Encoding of the Halting Problem into the Monster Type and Applications

Thierry Joly

Équipe Preuves, Programmes et Systèmes, Université Paris VII,
2, place Jussieu, 75251 Paris cedex 05, France**
joly@pps.jussieu.fr

Abstract. The question whether a given functional of a full type structure (FTS for short) is λ-definable by a closed λ-term, raised by G. Huet in [Hue75] and known as the Definability Problem, was proved to be undecidable by R. Loader in 1993. More precisely, R. Loader proved that the problem is undecidable for every FTS over at least 7 ground elements (cf [Loa01]).

We solve here the remaining non trivial cases and show that the problem is undecidable whenever there are more than one ground element. The proof is based on a direct encoding of the Halting Problem for register machines into the Definability Problem restricted to the functionals of the Monster type $\mathbb{M} = (((o \to o) \to o) \to o) \to (o \to o)$. It follows that this new restriction of the Definability Problem, which is orthogonal to the ones considered so far, is also undecidable. Another consequence of the latter fact, besides the result stated above, is the undecidability of the β-Pattern Matching Problem, recently established by R. Loader in [Loa03].

1 Introduction

Notations. We consider here the simply typed λ-calculus λ^τ *à la Church* over a single atomic type o. The λ-terms and the variables of any type A are respectively denoted by capital letters and small letters: M^A, x^A. Their type may be omitted if it is of no importance or clear from the context.
The λ-terms are considered up to the renaming of their bound variables, so that $=$ actually denotes α-equivalence. The other kinds of λ-terms equalities are always explicitly written: $=_\beta$, $=_{\beta\eta}, \ldots$
Λ (resp. $\Lambda^\varnothing$) is the set of the λ-terms (resp. of the closed λ-terms) of λ^τ.
$\mathbf{I}^{A\to A}$ denotes the combinator $\lambda x^A.x$ and $M \circ N$ the λ-term $\lambda x.M(Nx)$.
Types are implicitely parenthesized to the right, *i.e.* $A_1 \to \ldots \to A_n \to B$ stands for $A_1 \to (\ldots \to (A_n \to B)\ldots)$. We also use for concision the type notations: $1 = o \to o$, $2 = 1 \to o, \ldots, n{+}1 = n \to o, \ldots$ and $1_2 = o \to 1, \ldots, 1_{n+1} = o \to 1_n, \ldots$

Recall that a hereditarily finite full type structure of λ^τ (FTS for short) is a collection $\mathcal{F} = (\mathcal{F}^A)_{A \in \mathrm{Type}(\lambda^\tau)}$ such that $\mathcal{F}^o \neq \varnothing$ is finite and $\mathcal{F}^{A\to B} = (\mathcal{F}^B)^{\mathcal{F}^A}$

** The present work was done at the University of Nijmegen, The Netherlands.

M. Hofmann (Ed.): TLCA 2003, LNCS 2701, pp. 153–166, 2003.

for all A, B. Recall also that a functional $\varphi \in \mathcal{F}^A$ is said λ-*definable* whenever for some closed λ-term M^A we have $[\![M]\!] = \varphi$, where $[\![M]\!]$ denotes the interpretation of M in $\mathcal{F}$ defined in the usual way (see e.g. [Fri75]). The basic question:

DP: *"Given a functional φ of any FTS, is φ λ-definable?"*

is known as the Definability Problem, and the most natural restrictions of this very general problem are with no doubt:

$\mathrm{DP}_{\mathcal{F}_n}$: *"Given a functional φ of $\mathcal{F}_n$, is φ λ-definable?"*

where $\mathcal{F}_n$ ($n \geqslant 1$) denotes the unique FTS (up to isomorphism) such that $\mathcal{F}_n^o$ has n elements. We may also consider the following restrictions of DP, which are somehow orthogonal to the previous ones:

DP_A: *"Given a functional $\varphi \in \mathcal{F}_n^A$ of any FTS $\mathcal{F}_n$, is φ λ-definable?"*

The Statman conjecture was that DP is decidable (*cf* [Sta82]) and was refuted by R. Loader, who proved that $\mathrm{DP}_{\mathcal{F}_n}$ is undecidable for every $n \geqslant 7$ (*cf* [Loa01]). The Definability Problem is related to another famous one, the Higher Order Pattern Matching Problem, raised by Huet in [Hue75]:

MP: *"Given $P, Q \in \Lambda^{\varnothing}$, is there $X \in \Lambda^{\varnothing}$ such that $P\,X =_{\beta\eta} Q$?"*

It is well known that the decidability of MP, which is still an open question, would have been a consequence of the Statman conjecture (*cf* [Sta82]). DP is actually equivalent to the following generalization of MP:

MP^+: *"Given a type substitution $+ = [o{:=}1_n]$ and $P, Q \in \Lambda^{\varnothing}$,
is there a closed λ-term X such that $P\,X^+ =_{\beta\eta} Q$?"*

More precisely, for every type A the problem DP_A proves to be equivalent to:

MP^+_A: *"Given $P^{A^+ \to o^+}, Q^{o^+} \in \Lambda^{\varnothing}$ where $+$ is any substitution $[o{:=}1_n]$,
is there a closed λ-term X^A such that $P\,X^+ =_{\beta\eta} Q$?"*

We encode below (Section 2) the Halting Problem of a simple kind of register machines into functionals of the Monster type $\mathbb{M} = 3 \to 1$. This yields the undecidability of new restricted cases of the problems DP, MP^+, namely $\mathrm{DP}_{\mathbb{M}}$ and $\mathrm{MP}^+_{\mathbb{M}}$, which prove to be fruitful. E.g. from the undecidability of $\mathrm{MP}^+_{\mathbb{M}}$, we derive at once the undecidability of the following other generalization[1] of MP:

β-MP: *"Given $P, Q \in \Lambda^{\varnothing}$, is there $X \in \Lambda^{\varnothing}$ such that $P\,X =_{\beta} Q$?"*

which has already been established by R. Loader in [Loa03]. This alternative argument (Section 3), where the restriction of the equality of the calculus to $=_\beta$

[1] MP is indeed the particular case of β-MP where the given λ-terms P, Q are in long normal form, since for such P, Q, the equations $PX =_\beta Q$ and $PX =_{\beta\eta} Q$ have the same solutions $X \in \Lambda^{\varnothing}$.

is introduced at the latest, enlights the crucial role played by this restriction and the fact that the question whether MP is decidable is still wide open.

We also derive from the undecidability of $\mathrm{MP}^{+}_{\mathbb{M}}$ the one of $\mathrm{DP}_{\mathcal{F}_n}$ for every $n \geqslant 2$ (Sections 4,5). This closes the question of the decidability of $\mathrm{DP}_{\mathcal{F}_n}$ for all n, since $\mathrm{DP}_{\mathcal{F}_1}$ is equivalent to the Type Inhabitation Problem and is therefore easily decided (indeed $\varphi \in \mathcal{F}_1^A$ is λ-definable iff there is a closed λ-term M^A).

More applications of the encoding of the Halting Problem below will be detailed in another paper, where we will complete the present study of DP and establish for the larger λ-calculus $\lambda^{\rightarrow}$ with infinitely many atomic types that the decidability of DP_A is equivalent to several other statements about A. One of these statements (*"all the closed normal forms of type A can be written from a finite supply of variables"*) yields a procedure deciding for any given A whether DP_A is decidable. Another one (*"all the closed λ-terms of type A can be generated from a finite set of combinators"*) extends to $\lambda^{\rightarrow}$ the solution for λ^{τ} (cf [Jol01]) of the Finite Generation Problem raised by R. Statman in [Sta85].

2 The Definability Problem at the Monster Type

Every register machine M, as considered here, is completely determined by the (finite) number $p \geqslant 2$ of its registers, a finite set of states Σ, the choice of two different elements s, h in Σ and for all $q \in \Sigma \setminus \{h\}$, an instruction $\mathrm{Instr}(q)$. It is intended that the machine starts computations *with blank registers* from state s on, halts whenever it reaches state h and that $\mathrm{Instr}(q)$ is the next instruction to be executed when the machine is in state q. For all $q \in \Sigma \setminus \{h\}$, $\mathrm{Instr}(q)$ may only have one of the two following forms:

- $\underline{\mathrm{Inc}(k, q)}$: Increment register n° k and go to state q.
- $\underline{\mathrm{Jzd}(k, q_1, q_2)}$ (Jump if Zero & Decrement): If register n° k has content 0, then go to state q_1 else decrement register n° k and go to state q_2.

