PARTIAL WELL-ORDERING OF SETS OF VECTORS

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§1. Graham Higman (1) has investigated quasi-order [(2), p. 4] relations $a \leqslant b$ on a set S which, in his terminology, have the following *finite basis property*:

If $A \subset S$, then there is a finite set $B \subset A$ such that, given any $a \varepsilon A$, there is some $b \varepsilon B$ satisfying $b \leqslant a$.

In this note such a set S is said to be *partially well-ordered*. Such order relations were considered by Kaplansky [(2), example 8, p. 39] and, in special cases, by Erdös and Rado (3) and J. Kruskal, Jr. (4). It is not postulated that $a \leqslant b \leqslant a$ implies $a = b$, nor† that $a \nleqslant b$ implies $b \leqslant a$. The relation $a < b$ is, by definition, equivalent to $a \leqslant b \nleqslant a$.

For any ordinal m we denote‡ by $W_S(m)$ the set of all mappings $\nu \to x(\nu)$ of $[0, m]$ into S, and we put

$$W_S(<n) = \Sigma[m<n]\, W_S(m).$$

One may think of $W_S(m)$ as the set of all vectors over S of "length" m. Following Higman (1), a quasi-order is introduced in $W_S(<n)$ as follows. Let $x, y \varepsilon W_S(<n)$. Then there are unique ordinals $k, l < n$ such that $x \varepsilon W_S(k)$; $y \varepsilon W_S(l)$. We put $x \leqslant y$ if, and only if, there is a mapping $\nu \to \phi(\nu)$ of $[0, k]$ into $[0, l]$ such that

$$x(\nu) \leqslant y\big(\phi(\nu)\big) \quad (\nu < k),$$

$$\phi(\mu) < \phi(\nu) \quad (\mu < \nu < k).$$

This order relation induces an order relation in $W_S(m)$ for every $m < n$. In what follows, S is a fixed partially well-ordered set, and all orderings of vector sets over S refer to this given order of S. The letters a and b denote elements of S, and u, v, w, x, y, z vectors over S. If $x \varepsilon W_S(m)$ then, for $\nu < m$, $x(\nu)$ denotes the ν-th component of x.

Higman [(1), Theorem 4.3] proved the following theorem.

Theorem 1. *$W_S(<\omega_0)$ is partially well-ordered.*

The purpose of this note is to investigate the possibility of extending Theorem 1 to other vector sets. The first result is negative.

† $a \nleqslant b$ is the negation of $a \leqslant b$.

‡ Σ and Π denote union and intersection of sets. The symbol $\sum_{m<n} W_S(m)$, and similarly in other cases, is replaced, for typographical convenience, by $\Sigma[m<n]\, W_S(m)$. If m and n are ordinals such that $m \leqslant n$ we put $[m, n] = \Sigma[m \leqslant \nu < n]\{\nu\}$. For every set A, the symbol $|A|$ denotes the cardinal of A, and $A \subset B$ set inclusion, in the wide sense; $A_1 - B_1$ is the set of all $a \varepsilon A_1$ such that $a \notin B_1$.

[Mathematika, 1 (1954), 89–95]

THEOREM 2. *There are partially well-ordered sets S such that $W_S(\omega_0)$ is not partially well-ordered.*

This result was obtained independently of the present author by J. Kruskal, Jr. (but not published).

The next theorem gives conditions on S in order that $W_S(\omega_0)$ should be partially well-ordered.

THEOREM 3. *$W_S(\omega_0)$ is partially well-ordered if, and only if, the following condition holds.*

Condition (α). *If* $a_{rs} < a_{r,s+1}$ $(r < s < \omega_0)$ *then there are numbers* r, s, t *such that* $r < s < t < \omega_0$; $a_{rs} < a_{st}$.

COROLLARY. *If S is totally well-ordered then $W_S(\omega_0)$ is partially well-ordered.*

In order to obtain an extension of Theorem 1 to vectors of infinite length we define a certain subset of $W_S(m)$. Let $V_S(m)$ denote the set of all mappings of $[0, m]$ on *finite* subsets of S, *i.e.* the set of all $x \varepsilon W_S(m)$ such that $|\Sigma[\nu < m]\{x(\nu)\}| < \aleph_0$. We put $V_S(<n) = \Sigma[m < n] V_S(m)$. The following theorem is an easy consequence of Theorem 1.

