{\mathbf{M}}$.

(ii) $\mathrm{App}(B \cup \{\varphi_\top z\}, \alpha, M)$ and $\mathrm{Comp}(B', \varphi_\top, N)$ imply $\mathrm{App}(B \cup B', \alpha, M[z := N])$.

Proof. Let $B'' = B \cup \{\varphi_\top z\}$.

(i) $B'' \vdash^* \alpha z\vec{\mathbf{M}} \Rightarrow \exists \beta_1, ..., \beta_n$ such that $B'' \vdash^* \beta_1 \to \cdots \to \beta_n \to \alpha z$ and $B'' \vdash^* \beta_i M_i$ for $1 \leqslant i \leqslant n$ by repeated applications of 1.7(ii). Then $\varphi_\top \leqslant_* \beta_1 \to \cdots \to \beta_n \to \alpha$ by 1.7(i) and, since

$$\varphi_\top \sim_* \underbrace{\varphi_* \to \cdots \to \varphi_*}_{n \text{ times}} \to \varphi_\top,$$

we have $\beta_i \leqslant_* \varphi_*$ for $1 \leqslant i \leqslant n$ and $\varphi_\top \leqslant_* \alpha$ by n applications of 3.4 (iii) (note that by hypothesis $\alpha \not\sim_* \omega$).

(ii) $\mathrm{App}(B'', \alpha, M)$ implies, by definition, that there is $A \precsim_\eta M$ such that $B'' \vdash^* \alpha A$. We prove the lemma by induction on A.

$A \equiv y \not\equiv z$ or $A \equiv \Omega$. Trivial.

$A \equiv z$. In this case $M[z := N] \equiv N$. $\mathrm{Comp}(B', \varphi_\top, N)$ implies $\mathrm{App}(B', \varphi_\top, N)$ by A.1(ii)(3) and then $\mathrm{App}(B \cup B', \varphi_\top, N)$.

$A \equiv z\vec{\mathbf{A}}$. Then we have that $M =_{\beta\eta} z\vec{\mathbf{M}}$ where $A_i \precsim_\eta M_i$ for $1 \leqslant i \leqslant n$. Moreover by point (i), we have $B'' \vdash^* \varphi_* A_i$ and $B'' \vdash^* \varphi_* M_i$. Then $\mathrm{App}(B'', \varphi_*, M_i)$ and, by the induction hypothesis, $\mathrm{App}(B \cup B', \varphi_*, M_i[z := N])$ for $1 \leqslant i \leqslant n$. Last, observe that $M[z := N] \equiv NM_1[z := N] \cdots M_n[z := N]$ and, by Definition A.1(ii)(3), we have $\mathrm{App}(B \cup B', \varphi_\top, M[z := N])$. The proof follows by observing that, by (i), $\varphi_\top \leqslant_* \alpha$.

The case $A \equiv y\vec{\mathbf{A}}$ $(y \not\equiv z)$ is simpler.

$A \equiv \lambda y.A'$. In this case $M =_{\beta\eta} \lambda y.M'$ and $A' \precsim_\eta M'$. Let $\alpha \sim_* \bigwedge_{1 \leqslant i \leqslant n} \beta_i \to \gamma_i$ (see 2.3). We can assume $y \notin FV(B' \cup B'')$.

$$\begin{aligned}
B'' \vdash^* \bigwedge_{1 \leqslant i \leqslant n} \beta_i \to \gamma_i \lambda y.A' &\Rightarrow B'' \vdash^* \beta_i \to \gamma_i \lambda y.A' && \text{by } (\wedge E) \text{ for } 1 \leqslant i \leqslant n \\
&\Rightarrow B'' \cup \{\beta_i y\} \vdash^* \gamma_i A' && \text{by 3.4(ii) for } 1 \leqslant i \leqslant n \\
&\Rightarrow \mathrm{App}(B \cup B' \cup \{\beta_i y\}, \gamma_i, M'[z := N]) \\
&\qquad \text{by the induction hypothesis for } 1 \leqslant i \leqslant n \\
&\Rightarrow \mathrm{App}(B \cup B', \beta_i \to \gamma_i, M[z := N]) \\
&&& \text{by } (\to I) \text{ for } 1 \leqslant i \leqslant n.
\end{aligned}$$