The complete configuration of the machine is described at any time by a tuple $(q, n_1, \ldots, n_p)$, where q is the current state and n_i $(1 \leqslant i \leqslant p)$ the current content of register n° i. More formally:

Definition I. Let $\mathrm{M} = (p, \Sigma, s, h, \mathrm{Instr})$ be any register machine.

(i) The sequence of the configurations $\mathbf{c}_n \in \Sigma \times \mathbb{N}^p$ of M during its computation is defined by $\mathbf{c}_0 = (s, 0, \ldots, 0)$ and, in the case where say $\mathbf{c}_n = (q, n_1, \ldots, n_p)$:

- If $\mathrm{Instr}(q) = \mathrm{Inc}(k, q')$, then $\mathbf{c}_{n+1} = (q', n_1, \ldots, n_{k-1}, n_k + 1, n_{k+1}, \ldots, n_p)$.
- If $\mathrm{Instr}(q) = \mathrm{Jzd}(k, q_1, q_2)$, then:
  - $\mathbf{c}_{n+1} = (q_1, n_1, \ldots, n_p)$ in case $n_k = 0$,
  - $\mathbf{c}_{n+1} = (q_2, n_1, \ldots, n_{k-1}, n_k - 1, n_{k+1}, \ldots, n_p)$ otherwise.
- If $q = h$, then the machine halts and $\mathbf{c}_{n+1}$ is consequently not defined.

(ii) Let P_n be the λ-terms of type o inductively defined by $P_0 = e^o$ and:

- If $\mathbf{c}_n = (q, \ldots)$ and $\mathrm{Instr}(q) = \mathrm{Inc}(k, \ldots)$, then $P_{n+1} = x_k^1 P_n$.

– If $\mathbf{c}_n = (q, \ldots)$ and $\mathrm{Instr}(q) = \mathrm{Jzd}(k, \ldots)$, then $P_{n+1} = l^3(\lambda v^1.Q)$, where the λ-term Q is obtained from P_n by replacing its rightmost occurrence of x_k^1 with v if any, and $Q = P_n$ otherwise.
– If $\mathbf{c}_n = (h, \ldots)$, then the machine halts and P_{n+1} is not defined.

(iii) let $\mathcal{F}$ be the FTS of λ^τ over the ground domain:

$$\mathcal{F}^o = (\Sigma \cup \{\bot, \top\}) \times \{0,1\}^p.$$

The basic role played by the terms P_n in the sequel is to encode the sequence of the instructions performed by M. It will be the duty of some valuation ρ into $\mathcal{F}$ to check from the interpretation $[\![P]\!]_\rho$ of any λ-term P^o whether P is a correct encoding of the computations of M until some step n (*i.e.* $P = P_n$) and whether the halting state h is reached at this step. All this works so far in a similar way as in the former refutation of the Statman conjecture by R. Loader (*cf* [Loa01]), where the Word Problem is encoded, λ-terms are used to denote the sequences of rewritings and where a FTS checks whether a λ-term denotes a sequence of rewritings reaching a given word.

The λ-terms P_n have to play here another important role. Since the computations are encoded whithin the single type $\mathbb{M}$ (note that the terms P_n are just subterms of closed normal λ-terms $N^{\mathbb{M}}$), the FTS $\mathcal{F}$ may only keep track of a finitely bounded information at every step of the computation (this is not the case in R. Loader's encoding, which uses infinitely many types instead of $\mathbb{M}$). Therefore, the terms P_n have to memorize what the FTS $\mathcal{F}$ cannot, namely the content of the registers of M. It is so by virtue of the following.

Remark. For every n such that $\mathbf{c}_n = (q, n_1, \ldots, n_p)$ is defined, n_i is the number of occurrences of x_i in P_n. This can be checked by a straightforward induction on n.

But the main compensation for the restriction to the single type $\mathbb{M}$ is the essential fact that $\mathcal{F}$ depends on the register machine M, whereas the FTS used in R. Loader's encoding is quite independent of the rewriting system considered. The first component of an element of $\mathcal{F}^o$ recalls the current state, $\bot$ is for rejection (in case the term read is not one of the terms P_n) and the additional state $\top$ is used to test the occurrences of the variable v in a term Q such that $Q[v := x_k] = P_n$, in order to check whether $l^3(\lambda v^1.Q) = P_{n+1}$. The other components are flags recalling whether the p registers currently contain 0 or not.

Lemma 1. *There is a valuation ρ in $\mathcal{F}$ which is recursively defined from the register machine* M *and such that for every normal λ-term P^o with free variables among $l^3, e^o, x_1^1, \ldots, x_p^1$ and interpretation say $[\![P]\!]_\rho = (c, f_1, \ldots, f_p)$:*

$$c \in \Sigma \ \textit{ iff for some } n,\ P = P_n,\ \boldsymbol{c}_n = (c, n_1, \ldots, n_p) \textit{ and } \forall i\ f_i = 0 \Leftrightarrow n_i = 0.$$

Proof. Let INC_i and TEST_i $(1 \leqslant i \leqslant p)$ be the functionals of $\mathcal{F}^1$ such that:

$$\mathrm{INC}_i(a, f_1, \ldots, f_p) = (c, f_1, \ldots, f_{i-1}, 1, f_{i+1}, \ldots, f_p),$$
$$\text{where } c = \left[\begin{array}{ll} q & \text{if } a \in \Sigma \setminus \{h\} \text{ and } \mathrm{Instr}(a) = \mathrm{Inc}(k, q), \\ \top & \text{if } a = \top, \\ \bot & \text{otherwise.} \end{array}\right.$$
$$\mathrm{TEST}_i(a, f_1, \ldots, f_p) = (c, f_1, \ldots, f_p), \text{ where } c = \left[\begin{array}{ll} \top & \text{if } a \notin \{\bot, \top\} \text{ and } f_i = 0, \\ \bot & \text{otherwise.} \end{array}\right.$$

Let us define a functional JZD of $\mathcal{F}^3$ by the following. For any $\phi \in \mathcal{F}^2$, let $\phi\,\mathrm{Id}_{\mathcal{F}^o} = (a, f_1, \ldots, f_p)$ and for all i, let $\phi\,\mathrm{INC}_i = (a_i, \ldots)$, $\phi\,\mathrm{TEST}_i = (b_i, \ldots)$. Then $\mathrm{JZD}\,\phi = (c, f_1, \ldots, f_p)$ where:

$$c = \left[\begin{array}{ll} q_1 & \text{if } a_1 = \ldots = a_p = q \in \Sigma \setminus \{h\},\ \mathrm{Instr}(q) = \mathrm{Jzd}(k, q_1, q_2) \text{ and } f_k = 0, \\ q_2 & \text{if for some } k,\ \mathrm{Instr}(a_k) = \mathrm{Jzd}(k, q_1, q_2),\ \forall i \neq k\ \ a_i = \bot \text{ and } b_k = \top, \\ \top & \text{if } a_k = \top \text{ for some } k, \\ \bot & \text{if none of the previous cases applies.} \end{array}\right.$$

At last, let $\mathcal{S}$ be the set of the normal λ-terms of type o whose free variables are among $l^3, e^o, x_1^1, \ldots, x_p^1, z_1^1, \ldots, z_p^1$ and let ρ be a valuation such that $\rho(l^3) = \mathrm{JZD}$, $\rho(e^o) = (s, 0, \ldots, 0)$, $\rho(x_i^1) = \mathrm{INC}_i$ and $\rho(z_i^1) = \mathrm{TEST}_i$ $(1 \leqslant i \leqslant p)$.

Let us first prove for every $P \in \mathcal{S}$ with interpretation $[\![P]\!]_\rho = (c, f_1, \ldots, f_p)$:

$$f_i = 1 \ \Leftrightarrow\ x_i^1 \text{ is free in } P \tag{1}$$

by induction on P. Since it comes at once from the definitions of $\rho(e^o)$, INC_k, TEST_k if $P = e^o$, $P = x_k Q$ or $P = z_k Q$, we only consider here the remaining case: $P = l^3(\lambda v^1.Q^o)$. The λ-term $(\lambda v^1.Q)(\lambda y^o.y)$ reduces to a λ-term P_0 of $\mathcal{S}$ having exactly the same free variables x_i as P and by induction hypothesis, $[\![P_0]\!]_\rho$ has the form $(a, g_1, \ldots, g_p)$ where $g_i = 1$ iff x_i is free in P_0, so that we only have to check: $f_i = g_i$ for all i. But this follows at once from the definition of JZD, because $[\![P]\!]_\rho = \mathrm{JZD}\,[\![\lambda v^1.Q]\!]_\rho$ and $[\![\lambda v^1.Q]\!]_\rho \mathrm{Id}_{\mathcal{F}^o} = [\![(\lambda v^1.Q)(\lambda y^o.y)]\!]_\rho = [\![P_0]\!]_\rho$.

