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A transfer function approach to the realisation problem of nonlinear systems

Miroslav Halás [a] & Ülle Kotta [b]

[a] Institute of Control and Industrial Informatics, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, Ilkovičova 3, 81219 Bratislava, Slovakia

[b] Institute of Cybernetics, Tallinn University of Technology, Akademia tee 21, Tallinn, 12618, Estonia

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Miroslav Halás[a]* and Ülle Kotta[b]

[a]*Institute of Control and Industrial Informatics, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, Ilkovičova 3, 81219 Bratislava, Slovakia;* [b]*Institute of Cybernetics, Tallinn University of Technology, Akademia tee 21, Tallinn, 12618, Estonia*

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This article studies the nonlinear realisation problem, i.e. the problem of finding the state equations of a nonlinear system from the transfer function representation being easily computable from the higher order input–output differential equation. The realisation in both observer and controller canonical forms is studied. The results demonstrate a clear connection with those from linear theory. In the solution the concept of adjoint polynomials, adjoint transfer function and right factorisation of the transfer function play a key role. Finally, the results are applied for system linearisation up to input–output injection used in the observer design.

Keywords: nonlinear system; realisation problem; transfer function; adjoint polynomials; right factorisation; canonical forms

1. Introduction

In the control theory, systems are typically described either by higher order input–output differential equations or by state equations. While most results on parameter identification of real systems are available for the systems described by input–output differential equations, the majority of techniques for system analysis and control are based on the state-space description. In the linear case, any control system, described by an input–output differential equation, can be equivalently described by the state equations. However, this is, in general, not valid for nonlinear systems and the problem here is to find conditions under which a state-space realisation of a higher order input–output differential equation can be found. Three different intrinsic realisability conditions were given in terms of integrability of the subspaces of differential one-forms (Conte, Moog, and Perdon 2007), involutivity of the conditionally invariant distributions of the vector fields (Van der Schaft 1987) and commutativity of iterative Lie brackets of vector fields (Delaleau and Respondek 1995). Moreover, the algorithm-based solutions (Crouch and Lamnabhi-Lagarrigue 1988; Glad 1989) are demonstrated to compute the integrable bases of the subspaces of one-forms in Kotta and Mullari (2005).

In the linear case, the state equations in the observer or controller canonical form can directly be written out from the transfer function of a system. Since the transfer function formalism was recently developed also for nonlinear systems (Zheng and Cao 1995; Halás 2008), it is natural to be interested in finding a solution to the realisation problem by means of such a formalism which forms the main scope of our interest in this work. Adjoint polynomials, adjoint transfer function and right factorisation of the transfer function play a key role in our solution. In particular, we focus on finding the realisations in observer and controller canonical forms starting directly from the transfer function of a nonlinear system of which can easily be computed from its input–output differential equation. In doing so, one associates with the system two polynomials, as in Zheng, Willems, and Zhang (2001), defined over the field of meromorphic functions. Then, after defining the quotients of such polynomials (Ore 1931, 1933) one can speak of transfer functions of nonlinear systems. This can also be understood by the way that to a nonlinear system the so-called tangent linear system is associated (see, for instance, Fliess, Lévine, Martin, and Rouchon 1995), by using Kähler differentials (Johnson 1969) and then the ideas similar to those applied for linear time-varying systems in Fliess (1994) are applicable. Then the linearised system description resembles the time-varying linear system description except that now the time-varying coefficients of the polynomials are not necessarily independent (Li, Ondera, and Wang 2008).

Our results on the realisation problem by means of observer and controller canonical form establish the clear connection to those of linear theory. Besides, the

*Corresponding author. Email: miroslav.halas@stuba.sk

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byproduct of the solution is the availability of the less-restrictive non-canonical realisation which can be found computationally in a more direct manner than those in Conte et al. (2007), Van der Schaft (1987) and Delaleau and Respondek (1995) where one has to compute the necessary vector spaces of one-forms or vector fields step-by-step via certain algorithms. Then, the state equations can be found by integrating the respective one-forms. Finally, the solution to the realisation problem is applied to the linearisation of the state equations by the input–output injection employed in the standard Luenberger observer design. The differential geometric solution of the problem has been known for a long time (Krener and Isidori 1983; Krener and Respondek 1985; Xia and Gao 1989). The alternative approach (Glumineau, Moog, and Plestan 1996), based on the algebraic formalism of differential forms, assumes the knowledge of the input–output equation corresponding to the state space description. Then the solvability conditions require certain one-forms, associated with the input–output equation and defined iteratively via a certain algorithm, to be exact. However, this problem can be understood as a realisation in the observer canonical form. The coefficients of the one-forms equal the respective coefficients of the adjoint polynomials.

This article is organised as follows. In Section 2 the transfer function formalism of nonlinear systems is recalled. The realisation problem employing such a formalism is then studied in Section 3, by means of both observer and controller canonical form, and followed by the discussion on minimal realisations in Section 4. The application of the results to the observer design is demonstrated in Section 5 and, finally, this article is concluded in Section 6.

2. Transfer functions of nonlinear systems

To begin with, we briefly recall the transfer function formalism of nonlinear systems introduced in Halás (2008) using the algebraic setting of Conte et al. (2007).

Consider the single-output nonlinear system defined either by the state equations of the form

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and $y \in \mathbf{R}$, or by the input–output differential equation of the form

$$y^{(n)} = F(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(n-1)}) \tag{2}$$

In (1) and (2) the functions f, h and, respectively, F are assumed to be the elements of the differential field of meromorphic functions of variables $\{x, u^{(k)};\ k \geq 0\}$ or $\{y, \dot{y}, \ldots, y^{(n-1)}, u^{(k)};\ k \geq 0\}$, respectively, denoted by $\mathcal{K}$.

Define the formal vector space of differential one-forms $\mathcal{E} = \text{span}_{\mathcal{K}}\{\mathrm{d}\xi;\ \xi \in \mathcal{K}\}$. The left skew polynomial ring $\mathcal{K}[s]$ of polynomials in s over $\mathcal{K}$ with the usual addition, and the (non-commutative) multiplication defined by the commutation rule

$$sg = gs + \dot{g} \tag{3}$$

where $g \in \mathcal{K}$, represents the ring of linear ordinary differential operators that act on any $v \in \mathcal{E}$ as follows:

$$\left(\sum_{i=0}^{k} \alpha_i s^i\right) v = \sum_{i=0}^{k} \alpha_i v^{(i)}$$

Note that s is interpreted as a time derivative $\frac{\mathrm{d}}{\mathrm{d}t}$ and we thus have $sg = gs + \dot{g}$ for any $g \in \mathcal{K}$ but $s\,\mathrm{d}\xi = \mathrm{d}\dot{\xi}$ for any $\mathrm{d}\xi \in \mathcal{E}$.

Lemma 2.1 (Ore 1931)**:** *For all non-zero a, $b \in \mathcal{K}[s]$, there exist non-zero a_1, $b_1 \in \mathcal{K}[s]$ such that $a_1 b = b_1 a$.*

Thus, the ring $\mathcal{K}[s]$ can be embedded into the non-commutative quotient field $\mathcal{K}\langle s\rangle$ by defining quotients as (Ore 1931, 1933)

$$\frac{a}{b} = b^{-1} \cdot a \tag{4}$$

Once the fraction of two skew polynomials is defined, we can introduce the transfer function matrix of the nonlinear system (1), respectively (2), as an element $G(s) \in \mathcal{K}^{1\times m}\langle s\rangle$ such that $\mathrm{d}y = G(s)\mathrm{d}u$.

