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Progress measures, immediate determinacy, and a subset construction for tree automata★

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Abstract

Using the concept of progress measure, we give a new proof of Rabin's fundamental result that the languages defined by tree automata are closed under complementation.

To do this we show that for certain infinite games based on tree automata, an *immediate determinacy* property holds for the player who is trying to win according to a Rabin acceptance condition. Immediate determinancy is stronger than the *forgetful determinacy* of Gurevich and Harrington, which depends on more information about the past, but applies to another class of games.

Next, we show a graph theoretic duality theorem for winning conditions. Finally, we present an extended version of Safra's determinization construction. Together, these ingredients and the determinacy of Borel games yield a straightforward recipe for complementing tree automata.

Our construction is almost optimal, i.e. the state space blow-up is essentially exponential – thus roughly the same as for automata on finite or infinite words. To our knowledge, no prior constructions have been better than double exponential.

1. Introduction

The complementation problem has long been a central topic in the theory of automata on infinite objects. Its solution is crucial to decidability results for restricted second order logics [25]. More recently, this problem – and especially the need to solve it as efficiently as possible – has also been a focus of research in temporal logics [3, 8, 22, 23], where decision problems can often be reduced to automata-theoretic problems.

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Using the classical subset construction [17], one can easily complement nondeterministic automata on finite words. The complexity of this procedure, i.e. the increase in the size of the automaton, is exponential, and this can be shown to be a lower bound. For automata defining languages of infinite words, the complementation problem is substantially more challenging. Although Büchi solved the problem in 1962 [1], it was only recently that methods of essentially exponential complexity were given [20, 23] (by an *essentially exponential* method we mean that for some $p > 0$, the size of the complemented automaton is $2^{O(n^p)}$, where n is the size of the original automaton). For automata on infinite, labeled trees, the complementation problem was first solved in a long and very difficult proof by Rabin in 1969 [18]. Because of its significance, the complementation problem for tree automata has since been addressed in several articles [2, 4, 6, 15, 16, 24, 27]. To the author's knowledge, an exponential upper bound has not been achieved before. Analyses of earlier published work appear at best to give double exponential methods [26].

In this article we show that complementation of nondeterministic automata on infinite trees can be carried out with only an essentially exponential increase in size. This method is based on the technique of *progress measures* [7, 8, 10], which allow to reason locally about global properties of infinite paths in a graph. Our method relies on the following results.

Firstly, we generalize the result by Emerson and Julta [4] that with certain restrictions on the winning conditions, the *forgetful determinacy* of Gurevich and Harrington [6] – that winning strategies need only finite memory about the past – can be strengthened to *immediate determinacy* – that winning strategies can be made memoryless.

Our results applies to strategies for a player who tries to win according to a *Rabin condition* (or *pairs condition*) [18], which is a special disjunctive normal form of conditions expressing that designated subsets of states are encountered infinitely often (the related *Muller condition* is preferred by some authors). For the inverse condition, called a *Streett condition*, forgetful determinacy seems to require an amount of finite memory exponential in the size of the condition. (Also, the translation from one kind of condition to the other seems to require at least an exponential blow-up with respect to sizes of the automata.)

Secondly, we show a graph theoretic duality theorem for Rabin and Streett conditions. This result implies that to establish the global property of a Rabin condition, it suffices to guess a finitary approximation to a progress measure and then verify a Streett condition.

Thirdly, we present an extended version of Safra's determinization construction [20]. Together these results yield a rather straightforward method for complementing tree automata equipped with a Streett acceptance condition. Our approach is motivated by the idea of directly using the subset construction. But in contrast to most of the previous approaches, we do not here use mathematical induction (except for what is built into Rabin progress measures) or translate automata to and from second-order or fixed point logics.

According to results in [22] the lower bound for complementation of Streett automata on infinite words is essentially exponential. This bound carries over to Streett tree automata, making our method optimal to within an at most polynomial gap in the exponent.

1.1. Previous approaches

For Büchi automata on infinite words Sistla, et al. [23] gave an essentially exponential complementation construction based on ideas in Büchi's original proof. Safra [20] showed that the subset construction could serve to determinize and complement. He obtained an optimal method of complexity $2^{O(n \log n)}$, which in addition greatly simplifies McNaughton's determinization proof from 1966 [12]. Another optimal method for complementing, but not determinizing, Büchi automata is based on progress measures [8].

Rabin's original proof [18] of the complementation property for tree automata relies on complicated ordinal induction. The proof by Muchnik [15] is based on an elegant induction on the number of states.

Using infinite games, Büchi [2], and later Gurevich and Harrington [6], made significant advances in understanding the problem. The idea behind the game is to view a run of the automaton as a strategy for an *automaton player*, who tries to satisfy the acceptance condition; the other player, the *pathfinder*, tries to find a path in the input tree dissatisfying the acceptance condition. Given a labeled input tree, the players carry out their moves as follows. The automaton player starts by selecting an initial automaton state for the root. The pathfinder answers by choosing a child of the root in direction left or right. Then the automaton player selects a state for the node just chosen such that the transition relation is satisfied (thus the label of the root is used for this step). The pathfinder answers by proceeding to a node at the next level of the input tree, etc. After ω moves the game ends, and the automaton player *wins* if the acceptance condition of the automaton is satisfied for the infinite sequence of states played; otherwise the pathfinder wins. A *strategy* for a player determines his moves as a function of the input tree and the previous moves. A *winning* strategy is one that makes the player win whatever moves the opponent chooses. A *forgetful* strategy for a player is such that his choices depend only on

- the future, i.e. the subtree of the input tree determined by the current node, and
- a fixed amount of information about the history of previous automaton states played.

Gurevich and Harrington proved that it is always the case that one of the players has a winning forgetful strategy. Such strategies, also called *finite-memory* strategies, allow a subset construction to be carried through for the complementation. The article contains some complicated inductive arguments, which were later treated in more detail in an unpublished note by Monk [13].

Yakhnis and Yakhnis [27] investigated the Gurevich–Harrington method when certain restraints are placed on the players' strategies – a notion that is also present in

Monk's note – and, importantly, gave a more explicit description of the strategy for the winning player than had occurred before. The complexity of their method, however, has not been analyzed.

Muller and Schupp [16] avoided the complementation obstacles by using alternating automata for which the operation becomes trivial. In addition to having branching transitions, these automata may proceed along each branch to Boolean combinations of new states. The problem with this approach is that closure under projection – a fundamental property used for modeling existential quantifiers in second order logics – is about as hard as complementing usual nondeterministic tree automata.

The approach by Emerson and Jutla [4] uses a simple acceptance condition, called the *parity condition*. This allows them to show that for the pathfinder more than forgetful determinacy holds: a winning strategy can be made *memoryless*, which means that the only information needed about the past is the last state played by the automaton player. We call this property *immediate determinacy*. Their tree automaton complementation method is based on translations between alternating tree automata and the logic of μ-calculus. The parity condition appears to be exponentially less succinct than the usual Rabin or Streett conditions.

The immediate determinacy property for games with the parity condition was discovered independently by Mostowski [14].

Safra's construction has been generalized previously in [3], where it is shown how to express the property "a Streett condition holds along all paths calculated by a subset automaton" by means of a Rabin condition on subsets. We show how to express this property by means of a Streett condition on subsets.

[A new determinization construction by Safra [21] also allows one to express by means of a Streett condition that a Rabin condition is satisfied on all paths. This construction may be used in the complementation proof of the present paper as an alternative to the use of our duality theorem and our extension of the original construction by Safra.]

2. Rabin and Streett conditions

A *graph* $G = (V, E)$ consists of a countable set of vertices (or states) V and a set of directed edges $E \subseteq V \times V$. A *basic pair* (R, I) on V consists of a set $R \subseteq V$ of *reconfirming* states and a set $I \subseteq V$ of *invalidating* states. A *pairs set* C is a set $\{(R_\chi, I_\chi) \mid \chi \in X\}$ of basic pairs; here X is a finite set of *colors*, and basic pair (R_χ, I_χ) is said to have color χ. We assume that no pair in C is repeated and say that $|C| = |X|$ is the *size* of C. For technical reasons, we always assume without loss of generality that $0 \in X$ and that $I_0 = \emptyset$ (one can always add the pair $(\emptyset, \emptyset)$ without changing the semantics of satisfaction defined next). We say that v and v' are *equivalent with respect to C*, and we write $v \equiv_C v'$, if for all $\chi \in X$, $v \in R_\chi$ iff $v' \in R_\chi$ (note that R_χ sets alone determine equivalence).