Now let $\mathscr{A} = \{A_i \mid A_i \precsim_\eta M[z := N]$ and $B \cup B' \vdash^* \beta_i \to \gamma_i A_i$ for $1 \leqslant i \leqslant n\}$ and $A = \bigsqcup \mathscr{A}$. Since $\mathscr{A} \subseteq \mathscr{A}_\eta(M[z := N])$ and $\mathscr{A}_\eta(P)$ is directed for all P, A must exist. Moreover $A \precsim_\eta M[z := N]$, $B \cup B' \vdash^* \alpha A$, proving $\mathrm{App}(B \cup B', \alpha, M[z := N])$. ∎

A.4. Lemma. (i) $\mathrm{Comp}(B, \varphi_\top, M) \Rightarrow \mathrm{Comp}(B, \varphi_*, M)$.

(ii) $\mathrm{Comp}(B, \varphi_\top, M) \Leftrightarrow \mathrm{Comp}(B, \varphi_* \to \varphi_\top, M)$.

(iii) $\mathrm{Comp}(B, \varphi_*, M) \Leftrightarrow \mathrm{Comp}(B, \varphi_\top \to \varphi_*, M)$.

(iv) $\mathrm{Comp}(B, \alpha, M)$ and $\alpha \leqslant_* \beta \Rightarrow \mathrm{Comp}(B, \beta, M)$.

Proof. (i) Trivial.

(ii) ($\Rightarrow$) Let B', N be such that $\mathrm{Comp}(B', \varphi_*, N)$. By Definition A.1(ii)(3) we have $\mathrm{App}(B \cup B', \varphi_\top, MN)$. Moreover let B'', $\vec{\mathbf{P}}$ be such that $\mathrm{App}(B'', \varphi_*, \vec{\mathbf{P}})$. By the same point of Definition A.1 we have $\mathrm{App}(B \cup B' \cup B'', \varphi_\top, MN\vec{\mathbf{P}})$ proving $\mathrm{Comp}(B \cup B', \varphi_\top, MN)$ which implies by A.1(ii)(4), $\mathrm{Comp}(B, \varphi_* \to \varphi_\top, M)$.

($\Leftarrow$) By Definition A.1(ii)(3) we must prove $\mathrm{App}(B, \varphi_\top, M)$ and $[\mathrm{App}(B', \varphi_*, \vec{\mathbf{N}}) \Rightarrow \mathrm{App}(B \cup B', \varphi_\top, M\vec{\mathbf{N}})]$. Observe that

$$\begin{aligned} \mathrm{Comp}(B, \varphi_* \to \varphi_\top, M) &\Rightarrow \mathrm{App}(B, \varphi_* \to \varphi_\top, M) && \text{by A.2(ii)} \\ &\Rightarrow \mathrm{App}(B, \varphi_\top, M) && \text{by } (\leqslant_*). \end{aligned}$$

Now let $\vec{\mathbf{N}} = N_1, ..., N_n$ be such that $\mathrm{App}(B', \varphi_*, \vec{\mathbf{N}})$. By A.1(ii)(2) we have $\mathrm{Comp}(B', \varphi_*, N_1)$ and, by A.1(ii)(4), $\mathrm{Comp}(B \cup B', \varphi_\top, MN_1)$ which implies, by A.1(ii)(3), $\mathrm{App}(B \cup B', \varphi_\top, MN_1 \cdots N_n)$.

$$\begin{aligned} \text{(iii) } (\Rightarrow)\ \mathrm{Comp}(B, \varphi_*, M) &\Rightarrow \mathrm{App}(B, \varphi_*, M) && \text{by definition} \\ &\Rightarrow \exists A \precsim_\eta M\ B \vdash^* \varphi_\top \to \varphi_* A && \text{by } (\leqslant_*). \end{aligned}$$

Suppose that A is of the form $\lambda x.A'$ where, without loss of generality, we can assume $x \notin FV(B)$. In this case $M =_{\beta\eta} \lambda x.M'$ where $A' \precsim_\eta M'$ and we have immediately $\mathrm{App}(B \cup \{\varphi_\top x\}, \varphi_*, M')$.