Remark 1: By definition, the elements of the quotient field $\mathcal{K}\langle s\rangle$ are equivalence classes of pairs (a, b), where $b \neq 0$, with respect to the equivalence relation given by $(a_1, b_1) \sim (a_2, b_2)$ iff $\beta_2 a_1 = \beta_1 b_2$ where $\beta_2 b_1 = \beta_1 b_2$ by the Ore condition. Thus the transfer function, as an element of the quotient field, is naturally understood as the whole equivalence class and the respective quotient $\frac{a}{b}$ only as its representative. Note that the operations in the quotient field do not depend on which representatives from the equivalence class have been used. In the majority of cases, though not always, the simplest representative in the equivalence class (in our case the irreducible transfer function) is identified with the whole equivalence class. This point of view is well accepted also in the linear case (see, for instance, Kailath 1980).

The transfer function of a nonlinear system can be computed from both the state-space representation (1) and the input–output Equation (2). If we start with the state space description (1), after differentiating, we get

$$\begin{aligned} \mathrm{d}\dot{x} &= A\,\mathrm{d}x + B\,\mathrm{d}u \\ \mathrm{d}y &= C\,\mathrm{d}x \end{aligned} \tag{5}$$

where $A = (\partial f/\partial x)$, $B = (\partial f/\partial u)$ and $C = (\partial h/\partial x)$. Then $s\,\mathrm{d}x = A\,\mathrm{d}x + B\,\mathrm{d}u$ from which

$$G(s) = C(sI - A)^{-1}B$$

Note that one has to invert the matrix $(sI - A)$ over the non-commutative quotient field $\mathcal{K}\langle s\rangle$, as its entries are (non-commutative) skew polynomials from $\mathcal{K}[s]$, which might not be a trivial task. However, in the particular cases of the observer and controller canonical forms employed in Sections 3.1 and 3.2 the inversion can be found in a compact form.

When starting from the input–output description (2) we get

$$\mathrm{d}y^{(n)} - \sum_{i=0}^{n-1} \frac{\partial F}{\partial y^{(i)}} \mathrm{d}y^{(i)} = \sum_{j=1}^{m} \sum_{i=0}^{n-1} \frac{\partial F}{\partial u_j^{(i)}} \mathrm{d}u_j^{(i)}$$

after differentiating or alternatively

$$a(s)\mathrm{d}y = \sum_{j=1}^{m} b_j(s)\mathrm{d}u_j \tag{6}$$

where $a(s) = s^n - \sum_{i=1}^{n-1} \frac{\partial F}{\partial y^{(i)}} s^i$, $b_j(s) = \sum_{i=0}^{n-1} \frac{\partial F}{\partial u_j^{(i)}} s^i$ and $a(s)$, $b_j(s) \in \mathcal{K}[s]$ for $j = 1, \ldots, m$. The transfer function matrix is then

$$G(s) = \left(\frac{b_1(s)}{a(s)} \cdots \frac{b_m(s)}{a(s)} \right) \tag{7}$$

Example 2.2: Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 u \\ y &= x_1 \end{aligned}$$

After differentiating the system equations we get (5) with

$$A = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Then

$$(sI - A) = \begin{pmatrix} s & -1 \\ -u & s \end{pmatrix} \quad \text{and}$$

$$(sI - A)^{-1} = \begin{pmatrix} \dfrac{s}{s^2 - u} & \dfrac{1}{s^2 - u} \\ \dfrac{u}{s^2 - \dot{u}/u - u} & \dfrac{s - \dot{u}/u}{s^2 - \dot{u}/u - u} \end{pmatrix}$$

Thus the transfer function is

$$G(s) = C(sI - A)^{-1}B = \frac{x_1}{s^2 - u} = \frac{y}{s^2 - u} \tag{8}$$

Note that as the transfer function is supposed to display input–output properties, it is appropriate to replace the state variables by functions of system input, output and their derivatives. In this example one has $x_1 = y$.

Example 2.3: Consider the system

$$y^{(3)} = \ddot{y}u + \frac{\ddot{y}\dot{u}}{u}$$

After differentiating we get

$$\mathrm{d}y^{(3)} = \left(u + \frac{\dot{u}}{u}\right)\mathrm{d}\ddot{y} + \frac{\ddot{y}}{u}\mathrm{d}\dot{u} + \left(\ddot{y} - \frac{\ddot{y}\dot{u}}{u^2}\right)\mathrm{d}u$$

or, in terms of polynomials from $\mathcal{K}[s]$

$$\left(s^3 - \left(u + \frac{\dot{u}}{u}\right)s^2\right)\mathrm{d}y = \left(\frac{\ddot{y}}{u}s + \left(\ddot{y} - \frac{\ddot{y}\dot{u}}{u^2}\right)\right)\mathrm{d}u$$

Then the transfer function reads

$$G(s) = \frac{\frac{\ddot{y}}{u}s + \ddot{y} - \frac{\ddot{y}\dot{u}}{u^2}}{s^3 - \left(u + \frac{\dot{u}}{u}\right)s^2} \tag{9}$$

Example 2.4: Consider the system

$$\ddot{y} = \dot{y}\dot{u}_1 + u_2$$

This time, after differentiating we get

$$\mathrm{d}\ddot{y} = \dot{u}_1\,\mathrm{d}\dot{y} + \dot{y}\,\mathrm{d}\dot{u}_1 + \mathrm{d}u_2$$

or

$$(s^2 - \dot{u}_1 s)\mathrm{d}y = \dot{y}s\,\mathrm{d}u_1 + \mathrm{d}u_2$$

Finally, the transfer function matrix is given as

$$G(s) = \left(\frac{\dot{y}s}{s^2 - \dot{u}_1 s} \quad \frac{1}{s^2 - \dot{u}_1 s} \right)$$

We now recall how the notions of controllability and observability of the nonlinear system are related to the transfer function formalism, for we are interested in finding controllable and observable realisations of the system (2).

2.1 *Controllability*

In this article a special notion of controllability, called accessibility and defined through the non-existence of the so-called autonomous elements (Pommaret 1986), is used. However, to be consistent with the notion of controller canonical form and linear theory, we refer to this notion just as a controllability. The controllability of the system is guaranteed by the non-existence of (non-trivial) common left factors in the nonlinear system polynomial description (6) Zheng et al. (2001), i.e. by the irreducible transfer function, as in the linear case.

Definition 2.5: A one-form $\omega \in \mathcal{E}$ is called an autonomous element for the system (1), respectively (2), if there exist an integer $\nu \geq 1$ and functions $\alpha_i \in \mathcal{K}$, $i = 0, \dots, \nu$ such that

$$\alpha_0 \omega + \alpha_1 \dot{\omega} + \cdots + \alpha_\nu \omega^{(\nu)} = 0$$

Definition 2.6: The system (1), respectively (2), is said to be controllable if and only if there does not exist any non-zero autonomous element.

Theorem 2.7: *The system* (1), *respectively* (2), *is controllable if and only if the polynomials* $a(s)$, $b_i(s)$, $i = 1, \dots, m$ *in the transfer function* (7) *do not have non-trivial common left factor.*

Example 2.8: Consider the system from Example 2.3. The polynomials in the transfer function have a common left factor, since (9) may be rewritten as

$$G(s) = \frac{(s - \frac{\dot{u}}{u})\frac{\ddot{y}}{u}}{(s - \frac{\dot{u}}{u})(s^2 - us)} = \frac{\frac{\ddot{y}}{u}}{s^2 - us}$$

meaning that the system is not controllable. The irreducible (controllable) input–output differential equation can be found from $\mathrm{d}\ddot{y} - u\,\mathrm{d}\dot{y} = \frac{\ddot{y}}{u}\mathrm{d}u$ yielding $\ddot{y} = \dot{y}u$. Thus, the reduced transfer function can be rewritten as

$$G(s) = \frac{\dot{y}}{s^2 - us} \tag{10}$$

2.2 *Observability*

The definition of the observability of the system (1) in terms of the well-known observability rank condition

$$\operatorname{rank}_{\mathcal{K}}\left[\frac{\partial(y, \dot{y}, \dots, y^{(n-1)})}{\partial x}\right] = n$$

was related to the transfer function formalism in Perdon, Moog, and Conte (2007).

