Weighted Automata and Weighted Logics on Infinite Words

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Abstract. We introduce weighted automata over infinite words with Muller acceptance condition and we show that their behaviors coincide with the semantics of weighted restricted MSO-sentences. Furthermore, we establish an equivalence property of weighted Muller and weighted Büchi automata over certain semirings.

Keywords: Weighted logics, Weighted Muller automata, Infinitary formal power series.

1 Introduction

One of the cornerstones of automata theory is Büchi's theorem [6] on the coincidence of the class of regular languages of infinite words with the family of languages definable by monadic second order logic (MSO logic for short). This led to the development of several models of automata acting on infinite words, like Büchi, Muller, Rabin and Streett, cf. [29, 33, 34] for surveys; it also led to practical applications in model checking and for non-terminating processes, cf. [1, 25, 26]. On the other hand, Schützenberger [32] introduced finite automata with weights which can model quantitative aspects of transitions like use of resources, reliability or capacity. Schützenberger characterized the behavior of such automata as rational formal power series. For the theory of weighted automata, see [3, 21, 24, 31] for surveys. Recently, weighted automata were applied in digital image compression [7, 17, 18, 19] as well as in speech-to-text processing [27, 28].

It is the goal of this paper to extend Büchi's theorem mentioned above into the context of weighted automata, thereby obtaining a quantitative version. Furthermore, we obtain an equivalence result to a model investigated recently by Ésik and Kuich [16]. The last few years weighted automata over infinite words have attracted the interest of several researchers. This effort is not a simple generalization of the finitary case since convergence problems arise depending

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on the underlying semiring. This issue is dealt with either by considering special classes of automata [8, 11] or by restricting the underlying semirings so that convergence problems can be solved [12, 16, 20, 30].

Very recently, Droste and Gastin [10] extended the result of Büchi and Elgot [5, 14] to weighted automata over finite words. They introduced an MSO logic with weights and described the semantics of the formulas obtained as formal power series. The main result of their paper states that the recognizable formal power series over commutative semirings coincide with the series definable by certain weighted MSO-sentences.

In this paper, we will introduce weighted Muller automata acting on infinite words, and we will extend the weighted MSO logic of [10] to infinite words. We describe the behavior of weighted Muller automata as formal power series on infinite words. Our first main result states the coincidence of these ω-Muller-recognizable series with the semantics of a restricted weighted MSO logic and also with the semantics of a restricted existential MSO logic. Furthermore, we prove an equivalence to the important model of weighted Büchi automata investigated in Ésik and Kuich [16]. They have characterized the behaviors of weighted Büchi automata precisely as the ω-rational formal power series; for further work on this model, see [22, 23]. Combining these results, we thus obtain a robust notion of weighted automata, logics and rational series on infinite words. As in [16], we assume our semiring of weights to permit infinite sum and product operations. Such "complete" semirings have been investigated in detail in the literature, cf. [4, 13, 21]. However, we derive from this a version of our result for semirings which are *not* complete; this includes all Boolean algebras and also max-min semirings used for capacity models. In particular, when considering the Boolean semiring, we obtain Büchi's result as a very special consequence.

Next we briefly describe the structure of our paper. In Section 2, we introduce the notions of totally commutative complete semirings and weighted Muller automata and we state their basic properties. In Section 3 we recall weighted MSO logic from [10], but we interpret the semantics of weighted MSO-formulas as formal power series over infinite words. The main result of the paper in Section 4 states that a formal power series is Muller recognizable iff it is definable in our restricted weighted MSO logic iff it is definable in existential restricted weighted MSO logic. Its proof requires, in particular, a construction of specific weighted Muller automata for the universal quantifier. Then in Section 5, we relate our weighted Muller automata to the weighted Büchi automata of Ésik and Kuich [16], and we show that these two models are equivalent. Finally, in Section 6, we deal with bi-aperiodic semirings which were introduced by Droste and Gastin in [9, 10]. We show that our main result remains true if the underlying semiring is just commutative and weakly bi-aperiodic. Büchi's classical theorem follows as a very special case.

2 Semirings and Weighted Muller Automata

In this section, we introduce totally commutative complete semirings, infinitary formal power series and weighted Muller automata. The reader is referred to [3, 13, 21, 24, 31] for semirings, and to [29, 33, 34] for classical Muller automata.

