A Lyapunov Approach to Incremental Stability Properties

David Angeli

***Abstract*—This paper deals with several notions of incremental stability. In other words, the focus is on stability of trajectories with respect to one another, rather than with respect to some attractor. The aim is to present a framework for understanding such questions fully compatible with the well-known input-to-state stability (ISS) approach. Applications of the newly introduced stability notions are also discussed.**

***Index Terms*—Lyapunov methods, observers, stability, synchronization.**

I. Introduction

INPUT-TO-STATE stability (ISS) has proven a valid instrument in order to study questions of robust stability for finite-dimensional nonlinear systems. One reason for that is the possibility of dealing, at the same time, with a body of theory which nicely extends to nonautonomous systems the classic Lyapunov while still allowing for input–output descriptions of the system behavior, [11], [21], [27], [29]. In this way tools such as small-gain theorems [12] and Lyapunov dissipation inequalities [26] have come together in a unified framework which bridges the gap between the state-space and input–output approaches.

Stability properties are described through the use of comparison functions, the so called class $\mathcal{K}$ and $\mathcal{KL}$ functions, which can be thought of as nonlinear versions of linear gains and exponentially fading transients. This approach naturally leads to stability notions which are invariant with respect to nonlinear changes of coordinates [9] and at the same time avoids the use of the $\varepsilon - \delta$ formalism which is usually less intuitive. A similar way of thinking was exploited in order to study detectability questions, [15], [28].

The quest for nonlinear analogs of the separation principle already involved input-to-state stability as one of the ingredients, [20], [32], [30]. As a matter of fact, it is especially looking at the issue of state-detection and observer synthesis that it becomes relevant to understand which systems may enjoy incremental stability properties. In other words our focus is on systems whose trajectories converge to one another, besides being attracted toward some equilibrium position. Works along these lines have recently appeared in the literature, [5], [23], together with some examples of applications, [8], [18]. As a matter of fact, the notion of incremental input-to-state stability that will be introduced, can be thought of also as an "open-loop observability" property, that is as the possibility of designing an observer for the system which only processes past input data. It is well-known that for linear systems such a property is equivalent to asymptotic stability. It is indeed a much stronger property when dealing with nonlinear ones.

As already pointed out, our aim is to present such notions in a framework compatible with the ISS approach. The paper is organized as follows: in Sections II and III we study robust incremental global asymptotic stability and prove a Lyapunov characterization of such a property; in Section IV we introduce incremental ISS and give some results of general interest, whereas a Lyapunov characterization of the property is provided in Section V; Section VI shows a couple of examples involving incremental ISS and conclusions are given in Section VII.

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The author is with the Dipartimento di Sistemi e Informatica, Università di Firenze, 50139 Firenze, Italy (e-mail: angeli@dsi.unifi.it).

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II. Lyapunov Characterizations of Incremental Stability

Let us consider dynamical systems of the following form:

$$\dot{x} = f(x, d) \tag{1}$$

with state $x \in \mathbb{R}^n$ and input d, here seen as a disturbance rather than a control, taking values on a closed set $\mathcal{D} \subset \mathbb{R}^m$. By input signal we mean any measurable, locally essentially bounded function of time and we denote the set of such functions by $\mathcal{M}_{\mathcal{D}}$.

We are interested in characterizing in terms of Lyapunov dissipation inequalities the following property of solutions of (1).

Definition 2.1: We say that (1) is incremental globally asymptotically stable (δGAS) if there exists a function β of class $\mathcal{KL}$ so that for all $d \in \mathcal{M}_{\mathcal{D}}$, all $\xi, \eta \in \mathbb{R}^n$ and all $t \geq 0$ the following holds

$$|x(t, \xi, d) - x(t, \eta, d)| \leq \beta(|\xi - \eta|, t). \tag{2}$$

□

It is convenient to recast the notion of incremental stability as a standard problem of uniform global asymptotic stability with respect to sets. For the sake of completeness we recall the definition of uniform GAS.

Definition 2.2: We say that (1) is globally asymptotically stable (GAS) with respect to a closed set $\mathcal{A}$ if there exists a function β of class $\mathcal{KL}$ so that for all $d \in \mathcal{M}_{\mathcal{D}}$, all $\xi \in \mathbb{R}^n$ and all $t \geq 0$ the following holds:

$$|x(t, \xi, d)|_{\mathcal{A}} \leq \beta(|\xi|_{\mathcal{A}}, t). \tag{3}$$

□

In (3) the symbol $|\cdot|_{\mathcal{A}}$ denotes the Euclidean point-to-set distance, *viz.* $|x|_{\mathcal{A}} \doteq \inf_{z\in\mathcal{A}} |x-z|$. With this notation in mind the following lemma holds.

Lemma 2.3: System (1) is δGAS if and only if the auxiliary system

$$\begin{cases} \dot{x}_1 = f(x_1, d) \\ \dot{x}_2 = f(x_2, d) \end{cases} \tag{4}$$

with state $\chi \doteq [x_1', x_2']'$ is uniformly GAS with respect to the diagonal Δ

$$\Delta = \{\chi \in \mathbb{R}^{2n}: \exists\, x \in \mathbb{R}^n: \chi = [x', x']'\}. \tag{5}$$

□

Proof: What we need to show is that the point-to-set distance between an arbitrary vector $\chi = [\xi', \eta']' \in \mathbb{R}^{2n}$ and the diagonal Δ is proportional to the Euclidean norm of $\xi - \eta$. Notice that the diagonal is spanned by the following unitary vectors $v_i = [e_i', e_{i+n}']'/\sqrt{2}$ for $i = 1\cdots n$. Hence

$$\begin{aligned} |\chi|_\Delta \doteq \inf_{v\in\Delta} |\chi - v| &= \left|\chi - \sum_{i=1}^n (\chi' v_i) v_i\right| \\ &= |\chi - [\xi_1+\eta_1, \xi_2+\eta_2, \ldots, \xi_n+\eta_n \\ &\qquad \xi_1+\eta_1, \ldots, \xi_n+\eta_n]'/2| \\ &= |[\xi_1-\eta_1, \xi_2-\eta_2, \ldots, \xi_n-\eta_n \\ &\qquad \eta_1-\xi_1, \ldots, \eta_n-\xi_n]'/2| \\ &= \frac{1}{2}\sqrt{2\sum_{i=1}^n (\xi_i-\eta_i)^2} = \frac{\sqrt{2}}{2}|\xi-\eta|. \end{aligned} \tag{6}$$

Substituting the above equality in definitions (3) and (2) yields the desired equivalence. ■

Remark 2.4: Notice that, for $\mathcal{D}$ compact, by virtue of Lemma 2.3 and by applying the converse Lyapunov theorem in [31] we obtain a Lyapunov characterization of δGAS. Thanks to the special form of Δ, we have that δGAS is equivalent to the existence of a smooth function $V(x_1, x_2)$ and $\mathcal{K}_\infty$ functions α_1, α_2 such that

$$\alpha_1(|x_1 - x_2|) \le V(x_1, x_2) \le \alpha_2(|x_1 - x_2|)$$

and along trajectories of (4)

$$\frac{\partial V(x_1, x_2)}{\partial x_1} f(x_1, d) + \frac{\partial V(x_1, x_2)}{\partial x_2} f(x_2, d) \le -\alpha(|x_1 - x_2|) \quad \forall\, x_1, x_2 \in \mathbb{R}^n, \qquad \forall\, d \in \mathcal{D}. \tag{7}$$

It is known that the converse Lyapunov theorems in [17] and [31] do not extend to $\mathcal{D}$ being a generic closed set even for the apparently simpler case of equilibrium points as it is pointed out in [17, Sec. 8]. Nevertheless, for the special case of incremental stability, the theorem holds provided that we allow for *continuous* Lyapunov functions V and rewrite (7) in integral form. □

Theorem 1: System (1) is δGAS as in Definition 2.1 if and only if there exist a continuous function $U(x_1, x_2)$ and $\mathcal{K}_\infty$ functions α_1, α_2 such that

$$\alpha_1(|x_1 - x_2|) \le U(x_1, x_2) \le \alpha_2(|x_1 - x_2|) \tag{8}$$

and along trajectories of (4) satisfies for any ξ_1, ξ_2 in $\mathbb{R}^n$, any $t \ge 0$ and any $d \in \mathcal{M}_{\mathcal{D}}$

$$U(x(t, \xi_1, d), x(t, \xi_2, d)) - U(\xi_1, \xi_2) \le -\int_0^t \alpha(|x(\tau, \xi_1, d) - x(\tau, \xi_2, d)|)\, d\tau \tag{9}$$

with α positive definite. ■

Proof: The sufficiency part is not surprising and follows by a standard comparison principle, see for instance [14, Lemma 3.1]. The converse implication is more interesting. We prove existence of a continuous Lyapunov function in two steps.

Fact 2.5: Consider the function $g(\cdot): \mathbb{R}^{2n} \to \mathbb{R}_{\ge 0}$ defined as

$$g(\chi_0) = \sup_{t\ge 0,\, d} |\chi(t, \chi_0, d)|_\Delta. \tag{10}$$

Then there exist functions α_1, α_2 of class $\mathcal{K}_\infty$ such that $\alpha_1(|\chi|_\Delta) \le g(\chi) \le \alpha_2(|\chi|_\Delta)$. Moreover, g satisfies the following continuity condition for some $\gamma \in \mathcal{K}_\infty$:

$$|g(\chi_1) - g(\chi_2)| \le \gamma(|\chi_1 - \chi_2|). \tag{11}$$

□

Proof: By definition of g and recalling (2) we have

$$|\chi|_\Delta \le g(\chi) \le \beta(|\chi|_\Delta, 0) \tag{12}$$

which proves the first part of the claim. Therefore, we only need to show that (11) holds. Pick $\chi_1 = [\xi_1', \eta_1']'$ arbitrary in $\mathbb{R}^{2n}$. Then, for any $\varepsilon > 0$, there exist t_{ε,χ_1}, d_{ε,χ_1} such that

$$g(\chi_1) \le \varepsilon + |\chi(t_{\varepsilon,\chi_1}, \chi_1, d_{\varepsilon,\chi_1})|_\Delta. \tag{13}$$

