Logics for Weighted Timed Pushdown Automata

Manfred Droste and Vitaly Perevoshchikov

Leipzig University

YURIFEST 2015

Chapter XIII
Monadic Second-Order Theories

by Y. Gurevich

In the present chapter we will make a case for the monadic second-order logic (that is to say, for the extension of first-order logic allowing quantification over monadic predicates) as a good source of theories that are both expressive and manageable. We will illustrate two powerful decidability techniques here—the one makes use of automata and games while the other uses generalized products *à la* Feferman–Vaught. The latter is, of course, particularly relevant, since monadic logic definitely appears to be the proper framework for examining generalized products.

Undecidability proofs must be thought out anew in this area; for, whereas true first-order arithmetic is *reducible* to the monadic theory of the real line R, it is nevertheless not *interpretable* in the monadic theory of R. Thus, the examination of a quite unusual undecidability method is another subject that will be explained in this chapter. In the last section we will briefly review the history of the methods thus far developed and give a description of some further results.

1. Monadic Quantification

Monadic (second-order) logic is the extension of the first-order logic that allows quantification over monadic (unary) predicates. Thus, although binary, ternary, and other predicates, as well as functions, may appear in monadic (second-order) languages, they may nevertheless not be quantified over.

1.1. Formal Languages for Mathematical Theories

We are interested less in monadic (second-order) logic itself than in the applications of this logic to mathematical theories. We are interested in the monadic formalization of the language of a mathematical theory and in monadic theories of corresponding mathematical objects. Before we explore this line of thought in more detail, let us argue that formalizing a mathematical language—not necessarily in monadic logic, but rather in first-order logic or in any other formal logic for that matter—can be useful.

2.1.4 Theorem. *There is an algorithm that, given a formula $\phi(X_1, \ldots, X_n)$ in the monadic language of one successor (with free variables as shown), constructs a Σ_n-automaton A such that for every finite chain C and any subsets $X_1, \ldots, X_n$ of C, we have that*

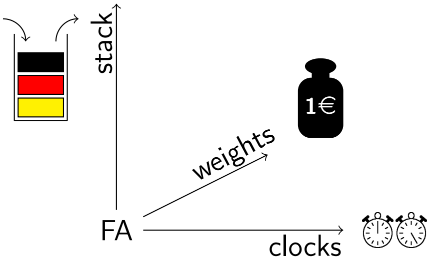
$$C \models \phi(X_1, \ldots, X_n) \quad \textit{iff} \quad A \textit{ accepts } \mathrm{Word}(C, X_1, \ldots, X_n).$$

Proof. We will merely sketch the proof. The automaton is built by induction on the formula. The atomic cases and the case of disjunction are quite easy. As to the case in which $\phi = \exists X_{n+1}\psi$, the desired Σ_n-automaton guesses X_{n+1} and mimics the Σ_{n+1}-automaton corresponding to ψ. The case of negation is easy for deterministic automata. We will now use Theorem 2.1.1 and the result will follow. $\square$

Weighted Timed Pushdown Automata[1] (WTPDA)

WTPDA are nondeterministic finite automata equipped with:

- real-valued global clocks
- timed stack
- weights (of transitions and stack letters)

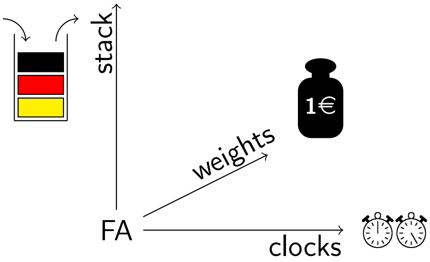


[1] Abdulla, Atig, Stenman '14

Weighted Timed Pushdown Automata[1] (WTPDA)

WTPDA are nondeterministic finite automata equipped with:

- real-valued global clocks
- timed stack
- weights (of transitions and stack letters)



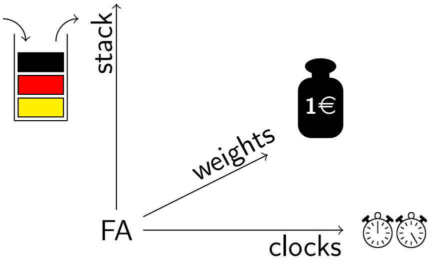
Optimal reachability costs in WTPDA are computable[1]

[1]Abdulla, Atig, Stenman '14

Weighted Timed Pushdown Automata[1] (WTPDA)

WTPDA are nondeterministic finite automata equipped with:

- real-valued global clocks
- timed stack
- weights (of transitions and stack letters)



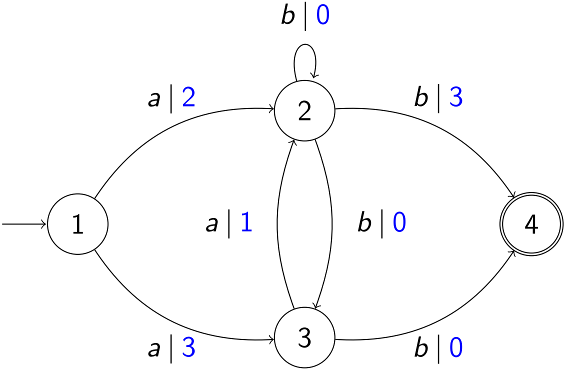
Optimal reachability costs in WTPDA are computable[1]

In this talk: no global clocks!

[1]Abdulla, Atig, Stenman '14

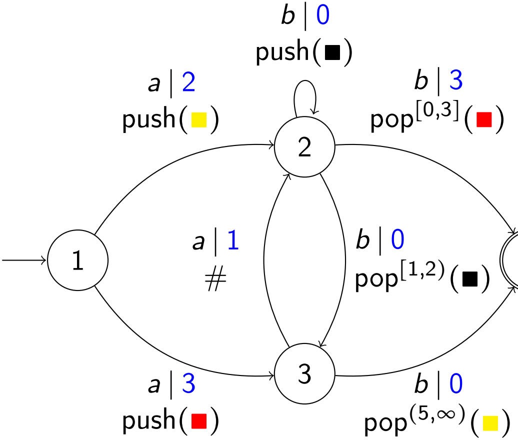
Weighted Timed Pushdown Automata (WTPDA)

Weighted automata:



Weighted Timed Pushdown Automata (WTPDA)

Weighted timed pushdown automata:

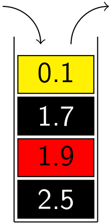


Stack letter	Weight
■	2
■	3
■	10

WTPDA: Behavior

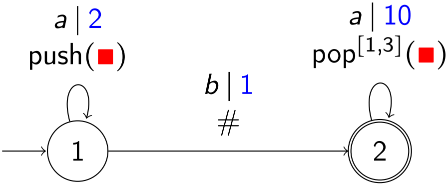
Configuration of a WTPDA:

1. state q
2. timed stack st



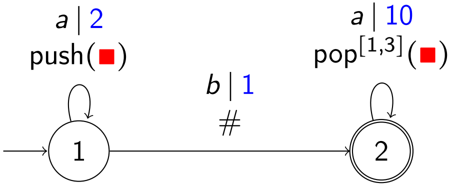
3. accumulated weight $\mathrm{wt} \in \mathbb{R}_{\geq 0}$

WTPDA: Behavior



Stack letter	Weight
■	5

WTPDA: Behavior



Stack letter	Weight
■	5

a	0.2	*a*	0.7	*b*	0.3	*a*	1.0	*a*

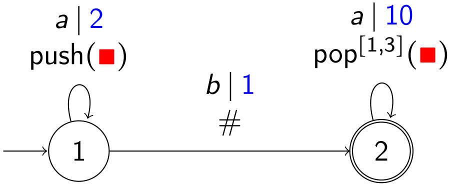
$q = 1$

st =

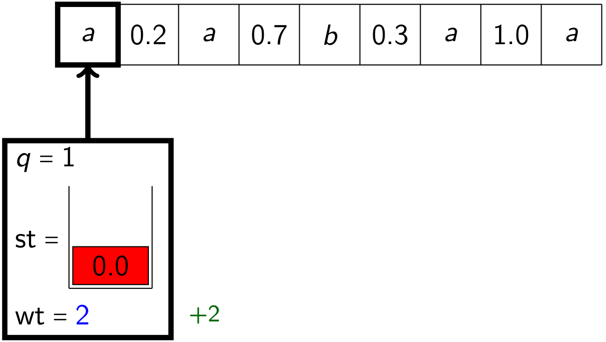
wt = 0

initial

WTPDA: Behavior

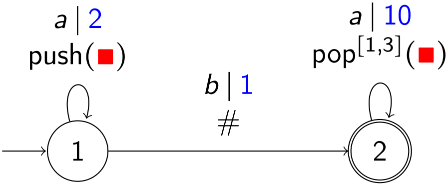


Stack letter	Weight
■	5



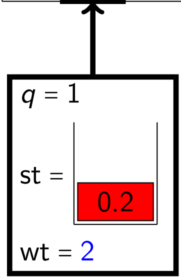
switch: $1 \xrightarrow{a \mid 2 \mid \text{push}(■)} 1$

