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Paperfolding infinite products and the gamma function

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ABSTRACT

Taking the product of $(2n+1)/(2n+2)$ raised to the power $+1$ or -1 according to the n-th term of the Thue–Morse sequence gives rise to an infinite product P while replacing $(2n+1)/(2n+2)$ with $(2n)/(2n+1)$ yields an infinite product Q, where

$$P = \left(\frac{1}{2}\right)^{+1}\left(\frac{3}{4}\right)^{-1}\left(\frac{5}{6}\right)^{-1}\left(\frac{7}{8}\right)^{+1}\cdots$$

and

$$Q = \left(\frac{2}{3}\right)^{+1}\left(\frac{4}{5}\right)^{-1}\left(\frac{6}{7}\right)^{-1}\left(\frac{8}{9}\right)^{+1}\cdots$$

Though it is known that $P = 2^{-1/2}$, nothing is known about Q. Looking at the corresponding question when the Thue–Morse sequence is replaced by the regular paperfolding sequence, we obtain two infinite products A and B, where

$$A = \left(\frac{1}{2}\right)^{+1}\left(\frac{3}{4}\right)^{+1}\left(\frac{5}{6}\right)^{-1}\left(\frac{7}{8}\right)^{+1}\cdots$$

and

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$$B = \left(\frac{2}{3}\right)^{+1}\left(\frac{4}{5}\right)^{+1}\left(\frac{6}{7}\right)^{-1}\left(\frac{8}{9}\right)^{+1}\cdots$$

Here nothing is known for A, but we give a closed form for B that involves the value of the gamma function at $1/4$. We then prove general results where $(2n+1)/(2n+2)$ or $(2n)/(2n+1)$ are replaced by specific rational functions. The corresponding infinite products have a closed form involving gamma values. In some cases there is no explicit gamma value occurring in the closed-form formula, but only trigonometric functions.

1. Introduction

Unexpected explicit values for infinite series or infinite products are somehow fascinating. One of the most famous examples is the celebrated *tour de force* of Euler addressing the Basel (or Mengoli) problem and finding the closed form

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

This formula is indeed unexpected: How could we guess that the square of the area of a circle with radius 1 would be occurring here? Of course this suggests looking at the more general sums $\zeta(k) = \sum_n \frac{1}{n^k}$: We know that Euler himself gave their values (also unexpected) for k an even integer, while the case where k is odd is still largely open, even after Apéry's proof of the irrationality of $\zeta(3)$ (see [30]), Ball–Rivoal's proof that infinitely many $\zeta(k)$ with k odd are irrational [6,26], and Zudilin's proof that at least one of the reals $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational [35].

Another classical closed formula, which also looks intriguing at first sight, is the infinite product discovered by Wallis about 80 years before Euler's result, namely

$$\frac{\pi}{2} = \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \frac{6}{7} \times \frac{8}{7} \times \frac{8}{9} \times \cdots = \prod_{n=1}^{\infty} \frac{(2n)(2n)}{(2n-1)(2n+1)}.$$

Again there is no immediate intuition why π should occur when multiplying the even numbers twice and dividing by the odd numbers twice.

Much less famous, but unexpected as well, is the Woods–Robbins infinite product [34,28]. Define $s(n)$ to be the sum of the binary digits of the integer n, and consider the sequence $(-1)^{s(n)}$. In other words $m_n = (-1)^{s(n)}$ can be defined by

$$m_0 = 1 \quad \text{and,} \quad \text{for all } n \geq 0, \quad m_{2n} = m_n \quad \text{and} \quad m_{2n+1} = -m_n.$$

This sequence is known as the (Prouhet–)Thue–Morse sequence, and its first terms are $+1, -1, -1, +1, \ldots$. Then

$$\left(\frac{1}{2}\right)^{+1}\left(\frac{3}{4}\right)^{-1}\left(\frac{5}{6}\right)^{-1}\left(\frac{7}{8}\right)^{+1}\cdots = \prod_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{m_n} = \frac{\sqrt{2}}{2}. \tag{1}$$

More general products of the form $\prod_n R(n)^{u_n}$ where $(u_n)_{n\geq 0}$ is a sequence with "regularity properties", and $R(n)$ is an ad hoc rational function can be looked at. Among possible sequences $(u_n)_{n\geq 0}$, one can think of *automatic sequences*: A sequence $(u_n)_{n\geq 0}$ is q-automatic for some integer $q \geq 2$ if its q-kernel, i.e., the set of subsequences $\{(u_{q^k n+j})_{n\geq 0},\ k \geq 0,\ j \in [0, q^k - 1]\}$ is finite (see, e.g., [4] to read more on automatic sequences). While finding closed forms of $\prod_n R(n)^{u_n}$ where $(u_n)_{n\geq 0}$ is any automatic sequence seems out of reach, closed forms for particular 2-automatic sequences were found (see, e.g., [1,2,5,21]). These are typically sequences $((-1)^{w_n})_{n\geq 0}$ where w_n counts the number of occurrences of some pattern in the binary expansion of n: For example the Thue–Morse sequence $((-1)^{s(n)})_{n\geq 0}$ (the sum of the binary digits of n is also the number of 1's in the binary expansion of n). Another classical, but totally different, 2-automatic sequence is the regular ± 1 paperfolding sequence ε_n which is obtained by iteratively folding a rectangular piece of paper, see, e.g., [4, pp. 155–156]. The regular paperfolding sequence can be defined by

$$\text{for all } n \geq 0, \quad \varepsilon_{2n} = (-1)^n \quad \text{and} \quad \varepsilon_{2n+1} = \varepsilon_n.$$

