

Normalisation by Traversals

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Abstract

We present a novel method of computing the β -normal η -long form of a simply-typed λ -term by constructing *traversals* over a variant abstract syntax tree of the term. In contrast to β -reduction, which changes the term by substitution, this method of normalisation by traversals leaves the original term intact. We prove the correctness of the normalisation procedure by game semantics. As an application, we establish a path-traversal correspondence theorem which is the basis of a key decidability result in higher-order model checking.

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1 Introduction

This paper is about a method of computing the normal form of a lambda-term by traversing a slightly souped up version of the abstract syntax tree of the term, called its *long form*. A *traversal* is a certain *justified sequence* of nodes of the tree i.e. sequence of nodes such that each (non-initial) node is equipped with a *justification pointer* to an earlier node. Each traversal may be viewed as computing a path in the abstract syntax tree of the β -normal η -long form of the term. Note that a term-tree, such as the normal form of a term, is determined by the set of its paths. The usual (normalisation by) β -reduction changes the term by substitution. By contrast, our method of normalisation by traversals does not perform β -reduction, thus leaving the original term intact. In this sense, normalisation by traversals uses a form of reduction that is *non-destructive* and *local* [Danos and Regnier, 1993].

1.1 Traversals: an example

We first illustrate traversals with an example. Take the term-in-context,

$$g : (o \rightarrow o) \rightarrow o \rightarrow o, a : o \vdash N P R : o,$$

where

$$\begin{aligned} N &= \lambda\varphi^{(o \rightarrow o) \rightarrow o \rightarrow o} z^{o \rightarrow o} . \varphi (\lambda x . \varphi (\lambda x' . x) a) (z a) \\ P &= \lambda f^{o \rightarrow o} y^o . f (g (\lambda b . b) y) \\ R &= g (\lambda b' . b') \end{aligned}$$

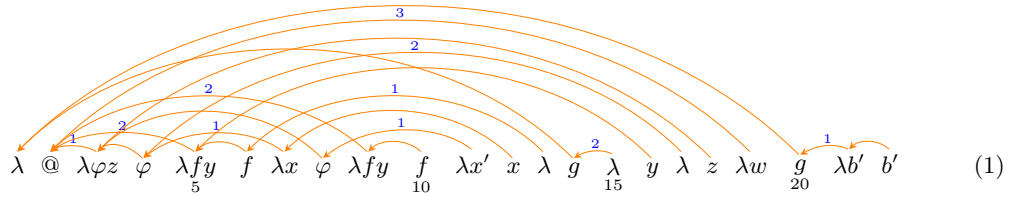
which has normal form $g (\lambda b . b) (g (\lambda b' . b') a)$. To normalise the term $N P R$ by traversal, we first construct its long form, written $\langle N P R \rangle$, which is the following term

$$\lambda . @ \left(\lambda \varphi z . \varphi (\lambda x . \varphi (\lambda x' . x) (\lambda . a)) (\lambda . z (\lambda . a)) \right) \left(\lambda f y . f (\lambda . g (\lambda b . b) (\lambda . y)) \right) \left(\lambda w . g (\lambda b' . b') (\lambda . w) \right)$$

The long form is obtained by η -expanding the term fully¹, and then replacing the (implicit) binary application operator of each redex by the *long application* operator @.

Now consider the abstract syntax tree of $\langle N P R \rangle$, as shown in Figure 1. Notice that nodes on levels 0, 2, 4, etc., are labelled by lambdas; and those on levels 1, 3, 5, etc., are labelled by either variables or the long application symbol @. The dotted arrows (pointing from a variable to its λ -binder) indicate an enabling relation between nodes of the tree: $n \vdash n'$ (read “ n' is enabled by n ”) just if $n \xrightarrow{\dots} n'$. By convention, (nodes labelled by) free variables are enabled by the root node, as indicated by the dotted arrows. Further, every lambda-labelled node, except the root, is enabled by its parent node in the tree (we omit all such dotted arrows from the figure to avoid clutter).

Traversals are *justified sequences* (i.e., sequences of nodes whereby each (non-initial) node has a justification pointer to an earlier node) that strictly alternate between lambda and non-lambda labels. The long form $\langle N P R \rangle$ has three maximal traversals, one of which is the following:



The five rules that define traversals are displayed in Table 1. The rule (Root) says that the root node is a traversal. The rule (Lam) says if a traversal t ends in a λ -labelled node n , then t extended with the child node of n , n' , is also a traversal. Note that every node of a traversal in an even position is constructed by rule (Lam). The rule (App) justifies the construction of the third node of traversal (1). If a traversal ends in a node labelled with a variable ξ_i , then

¹Somewhat nonstandardly, every ground-type subterm that is *not* in a function position is also expanded (to a term with a “dummy lambda”) $t \mapsto \lambda . t$. For example, $\lambda x^{o \rightarrow o \rightarrow o} . x a$ fully η -expands to $\lambda x^{o \rightarrow o \rightarrow o} . x (\lambda . a) (\lambda . z)$; and $g (\lambda b . b) (g (\lambda b' . b') a)$ fully η -expands to $\lambda . g (\lambda b . b) (\lambda . g (\lambda b' . b') \lambda . a)$.

there are two cases, corresponding to whether ξ_i is (hereditarily justified² by) a bound (BVar) or free (FVar) variable in the long form.

(BVar): If the traversal has the form $t \cdot n \cdot \lambda \bar{\xi} \cdots \xi_i$ where $\bar{\xi} = \xi_1 \cdots \xi_n$, and ξ_i is hereditarily justified by a @, then there are two subcases.

- If n is (labelled by) a variable then $t \cdot n \cdot \lambda \bar{\xi} \cdots \xi_i \cdot \lambda \bar{\eta}$ is a traversal, whereby the pointer label i means that the node $\lambda \bar{\eta}$ is the i -th child of n . For example, this rule justifies the construction of the 11th node $\lambda x'$ and 13th node λ of traversal (1).

- If n is (labelled by) @ then $t \cdot @ \cdot \lambda \bar{\xi} \cdots \xi_i \cdot \lambda \bar{\eta}$ is a traversal. For example, this rule justifies the construction of the 5th node $\lambda f y$, 9th node $\lambda f y$ and 19th node λw of traversal (1). meaning that the node is the $(i + 1)$ -th child of @.

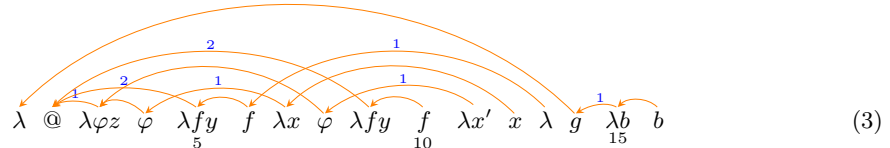
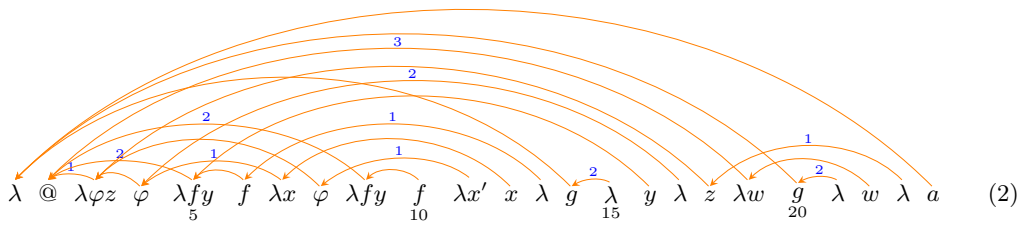
Intuitively the rule (BVar) captures the switching of control between caller and callee, or between formal and actual parameters. See Remark 3.15 for further details.

(FVar): If the traversal has the form $t \cdot \lambda \bar{\xi} \cdots \xi_i$ and ξ_i is hereditarily justified by the opening node ϵ , then $t \cdot \lambda \bar{\xi} \cdots \xi_i \cdot \lambda \bar{\eta}$ is a traversal, for each child-node $\lambda \bar{\eta}$ of ξ_i (so j ranges over $\{1, \dots, ar(\xi_i)\}$ where $ar(\xi_i)$ is the arity (branching factor) of ξ_i). For example, the 15th node λ and the 21st node λ of traversal (1) are constructed by this rule.

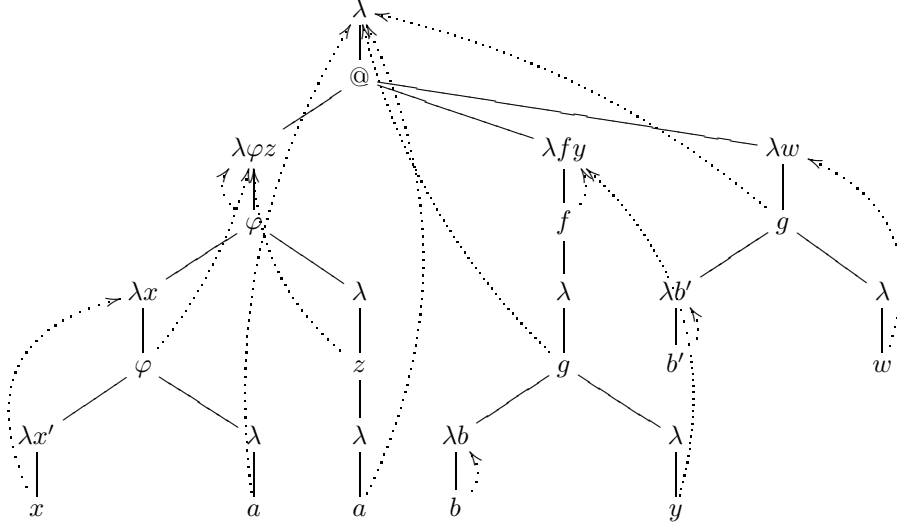
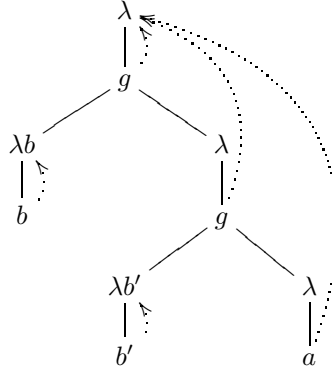
As mentioned earlier, each traversal computes a path in the abstract syntax tree of the β -normal η -long form of the term NPR , which is shown in Figure 2. With reference to the Figure, notice that each path of the tree is actually an alternating justified sequence, in fact, a P-view (about which more anon). Such a justified path is obtained from the traversal by projecting to those nodes that are hereditarily justified by the root node. Thus we obtain

the following projected justified subsequence from traversal (1), $\lambda \ g \ \lambda \ g \ \lambda b' \ b'$, which is a maximal path of $\lambda.g(\lambda b.b)(\lambda.g(\lambda b'.b'))(\lambda.a)$, the β -normal η -long form of NPR .

The other maximal traversals of $\langle NPR \rangle$ are:



²We say that a node-occurrence n in a justified sequence is hereditarily justified by another n' if there is a chain of pointers from n to n' .

Figure 1: The abstract syntax tree of the long form $\langle N P R \rangle$ in Section 1.1.Figure 2: The abstract syntax tree of $\langle g(\lambda b.b)(g(\lambda b'.b')a) \rangle$ in Section 1.1.

which respectively project to the maximal paths, $\lambda \xrightarrow{2} g \xrightarrow{2} \lambda \xrightarrow{2} g \xrightarrow{2} \lambda \xrightarrow{2} a$ and $\lambda \xrightarrow{1} g \xrightarrow{1} \lambda b \xrightarrow{1} b$, of the β -normal η -long form of $N P R$.

1.2 Correctness of normalisation by traversals

We state the correctness theorem.

Theorem 1.1 (Correctness). *Given a term-in-context $\Gamma \vdash M : A$, there is a bijection between the following sets of justified sequences:*

- $\mathfrak{Trav}\langle M \rangle \upharpoonright \epsilon$, traversals over $\langle M \rangle$ projected to nodes hereditarily justified by the root node ϵ

- $\text{Path}\langle\beta(M)\rangle$, justified paths in the abstract syntax tree of the long form $\langle\beta(M)\rangle$, where $\beta(M)$ is the β -normal form of M .

Furthermore the bijective map is strong in the sense that every projected traversal and its image as path are isomorphic as justified sequences. Thus, normalisation by traversals is correct: the set of projected traversals over $\langle M \rangle$ determine the β -normal η -long form of M .

The proof is via game semantics [Hyland and Ong, 2000]. The game-semantic denotation of a sequent, $\llbracket \Gamma \vdash M : A \rrbracket$, is an *innocent* strategy, represented as a certain prefix-closed set of justified sequences called *plays*. Innocent means that the strategy is generated by the subset $\ulcorner \llbracket \Gamma \vdash M : A \rrbracket \urcorner$ of plays which are *P-views*. (Intuitively the P-view of a play is a certain justified subsequence consisting only of those moves which player P considers relevant for determining his next move. See Section 2.1 for the definitions.) We prove the correctness theorem by showing that the following three sets of justified sequences are strongly bijective:

$$\mathfrak{Trav}\langle M \rangle \upharpoonright \epsilon \xrightarrow{\hat{\ell}^*} \ulcorner \llbracket \Gamma \vdash M : A \rrbracket \urcorner \xrightarrow{\mathcal{G}} \text{Path}\langle\beta(M)\rangle \quad (4)$$

The strong bijection on the right, \mathcal{G} , is a well-known fundamental result of game semantics, and the essence of the definability result [Hyland and Ong, 2000]. The strong bijection on the left, $\hat{\ell}^*$, is the main technical result of the paper, Theorem 3.18. The key intuition is that traversals correspond to a certain collection of *uncovered plays* (of an innocent strategy) or plays-without-hiding. Given a term-in-context $\Gamma \vdash M : A$, we formalise the long form, $\langle M \rangle$, as a Σ -labelled binding tree, in the sense of Stirling [2009]. We then identify two arenas associated with $\langle M \rangle$, viz., *explicit arena* $\text{ExpAr}\langle M \rangle$, and *succinct arena* $\text{SucAr}\langle M \rangle$. The enabling relation \vdash between nodes of the long form $\langle M \rangle$ (as discussed in Section 1.1) is defined as the enabling relation of the arena $\text{ExpAr}\langle M \rangle$, whose underlying set consists of nodes of the (binding) tree $\langle M \rangle$. Traversals over $\langle M \rangle$ are then defined as justified sequences over $\text{ExpAr}\langle M \rangle$ by induction over a number of rules.

To interpret traversals over $\langle M \rangle$ as uncovered plays, we define the succinct arena $\text{SucAr}\langle M \rangle$ as a disjoint union of

- a “revealed” arena, consisting of the arena over which $\llbracket \Gamma \vdash M : A \rrbracket$ is defined as a strategy, and
- a “hidden” arena, for interpreting the moves that are hereditarily justified by an @ in $\langle M \rangle$.