Theorem 2.9: *The system* (1) *is observable if and only if in the transfer function* (7) $\deg a(s) = n$.

Example 2.10: Consider the system from Example 2.2. Since in (8) $\deg(s^2 - u) = 2$, the system is observable.

3. Realization problem

In the nonlinear case there exists a class of input–output differential equations of the form (2) for which the state-space representation of the form (1) does not exist. A typical example is given by the system $\ddot{y} = \dot{u}^2$ since the necessary condition for realisability of an input–output equation is linearity in the highest derivative in control (see, for instance, Crouch, Lamnabhi-Lagarrigue, and Pinchon 1995a). The above result also follows from Theorem 1 in Freedman and Willems (1978). Thus, the necessary and sufficient conditions, under which such a realisation problem would be solvable in the nonlinear case, had to be found. Since the transfer function formalism for nonlinear control systems was not developed until recently, the problem was solved within different approaches. Here, we suggest the solution from the transfer function description.

The basic idea for solution is as simple as in the linear case; for a given transfer function matrix $G(s)$ find the matrices A, B and C such that $G(s) = C(sI - A)^{-1}B$. However, the computation are not as straightforward as in the linear case, for now the coefficients of polynomials in the transfer function matrix are not reals but meromorphic functions from the differential field $\mathcal{K}$. We demonstrate the computations and the necessary modifications on the simple example of a second-order nonlinear system with the transfer function

$$G(s) = \frac{1}{s^2 + a_1 s + a_0} \tag{11}$$

where a_0, $a_1 \in \mathcal{K}$. Then, if assuming a realisation in terms of the one-forms (5) with

$$A = \begin{pmatrix} 0 & -a_0 \\ 1 & -a_1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

in the observer canonical form like in the linear case, we get $\mathrm{d}y = \mathrm{d}x_2$, $\mathrm{d}\dot{y} = \mathrm{d}x_1 - a_1\,\mathrm{d}y$ and $\mathrm{d}\ddot{y} = -a_0\,\mathrm{d}y + \mathrm{d}u - a_1\,\mathrm{d}\dot{y} - \dot{a}_1\,\mathrm{d}y$ which corresponds to the transfer function

$$H(s) = \frac{1}{s^2 + a_1 s + \dot{a}_1 + a_0}$$

being different from (11). The mismatch is caused by the fact that the coefficients a_0, a_1 are functions from $\mathcal{K}$ and not just real numbers. Thus, a different commutation rule applies, $sa = as + \dot{a}$, while in the linear case, when $a \in \mathbf{R}$ and $\dot{a} = 0$, $sa = as$. Nevertheless, the mismatch may easily be avoided by constructing the matrices A, B and C in (5) from the different representation of the transfer function

$$G(s) = \frac{1}{s^2 + sa_1^* + a_0^*} = \frac{1}{s^2 + sa_1 + (a_0 - \dot{a}_1)}$$

where the polynomials in the transfer function have the indeterminate s on the *left* and not on the right as in (11). Now (5) with

$$A = \begin{pmatrix} 0 & -a_0^* \\ 1 & -a_1^* \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

is a realisation in terms of one-forms, for $\mathrm{d}y = \mathrm{d}x_2$, $\mathrm{d}\dot{y} = \mathrm{d}x_1 - a_1^*\,\mathrm{d}y$ and $\mathrm{d}\ddot{y} = -a_0^*\,\mathrm{d}y + \mathrm{d}u - a_1^*\,\mathrm{d}\dot{y} - \dot{a}_1^*\,\mathrm{d}y = -a_0\,\mathrm{d}y - a_1\,\mathrm{d}\dot{y} + \mathrm{d}u$ yielding the original transfer function (11).

The idea of moving the indeterminate s to the *left* of each summand in original polynomials, using the commutation rule (3), may be formalised and implemented transparently by introducing the notion of adjoint polynomials (Abramov, Le, and Li 2005) that represent the dual objects to skew polynomials. Such an idea has already been used before (Kamen 1976), where the respective operation was called adjoint, though the adjoint polynomials have not been defined formally. The process of moving the indeterminate to the left is also close to repeated integration by parts used in Crouch, Lamnabhi-Lagarrigue, and Van der Schaft (1995b).

Definition 3.1: The adjoint of a skew polynomial ring $\mathcal{K}[s]$ is defined as a skew polynomial ring $\mathcal{K}[s^*]$ with the commutation rule

$$s^*g = gs^* - \dot{g} \tag{12}$$

where $g \in \mathcal{K}$.

If $p(s) = p_n s^n + \cdots + p_1 s + p_0$ is a polynomial in the original ring $\mathcal{K}[s]$ then the adjoint polynomial $p^*(s^*)$ in $\mathcal{K}[s^*]$ can be computed from the coefficients of $p(s)$ by the formula

$$p^*(s^*) = s^{*n} p_n + \cdots + s^* p_1 + p_0 \tag{13}$$

where the products ${s^*}^i p_i$ must be computed according to the commutation rule (12).

Note that the adjoint is a bijective mapping and $(p^*)^* = p$, $(pq)^* = q^* p^*$ for any $p, q \in \mathcal{K}[s]$ (see Abramov et al. 2005 for details). Also note that in the commutative case, that is, in the case of linear time-invariant systems where all the coefficients are in $\mathbf{R}$, a polynomial and its adjoint are identical objects.

Example 3.2: Consider the system from Example 2.3 where

$$\begin{aligned} a(s) &= s^3 - \left(u + \frac{\dot{u}}{u}\right)s^2 \\ b(s) &= \frac{\ddot{y}}{u}s + \ddot{y} - \frac{\ddot{y}\dot{u}}{u^2} \end{aligned}$$

are polynomials in $\mathcal{K}[s]$. According to (13) and (12), the adjoint polynomials in $\mathcal{K}[s^*]$ can be computed as

$$\begin{aligned} a^*(s^*) &= s^{*3} - s^{*2}\left(u + \frac{\dot{u}}{u}\right) \\ &= s^{*3} - \left(u + \frac{\dot{u}}{u}\right)s^{*2} + 2\left(\frac{\ddot{u}}{u} - \frac{\dot{u}^2}{u^2} + \dot{u}\right)s^* \\ &\quad - \frac{u^{(3)}}{u} - \ddot{u} + 3\frac{\dot{u}\ddot{u}}{u^2} - 2\frac{u^{(3)}}{u^3} \\ b^*(s^*) &= s^*\frac{\ddot{y}}{u} + \ddot{y} - \frac{\ddot{y}\dot{u}}{u^2} = \frac{\ddot{y}}{u}s^* - \frac{\ddot{y}\dot{u}}{u^2} \end{aligned}$$

This is, in fact, the formalisation of the idea of moving the indeterminate s to the left of each summand in the original polynomials from $\mathcal{K}[s]$

$$\begin{aligned} a(s) &= s^3 - s^2\left(u + \frac{\dot{u}}{u}\right) + s\,2\left(\frac{\ddot{u}}{u} - \frac{\dot{u}^2}{u^2} + \dot{u}\right) \\ &\quad - \frac{u^{(3)}}{u} - \ddot{u} + 3\frac{\dot{u}\ddot{u}}{u^2} - 2\frac{u^{(3)}}{u^3} \\ b(s) &= s\frac{\ddot{y}}{u} - \frac{\ddot{y}\dot{u}}{u^2} \end{aligned}$$

following the commutation rule (3) in the original ring $\mathcal{K}[s]$.