In case $P = P_n$ and $\mathbf{c}_n = (c, n_1, \ldots, n_p)$, we get from (1) and the remark above: $f_i = 0 \Leftrightarrow n_i = 0$ for all i.

Let us now prove that the first component of $[\![P]\!]_\rho$ is given by:

$$c = \left[\begin{array}{ll} q \in \Sigma & \text{if for some } n,\ P = P_n \text{ and } \mathbf{c}_n = (q, \ldots), \\ \top & \text{if } P \in \mathcal{T}, \\ \bot & \text{in every other case.} \end{array}\right. \tag{2}$$

where $\mathcal{T} \subset \mathcal{S}$ is the set of the terms M such that for some k, M has exactly one free occurrence of z_k, none of the other variables z_i and such that the largest subterm in which z_k is not free can be given the form P_n where $x_k \notin P_n$ by substituting variables $x_1, \ldots, x_p$ for its free variables that are bound in M.

Here again, the induction on $P \in \mathcal{S}$ is straightforward from the above definitions whenever $P = e^o$, $P = x_k Q$ or $P = z_k Q$, and we just detail the case where $P = l^3(\lambda v^1.Q^o)$. Let $\phi = [\![\lambda v^1.Q]\!]_\rho$ and let us put like in the definition of JZD: $\phi\,\mathrm{Id}_{\mathcal{F}^o} = [\![Q]\!]_{\rho[v\leftarrow \mathrm{Id}_{\mathcal{F}^o}]} = (a, f_1, \ldots, f_p)$, $\phi\,\mathrm{INC}_i = [\![Q[v\!:=\!x_i]]\!]_\rho = (a_i, \ldots)$, $\phi\,\mathrm{TEST}_i = [\![Q[v\!:=\!z_i]]\!]_\rho = (b_i, \ldots)$.

By the definition of $\mathcal{T}$, we have $P \in \mathcal{T}$ iff $Q[v := x_i] \in \mathcal{T}$ for some i, i.e. by induction hypothesis, iff $a_i = \top$ for some i; moreover, the latter is equivalent to $c = \top$. We may therefore assume $P \notin \mathcal{T}$, $Q[v\!:=\!x_i] \notin \mathcal{T}$ and $a_i \in \Sigma \cup \{\bot\}$ for all i. In the case where no λ-term $Q[v\!:=\!x_i]$ is of the form P_n, neither is P and $a_i = \bot$ for all i by induction hypothesis, so that $[\![P]\!]_\rho = \bot$ and we are done again. Let us then put $Q[v\!:=\!x_k] = P_n$; by induction hypothesis, it follows $\mathbf{c}_n = (a_k, \ldots)$. If $a_k = h$ or $\mathrm{Instr}(a_k)$ has not the form $\mathrm{Jzd}(k, q_1, q_2)$, then P is not one of the terms P_i and the definition of JZD yields $[\![P]\!]_\rho = \bot$, hence (2) holds. We may therefore assume $\mathrm{Instr}(a_k) = \mathrm{Jzd}(k, q_1, q_2)$.

- If v does not occur free in Q, then we obtain $a_1 = \ldots = a_p = [\![Q]\!]_\rho$. In the case where x_k is not free either in $Q = P_n$, we get $f_k = n_k = 0$ by (1), hence $[\![P]\!]_\rho = q_1$ and $P = P_{n+1}$. Otherwise, $f_k = 1$, $[\![P]\!]_\rho = \bot$ and P has not the form P_i. This yields (2) in both cases.
- If v occurs free in Q, then there is no λ-term $Q[v\!:=\!x_i]$ of the form P_m except $Q[v\!:=\!x_k] = P_n$; hence by induction hypothesis, we get $a_i = \bot$ for all $i \neq k$.
  - If Q is obtained from $Q[v\!:=\!x_k]$ by replacing its rightmost occurrence of x_k, then $Q[v\!:=\!z_k] \in \mathcal{T}$ by the definition $\mathcal{T}$ and by induction hypothesis, $b_k = \top$. Hence, $[\![P]\!]_\rho = q_2$ and (2) follows since in this case, $P = P_{n+1}$.
  - Otherwise, there are several occurrences of v in Q or x_k occurs free on the right of v, so that P has not the form P_i and $Q[v := z_k] \notin \mathcal{T}$. The induction hypothesis gives then $b_k \neq \top$, $[\![P]\!]_\rho = \bot$ and therefore (2).

□

Proposition 2. *For every register machine* M *with* m *states and* p *registers say, there are* JZD $\in \mathcal{F}^3_k$, S $\in \mathcal{F}^o_k$ *and* H $\in \mathcal{F}^o_k$ *where* $k = 2^p(m+2)$ *which are recursively defined from* M *and such that:*

$$\mathrm{M}\ \textit{halts with cleared registers} \quad \Leftrightarrow \quad \exists X^{\mathbb{M}} \in \Lambda^{\varnothing} \quad [\![X]\!]\,\mathrm{JZD}\,\mathrm{S} = \mathrm{H}.$$

Proof. The FTS $\mathcal{F}_k$ where $k = 2^p(m+2)$ may be identified with the FTS $\mathcal{F}$ defined above and for any register machine M with m states and p registers, we may consider a valuation ρ in $\mathcal{F}_k$ as in the lemma. From the latter and the remark made above, it follows that M halts with cleared registers iff there is a normal λ-term P^o with free variables among l^3, e^o such that $[\![P]\!]_\rho = (h, 0, \ldots, 0)$. We may therefore take $\mathrm{JZD} = \rho(l^3)$, $\mathrm{S} = \rho(e^o)$ and $\mathrm{H} = (h, 0, \ldots, 0)$. □

The point of halting with cleared registers is not a serious modification of the Halting Problem. Indeed, we may transform uniformly every machine M into a machine M′ which halts with cleared registers iff M halts, *e.g.* by replacing the halting state of M with new states $q_1, \ldots, q_{p+1}$ such that $\mathrm{Instr}(q_i) = \mathrm{Jzd}(i, q_{i+1}, q_i)$ $(1 \leqslant i \leqslant p)$, where p is the number of registers of M and q_{p+1} the new halting state. In the same way, the fact that we consider only machines starting with clear registers is not a restriction, since such a single-computation machine can be obtained for every more general machine and every input data by inserting instructions Inc at the initial state of the latter machine. Therefore, we get from the previous proposition:

Proposition 3. *The following problem is undecidable.*

> *"Given elements $s, h \in \mathcal{F}_n^o$ and $\varphi \in \mathcal{F}_n^3$ of any full type structure $\mathcal{F}_n$, is there a closed λ-term $X^{\mathbb{M}}$ such that $[\![X]\!]\, \varphi\, s = h$?"*

Theorem 4. $\mathrm{DP}_{\mathbb{M}}$ *is undecidable.*

Proof. Indeed, otherwise we could check out for any $s, h \in \mathcal{F}_n^o$, $\varphi \in \mathcal{F}_n^3$ whether one of the finitely many $\psi \in \mathcal{F}_n^{\mathbb{M}}$ such that $\psi\varphi s = h$ is λ-definable, but this contradicts Proposition 3.[2] □

