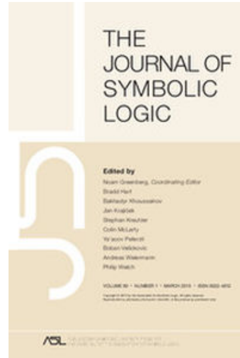

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§0. **Introduction.** The study of preservation theorems for first order logic was the focus of much research by model theorists in the 1960's. These theorems, which came to form the foundation for classical model theory, characterize first order sentences and theories that are preserved under operations such as the taking of unions or submodels (see Chang and Keisler [5] for a discussion of preservation theorems for first order logic). In current model theoretic research, logics richer than first order logic and applications of logic to other parts of mathematics have assumed the central position. In the former area, preservation theorems are not so important; in the latter, especially in applications to algebra, many of the techniques developed for proving these theorems have been useful.

In this paper I prove several preservation theorems for first order logic which I discovered while investigating the asymptotic growth of classes of finite combinatorial structures. The significance of these theorems lies in their applications to problems in finite combinatorics. Since the applications require combinatorial and analytical techniques that are not pertinent to logical questions discussed here, I shall present them in another paper [7].

Here my concern is to syntactically characterize certain classes of structures that have played an important role in combinatorics. The organization of the paper is as follows. §1 contains preliminaries; §2 contains a discussion of the combinatorial ideas involved and examples of the classes; §§3 and 4 contain the preservation theorems; and §5 contains some useful results that follow from theorems in previous sections. Much of the material in this paper appeared in my Ph.D. thesis [6]. I would like to express my appreciation to my advisor, H.J. Keisler, for his encouragement.

§1. **Preliminaries.** I will assume a knowledge of the rudiments of model theory. The reference for all matters of notation and terminology is Chang and Keisler [5].

Throughout the paper L and all other languages used will contain no function symbols. L-structures are denoted by upper case Fraktur letters: $\mathfrak{A}$, $\mathfrak{A}^*$, $\mathfrak{B}$, $\mathfrak{K}$, etc., and their universes by the corresponding upper case Latin letters: A, A^*, B, K, etc.

I will also use some notions that are not standard in model theory texts.

DEFINITION. Suppose $\mathfrak{A}$ is an L_0-structure and $\mathfrak{B}$ is an L_1-structure. Define the *model pair* $\langle \mathfrak{A}, \mathfrak{B} \rangle$ as follows. Form a new language L which contains a symbol R^A for every relation symbol R in L_0, a constant symbol c^A for every constant

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symbol c in L_0, relation and constant symbols R^B and c^B defined similarly for L_1, and two new unary symbols A and B. $\langle \mathfrak{A}, \mathfrak{B} \rangle$ has $A \perp\!\!\!\perp B$, the disjoint union of A and B, as its universe. R^A is interpreted by the set which interprets R in $\mathfrak{A}$, and the analogous statements hold for c^A, R^B, and c^B. A and B are interpreted by A and B (it should be clear from the context when A and B are symbols and when they are sets).

Now for any sentence φ in the language L_0, form the sentence $\varphi^{(A)}$ by replacing each occurrence of R with R^A, of c with c^A, of $(\forall x)$ with $(\forall x \in A)$, and of $(\exists x)$ with $(\exists x \in A)$. Here the meaning of the new quantifiers is defined by:

$\models (\forall x \in A)\, \psi \leftrightarrow \forall x(A(x) \rightarrow \psi)$, and

$\models (\exists x \in A)\, \psi \leftrightarrow \exists x(A(x) \wedge \psi)$.

Form the sentence $\varphi^{(B)}$ in the same way.

I will also need to consider *model sequences* at one point. If $\mathfrak{A}$ is an L_0-structure and $\mathfrak{B}_i$ is an L_1-structure for each i in an index set I, the intended meaning of $\langle \mathfrak{B}, \mathfrak{A} \rangle_{i \in I}$ should be clear.

Another useful concept is that of a recursively saturated model, due to Barwise and Schlipf [2].

DEFINITION. Let L be a countable language. An L-structure $\mathfrak{A}$ is *recursively saturated* if for every finite $\bar{c} = \langle c_0, \ldots, c_{n-1} \rangle$, every set $\Gamma(x, \bar{c})$ of formulas in $L \cup \{c_0, \ldots, c_{n-1}\}$ recursive in L, and every $\bar{a} = \langle a_0, \ldots, a_{n-1} \rangle$ from A, if $\Gamma(x, \bar{c})$ is finitely satisfiable in $\langle \mathfrak{A}, \bar{a} \rangle$ then it is realized in $\langle \mathfrak{A}, \bar{a} \rangle$.

The next theorem is a collection of sundry results that I will need later.

THEOREM 1.1. *In the following $\mathfrak{A}$, $\mathfrak{A}^*$ are L_0-structures, $\mathfrak{B}$, $\mathfrak{B}^*$ are L_1-structures.*

(i) If $\langle \mathfrak{A}, \mathfrak{B} \rangle \prec \langle \mathfrak{A}^*, \mathfrak{B}^* \rangle$ *then* $\mathfrak{A} \prec \mathfrak{A}^*$ *and* $\mathfrak{B} \prec \mathfrak{B}^*$.

(ii) *If $L_0 \subseteq L_1$ and $\mathfrak{A}$ is the reduct of $\mathfrak{B}$ to L_0 with $\mathfrak{B}$ recursively saturated, then $\mathfrak{A}$ is recursively saturated.*

(iii) *If $\mathfrak{A}$ is recursively saturated, and $\bar{a} = \langle a_0, \ldots, a_{n-1} \rangle$ is a finite sequence of elements in A, then $\langle \mathfrak{A}, \bar{a} \rangle$ is recursively saturated.*

(iv) *If L_0 and $\mathfrak{A}$ are countable then $\mathfrak{A}$ has a countable, recursively saturated elementary extension.*

The proofs of these assertions are not difficult. Only (iv) does not follow immediately from the definitions: it is proved along the same lines as Theorem 2.3.7 in [5].

I shall also require the following lemma, which is the starting point for many preservation theorems. (This theorem remains true even if the prohibition against function symbols in L is dropped.)

LEMMA 1.2. *Let Σ be a set of L-sentences closed under finite disjunctions.*

(i) *For a consistent theory T the following are equivalent:*

(a) *T has a set of axioms $\subseteq \Sigma$.*

(b) *If $\mathfrak{A} \models T$ and every sentence in Σ which holds in $\mathfrak{A}$ holds in $\mathfrak{B}$, then $\mathfrak{B} \models T$.*

(ii) *If, in addition, Σ is closed under conjunctions, then for a consistent sentence φ the following are equivalent:*

(a) *φ is logically equivalent to some sentence in Σ.*

(b) *If $\mathfrak{A} \models \varphi$ and every sentence in Σ which holds in $\mathfrak{A}$ holds in $\mathfrak{B}$, then $\mathfrak{B} \models \varphi$.*

The proof of this theorem is standard in model theory texts and will not be given here (see Chang and Keisler [5, Theorem 3.2.1], for details).