THEOREM 4. *If $V_S(m)$ is partially well-ordered then so is $V_S(< m\omega_0)$.*

It does not appear unreasonable to conjecture that $V_S(m)$ is, in fact, partially well-ordered for every m. In this direction the following result will be proved.

THEOREM 5. *$V_S(< \omega_0^3)$ is partially well-ordered.*

The proof of Theorem 5 uses ideas found jointly with P. Erdös in attempts to extend Theorem 1 in a different direction.

§2. *Proof of Theorem* 2. Let $S = \Sigma[\mu < \nu < \omega_0]\{(\mu, \nu)\}$. A partial order is defined in S by stipulating that all relations $a < b$ are given by

$$(\mu, \nu) < (\mu', \nu'),$$
$$(\mu, \nu) < (\mu, \nu') \quad (\mu < \nu \leqslant \mu' < \nu' < \omega_0).$$

For this binary relation is, clearly, transitive and non-reflexive. This order is a partial well-order. For let

$$\mu_\alpha < \nu_\alpha < \omega_0 \quad (\alpha < \omega_0),$$
$$(\mu_\alpha, \nu_\alpha) \nleqslant (\mu_\beta, \nu_\beta) \quad (\alpha < \beta < \omega_0). \tag{1}$$

Then, by (1), Theorem 2.1, there is a sequence $\alpha_0, \alpha_1, \ldots,$ such that $\alpha_0 < \alpha_1 < \ldots$; $\mu_{\alpha_0} \leqslant \mu_{\alpha_1} \leqslant \ldots$; $\nu_{\alpha_0} \leqslant \nu_{\alpha_1} \leqslant \ldots$. Then $\mu_{\alpha_0} < \mu_{\alpha_1} < \ldots$, and there is $n < \omega_0$ such that $\mu_{\alpha_0} < \nu_{\alpha_0} \leqslant \mu_{\alpha_n} < \nu_{\alpha_n}$. But this contradicts (1). Hence, by (1), Theorem 2.1, S is partially well-ordered. Let x_t, for $t < \omega_0$, be that element of $W_S(\omega_0)$ for which $x_t(\nu) = (t, t+\nu+1)$ $(\nu < \omega_0)$. If we now assume that $W_S(\omega_0)$ is partially well-ordered then there are numbers r, s such that $r < s < \omega_0$; $x_r \leqslant x_s$. Then $x_r(s-r) \leqslant x_s(t)$ for some $t < \omega_0$, *i.e.* $(r, s+1) \leqslant (s, s+t+1)$, which is false. This proves Theorem 2.

Proof of Theorem 3.

1. Suppose that $W_S(\omega_0)$ is partially well-ordered. Let

$$a_{rs} < a_{r,s+1} \quad (r < s < \omega_0).$$

Define $x_r \varepsilon W_S(\omega_0)$ by means of $x_r(\lambda) = a_{r,r+\lambda+1}$ $(r, \lambda < \omega_0)$. By hypothesis, there is $r < s < \omega_0$ such that $x_r \leqslant x_s$. Then, for some $t < \omega_0$,

$$a_{rs} = x_r(s-r-1) \leqslant x_s(t) = a_{s,s+t+1} < a_{s,s+t+2},$$

so that condition (α) is satisfied.

2. Suppose that S satisfies the condition (α). Let

$$x_s \varepsilon W_S(\omega_0); \quad x_s \nleqslant x_t \quad (s < t < \omega_0).$$

We shall deduce a contradiction.

Throughout the remainder of this paper we may assume, without loss of generality, that the quasi-order of S is a partial order [see (**2**), p. 4, Theorem 3].