We then define a map $\hat{\ell} : \text{ExpAr}\langle M \rangle \rightarrow \text{SucAr}\langle M \rangle$, called *direct arena morphism*, which preserves initial moves, and preserves and reflects the enabling relation. Furthermore the morphism extends to a function $\hat{\ell}^*$ that maps justified sequences of $\text{ExpAr}\langle M \rangle$ to those of $\text{SucAr}\langle M \rangle$. Theorem 3.18 then asserts that the map $\hat{\ell}^*$ defines a strong bijection from $\mathfrak{Trav}\langle M \rangle \upharpoonright \epsilon$ to $\ulcorner \llbracket \Gamma \vdash M : A \rrbracket \urcorner$.

Application to higher-order model checking

We apply normalisation by traversals to higher-order model checking [Ong, 2015]. The Higher-order Model Checking Problem [Knapik et al., 2002] asks, given a higher-order recursion scheme \mathcal{G} and a monadic second-order formula φ , whether the tree generated by \mathcal{G} , written $\llbracket \mathcal{G} \rrbracket$, satisfies φ ?

This problem was first shown to be decidable by Ong [2006]. Ong’s proof uses a *transference principle*: instead of reasoning about the parity winning condition of infinite paths in the generated tree $\llbracket \mathcal{G} \rrbracket$, he considers *traversals* over the computation tree of \mathcal{G} , $\lambda(\mathcal{G})$, which is a tree

obtained from \mathcal{G} by first transforming the rewrite rules into long forms, then unfolding these transformed rules *ad infinitum*, but without performing any β -reduction (i.e. substitution of actual parameters for formal parameters). The argument uses a key technical lemma, presented in the following as Theorem 4.5, which states that paths in the generated tree $\llbracket \mathcal{G} \rrbracket$ on the one hand, and traversals over the computation tree $\lambda(\mathcal{G})$ projected to the terminal symbols from Σ on the other, are the same set of finite and infinite sequences over Σ . In this paper, we apply Theorem 3.18 to prove Theorem 4.5.

2 Technical preliminaries

We write $\mathbb{N} = \{1, 2, \dots\}$, $X_0 = \{0\} \cup X$ for $X \subseteq \mathbb{N}$, $[n] = \{1, \dots, n\}$ for $n \in \mathbb{N}$, X^* for the set of finite sequences of elements of X , \leq for the prefix ordering over sequences, and $|x_1 \cdot x_2 \cdots x_n| = n$ for the length of sequences. By a *tree* T , we mean a subset of \mathbb{N}^* that is prefix-closed (i.e. if $\alpha \in T$ and $\alpha' \leq \alpha$ then $\alpha' \in T$) and order-closed (i.e. if $\alpha \cdot n \in T$ and $1 \leq n' < n$ then $\alpha \cdot n' \in T$). A *path* in T is a sequence of elements of T , $\alpha_1 \cdot \alpha_2 \cdots \alpha_n$, such that $\alpha_1 = \epsilon$, $\alpha_n = i_1 \cdot i_2 \cdots i_{n-1}$, and $\alpha_{j+1} = \alpha_j \cdot i_j$ for each $1 \leq j \leq n-1$. We write $\text{Path}(T)$ for the set of paths in the tree T . A *ranked alphabet* Σ is a set of symbols such that each symbol $f \in \Sigma$ has an arity $\text{ar}(f) \geq 0$. A Σ -*labelled tree* is a function $F : T \rightarrow \Sigma$ such that (i) T is a tree, and (ii) for each $\alpha \in T$, if $\text{ar}(F(\alpha)) = n$ then $\{1, \dots, n\} = \{i \mid \alpha \cdot i \in T\}$. By definition, Σ -labelled trees are *ordered*, i.e., the set of children of each node is a (finite) linear order. Let T be a tree, and let $\alpha \in T$. The tree T *rooted at* α , denoted $T_{@ \alpha}$, is the set $\{\gamma \in \mathbb{N}^* \mid \alpha \cdot \gamma \in T\}$.

Types (ranged over by A, B , etc.) are defined by the grammar: $A ::= o \mid (A \rightarrow B)$. A type can be written uniquely as (by convention \rightarrow associates to the right), $A_1 \rightarrow \cdots \rightarrow A_n \rightarrow o$, which we abbreviate to (A_1, \dots, A_n, o) . The *order* of a type A , $\text{ord}(A)$, which measures how deeply nested a type is on the left of the arrow, is defined as $\text{ord}(o) := 0$, and $\text{ord}(A \rightarrow B) := \max(\text{ord}(A) + 1, \text{ord}(B))$. The *arity* of a type $A = (A_1, \dots, A_n, o)$, written $\text{ar}(A)$, is defined to be n .

We assume an infinite set Var of typed variables, ranged over by $\Phi, \Psi, \varphi, \psi, x, y, z$, etc. Raw terms (ranged over by M, N, P, Q , etc.) of the pure lambda calculus are defined by the grammar: $M ::= x \mid \lambda x^A. M \mid (M N)$; by convention, applications associate to the left. Typing judgements (or *terms-in-context*) have the form, $\Gamma \vdash M : A$, where the environment Γ is a list of variable bindings of the form $x : A$. Henceforth, by a *term* M (respectively, $M : A$) we mean a well-typed term (respectively, of type A), i.e., $\Gamma \vdash M : A$ is provable for some environment Γ and type A . We write $\text{FV}(M)$ for the set of variables that occur free in M .

2.1 Arenas, direct arena morphisms and justified sequences

Definition 2.1 (Arena). An *arena* is a triple $A = (|A|, \vdash_A, \lambda_A)$ such that $|A|$ is a set (of moves), $\vdash_A \subseteq (|A| + \{\star\}) \times |A|$ is the enabling relation, and $\lambda_A : |A| \rightarrow \{O, P\}$ is the ownership function that partitions moves into *O-moves* and *P-moves*, satisfying:

- For every $m \in |A|$, there exists a unique $m' \in (|A| \cup \{\star\})$ such that $m' \vdash_A m$; we call m' the *enabler* of m , or m' *enables* m . Writing $\widehat{\vdash}_A$ for the inclusion of \vdash_A in $(\{\star\} \cup |A|)^2$, we say that m is *hereditarily enabled* by m' if $m' \widehat{\vdash}_A^* m$, writing ρ^* for the reflexive, transitive closure of a binary relation ρ .
- Whenever $m \vdash_A m'$ then $\lambda_A(m) \neq \lambda_A(m')$.

We call a move *initial* if its enabler is \star , and write Init_A for the set of initial moves of A . An arena A is said to be *O-initial* if $\lambda_A(m) = O$ for every initial m ; and *pointed* if Init_A is a

singleton set. We define the set of *O-moves* as $|A|^O := \{m \in |A| \mid \lambda_A(m) = O\}$, and the set of *P-moves* as $|A|^P := \{m \in |A| \mid \lambda(m) = P\}$. The *opposite arena* of A is $A^\perp := (|A|, \vdash_A, \lambda_A^\perp)$ where $\lambda_A^\perp(m) = P$ if, and only if, $\lambda_A(m) = O$. We say that A is *ordered* if for each $m \in |A|$, the set $\{m' \mid m \vdash_A m'\}$ is a linear order; and if so, we write $m \vdash_A^i m'$, where $i \geq 1$, to mean m' is the i -th child of m .

There is an obvious one-one correspondence between types and finite, ordered trees. For example, the type $((o, o), o), o, (o, o, o), o$ corresponds to the tree whose maximal (with respect to \leq) elements are $1 \cdot 1 \cdot 1$, 2 , $3 \cdot 1$ and $3 \cdot 2$. We write $Tree_A$ to mean the tree that corresponds to the type A .

Definition 2.2 (Arena determined by type A). Let A be a type. The *arena determined by A* , written $Ar(A)$, is defined as follows:

- $|Ar(A)| := Tree_A$
- $\star \vdash_{Ar(A)} \epsilon$, and for all $\alpha, \beta \in |Ar(A)|$, $\alpha \vdash_{Ar(A)} \beta \iff (\alpha \leq \beta \wedge |\beta| = |\alpha| + 1)$
- $\lambda_{Ar(A)}(\alpha) = O \iff |\alpha| \text{ even}$.

Plainly $Ar(A)$ is a finite arena which is ordered, pointed and O-initial.

We fix a scheme for naming nodes of a given tree (and hence moves of an arena determined by a type) using symbols of the following infinite ranked alphabet

$$\mathbf{\Lambda} := Var \cup \{\lambda^\alpha \mid \alpha \in \mathbb{N}^*\} \cup \{\lambda x_1^{A_1} \cdots x_{n+1}^{A_{n+1}} \mid x_i^{A_i} \in Var, n \geq 0\}$$

such that $ar(x^A) := ar(A)$, $ar(\lambda x_1^{A_1} \cdots x_{n+1}^{A_{n+1}}) := 1$ and $ar(\lambda^\alpha) := 0$. By abuse of language, we say that the *lambda* $\lambda \xi_1^{A_1} \cdots \xi_n^{A_n}$ has type (A_1, \dots, A_n, o) , and the *dummy lambda* λ^α has type o .

Given a type A and an injective function, $\nu_A : \{\alpha \in Tree_A \mid |\alpha| \text{ odd}\} \rightarrow Var$, such that whenever $\nu_A(\alpha) = x^B$ then $(Tree_A)_{@ \alpha} = Tree_B$, we extend ν_A to a function $Tree_A \rightarrow \mathbf{\Lambda}$ as follows. Let $\alpha \in Tree_A$ be of even length. Suppose for each $\alpha \cdot i \in Tree_A$, we have $\nu_A(\alpha \cdot i) = x_i^{B_i}$, and $n = |\{i \mid \alpha \cdot i \in Tree_A\}|$, we define

$$\nu_A(\alpha) := \begin{cases} \lambda x_1^{B_1} \cdots x_n^{B_n} & \text{if } n > 0 \\ \lambda^\alpha & \text{if } n = 0 \end{cases}$$

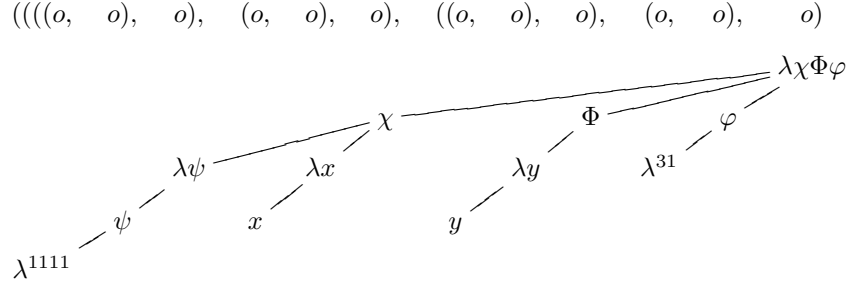
It is straightforward to see that for every type A , the function $\nu_A : Tree_A \rightarrow \mathbf{\Lambda}$ satisfies the following: 1. ν_A is injective, 2. ν_A defines a $\mathbf{\Lambda}$ -labelled tree, 3. for all $\alpha \in Tree_A$, if $\nu_A(\alpha)$ has type B then $(Tree_A)_{@ \alpha} = Tree_B$. We call such a function ν_A a $\mathbf{\Lambda}$ -*representation* of $Tree_A$ (and for arena $Ar(A)$, and type A).

Example 2.3. Take the type $A \rightarrow A$ where $A = (((o, o), o), (o, o), o)$. We display the nodes of $Tree_{A \rightarrow A}$ via a naming scheme $\nu_{A \rightarrow A} : Tree_{A \rightarrow A} \rightarrow \mathbf{\Lambda}$ in Figure 3.

Product

For arenas A and B , we define the *product arean* $A \times B$ by:

- $|A \times B| := |A| + |B|$,
- $\star \vdash_{A \times B} m \iff \star \vdash_A m \text{ or } \star \vdash_B m$,
- $m \vdash_{A \times B} m' \iff m \vdash_A m' \text{ or } m \vdash_B m'$,

Figure 3: A Λ -representation of $Tree_{A \rightarrow A}$ where $A = (((o, o), o), (o, o), o)$.

$$\bullet \lambda_{A \times B}(m) := \begin{cases} \lambda_A(m) & \text{if } m \in |A| \\ \lambda_B(m) & \text{if } m \in |B| \end{cases}$$

Thus $A \times B$ is just the disjoint union of A and B *qua* labelled directed graphs. For an indexed set $\{A_i\}_{i \in I}$ of arenas, their product $\prod_{i \in I} A_i$ is defined similarly.

Function space

For arenas A and B , we define *the function space arena* $A \Rightarrow B$ by:

- $|A \Rightarrow B| := |A| \times \text{Init}_B + |B|$,
- $\star \vdash_{A \Rightarrow B} m \iff \star \vdash_B m$,
- $m \vdash_{A \Rightarrow B} m' \iff$
 - $m \vdash_B m'$, or
 - $\star \vdash_B m$ and $m' = (m'_A, m)$ and $\star \vdash_A m'_A$, or
 - $m = (m_A, m_B)$ and $m' = (m'_A, m_B)$ and $m_A \vdash_A m'_A$
- $\lambda_{A \Rightarrow B}(m) := \begin{cases} \lambda_A^\perp(m_A) & \text{if } m = (m_A, m_B) \in |A| \times \text{Init}_B \\ \lambda_B(m) & \text{if } m \in |B| \end{cases}$

Observe that if A and B are types, then $Ar(A \rightarrow B) = Ar(A) \Rightarrow Ar(B)$.

Definition 2.4 (Justified sequence). A *justified sequence of an arena* A is a finite sequence of moves, $m_1 \cdot m_2 \cdot \dots \cdot m_n$, such that for each j , if m_j is non-initial then m_j has a pointer to m_i such that $i < j$ and $m_i \vdash_A m_j$. Formally it is a triple $s = (\#s, s, \rho_s)$ consisting of a number $\#s \in \mathbb{N}_0$ (which is the length of the justified sequence), and total functions $s : [\#s] \rightarrow |A|$ (moves function) and $\rho_s : [\#s] \rightarrow [\#s]_0$ (pointers function) such that

- $\rho_s(k) < k$ for every $k \in [\#s]$, and
- ρ_s respects the enabling relation: $\rho_s(k) = 0$ implies $\star \vdash_A s(k)$, and $\rho_s(k) \neq 0$ implies $s(\rho_s(k)) \vdash_A s(k)$.

As usual, by abuse of notation, we often write $m_1 \cdot m_2 \cdot \dots \cdot m_n$ for a justified sequence such that $s(i) = m_i$ for every i , leaving the justification pointers implicit. Further we use m and m_i as meta-variables of *move occurrences* in justified sequences. We write $m_i \curvearrowright m_j$ if $\rho_s(j) = i > 0$ and $\star \curvearrowright m_j$ if $\rho_s(j) = 0$. We call m_i the *justifier* of m_j , and say m_j is justified by m_i whenever $m_i \curvearrowright m_j$. We say that m_j is *hereditarily justified* by m_i if $m_i \curvearrowright^* m_j$. In case A is an ordered arena, we write $m \overset{i}{\curvearrowright} m'$ to mean $m \curvearrowright m'$ and $m \vdash_A^i m'$.

