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Some Results on Analytic and Meromorphic Solutions of Algebraic Differential Equations

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1. INTRODUCTION

This paper contains several results concerning the growth of analytic and meromorphic solutions of nth order algebraic differential equations having polynomial coefficients (i.e., equations of the form

$$\Omega(z, y, y', \ldots, y^{(n)}) = 0,$$

where Ω is a polynomial in $z, y, y', \ldots, y^{(n)}$, which is not identically zero).

The paper is divided into three parts. The first part deals mainly with meromorphic functions $y_0(z)$, defined on the plane, which satisfy second-order equations. In the case of first-order algebraic differential equations, it was shown by Valiron [19] that all entire solutions are of finite order of growth. In [12], A. A. Gol'dberg showed that the same conclusion holds for all meromorphic solutions $y_0(z)$ on the plane of first-order equations (i.e., the Nevanlinna characteristic $T(r, y_0)$ for such a solution satisfies $T(r, y_0) = O(r^A)$ as $r \to +\infty$ for some $A \geqslant 0$). It seems that a reasonable conjecture for meromorphic solutions on the plane for second-order equations would be $T(r, y_0) = O(\exp r^A)$ as $r \to +\infty$ for some $A \geqslant 0$. In [1], this conclusion was verified for all meromorphic solutions $y_0(z)$ of second-order equations, which have the property that for two distinct values of λ (finite or infinity), the sequence of roots of the equation $y_0(z) = \lambda$ has a finite exponent of convergence. (This property is equivalent to the condition that for the two values of λ, the counting functions $N(r, \lambda)$ for the roots of $y_0(z) = \lambda$ (see [16, pp. 6, 27]), be $O(r^A)$ as $r \to +\infty$ for some $A > 0$.) In this paper (Section 4 below), we verify the conjecture for those meromorphic solutions of second-order equations which have the property that for two distinct values of λ, the counting functions $N(r, \lambda)$ are $O(\exp r^A)$ as $r \to +\infty$, for some $A > 0$. Of course, this result is a substantial improvement over the result in [1], since it deals with certain solutions y_0 for which the exponent of convergence of the roots of $y_0(z) = \lambda$ is infinite for every λ (e.g., the function, $P(e^z)$, where $P(u)$ is the Weierstrass P-function, is such a solution). The result here follows from a preliminary result (Section 3) which provides an estimate on the growth of certain meromorphic solutions of nth order equations and all solutions of second-order equations in terms of the counting functions for the zeros and the poles. Part of this preliminary result was proved in [1], but the rest requires a very recent result of the author [3] and the theorem of Gol'dberg cited above.

The starting point for the second part of the paper is a theorem which was proved in [2] and which is restated in Section 5 Part (1) below for the reader's convenience. This result deals with solutions $y_0(z)$, of an nth order equation $\Omega = 0$, which are analytic in a region, and which do not satisfy some equation $\Omega_q = 0$, where Ω_q is the homogeneous part of Ω of degree q in the indeterminates $y, y', \ldots, y^{(n)}$. It was shown that for any simply connected region R of analyticity (not the whole plane), in which the zeros of the solution y_0 are "sparse," the solution is majorized on R by a function of the form $B \exp((1 - |f(z)|)^{-A})$, where A and B are constants and f is a univalent, analytic mapping of R onto the unit disk. (The precise notion of R being a "zero-sparse" region for y_0 is that for some univalent, analytic mapping f of R onto the disk, the image under f of the sequence of zeros of y_0 in R has a finite exponent of convergence in the disk [18, p. 7].) By examining the conformal mappings of sectors and semi-infinite strips onto the disk, it was shown [2, p. 95] that if a sector (respectively, a semi-infinite strip) is a zero-sparse region for a solution of the type being considered, then the solution in subsectors (respectively, substrips) is majorized by a function of the form $\exp |z|^B$ (respectively, $\exp(\exp B|z|)$), where B is a constant. In this paper, we continue this investigation. First, we obtain (Section 5 Part (2)) a majorant in zero-sparse regions for solutions of second-order equations which do satisfy each equation $\Omega_q = 0$. (Majorants for solutions of first-order equations were obtained in [4].) Then after some preliminary results about zero-sparse regions (Section 7), we investigate the magnitude of the majorants for certain types of regions. Specifically, we consider regions R_φ, which is the region bounded by the curves $y = \varphi(x)$, $y = -\varphi(x)$, and $x = 1$, where φ is a positive, monotone nonincreasing, convex function on $[1, +\infty)$. Using the Koebe–Faber distortion theorem [9, Vol. 2, p. 68] and a simple partial converse of it, we obtain

(Section 10) precise upper and lower estimates on the magnitude of $(1 - |f(z)|)^{-1}$ along the positive real axis, where f is a specific univalent, analytic mapping of R_φ onto the unit disk. We thus obtain (Section 11) an estimate on the growth of the above-mentioned solutions along the positive real axis if the solution has R_φ for a zero-sparse region. (It is shown in Section 12 that this estimate cannot be greatly improved.) Of course, in the case of entire solutions $\sum a_k z^k$, where $a_k \geqslant 0$, the growth along the positive real axis is the growth of the entire solution and so our results are very well suited to this type of solution.

The final part of the paper deals with solutions in the real domain, of algebraic differential equations having polynomial coefficients, but the main result (Section 15) has application to certain entire solutions. In [8], E. Borel treated solutions which are defined and real-valued on an interval $(x_0, +\infty)$, and he proved [8, p. 27] that any such solution of a first-order equation is majorized by $\exp(\exp x)$ for all sufficiently large x. (This result was later improved by Lindelöf [15] and Hardy [13].) Borel also considered higher-order equations, and he indicated a line of reasoning which would show that such solutions of nth order equations are eventually majorized by $\exp_{n+1} x$ (where $\exp_k x$ is the kth iterate of the exponential function). (We remark here that at the outset [8, p. 26], Borel states that he is considering increasing solutions, but this is not indicated in his proof.) However, as pointed out by several authors (e.g., Fowler [11], Vijayaraghavan [20]), Borel's proof in the higher-order case was incomplete. (One can see evidence of this in the footnote on p. 34 of [8].) In [6] and [20], Vijayaraghavan and others constructed examples to show that second-order equations can possess real-valued solutions which dominate any preassigned function at a sequence of x tending to $+\infty$. However, none of these examples were increasing solutions, and it is not clear whether increasing solutions of second-order equations can have this property. In this paper (Section 14), we use the examples constructed in [6] to show that third-order equations can possess increasing solutions which dominate any preassigned function at a sequence tending to $+\infty$. Finally, we return to the methods of Borel, and we show (Section 15) that under suitable hypothesis on the solutions considered, these methods can be used to obtain a useful necessary condition that certain real-valued increasing functions on $(x_0, +\infty)$ be solutions of second-order algebraic differential equations. The functions to which this result applies are those positive real-valued functions $y_0(x)$ on $(x_0, +\infty)$ such that $y_0(x)/x^\alpha \to +\infty$ for all $\alpha \geqslant 0$ as $x \to +\infty$, and such that $\log y_0(x)$ is an increasing, convex function of $\log x$. Of course, by the Hadamard three circles theorem [14, p. 410] (and Cauchy's estimate), any entire, transcendental function $\sum a_k z^k$, with $a_k \geqslant 0$, has this property on $(0, +\infty)$. It is also interesting to note that if $y_0(x)$ has the properties mentioned above, then by a result of Clunie [10, p. 396], there is an entire transcendental function $g(z) = \sum a_k z^k$, with $a_k \geqslant 0$, such that $\log M(r, g)$ is asymptotically equivalent to $\log y_0(r)$ as $r \to +\infty$.