3 Around the Higher Order Pattern Matching Problem

Let us explain briefly the equivalence of MP_A^+ and DP_A stated in the introduction. Let $PX^+ =_{\beta\eta} Q$ $(+ = [o := 1_n])$ be any instance of MP_A^+. By virtue of a famous theorem by R. Statman (*cf* [Sta82]), we may compute k such that for all $M \in \Lambda^\varnothing$, $\mathcal{F}_k \vDash M = Q \;\Leftrightarrow\; M =_{\beta\eta} Q$, so that the question is whether some functional of $\mathcal{E} = \{\varphi \in \mathcal{F}_k^{A^+} ; [\![P]\!]^{\mathcal{F}_k}\varphi = [\![Q]\!]^{\mathcal{F}_k}\}$ has the form $[\![X^+]\!]^{\mathcal{F}_k}$ $(X \in \Lambda^\varnothing)$. Moreover, we may identify $\mathcal{F}_k^{1_n}$ with the ground domain $\mathcal{F}_{k^{k^n}}^o$ and check that we have $[\![X^+]\!]^{\mathcal{F}_k} = [\![X]\!]^{\mathcal{F}_{k^{k^n}}}$ for all $X \in \Lambda^\varnothing$, so that the question is now whether some $\varphi \in \mathcal{E} \subseteq \mathcal{F}_{k^{k^n}}^A$ is λ-definable. Therefore, if DP_A is decidable, then so is MP_A^+. The converse comes from a well known reconstruction of the FTS $\mathcal{F}_n$ as the type structure $\mathcal{S}/{\sim} = \bigcup_A (\mathcal{S}^A/{\sim})$ where $\mathcal{S}^A$ is the set of the closed λ-terms of type A^+ $(+ = [o := 1_n])$ and where $\sim$ is the observational equivalence defined by $M \sim N \Leftrightarrow \forall C^{o^+}[\ ]\ C[M] =_{\beta\eta} C[N]$. Indeed, the usual argument by induction on the reduction length of $C[M]$ establishes $\forall N \in \mathcal{S}^A\ MN \sim M'N \;\Rightarrow\; M \sim M'$, and therefore the extensionality of the structure $\mathcal{S}/{\sim}$. Moreover, we may easily build by a single induction on the type A λ-terms $R_\varphi \in \mathcal{S}^A$, $P_\varphi \in \mathcal{S}^{A \to o}$ for all $\varphi \in \mathcal{F}_n^A$ such that $\{R_i ; i \in \mathcal{F}_n^o\} = \mathcal{S}^o$, $R_\varphi R_\psi \sim R_{\varphi\psi}$ and $P_\varphi R_\psi \sim R_1 \;\Leftrightarrow\; R_\varphi \sim R_\psi$; it follows that we get an isomorphism of $\mathcal{F}_n$ onto $\mathcal{S}/{\sim}$ by mapping every φ to the observational class of R_φ. At last, we may check that the interpretation of any $X \in \Lambda^\varnothing$ in $\mathcal{S}/{\sim}$ is just the observational class of X^+. Hence for all $X^A \in \Lambda^\varnothing$, $[\![X]\!] = \varphi \;\Leftrightarrow\; X^+ \sim R_\varphi \;\Leftrightarrow\; P_\varphi X^+ \sim R_1 \;\Leftrightarrow\; P_\varphi X^+ =_{\beta\eta} R_1$. This reduces DP_A to MP_A^+ and we may conclude:

$$\mathrm{DP}_A \text{ decidable} \;\Leftrightarrow\; \mathrm{MP}_A^+ \text{ decidable.}$$

[2] Note that Proposition 3 may in turn be easily derived from this apparently weaker theorem by considering the cartesian products $(\mathcal{F}_n)^k$. Indeed, the λ-definability of $\psi \in \mathcal{F}_n^{\mathbb{M}}$ is equivalent to $\exists X \in \Lambda^\varnothing\ [\![X]\!]\boldsymbol{\varphi s} = \boldsymbol{\psi\varphi s}$ where $\boldsymbol{\psi} = (\psi, \ldots, \psi) \in (\mathcal{F}_n^{\mathbb{M}})^k$, $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_k)$, $\boldsymbol{s} = (s_1, \ldots, s_k)$, $\{(\varphi_i, s_i); 1 \leqslant i \leqslant k\} = \mathcal{F}_n^3 \times \mathcal{F}_n^o$ and the latter is equivalent to the existence of $\varphi \in \mathcal{F}_{n^k}^{\mathbb{M}}$ such that $\exists X \in \Lambda^\varnothing\ [\![X]\!]\varphi\boldsymbol{s} = \boldsymbol{\psi\varphi s}$ and $f(\varphi) = \boldsymbol{\varphi}$ where $f : \mathcal{F}_{n^k} \to (\mathcal{F}_n)^k$ is the partial surjective morphism defined in [Fri75].

In particular, Proposition 3 and Theorem 4 can also be restated as follows.

Proposition 5. *The following problem is undecidable.*

> *"Given $M^{3^+}, S^{o^+}, H^{o^+} \in \Lambda^{\varnothing}$ where $+$ is any type substitution $[o{:=}1_n]$, is there a closed λ-term $X^{\mathbb{M}}$ such that $X^+MS =_{\beta\eta} H$?"*

Proof. Let $\mathcal{F}_n^o = \{1, \ldots, n\}$. For any functional c of the 1-section of $\mathcal{F}_n$, say $c \in \mathcal{F}_n^{1_k}$, we define by induction on k a closed λ-term $|c|^{1_k^+}$ as follows: if $k = 0$ then $|c| = \lambda z_1^o \ldots z_n^o.z_c$ else $|c| = \lambda x^{o^+}\vec{a}.x(|c\,1|\vec{a})\ldots(|c\,n|\vec{a})$ where $\vec{a} = x_2^{o^+}\ldots x_k^{o^+} z_1^o \ldots z_n^o$. Let us put $\mathcal{F}_n^1 = \{c_1, \ldots, c_r\}$ $(r = n^n)$ and let us define $|\varphi|^{3^+}$ for all $\varphi \in \mathcal{F}_n^3$ by $|\varphi|^{3^+} = \lambda f^{2^+}.|c|(f|c_1|)\ldots(f|c_r|)$, where c is any element of $\mathcal{F}_n^{1_r}$ such that $c(ac_1)\ldots(ac_r) = \varphi a$ for all $a \in \mathcal{F}_n^2$. From these definitions, we may check by induction on the normal λ-terms N^o with free variables among l^3, e^o, x_i^1 $(i \geqslant 0)$:

$$|[\![N]\!]_\rho| =_{\beta\eta} N^+[l{:=}|\rho(l)|,\, e{:=}|\rho(e)|,\, x_i{:=}|\rho(x_i)|\,]_{i\geqslant 0}.$$

It follows $|[\![X]\!]\,\varphi\,s| =_{\beta\eta} X^+|\varphi|\,|s|$ for all $X^{\mathbb{M}} \in \Lambda^{\varnothing}$, $\varphi \in \mathcal{F}_n^3$, $s \in \mathcal{F}_n^o$, hence any equation $[\![X]\!]\,\varphi\,s = h$ is equivalent to $X^+|\varphi|\,|s| =_{\beta\eta} |h|$ and we are done by Proposition 3. $\square$

Recall that we may replace the equation in either Matching Problem (MP, MP$^+$, β-MP) with several ones of the same kind without generalizing the problem. Indeed, a system of finitely many equations $P_iX^{(+)} =_{\beta(\eta)} Q_i$ $(1 \leqslant i \leqslant n)$ is equivalent to the single equation $PX^{(+)} =_{\beta(\eta)} Q$ where $P = \lambda xc.c(P_1x)\ldots(P_nx)$ and $Q = \lambda c.cQ_1\ldots Q_n$. By virtue of this remark, Proposition 5 and the following lemma yield a fairly simple proof of the undecidability of β-MP.

Lemma 6. *For any type substitution $+$ and any $Y^{\mathbb{M}^+} \in \Lambda^{\varnothing}$,*

$$\exists X^{\mathbb{M}} \in \Lambda^{\varnothing} \quad Y =_\beta X^+ \quad \Leftrightarrow \quad Y(\lambda f^{2^+}.f\mathbf{I}^{1^+}) =_\beta \mathbf{I}^{1^+}.$$

Proof. Let $B^3 = \lambda f^2.f\mathbf{I}^1$. If $Y =_\beta X^+$ $(X \in \Lambda^{\varnothing})$, then XB is a closed λ-term of type 1, hence $XB =_\beta \mathbf{I}^1$ and $YB^+ =_\beta (XB)^+ =_\beta \mathbf{I}^{1^+}$.

For the converse, we may assume w.l.o.g. that Y is β-normal and consider a particular reduction sequence $C \twoheadrightarrow_\beta \mathbf{I}^{1^+}$, where C is the contractum of the redex YB^+. Let us say that a β-redex is *nice* whenever it has one of the forms B^+M, $M^{2^+}\mathbf{I}^{1^+}$, $\mathbf{I}^{1^+}M$ or M^TN where T is a subtype of o^+ (possibly $T = o^+$). We may easily check that if $P \to_\beta Q$ and if all the β-redexes of P are nice then so are the ones of Q and that if moreover Q is of the form X^+, then so is P. It follows that all the β-redexes of the terms of the sequence $C \twoheadrightarrow_\beta \mathbf{I}^{1^+}$ are nice and that C has the form X^+. By the definition of C, Y is then also of the form X^+. $\square$

Theorem 7. (R. Loader, [Loa03]) *The problem* β-MP *is undecidable.*

Proof. Indeed, remark that for all $G^{o^+}, H^{o^+} \in \Lambda^{\varnothing}$ where $+ = [o{:=}1_n]$, we have $G =_{\beta\eta} H \Leftrightarrow G =_\beta H$. By Lemma 6, we then get for any given $M^{3^+}, S^{o^+}, H^{o^+} \in \Lambda^{\varnothing}$:

$$\exists X^{\mathbb{M}} \in \Lambda^{\varnothing} \quad X^+MS =_{\beta\eta} H \quad \Leftrightarrow \quad \exists Y^{\mathbb{M}^+} \in \Lambda^{\varnothing} \quad \begin{bmatrix} YMS =_\beta H \\ Y(\lambda f^{2^+}.f\mathbf{I}) =_\beta \mathbf{I}^{1^+}. \end{bmatrix}$$