It is convenient to relax the domain $[\#s] = \{1, 2, \dots, \#s\}$ of justified sequences to arbitrary linearly-ordered finite sets such as a subset of $[\#s]$. For example, given a justified sequence $(\#s, s : [\#s] \rightarrow |A|, \rho_s : [\#s] \rightarrow [\#s]_0)$, consider a subset $I \subseteq [\#s]$ that respects the justification pointers, i.e., $k \in I$ implies $\rho_s(k) \in I \cup \{0\}$. Then the restriction $(I, s|_I : I \rightarrow |A|, \rho_s|_I : I \rightarrow \{0\} \cup I)$ is a justified sequence in the relaxed sense. A justified sequence in the relaxed sense is identified with that in the strict sense through the unique monotone bijection $\alpha : I \rightarrow [n]$.

A justified sequence is *alternating* just if $s(k) \in |A|^O \iff k$ is odd. Henceforth we assume that justified sequences are alternating.

Definition 2.5 (P-View / O-view). Let $m_1 \dots m_n$ be a justified sequence over an arena A . Its *P-view* $\lceil m_1 \dots m_n \rceil$ is a subsequence defined inductively by:

$$\begin{aligned} \lceil m_1 \dots m_n \rceil &:= \lceil m_1 \dots m_{n-1} \rceil m_n && (\text{if } m_n \in |A|^P) \\ \lceil m_1 \dots m_n \rceil &:= m_n && (\text{if } \star \curvearrowright m_n \in |A|^O) \\ \lceil m_1 \dots m_n \rceil &:= \lceil m_1 \dots m_k \rceil m_n && (\text{if } m_k \curvearrowright m_n \in |A|^O). \end{aligned}$$

Its *O-view* $\lfloor m_1 \dots m_n \rfloor$ is a subsequence defined inductively by:

$$\begin{aligned} \lfloor m_1 \dots m_n \rfloor &:= \lfloor m_1 \dots m_{n-1} \rfloor m_n && (\text{if } m_n \in |A|^O) \\ \lfloor m_1 \dots m_n \rfloor &:= m_n && (\text{if } \star \curvearrowright m_n \in |A|^P) \\ \lfloor m_1 \dots m_n \rfloor &:= \lfloor m_1 \dots m_k \rfloor m_n && (\text{if } m_k \curvearrowright m_n \in |A|^P). \end{aligned}$$

Formally the P-view of a justified sequence s is a subset $I \subseteq [\#s]$. Then $\lceil s \rceil$ is the restriction of s to I ; similarly for $\lfloor s \rfloor$. Henceforth by a *P-view*, we mean a justified sequence s such that $\lceil s \rceil = s$.

A P-move m_k in the sequence $m_1 \dots m_n$ ($n \geq k$) is *P-visible* just if $\star \curvearrowright m_k$ or its justifier is in $\lceil m_1 \dots m_k \rceil$. Similarly, an O-move m_k in the sequence $m_1 \dots m_n$ ($n \geq k$) is *O-visible* just if $\star \curvearrowright m_k$ or its justifier is in $\lfloor m_1 \dots m_k \rfloor$. A justified sequence s is *P-visible* (respectively, *O-visible*) just if each P-move (respectively, O-move) occurrence in s is P-visible (respectively, O-visible); s is *visible* just if it is both P- and O-visible. If s is a visible justified sequence, then so are $\lceil s \rceil$ and $\lfloor s \rfloor$ [Hyland and Ong, 2000].

Definition 2.6 (Direct arena morphism). A *direct arena morphism*, $\mathcal{I} : (|A|, \vdash_A, \lambda_A) \rightarrow (|B|, \vdash_B, \lambda_B)$, is a function $\mathcal{I} : |A| \rightarrow |B|$ that respects:

- enabling relation: for all $m, m' \in |A|$, $\star \vdash_A m \iff \star \vdash_B \mathcal{I}(m)$, and $m \vdash_A m' \iff \mathcal{I}(m) \vdash_B \mathcal{I}(m')$;
- ownership: for all $m \in |A|$, $\lambda_A(m) = \lambda_B(\mathcal{I}(m))$.

A direct arena morphism $\mathcal{I} : A \rightarrow B$ induces a function on justified sequences by extension in the obvious way: take a justified sequence t over A , define $\mathcal{I}^*(t) := (\#t, t', \rho_t)$ where $t'(i) := \mathcal{I}(t(i))$ for $i \in [\#t]$. It is straightforward to see that $\mathcal{I}^*(t)$ is a justified sequence over B . In fact, t and $\mathcal{I}^*(t)$ are *isomorphic as justified sequences*, by which we mean that they are isomorphic directed graphs (viewing a justified sequence as a linear tree with justification pointers represented as back edges). I.e. writing $\mathcal{I}^*(t) = (\#u, u, \rho_u)$, we have $\#t = \#u$, and $\rho_t = \rho_u$. (We do *not*, however, require the map ι from the image of t to the image of u , whereby $u(i) = \iota(t(i))$ for all $i \in [\#t]$, to be bijective.) Thus it follows that t is visible if, and only if, $\mathcal{I}^*(t)$ is.

Definition 2.7 (Strong bijection induced by direct arena morphism). Let \mathcal{L} and \mathcal{L}' be sets of justified sequences over arenas A and A' respectively. Given a direct arena morphism $\mathcal{I} : A \rightarrow A'$, we say that the map $\mathcal{I}^* : \mathcal{L} \rightarrow \mathcal{L}'$ is a *strong bijection (induced by \mathcal{I})* if \mathcal{I}^* restricted to \mathcal{L} , $\mathcal{I}^* \upharpoonright \mathcal{L}$, is injective, and the image of $\mathcal{I}^* \upharpoonright \mathcal{L}$ is \mathcal{L}' . The adjective strong emphasises that, in addition to the bijectivity of \mathcal{I}^* , for every $s \in \mathcal{L}$, we have s and $\mathcal{I}^*(s)$ are isomorphic as justified sequences.

Henceforth, whenever it is clear from the context, we write \mathcal{I}^* simply as \mathcal{I} .

2.2 Game semantics of the lambda calculus

A *play* over an arena A is a visible justified sequence. A *P-strategy* over arena A , or just *strategy* for short, is a non-empty prefix-closed set σ of plays over A satisfying:

- *Determinacy*. For every odd-length $s \in \sigma$, if $s \cdot a, s \cdot b \in \sigma$ then $a = b$.
- For every even-length $s \in \sigma$, for every O-move a , if $s \cdot a$ is a play, then it is in σ .

We say that σ is *total* if for every odd-length $s \in \sigma$, there exists a such that $s \cdot a \in \sigma$. We say that σ is *innocent* if for every even-length $s \cdot a \in \sigma$ and for every odd-length $t \in \sigma$ if $\ulcorner s \urcorner = \ulcorner t \urcorner$ then $t \cdot a \in \sigma$. It follows from the definition that an innocent strategy σ is determined by the set of even-length P-views in σ , written $\ulcorner \sigma \urcorner$; we say that σ is *compact* if $\ulcorner \sigma \urcorner$ is a finite set.

We can now organise arenas and innocent strategies into a category \mathbb{I} : objects are O-initial arenas; and maps $\sigma : A \rightarrow B$ are innocent strategies over arena $A \Rightarrow B$.

Theorem 2.8. *The category \mathbb{I} is cartesian closed and enriched over CPOs.*

Given a typing context $\Gamma = x_1 : A_1, \dots, x_n : A_n$, we define $\llbracket \Gamma \rrbracket := ((\llbracket A_1 \rrbracket \times \llbracket A_2 \rrbracket) \cdots \llbracket A_{n-1} \rrbracket) \times \llbracket A_n \rrbracket$, with the empty context interpreted as the terminal object. The interpretation of a given term-in-context $\Gamma \vdash M : A$ as a \mathbb{I} -map $\llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$ is standard, and we omit the definition.

Theorem 2.9 (Definability). *Let $A = (A_1, \dots, A_n, o)$ be a type. For every compact, total, innocent strategy σ over $\llbracket A \rrbracket$, there is a unique η -long β -normal form, $x_1 : A_1, \dots, x_n : A_n \vdash M_\sigma : o$, whose denotation is σ .*

Proof. See [Hyland and Ong, 2000]. □

3 Traversals over long forms

3.1 Long form of a lambda term

The *long form* of a term is a kind of canonical form, which is obtained by first constructing the η -long form of the term, and then replacing the standard binary application operators by full application operators $@^{\hat{B}}$, which we called *long application*. The latter is achieved by replacing every subterm of the η -long form that has the form $(\lambda x.P) Q_1 \cdots Q_m : o$ by $@^{\hat{B}}(\lambda x.P) Q_1 \cdots Q_m : o$ where $m \geq 1$. The type of the function symbol $@^{\hat{B}}$ is \hat{B} , which is a shorthand for $((B_1, \dots, B_m, o), B_1, \dots, B_m, o)$ where $B = (B_1, \dots, B_m, o)$.

Definition 3.1 (Long form in concrete syntax). Assume $\Gamma \vdash M : (A_1, \dots, A_n, o)$. The concrete syntax of the *long form* of M , written $\langle M \rangle$, is defined by cases as follows.

1. M is an application headed by a variable:

$$\langle x N_1 \cdots N_m \rangle := \lambda z_1^{A_1} \cdots z_n^{A_n} . x \langle N_1 \rangle \cdots \langle N_m \rangle \langle z_1 \rangle \cdots \langle z_n \rangle$$

where $z_i \notin \{x\} \cup \bigcup_{j=1}^m \text{FV}\langle N_j \rangle$ for each i .

2. M is an application headed by an abstraction with $m \geq 1$:

$$\langle (\lambda x.P) Q_1 \cdots Q_m \rangle := \lambda z_1^{A_1} \cdots z_n^{A_n} . @^{\hat{\Xi}} \langle \lambda x.P \rangle \langle Q_1 \rangle \cdots \langle Q_m \rangle \langle z_1 \rangle \cdots \langle z_n \rangle$$

where $\Xi = (B_1, \dots, B_m, A_1, \dots, A_n, o)$ which is the type of $\lambda x.P$, and $z_i \notin \text{FV}\langle \lambda x.P \rangle \cup \bigcup_{j=1}^m \text{FV}\langle Q_j \rangle$ for each i .

3. M is an abstraction: $\langle \lambda x_1^{A_1} \cdots x_i^{A_i} . P \rangle := \lambda x_1^{A_1} \cdots x_i^{A_i} . \langle P \rangle$.

We shall elide the type superscript from variables x^A and long application symbols $@^{\hat{\Xi}}$, whenever it is clear from the context. We assume that bound variables in $\langle M \rangle$ are renamed afresh where necessary, so that if $\lambda \bar{x}.P$ and $\lambda \bar{y}.Q$ are distinct subterms (i.e. they have different occurrences) then $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_n\}$ are disjoint. It is easy to verify that $\Gamma \vdash \langle M \rangle : A$.

If M is β -normal, and so $\langle M \rangle$ has no occurrences of $@$, then $\langle M \rangle$ is essentially the η -long β -normal form of M . Note that in $\langle M \rangle$ we additionally η -expand every ground-type subterm P of M to $\lambda.P$ (we call λ a “dummy lambda”) provided P occurs at an *operand* position (meaning that LP is a subterm of M for some L). By a *long form*, we mean the long form of a term.

Example 3.2. Consider the term-in-context $\varphi : ((o, o), o, o), o, o, o, a : o \vdash M : (o, o)$ where $M = \varphi(\lambda x^{(o,o)}.x)((\lambda y^o.y)a)$. We have

$$\langle M \rangle = \lambda z_1 . \varphi(\lambda x^{(o,o)} z^o . x(\lambda z) (\lambda . @(\lambda y^o.y)(\lambda a)) (\lambda . z_1)) : (o, o).$$

We organise the abstract syntax tree (AST) of a long form, somewhat non-standardly, as a $\Lambda(@)$ -labelled (binding) tree where

$$\Lambda(@) := \underbrace{\{\lambda \bar{x} \mid \bar{x} = x_1 \cdots x_n \in \text{Var}^*\}}_{\text{lambdas}} \cup \underbrace{\{\text{Var} \cup \{ @^{\hat{A}} \mid A \in \text{Types}, ar(A) > 0 \}}}_{\text{non-lambdas}}$$

is a ranked alphabet such that $ar(x^A) := ar(A)$; $ar(\lambda \bar{x}) = 1$, and $ar(@^{\hat{A}}) := ar(A) + 1$. By construction, in the AST of a long form, nodes on levels 0, 2, 4, etc., are labelled by *lambdas*, and nodes on levels 1, 3, 5, etc., are labelled by *non-lambdas*.

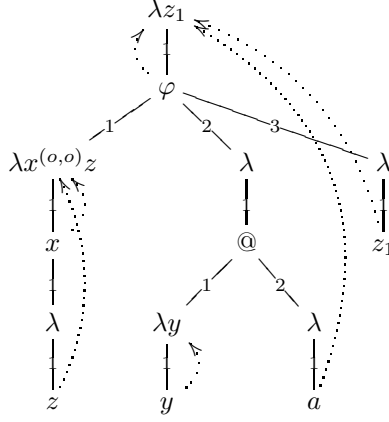
Definition 3.3 (Σ -labelled binding tree). 1. A λ -*alphabet* is a ranked alphabet Σ which is partitioned into $\Sigma_\lambda, \Sigma_{\text{Var}}$ and Σ_{aux} , such that Σ_λ consists of binders which have arity 1, Σ_{Var} consists of variables (whether bound or not), and Σ_{aux} consists of the remaining auxiliary symbols.

2. A Σ -*labelled binding tree* is a triple (T, B, ℓ) where Σ is a λ -alphabet, $\ell : T \rightarrow \Sigma$ is a Σ -labelled tree, and $B : T \rightarrow T$ is a partial function called *binder*, satisfying: for all $\beta \in T$ (**Bind**) $\ell(\beta) \in \Sigma_{\text{Var}} \iff \beta \in \text{dom}(B)$; and if $\beta \in \text{dom}(B)$ then $B(\beta) < \beta$ and $\ell(\beta) \in \Sigma_\lambda$.
By convention, if $\ell(\beta)$ is a free variable then $B(\beta) = \epsilon$.

(**Label**) The labelling function ℓ maps elements in T of even lengths (including 0) into Σ_λ , and elements in T of odd lengths into $\Sigma_{\text{Var}} \cup \Sigma_{\text{aux}}$.

Remark 3.4. Our definition of binding tree is slightly more permissive than the original [Stirling, 2009]: unlike Stirling, we do not assume that terms are closed.

Observe that $\Lambda(@)$ is a λ -alphabet: $\Lambda(@)_\lambda$ consists of the *lambdas*, $\Lambda(@)_{\text{Var}}$ consists of the variables, and $\Lambda(@)_{\text{aux}}$ consists of the long application symbols. The AST of a long form is a $\Lambda(@)$ -labelled binding tree.

Figure 4: The long form of the term M in Example 3.2

Example 3.5. Take $M = \varphi(\lambda x^{(o,o)}.x)((\lambda y^o.y)a)$ of Example 3.2. The abstract syntax tree of the long form

$$\langle M \rangle = \lambda z_1. \varphi(\lambda x^{(o,o)} z^o. x(\lambda z. z))(\lambda. @(\lambda y^o. y)(\lambda. a))(\lambda. z_1) : (o, o).$$

is displayed in Figure 4. Let (T, B, ℓ) be the $\mathbf{\Lambda}(@)$ -labelled binding tree representation of $\langle M \rangle$. The binder function B is indicated by the dotted arrows in the figure i.e. $m \xrightarrow{\text{dotted}} m'$ means $B(m') = m$. By convention (nodes that are labelled with) free variables are mapped by B to the root node: thus $B : 1 \mapsto \epsilon, 12121 \mapsto \epsilon$.