Now assume $\mathrm{Comp}(B', \varphi_\top, N)$. By A.3(ii) we have $\mathrm{App}(B \cup B', \varphi_*, M'[x := N])$. By Definition A.1(ii)(2) and conversion we conclude $\mathrm{Comp}(B \cup B', \varphi_*, MN)$. The case $A \equiv z\vec{\mathbf{A}}$ is simpler.

($\Leftarrow$) By A.2(ii) and $\varphi_\top \to \varphi_* \leqslant_* \varphi_*$.

(iv) Induction on the definition of $\leqslant_*$ using (i), (ii), and (iii). ∎

A.5. Lemma. *Let* $B = \{\beta_1 x_1, ..., \beta_n x_n\}$ *and* $\mathrm{Comp}(B_i, \beta_i, N_i)$ *for* $1 \leqslant i \leqslant n$. *Then* $B \vdash^* \alpha M$ *implies* $\mathrm{Comp}(B_1 \cup \cdots \cup B_n, \alpha, M[x_1 := N_1, ..., x_n := N_n])$.

Proof. Induction on the derivation showing $B \vdash^* \alpha M$.
If the last applied rule is $(\leqslant_*)$ use A.4(iv).
If the last applied rule is $(\to I)$ and $M \equiv \lambda x.M'$ we have

$$\dfrac{\begin{array}{c}[\beta x]\\ \vdots \\ \gamma M'\end{array}}{\beta \to \gamma \lambda x.M'}\ (\to I).$$

Let B', N be such that $\mathrm{Comp}(B', \beta, N)$.

$$\begin{aligned}&\mathrm{Comp}(B', \beta, N)\\ &\quad \Rightarrow \mathrm{Comp}(B' \cup B_1 \cup \cdots \cup B_n, \gamma, M'[x := N, x_1 := N_1, ..., x_n := N_n])\\ &\qquad\quad \text{by the induction hypothesis}\\ &\quad \Rightarrow \mathrm{Comp}(B' \cup B_1 \cup \cdots \cup B_n, \gamma, (\lambda x.M'[x_1 := N_1, ..., x_n := N_n])\, N)\end{aligned}$$

since Comp is invariant under $\beta\eta\Omega$-conversion of terms (note that $x \notin FV(N)$).

Therefore we can conclude $\mathrm{Comp}(B_1 \cup \cdots \cup B_n, \beta \to \gamma, M[x_1 := N_1, ..., x_n := N_n])$. The other cases are trivial. ∎

Proof of Theorem 4.1. $(\Rightarrow)$ Note that $\beta x \in B \Rightarrow \mathrm{Comp}(B, \beta, x)$ by A.2(i).

$$\begin{aligned}B \vdash^* \alpha M &\Rightarrow \mathrm{Comp}(B, \alpha, M) && \text{by A.5}\\ &\Rightarrow \mathrm{App}(B, \alpha, M) && \text{by A.2(ii).}\end{aligned}$$ ∎

APPENDIX B: Proof of Theorem 4.3 (iii) $\Rightarrow$ (i)

B.1. Definition. Let M be a normal form. We assume, without loss of generality, that all free and bound variables of M have distinct names.

(i) The *replacement path* $\pi(x, M)$ $(\in \mathrm{Var} \times (\mathbf{N} \times \mathbf{N})^*)$ of a variable x in M is defined by:

$\pi(x, M) = x$ if x is free in M.

$\pi(x, M) = \pi(y, M)\langle j, n \rangle$ if x is bound in a subterm of M of the shape

$$yN_1 \cdots N_{j-1}(\lambda z_1 \cdots z_{n-1} x.P) \quad (n, j > 0).$$

(ii) Two variables x and y (not necessarily distinct) are *adjacent* in M iff $xN_1 \cdots N_j(\lambda z_1 \cdots z_n.yP_1 \cdots P_l)$ is a subterm of M $(n, j, l \geqslant 0)$.