A *Rabin condition* $\mathbf{R}C$ is defined by a pairs set C. We say that an infinite sequence $v_0 v_1 \cdots$ *satisfies* $\mathbf{R}C$ and we write $v_0 v_1 \cdots \vDash \mathbf{R}C$, if for some color χ, $v_k \in R_\chi$ infinitely often and $v_k \in I_\chi$ only finitely often. In this case we say that color χ is *reconfirmed* infinitely often and *invalidated* only finitely often. We say that a graph $G = (V, E)$ satisfies a Rabin condition $\mathbf{R}C$ on V, and we write $G \vDash \mathbf{R}C$, if every infinite path $v_0 v_1 \cdots$ in G satisfies $\mathbf{R}C$. A *Streett condition* $\mathbf{S}C$ is the dual of Rabin condition and also given by a pairs set C. We let $v_0 v_1 \cdots \vDash \mathbf{S}C$ denote that $v_0 v_1 \cdots \nvDash \mathbf{R}C$, and let $G \vDash \mathbf{S}C$ denote that for all paths $v_0 v_1 \cdots$ in G, $v_0 v_1 \cdots \vDash \mathbf{S}C$.

3. Measures, surgery, and duality

Rabin progress measures are defined in terms of pointer trees, also known as direction trees. A *pointer tree* T is a countable, prefix-closed subset of ω_1^*, the set of finite sequences of countable ordinals.[1] Each sequence $t = t^1 \cdots t^\ell$ in T represents a *node*, which has *children* $td \in T$, where td denotes t concatenated with the single element sequence d. Here $d \in \omega_1$ is the *pointer* to td from t. The root of T is the empty sequence denoted ε. We visualize pointer trees as growing upwards; see Fig. 1, where children are depicted from left to right in descending order. If t' is a prefix of $t \in T$, denoted $t' \leqslant t$, then t' is called an *ancestor* of t. The *highest common ancestor* $t \uparrow t'$ of nodes $t = t^1 \cdots t^\ell$ and $t' = t'^1 \cdots t'^{\ell'}$ is the node $t^1 \cdots t^\lambda$, where λ is maximal such that $t^1 \cdots t^\lambda = t'^1 \cdots t'^\lambda$. The *level* $|t|$ of a node $t = t^1 \cdots t^\ell$ is the number ℓ; the level of ε is 0. The *prefix up to level* λ of $t = t^1 \cdots t^n$ is $t^1 \cdots t^{\min\{n,\lambda\}}$, denoted $t \downarrow \lambda$. The *height* of T is the maximum node level (if it exists). T is *finite-path* if there are no infinite paths in T.

Definition 1 (*Kleene–Brouwer ordering*). The ordering $\succ$ on T is defined by: $t \succ t'$ if there is a λ such that $t \downarrow \lambda = t' \downarrow \lambda$ and either $\lambda = |t| < |t'|$ or $\lambda < |t|, |t'|$ and $t^{\lambda+1} > t'^{\lambda+1}$. The ordering $\succeq$ is defined as $t \succeq t'$ if $t \succ t'$ or $t = t'$.

In other words $t \succeq t'$ if t is an ancestor of t' or if t' branches off to the right of t (assuming T is depicted as in Fig. 1).

Lemma 1 (*Kleene–Brouwer ordering*). *If T is finite-path, then $\succ$ is well-ordered.*

Proof. See [19]. □

The Kleene–Brouwer ordering has a property that may be explained informally as follows: if t' is to the left of t'' and if t is such that t'' branches off from t below where t' branches off from t, then t is to the left of t'' – as illustrated by two examples in Fig. 2. Formally, we have the following lemma.

[1] X^* denotes the set of sequences of elements of X. X^+ denotes the set of non-empty sequences of elements of X.

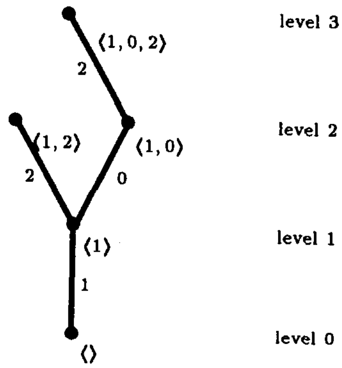

Fig. 1. A pointer tree.

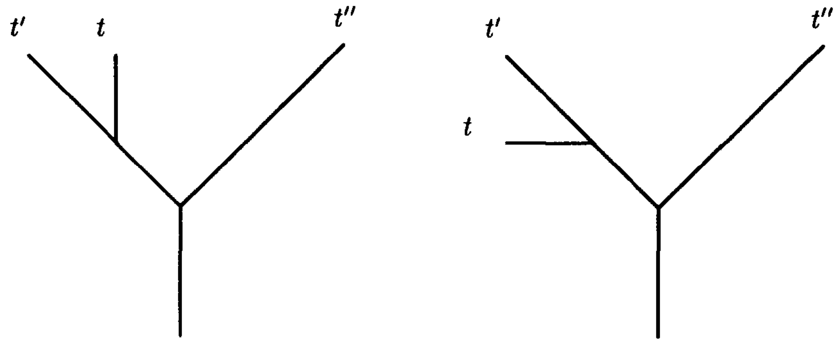

Fig. 2. A property of the Kleene–Brouwer ordering.

Lemma 2. *$t' \succeq t''$ and $t \uparrow t'' \underset{\cdot}{<} t \uparrow t'$ implies $t \succ t''$.*

Proof. Assume $t' \succeq t''$ and $t \uparrow t'' \underset{\cdot}{<} t \uparrow t'$. Let $\lambda = |t' \uparrow t''|$. From $t \uparrow t'' \underset{\cdot}{<} t \uparrow t'$, it follows that $|t|$, $|t'| > \lambda$ and that $t \downarrow \lambda + 1 = t' \downarrow \lambda + 1$. Moreover, since $t' \succeq t''$, it holds that $|t''| > \lambda$. Since also $|t'| > \lambda$, it follows that $t' \downarrow \lambda + 1 \succ t'' \downarrow \lambda + 1$. Thus $t \downarrow \lambda + 1 = t' \downarrow \lambda + 1 \succ t'' \downarrow \lambda + 1$. It follows that $t \succ t''$. $\square$

A *coloring* $\xi : T \rightharpoonup X$ of a pointer tree T assigns a color to a subset of nodes of T such that the following conditions are satisfied:

- the root receives color 0, i.e. $\xi(\varepsilon) = 0$,
- each non-leaf node t must receive a color, i.e. $t \in \mathrm{dom}(\xi)$, and
- all colors along any path starting in the root are distinct.[2]

[2] This condition was not imposed in [10], but it may help in understanding the concepts presented. The completeness proof in [10] gives a coloring satisfying this condition.

For color χ we write "$\chi \in t$" as a shorthand for "$\exists \ell : \chi = \xi(t \downarrow \ell)$." Note that a colored pointer tree (T, ξ) has height at most $|X|$ (and is hence finite-path) and that all nodes at this level receive no color.

From [10] we slightly modify the notion of a Rabin progress measure:

Definition 2. A *Rabin progress measure* (or just *Rabin measure*) (μ, T, ξ) for $(G, \mathbf{R}C)$ is a mapping $\mu : V \to T$, where (T, ξ) is a colored pointer tree, such that

(I) for all $v \in V$ and all $\chi \in \mu(v)$, $v \notin I_\chi$, and

(R) for all $(u, v) \in E$, $u \rhd_\mu v$,

where

$(\rhd_\mu)$ $u \rhd_\mu v$ if $\mu(u) \succ \mu(v)$, or

if there exists $\chi \in \mu(u) \uparrow \mu(v)$ such that $v \in R_\chi$.

The value $\mu(v)$ of a Rabin progress measure μ is best regarded as representing the path from the root ε to the node $\mu(v)$. The nodes on this path are all colored, except possibly for $\mu(v)$ itself. Condition (I) states that none of these are invalidating. Condition (R) states that progress according to $\rhd_\mu$ takes place across any edge. Here the relation $u \rhd_\mu v$ holds if some node common to (the paths represented by) $\mu(u)$ and $\mu(v)$ has a color χ for which v is a reconfirming state or if the value of μ decreases according to the Kleene–Brouwer ordering from u to v. Intuitively, the value of the progress measure represents a prioritized list of hypotheses about which pair is going to be satisfied in the limit. According to this view, hypothesis at low levels are the least likely to be voided by the occurrence of an invalidating state. In particular, the hypothesis at the bottom – corresponding to the root of the tree, which is colored 0 – is never invalidated by our assumption that there are no invalidating states for color 0.

The main result of [10] can be formulated as follows.

