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Janet's Algorithm

by

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Abstract

We have introduced the Janet's algorithm for the Stanley decomposition of a monomial ideal $I \subset S = K[x_1, ..., x_n]$ and prove that Janet's algorithm gives the squarefree Stanley decomposition of S/I for a squarefree monomial ideal I. We have also shown that the Janet's algorithm gives a partition of a simplicial complex.

Key Words: Stanley decomposition, squarefree Stanley decomposition, partition of a simplicial complex.
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1 Introduction

Let K be a field, $S = K[x_1, ..., x_n]$ the polynomial ring in n variables. Let $u \in S$ be a monomial and Z a subset of $\{x_1, ..., x_n\}$. We denote by $uK[Z]$ the K-subspace of S whose basis consists of all monomials uv where v is a monomial in $K[Z]$. The K-subspace $uK[Z] \subset S$ is called a Stanley space of dimension $|Z|$. Stanley decomposition has been discussed in various combinatorial and algebraic contexts see [1], [2], [3], [6], [8], [12] and [16].

Let $I \subset S$ be a monomial ideal, and denote by $I^c \subset S$ the K-linear subspace of S spanned by all monomials which do not belong to I. Then $S = I^c \oplus I$ as a K-vector space, and the residues of the monomials in I^c form a K-basis of S/I. One way to obtain the Stanley decomposition for S/I is prime filtration for instance see proof of [7, Theorem 6.5], but not all the Stanley decompositions can be obtained from prime filtrations see [12] and [9].

Let Δ be a simplicial complex of dimension $d - 1$ on the vertex set $V = x_1, ..., x_n$. A subset $\mathcal{I} \in \Delta$ is called an interval, if there exists faces $F, G \subset \Delta$ such that $\mathcal{I} = \{H \in \Delta : F \subseteq H \subseteq G\}$. We denote this interval given by F and G also by $[F, G]$ and call $dim\ (G) - dim\ (F)$ the rank of the interval. A partition $\mathcal{P}$ of Δ is a presentation of Δ as a disjoint union of intervals. The r-vector of $\mathcal{P}$ is the integer vector $r = (r_0, r_1, ..., r_d)$ where r_i is the number of intervals of rank i. Let Δ be a simplicial complex and $\mathcal{F}(\Delta)$ its set of facets. Stanley calls a simplicial complex Δ partitionable if there exists a partition $\Delta = \bigcup_{i=1}^{r} [F_i, G_i]$ with $\mathcal{F}(\Delta) = \{G_1, ..., G_r\}$. We call a partition with this property a nice partition. If a Cohen Macaulay simplicial complex Δ is partitionable then the square free ideal I_Δ will be a Stanley ideal see [6, corollary 3.5].

We have described the Janet's algorithm to obtain the Stanley decomposition of a monomial ideal I. More importantly to obtain a square free Stanley decomposition of I^c for a square free ideal I see Lemma 2.3. When we have an algorithm for the squarefree Stanley decomposition of I^c then from [6, Proposition 3.2] we get a motivation to develop the Janet's algorithm for the partition of a simplicial complex Δ see Lemma 3.1.

Here I would like to give a short description on the history of this subject. The French mathematician Maurice Janet presented an algorithm to construct a special basis (Janet's Basis) for a finitely generated module over $K < \partial_1, ..., \partial_n >$ (where K is a differential field and ∂_i's are partial derivatives) after a longer visit to Hilbert in Göttingen in the early twenties of the last century, cf.[10], [11]. Independently W. Gröbner introduced a device now a days known as Gröbner basis, to compute in residue class rings of polynomial rings in the late thirties, cf.[4], [5], at that time restricted to the zero-dimensional case. In the 1960s, Gröbner basis techniques to compute with modules over the polynomial ring had an enormous boom as a consequence of both, B.Buchberger's thesis constructing Gröbner bases, and the general development of powerful computing devices. By 1980, F.-O. Schreyer proved that Buchberger's so called S-polynomial come very close to a Gröbner bases of the syzygy module. After Janet work has been ignored by the mathematical community more than fifty years, J.-F. Pommaret, working on Spencer cohomology, became aware of Janet's work and pointed out that Janet's algorithm when applied to linear partial differential equations with constant coefficients is a variant of Buchberger's algorithm and the Janet bases is a special case of Gröbner bases in this case, though Janet's philosophy is completely different from Gröbner's philosophy. V. Gerdt and collaborators have shown that Janet's constructive ideas lead to very effective methods. They created an axiomatic framework for Janet's approach called involutive division algorithm. For instance, the Singular package, recently has started to use the Janet or involutive division algorithm to construct the Gröbner bases.

