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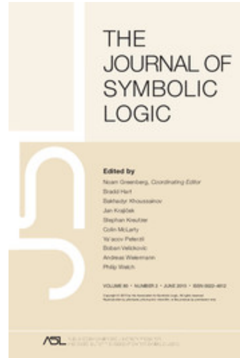

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THE DETERMINACY OF BLACKWELL GAMES

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§0. Introduction. Games of infinite length and perfect information have been studied for many years. There are numerous determinacy results for these games, and there is a wide body of work on consequences of their determinacy.

Except for games with very special payoff functions, games of infinite length and imperfect information have been little studied. In 1969, David Blackwell [1] introduced a class of such games and proved a determinacy theorem for a subclass. During the intervening time, there has not been much progress in proving the determinacy of Blackwell's games. Orkin [17] extended Blackwell's result to a slightly wider class. Blackwell [2] found a new proof of his own result. Maitra and Sudderth [9, 10] improved Blackwell's result in a different direction from that of Orkin and also generalized to the case of stochastic games. Recently Vervoort [18] has obtained a substantial improvement. Nevertheless, almost all the basic questions have remained open.

In this paper we associate with each Blackwell game a family of perfect information games, and we show that the (mixed strategy) determinacy of the former follows from the (pure strategy) determinacy of the latter. The complexity of the payoff function for the Blackwell game is approximately the same as the complexity of the payoff sets for the perfect information games. In particular, this means that the determinacy of Blackwell games with Borel measurable payoff functions follows from the known determinacy of perfect information games with Borel payoff sets. This result confirms a conjecture of Blackwell [1]. Another consequence of our theorem is that the Axiom of Determinacy implies the analogous Axiom of Determinacy for Blackwell Games introduced in Vervoort [18]. Maitra and Sudderth have pointed out to us that the proof of our theorem works not just for Blackwell games but also for the wider class of (two-person, zero-sum) stochastic games with countable state spaces.

This section will be devoted to definitions and background. In §1 we shall prove our main result. At the end of the section, we shall indicate briefly how the main result extends to stochastic games. In §2 we shall prove variants of the main result. In §3 we shall give consequences of the theorems of §1 and §2, and we shall discuss open problems.

We first describe the games of perfect information that we shall consider.

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A *game tree* is a set T of of finite sequences such that

$$(q \in T \wedge p \subseteq q) \rightarrow p \in T.$$

(We think of finite and infinite sequences as the graphs of the corresponding functions, so that "$p \subseteq q$" means that q extends p.)

From a game tree T, we get a two-person perfect information game. Play in this game proceeds as follows:

$$\begin{array}{lcccccccc} \mathrm{I} & a_0 & & a_2 & & a_4 & & \dots & \\ \mathrm{II} & & a_1 & & a_3 & & a_5 & & \dots \end{array}$$

It is required that all positions $\langle a_i \mid i < n \rangle$ belong to T. Thus T is the set of legal positions in the game. If a play reaches a terminal position, i.e., a $p \in T$ such that no proper exension of p belongs to T, then the play is finite. Otherwise it is infinite. None of the perfect information games we consider in this paper will have terminal positions.

We shall use the phrase "perfect information game" ambiguously, both for the game given by a game tree T as in the last paragraph and for the game given by T together with a *payoff set*, a set of plays of the game given by T. The payoff set gives the winning condition for the latter game. If A is a payoff set and x is a play, then

$$x \text{ is a win for I} \longleftrightarrow x \in A.$$

CONVENTION. Upper case letters G and H, possibly with subscripts and/or superscripts, will be used to denote games in the second sense, games given by a game tree plus a payoff set.

Let G be a perfect information game. A *strategy* for player I or II for G is a function that assigns a legal move to each position in G in which it is that player's turn to move. A position or play is *consistent* with a strategy for one of the players if the moves made by that player are the moves given by the strategy. A strategy is *winning* if all plays consistent with the strategy are wins for the player whose strategy it is. The game G is *determined* if one of the players has a winning strategy.

Let T be a game tree. For $p \in T$, let N_p be the set of all plays x in the game given by T such that $p \subseteq x$. We put a topology on the set of all plays by taking as a base $\{ N_p \mid p \in T \}$. A perfect information game G is *open*, *Borel*, etc., if and only if its payoff set is open, Borel, etc.

The determinacy of all Borel games of perfect information is proved in Martin [12]. (See also Martin [13] and Martin [14].) It follows from results of Davis [3] and Gödel [5] that the determinacy of all $\boldsymbol{\Sigma}^1_1$ (analytic) perfect information games, even for countable T, is not provable from the standard ZFC axioms of set theory. But the determinacy of all $\boldsymbol{\Sigma}^1_1$ games, indeed of all projective games and even of all games in $L(\mathbb{R})$, follows from large cardinal axioms. See Martin [11], Martin–Steel [15], Woodin [20], and Neeman [16].

We now turn to games of imperfect information. Let T be a game tree. Assume that

(a) the members of T are finite sequences of ordered pairs;
(b) if $p \in T$ and p has length i, then there are finite sets X_p and Y_p such that the length $i+1$ extensions of p that belong to T are precisely the $p^\frown\langle a, b\rangle$ with $a \in X_p$ and $b \in Y_p$.

From T we get a *Blackwell game* Γ, which is played as follows.

$$\begin{array}{lllll} \mathrm{I} & a_0 & a_1 & a_2 & \dots \\ \mathrm{II} & b_0 & b_1 & b_2 & \dots \end{array}$$

In other words, for each i the moves a_i and b_i are made simultaneously. It is required that all positions $\langle \langle a_i, b_i \rangle \mid i < n \rangle$ belong to T.

REMARK. Condition (b) on T means that each player has a finite set of legal moves in each position. If infinitely many legal moves are allowed for both players, then determinacy fails even for games in which the players make only one move each. Very little in this paper would be affected, however, if (b) were relaxed to the requirement that in each position at least one of the players has only finitely many legal moves and neither player has more than countably many legal moves. See the last paragraph of §2.

Let Γ be a Blackwell game. A *strategy* for player I or II for Γ is a function that assigns to each position in Γ a probability distribution on the finite set of legal moves for that player in the position.

A *payoff function* for a Blackwell game Γ is a function f from the set of all plays of Γ into a bounded subset of the real numbers.

We shall use the phrase "Blackwell game" ambiguously, both for the game Γ given by a game tree T satisfying (a) and (b) and for the game $\Gamma(f)$ given by T together with a payoff function for Γ. We have just indicated how we shall distinguish notationally between the two sorts of games.

