Total Dual Integrality and Integer Polyhedra*

F. R. Giles†
Department of Operations Research
Cornell University
Ithaca, New York

and

W. R. Pulleyblank‡
CORE, de Croylaan 54
3030 Heverlee
Belgium

Submitted by Hans Schneider

ABSTRACT

A linear system $Ax \leqslant b$ (A, b rational) is said to be totally dual integral (TDI) if for any integer objective function c such that max $\{cx: Ax \leqslant b\}$ exists, there is an integer optimum dual solution. We show that if P is a polytope all of whose vertices are integer valued, then it is the solution set of a TDI system $Ax \leqslant b$ where b is integer valued. This was shown by Edmonds and Giles [4] to be a sufficient condition for a polytope to have integer vertices.

1. INTRODUCTION

Let $Ax \leqslant b$ be a linear system with A and b rational. We say that this linear system is *totally dual integral* (or TDI) if for any integer valued c such that the linear program

$$\text{maximize} \quad \{cx: Ax \leqslant b\}$$

*Research supported in part by the National Research Council of Canada.

†Current address: Department of Mathematics, University of Kentucky, Lexington, Kentucky.

‡Current address: Department of Computer Science, University of Calgary, Calgary, Alberta, Canada.

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has an optimum solution, the corresponding dual linear program has an *integer* optimal solution. This concept was introduced by Edmonds and Giles [4], who showed:

THEOREM 1.1. *If a polyhedron is the solution set of a* TDI *system* $Ax \leqslant b$, *where* b *is integer valued, then every nonempty face of* P *contains an integer point—in particular, every vertex of* P *is integer valued.*

In Sec. 3 we observe that for any rational linear system $Ax \leqslant b$ there exists a rational α such that $(\alpha A)x \leqslant \alpha b$ is a TDI system. Of course, if the polyhedron defined by the system has some nonempty face which contains no integer point, then for any α which makes $(\alpha A)x \leqslant \alpha b$ a TDI system, we will have αb fractional. However, we also prove a converse to the Edmonds-Giles theorem, namely that if P is a polyhedron such that every nonempty face contains an integer point, then there exists a TDI system $Ax \leqslant b$ with b integer such that $P = \{x : Ax \leqslant b\}$. We give a proof of this result which is based on the fact that the set of objective functions maximized over a face of a polyhedron forms a convex cone. In Sec. 2 we present the basic definitions and results on cones which we require. In particular, we give a new short proof of a classical theorem of Hilbert which shows that a rational cone has a finite integer basis (see Theorem 2.1)).

2. RATIONAL CONES

Let D be a finite subset of $\mathbb{R}^J$. The cone $K(D)$ generated by D is the set of all vectors $x \in \mathbb{R}^J$ such that $x = \Sigma(\lambda_d d : d \in D)$, where for each $d \in D$, λ_d is a nonnegative real number. We call K a *rational cone* if $K = K(D)$ for some $D \subseteq \mathbb{R}^J$ such that every member of D is rational. We now prove a classical theorem of Hilbert [7] which shows that for any rational cone K there exists a finite set Z of integer members of K such that every integer $x \in K$ can be expressed as a nonnegative *integer* linear combination of members of Z.

This is not in general true for a cone with irrational generators. For example, consider the cone K in $\mathbb{R}^2$ generated by $(0,1)$ and $(1,z)$, where z is some positive irrational. The line $\alpha \cdot (1,z)$ for $\alpha \geqslant 0$ contains no rational points, and hence for any finite subset Z of integer members of K, there must exist some rational $p \in K - K(Z)$. But then if we multiply p by a sufficiently large integer, we obtain an integer $\hat{p} \in K - K(Z)$, and so $\hat{p}$ is certainly not a positive integer linear combination of members of Z.

THEOREM 2.1. *Let* K *be a rational cone. Then there exists a finite set* Z *of integer members of* K *such that every integer* $x \in K$ *is a nonnegative integer linear combination of members of* Z.

Proof. Let $K=K(D)$. We may assume that D is a set of integer vectors. Let

$$\begin{aligned} Z \equiv \{x \in \mathbb{R}^J : {} & x \text{ is an integer vector,} \\ & x=\Sigma(\lambda_d d : d \in D), \\ & 0 \leqslant \lambda_d \leqslant 1 \text{ for all } d \in D\}. \end{aligned}$$

Then Z is a finite subset of integer vectors in K. For any integer $x \in K$, there exists $(\lambda_d \geqslant 0 : d \in D)$ such that

$$x=\sum(\lambda_d d : d \in D)=\sum(\lfloor\lambda_d\rfloor d : d \in D)+\sum((\lambda_d-\lfloor\lambda_d\rfloor)d : d \in D),$$

where for any real number t, $\lfloor t \rfloor$ denotes the greatest integer no greater than t. Because x and $\Sigma(\lfloor\lambda_d\rfloor d : d \in D)$ are integer vectors, $\Sigma((\lambda_d-\lfloor\lambda_d\rfloor)d : d \in D)$ is an element of Z. Since $D \subseteq Z$, x is a nonnegative integer combination of members of Z. ■

