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CONCATENATION AS A BASIS FOR ARITHMETIC

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1. Introduction. General syntax, the formal part of the general theory of signs, has as its basic operation the operation of *concatenation*, expressed by the connective '$\frown$' and understood as follows: where x and y are any expressions, $x \frown y$ is the expression formed by writing the expression x immediately followed by the expression y. E.g., where 'alpha' and 'beta' are understood as names of the respective signs 'α' and 'β', the syntactical expression 'alpha $\frown$ beta' is a name of the expression '$\alpha\beta$'.

Tarski[1] and Hermes[2] have presented axioms for concatenation, and definitions of derivative syntactical concepts. Hermes has also related concatenation theory to the arithmetic of natural numbers, constructing a model of the latter within the former. Conversely, Gödel's proof of the impossibility of a complete consistent systematization of arithmetic[3] depended on constructing a model of concatenation theory within arithmetic.

Like Tarski and Hermes, I also have used concatenation as a basis for various constructions;[4] but my constructions differ essentially from theirs in presupposing no auxiliary logical machinery beyond the *elementary* level: truth-functions, quantification, and identity. It can be easily shown to follow from my constructions that the elementary arithmetic of natural numbers (elementary in the sense of using none but the aforementioned logical auxiliaries) can be embedded in the elementary theory of concatenation. The only proviso is that the atomic components from which concatenations can be formed must be at least two in number, and distinguishable by name; e.g., alpha and beta.

The present paper, undertaken at Professor Tarski's suggestion, will establish the above more explicitly and in a strengthened form: it will be shown not only that the elementary arithmetic of natural numbers can be embedded in the elementary theory of concatenation, but that it can be so embedded as to exhaust the latter, rendering the elementary theory of concatenation and the elementary arithmetic of natural numbers identical.

Actually it will be more convenient to consider, in place of the arithmetic of natural numbers, that of positive integers. The conclusions reached can afterward be transferred to the arithmetic of natural numbers, as will be seen.

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[1] Alfred Tarski, *Der Wahrheitsbegriff in den formalisierten Sprachen*, ***Studia philosophica*** (Lwów), vol. 1 (1935), pp. 261–405, esp. pp. 287 ff.

[2] Hans Hermes, ***Semiotik. Eine Theorie der Zeichengestalten als Grundlage für Untersuchungen von formalisierten Sprachen.*** Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, new series, no. 5. Leipzig, 1938, 22 pp.

[3] Kurt Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme*, ***Monatshefte für Mathematik und Physik,*** vol. 38 (1931), pp. 173–198.

[4] W. V. Quine, *Definition of substitution*, ***Bulletin of the American Mathematical Society,*** vol. 42 (1936), pp. 561–569; *On derivability*, this JOURNAL, vol. 2 (1937), pp. 113–119; ***Mathematical logic*** (New York, 1940), Ch. 7.

The paper will presuppose no acquaintance with previous work on concatenation.

Let us now turn to a more exact statement of what is meant by the elementary theory of concatenation and the elementary arithmetic of positive integers, and a more exact statement of what is to be proved about them.

The theory of concatenation need not be thought of as syntactical in subject matter; it may be regarded as having to do with finite sequences of any manner of objects. The objects, called *atoms*, are themselves considered sequences, viz., sequences of length one; and concatenation is the operation of laying sequences end to end to form new sequences. Thus, where x is the sequence consisting successively of the atoms a, b, b, a, and c, and where y is the sequence consisting successively of b and a, the sequence $x \frown y$ will consist successively of a, b, b, a, c, b, a. Parentheses will be suppressed by construing '$x \frown y \frown z$' as '$(x \frown y) \frown z$'; clearly the grouping is in fact indifferent.

The *elementary theory of concatenation* comprises that body of theory which can be expressed in terms of concatenation, identity, names of the several atoms, truth-functions, and quantification with respect to variables 'x', 'y', 'z', $\cdots$ whose values are sequences. The theory varies in detail according to the number of atoms assumed. Thus, if we suppose the atoms to be just two in number and designated by 'a' and 'b', the formulas of the elementary theory of concatenation are the following:

(i) identities, consisting of '$=$' flanked by terms each of which is either a simple constant 'a', or 'b', or a variable 'x', 'y', 'z', $\cdots$, or an expression compounded of such simple signs by one or more applications of the concatenation sign;

(ii) all compounds formed from such identities by means of truth-functional connectives and quantification with respect to variables 'x', 'y', etc.