Let Γ be a Blackwell game. Let σ and τ be strategies for I and II respectively for Γ. The strategies σ and τ give, in an obvious manner, a probability measure $\mu_{\sigma,\tau}$ on the set of all plays of Γ. If f is a payoff function for Γ that is $\mu_{\sigma,\tau}$-measurable, then set

$$E_{\sigma,\tau}(f) = \int f \, d\mu_{\sigma,\tau}.$$

Note that f is $\mu_{\sigma,\tau}$-measurable if f is Borel measurable. For arbitrary payoff functions f, set

$$E^-_{\sigma,\tau}(f) = \sup\{ E_{\sigma,\tau}(g) \mid g \text{ is Borel measurable} \wedge (\forall x)\; g(x) \leq f(x) \};$$
$$E^+_{\sigma,\tau}(f) = \inf\{ E_{\sigma,\tau}(g) \mid g \text{ is Borel measurable} \wedge (\forall x)\; g(x) \geq f(x) \}.$$

If f is $\mu_{\sigma,\tau}$-measurable, then $E^-_{\sigma,\tau}(f) = E^+_{\sigma,\tau}(f) = E_{\sigma,\tau}(f)$.

Let Γ be a Blackwell game and let f be a payoff function for Γ. If σ is a strategy for I for Γ then the *value* of σ in $\Gamma(f)$ is

$$\inf\{ E^-_{\sigma,\tau}(f) \mid \tau \text{ is a strategy for II} \}.$$

If τ is a strategy for II for Γ then the *value* of τ in $\Gamma(f)$ is

$$\sup\{ E^+_{\sigma,\tau}(f) \mid \sigma \text{ is a strategy for I} \}.$$

Let $\mathrm{val}_\downarrow(\Gamma(f))$ be the supremum over all strategies σ for I for Γ of the value of σ in $\Gamma(f)$ and let $\mathrm{val}^\uparrow(\Gamma(f))$ be the infinum over all strategies τ for II for Γ of the value of τ in $\Gamma(f)$. The game $\Gamma(f)$ is *determined* if

$$\mathrm{val}_\downarrow(\Gamma(f)) = \mathrm{val}^\uparrow(\Gamma(f)).$$

If $\Gamma(f)$ is determined, then

$$\text{the } \textit{value} \text{ of } \Gamma(f) = \text{val}(\Gamma(f)) = \text{val}_{\downarrow}(\Gamma(f)) = \text{val}^{\uparrow}(\Gamma(f)).$$

REMARKS.

(a) We are using the term "Blackwell games" to cover a rather wide class. It might be more accurate to reserve the term "Blackwell games" for infinite length games. Moreover Blackwell considered only measurable payoff functions. The definitions for the non-measurable case are due to Vervoort [18].

(b) One can easily extend the notion of perfect information games to allow for payoff functions instead of just payoff sets. There is, however, little independent interest in this generalization.

Somewhat artificially, we say that a Blackwell game $\Gamma(f)$ is *open*, *closed*, *Borel*, etc., if, for all rationals y, the set of all plays x such that $y \leq f(x)$ is open, closed, Borel, etc.

Suppose that Γ is a Blackwell game and that A is a set of plays of Γ. Let $\chi(A)$ be the *characteristic function* of A, the function f such that $f(x) = 1$ for $x \in A$ and $f(x) = 0$ for $x \notin A$. According to the definition of the preceding paragraph, $\Gamma(\chi(A))$ is open, closed, Borel, etc., just in case A is open, closed, Borel, etc.

Note that a Blackwell game is Borel just in case its payoff function is Borel measurable. The analogous statement holds for projective Blackwell games.

The basic von Neumann Minimax Theorem of [19] states that all one-move Blackwell games are determined in the strong sense that each player has an optimal strategy, a strategy whose value is the value of the game. From this it follows that all Blackwell games of finite length are determined and have optimal strategies. Blackwell [1] proves that all $\mathbf{G}_\delta$ Blackwell games of the form $\Gamma(\chi(A))$ are determined. Orkin [17] extends this result to Boolean combinations of $\mathbf{G}_\delta$'s. Maitra and Sudderth [9] prove determinacy for games $\Gamma(f)$ where, for some function h, each $f(x) = \limsup_i h(x \restriction i)$. (These games can be shown to be the same as what we are calling $\mathbf{G}_\delta$ games.) Moreover their theorem applies to certain stochastic games and not just to Blackwell games. In [10], they extend the result to a wider class of stochastic games. Vervoort [18] betters Blackwell's result by a whole level of the Borel hierarchy, proving the determinacy of all $\mathbf{G}_{\delta\sigma}$ $(\mathbf{\Sigma}^0_3)$ Blackwell games of the form $\Gamma(\chi(A))$.

It follows from facts we shall mention near the end of the paper that the strongest form of Blackwell determinacy provable in ZFC is the determinacy of Borel Blackwell games, just as the strongest form of perfect information determinacy provable in ZFC is Borel determinacy. Using so-called large cardinal axioms, stronger forms of perfect information determinacy have been proved. The basis of many of these proofs is the ZFC theorem that all *homogeneously Souslin* perfect information games are determined. (See, for example, [15].) Large cardinal axioms are then used to prove that sets belonging to various classes are homogeneously Souslin. For, example the existence of a measurable cardinal implies that every $\mathbf{\Sigma}^1_1$ set is homogeneously Souslin. (See [15] for this also.) Before proving the results of this paper (and before learning of the results of Vervoort [18]), the author noticed that the proof of the determinacy of homogeneously Souslin games adapts to show that every homogeneously Souslin Blackwell game is determined. This made the author confident that Borel Blackwell determinacy could be proved in ZFC, and Vervoort's result strengthened that confidence.

§1. The main result. Throughout this section and §2, let Γ be a Blackwell game with no terminal positions and let f be a payoff function for Γ such that $0 \leq f(x) \leq 1$ for every x.

For each $v \in (0, 1]$, we define a perfect information game G_v.

Play in the game G_v is as follows:

$$\begin{array}{lccccccc} \mathrm{I} & h_0 & & h_1 & & h_2 & & \cdots \\ \mathrm{II} & & p_1 & & p_2 & & p_3 & \cdots \end{array}$$

Set $p_0 = \emptyset$, the starting position in Γ. For $i \geq 1$, p_i must be a position in Γ of length i. It is required that $p_0 \subseteq p_1 \subseteq \cdots$. For each i, h_i must be a function into $[0, 1]$ from the set of positions in Γ that are length $i + 1$ extensions of p_i. Let $v_0 = v$ and for $i \geq 0$ let

$$v_{i+1} = h_i(p_{i+1}).$$

For each i, let Δ_i be the game in which the players start at p_i and simultaneously make one move legal in Γ. We may think of h_i as a payoff function for Δ_i. The final requirement on h_i is that

$$\mathrm{val}(\Delta_i(h_i)) \geq v_i.$$

Note that I always has a legal move that fulfills this requirement. For example, I can set $h_i(q) = 1$ for all q. The final requirement on p_{i+1} is that

$$v_{i+1} > 0.$$

It is easy to see by induction on i that II always has a legal move that fulfills this requirement.

