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Fundamental Study

The monadic second-order logic of graphs XIII: Graph drawings with edge crossings

Bruno Courcelle [1]

Université Bordeaux-I, LaBRI (CNRS, UMR 5800), 351, Cours de la Libération, 33405 Talence cedex, France

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Abstract

We introduce finite relational structures called *sketches*, that represent edge crossings in drawings of finite graphs. We consider the problem of characterizing sketches in Monadic Second-Order logic. We answer positively the question for *framed* sketches, i.e., for those representing drawings of graphs consisting of a planar connected spanning subgraph (the *frame*) augmented with additional edges that may cross one another and that may cross the edges of the frame. We prove the 3-*Edge Theorem* stating that a structure of appropriate type with a frame is a sketch if and only if every induced substructure representing the frame and at most 3 edges not in the frame is a sketch. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Planar graph; Combinatorial map; Graph drawing; Monadic second-order logic; Edge crossing

Contents

[1] WWW: http://dept-info.labri.u-bordeaux.fr/∼ courcell/ActSci.html
E-mail address: courcell@labri.u-bordeaux.fr (B. Courcelle).

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0. Introduction

We are interested in space economical machine representations of graph drawings for the purpose of efficient storing, processing, and transmission over computer networks. According to the intended use, the representation can be more or less precise. There is actually a hierarchy of representations, the most precise one being the drawing itself stored as an image.

At the bottom level lies the *simple graph*, consisting of the set of vertices and the binary relation of adjacency. Just above lies the *graph*, in which the multiplicity of edges is stored in one way or another. In both cases, all topological information is lost.

The (*combinatorial*) *map*, a notion introduced by Edmonds [7] (see [10, chapter 3]), retains from the drawing the circular order of incidences of edges around each vertex. From the map of a *planar embedding* of a connected graph (i.e., a drawing such that edges do not cross), and a pair of edges forming a "corner" of the infinite region of the plane delimited by the curve segments representing the edges, one can reconstruct the embedding, *up to a homeomorphism* of the plane. (This result is proved as Theorem 3.2.4 in [10].) Hence, this notion is very well suited for representing planar embeddings of connected graphs.

However, not all graphs are planar. Furthermore, drawings of planar graphs *with edge crossings* can also be useful since they may exhibit some symmetries of the graph better than a planar embedding.

Consider a map of a nonplanar graph like $K_{3,3}$. It may be associated with several drawings, such that edges cross in different ways; hence these are not homeomorphic. In order to capture the information about relative positions of crossings, we introduce the notion of a *sketch*. The sketch of a drawing consists of the underlying map together with one relation on edges that represents the linear order of crossings on each edge and another relation that expresses whether an edge e crosses an edge f from the right or from the left. (We assume that edges are directed and that the plane is oriented.)

The sketch is thus more precise than the map while remaining a finite logical structure (all graphs are finite in this paper). A drawing of a connected graph such that no edge crosses itself, no two edges cross more than once and no three edges cross at a same point can be reconstructed, up to homeomorphism, from its sketch and two edges specifying the infinite region.

For applications to geographical information systems, it is important for efficiency purposes to query properties at an appropriate abstraction level, as in [12]. For an example, one can search the graph for a shortest route avoiding a crowded road section, or a sketch for the existence of a route for transporting nuclear wastes avoiding areas where some rare bird species are protected. All these queries are more efficiently processed at the levels of graphs or sketches than at the level of images. A similar idea is used by Segoufin and Vianu in [12].

We have thus a hierarchy of finite logical structures that give more and more information about drawings of finite graphs.

However, instead of adding new relations representing edge crossings to the map of a graph G, one could alternatively formalize crossings as new vertices of a planar graph H "containing" G, the edges of which are the portions of edges of G delimited by the crossings. However, this method modifies the considered graph G. We prefer to keep it untouched and add auxiliary information relative to its drawings, which are not unique, even up to homeomorphism. In particular, if the map of a nonplanar graph is fixed, but the edge crossings are not, the information to be added is limited to two relations, and those defining the graph and its map need not be modified. This would not be the case if we represented crossings by new vertices.

Hence, one goes from one level of the hierarchy graph/map/sketch to the next one below by discarding information, and to the next one above by determining one or more new relations. "Determining" means either computing by an algorithm or specifying by a logical formula. For instance, it is proved in [4, Theorem 3.1] that the unique planar map of a planar 3-connected graph can be defined from the graph itself, by a Monadic Second-Order formula.

We come to the motivation for the use of Monadic Second-Order logic. Graphs, and related structures like hypergraphs and partial orders can be represented by logical structures and their properties can be expressed by logical formulas. *Monadic second-order* (MS) logic (i.e., the extension of first-order logic with quantifications over set variables) is especially interesting for this purpose, because every property expressible by an MS formula is decidable in linear time on graphs of bounded tree-width, or on graphs structured hierarchically in a similar way. Furthermore, optimization functions and counting problems based on MS formulas are also evaluable efficiently on these graphs. We refer the reader to the survey [5] on these algorithmic issues.

Quite a number of graph theoretic notions, like colorings, paths, spanning trees can be expressed in MS logic. On the one hand, the expressive power is "large" but, on the other, the construction of formulas requires sometimes a fine analysis of the properties and may give new insight. For instance, a result of [4, Proposition 2.3] establishes that the planarity of a map is MS expressible, by using a characterization of planarity by forbidden configurations (Proposition 2.2). The characterization based on the number of cycles in certain permutations associated with the map (a theorem of Jacques [9] which is based on Euler's relation, see the survey [1]) is not (as such) MS expressible, because one cannot handle arithmetic in MS logic (at least as we handle it).

In the definition of a sketch, we use several relations that in total contain redundant information. It follows that not every logical structure of the appropriate type is a sketch, even if the underlying map is planar. Our main problem will be to find a combinatorial characterization of sketches that is expressible in monadic second-order logic.

We introduce *protosketches*. A protosketch is a logical structure R that contains the specification of a graph denoted by $G(R)$ (to be a little more concrete, let us say that the domain of R consists of the vertices and the edges of this graph), the specification of a map $M(R)$ of $G(R)$ (that is not necessarily planar) and, in addition, two relations $cross_R$ and $before_R$ that describe edge crossings. A protosketch R is a sketch if and only if it describes a drawing of $G(R)$, with edge crossings as specified by these relations. If R is a sketch and $cross_R$ and $before_R$ are empty, then $M(R)$ is a planar map. However, there exist drawings with edge crossings such that the corresponding map is planar. Hence, we may have a sketch R with nonempty relations $cross_R$ and $before_R$ such that $M(R)$ is planar.

Whether a structure R is a sketch is (easily) expressible in second-order logic, with quantifications over relations of arity more than one. We raise the problem whether this property is MS-expressible (i.e., expressible in monadic second-order logic), and we prove that it is in the special case of *framed protosketches*.

A framed protosketch represents a planar embedding of a connected graph (called the *frame*) augmented with additional edges that may cross one another and that may cross the edges of the frame. We prove the 3-*Edge Theorem* stating that a framed protosketch R is a sketch if every substructure of R induced by the frame together with any set of at most three edges not in the frame is a sketch. The desired logical characterization follows easily.

Most of the paper is devoted to graph theoretical constructions, hopefully interesting on their own. Furthermore, a sketch, which is a finite logical structure, can be given as input to a graph drawing system. However, we do not develop here the applications to graph drawing algorithms.

The construction of MS formulas will follow immediately from the combinatorial results. We assume that the reader who is interested in the logical aspects knows MS logic, say from [3], [4] or [5]; hence we do not repeat the definitions.

The paper is organized as follows. Sections 1 and 2 introduce notation and definitions. Section 3 introduces framed protosketches. The 3-Edge Theorem for protosketches having a Hamiltonian cycle as frame is proved in Section 4, and then extended in Section 5 to the case of those having a tree as frame, which gives the general case.

1. Preliminaries

1.1. Notation

If $S \subseteq D^n$, $n \geqslant 2$, and $d_1, \ldots, d_m \in D$ with $m < n$, we let

$$S[d_1, \ldots, d_m] = \{(d_{m+1}, \ldots, d_n)/(d_1, \ldots, d_n) \in S\}.$$

If $\leqslant_\alpha$ is a partial order on a set D, we denote by $<_\alpha$ this corresponding strict partial order. If $\leqslant_\alpha$ is linear, we let

$$x_1|_\alpha x_2|_\alpha x_3|_\alpha \ldots |_\alpha x_n$$

mean that either

$$x_1 <_\alpha x_2 <_\alpha x_3 <_\alpha \cdots <_\alpha x_n$$

or

$$x_n <_\alpha x_{n-1} <_\alpha \cdots <_\alpha x_2 <_\alpha x_1.$$

We write

$$x_1\|_\alpha x_2\|_\alpha x_3\|_\alpha \ldots \|_\alpha x_n$$

if for some $i = 1,\ldots,n$ we have

$$x_i|_\alpha x_{i+1}|_\alpha \ldots |_\alpha x_n|_\alpha x_1|_\alpha x_2|_\alpha \ldots |_\alpha x_{i-1}.$$

We write

$$x_1 \lessdot_\alpha x_2 \lessdot_\alpha x_3 \lessdot_\alpha \cdots \lessdot_\alpha x_n$$

if $n \geqslant 3$ and for some $i = 1,\ldots,n$ we have

$$x_i <_\alpha x_{i+1} <_\alpha \cdots <_\alpha x_n <_\alpha x_1 <_\alpha x_2 <_\alpha \cdots <_\alpha x_{i-1}.$$

Hence $\lessdot_\alpha$ is a relation of variable arity at least 3.

1.2. Graphs

By a graph we will always mean a finite, directed, loop-free graph.

We will denote by V_G the set of vertices of a graph G, by E_G its set of edges; we will write $e : x \to y$ if e is an edge linking x to y; the vertex x is the *source* of e, denoted by $s(e)$, and y is its *target*, denoted by $t(e)$. We denote by $E_G(x)$ the set of edges incident with x. We say that an edge links x *and* y if it links x to y or y to x.

We let $inc_G = \{(e,x,y) \mid e : x \to y\}$. A *path* from x to y is a sequence of edges $(e_1, e_2, \ldots, e_n)$ such that for some $x_1,\ldots,x_n \in V_G$ we have $x_1 = x$, e_i links x_i and x_{i+1} for $i = 1,\ldots,n-1$, e_n links x_n and y, and the vertices x_i are pairwise distinct and each x_i for $i > 1$, is distinct from y. However, x and y may be equal, and the path is called a *cycle.* A *directed path* is similar with $e_i : x_i \to x_{i+1}$ and $e_n : x_n \to y$. Occasionally, we will consider empty paths, denoted as ( ). A *circuit* is a directed path from a vertex to itself.

We let $deg(G)$ denote the maximum degree of a vertex of G.

If X is a set of edges of G, we denote by $G[X]$ the subgraph of G consisting of the edges of X and their vertices. If Y is a set of vertices, we denote by $G[Y]$ the graph $G[X]$ where X is the set of all edges of G having there vertices in Y. Such a graph is an *induced subgraph* of G.

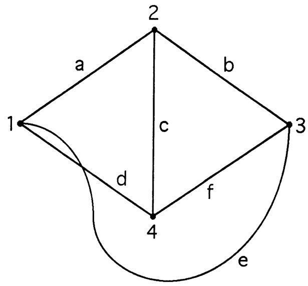


Fig. 1.