Theorem 1. *$G \models \mathbf{R}C$ if and only if there is a Rabin progress measure μ for $(G, \mathbf{R}C)$.*

Proof. "$\Leftarrow$". Let $v_0 v_1 \cdots$ be an infinite path in G. Then by (R), $v_0 \rhd_\mu v_1 \rhd_\mu \cdots$. It is not hard to see that there is a unique node $t \in T$ such that almost always $t \leqslant \mu(v_k)$ and infinitely often $t = \mu(v_k) \uparrow \mu(v_{k+1})$. Let $\ell = |t|$. Suppose for a contradiction that there is no $\hat{t} \leqslant t$ such that $v_k \in R_{\xi(\hat{t})}$ infinitely often. Then by (R) and the definition of ℓ, it can be seen that almost always, it holds that $|\mu(v_k)| > \ell$ and $\mu(v_k) \downarrow (\ell + 1) \succeq \mu(v_{k+1}) \downarrow (\ell + 1)$. Moreover, since $t = \mu(v_k) \uparrow \mu(v_{k+1})$ holds infinitely often, also $\mu(v_k) \downarrow (\ell + 1) \succ \mu(v_{k+1}) \downarrow (\ell + 1)$ holds infinitely often. Thus there exists a K such that $\mu(v_K) \downarrow (\ell + 1) \succeq \mu(v_{K+1}) \downarrow (\ell + 1) \succeq \cdots$, where infinitely many of the inequalities are strict. This contradicts Lemma 1 (Kleene–Brouwer Ordering). Thus there is a $\hat{t} \leqslant t$ such that $v_k \in R_{\xi(\hat{t})}$ infinitely often. By (I) and definition of t, it holds that

$v_k \in I_{\xi(\hat{t})}$ only finitely often. We conclude that $v_0 v_1 \cdots \models \mathbf{R}(R_{\xi(\hat{t})}, I_{\xi(\hat{t})})$, whence $v_0 v_1 \cdots \models \mathbf{R}C$.

"$\Rightarrow$". See [10] (the assumption that for color $\chi = 0$, $I_\chi = \emptyset$, is used here, which makes it possible to assign $\xi(\varepsilon)$ the color 0). $\square$

From the proof of Theorem 1 the following Lemma follows.

Lemma 3. *If $(G, \mathbf{R}C)$ has Rabin measure (μ, T, ξ) and $v_0 v_1 \cdots$ is a path in G, then there is a colored $t \in T$ such that almost always $\mu(v_k) \gtrsim t$ and $v_0 v_1 \cdots \mathbf{R} \models (R_{\xi(t)}, I_{\xi(t)})$.*

The next lemma is instrumental for the proof of immediate determinacy.

Lemma 4 (*Surgery*).
(1) *If $v \rhd_\mu v'$, $v' \equiv_C v''$, and $\mu(v') \succeq \mu(v'')$, then $v \rhd_\mu v''$.*
(2) *Thus if in addition $G \models \mathbf{R}C$, then all edges (v, v'') for v, v', and v'' satisfying* (1) *can be added to G, and $G \models \mathbf{R}C$ will still hold.*

Proof. Assume $v \rhd_\mu v'$, $v' \equiv_C v''$ and $\mu(v') \succeq \mu(v'')$.

(1) Case 1: $\mu(v) \succ \mu(v')$. Since $\succeq$ is an order, $\mu(v) \succeq \mu(v'')$, and it follows that ($\rhd$) is satisfied for v and v''. Case 2: There is a $\hat{t} \leqslant \mu(v) \uparrow \mu(v')$ such that $v' \in R_{\xi(\hat{t})}$. If $\hat{t} \leqslant \mu(v) \uparrow \mu(v'')$, then ($\rhd$) is satisfied for v and v'', because $v' \equiv_C v''$ and thus $v'' \in R_{\xi(\hat{t})}$; otherwise, $\mu(v) \uparrow \mu(v'') < \hat{t} \leqslant \mu(v) \uparrow \mu(v')$ and since $\mu(v') \succeq \mu(v'')$, application of Lemma 2 gives $\mu(v) \succ \mu(v'')$, whence ($\rhd$) also holds for v and v''.

(2) After adding all such edges, μ is still a Rabin measure. Thus, by Theorem 1, $G \models \mathbf{R}C$ still holds. $\square$

(A variant of this lemma is used in [9] for a proof of a version of Theorem 1 formulated for reasoning about fairness of programs.)

The variation on Rabin measures introduced next is a further simplification of the *quasi Rabin measures* in [8]. The idea is to get rid of the infinite range of a Rabin progress measure so that the values of the measure can be guessed by a finite-state machine. This allows us to formulate a duality theorem between Rabin and Streett conditions on graphs.

Definition 3. The *edge graph* $\mathscr{E}G$ of $G = (V, E)$ is the graph whose vertices are E and whose edges are of the form $((u, v), (u', v'))$ with $v = u'$.

Definition 4. A *color list* is a nonempty list of colors indexed by 0, 1, ... Such lists are used to describe colorings of paths originating in the root of a colored pointer tree. It is convenient to define the *level* of a color list $\chi^0 \cdots \chi^\ell$ as ℓ and to use the notation $|\chi^0 \cdots \chi^\ell|$ to denote ℓ. Also, we use the notation $\uparrow$ to denote the longest common prefix of two color lists. Note that this prefix is always non-empty and begins with color 0.

A *reduced quasi Rabin measure* $\mu: V \to X^*$ for $(G\mathbf{R}C)$ assigns to each $v \in V$ a color list $\mu(v) = \chi^0 \cdots \chi^\ell$, $\ell < |X|$, such that

- $\chi^0 = 0$,
- all colors are distinct, and
- for every $\chi \in \mu(v)$, $v \notin I_\chi$.

Thus a reduced quasi Rabin measure holds just the color information of a Rabin measure. Note that for any infinite path $v_0 v_1 \ldots$, $0 \leqslant \underline{\lim}_i |\mu(v_i) \uparrow \mu(v_{i+1})| < |X|$, where $\underline{\lim}_i h_i$, the lim inf of h_i, is the least h such that infinitely often $h_i = h$.

Given a reduced quasi Rabin measure, we may convert a Rabin condition to a Streett condition on the edge graph by defining the new colors to be $\{0, \ldots, |X| - 1\}$ and the new pairs set to be as follows.

Definition 5. Given a reduced quasi Rabin measure μ for $(G, \mathbf{R}C)$, we define a pairs set $C^\mu = \{(\bar{R}_\ell, \bar{I}_\ell) \mid 0 \leqslant \ell < |X|\}$ on $\mathscr{E}G$ by:

$$(u,v) \in \bar{R}_\ell \text{ iff } |\mu(u) \uparrow \mu(v)| \leqslant \ell$$

$$(u,v) \in \bar{I}_\ell \text{ iff } |\mu(u) \uparrow \mu(v)| \leqslant \ell, \text{ or } |\mu(u) \uparrow \mu(v)| \geqslant \ell \text{ and}$$
$$v \in R_{\mu(v)^{\hat{\ell}}} \text{ for some } \hat{\ell} \leqslant \ell.$$

C^μ is called the *dual pairs set of C with respect to μ.*

Intuitively, the aim of this definition is to construct the new pair at level ℓ such that if $\underline{\lim}_i |\mu(v_i) \uparrow \mu(v_{i+1})| = \ell$, then the pair is satisfied if and only if one of the $\ell + 1$ eventually constant colors below or at ℓ is infinitely often reconfirmed.

The set $\bar{R}_\ell$ is constructed so that for an infinite sequence $v_0 v_1 \cdots$, $(v_i, v_{i+1}) \in \bar{R}_\ell$ holds infinitely often if and only if $\ell \geqslant \underline{\lim}_i |\mu(v_i) \uparrow \mu(v_{i+1})|$, i.e. if ℓ is, or is above, the highest level with an eventually constant color.

The set $\bar{I}_\ell$ is constructed so that for an infinite sequence $v_0 v_1 \cdots$, $(v_i, v_{i+1}) \in \bar{I}_\ell$ holds infinitely often if and only if either $\ell > \underline{\lim}_i |\mu(v_i) \uparrow \mu(v_{i+1})|$, i.e. if ℓ is above the highest level with an eventually constant color, or if $\ell \leqslant \underline{\lim}_i |\mu(v_i) \uparrow \mu(v_{i+1})|$ and infinitely often some color at or below ℓ is reconfirmed.

A Streett condition $\mathbf{S}C^\mu$ on *all* paths in the edge graph can be used to check that *no* path in the original graph satisfies a given Streett condition $\mathbf{S}C$ (i.e. that all paths satisfy $\mathbf{R}C$).

Lemma 5 (*Rabin duality*). *$G \models \mathbf{R}C$ if and only if there is a reduced quasi Rabin measure μ such that $\mathscr{E}G \models \mathbf{S}C^\mu$.*

Proof. "$\Leftarrow$". Assume that $\mathscr{E}G \models \mathbf{S}C^\mu$. Let $v_0 v_1 \cdots$ be an infinite path in G. Note that $(v_0, v_1)(v_1, v_2) \cdots$ is an infinite path in $\mathscr{E}G$ and let $\ell_k = |\mu(v_k) \uparrow \mu(v_{k+1})|$ and $\ell_{\underline{\lim}} = \underline{\lim}_k \ell_k$. Also for $\ell \leqslant l_{\underline{\lim}}$, let χ^ℓ be the eventually constant value of $\mu(v_k)^\ell$.