For each position p^* in G_v, let $\pi(p^*)$ be the union of all the moves made by II in arriving at p^*. (If length$(p^*) \leq 1$, then $\pi(p^*) = \emptyset$; otherwise $\pi(p^*)$ is the last move made by II.) For any play x^* of G_v, let $\pi(x^*) = \bigcup_i \pi(x^* \restriction i)$, i.e., let $\pi(x^*)$ be the play of Γ extending all the p_i. A play x^* is a win for I if and only if

$$\limsup{}_i v_i \leq f(\pi(x^*)).$$

REMARK. One way to think of the game G_v is to imagine that player I is trying to show that $\mathrm{val}(\Gamma(f)) \geq v$. This account takes I to be asserting generally that $\mathrm{val}(\Gamma(f; p_i)) \geq v_i$, where $\Gamma(f; p_i))$ is like $\Gamma(f)$ except that play begins at p_i. To substantiate this assertion, I chooses the $h_i(q)$. If $\mathrm{val}(\Gamma(f; q)) \geq h_i(q)$ for each q, then the fact that $\mathrm{val}(\Delta(h_i)) \geq v_i$ shows that I's assertion is correct. Player II is therefore required to choose some q as p_{i+1}, thus asking I to show that $\mathrm{val}(\Gamma(f; q) \geq h_i(q)$.

This account of G_v is not entirely accurate, however. Suppose that the legal moves in Γ are 0 and 1 in each position. Suppose also that $f(x) = 0$ if 1 is never played or if the two players first play 1 simultaneously and that $f(x) = 1$ otherwise. This game has value 1, but II has a winning strategy for G_1. (Vervoort [18] uses this $\Gamma(f)$ to illustrate a different, though related, point. There are other choices for $\Gamma(f)$ such that I has a winning strategy for $G_{\mathrm{val}(\Gamma(f))}$.)

Suppose first that σ^* is a winning strategy for I for G_v. We simultaneously define

(i) a strategy σ for I for Γ;
(ii) the notion of an *acceptable* position in Γ;
(iii) for each acceptable position p in Γ, a position $\psi(p)$ in G_v such that

$$\text{length}(\psi(p)) = 2\ \text{length}(p) + 1,$$

$\psi(p)$ is consistent with σ^*, and $\pi(\psi(p)) = p$.

The function $p \mapsto \psi(p)$ will satisfy the condition

$$p \subseteq q \rightarrow \psi(p) \subseteq \psi(q).$$

Thus, for each play x of Γ that contains only acceptable positions, we shall have a play $\psi(x) = \bigcup_{p \subseteq x} \psi(p)$ of Γ such that $\psi(x)$ is consistent with σ^* and $\pi(\psi(x)) = x$.

The starting position $\emptyset$ is acceptable. Every position extending an unacceptable position is unacceptable.

For unacceptable positions p, we define $\sigma(p)$ in an arbitrary fashion.

Let $\psi(\emptyset) = \langle h_0 \rangle$, where h_0 is given by σ^*.

Suppose inductively that we are given an acceptable p of length i and that either (a) $i = 0$ or else (b) $i > 0$ and we have defined

$$\psi(p) = \langle h_0, \ldots, p_i, h_i \rangle,$$

a position in G_v consistent with σ^* and with $p_i = p$.

For positions q of length $i + 1$ that extend p, define q to be acceptable if and only if $h_i(q) > 0$.

Because $\text{val}(\Delta_i(h_i)) \geq v_i$, there is by von Neumann's theorem a strategy for I for Δ_i whose value in $\Delta_i(h_i)$ is $\geq v_i$. Let $\sigma(p)$ be the probability distribution given by such a strategy. Given any acceptable q of length $i + 1$ that extends p, set $\psi(q) = \psi(p)^\frown\langle p_{i+1}, h_{i+1} \rangle$, where $p_{i+1} = q$ and where h_{i+1} is given by σ^*.

For acceptable positions p in Γ let h_i^p, $0 \leq i \leq \text{length}(p)$, be the moves made by I in reaching the position $\psi(p)$. Let $v_0^p = v$ and, for $i < \text{length}(p)$, let $v_{i+1}^p = h_i^p(p \restriction i + 1)$. For plays x of Γ, set

$$g(x) = \begin{cases} \limsup_i v_i^{x \restriction i} & \text{if } (\forall p \subseteq x)\ \ p \text{ is acceptable;} \\ 0 & \text{otherwise.} \end{cases}$$

Note that g is Borel measurable and that $\text{range}(g) \subseteq [0, 1]$.

LEMMA 1.1. *For any strategy τ for* II *for* Γ, $E_{\sigma,\tau}(g) \geq v$.

PROOF. Given τ, let $\mu = \mu_{\sigma,\tau}$. Assume that $E_{\sigma,\tau}(g) < v$. Thus $\int g\, d\mu < v$. Let $\varepsilon > 0$ be such that $\int g\, d\mu < v - \varepsilon$. Hence $\int(1 - g)\, d\mu > 1 - v + \varepsilon$. There is a closed set C such that g is continuous on C and $\int_C (1 - g)\, d\mu > 1 - v + \varepsilon$. (See [6], Theorem 17.12.)

We shall define a play x of Γ such that, for all i, $x \restriction i$ is acceptable and

$$\int_{C \cap N_{x \restriction i}} (1 - g)\, d\mu > (1 - v_i^{x \restriction i} + \varepsilon)\, \mu(N_{x \restriction i}).$$

Suppose that $x \restriction i$ has been defined so that $x \restriction i$ is acceptable and the inequality just stated holds. If there is an acceptable q of length $i + 1$ that extends $x \restriction i$ and is such that $\int_{C\cap N_q}(1-g)\,d\mu > (1-h_i^{x\restriction i}(q)+\varepsilon)\,\mu(N_q)$, then let $x\restriction i+1$ be such a q. Assume, in order to derive a contradiction, that the inequality

$$\int_{C\cap N_q}(1-g)\,d\mu \le (1-h_i^{x\restriction i}(q)+\varepsilon)\,\mu(N_q)$$

holds for every acceptable q. This inequality holds also for unacceptable q, since for them $h_i^{x\restriction i}(q)=0$. Thus

$$\begin{aligned}\int_{C\cap N_{x\restriction i}}(1-g)\,d\mu &= \sum_q \int_{C\cap N_q}(1-g)\,d\mu\\ &\le \sum_q (1-h_i^{x\restriction i}(q)+\varepsilon)\,\mu(N_q)\\ &\le (1-v_i^{x\restriction i}+\varepsilon)\,\mu(N_{x\restriction i}),\end{aligned}$$

contradicting our induction hypothesis for i. (The last inequality holds because $\sigma(x\restriction i)$ is given by a strategy with value $\ge v_i^{x\restriction i}$ in $\Delta_i^{x\restriction i}(h_i^{x\restriction i})$, where $\Delta_i^{x\restriction i}$ is the obvious game.)