Let $(K, +, \cdot, 0, 1)$ be a *complete semiring*[13, 21, 16], i.e, a semiring that permits infinite sums extending the associativity, the commutativity and the distributivity laws of the finite sum operation. Then K is called *totally complete*[15], if it is endowed with countably infinite product operations satisfying for all sequences $(a_i \mid i \geq 1)$ of elements of K the following conditions:

$$a_1 \cdot \prod_{i\geq 1} a_{i+1} = \prod_{i\geq 1} a_i, \qquad \prod_{i\geq 1} a_i = \prod_{i\geq 1} (a_{n_{i-1}+1} \cdot \dots \cdot a_{n_i}),$$

$$\prod_{i\geq 0} 1 = 1, \qquad \prod_{j\geq 1} \sum_{i_j \in I_j} a_{i_j} = \sum_{(i_1, i_2, \dots) \in I_1 \times I_2 \times \dots} \prod_{j\geq 1} a_{i_j}$$

where $0 = n_0 \leq n_1 \leq n_2 \leq \dots$ and $I_1, I_2, \dots$ are arbitrary index sets.

Furthermore, we will call a totally complete semiring *totally commutative complete* if it is commutative and satisfies the statement:

$$\prod_{j\in J} \left(\prod_{i_j \in I_j} a_{i_j} \right) = \prod_{i\geq 0} a_i$$

where $\bigcup_{j\in J} I_j = \mathbb{N}$ and $I_j \cap I_{j'} = \varnothing$ for $j \neq j'$.

Concrete examples of totally commutative complete semirings are the semiring $(\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ of *extended natural numbers* [16], the *fuzzy semiring* $F = ([0, 1], \sup, \inf, 0, 1)$ [30, 23], and each *completely distributive lattice* (cf. [2]) with the operations *supremum* and *infimum*. Further examples will be given in Section 6.

Let A be a finite alphabet. As usual we denote by A^ω the set of all infinite words over A. An infinite word $w = a_0 a_1 \ldots \in A^\omega$ is written as $w = w(0)w(1)\ldots$ with $w(i) = a_i$, $i \geq 0$. We shall denote the set of natural numbers $\mathbb{N}$ also by ω.

Given a finite alphabet A and a semiring K, an *infinitary formal power series* or *series* for short, is a mapping $S : A^\omega \to K$. The class of all power series over A and K is denoted by $K \langle\langle A^\omega \rangle\rangle$. The *sum* $S + T$, the *scalar products* kS and Sk, the *Hadamard product* $S \odot T$ for $S, T \in K \langle\langle A^\omega \rangle\rangle$ and $k \in K$, as well as the *characteristic series* $1_L : A^\omega \to K$ *of* $L \subseteq A^\omega$, are defined in $K \langle\langle A^\omega \rangle\rangle$ pointwise as in the finitary case.

Consider two alphabets A, B and an homomorphism $h : A^* \to B^*$. Then h can be extended to a mapping $h : A^\omega \to B^\omega$ in the obvious way. For any power series $T \in K \langle\langle B^\omega \rangle\rangle$ the series $h^{-1}(T) \in K \langle\langle A^\omega \rangle\rangle$ is defined by $(h^{-1}(T), u) = (T, h(u))$ for $u \in A^\omega$. Furthermore, if h is non-deleting, i.e., $h(a) \neq \varepsilon$ for each $a \in A$, and K is complete, then for any $S \in K \langle\langle A^\omega \rangle\rangle$ the series $h(S) \in K \langle\langle B^\omega \rangle\rangle$ is specified by $(h(S), w) = \sum_{u \in h^{-1}(w)} (S, u)$ for $w \in B^\omega$. The homomorphism h is *strict alphabetic,* if $h(a) \in B$ for each $a \in A$.

For the rest of this section, let A be a finite alphabet and K be a totally complete semiring. We shall simply denote the operation $\cdot$ by concatenation.

Definition 1. *A* weighted Muller automaton *(WMA for short)* over A and K *is a quadruple* $\mathcal{A} = (Q, in, wt, \mathcal{F})$, *where* Q *is the* finite state set, $in : Q \to K$ *is* *the* initial distribution, $wt : Q \times A \times Q \to K$ *is a* mapping assigning weights to the transitions *of the automaton, and* $\mathcal{F} \subseteq 2^Q$ *is the* family of final state sets.