As a consequence, letting $\chi_2 = [\xi_2', \eta_2']'$ and taking increments of g yields

$$\begin{aligned} &g(\chi_1) - g(\chi_2) \\ &\quad \le \varepsilon + |\chi(t_{\varepsilon,\chi_1}, \chi_1, d_{\varepsilon,\chi_1})|_\Delta - |\chi(t_{\varepsilon,\chi_1}, \chi_2, d_{\varepsilon,\chi_1})|_\Delta \\ &\quad = \varepsilon + \frac{\sqrt{2}}{2}\big(|x(t_{\varepsilon,\chi_1}, \xi_1, d_{\varepsilon,\chi_1}) - x(t_{\varepsilon,\chi_1}, \eta_1, d_{\varepsilon,\chi_1})| \\ &\qquad\qquad - |x(t_{\varepsilon,\chi_1}, \xi_2, d_{\varepsilon,\chi_1}) - x(t_{\varepsilon,\chi_1}, \eta_2, d_{\varepsilon,\chi_1})|\big) \\ &\quad \le \varepsilon + \frac{\sqrt{2}}{2}\big(|x(t_{\varepsilon,\chi_1}, \xi_1, d_{\varepsilon,\chi_1}) - x(t_{\varepsilon,\chi_1}, \xi_2, d_{\varepsilon,\chi_1})| \\ &\qquad\qquad + |x(t_{\varepsilon,\chi_1}, \eta_1, d_{\varepsilon,\chi_1}) - x(t_{\varepsilon,\chi_1}, \eta_2, d_{\varepsilon,\chi_1})|\big) \\ &\quad \le \varepsilon + \frac{\sqrt{2}}{2}(\beta(|\xi_1 - \xi_2|, 0) + \beta(|\eta_1 - \eta_2|, 0)) \\ &\quad \le \varepsilon + \sqrt{2}\beta(|\xi_1 - \xi_2| + |\eta_1 - \eta_2|, 0) \\ &\quad \le \varepsilon + \sqrt{2}\beta(2|\chi_1 - \chi_2|, 0). \end{aligned} \tag{14}$$

Since ε is arbitrary, $g(\chi_1) - g(\chi_2) \le \sqrt{2}\beta(2|\chi_1 - \chi_2|, 0)$, for all $\chi_1, \chi_2 \in \mathbb{R}^{2n}$. By a symmetric argument it is easy to see that g satisfies (11) with $\gamma(r) \doteq \sqrt{2}\beta(2r, 0)$. This concludes the proof of the claim. ■

It is straightforward, from definition (10), to see that $g(\chi)$ be a nonincreasing function along trajectories of (4). We show next how to get a strictly decreasing Lyapunov function by suitably modifying $g(\chi)$.

Fact 2.6: Let $U(\xi)$ be defined as

$$U(\xi) = \sup_{t\geq 0, d} g(\chi(t, \xi, d))k(t) \tag{15}$$

where $k(t)$ is any positive, increasing function satisfying the following properties:

- $\exists\, c_1$ and c_2 such that $0 < c_1 < c_2$ and $k(t) \in [c_1, c_2] \qquad \forall\, t \geq 0,$ (16)
- There is a bounded, positive and decreasing function $\tau(\cdot)$, such that: $k'(t) \geq \tau(t) \qquad$ for all $t \geq 0.$ (17)

Then U is continuous according to same Lipschitz-like condition (11) valid for g and satisfies (8), (9). □

Proof: Let $\chi_1 \in \mathbb{R}^{2n}$ be arbitrary. Then, for all $\varepsilon > 0$, there exists t_{ε,χ_1}, d_{ε,χ_1} such that

$$U(\chi_1) \leq \varepsilon + g(\chi(t_{\varepsilon,\chi_1}, \chi_1, d_{\varepsilon,\chi_1}))k(t_{\varepsilon,\chi_1}). \tag{18}$$

Thus, taking increments of U, with respect to an arbitrary $\chi_2 \in \mathbb{R}^{2n}$ yields

$$\begin{aligned} &U(\chi_1) - U(\chi_2) \\ &\quad \leq \varepsilon + [g(\chi(t_{\varepsilon,\chi_1}, \chi_1, d_{\varepsilon,\chi_1})) \\ &\qquad\quad - g(\chi(t_{\varepsilon,\chi_1}, \chi_2, d_{\varepsilon,\chi_1}))]k(t_{\varepsilon,\chi_1}) \\ &\quad \leq \varepsilon + \gamma(|\chi(t_{\varepsilon,\chi_1}, \chi_1, d_{\varepsilon,\chi_1}) - \chi(t_{\varepsilon,\chi_1}, \chi_2, d_{\varepsilon,\chi_1})|)c_2 \\ &\quad \leq \varepsilon + \gamma(\beta(|\chi_1 - \chi_2|, 0))c_2 \end{aligned} \tag{19}$$

and, hence, being ε arbitrary, we have $|U(\chi_1) - U(\chi_2)| \leq \tilde{\gamma}(|\chi_1 - \chi_2|)$, where $\tilde{\gamma}(r) = \gamma(\beta(r, 0))c_2$ is still a $\mathcal{K}_\infty$ function. We show next that U decreases along trajectories. Pick any $M > 0$ and $\chi_0 \in \mathbb{R}^{2n}$ such that $M/2 \leq |\chi_0|_\Delta \leq M$. For each $h > 0$ and each bounded and measurable $\overline{d}(\cdot)$: $[0, h] \to \mathcal{D}$, we let $\hat{\chi}_{h,\overline{d}} = \chi(h, \chi_0, \overline{d})$. By definition (15), for any $\varepsilon > 0$, there exists some $t_{\varepsilon,h,\overline{d}}$, $d_{\varepsilon,h,\overline{d}}$ such that

$$\begin{aligned} U(\hat{\chi}_{h,\overline{d}}) &\leq g(\chi(t_{\varepsilon,h,\overline{d}}, \hat{\chi}_{h,\overline{d}}, d_{\varepsilon,h,\overline{d}}))k(t_{\varepsilon,h,\overline{d}}) + \varepsilon \\ &\leq g(\chi(t_{\varepsilon,h,\overline{d}} + h, \chi_0, \overline{d}\# d_{\varepsilon,h,\overline{d}}))k(t_{\varepsilon,h,\overline{d}} + h) \\ &\quad \cdot \left(1 - \frac{k(t_{\varepsilon,h,\overline{d}} + h) - k(t_{\varepsilon,h,\overline{d}})}{k(t_{\varepsilon,h,\overline{d}} + h)}\right) + \varepsilon \\ &\leq U(\chi_0)\left(1 - \frac{k(t_{\varepsilon,h,\overline{d}} + h) - k(t_{\varepsilon,h,\overline{d}})}{c_2}\right) + \varepsilon \end{aligned} \tag{20}$$

where $\#$ denotes the concatenation of functions. It appears from (20) that, in order to estimate the derivative of U along trajectories we need to find an upper bound for $t_{\varepsilon,h,\overline{d}}$. We claim that, there exists T_M such that $t_{\varepsilon,h,\overline{d}} < T_M$ for all χ_0 and all $\varepsilon \leq c_1 M/8$ (notice that, though not explicitly pointed out by the notation, $t_{\varepsilon,h,\overline{d}}$ is also dependent of χ_0). In fact

$$\begin{aligned} U(\chi_0) &\leq g(\chi(t_{\varepsilon,h,\overline{d}} + h, \chi_0, \overline{d}\# d_{\varepsilon,h,\overline{d}}))k(t_{\varepsilon,h,\overline{d}}) + \varepsilon \\ &\leq c_2\beta(|\chi(t_{\varepsilon,h,\overline{d}} + h, \chi_0, \overline{d}\# d_{\varepsilon,h,\overline{d}})|_\Delta, 0) + \varepsilon \\ &\leq c_2\beta(\beta(|\chi_0|_\Delta, t_{\varepsilon,h,\overline{d}}), 0) + c_1 M/8 \\ &\leq c_2\beta(\beta(M, t_{\varepsilon,h,\overline{d}}), 0) + c_1 M/8. \end{aligned} \tag{21}$$

Then, since $\beta(\cdot, t) \to 0$ as $t \to +\infty$, it follows that for all $M > 0$, there exists a positive constant T_M such that $\beta(\beta(M, T_M), 0) \leq c_1 M/(8c_2)$. Thus, assuming by contradiction $t_{\varepsilon,h,\overline{d}} \geq T_M$ we obtain

$$\begin{aligned} U(\chi_0) &\leq c_2\beta(\beta(M, t_{\varepsilon,h,\overline{d}}), 0) + c_1 M/8 \\ &\leq c_1(M/8 + M/8) < c_1 M/2 \leq c_1|\chi_0|_\Delta \leq U(\chi_0) \end{aligned} \tag{22}$$

which is a contradiction.

Now that we have an upper bound for $t_{\varepsilon,h,\overline{d}}$, the derivative of U along trajectories can be estimated. By virtue of (20) and recalling property (17) we obtain

$$\begin{aligned} &U(\chi(h, \chi_0, d)) - U(\chi_0) \\ &\quad \leq -U(\chi_0)\frac{k(t_{\varepsilon,h,\overline{d}} + h) - k(t_{\varepsilon,h,\overline{d}})}{c_2} + \varepsilon \\ &\quad = -\frac{U(\chi_0)}{c_2}\, k'(t_{\varepsilon,h,\overline{d}} + \theta h)h + \varepsilon \end{aligned} \tag{23}$$

for certain $0 < \theta < 1$. Hence, by the assumptions made for the function k we have

$$\begin{aligned} U(\chi(h, \chi_0, d)) - U(\chi_0) &\leq -\frac{U(\chi_0)}{c_2}\,\tau(t_{\varepsilon,h,\overline{d}} + \theta h)h + \varepsilon \\ &\leq -\frac{U(\chi_0)}{c_2}\,\tau(T_M + h)h + \varepsilon. \end{aligned} \tag{24}$$

Since ε can be chosen arbitrarily small (in (24), we have only assumed $\varepsilon \leq c_1 M/8$), we obtain

$$U(\chi(h, \chi_0, d)) - U(\chi_0) \leq -\frac{U(\chi_0)}{c_2}\tau(T_M + h)h. \tag{25}$$

We define $\dot{U}$ as

$$\dot{U}(\chi_0, d) = \limsup_{h\to 0} \frac{U(\chi(h, \chi_0, d)) - U(\chi_0)}{h}. \tag{26}$$

Obviously, from (25) and (26), it follows:

$$\dot{U}(\chi_0, d) \leq -\frac{U(\chi_0)}{c_2}\,\tau(T_M). \tag{27}$$

Using (27) and a standard partition of unity argument we can establish the existence of a positive–definite function α such that

$$\dot{U}(\chi_0, d) \leq -\alpha(|\chi_0|_\Delta), \qquad \forall\, \chi_0 \in \mathbb{R}^{2n}. \tag{28}$$

Since $U(\chi(t))$ need not be absolutely continuous, (28) is not enough to establish (9). Nevertheless, $U(\chi(t))$ is by definition nonincreasing, thus we can decompose $U(\chi(t))$ into the sum of two nonincreasing functions, $V(t)$ and $H(t)$, of which V is absolutely continuous and $\dot{H} = 0$ almost everywhere, see [22]. Then U is differentiable almost everywhere (though it need not be the integral of its derivative), and

$$\dot{V} = \dot{U}(\chi(t)) \leq -\alpha(|\chi(t)|_\Delta) \qquad \text{a.e.} \tag{29}$$

which, by absolute continuity of V, implies

$$V(t) - V(0) \leq -\int_0^t \alpha(|\chi(s, \chi(0), d)|_\Delta)\, ds. \tag{30}$$

Finally, since $H(t)$ is nonincreasing, (30) yields

$$\begin{aligned} U(\chi(t, \chi_0, d)) - U(\chi_0) &= V(t) + H(t) - V(0) - H(0) \\ &\le V(t) - V(0) \\ &\le -\int_0^t \alpha(|\chi(s, \chi_0, d)|_\Delta)\, ds. \end{aligned} \tag{31}$$

This also concludes the proof of the Proposition. ∎

III. On Local, Semiglobal, and Global Aspects of Incremental Stability

In this section, we derive some results concerning the relationships between local, semiglobal and global versions of the newly introduced stability notions.