If β-MP were decidable, then we could decide the right hand side of this equivalence, but this contradicts Proposition 5. $\square$

The absence of the η-rule is strongly used above in order to sift out the λ-terms not of the form X^+, and an equation like the one of Lemma 6 does not seem to exist in the case of the $\beta\eta$-equality, even if we allow an additional variable substitution[3] "$Y = FZ$", i.e. even if we look for F, P, Q such that:

$$\exists X^{\mathbb{M}} \in \Lambda^{\varnothing} \quad Y =_{\beta\eta} X^+ \quad \Leftrightarrow \quad \exists Z^A \in \Lambda^{\varnothing} \quad \left[\begin{array}{l} Y =_{\beta\eta} FZ \\ PZ =_{\beta\eta} Q. \end{array}\right.$$

4 Back to the Definability Problem

Although this task of distinguishing the λ-terms $X^{\mathbb{M}^+}$ of the form X^+ does not seem possible by means of $\beta\eta$-equations of λ^{τ}, it is done by any FTS $\mathcal{F}_q$ such that $q \geqslant 3$ (see Lemma 11 below), so that another consequence of Proposition 5 is the undecidability of the Definability Problem for these FTS. For convenience, we adopt in the sequel the following conventions and notations.

Definition II. (i) We choose arbitrarily in every FTS $\mathcal{F}_n$ such that $n \geqslant 2$ two distinct elements of $\mathcal{F}_n^o$ that we denote $\top, \bot$.
(ii) Let $\Lambda_{\bot}$ (resp. $\Lambda_{\bot}^{\varnothing}$) be the set of the λ-terms (resp. the closed λ-terms) where an additional constant $\bot$ of type o may occur. This constant $\bot$ is interpreted in every FTS $\mathcal{F}_n$ $(n \geqslant 2)$ by $\bot \in \mathcal{F}_n^o$.
(iii) Let ι be the type substitution $[o := 1]$ and let $\mathbb{N} = 1^{\iota} = 1 \to 1$.
(iv) For any type substitution τ, let $\mathcal{S}_{\tau} = \{M \in \Lambda^{\varnothing} \ ; \ \exists X \in \Lambda^{\varnothing} \quad M =_{\beta\eta} X^{\tau}\}$.

Lemma 8. *For any $q \geqslant 3$, there is $\psi \in \mathcal{F}_q^{\mathbb{N}^{\iota} \to o}$ such that for all $M^{\mathbb{N}^{\iota}} \in \Lambda_{\bot}^{\varnothing}$,*

$$M \in \mathcal{S}_{\iota} \quad \Leftrightarrow \quad \psi \, [\![M]\!] = \top.$$

Proof. Let $a \in \mathcal{F}_q^o \setminus \{\top, \bot\}$ and let ρ be a valuation into $\mathcal{F}_q$ such that for every variable z of type o, we have $\rho(z) = a$ and for some variables $e^1, x^{\mathbb{N}}$:

$$\left[\begin{array}{l} \rho(e^1)\, a = \top, \\ \rho(e^1)\, c = \bot \ \text{ if } c \neq a, \end{array}\right. \qquad \left[\begin{array}{l} \rho(x^{\mathbb{N}})\, \rho(e^1) = \rho(e^1), \\ \rho(x^{\mathbb{N}})\, \varphi = \lambda\!\!\lambda d.\bot \ \text{ if } \varphi \neq \rho(e^1). \end{array}\right.$$

We may check by a straightforward induction on all the long normal λ-terms $G^1 \in \Lambda_{\bot} \setminus \{\mathbf{I}^1\}$ with free variables among $x^{\mathbb{N}}, e^1, z_i^o$ $(i \geqslant 0)$, that if G η-reduces to a λ-term of the form X^{ι}, then $[\![G]\!]_{\rho} = \rho(e)$ else $[\![G]\!]_{\rho} = \lambda\!\!\lambda d.c$ for some $c \in \mathcal{F}_q^o$. In particular we get for all $M^{\mathbb{N}^{\iota}} \in \Lambda_{\bot}^{\varnothing}$, $M \in \mathcal{S}_{\iota} \Leftrightarrow [\![M]\!]\, \rho(x)\, \rho(e) = \rho(e)$, so that we are done by taking any $\psi \in \mathcal{F}_q^{\mathbb{N}^{\iota} \to o}$ such that $\varphi\, \rho(x)\, \rho(e) = \rho(e) \ \Leftrightarrow \ \psi\, \varphi = \top$ for all $\varphi \in \mathcal{F}_q^{\mathbb{N}^{\iota}}$. $\square$

[3] This technique is used in [Loa03] to get the undecidability of β-PM.

Definition III. Let us define the λ-terms:

- $\mathrm{R}^{\mathbb{M}\to\mathbb{N}} = \lambda r^{\mathbb{M}} x^1.r(\lambda f^2.x(fx)) \in \Lambda^{\varnothing}$,
- $\mathrm{P}_{n,k}^{1_n\to 1} = \lambda f^{1_n} z^o.f\bot\ldots\bot z\bot\ldots\bot$ where z is the k^{th} argument of f,
- $\mathrm{R}_{n,k}^{\mathbb{N}^+\to\mathbb{N}^\iota} = \lambda r^{\mathbb{N}^+} x^{1^\iota} e^{o^\iota}.\mathrm{P}_{n,k}(rX_{n,k}E_{n,k}) \quad (1\leqslant k\leqslant n)$,

where $+ = [o{:=}1_n]$, $X_{n,k}^{1^+} = \lambda f^{1_n} z_1^o\ldots z_n^o.x^{1^\iota}(\mathrm{P}_k f)z_k$ and $E_{n,k}^{o^+} = \lambda z_1^o\ldots z_n^o.e^{o^\iota}z_k$.

Lemma 9. *Let* $+ = [o{:=}1_n]$ $(n\geqslant 1)$. *We have for any* $M^{\mathbb{N}^+}\in\Lambda^{\varnothing}$,

$$M \in \mathcal{S}_+ \quad\Leftrightarrow\quad \forall k\in\{1,\ldots,n\}\ \ \mathrm{R}_{n,k}M \in \mathcal{S}_\iota.$$

Proof. Let $\sigma_k = [x^{1^+}{:=}X_{n,k},\ e^{o^+}{:=}E_{n,k}]$ and let N^{o^+} be any $\beta\eta$-normal λ-term with free variables among x^{1^+}, e^{o^+}, z_i^o $(i\geqslant 0)$. Through an easy case study, we may check by induction on the size of N that N has the form X^+ iff for every $k\in\{1,\ldots,n\}$, $\mathrm{P}_k N^{\sigma_k}$ $\beta\eta$-reduces to a λ-term of the form X^ι. It follows for any λ-term $M^{\mathbb{N}^+}$ that $M^{\mathbb{N}^+}\in\mathcal{S}_+$ iff for all k, $\mathrm{P}_k(MX_{n,k}E_{n,k})\in\mathcal{S}_\iota$. □

Lemma 10. *For any type substitution* τ *and any* $M^{\mathbb{M}^\tau}\in\Lambda^{\varnothing}$, *we have*

$$M \in \mathcal{S}_\tau \quad\Leftrightarrow\quad \mathrm{R}^\tau M \in \mathcal{S}_\tau.$$

Proof. Clearly, if $M\in\mathcal{S}^\tau$, say $M =_{\beta\eta} X^\tau$ $(X\in\Lambda^{\varnothing})$, then $R^\tau M =_{\beta\eta} (RX)^\tau\in\mathcal{S}^\tau$. Conversely, let σ be the variable substitution $[l^{3^\tau}{:=}\lambda f^{2^\tau}.x^{1^\tau}(fx^{1^\tau}),\ x_i^{1^\tau}{:=}x^{1^\tau}]_{i\geqslant 0}$ and let N^{o^τ} be any $\beta\eta$-normal λ-term with free variables among l^{3^τ}, e^{o^τ}, $x_i^{1^\tau}$, z_i^o $(i\geqslant 0)$. We may check by a straightforward induction on the size of N that if N^σ $\beta\eta$-reduces to a λ-term of the form X^τ then N has the form X^τ. It follows that a closed $\beta\eta$-normal λ-term $M^{\mathbb{M}^\tau} = \lambda l^{3^\tau}e^{o^\tau}.N^{o^\tau}$ has the form X^τ whenever $\mathrm{R}^\tau M =_{\beta\eta} \lambda x^{1^\tau}e^{o^\tau}.N^\sigma$ $\beta\eta$-reduces to a λ-term of the form X^τ. □