Remark 1: In the algebraic setting used in this article a polynomial is defined as an object having coefficients on the left. In some works this representation is called the left standard representation (see, for instance, Bueso, Gomez-Torrecillas, and Verschoren 2003). Though for polynomials from the ring $\mathcal{K}[s]$ also the right standard representation exists, with coefficients on the right, we prefer, like in Abramov et al. (2005), instead to define another polynomial in a different indeterminate, called adjoint, which has the same coefficients as the right standard representation but is represented by the left standard form (see Definition 3.1 and formula (13)). Such a formalism is much better suited for computer algebra implementations, for example in Maple and Mathematica.

However, for the sake of simplicity, we often abandon this convention and simply write the indeterminate s on the left in the original polynomials, for instance in the transfer function from Example 2.3

$$\begin{aligned} G(s) &= \frac{\frac{\ddot{y}}{u}s + \ddot{y} - \frac{\ddot{y}\dot{u}}{u^2}}{s^3 - (u + \frac{\dot{u}}{u})s^2} \\ &= \frac{s\frac{\ddot{y}}{u} - \frac{\ddot{y}\dot{u}}{u^2}}{s^3 - s^2(u + \frac{\dot{u}}{u}) + s\,2(\frac{\ddot{u}}{u} - \frac{\dot{u}^2}{u^2} + \dot{u}) - \frac{u^{(3)}}{u} - \ddot{u} + 3\frac{\dot{u}\ddot{u}}{u^2} - 2\frac{u^{(3)}}{u^3}} \end{aligned} \tag{14}$$

while formally speaking of adjoint polynomials and, respectively, adjoint transfer functions.

Now, we are ready to present the main results of this article, that is how we find realisations in observer and controller canonical forms from the transfer function of a nonlinear system.

3.1 *Observer canonical form*

First, we derive the realisation in terms of differentials (5) using the observer canonical form. Our results have contact points with those of Kamen (1976) for linear time-varying systems.

Lemma 3.3: *Let*

$$G(s)=\left(\frac{b_1(s)}{a(s)}\cdots\frac{b_m(s)}{a(s)}\right)=\left(\frac{s^{n-1}b^*_{1,n-1}+\cdots+b^*_{1,0}}{s^n+s^{n-1}a^*_{n-1}+\cdots+a^*_0}\cdots\frac{s^{n-1}b^*_{m,n-1}+\cdots+b^*_{m,0}}{s^n+s^{n-1}a^*_{n-1}+\cdots+a^*_0}\right)$$

be the transfer function matrix and, respectively, the adjoint transfer function matrix of the nonlinear system (2). *Then* (5) *with*

$$A=\begin{pmatrix}0 & 0 & \cdots & 0 & -a^*_0\\ 1 & 0 & \cdots & 0 & -a^*_1\\ \vdots & & \vdots & & \\ 0 & 0 & \cdots & 1 & -a^*_{n-1}\end{pmatrix},\quad B=\begin{pmatrix}b^*_{1,0} & \cdots & b^*_{m,0}\\ \vdots & & \\ b^*_{1,n-1} & \cdots & b^*_{m,n-1}\end{pmatrix},\quad C=\begin{pmatrix}0 & 0 & 1\end{pmatrix} \tag{15}$$

is the realisation in terms of one-forms.

Proof: It suffices to show that $G(s)=C(\mathrm{s}I-A)^{-1}B$.

Let $C(sI-A)^{-1}=E$ with E being a row vector, $E=(e_1(s)\dots e_n(s))$. Note that $e_i(s)\in\mathcal{K}\langle s\rangle$, $i=1,\dots,n$. Thus, $C=E(sI-A)$ and by (15) we get the set of equations $0=e_1(s)s-e_2(s),\dots,0=e_{n-1}(s)s-e_n(s)$ and $1=e_n(s)s+\sum_{i=0}^{n-1}e_{i+1}(s)a_i^*$. The first $n-1$ equations imply that $e_i(s)=e_1(s)s^{i-1}$, $i=1,\dots,n$ which, when inserted into the last equation, gives $1=e_1(s)s^n+\sum_{i=0}^{n-1}e_1(s)s^ia_i^*=e_1(s)(s^n+\sum_{i=0}^{n-1}s^ia_i^*)$. By the definition of the adjoint polynomial ring $\mathcal{K}[s^*]$, we can thus write $1=e_1(s)(s^n+\sum_{i=0}^{n-1}a_is^i)=e_1(s)a(s)$ which implies $e_1(s)=a^{-1}(s)$. Now

$$\begin{aligned}C(sI-A)^{-1}B&=EB\\&=\left(\sum_{i=0}^{n-1}e_{i+1}(s)b^*_{1,i}\quad\cdots\quad\sum_{i=0}^{n-1}e_{i+1}(s)b^*_{m,i}\right)\\&=\left(\sum_{i=0}^{n-1}e_1(s)s^ib^*_{1,i}\quad\cdots\quad\sum_{i=0}^{n-1}e_1(s)s^ib^*_{m,i}\right)\\&=e_1(s)\left(\sum_{i=0}^{n-1}b_{1,i}s^i\quad\cdots\quad\sum_{i=0}^{n-1}b_{m,i}s^i\right)\\&=a^{-1}(s)\cdot(b_1(s)\ \cdots\ b_m(s))\end{aligned}$$

that completes the proof. □

For the input–output equation (2) to be realisable, the equations in (5) with (A,B,C) defined by (15) have to result in a certain set of exact/integrable one-forms. Recall that the exactness of a one-form $\nu\in\mathcal{E}$ means that there exists a function $\xi\in\mathcal{K}$ such that $\mathrm{d}\xi=\nu$, while the integrability of a vector space of one-forms $\mathcal{V}=\mathrm{span}_{\mathcal{K}}\{\nu_1,\dots,\nu_r\}$ of $\mathcal{E}$ means that there exist, at least locally, functions $\xi_1,\dots,\xi_r\in\mathcal{K}$ such that $\mathcal{V}=\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi_1,\dots,\mathrm{d}\xi_r\}$.

Theorem 3.4: *Let*

$$G(s)=\left(\frac{b_1(s)}{a(s)}\cdots\frac{b_m(s)}{a(s)}\right)=\left(\frac{s^{n-1}b^*_{1,n-1}+\cdots+b^*_{1,0}}{s^n+s^{n-1}a^*_{n-1}+\cdots+a^*_0}\cdots\frac{s^{n-1}b^*_{m,n-1}+\cdots+b^*_{m,0}}{s^n+s^{n-1}a^*_{n-1}+\cdots+a^*_0}\right)$$

be the transfer function matrix and, respectively, the adjoint transfer function matrix of the nonlinear system (2). *Let* $\omega_i=\sum_{j=1}^m b^*_{j,i-1}\mathrm{d}u_j-a^*_{i-1}\mathrm{d}y$ *for* $i=1,\dots,n$. *Then*:

(i) *there exists a state space realisation in the observer canonical form*

$$\begin{aligned}\dot x_1&=\varphi_1(x_n,u)\\\dot x_2&=x_1+\varphi_2(x_n,u)\\&\ \vdots\\\dot x_n&=x_{n-1}+\varphi_n(x_n,u)\\y&=x_n\end{aligned}\tag{16}$$

if and only if all ω_i*'s are exact*; *that is*, $\mathrm{d}\omega_i=0$ *for* $i=1,\dots,n$.