Definition 3.1: We say that system (1) is *semiglobally asymptotically stable* with respect to a closed set $\mathcal{A}$, if for any $M > 0$ there exists β_M of class $\mathcal{KL}$ such that

$$\begin{aligned} &|x(t, \xi, d)|_{\mathcal{A}} \le \beta_M(|\xi|_{\mathcal{A}}, t) \\ &\qquad \forall t \ge 0, \quad \forall d \in \mathcal{M}_{\mathcal{D}}, \quad \forall |\xi|_{\mathcal{A}} \le M. \end{aligned} \tag{32}$$

□

Definition 3.2: We say that system (1) is *incremental semiglobally asymptotically stable* if for any $M > 0$ there exists $\beta_M \in \mathcal{KL}$ such that

$$\begin{aligned} &|x(t, \xi_1, d) - x(t, \xi_2, d)| \le \beta_M(|\xi_1 - \xi_2|, t) \\ &\quad \forall t \ge 0, \quad \forall d \in \mathcal{M}_{\mathcal{D}} \\ &\quad \forall \xi_1, \xi_2 \colon |\xi_1| \le M, \quad |\xi_2| \le M. \end{aligned} \tag{33}$$

□

Notice that incremental semiglobal asymptotic stability is not the same as semiglobal asymptotic stability of (4) with respect to the diagonal Δ.

Definition 3.3: A system (1) is *incremental locally asymptotically stable* if there exist $\varepsilon > 0$ and $\beta \in \mathcal{KL}$ such that

$$\begin{aligned} &|x(t, \xi_1, d) - x(t, \xi_2, d)| \le \beta(|\xi_1 - \xi_2|, t), \\ &\qquad \forall t \ge 0, \quad \forall d \in \mathcal{M}_{\mathcal{D}}, \quad \forall \xi_1, \xi_2 \colon |\xi_1 - \xi_2| < \varepsilon. \end{aligned} \tag{34}$$

□

The next Proposition summarizes the main results of this section.

Proposition 3.4: The following equivalences hold.

- A system is semiglobally asymptotically stable (with respect to a set $\mathcal{A}$) if and only if it is GAS (with respect to $\mathcal{A}$).
- For a system $\dot{x} = f(x, d)$, with $f(0, d) = 0$ for all $d \in \mathcal{D}$ the following notions of stability are equivalent:
  1) semiglobal asymptotic stability with respect to the origin;
  2) global asymptotic stability with respect to the origin;
  3) semiglobal incremental asymptotic stability.
- A system is incremental locally asymptotically stable if and only if it is incremental globally asymptotically stable.

□

Proof: As far as the first item of Proposition 3.4, one direction of the proof is obvious, namely the "global implies semiglobal" implication; the other is a corollary of [17, Lemma 3.1]. Consider now the second item of Proposition 3.4. Equivalence of 1 and 2 follows as a particular case the previous point, (when the set $\mathcal{A}$ is the origin). Also, since $x(t) \equiv 0$ is a solution of (1), regardless of the input disturbance, it follows that 3 implies 1 just by definition of incremental semiglobal asymptotic stability for a generic trajectory with respect to the $x(t) \equiv 0$ solution. Hence, we are only left to show that 2 implies 3. Let $\beta_M \colon \mathbb{R}^2_{\ge 0} \to \mathbb{R}_{\ge 0}$ be the following function:

$$\beta_M(r, t) = \sup_{\substack{d \in \mathcal{M}_{\mathcal{D}}, \tau \ge t \\ |\xi_1 - \xi_2| \le r \\ |\xi_1|, |\xi_2| \le M}} |x(\tau, \xi_1, d) - x(\tau, \xi_2, d)|. \tag{35}$$

Notice, because of GAS and by the triangular inequality, that the function is well defined. In fact $\beta_M(r, t) \le 2\beta(M, t)$. Besides, the nesting of the domains over which the supremum is taken implies that $\beta_M(r, t)$ is, for each fixed value of M, nonincreasing with respect to t and nondecreasing with respect to r. Furthermore, $\beta_M(r, t) \to 0$ as $t \to +\infty$ (for each fixed value of r). In order to show that β_M is the sought family of $\mathcal{KL}$ functions and modulo some technicalities to make it strictly monotone with respect to its arguments, we are only left to show that for each fixed value of t the following holds:

$$\lim_{r \to 0^+} \beta_M(r, t) = 0. \tag{36}$$

Monotonicity implies that the limit exists; assume by contradiction that for any $r > 0$, $\beta_M(r, t) > \bar{\varepsilon}$ for some strictly positive $\bar{\varepsilon}$. By definition of β_M we also have

$$\begin{aligned} &\forall \varepsilon > 0\, \forall r > 0, \quad \exists \xi^1_{r,\varepsilon}, \xi^2_{r,\varepsilon} \in \mathcal{B}_M, \quad d_{\varepsilon,r} \in \mathcal{M}_{\mathcal{D}} \\ &\tau_{\varepsilon,r} \ge t \colon |\xi^1_{r,\varepsilon} - \xi^2_{r,\varepsilon}| \le r \\ &|x(\tau_{\varepsilon,r}, \xi^1_{r,\varepsilon}, d_{\varepsilon,r}) - x(\tau_{\varepsilon,r}, \xi^2_{r,\varepsilon}, d_{\varepsilon,r})| \ge \beta_M(r, t) - \varepsilon. \end{aligned} \tag{37}$$

Letting $\varepsilon = \bar{\varepsilon}/2$ and $r = 1/n$ for all $n \in \mathbb{N}$ yields

$$\begin{aligned} &\forall n \in \mathbb{N}, \quad \exists \xi^1_n, \quad \xi^2_n \in \mathcal{B}_M, \quad d_n \in \mathcal{M}_{\mathcal{D}} \\ &\tau_n \ge t \colon |\xi^1_n - \xi^2_n| \le 1/n \\ &|x(\tau_n, \xi^1_n, d_n) - x(\tau_n, \xi^2_n, d_n)| \ge \beta_M(r, t) - \bar{\varepsilon}/2 \ge \bar{\varepsilon}/2. \end{aligned} \tag{38}$$

By compactness we can assume without loss of generality $\xi^1_n, \xi^2_n \to \bar{\xi}$ as $n \to +\infty$. Hence we are left to consider two cases; τ_n bounded or τ_n unbounded; in particular we can without loss of generality restrict to the case $\tau_n \to \bar{t} < +\infty$ or $\tau_n \to +\infty$. In the first case, by continuous dependence from initial conditions, the inequality in (38) gives a contradiction. If instead $\tau_n \to +\infty$ we still get a contradiction by virtue of global asymptotic stability.

Hence, we conclude that the function β_M can be majorized by a $\mathcal{KL}$ function of r and t for each fixed value of M, thus proving semiglobal incremental asymptotic stability, since

$$\begin{aligned} &|x(t, \xi_1, d) - x(t, \xi_2, d)| \le \beta_M(|\xi_1 - \xi_2|, t) \\ &\qquad \forall |\xi_1|, |\xi_2| \le M. \end{aligned} \tag{39}$$

This completes the proof of the second item.

Consider now the last item of Proposition 3.4. Let system (1) be locally incrementally asymptotically stable. Then, there exists $\varepsilon > 0$ in such a way that (34) holds. Let $\xi_1, \xi_2 \in \mathbb{R}^n$ be such that $|\xi_1 - \xi_2| \le 2\varepsilon$. Then we can let $\bar{\xi} = (\xi_1 + \xi_2)/2$. Clearly $|\xi_{i=1,2} - \bar{\xi}| \le \varepsilon$. Hence, we exploit incremental local asymptotic stability to get the following estimate:

$$\begin{aligned}
&|x(t, \xi_1, d) - x(t, \xi_2, d)| \\
&\quad \le |x(t, \xi_1, d) - x(t, \bar{\xi}, d)| + |x(t, \bar{\xi}, d) - x(t, \xi_2, d)| \\
&\quad \le \beta(|\xi_1 - \bar{\xi}|, t) + \beta(|\xi_2 - \bar{\xi}|, t) \\
&\quad = 2\beta(|\xi_1 - \xi_2|/2, t) \\
&\quad \doteq \tilde{\beta}(|\xi_1 - \xi_2|, t).
\end{aligned}$$

Clearly $\tilde{\beta}$ is again a class $\mathcal{KL}$ function; moreover we have been able to make the domain of local incremental asymptotic stability twice as big. Then, by repeating this argument for each given M sufficiently many times, and using Lemma 2.3 to express incremental stability notions in terms of set-point distances we proved

$$\begin{aligned}
\forall M > 0 \quad \exists \beta_M\colon |\chi(t, [\xi_1', \xi_2']', d)|_\Delta \le \beta_M(|[\xi_1', \xi_2']'|_\Delta, t) \\
\forall \xi_1, \xi_2\colon |[\xi_1', \xi_2']'|_\Delta \le M. \quad (40)
\end{aligned}$$

Now, equation (40) is nothing more than semiglobal asymptotic stability of (4) with respect to the set Δ. Hence, by item 1, we can conclude that (4) is GAS with respect to Δ and by virtue of Lemma 2.3, δGAS of (1) is implied. ∎

Remark 3.5: Semi-global stability notions are somewhat new in the literature. Such a terminology was originally introduced for stabilizability properties instead. Because of the results in Proposition 3.4 it is evident that incremental asymptotic stability becomes of interest only when considered from a global point of view (otherwise is the same as asymptotic stability). □

Corollary 3.6: Let system (1) be globally asymptotically stable; then a sufficient condition for incremental GAS is the existence of two strictly positive real M, ε and of a differentiable function $V(x_1, x_2)$, with $\alpha_1(|x_1 - x_2|) \le V(x_1, x_2) \le \alpha_2(|x_1 - x_2|)$ for some α_1, α_2 of class $\mathcal{K}_\infty$, such that the following dissipation inequality holds:

$$\begin{aligned}
&\frac{\partial V}{\partial x_1}(x_1, x_2)f(x_1, d) + \frac{\partial V}{\partial x_2}(x_1, x_2)f(x_2, d) \\
&\quad \le -\rho(|x_1 - x_2|) \\
&\qquad \forall d \in \mathcal{D}, \quad \forall x_1, x_2 \in \mathbb{R}^n\colon |x_1 - x_2| \le \varepsilon \\
&\qquad |x_1| \ge M, \quad |x_2| \ge M. \qquad (41)
\end{aligned}$$

□

Proof: The result follows by Proposition 3.4 once the following general fact about $\mathcal{KL}$ functions is understood.

