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A well-structured framework for analysing petri net extensions

Alain Finkel[a,*], Pierre McKenzie[b,1], Claudine Picaronny[a]

[a] *Laboratoire Spécification et Vérification (CNRS URA 8643), École Normale Supérieure de Cachan, 61, Avenue du Président Wilson, 94235 Cachan Cedex, France*
[b] *Informatique et recherche opérationnelle, Université de Montréal, C.P. 6128, succ. Centre-Ville, Montréal, Que., Canada H3C 3J7*

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Abstract

Transition systems defined from recursive functions $\mathbb{N}^p \to \mathbb{N}^p$ are introduced and named WSNs, or *well-structured nets*. Such nets sit conveniently between Petri net extensions and general transition systems. In the first part of this paper, we study decidability properties of WSN classes obtained by imposing natural restrictions on their defining functions, with respect to termination, coverability, and four variants of the boundedness problem. We are able to precisely answer almost all the questions which arise, thus gaining much insight into old and new generalized Petri net decidability results. In the second part, we specialize our analysis to WSNs defined from affine functions, which elegantly encompass most Petri net extensions studied in the literature. Again, we study decidability properties of natural classes of affine WSN with respect to the above six computational problems. In particular, we develop an algorithm computing limits of iterated nonnegative affine functions, in order to decide the *path-place* variant of the boundedness problem for *non-negative* affine WSN.
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* Corresponding author. Fax: +33 1 47 40 75 29.
E-mail address: finkel@lsv.ens-cachan.fr (A. Finkel).
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1. Introduction

A *transition system* is a set S endowed with a transition relation "$\rightarrow$," i.e., with a binary relation on the set S. It is a WSTS, i.e., a *well-structured* transition system, if a reflexive and transitive well-quasi-ordering "$\leqslant$" on S exists which is compatible with "$\rightarrow$" (i.e., such that $s_1 \leqslant t_1$ and $s_1 \rightarrow s_2$ imply the existence of a sequence of transitions leading from t_1 to $t_2 \geqslant s_2$ [9,1,11]). We deal in this paper with infinite state transition systems, with WSTS, and with Petri net extensions [3,4,14,17,20].

Our motivating theme is that Petri net extensions can naturally and fruitfully be defined and studied from the perspective of WSTS. This theme is only marginally new: WSTS were introduced as abstractions of Petri nets in the first place, as a means of identifying the essential properties of Petri net problems, algorithms, and extensions. But WSTS in fact capture many other infinite state systems [11], and WSTS are so general that (1) only rather coarse Petri net extension decidability results, concerning boundedness detection, for example, can be dealt with from WSTS, and (2) the classification of Petri net extensions afforded by WSTS is not as complete as one might have hoped.

In this paper, we respond to both weaknesses of the WSTS with the introduction of the WSN, or *well-structured net*. A WSN is a WSTS in which the set S is $\mathbb{N}^p$ and the transition relation "$\rightarrow$" is given by a finite set of recursive functions with upward-closed domains, i.e., with domains $K \subseteq \mathbb{N}^p$ such that $u \in K$ and $u \leqslant v$ imply $v \in K$. We establish the suitability of the WSN model for a systematic study of Petri net extensions and their decidability properties.

Our paper is formed of two parts. The first part deals with WSN in general. The second part deals with WSN defined from affine functions.

In the first part, we define progressively stronger properties of functions used in the definitions of WSNs: nondecreasing (i.e., $u < v \Rightarrow f(u) \leqslant f(v)$), increasing (i.e., $u < v \Rightarrow f(u) < f(v)$), ρ-increasing (i.e., increasing, with the added requirement that knowledge of the components along which $u < v$ provides information on the components along which $f(u) < f(v)$), strongly increasing (i.e., increasing componentwise), each with or without the ω-*recursivity* hypothesis that the extension of the function to the domain $(\mathbb{N} \cup \{\omega\})^p$ is recursive. These properties give rise to two groups of four WSN classes, and we exhaustively study these classes with respect to six decision problems. These problems are the usual termination problem (is the reachability tree of a given state s_0 finite?), coverability problem (is there a state s' in the reachability set of a given state s_0 which covers a given state s?), boundedness problem (is the reachability set of a given state s_0 finite?), place-boundedness problem (is the component i bounded, given a start state s_0?), together with two new variants of the boundedness problem introduced here to demonstrate the subtleties of boundedness detection. Let us note $\sigma(s_0)$ the state reached from s_0 by applying σ. These two variants are the path-unbounded-witness problem (given s_0 and a sequence of transitions σ such that $s_0 < \sigma(s_0)$, is there an unbounded component in the sequence of states starting at s_0 and obtained by repeatedly applying σ?) and the path-place-boundedness problem (given s_0 and a sequence of transitions σ such that $s_0 < \sigma(s_0)$, is the component i unbounded in the sequence of states starting at s_0 and obtained by repeatedly applying σ?).

Our results are summarized on Figs. 1 and 2: only four of the 48 relevant decidability questions (for some of them, answers were already implicitly or explicitly known) remain open. We also claim new results on ω-recursive functions.

S_1	WSN (well-structured nets)
S_2	increasing WSN
S_3	ρ-increasing WSN
S_4	strongly increasing WSN
$\bar{S}_1$	ω-WSN
$\bar{S}_2$	increasing ω-WSN
$\bar{S}_3$	ρ-increasing ω-WSN
$\bar{S}_4$	strongly increasing ω-WSN

Fig. 1. Abbreviations.

	S_1	S_2	S_3	S_4	$\bar{S}_1$	$\bar{S}_2$	$\bar{S}_3$	$\bar{S}_4$
Termination	D	D	D	D	D	D	D	D
Coverability	U	U	U	U	D	D	D	D
Boundedness	U	D	D	D	U	D	D	D
Path-unbounded-witness	U	U	D	D	?	?	D	D
Path-place-boundedness	U	U	U	U	U	?	?	D
Place-boundedness	U	U	U	U	U	U	U	D

Fig. 2. Decidability (D) or undecidability (U) for well-structured nets.

We single out, from the first part of our paper, the following results:

- Coverability for ω-WSN, i.e., WSN whose functions satisfy the ω-recursivity hypothesis, is shown decidable, while coverability is shown undecidable for our most restrictive WSN class (i.e., strongly increasing WSN) in the absence of the ω-recursivity hypothesis.
- The decidability status of the four boundedness problems for WSN is resolved. This provably separates three of our WSN classes, and the fourth class is distinguished from the others when viewed as a ω-WSN class.
- Boundedness is shown decidable for increasing WSN but undecidable for ω-WSN.
- Path-unbounded-witness is shown decidable for ρ-increasing WSN but undecidable for increasing WSN.
- Path-place-boundedness, known to be decidable for reset Petri nets (and for transfer Petri nets), is shown undecidable for strongly increasing WSN and for ω-WSN.
- Place-boundedness is shown decidable for strongly increasing ω-WSN. This is proved by showing that the coverability tree algorithm of Karp and Miller [10,15,16,18] still applies when the exact computation of the limit of an infinite repetitive sequence is replaced by the computation of a good enough approximation. Yet, the place-boundedness is shown undecidable for both strongly increasing WSN and ρ-increasing ω-WSN.

In the second part of our paper, we specialize our study to affine WSN, i.e., to WSN defined from affine functions. We do this because testing for nontrivial properties of general recursive functions is undecidable, and because, in the concrete cases likely to be encountered in practice, functions are either affine, or their properties are known a priori, or these are easy to test. In Fig. 3, we record some easy (but appealing) connections between affine WSN, Petri net extensions, and general WSN. For example, generalized transfer Petri nets can be seen as WSN defined from affine functions $f(X) = AX + B$ satisfying $A \geqslant \mathbf{0}$ with every column of the matrix A different from $\mathbf{0}$. Affine WSN thus capture a wealth of known Petri net extensions, including Petri nets themselves [23], Reset Petri nets [7], Post SM nets [27], Double Petri nets, and they suggest new extensions of independent interest.

We then developed a nontrivial algorithm to solve the following problem: given a nonnegative $p \times p$-matrix A and two nonnegative p-vectors B and X, compute $d \geqslant 1$ such that $\lim_{n\to\omega} f^{nd}(X)$ exists, where f is the affine function $f(X) = AX + B$, and compute this limit. We use this algorithm to solve the path-place-boundedness problem for affine WSN defined from nonnegative matrices and vectors. This WSN class is incomparable with Petri Nets and it does not have an immediate practical value but it provides an interesting instance in which the path-place boundedness problem is decidable.

Finally, we record the decidability status of our six computational problems, for five prominent classes of affine WSN, including Petri nets and Self-Modifying nets [27,28]: only two out of the 30 resulting questions (whose answers were known for Petri nets and Self-Modifying nets) remain open. Fig. 4 reports these results.

Section 2 in this paper introduces our various functions, the notion of limit, ω-recursivity, and proves relevant undecidability results. Section 3 formally defines well-structured nets and our computational problems. Section 4 examines decidability of termination and coverability for WSN. Section 5 does the same for the four boundedness problems, Section 6 summarizes our decidability

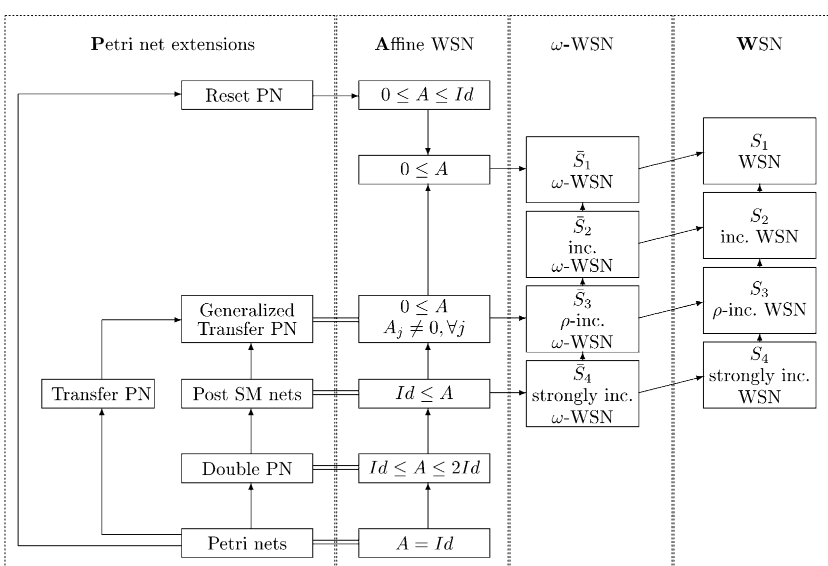

Fig. 3. The world of well-structured nets. An arrow $C \to D$ indicates $C \subset D$ and a double line indicates equality.

	$A \geq \mathbf{0}$	$A \geq \mathbf{0}, B \geq \mathbf{0}$	$A_j > \mathbf{0}, \forall j$	$A \geq Id$	$A = Id$
Termination	D	D	D	D	D
Coverability	D	D	D	D	D
Boundedness	U	?	D	D	D
Path-unbounded-witness	D	D	D	D	D
Path-place boundedness	D	D	D	D	D
Place-boundedness	U	?	U	D	D

Fig. 4. Affine ω-well-structured nets.

results for WSN. Section 7 discusses affine WSN, their relationship with WSN and extended Petri nets, and their decidability properties. Section 8 concludes.

2. Notation and preliminaries

Sets. When X is a set, a *word* $v \in X^*$ is a possibly empty finite sequence of elements of X; we write $|v|$ for the length of v.

We write $[p]$ for $\{1, 2, \ldots, p\}$, $\mathbb{N}^+$ for $\mathbb{N} \setminus \{0\}$, and $\mathbb{N}_\omega$ for the standard completion $\mathbb{N} \cup \{\omega\}$ of the set $\mathbb{N}$ of nonnegative integers (where $n < \omega$ for any $n \in \mathbb{N}$).

Upward-closed sets. Fix any integer $p \geqslant 1$. Let $\leqslant$ be the usual ordering on $\mathbb{N}_\omega$. We also write $\leqslant$ for the ordering on $\mathbb{N}_\omega^p$ defined by: for all $u, v \in \mathbb{N}_\omega^p$, $u \leqslant v$ if for every $i \in [p]$, $u(i) \leqslant v(i)$, where $u(i)$ is the ith component of u. We write $u < v$ if $u \leqslant v$ and $u \neq v$. Addition in $\mathbb{N}_\omega$ and in $\mathbb{N}_\omega^p$ is defined componentwise, with $x + \omega = \omega + x = \omega$ for any $x \in \mathbb{N}_\omega$. Let $U + V$ denote the *sum* of two subsets $U, V \subseteq \mathbb{N}_\omega^p$: $U + V = \{u + v : u \in U, v \in V\}$. We say that a set $K \subseteq \mathbb{N}^p$ is *upward-closed* (for $\leqslant$) if $K = K + \mathbb{N}^p$.