WTPDA: Behavior



Stack letter	Weight
■	5

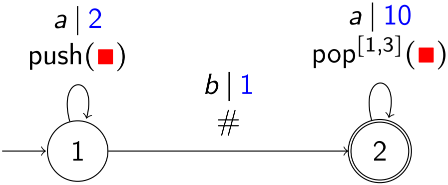
a	**0.2**	*a*	0.7	*b*	0.3	*a*	1.0	*a*



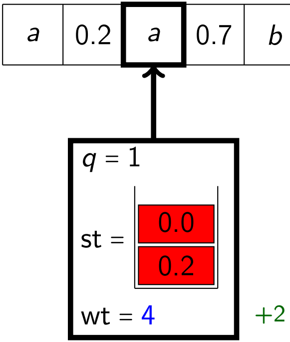
+0.2

delay: 0.2

WTPDA: Behavior

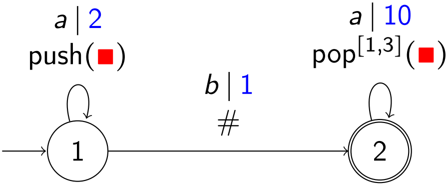


Stack letter	Weight
■	5

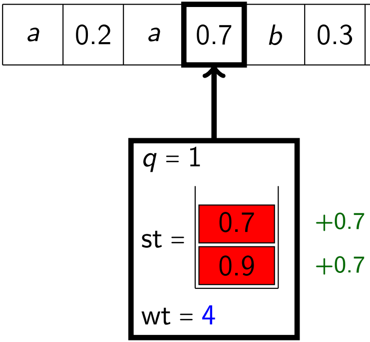


switch: $1 \xrightarrow{a \mid 2 \mid \text{push}(■)} 1$

WTPDA: Behavior

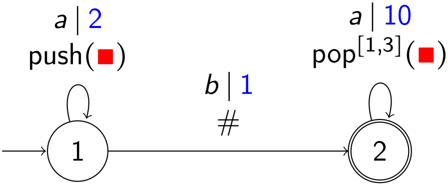


Stack letter	Weight
■	5

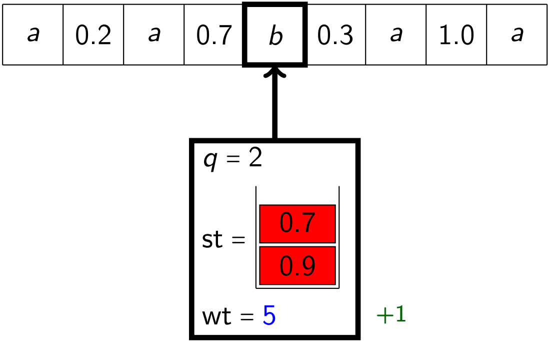


delay: 0.7

WTPDA: Behavior



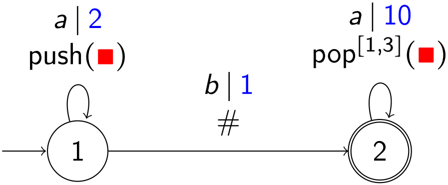
Stack letter	Weight
■	5



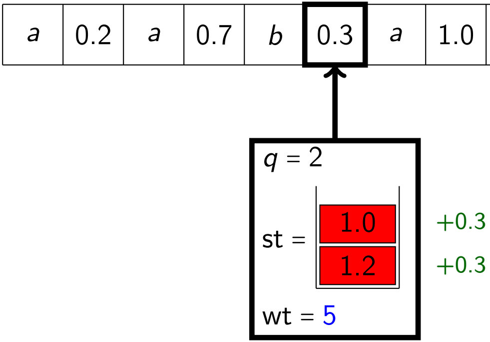
+1

switch: $1 \xrightarrow{b \mid 1 \mid \#} 2$

WTPDA: Behavior

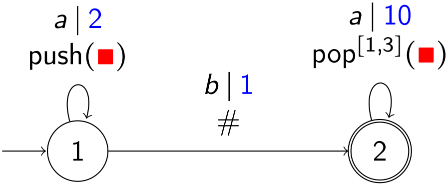


Stack letter	Weight
■	5

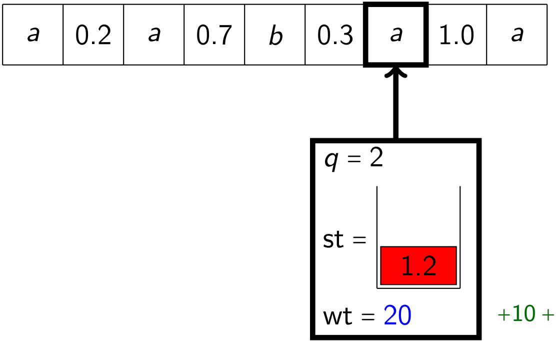


delay: 0.3

WTPDA: Behavior



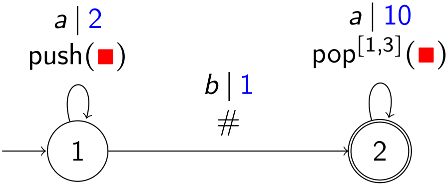
Stack letter	Weight
■	5



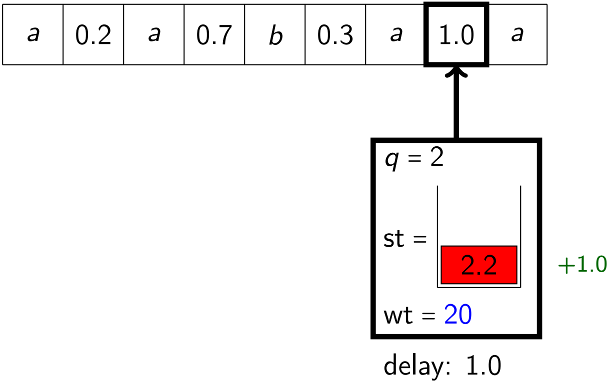
$+10 + (1.0 * 5)$

switch: $1 \xrightarrow{a \mid 10 \mid \text{pop}^{[1,3]}(■)} 2$

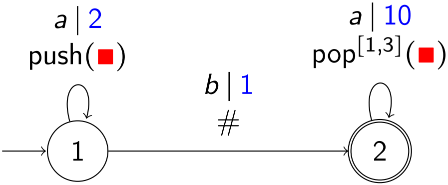
WTPDA: Behavior



Stack letter	Weight
■	5



WTPDA: Behavior



Stack letter	Weight
■	5

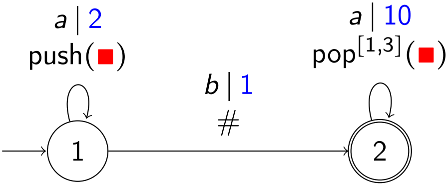
a	0.2	a	0.7	b	0.3	a	1.0	$\boxed{a}$

$q = 2$

st =

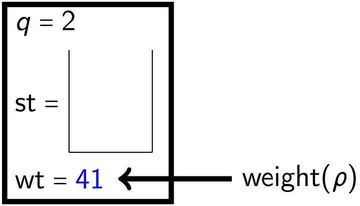
wt = 41 $\quad +10 + (2.2 * 5)$

switch: $1 \xrightarrow{a \mid 10 \mid \text{pop}^{[1,3]}(\blacksquare)} 2$



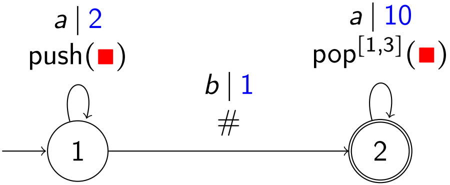
Stack letter	Weight
■	5

a	0.2	a	0.7	b	0.3	a	1.0	a



final

WTPDA: Behavior



Stack letter	Weight
■	5

Behavior:

$$
\begin{aligned}
[\![\mathcal{A}]\!] : \mathbb{T}\Sigma^{+} &\to \mathbb{R}_{\geq 0} \cup \{\infty\} \\
w &\mapsto \min\{\mathrm{weight}(\rho) \mid \rho \text{ is a run on } w\}
\end{aligned}
$$

Algebraic Framework for WTPDA

Definition[1]

A **timed semiring** $\mathbb{S} = \langle (S, +, \times, \mathbb{0}, \mathbb{1}), \mathcal{F} \rangle$ consists of:

- a semiring $(S, +, \times, \mathbb{0}, \mathbb{1})$;
- a class of functions $\mathcal{F} \subseteq S^{\mathbb{R}_{\geq 0}}$ with $\mathbb{1}^{\mathbb{R}_{\geq 0}} \in \mathcal{F}$.

[1] Quaas '10

Algebraic Framework for WTPDA

Definition[1]

A **timed semiring** $\mathbb{S} = \langle (S, +, \times, \mathbb{0}, \mathbb{1}), \mathcal{F} \rangle$ consists of:

- a semiring $(S, +, \times, \mathbb{0}, \mathbb{1})$;
- a class of functions $\mathcal{F} \subseteq S^{\mathbb{R}_{\geq 0}}$ with $\mathbb{1}^{\mathbb{R}_{\geq 0}} \in \mathcal{F}$.

Example: $\mathbb{S} = \mathrm{Trop}^{\mathrm{Lin}} = \langle (\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +, \infty, 0), \mathcal{F} \rangle$ with $\mathcal{F} = \{ t \mapsto c \cdot t \mid c \in \mathbb{R}_{\geq 0} \}$

[1] Quaas '10

Algebraic Framework for WTPDA

Definition[1]

A **timed semiring** $\mathbb{S} = \langle (S, +, \times, \mathbb{0}, \mathbb{1}), \mathcal{F} \rangle$ consists of:

- a semiring $(S, +, \times, \mathbb{0}, \mathbb{1})$;
- a class of functions $\mathcal{F} \subseteq S^{\mathbb{R}_{\geq 0}}$ with $\mathbb{1}^{\mathbb{R}_{\geq 0}} \in \mathcal{F}$.

Example: $\mathbb{S} = \mathrm{Trop}^{\mathrm{Lin}} = \langle (\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +, \infty, 0), \mathcal{F} \rangle$ with $\mathcal{F} = \{t \mapsto c \cdot t \mid c \in \mathbb{R}_{\geq 0}\}$

WTPDA $\mathcal{A}$ over timed semirings:

- Weights of transitions: in S; weights of stack letters: in $\mathcal{F}$

[1] Quaas '10

Algebraic Framework for WTPDA

Definition[1]

A **timed semiring** $\mathbb{S} = \langle (S, +, \times, \mathbb{0}, \mathbb{1}), \mathcal{F} \rangle$ consists of:

- a semiring $(S, +, \times, \mathbb{0}, \mathbb{1})$;
- a class of functions $\mathcal{F} \subseteq S^{\mathbb{R}_{\geq 0}}$ with $\mathbb{1}^{\mathbb{R}_{\geq 0}} \in \mathcal{F}$.

