


Simulations in Rank-Based Büchi Automata Complementation

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Abstract

The long search for an optimal complementation construction for Büchi automata climaxed with the work of Schewe, who proposed a worst-case optimal rank-based procedure that generates complements of a size matching the theoretical lower bound of $(0.76n)^n$, modulo a polynomial factor of $\mathcal{O}(n^2)$. Although worst-case optimal, the procedure in many cases produces automata that are unnecessarily large. In this paper, we propose several ways of how to use the direct and delayed simulation relations to reduce the size of the automaton obtained in the rank-based complementation procedure. Our techniques are based on either (i) ignoring macrostates that cannot be used for accepting a word in the complement or (ii) saturating macrostates with simulation-smaller states, in order to decrease their total number. We experimentally showed that our techniques can indeed considerably decrease the size of the output of the complementation.

2012 ACM Subject Classification Theory of computation → Automata over infinite objects

Keywords and phrases Büchi automata, infinite words, simulation, delayed simulation, rank-based complementation, subsumption

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

1 Introduction

Büchi automata (BA) complementation is a fundamental problem in formal verification, from both theoretical and practical angles. It is, for instance, a critical step in language inclusion testing, which is used in automata-based program termination analysis [15, 8], or a component of decision procedures of some logics, such S1S capturing a decidable fragment of second-order arithmetic [5] or the temporal logics ETL and QPTL [32].

The study of the BA complementation problem can be traced back to 1962, when Büchi introduced his automaton model in the seminal paper [5] in the context of a decision procedure for the S1S fragment of second-order arithmetic. In the paper, a doubly exponential complementation algorithm based on the infinite Ramsey theorem is proposed. In 1988, Safra [29] introduced a complementation procedure with an $n^{\mathcal{O}(n)}$ upper bound and, in the same year, Michel [25] established an $n!$ lower bound. From the traditional theoretical point of view, the problem was already solved, since exponents in the two bounds matched under the \mathcal{O} notation (recall that $n!$ is approximately $(n/e)^n$). From a more practical point of view, a linear factor in an exponent has a significant impact on real-world applications. It was established that the upper bound of Safra's construction is 2^{2n} , so the hunt for an



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42nd Conference on Very Important Topics (CVIT 2016).

Editors: John Q. Open and Joan R. Access; Article No. 23; pp. 23:1–23:17

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

optimal algorithm continued [35]. A series of research efforts participated in narrowing the gap [20, 13, 36, 19, 38]. The long journey climaxed with the result of Schewe [30], who proposed an optimal rank-based procedure that generates complements of a size matching the theoretical lower bound of $(0.76n)^n$ found by Yan [38], modulo a polynomial factor of $\mathcal{O}(n^2)$.

Although the algorithm of Schewe is worst-case optimal, it often generates unnecessarily large complements. The standard approach to alleviate this problem is to decrease the size of the input BA before the complementation starts. Since minimization of (nondeterministic) BAs is a PSPACE-complete problem, more lightweight reduction methods are necessary. The most prevalent approaches are those based on various notions of *simulation-based reduction*, such as reductions based on *direct simulation* [6, 33], a richer *delayed simulation* [11], or their *multi-pebble* variants [12]. These approaches first compute a simulation relation over the input BA—which can be done with the time complexity $\mathcal{O}(mn)$ [16, 18, 27, 28, 7] and $\mathcal{O}(mn^3)$ [11] for direct and delayed simulation respectively, with the number of states n and transitions m —and then construct a *quotient* BA by merging simulation-equivalent states, while preserving the language of the input BA. Then the reduced BA is used as the input of the complementation, which often significantly reduces the size of the result.

In this paper, we propose several ways of how to exploit the direct and delayed simulations in BA complementation even further to obtain smaller complements and shorter running times. We focus, in particular, on the optimal *rank-based* complementation procedure of Schewe [30]. Essentially, the rank-based construction is an extension of traditional subset construction for determinizing finite automata, with some additional information kept in each macrostate (a state in the complemented BA) to track the acceptance condition of all runs of the input automaton on a given word. In particular, it stores the *rank* of each state in a macrostate, which, informally, measures the distance to the last accepting state on the corresponding run in the input BA. The main contributions of this paper are the following optimisations of rank-based complementation algorithms for BAs, for an input BA \mathcal{A} and the output of the rank-based complementation algorithm \mathcal{B} .

1. *Purging*: We use simulation relations over \mathcal{A} to remove some useless macrostates during the construction of \mathcal{B} . In particular, if a state p is simulated by q in \mathcal{A} , this puts a restriction on the relation between the ranks of runs from p and from q . As a consequence, macrostates that assign ranks violating this restriction can be purged from \mathcal{B} .
2. *Saturation*: We saturate macrostates with states that are simulated by the macrostate; this can reduce the total number of states of \mathcal{B} because two or more macrostates can be mapped to a single saturated macrostate. This is inspired by the technique of Glabbeek and Ploeger that uses *closures* in finite automata determinization [14].

The proposed optimizations are orthogonal to simulation-based size reduction mentioned above. Since the quotienting methods are based on taking only the symmetric fragment of the simulation, i.e., they merge states that simulate *each other*, after the quotienting, there might still be many pairs where the simulation holds in only one way, and can therefore be exploited by our techniques. Since the considered notions of simulation-based quotienting preserve the respective simulations, our techniques can be used to optimize the complementation *at no additional cost*. Our experimental evaluation of the optimizations showed that in many cases, they indeed significantly reduce the size of the complemented BA.

2 Preliminaries

We fix a finite nonempty alphabet Σ and the first infinite ordinal $\omega = \{0, 1, \dots\}$. For $n \in \omega$, by $[n]$ we denote the set $\{0, \dots, n\}$. An (infinite) word α is represented as a function

$\alpha : \omega \rightarrow \Sigma$ where the i -th symbol is denoted as α_i . A finite word w of the length $n + 1$ is represented as a function $w : [n] \rightarrow \Sigma$. The finite word of the length 0 is denoted as ϵ . We abuse notation and sometimes also represent α as an infinite sequence $\alpha = \alpha_0\alpha_1\ldots$ and w as a finite sequence $w = w_0\ldots w_{n-1}$. The suffix $\alpha_i\alpha_{i+1}\ldots$ of α is denoted by $\alpha_{i:\omega}$. We use Σ^ω to denote the set of all infinite words over Σ and Σ^* to denote the set of all finite words. For $L \subseteq \Sigma^*$ we define $L^* = \{u \in \Sigma^* \mid u = w_1\cdots w_n \wedge \forall 1 \leq i \leq n : w_i \in L\}$ and $L^\omega = \{\alpha \in \Sigma^\omega \mid \alpha = w_1w_2\cdots \wedge \forall i \geq 1 : w_i \in L\}$ (note that $\{\epsilon\}^\omega = \emptyset$). Given $L_1, L_2 \subseteq \Sigma^*$, we use L_1L_2 to denote the set $\{w_1w_2 \mid w_1 \in L_1, w_2 \in L_2\}$.

A (nondeterministic) *Büchi automaton* (BA) over Σ is a quadruple $\mathcal{A} = (Q, \delta, I, F)$ where Q is a finite set of *states*, δ is a *transition function* $\delta : Q \times \Sigma \rightarrow 2^Q$, and $I, F \subseteq Q$ are the sets of *initial* and *accepting* states respectively. We sometimes treat δ as a set of transitions $p \xrightarrow{a} q$, for instance, we use $p \xrightarrow{a} q \in \delta$ to denote that $q \in \delta(p, a)$. Moreover, we extend δ to sets of states $P \subseteq Q$ as $\delta(P, a) = \bigcup_{p \in P} \delta(p, a)$. A *run* of \mathcal{A} from $q \in Q$ on an input word α is an infinite sequence $\rho : \omega \rightarrow Q$ that starts in q and respects δ , i.e., $\rho_0 = q$ and $\forall i \geq 0 : \rho_i \xrightarrow{\alpha_i} \rho_{i+1} \in \delta$. We say that ρ is *accepting* iff it contains infinitely many occurrences of some accepting state, i.e., $\exists q_f \in F : |\{i \in \omega \mid \rho_i = q_f\}| = \omega$. A word α is *accepted* by \mathcal{A} from a state $q \in Q$ if there is an accepting run ρ of \mathcal{A} from q , i.e., $\rho_0 = q$. The set $\mathcal{L}_{\mathcal{A}}(q) = \{\alpha \in \Sigma^\omega \mid \mathcal{A} \text{ accepts } \alpha \text{ from } q\}$ is called the *language* of q (in \mathcal{A}). Given a set of states $R \subseteq Q$, we define the language of R as $\mathcal{L}_{\mathcal{A}}(R) = \bigcup_{q \in R} \mathcal{L}_{\mathcal{A}}(q)$ and the language of \mathcal{A} as $\mathcal{L}(\mathcal{A}) = \mathcal{L}_{\mathcal{A}}(I)$. For a pair of states p and q in \mathcal{A} , we use $p \subseteq_{\mathcal{L}} q$ to denote $\mathcal{L}_{\mathcal{A}}(p) \subseteq \mathcal{L}_{\mathcal{A}}(q)$.

Without loss of generality, in this paper, we assume \mathcal{A} to be complete, i.e., for every state q and symbol a , it holds that $\delta(q, a) \neq \emptyset$. A *trace* over a word α is an infinite sequence $\pi = q_0 \xrightarrow{\alpha_0} q_1 \xrightarrow{\alpha_1} \cdots$ such that $\rho = q_0q_1\ldots$ is a run of \mathcal{A} over α from q_0 . We say π is *fair* if it contains infinitely many final states. Moreover, we use $p \xrightarrow{w} q$ for $w \in \Sigma^*$ to denote that q is reachable from p over the word w ; if the path from p to q contains an accepting state, we can write $p \xrightarrow[F]{w} q$. In this paper, we fix a complete BA $\mathcal{A} = (Q, \delta, I, F)$.