From the definition of $\ell_{\underline{\lim}}$ and $\bar{R}_{\ell_{\underline{\lim}}}$, it follows that $(v_k, v_{k+1}) \in \bar{R}_{\ell_{\underline{\lim}}}$ holds infinitely often. Thus by assumption that $\mathscr{E}G \models SC^{\mu}$, $(v_k, v_{k+1}) \in \bar{I}_{\ell_{\underline{\lim}}}$ holds infinitely often. By definition of $\bar{I}_{\ell_{\underline{\lim}}}$ and since $\ell_k \geqslant \ell_{\underline{\lim}}$ holds almost always, $v_k \in R_{\chi^{\ell}}$ holds infinitely often for some $\ell \leqslant \ell_{\underline{\lim}}$. Also, since $v_k \notin I_{\chi^{\ell}}$ holds almost always, it follows that $v_0 v_1 \cdots \models \mathbf{R}(R_{\chi^{\ell}}, I_{\chi^{\ell}})$, whence $v_0 v_1 \cdots \models \mathbf{R}C$.

"$\Rightarrow$". Assume $G \models \mathbf{R}C$. By Theorem 1, there is a Rabin measure $(\tilde{\mu}, T, \xi)$ for $(G, \mathbf{R}C)$. The mapping $\tilde{\mu}: V \to T$ naturally induces a mapping $\mu: V \to X^*$ defined by $\mu(v) = \xi(\varepsilon) \cdots \xi(t)$, where t is defined as follows: if the node $\tilde{\mu}(v)$ is colored, then t is $\tilde{\mu}(v)$; otherwise, the parent of $\tilde{\mu}(v)$ is colored and t is defined to be this node. The mapping μ is easily seen to be a reduced quasi Rabin measure.

To see that $\mathscr{E}G \models SC^{\mu}$, consider a path $(v_0 v_1)(v_1, v_2) \cdots$ in $\mathscr{E}G$. Let $\ell_k = |\mu(v_k) \uparrow \mu(v_{k+1})|$ and $\ell_{\underline{\lim}} = \underline{\lim}_k \ell_k$. For $\ell \leqslant \ell_{\underline{\lim}}$, let χ^{ℓ} be the eventually constant value of $\mu(v_k)^{\ell} = \xi(\tilde{\mu}(v_k) \downarrow \ell)$. Fix $\lambda < |X|$. Assume that $(v_k, v_{k+1}) \in \bar{R}_{\lambda}$ holds infinitely often. We must prove that $(v_k, v_{k+1}) \in \bar{I}_{\lambda}$ holds infinitely often.

Case 1: $\lambda > \ell_{\underline{\lim}}$. By definitions of λ, $\ell_{\underline{\lim}}$, and $\bar{I}_{\lambda}$, it holds infinitely often that $(v_k, v_{k+1}) \in \bar{I}_{\lambda}$.

Case 2: $\lambda \leqslant \ell_{\underline{\lim}}$. It must hold that $\ell_{\underline{\lim}}. = \lambda$; for if $\lambda < \ell_{\underline{\lim}}$ then it holds almost always that $\ell_k \geqslant \ell_{\underline{\lim}} > \lambda$, whence v_k would be in $\bar{R}_{\lambda}$ only finitely often. Since $\tilde{\mu}$ is a Rabin measure, we have by Lemma 3 that $v_0 v_1 \cdots \models \mathbf{R}(R_{\xi(t)}, I_{\xi(t)})$ for some colored $t \in T$ such that almost always $t \leqslant \tilde{\mu}(v_k)$. Let $\hat{\ell} = |t|$. By definition of μ, $\hat{\ell} \leqslant \ell_{\underline{\lim}}$ and $\chi^{\hat{\ell}} = \xi(t)$. Thus, $v_0 v_1 \cdots \models \mathbf{R}(R_{\chi^{\hat{\ell}}}, I_{\chi^{\hat{\ell}}})$; in particular, $v_k \in R_{\chi^{\hat{\ell}}}$ infinitely often. It follows that $(v_k, v_{k+1}) \in \bar{I}_{\lambda}$ holds infinitely often. $\square$

4. Tree automata and games

An infinite word is an infinite sequence of letters. Labeled trees are a generalization that allow a branching structure to the infinite objects.

Definition 6. A *Σ-labeled tree* $\tau: \mathbb{B}^* \to \Sigma$ assigns a letter $\tau(\delta)$ to each $\delta \in \mathbb{B}^*$.

We will sometimes call 1 the *left direction* and 0 the *right direction.* Tree automata are simple machine models designed to give a finite presentation of sets of infinite labeled trees.

Definition 7. A *tree automaton* $\mathfrak{U} = (\Sigma, V, \to, V^0)$ consists of an *alphabet* Σ, *state set* V, *transition relation* $\to \subseteq V \times \mathbb{B} \times \Sigma \times V$, and a set of *initial states* $V^0 \subseteq V$. The *size of* $\mathfrak{U}$ is denoted $|\mathfrak{U}|$ and is the number of states, i.e. $|V|$. An *acceptance condition* is a condition on infinite state sequences. Automaton $\mathfrak{U}$ equipped with acceptance condition SC is denoted $\mathfrak{U}SC$ and is called a *Streett tree automaton.*

4. Games

Given a tree automaton $\mathfrak{U}$ and a Σ-labeled tree τ, called the *input tree*, we define the *game* $(\mathfrak{U}, \tau)$, which is played between the *automaton player*, denoted A, and the *pathfinder player*, denoted PF. The players A and PF alternate. A plays states according to the transition relation, and PF plays directions indicating a path through τ. More precisely, the game goes as follows:

Round 0: $\{$ • A plays an initial state $v_0 \in V^0$.

Round 1: $\begin{cases} \text{• PF plays a direction } d_1 \in \mathcal{B}, \text{ and} \\ \text{• A plays a state } v_1 \text{ such that } v_0, d_1 \xrightarrow{\tau(\varepsilon)} v_1. \end{cases}$

...

Round i: $\begin{cases} \text{• PF plays } d_i \in \mathbb{B}, \text{ and} \\ \text{• A plays } v_i \text{ such that } v_{i-1}, d_i \xrightarrow{\tau(d_1 \cdots d_{i-1})} v_i. \end{cases}$

...

Note the asymmetry in the game $(\mathfrak{U}, \tau)$: any $d \in \mathbb{B}$ is a legal move for player PF, whereas player A must obey the transition relation of $\mathfrak{U}$. At any point during the course of a game, the *position* is a pair (v, δ) describing the moves leading up that point. A *play* is the sequence of positions encountered, namely $(\varepsilon, \varepsilon)$, (v_0, ε), (v_0, d_1), $(v_0 v_1, d_1)$, ... In general a position is of the form (v, δ) if it is A's turn and (vv, δ) if it is PF's turn, where in both cases $|v| = |\delta|$. In round i, the state *visited* is v_i, and if $i > 0$, the *input position* is $d_1 \cdots d_{i-1}$, and the letter *read* by $\mathfrak{U}$ is $\tau(d_1 \cdots d_{i-1})$.

In the *acceptance game* $(\mathfrak{U}SC, \tau)$, A's goal is to play states satisfying SC. Thus A *wins* the play if $v_0 v_1 \cdots \models SC$; otherwise PF *wins*.

4.2. Strategies for player A

An A *strategy* α in the game $(\mathfrak{U}, \tau)$ is a V-labeled tree that determines A's moves as a function of PF's moves; player A *plays according to* α if A's move in round i is $\alpha(d_1 \cdots d_i)$. Since the strategy α must denote legal moves, it must satisfy:

(α0) $\quad \alpha(\varepsilon) \in V^0$ and

(α1) $\quad \alpha(\delta), d \xrightarrow{\tau(\delta)} \alpha(\delta d)$.

The strategy α is a *winning strategy for* A in the acceptance game $(\mathfrak{U}SC, \tau)$ if A wins all plays according to α. Automaton $\mathfrak{U}SC$ *accepts* τ *if* A has a winning strategy in the acceptance game. A winning strategy is also called a *run* over τ. The *language* $L(\mathfrak{U}SC)$ defined by $\mathfrak{U}SC$ is the set of all τ accepted by $\mathfrak{U}SC$.

Streett tree automata are as powerful as tree automata with the Rabin or Muller acceptance condition [25]. It is not hard to see that languages accepted by Streett automata are closed under union, intersection, and projection (homomorphism).

5. Complementing tree automata

The problem that we are to solve is: given $\mathfrak{U}SC$, find $\bar{\mathfrak{U}}S\bar{C}$ such that $\tau \notin L(\mathfrak{U}SC)$ if and only if $\tau \notin L(\mathfrak{U}SC)$. The proof follows the following recipe.

