Annals of Pure and Applied Logic 62 (1993) 21–49
North-Holland

Rudimentary Kripke models for the intuitionistic propositional calculus

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Communicated by D. van Dalen
Received 30 October 1991
Revised 12 December 1992

Abstract

Došen, K., Rudimentary Kripe models for the intuitionistic propositional calculus, Annals of Pure and Applied Logic 62 (1993) 21–49.

It is shown that the intuitionistic propositional calculus is sound and complete with respect to Kripke-style models that are not quasi-ordered. These models, called *rudimentary Kripke models*, differ from the ordinary intuitionistic Kripke models by making fewer assumptions about the underlying frames, but have the same conditions for valuations. However, since accessibility between points in the frames need not be reflexive, we have to assume, besides the usual intuitionistic heredity, the converse of heredity, which says that if a formula holds in all points accessible to a point x, then it holds in x. Among frames of rudimentary Kripke models, particular attention is paid to those that guarantee that the assumption of heredity and converse heredity for propositional variables implies heredity and converse heredity for all propositional formulae. These frames need to be neither reflexive nor transitive.

The intuitionistic propositional calculus, which we call **H** (after Heyting), is sound and complete with respect to Kripke models based on quasi-ordered frames. Besides this class of Kripke models there are many smaller classes of Kripke models with respect to which **H** is sound and complete. For example, we may require from the frames of Kripke models that in addition to being quasi-ordered they satisfy one or more of the following:

– the frame is partially ordered (i.e., we have added antisymmetry),
– the frame is generated (i.e., there is a point which is lesser than or equal to every point),
– the frame is a tree,
– the frame is a Jaśkowski tree,
– the frame is finite.

The propositional calculus **H** is also sound and complete with respect to classes of Kripke models which are all based on a single frame (for example, this frame may be the disjoint union of all finite quasi-ordered frames), or with respect to classes which contain a single Kripke model (like the class whose only member is the canonical model for **H**, familiar from the Henkin-style completeness proof for **H**).

For all these classes of Kripke models the class of all quasi-ordered Kripke models is the largest class, in which all are included. Here we will consider classes of models with respect to which **H** can be shown sound and complete in which the class of all ordinary quasi-ordered Kripke models is properly included. In producing models in these wider classes we will feel free to tamper as much as we can with the conditions on the underlying frames, while the conditions concerning valuations on these frames and the definition of holding in a model will be practically identical to those in ordinary Kripke models for **H**. These new models are not meant to replace ordinary Kripke models for the investigation of **H**. Neither are they meant to be philosophically significant. We want to have them rather as an instrument for the analysis of the inner mechanism of Kripke models. But they might also raise some interesting technical questions.

Our models delimit a field within which we can look for new models without ever leaving standard Kripke models too far behind. Since Kripke models for intuitionistic logic are often taken as a paradigm when we try to model other nonclassical logics, it might be worth knowing all the possibilities inherent in this paradigm, lest we should be stranded by a too narrow imitation of features that are perhaps accidental. (In [6] one can find models for logics based on weaker implications, like relevant and linear implication, that are in some respect analogous to models introduced here.)

The spirit of this paper will be close to the spirit of correspondence theory (see [1] and [11]). A number of our results will be of the form that a frame satisfies certain conditions concerning its relation iff it satisfies certain conditions concerning valuations on it, or something similar. However, this is not a paper at the level of correspondence theory, because it does not go far enough. It only introduces notions, and proves for them rather straightforward matters which perhaps could lead to a more advanced theory.

We will concentrate here only on propositional logic and leave aside a possible extension of our approach to predicate logic. The paper will be divided into three sections. In the first section we introduce our main generalization of Kripke models for **H**, called *rudimentary* Kripke models. The frames of these models must be only serial, but in the absence of reflexivity we assume for valuations on these frames a condition converse to the usual heredity condition of ordinary Kripke models for **H**. We prove that **H** is sound and complete with respect to rudimentary Kripke models, and consider questions related to completeness. We show that in a certain sense rudimentary Kripke models make the largest class of Kripke-type models with respect to which **H** is *strongly* sound and complete.

In the second section we present a canonical Kripke model for **H** which, though serial and transitive, is not reflexive, and is hence not an ordinary Kripke model, but rudimentary. We also consider briefly at the end of this section a representation for Heyting algebras which is in the background of our canonical model.

In the third section we consider rudimentary Kripke models where valuations are defined inductively. We find necessary and sufficient conditions on frames for the inductive character of rudimentary Kripke models of this type, which make a proper subclass of the class of all rudimentary Kripke models. We also consider at the end of this section some questions related to modal logic.

In a sequel to this paper [4] we shall consider three related topics. First, we shall introduce a very general notion of Beth models for **H**, such that rudimentary Kripke models may be conceived as a particular type of these models. These Beth models are interesting because for them we make another assumption analogous to the converse heredity of rudimentary Kripke models. We shall also consider such Beth models where valuations are defined inductively.

Next we shall present in [4] the correspondence between on the one hand conditions on frames of various types of rudimentary Kripke models and on the other hand the characteristic schemata of Dummett's logic, the logic of weak excluded middle and classical propositional logic.

Finally, we shall consider in [4] a generalization of rudimentary Kripke models which consists in restricting the conditions for rudimentary Kripke models only to those points of our frames which are accessible from some point. In a rather natural sense this makes the largest class of Kripke-type models with respect to which **H** is sound and complete, though even larger classes may be envisaged.

Some further topics related to rudimentary Kripke models will be considered in [5].

1. Rudimentary Kripke models

Our propositional language has infinitely many propositional variables, the propositional constant $\bot$, and the binary connectives $\rightarrow$, $\wedge$ and $\vee$. For propositional variables we use the schematic letters $p, q, r, \ldots, p_1, \ldots$, for formulae the schematic letters $A, B, C, \ldots, A_1, \ldots$, and for sets of formulae the schematic letters $\Gamma, \Delta, \Theta, \ldots, \Gamma_1, \ldots$. As usual, $A \leftrightarrow B$ is defined as $(A \rightarrow B) \wedge (B \rightarrow A)$ and $\neg A$ as $A \rightarrow \bot$. We denote the set of all formulae by $\mathbf{L}$, the set of all formulae in which $\bot$ does not occur by $\mathbf{L}^+$, and the set of all formulae in which only propositional variables and $\rightarrow$ occur by $\mathbf{L}^{\rightarrow}$. In the metalanguage we use $\Rightarrow, \Leftrightarrow, \&,$ *or*, *not*, $\forall$, $\exists$ and set-theoretical symbols, with their usual meaning in classical logic.

The intuitionistic propositional calculus **H** in **L** is axiomatized by the following familiar axiom-schemata:

$$A \to (B \to A),$$
$$(A \to (B \to C)) \to ((A \to B) \to (A \to C)),$$
$$(C \to A) \to ((C \to B) \to (C \to (A \wedge B))),$$
$$(A \wedge B) \to A, \qquad (A \wedge B) \to B,$$
$$A \to (A \vee B), \qquad B \to (A \vee B),$$
$$(A \vee B) \to ((A \to C) \to ((B \to C) \to C)),$$
$$\bot \to A,$$

and the rule *modus ponens*. It is well known that this axiomatization is separative, in the sense that an axiomatization of a fragment of **H** involving some connectives, among which we must have $\to$, is obtained by assuming *modus ponens* and all those axiom-schemata from the list above in which the connectives of the fragment in question occur. So the positive intuitionistic propositional calculus $\mathbf{H}^+$ in $\mathbf{L}^+$ is axiomatized by deleting $\bot \to A$, and the implicational fragment of **H**, i.e. the system $\mathbf{H}^\to$ in $\mathbf{L}^\to$, is axiomatized by the first two axiom-schemata and *modus ponens*.

A *frame* is a pair $\langle W, R\rangle$ where W is a nonempty set and R a binary relation on W. Members of W, which in modal logic are called *worlds*, will here be called more neutrally *points*. We use $x, y, z, \ldots, x_1, \ldots$ for members of W, and $X, Y, Z, \ldots, X_1, \ldots$ for subsets of W. For a frame $\langle W, R\rangle$ a subset X of W will be called *hereditary* iff for every x

$$x \in X \Rightarrow \forall y\, (x R y \Rightarrow y \in X),$$

and it will be called *conversely hereditary* iff for every x

$$\forall y\, (x R y \Rightarrow y \in X) \Rightarrow x \in X.$$

For a frame $\langle W, R\rangle$ and $X, Y \subseteq W$, the binary operation $\to_R$ is defined by

$$X \to_R Y = \{x\colon\ \forall y\, (x R y \Rightarrow (y \in X \Rightarrow y \in Y))\}.$$

A *pseudo-valuation* v on a frame $\langle W, R\rangle$ is a function from **L** into $\mathbf{P}W$, i.e. the power set of W, which satisfies the following conditions for every $A, B \in \mathbf{L}$:

($v\bot$) $\quad v(\bot) = \emptyset$,
($v\to$) $\quad v(A \to B) = v(A) \to_R v(B)$,
($v\wedge$) $\quad v(A \wedge B) = v(A) \cap v(B)$,
($v\vee$) $\quad v(A \vee B) = v(A) \cup v(B)$.

A *valuation* v on a frame $\langle W, R\rangle$ is a pseudo-valuation which satisfies

(*General Heredity*) *for every formula A the set $v(A)$ is hereditary,*
(*Converse General Heredity*) *for every formula A the set $v(A)$ is conversely hereditary.*

(In [4] and [5] *General Heredity* and *Converse General Heredity* are called respectively *A-Heredity* and *Converse A-Heredity*, whereas *Atomic Heredity* and *Converse Atomic Heredity* below are called respectively *p-Heredity* and *Converse p-Heredity*.)

A *rudimentary Kripke model* is a triple $\langle W, R, v\rangle$ where $\langle W, R\rangle$ is a frame and v a valuation on this frame. A formula A *holds in* $\langle W, R, v\rangle$ iff $v(A) = W$.

With our usual experience with Kripke models, instead of General Heredity and Converse General Heredity we would expect only the following conditions:

(*Atomic Heredity*) *for every propositional variable p the set $v(p)$ is hereditary,*

(*Converse Atomic Heredity*) *for every propositional variable p the set $v(p)$ is conversely hereditary.*

Valuations would be defined by specifying v for propositional variables and using $(v\bot)$, $(v\rightarrow)$, $(v\wedge)$ and $(v\vee)$ as clauses in an inductive definition. That General Heredity and Converse General Heredity obtain would be derived by induction on the complexity of A. Rudimentary Kripke models whose frames have properties which guarantee that every v defined on them in such an inductive way satisfies General Heredity and Converse General Heredity form an important proper subclass of the class of all rudimentary Kripke models (we will study this subclass in the third section). Here, however, we deal first with frames where there is no guarantee that a pseudo-valuation which satisfies Atomic Heredity and Converse Atomic Heredity will satisfy also General Heredity and Converse General Heredity. In arbitrary rudimentary Kripke models General Heredity and Converse General Heredity are not derived but stipulated; namely, we restrict ourselves to the pseudo-valuations where these two heredity conditions have somehow been secured.