Fact: Let β_1, β_2 be $\mathcal{KL}$ functions; then $\beta\colon \mathbb{R}^2_{\ge 0} \to \mathbb{R}_{\ge 0}$, defined as

$$\beta(s, t) = \max_{\tau \in [0, t]} \beta_1(\beta_2(s, \tau), t - \tau) \qquad (42)$$

can be bounded from above by a class $\mathcal{KL}$ function. Let $\tau_{s,t} = \arg\max_{\tau \in [0,t]} \beta_1(\beta_2(s, \tau), t - \tau)$. It is clear that $\max\{t - \tau_{s,t}, \tau_{s,t}\} \ge t/2$. Hence, we have

$$\begin{aligned}
\beta(s, t) &= \beta_1(\beta_2(s, \tau_{s,t}), t - \tau_{s,t}) \\
&\le \max\{\beta_1(\beta_2(s, 0), t/2), \beta_1(\beta_2(s, t/2), 0)\} \\
&\in \mathcal{KL}. \qquad (43)
\end{aligned}$$

In order to prove the result assume without loss of generality $\varepsilon \le M$. We consider the set $\tilde{\Delta} = \{(x_1, x_2) \in \mathbb{R}^{2n}\colon |x_1| + |x_2| \ge 3M, |x_1 - x_2| \le \varepsilon\}$. Clearly $3M \le |x_1| + |x_2| \le 2|x_1| + |x_2 - x_1| \le 2|x_1| + M$, which implies $|x_1| \ge M$, and by a symmetric argument $|x_2| \ge M$. Hence, by (41) we have

$$\begin{aligned}
&\frac{\partial V}{\partial x_1}(x_1, x_2)f(x_1, d) + \frac{\partial V}{\partial x_2}(x_1, x_2)f(x_2, d) \\
&\quad \le -\rho(|x_1 - x_2|) \\
&\qquad \forall d \in \mathcal{D}, \qquad \forall (x_1, x_2) \in \tilde{\Delta}. \qquad (44)
\end{aligned}$$

Pick any $\xi_1, \xi_2 \in \mathbb{R}^n$ such that $|\xi_1 - \xi_2| \le \alpha_2^{-1}(\alpha_1(\varepsilon/2))$. We claim that, as long as $|x(t, \xi_1, d)| + |x(t, \xi_2, d)| \ge 3M$, $|x(t, \xi_1, d) - x(t, \xi_2, d)| \le \varepsilon$. Assume by contradiction that there exists $\bar{t} > 0, \bar{d} \in \mathcal{M}_\mathcal{D}$, such that the previous inequality be violated and $|x(t, \xi_1, \bar{d})| + |x(t, \xi_2, \bar{d})| \ge 3M$ for all $t \in [0, \bar{t}]$. Let, without loss of generality, $\bar{t}$ be the infimum of such $\bar{t}$s. Since V is decreasing along trajectories whenever $(x_1, x_2) \in \tilde{\Delta}$, we get a contradiction through the following inequalities:

$$\begin{aligned}
\alpha_1(\varepsilon) &\le \alpha_1(|x(\bar{t}, \xi_1, \bar{d}) - x(\bar{t}, \xi_2, \bar{d})|) \\
&\le V(x(\bar{t}, \xi_1, \bar{d}), x(\bar{t}, \xi_2, \bar{d})) \\
&\le V(\xi_1, \xi_2) \le \alpha_2(|\xi_1 - \xi_2|) \le \alpha_1(\varepsilon/2). \qquad (45)
\end{aligned}$$

By Lemma 4.4 in [17], as long as $|x(t, \xi_1, \bar{d})| + |x(t, \xi_2, \bar{d})| \ge 3M$, there exists $\beta_1 \in \mathcal{KL}$ such that $|x(t, \xi_1, d) - x(t, \xi_2, d)| \le \beta_1(|\xi_1 - \xi_2|, t)$. Let $\bar{\tau}$ be the first time that $|x(t, \xi_1, \bar{d})| + |x(t, \xi_2, \bar{d})| \le 3M$. By virtue of the second item of Proposition 3.4, GAS implies semiglobal incremental asymptotic stability. Hence, there exists $\beta_2 \in \mathcal{KL}$ such that $|x(t, \xi_1, d) - x(t, \xi_2, d)| \le \beta_2(\beta_1(|\xi_1 - \xi_2|, \bar{\tau}), t - \bar{\tau})$ for all $t \ge \bar{\tau}$. By the above arguments, we have for any $d \in \mathcal{M}_\mathcal{D}$,

$$\begin{aligned}
&|x(t, \xi_1, d) - x(t, \xi_2, d)| \\
&\quad \le \max\{\beta_1(|\xi_1 - \xi_2|, t), \beta_2(\beta_1(|\xi_1 - \xi_2|, \bar{\tau}), t - \bar{\tau})\} \\
&\qquad \forall |\xi_1 - \xi_2| \le \alpha_2^{-1}(\alpha_1(\varepsilon/2)).
\end{aligned}$$

Hence, local incremental asymptotic stability follows by the fact on composition of $\mathcal{KL}$ functions. Finally, δGAS is implied by the third item of Proposition 3.4. ∎

IV. Incremental Input-to-State Stability

Throughout this section we consider systems of the following form:

$$\dot{x} = f(x, u) \qquad (46)$$

with state $x \in \mathbb{R}^n$ and input $u \subset \mathcal{U}$, with $\mathcal{U}$ a closed and convex set of $\mathbb{R}^m$ containing the origin. The function $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz continuous and such that $f(0, 0) = 0$. We are interested in studying systems which forget past inputs and states as made precise by the following definition.

Definition 4.1: We call a system (46) *incrementally input-to-state stable* (δISS) if there exists a $\mathcal{KL}$ function β and $\gamma \in \mathcal{K}_\infty$ such that for any $t \geq 0$, any $\xi_1, \xi_2 \in \mathbb{R}^n$ and any couple of input signals u_1, u_2 the following is true:

$$|x(t, \xi_1, u_1) - x(t, \xi_2, u_2)| \leq \beta(|\xi_1 - \xi_2|, t) + \gamma(\|u_1 - u_2\|_\infty). \tag{47}$$

□

Since $f(0, 0) = 0$ it is easy to check that δISS implies ISS just comparing an arbitrary trajectory with $x(t) \equiv 0$. A first interesting fact about δISS is stated in the next proposition.

Proposition 4.2: Let system (46) be δISS. Then, for all constant input signals $\overline{u}$ there exists a unique, globally asymptotically stable, equilibrium point. □

Proof: Since $f(0, 0) = 0$, δISS implies ISS and in particular the so called bounded-input bounded-state property, see [27]. Hence, for any constant $\overline{u}$, the trajectory $x(t, 0, \overline{u})$ is uniformly bounded and has a nonempty ω-limit set Ω. It is known that Ω is compact. We would like to show that Ω can only be an equilibrium point. Let $\omega \in \Omega$ and $\xi \in \mathbb{R}^n$ be arbitrary. By δISS we have

$$|x(t, \xi, \overline{u}) - x(t, \omega, \overline{u})| \leq \beta(|\xi - \omega|, t). \tag{48}$$

Hence, taking the infimum in both sides of (48), and by invariance of Ω, we obtain

$$\begin{aligned}\inf_{\omega \in \Omega} |x(t, \xi, \overline{u}) - \omega| &\leq \inf_{\omega \in \Omega} |x(t, \xi, \overline{u}) - x(t, \omega, \overline{u})| \\ &\leq \beta\left(\inf_{\omega \in \Omega} |\xi - \omega|, t\right).\end{aligned} \tag{49}$$

This amounts to say that $|x(t, \xi, \overline{u})|_\Omega \leq \beta(|\xi|_\Omega, t)$, thus showing global asymptotic stability of Ω. It is shown in [3, Corollary 3.11] that the only compact, minimal global attractor in Euclidean space can be an equilibrium point. Thus, in order to complete the proof of the lemma we are only left to show that Ω is minimal. As a matter of fact ω-limit sets need not be minimal; therefore δISS is again crucial in order to show minimality. By [3, Th. 3.4], we have that a set $\mathcal{X}$ is minimal if and only if $\Omega(\xi) = \mathcal{X}$ for all $\xi \in \mathcal{X}$. We clearly have $\Omega(\xi) \subset \Omega$ for all $\xi \in \Omega$, thus we only need to show $\Omega \subset \Omega(\xi)$, for all $\xi \in \Omega$. Let $\omega \in \Omega$ be arbitrary. Then there exists a sequence $t_n \to +\infty$ such that $x(t_n, 0, \overline{u}) \to \omega$ as $n \to +\infty$. Pick an arbitrary $\xi \in \Omega$. We claim that $\omega \in \Omega(\xi)$. In fact, by δISS

$$\begin{aligned}&|x(t_n, \xi, \overline{u}) - \omega| \\ &\quad\leq |x(t_n, \xi, \overline{u}) - x(t_n, 0, \overline{u})| + |x(t_n, 0, \overline{u}) - \omega| \\ &\quad\leq \beta(|\xi|, t_n) + |x(t_n, 0, \overline{u}) - \omega| \to 0.\end{aligned} \tag{50}$$

This completes the proof of the proposition. ■

Remark 4.3: It follows from Proposition 4.2 that a necessary condition for δISS is the following:

$$\forall\, u \in \mathcal{U}, \qquad \exists \text{ unique } x_u\colon f(x_u, u) = 0. \tag{51}$$

Furthermore, x_u is a continuous function of u. In fact, let u_1, u_2 be arbitrary and such that $|u_1|, |u_2| \leq M$ for some $M > 0$. Hence, by δISS we have $|x_{u_1}| \leq \gamma(M)$ and $|x_{u_2}| \leq \gamma(M)$. Moreover

$$\begin{aligned}|x_{u_1} - x_{u_2}| &= |x(t, x_{u_1}, u_1) - x(t, x_{u_2}, u_2)| \\ &\leq \beta(|x_{u_1} - x_{u_2}|, t) + \gamma(|u_1 - u_2|) \\ &\leq \beta(2\gamma(M), t) + \gamma(|u_1 - u_2|).\end{aligned} \tag{52}$$

Since t is arbitrary we obtain $|x_{u_1} - x_{u_2}| \leq \gamma(|u_1 - u_2|)$, thus implying continuity of x_u. □

Trajectories of δISS systems all converge to one another. This property implies that for periodic input signals the state response tends to a periodic orbit (An analogous result, for slightly different definitions of incremental stability, was already proved in [5] and [18]).