- We define strategies for PF. By the determinacy of Borel games, either A or PF has a winning strategy.
- To each PF strategy ρ we construct a graph $\mathscr{G}_\rho$, called a ρ-graph, that describes all possible games when PF plays according to ρ and A plays in any legal way. We show that a PF strategy ρ is winning if and only if $\mathscr{G}_\rho \models \mathbf{R}C$.
- We prove the immediate determinacy property that if PF has a winning strategy, then PF has a *memoryless winning strategy*, i.e. one that depends only on the position in the input tree and the last move by A.
- By the Rabin Duality Lemma, checking that $\mathscr{G}_\rho \models \mathbf{R}C$ is equivalent to checking that for some reduced quasi Rabin measure μ on $\mathscr{G}_\rho$, $\mathscr{E}\mathscr{G}_\rho \models \mathbf{S}C^\mu$. We show how the edge graph of $\mathscr{G}_\rho$, where ρ is now assumed memoryless, can be represented as a ρ-graph of an automaton $\mathscr{E}\mathfrak{U}$, called the *edge automaton.*
- To complement, we apply a *subset construction* and obtain an automaton $\mathscr{P}\mathscr{E}\mathfrak{U}$. This automaton guesses a memoryless PF strategy ρ for $\mathscr{E}\mathfrak{U}$ and calculates piecemeal the graph describing all possible games that can be played when PF plays ρ.
- In order to make $\mathscr{P}\mathscr{E}\mathfrak{U}$ verify that for some μ, $\mathscr{E}\mathscr{G}_\rho \models \mathbf{S}C^\mu$ holds, we modify it into an automaton $\mathscr{M}\mathscr{P}\mathscr{E}\mathfrak{U}$, called the *measure automaton* that guesses the value of a reduced quasi Rabin measure μ for each vertex in $\mathscr{G}_\rho$. A Streett condition $\mathbf{S}C^{\mathscr{M}}$ on the graph calculated by $\mathscr{P}\mathscr{E}\mathfrak{U}$ expresses $\mathbf{S}C^\mu$ according to the progress values guessed by $\mathscr{M}\mathscr{P}\mathscr{E}\mathfrak{U}$.
- The final step consists of changing $\mathscr{M}\mathscr{P}\mathscr{E}\mathfrak{U}$ into a Streett automaton that verifies that along all paths through the subsets, the Streett condition $\mathbf{S}C^{\mathscr{M}}$ holds. For each basic pair $(\bar{R}, \bar{I})$, in $C^{\mathscr{M}}$, we use an extended version of Safra's construction that allows a Streett condition on the states of $\mathscr{M}\mathscr{P}\mathscr{E}\mathfrak{U}$ to express that for all paths, $\mathrm{S}(\bar{R}, \bar{I})$ holds. We use a cross product on the Safra constructions to express the conjuction of the basic pairs in $C^{\mathscr{M}}$.

5.1. *Strategies for player* PF

Strategies for PF specify PF's moves as function of A's moves. We could define a strategy ρ for PF to be a function from V^+ to $\mathbb{B}$, but in order later to introduce memoryless strategies, we define ρ to be partial function $V^+ \times \mathbb{B}^* \to \mathbb{B}$. Each element in $\mathrm{dom}(\rho)$ is of the form (vv, δ), $|v| = |\delta|$, and denotes the position after A has played vv and PF has played δ. The value of $\rho(vv, \delta)$ denotes PF's next move according to ρ. Formally, ρ must satisfy

$(\rho 0)$ $\quad (v, \varepsilon) \in \mathrm{dom}(\rho)$ iff $v \in V^0$ and

$$(\rho 1) \qquad \left.\begin{array}{l} (vv,\delta) \in \mathrm{dom}(\rho), \\ d = \rho(vv,\delta), \quad \text{and} \\ v, d \xrightarrow{\tau(\delta)} v' \end{array}\right\} \quad \text{iff } (vvv',\delta d) \in \mathrm{dom}(\rho).$$

Condition ($\rho 0$) states that ρ defines a move for PF after any initial move by A. Condition ($\rho 1$) states that from any position (vv,δ) in a game played according to ρ, PF plays $d = \rho(vv,\delta)$ and for any move v' that A then may play, the position $(vvv',\delta d)$ must be described by ρ; in the other direction, condition ($\rho 1$) states that any position $(vvv',\delta d)$ described must be accessible by moves according to the strategy.

Lemma 6 (*Determinacy*). *Either* A *or* PF *has a winning strategy in the acceptance game* $(\mathfrak{U}SC,\tau)$.

Proof. The set of sequences satisfying a Streett condition is a Boolean combination of Π_2^0 sets (sets at the second level of the Borel hierarchy also called $\mathbf{G}_\delta$ sets). Such combinations are contained in the third level of the Borel hierarchy. Then by Martin's Theorem [11] (or the simpler result in [5]) the acceptance game is determined, i.e. one of the players A or PF has a winning strategy. $\square$

Definition 8. The graph $\mathscr{G}_\rho(\mathfrak{U},\tau)$ called the *ρ-graph*, of a strategy ρ in the game $(\mathfrak{U},\tau)$ is the graph on $\mathrm{dom}(\rho)$ whose edges are $((vv,\delta),(vvv',\delta d))$, where $d = \rho(vv,\delta)$ and $v,d \xrightarrow{\tau(\delta)} v'$, corresponding to the condition ($\rho 1$). This graph describes all possible ways that A can play given that PF plays according to ρ. $\mathscr{G}_\rho(\mathfrak{U},\tau)$ inherits a pairs set C in the natural way; for example, $R \subseteq V$ is inherited as $\{(vv,\delta) \mid v \in R, |v| = |\delta|\}$. The inherited pairs set is also denoted C.

Example. In Fig. 3 we have depicted the ρ-graph of a strategy for PF in a game $(\mathfrak{U},\tau)$. The input tree τ is shown by representing the edges as "dividing screens." The only initial state v^0 is shown to the left; the black square here denotes the game position (v^0,ε) after A has played v^0 in Round 0. In Round 1, PF's strategy is to play "left," i.e. $\rho(v^0,\varepsilon) = 1$. There are three possible states that A may play in turn, namely v', v'', and v'''; that is, these states are all v such that $v^0, 1 \xrightarrow{\tau(\delta)} v$. The move "left" by PF and the choices of A in Round 1 are depicted as three edges along the "dividing screen" along the left direction from the root. There are three possible game positions corresponding to the position 1 in the input tree, namely $(v^0 v',1)$, $(v^0 v'',1)$ and $(v^0 v''',1)$. In Round 2, PF plays "left" in the first position, giving A two choices (the dotted edge will be explained later); "left" in the second position, giving A three choices; and "right" in the third position, giving A two choices. Note that in the position 11 of the input tree, there are *two* ways of reaching the state $\hat{v}$: through v' and through v''. The strategy is conflicting: in the first case it specifies that PF play "left," in the second "right". In general, there is no finite bound on the number of different copies of a state that

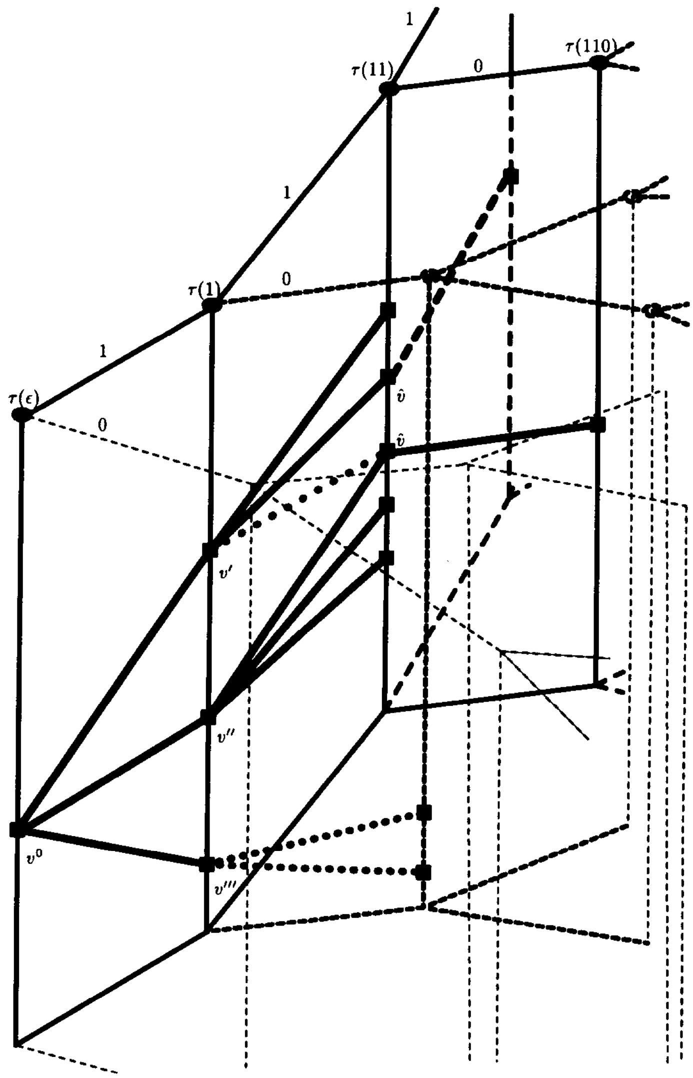

Fig. 3. Immediate determinacy.

correspond to a given position of the input tree. This is the fundamental problem that is overcome by means of finite-memory strategies.