Example: $\mathbb{S} = \mathrm{Trop}^{\mathrm{Lin}} = \langle (\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +, \infty, 0), \mathcal{F} \rangle$ with $\mathcal{F} = \{t \mapsto c \cdot t \mid c \in \mathbb{R}_{\geq 0}\}$

WTPDA $\mathcal{A}$ over timed semirings:

- Weights of transitions: in S; weights of stack letters: in $\mathcal{F}$
- Weight of the removal of a top stack letter of age $t \in \mathbb{R}_{\geq 0}$ and weight $f \in \mathcal{F}$: $f(t) \in S$ (e.g., $c \cdot t$)

[1] Quaas '10

Definition[1]

A **timed semiring** $\mathbb{S} = \langle (S, +, \times, \mathbb{0}, \mathbb{1}), \mathcal{F} \rangle$ consists of:

- a semiring $(S, +, \times, \mathbb{0}, \mathbb{1})$;
- a class of functions $\mathcal{F} \subseteq S^{\mathbb{R}_{\geq 0}}$ with $\mathbb{1}^{\mathbb{R}_{\geq 0}} \in \mathcal{F}$.

Example: $\mathbb{S} = \mathrm{Trop}^{\mathrm{Lin}} = \langle (\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +, \infty, 0), \mathcal{F} \rangle$ with $\mathcal{F} = \{ t \mapsto c \cdot t \mid c \in \mathbb{R}_{\geq 0} \}$

WTPDA $\mathcal{A}$ over timed semirings:

- Weights of transitions: in S; weights of stack letters: in $\mathcal{F}$
- Weight of the removal of a top stack letter of age $t \in \mathbb{R}_{\geq 0}$ and weight $f \in \mathcal{F}$: $f(t) \in S$ (e.g., $c \cdot t$)
- Accumulation of weights: using $\times$

[1] Quaas '10

Algebraic Framework for WTPDA

Definition[1]

A **timed semiring** $\mathbb{S} = \langle (S, +, \times, \mathbb{0}, \mathbb{1}), \mathcal{F} \rangle$ consists of:

- a semiring $(S, +, \times, \mathbb{0}, \mathbb{1})$;
- a class of functions $\mathcal{F} \subseteq S^{\mathbb{R}_{\geq 0}}$ with $\mathbb{1}^{\mathbb{R}_{\geq 0}} \in \mathcal{F}$.

Example: $\mathbb{S} = \mathrm{Trop}^{\mathrm{Lin}} = \langle (\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +, \infty, 0), \mathcal{F} \rangle$ with $\mathcal{F} = \{t \mapsto c \cdot t \mid c \in \mathbb{R}_{\geq 0}\}$

WTPDA $\mathcal{A}$ **over timed semirings**:

- Weights of transitions: in S; weights of stack letters: in $\mathcal{F}$
- Weight of the removal of a top stack letter of age $t \in \mathbb{R}_{\geq 0}$ and weight $f \in \mathcal{F}$: $f(t) \in S$ (e.g., $c \cdot t$)
- Accumulation of weights: using $\times$
- Nondeterminism resolving: using $+$ (e.g., min)

[1] Quaas '10

Algebraic Framework for WTPDA

Definition[1]

A **timed semiring** $\mathbb{S} = \langle (S, +, \times, \mathbb{0}, \mathbb{1}), \mathcal{F} \rangle$ consists of:

- a semiring $(S, +, \times, \mathbb{0}, \mathbb{1})$;
- a class of functions $\mathcal{F} \subseteq S^{\mathbb{R}_{\geq 0}}$ with $\mathbb{1}^{\mathbb{R}_{\geq 0}} \in \mathcal{F}$.

Example: $\mathbb{S} = \mathrm{Trop}^{\mathrm{Lin}} = \langle (\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +, \infty, 0), \mathcal{F} \rangle$ with $\mathcal{F} = \{t \mapsto c \cdot t \mid c \in \mathbb{R}_{\geq 0}\}$

WTPDA $\mathcal{A}$ over timed semirings:

- Weights of transitions: in S; weights of stack letters: in $\mathcal{F}$
- Weight of the removal of a top stack letter of age $t \in \mathbb{R}_{\geq 0}$ and weight $f \in \mathcal{F}$: $f(t) \in S$ (e.g., $c \cdot t$)
- Accumulation of weights: using $\times$
- Nondeterminism resolving: using $+$ (e.g., min)
- Accepted weighted language: $[\![\mathcal{A}]\!] : \mathbb{T}\Sigma^+ \to S$

[1] Quaas '10

Algebraic Framework for WTPDA

Definition[1]

A **timed semiring** $\mathbb{S} = \langle (S, +, \times, \mathbb{0}, \mathbb{1}), \mathcal{F} \rangle$ consists of:

- a semiring $(S, +, \times, \mathbb{0}, \mathbb{1})$;
- a class of functions $\mathcal{F} \subseteq S^{\mathbb{R}_{\geq 0}}$ with $\mathbb{1}^{\mathbb{R}_{\geq 0}} \in \mathcal{F}$.

Example: $\mathbb{S} = \mathrm{Trop}^{\mathrm{Lin}} = \langle (\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +, \infty, 0), \mathcal{F} \rangle$ with $\mathcal{F} = \{t \mapsto c \cdot t \mid c \in \mathbb{R}_{\geq 0}\}$

WTPDA $\mathcal{A}$ over timed semirings:

- Weights of transitions: in S; weights of stack letters: in $\mathcal{F}$
- Weight of the removal of a top stack letter of age $t \in \mathbb{R}_{\geq 0}$ and weight $f \in \mathcal{F}$: $f(t) \in S$ (e.g., $c \cdot t$)
- Accumulation of weights: using $\times$
- Nondeterminism resolving: using $+$ (e.g., min)
- Accepted weighted language: $[\![\mathcal{A}]\!] : \mathbb{T}\Sigma^+ \to S$

[1] Quaas '10

Logics for Timed Pushdown Automata[1]

Timed extension of MSO with matchings[2].

Definition

Let Σ be an alphabet.

1. $\mathrm{TMSO}(\Sigma, \leq, \mu)$: defined by the grammar

$$\varphi ::= P_a(x) \mid x \leq y \mid x \in X \mid \mu(x,y) \in I \mid \varphi \vee \varphi \mid \neg\varphi \mid \exists x.\varphi \mid \exists X.\varphi$$

where $a \in \Sigma$ and I is an interval.

