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Linear complete differential resultants and the implicitization of linear DPPEs

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ABSTRACT

The linear complete differential resultant of a finite set of linear ordinary differential polynomials is defined. We study the computation by linear complete differential resultants of the implicit equation of a system of n linear differential polynomial parametric equations in $n-1$ differential parameters. We give necessary conditions to ensure properness of the system of differential polynomial parametric equations.

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1. Introduction

The implicitization problem of unirational algebraic varieties has been widely studied and the results on the computation of the implicit equation of a system of algebraic rational parametric equations by algebraic resultants are well known (Cox et al., 1997, 1998). This work was motivated by the paper (Gao, 2003), where computational issues, related to the implicitization problem of differential rational parametric equations, are treated by characteristic set methods, making use of the differential algebra theory developed by Ritt (1950) and Kolchin (1973). The paper by Gao (2003), establishes the background for the generalization to the differential case of the results in algebraic geometry on implicit and parametric representations of unirational varieties, conversion algorithms, etc (see, for instance, (Cox et al., 1997, 1998; Sendra et al., 2007)).

We explore the first steps of the generalization to the differential case of the results in algebraic geometry on implicit representations of unirational varieties. To be more precise, we are interested in finding a differential resultant that would solve the differential implicitization problem. The implicitization problem of differential rational parametric equations is a differential elimination

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problem. There is a wide range of applications of the differential elimination method to computer algebra and applied mathematics. For a survey on differential elimination techniques and their application to biological modeling we refer to Boulier (2007).

We defined the implicit equation of a system of n differential rational parametric equations in $n-1$ differential parameters in Rueda and Sendra (2008). In this paper, we study the computation by differential resultants of the implicit equation of a system $\mathcal{P}(X,U)$ of n linear differential polynomial parametric equations (linear DPPEs) in $n-1$ differential parameters $u_1,\dots,u_{n-1}$ (we give a precise statement of the problem in Section 2),

$$\mathcal{P}(X,U)=\begin{cases} x_1 &= P_1(U)\\ &\vdots\\ x_n &= P_n(U).\end{cases}$$

The differential resultant of a set of ordinary differential polynomials was introduced and studied in Carra'Ferro (1997a) for two differential polynomials and in Carra'Ferro (1997b) for a finite number n of differential polynomials in $n-1$ differential variables. The generalized differential resultant was also defined by Carra'Ferro (2005) for $n+s$ differential polynomials in $n-1$ differential variables, $s\geq 0$. Previous definitions of the differential resultant for differential operators are due to Berkovich and Tsirulik (1986) and Chardin (1991).

Let us consider the linear ordinary differential polynomials $F_i(X,U) = x_i - P_i(U)$ of order o_i, $i=1,\dots,n$. Let $L=\sum_{i=1}^{n}((\sum_{k=1}^{n} o_k)-o_i+1)$, as we will explain in Section 3.1 the differential resultant $\partial\mathrm{Res}(F_1,\dots,F_n)$ is the determinant of an $L\times L$ matrix $M(L)$. In Section 4, we prove that if $\partial\mathrm{Res}(F_1,\dots,F_n)\neq 0$ then the implicit equation of the system $\mathcal{P}(X,U)$ is given by the differential polynomial $\partial\mathrm{Res}(F_1,\dots,F_n)(X)$. Then we analyze some of the reasons for $\partial\mathrm{Res}(F_1,\dots,F_n)=0$.

The differential resultant $\partial\mathrm{Res}(F_1,\dots,F_n)$ is the Macaulay's algebraic resultant of a differential polynomial set PS with L elements. One reason for $\partial\mathrm{Res}(F_1,\dots,F_n) = 0$ is the following: if the polynomials in PS are not complete in all its variables then the resultant will be zero. If for some $j\in\{1,\dots,n-1\}$ the differential variable u_j has order less than o_i in F_i for all $i\in\{1,\dots,n\}$ then the matrix $M(L)$ has one or more columns of zeros. Let γ be the number of zero columns in $M(L)$ due to this reason. Let us see an illustrating example.

Example 1. Let us consider the following system of linear DPPEs

$$\begin{cases} x_1 &= u_1+u_2+u_{21}\\ x_2 &= tu_{11}+u_{22}\\ x_3 &= u_1+u_{21}\end{cases}$$

where $u_{jk}=\partial^k u_j/\partial t^k, j=1,2$ and $k\in\mathbb{N}$. The differential resultant of $F_1(X,U)=x_1-u_1-u_2-u_{21}$, $F_2(X,U)=x_2-tu_{11}-u_{22}$, $F_3(X,U)=x_3-u_1-u_{21}$ is zero because the order of u_1 in every polynomial is less than its order, $\gamma=1$. The implicit equation of this systems is $(t-1)x_{12}-tx_{31}-(t-1)x_{32}+x_2$, $x_{ik}=\partial^k x_i/\partial t^k, i=1,2,3, k\in\mathbb{N}$.

We define in Section 5 the linear complete differential resultant of a finite set of linear ordinary differential polynomials, generalizing Carra'Ferro's differential resultant in the linear case. We prove the next theorem in Section 5.

Theorem. *Given a system $\mathcal{P}(X,U)$ of differential polynomial parametric equations. If the linear complete differential resultant $\partial\mathrm{CRes}(F_1,\dots,F_n)\neq 0$ then the implicit equation of $\mathcal{P}(X,U)$ is $\partial\mathrm{CRes}(F_1,\dots,F_n)(X)=0$.*

In Section 7, we study whether the system $\mathcal{P}(X,U)$ of linear DPPEs is proper, problem closely related to the existence of inversion maps. If the system $\mathcal{P}(X,U)$ is not proper we prove that $\partial\mathrm{CRes}(F_1,\dots,F_n)=0$. The homogeneous part of the linear differential polynomials $F_1,\dots,F_n$ can be written in terms of differential operators. We obtain necessary conditions on these differential operators so that $\partial\mathrm{CRes}(F_1,\dots,F_n)\neq 0$.

Computations throughout this paper were carried out with our Maple implementation of functions to compute the differential resultant of Carra'Ferro (1997b) and the linear complete differential resultant defined in Section 5 of this paper. This is the tool used to perform our experiments and in particular the computations in the examples of this paper, our implementation is available at Rueda (2008).

The paper is organized as follows. In Section 2 we introduce the main notions and notation. Next we review the definition of the differential resultant defined by Carra'Ferro in Section 3 and we define the differential homogeneous resultant. In Section 4 we explain the computation of the implicit equation by Carra'Ferro's differential resultant. We define the linear complete differential resultant in Section 5. In Section 6 we give our main results on the implicitization of linear DPPEs by linear complete differential resultants. Our results on properness appear in Section 7. We finish the paper discussing the special cases $n = 2$ and $n = 3$ in Section 8.

2. Basic notions and notation

In this section, we introduce the basic notions related to the problem we deal with, as well as the notation and terminology used throughout the paper. For further concepts and results on differential algebra we refer to Kolchin (1973) and Ritt (1950).

Let $\mathbb{K}$ be an ordinary differential field with derivation ∂, (e.g. $\mathbb{Q}(t)$, $\partial = \frac{\partial}{\partial t}$). Let $X = \{x_1, \ldots, x_n\}$ and $U = \{u_1, \ldots, u_{n-1}\}$ be sets of differential indeterminates over $\mathbb{K}$. Let $\mathbb{E}$ be a universal extension field of $\mathbb{K}$ containing the set of indeterminates U. Let $\mathbb{N}_0 = \{0, 1, 2, \ldots, n, \ldots\}$. For $k \in \mathbb{N}_0$ we denote by x_{ik} the kth derivative of x_i. Given a set Y of differential indeterminates over $\mathbb{K}$ we denote by $\{Y\}$ the set of derivatives of the elements of Y, $\{Y\} = \{\partial^k y \mid y \in Y,\ k \in \mathbb{N}_0\}$, and by $\mathbb{K}\{X\}$ the ring of differential polynomials in the differential indeterminates $x_1, \ldots, x_n$, that is

$$\mathbb{K}\{X\} = \mathbb{K}[x_{ik} \mid i = 1, \ldots, n,\ k \in \mathbb{N}_0].$$

Analogously for $\mathbb{K}\{U\}$.

A system of differential rational parametric equations (system of DRPEs) is a system of the form

$$\mathcal{Q}(X, U) = \begin{cases} x_1 = \frac{P_1(U)}{Q_1(U)} \\ \quad\vdots \\ x_n = \frac{P_n(U)}{Q_n(U)} \end{cases} \tag{1}$$

where $P_1, \ldots, P_n, Q_1, \ldots, Q_n \in \mathbb{K}\{U\}$, Q_i are nonzero, not all $P_i, Q_i \in \mathbb{K}$, $i = 1, \ldots, n$ and $\gcd(P_i, Q_i) = 1$. We call the indeterminates U a set of parameters of $\mathcal{Q}(X, U)$; they are not necessarily independent. When all differential polynomials P_i, Q_i are of degree at most 1, we say that (1) is a linear system. Moreover, if all $Q_i \in \mathbb{K}$, we say that (1) is a system of differential polynomial parametric equations (system of DPPEs). Associated with the system (1) we consider the differential ideal (see (Gao, 2003), Section 3)

$$\mathrm{ID} = \{f \in \mathbb{K}\{X\} \mid f(P_1(U)/Q_1(U), \ldots, P_n(U)/Q_n(U)) = 0\},$$

and we call it the implicit ideal of the system (1). By Gao (2003), Lemma 3.1, the implicit ideal ID is a differential prime ideal. Moreover, given a characteristic set $\mathcal{C}$ of ID then $n- \mid \mathcal{C} \mid$ is the (differential) dimension of ID, by abuse of notation, we will also speak about the dimension of a DRPEs system meaning the dimension of its implicit ideal. The parameters U are independent if $\dim(\mathcal{Q}(X, U)) = |U|$ (see (Gao, 2003), Section 5).

The differential variety defined by

$$\mathrm{Zero(ID)} = \{\eta \in \mathbb{E}^n \mid \forall f \in \mathrm{ID}, f(\eta) = 0\}$$

is called the implicit variety of $\mathcal{Q}(X, U)$. If the parameters of $\mathcal{Q}(X, U)$ are not independent and $\mathbb{K} = \mathbb{Q}(t)$, there exists a set of new DRPEs with the same implicit variety as $\mathcal{Q}(X, U)$ but with independent parameters, (see (Gao, 2003), Theorem 5.1).

If $\dim(\mathrm{ID}) = n - 1$, then $\mathcal{C} = \{A(X)\}$ for some irreducible differential polynomial $A \in \mathbb{K}\{X\}$. The polynomial A is called a characteristic polynomial of ID. Furthermore, if B is another characteristic polynomial of ID then $A = bB$ with $b \in \mathbb{K}$.

We introduced the notion of implicit equation in Rueda and Sendra (2008) and we include it here for completion.

Definition 2. The *implicit equation* of a $(n-1)$-dimensional system of DRPEs, in n differential indeterminates $X = \{x_1, \ldots, x_n\}$, is defined as the equation $A(X) = 0$, where A is any characteristic polynomial of the implicit ideal ID of the system.

Throughout the remaining parts of this paper, we consider the linear system of DPPEs

$$\mathcal{P}(X, U) = \begin{cases} x_1 = P_1(U) \\ \quad \vdots \\ x_n = P_n(U). \end{cases} \tag{2}$$

So, $P_1, \ldots, P_n \in \mathbb{K}\{U\}$ with degree at most 1, and not all $P_i \in \mathbb{K}$, $i = 1, \ldots, n$.