Definition 9. If $w = d_1 d_2 \cdots$ is an infinite path in τ, then the subgraph of $\mathscr{G}_\rho(\mathfrak{U}, \tau)$ consisting of all nodes of the form $(vv, d_1 \cdots d_k)$, where $|v| = |k|$, is called the *w-bundle of* $\mathscr{G}_\rho(\mathfrak{U}, \tau)$.

Note that each w-bundle is a forest; in fact, all of $\mathscr{G}_\rho(\mathfrak{U}, \tau)$ is a forest.

Lemma 7. *In the game* $(\mathfrak{U}SC, \tau)$, ρ *is a winning strategy for* PF *if and only if* $\mathscr{G}_\rho(\mathfrak{U}, \tau) \models$ **R**C.

Proof. Follows from the fact that the infinite paths starting in (v^0, ε), where $v^0 \in V^0$, of $\mathscr{G}_\rho(\mathfrak{U}, \tau)$ correspond to all possible outcomes of the game when PF plays according to ρ. □

5.2. Memoryless PF *strategies*

A *memoryless* PF *strategy* ρ depends on all the previous directions played by PF but only on the very last move by A. Thus it can be described by a partial function $V \times \mathbb{B}^* \rightharpoonup \mathbb{B}$. An element $(v, \delta) \in \mathrm{dom}(\rho)$ describes a position in a game where PF has played δ according to ρ and A has played vv for some v, where $|v| = |\delta|$; PF's next move is then $\rho(v, \delta)$. A memoryless strategy ρ defines a strategy $\bar{\rho}$ by $\bar{\rho}(vv, \delta) = \rho(v, \delta)$. Thus ρ must satisfy:

$$(\rho ML0) \qquad (v, \varepsilon) \in \mathrm{dom}(\rho) \text{ iff } v \in V^0 \quad \text{and}$$

$$(\rho ML1) \qquad \left.\begin{array}{l} \exists v : (v, \delta) \in \mathrm{dom}(\rho), \\ d = \rho(v, \delta), \quad \text{and} \\ v, d \xrightarrow{\tau(\delta)} v' \end{array}\right\} \text{ iff } (v', \delta d) \in \mathrm{dom}(\rho).$$

Definition 10. The graph $\mathscr{G}_\rho^{\mathrm{ML}}(\mathfrak{U}, \tau)$, called the *memoryless* ρ*-graph*, of a memoryless strategy ρ in the game $(\mathfrak{U}, \tau)$ is the graph on $\mathrm{dom}(\rho)$ whose edges are $((v, \delta), (v', \delta d))$, where $d = \rho(v, \delta)$ and $v, d \xrightarrow{\tau(\delta)} v'$, corresponding to the condition $(\rho ML1)$. As in Definition 8, a Rabin condition C on V is inherited and also denoted C.

Definition 11. If $w = d_1 d_2 \cdots$ is an infinite path in τ, then the subgraph of $\mathscr{G}_\rho^{\mathrm{ML}}(\mathfrak{U}, \tau)$ consisting of all nodes of the form $(v, d_1 \cdots d_k)$ is called the w-bundle of $\mathscr{G}_\rho^{\mathrm{ML}}(\mathfrak{U}, \tau)$.

Note that a w-bundle of $\mathscr{G}_\rho^{\mathrm{ML}}(\mathfrak{U}, \tau)$ is not a forest, but a graph whose "width" is at most $|V|$; in other words for each δ, there are at most $|V|$ vertices of the form (v, δ).

Lemma 8 (*Memoryless* ρ*-graph*). *In the game* $(\mathfrak{U}SC, \tau)$ *the memoryless* PF *strategy* ρ *is a winning strategy if and only if* $\mathscr{G}_\rho^{\mathrm{ML}}(\mathfrak{U}, \tau) \models$ **R**C.

Proof. Cf. proof of Lemma 7. □

Example (*Continued*). Consider again the ρ-graph in Fig. 3. Assume that the strategy for PF is winning with respect to some Rabin condition on the automaton states. Then there is a Rabin measure μ for the graph. Suppose that $\mu(v^0 v' \hat{v}, 11) \geqslant \mu(v^0 v'' \hat{v}, 11)$. Now according to Lemma 4 (Surgery), an edge from $(v^0 v', 1)$ to $(v^0 v'' \hat{v}, 11)$ can be added with μ remaining a Rabin measure. The edge from $(v^0 v', 1)$ to $(v^0 v' \hat{v}, 11)$ can now be deleted (together with the unattached subtree rooted in $(v^0 v' \hat{v}, 11)$). The graph still represent a winning strategy since it has a Rabin measure. The strategy ρ at 11 has become memoryless: when $\hat{v}$ was the last state played by A, ρ specifies that PF plays "right" regardless of how $\hat{v}$ was reached.

By using the idea in the example above, we can easily prove the following result.

Lemma 9 (*Immediate determinacy*). *In the game* $(\mathfrak{U}\mathbf{S}C, \tau)$, *there is a winning* PF *strategy if and only if there is a memoryless winning* PF *strategy*.

Proof. "$\Leftarrow$". Obvious.

"$\Rightarrow$". Assume that ρ is a winning strategy for PF. We will define a memoryless winning strategy $\bar{\rho}$ by applying surgery to the ρ-graph representation $(\hat{V}, \hat{E}) = \mathcal{G}_\rho(\mathfrak{U}, \tau)$. For $\hat{v} = (vv, \delta) \in \hat{V}$, define $\mathbf{v}\hat{v} = v$ and $\boldsymbol{\delta}\hat{v} = \delta$. The idea is for each (v, δ) to select a canonical position $\hat{v} = m(v, \delta)$ in $(\hat{V}, \hat{E})$ with $v = \mathbf{v}\hat{v}$ and $\delta = \boldsymbol{\delta}\hat{v}$. Then $\bar{\rho}(v, \delta)$ is defined to be $\rho(\hat{v})$. According to Lemma 7 and Theorem 1 there is a Rabin measure μ for $(\mathcal{G}_\rho(\mathfrak{U}, \tau), \mathbf{R}C)$. The canonical position $m(v, \delta)$ is now chosen according to μ: define $m(v, \delta) = \hat{v}$ such that $v = \mathbf{v}\hat{v}$, $\delta = \boldsymbol{\delta}\hat{v}$, and $\mu(\hat{v})$ is minimal with respect to the Kleene-Brouwer ordering (if several $\hat{v}$ qualify, choose one).

The graph $(\hat{V}, \hat{E})$ restricted to canonical positions would not correspond to a strategy: from $\hat{v} = m(v, \delta)$ there may emanate an edge $(\hat{v}, \hat{v}')$ to a position $\hat{v}'$ that is not a canonical position. The position $\hat{v}'$ is such that $\mathbf{v}\hat{v}' = v'$ for some v' with $v, d \xrightarrow{\tau(\delta)} v'$ and $\boldsymbol{\delta}\hat{v}' = \delta d$, where $d = \rho(\hat{v})$. Now consider the vertex $\hat{v}'' = m(v', \delta d)$. Since $(\hat{v}, \hat{v}') \in \hat{E}$, it holds that $\hat{v} \rhd_\mu \hat{v}'$; since the pairs set C on $\hat{V}$ is inherited from V and $\mathbf{v}\hat{v}' = \mathbf{v}\hat{v}'' = v'$, it holds that $\hat{v}' \equiv_C \hat{v}''$; and by the definition of $\hat{v}''$, it holds that $\mu(\hat{v}') \succeq \mu(\hat{v}'')$.

All edges $(\hat{v}, \hat{v}'')$, where $\hat{v}$, $\hat{v}''$ are gotten as above, are added to $(\hat{V}, \hat{E})$. By Lemma 4 (Surgery), the resulting graph, also denoted $(\hat{V}, \hat{E})$, has a Rabin measure; moreover, $(\hat{V}, \hat{E})$ restricted to canonical positions now describes a memoryless strategy $\bar{\rho}$. Thus by Theorem 1, $\bar{\rho}$ is a winning strategy. $\square$

The kind of forgetful determinacy gotten in this way does not quite correspond to the informal explanation in Section 1. Using a slightly more elaborate surgical technique, we can strengthen the Lemma so that the pathfinder's move is dependent only on the last state played and the subtree rooted at the current input position (called the *node-residue* in [6]). This, however, is not necessary for the complementation result.

5.3. Edge automata

Definition 12. Given $\mathfrak{U}$, define the *edge automaton* $\mathscr{E}\mathfrak{U}$ by

$$\mathscr{E}\mathfrak{U} = (\Sigma, (V \cup \{\bot\}) \times V, \rightarrow_{\sim}, \{\bot\} \times V^0),$$

where $(u, v), d \overset{\sigma}{\rightarrow}_{\sim} (u', v')$ if and only if $v = u'$ and $u', d \overset{\sigma}{\rightarrow} v'$.

