JOURNAL OF ALGEBRA **198,** 101–119 (1997)
ARTICLE NO. JA977143

Quantum Convolution of Linearly Recursive Sequences

Siu-Hung Ng* and Earl J. Taft

Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

Communicated by Susan Montgomery

Received March 18, 1996

Contents.

1. INTRODUCTION

Let k be a field, $A = k[x]$. We identify an element f in the dual space A^* with the sequences $(f_n)_{n \geq 0} = (f_0, f_1, f_2, \dots)$, where $f_n = f(x^n)$ for $n \geq 0$. A is a Hopf algebra with x primitive, i.e., the comultiplication Δ is given by putting $\Delta x = 1 \otimes x + x \otimes 1$ and requiring Δ to be an algebra morphism from A to $A \otimes A$. Thus

$$\Delta(x^n) = \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i}, \qquad \text{for all } n \geq 0,$$

$\binom{n}{i}$ the binomial coefficient. The composite map $A^* \otimes A^* \to (A \otimes A)^* \to A^*$, defines a (convolution) product on A^*, where the first arrow is the canonical identification $(f \otimes g)(a \otimes b) = f(a)g(b)$, and the second arrow is Δ^*. Concretely A^* is an algebra under the convolution product $(f_n) * (g_n) = (h_n)$, where $h_n = \sum_{i=0}^{n} \binom{n}{i} f_i g_{n-i}$ for $n \geq 0$.

The algebra A has a (continuous) dual coalgebra $A^\circ = \{f \text{ in } A^* \mid f(J)$

* E-mail address: shng@math.ucsc.edu, jcia@math.rutgers.edu.

0021-8693/97 $25.00

$= 0$ for some cofinite ideal J of $A\}$, i.e., A/J is finite dimensional. $\Delta f = \Sigma g_i \otimes h_i$ for g_i, h_i in A° means that $f(ab) = \Sigma g_i(a)h_i(b)$ for all a, b in A. It is well known that if A is a Hopf algebra, then A° is also a Hopf algebra. In particular, A° is closed under the convolution product on A^*.

Since a cofinite ideal J of $A = k[x]$ is just a nonzero ideal generated by a monic polynomial $h(x) = x^r - h_1 x^{r-1} - \cdots - h_r$, the condition $f(J) = 0$ means that $f_n = h_1 f_{r-1} + \cdots + h_r f_{n-r}$ for all $n \geq r$, i.e., f is linearly recursive, satisfying the recursive relation $h(x)$. Thus the space of linearly recursive sequences forms a Hopf algebra, whose product is the convolution product. See [P-T] and [T3] for a development of this structure, and [K, Mo, S] for Hopf algebra background.

Let q be a nonzero element of k, $\binom{n}{i}_q$ the Gaussian polynomial $(n)_q!/((i)_q!(n-i)_q!)$, where

$$(n)_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1} \text{ and } (n)_q! = (n)_q (n-1)_q \cdots (1)_q.$$

If $q = 1$, then $\binom{n}{i}$ is the ordinary binomial coefficient. In this paper, we will show that if q is a root of unity, then the space of linearly recursive sequences is closed under the quantum convolution product $(f_n) *_q (g_n) = (h_n)$, where $h_n = \Sigma_{i=0}^n \binom{n}{i}_q f_i g_{n-i}$. We also show that this is not the case when q is not a root of unity (Section 6).

We first obtain this result in the framework of a duality result concerning bialgebras in the braided monoidal category of modules over a quasitriangular Hopf algebra (Sections 2 and 3). In Section 5, we obtain the closure result by a direct combinatorial computation. In particular, we can display a recursive relation satisfied by a quantum product in terms of relations satisfied by the factors.

2. QUASITRIANGULAR BIALGEBRAS

A bialgebra H is called *quasitriangular* if there is an invertible element R in $H \otimes H$ such that

$$\Delta^{\mathrm{op}}(h) = R\Delta(h)R^{-1}, \qquad \text{for } h \text{ in } H, \tag{1}$$

$$(\Delta \otimes I)R = R_{13}R_{23}, \tag{2}$$

$$(I \otimes \Delta)R = R_{13}R_{12}. \tag{3}$$

Here, $\Delta^{\mathrm{op}}(h) = \Sigma h_2 \otimes h_1$, where $\Delta h = \Sigma h_1 \otimes h_2$ (Sweedler notation). We also use (e.g., in Section 3) the notation $\Delta h = \sum h_{(1)} \otimes h_{(2)}$. If $R = \sum R_i \otimes R_i'$, then $R_{12} = \sum R_i \otimes R_i' \otimes 1$, $R_{23} = \sum 1 \otimes R_i \otimes R_i'$, and $R_{13} = \sum R_i \otimes 1 \otimes R_i'$. The terminology is due to Drinfeld [D]. See also [K], where H is called a *braided* bialgebra.

We consider the category ${}_H\mathsf{M}$ of left H-modules. ${}_H\mathsf{M}$ is a braided monoidal category. The monoidal structure uses only the bialgebra structure of H. If $M, N \in {}_H\mathsf{M}$, then $M \otimes N \in {}_H\mathsf{M}$, where $h \cdot (m \otimes n) = \sum (h_1 \cdot m) \otimes (h_2 \cdot n)$. The braiding in ${}_H\mathsf{M}$ uses the matrix $R = \sum R_i \otimes R_i'$ and is given by the twist map $t_{M,N}\colon M \otimes N \to N \otimes M$, where $t_{M,N}(m \otimes n) = \sum (R_i' \cdot n) \otimes (R_i \cdot m)$. If H is cocommutative and $R = 1 \otimes 1$, then $t_{M,N}$ is the usual twist map.

A bialgebra H is called *coquasitriangular* (see [Mo]) if there is a bilinear form $\langle\ ,\ \rangle$ in $(H \otimes H)^*$ which is convolution invertible in $(H \otimes H)^*$, such that

$$\sum \langle h_1, k_1 \rangle k_2 h_2 = \sum \langle h_2, k_2 \rangle h_1 k_1, \tag{1$'$}$$

$$\langle h, kl \rangle = \sum \langle h_1, k \rangle \langle h_2, l \rangle, \tag{2$'$}$$

$$\langle hk, l \rangle = \sum \langle h, l_2 \rangle \langle k, l_1 \rangle. \tag{3$'$}$$

Let M^H be the category of right H-comodules. If $M \in \mathsf{M}^H$ with comodule structure map $\rho\colon M \to M \otimes H$, we use that notation $\rho(m) = \sum m_0 \otimes m_1$, $m_0 \in M$, $m_1 \in H$. Then M^H is also a braided monoidal category. The monoidal structure uses only the bialgebra structure of H and is given by $\rho(m \otimes n) = \sum m_0 \otimes n_0 \otimes m_1 n_1$, and the braiding is given by $t_{M,N}(m \otimes n) = \sum \langle n_1, m_1 \rangle n_0 \otimes m_0$.

An algebra (coalgebra, bialgebra, Hopf algebra) in a braided monoidal category is one whose structure maps are morphisms in the category. Tensoring algebras or coalgebras involves the braiding. If A and B are algebras in the category, then so is $A \otimes B$ (see [Ma2]) via

$$A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{I} \otimes t_{B,A} \otimes \mathrm{I}} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B.$$

Similarly, if C, D are coalgebras in the category, then so is $C \otimes D$ via

$$C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\mathrm{I} \otimes t_{C,D} \otimes \mathrm{I}} C \otimes D \otimes C \otimes D.$$

In particular, if A is a bialgebra in the category, then $\Delta_A\colon A \to A \otimes A$ is an algebra map in the category, where $A \otimes A$ has the above algebra structure using $t_{A,A}$.