Note that, if $\pi(y, M) = x\langle j, n\rangle\, \sigma$, then M has a subterm of the shape

$$xN_1 \cdots N_{j-1}(\lambda x_1 \cdots x_n.P) \qquad \text{and} \qquad \pi(y, P) = x_n\sigma.$$

For example, if $M \equiv \lambda t.x(\lambda z.z(\lambda uv.yv)\, t)$, then $\pi(v, M) = x\langle 1, 1\rangle\langle 1, 2\rangle$, the replacement path of t in M is undefined and z, y are adjacent in M.

B.2. Lemma. *Let M be a normal form, z and t two variables which are adjacent in M and $\pi(z, M) = x\sigma$, $\pi(t, M) = y\tau$. Then there are normalizable terms X, Y such that $M[x := X, y := Y]$ does not have a normal form (possibly $x \equiv y$ and, in this case, $X \equiv Y$).*

Proof. Let $|\sigma|$ be the length of $\sigma \in (\mathbf{N} \times \mathbf{N})^*$. The proof is by induction on $|\sigma| + |\tau|$.

First step. If $\sigma = \tau = \varepsilon$ then $z \equiv x$, $t \equiv y$. By definition M has a subterm of the shape $xN_1 \cdots N_{j-1}(\lambda z_1 \cdots z_n.yP_1 \cdots P_l)$. Then if $x \not\equiv y$ a possible choice is

$$X \equiv \lambda z_1 \cdots z_j.az_1 \cdots z_{j-1}(z_j b_1 \cdots b_n \mathbf{\Delta}), \qquad Y \equiv \lambda t_1 \cdots t_l u.uut_1 \cdots t_l,$$

where $\mathbf{\Delta} \equiv \lambda w.ww$.

If $x \equiv y$ we can choose.

$$X \equiv \lambda z_1 \cdots z_i.az_1 \cdots z_{i-1}(z_{l+1} z_{l+1})(z_j b_1 \cdots b_n(z_j b_1 \cdots b_n)),$$

where $i = \max[j, l+1]$.

Note that X, Y are λ–**I**-terms (Barendregt, 1984, 9) to avoid the possibility of erasing the subterm of $M[x := X, y := Y]$ which does not have a normal form.

Induction step. If $\sigma = \langle j, n\rangle\, \sigma'$ then M has a subterm of the shape $xN_1 \cdots N_{j-1}(\lambda u_1 \cdots u_n.P)$ and $\pi(z, P) = u_n\sigma'$. Let $X' \equiv \lambda v_1 \cdots v_j.xv_1 \cdots v_{j-1}(v_j u_1 \cdots u_n)$ and $M' \equiv M[x := X']$.

Observe that:

(a) M' reduces to a normal form M'',

(b) the variables which are adjacent in M' are such also in M'',

(c) if $\pi(w, M) = x\langle j, i\rangle\,\rho$ with $1 \leqslant i \leqslant n$ then $\pi(w, M'') = u_i\rho$,
if $\pi(w, M) = x\langle j, i\rangle\,\rho$ with $i > n$ then $\pi(w, M'') = x\langle j, i-n\rangle\,\rho$,
otherwise $\pi(w, M) = \pi(w, M'')$.

Therefore, in M'', z and t are adjacent and $\pi(z, M'') = u_n\sigma'$. Let $\pi(t, M'') = s\rho$. By point (c) $s \in \{y, u_1, ..., u_n\}$ and $|\rho| \leqslant |\tau|$. By the induction hypothesis there are U_n, T such that $M''[u_n := U_n, s := T]$ (or $M''[u_n := U_n]$ if $s \equiv u_n$) does not have a normal form. Then we choose

$$X \equiv \begin{cases} X'[u_n := U_n] \text{ and } Y \equiv T & \text{if } x \not\equiv y, \\ X'[u_n := U_n, s := T] & \text{if } x \equiv y \text{ and } s \not\equiv u_n, \\ X'[u_n := U_n] & \text{otherwise.} \end{cases}$$