If we try to mimic one proof (see [3, Proposition 5, p. 6]) of equality (1) above, we can proceed as follows. Let A and B be the two infinite products (they are convergent as will be seen in Proposition 3 below) defined by

$$A := \prod_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{\varepsilon_n} \quad \text{and} \quad B := \prod_{n\geq 1}\left(\frac{2n}{2n+1}\right)^{\varepsilon_n}.$$

Multiplying these two products, we can write

$$AB = \frac{1}{2}\prod_{n\geq 1}\left(\frac{(2n+1)(2n)}{(2n+2)(2n+1)}\right)^{\varepsilon_n} = \frac{1}{2}\prod_{n\geq 1}\left(\frac{n}{n+1}\right)^{\varepsilon_n}.$$

Splitting the product on the right into odd and even indexes, we thus have

$$\begin{aligned} AB &= \frac{1}{2}\prod_{n\geq 1}\left(\frac{2n}{2n+1}\right)^{\varepsilon_{2n}}\prod_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{\varepsilon_{2n+1}} \\ &= \frac{1}{2}\prod_{n\geq 1}\left(\frac{2n}{2n+1}\right)^{(-1)^n}\prod_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{\varepsilon_n}. \end{aligned}$$

Since the last product is A ($\neq 0$), we have

$$\begin{aligned}
B &= \frac{1}{2}\prod_{n\geq 1}\left(\frac{2n}{2n+1}\right)^{(-1)^n} = \frac{1}{2}\prod_{n\geq 1}\left(\frac{4n}{4n+1}\right)\prod_{n\geq 0}\left(\frac{4n+3}{4n+2}\right) \\
&= \frac{1}{2}\prod_{n\geq 0}\left(\frac{(4n+4)(4n+3)}{(4n+5)(4n+2)}\right).
\end{aligned}$$

Using classical results on the gamma function (Proposition 1 below and the reflection formula) we obtain

$$\prod_{n\geq 1}\left(\frac{2n}{2n+1}\right)^{\varepsilon_n} = \frac{\Gamma(1/4)^2}{8\sqrt{2\pi}}. \tag{2}$$

The purpose of this paper is to show how to obtain closed forms for certain infinite products $\prod_n R(n)^{\varepsilon_n}$, where $R(n)$ belongs to some families of rational functions. These closed forms involve, as in equality (2), values of the gamma function (and/or of trigonometric functions).

2. Three preliminary results

In this section we give three preliminary results. The first two are classical.

Proposition 1. *(See, e.g., [33, Section 12-13, pp. 238–239].) Let d be a positive integer. Let $(a_i)_{1\leq i\leq d}$ and $(b_j)_{1\leq j\leq d}$ be complex numbers such that no a_i and no b_j belongs to $\{0,-1,-2,\ldots\}$. If $a_1+a_2+\cdots+a_d = b_1+b_2+\cdots+b_d$, then*

$$\prod_{n\geq 0}\frac{(n+a_1)\cdots(n+a_d)}{(n+b_1)\cdots(n+b_d)} = \frac{\Gamma(b_1)\cdots\Gamma(b_d)}{\Gamma(a_1)\cdots\Gamma(a_d)}.$$

Proposition 2. *(See, e.g., [33, Sections 12-14 and 12-15, pp. 239–240].) The gamma function satisfies the "reflection formula" and the "duplication formula" respectively given by*

$$\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z) \quad \textit{and} \quad 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2z).$$

The third result that we need is the asymptotic behavior of the summatory function of the (regular) ± 1 paperfolding sequence and an application to the convergence of certain series.

Proposition 3. *Let $(\varepsilon_n)_{n\geq 0}$ be the ± 1 paperfolding sequence.*

(i) *We have the upper bound*

$$\sum_{0\leq k<n}\varepsilon_k = O(\log n).$$

(ii) *Let f be a map from $\mathbb{R}$ to $\mathbb{C}$ such that, when x tends to infinity, $f(x)$ tends to zero and $|f(x+1) - f(x)| = O(1/x^a)$ for some $a > 1$. Then the series $\sum \varepsilon_n f(n)$ is convergent.*

Proof. (i) (Also see [4, Exercise 28, p. 206].) Let $S(n) = \sum_{0 \le k < n} \varepsilon_k$. Looking at $S(2n)$ and splitting the sum into even and odd indexes yields

$$\begin{aligned} S(2n) &= \sum_{0 \le k < 2n} \varepsilon_k = \sum_{0 \le k < n} \varepsilon_{2k} + \sum_{0 \le k < n} \varepsilon_{2k+1} = \sum_{0 \le k < n} (-1)^k + \sum_{0 \le k < n} \varepsilon_k \\ &= \sum_{0 \le k < n} (-1)^k + S_n = \frac{1 - (-1)^n}{2} + S_n. \end{aligned} \tag{3}$$

