The Fan Theorem and Uniform Continuity

Josef Berger*

Mathematisches Institut,
Universität München, Theresienstraße 39,
D-80333 München, Germany
Josef.Berger@mathematik.uni-muenchen.de

Abstract. In presence of continuous choice the fan theorem is equivalent to each pointwise continuous function f from the Cantor space to the natural numbers being uniformly continuous. We investigate whether we can prove this equivalence without the use of continuous choice. By strengthening the assumption of pointwise continuity of f to the assertion that f has a modulus of pointwise continuity which itself is pointwise continuous, we obtain the desired equivalence.

We work entirely in the system **BISH** of Bishop's constructive mathematics [2], which means we are using intuitionistic logic and an appropriate set-theoretic foundation like Aczel's CZF, see [1]. We are interested in the constructive content of the following assertion:

UC Each pointwise continuous function $f : 2^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous.

For a binary sequence[1] $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots)$ we write $\overline{\alpha}n$ for the initial segment $(\alpha_0, \dots, \alpha_{n-1})$. The **Cantor space** $2^{\mathbb{N}}$ is the set of binary sequences equipped with the compact metric

$$d(\alpha, \beta) = \inf \left\{ 2^{-n} \mid \overline{\alpha}n = \overline{\beta}n \right\}.$$

By Theorem 1.2 in Chapter 5 of [4], a function $f : 2^{\mathbb{N}} \to \mathbb{N}$ is pointwise continuous if and only if

$$\forall \alpha \exists N \forall \beta \left(\overline{\alpha}N = \overline{\beta}N \Rightarrow f(\alpha) = f(\beta) \right).$$

In this case N is called a **witness for f being continuous at α**. Another version of this theorem is that a function $f : 2^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous if and only if

$$\exists N \forall \alpha, \beta \left(\overline{\alpha}N = \overline{\beta}N \Rightarrow f(\alpha) = f(\beta) \right).$$

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[1] We use Greek letters α, β, γ for infinite binary sequences. For finite binary sequences we use the letters u, v, w. We use the letters n and N for natural numbers.

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We now formulate two major principles of Brouwer's intuitionism, namely the **fan theorem** for detachable bars and the principle of **continuous choice**. The fan theorem reads as follows:

FAN Each detachable bar is uniform.

A set B of finite binary sequences is **detachable** if

$$\forall u \, (u \in B \vee u \notin B) \, .$$

A detachable set B is a **bar** if

$$\forall \alpha \exists n \, (\overline{\alpha} n \in B) \, .$$

A bar is **uniform** if

$$\exists N \forall \alpha \exists n \leq N \, (\overline{\alpha} n \in B) \, .$$

Note that by the compactness of the Cantor space, **FAN** holds classically. Note further that **FAN** does not hold recursively[2]; see Corollary 4.7.6 in [8].

The **principle of continuous choice** consists of two parts:[3]

$\mathbf{CC}_1$ Each function from $2^{\mathbb{N}}$ to $\mathbb{N}$ is pointwise continuous.

$\mathbf{CC}_2$ If $P \subset 2^{\mathbb{N}} \times \mathbb{N}$, and for each α there exists n such that $(\alpha, n) \in P$, then there is a function $f : 2^{\mathbb{N}} \to \mathbb{N}$ such that $(\alpha, f(\alpha)) \in P$ for all α.

Thus continuous choice divides into a continuity part $\mathbf{CC}_1$ and a choice part $\mathbf{CC}_2$. Using *tertium non datur*, we can define a discontinuous function; thus continuous choice fails classically. It also fails recursively; see Theorem 2.2 in Chapter 5 of [4] or Proposition 4.6.7 in [8]. Bridges and Richman showed that under continuous choice, **FAN** and **UC** are equivalent; see Section 3 of Chapter 5 in [4]. Assuming continuous choice implicitly in their concept of function, Iris Loeb [3] and Wim Veldman [9] obtain the equivalence of **FAN** and **UC** as well.

Proposition 1. *Under continuous choice, **UC** and **FAN** are equivalent.*

We want to investigate how far we can get without continuous choice. First we mention that the proof of **UC** $\Rightarrow$ **FAN** does not require this additional choice assumption; see Theorem 3.3 in Chapter 5 of [4].

Lemma 2. ***UC** implies **FAN**.*

Proof. Let B be a detachable bar. We define

$$f : 2^{\mathbb{N}} \to \mathbb{N}, \, \alpha \mapsto \min\{n \mid \overline{\alpha} n \in B\} \, .$$

[2] That means, when you add the Church-Markov-Turing thesis to **BISH**.

[3] This principle is often formulated in terms of sequences of natural numbers, instead of binary sequences.

We stress that this definition only makes sense because B is detachable. Note that f is pointwise continuous, since

$$\forall \alpha, \beta \left(\overline{\alpha} f(\alpha) = \overline{\beta} f(\alpha) \Rightarrow f(\alpha) = f(\beta) \right). \tag{1}$$

Thus by **UC**, f is even uniformly continuous. By Proposition 4.2 in Chapter 4 of [2], f is bounded. Let N be a bound for the range of f. This implies that

$$\forall \alpha \exists n \leq N \left(f(\alpha) = n \right),$$

which in turn implies that

$$\forall \alpha \exists n \leq N \left(\overline{\alpha} n \in B \right).$$

Thus B is a uniform bar.