2   Janet's algorithm and Stanley decomposition

In this section, I have given a description on the Janet's algorithm for the Stanley decompositions, note that it is a recursive procedure to find the Stanley decomposition. Also Janet's algorithm give a unique Stanley decomposition after fixing the order of the variables.

Lemma 2.1. *Let $I \subset S = K[x_1, x_2, ..., x_n]$ be a monomial ideal, Janet's algorithm gives a Stanley decomposition of I.*

Proof: By Janet's algorithm, we can write

$$I \cap x_n^k K[x_1, x_2, ..., x_{n-1}] = x_n^k I_k$$

where $I_k \subset K[x_1, x_2, ..., x_{n-1}]$ is a monomial ideal and from construction it is clear that

$$I_0 \subseteq I_1 \subseteq \quad . \quad . \quad . \quad I_k \subseteq I_{k+1} \subseteq . .$$

Let us define

$$\alpha = min\{k \ |I_k \neq 0\}$$

and

$$\beta = min\{k \ |I_k = I_\gamma \ for \ all \ \gamma \geq k\}$$

there exists such a β because $S' = K[x_1, x_2, ..., x_{n-1}]$ is Noetherian so the ascending chain of ideals mentioned above will stabilize at some point.
We will prove it by using induction on n.
For $n = 1$, it is clear.
Suppose all the monomial ideals in $S' = K[x_1, x_2, ..., x_{n-1}]$ has a Stanley decomposition. Now consider $I \subset S = K[x_1, x_2, ..., x_n]$, from above it is clear that

$$I = \bigoplus_k x_n^k I_k$$

where I_k is a monomial ideal in $S' = K[x_1, x_2, ..., x_{n-1}]$ so it has a Stanley decomposition as

$$I_k = \bigoplus_{i_k=1}^{r_k} u_{i_k} K[Z_{i_k}] \qquad for \ all \ k.$$

Now by Janet's algorithm we have the Stanley decomposition of I as follows:

$$I = (\bigoplus_{\alpha \leq k < \beta} x_n^k I_k) \quad \bigoplus \quad (\bigoplus_{k \geq \beta} x_n^k I_k)$$

$$I = \bigoplus_{\alpha \leq k < \beta} (\bigoplus_{i_k=1}^{r_k} u_{i_k} x_n^k K[Z_{i_k}]) \quad \bigoplus \quad (\bigoplus_{i_\beta=1}^{r_\beta} u_{i_\beta} x_n^\beta K[Z_{i_\beta}, x_n]),$$

which is a Stanley decomposition of I. $\qquad \square$

A Stanley space $uK[Z]$ is called a squarefree Stanley space, if u is a squarefree monomial and $supp(u) \subset Z$. Now we will show that in the case of a square free monomial ideal I, Janet's algorithm gives a square free Stanley decomposition recursively in the following lemma;

Lemma 2.2. *If $I \subset S = K[x_1, x_2, ..., x_n]$ is a square free monomial ideal, Janet's algorithm gives a square free Stanley decomposition of I.*

Proof: From above lemma, we can write

$$I \cap x_n^k K[x_1, x_2, ..., x_{n-1}] = x_n^k I_k$$

define α and β as above. For any square free monomial ideal I it is easy to see that $\alpha, \beta \leq 1$ since $I_1 = I_\gamma$ for all $\gamma \geq 1$. As we know that $I_1 \subseteq I_\gamma$ for $\gamma \geq 1$, let us take a monomial $u \in I_\gamma$ then $u \in K[x_1, x_2, ..., x_{n-1}]$ so $ux_n^\gamma \in I \Rightarrow \sqrt{u}.x_n \in I$ $\Rightarrow ux_n \in I$, hence $u \in I_1$.