2.1 Simulations

We introduce simulation relations between states of a BA \mathcal{A} using the game semantics in a similar manner as in [22]. In particular, in a *simulation game* between two players (called Spoiler and Duplicator) from a pair of states (p_0, r_0) , for any (infinite) trace over a word α that Spoiler takes starting from p_0 , Duplicator tries to mimic the trace starting from r_0 . On the other hand, Spoiler tries to find a trace that Duplicator cannot mimic. The game starts in the configuration (p_0, r_0) and every i -th round proceeds by, first, Spoiler choosing a transition $p_i \xrightarrow{\alpha_i} p_{i+1}$ and, second, Duplicator mimicking Spoiler by choosing a matching transition $r_i \xrightarrow{\alpha_i} r_{i+1}$ over the same symbol α_i . The next game configuration is (p_{i+1}, r_{i+1}) . Suppose that $\pi_p = p_0 \xrightarrow{\alpha_0} p_1 \xrightarrow{\alpha_1} \cdots$ and $\pi_r = r_0 \xrightarrow{\alpha_0} r_1 \xrightarrow{\alpha_1} \cdots$ are the two (infinite) traces constructed during the game. Duplicator *wins* the simulation game if $\mathcal{C}^x(\pi_p, \pi_r)$ holds, where $\mathcal{C}^x(\pi_p, \pi_r)$ is a condition that depends on the particular simulation. In the current paper, we consider the following simulation relations:

- **direct** [10]: $\mathcal{C}^{di}(\pi_p, \pi_r) \stackrel{\text{def}}{\iff} \forall i : p_i \in F \Rightarrow r_i \in F$,
- **delayed** [11]: $\mathcal{C}^{de}(\pi_p, \pi_r) \stackrel{\text{def}}{\iff} \forall i : p_i \in F \Rightarrow \exists k \geq i : r_k \in F$, and
- **fair** [17]: $\mathcal{C}^f(\pi_p, \pi_r) \stackrel{\text{def}}{\iff}$ if π_p is fair, then π_r is fair.

A (maximum) x -simulation relation $\preceq_x \subseteq Q \times Q$, for $x \in \{di, de, f\}$, is defined such that $p \preceq_x r$ iff Duplicator has a winning strategy in the simulation game with the winning condition \mathcal{C}^x starting from (p, r) . Formally, we define a strategy to be a (total) mapping

$\sigma : Q \times (Q \times \Sigma \times Q) \rightarrow Q$ such that $\sigma(r, p \xrightarrow{a} p') \in \delta(r, a)$, i.e., if Duplicator is in state r and Spoiler selects a transition $p \xrightarrow{a} p'$, the strategy picks a state r' such that $r \xrightarrow{a} r' \in \delta$ (and because \mathcal{A} is complete, such a transition always exists). Note that Duplicator cannot look ahead at Spoiler's future moves. We use σ_x to denote any winning strategy of Duplicator in the \mathcal{C}^x simulation game. Let σ_x and σ'_x be a pair of winning strategies in the \mathcal{C}^x simulation game. We say that σ_x is *dominated* by σ'_x if for all states p and all transitions $q \xrightarrow{a} q'$ it holds that $\sigma_x(p, q \xrightarrow{a} q') \preceq_x \sigma'_x(p, q \xrightarrow{a} q')$, and that σ_x is *strictly dominated* by σ'_x if σ_x is dominated by σ'_x and σ_x does not dominate σ'_x . A strategy is *dominating* if it is not strictly dominated by any other strategy. Strategies are also lifted to traces as follows: let π_p be as above, then $\sigma(r_0, \pi_p) = r_0 \xrightarrow{\alpha_0} r_1 \xrightarrow{\alpha_1} \dots$ where for all $i \geq 0$ it holds that $\sigma(r_i, p_i \xrightarrow{\alpha_i} p_{i+1}) = r_{i+1}$. The considered simulation relations form the following hierarchy: $\preceq_{di} \subseteq \preceq_{de} \subseteq \preceq_f \subseteq \preceq_{\mathcal{L}}$. Note that every simulation relation is a preorder, i.e., a reflexive and transitive relation.

2.2 Run DAGs

In this section, we recall the terminology from [30] (which is a minor modification of the terminology from [20]). We fix the definition of the *run DAG* of \mathcal{A} over a word α to be a DAG (directed acyclic graph) $\mathcal{G}_\alpha = (V, E)$ of vertices V and edges E where

- $V \subseteq Q \times \omega$ s.t. $(q, i) \in V$ iff there is a run ρ of \mathcal{A} over α with $\rho_i = q$,
- $E \subseteq V \times V$ s.t. $((q, i), (q', i')) \in E$ iff $i' = i + 1$ and $q' \in \delta(q, \alpha_i)$.

Given \mathcal{G}_α as above, we will write $(p, i) \in \mathcal{G}_\alpha$ to denote that $(p, i) \in V$. We call a vertex *accepting* if it contains an accepting state (i.e., it is from $F \times \omega$). \mathcal{G}_α is *rejecting* if it contains no path with infinitely many accepting vertices. A vertex $(p, i) \in \mathcal{G}_\alpha$ is *finite* if the set of vertices reachable from (p, i) is finite, *infinite* if it is not finite, and *endangered* if (p, i) cannot reach an accepting vertex.

We assign ranks to vertices of run DAGs using the following procedure. Let $\mathcal{G}_\alpha^0 = \mathcal{G}_\alpha$ and $j = 0$. Repeat the following steps until the fixpoint or for at most $2n + 1$ steps.

- Set $\text{rank}_\alpha(p, i) := j$ for all finite vertices (p, i) of \mathcal{G}_α^j and let \mathcal{G}_α^{j+1} be \mathcal{G}_α^j minus the vertices with the rank j .
- Set $\text{rank}_\alpha(p, i) := j + 1$ for all endangered vertices (p, i) of \mathcal{G}_α^{j+1} and let \mathcal{G}_α^{j+2} be \mathcal{G}_α^{j+1} minus the vertices with the rank $j + 1$.
- Set $j := j + 2$.

For all vertices v that have not been assigned a rank yet, we assign $\text{rank}_\alpha(v) := \omega$.

► **Lemma 1.** *If $\alpha \notin \mathcal{L}(\mathcal{A})$, then $0 \leq \text{rank}_\alpha(v) \leq 2n$ for all $v \in \mathcal{G}_\alpha$. Moreover, if $\alpha \in \mathcal{L}(\mathcal{A})$, then there is a vertex $(p, 0) \in \mathcal{G}_\alpha$ s.t. $\text{rank}_\alpha(p, 0) = \omega$.*

Proof. Follows from Corollary 3.3 in [20]. ◀

3 Complementing Büchi Automata

We use as the starting point the complementation procedure of Schewe [30, Section 3.1], which we denote as COMP_S . The procedure works with the notion of level rankings. Given $n = |Q|$, a (*level*) *ranking* is a function $f : Q \rightarrow [2n]$ such that $\{f(q_f) \mid q_f \in F\} \subseteq \{0, 2, \dots, 2n\}$, i.e., f assigns even ranks to accepting states of \mathcal{A} . For a ranking f , the *rank* of f is defined as $\text{rank}(f) = \max\{f(q) \mid q \in Q\}$. For a set of states $S \subseteq Q$, we call f to be *S-tight* if (i) it has an odd rank r , (ii) $\{f(s) \mid s \in S\} \supseteq \{1, 3, \dots, r\}$, and (iii) $\{f(q) \mid q \notin S\} = \{1\}$.

A ranking is *tight* if it is Q -tight; we use \mathcal{T} to denote the set of all tight rankings. For a pair of rankings f and f' , a set $S \subseteq Q$, and a symbol $a \in \Sigma$, we use $f' \leq_a^S f$ iff for every $q \in S$ and $q' \in \delta(q, a)$ it holds that $f'(q') \leq f(q)$.