A *basis* of an upward-closed set K is a set $\mathcal{B}$ such that $K = \mathcal{B} + \mathbb{N}^p$. We define the *downward-closure* of $u \in \mathbb{N}_\omega^p$, written $\downarrow u$, as $\downarrow u = \{v \in \mathbb{N}^p : v \leqslant u\}$. The two well-known facts which follow can be found in [12], or in [11] where their proofs are briefly recalled:

1. Every upward-closed set $K \subseteq \mathbb{N}^p$ has a finite basis.
2. Any strictly increasing sequence $K_0 \subset K_1 \subset K_2 \cdots$ of upward-closed sets in $\mathbb{N}^p$ is finite.

We will require another result due to Valk and Jantzen:

Theorem 2.1 ([30]). *A finite basis of an upward-closed set $K \subseteq \mathbb{N}^p$ is effectively computable iff for any vector $u \in \mathbb{N}_\omega^p$ the predicate $\downarrow u \cap K \neq \emptyset$ is decidable.*

Transition systems. A *transition system* (TS) is a structure $\mathcal{S} = \langle S, \rightarrow, \ldots \rangle$ where $S = \{s, t, \ldots\}$ is a set of *states*, and $\rightarrow \subseteq S \times S$ is a set of *transitions*. For every $s \in S$, we write *Succ*(s) (respectively *Pred*(s)) for the set $\{t \in S : s \rightarrow t\}$ of immediate *successors* of s (respectively for the set $\{t \in S : t \rightarrow s\}$ of immediate *predecessors* of s). We write $s \xrightarrow{+} t$ to mean that there exist $k > 0$ and a sequence of states $s_0 = s, s_1, \ldots, s_k = t$ such that $s_0 \rightarrow s_1 \rightarrow \cdots \rightarrow s_k$. We write $s \xrightarrow{*} t$ to mean that $s \xrightarrow{+} t$ or $s = t$. We say that $\mathcal{S}$ is *finitely branching* if *Succ*(s) is finite for each $s \in S$; the usefulness of this property is tied to König's lemma [19] stating that an infinite tree, finitely branching, has an infinite path.

Reachability. A state t of a TS $\mathcal{S}$ is *reachable* from a state s if $s \xrightarrow{*} t$.

- The *reachability set* of $\mathcal{S}$ from s_0 is denoted $RS(\mathcal{S}, s_0)$ and is defined as the set of states reachable from s_0.
- The reachability tree $RT(\mathcal{S}, s_0)$ of $\mathcal{S}$ with the initial state s_0 is a directed unordered tree where nodes are labelled by the states of S. The root node is labelled by s_0. A node labelled by s has a son labelled by t iff $t \in Succ(s)$. Moreover, any two different sons of a node must have different labels.

Orders and limits. Consider an infinite nondecreasing sequence $(u_n)_{n \geqslant 1}$ of natural numbers. This sequence is *bounded* if there exists $v \in \mathbb{N}$ such that $u_n \leqslant v$ holds for every $n \in \mathbb{N}$. We define

$$\lim_{n\to\omega} u_n = \begin{cases} \omega & \text{if } (u_n)_{n\geqslant 1} \text{ is not bounded,} \\ \max\{u_n : n \geqslant 1\} & \text{otherwise.} \end{cases}$$

Now consider a nondecreasing infinite sequence $(u_n)_{n \geqslant 1}$ of p-tuples $u_n \in \mathbb{N}^p$. We also define:

$$\lim_{n\to\omega} u_n = \left(\lim_{n\to\omega} u_n(1), \lim_{n\to\omega} u_n(2), \ldots, \lim_{n\to\omega} u_n(p)\right).$$

For our purpose, we shall need the following orders: for any nonempty subset $Q \subseteq [p]$, let $<_Q$ be the strict order relation, called *Q-strict order*, defined on $\mathbb{N}_\omega^p$ by:

$$u <_Q v \quad \text{if} \quad (u \leqslant v) \quad \text{and} \quad (\forall i \in Q, [u(i) < v(i)]).$$

Note that $u <_Q v$ implies $u <_{Q'} v$ for any $\emptyset \neq Q' \subseteq Q$.

The Q-strict order relation is strictly contained in the usual order $<$. It is well-known that the ordering $<$ on $\mathbb{N}_\omega^p$ is a well-ordering [6]: for any infinite sequence $u_0, u_1, \ldots \in \mathbb{N}_\omega^p$, there exist two indices $i < j$ with $u_i \leqslant u_j$. The Q-strict orders (for all Q's) can be used to refine the latter statement:

Lemma 2.2. *For any infinite sequence $u_0 < u_1 < u_2 \ldots$ in $\mathbb{N}_\omega^p$, there exists a nonempty maximal (w.r.t. $\subseteq$) set $Q \subseteq [p]$ such that there is an infinite subsequence $u_{i_0} <_Q u_{i_1} <_Q u_{i_2} \ldots$ (with $i_0 < i_1 < i_2 \ldots$).*

Proof. Let $u_0 < u_1 < u_2 \ldots$ be an infinite sequence in $\mathbb{N}_\omega^p$. Recall that any increasing sequence $n_0 \leqslant n_1 \leqslant n_2 \ldots$ in $\mathbb{N}_\omega$ either is stationary (i.e., there exists an integer k such that $n_k = n_l, \forall l \geqslant k$) or tends to ω. As the sequence $(u_n)_{n \geqslant 0}$ takes an infinity of values, not all non-decreasing sequences $(u_n(i))_{n \geqslant 0}$ ($\forall i \in [p]$), can be stationary. Therefore, we let Q be the (non empty) set of i such that $(u_n(i))_{n \geqslant 0}$ is non stationary. □

2.1. Transitions as functions

Throughout this paper, we shall consider transition systems whose sets of states are subsets of $\mathbb{N}^p$, for some p in $\mathbb{N}$. Transitions are considered as recursive functions from $\mathbb{N}^p$ to $\mathbb{N}^p$. Thus, transition functions are not necessarily total. Consider a function $f : X \to Y$. We write *Dom* f for $\{x \in X \mid f(x) \text{ is defined}\}$. We adopt the convention that we compose functions from right to left, i.e.,

$gf(x) = g(f(x))$. The notion of growth signature will help us define classes of functions which will illustrate the subtleties of the boundedness question:

Growth signature. The *growth signature* of an arbitrary function $f : \mathbb{N}^p \to \mathbb{N}^p$ is the function ρ_f defined as follows:

$$\begin{aligned}\rho_f : \{Q \subseteq [p] : Q \neq \emptyset\} &\longrightarrow 2^{\{Q \subseteq [p] : Q \neq \emptyset\}}\\ Q &\longmapsto \{Q' \subseteq [p] : \forall u, v \in Dom\ f, [u <_Q v \Longrightarrow f(u) <_{Q'} f(v)]\}.\end{aligned}$$

Thus the knowledge that $Q' \in \rho_f(Q)$ provides the information that for any $u, v \in Dom\ f$, $f(u) <_{Q'} f(v)$ whenever $u <_Q v$.
Types of functions. A function $f : \mathbb{N}^p \to \mathbb{N}^p$ is

(type 1) *nondecreasing* if $u < v \Rightarrow f(u) \leqslant f(v)$ for all $u, v \in Dom\ f$,
(type 2) *increasing* if $u < v \Rightarrow f(u) < f(v)$ for all $u, v \in Dom\ f$,
(type 3) *ρ-increasing* if $\rho_f(Q) \neq \emptyset$ for all nonempty $Q \subseteq [p]$,
(type 4) *strongly increasing* if $Q \in \rho_f(Q)$ for all nonempty $Q \subseteq [p]$.

Intuition. A function is strongly increasing if whenever $u, v \in Dom\ f$ and $u <_Q v$, it is guaranteed that $f(u) <_Q f(v)$. The function is only ρ-increasing[1] if the guarantee is only that $f(u) <_{Q'} f(v)$ for the non-empty sets found in $\rho_f(Q)$; these Q' may be incomparable with Q. Yet weaker, the increasing condition only offers the guarantee that $f(u) <_{Q'} f(v)$ for some Q', but the latter Q' may depend on the particular choice of u and v. Observe that there exist increasing functions that are not ρ-increasing. Indeed, the total (hence upward-closed) increasing function $f : \mathbb{N}^2 \to \mathbb{N}^2$ given by

$$f(i, j) = \begin{cases} (0, 0) & \text{if } i = j = 0,\\ (i, i-1) & \text{if } j = 0 \text{ and } i \text{ is odd},\\ (i-1, i) & \text{if } j = 0 \text{ and } i > 0 \text{ is even},\\ (i+1, i+j) & \text{if } j > 0 \end{cases}$$

is not ρ-increasing (this is because, for each $n > 0$, $(2n, 0) <_{\{1\}} (2n+1, 0) <_{\{1\}} (2n+2, 0)$ and applying f to these pairs yields $(2n-1, 2n) <_{\{1\}} (2n+1, 2n) <_{\{2\}} (2n+1, 2n+2)$, but $\neg((2n+1, 2n) <_{\{1\}} (2n+1, 2n+2))$ and $\neg((2n-1, 2n) <_{\{2\}} (2n+1, 2n))$, so that no nonempty set $R \subseteq \{1, 2\}$ fulfills the condition that, for each $u <_{\{1\}} v$ with (u, v) in $\mathbb{N} \times \mathbb{N}$, $f(u) <_R f(v)$).

Proposition 2.3. *Let f and g be two functions from $\mathbb{N}^p$ to $\mathbb{N}^p$.*

(1) *If f and g are of type t above, $1 \leqslant t \leqslant 4$, then gf is again of type t.*
(2) *If f is nondecreasing and both f and g have upward-closed domains, then the domain of gf is upward-closed.*

[1] The name ρ-increasing was given to these functions because the letter ρ resembles the shape of a path in a finite graph which ends in a cycle; see Theorem 5.3.

(3) *If f and g are recursive, then gf is recursive.*
(4) *If f is of type $t, 1 \leqslant t \leqslant 4$, then f is of type $s, \forall 1 \leqslant s \leqslant t$.*

Proof. Straightforward. We nonetheless argue (1) for the case of ρ-increasing functions. Let f and g be ρ-increasing functions. Consider $\emptyset \neq Q \subseteq [p]$. Pick $u, v \in Dom\ gf$. Then $u, v \in Dom\ f$ and $f(u), f(v) \in Dom\ g$. Hence by the ρ-increasing properties of f and g,

$$u <_Q v \Longrightarrow f(u) <_{Q'} f(v) \Longrightarrow g(f(u)) <_{Q''} g(f(v)),$$

where Q' is any nonempty set in $\rho_f(Q)$ and where Q'' is any nonempty set in $\rho_f(Q')$. Hence for any Q, $\rho_{gf}(Q) \neq \emptyset$. $\square$

Extension of a nondecreasing function with upward-closed domain.
The *extension* $\bar{f} : \mathbb{N}_\omega^p \to \mathbb{N}_\omega^p$ of a nondecreasing function $f : \mathbb{N}^p \to \mathbb{N}^p$ with upward-closed domain is defined by setting $Dom\ \bar{f} = Dom\ f + \mathbb{N}_\omega^p$ and, for any $v \in Dom\ \bar{f}$:

$$\bar{f}(v) = \lim_{n \to \omega} f(v_n),$$

where $(v_n)_{n \geqslant 0}$ is any infinite nondecreasing sequence such that $v = \lim_{n\to\omega} v_n$. This definition is correct because since f is nondecreasing, $(f(v_n))_{n \geqslant 0}$ is also a nondecreasing sequence and it admits a limit in $\mathbb{N}_\omega^p$; moreover for every $v \in \mathbb{N}_\omega^p$, there exists an infinite nondecreasing sequence $(v_n)_{n \geqslant 0}$, $v_n \in \mathbb{N}^p$ such that $v = \lim_{n\to\omega} v_n$ (in fact, there are an infinite number of sequences having v as limit). Finally, the evaluation of $\bar{f}(v)$ does not depend on the choice of the sequence $(v_n)_{n \geqslant 0}$: given two infinite nondecreasing sequences $(v_n)_{n \geqslant 0}$ and $(v'_n)_{n \geqslant 0}$ such that $v = \lim_{n\to\omega} v_n = \lim_{n\to\omega} v'_n$, one has that for every n, there exist p, q such that: $v_n \leqslant v'_{n+p} \leqslant v_{n+p+q}$, hence we also have: for every n, $f(v_n) \leqslant f(v'_{n+p}) \leqslant f(v_{n+p+q})$. Then, $\lim_{n\to\omega} f(v_n) = \lim_{n\to\omega} f(v'_n)$. Note that if $v \in Dom\ f$ then $\bar{f}(v) = f(v)$.

