Reductions via representation

Work in progress

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Reductions in the abstract

Let X be a set of objects, $A, B \subseteq X$ be two properties.

Does x satisfy A?

Does $f(x)$ satisfy B?

$$x \longrightarrow \boxed{\text{a simple method}} \longrightarrow f(x)$$

The motto is:

$$\text{complexity of a property} = \begin{array}{c}\text{difficulty of determining}\\ \text{when it holds.}\end{array}$$

Reductions in topological spaces

Let X be a topological space, $A, B \subseteq X$.

A is Wadge reducible to B, in symbols $A \leq_W B$, if there is a continuous function $f : X \to X$ such that $f^{-1}(B) = A$, or equivalently, for all $x \in X$

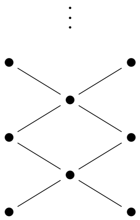
$$x \in A \longleftrightarrow f(x) \in B$$

[...] *continuous functions can for many reasons be considered as 'simple' or 'natural' and so 'easy to compute'.*

Bill Wadge, Phd Thesis, 1977.

Hierarchies?

On the Cantor space 2^ω,
the relation $\leq_W$ yields
a nice and useful hierarchy,
by results of Wadge, Martin, Monk,
Louveau, Duparc and others.

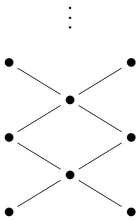


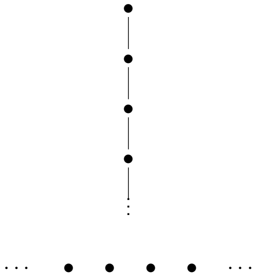
Thanks to a game theoretic
formulation of the reduction.

Hierarchies?

On the Cantor space 2^ω, the relation $\leq_W$ yields a nice and useful hierarchy, by results of Wadge, Martin, Monk, Louveau, Duparc and others.

On $\mathbb{R}$ and many other spaces, the relation $\leq_W$ yields no hierarchy at all, by results of Schlicht, Ikegami, Tanaka and others.





Thanks to a game theoretic formulation of the reduction.

No game theoretic formulation...

Another approach to reductions: Representability

Let X be an abstract topological space and $A \subseteq X$.

- A representation of a space X is a partial continuous and surjective map $\rho :\subseteq 2^\omega \to X$.
- A $p \in 2^\omega$ with $\rho(p) = x$ is a (ρ)-name for x.

Continuity means: for all name $p \in 2^\omega$ of some $x = \rho(p) \in X$, the larger n is, the best $(p_0, \ldots, p_n)$ approximates x.

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Theorem (Weihrauch, See also Schröder)

Every second countable T_0 space admits a unique (up to equivalence) best continuous representation, called admissible representation.

An continuous representation is admissible if it makes the membership problem in the codes for any subset as complicated as it can be.

Reductions in the codes

For an admissible representation $\rho :\subseteq 2^\omega$:

Definition

For $A, B \subseteq X$, an R-reduction of A to B is a partial function $f :\subseteq 2^\omega \to 2^\omega$ such that for every $p \in \operatorname{dom} \rho$

$$\rho(p) \in A \longleftrightarrow \rho \circ f(p) \in B.$$

Say that A is R-reducible to B, in symbols $A \leq_R B$, if there exists a continuous R-reduction of A to B.

- The notion of R-reducibility is actually independent of the chosen admissible representation.
- Two different names of the same point may be sent to names of different points, i.e. in general $\rho(p) = \rho(q) \not\Rightarrow \rho(f(p)) = \rho(f(q))$.

The R-reducibility relation

- The R-reducibility admits a game formulation.
- Borel determinacy can be used.

Some properties of $\leq_R$ in the framework of second countable T_0 spaces are:

- $A \leq_W B$ implies $A \leq_R B$.
- On 0-dimensional spaces, $\leq_W$ and $\leq_R$ coincide.
- Borel sets are well quasiordered by $\leq_R$, antichains are of length at most 2.
- On $\mathbb{R}$ and $\mathcal{P}\omega$, $\leq_W$ and $\leq_R$ are different.

This notion of reducibility was first studied in the particular case of the Scott Domain $\mathcal{P}\omega$ by A. Tang (1981), a student of Dana Scott.