Let $s < \omega_0$. Denote by N_s the set of all numbers $\mu < \omega_0$ which have the property that $x_s(\mu) \leqslant x_s(\nu)$ for only a finite number of numbers ν. Then $|N_s| < \aleph_0$. For, if† $N_s = \{\mu_0, \mu_1, \ldots\}_<$ then, given any $\mu \varepsilon N_s$, there is $\mu' \varepsilon N_s$ such that $x_s(\mu) \nleqslant x_s(\nu)$ for all $\nu \geqslant \mu'$. If this is used repeatedly, a set $\{\lambda_0, \lambda_1, \ldots\}_< \subset N_s$ is found such that $x_s(\lambda_\alpha) \nleqslant x_s(\lambda_\beta)$ $(\alpha < \beta < \omega_0)$. This is the desired contradiction. Hence $N_s \subset [0, n_s]$ for some $n_s < \omega_0$, and we can write $x_s = y_s z_s$, where $y_s \varepsilon W_S(n_s)$. Here, as in similar cases later on, the operation expressed by juxtaposition of vectors y_s and z_s to form a vector $x_s = y_s z_s$ is defined in the obvious way:

$$x_s(\nu) = y_s(\nu) \ \ (\nu < n_s); \quad x_s(n_s+\lambda) = z_s(\lambda) \ \ (\lambda < \omega_0).$$

Then, given any $\mu < \omega_0$, there are infinitely many $\nu < \omega_0$ such that $z_s(\mu) \leqslant z_s(\nu)$.

By Theorem 1, there is $\{s_0, s_1, \ldots\}_<$ such that $y_{s_0} \leqslant y_{s_1} \leqslant \ldots$. Without loss of generality, we may assume that $y_0 \leqslant y_1 \leqslant \ldots$. Put

$$A(s) = \Sigma[a \leqslant z_s(\nu) \text{ for at least one } \nu]\{a\} \quad (s < \omega_0).$$

Case 1. There is $s < t < \omega_0$ such that $A(s) \subset A(t)$. Then, for $\mu < \omega_0$, we have $z_s(\mu) \varepsilon A(s) \subset A(t)$ and therefore $z_s(\mu) \leqslant z_t(\nu)$ for some $\nu < \omega_0$. Then there are infinitely many ν' such that $z_t(\nu) \leqslant z_t(\nu')$ and hence

† The symbol $\{\mu_0, \mu_1, \ldots\}_<$ denotes the set whose elements are $\mu_0, \mu_1, \ldots$ and, at the same time, states that $\mu_0 < \mu_1 < \ldots$.

$z_s(\mu) \leqslant z_t(\nu')$. This clearly implies $z_s \leqslant z_t$ and therefore

$$x_s = y_s z_s \leqslant y_t z_t = x_t,$$

which is a contradiction.

Case 2. $A(s) \not\subset A(t)$ $(s < t < \omega_0)$. Then there is $\{s_0, s_1, \ldots\}_< \subset [0, \omega_0]$ such that

$$\Sigma\,[\lambda < \omega_0]\, A(s_\lambda) = \Sigma\,[\mu < \omega_0]\, \Pi\,[\nu \geqslant \mu]\, A(s_\nu). \tag{2}$$

For, let us suppose that this is not so. The letter N will always denote infinite subsets of $[0, \omega_0]$. Then one can find the following sets and elements.

There is ν_0, N_0, x_0 such that $x_0 \,\varepsilon\, A(\nu_0) - \Sigma\,[\nu \,\varepsilon\, N_0]\, A(\nu)$.

There is $\nu_1 \,\varepsilon\, N_0$; $N_1 \subset N_0$; x_1 such that $x_1 \,\varepsilon\, A(\nu_1) - \Sigma\,[\nu \,\varepsilon\, N_1]\, A(\nu)$.

There is $\nu_2 \,\varepsilon\, N_1$; $N_2 \subset N_1$; x_2 such that $x_2 \,\varepsilon\, A(\nu_2) - \Sigma\,[\nu \,\varepsilon\, N_2]\, A(\nu)$, etc.