The following proposition shows that frames of rudimentary Kripke models cannot be completely arbitrary:

Proposition 1. *For every rudimentary Kripke model $\langle W, R, v\rangle$, the relation R is* ***serial****, i.e., $\forall x\, \exists y\, (x R y)$.*

Proof. Since for every x we have $x \notin v(\bot)$, by Converse General Heredity there is a y such that $x R y$ and $y \notin v(\bot)$. $\square$

The next proposition shows that we need not assume anything besides seriality for frames of rudimentary Kripke models:

Proposition 2. *If in the frame $\langle W, R\rangle$ the relation R is serial, then there is a valuation v on $\langle W, R\rangle$.*

Proof. On $\langle W, R\rangle$ where R is serial let $v(p)$ be either W or $\emptyset$ and let $v(\bot) = \emptyset$. Then using the conditions $(v\rightarrow)$, $(v\wedge)$ and $(v\vee)$ we define $v(A)$ for every formula A. It is easy to check by induction on the complexity of A that $v(A)$ is either W or $\emptyset$. (The only interesting case in this induction is when A is of the form $A_1 \rightarrow A_2$, and we have $v(a_1) = W$ and $v(a_2) = \emptyset$; then $v(A_1 \rightarrow A_2) = \{x: \textit{not } \exists y\, (x R y)\} = \emptyset$, by using the seriality of R.) It is clear that General Heredity and Converse General Heredity obtain if $v(A) = W$. If $v(A) = \emptyset$, then General Heredity is vacuously satisfied and Converse General Heredity follows from the seriality of R. □

This proof shows that for every serial frame $\langle W, R \rangle$ the set $\{W, \emptyset\}$ is closed under the operations $\rightarrow_R$, $\cap$ and $\cup$. (We call a frame $\langle W, R \rangle$ *serial* iff R is serial, and similarly with models and other properties.)

A *positive* valuation v^+ on a frame $\langle W, R \rangle$ is a function from $\mathbf{L}^+$ into $\mathbf{P}W$ which satisfies all the conditions for valuations (except $(v\bot)$, which does not apply any more). A *positive* rudimentary Kripke model is a triple $\langle W, R, v^+ \rangle$. In a positive rudimentary Kripke model R can be completely arbitrary, even empty, as the following proposition shows:

Proposition 3. *For every frame $\langle W, R \rangle$ there is a positive valuation v^+ on $\langle W, R \rangle$.*

Proof. For every $A \in \mathbf{L}^+$ let $v^+(A) = W$, and check that v^+ is a positive valuation. □

This proof is based on the simple fact that for an arbitrary frame $\langle W, R \rangle$ the set $\{W\}$ is closed under $\rightarrow_R$, $\cap$ and $\cup$. We can similarly show that in *implicative* rudimentary Kripke models $\langle W, R, v^{\rightarrow} \rangle$, where $v^{\rightarrow}$ maps $\mathbf{L}^{\rightarrow}$ into $\mathbf{P}W$ and satisfies $(v\rightarrow)$, General Heredity and Converse General Heredity, R can also be completely arbitrary.

The valuations defined in the proof of Proposition 2 and the positive valuation defined in the proof of Proposition 3 are trivial, since in every rudimetnary Kripke model of the first proof every two-valued tautology holds, and in the positive rudimentary Kripke model of the second proof every formula of $\mathbf{L}^+$ holds. Of course, not all rudimentary, or positive rudimentary, Kripke models are trivial in this way.

In a *quasi-ordered* frame $\langle W, R \rangle$ the relation R is reflexive and transitive, and because of reflexivity these frames are serial. Rudimentary Kripke models based on such frames, which we will call *quasi-ordered Kripke models,* are the ordinary Kripke models for $\mathbf{H}$. The conditions for valuations which we have given above are necessary and sufficient for valuations in these ordinary Kripke models, though with ordinary Kripke models they are usually introduced in a different way. Namely, instead of General Heredity we assume only Atomic Heredity, whereas Converse General Heredity is not assumed in any form. We also assume the conditions $(v\bot)$, $(v\rightarrow)$, $(v\wedge)$ and $(v\vee)$ formulated exactly as above. By induction on the complexity of A we can then demonstrate General Heredity, whereas Converse General Heredity is an immediate consequence of the reflexivity of R. So every quasi-ordered Kripke model is a rudimentary Kripke model, but not vice versa, as Proposition 2 shows. As we have remarked, the rudimentary Kripke models of the proof of Proposition 2 are trivial, but we will see below in Proposition 7 and in the next section that there are nontrivial rudimentary Kripke models which are not quasi-ordered.

We will now demonstrate that **H** is sound and complete with respect to rudimentary Kripke models. For soundness we have the following proposition:

Proposition 4. *If B is provable in* **H**, *then B holds in every rudimentary Kripke model.*

Proof. We proceed by induction on the length of proof of B in **H**. If B is an axiom, the only case where we must invoke Converse General Heredity is when B is of the form $(B_1 \rightarrow (B_2 \rightarrow B_3)) \rightarrow ((B_1 \rightarrow B_2) \rightarrow (B_1 \rightarrow B_3))$, and this is why in the basis of the induction we will consider only this case as an example.

Suppose for B of the form above that for some x we have $x \notin v(B)$. We easily infer that we must have y, z and t such that

$$
\begin{array}{lll}
x\,R\,y & \text{and} & y \in v(B_1 \rightarrow (B_2 \rightarrow B_3)), \\
y\,R\,z & \text{and} & z \in v(B_1 \rightarrow B_2), \\
z\,R\,t, & t \in v(B_1) & \text{and} \quad t \notin v(B_3).
\end{array}
$$

So $t \in v(B_2)$ and by General Heredity $t \in v(B_2 \rightarrow B_3)$. Then from $t \notin v(B_3)$, by Converse General Heredity, it follows that there must be a u such that $t\,R\,u$ and $u \notin v(B_3)$. By General Heredity $u \in v(B_2)$, but since $t \in v(B_2 \rightarrow B_3)$ we obtain a contradiction.

For the induction step suppose that $v(B_1) = W$ and $v(B_1 \rightarrow B_2) = W$. Next suppose $x\,R\,y$. Since $x \in v(B_1 \rightarrow B_2)$ and $y \in v(B_1)$ we obtain $y \in v(B_2)$. So $\forall y\,(x\,R\,y \Rightarrow y \in v(B_2))$, from which $x \in v(B_2)$ follows by Converse General Heredity. □

For completeness it is enough to appeal to the completeness of **H** with respect to quasi-ordered Kripke models. Indeed, if B holds in all rudimentary Kripke models, then B holds in all quasi-ordered Kripke models, and hence B is provable in **H**. So we have:

Proposition 5. *A formula B is provable in* **H** *iff B holds in every rudimentary Kripke model.*

We can similarly demonstrate the soundness and completeness of $\mathbf{H}^{+}$ with respect to positive rudimentary Kripke models, and of $\mathbf{H}^{\rightarrow}$ with respect to implicative rudimentary Kripke models.

In the background of the soundness of **H** with respect to rudimentary Kripke models is the following algebraic fact. For every rudimentary Kripke model $\langle W, R, v\rangle$, the set $\{v(A)\colon A \in \mathbf{L}\}$ contains $\emptyset$ and is closed under the operations $\rightarrow_R$, $\cap$ and $\cup$; the algebra $\langle\{v(A)\colon A \in \mathbf{L}\}, \rightarrow_R, \cap, \cup, \emptyset\rangle$ is a Heyting algebra. In terms of frames, for every frame $\langle W, R\rangle$, every set $\mathscr{A}$ of hereditary and conversely hereditary subsets of W which contains $\emptyset$ and is closed under the operations $\rightarrow_R$, $\cap$ and $\cup$ is a Heyting algebra. (For $\emptyset$ to be conversely hereditary our frame must be serial.) When we verify for $X, Y, Z \in \mathscr{A}$ that

$$X \cap Y \subseteq Z \Leftrightarrow X \subseteq Y \rightarrow_R Z$$

we use the hereditariness of X from left to right, whereas the hereditariness of Y and converse hereditariness of Z are used from right to left. Our frame may be such that $\mathscr{A}$ never coincides with the set of *all* hereditary and conversely hereditary subsets of W (this will become clear in the third section; see Proposition 19 and the comments following this proposition).

For a frame $\langle W, R\rangle$, let us define R^k, where $k \geq 0$, by the following recursive clauses:

$$x\, R^0\, y \Leftrightarrow x = y,$$

$$x\, R^{k+1}\, y \Leftrightarrow \exists z\, (x\, R^k\, z\ \&\ z\, R\, y).$$

It is clear that $x\, R^1\, y \Leftrightarrow x\, R\, y$. Note that General Heredity is equivalent with the conditions that for every A and every x

$$x \in v(A) \Rightarrow \forall y\, ((\exists k \geq m)\, x\, R^k\, y \Rightarrow y \in v(A)),$$

where $m = 0$ or $m = 1$. On the other hand, Converse General Heredity is not equivalent with the converse conditions, namely, that for every A and every x the converse implication obtains, with either $m = 0$ or $m = 1$, though it implies these converse conditions, for both $m = 0$ and $m = 1$ (the converse condition where $m = 0$ is vacuously true). However, in the presence of General Heredity, Converse General Heredity is equivalent with the converse condition where $m = 1$.

The soundness of **H** with respect to rudimentary Kripke models can be inferred from the following proposition too:

Proposition 6. *For every rudimentary Kripke model $\langle W, R, v\rangle$ there is a quasi-ordered Kripke model $\langle W, R', v'\rangle$ with the same W such that for every A we have $v(A) = v'(A)$.*

Proof. For a rudimentary Kripke model $\langle W, R, v\rangle$ we define $\langle W, R', v'\rangle$ by stipulating that $x\, R'\, y$ iff $(\exists k \geq 0)\, x\, R^k\, y$, and $v'(A) = v(A)$. In other words, R' is the reflexive and transitive closure of R, and v and v' coincide. Then we verify that $\langle W, R', v'\rangle$ is indeed a quasi-ordered Kripke model.