Proposition 4.4: A δISS system, forced with a periodic input u (of period T), has a state response which asymptotically tends to a periodic function of the same period. Moreover there exists initial conditions $\overline{\xi}$ such that $x(t, \overline{\xi}, u)$ is periodic. □

Proof: Let $u(t)$ be periodic of period $T > 0$, *viz.* $u(t) = u(t + kT)$ for all $t \geq 0$ and all $k \in \mathbb{N}$. Pick ξ in $\mathbb{R}^n$. By periodicity u is globally essentially bounded and therefore δISS yields, for any $t \geq 0$ and some positive constant M, $|x(t, \xi, u)| < M$. Let us define the sequence of functions $x_k(\cdot) = x(\cdot + kT, \xi, u)$. We claim that x_k is a Cauchy sequence in the uniform topology. In fact, taking arbitrary $0 < n < m$ in $\mathbb{N}$ and exploiting the semigroup property $x(\tau + t, \xi, u(\cdot)) = x(\tau, x(t, \xi, u(\cdot)), u(\cdot + t))$ we obtain

$$\begin{aligned}&\|x_n - x_m\|_\infty \\ &\quad= \operatorname*{ess\,sup}_{t \geq 0} |x(t + nT, \xi, u(\cdot)) - x(t + mT, \xi, u(\cdot))| \\ &\quad= \operatorname*{ess\,sup}_{t \geq 0} |x(nT, x(t, \xi, u(\cdot)), u(\cdot + t)) \\ &\qquad\qquad - x(nT, x(t + (m-n)T, \xi, u(\cdot)) \\ &\qquad\qquad\quad u(\cdot + t + (m-n)T))| \\ &\quad= \operatorname*{ess\,sup}_{t \geq 0} |x(nT, x(t, \xi, u(\cdot)), u(\cdot + t)) \\ &\qquad\qquad - x(nT, x(t + (m-n)T, \xi, u(\cdot)), u(\cdot + t))| \\ &\quad\leq \beta(|x(t, \xi, u(\cdot)) - x(t + (m-n)T, \xi, u(\cdot))|, nT) \\ &\quad\leq \beta(2M, nT).\end{aligned} \tag{53}$$

Thus, increments $\|x_n - x_m\|_\infty$ can be made arbitrarily small just by taking , $\min\{n, m\}$ sufficiently large. By completeness of the space of continuous functions defined over $[0, +\infty)$ there exists a continuous function $\overline{x}(\cdot)$ such that $x_k(\cdot) \to \overline{x}(\cdot)$. Moreover, it is straightforward from the definition

$$\begin{aligned}\overline{x}(t + T) &= \lim_{k \to +\infty} x(t + (1 + k)T, \xi, u) \\ &= \lim_{k \to +\infty} x(t + kT, \xi, u) = \overline{x}(t)\end{aligned} \tag{54}$$

and therefore $\overline{x}(\cdot)$ is periodic. We are only left to show that $\overline{x}(t)$ is a solution of (46), namely $x(t, \overline{x}(0), u) = \overline{x}(t)$. The equality clearly holds for $t = 0$, furthermore taking derivatives with respect to time and by virtue of uniform convergence yields

$$\begin{aligned}\frac{d}{dt}\,\overline{x}(t) &= \frac{d}{dt} \lim_{k \to +\infty} x_k(t) = \lim_{k \to +\infty} \frac{d}{dt}\, x_k(t) \\ &= \lim_{k \to +\infty} f(x(t + KT, \xi, u), u(t)) \\ &= f(\overline{x}(t), u(t)).\end{aligned} \tag{55}$$

■

The converging-input–converging-state property has its counter-part in δISS in the fact that $u_1 - u_2 \to 0$ implies $x(t, \cdot, u_1) - x(t, \cdot, u_2) \to 0$.

Proposition 4.5: Consider a δISS system as in (46) and let u_1 and u_2 be inputs signals such that

$$\lim_{t \to +\infty} |u_1(t) - u_2(t)| = 0.$$

Pick ξ_1, ξ_2 arbitrary in $\mathbb{R}^n$. Then $|x(t, \xi_1, u_1) - x(t, \xi_1, u_2)| \to 0$. □

Proof: Since $|u_1 - u_2| \to 0$ as $t \to +\infty$ and $|u_1 - u_2|$ is locally essentially bounded we have that $|u_1 - u_2|$ is uniformly essentially bounded. Let M be such that $|u_1(t) - u_2(t)| \leq M$ almost everywhere. By δISS we have

$$|x(t, \xi_1, u_1) - x(t, \xi_2, u_2)| \leq \beta(|\xi_1 - \xi_2|, t) + \gamma(M) \quad (56)$$

for all $t \geq 0$. We know that

$$\forall \epsilon > 0 \, \exists T_\varepsilon > 0: t > T_\varepsilon \Rightarrow |u_1(t) - u_2(t)| \leq \gamma^{-1}(\varepsilon/2) \quad \text{a.e.}$$

By definition of $\mathcal{KL}$ function for all $\varepsilon > 0$ there exists τ_ε such that $\beta(\beta(|\xi_1 - \xi_2|, 0) + \gamma(M), \tau_\varepsilon) \leq \varepsilon/2$. Hence, for $t \geq T_\varepsilon + \tau_\varepsilon$ we obtain

$$\begin{aligned} &|x(t, \xi_1, u_1) - x(t, \xi_2, u_2)| \\ &\quad \leq \beta(|x(T_\varepsilon, \xi_1, u_1) - x(T_\varepsilon, \xi_2, u_2)|, t - T_\varepsilon) \\ &\qquad + \gamma(\|u_1([T_\varepsilon, t]) - u_2([T_\varepsilon, t])\|_\infty) \\ &\quad \leq \beta(\beta(|\xi_1 - \xi_2|, 0) + \gamma(M), \tau_\varepsilon) + \varepsilon/2 \leq \varepsilon. \end{aligned} \quad (57)$$

■

Some examples follow in order to develop an intuition for which systems might or not be incrementally ISS. For finite-dimensional linear systems δISS is equivalent to asymptotic stability. In particular, the simple scalar system $\dot{x} = -x + u$ is such. Rather surprisingly $\dot{x} = -x + u^3$ is not δISS.

Counter-Example 1: The system $\dot{x} = -x + u^3$ is ISS with respect to the equilibrium point $x_u = \overline{u}^3$ and the input signal $u - \overline{u}$ for all $\overline{u} \in \mathbb{R}$. Moreover it is δGAS (with respect to the "disturbance" u in any compact subset of $\mathbb{R}$). Nevertheless, it is not δISS.

In order to show ISS with respect to constant values of u it is enough to choose as a Lyapunov function $V_{\overline{u}}(x) = (x - \overline{u}^3)^2$. Robust incremental GAS follows considering that (looking at u as a disturbance):

$$\frac{d}{dt} x(t, \xi_1, u) - x(t, \xi_2, u) = -(x(t, \xi_1, u) - x(t, \xi_2, u)). \quad (58)$$

Hence, $|x(t, \xi_1, u) - x(t, \xi_2, u)| \leq |\xi_1 - \xi_2| e^{-t}$. In order to show that the system is not δISS we pick u_1 and u_2 as given by the closed-loop feedback that makes the system into $\dot{x} = 1$. In particular then $u_1^3 = x(t, \xi_1, u_1) + 1$ and $u_2^3 = x(t, \xi_2, u_2) + 1$. With this definition we have $x(t, \xi_1, u_1) = t + \xi_1$ and $x(t, \xi_2, u_2) = \xi_2 + t$. Hence

$$u_1(t) = (t + \xi_1 + 1)^{1/3} \qquad u_2(t) = (t + \xi_2 + 1)^{1/3}.$$

In particular then $x(t, \xi_1, u_1) - x(t, \xi_2, u_2) = \xi_1 - \xi_2$ is constant for all t whereas $u_1(t) - u_2(t) \to 0$. This contradicts the converging-inputs–converging-states property of δISS.

Counterexample 2: A similar argument can be repeated to show that the system

$$\dot{x} = -\frac{x}{(1 + x^2)^{1/3}} + u \quad (59)$$

is again ISS for all constant inputs $\overline{u}$, δGAS but not δISS.

We remark that the function of x in (59) is globally Lipschitz. As a matter of fact, it is true that δGAS is preserved under globally Lipschitz changes of coordinates provided that the inverse is again a globally Lipschitz function. More precisely the following Proposition holds.

Proposition 4.6: Let ϕ: $\mathbb{R}^n \to \mathbb{R}^n$ be a global diffeomorphism such that

$$\begin{aligned} |\phi(x_1) - \phi(x_2)| &\leq \gamma(|x_1 - x_2|) \\ |\phi^{-1}(x_1) - \phi^{-1}(x_2)| &\leq \delta(|x_1 - x_2|) \end{aligned} \quad (60)$$

for some functions δ, γ of class $\mathcal{K}_\infty$. If the system $\dot{x} = f(x, u)$ is δISS, then

$$\dot{z} = \nabla\phi(\phi^{-1}(z)) f(\phi^{-1}(z), u) \quad (61)$$

is also δISS. □

Proof: In the new coordinates we have for trajectories of (61)

$$\begin{aligned} &|z(t, \xi_1, u_1) - z(t, \xi_2, u_2)| \\ &\quad = |\phi(x(t, \phi^{-1}(\xi_1), u_1)) - \phi(x(t, \phi^{-1}(\xi_2), u_2))| \\ &\quad \leq \delta(|x(t, \phi^{-1}(\xi_1), u_1) - x(t, \phi^{-1}(\xi_2), u_2)|) \\ &\quad \leq \delta(\beta(|\phi^{-1}(\xi_1) - \phi^{-1}(\xi_2)|, t) + \gamma(\|u_1 - u_2\|_\infty)) \\ &\quad \leq \delta(2\beta(\delta(|\xi_1 - \xi_2|), t)) + \delta(2\gamma(\|u_1 - u_2\|_\infty)). \end{aligned} \quad (62)$$

■

The next is the analogous, for incremental ISS, of a well known result on cascaded ISS systems which first appeared in [24] and [25].