Let $\mathbb{K}[\partial]$ be the ring of differential operators with coefficients in $\mathbb{K}$. For $i = 1, \ldots n$ and $j = 1, \ldots n-1$, there exist differential operators $\mathcal{L}_{ij} \in \mathbb{K}[\partial]$ and $a_i \in \mathbb{K}$ such that

$$P_i(U) = a_i - \sum_{j=1}^{n-1} \mathcal{L}_{ij}(u_j).$$

We define the differential polynomials

$$T_i(X) = x_i - a_i, \qquad H_i(U) = \sum_{j=1}^{n-1} \mathcal{L}_{ij}(u_j), \quad \text{and} \quad F_i(X, U) = T_i(X) + H_i(U).$$

Given $P \in \mathbb{K}\{X \cup U\}$ and $y \in X \cup U$, we denote by $\mathrm{ord}(P, y)$ the order of P in the variable y. If P does not have a term in y then we define $\mathrm{ord}(P, y) = -1$.

Remark 3. To ensure that the number of parameters is $n-1$, we assume that for each $j \in \{1, \ldots, n-1\}$ there exists $i \in \{1, \ldots, n\}$ such that the differential operator $\mathcal{L}_{ij} \neq 0$. That is, for each $j \in \{1, \ldots, n-1\}$ there exists $i \in \{1, \ldots, n\}$ such that $\mathrm{ord}(F_i, u_j) \geq 0$.

In this situation, the problem we deal with in this paper is: given a system $\mathcal{P}(X, U)$ of linear DPPEs, to compute an implicit equation using differential resultants. In Gao (2003), algorithmic methods for solving this problem in a more general case are presented in the language of characteristic sets. Our candidate to be the implicit equation of $\mathcal{P}(X, U)$ is the differential resultant of $F_1, \ldots F_n$ that we describe in the next section.

3. Differential resultants

Let $\mathbb{D}$ be a differential integral domain, and let $f_i \in \mathbb{D}\{U\}$ be an ordinary differential polynomial of order o_i, $i = 1, \ldots, n$. A *differential resultant* $\partial\mathrm{Res}(f_1, \ldots, f_n)$ of n ordinary differential polynomials $f_1, \ldots, f_n$ in $n-1$ differential variables $u_1, \ldots, u_{n-1}$ was introduced in Carra'Ferro (1997b). Such a notion coincides with the Macaulay's algebraic resultant (Macaulay, 1916) of the differential polynomial set

$$\mathrm{PS}(f_1, \ldots, f_n) := \left\{ \partial^{N-o_i} f_i, \ldots, \partial f_i, f_i \mid i = 1, \ldots, n, \quad \text{where } N = \sum_{i=1}^{n} o_i \right\}.$$

Now, let $h_i \in \mathbb{D}\{U\}$ be an ordinary differential homogeneous polynomial of order o_i, $i = 1, \ldots, n$. We define the *differential homogeneous resultant* $\partial\mathrm{Res}^h(h_1, \ldots, h_n)$ of n ordinary differential homogeneous polynomials $h_1, \ldots, h_n$ in $n-1$ variables as the Macaulay's algebraic resultant of the differential polynomial set

$$\mathrm{PS}^h(h_1, \ldots, h_n) := \left\{ \partial^{N-o_i-1} h_i, \ldots, \partial h_i, h_i \mid i = 1, \ldots, n, \quad \text{where } N = \sum_{i=1}^{n} o_i \right\}.$$

A differential homogeneous resultant was defined also by Carra'Ferro (1997a) for $n = 2$. In addition, when the homogeneous polynomials have degree one and $n = 2$ the differential homogeneous resultant coincides with the differential resultant of two differential operators defined by Berkovich and Tsirulik (1986) and studied also by Chardin (1991).

We implemented in Maple a package of functions that allows the computation of the differential resultant defined by Carra'Ferro (1997b), our implementation is available at Rueda (2008).

Differential resultants are Macaulay's algebraic resultants therefore with some previous computations, the implementation of the Macaulay's algebraic resultant (available at Minimair (2005)) could be also used to compute differential resultants.

3.1. Differential resultant of $F_1, \ldots, F_n$

The computation by means of determinants of differential resultants was presented in Carra'Ferro (1997b) and in Carra'Ferro (1997a). In Carra'Ferro (1997a) only the case $n = 2$ is treated but the computation is also shown for differential homogeneous resultants. We consider now the polynomials F_i and H_i, introduced in Section 2, and we set $\mathbb{D} = \mathbb{K}\{X\}$. In this section, we give details on the computation of $\partial\mathrm{Res}(F_1, \ldots, F_n)$ and $\partial\mathrm{Res}^h(H_1, \ldots, H_n)$, since they will be important tools in this paper. We think of $F_1, \ldots, F_n$ as polynomials in the $n-1$ variables $u_1, \ldots, u_{n-1}$ and coefficients in the differential domain $\mathbb{D}$; recall that $F_1, \ldots, F_n$ are of orders o_i and degree one. We review below the computation of $\partial\mathrm{Res}(F_1, \ldots, F_n)$ by means of determinants as in Carra'Ferro (1997b).

We define rankings on the sets of variables X and U so that the matrix we use to compute the differential resultant $\partial\mathrm{Res}(F_1, \ldots, F_n)$ equals the one used in Carra'Ferro (1997b).

- The order $x_n < \cdots < x_1$ induces a ranking on X (i.e. an order on $\{X\}$) as follows (see (Kolchin, 1973), page 75): $x < \partial x$ and $x < x^\star \Rightarrow \partial^k x < \partial^{k^\star} x^\star$, for all $x, x^\star \in X$, $k, k^\star \in \mathbb{N}_0$.
- The order $u_1 < \cdots < u_{n-1}$ induces an orderly ranking on U as follows (see (Kolchin, 1973), page 75): $u < \partial u$, $u < u^\star \Rightarrow \partial u < \partial u^\star$ and $k < k^\star \Rightarrow \partial^k u < \partial^{k^\star} u^\star$, for all $u, u^\star \in U$, $k, k^\star \in \mathbb{N}_0$. We set $1 < u_1$.

We call $\mathcal{R}$ the ranking on $X \cup U$ that eliminates X with respect to U, that is $\partial^k x > \partial^{k^\star} u$, for all $x \in X$, $u \in U$ and $k, k^\star \in \mathbb{N}_0$. Now, the set $\mathrm{PS} = \mathrm{PS}(F_1, \ldots, F_n)$ is ordered by $\mathcal{R}$. Note that, because of the particular structure of F_i, one has that:

$$F_n < \partial F_n < \cdots < \partial^{N-o_n} F_n < \cdots < F_2 < \partial F_2 < \cdots < F_1 < \cdots < \partial^{N-o_1} F_1.$$

That is, PS is a chain (see (Ritt, 1950), page 3) of differential polynomials $\{G_1, \ldots, G_L\}$ with $L = \sum_{i=1}^n (N - o_i + 1) = (n-1)N + n$; recall that $N = \sum_{i=1}^n o_i$.

Then, let $M(L)$ be the $L \times L$ matrix whose kth row contains the coefficients of the $(L-k+1)$th polynomial in PS, as a polynomial in $\mathbb{D}\{U\}$, and where the coefficients are written in decreasing order with respect to the orderly ranking on U. Hence, $M(L)$ is a matrix over $\mathbb{K}\{X\}$ that we call the differential resultant matrix of $F_1, \ldots, F_n$. In this situation:

$$\partial\mathrm{Res}(F_1, \ldots, F_n) = \det(M(L)).$$

Analogously, we use determinants to compute $\partial\mathrm{Res}^h(H_1, \ldots, H_n)$; recall that the homogeneous differential polynomial $H_i \in \mathbb{K}\{U\}$ has order o_i and degree one. Let $L^h = L - n$ and consider $\mathrm{PS}^h = \mathrm{PS}^h(H_1, \ldots, H_n)$ as the polynomial set obtained from PS by subtracting from the chosen polynomials its monomial in $\mathbb{D}$ (i.e. $x_i - a_i$), we maintain in PS^h the ordering inherited from PS. Let $M(L^h)$ be the $L^h \times L^h$ matrix whose $(L^h - k + 1)$th row contains the coefficients of the kth polynomial in PS^h, as a polynomial in $\mathbb{D}\{U\}$, and where the coefficients are written in decreasing order with respect to the orderly ranking on U. Hence, $M(L^h)$ is a matrix over $\mathbb{K}$ that we call the differential homogeneous resultant matrix of $H_1, \ldots, H_n$. In this situation:

$$\partial\mathrm{Res}^h(H_1, \ldots, H_n) = \det(M(L^h)).$$

Example 4. Let $\mathbb{K} = \mathbb{Q}(t)$, $\partial = \frac{\partial}{\partial t}$ and consider the set of differential polynomials in $\mathbb{K}\{x_1, x_2, x_3\}\{u_1, u_2\}$,

$$F_1(X, U) = x_1 - 7t - u_1 - 3u_{11} + 3u_2 - tu_{21}$$
$$F_2(X, U) = x_2 - u_1 + u_{12} - 5u_{22}$$
$$F_3(X, U) = x_3 - u_1 + u_{12} - t^2 u_{21}.$$

Then the set $\mathrm{PS}(F_1, F_2, F_3)$ contains $L = 13$ differential polynomials and $\partial\mathrm{Res}(F_1, F_2, F_3)$ equals $\det(M(L))$ where $M(L)$ is the following $L \times L$ coefficient matrix of $\mathrm{PS}(F_1, F_2, F_3)$.

$$\begin{bmatrix}
-t & -3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{14} \\
0 & 0 & -t & -3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & x_{13} \\
0 & 0 & 0 & 0 & -t & -3 & 1 & -1 & 0 & 0 & 0 & 0 & x_{12} \\
0 & 0 & 0 & 0 & 0 & 0 & -t & -3 & 2 & -1 & 0 & 0 & x_{11}-7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t & -3 & 3 & -1 & x_1-7t \\
-5 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & x_{23} \\
0 & 0 & -5 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & x_{22} \\
0 & 0 & 0 & 0 & -5 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & x_{21} \\
0 & 0 & 0 & 0 & 0 & 0 & -5 & 1 & 0 & 0 & 0 & -1 & x_2 \\
0 & 1 & -t^2 & 0 & -6t & -1 & -6 & 0 & 0 & 0 & 0 & 0 & x_{33} \\
0 & 0 & 0 & 1 & -t^2 & 0 & -4t & -1 & -2 & 0 & 0 & 0 & x_{32} \\
0 & 0 & 0 & 0 & 0 & 1 & -t^2 & 0 & -2t & -1 & 0 & 0 & x_{31} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -t^2 & 0 & 0 & -1 & x_3
\end{bmatrix}.$$

Let $M(L^h)$ be the square submatrix of size $L^h = 10$ of $M(L)$ obtained by removing columns 1, 2, 13 and rows 1, 6, 10. Then $\partial\mathrm{Res}^h(H_1, H_2, H_3) = \det(M(L^h))$.

We prove next some properties of $\partial\mathrm{Res}(F_1, \ldots, F_n)$ and $\partial\mathrm{Res}^h(H_1, \ldots, H_n)$ that will be used later in the paper.