The COMP_S procedure constructs the BA $\mathcal{B}_S = (Q', \delta', I', F')$ whose components are defined as follows:

- $Q' = Q_1 \cup Q_2$ where
 - $Q_1 = 2^Q$ and
 - $Q_2 = \{(S, O, f, i) \in 2^Q \times 2^Q \times \mathcal{T} \times \{0, 2, \dots, 2n-2\} \mid$
 $f \text{ is } S\text{-tight}, O \subseteq S, \exists i \in \omega : O \subseteq f^{-1}(i)\},$
- $I' = \{I\},$
- $\delta' = \delta_1 \cup \delta_2 \cup \delta_3$ where
 - $\delta_1 : Q_1 \times \Sigma \rightarrow 2^{Q_1}$ with $\delta_1(S, a) = \{\delta(S, a)\},$
 - $\delta_2 : Q_1 \times \Sigma \rightarrow 2^{Q_2}$ with $\delta_2(S, a) = \{(S', \emptyset, f, 0) \mid S' = \delta(S, a)\},$ and
 - $\delta_3 : Q_2 \times \Sigma \rightarrow 2^{Q_2}$ with $(S', O', f', i') \in \delta_3((S, O, f, i), a)$ iff $S' = \delta(S, a), f' \leq_a^S f,$
 $\text{rank}(f) = \text{rank}(f'),$ and
 - * $i' = (i+2) \bmod (\text{rank}(f') + 1)$ and $O' = f'^{-1}(i')$ if $O = \emptyset$ or
 - * $i' = i$ and $O' = \delta(O, a) \cap f'^{-1}(i)$ if $O \neq \emptyset,$ and
- $F' = \{\emptyset\} \cup ((2^Q \times \{\emptyset\} \times \mathcal{T} \times \omega) \cap Q_2).$

Intuitively, COMP_S is an extension of the classical subset construction for determinization of finite automata. In particular, $Q_1, \delta_1,$ and I_1 constitute the deterministic finite automaton obtained from \mathcal{A} using the subset construction. The automaton can, however, nondeterministically guess a point at which it will make a transition to a *macrostate* (S, O, f, i) in the Q_2 part; this guess corresponds to a level in the run DAG of the accepted word from which the ranks of all levels form an S -tight ranking, where the S component of the macrostate is again a subset from the subset construction. In the Q_2 part, \mathcal{B}_S makes sure that in order for a word to be accepted by \mathcal{B}_S , all runs of \mathcal{A} over the word need to touch an accepting state only finitely many times. This is ensured by the f component, which, roughly speaking, maps states to ranks of corresponding vertices in the run DAG over the given word. The O component is used for a standard cut-point construction, and is used to make sure that all runs that have reached an accepting state in \mathcal{A} will eventually leave it (this can happen for different runs at a different point). The $S, O,$ and f components were already present in [20]. The i component was introduced by Schewe to improve the complexity of the construction; it is used to cycle over phases, where in each we focus on cut-points of a different rank. See [30] for a more elaborate exposition.

► **Proposition 2** (Corollary 3.3 in [30]). $\mathcal{L}(\mathcal{B}_S) = \overline{\mathcal{L}(\mathcal{A})}.$

4 Purging Macrostates with Incompatible Rankings

Our first optimisation is based on removing from \mathcal{B}_S macrostates $(S, O, f, i) \in Q_2$ whose level ranking f assigns some states of S an unnecessarily high rank. Intuitively, when S contains a state p and a state q such that p is (directly) simulated by q , i.e. $p \preceq_{di} q$, then $f(p)$ needs to be at most $f(q)$. This is because in any word α and its run DAG \mathcal{G}_α in \mathcal{A} , if p and q are at the same level i of \mathcal{G}_α , then the ranks of their vertices v_p and v_q at the given level are either both ω (when $\alpha \in \mathcal{L}(\mathcal{A})$), or such that $\text{rank}_\alpha(v_p) \leq \text{rank}_\alpha(v_q)$ otherwise. This is because the tree rooted in v_p in \mathcal{G}_α is structurally *embedded* within a tree rooted in v_q .

Formally, consider the following predicate on macrostates of \mathcal{B}_S :

$$\mathcal{P}_{di}(S, O, f, i) \quad \text{iff} \quad \exists p, q \in S : p \preceq_{di} q \wedge f(p) > f(q). \quad (1)$$

We modify COMP_S to purge macrostates that satisfy \mathcal{P}_{di} . That is, we create a new procedure PURGE_{di} obtained from COMP_S by modifying the definition of \mathcal{B}_S such that all occurrences of Q_2 are substituted by Q_2^{di} and

$$Q_2^{di} = Q_2 \setminus \{(S, O, f, i) \in Q_2 \mid \mathcal{P}_{di}(S, O, f, i)\}. \quad (2)$$

We denote the BA obtained from PURGE_{di} as \mathcal{B}_S^{di} . The following lemma, proved in Section 4.1 states the correctness of this construction.

► **Lemma 3.** $\mathcal{L}(\mathcal{B}_S^{di}) = \mathcal{L}(\mathcal{B}_S)$

The following natural question arises: Is it possible to extend the purging technique from direct simulation to other notions of simulation? For *fair* simulation, this cannot be done. The reason is that for a pair of states p and q s.t. $p \preceq_f q$, it can happen that for a word $\beta \in \Sigma^\omega$, there can be a trace from p over β that several, but finitely many times touches an accepting state (i.e., a vertex of p in the corresponding run DAG can have any rank between 0 and $2n$), while all traces from q over β can completely avoid touching any accepting state. From the point of view of fair simulation, these are both unfair traces, and, therefore, disregarded.

On the other hand, *delayed* simulation—which is often much richer than direct simulation—can be used, with a small caveat. Intuitively, the delayed simulation can be used because $p \preceq_{de} q$ guarantees that on every level of trees in \mathcal{G}_α rooted in v_p and in v_q , the ranks of corresponding vertices differ at most by one. Formally, let \mathcal{P}_{de} be the following predicate on macrostates of \mathcal{B}_S :

$$\mathcal{P}_{de}(S, O, f, i) \quad \text{iff} \quad \exists p, q \in S : p \preceq_{de} q \wedge f(p) > \llbracket f(q) \rrbracket, \quad (3)$$

where $\llbracket x \rrbracket$ for $x \in \omega$ denotes the smallest even number greater or equal to x . Similarly as above, we create a new procedure, called PURGE_{de} , which is obtained from COMP_S by modifying the definition of \mathcal{B}_S such that all occurrences of Q_2 are substituted by Q_2^{de} and

$$Q_2^{de} = Q_2 \setminus \{(S, O, f, i) \in Q_2 \mid \mathcal{P}_{de}(S, O, f, i)\}. \quad (4)$$

We denote the BA obtained from PURGE_{de} as \mathcal{B}_S^{de} .

► **Lemma 4.** $\mathcal{L}(\mathcal{B}_S^{de}) = \mathcal{L}(\mathcal{B}_S)$

The use of $\llbracket f(q) \rrbracket$ in \mathcal{P}_{de} results in the fact that the two purging techniques are incomparable. For instance, consider a macrostate $(\{p, q\}, \emptyset, \{p \mapsto 2, q \mapsto 1\}, 0)$ such that $p \preceq_{di} q$ and $p \preceq_{de} q$. Then the macrostate will be purged in PURGE_{di} , but not in PURGE_{de} .

The two techniques can, however, be easily combined into a third procedure PURGE_{di+de} , when Q_2 is substituted in COMP_S with Q_2^{di+de} defined as

$$Q_2^{di+de} = Q_2 \setminus \{(S, O, f, i) \in Q_2 \mid \mathcal{P}_{di}(S, O, f, i) \vee \mathcal{P}_{de}(S, O, f, i)\}. \quad (5)$$

We denote the resulting BA as \mathcal{B}_S^{di+de} . Note that, practically, direct simulation can be computed from delayed simulation much more efficiently by taking the delayed simulation as the initial preorder.

► **Lemma 5.** $\mathcal{L}(\mathcal{B}_S^{di+de}) = \mathcal{L}(\mathcal{B}_S)$

Proof. Follows directly from Lemma 3 and Lemma 4. ◀

4.1 Proofs of Lemmas 3 and 4

We first give a lemma that an x -strategy σ_x preserves an x -simulation \preceq_x .

► **Lemma 6.** *Let \preceq_x be an x -simulation (for $x \in \{di, de, f\}$). Then, the following holds: $\forall p, q \in Q : p \preceq_x q \wedge p \xrightarrow{a} p' \in \delta \Rightarrow \exists q' \in Q : q \xrightarrow{a} q' \in \delta \wedge p' \preceq_x q'$.*

Proof. Let $p, q \in Q$ such that $p \preceq_x q$ and $p \xrightarrow{a} p' \in \delta$, and let π_p be a trace starting from p with the first transition $p \xrightarrow{a} p'$. From the definition of x -simulation, there is a winning Duplicator strategy σ_x ; let $\pi_q = \sigma_x(q', \pi_p)$ and let $q \xrightarrow{a} q'$ be the first transition of π_q . Let $\pi_{p'}$ and $\pi_{r'}$ be traces obtained from π_p and π_r by removing their first transitions. It is easy to see that if $\mathcal{C}^x(\pi_p, \pi_r)$ then also $\mathcal{C}^x(\pi_{p'}, \pi_{r'})$ for any $x \in \{di, de, f\}$. It follows that σ_x is also a winning Duplicator strategy from (p', r') . ◀

Next, we focus on delayed simulation and the proof of Lemma 4. In the next lemma, we show that if there is a pair of vertices on some level of the run DAG where one vertex delay-simulates the other one, there exists a relation between their rankings. This will be used to purge some useless rankings from the complemented BA.

► **Lemma 7.** *Let $p, q \in Q$ such that $p \preceq_{de} q$ and $\mathcal{G}_\alpha = (V, E)$ be the run DAG of \mathcal{A} over α . For all $i \geq 0$, it holds that $(p, i) \in V \wedge (q, i) \in V \Rightarrow \text{rank}_\alpha(p, i) \leq \llbracket \text{rank}_\alpha(q, i) \rrbracket$.*

Proof. Consider some $(p, i) \in V$ and $(q, i) \in V$. First, suppose that $\text{rank}_\alpha(q, i) = \omega$. Since rank can be at most ω , it will always hold that $\text{rank}_\alpha(p, i) \leq \llbracket \text{rank}_\alpha(q, i) \rrbracket$.

On the other hand, suppose that $\text{rank}_\alpha(q, i)$ is finite, i.e., $\alpha_{i:\omega}$ is not accepted by q . Then, due to Lemma 1, $0 \leq \text{rank}_\alpha(q, i) \leq 2n$. Because $p \preceq_{de} q$, it holds that $\alpha_{i:\omega}$ is also not accepted by p , and therefore also $0 \leq \text{rank}_\alpha(p, i) \leq 2n$. We now need to show that $0 \leq \text{rank}_\alpha(p, i) \leq \text{rank}_\alpha(q, i) \leq 2n$.

