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NOTE

ON THE FINITE CONTAINMENT PROBLEM FOR PETRI NETS

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Abstract. We prove that the Finite Containment Problem (FCP) for Petri nets is DTIME(Ackermann) complete for the reducibility $\leqslant_T^P$, thus sharpening previous results due to McAloon (1984) and to Mayr and Meyer (1981). Our principal technique is to replace an application of the infinite Ramsey Theorem by a certain finite Ramsey Theorem previously studied by Paris (1980) and by Ketonen and Solovay (1981). Such techniques may have further applications in obtaining upper bounds for combinatorial problems.

Key Words. Petri net, vector addition set, finite containment problem, Ackermann function, Ramsey's Theorem.

Introduction

Since it is well known (see [4]) that Petri nets and vector addition systems are equivalent, we choose to use the latter, notationally more elegant notion. As usual, $\mathbb{N}$ denotes the set of nonnegative integers and $\mathbb{Z}$ the set of rational integers. A *k-dimensional vector addition system* or k-VAS is an ordered pair (v, A), where $v \in \mathbb{N}^k$ and A is a finite subset of $\mathbb{Z}^k$. A VAS is a k-VAS for some k.

The *reachability set* $R(v, A)$ is the collection

$$\{w \in \mathbb{N}^k : \text{there exist } n \text{ in } \mathbb{N} \text{ and } w_1, \ldots, w_n \text{ in } A \text{ such that for all } m \leqslant n \\ v + \textstyle\sum_{i=1}^m w_i \in \mathbb{N}^k \text{ and } w = v + \sum_{i=1}^n w_i\}.$$

Thus, a vector w is reachable from (v, A) exactly when there is a path from v to w lying entirely within the positive 'orthant'. The Boundedness Problem (BP) is to determine, given (v, A), whether $R(v, A)$ is finite.

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1. Ramsey's Theorem

For a review on Ramsey's work, we refer to [3].

Whenever the complete graph on the collection $\mathbb{N}$ of vertices is colored with m colors, there exists an infinite subset X of $\mathbb{N}$ of vertices whose induced subgraph is monochromatic. Following Erdös–Rado's notation, this is denoted

$$\mathbb{N} \to (\mathbb{N})^2_m.$$

We give the following proof to illustrate a classical application of Ramsey's Theorem. If $w, w' \in \mathbb{N}^k$, then $w \leqslant w'$ if $\forall i < k[w(i) \leqslant w'(i)]$, and $w < w'$ if $w \leqslant w'$ but $w' \not\leqslant w$.

Fact 1.1 ([5]). BP *is decidable.*

Proof. Given a k-VAS (v, A), associate the Karp–Miller covering tree T, defined inductively. Let v be the *root of* T. If w is a non-infinite node belonging to T and a belongs to A, then $w+a$ belongs to T and is an *immediate successor of* w provided that

(i) there is no predecessor of w equal to w,

(ii) $w+a \in \mathbb{N}^k$.

Furthermore, $w+a$ is an *infinite node* if there is a predecessor r of $w+a$ satisfying $r < w+a$.

Claim 1.2. *$R(v, A)$ is infinite if and only if T contains an infinite node.*

Proof. ($\Leftarrow$): Suppose $r < r_1$, r_1 is an infinite node, and

$$r_1 = r + \sum_{i=1}^{m} w_i.$$

Let $r_k = r + k \sum_{i=1}^{m} w_i$. Then each r_k is reachable from (v, A) and so $R(v, A)$ is infinite.

($\Rightarrow$): Suppose that $R(v, A)$ is infinite but that T contains no infinite node. Then it is clear that every element of $R(v, A)$ is in T, so T is an infinite, finite branching tree (finiteness of the addition set is used here). By König's Infinity Lemma, there is an infinite branch

$$B = \{v_0, v_1, \ldots\}$$

of T, where $v_0 = v$ and v_i is on the ith level of the tree. Define the $k+1$ coloring of the complete graph of $\mathbb{N}$ by assigning, for $a < b$,

$$F(a, b) = \begin{cases} k & \text{if } v_a \leqslant v_b, \\ i & \text{if } \forall j < i[v_a(j) \leqslant v_b(j) \text{ and } v_a(i) > v_b(i)]. \end{cases}$$