ω-recursive functions. Let f be a nondecreasing function $f : \mathbb{N}^p \to \mathbb{N}^p$ with upward-closed domain. We say f is *ω-recursive* if its extension $\bar{f}$ is recursive.

We will see in Section 4 that computing $\bar{f}$ plays a crucial role in deciding the coverability problem and the place-boundedness problem. Here, we note that not all recursive functions f have a computable extension $\bar{f}$. Indeed, although $\bar{f}$ is recursive for any fixed nondecreasing recursive function $f : \mathbb{N} \to \mathbb{N}$ (because $\bar{f}$ agrees with f on $\mathbb{N}$, and because $\bar{f}(\omega)$ can be hardwired into a machine, i.e., there exists a machine computing $\bar{f}$ though we may not know which one), even for a fixed total recursive function $g : \mathbb{N}^2 \to \mathbb{N}^2$ of type 4, it may not be possible to compute $\bar{g}$.

We write TM_j for the jth Turing machine (in a classical enumeration) which moreover begins by writing the integer j on its tape. We say that TM_j halts if TM_j halts on input j.

Proposition 2.4. *There is a strongly increasing total recursive function $g : \mathbb{N}^2 \to \mathbb{N}^2$ which is not ω-recursive.*

Proof . Let $g(m, n) = (m + |\{j \leqslant m\ : \mathrm{TM}_j \text{ halts in at most } n \text{ steps}\}|, n)$. Then g is a strongly increasing total recursive function, but $\bar{g}$ is not computable because

$$\overline{g}(m, \omega) = \overline{g}(m-1, \omega) + \begin{cases} (2,0) & \text{if the } \mathrm{TM}_m \text{ halts,} \\ (1,0) & \text{otherwise.} \end{cases} \qquad \square$$

The rest of the section studies the impact of the properties of f on the decidability properties of $\bar{f}$.

Consider adapting in the natural way the nondecreasing, increasing, and strongly increasing properties to the case of functions $f : \mathbb{N}^p \to \mathbb{N}$ for $p > 1$. Then, increasing and strongly increasing become synonymous properties. Furthermore, as for any u in $\mathbb{N}^p_\omega$ having at least one component equal to ω, we must have $\bar{f}(u) = \omega$, the extension $\bar{f}$ of any increasing total recursive function $f : \mathbb{N}^p \to \mathbb{N}$ is recursive. On the other hand, projecting onto its first component the function g used in proving Proposition 2.4 yields a nondecreasing total recursive function $g' : \mathbb{N}^2 \to \mathbb{N}$ that is not ω-recursive.

Returning to nondecreasing total recursive functions $f : \mathbb{N} \to \mathbb{N}$, whose extensions $\bar{f}$ are recursive as we have seen, we might hope to be able to compute the description of a Turing machine computing $\bar{f}$, given one computing a nondecreasing total f (this is clearly possible when f is increasing). The next proposition shows that this is impossible. Let *NDEC* be the set of Turing machines m computing a nondecreasing total recursive function $f_m : \mathbb{N} \to \mathbb{N}$, and let $NDEC_\omega \subset NDEC$ comprise those m for which $\bar{f}_m(\omega) = \omega$. Now assume to the contrary, that there is a total recursive function g such that $g(m)$ is a machine computing $\bar{f}_m$ whenever $m \in NDEC$. Let L be the language of all m such that the machine $g(m)$ halts on ω and outputs "ω." Then L is a recursively enumerable language such that $L \cap NDEC = NDEC_\omega$, which contradicts the following:

Proposition 2.5. *Any language L such that*

$$L \cap NDEC = NDEC_\omega$$

is not recursively enumerable.

Proof. Consider such an L. Note that L is not necessarily an index set so that Rice's theorem (for recursively enumerable index sets) does not apply. Consider for each m the Turing machine $\mathrm{M}_{\varphi(m)}$ computing the function $f_{\varphi(m)} : \mathbb{N} \to \mathbb{N}$ defined as

$$f_{\varphi(m)}(n) = \begin{cases} t & \text{if the } \mathrm{TM}_m \text{ halts in exactly } t \text{ steps and } t < n, \\ n & \text{otherwise.} \end{cases}$$

Note that $\varphi(m) \in NDEC$ for every m. Now the TM_m does not halt iff $\bar{f}_{\varphi(m)}(\omega) = \omega$ iff $\varphi(m) \in NDEC_\omega$ iff $\varphi(m) \in L \cap NDEC$ iff $\varphi(m) \in L$. Hence, the recursive map φ is a many-one reduction from the complement of the halting problem to L, proving that the latter is not recursively enumerable. $\square$

2.2. Transitions as affine functions

An affine[2] function $f : \mathbb{N}^p \to \mathbb{N}^p$ is given by $f(X) = AX + B$ for some $A = (A_{i,j}) \in \mathbb{N}^{p \times p}$ and $B \in \mathbb{Z}^p$ and a recursive definition domain $Dom\ f \subseteq \{X \in \mathbb{N}^p : AX + B \geqslant 0\}$.

[2] We restrict the usual sense of affine functions. In our context, all affine functions have a non negative linear part.

We allow $Dom\ f$ to differ from $\{X \in \mathbb{N}^p : AX + B \geqslant 0\}$, to be able to simulate transitions of Petri nets. For instance, suppose that, in a Petri net with p places, the transition t only tests whether there is a token in place 1. This transition is the identity function $t(X) = X$ (hence $A = Id$ and $B = 0$) and $t(X) \geqslant 0$ for every $X \in \mathbb{N}^p$. Yet its definition domain is $Dom\ t = \mathbb{N}^+ \times \mathbb{N}^{p-1}$ and it is not equal to $\mathbb{N}^p$.

We denote by A_j the jth column of the matrix A, for any j in $[p]$. We write $A \geqslant 0$ to mean that $A_{i,j} \geqslant 0$, for all i, j and $A \geqslant Id$ to mean that for all $i, A_{ii} \geqslant 1$ and $A \geqslant 0$.

The following holds for every affine function f defined by $f(X) = AX + B$:

1. f is nondecreasing,
2. f is increasing iff f is ρ-increasing iff $A_j \neq 0$, for every j in $[p]$,
3. f is strongly increasing iff $A \geqslant Id$.

Hence, given an affine function f by way of its defining matrix A and vector B, it is straightforward to determine its "increasing" property. Moreover, belonging to the set $\{X \in \mathbb{N}^p : AX + B \geqslant 0\}$ is easily tested, and it is easy to see that this set is upward-closed whenever f is nondecreasing ($A \geqslant 0$).

The composition of two affine functions is affine and every affine function is recursive. Furthermore, the extension of a nondecreasing affine function f is recursive as the extension can also be defined by its affine form $AX + B$ on $X \in \mathbb{N}^p_\omega$ with the usual conventions $0\omega = 0$ and $m\omega = \omega$, for all $m \in \mathbb{N}$ such that $m > 0$.

3. Well-structured nets

We recall [11] that a *well-structured transition system* (WSTS) is a transition system $S = \langle D, \rightarrow, \leqslant \rangle$ equipped with a reflexive and transitive relation $\leqslant \subseteq D \times D$ fulfilling the following two conditions:

(1) *well-quasi-ordering*: for any infinite sequence $s_0, s_1, \ldots \in D$, there exist two indices $i < j$ with $s_i \leqslant s_j$.
(2) *compatibility*: $\leqslant$ is (upward) compatible with $\rightarrow$, i.e., for any $s_1 \leqslant t_1$ and transition $s_1 \rightarrow s_2$, there exists t_2 such that $t_1 \xrightarrow{*} t_2$ and $s_2 \leqslant t_2$.

Well-structured nets. A *well-structured net* (WSN) is a triple $S = \langle \mathbb{N}^p, F, \leqslant \rangle$ for some p, where F is a finite set of nondecreasing recursive functions defined on upward-closed subsets of $\mathbb{N}^p$.

Intuitively, an ω-well-structured net is a generalization of a WSN defined on $\mathbb{N}^p_\omega$. *ω-Well-structured nets.* A *ω-well-structured net* (ω-WSN) is a triple $S = \langle \mathbb{N}^p_\omega, F, \leqslant \rangle$ for some p, where F is a finite set of nondecreasing recursive functions defined on upward-closed subsets of $\mathbb{N}^p_\omega$ such that for every function $f \in F$, we have $f = \bar{f}_{\mathbb{N}^f}$ where $f_{\mathbb{N}^f} : \mathbb{N}^f \rightarrow \mathbb{N}^f$ is the restriction of f on $\mathbb{N}^f$.
A WSN $S = \langle \mathbb{N}^p, F, \leqslant \rangle$ or a ω-WSN $S = \langle \mathbb{N}^p_\omega, F, \leqslant \rangle$ are, respectively,

- *increasing* if every function in F is increasing,
- *ρ-increasing* if every function $f \in F$ is ρ-increasing and an algorithm which computes the growth signature of f is given,
- *strongly increasing* if every function in F is strongly increasing.

Note that a WSN $S = \langle \mathbb{N}^p, F, \leqslant \rangle$ gives rise to the WSTS $\langle \mathbb{N}^p, \xrightarrow{F}, \leqslant \rangle$, where $s \xrightarrow{F} t$ for $(s,t) \in \mathbb{N}^p \times \mathbb{N}^p$ is defined to mean that $f(s) = t$ for some $f \in F$. Therefore, we use for S all notions defined for $\langle \mathbb{N}^p, \xrightarrow{F}, \leqslant \rangle$ (reachability...). In the context of a WSN, a place is an integer; we can speak of the boundedness of a place: given S and an initial state s_0, the place i is *bounded* if there exists $k \in \mathbb{N}$ such that for each $s \in RS(S, s_0), s(i) \leqslant k$. Note also that every WSN is finitely branching, because F is finite.

Remark 3.1. Every Petri net N with p place is a WSN $S = \langle \mathbb{N}^p, F, \leqslant \rangle$ such that every function (called transition) $t \in F$ may be written as $t(X) = X + v_t$, where v_t is an integer vector of dimension p. Reset Petri nets are extended Petri nets in which a transition may clear (reset) a place. Formally, a reset Petri net N with p places is a WSN such that every function t in F is an affine function $t(X) = A_t X + v_t$ in which the matrix (of integers) A_t satisfies $Id \geqslant A_t \geqslant 0$, where Id and 0 are the identity and the null matrices respectively, and inequality is taken componentwise.

Let $S = \langle \mathbb{N}^p_\omega, F, \leqslant \rangle$ a ω-WSN, $s_1, s_2 \in \mathbb{N}^p_\omega$ and $\sigma = f_k f_{k-1} \ldots f_1$, with $f_1, \ldots, f_k \in F$. We write $s_1 \xrightarrow{\sigma} s_2$ to mean that there exist $t_1, \ldots, t_{k+1}$ such that $s_1 = t_1 \xrightarrow{f_1} t_2 \xrightarrow{f_2} t_3 \rightarrow \cdots \xrightarrow{f_k} t_{k+1} = s_2$.

Given a state s_o in $\mathbb{N}^p$ and $\sigma = f_k f_{k-1} \ldots f_1$, with $f_1, \ldots, f_k \in F$, we define $RS(S, s_0, \sigma) = \{s \in \mathbb{N}^p \mid s_0 \xrightarrow{\sigma^n \tau} s, n \in \mathbb{N}, \tau \text{ prefix of } \sigma\}$.