Proposition 4.7: Consider the interconnected system

$$\begin{cases} \dot{x} = f(x, y, u) \\ \dot{y} = g(y, u). \end{cases} \quad (63)$$

Let the x-subsystem be δISS with respect to u, y and the y-subsystem with respect to u. Then, the overall system (63) is δISS with respect to the input u. □

Proof: Define $\chi = [x', y']' \in \mathbb{R}^{n_x + n_y}$. For arbitrary χ_1, χ_2 in $\mathbb{R}^{n_x + n_y}$ we have $|x_1 - x_2|/2 + |y_1 - y_2|/2 \leq |\chi_1 - \chi_2| \leq |x_1 - x_2| + |y_1 - y_2|$. Thus, a suitable estimate for trajectories of (63) can be obtained considering the x, y-subsystems separately. By δISS we have

$$\begin{aligned} &|x(t, \xi_1, u_1, y_1) - x(t, \xi_2, u_2, y_2)| \\ &\quad \leq \beta_x(|\xi_1 - \xi_2|, t) + \gamma_u(\|u_1 - u_2\|) + \gamma_y(\|y_1 - y_2\|) \\ &|y(t, \eta_1, u_1) - y(t, \eta_2, u_2)| \\ &\quad \leq \beta_y(|\eta_1 - \eta_2|, t) + \tilde{\gamma}_u(\|u_1 - u_2\|) \end{aligned} \quad (64)$$

where the $\eta_i s$ are the initial conditions for the y component of the state. Exploiting (64) and the weak triangular inequality the following series of inequalities is obtained:

$$\begin{aligned}
&|x(t,\xi_1,u_1,y_1)-x(t,\xi_2,u_2,y_2)|\\
&\quad\le \beta_x(|x(t/2,\xi_1,u_1,y_1)-x(t/2,\xi_2,u_2,y_2)|,t/2)\\
&\qquad+\gamma_u(\|u_1-u_2\|_{[t/2,t)})+\gamma_y(\|y_1-y_2\|_{[t/2,t)})\\
&\quad\le \beta_x(\beta_x(|\xi_1-\xi_2|,t/2)+\gamma_u(\|u_1-u_2\|)\\
&\qquad+\gamma_y(\|y_1-y_2\|),t/2)+\gamma_u(\|u_1-u_2\|_{[t/2,t)})\\
&\qquad+\gamma_y(\|y_1-y_2\|_{[t/2,t)})\\
&\quad\le \beta_x(3\beta_x(|\xi_1-\xi_2|,t/2),t/2)+\beta_x(3\gamma_u(\|u_1-u_2\|),0)\\
&\qquad+\beta_x(3\gamma_y(\beta_y(|\eta_1-\eta_2|,0)+\tilde\gamma_u(\|u_1-u_2\|)),t/2)\\
&\qquad+\gamma_u(\|u_1-u_2\|_{[t/2,t)})+\gamma_y(\|y_1-y_2\|_{[t/2,t)})\\
&\quad\le \beta_x(3\beta_x(|\xi_1-\xi_2|,t/2),t/2)+\beta_x(3\gamma_u(\|u_1-u_2\|),0)\\
&\qquad+\beta_x(3\gamma_y(2\beta_y(|\eta_1-\eta_2|,0)),t/2)\\
&\qquad+\beta_x(3\gamma_y(2\tilde\gamma_u(\|u_1-u_2\|)),0)+\gamma_u(\|u_1-u_2\|)\\
&\qquad+\gamma_y(\beta_y(|\eta_1-\eta_2|,t/2)+\tilde\gamma_u(\|u_1-u_2\|))\\
&\quad\le \hat\beta(|\xi_1-\xi_2|,t)+\overline{\beta}(|\eta_1-\eta_2|,t)+\hat\gamma(\|u_1-u_2\|)
\end{aligned}\tag{65}$$

where $\hat\gamma\in\mathcal{K}_\infty$ and $\hat\beta,\overline{\beta}\in\mathcal{KL}$ are defined as

$$\begin{aligned}
\hat\gamma(r)&=\beta_x(3\gamma_u(r),0)+\beta_x(3\gamma_y(2\tilde\gamma_u(r)),0)\\
&\quad+\gamma_u(r)+\gamma_y(2\tilde\gamma_u(r))\\
\hat\beta(r,t)&=\beta_x(3\beta_x(r,t/2),t/2)\\
\overline{\beta}(r,t)&=\beta_x(3\gamma_y(2\beta_y(r,0)),t/2)+\gamma_y(2\beta_y(r,t/2)).
\end{aligned}\tag{66}$$

■

A small gain theorem for δISS systems in feedback interconnection can be proved along the same lines as in [12]. A precise statement of the Theorem is given later.

Proposition 4.8: Consider the following interconnection of nonlinear systems:

$$\begin{aligned}
\dot x_1&=f_1(x_1,x_2,u)\\
\dot x_2&=f_2(x_1,x_2,u).
\end{aligned}\tag{67}$$

Assume that each of the x_i-subsystems ($i=1,2$) be δISS with respect to u and to the other state component (x_{1-i}) seen as an input, *viz.*

$$\begin{aligned}
&|x_1(t,\xi_{a1},x_{2a},u_a)-x_1(t,\xi_{b1},x_{2b},u_b)|\\
&\quad\le\beta_1(|\xi_{a1}-\xi_{b1}|,t)+\gamma_1(\|x_{2a}-x_{2b}\|_\infty)\\
&\qquad+\tilde\gamma_1(\|u_a-u_b\|_\infty)\\
&|x_2(t,\xi_{a2},x_{1a},u_a)-x_1(t,\xi_{b2},x_{1b},u_b)|\\
&\quad\le\beta_2(|\xi_{a2}-\xi_{b2}|,t)+\gamma_2(\|x_{1a}-x_{1b}\|_\infty)\\
&\qquad+\tilde\gamma_2(\|u_a-u_b\|_\infty)
\end{aligned}\tag{68}$$

for some functions $\gamma_1,\gamma_2,\tilde\gamma_1,\tilde\gamma_2$ of class $\mathcal{K}_\infty$ and β_1,β_2 of class $\mathcal{KL}$. If the following small-gain condition is satisfied:

$$(\gamma_1+\rho)\circ(\gamma_2+\rho)(r)\le r\qquad\forall r\ge 0\tag{69}$$

for some function ρ of class $\mathcal{K}_\infty$, then the interconnected system (67) is δISS with respect to u. □

V. Lyapunov Conditions for Incremental ISS

Definition 5.1: A smooth function $V(x_1,x_2)$: $\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_{\ge0}$, is called a δISS-*Lyapunov function* if $\alpha_1(|x_1-x_2|)\le V(x_1,x_2)\le\alpha_2(|x_1-x_2|)$ for some $\alpha_1,\alpha_2\in\mathcal{K}_\infty$ and there exists $\kappa\in\mathcal{K}_\infty$ so that for any $u_1,u_2\in\mathcal{U}$ and any $x_1,x_2\in\mathbb{R}^n$

$$\begin{aligned}
\kappa(|x_1-x_2|)\ge|u_1-u_2|\Longrightarrow\frac{\partial V}{\partial x_1}f(x_1,u_1)+\frac{\partial V}{\partial x_2}f(x_2,u_2)\\
<-\rho(|x_1-x_2|)
\end{aligned}\tag{70}$$

is satisfied with ρ positive definite. □

We are now ready to state the main result of this Section.

Theorem 2: System (46) is δISS provided that it admits a δISS-Lyapunov function. Moreover, if $\mathcal{U}$ is compact, existence of a δISS-Lyapunov function is equivalent to δISS. ■

Remark 5.2: It is not known if the Lyapunov condition (70) is necessary for δISS in the case of unbounded $\mathcal{U}$. As a matter of fact, smooth converse Lyapunov results for robust incremental stability with disturbances in arbitrary closed sets are still not available. In Section II a continuous Lyapunov function for δGAS is constructed, but, unlike the compact case, there might be a gap between continuity and smoothness. □

The main result will be derived as a corollary of the following Proposition 5.3. In order to state the result we need to define the function

$$\mathrm{sat}_{\mathcal{U}}(u)=\begin{cases}u & \text{if } u\in\mathcal{U}\\ \arg\min\limits_{\nu\in\mathcal{U}}|\nu-u| & \text{if } u\notin\mathcal{U}.\end{cases}\tag{71}$$

Since $\mathcal{U}$ is closed and convex and $|\cdot|$ is a proper, convex function, definition (71) is well-posed, in particular the minimizer of $|\nu-u|$ with $\nu\in\mathcal{U}$ is unique. Moreover, by convexity of $\mathcal{U}$ we have

$$|\mathrm{sat}_{\mathcal{U}}(u_1)-\mathrm{sat}_{\mathcal{U}}(u_2)|\le|u_1-u_2|,\quad\forall u_1,u_2\in\mathbb{R}^m.\tag{72}$$

Proposition 5.3: System (46) is δISS if and only if there exists a smooth gain margin ρ of class $\mathcal{K}_\infty$ that makes the auxiliary system

$$\begin{cases}\dot x_1=f(x_1,\mathrm{sat}_{\mathcal{U}}(d_1+\rho(|x_1-x_2|)d_2))\\ \dot x_2=f(x_2,\mathrm{sat}_{\mathcal{U}}(d_1-\rho(|x_1-x_2|)\,d_2))\end{cases}\tag{73}$$

with state $\chi=[x_1',x_2']'\in\mathbb{R}^{2n}$ and input $d=[d_1,d_2]$ taking value in $\mathcal{D}\doteq\mathcal{U}\times\mathcal{B}$ uniformly GAS with respect to the diagonal set Δ (where $\mathcal{B}$ denotes the closed unit ball in $\mathbb{R}^m$). □

Proof: The proof is based on a small gain argument analogous to the one in [26, Lemma 2.12]. Let the system (1) be δISS. Then, the following estimate holds for some function $\beta\in\mathcal{KL}$ and $\gamma\in\mathcal{K}_\infty$:

$$\begin{aligned}
&|x(t,\xi_1,u_1)-x(t,\xi_2,u_2)|\\
&\qquad\le\max\{\beta(|\xi_1-\xi_2|,t),\gamma(\|u_1-u_2\|_\infty)\}.
\end{aligned}\tag{74}$$

We recall that $|\xi_1-\xi_2|=\sqrt{2}|\chi|_\Delta$. Without loss of generality we will assume $\alpha(r)\doteq\beta(r,0)>r$ for all $r>0$. Let ρ be a $\mathcal{K}_\infty$ function satisfying $\rho(r)\le(1/2)\gamma^{-1}\circ((\alpha^{-1}(r))/4\sqrt{2})$. To prove GAS with respect to Δ, we first show that

$$\gamma(|d_2(t)2\rho(\sqrt{2}|\chi(t,[\xi_1',\xi_2']',d)|_\Delta)|)\le\frac{1}{\sqrt{2}}|[\xi_1',\xi_2']'|_\Delta,\qquad\text{a.e. } t\ge0\tag{75}$$

for any $\xi_1, \xi_2 \in \mathbb{R}^n$ and any $d \in \mathcal{M}_\mathcal{D}$. For this it is enough to show, because of the monotonicity of γ, that for $t \geq 0$

$$\gamma(2\rho(\sqrt{2}|\chi(t, [\xi_1', \xi_2']', d)|_\Delta)) \leq \frac{1}{\sqrt{2}}|[\xi_1', \xi_2']'|_\Delta, \quad \text{a.e.} \tag{76}$$