Lemma 11. *Let* $+ = [o{:=}1_n]$ $(n\geqslant 1)$. *For every* $q\geqslant 3$, *there is a functional* $\phi\in\mathcal{F}_q^{\mathbb{M}^+\to o}$ *such that for all* $M^{\mathbb{M}^+}\in\Lambda^{\varnothing}$,

$$M \in \mathcal{S}_+ \quad\Leftrightarrow\quad \phi\,[\![M]\!] = \top.$$

Proof. Indeed, for any $M^{\mathbb{M}^+}\in\Lambda^{\varnothing}$ we have by Lemmata 8, 9 and 10, $M\in\mathcal{S}_+$ iff for all $k\in\{1,\ldots,n\}$ $\psi\,[\![\mathrm{R}^+(\mathrm{R}_{n,k}M)]\!] = \top$, where ψ is like in Lemma 8. Hence, we are done by choosing ϕ such that for all $\varphi\in\mathcal{F}_q^{\mathbb{M}^+}$, $\phi\,\varphi = \top$ iff $\psi([\![\mathrm{R}^+\circ\mathrm{R}_{n,k}]\!]\,\varphi) = \top$ for all $k\in\{1,\ldots,n\}$. □

Theorem 12. *For every* $q\geqslant 3$, $\mathrm{DP}_{\mathcal{F}_q}$ *is undecidable.*

Proof. Note that for all $G^{o^+}, H^{o^+}\in\Lambda^{\varnothing}$ where $+ = [o{:=}1_n]$ $(n\geqslant 1)$, we have $G =_{\beta\eta} H$ iff $[\![G]\!]^{\mathcal{F}_q} = [\![H]\!]^{\mathcal{F}_q}$. Hence, we get for all $M^{3^+}, S^{o^+}, H^{o^+}\in\Lambda^{\varnothing}$:

$$\exists X^{\mathbb{M}}\in\Lambda^{\varnothing}\ \ X^+MS =_{\beta\eta} H \quad\Leftrightarrow\quad \exists Y^{\mathbb{M}^+}\in\Lambda^{\varnothing}\ \ \left[\begin{array}{l}[\![YMS]\!] = [\![H]\!]\\ \phi\,[\![Y]\!] = \top\end{array}\right.$$

where $\phi\in\mathcal{F}_q^{\mathbb{M}^+\to o}$ is like in the previous lemma. Now, the second member of this equivalence holds iff one of the finitely many $\varphi\in\mathcal{F}_q^{\mathbb{M}^+}$ such that $\varphi\,[\![M]\!]\,[\![S]\!] = [\![H]\!]$ and $\phi\,\varphi = \top$ is λ-definable. If $\mathrm{DP}_{\mathcal{F}_q}$ were decidable, then we could decide it, but this contradicts Proposition 5. □

5 Undecidability of $\mathrm{DP}_{\mathcal{F}_2}$

Unfortunately, Lemma 11 does not extend to the FTS $\mathcal{F}_2$. Indeed, every functional $\varphi \in \mathcal{F}_2^1$ satisfies $\varphi\circ\varphi\circ\varphi = \varphi$, so that e.g. $\mathcal{F}_2$ does not distinguish between the λ-terms (of type $\mathbb{M}^\iota$) $\lambda l^{3^\iota} e^1.e^1 \in \mathcal{S}_\iota$ and $\lambda l^{3^\iota} e^1.(e^1\circ e^1\circ e^1) \notin \mathcal{S}_\iota$. In order to derive the undecidability of $\mathrm{DP}_{\mathcal{F}_2}$ from Proposition 5, we will borrow a useful tool of [Loa03] that overcomes the lack of points in $\mathcal{F}_2$: surjective λ-terms $\mathsf{proj}_A^{A^\iota\to A}$ which allow us to replace artificially every domain $\mathcal{F}_2^A$ with the larger domain $\mathcal{F}_2^{A^\iota}$. We then may obtain in replacement of the functional of Lemma 11 a functional $\phi \in \mathcal{F}_2^{\mathbb{M}^{+\iota}\to o}$ such that:

$$Y^{\mathbb{M}^+}\in\mathcal{S}_+ \quad\Leftrightarrow\quad \exists X^{\mathbb{M}^{+\iota}}\in\Lambda^{\varnothing}\quad \begin{bmatrix} Y =_{\beta\eta} \mathsf{proj}_{\mathbb{M}^+}X \\ \phi[\![X]\!] = \top. \end{bmatrix}$$

Definition IV. (i) Let $\mathrm{N}_\perp^\iota$ be the set of the β-normal λ-terms $M \in \Lambda_\perp$ such that the type of every subterm (including M) has the form A^ι or is equal to o.
(ii) Let $\pi : \mathrm{N}_\perp^\iota \to \Lambda_\perp$ be the map such that every $\pi(M^{A^\iota})$ has type A and every $\pi(M^o)$ has type o, defined by:

- $\pi(x^{A^\iota}) = x^A$,
- $\pi(P^{A^\iota\to B^\iota}Q^{A^\iota}) = \pi(P)\pi(Q)$,
- $\pi(\lambda x^{A^\iota}.P^{B^\iota}) = \lambda x^A.\pi(P)$,
- $\pi(x^o) = \perp$,
- $\pi(P^1Q^o) = \pi(P)$,
- $\pi(\lambda x^o.P^o) = \pi(P)$.

(iii) For every type A of λ^τ, let $\mathsf{proj}_A^{A^\iota\to A}$ and $\mathsf{inj}_A^{A\to A^\iota}$ be the closed λ-terms mutually defined by:

- $\mathsf{proj}_o = \lambda x^1.x\perp,\ \ \mathsf{inj}_o = \lambda x^o d^o.x$,
- $\mathsf{proj}_{A\to B} = \lambda f^{A^\iota\to B^\iota}x^A.\mathsf{proj}_B(f(\mathsf{inj}_A x))$,
- $\mathsf{inj}_{A\to B} = \lambda f^{A\to B}x^{A^\iota}.\mathsf{inj}_B(f(\mathsf{proj}_A x))$.

The following lemma is essentially lemma 18 of [Loa03].

Lemma 13. *For any β-normal λ-term $M^{A^\iota}\in\Lambda_\perp^\varnothing$, we have* $\mathsf{proj}_A M =_{\beta\eta} \pi(M)$.

Proof. We obtain $\mathsf{proj}_A \circ \mathsf{inj}_A =_{\beta\eta} \mathbf{I}^{A\to A}$ from the definition of $\mathsf{proj}_A, \mathsf{inj}_A$ by a straightforward induction on A. Let σ be the substitution such that $\sigma(x^{A^\iota}) = \mathsf{inj}_A\, x^A$, $\sigma(y^o) = \perp$ for all the variables x^{A^ι}, y^o. We may check by a single induction on a λ-term $M^{A^\iota}\in \mathrm{N}_\perp^\iota$ that $\mathsf{proj}_A M^\sigma =_{\beta\eta} \pi(M)$ and moreover in the case where M is not an abstraction $M =_{\beta\eta} \mathsf{inj}_A\pi(M)$ (the latter relation implies clearly the previous one by virtue of the relation $\mathsf{proj}_A\circ\mathsf{inj}_A =_{\beta\eta} \mathbf{I}$). The lemma follows. □

Let us now introduce cousins[4] $\phi_A \in \mathcal{F}_2^{A\to o}$ of the λ-terms proj_A that distinguish the λI-terms.

[4] These functionals ϕ_A are actually interpretations of λ-terms θ_A and the λ-terms proj_A are instantiations of the same terms θ_A. The latters are themselves extracted from an intuitionistic proof of weak normalisation for λ^τ. Unsurprisingly, they compute

Definition V. (i) Let Λ_I be the set of the (possibly open) λI-terms where the constant $\perp$ does not occur.
(ii) Let $\mathrm{And} \in \mathcal{F}_2^{1_2}$, $\chi \in \mathcal{F}_2^2$ be the functionals such that: $\mathrm{And}\, a\, b = \top$ iff $a = b = \top$ and $\chi\, \varphi = \top$ iff $\varphi = \mathrm{Id}_{\mathcal{F}_2^o}$ and for any type A of λ^τ, let $\phi_A \in \mathcal{F}_2^{A\to o}$ and $\psi_A \in \mathcal{F}_2^{o\to A}$ be mutually defined by:

- $\phi_o = \psi_o = \mathrm{Id}_{\mathcal{F}_2^o}$,
- $\phi_{A\to B} = \lambda\!\!\lambda \varphi \in \mathcal{F}_2^{A\to B} . \chi(\phi_B \circ \varphi \circ \psi_A)$,
- $\psi_{A\to B} = \lambda\!\!\lambda a \in \mathcal{F}_2^o . \psi_B \circ (\mathrm{And}\, a) \circ \phi_A$.