The only part of this verification which is not quite straightforward is when in the verification that v' is a valuation on $\langle W, R' \rangle$ we have to check that v' satisfies $(v\rightarrow)$, i.e., when we show that

$$\forall y\,((\exists k \geqslant 0)\, x\, R^k\, y \Rightarrow (y \in v(B) \Rightarrow y \in v(C)))$$
$$\text{iff}\quad \forall y\,(x\, R\, y \Rightarrow (y \in v(B) \Rightarrow y \in v(C))).$$

From left to right we just appeal to the fact that $x\, R\, y \Rightarrow (\exists k \geqslant 0)\, x\, R^k\, y$. For the other direction suppose that for some y we have $(\exists k \geqslant 0)\, x\, R^k\, y$, $y \in v(B)$ and $y \notin v(C)$. If $k = 0$, then $x \in v(B)$ and $x \notin v(C)$. Hence, by the Converse General Heredity of v in $\langle W, R, v \rangle$ we have a y such that $x\, R\, y$ and $y \notin v(C)$, and by the General Heredity of v in $\langle W, R, v \rangle$ we also have $y \in v(B)$. If $k > 0$, then for some z we have $x\, R^{k-1}\, z$ and $z\, R\, y$. It follows that $z \notin v(B \rightarrow C)$. Either $x = z$, in which case $x \notin v(B \rightarrow C)$, or $x \neq z$, in which case by the General Heredity of v in $\langle W, R, v \rangle$ it again follows that $x \notin v(B \rightarrow C)$. Hence there is a y such that $x\, R\, y$, $y \in v(B)$ and $y \notin v(C)$. □

As a kind of converse of Proposition 6 we can demonstrate the following:

Proposition 7. *For every quasi-ordered Kripke model $\langle W, R, v \rangle$ there is a rudimentary Kripke model $\langle W^*, R^*, v^* \rangle$ which is not quasi-ordered such that for every A we have $v(A) = W$ iff $v^*(A) = W^*$.*

Proof. If $W' = \{x' \colon x \in W\}$ and $W' \cap W = \emptyset$, let $W^* = W \cup W'$. On W' we define R' by

$$x'\, R'\, y' \Leftrightarrow (x\, R\, y \;\&\; x \neq y),$$

and we let $R^* = R \cup R' \cup \{\langle x', x \rangle \colon x \in W\}$. So the frame $\langle W^*, R^* \rangle$ consists of $\langle W, R \rangle$ plus an irreflexive copy $\langle W', R' \rangle$ of $\langle W, R \rangle$ such that for every $x' \in W'$ and $x \in W$ we have $x'\, R^*\, x$. The frame $\langle W^*, R^* \rangle$ is not reflexive since $\langle W', R' \rangle$ is irreflexive, and if there is a $y \in W$ distinct from $x \in W$ such that $x\, R\, y$, then $\langle W^*, R^* \rangle$ is not transitive, since though we have $x'\, R^*\, x$ and $x\, R^*\, y$, we don't have $x'\, R^*\, y$.

If $v'(A) = \{x' \in W' \colon x \in v(A)\}$, let $v^*(A) = v(A) \cup v'(A)$. It is straightforward to check that v^* is a valuation on $\langle W^*, R^* \rangle$ and that $\langle W^*, R^*, v^* \rangle$ is a rudimentary Kripke model such that our proposition is satisfied. □

For W and W^* as in the proof of Proposition 7, let f be a function from W^* *onto* W defined by $f(x) = x$ and $f(x') = x$. Then f is a pseudo-epimorphism, or zigzag morphism, from $\langle W^*, R^* \rangle$ onto $\langle W, R \rangle$ (see [9, pp. 70–75], [1, pp. 174, 187], or [11, 2.4.2]) since we have

$$(\forall z, t \in W^*)(z\, R^*\, t \Rightarrow f(z)\, R\, f(t)),$$

$$(\forall z \in W^*)(\forall y \in W)(f(z)\, R\, y \Rightarrow (\exists t \in W^*)(f(t) = y \;\&\; z\, R^*\, t)).$$

We also have for every $z \in W^*$ and every formula A that

$$z \in v^*(A) \Leftrightarrow f(z) \in v(A).$$

More generally, we can prove Proposition 7 by letting W^* be the disjoint union of two or more sets W_i, each in *one–one* correspondence with W. On W_i for every $x, y \in W$ such that $x \neq y$ we have $x_i\, R_i\, y_i$ iff $x\, R\, y$, but for some $x_i \in W_i$ we may lack $x_i\, R_i\, x_i$, which makes $\langle W_i, R_i \rangle$ nonreflexive. In the relation R^* on W^* is included the union of all the relations R_i and moreover for every $x_i \in W_i$ we have an $x_j \in W_j$ such that $x_i\, R^*\, x_j$. If one of the frames $\langle W_i, R_i \rangle$ is nonreflexive, then $\langle W^*, R^* \rangle$ is nonreflexive, whereas transitivity will fail if $x_i\, R^*\, x_j$ and $x_j\, R^*\, x_k$ but *not* $x_i\, R^*\, x_k$ (in the proof of Proposition 7 above, transitivity fails for a different reason). The set $v^*(A)$ is the union of all the sets $v_i(A) = \{x_i \in W_i\colon\ x \in v(A)\}$. The frame $\langle W, R \rangle$ is a pseudo-epimorphic image of $\langle W^*, R^* \rangle$ under $f: W^* \to W$ defined by $f(x_i) = x$, and we have $x_i \in v^*(A)$ iff $f(x_i) \in v(A)$.

Proposition 6 says that for every rudimentary Kripke model there is a quasi-ordered Kripke model in which the same formulae hold, and since the converse is trivially satisfied, it might seem that rudimentary Kripke models do not bring anything new. However, they may bring something new if instead of holding in a model we consider holding in a frame. We say that A *holds in a frame* $\langle W, R \rangle$ iff for every valuation v on $\langle W, R \rangle$ we have that A holds in $\langle W, R, v \rangle$. Our soundness and completeness result of Proposition 5 can equivalently be expressed in terms of holding in frames; namely, B is provable in **H** iff B holds in every serial frame.

It is also true that B is provable in **H** iff B holds in *every* frame, for if a frame $\langle W, R \rangle$ is not serial, then, since there are no valuations on $\langle W, R \rangle$, it is vacuously satisfied that for every valuation v on $\langle W, R \rangle$ every formula A holds in $\langle W, R, v \rangle$. This is like accepting among our rudimentary Kripke models also models where W is empty, since if W is empty, every formula A holds vacuously in $\langle W, R, v \rangle$. The problem with these vacuous holdings is that every formula, even $\perp$, will have a model, or a frame in which it holds. So, we should exclude vacuous holdings from our considerations.

In [12] Shekhtman gave an example of an intermediate propositional logic **S** incomplete with respect to any class of partially ordered frames; i.e., there is no class $\mathscr{C}$ of partially ordered frames such that B is provable in **S** iff B holds in every frame in $\mathscr{C}$. It follows easily that there is no class $\mathscr{C}$ of quasi-ordered frames such that B is provable in **S** iff B holds in every frame in $\mathscr{C}$, since for every quasi-ordered frame there is a partially ordered frame in which the same formulae hold (for the quasi-ordered frame $\langle W, R \rangle$ we take the partially ordered frame $\langle W', R' \rangle$ where with $[x] = \{y\colon\ x\, R\, y\ \&\ y\, R\, x\}$ we have $W' = \{[x]\colon\ x \in W\}$ and $[x]\, R'\, [y]$ iff $x\, R\, y$). However, for all we know, it seems possible that a logic like **S** be incomplete with respect to any class of quasi-ordered frames but nevertheless complete with respect to a class of serial frames, where the holding of formulae in serial frames is defined in terms of rudimentary Kripke models. In the proof of Proposition 6 we produced out of a rudimentary Kripke model $\langle W, R, v\rangle$ a quasi-ordered Kripke model $\langle W, R', v'\rangle$ in which the same formulae hold, but this does not mean that in $\langle W, R\rangle$ and $\langle W, R'\rangle$ the same formulae will hold. Every valuation v on $\langle W, R\rangle$ will induce an equivalent valuation v' on $\langle W, R'\rangle$, as in the proof of Proposition 6, but on $\langle W, R'\rangle$ we might have valuations to which no valuation corresponds on $\langle W, R\rangle$. For example, let in $\langle W, R\rangle$ *not* $x\,R\,x$; then for a v' on $\langle W, R'\rangle$ we can have

$$x \notin v'(A) \quad \text{and} \quad \forall y\,((\exists k \geqslant 1)\, x\,R^k\,y \Rightarrow y \in v'(A)),$$

but for no v on $\langle W, R\rangle$ we can have

$$x \notin v(A) \quad \text{and} \quad \forall y\,((\exists k \geqslant 1)\, x\,R^k\,y \Rightarrow y \in v(A)).$$

So we ask the following question:

(1) Is there an intermediate propositional logic incomplete with respect to any class of quasi-ordered frames but complete with respect to a class of serial frames?

When we show that a logic like Shekhtman's **S** is incomplete with respect to any class of quasi-ordered frames we find a formula B which is not a theorem of **S** but which holds in every quasi-ordered frame in which all the theorems of **S** hold. In order to show that B is not a theorem of **S**, or of a similar logic, we can use a more general type of frames (like the *general*, or *first-order*, frames in modal logic; see [9, pp. 62–67]). Can serial frames be used for the same purpose, namely:

(2) Is there a set of formulae Γ and a formula B such that in every quasi-ordered frame in which all the members of Γ hold B holds too, whereas there is a serial frame in which all the members of Γ hold and B does not hold?

A positive answer to (1) entails a positive answer to (2), but (2) seems to be a weaker question.

A question related to (1) is:

(3) Is there an intermediate propositional logic incomplete with respect to any class of serial frames?

Let $\Gamma \vdash \mathrm{B}$ mean as usual that there is a proof of B in **H** from hypotheses in Γ. A positive answer to (3) would show that it is impossible to prove for **H** the completeness direction of the following *strong* soundness and completeness for every Γ and B:

$\Gamma \vdash B$ iff for every serial frame $\langle W, R\rangle$, if all the members of Γ hold in $\langle W, R\rangle$, then B holds in $\langle W, R\rangle$.

Shekhtman's result mentioned above shows that such a strong completeness fails when we replace serial by quasi-ordered or partially ordered frames. However, we can easily establish the following *strong* soundness and completeness for **H**:

Proposition 8. *For every Γ and B:*

$\Gamma \vdash B$ iff for every rudimentary Kripke model $\langle W, R, v\rangle$, if all the members of Γ hold in $\langle W, R, v\rangle$, then B holds in $\langle W, R, v\rangle$.

Before proving this proposition let us note that its right-hand side:

$$(*) \qquad \text{for every } \langle W, R, v\rangle, \quad \bigcap_{C\in\Gamma} v(C) = W \Rightarrow v(B) = W$$

is equivalent for rudimentary Kripke models with the seemingly stronger assertion

$$(**) \qquad \text{for every } \langle W, R, v\rangle, \quad \bigcap_{C\in\Gamma} v(C) \subseteq v(B).$$

That $(*)$ implies $(**)$ follows from the fact that for every rudimentary Kripke model $\langle W, r, v\rangle$ and $x \in W$, the *submodel generated by* x, i.e., $\langle W_x, R_x, v_x\rangle$ where

$$W_x = \{y \in W\colon\ (\exists k \geqslant 0)\, x\, R^k\, y\},$$

$$(\forall y, z \in W_x)(y\, R_x\, z \Leftrightarrow y\, R\, z),$$

$$v_x(A) = v(A) \cap W_x,$$

is a rudimentary Kripke model. Suppose that in $\langle W, R, v\rangle$ we have $x \in \bigcap_{C\in\Gamma} v(C)$. Then by the General Heredity of v in $\langle W, R, v\rangle$, in $\langle W_x, R_x, v_x\rangle$ we have $\bigcap_{C\in\Gamma} v_x(C) = W_x$, and by $(*)$ we obtain $v_x(B) = W_x$. So $x \in v(B)$. That $(**)$ implies $(*)$ follows immediately from the definitions and does not rely on either General Heredity or Converse General Heredity.