The *elementary arithmetic of positive integers* comprises that body of theory which can be expressed in terms of sum, product, power, identity, names of the several positive integers, truth-functions, and quantification with respect to variables 'x', 'y', 'z', $\cdots$ whose values are positive integers. Thus the formulas of the elementary arithmetic of positive integers are the following:

(i) identities, consisting of '$=$' flanked by terms each of which is either a simple constant '1', '2', $\cdots$, or a variable 'x', 'y', 'z', $\cdots$, or an expression compounded of such simple signs by one or more applications of the notational forms '$x + y$', '$x \times y$', 'x^y' of sum, product, and power;

(ii) all compounds formed from such identities by means of the truth-functional connectives and quantification with respect to variables 'x', 'y', etc.

It will be proved that a model of arithmetic can be constructed, in terms of finite sequences, in such a way that all the notations of the elementary arithmetic of positive integers become definable in the notation of the elementary theory of concatenation of two or more atoms.

Identification of the positive integers with sequences may be either *unilateral*, in the sense that the integers are exhaustively identified with an inexhaustive sub-class of sequences, or *bilateral*, in the sense that the integers and sequences are so identified as to exhaust both the integers and the sequences. The construction of arithmetic within the theory of concatenation will be carried through first in unilateral fashion (§3), since this is easier. But the bilateral construction (§4) is of especial interest in that it makes the elementary theory of concatenation and the elementary arithmetic of positive integers completely intertranslatable. And it shows that concatenation can be interpreted as a purely arithmetical operation, in terms of which the other operations of arithmetic are definable.

After proving these things with regard to the arithmetic of positive integers, we can infer the same with regard to the arithmetic of natural numbers; for it is known (and will be reëstablished briefly in §5) that the arithmetic of natural numbers is bilaterally constructible within that of positive integers.

The definitions comprised in all these constructions will commonly be, in a certain sense, indirect; that is, free use will be made of the descriptive notation '$(\imath x)(\cdots)$', which is not eliminable by any direct substitution of expressions built up of primitive terms. This notation has not been cited in the above inventory of the notations of the theories under consideration, because it is well known that the presence of the notational machinery of quantification, truth functions, and identity renders the descriptive notation eliminable from any theorem or other formula in which it occurs.

2. Finite relations. The chief circumstance which makes possible the construction of arithmetic from concatenation theory is that single sequences can be made to do the work of all finite relations of sequences, by a method which will now be set forth. I shall assume here that our theory of concatenation involves at least two atoms, a and b; there may be more.

A finite relation may be thought of first as a finite collection of ordered pairs, and represented graphically as a two-column list. Consider now any such relation; let it pair u_1 with v_1, and u_2 with v_2, and so on to u_n and v_n. (The elements u_1, v_1, u_2, etc. are themselves any sequences.) Let z be a certain *tally*, i.e., a certain sequence consisting wholly of occurrences of a. (The reason for calling such a sequence a tally will become evident in §4.) Further, let this tally z be longer than any tally appearing within any of the sequences u_1, v_1, u_2, etc. Now form n sequences, one corresponding to each of the n pairs of the original relation, as follows:

$$\begin{aligned} w_1 &= b^\frown z^\frown b^\frown u_1{}^\frown b^\frown z^\frown b^\frown v_1{}^\frown b^\frown z, \\ w_2 &= b^\frown z^\frown b^\frown u_2{}^\frown b^\frown z^\frown b^\frown v_2{}^\frown b^\frown z, \\ &\ \vdots \\ w_n &= b^\frown z^\frown b^\frown u_n{}^\frown b^\frown z^\frown b^\frown v_n{}^\frown b^\frown z, \end{aligned}$$

and lay all these end to end in turn to form a single sequence:

$$w = w_1{}^\frown w_2{}^\frown \cdots {}^\frown w_n .$$

Next let us observe certain peculiarities of w. (i) *The only occurrences of z in w are the $3n$ occurrences of z which are shown explicitly in the above expansions of w_1, w_2, $\cdots$, w_n*; there are no additional hidden occurrences of z. This is seen as follows: There can be no additional occurrence of z hidden within any of the occurrences of u_1, v_1, u_2, etc., because z exceeds any tally therein; nor can there be any additional occurrence of the tally z partially overlapping upon one of the explicitly shown occurrences of z or of u_1, v_1, u_2, etc., because each of those explicitly shown occurrences is insulated in front and back by the atom b, which cannot occur in a tally. The possibilities are thus exhausted.

