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ON CONGRUENCES AND CONTINUED FRACTIONS FOR SOME CLASSICAL COMBINATORIAL QUANTITIES

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Several classical combinatorial quantities—including factorials, Bell numbers, tangent numbers. . .—have been shown to form eventually periodic sequences modulo any integer. We relate this phenomenon to the existence of continued fraction expansions for corresponding ordinary (and divergent) generating functions. This leads to a class of congruences obtained in a uniform way.

1. Introduction

It has been shown by Gessel [6] that a number of classical combinatorial quantities constitute sequences that are ultimately periodic modulo any integer. The most trivial example is certainly the sequence of factorials $\{n!\}_{n\geq 0}$ which modulo any fixed integer m eventually reduces to 0. Examples given by Gessel include the tangent numbers, the Bell numbers, the preferential arrangement numbers and the derangement numbers; some cases have also been previously considered in relation to the arithmetical properties of Bernoulli numbers [9] or p-adic analysis [1]. This note was motivated by the fact that all of Gessel's examples are also sequences whose ordinary generating functions have a continued fraction expansion with integer coefficients of a simple form [4].

As an illustration, the sequence of Euler numbers (secant numbers) defined by[1]:

$$E_{2n} = \left[\frac{z^{2n}}{(2n)!}\right]\frac{1}{\cos z} \tag{1}$$

which starts as

$$1, 1, 5, 61, 1385, 50521, 2702765 \ldots \tag{2}$$

yields when reduced modulo 4 the periodic sequence

$$1, 1, 1, 1, 1, 1, 1, \ldots \tag{3}$$

[1] The notation $[z^n]f(z)$ is used to denote the coefficient of z^n in the power series f; $[z^n/n!]f(z)$ similarly represents the coefficient of $z^n/n!$ in $f(z)$. . . .

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and reduced modulo 36 the eventually periodic sequence:

$$1, 1, 5, -11, 17, 13, -7, 1, 5, -11, 17 \dots. \tag{4}$$

Existing proofs are based on induction or on ad hoc addition formulae [6].

The relation to continued fraction can be illustrated by a simple example. It was shown combinatorially in [4] (and indeed known since Stieltjes and Rogers) that

$$E(z) = \sum E_{2n} z^{2n} = \cfrac{1}{1 - \cfrac{1^2 z^2}{1 - \cfrac{2^2 z^2}{1 - \cfrac{3^2 z^2}{\dots}}}} \tag{5}$$

(see also [7] for related problems).

For two series with integer coefficients $a(z)$ and $b(z)$, let us write

$$a(z) \equiv b(z) \ (\mathrm{mod}\ m) \quad \text{iff} \quad \forall n\ a_n \equiv b_n \ (\mathrm{mod}\ m)$$

where $a_n = [z^n]a(z)$ and $b_n = [z^n]b(z)$. It is straightforward that

$$(1 - 4z^2 c(z))^{-1} \equiv 1 \ (\mathrm{mod}\ 4)$$

for any $c(z)$ with integer coefficients. We thus see, using this property in (5), that

$$E(z) \equiv (1 - z^2)^{-1} = 1 + z^2 + z^4 + \cdots \ (\mathrm{mod}\ 4),$$

which is strictly equivalent to (3).

In this paper, we propose to elaborate on congruences derived in this way, making moduli explicit and showing a general method for computing coefficients. This paper is a continuation of a general combinatorial study of continued fractions that was started in [4] and to which the reader is referred for complete definitions.

2. Congruences and continued fractions

We restrict attention here to integer sequences $\{a_n\}_{n \geqslant 0}$ such that the corresponding ordinary generating function

$$a(z) = \sum_{n \geqslant 0} a_n z^n$$

has a continued fraction expansion of the form

$$a(z) = \cfrac{1}{1 - \kappa_0 z - \cfrac{\lambda_1 z^2}{1 - \kappa_1 z - \cfrac{\lambda_2 z^2}{1 - \kappa_2 z - \cfrac{\lambda_3 z^2}{\dots}}}} \tag{6}$$

with $\kappa_j, \lambda_j \in \mathbb{Z}$. This is the classical J-fraction expansion of power series [10].