2. Notation

For $0 < t \leqslant +\infty$, and a meromorphic function $y_0(z)$ in $|z| < t$, we will use the standard notation for the Nevanlinna functions $m(r, y_0)$, $N(r, y_0)$, $N(r, \lambda)$ (where λ is a complex number or ∞) and $T(r, y_0)$ introduced in [16, pp. 6, 12]. We will also use the notation $n(r, y_0)$, for $r < t$, to denote the number of poles (counting multiplicity) of y_0 in $|z| \leqslant r$. We use the abbreviation "n.e." (*nearly everywhere*) to mean "everywhere in $[0, +\infty)$ except in a set of finite measure." If $\Omega(z, y, y', \ldots, y^{(n)})$ is a polynomial in $z, y, \ldots, y^{(n)}$, then for each nonnegative integer q, we denote by Ω_q the homogeneous part of Ω of degree q in the indeterminates $y, y', \ldots, y^{(n)}$. Finally, the notation $\exp_k x$ will mean the kth iterate of the exponential function if $k > 0$, and $\exp_0 x = x$.

Part A: Meromorphic Solutions

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Theorem 1. *Let $\Omega = 0$ be an n-th order algebraic differential equation with polynomial coefficients, and let $y_0(z)$ be a nonconstant meromorphic function on the plane which satisfies $\Omega = 0$. Then:*

(A) *If for some nonnegative integer q, y_0 is not a solution of $\Omega_q = 0$, then for any real number $a > 1$, there exist positive constants K and r_0 such that for all $r > r_0$, we have,*

$$T(r, y_0) \leqslant K[N(ar, y_0) + N(ar, 1/y_0) + \log(ar)]. \tag{1}$$

(B) *If $n = 2$ and y_0 is a solution of some equation $\Omega_q = 0$ (where the polynomial Ω_q is not identically zero), then for any $a > 1$, there* *exist positive constants K, r_0, and b such that for all $r > r_0$,*

$$T(r, y_0) \leqslant K(\exp(r^b) + rN(ar, y_0)). \tag{2}$$

Proof. To prove (A), let us denote by (1′) the inequality (1) when $a = 1$. Noting that $T(r, y_0) \to +\infty$ as $r \to +\infty$, it follows from [1, Theorem 2, p. 792] that for some $K > 0$, inequality (1′) holds n.e. If $a > 1$ and σ is the measure of the exceptional set for (1′), then for any $r > \sigma/(a-1)$, the interval $[r, ar]$ must clearly contain a point at which (1′) holds. Noting that both sides of (1′) are increasing functions of r, it easily follows that (1) holds for all $r > \sigma/(a-1)$.

To prove (B), we suppose $n = 2$ and that y_0 is a solution of some $\Omega_q = 0$. Setting $w_0 = y_0'/y_0$ and dividing the relation

$$\Omega_q(z, y_0(z), y_0'(z), y_0''(z)) \equiv 0$$

by $y_0{}^q$ (and noting that $y_0''/y_0 = w_0' + w_0{}^2$), it easily follows that the meromorphic function w_0 is a solution of a first-order algebraic differential equation having polynomial coefficients. Thus by the theorem of Gol'dberg [12], w_0 is a meromorphic function of finite order and hence may be written $w_0 = f/g$, where f and g are entire functions of finite order. Thus y_0 is a solution of the equation

$$gy' - fy = 0. \tag{3}$$

Now $M(r, f)$ and $M(r, g)$ are each $O(\exp r^c)$ for some $c > 0$ as $r \to +\infty$. Furthermore, by [17, p. 336], except for a set of r-values of finite measure, $|g(z)| \geqslant \exp(-r^{c+1})$ on the circle $|z| = r$. Using these estimates on the coefficients of equation (3), an application of [3, Section 3] now shows that the solution y_0 of (3) must satisfy inequality (2) for some K and r_0 (depending on a), and where b may be taken to be any number greater than $c + 1$.

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COROLLARY. *Let $\Omega = 0$ be a second-order algebraic differential equation having polynomial coefficients. Then if y_0 is any meromorphic function on the plane which satisfies $\Omega = 0$, and for which there are two distinct values of λ (finite or infinity) such that for some $c > 0$, $N(r, \lambda) = O(\exp r^c)$ as $r \to +\infty$, then for some $d > 0$, we have $T(r, y_0) = O(\exp r^d)$ as $r \to +\infty$.*

Proof. We transform to the case where the two values of λ are zero and infinity by using a suitable linear fractional transform u_0 of y_0 in place of y_0. (If λ_1 and λ_2 are the values of λ, set $u_0 = (y_0 - \lambda_2)/(y_0 - \lambda_1)$ if both λ_1 and λ_2 are finite, while if $\lambda_1 = \infty$, set $u_0 = y_0 - \lambda_2$.) Then clearly u_0 also satisfies a second-order algebraic differential equation $\Lambda = 0$ having polynomial coefficients and by [16, p. 14], $T(r, y_0) = T(r, u_0) + O(1)$ as $r \to +\infty$. By applying Theorem 1 to the solution u_0 of $\Lambda = 0$, the result easily follows for y_0.

Part B: Analytic Solutions

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The starting point for this section is the following result, the first part of which was proved in [2].

THEOREM 2. *Let R be a simply connected region in the plane which is not the whole plane, and let f be a univalent analytic mapping of R onto the unit disk. Let $y_0(z)$ be a function which is defined, analytic, and not identically zero on R, and assume $y_0(z)$ is a solution of an n-th order algebraic differential equation $\Omega = 0$ having polynomial coefficients. Let $(b_1, b_2, \ldots)$ be the zeros of y_0 in R arranged in a sequence (multiple zeros appearing as many times as their multiplicity indicates), and assume that for some $a > 0$, $\sum_{m\geqslant 1} (1 - |f(b_m)|)^a < +\infty$. Then:*

(a) *If for some integer $q \geqslant 0$, y_0 is not a solution of the equation $\Omega_q = 0$, then there exist a positive constant A and a compact set J contained in R such that the inequality $|y_0(z)| \leqslant \exp((1 - |f(z)|)^{-A})$ holds on the set $R - J$.*

(b) *If $n = 2$, and if for some positive integer s such that the polynomial Ω_s is not identically zero, the function y_0 is a solution of $\Omega_s = 0$, then there exist a positive constant A and a compact set J contained in R such that the inequality $|y_0(z)| \leqslant \exp_2((1 - |f(z)|)^{-A})$ holds on $R - J$.*

