

Fixpoint operators for domain equations

John Power^{a,1}, Giuseppe Rosolini^{b,*}

^a*Department of Computer Science, University of Edinburgh, King's Buildings,
Edinburgh EH9 3JZ, Scotland, UK*

^b*DISI, University di Genova, via Dodecaneso 35, 16146 Genova, Italy*

Dedicated to Peter Freyd on the occasion of his 60th birthday

Abstract

We investigate fixpoint operators for domain equations. It is routine to verify that if every endofunctor on a category has an initial algebra, then one can construct a fixpoint operator from the category of endofunctors to the category. That construction does not lift routinely to enriched categories, using the usual enriched notion of initiality of an endofunctor. We show that by embedding the 2-category of small enriched categories into the 2-category of internal categories of a presheaf topos, we can recover the fixpoint construction elegantly. Also, we show that in the presence of cotensors, an enriched category allows the fixpoint construction. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Enriched category; Initial algebra; Internal category; Yoneda lemma

1. Introduction

Given a functor $H : C \rightarrow C$, an initial algebra for H is an initial object of the category $H\text{-Alg}$, an object of which is an object of C together with an H -action, and an arrow is an arrow in C that respects the H -actions. More explicitly, an initial H -algebra consists of an object I of C together with a map $i : HI \rightarrow I$ such that for any other such pair $(A, a : HA \rightarrow A)$, there is a unique map $f : I \rightarrow A$ in C such that $a \cdot Hf = f \cdot i$.

The condition that (I, i) is an initial H -algebra can be re-expressed as the condition that for each object A of C , there is a unique function from $C(HA, A)$ to $C(I, A)$ such that the diagram

$$\begin{array}{ccccc}
 C(HA, A) & \xrightarrow{\hspace{10em}} & C(HA, A) \times C(I, A) \\
 \downarrow & & \downarrow \\
 C(I, A) \times C(HI, I) & \xrightarrow{\hspace{2em}} & C(HI, A) & \xleftarrow{\hspace{2em}} & C(HA, A) \times C(HI, HA)
 \end{array}$$

* Corresponding author. Tel.: +39-0-10-353-6630; fax: +39-0-10-353-6699.

E-mail address: rosolini@disi.unige.it (G. Rosolini).

¹ This work is supported by EPSRC Grant GR/J84205: Frameworks for programming language semantics and logic.

commutes, where the top arrow of the diagram is the pair consisting of the identity and the function, the left-hand vertical arrow is determined by the function together with the initial object, the right-hand arrow is given by the behaviour of H on homs, and the bottom arrows are given by composition in C .

Such equivalent expression allows us to make the following analysis: suppose we seek a category C for which every endofunctor has a fixed point, as for instance advocated by Freyd in [3, 4] and further developed by others such as Fiore and Plotkin [2]. That can be neatly expressed as the criterion that there is a functor $Y: [C, C] \rightarrow C$ such that the diagram

$$\begin{array}{ccc} [C, C] & \longrightarrow & [C, C] \times C \\ & \searrow & \downarrow \\ & & C \end{array}$$

with horizontal map determined by Y and the identity, diagonal map given by Y , and vertical map the evaluation map, commutes. This diagram expresses the fixpoint property of the operator Y . So these data and axiom assert that C is equipped with a fixpoint operator, as is used to model solutions to positively defined recursive domain equations.

By the Yoneda lemma applied to Cat , to give such a fixpoint operator is to give a function, for each small category D , that assigns a functor $Y_h: D \rightarrow C$ to each functor $h: D \times C \rightarrow C$, naturally in D , and satisfying the fixpoint condition. To obtain such a functor, for each object W of D , one considers the endofunctor $h(W, —): C \rightarrow C$, and takes an initial fixpoint. Then, one uses the above condition together with some routine calculation to make that construction into a functor from D to C : given a map $f: W \rightarrow Z$ in D , one takes an initial fixpoint $(J, j: h(Z, J) \rightarrow J)$ for $h(Z, —)$; pre-composing with $h(f, J)$ gives an $h(W, —)$ -algebra, and using initiality of our choice of initial $h(W, —)$ -algebra gives us the desired map in C . One must then do some checking to verify functoriality.

It is interesting to note that the presentation given via the Yoneda lemma allows us to prove that the product of two categories which admit a fixed point for every endofunctor also enjoys that property.