§2. Combinatorial definitions and examples. In this section I extend some common notions of combinatorics and examine some classes of structures which often arise in combinatorics. I also introduce the idea of a *connecting quantifier* in order to investigate the language of these classes.

The following definitions are extensions of the graph theoretic concepts of *component* and *connectedness.*

DEFINITIONS. Let $\mathfrak{A}$ be an L-structure. For $a, b \in A$ define $\sim$ by: $a \sim b$ if for some relation symbol R in L and sequences $\bar{x}, \bar{y}, \bar{z}$ of variables,

$$\mathfrak{A} \models (\exists \bar{x}, \bar{y}, \bar{z})\ R\ (\bar{x}, a, \bar{y}, b, \bar{z}).$$

Let $\sim^*$ be the least equivalence relation extending $\sim$. The $\sim^*$ equivalence classes are called the *components* of $\mathfrak{A}$. If $\mathfrak{A}$ is a graph this coincides with the graph theoretic definition.

A structure $\mathfrak{K}$ is a component of $\mathfrak{A}$ if it is a substructure of $\mathfrak{A}$ and K is a component of $\mathfrak{A}$. $\mathfrak{A}$ is *connected* if it has just one component.

The next set of definitions do not have a standard terminology within combinatorics, but they will be useful later.

DEFINITIONS. Let $\mathfrak{A}$ and $\mathfrak{K}$ be L-structures. $\mathfrak{A}$ is a *closed submodel* of $\mathfrak{B}$ (equivalently, $\mathfrak{B}$ is a *closed extension* of $\mathfrak{A}$) if $\mathfrak{A}$ is a submodel of $\mathfrak{B}$ and is a union of components of $\mathfrak{B}$.

An embedding of $\mathfrak{A}$ into $\mathfrak{B}$ is a *closed embedding* if the image of $\mathfrak{A}$ is a closed submodel of $\mathfrak{B}$.

$\mathfrak{A} = \bigcup_{i \in I} \mathfrak{A}_i$ is a *closed union* of the $\mathfrak{A}_i$'s if each $\mathfrak{A}_i$ is a closed submodel of $\mathfrak{A}$.

My primary interest will be with classes of L-structures which are closed under disjoint unions and components, i.e., classes $\mathscr{C}$ which satisfy the following.

(i) If $\mathfrak{A}, \mathfrak{B} \in \mathscr{C}$ then $\mathfrak{A} \amalg \mathfrak{B}$, the disjoint union of $\mathfrak{A}$ and $\mathfrak{B}$, is in $\mathscr{C}$.

(ii) If $\mathfrak{A} \in \mathscr{C}$ and $\mathfrak{K}$ is a component of $\mathfrak{A}$ then $\mathfrak{K} \in \mathscr{C}$.

Combinatorialists have studied classes with these closure properties extensively, much of their work dealing with the enumeration of these classes. One reason that these classes have received so much attention is that there is a particularly nice relationship between generating series for a class closed under disjoint unions and components, and the generating series for the subclass of connected structures in that class (see [6] or [7]). Another reason is that many familiar classes of combinatorial structures possess these closure properties. Listed below are some examples. With each example is a first order axiomatization. The following definition is useful for specifying the axiomatizations.

DEFINITION. A *connecting quantifier* is a quantifier of the form $(\forall \bar{x}$ s.t. $R(\bar{t}))$ or $(\exists \bar{x}$ s.t. $R(\bar{t}))$, where R is a relation in L, $\bar{x}$ is a sequence of variables, $\bar{t}$ is a sequence of variables and constant symbols from L, and the set of variables occurring in $\bar{x}$ is properly contained in the set of variables and constant symbols occurring in $\bar{t}$. The interpretations of these quantifiers are given by

Guarded quantification

$$\models (\forall \bar{x} \text{ s.t. } R(\bar{t}))\ \varphi(\bar{t}, \bar{y}) \leftrightarrow \forall \bar{x}(R(\bar{t}) \rightarrow \varphi(\bar{t}, \bar{y})),$$

$$\models (\exists \bar{x} \text{ s.t. } R(t))\ \varphi(\bar{t}, \bar{y}) \leftrightarrow \exists \bar{x}(R(\bar{t}) \wedge \varphi(\bar{t}, \bar{y})).$$

Connecting quantifiers connect the variables they bind to either unbound variables or constant symbols via some relation symbol. The most well-known examples of connecting quantifiers are the bounded quantifiers $(\forall x \leq y)$ and $(\exists x \leq y)$. They are the same as $(\forall x \text{ s.t. } x \leq y)$ and $(\exists x \text{ s.t. } x \leq y)$.

EXAMPLE 2.1. THE CLASS OF GRAPHS. L contains just one relation symbol R, a binary relation symbol which denotes the edge relation. Axioms for this class say simply that R is irreflexive and symmetric:

$\forall x \neg R(x, x)$,

$\forall x(\forall y \text{ s.t. } R(x, y))\, R(y, x)$.

EXAMPLE 2.2. THE CLASS OF UNARY FUNCTIONS. Since L may not contain function symbols, the function is represented by a binary relation symbol R. The axiom for the class is

$$\forall x(\forall y \text{ s.t. } R(x, y))(\forall z \text{ s.t. } R(x, z))\, y = z \wedge \forall x(\exists y \text{ s.t. } R(x, y))\, y = y.$$

Metropolis and Ulam [14] posed the problem of computing for this class the number of connected structures of each finite cardinality. The solution was given by Katz [10]. Harary and Palmer [9] consider a related enumeration problem involving this class.

EXAMPLE 2.3. THE CLASS OF PERMUTATIONS. To the axiom of the previous example add

$$\forall x(\forall y \text{ s.t. } R(y, x))(\forall z \text{ s.t. } R(z, x))\, y = z.$$

This says that R is a one-to-one function, which, for finite structures, is equivalent to being a permutation. The axiom which says R is onto is

$$\forall x(\exists y \text{ s.t. } R(y, x))\, y = y.$$

Various asymptotic properties of this class have been studied by Goncharov [8] and Shepp and Lloyd [15].