Lemma 3.6. A finite $\mathbf{\Lambda}(@)$ -labelled binding tree (T, B, ℓ) is the long form of a term if, and only if, it satisfies the following labelling axioms:

- (**Lam**) If $\ell(\beta) = y$ and $\ell(B(\beta)) = \lambda \bar{x}$ then $y \in \bar{x}$ (i.e. y is bound in the term) or $B(\beta) = \epsilon$ (y is a free variable).
- (**Leaf**) A node α is maximal in T if, and only if, $\ell(\alpha)$ is a ground-type variable.
- (**TVar**) A node labelled by $@^A$ where $A = (A_1, \dots, A_n, o)$ has $n+1$ children with lambda labels of types A, A_1, \dots, A_n respectively (see left of Figure 5).
- (**T@**) A node labelled by a variable $\varphi : (A_1, \dots, A_n, o)$ has n children with lambda labels of types A_1, \dots, A_n respectively (see right of Figure 5)

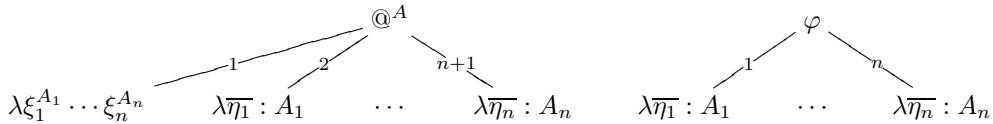


Figure 5: Labelling rules for long application symbols and variables.

Proof. The direction “ \Rightarrow ” can be proved straightforwardly by induction on the rules that define $\langle M \rangle$. For “ \Leftarrow ”, take a $\mathbf{\Lambda}(@)$ -labelled binding tree (T, B, ℓ) that satisfies the axioms. Because of

(Label), $\ell(\epsilon)$ has the form $\lambda x_1 \cdots x_m$; and because of (Leaf), every maximal element in T has an odd length. The base cases are therefore long forms of the shape $\lambda \bar{x}.y$ with $B : 1 \mapsto \epsilon$. For the inductive cases, $\ell(1)$ is either $@^{\hat{A}}$ or x^A where $A = (A_1, \dots, A_n, o)$ with $n \geq 1$. Suppose the former. By assumption, for all $\beta \in \text{dom}(B)$, $B(\beta) < \beta$. Let $\alpha \in T$. The tree T rooted at α , denoted $T_{@ \alpha}$, is the set $\{\gamma \mid \alpha \cdot \gamma \in T\}$. Define $B_{@ \alpha} : T_{@ \alpha} \rightarrow T_{@ \alpha}$ by setting $\text{dom}(B_{@ \alpha}) := \{\beta \mid \alpha \cdot \beta \in \text{dom}(B)\}$; and $B_{@ \alpha}(\beta) := \beta'$ if $B(\alpha \cdot \beta) = \alpha \cdot \beta'$, and $B_{@ \alpha}(\beta) := \epsilon$ if $B(\alpha \cdot \beta) < \alpha$. Next define $\ell_{@ \alpha} : T_{@ \alpha} \rightarrow \mathbf{\Lambda}(@)$ by $\gamma \mapsto \ell(\alpha \cdot \gamma)$. It follows that for each $i \in [n+1]$, $(T_{@ 1 \cdot i}, B_{@ 1 \cdot i}, \ell_{@ 1 \cdot i})$ is a $\mathbf{\Lambda}(@)$ -labelled binding tree that satisfies the axioms. By the induction hypothesis, suppose the binding trees are the ASTs of the long forms $\langle M \rangle, \langle N_1 \rangle, \dots, \langle N_n \rangle$ respectively. Then (T, B, ℓ) is the AST of the long form $\lambda \bar{x}.@ \langle M \rangle \langle N_1 \rangle \cdots \langle N_n \rangle$ where $\ell(\epsilon) = \lambda \bar{x}$. The latter case of $\ell(1) = x^A$ is similar. \square

3.2 Explicit arenas, succinct arenas and succinct long form

Given a term, we identify two arenas that are associated with $\langle M \rangle$, namely, explicit arena $\text{ExpAr}\langle M \rangle$, and succinct arena $\text{SucAr}\langle M \rangle$. The former arena provides the setting for traversals: we will define traversals over $\langle M \rangle$ as justified sequences over the explicit arena $\text{ExpAr}\langle M \rangle$ that satisfy certain constraints.

Fix a term-in-context $z_1 : A_1, \dots, z_i : A_i \vdash M : (A_{i+1}, \dots, A_n, o)$, and write $A = (A_1, \dots, A_n, o)$ and $\langle M \rangle = (T, B, \ell)$. Furthermore assume that $\{(\gamma_1, \Xi_1), \dots, (\gamma_r, \Xi_r)\} = \{(\gamma, \Xi) \mid \ell(\gamma) = @^{\hat{\Xi}}\}$.

Definition 3.7 (Explicit Arena of $\langle M \rangle$, $\text{ExpAr}\langle M \rangle$). The *explicit arena* of $\langle M \rangle = (T, B, \ell)$, $\text{ExpAr}\langle M \rangle$, is defined as follows. The underlying move-set $|\text{ExpAr}\langle M \rangle| := T$. The enabling relation, $\vdash_{\text{ExpAr}\langle M \rangle}$, is defined as follows:

- $\star \vdash_{\text{ExpAr}\langle M \rangle} \epsilon$; and for all $i \in [r]$, $\star \vdash_{\text{ExpAr}\langle M \rangle} \gamma_i$.
- If $\ell(\alpha) = x^{(B_1, \dots, B_m, o)}$ then for each $i \in [m]$, $\alpha \vdash_{\text{ExpAr}\langle M \rangle} \alpha \cdot i$.
- If $\ell(\alpha) = @^{\hat{C}}$ where $C = (C_1, \dots, C_l, o)$ then for each $i \in [l+1]$, $\alpha \vdash_{\text{ExpAr}\langle M \rangle} \alpha \cdot i$.
- If $\ell(\alpha) = \lambda x_1 \cdots x_m$ then: for all $\alpha' \in B^{-1}(\alpha)$
 - if $\ell(\alpha') = x_i$ for some $i \in [m]$ then $\alpha \vdash_{\text{ExpAr}\langle M \rangle} \alpha'$.
 - if $\ell(\alpha') \notin \{x_1, \dots, x_m\}$ (it follows from (Lam) that $\alpha = \epsilon$ and $\ell(\alpha') = z_k \in \text{FV}(M)$ for some $k \in [i]$) then $\epsilon \vdash_{\text{ExpAr}\langle M \rangle} \alpha'$.

Finally $\lambda_{\text{ExpAr}\langle M \rangle}(\alpha) = \text{O} \iff |\alpha|$ is even.

Thus the explicit arena $\text{ExpAr}\langle M \rangle$ has the same underlying node-set T as the long form $\langle M \rangle = (T, B, \ell)$. Every lambda-labelled node is enabled by its predecessor in T . A variable-labelled node α is enabled by its binder $B(\alpha)$; by convention, a node α labelled by a free variable is enabled by the root $\epsilon = B(\alpha)$. A node is an O-move if and only if its label is a lambda.

The succinct arena of a long form is part of a compact representation of the long form as a binding tree, called succinct long form.

Definition 3.8 (Succinct Arena of $\langle M \rangle$, $\text{SucAr}\langle M \rangle$). The *succinct arena* of $\langle M \rangle$, $\text{SucAr}\langle M \rangle$, is defined to be the arena $\text{Ar}(A) \times \prod_{i=1}^r \text{Ar}^\perp(\hat{\Xi}_i)$.

The arena $\text{SucAr}\langle A \rangle$ is a disjoint union of $\text{Ar}(A), \text{Ar}^\perp(\hat{\Xi}_1), \dots, \text{Ar}^\perp(\hat{\Xi}_r)$, *qua* labelled directed graphs. Notice that $\text{SucAr}\langle M \rangle$ depends only on the list of types (viz. A, Ξ_1, \dots, Ξ_r) that occur in $\langle M \rangle$, and not on the size of M ; furthermore, in case M is β -normal, $\text{SucAr}\langle M \rangle = \text{Ar}(A)$.

Next we define a function $\hat{\ell} : |ExpAr\langle M \rangle| \rightarrow |SucAr\langle M \rangle|$ where

$$|SucAr\langle M \rangle| := \{ \epsilon \} \times |Ar(A)| \cup \bigcup_{i=1}^r \{ \gamma_i \} \times |Ar^\perp(\hat{\Xi}_i)|$$

is a (convenient) representation of the underlying set of the arena $SucAr\langle M \rangle$. To aid the definition of $\hat{\ell}$, we use a predicate $S \subseteq T \times \mathbf{\Lambda}(@) \times |SucAr\langle M \rangle|$. The idea is that $(\alpha, s, \gamma, \beta) \in S$ means: (i) $\alpha \in T$ and $\ell(\alpha) = s$, and (ii) α is hereditarily enabled by γ in the arena $ExpAr\langle M \rangle$ where $\gamma \in \{ \epsilon, \gamma_1, \dots, \gamma_r \}$, and (iii) α is mapped by $\hat{\ell}$ to (γ, β) .

The predicate S is defined by induction over the following rules:

- $(\epsilon, \ell(\epsilon), \epsilon, \epsilon) \in S$; for all $i \in [r]$, $(\gamma_i, \ell(\gamma_i), \gamma_i, \epsilon) \in S$.
- If $(\alpha, x^{(B_1, \dots, B_r, o)}, \gamma, \beta) \in S$ then for all $i \in [m]$, $(\alpha \cdot i, \ell(\alpha \cdot i), \gamma, \beta \cdot i) \in S$.
- If $(\gamma, @^C, \gamma, \epsilon) \in S$ where $C = (C_1, \dots, C_l, o)$ then for all $i \in [l+1]$, $(\gamma \cdot i, \ell(\gamma \cdot i), \gamma, i) \in S$.
- If $(\alpha, \lambda x_1^{B_1} \dots x_m^{B_m}, \gamma, \beta) \in S$ then: for all $\alpha' \in B^{-1}(\alpha)$
 - if $\ell(\alpha') = x_i^{B_i}$ for some $i \in [m]$ then $(\alpha', \ell(\alpha'), \gamma, \beta \cdot i) \in S$
 - if $\ell(\alpha') \notin \{ x_1, \dots, x_m \}$ then (by (Lam)) $\alpha = \epsilon$ and $\ell(\alpha') = z_k^{A_k} \in FV(M)$ for some $k \in [i]$, and we have $(\alpha', \ell(\alpha'), \epsilon, k) \in S$

Then, by a straightforward induction over the length of α , we have: for all $\alpha \in T$, there exist unique γ and β such that $(\alpha, \ell(\alpha), \gamma, \beta) \in S$.

Definition 3.9 (Succinct Long Form). We organise $|SucAr\langle M \rangle|$ into a λ -alphabet as follows:

$$\begin{aligned} |SucAr\langle M \rangle|_\lambda &:= \{ (\epsilon, \alpha) \mid \alpha \in |Ar(A)|, |\alpha| \text{ even} \} \cup \\ &\quad \bigcup_{i=1}^r \{ (\gamma_i, \alpha) \mid \alpha \in |Ar^\perp(\hat{\Xi}_i)|, |\alpha| \text{ odd} \} \\ |SucAr\langle M \rangle|_{var} &:= \{ (\epsilon, \alpha) \mid \alpha \in |Ar(A)|, |\alpha| \text{ odd} \} \cup \\ &\quad \bigcup_{i=1}^r \{ (\gamma_i, \alpha) \mid \alpha \in |Ar^\perp(\hat{\Xi}_i)|, |\alpha| > 0, |\alpha| \text{ even} \} \\ |SucAr\langle M \rangle|_{aux} &:= \{ (\gamma_1, \epsilon), \dots, (\gamma_r, \epsilon) \} \end{aligned}$$

The *succinct long form* of a term M is the $|SucAr\langle M \rangle|$ -labelled binding tree, $(T, B, \hat{\ell})$, where $\hat{\ell} : T \rightarrow |SucAr\langle M \rangle|$ is given by $\hat{\ell}(\alpha) = (\gamma, \beta)$ just if $(\alpha, \ell(\alpha), \gamma, \beta) \in S$.

Lemma 3.10. *For every term M , the function $\hat{\ell} : |ExpAr\langle M \rangle| \rightarrow |SucAr\langle M \rangle|$ gives a direct arena morphism from $ExpAr\langle M \rangle$ to $SucAr\langle M \rangle$. In general $\hat{\ell}$ is neither injective nor surjective.*

Proof. It follows from the definition that $\star \vdash_{ExpAr\langle M \rangle} \alpha \iff \star \vdash_{SucAr\langle M \rangle} \hat{\ell}(\alpha)$.

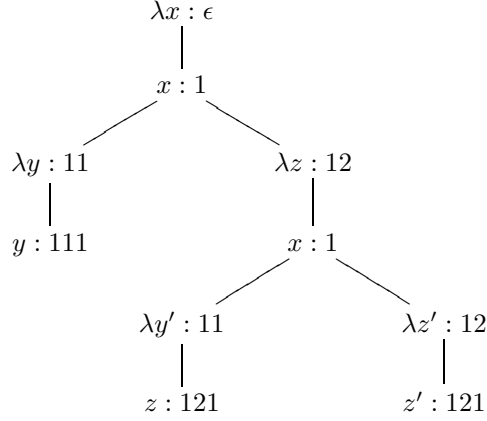
To verify $\alpha \vdash_{ExpAr\langle M \rangle} \alpha' \iff \hat{\ell}(\alpha) \vdash_{SucAr\langle M \rangle} \hat{\ell}(\alpha')$, we analyse the cases of $\ell(\alpha)$. To illustrate, consider the case of $\ell(\alpha) = \lambda x_1 \dots x_m$. Take $\alpha' \in B^{-1}(\alpha)$. If $\ell(\alpha') = x_i$ for some $i \in [m]$ then $\alpha \vdash_{ExpAr\langle M \rangle} \alpha'$. Suppose $\hat{\ell}(\alpha) = (\gamma, \beta)$, say. Then $\hat{\ell}(\alpha') = (\gamma, \beta \cdot i)$, and we have $\hat{\ell}(\alpha) \vdash_{SucAr\langle M \rangle} \hat{\ell}(\alpha')$ as desired. If for some $k \in [i]$, $\ell(\alpha') = z_k \notin \{ x_1, \dots, x_m \}$, then $\alpha = \epsilon$, and $\alpha \vdash_{ExpAr\langle M \rangle} \alpha'$. Now $\hat{\ell}(\alpha) = (\epsilon, \epsilon)$ and $\hat{\ell}(\alpha') = (\epsilon, k)$, and we have $\hat{\ell}(\alpha) \vdash_{SucAr\langle M \rangle} \hat{\ell}(\alpha')$ as desired.