(ii) *The only segments of w which have $z \frown b$ just before them and $b \frown z$ just after, and do not themselves contain any occurrence of z, are the occurrences of u_1, v_1, u_2, etc. which are shown explicitly in the above expansions of $w_1, w_2, \cdots, w_n$.* This follows from (i) on inspection of the expansions of $w_1, w_2, \cdots, w_n$.

(iii) *If x and y are sequences neither of which contains any occurrence of z, and if*

$$z \frown b \frown x \frown b \frown z \frown b \frown y \frown b \frown z \tag{1}$$

occurs as part of w, then x and y must be respectively u_i and v_i for some i. For, by (ii), x must be one of u_1, v_1, u_2, etc., and y must be a later one of them. Yet the occurrences of x and y in question cannot be within different ones of the segments $w_1, w_2, \cdots, w_n$ of w, because if they were they would be separated by at least $b \frown z \frown b \frown z \frown b$ instead of just $b \frown z \frown b$. So x and y must be respectively u_i and v_i for some one i.

In view of (iii) we have the following result: *instead of saying that x stands to y in our original finite relation, we can omit mention of that relation and speak instead of the sequence w*, saying that (1) is part of w and that x and y contain no occurrences of z (z being explained in turn as the longest tally in w).

Let us write '$w(x, y)$' to mean that (1) is part of w and that x and y contain no occurrences of z (where z is explained as the longest tally in w); then what we have found is that we can construct a sequence w such that $(x)(y)[w(x, y) \equiv . \ x$ stands to y in our original finite relation].

Preparatory to setting up the formal definition of '$w(x, y)$' we shall need to introduce the notation '$\dot{\subset}$', meaning 'occurs in', 'is part of'. It is definable within concatenation theory as follows:

D1. $x \mathbin{\dot{\subset}} y =_{\mathrm{df}} (\exists z)(\exists w)(x = y \ .\vee . \ z \frown x = y \ .\vee . \ x \frown w = y \ .\vee . \ z \frown x \frown w = y)$.

Thus '$x \mathbin{\dot{\subset}} y$' means that the sequence x is a continuous part (or all) of the sequence y.[5]

Next the notation '$\mathrm{T}z$', meaning that z is a tally, i.e., contains only a's, is readily defined:

D2. $$\mathrm{T}z =_{\mathrm{df}} (x)(x \mathbin{\dot{\subset}} z \ .\supset . \ a \mathbin{\dot{\subset}} x).$$

To say that z is the longest tally in w is to way that z is a tally and is part of w and that every tally occurring anywhere in w reappears as part of z. In symbols this becomes:

$$\mathrm{T}z \ . \ z \mathbin{\dot{\subset}} w \ . \ (v)(\mathrm{T}v \ . \ v \mathbin{\dot{\subset}} w \ .\supset . \ v \mathbin{\dot{\subset}} z),$$

[5] The definiens in D1 could be simplified to '$(\exists z)(\exists w)(z \frown x \frown w = y)$' if we were to assume a null sequence among the values of our variables. However, I have chosen rather to repudiate the null sequence throughout the present paper, lest it be thought to be an essential assumption.

which is equivalent to:

$$(2) \qquad (v)(\mathrm{T}v \;.\; v \mathrel{\dot{\subset}} w \;.{\equiv}.\; v \mathrel{\dot{\subset}} z).$$

So now we are ready to define '$w(x, y)$' as meaning that there is a sequence z which satisfies (2), and is not part of x or y, and is such further that the sequence expressed in (1) is part of w.

D3. $w(x, y) =_{\mathrm{df}} (\exists z)[(v)(\mathrm{T}v \;.\; v \mathrel{\dot{\subset}} w \;.{\equiv}.\; v \mathrel{\dot{\subset}} z) \;.\; \sim (z \mathrel{\dot{\subset}} x) \;.\; \sim (z \mathrel{\dot{\subset}} y) \;.$
$z^\frown b^\frown x^\frown b^\frown z^\frown b^\frown y^\frown b^\frown z \;\mathrel{\dot{\subset}}\; w].$

The reasoning of the present section has shown that, for *every* finite relation of sequences, *there is* a sequence w such that

$$(x)(y)[w(x, y) \equiv .\ x \text{ bears the relation to } y].$$

We have seen how, given a list of the pairs comprising the relation, such a sequence w can actually be constructed.