Proof. Part (a) is just [2, Theorem 2, p. 92]. To prove (b), we observe first that by dividing the relation $\Omega_s(z, y_0(z), y_0'(z), y_0''(z)) \equiv 0$ by $(y_0(z))^s$, it follows that $h_0 = y_0'/y_0$ is a solution of a first-order equation,

$$\sum H_{kj}(z)\, h^k (h')^j = 0, \tag{4}$$

where $H_{kj}(z)$ are polynomials. Let g be the inverse of f, and for $|\zeta| < 1$, let $\varphi(\zeta) = h_0(g(\zeta))$. Then from (4), $\varphi(\zeta)$ satisfies the first-order equation

$$\sum F_{kj}(\zeta)(\varphi(\zeta))^k (\varphi'(\zeta))^j \equiv 0, \qquad \text{on} \quad |\zeta| < 1, \tag{5}$$

where

$$F_{kj}(\zeta) = H_{kj}(g(\zeta))/(g'(\zeta))^j \qquad \text{for each } (k, j). \tag{6}$$

For each (k, j) such that $H_{kj} \not\equiv 0$, let $d(k, j)$ be the degree of H_{kj}. Let $q = 1 + \max\{j + 2d(k, j)\colon H_{kj} \not\equiv 0\}$. Then it is proved in [4, Lemma A, p. 575] that for some positive constant K_1, we have for all (k, j) and all $r < 1$,

$$|F_{kj}(\zeta)| \leqslant K_1(1 - r)^{-q} \qquad \text{on} \quad |\zeta| = r. \tag{7}$$

Set $p = \max\{k + j\colon F_{kj} \not\equiv 0\}$ and $m = \max\{j\colon F_{p-j,j} \not\equiv 0\}$. It is proved in [4, Lemma B, p. 576] that there exist constants $K_2 > 0$, $\sigma \geqslant 0$, and r_0 in $[0, 1)$ such that for $r_0 < r < 1$, we have

$$|F_{p-m,m}(\zeta)| \geqslant K_2(1 - r)^{\sigma} \qquad \text{on} \quad |\zeta| = r. \tag{8}$$

We now investigate the poles of $\varphi(\zeta)$. Let $u(\zeta) = y_0(g(\zeta))$ for $|\zeta| < 1$. Then clearly,

$$\varphi(\zeta) \equiv (1/g'(\zeta))(u'(\zeta)/u(\zeta)). \tag{9}$$

Since g' is never zero in $|\zeta| < 1$ and since u is analytic in $|\zeta| < 1$, it follows from (9) that the sequence of poles of φ in $|\zeta| < 1$ is the sequence consisting of the distinct zeros of u in $|\zeta| < 1$. But $(f(b_1), f(b_2),\ldots)$ is the sequence of all zeros of u in $|\zeta| < 1$, and by hypothesis, $\sum_{m\geqslant 1} (1 - |f(b_m)|)^{\alpha} < +\infty$. Thus if $(c_1, c_2, \ldots)$ is the sequence of poles of φ in $|\zeta| < 1$, we have $\sum_{k\geqslant 1} (1 - |c_k|)^{\alpha} < +\infty$. It then follows easily from [16, p. 139] that as $r \to 1$,

$$N(r, \varphi) = O((1 - r)^{-\alpha}) \qquad \text{and} \qquad n(r, \varphi) = O((1 - r)^{-(\alpha+1)}). \tag{10}$$

We now apply [3, Section 3(F)] to the solution φ of Eq. (5). In view of the estimates (7) and (8) on the coefficients of (5) and the estimate on the poles of φ given in (10), it follows from this result that φ is of finite order of growth in the unit disk. By the Koebe distortion theorem [14, p. 351], the analytic function g' is also of finite order in the disk, so by (9), the meromorphic function u'/u is of finite order in the disk.

Hence by [18, p. 11], u'/u can be written as the quotient ψ_1/ψ_2 of two analytic functions of finite order in the disk. Thus the analytic function u in the disk is a solution of the first-order equation, $\psi_2 u' - \psi_1 u = 0$, whose coefficients are analytic functions of finite order in the disk. It follows from [5, Section 2] that there exist $A > 0$ and r_0 in $[0, 1)$ such that $|u(\zeta)| \leqslant \exp_2((1-r)^{-A})$ on $|\zeta| = r$ if $r_0 < r < 1$. Since $y_0(z) = u(f(z))$ for z in R, conclusion (b) now follows immediately if we take J to be the image under g of the compact disk $|\zeta| \leqslant r_0$.

Remark. It is easy to see that the condition in (a) that y_0 not satisfy some equation $\Omega_q = 0$ cannot be omitted if y_0 is to be majorized by the function stated in (a). (See [2, Section 8, Remark (2), p. 96].)

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In view of the previous theorem, we make the following definition:

DEFINITION. Let R be a simply connected region which is not the whole plane, and let $y_0(z)$ be defined, analytic, and not identically zero on R. Then R is called a *zero-sparse region* for y_0 if for some univalent analytic mapping f of R onto the unit disk, the sequence $(b_1, b_2, \ldots)$ of zeros (counting multiplicity) of y_0 in R satisfies the condition

$$\sum_{m \geqslant 1} (1 - |f(b_m)|)^a < +\infty, \tag{11}$$

for some $a > 0$. If y_0 has no zeros in R, we say R is a *zero-free* region for y_0.

We now show that this definition is independent of the mapping used.

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LEMMA. *Let R be a simply connected region which is not the whole plane. Then,*

(A) *If f and h are univalent analytic mappings of R onto the unit disk, then there is a constant $K > 0$ such that $1 - |f(z)| \geqslant K(1 - |h(z)|)$ for all z in R.*

(B) *If R is a zero-sparse region for a function y_0, then every univalent analytic mapping of R onto the unit disk has property* (11).

(C) *If R is a zero-sparse region for y_0, and R_1 is a simply-connected region contained in R, then R_1 is also a zero-sparse region for y_0.*

Proof. Part (A). The function $L = f \circ h^{-1}$ is a linear fractional transformation of the form

$$L(\zeta) = e^{i\theta}((\zeta - b)/(1 - \bar{b}\zeta)) \tag{12}$$

for some real θ and $|b| < 1$. By a well-known elementary inequality [9, Vol. 1, p. 13], for $|\zeta| < 1$,

$$|L(\zeta)| \leqslant (|\zeta| + |b|)/(1 + |b||\zeta|) \leqslant 1. \tag{13}$$

Thus for z in R, $|f(z)| \leqslant (|h(z)| + |b|)/(1 + |b||h(z)|)$, from which Part (A) immediately follows with $K = (1 - |b|)/2$.

Part (B). This follows immediately from Part (A).

Part (C). Let z_0 be a point of R_1. By the Riemann mapping theorem [9, Vol. 2, p. 62], there exist univalent analytic mappings f and h of R and R_1, respectively, onto the unit disk such that $f(z_0) = h(z_0) = 0$. Then $v = f \circ h^{-1}$ maps the disk into itself and $v(0) = 0$. Thus by Schwarz's lemma, $|v(\zeta)| \leqslant |\zeta|$ for all $|\zeta| < 1$. Thus $|f(z)| \leqslant |h(z)|$ on R_1. Since the sequence of zeros of y_0 in R_1 is either finite or a subsequence of the sequence of zeros of y_0 in R, and since f has property (11) for some $a > 0$ by Part (B), it thus follows that R_1 is a zero-sparse region for y_0.