3. INTEGER POLYHEDRA AND TOTAL DUAL INTEGRALITY

Let $P=\{x \in \mathbb{R}^n : Ax \leqslant b\}$ be a polyhedron. We say that P is an *integer polyhedron* if every nonempty face of P contains an integer point. If P is pointed, that is, has at least one vertex, this is equivalent to the assertion that every vertex of P is integer valued. For any nonempty face F of P, there exists a linear objective function c such that cx is maximized over P by precisely the members of F. Conversely, for any linear objective function c such that cx has a maximum over P, there exists a nonempty face F of P such that cx is maximized over P by the members of F. Thus it follows immediately that P is an integer polyhedron if and only if for every linear objective function c such that cx has a maximum over P, there is an integer member of P for which the maximum is attained. Edmonds and Giles [4] prove the following strengthening of this result.

THEOREM 3.1. *Let $P=\{x \in \mathbb{R}^n : Ax \leqslant b\}$ where A and b are rational. Then P is an integer polyhedron if and only if* $\max\{cx : x \in P\}$ *is integer valued for every integer objective function c having a maximum over P.*

The important fact about Theorem 3.1 is the assertion that if P is not an integer polyhedron, then there exists an *integer* objective function which when maximized over P takes on a *fractional* value.

If P has vertices, then this result can be proved using the Gomory cutting-plane algorithm [6], since if this algorithm finds a fractional vertex v of polyhedron P, it generates a cut $ax \leqslant \beta$ where a is integer valued, a is maximized over P at v, and av is not integer valued.

If a polyhedron P is the solution set of a TDI system with integer right hand sides, then for any integer objective function c which has a maximum over P, by linear programming duality this maximum value is an integer. Hence from Theorem 3.1 P is an integer polyhedron, and so Theorem 1.1 follows directly from Theorem 3.1. We now prove two converse results.

THEOREM 3.2. *For any rational linear system $Ax \leqslant b$ there is a positive rational number α such that $\alpha Ax \leqslant \alpha b$ is* TDI.

Proof. It is trivial to show that for a fixed integer valued c there exists an α such that the polyhedron $\{y \in \mathbb{R}^m : y(\alpha A) = c,\ y > 0\}$ is an integer polyhedron, since multiplying A by α has the effect of multiplying all extreme points by $1/\alpha$. However, we must show that it is possible to choose a single α which will work for every possible integer c.

We may assume that $A = (a_{ij} : i \in I,\ j \in J)$ is integer valued. Let N be the set of all $|I| \times |I|$ nonsingular submatrices of the concatenation of A with a $|I| \times |I|$ identity matrix. Let

$$\beta \equiv \left| \prod (\det(B) : B \in N) \right| \quad \text{and} \quad \alpha \equiv 1/\beta.$$

A simple application of Cramer's rule now shows that every component of every basic feasible solution y^* of $y(\alpha A) = c$, $y \geqslant 0$, is integer valued for any integer c. ■

This theorem makes clear the importance of the hypothesis in Theorem 1.1 that b should be integer valued. If $P = \{x : Ax \leqslant b\}$ is not an integer polyhedron, then for any rational α which makes $\alpha Ax \leqslant \alpha b$ a TDI system, we must have αb fractional. However, we show the following.

THEOREM 3.3. *Let $P = \{x : Ax \leqslant b\}$, where A and b are rational. If P is an integer polyhedron, then there exists a* TDI *linear system $A'x \leqslant b'$ with b' integer such that $P = \{x \in \mathbb{R}^J : A'x \leqslant b'\}$.*

Before proving Theorem 3.3 we give some definitions and a lemma. Let $A = (a_{ij} : i \in I,\ j \in J)$ and $b = (b_i : i \in I)$. For any $H \subseteq I$ we let $A[H] \equiv (a_{ij} : i \in H,\ j \in J)$ and $b[H] \equiv (b_i : i \in H)$. For any $i \in I$ we let $A[i]$ denote $(a_{ij} : j \in J)$. Let F be any nonempty face of the polyhedron $P = \{x \in \mathbb{R}^E : Ax \leqslant b\}$. Then there is a unique maximal subset H of I such that $F = \{x \in P : A[H]x = b[H]\}$. We call H the *equality set* of F (relative to the system $Ax \leqslant b$). Finally, we let $C(F)$ be the set of all $c \in \mathbb{R}^J$ such that every $x \in F$ maximizes cx over P.