3. DUALITY

The object of this section is to show that if H is a quasitriangular Hopf algebra with a bijective antipode S, ${}_H M$ the braided monoidal category of left H-modules, and A a bialgebra in ${}_H M$, then A° is also a bialgebra in ${}_H M$. This extends to Hopf algebras in ${}_H M$.

For any $V \in {}_H M$, V^* admits a left H-module structure defined by $(hf)(x) = f(S(h)x)$. In general, the dual of a morphism in ${}_H M$ is also a morphism in ${}_H M$. For $h \in H$, denote by $\hat{h}$ the linear endomorphism on V which maps $x \mapsto hx$ for $x \in V$. This notation will be used in the proof without further explanation.

Let $(A, m_A, \mu_A, \Delta_A, \epsilon_A)$ be a bialgebra in ${}_H M$ and

$$A^\circ = \{ f \in A^* | \ker f \supset \text{ an ideal of } A \text{ of finite codimension} \},$$

where A is an H-module and m_A, Δ_A, μ_A, and ϵ_A are the multiplication, comultiplication, unit, and counit of A, respectively, and are morphisms in ${}_H M$. It is generally true that $(A^\circ, m_A^*, \mu_A^*)$ is a coalgebra and $(A^*, \Delta_A^*, \epsilon_A^*)$ is an algebra (see [S]). Actually, A° is a subalgebra of $(A^*, \Delta_A^*, \epsilon_A^*)$. Before proving this lemma, let us recall some basic facts about an algebra B. For $f \in B^*$ and $a, b \in B$, define

$$(a \rightharpoonup f)(b) = f(ba),$$
$$(f \leftharpoonup a)(a) = f(ab).$$

$\leftharpoonup$ and $\rightharpoonup$ defines a B-B-bimodule structure on B^*. We have the following lemma (see [Mo]).

LEMMA 3.1. *Let B be an algebra and $f \in B^*$. Then the following are equivalent*:

(i) $\ker f \supseteq$ *a finite codimensional ideal of B*.

(ii) $\dim(B \rightharpoonup f) < \infty$.

(iii) $\dim(f \leftharpoonup B) < \infty$.

LEMMA 3.2. *A° is a subalgebra of $(A^*, \Delta_A^*, \epsilon_A^*)$.*

Proof. Suppose $R = \sum_{i=1}^n r_{1i} \otimes r_{2i}$. For $f, g \in A^*$ and $a, b \in A$,

$$\begin{aligned}(fg \leftharpoonup a)(b) &= (fg)(ab)\\ &= (f \otimes g)\Delta_A(ab)\\ &= (f \otimes g)(\Delta_A(a)\Delta_A(b))\\ &= \sum (f \otimes g)\big(a_{(1)}(r_{2i}b_{(1)}) \otimes (r_{1i}a_{(2)})b_{(2)}\big),\end{aligned}$$

where $\Delta_A(a) = \sum a_{(1)} \otimes a_{(2)}$ and $\Delta_A(b) = \sum b_{(1)} \otimes b_{(2)}$. Therefore,

$$\begin{aligned}(fg \leftharpoonup a)(b) &= \sum f\big(a_{(1)}(r_{2i}b_{(1)})\big)g\big((r_{1i}a_{(2)})b_{(2)}\big)\\ &= \sum \big((f \leftharpoonup a_{(1)}) \otimes (g \leftharpoonup r_{1i}a_{(1)})(\hat{r}_{2i} \otimes \mathrm{id}_A)\Delta_A(b).\end{aligned}$$

Therefore,

$$fg \leftharpoonup A \subseteq \sum_{i=1}^{n} \Delta_A^*(\hat{r}_{2i} \otimes \mathrm{id}_A)^*((f \leftharpoonup A) \otimes (g \leftharpoonup A)).$$

Since $f, g \in A^\circ$, the left-hand side of the above containment is finite dimensional. Hence, by Lemma 3.1, $fg \in A^\circ$. For $a, b \in A$,

$$\begin{aligned}(a \rightharpoonup \epsilon_A^*(1))(b) &= \epsilon_A^*(1)(ba) = \epsilon_A(ba) = \epsilon_A(b)\epsilon_A(a)\\ &= (\epsilon_A(a)\epsilon_A^*(1))(b).\end{aligned}$$

Therefore, $A \rightharpoonup \epsilon_A^*(1) \subseteq k\epsilon_A^*(1)$. Hence, $\epsilon_A^*(k) \subseteq A^\circ$. ∎

Since A° is an algebra as well as a coalgebra, $(A^\circ)^{\mathrm{op}}$ and $(A^\circ)^{\mathrm{cop}}$ are an algebra and a coalgebra, respectively. We denote by $(m_{A^\circ})^{\mathrm{op}}$ the multiplication in $(A^\circ)^{\mathrm{op}}$ and by $(\Delta_{A^\circ})^{\mathrm{op}}$ the coproduct on $(A^\circ)^{\mathrm{cop}}$. Clearly,

$$(m_{A^\circ})^{\mathrm{op}} = \Delta_A^*\tau \quad \text{and} \quad (\Delta_{A^\circ})^{\mathrm{op}} = \tau m_A^*,$$

where τ is the usual twist map.

By the above remark, A^* is an H-module and so are $A^* \otimes A^*$ and $(A \otimes A)^*$. Let j: $A^* \otimes A^* \to (A \otimes A)^*$ be the natural embedding, defined by

$$(j(f \otimes g))(a \otimes b) = f(a)g(b)$$

for $f, g \in A^*$ and $a, b \in A$. In general, j is not an H-module homomorphism unless H is cocommutative. However, $j\tau$ is a morphism in ${}_H\mathcal{M}$.

LEMMA 3.3. *$j\tau$: $(A^*)^{\mathrm{op}} \otimes (A^*)^{\mathrm{op}} \to (A \otimes A)^{*\mathrm{op}}$ is an algebra map in ${}_H\mathcal{M}$*

Proof. It is straightforward to check that $j\tau$ is an H-module map. Let $u \otimes v, f \otimes g \in (A^*)^{\mathrm{op}} \otimes (A^*)^{\mathrm{op}}$ and $a \otimes b \in A \otimes A$.

$$\begin{aligned}&[j\tau((f \otimes g)\cdot(u \otimes v))](a \otimes b)\\ &\quad= \sum_i [j\tau((r_{2i}u)f \otimes v(r_{1i}g))](a \otimes b)\\ &\quad= \sum_i [j(v(r_{1i}g) \otimes (r_{2i}u)f)](a \otimes b)\\ &\quad= \sum v(a_1)g(S(r_{1i})a_2)u(S(r_{2i})b_1)f(b_2)\\ &\quad= \sum v(a_1)g(r_{1i}a_2)u(r_{2i}b_1)f(b_2),\end{aligned}$$

by Theorem VIII.2.4 of [K],

$$
\begin{aligned}
&[j\tau(f\otimes g)j\tau(u\otimes v)](a\otimes b)\\
&\qquad = [j(g\otimes f)\cdot j(v\otimes u)](a\otimes b)\\
&\qquad = \sum (v\otimes u\otimes g\otimes f)(a_1\otimes r_{2i}b_1\otimes r_{1i}a_2\otimes b_2)\\
&\qquad = \sum v(a_1)u(r_{2i}b_1)g(r_{1i}a_2)f(b_2)\\
&\qquad = [j\tau((f\otimes g)\cdot(u\otimes v))](a\otimes b).
\end{aligned}
$$