1.3. Maps

A *map* is a pair $M = \langle G, sigma \rangle$ where G is a graph (satisfying the general requirement of (1.2)) and *sigma* is a mapping: $V_G \to \mathscr{P}(E_G \times E_G)$ associating with every $v \in V_G$ a circular permutation of $E_G(v)$. We will frequently specify $sigma(v)$ as $\{e_1 \to e_2 \to e_3 \to \cdots \to e_{k-1} \to e_k \to e_1\}$ where $\{e_1, e_2, e_3, \ldots, e_{k-1}, e_k\}$ is an enumeration of $E_G(v)$. ($\mathscr{P}(E)$ denotes the set of subsets of E). Maps, surveyed in [1], and used in [8] for graph drawing algorithms, are defined as pairs of permutations over sets of "half-edges", i.e., of pairs consisting of one vertex and one incident edge, satisfying additional properties. Our definition is essentially equivalent, but it defines the map as an extension of the graph by some new information. Hence, it fits the requirement presented in the introduction of having a hierarchy of richer and richer structures where each level adds information without modifying the previous level.

In a *drawing* of a graph G, the curve segments representing edges may cross. By an *embedding* of G, we mean a drawing without edge crossings, i.e., a *planar embedding* in the terminology of books like [6] or [10]. Since we will not consider surfaces other than the plane, we will not specify "planar" when discussing embeddings.

To every drawing of a graph G corresponds a map $M = \langle G, sigma \rangle$ where $sigma(v)$ represents the order in which the edges incident with v appear in this drawing if we sweep the plane around from v in the trigonometric sense. If we do that in the clockwise sense, we get the map $M' = \langle G, sigma' \rangle$ where $sigma'(v) = sigma(v)^{-1}$ for each v.

Fig. 1 shows a drawing D of a graph G; the corresponding map is $M = \langle G, sigma \rangle$ where $sigma(1) = \{a \to d \to e \to a\}$, $sigma(2) = \{a \to c \to b \to a\}$, $sigma(3) = \{b \to f \to e \to b\}$, $sigma(4) = \{c \to d \to f \to c\}$.

A map is *planar* if it is associated with an embedding of a graph.

If $M = \langle G, sigma \rangle$ is a map, we denote by $\leqslant_{M,v}$ the relation (of variable arity at least 3) on the set $E_G(v)$, associated with the linear order $\{e_1 < e_2 < e_3 < \cdots < e_{k-1} < e_k\}$ where $sigma(v) = \{e_1 \to e_2 \to e_3 \to \cdots \to e_{k-1} \to e_k \to e_1\}$. If M is a map of G and G' is a subgraph of G, we will denote by $M[G']$ the unique map M' of G' such that $\leqslant_{M'}$ and $\leqslant_M$ coincide on the sets $E_{G'}(v)$ for $v \in V_{G'}$. If M is planar, representing an embedding D of G, then M' is planar and represents the embedding D' of G' obtained by removing from D the edges and vertices not in G'.

A map $M = \langle G, sigma\rangle$ will be represented by the logical structure $|M|_2 = \langle V_G \cup E_G, inc_G, sig_M\rangle$ where $sig_M := \{(v,e,f) \mid v \in V_G, (e,f) \in sigma(v)\}$. (In the above example, sig_M contains $(1,a,d)$, $(1,d,e)$, $(1,e,a)$.) This logical structure contains the structure $|G|_2 = \langle V_G \cup E_G, inc_G\rangle$ representing G. It is proved in [4, Proposition 2.3] that the property "M is a planar map" is expressible in MS logic over the structure $|M|_2$.

The map of an embedding of a connected graph characterizes up to homeomorphism its embeddings (without edge crossings) on the *sphere* (we recall that the plane is homeomorphic to the sphere minus one point). In order to have a combinatorial representation of embeddings in the plane, one needs to only specify the external face, that is the infinite region of the plane delimited by the curve segments forming the embedding. This can be done by a pair of edges (e, f) that share a vertex v of degree at least 3 and are such that f follows e in the circular order $sigma(v)$. This pair represents a "corner" of the infinite region. If in the drawing of Fig. 1 we omit edge e, we obtain an embedding. The external face can be specified by the pair (b, a) or by (d, f). The corresponding map together with such a pair of edges specifies a unique embedding, where unicity holds up to homeomorphisms of the plane. (See [10] on topological background, especially Chapter 3.)

1.4. Monadic second-order transductions

All logical structures will be finite because they are intended to represent graphs that will be themselves always finite.

An essential tool is the notion of an *MS-transduction*, an adaptation to monadic second-order logic (called MS logic for short) of the general notion of an *interpretation*, used in particular by Rabin [11]. The idea of an interpretation is to build a structure T from a given structure S, by defining the domain of T as a subset of a Cartesian power of that of S by means of a fixed logical formula written with the symbols of S, and by defining the relations of T by fixed logical formulas of the same language. A fundamental fact is the *Backwards Translation Lemma* saying that a property of the object structure T can be expressed in the input structure S by a logical formula constructed from that expressing the property of T and those expressing the transformation of S into T, in such a way that this construction is valid for all structures S and T of appropriate type. This holds true if formulas are either all first-order or all second-order. However, if we want to translate an MS property of T into one of S, assuming that the transformation of S into T is also specified by MS formulas, we must restrict the domain of T to be a subset of that of S (and not of a Cartesian power of the domain of S).

To overcome this restriction, we use variants of interpretations called *MS transductions* that transform a structure S with domain D into a structure T with domain a subset of $D \times \{0, \dots, n\}$, for some fixed integer n. The domain and the relations of T are specified by MS formulas evaluated in S. Thus, T can have a *larger* domain than S. These transformations satisfy the *Backwards Translation Lemma* for properties of S and T expressed in MS logic. See [2] or [3] (especially Section 5.5.2) about MS-transductions.

A class of structures is *MS-definable* if it is the class of finite models of an MS formula. The inverse image of an MS definable class of structures by an MS transduction is MS definable, by the Backwards Translation Lemma. For more about monadic second-order logic, the reader is referred to [2–4].

2. Drawings

The notion of a map is appropriate to represent embeddings of (planar) graphs. Our objective is to enrich it with additional relations representing crossings of edges in drawings of graphs. In the example of Fig. 1, the edge e crosses the edge d, but one could also draw the *same map* with edge e crossing c and f, and get thus a different drawing.

We will not handle arbitrary drawings. Edge crossings will be limited in the following ways:

(1) an edge does not cross itself;

(2) two edges cross at most once (since we admit multiple edges, the Jordan arcs representing two edges can have at most 3 points in common);

(3) no three edges cross at any point.

2.1. Definitions: sketch, protosketch

Let G be a graph (which is, by the general assumption of Section 1.2, finite, directed, loop-free); let D be a drawing of G in the oriented plane. The graph is defined by V_G, E_G, inc_G. From this drawing we get a map $M(D)$ as recalled in Section 1.3.

We let $cross_D \subseteq E_G \times E_G$ be such that $(e,f) \in cross_D$ if and only if e and f cross with f going from the left of e to its right (we recall that edges are directed). Hence $(e,f) \in cross_D \Rightarrow (f,e) \notin cross_D$. If e and f cross in a drawing D, we denote by $c_D(e,f)$ their crossing point in the plane (thus $c_D(e,f) = c_D(f,e)$).

We let $before_D \subseteq E_G \times E_G \times E_G$ be such that (e,x,y) belongs to $before_D$ if and only if x and y are edges crossing e, $x \neq y$ (whence $c_D(e,x) \neq c_D(e,y)$), and $c_D(e,x)$ is before $c_D(e,y)$ on the arc representing e.

Hence $cross_D$ is empty if and only if D has no edge crossing, and these conditions imply that $before_D$ is empty.

With D, we associate the relational structure $R(D) = \langle V_G \cup E_G, inc_G, sig_{M(D)}, cross_D, before_D \rangle$ called its *sketch*. Our main question will be to characterize sketches by a logical formula, hopefully of monadic second-order logic.

A relational structure $R = \langle W, inc, sig, cross, before \rangle$ where *cross* is binary and the other relation symbols are ternary is a *protosketch* if it satisfies the following conditions:

(P_1) $\langle W, inc \rangle = |G|_2$ for some graph G.

If G exists, it is unique. Assuming (P_1) the next conditions are:

(P_2) $\langle W, inc, sig\rangle$ is a map of G;

(P_3) $\forall e, e' \in W,\ cross(e,e') \Rightarrow e, e' \in E_G \wedge e \neq e' \wedge \neg cross(e', e)$.

For every $e \in E_G$, we let $K(R,e) := \{f \mid cross_R(e,f) \vee cross_R(f,e)\}$. The last condition is:

(P_4) For every e in W, the relation $before_R[e]$ is a strict linear order on the set $K(R,e)$ (this set may be empty).

It is clear from the definitions that every sketch is a protosketch.

With a protosketch R we associate the underlying graph $G(R)$ (it exists by P_1), the underlying map $M(R)$ of $G(R)$ (it exists by P_2), the set $X(R) := \{\{e,f\} \mid cross_R(e,f)\}$, called the set of *crossings* of R. (A crossing is an unordered pair.)

We will associate with every protosketch R a map $M(\hat{R})$ such that R is a sketch, i.e. $R = R(D)$ for some drawing D if and only if $M(\hat{R})$ is planar. The definition of $M(\hat{R})$ is based on the following observation. Every drawing D is actually a drawing without edge crossings, call it $\hat{D}$, of a graph H having as vertices those of G together with the edge crossings of D, and having as edges the portions of edges of G delimited by the crossings. This drawing $\hat{D}$ can be handled as a planar map of H.

This observation is now made into a formal definition (slightly more general than described here; the generalization will be used in Section 4).

2.2. Definition: the map representing a sketch

Our objective is to associate a map with a sketch by introducing new vertices for crossings.

Let R be a protosketch, let $G = G(R)$. For every e in E_G, we let $K^+(R,e)$ be the sequence $(s(e), c_1, \ldots, c_k, t(e))$ where $c_1, \ldots, c_k$ is the sequence of crossings of e ordered according to $before_R[e]$.

Let $U \subseteq X(R)$ be a set of crossings of R. (U is a set of "formal" crossings; we do not assume here the existence of a drawing D of which R is a sketch.) We define a protosketch R' that, intuitively, turns the elements of U into new vertices.

We let G' be a directed graph such that $V_{G'} = V_G \cup U$, and $E_{G'} = \{(e,x,y) \mid e \in E_G, y$ follows x in the restriction of the sequence $K^+(R,e)$ to the set $U \cup \{s(e), t(e)\}\}$.

In G', the source of (e,x,y) is x and its target is y. Hence the edges of G' are obtained from those of G by subdivisions by means of the "new" vertices from U. We now define a map M' of G'. We first define $sig_{M'}[x]$ for $x \in V_G$, (x is an "old" vertex):

$$sig_{M'}[x] = \{(e,f) \mid e,f \in E_{G'}(x),\ (\pi_1(e), \pi_1(f)) \in sig_M[x]\}.$$

(The edges of G' are triples (h,u,v), $h \in E_G$; we let $\pi_1(h,u,v) = h$, $\pi_2(h,u,v) = u$, $\pi_3(h,u,v) = v$; π_i is the "ith projection".)