Proposition 5. *If the system $\{H_1 = 0, \ldots, H_n = 0\}$ has a nonzero solution then $\partial\mathrm{Res}^h(H_1, \ldots, H_n) = 0$.*

Proof. Let $\mathbb{F}$ be a differential field extension of $\mathbb{K}$. Then every nonzero solution of $\{H_1 = 0, \ldots, H_n = 0\}$ in $\mathbb{F}^{n-1}$ is a nonzero solution of the system $\{\partial^{N-o_i-1}H_i = 0, \ldots, \partial H_i = 0, H_i = 0 \mid i = 1, \ldots, n\}$. If such a solution exists then the columns of $M(L^h)$ are linearly dependent on $\mathbb{F}$. □

Remark 6. For $n = 2$, $\partial\mathrm{Res}^h(H_1, H_2) = 0$ if and only if $\{H_1 = 0, H_2 = 0\}$ has a nonzero solution in $\mathbb{F}$, a differential field extension of $\mathbb{K}$ (see (Berkovich and Tsirulik, 1986), Theorem 3.1). Unfortunately, for $n > 2$ the condition $\partial\mathrm{Res}^h(H_1, \ldots, H_n) = 0$ is not sufficient for the existence of nonzero solutions of the system $\{H_1 = 0, \ldots, H_n = 0\}$. Let $n = 3$ and

$$H_1(U) = u_{11} + u_{21}, \qquad H_2(U) = u_1 + u_2, \qquad H_3(U) = u_1 + u_{11} + u_{21}.$$

The first two columns of $M(L^h)$ are equal and therefore $\partial\mathrm{Res}^h(H_1, H_2, H_3) = 0$. The system $\{H_1 = 0, H_2 = 0, H_3 = 0\}$ has only the zero solution.

We introduce some matrices that will be used in the next result and also in later results in the paper. Let S be the $n \times (n-1)$ matrix whose entry (i, j) is the coefficient of $u_{n-j\,o_i}$ in F_i, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n-1\}$. Let S_i be the matrix obtained by removing the ith row of S.

Remark 7. Note that the nonzero rows of the columns of $M(L)$ (resp. $M(L^h)$) corresponding to the coefficients of $u_{n-j\,N}$ (resp. $u_{n-j\,N-1}$), $j = 1, \ldots, n-1$ are the rows of S.

Let (PS) be the ideal in $\mathbb{K}[x_i, \ldots, x_{iN-o_i}, u_j, \ldots, u_{jN} \mid i = 1, \ldots n, j = 1, \ldots n-1]$ generated by PS. Observe that, if $\partial\mathrm{Res}(F_1, \ldots, F_n) \neq 0$ then it is a linear differential polynomial and by Carra'Ferro (1997b), Proposition 12 it belongs to $(\mathrm{PS}) \cap \mathbb{K}\{X\}$. The next theorem gives an explicit formula of $\partial\mathrm{Res}(F_1, \ldots, F_n)$ in terms of $\partial\mathrm{Res}^h(H_1, \ldots, H_n)$.

Theorem 8. *Let $P \in (\mathrm{PS}) \cap \mathbb{K}\{X\}$ be a linear differential polynomial such that for some $k \in \{1, \ldots, n\}$, $\mathrm{ord}(P, x_k) = N - o_k$. It holds*

$$\partial\mathrm{Res}(F_1, \ldots, F_n) = \frac{1}{\alpha}\det(S_k)\partial\mathrm{Res}^h(H_1, \ldots, H_n)P(X),$$

with $\alpha = (-1)^a \frac{\partial P}{\partial x_{k\,N-o_k}}$, $a \in \mathbb{N}$.

Proof. Given a linear differential polynomial $P \in (\mathrm{PS}) \cap \mathbb{K}\{X\}$ there exist $\mathcal{F}_1, \ldots, \mathcal{F}_n \in \mathbb{K}[\partial]$, $\deg(\mathcal{F}_i) \leq N - o_i$ such that $P(X) = \mathcal{F}_1(F_1(X,U)) + \cdots + \mathcal{F}_n(F_n(X,U))$. Then $P(X) = \mathcal{F}_1(T_1(X)) + \cdots + \mathcal{F}_n(T_n(X))$ and $\mathcal{F}_1(H_1(U)) + \cdots + \mathcal{F}_n(H_n(U)) = 0$. As a consequence, we can perform row operations on $M(L)$ to obtain a matrix of the kind

$$\begin{bmatrix} 0\ldots0 & 0\ldots0 & P(X) \\ & *\ldots* & * \\ S_k & \ddots & \vdots \\ & *\ldots* & * \\ 0\ldots0 & & * \\ \ddots & M(L^h) & \vdots \\ 0\ldots0 & & * \end{bmatrix}.$$

To be more precise, we reorder the rows of $M(L)$ so that the coefficients of $\partial^{N-o_k}F_k$ are in the first row, rows 2 to n are the rows containing the entries of the matrix S_k and rows $n+1$ to L are the rows containing the entries of $M(L^h)$. Then multiply the first row of the obtained matrix by $\frac{\partial P}{\partial x_{k\,N-o_k}} \neq 0$. Finally, replace the first row by the coefficients of $P(X)$ as a polynomial in $\mathbb{D}\{U\}$ written in decreasing order w.r.t. the ranking on U, that is, all zeros and the last entry equal to $P(X)$.

Let $\alpha = (-1)^a \frac{\partial P}{\partial x_{k\,N-o_k}}$, for some $a \in \mathbb{N}$ then

$$\alpha \det(M(L)) = \det(S_k)\partial\mathrm{Res}^h(H_1, \ldots, H_n)P(X). \quad \square$$

Let M_{L-1} be the $L \times (L-1)$ principal submatrix of $M(L)$. Let us consider the set $\mathcal{X} = \{x_i, \partial x_i, \ldots, \partial^{N-o_i}x_i \mid i = 1, \ldots, n\}$. Given $x \in \mathcal{X}$, say $x = x_{ik}$ with $k \in \{0, 1, \ldots, N - o_i\}$, let us call M_x the submatrix of M_{L-1} obtained by removing the row corresponding to the coefficients of $\partial^k F_i = x_{ik} + \partial^k(H_i(U) - a_i)$. Then, developing the determinant of $M(L)$ by the last column we obtain

$$\partial\mathrm{Res}(F_1, \ldots, F_n) = \sum_{i=1}^{n}\sum_{k=0}^{N-o_i} b_{ik}\det(M_{x_{ik}})(x_{ik} - \partial^k a_i), \tag{3}$$

with $b_{ik} = \pm 1$ according to the row index of $x_{ik} - \partial^k a_i$ in the matrix $M(L)$. Also, for every $i \in \{1, \ldots, n\}$, there exists $a \in \mathbb{N}$ such that

$$\det(M_{x_{iN-o_i}}) = (-1)^a \partial\mathrm{Res}^h(H_1, \ldots, H_n)\det(S_i). \tag{4}$$

Remark 9. If $\partial\mathrm{Res}^h(H_1, \ldots, H_n) \neq 0$ then by Remark 7 there exists $k \in \{1, \ldots, n\}$ such that $\det(S_k) \neq 0$. If $\partial\mathrm{Res}^h(H_1, \ldots, H_n) \neq 0$ then by formula (4) the following statements hold.

(1) For every $i \in \{1, \ldots, n\}$, $\det(M_{x_{iN-o_i}}) = 0$ if and only if $\det(S_i) = 0$.
(2) There exists $k \in \{1, \ldots, n\}$ such that $\det(M_{x_{kN-o_k}}) \neq 0$.

Proposition 10. *The following statements are equivalent.*

(1) $\partial\mathrm{Res}(F_1, \ldots, F_n) \neq 0$.
(2) $\partial\mathrm{Res}^h(H_1, \ldots, H_n) \neq 0$.
(3) $\mathrm{rank}(M_{L-1}) = L - 1$.

Proof. (1)⇒(2) Let $B = \partial\mathrm{Res}(F_1, \ldots, F_n) \neq 0$, then $\mathrm{ord}(B, x_i) \leq N - o_i$ for $i = 1, \ldots, n$ and as mentioned earlier B is a linear differential polynomial in $(\mathrm{PS}) \cap \mathbb{K}\{X\}$ so $B = \sum_{i=1}^{n}\sum_{k=0}^{\mathrm{ord}(B,x_i)} c_{ik}\partial^k F_i$, with $c_{ik} \in \mathbb{K}$. Given $\beta = \min\{N - o_i - \mathrm{ord}(B, x_i) \mid i = 1, \ldots, n\}$ then $\beta = N - o_l - \mathrm{ord}(B, x_l)$ for some $l \in \{1, \ldots, n\}$ and $P = \partial^\beta B$ verifies $\mathrm{ord}(P, x_l) = N - o_l$ and $P \in (\mathrm{PS}) \cap \mathbb{K}\{X\}$. By Theorem 8 then $\partial\mathrm{Res}^h(H_1, \ldots, H_n) \neq 0$.

(2)⇒(3) If $\partial\mathrm{Res}^h(H_1, \ldots, H_n) \neq 0$ then by Remark 9 there exists $k \in \{1, \ldots, n\}$ such that $\det(M_{x_{kN-o_k}}) \neq 0$ and hence $\mathrm{rank}(M_{L-1}) = L - 1$.

(3)⇒(1) If $\mathrm{rank}(M_{L-1}) = L - 1$ then there exists $x \in \mathcal{X}$ such that $\det(M_x) \neq 0$. By (3) then $\partial\mathrm{Res}(F_1, \ldots, F_n) \neq 0$. $\square$

4. Implicitization of linear DPPEs by Carra'Ferro's differential resultant

Let $\mathcal{P}(X, U)$, F_i, H_i be as in Section 2. Let $\mathbb{D} = \mathbb{K}\{X\}$, $\mathrm{PS} = \mathrm{PS}(F_1, \ldots, F_n)$ and let ID be the implicit ideal of $\mathcal{P}(X, U)$. In this section, we use the implicitization results in terms of characteristic sets given in Gao (2003) to obtain implicitization results in terms of differential resultants. We prove that if $\partial\mathrm{Res}(F_1, \ldots, F_n) \neq 0$ then $\partial\mathrm{Res}(F_1, \ldots, F_n)(X) = 0$ is the implicit equation of $\mathcal{P}(X, U)$.

Recall that PS is a set of linear differential polynomials. Let [PS] be the differential ideal generated by PS, then it holds $\mathrm{ID} = [\mathrm{PS}] \cap \mathbb{K}\{X\}$ by Gao (2003), Lemma 3.2. Let $\mathcal{A}$ be a characteristic set of [PS] and $\mathcal{A}_0 = \mathcal{A} \cap \mathbb{K}\{X\}$. By Gao (2003), Theorem 3.1, the implicit ideal is

$$\mathrm{ID} = [\mathrm{PS}] \cap \mathbb{K}\{X\} = [\mathcal{A}_0].$$

Furthermore, if $|\mathcal{A}_0| = 1$ then $\mathcal{A}_0 = \{A(X)\}$ where A is a characteristic polynomial of ID. By Carra'Ferro (1997b), Proposition 12, $\partial\mathrm{Res}(F_1, \ldots, F_n) \in \mathrm{ID}$ and it is our candidate to be a characteristic polynomial of ID.

To compute a characteristic set of ID we will use a Groebner basis $\mathcal{G}$ of (PS) with respect to lex monomial order induced by the ranking $\mathcal{R}^\star$ on $X \cup U$ that eliminates U with respect to X, that is $\partial^k x < \partial^{k^\star} u$, for all $x \in X$, $u \in U$ and $k, k^\star \in \mathbb{N}_0$. Obviously, we are dealing with a linear system of polynomials and computing a Groebner basis is equivalent to performing gaussian elimination.