Therefore, $j\tau$ is an algebra map. ∎

THEOREM 3.4. *Let (H,R) be a quasitriangular Hopf algebra with a bijective antipode S. If $(A, m_A, \mu_A, \Delta_A, \epsilon_A)$ is a bialgebra in ${}_H\mathsf{M}$, then $(A^\circ, (m_{A^\circ})^{\mathrm{op}}, \epsilon^*, (\Delta_{A^\circ})^{\mathrm{op}}, \mu_A^*)$ is a bialgebra in ${}_H\mathsf{M}$. Moreover, if A is a Hopf algebra in ${}_H\mathsf{M}$ with antipode S_A, then A° is also a Hopf algebra in ${}_H\mathsf{M}$ with antipode S_A^*.*

Proof. (i) A° is an H-submodule of A^*. By means of Lemma 3.1, it suffices to show that $A \rightharpoonup (hf)$ is finite dimensional for any $h \in H$ and $f \in A^\circ$. For $a, b \in A$,

$$
\begin{aligned}
&(a \rightharpoonup hf)(b)\\
&\qquad = f(S(h)(ba))\\
&\qquad = \sum f\big((S(h)_{(1)}b)(S(h)_{(2)}a)\big) \qquad \text{(since } A \text{ is an algebra in } {}_H\mathsf{M})\\
&\qquad = \sum f\Big(\big(S(h_{(2)})b\big)\big(S(h_{(1)})a\big)\Big)\\
&\qquad = \sum \Big(\big(S(h_{(1)})a\big) \rightharpoonup f\Big)\widehat{S(h_{(2)})}(b)\\
&\qquad = \sum \widehat{S(h_{(2)})}^*\Big(\big(S(h_{(1)})a\big) \rightharpoonup f\Big)(b).
\end{aligned}
$$

Therefore, $A \rightharpoonup hf \subseteq \sum \widehat{S(h_{(2)})}^*(A \rightharpoonup f)$. As the right-hand side of the previous inclusion is finite dimensional, $A \rightharpoonup hf$ is finite dimensional.

(ii) $(m_{A^\circ})^{\mathrm{op}}$ is the composite map

$$A^\circ \otimes A^\circ \xrightarrow{j\tau} (A\otimes A)^* \xrightarrow{\Delta_A^*} A^*,$$

and Δ_A^* is an H-module map. Hence, by Lemma 3.3, $(m_{A^\circ})^{\mathrm{op}}$ is a morphism in ${}_H\mathsf{M}$. Since $m_A^*(A^\circ) \subseteq j(A^\circ\otimes A^\circ)$ and $j\tau\colon A^\circ\otimes A^\circ \to j(A^\circ\otimes A^\circ)$ is an isomorphism, the composition map

$$A^\circ \xrightarrow{m_A^*} j(A^\circ\otimes A^\circ) \xrightarrow{(j\tau)^{-1}} A^\circ\otimes A^\circ$$

is a morphism in ${}_H\mathsf{M}$. However, the composition map is the same as $(\Delta_{A^\circ})^{\mathrm{op}}$ and so $(\Delta_{A^\circ})^{\mathrm{op}}$ is a morphism in ${}_H\mathsf{M}$. The maps $A^\circ \to^{\mu_A^*} k$ and

$k \to^{\epsilon_A^*} A^\circ$ are obviously H-module homomorphisms. This proves that $(A^\circ, (\Delta_{A^\circ})^{\mathrm{op}}, \epsilon_A^*)$ is an algebra in ${}_H M$ and $(A^\circ, (m_{A^\circ})^{\mathrm{op}}, \mu_A^*)$ is a coalgebra in ${}_H M$.

(iii) The map $(\Delta_{A^\circ})^{\mathrm{op}}: (A^\circ)^{\mathrm{op}} \to (A^\circ)^{\mathrm{op}} \otimes (A^\circ)^{\mathrm{op}}$ is an algebra map.

Since $m_A: A \otimes A \to A$ is a coalgebra map, $m_A^*: A^{*\mathrm{op}} \to (A \otimes A)^{*\mathrm{op}}$ is an algebra map. Note that $A^\circ = (m_A^*)^{-1} j(A^* \otimes A^*)$, (cf. [S]). Hence $(A^\circ)^{\mathrm{op}} = (m_A^*)^{-1} j\tau(A^{*\mathrm{op}} \otimes A^{*\mathrm{op}})$ is a subalgebra of $A^{*\mathrm{op}}$ by Lemma 3.3. Since $j\tau(\Delta_{A^\circ})^{\mathrm{op}} = m_A^*|_{(A^\circ)^{\mathrm{op}}}$ is an algebra map, the injectivity of $j\tau$ yields that $(\Delta_{A^\circ})^{\mathrm{op}}: (A^\circ)^{\mathrm{op}} \to (A^\circ)^{\mathrm{op}} \otimes (A^\circ)^{\mathrm{op}}$ is an algebra map in ${}_H M$.

Finally, suppose S_A is the antipode of A in ${}_H M$. Then for $f \in A^\circ$, $a \in A$,

$$\begin{aligned}
(a \rightharpoonup S_A^*(f))(b) &= f(S_A(ba)) \\
&= \sum_{i=1}^{n} f(r_{2i} S_A(a) r_{1i} S_A(b)) \qquad (\text{by [Ma2, Lemma 2.3]}) \\
&= \sum_{i=1}^{n} (f \leftharpoonup r_{2i} S_A(a)) \hat{r}_{1i} S_A(b).
\end{aligned}$$

Therefore,

$$A \rightharpoonup S_A^*(f) \subseteq \sum_{i=1}^{n} S_A^* \hat{r}_{1i}^*(f \leftharpoonup A).$$

As the right-hand side is finite dimensional, $S_A^*(f) \in A^\circ$ and hence $S_A^*(A^\circ) \subseteq A^\circ$. By (i), $A^\circ \to^{S_A^*} A^\circ$ is an H-module homomorphism. Furthermore,

$$\begin{aligned}
\left(\sum S_A^*(f_{(2)}) f_{(1)}\right)(a) &= \sum S_A^*(f_{(2)})(a_{(2)}) f_{(1)}(a_{(1)}) \\
&= \sum (f_{(1)} \otimes f_{(2)})(a_{(1)} \otimes S_A(a_{(2)})) \\
&= \sum f(a_{(1)} S_A(a_{(2)})) \\
&= \sum f(\epsilon_A(a) 1) \\
&= \sum (\epsilon_A^* \mu_A^*(f))(a).
\end{aligned}$$

Therefore, $(m_{A^\circ})^{\mathrm{op}} (S_A^* \otimes \mathrm{id}_{A^\circ})(\Delta_{A^\circ})^{\mathrm{op}} = \epsilon_A^* \mu_A^*$. Similarly, one can prove $(m_{A^\circ})^{\mathrm{op}} (\mathrm{id}_{A^\circ} \otimes S_A^*)(\Delta_{A^\circ})^{\mathrm{op}} = \epsilon_A^* \mu_A^*$.

This completes the proof of the theorem. ∎

Remarks. Lyubashenko considered this kind of duality in the cocompletion C of a rigid braided monoidal category D. Our category ${}_H M$ is the category of all left modules over a quasitriangular Hopf algebra H. Our category is not, in general, such a cocompletion. For example, if $H = U(L)$,

L a finite dimensional semisimple complex Lie algebra, the cocompletion of the finite dimensional H-modules is not all the H-modules. Thus, our setting is not a particular case of the framework of [Ly].

Let H be finite dimensional. Then our category of left H-modules is the cocompletion of the category of finite dimensional left H-modules, which is a rigid braided monoidal category. For A an algebra in ${}_H\mathsf{M}$, H a quasitriangular Hopf algebra, our A° is the colimits of all finite dimensional quotients A/J, which we show is indeed in ${}_H\mathsf{M}$. The construction of A° in [Ly] would involve only finite dimensional quotients A/J, where J is an object in ${}_H\mathsf{M}$. However, we note that in this setting the two versions of A° coincide. This follows from the following proposition.