We now define $sig_{M'}[u]$, for $u \in U$. We let $u = \{e,f\}$ where $cross_R(e,f)$ holds. There exist edges $e_1, e_2, f_1, f_2 \in E_{G'}$ such that $\pi_1(e_1) = \pi_1(e_2) = e$, $\pi_1(f_1) = \pi_1(f_2) = f$, $t(f_1) = t(e_1) = s(e_2) = s(f_2) = u$. We let $sig_{M'}[u] := \{(e_1, f_2), (f_2, e_2), (e_2, f_1), (f_1, e_1)\}$.

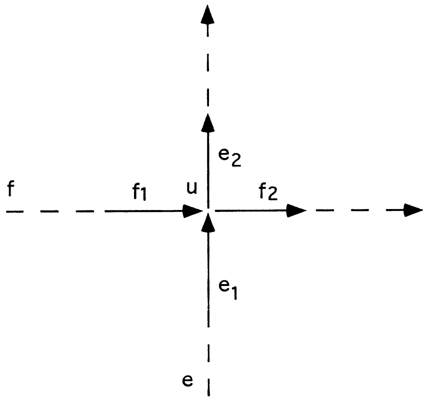


Fig. 2.

See Fig. 2. For latter use, the edges e_1, f_2, e_2, f_1 will be denoted, respectively, by $edg_1(u), edg_2(u), edg_3(u), edg_4(u)$.

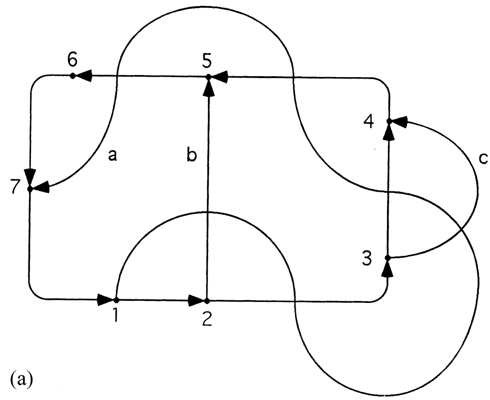
In order to have a protosketch R' such that $G(R') = G'$, $M(R') = M'$, we need also define $cross_{R'}$, and $before_{R'}$. We let $cross_{R'}(e', f')$ hold iff
(1) $e', f' \in E_{G'}$,
(2) $cross_R(\pi_1(e'), \pi_1(f'))$ holds,
(3) $\{\pi_1(e'), \pi_1(f')\} \notin U$,
(4) $\pi_1(f')$ is between $\pi_2(e')$ and $\pi_3(e')$ on $K^+(R, \pi_1(e'))$,
(5) $\pi_1(e')$ is between $\pi_2(f')$ and $\pi_3(f')$ on $K^+(R, \pi_1(f'))$.

For every (e, f) such that $cross_R(e, f)$ holds and $\{e, f\} \notin U$, there exists a unique pair (e', f') such that $cross_{R'}(e', f')$ holds and $e = \pi_1(e'), f = \pi_1(f')$. We have in particular $e' = (e, \alpha, \beta)$, $f' = (f, \gamma, \delta)$ where $\alpha, \beta, \gamma, \delta$ are determined in a unique way by (4) and (5).

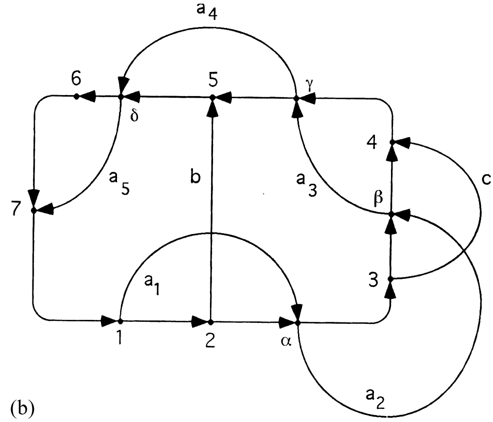
It remains to define $before_{R'}$. If $e' \in E_{G'}$, $e' = (e, \alpha, \beta)$ then we obtain $before_{R'}[e']$ from the sequence $before_R[e] = (f_1, \ldots, f_k)$ as follows: we let $before_{R'}[e'] = (f'_{i_1}, \ldots, f'_{i_\ell})$ where $i_1, \ldots, i_\ell$ is the subsequence of elements i of $1, \ldots, k$ such that $\{e, f_i\} \notin U$ and f'_i is the unique edge of G' such that $\pi_1(f'_i) = f_i$ and either $cross_{R'}(e', f'_i)$ or $cross_{R'}(f'_i, e')$ holds. We will denote R' by $R + U$.

In the example of Figs. 3(a) and (b) the protosketch R' is obtained from R by letting $U = \{\{a, (2,3)\}, \{a, (3,4)\}, \{a, (4,5)\}, \{a, (5,6)\}\}$. The corresponding new vertices are named $\alpha, \beta, \gamma, \delta$, respectively. The edge a is subdivided into a_1, a_2, a_3, a_4, a_5 where $a_1 = (a, 1, (2,3))$, $a_2 = (a, (2,3), (3,4))$, $a_3 = (a, (3,4), (4,5))$, $a_4 = (a, (4,5), (5,6))$, $a_5 = (a, (5,6), 7)$. We have $cross_R = \{(b, a), (a, c), ((2,3), a), (a, (3,4)), ((4,5), a), (a, (5,6))\}$ and $cross_{R'} = \{(b, a_1), (a_2, c)\}$.

Proposition 2.1. *Let R be a protosketch and U be a set of crossings. Then R is a sketch if and only if $R + U$ is a sketch.*



(a)



(b)

Fig. 3.

Proof. Let $R = R(D)$ where D is a drawing. The lines forming D yield a drawing D' of $G(R+U)$ and from the construction of $R+U$ we have $R(D') = R+U$.

Let us conversely assume that D' is a drawing of $R+U$. The edges of $G(R)$ are paths in $G(R+U)$. A curve segment representing e in D can be obtained by concatenating the segments of D' representing the edges (e,x,y) of $G(R+U)$. From the definition of $sig_{R+U}[u]$ for $u = \{e,f\} \in U$, it follows that the segments representing e and f cross. Hence D is a drawing of $G(R)$, and $R = R(D)$. $\square$

We let $\hat{R}$ denote $R + X(R)$. Hence, $\hat{R}$ is made from R by turning *all crossings* into new vertices, and $X(\hat{R}) = \emptyset$. The following corollary is immediate.

Corollary 2.2. *A protosketch R is a sketch if and only if the map $M(\hat{R})$ is planar.*

One can use this construction in an algorithm testing whether a protosketch is a sketch. Let the size of a structure R be the cardinality of its domain plus the sum of cardinalities of its relations. The size of $\hat{R}$ is $\mathbf{size}(R) + O(\mathbf{Card}(U))$, hence it is $O(\mathbf{size}(R))$. Since one can decide in linear time whether a map is planar (one can use

Euler's formula, because the number of faces is easy to obtain from R, see [1]), one can decide in linear time (in terms of its size) whether a protosketch R (or even a structure R) is a sketch.

For drawing a graph according to a given sketch R: it suffices to build the planar map $M(\hat{R})$ (which can be done in linear time) and to give it as input to a drawing algorithm like that given by Hanusse [8] which draws graphs from planar maps.

Corollary 2.3. *The sketch R of a drawing of a connected graph and a pair of edges of $M(\hat{R})$ defining an external face specify the drawing up to a homeomorphism of the plane.*

This corollary is an immediate consequence of the proof of Proposition 2.1 and the analogous result for maps and planar embeddings of connected graphs.

Proposition 2.4. (1) *That a structure R is a protosketch can be expressed by a monadic second-order formula.*

(2) *That a protosketch is a sketch is expressible by a second-order formula.*

Proof. (1) Conditions P_1, P_3, P_4 are first order. Condition P_2 is not first order because one needs to check that certain binary relations are circuits. However, it is MS-expressible. (The construction is as follows. A binary relation A on a finite set X is a circuit if and only if for every x in X there are unique y and z such that $A(x,y)$ and $A(z,x)$ hold, and X has no nonempty proper subset Y such that for all y in Y and x in X, if $A(y,x)$ holds then x is also in Y.) Hence, the property that R is a protosketch is MS-expressible.

(2) Consider the transformation of structures: $R \mapsto |M(\hat{R})|_2$ where $R = \langle W, inc, sig, cross, before \rangle$. The domain of $M(\hat{R})$ is defined as a subset of $W' \cup W''$, where $W' = W \cup W \times W$ (for the vertices) and $W'' = W \times W \times W \cup W \times (W \times W) \times W \cup W \times W \times (W \times W) \cup W \times (W \times W) \times (W \times W)$ (for the edges).

The relations of the structure $M(\hat{R})$ can be defined by MS-formulas on R. The planarity of the map defined by $M(\hat{R})$ is MS-expressible [4] (Proposition 2.3) hence, by the Backwards Translation Lemma discussed in Section 1.4, it is expressible in R by a second-order formula. The details of the construction would be lengthy (and will not be used in the sequel). $\square$

We use here the method of interpretations (recalled in Section 1.4) but for a transformation which is not an MS transduction, because the domain of $M(\hat{R})$ is a union of subsets of Cartesian powers of W and not a subset of $W \times \{1,\dots,n\}$ for any fixed n.

An example can show that this cannot be avoided by another construction. We let G_n denote a square grid with internal vertices defined as edge crossings. We show the grid G_3 in Fig. 4. The grid G_n has vertices $a,b,c,d,\ a_1,\dots,a_n,\ b_1,\dots,b_n,\ c_1,\dots,c_n,\ d_1,\dots,d_n$, and $4(n+1)+2n$ edges that we do not name. The cardinality of the domain of $M(\hat{G}_n)$ is $(n+2)^2+2(n+1)(n+2)$; hence it is not linearly bounded in terms of that of G_n.

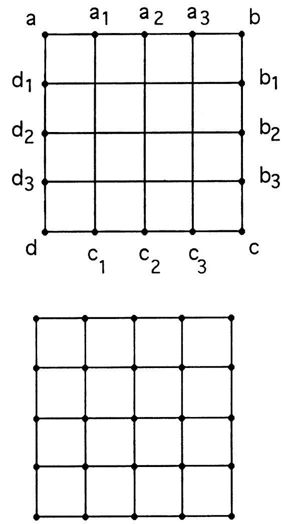


Fig. 4.

This example shows that the cardinality of the domain of $|M(\hat{R})|_2$ is not linearly bounded in terms of the cardinality of the domain of R. Hence the mapping $R \mapsto |M(\hat{R})|_2$ is not an MS-transduction.

Moreover, and even worse, some MS properties of $\hat{R}$ cannot be translated into MS properties of R. To prove this, we consider G_n, and we recall that the MS-theory of $\{M(\hat{G}_n) \mid n \geqslant 1\}$, an infinite set of square grids, is undecidable (one cannot decide whether a given MS formula is satisfied by some structure in the set). On the other hand, each structure G_n can be obtained by an MS-transduction from the structure representing the finite word $pq^n r$. Since the MS-theory of these latter structures is decidable, so is that of the structures $\{G_n \mid n \geqslant 1\}$ by the Backwards Translation Lemma recalled in Section 1.4. If there existed an algorithm transforming an MS formula to be interpreted in $|M(\hat{R})|_2$ into an equivalent MS formula in R, then the MS-theory of $\{|M(\hat{G}_n)|_2 \mid n \geqslant 1\}$, would be decidable, which is not the case.

We now ask the following question:

Question 2.5. *Is the class of sketches MS definable*?

We will not be able to answer this question but we will establish the MS-definability of two special classes of sketches, first of the class of k-bounded sketches and later, of the class of framed sketches. These two classes are incomparable.