Let $w = a_0 a_1 \ldots \in A^\omega$. A *path of* $\mathcal{A}$ *over* w is an infinite sequence of transitions $P_w := (t_i)_{i \geq 0}$, so that $t_i = (q_i, a_i, q_{i+1})$ for all $i \geq 0$. The *weight of* P_w is defined by $weight(P_w) := in(q_0) \cdot \prod\limits_{i \geq 0} wt(t_i)$. The path P_w is called *successful* if the set of states that appear infinitely often along P_w constitute a final state set. The *behavior* of $\mathcal{A}$ is the formal power series $\|\mathcal{A}\| : A^\omega \to K$ whose coefficients are determined by $(\|\mathcal{A}\|, w) = \sum\limits_{P_w} weight(P_w)$ for $w \in A^\omega$, where the sum is taken over all successful paths P_w of $\mathcal{A}$ over w. A series $S : A^\omega \to K$ is said to be *Muller recognizable* if there is a WMA $\mathcal{A}$ so that $S = \|\mathcal{A}\|$. We shall denote the family of all such series over A and K by $K^{M-rec}\langle\langle A^\omega \rangle\rangle$.

The next result states closure properties of the family $K^{M-rec}\langle\langle A^\omega \rangle\rangle$.

Theorem 1. *The class* $K^{M-rec}\langle\langle A^\omega \rangle\rangle$ *is closed under:*

- *sum and scalar products; furthermore, if* K *is totally commutative complete, then* $K^{M-rec}\langle\langle A^\omega \rangle\rangle$ *is also closed under Hadamard products,*
- *non-deleting homomorphisms,*
- *inverse image of strict alphabetic homomorphisms.*

Proposition 1. *The characteristic series* $1_L : A^\omega \to K$ *of any* ω*-recognizable language* $L \subseteq A^\omega$ *is Muller recogizable.*

We will call a power series $S : A^\omega \to K$ a *Muller recognizable step function* if $S = \sum\limits_{1 \leq j \leq n} k_j 1_{L_j}$ where $k_j \in K$ and $L_j \subseteq A^\omega$ $(1 \leq j \leq n$ and $n \in \mathbb{N})$ are ω-recognizable languages. Then by Theorem 1 and Proposition 1, S is Muller recognizable.

3 Weighted Monadic Second Order Logic

Weighted MSO logic was introduced by Droste and Gastin in [10] in order to obtain a logical characterization of recognizable formal power series over finite words.

Let A be a finite alphabet and $\mathcal{V}$ a finite set of first and second order variables. An infinite word $w \in A^\omega$ is represented by the relational structure $(\omega, \leq, (R_a)_{a \in A})$ where $R_a = \{i \mid w(i) = a\}$ for $a \in A$. A $(w, \mathcal{V})$*-assignment* σ is a mapping associating first order variables from $\mathcal{V}$ to elements of ω, and second order variables from $\mathcal{V}$ to subsets of ω. If x is a first order variable and $i \in \omega$, then $\sigma[x \to i]$ denotes the $(w, \mathcal{V} \cup \{x\})$-assignment which associates i to x and acts as σ on $\mathcal{V} \setminus \{x\}$. For a second order variable X and $I \subseteq \omega$, the notation $\sigma[X \to I]$ has a similar meaning. In order to encode pairs (w, σ) for all $w \in A^\omega$ and any $(w, \mathcal{V})$-assignment σ, we use an extended alphabet $A_\mathcal{V} = A \times \{0,1\}^\mathcal{V}$ (cf. [33, 34, 10])

It is not difficult to see that the set $N_{\mathcal{V}} = \{(w, \sigma) \in A_{\mathcal{V}}^{\omega} \mid \sigma \text{ is a valid } (w, \mathcal{V})\text{-assignment}\}$ is ω-recognizable. Let now φ be an MSO-formula [33, 34]. We shall write A_{φ} for $A_{Free(\varphi)}$ and $N_{\varphi} = N_{Free(\varphi)}$. The fundamental Büchi theorem [6] states that for $Free(\varphi) \subseteq \mathcal{V}$ the language $\mathcal{L}_{\mathcal{V}}(\varphi) = \{(w, \sigma) \in N_{\mathcal{V}} \mid (w, \sigma) \models \varphi\}$ defined by φ over $A_{\mathcal{V}}$ is ω-recognizable. We simply write $\mathcal{L}(\varphi) = \mathcal{L}_{Free(\varphi)}(\varphi)$. Conversely, each ω-recognizable language $L \subseteq A^{\omega}$ is definable by an MSO-sentence φ, i.e., $L = \mathcal{L}(\varphi)$.