We also define the product

$$M_h = \lambda_1\lambda_2 \cdots \lambda_h \tag{7}$$

called the hth modulus of fraction (6).

Theorem 1. *Assume a sequence $\{a_n\}_{n\geq 0}$ has an ordinary generating function with an integral continued fraction expansion* (6), *where the moduli M_h are defined by* (7). *Then the sequence is eventually periodic modulo M_h, and satisfies modulo M_h a linear recurrence of order h at most.*

We shall give two proofs of this fact: a combinatorial one, and an algebraic one.

Proof 1 (Combinatorial). From the continued fraction theorem of [4, Theorem 1] and [7], we know that a_n is the number of path diagrams (=weighted ballot sequences in [7]) of length n with weight w defined as follows:
κ_j is associated with level steps starting at altitude[2] j,
λ_j is associated with ascents starting at altitude j,
1 is associated with descents.

Denoting by $\mathcal{P}$ the set of all positive paths, and by $\mathcal{P}^{[h]}$ the subset of those that have height $\leq h$, we thus have:

$$a_n = \sum_{\pi\in\mathcal{P}} w(\pi)$$

(the weight is extended multiplicatively from segments to paths) which can be decomposed into:

$$a_n = \sum_{\pi\in\mathcal{P}^{[h-1]}} w(\pi) + \sum_{\pi\in\mathcal{P}\setminus\mathcal{P}^{[h-1]}} w(\pi).$$

The second sum thus represents the cumulated weight of all paths that have height $\geq h$. Now, each π in $\mathcal{P}\setminus\mathcal{P}^{[h]}$ must comprise at least one ascent from altitude 0, one ascent from altitude $1, \ldots,$ until altitude $h-1$. Therefore

$$\forall \pi \in \mathcal{P}\setminus\mathcal{P}^{[h-1]}: \quad w(\pi) \equiv 0 \pmod{M_h}$$

and thus

$$a_n \equiv \sum_{\pi\in\mathcal{P}^{[h-1]}} w(\pi) \pmod{M_h}.$$

Let $a_n^{[h]}$ denote this last sum. Appealing to the continued fraction theorem again, one finds

$$a^{[h]}(z) = \sum_{n\geq 0} a_n^{[h]} z^n = \cfrac{1}{1-\kappa_0 z - \cfrac{\lambda_1 z^2}{1-\kappa_1 z - \cfrac{\lambda_2 z^2}{\cfrac{\cdots}{1-\kappa_{h-1} z}}}} \tag{8}$$

[2] The altitude of a point in a path diagram is the ordinate of that point.

which is exactly the hth convergent of the continued fraction (7). Now this finite fraction can be put under the normal form

$$a^{[h]}(z)=P_h(z)/Q_h(z) \tag{9}$$

where the P and Q polynomials are determined by the classical recurrences [10]:

$$\begin{aligned} P_{h+1}(z)&=(1-\kappa_h z)P_h(z)-\lambda_h z^2 P_{h-1}(z); \qquad P_{-1}(z)=0, \quad P_0(z)=1;\\ Q_{h+1}(z)&=(1-\kappa_h z)Q_h(z)-\lambda_h z^2 Q_0(z); \qquad Q_{-1}(z)=1, \quad Q_0(z)=1. \end{aligned} \tag{10}$$

This means that the $\{a_n^{[h]}\}_{n\geq 0}$ satisfy a recurrence of order h at most with characteristic polynomial Q_h. Thus the a_n satisfy the same recurrence modulo M_h, and they must form an eventually periodic sequence modulo M_h.