Then $x_m \leqslant x_n$, for some $m < n < \omega_0$. Hence $x_m \leqslant x_n \,\varepsilon\, A(\nu_n)$ and so, by definition of $A(\nu_n)$, $x_m \,\varepsilon\, A(\nu_n)$. But this is a contradiction, since $\nu_n \,\varepsilon\, N_{n-1} \subset N_m$, and

$$x_m \,\varepsilon\, A(\nu_m) - \Sigma\,[\nu \,\varepsilon\, N_m]\, A(\nu) \subset A(\nu_m) - A(\nu_n).$$

This proves that (2) holds for some s_λ. We may assume that $s_\lambda = \lambda$, so that

$$\Sigma\,[\lambda < \omega_0]\, A(\lambda) = \Sigma\,[\mu < \omega_0]\, \Pi\,[\mu \leqslant \nu < \omega_0]\, A(\nu). \tag{3}$$

We can choose $b_0(\lambda) \,\varepsilon\, A(0) - A(\lambda)$ $(0 < \lambda < \omega_0)$.

By (3), $b_0(\lambda) \,\varepsilon\, A(\nu)$ if ν is sufficiently large. Hence, for fixed λ, $b_0(\lambda) \neq b_0(\nu)$ for every large ν. Therefore there is a set

$$\{\kappa_1, \kappa_2, \ldots\}_< \subset \{1, 2, \ldots\}$$

such that $b_0(\kappa_1) < b_0(\kappa_2) < \ldots$. We can choose

$$b_1(\lambda) \,\varepsilon\, A(\kappa_1) - A(\lambda) \quad (\kappa_1 < \lambda < \omega_0)$$

and, similarly, $\{\sigma_2, \sigma_3, \ldots\}_< \subset \{\kappa_2, \kappa_3, \ldots\}$ such that $b_1(\sigma_2) < b_1(\sigma_3) < \ldots$. There is $b_2(\lambda) \,\varepsilon\, A(\sigma_2) - A(\lambda)$ $(\sigma_2 < \lambda < \omega_0)$ and $\{\tau_3, \tau_4, \ldots\}_< \subset \{\sigma_3, \sigma_4, \ldots\}$ such that $b_2(\tau_3) < b_2(\tau_4) < \ldots$, etc. We put $\rho_0 = 0$; $\rho_1 = \kappa_1$; $\rho_2 = \sigma_2$; $\rho_3 = \tau_3$; $\ldots$ and $a_{rs} = b_r(\rho_s)$ $(r < s < \omega_0)$. Then $a_{rs} < a_{r,s+1}$ $(r < s < \omega_0)$ and, by condition (α), there are numbers $r < s < t < \omega_0$ such that $a_{rs} < a_{st}$. Then $a_{rs} < a_{st} \,\varepsilon\, A(\rho_s)$; $a_{rs} \,\varepsilon\, A(\rho_s)$. On the other hand, $a_{rs} \,\varepsilon\, A(\rho_r) - A(\rho_s)$, which is the desired contradiction. This proves Theorem 3.

Proof of Theorem 4. We begin by showing that $V_S(< m)$ is partially well-ordered. We may assume that $m > 0$; $S \neq 0$. Then S possesses at least one minimal element, *i.e.* an element a^* such that $a \leqslant a^*$ implies $a = a^*$. Now let $x_t \,\varepsilon\, V_S(n_t)$; $n_t < m (t < \omega_0)$. Then we can write $x_t = y_t z_t$, where $y_t \,\varepsilon\, V_S(k_t)$; $z_t \,\varepsilon\, V_S(l_t)$; $z_t(\nu) = a^*(\nu < l_t)$ and, given any $\mu < k_t$, there is $\nu \,\varepsilon\, [\mu, k_t]$ such that $y_t(\nu) \neq a^*$. Then there is u_t such that $y_t z_t u_t \,\varepsilon\, V_S(m)$;

$u_t \varepsilon V_S(m_t)$; $u_t(\nu) = a^*$ $(\nu < m_t)$. By hypothesis and (1), Theorem 2.3, there is $s < t < \omega_0$ such that $y_s z_s u_s \leqslant y_t z_t u_t$; $z_s \leqslant z_t$. Then $y_s \leqslant y_t$. For, otherwise, there is $\nu_0 < k_s$ such that $y_s(\nu) \leqslant a^*(\nu_0 \leqslant \nu < k_s)$ and hence, by definition of a^*, $y_s(\nu) = a^*(\nu_0 \leqslant \nu < k_s)$. This contradicts the definition of k_s. We have therefore proved that $x_s = y_s z_s \leqslant y_t z_t = x_t$, *i.e.* that $V_S(<m)$ is partially well-ordered. Put $V_S(<m) + V_S(m) = T$.