Let us prove that for all $n \ge 1$ we have $|S(n)| \le 1 + \log_2(n)$ (where $\log_2$ is the base 2 logarithm). Since $S(1) = \varepsilon_0 = 1 = 1 + \log_2 1$, it suffices to prove that if the inequality is true for n, then it is true for both $2n$ and $2n+1$. (Hint: This implies by induction on N that, if the property is true on $[1, 2^N - 1]$ then it is true on $[1, 2^{N+1} - 1]$.) Equality (3) above implies

$$\begin{aligned} |S_{2n}| &\le 1 + |S_n| \le 2 + \log_2(n) = 1 + \log_2(2n), \\ |S_{2n+1}| &= \left|S_{2n} + (-1)^n\right| = \frac{1 + (-1)^n}{2} + S_n \\ &\le 1 + |S_n| \le 2 + \log_2(n) = 1 + \log_2(2n) < 1 + \log_2(2n+1). \end{aligned}$$

(ii) The hypothesis on f implies $f(n) = O(1/n^{a-1})$ by telescopic cancellation, thus $S(N)f(N)$ tends to zero as N tends to infinity. The result is then an easy consequence of (i) by Abel summation. □

3. A general product involving the paperfolding sequence

We give here a general result, involving the ± 1 paperfolding sequence. (We use the notation $\mathbb{R}^+$ for the set of nonnegative real numbers and $\mathbb{R}^{+*}$ for the set of positive real numbers.)

Lemma 1. *Let g be a map from $\mathbb{R}^+$ to $\mathbb{R}^{+*}$ such that, when x tends to infinity, $g(x)$ tends to 1 and $g(x+1)/g(x) = 1 + O(1/x^a)$ for some $a > 1$. Then*

$$\prod_{n \ge 0} \left(\frac{g(n)}{g(2n+1)} \right)^{\varepsilon_n} = \prod_{n \ge 0} \frac{g(4n)}{g(4n+2)}.$$

Proof. Convergence of the infinite products results from Proposition 3(ii) with $f = \log g$. We then write

$$\prod_{n\geq 0} g(n)^{\varepsilon_n} = \prod_{n\geq 0} g(2n)^{\varepsilon_{2n}} \prod_{n\geq 0} g(2n+1)^{\varepsilon_{2n+1}} = \prod_{n\geq 0} g(2n)^{(-1)^n} \prod_{n\geq 0} g(2n+1)^{\varepsilon_{2n+1}}.$$

Hence

$$\prod_{n\geq 0} \left(\frac{g(n)}{g(2n+1)}\right)^{\varepsilon_n} = \prod_{n\geq 0} g(2n)^{(-1)^n} = \prod_{n\geq 0} \frac{g(4n)}{g(4n+2)}. \qquad \square$$

Example. Taking $g : \mathbb{R}^+ \to \mathbb{R}^{+*}$ defined by $g(x) = \frac{x}{x+1}$ if $x > 0$ and $g(0) = 1$, and using Propositions 1 and 2, we obtain equality (2) of the introduction:

$$\prod_{n\geq 1} \left(\frac{2n}{2n+1}\right)^{\varepsilon_n} = \frac{\Gamma(1/4)^2}{8\sqrt{2\pi}}.$$

Remark 1. Using a classical result on elliptic integrals (see, e.g., [11, p. 373]), it is amusing though anecdotical to note that

$$\prod_{n\geq 1} \left(\frac{2n}{2n+1}\right)^{\varepsilon_n} = \frac{1}{2} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{2 - \sin^2 \varphi}}.$$

Remark 2. The quest for closed-form values is teasing but endless. Still focusing on the (regular) paperfolding sequence, we would like to cite a result (due to von Haeseler in the case $s = 1$, see [4, Exercise 27, p. 205]):

$$\sum_{n\geq 0} \frac{\varepsilon_n}{(n+1)^s} = \frac{2^s}{2^s - 1} \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^s} \quad \text{in particular} \quad \sum_{n\geq 0} \frac{\varepsilon_n}{n+1} = \frac{\pi}{2}.$$

Note that this result can also be obtained from an additive version of Lemma 1.

4. Paperfolding infinite products and the gamma function

In this section we give applications of Lemma 1 above to finding closed-form expressions for “simple” infinite products involving the paperfolding sequence.

Theorem 1. *Let b be a positive real number. We have the following expression:*

$$\prod_{n\geq 0} \left(\frac{n+b}{n+\frac{1+b}{2}}\right)^{\varepsilon_n} = \frac{\Gamma(\frac{1}{4})^2}{\pi\sqrt{2}} \times \frac{\Gamma(\frac{1}{2}+\frac{b}{4})}{\Gamma(\frac{b}{4})} = 2^{\frac{1-b}{2}} \left(\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{b}{4})}\right)^2 \times \frac{\Gamma(\frac{b}{2})}{\sqrt{\pi}}.$$

Proof. Apply Lemma 1 with $g(x) = \frac{x+b}{x+1}$ and Propositions 1 and 2. $\square$