Finally, to show $\lambda_{SucAr\langle M \rangle}(\hat{\ell}(\alpha)) = \lambda_{ExpAr\langle M \rangle}(\alpha)$, notice that $\lambda_{ExpAr\langle M \rangle}(\alpha) = O \iff |\alpha|$ even. Let $\hat{\ell}(\alpha) = (\gamma, \beta)$ and suppose $|\alpha|$ is even. It is straightforward to see that if $\gamma = \epsilon$ then

$|\alpha| \equiv |\beta| \pmod{2}$, and so $\lambda_{Ar(A)}(\beta) = O = \lambda_{SucAr\langle M \rangle}(\hat{\ell}(\alpha))$. And if $\gamma = \gamma_i$ for some $i \in [r]$ then $|\alpha| \equiv |\beta| + 1 \pmod{2}$, and so $\lambda_{Ar^\perp(\Xi_i)}(\beta) = O = \lambda_{SucAr\langle M \rangle}(\hat{\ell}(\alpha))$. The other case (i.e. $|\alpha|$ is odd) is symmetric. \square

Take a term-in-context $\vdash M : A$. The direct arena morphism $\hat{\ell} : ExpAr\langle M \rangle \rightarrow SucAr\langle M \rangle$ is closely related to the game semantics of M , $\llbracket \vdash M \rrbracket$, an innocent strategy over the arena $\llbracket A \rrbracket$. The function $\hat{\ell}$ maps nodes of the tree $|ExpAr\langle M \rangle|$ to moves of the arena $\llbracket A \rrbracket$. We illustrate this map in the following example.

Example 3.11. Take $\vdash M : A$ where $M = \lambda x.x (\lambda y.y) (\lambda z.x (\lambda y'.y') (\lambda z'.z'))$ and $A = (((o, o), (o, o), o), o)$. Since M is β -normal, we have $SucAr\langle M \rangle = Ar(A)$. In the following, we display the direct arena morphism $\hat{\ell} : |ExpAr\langle M \rangle| \rightarrow |SucAr\langle M \rangle|$ by annotating $\hat{\ell}(\alpha)$ next to the node α , separated by $:$.



Remark 3.12 (Every path in the long form is a P-view). Take a long form $\langle M \rangle = (T, B, \ell)$. Every path $\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_n$ in the tree T is the underlying sequence of a (unique) P-view p over the arena $ExpAr\langle M \rangle$, whose pointers are defined as follows. Suppose $i \geq 3$ is odd; then the O-move α_i (where $\ell(\alpha_i)$ is necessarily a lambda) is justified by α_{i-1} ; note that we have $\alpha_{i-1} \vdash_{ExpAr\langle M \rangle} \alpha_i$. Suppose i is even; if $\ell(\alpha_i)$ is a variable, then the P-move α_i is justified by $B(\alpha_i)$ – note that $B(\alpha_i) < \alpha_i$; otherwise, $\ell(\alpha_i) = @$, and α_i is initial in $ExpAr\langle M \rangle$. Since $\hat{\ell}$ is a direct arena morphism, $\hat{\ell}(p)$ is a P-view over the arena $SucAr\langle M \rangle$.

Example 3.13. Consider $\langle K \rangle = \lambda f.f(\lambda x.f(\lambda y.f(\lambda z.\lambda x))) : (((o, o), o), o)$. *Qua* $\Lambda(@)$ -labelled binding tree (T, B, ℓ) , the tree T consists of all prefixes of $1^7 = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$ and $B : 1 \mapsto \epsilon, 1^3 \mapsto \epsilon, 1^5 \mapsto \epsilon, 1^7 \mapsto 1^2$. Take the P-view p whose underlying sequence of moves is the maximal path of the tree T . (In the following, pointers from O-moves are not displayed.) The P-view p over the explicit arena $ExpAr\langle K \rangle$ is

$$p = \epsilon \cdot 1 \cdot 1^2 \cdot 1^3 \cdot 1^4 \cdot 1^5 \cdot 1^6 \cdot 1^7 \quad (5)$$

The “justified sequence” of the $\Lambda(@)$ -labels traced out by p , $\ell^*(p)$, is

$$\ell^*(p) = \lambda f \cdot f \cdot \lambda x \cdot f \cdot \lambda y \cdot f \cdot \lambda z \cdot x \quad (6)$$

The P-view, $\hat{\ell}^*(p)$, over the succinct arena $SucAr\langle M \rangle = Ar(((o, o), o), o)$ is

$$\hat{\ell}^*(p) = \epsilon \cdot 1 \cdot 1^2 \cdot 1 \cdot 1^2 \cdot 1 \cdot 1^2 \cdot 1^3 \quad (7)$$

The P-view, $\hat{\ell}^*(p)$, over a Λ -representation of $SucAr\langle M \rangle$ is

$$\hat{\ell}^*(p) = \lambda f \cdot f \cdot \lambda x \cdot f \cdot \lambda x \cdot f \cdot \lambda x \cdot x \quad (8)$$

Note that the set of elements that occur in p (respectively $\ell^*(p)$ and $\hat{\ell}^*(p)$) has size 8 (respectively 6 and 4).

3.3 Traversals over a long form

Henceforth we write a long form qua $\Lambda(@)$ -labelled binding tree as $\langle M \rangle = (T_M, B_M, \ell_M)$.

Definition 3.14 (Traversals). *Traversals* over a long form $\langle M \rangle$ are justified sequences over the arena $ExpAr\langle M \rangle$ defined by induction over the rules in Table 1. We write $\mathfrak{Trab}\langle M \rangle$ for the set of traversals over $\langle M \rangle$. It is convenient to refer to elements $\alpha \in |ExpAr\langle M \rangle| = T_M$ by their labels $\ell_M(\alpha)$. We shall do so in the following whenever what we mean is clear from the context.

Remark 3.15. (i) The rule (App) says that if a traversal ends in a $@$ -labelled node n , then the traversal extended with the first (left-most) child of n is a traversal.

(ii) The rule (Lam) says that if a traversal t ends in a λ -labelled node n , then t extended with the child node of n , n' , is also a traversal. To illustrate how the pointer of n' in $t \cdot n'$ is determined, first note that every path in the (abstract syntax tree of the) long form $\langle M \rangle$ is a justified sequence which is a P-view (Remark 3.12). Take, for example, traversal (1) truncated at the 15-th move—call it $t \cdot \lambda$. By (Lam), $t \cdot \lambda \cdot y$ is a traversal, where y is the child node of λ . Notice that there are two occurrences of $\lambda f y$ (5th and 9th move respectively) in t to which y

could potentially point. However, in $\lceil t \cdot \lambda \cdot y \rceil = \lambda @ \lambda f y f \lambda g \lambda y$, which is a path in the long form $\langle N P R \rangle$, y is bound by the 3rd move. Since it is the 5th move of $t \cdot \lambda \cdot y$ which is mapped by $\lceil - \rceil$ to the 3rd move of $\lceil t \cdot \lambda \cdot y \rceil$, (Lam) says that y in $t \cdot \lambda \cdot y$ points to the 5th-move.

(iii) If a traversal ends in a node labelled with a variable ξ_i , then there are two cases, corresponding to whether ξ_i is hereditarily justified by a bound (BVar) or free (FVar) variable in

the long form $\langle M \rangle$. Observe that some pointers in (BVar) are labelled; for example, $n \cdots \lambda \bar{\eta}$ means that the node $\lambda \bar{\eta}$ is the $(i+1)$ -th child of n . Intuitively, the rules (BVar) capture the switching of control between caller and callee, or between formal and actual parameters. Thus,

in rule (BVar).2, ξ_i is the i -th formal parameter, and $\lambda\bar{\eta}$ —the i -th child of n —the (root of the) i -th actual parameter. Note, however, that in rule (Bvar).1, although $\lambda\bar{\eta}$ is the $(i + 1)$ -th child, it is actually the i -th actual parameter because the 1st-child of $@$ is not the 1st actual parameter, but rather the body of the function call itself.

(iv) The rule (FVar) is the only rule that permits traversals to branch, and grow in different directions.





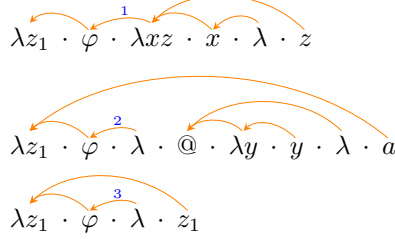
(Root)	$\epsilon \in \mathfrak{Trav}\langle M \rangle$.
(App)	If $t \cdot @ \in \mathfrak{Trav}\langle M \rangle$ then $t \cdot @ \cdot \lambda\bar{\xi} \in \mathfrak{Trav}\langle M \rangle$. 
(BVar)	If $t \cdot n \cdot \lambda\bar{\xi} \cdots \xi_i \in \mathfrak{Trav}\langle M \rangle$ where $\bar{\xi} = \xi_1 \cdots \xi_n$ and ξ_i is hereditarily justified by an $@$ then <ol style="list-style-type: none"> 1. if n is (labelled by) $@$ then $t \cdot n \cdot \lambda\bar{\xi} \cdots \xi_i \cdot \lambda\bar{\eta} \in \mathfrak{Trav}\langle M \rangle$  2. if n is (labelled by) a variable then $t \cdot n \cdot \lambda\bar{\xi} \cdots \xi_i \cdot \lambda\bar{\eta} \in \mathfrak{Trav}\langle M \rangle$. 
(FVar)	If $t \cdot \xi \in \mathfrak{Trav}\langle M \rangle$ and the variable ξ is <i>not</i> hereditarily justified by an $@$ (equivalently, ξ is hereditarily justified by the opening node ϵ) then $t \cdot \xi \cdot \lambda\bar{\eta} \in \mathfrak{Trav}\langle M \rangle$, for every $1 \leq j \leq ar(\xi)$. 
(Lam)	If $t \cdot \lambda\bar{\xi} \in \mathfrak{Trav}\langle M \rangle$ and let n be the (unique) child node of $\lambda\bar{\xi}$ in T_M , then $t \cdot \lambda\bar{\xi} \cdot n \in \mathfrak{Trav}\langle M \rangle$. By a straightforward induction, $\ulcorner t \cdot \lambda\bar{\xi} \cdot n \urcorner$ is a (justified) path in the tree T_M . If n is labelled by a variable (as opposed to $@$) then its pointer in $t \cdot \lambda\bar{\xi} \cdot n$ is determined by the justified sequence $\ulcorner t \cdot \lambda\bar{\xi} \cdot n \urcorner$ which is guaranteed to be a path in T_M . Precisely if in the P-view $\ulcorner t \cdot \lambda\bar{\xi} \cdot n \urcorner$, n points to the i -th move, then n in $t \cdot \lambda\bar{\xi} \cdot n$ points to the j -th move, where the j -th move is the necessarily unique move-occurrence that is mapped to the i -th move under the P-view transformation: $\ulcorner - \urcorner : t \cdot \lambda\bar{\xi} \cdot n \mapsto \ulcorner t \cdot \lambda\bar{\xi} \cdot n \urcorner$.

Table 1: Rules that define traversals over a long form

Example 3.16. There are three maximal traversals over the long form defined in Example 3.2

(Figure 4) as follows:



Lemma 3.17. *Let t be a traversal over $\langle M \rangle = (T_M, B_M, \ell_M)$. Then t is a well-defined justified sequence over $\text{ExpAr}\langle M \rangle$. Further*

1. *The sequence underlying $\ulcorner t \urcorner$ is a path in the tree T_M , and the P -view determined by this path (Remark 3.12) is exactly $\ulcorner t \urcorner$.*
2. *If t is maximal then the last node of t is labelled by a variable of ground type.*
3. *If M is β -normal then t is a P -view over $\text{ExpAr}\langle M \rangle$ (i.e. $t = \ulcorner t \urcorner$).*

Proof. (1) and (2) can be proved by a straightforward induction on the length of t ; for (2), we appeal to the labelling axioms of Lemma 3.6. For (3), since $\langle M \rangle$ does not have any @-labelled nodes, the traversal t is constructed using only the rules (Root), (FVar) and (Lam). Thus t is a path, which, thanks to Remark 3.12, determines a P -view. \square

Let $t \in \mathfrak{Trav}\langle M \rangle$ and $\Theta \subseteq |\text{ExpAr}\langle M \rangle|$, define $t \upharpoonright \Theta$ to be the (justified) subsequence of t consisting of nodes that are hereditarily justified by some occurrence of an element of Θ in t . If Θ is a set of initial moves, then $\mathfrak{Trav}\langle M \rangle \upharpoonright \Theta := \{t \upharpoonright \Theta \mid t \in \mathfrak{Trav}\langle M \rangle\}$ is a well-defined set of justified sequences over $\text{ExpAr}\langle M \rangle$ (see e.g. [McCusker, 2000, Lemma 2.6]). Let $\alpha \in |\text{ExpAr}\langle M \rangle|$, we write $\mathfrak{Trav}\langle M \rangle \upharpoonright (\alpha, \Theta)$ to mean $\mathfrak{Trav}\langle M \rangle \upharpoonright (\{\alpha\} \cup \Theta)$.

The rest of the section is about the following theorem and its proof. Given a term-in-context $\Gamma \vdash M : A$ where $\Gamma = x_1 : C_1, \dots, x_n : C_n$, recall that $\hat{\ell} : \text{ExpAr}\langle M \rangle \rightarrow \text{SucAr}\langle M \rangle$ is a direct arena morphism, and $\ulcorner \Gamma \vdash M : A \urcorner$ is a set of justified sequences over the arena $\llbracket C_1 \rightarrow \dots \rightarrow C_n \rightarrow A \rrbracket$ and hence over the succinct arena $\text{SucAr}\langle M \rangle$ (the former is a subarena of the latter).

Theorem 3.18 (Strong Bijection). *Let $\Gamma \vdash M : A$ be a term-in-context, with long form $\langle M \rangle = (T, B, \ell)$. The extension $\hat{\ell}^* : \mathfrak{Trav}\langle M \rangle \upharpoonright \epsilon \xrightarrow{\sim} \ulcorner \Gamma \vdash M : A \urcorner$ induced by the direct arena morphism $\hat{\ell}$ is a strong bijection.*

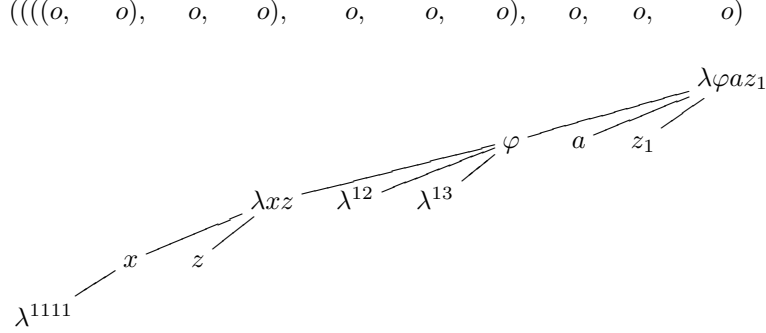
In general $\hat{\ell}$ is neither injective nor surjective; nevertheless $\hat{\ell}^*$ defines a bijection from $\mathfrak{Trav}\langle M \rangle \upharpoonright \epsilon$ to $\ulcorner \Gamma \vdash M : A \urcorner$, which is *strong*, in the sense that for each $t \in \mathfrak{Trav}\langle M \rangle \upharpoonright \epsilon$, the two justified sequences t and $\hat{\ell}^*(t)$ are isomorphic.