We next observe that for any i there is a $y_i \in C\cap N_{x\restriction i}$ such that

$$g(y_i) < v_i^{x\restriction i} - \varepsilon.$$

Assume to the contrary that $g(y) \ge v_i^{x\restriction i}-\varepsilon$ for every $y\in C\cap N_{x\restriction i}$. Then

$$\int_{C\cap N_{x\restriction i}}(1-g)\,d\mu \le (1-v_i^{x\restriction i}+\varepsilon)\,\mu(N_{x\restriction i}),$$

contradicting what we have just proved.

Since $x=\lim_i y_i$, we have that $x\in C$ and so that $g(x)=\lim_i g(y_i)$. Let j be such that

$$(\forall i\ge j)\ |g(x)-g(y_i)|<\varepsilon/2.$$

Thus

$$(\forall i\ge j)\ g(x)<g(y_i)+\varepsilon/2<v_i^{x\restriction i}-\varepsilon/2.$$

Therefore

$$g(x)\le \limsup{}_i v_i^{x\restriction i}-\varepsilon/2 = g(x)-\varepsilon/2.$$

This contradiction completes the proof of the lemma. ⊣

LEMMA 1.2. *The value of σ in $\Gamma(f)$ is $\ge v$.*

PROOF. By the fact that $g(x)\le f(x)$ for all x and by Lemma 1.1, the value of σ in $\Gamma(f)$ is $\ge$ the value of σ in $\Gamma(g)$, which is $\ge v$. ⊣

Dropping our assumption about σ^*, we get the following.

LEMMA 1.3. *If* I *has a winning strategy for G_v, then* $\mathrm{val}_\downarrow(\Gamma(f))\ge v$.

Now assume that τ^* is a winning strategy for II for G_v. Let $\delta>0$. We simultaneously define

(i) a strategy τ for II for Γ;
(ii) the notion of an *acceptable* position in Γ;

(iii) for each acceptable position p, a function u_p into $[0, 1]$ from the set of all q extending p such that $\text{length}(q) = \text{length}(p) + 1$;

(iv) for each acceptable position p in Γ, a position $\psi(p)$ in G_v such that

$$\text{length}(\psi(p)) = 2\ \text{length}(p),$$

$\psi(p)$ is consistent with τ^*, and $\pi(\psi(p)) = p$.

The function $p \mapsto \psi(p)$ will satisfy the condition

$$p \subseteq q \to \psi(p) \subseteq \psi(q).$$

Thus, for each play x of Γ that contains only acceptable positions, we shall have a play $\psi(x) = \bigcup_{p \subseteq x} \psi(p)$ of Γ such that $\psi(x)$ is consistent with τ^* and $\pi(\psi(x)) = x$.

The starting position $\emptyset$ is acceptable. Every position extending an unacceptable position is unacceptable.

For unacceptable positions p, we define $\tau(p)$ in an arbitrary fashion.

Let $\psi(\emptyset) = \emptyset$.

Suppose inductively that we are given an acceptable p of length i and that either (a) $i = 0$ or (b) $i > 0$ and we have defined

$$\psi(p) = \langle h_0, \ldots, p_i \rangle,$$

a position in G_v consistent with τ^* and with $p_i = p$. For positions q of length $i + 1$ that extend p, define q to be acceptable if and only if there is a legal move h for I in G_v at $\psi(p)$ such that $\tau^*(\psi(p)^\frown\langle h \rangle) = q$.

For acceptable q, set

$$u_p(q) = \inf\{\, h(q) \mid h \text{ is legal in } G_v \text{ at } \psi(p) \wedge \tau^*(\psi(p)^\frown\langle h \rangle) = q \,\}.$$

For unacceptable q, set $u_p(q) = 1$.

We show that

$$\text{val}(\Delta_i(u_p)) \leq v_i.$$

Assume, to the contrary, that $\text{val}(\Delta_i(u_p)) > v_i$. Let $\varepsilon > 0$ be such that

$$\text{val}(\Delta_i(u_p)) \geq v_i + \varepsilon.$$

Define a function h by

$$h(q) = \begin{cases} u_p(q) - \varepsilon & \text{if } u_p(q) > \varepsilon; \\ 0 & \text{if } u_p(q) \leq \varepsilon. \end{cases}$$

Then h is a is a legal move for I at the position $\psi(p)$. Hence there is some q such that $\tau^*(\psi(p)^\frown\langle h \rangle) = q$. If $u_p(q) \leq \varepsilon$ then $h(q) = 0$, and so q is not a legal move. If $u_p(q) > \varepsilon$ then $h(q) < u_p(q)$, and the definition of $u_p(q)$ is contradicted.

Let $\tau(p)$ be the probability distribution given by some strategy for II for Δ_i whose value in $\Delta_i(u_p)$ is $\leq v_i$.

Let q have length $i + 1$, be acceptable, and extend p. Let h_i be a legal move for I at $\psi(p)$ such that $h_i(q) \leq u_p(q) + \delta/2^{i+1}$ and such that $\tau^*(\psi(p)^\frown\langle h_i \rangle) = q$. Set $\psi(q) = \psi(p)^\frown\langle h_i, p_{i+1} \rangle$, where $p_{i+1} = q$.

For acceptable positions p in Γ let h_i^p, $0 \leq i < \text{length}(p)$, be the moves made by I in reaching the position $\psi(p)$. Let $v_0^p = v$ and, for $i < \text{length}(p)$, $v_{i+1}^p = h_i^p(p \restriction i{+}1)$.