Lemma 11. *Let $\mathcal{G}$ be the reduced Groebner basis of* (PS) *with respect to lex monomial order induced by the ranking $\mathcal{R}^\star$.*

(1) *$\mathcal{G} = \{B_0, B_1, \ldots, B_{L-1}\}$ is a linear set of differential polynomials where $B_0 < B_1 < \cdots < B_{L-1}$ with respect to the ranking $\mathcal{R}^\star$ and $\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$ is not empty, $B_0 \in \mathcal{G}_0$.*
(2) *Let $E(L)$ be the $L \times L$ matrix whose kth row contains the coefficients of B_{L-k}, $k = 1, \ldots, L$ as a polynomial in $\mathbb{D}\{U\}$, and where the coefficients are written in decreasing order with respect to the orderly ranking on U. Given the differential resultant matrix $M(L)$ of $F_1, \ldots F_n$, then* $\det(M(L)) = b\det(E(L))$ *for some $b \in \mathbb{K}$.*
(3) *The cardinality of $\mathcal{G}_0$ is one if and only if $\partial\mathrm{Res}^h(H_1, \ldots, H_n) \neq 0$.*
(4) *Let $\mathcal{B} = \{B_1, \ldots, B_{L-n}\}$. If $\partial\mathrm{Res}^h(H_1, \ldots, H_n) \neq 0$ then for every $B \in \mathcal{B}$ then there exists $\mu_B \in \mathbb{K}$ such that* $\mathrm{ord}(B - \mu_B B_0, x_i) < N - o_i$, $i = 1, \ldots, n$.

Proof. Let M_{2L} be the $L \times (2L)$ matrix whose kth row contains the coefficients, as a polynomial in $\mathbb{K}[x_i, \ldots, x_{iN-o_i}, u_j, \ldots, u_{jN} \mid i = 1, \ldots n, j = 1, \ldots n-1]$, of the $(L-k+1)$th polynomial in PS and where the coefficients are written in decreasing order w.r.t. $\mathcal{R}^\star$.

$$M_{2L} = \begin{bmatrix} & 1 & & & & & & \partial^{N-o_1} a_1 \\ & & \ddots & & & & & \vdots \\ & & & 1 & & & & a_1 \\ M_{L-1} & & & & \ddots & & & \vdots \\ & & & & & 1 & & \partial^{N-o_n} a_n \\ & & & & & & \ddots & \vdots \\ & & & & & & 1 & a_n \end{bmatrix}.$$

We can find a Groebner basis of (PS) performing gaussian elimination on the rows of M_{2L}, see (Cox et al., 1997), Chapter 2, Section 7, Exercise 10. In particular, the polynomials corresponding to the rows of the reduced echelon form E_{2L} of M_{2L} are the elements of the reduced Groebner basis $\mathcal{G}$ of (PS).

(1) Observe that M_{2L} has rank L. Therefore $\mathcal{G}$ contains L elements $B_0 < B_1 < \cdots < B_{L-1}$. The submatrix formed by the first $L-1$ columns of M_{2L} is M_{L-1} which has rank less than or equal to $L-1$, therefore the set $\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$ is not empty, at least $B_0 \in \mathcal{G}_0$.
(2) The matrix $E(L)$ is obtained by performing on $M(L)$ the same row operations as the operations performed on M_{2L} to obtain E_{2L}. Therefore $\det(M(L)) = b\det(E(L))$ for some $b \in \mathbb{K}$.
(3) The cardinality of $\mathcal{G}_0$ is one if and only if $\mathrm{rank}(M_{L-1}) = L-1$. Equivalently, $\partial\mathrm{Res}^h(H_1, \ldots, H_n) \neq 0$ by Proposition 10.

(4) If $\partial\mathrm{Res}^h(H_1,\ldots,H_n)\neq 0$ then $E(L)$ is an upper triangular matrix of rank L whose $L-1$ principal submatrix is the identity. Then

$$\mathcal{B}=\{u_j-U_j(X),u_{jk}-U_{jk}(X)\mid j=1,\ldots,n-1,k=1,\ldots,N-1\}$$

with linear differential polynomials $U_j,U_{jk}\in\mathbb{K}\{X\}$.

The differential polynomial B_0 is linear and belongs to (PS) $\cap\,\mathbb{K}\{X\}$ therefore $B_0(X,U)=\sum_{i=1}^n\mathcal{F}_i(F_i(X,U))$ with $\mathcal{F}_i\in\mathbb{K}[\partial]$, $\deg(\mathcal{F}_i)\leq N-o_i$, $i=1,\ldots,n$. For $i\in\{1,\ldots,n\}$ let α_i be the coefficient of ∂^{N-o_i} in $\mathcal{F}_i$. Similarly, given $B\in\mathcal{B}$ then $B(X,U)=\sum_{i=1}^n\mathcal{G}_i(F_i(X,U))$ with $\mathcal{G}_i\in\mathbb{K}[\partial]$, $\deg(\mathcal{G}_i)\leq N-o_i$, $i=1,\ldots,n$. For $i\in\{1,\ldots,n\}$ let β_i be the coefficient of ∂^{N-o_i} in $\mathcal{G}_i$.

Let S be the matrix defined right before Remark 7. Note that for all $i\in\{1,\ldots,n\}$ the ith row of S consists of the coefficients of u_{n-jN}, $j\in\{1,\ldots,n-1\}$ in $\partial^{N-o_i}F_i(X,U)$. Since $B_0\in\mathbb{K}\{X\}$ then $\mathrm{ord}(B_0,u_j)<N$, hence

$$\begin{pmatrix}\alpha_1&\cdots&\alpha_n\end{pmatrix}S=\bar{0}. \tag{5}$$

By definition of $\mathcal{B}$, $\mathrm{ord}(B,u_j)<N$, $j=1,\ldots,n-1$, then

$$\begin{pmatrix}\beta_1&\cdots&\beta_n\end{pmatrix}S=\bar{0}. \tag{6}$$

From statement 2 there exists $b\neq 0$ such that $B_0=\det(E(L))=(1/b)\det(M(L))$. Then by Eq. (3) and Remark 9 it holds: $\alpha_i=0$ if and only if $\det(M_{x_{iN-o_i}})=0$ if and only if $\det(S_i)=0$.

Let us call w_i the ith row vector of S. Now, by Remark 9 there exists $k\in\{1,\ldots,n\}$ such that $\det(M_{x_{kN-o_k}})\neq 0$. We have $\alpha_k\neq 0$ and by Eq. (5) then $\alpha_k w_k=-\sum_{t=1,t\neq k}^n\alpha_t w_t$ which is not zero since no row of S is zero. From Eq. (6) it follows $\sum_{i=1,i\neq k}^n(\beta_i-\frac{\beta_k}{\alpha_k}\alpha_i)w_i=\bar{0}$. Furthermore $\det(S_k)\neq 0$ implies $\beta_i=(\beta_k/\alpha_k)\alpha_i$, for all $i=1,\ldots,n$. Finally, for every $i\in\{1,\ldots n\}$, $\beta_i-(\beta_k/\alpha_k)\alpha_i=0$, implying $\mathrm{ord}(B-\mu_B B_0)<N-o_i$, with $\mu_B=\beta_k/\alpha_k$. □

To compute a characteristic set of [PS] we apply the algorithm given in Boulier et al. (1995), Theorem 6, that we briefly include below for completion. Given $P\in\mathbb{K}\{X\cup U\}$, the lead of P is the highest derivative present in P w.r.t. $\mathcal{R}^\star$, we denote it by $\mathrm{lead}(P)$. Given $P,Q\in\mathbb{K}\{X\cup U\}$ we denote by $\mathrm{prem}(P,Q)$ the pseudo-remainder of P with respect to Q, (Ritt, 1950), page 7. Given a chain $\mathcal{A}=\{A_1,\ldots,A_t\}$ of elements of $\mathbb{K}\{X\cup U\}$ then $\mathrm{prem}(P,\mathcal{A})=\mathrm{prem}(\mathrm{prem}(P,A_t),\{A_1,\ldots,A_{t-1}\})$ and $\mathrm{prem}(P,\emptyset)=P$.

Algorithm 1. Given the set of polynomials PS the next algorithm returns a characteristic set of [PS].

(1) Compute the reduced Groebner basis $\mathcal{G}$ of (PS) with respect to lex monomial order induced by $\mathcal{R}^\star$.
(2) Assume that the elements of $\mathcal{G}$ are arranged in increasing order $B_0<B_1<\cdots<B_{L-1}$ w.r.t. $\mathcal{R}^\star$. Let $\mathcal{A}=\{B_0\}$. For i from 1 to $L-1$ do, if $\mathrm{lead}(B_i)\neq\mathrm{lead}(B_{i-1})$ then $\mathcal{A}:=\mathcal{A}\cup\{\mathrm{prem}(B_i,\mathcal{A})\}$.

Lemma 12. *If $\partial\mathrm{Res}^h(H_1,\ldots,H_n)\neq 0$ then $\mathcal{A}=\{B_0,B_1,\ldots,B_{n-1}\}$ is a characteristic set of* [PS] *and $\mathcal{A}_0=\mathcal{A}\cap\mathbb{K}\{X\}=\{B_0\}$.*

Proof. We will prove that for all $B\in\{B_n,\ldots,B_{L-1}\}$ then B is reduced to zero by $\{B_0,\ldots,B_{n-1}\}$. Note that

$$\{B_n,\ldots,B_{L-1}\}=\{u_{jk}-U_{jk}(X)\mid j=1,\ldots,n-1,k=1,\ldots,N\}.$$

Let us prove that for $j=1,\ldots,n-1$ and $k=1,\ldots,N$ then

$$u_{jk}-U_{jk}(X)=\partial^k(B_j)+\sum_{t=1}^{k}\partial^{k-t}(c_{jt}B_0)-\partial^{k+1-t}(\mu_{jt-1}B_0) \tag{7}$$

for some $c_{jt},\mu_{jt-1}\in\mathbb{K}$. Then by Algorithm 1 the result is obtained.

Given $j\in\{1,\ldots,n-1\}$, by Lemma 11(4) and for every $i\in\{1,\ldots,n\}$ it holds $\mathrm{ord}(B_j-\mu_{j0}B_0,x_i)<N-o_i$ for some $\mu_{j0}\in\mathbb{K}$, therefore $\partial(B_j-\mu_{j0}B_0)\in(PS)$. Moreover, $u_{j1}-U_{j1}(X)-\partial(B_j-\mu_{j0}B_0)=u_{j1}-U_{j1}(X)-\partial(u_j-U_j(X)-\mu_{j0}B_0)=-U_{j1}(X)+\partial(U_j(X)-\mu_{j0}B_0)\in$ (PS) $\cap\,\mathbb{K}\{X\}$ is a linear polynomial. Under our assumption $\mathcal{G}_0=\{B_0\}$, then there exists $c_{j1}\in\mathbb{K}$ such that $u_{j1}-U_{j1}(X)=\partial(B_j-\mu_{j0}B_0)+c_{j1}B_0$.