EXAMPLE 2.4. THE CLASS OF PARTIAL ORDERS. The reflexive, antisymmetric and transitive axioms are (interpreting $R(x, y)$ to mean $x \leq y$):

$\forall x\, R(x, x)$,

$\forall x\, (\forall y \text{ s.t. } R(x, y))\, R(y, x) \rightarrow x = y$,

$\forall x\, (\forall y \text{ s.t. } R(x, y))\, (\forall z \text{ s.t. } R(y, z))\, R(x, z)$.

Kleitman and Rothschild [12] have computed the asymptotic growth of this class.

EXAMPLE 2.5. THE CLASS OF ORIENTED FORESTS. An *oriented tree* (this term is from Knuth [13]) is a tree with a distinguished node called the root. These trees are often called *rooted trees* in the literature. A forest is just a set of trees. Identify oriented forests with partial orders that have the property that the set of all elements below any given element are linearly ordered (hence the term "oriented"). To the axioms of the previous example add

$$\forall x(\forall y \text{ s.t. } R(y, x))(\forall z \text{ s.t. } R(z, x))\, R(y, z) \vee R(z, y).$$

Oriented forests arise in the enumeration of oriented trees. As mentioned earlier, there is a nice relationship between the generating series for a class of structures closed under disjoint unions and components, and the subclass of connected structures, in this case, between the generating series for oriented forests and oriented trees. The enumeration of both classes is fairly simple using this relationship and some observations about forests and trees (see Harary and Palmer [9] for details).

EXAMPLE 2.6. THE CLASS OF EQUIVALENCE RELATIONS OR PARTITIONS. The familiar axiomatization is

$\forall x R(x, x)$,

$\forall x(\forall y \text{ s.t. } R(x, y))\ R(y, x)$,

$\forall x(\forall y \text{ s.t. } R(x, y))\ (\forall z \text{ s.t. } R(y, z))\ R(x, z)$.

This is probably the most studied of all our examples. The asymptotics of partitions is worked out in De Bruijn [4], Bender [3] and Andrews [1].

§3. Theories and sentences preserved under closed submodels and closed extensions. The main theorems of this section are Theorems 3.3 and 3.4 in which I characterize theories and sentences preserved under closed submodels, closed extensions, and both. In each case I use Lemma 1.2 and the basic properties of recursively saturated models detailed in Theorem 1.1.

Before stating the theorems, I need to define certain sets of first order formulas that are required for the characterization.

DEFINITION. The set S of *basic connected* formulas is defined inductively by:

(0) Atomic and negated atomic formulas are elements of S.

(1) S is closed under finite disjunctions and finite conjunctions.

(2) S is closed under connecting quantifiers.

The set of *universal connected* formulas is defined inductively by conditions (1)–(2) together with

(3$\forall$) S is closed under universal quantifiers.

The set of *existential connected* formulas is defined inductively by conditions (0)–(2) together with

(3$\exists$) S is closed under existential quantifiers.

First, I state the easy directions of the main theorems of this section.

PROPOSITION 3.1. (i) *Basic connected formulas are preserved under closed submodels and closed extensions.*

(ii) *Existential connected formulas are preserved under closed extensions (equivalently, under closed embeddings).*

(iii) *Universal connected formulas are preserved under closed submodels.*

PROOF. A simple induction on formulas suffices in each case. ⊣

To develop a good intuition about the formulas defined above, it is helpful to think of game theoretic interpretations of truth conditions for the formulas. Suppose that $\mathfrak{A}$ is an L-structure and φ is a formula with parameters from $\mathfrak{A}$ which is built from atomic and negated atomic formulas using only conjunctions, disjunctions, and universal, existential, and connecting quantifiers. The game begins with the formula φ written on a blackboard. During the play of the game, two players $\forall$ and $\exists$ erase the formula on the board, replacing it with a simpler formula in accordance with these rules:

(1) If the formula on the board is of the form $\psi \wedge \theta$ then $\forall$ erases it and has the option of replacing it with either ψ or θ; if the formula is $\psi \vee \theta$ then $\exists$ erases it and has the same option.

(2) If the formula is of the form $\forall \bar{x}\psi(\bar{x})$ then $\forall$ erases it and is allowed to replace

it with a formula of the form $\psi(\bar{a})$, where $\bar{a}$ is a sequence of parameters from A; the analogous statement holds if the formula is of the form $\exists\bar{x}\psi(\bar{x})$.

(3) If the formula on the board begins with a connecting quantifier rather than an ordinary quantifier, $\forall$ and $\exists$ play as before, except that they must choose parameters that relate to previously chosen parameters and constants in the manner prescribed by the quantifier.

Eventually there will be an atomic or negated atomic formula on the board. $\exists$ wins if this formula is true—otherwise $\forall$ wins. Now φ is true precisely when $\exists$ has a winning strategy for this game.

Apply this to part (ii) of the above proposition. Suppose that $\mathfrak{B}$ is a closed extension of $\mathfrak{A}$ and that the existential connected formula φ holds in $\mathfrak{A}$. Then $\exists$ has a winning strategy if $\mathfrak{A}$ is used for the game. But $\exists$ will have a winning strategy when $\mathfrak{B}$ is used for the game simply by playing as she did when $\mathfrak{A}$ was used. $\forall$ has the same choices as before because the only time he chooses parameters the formula on the board begins with a universal connecting quantifier, and he must choose them in $\mathfrak{A}$. Thus, φ is preserved under closed extensions.

Parts (i) and (iii) may be proved similarly.

The next theorem is a technical result used in proving the converses to the parts of Proposition 3.1.

THEOREM 3.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable L-structures, L countable, and suppose that $\langle\mathfrak{A}^*, \mathfrak{B}^*\rangle$ is a countable recursively saturated elementary extension of $\langle\mathfrak{A}, \mathfrak{B}\rangle$.*

(i) *Assume that L contains some constant symbols. If every basic connected sentence true in $\mathfrak{A}$ is true in $\mathfrak{B}$, there is a closed embedding of the smallest closed submodel of $\mathfrak{A}^*$* into *$\mathfrak{B}^*$.*

(ii) *If every existential connected sentence true in $\mathfrak{A}$ is true in $\mathfrak{B}$, then there is a closed embedding of $\mathfrak{A}^*$ into $\mathfrak{B}^*$.*

PROOF. Because the proof of the two parts are similiar, I prove only part (ii).

Order the universes of $\mathfrak{A}^*$ and $\mathfrak{B}^*$:

$$A^* = \{a'_n : n \in \omega\}, \qquad B^* = \{b'_n : n \in \omega\}.$$

I will choose elements a_n, $n \in \omega$, from A^*, and b_n, $n \in \omega$, from B^* in ω steps. They will be chosen so that for each $n \in \omega$ and each existential connected formula $\varphi(x_0, \ldots, x_{n-1})$,

$$\langle\mathfrak{A}^*, \mathfrak{B}^*\rangle \models \varphi^{(A)}(a_0, \ldots, a_{n-1}) \rightarrow \varphi^{(B)}(b_0, \ldots, b_{n-1}).$$

The process by which I choose these elements varies according to whether the step number is even or odd.