(ii) *there exists a state space realisation of the form* (1) *if and only if* $\mathcal{V}=\mathrm{span}_{\mathcal{K}}\{\mathrm{d}y,\ \mathrm{d}\dot y-\omega_n,\ \mathrm{d}\ddot y-\dot\omega_n-\omega_{n-1},\dots,\mathrm{d}y^{(n-1)}-\omega_n^{(n-2)}-\omega_{n-1}^{(n-3)}-\cdots-\omega_2\}$ *is integrable.*

Proof:

(i) By (5) and (15) one has

$$\begin{aligned}\mathrm{d}\dot x_1&=\omega_1\\\mathrm{d}\dot x_2&=\mathrm{d}x_1+\omega_2\\&\ \vdots\\\mathrm{d}\dot x_n&=\mathrm{d}x_{n-1}+\omega_n\\\mathrm{d}y&=\mathrm{d}x_n\end{aligned}$$

The only thing to show is that $\mathrm{d}x_i$'s are exact if and only if ω_i's are exact for $i=1,\dots,n$ (Note that the one-forms $\mathrm{d}x_i$ have to be understood as the notations and are not necessarily exact. Only in case they are exact, their integration will yield the states coordinates x_i, $i=1,\dots,n$.). Note that $\mathrm{d}y=\mathrm{d}x_n$, $\mathrm{d}\dot y=\mathrm{d}x_{n-1}+\omega_n$, $\mathrm{d}\ddot y=\mathrm{d}x_{n-2}+\omega_{n-1}+\dot\omega_n,\dots$, $\mathrm{d}y^{(n-1)}=\mathrm{d}x_1+\omega_2+\cdots+\omega_{n-1}^{(n-3)}+\omega_n^{(n-2)}$ yielding $\mathrm{d}x_n=\mathrm{d}y$, $\mathrm{d}x_{n-1}=\mathrm{d}\dot y-\omega_n,\dots$, $\mathrm{d}x_1=\mathrm{d}y^{(n-1)}-\omega_n^{(n-2)}-\omega_{n-1}^{(n-3)}-\cdots-\omega_2$. Since $\mathrm{d}y^{(i)}$, $i=0,\dots,n-1$ are exact and since the derivative operator $\dot{}$ commutes with the differential operator d, the one-forms $\mathrm{d}x_i$, $i=1,\dots,n$ are exact if and only if so are ω_i, $i=1,\dots,n$. Then there exist functions $\varphi_i\in\mathcal{K}$, $i=1,\dots,n$ such that $\mathrm{d}\varphi_i=\omega_i$, $i=1,\dots,n$ yielding the realisation in the observer canonical form (16).

(ii) From the proof of part (i), we have $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1,\dots,\mathrm{d}x_n\}=\mathcal{V}$.

□

Remark 2: Note that $\mathcal{V}$ in Theorem 3.4 (ii) is, in fact, the vector space $\mathcal{H}_{s+2}$ from Theorem 2.16 in Conte et al. (2007). We can thus write down $\mathcal{H}_{s+2}$ directly from the coefficients of the adjoint transfer function without computing the intermediate vector spaces $\mathcal{H}_1,\dots,\mathcal{H}_{s+1}$ first.

Remark 3: In the linear time-invariant case, when the coefficients of polynomials in transfer function are real numbers, the commutation rule (3) reduces to $sa=as$, as $\dot{a}=0$ for $a\in\mathbf{R}$. Thereby, a polynomial and its adjoint are identical objects, $a_i=a_i^*$, $b_{j,i}=b_{j,i}^*$, $i=0,\dots,n-1$, $j=1,\dots,m$. The results, therefore, reduce to the well-known observer canonical form

$$A_O=\begin{pmatrix}0 & 0 & \cdots & 0 & -a_0\\ 1 & 0 & \cdots & 0 & -a_1\\ \vdots & & & & \vdots\\ 0 & 0 & \cdots & 1 & -a_{n-1}\end{pmatrix},$$

$$B_O=\begin{pmatrix}b_{1,0} & \cdots & b_{m,0}\\ \vdots & & \\ b_{1,n-1} & \cdots & b_{m,n-1}\end{pmatrix},\quad C_O=\begin{pmatrix}0 & \cdots & 0 & 1\end{pmatrix} \tag{17}$$

that can be written out directly from the transfer function matrix.

3.2 *Controller canonical form*

The natural question can be asked whether there exists an analogous result for the controller canonical form, like in the linear time-invariant case when the observer and controller canonical forms are dual objects. To find the realisation in the nonlinear controller canonical form we need to start from, the right factorisation of the transfer function matrix.

By the duality concept the result holds for single-input systems with possibly more than one output

$$F(y,\dot{y},\dots,y^{(n)},u,\dot{u},\dots,u^{(n-1)})=0 \tag{18}$$

where $y\in\mathbf{R}^p$, $u\in\mathbf{R}$ and $F\in\mathcal{K}$.

Remark 4: Note that the transfer function of a nonlinear system is defined as an element of $\mathcal{K}\langle s\rangle$, i.e. a left quotient (4). However, any left quotient can equivalently be rewritten as a right quotient

$$\frac{b}{a}=\frac{1}{a}\cdot b=\bar{b}\cdot\frac{1}{\bar{a}}$$

where a, b, $\bar{a}$ and $\bar{b}$ are in $\mathcal{K}[s]$, which is, in what follows, called the right factorisation. To simplify the computation of the right factorisation, one may use the adjoint polynomials, since it is easier to find the least common left multiple than the right one. Note that the equality $a^{-1}\cdot b=\bar{b}\cdot\bar{a}^{-1}$ implies $b\bar{a}=a\bar{b}$, the latter being the right Ore condition, and we are looking for the least common right multiple of a and b. In order to find the polynomials $\bar{a}$ and $\bar{b}$ directly from $b\bar{a}=a\bar{b}$, one has to solve a set of differential equations for the coefficients of polynomials $\bar{a}$ and $\bar{b}$. As the adjoint satisfies $(pq)^*=q^*p^*$ for any p, $q\in\mathcal{K}[s]$, one may find instead $\bar{a}^*$ and $\bar{b}^*$ from $\bar{a}^*b^*=\bar{b}^*a^*$ which is the left Ore condition in $\mathcal{K}[s^*]$ and one has to solve only a set of algebraic equations for the coefficients of polynomials $\bar{a}^*$ and $\bar{b}^*$. Then $(\bar{a}^*)^*=\bar{a}$ and $(\bar{b}^*)^*=\bar{b}$.

Lemma 3.5: *Let*

$$G(s)=\begin{pmatrix}\dfrac{b(s)}{a_1(s)}\\ \vdots\\ \dfrac{b(s)}{a_p(s)}\end{pmatrix}=\begin{pmatrix}\bar{b}_{1,n-1}s^{n-1}+\cdots+\bar{b}_{1,0}\\ \vdots\\ \bar{b}_{p,n-1}s^{n-1}+\cdots+\bar{b}_{p,0}\end{pmatrix}\cdot\frac{1}{s^n+\bar{a}_{n-1}s^{n-1}+\cdots+\bar{a}_0}$$

be the transfer function matrix of the nonlinear system (18) *and, respectively, its right factorisation. Then* (5) *with*

$$A=\begin{pmatrix}0 & 1 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 1\\ -\bar{a}_0 & -\bar{a}_1 & \cdots & -\bar{a}_{n-1}\end{pmatrix},$$

$$B=\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix},\quad C=\begin{pmatrix}\bar{b}_{1,0} & \cdots & \bar{b}_{1,n-1}\\ \vdots & & \\ \bar{b}_{p,0} & \cdots & \bar{b}_{p,n-1}\end{pmatrix} \tag{19}$$

is the realisation in terms of one-forms.