Proof of Proposition 8. The soundness direction is a simple corollary of Proposition 4. For if $\Gamma \vdash B$, then by the deduction theorem either B is provable in **H** or for some $n \geqslant 1$ and some $C_1, \ldots, C_n \in \Gamma$ we have that $C_1 \to (C_2 \to \cdots \to (C_n \to B) \cdots)$ is provable in **H**. In either case $(**)$ obtains (in the latter case we apply General Heredity and Converse General Heredity). The completeness direction follows immediately from the fact that this implication obtains when we replace rudimentary Kripke models by quasi-ordered Kripke models. □

An analogous strong soundness and completeness can also be proved for $\mathbf{H}^+$ with respect to positive rudimentary Kripke models, and for $\mathbf{H}^{\to}$ with respect to implicative rudimentary Kripke models.

Let us say that $\langle W, R, v\rangle$ is a *pseudo-Kripke model* iff $\langle W, R\rangle$ is a frame and v a pseudo-valuation; A holds in $\langle W, r, v\rangle$ iff $v(A) = W$. (The logic in **L** sound and complete with respect to all pseudo-Kripke models is axiomatized in [2].) We can interpret Proposition 8 as saying that General Heredity and Converse General Heredity are sufficient conditions on pseudo-Kripke models for obtaining the strong soundness and completeness of **H**. That these two heredity conditions are also in a certain sense necessary will be inferred from the following two propositions, which follow immediately from the definitions:

Proposition 9. *A pseudo-valuation v on a frame $\langle W, R\rangle$ satisfies General Heredity iff for every B and C we have $v(B) \subseteq v((C \to C) \to B)$.*

Proposition 10. *A pseudo-valuation v on a frame $\langle W, R\rangle$ satisfies Converse General Heredity iff for every B and C we have $v((C \to C) \to B) \subseteq v(B)$.*

Now we can prove the following:

Proposition 11. *The class of all rudimentary Kripke models is the largest class of pseudo-Kripke models with respect to which **H** is strongly sound and complete in the sense that for every Γ and B, $\Gamma \vdash B$ iff* (**).

Proof. The sufficiency of General Heredity and Converse General Heredity follows from Proposition 8. Next we show their necessity. Since for **H**, for every B and C we have $\{B\} \vdash (C \to C) \to B$ and $\{(C \to C) \to B\} \vdash B$, for each of our pseudo-Kripke models $\langle W, R, v\rangle$, for every B and C we must have $v(B) = v((C \to C) \to B)$. Then we apply Propositions 9 and 10. $\square$

(Note that $\{B\} \vdash (C \to C) \to B$ is related to the deduction theorem, whereas $\{(C \to C) \to B\} \vdash B$ is related to *modus ponens*.)

Though (*) and (**) are equivalent for rudimentary Kripke models, (*) does not imply (**) for every pseudo-Kripke model. For example, that $v(B) = W$ implies $v((C \to C) \to B) = W$ is satisfied for every pseudo-Kripke model, but $v(B) \subseteq v((C \to C) \to B)$ may fail in the absence of General Heredity. So we cannot replace (**) by (*) in Proposition 11. We will see in the last section of [4] that the class of rudimentary Kripke models is properly included in the largest class of pseudo-Kripke models with respect to which **H** is strongly sound and complete in the sense that for every Γ and B, $\Gamma \vdash B$ iff (*); and the latter class is properly included in the largest class of pseudo-Kripke models with respect to which we can prove the ordinary soundness and completeness of **H**.

The Kolmogorov–Johansson, or *minimal*, intuitionistic propositional calculus **J** in **L** is obtained by deleting $\bot \to A$ from our axiomatization of **H**. This system does not differ essentially from $\mathbf{H}^+$, and it is not difficult to obtain a soundness and completeness result for **J** with respect to 'rudimentary' Kripke models which differ from rudimentary Kripke models for **H** only in not requiring $v(\bot) = \emptyset$; the set $v(\bot)$ can be an arbitrary hereditary and conversely hereditary set. 'Rudimentary' Kripke models for **J** need not be serial, and their frames may be completely arbitrary. (So Johansson may after all have been right in calling **J** *minimal.*)

2. A canonical rudimentary Kripke model

We shall now consider a nontrivial rudimentary Kripke model which is not quasi-ordered, but is analogous to the canonical partially ordered Kripke model familiar from the Henkin-style completeness proof for **H**.

A set of formulae Γ is *consistent* iff for some A *not* $\Gamma \vdash A$; the set Γ is *deductively closed* iff for every A we have that $\Gamma \vdash A$ implies $A \in \Gamma$; and Γ has the *disjunction property* iff for every A and B we have that $A \vee B \in \Gamma$ implies $A \in \Gamma$ or $B \in \Gamma$. A set of formulae which is consistent, deductively closed and has the disjunction property will be called a *prime theory*.

A set of formulae Γ will be called *A-maximal* iff $A \notin \Gamma$ and for every B, either $B \in \Gamma$ or $B \to A \in \Gamma$. It is easy to check that a prime theory Γ is maximal (in the sense that for every prime theory Δ if $\Gamma \subseteq \Delta$, then $\Gamma = \Delta$) iff Γ is $\bot$-maximal (the same holds when we replace prime theories by consistent deductively closed sets).

Let $W_c = \{\Gamma\colon\ \Gamma \text{ is a prime theory}\}$, and let us define on W_c the relation R_c by

$$\Gamma R_c \Delta \Leftrightarrow (\Gamma = \Delta \;\&\; (\exists A)\, \Gamma \text{ is } A\text{-maximal}) \textit{ or } \Gamma \subset \Delta$$

where $\Gamma \subset \Delta$ means that Γ is a proper subset of Δ. Next let $v_c(A) = \{\Gamma \in W_c\colon\ A \in \Gamma\}$. We shall call $\langle W_c, R_c, v_c \rangle$ the *canonical rudimentary Kripke model* for **H**. This model differs from the usual canonical Kripke model for **H** only in the definition of R_c; in the usual canonical Kripke model $\Gamma R_c \Delta$ is defined as $\Gamma \subseteq \Delta$. Let us first prove the following proposition:

Proposition 12. *The canonical rudimentary Kripke model for* **H** *is a rudimentary Kripke model.*

Proof. It is clear that W_c is nonempty and that $\langle W_c, R_c \rangle$ is a frame. That the conditions $(v\bot)$, $(v\wedge)$ and $(v\vee)$ are satisfied for v_c follows immediately from the definition of prime theories. To verify $(v\to)$ for v_c we show that for every prime theory Γ and every A and B

(I) $\quad A \to B \in \Gamma \Leftrightarrow \forall\Delta\,(\Gamma R_c \Delta \Rightarrow (A \in \Delta \Rightarrow B \in \Delta)).$

From left to right this follows immediately from the fact that $\Gamma R_c \Delta$ implies $\Gamma \subseteq \Delta$. For the other direction suppose $A \to B \notin \Gamma$; hence $B \notin \Gamma$. If $A \in \Gamma$ and for some C the set Γ is C-maximal, we have $\Gamma R_c \Gamma$, $A \in \Gamma$ and $B \notin \Gamma$. If $A \in \Gamma$ and there is no C such that Γ is C-maximal, then for some D we have $D \notin \Gamma$ and $D \to B \notin \Gamma$, and we extend $\Gamma \cup \{D\}$ to a prime theory Δ such that $A \in \Delta$ and $B \notin \Delta$. This may be done by taking a maximal element of the set of all sets of formulae Θ such that $\Gamma \cup \{D\} \subseteq \Theta$ and *not* $\Theta \vdash B$ (this set is nonempty, since $\Gamma \cup \{D\}$ is in it, and it is closed under unions of nonempty chains with respect to set inclusion; so, by Zorn's Lemma it has a maximal element Δ, for which we can check that it is the required prime theory). Similarly, if $A \notin \Gamma$, we extend $\Gamma \cup \{A\}$ to a prime theory Δ such that $A \in \Delta$ and $B \notin \Delta$.

If in (I) we let A be $C\rightarrow C$, then since in **H** we have $B\leftrightarrow((C\rightarrow C)\rightarrow B)$ we immediately obtain General Heredity and Converse General Heredity for v_c. □

To prove the strong completeness of **H** with respect to rudimentary Kripke models we could use the canonical rudimentary Kripke model instead of the usual canonical Kripke model for **H**. As for the usual canonical model, if *not* $\Gamma\vdash B$, then there is a prime theory Δ such that $\Gamma\subseteq\Delta$ and $B\notin\Delta$.

The canonical rudimentary Kripke model for **H** is serial, as it follows from Proposition 1 (and as can directly be proved by copying the argument in the proof of Proposition 12). This model is also transitive, but it is not reflexive. Let $\Gamma_{\mathbf{H}}$ be the set of theorems of **H**. The set $\Gamma_{\mathbf{H}}$ is a prime theory for which there is no A such that $\Gamma_{\mathbf{H}}$ is A-maximal. Otherwise there would be an A which is not a theorem of **H** such that for a propositional variable p foreign to A we would have that $p\vee(p\rightarrow A)$ is a theorem of **H**. So we don't have $\Gamma_{\mathbf{H}}\,R_c\,\Gamma_{\mathbf{H}}$.

The set $\Gamma_{\mathbf{H}}$ is not the only prime theory Γ for which there is no A such that Γ is A-maximal. Such are also all the prime theories $\Gamma_B=\{C\colon\ \{B\}\vdash C\}$ where B is a *Harrop formula* nonequivalent to $\bot$ in **H**; that Γ_B is a prime theory follows from the fact for Harrop formulae B we have that if $\{B\}\vdash C_1\vee C_2$, then either $\{B\}\vdash C_1$ or $\{B\}\vdash C_2$ (see [8] or [10, p. 55]). That there is no A such that Γ_B is A-maximal is shown as follows. Suppose $A\notin\Gamma_B$, i.e., *not* $\{B\}\vdash A$; then for a p foreign to both B and A we have that *not* $\{B\}\vdash p$ (otherwise we would have $\{B\}\vdash\bot$, and hence also $\{B\}\vdash A$) and *not* $\{B\}\vdash p\rightarrow A$ (otherwise we would have $\{B\}\vdash(C\rightarrow C)\rightarrow A$, and hence $\{B\}\vdash A$). The prime theory $\Gamma_{\mathbf{H}}$ is a particular case of a Γ_B where B is the Harrop formula $q\rightarrow q$.

If in the building of the canonical Kripke model for **H** we start by assuming the definition of the canonical valuation v_c, then General Heredity implies for every Γ and Δ

(II) $\quad \Gamma R_c\Delta\Rightarrow\Gamma\subseteq\Delta,$

Converse General Heredity implies for every Γ and every B

(III) $\quad B\notin\Gamma\Rightarrow\exists\Delta\,(\Gamma R_c\Delta\,\&\,B\notin\Delta),$

and one direction of the condition $(v\rightarrow)$ implies for every Γ and every A and B

(I$\Leftarrow$) $\quad A\rightarrow B\notin\Gamma\Rightarrow\exists\Delta\,(\Gamma R_c\Delta\,\&\,A\in\Delta\,\&\,B\notin\Delta).$

The other direction of $(v\rightarrow)$ and the conditions $(v\bot)$, $(v\wedge)$ and $(v\vee)$, together with (II), (III) and the requirement that $\Gamma_{\mathbf{H}}$ be included in every Γ, imply the definition of prime theories. For prime theories Γ and Δ we have that (I$\Leftarrow$) implies (III), and (II) is equivalent with the converse of (I$\Leftarrow$). So (II) and (I$\Leftarrow$) are equivalent with (I) of the proof of Proposition 12, and as this proof shows, (I) is necessary and sufficient for verifying that $\langle W_c, R_c, v_c\rangle$ is a rudimentary Kripke model.