Let $\{\mathcal{G}_\alpha^k\}_{k=0}^{2n+1}$ be the sequence of run DAGs obtained from \mathcal{G}_α in the ranking procedure from Section 2.2. In the following text we use abbreviation $v \in \mathcal{G}_\alpha^m \setminus \mathcal{G}_\alpha^n$ for $v \in \mathcal{G}_\alpha^m \wedge v \notin \mathcal{G}_\alpha^n$. Since the rank of a node (r, j) is given as the number l s.t. $(r, j) \in \mathcal{G}_\alpha^l \setminus \mathcal{G}_\alpha^{l+1}$, we will finish the proof of this lemma by proving the following claim:

▷ **Claim 8.** Let k and l be s.t. $(p, i) \in \mathcal{G}_\alpha^k \setminus \mathcal{G}_\alpha^{k+1}$ and $(q, i) \in \mathcal{G}_\alpha^l \setminus \mathcal{G}_\alpha^{l+1}$. Then $k \leq \llbracket l \rrbracket$.

Proof. We prove the claim by induction on l .

- **Base case:** ($l = 0$) Since we assume \mathcal{A} is complete, no vertex in \mathcal{G}_α^0 is finite.
 ($l = 1$) We prove that if (q, i) is endangered in \mathcal{G}_α^1 , then (p, i) is endangered in \mathcal{G}_α^1 as well (and therefore both would be removed in \mathcal{G}_α^2). For the sake of contradiction, assume that (q, i) is endangered in \mathcal{G}_α^1 and (p, i) is not. Therefore, since \mathcal{G}_α^1 contains no finite vertices, there is an infinite path π from (p, i) s.t. π contains at least one final state. In the following, we abuse notation and, given a strategy σ_{de} and a state $s \in Q$, use $\sigma_{de}((s, i), \pi)$ to denote the path $(s_0, i)(s_1, i+1)(s_2, i+2) \dots$ such that $s_0 = s$ and $\forall j \geq 0$, it holds that $s_{j+1} = \sigma_{de}(s_j, r_{i+j} \xrightarrow{\alpha_{i+j}} r_{i+j+1})$ where $\pi_x = (r_x, x)$ for every $x \geq 0$. Since $p \preceq_{de} q$, there is a corresponding infinite path $\pi' = \sigma_{de}((q, i), \pi)$ that also contains at least one final state. Therefore, (q, i) is not endangered, which is in contradiction to the assumption, so we can conclude that $l = 1 \Rightarrow k = 1$.
- **Inductive step:** We assume the claim holds for all $l < 2j$ and prove the inductive step for even and odd steps independently.
 ($l = 2j$) We prove that if (q, i) is finite in \mathcal{G}_α^l (and therefore would be removed in \mathcal{G}_α^{l+1}), then either $(p, i) \notin \mathcal{G}_\alpha^l$, or (p, i) is also finite in \mathcal{G}_α^l . For the sake of contradiction, we

assume that (q, i) is finite in \mathcal{G}_α^l and that (p, i) is in \mathcal{G}_α^l , but is not finite there (and, therefore, $k > l$). Since (p, i) is not finite in \mathcal{G}_α^l , there is an infinite path π from (p, i) in \mathcal{G}_α^l . Because $p \preceq_{de} q$, it follows that there is an infinite path $\pi' = \sigma_{de}((q, i), \pi)$ in \mathcal{G}_α^0 (π' is not in \mathcal{G}_α^l because (q, i) is finite there). Using Lemma 6 (possibly multiple times) and the fact that (q, i) is finite, we can find vertices (p', x) in π and (q', x) in π' s.t. $p' \preceq_{de} q'$ and (q', x) is not in \mathcal{G}_α^l , therefore, $(q', x) \in \mathcal{G}_\alpha^e \setminus \mathcal{G}_\alpha^{e+1}$ for some $e < l$. Because $(p', x) \in \mathcal{G}_\alpha^l$ and it is not finite (π is infinite), it follows that $(p', x) \in \mathcal{G}_\alpha^f \setminus \mathcal{G}_\alpha^{f+1}$ for some $f > l$, and since $e < l < f$, we have that $f \not\leq e + 1$, implying $f \not\leq \llbracket e \rrbracket$, which is in contradiction to the induction hypothesis.

($l = 2j + 1$) We prove that if (q, i) is endangered in \mathcal{G}_α^l (and therefore would be removed in \mathcal{G}_α^{l+1}), then either $(p, i) \notin \mathcal{G}_\alpha^l$, or (p, i) is removed at the latest in \mathcal{G}_α^{l+1} . For the sake of contradiction, assume that (q, i) is endangered in \mathcal{G}_α^l while (p, i) is removed later than in \mathcal{G}_α^{l+1} . Therefore, since \mathcal{G}_α^l contains no finite vertices (they were removed in the $(l - 1)$ -th step), there is an infinite path π from (p, i) s.t. π contains at least one final state. Because $p \preceq_{de} q$, there is a corresponding path $\pi' = \sigma_{de}((q, i), \pi)$ from (q, i) in \mathcal{G}_α^0 that also contains at least one final state and moreover $\pi' \notin \mathcal{G}_\alpha^l$. Since π' has an infinite number of states (and at least one final), not all states from π' were removed in \mathcal{G}_α^{l-1} , i.e., there is at least one node with rank less or equal to $l - 2$. Using Lemma 6 (also possibly multiple times) we can hence find states (p', x) in π and (q', x) in π' s.t. $p' \preceq_{de} q'$ and (q', x) is not in \mathcal{G}_α^l and has a rank less or equal to $l - 2$, therefore, $(q', x) \in \mathcal{G}_\alpha^e \setminus \mathcal{G}_\alpha^{e+1}$ for some $e < l - 1$. Because $(p', x) \in \mathcal{G}_\alpha^l$, it follows that $(p', x) \in \mathcal{G}_\alpha^f \setminus \mathcal{G}_\alpha^{f+1}$ for some $f \geq l$, and, therefore, $f \not\leq e + 1$, which is in contradiction to the induction hypothesis. \triangleleft

This concludes the proof. \blacktriangleleft

► **Lemma 9.** *Let $p, q \in Q$ such that $p \preceq_{di} q$ and $\mathcal{G}_\alpha = (V, E)$ be the run DAG of \mathcal{A} over α . For all $i \geq 0$, it holds that $(p, i) \in V \wedge (q, i) \in V \Rightarrow \text{rank}_\alpha(p, i) \leq \text{rank}_\alpha(q, i)$.*

Proof. The proof can be obtained as a simplified version of the proof of Lemma 7. \blacktriangleleft

We are now ready to prove Lemma 4.

► **Lemma 4.** $\mathcal{L}(\mathcal{B}_S^{de}) = \mathcal{L}(\mathcal{B}_S)$

Proof. (\subseteq) Follows directly from the fact that \mathcal{B}_S^{de} is obtained by removing states from \mathcal{B}_S . (\supseteq) Let $\alpha \in \mathcal{L}(\mathcal{B}_S)$. As shown in the proof of Lemma 3.2 in [30], there are two cases:

1. (all vertices of \mathcal{G}_α are finite) In this case there is a run $\rho = S_0 S_1 \dots$ in \mathcal{B}_S with $S_0 = I$ and $S_{i+1} \in \delta_1(S_i, \alpha_i)$ for all $i \geq 0$ such that from some p it holds that $S_{p+j} = \emptyset$ for every j . Therefore, ρ is also an accepting run in \mathcal{B}_S^{de} .
2. (\mathcal{G}_α contains an infinite vertex) In this case \mathcal{B}_S contains an accepting run

$$\rho = S_0 S_1 \dots S_p(S_{p+1}, O_{p+1}, f_{p+1}, i_{p+1})(S_{p+2}, O_{p+2}, f_{p+2}, i_{p+2}) \dots$$

with

- $S_0 = I, O_{p+1} = \emptyset$, and $i_{p+1} = 0$,
- $S_{j+1} = \delta(S_j, \alpha_j)$ for all $j \in \omega$,

and, for all $j > p$,

- $O_{j+1} = f_{j+1}^{-1}(i_{j+1})$ if $O_j = \emptyset$ or
 $O_{j+1} = \delta(O_j, \alpha_j) \cap f_{j+1}^{-1}(i_{j+1})$ if $O_j \neq \emptyset$, respectively,
- f_j is the S_j -tight level ranking that maps each $q \in S_j$ to the rank of $(q, j) \in \mathcal{G}_\alpha$,
- $i_{j+1} = i_j$ if $O_j \neq \emptyset$ or
 $i_{j+1} = (i_j + 2) \bmod (\text{rank}(f) + 1)$ if $O_j = \emptyset$, respectively.

Because the ranks assigned by f_j to states of S_j match the ranks of the corresponding vertices in \mathcal{G}_α , using Lemma 7, we can conclude that ρ contains no macrostate of the form (S, O, f, j) , where $f(p) > \llbracket f(q) \rrbracket$ and $p \preceq_{de} q$ for $p, q \in S$. Therefore, ρ is also an accepting run in \mathcal{B}_S^{de} . ◀

► **Lemma 3.** $\mathcal{L}(\mathcal{B}_S^{di}) = \mathcal{L}(\mathcal{B}_S)$

Proof. Similar as the proof of Lemma 4. ◀

5 Saturation of Macrostates

Our second optimisation is inspired by an optimisation of determinisation of classical finite automata from [14, Section 5]. Their optimisation is based on saturating every constructed macrostate in the classical subset construction with all direct-simulation-smaller states. This can reduce the total number of states of the determinized automaton because two or more macrostates can be mapped to a single saturated macrostate. (In Appendix A, we show why an analogue of their *compression* cannot be used.)

