Presentation-invariant definability

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Elementary definability

Simple graph: $- \subseteq V^2$ $\forall x, y \in V$

- $\neg(x - x) \wedge (x - y \rightarrow y - x)$

Total ordering: $< \subseteq D^2$ $\forall x, y, z \in D$

- $\neg(x < x) \wedge (x \neq y \rightarrow x < y \vee y < x)$

  $(x < y < z \rightarrow x < z)$

Undefinability

even: The number of vertices is even.

connected: The graph is connected.

acyclic: The graph is acyclic.

None of these are elementary over *finite* graphs.

Because first-order logic is *local* (compactness).

Order invariance

Augment each graph with an arbitrary ordering:

$$(G, <)$$

Elementary definability invariant of particular $<$:

$$(G, <) \models \theta \;\Leftrightarrow (G, <') \models \theta$$

But: **even**, **connected**, **acyclic** $\notin$ FO($<$) $\neq$ FO

Presentation invariance

Expand each graph by a definable relation R:

$$(\exists R)\ (G, R) \models \sigma$$

<u>Special case</u>: σ depends only on $|G|$ and R.

Using R, define a graph query Q, invariant of R:

$$(\forall R\ (G, R) \models \sigma)\ \ [(G, R) \models \theta \Leftrightarrow G \in Q]$$

Examples for P ⊆ S

Degree:	zero	one	two
	isolated	barbell	chain

even: barbells with at most one isolated point.

parity: barbells where both ends are in P.

majority: barbells with ends in P and $\neg P$.

Fact: Distance is not bounded-degree invariant.

Graph traversals

An ordering of its components, each with the property that every initial segment is connected:

$$[\quad ..\quad]\quad ..\quad [x, y]\quad ..\quad [\quad ..\quad]$$

$$(\forall v)(\exists x)(\exists y)[x \le v \le y](\forall z)(x \le z \le y)$$

$$\{(\forall w - z)[x \le w \le y]\} \wedge \{z \ne x \to (\exists w - z)[w < z]\}$$

Traversal invariance

Connected: consists of one component interval

Acyclic: no node with two prior neighbors

Reachable: both nodes are in same component

- Can use breadth-first and depth-first traversals
- Can also define biconnected and bipartite