Examples. Taking $b = 2$, $b = 3$, and $b = 5$ in Theorem 1 above, we obtain respectively

$$\prod_{n\geq 0}\left(\frac{2n+4}{2n+3}\right)^{\varepsilon_n} = \frac{\Gamma(1/4)^2}{\pi^{3/2}\sqrt{2}}, \qquad \prod_{n\geq 0}\left(\frac{n+3}{n+2}\right)^{\varepsilon_n} = \frac{\Gamma(1/4)^4}{8\pi^2},$$

$$\prod_{n\geq 0}\left(\frac{n+5}{n+3}\right)^{\varepsilon_n} = 3.$$

Remark 3. Letting $F(b)$ denote the product $F(b) = \prod_{n\geq 0}(\frac{n+b}{n+\frac{1+b}{2}})^{\varepsilon_n}$, Theorem 1 and the second of the examples just above give the simple functional equations

$$\forall b > 0, \quad F(b+2)F(b) = bF(3) \quad \text{hence} \quad \forall b > 0, \quad F(b+4) = \frac{b+2}{b}F(b).$$

Remark 4. A totally unexpected relation can be obtained from Theorem 1 above with $b = 1 + \frac{2x}{\pi}$ and from relation (14) in the paper [7] of Blagouchine

$$\log \prod_{n\geq 0}\left(\frac{n+1+\frac{2x}{\pi}}{n+1+\frac{x}{\pi}}\right)^{\varepsilon_n} = \frac{1}{2\pi}\int_0^{\infty} \frac{\log(1+\frac{x^2}{u^2})}{\cosh u}\mathrm{d}u.$$

The paper [7] refers in particular to a nice paper of Malmstén [22] that contains very interesting infinite series and products involving the gamma function.

Remark 5. Taking a nonnegative integer k, and replacing b by $\frac{b+2^k-1}{2^k}$ in Theorem 1 above, we get

$$\begin{aligned} T_k(b) = \prod_{n\geq 0}\left(\frac{n+\frac{b+2^k-1}{2^k}}{n+\frac{b+2^{k+1}-1}{2^{k+1}}}\right)^{\varepsilon_n} &= \frac{\Gamma(\frac{1}{4})^2}{\pi\sqrt{2}} \times \frac{\Gamma(\frac{3\times 2^k-1+b}{2^{k+2}})}{\Gamma(\frac{2^k-1+b}{2^{k+2}})} \\ &= 2^{\frac{1-b}{2^{k+1}}}\left(\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{2^k-1+b}{2^{k+2}})}\right)^2 \times \frac{\Gamma(\frac{2^k-1+b}{2^{k+1}})}{\sqrt{\pi}}. \end{aligned}$$

By multiplying together these formulas for consecutive values of k, we can obtain a closed formula for the infinite products $T_{k,\ell}(b)$ (where k and ℓ are nonnegative integers with $k < \ell$, and where b is a positive real)

$$T_{k,\ell}(b) = \prod_{n\geq 0}\left(\frac{n+\frac{b+2^k-1}{2^k}}{n+\frac{b+2^\ell-1}{2^\ell}}\right)^{\varepsilon_n} = T_k(b)T_{k+1}(b)\cdots T_{\ell-1}(b).$$

Thus, by replacing in $T_{k,\ell}(b)$ the quantity $\frac{b+2^\ell-1}{2^\ell}$ by c, we have a closed form formula for the infinite product $U_j(c)$, defined by (where j is a positive integer, and where c is a real $> 1 - \frac{1}{2^j}$)

$$U_j(c) = \prod_{n\geq 0}\left(\frac{n+2^j c+1-2^j}{n+c}\right)^{\varepsilon_n}.$$

Example. An example of an infinite paperfolding product obtained with Remark 5 is

$$\prod_{n\geq 0}\left(\frac{2n+6}{2n+3}\right)^{\varepsilon_n} = \frac{\Gamma(1/4)^6}{8\sqrt{2}\pi^{7/2}}.$$

Namely the left side of the above equality can be computed as follows

$$\prod_{n\geq 0}\left(\frac{2n+6}{2n+3}\right)^{\varepsilon_n} = \prod_{n\geq 0}\left(\frac{n+3}{n+\frac{3}{2}}\right)^{\varepsilon_n} = T_{0,2}(3) = T_0(3)T_1(3)$$
$$= \frac{\Gamma(1/4)^4}{8\pi^2}\frac{\Gamma(1/4)^2}{\sqrt{2}\pi^{3/2}} = \frac{\Gamma(1/4)^6}{8\sqrt{2}\pi^{7/2}}.$$

Remark 6. We do not know how to characterize the positive integers u, v, w for which the infinite product $\prod_{n\geq 0}(\frac{un+v}{un+w})^{\varepsilon_n}$ can be expressed in a closed form (also see Remark 12).

Remark 7. For large a (e.g., $a > 2$) in Proposition 3(ii), the series $\sum \varepsilon_n f(n)$ is absolutely convergent. The larger the value of a, the quicker the convergence of the series. Taking a large value of a would also reflect in Lemma 1.

Starting with Theorem 1 we give a more symmetric expression which can yield "gamma-free" values of paperfolding infinite products, as shown in Section 5 below.