Intuitively, this automaton works as $\mathfrak{U}$ except that the previous $\mathfrak{U}$ state visited is remembered by the current state of $\mathscr{E}\mathfrak{U}$. A memoryless strategy for $(\mathscr{E}\mathfrak{U}, \tau)$ is then a partial function $V \times V \times \mathbb{B}^* \subsetneq \mathbb{B}$, since it depends also on the previous $\mathfrak{U}$ state visited.

In the sequel we always consider a reduced quasi measure μ as a function that assigns a color list to pairs of form (v, δ). So given a memoryless strategy ρ for $(\mathscr{E}\mathfrak{U}, \tau)$ and a reduced quasi measure μ, we may construct a pairs set C^μ on the nodes of $\mathscr{G}_\rho^{\mathrm{ML}}(\mathscr{E}\mathfrak{U}, \tau)$ as follows: we do not consider the initial ones, which are finite in number and therefore can be neglected, so a node is of the form $((u, v), \delta \cdot d)$ and we can apply Definition 5 to (u, δ) and $(v, \delta \cdot d)$. For this to work, it is necessary to assume that the domain of μ includes all such nodes.

The relationships between $(\mathfrak{U}, \tau)$ and $(\mathscr{E}\mathfrak{U}, \tau)$ can now be stated formally.

Lemma 10 (*Edge automaton*). (1) *To each* PF *strategy ρ in the game $(\mathfrak{U}, \tau)$ corresponds a* PF *strategy $\tilde{\rho}$ in the game $(\mathscr{E}\mathfrak{U}, \tau)$, and this correspondence is bijective. Also, if ρ is memoryless, then $\tilde{\rho}$ is memoryless.*

(2) *$\mathscr{E}\mathscr{G}_\rho(\mathfrak{U}, \tau)$ and $\mathscr{G}_{\tilde{\rho}}(\mathscr{E}\mathfrak{U}, \tau)$ are essentially isomorphic. More precisely, $\mathscr{E}\mathscr{G}_\rho(\mathfrak{U}, \tau) \simeq \mathscr{G}_{\tilde{\rho}}(\mathscr{E}\mathfrak{U}, \tau) \setminus \tilde{V}^0$, where $\tilde{V}^0 = \{((\perp, v^0), \varepsilon) \mid v^0 \in V^0\}$.*

(3) *The following two conditions are then equivalent.*

(a) *There is a memoryless strategy ρ in $(\mathfrak{U}, \tau)$ and a reduced quasi measure $\mu: V \times \mathbb{B}^* \subsetneq \mathbb{B}$ such that $\mathscr{E}\mathscr{G}_\rho^{\mathrm{ML}}(\mathfrak{U}, \tau) \models \mathbf{S}C^\mu$.*

(b) *There is a memoryless strategy $\tilde{\rho}$ in $(\mathscr{E}\mathfrak{U}, \tau)$ and a reduced quasi measure $\mu: V \times \mathbb{B}^* \subsetneq \mathbb{B}$ such that $\mathscr{G}_{\tilde{\rho}}^{\mathrm{ML}}(\mathscr{E}\mathfrak{U}, \tau) \models \mathbf{S}C^\mu$.*

Proof. (1) The correspondence is given by

$$\tilde{\rho}((\perp, v_0), \varepsilon) = \rho(v_0, \varepsilon)$$

$$\tilde{\rho}((\perp, v_0)(v_0, v_1) \cdots (v_{k-1}, v_k), \delta) = \rho(v_0 \cdots v_k, \delta), \quad k \geqslant 1, \quad |\delta| = k.$$

Also, it is not hard to see that if ρ is memoryless, then $\tilde{\rho}$ is memoryless.

(2) An isomorphism $h: \mathscr{G}_{\tilde{\rho}}(\mathscr{E}\mathfrak{U}, \tau)/\tilde{V}^0 \to \mathscr{E}\mathscr{G}_\rho(\mathfrak{U}, \tau)$ is defined by:

$$h((\perp, v_0)(v_0, v_1) \cdots (v_{k-1}, v_k)\delta d) = ((v_0 \cdots v_{k-1}, \delta), (v_0 \cdots v_k, \delta d)).$$

where $k \geqslant 1, |\delta| = k - 1$.

(3) "(a) $\Rightarrow$ (b)". This follows from (1) and (2).

"(b) $\Rightarrow$ (a)." Assume (b). Then it follows from (2) that $\mathscr{E}\mathscr{G}_\rho(\mathfrak{U}, \tau) \models \mathbf{S}C^\mu$ and then from Lemma 5 (Rabin Duality) that $\mathscr{G}_\rho(\mathfrak{U}, \tau) \models \mathbf{R}C$. Note that ρ is not necessarily memoryless in $(\mathfrak{U}, \tau)$. But by Lemma 9 (Immediate Determinacy), there is a memoryless strategy $\hat{\rho}$ such that $\mathscr{G}_{\hat{\rho}}(\mathfrak{U}, \tau) \models \mathbf{R}C$. Then yet an application of Lemma 5 yields a reduced quasi measure $\hat{\mu}$ such that $\mathscr{E}\mathscr{G}_{\hat{\rho}}^{\mathrm{ML}}(\mathfrak{U}, \tau) \models \mathbf{S}C^{\hat{\mu}}$. $\square$

5.4. Subset automata

We define an automaton $\mathscr{P}\mathfrak{U}$ whose runs or A strategies are the memoryless PF strategies of $\mathfrak{U}$. To do this, $\mathscr{P}\mathfrak{U}$ piecemeal guesses a memoryless strategy ρ, and along any path $w = d_0 d_1 \cdots$ in the input tree, it piecemeal constructs the w-bundle by determining the subset of states of $\mathfrak{U}$ that are consistent with ρ. More precisely, $\mathscr{P}\mathfrak{U}$ is defined as follows.

Definition 13. Given $\mathfrak{U} = (\Sigma, V, \rightarrow, V^0)$, define the *subset automaton* $\mathscr{P}\mathfrak{U}$ of A by

$$\mathscr{P}\mathfrak{U} = (\Sigma, (V \rightharpoonup \mathbb{B}), \rightarrow_{\mathscr{P}}, \{\Delta \mid \mathrm{dom}(\Delta) = V^0\}),$$

where $\Delta, d \xrightarrow{\sigma}_{\mathscr{P}} \Delta'$ if and only if $\mathrm{dom}(\Delta') = \{v' \mid \exists\, v \in \mathrm{dom}(\Delta) : \Delta(v) = d \text{ and } v, d \xrightarrow{\sigma} v'\}$.

Intuitively, each $\Delta : V \rightharpoonup \mathbb{B}$ describes the current subset of the bundle as $\mathrm{dom}(\Delta)$ and prescribes the direction $\Delta(v)$ that PF chooses for $v \in \mathrm{dom}(\Delta)$.

Definition 14. If α is an A strategy in $(\mathscr{P}\mathfrak{U}, \tau)$, define the *$\alpha$-graph* $\mathscr{G}_\alpha(\mathscr{P}\mathfrak{U}, \tau)$ as the graph whose vertices are (v, δ), where $v \in \mathrm{dom}(\alpha(\delta))$, and whose edges are $((v, \delta), (v', \delta d))$, where $v, d \xrightarrow{\tau(\delta)} v'$ and $d = \alpha(\delta)(v)$.

Lemma 11 (*Subset construction*).
(1) *To each memoryless* PF *strategy* ρ *in the game* $(\mathfrak{U}, \tau)$ *corresponds an* A *strategy* α *in the game* $(\mathscr{P}\mathfrak{U}, \tau)$ *and this correspondence is bijective.*
(2) *Moreover,* $\mathscr{G}_\rho^{\mathrm{ML}}(\mathfrak{U}, \tau) \simeq \mathscr{G}_\alpha(\mathscr{P}\mathfrak{U}, \tau)$.

Proof (*sketch*). The proof consists of straightforward applications of the natural isomorphism between $V \times \mathbb{B}^* \rightharpoonup \mathbb{B}$ and $\mathbb{B}^* \rightarrow (V \rightharpoonup \mathbb{B})$, since memoryless PF strategies for $(\mathfrak{U}, \tau)$ are functions of the first kind and A strategies for $(\mathscr{P}\mathfrak{U}, \tau)$ are functions of the second kind. □

Note that for a strategy α in $(\mathscr{P}\mathfrak{U}, \tau)$ each path $w = d_1 d_2 \cdots$ defines a w-bundle consisting of all paths that go through subsets along w according to the transition relation.