Proof 2 (Algebraic). Let again $a^{[h]}(z)$ be the hth convergent of fraction (7) as given by (8). One can write formally

$$a(z)=\sum_{k\geq 1}(a^{[k]}(z)-a^{[k-1]}(z)). \tag{11}$$

Using (9), we find that

$$\begin{aligned} a^{[k]}(z)-a^{[k-1]}(z)&=\frac{P_k(z)}{Q_k(z)}-\frac{P_{k-1}(z)}{Q_{k-1}(z)}\\ &=\frac{P_k(z)Q_{k-1}(z)-Q_k(z)P_{k-1}(z)}{Q_k(z)Q_{k-1}(z)}. \end{aligned}$$

Using the classical 'determinant identity' of continued fractions (which can easily be checked by recurrence), namely

$$P_k(z)Q_{k-1}(z)-P_{k-1}(z)Q_k(z)=(-1)^k\lambda_1\lambda_1\cdots\lambda_k z^{2k}$$

and splitting the sum (11) according to $k\leq h$, $k>h$ we find:

$$\begin{aligned} a(z)&\equiv\sum_{k\leq h}(a^{[k]}(z)-a^{[k-1]}(z)) \pmod{M_h}\\ &\equiv a^{[h]}(z) \pmod{M_h}. \end{aligned}$$

The proof then continues as above. □

A few observations are in order: the combinatorial and the algebraic proof mimic each other rather closely. The interest of the combinatorial proof is to show congruences to be related to a simple combinatorial property of path diagrams, viz. to reach altitude h, one has to traverse at least once levels $0, 1, \ldots$ to $h-1$. The easy type of symmetry argument involved on path diagrams need not be apparent on more classical presentations of combinatorial structures counted by the a_n (as for instance alternating permutations counted by E_{2n} in our previous example).

Furthermore the reverse polynomials of the $Q_h(z)$ defined by

$$\bar{Q}_h(z) = z^h Q_h(z^{-1})$$

are otherwise known to be orthogonal polynomials. We thus see that the coefficients of the linear recurrence modulo M_h satisfied by the a_n are coefficients of orthogonal polynomials. For instance the denominator polynomials of the convergents of (5) are Meixner (or Mittag–Leffler) polynomials.

Returning to this example, the proof of the theorem shows that

$$E(z) \equiv \frac{1}{1 - z^2/(1-4z^2)} = \frac{1-4z^2}{1-5z^2} \pmod{36},$$

whence

$$E_{2n} \equiv 5^{n-1} \pmod{36} \quad \text{for } n \geqslant 1$$

which accounts precisely for congruences (4).

It is to be noticed last, that if the set $\{\lambda_i\}_{i \geqslant 1}$ contains the set of all positive integers, then the sequence a_n is eventually periodic modulo any integer. Such is the case for all examples discussed in the next section.

3. Sample applications

Several continued fractions for classical combinatorial quantities have been given in [4] where the corresponding Q-polynomials are identified (see also [5] for further enumerative uses of the polynomials). To avoid redundancy, we shall only present a table of fractions including the exponential generating function of Q-polynomials which is a concise way of describing the coefficients of the modular recurrences. Our method accounts for the origin of moduli like $(k!)^2$, $(k-1)!\,k!\ldots$ in a simple way.

Table 1 covers all examples given in [6] with the addition of the secant

Table 1. For each number sequence $\{a_n\}_{n \geqslant 0}$ considered, column 2 gives the corresponding exponential generating function $a(z) = \sum_{n \geqslant 0} a_n (z^n/n!)$; column 3 shows the integer coefficients $\kappa_k \lambda_k$ in the continued fraction expansion of $\sum a_n z^n$ together with the modulus M_k. Column 4 provides the exponential generating function of the $\bar{Q}$ polynomials: $K(z,t) = \sum_{n \geqslant 0} \bar{Q}_n(z)(t^n/n!)$ with $\bar{Q}_n(z) = z^n \bar{Q}_n(z^{-1})$.