A protosketch R is *k-bounded* if there exist two disjoint sets $Y, Z \subseteq E_R$ such that:
(i) $cross_R \subseteq Y \times E_R \cup E_R \times Y \cup Z \times Z$,
(ii) Z has cardinality at most k,
(iii) every edge of Y is crossed by at most k edges.
The set $\{G_n \mid n \geqslant 1\}$ considered above is not k-bounded for any k.

Proposition 2.6. *For every k, the class of k-bounded sketches is* MS *definable.*

Proof. One can construct an MS formula $\varphi(Y,Z)$ expressing that Y and Z are sets of edges witnessing that the considered protosketch R is k-bounded.

From such a pair (Y,Z), one can define the set $U = X(R)$ of the definition of $M(\hat{R})$ as a set P of pairs (e,i) such that $1 \leqslant i \leqslant k$ and, either $e \in Y$, and (e,i) encodes the ith crossing of e with an edge, in the order specified by $before_R[e]$, or $e \in Z$ and (e,i) encodes the ith crossing of e with an edge of Z, in the order specified by $before_R[e]$.

It may happen that (e,i) and (e',j) encode the same crossing. This can be expressed by an MS-formula $\tau_{i,j}(e,e')$. Hence, U can be defined as a quotient set of P by an MS-definable equivalence relation. We omit details.

To every crossing u correspond four new edges denoted by $edg_j(u)$, for $j = 1,\ldots,4$, see Definition 2.2 and Fig. 2. Hence, if (e,i) encodes a crossing u of R, the four edges of $M(\hat{R})$ incident to the vertex u can be encoded by $(e,(i,j))$, for $j = 1,\ldots,4$. These pairs (i,j) can be mapped onto integers between $k+1$ and $5k$. Again, certain edges can be encoded by several pairs $(e,(i,j))$. This can be detected by MS formulas and a quotient set for the appropriate equivalence relation can be defined by an MS transduction.

It follow that the domain of the structure $\hat{R}$ can be defined as a subset of $D_R \times \{0,\ldots,5k\}$. (An element x of the domain of R that is also in that of $\hat{R}$ will be handled as the pair $(x,0)$. This is the case of the vertices of $G(R)$ and of its edges that are not crossed.)

It remains to specify the relations *inc* and *sig* of $\hat{R}$ by MS formulas. (The relations *cross* and *before* are empty.) This is possible by translating adequately the definitions of Section 2.2.

Hence the transformation $R \mapsto |M(\hat{R})|_2$ restricted to k-bounded protosketches is an MS transduction. It follows from the *Backwards Translation Lemma* recalled in Section 1.4 that the planarity of $M(\hat{R})$, which is expressible by an MS formula evaluated in $|M(\hat{R})|_2$ is also expressible in R by an MS formula. This is equivalent to the fact that R is a sketch by Corollary 2.2 and completes the proof. □

3. Framed sketches

Let R be a protosketch. We recall that $G(R)$ denotes the underlying graph and $M(R)$ the underlying map of $G(R)$. They both exist by conditions P_1 and P_2 of Definition 2.1.

We say that R is *planar* if $X(R) = \emptyset$, and $M(R)$ is a planar map. If $Y \subseteq E_R$ we denote by $R[Y]$ the restriction of R to the set Y: it is the protosketch R' such that $G(R') = G(R)[Y]$, $M(R') = M(R)[G(R)[Y]]$, $cross_{R'} = cross_R \cap (Y \times Y)$, and $before_{R'} = before_R \cap (Y \times Y \times Y)$. It is clear that $R[Y]$ is a sketch if R is. If H is a subgraph of $G(R)$ without isolated vertices, we will denote $R[E_H]$ by $R[H]$.

A subgraph F of $G(R)$ is a *frame* of R if it is connected, $V_F = V_{G(R)}$ and $R[F]$ is planar. We have, in particular, $cross_R \cap (E_F \times E_F) = \emptyset$ and $before_R \cap (E_F \times E_F \times E_F) = \emptyset$, and the graph $G(R)$ is connected. An example of a sketch without frame is that associated with a drawing of a path a—b—c—d such that edges a—b and c—d cross. The next two sections will be devoted to the proof of the following result.

Theorem 3.1 (The 3-edge theorem). *A protosketch R with frame F is a sketch if and only if $R[E_F \cup X]$ is a sketch for every set X of at most 3 edges in $E_{G(R)} - E_F$.*

We can state and prove immediately the following:

Corollary 3.2. *The class of framed protosketches is* MS-*definable.*

Proof. One can construct an MS-formula $\varphi_1(U)$ expressing in the structure R (assumed to be a protosketch) that U is the set of edges of a subgraph F of $G(R)$ that is a frame.

Let $X \subseteq E_R - U$ have cardinality at most 3. The protosketch $R[U \cup X]$ is 3-bounded: one takes $U = E_F$ as set Y and X as set Z in the definition of k-boundedness; since F is planar, no two edges in Y can cross, hence each of them can only cross the edges in X which are at most three; finally, all other crossings are between edges in X. Hence, by Proposition 2.6, one can express by an MS formula interpreted in R, call it $\varphi_2(X, U)$, that $R[U \cup X]$ is a sketch. By the 3-edge theorem, the fact that R, assumed to be a framed protosketch, is a sketch is expressed by the MS-formula:

$$\exists U\ [\varphi_1(U) \wedge \forall X(\text{“}\mathbf{Card}(X) \leqslant 3\text{”} \wedge \text{“}X \subseteq E_R - U\text{”} \Rightarrow \varphi_2(X, U))]. \qquad \Box$$

4. Hamiltonian protosketches

This section is the most technical part of the paper. We establish the 3-edge theorem for protosketches R having a frame F which is a Hamiltonian cycle of $G(R)$.

A Hamiltonian cycle F of a graph G will be *oriented*, that is, given by an enumeration $(x_1, x_2, \ldots, x_n)$ of V_G and a sequence of edges $(e_1, e_2, \ldots, e_n)$ such that e_i links x_i and x_{i+1} for $i = 1, \ldots, n$ (we let $x_{n+1} = x_1$). We let $E_F = \{e_1, \ldots, e_n\}$. (Note that F is not necessarily a circuit, because its edges can be directed in either direction.)

Letting F be so, a drawing of G is *F-internal* if F is drawn without edge crossings, $(x_1, \ldots, x_n)$ is a traversal of F in the trigonometric sense of the plane, and all edges not in F are inside the finite region of the plane defined by F. (They can cross one another but not those of F.) A protosketch R is *F-internal* if F is a Hamiltonian cycle of $G(R)$ and $R = R(D)$ for some F-internal drawing D of $G(R)$.

We say that D is *F-outerplanar* (resp. that R is *F-outerplanar*) if D is F-internal without edge crossings (resp. $R = R(D)$ for an F-outerplanar drawing D). Hence, $G(R)$ is here planar.

Proposition 4.1. *Let R be a protosketch such that $G(R)$ has a Hamiltonian cycle F. It is F-internal if $R[E_F \cup X]$ is F-internal for every set X of at most 3 edges in $E_{G(R)} - E_F$.*

The proof is relatively long. We let $G=G(R)$ and $M=M(R)$. If R satisfies the hypotheses of Proposition 4.1 then we have necessarily:

$$(\mathrm{H}_0)\qquad \begin{cases} cross_R \subseteq (E_G - E_F)\times(E_G - E_F),\\ before_R \subseteq (E_G - E_F)\times(E_G - E_F)\times(E_G - E_F).\end{cases}$$

The crux of the proof will concern F-internal protosketches R such that $deg(G(R))\leqslant 3$.

We define four conditions, which are necessary for such a protosketch R to be F-internal. Their meaning is clear from the consideration of an F-internal drawing D, such that $R=R(D)$. The following notation will be used:

$<_F$ for the strict linear order on V_G such that $x_1 <_F x_2 <_F \cdots <_F x_n$, where F is given by $(x_1,\ldots,x_n)$ and $E_F=\{e_1,\ldots,e_n\}$; see above;

M for the map of G associated with R; we will use the relation $\ll_{M,x}$ defined in Definition 1.3;

$<_f$ for the strict linear order of the sequence $K^+(R,f)$ defined from $before_R[f]$ (Definition 2.2; the associated notation $\|_f$ will be used, see Section 1.1).

We are ready to define the four conditions.

(H_1) For every $f\in E_G-E_F$, if f is incident with $x=x_{i+1}$ $(1\leqslant i\leqslant n)$ we have, letting $e_{n+1}=e_1$:

$$e_i \ll_{M,x} e_{i+1} \ll_{M,x} f.$$

(H_2) For every $f,g\in E_G - E_F$, $f\neq g$, where f links u to w and g links x to v, we have

$$u \ll_F v \ll_F w \ll_F x \quad\Leftrightarrow\quad cross_R(f,g)$$

and

$$u \ll_F x \ll_F w \ll_F v \quad\Leftrightarrow\quad cross_R(g,f).$$

The first case is shown in Fig. 5(a).

(H_3) For every pairwise distinct $f,g,h\in E_G - E_F$, where f links u to w, g links x and y, h links x' and y', if

$$u \ll_F x \ll_F x' \ll_F w \ll_F y' \ll_F y$$

then $before_R(f,g,h)$ holds. This case is shown in Fig. 5(b).

(H_4) For every pairwise distinct $f,g,h\in E_G - E_F$, where f links u and w, g links x' and y, h links x and y', if

$$u \ll_F x \ll_F x' \ll_F w \ll_F y' \ll_F y$$

then either

$$u \|_f\, g \|_f\, h \|_f\, w \quad\text{and}\quad y \|_g\, f \|_g\, h \|_g\, x' \quad\text{and}\quad y' \|_h\, f \|_h\, g \|_h\, x$$

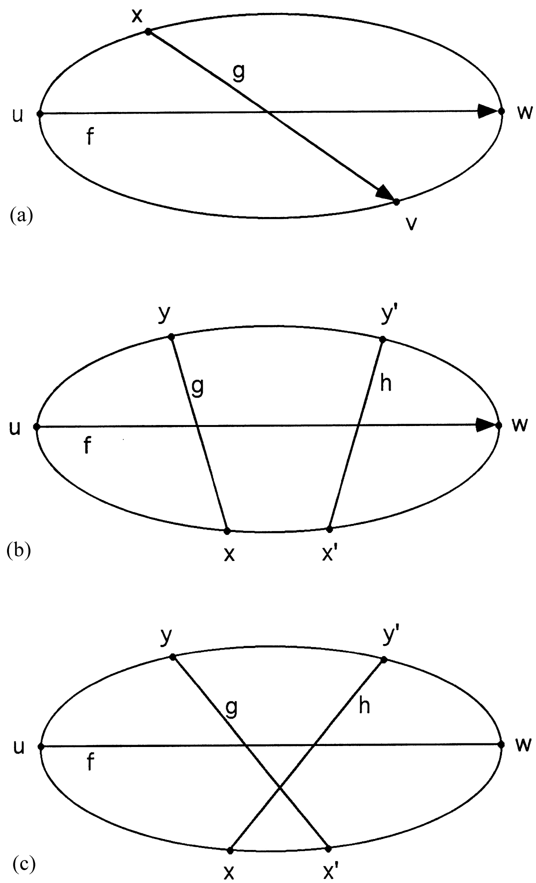


Fig. 5.

or

$$u\|_f h\|_f g\|_f w \quad \text{and} \quad x'\|_g f\|_g h\|_g y \quad \text{and} \quad x\|_h f\|_h g\|_h y'.$$

See Fig. 5(c) for the first case.