2. **Timed matching logic** $\mathrm{TML}(\Sigma)$: the set of all formulas $\exists^{\mathrm{match}} \mu.\varphi$ with $\varphi \in \mathrm{TMSO}(\Sigma, \leq, \mu)$.

[1] Droste, Perevoshchikov '15

[2] Lautemann, Schwentick, Thérien '94

Logics for Timed Pushdown Automata[1]

Timed extension of MSO with matchings[2].

Definition

Let Σ be an alphabet.

1. $\mathrm{TMSO}(\Sigma, \leq, \mu)$: defined by the grammar

   $$\varphi ::= P_a(x) \mid x \leq y \mid x \in X \mid \mu(x,y) \in I \mid \varphi \vee \varphi \mid \neg\varphi \mid \exists x.\varphi \mid \exists X.\varphi$$

   where $a \in \Sigma$ and I is an interval.

2. **Timed matching logic** $\mathrm{TML}(\Sigma)$: the set of all formulas $\exists^{\text{match}} \mu.\varphi$ with $\varphi \in \mathrm{TMSO}(\Sigma, \leq, \mu)$.

[1] Droste, Perevoshchikov '15

[2] Lautemann, Schwentick, Thérien '94

Definition (Matching).

A relation $M \subseteq \{1, \ldots, n\}^2$ is a matching if:

1. $(x, y) \in M \Rightarrow x < y$;
2. every $x \in \{1, \ldots, n\}$ belongs to at most one pair in M;

Logics for Timed Pushdown Languages

Definition (Matching).

A relation $M \subseteq \{1, \dots, n\}^2$ is a matching if:

1. $(x, y) \in M \Rightarrow x < y$;
2. every $x \in \{1, \dots, n\}$ belongs to at most one pair in M;
3. M is non-crossing:

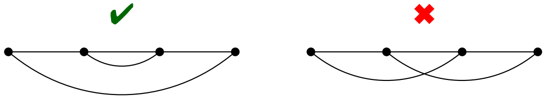
✔

✖

Definition (Matching).

A relation $M \subseteq \{1, \ldots, n\}^2$ is a matching if:

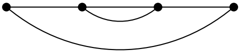
1. $(x, y) \in M \Rightarrow x < y$;
2. every $x \in \{1, \ldots, n\}$ belongs to at most one pair in M;
3. M is non-crossing:



Definition (Matching).

A relation $M \subseteq \{1, \dots, n\}^2$ is a matching if:

1. $(x, y) \in M \Rightarrow x < y$;
2. every $x \in \{1, \dots, n\}$ belongs to at most one pair in M;
3. M is non-crossing:

✔ 

✖ 

For $w = (a_1, t_1) \dots (a_n, t_n) \in \mathbb{T}\Sigma^+$, we let $(w, \sigma) \models \mu(x, y) \in I$ iff:

1. $(\sigma(x), \sigma(y)) \in \sigma(\mu)$;
2. $(t_{\sigma(y)} - t_{\sigma(x)}) \in I$.

Weighted Timed Matching Logics

Weighted extension of TML[1].
Let Σ be an alphabet and $\mathbb{S} = \langle (S, +, \times, \mathbb{0}, \mathbb{1}), \mathcal{F} \rangle$ a timed semiring.

Definition

Weighted timed matching logic $\mathrm{WTML}(\Sigma, \mathbb{S})$: consists of formulas $\bigoplus^{\mathrm{match}} \mu.\varphi$ with

$$\varphi ::= \beta \mid s \mid f(\mu - x) \mid \varphi \oplus \varphi \mid \varphi \otimes \varphi \mid \bigoplus x.\varphi \mid \bigotimes x.\varphi \mid \bigoplus X.\varphi \mid \bigotimes X.\varphi$$

where $\beta \in \mathrm{TMSO}(\Sigma, \leq, \mu)$, $s \in S$ and $f \in \mathcal{F}$.

[1] Droste, Gastin '07

Weighted Timed Matching Logics: Semantics

Semantics: $[\![\varphi]\!] : \mathbb{T}\Sigma^+_{\mathsf{Var}} \to S$.

Let $w = (a_1, t_1)\dots(a_n, t_n) \in \mathbb{T}\Sigma^+$.

$$[\![\beta]\!](w,\sigma) = \begin{cases} \mathbb{1}, & \text{if } (w,\sigma) \models \beta \\ \mathbb{0}, & \text{otherwise} \end{cases}$$

Weighted Timed Matching Logics: Semantics

Semantics: $[\![\varphi]\!] : \mathbb{T}\Sigma^+_{\mathsf{Var}} \to S$.

Let $w = (a_1, t_1)\ldots(a_n, t_n) \in \mathbb{T}\Sigma^+$.

$$[\![\beta]\!](w, \sigma) = \begin{cases} \mathbb{1}, & \text{if } (w, \sigma) \models \beta \\ \mathbb{0}, & \text{otherwise} \end{cases}$$

$$[\![s]\!](w, \sigma) = s$$

Weighted Timed Matching Logics: Semantics

Semantics: $[\![\varphi]\!] : \mathbb{T}\Sigma^+_{\mathsf{Var}} \to S$.

Let $w = (a_1, t_1)\ldots(a_n, t_n) \in \mathbb{T}\Sigma^+$.

$$
\begin{aligned}
[\![\beta]\!](w,\sigma) &= \begin{cases} \mathbb{1}, & \text{if } (w,\sigma) \models \beta \\ \mathbb{0}, & \text{otherwise} \end{cases} \\
[\![s]\!](w,\sigma) &= s \\
[\![\varphi_1 \oplus \varphi_2]\!](w,\sigma) &= [\![\varphi_1]\!](w,\sigma) + [\![\varphi_2]\!](w,\sigma)
\end{aligned}
$$

Weighted Timed Matching Logics: Semantics

Semantics: $[\![\varphi]\!] : \mathbb{T}\Sigma^+_{\mathsf{Var}} \to S$.

Let $w = (a_1, t_1)\ldots(a_n, t_n) \in \mathbb{T}\Sigma^+$.

$$[\![\beta]\!](w,\sigma) = \begin{cases} \mathbb{1}, & \text{if } (w,\sigma) \models \beta \\ \mathbb{0}, & \text{otherwise} \end{cases}$$

$$[\![s]\!](w,\sigma) = s$$

$$[\![\varphi_1 \oplus \varphi_2]\!](w,\sigma) = [\![\varphi_1]\!](w,\sigma) + [\![\varphi_2]\!](w,\sigma)$$

$$[\![\varphi_1 \otimes \varphi_2]\!](w,\sigma) = [\![\varphi_1]\!](w,\sigma) \times [\![\varphi_2]\!](w,\sigma)$$

Weighted Timed Matching Logics: Semantics

Semantics: $[\![\varphi]\!] : \mathbb{T}\Sigma^+_{\mathsf{Var}} \to S$.