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DEFINITION. Let $\varphi(x)$ be a twice differentiable function on $[1, +\infty)$ such that $\varphi > 0$, $\varphi' \leqslant 0$, and $\varphi'' \geqslant 0$ on $[1, +\infty)$. (We call such a function *admissible.*) We denote by R_φ the region bounded by the line $x = 1$ and the curves $y = \varphi(x)$ and $y = -\varphi(x)$ for $x \geqslant 1$.

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LEMMA. *Let φ be admissible. For each $r > 1$, let $\Delta(r)$ denote the distance from r to the boundary of R_φ. Then:*

(a) $\Delta(r) \leqslant \varphi(r)$ *for all* $r > 1$;

(b) *for any* $\epsilon > 0$, *there exists* $r_0(\epsilon) > 1$ *such that for all* $r > r_0(\epsilon)$, *we have*

$$\Delta(r) \geqslant \varphi(r - \varphi(r)\,\varphi'(r)) \geqslant \varphi(r + \epsilon). \tag{14}$$

Proof. Part (a) is obvious. To prove Part (b), let $\Gamma = \{(x, \varphi(x)) : x \geqslant 1\}$. Since φ is monotone nonincreasing, clearly there exists $r_1 > 1$ such that $r - 1 > \varphi(r)$ for all $r > r_1$. Hence for $r > r_1$, $\Delta(r)$ is the distance from r to Γ. Thus if $r > r_1$, then $\Delta(r)$ is the infimum, over all $x \geqslant 1$, of the function $u(x; r) = ((x - r)^2 + (\varphi(x))^2)^{1/2}$. For fixed $r > r_1$, clearly there exists $x_0(r)$ such that

$$u(x; r) > \varphi(r) = u(r; r) \qquad \text{for} \quad x \geqslant x_0(r). \tag{15}$$

In view of (15), it follows that $r < x_0(r)$ and that $\Delta(r)$ is the infimum, over all x in the interval $[1, x_0(r)]$, of $u(x; r)$. If $x_1 = x_1(r)$ is a value of x in this interval at which the infimum is assumed, it follows from (15) that $x_1 \neq x_0(r)$. But since $u(1; r) > \varphi(1) \geqslant \varphi(r) = u(r; r)$, it is also clear that $x_1 \neq 1$. Thus x_1 is an interior point of the interval, and hence by elementary calculus, $\partial u(x; r)/\partial x$ must vanish at the point x_1. From this we see that $x_1 = r - \varphi(x_1)\,\varphi'(x_1)$ and therefore $x_1 \geqslant r$. Since $\varphi\varphi'$ has a nonnegative derivative, we obtain $x_1 \leqslant r - \varphi(r)\,\varphi'(r)$, and hence,

$$\Delta(r) = u(x_1; r) \geqslant \varphi(x_1) \geqslant \varphi(r - \varphi(r)\,\varphi'(r)) \qquad \text{for} \quad r > r_1. \tag{16}$$

Since φ is monotone nonincreasing, φ tends to a nonnegative limit as $x \to +\infty$. Since φ' is monotone nondecreasing and nonpositive, φ' tends to a limit $\sigma \leqslant 0$ as $x \to +\infty$. If $\sigma < 0$, then simple integration would show $\varphi \to -\infty$ as $x \to +\infty$ which is not true. Thus $\sigma = 0$ and hence $\varphi\varphi' \to 0$ as $x \to +\infty$. Thus for any $\epsilon > 0$, there exists $r_0(\epsilon) > r_1$ such that $\varphi(r)\,\varphi'(r) \geqslant -\epsilon$ for $r > r_0(\epsilon)$. From (16), we then obtain (14), proving Part (b).

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LEMMA. *Let* φ *be admissible, and let* r_0 *be the number* $r_0(\epsilon)$ *in Section* 9(b) *corresponding to* $\epsilon = 1$. *Let* f *be the univalent analytic mapping of* R_φ *onto the unit disk such that* $f(r_0) = 0$ *and* $f'(r_0) > 0$. *Then for all* $r > r_0$ *we have* $0 < f(r) < 1$, *and furthermore,*

(a) $(1 - f(r))^{-1} \leqslant \exp(2r/\varphi(r+1))$ *for* $r > r_0$, *and*

(b) $(1 - f(r))^{-1} \geqslant (1/2)\exp(1/(2\varphi(r-1)))$ *for* $r > r_0 + 1$.

Proof. Let g be the inverse of f, and for z in R_φ, let $\Delta(z)$ denote the distance from z to the boundary of R_φ. By the Koebe–Faber distortion theorem [9, Vol. 2, p. 68], we have

$$| g'(\zeta)| \leqslant 4\Delta(g(\zeta))/(1 - |\zeta|^2) \qquad \text{for} \quad |\zeta| < 1. \tag{17}$$

We also claim that

$$| g'(\zeta)| \geqslant \Delta(g(\zeta))/(1 - |\zeta|^2) \qquad \text{for} \quad |\zeta| < 1. \tag{18}$$

To prove (18), let $\sigma = \Delta(g(\zeta))$. Then for $|w| < 1$, the point $g(\zeta) + w\sigma$ lies in R_φ so that $h(w) = f(g(\zeta) + w\sigma)$ is an analytic function from the disk into the disk and $h(0) = \zeta$. Let $L(u) = (u - \zeta)/(1 - \bar{\zeta}u)$ and let $\psi = L \circ h$. Then ψ maps the disk into the disk and $\psi(0) = 0$. Thus by Schwarz's lemma, $|\psi'(0)| \leqslant 1$ from which (18) follows easily.

Now R_φ is symmetric with respect to the real axis, so that $v(z) = \overline{f(\bar{z})}$ is also a univalent analytic mapping of R_φ onto the disk with $v(r_0) = 0$ and $v'(r_0) > 0$. By uniqueness of the map [9, Vol. 2, p. 62], $v \equiv f$ so that $f(r)$ is real for $r > 1$. Thus f' is also real on $(1, +\infty)$, and since $f'(r_0) > 0$ and f' is nowhere zero, we have

$$f'(r) > 0 \qquad \text{for} \quad r > 1. \tag{19}$$

Thus f is strictly increasing for $r > 1$, so

$$0 < f(r) < 1 \qquad \text{for} \quad r > r_0. \tag{20}$$

Now let $r > 1$. Evaluating (17) and (18) at $\zeta = f(r)$, and using (19) (and the fact that $f(r)$ is real), it follows that

$$(4\Delta(r))^{-1} \leqslant f'(r)/(1 - f(r)^2) \leqslant \Delta(r)^{-1} \qquad \text{for} \quad r > 1. \tag{21}$$