Lemma 4.2. *Let G be a graph of degree 3 with Hamiltonian cycle F. If D is an F-internal drawing of G then the sketch $R(D)$ and F satisfy conditions* H_0–H_4.

Proof. Clear from the definitions. □

Our next lemma needs further definitions. We let R be a protosketch with $deg(G(R)) \leqslant 3$, and we assume that conditions H_0–H_4 hold. We let $N = E_G - E_F$.

A *small triangle* is a tuple $\Delta = (e, x, y_1, \dots, y_n, z, e')$ where $(x, y_1, \dots, y_n, z)$ is a path in F with $x \leqslant_F y_1 \leqslant_F \cdots y_n \leqslant_F z$, $e \in N, e$ links x and x', $x' \notin V(\Delta) = \{x, y_1, \dots,$

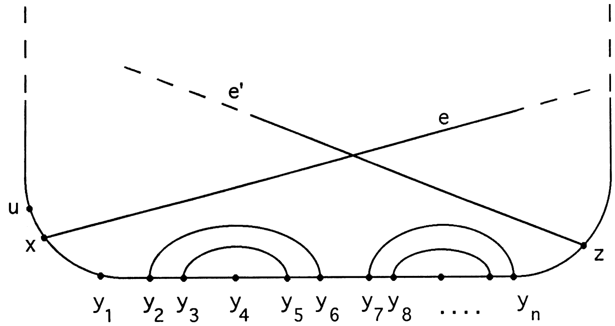


Fig. 6.

$y_n, z\}$, $e' \in N$, e' links z and $z', z' \notin V(\Delta)$, e and e' cross, there is no crossing on e between x and e', there is no crossing on e' between z and e, and finally:

(∗) if an edge in N is incident with some y_i, then its other end is in $V(\Delta)$ and it crosses no other edge of N.

A small triangle is shown in Fig. 6.

Lemma 4.3. *Let R be a protosketch with a frame F which is a Hamiltonian cycle of $G(R)$, such that $G(R)$ has degree at most* 3, *and conditions* H_0–H_4 *hold*:
(1) *If $X(R)=\emptyset$, then $R=R(D)$ for some F-outerplanar drawing D of $G(R)$.*
(2) *If $X(R)\neq\emptyset$, then R has a small triangle.*
(3) *If $X(R)\neq\emptyset$, then R is F-internal.*

Proof. (1) Let $G=G(R)$ with $X(R)=\emptyset$ and R is a protosketch. We have: $before_R=\emptyset$. If $f,g\in E_G-E_F$, $g\neq f, f$ links u to w and g links x to v, then we have

$$u \ll_F w \ll_F v \ll_F x \quad \text{or} \quad u \ll_F w \ll_F x \ll_F v, \text{ or}$$

$$u \ll_F v \ll_F x \ll_F w \quad \text{or} \quad u \ll_F x \ll_F v \ll_F w,$$

because in the other cases, we must have a crossing by H_2. It follows that G is outerplanar with Hamiltonian cycle F. (The proof is by induction on the number of edges not in F. There is in G an edge not in F linking vertices z and z', such that every vertex on one of the two paths in F linking z and z' has degree 2. One removes this edge and the same property holds for the subgraph obtained. One can add to an F-outerplanar drawing of this graph a curve segment to obtain an F-outerplanar drawing of G.)

Let us draw F on the plane in the trigonometric sense. There is a unique way (up to homeomorphism) to draw the edges of E_G-E_F in the finite region defined by F. By condition H_1 the map $M(D)$ of this drawing coincides with $M(R)$. Finally we get $R(D)=R$. Hence R is F-outerplanar.

(2) We now assume that $X(R)\neq\emptyset$, and we want to prove that R has a small triangle.

We choose an edge $e \in N$ and a path $(x, y_1, \ldots, y_m, x')$ in F such that e links x and x' and is crossed by at least one edge $f \in N$ and m is minimal such that these conditions can be satisfied. By H_2 one end of any such f is in $\{y_1, \ldots, y_m\}$ and the other is outside of $\{x, y_1, \ldots, y_m, x'\}$.

We let e' be the edge of N that crosses e next to x. It links some $y_{n+1}, 0 \leqslant n < m$ and some z' outside of $\{x, y_1, \ldots, y_m, x'\}$. We let $z = y_{n+1}$ and we consider $\Delta = (e, x, y_1, \ldots, y_n, z, e')$.

Case 1: There is no crossing on e' between z and e. We claim that Δ is in this case a small triangle. We need only consider the condition $(*)$ of the definition. Let g be an edge in N incident with y_i, $1 \leqslant i \leqslant n$. Its other end is y'. We have $x \lll_F y_i \lll_F z \lll_F x' \lll_F z'$. The vertex y' must be some y_j, $j \neq i$, $1 \leqslant j \leqslant n$ because otherwise, we have several cases leading to contradictions:

(i) $x \lll_F y_i \lll_F z \lll_F y' \lll_F x' \lll_F z'$. By H_3, g must cross e' between z and e but this contradicts the hypothesis;

(ii) $x \lll_F y_i \lll_F z \lll_F x' \lll_F y' \lll_F z'$. The edge g crosses e and e' by H_2, and then by H_4, one of these crossings must be on e between x and e' (but this contradicts the choice of e') or on between z and e but this contradicts the hypothesis;

(iii) $x \lll_F y_i \lll_F z \lll_F x' \lll_F z' \lll_F y' \lll_F x$; by H_3, the edge g crosses e between x and e', but this contradicts the choice of e', or on e' between z and e but this contradicts the hypothesis.

If this edge g is crossed by an edge of N, then it could have been chosen instead of e and this contradicts the minimality of m. Hence g is not crossed and we have a small triangle as desired.

Case 2: There is an edge in N crossing e' between z and e. Any such edge has one end in $\{y_1, \ldots, y_n\}$, otherwise by H_4, it would cross e between x and e' (contradicting the choice of e'). We let f be the unique one that crosses e' next to z. One of its ends is y_p. We now consider the tuple:

$$\Delta' = (f, y_p, y_{p+1}, \ldots, y_n, z, e').$$

If f has no crossing between y_p and e', then Δ' is a small triangle (the proof is completed as in the first case). Otherwise, we observe that we are in a similar situation as with Δ, and $V(\Delta')$ a strict subset of $V(\Delta)$. We can thus replace Δ' by a smaller candidate Δ'' with $V(\Delta'')$ strictly included in $V(\Delta)$. Since $V(\Delta)$ is finite, this construction of smaller and smaller candidates cannot continue forever and we must reach a situation where the first case holds, which yields a small triangle. $\square$

See Fig. 7 for an example. We obtain the small triangle $\Delta'' = (f, y_p, y_{p+1}, \ldots, y_q, f')$ after using twice the construction of the second case.

To conclude the proof of Lemma 4.3, it remains to prove (3) i.e., that R is an F-internal sketch. The proof is by induction on $Card(X(R))$.

The case $X(R) = \emptyset$ holds by assertion (1).

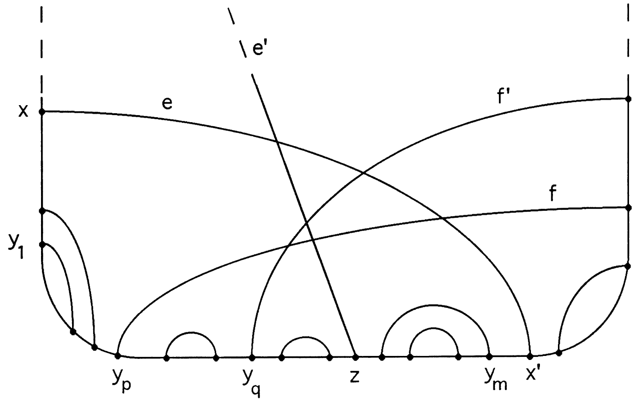


Fig. 7.

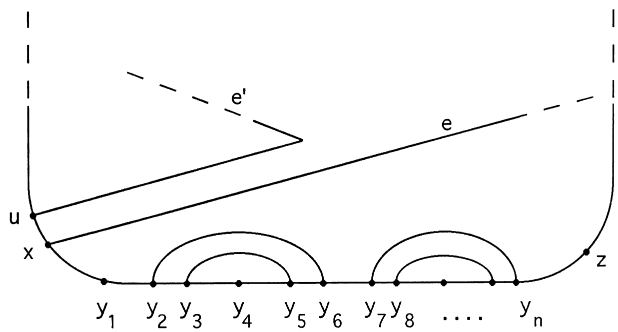


Fig. 8.

Otherwise, there is by (2) a small triangle Δ. Without loss of generality we assume that $\Delta = (e, x, y_1, y_2, \dots, y_n, z, e')$, e links x and x', e' links z and z', $z', x' \notin V(\Delta)$, $(u, x, y_1, y_2, \dots, y_n, z)$ is a path in F (see Fig. 6). We assume also that the vertex u before x has degree 2: this condition will be lifted at the end of the proof.

We transform R into R' in the following way:

– we let e' link z' and u instead of z' and z (we change one tuple in inc_R),
– we define $cross_{R'}$ from $cross_R$ by deleting the crossing of e and e', and accordingly, we transform $before_R$ into $before_{R'}$ (the changes concern $before_R[e]$ and $before_R[e']$),
– we transform sig_R into $sig_{R'}$ so that H_1 holds in R' (these changes only concern $sig_R[z]$ and $sig_R[u]$).

See Fig. 8 for an example.

Claim. *R' satisfies* H_0–H_4.

Since $Card(X(R')) = Card(X(R)) - 1$, there exists a drawing D' such that $R(D') = R'$. From this drawing, we get a drawing D such that $R(D) = R$. See Figs. 9(a) and (b).

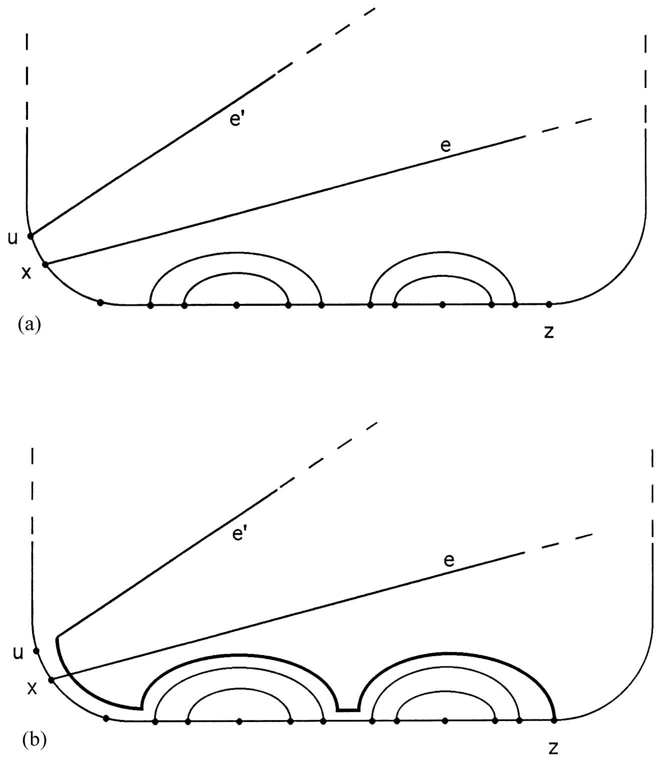


Fig. 9. (a) A drawing D' of R'; (b) A drawing D of R obtained by modifying e'.