Given $k\in\{1,\ldots,N-1\}$, we can prove analogously that $u_{jk+1}-U_{jk+1}(X)-\partial(u_{jk}-U_{jk}(X)-\mu_{jk}B_0)\in$ (PS) $\cap\,\mathbb{K}\{X\}=(B_0)$ for $\mu_{jk}\in\mathbb{K}$ and it is a linear polynomial. Then there exists $c_{jk+1}\in\mathbb{K}$ such that $u_{jk+1}-U_{jk+1}(X)=\partial(u_{jk}-U_{jk}(X)-\mu_{jk}B_0)+c_{jk+1}B_0$. By induction on k formula (7) is proved. □

Theorem 13. *Given a system $\mathcal{P}(X, U)$ of linear DPPEs with implicit ideal* ID. *If* $\partial\text{Res}^h(H_1, \ldots, H_n) \neq 0$ *then* ID *has dimension* $n - 1$ *and*

$$\partial\text{Res}(F_1, \ldots, F_n)(X) = 0$$

is the implicit equation of $\mathcal{P}(X, U)$.

Proof. Let $\mathcal{G}$ be the reduced Groebner basis of (PS) with respect to lex monomial order induced by $\mathcal{R}^\star$ and let $B_0 < B_1 < \cdots < B_{L-1}$ be the elements of $\mathcal{G}$. By Lemma 12 there exists a characteristic set $\mathcal{A}$ of [PS] such that $\mathcal{A}_0 = \{B_0\}$. Consequently, the dimension of ID is $n - 1$.

By Lemma 11 and the definition of the differential resultant,

$$\partial\text{Res}(F_1, \ldots, F_n) = \det(M(L)) = b\det(E(L)),$$

for some $b \in \mathbb{K}$, therefore $\partial\text{Res}(F_1, \ldots, F_n) = bB_0$. Thus $\partial\text{Res}(F_1, \ldots, F_n)$ is a characteristic polynomial of ID and therefore $\partial\text{Res}(F_1, \ldots, F_n)(X) = 0$ is the implicit equation of $\mathcal{P}(X, U)$. □

5. Linear complete differential resultants

Let $\mathbb{D}$ be a differential integral domain, and let $f_i \in \mathbb{D}\{U\}$ be a linear ordinary differential polynomial of order o_i, $i = 1, \ldots, n$. For each $j \in \{1, \ldots, n-1\}$ we define the positive integer

$$\gamma_j(f_1, \ldots, f_n) := \min\{o_i - \text{ord}(f_i, u_j) \mid i \in \{1, \ldots, n\}\}.$$

Observe that, $0 \leq \gamma_j(f_1, \ldots, f_n) \leq o_i$ for each $i \in \{1, \ldots, n\}$. We also define

$$\gamma(f_1, \ldots, f_n) := \sum_{j=1}^{n-1} \gamma_j(f_1, \ldots, f_n)$$

and it is easily proved that $0 \leq \gamma(f_1, \ldots, f_n) \leq N - o_i$, for all $i \in \{1, \ldots, n\}$.

Observe that, when $\gamma(f_1, \ldots, f_n) \neq 0$ then the set $\text{PS}(f_1, \ldots, f_n)$ defined in Section 3, is a set of L differential polynomials in $L - \gamma(f_1, \ldots, f_n) - 1$ differential variables. Then $\partial\text{Res}(f_1, \ldots, f_n)$ is the Macaulay's resultant of a set of polynomials which are not complete in all its variables, thus $\partial\text{Res}(f_1, \ldots, f_n) = 0$. If $\gamma(f_1, \ldots, f_n) = 0$ we will say that the set of differential polynomials $\{f_1, \ldots, f_n\}$ is complete. We will call $\gamma(f_1, \ldots, f_n)$ the completeness index of $\{f_1, \ldots, f_n\}$.

We define next a differential resultant that generalizes Carra'Ferro's differential resultant in the linear case. Observe that the generalized differential resultant in Carra'Ferro (2005) deals with a different aspect.

Definition 14. The *linear complete differential resultant* $\partial\text{CRes}(f_1, \ldots, f_n)$, of n linear ordinary differential polynomials $f_1, \ldots, f_n$ in $n - 1$ differential variables $u_1, \ldots, u_{n-1}$, is defined as the Macaulay's algebraic resultant of the differential polynomial set

$$\text{PS}_\gamma(f_1, \ldots, f_n) := \left\{ \partial^{N-o_i-\gamma} f_i, \ldots, \partial f_i, f_i \;\middle|\; i = 1, \ldots, n,\;\; N = \sum_{i=1}^{n} o_i,\;\; \gamma = \gamma(f_1, \ldots, f_n) \right\}.$$

The set $\text{PS}_\gamma(f_1, \ldots, f_n)$ contains $L_\gamma = \sum_{i=1}^{n}(N - o_i - \gamma + 1)$ polynomials in the following set $\mathcal{V}$ of $L_\gamma - 1$ differential variables

$$\mathcal{V} = \{u_j, u_{j1} \ldots, u_{jN-\gamma_j-\gamma} \mid \gamma_j = \gamma_j(f_1, \ldots, f_n), j = 1, \ldots, n-1\}. \tag{8}$$

Observe that $\sum_{j=1}^{n-1}(N - \gamma_j - \gamma + 1) = (n-1)N - n\gamma + n - 1 = L_\gamma - 1$.

Let $h_i \in \mathbb{D}\{U\}$ be a linear ordinary differential homogeneous polynomial of order o_i, $i = 1, \ldots, n$.

Definition 15. We define the *linear complete differential homogeneous resultant*, of n linear ordinary differential homogeneous polynomials $h_1, \ldots, h_n$ in $n-1$ differential variables, $\partial\mathrm{CRes}^h(h_1, \ldots, h_n)$ as the Macaulay's algebraic resultant of the differential polynomial set

$$\mathrm{PS}^h_\gamma(h_1, \ldots, h_n) := \left\{ \partial^{N-o_i-\gamma-1} h_i, \ldots, \partial h_i, h_i \,\middle|\, i = 1, \ldots, n, \quad N = \sum_{i=1}^{n} o_i, \quad \gamma = \gamma(h_1, \ldots, h_n) \right\}.$$

The set $\mathrm{PS}^h_\gamma(h_1, \ldots, h_n)$ contains $L^h_\gamma = \sum_{i=1}^n (N - o_i - \gamma)$ polynomials in the following set $\mathcal{V}^h$ of $L^h_\gamma - 1$ differential variables

$$\mathcal{V}^h = \{u_j, u_{j1} \ldots, u_{jN-\gamma_j-\gamma-1} \mid \gamma_j = \gamma_j(h_1, \ldots, h_n), j = 1, \ldots, n-1\}.$$

Observe that for $\gamma(f_1, \ldots, f_n) = 0$ (resp. $\gamma(h_1, \ldots, h_n) = 0$) it holds

$$\partial\mathrm{CRes}(f_1, \ldots, f_n) = \partial\mathrm{Res}(f_1, \ldots, f_n),$$
$$\partial\mathrm{CRes}^h(h_1, \ldots, h_n) = \partial\mathrm{Res}^h(h_1, \ldots, h_n).$$

We introduce next the matrices that will allow the use of determinants to compute $\partial\mathrm{CRes}(f_1, \ldots, f_n)$ and $\partial\mathrm{CRes}^h(h_1, \ldots, h_n)$. For $i = 1, \ldots, n$, $\gamma = \gamma(f_1, \ldots, f_n)$ (resp. $\gamma = \gamma(h_1, \ldots, h_n)$) and $k = 0, \ldots, N - o_i - \gamma$ (resp. $k = 0, \ldots, N - o_i - \gamma - 1$) define the positive integers,

$$l(i, k) = (i-1)(N-\gamma) - \sum_{h=1}^{i-1} o_i + i + k,$$

$$l^h(i, k) = (i-1)(N-\gamma-1) - \sum_{h=1}^{i-1} o_i + i + k.$$

Then $l(i, k) \in \{1, \ldots, L_\gamma\}$ and $l^h(i, k) \in \{1, \ldots, L^h_\gamma\}$, with the appropriate value of γ in each case.

Let $M(L_\gamma)$ (resp. $M(L^h_\gamma)$) be the $L_\gamma \times L_\gamma$ (resp. $L^h_\gamma \times L^h_\gamma$) matrix containing the coefficients of $\partial^{N-o_i-\gamma-k} f_i$ (resp. $\partial^{N-o_i-\gamma-k-1} h_i$) as a polynomial in $\mathbb{D}[\mathcal{V}]$ (resp. in $\mathbb{D}[\mathcal{V}^h]$) in the $l(i, k)$th row (resp. $l^h(i, k)$th row), where the coefficients are written in decreasing order with respect to the orderly ranking on U. Hence, $M(L_\gamma)$ (resp. $M(L^h_\gamma)$) is a matrix over $\mathbb{D}$ that we call the complete differential (homogeneous) resultant matrix of $f_1, \ldots, f_n$ (resp. $h_1, \ldots, h_n$). In this situation:

$$\partial\mathrm{CRes}(f_1, \ldots, f_n) = \det(M(L_\gamma)),$$
$$\partial\mathrm{CRes}^h(h_1, \ldots, h_n) = \det(M(L^h_\gamma)).$$

Proposition 16. (1) *Let* $(\mathrm{PS}_\gamma(f_1, \ldots, f_n))$ *be the ideal generated by* $\mathrm{PS}_\gamma(f_1, \ldots, f_n)$ *in* $\mathbb{D}[\mathcal{V}]$. *Then* $\partial\mathrm{CRes}(f_1, \ldots, f_n) \in (\mathrm{PS}_\gamma(f_1, \ldots, f_n))$.

(2) *If the system* $\{h_1 = 0, \ldots, h_n = 0\}$ *has a nonzero solution then* $\partial\mathrm{CRes}^h(h_1, \ldots, h_n) = 0$.

Proof. (1) The proof of 1 is analogous to the proof of Carra'Ferro (1997b), Proposition 12. To be more precise, the first $L_\gamma - 1$ columns of $M(L_\gamma)$ are indexed by the elements of $\mathcal{V}$. Let M be the matrix obtained by replacing the last column of $M(L_\gamma)$ by its sum with each one of the previous columns multiplied by its corresponding element in $\mathcal{V}$. Then $\det(M) = \sum_{i=1}^n \sum_{k=1}^{N-o_i-\gamma} d_{ik} \partial^k f_i$, with $d_{ik} \in \mathbb{D}$. Thus $\partial\mathrm{CRes}(f_1, \ldots, f_n) \in (\mathrm{PS}_\gamma(f_1, \ldots, f_n))$.

(2) The proof is analogous to the proof of Proposition 5. □

Let F_i and H_i be as in Section 2 and $\gamma = \gamma(F_1, \ldots, F_n) = \gamma(H_1, \ldots, H_n)$. Let $\mathrm{PS}_\gamma = \mathrm{PS}_\gamma(F_1, \ldots, F_n)$, $\mathbb{D} = \mathbb{K}\{X\}$ and let (PS_γ) be the ideal generated by PS_γ in $\mathbb{K}[x_i, \ldots, x_{iN-o_i-\gamma}, u_j, \ldots, u_{jN-\gamma_j-\gamma} \mid i = 1, \ldots n, j = 1, \ldots n-1]$.

The next matrices will be used in the results in this section and also in later results in the paper. Let S_γ be the $n \times (n-1)$ matrix whose entry (i, j) is the coefficient of $u_{n-j\, o_i-\gamma_j}$ in F_i, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n-1\}$. For $i \in \{1, \ldots n\}$ let $S_{\gamma i}$ be the $(n-1) \times (n-1)$ matrix obtained by removing the ith row of S_γ.