Let $w = (a_1, t_1)\dots(a_n, t_n) \in \mathbb{T}\Sigma^+$.

$$
\begin{aligned}
[\![\beta]\!](w,\sigma) &= \begin{cases} \mathbb{1}, & \text{if } (w,\sigma) \models \beta \\ \mathbb{0}, & \text{otherwise} \end{cases} \\
[\![s]\!](w,\sigma) &= s \\
[\![\varphi_1 \oplus \varphi_2]\!](w,\sigma) &= [\![\varphi_1]\!](w,\sigma) + [\![\varphi_2]\!](w,\sigma) \\
[\![\varphi_1 \otimes \varphi_2]\!](w,\sigma) &= [\![\varphi_1]\!](w,\sigma) \times [\![\varphi_2]\!](w,\sigma) \\
[\![\bigoplus x.\varphi]\!](w,\sigma) &= \sum_{i \in \{1,\dots,n\}} [\![\varphi]\!](w,\sigma[x/i])
\end{aligned}
$$

Weighted Timed Matching Logics: Semantics

Semantics: $[\![\varphi]\!] : \mathbb{T}\Sigma^+_{\mathsf{Var}} \to S$.

Let $w = (a_1, t_1)\ldots(a_n, t_n) \in \mathbb{T}\Sigma^+$.

$$
\begin{aligned}
[\![\beta]\!](w,\sigma) &= \begin{cases} \mathbb{1}, & \text{if } (w,\sigma) \models \beta \\ \mathbb{0}, & \text{otherwise} \end{cases} \\
[\![s]\!](w,\sigma) &= s \\
[\![\varphi_1 \oplus \varphi_2]\!](w,\sigma) &= [\![\varphi_1]\!](w,\sigma) + [\![\varphi_2]\!](w,\sigma) \\
[\![\varphi_1 \otimes \varphi_2]\!](w,\sigma) &= [\![\varphi_1]\!](w,\sigma) \times [\![\varphi_2]\!](w,\sigma) \\
[\![\bigoplus x.\varphi]\!](w,\sigma) &= \sum\nolimits_{i \in \{1,\ldots,n\}} [\![\varphi]\!](w,\sigma[x/i]) \\
[\![\bigoplus X.\varphi]\!](w,\sigma) &= \sum\nolimits_{I \subseteq \{1,\ldots,n\}} [\![\varphi]\!](w,\sigma[X/I])
\end{aligned}
$$

Weighted Timed Matching Logics: Semantics

Semantics: $[\![\varphi]\!] : \mathbb{T}\Sigma^+_{\mathrm{Var}} \to S$.

Let $w = (a_1, t_1)\dots(a_n, t_n) \in \mathbb{T}\Sigma^+$.

$$
\begin{aligned}
[\![\beta]\!](w,\sigma) &= \begin{cases} \mathbb{1}, & \text{if } (w,\sigma) \models \beta \\ \mathbb{0}, & \text{otherwise} \end{cases} \\
[\![s]\!](w,\sigma) &= s \\
[\![\varphi_1 \oplus \varphi_2]\!](w,\sigma) &= [\![\varphi_1]\!](w,\sigma) + [\![\varphi_2]\!](w,\sigma) \\
[\![\varphi_1 \otimes \varphi_2]\!](w,\sigma) &= [\![\varphi_1]\!](w,\sigma) \times [\![\varphi_2]\!](w,\sigma) \\
[\![\bigoplus x.\varphi]\!](w,\sigma) &= \sum_{i \in \{1,\dots,n\}} [\![\varphi]\!](w,\sigma[x/i]) \\
[\![\bigoplus X.\varphi]\!](w,\sigma) &= \sum_{I \subseteq \{1,\dots,n\}} [\![\varphi]\!](w,\sigma[X/I]) \\
[\![\bigotimes x.\varphi]\!](w,\sigma) &= \prod_{i \in \{1,\dots,n\}} [\![\varphi]\!](w,\sigma[x/i]) \\
[\![\bigotimes X.\varphi]\!](w,\sigma) &= \prod_{I \subseteq \{1,\dots,n\}} [\![\varphi]\!](w,\sigma[X/I])
\end{aligned}
$$

Weighted Timed Matching Logics: Semantics

Semantics: $[\![\varphi]\!] : \mathbb{T}\Sigma^+_{\mathsf{Var}} \to S$.

Let $w = (a_1, t_1)\ldots(a_n, t_n) \in \mathbb{T}\Sigma^+$.

$$[\![\, f(\mu - x) \,]\!](w, \sigma) = \begin{cases} f(t_j - t_{\sigma(x)}), & \text{if } (\sigma(x), j) \in \sigma(\mu), \\ \mathbb{0}, & \text{otherwise} \end{cases}$$

Weighted Timed Matching Logics: Semantics

Semantics: $[\![\varphi]\!] : \mathbb{T}\Sigma^+_{\mathrm{Var}} \to S$.

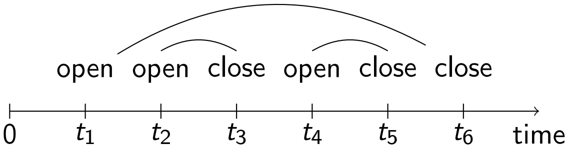
Let $w = (a_1, t_1)\ldots(a_n, t_n) \in \mathbb{T}\Sigma^+$.

$$[\![\, f(\mu - x) \,]\!](w, \sigma) = \begin{cases} f(t_j - t_{\sigma(x)}), & \text{if } (\sigma(x), j) \in \sigma(\mu), \\ \mathbb{0}, & \text{otherwise} \end{cases}$$

$$[\![\, \bigoplus^{\mathrm{match}} \mu.\varphi \,]\!](w, \sigma) = \sum \left([\![\varphi]\!](w, \sigma[\mu/M]) \mid M \subseteq \{1, \ldots, n\}^2 \text{ matching} \right)$$

Example

For $\Sigma = \{\text{open}, \text{close}\}$, let $\mathcal{D} \subseteq \Sigma^+$ be the Dyck language, i.e., the set of all correctly nested sequences of brackets.



Example.