It was recognized by Freyd and others that the situation of ordinary categories was too limited. One requires an account of this situation for internal categories in a topos, as arise in synthetic domain theory (see Rosolini's notes [8] for a recent account), and for categories enriched in a cartesian closed category, as has begun to be developed by Fiore in his book [1] and others under the guise of axiomatic domain theory. The former, the situation internally in a topos, is straightforward provided one takes a little care with the definitions and constructions. The latter, the generalisation to enriched categories, is less so. Here, we show that in fact, the latter may be seen as

a special case of the former, by means of a construction that embeds the 2-category of small enriched categories into the 2-category of internal categories of a topos. In particular, this accounts for the *a priori* strange conditions that have been put on enriched categories to explain the above analysis, and yields a simple criterion on an enriched category giving an enriched version of this analysis.

The paper is organized as follows. In Section 2, we recall the basic definitions of enriched category theory and explain the problem of extending the above analysis. In Section 3, we describe the main construction of the paper, that of the full embedding of the 2-category of small enriched categories into a 2-category of the form $[V^{\text{op}}, \text{Cat}]$. In Section 4, we recall the basic definitions of internal categories, explain some of the issues therein, and show how our main construction yields the elegant analysis we desire. In Section 5, we show that in the presence of cotensors, which we define, an enriched category has the required property. Finally, in Section 6, we outline an extension of this work to account for domain equations of mixed variance.

2. The question

Let V be a cartesian category. The specific category we have in mind is that of ω -cpo's with least element and maps that preserve the ω -cpo structure but not necessarily the least element. Such arise in modelling domain equations, for instance in [10].

Definition 2.1. A V -category C consists of

- a set $ob(C)$ of objects;
- for each pair (X, Y) of objects of C , a homobject $C(X, Y)$ of V ;
- a family of maps $\cdot : C(Y, Z) \times C(X, Y) \rightarrow C(X, Z)$ in V , called composition maps; and
- identities $1 \rightarrow C(X, X)$

satisfying the associativity law for composition, and left- and right-unit laws with respect to the composition for the identities, see [6].

In the case that V is Set , a V -category is exactly a small category. If V is the cartesian closed category mentioned above, we have exactly the situation of Smyth and Plotkin's O -categories, in which they modelled domain equations.

Definition 2.2. A V -functor between V -categories C and D consists of an object function $H : ob(C) \rightarrow ob(D)$, together with maps in V of the form $C(X, Y) \rightarrow D(HX, HY)$, subject to two axioms asserting preservation of composition and identities, see [6].

One can similarly define V -natural transformations, yielding the 2-category $V\text{-Cat}$ of V -categories. These definitions are fundamental to axiomatic domain theory, and form the conceptual basis for Fiore's book [1] on the subject. Here, we seek to extend the analysis of the introduction to this section; for it is here that domain equations have

traditionally been studied, so we require an analysis in this enriched setting to provide an axiomatic basis for solving domain equations.

So assume that we have not just a category C and an endofunctor H on it, but a V -category C and a V -enriched endofunctor H on it for a cartesian closed category V . In this more general setting, the condition that is required for the analysis of Section 1 is the latter of the two conditions cited, where the diagram is interpreted as a diagram in V rather than in Set : for each object A of C , there is a unique map in V from $C(HA, A)$ to $C(I, A)$ such that the diagram

$$\begin{array}{ccccc}
 C(HA, A) & \xrightarrow{\quad\quad\quad} & C(HA, A) \times C(I, A) & & \\
 \downarrow & & \downarrow & & \\
 C(I, A) \times C(HI, I) & \longrightarrow & C(HI, A) & \longleftarrow & C(HA, A) \times C(HI, HA)
 \end{array}$$

commutes. But not only is that strictly stronger than the condition that one has an initial algebra in the ordinary category of H -algebras, but a priori seems to have nothing to do with the notion of (I, i) being initial in a V -enriched category $H\text{-Alg}$ either. So we seek some account of this condition that draws it into line with the standard development of category theory.