We will prove it by using induction on n.

For $n = 1$ it is trivial. Suppose every square free monomial ideal I in $S' = K[x_1, x_2, \ldots, x_{n-1}]$ has a square free Stanley decomposition.

Now take $I \subseteq S = K[x_1, x_2, \ldots, x_n]$, by Janet's algorithm we can write

$$I = \bigoplus_k x_n^k I_k \ ,$$

where each I_k is a square free monomial ideal in S' and so, it has a square free Stanley decomposition as follows

$$I_k = \bigoplus_{i_k=1}^{r_k} u_{i_k} K[Z_{i_k}] \ ,$$

for all k. Now by Janet's algorithm we have the Stanley decomposition of I as follows:

$$I = \bigoplus_{k \geq \alpha} x_n^k I_k$$

Janet algorithm gives the Stanley decomposition for different cases as follows:
When $\alpha \neq \beta$ then $\alpha = 0$ and $\beta = 1$, so the Stanley decomposition of I will be of the form:

$$I = (\bigoplus_{i_\alpha=1}^{r_\alpha} u_{i_\alpha} K[Z_{i_\alpha}]) \bigoplus (\bigoplus_{i_\beta=1}^{r_\beta} u_{i_\beta} x_n K[Z_{i_\beta}, x_n])$$

as $supp(u_{i_\beta}) \in Z_{i_\beta} \Rightarrow supp(u_{i_\beta} x_n) \in \{Z_{i_\beta}, x_n\}$ and $u_{i_\beta} x_n$ remain square free as u_{i_β} is square free in S'. Hence it is a square free Stanley decomposition of I.
When $\alpha = \beta (\leq 1)$, then Stanley decomposition of I will be

$$I = \bigoplus_{i_\beta=1}^{r_\beta} u_{i_\beta} x_n^\beta K[Z_{i_\beta}, x_n],$$

where $\beta \leq 1$, this is clearly a square free Stanley decomposition. $\qquad \square$

Now we will describe the Janet's algorithm for a squarefree Stanley decomposition of I^c when $I \subset S = K[x_1, x_2, ..., x_n]$ is a squarefree monomial ideal.

Lemma 2.3. *If $I \subset S = K[x_1, x_2, ..., x_n]$ is a square free monomial ideal, Janet's algorithm gives a square free Stanley decomposition of I^c recursively.*

Proof: For a monomial ideal $I \subset S$, we can write

$$I_k^c x_n^k = I^c \cap x_n^k K[x_1, x_2, \ldots, x_{n-1}] = x_n^k(K[x_1, x_2, \ldots, x_{n-1}] - I_k),$$

where I_k is same as above and we have the inclusions other way around

$$I_0^c \supseteq I_1^c \supseteq \quad \cdots \quad I_k^c \supseteq I_{k+1}^c \supseteq \cdots$$

We will use induction on n;
For $n = 1$, it is trivial.
Suppose there exist a square free Stanley decomposition of J^c for a square free monomial ideal $J \subset S' = K[x_1, x_2, ..., x_{n-1}]$.
Consider $I \subset S = K[x_1, x_2, ..., x_n]$ be a square free monomial ideal, by Janet's algorithm

$$I^c = \bigoplus_k x_n^k I_k^c \quad ,$$

where each $I_k^c \subset S' = K[x_1, x_2, ..., x_{n-1}]$, it has a square free Stanley decomposition as

$$I_k^c = \bigoplus_{i_k=1}^{r_k} u_{i_k} K[Z_{i_k}] \qquad for\ all\ k.$$

Janet algorithm gives the Stanley decomposition for different cases as follows:
(C1) When $\alpha \neq \beta$ ($\alpha = 0$ and $\beta = 1$), so the Stanley decomposition of I^c will be of the form:

$$I^c = (\bigoplus_{i_\alpha=1}^{r_\alpha} u_{i_\alpha} K[Z_{i_\alpha}]) \bigoplus (\bigoplus_{i_\beta=1}^{r_\beta} u_{i_\beta} x_n K[Z_{i_\beta}, x_n])$$

as $supp(u_{i_\beta}) \in \{Z_{i_\beta}\} \Rightarrow supp(u_{i_\beta} x_n) \in \{Z_{i_\beta}, x_n\}$ and $u_{i_\beta} x_n$ remain square free as u_{i_α} is square free in S'. Hence it is a square free Stanley decomposition of I^c.
(C2) When $\alpha = \beta = 0$, the Stanley decomposition of I^c will be of the form:

$$I^c = \bigoplus_{i_\beta=1}^{r_\beta} u_{i_\beta} K[Z_{i_\beta}, x_n]$$

It is clearly a square free Stanley decomposition of I^c.
(C3) When $\alpha = \beta = 1$, the Stanley decomposition of I^c will be of the form:

$$I^c = K[x_1, x_2, ..., x_{n-1}] \bigoplus (\bigoplus_{i_\beta=1}^{r_\beta} u_{i_\beta} x_n K[Z_{i_\beta}, x_n]).$$

$\square$

This lemma gives a motivation to describe the Janet's algorithm for the partitions of simplicial complexes.

3 Janet's algorithm for the partition of simplicial complexes

We will describe the algorithm for the partition of simplicial complex Δ on $[n]$ in the view of above lemma.

Lemma 3.1. *Janet's algorithm gives a partition of a simplicial complex Δ on $[n]$ recursively.*

Proof: For any simplicial complex Δ on $[n]$, we can write

$$\Delta_0 = \Delta \cap \Delta_{[n-1]}$$

$$n\Delta_1 = \Delta \cap n\Delta_{[n-1]}$$

where $\Delta_{[n-1]} = [\emptyset\ ,\ \{123...(n-1)\}]$ and $n\Delta_{[n-1]}$ is the interval $\Delta_{[n-1]}$ shifted with n, namely $n\Delta_{[n-1]} = [n\ ,\ \{12..(n-1)n\}]$.
It should be noted that Δ_0 and Δ_1 are the simplicial complexes on $[n-1]$. We use induction on n.
For $n = 1$, there is nothing to prove.
Suppose the result holds for $n - 1$ i.e, every simplicial complex in $[n - 1]$ has a computed partition.
Consider Δ on $[n]$, by the Janet's algorithm

$$\Delta = \Delta_0 \sqcup n\Delta_1$$

where Δ_0 and Δ_1 are the simplicial complexes on $[n - 1]$, so there exist their partitions:

$$\Delta_0 = \bigsqcup_{i_0=1}^{r_0} [F_{i_0}\ ,\ G_{i_0}]$$

$$\Delta_1 = \bigsqcup_{i_1=1}^{r_1} [F_{i_1}\ ,\ G_{i_1}].$$

Janet's algorithm gives the partition of Δ for different cases as follows:
(C1) When $\Delta_0 \neq \Delta_1$ and $\Delta_0 \neq \Delta_{[n-1]}$, then the partition of Δ will be of the form

$$\Delta = (\bigsqcup_{i_0=1}^{r_0} [F_{i_0}\ ,\ G_{i_0}]) \bigsqcup (\bigsqcup_{i_1=1}^{r_1} [nF_{i_1}\ ,\ nG_{i_1}]).$$

(C2) When $\Delta_0 = \Delta_1$, then the partition of Δ will be of the form

$$\Delta = (\bigsqcup_{i_0=1}^{r_0} [F_{i_0}\ ,\ nG_{i_0}]).$$

(C3) When $\Delta_0 = \Delta_{[n-1]}$, then the partition of Δ will be of the form

$$\Delta = [\emptyset, \{123...(n-1)\}] \bigsqcup (\bigsqcup_{i_1=1}^{r_1} [nF_{i_1}, nG_{i_1}]).$$

$\square$

The following example shows how the Janet's algorithm works to compute the partition of a simplicial complex Δ.