3. Unilateral construction. A convenient way of getting a model of the realm of positive integers within the realm of sequences based on two or more atoms $a, b, \cdots$ is by identifying each integer n with the tally which comprises n occurrences of a. Thus $1, 2, 3, \cdots$ become definable respectively as a, $a^\frown a, a^\frown a^\frown a, \cdots$; and to say that x is a positive integer is to say simply that $\mathrm{T}x$.

Where x and y are positive integers, clearly '$x \mathrel{\dot{\subset}} y$' amounts to '$x \leqq y$'; and $x^\frown y$ is the *sum* of x and y.

The *product* $x \times y$ is the sequence which we would get by laying down the sequence x over and over, end to end, y times—i.e., as many times as a occurs in y. (This explanation makes good sense even where x is not an integer; but this is an incidental extension that need not interest us.) Let us now turn to the problem of constructing an actual definition to this effect.

It will first be shown that, for any sequence z, the following condition implies that z is the product $x \times y$ in the desired sense:

$$(3) \quad (\exists w)\{w(y, z) \;.\; (s)(t)[w(s, t) \supset : s = a \;.\; t = x \;.{\vee}\; (\exists u)(\exists v)(w(u, v) \;.\; s = a^\frown u \,.\, t = x^\frown v)]\}.$$

This is seen as follows. (3) says there is a sequence w such that

$$(4) \qquad w(y, z),$$

$$(5) \quad (s)(t)\{w(s, t) \supset : s = a \;.\; t = x \;.{\vee}(\exists u)(\exists v)[w(u, v) \;.\; s = a^\frown u \;.\; t = x^\frown v]\}.$$

By (4) and (5),

$$(6) \qquad y = a \;.\; z = x \;.{\vee}\; (\exists u)(\exists v)[w(u, v) \;.\; y = a^\frown u \;.\; z = x^\frown v].$$

Now if the sequence y is an atom, and hence no concatenation, the second alternative in the alternation (6) must fail; and then the first alternative in (6) tells us that y is a and z is x. If on the other hand y is longer than an atom, then the second alternative in (6) tells us that y and z begin respectively with a and x, and also that the remainders y' and z' of y and z are such that

(7) $$w(y', z').$$

By (7) and (5),

(8) $$y' = a \,.\, z' = x \,.\!\vee\, (\exists u)(\exists v)[w(u, v) \,.\, y' = a^\frown u \,.\, z' = x^\frown v].$$

Continuing thus, for as many steps as there are places in the original sequence y, we see finally that y must consist wholly of a's and that z must consist of a concatenation of a like number of x's; in brief, y is a positive integer and

(9) $$z = x \times y.$$

It will now be shown conversely that, for all integers x and y, (9) implies (3). Consider the following list of pairs:

$$\begin{array}{ll} a & x \\ a^\frown a & x^\frown x \\ a^\frown a^\frown a & x^\frown x^\frown x \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ y & x \times y \end{array}$$

(We know, from the intended sense of '$x \times y$', that the sequence reached in the right-hand column opposite y will in fact be $x \times y$.) Now the principle stated at the end of §2 assures us that there is a sequence w such that, for any sequences s and t, '$w(s, t)$' holds if and only if s and t are paired in the above list. From inspection of the list, then, it is evident that (5) holds, and also that

(10) $$w(y, x \times y).$$

From (10) and (9) we get (4), and from (5) and (4) we get (3).

The product $x \times y$, therefore, is definable as the one and only sequence z fulfilling (3).

D4. $$x \times y =_{\mathrm{df}} (\imath z)(\exists w)\{w(y, z) \,.\, (s)(t)[w(s, t) \supset : s = a \,.\, t = x \,.\!\vee (\exists u)(\exists v)(w(u, v) \,.\, s = a^\frown u \,.\, t = x^\frown v)]\}.$$

The *power* x^y can be defined quite analogously. The only difference is that whereas the product $x \times y$ was a continued concatenation $x^\frown x^\frown \cdots {}^\frown x$ to y occurrences of x, the power x^y is a continued *product* $x \times x \times \cdots \times x$ to y occurrences of x. So the definition of x^y differs from D4 only in using '$x \times v$' instead of '$x^\frown v$' at the end.