Suppose the step number is even and $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}$ have been chosen to satisfy the induction hypothesis (if the step number is 0, no elements have been chosen and the induction hypothesis is true by virtue of $\mathfrak{A} \prec \mathfrak{A}^*$, $\mathfrak{B} \prec \mathfrak{B}^*$, which follows from Theorem 1.1(i)). Let a_n be the first item in the sequence $a'_0, a'_1, \ldots$ not yet chosen. Now consider the set of formulas

$$\Gamma(x) = \{\varphi^{(A)}(a_0, \ldots, a_n) \rightarrow \varphi^{(B)}(b_0, \ldots, b_{n-1}, x) \wedge B(x) : \varphi \text{ is existential connected}\}.$$

$\Gamma(x)$ is clearly recursive. It is also finitely satisfiable in $\langle\langle\mathfrak{A}^*, \mathfrak{B}^*\rangle, a_0, \ldots, a_n, b_0, \ldots, b_{n-1}\rangle$. For suppose that $\varphi_0, \ldots, \varphi_{k-1}$ are existential connected formulas with

$$\mathfrak{A}^* \models \varphi_i(a_0, \ldots, a_n), \qquad i < k.$$

Then

$$\mathfrak{A}^* \models \exists x \bigwedge_{i<k} \varphi_i(a_0, \ldots, a_{n-1}, x).$$

This sentence is existential connected, so by the induction hypothesis $\mathfrak{B}^*$ satisfies this sentence with b_i's substituted for a_i's, and consequently there is a b is B^* such that

$$\mathfrak{B}^* \models \bigwedge_{i<k} \varphi_i(b_0, \ldots, b_{n-1}, b).$$

Hence

$$\langle\mathfrak{A}^*, \mathfrak{B}^*\rangle \models \bigwedge_{i<k} (\varphi_i^{(A)}(a_0, \ldots, a_n) \to \varphi_i^{(B)}(b_0, \ldots, b_{n-1}, b) \wedge B(b)).$$

By recursive saturation, there is a b_n which satisfies $\Gamma(x)$ in $\langle\mathfrak{A}^*\ \mathfrak{B}^*\rangle$. The sequences $a_0, \ldots, a_n, b_0, \ldots, b_n$ satisfy the induction hypothesis.

Suppose that the step number is odd and that $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}$ have been chosen to satisfy the induction hypothesis. Choose $b_n, b_{n+1}, \ldots, b_{n+m-1}$ so that they are related to some of the previously chosen b_i's (for example, it might be true that $\mathfrak{B}^* = R(b_{n-1}, b_n, \ldots, b_{n+m-1}))$ and they include the first item in the sequence $b_0', b_1', \ldots$ which is not among $b_0, \ldots, b_{n-1}$ but is related to some of them (if there is no such element, do nothing at this step). Let

$$\begin{aligned}\Delta(x_0, \ldots, x_{m-1}) \\ = \{\varphi^{(B)}(b_0, \ldots, b_{n+m-1}) \to \varphi^{(A)}(a_0, \ldots, a_{n-1}, x_0, \ldots, x_{m-1}) \wedge \bigwedge_{i<m} A(x_i): \\ \varphi \text{ is universal connected}\}.\end{aligned}$$

I claim $\Delta(x_0, \ldots, x_{m-1})$ is finitely satisfiable. Suppose not. Then there are universal connected formulas $\varphi_0, \ldots, \varphi_{k-1}$ such that

$$\mathfrak{B}^* \models \varphi_i(b_0, \ldots, b_{n+m-1}), \qquad i < k,$$

but

$$\mathfrak{A}^* \models \forall x_0, \ldots, x_{m-1}(\neg \bigwedge_{i<k} \varphi_i(a_0, \ldots, a_{n-1}, x_0, \ldots, x_{m-1})).$$

This implies (for the case suggested above where it happens that $\mathfrak{B}^* \models R(b_{n-1}, \ldots, b_{n+m-1}))$ that

$$\begin{aligned}\mathfrak{A}^* \models (\forall x_0, \ldots, x_m \text{ s.t. } R(b_{n-1}, x_0, \ldots, x_{m-1})) \\ (\neg \bigwedge_{i<k} \varphi_i(b_0, \ldots, b_{n-1}, x_0, \ldots, x_{m-1})),\end{aligned}$$

a contradiction. Once again, the sequences $a_0, \ldots, a_{n+m-1}$ and $b_0, \ldots, b_{n+m-1}$ satisfy the induction hypothesis.

I claim that the mapping $a_n \to b_n$ is a closed embedding. It is well defined and one-to-one because $x_0 = x_1$, $x_0 \neq x_1$ are existential connected formulas and consequently $\mathfrak{A}^* \models a_n = a_m$ iff $\mathfrak{B}^* \models b_n = b_m$. It is a homomorphism because all atomic formulas are existential connected. Finally, it is a closed embedding because the procedure for choosing elements in the odd steps insures that anything in B^* which relates to any of the b's will eventually be chosen (to see this, simply consider the first such in the sequence $b_0', b_1', \ldots$ which is not chosen). $\dashv$

Now I have everything required to prove the preservation theorems.

THEOREM 3.3. *Assume that L is countable and contains some constant symbols.*

(i) *A theory T is preserved under closed submodels and closed extensions iff it has a basic connected set of axioms.*

(ii) *A sentence φ is preserved under closed submodels and closed extensions iff it is logically equivalent to a basic connected sentence.*

PROOF. Proposition 3.1 is the forward direction of these statements.

To prove the other direction for (i), Lemma 1.2 says that I need only show that if $\mathfrak{A} \models T$ and every basic connected sentence true in $\mathfrak{A}$ is true in $\mathfrak{B}$, then $\mathfrak{B} \models T$.

Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are countable. By Theorem 1.1(iv), the model pair $\langle \mathfrak{A}, \mathfrak{B} \rangle$ has a countable recursively saturated elementary extension $\langle \mathfrak{A}^*, \mathfrak{B}^* \rangle$. Use Theorem 3.2(i) to obtain a closed embedding from $\mathfrak{A}^-$, the smallest closed submodel of $\mathfrak{A}^*$, into $\mathfrak{B}^*$. Now any basic connected sentence true in $\mathfrak{A}$ is true in $\mathfrak{A}^*$ since $\mathfrak{A} \prec \mathfrak{A}^*$, hence true in $\mathfrak{A}^-$ by Proposition 3.1(i), hence true in $\mathfrak{B}^*$, again by Proposition 3.1(i), and finally true in $\mathfrak{B}$ since $\mathfrak{B} \prec \mathfrak{B}^*$. Part (ii) follows similarly using Lemma 1.2(ii). $\dashv$

THEOREM 3.4. *Let L be a countable language.*

(i) *A theory T is preserved under closed extensions iff it has an existential connected set of axioms.*

(ii) *A sentence φ is preserved under closed extensions iff it is logically equivalent to an existential connected sentence.*

(iii) *A theory T is preserved under closed submodels iff it has a universal connected set of axioms.*

(iv) *A sentence φ is preserved under closed submodels iff it is logically equivalent to a universal connected sentence.*

PROOF. The proofs of (i) and (ii) proceed as in Theorem 3.3, using Theorem 3.2(ii) rather than 3.2(i); (iii) and (iv) are immediate consequences of (i) and (ii). $\dashv$

§4. Theories and sentences preserved under closed unions, and under components and disjoint unions. This section contains more preservation theorems. Theorem 4.4 is of the greatest interest. It syntactically characterizes classes closed under disjoint unions and components.

I first state the easy directions of the preservation theorems of this section.

PROPOSITION 4.1. (i) *Any sentence ψ of the form $\forall x \varphi(x)$, where φ is existential connected, is preserved under closed unions.*

(ii) *Assume L has no constant symbols. Any sentence ψ of the form $\forall x \varphi(x)$, where φ is basic connected, is preserved under disjoint unions and components.*

PROOF. Part (i) is easy to see using the game theoretic interpretation of truth conditions for a sentence. Suppose $\mathfrak{A} = \bigcup_{i<k} \mathfrak{A}_i$ is a closed union and each $\mathfrak{A}_i \models \psi$.

Then $\exists$ has a winning strategy when the game begins with ψ and uses $\mathfrak{A}_i$. Suppose the game uses $\mathfrak{A}$ instead. Then $\forall$ plays first, erasing ψ and replacing it with $\varphi(a)$, for some $a \in A$. Now a must be an element of some A_i and $\exists$ can win by pursuing the winning strategy she would use with $\mathfrak{A}_i$. Clearly this is a winning strategy for $\exists$ on $\mathfrak{A}$ so $\mathfrak{A} \models \psi$.

Part (ii) now follows easily. Certainly ψ is preserved under components since it is universal connected. It is preserved under disjoint unions by part (i). $\dashv$

The difficulty in proving the converses to Proposition 4.1 is that the classes of sentences presented there are not closed under disjunctions. The following technical lemma gets around this problem.

LEMMA 4.2. *Let L be a countable language and $\mathfrak{A}$ a countable L-structure. Let $\langle \mathfrak{A}, A_\alpha \rangle_{\alpha<\kappa}$ be a structure for the language obtained by adding finitely or countably many unary relation symbols S_α, $\alpha < \kappa$, to the language.*

(i) *If T is a theory in the language L which is preserved under closed unions, and $\mathfrak{A}$ is a model of $\{\forall x \psi(x) : \psi$ is existential connected and $T \models \forall x \psi(x)\}$ then there is a countable elementary extension $\langle \mathfrak{A}^*, A^*_\alpha \rangle_{\alpha<\kappa}$ of $\langle \mathfrak{A}, A_\alpha \rangle_{\alpha<\kappa}$ and closed submodels $\mathfrak{B}_\alpha$, $\alpha < \lambda$, of $\mathfrak{A}^*$ such that each $\mathfrak{B}_\alpha \models T$ and $A \subseteq \bigcup_{\alpha<\lambda} B_\alpha$.*

(ii) *Assume that A_α is a union of components of $\mathfrak{A}$ and L has no constant symbols. If T is a theory in L preserved under disjoint unions and components, and $\mathfrak{A}$ is a model of $\{\forall x \psi(x) : \psi$ is basic connected and $T \models \forall x \psi(x)\}$ then there is a countable elementary extension $\langle \mathfrak{A}^*, A^*_\alpha \rangle_{\alpha<\kappa}$ of $\langle \mathfrak{A}, A_\alpha \rangle_{\alpha<\kappa}$ and components $\mathfrak{B}_\alpha$, $\alpha < \lambda$, of $\mathfrak{A}^*$ such that each $\mathfrak{B}_\alpha \models T$ and*

$$A - \bigcup_{\alpha<\kappa} A_\alpha \subseteq \bigcup_{\alpha<\lambda} B_\alpha,$$

$$\bigcup_{\alpha<\kappa} A^*_\alpha \cap \bigcup_{\alpha<\lambda} B_\alpha = \varnothing.$$

PROOF. First prove part (i).

For each $a \in A$ let

$$\Gamma_a(c) = \{\theta(c) : \mathfrak{A} \models \theta(a) \text{ and } \theta(x) \text{ is universal connected}\},$$

where c is a constant symbol not in L. Then $T \cup \Gamma_a(c)$ is consistent; for, if not, there are sentences $\theta_i(c)$, $i < k$, in $\Gamma_a(c)$ such that

$$T \models \neg \bigwedge_{i<k} \theta_i(c)$$

whence

$$T \models \forall x \neg \bigwedge_{i<k} \theta_i(x).$$

The latter sentence is equivalent to one of the form $\forall x \psi(x)$, for some existential connected ψ, so it is true in $\mathfrak{A}$, a contradiction.

Let $\langle \mathfrak{B}'_a, b_a \rangle \models T \cup \Gamma_a(c)$, B'_a countable, for each $a \in A$. Form the model sequence $\langle \mathfrak{B}'_a, \langle \mathfrak{A}, A_\alpha \rangle_{\alpha<\kappa} \rangle_{a \in A}$ and let $\langle \mathfrak{B}^*_a, \langle \mathfrak{A}^*, A^*_\alpha \rangle_{\alpha<\kappa} \rangle_{a \in A}$ be a countable, recursively saturated elementary extension of it. Notice that for each a, $\langle \mathfrak{B}^*_a, \mathfrak{A}^* \rangle$ is recursively saturated and that if a new constant symbol c is added to L and interpreted by b_a in $\mathfrak{B}^*_a$ and a in $\mathfrak{A}^*$, then the model pair $\langle \langle \mathfrak{B}^*_a, b_a \rangle, \langle \mathfrak{A}^*, a \rangle \rangle$

is recursively saturated (this follows from Theorem 1.1). Now b_a is chosen so that any existential formula $\psi(x)$ which satisfies $\langle \mathfrak{B}_a^*, b_a\rangle \models \psi(c)$ also satisfies $\langle \mathfrak{A}^*, a\rangle \models \psi(c)$. Thus, by Theorem 3.2(ii), there is a closed embedding of $\langle \mathfrak{B}_a^*, b_a\rangle$ into $\langle \mathfrak{A}^*, a\rangle$; in other words, there is a closed embedding of $\mathfrak{B}_a^*$ into $\mathfrak{A}^*$ which maps b_a to a. Let $\mathfrak{B}_a$ be the image of $\mathfrak{B}_a^*$ under this embedding. Then $\mathfrak{B}_a \models T$ and $A \subseteq \bigcup_{a\in A} B_a$.