Now let $v_t \varepsilon V_S(<m\omega_0)$ $(t < \omega_0)$. Then $v_t = v_{t0} v_{t1} \ldots v_{t, r_t - 1}$ where $r_t < \omega_0$; $v_{t\mu} \varepsilon T$. Then we have $v_t \varepsilon V_T(<\omega_0)$ and, by Theorem 1 and what has been proved above, there is $s < t$ such that, in the order given in $V_T(<\omega_0)$, $v_s \leqslant v_t$. Then the definition of the order relation in vector sets implies that $v_s \leqslant v_t$ in the order given in $V_S(<m\omega_0)$. This proves Theorem 4.

§3. *Proof of Theorem* 5. First of all, we show that $V_S(\omega_0)$ is partially well-ordered. Let $x_t \varepsilon V_S(\omega_0)$ $(t < \omega_0)$, and put

$$M_t(k, l) = \Sigma[k \leqslant \nu < l]\{x_t(\nu)\} \quad (k \leqslant l < \omega_0).$$

Then, by definition of $V_S(\omega_0)$, there is $m_t < \omega_0$ such that

$$M_t(m_t, \omega_0) = M_t(m, \omega_0) \quad (m_t \leqslant m < \omega_0).$$

Now we can find $n_t \varepsilon [m_t, \omega_0]$ such that $M_t(m_t, n_t) = M_t(m_t, \omega_0)$. We can write $x_t = u_t v_t w_t$, where $u_t \varepsilon V_S(m_t)$; $u_t v_t \varepsilon V_S(n_t)$. By Theorem 1 and (1), Theorem 2.3, there is $s < t < \omega_0$ such that $u_s \leqslant u_t$; $v_s \leqslant v_t$. It now suffices to prove $w_s \leqslant w_t$.

Let $\nu < \omega_0$. Then $w_s(\nu) = v_s(\mu)$ for some μ, and $v_s(\mu) \leqslant v_t(\rho)$ for some ρ. Then $v_t(\rho) = w_t(\sigma)$ for infinitely many σ, and hence $w_s(\nu) \leqslant w_t(\sigma)$ for infinitely many σ. Since ν is arbitrary, this proves $w_s \leqslant w_t$, and therefore establishes that $V_S(\omega_0)$ is partially well-ordered.

In order to prove Theorem 5 it is sufficient, by Theorem 4, to show that $V_S(\omega_0^2)$ is partially well-ordered. Let $x_t \varepsilon V_S(\omega_0^2)$ $(t < \omega_0)$.

If $k \leqslant k + l < \omega_0^2$ then $x_t(k, k+l)$ denotes the element of $V_S(l)$ defined by

$$x_t(k, k+l) \ (\nu) = x_t(k+\nu) \ (\nu < l).$$

We put
$$M_t(p, q) = \Sigma\, [p \leqslant \nu < q]\{x_t(\nu)\} \quad (p \leqslant q < \omega_0^2),$$

$$D_t(\lambda) = \Pi\, [\nu < \omega_0]\, M_t(\lambda + \nu, \lambda + \omega_0) \quad (\lambda < \omega_0^2).$$

Then there is $\nu_t(\lambda) < \omega_0$ such that

$$D_t(\lambda) = M_t\big(\lambda + \nu_t(\lambda),\ \lambda + \omega_0\big).$$

Finally, put

$$N_t(p, q) = \Sigma[p \leqslant \nu < q]\{D_t(\nu)\}.$$

Then there is $m_t < \omega_0^2$ *such that, for* $m_t \leqslant m < \omega_0^2$,

$$M_t(m_t, \omega_0^2) = M_t(m, \omega_0^2),$$

$$N_t(m_t, \omega_0^2) = N_t(m, \omega_0^2).$$

There is a limit number $n_t \varepsilon [m_t, \omega_0^2]$ such that

$$M_t(m_t, n_t) = M_t(m_t, \omega_0^2),$$

$$N_t(m_t, n_t) = N_t(m_t, \omega_0^2).$$

Then $$x_t = u_t v_t w_t; \quad u_t \varepsilon V_S(m_t); \quad u_t v_t \varepsilon V_S(n_t).$$