The next question is whether **FAN** implies **UC**. A first step in this direction is to show that a stronger form of **FAN** implies **UC**.

F-FAN Each bar is uniform.

The **F** in **F-FAN** stands for *full* and indicates that **F-FAN** is the full form of the fan theorem. From the very formulation it is clear that **F-FAN** implies **FAN**. Since classically every set is detachable, the statements **FAN** and **F-FAN** are equivalent, hence true. For a deeper discussion about this implication, see [7]. It is well-known that **F-FAN** implies **UC**; see for example Theorem 3.6 in Chapter 6 of [8].

Lemma 3. ***F-FAN*** *implies* ***UC****.*

Proof. Let $f : 2^{\mathbb{N}} \to \mathbb{N}$ be pointwise continuous. Set

$$B = \{ \overline{\alpha} n \mid \alpha \in 2^{\mathbb{N}} \text{ and } n \text{ is a witness for } f \text{ being continuous at } \alpha \}.$$

This B is a bar because f is pointwise continuous. By **F-FAN**, B is uniform; that is, there is N such for each α there is a witness n for f being continuous at α with $n \leq N$. But this amounts to the uniform continuity of f.

Thus **UC** lies somewhere in the no-man's-land between **FAN** and **F-FAN**. What we are doing now is to strengthen the property of f being continuous in the formulation of **UC**. This gives rise to a weaker condition **MUC** which still implies **FAN**, but now is derivable from **FAN**. This is in the spirit of constructive reverse mathematics as practised by Hajime Ishihara [5], [6]. See also the recent papers of Iris Loeb [3] and Wim Veldman [9].

For this purpose we define:

MUC Each function $f : 2^{\mathbb{N}} \to \mathbb{N}$ which has a modulus of continuity which itself is pointwise continuous is uniformly continuous.

If $f, g : 2^{\mathbb{N}} \to \mathbb{N}$ are functions, then g is a **modulus of pointwise continuity of** f if the following holds:

$$\forall \alpha, \beta \left(\overline{\alpha} g(\alpha) = \overline{\beta} g(\alpha) \Rightarrow f(\alpha) = f(\beta) \right).$$

Classically, each pointwise continuous function has a modulus of pointwise continuity. But this step requires a strong form of choice, namely $\mathbf{CC}_2$. Thus inside **BISH** it makes a significant difference to require that the witnesses of pointwise continuity are given by a function.

We obtain the equivalence of **MUC** and **FAN** inside **BISH**, without any use of continuous choice.

Proposition 4. ***MUC*** *and* ***FAN*** *are equivalent.*

Proof. First we prove that **MUC** implies **FAN**. Considered carefully, our proof of Lemma 2 still works in this altered situation. Let B be a detachable bar. Define again

$$f : 2^{\mathbb{N}} \to \mathbb{N},\, \alpha \mapsto \min\{n \mid \overline{\alpha}n \in B\}\,.$$

By (1) we obtain that $g \equiv f$ is a modulus of continuity for f. By **MUC**, f is uniformly continuous. Now the proof that B is uniform proceeds just like the proof of Lemma 2.

Now we show that **FAN** implies **MUC**. Our proof is similar to the proof of Theorem 3.2 in Chapter 5 of [4], where the authors show that under **FAN** and continuous choice each function $f : 2^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous. Here we have weakened the assumption of continuous choice to the assumption that f has a continuous modulus of continuity. Assume that functions f, g from $2^{\mathbb{N}}$ to $\mathbb{N}$ are given such that g is a modulus of continuity for f and that g is pointwise continuous. We show that f is uniformly continuous. Set

$$B = \{u \mid g(u^*) \le |u|\}.$$

We must explain what we mean by u^* and by $|u|$. Suppose that $u = (u_0, \dots, u_{n-1})$. Then

$$u^* = (u_0, \dots, u_{n-1}, 0, 0, 0, \dots)$$

and $|u|$ is the length of u, that means n. Clearly B is detachable. We show that B is a bar. Fix any α. Let n be a witness for g being continuous at α, and assume that $n \ge g(\alpha)$. Then

$$g(\overline{\alpha}n^*) = g(\alpha) \le n = |\overline{\alpha}n|\,,$$

thus $\overline{\alpha}n \in B$. By **FAN**, there exists N such that

$$\forall \alpha \exists n \le N\, (\overline{\alpha}n \in B)\,.$$

Now we can show that

$$\forall \alpha, \beta\, \left(\overline{\alpha}N = \overline{\beta}N \Rightarrow f(\alpha) = f(\beta)\right),$$

which is just the uniform continuity of f. To this end fix α, β with $\overline{\alpha}N = \overline{\beta}N$. There is $n \le N$ such that $\overline{\alpha}n \in B$. Set $\gamma \equiv \overline{\alpha}n^*$. Thus $g(\gamma) \le n \le N$. We obtain

$$\overline{\gamma}g(\gamma) = \overline{\alpha}g(\gamma) = \overline{\beta}g(\gamma),$$

which implies that

$$f(\alpha) = f(\gamma) = f(\beta).$$

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