Note that X has a normal form and X, Y are $\lambda - \mathbf{I}$-terms. ▮

B.3. Lemma. *Let* $M \equiv \lambda x_1 \cdots x_n . yM_1 \cdots M_m \equiv \lambda x_1 \cdots x_n . M'$ *be a normal form such that for all normalizable* N NM *has a normal form. Then* y *occurs free in* M *and all variables* z, t *such that* $\pi(z, M') = x_i\sigma$, $\pi(t, M') = x_j\tau$ *with* $1 \leqslant i, j \leqslant n$ *(possibly* $i = j$*) are not adjacent in* M'.

Proof. If one of these two conditions is not satisfied we exhibit a normalizable term N such that NM does not have a normal form. If $M \equiv \lambda x_1 \cdots x_n . x_i M_1 \cdots M_m$ with $1 \leqslant i \leqslant n$ we can choose $N \equiv \lambda z . zx_1 \cdots x_{i-1}(\lambda y_1 \cdots y_m . \Delta)\, x_{i+1} \cdots x_n \Delta$. If the second condition is not satisfied, then we can build X_i, X_j according to Lemma B.2 and choose (for $i < j$) $N \equiv \lambda z . zx_1 \cdots x_{i-1} X_i x_{i+1} \cdots x_{j-1} X_j x_{j+1} \cdots x_n$. The case $i = j$ is treated similarly. ▮

A similar result with a different proof technique was done in Böhm and Dezani (1975).

Proof of Theorem 4.3. (iii) $\Rightarrow$ (i). Obviously M is normalizable. We can show that $B_\top \vdash^* \varphi_\top M$ by induction on the normal form of M using the conditions given in B.3. ▮

APPENDIX C: Proof of Lemma 4.5

Let $\|\alpha\|$ be the number of occurrences of "$\rightarrow$" in $\alpha \in T_\rightarrow$. If $B = \{\alpha_i x_i \mid 1 \leqslant i \leqslant n\}$, let $\|B\| = \sum_{1 \leqslant i \leqslant n} \|\alpha_i\|$.

Proof of Lemma 4.5. The proof is by induction on $\|B\uparrow M\| + \|\alpha\|$.

First step. $\|B\uparrow M\| + \|\alpha\| = 0$. Let $\alpha_i \equiv \varphi_*$ for $1 \leqslant i \leqslant h$ and $\alpha_i \equiv \omega$ for $h+1 \leqslant i \leqslant n$. Define $M' = M[x_{h+1} := (\Delta\Delta),..., x_n := (\Delta\Delta)]$ where $\Delta = \lambda x.xx$. Obviously $B_\top \vdash^* \omega(\Delta\Delta)$. By 3.5 and 3.8(ii) $B \nvdash^* \alpha M'$. We distinguish two cases according to $\alpha \equiv \varphi_*$ or $\alpha \equiv \varphi_\top$ (note that $\alpha \equiv \omega$ is impossible).

Case 1. $\alpha \equiv \varphi_\top$, $B \nvdash^* \varphi_\top M'$ implies $B_\top \nvdash^* \underbrace{\varphi_* \to \cdots \to \varphi_*}_{h \text{ times}} \to \varphi_\top \lambda x_1 \cdots x_h.M'$, i.e., $B_\top \nvdash^* \varphi_\top \lambda x_1 \cdots x_h.M'$. By 4.3, then, $\exists k > 0$ and $N_1,..., N_k$ normalizable such that $(\lambda x_1 \cdots x_h.M') N_1 \cdots N_k$ has no normal form. We can assume $k \geqslant h$ (else take $N_i \equiv z$, where $z \notin FV(M')$, for $k < i \leqslant h$). Then, by 4.1(ii), $B_\top \vdash^* \varphi_* N_i$ $(1 \leqslant i \leqslant k)$ and, by 4.3, $B_\top \nvdash^* \varphi_\top M'[x_1 := N_1,..., x_h := N_h]$ (note that $(\lambda x_1 \cdots x_h.M') N_1 \cdots N_k =_\beta M'[x_1 := N_1,..., x_h := N_h] N_{h+1} \cdots N_k$).