Observe that by the Yoneda lemma, the condition may be expressed by

Condition 2.3. For every object A in C , for each object X of V , and each map in V from X to $C(HA, A)$, there is a unique map from X to $C(I, A)$ making

$$\begin{array}{ccccc}
 X & \xrightarrow{\quad\quad\quad} & C(HA, A) \times C(I, A) & & \\
 \downarrow & & \downarrow & & \\
 C(I, A) \times C(HI, I) & \longrightarrow & C(HI, A) & \longleftarrow & C(HA, A) \times C(HI, HA)
 \end{array}$$

commute, naturally in X , where the top arrow of the diagram is the pair consisting of the two maps, the left-hand vertical arrow is determined by the latter map together with the initial algebra, the right-hand arrow is given by the behaviour of H on homs, and the bottom arrows are given by composition in C .

By taking the case of $X = 1$, the terminal object and unit of V , we see that this says that (I, i) is an initial H -algebra, and a little more than that: one may say that (I, i) is a *parameterized* initial H -algebra. So, essentially, we need some natural account of how such a condition might be handled.

This condition has been used recently by Plotkin in his invited lecture at the Darmstadt LDPL conference in celebration of Dana Scott’s honorary doctorate, and has appeared either implicitly or explicitly in the work of various colleagues, in particular Alex Simpson. We seek to understand it in more natural category-theoretic terms.

This observation gives us a hint that we need some sort of parametrization to analyse the condition. Parametrization means that we are implicitly passing to a functor 2-category of the form $[V^{\text{op}}, \text{Cat}]$. This hint leads us to the construction of the next section, which is the fundamental construction of the paper.

3. The construction

Given a cartesian category V , there is a 2-functor from the 2-category $V\text{-Cat}$ of V -categories, V -functors, and V -natural transformations, into the functor 2-category $[V^{\text{op}}, \text{Cat}]$. The 2-functor takes a V -category C to the functor for which the X -component has as objects the objects of C and as a map from A to B , an arrow in V from X to $C(A, B)$. Composition is given by using the diagonal on X ; and this functor behaves on arrows of V^{op} by precomposition. We shall call this functor $\text{Para} : V\text{-Cat} \rightarrow [V^{\text{op}}, \text{Cat}]$.

Theorem 3.1. *The 2-functor Para is fully faithful.*

Proof. Consider an indexed functor, i.e. an arrow in $[V^{\text{op}}, \text{Cat}]$, of the form $h : \text{Para}(C) \rightarrow \text{Para}(D)$. Taking its component at 1 determines the object function of a V -functor H from C to D : by naturality, the object function for each h_X must agree with that for h_1 because each $\text{Para}(C)(!)$ is the identity on objects for the unique map $! : X \rightarrow 1$. Now, to each map from X to $C(A, B)$ in V , the functor h_X assigns a map from X to $D(HA, HB)$, and this is natural in X . So by the Yoneda lemma, this yields a map in V from $C(A, B)$ to $D(HA, HB)$. Thus we have the data for a V -functor H . Two other applications of the Yoneda lemma prove that it satisfies the axioms for a V -functor. The situation for V -natural transformations is similar. \square

We also have

Proposition 3.2. *The 2-functor Para preserves finite products.*

Proof. Routine checking. \square

In fact, Para preserves all conical limits, provided we take a little care with size: since V is not assumed to be small, the 2-category $[V^{\text{op}}, \text{Cat}]$ need not be locally small, so we must be a little careful about completeness. We remark, for those familiar with 2-categories, that Para does not preserve all 2-limits: if it preserved inserters, we would gain nothing from our analysis here, as will become clear in the next section.

Full faithfulness of Para tells us that to give an enriched endofunctor H on a V -category C is equivalent to giving a V^{op} -indexed endofunctor on $\text{Para}(C)$. Moreover, since Para preserves finite products, to give a V -functor $h: D \times C \rightarrow C$ is to give a V^{op} -indexed functor from $\text{Para}(D) \times \text{Para}(C)$ to $\text{Para}(C)$. Thus, it suffices to consider conditions under which, given a V^{op} -indexed functor from $D \times C$ to C for arbitrary objects C and D of the 2-category $[V^{\text{op}}, \text{Cat}]$, we can obtain a V^{op} -functor from D to C , naturally in D and satisfying the fixpoint condition. In fact, alas, that is a little too strong for our desired result: we can work with an arbitrary C , but need to restrict attention at one key point to those D in the image of Para .