By Ramsey's Theorem, let $X = \{a_0, a_1, \ldots\}$ be an infinite monochromatic set. If X has color k, then $v_{a_0} \leqslant v_{a_1} \leqslant v_{a_2}$, contradicting the construction principle for the tree T. If x has a color $i < k$, then $v_{a_0}(i) > v_{a_1}(i) > \cdots$ yielding an infinite decreasing sequence of integers. Thus our original hypothesis is false, and so T contains an infinite node. $\square$

Proof of Fact 1.1 (*continued*). By König's Infinity Lemma and the definition of the tree T, it is clear that T is finite. Now, by exhaustively searching for an infinite node, one can decide BP. □

We remark that the proof in [5] does not use Ramsey's Theorem but rather a trivial combinatorial property which was 'finitized' in [7]. Note that Rackoff [10] has shown that BP is in $\mathrm{DSPACE}(2^{cn\log(n)})$. The Containment Problem (CP) is to determine, given two VAS (v, A) and (w, B) whether $R(v, A) \subseteq R(w, B)$. Rabin has shown that CP is undecidable (see [4]). The k-dimensional Finite Containment Problem or k-FCP is to determine, given two k-VAS (v, A) and (w, B) whose reachability sets are finite, whether $R(v, A) \subseteq R(w, B)$. When the dimension is arbitrary, we write FCP. By the proof of Fact 1.1, we have the following fact.

Fact 1.3 ([5]). *The* FCP *is decidable.*

However, we also have the following fact.

Fact 1.4 ([8]). *The* FCP *is not primitive recursively decidable.*

Our principal objective is to give a sharpening and a much easier proof (modulo work of Ketonen–Solovay [6]) of the following fact.

Fact 1.5 ([7], see also [2]). *For each k, the k-*FCP *admits a primitive recursive decision procedure, while* FCP *is primitive recursive in the Ackermann function.*

Following [6], let

$$f_1(x) := x + 1,$$

$$f_{n+1}(x) := f_n^{(x+1)}(x) := \underbrace{f_n(f_n(\ldots f_n(x)\ldots))}_{(x+1)\text{ many times}},$$

$$\mathrm{Ack}(n) := f_n(n).$$

For $k \geqslant 3$, the Grzegorczyk class $\mathscr{E}^k$ is the closure of the successor, constant, projection functions, and f_k under substitution and limited recursion, while $\mathscr{E}^k_*$ is the class of relations whose characteristic function is in $\mathscr{E}^k$. It should be noted that McAloon's proof, though not ours, immediately yields the following fact, though not explicitly stated in [7].

Fact 1.6 ([7]). *k-FCP is in $\mathscr{E}^{k+2}_*$.*

This suggests the question whether Fact 1.6 is the best possible.

Our goal is to modify the simple proof of Karp–Miller's Fact 1.3 by substituting a certain finite version of Ramsey's Theorem in the place of the infinite Ramsey

Theorem. A finite set X of nonnegative integers is *large* if its cardinality is greater than or equal to its minimum element. Essentially, following the notation of [9],

$$[x, x+n] \overset{\to}{*} (3)^2_m$$

means that, for any coloring with m colors of the complete graph on the vertices $\{x, x+1, \ldots, x+n\}$, there is a *large* monochromatic set containing at least three elements. Following [6],

$$\sigma_{2,m}(x) = \text{least } y \text{ satisfying } [x, y] \overset{\to}{*} (3)^2_m.$$

Given a k-VAS (v, A), let

$$\text{Amax} := \max\{|w(i)| : w \in A,\ i < k\},$$

$$v\max := \max\{v(i) : i < k\},$$

$$x := x_{(v,A)} := \max\{\text{Amax}+1,\ v\max+2\}.$$

Let $h(\langle v, A\rangle) :=$ least n satisfying

$$[x, x+n] \overset{\to}{*} (3)^2_{k+2}$$

so that $h(\langle v, A\rangle) = \sigma_{2,k+2}(x) - x$.