Proof: Now let $(sI-A)^{-1}B=E$ with E be a column vector written as $E=(e_1(s)\dots e_n(s))^T$ and $e_i(s)\in\mathcal{K}\langle s\rangle$, $i=1,\dots,n$. Thus, $B=(sI-A)E$ and by (19) we get the set of equations $0 = se_1(s) - e_2(s), \ldots, 0 = se_{n-1}(s) - e_n(s)$ and $1 = se_n(s) + \sum_{i=0}^{n-1} \bar{a}_i e_{i+1}(s)$. The first $n-1$ equations now imply that $e_i(s) = s^{i-1}\, e_1(s)$, $i = 1, \ldots, n$ which, when inserted into the last equation, gives us $1 = s^n e_1(s) + \sum_{i=0}^{n-1} \bar{a}_i s^i e_1(s) = (s^n + \sum_{i=0}^{n-1} \bar{a}_i s^i) \quad e_1(s) = \bar{a}(s) e_1(s)$ which implies that $e_1(s) = \bar{a}^{-1}(s)$. Now

$$
\begin{aligned}
C(sI - A)^{-1}B &= CE \\
&= \begin{pmatrix} \sum_{i=0}^{n-1} \bar{b}_{1,i} e_{i+1}(s) & \cdots & \sum_{i=0}^{n-1} \bar{b}_{p,i} e_{i+1}(s) \end{pmatrix}^T \\
&= \begin{pmatrix} \sum_{i=0}^{n-1} \bar{b}_{1,i} s^i e_1(s) & \cdots & \sum_{i=0}^{n-1} \bar{b}_{p,i} s^i e_1(s) \end{pmatrix}^T \\
&= \begin{pmatrix} \sum_{i=0}^{n-1} \bar{b}_{1,i} s^i & \cdots & \sum_{i=0}^{n-1} \bar{b}_{p,i} s^i \end{pmatrix}^T e_1(s) \\
&= (\bar{b}_1(s) \ \cdots \ \bar{b}_p(s))^T \cdot \bar{a}^{-1}(s)
\end{aligned}
$$

that is the right factorisation of the transfer function matrix $G(s)$. □

Theorem 3.6: *Let*

$$
G(s) = \begin{pmatrix} \dfrac{b(s)}{a_1(s)} \\ \vdots \\ \dfrac{b(s)}{a_p(s)} \end{pmatrix}
= \begin{pmatrix} \bar{b}_{1,n-1} s^{n-1} + \cdots + \bar{b}_{1,0} \\ \vdots \\ \bar{b}_{p,n-1} s^{n-1} + \cdots + \bar{b}_{p,0} \end{pmatrix} \cdot \frac{1}{s^n + \bar{a}_{n-1} s^{n-1} + \cdots + \bar{a}_0}
$$

be the transfer function matrix of the nonlinear system (18) *and its right factorisation, respectively. Let* $dy = C\,dx$ *with C defined by* (19) *and denote by k the least non-negative integer such that* $\dim \operatorname{span}_{\mathcal{K}}\{dy, \ldots, dy^{(k-1)}\} = \dim \operatorname{span}_{\mathcal{K}}\{dy, \ldots, dy^{(k)}\}$. *Let* ϖ_i, $i = 1, \ldots, n$ *be the solution of the set of linear equations* $\{dy = C\,dx, \ldots, dy^{(k-1)} = (C\,dx)^{(k-1)}\}$ *with respect to* dx_i, $i = 1, \ldots, n$ *where* $d\dot{x} = A\,dx + B\,du$ *with A and B defined by* (19). *Then*:

(i) *there exists a state space realisation in the controller canonical form*

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= \phi(x) + u \\
y &= \psi(x)
\end{aligned}
\tag{20}
$$

if and only if all ϖ_i*'s are exact; that is,* $d\varpi_i = 0$ *for* $i = 1, \ldots, n$.

(ii) *there exists a state space realisation of the form* (1) *if and only if* $\operatorname{span}_{\mathcal{K}}\{\varpi_1, \ldots, \varpi_n\}$ *is integrable.*

Proof:

(i) By the assumption of the theorem ϖ_i, $i = 1, \ldots, n$ is the set of solutions to the system (5) with (19)

$$
\begin{aligned}
d\dot{x}_1 &= dx_2 \\
&\vdots \\
d\dot{x}_{n-1} &= dx_n \\
d\dot{x}_n &= -\bar{a}_0 dx_1 - \cdots - \bar{a}_{n-1} dx_n + du \\
dy &= C\,dx
\end{aligned}
$$

That is, $dx_i = \varpi_i$, $i = 1, \ldots, n$. Then there exist functions $\varphi_i \in \mathcal{K}$ such that $d\varphi_i = \varpi_i$, $i = 1, \ldots, n$ if and only if ϖ_i's are exact for $i = 1, \ldots, n$. Since the derivative operator $\dot{}$ commutes with the differential operator d, also $\dot{\varpi}_i$, $i = 1, \ldots, n$ are exact and the choice $x_i = \varphi_i$, $i = 1, \ldots, n$ thus yields the controller canonical form (20).

(ii) Obviously, $\operatorname{span}_{\mathcal{K}}\{dx_1, \ldots, dx_n\} = \operatorname{span}_{\mathcal{K}}\{\varpi_1, \ldots, \varpi_n\}$.

□

Remark 5: In the linear time-invariant case the transfer function and its right factorisation are identical objects, $\frac{b}{a} = \frac{1}{a} \cdot b = b \cdot \frac{1}{a}$. The results, therefore, reduce to the well-known controller canonical form with $A_C = A_O^T$, $B_C = C_O^T$ and $C_C = B_O^T$ in (17).

Remark 6: In Van der Schaft (1989) the realisation in nonlinear controller canonical form has been obtained from the moving average representation of the nonlinear system, $y = \varphi(\xi, \dot{\xi}, \ldots)$, $u = \psi(\xi, \dot{\xi}, \ldots)$, though it has not been shown when this realisation exists and how to find it. Note that the right factorisation of the transfer function may be understood as the procedure for finding the moving average representation on the level of differential one-forms, $dy = \bar{b}(s)d\xi$, $du = \bar{a}(s)d\xi$ from which follows $dy = \bar{b}(s) \cdot \frac{1}{\bar{a}(s)} du$.

3.3 *Examples*

Example 3.7: Consider the system $\ddot{y} = \dot{u}^2$ with the transfer function and, respectively, the adjoint transfer function

$$
G(s) = \frac{2\dot{u}s}{s^2} = \frac{s2\dot{u} - 2\ddot{u}}{s^2}
$$

The realisation in terms of one-forms is

$$
\begin{aligned}
\mathrm{d}\dot{x}_1 &= -2\ddot{u}\,\mathrm{d}u \\
\mathrm{d}\dot{x}_2 &= \mathrm{d}x_1 + 2\dot{u}\,\mathrm{d}u \\
\mathrm{d}y &= \mathrm{d}x_2
\end{aligned}
$$

Neither $\omega_1 = -2\ddot{u}\,\mathrm{d}u$ nor $\omega_2 = 2\dot{u}\,\mathrm{d}u$ are exact, therefore the realisation in the observer canonical form does not exist. Additionally, $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y} - 2\dot{u}\,\mathrm{d}u\}$ is not integrable which means there does not exist any system of the form (1) which generates the input–output equation $\ddot{y} = \dot{u}^2$.