As we shall explain in the next section, $[V^{\text{op}}, \text{Cat}]$ is exactly $\text{Cat}([V^{\text{op}}, \text{Set}])$, the 2-category of internal categories in the category $[V^{\text{op}}, \text{Set}]$, so modulo the *caveat* regarding D above, we are reduced to generalising the analysis of Section 1 to $\text{Cat}(E)$ for a reasonable category E . Of course, we have a very special category E as it is a presheaf category with large exponent. But we need much less than that: really just finite limits for most purposes. We pursue that study in the next section.

4. Fixpoint operators to solve domain equations in $\text{Cat}(E)$

In Section 1, we already have an analysis of fixpoint operators for domain equations in the 2-category Cat . A careful study of our argument there allows us to extend from Cat to a 2-category of the form $\text{Cat}(E)$ if E has finite limits.

Definition 4.1. Given any category E with finite limits, an *internal category* in E consists of

- a parallel pair of maps $\text{dom}, \text{cod}: C_{\text{ar}} \rightarrow C_{\text{ob}}$,
- a composition map $\cdot: C_{\text{tr}} \rightarrow C_{\text{ar}}$, where C_{tr} is given by the pullback of cod along dom ,
- a map to give identities $\iota: C_{\text{ob}} \rightarrow C_{\text{ar}}$

subject to one equation to represent associativity of composition, and two equations to ensure that the map for identities yields a left and right identity for composition, see e.g. [7, 5].

The idea is that C_{ob} is the object of E consisting of the objects of the internal category, C_{ar} represents the arrows of the internal category, and dom and cod are the domain and codomain maps. One also has data for composition and for identities, subject to the inevitable axioms. The paradigmatic example of a possible category E is that of Set , and the data and axioms yield that an internal category in Set is precisely a small category.

For any category E with finite limits, one can similarly define internal functors and internal natural transformations, see [7], yielding a 2-category $\text{Cat}(E)$, the 2-category of internal categories in E . Since the definition of internal category depends only upon

pullbacks, and limits are given pointwise in functor categories for which the base category has limits, it follows that we have

Proposition 4.2. *The 2-category $Cat(V^{\text{op}}, Set)$ is $[V^{\text{op}}, Cat]$ for any V .*

We remarked upon this result at the end of the previous section. Together with our full embedding, it is what allows us to reduce the study of fixpoints in $V\text{-Cat}$ to that of fixpoints in $Cat(E)$.

For an idea of how the study of internal categories works, it is straightforward to verify that if E is cartesian closed, then so is $Cat(E)$, the construction and proof being just as for Cat . So one can speak of functor categories internally. In fact, a considerable amount of the structure of Cat generalises to $Cat(E)$ without difficulty: those structures involving limits, colimits, cartesian closure, and the construction $(—)^{\text{op}}$ extend with ease. So here, we continue in that spirit, seeking to generalise our analysis of Section 1.

For an ordinary category, it is clear what is meant by an object of the category. However, for a category in E , it is less clear. If one defines an object of an internal category C to be an arrow in E from 1 to C_{ob} , one obtains a limited analysis: one is effectively taking the functor from $Cat(E)$ to Cat given by the representable $E(1, —)$ and restricting attention to the ordinary category thus produced. That is typically not of great help. For instance, for us, it would amount to taking our V -category C and taking its underlying ordinary category C_0 , but we already know that giving an initial algebra for the ordinary endofunctor on the ordinary category C_0 is not sufficient for our purposes.

In fact, one knows that it is the family of all representable functors $E(X, —)$ which is collectively faithful. So a more fruitful definition is to say

Definition 4.3. For an arbitrary object X of E , an *object at X* of an internal category C is an arrow in E from X into C_{ob} . A *map at X* of C is an arrow in E from X into C_{ar} . By postcomposition with the structure maps of the internal category C , one gets a category C_X of the objects at X of C . For instance, given an arrow a of C at X , the domain of a is $\text{dom } a$.

Having made this definition, it follows that for each internal functor of the form $H : D \times C \rightarrow C$, and each object $d : X \rightarrow D_{\text{ob}}$ of D , we obtain an ordinary endofunctor on the ordinary category C_X . We denote this functor $H_X(d, —)$. So if we assume each endofunctor on each C_X has an initial algebra, we are back in the position of our original analysis, yielding, for each object X of E , a functor from D_X to C_X that assigns a fixpoint to each object of D_X . Finally, we want to take that family of functors and turn it into a single functor in $Cat(E)$ from D to C . To do that, we demand that this family be natural in X , so we can apply the Yoneda lemma to deduce the result.