Weighted timed Dyck language $\mathbb{D} : \mathbb{T}\Sigma^+ \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is defined for all $w = (a_1, t_1)\ldots(a_n, t_n) \in \mathbb{T}\Sigma^+$ by

$$\mathbb{D}(w) = \begin{cases} \text{minimal time between matching brackets}, & \text{if } a_1 \ldots a_n \in \mathcal{D}, \\ \infty, & \text{otherwise} \end{cases}$$

Example

Example.

Weighted timed Dyck language $\mathbb{D} : \mathbb{T}\Sigma^+ \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is defined for all $w = (a_1, t_1)\ldots(a_n, t_n) \in \mathbb{T}\Sigma^+$ by

$$\mathbb{D}(w) = \begin{cases} \text{minimal time between matching brackets,} & \text{if } a_1 \ldots a_n \in \mathcal{D}, \\ \infty, & \text{otherwise} \end{cases}$$

$\mathbb{D}$ is defined by the WTML(Σ, Trop$^{\mathrm{Lin}}$)-formula:

$$\varphi = \bigoplus^{\mathrm{match}} \mu.(\beta \otimes \bigoplus x.\mathrm{id}(\mu - x))$$

where

$$\beta = \forall x.[(P_{\mathrm{open}}(x) \to \exists y.\mu(x,y)) \wedge (P_{\mathrm{close}}(x) \to \exists y.\mu(y,x))]$$

Restricted WTML

Formulas with unrecognizable semantics:

- formulas with nested $\bigotimes x.$-quantifiers;
- formulas with $\bigotimes X.$-quantifiers;

Restricted WTML

Formulas with unrecognizable semantics:

- formulas with nested $\bigotimes x.$-quantifiers;
- formulas with $\bigotimes X.$-quantifiers;
- formulas $\varphi_1 \otimes \varphi_2$;

Restricted WTML

Formulas with unrecognizable semantics:

- formulas with nested $\bigotimes x.$-quantifiers;
- formulas with $\bigotimes X.$-quantifiers;
- formulas $\varphi_1 \otimes \varphi_2$;
- formulas $\bigotimes x.\varphi$ where φ contains $f(\mu - y)$ with $y \neq x$.

Restricted WTML

Formulas with unrecognizable semantics:

- formulas with nested $\bigotimes x.$-quantifiers;
- formulas with $\bigotimes X.$-quantifiers;
- formulas $\varphi_1 \otimes \varphi_2$;
- formulas $\bigotimes x.\varphi$ where φ contains $f(\mu - y)$ with $y \neq x$.

Restricted WTML

Formulas with unrecognizable semantics:

- formulas with nested $\bigotimes x.$-quantifiers;
- formulas with $\bigotimes X.$-quantifiers;
- formulas $\varphi_1 \otimes \varphi_2$;
- formulas $\bigotimes x.\varphi$ where φ contains $f(\mu - y)$ with $y \neq x$.

Let $\mathbb{S} = \langle (S, +, \times, \mathbb{0}, \mathbb{1}), \mathcal{F} \rangle$.

Definition (restricted WTML).

$\mathrm{WTML}^{\mathrm{res}}(\Sigma, \mathbb{S})$: the set of all formulas $\bigoplus^{\mathrm{match}} \mu.\varphi$ with

$$
\begin{aligned}
\gamma_x &::= \beta \mid s \otimes f(\mu - x) \mid \gamma_x \oplus \gamma_x \mid \beta \otimes \gamma_x \\
\varphi &::= \beta \mid s \otimes f(\mu - x) \mid \varphi \oplus \varphi \mid \beta \otimes \varphi \mid \bigoplus x.\varphi \mid \bigoplus X.\varphi \mid \bigotimes x.\gamma_x
\end{aligned}
$$

where $\beta \in \mathrm{TMSO}(\Sigma)$, $s \in S$ and $f \in \mathcal{F}$.

Main Result

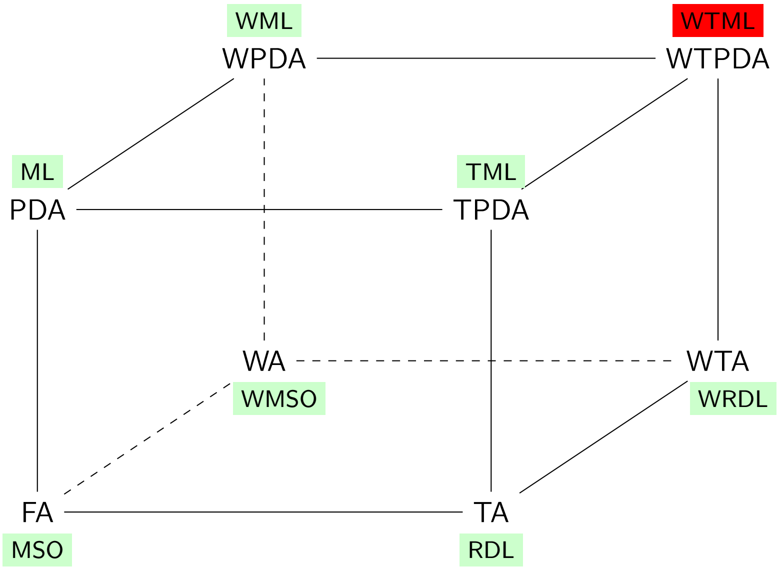
Let Σ be an alphabet and $\mathbb{S} = \langle (S, +, \times, \mathbb{0}, \mathbb{1}), \mathcal{F} \rangle$ a timed semiring.

Theorem.

Let $\mathbb{W} : \mathbb{T}\Sigma^+ \to S$ be a weighted timed language. TFAE:

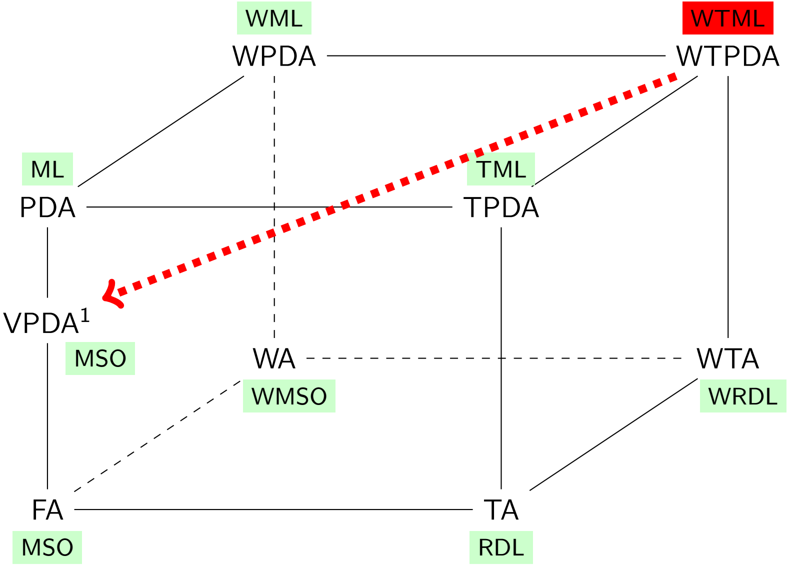
1. $\mathbb{W}$ is **recognizable** by a weighted timed pushdown automaton (WTPDA) over Σ and $\mathbb{S}$.
2. $\mathbb{W}$ is **definable** by a restricted weighted timed matching sentence in $\mathrm{WTML}^{\mathrm{res}}(\Sigma, \mathbb{S})$.