Pick arbitrary $\xi_1, \xi_2 \in \mathbb{R}^n$ and $d \in \mathcal{M}_\mathcal{D}$ and use simply $\chi(t)$ to denote $\chi(t, [\xi_1', \xi_2']', d)$. Notice that $\gamma(2\rho(\sqrt{2}|\chi(t)|_\Delta)) \leq |[\xi_1', \xi_2']'|_\Delta/4$ for all t small enough, since $\gamma(2\rho(\sqrt{2}|\chi(0)|_\Delta)) \leq (1/4\sqrt{2})\alpha^{-1}(\sqrt{2}|[\xi_1, \xi_2]|_\Delta) < |[\xi_1', \xi_2']'|_\Delta/4$. Now, let $t_1 = \inf\{t > 0\colon \gamma(2\rho(\sqrt{2}|\chi(t)|_\Delta)) > |[\xi_1, \xi_2]|_\Delta/\sqrt{2}\}$. Clearly $t_1 > 0$. Assume by contradiction $t_1 < \infty$. Then (76) holds for all $t \in [0, t_1)$, from which it follows that for almost all $t \in [0, t_1)$:

$$\begin{aligned}\gamma\left(\left|d_2 2\rho\left(\sqrt{2}|\chi(t)|_\Delta\right)\right|\right) &\leq \gamma\left(2\rho\left(\sqrt{2}|\chi(t)|_\Delta\right)\right)\\ &\leq \frac{1}{\sqrt{2}}|[\xi_1', \xi_2']'|_\Delta\\ &< \frac{1}{2}\alpha\left(\sqrt{2}|[\xi_1', \xi_2']'|_\Delta\right).\end{aligned} \tag{77}$$

Let u_1 and u_2 be defined as

$$\begin{aligned}u_1(t) &= \mathrm{sat}_\mathcal{U}\left(d_1(t) + \rho\left(\sqrt{2}|\chi(t)|_\Delta\right)d_2(t)\right)\\ u_2(t) &= \mathrm{sat}_\mathcal{U}\left(d_1(t) - \rho\left(\sqrt{2}|\chi(t)|_\Delta\right)d_2(t)\right).\end{aligned}$$

By virtue of (72) we have $|u_1(t) - u_2(t)| \leq |d_2(t)2\rho(\sqrt{2}|\chi(t)|_\Delta)|$. Hence, from (74) and (77) one sees that $\sqrt{2}|\chi(t)|_\Delta \leq \beta(\sqrt{2}|[\xi_1', \xi_2']'|_\Delta, 0) = \alpha(\sqrt{2}|[\xi_1', \xi_2']'|_\Delta)$, for all $0 \leq t \leq t_1$ which, in turn implies that $\gamma(2\rho(\sqrt{2}|\chi(t_1)|_\Delta)) \leq |[\xi_1, \xi_2]|_\Delta/4$. This contradicts the definition of t_1. Thus $t_1 = \infty$, and (76) is proved.

Claim: For each $r > 0$ there is some $T_r \geq 0$ so that

$$t \geq T_r,\ |[\xi_1', \xi_2']_\Delta'| \leq r \Longrightarrow |\chi(t, [\xi_1', \xi_2']', d)|_\Delta \leq r/2. \tag{78}$$

To establish this claim, note that, from (74) and (75), it follows that:

$$|\chi(t, [\xi_1', \xi_2']', d)|_\Delta \leq \max\left\{\tilde{\beta}(|[\xi_1', \xi_2']'|_\Delta, t),\ |[\xi_1', \xi_2']'|_\Delta/2\right\}$$

holds for all ξ_1, ξ_2, with $\tilde{\beta}(r, t) = (1/\sqrt{2})\beta(\sqrt{2}r, t)$. On the other hand, since $\tilde{\beta} \in \mathcal{KL}$, for each $r > 0$ there exists $T_r > 0$ such that $\tilde{\beta}(r, t) < r/2$ for all $t \geq T_r$. This T_r satisfies the requirements of the claim.

Now, pick any $\varepsilon > 0$. Let k be a positive integer such that $2^{-k}r < \varepsilon$. Let $r_1 = r$ and $r_i = r_{i-1}/2$ for $i \geq 2$, and let $\tau = T_{r1} + T_{r2} + \cdots + T_{rk}$. Then, for $t \geq \tau$, it holds that $|\chi(t, [\xi_1, \xi_2], d)|_\Delta < r/2^k < \varepsilon$ for all $|[\xi_1, \xi_2]|_\Delta \leq r$, all $d \in \mathcal{M}_\mathcal{D}$ and all $t \geq \tau$. This shows that the diagonal is a uniform attractor for system (73). To show the uniform stability for the system notice that $|\chi(t, [\xi_1', \xi_2']', d)| \leq \tilde{\beta}(|[\xi_1', \xi_2']'|_\Delta, 0)$ for all $t \geq 0$, all $\xi_1, \xi_2 \in \mathbb{R}^n$, and all $d \in \mathcal{M}_\mathcal{D}$. We conclude that system (73) is uniformly globally asymptotically stable.

We show next the converse implication. Let the system (73) be uniformly GAS with respect to the diagonal set Δ. Pick $\xi_1, \xi_2 \in \mathbb{R}^n$ and let u_1, u_2 be arbitrary input signals. Suppose that for all τ belonging to $[0, t]$ we have $|u_1(\tau) - u_2(\tau)| \leq 2\rho(|x(\tau, \xi_1, u_1) - x(\tau, \xi_2, u_2)|)$. Then there exists $d_1(\tau) \doteq (u_1 + u_2)/2$ taking values in $\mathcal{U}$ and $d_2(\tau) \doteq [u_1(\tau) - u_2(\tau)]/2\rho(|x(\tau, \xi_1, u_1) - x(\tau, \xi_2, u_2)|) \in \mathcal{B}$ such that $x_1(\tau) \doteq x(\tau, \xi_1, u_1)$ and $x_2(\tau) \doteq x(\tau, \xi_2, u_2)$ satisfy equations (73). Hence, by robust GAS we have

$$\begin{aligned}|x(t, \xi_1, u_1) - x(t, \xi_2, u_2)| &= \sqrt{2}|\chi(t, [\xi_1', \xi_2']', d)|_\Delta\\ &\leq \sqrt{2}\beta\left(|\xi_1 - \xi_2|/\sqrt{2}, t\right).\end{aligned} \tag{79}$$

As a matter of fact, we are only left to deal with the case in which there exists $\bar{t} \in [0, t]$ such that $|u_1(\bar{t}) - u_2(\bar{t})| > 2\rho(|x(\bar{t}, \xi_1, u_1) - x(\bar{t}, \xi_2, u_2)|)$. Let $\overline{\tau}$ be the supremum of such $\bar{t}$s. Then

$$\begin{aligned}&|x(\overline{\tau}, \xi_1, u_1) - x(\overline{\tau}, \xi_2, u_2)|\\ &\quad\leq \rho^{-1}(|u_1(\overline{\tau}) - u_2(\overline{\tau})|)\\ &\quad\leq \rho^{-1}\left(\sup_{t \in [0, \overline{\tau}]} |u_1(t) - u_2(t)|\right).\end{aligned}$$

Exploiting again the estimate provided by (79) initialized at time $\overline{\tau}$ yields

$$\begin{aligned}&|x(t, \xi_1, u_1) - x(t, \xi_2, u_2)|\\ &\quad\leq \sqrt{2}\beta(|x(\overline{\tau}, \xi_1, u_1) - x\left(\overline{\tau}, \xi_2, u_2)|/\sqrt{2}, t - \overline{\tau}\right)\\ &\quad\leq \sqrt{2}\beta\left(\rho^{-1}(\|u_1 - u_2\|_\infty)/\sqrt{2}, 0\right).\end{aligned} \tag{80}$$

Since the functions ρ and β do not depend on t, δISS follows combining (79) and (80). ∎

We go back to the proof of Theorem 2. By Proposition 5.3, incremental ISS is equivalent to uniform global asymptotic stability of (73) with respect to the diagonal set Δ. For compact $\mathcal{U}$, by virtue of the converse Lyapunov theorem in [31], this is the case if and only if there exist a smooth $V(x_1, x_2)$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, with $\alpha_1(|x_1 - x_2|) \leq V(x_1, x_2) \leq \alpha_2(|x_1 - x_2|)$, so that

$$\begin{aligned}&\frac{\partial V}{\partial x_1} f(x_1, \mathrm{sat}_\mathcal{U}(d_1 + \rho(|x_1 - x_2|)d_2))\\ &\quad+ \frac{\partial V}{\partial x_2} f(x_2, \mathrm{sat}_\mathcal{U}(d_1 - \rho(|x_1 - x_2|)d_2))\\ &\leq -\alpha(|x_1 - x_2|)\end{aligned} \tag{81}$$

holds for any $x_1, x_2 \in \mathbb{R}^n$ and any $[d_1, d_2] \in \mathcal{D}$. Then, whenever $\rho^{-1}(|x_1 - x_2|/2) \geq |u_1 - u_2|$, the input disturbances $d_1 = (u_1 + u_2)/2$ and $d_2 = (u_1 - u_2)/2\rho(|x_1 - x_2|)$ are such that $[d_1, d_2]$ belongs to $\mathcal{D} = \mathcal{U} \times \mathcal{B}$. Then (81) implies

$$\begin{aligned}\kappa(|x_1 - x_2|) \geq |u_1 - u_2| \Rightarrow \frac{\partial V}{\partial x_1} f(x_1, u_1) + \frac{\partial V}{\partial x_2} f(x_2, u_2)\\ \leq -\alpha(|x_1 - x_2|)\end{aligned} \tag{82}$$

with $\kappa(r) \doteq \rho^{-1}(r/2)$. This completes the converse implication for $\mathcal{U}$ compact. The sufficiency part can be proved along the same lines as in [26] even dropping the compactness assumption on $\mathcal{U}$.

VI. A Few Examples

As already pointed out by Slotine and Lohmiller, see. for instance [18] and [23], incremental stability notions are particularly suited to address problems of synchronization of coupled systems. Such issues arise for instance in observers design or in synchronization of coupled chaotic behaviors for secure communications. In this section, we describe two examples of such applications.

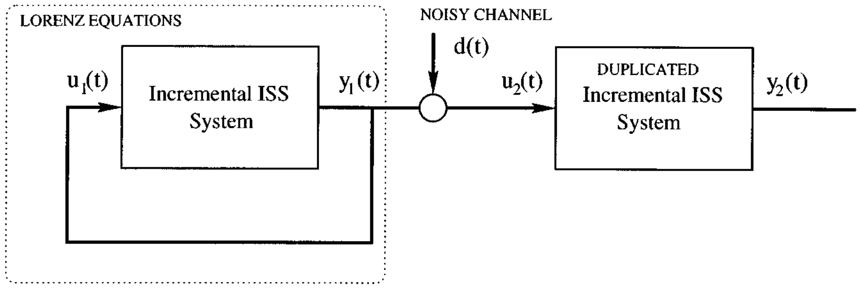

Fig. 1. Interconnection of the coupled systems.