Theorem 2. *Let b and c be two positive real numbers. Then*

$$\prod_{n\geq 0}\left(\frac{(n+b)(n+\frac{1+c}{2})}{(n+c)(n+\frac{1+b}{2})}\right)^{\varepsilon_n} = \frac{\Gamma(\frac{c}{4})\Gamma(\frac{1}{2}+\frac{b}{4})}{\Gamma(\frac{b}{4})\Gamma(\frac{1}{2}+\frac{c}{4})}.$$

Proof. Apply Theorem 1 for b and c and compute the quotient. $\square$

Remark 8. Another possibly anecdotal remark is that the right side of the equality in Theorem 2 above can be seen as a coefficient in a connection formula for algebraic hypergeometric functions (see, e.g., [13, p. 107]; also see [17,18]): With the notation of [17], $\alpha_1^2(u, v, w) := \frac{\Gamma(w)\Gamma(w-u-v)}{\Gamma(w-u)\Gamma(w-v)}$, we have

$$\prod_{n\geq 0}\left(\frac{(n+b)(n+\frac{1+c}{2})}{(n+c)(n+\frac{1+b}{2})}\right)^{\varepsilon_n} = \alpha_1^2\left(\frac{1}{2}, \frac{b-c}{4}, \frac{1}{2}+\frac{b}{4}\right).$$

5. "Gamma-free" paperfolding products

In this section we describe "gamma-free" paperfolding infinite products, i.e., products where the gamma function "does not appear explicitly". We begin with a "trigonometric" corollary of Theorem 2 above.

Corollary 1.

(i) *Let b be a real number in $(0, 2)$. Then*

$$\prod_{n \geq 0} \left(\frac{(n+b)(2n+3-b)}{(n+2-b)(2n+1+b)} \right)^{\varepsilon_n} = \tan \frac{\pi b}{4}.$$

(ii) *Let k be an integer ≥ 3. Then*

$$\prod_{n \geq 0} \left(\frac{(kn+k-1)(2kn+2k+1)}{(kn+k+1)(2kn+2k-1)} \right)^{\varepsilon_n} = \tan \frac{(k-1)\pi}{4k},$$

$$\prod_{n \geq 0} \left(\frac{(kn+k-2)(kn+k+1)}{(kn+k+2)(kn+k-1)} \right)^{\varepsilon_n} = \tan \frac{(k-2)\pi}{4k},$$

$$\prod_{n \geq 0} \left(\frac{(kn+k-2)(2kn+2k+1)}{(kn+k+2)(2kn+2k-1)} \right)^{\varepsilon_n} = \tan \frac{(k-1)\pi}{4k} \tan \frac{(k-2)\pi}{4k}.$$

Proof. Relation (i) is a consequence of Theorem 2 above with $c = 2 - b$, and of the reflection formula $\Gamma(z)\Gamma(1-z) = \pi / \sin(\pi z)$ (Proposition 2 above). The first two relations in (ii) are obtained from (i) by taking $b = (k-1)/k$ and $b = (k-2)/k$; multiplying them together gives the last relation. □

Relations obtained for some particular values of b or of k are of interest. In particular it has been known since Lambert (see [19, pp. 133–139]; or see, e.g., [15]) that trigonometric functions at angles whose values in degrees are $0, 3, 6, 9, \ldots$ can be written using only addition, subtraction, multiplication, division, and (possibly nested) square roots, of rational numbers. Also if the trigonometric functions at angle α can be obtained this way, then $\alpha/2$ has the same property. Using these expressions in (i) and then in (ii) yields, e.g., the following paperfolding infinite products.

Examples.

- For $b = \frac{3}{2}$ we obtain

$$\prod_{n \geq 0} \left(\frac{(2n+3)(4n+3)}{(2n+1)(4n+5)} \right)^{\varepsilon_n} = \tan \frac{3\pi}{8} = \cot \frac{\pi}{8} = 1 + \sqrt{2}.$$

- For $b = \frac{1}{5}$ we obtain

$$\prod_{n\geq 0}\left(\frac{(5n+1)(5n+7)}{(5n+9)(5n+3)}\right)^{\varepsilon_n} = \tan\frac{\pi}{20} = \sqrt{5}+1-\sqrt{5+2\sqrt{5}}.$$

- For $b = \frac{2}{5}$ we obtain

$$\prod_{n\geq 0}\left(\frac{(5n+2)(10n+13)}{(5n+8)(10n+7)}\right)^{\varepsilon_n} = \tan\frac{\pi}{10} = \frac{1}{5}\sqrt{5(5-2\sqrt{5}\,)}.$$

- For $k = 3$ we obtain

$$\prod_{n\geq 0}\left(\frac{(3n+2)(6n+7)}{(3n+4)(6n+5)}\right)^{\varepsilon_n} = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3},$$

$$\prod_{n\geq 0}\left(\frac{(3n+1)(3n+4)}{(3n+5)(3n+2)}\right)^{\varepsilon_n} = \tan\frac{\pi}{12} = 2-\sqrt{3}, \quad \text{thus}$$

$$\prod_{n\geq 0}\left(\frac{(3n+1)(6n+7)}{(3n+5)(6n+5)}\right)^{\varepsilon_n} = \tan\frac{\pi}{6}\tan\frac{\pi}{12} = \frac{2\sqrt{3}}{3}-1.$$