Proposition 3.2 ([9]). *Let $S = \langle \mathbb{N}^p, F, \leqslant \rangle$ be a WSN, s_1 and s_2 two states in $\mathbb{N}^p$, and σ a finite sequence such that $s_1 \xrightarrow{\sigma} s_2$ and $s_1 \leqslant s_2$. Then there is a sequence of states $(s_n)_{n \geqslant 1}$ such that for all n, $s_n \xrightarrow{\sigma} s_{n+1}$ and $s_n \leqslant s_{n+1}$.*

Problems. We will consider the following computational problems, where in each case, a WSN $S = \langle \mathbb{N}^p, F, \leqslant \rangle$ or a ω-WSN $S = \langle \mathbb{N}^p_\omega, F, \leqslant \rangle$ is given as part of the input:

1. *Termination*. Given a state s_0 in $\mathbb{N}^p$, is the reachability tree $RT(S, s_0)$ finite?
2. *Coverability*. Given states s_0 and s in $\mathbb{N}^p$, does there exist a state s' in $RS(S, s_0)$ which covers s, i.e., which satisfies $s \leqslant s'$?
3. *Boundedness*. Given a state s_0 in $\mathbb{N}^p$, is the reachability set $RS(S, s_0)$ finite?
4. *Path-unbounded-witness*. This problem is not a decision problem, but rather a search problem in the sense of [22]: given a state s_0 in $\mathbb{N}^p$ and a finite non empty sequence σ such that $s_0 \xrightarrow{\sigma} \sigma(s_0)$ and $s_0 < \sigma(s_0)$, determine if there is an unbounded place in $RS(S, \sigma, s_0)$ (i.e., a place i satisfying $\forall k \in \mathbb{N}$, $\exists s \in RS(S, \sigma, s_0)$ such that $s(i) \geqslant k$) and if so identify[3] at least one such an unbounded place i.
5. *Path-place-boundedness*: Given a state s_0 in $\mathbb{N}^p$, a sequence σ such that $s_0 \xrightarrow{\sigma} \sigma(s_0)$ and $s_0 < \sigma(s_0)$, and a place i, is the place i bounded in $RS(S, \sigma, s_0)$ (i.e., does there exist $k \in \mathbb{N}$ such that $s(i) \leqslant k$, $\forall s \in RS(S, \sigma, s_0)$)?
6. *Place-boundedness*: Given an initial state s_0 and a place i, is i bounded?

[3] Although this problem is not a decision problem, we say that the problem is decidable if there exists a Turing machine which solves it (and always halts).

In this list, we have introduced two new problems related to the boundedness problem. The *Path-place-boundedness* can be seen as the place-boundedness problem restricted to a particular path of the reachability tree. It may be decidable when the boundedness problem is not (e.g. reset Petri Nets [7]). Note that decidability of the path-place-boundedness problem implies decidability of the path-unbounded-witness problem. As we show later, the reverse implication is false.

We do not mention the usual *Reachability problem* (given two states s and s', does $s \xrightarrow{*} s'$ hold?), as it is irrelevant to our paper, being undecidable in all the eight Petri Net extensions we shall study (cf. Fig. 1).

4. When are termination and coverability decidable?

For every state s we denote by $\uparrow s$ the set $\{t \in D : t \geqslant s\}$.

Theorem 4.1. *Termination is decidable for well-structured nets.*

Proof. Let $S = \langle \mathbb{N}^p, F, \leqslant \rangle$ be a WSN. If $s_1 \xrightarrow{f} s_2$ for a function $f \in F$ and $s_1 \leqslant t_1$, then because f is nondecreasing and $Dom\ f = \uparrow Dom\ f$, we have $t_1 \xrightarrow{f} t_2$ and $s_2 \leqslant t_2$. This property and the recursivity of f allow us to apply Theorem 4.6 from [11] and then to decide termination. □

Theorem 4.2. *Coverability is decidable* [4] *for ω-well-structured nets.*

Proof. We will show that it is possible to compute a finite basis of $\uparrow Pred(\uparrow s)$ for any $s \in \mathbb{N}^p$. Then by using Theorem 3.6 from [11], we will conclude that coverability is decidable. Fix $s \in \mathbb{N}^p$ and write $Pred_f(\uparrow s)$ for $\{t \in \mathbb{N}^p \mid f(t) \geqslant s\}$. We have:

$$\uparrow Pred(\uparrow s) = \uparrow \bigcup_{f \in F} Pred_f(\uparrow s) = \bigcup_{f \in F} \uparrow Pred_f(\uparrow s) = \bigcup_{f \in F} Pred_f(\uparrow s),$$

where the rightmost equality uses the fact that $\uparrow Pred_f(\uparrow s) = Pred_f(\uparrow s)$, which holds because $Dom\ f = \uparrow Dom\ f$ and f is nondecreasing, for every $f \in F$. Because F is finite, it suffices to be able to compute a finite basis of $Pred_f(\uparrow s)$ for $f \in F$. By Theorem 2.1, namely the Valk and Jantzen result, a finite basis of $Pred_f(\uparrow s)$ is computable iff the predicate $\downarrow t \cap Pred_f(\uparrow s) \neq \emptyset$ is computable for all $t \in \mathbb{N}^p_\omega$. And this holds because

$$\downarrow t \cap Pred_f(\uparrow s) \neq \emptyset \text{ iff } (\exists t' \in \mathbb{N}^p, t' \leqslant t)[f(t') \geqslant s] \text{ iff } \bar{f}(t) \geqslant s,$$

where the left to right implication in the second "iff"relies on the non decreasing property of $\bar{f}$. Now the inequality $\bar{f}(t) \geqslant s$ can be checked because $\bar{f}$ is recursive. □

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[4] Such a statement means: there is a TM deciding the coverability problem, given the premise that the input WSN is an ω-WSN.

The computability of $\bar{f}$ is a hypothesis needed to decide coverability, since we have as a counterpart to Theorem 4.2 that our strongly increasing WSNs have an undecidable coverability when this hypothesis is absent:

Theorem 4.3. *The coverability problem for strongly increasing well-structured nets is undecidable.*[5]

Proof. Consider the family $\{f_j : \mathbb{N}^2 \to \mathbb{N}^2\}$ of strongly increasing recursive functions

$$f_j(n,k) = \begin{cases} (n,0) & \text{if } k = 0 \text{ and TM}_j \text{ runs for more than } n \text{ steps} \\ (n, n+k) & \text{otherwise} \end{cases}$$

and the strongly increasing function $g : \mathbb{N}^2 \to \mathbb{N}^2$ defined by $g(n,k) = (n+1,k)$. The strongly increasing WSN $S_j = \langle \mathbb{N}^2, \{f_j, g\}, \leqslant \rangle$ with initial state $(0,0)$ has the property that the state $(1,1)$ is coverable iff the TM$_j$ halts. Hence there is no Turing machine which correctly determines coverability and halts whenever its input is a strongly increasing WSN. $\square$

5. When are boundedness problems decidable?

5.1. The boundedness problem

An increasing WSN verifies the property that if $s_1 \to s_2$ and $t_1 > s_1$, then there exists t_2 such that $t_1 \to t_2$ with $t_2 > s_2$, while we can only conclude $t_2 \geqslant s_2$ in the case of a WSN. Boundedness for increasing WSNs can thus be decided by searching for a sequence $s_0 \xrightarrow{*} s_1 \xrightarrow{\sigma} s_2$ with $s_1 < s_2$. Then, σ can be iterated to produce an increasing sequence of reachable states $s_0 \xrightarrow{*} s_1 \xrightarrow{\sigma} s_2 \xrightarrow{\sigma} s_3 \ldots$ with $s_1 < s_2 < s_3 \ldots$, and the system is unbounded. Because any WSN by definition has a recursive Succ (i.e., Succ(s) is computable, for every s in $\mathbb{N}^p$), Theorem 4.11 from [11] yields:

Theorem 5.1. *Boundedness is decidable for increasing well-structured nets.*

We note further that Theorem 5.1 is tight. If we omit the "increasing" property from the hypothesis of Theorem 5.1, and yet add the "ω-recursivity" property, the boundedness problem becomes undecidable:

Theorem 5.2. *Boundedness is undecidable for ω-well-structured nets.*

Proof. Boundedness is undecidable for reset Petri nets [7], which are affine WSNs and every affine WSN is a ω-WSN (see Section 2.2). $\square$

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[5] The precise technical meaning of this statement is that there is no Turing machine which correctly decides whether some state in $RS(S, s_0)$ covers s, and halts, whenever its input S is indeed a strongly increasing WSN (with its defining functions given by Turing machines) together with s_0 and s.

5.2. The path-unbounded-witness problem

In this section, we show how to detect at least one unbounded place in some unbounded WSNs. Indeed we show the problem decidable for ρ-increasing WSNs. This decidability result provably does not extend to increasing WSNs, as we also prove in this section.

Theorem 5.3. *The path-unbounded-witness problem is decidable for ρ-increasing WSN.*

Proof. Consider a ρ-increasing WSN $S = \langle \mathbb{N}^p, F, \leqslant \rangle$. Consider the edge-labelled graph

$$G = (\{Q \subseteq [p] : Q \neq \emptyset\}, \{(Q, f, Q') : Q' \in \rho_f(Q)\}),$$

where ρ_f is the (computable) growth signature of f. Consider any non empty finite path in this graph from a node Q to a node Q'. Let $f_1, f_2, \ldots, f_k$ be the sequence of edge labels encountered along the path. A straigthforward induction on k proves that

$$\forall u, v \in Dom\ f_k f_{k-1} \cdots f_1, [u <_Q v \Longrightarrow f_k f_{k-1} \cdots f_1(u) <_{Q'} f_k f_{k-1} \cdots f_1(v)]. \tag{1}$$

Now let a state s_0 and a sequence $\sigma = f_1 f_2 \cdots f_k$ be given such that $s_0 < \sigma(s_0)$. Let Q be the nonempty set such that $s_0 <_Q \sigma(s_0)$. Our goal is to determine a place along which iterating σ from s_0, (which is always possible because $Dom\,\sigma$ is upward-closed), grows unbounded. Because each $f \in F$ is ρ-increasing, each node in G has at least one outgoing edge labelled f. Because G is finite, there exists a computable finite path π in G with the following properties:

- π starts at node Q,
- π visits nodes $Q = Q_0, Q_1, Q_2, \ldots, Q_{k\cdot j}, Q_{k\cdot j+1}, \ldots, Q_{k\cdot l}$ for some j, such that $1 \leqslant j < l$ and $Q_{k\cdot j} = Q_{k\cdot l} (= Q')$,
- π traverses the edges labelled $f_1, f_2, \ldots, f_k, f_1, f_2, \ldots, f_k, f_1, f_2, \ldots, f_k, f_1, \ldots, f_k$.

Applying (1) inductively shows that $\sigma^i(s_0) <_{Q_{k\cdot i}} \sigma^{i+1}(s_0)$ for $1 \leqslant i < l$.
In particular,

$$s_0 \leqslant \sigma^j(s_0) \leqslant \sigma^{l-1}(s_0) <_{Q'} \sigma^l(s_0).$$

Write $d = l - j$. Because the last $k \cdot d$ steps of π form a cycle c around Q', this cycle can be repeated arbitrarily often to extend π. Since the sequence of edge labels forming c spells σ^d, each extension of π by c reflects the effect of further applying σ^d. Applying (1) again,

$$\begin{aligned}
\sigma^l(s_0) &\leqslant \sigma^{l+d-1}(s_0) \\
&<_{Q'} \sigma^{l+d}(s_0) \\
&\leqslant \sigma^{l+2\cdot d-1}(s_0) \\
&<_{Q'} \sigma^{l+2\cdot d}(s_0) \\
&\leqslant \cdots \\
&<_{Q'} \sigma^{l+m\cdot d}(s_0) \\
&\leqslant \cdots
\end{aligned}$$

This implies that all the places in the non-empty set Q' grow unbounded when σ is iterated from s_0. Hence any place in Q' is a satisfactory answer. □

Theorem 5.3 is tight. It provably extends neither to the guaranteed identification of at least one unbounded place in increasing well-structured nets, as we now show, nor to the decidability of path-place-boundedness in ρ-increasing WSN, as we will show in the next section.

Theorem 5.4. *The path-unbounded-witness problem is undecidable for increasing well-structured nets.*

Proof. For each m, we define $f'_m : \mathbb{N}^2 \to \mathbb{N}^2$ by

$$f'_m(k,n) = \begin{cases} (k, n+1) & \text{if the TM}_m \text{ halts in at most } k-1 \text{ steps} \\ (k+1, n) & \text{otherwise.} \end{cases}$$

Every f'_m is total (hence upward-closed), recursive, and increasing.

Now, from any initial state $(k,n) \in \mathbb{N}^2$, the increasing WSN $S_m = \langle \mathbb{N}^2, \{f'_m\}, \leqslant \rangle$ possesses one and only one unbounded place, which is the second place iff the TM$_m$ halts. Suppose to the contrary that there is a TM M which solves the path-unbounded-witness problem. On input $s_1 = (0,0)$, $\sigma = f'_m$, and $s_2 = (1,0)$, M will necessarily come up with the unique unbounded place resulting from the iteration of f'_m from s_1. This place is the second place if TM$_m$ halts, and the first place otherwise, which means that M solves the halting problem. Hence no such M exists, and so path-unbounded-witness problem is undecidable. □

5.3. The path-place-boundedness problem

Theorem 5.5. *Path-place-boundedness is undecidable for strongly increasing well-structured nets and for ω-well-structured nets.*

Proof.