Numbers	gen. fun.	κ_k	λ_k	M_k	$K(z,t)$
Secant	$\sec z$	0	k^2	$(k!)^2$	$(1+t^2)^{-1}\exp(z \operatorname{Arctan} t)$
Tangent	$\tan z$	0	$k(k+1)$	$(k+1)!\,k!$	$\exp(z \operatorname{Arctan} t)$
Bell	$\exp(e^z - 1)$	$k+1$	k	$k!$	$e^{-t}\exp(z \log(1+t))$
2-Bell	$\exp(e^z - z - 1)$	k	k	$k!$	$e^{-t}(1+t)^{z+1}$
Involution	$\exp(z + \frac{1}{2}k^2)$	1	k	$k!$	$\exp(-t - \frac{1}{2}t^2 + zt)$
Derangement	$e^{-z}(1+z)^{-1}$	$2k$	k^2	$(k!)^2$	$(1+t)^{-1}\exp(t(z+1)(1+t)^{-1})$
Pref. Arrangements	$(2-e^z)^{-1}$	$3k+1$	$2k^2$	$2^k k!$	$(1+t)^{-z-1}(1+2t)^z$

numbers, and the associated (2-) Bell numbers (counting partitions without singletons [2]). Since the continued fraction relative to preferential arrangements appears to be new as is a related fraction relative to Stirling numbers, we shall devote the next section to a short proof of it. It also reveals the way the entries in column 4 of the table are systematically determined.

4. A continued fraction relative to preferential arrangements and Stirling numbers

Let P_n be the number of preferential arrangements of n elements, i.e. partitions into blocks where the order of appearance of blocks is taken into account (but not the order of elements inside blocks). It is well known (see e.g. [11]) that

$$\sum_{n\geq 0} P_n \frac{z^n}{n!} = (2-e^z)^{-1} \tag{12}$$

and $P_n = \sum_k S_{n,k} k!$, with $S_{n,k}$ a Stirling number of the second kind. The latter identity follows from the fact that a preferential arrangement with k blocks can be built from a partition into k blocks (there are $S_{n,k}$ of these over n elements) and a permutation of the k blocks.

We state:

Theorem 2. *The generating functions of P_n and $k!\,S_{n,k}$ have the continued fraction expansions*

(i) $$\sum_{n\geq 0} P_n z^n = \cfrac{1}{1-1z-\cfrac{2\cdot 1^2z^2}{1-4z-\cfrac{2\cdot 2^2z^2}{1-7z-\cfrac{2\cdot 3^2z^2}{\cdots}}}}$$

(ii) $$\sum_{n,k\geq 0} k!\,S_{n,k}u^kz^n = \cfrac{1}{1-uz-\cfrac{u(1+u)1^2z^2}{1-(1+3u)-\cfrac{u(1+u)2^2z^2}{1-(2+5u)-\cfrac{u(1+u)3^2z^2}{\cdots}}}}$$

Proof. Let $\pi = \{B_1, B_2, \ldots, B_k\}$ be a preferential arrangement of $[1, \ldots, n]$ with block $B_i = \{b_{i,1} < b_{i,2} < \cdots < b_{i,j(i)}\}$. Associate with π the permutation

$$\sigma = b_{1,1}b_{2,2}\cdots b_{1,j(1)}b_{2,1}b_{2,2}\cdots b_{j,2}\cdots b_{k,1}b_{k,2}\cdots b_{k,j(k)}.$$

Each fall in σ, i.e. element such that $\sigma_{l-1} < \sigma_l$, corresponds to a minimal (initial) element of a block in π. To obtain an encoding of π we must further mark (say by underlining them) those elements in σ that open a block without being falls in σ.

For instance with $\pi=\{1\}\{3,6\}\{7\}\{2,4,8\}\{5\}$, we have $\sigma = 1\ 3\ 6\ 7\ 2\ 4\ 8\ 5$ and the marked permutation reads $\sigma^* = 1\ \underset{\sim}{3}\ 6\ \underset{\sim}{7}\ 2\ 4\ 8\ 5$.

The mapping described thus associates to each preferential arrangement a permutation with some rises (elements such that $\sigma_{l-1} > \sigma_l$) marked. It is clearly invertible and we have proved by a combinatorial argument the identity

$$P_n = \sum_{k\geq 0} k!\, S_{n,k} = \sum_{m\geq 0} a_{n,m} 2^m, \tag{13}$$

where $a_{n,t}$ is a Eulerian number with generating function

$$\sum a_{n,t} u^m \frac{z^n}{n!} = \frac{u-1}{u-e^{z(u-1)}}.$$

These numbers are known to count permutations partitioned by number of rises (or falls).