Theorem 1.7. *With the preceding notations, if (v, A) is a k-VAS having finite reachability set, then the value $h(\langle v, A\rangle)$ is greater than or equal to the height of the Karp–Miller covering tree T associated with (v, A).*

Proof. We show that, in fact, $m := \lfloor h(\langle v, A\rangle)/\text{Amax} \rfloor$ is greater than or equal to the height of T. Suppose not, in order to obtain a contradiction. Let $B = \{v_0, v_1, \ldots, v_m\}$ be a branch of T, where v_i is at level i. Create 'Amax blocks' as follows

$$\underset{x}{|}\overbrace{\hspace{5em}}^{\text{Amax}}|\overbrace{\hspace{5em}}^{\text{Amax}}|\overbrace{\hspace{5em}}^{\text{Amax}}|\overbrace{\hspace{5em}}^{\text{Amax}}\hspace{2em}\underset{x+n}{|}$$

Let $|w| = \max\{|w(i)| : i < k\}$. Note that

$$|v_i| \leq v\max + i\,\text{Amax}. \tag{$*$}$$

For a in $[x, x+n]$, let

$$\text{index}(a) := \lfloor (a-x)/\text{Amax} \rfloor$$

so that index(a) indicates the Amax block in which it lies. Our coloring of the edges of the complete graph on $[x, x+n]$, analogous to the coloring in the proof of Fact 1.1, is given by the following. For $x \leq a < b \leq x+n$, let

$$F(a, b) = \begin{cases} k+1 & \text{if index}(a) = \text{index}(b), \\ k & \text{if not the case above and } v_{\text{index}(a)} \leq v_{\text{index}(b)}, \\ i & \text{where } \forall j < i[v_{\text{index}(a)}(j) \leq v_{\text{index}(b)}(j) \\ & \text{and } v_{\text{index}(a)}(i) > v_{\text{index}(b)}(i)]. \end{cases}$$

Now let $X = \{a_0, \ldots, a_{a_0-1}\} \subseteq [x, x+n]$ be a large monochromatic set having at least three elements.

Claim 1.8. *X is not colored $k+1$.*

Proof. $\text{Amax}+1 \leqslant x \leqslant a_0 \leqslant \text{card}(X)$. $\square$

Claim 1.9. *X is not colored k.*

Proof. If $a<b<c$ are in X and $F(a, b)=k=F(b, c)$, then $v_{\text{index}(a)} \leqslant v_{\text{index}(b)} \leqslant v_{\text{index}(c)}$, contradicting the definition of the covering tree. $\square$

Claim 1.10. *X is not colored i, for any $i<k$.*

Proof. If so, then

$$v_{\text{index}(a_0)}(i) > v_{\text{index}(a_0+1)}(i) > \cdots > v_{\text{index}(a_{a_0-1})}(i),$$

thus, $v_{\text{index}(a_0)}(i) \geqslant a_0-1$.

But by (*), it follows that

$$v\text{max}+\text{index}(a_0)\ \text{Amax} \geqslant |v_{\text{index}(a_0)}| \geqslant v_{\text{index}(a_0)}(i) \geqslant a_0-1$$

and since $x \geqslant v\text{max}+2$, it follows that

$$x+\text{index}(a_0)\ \text{Amax} \geqslant a_0+1.$$

However,

$$a_0 \geqslant x+\lfloor (a_0-x)/\text{Amax} \rfloor\ \text{Amax} = x+\text{index}(a_0)\ \text{Amax} \geqslant a_0+1$$

yielding a contradiction. $\square$

Proof of Theorem 1.7 (*continued*). Since X cannot be monochromatic, our hypothesis must be false and so the theorem is proved. $\square$

By [6, Theorem 6.8, p. 313] (see [6, Definition 2.9, p. 286] for a definition of $\gamma_{n,c}$), it follows that

$$\sigma_{2,m}(x) \leqslant f_{m+5}(x).$$

Corollary 1.11. *k-FCP is primitive-recursive and in fact in $\mathscr{E}_*^{k+7}$.*

Proof. k-FCP is clearly elementary in the function $h(\langle v, A\rangle) \leqslant f_{k+7}(x)$. $\square$

Corollary 1.12. FCP *is in* DTIME(Ack(n)).