- Consider the strongly increasing WSN $S_j = \langle \mathbb{N}^2, \{f_j, g\}, \leqslant \rangle$ from the proof of Theorem 4.3, with initial state $(0,0)$ and the sequence $\sigma = g f_j$. Then, place 2 is unbounded along the infinite path $(0,0) \xrightarrow{\sigma}\xrightarrow{\sigma}\xrightarrow{\sigma} \ldots$ if and only if TM$_j$ halts.
- Consider the family of functions $g_j : \mathbb{N} \to \mathbb{N}$ defined by

$$g_j(n) = \begin{cases} n & \text{if TM}_j \text{ halts in at most } n \text{ steps} \\ n+1 & \text{otherwise.} \end{cases}$$

  For every $j \geqslant 1$, the function g_j is nondecreasing, total, and ω-recursive ($\bar{g}_j(\omega) = \omega$). Let $T_j = \langle \mathbb{N}, \{g_j\}, \leqslant \rangle$ with initial state 0. The only place is bounded along the infinite path $0 \xrightarrow{g_j} \ldots \xrightarrow{g_j} \ldots$ if and only if the TM$_j$ halts. □

The following proposition justifies our interest in the path-place boundedness problem. Although boundedness in undecidable for Reset Petri nets [7], path-place boundedness is decidable.

Proposition 5.6. *Path-place-boundedness is decidable for reset Petri nets.*

Proof . Given a reset Petri net N with p places, a marking m, a place i and a sequence σ of transitions such that $m \xrightarrow{\sigma} m_1$ with $m < m_1$. By monotonicity, (σ is non-decreasing), the sequence σ may be infinitely repeated; let us write the infinite path $m \xrightarrow{\sigma} m_1 \xrightarrow{\sigma} m_2 \xrightarrow{\sigma} \cdots$.

Let us write $\sigma = f_k f_{k-1} \ldots f_1$. Because each affine function $f_i(X) = A_i X + v_i$ satisfies $0 \leqslant A_i \leqslant Id$ and v_i is a vector of integers, we deduce that there exist A_σ and v_σ such that σ is the affine function $\sigma(X) = A_\sigma X + v_\sigma$ with still $0 \leqslant A_\sigma \leqslant Id$ and v_σ a vector of integers (where A_σ is computable from the A_i's and v_σ from the v_i's and A_i's).

For every n and $1 \leqslant j \leqslant p$, we have: if σ does not contain any reset operation on j (i.e., $A_\sigma(j, j) = 1$) then $m_n(j) = m(j) + nv_\sigma(j)$; moreover, if σ contains a reset operation on place j, then the sequence $(m_n(j))_{n \geqslant 1}$ is stationary and for every n, $2 \leqslant n$, $m_n(j) = m_1(j)$ and hence place j is bounded along the path $m \xrightarrow{\sigma} m_1 \xrightarrow{\sigma} m_2 \xrightarrow{\sigma} \cdots$.

We will now prove that i is unbounded along the infinite path $m \xrightarrow{\sigma} m_1 \xrightarrow{\sigma} m_2 \xrightarrow{\sigma} \cdots$ iff $m_1 \leqslant m_2$ and $m_1(i) < m_2(i)$.

(1) If i is unbounded along the infinite path $m \xrightarrow{\sigma} m_1 \xrightarrow{\sigma} m_2 \xrightarrow{\sigma} \cdots$, then σ cannot contain any reset operation on place i (hence $A_\sigma(i, i) = 1$), v_σ must be positive (otherwise the path would be finite); hence, we obtain that for every n, $1 \leqslant n$, $m_n \leqslant m_{n+1}$. Moreover, we can find two integers r and k such that: $1 \leqslant r$, $1 \leqslant k$, $m_r \leqslant m_{r+k}$, and $m_r(i) < m_{r+k}(i)$.
Hence, from $m_r(i) < m_{r+k}(i)$ and $0 < k$, we obtain $0 < v_\sigma(i)$. Hence, we also have: $m_1(i) < m_2(i)$.

(2) Let us prove that if $m_1 \leqslant m_2$ and $m_1(i) < m_2(i)$ then i is unbounded along the infinite path $m \xrightarrow{\sigma} m_1 \xrightarrow{\sigma} m_2 \xrightarrow{\sigma} \cdots$. First, by monotonicity, (σ is non-decreasing), the path can be infinitely continued and then $0 \leqslant v_\sigma$. Because $m_1(i) < m_2(i)$, the sequence σ cannot contain any reset operation on place i and then $1 \leqslant v_\sigma(i)$. From $m_n(i) = m(i) + nv_\sigma(i)$, we obtain that $m_n(i)$ goes to infinity when n goes to infinity. Hence, the place i is not bounded on the infinite path $m \xrightarrow{\sigma} m_1 \xrightarrow{\sigma} m_2 \xrightarrow{\sigma} \cdots$. $\square$

5.4. The place-boundedness problem

Let us recall the construction of the coverability tree of a Petri net, which is used to solve the place-boundedness problem. The strategy of the construction is to develop every path of the reachability tree until one meets two markings m and m' such that $m_0 \xrightarrow{*} m \xrightarrow{\sigma} m'$ with $m <_Q m'$ and Q the maximal nonempty subset of places for this property. When one meets two such markings m and m', one replaces m' by the limit of the infinite increasing sequence of markings obtained from m by iterating σ. This limit is exactly and effectively computable for Petri nets (and for transfer and reset Petri nets as well):

$$\lim_{n \to \omega} (\sigma^n(m))(i) = \begin{cases} \omega & \text{if } i \in Q \\ m(i) & \text{if } i \notin Q. \end{cases}$$

Then one continues using the same strategy with all transitions extended from $\mathbb{N}^p$ to $\mathbb{N}^p_\omega$. Termination is guaranteed because $\leqslant$ is still a well-ordering on $\mathbb{N}^p_\omega$.

To apply the above strategy to construct a coverability tree for a WSN, it is first necessary to be able to effectively extend functions to $\mathbb{N}^p_\omega$. This means that functions have to be ω-recursive. Furthermore, the extension $\bar{f}$ of any nondecreasing function f is nondecreasing, but any type-i property of f may not be preserved on $\bar{f}$. In particular, strongly increasing functions extended to $\mathbb{N}^p_\omega$ may no longer be increasing (for example, when f is defined as $f(m,n) = (m, m+n)$, $f(\omega,0) = f(\omega,1)$).

Moreover, either $\lim_{n\to\omega} \sigma^n(u)$ must be computable (but this is not true for nondecreasing ω-recursive functions), or a good enough approximation l of $\lim_{n\to\omega} \sigma^n(u)$ must be computable such that $u < l \leqslant \lim_{n\to\omega} \sigma^n(u)$ and l has at least one more ω-component than u, for every infinite increasing sequence $(\sigma^n(u))_{n\geqslant 1}$.

Let ω-**max** be the following function which takes two elements s_1 and s_2 in $\mathbb{N}^p_\omega$ and returns an element in $\mathbb{N}^p_\omega$, defined by:

function ω-**max**(s_1, s_2): returns $s : \mathbb{N}^p_\omega$

$s \leftarrow s_2$
`for` each i such that$s_1(i) < s_2(i)$ and $s_2(i) \neq \omega$ `do`$s(i) \leftarrow \omega$;

We will only apply this function to pairs of states (s_1, s_2) satisfying $s_1 \leqslant s_2$. Note that in general, ω-**max** can satisfy ω-**max**$(u, \sigma(u)) < \lim_{n\to\omega} \sigma^n(u)$, for example when $f(m,n) = (m+n, n+1)$ and $u = (0,0)$, in which case $f(u) = (0,1)$ and ω-**max**$(u, f(u)) = (0,\omega)$, while $\lim_{n\to\omega} f^n(u) = (\omega,\omega)$.

Nevertheless, any strongly increasing WSN satisfies the interesting property:

Proposition 5.7. *Let $\langle \mathbb{N}^p, F, \leqslant \rangle$ be a strongly increasing WSN and let u, v be in $\mathbb{N}^p_\omega$. If there exist $f_1, \ldots, f_k$ in F such that $u \xrightarrow{\sigma} v$ for $\sigma = \bar{f}_k \bar{f}_{k-1} \ldots \bar{f}_1$, and there exists a nonempty Q such that $u <_Q v$, then $\lim_{n\to\omega} \sigma^n(u) \geqslant \omega$-**max**$(u,v) > v$, and ω-**max**(u,v) contains $|Q|$ more ω's than u.*

Proof. As $(\sigma^n(u))_{n\geqslant 1}$ is a nondecreasing sequence, $\lim_{n\to\omega} \sigma^n(u)$ exists. As a composition of strongly increasing functions, σ is strongly increasing. So $u \xrightarrow{\sigma} v$ and $u <_Q v$ imply $\sigma^n(u) <_Q \sigma^{n+1}(u)$, for any positive integer n. So for i in Q, $\lim_{n\to\omega} \sigma^n(u)(i) = \omega$ and for i not in Q, we do not know if this limit is finite or not, but at least we have $\lim_{n\to\omega} \sigma^n(u) \geqslant \omega$-**max**$(u,v) \geqslant v$. Note that ω-**max**(u,v) contains $|Q|$ more ω than u. $\square$

It is possible to apply (after a natural generalization which replaces the Petri net transitions by ω-recursive functions) the Karp–Miller algorithm (originally designed to compute a coverability tree of a Petri net) [18] to the more abstract model, ω-WSNs.

Let $S = \langle \mathbb{N}^p_\omega, F, \leqslant \rangle$ be an ω-WSN and s_0 an initial state in $\mathbb{N}^p_\omega$. The *Karp–Miller tree* of S starting at s_0, denoted KMT(S, s_0), is the tree produced by the following extended Karp–Miller algorithm. It is a labelled tree consisting of a set of nodes NODES $\subseteq \mathbb{N}$ and a set of arcs ARCS $\subseteq$ NODES $\times$ NODES; each node is labelled with a state from $\mathbb{N}^p_\omega$ and each arc is labelled with a function from F.

(We refer to a node as a pair (i,s) to indicate that the node $i \in \mathbb{N}$ is labelled $s \in \mathbb{N}^p_\omega$; we refer to an arc as a triple (i,f,j) to indicate that (i,j) is an arc labelled $f \in F$.) In the algorithm and later, we will use the notations $(n_1, s_1) \xrightarrow{\sigma}_{KMT} (n,s)$, (respectively $(n_1, s_1) \xrightarrow{*}_{KMT} (n,s)$, respectively $(n_1, s_1) \xrightarrow{+}_{KMT} (n, s)$) for saying that there is path labelled by σ (respectively a possibly empty, respectively non-empty) path in KMT from node (n_1, s_1) to the node (n, s).

Karp–Miller_tree $(\langle \mathbb{N}^p_\omega, F, \leqslant\rangle : \omega\text{-WSN},\ s_0 :$ state in $\mathbb{N}^p_\omega)$ *returns* KMT: tree

NODES $\leftarrow \emptyset$; ARCS $\leftarrow \emptyset$;
UNPROCESSED_NODES $\leftarrow \{(0, s_0)\}$; {* INITIAL SINGLETON SET OF UNPROCESSED NODES, EACH NODE REFERRED TO AS "(NODE NUMBER, TEMPORARY LABEL)" *}
`while` a node (n, s) is in UNPROCESSED_NODES `do`
  UNPROCESSED_NODES $\leftarrow$ UNPROCESSED_NODES $\setminus$ $\{(n, s)\}$;
  `for all` nodes (n_i, s_i) such that $(n_i, s_i) \xrightarrow{*}_{KMT} (n, s)$ and $s_i < s$ `do`
    $s \leftarrow \omega\text{-}\mathbf{max}(s_i, s)$;
  NODES $\leftarrow$ NODES $\cup \{(n, s)\}$; {* ADD NODE AND FIX LABEL FOREVER *}
  `if` no node (n_1, s) exists such that $(n_1, s) \xrightarrow{+}_{KMT} (n, s)$ `then`
    `for` each function $f \in F$ such that $s \in Dom\ \bar{f}$ `do`
      $n' \leftarrow$ new node number;
      UNPROCESSED_NODES $\leftarrow$ UNPROCESSED_NODES $\cup \{(n', \bar{f}(s))\}$;
      ARCS $\leftarrow$ ARCS $\cup \{(n, f, n')\}$;
KMT $\leftarrow$ (*NODES*, *ARCS*);

Before proving that the algorithm terminates on all ω-WSNs, we remark some useful elementary facts about the finite paths of the Karp–Miller tree.