LEMMA 3.4. *Let F be a nonempty face of $P = \{x : Ax \leqslant b\}$, and let H be the equality set of F. Then, if R is the set of rows of* $A[H]$,

(a) $C(F)=K(R)$ (*that is, $C(F)$ is the cone generated by the rows of A indexed by the members of the equality set of F*);

(b) *for any $c\in C(F)$, if $c=\Sigma(\lambda_h A[h]:h\in H)$ where $\lambda_h\geqslant 0$ for all $h\in H$, then* $\max\{cx:x\in P\}=\Sigma(\lambda_h b_h:h\in H)$.

Proof. If $c\in K(R)$, then $c=\Sigma(\lambda_h A[h]:h\in H)$, where $\lambda_h\geqslant 0$ for all $h\in H$. For any $x\in F$, $cx=\Sigma(\lambda_h A[h]x:h\in H)=\Sigma(\lambda_h b_h:h\in H)$. For any $x\in P$, $cx=\Sigma(\lambda_h A[h]x:h\in H)\leqslant\Sigma(\lambda_h b_h:h\in H)$. Hence $c\in C(F)$. Moreover (b) is established for every $c\in K(R)$.

Conversely, for any $c\in C(F)$ consider the dual linear program to $\max\{cx:x\in P\}$, namely $\min\{by:y\geqslant 0,\ yA=c\}$. There exists an optimal solution y^*, since $c\in C(F)$; and moreover, if $y_i^*>0$, then by complementary slackness, $A[i]x=b_i$ for any $x\in P$ which maximizes cx over P. Hence $A[i]x=b_i$ for all $x\in F$, and so $i\in H$. Therefore y expresses c as a nonnegative linear combination of members of R, so $c\in K(R)$. Thus $C(F)=K(R)$, and the proof is complete. ■

Proof of Theorem 3.3. We may assume that A and b are integer. If c is an objective function that has a maximum value over P, then there is a minimal nonempty face F of P for whose members the maximum is attained, so it will be sufficient to show:

(c) *For any minimal nonempty face F of P there is a linear system $D_F x\leqslant d_F$, with d_F integer, satisfied by every $x\in P$ and such that for any $c\in C(F)$ there exists an integer optimal dual solution to the linear program* $\max\{cx:Ax\leqslant b,\ D_F x\leqslant d_F\}$.

Let F be a minimal nonempty face of P, with equality set H. By Lemma 3.4(a), $C(F)$ is the rational cone generated by the rows of $A[H]$. By Theorem 2.1 there is a finite "integer basis" Z of $C(F)$ such that every $z\in Z$ is an integer and every integer $c\in C(F)$ is a nonnegative integer linear combination of members of Z. For any $z\in Z$ let $\zeta_z=\max\{zx:x\in P\}=z\bar{x}$ for any $\bar{x}\in F$. Since F contains an integer point, ζ_z is integer valued. Moreover every $x\in P$ satisfies $zx\leqslant\zeta_z$. We let $D_F x\leqslant d_F$ be the linear system $(zx\leqslant\zeta_z:z\in Z)$. Now since every integer $c\in C(F)$ is a nonnegative integer linear combination of rows of D_F, (c) follows from (b). Hence a TDI linear system defining P is $(D_F x\leqslant d_F:F$ is a minimal nonempty face of $P)$, which is a finite system because P has a finite number of minimal nonempty faces. ■

For a given integer polyhedron P, a popular problem is that of finding a minimal linear system $Ax\leqslant b$ such that $P=\{x:Ax\leqslant b\}$. In view of Theorem 3.3 a second question that can be asked is the following. What is a minimal TDI system $A'x\leqslant b'$ with integer right-hand sides such that $P=\{x:A'x\leqslant b'\}$? For some classes of polyhedra these two linear systems can be "identical"; we can find a minimal linear system $Ax\leqslant b$ which defines P and which is TDI with integer b. For example, if $G=(V,E)$ is a graph and $b=(b_i : i \in V)$ is a vector of positive integers, the matching polyhedron $P(G,b)$ is defined to be the convex hull of all nonnegative integer vectors $x=(x_j : j \in E)$ such that for each node i of G, the sum of the x_j over the edges of G incident with i is at most b_i. Let $\bar{1}$ be the unit vector indexed by V.

THEOREM 3.5 (Cunningham and Marsh [2]). *The minimal linear system $Ax \leqslant d$ such that $P(G,\bar{1})=\{x : Ax \leqslant d\}$ (scaled in the "natural way" so that A is 0-1 valued) is a TDI system with d integer.*

However, this result does not generalize to matching problems with arbitrary b. In general a TDI linear system with integer right-hand side that defines $P(G,b)$ will be larger than a minimal linear system necessary to define $P(G,b)$.

Other examples of polyhedra for which the two linear systems above can be identical are the intersection of integral polymatroids (see Edmonds [3] and Giles [5]) and the convex hull of the incidence vectors of independent sets of nodes in a perfect graph (see Chvátal [1]).

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