In the usual canonical Kripke model for **H** we take for R_c the largest relation possible, and we identify R_c with the subset relation $\subseteq$. But, as our canonical rudimentary Kripke model shows, we need not do that. We can take a relation on prime theories properly included in $\subseteq$ which will also satisfy (I). Our new relation R_c is serial, but it is not reflexive. It is also transitive, though this is not needed for rudimentary Kripke models. We leave open the following question:

(4) Can we define on W_c a relation R_c which satisfies (I) and is not transitive, or neither transitive nor reflexive?

Let now W_c be the set of all prime theories which for some A are A-maximal, and define on this W_c the relation R_c and v_c as for our canonical rudimentary Kripke model for **H**. Then R_c coincides with $\subseteq$, and $\langle W_c, R_c, v_c\rangle$ is a partially ordered Kripke model for **H**. To verify this we establish that if *not* $\Gamma \vdash A$, then there is a prime theory Δ such that $\Gamma \subseteq \Delta$ and Δ is A-maximal. So our canonical rudimentary Kripke model for **H** has a partially ordered canonical Kripke model for **H** as a proper submodel.

In the background of our canonical rudimentary Kripke model for **H** lies a representation theorem for Heyting algebras. For a Heyting algebra $\langle \mathscr{A}, \rightarrow, \wedge, \vee, \bot\rangle$, let $W_{\mathscr{A}} = \{x\colon x \text{ is a prime filter of } \mathscr{A}\}$. For $a \in \mathscr{A}$, a prime filter x will be called *a-maximal* iff $a \notin x$ and $(\forall b \in \mathscr{A})(b \in x \text{ or } b \rightarrow a \in x)$. Then we define on $W_{\mathscr{A}}$ the following relation analogous to our R_c:

$$x R_{\mathscr{A}} y \Leftrightarrow (x = y \;\&\; (\exists a \in \mathscr{A})\, x \text{ is } a\text{-maximal}) \textit{ or } x \subset y.$$

If $f(a) = \{x \in W_{\mathscr{A}}\colon a \in x\}$, then $\langle \{f(a)\colon a \in \mathscr{A}\}, \rightarrow_{R_{\mathscr{A}}}, \cap, \cup, \emptyset\rangle$ is a Heyting algebra isomorphic to our initial Heyting algebra $\mathscr{A}$ by the mapping f. As before, the relation $R_{\mathscr{A}}$, though it must be transitive, need not be reflexive. If $\mathscr{A}$ is the Lindenbaum algebra of **H** and x is the principal filter generated by the equivalence class of a Harrop formula nonequivalent to $\bot$, then x is a prime filter for which there is no $a \in \mathscr{A}$ such that x is a-maximal (which is shown by an argument analogous to what we had above for the prime theories Γ_B).

3. Inductive Kripke models

In this section we study frames for which it is enough to assume that pseudo-valuations on them satisfy Atomic Heredity and Converse Atomic Heredity in order to infer by induction on the complexity of A that General Heredity and Converse General Heredity are satisfied. These frames and the corresponding rudimentary Kripke models, which we will call *inductive*, will be more like ordinary frames and Kripke models for **H**, but we shall see that they need not be quasi-ordered.

In ordinary Kripke models for **H** besides the conditions for pseudo-valuations we assume only Atomic Heredity. That General Heredity is satisfied in full generality is then proved by induction on the complexity of A. The transitivity of R is a sufficient condition for this induction to go through. (Actually, in the induction step we use the transitivity of R only for the case when A is of the form $A_1 \rightarrow A_2$, and then we don't need the induction hypothesis; see Proposition 23 below. The instance of General Heredity where A is $\perp$ is satisfied vacuously.) Before showing that the necessary and sufficient condition on frames for this induction to go through is weaker than transitivity we introduce the following notions.

For a frame $\langle W, R \rangle$ and an arbitrary $X \subseteq W$ let

$$\mathbf{Cone}\, X = \{y\colon\ (\exists x \in X)(\exists k \geqslant 0)\, x\, R^k\, y\},$$

$$\mathbf{Cone}^-\, X = \{y\colon\ \textit{not}\ (\exists x \in X)(\exists k \geqslant 0)\, y\, R^k\, x\}.$$

The operations **Cone** and **Cone**$^-$ are connected with hereditary sets by the following two propositions, whose straightforward proofs will be omitted:

Proposition 13. *For every frame $\langle W, R \rangle$ and every $X \subseteq W$, the set* **Cone** *X is the least hereditary superset of X.*

As a corollary of this proposition we obtain that X is hereditary iff **Cone** $X = X$.

Proposition 14. *For every frame $\langle W, R \rangle$ and every $X \subseteq W$, the set* **Cone**$^-$ *X is the greatest hereditary set disjoint from X.*

So, in particular, the sets

$$\mathbf{Cone}\{x\} = \{y\colon\ (\exists k \geqslant 0)\, x\, R^k\, y\},$$

$$\mathbf{Cone}^-\{x\} = \{y\colon\ \textit{not}\ (\exists k \geqslant 0)\, y\, R^k\, x\}$$

are hereditary. It is clear that $y \in \mathbf{Cone}\{x\}$ iff $x \notin \mathbf{Cone}^-\{y\}$. It is also clear that for every $X \subseteq W$ in a frame $\langle W, R \rangle$ the sets

$$\mathbf{Cone}_m\, X = \{y\colon\ (\exists x \in X)(\exists k \geqslant m)\, x\, R^k\, y\},$$

$$\mathbf{Cone}_m^-\, X = \{y\colon\ \textit{not}\ (\exists x \in X)(\exists k \geqslant m)\, y\, R^k\, x,$$

where $m \geqslant 0$, are hereditary.

We shall say that a relation R in a frame $\langle W, R \rangle$ is *weakly transitive* iff

$$\forall x, z\ (x\, R^2\, z\ \Rightarrow\ \exists t\ (x\, R\, t\ \&\ t \in \mathbf{Cone}\{z\}\ \&\ z \in \mathbf{Cone}\{t\})).$$

Then we can prove:

Proposition 15. *In a frame $\langle W, R \rangle$ the relation R is weakly transitive iff for every pseudo-valuation v on $\langle W, R \rangle$, if v satisfies Atomic Heredity, then v satisfies General Heredity.*

Proof. From left to right we proceed by induction on the complexity of A in order to show that v satisfies General Heredity. The crucial case in the induction step is when A is of the form $A_1 \rightarrow A_2$. Suppose for some x and y that $x \in v(A_1 \rightarrow A_2)$, $x R y$ and $y \notin v(A_1 \rightarrow A_2)$. Then for some z we have $y R z$, $z \in v(A_1)$ and $z \notin v(A_2)$. So there is a t such that $x R t$, $(\exists k \geq 0)\, z R^k t$ and $(\exists m \geq 0)\, t R^m z$. By the induction hypothesis we get $t \in v(A_1)$, but since $x R t$ we also have $t \in v(A_2)$. Then again by the induction hypothesis $z \in v(A_2)$, which is a contradiction.

For the other direction suppose that for some x, y and z we have $x R y$, $y R z$ and

$$\forall t\, ((x R t \;\&\; t \in \mathbf{Cone}\{z\}) \Rightarrow z \notin \mathbf{Cone}\{t\}).$$

Then by Propositions 13 and 14 it is clear that there is a pseudo-valuation v which satifies Atomic Heredity such that $v(p_1) = \mathbf{Cone}\{z\}$ and $v(p_2) = \mathbf{Cone}^-\{z\}$. We know that $z \notin \mathbf{Cone}\{t\}$ iff $t \in \mathbf{Cone}^-\{z\}$. It follows that $x \in v(p_1 \rightarrow p_2)$, but since $z \in v(p_1)$ and $z \notin v(p_2)$, we have $y \notin v(p_1 \rightarrow p_2)$. So v does not satisfy General Heredity. □

This proposition shows that in every weakly transitive frame $\langle W, R \rangle$ the set $\mathscr{A}$ of *all* hereditary subsets of W contains $\emptyset$ and is closed under the operations $\rightarrow_R$, $\cap$ and $\cup$. Moreover, the weak transitivity of R is not only sufficient but also necessary for that to be the case. The algebra $\langle \mathscr{A}, \rightarrow_R, \cap, \cup, \emptyset \rangle$ is a distributive lattice with zero which for every $X, Y, Z \in \mathscr{A}$ satisfies

$$X \cap Y \subseteq Z \Rightarrow X \subseteq Y \rightarrow_R Z.$$

The converse implication may fail. For example, let $X = W \rightarrow_R \emptyset$, $Y = W$ and $Z = \emptyset$; then a *dead end* x, i.e. a point x such that there is no y for which $x R y$, will belong to $(W \rightarrow_R \emptyset) \cap W$ but cannot belong to $\emptyset$. So our algebra $\mathscr{A}$ is not necessarily a Heyting algebra.

Weak transitivity is satisfied by transitive frames, but it is clear that this is a weaker condition than transitivity. This condition is exclusively tied to the connective $\rightarrow$ and is not invoked in any other part of the proof of Proposition 15, not involving $\rightarrow$.

However, in connexion with rudimentary Kripke models we are not interested in inferring General Heredity from Atomic Heredity, as we did in the proof of Proposition 15, but we want to infer General Heredity and Converse General Heredity from Atomic Heredity and Converse Atomic Heredity. In other words, we want to infer that a pseudo-valuation which satisfies Atomic Heredity and Converse Atomic Heredity is a valuation. In order to show what are the necessary and sufficient conditions on frames for this, we shall introduce the following notions.

For a frame $\langle W, R \rangle$ a nonempty subset X of W will be called an *ω-chain from* x iff there is a mapping f from the ordinal ω *onto* X such that $f(0) = x$ and

$(\forall n \in \omega)\ f(n) R f(n+1)$. Let $\omega(x) = \{X \subseteq W\colon X$ is an ω-chain from $x\}$. An ω-chain from x makes an infinite sequence $x_0x_1x_2 \cdots$ such that $x_0 = x$ and for every $n \geqslant 0$ we have $x_n \, R \, x_{n+1}$. Since f in the definition of ω-chains need not be *one–one,* there may be repetitions in the sequence $x_0x_1x_2 \cdots$, and an ω-chain need not be infinite; it may actually be the singleton $\{x\}$ if $x R x$. In arbitrary frames there may be points x such that $\omega(x)$ is empty; for example, x may be a dead end. For every x the set $\omega(x)$ is nonempty iff our frame is serial.