Lemma 5.8. *Let $S = \langle \mathbb{N}^p_\omega, F, \leqslant\rangle$ be an ω-WSN, s_0 an initial state in $\mathbb{N}^p_\omega$ and $KMT(S, s_0)$ the Karp–Miller tree of (S, s_0). We have*:

- *If* $(0, s_0) \xrightarrow{*}_{KMT} (n_1, s) \xrightarrow{*}_{KMT} (n_2, s)$ *then the node n_2 has no successor* (*along a branch of the KMT, the same state cannot appear more than twice*).
- *if* $(n, s) \xrightarrow{\sigma}_{KMT} (n', s')$ *then* $\bar{\sigma}(s) \leqslant s'$.
- *if* $(n, s) \xrightarrow{*}_{KMT} (n', s')$ *and* $s < s'$ *then s' has at least one ω more than s.*

Proposition 5.9. *The Karp–Miller tree algorithm terminates when it is applied to an ω-WSN.*

Proof. Suppose that the Karp–Miller tree algorithm does not terminate and so KMT is infinite. Because the set of functions is finite, the infinite tree KMT is finitely branching and there is an infinite branch in KMT:

$$(0, s_0) \longrightarrow_{KMT} (n_1, s_1) \longrightarrow_{KMT} \dots \longrightarrow_{KMT} (n_k, s_k) \longrightarrow_{KMT} \cdots$$

Because $\leqslant$ is a well-ordering on $\mathbb{N}^p_\omega$, we may extract an infinite non-decreasing sequence $\{s_{k_i}\}$ that cannot be stationary (by Lemma 5.8).

So we may also choose an infinite increasing sequence along this branch:

$$(n_{k_0}, s_{k_0}) \xrightarrow{*}_{KMT} (n_{k_1}, s_{k_1}) \xrightarrow{*}_{KMT} \dots \quad \text{with} \quad s_{k_i} < s_{k_{i+1}}$$

We recall that $(n,s) \xrightarrow{*}_{KMT} (n',s')$ and $s < s'$ imply that s' must contain at least one ω more than s, by Lemma 5.8.

After a finite number (at least p) of such executions, no "new ω" can be inserted. Hence, for every n, $s_{k_p} = s_{k_{p+n}}$ and the sequence $(s_{k_n})_{n\geqslant 1}$ is stationary, which produces a contradiction. Hence KMT is finite. $\square$

In the following, we will say "a state s in $KMT(S, s_0)$," for "a state s such that there exists a node (n,s) in $KMT(S, s_0)$."

When the Karp–Miller algorithm is applied to an ω-WSN, it produces a finite tree such that every reachable state is smaller than (or equal to) a state in KMT (see Proposition 5.11). But in general, the KMT may very roughly cover the reachability set, in the following sense: some state in the KMT may be strictly greater than the limit of any infinite sequence of reachable states. When the Karp–Miller algorithm is applied to a *strongly increasing* ω-WSN, it produces a tree, often called a coverability tree, which finely covers the reachability set: each state in KMT is smaller than (or equal to) the limit of an infinite sequence of *reachable states.* The following property of strongly increasing functions (with upward-closed domain) is used for proving that the cover is fine.

Lemma 5.10. *Let $f : \mathbb{N}^p \to \mathbb{N}^p$ be any strongly increasing function with upward-closed domain. Let $(u_n)_{n\geqslant 1}$ be a nondecreasing sequence in Dom f and $\bar{u}$ its limit in $\mathbb{N}^p_\omega$. There exists a strictly increasing sequence of natural numbers $(\varphi(n))_{n\geqslant 1}$ such that* $\lim_{n\to\omega} \bar{f}^n(\bar{u}) = \lim_{n\to\omega} f^n(u_{\varphi(n)})$.

Proof. Let us recall that any nondecreasing sequence in $\mathbb{N}_\omega$ either goes to ω, either is stationary. For all m, we define I_m as the set of places i such that the nondecreasing sequence $(f^m(u_n)(i))_{n\geqslant 1}$ is non stationary. Let $i \in I_m$. Then for all integer n, there exists $p > n$ such that $f^m(u_n)(i) < f^m(u_p)(i)$. Because f is strongly increasing, $f^{m+1}(u_n)(i) < f^{m+1}(u_p)(i)$. Therefore $i \in I_{m+1}$. We have shown the sequence $(I_m)_{m\geqslant 1}$ is nondecreasing (for $\subseteq$), therefore there exist a subset I of places and a natural number N such that $I_m = I$ for all $m \geqslant N$. These I and N satisfy:

(1) For $i \in I$ and $m \geqslant N$, the sequence $(f^m(u_n)(i))_{n\geqslant 1}$ is non stationary, thus its limit value is ω. We may write:
$\forall m \geqslant N, \exists \alpha_m > 0$ such that $\forall i \in I, \forall n \geqslant \alpha_m$, we have $f^m(u_n)(i) > m$.
(2) For $i \notin I$ and $m \geqslant N$, the sequence $(f^m(u_n)(i))_{n\geqslant 1}$ is stationary. Thus the limit value must be $\bar{f}^m(\bar{u})(i)$. In that case, we may write:
$\forall m \geqslant N, \exists \beta_m > 0$ such that $\forall i \notin I, \forall n \geqslant \beta_m$, we have $f^m(u_n)(i) = \bar{f}^m(\bar{u})(i)$.

For all $m \geqslant N$, we define by induction $\varphi(m)$ as the maximum of A_m, B_m, and $\varphi(m-1)+1$. We then have:

(1) For $i \in I$ and $m \geqslant N$, $f^m(u_{\varphi(m)})(i) > m$.
(2) For $i \notin I$ and $m \geqslant N$, $f^m(u_{\varphi(m)})(i) = \bar{f}^m(\bar{u})(i)$.

Hence, we have $\lim_{n\to\omega} \bar{f}^n(\bar{u}) = \lim_{n\to\omega} f^n(u_{\varphi(n)})$. $\square$

Proposition 5.11.
For every ω-WSN $S = \langle \mathbb{N}^p_\omega, F, \leqslant\rangle$ with initial state s_0, we have:

1. *for every state $s \in RS(S, s_0)$, there exists a state $s' \in KMT(S, s_0)$ such that $s \leqslant s'$,*
2. *If, moreover S is strongly increasing, we have: for every state $s \in KMT(S, s_0)$, there exists an infinite non decreasing sequence $(s_n)_{n\geqslant 1}$ of reachable states $s_n \in RS(S, s_0)$ such that $s \leqslant \lim_{n\to\omega} s_n$.*

Proof .

1. Let s be a reachable state in $RS(S, s_0)$; we will use induction on the length of σ with $s_0 \xrightarrow{\sigma} s$. When σ is the empty word, the statement holds because it is always true that $s_0 \in KMT(S, s_0)$.
   Consider a sequence (in the reachability tree) σf of length $k+1$. We define inductively $\overline{\sigma f} = \bar{\sigma}\bar{f}$.
   From $s_0 \xrightarrow{\sigma} s \xrightarrow{f} f(s)$ and the induction hypothesis, we deduce that there exists $(0, s_0) \longrightarrow^*_{KMT} (n', s')$ with $s' \in \mathbb{N}^p_\omega$ and $s \leqslant s'$. Because $s \leqslant s'$ and $\bar{f}$ is nondecreasing, we obtain $\bar{f}(s) \leqslant \bar{f}(s')$.
   Now there are two possible cases:

   - The node (n', s') has no successor. This is necessarily because there exists a path $(n_1, s') \longrightarrow^*_{KMT} (n', s')$.
     Because $\bar{f}(s')$ exists, the node (n_1, s') has at least one successor (n'', t) such that: $(n_1, s') \xrightarrow{f}_{KMT} (n'', t)$ with $\bar{f}(s') \leqslant t$.
   - The node (n', s') has at least one successor then there is an arc labelled f: $(n', s') \xrightarrow{f}_{KMT} (n'', t)$ with $\bar{f}(s') \leqslant t$.

   In both cases, we have: there exists a state t in KMT such that $t \geqslant \bar{f}(s') \geqslant f(s)$.
2. We use induction on the number of times k one has used the function ω-**max** on a sub-path (in KMT) from the initial state to state s.

   - Let $k = 0$; suppose that there is a branch $(0, s_0) \longrightarrow^*_{KMT} (n, s)$ such that the function ω-**max** has not been used along this branch. Then s is a reachable state and we may write $s_n = s$ for every n.
   - Now let $k = n+1$, and consider $(0, s_0) \longrightarrow^*_{KMT} (n_1, s) \xrightarrow{\sigma}_{KMT} (n_2, t)$ in which the last application (in the Karp–Miller algorithm) of the function ω-**max** occurred directly before n_2, whence: $t = \omega\text{-}\mathbf{max}(s, \bar{\sigma}(s))$. Then by the induction hypothesis, there exists an infinite nondecreasing sequence of reachable states $(s_n)_{n\geqslant 1}$ such that $s \leqslant \lim_{n\to\omega} s_n$; and from the algorithm, we may deduce that $s <_Q \bar{\sigma}(s)$ for a maximal non-empty subset Q of $[p]$.
     From $s <_Q \bar{\sigma}(s)$, we deduce that the sequence $(\bar{\sigma}^n(s))_{n\geqslant 1}$ is an infinite nondecreasing sequence whose limit is written $l = \lim_{n\to\omega} \bar{\sigma}^n(s)$. From Proposition 5.7, we have $l \geqslant t$ and from Lemma 5.10, there exists a function φ such that $l \leqslant \lim_{n\to\omega} \sigma^n(s_{\varphi(n)})$ because $s \leqslant \lim_{n\to\omega} s_n$. Hence we have found an infinite nondecreasing sequence of reachable states $(\sigma^n(s_{\varphi(n)}))_{n\geqslant 1}$ whose limit is greater than t. $\square$

Remark 5.12. From the last proposition, we may prove that for every infinite nondecreasing sequence of reachable states $(s_n)_{n \geqslant 1}$, there exists a state t in the Karp–Miller tree such that $\lim_{n \to \omega} s_n \leqslant t$; in general, there is no reason for having: $\lim_{n \to \omega} s_n = t$. Moreover, every state t in KMT is not necessarily the limit of an infinite nondecreasing sequence of reachable states. Hence for strongly increasing ω-WSN, the Karp–Miller tree does not exactly enjoy the same properties as those defined for Petri nets. But we still have a strong relation between the Karp–Miller tree and the reachability set: $\downarrow KMT(S, s_0) = \downarrow RS(S, s_0)$ where $\downarrow KMT(S, s_0)$ is the downward-closure of the set of states occurring in $KMT(S, s_0)$; this is sufficient for deciding the place-boundedness problem.

We thus obtain:

Theorem 5.13. *For a strongly increasing ω-well-structured net $\langle \mathbb{N}^p_\omega, F, \leqslant \rangle$, place-boundedness is decidable.*

Proof. From Proposition 5.11, we show that a place i is not bounded in $S = \langle \mathbb{N}^p_\omega, F, \leqslant \rangle$ with the initial state s_0 iff there is a node (n, t) in $KMT(S, s_0)$ such that $t(i) = \omega$. By definition, i is not bounded in (S, s_0) means that there exists an infinite sequence of reachable states $(t_n)_{n \geqslant 1}$ such that $\lim_{n \to \omega} t_n(i) = \omega$. Hence, from Proposition 5.11 (1), we obtain that if i is not bounded in (S, s_0) then there exists $t' \in KMT(S, s_0)$ such that $t'(i) = \omega$. Conversely: suppose there is a $t \in KMT(S, s_0)$ such that $t(i) = \omega$. From Proposition 5.11 (2), there exists an infinite non-decreasing sequence of reachable states $(s_n)_{n \geqslant 1}$ such that: $t \leqslant \lim_{n \to \omega} s_n$. Hence, $\lim_{n \to \omega} s_n(i) = \omega$ and i is not bounded in (S, s_0). Because the KMT algorithm terminates (cf. Proposition 5.9) for ω-well-structured nets, we conclude that place-boundedness is decidable. $\square$

Theorem 5.13 is optimal in the following sense:

Theorem 5.14. *Place boundedness is undecidable for both strongly increasing well-structured nets and for ρ-increasing ω-well-structured nets.*

Proof. Undecidability of place-boundedness holds for ρ-increasing ω-WSN because transfer Petri nets [7] are ρ-increasing ω-WSN and undecidability of place-boundedness holds for those. For proving the undecidability for strongly increasing WSN, we again use the family $\{S_j\}$ with initial state $(0, 0)$ as in Theorem 4.3. The place 2 is unbounded if and only if TM_j halts. $\square$

6. Summary

Recall the six problems defined in Section 3. We have shown in Sections 4 and 5 the following results:

Theorem 6.1. *The decidability status of the six problems defined in Section 3 of this paper, for the eight classes depicted in Fig. 1, is summarized on Fig. 3.*

7. Affine realizations of well-structured nets

7.1. On the difficulty of testing general recursive functions

Given a set F of functions $\mathbb{N}^p \to \mathbb{N}^p$ prescribed by their respective Turing machines, deciding whether F gives rise to a WSN is undecidable, a consequence of Rice's theorem [25] which we state as Proposition 7.1:

Proposition 7.1. *For each type* (*t*) *of function chosen from* (1) *nondecreasing,* (2) *increasing*, (3) ρ*-increasing*, *and* (4) *strongly increasing*, *the following languages*

(a) $\{k \mid \textit{the } k\textit{th TM computes } f : \mathbb{N} \to \mathbb{N} \textit{ of type } (t)\}$
(b) $\{k \mid \textit{the } k\textit{th TM computes } f : \mathbb{N} \to \mathbb{N} \textit{ of type } (t) \textit{ whose domain is upward-closed}\}$
(c) $\{k \mid \textit{the } k\textit{th TM computes } f : \mathbb{N} \to \mathbb{N} \textit{ whose domain is upward-closed}\}$

are undecidable.