Remark 17. Note that the nonzero rows of the columns of $M(L_\gamma)$ (resp. $M(L^h_\gamma)$) corresponding to the coefficients of $u_{n-jN-\gamma_j-\gamma}$ (resp. $u_{n-jN-\gamma_j-\gamma-1}$), $j=1,\ldots,n-1$ are the rows of S_γ.

Observe that $\partial\mathrm{CRes}(F_1,\ldots,F_n)$ is a linear differential polynomial and by the previous proposition it belongs to $(\mathrm{PS}_\gamma)\cap\mathbb{K}\{X\}$. The following theorem shows that the differential resultants $\partial\mathrm{CRes}(F_1,\ldots,F_n)$ and $\partial\mathrm{CRes}^h(H_1,\ldots,H_n)$ are closely related.

Theorem 18. *Let F_i and H_i be as in Section 2. The following statements hold.*

(1) *Let $P\in(\mathrm{PS}_\gamma)\cap\mathbb{K}\{X\}$ be a linear polynomial such that $\mathrm{ord}(P,x_k)=N-o_k-\gamma$ for some $k\in\{1,\ldots,n\}$. Then*

$$\partial\mathrm{CRes}(F_1,\ldots,F_n)=\frac{1}{\alpha}\det(S_{\gamma k})\partial\mathrm{CRes}^h(H_1,\ldots,H_n)P(X)$$

with $\alpha=(-1)^a\frac{\partial P}{\partial x_{k\,N-o_k-\gamma}}$, $a\in\mathbb{N}$.

(2) *Let $M_{L_\gamma-1}$ be the $L_\gamma\times(L_\gamma-1)$ principal submatrix of $M(L_\gamma)$. The following statements are equivalent.*
   (a) $\partial\mathrm{CRes}(F_1,\ldots,F_n)\neq 0$.
   (b) $\partial\mathrm{CRes}^h(H_1,\ldots,H_n)\neq 0$.
   (c) $\mathrm{rank}(M_{L_\gamma-1})=L_\gamma-1$.

Proof. The proof of 1 and 2 is analogous to the proof of Theorem 8 and Proposition 10 respectively. In this case take $\mathcal{X}=\{x_i,\partial x_i,\ldots,\partial^{N-o_i-\gamma}x_i\mid i=1,\ldots,n\}$ and observe that if $\partial\mathrm{CRes}^h(H_1,\ldots,H_n)\neq 0$ then by Remark 17 there exists $k\in\{1,\ldots,n\}$ such that $\det(S_{\gamma k})\neq 0$. Furthermore, there exists $a\in\mathbb{N}$ such that

$$\det(M_{x_{kN-o_k-\gamma}})=(-1)^a\partial\mathrm{CRes}^h(H_1,\ldots,H_n)\det(S_{\gamma k})\neq 0.\quad\square$$

Our Maple implementation of the linear complete differential (homogeneous) resultant can be found in Rueda (2008).

Example 19. Let $\mathbb{K}=\mathbb{Q}(t)$, $\partial=\frac{\partial}{\partial t}$ and consider the set of differential polynomials in $\mathbb{K}\{x_1,x_2,x_3\}\{u_1,u_2\}$,

$$\begin{aligned}F_1(X,U)&=x_1-t-u_1-u_2-2u_{22}\\F_2(X,U)&=x_2-t^2-2u_1-u_2-u_{22}\\F_3(X,U)&=x_3-5-u_1-3u_{11}-u_{21}-u_{23}.\end{aligned}$$

Then the set $\mathrm{PS}(F_1,F_2,F_3)$ contains $L=17$ differential polynomials and $\partial\mathrm{Res}(F_1,F_2,F_3)$ equals $\det(M(L))=0$ because the columns of $M(L)$ corresponding to the coefficients of u_{17} and u_{16} are columns of zeros.

We have $\gamma_1=2$ and $\gamma_2=0$ so $\gamma=2$. Let us compute $\partial\mathrm{CRes}(F_1,F_2,F_3)$. The set $\mathrm{PS}_\gamma(F_1,F_2,F_3)$ contains $L_\gamma=11$ differential polynomials in the differential variables $\mathcal{V}=\{u_{25},u_{24},u_{23},u_{13},u_{22},u_{12},u_{21},u_{11},u_2,u_1\}$ written in decreasing order w.r.t. the ranking on U.

$$M(L_\gamma)=\begin{bmatrix}-2&0&-1&-1&0&0&0&0&0&0&x_{13}\\0&-2&0&0&-1&-1&0&0&0&0&x_{12}\\0&0&-2&0&0&0&-1&-1&0&0&x_{11}-1\\0&0&0&0&-2&0&0&0&-1&-1&x_1-t\\-1&0&-1&-2&0&0&0&0&0&0&x_{23}\\0&-1&0&0&-1&-2&0&0&0&0&x_{22}-2\\0&0&-1&0&0&0&-1&-2&0&0&x_{21}-2t\\0&0&0&0&-1&0&0&0&-1&-2&x_2-t^2\\-1&0&-1&-3&0&-1&0&0&0&0&x_{32}\\0&-1&0&0&-1&-3&0&-1&0&0&x_{31}\\0&0&-1&0&0&0&-1&-3&0&-1&x_3-5\end{bmatrix}.$$

The matrix $M(L^h_\gamma)$ is obtained by removing rows 1, 5, 9 and columns 1, 4, 11 of $M(L_\gamma)$. Then,

$$\partial\mathrm{CRes}(F_1,F_2,F_3) = 4x_3 - 8 - 8x_{21} + 12t + 12x_{32} + 4x_{13} + 4x_{11} - 20x_{23}$$
$$-8x_{22} + 4x_{12} + 4x_1 - 4x_2 + 4t^2$$
$$\partial\mathrm{CRes}^h(H_1,H_2,H_3) = -4.$$

6. Implicitization of linear DPPEs by linear complete differential resultants

Let $\mathcal{P}(X,U)$, F_i, H_i be as in Section 2 and let ID be the implicit ideal of $\mathcal{P}(X,U)$. In this section, we prove the main result of this paper, namely if $\partial\mathrm{CRes}^h(H_1,\dots,H_n) \neq 0$ then $\partial\mathrm{CRes}(F_1,\dots,F_n)(X) = 0$ is the implicit equation of $\mathcal{P}(X,U)$ and hence the results in Section 4 are extended.

Let $\gamma = \gamma(F_1,\dots,F_n) = \gamma(H_1,\dots,H_n)$, $\mathrm{PS}_\gamma = \mathrm{PS}_\gamma(F_1,\dots,F_n)$ and let $\mathcal{R}^\star$ be the ranking on $X \cup U$ defined in Section 4. Note that $\mathrm{ID} = [\mathrm{PS}_\gamma] \cap \mathbb{K}\{X\}$.

Lemma 20. *Let $\mathcal{G}$ be the reduced Groebner basis of (PS_γ) with respect to lex monomial order induced by the ranking $\mathcal{R}^\star$.*

(1) *$\mathcal{G} = \{B_0, B_1, \dots, B_{L_\gamma-1}\}$ is a set of linear differential polynomials where $B_0 < B_1 < \dots < B_{L_\gamma-1}$ with respect to the ranking $\mathcal{R}^\star$ and $\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$ is not empty, $B_0 \in \mathcal{G}_0$*

(2) *Let $E(L_\gamma)$ be the $L_\gamma \times L_\gamma$ matrix whose kth row contains the coefficients of $B_{L_\gamma-k}$, $k = 1,\dots,L_\gamma$ as a polynomial in $\mathbb{D}[\mathcal{V}]$, and where the coefficients are written in decreasing order with respect to the orderly ranking on U. Given the differential resultant matrix $M(L_\gamma)$ of $F_1,\dots F_n$, then $\det(M(L_\gamma)) = b\det(E(L_\gamma))$ for some $b \in \mathbb{K}$.*

(3) *The cardinality of $\mathcal{G}_0$ is one if and only if $\partial\mathrm{CRes}^h(H_1,\dots,H_n) \neq 0$.*

(4) *If $\partial\mathrm{CRes}^h(H_1,\dots,H_n) \neq 0$ then $\mathcal{A} = \{B_0, B_1, \dots, B_{n-1}\}$ is a characteristic set of $[\mathrm{PS}_\gamma]$ and $\mathcal{A}_0 = \mathcal{A} \cap \mathbb{K}\{X\} = \{B_0\}$.*

Proof. The proof of 1, 2 and 3 is analogous to the proof of Lemma 11(1),(2) and (3).

Let $\mathcal{B}_\gamma = \{B_1,\dots,B_{L_\gamma-n}\}$. If $\partial\mathrm{CRes}^h(H_1,\dots,H_n) \neq 0$ then it can be proved as Lemma 11(4) that for all $B \in \mathcal{B}_\gamma$ then there exists $\mu \in \mathbb{K}$ such that $\mathrm{ord}(B - \mu B_0, x_i) < N - o_i - \gamma$, $i = 1,\dots,n$.

The proof of 4 is now analogous to the proof of Lemma 12. Given $B \in \{B_n,\dots,B_{L_\gamma-1}\}$ then B is reduced to zero by $\{B_0,\dots,B_{n-1}\}$. Note that

$$\{B_n,\dots,B_{L_\gamma-1}\} = \{u_{jk} - U_{jk}(X) \mid j = 1,\dots,n-1, k = 1,\dots,N-\gamma_j-\gamma\}.$$

To adapt the proof of Lemma 12 we show that for $j = 1,\dots,n-1$ and $k = 1,\dots,N-\gamma_j-\gamma$ then (7) holds for some $c_{jt}, \mu_{jt-1} \in \mathbb{K}$. Then by Algorithm 1 the result is obtained. □

Theorem 21. *Given a system $\mathcal{P}(X,U)$ of linear DPPEs with implicit ideal* ID. *If $\partial\mathrm{CRes}^h(H_1,\dots,H_n) \neq 0$ then* ID *has dimension $n-1$ and*

$$\partial\mathrm{CRes}(F_1,\dots,F_n)(X) = 0$$

is its implicit equation.

Proof. By Lemma 20(4), a characteristic set of ID is $\mathcal{A}_0 = \{B_0\}$. Then the dimension of ID is $n-1$. By Lemma 20(2), and the definition of the linear complete differential resultant,

$$\partial\mathrm{CRes}(F_1,\dots,F_n) = \det(M(L_\gamma)) = b\det(E(L_\gamma)),$$

for some $b \in \mathbb{K}$. Therefore $\partial\mathrm{CRes}(F_1,\dots,F_n) = bB_0$. This proves that $\partial\mathrm{CRes}(F_1,\dots,F_n)$ is a characteristic polynomial of ID and therefore the implicit equation of $\mathcal{P}(X,U)$ is $\partial\mathrm{CRes}(F_1,\dots,F_n)(X) = 0$. □

7. Some results on properness

Let $\mathcal{P}(X, U)$, H_i, P_i and $\mathcal{L}_{ij}$ be as in Section 2. In this section, we give some results related with the inversion problem. To be more precise, we study conditions on the differential operators $\mathcal{L}_{ij}$ so that $\mathcal{P}(X, U)$ is a set of proper DPPEs.