D5. $$x^y =_{\mathrm{df}} (\imath z)(\exists w)\{w(y, z) \,.\, (s)(t)[w(s, t) \supset : s = a \,.\, t = x \,.\!\vee (\exists u)(\exists v)(w(u, v) \,.\, s = a^\frown u \,.\, t = x \times v)]\}.$$

The definitions D4 and D5 are of course of interest only where the sequences x and y are positive integers; what they generate in other cases need not concern us. It may, however, be remarked in passing that, where x is not an integer but y is, $x \times y$ continues to be the sequence $x^\frown x^\frown \cdots {}^\frown x$ to y occurrences of x.

4. Bilateral construction. We now turn to an alternative method of identifying the positive integers with sequences; this time a bilateral method, i.e., one which uses up all the sequences.

Up to now we have been able to leave the number of atoms unspecified, requiring only that there be at least two. For the bilateral construction, however, we need to know how many there are. Any specified finite number of them would serve, provided there are at least two. The construction will first be set forth for the case where there are exactly two atoms, a and b. Afterward we shall see how to adapt the method to any larger finite number of atoms.

Bilateral identification of the positive integers with the finite sequences of atoms a and b can be accomplished by arranging the sequences *lexicographically*; i.e., in order of length and alphabetically within each length:

$$(11)\quad \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \cdots \\ a & b & a^\frown a & a^\frown b & b^\frown a & b^\frown b & a^\frown a^\frown a & a^\frown a^\frown b & a^\frown b^\frown a & a^\frown b^\frown b & b^\frown a^\frown a & \cdots \end{array}$$

In this way not only do the integers receive an interpretation as sequences, but also conversely, the finite sequences of a and b all receive an interpretation as positive integers, and concatenation thus itself becomes an arithmetical operation on positive integers. In fact, concatenation becomes expressible within elementary arithmetic, as follows:

$$x^\frown y = y + [x \times (\imath z)(\exists w)(z = 2^w \,.\, z \leqq y + 1 < z + z)].$$

This identity will be left unsubstantiated;[6] our avowed business lies in the opposite direction, viz., to find ways of defining $x + y$, $x \times y$, and x^y, conformable to (11), within the theory of concatenation. Our business is to show that the new arithmetical operation of concatenation is an arithmetical operation in terms of which the more familiar operations of sum, product, and power can be defined.

Though the sequences composed purely of occurrences of a no longer exhaust the positive integers as they did in §3, they will continue to play an important rôle under the earlier name of *tallies*. Definitions D1–3 will be retained. D4 and D5, no longer available as definitions of products and powers of positive integers, will still be useful in an auxiliary status as defining what may be called the tally-products and tally-powers of tallies; they are accordingly reproduced here with accented designations and modified notations:

D4′. $x \times_T y =_{\mathrm{df}} (\imath z)(\exists w)\{w(y, z) \,.\, (s)(t)[w(s, t) \supset : s = a \,.\, t = x \,.\vee$
$(\exists u)(\exists v)(w(u, v) \,.\, s = a^\frown u \,.\, t = x^\frown v)]\}.$

D5′. $x \wedge_T y =_{\mathrm{df}} (\imath z)(\exists w)\{w(y, z) \,.\, (s)(t)[w(s, t) \supset : s = a \,.\, t = x \,.\vee$
$(\exists u)(\exists v)(w(u, v) \,.\, s = a^\frown u \,.\, t = x \times_T v)]\}.$

[6] The reasoning behind it is evident from *On derivability*, this JOURNAL, vol. 2 (1937), p. 115, where the present identification of integers with sequences is so rephrased as to exhibit its relation to the dual system of numeration.

A tally consisting of n occurrences of a will be called the *tally of n*. Where x is any positive integer (hence any sequence whatever, under the method of the present section), the tally of x will be referred to as τx. It is definable in concatenation theory in the light of the following considerations.