(iii) Let ρ_I be the valuation into $\mathcal{F}_2$ such that $\rho_I(x^A) = \psi_A \top$ for all x^A.

Lemma 14. *For every β-normal λ-term $M^A \in \Lambda_\perp$,*

$$M^A \in \Lambda_I \quad \Leftrightarrow \quad \phi_A\, [\![M]\!]_{\rho_I} = \top\,.$$

Proof. We get at once $\phi_A \circ \psi_A = \mathrm{Id}_{\mathcal{F}_2^o}$ by induction on A. Let $\mathcal{V}$ be the set of the valuations ρ into $\mathcal{F}_2$ such that $\rho(x^A) = \psi_A \top$ or $\rho(x^A) = \psi_A \perp$ for all variables x^A. For any $\rho \in \mathcal{V}$, let us define $f_\rho : \Lambda_\perp \to \mathcal{F}_2^o$ by $f_\rho(M) = \top$ iff $M \in \Lambda_I$ and $\rho(x^A) = \psi_A \top$ for all the free variables x^A of M. We may check by a single induction on a β-normal λ-term $M^A \in \Lambda_\perp$ that we have for every $\rho \in \mathcal{V}$, $\phi_A\, [\![M]\!]_\rho = f_\rho(M)$ and moreover in the case where M is not an abstraction $[\![M]\!]_\rho = \psi_A(f_\rho(M))$ (note that the latter relation implies the previous one by virtue of the relation $\phi_A \circ \psi_A = \mathrm{Id}_{\mathcal{F}_2^o}$). The lemma follows from the particular case where $\rho = \rho_I$. $\square$

Lemma 15. *There is a functional $\psi \in \mathcal{F}_2^{\mathbb{N}^{\iota\iota}\to o}$ such that for every $M^{\mathbb{N}^{\iota\iota}} \in \Lambda_\perp^\varnothing$,*

$$M \in \mathcal{S}_{\iota\iota} \quad \Rightarrow \quad \psi\, [\![M]\!] = \top \quad \Rightarrow \quad \mathsf{proj}_{\mathbb{N}^\iota} M \in \mathcal{S}_\iota\,.$$

Proof. Let $\psi_k \in \mathcal{F}_2^{\mathbb{N}^{\iota\iota}\to o}$ $(0 \leqslant k \leqslant 2)$ be the functionals defined by:

- $\psi_0\, \varphi = \phi_{\mathbb{N}^\iota}(\varphi(\rho_I(x^{\mathbb{N}^\iota})))$,
- $\psi_1\, \varphi = \phi_{\mathbb{N}}([\![\, \mathsf{proj}_{\mathbb{N}^\iota}]\!]\, \varphi\, (\rho_I(x^{\mathbb{N}})))$,
- $\psi_2\, \varphi = \phi_1([\![\, \mathsf{proj}_{\mathbb{N}} \circ \mathsf{proj}_{\mathbb{N}^\iota}]\!]\, \varphi\, (\rho_I(x^1)))$,

From Lemmata 13 and 14, we get for any β-normal λ-term $M^{\mathbb{N}^{\iota\iota}} = \lambda x.N \in \Lambda_\perp^\varnothing$:

$$\psi_k\, [\![M]\!] = \top \quad \Leftrightarrow \quad \pi^k(N) \in \Lambda_I \qquad (0 \leqslant k \leqslant 2).$$

---

for every $M^A \in \Lambda^\varnothing$ some code $\underline{N}$ of its long normal form N^A: $\theta_A M^A \twoheadrightarrow_\beta \underline{N}$ (see [Jol00], Chapter 1 for details). Their interest lies in that the code $\underline{N}$ produced allows to interpret N in all kinds of λ-models and to make various syntactical treatments on it. E.g. the above functionals ϕ_A are obtained from a 1-element partial model of the untyped λ-calculus, as defined in [Sel96], in which the λ-terms interpreted are exactly the λI-terms: ϕ_A is just the interpretation of θ_A in a model environment which evaluates in this finite partial model the terms N through their code $\underline{N}$. Other interpretation and instantiation of the very same λ-terms θ_A are the functionals of [BS91] and the λ-terms $\lambda z^A.\tau_z^o$ of [Jol01a].

Let ρ be a valuation into $\mathcal{F}_2$ such that:

$$\begin{bmatrix} \rho(x^{\mathbb{N}})\,\lambda\!\!\lambda d.\top = \lambda\!\!\lambda d.\top, \\ \rho(x^{\mathbb{N}})\,\varphi = \lambda\!\!\lambda d.\bot \ \text{ if } \varphi \neq \lambda\!\!\lambda d.\top, \end{bmatrix} \qquad \begin{bmatrix} \rho(e^{\mathbb{N}})\,\mathrm{Id}_{\mathcal{F}_2^o} = \lambda\!\!\lambda d.\top, \\ \rho(e^{\mathbb{N}})\,\varphi = \lambda\!\!\lambda d.\bot \ \text{ if } \varphi \neq \mathrm{Id}_{\mathcal{F}_2^o}. \end{bmatrix}$$

At last, let us define $\psi_3 \in \mathcal{F}_2^{\mathbb{N}^{\iota\iota}\to o}$ by $\psi_3\,\varphi = \varphi\,[\![\lambda f^{\mathbb{N}}.\,x^{\mathbb{N}}\circ f]\!]_\rho\,\rho(e^{\mathbb{N}})\,\mathrm{Id}_{\mathcal{F}_2^o}\top$ and $\psi \in \mathcal{F}_2^{\mathbb{N}^{\iota\iota}\to o}$ by $\psi\,\varphi = \top$ iff $\psi_k\,\varphi = \top$ for all $k \leqslant 3$.

We may check by a straightforward induction on the long normal λ-terms $G^1 \in \Lambda_I \smallsetminus \{\mathbf{I}^1\}$ with free variables among $x^{\mathbb{N}}, e^{\mathbb{N}}, a_i^o$ $(i \geqslant 0)$, that if G has the form $\lambda v_n^o.x(\ldots(\lambda v_1^o.x(\lambda v_0^o.e\,\mathbf{I}^1 H_0^o)H_1^o)\ldots)H_n^o$, then $[\![G]\!]_\rho = \lambda\!\!\lambda d.\top$ else $[\![G]\!]_\rho = \lambda\!\!\lambda d.\bot$.

For every Church numeral $M^{\mathbb{N}^{\iota\iota}} = \lambda x^{\mathbb{N}^\iota} e^{\mathbb{N}}.x^n e$, we have $\pi^k(M) = \lambda xe.x^n e$ hence $\psi_k\,[\![M]\!] = \top$ for all $k \leqslant 2$. Moreover, $M(\lambda f^{\mathbb{N}}.\,x^{\mathbb{N}}\circ f)e^{\mathbb{N}}\mathbf{I}^1 \twoheadrightarrow x^n(e\,\mathbf{I}^1)$, hence $\psi_3\,[\![M]\!] = [\![M(\lambda f^{\mathbb{N}}.\,x^{\mathbb{N}}\circ f)e^{\mathbb{N}}\mathbf{I}^1]\!]_\rho\top = [\![x^n(e\,\mathbf{I}^1)]\!]_\rho\top = \top$ and therefore $\psi\,[\![M]\!] = \top$. This proves the first implication of the proposition.