Now we turn to weighted logics.

Definition 2. *The syntax of formulas of the weighted MSO logic is given by*

$$\varphi := k \mid P_a(x) \mid \neg P_a(x) \mid x \leq y \mid \neg(x \leq y) \mid x \in X \mid \neg(x \in X) \\ \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \exists x \,.\, \varphi \mid \exists X \,.\, \varphi \mid \forall x \,.\, \varphi \mid \forall X \,.\, \varphi$$

where $k \in K$, $a \in A$. We shall denote by $MSO(K, A)$ the set of all such weighted MSO-formulas φ.

Next we represent the semantics of the formulas in $MSO(K, A)$ as formal power series over the extended alphabet $A_{\mathcal{V}}$ and the semiring K. We assume K to be a totally commutative complete semiring.

Definition 3. *Let $\varphi \in MSO(K, A)$ and $\mathcal{V}$ be a finite set of variables with $Free(\varphi) \subseteq \mathcal{V}$. The semantics of φ is a formal power series $\|\varphi\|_{\mathcal{V}} \in K \langle\langle A_{\mathcal{V}}^{\omega} \rangle\rangle$. Consider an element $(w, \sigma) \in A_{\mathcal{V}}^{\omega}$. If σ is not a valid assignment, then we put $\|\varphi\|_{\mathcal{V}}(w, \sigma) = 0$. Otherwise, we inductively define $\|\varphi\|_{\mathcal{V}}(w, \sigma) \in K$ as follows:*

- $\|k\|_{\mathcal{V}}(w, \sigma) = k$
- $\|P_a(x)\|_{\mathcal{V}}(w, \sigma) = \begin{cases} 1 & \textit{if } w(\sigma(x)) = a \\ 0 & \textit{otherwise} \end{cases}$
- $\|x \leq y\|_{\mathcal{V}}(w, \sigma) = \begin{cases} 1 & \textit{if } \sigma(x) \leq \sigma(y) \\ 0 & \textit{otherwise} \end{cases}$
- $\|x \in X\|_{\mathcal{V}}(w, \sigma) = \begin{cases} 1 & \textit{if } \sigma(x) \in \sigma(X) \\ 0 & \textit{otherwise} \end{cases}$
- $\|\neg\varphi\|_{\mathcal{V}}(w, \sigma) = \begin{cases} 1 & \textit{if } \|\varphi\|_{\mathcal{V}}(w, \sigma) = 0 \\ 0 & \textit{if } \|\varphi\|_{\mathcal{V}}(w, \sigma) = 1 \end{cases}$, *provided that φ is of the form $P_a(x)$, $(x \leq y)$ or $(x \in X)$*
- $\|\varphi \vee \psi\|_{\mathcal{V}}(w, \sigma) = \|\varphi\|_{\mathcal{V}}(w, \sigma) + \|\psi\|_{\mathcal{V}}(w, \sigma)$
- $\|\varphi \wedge \psi\|_{\mathcal{V}}(w, \sigma) = \|\varphi\|_{\mathcal{V}}(w, \sigma) \cdot \|\psi\|_{\mathcal{V}}(w, \sigma)$
- $\|\exists x \,.\, \varphi\|_{\mathcal{V}}(w, \sigma) = \sum_{i \in \omega} \|\varphi\|_{\mathcal{V} \cup \{x\}}(w, \sigma[x \to i])$
- $\|\exists X \,.\, \varphi\|_{\mathcal{V}}(w, \sigma) = \sum_{I \subseteq \omega} \|\varphi\|_{\mathcal{V} \cup \{X\}}(w, \sigma[X \to I])$
- $\|\forall x \,.\, \varphi\|_{\mathcal{V}}(w, \sigma) = \prod_{i \in \omega} \|\varphi\|_{\mathcal{V} \cup \{x\}}(w, \sigma[x \to i])$ *and*
- $\|\forall X \,.\, \varphi\|_{\mathcal{V}}(w, \sigma) = \prod_{I \subseteq \omega} \|\varphi\|_{\mathcal{V} \cup \{X\}}(w, \sigma[X \to I])$.