- For $k = 5$ we obtain

$$\prod_{n\geq 0}\left(\frac{(5n+3)(5n+6)}{(5n+7)(5n+4)}\right)^{\varepsilon_n} = \tan\frac{3\pi}{20} = \sqrt{5}-1-\sqrt{5-2\sqrt{5}},$$

$$\prod_{n\geq 0}\left(\frac{(5n+4)(10n+11)}{(5n+6)(10n+9)}\right)^{\varepsilon_n} = \tan\frac{\pi}{5} = \sqrt{5-2\sqrt{5}}, \quad \text{thus}$$

$$\prod_{n\geq 0}\left(\frac{(5n+3)(10n+11)}{(5n+7)(10n+9)}\right)^{\varepsilon_n} = \sqrt{25-10\sqrt{5}}-\sqrt{5-2\sqrt{5}}-5+2\sqrt{5}.$$

One may ask whether it is possible to find other cases of "gamma-free" paperfolding products. In particular the so-called "short gamma products" may give paperfolding products where the gamma function does not occur explicitly. Results on short gamma products are due to Sándor and Tóth [29] and to Nijenhuis [24], while "isolated" examples can be found in the literature (see the paper of Borwein and Zucker [8]; also see [24]). Before stating these results we give some notation: φ is the Euler function, i.e., the cardinality of $\Phi(m)$, the set of integers in $[1, m]$ that are relatively prime to m. We note that $\Phi(m)$ is a group with respect to multiplication modulo m.

Theorem 3.

(i) (Sándor and Tóth [29]; also see [10,23].) *The following relation holds:*

$$\prod_{\substack{1\le k\le m\\ \gcd(k,m)=1}} \Gamma\left(\frac{k}{m}\right) = \begin{cases} \frac{(2\pi)^{\varphi(m)/2}}{\sqrt{p}}, & \text{if } m \text{ is a prime power;}\\ (2\pi)^{\varphi(m)/2}, & \text{otherwise.}\end{cases}$$

(ii) (Nijenhuis [24].) *Let $n > 1$ be an odd integer. Let A_n be the cyclic subgroup of $\Phi(2n)$ generated by $(n+2)$ or any one of its cosets. Let $\nu(n)$ denote the cardinality of A_n, and $b(A_n)$ the number of elements of A_n that are larger than n. Then*

$$\prod_{x\in A_n} \Gamma\left(\frac{x}{2n}\right) = 2^{b(A)}\pi^{\nu(n)/2}.$$

(iii) (Borwein and Zucker [8]; also see [24, Section 5].) *The following relations hold:*

$$\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{11}{24}\right)\Gamma\left(\frac{19}{24}\right)\Gamma\left(\frac{17}{24}\right) = 4\pi^2 5^{1/4},$$
$$\Gamma\left(\frac{1}{20}\right)\Gamma\left(\frac{9}{20}\right)\Gamma\left(\frac{13}{20}\right)\Gamma\left(\frac{17}{20}\right) = 4\pi^2\sqrt{3}.$$

Remark 9. The "sporadic" formulas in Theorem 3(iii) can be retrieved from the relations for the values $\Gamma(p/q)$ with $q|60$ given in [31].

In order to use such a "short" gamma product in the infinite products given in Theorem 2 we would like to have four terms in the theorem above (up to multiplying the denominator $\Gamma(\frac{b}{4})\Gamma(\frac{1}{2}+\frac{c}{4})$ in Theorem 2 by $\Gamma(1-\frac{b}{4})\Gamma(\frac{1}{2}-\frac{c}{4})$ and using the reflection formula). Since the number of terms in the short product of Theorem 3(i) is $\varphi(n)$, we need to restrict to the integers m such that $\varphi(m) = 4$ (which is easily proven to be the set $\{5, 8, 10, 12\}$). We did not obtain new products this way. Using Theorem 3(ii) gives new results but they are not gamma-free. Finally we can deduce from Theorem 3(iii) the following result.

Theorem 4. *The following relations hold:*

$$\prod_{n\ge 0}\left(\frac{(6n+5)(12n+7)}{(6n+1)(12n+11)}\right)^{\varepsilon_n} = 4\times 5^{1/4}\times \sin\frac{5\pi}{24}\times\sin\frac{11\pi}{24} = 5^{1/4}(1+\sqrt{2});$$

$$\prod_{n\ge 0}\left(\frac{(5n+3)^2}{(5n+1)(5n+4)}\right)^{\varepsilon_n} = 4\sqrt{3}\times\sin\frac{3\pi}{20}\times\sin\frac{11\pi}{20} = \frac{\sqrt{6}\sqrt{5-\sqrt{5}}+\sqrt{15}-\sqrt{3}}{2}.$$

Proof. Put $b = 5/6$ and $c = 1/6$ (resp. $b = 3/5$ and $c = 1/5$) in Corollary 2. Use Theorem 3(iii) and the reflection formula. $\square$

Remark 10. Other gamma products can be found in the literature, see, e.g., [16,25]. We would like to cite in particular a result due to Stadler (see [16]).