Let $F(r) = (1/2)\log((1 + f(r))/(1 - f(r)))$. Then F is a primitive of $f'/(1 - f^2)$ and $F(r_0) = 0$, so that from (21), we obtain for $r \geqslant r_0$,

$$\int_{r_0}^{r} (4\Delta(s))^{-1}\, ds \leqslant F(r) \leqslant \int_{r_0}^{r} (\Delta(s))^{-1}\, ds. \tag{22}$$

Now by definition of r_0, for $r_0 \leqslant s \leqslant r$,

$$\Delta(s) \geqslant \varphi(s+1) \geqslant \varphi(r+1). \tag{23}$$

Also, if $r \geqslant r_0 + 1$, then the left side of (22) is $\geqslant \int_{r-1}^{r} (4\Delta(s))^{-1}\, ds$, which is $\geqslant (4\varphi(r-1))^{-1}$ since for $r - 1 \leqslant s \leqslant r$, we have $\Delta(s) \leqslant \varphi(s) \leqslant \varphi(r-1)$ by Section 9(a). Using this estimate and (23) in (22), we obtain

$$(4\varphi(r-1))^{-1} \leqslant F(r) \leqslant r/\varphi(r+1), \tag{24}$$

the first inequality holding for $r \geqslant r_0 + 1$ and the second holding for $r \geqslant r_0$. Multiplying (24) by two and taking the exponential (and using $1 < 1 + f(r) < 2$ from (20)), we obtain the conclusions (a) and (b) of the lemma.

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THEOREM 3. *Let φ be admissible. Let $y_0(z)$ be defined, analytic, and not identically zero on R_φ, and assume $y_0(z)$ is a solution of an n-th order algebraic differential equation $\Omega = 0$ having polynomial coefficients. Suppose R_φ is a zero-sparse region for y_0. Then:*

(a) *If for some integer $q \geqslant 0$, y_0 is not a solution of the equation $\Omega_q = 0$, then there exist constants $B \geqslant 0$ and $r_1 > 1$ such that for all $r > r_1$, we have*

$$|y_0(r)| \leqslant \exp_2(Br/\varphi(r+1)). \tag{25}$$

(b) *If $n = 2$, and if for some positive integer s such that the polynomial Ω_s is not identically zero, the function y_0 is a solution of $\Omega_s = 0$, then there exist constants $B \geqslant 0$ and $r_1 > 1$ such that for all $r > r_1$, we have $|y_0(r)| \leqslant \exp_3(Br/\varphi(r+1))$.*

Proof. In view of Section 7(B), the univalent analytic mapping f investigated in Section 10 can be used in Section 5, and hence the theorem follows readily from Sections 5 and 10.

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Remark. It is easy to see that the estimate (25) in Part (a) of Theorem 3 cannot be greatly improved. For example, if such a solution y_0 has the semi-infinite strip R_φ, where φ is a constant function, as a zero-sparse region, then the estimate (25) takes the form $|y_0(r)| \leqslant \exp_2(B_1 r)$, where B_1 is a constant. Now the function $y_0(z) = \exp_2 z - 1$

has R_φ, where $\varphi(x) \equiv \pi/2$, for a zero-free region and y_0 is a solution satisfying Part (a), and in this case $|y_0(r)| = \exp_2 r - 1$ for $r > 0$. Similarly, when φ is of the form $\varphi(x) \equiv Ke^{-x}$ (where K is a constant), the estimate (25) gives $|y_0(r)| \leqslant \exp_2(B_1 re^{r+1})$. The solution $y_0(z) = \exp_3 z - 1$ satisfies Part (a) and has the region R_φ, where $\varphi(x) = (\pi/2)\, e^{-x}$, for a zero-free region, and in this case $|y_0(r)| = \exp_3 r - 1$. (Of course the estimate (25) shows that the solution $\exp_3 z - 1$ cannot have a zero-sparse region of the form R_φ where $\varphi(x) = Kx^{-m}$, where m is a nonnegative integer.)

13. Application to Certain Entire Solutions

In the special case of entire transcendental solutions, $y_0(z) = \sum a_k z^k$, with $a_k \geqslant 0$, we have $|y_0(r)| = M(r, y_0)$, so the estimate (25) is an estimate on the growth of the maximum modulus. Every such solution clearly has a zero-free region containing $[1, +\infty)$. Now a reasonable conjecture for the growth of entire solutions y_0 of nth order equations with polynomial coefficients is $M(r, y_0) = O(\exp_n r^A)$ for some $A > 0$. Our estimate (25) shows that this conjecture holds for those entire solutions $\sum a_k z^k$, with $a_k \geqslant 0$, of an nth order equation $\Omega = 0$ having polynomial coefficients, which fail to satisfy some equation $\Omega_q = 0$, and which have a zero-sparse region R_φ, where $\varphi(r) \geqslant (\exp_{n-2} r^d)^{-1}$ for some $d > 0$ and all sufficiently large r. In the case of entire solutions of second-order equations $\Omega = 0$, which do satisfy some nontrivial equation $\Omega_q = 0$, it follows from Section 3(b) that $M(r, y_0) = O(\exp_2 r^A)$ for some $A > 0$. Thus in this case, the conjecture is verified without regard to the size of the zero-sparse region of the solution.

Part C: Solutions in the Real Domain

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Theorem 4. *Let $\Phi(x)$ be an increasing function on $(0, +\infty)$ such that $\Phi \to +\infty$ as $x \to +\infty$. Then there exists a positive irrational number α such that if $u(x) = (2 - \cos x - \cos \alpha x)^{-1}$ and $y_0(x) = \int_1^x u(t)\, dt$, then y_0 is a positive, increasing, infinitely differentiable function on $(1, +\infty)$, which satisfies a third-order algebraic differential equation with polynomial coefficients, and which has the property that $y_0(x) > \Phi(x)$ at a sequence of real x tending to $+\infty$.*

Proof. It is clear that for any irrational α, the function y_0 is positive, increasing, and infinitely differentiable on $(1, +\infty)$. It is shown in [6] (and also in [7, p. 97]) that u satisfies a second-order equation with polynomial coefficients so it follows that y_0 satisfies a third-order equation. It remains to show that α can be chosen so that $y_0 > \Phi$ at a sequence tending to $+\infty$. Let $\varphi = 2\Phi$. We construct the same α as was constructed in [6] where it was shown that $u > \varphi$ at a sequence tending to $+\infty$, but it will require a deeper analysis to show that $y_0 > \Phi$ at such a sequence. Set $q_0 = 1$, and let $\{d_n\}$ be a strictly increasing sequence of positive integers greater than one such that $d_r > 4\pi\varphi(2\pi q_{r-1})$, for $r = 1, 2,\ldots$, where $q_r = d_1 d_2 \cdots d_r$ for $r = 1, 2,\ldots$. We set $\alpha = \sum_{r=1}^{\infty} (1/q_r)$ and $\delta_n = q_n \sum_{r=n+1}^{\infty} (1/q_r)$. We may write $\sum_{r=1}^{n} (1/q_r) = p_n/q_n$, where p_n is a positive integer. Thus we have

$$\delta_n = \alpha q_n - p_n \qquad \text{for} \quad n = 1, 2,\ldots, \tag{26}$$

and it is proved in [7, p. 98] and [6, p. 252] that α is irrational and that for all n greater than some n_0,

$$0 < 2\pi\delta_n < 1/\varphi(2\pi q_n). \tag{27}$$

Hence, we see that

$$\delta_n \to 0 \qquad \text{as} \quad n \to +\infty. \tag{28}$$

Since $d_r \geqslant r + 1$, it follows that $q_r \geqslant (r + 1)!$, and hence

$$0 < \alpha \leqslant e - 2 < 1. \tag{29}$$

We will require the standard inequalities

$$\sin x \leqslant x \qquad \text{and} \qquad 1 - \cos x \leqslant x^2 \qquad \text{for} \quad x \geqslant 0. \tag{30}$$