The reader may notice that the product in universal second order quantification is uncountable. But this is not a problem since later we exclude it from our constructions. Also as in [10], we have restricted negation to atomic formulas.

The reason is that if K is not a Boolean algebra, then it is difficult to define the semantics of the negation of an arbitrary formula. Our restriction is not essential in comparison to classical MSO logics, since any MSO-formula φ is equivalent (both logically and in the sense of defining the same ω-language) to one in which negation is applied only to atomic formulas. We simply write $\|\varphi\|$ for $\|\varphi\|_{Free(\varphi)}$. If φ has no free variables, i.e., if it is a sentence, then $\|\varphi\| \in K\langle\langle A^\omega\rangle\rangle$. Next, we present several examples of possible interpretations for weighted formulas, for details see [10].

(i) Consider the semiring $K = (\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ and assume that φ does not contain constants $k \in K$. Then we may interpret $\|\varphi\|(w, \sigma)$ as the number of proofs we have that (w, σ) satisfies formula φ.
(ii) The formula $\exists x \,.\, P_a(x)$ counts *how often* (depending on the semiring) the letter a occurs in the word.
(iii) For any formula φ over the fuzzy semiring F, we have that $\|\varphi\|(w, \sigma) \neq 0$ iff (w, σ) satisfies φ.
(iv) Let K be an arbitrary Boolean algebra $(B, \vee, \wedge, ^-, 0, 1)$. In this case, infinite sums correspond to suprema and infinite products to infima. For any formula φ, we can define the semantics of $\neg\varphi$, by $\|\neg\varphi\|(w, \sigma) := \overline{\|\varphi\|(w, \sigma)}$. Especially, for $K = \mathbf{B}$ the 2-valued Boolean algebra our semantics coincides with the usual semantics of classical MSO-formulas, identifying characteristic series with their supports.

The reader may observe that the above definition is valid for each formula $\varphi \in MSO(K, A)$ and each finite set $\mathcal{V}$ of variables containing $Free(\varphi)$. According to the next proposition the semantics $\|\varphi\|_{\mathcal{V}}$ depends only on $Free(\varphi)$.

Proposition 2. *Let $\varphi \in MSO(K, A)$ and $\mathcal{V}$ be a finite set of variables such that $Free(\varphi) \subseteq \mathcal{V}$. Then*

$$\|\varphi\|_{\mathcal{V}}(w, \sigma) = \|\varphi\|(w, \sigma|_{Free(\varphi)})$$

for each $(w, \sigma) \in A^\omega_{\mathcal{V}}$, where σ is a valid $(w, \mathcal{V})$-assignment. Furthermore, $\|\varphi\|$ is Muller recognizable iff $\|\varphi\|_{\mathcal{V}}$ is Muller recognizable.

Let now $Z \subseteq MSO(K, A)$. A series $S : A^\omega \to K$ is called *Z-definable* if there is a sentence $\varphi \in Z$ so that $S = \|\varphi\|$.

It has been proved in [10] that universal quantifiers do not preserve in general the recognizability property of power series over finite words. Thus the authors worked on a restricted framework of weighted MSO logics, which we also adopt here. More precisely, a formula $\varphi \in MSO(K, A)$ will be called *restricted* (cf. [10]) if it contains no universal quantification of the form $\forall X \,.\, \psi$, and whenever φ contains a universal first order quantification $\forall x \,.\, \psi$, then $\|\psi\|$ is a Muller recognizable step function. The subclass of all restricted formulas of $MSO(K, A)$ will be denoted by $RMSO(K, A)$. Moreover, a formula $\varphi \in RMSO(K, A)$ is *restricted existential* if it is of the form $\exists X_1, \ldots, X_n \,.\, \psi$ with $\psi \in RMSO(K, A)$ and ψ contains no set quantification. All such restricted existential formulas will compose the class $REMSO(K, A)$. We let $K^{rmso}\langle\langle A^\omega\rangle\rangle$ (resp. $K^{remso}\langle\langle A^\omega\rangle\rangle$) comprise all series from $K\langle\langle A^\omega\rangle\rangle$ which are definable by some sentence in $RMSO(K, A)$ (resp. in $REMSO(K, A)$).