PROPOSITION 3.5. *Let H be a finite dimensional Hopf algebra, A an algebra in the category ${}_H\mathsf{M}$. If $f \in A^*$ vanishes on an ideal of A of finite codimension, then f vanishes on an H-invariant ideal of A of finite codimension.*

Proof. In order to prove this statement, it suffices to show that for each finite codimensional ideal I of an H-module algebra A, I contains a finite codimensional H-invariant ideal. Denote by A° the usual dual coalgebra of the algebra A. Since A is a left H-module algebra, A° is then a right H-module coalgebra where the right H-module action is given by

$$(fh)(x) = f(hx)$$

for $f \in A^\circ$, $h \in H$, and $x \in A$. The proof of this is essentially the same as the proof of Theorem 3.4(i), with $S(h)$ replaced by h. Let I be a finite codimensional ideal of A. Then $I^\perp = \{f \in A^* \mid f(x) = 0 \text{ for } x \in I\}$ is a finite dimensional subcoalgebra of A°. Then $I^\perp H$ is also a finite dimensional subcoalgebra of A°. Hence, $J = (I^\perp H)^\perp$ is a finite codimensional ideal in A and $J \subset I$. Moreover, J is H-invariant. ∎

4. THE QUANTUM PRODUCT OF LINEARLY RECURSIVE SEQUENCES

Let q be a primitive nth root of unity in k. Let $A = k[x]$. In this section, we show that the space A° of linearly recursive sequences is closed under quantum convolution $(f) *_q (g) = (h)$, where $h_m = \sum_{i=0}^{m} \binom{m}{i}_q f_i g_{m-i}$ for $m \geq 0$.

Let G be the cyclic group of order n generated by g. The group algebra $H = k[G]$ is a Hopf algebra in the usual way, with $\Delta g = g \otimes g$ for g in G. H is a quasitriangular Hopf algebra with respect to $R =$

$(1/n)\sum_{i,j=0}^{n-1} q^{-ij}(g^i \otimes g^j)$ (see [Ma1, Prop. 2.1] or [Ma2, Example 1.7]). Thus ${}_H M$ is a braided monoidal category.

We consider $A = k[x]$ in ${}_H M$ by $g^i \cdot x^j = q^{ij}x^j$. It is easy to see that A is an algebra in ${}_H M$. The braiding $t_{A,A}$ is given by $t_{A,A}(x^l \otimes x^m) = q^{lm}(x^m \otimes x^l)$. For

$$t_{A,A}(x^l \otimes x^m) = \frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij}(g^i \cdot x^m) \otimes (g^j \cdot x^l)$$

$$= \left(\frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij}q^{im}q^{jl}\right)(x^m \otimes x^l).$$

Write $l = an + r$, $m = bn + s$ with $0 \le r, s < n$. Then

$$\frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij}q^{im}q^{jl} = \frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij}q^{is}q^{jr}$$

$$= \frac{1}{n}\sum_{i=0}^{n-1} q^{is}\left(\sum_{j=0}^{n-1} q^{j(r-i)}\right).$$

If $i \ne r$,

$$\sum_{j=0}^{n-1} q^{j(r-i)} = \frac{(q^{r-i})^n - 1}{q^{r-i} - 1} = 0.$$

Hence $t_{A,A}(x^l \otimes x^m) = q^{rs}(x^m \otimes x^l) = q^{lm}(x^m \otimes x^l)$. Then $A \otimes A$ is an algebra with $(x^i \otimes x^j)(x^l \otimes x^m) = q^{jl}x^{i+l} \otimes x^{j+m}$. We consider A as a bialgebra in ${}_H M$ by putting $\Delta x = 1 \otimes x + x \otimes 1$ and requiring Δ to be an algebra map in ${}_H M$. Note that $(x \otimes 1)(1 \otimes x) = x \otimes x$ but that $(1 \otimes x)(x \otimes 1) = q(x \otimes x)$. The q-binomial theorem (see [K]) then gives that $\Delta x^m = \sum_{i=0}^{m} \binom{m}{i}_q x^i \otimes x^{m-i}$ for all $m \ge 0$. Note that x^n is primitive, since $\binom{n}{i}_q = 0$ for $1 \le i \le n-1$. The counit ϵ of A is given as usual by $\epsilon(x^i) = \delta_{0,i}$. One checks easily that Δ and ϵ are morphisms in ${}_H M$, ϵ is an algebra map in ${}_H M$, and thus A is a bialgebra in ${}_H M$. A is a Hopf algebra in ${}_H M$ with antipode S defined inductively. $S(1) = 1$ and $S(x) = -x$. Then $S(x^m) = -\sum_{i=0}^{m-1}\binom{m}{i}_q S(x^i)x^{m-i}$ for $m > 1$.

We stress that A is not a bialgebra (Hopf algebra) in the usual sense, but rather is such in the braided monoidal category ${}_H M$. As an algebra in ${}_H M$, A is not commutative. In fact, x does not "commute" with itself, since $x \otimes x \to x^2$, but $t_{A\otimes A}(x \otimes x) = q(x \otimes x) \to qx^2$. Similarly, A is not generally cocommutative as a coalgebra in ${}_H M$.

The results of Section 3 now show that the space A° of linearly recursive sequences is a Hopf algebra in ${}_H M$. The convolution product in A^* is given by $(f_m) *_q (g_m) = (h_m)$, where $h_m = \sum_{i=0}^{m} \binom{m}{i}_q f_i g_{m-i}$. Thus we have:

THEOREM 4.1. *Let q be a root of unity in k. Then the space of linearly recursive sequences is closed under the product $(f_n) * (g_n) = (h_n)$, where $h_n = \sum_{i=0}^{n} \binom{n}{i}_q f_i g_{n-i}$.*

Remark. Theorem 4.1 also follows from the corollary in Section 3.5 of [Ly]. See the remarks at the end of Section 3 and Proposition 3.5.

Since H is quasitriangular, H^* is coquasitriangular. Let $\{h_0, \dots, h_{n-1}\}$ be the basis of H^* dual to $\{1, g, g^2, \dots, g^{n-1}\}$, i.e., $h_i(g^j) = \delta_{ij}$. Then H^* is coquasitriangular via $\langle h_i, h_j \rangle = (h_i \otimes h_j)(R) = (1/n) q^{-ij}$. $A = k[x]$ is in M^{H^*} via $\rho(x^i) = x^i \otimes \sum_{j=0}^{n-1} q^{ij} h_j$. Note that the structure of A in ${}_H M$ is the (rational) left H-module structure induced by ρ. The twist on $A \otimes A$ in M^{H^*} is the same as that in ${}_H M$. In this case, we know from ${}_H M$ that A° is closed under quantum convolution.

We consider H as coquasitriangular via $\langle g^i, g^j \rangle = q^{ij}$. This comes from the bicharacter on G with $\langle g, g \rangle = q$. (See [Mo, Examples 10.2.6 and 10.2.7]). A is in M^H via $\rho'(x^j) = x^j \otimes g^j$ and the twist $t_{A \otimes A}$ in M^H remains as before.

The dual quasitriangular structure on H^* is given by $R' = \sum_{i,j=0}^{n-1} q^{ij} (h_i \otimes h_j)$. A is in ${}_{H^*} M$ via $h_i \cdot x^j = \delta_{i\bar{j}} x^j$, where $\bar{j} \equiv j \bmod n$, $0 \le \bar{j} \le n-1$. This is the (rational) left H^*-module structure induced by ρ'. The twist $t_{A,A}$ in ${}_H M$ again does not change.