We gather below some definitions that will be needed in this section and that were used in Gao (2003), Section 6, to study the inversion problem in terms of characteristic sets. The image of $\mathcal{P}(X, U)$ is the set

$$\mathrm{IM} = \{(\eta_1, \dots, \eta_n) \in \mathbb{E}^n \mid \exists(\tau_1, \dots, \tau_{n-1}) \in \mathbb{E}^{n-1} \text{ with } \eta_i = P_i(\tau_1, \dots, \tau_{n-1})\}.$$

The inversion problem says: given $(\eta_1, \dots, \eta_n) \in \mathrm{IM}$, find $(\tau_1, \dots, \tau_{n-1}) \in \mathbb{E}^{n-1}$ such that $\eta_i = P_i(\tau_1, \dots, \tau_{n-1})$. We call inversion maps for $\mathcal{P}(X, U)$ to a set of functions $g_1, \dots, g_{n-1}$ in $\{X\}$ such that

$$u_j = g_j(x_1, \dots, x_n), \quad j = 1, \dots, n-1.$$

A set $\mathcal{P}(X, U)$ of DPPEs is proper if for a generic zero $(\Gamma_1, \dots, \Gamma_n)$ (and hence most of the points) of the implicit variety Zero(ID), there exists only one $(\tau_1, \dots, \tau_{n-1}) \in \mathbb{E}^{n-1}$ such that $\Gamma_i = P_i(\tau_1, \dots, \tau_{n-1})$. By Gao (2003), Lemma 3.1, $(P_1(U), \dots, P_n(U))$ is a generic zero of the implicit ideal ID.

Proposition 22. *Let us suppose that $\partial\mathrm{CRes}^h(H_1, \dots, H_n) \neq 0$, then the next statements hold.*

(1) *The set $\mathcal{P}(X, U)$ of DPPEs is proper.*
(2) *Furthermore, there exists a set of linear inversion maps for $\mathcal{P}(X, U)$,*

$$U_1, \dots, U_{n-1} \in \mathbb{K}\{X\}.$$

Proof. By Lemma 20(4) then $\mathcal{A} = \{B_0, B_1, \dots, B_{n-1}\}$ is a characteristic set of $[\mathrm{PS}_\gamma]$ where

$$B_j(X, U) = u_j - U_j(X), \quad j = 1, \dots, n-1,$$

for linear differential polynomials $U_j \in \mathbb{K}\{X\}$. By Gao (2003), Theorem 6.1, the set $\mathcal{P}(X, U)$ of DPPEs is proper and $U_1, \dots, U_{n-1}$ is a set of inversion maps of $\mathcal{P}(X, U)$. □

If $\mathbb{K}$ is not a field of constants with respect to ∂, then $\mathbb{K}[\partial]$ is not commutative but

$$\partial k - k\partial = \partial(k)$$

for all $k \in \mathbb{K}$. The ring $\mathbb{K}[\partial]$ of differential operators with coefficients in $\mathbb{K}$ is right euclidean (and also left euclidean). Given $\mathcal{L}, \mathcal{L}' \in \mathbb{K}[\partial]$, by applying the right division algorithm we obtain $q, r \in \mathbb{K}[\partial]$, the right quotient and the right reminder of $\mathcal{L}$ and $\mathcal{L}'$ respectively, such that $\mathcal{L} = q\mathcal{L}' + r$ where $\deg(r) < \deg(\mathcal{L}')$.

If $\mathcal{L}_j \in \mathbb{K}[\partial]$ is the greatest common right divisor of $\mathcal{L}_{1j}, \dots, \mathcal{L}_{nj}$ then there exists $\mathcal{L}'_{ij} \in \mathbb{K}[\partial]$ such that $\mathcal{L}_{ij} = \mathcal{L}'_{ij}\mathcal{L}_j$, $i = 1, \dots, n$. Then we write

$$\mathcal{L}_j = \mathrm{gcrd}(\mathcal{L}_{1j}, \dots, \mathcal{L}_{nj}), \quad j = 1, \dots, n-1.$$

By Remark 3, then $\mathcal{L}_j \neq 0$. If $\mathcal{L}_j \in \mathbb{K}$, then we say that $\mathcal{L}_{1j}, \dots, \mathcal{L}_{nj}$ are coprime and we write

$$(\mathcal{L}_{1j}, \dots, \mathcal{L}_{nj}) = 1.$$

Theorem 23. *A necessary condition for the set $\mathcal{P}(X, U)$ of DPPEs to be proper is*

$$(\mathcal{L}_{1j}, \dots, \mathcal{L}_{nj}) = 1, \quad j = 1, \dots, n-1.$$

Proof. Let us suppose that there exists $k \in \{1, \dots, n-1\}$ such that the differential operator $\mathcal{L}_k = \mathrm{gcrd}(\mathcal{L}_{1k}, \dots, \mathcal{L}_{nk})$ is nonconstant. Then there exists a nonzero element $\eta \in \mathbb{E}$ such that $\mathcal{L}_{ik}(\eta) = 0$, $i = 1, \dots, n$. Define the element

$$U + \eta = (u_1, \dots, u_k + \eta, \dots, u_{n-1}) \in \mathbb{E}^{n-1}.$$

Recall that $P_i(U) = a_i - \sum_{j=1}^{n-1} \mathcal{L}_{ij}(u_j)$, then

$$(P_1(U + \eta), \dots, P_n(U + \eta)) = (P_1(U), \dots, P_n(U)).$$

Thus by definition, $\mathcal{P}(X, U)$ is not proper. □

Remark 24. For $n = 2$ the condition in Theorem 23 is also sufficient but for $n \geq 3$ this is not true. Let $\mathbb{K} = \mathbb{C}(t)$ and $\partial = \frac{\partial}{\partial t}$. Let us consider the system of linear DPPEs

$$\begin{cases} x_1 = 2u_1 + u_{11} + u_2 + u_{22} \\ x_2 = u_1 + u_{11} + u_{12} + u_2 + u_{22} \\ x_3 = u_1 + 2u_{11} + u_2 + u_{21}, \end{cases}$$

where

$$\begin{array}{lll} \mathcal{L}_{11} = 2 + \partial & \mathcal{L}_{21} = 1 + \partial + \partial^2 & \mathcal{L}_{31} = 1 + 2\partial \\ \mathcal{L}_{12} = 1 + \partial^2 & \mathcal{L}_{22} = 1 + \partial^2 & \mathcal{L}_{32} = 1 + \partial. \end{array}$$

We compute a characteristic set $\mathcal{A}$ of $[\mathrm{PS}_\gamma]$ with Algorithm 1

$$\begin{aligned} \mathcal{A} = \{&x_{12} - 2x_{22} - 2x_{21} - x_2 + x_{33} + x_{32} + x_{31} + x_3, \\ &u_2 + 3/2u_1 + x_{11} - 1/2x_1 - 2x_{21} - x_2 + x_{32} + 1/2x_{31} + 1/2x_3, \\ &u_{11} - u_1 - x_{11} + x_1 + 2x_{21} - x_{32} - x_3\}. \end{aligned}$$

Then by Gao (2003), Theorem 6.1, the system is not proper but $(\mathcal{L}_{1j}, \mathcal{L}_{2j}, \mathcal{L}_{3j}) = 1, j = 1, 2.$

We define a new system of DPPEs having the same implicit ideal than $\mathcal{P}(X, U)$

$$\mathcal{P}'(X, U) = \begin{cases} x_1 = P_1'(U) = a_1 - H_1'(U) \\ \vdots \\ x_n = P_n'(U) = a_n - H_i'(U), \end{cases} \tag{9}$$

where $H_i'(U) = \sum_{j=1}^{n-1} \mathcal{L}_{ij}'(u_j)$ and $F_i'(U) = T_i(X) + H_i'(U)$.

Proposition 25. *Let* ID' *be the implicit ideal of the set* $\mathcal{P}'(X, U)$ *of DPPEs. Then* $\mathrm{ID} = \mathrm{ID}'$.

Proof. Given $f \in \mathrm{ID}'$ then $f(P_1'(U), \ldots, P_n'(U)) = 0$. For $\eta = (\mathcal{L}_1(u_1), \ldots, \mathcal{L}_{n-1}(u_{n-1}))$ then $0 = f(P_1'(\eta), \ldots, P_n'(\eta)) = f(P_1(U), \ldots, P_n(U))$. Therefore $\mathrm{ID}' \subseteq \mathrm{ID}$.

Now let us suppose that $f \in \mathrm{ID} = [\mathrm{PS}_\gamma] \cap \mathbb{K}\{X\}$ and it is linear. Then, there exists $\mathcal{F}_i \in \mathbb{K}[\partial], i = 1, \ldots, n$ such that $f(X) = \mathcal{F}_1(F_1(X, U)) + \cdots + \mathcal{F}_n(F_n(X, U))$. Thus, $\mathcal{F}_1(H_1(U)) + \cdots + \mathcal{F}_n(H_n(U)) = 0$ and $f(X) = \mathcal{F}_1(T_1(X)) + \cdots + \mathcal{F}_n(T_n(X))$. As a consequence, for each $j \in \{1, \ldots, n-1\}$ we have $\mathcal{F}_1(\mathcal{L}_{1j}(u_j)) + \cdots + \mathcal{F}_n(\mathcal{L}_{nj}(u_j)) = 0$. Thus $(\mathcal{F}_1\mathcal{L}_{1j}' + \cdots + \mathcal{F}_n\mathcal{L}_{nj}')\mathcal{L}_j = 0$ and by Remark 3 $\mathcal{L}_j \neq 0$ so the differential operator $\mathcal{F}_1\mathcal{L}_{1j}' + \cdots + \mathcal{F}_n\mathcal{L}_{nj}' = 0$. We conclude that $\mathcal{F}_1(H_1'(U)) + \cdots + \mathcal{F}_n(H_n'(U)) = 0$ which implies $f(P_1'(U), \ldots, P_n'(U)) = 0$ and proves $f \in \mathrm{ID}'$.

By Lemma 20(1) and Algorithm 1 there exists a characteristic set $\mathcal{A}_0$ of ID whose differential polynomials are linear. Then $\mathcal{A}_0 \subset \mathrm{ID}'$ and $\mathrm{ID} = [\mathcal{A}_0] \subset \mathrm{ID}'$. $\square$

The following corollary follows directly from Theorem 21 and Proposition 25.

Corollary 26. *Given a system* $\mathcal{P}(X, U)$ *of linear DPPEs with implicit ideal* ID. *If* $\partial\mathrm{CRes}^h(H_1', \ldots, H_n') \neq 0$ *then* ID *has dimension* $n - 1$ *and*

$$\partial\mathrm{CRes}(F_1', \ldots, F_n')(X) = 0$$

is its implicit equation.

Example 27. The method to compute the implicit equation of the system $\mathcal{P}(X, U)$ given by the equations $x_1 = u_1 + u_{11} + u_2 + u_{21}$; $x_2 = t(u_{11} + u_{12}) + u_{22}$; $x_3 = u_1 + u_{11} + u_{21}$ would be the following. First, compute the system $\mathcal{P}'(X, U)$ as in (9), which is the system in Example 1. Then the implicit equation of $\mathcal{P}(X, U)$ is $\partial\mathrm{CRes}(F_1', F_2', F_3')(X) = (t-1)x_{12} - tx_{31} - (t-1)x_{32} + x_2 = 0$.

8. Treatment of special cases

We give an explicit expression of the implicit equation of $\mathcal{P}(X, U)$ in terms of the differential operators defining the linear DPPEs for $n = 2$ and with some restrictions for $n = 3$.