For a frame $\langle W, R\rangle$ and $X \subseteq W$ let

$$\mathbf{Cl}_\omega X = \{y\colon\ (\forall Y \in \omega(y))\ Y \cap X \neq \emptyset\}.$$

The set $\mathbf{Cl}_\omega X$ contains all the points y such that every ω-chain from y intersects X. Every y such that $\omega(y)$ is empty will also belong to $\mathbf{Cl}_\omega X$, since for such a y it is vacuously satisfied that every ω-chain from y intersects X. The operation $\mathbf{Cl}_\omega$ satisfies for every $X, Y \subseteq W$:

$$X \subseteq \mathbf{Cl}_\omega X,$$

$$\mathbf{Cl}_\omega \mathbf{Cl}_\omega X = \mathbf{Cl}_\omega X,$$

$$\mathbf{Cl}_\omega X \cup \mathbf{Cl}_\omega Y \subseteq \mathbf{Cl}_\omega (X \cup Y),$$

and in serial frames we also have $\mathbf{Cl}_\omega \emptyset = \emptyset$, but we need not have

$$\mathbf{Cl}_\omega (X \cup Y) \subseteq \mathbf{Cl}_\omega X \cup \mathbf{Cl}_\omega Y.$$

So $\mathbf{Cl}_\omega$ is not quite a topological closure operation. If X and Y are hereditary subsets of W, we also have

$$\mathbf{Cl}_\omega (X \cap Y) = \mathbf{Cl}_\omega X \cap \mathbf{Cl}_\omega Y.$$

The operation $\mathbf{Cl}_\omega$ is analogous to an operation which naturally arises in connexion with Beth models (see [7, 3.2], and the section on rudimentary Beth models in [4]).

It is clear that if X is a conversely hereditary subset of W in a frame $\langle W, R\rangle$ and $y \notin X$, then $(\exists Y \in \omega(y))\ Y \cap X = \emptyset$. The operation $\mathbf{Cl}_\omega$ is connected with conversely hereditary sets by the following proposition:

Proposition 16. *For every frame $\langle W, R\rangle$ and every $X \subseteq W$, the set $\mathbf{Cl}_\omega X$ is the least conversely hereditary superset of X.*

Proof. To show that $\mathbf{Cl}_\omega X$ is conversely hereditary suppose $y \notin \mathbf{Cl}_\omega X$. Then there is a $Y \in \omega(y)$ such that $Y \cap X = \emptyset$ and a $z \in Y$ such that $y R z$. The set $Y \cap \mathbf{Cone}\{z\}$ is an ω-chain from z disjoint from X, i.e., $z \notin \mathbf{Cl}_\omega X$.

To show that $\mathbf{Cl}_\omega X$ is the least conversely hereditary superset of X suppose Y is conversely hereditary and $X \subseteq Y$, and let there be an x such that $x \in \mathbf{Cl}_\omega X$, i.e., $(\forall Z \in \omega(x))\ Z \cap X \neq \emptyset$, and $x \notin Y$. Since Y is conversely hereditary there is a $Z' \in \omega(x)$ such that $Z' \cup X = \emptyset$, which is a contradiction. So $\mathbf{Cl}_\omega X \subseteq Y$. $\square$

As a corollary of this proposition we obtain that X is conversely hereditary iff $\mathbf{Cl}_\omega X = X$.

Propositions 16 and 13 show that $\mathbf{Cl}_\omega$ is analogous to **Cone**. Is there an operation analogous to $\mathbf{Cone}^-$, which applied to X would give the greatest conversely hereditary set disjoint from X? The following example shows that such an operation need not exist. Let $W = \{0, 1, 2\}$ and $R = \{\langle 0, 1\rangle, \langle 0, 2\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle\}$. Then *the* greatest conversely hereditary set disjoint from $\{0\}$ does not exist ($\{1\}$ and $\{2\}$ are conversely hereditary, but $\{1, 2\}$ is not).

The following proposition connects the operation $\mathbf{Cl}_\omega$, and conversely hereditary sets, with reflexivity:

Proposition 17. *In a frame $\langle W, R\rangle$ the relation R is reflexive iff for every $X \subseteq W$ we have $\mathbf{Cl}_\omega X = X$.*

Proof. ($\Rightarrow$) Suppose R is reflexive and $x \in \mathbf{Cl}_\omega X$. Then $\{x\} \in \omega(x)$, and hence $x \in X$.

($\Leftarrow$) Suppose for some x *not* xRx. Then for every $Y \in \omega(x)$ we have $Y \cap \{y\colon xRy\} \neq \emptyset$, i.e., $x \in \mathbf{Cl}_\omega\{y\colon xRy\}$, but $x \notin \{y\colon xRy\}$. $\square$

The following proposition connects the operation $\mathbf{Cl}_\omega$ with hereditary sets:

Proposition 18. *For every frame $\langle W, R\rangle$ and every hereditary $X \subseteq W$, the set $\mathbf{Cl}_\omega X$ is hereditary.*

Proof. Suppose $x \in \mathbf{Cl}_\omega X$, xRy and $Y \in \omega(y)$. Then $\{x\} \cup Y \in \omega(x)$, and hence $(\{x\} \cup Y) \cap X \neq \emptyset$, i.e., for some $z \in \{x\} \cup Y$ we have $z \in X$. If $z \in Y$, then $Y \cap X \neq \emptyset$. If $z \notin Y$, then $z = x$, and $y \in Y \cap X$ since X is hereditary. So $y \in \mathbf{Cl}_\omega X$. $\square$

Hence, $\mathbf{Cone}\,\mathbf{Cl}_\omega\,\mathbf{Cone}\,X = \mathbf{Cl}_\omega\,\mathbf{Cone}\,X$. However, we don't always have $\mathbf{Cl}_\omega\,\mathbf{Cone}\,\mathbf{Cl}_\omega X \subseteq \mathbf{Cone}\,\mathbf{Cl}_\omega X$, i.e., we can have a conversely hereditary set X such that $\mathbf{Cone}\,X$ is not conversely hereditary. For example, let our frame have $W = \{a, b\} \cup \{0, 1, 2, \ldots\}$ and let $R = \{\langle a, 0\rangle, \langle b, 0\rangle\} \cup \{\langle n, n+1\rangle\colon n \in \omega\}$. Then $\{a\}$ is conversely hereditary, but $\mathbf{Cone}\{a\}$ is not, since $b \notin \mathbf{Cone}\{a\}$ and $b \in \mathbf{Cl}_\omega\,\mathbf{Cone}\{a\}$.

We are now ready to show what are the necessary and sufficient conditions on frames for inferring that every pseudo-valuation which satisfies Atomic Heredity and Converse Atomic Heredity is a valuation. We shall say that in a frame $\langle W, R\rangle$ the relation R is *prototransitive* iff

$$\forall x, z\, (xR^2z \Rightarrow (\forall Z \in \omega(z))\, \exists t\, (xRt \;\&\; t \in \mathbf{Cl}_\omega\,\mathbf{Cone}\{z\} \;\&\; t \notin \mathbf{Cl}_\omega\,\mathbf{Cone}^-\,Z)).$$

This condition says that if $x\,R^2\,z$, then for every ω-chain Z from z there is a t such that $x\,R\,t$, every ω-chain from t intersects $\mathbf{Cone}\{z\}$, and there is an ω-chain from t which is disjoint from $\mathbf{Cone}^-\,Z$. If for $X, Y \subseteq W$ we have that for every $x \in X$ there is a $y \in Y$ such that $(\exists k \geq 0)\, x\,R^k\,y$ (i.e., for every $x \in X$ we have $Y \cap \mathbf{Cone}\{x\} \neq \emptyset$), we shall say that X is a *shadow* of Y. The last conjunct above, which claims that there is an ω-chain from t which is disjoint from $\mathbf{Cone}^-\,Z$, says that this ω-chain from t is a shadow of the ω-chain Z. Note that the consequent of the condition of prototransitivity is satisfied vacuously if $\omega(z)$ is empty.

Every transitive relation is prototransitive. For suppose R is transitive, $x\,R^2\,z$ and Z is an ω-chain from z. Then we have $x\,R\,z$, $z \in \mathbf{Cl}_\omega\,\mathbf{Cone}\{z\}$ and there is an ω-chain from z, namely Z itself, which is a shadow of the ω-chain Z. We also have that every weakly transitive relation is prototransitive, but, of course, prototransitivity entails neither weak transitivity nor transitivity.

We shall say that in a frame $\langle W, R\rangle$ the relation R is *protoreflexive* iff

$$\forall x\,(\forall X_1, X_2 \in \omega(x))\,\exists y\,(x\,R\,y\ \&\ y \notin \mathbf{Cl}_\omega\,\mathbf{Cone}^-\,X_1 \\ \&\ y \notin \mathbf{Cl}_\omega\,\mathbf{Cone}^-\,X_2).$$

This condition says that if X_1 and X_2 are ω-chains from x, not necessarily distinct, then there is a y such that $x\,R\,y$ and from y we have an ω-chain which is a shadow of X_1 and an ω-chain which is a shadow of X_2. Every reflexive relation is protoreflexive. For if R is reflexive, then for ω-chains X_1 and X_2 from x we have $x\,R\,x$, and X_1 is a shadow of X_1 and X_2 a shadow of X_2. Of course, protoreflexivity does not entail reflexivity.

We can now prove the proposition for which we have been preparing all along:

Proposition 19. *In a frame $\langle W, R\rangle$ the relation R is serial, prototransitive and protoreflexive iff for every pseudo-valuation v on $\langle W, R\rangle$, if v satisfies Atomic Heredity and Converse Atomic Heredity, then v satisfies General Heredity and Converse General Heredity.*

Proof. ($\Rightarrow$) We proceed by induction on the complexity of A in order to show that v satisfies General Heredity and Converse General Heredity. In the basis of this induction we use the *seriality* of R in order to demonstrate that $v(\bot)$ is conversely hereditary. That $v(\bot)$ is hereditary is satisfied vacuously.

In the induction step we first prove that $v(A_1 \to A_2)$ is hereditary. So suppose for some x and y that $x \in v(A_1 \to A_2)$, $x\,R\,y$ and $y \notin v(A_1 \to A_2)$. Then for some z we have $y\,R\,z$, $z \in v(A_1)$ and $z \notin v(A_2)$. By the Converse General Heredity of the induction hypothesis there is a $Z \in \omega(z)$ such that $Z \cap v(A_2) = \emptyset$. So by the *prototransitivity of R* there is for this Z a t such that $x\,R\,t$, $t \in \mathbf{Cl}_\omega\,\mathbf{Cone}\,\{z\}$ and $t \notin \mathbf{Cl}_\omega\,\mathbf{Cone}^-\,Z$. It $t \notin v(A_1)$, then there is a $U \in \omega(t)$ such that $U \cap v(A_1) = \emptyset$, which contradicts $t \in \mathbf{Cl}_\omega\,\mathbf{Cone}\,\{z\}$ and $\mathbf{Cone}\,\{z\} \subseteq v(A_1)$; we used the Converse General Heredity and General Heredity of the induction hypothesis. So $t \in v(A_1)$, and since $x\,R\,t$, we obtain $t \in v(A_2)$. But there is a $U \in \omega(t)$ which is a shadow of Z, and $U \subseteq v(A_2)$ by the General Heredity of the induction hypothesis. This is in contradiction with $Z \cap v(A_2) = \emptyset$ and the General Heredity of the induction hypothesis.