Proof



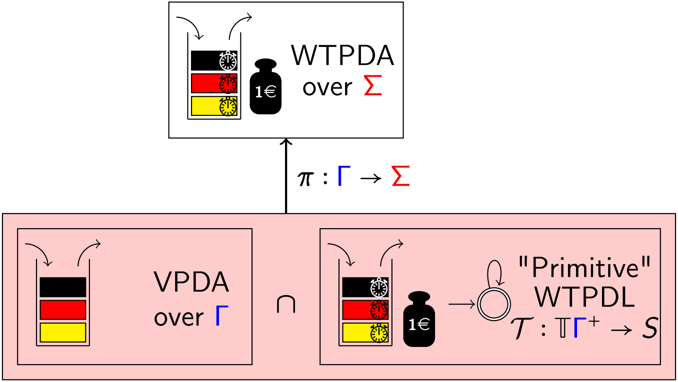
WML
WTML
WPDA
WTPDA
ML
TML
PDA
TPDA
WA
WTA
WMSO
WRDL
FA
TA
MSO
RDL

Proof



[1]Visibly pushdown automata (Alur, Madhusudan '04)

Decomposition of WTPDA

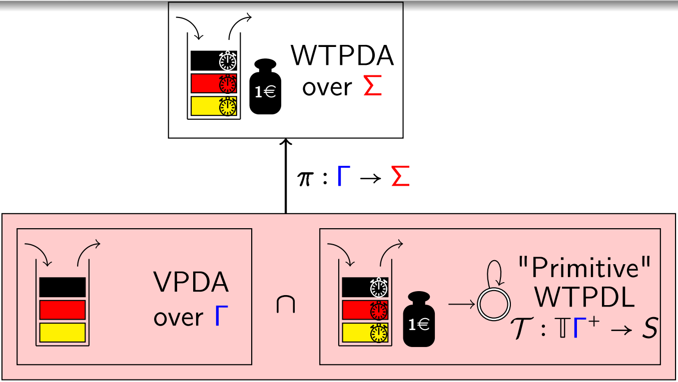


Extended alphabet

$$\Gamma = \underbrace{\Delta}_{\text{transitions}} \times \underbrace{\mathbb{P}(k)}_{\text{stack constraints}} \times \underbrace{\hat{\mathcal{S}}}_{\mathcal{S}\text{-constants}} \times \underbrace{\hat{\mathcal{F}}}_{\mathcal{F}\text{-constants}} \times \underbrace{\{\text{push}, \#, \text{pop}\}}_{\text{stack commands}}$$

- $k :=$ maximal number appearing in constraints
- $\mathbb{P}(k) := \{[0,0], (0,1), [1,1], \ldots, (k-1,k), [k,k], (k,\infty)\}$

Decomposition of WTPDA



Theorem

Let $\mathbb{W} : \mathbb{T}\Sigma^+ \to S$. *TFAE:*

1. $\mathbb{W}$ *is recognizable by a WTPDA.*
2. *There exist* $k \in \mathbb{N}$, *alphabets* Δ, $\hat{S} \subseteq S$ *and* $\hat{\mathcal{F}} \subseteq \mathcal{F}$, *and a VPDL* $\mathcal{L} \subseteq (\Gamma(k, \Delta, \hat{S}, \hat{\mathcal{F}}))^+$ *with*

$$\mathbb{W} = \pi(\mathcal{L}' \cap \mathcal{T})$$

Conclusions

1. Effective translation $\Rightarrow$ decidability results.
2. WTPDA with global clocks.

Conclusions

1. Effective translation $\Rightarrow$ decidability results.
2. WTPDA with global clocks.
3. Future work:

Conclusions

1. Effective translation $\Rightarrow$ decidability results.
2. WTPDA with global clocks.
3. Future work:
   - WTPDA with additional properties of $\mathcal{F}$.

Conclusions

1. Effective translation $\Rightarrow$ decidability results.
2. WTPDA with global clocks.
3. Future work:
    - WTPDA with additional properties of $\mathcal{F}$.
    - WTPDA with location weights.

Conclusions

1. Effective translation $\Rightarrow$ decidability results.
2. WTPDA with global clocks.
3. Future work:
   - WTPDA with additional properties of $\mathcal{F}$.
   - WTPDA with location weights.
   - $\exists^{\mathsf{match}}\mu.\mathsf{FO} \equiv \mathsf{TPDA}$?

Conclusions

1. Effective translation $\Rightarrow$ decidability results.
2. WTPDA with global clocks.
3. Future work:
   - WTPDA with additional properties of $\mathcal{F}$.
   - WTPDA with location weights.
   - $\exists^{\mathsf{match}}\mu.\mathsf{FO} \equiv \mathsf{TPDA}$?
   - Applications

Conclusions

1. Effective translation $\Rightarrow$ decidability results.
2. WTPDA with global clocks.
3. Future work:
   - WTPDA with additional properties of $\mathcal{F}$.
   - WTPDA with location weights.
   - $\exists^{\mathsf{match}}\mu.\mathsf{FO} \equiv \mathsf{TPDA}$?
   - Applications

Conclusions

1. Effective translation $\Rightarrow$ decidability results.
2. WTPDA with global clocks.
3. Future work:
   - WTPDA with additional properties of $\mathcal{F}$.
   - WTPDA with location weights.
   - $\exists^{\mathsf{match}}\mu.\mathsf{FO} \equiv \mathsf{TPDA}$?
   - Applications

Conclusions

1. Effective translation $\Rightarrow$ decidability results.
2. WTPDA with global clocks.
3. Future work:
   - WTPDA with additional properties of $\mathcal{F}$.
   - WTPDA with location weights.
   - $\exists^{\mathsf{match}}\mu.\mathsf{FO} \equiv \mathsf{TPDA}$?
   - Applications

THANK YOU!