The last two paragraphs might appear redundant, since $H = k\mathbb{Z}_n$ is self-dual (i.e., isomorphic to H^* as Hopf algebras). However, we note that the structure of H as coquasitriangular Hopf algebra works also for $k\mathbb{Z}$, where we do not require that q be a root of unity. We write $\mathbb{Z} = \langle g \rangle$ with g of infinite order, to compare with the previous situation. Then we define $\langle g^i, g^j \rangle = q^{ij}$ for all $i, j \ge 0$. A is a right $k\mathbb{Z}$-comodule via $\rho'(x^j) = x^j \otimes g^j$ for $j \ge 0$. The twist $t_{A,A}$ remains the same, i.e., $t_{A,A}(x^i \otimes x^j) = q^{ij} (x^j \otimes x^i)$ for all $i, j \ge 0$. In Section 6, we will show that if q is not a root of unity, then A° is not closed under quantum convolution.

For a Hopf algebra A in ${}_H M$ for a quasitriangular Hopf algebra H, one can construct an ordinary Hopf algebra called the bosonization $B(A)$ of A. (See [Ma2, Theorem 4.11], [Ma3], and [R].) We compute $B(A)$ for $A = k[x]$ and $H = k\mathbb{Z}_n$ as above. As an algebra, $B(A) = A \# H$ is the smash product of A and H. Thus $A = A\#1$ and $H = 1\#H$ are subalgebras, and $(1\#g)(x\#1) = (g \cdot x)\#g = q(x\#g) = q(x\#1)(1\#g)$. Identifying

x with $x\#1$, and g with $1\#g$, then $gx = qxg$. The coalgebra structure depends on $R = (1/n)\sum_{i,j=0}^{n-1} q^{-ij}(g^i \otimes g^j)$. Then

$$\begin{aligned}\Delta(1\#g) &= \frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij}\big(1\#(g^j g)\big) \otimes \big((g^i \cdot 1)\#g\big)\\ &= \left(1\#\frac{1}{n}\sum_{i,j=0}^{n-1} q^{-ij}g^{j+1}\right) \otimes (1\#g).\end{aligned}$$

Now $\sum_{i,j=0}^{n-1} q^{-ij}g^{j+1} = \sum_j g^{j+1}(\sum_i (q^{-j})^i)$. The inner sum is 0 unless $j = 0$ and then it is n. So, with our identification, $\Delta g = g \otimes g$. Similarly,

$$\begin{aligned}\Delta(x\#1) &= \frac{1}{n}\sum_{i,j}(1\#q^{-ij}g^j) \otimes \big((g^i \cdot x)\#1\big)\\ &\quad + \frac{1}{n}\sum_{i,j}(x\#q^{-ij}g^j) \otimes \big((g^i \cdot 1)\#1\big)\\ &= \left(1\#\frac{1}{n}\sum q^{-ij+i}g^j\right) \otimes (x\#1) + \left(x\#\frac{1}{n}\sum_{i,j} q^{-ij}g^j\right) \otimes (1\#1).\end{aligned}$$

$\sum_{i,j} q^{-ij+i}g^j = \sum_j g^j(\sum_i (q^{1-j})^i)$. The inner sum is 0 unless $j = 1$, and it is n if $j = 1$. Thus the sum is ng. Also, $\sum_{i,j} q^{-ij}g^j = \sum_j g^j \sum_i (q^{-j})^i$. The inner sum is 0 unless $j = 0$, so the sum is n. So, in the shortened notation, $\Delta x = g \otimes x + x \otimes 1$. Thus $B(A) = k\langle g, x\rangle$ with $g^n = 1$ and $gx = qxg$, $\Delta g = g \otimes g$, and $\Delta x = g \otimes x + x \otimes 1$. This is related to the finite dimensional Hopf algebra constructed in [T1], which is now denoted $H_n(q^{-1})$, where the additional relation $x^n = 0$ is imposed.

One can also compute the bosonization of A in ${}_{H^*}\mathcal{M}$, and the cobosonization of A in $\mathcal{M}^H$ and $\mathcal{M}^{H^*}$ (see Theorem 4.14 of [Ma2]).

The cobosonization of A in $\mathcal{M}^H$ is isomorphic to the bosonization of A in ${}_H\mathcal{M}$ described above. The bosonization of A in ${}_{H^*}\mathcal{M}$ and its cobosonization in $\mathcal{M}^{H^*}$ are isomorphic. One finds that $h_i x = xh_{i-1}$, $\Delta h_i = \sum_{j+l=i} h_j \otimes h_l$, and $\Delta x = (\sum_{i=0}^{n-1} q^i h_i) \otimes x + x \otimes 1$ for $B(A)$ in ${}_{H^*}\mathcal{M}$ (all indices are modulo n, i.e., between 0 and $n-1$ inclusively). Since $y = \sum_{i=0}^{n-1} q^i h_i$ is a grouplike element of H^* with $y^n = 1$, and $yx = qxy$, the four constructions are all isomorphic. This is not surprising, since H and H^* are dual Hopf algebras, and the left module structures are rational duals of right comodule structures. See [Ma4] for details.

5. COMBINATORIAL ASPECTS

Let q be a primitive nth root of 1. We give here a direct proof that if (f_m) and (g_m) are linearly recursive, then so is $(f_m) *_q (g_m) = (h_m)$, where $h_m = \sum_{i=0}^{m} \binom{m}{i}_q f_i g_{m-i}$. In particular, (h_m) is an interlacing of n linearly recursive sequences. The main ingredients are certain subsequences of linearly recursive sequences, and reduction of Gaussian polynomials modulo n. Our methods will also enable us to determine which linearly recursive sequences are invertible under the quantum product, and which are zero-divisors. The following Lemmas 5.1 and 5.2 are well known. We give proofs in the spirit of earlier parts of the paper.

LEMMA 5.1. *Let $f = (f_m)$ be a linearly recursive sequence over k satisfying the relation $h(x)$. Let $\alpha_1, \ldots, \alpha_l$ be the roots of $h(x)$ in $\bar{k}$, the algebraic closure of k. Then, for any $i, j \geq 0$, the subsequence $f^{(i,j)} = (f_i, f_{i+j}, f_{i+2j}, \ldots)$ is linearly recursive and satisfies $h_j(x) = (x - \alpha_1^j) \cdots (x - \alpha_l^j)$.*

Proof. Let $A = k[x]$, so we have identified $A°$ with the space of linearly recursive sequences. $A°$ is an A-module by $(a \rightharpoonup f)(b) = f(ba)$ for a, b in A, f in $A°$. Then $f = (f_m)$ satisfying $h(x)$ means that $h(x) \rightharpoonup f = 0$ in $A°$.

Note that $f^{(i,j)}(x^t) = f_{i+tj} = f(x^{i+tj})$ for $t \geq 0$. Hence, for $g(x)$ in A, $f^{(i,j)}(g(x)) = f(x^i g(x^j))$. Thus $(h_j(x) \rightharpoonup f^{(i,j)})(x^t) = f^{(i,j)}(x^t h_j(x)) = f(x^i x^{jt} h_j(x^j)) = f(x^{i+jt} h_j(x^j))$. Since $h_j(x^j) = h(x)g(x)$ for some $g(x)$ in $k[x]$, and $h(x) \rightharpoonup f = 0$, we have $f(x^{i+jt} h_j(x^j)) = f(x^{i+jt} h(x) g(x)) = (h(x) \rightharpoonup f)(x^{i+jt} q(x)) = 0$. So $h_j(x) \rightharpoonup f^{(i,j)} = 0$, i.e., $f^{(i,j)}$ satisfies $h_j(x)$. ∎

LEMMA 5.2. *Let q be a primitive nth root of 1. For integers $a \geq b \geq 0$, write $a = a'n + r$, $b = b'n + s$ for $0 \leq r, s < n$. Then $\binom{a}{b}_q = \binom{a'}{b'}\binom{r}{s}_q$, where $\binom{r}{s}_q = 0$ if $r < s$.*