The proof of (ii) proceeds in much the same way. For each $a \in A - \bigcup_{\alpha<\kappa} A_\alpha$ (call this set A') let

$$\Gamma_a(c) = \{\theta(c) \colon \theta \text{ is basic connected and } \mathfrak{A} \models \theta(a)\}.$$

Again $T \cup \Gamma_a(c)$ is consistent and has a countable model $\langle \mathfrak{B}_a', b_a\rangle$. Find a countable recursively saturated elementary extension $\langle \mathfrak{B}_a^*, \langle \mathfrak{A}^*, A_\alpha^*\rangle_{\alpha<\kappa}\rangle_{a\in A'}$. This time use Theorem 3.2(i) to show that there is a closed embedding of the component of $\mathfrak{B}_a^*$ containing b_a into $\mathfrak{A}^*$ which maps b_a to a. Let $\mathfrak{B}_a$ be the image of the component under this embedding. Since T is closed under components, $\mathfrak{B}_a \models T$. Clearly $A - \bigcup_{\alpha<\kappa} A_\alpha \subseteq \bigcup_{a\in A'} B_a$. To see that $\bigcup_{\alpha<\kappa} A_\alpha^* \cap \bigcup_{a\in A'} B_a = \varnothing$, observe that for any relation symbol R in L,

$$\langle \mathfrak{A}, A_\alpha\rangle \models (\forall x_0, \ldots, x_{m-1})(\bigvee_{i<m} S_\alpha(x_i) \wedge \bigvee_{i<m} \neg S_\alpha(x_i)) \rightarrow \neg R(x_0, \ldots, x_{m-1}).$$

These sentences (taken over all R in L) express the fact that A_α is a union of components of $\mathfrak{A}$. Since $\langle \mathfrak{A}, A_\alpha\rangle \prec \langle \mathfrak{A}^*, A_\alpha^*\rangle$, A_α^* is a union of components of $\mathfrak{A}^*$. Also, for each $a \in A'$, $\mathfrak{A} \models \neg S_\alpha(a)$ and hence $\mathfrak{A}^* \models \neg S_\alpha(a)$. Therefore $a \notin A_\alpha^*$ so $B_a \cap A_\alpha^* = \varnothing$ since $\mathfrak{B}_a$ is a component. This shows that $\bigcup_{\alpha<\kappa} A_\alpha^* \cap \bigcup_{a\in A'} B_a = \varnothing$. $\dashv$

The next theorem is reminiscent of a theorem of Keisler which says that sentences preserved under unions of models are precisely the ones equivalent to sentences of the form $\forall x\psi(x)$, where ψ is existential (cf. [11]).

THEOREM 4.3. *Assume that L is countable.*

(i) *A theory T is preserved under closed unions iff it has a set of axioms of the form*

$$(*) \qquad \forall x\psi(x),\ \psi \text{ existential connected.}$$

(ii) *A sentence φ is preserved under closed unions iff it is logically equivalent to a sentence of the form* (*).

PROOF. (i) Let Γ be the set of sentences of the form (*) which are consequences of T. By the Compactness Theorem, it is enough to show that $\Gamma \models T$.

Let $\mathfrak{A}^0$ be a countable model of Γ. By Lemma 4.2(i) there is an elementary extension $\mathfrak{A}^1$ of $\mathfrak{A}^0$ and closed submodels $\mathfrak{B}_{0\alpha}^1$, $\alpha < \lambda(0)$, of $\mathfrak{A}^1$ such that $A^0 \subseteq \bigcup_{\alpha<\lambda(0)} B_{0\alpha}^1$ and each $\mathfrak{B}_{0\alpha}^1 \models T$. Add new unary relation symbols $S_{0\alpha}$ to the language and take $\langle \mathfrak{A}^1, B_{0\alpha}^1\rangle_{\alpha<\lambda(0)}$ to be a structure for the expanded language. Repeat this process for each $n \in \omega$, i.e., assume that at step n there is a structure $\langle \mathfrak{A}^n, B_{k\alpha}^n\rangle_{k<n,\alpha<\lambda(k)}$, with each $B_{k\alpha}^n$ interpreting a unary relation symbol $S_{k\alpha}$ which was added to the language at step k. Use Lemma 4.2(i) to produce a countable elementary extension $\langle \mathfrak{A}^{n+1}, B_{k\alpha}^{n+1}\rangle_{k<n,\alpha<\lambda(k)}$ and closed submodels $\mathfrak{B}_{n\alpha}^{n+1}$, $\alpha < \lambda(n)$, of $\mathfrak{A}^{n+1}$ such that each $\mathfrak{B}_{n\alpha}^{n+1} \models T$ and $A^n \subseteq \bigcup_{\alpha<\lambda(n)} B_{n\alpha}^{n+1}$. Add unary relation symbols $S_{n\alpha}$, $\alpha < \lambda(n)$, to the language and let $\langle \mathfrak{A}^{n+1}, B_{n\alpha}^{n+1} \rangle$ be a structure for the expanded language.

Thus, $\mathfrak{A}^0 \prec \mathfrak{A}^1 \prec \dots$ and for each $k \in \omega$, $\alpha < \lambda(k)$, $\mathfrak{B}_{k\alpha}^{k+1} \prec \mathfrak{B}_{k\alpha}^{k+2} \prec \cdots$. Let $\mathfrak{A} = \bigcup_{n\in\omega} \mathfrak{A}^n$ and $\mathfrak{B}_{k\alpha} = \bigcup_{n>k} \mathfrak{B}_{k\alpha}^n$. By the Elementary Chain Theorem (see [5, Theorem 3.1.13]) $\mathfrak{A}^0 \prec \mathfrak{A}$ and each $\mathfrak{B}_{k\alpha} \models T$.