Example 3.8: Consider the system from Example 2.3 with the transfer function and, respectively, adjoint transfer function (14). Since none of the $\omega_1 = -\frac{\ddot{y}\ddot{u}}{u^2}\mathrm{d}u + (\frac{u^{(3)}}{u} + \ddot{u} - 3\frac{\dot{u}\ddot{u}}{u^2} + 2\frac{u^{(3)}}{u^3})\mathrm{d}y$, $\omega_2 = \frac{\ddot{y}}{u}\mathrm{d}u - 2(\frac{\ddot{u}}{u} - \frac{\dot{u}^2}{u^2} + \dot{u})\mathrm{d}y$ and $\omega_3 = (u + \frac{\dot{u}}{u})\mathrm{d}y$ is exact, the realisation in the observer canonical form does not exist. However, an observable realisation of the form (1) exists, for $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y} - \omega_3, \mathrm{d}\ddot{y} - \dot{\omega}_3 - \omega_2\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}\ddot{y} - \frac{\ddot{y}}{u}\mathrm{d}u\}$ is integrable and the choice $x_1 = y$, $x_2 = \dot{y}$ and $x_3 = \frac{\ddot{y}}{u}$ yields

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 u \\
\dot{x}_3 &= x_3 u \\
y &= x_1
\end{aligned}
\tag{21}
$$

Example 3.9: Consider the system from Example 2.4 with the transfer function matrix and, respectively, the adjoint transfer function matrix that can be computed as

$$
\begin{aligned}
G(s) &= \begin{pmatrix} \dfrac{\dot{y}s}{s^2 - \dot{u}_1 s} & \dfrac{1}{s^2 - \dot{u}_1 s} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{s\dot{y} - \dot{y}\dot{u}_1 - u_2}{s^2 - s\dot{u}_1 + \ddot{u}_1} & \dfrac{1}{s^2 - s\dot{u}_1 + \ddot{u}_1} \end{pmatrix}
\end{aligned}
$$

Here, again, neither $\omega_1 = (-\dot{y}\dot{u}_1 - u_2)\mathrm{d}u_1 + \mathrm{d}u_2 - \ddot{u}_1\mathrm{d}y$ nor $\omega_2 = \dot{y}\,\mathrm{d}u_1 + \dot{u}_1\mathrm{d}y$ are exact. Therefore the realisation in the observer canonical form does not exist. However, an observable realisation of the form (1) exists, for $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y} - \dot{y}\,\mathrm{d}u_1 - \dot{u}_1\mathrm{d}y\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y} - \dot{y}\,\mathrm{d}u_1\}$ is integrable. The choice $x_1 = y$, $x_2 = \ln\dot{y} - u_1$ yields

$$
\begin{aligned}
\dot{x}_1 &= \exp(x_2 + u_1) \\
\dot{x}_2 &= u_2/\exp(x_2 + u_1) \\
y &= x_1
\end{aligned}
$$

Example: Consider the system

$$
\begin{aligned}
\dot{y}_1 &= y_2^2 + \frac{y_1^2}{y_2} + \frac{y_1}{y_2}u \\
\dot{y}_2 &= y_1 + u
\end{aligned}
$$

for which a realisation of the form (1) can easily be found by setting $x_1 = y_1$ and $x_2 = y_2$ being, however, not in the controller canonical form (20). If we compute the transfer function matrix and its right factorisation

$$
\begin{aligned}
G(s) &= \begin{pmatrix} s - \dfrac{2y_1 + u}{y_2} & -2y_2 - \dfrac{y_1^2 + y_1 u}{y_2^2} \\ -1 & s \end{pmatrix}^{-1} \cdot \begin{pmatrix} \dfrac{y_1}{y_2} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{y_1}{y_2}s + y_2 \\ s \end{pmatrix} \cdot \frac{1}{s^2 - \frac{y_1}{y_2}s - y_2}
\end{aligned}
$$

the realisation in terms of one-forms is

$$
\begin{aligned}
\mathrm{d}\dot{x}_1 &= \mathrm{d}x_2 \\
\mathrm{d}\dot{x}_2 &= y_2\,\mathrm{d}x_1 + \frac{y_1}{y_2}\mathrm{d}x_2 + \mathrm{d}u \\
\mathrm{d}y_1 &= y_1\,\mathrm{d}x_1 + \frac{y_1}{y_2}\,\mathrm{d}x_2 \\
\mathrm{d}y_2 &= \mathrm{d}x_2
\end{aligned}
$$

From the last two equations we get $\mathrm{d}x_1 = \frac{1}{y_2}\mathrm{d}y_1 - \frac{y_1}{y_2^2}\mathrm{d}y_2$, and $\mathrm{d}x_2 = \mathrm{d}y_2$, both are exact, giving us $x_1 = \frac{y_1}{y_2}$ and $x_2 = y_2$. Then the controller canonical form reads

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1 x_2 + u \\
y_1 &= x_1 x_2 \\
y_2 &= x_2
\end{aligned}
$$

4. Minimal realisation

In the nonlinear case the question of minimality of the system realisation is more subtle than in the linear case, where a realisation is called minimal if it is controllable and observable. This definition also guarantees the minimality of the state dimension. However, in the nonlinear case those two definitions do not always coincide (see, for instance, Zhang, Moog, and Xia 2010).

In this article we rely on the definition based on the controllability and observability, for it is consistent with the transfer function formalism.

Definition 4.1: The system (1) is called a minimal realisation of the input–output Equation (2) if it is controllable and observable.

The transfer function approach suggested here always yields an observable realisation (see Theorem 2.9) as we, obviously, cannot obtain more than n state equations for the input–output equation (2), respectively (18). However, the realisation is not necessarily controllable. To get the controllable realisation we have two options,

- either we start from irreducible (controllable) input–output differential equation by cancelling all the possible common left factors $\rho(s)$ in the transfer function description (see Remark 1 and Theorem 2.7)

$$G(s) = \frac{b(s)}{a(s)} = \frac{\rho(s)\tilde{b}(s)}{\rho(s)\tilde{a}(s)} = \frac{\tilde{b}(s)}{\tilde{a}(s)}$$

- or we find the realisation from the controller canonical form where the possible presence of common left factors plays no role in finding the right factorisation

$$\rho(s)\tilde{b}(s)\bar{a}(s) = \rho(s)\tilde{a}(s)\bar{b}(s)$$

assuming we are interested in finding the least common right multiple here. In this case the realisation is always controllable and observable.

Example 4.2: Consider the system from Example 2.3 which is not controllable (see also Example 2.8). Therefore, the realisation we found in Example 3.8 is not minimal. To find a minimal realisation we can start from the reduced transfer function (Example 2.8) and its adjoint transfer function

$$G(s) = \frac{\dot{y}}{s^2 - us} = \frac{\dot{y}}{s^2 - su + \dot{u}}$$

Then we get $\omega_1 = \dot{y}\,\mathrm{d}u - \dot{u}\,\mathrm{d}y$ and $\omega_2 = u\,\mathrm{d}y$, none of them are exact. Hence, the realisation in the observer canonical form does not exist. However, an observable and controllable realisation of the form (1) exists, for $G(s)$ is irreducible transfer function and $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y} - \omega_2\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y} - u\,\mathrm{d}y\}$ is integrable. The choice $x_1 = y$ and $x_2 = \dot{y}$ yields

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_2 u \\ y &= x_1 \end{aligned} \tag{22}$$

which is a minimal realisation of the system.