We show that a similar technique can be applied to BAs. We do not restrain ourselves to direct simulation, though, and generalize the technique to delayed simulation. In particular, in our optimisation, we saturate the S components of macrostates (S, O, f, i) obtained in COMP_S with all \preceq_{de} -smaller states. Formally, we modify COMP_S by substituting the definition of the constructed transition function δ' with δ'_{Sat} defined as follows:

- $\delta'_{Sat} = \delta_1^{Sat} \cup \delta_2^{Sat} \cup \delta_3^{Sat}$ where
 - $\delta_1^{Sat} : Q_1 \times \Sigma \rightarrow 2^{Q_1}$ with $\delta_1^{Sat}(S, a) = \{cl[\delta(S, a)]\}$,
 - $\delta_2^{Sat} : Q_1 \times \Sigma \rightarrow 2^{Q_2}$ with $\delta_2^{Sat}(S, a) = \{(S', \emptyset, f, 0) \mid S' = cl[\delta(S, a)]\}$, and
 - $\delta_3^{Sat} : Q_2 \times \Sigma \rightarrow 2^{Q_2}$ with $(S', O', f', i') \in \delta_3^{Sat}((S, O, f, i), a)$ iff $S' = cl[\delta(S, a)]$, $f' \leq_a^S f$, $rank(f) = rank(f')$, and
 - * $i' = (i + 2) \bmod (rank(f') + 1)$ and $O' = f'^{-1}(i')$ if $O = \emptyset$ or
 - * $i' = i$ and $O' = \delta(O, a) \cap f'^{-1}(i)$ if $O \neq \emptyset$,

where $cl[S] = \{q \in Q \mid \exists s \in S : q \preceq_{de} s\}$. We denote the obtained procedure as **SATURATE** and the obtained BA as \mathcal{B}_{Sat} .

► **Lemma 10.** $\mathcal{L}(\mathcal{B}_{Sat}) = \mathcal{L}(\mathcal{B}_S)$

Obviously, as direct simulation is stronger than delayed simulation, the previous technique can also use direct simulation only (e.g., when computing the full delayed simulation is computationally too demanding). Moreover, **SATURATE** is also compatible with all PURGE_x algorithms for $x \in \{di, de, di + de\}$ (because they just remove macrostates with incompatible rankings from Q_2)—we call the combined versions $\text{PURGE}_x + \text{SATURATE}$ and the complement BAs they output \mathcal{B}_{Sat}^x .

► **Proposition 11.** $\mathcal{L}(\mathcal{B}_{Sat}^{di}) = \mathcal{L}(\mathcal{B}_{Sat}^{de}) = \mathcal{L}(\mathcal{B}_{Sat}^{di+de}) = \mathcal{L}(\mathcal{B}_S)$

5.1 Proof of Lemma 10

We start with a lemma, used later, that talks about languages of states related by delayed simulation when there is a path between them.

► **Lemma 12.** For $p, q \in Q$ such that $p \preceq_{de} q$, let $L_\top = \{w \in \Sigma^* \mid p \xrightarrow[F]{w} q\}$ and $L_\perp = \{w \in \Sigma^* \mid p \xrightarrow{w} q\}$. Then $L(q) \supseteq (L_\perp^* L_\top)^\omega$.

Proof. First we prove the following claim:

▷ **Claim 13.** For every word $\alpha = w_0 w_1 w_2 \dots \in \Sigma^\omega$ where $w_i \in L_\top \cup L_\perp$, we can construct a trace $\pi = p \xrightarrow{w_0} q_0 \xrightarrow{w_1} q_1 \xrightarrow{w_2} \dots$ over α such that $p \preceq_{de} q_0$ and $q_i \preceq_{de} q_{i+1}$ for all $i \geq 0$.

Proof. We assign $q_0 := q$ and construct the rest of π by the following inductive construction.

- Base case: ($i = 0$) From the assumption it holds that $p \xrightarrow{w_1} q_0$ and $p \preceq_{de} q_0$. From Lemma 6 there is some $r \in Q$ s.t. $q_0 \xrightarrow{w_1} r$ and $q_0 \preceq_{de} r$. We assign $q_1 := r$, so $q_0 \preceq_{de} q_1$.
- Inductive step: Let $\pi' = p \xrightarrow{w_0} q_0 \xrightarrow{w_1} \dots \xrightarrow{w_i} q_i$ be a prefix of a trace such that $q_j \preceq_{de} q_{j+1}$ for every $j < i$. From the transitivity of \preceq_{de} , it follows that $p \preceq_{de} q_i$. From Lemma 6 there is some $r \in Q$ s.t. $q_i \xrightarrow{w_{i+1}} r$ and $q_i \preceq_{de} r$. We assign $q_{i+1} := r$, so $q_i \preceq_{de} q_{i+1}$. ◀

Consider a word $\alpha \in (L_\perp^* L_\top)^\omega$ such that $\alpha = w_0 w_1 w_2 \dots$ for $w_i \in L_\top \cup L_\perp$. We show that $\alpha \in \mathcal{L}(q)$. According to the previous claim, we can construct a trace $\pi = p \xrightarrow{w_0} q = q_0 \xrightarrow{w_1} q_1 \xrightarrow{w_2} \dots$ over α s.t. $p \preceq_{de} q_0$ and $q_i \preceq_{de} q_{i+1}$ for all $i \geq 0$. Since $p \preceq_{de} q$, from Lemma 6 it follows that we can construct a trace $\pi' = q \xrightarrow{w_0} r_0 \xrightarrow{w_1} r_1 \xrightarrow{w_2} \dots$ s.t. $q_i \preceq_{de} r_i$ for every $i \geq 0$. Because α contains infinitely often a subword from L_\top , there is some $\ell \in \omega$ such that $q_\ell \xrightarrow{w_\ell} q_{\ell+1}$ and $r_\ell \xrightarrow{w_\ell} r_{\ell+1}$ for $w_\ell \in L_\top$. Note that it holds that $p \preceq_{de} q_\ell \preceq_{de} r_\ell$. We can again use the claim above to construct a trace $\pi^* = p \xrightarrow[F]{w_\ell} q = s_0 \xrightarrow{w_{\ell+1}} s_1 \xrightarrow{w_{\ell+2}} \dots$ over $\alpha_\ell = w_\ell w_{\ell+1} w_{\ell+2} \dots$ such that $p \preceq_{de} s_0$ and $s_i \preceq_{de} s_{i+1}$ for all $i \geq 0$. Since $p \preceq_{de} r_\ell$, we can simulate π^* from r_ℓ by a trace $\pi^{*'}$, and because $p \xrightarrow[F]{w_\ell} q$, we know that $\pi^{*'}$ will touch an accepting state in finitely many steps. Consider $m \geq \ell$ such that s_m is the first state after the accepting state that is one of the $\{s_0, s_1, \dots\}$ in $\pi^{*'}$. This reasoning could be repeated for all occurrences of a subword from L_\top in π^* , therefore $\alpha \in \mathcal{L}(q)$. ◀

Next, we give a lemma used for establishing correctness of saturating macrostates with \preceq_{de} -smaller states.

► **Lemma 14.** Let $p, q, r \in Q$ such that $r \xrightarrow{a} q \in \delta$ and $p \preceq_{de} q$. Further, let $\mathcal{A}' = (Q, \delta', I, F)$ where $\delta' = \delta \cup \{r \xrightarrow{a} p\}$. Then $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.

Proof. (\subseteq) Clear.

(\supseteq) Consider some $\alpha \in \mathcal{L}(\mathcal{A}')$ and an accepting trace π in \mathcal{A}' over α . There are two cases:

1. (π contains only finitely many transitions $r \xrightarrow{a} p$)

In this case, π is of the form $\pi = \pi_i \pi_\omega$ where π_i is a finite prefix $\pi_i = q_0 \xrightarrow{w_0} r \xrightarrow{a} p \xrightarrow{w_1} r \xrightarrow{a} p \xrightarrow{w_2} \dots \xrightarrow{w_n} r \xrightarrow{a} p$, for $q_0 \in I$, and π_ω is an infinite trace from p that does not contain any occurrence of the transition $r \xrightarrow{a} p$. We construct in \mathcal{A} a trace $\pi' = q_0 \xrightarrow{w_0} r \xrightarrow{a} q \xrightarrow{w_1} r_1 \xrightarrow{a} q_1 \xrightarrow{w_2} \dots \xrightarrow{w_n} r_n \xrightarrow{a} q_n \cdot \pi'_\omega$ as follows. Let σ_{de} be a strategy for \preceq_{de} . We set $r_1 := \sigma_{de}(q, p \xrightarrow{w_1} r)$, so $r \preceq_{de} r_1$. Since $r \xrightarrow{a} q \in \delta$, it follows that there is $r_1 \xrightarrow{a} q_1 \in \delta$ such that $p \preceq_{de} q_1$. For $i > 1$, we set $r_i := \sigma_{de}(q_{i-1}, p \xrightarrow{w_i} r)$. By induction, it follows that $\forall 1 \leq i \leq n : p \preceq_{de} q_i$, in particular $p \preceq_{de} q_n$. We set $\pi'_\omega := \sigma_{de}(q_n, \pi_\omega)$. Since π_ω starts in p and contains infinitely many accepting states and π'_ω starts in q_n and $p \preceq_{de} q_n$, then π'_ω also contains infinitely many accepting states. It follows that π' is accepting, so $\alpha \in \mathcal{L}(\mathcal{A})$.