4 The Main Result

In this section we establish our main result:

Theorem 2. *Let A be an alphabet and K any totally commutative complete semiring. Then*

$$K^{M-rec}\langle\langle A^\omega\rangle\rangle = K^{rmso}\langle\langle A^\omega\rangle\rangle = K^{remso}\langle\langle A^\omega\rangle\rangle .$$

Proof. Let us present a sketch of the proof. First, using Theorem 1 and Proposition 1, we show by induction on the structure of $RMSO$-formulas that $K^{rmso}\langle\langle A^\omega\rangle\rangle \subseteq K^{M-rec}\langle\langle A^\omega\rangle\rangle$. The most difficult case arises with first order universal quantification. Let $\mathcal{W} = Free(\varphi)$ and $\mathcal{V} = Free(\forall x \,\text{.}\, \varphi) = \mathcal{W} \setminus \{x\}$. Let also $\|\varphi\| = \sum_{1\leq j\leq n} k_j 1_{L_j}$ with ω-recognizable languages $L_j \subseteq A^\omega_{\mathcal{W}}$ $(1 \leq j \leq n)$. We claim that $\|\forall x \,\text{.}\, \varphi\|$ is Muller recognizable. Without any loss, we can assume that the family $(L_j)_{1\leq j\leq n}$ is a partition of $A^\omega_{\mathcal{W}}$. We distinguish two cases.

Case 1: $x \in \mathcal{W}$.
We consider the alphabet $\widetilde{A} = A \times \{1,\ldots,n\}$, and the language $\widetilde{L} \subseteq \widetilde{A}^\omega_{\mathcal{V}}$ to be the collection of all words $(w, v, \sigma) \in \widetilde{A}^\omega_{\mathcal{V}}$, so that for all $i \in \omega$ and $j \in \{1,\ldots,n\}$, then $v(i) = j$ implies $(w, \sigma[x \to i]) \in L_j$. The languages L_j are ω-recognizable by deterministic Muller automata, and from these we construct a deterministic Muller automaton $\widetilde{\mathcal{A}}$ recognizing $\widetilde{L}$. In the sequel, we convert $\widetilde{\mathcal{A}}$ to a WMA $\mathcal{A}$ over $\widetilde{A}_{\mathcal{V}}$, and we show that $\|\forall x \,\text{.}\, \varphi\| = h(\|\mathcal{A}\|)$, where h is the projection of $\widetilde{A}^\omega_{\mathcal{V}}$ to$A^\omega_{\mathcal{V}}$. Thus by Theorem 1 the series $\|\forall x \,\text{.}\, \varphi\|$ is Muller recognizable.

Case 2: If $x \notin \mathcal{W}$, then we consider the formula $\varphi' = \varphi \wedge (x \leq x)$, and the result comes by Case 1.

Now we state the converse inclusion. Given a WMA $\mathcal{A}$ over A and K we effectively construct a $RMSO(K, A)$-formula ψ representing the paths of $\mathcal{A}$. Next we equip ψ with weights, so that the semantics $\|\varphi\|$ of the obtained formula φ takes as values the weights of the corresponding paths of $\mathcal{A}$. Finally, from φ we obtain a formula ξ in $REMSO(K, A)$ whose semantics equals the behavior of the automaton $\mathcal{A}$.

5 Weighted Büchi Automata

In this section, we consider weighted automata over infinite words with Büchi acceptance condition which were introduced by Ésik and Kuich [16]. We show their equivalence to our model with Muller acceptance condition.

Let A be any alphabet and K be a totally complete semiring.

Definition 4. *([16]) A* weighted Büchi automaton *(WBA for short) over A and K is a quadruple $\mathcal{A} = (Q, in, wt, F)$, where Q is the* finite state set*, $in : Q \to K$ is the* initial distribution, *$wt : Q \times A \times Q \to K$ is a* mapping assigning weights to the transitions of $\mathcal{A}$, *and F is the* final state set.