By (26), for all n greater than n_0, we have $\cos(2\pi\alpha q_n) = \cos(2\pi\delta_n)$, and hence by (30) and (27),

$$1 - \cos(2\pi\alpha q_n) < 2\pi\delta_n/\varphi(2\pi q_n) \qquad \text{for} \quad n > n_0. \tag{31}$$

For n sufficiently large, say $n > n_1$ (where $n_1 > n_0$), the right side of (31) is <1 (by (27)) and hence $\cos(2\pi\alpha q_n) > 0$ for $n > n_1$. Thus for each $n > n_1$, there is an integer $m(n)$ such that

$$2\pi m(n) - (\pi/2) < 2\pi\alpha q_n < 2\pi m(n) + (\pi/2). \tag{32}$$

Thus $| \alpha q_n - m(n)| < 1/4$ for $n > n_1$. But by (26), $| \alpha q_n - p_n | = \delta_n$, and by (27), there is an $n_2 > n_1$ such that $\delta_n < (1/4)$ for $n > n_2$. Thus, for $n > n_2$, $| m(n) - p_n | < 1/2$, so since both $m(n)$ and p_n are integers, we have

$$m(n) = p_n \qquad \text{for} \quad n > n_2 . \tag{33}$$

We now let J_n be the interval $[q_n - \delta_n , q_n]$. For t in J_n, it follows from (26) and (29) that $2\pi\alpha t \geqslant 2\pi p_n$, and from (32) and (33) it follows that

$$2\pi\alpha t \leqslant 2\pi\alpha q_n \leqslant 2\pi p_n + (\pi/2) \qquad \text{for} \quad n > n_2 . \tag{34}$$

But on the interval $[2\pi p_n , 2\pi p_n + (\pi/2)]$, $\cos x$ is decreasing. Hence from (34), we obtain $\cos(2\pi\alpha t) \geqslant \cos(2\pi\alpha q_n)$, and thus by (31), if $n > n_2$,

$$1 - \cos(2\pi\alpha t) < 2\pi\delta_n/\varphi(2\pi q_n) \qquad \text{for} \quad t \quad \text{in} \quad J_n . \tag{35}$$

Since $\cos(2\pi q_n) = 1$, we have by the law of the mean, if t belongs to J_n,

$$1 - \cos(2\pi t) = (-\sin x)(2\pi q_n - 2\pi t), \tag{36}$$

where $2\pi t < x < 2\pi q_n$. Now $t \geqslant q_n - \delta_n$, and since $\delta_n < 1/4$ for $n > n_2$, we thus have for t in J_n and $n > n_2$,

$$2\pi q_n - (\pi/2) < 2\pi(q_n - \delta_n) < x < 2\pi q_n . \tag{37}$$

Now on the interval $(2\pi q_n - (\pi/2), 2\pi q_n)$, the sin function is negative and increasing. Thus from (37), we have $\sin(2\pi(q_n - \delta_n)) < \sin x$. Since q_n is an integer, we thus obtain, $-\sin x < \sin(2\pi\delta_n)$. In view of (30) and (27), we therefore see that for t in J_n and $n > n_2$,

$$1 - \cos(2\pi t) < 2\pi\delta_n/\varphi(2\pi q_n). \tag{38}$$

In view of (35) and (38), we thus have for t in J_n and $n > n_2$, $(u(2\pi t))^{-1} \leqslant 4\pi\delta_n/\varphi(2\pi q_n)$, and hence,

$$u(2\pi t) \geqslant \varphi(2\pi q_n)/4\pi\delta_n \qquad \text{for} \quad t \quad \text{in} \quad J_n \quad \text{and} \quad n > n_2 . \tag{39}$$

Now for $n > n_2$, we have $\delta_n < 1/4$ and $q_n \geqslant 2$, so $q_n - \delta_n > 1$. Thus, from (39), we have for $n > n_2$,

$$y_0(2\pi q_n) = \int_{(2\pi)^{-1}}^{q_n} 2\pi u(2\pi t)\, dt > \int_{q_n - \delta_n}^{q_n} 2\pi u(2\pi t)\, dt \geqslant \varphi(2\pi q_n)/2, \tag{40}$$

and hence $y_0 > \Phi$ at the sequence $2\pi q_n$ for $n > n_2$. This proves the theorem.

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THEOREM 5. *Let* $\Omega(x, y, y', y'') = \sum f_{ijk}(x)\, y^i(y')^j(y'')^k$ *be a nonzero polynomial in* x, y, y', *and* y''. *Let* $y_0(x)$ *be a positive solution of* $\Omega = 0$ *on an interval* $(x_0, +\infty)$ *where* $x_0 > 0$ *and let* y_0 *have the following two properties*: (i) *For every* $\alpha \geqslant 0$, $y_0/x^\alpha \to +\infty$ *as* $x \to +\infty$, *and* (ii) $\log y_0(x)$ *is an increasing convex function of* $\log x$ *on* $(x_0, +\infty)$. *Let* $v_0(x) = xy_0'(x)/y_0(x)$. *Then either there is a constant* $b \geqslant 0$ *such that* $y_0 = O(\exp x^b)$ *as* $x \to +\infty$, *or there is a constant* $b \geqslant 0$ *such that* $v_0'/v_0 = O(x^b)$ n.e. *as* $x \to +\infty$.