Case 2. $\alpha \equiv \varphi_*$. $B \nvdash^* \varphi_* M'$ implies $B \cup \{\varphi_\top y\} \nvdash^* \varphi_\top yM'$, where $y \notin FV(M')$, since $\varphi_\top \sim_* \varphi_* \to \varphi_\top$. By case 1, then, $\exists N_1,..., N_h$ such that $B_\top \vdash^* \varphi_* N_i$ $(1 \leqslant i \leqslant h)$ and $B_\top \nvdash^* \varphi_\top yM'[x_1 := N_1,..., x_h := N_h]$. But this implies $B_\top \nvdash^* \varphi_* M'[x_1 := N_1,..., x_h := N_h]$.

Induction step. We distinguish the following cases:

Case 1. $\|\alpha\| \neq 0$. Let $\alpha \equiv \beta \to \gamma$.

$B \nvdash^* \beta \to \gamma M \Rightarrow B \cup \{\beta y\} \nvdash^* \gamma(My)$,

where $y \notin FV(M)$, by 1.7 (iii)

$\Rightarrow \exists N_1,..., N_n, P$

such that $B_\top \vdash^* \alpha_i N_i$ for $1 \leqslant i \leqslant n$, $B_\top \vdash^* \beta P$ and

$B_\top \nvdash^* \gamma(My)[x_1 := N_1,..., x_n := N_n, y := P]$

by the induction hypothesis

$\Rightarrow B_\top \nvdash^* \beta \to \gamma M[x_1 := N_1,..., x_n := N_n]$.

In fact $B_\top \vdash^* \beta \to \gamma M[x_1 := N_1,..., x_n := N_n]$ would imply, by rule $(\to E)$, $B_\top \vdash^* \gamma M[x_1 := N_1,..., x_n := N_n] P$, since $My[y := P] \equiv MP$.

Case 2. $\|\alpha\| = 0$ *and* $\|\alpha_j\| \neq 0$ *for some* j $(1 \leqslant j \leqslant n)$.

Case 2.1. $\alpha_j \equiv \gamma_1 \to \cdots \to \gamma_k \to \psi$, where $\|\gamma_k\| \neq 0$ and $\|\psi\| = 0$. Let $\gamma_k \equiv \beta_1 \to \beta_2$.

$$\begin{aligned}
B \not\vdash^* \alpha M \Rightarrow\ & B/x_j \cup \{\gamma_1 \to \cdots \to \gamma_{k-1} \to \beta_2 \to \psi y, \beta_1 z\} \not\vdash^* \alpha M' \\
& \text{where } y, z \notin FV(M) \text{ and} \\
& M' \equiv M[x_j := \lambda y_1 \cdots y_{k-1} t . y y_1 \cdots y_{k-1}(tz)] \quad \text{by 1.7} \quad \text{and 3.4(ii)} \\
\Rightarrow\ & \exists N_1, \ldots, N_{j-1}, N_{j+1}, \ldots, N_n, P, Q \\
& \text{such that } B_\top \vdash^* \alpha_i N_i \text{ for } 1 \leqslant i \leqslant n \text{ and } i \neq j \\
& B_\top \vdash^* \gamma_1 \to \cdots \to \gamma_{k-1} \to \beta_2 \to \psi P,\ B_\top \vdash^* \beta_1 Q
\end{aligned}$$

and

$$\begin{aligned}
B_\top \not\vdash^* \alpha M'[& x_1 := N_1, \ldots, x_{j-1} := N_{j-1}, x_{j+1} := N_{j+1}, \ldots, \\
& x_n := N_n, y := P, z := Q] \qquad \text{by the induction hypothesis.}
\end{aligned}$$

We can choose $N_j \equiv \lambda y_1 \cdots y_{k-1} t . P y_1 \cdots y_{k-1}(tQ)$. In fact $B_\top \vdash^* \alpha_j N_j$ and

$$\begin{aligned}
M[x_1 := N_1, \ldots, x_n := N_n] \equiv M'[& x_1 := N_1, \ldots, x_{j-1} := N_{j-1}, \\
& x_{j+1} := N_{j+1}, \ldots, x_n := N_n, y := P, z := Q].
\end{aligned}$$