Example 3.2. Let Δ be a simplicial complex given by the facets;

$$\Delta = < \{124\}, \{126\}, \{135\}, \{143\}, \{156\}, \{245\}, \{236\}, \{235\}, \{346\}, \{456\} >$$

Now by applying the Janet's algorithm,

$$\Delta_0 = \Delta \cap \Delta_{[5]} = < \{124\}, \{135\}, \{143\}, \{245\}, \{235\} >$$
$$6\Delta_1 = \Delta \cap 6\Delta_{[5]} = < \{126\}, \{156\}, \{236\}, \{346\}, \{456\} >$$

Now consider Δ_0 in $[5]$, we will use the Janet's algorithm to find its partition.

$$\Delta_0 = < \{124\}, \{135\}, \{143\}, \{245\}, \{235\} >$$

by applying Janet's algorithm,

$$\Delta'_{00} = \Delta_0 \cap \Delta_{[4]} = < \{124\}, \{143\}, \{23\} >$$
$$5\Delta'_{01} = \Delta_0 \cap 5\Delta_{[4]} = < \{135\}, \{245\}, \{235\} >$$

Partition of Δ'_{00} will be as follows;

$$\Delta'_{00} = [\emptyset, \{124\}] \sqcup [\{3\}, \{143\}] \sqcup [\{23\}, \{23\}]$$

Partition of $\Delta'_{01} = < \{13\}, \{24\}, \{23\} >$ will be as follows;

$$\Delta'_{01} = [\emptyset, \{13\}] \sqcup [\{4\}, \{24\}] \sqcup [\{2\}\{23\}]$$

Hence the partition of Δ_0 by Janet's algorithm is as follows;

$$\Delta_0 = [\emptyset, \{124\}] \sqcup [\{3\}, \{143\}] \sqcup [\{23\}, \{23\}] \sqcup [\{5\}, \{135\}] \sqcup [\{45\}, \{245\}] \sqcup [\{25\}\{235\}]$$

Now consider Δ_1 in $[5]$, we will use the Janet's algorithm to find its partition.

$$\Delta_1 = < \{12\}, \{15\}, \{23\}, \{34\}, \{45\} >$$

by applying Janet's algorithm,

$$\Delta'_{10} = \Delta_1 \cap \Delta_{[4]} \quad and \quad 5\Delta'_{11} = \Delta_1 \cap 5\Delta_{[4]}$$

$$\Delta'_{10} = <\{12\}, \{23\}, \{34\}> \quad \Rightarrow \quad \Delta'_{10} = [\emptyset, \{12\}] \sqcup [\{3\}, \{23\}] \sqcup [\{4\}, \{34\}]$$

$$5\Delta'_{11} = <\{15\}, \{45\}> \quad \Rightarrow \quad \Delta'_{11} = [\emptyset, \{1\}] \sqcup [\{4\}, \{4\}]$$

Hence the partition of Δ_1 by Janet's algorithm is as follows;

$$\Delta_1 = [\emptyset, \{12\}] \sqcup [\{3\}, \{23\}] \sqcup [\{4\}, \{34\}] \sqcup [\{5\}, \{15\}] \sqcup [\{45\}, \{45\}]$$

consequently, we have the partition of Δ

$$\Delta = [\emptyset, \{124\}] \sqcup [\{3\}, \{143\}] \sqcup [\{23\}, \{23\}] \sqcup [\{5\}, \{135\}] \sqcup [\{45\}, \{245\}] \sqcup [\{25\}\{235\}]$$

$$\sqcup [\{6\}, \{126\}] \sqcup [\{36\}, \{236\}] \sqcup [\{46\}, \{346\}] \sqcup [\{56\}, \{156\}] \sqcup [\{456\}, \{456\}] \quad \square$$

Remark 3.3. In the above example, it is clear that the partition obtained from Janet's algorithm is not a *nice partition*. Note that Δ in the above example is in fact the simplicial complex given by the triangulation of the real projective plane and it has a *nice partition* see [14, Example 22]. So it is not possible to obtain always a *nice partition* by Janet's algorithm.

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