Example 3.19. To illustrate Theorem 3.18, consider the term-in-context of Example 3.2, $\Gamma \vdash M : (o, o)$, where $\Gamma = \{\varphi : ((o, o), o, o), o, o, o\}$, $a : o$ and

$$\langle M \rangle = \lambda z_1^o. \varphi (\lambda x^{(o,o)} z^o. x (\lambda. z)) (\lambda. @ (\lambda y^o. y) (\lambda. a)) (\lambda. z_1)$$

The interpretation, $\llbracket \Gamma \vdash M : (o, o) \rrbracket$, is an innocent strategy over the arena $\text{Ar}(B)$ where $B =$

$(((((o, o), o, o), o, o, o), o, o, o), o)$. Take a Λ -representation of $Ar(B)$ as follows:



Note that when restricted to $\mathfrak{T}\mathbf{rav}\langle M \rangle \upharpoonright \epsilon$, the image of $\hat{\ell}^*$ consists of justified sequences over $Ar(B)$, a subarena of $SucAr\langle M \rangle$. Thus maximal justified sequences in $\hat{\ell}(\mathfrak{T}\mathbf{rav}\langle M \rangle \upharpoonright \epsilon)$ are as follows (omitting the pointers)

$$\begin{aligned} & \lambda \phi a z_1 \cdot \phi \cdot \lambda x z \cdot x \cdot \lambda^{1111} \cdot z \\ & \lambda \phi a z_1 \cdot \phi \cdot \lambda^{12} \cdot a \\ & \lambda \phi a z_1 \cdot \phi \cdot \lambda^{13} \cdot z_1 \end{aligned}$$

coninciding with the maximal P-views in the strategy denotation $\llbracket \Gamma \vdash M : (o, o) \rrbracket$.

3.4 Proof of the strong bijection theorem

We first state and prove a useful lemma. Let t be a traversal over the long form

$$\langle M \rangle = \lambda. @ (\lambda \bar{\xi}. P) (\lambda \bar{\eta}_1. Q_1) \cdots (\lambda \bar{\eta}_n. Q_n) : o$$

Let $\theta_1, \dots, \theta_l$ be a list of all the occurrences of the nodes, $\lambda \bar{\xi}, \lambda \bar{\eta}_1, \dots, \lambda \bar{\eta}_n$, in t . Then, except for the first two nodes (i.e. λ and $@$) of the traversal t , every node occurrence m in t belongs to one of the *components*, $Comp_{\theta_1}(t), \dots, Comp_{\theta_l}(t)$, defined as follows:

- If m is hereditarily justified by θ_i then m belongs to $Comp_{\theta_i}(t)$.
- If m is hereditarily justified by an *internal* $@$ (as opposed to the top-level $@$), or by the root ϵ (i.e. by a free variable), let m' be the *last* node occurrence in t which precedes m and which is hereditarily justified by θ_i for some i , then m belongs to $Comp_{\theta_i}(t)$.

Henceforth, by abuse of notation, by $Comp_{\theta_i}(t)$ we mean the subsequence of t determined by the set of node occurrences $Comp_{\theta_i}(t)$.

Lemma 3.20 (Component Projection). *Using the preceding notation, let t be a traversal over the long form*

$$\langle M \rangle = \lambda. @ (\lambda \bar{\xi}. P) (\lambda \bar{\eta}_1. Q_1) \cdots (\lambda \bar{\eta}_n. Q_n) : o$$

If the last node of t is a lambda node which belongs to $Comp_{\theta}(t)$ then

- (i) $Comp_{\theta}(t) \in \mathfrak{T}\mathbf{rav}\langle \theta.M \rangle$
- (ii) $\ulcorner t \urcorner = \lambda \cdot @ \cdot \ulcorner Comp_{\theta}(t) \urcorner$

where $\theta.M := \lambda \bar{\xi}. P$ (respectively $\lambda \bar{\eta}_j. Q_j$) in case θ is an occurrence of $\lambda \bar{\xi}$ (respectively $\lambda \bar{\eta}_j$).

Proof. (i) Let t be a traversal satisfying the premises of the lemma. It follows from the rules of Definition 3.14 that t is a prefix of a justified sequence of the following shape:

$$t = \lambda \cdot @ \cdots \theta \cdot B_1 \cdots B_l \quad (9)$$

where θ is justified by $@$, and each B_i is a block of nodes of one of two types:

I. Two-node block $n_i \cdot l_i$ where the non-lambda node n_i and the lambda node l_i are hereditarily justified by an $@$ that belongs to $\text{Comp}_\theta(t)$.

II. $n_i \ m_1 \cdots m_r \ l_i$ where the lambda node l_i is justified by n_i , which belongs to $\text{Comp}_\theta(t)$ and is hereditarily justified by θ or by ϵ .

It follows that for each i , both n_i and l_i are occurrences of nodes from the long form $\langle \theta.M \rangle$ *qua* subtree of $\langle M \rangle$. Now define B'_i to be $n_i \cdot l_i$ for each i . Observe that $\text{Comp}_\theta(t) = \theta \cdot B'_1 \cdots B'_l$. It then follows that $\text{Comp}_\theta(t) \in \mathfrak{T}\mathfrak{rav}\langle \Gamma \vdash \theta.M \rangle$.

(ii) Immediate consequence of (9) and $\text{Comp}_\theta(t) = \theta \cdot B'_1 \cdots B'_l$. \square

Lemma 3.21. *Let $\Gamma \vdash M : A$ be in β -normal form with long form $\langle M \rangle = (T, B, \ell)$. The extension $\hat{\ell}^* : \mathfrak{T}\mathfrak{rav}\langle M \rangle \vdash \epsilon \xrightarrow{\sim} \ulcorner \llbracket \Gamma \vdash M : A \rrbracket \urcorner$ induced by $\hat{\ell}$ is a strong bijection.*

Proof. Take the term-in-context $\Gamma \vdash M : o$ where $\Gamma = x_1 : A_1, \dots, x_n : A_n$, $A_i = (B_1, \dots, B_m, o)$, and $M = x_i (\lambda \overline{y}_1. P_1) \cdots (\lambda \overline{y}_m. P_m)$. We prove by induction on the size of M . Observe that, since M is β -normal, $\mathfrak{T}\mathfrak{rav}\langle M \rangle \vdash \epsilon = \mathfrak{T}\mathfrak{rav}\langle M \rangle$. First we show that the map $\hat{\ell}^*$ is injective. Let $\lambda \cdot x_i \cdot t, \lambda \cdot x_i \cdot t' \in \mathfrak{T}\mathfrak{rav}\langle M \rangle$ such that $\hat{\ell}^*(\lambda \cdot x_i \cdot t) = \hat{\ell}^*(\lambda \cdot x_i \cdot t')$, which implies that $\hat{\ell}^*(t) = \hat{\ell}^*(t')$, and $t, t' \in \mathfrak{T}\mathfrak{rav}\langle \lambda \overline{y}_j. P_j \rangle$ for some $j \in [m]$. Since $\hat{\ell}^* : \mathfrak{T}\mathfrak{rav}\langle \lambda \overline{y}_j. P_j \rangle \xrightarrow{\sim} \ulcorner \llbracket \Gamma \vdash \lambda \overline{y}_j. P_j : B_j \rrbracket \urcorner$ is injective, we have $t = t'$ and hence $\lambda \cdot x_i \cdot t = \lambda \cdot x_i \cdot t'$ as desired.

For surjectivity of $\hat{\ell}^*$, take a P-view $p \in \ulcorner \llbracket \Gamma \vdash M : o \rrbracket \urcorner$. Notice that p is a justified sequence over $\text{Ar}(A_1, \dots, A_n, o)$. Since the head variable of M is x_i , we have $p = \epsilon \cdot i \cdot (p' \uparrow i)$ and $p' \in \ulcorner \llbracket \Gamma \vdash \lambda \overline{y}_j. P_j : B_j \rrbracket \urcorner$ for some $j \in [m]$. (Given a sequence of nodes, $p = \alpha_1, \dots, \alpha_n$, we write $p \uparrow i$ for the sequence $i \cdot \alpha_1, \dots, i \cdot \alpha_n$.) By the induction hypothesis, there exists $t \in \mathfrak{T}\mathfrak{rav}\langle \lambda \overline{y}_j. P_j \rangle$ such that $\hat{\ell}^*(t) = p'$. Thus we have $\lambda \cdot x_i \cdot t \in \mathfrak{T}\mathfrak{rav}\langle M \rangle$, and $\hat{\ell}^*(\lambda \cdot x_i \cdot t) = p$ as desired. \square

Corollary 3.22 (Variable). *Let $\langle \varphi \rangle = (T, B, \ell)$ where φ is a variable. The extension $\hat{\ell}^* : \mathfrak{T}\mathfrak{rav}\langle \varphi \rangle \vdash \epsilon \xrightarrow{\sim} \ulcorner \llbracket \Gamma \vdash \varphi : A \rrbracket \urcorner$ induced by $\hat{\ell}$ is a strong bijection.*

Example 3.23. Consider the long form

$$\langle \lambda \chi. \chi \rangle = \lambda \chi. \Phi \varphi. \chi (\lambda \psi. \Phi (\lambda y. \psi (\lambda y))) (\lambda x. \varphi (\lambda x)).$$

Since there is no occurrence of $@$ in $\langle \lambda \chi. \chi \rangle$, traversals over it coincide with paths from the root. For example, the traversal $\lambda \chi. \Phi \varphi \cdot \chi \cdot \lambda \psi \cdot \Phi \cdot \lambda y \cdot \psi \cdot \lambda \cdot y$ (pointers are omitted) represents a P-view in the copycat strategy $\llbracket \vdash \lambda \chi. \chi : A \rightarrow A \rrbracket$. This illustrates the strong bijection of Lemma 3.22.

We recall the notion of interaction sequences and the associated notation from [Hyland and Ong, 2000]. Let $\sigma : A \rightarrow B$ and $\tau : B \rightarrow C$ be innocent strategies, and let us write $\ulcorner \sigma \urcorner$ for the collection of P-views in σ . Given a justified sequence t over the triple (A, B, C) of arenas, let

X range over the *components* (B, C) and $(A, B)_b$ where b ranges over the occurrences of initial moves of B in t ; set

$$\rho_X := \begin{cases} \sigma & \text{if } X = (A, B)_b \\ \tau & \text{if } X = (B, C) \end{cases}$$

Similarly we define $\ulcorner \rho_X \urcorner$ to mean $\ulcorner \sigma \urcorner$ or $\ulcorner \tau \urcorner$ depending on what X is. The set of *interaction sequences between σ and τ* , $\mathbf{IntSeq}(\sigma, \tau)$, consists of justified sequences t over (A, B, C) , which are defined by induction over the rules (IS1), (IS2) and (IS3):

(IS1) $c \in \mathbf{IntSeq}(\sigma, \tau)$ where c ranges over the initial moves of C .

(IS2) If $t \cdot m \in \mathbf{IntSeq}(\sigma, \tau)$, and m is a *generalised O-move* of the component X (i.e. either an O-move of $A \Rightarrow C$ or a move of B), and $\ulcorner t \cdot m \urcorner \upharpoonright X \urcorner \cdot m' \in \rho_X$, then $t \cdot m \cdot m' \in \mathbf{IntSeq}(\sigma, \tau)$.

(IS3) If $t \cdot m \in \mathbf{IntSeq}(\sigma, \tau)$, and m is a P-move of $A \Rightarrow C$, and $(t \cdot m \upharpoonright (A, C)) \cdot m'$ is a play of $A \Rightarrow C$, then $t \cdot m \cdot m' \in \mathbf{IntSeq}(\sigma, \tau)$.

Definition 3.24. The set of *P-visible interaction sequences between σ and τ* , $\mathbf{IntSeq}^{\text{PV}}(\sigma, \tau)$, consists of justified sequences over (A, B, C) , defined by induction over the rules (IS1), (IS2), and (IS4) as follows:

(IS4) If $t \cdot m \in \mathbf{IntSeq}^{\text{PV}}(\sigma, \tau)$, and m is a P-move of $A \Rightarrow C$, and m' is an O-move justified by m , then $t \cdot m \cdot m' \in \mathbf{IntSeq}^{\text{PV}}(\sigma, \tau)$.

It is straightforward to see that $\mathbf{IntSeq}^{\text{PV}}(\sigma, \tau) \upharpoonright (A, C) = \ulcorner \sigma; \tau \urcorner$. Note that the definition would still make sense if σ and τ in (IS1), (IS2) and (IS4) are replaced by $\ulcorner \sigma \urcorner$ and $\ulcorner \tau \urcorner$ respectively, and ρ_X replaced by $\ulcorner \rho_X \urcorner$. In other words $\mathbf{IntSeq}^{\text{PV}}(\ulcorner \sigma \urcorner, \ulcorner \tau \urcorner)$ is well-defined, and coincides with $\mathbf{IntSeq}^{\text{PV}}(\sigma, \tau)$. Thus we have $\mathbf{IntSeq}^{\text{PV}}(\ulcorner \sigma \urcorner, \ulcorner \tau \urcorner) \upharpoonright (A, C) = \ulcorner \sigma; \tau \urcorner$.

Lemma 3.25. (i) Let $\Gamma \vdash M : A$ be a term-in-context with long form $\langle M \rangle = (T, B, \ell)$. There is a $\hat{\ell}$ -induced strong bijection $\hat{\ell}^* : \mathfrak{Trav}\langle M \rangle \upharpoonright \epsilon \xrightarrow{\sim} \ulcorner \llbracket \Gamma \vdash M : A \rrbracket \urcorner$.

(ii) Suppose $\Gamma \vdash \langle M \rangle = \lambda.\text{@}(\lambda\bar{\xi}.P)(\lambda\bar{\eta}_1.Q_1) \cdots (\lambda\bar{\eta}_n.Q_n) : o$, and $\langle M \rangle = (T, B, \ell)$. There is a $\hat{\ell}$ -induced bijection.

$$\hat{\ell}^* : \mathfrak{Trav}\langle M \rangle \upharpoonright (\epsilon, \Theta) \xrightarrow{\sim} \mathbf{IntSeq}^{\text{PV}}(\langle p, q_1, \dots, q_n \rangle, \ulcorner ev \urcorner) \quad (10)$$

where $\Theta := \{ \alpha \in T \mid \exists i. \alpha = 1 \cdot i \}$, $p = \ulcorner \llbracket \Gamma \vdash \lambda\bar{\xi}.P : (\prod_{i=1}^n B_i) \Rightarrow o \rrbracket \urcorner$, $q_i = \ulcorner \llbracket \Gamma \vdash \lambda\bar{\eta}_i.Q_i : B_i \rrbracket \urcorner$ for each i , and $((\prod_{i=1}^n B_i) \Rightarrow o) \times \prod_{i=1}^n B_i \xrightarrow{ev} o$ is the obvious copycat strategy.