Hereunder are listed the successive positive integers from 1 to x, paired off with their tallies.

$$\begin{array}{ll}
a & a \\
b & a^\frown a \\
a^\frown a & a^\frown a^\frown a \\
a^\frown b & a^\frown a^\frown a^\frown a \\
b^\frown a & a^\frown a^\frown a^\frown a^\frown a \\
b^\frown b & a^\frown a^\frown a^\frown a^\frown a^\frown a \\
a^\frown a^\frown a & a^\frown a^\frown a^\frown a^\frown a^\frown a^\frown a \\
a^\frown a^\frown b & a^\frown a^\frown a^\frown a^\frown a^\frown a^\frown a^\frown a \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
x & \tau x
\end{array}$$

A method of generating the column on the left, comprising the integers in the normal order 1, 2, 3, $\cdots$, is described by the following rules:

(i) start with a and b;

(ii) if u appears in the column, put $u^\frown a$ one place more than twice as far down the column, and put $u^\frown b$ next after (i.e., $u^\frown a = 2u + 1$ and $u^\frown b = 2u + 2$). From (ii), together with the nature of the right-hand column (the tallies), it is evident that, where v appears opposite u (so that $v = \tau u$), what will appear opposite $u^\frown a$ is $v^\frown v^\frown a$; and what will appear opposite $u^\frown b$ is $v^\frown v^\frown a^\frown a$.

In view of these considerations, it can be shown that the definition:

$$\textbf{D6.}\quad \tau x =_{\mathrm{df}} (\imath y)(\exists w)\{w(x, y)\ .\ (s)(t)[w(s, t) \supset : s = a\ .\ t = a\ .\vee .\ s = b\ .\ t = a^\frown a\ .\vee\ (\exists u)(\exists v)(w(u, v)\ :\ s = u^\frown a\ .\ t = v^\frown v^\frown a\ .\vee .\ s = u^\frown b\ .\ t = v^\frown v^\frown a^\frown a)]\}$$

defines the tally of x in the intended sense. The reasoning which shows that the condition following '$(\imath y)$' in D6 implies that y is τx in the intended sense, and vice versa, is parallel to the reasoning which showed in §3 that (3) implied (9) and vice versa.

Now the definitions of $x + y$, $x \times y$, and x^y are immediate. We can define $x + y$ as the integer whose tally is the concatenate of the tallies of x and y; we can define $x \times y$ as the integer whose tally is the tally-product (as defined in D4′) of the tallies of x and y; and we can define x^y analogously.

D7. $$x + y =_{\mathrm{df}} (\imath z)(\tau z = \tau x \frown \tau y).$$

D8. $$x \times y =_{\mathrm{df}} (\imath z)(\tau z = \tau x \times_\tau \tau y).$$

D9. $$x^y =_{\mathrm{df}} (\imath z)(\tau z = \tau x \wedge_\tau \tau y).$$

The methods of construction which have just now been explained for the case of two atoms a and b will now be adapted to the general case of k atoms a_1, a_2, $\cdots$, a_k. We are to think of k as some specified number, finite and greater than 1.

Bilateral identification of the positive integers with the finite sequences of atoms $a_1, a_2, \cdots, a_k$ can still be accomplished by the old expedient of arranging the sequences in order of length and alphabetically within each length:

$$\begin{array}{ccccccccccccc} 1 & 2 & \cdots & k & k+1 & k+2 & \cdots & 2k & 2k+1 & \cdots & k^2+k & k^2+k+1 & \cdots \\ a_1 & a_2 & \cdots & a_k & a_1\frown a_1 & a_1\frown a_2 & \cdots & a_1\frown a_k & a_2\frown a_1 & \cdots & a_k\frown a_k & a_1\frown a_1\frown a_1 & \cdots \end{array}$$

This generalization requires us to generalize the observations (i) and (ii) above to read as follows:

(i) start with a_1, a_2, $\cdots$, and a_k;

(ii) if u appears in the column, put $u\frown a_1$ one place more than k times as far down the column, then put $u\frown a_2$ next, and so on to $u\frown a_k$.