If u is not of degree 2, we first insert a new vertex u' on the edge between u and x. This preliminary modification of R preserves satisfaction of H_0–H_4. For constructing R', we use u' instead of u. $\square$

Proof of Proposition 4.1. We let R be a protosketch, F be a Hamiltonian cycle of $G = G(R)$ and we assume that $R[E_F \cup X]$ is F-internal for every $X \subseteq E_G - E_F$ of cardinality at most 3. We want to prove that $R = R(D)$ for some F-internal drawing D.

We first consider the case where $deg(G) \leqslant 3$.

We check conditions H_0–H_4. The result follows by Lemma 4.3(3). Condition H_0 holds because $R[E_F]$ is F-internal. Condition H_1 holds because $R[E_F \cup \{f\}]$ is F-internal for every $f \in E_G - E_F$, condition H_2 holds because $R[E_F \cup \{f,g\}]$ is F-internal for any two $f,g \in E_G - E_F$ and conditions H_3 and H_4 hold because $R[E_F \cup \{f,g,h\}]$ is F-internal for any three f,g,h in $E_G - E_F$.

Here is the general case. If $G(R)$ has vertices of degree more than 3 then we transform (R,F) into (R',F') which satisfies the hypothesis of Proposition 4.1, is such that $deg(G(R')) \leqslant 3$ and $G(R)$ is obtained from $G(R')$ by edge contractions. An F'-internal drawing of $G(R')$ exists by the special case, and we will obtain from it an F-internal drawing D such that $R(D) = R$.

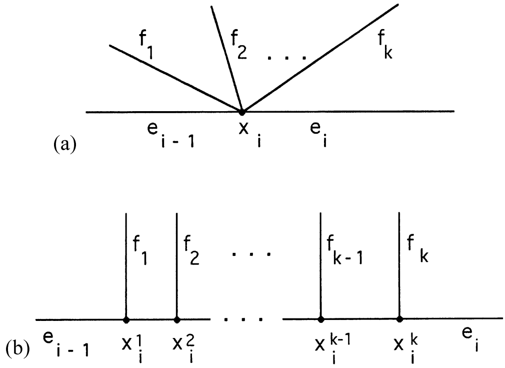


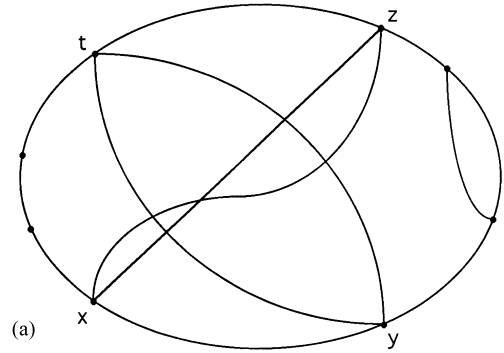
Fig. 10. (a) The neighborhood of x_i in $G(R)$; (b) The result of the replacement of x_i by a path.

Let us describe the construction of R'. As before, we let F be given by $(x_1,\dots,x_n)$ with the corresponding list of edges $e_1,e_2,\dots,e_n$, and for convenience, we will use $x_{n+1}=x_1$. We define R' by "splitting" as follows each vertex x_i of degree more than 3. We assume that $E_{G(R)}(x_i)=\{e_i,f_1,\dots,f_k,e_{i-1}\}$, $k\geqslant 2$ and $sig_{M(R)}[x_i]=\{e_i\to f_k\to f_{k-1}\to\cdots\to f_2\to f_1\to e_{i-1}\to e_i\}$.

We replace x_i by a path $x_i^1\to x_i^2\to\cdots\to x_i^k$ and we make f_j incident to x_i^j instead of to x_i. See Fig. 10(b). The structure R' is defined accordingly. In particular, $E_{G(R')}$ is equal to $E_{G(R)}$ augmented with the new edges resulting from the vertex splitting, $cross_{R'}=cross_R$ and $before_{R'}=before_R$. Let F' consist of F and the new edges. (See Figs. 11(a) and (b).) Consider an arbitrary set $X\subseteq E_{G(R')}-E_{F'}=E_{G(R)}-E_F$ of cardinality at most 3. We assume that $R[E_F\cup X]$ has an F-internal drawing D_X from which we obtain an F'-internal drawing D'_X of $R'[E_{F'}\cup X]$; hence $R'[E_{F'}\cup X]$ satisfies conditions H_0–H_4 relative to F'. Since $G(R')$ has degree 3 and by the special case, R' is F'-internal. From an F'-internal drawing D' of R' and by contracting the edges of $E_{F'}-E_F$ (i.e. those introduced for splitting some vertices) one obtains an F-internal drawing of R. Hence R is F-internal. □

Theorem 4.4. *Let R be a protosketch such that $G(R)$ has a Hamiltonian cycle F and $R[F]$ is planar (i.e. $cross_R\cap(E_F\times E_F)=\emptyset$). Then R is a sketch if $R[E_F\cup X]$ is a sketch for every set X of at most 3 edges in $E_{G(R)}-E_F$.*

Proof. We let R and F be as stated. We first consider the special case where $cross_R\subseteq(E_{G(R)}-E_F)\times(E_{G(R)}-E_F)$. This means that no edge not found in F crosses an edge of F. We assume that $R[E_F\cup X]$ is a sketch for every set $X\subseteq E_{G(R)}-E_F$, $card(X)\leqslant 3$. Consider $f\in E_{G(R)}-E_F$; the protosketch $R[E_F\cup\{f\}]$ is a sketch: let D_f be a drawing such that $R(D_f)=R[E_F\cup\{f\}]$. It consists of one closed curve representing F in the trigonometric sense and a curve segment between two points of this curve. Since it does not cross an edge of F, it is either in the finite region or in the infinite one. In the first case, we will say that f is *F-internal*. In the second case it is *F-external*.



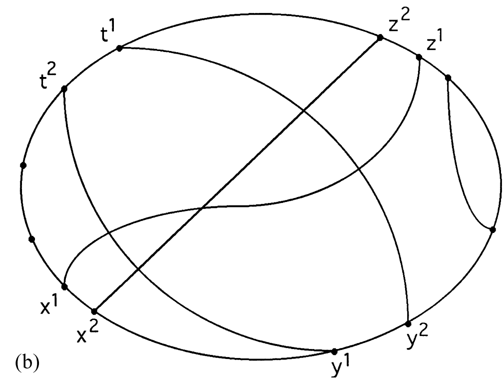


Fig. 11. (a) A drawing D of R; (b) A drawing D' of R'.

(In the example of Fig. 12(a), a and b are F-internal and c and d are F-external.) Whether f is F-internal or F-external depends only on sig_R and the chosen ordering of F. Precisely, by Condition H_1, an edge $f \in E_{G(R)} - E_F$ is F-internal if for each of its ends x, if $x = x_{i+1}$, $1 \leqslant i \leqslant n$, we have

$$e_i \lessdot_{M(R),x} e_{i+1} \lessdot_{M(R),x} f.$$

It is F-external if for each of its ends x, if $x = x_{i+1}$, $1 \leqslant i \leqslant n$, we have

$$e_i \lessdot_{M(R),x} f \lessdot_{M(R),x} e_{i+1}.$$

We denote by Int (resp. Ext) the set of F-internal (resp. F-external) edges. We will define a drawing D of R as the union of a drawing D_{Ext} of $R[E_F \cup Ext]$ and a drawing D_{Int} of $R[E_F \cup Int]$. The existence of D_{Int} follows from Proposition 4.1. We can even construct it such that F is drawn as a circle of radius 1 centered at O, the origin of the Euclidian plane $\mathbf{P}$. We now let $R' = R[E_F \cup Ext]$; we transform it into R'' by letting: $sig_{R''} = \{(x,e,f) \mid (x,f,e) \in sig_{R'}\}$ and $cross_{R''} = \{(e,f) \mid (f,e) \in cross_{R'}\}$ and by letting the other components be as in R'. The set Ext is now the set of F-internal edges of R''. By Proposition 4.1, one can find a drawing D'' of R'', and furthermore,

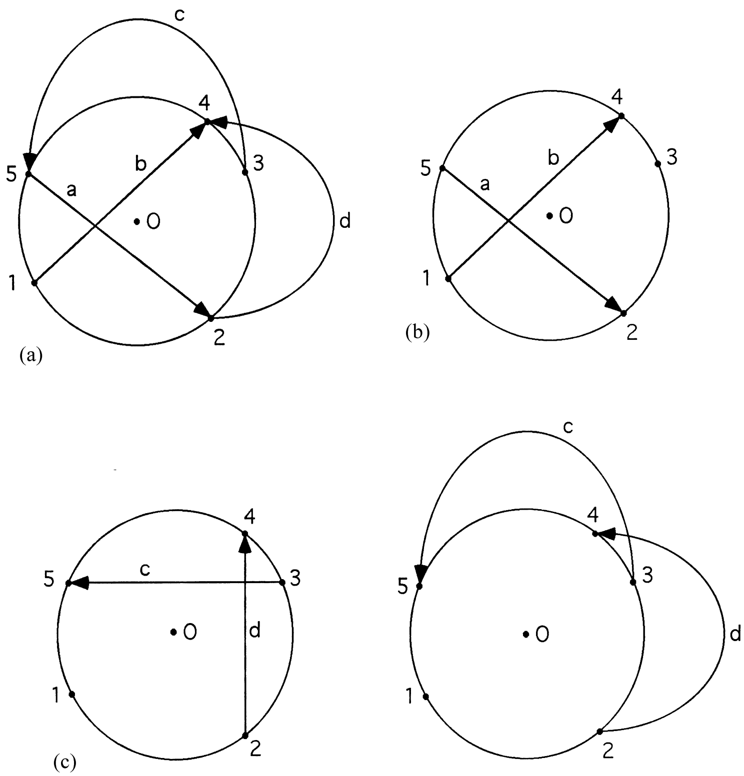


Fig. 12. (a) A drawing D of R as union of D_{Int} and D_{Ext}; (b) The drawing D_{Int}. (c) A drawing D'' of R'' and the drawing D_{Ext}.

F can be drawn exactly as in D_{Int}, and one can assume that O does not belong to any edge. We now transform D'' into D_{Ext} by using the homeomorphism of $\mathbf{P} - \{O\}$ onto itself that maps M to Q such that $\vec{OQ} = \vec{OM}/OM^2$. Since this homeomorphism leaves invariant each point of the circle of radius 1, F is the same in D_{Int} and in D_{Ext}. (The assumption that O belongs to no edge of D_{Int} is here important). Their union yields the desired drawing of R. See Figs. 12(a)–12(c).

General Case: We assume that

$$cross_R \subseteq (N \times E_F) \cup (E_F \times N) \cup (N \times N)$$

where $N = E_{G(R)} - E_F$. We let $U = \{\{e,f\} \in X(R) \mid e \in N, f \in E_F\}$. By Proposition 2.1 we need only prove that $R + U$ is a sketch. The protosketch $R' = R + U$ (see Section 2 for notation) is obtained from R by the insertion of new vertices on the edges of F. Hence, F is transformed into a Hamiltonian cycle F' of $G(R')$. From the definition of U, the protosketch R' satisfies the condition:

$$cross_{R'} \subseteq (E_{G(R')} - E_{F'}) \times (E_{G(R')} - E_{F'}).$$

Let $X \subseteq (E_{G(R')} - E_{F'})$, $Card(X) \leqslant 3$. Let $Y = \pi_1(X) \subseteq (E_{G(R)} - E_F)$ (i.e., Y is the set of edges of $G(R)$ from which the edges of X are defined by subdivision, see Section 2). Clearly, $Card(Y) \leqslant 3$ (Y can have less elements than X if two edges of X come from the subdivision of a single edge of Y). Hence, $R[E_F \cup Y]$ is a sketch.