Let p be an odd prime. Let $(\frac{k}{p})$ denote the Legendre symbol. Let $e(p)$ and $h(p)$ respectively denote the fundamental unit and the class number of the real quadratic field $\mathbb{Q}(\sqrt{p})$. Then

$$\prod_{1\le k\le p} \Gamma\left(\frac{2k-1}{2p}\right)^{(\frac{2k-1}{p})} = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 8, \\ e(p)^{h(p)} & \text{if } p \equiv 5 \bmod 8, \\ 2^{-\sum_{1\le k\le p-1}(\frac{k}{p})\frac{k}{p}} & \text{if } p \equiv 7 \bmod 8. \end{cases}$$

6. Arithmetical nature of the paperfolding products

So far trying to give closed-form expressions for infinite products of the type $\prod_n R(n)^{u_n}$ where $(u_n)_{n\ge 0}$ is some automatic sequence, is possible when $u_n = (-1)^{w_n}$ (where w_n counts the number of occurrences of some pattern in the binary expansion of n) as recalled in the introduction, or, as done above, when $u_n = \varepsilon_n$ is the ± 1 paperfolding sequence. But the choice for the corresponding $R(n)$'s is in both cases drastically limited: This is not the case where $u_n = 1$ for all n; see Proposition 1 where the only limitation is that the infinite product converges.

A partly related question is the arithmetical nature of these infinite products: When are they algebraic, and where are they transcendental? The question is certainly not easy, even when we have a closed-form expression. In the examples above, we know that:

- all infinite products given in Corollary 1(i) for b rational, and in Corollary 1(ii) are algebraic;
- both products in Theorem 4 are algebraic;
- the two examples of products following Theorem 1 as well as the example following Remark 5 are transcendental from a result of Čudnovs'kiĭ (see [12]; also see, e.g., [32, Theorem 14, p. 441]) stating that the numbers $\Gamma(1/4)$ and π are algebraically independent;

Remark 11. In order to prove the transcendence of the infinite product (as previously $(\varepsilon_n)_n$ is the paperfolding sequence) $\prod_{n\ge 1}(\frac{2n}{2n+1})^{\varepsilon_n} = \frac{\Gamma(1/4)^2}{8\sqrt{2\pi}}$, we do not need the full strength of Čudnovs'kiĭ's result; it suffices to use, e.g., [9, Corollary 7.4, p. 200]: The transcendence of $\Gamma^2(1/4)/\sqrt{\pi}$ can be proved by relating this constant to a nonzero period of the Weierstrass $\mathcal{P}$-function, i.e., of the elliptic curve $y^2 = 4x^3 - 4x$.

We would like to cite two such problems in this section: The nature of a very particular product on one hand, and a hard general problem on the other hand.

6.1. The Flajolet–Martin constant

In the 1985 paper [14, Theorem 3.1, p. 193] Flajolet and Martin came across the following constant:

$$R := \frac{e^\gamma\sqrt{2}}{3}\prod_{n\ge 1}\left(\frac{(4n+1)(4n+2)}{4n(4n+3)}\right)^{(-1)^{s(n)}}$$

where γ is the Euler constant and, as in the introduction, $s(n)$ is the sum of the binary digits of n. It is easily proven that $R = \frac{2^{-1/2}e^\gamma}{Q}$, where

$$Q := \prod_{n\ge 1}\left(\frac{2n}{2n+1}\right)^{(-1)^{s(n)}}.$$

This product resembles the (algebraic) Woods–Robbins product quoted in the introduction very much and it is not extraordinarily different from the third example (transcendental) given after Remark 5, respectively

$$\prod_{n\ge 0}\left(\frac{2n+1}{2n+2}\right)^{(-1)^{s(n)}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \prod_{n\ge 0}\left(\frac{2n+6}{2n+3}\right)^{\varepsilon_n} = \frac{\Gamma(1/4)^6}{8\sqrt{2}\pi^{7/2}}.$$

But not only we do not know any closed-form expression for the infinite product Q, but also we do not know its arithmetical nature, nor the arithmetical nature of the Flajolet–Martin constant.

Remark 12. In the same spirit, looking at the infinite products A and B given in the introduction

$$A = \prod_{n\ge 0}\left(\frac{2n+1}{2n+2}\right)^{\varepsilon_n} \quad \text{and} \quad B = \prod_{n\ge 1}\left(\frac{2n}{2n+1}\right)^{\varepsilon_n} = \frac{\Gamma(1/4)^2}{8\sqrt{2\pi}},$$

we do not know any closed-form formula for A, nor whether this is a transcendental or algebraic number.

6.2. The Rohrlich and Rohrlich–Lang conjectures

A strong conjecture known as the Rohrlich conjecture predicts that the algebraicity of any products and quotients of normalized gamma values (the normalized gamma function is $\Gamma/\sqrt{\pi}$) must derive from the properties $\Gamma(x+1) = \Gamma(x)$, the reflection formula and the multiplication formula for Γ (see, e.g., [20, p. 418] or [32, pp. 444–445]). An explicit formulation is given, e.g., in [27]:

Conjecture *(Rohrlich). Let a_1, a_2, $\dots$, a_r be rational numbers that are not in $\{0, -1, -2, \dots\}$. Let D be a common denominator of the a_i's. Then the product $\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_r)$ is an algebraic multiple of $\pi^{r/2}$ if and only if for all $m \in \{1, 2, \dots, D-1\}$ relatively prime to D we have $\sum_{1 \le i \le r}\{ma_i\} = r/2$, where $\{x\}$ denotes the fractional part of the real x.*

Remark 13. There is an even stronger conjecture, known as the Rohrlich–Lang conjecture. The interested reader can look, e.g., at [32, p. 445].