For the converse hereditariness of $v(A_1 \to A_2)$ suppose $x \notin v(A_1 \to A_2)$, i.e., there is a y such that $x R y$, $y \in v(A_1)$ and $y \notin v(A_2)$. By the converse General Heredity of the induction hypothesis there is a z such that $y R z$ and $z \notin v(A_2)$. By the General Heredity of the induction hypothesis $z \in v(A_1)$, and so $y \notin v(A_1 \to A_2)$. Note that we did not appeal to any particular property of R in this paragraph.

For the hereditariness and converse hereditariness of $v(A_1 \wedge A_2)$, and for the hereditariness of $v(A_1 \vee A_2)$, we do not appeal to any particular properties of R, and we will omit these easy cases. It remains to consider the converse hereditariness of $v(A_1 \vee A_2)$. So suppose $x \notin v(A_1 \vee A_2)$, i.e., $x \notin v(A_1)$ and $x \notin v(A_2)$. By the Converse General Heredity of the induction hypothesis there is an $X_1 \in \omega(x)$ such that $X_1 \cap v(A_1) = \emptyset$ and an $X_2 \in \omega(x)$ such that $X_2 \cap v(A_2) = \emptyset$. So by the *protoreflexivity* of R there is a y such that $x R y$, $y \notin \mathbf{Cl}_\omega\, \mathbf{Cone}^- X_1$ and $y \notin \mathbf{Cl}_\omega\, \mathbf{Cone}^- X_2$. If $y \in v(A_1)$, then for the $Y \in \omega(y)$ which is a shadow of X_1 we would have $Y \subseteq v(A_1)$, which is in contradiction with $X_1 \cup v(A_1) = \emptyset$; we used the General Heredity of the induction hypothesis. So $y \notin v(A_1)$, and we obtain analogously $y \notin v(A_2)$, which means $y \notin v(A_1 \vee A_2)$.

($\Leftarrow$) If R is not *serial*, then $v(\bot)$ is not conversely hereditary. Suppose R is not *prototransitive*, i.e., for some x, y and z we have $x R y$, $y R z$ and there is a $Z \in \omega(z)$ such that

$$\forall t\, ((x R t \;\&\; t \in \mathbf{Cl}_\omega\, \mathbf{Cone}\{z\}) \Rightarrow t \in \mathbf{Cl}_\omega\, \mathbf{Cone}^- Z).$$

Then by Propositions 13, 14, 16 and 18 it is clear that there is a pseudo-valuation v which satisfies Atomic Heredity and Converse Atomic Heredity such that $v(p_1) = \mathbf{Cl}_\omega\, \mathbf{Cone}\{z\}$ and $v(p_2) = \mathbf{Cl}_\omega\, \mathbf{Cone}^- Z$. It follows that $x \in v(p_1 \to p_2)$. We also have $z \in v(p_1)$ and $z \notin v(p_2)$, because $Z \cap \mathbf{Cone}^- Z = \emptyset$ (otherwise for some $t \in Z$ we would have $t \in \mathbf{Cone}^- Z$, but $t R^0 t$). So $y \notin v(p_1 \to p_2)$, and General Heredity fails.

Suppose R is not *protoreflexive*, i.e., for some x there are $X_1, X_2 \in \omega(x)$ such that

$$\forall y\, (x R y \Rightarrow (y \in \mathbf{Cl}_\omega\, \mathbf{Cone}^- X_1 \text{ or } y \in \mathbf{Cl}_\omega\, \mathbf{Cone}^- X_2)).$$

Then by Propositions 14, 16 and 18 it is clear that there is pseudo-valuation v which satisfies Atomic Heredity and Converse Atomic Heredity such that $v(p_1) = \mathbf{Cl}_\omega\, \mathbf{Cone}^- X_1$ and $v(p_2) = \mathbf{Cl}_\omega\, \mathbf{Cone}^- X_2$. It follows that $\forall y\, (x R y \Rightarrow y \in v(p_1 \vee p_2))$. However, $x \notin v(p_1)$, because $X_1 \cap \mathbf{Cone}^- X_1 = \emptyset$, and analogously $x \notin v(p_2)$. So $x \notin v(p_1 \vee p_2)$, and Converse General Heredity fails. □

Proposition 19 shows that in every frame $\langle W, R \rangle$ which is serial, prototransitive and protoreflexive, the set $\mathcal{A}$ of *all* hereditary and conversely hereditary subsets of W contains $\emptyset$ and is closed under the operations $\rightarrow_R$, $\cap$ and $\cup$. Moreover, the seriality, prototransitivity and protoreflexivity of R are not only sufficient but also necessary for that to be the case. As we know, $\langle \mathscr{A}, \rightarrow_R, \cap, \cup, \emptyset \rangle$ is a Heyting algebra.

The seriality of R is exclusively tied to $\perp$, so that if we restrict ourselves to pseudo-valuations v from $\mathbf{L}^+$, we can omit the requirement of seriality from the left-hand side of Proposition 19. Similarly, prototransitivity is exclusively tied to $\rightarrow$ and protoreflexivity to $\vee$. So, if we restrict ourselves to pseudo-valuations v from $\mathbf{L}^{\rightarrow}$, we need to keep only the requirement of prototransitivity on the left-hand side of Proposition 19. The same holds if pseudo-valuations are from the $(\rightarrow, \wedge)$ fragment of $\mathbf{L}$, and if they are from the $(\rightarrow, \perp)$ or $(\rightarrow, \wedge, \perp)$ fragment, we need seriality and prototransitivity. Seriality is equivalent with the condition that $\emptyset$ is conversely hereditary, prototransitivity with the condition that for every $X, Y \subseteq W$ the set $\mathbf{Cl}_\omega\, \mathbf{Cone}\, X \rightarrow_R \mathbf{Cl}_\omega\, \mathbf{Cone}\, Y$ is hereditary, and protoreflexivity with the condition that for every $X, Y \subseteq W$ the set $\mathbf{Cl}_\omega\, \mathbf{Cone}\, X \cup \mathbf{Cl}_\omega\, \mathbf{Cone}\, Y$ is conversely hereditary. This is shown as in the proof of Proposition 19.

We shall call frames $\langle W, R \rangle$ where R is serial, prototransitive and protoreflexive *inductive frames*. An *inductive Kripke model* is then defined as a $\langle W, R, v \rangle$ such that $\langle W, R \rangle$ is an inductive frame and v, called an *inductive valuation*, is a pseudo-valuation which satisfies Atomic Heredity and Converse Atomic Heredity. Proposition 19 guarantees that inductive valuations on inductive frames are valuations, i.e., that inductive Kripke models are rudimentary Kripke models. In every inductive Kripke model $\langle W, R, v \rangle$, for every propositional variable p, the set $v(p)$ may be any hereditary and conversely hereditary subset of W we choose. In an arbitrary rudimentary Kripke model this is not the case, since the set of *all* hereditary and conversely hereditary subsets of W need not be closed under the operations $\rightarrow_R$, $\cap$ and $\cup$. A fortiori, it will not be a Heyting algebra with these operations.

If holding in frames is defined in terms of inductive valuations, instead of valuations of rudimentary Kripke models, then $\mathbf{H}$ is not sound with respect to serial frames but it is sound and complete with respect to inductive frames. Inductive frames do not make the largest class of frames which would give this soundness and completeness, because pseudo-valuations need not satisfy exactly General Heredity and Converse General Heredity in order to secure the soundness of $\mathbf{H}$. As we will show in the last section of [4], somewhat weaker forms of these conditions will also do. However, we know that General Heredity and Converse General Heredity are necessary for the strong soundness and completeness of Proposition 11.

We may also envisage frames of the type $\langle W, R, \mathscr{A}' \rangle$, called *general inductive frames*, where $\langle W, R \rangle$ is an inductive frame and $\mathscr{A}'$ is a subalgebra of the Heyting algebra $\langle \mathscr{A}, \rightarrow_R, \cap, \cup, \emptyset \rangle$ of *all* hereditary and conversely hereditary subsets of W. These frames are analogous to the *general*, or *first-order*, frames in modal logic (see [9, pp. 62–67]). If we restrict inductive valuations on $\langle W, R, \mathscr{A}' \rangle$ to those which take values in $\mathscr{A}'$, we obtain that **H** is sound and complete with respect to all general inductive frames. Ordinary inductive frames may be conceived as a particular type of general inductive frames where $\mathscr{A}'$ is the Heyting algebra of *all* hereditary and conversely hereditary subsets of W. At an even more general level we would have *general rudimentary frames* of the type $\langle W, R, \mathscr{A} \rangle$ where $\langle W, R \rangle$ is a serial frame and $\mathscr{A}$ a particular set of hereditary and conversely hereditary subsets of W which contains $\emptyset$ and is closed under the operations $\rightarrow_R$, $\cap$ and $\cup$. Valuations on these frames would be restricted to those which take values in $\mathscr{A}$.

If holding in frames is defined in terms of pseudo-valuations which satisfy only Atomic Heredity, as for ordinary Kripke models for **H**, then **H** is not sound with respect to inductive frames. We know that in this sense **H** is sound and complete with respect to quasi-ordered frames, but there is an interesting class of frames properly in between the class of inductive frames and the class of quasi-ordered frames with respect to which **H** is also sound and complete in this sense. This is the largest class of frames such that every pseudo-valuation on a frame in this class which satisfies Atomic Heredity will also satisfy General Heredity and Converse General Heredity. Frames in this class, called *weakly quasi-ordered frames*, satisfy *weak reflexivity*:

$$\forall x\, (\exists k \geq 1)\, x\, R^k\, x$$

and *weak transitivity* from Proposition 15:

$$\forall x, z\, (x\, R^2\, z \Rightarrow \exists t\, (x\, R\, t\ \&\ t \in \mathbf{Cone}\{z\}\ \&\ z \in \mathbf{Cone}\{t\})).$$

Reflexivity of course entails weak reflexivity but not vice versa. Also, every quasi-ordered frame is weakly quasi-ordered, but not vice versa.

The following proposition about weak reflexivity is analogous to Proposition 17:

Proposition 20. *In a frame $\langle W, R \rangle$ the relation R is weakly reflexive iff for every hereditary $X \subseteq W$ we have $\mathbf{Cl}_\omega X = X$.*

Proof. ($\Rightarrow$) Suppose R is weakly reflexive, $X \subseteq W$ is hereditary and $x \in \mathbf{Cl}_\omega X$. From weak reflexivity it follows that there is an ω-chain Z from x in which x is cyclically repeated. Since $Z \cap X \neq \emptyset$ and X is hereditary we get $x \in X$.