Each $\mathfrak{B}_{k\alpha}$ is a closed submodel of $\mathfrak{A}$. To prove this, I show for each relation symbol R in L

$$(**) \qquad \langle \mathfrak{A}, B_{k\alpha} \rangle = (\forall x_0, \dots, x_{m-1}) \left(\bigvee_{i<m} S_{k\alpha}(x_i) \wedge \bigvee_{i<m} \neg S_{k\alpha}(x_i)\right) \rightarrow \neg R(x_0, \dots, x_{m-1}).$$

But $\langle \mathfrak{A}^{k+1}, B_{k\alpha}^{k+1} \rangle \prec \langle \mathfrak{A}^{k+2}, B_{k\alpha}^{k+2} \rangle \prec \cdots$ so again by the Elementary Chain Theorem $\langle \mathfrak{A}^{k+1}, B_{k\alpha}^{k+1} \rangle \prec \langle \mathfrak{A}, B_{k\alpha} \rangle$. $\langle \mathfrak{A}_{k\alpha}^{k+1}, B_{k\alpha}^{k+1} \rangle$ satisfies the sentence on the left side of $(**)$ so $\langle \mathfrak{A}, B_{k\alpha} \rangle$ must also satisfy it.

Thus, $\mathfrak{A}$ is a closed union of the $\mathfrak{B}_{k\alpha}$'s and each $\mathfrak{B}_{k\alpha} \models T$, so $\mathfrak{A} \models T$. Finally, since $\mathfrak{A}^0 \prec \mathfrak{A}$, $\mathfrak{A}^0 \models T$. It follows that $\Gamma \models T$.

(ii) Let $T = \{\varphi\}$ and use part (i). There is a set Γ of sentences of the form $(*)$ such that $\varphi \models \Gamma$, $\Gamma \models \varphi$. There is a finite subset of Γ which proves φ, by the Compactness Theorem. In fact, there is a sentence θ in Γ which proves φ because Γ is closed under conjunctions (up to logical equivalence). It follows that $\models \varphi \leftrightarrow \theta$. $\dashv$

The next theorem is the characterization of classes closed under disjoint union and components.

THEOREM 4.4. (i) *A theory T is preserved under disjoint unions and components iff it has a set of axioms of the form*

$$(\sharp) \qquad \forall x \psi(x), \quad \psi \textit{ basic connected.}$$

PROOF. (i) Use the same approach as in the proof of Theorem 4.3. Let Γ be the set of sentences of the form $(\sharp)$ that are consequences of T. I show that $\Gamma \models T$.

Let $\mathfrak{A}^0 \models \Gamma$ be a countable model. Again, construct an elementary chain of models with $\mathfrak{A}^0$ at the bottom. Assume that at step n there is a structure $\langle \mathfrak{A}^n, B_{k\alpha}^n \rangle_{k<n, \alpha<\lambda(n)}$ with each $B_{k\alpha}^n$ an interpretation of a unary relation symbol $S_{k\alpha}$ which was added to the language at step k. Suppose, furthermore, that $B_{k\alpha}^n$ is a union of components of $\mathfrak{A}^n$. By Lemma 4.2(ii), there is an elementary extension $\langle \mathfrak{A}^{n+1}, B_{k\alpha}^{n+1} \rangle_{k<n, \alpha<\lambda(k)}$ of $\langle \mathfrak{A}^n, B_{k\alpha}^n \rangle_{k<n, \alpha<\lambda(k)}$, and components $\mathfrak{B}_{n\alpha}^{n+1}$, $\alpha < \lambda(n)$, of $\mathfrak{A}^{n+1}$ such that each $\mathfrak{B}_{n\alpha}^{n+1} \models T$, $A^n - \bigcup_{k<n} \bigcup_{\alpha<\lambda(k)} B_{k\alpha}^n \subseteq \bigcup_{\alpha<\lambda(n)} B_{n\alpha}^{n+1}$, and

$$(\sharp\sharp) \qquad \bigcup_{k<n} \bigcup_{\alpha<\lambda(k)} B_{k\alpha}^n \cap \bigcup_{\alpha<\lambda(n)} B_{n\alpha}^{n+1} = \varnothing.$$

The $B_{n\alpha}^{n+1}$'s may be assumed to be disjoint. Add unary relation symbols $S_{n\alpha}$, $\alpha < \lambda(n)$, and let $\langle \mathfrak{A}^{n+1}, B_{k\alpha}^{n+1} \rangle_{k<n+1, \alpha<\lambda(k)}$ be a structure for the expanded language.

Once again, let $\mathfrak{A} = \bigcup_{n\in\omega} \mathfrak{A}^n$, $B_{k\alpha} = \bigcup_{n>k} B_{k\alpha}^n$. Then $\mathfrak{A}^0 \prec \mathfrak{A}$, $\mathfrak{B}_{k\alpha} \models T$. Then the $B_{k\alpha}$'s are disjoint. For consider $\mathfrak{B}_{k\alpha}$ and $\mathfrak{B}_{l\beta}$ for $\langle k, \alpha \rangle \neq \langle l, \beta \rangle$, $k \geq l$. Then

$$\langle \mathfrak{A}^{k+1}, B_{k\alpha}^{k+1}, B_{l\beta}^{k+1} \rangle \models \forall x \neg (S_{k\alpha}(X) \wedge S_{l\beta}(x)).$$

This is true when $k = l$ by the supposition that the $B_{k\alpha}^{k+1}$'s are disjoint, and true when $k > l$ by (##). Thus $\mathfrak{A}$ is a disjoint union of the $\mathfrak{B}_{k\alpha}$'s so $\mathfrak{A} \models T$. Therefore, $\mathfrak{A}^0 \models T$ and $\Gamma \models T$.

(ii) The proof is exactly the same as for Theorem 4.3(ii). $\dashv$

The axioms for all the examples in §2 have been stated in the form (#).

§5. **Related theorems.** I conclude this paper with some interesting consequences of theorems in §3. Throughout the section I assume that L has no constant symbols.

I begin with a simple proposition.