Proof. We consider the quantum plane $k\langle x, y\rangle$ with $yx = qxy$. For $m \geq 1$, $(x+y)^m = \sum_{l=0}^{m} \binom{m}{l}_q x^l y^{m-l}$. Note that $\binom{n}{r}_q = 0$ for $0 < r < n$ and that x^n and y^n are in the center of $k\langle x, y\rangle$. Therefore

$$(x+y)^a = (x+y)^{a'n}(x+y)^r = (x^n + y^n)^{a'}(x+y)^r$$
$$= \left(\sum_{i=0}^{a'} \binom{a'}{i} x^{ni} y^{n(a'-i)}\right)\left(\sum_{j=0}^{r} \binom{r}{j}_q x^j y^{r-j}\right)$$
$$= \sum_{i=0}^{a'} \sum_{j=0}^{r} \binom{a'}{i}\binom{r}{j}_q x^{ni+j} y^{a-ni-j}.$$

But also $(x+y)^a = \sum_{t=0}^{a} \binom{a}{t}_q x^t y^{a-t}$. Compare the coefficients of $x^b y^{a-b}$, i.e., let $t = b$ in the second sum. If $s \le r$, take $i = b'$ and $j = s$ in the double sum. If $r < s$, the term $x^b y^{a-b}$ does not appear in the double sum, so $\binom{a}{b}_q = 0$.

THEOREM 5.3. *Let q be a primitive nth root of unity. Let $f = (f_m)$ and $g = (g_m)$ be linearly recursive, satisfying $h(x) = \prod_i (x - \alpha_i)^{m_i}$ and $l(x) = \prod_j (x - \beta_j)^{n_j}$, respectively. Then the quantum product $f *_q g = h = (h_m)$, where $h_m = \sum_{t=0}^{m} \binom{m}{t}_q f_t g_{m-t}$, is linearly recursive, satisfying $u(x^n)$, where $u(x) = \prod_{i,j} (x - (a_i^n + \beta_j^n)^{m_i + n_j - 1})$.*

Proof. For $0 \le r < n$ and $d \ge 0$,

$$
\begin{aligned}
h_{r+dn} &= \sum_{b=0}^{dn+r} \binom{dn+r}{b}_q f_b g_{r+dn-b} \\
&= \sum_{s=0}^{r} \sum_{j=0}^{d} \binom{d}{j} \binom{r}{s}_q f_{s+jn} g_{r-s+n(d-j)} \quad \text{(by Lemma 5.2)} \\
&= \sum_{s=0}^{r} \binom{r}{s}_q \left(\sum_{j=0}^{d} \binom{d}{j} (f^{(s,n)})_j (g^{(r-s,n)})_{d-j} \right).
\end{aligned}
$$

Therefore, $h^{(r,n)} = \sum_{s=0}^{r} \binom{r}{s}_q f^{(s,n)} * g^{(r-s,n)}$, where $*$ is the usual convolution product. By Lemma 5.1, $f^{(s,n)}$ and $g^{(r-s,n)}$ are linearly recursive, satisfying $f_n(x) = \prod_i (x - \alpha_i^n)^{m_i}$ and $g_n(x) = \prod_j (x - \beta_j^n)^{n_j}$, respectively. By classical results, $h^{(r,n)}$ is linearly recursive, satisfying $u(x)$. Now h is the interlacing of $h^{(0,n)}, h^{(1,n)}, \dots, h^{(n-1,n)}$ (see [La-T]), so that h is linearly recursive and satisfies $u(x^n)$. ∎

Lemma 5.2 appeared in the thesis of Gloria Olive [O]. The idea of using it to prove Theorem 5.3 was suggested by Ira Gessel, and the strategy of our proof of Theorem 5.3 was suggested by J.-P. Bézivin. We thank these mathematicians for their suggestions.

We turn now to the group of units under $*_q$. Recall that if $q = 1$, the result depends on the characteristic of k. If k has characteristic zero, the $*$-invertible linearly recursive sequences are the nonzero geometric sequences $(a, ar, ar^2, ar^3, \dots)$. If k has positive characteristic, then a linearly recursive sequence $f = (f_m)$ is invertible if and only if $f_0 = 0$. See [T2] for a discussion of these results.

Let q be a primitive nth root of 1, $*_q$ the quantum product on the space L of linearly recursive sequences. Let ϕ be the linear map of L to L given by $\phi(f) = f^{(0,n)}$, i.e., $\phi(f_0, f_1, f_2, \dots) = (f_0, f_n, f_{2n}, \dots)$.

LEMMA 5.4. *ϕ is a surjective algebra map of $(L, *_q)$ onto $(L, *)$.*

Proof. Let $f, g \in L$, $h = f *_q g$. The proof of Theorem 5.3 shows that h is the interlacing of $h^{(0,n)}, h^{(1,n)}, \ldots, h^{(n-1,n)}$, where $h^{(0,n)} = f^{(0,n)} * g^{(0,n)}$. Hence $\phi(f *_q g) = \phi(h) = h^{(0,n)} = f^{(0,n)} * g^{(0,n)} = \phi(f) * \phi(g)$. Clearly ϕ is surjective. ∎

LEMMA 5.5. *The kernel of ϕ is a nilpotent ideal of $(L, *_q)$.*

Proof. Let $Z^{(i)}$ be the sequence with 1 in the ith coordinate and zero elsewhere. Then $Z^{(i)} *_q Z^{(j)} = \binom{i+j}{i}_q Z^{(i+j)}$. An element f in $\ker \phi$ is of the form $\sum_{i \not\equiv 0 \pmod n} \alpha_i Z^{(i)}$ (an infinite sum), so an element in $(\ker \phi)^n$ is of the form

$$\sum_{i_1, \ldots, i_n \not\equiv 0 \pmod n} \alpha_{i_1, \ldots, i_n} Z^{(i_1)} *_q \cdots *_q Z^{(i_n)}.$$

We claim that $Z^{(i_1)} *_q \cdots *_q Z^{(i_n)} = 0$ if $i_1, \ldots, i_n \not\equiv 0 \pmod n$. Write $i_j = a_j n - r_j$ with $0 < r_j < n$ for $j = 1, \ldots, n$. There is a $j' \leq n - 1$ such that $r_1 + \cdots + r_{j'} < n$ but $r_1 + \cdots + r_{j'} + r_{j'+1} \geq n$. So, for some β in k,

$$\begin{aligned} Z^{(i_1)} *_q \cdots *_q Z^{(i_{j'+1})} &= \beta Z^{(i_1 + \cdots + i_{j'})} *_q Z^{(i_{j'+1})} \\ &= \beta \binom{i_1 + \cdots + i_{j'} + i_{j'+1}}{i_{j'+1}}_q Z^{(i_1 + \cdots + i_{j'} + i_{j'+1})}. \end{aligned}$$

Now $i_1 + \cdots + i_{j'} + i_{j'+1} = n(a_1 + \cdots + a_{j+1}) + (r_1 + \cdots + r_{j'+1} - n) + n = n(a_1 + \cdots + a_{j'+1} + 1) + (r_1 + \cdots + r_{j+1} - n)$, where $0 \leq r_1 + \cdots + r_{j+1} - n < n$. By Lemma 5.2,

$$\binom{i_1 + \cdots + i_{j'+1}}{i_{j'+1}}_q = \binom{a_1 + \cdots + a_{j'+1} + 1}{a_{j'+1}} \binom{r_1 + \cdots + r_{j'+1} - n}{r_{j'+1}}_q.$$

The second factor is 0 by Lemma 5.2, since $r_1 + \cdots + r_{j'} < n$. Thus $(\ker \phi)^n = 0$. ∎