($\Leftarrow$) Suppose for some x *not* $(\exists k \geq 1)\, x\, R^k\, x$. Then the set $\mathbf{Cone}_1\{x\} = \{y\colon (\exists k \geq 1)\, x\, R^k\, y\}$ is hereditary and $x \in \mathbf{Cl}_\omega\, \mathbf{Cone}_1\{x\}$, but $x \notin \mathbf{Cone}_1\{x\}$. □

As a corollary we obtain:

Proposition 21. *In a frame $\langle W, R \rangle$ the relation R is weakly reflexive iff for every pseudo-valuation v on $\langle W, R \rangle$, if v satisfies Atomic Heredity, then v satisfies Converse Atomic Heredity.*

Propositions 15 and 21 from right to left show that weak reflexivity and weak transitivity are necessary if we want to infer General Heredity and Converse General Heredity from Atomic Heredity. That these conditions are also sufficient follows from Propositions 15 and 20 from left to right.

We can easily verify that weakly quasi-ordered frames could alternatively be defined by assuming weak reflexivity and prototransitivity. So weak reflexivity, which entails seriality and protoreflexivity, is really the new assumption we make when we pass from the class of inductive frames to its proper subclass made of all weakly quasi-ordered frames.

We have already shown in Proposition 15 that the weak transitivity of R in a frame $\langle W, R\rangle$ is necessary and sufficient for the set $\mathscr{A}$ of *all* hereditary subsets of W to contain $\emptyset$ and be closed under the operations $\rightarrow_R$, $\cap$ and $\cup$. However, $\langle \mathscr{A}, \rightarrow_R, \cap, \cup, \emptyset\rangle$ need not have been a Heyting algebra. Proposition 20 shows that the weak reflexivity of R is necessary and sufficient to make every member of $\mathscr{A}$ conversely hereditary. So for every weakly quasi-ordered frame, $\langle \mathscr{A}, \rightarrow_R, \cap, \cup, \emptyset\rangle$ is a Heyting algebra.

We also prove the following opposite of Proposition 21:

Proposition 22. *In a frame $\langle W, R\rangle$ we have*

$$\forall x, y\, (x R y \Rightarrow (\forall Y \in \omega(y))\, x \in Y)$$

iff for every pseudo-valuation v on $\langle W, R\rangle$, if v satisfies Converse Atomic Heredity, then v satisfies Atomic Heredity.

Proof. ($\Rightarrow$) Suppose $x \in v(p)$, $x R y$ and $y \notin v(p)$. Then by Converse Atomic Heredity there is a $Y \in \omega(y)$ such that $Y \cap v(p) = \emptyset$. But since $x R y$ we have that $x \in Y$, which is a contradiction.

($\Leftarrow$) Suppose for some x and y that $x R y$ and there is a $Y \in \omega(y)$ such that $x \notin Y$. Then by Proposition 16 it is clear that there is a pseudo-valuation which satisfies Converse Atomic Heredity such that $v(p) = \mathbf{Cl}_\omega\{x\}$. We infer that $x \in v(p)$ and $y \notin v(p)$, i.e., Atomic Heredity fails. $\square$

This proposition shows what happens if we define holding in frames in terms of pseudo-valuations which satisfy only Converse Atomic Heredity and expect these pseudo-valuations to give rise to rudimentary Kripke models. We know that frames for rudimentary Kripke models must be serial, and with seriality the condition of Proposition 22 entails that if $x R y$, then $x \in \mathbf{Cone}\{y\}$, which with General Heredity would give that for every A we have $x \in v(A)$ iff $y \in v(A)$. But with that, every theorem of the classical propositional calculus would hold. Of course, the condition of Proposition 22 need not be satisfied by quasi-ordered frames.

We have characterized inductive frames and weakly quasi-ordered frames by conditions on pseudo-valuations which indicate that we can define inductively valuations on these frames. Is there a similar characterization of quasi-ordered frames in terms of conditions necessary and sufficient to make valuations inductively definable in some way? The condition on pseudo-valuations which corresponds to reflexivity is contained in Proposition 17, which says that in reflexive frames $\langle W, R\rangle$, and only in reflexive frames, every subset of W is conversely hereditary. So reflexivity implies Converse General Heredity. On the other hand, transitivity secures the hereditariness of $v(A_1 \rightarrow A_2)$, as the following proposition shows:

Proposition 23. *In a frame $\langle W, R\rangle$ the relation R is transitive iff for every pseudo-valuation v on $\langle W, R\rangle$ and every A_1 and A_2 the set $v(A_1 \rightarrow A_2)$ is hereditary.*

Proof. ($\Rightarrow$) Suppose $x \in v(A_1 \rightarrow A_2)$, $x\,R\,y$, $y\,R\,z$ and $z \in v(A_1)$. Then by the transitivity of R we have $x\,R\,z$, and hence $z \in v(A_2)$.

($\Leftarrow$) Suppose $x\,R\,y$, $y\,R\,z$ and *not* $x\,R\,z$. Let $v(p_1) = \{z\}$ and $v(p_2) = \emptyset$ (we may also take $v(p_2) = W - \{z\}$). It follows that $x \in v(p_1 \rightarrow p_2)$ and $y \notin v(p_1 \rightarrow p_2)$. □

This proposition from left to right shows that when for transitive frames we prove by induction on the complexity of A that pseudo-valuations on them which satisfy Atomic Heredity satisfy General Heredity, in the induction step we don't need the induction hypothesis for the case when A is of the form $A_1 \rightarrow A_2$.

Though the conditions corresponding to reflexivity and transitivity are sufficient for the inductive character of valuations, it is not clear what conception of this inductive character would make reflexivity and transitivity also necessary. No doubt, quasi-ordered Kripke models stand out by their simplicity and naturalness, and they are not very far from weakly quasi-ordered rudimentary Kripke models. However, it seems that the preponderance of quasi-ordered Kripke models could not be justified by saying that only in these models valuations are inductively defined.

The previous results show that the assumptions of reflexivity and transitivity for ordinary quasi-ordered Kripke models for **H** are not exactly in the same position. Transitivity secures prototransitivity and weak transitivity, which are tied to implication. Reflexivity secures protoreflexivity, which is tied to disjunction, but it secures also seriality and weak reflexivity, which are not tied to disjunction. Reflexivity also secures at one stroke Converse General Heredity. With reflexivity we have reduced an assumption about valuations to an assumption purely about frames, which does not mention valuations.

If reflexivity is written as $R^0 \subseteq R$, the converse condition $R \subseteq R^0$ would be sufficient for General Heredity as reflexivity is sufficient for Converse General Heredity. However, though we can replace Converse General Heredity by reflexivity, we cannot replace general Heredity by $R \subseteq R^0$. By assuming $R \subseteq R^0$ we would immediately bring in classical propositional logic.

Among inductive frames we find frames in which R is serial, transitive and satisfies *branching density*:

$$\forall x, x_1, x_2 ((x R x_1 \,\&\, x R x_2) \Rightarrow \exists y (x R y \,\&\, y R x_1 \,\&\, y R x_2)),$$

which is a stronger version of protoreflexivity. These inductive frames need not be weakly quasi-ordered. That **H** is sound and complete with respect to these frames, with inductive valuations, was shown in [3]. These frames are interesting because they are the frames with respect to which the normal modal propositional logic **K4N** can be shown sound and complete (of course, with usual modal valuations on these frames). The system **K4N** is axiomatized by adding to the weakest normal modal propositional logic **K** the following axiom-schemata:

(s) $\neg\Box\neg(A \rightarrow A)$,

(t) $\Box A \rightarrow \Box\Box A$,

(bd) $\Box(\Box A \vee \Box B) \rightarrow (\Box A \vee \Box B)$.

This system is the weakest normal modal propositional logic in which **H** can be embedded by the modal translation which prefixes $\Box$ to every proper subformula which is a propositional variable or an implication. (This is shown in [3]; the language in [3] has $\neg$ as primitive instead of $\bot$, but the modal translation just mentioned does not differ essentially from the translation considered there in connexion with the minimality of **K4N** since here $\Box$ is not prefixed to $\bot$.) The schema **(bd)** defines branching density on frames, in the sense that a frame satisfies branching density iff every instance of **(bd)** holds in this frame (with respect to usual modal valuations). That in the same sense **(s)** defines seriality, and **(t)** transitivity, are among the oldest examples in the correspondence theory of modal logic (see [1]).

It is not clear whether the other conditions we have met in connexion with inductive frames: weak reflexivity, weak transitivity, protoreflexivity and prototransitivity, may be defined by modal schemata. The sentences by which we have introduced these conditions are not first-order. The following first-order condition related to weak reflexivity:

$$\forall x (x R^k x),$$

where $k \geq 0$, is defined by the modal schema $\Box^k A \rightarrow A$, where $\Box^0 A$ is A and $\Box^{k+1} A$ is $\Box\Box^k A$. Similarly, the following first-order condition related to weak transitivity:

$$\forall x, z (x R^2 z \Rightarrow \exists t (x R t \,\&\, z R^k t \,\&\, t R^m z)),$$

where $k, m \geq 0$, is defined by:

$$\Box(B \rightarrow \Box^m C) \rightarrow \Box^2(\Box^k B \rightarrow C).$$

The schema **(t)**, i.e. $\Box A \rightarrow \Box^2 A$, is equivalent to this schema when $k = m = 0$.

However, this does not yet answer the question:

(5) Are weak reflexivity, weak transitivity, protoreflexivity and prototransitivity definable by modal schemata?

A similar, but different, question is:

(6) Can we axiomatize sets of modal formulae which hold in inductive frames, or weakly quasi-ordered frames?

If a modal system **M** is sound and complete with respect to a class of frames $\mathscr{C}$ (via modal valuations) and **H** is also sound and complete with respect to $\mathscr{C}$ (via rudimentary Kripke model valuations), we cannot immediately conclude that **H** must be embeddable in **M** by a modal translation. For example, the modal system sound and complete with respect to serial frames is **D**, i.e. **K** + (**s**), but **H** cannot be embedded in **D** by the modal translation which embeds **H** in **K4N**; we obtain the same thing with several other natural modal translations (as will be shown in a paper devoted to modal translations in normal modal logics), and it is unlikely that any modal translation would work. To put it roughly, it is as if Atomic Heredity and Converse Atomic Heredity require an infinity of operators $\Box$ to be prefixed to every p, and in the absence of modality reduction principles like $\Box A \leftrightarrow \Box\Box A$, which is provable in **K4N**, no finite amount of operators $\Box$ would do.

The canonical rudimentary Kripke model for **H** of the previous section, though serial and transitive, is not reflexive. It is clear that it is also not weakly reflexive. However, we leave open the following question:

(7) Does the frame of the canonical rudimentary Kripke model for **H** satisfy branching density, or at least protoreflexivity?

With this question we conclude our preliminary investigation of rudimentary Kripke models. As announced in the introduction, we shall consider some further topics related to rudimentary Kripke models in [4] and [5].

Acknowledgments

I would like to thank Djordje Čubrić, Kit Fine, Silvio Ghilardi, Valentin Shekhtman, Predrag Tanović and Slobodan Vujošević for stimulating discussions about several points of this paper and of [4]. I would also like to thank the organizers of *Logic Colloquium '89* for inviting me to present my results in Berlin in July, 1989. Peter Schroeder-Heister has kindly enabled me to present these results at the *SNS Logic Colloquium* in Tübingen in March, 1990. The preparation of this paper was generously supported by a research scholarship of the Alexander von Humboldt Foundation and by Grant 0401A of the Science Fund of Serbia.

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