A. Chaos Synchronization

We look at synchronization of coupled chaotic systems from the perspective of incremental ISS. See [4] for frequency domain condition for synchronization of nonlinear systems in the Lur'e form. We consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1 &= -\beta x_1 + \mathrm{sat}(x_2)\mathrm{sat}(x_3)\\ \dot{x}_2 &= -\sigma x_2 + \sigma x_3\\ \dot{x}_3 &= -x_3 + u\\ y &= \rho x_2 - x_1 x_2\end{aligned} \tag{83}$$

where β, σ, ρ are given constant parameters and $\mathrm{sat}(\cdot)$ is a piecewise linear saturation. It is not difficult to verify, by virtue of Proposition 4.7 on cascaded systems, that (83) is δISS. Besides we chose the output signal y in such a way that under unitary feedback $u = y$, the system evolution is governed (in the linear region of the saturation function), by the celebrated Lorenz equations. Since we were able to isolate a δISS subsystem we can easily synchronize two such systems providing that we force them with the same input signal. In particular we will consider the interconnection in Fig. 1, whose equations are

$$\begin{aligned}\dot{x}_1 &= -\beta x_1 + \mathrm{sat}(x_2 x_3)\\ \dot{x}_2 &= -\sigma x_2 + \sigma x_3\\ \dot{x}_3 &= -x_3 + \rho x_2 - x_1 x_2\\ y &= \rho x_2 - x_1 x_2\\ \dot{\hat{x}}_1 &= -\beta \hat{x}_1 + \mathrm{sat}(x_2 x_3)\\ \dot{\hat{x}}_2 &= -\sigma \hat{x}_2 + \sigma \hat{x}_3\\ \dot{\hat{x}}_3 &= -\hat{x}_3 + \rho \hat{x}_2 - \hat{x}_1 \hat{x}_2 + (y - \hat{y}) + d\\ \hat{y} &= \rho \hat{x}_2 - \hat{x}_1 \hat{x}_2.\end{aligned} \tag{84}$$

By virtue of incremental ISS of (83) and letting $x = [x_1, x_2, x_3]$ and $\hat{x} = [\hat{x}_1, \hat{x}_2, \hat{x}_3]$ we have the following estimate:

$$|x(t, \xi) - \hat{x}(t, \hat{\xi}, d)| \leq \beta(|\xi - \hat{\xi}|, t) + \gamma(\|d\|_\infty) \tag{85}$$

which in turns guarantees exact synchronization of the two systems in the ideal case of a noise-free channel, and robustness in the presence of persistent channel disturbances. In the simulation we show the behavior of the system for $\beta = 8/3$, $\sigma = 10$ and $\rho = 28$; this values of the parameter give rise to a chaotic attractor; we show how the systems trajectories follow each other also in the presence of significant signal-to-noise ratios (see Fig. 2).

B. Observer Design

In a recent paper on detectability notions for nonlinear systems [28], Sontag and Wang give a definition of full-state observer, robust with respect to sensor and input noise, for systems

$$\begin{aligned}\dot{x} &= f(x, u)\\ y &= h(x).\end{aligned} \tag{86}$$

It follows from their definition that such an observer must take the "output injection" form

$$\dot{z} = f(z, u + d_u) + L(z, u + d_u, y - h(z) + d_y) \tag{87}$$

where the vector field L satisfies $L(a, b, 0) = 0$ for all a, b. In (87), the vector $z \in \mathbb{R}^n$ is the estimate of $x(t)$, whereas d_u and d_y are input and sensor noises. In order for (87) to be an observer it is required that feeding the output y of (86) to (87) the following estimate holds:

$$\begin{aligned}&|x(t, \xi, u) - z(t, \zeta, u + d_u, h(x) + d_y)|\\ &\quad \leq \beta(|\xi - \zeta|, t) + \gamma_1\left(\|d_u\|_{[0,t)}\right) + \gamma_2\left(\|d_y\|_{[0,t)}\right)\end{aligned} \tag{88}$$

for some function β of class $\mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$. It is proved in [28] that a necessary condition for the existence of an observer is incremental input–output to state stability of (86). Hereby, we present a sufficient condition in terms of incremental ISS of the output-injected system.

Proposition 6.1: System (86) admits an observer if there exists an output injection such that system

$$\dot{z} = f(z, u) + L(z, u, y - h(z)) \tag{89}$$

is δISS with respect to u and y, [*viz.* u and y are seen here as inputs of (89)]. □

Proof: Definition of δISS yields, for any $\zeta_1, \zeta_2 \in \mathbb{R}^n$ and any input u_1, u_2, y_1, y_2

$$\begin{aligned}&|z(t, \zeta_1, u_1, y_1) - z(t, \zeta_2, u_2, y_2)|\\ &\quad \leq \beta(|\zeta_1 - \zeta_2|, t) + \gamma_1(\|u_1 - u_2\|) + \gamma_2(\|y_1 - y_2\|).\end{aligned} \tag{90}$$

Notice that, by the output injection form of (89), $x(t, \xi, u) = z(t, \xi, u, h(x(\cdot, \xi, u)))$. With this in mind and recalling (90) it is straightforward to see

$$\begin{aligned}&|x(t, \xi, u) - z(t, \zeta, u + d_u, h(x) + d_y)|\\ &\quad = |z(t, \xi, u, h(x(\cdot))) - z(t, \zeta, u + d_u, h(x(\cdot)) + d_y)|\\ &\quad \leq \beta(|\xi - \zeta|, t) + \gamma_1(\|d_u\|) + \gamma_2(\|d_y\|).\end{aligned} \tag{91}$$

■

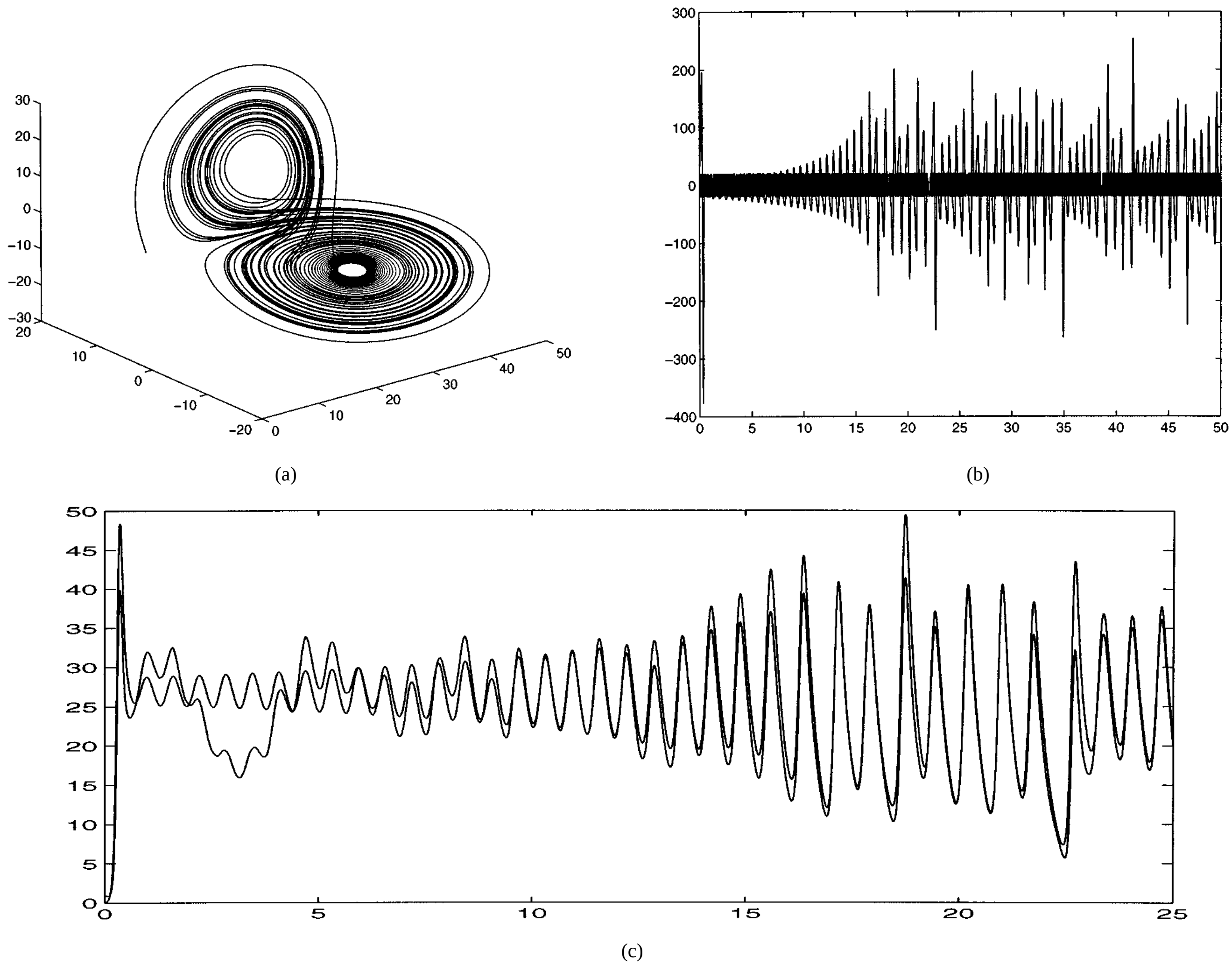

Fig. 2. (a) Lorentz attractor. (b) $y(t)$ and channel noise. (c) x_1 and $\hat{x}_1$.

Remark 6.2: In [28] a counterexample was given to show that incremental IOSS does not guarantee the existence of an observer in general. Therefore, there is a gap between incremental IOSS of (86) and the existence of an output injection (89) which renders the system incremental ISS. Nevertheless, for the special case of linear systems both conditions are equivalent to detectability. In particular, condition (89), can be seen as a robustified version of the detectability notion introduced in [32]. □

VII. Conclusion

We introduced and discussed several versions of robust incremental stability together with their characterizations in terms of Lyapunov dissipation inequalities. Such properties extend the input-to-state stability paradigm to study stability of solutions with respect to one another rather than with respect to some equilibrium position. The central idea is that incremental stability notions can be seen as standard asymptotic stability properties of some duplicated auxiliary system with respect to the diagonal set. Hopefully, this paper will serve as a starting point for the extension of more specific analysis techniques to the context of incremental stability and for the developing of synthesis tools, such as control-Lyapunov functions, backstepping, feedforwarding, [16], [13], which are well-known in nonlinear control theory but usually limited to more traditional stability properties.

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David Angeli received the degree in computer science engineering and the Ph.D. degree from the Università di Firenze, Italy, in 1996 and 2000.

Since then, he has been with the Dipartimento di Sistemi e Informatica, the Università di Firenze, where he is currently an Assistant Professor. During his doctoral studies, he spent semesters as a Visiting Fellow in the Department of Automatic Control of Lund Institute of Technology, Sweden, and in the Department of Mathematics, Rutgers University, New Brunswick, NJ. His research interests include constrained and hybrid control, and stability of nonlinear systems.