8.1. Case $n = 2$

Given differential operators $\mathcal{L}_1, \mathcal{L}_2 \in \mathbb{K}[\partial]$ we consider the set of DPPEs

$$\mathcal{P}_2(x_1, x_2, u) = \begin{cases} x_1 = a_1 - \mathcal{L}_1(u) \\ x_2 = a_2 - \mathcal{L}_2(u). \end{cases} \tag{10}$$

Let $H_1(u) = \mathcal{L}_1(u), H_2(u) = \mathcal{L}_2(u)$ and $F_1(x_1, x_2, u) = x_1 - a_1 + H_1(u), F_2(x_1, x_2, u) = x_2 - a_2 + H_2(u)$.

Remark 28. By Chardin (1991), Theorem 2, it holds $\partial\mathrm{Res}^h(H_1, H_2) \neq 0$ if and only if $(\mathcal{L}_1, \mathcal{L}_2) = 1$.

If $\mathbb{K}$ is a field of constants with respect to ∂, that is $\partial(k) = 0$ for all $k \in \mathbb{K}$ (for example $\partial = \frac{\partial}{\partial t}$ and $\mathbb{K} = \mathbb{C}$) then $\mathcal{L}_1\mathcal{L}_2 - \mathcal{L}_2\mathcal{L}_1 = 0$, i.e. the differential operators commute. If $\mathbb{K}$ is not a field of constants with respect to ∂, then $\mathbb{K}[\partial]$ is not commutative. We need the next commutativity result to prove the results in this section. Most likely it was studied by Tsirulik (1981) but we have not been able to find this paper. We give an algorithmic proof.

Lemma 29. *Let $\mathcal{L}_1, \mathcal{L}_2 \in \mathbb{K}[\partial]$ such that $(\mathcal{L}_1, \mathcal{L}_2) = 1$. Then there exist $\mathcal{D}_1, \mathcal{D}_2 \in \mathbb{K}[\partial]$ with $deg(\mathcal{D}_i) \leq deg(\mathcal{L}_i) - 1$, $i = 1, 2$ such that*

$$(\mathcal{L}_2 - \mathcal{D}_2)\mathcal{L}_1 - (\mathcal{L}_1 - \mathcal{D}_1)\mathcal{L}_2 = 0.$$

Proof. If $\mathcal{L}_1$ and $\mathcal{L}_2$ commute then $\mathcal{D}_i = 0$, $i = 1, 2$. Let us suppose that $\mathcal{L}_1\mathcal{L}_2 - \mathcal{L}_2\mathcal{L}_1 \neq 0$. Let $\mathcal{L}_1 = \sum_{i=0}^{o_1} \phi_i\partial^i$ and $\mathcal{L}_2 = \sum_{j=0}^{o_2} \psi_j\partial^j$ with $\phi_i, \psi_j \in \mathbb{K}$. By equation (1.2) in Berkovich and Tsirulik (1986), we have

$$\mathcal{L}_1\mathcal{L}_2 - \mathcal{L}_2\mathcal{L}_1 = \sum_{k=0}^{o_1+o_2} (c_k - \bar{c}_k)\partial^k \quad \text{where}$$

$$c_k = \sum_{s=\max(0,k-o_2)}^{\min(o_1,k)} \sum_{i=s}^{o_1} \phi_i\partial^{(i-s)}(\psi_{k-s}), \qquad \bar{c}_k = \sum_{s=\max(0,k-o_1)}^{\min(o_2,k)} \sum_{j=s}^{o_2} \psi_j\partial^{(j-s)}(\phi_{k-s}).$$

Then $c_{o_1+o_2} = \bar{c}_{o_1+o_2} = \phi_{o_1}\psi_{o_2}$. Therefore the degree of the operator $\mathcal{L}_1\mathcal{L}_2 - \mathcal{L}_2\mathcal{L}_1$ is less than or equal to $o_1 + o_2 - 1$.

We want to find $\mathcal{D}_1 = \sum_{i=0}^{o_1-1} \alpha_i\partial^i$ and $\mathcal{D}_2 = \sum_{j=0}^{o_2-1} \beta_j\partial^j$ with $\alpha_i, \beta_j \in \mathbb{K}$ such that $\mathcal{D}_1\mathcal{L}_2 - \mathcal{D}_2\mathcal{L}_1 = \mathcal{L}_1\mathcal{L}_2 - \mathcal{L}_2\mathcal{L}_1$. We have

$$\mathcal{D}_1\mathcal{L}_2 - \mathcal{D}_2\mathcal{L}_1 = \sum_{k=0}^{o_1+o_2-1} (\gamma_k - \bar{\gamma}_k)\partial^k \qquad \text{where}$$

$$\gamma_k = \sum_{s=\max(0,k-o_2)}^{\min(o_1-1,k)} \sum_{i=s}^{o_1-1} \alpha_i\partial^{(i-s)}(\psi_{k-s}), \qquad \bar{\gamma}_k = \sum_{s=\max(0,k-o_1)}^{\min(o_2-1,k)} \sum_{j=s}^{o_2-1} \beta_j\partial^{(j-s)}(\phi_{k-s}).$$

Let us consider the system of $o_1 + o_2$ equations

$$\gamma_k - \bar{\gamma}_k = c_k - \bar{c}_k, \quad k = 0, \dots, o_1 + o_2 - 1 \tag{11}$$

in the $N = o_1 + o_2$ unknowns $\alpha_i, i = 0, \dots, o_1$ and $\beta_j, j = 0, \dots, o_2$. The coefficient matrix of the system (11) is the differential homogeneous resultant matrix $M(L^h)$ of H_1, H_2, with $L^h = N$. Thus the system has a unique solution since $\det(M(L^h)) = \partial\mathrm{Res}^h(H_1, H_2) \neq 0$, by Remark 28. □

Theorem 30. *Given a system $\mathcal{P}_2(x_1, x_2, u)$ with implicit ideal* ID.

(1) *The following statements are equivalent.*
 (a) *$\mathcal{P}_2(x_1, x_2, u)$ is proper.*
 (b) $(\mathcal{L}_1, \mathcal{L}_2) = 1$.
 (c) $\partial\mathrm{Res}^h(H_1, H_2) \neq 0$.

(2) *If* $(\mathcal{L}_1, \mathcal{L}_2) = 1$ *then the dimension of* ID *is* 1 *and* $\partial\mathrm{Res}(F_1, F_2)(x_1, x_2) = 0$ *is the implicit equation of* $\mathcal{P}_2(x_1, x_2, u)$. *Furthermore, there exist* $\mathcal{D}_1, \mathcal{D}_2 \in \mathbb{K}[\partial]$ *such that*

$$\partial\mathrm{Res}(F_1, F_2)(x_1, x_2) = (-1)^a \partial\mathrm{Res}^h(H_1, H_2)[(\mathcal{L}_2 - \mathcal{D}_2)(x_1 - a_1) - (\mathcal{L}_1 - \mathcal{D}_1)(x_2 - a_2)],$$

for some $a \in \mathbb{N}$.

Proof. (1) It follows from Proposition 22, Theorem 23 and Remark 28.

1. Observe that for $n = 2$, $\gamma(F_1, F_2) = \gamma(H_1, H_2) = 0$. By statement 1 and Theorem 21, $\partial\mathrm{Res}(F_1, F_2)(x_1, x_2) = 0$ is the implicit equation of $\mathcal{P}_2(x_1, x_2, u)$. By Lemma 29 there exist $\mathcal{D}_1, \mathcal{D}_2 \in \mathbb{K}[\partial]$ such that $(\mathcal{L}_2 - \mathcal{D}_2)\mathcal{L}_1(u) - (\mathcal{L}_1 - \mathcal{D}_1)\mathcal{L}_2(u) = 0$. Let

$$P(x_1, x_2) = (\mathcal{L}_2 - \mathcal{D}_2)(x_1 - a_1) - (\mathcal{L}_1 - \mathcal{D}_1)(x_2 - a_2).$$

Since $\partial\mathrm{Res}^h(H_1, H_2) \neq 0$ then there exists $k \in \{1, 2\}$ such that the determinant $\det(S_k) = \partial P/\partial x_{kN-o_k} \neq 0$. The result follows from Theorem 8. □

Algorithm 2. Given the system $\mathcal{P}_2(x_1, x_2, u)$ of DPPEs this algorithm returns its implicit equation $A(X)$.

(1) $\mathcal{L} := \mathrm{gcrd}(\mathcal{L}_1, \mathcal{L}_2)$.
(2) If $\mathcal{L} \notin \mathbb{K}$ then compute $\mathcal{L}'_i$ such that $\mathcal{L}_i = \mathcal{L}'_i\mathcal{L}$ and set $\mathcal{L}_i := \mathcal{L}'_i$, $i = 1, 2$.
(3) Compute $\mathcal{D}_1$ and $\mathcal{D}_2$.
(4) $A(X) := (\mathcal{L}_2 - \mathcal{D}_2)(x_1 - a_1) - (\mathcal{L}_1 - \mathcal{D}_1)(x_2 - a_2)$.

The Maple packages OreTools, DEtools and Ore-algebra can be used to compute $\mathrm{gcrd}(\mathcal{L}_1, \mathcal{L}_2)$, (Abramov et al., 2005). Also, we can compute $\mathcal{D}_1$ and $\mathcal{D}_2$ following the algorithm in the proof of Lemma 29.

8.2. Case n = 3

We restrict to the case $\mathbb{K} = \mathbb{C}$ and consider the system of DPPEs

$$\mathcal{P}_3(X, U) = \begin{cases} x_1 = a_1 - \mathcal{L}_{11}(u_1) - \mathcal{L}_{12}(u_2) \\ x_2 = a_2 - \mathcal{L}_{21}(u_1) - \mathcal{L}_{22}(u_2) \\ x_3 = a_3 - \mathcal{L}_{31}(u_1) - \mathcal{L}_{32}(u_2) \end{cases} \tag{12}$$

where $X = \{x_1, x_2, x_3\}$, $U = \{u_1, u_2\}$ and $\mathcal{L}_{ij} \in \mathbb{C}[\partial]$. These differential operators commute, therefore the polynomial

$$\begin{aligned} P(X) &= \mathcal{L}_{21}\mathcal{L}_{32}(x_1 - a_1) - \mathcal{L}_{22}\mathcal{L}_{31}(x_1 - a_1) - \mathcal{L}_{11}\mathcal{L}_{32}(x_2 - a_2) \\ &\quad + \mathcal{L}_{12}\mathcal{L}_{31}(x_2 - a_2) + \mathcal{L}_{11}\mathcal{L}_{22}(x_3 - a_3) - \mathcal{L}_{12}\mathcal{L}_{21}(x_3 - a_3) \end{aligned}$$

belongs to the implicit ideal of $\mathcal{P}_3(X, U)$.

For $i = 1, 2, 3$, we have $F_i(X, U) = x_i - a_i + H_i(U)$ and $H_i(U) = \mathcal{L}_{i1}(u_1) + \mathcal{L}_{i2}(u_2)$ of order o_i. Let $\gamma = \gamma(F_1, F_2, F_3)$ and let $S_{\gamma i}$ be the matrix defined in Section 5. It is easily obtained that

$$\det(S_{\gamma i}) = \frac{\partial P}{\partial x_{iN-o_i-\gamma}}.$$

If $\det(S_{\gamma i}) \neq 0$ for some $i \in \{1, 2, 3\}$ then it follows from Theorem 8 that

$$\partial\mathrm{CRes}(F_1, F_2, F_3)(X) = (-1)^a \partial\mathrm{CRes}^h(H_1, H_2, H_3)P(X),$$

$a \in \mathbb{N}$.

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