For plays x of Γ, set

$$g(x) = \begin{cases} \limsup_i v_i^{x\restriction i} & \text{if } (\forall p \subseteq x)\ p \text{ is acceptable;} \\ 1 & \text{otherwise.} \end{cases}$$

LEMMA 1.4. *For any strategy σ for* I *for* Γ,

$$E_{\sigma,\tau}(g) \leq v + \delta.$$

PROOF. Given σ, let $\mu = \mu_{\sigma,\tau}$. Assume that $E_{\sigma,\tau}(g) > v + \delta$. Let $\varepsilon > 0$ be such that $E_{\sigma,\tau}(g) > v + \delta + \varepsilon$. Thus $\int g\, d\mu > v + \delta + \varepsilon$. Let C be a closed set such that g is continuous on C and such that $\int_C g\, d\mu > v + \delta + \varepsilon$.

We shall define a play x of Γ such that, for all i, $x\restriction i$ is acceptable and

$$\int_{C\cap N_{x\restriction i}} g\, d\mu > (v_i^{x\restriction i} + \delta/2^i + \varepsilon)\, \mu(N_{x\restriction i}).$$

Suppose that $x\restriction i$ has been defined so that $x\restriction i$ is acceptable and the inequality just stated holds. If there is an acceptable q such that

$$\int_{C\cap N_q} g\, d\mu > (h_i^q(q) + \delta/2^{i+1} + \varepsilon)\, \mu(N_q),$$

then let $x\restriction i+1$ be such a q. If, for every acceptable q,

$$\int_{C\cap N_q} g\, d\mu \leq (h_i^q(q) + \delta/2^{i+1} + \varepsilon)\, \mu(N_q),$$

then

$$\begin{aligned} \int_{C\cap N_{x\restriction i}} g\, d\mu &= \sum_q \int_{C\cap N_q} g\, d\mu \\ &\leq \sum_{q \text{ acceptable}} (h_i^q(q) + \delta/2^{i+1} + \varepsilon)\, \mu(N_q) + \sum_{q \text{ unacceptable}} \mu(C \cap N_q) \\ &\leq \sum_q (u_{x\restriction i}(q) + \delta/2^{i+1} + \delta/2^{i+1} + \varepsilon)\, \mu(N_q) \\ &\leq (v_i^{x\restriction i} + \delta/2^i + \varepsilon)\, \mu(N_{x\restriction i}), \end{aligned}$$

contradicting our induction hypothesis for i. (The second inequality holds because $h_i^q(q) \leq u_{x\restriction i}(q) + \delta/2^{i+1}$ for q acceptable and $u_{x\restriction i}(q) = 1$ for q unacceptable. The last inequality holds because $\tau(x\restriction i)$ is given by a strategy with value $\leq v_i^{x\restriction i}$ in $\Delta_i^{x\restriction i}(u_{x\restriction i})$.)

We next observe that for any i there is a $y_i \in C \cap N_{x\restriction i}$ such that

$$g(y_i) > v_i^{x\restriction i} + \delta/2^i + \varepsilon.$$

Assume to the contrary that $g(y) \leq v_i^{x\restriction i} + \delta/2^i + \varepsilon$ for every $y \in C \cap N_{x\restriction i}$. Then

$$\int_{C\cap N_{x\restriction i}} g\, d\mu \leq (v_i^{x\restriction i} + \delta/2^i + \varepsilon)\, \mu(N_{x\restriction i}),$$

contradicting what we have just proved.

Since $x = \lim_i y_i$, we have that $x \in C$ and so that $g(x) = \lim_i g(y_i)$. Let j be such that

$$(\forall i \geq j)\, |g(x) - g(y_i)| < \varepsilon/2.$$

Thus

$$(\forall i \geq j)\, g(x) > g(y_i) - \varepsilon/2 > v_i^{x\restriction i} + \delta/2^i + \varepsilon/2.$$

Therefore

$$g(x) \geq \limsup_i v_i^{x\restriction i} + \varepsilon/2 = g(x) + \varepsilon/2.$$

This contradiction completes the proof of the lemma. ⊣

LEMMA 1.5. *The value of* τ *in* $\Gamma(f)$ *is* $\leq v + \delta$.

PROOF. By the fact that $g(x) \geq f(x)$ for all x and by Lemma 1.4, the value of τ in $\Gamma(f)$ is $\leq$ the value of τ in $\Gamma(g)$, which is $\leq v + \delta$. ⊣

Using the fact that δ was an arbitrary positive number and dropping our assumption about τ^*, we get the following.

LEMMA 1.6. *If* II *has a winning strategy for* G_v, *then* $\mathrm{val}^\uparrow(\Gamma(f)) \leq v$.

THEOREM 1. *If* G_v *is determined for every* $v \in (0, 1]$, *then* $\Gamma(f)$ *is determined.*

PROOF. Assume that all the G_v are determined. Let w be the least upper bound of all the v such that I has a winning strategy for G_v. By Lemma 1.3, $\mathrm{val}_\downarrow(\Gamma(f)) \geq w$. By Lemma 1.6, $\mathrm{val}^\uparrow(\Gamma(f)) \leq w$. Thus $\mathrm{val}(\Gamma(f)) = w$. ⊣

We close the section by indicating how to extend its results to stochastic games. In doing so we are reporting an observation of Maitra and Sudderth.

Stochastic games $\tilde{\Gamma}$ are played like Blackwell games, except that each pair of moves of I and II is followed by a move of a third player, whom we shall call Nature. We shall restrict ourselves to the case that Nature has a countable set of legal moves in each postion in which she must make a move. Payoff functions for $\tilde{\Gamma}$ are functions of the entire play, including Nature's moves. The analogue of $\Gamma(f)$ is $\tilde{\Gamma}(\tilde{f}, \rho)$ where $\tilde{f}$ is a payoff function and ρ is a mixed strategy for Nature. If σ and τ are strategies for I and II respectively, then σ, τ, and ρ give a probability measure $\mu_{\sigma,\tau,\rho}$ on the set of plays of $\tilde{\Gamma}$. Using this measure, we define $E_{\sigma,\tau,\rho}(\tilde{f})$, $E^-_{\sigma,\tau,\rho}(\tilde{f})$, $E^+_{\sigma,\tau,\rho}(\tilde{f})$, $\mathrm{val}_\downarrow(\tilde{\Gamma}(\tilde{f}, \rho))$, $\mathrm{val}^\uparrow(\tilde{\Gamma}(\tilde{f}, \rho))$, *determinacy* of $\tilde{\Gamma}(\tilde{f}, \rho)$, and the *value* of $\tilde{\Gamma}(\tilde{f}, \rho)$ in the obvious way.