2. (π contains infinitely many transitions $r \xrightarrow{a} p$)

In this case, π is of the form $\pi = q_0 \xrightarrow{w_0} r \xrightarrow{a} p \xrightarrow{w_1} r \xrightarrow{a} p \xrightarrow{w_2} \dots \xrightarrow{w_n} r \xrightarrow{a} p \xrightarrow{w_{n+1}} \dots$, for $q_0 \in I$ and $\alpha = w_0 a w_1 a w_2 \dots$. Since π is accepting, for infinitely many $i \in \omega$, we have

$p \xrightarrow[F]{w_i a} p$ in \mathcal{A}' and hence also $p \xrightarrow[F]{w_i a} q$ in the original BA \mathcal{A} . Using Lemma 12 and the fact that $p \preceq_{de} q$, we have $w_1 a w_2 a \dots \in L(q)$ and hence $\alpha = w_0 a w_1 a w_2 a \dots \in \mathcal{L}(\mathcal{A})$. ◀

The following lemma guarantees that adding transitions in the way of Lemma 14 does not break the computed delayed simulation and can, therefore, be performed repeatedly, without the need to recompute the simulation.

► **Lemma 15.** *Let \preceq_{de} be a delayed simulation on \mathcal{A} . Further, let $p, q, r \in Q$ be such that $r \xrightarrow{a} q \in \delta$ and $p \preceq_{de} q$, and let $\mathcal{A}' = (Q, \delta', I, F)$ where $\delta' = \delta \cup \{r \xrightarrow{a} p\}$. Then \preceq_{de} is also a delayed simulation on \mathcal{A}' .*

Proof. Let σ_{de} be a dominating strategy compatible with \preceq_{de} and σ'_{de} be a strategy defined for all $s \in Q$ such that $r \preceq_{de} s$ as $\sigma'_{de}(s, x) = \sigma_{de}(s, x)$ when $x \neq (r \xrightarrow{a} p)$ and $\sigma'_{de}(s, r \xrightarrow{a} p) = \sigma_{de}(s, r \xrightarrow{a} q)$. Note that σ'_{de} is also dominating wrt \preceq_{de} . Further, let $t, u \in Q$ be such that $t \preceq_{de} u$. Let $\pi_t = t \xrightarrow{w_1} t_f \xrightarrow{w_2} r \xrightarrow{a} p. \pi'_t$ be a trace over $\alpha = w_1 w_2 a w_\omega \in \Sigma^\omega$ in \mathcal{A}' such that t_f is an accepting state and $t_f \xrightarrow{w_2} r$ does not contain any occurrence of $r \xrightarrow{a} p$. Further, let $\pi_u = u_0 \xrightarrow{w_1} u_f \xrightarrow{w_2} u_i \xrightarrow{a} u_{i+1}. \pi'_u$ be a trace corresponding to a run $u_0 u_1 u_2 \dots$ over α in \mathcal{A} , where $u_0 = u$, constructed as $\pi_u = \sigma'_{de}(u, \pi_t)$.

► **Claim 16.** There is a trace $\pi_v = t \xrightarrow{w_1} v_f. \pi'_v$ over α such that π'_v contains an accepting state and π_v is \preceq_{de} -simulated by π_u at every position.

Proof. We have the following two cases:

- $(t \xrightarrow{w_1} t_f \text{ does not contain any occurrence of } r \xrightarrow{a} p)$
Let $\pi_v = t \xrightarrow{w_1} t_f \xrightarrow{w_2} r \xrightarrow{a} q. \pi'_v$ be a trace in \mathcal{A} over α obtained from π_t by starting with its prefix up to r , taking $r \xrightarrow{a} q$, and continuing with $\pi'_v = \sigma'_{de}(q, \pi'_t)$. Since in π_v , it holds that t_f is at the same position as t_f in π_t , the first part of the claim holds. Further, π_u clearly \preceq_{de} -simulates π_v on $t \xrightarrow{w_1} t_f \xrightarrow{w_2} r$, and because σ'_{de} simulates $r \xrightarrow{a} p$ by a transition to a state u_{i+1} such that $q \preceq_{de} u_{i+1}$ and π'_v is constructed using σ'_{de} , then also the second part of the claim holds.
- $(t \xrightarrow{w_1} t_f \text{ contains at least one occurrence of } r \xrightarrow{a} p)$
Suppose that π_t starts with $t \xrightarrow{w_1} r \xrightarrow{a} p \xrightarrow{w_2} t_f$ such that $t \xrightarrow{w_1} r$ does not contain any $r \xrightarrow{a} p$. Then let us start building π_v such that it starts with $t \xrightarrow{w_1} r \xrightarrow{a} q$. On this prefix, π_v is clearly \preceq_{de} -simulated by the corresponding prefix of π_u . We continue from q using the strategy σ'_{de} . In particular, the next time we reach $r \xrightarrow{a} p$ in π_t while we are at some state v_1 such that $r \preceq_{de} v_1$, we simulate the transition by $\sigma'_{de}(v_1, r \xrightarrow{a} p)$ and so on. We can observe that when we arrive to t_f in π_t , we also arrive to v_f in π_v such that $t_f \preceq_{de} v_f$. Therefore, π'_v contains an accepting state. Moreover, since σ'_{de} is dominating, the second part of the claim also holds. ◀

From the claim above, it follows that the trace $u_f \xrightarrow{w_2} u_i \xrightarrow{a} u_{i+1}. \pi'_u$ contains an accepting state, so $\mathcal{C}^{de}(\pi_t, \pi_u)$. ◀

Finally, we are ready to prove Lemma 10.

► **Lemma 10.** $\mathcal{L}(\mathcal{B}_{Sat}) = \mathcal{L}(\mathcal{B}_S)$

Proof. (\subseteq) Let $\alpha \in \mathcal{L}(\mathcal{B}_{Sat})$ and ρ be an arbitrary accepting run over α in \mathcal{B}_{Sat} such that $\rho = S_0 S_1 \dots S_{n-1} (S_n, O_n, f_n, i_n) (S_{n+1}, O_{n+1}, f_{n+1}, i_{n+1}) \dots$. For the sake of contradiction, assume that $\alpha \in \mathcal{L}(\mathcal{A})$, therefore, there is a run ρ' on α in \mathcal{A} having infinitely many final states. From the monotonicity of tight level rankings, we have that $f_n(\rho'(n)) \geq$

$f_{n+1}(\rho'(n+1)) \geq \dots$. This sequence eventually stabilizes and from the property of level rankings and the fact that ρ' is accepting, it stabilizes in some ℓ such that $f_\ell(\rho'(\ell))$ is even. This, however, means that the O component of macrostates in ρ cannot be emptied infinitely often, and, therefore, ρ is not accepting, which is a contradiction. Hence $\alpha \notin \mathcal{L}(\mathcal{A})$, so (from Proposition 2) $\alpha \in \mathcal{L}(\mathcal{B}_S)$.

(\supseteq) Consider some $\alpha \in \mathcal{L}(\mathcal{B}_S)$. Let \mathcal{A}' be a BA obtained from \mathcal{A} by adding transitions according to Lemma 15. Then from Lemma 14, we have that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$. Therefore, $\alpha \in \mathcal{L}(\mathcal{B}'_S)$ where \mathcal{B}'_S is a BA obtained from \mathcal{A}' using COMP_S . It is easy to see that we can construct a run in \mathcal{B}_{Sat} that mimics the levels of run DAG of α in \mathcal{A}' (i.e., we are able to empty the O component infinitely often). Hence $\alpha \in \mathcal{L}(\mathcal{B}_{Sat})$. \blacktriangleleft

6 Use after Simulation Quotienting

In this short section, we establish that our optimizations introduced in Sections 4 and 5 can be applied with no additional cost in the setting when BA complementation is preceded with simulation-based reduction of the input BA (which is usually helpful), i.e., when the simulation is already computed beforehand for another purpose. In particular, we show that simulation-based reduction preserves the simulation (when naturally extended to the quotient automaton). First, let us formally define the operation of quotienting.

Given an x -simulation \preceq_x for $x \in \{di, de\}$, we use \approx_x to denote the x -similarity relation (i.e., the symmetric fragment) $\approx_x = \preceq_x \cap \preceq_x^{-1}$. Note that since \preceq_x is a preorder, it holds that \approx_x is an equivalence. We use $[q]_x$ to denote the equivalence class of q wrt \approx_x . The *quotient* of a BA $\mathcal{A} = (Q, \delta, I, F)$ wrt \approx_x is the automaton

$$\mathcal{A}/\approx_x = (Q/\approx_x, \delta_{\approx_x}, I_{\approx_x}, F_{\approx_x}) \quad (6)$$

where $\delta_{\approx_x}([q]_x, a) = \{[r]_x \mid r \in \delta([q]_x, a)\}$ and $S_{\approx_x} = \{[q]_x \in Q/\approx_x \mid q \in S\}$ for $S \in \{I, F\}$.

► **Proposition 17** ([6], [11]). *If $x \in \{di, de\}$, then $\mathcal{L}(\mathcal{A}/\approx_x) = \mathcal{L}(\mathcal{A})$.*

► **Remark 18** ([11]). $\mathcal{L}(\mathcal{A}/\approx_f) \neq \mathcal{L}(\mathcal{A})$

The following lemma shows that quotienting preserves direct and delayed simulations.