A drawing D_Y such that $R(D_Y) = R[E_F \cup Y]$ yields a drawing of $R'[E_{F'} \cup X']$ where X' is the set of edges of $E_{G(R')} - E_{F'}$ coming from the subdivision of the edges of Y. Hence $X \subseteq X'$. Hence $R'[E_{F'} \cup X]$ is a sketch.

The result of the special case yields that R' is a sketch, hence so is R by Proposition 2.1. $\square$

5. Tree-framed protosketches

The proof of the 3-Edge Theorem (Theorem 3.1) will follow easily from the next special case.

Theorem 5.1. *A protosketch R with a frame F that is a tree is a sketch if $R[E_F \cup X]$ is a sketch for every set X of at most three edges in $E_{G(R)} - E_F$.*

The proof idea is as follows. We will transform R into R' by duplicating each edge of F and making F into a Hamiltonian cycle of the graph $G(R')$. We will apply Theorem 4.4 to R' and obtain a drawing D' such that $R' = R(D')$. By fusing any two edges coming from a same edge of F, we will get a drawing D such that $R(D) = R$. This construction is illustrated in Figs. 13(a) and (b).

Definition 5.1 (*Fattening a tree*). The construction will use two steps.

Step 1: We define formally the transformation of R into R' illustrated in Figs. 13(a) and (b). We call it the *fattening* of the tree F. Since F is assumed to be a frame, it is a spanning tree of $G(R)$ and $cross_R \cap (E_F \times E_F) = \emptyset$. Without loss of generality, we assume that F is directed in such a way that every vertex is reachable from a vertex x_1 of degree 1 by a directed path in F. Thus F is a rooted tree with root x_1, and the root has only one son.

We let $d_F(y)$ denote the degree of y with respect to F. We let $V_{G(R')} = \{(y,i) \mid y \in V_F, 1 \leqslant i \leqslant d_F(y)\}$. We let $E_{F'} = \{e^+, e^- \mid e \in E_F\}$, $E_{G(R')} = E_{F'} \cup (E_{G(R)} - E_F)$.

We let $e(x_1, 1)$ denote the unique edge of F incident with x_1.

For each $y \in V_F - \{x_1\}$ we enumerate $E_F(y)$ as: $e(y,1), \dots, e(y,k)$, with $k = d_F(y)$, in such a way that $e(y,k)$ is directed towards y (the others are thus directed from y) and

$$e(y,1) \leqslant_{M,y} e(y,2) \leqslant_{M,y} \cdots \leqslant_{M,y} e(y,k)$$

where $M = M(R)$. (See Fig. 14(a) which shows a fragment of a graph.)

Let $e = e(y,i)$, $1 \leqslant i < d_F(y)$, link y to z. Then we let e^+ link (y,i) to $(z,1)$ and we let e^- link $(z, d_F(z))$ to $(y, i+1)$. If e links x_1 to z, we let e^+ link $(x_1, 1)$ to $(z,1)$ and e^- link $(z, d_F(z))$ to $(x_1, 1)$.

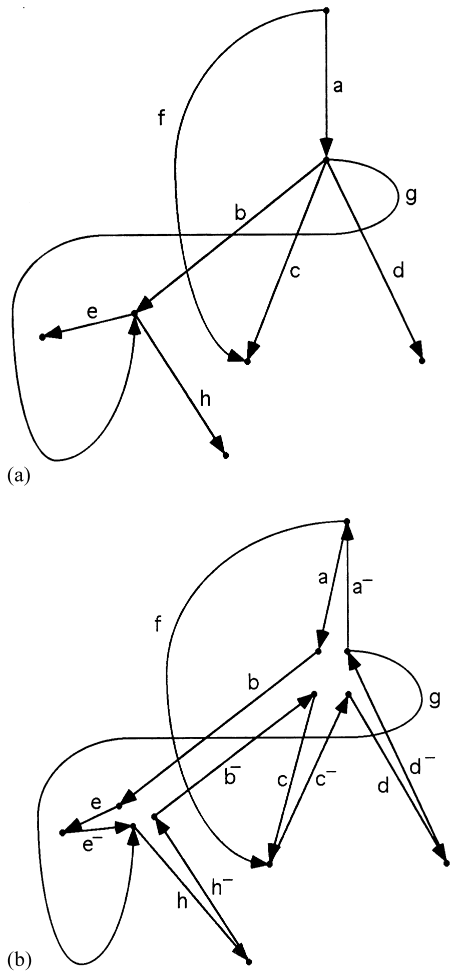


Fig. 13. (a) A protosketch R with frame F consisting of a, b, c, d, e, h (b); The drawing of R' obtained from R by duplicating the edges of F, which makes F into a Hamiltonian cycle.

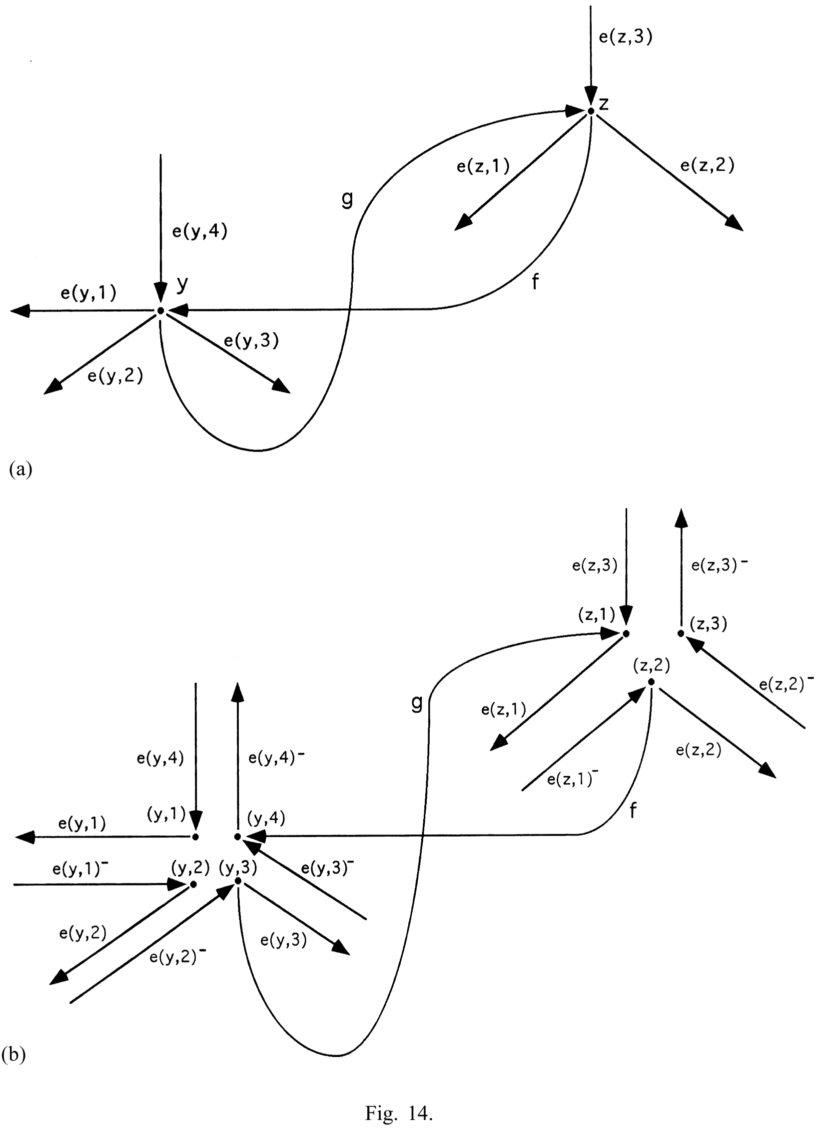
We now consider an edge $g \in E_{G(R)} - E_F$, such that g links y to z (see Fig. 14(a)). Let us assume that

$$e(y,i) \leqslant_{M,y} g \leqslant_{M,y} e(y,i+1)$$

(with $e(y,d_F(y)+1) = e(y,1)$ and

$$e(z,j) \leqslant_{M,z} g \leqslant_{M,z} e(z,j+1)$$

(with $e(z,d_F(z)+1) = e(z,1)$). Then, in R', we let g link $(y,i+1)$ to $(z,j+1)$ (with $(y,d_F(y)+1) = (y,1)$ and $(z,d_F(z)+1) = (z,1)$ in order to uniformize the notation).



(a) (b)

Fig. 14.

(In Fig. 14(b), we have for g, y and z: $i=2$ and $j=3$.) We define R' in such a way that

$$e(y,i)^{\alpha} \leqslant_{M',(y,i+1)} g \leqslant_{M',(y,i+1)} e(y,i+1)^{\beta},$$

$$e(z,j)^{\gamma} \leqslant_{M',(z,j+1)} g \leqslant_{M',(z,j+1)} e(z,j+1)^{\delta},$$

where

$$\begin{aligned} &\alpha = -, \quad \beta = + \quad \text{if } 1 \leqslant i < d_F(y) - 1,\\ &\alpha = -, \quad \beta = - \quad \text{if } i = d_F(y) - 1,\\ &\alpha = +, \quad \beta = + \quad \text{if } i = d_F(y), \end{aligned}$$

and γ, δ are defined similarly in terms of j and z. See Fig. 14(b) where the + marks on edges have been omitted.

Hence if

$$\begin{aligned} sigma_R[y] = e(y,1) &\to f_1^1 \to f_2^1 \to \cdots \to f_{n_1}^1 \to \\ e(y,2) &\to f_1^2 \to f_2^2 \to \cdots \to f_{n_2}^2 \to \\ e(y,3) &\to \cdots \to \\ e(y,k) &\to f_1^k \to \cdots \to f_{n_k}^k \to e(y,1), \end{aligned}$$

we let:

$$\begin{aligned} sigma_{R'}[(y,1)] &= e(y,k)^+ \to f_1^k \to \cdots \to f_{n_k}^k \to e(y,1)^+ \to e(y,k)^+, \\ sigma_{R'}[(y,2)] &= e(y,1)^- \to f_1^1 \to \cdots \to f_{n_1}^1 \to e(y,2)^+ \to e(y,1)^-, \\ sigma_{R'}[(y,i)] &= e(y,i-1)^- \to f_1^{i-1} \to \cdots \to f_{n_{i-1}}^{i-1} \to e(y,i)^+ \to e(y,i-1)^-, \end{aligned}$$

for $i = 3,\ldots,k-1$, and finally

$$\begin{aligned} sigma_{R'}[(y,k)] = e(y,k-1)^- &\to f_1^{k-1} \to \cdots \to f_{n_{k-1}}^{k-1} \\ &\to e(y,k)^- \to e(y,k-1)^-. \end{aligned}$$

On the example of Fig. 14, we have in particular:

$$sigma_R[y] = e(y,1) \to e(y,2) \to g \to e(y,3) \to f \to e(y,4) \to e(y,1)$$

which gives

$$\begin{aligned} sigma_{R'}[(y,1)] &= e(y,4)^+ \to e(y,1)^+ \to e(y,4)^+, \\ sigma_{R'}[(y,2)] &= e(y,1)^- \to e(y,2)^+ \to e(y,1)^-, \\ sigma_{R'}[(y,3)] &= e(y,2)^- \to g \to e(y,3)^+ \to e(y,2)^-, \\ sigma_{R'}[(y,4)] &= e(y,3)^- \to f \to e(y,4)^- \to e(y,3)^-. \end{aligned}$$

We now define the relation $cross_{R'}$. If $cross_R(e,f)$ holds, $e \in E_F$, $f \in E_{G(R)} - E_F$ then we let $cross_{R'}(e^+,f)$ and $cross_{R'}(f,e^-)$ hold. If $cross_R(f,e)$ holds, with e,f as above, we let $cross_{R'}(f,e^+)$ and $cross_{R'}(e^-,f)$ hold. If $cross_R(f,g)$ with $f,g \in E_{G(R)} - E_F$, we let $cross_{R'}(f,g)$ hold.