Fix $\tilde{\Gamma}$ with no terminal positions. Fix $\tilde{f}$ and ρ. For $v \in (0, 1]$, let $\tilde{G}_v$ be the perfect information game played as follows. Set $p_0 = \emptyset$. I's moves are functions h_0, $h_1, \ldots$ and II's moves are positions $p_1, p_2, \ldots$ in $\tilde{\Gamma}$. For each i, h_i is a function into $[0, 1]$ from the set of all length $2i + 2$ extensions of p_i, which has length $2i$. Let $v_0 = v$ and for $i \geq 0$ let $v_{i+1} = h_i(p_{i+1})$. For each i, let $\tilde{\Delta}_i$ be the game in which, starting at p_i, the two players and then Nature make legal moves in $\tilde{\Gamma}$. The final requirement on h_i is that

$$\mathrm{val}(\tilde{\Delta}_i(h_i, \rho_{p_i})) \geq v_i,$$

where ρ_{p_i} is the strategy for Nature for $\tilde{\Delta}_i$ that is given by ρ. The final requirement on p_{i+1} is that $v_{i+1} > 0$.

For positions p^* in $\tilde{G}_v$, define $\pi(p^*)$, a position in $\tilde{G}$ of length $2\,\mathrm{length}(p^*)$, in the obvious way. Also call π the function induced by π from plays of $\tilde{G}_v$ to plays of $\tilde{\Gamma}$. A play x^* is a win for I if and only if $\limsup_i v_i \leq \tilde{f}(\pi(x^*))$.

The constructions, lemmas, and proofs of the earlier part of this section adapt in obvious ways to $\tilde{G}_v$ and $\tilde{\Gamma}(\tilde{f}, \rho)$. (The first draft of our paper had a slightly different definition of the function h on page 1572. Maitra and Sudderth remarked that the current definition, unlike the original one, would work for stochastic games.) Thus we get the following theorem.

THEOREM 2. *If $\tilde{G}_v$ is determined for every $v \in (0, 1]$, then $\tilde{\Gamma}(\tilde{f}, \rho)$ is determined.*

For more details, see [8]. There Maitra and Sudderth adapt our proof to demonstrate the determinacy of a wider class of stochastic games. They work in the context of finitely additive probabililty measures, removing the restrictions that I and II choose their moves from finite sets and that Nature's moves are chosen from countable sets.

§2. **Variations.** In this section, we consider some variants of the games G_v of §1. We leave variants of the games $\tilde{G}_v$ to the reader.

If we replace "lim sup" by "lim inf" in stating the winning condition for G_v, then our proofs still go through, provided that we everywhere replace "lim sup" by "lim inf."

For $v \in (0, 1]$, let $\bar{G}_v$ differ from G_v only in an additional requirement that all values $h_i(q)$ be rational. Our proofs go through unchanged for $\bar{G}_v$ in place of G_v. We state this formally as the following theorem.

THEOREM 3. *If $\bar{G}_v$ is determined for every real (indeed, for every rational) $v \in (0, 1]$, then $\Gamma(f)$ is determined.*

If f is the characteristic function of a set, then there is a modification of the games G_v that gives our results with slightly simpler proofs. For the rest of this section, let A be any set of plays of Γ. For $v \in (0, 1]$ let G'_v be played exactly as is G_v, but let a play x^* of G'_v be a win for I if and only if $\pi(x^*) \in A$.

Suppose first that σ^* is a winning strategy for I for G'_v. Define σ, *acceptable* positions, and ψ, exactly as in §1.

Let C_1 be the closed set of all plays of Γ that contain only acceptable positions. Since $x = \pi(\psi(x))$ for $x \in C_1$, $C_1 \subseteq A$.

LEMMA 2.1. *For any strategy τ for II for Γ, $\mu_{\sigma,\tau}(C_1) \geq v$.*

PROOF. Given τ, assume that $\mu_\sigma(C_1) < v$. It follows that there is a closed set C disjoint from C_1 such that $\mu_{\sigma,\tau}(C) > 1 - v$. By a construction like that in the proof of Lemma 1.1, there is an $x \in C_1$ such that, for all i, $\mu_{\sigma,\tau}(C \cap N_{x\restriction i}) > 1 - v_i^{x\restriction i}$. But this is a contradiction, for such an x must belong to $C_1 \cap C$. ⊣

LEMMA 2.2. *The value of σ in $\Gamma(\chi(A))$ is $\geq v$.*

PROOF. The lemma follows from Lemma 2.1.

Here is a direct proof of the lemma. For each i, consider the game Γ^i which is played in the same way as Γ except that play terminates when the position p has length i. For plays p of Γ^i, let

$$h^i(p) = \begin{cases} v_i^p & \text{if } p \text{ is acceptable;} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to prove by induction on i that the value of the appropriate fragment σ^i of σ in $\Gamma^i(h^i)$ is $\geq v$. Thus the value of σ^i in $\Gamma^i(\chi(C_1^i))$ is $\geq v$, where C_1^i is the set of all acceptable plays of Γ^i. Thus the value of σ in $\Gamma(\chi(C_1))$ is $\geq v$. ⊣

Dropping our assumption about σ^*, we get the following.

Lemma 2.3. *If* I *has a winning strategy for* G_v', *then* $\mathrm{val}_\downarrow(\Gamma(\chi(A))) \geq v$.

Now assume that τ^* is a winning strategy for II for G_v'. Let $\delta > 0$. Define τ, *acceptable* positions, and ψ, exactly as in §1.

Let C_2 be the closed set of all plays of Γ that contain only acceptable positions. Since $x = \pi(\psi(x0)$ for $x \in C_2$, $C_2 \cap A = \emptyset$.

Lemma 2.4. *For any strategy* σ *for* I *for* Γ, $\mu_{\sigma,\tau}(C_2) \leq v + \delta$.

Lemma 2.5. *The value of* τ *in* $\Gamma(\chi(A))$ *is* $\leq v + \delta$.

Proof. The lemma follows from Lemma 2.4.

There is also a direct proof of the lemma, analogous to the direct proof of Lemma 2.2. ⊣

Dropping our assumption about τ^*, we get the following.

Lemma 2.6. *If* II *has a winning strategy for* G_v', *then* $\mathrm{val}^\uparrow(\Gamma(\chi(A))) \leq v$.

Theorem 4. *If* G_v' *is determined for every* $v \in (0, 1]$, *then* $\Gamma(\chi(A))$ *is determined.*

The proof of Theorem 4 gives the following stronger result.