THEOREM 5.6. *f in L is $*_q$-invertible if and only if $\phi(f)$ is $*$-invertible. If k has characteristic zero, f is $*_q$-invertible if and only if $(f_0, f_n, f_{2n}, \ldots)$ is a nonzero geometric sequence. If k has positive characteristic, f is $*_q$-invertible if and only if $f_0 \neq 0$.*

Proof. If f is $*_q$-invertible, then $\phi(f)$ is $*$-invertible by Lemma 5.4. Conversely, if $\phi(f)$ is $*$-invertible, with $*$-inverse $\phi(g)$, then $\phi(f) * \phi(g) = \epsilon$ $(= (1, 0, 0, 0, \ldots))$, the unit element of L under $*$ and $*_q$. Then $f *_q g - \epsilon$ is in $\ker \phi$ so $f *_q g = \epsilon + h$, where h is $*_q$-nilpotent by

Lemma 5.5. Since $\epsilon + h$ is $*_q$-invertible with $*_q$-inverse $\epsilon - h + h^2 + \cdots + (-1)^{n-1}h^{n-1}$ (where h^i is the i-fold $*_q$-product of h with itself), it follows that f is $*_q$-invertible. The specific form of f follows from the case $q = 1$ described before Lemma 5.4. ∎

PROPOSITION 5.7. (a) *If k has characteristic zero, then* $\ker\phi$ *is the set of zero-divisors of* $(L, *_q)$. (b) *If k has positive characteristic, then* $f = (f_m)$ *in L is a zero-divisor in* $(L, *_q)$ *if and only if* $f_0 = 0$.

Proof. (a) The elements of $\ker\phi$ are nilpotent, hence zero-divisors. If $f *_q g = 0$ in L, with $f \neq 0$ and $g \neq 0$, then $\phi(f) * \phi(g) = 0$. If $\phi(f) \neq 0$, then $\phi(g) = 0$, since $(L, *)$ is an integral domain. So $g_i = 0$ if $i \equiv 0 \pmod n$, i.e., $g^{(0,n)} = 0$, in the notation of Lemma 5.1. By the proof of Theorem 5.3, $0 = f *_q g$ is the interlacing of $w^{(0)}, \ldots, w^{(n-1)}$, where $w^{(i)} = \sum_{j=0}^{i} \binom{i}{j}_q f^{(j,n)} * g^{(i-j,n)}$. Thus each $w^{(i)} = 0$. Now $w^{(1)} = f^{(0,n)} * g^{(1,n)}$, since $g^{(0,n)} = 0$. Since $f^{(0,n)} = \phi(f) \neq 0$, we have $g^{(1,n)} = 0$. Suppose $0 = g^{(0,n)} = g^{(1,n)} = \cdots = g^{(l,n)}$ for $l \leq n-2$. We show that $g^{(l+1,n)} = 0$, which implies that $g = 0$, a contradiction.

$$0 = w^{(l+1)} = \sum_{j=0}^{l+1} \binom{l+1}{j}_q f^{(j,n)} * g^{(l+1-j,n)} = f^{(0,n)} * g^{(l+1,n)}$$

by the induction assumption. Thus $g^{(l+1,n)} = 0$.

(b) If $f \in L$ with $f_0 \neq 0$, then f is $*_q$-invertible, and so not a zero-divisor. Conversely, if $f_0 = 0$, then $\phi(f)$ has a zero initial coordinate, i.e., $\phi(f) = \sum_{i=1}^{\infty} \alpha_i Z^{(i)}$. Then

$$\phi(f)^p = \left(\sum_{i=1}^{\infty} \alpha_i Z^{(i)}\right)^p = \sum_{i=1}^{\infty} \alpha_i^p \binom{pi}{i, i, \ldots, i} Z^{ip},$$

where the multinominal coefficient has p is in the denominator. This coefficient $= p\binom{pi-1}{i-1, i, \ldots, i} = 0$ in k (see [T2]). Then $0 = \phi(f)^p = \phi(f^p)$. So $f^p \in \ker\phi$ and is nilpotent. Thus f is nilpotent and a zero-divisor. ∎

6. THE CASE WHEN q IS NOT A ROOT OF UNITY

Let $q \neq 0$ in k be a nonroot of unity. Let $a \neq 0$ in k. Denote by $e(a)$ the geometric sequence $e(a) = (1, a, a^2, a^3, \ldots) = (a^n)_{n \geq 0}$. $e(a)$ is linearly recursive, satisfying $x - a$. We will show that if a and b are nonzero elements of k, then $e(a) *_q e(b)$ is not linearly recursive. Let $f(a,b) = (f_n(a,b)) = e(a) *_q e(b)$. Then $f_n(a,b) = \sum_{i=0}^{n} \binom{n}{i}_q a^i b^{n-i}$.

We claim that

$$f_{n+1}(a,b) = (a+b)f_n(a,b) + ab(q^n - 1)f_{n-1}(a,b), \qquad \text{for all } n \geq 1. \tag{†}$$

(Note, this is a recursion relation with a nonconstant coefficient.)

Note that $f_0(a,b) = 1$ and $f_1(a,b) = a + b$. We will give a direct proof of (†) for $n \geq 1$, based on the following two properties of Gaussian polynomials [A, (3.3.3) and (3.3.4)]:

$$\text{(i)} \qquad \binom{n}{m}_q = \binom{n-1}{m}_q + q^{n-m}\binom{n-1}{m-1}_q,$$

$$\text{(ii)} \qquad \binom{n}{m}_q = \binom{n-1}{m-1}_q + q^{m}\binom{n-1}{m}_q.$$

$$f_n(a,b) = \sum_{i=0}^{n}\binom{n}{i}_q a^i b^{n-i} = \sum_{i=1}^{n}\binom{n-1}{i-1}_q a^i b^{n-i} + \sum_{i=0}^{n-1}\binom{n-1}{i}_q q^i a^i b^{n-i} \qquad \text{(by (ii))}$$

$$= a\sum_{i=0}^{n-1}\binom{n-1}{i}_q a^i b^{n-i-1} + b\sum_{i=0}^{n-1}\binom{n-1}{i}_q (qa)^i b^{n-i-1}$$

$$= af_{n-1}(a,b) + bf_{n-1}(qa,b).$$

But

$$f_n(qa,b) - bf_{n-1}(qa,b)$$

$$= q^n a^n + \sum_{i=0}^{n-1}\binom{n}{i}_q q^i a^i b^{n-i} - b\sum_{i=0}^{n-1}\binom{n-1}{i}_q q^i a^i b^{n-1-i}$$

$$= q^n a^n + \sum_{i=0}^{n-1}\left(\binom{n}{i}_q - \binom{n-1}{i}_q\right) q^i a^i b^{n-i}$$

$$= q^n a^n + \sum_{i=1}^{n-1} q^{n-i}\binom{n-1}{i-1}_q q^i a^i b^{n-i} \qquad \text{(by (i))}$$

$$= aq^n\left(\sum_{i=1}^{n-1}\binom{n-1}{i}_q a^i b^{n-i-1}\right) = aq^n f_{n-1}(a,b).$$

Thus we have that $f_n(a,b) = af_{n-1}(a,b) + bf_{n-1}(qa,b)$ and $f_n(qa,b) = bf_{n-1}(qa,b) + aq^n f_{n-1}(a,b)$.