Remark 3.26. In (ii), since $\ell(1) = \text{@}$, each $t \in \mathfrak{Trav}\langle M \rangle \upharpoonright (\epsilon, \Theta)$ is a subsequence of a traversal over $\langle M \rangle$ consisting of nodes that are hereditarily justified by ϵ , or by an occurrence of one of $\lambda\bar{\xi}, \lambda\bar{\eta}_1, \dots, \lambda\bar{\eta}_n$ (each being a child of node 1 in the tree $\langle M \rangle$). The bijection $\hat{\ell}$ in (ii) would be strong if for each $t \in \mathfrak{Trav}\langle M \rangle \upharpoonright (\epsilon, \Theta)$, pointers were added from every occurrence of $\lambda\bar{\xi}, \lambda\bar{\eta}_1, \dots, \lambda\bar{\eta}_n$ to the opening node ϵ .

Proof. We shall prove (i) and (ii) by mutual induction.

(i) The term M has one of the following shapes:

- (a) abstraction $\lambda\bar{\xi}.P$
- (b) variable φ
- (c) application $N L_1 \cdots L_n$ where $n \geq 1$ and N has shape (a) or (b).

First we reduce case (a) to case (b) or case (c). Let $A = (A_1, \dots, A_n, o)$ and the η -long normal form of $\lambda\xi.P$ be $\lambda x_1 \dots x_n.R$ where R is a term of either case (b) or (c). Plainly $\mathfrak{T}\mathbf{rav}\langle\lambda\xi.P\rangle = \mathfrak{T}\mathbf{rav}\langle\lambda\bar{x}.R\rangle$. It remains to observe that, on the one hand, there is a strong bijection between $\mathfrak{T}\mathbf{rav}\langle\lambda\bar{x}.R\rangle$ and $\mathfrak{T}\mathbf{rav}\langle R\rangle$ (note that pointers from node occurrences labelled with free variables are to the opening node); and on the other, there is a strong bijection between $\ulcorner \llbracket \Gamma \vdash \lambda\bar{x}.R : A \rrbracket \urcorner$ and $\ulcorner \llbracket \Gamma, \bar{x} : \bar{A} \vdash R : o \rrbracket \urcorner$.

Case (b) is just Corollary 3.22.

As for case (c), suppose N is an abstraction. W.l.o.g. assume

$$\langle M \rangle = \lambda. @ (\lambda \bar{\xi}. P) (\lambda \bar{\eta}_1. Q_1) \dots (\lambda \bar{\eta}_n. Q_n) : o.$$

Then, by (ii) and using the notation therein, we have a bijection

$$\hat{\ell} : \mathfrak{T}\mathbf{rav}\langle M \rangle \upharpoonright (\epsilon, \Theta) \xrightarrow{\sim} \mathbf{IntSeq}^{\text{PV}}(\langle p, q_1, \dots, q_n \rangle, \ulcorner ev \urcorner) \quad (11)$$

Observe that applying $(-) \upharpoonright \epsilon$ on the LHS of (11) corresponds to applying $(-) \upharpoonright (\llbracket \Gamma \rrbracket, o)$ on the RHS. Thus, in view of Remark 3.26, $\mathfrak{T}\mathbf{rav}\langle M \rangle \upharpoonright (\epsilon, \Theta) \upharpoonright \epsilon = \mathfrak{T}\mathbf{rav}\langle M \rangle \upharpoonright \epsilon$ is in strong bijection with $\mathbf{IntSeq}^{\text{PV}}(\langle p, q_1, \dots, q_n \rangle, \ulcorner ev \urcorner) \upharpoonright (\llbracket \Gamma \rrbracket, o) = \ulcorner \llbracket \Gamma \vdash M : o \rrbracket \urcorner$, as desired.

Finally suppose N is a variable φ . W.l.o.g. assume $M = \varphi(\lambda \bar{\eta}_1. Q_1) \dots (\lambda \bar{\eta}_m. Q_m) : o$. Then it follows from the respective definitions that

$$\begin{aligned} \mathfrak{T}\mathbf{rav}\langle M \rangle &= \bigcup_i \{ \lambda \cdot \varphi \cdot t \mid t \in \mathfrak{T}\mathbf{rav}\langle \Gamma \vdash \lambda \bar{\eta}_i. Q_i \rangle \} \cup \{ \epsilon, \lambda \} \\ \ulcorner \llbracket \Gamma \vdash M \rrbracket \urcorner &= \bigcup_i \{ \hat{\ell}(\lambda) \cdot \hat{\ell}(\varphi) \cdot t \mid t \in \ulcorner \llbracket \Gamma \vdash \lambda \bar{\eta}_i. Q_i \rrbracket \urcorner \} \cup \{ \epsilon, \hat{\ell}(\lambda) \} \end{aligned}$$

It follows from the induction hypothesis that there is a strong bijection

$$\hat{\ell} : \mathfrak{T}\mathbf{rav}\langle M \rangle \upharpoonright \epsilon \xrightarrow{\sim} \ulcorner \llbracket \Gamma \vdash M : o \rrbracket \urcorner.$$

(ii) We first show that for every $t \in \mathfrak{T}\mathbf{rav}\langle M \rangle$, we have $\hat{\ell}(t \upharpoonright (\epsilon, \Theta)) \in \mathbf{IntSeq}^{\text{PV}}(\langle p, q_1, \dots, q_n \rangle, \ulcorner ev \urcorner)$. The proof is by induction on the length of t , with case distinction on the last node of t , using the notation of Lemma 3.20.

Case 1. The last node of t is in the component $\text{Comp}_\theta(t)$ where θ is the unique occurrence of $\lambda\xi$ in t . Let m be the last lambda node in t that is hereditarily justified by ϵ or θ . There are two subcases.

Case 1.1. The traversal $t = \dots m \cdot d_1 d_2 \dots d_l$ where $l \geq 0$ and each d_i is a Type-I node (i.e. not hereditarily justified by ϵ or θ), as defined in the proof of Lemma 3.20. Let m' be a Type-II node (i.e. hereditarily justified by ϵ or θ) such that $t \cdot d_{l+1} \dots d_{l'} \cdot m' \in \mathfrak{T}\mathbf{rav}\langle M \rangle$ where $l' \geq l$ and each d_j is Type I. Writing $t' = t \cdot d_{n+1} \dots d_{n'}$, we claim that $\hat{\ell}((t' \cdot m') \upharpoonright (\epsilon, \Theta)) = \hat{\ell}(t' \upharpoonright (\epsilon, \Theta)) \cdot \hat{\ell}(m') \in \mathbf{IntSeq}^{\text{PV}}(\langle p, \bar{q} \rangle, \ulcorner ev \urcorner)$ by rule (IS2)- σ . By assumption $\hat{\ell}(m')$, which is a P-move, belongs to the component

$$(\llbracket \Gamma \rrbracket, (\prod_{i=1}^n B_i \Rightarrow o) \times \prod_{i=1}^n B_i)_o = (\llbracket \Gamma \rrbracket, \prod_{i=1}^n B_i \Rightarrow o)_o$$

in the sense of the strategy composition $\llbracket \Gamma \rrbracket \xrightarrow{\langle p, \bar{q} \rangle} (\prod_{i=1}^n B_i \Rightarrow o) \times \prod_{i=1}^n B_i \xrightarrow{ev} o$. Observe that

- (A) $\hat{\ell}(t' \upharpoonright (\epsilon, \Theta)) \upharpoonright (\llbracket \Gamma \rrbracket, \prod_{i=1}^n B_i \Rightarrow o)_o = \hat{\ell}(t' \upharpoonright (\epsilon, \Theta) \upharpoonright \theta),$
 (B) $t' \upharpoonright (\epsilon, \Theta) \upharpoonright \theta = t' \upharpoonright \theta = \text{Comp}_\theta(t') \upharpoonright \theta$, because all node occurrences in t' hereditarily justified by θ are in $\text{Comp}_\theta(t')$.

Thus we have

$$\begin{aligned}
 & \ulcorner \hat{\ell}(t' \upharpoonright (\epsilon, \Theta)) \upharpoonright (\llbracket \Gamma \rrbracket, \prod_{i=1}^n B_i \Rightarrow o)^\top \cdot \hat{\ell}(m') \\
 = & \ulcorner \hat{\ell}(t' \upharpoonright (\epsilon, \Theta) \upharpoonright \theta)^\top \cdot \hat{\ell}(m') & \text{(A)} \\
 = & \ulcorner \hat{\ell}(\text{Comp}_\theta(t') \upharpoonright \theta)^\top \cdot \hat{\ell}(m') & \text{(B)} \\
 = & \ulcorner \hat{\ell}(\text{Comp}_\theta(t' \cdot m') \upharpoonright \theta)^\top & m' \text{ is a non-lambda node} \\
 \in & \ulcorner \hat{\ell}(\text{Comp}_\theta(t' \cdot m') \upharpoonright \theta)^\top & \text{Lemma 3.20(i) \& I.H.(i)} \\
 & \ulcorner \llbracket \Gamma \vdash \lambda \bar{\xi}. P \rrbracket^\top
 \end{aligned}$$

as desired.

Case 1.2. The traversal $t = \dots m \cdot d_1 \dots d_l \cdot m'$ where $l \geq 0$ and each d_i is of Type I (i.e. not hereditarily justified by ϵ or θ), and m' is a non-lambda node.

There are two subcases.

Case 1.2.1. The non-lambda node m' is hereditarily justified by ϵ .

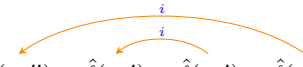
By rule (FVar) of the definition of traversal, for each lambda node m'' that is justified by m' , we have $t \cdot m'' \in \mathfrak{T}\mathfrak{rav}\langle M \rangle$. We claim that $\hat{\ell}(t \cdot m'' \upharpoonright (\epsilon, \Theta)) \in \mathbf{IntSeq}^{\text{PV}}(\langle p, \bar{q} \rangle, \ulcorner ev \urcorner)$.

By the induction hypothesis, $t \upharpoonright (\epsilon, \Theta) = (t_{\leq d_l} \upharpoonright (\epsilon, \Theta)) \cdot m'$ is mapped by $\hat{\ell}$ into $\mathbf{IntSeq}^{\text{PV}}(\langle p, \bar{q} \rangle, \ulcorner ev \urcorner)$.

Since $\hat{\ell}(m')$ is an P-move of the arena $\llbracket \Gamma \rrbracket \Rightarrow o$, it follows from rule (IS4) of the definition of P-visible interaction sequence that $\hat{\ell}(t_{\leq d_l} \upharpoonright (\epsilon, \Theta) \cdot m') \cdot \hat{\ell}(m'') = \hat{\ell}(t \cdot m'' \upharpoonright (\epsilon, \Theta)) \in \mathbf{IntSeq}^{\text{PV}}(\langle p, \bar{q} \rangle, \ulcorner ev \urcorner)$ as required.

Case 1.2.2. The non-lambda node m' is hereditarily justified by θ .

Suppose $t = \dots \underline{m}'' \cdot \underline{m}' \dots m \cdot \bar{d} \cdot m'$ such that m' is explicitly i -justified by the lambda node \underline{m}' . By rule (BVar) of the definition of traversal, \underline{m}'' is a non-lambda node in $\text{Comp}_{\theta'}(t)$ where θ' is an occurrence of some $\lambda \bar{\eta}_j$ in t , and $t \cdot m'' \in \mathfrak{T}\mathfrak{rav}\langle M \rangle$ where m'' is i -justified by \underline{m}'' . We claim that $t \cdot m'' \upharpoonright (\epsilon, \Theta) = (t \upharpoonright (\epsilon, \Theta)) \cdot m''$ is mapped by $\hat{\ell}$ into $\mathbf{IntSeq}^{\text{PV}}(\langle p, \bar{q} \rangle, \ulcorner ev \urcorner)$. Writing $\tilde{t} = t \upharpoonright (\epsilon, \Theta)$, by the induction hypothesis, we have $\hat{\ell}(\tilde{t}) \in \mathbf{IntSeq}^{\text{PV}}(\langle p, \bar{q} \rangle, \ulcorner ev \urcorner)$. By rule (IS2)- τ of the definition of interaction sequence, it suffices to show $\ulcorner \hat{\ell}(\tilde{t}) \upharpoonright ((\prod_{i=1}^n B_i \Rightarrow o) \times \prod_{i=1}^n B_i, o)^\top \cdot \hat{\ell}(m'') \in \ulcorner ev \urcorner$. Since projecting to $((\prod_{i=1}^n B_i \Rightarrow o) \times \prod_{i=1}^n B_i, o)$ reverses the P/O polarity of moves, we have

$$\ulcorner \hat{\ell}(\tilde{t}) \upharpoonright ((\prod_{i=1}^n B_i \Rightarrow o) \times \prod_{i=1}^n B_i, o)^\top \cdot \hat{\ell}(m'') = \dots \hat{\ell}(\underline{m}'') \cdot \hat{\ell}(\underline{m}') \cdot \hat{\ell}(m') \cdot \hat{\ell}(m'')$$


which is in $\ulcorner ev \urcorner$ as required.

It remains to show that for every $u \in \mathbf{IntSeq}^{\text{PV}}(\langle p, \bar{q} \rangle, \ulcorner ev \urcorner)$, there exists a unique $t_u \in \mathfrak{T}\mathfrak{rav}\langle M \rangle$ such that $\hat{\ell}(t_u \upharpoonright (\epsilon, \Theta)) = u$.

We argue by induction on the length of u . The base case of (IS1) is trivial. For the inductive case, suppose $u \cdot n \in \mathbf{IntSeq}^{\text{PV}}(\langle p, \bar{q} \rangle, \ulcorner ev \urcorner)$. With reference to Definition 3.24, there are three cases, namely, (IS2)- σ , (IS2)- τ and (IS4), which correspond to the preceding cases of

1.1, 1.2.2 and 1.2.1 respectively. Here we consider the case of (IS2)- σ for illustration; the other cases are similar and simpler. I.e. by assumption, we have $\ulcorner u \upharpoonright (\llbracket \Gamma \rrbracket, \prod_{i=1}^n B_i \Rightarrow o) \urcorner \cdot n \in \ulcorner \llbracket \Gamma \vdash \lambda \bar{\xi}.P \rrbracket \urcorner$ where n is a P-move of $\llbracket \Gamma \vdash \lambda \bar{\xi}.P \rrbracket$. By the induction hypothesis, there exists a unique maximal $t_u \in \mathfrak{Trav}\langle M \rangle$ such that $\hat{\ell}(t_u \upharpoonright (\epsilon, \Theta)) = u$. Since $\hat{\ell}(Comp_\theta(t_u) \upharpoonright \theta) = \hat{\ell}(t_u \upharpoonright (\epsilon, \Theta)) \upharpoonright (\llbracket \Gamma \rrbracket, \prod_{i=1}^n B_i \Rightarrow o)_o$ (which is (B) of Case 1.1), we have $\ulcorner \hat{\ell}(Comp_\theta(t_u) \upharpoonright \theta) \urcorner \cdot \hat{\ell}(m') \in \ulcorner \llbracket \Gamma \vdash \lambda \bar{\xi}.P \rrbracket \urcorner$ where $n = \hat{\ell}(m')$ for some m' . Thanks to the strong bijection of (i), we have $\ulcorner Comp_\theta(t_u) \upharpoonright \theta \cdot m' \urcorner \in \mathfrak{Trav}\langle \lambda \bar{\xi}.P \rangle \upharpoonright \epsilon$ for a unique m' . Then, because $Comp_\theta(t_u) \in \mathfrak{Trav}\langle \lambda \bar{\xi}.P \rangle$ by Lemma 3.20(i) and because the last node of t_u is a lambda node, we have $Comp_\theta(t_u) \cdot m' \in \mathfrak{Trav}\langle \lambda \bar{\xi}.P \rangle$. By rule (Lam) of the definition of traversals, we have the $\ulcorner Comp_\theta(t_u) \urcorner \cdot m'$ is a path in the tree $\langle \lambda \bar{\xi}.P \rangle$. It follows that $\lambda \cdot @ \cdot \ulcorner Comp_\theta(t_u) \urcorner \cdot m'$ is a path in the tree $\langle M \rangle$. But, by Lemma 3.20(ii), $\ulcorner t_u \urcorner \cdot m' = \lambda \cdot @ \cdot \ulcorner Comp_\theta(t_u) \urcorner \cdot m'$. Hence, by rule (Lam), $t_u \cdot m' \in \mathfrak{Trav}\langle M \rangle$ with $\hat{\ell}(t_u \cdot m' \upharpoonright (\epsilon, \Theta)) = u \cdot n$ as desired.