Nevertheless, in concrete cases, it is generally known or easy to decide in which class the given set of functions lies. In many Petri net extensions, this given set is in fact contained in the set of affine functions. Many connections between affine functions and Petri net extensions are shown in Fig. 2. Section 7.2 describes WSN for which deciding the properties of relevant functions is easy. Section 7.4 summarizes the decidability status of problems involving affine functions.

7.2. Petri net extensions and affine well-structured nets

In Section 2.2, we discussed the fact that every affine WSN is a ω-WSN.

Figs. 3 and 4 locate Petri net extensions within our classes of affine functions. Fig. 3 also locates the affine classes within our general framework of well-structured nets.

The Double Petri nets [7] appearing on Fig. 3 are extended Petri nets in which some arc from a transition to a place p may double the content of p. Double Petri nets are then affine WSN such that every affine function $f(X) = AX + B$ satisfies $Id \leqslant A \leqslant 2Id$. Transfer Petri nets [7] are extended Petri nets in which some transitions may transfer the content of a place p to another place p'. Transfer Petri nets are then affine WSN such that every affine function $f(X) = AX + B$ satisfies $A \geqslant 0$ and every column A_j satisfies $\Sigma_i A_{ij} = 1$. Generalized Transfer Petri nets may transfer and duplicate, so that they are affine WSN such that $A \geqslant 0$ and every column A_j is nonnull. Reset Petri nets [2,7] are extended Petri nets in which an arc may clear a place. This class corresponds to affine WSN such that every affine function $f(X) = AX + B$ satisfies $Id \geqslant A \geqslant 0$.

Many Petri net extensions are thus surprisingly well parametrized by affine functions.

7.3. An algorithm for path-place-boundedness

A nonnegative affine function is an affine function $f : \mathbb{N}^p \to \mathbb{N}^p$ such that $f(X) = AX + B$ with $B \geqslant 0$.

A nonnegative affine WSN is a WSN such that every function $f \in F$ is nonnegative affine.[6] We prove here that path-place-boundedness is decidable for nonnegative affine WSN.

For f a nonnegative affine function and X a nonnegative vector in $\mathbb{N}^p$, we prove that the limit of $f^{nd}(X)$ for a suitable d is computable. We can determine precisely which entries in $f^{nd}(X)$ go to infinity and therefore which entries in $f^{nd+r}(X)$.[7] go to infinity, for all r such that $0 \leqslant r \leqslant d-1$. Furthermore, we show that this limit is computable by searching a computable graph. Let $f(X) = AX + B$ be a nonnegative affine function. Note that, for $n \in \mathbb{N}$, f^n is also a non negative affine function, precisely: $f^n(X) = A^nX + (A^{n-1} + A^{n-2} + \cdots + A + I)B$.

Recall that multiplication is extended by $0\omega = 0$ and $x\omega = \omega$, $\forall x \in \mathbb{N}_\omega - \{0\}$. Let's first remark the following:

Lemma 7.2. *Let $(A_n)_{n \geqslant 0}$ be a sequence of non negative matrices having a limit A with possibly ω coefficients. Let X be any non negative vector. Then* $\lim_{n\to\omega} A_nX = AX$.

So we may restrict the problem to the study of the sequences $(A^n)_{n\geqslant 0}$ and $(I + A + \cdots + A^n)_{n\geqslant 0}$.

Notations. Let A be a matrix. We write $A = 0$ or $A = \omega$ when each coefficient of A is respectively 0 or ω. We write A^ω for $\lim_{n\to\omega} A^n$ when the limit exists.

Let $A = (A_{i,j})_{(i,j)\in[p]\times[p]}$ be a matrix with nonnegative integers coefficients. Let us denote $A^n = (A^{(n)}_{i,j})$ (where the brackets in the exponent are used to avoid confusion with the (i, j)-entry of A raised to the nth power).

We associate to A a directed graph G_A whose vertices are $1, \ldots, p$ and whose set of edges is defined by: there is an edge leaving vertex i and entering vertex j if and only if $A_{i,j} \neq 0$; then, the coefficient $A_{i,j}$ is called the *weight* of this edge. For a subgraph C of G_A, we shall denote by A_C the submatrix $(A_{i,j})_{(i,j)\in C\times C}$. We shall need the general decomposition of a non-negative matrix A in term of its associate graph G_A, as it is usually done in the study of finite homogeneous Markov chains or the proof of Perron-Frobenius theorem (cf. [13], [21], [26]).

A *path* of *length* n from a vertex i to a vertex j is a sequence $i = i_0, i_1, \ldots, i_{n-1}, i_n = j$ such that (i_k, i_{k+1}) is an edge in G_A, for $k = 0, 1, \ldots, n-1$. The *weight* of this path is the product $A_{i,i_1}A_{i_1,i_2}\ldots A_{i_{n-1},j}$. The path $i = i_0, i_1, \ldots, i_{n-1}, i_n = j$ is called a *cycle* whenever $n \neq 0$ and $i = j$. The cycle $i = i_0, i_1, \ldots, i_{n-1}, i_n = i$ is called a *circuit* if $i_0, i_1, \ldots, i_{n-1}$ are distinct. The *index of imprimitivity* of a graph is the g.c.d. of the lengths of all circuits of the graph. When this index of imprimitivity is equal to 1, we say the graph is *primitive*. We first recall two properties of this index of imprimitivity:

Lemma 7.3. *Let G be a graph whose index of imprimitivity is denoted by d. Let i be a vertex of G. Then all cycles going through i in G have lengths multiple of d.*

---

[6] A nonnegative affine function $f : X \mapsto AX + B$ in a WSN satisfies $A \geqslant 0$ and $B \geqslant 0$, as f is required to be nondecreasing.

[7] If we denote by $L(X)$ the limit of $(f^{nd}(X))_{n\geqslant 0}$, whenever $d > 1$, the d sub-sequences $(f^{nd}(X))_{n\geqslant 0}$, $(f^{nd+1}(X))_{n\geqslant 0}, \ldots, (f^{nd+d-1}(X))_{n\geqslant 0}$ are convergent with respective limits $L(X), f(L(X)), \ldots, f^{d-1}(L(X))$.

Proof. Let $i = i_0, i_1, \ldots, i_{n-1}, i_n = i$ be a cycle of length n going through i. Let us show that n is a multiple of d by induction: If $i_0, i_1, \ldots, i_{n-1}$ are distinct, then this cycle is a circuit and n is a multiple of d by definition. Otherwise, let k, l with $0 \leqslant k < l \leqslant n$ be such that $i_k = i_l$. Take l as small as possible. So $i_k, i_{k+1}, \ldots, i_l$ is a circuit of length $l - k$, which implies $l - k$ is a multiple of d. And $i_0, i_1, i_k, i_{l+1} \ldots, i_{n-1}$ is a cycle of length $n - l + k < n$, thus a multiple of d by induction. □

Lemma 7.4. *Let G be a strongly connected graph whose index of imprimitivity is denoted by d. Let i be a vertex of G. Then d is the g.c.d. of the lengths of all cycles going through i in G.*

Proof. Let e be the g.c.d. of the lengths of all cycles going through i in G. Thanks to Lemma 7.3 e is a multiple of d. Let $j = j_0, j_1, \ldots, j_{n-1}, j_n = j$ be a circuit in G. Let $i = i_0, i_1, \ldots, i_l = j$ be a path going from i to j and let $j = i_{l+1}, i_{l+2}, \ldots, i_m = i$ be a path going from j to i in G. Then, $i = i_0, i_1, \ldots, i_l = j_0, j_1, \ldots, j_n = i_{l+1}, i_{l+2}, \ldots, i_m = i$ and $i = i_0, i_1, \ldots, i_l = i_{l+1}, i_{l+2}, \ldots, i_m = i$ are cycles going through i, thus their lengths are multiple of e. thus, the length of the circuit is a multiple of e. This shows d is a multiple of e. □

As we shall see, the asymptotic of the sequence $(A^n)_{n \geqslant 0}$ may be completely described by the graph G_A. The proof is based on the fact that $A^{(n)}_{i,j}$ is the sum of the weights of all paths of length n from i to j.

The next two results correspond to particular cases of the final result of this section.

Lemma 7.5. *Let C be a strongly connected component of G_A whose index of imprimitivity is equal to* 1. *Then, either C is reduced to a singleton and A_C is one of the* 1×1 *matrices* (0) *or* (1), *or* $A_C^{\omega} = \omega$.

Proof . Recall some facts about primitive strongly connected graphs : We discard the case where the graph C has no edge at all, i.e., C is a singleton and $A_C = (0)$. Let i, j in C. Note that because C is strongly connected there exists a path from i to j (so there exists t such that $A^{(t)}_{i,j} > 0$) and there exists a path from i to j which goes through all vertices of C. Because C is maximal for this property, all paths from i to j in G_A are in fact in C. Furthermore, we may find circuits in C with lengths l_1,l_2,..., l_k such that g.c.d.$(l_1, l_2, \ldots, l_k) = 1$. With a path of length l_0 from i to j crossing all these circuits, we can build a path in C of length $l_0 + n_1 l_1 + n_2 l_2 + \cdots + n_k l_k$, for any non negative integers $n_1, n_2, \ldots n_k$. It means there exists a path from i to j of any large enough length, i.e., $\exists q \in \mathbb{N}$ such that $A^{(n)}_{i,j} > 0$ for $n > q$.