So a sufficient condition for our result would be

Condition 4.4. Each endofunctor on C_X has an initial algebra, and for any family of such endofunctors natural in X , the family of initial algebras is natural in X .

This is a priori a strong condition: for instance, if $E = \mathbf{Set}$, it implies directly that every endofunctor on each power C^κ has an initial algebra for any cardinal κ . So we should like to weaken this condition while retaining the same conclusion.

We can weaken the condition in two ways. First, observe that we do not need every endofunctor on each C_X to have an initial object, but can restrict to enough X 's to determine a map in E . This means that we can restrict to a collection of X 's that form a full *dense* subcategory. We will not develop the concept of density here, but point out that 1 is dense in \mathbf{Set} , and the representables are dense in a presheaf category. So in our particular case of $\mathbf{Cat}(E)$, the representables form a dense subcategory of $E = [V^{\text{op}}, \mathbf{Cat}]$, so it suffices to consider those X that are representables. Second, for our particular D , i.e. one in the image of \mathbf{Para} , the maps from any representable to D_{ob} are the same as the maps from 1 to D_{ob} . So it is more than enough to ask that for each map from 1 to D_{ob} , and for every X , the corresponding natural family of endofunctors on the C_X 's has a family of initial objects, natural in X . But, by the Yoneda lemma, to give a natural family of endofunctors on C_X is equivalent to giving an internal endofunctor on C . Moreover, to give a natural family of objects of the C_X 's is equivalent to giving a single object of the single category C_1 . So, leading to Condition 4.6, we make the following definition.

Definition 4.5. Given an endofunctor H on an internal category C of E , an *initial object* of H is an initial algebra $H_1 : C_1 \rightarrow C_1$ which is preserved by each $H_1\text{-Alg} : H_1\text{-Alg} \rightarrow H_X\text{-Alg}$.

This definition is equivalent to the natural 2-categorical definition of initiality: $\mathbf{Cat}(E)$ is a 2-category with finite limits. So, for any 1-cell of the form $H : C \rightarrow C$, i.e. for any internal endofunctor, one has a definitive notion of the 0-cell $H\text{-Alg}$ of algebras for that 1-cell: it is the *inserter* from the 1-cell to the identity 1-cell, cf. [11]. One can then ask for the unique map from $H\text{-Alg}$ to 1 to have a left adjoint. In the case of the 2-category \mathbf{Cat} , that is equivalent to asking for an initial algebra. It agrees with the above definition.

Then we may weaken Condition 4.4 to the assertion

Condition 4.6. Each endofunctor on the internal category C has an initial object.

As implicit above, in our particular case of E being a presheaf category, we have

Proposition 4.7. *Given an endofunctor H on an internal category, if each H_U has an initial algebra for each representable U , naturally in U , then H has an initial algebra.*

It is routine to express this condition directly in terms of V -categories when C is of the form $\text{Para}(C)$, and it is precisely Condition 2.3 about parametrized initial algebras of the end of Section 2.

5. In the presence of cotensors

First, observe the following proposition about ordinary categories and ordinary endofunctors on them.

Proposition 5.1. *Let (C, H) and (D, K) each consist of a small category together with an endofunctor on it. Let $F : C \rightarrow D$ be a functor that commutes with the endofunctors, and suppose F has a right adjoint G . Then the induced functor $F\text{-Alg} : H\text{-Alg} \rightarrow K\text{-Alg}$ has a right adjoint.*

Proof. This result holds not only in Cat , but in an arbitrary 2-category with some finite limits, specifically inserters. The equality $FH = KF$ induces a natural transformation $\sigma : HG \rightarrow GK$. This may be used to define the right adjoint to $F\text{-Alg}$: the right adjoint takes a K -algebra (A, a) to $(GA, KGA \rightarrow GKA \rightarrow GA)$, where the composite is defined by σ and $K(a)$. \square

When V is cartesian closed, the category V may itself be seen as a V -category: the V -category associated to V has the same set of objects as V , and for homobject, one has the exponential $[X, Y]$ of V . Composition and identities are given by the cartesian closedness of V , using the evaluation map. That allows us to make the following definition, see [6].

Definition 5.2. A V -category C has *cotensors* if, for every object X of V and every object A of C , the V -functor $[X, C(-, A)] : C^{\text{op}} \rightarrow V$ is representable.