The Rohrlich conjecture implies a criterion for the algebraicity of the infinite products in Theorem 2. We first need an easy lemma.

Lemma 2. *Let $\{x\}$ denote the fractional part of the real number x. Then we have*

$$\{-x\} = 1 - \{x\} \quad \text{if } x \text{ is not an integer,}$$
$$\left\{\frac{1}{2} + x\right\} = \begin{cases} \{x\} + \frac{1}{2} & \text{if } \{x\} < \frac{1}{2}, \\ \{x\} - \frac{1}{2} & \text{if } \{x\} \ge \frac{1}{2}, \end{cases}$$
$$\left\{\frac{1}{2} - x\right\} = \begin{cases} \frac{3}{2} - \{x\} & \text{if } \{x\} > \frac{1}{2}, \\ \frac{1}{2} - \{x\} & \text{if } \{x\} \le \frac{1}{2}. \end{cases}$$

Proof. Left to the reader. □

Corollary 2.

(i) *Let b be a positive rational number with denominator a such that $b/4$ is not an integer. Then, under the conjecture of Rohrlich, we have that the infinite product*

$$\prod_{n \ge 0}\left(\frac{n+b}{n+\frac{1+b}{2}}\right)^{\varepsilon_n}$$

is algebraic if and only if for all $m \in \{1, 2, \dots, 4a-1\}$ such that m is relatively prime to $4a$ we have either ($\{mb/4\} < 1/2$ and $\{m/4\} \le 1/2$) or ($\{mb/4\} \ge 1/2$ and $\{m/4\} > 1/2$). In other words this condition is equivalent to saying that for all $m \equiv 1 \bmod 4$ and relatively prime to a we have $\{mb/4\} < 1/2$, and for all $m \equiv 3 \bmod 4$ and relatively prime to a we have $\{mb/4\} \ge 1/2$.

(ii) *Let b and c be positive rational numbers with common denominator a. We also suppose that neither $b/4$ nor $(c-2)/4$ are integers. Then, under the conjecture of Rohrlich, we have that the infinite product*

$$A(b, c) := \prod_{n \ge 0}\left(\frac{(n+b)(n+\frac{1+c}{2})}{(n+c)(n+\frac{1+b}{2})}\right)^{\varepsilon_n}$$

is algebraic if and only if for all $m \in \{1, 2, \dots, 4a-1\}$ such that m is relatively prime to $4a$ we have either ($\{mb/4\} < 1/2$ and $\{mc/4\} \le 1/2$) or ($\{mb/4\} \ge 1/2$ and $\{mc/4\} > 1/2$).

Proof. Since assertion (i) is the particular case $c = 1$ of assertion (ii), it suffices to prove (ii). Applying Theorem 2 and the reflection formula for the gamma function, we have

$$\begin{aligned} A(b,c) &= \frac{\Gamma(\frac{c}{4})\Gamma(\frac{1}{2}+\frac{b}{4})}{\Gamma(\frac{b}{4})\Gamma(\frac{1}{2}+\frac{c}{4})} \\ &= \frac{1}{\pi^2}\sin\frac{b\pi}{4}\cos\frac{c\pi}{4}\Gamma\left(\frac{c}{4}\right)\Gamma\left(\frac{1}{2}-\frac{c}{4}\right)\Gamma\left(\frac{1}{2}+\frac{b}{4}\right)\Gamma\left(1-\frac{b}{4}\right). \end{aligned}$$

Since b and c are rational, the product $\sin\frac{b\pi}{4}\cos\frac{c\pi}{4}$ is algebraic. Hence $A(b,c)$ is algebraic if and only if the quantity

$$\frac{1}{\pi^2}\Gamma\left(\frac{c}{4}\right)\Gamma\left(\frac{1}{2}-\frac{c}{4}\right)\Gamma\left(\frac{1}{2}+\frac{b}{4}\right)\Gamma\left(1-\frac{b}{4}\right)$$

is algebraic (recall the conditions on b and c). Under the conjecture of Rohrlich, we thus see that $A(b,c)$ is algebraic if and only if for all $m \in \{1, 2, \dots 4a-1\}$ which is relatively prime to $4a$ we have

$$\left\{\frac{mc}{4}\right\} + \left\{m\left(\frac{1}{2}-\frac{c}{4}\right)\right\} + \left\{m\left(\frac{1}{2}+\frac{b}{4}\right)\right\} + \left\{m\left(1-\frac{b}{4}\right)\right\} = 2.$$

Since m is relatively prime to $4a$, m must be odd, thus the condition becomes

$$\left\{\frac{mc}{4}\right\} + \left\{\frac{1}{2}-\frac{mc}{4}\right\} + \left\{\frac{1}{2}+\frac{mb}{4}\right\} + \left\{-\frac{mb}{4}\right\} = 2.$$

Applying Lemma 2 this is equivalent to

$$\begin{aligned} &\left(\left\{\frac{mb}{4}\right\} < \frac{1}{2} \quad \text{and} \quad \left\{\frac{mc}{4}\right\} \le \frac{1}{2}\right) \quad \text{or} \\ &\left(\left\{\frac{mb}{4}\right\} \ge \frac{1}{2} \quad \text{and} \quad \left\{\frac{mc}{4}\right\} > \frac{1}{2}\right). \qquad \square \end{aligned}$$

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