Proof. By assumption, $\varphi(t) = \log y_0(e^t)$ is increasing and convex on $(\log x_0, +\infty)$, so $\varphi' \geqslant 0$ and $\varphi'' \geqslant 0$ on this interval. Clearly, $y_0(x)$ is increasing on $(x_0, +\infty)$ and we see clearly from the hypothesis of the theorem that $y_0 \to +\infty$ when $x \to +\infty$. Now for any $\epsilon > 0$, $\int_{x_0+1}^{+\infty} y_0'/y_0^{1+\epsilon}$ clearly converges, so since $y_0' \geqslant 0$ it follows easily that for any $\epsilon > 0$, there is a set I_ϵ in $(x_0, +\infty)$ of finite measure such that for $x > x_0$ and x not in I_ϵ, we have

$$0 \leqslant y_0'(x)/(y_0(x))^{1+\epsilon} \leqslant 1. \tag{41}$$

Since $\varphi'(t) = v_0(e^t)$, it follows from assumption (ii) that $v_0(x) \geqslant 0$ and $v_0'(x) \geqslant 0$ on $(x_0, +\infty)$. We may assume that $v_0 \to +\infty$ as $x \to +\infty$, or otherwise (i.e., if v_0 is bounded on $(x_0, +\infty)$), the theorem follows easily since integration would yield $y_0 = O(x^a)$ for some $a > 0$ as $x \to +\infty$. Thus $xy_0' = y_0 v_0$ is increasing and $\to +\infty$ as $x \to +\infty$. We now assert that for any $\epsilon > 0$, there is a set J_ϵ in $(x_0, +\infty)$ of finite measure such that if $x > x_0$ and x is not in J_ϵ, we have

$$0 \leqslant y_0''(x) \leqslant x^\epsilon (y_0(x))^{1+\epsilon}. \tag{42}$$

To prove (42), let $\delta > 0$ be such that $(1 + \delta)^2 = 1 + \epsilon$. Now $u(x) = xy_0'$ is increasing and $\to +\infty$, so as in (41), the inequality $u' \leqslant u^{1+\delta}$ holds n.e. Since $y_0' \geqslant 0$, it follows that $y_0'' \leqslant x^\delta (y_0')^{1+\delta}$ n.e. But by (41), $y_0' \leqslant y_0^{1+\delta}$ n.e., so since $\delta < \epsilon$, we have $y'' \leqslant x^\epsilon y_0^{1+\epsilon}$ n.e.

To complete the proof of (42), we must show $y'' \geqslant 0$ n.e. But $\varphi''(t) \geqslant 0$ yields

$$y_0''(x) \geqslant x^{-2} y_0(x)(v_0(x)^2 - v_0(x)) \qquad \text{for} \quad x > x_0, \tag{43}$$

and since $v_0(x) \to +\infty$, the result follows.

We return to the equation $\Omega = 0$, and let $p = \max\{i + j + k : f_{ijk} \not\equiv 0\}$. By isolating the terms of degree p and dividing the relation $\Omega(x, y_0(x), y_0'(x), y''(x)) \equiv 0$ by $(y_0(x))^p$, we obtain

$$\sum_{i+j+k=p} f_{ijk}(y_0'/y_0)^j \, (y_0''/y_0)^k = -\Phi(x), \tag{44}$$

where $\Phi(x) = \sum_{i+j+k<p} h_{ijk}$, and where

$$h_{ijk} = f_{ijk}(y_0')^j \, (y_0'')^k \, y_0^{i-p} \qquad \text{for} \quad i + j + k < p. \tag{45}$$

We assert that for every $\alpha > 0$,

$$x^\alpha \Phi(x) = O(1) \qquad \text{n.e. as} \quad x \to +\infty. \tag{46}$$

To prove (46), it clearly suffices to prove that for every $\alpha > 0$ and each (i, j, k) with $i + j + k < p$, we have

$$x^\alpha h_{ijk} = O(1) \qquad \text{n.e. as} \quad x \to +\infty. \tag{47}$$

Since the f_{ijk} are polynomials, there exists $c' > 0$ such that

$$f_{ijk} = O(x^{c'}) \qquad \text{as} \quad x \to +\infty, \quad \text{for each} \quad (i, j, k). \tag{48}$$

If $j = k = 0$, then (47) follows from (i) of the hypothesis. For the case when $j + k > 0$, we observe that we may write

$$h_{ijk} = f_{ijk}(y_0'/y_0^{1+\epsilon_1})^j \, (y_0''/y_0^{1+\epsilon_1})^k, \tag{49}$$

where

$$\epsilon_1 = (p - (i + j + k))/(j + k) > 0. \tag{50}$$

Let $\epsilon > 0$ be smaller than each of the numbers ϵ_1 in (50). For this $\epsilon > 0$, the inequalities (41) and (42) hold n.e., and thus (using (48) and (49)), we have n.e.,

$$| h_{ijk}(x) | \leqslant K x^{c' + \epsilon k} / (y_0(x))^{(j+k)(\epsilon_1 - \epsilon)} \qquad \text{for some} \quad K > 0. \tag{51}$$

Since $j + k > 0$ and $\epsilon < \epsilon_1$, (47) now follows immediately from (51) and (i) of the hypothesis, thus proving (46).

Returning now to Eq. (44) and noting that $y_0'/y_0 = v_0/x$ and $y_0''/y_0 = x^{-2}(v_0^2 + xv_0' - v_0)$, we may write (44) in the form

$$\sum g_{mn}(x)\, v_0^m (v_0')^n = -x^d \Phi(x), \tag{52}$$

where the g_{mn} are polynomials and d is a positive integer. It is easy to see that not all g_{mn} can be identically zero, and we set

$$q = \max\{m + n : g_{mn} \not\equiv 0\} \qquad \text{and} \qquad \sigma = \max\{n : g_{q-n,n} \not\equiv 0\}.$$

Also, let N be an integer greater than the number of nonzero g_{mn}. In view of (46), there is a constant $K > 0$ such that n.e., $x^{d+1} \mid \Phi(x) \mid \leqslant K$. This shows that $q > 0$, for if $q = 0$ then (52) would imply $\mid g_{00}(x) \mid \leqslant K/x$ n.e., which is certainly impossible since the function g_{00} is a polynomial which is not identically zero. We now isolate the terms of degree q on the left side of (52), and divide (52) by v_0^q. Since $v_0 \geqslant 0$ and $v_0' \geqslant 0$, we obtain n.e.,

$$\mid \Gamma(x) \mid \leqslant \sum_{m+n<q} \mid g_{mn} \mid (v_0')^n\, v_0^{m-q} + Kx^{-1} v_0^{-q}, \tag{53}$$

where $\Gamma(x) = \sum_{n=0}^{\sigma} g_{q-n,n} (v_0'/v_0)^n$. Now $v_0 \to +\infty$ as $x \to +\infty$, and $v_0' \geqslant 0$, so it follows as in (41) that for any $\epsilon > 0$,

$$v_0' \leqslant v_0^{1+\epsilon} \text{ n.e.} \tag{54}$$

Looking at the right side of (53), we have if $n > 0$,

$$(v_0')^n\, v_0^{m-q} = (v_0'/v_0^{1+\epsilon_1})^n \qquad \text{where} \quad \epsilon_1 = (q - (m + n))/n > 0. \tag{55}$$

Let ϵ be a positive number which is less than one-half of each of the numbers ϵ_1 in (55) and which is also less than 1. Since the g_{mn} are polynomials, there exists $c > 0$ such that for all (m, n),

$$\mid g_{mn}(x) \mid \leqslant x^c \qquad \text{for all sufficiently large } x. \tag{56}$$

In view of (54) applied to ϵ, (55), and (56), it follows from (53) that n.e.,

$$\mid \Gamma(x) \mid \leqslant \sum_1 x^c v_0^{-\epsilon n} + \sum_2 x^c v_0^{m-q} + Kx^{-1} v_0^{-q}, \tag{57}$$

where $\sum_1$ is taken over all (m, n) such that $m + n < q$, $n > 0$ and

$g_{mn} \not\equiv 0$, while $\sum_2$ is taken over all (m, n) such that $m + n < q$, $n = 0$, and $g_{mn} \not\equiv 0$. We now distinguish two cases:

Case 1. $\sigma = 0$. Then $\Gamma(x) = g_{q0}$ which is a polynomial which is not identically zero. Thus, there exists $K_2 > 0$ such that for all sufficiently large x,

$$| \Gamma(x)| = | g_{q0}(x)| > K_2 . \tag{58}$$

Since $c > 0$ and $v_0 \to +\infty$, we have for all sufficiently large x,

$$v_0(x) > 1, \qquad Nx^c/K_2 > 1 \qquad \text{and} \qquad x^{c+1} > K. \tag{59}$$

Let x be a point at which (57), (58), and (59) hold. There are at most N individual terms on the right side of (57). In view of (57) and (58), it is clearly impossible that at x, each term on the right of (57) be $<K_2/N$. Thus some term (depending on x, of course) is $\geqslant K_2/N$. If it is a term from $\sum_1$, then $x^c(v_0(x))^{-\epsilon n} \geqslant K_2/N$, so since $n \geqslant 1$, it follows using (59) that

$$v_0(x) \leqslant (N/K_2)^{1/\epsilon}\, x^{c/\epsilon}. \tag{60}$$

If it is a term from $\sum_2$, say $x^c(v_0(x))^{m-q} \geqslant K_2/N$, then since $q - m \geqslant 1$, it follows using (59) that we again obtain (60) since $\epsilon < 1$. Finally, if it is the last term of (53), we have $Kx^{-1}(v_0(x))^{-q} \geqslant K_2/N$, and we again obtain (60) using (59) and the fact that $\epsilon < 1$ and $q \geqslant 1$. Thus (60) holds for all x for which (57), (58), and (59) are valid. Thus (60) holds n.e. If λ is the measure of the exceptional set for (60), then for any $x > \lambda$, the interval $[x, 2x]$ must contain a point t for which (60) holds. Since both sides of (60) are monotone nondecreasing, we thus see that for all $x > \max(\lambda, x_0)$, we have

$$v_0(x) \leqslant K_3 x^{c/\epsilon}, \qquad \text{where} \quad K_3 = (N/K_2)^{1/\epsilon}\, 2^{c/\epsilon}. \tag{61}$$

Since $v_0 = xy_0'/y_0$, a simple integration now shows that $y_0 = O(\exp x^b)$ as $x \to +\infty$, where b is any number larger than c/ϵ. Thus the theorem is proved in Case 1.

Case 2. $\sigma \geqslant 1$. Again, since $v_0 \to +\infty$ and $c > 0$, we have for all sufficiently large x,

$$v_0(x) > 1, \qquad x^{c+1} > K \qquad \text{and} \qquad x > 1. \tag{62}$$

Since $g_{q-\sigma,\sigma}$ is a polynomial which is not identically zero, there exists $K_3 > 0$ such that

$$|g_{q-\sigma,\sigma}(x)| \geqslant K_3 \qquad \text{for all sufficiently large } x. \tag{63}$$

Finally, since $N \geqslant 1$, we have that

$$K_4 = \min(K_3, (\sigma + 1)N) > 0. \tag{64}$$

We now assert that for all x for which (56), (57), (62), and (63) hold, we have

$$|v_0'(x)/v_0(x)| \leqslant ((\sigma + 1)N/K_4)\, x^c. \tag{65}$$

To prove (65), let $x > x_0$ be a point at which (56), (57), (62), (63) hold. From (57) and (62), we have

$$|\Gamma(x)| \leqslant Nx^c. \tag{66}$$

Now, we may write

$$\Gamma(x) = g_{q-\sigma,\sigma}(x)(v_0'(x)/v_0(x))^\sigma \left(1 + \sum_{n=0}^{\sigma-1} \Psi_n(x)\right), \tag{67}$$

where $\Psi_n(x) = (g_{q-n,n}(x)/g_{q-\sigma,\sigma}(x))(v_0'(x)/v_0(x))^{n-\sigma}$ for $n < \sigma$. Let us define a set A as follows:

$$A = \{r : r > x_0,\ r^c(\sigma + 1)/K_3 < |v_0'(r)/v_0(r)|^{\sigma-n} \text{ for } n = 0, 1,..., \sigma - 1\}. \tag{68}$$

Now if x belongs to A, then in view of (56) and (63), we clearly have $|\Psi_n(x)| < 1/(\sigma + 1)$ for $n = 0, 1,..., \sigma - 1$, and hence from (67) (and (64)), we have

$$|\Gamma(x)| > (K_4/(\sigma + 1))\, |v_0'(x)/v_0(x)|^\sigma. \tag{69}$$

Together with (66), we obtain

$$|v_0'(x)/v_0(x)| \leqslant ((\sigma + 1)N/K_4)^{1/\sigma}\, x^{c/\sigma}. \tag{70}$$

Since $\sigma \geqslant 1$, $x > 1$ and $(\sigma + 1)N/K_4 \geqslant 1$ by (64), we thus obtain (65) if x belongs to A.

If x does not belong to A, then for some $n < \sigma$, we have

$$x^c(\sigma + 1)/K_3 \geqslant |v_0'(x)/v_0(x)|^{\sigma-n}. \tag{71}$$

Since $N \geqslant 1$, $K_3 \geqslant K_4$ and $\sigma - n \geqslant 1$, we thus obtain

$$|v_0'(x)/v_0(x)| \leqslant ((\sigma + 1) Nx^c/K_4)^{1/(\sigma-n)}, \tag{72}$$

and since $(\sigma + 1)N/K_4 \geqslant 1$ by (64) and $x > 1$, we again obtain (65). Thus (65) is proved, and hence $v_0'/v_0 = O(x^c)$ n.e. as $x \to +\infty$ which proves the theorem in Case 2, and thus the proof is complete.

Remarks. (1) Concerning the possible exceptional set in the above theorem, it is not known for a solution y_0, which is not $O(\exp x^b)$ as $x \to +\infty$ for any $b > 0$ (and which satisfies the hypothesis), whether the exceptional set in the estimate $v_0'/v_0 = O(x^b)$ can be removed. If the exceptional set can be removed so that we have $v_0'/v_0 = O(x^b)$ as $x \to +\infty$, for some $b > 0$, then since $v_0'/v_0 \geqslant 0$, two integrations would yield, $y_0 = O(\exp_2 x^{b+1+\epsilon})$ as $x \to +\infty$, for any $\epsilon > 0$. Of course, if we omit the hypothesis that y_0 be a solution of a second-order equation, then it is very easy to construct positive functions y_0, which have properties (i) and (ii) stated in the theorem, which are not $O(\exp x^b)$ as $x \to +\infty$ for any $b > 0$, and for which $v_0'/v_0 = O(x^c)$ n.e. as $x \to +\infty$, for some $c > 0$, but such that for no $c > 0$ is $v_0'/v_0 = O(x^c)$ as $x \to +\infty$.

(2) As stated in Section 1, the above theorem clearly applies to entire transcendental solutions whose power series about the origin have nonnegative coefficients.

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