For example, regarding V as a V -category, V always has cotensors, and they are given by the exponentials of V . If $V = \text{Set}$, then a V -category has cotensors if it has products, because a cotensor of a set X with an object A of a category C is given by the product of X copies of A . Cotensors are a fundamental feature of the study of enriched categories, and appear in such a role in Fiore's book [1]. It is a mild completeness condition on a V -category to demand that it have cotensors: for instance, in the case $V = \text{Set}$ it follows from the existence of products.

Now, suppose we have a V -category C with cotensors. Then the $[V^{\text{op}}, \text{Cat}]$ -category $\text{Para}(C)$ has a special property: recall that the value of the functor $\text{Para}(C)$ at an object X of V is given by the category with the same objects as C and with an arrow from A to B being a map in V from X to $C(A, B)$. The functor $\text{Para}(C)(!) : \text{Para}(C)(1) \rightarrow \text{Para}(C)(X)$ determined by functoriality of $\text{Para}(C)$ applied to the unique map from X to 1 , has a right adjoint given by the cotensor of X with any object of C : that follows immediately from the definition of cotensor.

Thus, the condition of Proposition 5.1 is satisfied, so since left adjoints preserve colimits, any initial object of H is sent to an initial algebra of H_X , so together with Proposition 4.7 and the analysis of the last section, we may conclude

Theorem 5.3. *Let V be a cartesian closed category, let C be a V -category with cotensors, and let H be any V -enriched endofunctor on C . Then any initial algebra for H is a parametrized initial algebra.*

We can give an explicit description of the map we require for the parametrization without explicit reference to the embedding of $V\text{-Cat}$ into $[V^{\text{op}}, \text{Cat}]$. Assume the conditions of the theorem. Then the construction is as follows:

Construction 5.4. We need a map $C(HA, A) \rightarrow C(I, A)$ in V . This is the same as a map $I \rightarrow A^{C(HA, A)}$ in C . So finding an H -action on the object $A^{C(HA, A)}$ would suffice.

The structure map must be of the form $H(A^{C(HA, A)}) \rightarrow A^{C(HA, A)}$ or equivalently, $C(HA, A) \rightarrow C(H(A^{C(HA, A)}), A)$, and here it is

$$\begin{array}{ccc} C(HA, A) & & C(H(A^{C(HA, A)}), A) \\ \downarrow & & \uparrow \\ C(A^{C(HA, A)}, A) \times C(HA, A) & \rightarrow & C(H(A^{C(HA, A)}), HA) \times C(HA, A). \end{array}$$

The first component of the first arrow is the unit of the cotensor representation. The others are evident.

There is a dual result.

Corollary 5.5. *Suppose $H : C \rightarrow C$ is V -enriched where V is cartesian closed, and C has tensors. Suppose, moreover, that there is a final H -coalgebra $f : F \rightarrow HF$. Then one has a parametrized final coalgebra dually to the above.*

There is something very close to this in Alex Simpson's notes [9] (Lemma 5.4), and in Plotkin's Darmstadt talk.

6. Extensions

In the bulk of this paper, we have restricted attention to positively recursively defined domain equations. The analysis of domain equations of mixed variance is not complete yet, but one approach, pioneered by Freyd, is to consider functors on categories of the form $C^{\text{op}} \times C$. So, in connection with that, we remark that the 2-functor from $V\text{-Cat}$ to $[V^{\text{op}}, \text{Cat}]$ preserves $(-)^{\text{op}}$, where $(-)^{\text{op}}$ is defined pointwise in the latter 2-category (which is the usual way there). So the Freydian approach may well extend when fully developed, thus reducing the enriched version to the usual version plus naturality.

Second, note that we have consistently restricted attention to enrichment over a cartesian closed category V . That includes our leading examples, and implicit in the conditions we have considered, we have used diagonal maps $\Delta : X \rightarrow X \times X$ freely, and the fact that the unit is terminal. One does not need to be quite so rigid. One certainly seems to need that data, but one might find value in restricting naturality a little: so for instance, one might have a subcategory of a symmetric monoidal closed category, in which the subcategory has the same objects and has finite products. That situation, and analysis along the lines of this paper, is currently appearing in the work of Adam Eppendahl and Edmund Robinson; and seems likely to be of continued interest.

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