- If $A^{(n)}_{i,i} = 1$, for sufficiently large integers n, then there is only one circuit in A_C with all edges labelled by 1. So A_C is a permutation matrix, which means G_{A_C} is a disjoint union of circuits. But $G_{A_C} = C$ is strongly connected and the index of imprimitivity of C is equal to 1, so $A_C = (1)$.
- Otherwise, there exists s such that $A^{(s)}_{i,i} \geqslant 2$. Let $i, i_1, \ldots, i_{r-1}, j$ a path of length r from i to j and let q be such that $A^{(n)}_{j,j} > 0$, if $n \geqslant q$. Now, if $n \geqslant s + q + r$, we may decompose n as $u + r + ts$ with $u \in \{q, \ldots, q + s - 1\}$ such that $u \equiv n - r$ modulo s and $t > \frac{n-q-s-r}{s}$. Then $A^{(n)}_{i,j} \geqslant (A^{(s)}_{i,i})^t A_{i,i_1} \ldots A_{i_{r-1},j} A^{(u)}_{j,j} \geqslant 2^t$ which shows $\lim_{n \to \omega} A^{(n)}_{i,j} = \omega$. □

Lemma 7.6. *Let C be a strongly connected component of G_A with index of imprimitivity equal to $c > 1$. Let D, E two strongly connected components of $G_{A^c_C}$ and $(i,j) \in D \times E$. Then*

$$(A^c)^{\omega}_{i,j} = \begin{cases} 0 & \textit{if } D \neq E \\ 1 & \textit{if } \{i\} = D = E = \{j\} \textit{ and } A^{(c)}_{i,i} = 1 \\ \omega & \textit{otherwise.} \end{cases}$$

Proof. Let us first remark that there exists a path of length l from i to j in $G_{A^c_C}$ if and only if there exists a path of length cl from i to j in G_A. Let D be a strongly connected component of $G_{A^c_C}$ and i be a vertex of D. Thanks to Lemma 7.4, the index of imprimitivity of D is the g.c.d. of the lengths of all cycles in $G_{A^c_C}$ going through i, i.e., the quotient by c of the g.c.d. of the lengths of all cycles in G_A going through i. Therefore, it is equal to 1. Let D, E two strongly connected components of $G_{A^c_C}$ and $(i,j) \in D \times E$. If $D \neq E$, there cannot exist a path from i to j and a path from j to i in $G_{A^c_C}$. Let us suppose, for example, that i cannot be joined to j by a path in $G_{A^c_C}$, which means i cannot be joined to j by a path of length multiple of c in G_A. As all cycles have lengths multiple of c, j cannot be joined either to i by a path of length multiple of c. So $\forall n, A^{(nc)}_{i,j} = A^{(nc)}_{j,i} = 0$. When $D = E$, we apply Lemma 7.5 to $(A^{(c)}_{(i,j)})_{(i,j) \in D \times D}$ which defines a strongly connected graph with index of imprimitivity equal to 1. □

The general case: The next theorem explains how to describe the behaviour of the sequence $(A^n)_{n \geqslant 0}$, without making any assumption of strong connectivity or on the index of imprimitivity of the graph G_A associated to the matrix A. We compute the strongly connected components of G_A. Grouping nodes by components, we find a permutation matrix P such that PAP^{-1} is a triangular-blocks matrix where each diagonal block is irreducible (corresponding to a strongly connected component of G_A). Let d be the l.c.m. of the indices of imprimitivity of all components of G_A. We already know by the two previous lemma how to compute the limit of the powers of each irreducible diagonal block of A^d. Let r be the number of strongly connected components of G_{A^d}. We partition the matrix $B = PA^dP^{-1} = (B_{i,j})_{(i,j) \in [r] \times [r]}$ such that each block $B_{i,i}$ corresponds to a strongly connected component of G_{A^d}. We shall call this component a $0, 1$ or ω block according to the nature of its limit in Lemma 7.5.

We consider the graph G_B: vertices are the strongly connected components of G_{A^d}, and edges are defined by: there is an edge leaving vertex i and entering vertex j if and only if $B_{i,j} \neq 0$. As the matrix B is upper-triangular, each path $i, i_1, \cdots, i_t, j$ satisfies $i \leqslant i_1 \leqslant \cdots \leqslant i_t \leqslant j$. We say that such a path is *increasing*. We say this path is *strictly increasing* if $i < i_1 < \cdots < i_t < j$. If $1 \leqslant i < j \leqslant r$, we denote by $P_{i,j}$ the set of strictly increasing paths from i to j; this set contains only paths of length $\leqslant j - i$ so is finite and computable. We may decompose $P_{i,j} = \cup_{t=1}^{j-i} P^t_{i,j}$ where $P^t_{i,j}$ is the set of paths of length t in $P_{i,j}$, for all t such that $1 \leqslant t \leqslant j - i$. All these sets allow us to express the powers of B:

$$B^{(n)}_{i,j} = \sum_{t=1}^{\min(j-i,n)} \sum_{(i,i_1,\ldots,i_{t-1},j) \in P^t_{i,j}} \sum_{p_0+\cdots+p_t=n-t} B^{p_0}_{i,i} B_{i,i_1} B^{p_1}_{i_1,i_1} \ldots B_{i_{t-1},j} B^{p_t}_{j,j}.$$

So we get the next result which describe the behaviour of the sequence $(A^{nd})_{n \geqslant 0}$:

Theorem 7.7. *Let $i < j \leqslant r$.*

- *If there exists a strictly increasing path in G_B connecting i to j which goes through a node in a ω block or through two nodes in different 1 blocks, then $B^{\omega}_{i,j} = \omega$.*
- *If all strictly increasing paths in G_B connecting i to j go through nodes in 0 blocks or if there exists no path connecting i to j, then $B^{\omega}_{i,j} = 0$.*
- *Otherwise, all strictly increasing paths in G_B connecting i to j go through nodes in 0 blocks except one 1 block $B_{l,l}$. Then $B^{\omega}_{i,j} = \sum_{(i,i_1,\ldots,l,\ldots,i_{t-1},j)\in P_{i,j}} B_{i,i_1}B_{i_1,i_2}\ldots B^{\omega}_{l,l}\ldots B_{i_{t-1},j}$.*

The same considerations allow to compute $L = \lim_{n\to\omega}(A^{nd} + A^{(n-1)d} + \cdots + I)$. If $\lim_{n\to\omega} A^{(nd)}_{i,j} \neq 0$, the i,j-term in L is ω. If $\lim_{n\to\omega} A^{(nd)}_{i,j} = 0$, the number of paths from i to j is finite ; in particular, they do not cross any cycle and they must be of length $\leqslant p$. So the i,j-term in L is $1 + A^{d}_{i,j} + \cdots + A^{kd}_{i,j}$ where k is such that $kd \leqslant p \leqslant (k+1)d$ if $i \neq j$ or 1 if $i = j$.

Remark 7.8. This result may in fact be extended to matrices with non negative real coefficients (see [24]).

Theorem 7.9. *Let $f(X) = AX + B$ be a nonnegative affine function from $\mathbb{N}^p$ to $\mathbb{N}^p$. Let X_0 be a nonnegative vector. Then there exists an integer $d \geqslant 1$ such that $\lim_{n\to\omega} f^{nd}(X_0)$ exists in $\mathbb{N}^p_{\omega}$. Furthermore, d and $\lim_{n\to\omega} f^{nd}(X_0)$ are computable.*

Proof . Let us write the informal algorithm which computes d and the limit $\lim_{n\to\omega} f^{nd}(X_0)$:

1. Get the strongly connected components (C) of G_A and their imprimitivity indices (d_C)
2. Compute $C^{d_C{}^{\omega}}$ for each irreducible diagonal block of A, corresponding to a strongly connected component of G_A
   (i) Search for connected components (E) of C^{d_C}
   (ii) If one of $E = (1)$, then $E^{\omega} = (1)$ for all E
   (iii) Else, $E^{\omega} = \omega$ for all E
3. Compute the index d and $A^{d^{\omega}}$ (*using Theorem 7.7*)
4. Compute $\sum_n A^{nd}$
5. Compute $(A^{d^{\omega}})X_0 + (\sum_n A^{nd})B$ (*using Lemma 7.2*) □

Remark 7.10. Even if its interest seems to be more theoretical than practical, we may very roughly study the complexity of this algorithm, in the dimension p of the matrix A and in the size of the coefficients of A, B, X_0, denoted respectively by $m(A)$, $m(B)$, $m(X_0)$. Computing strongly connected components [5] and indices of imprimitivity of strongly connected components in G_A may be done by inspecting the matrix A and its ith powers (for $i \leqslant p$), thus is polynomial in p. Step 3 in the algorithm may be most expensive as we have to compute the dth-power of A, where d is the l.c.m. of the indices of imprimitivity of strongly connected components. As all these indices can be bounded by p, we roughly bound d by p! The cost of such a computation is therefore bounded by $p!\mathrm{O}(p^3)$ in operations and $p!\mathrm{O}(p^3)m(A)$ in the size of coefficients. The last steps are less expensive.

Corollary 7.11. *Path-place boundedness is decidable for nonnegative affine WSN.*

Proof. Using notations of theorem 7.9, the sequence $(f^n(X_0))_{n \geqslant 0}$ is bounded on the ith-component if and only if all vectors $\lim_{n\to\omega} f^{nd}(X_0)$, $\lim_{n\to\omega} f^{nd+1}(X_0)$,..., $\lim_{n\to\omega} f^{nd+d-1}(X_0)$ have a finite ith-component. □

Interestingly, although the procedure in Theorem 7.9 computes limits and always terminates, we do not see how to bound the number of limit computations required by the coverability tree strategy from Section 5.

Note that this method cannot be applied if we do not suppose $B \geqslant 0$ because of Lemma 7.2, in which the hypothesis $X \geqslant 0$ is crucial. We could give partial results with other kinds of hypotheses ($AB = B$ which captures the Petri net model, $\{A^n \mid n \in \mathbb{N}\}$ is a finite set which captures the transfer Petri net model...).

7.4. Affine well-structured nets

Theorem 7.12. *The decidability status of the six problems defined in Section 3 of this paper, for five classes of affine ω-WSN, is summarized on Fig. 4.*

Proof. First column: termination and coverability are decidable because affine WSN with $A \geqslant 0$ are ω-WSN (i.e., in $\bar{S}_1$); then see Fig. 2. Path-place boundedness, and hence path-unbounded-witness, are decidable because path-place boundedness for affine WSN with $A \geqslant 0$ reduces to the path-place boundedness for affine WSN with $A \geqslant 0$ and $B \geqslant 0$: let us write $\sigma(X) = AX + B$ and $\delta = \sigma(s_0) - s_0$; we have $\delta = As_0 + B - s_0 \geqslant 0$. Now, $\sigma^n(s_0) = s_0 + \delta \times (1 + A + A^2 + \cdots + A^{n-1})$. Let us note f_σ the affine function $f_\sigma(X) = AX + \delta$ We remark that for every n, $f_\sigma^n(0) = \delta \times (1 + A + A^2 + \cdots + A^{n-1}) = \sigma^n(s_0) - s_0$; hence place i is bounded along the infinite path obtained in iterating σ from the initial state s_0 iff i is bounded along the infinite path obtained in iterating f_σ from the initial state 0. We now remark that the affine function f_σ satisfies $A \geqslant 0$ and $\delta \geqslant 0$; hence the path-place boundedness is decidable for f_σ, so we may decide whether i is bounded on $f_\sigma^n(0)$; hence we may also decide the path-place boundedness on $\sigma^n(s_0)$ too.

Boundedness, and hence place-boundedness, are undecidable because boundedness is already undecidable for reset Petri nets [7] and reset Petri nets are affine WSN with $0 \leqslant A \leqslant Id$; then see Fig. 3.

Second column: path-place-boundedness, hence path-unbounded-witness, is decidable (Corollary 7.11). The case of boundedness is open; boundedness does not reduce to the path-place-boundedness problem for ($A \geqslant 0, B \geqslant 0$)-affine WSN directly because one could imagine detecting a finite path $s \xrightarrow{*} s_1 \xrightarrow{\sigma} s_2$ with $s_1 < s_2$ and $\lim_{n\to\omega} s_n \in \mathbb{N}^p$.

Third column: boundedness is decidable because affine WSN with each column of each matrix greater than 0 are increasing ω-WSN (i.e., in $\bar{S}_2$).

Fourth column: follows from [27–29].

Fifth column: affine ω-WSN with $A = Id$ are Petri nets! □

8. Conclusion

We have answered all but six decidability questions concerning termination, coverability, boundedness, path-unbounded-witness, path-place boundedness, and place-boundedness, for our eight classes of general WSN, and for five classes of affine WSN. These results are summarized on Figs. 2 and 4. We highlight specific contributions:

- The coverability problem (and the termination problem) are shown decidable for every ω-WSN. This uses a characterization of effective upward-closed subsets of $\mathbb{N}^p$.
- To clarify the delicate role of model variations on the difficulty of deciding boundedness, we introduced two new boundedness problems, called the path-unbounded-witness and the path-place-boundedness problems, as companions to the boundedness and the place-boundedness problems classically studied. Our WSN classes provably distinguish the decidability properties of these four boundedness problem variants.
- The place-boundedness problem remains decidable for any class of WSN in which each function is ω-recursive and increases in the strong sense.
- Reset Petri nets, Generalized transfer Petri nets, transfer Petri nets, post-SM nets, double Petri nets, and Petri nets (!) arise as particular cases of WSN in which every function is affine and is defined by matrices satisfying simple conditions: see Fig. 3. This point of view is new and puts the decidability results of [29,7] in a clearer perspective.
- An algorithm of independent interest was developed to compute the limit of a nonnegative affine function (the algorithm extends to nonnegative affine functions with real coefficients).

We leave open the questions for which "?" appears in Figs. 2 and 4. It would be interesting to complete the global picture afforded by WSN and affine WSN by answering these questions.

Finally, we have mentioned that well-structured transition systems capture many infinite state systems beyond generalized Petri nets (see for instance [11]). Clearly, any useful such transition system ought to be defined from recursive functions over some domain, so that WSN could in principle represent such systems as well. But there is one limitation of WSN, namely finiteness of their defining sets of functions, which could be lifted to make WSN more expressive. Are there infinite state systems which would require this generality to be modelled conveniently by WSN, and if so, what would become of our Figs. 2 and 4 in this more general context?

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