Theorem 5. *Assume that all* G_v' *are determined. Then*

$$\mathrm{val}(\Gamma(\chi(A))) = \sup\{\, \mathrm{val}(\Gamma(\chi(C))) \mid C \text{ closed and } C \subseteq A \,\}.$$

Proof. Let $v < \mathrm{val}(\Gamma(\chi(A)))$. Let σ^* be a winning strategy for I for G_v'. Let σ be the strategy defined from σ^* as above. Let C be the set C_1 defined just before the statement of Lemma 2.1. The proofs of Lemma 2.2 both show that the value of σ in $\Gamma(\chi(C))$ is $\geq v$. ⊣

Remarks.

(a) For $\mathbf{F}_{\sigma\delta}$ sets A, Vervoort [18] directly proves a strengthening of the conclusion of Theorem 5. He conjectures that the conclusion of Theorem 5 holds for all Borel sets A. Since the hypothesis of Theorem 5 holds for Borel A, his conjecture is confirmed.

(b) Applying Theorem 5 to the complement of A, we see that the theorem's hypothesis implies that

$$\mathrm{val}(\Gamma(\chi(A))) = \inf\{\, \mathrm{val}(\Gamma(\chi(B))) \mid B \text{ open and } B \supseteq A \,\}.$$

One can also get this conclusion directly from the proofs of Lemma 2.5.

(c) There is an analogous strengthening of Theorem 1. If all G_v are determined, then $\mathrm{val}(\Gamma(f))$ is the supremum of the $\mathrm{val}(\Gamma(g))$ for functions $g \leq f$ that are lim sup's of functions defined on positions in Γ.

(d) For Borel sets A, Maitra, Purves, and Sudderth [7] show that the determinacy of $\Gamma(\chi(A))$ implies the conclusion of Theorem 5. As mentioned in (a) above, their result follows *a fortiori* from Theorem 5 and the determinacy of all Borel perfect information games. It is not true that for arbitrary A the determinacy of $\Gamma(\chi(A))$ implies the conclusion of Theorem 5. For a counterexample, see the next-to-last paragraph of the present paper. The last paragraph of the paper also discusses issues related to the theorem of [7].

Let $\bar{G}'_v$ be like G'_v except that all $h_i(q)$ must be rational.

THEOREM 6. *If $\bar{G}'_v$ is determined for every rational $v \in (0,1]$, then $\Gamma(\chi(A))$ is determined.*

THEOREM 7. *Assume that $\bar{G}'_v$ is determined for every rational $v \in (0,1]$. Then*

$$\mathrm{val}(\Gamma(\chi(A))) = \sup\{\, \mathrm{val}(\Gamma(\chi(C))) \mid C \text{ closed} \wedge C \subseteq A\,\}.$$

Combining the proof of Theorem 4 with a proof of Vervoort [18] that Blackwell determinacy implies that all sets are Lebesgue measurable, we get, on eliminating the Blackwell game, a new way to deduce Lebesgue measurability from the determinacy of perfect information games.

For convenenience, we think of Lebesgue measure as being the coin-flipping measure on $2^{\mathbb{N}}$.

For the rest of this section, let $B \subseteq 2^{\mathbb{N}}$.

Let H_v be played as follows:

$$\begin{array}{lcccccccc} \mathrm{I} & h_0 & & h_1 & & h_2 & & \dots & \\ \mathrm{II} & & p_1 & & p_2 & & p_3 & & \dots \end{array}$$

Set $p_0 = \emptyset$. For $i \geq 1$, p_i must be a sequence of 0's and 1's of length i. It is required that $p_0 \subseteq p_1 \subseteq \cdots$. For each i, h_i must be a function into $[0,1]$ from $\{p_i{}^\frown\langle 0\rangle, p_i{}^\frown\langle 1\rangle\}$. Let $v_0 = v$ and for $i \geq 0$ let

$$v_{i+1} = h_i(p_{i+1}).$$

The final requirement on h_i is that

$$\tfrac{1}{2}h_i(p_i{}^\frown\langle 0\rangle) + \tfrac{1}{2}h_i(p_i{}^\frown\langle 1\rangle) \geq v_i.$$

The final requirement on p_{i+1} is that $v_{i+1} > 0$.

For any play x^* of H_v, let $\pi(x^*)$ be the member of $2^{\mathbb{N}}$ extending all the p_i. The play x^* is a win for I if and only if $\pi(x^*) \in B$.

THEOREM 8. *If H_v is determined for every v, then B is Lebesgue measurable.*

PROOF. Analogues of Lemmas 2.3 and 2.6 give that the inner measure of B is $\geq v$ if I has an winning strategy for H_v and that the outer measure of B is $\leq v$ if II has a winning strategy for H_v. ⊣

Let $\bar{H}_v$ be like H_v except that all $h_i(q)$ must be rational.

THEOREM 9. *If $\bar{H}_v$ is determined for every rational $v \in (0,1]$, then B is Lebesgue measurable.*

Our definition of Blackwell games requires that each player has only finitely many legal moves in each position. We could relax this requirement by demanding only that, in each position, each player has only countably many legal moves and at least one of the players has only finitely many legal moves. All our determinacy proofs would still work for this more general concept. Some proofs would change in a very minor way, because 1-move games would no longer have optimal strategies. Of course, one could generalize further by allowing positions in which one or the other player makes a move alone, from a countable set of possibilities.

§3. Consequences and open problems. In §1 and §2, we dealt only with Blackwell games $\Gamma(f)$ such that all plays of Γ are infinite and such that the range of f is a subset of $[0, 1]$ (though we made no real use of the former hypothesis). It is clear that our proofs work with only trivial modifications for general Blackwell games. We shall therefore cite the theorems of the earlier sections as if they were their generalizations.

THEOREM 10. *All Borel Blackwell games are determined.*

PROOF. For Borel measurable f, the games G_v and $\bar{G}_v$ have Borel payoff sets. By [12] or [13], Borel games of perfect information are determined. ⊣

THEOREM 11. *All Borel stochastic games (of the kind defined in §1) are determined.*

PROOF. This follows from Theorem 2 and Borel perfect information determinacy. ⊣

As we said earlier, it was Maitra and Sudderth who noticed that our methods yield Theorem 11, and in [8] they prove a more general version of it.

Borel perfect information determinacy for the case of countable game trees can be stated in, for example, formal second order number theory. The same is true of Borel Blackwell determinacy. Results of Friedman [4] show, in a technical sense, that Borel perfect information determinacy cannot be proved without appealing to uncountably many uncountable cardinal numbers. Indeed, for every new level of the Borel hierarchy beyond the third level, a new cardinal number is needed. Thus it is of interest that the proof of Theorem 10 goes through in second order number theory and that the proof is "local," i.e., Blackwell determinacy for a given Borel level needs only perfect information Borel determinacy for the same level.