Case 2. The last node of t belongs to $Comp_{\theta'}(t)$ where θ' is an occurrence of $\lambda \bar{\eta}_i$ in t . This case is symmetrical to Case 1. \square

4 Application

4.1 Interpreting higher-order recursion schemes

We assume the standard notion of higher-order recursion scheme [Knapik et al., 2002; Ong, 2006]. Fix a (possibly infinite) higher-order recursion scheme $\mathcal{G} = (\Sigma, \mathcal{N}, \mathcal{R}, F_1)$ over a ranked alphabet $\Sigma = \{a_1 : r_1, \dots, a_l : r_l\}$ where r_i is the arity of the terminal a_i ; with non-terminals $\mathcal{N} = \{F_i : A_i \mid i \in \mathcal{I}\}$ and rules $F_i \rightarrow \lambda \bar{x}_i. M_i$ for each $i \in \mathcal{I}$, and $F_1 : o$ is the start symbol. Note that we do not assume \mathcal{I} to be finite. Henceforth we assume $\mathcal{I} = \omega$ for convenience, and regard Σ as a set of free variables.

We first give the semantics of \mathcal{G} . Writing $\llbracket \Sigma \rrbracket := \prod_{i=1}^l \llbracket o^{r_i} \rightarrow o \rrbracket$ and $\llbracket \mathcal{N} \rrbracket := \prod_{i \in \omega} \llbracket A_i \rrbracket$, the semantics of \mathcal{G} , $\llbracket \Sigma \vdash F_1 : o \rrbracket : \llbracket \Sigma \rrbracket \longrightarrow \llbracket o \rrbracket$, is the composite

$$\llbracket \Sigma \rrbracket \xrightarrow{\Lambda(\mathbf{g})} (\llbracket \mathcal{N} \rrbracket \Rightarrow \llbracket \mathcal{N} \rrbracket) \xrightarrow{\mathcal{Y}_{\llbracket \mathcal{N} \rrbracket}} \llbracket \mathcal{N} \rrbracket \xrightarrow{\pi_1} \llbracket o \rrbracket$$

in the category \mathbb{I} of arenas and innocent strategies, where

- $\mathbf{g} : \llbracket \Sigma \rrbracket \times \llbracket \mathcal{N} \rrbracket \longrightarrow \llbracket \mathcal{N} \rrbracket$ is $\llbracket \Sigma, \mathcal{N} \vdash (\lambda \bar{x}_1. M_1, \dots, \lambda \bar{x}_m. M_m, \dots) : \prod_{i \in \omega} A_i \rrbracket$, and $\Lambda(-)$ is currying
- $\mathcal{Y}_A : (A \Rightarrow A) \rightarrow A$ is the fixpoint strategy (see [Hyland and Ong, 2000, §7.2]) over an arena A
- π_1 is the projection map.

Remark 4.1. Since $\ulcorner \llbracket \Sigma \vdash F_1 : o \rrbracket \urcorner$ coincide with the branch language³ of the Σ -labelled tree generated by \mathcal{G} , we can identify $\ulcorner \llbracket \Sigma \vdash F_1 : o \rrbracket \urcorner$ with $\llbracket \mathcal{G} \rrbracket$, the tree generated by \mathcal{G} .

³Let m be the maximum arity of the Σ -symbols, and write $[m] := \{1, \dots, m\}$. The *branch language* of $t : \text{dom}(t) \longrightarrow \Sigma$ consists of (i) infinite words $(f_1, d_1)(f_2, d_2) \dots$ such that there exists $d_1 d_2 \dots \in [m]^\omega$ such that $t(d_1 \dots d_i) = f_{i+1}$ for every $i \in \omega$ and (ii) finite words $(f_1, d_1) \dots (f_n, d_n) f_{n+1}$ such that there exists $d_1 \dots d_n \in [m]^*$ such that $t(d_1 \dots d_i) = f_{i+1}$ for $0 \leq i \leq n$, and the arity of f_{n+1} is 0.

4.2 The traversal-path correspondence theorem

Fix a higher-order infinite recursion scheme $\mathcal{G} = (\Sigma, \mathcal{N}, \mathcal{R}, F_1)$, using the same notation as before. Define an ω -indexed family of λ -terms, $\mathbf{G}^{(n)} : \prod_{i \in \omega} A_i$ with n ranging over ω , as follows:

$$\begin{aligned} \mathbf{G}^{(0)} &:= (\lambda \overline{x_1}. \perp^o, \dots, \lambda \overline{x_m}. \perp^o, \dots) \\ \mathbf{G}^{(n+1)} &:= (\lambda \overline{x_1}. M_1[\mathbf{G}^{(n)} / \overline{F}], \dots, \lambda \overline{x_m}. M_m[\mathbf{G}^{(n)} / \overline{F}], \dots) \end{aligned}$$

where \perp^A is a constant symbol of type A , and $(-)[\mathbf{G}^{(n)} / \overline{F}]$ means the simultaneous substitution $(-)[\pi_1 \mathbf{G}^{(n)} / F_1, \dots, \pi_m \mathbf{G}^{(n)} / F_m, \dots]$, and $\pi_i(s_1, s_2, \dots)$ is a short hand for s_i . Write $G^{(n)} := \pi_1 \mathbf{G}^{(n)}$ for each $n \in \omega$. Note that each $G^{(n)}$ is a (recursion-free) λ -term of type o .

Lemma 4.2. $\llbracket \Sigma \vdash F_1 : o \rrbracket = \bigsqcup_{n \in \omega} \llbracket \Sigma \vdash G^{(n)} : o \rrbracket$.

Proof. Because \mathbb{A} is a CCC that is enriched over the CPOs, for each type A , there is a fixpoint strategy $\mathcal{Y}_A : (A \Rightarrow A) \Rightarrow A$ which is the least (with respect to the enriching order, namely, set inclusion) fixpoint of the map $(A \Rightarrow A) \Rightarrow A \longrightarrow (A \Rightarrow A) \Rightarrow A$ that is the denotation of the λ -term

$$\lambda F : (A \Rightarrow A) \Rightarrow A. \lambda f : A \Rightarrow A. f (F f)$$

Define the following family of λ -terms, $Y_A^{(i)} : (A \Rightarrow A) \Rightarrow A$ with $i \in \omega$

$$\begin{aligned} Y_A^{(0)} &:= \lambda f : A \Rightarrow A. \perp^A \\ Y_A^{(n+1)} &:= \lambda f : A \Rightarrow A. f (Y_A^{(n)} f) \end{aligned}$$

Note that $\mathbf{G}^{(n)} = Y_{\mathcal{N}}^{(n)} (\lambda \overline{x_1}. M_1, \dots, \lambda \overline{x_m}. M_m, \dots) : \mathcal{N}$, writing $\mathcal{N} = \prod_{i \in \omega} A_i$ by abuse of notation. By interpreting \perp^A as the least element of the homset $\mathbb{A}_\Sigma(\mathbf{1}, A)$, we have $\llbracket \Sigma \vdash G^{(n)} : o \rrbracket = \Lambda(\mathbf{g}); \llbracket Y_{\mathcal{N}}^{(n)} \rrbracket; \pi_1$. By Knaster-Tarski Fixpoint Theorem, $\mathcal{Y}_A = \bigsqcup_{n \in \omega} \llbracket Y_A^{(n)} \rrbracket$. Since composition is continuous, we have $\llbracket \Sigma \vdash F_1 : o \rrbracket = \Lambda(\mathbf{g}); \mathcal{Y}_{\llbracket \mathcal{N} \rrbracket}; \pi_1 = \Lambda(\mathbf{g}); \bigsqcup_{n \in \omega} \llbracket Y_{\mathcal{N}}^{(n)} \rrbracket; \pi_1 = \bigsqcup_{n \in \omega} (\Lambda(\mathbf{g}); \llbracket Y_{\mathcal{N}}^{(n)} \rrbracket; \pi_1) = \bigsqcup_{n \in \omega} \llbracket \Sigma \vdash G^{(n)} : o \rrbracket$ as desired. \square

Lemma 4.3 (P-view Decomposition). (i) For every (possibly infinite) P-view $p = \bigsqcup_{i \in \omega} p_i \in \llbracket \Sigma \vdash F_1 : o \rrbracket$ where each p_i is a finite P-view such that $p_0 \leq p_1 \leq p_2 \leq \dots$, there is an increasing sequence of natural numbers $n_0 < n_1 < n_2 < \dots$ such that each $p_i \in \llbracket \Sigma \vdash G^{(n_i)} : o \rrbracket$.

(ii) For every ω -indexed family of finite P-views, $p_i \in \llbracket \Sigma \vdash G^{(n_i)} : o \rrbracket$ with $i \in \omega$, such that $p_0 \leq p_1 \leq p_2 \leq \dots$ and an infinite sequence of natural numbers $n_0 < n_1 < n_2 < \dots$, the (possibly infinite) P-view $\bigsqcup_{i \in \omega} p_i \in \llbracket \Sigma \vdash F_1 : o \rrbracket$.

Proof. (i) Thanks to Lemma 4.2, for every i , there exists $n_i \geq 1$ such that $p_i \in \llbracket \Sigma \vdash \mathbf{G}^{(n_i)} : o \rrbracket$. (ii) An immediate consequence of Lemma 4.2. \square

Given a higher-order recursion scheme \mathcal{G} , the *computation tree* $\lambda(\mathcal{G})$ is obtained by first transforming the rewrite rules into long forms, and then unfolding the transformed rules *ad infinitum*, starting from F_1 , and without performing any β -reduction (i.e. substitution of actual parameters for formal parameters); see [Ong, 2006] for a definition. By construction, the tree $\lambda(\mathcal{G})$ is a (possibly infinite) $\Lambda(@)$ -labelled binding tree that satisfies the labelling axioms (Lam), (Leaf), (TVar) and (T@). We write $\mathfrak{T}^{\text{trav}}(\mathcal{G})$ be the set of finite and infinite traversals over $\lambda(\mathcal{G})$,

whereby an infinite traversal is just an infinite justified sequence over $\lambda(\mathcal{G})$ such that every finite prefix is a traversal.

Next we prove a similar decomposition lemma for traversals. First, notice that each $\langle G^{(i)} \rangle$ is a $\mathbf{\Lambda}(\textcircled{a})_{\perp}$ -labelled binding tree, where $\mathbf{\Lambda}(\textcircled{a})_{\perp}$ is $\mathbf{\Lambda}(\textcircled{a})$ augmented by an auxiliary symbol \perp of arity 0. Given $\mathbf{\Lambda}(\textcircled{a})_{\perp}$ -labelled trees T and T' , we define $T \sqsubseteq T'$ if $\text{dom}(T) \subseteq \text{dom}(T')$, and for all $\alpha \in \text{dom}(T)$, if $T(\alpha) \neq \perp$ then $T(\alpha) = T'(\alpha)$. Thus if $i < j$ then $\langle G^{(i)} \rangle \sqsubseteq \langle G^{(j)} \rangle$ and $\langle G^{(i)} \rangle \sqsubseteq \langle G \rangle$. It follows that if t is a \perp -free traversal over $\langle G^{(i)} \rangle$ then t is also a traversal over $\langle G^{(j)} \rangle$ and over $\langle G \rangle$. Conversely if t is a finite traversal over $\langle G \rangle$, then for every n greater than the length of t , t is also a traversal over $\langle G^{(n)} \rangle$. To summarise, we have the following.

Lemma 4.4 (Traversal Decomposition). *(i) For every (possibly infinite) traversal $t = \bigsqcup_{i \in \omega} t_i \in \mathfrak{Trav}\langle \mathcal{G} \rangle$ where each t_i is a finite traversal and $t_0 \leq t_1 \leq t_2 \leq \dots$, there is an increasing sequence of natural numbers $n_0 < n_1 < n_2 < \dots$ such that each $t_i \in \mathfrak{Trav}\langle G^{(n_i)} : o \rangle$.*
(ii) Given an ω -indexed family of \perp -free traversals, $t_i \in \mathfrak{Trav}\langle G^{(n_i)} : o \rangle$ with $i \in \omega$, such that $t_0 \leq t_1 \leq t_2 \leq \dots$, and an infinite sequence of natural numbers $n_0 < n_1 < n_2 < \dots$, the (possibly infinite) traversal $\bigsqcup_{i \in \omega} t_i \in \mathfrak{Trav}\langle \mathcal{G} \rangle$. \square

Theorem 4.5 (Traversal-Path Correspondence). *Let $\mathcal{G} = (\Sigma, \mathcal{N}, \mathcal{R}, F_1)$ be a possibly infinite higher-order recursion scheme. Paths in $\llbracket \mathcal{G} \rrbracket$ and traversals over $\lambda(\mathcal{G})$ projected to symbols from Σ are the same set of finite and infinite sequences over Σ .*

Proof. By combining Lemma 4.3, Lemma 4.4 and Theorem 3.18, we obtain a bijection $\varphi : \mathfrak{Trav}\langle \mathcal{G} \rangle \upharpoonright \epsilon \xrightarrow{\sim} \ulcorner \llbracket \Sigma \vdash F_1 : o \rrbracket \urcorner$ which is strong, in that for every $t \in \mathfrak{Trav}\langle \mathcal{G} \rangle \upharpoonright \epsilon$, we have t and $\varphi(t)$ are isomorphic as justified sequences. Since terminal symbols from Σ are assumed to be of order 1, justified sequences from $\mathfrak{Trav}\langle \mathcal{G} \rangle \upharpoonright \epsilon$ are completely determined by their underlying sequence over Σ ; similarly for $\ulcorner \llbracket \Sigma \vdash F_1 : o \rrbracket \urcorner$. On the one hand, there is a one-one correspondence between P-views in $\llbracket \Sigma \vdash F_1 : o \rrbracket$ and paths in the generated tree $\llbracket \mathcal{G} \rrbracket$: given a P-view, the corresponding path is obtained by erasing the O-moves. On the other, there is a one-one correspondence between $\mathfrak{Trav}\langle \mathcal{G} \rangle \upharpoonright \epsilon$ and traversals over $\lambda(\mathcal{G})$ projected to symbols from Σ . Hence we have the desired set equality. \square

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