Case 2.2. $\alpha_j \equiv \gamma_1 \to \cdots \to \gamma_{k-1} \to \psi \to \psi$, where $\psi \in \{\varphi_*, \varphi_\top\}$. Define

$$\bar{\psi} = \begin{cases} \varphi_\top & \text{if} \quad \psi \equiv \varphi_* \\ \varphi_* & \text{if} \quad \psi \equiv \varphi_\top . \end{cases}$$

$$\begin{aligned}
B \not\vdash^* \alpha M \Rightarrow\ & B/x_j \cup \{\gamma_1 \to \cdots \gamma_{k-1} \to \psi y, \bar{\psi} z\} \not\vdash^* \alpha M', \\
& \text{where } y, z \notin FV(M) \text{ and} \\
& M' \equiv M[x_j := \lambda y_1 \cdots y_{k-1} t . y y_1 \cdots y_{k-1}(zt)] \\
\Rightarrow\ & \exists N_1, \ldots, N_{j-1}, N_{j+1}, \ldots N_n, P, Q \\
& \text{such that } B_\top \vdash^* \alpha_i N_i \quad \text{for} \quad 1 \leqslant i \leqslant n \quad \text{and} \quad i \neq j, \\
& B_\top \vdash^* \gamma_1 \to \cdots \to \gamma_{k-1} \to \psi P,\ B_\top \vdash^* \bar{\psi} Q \quad \text{and} \\
& B_\top \not\vdash^* \alpha M'[x_1 := N_1, \ldots, x_{j-1} := N_{j-1}, \\
& x_{j+1} := N_{j+1}, \ldots, x_n := N_n, y := P, z := Q] \\
& \text{by the induction hypothesis.}
\end{aligned}$$

We can choose $N_j \equiv \lambda y_1 \cdots y_{k-1} t . P y_1 \cdots y_{k-1}(Qt)$.

Case 2.3. $\alpha_j \equiv \gamma_1 \to \cdots \to \gamma_{k-1} \to \omega \to \psi$, where $\psi \in \{\varphi_*, \varphi_\top\}$.

$$
\begin{aligned}
B \not\vdash^* \alpha M \Rightarrow B/x_j \cup \{\gamma_1 \to \cdots \to \gamma_{k-1} \to \psi y\} \not\vdash^* \alpha M', \\
&\text{where} \quad y \notin FV(M) \quad \text{and} \\
&M' \equiv M[x_j := \lambda y_1 \cdots y_{k-1} t . y y_1 \cdots y_{k-1}] \\
\Rightarrow \exists N_1, ..., N_{j-1}, N_{j+1}, ..., N_n, P \\
&\text{such that} \quad B_\top \vdash^* \alpha_i N_i \quad \text{for} \quad 1 \leqslant i \leqslant n \\
&\text{and} \quad i \neq j,\ B_\top \vdash^* \gamma_1 \to \cdots \to \gamma_{k-1} \to \psi P \quad \text{and} \\
&B_\top \not\vdash^* \alpha M'[x_1 := N_1, ..., x_{j-1} := N_{j-1}, \\
&x_{j+1} := N_{j+1}, ..., x_n := N_n, y := P] \\
&\text{by the induction hypothesis.}
\end{aligned}
$$

We can choose $N_j \equiv \lambda y_1 \cdots y_{k-1} t . P y_1 \cdots y_{k-1}$.

Note that there are no other cases. In fact if $\alpha_j \equiv \gamma_1 \to \cdots \to \gamma_k \to \omega$ then $\alpha_j \sim_* \omega$ and if $\alpha_j \equiv \gamma_1 \to \cdots \to \gamma_{k-1} \to \bar{\psi} \to \psi$ then $\alpha_j \sim_* \gamma_1 \to \cdots \to \gamma_{k-1} \to \psi$. ∎

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