THEOREM 12. *Work in formal second order number theory. Let α be a countable ordinal. Assume that, for countable game trees, every $\mathbf{\Pi}^0_\alpha$ perfect information game is determined. Then every $\mathbf{\Pi}^0_\alpha$ Blackwell game is determined. (For what we mean by a "$\mathbf{\Pi}^0_\alpha$ Blackwell game," see page 1568.)*

PROOF. If $\Gamma(f)$ is $\mathbf{\Pi}^0_\alpha$, then the games $\bar{G}_v$ are $\mathbf{\Pi}^0_\alpha$ as long as $\alpha > 2$. ⊣

REMARKS.

(a) For all $\alpha \geq 1$, the games G'_v and $\bar{G}'_v$ are $\mathbf{\Pi}^0_\alpha$ if the set A is $\mathbf{\Pi}^0_\alpha$.

(b) Theorem 12 holds for the stochastic games defined in §1, since the proof of Theorem 11 is local in the same way as the proof of Theorem 10.

Going beyond the Borel sets, we can derive from the results of §1 and §2 local results for pretty much any natural classes. Here are just two examples. Projective perfect information determinacy for countable game trees implies projective Blackwell determinacy. For each positive integer n, $\mathbf{\Sigma}^1_n$ perfect information determinacy implies $\mathbf{\Sigma}^1_n$ Blackwell determinacy.

As we have already said, the determinacy of many classes of perfect information games has been deduced from so-called large cardinal axioms. With the aid of our theorem, we get corresponding determinacy results for Blackwell games. For example, for all $n \geq 0$, Σ^1_{n+1} Blackwell determinacy follows from the existence of n Woodin cardinals with a measurable cardinal above them.

Vervoort [18] introduces the *Axiom of Determinacy for Blackwell Games* (AD-Bl), the assertion that all Blackwell Games are determined. He shows that AD-Bl, like AD, contradicts the Axiom of Choice. He deduces from AD-Bl an important known consequence of AD: that all sets of reals are Lebesgue measurable.

Itay Neeman pointed out to us that there are several variants of AD-Bl that are not obviously equivalent to one another. One could restrict to games of the form $\Gamma(\chi(A))$. Whether or not one did this, one could require that each player has exactly 2 legal moves in each position, or one could replace 2 by some other number n. In the opposite direction, one could allow that in each position one of the players has countably infinitely many legal moves. We know of no simple argument that any two of the possible versions of AD-Bl are equivalent. Nonetheless, it can be shown that they are all equivalent. The games $\bar{G}_v$ of §2 can easily be turned into equivalent games in which only two legal moves are available to each player in each position. Our proofs adapt to show that the mixed strategy determinacy of these games is enough to yield the determinacy of the given game $\Gamma(f)$.

Vervoort [18] asks whether either of AD and AD-Bl implies the other. Our results obviously give an implication in one direction.

THEOREM 13. *Work in* ZF *without the Axiom of Choice.* AD *implies* AD-Bl.

What about the converse? We mention three routes one might take in trying to prove it and in trying to prove that forms of Blackwell determinacy consistent with the Axiom of Choice imply the corresponding forms of perfect information determinacy.

The first route is an indirect one. A consequence of AD-Bl is that every perfect information game is determined in the sense of mixed strategies. Many of the proofs of consequences of perfect information determinacy still work if the existence of mixed strategies replaces that of pure strategies. Among the consequences of perfect information determinacy is the existence of good inner models for various large cardinal axioms. Many of the proofs of perfect information determinacy from large cardinal axioms need as hypotheses only the existence of good inner models of the large cardinal axioms. In this way one often gets the equivalence of forms of determinacy and the existence of good inner models of large cardinal axioms. These facts provide a method for proving perfect information determinacy from Blackwell determinacy. A sample theorem that can be proved in this way is the following. Σ^1_1 Blackwell determinacy—even just for games of the form $\Gamma(\chi(A))$—implies Σ^1_1 perfect information determinacy. We suspect that virtually every form of perfect information determinacy that has been related to large cardinals can be shown in this way to follow from the corresponding form of Blackwell determinacy. In particular, we suspect that one can show that the consistency of AD-Bl implies that consistency of AD.

The second route is the direct route. The most direct way to proceed would be to show that any countable-tree perfect information game that is determined in the sense of mixed strategies is determined in the sense of pure strategies. Unfortunately, this is false, as the following example of Greg Hjorth shows. Let A be any uncountable subset of $2^{\mathbb{N}}$ such that $\mu(A) = 0$ for every atomless Borel probability measure μ. (For example, let the members of A code wellorderings, exactly one of the order type of each countable ordinal.) Consider the game $G^*(A)$ defined on page 149 of [6]. Player II has a mixed strategy whose value in $G^*(A)$ is 1: in each position, assign $1/2$ to each of the two legal moves. But II has no winning pure strategy. (See part (ii) of Theorem 21.1 of [6].) This counterexample does not destroy all branches of the direct route. For example, Vervoort's theorem lets one assume that all sets are Lebesgue measurable, and this rules out counterexamples of the kind described in parentheses above. Moreover, although mixed strategy determinacy for a perfect information game does not imply pure strategy determinacy, there are useful constraints on the values of perfect information games. We have been able to prove that the upper or lower value (in the mixed strategy sense) of a perfect information game in our sense (i.e., one whose winning condition is given by a set of plays) is either 0 or 1.

Theorems 5 and 7 and the related result for general payoff functions give a strong version of Blackwell determinacy. It is easy to see that this strong version implies, level by level, perfect information determinacy. Thus another route to our goal would be to show that Blackwell determinacy implies strong Blackwell determinacy. As we mentioned earlier, Maitra, Purves, and Sudderth [7] have shown that, for Borel A, the determinacy of $\Gamma(\chi(A))$ implies the strong determinacy of $\Gamma(\chi(A))$. Hjorth's example given in the preceding paragraph shows that, under Choice, this is not true for arbitrary A. Nevertheless, their proof may still be relevant. That proof uses the fact that Borel sets are universally measurable. The proof of the Lebesgue measurability result of Vervoort [18] shows that the universal measurability of a set follows from the Blackwell determinacy of sets of about the same complexity. The additional fact about Borel sets used in the proof of [7] is their universal capacitability. This does not generalize to all sets under AD-Bl, for there exist even Π^1_1 sets that are not universally capacitable. But to prove that AD-Bl implies strong AD-Bl it would be enough to prove from AD-Bl that all sets are capacitable with respect to the capacities of [7]. See Section 30 and Exercises 36.22 and 39.14 of [6] for some of the capacitability consequences of perfect information determinacy, consequences that are due independently to Busch, Shochat, and Mycielski.

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