Since u_t, $v_t \varepsilon V_S(<\omega_0^2)$, and $V_S(\omega_0)$ is partially well-ordered, it follows from Theorem 4 that there is $s < t < \omega_0$ such that $u_s \leqslant u_t; v_s \leqslant v_t$. It suffices, therefore, to prove that $w_s \leqslant w_t$. Our assumption about s and t implies that there is a mapping $\nu \to \phi(\nu)$ of $[m_s, n_s]$ into $[m_t, n_t]$ such that

$$x_s(\nu) \leqslant x_t\big(\phi(\nu)\big) \quad (m_s \leqslant \nu < n_s),$$

$$\phi(\mu) < \phi(\nu) \quad (m_s \leqslant \mu < \nu < n_s).$$

Let $n_s \leqslant \lambda < \omega_0^2$; $n_t \leqslant \lambda' < \omega_0^2$.

(i) Let $\nu < \omega_0$. Then $x_s(\lambda+\nu) \varepsilon M_s(n_s, \omega_0^2) = M_s(m_s, n_s)$,

$$x_s(\lambda+\nu) = x_s(\mu) \text{ for some } \mu \varepsilon [m_s, n_s],$$

$$x_s(\mu) \leqslant x_t\big(\phi(\mu)\big) = x_t(\rho) \text{ for some } \rho \geqslant \lambda'.$$

Hence $$x_s(\lambda+\nu) \leqslant x_t(\rho) \text{ for some } \rho \geqslant \lambda'.$$

By a finite number of applications of this result, with varying values of λ', one finds a number $\mu' < \omega_0^2$ such that $x_s\big(\lambda, \lambda+\nu_s(\lambda)\big) \leqslant x_t(\lambda', \mu')$.

(ii) We have $D_s(\lambda) = D_s(\tau)$ for some $\tau \varepsilon [m_s, n_s]$,

$$\lim [\mu < \omega_0] \, \phi(\tau+\mu) = \kappa \leqslant n_t.$$

Then κ is a limit number. There is $\xi \geqslant m_t$ such that $\kappa = \xi + \omega_0$. Then $D_t(\xi) = D_t(\eta)$ for some $\eta \geqslant \mu'$. Now let $\nu \varepsilon [\nu_s(\lambda), \omega_0]$. Then

$$x_s(\lambda+\nu) \varepsilon D_s(\lambda) = D_s(\tau);$$

$$x_s(\lambda+\nu) = x_s(\tau + l_{\nu\alpha}) \quad (\alpha < \omega_0)$$

for some numbers $l_{\nu\alpha}$ such that $l_{\nu\alpha} < l_{\nu,\alpha+1} < \omega_0$ $(\alpha < \omega_0)$. Then there is α such that

$$x_s(\lambda+\nu) = x_s(\tau + l_{\nu\alpha}) \leqslant x_t\big(\phi(\tau + l_{\nu\alpha})\big) \varepsilon D_t(\xi) = D_t(\eta).$$

Hence $$x_s(\lambda+\nu) \leqslant x_t(\eta + p_{\nu\beta}) \quad (\beta < \omega_0),$$

where $$p_{\nu\beta} < p_{\nu,\beta+1} < \omega_0 \quad (\beta < \omega_0).$$

Since ν is arbitrary our result implies that

$$x_s\big(\lambda+\nu_s(\lambda),\ \lambda+\omega_0\big) \leqslant x_t(\mu',\ \eta+\omega_0).$$

We deduce from (i) and (ii) that, given any numbers $\lambda \,\varepsilon\, [n_s,\ \omega_0^2]$ and $\lambda' \,\varepsilon\, [n_t,\ \omega_0^2]$, there is $\mu'' \,\varepsilon\, [\lambda',\ \omega_0^2]$ such that

$$x_s(\lambda,\ \lambda+\omega_0) \leqslant x_t(\lambda',\ \mu'').$$

Repeated application of this result, with varying values of λ and λ', leads to

$$w_s = x_s(n_s,\ \omega_0^2) \leqslant x_t(n_t,\ \omega_0^2) = w_t$$

which is the desired conclusion. This proves Theorem 5.

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