Given an infinite word $w = a_0a_1 \ldots \in A^\omega$, a *path P_w of $\mathcal{A}$ over w* and its *weight* are defined as for weighted Muller automata. The path P_w is called *successful* if at least one final state appears infinitely often. The *behavior of* $\mathcal{A}$ is the infinitary power series $\|\mathcal{A}\| : A^\omega \to K$, with coefficients specified for $w \in A^\omega$ by $(\|\mathcal{A}\|, w) = \sum_{P_w} weight(P_w)$, where the sum is taken over all successful paths P_w of $\mathcal{A}$ over w. A series $S : A^\omega \to K$ is called *ω-recognizable* if there is a WBA $\mathcal{A}$ such that $S = \|\mathcal{A}\|$. The class of all ω-recognizable series over A and K is denoted by $K^{\omega-rec}\langle\langle A^\omega\rangle\rangle$.

Theorem 3. *Let A be an alphabet and K any totally complete semiring. Then*

$$K^{\omega-rec}\langle\langle A^\omega\rangle\rangle = K^{M-rec}\langle\langle A^\omega\rangle\rangle .$$

6 Bi-aperiodic Semirings

In this section we state our main result for weakly bi-aperiodic [9, 10] and commutative semirings. A semiring $(K, +, \cdot, 0, 1)$ is called *bi-aperiodic* if there exists an integer $m \geq 0$ such that for all $a \in K$ $(m+1)a = ma$ and $a^{m+1} = a^m$. All distributive lattices with 0 and 1 are bi-aperiodic semirings with supremum and infimum as operations. Furthermore, all Boolean algebras, and the reals with max-min or min-max, constitute bi-aperiodic semirings. In this case, for any $a \in K, m \geq 0$ as above, and any infinite index set I, we can define the infinite sum and product of a's by letting $\sum_{i \in I} a = ma$ and $\prod_{i \in I} a = a^m$. Now assume that our semiring K is finite, bi-aperiodic and commutative. Then we can also define the infinite sum and product of any family of elements of K, by splitting them suitably and then taking the corresponding finite sums and products. We obtain:

Proposition 3. *Each finite bi-aperiodic commutative semiring* $(K, +, \cdot, 0, 1)$ *is totally commutative complete.*

Next, a semiring $(K, +, \cdot, 0, 1)$ is called *weakly bi-aperiodic* iff for each element $a \in K$ there exists $m \geq 0$ such that $(m+1)a = ma$ and $a^{m+1} = a^m$. Trivially, each bi-aperiodic semiring is weakly bi-aperiodic and each finite weakly bi-aperiodic semiring is bi-aperiodic. We refer the reader to [10] for examples of weakly bi-aperiodic semirings. Furthermore,

Example 1. Let $0 < c < 1$ and $K = \{0\} \cup [c, 1]$. We define in K the truncated multiplication $\cdot_c$ in the following way: for $x \neq 0$ and $y \neq 0$, $x \cdot_c y := x \cdot y$ if $x \cdot y \geq c$ and $x \cdot_c y := c$ if $x \cdot y \leq c$. The semiring $(K, \max, \cdot_c, 0, 1)$ is called the *truncated probabilistic semiring.* Obviously, it is weakly bi-aperiodic but not aperiodic.

Next we state that

Theorem 4. *Let A be an alphabet and K any weakly bi-aperiodic commutative semiring. Then*

$$K^{M-rec}\langle\langle A^\omega\rangle\rangle = K^{rmso}\langle\langle A^\omega\rangle\rangle = K^{remso}\langle\langle A^\omega\rangle\rangle .$$

Corollary 1 (Büchi's Theorem). *An infinitary language is ω-recognizable iff it is definable by a MSO-sentence.*

7 Conclusion

We introduced weighted Muller automata over totally complete semirings. We verified that the family of their behaviors coincides with the class of infinitary formal power series obtained as semantics of weighted restricted MSO-sentences, provided that the underlying semiring is totally commutative complete and also with the family of behaviors of weighted Büchi automata investigated by Ésik and Kuich [16]. We do not know if this family coincides with the class of series specified by all weighted MSO-sentences. Also, the question arises whether Theorem 2, in particular the construction of a WMA $\mathcal{A}$ for a given MSO-formula φ can be made effective. The problem is the universal quantifier: Given a WMA for φ as described in the proof of Theorem 2, how do we obtain the values k_j and WMA for the languages L_j? In the case of finite words and given a field K, Droste and Gastin [10] could use results from the literature on formal power series to obtain a construction. Therefore, also in our situation we should consider specific semirings.

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