Finally, we define $before_{R'}$. If $e \in E_F$, we define $before_{R'}[e^+]$ as equal to $before_R[e]$ and $before_{R'}[e^-]$ as its reversal. Note that all elements of these sequences are in $E_{G(R)} - E_F$. If $g \in E_{G(R)} - E_F$ and $before_R[g] = (f_1, f_2, \ldots, f_k)$, we let $before_{R'}[g] = w_1.w_2.\ldots.w_k$ be the concatenation of the sequences $w_1, w_2, \ldots, w_k$ where

$$\begin{aligned} w_\ell &= (f_\ell) \quad \text{if } f_\ell \in E_{G(R)} - E_F, \\ w_\ell &= (f_\ell^+, f_\ell^-) \quad \text{if } f_\ell \in E_F \text{ and } cross_R(g, f_\ell) \text{ holds}, \\ w_\ell &= (f_\ell^-, f_\ell^+) \quad \text{if } f_\ell \in E_F \text{ and } cross_R(f_\ell, g) \text{ holds}. \end{aligned}$$

The graph $G(R')$ has a Hamiltonian cycle defined by the sequence of vertices $v=((x_1,1),(x_2,1),\ldots)$ such that $v=(x_1,1)\sigma(x_2), x_2$ is the son of x_1 and σ is defined recursively as follows:

$$\sigma(y) = ((y,1)) \quad \text{if } y \text{ is a leaf}$$
$$\sigma(y) = (y,1)\sigma(z_1)(y,2)\sigma(z_2)\ldots(y,k)\sigma(z_k)(y,k+1)$$

if y has degree $k+1$ and list of sons $z_1, z_2, \ldots, z_k$.

This Hamiltonian cycle has v as a sequence of vertices and $\{e^+, e^- \mid e \in E_F\}$ as a set of edges. We will denote it by F'.

Step 2: We now augment R' into R'' by adding an edge denoted by $h(y,i)$ between (y,i) and $(y,i+1)$ for each y in $V_{G(R)}$, each $i=1,\ldots,d_F(y)$ (we consider $(y,1)$ as equal to $(y,d_F(y)+1)$). We let these new edges have no crossing. It remains to define the relation $sig_{R''}$. We do this in such a way that these edges form *faces*, i.e., regions of the plane delimited by the curves forming a drawing of R''. By contracting these edges, we will obtain the vertex y from which they come.

We define $sig_{R''}$ in such a way that, with the notation used for defining $sig_{R'}$ we have

$$\begin{aligned} sigma_{R''}[(y,1)] = {} & e(y,k)^+ \to f_1^k \to \cdots \to e(y,1)^+ \to h(y,1) \\ & \to h(y,k) \to e(y,k)^+, \end{aligned}$$

$$\begin{aligned} sigma_{R''}[(y,i)] = {} & e(y,i-1)^- \to f_1^{i-1} \to \cdots \to e(y,i)^+ \to h(y,i) \\ & \to h(y,i-1) \to e(y,i-1)^-, \end{aligned}$$

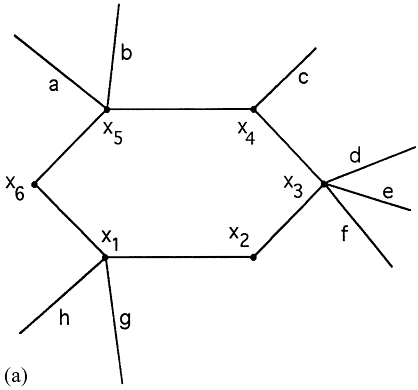
for $i = 2, \ldots, k-1$, and finally

$$\begin{aligned} sigma_{R''}[(y,k)] = {} & e(y,k-1)^- \to f_1^{k-1} \to \cdots \to e(y,k)^- \to h(y,k) \\ & \to h(y,k-1) \to e(y,k-1)^-. \end{aligned}$$

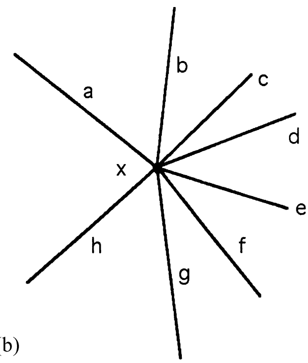
From the consideration of drawings we get that if R is a sketch, then so is R''. We now define *face contractions* without reference to drawings.

Definition 5.2 (*Simple face contractions*). We first consider maps. Let $M=(G, sigma)$ be a map. A *simple face* in M is a circuit $F=(e_1,e_2,\ldots,e_n)$ such that for every $i=1,\ldots,n$, we have $sigma(x_i,e_i,e_{i-1})$ where x_i is the vertex common to e_i and e_{i-1} (and $e_0=e_n$). We will define the result of the contraction of such a face, denoted by $M\backslash F$. We let $G\backslash F$ be the graph resulting from the contraction of all edges of F. Its set of edges is $E_G - E_F$ and its set of vertices is $(V_G - V_F) \cup \{x_F\}$ where x_F is the vertex resulting from the fusion of all vertices of V_F.

We now define $sig_{M\backslash F}$. If $x \notin V_F$ then we let $sig_{M\backslash F}(x,e,f)$ hold if and only if $sig_M(x,e,f)$ holds. We let $sig_{M\backslash F}[x_F]$ be empty if x_F is an isolated vertex. Otherwise, we may assume, without loss of generality, that x_1 is incident in G with one edge not



(a)



(b)

Fig. 15. The face F in G and the edges around $x = x_F$.

in F. We let $sig_{M\backslash F}[x_F]$ be the union of the following sets, for $i = 1,\ldots,n$ (we let e_i link x_i and x_{i+1} and $x_{n+1} = x_1$):

$sig_M[x_i] \cap (E_{G\backslash F} \times E_{G\backslash F})$ and

$$\{(g, f) \in E_{G\backslash F} \times E_{G\backslash F} / sig_M(x_i, g, e_i) \wedge sig_M(x_j, e_{j-1}, f)\}$$

where for each i at most n, j is the smallest integer at most $n+1$, that is larger than i and is such that x_j is incident to an edge not in F. (In the example of Fig. 15, $j = 3$ if $i = 1$, and $j = 7$ if $i = 5$.) It is clear that if F is a simple face of a map M, then $M\backslash F$ is a map which is planar if M is planar.

The extension to protosketches R such that the edges of F have no crossing is clear: we let $R\backslash F$ be such that $M(R\backslash F) = M(R)\backslash F$, $cross_{R\backslash F} = cross_R$, $before_{R\backslash F} = before_R$. If R is a sketch then so is $R\backslash F$.

With the hypotheses and notation of Definition 5.1, we have:

Lemma 5.2. *R'' is a sketch if and only if R is a sketch.*

Proof. Let $R = R(D)$ for some drawing D. One obtains a drawing D' such that $R(D') = R'$ by duplicating each edge of F as explained, and illustrated in Fig. 13. The tree F is made into a Hamiltonian cycle F' drawn in the trigonometric sense.

One also introduces new edges (not shown in the figures) such that each vertex of R yields a simple face. One obtains a drawing D'' such that $R(D'') = R''$. Hence R'' is a sketch.

Let us conversely assume that R'' obtained from R is a sketch. By fusing into a single vertex y, each set $\{(y,i) \mid y \in V_{G(R)}, 1 \leqslant i \leqslant d_F(y)\}$ (this is a simple face contraction), and by deleting the edges e^-, for each $e \in E_F$, one obtains a sketch which is R. □

Proof of Theorem 5.1. Let R be a protosketch with tree F as a frame such that $R[E_F \cup X]$ is a sketch for every subset $X \subseteq E_{G(R)} - E_F$ of cardinality at most 3. Let R'' be obtained from R by fattening F (according to Definition 5.1). Let E be the set of edges added at the second step, in order to build R'' from R'. From the definitions, for every X as above, $R''[E_{F'} \cup E \cup X]$ is obtained from $R[E_F \cup X]$ by fattening F; hence it is a sketch. Hence $R''[E_{F'} \cup X']$ is a sketch for every subset X' of $E \cup X$ of cardinality at most 3. Since F' is a Hamiltonian cycle in $G(R'')$, we can apply Theorem 4.4 and we obtain that R'' is a sketch. Hence R is a sketch by Lemma 5.2. □

Proof of the 3-edge theorem (Theorem (3.1)). Let R be a sketch with frame F. For every subset X of at most 3 edges in $E_{G(R)} - E_F$ the protosketch $R[E_F \cup X]$ is a sketch.

Let us conversely assume that R is a protosketch with frame F such that $R[E_F \cup X]$ is a sketch for each set X as above. Let us choose a spanning tree H of F. Then R is a protosketch with frame H. For every $Y \subseteq E_{G(R)} - E_H$ of cardinality at most 3, $R[E_H \cup Y]$ is a sketch since $R[E_H \cup Y]$ is obtained by edge deletions from $R[E_F \cup Y]$ which is itself a sketch by the assumption. Hence, R is a sketch by Theorem 5.1. □

6. Conclusion and open problems

We have defined a logical structure called a *sketch*, making it possible to represent (certain) drawings of graphs with edge crossings, up to homeomorphism.

It would remain to extend these notions in several ways: for handling drawings of nonconnected graphs, and drawings without the limitations on crossings introduced in Section 2.

Sketches can be used to economically store schematic views of graph drawings, and to transmit these data over computer networks. They can be given as input to graph drawing algorithms.

We have focused our attention on the combinatorial and logical characterization of sketches. We have only given partial answers to Question 2.5, in particular, for framed sketches. However, we think that this class is already significant because in many concrete cases a graph consists of a basic planar graph (for example a network of roads) augmented with additional edges (for example high-power lines or railway lines), that may cross them.

We have already indicated a few directions for future work. Here is another one. Let a *partial protosketch* be a structure P of the form $\langle V_G \cup E_G,\ inc_G,\ sig_M,\ cross,\ before\rangle$ for some graph G with map defined by sig_M, where *before* is a ternary relation and *cross* is binary. We say that P is *consistent* if there exists a sketch R such that $M(R) = M$, $cross \subseteq cross_R$, $before \subseteq before_R$. We thus consider P as a *partial specification* of a class of drawings of G, for which some edge crossings are left unspecified.

Question 6.1. *Is the consistency of partial protosketches expressible in monadic second-order logic*?

If this is not possible in general, a derived problem consists in finding constraints on partial protosketches making consistency monadic second-order expressible and efficiently testable.

Acknowledgements

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