Proof. Using depth-first search and pointers, straightforward analysis shows that computing whether $R(v, A) \subseteq R(w, B)$ for the two k-VAS with bounded reachability sets can be done in $\mathrm{DSPACE}(nh(2^n))$, where $\langle v, A\rangle \# \# \# \langle w, B\rangle$ is encoded by a word of length n. Now,

$$nh(2^n) \leqslant nf_{k+7}(2^n) \leqslant f_{k+9}(n)$$

and $\mathrm{DSPACE}(f_{k+9}(n)) \subseteq \mathrm{DTIME}(f_{k+10}(n))$. Looking closer at our encoding of $\langle v, A\rangle \# \# \# \langle w, B\rangle$, the input must be of the form

$$\tilde{v} \# \# \tilde{a}_1 \# \# \cdots \# \# \tilde{a}_r \# \# \# \tilde{w} \# \# \tilde{b}_1 \# \# \cdots \# \# \tilde{b}_s,$$

where, for instance,

$$\tilde{v} = \tilde{v}(0) \# \cdots \# \tilde{v}(k-1)$$

and $\tilde{v}(i)$ is the binary word corresponding to $v(i)$.

If n is the length of the encoding of $\langle v, A\rangle \# \# \# \langle w, B\rangle$, then

$$2(k+(k-1))+7 \leqslant n,$$

so $f_{k+10}(n) \leqslant f_n(n)$, except for finitely many cases.

Thus, FCP is in $\mathrm{DTIME}(\mathrm{Ack}(n))$. □

We now sharpen Fact 1.4.

Proposition 1.13. FCP *is hard for* $\mathrm{NTIME}(\mathrm{Ack}(n))$ *with respect to* $\leqslant^{\mathrm{P}}_{T}$.

Proof. By the Fundamental Theorem of [1]: There exists a positive interger m such that, for any nondeterministic Turing machine M bounded in time by $\mathrm{Ack}(n)$, there exists a polynomial $p_M(x, \boldsymbol{u})$ in $m+1$ variables and having *rational* integer coefficients such that

$$\forall x[\exists \boldsymbol{u} < 2^{2^{10\,\mathrm{Ack}(x)^2}}[p_M(x, \boldsymbol{u}) = 0] \Leftrightarrow M \text{ accepts } x].$$

Let $A := \{x : \exists \boldsymbol{u} < 2^{2^{10\,\mathrm{Ack}(x)^2}}[p_M(x, \boldsymbol{u}) = 0]\}$.

Then the complement

$$\mathbb{N} - A := \{x : \forall \boldsymbol{u} < 2^{2^{10\,\mathrm{Ack}(x)^2}}[(p_M(x, \boldsymbol{u}))^2 \geqslant 1]\}.$$

This is a special subcase of

$$\mathrm{BPI}^* := \{(n, q, r) : \forall \boldsymbol{u} \leqslant 2^{2^{10\,\mathrm{Ack}(x)^2}}[q(n, \boldsymbol{u}) \leqslant r(n, \boldsymbol{u})]\},$$

where q, r are polynomials with *nonnegative* integer coefficients.

Now, in [8, Fig. 1, p. 567], one may substitute a λ-WPNC to compute the function $2^{2^{10 f_n^2}}$ instead of A_n, and thus, as in [8], show that $A \leqslant^{\mathrm{P}}_{T} \mathrm{BPI}^* \leqslant^{\mathrm{P}}_{T}$ FPC. □

2. Summary and open questions

We have shown that the FCP is $\leqslant^{\mathrm{P}}_{T}$ complete for $\mathrm{DTIME}(\mathrm{Ack})$, thus sharpening results of McAloon and of Mayr and Meyer. While our technique does not provide the sharp result k-FCP $\in \mathscr{E}_*^{k+2}$, it is a simple modification of an easy application of the infinite Ramsey Theorem, and so may be applicable in other situations. Finally, it would be interesting to know whether k-FCP $\notin \mathscr{E}_*^{k+1}$ or whether it is complete for $\mathscr{E}_*^{k+2}$. As well, it is not hard to see that $\text{NTIME}(\text{Ack}(n)) \subseteq \text{DTIME}(\text{Ack}(n+1))$ and even that $\text{DTIME}(2(m, \text{Ack}(n))) \subseteq \text{DTIME}(\text{Ack}(n+1))$, where $2(0, x) = x$ and $2(m+1, x) = 2^{2(m,x)}$. Given this, what do complexity classes for such rapidly growing functions really mean?

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