► **Lemma 19.** *Let \preceq_x be an x -simulation on \mathcal{A} for $x \in \{di, de\}$. Then the relation \preceq_x^\approx defined as $[q]_x \preceq_x^\approx [r]_x$ iff $q \preceq_x r$ is an x -simulation on \mathcal{A}/\approx_x .*

Proof. First, we show that \preceq_x^\approx is well defined, i.e., if $q \preceq_x r$, then for all $q' \in [q]_x$ and $r' \in [r]_x$, it holds that $q' \preceq_x r'$. Indeed, this holds because $q' \approx_x q$ and $r \approx_x r'$, and therefore $q' \preceq_x q \preceq_x r \preceq_x r'$; the transitivity of simulation yields $q' \preceq_x r'$.

Next, let σ_x be a strategy that gives \preceq_x . Consider a trace $[\pi_q]_x = [q_0]_x \xrightarrow{\alpha_0} [q_1]_x \xrightarrow{\alpha_1} \dots$ over a word $\alpha \in \Sigma^\omega$ in \mathcal{A}/\approx_x . Then,

1. for $x = di$ there is a trace $\pi_q = q'_0 \xrightarrow{\alpha_0} q'_1 \xrightarrow{\alpha_1} \dots$ in \mathcal{A} s.t. $q'_0 \in [q_0]_x$ and $q_i \preceq_x q'_i$ for $i \geq 0$. Therefore, if $[q_i]_x$ is final then so is q'_i ;
2. for $x = de$ there is a trace $\pi_q = q'_0 \xrightarrow{\alpha_0} q'_1 \xrightarrow{\alpha_1} \dots$ in \mathcal{A} s.t. $q'_0 \in [q_0]_x$, $q_i \preceq_x q'_i$ for $i \geq 0$ and, moreover, if $[q_i]_x$ is final then there is q'_k for $k \geq i$ s.t. $q'_k \in F$.

Further, let $[q_0]_x \preceq_x^\approx [r_0]_x$. Then there is a trace $\pi_r = \sigma_x(r, \pi_q) = (r = r_0) \xrightarrow{\alpha_0} r_1 \xrightarrow{\alpha_1} \dots$ simulating π_q in \mathcal{A} from r . Further let $[\pi_r]_x = [r_0]_x \xrightarrow{\alpha_0} [r_1]_x \xrightarrow{\alpha_1} \dots$ be its projection into \mathcal{A}/\approx_x . For all $i \geq 0$, we have that $q_i \preceq_x r_i$, and therefore also $[q_i]_x \preceq_x^\approx [r_i]_x$. Since $\mathcal{C}^x(\pi_q, \pi_r)$, then also $\mathcal{C}^x([\pi_q]_x, [\pi_r]_x)$. \blacktriangleleft

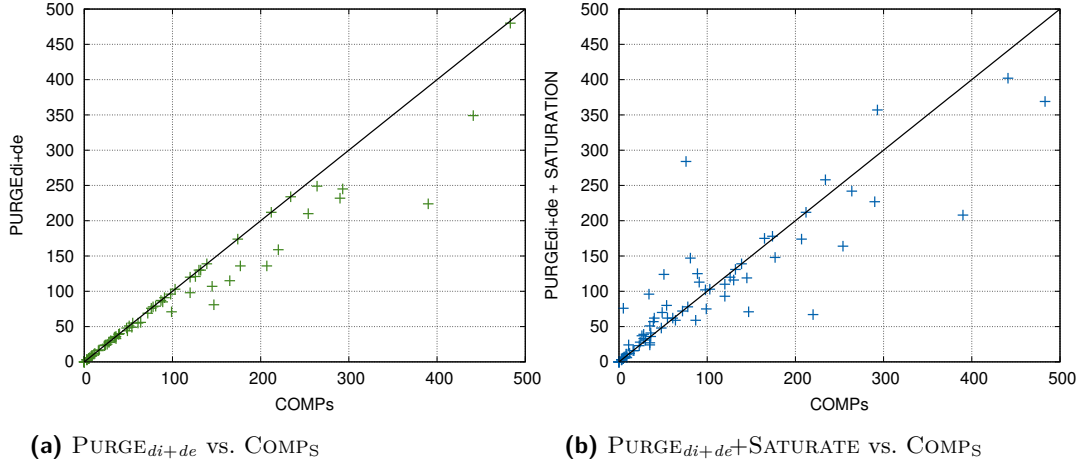


Figure 1 Comparison of the number of states of complement BAs generated by COMP_S and our optimizations (lower is better)

7 Experimental Evaluation

We implemented our optimisations in a prototype tool¹ written in Haskell and performed preliminary experimental evaluation on a set of 100 random BAs over a two-symbol alphabet generated using Tabakov and Vardi’s model [34]. The parameters of the model were set to the following bounds: number of states: 5–7, transition density: 1.2–1.3, and acceptance density: 0.35–0.5. The timeout was set to 300 s.

We present the results for our strongest optimizations for outputs of the size up to 500 states in Figure 1. As can be seen in Figure 1a, purging alone often significantly reduces the size of the output. The situation with saturation is, on the other hand, more complicated. This is expected, because saturating the S component of macrostates also means that more level rankings (the f component) need to be considered.

For outputs of a larger size (we had 11 of them), the results follow a similar trend, but the probability that saturation will increase the size of the result decreases. For some concrete results, for one BA, the size of the output BA decreased from 4065 (COMP_S) to 1017 (PURGE_{di+de}) to 998 ($\text{PURGE}_{di+de} + \text{SATURATE}$), and, for another BA, we obtained a reduction from 1290 (COMP_S) to 107 (PURGE_{di+de}) to 56 ($\text{PURGE}_{di+de} + \text{SATURATE}$) states, which yields a reduction to 8%, resp. 4%! Further, we observed that PURGE_{di} and PURGE_{de} usually give similar results, with the difference of only a few states (PURGE_{di} usually wins).

8 Related Work

BA complementation has a long research track. Known approaches can be roughly classified into Ramsey-based [31], determinization-based [29, 26], rank-based [30], slice-based [19, 36], learning-based [21], and the recently proposed subset-tuple construction [4]. Those approaches build on top of different concepts of capturing words accepted by a complement automaton. Some fundamental concepts can be translated into others, such as the slice-based approach, which can be translated to the rank-based approach [37]. Such a translation is highly valuable,

¹ <https://github.com/vhavlena/ba-complement>

because it can help to get deeper understand of the BA complementation problem and the relationship between optimization techniques for different complementation algorithms.

Because of the high computational complexity of complementing a BA, and, consequently, also checking BA inclusion and universality (which use complementation as their component), there has been some effort to develop heuristics that help to reduce the number of explored states in practical cases. The most prominent ones are heuristics that leverage various notions of simulation relations, which often provide a good compromise between the overhead they impose and the achieved state space reduction. Direct [6, 33], delayed [11], fair [11], and multi-pebble simulations [12] are the best-studied relations of this kind. Some of the relations can be used quotienting, but also for pruning transitions entering simulation-smaller states (which may cause some parts of the BA to become inaccessible). A series of results in this direction was recently developed by Clemente and Mayr [9, 23, 24].

Not only can the relations be used for reducing the size of the input BA, they can also be used for under-approximating inclusion of languages of states. For instance, during a BA inclusion test $\mathcal{L}(\mathcal{A}_S) \stackrel{?}{\subseteq} \mathcal{L}(\mathcal{A}_B)$, if every initial state of \mathcal{A}_S is simulated by an initial state of \mathcal{A}_B , the inclusion holds and no complementation needs to be performed. But simulations can also be used to reduce the explored state space within, e.g., the inclusion check itself, for instance in the context of Ramsey-based algorithms [1, 2]. Ramsey-based complementation algorithms [31] in the worst case produce $2^{\mathcal{O}(n^2)}$ states, which is a significant gap from the lower bound of Michel [25] and Yan [38]. The way simulations are applied in the Ramsey-based approach is fundamentally different from the current work, which is based on rank-based construction. Taking universality checking as an example, the algorithm checks if the language of the complement automaton is empty. They run the complementation algorithm and the emptiness check together, on the fly, and during the construction check if a macrostate with a larger language has been produced before; if yes, then they can stop the search from the language-smaller macrostate. Note that, in contract to our approach, their algorithm does not produce the complement automaton.

9 Conclusion and Future Work

We developed two optimizations of the rank-based complementation algorithm for Büchi automata that are based on leveraging direct and delayed simulation relations to reduce the number of states of the complemented automaton. The optimizations are directly usable in rank-based BA inclusion and universality checking. We conjecture that the decision problem of checking BA language inclusion might also bring another opportunities for exploiting simulation, such as in a similar manner as in [3]. Another, orthogonal, directions of future work are (i) applying simulation in other than the rank-based approach (in addition to the particular use within [1, 2]) and (ii) generalizing our techniques for richer simulations, such as the multi-pebble simulation [12] or various look-ahead simulations [22]. Since the richer simulations are usually harder to compute, it would be interesting to find the sweet spot between the overhead of simulation computation and the achieved state space reduction.

References

- 1 Parosh Aziz Abdulla, Yu-Fang Chen, Lorenzo Clemente, Lukáš Holík, Chih-Duo Hong, Richard Mayr, and Tomáš Vojnar. Simulation Subsumption in Ramsey-based Büchi Automata Universality and Inclusion Testing. In *International Conference on Computer Aided Verification*, pages 132–147. Springer, 2010.