PROPOSITION 5.1 (CANTOR-BERNSTEIN THEOREM FOR CLOSED EMBEDDINGS). *If there is a closed embedding of $\mathfrak{A}$ into $\mathfrak{B}$ and another of $\mathfrak{B}$ into $\mathfrak{A}$, then $\mathfrak{A} \cong \mathfrak{B}$.*

PROOF. All I need to do is show that for any connected model $\mathfrak{K}$, $\mathfrak{A}$ has the same number of components isomorphic to $\mathfrak{K}$ as $\mathfrak{B}$ does. (Here I am being a little sloppy—$\mathfrak{A}$ and $\mathfrak{B}$ are certainly identical on components that contain interpretations of constant symbols; when I say components of $\mathfrak{A}$ isomorphic to $\mathfrak{K}$, I consider them as structures for the language with constant symbols removed.) Let $X_{\mathfrak{A}}$ be the set of components in $\mathfrak{A}$ which are isomorphic to $\mathfrak{K}$ and define $X_{\mathfrak{B}}$ similarly. The closed embedding from $\mathfrak{A}$ into $\mathfrak{B}$ induces a one-to-one mapping of $X_{\mathfrak{A}}$ into $X_{\mathfrak{B}}$ and by the same token the closed embedding of $\mathfrak{B}$ into $\mathfrak{A}$ induces a one-to-one mapping of $X_{\mathfrak{B}}$ into $X_{\mathfrak{A}}$. By the Cantor-Bernstein Theorem $|X_{\mathfrak{A}}| = |X_{\mathfrak{B}}|$ so $\mathfrak{A}$ and $\mathfrak{B}$ have the same number of components isomorphic to $\mathfrak{K}$. $\dashv$

This proposition has the following theorem as a consequence.

THEOREM 5.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures. If the set of existential connected sentences satisfied by $\mathfrak{A}$ is the same as the set of existential connected sentences satisfied by $\mathfrak{B}$, then $\mathfrak{A} \equiv \mathfrak{B}$.*

PROOF. Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are countable. By Theorem 1.1(iv), there is a countable recursively saturated elementary extension $\langle \mathfrak{A}^*, \mathfrak{B}^* \rangle$ of $\langle \mathfrak{A}, \mathfrak{B} \rangle$. Every existential connected sentence true in $\mathfrak{A}^*$ is true in $\mathfrak{B}^*$ so by Theorem 3.2(ii) there is a closed embedding of $\mathfrak{A}^*$ into $\mathfrak{B}^*$. The same argument shows that there is a closed embedding of $\mathfrak{B}^*$ into $\mathfrak{A}^*$. The previous proposition shows that $\mathfrak{A}^* \cong \mathfrak{B}^*$. Since $\mathfrak{A} \prec \mathfrak{A}^*$ and $\mathfrak{B} \prec \mathfrak{B}^*$, $\mathfrak{A} \equiv \mathfrak{B}$. $\dashv$

COROLLARY 5.3. *Any sentence φ is logically equivalent to a Boolean combination of existential connected sentences (equivalently, to a Boolean combination of universal connected sentences).*

PROOF. By Lemma 1.2(ii), I need only show that for any sentence φ, if $\mathfrak{A} \models \varphi$ and every Boolean combination of existential connected sentences true in $\mathfrak{A}$ is true in $\mathfrak{B}$, then $\mathfrak{B} \models \varphi$. This clearly follows from the preceding theorem since $\mathfrak{A}$ and $\mathfrak{B}$ will satisfy the same existential connected sentences. $\dashv$

The next, and final, theorem has ramifications for finite combinatorics. The theorem says that a certain theory which is obtained from a class of structures closed under disjoint unions and components will be complete. This theory will, in some sense, be the set of sentences true in almost all finite structures in the class whenever the class grows slowly (see [6] or [7] for details). For a connected structure $\mathfrak{K}$ and integer j, $\theta_{\mathfrak{K}_j}$ will denote a first order sentence which says "there are exactly j components isomorphic to $\mathfrak{K}$."

THEOREM 5.4. *Suppose that $\mathscr{C}$ is a class of L-structures closed under disjoint* *unions and components, and that $\mathscr{C}$ contains at least one finite structure. The set of sentences*

$$T = \{\neg\theta_{\mathfrak{K}_j} : \mathfrak{K} \text{ is connected}, j \in \omega\} \cup \bigcup_{\substack{\mathfrak{A} \in \mathscr{C} \\ \text{finite}}} \mathrm{Th}(\mathfrak{A})$$

is a complete consistent theory. The deductive closure of this theory is

$$T^* = \bigcup_{\substack{\mathfrak{A} \in \mathscr{C} \\ \text{finite}}} \; \bigcap_{\substack{\mathfrak{B} \text{ finite closed} \\ \text{extension of } \mathfrak{A}}} \mathrm{Th}(\mathfrak{B}).$$

PROOF. It is obvious that T is consistent. To see that T is complete, I show that any two models $\mathfrak{A}$ and $\mathfrak{B}$ of T are elementarily equivalent. By Theorem 5.2 it is enough to show that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same existential connected sentences. Suppose $\mathfrak{A} \models \varphi$, φ existential connected. There must be a finite model $\mathfrak{A}' \in \mathscr{C}$ which satisfies φ, for otherwise each finite model would satisfy $\neg\varphi$ and hence $\neg\varphi \in T$. Since $\mathfrak{B} \models \neg\theta_{\mathfrak{K}_j}$ for each connected $\mathfrak{K} \in \mathscr{C}$ and $j \in \omega$, there is a closed embedding of each finite structure in $\mathscr{C}$ into $\mathfrak{B}$. In particular, there is a closed embedding of $\mathfrak{A}'$ into $\mathfrak{B}$. Closed embeddings preserve existential connected sentences so $\mathfrak{B} \models \varphi$. The same argument shows that $\mathfrak{A}$ satisfies every existential connected sentence satisfied by $\mathfrak{B}$.

Now to show that T^* is the deductive closure of T, I show that T^* is consistent, closed under deduction, and contains T. The result follows from the completeness of T.

First notice that a sentence φ is in T^* iff it is true in all the finite closed extensions of some finite structure in $\mathscr{C}$. Therefore, if $\varphi_0, \ldots, \varphi_{k-1}$ are in T^*, there are finite structures $\mathfrak{A}_0, \ldots, \mathfrak{A}_{k-1}$ in $\mathscr{C}$ such that φ_i is true in all finite closed extensions of $\mathfrak{A}_i$, $i < k$. Now the set containing $\varphi_0, \ldots, \varphi_{k-1}$ is consistent since each φ_i is true in $\mathfrak{A} = \mathfrak{A}_0 \amalg \cdots \amalg \mathfrak{A}_{k-1}$. This shows that T^* is consistent. Notice also that each φ_i is true in all finite closed extensions of $\mathfrak{A}$ so that any sentence φ which is provable from the φ_i's is true in all the finite closed extensions of some finite structure in $\mathscr{C}$, and hence is in T^*. This shows that T^* is closed under deduction. Finally, each sentence $\neg\theta_{\mathfrak{K}_j}$ is true in all the finite closed extensions of any finite structure in $\mathscr{C}$ with $k + 1$ components isomorphic to $\mathfrak{K}$, consequently is in T^*, and any sentence true in all finite structures in $\mathscr{C}$ is clearly in T^*. Therefore, T is contained in T^*. $\dashv$

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