Now, let $M^{\mathbb{N}^{\iota\iota}} = \lambda x.N \in \Lambda_\bot^\varnothing$ be any long normal λ-term such that $\psi\,[\![M]\!] = \top$. It follows $\psi_k\,[\![M]\!] = \top$ and then $\pi^k(N) \in \Lambda_I$ for all $k \leqslant 2$. We get in particular $\bot \notin \pi^2(N)$, hence $\pi^2(M)$ is a Church numeral, say $\pi^2(M) = \lambda x^1 e^o.x^n e$, and M has the form:

$$M = \lambda x^{\mathbb{N}^\iota} e^{\mathbb{N}} z_n^1 a_n^o.x(\ldots x(\lambda z_1^1 a_1^o.x(\lambda z_0^1 a_0^o.eP_0Q_0)P_1Q_1)P_2Q_2\ldots)P_nQ_n$$

for some λ-terms P_i^1, Q_i^o. Suppose for a contradiction that for some i the head variable of P_i is not z_i^1 and let k be the smallest such index i. The head variable of P_k could not be of type o, otherwise $\bot$ would occur in $\pi(N) \in \Lambda_I$. It could not be another variable z_i^1, otherwise $\pi(N)$ would not be a λI-term. Therefore, this head variable would be $x^{\mathbb{N}^\iota}$ or $e^{\mathbb{N}}$ and $M(\lambda f^{\mathbb{N}}.\,x^{\mathbb{N}}\circ f)e^{\mathbb{N}}\mathbf{I}^1$ would reduce into a long normal λ-term $G^1 = \lambda a_n^o.x(\ldots(\lambda a_1^o.x(\lambda a_0^o.eF^1H_0^o)H_1^o)\ldots)H_n^o$ where the head variable of F is $x^{\mathbb{N}}$ or $e^{\mathbb{N}}$. Moreover, we would have $G \in \Lambda_I$ since $N = \pi^0(N) \in \Lambda_I$ and $(N[x^{\mathbb{N}^\iota} := \lambda f^{\mathbb{N}}.\,x^{\mathbb{N}}\circ f])e^{\mathbb{N}}\mathbf{I}^1 \twoheadrightarrow G$. By the remark above, this would yield $[\![G]\!]_\rho = \lambda\!\!\lambda d.\bot$ and then $\psi_3\,[\![M]\!] = [\![G]\!]_\rho\top = \bot$, contradicting the assumption $\psi\,[\![M]\!] = \top$. It follows $\pi(P_i) = z_i^o$ for all i and by Lemma 13: $\mathsf{proj}_{\mathbb{N}^\iota}M =_{\beta\eta} \pi(M) = \lambda x^{\mathbb{N}} e^1 z_n^o.x(\ldots(\lambda z_1^o.x(\lambda z_0^o.e\,z_0)z_1)\ldots)z_n \twoheadrightarrow_\eta \lambda x^{\mathbb{N}} e^1.x^n e$. $\square$

Lemma 16. *Let $\iota_\bot = [o{:=}1,\ \bot{:=}\lambda d^o.\bot]$. We have:*

$$\begin{aligned} \mathsf{proj}_{o^\iota} \circ \mathrm{P}_{n,k}^{\iota_\bot} &=_{\beta\eta} \mathrm{P}_{n,k} \circ \mathsf{proj}_{o^+}, \\ \mathsf{proj}_{\mathbb{N}^\iota} \circ \mathrm{R}_{n,k}^{\iota_\bot} &=_{\beta\eta} \mathrm{R}_{n,k} \circ \mathsf{proj}_{\mathbb{N}^+}, \\ \mathsf{proj}_{\mathbb{N}^+} \circ \mathrm{R}^{+\iota} &=_{\beta\eta} \mathrm{R}^+ \circ \mathsf{proj}_{\mathbb{M}^+}. \end{aligned}$$

Proof. These $\beta\eta$-equalities are easily checked by reducing their members. $\square$

Theorem 17. $\mathrm{DP}_{\mathcal{F}_2}$ *is undecidable.*

Proof. Let $\psi \in \mathcal{F}_2^{\mathbb{N}^{\iota\iota}\to o}$ be like in Lemma 15 and let us prove for any closed λ-terms $M^{3^+}, S^{o^+}, H^{o^+}$:

$$\exists X^{\mathbb{M}} \in \Lambda^\varnothing\ \ X^+MS =_{\beta\eta} H \quad\Leftrightarrow\quad \exists Y^{\mathbb{M}^{+\iota}} \in \Lambda^\varnothing\ \begin{bmatrix} \mathsf{proj}_{\mathbb{M}^+}YMS =_{\beta\eta} H \\ \psi\,[\![\mathrm{R}_{n,k}^{\iota_\bot}(\mathrm{R}^{+\iota}Y)]\!] = \top \quad (1 \leqslant k \leqslant n) \end{bmatrix}$$

If some $X^{\mathbb{M}} \in \Lambda^{\varnothing}$ is such that $X^+MS =_{\beta\eta} H$ then $Y = X^{+\iota}$ satisfies the right hand side. Indeed, by Lemma 13 we then have $\mathsf{proj}_{\mathbb{M}^+} YMS =_{\beta\eta} H$. Moreover, for every k there is $Z \in \Lambda^{\varnothing}$ such that $\mathrm{R}_{n,k}(\mathrm{R}X)^+ =_{\beta\eta} Z^{\iota}$ by Lemma 9, hence $\mathrm{R}^{\iota\perp}_{n,k}(\mathrm{R}^{+\iota}Y) = (\mathrm{R}_{n,k}(\mathrm{R}X)^+)^{\iota\perp} =_{\beta\eta} Z^{\iota\iota}$ and by Lemma 15 $\psi\,[\![\mathrm{R}^{\iota\perp}_{n,k}(\mathrm{R}^{+\iota}Y)]\!] = \top$. Conversely, suppose that $Y^{\mathbb{M}^{+\iota}} \in \Lambda^{\varnothing}$ satisfies the right hand side. By Lemma 16, every λ-term $\mathrm{R}_{n,k}(\mathrm{R}^+(\mathsf{proj}_{\mathbb{M}^+} Y))$ converts to $\mathsf{proj}_{\mathbb{N}^{\iota}}(\mathrm{R}^{\iota\perp}_{n,k}(\mathrm{R}^{+\iota}Y))$ and has therefore the form Z^{ι} up to $\beta\eta$-equality by Lemma 15; hence by Lemmata 9 and 10 there is a closed λ-term $X^{\mathbb{M}}$ such that $\mathsf{proj}_{\mathbb{M}^+} Y = X^+$ and then $X^+MS =_{\beta\eta} H$.

Now, we have $\mathsf{proj}_{\mathbb{M}^+} YMS =_{\beta\eta} H$ iff $\mathcal{F}_2 \vDash \mathsf{proj}_{\mathbb{M}^+} YMS = H$ because the members of these relations have the type 1_n. Therefore, the right hand side of the equivalence above just states the existence of a λ-definable functional $\varphi \in \mathcal{F}_2^{\mathbb{M}^{+\iota}\to o}$ such that $[\![\mathsf{proj}_{\mathbb{M}^+}]\!]\,\varphi\,[\![M]\!]\,[\![S]\!] = [\![H]\!]$ and $(\psi \circ [\![\mathrm{R}^{\iota\perp}_k \circ \mathrm{R}^{+\iota}]\!])\varphi = \top$. It follows that if $\mathrm{DP}_{\mathcal{F}_2}$ were decidable, then we could decide whether there is $X^{\mathbb{M}} \in \Lambda^{\varnothing}$ such that $X^+MS =_{\beta\eta} H$ for any given $M^{3^+}, S^{o^+}, H^{o^+} \in \Lambda^{\varnothing}$, but this contradicts Proposition 5. □

Acknowledgement. I would like to thank Henk Barendregt, who offered me a long and peaceful time to achieve the present study. I am also greatly indebted to Ralph Loader, whose last work ([Loa03]) gave me the idea of Sections 3, 4 and 5.

References

[BS91] Ulrich Berger and Helmut Schwichtenberg. An inverse of the evaluation functional for typed λ-calculus. *6th Annual IEEE Symposium on Logic in Computer Science*, 203–211, 1991.

[Fri75] Harvey Friedman. Equality between functionals. *LNM*, 453:22–37, 1975.

[Hue75] Gérard Huet. A unification algorithm for typed λ-calculus. *TCS*, 1(1):27–57, 1975.

[Jol00] Thierry Joly. *Codages, séparabilité et représentation de fonctions en λ-calcul simplement typé et dans d'autres systèmes de types.* Thèse de Doctorat, Université Paris VII, January 2000.

[Jol01a] Thierry Joly. Constant time parallel computations in λ-calculus. *TCS*, 266:975–985, 2001.

[Jol01b] Thierry Joly. The finitely generated types of the λ-calculus. *LNCS*, 2044:240–252, 2001. (proc. TLCA'01).

[Loa01] Ralph Loader. The Undecidability of λ-Definability. In *Logic, Meaning and Computation: Essays in Memory of Alonzo Church*, 331–342, C.A. Anderson and M. Zelëny editors, Kluwer Academic Publishers, Dordrecht, 2001.

[Loa03] Ralph Loader. Higher β matching is undecidable. *Logic Journal of the IGPL*, Vol. 11, No. 1:51–68, 2003.

[Sel96] Peter Selinger. Order-Incompleteness and Finite Lambda Models. *11th Annual IEEE Symposium on Logic in Computer Science*, 432–439, 1996.

[Sta82] Richard Statman. Completeness, invariance and λ-definability. *JSL*, 47:17–26, 1982.

[Sta85] Richard Statman. Logical Relations and the Typed λ-calculus. *IC*, 65:85–97, 1985.