Alternatively, we can find the right factorisation of the non-reduced transfer function as follows:

$$G(s) = \frac{\frac{\ddot{y}}{u}s + \ddot{y} - \frac{\ddot{y}\dot{u}}{u^2}}{s^3 - (u + \frac{\dot{u}}{u})s^2} = \dot{y} \cdot \frac{1}{s^2 + us + \dot{u}}$$

Then the realisation in terms of one-forms is

$$\begin{aligned} \mathrm{d}\dot{x}_1 &= \mathrm{d}x_2 \\ \mathrm{d}\dot{x}_2 &= -\dot{u}\,\mathrm{d}x_1 - u\,\mathrm{d}x_2 + \mathrm{d}u \\ \mathrm{d}y &= \dot{y}\,\mathrm{d}x_1 \end{aligned}$$

By solving the set of equations $\mathrm{d}y = \dot{y}\mathrm{d}x_1$, $\mathrm{d}\dot{y} = \ddot{y}\,\mathrm{d}x_1 + \dot{y}\,\mathrm{d}x_2 = \dot{y}u\,\mathrm{d}x_1 + \dot{y}\,\mathrm{d}x_2$ for $\mathrm{d}x_1$ and $\mathrm{d}x_2$, we get $\mathrm{d}x_1 = \frac{1}{\dot{y}}\mathrm{d}y$ and $\mathrm{d}x_2 = \frac{1}{\dot{y}}\mathrm{d}\dot{y} - \frac{u}{\dot{y}}\mathrm{d}y$ none of them are exact. However, $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2\} = \mathrm{span}_{\mathcal{K}}\{\frac{1}{\dot{y}}\mathrm{d}y, \frac{1}{\dot{y}}\mathrm{d}\dot{y} - \frac{u}{\dot{y}}\mathrm{d}y\}$ is integrable and the choice $x_1 = y$ and $x_2 = \dot{y}$ yields (22) as above.

5. Linearisation of the state equations by the input–output injection

The solution of the realisation problem, as suggested in this article, may also be used to address the problem of linearisation of the state Equations (1) up to the input–output injection

$$\begin{aligned} \dot{\xi} &= A\xi + \varphi(y, u) \\ y &= C\xi \end{aligned} \tag{23}$$

which plays a key role in the standard Luenberger observer design. The algebraic solution to the problem was given in Glumineau et al. (1996). It assumes the knowledge of the input–output differential equation corresponding to the system (1) that can be computed by the state elimination algorithm in Conte et al. (2007). In such a case the solvability conditions require the one-forms ω_i, $i = 1, \ldots, n$, associated with the input–output equation, and defined iteratively via an algorithm, to be exact. However, this problem reduces to a realisation problem in the observer canonical form (16). The one-forms ω_i, $i = 1, \ldots, n$, can be found from the adjoint transfer function matrix of the system in one single step. That is, the problem of linearising the system Equations (1) up to the input–output injection (23) reduces to the realisation problem for which the solvability conditions are given by Theorem 3.4, part (i).

Remark 1: Note that the state elimination algorithm of Conte et al. (2007) ensures that for a system of the form (1) an input–output representation of the form (2), at least locally, always exists. However, the computation of the input–output equation might not be a trivial task and, for instance, in some cases it requires to consider also certain inequations (Diop 1991). In addition, it includes the question of observability as well. However, since the observability is a necessary condition to assure the existence of a suitable observer, we assume the system (1) to be observable in which case the state elimination algorithm of Conte et al. (2007) yields an input–output differential equation of n-th order. In terms of the transfer function matrix (7) this means that deg $a(s) = n$ (see Theorem 2.9).

Example 5.1: Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= (x_3 + u_1)/x_1\\ \dot{x}_3 &= (1 + x_2)(x_3 + u_1)/x_1 - x_2(x_2 + u_1)/x_1 + x_1^2 u_2\\ y &= x_1\end{aligned}$$

The input–output equation can be found as

$$y^{(3)} = (\ddot{y} + \dot{u}_1)/y - \dot{y}(\dot{y} + u_1)/y^2 + yu_2$$

The transfer function matrix and, respectively, the adjoint transfer function matrix are

$$\begin{aligned}G(s) &= \begin{pmatrix} \dfrac{\frac{1}{y}s - \frac{\dot{y}}{y^2}}{a(s)} & \dfrac{y}{a(s)} \end{pmatrix}\\ &= \begin{pmatrix} \dfrac{s\frac{1}{y}}{s^3 - s^2\frac{1}{y} + s\frac{u_1}{y^2} - u_2} & \dfrac{y}{s^3 - s^2\frac{1}{y} + s\frac{u_1}{y^2} - u_2} \end{pmatrix}\end{aligned}$$

where $a(s) = s^3 - \frac{1}{y}s^2 + \frac{1}{y^2}(2\dot{y} + u_1)s - \frac{2\dot{y}}{y^3}(\dot{y} + u_1) + \frac{1}{y^2}(\ddot{y} + \dot{u}_1) - u_2$. Since $\deg a(s) = 3$, the system is observable.

Here all $\omega_1 = y\,\mathrm{d}u_2 + u_2\,\mathrm{d}y$, $\omega_2 = \frac{1}{y}\,\mathrm{d}u_1 - \frac{u_1}{y^2}\,\mathrm{d}y$ and $\omega_3 = \frac{1}{y}\mathrm{d}y$ are exact, yielding $\varphi_1 = yu_2$, $\varphi_2 = u_1/y$ and $\varphi_3 = \ln y$. Hence, the system is transformable into the observer canonical form

$$\begin{aligned}\dot{\xi}_1 &= yu_2\\ \dot{\xi}_2 &= \xi_1 + u_1/y\\ \dot{\xi}_3 &= \xi_2 + \ln y\\ y &= \xi_3\end{aligned}$$

The corresponding state transformation can be found in the same way as in Glumineau et al. (1996), $(\xi_1, \xi_2, \xi_3) = ((x_3 - x_2)/x_1, x_2 - \ln x_1, x_1)$.

6. Conclusions

This article developed the procedures for finding the state space realisations of a nonlinear system in the observer and controller canonical form, whenever they exist, as well as necessary and sufficient conditions for their existence. The characteristic property of these procedures is that they require the minimal amount of computations and the realisations can practically be written down either from the adjoint transfer function or from the right factorisation of the transfer function. The (adjoint) transfer function itself is easily computable from the input–output differential equation, being often the end result of the system identification. The right factorisation of the transfer function, that requires to find the least common right multiple of two polynomials, may be replaced by much simpler problem of finding the least common left multiple of the respective adjoint polynomials. In doing so, one has to solve instead the differential equations only the algebraic equations. As a byproduct, the results of this article also allow to compute the differentials of the state coordinates of the non-canonical realisations in a very simple way. Compared with the canonical realisations the set of certain one-forms that had to be exact, now have to constitute completely integrable vector spaces. Finally, it was demonstrated that the linearisation of state equations up to the input–output injection may be understood as the realisation problem in the observer canonical form.

The results of this article can be extended to the discrete-time case without any major changes through the corresponding transfer function formalism (Halás and Kotta 2007, 2010). Following the ideas of Kamen (1976), one may find the approach suitable even for the nonlinear time-delay systems using the polynomial approach suggested in Halás (2009). Another topic for the future research is the extension of the results to the multi-input multi-output case. There is, however, no simple formula for finding a state space realisation from a transfer function matrix with several inputs and outputs even in the case of linear time-invariant systems (Glad and Ljung 2000). Additionally, the duality between the observer and controller canonical forms may be lost. The extension to the MIMO case is probably possible when the identification results in the set of input–output differential equations in specific forms, that mimic the Hermite, Popov or Guidorzi canonical forms in the linear case.

The polynomial approach to the realisation problem suggested in this article is implementable for instance in the computer algebra system Maple with no additional effort since the functions for handling the skew polynomials, including (but not limited to) the computations of least common left/right multiples, adjoint rings and adjoint polynomials, are available. The respective Maple procedures were used to find adjoint polynomials and/or right factorisations in the examples.

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