Now using the continued fraction expansion for the generating function of Eulerian numbers

$$\sum a_{n,t} u^m z^m = \cfrac{1}{1-1z-\cfrac{u1^2z^2}{1-(2+u)z-\cfrac{u2^2z^2}{1-(3+u)z-\cfrac{u3^2z^2}{\cdots}}}}$$

and substituting $u=2$ gives (i). The same argument is easily extended to prove (ii) also leading to a correspondence between preferential arrangements and path diagrams. $\square$

Notice that (i) and (ii) can also be derived using the Stieltjes–Rogers addition theorem from the identity

$$[1-u(e^{(x+y)}-1)]^{-1} = \sum_k (k!)^2 u^k (1+u)^k \theta_k(x)\theta_k(y)$$

where

$$\theta_k(z) = \frac{(e^z-1)^k}{k!\,(1-u(e^z-1))^{k+1}}.$$

To conclude with the entry corresponding to preferential arrangements in Table 1, we only need to compute the coefficients of the recurrence. The method is systematic and typical of the whole class given in this table.

The recurrence on the $\bar{Q}$ polynomials is here

$$\bar{Q}_{k+1}(z) = (z-(3k+1))\bar{Q}_k(z) - 2k^2\bar{Q}_{k-1}(z)$$

whence, for the exponential generating function

$$K(z,t) = \sum_{n\geq 0} Q_k(z)\frac{t^k}{k!}$$

the partial differential equation:

$$\frac{\partial K}{\partial t}=(z-1-2t)K-(3t+2t)^2)\frac{\partial K}{\partial t}$$

as obtained by multiplying by $t^k/k!$ and summing. The equation allows separation, of variables and solving one finds

$$K(z,t)=(1+t)^{-1}\left(\frac{1+2t}{1+t}\right)^z,$$

as remained to be shown.

5. Conclusion

We have demonstrated the relation between certain classes of congruences and continued fraction expansions for (usually divergent) ordinary generating functions. Actually all examples given above fall into the 'Meixner class' of path diagrams characterized essentially by the fact that κ_k is linear and λ_k is quadratic in k. For this class the Q polynomials (giving coefficients in the recurrences) can be determined systematically.

The method can also accomodate polynomial congruences; cases to which it applies are Stirling polynomials (of both kind), Eulerian polynomials....

Further arithmetical properties follow from consideration of these fractions. For instance a classical (and easy) result [10] expresses Hankel determinants of the a_n in terms of the moduli M_k, and one has[3]

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & & & \vdots \\ a_n & a_{n+1} & & \cdots & a_{2n} \end{vmatrix} = M_1M_2M_3\cdots M_n.$$

All expansions in Table 1 and others from [4] can be translated in this way. We mention the identity

$$\begin{vmatrix} B_0 & B_1 & \cdots & B_n \\ B_1 & B_2 & \cdots & B_{n+1} \\ \vdots & \vdots & & \vdots \\ B_n & B_{n+1} & \cdots & B_{2n} \end{vmatrix} = 1!\,2!\,3!\cdots n!$$

which leads to an alternative perhaps more combinatorial solution of the 'Radoux

[3] Recently Gerard Viennot (private communication) has also given a purely combinatorial proof of this identity together with a powerful generalization to the counting of path diagrams.

conjecture' (first settled in [3]), and the companion identity:

$$\begin{vmatrix} B_1 & B_2 & \cdots & B_n \\ B_2 & B_3 & \cdots & B_{n+1} \\ \vdots & \vdots & & \vdots \\ B_n & B_{n+1} & \cdots & B_{2n-1} \end{vmatrix} = 1!\,2!\,3!\cdots(n-1)!.$$

For a number of related topics, the reader is referred to the forthcoming book of Jackson and Goulden [8].

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