Thereupon D6 has to be generalized correspondingly. The following auxiliary notation will be used:

$$(x)_2 =_{\mathrm{df}} x\frown x, \qquad (x)_3 =_{\mathrm{df}} x\frown x\frown x, \qquad \text{etc.}$$

The generalized version of D6 then appears as follows:

$$\tau x =_{\mathrm{df}} (\imath y)(\exists w)\{w(x, y) \;.\; (s)(t)[w(s, t) \supset : s = a_1 \,.\, t = a_1 \,.\vee.\, s = a_2 \,.\, t = (a_1)_2$$
$$.\vee.\ \cdots\ .\vee.\, s = a_k \,.\, t = (a_1)_k \,.\vee\, (\exists u)(\exists v)(w(u, v) \;: s = u\frown a_1 \,.\, t = (v)_k\frown a_1$$
$$.\vee.\; s = u\frown a_2 \,.\, t = (v)_k\frown(a_1)_2 \,.\vee.\ \cdots\ .\vee.\; s = u\frown a_k \,.\, t = (v)_k\frown(a_1)_k)]\}.$$

The definitions D7–9 now carry over to the general case without modification·

5. Natural numbers. We saw in §3 how the elementary arithmetic of positive integers can be constructed unilaterally within any elementary theory of concatenation which provides at least two atoms. We then saw in §4 how that same arithmetic can be constructed bilaterally within any elementary theory of concatenation which provides an explicit finite number of atoms, at least two.

What has thus been shown for the elementary arithmetic of positive integers could be shown equally for the elementary arithmetic of *natural numbers* (i.e., positive integers and 0). This follows from the fact that the elementary arithmetic of natural numbers can be constructed in turn, and indeed in bilateral fashion, within the elementary arithmetic of positive integers. A method of carrying out the last-mentioned construction will now be presented.

The natural numbers, which I shall call 0_n, 1_n, 2_n, etc., may be arbitrarily identified with the positive integers (which I shall continue to call 1, 2, 3, etc.) thus:

$$\begin{array}{cccc} 0_n & 1_n & 2_n & \cdots \\ 1 & 2 & 3 & \cdots \end{array}$$

I shall call 1_n the *nominal* correspondent of 1, and 2_n of 2, and so on, alluding thus to the resemblance in names; but nominal correspondence is not to be confused with the *real* correspondence (*identity*) which, under our construction, holds rather between 0_n and 1, between 1_n and 2, and so on. In general, the nominal correspondent of any integer k is $k + 1$; 2_n, for example, is 3.

Now let us consider how to define the addition '$+_n$' of the arithmetic of natural numbers, in terms of the addition '$+$' of the arithmetic of positive integers. Where x' and y' are the nominal correspondents of x and y, we shall want $x' +_n y'$ to turn out as the nominal correspondent of $x + y$. (E.g., we want $5_n +_n 6_n$ to be 11_n.) Therefore, since the nominal correspondent of any integer k is $k + 1$, we want to realize the identity:

$$(x + 1) +_n (y + 1) = x + y + 1. \tag{12}$$

Similarly, where '$x \times_n y$' represents the product of natural arithmetic and '$x \wedge_n y$' the power, we want to realize the identities:

$$(x + 1) \times_n (y + 1) = (x \times y) + 1, \tag{13}$$

$$(x + 1) \wedge_n (y + 1) = x^y + 1, \tag{14}$$

where '$x \times y$' and 'x^y' represent the product and power as of positive arithmetic.

The desiderata (12)–(14) are not quite enough to determine our choice of the general definitions of '$+_n$', '$\times_n$' and '$\wedge_n$', because they describe these operations only in application to numbers of the form '$x + 1$' and '$y + 1$'; and there is one number, namely, 1 ($= 0_n$), which cannot be rendered in that form. So we must supplement (12)–(14) with stipulations of what we want when one or both operands are 0_n:

$$0_n +_n z = z, \tag{15}$$

$$z +_n 0_n = z, \tag{16}$$

$$0_n \times_n z = 0_n = 1, \tag{17}$$

$$z \times_n 0_n = 0_n = 1, \tag{18}$$

$$0_n \wedge_n (y + 1) = 0_n = 1, \tag{19}$$

$$z \wedge_n 0_n = 1_n = 2. \tag{20}$$

Verification that the following definitions meet the requirements is left to the reader.

$$x +_n y =_{df} (\iota z)(x + y = z + 1),$$

$$x \times_n y =_{df} (\iota z)[x + y + z = (x \times y) + 2],$$

$$x \wedge_n y =_{df} (\iota z)[y = 1 \,.\, z = 2 \,.\!\vee\!.\, y \neq 1 \,.\, x = z = 1 \,.\!\vee\, (\exists w)(x = w + 1 \,.\, z \times w = w^y + w)].$$

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