Combining these results,

$$\begin{aligned} f_{n+1}(a,b) &= af_n(a,b) + bf_n(qa,b) \\ &= af_n(a,b) + b^2 f_{n-1}(qa,b) + abq^n f_{n-1}(a,b) \\ &= af_n(a,b) + b\left[f_n(a,b) - af_{n-1}(a,b)\right] + abq^n f_{n-1}(a,b) \\ &= (a+b)f_n(a,b) + ab(q^n - 1)f_{n-1}(a,b). \qquad \text{This is } (\dagger). \end{aligned}$$

Now denote $f_n(a,b)$ by just f_n. If f were linearly recursive satisfying a relation of degree d, the Hankel matrix

$$H_d(f) = \begin{bmatrix} f_0 & f_1 & \cdots & f_d \\ f_1 & f_2 & \cdots & f_{d+1} \\ \vdots & \vdots & & \vdots \\ f_d & f_{d+1} & \cdots & f_{2d} \end{bmatrix}$$

of f would have determinant 0. We show that all Hankel matrices $H_n(f)$ of f have nonzero determinant, showing that $f = e(a) *_q e(b)$ is not linearly recursive.

$$H_1(f) = \begin{bmatrix} 1 & a+b \\ a+b & a^2 + (q+1)ab + b^2 \end{bmatrix}$$

has determinant $(q-1)ab \neq 0$. We proceed by induction, using $(\dagger)$ to perform elementary row operations on

$$H_{n+1}(f) = \begin{bmatrix} f_0 & f_1 & \cdots & f_{n+1} \\ f_1 & f_2 & \cdots & f_{n+2} \\ \vdots & \vdots & & \vdots \\ f_{n-1} & f_n & \cdots & f_{2n} \\ f_n & f_{n+1} & \cdots & f_{2n+1} \\ f_{n+1} & f_{n+2} & \cdots & f_{2n+2} \end{bmatrix}.$$

Call the rows of $H_{n+1}(f)$ by $R_0, R_1, \ldots, R_{n+1}$. Replacing R_{n+1} by $R_{n+1} - (a+b)R_n - ab(q^n - 1)R_{n-1}$ replaces R_{n+1} by

$$\left[0, (q^{n+1} - q^n)abf_n, (q^{n+2} - q^n)abf_{n+1}, \ldots, (q^{2n+1} - q^n)f_{2n}\right].$$

Then replacing R_n by $R_n - (a+b)R_{n-1} - ab(q^{n-1} - 1)R_{n-2}$ replaces R_n by

$$\left[0, (q^n - q^{n-1})abf_{n-1}, (q^{n+1} - q^{n-1})abf_n, \ldots, (q^{2n} - q^{n-1})abf_{2n-1}\right].$$

Continue in this way to clear out the first column beneath f_0 and f_1. Finally, replacing R_1 by $R_1 - (a+b)R_0$, yields the matrix

$$\begin{bmatrix} f_0 & f_1 & \cdots & f_{n+1} \\ 0 & (q-1)abf_0 & \cdots & (q^{n+1}-1)abf_n \\ 0 & (q^2-q)abf_1 & \cdots & (q^{n+2}-q)abf_{n+1} \\ \vdots & \vdots & & \vdots \\ 0 & (q^n - q^{n-1})abf_{n-1} & \cdots & (q^{2n}-q^{n-1})abf_{2n-1} \\ 0 & (q^{n+1}-q^n)abf_n & \cdots & ((q^{2n+1}-q^n)abf_{2n} \end{bmatrix}.$$

Thus

$$\det H_{n+1}(f) = (ab)^{n+1} \det \begin{bmatrix} (q-1)f_0 & \cdots & (q^{n+1}-1)f_n \\ (q^2-q)f_1 & \cdots & (q^{n+2}-q)f_{n+1} \\ \vdots & & \vdots \\ (q^{n+1}-q^n)f_n & \cdots & (q^{2n+1}-q^n)f_{2n} \end{bmatrix}.$$

Factoring $q-1$ from the first column, q^2-1 from the second column $,\dots, q^{n+1}-1$ from the last column, we get

$$\begin{aligned} &\det H_{n+1}(f) \\ &\quad = (ab)^{n+1}(q-1)(q^2-1)\cdots(q^{n+1}-1)\det \begin{bmatrix} f_0 & \cdots & f_n \\ qf_1 & \cdots & qf_{n+1} \\ \vdots & & \\ q^n f_n & \cdots & q^n f_{2n} \end{bmatrix} \\ &\quad = (ab)^{n+1}(q-1)(q^2-1)\cdots(q^{n+1}-1)\cdot q \cdot q^2 \cdots q^n \det H_n(f). \end{aligned}$$

Hence $\det H_{n+1}(f) \neq 0$ by induction.

It follows by (†) that $f_n(a,b)$ is a polynomial in a and b, since $f_0(a,b) = 1$ and $f_1(a,b) = a+b$. If $b = 1$, $f_n(a,1)$ is called the nth Rogers–Szegö polynomial in a, as noted in Example 3 on page 49 of [A]. In this case, our relation (†) appears there as Example 6.

As explained in Section 4, we can consider $H = k\mathbb{Z}$ as a coquasitriangular Hopf algebra, and A in M^H, the braided monoidal category of right H-comodules. Our example shows that A° is not closed under quantum convolution.

ACKNOWLEDGMENT

We thank the referee for his/her careful reading of the paper and useful comments.

REFERENCES

[A] G. E. Andrews, "The Theory of Partitions," Addison-Wesley, Reading, MA, 1976.

[D] V. G. Drinfeld, Quantum groups, *in* Proc. Int. Cong. Math., Berkeley 1986," 798–820, Am. Math. Soc., Providence, 1987.

[K] Ch. Kassel, "Quantum Groups," Springer-Verlag, New York/Berlin, 1995.

[La-T] R. G. Larson and E. J. Taft, The algebraic structure of linearly recursive sequences under Hadamard product, *Israel J. Math.* **72** (1990), 118–132.

[Ly] V. Lyubashenko, Tangles and Hopf algebras in braided categories, *J. Pure Appl. Algebra* **98** (1995), 245–278.

[Ma1] S. Majid, Anyonic quantum groups, *in* "Spinors, Twistors, Clifford Algebras and Quantum Deformations" (Z. Oziewicz et al., Ed.), pp. 327–336, Kluwer Academic, Dordrecht/Norwell, MA, 1993.

[Ma2] S. Majid, Algebras and Hopf algebras in braided categories, *in* "Advances in Hopf Algebras," Lecture Notes in Pure and Applied Mathematics, Vol. 158, pp. 55–105, Dekker, New York, 1994.

[Ma3] S. Majid, Cross products by braided groups and bosonization, *J. Algebra* **163** (1994), 165–190.

[Ma4] S. Majid, The quantum double as quantum mechanics, *J. Geom. Phys.* **13** (1994), 169–202.

[Mo] S. Montgomery, "Hopf Algebras and Their Actions on Rings," Am. Math. Soc. CBMS, Vol. 82, Providence, 1993.

[O] G. Olive, "Generalized Powers," Ph.D. thesis, Oregon State University, 1963.

[P-T] B. Peterson and E. J. Taft, The Hopf algebra of linearly recursive sequences, *Aequationes Math.* **20** (1980), 1–17.

[R] D. Radford, The structure of Hopf algebras with a projection, *J. Algebra* **92** (1985), 322–347.

[S] M. Sweedler, "Hopf Algebras," Benjamin, New York, 1969.

[T1] E. J. Taft, The order of the antipode of a finite-dimensional Hopf algebra, *Proc. Nat. Acad. Sci.* **68** (1971), 2631–2633.

[T2] E. J. Taft, Hurwitz invertibility of linearly recursive sequences, *Congr. Numer.* **73** (1990), 37–40.

[T3] E. J. Taft, Algebraic aspects of linearly recursive sequences, *in* "Advances in Hopf Algebras," Lecture Notes in Pure and Applied Mathematics, Vol. 158, pp. 299–317, Dekker, New York, 1994.