- 2 Parosh Aziz Abdulla, Yu-Fang Chen, Lorenzo Clemente, Lukáš Holík, Chih-Duo Hong, Richard Mayr, and Tomáš Vojnar. Advanced Ramsey-based Büchi Automata Inclusion Testing. In *International Conference on Concurrency Theory*, pages 187–202. Springer, 2011.
- 3 Parosh Aziz Abdulla, Yu-Fang Chen, Lukáš Holík, Richard Mayr, and Tomáš Vojnar. When Simulation Meets Antichains. In Javier Esparza and Rupak Majumdar, editors, *Tools and Algorithms for the Construction and Analysis of Systems*, pages 158–174, Berlin, Heidelberg, 2010. Springer Berlin Heidelberg.
- 4 Joel D. Allred and Ulrich Ultes-Nitsche. A Simple and Optimal Complementation Algorithm for Büchi Automata. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 46–55. ACM, 2018.
- 5 J. Richard Büchi. On a Decision Method in Restricted Second Order Arithmetic. In *Proceedings of International Congress on Logic, Method, and Philosophy of Science 1960*. Stanford Univ. Press, Stanford, Calif, 1962.
- 6 Doron Bustan and Orna Grumberg. Simulation-based Minimization. *ACM Transactions on Computational Logic (TOCL)*, 4(2):181–206, 2003.
- 7 G. Cécé. Foundation for a Series of Efficient Simulation Algorithms. In *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–12, 2017.
- 8 Yu-Fang Chen, Matthias Heizmann, Ondřej Lengál, Yong Li, Ming-Hsien Tsai, Andrea Turrini, and Lijun Zhang. Advanced Automata-based Algorithms for Program Termination Checking. In *Proc. of 39th ACM SIGPLAN Conference on Programming Language Design and Implementation — PLDI’18*, pages 135–150. ACM, 2018.
- 9 Lorenzo Clemente. Büchi Automata Can Have Smaller Quotients. In *International Colloquium on Automata, Languages, and Programming*, pages 258–270. Springer, 2011.
- 10 David L. Dill, Alan J. Hu, and Howard Wong-Toi. Checking for Language Inclusion Using Simulation Preorders. In Kim G. Larsen and Arne Skou, editors, *Computer Aided Verification*, pages 255–265, Berlin, Heidelberg, 1992. Springer.
- 11 K. Etessami, T. Wilke, and R. Schuller. Fair Simulation Relations, Parity Games, and State Space Reduction for Büchi Automata. *SIAM Journal on Computing*, 34(5):1159–1175, 2005.
- 12 Kousha Etessami. A hierarchy of polynomial-time computable simulations for automata. In *International Conference on Concurrency Theory*, pages 131–144. Springer, 2002.
- 13 Ehud Friedgut, Orna Kupferman, and Moshe Vardi. Büchi Complementation Made Tighter. 17:851–868, 08 2006.
- 14 Rob Glabbeek and Bas Ploeger. Five Determinisation Algorithms. In *Proceedings of the 13th International Conference on Implementation and Applications of Automata*, CIAA ’08, pages 161–170, Berlin, Heidelberg, 2008. Springer.
- 15 Matthias Heizmann, Jochen Hoenicke, and Andreas Podelski. Termination Analysis by Learning Terminating Programs. In *International Conference on Computer Aided Verification*, pages 797–813. Springer, 2014.
- 16 M. R. Henzinger, T. A. Henzinger, and P. W. Kopke. Computing Simulations on Finite and Infinite Graphs. In *Proceedings of the 36th Annual Symposium on Foundations of Computer Science*, FOCS ’95, pages 453–462, Washington, DC, USA, 1995. IEEE Computer Society.
- 17 Thomas A. Henzinger, Orna Kupferman, and Sriram K. Rajamani. Fair Simulation. *Inf. Comput.*, 173(1):64–81, 2002.
- 18 Lucian Ilie, Gonzalo Navarro, and Sheng Yu. On NFA Reductions. In *Theory Is Forever, Essays Dedicated to Arto Salomaa on the Occasion of His 70th Birthday*, pages 112–124, 2004.
- 19 Detlef Kähler and Thomas Wilke. Complementation, Disambiguation, and Determinization of Büchi Automata Unified. In *International Colloquium on Automata, Languages, and Programming*, pages 724–735. Springer, 2008.
- 20 Orna Kupferman and Moshe Y. Vardi. Weak Alternating Automata Are Not That Weak. *ACM Trans. Comput. Logic*, 2(3):408–429, July 2001.

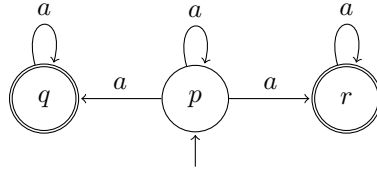
- 21 Yong Li, Andrea Turrini, Lijun Zhang, and Sven Schewe. Learning to Complement Büchi Automata. In *International Conference on Verification, Model Checking, and Abstract Interpretation*, pages 313–335. Springer, 2018.
- 22 Richard Mayr and Lorenzo Clemente. Advanced Automata Minimization. In *The 40th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL '13*, pages 63–74, 2013.
- 23 Richard Mayr and Lorenzo Clemente. Advanced Automata Minimization. In *ACM SIGPLAN Notices*, volume 48, pages 63–74. ACM, 2013.
- 24 Richard Mayr and Lorenzo Clemente. Efficient Reduction of Nondeterministic Automata with Application to Language Inclusion Testing. *Logical Methods in Computer Science*, 15, 2019.
- 25 Max Michel. Complementation is More Difficult with Automata on Infinite Words. *CNET, Paris*, 15, 1988.
- 26 Nir Piterman. From Nondeterministic Büchi and Streett Automata to Deterministic Parity Automata. In *21st Annual IEEE Symposium on Logic in Computer Science (LICS'06)*, pages 255–264. IEEE, 2006.
- 27 Francesco Ranzato and Francesco Tapparo. A New Efficient Simulation Equivalence Algorithm. In *22nd Annual IEEE Symposium on Logic in Computer Science (LICS 2007)*, pages 171–180, 2007.
- 28 Francesco Ranzato and Francesco Tapparo. An Efficient Simulation Algorithm based on Abstract Interpretation. *Information and Computation*, 208(1):1–22, 2010.
- 29 Shmuel Safra. On the Complexity of ω -automata. In *29th Annual Symposium on Foundations of Computer Science*, pages 319–327. IEEE, 1988.
- 30 Sven Schewe. Büchi Complementation Made Tight. In *26th International Symposium on Theoretical Aspects of Computer Science*, volume 3 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 661–672. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2009.
- 31 A. Prasad Sistla, Moshe Y. Vardi, and Pierre Wolper. The Complementation Problem for Büchi Automata with Applications to Temporal Logic. *Theoretical Computer Science*, 49(2-3):217–237, 1987.
- 32 Prasad A. Sistla, Moshe Y. Vardi, and Pierre Wolper. The complementation problem for Büchi automata with applications to temporal logic. In *Automata, Languages and Programming*, pages 465–474, Berlin, Heidelberg, 1985. Springer.
- 33 Fabio Somenzi and Roderick Bloem. Efficient Büchi automata from LTL formulae. In *International Conference on Computer Aided Verification*, pages 248–263. Springer, 2000.
- 34 Deian Tabakov and Moshe Y. Vardi. Experimental Evaluation of Classical Automata Constructions. In Geoff Sutcliffe and Andrei Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning*, pages 396–411, Berlin, Heidelberg, 2005. Springer.
- 35 Moshe Y. Vardi. The Büchi Complementation Saga. In *Annual Symposium on Theoretical Aspects of Computer Science*, pages 12–22. Springer, 2007.
- 36 Moshe Y. Vardi and Thomas Wilke. Automata: From Logics to Algorithms. *Logic and Automata*, 2:629–736, 2008.
- 37 Moshe Y. Vardi, Thomas Wilke, Orna Kupferman, and Seth J. Fogarty. Unifying Büchi Complementation Constructions. *Logical Methods in Computer Science*, 9, 2013.
- 38 Qiqi Yan. Lower Bounds for Complementation of ω -automata Via the Full Automata Technique. In *International Colloquium on Automata, Languages, and Programming*, pages 589–600. Springer, 2006.

A

 Remarks on Compression

An analogy to saturation of macrostates is their compression [14, Section 6]. This is based on removing simulation-smaller states from a macrostate. This is, however, not possible even for direct simulation, as we can see in the following example.

► **Example 20.** Consider the BA over $\Sigma = \{a\}$ given below.



For this BA we have $q \preceq_{di} r$ and $r \preceq_{di} q$. If we compress the macrostates obtained in COMP_S , there is the following trace in the output automaton:

$$\begin{aligned}
 \{p\} &\xrightarrow{a} (\{p, q\}, \emptyset, \{p \mapsto 3, q \mapsto 2, r \mapsto 1\}, 0) \xrightarrow{a} (\{p, r\}, \{r\}, \{p \mapsto 3, q \mapsto 1, r \mapsto 2\}, 2) \\
 &\xrightarrow{a} (\{p, q\}, \emptyset, \{p \mapsto 3, q \mapsto 2, r \mapsto 1\}, 2) \xrightarrow{a} (\{p, r\}, \{r\}, \{p \mapsto 3, q \mapsto 1, r \mapsto 2\}, 0) \\
 &\xrightarrow{a} (\{p, q\}, \emptyset, \{p \mapsto 3, q \mapsto 2, r \mapsto 1\}, 0) \xrightarrow{a} \dots
 \end{aligned}$$

This trace contains infinitely many final states (we flush the O -set infinitely often), hence we are able to accept the word a^ω , which is, however, in the language of the input BA. ◀