5.5. Automata extensions

We will need to augment the information carried by the states of an automaton $\mathfrak{U}$ by adding a component W to its state space while preserving its basic behavior. Formally, we define an *extension* $\mathfrak{X}\mathfrak{U}$ of $\mathfrak{U}$ to be an automaton of the form $(\Sigma, V \times W, \rightarrow_{\mathscr{X}}, V^0 \times \{w^0\})$ such that

$$v, d \xrightarrow{a} v' \quad \text{and} \quad w \in W \text{ implies } (v, w), d \xrightarrow{a}_{\mathfrak{X}} (v', w') \quad \text{for some unique } w' \in W.$$

A run α of $\mathfrak{U}$ corresponds to a unique run, also denoted α, of an extension $\mathscr{X}\mathfrak{U}$. The *size-factor* $|\mathfrak{X}|$ of $\mathfrak{X}\mathfrak{U}$ is defined as $|W|$.

If $\mathfrak{X}_i\mathfrak{U}$ where i ranges over some finite set, are extensions of $\mathfrak{U}$, then they can be combined in a cross product construction to form an extension denoted $(\otimes_i \mathfrak{X}_i)\mathfrak{U}$; this automaton has state space $V \times \prod W_i$. In the following we consider *Streett extensions* of the form $\mathfrak{X}\mathfrak{U}\mathbf{S}C$, where C describes a pairs set on the automaton $\mathfrak{X}\mathfrak{U}$. Assume that $\mathfrak{X}_i\mathfrak{U}\mathbf{S}C_i$ are Streett extensions. Then $(\otimes_i \mathfrak{X}_i)\mathfrak{U}\mathbf{S}\bigcup_i C_i$ denotes the Streett extension that has as acceptance condition the conjunction of the Streett conditions $\mathbf{S}\bar{C}_i$'s.

Lemma 12 (*And'ing extensions*). *The strategy α is a winning strategy for all Streett extensions $\mathfrak{X}_i\mathfrak{U}\mathbf{S}C_i$ if and only if α is a winning strategy for the Streett extension $(\otimes_i \mathfrak{X}_i)\mathfrak{U}\mathbf{S}\bigcup_i C_i$.*

Proof. By definition of $(\otimes_i \mathfrak{X}_i)\mathfrak{U}\mathbf{S}\bigcup_i C_i$ $\square$

5.6. Extended Safra construction

Safra's subset construction for automata on infinite words shows how to obtain a deterministic Rabin automaton accepting the language of a nondeterministic Büchi automaton [20]. By duality, if we are given an automaton on infinite words and a Streett condition of the form $\mathbf{S}(R, \emptyset)$ – which is satisfied if and only if R is encountered only finitely often – then Safra's result tells us how to use a subset construction for obtaining a deterministic Streett automaton that accepts a word if and only if *all* runs over the word satisfies $\mathbf{S}(R, \emptyset)$. It is easy to extend this subset construction to a tree automaton $\mathfrak{U}$: there exists a Streett extension $\mathfrak{X}\mathscr{P}\mathfrak{U}\mathbf{S}\mathscr{C}$ such that α is a winning memoryless strategy for A in $(\mathfrak{X}\mathscr{P}\mathfrak{U}\mathbf{S}\mathscr{C}, \tau)$ if and only if all paths in $\mathscr{G}_\alpha(\mathscr{P}\mathfrak{U}, \tau)$ satisfy $\mathbf{S}(R, \emptyset)$. Thus, $\mathfrak{X}\mathscr{P}\mathfrak{U}\mathbf{S}\mathscr{C}$ accepts τ if and only if there is a winning strategy for PF in the game $(\mathfrak{U}\mathbf{R}(R, \emptyset), \tau)$ (i.e. whatever A does, a play of $(\mathfrak{U}, \tau)$ satisfies $\mathbf{S}(R, \emptyset)$). Conditions of the form $\mathbf{S}(R, I)$ require a more detailed consideration.

Lemma 13 (*Extended Safra*). *Let $\mathfrak{U} = (\Sigma, V, \rightarrow, V^0)$ and let (R, I) be a basic pair on V. There is a Streett extension $\mathfrak{X}\mathscr{P}\mathfrak{U}\mathbf{S}\mathscr{C}$ of $\mathscr{P}\mathfrak{U}$ such that*

$$\mathscr{G}_\alpha(\mathscr{P}\mathfrak{U}, \tau) \models \mathbf{S}(R, I) \text{ iff}$$

α is a winning strategy for A in the game $(\mathfrak{X}\mathscr{P}\mathfrak{U}\mathbf{S}\mathscr{C}, \tau)$. Moreover, $|\mathfrak{X}| = 2^{O(n \cdot \log n)}$ and $|\mathscr{C}| = n$, where $n = |V|$.

Proof. Let $\mathfrak{U} = (\Sigma, V, \rightarrow, V^0)$ be a tree automaton and let (R, I) be a basic pair on V. Let τ be an infinite Σ-labeled tree and let α be a run of $\mathscr{P}\mathfrak{U}$ over τ, i.e. a memoryless PF strategy in $(\mathfrak{U}, \tau)$. We extend the subset automaton $\mathscr{P}\mathfrak{U}$ with a component W that traces paths through the bundles of $\mathscr{G}_\alpha(\mathscr{P}\mathfrak{U}, \tau)$. The following describes how W and the corresponding Streett extension $\mathfrak{X}\mathscr{P}\mathfrak{U}\mathbf{S}\mathscr{C}$ are to be defined.

The component W designates at each position a finite tree Z for which each node $z \in Z$ is labeled with a set $\ell(z)$ of states from the current subset as calculated by $\mathscr{P}\mathfrak{U}$. Each such node z (except the root) traces a part of $\mathscr{G}_\alpha(\mathscr{P}\mathfrak{U}, \tau)$ that corresponds to paths where R might occur infinitely often but I does not occur. Note that $\mathrm{S}(R, I)$ is satisfied on $\mathscr{G}_\alpha(\mathscr{P}\mathfrak{U}, \tau)$ if and only if no such part exists. Thus by defining $\mathscr{C} = \{(\hat{R}_z, \hat{I}_z) |$ z is a node (except the root)$\}$ in a certain way, we may express by a Streett condition $\mathrm{S}\mathscr{C}$ that for each node z, if R occurs again and again in the part designated by $\ell(z)$, then eventually $l(z)$ becomes empty. When $\ell(z)$ becomes empty the node z can be reused to trace another part of the graph. This will allow us to use only $n + 1$ different nodes.

Making the Streett condition $\mathrm{S}\mathscr{C}$ sufficient for $\mathrm{S}(R, I)$ to hold calls for a sophisticated management of the different parts traced. The tree Z is used to dynamically maintain a hierarchy expressing the possible parts of the graph where $\mathrm{S}(R, I)$ might not hold. Initially Z consists only of the root, designated z^0. The label of the root is always the current subset; thus initially $\ell(z^0) = \{V^0\}$. Except for the root, a node z can be removed and inserted repeatedly in Z. A node z not in Z is *free*. When a node is in Z, its *last insertion position* is the position in τ when it was last inserted. Note that these positions are ordered by $\geqslant$.

The tree Z always satisfies the following requirements, where the tree is viewed as growing upwards:

(a) The children of a node are ordered from left to right according to decreasing last insertion position. In general we say that a node z is to the *left* of a node z' if their highest common ancestor z'' is a proper ancestor of both z and z' and if the child of z'' on the path to z is to the left of the child of z'' on the path to z'.

(b) For any node z, all $\ell(z')$, where z' is a child of z, are disjoint.

(c) For any node $z \neq z^0$, the union of $\ell(z')$, where z' is a child of z, is properly contained in $\ell(z)$.

Note that when a node z is to the left of z', then $\ell(z)$ and $\ell(z')$ are disjoint. For a state v in the current subset, the *top-node* z of v is defined as the highest node z such that $v \in \ell(z)$. By the requirements above, the top-node of v is uniquely defined. Vice versa, the top-nodes of all v in the current subset uniquely determine the labeling ℓ. Note that any non-empty node is at a level less than or equal to n (where the level is defined as for pointer trees) and that there are at most $n + 1$ nodes in Z. Therefore it can be seen that the description of the shape of Z is $O(n \log n)$ bits and that the labeling ℓ can also be specified using $O(n \log n)$ bits. Thus the size factor $|\mathfrak{X}| = |W|$ is $O(2^{n \log n})$.

In the algorithm in Fig. 4 it is described how Z is changed along a transition from input position δ to δd. Also, it is indicated for each $z \neq z^0$ whether the resulting state of the extended tree automaton is in $\hat{R}_z$ ("flash $\hat{R}_z$") or $\hat{I}_z$ ("flash $\hat{I}_z$"). We use $\mathscr{R}(\delta d, \hat{V})$ to denote $\{v' | \exists v \in \hat{V}\colon \Delta(v) = d \text{ and } v, d \xrightarrow{\tau(\delta)} v'\}$, which is the set of successors of $\hat{V}$ in direction d of $\mathscr{G}_\alpha(\mathscr{P}\mathfrak{U}, \tau)$; here Δ is the state of the subset automaton at δ.

In Step (1) of the algorithm every set $\ell(z)$ is expanded to its set of successors. This step may violate requirements (b) and (c). In Step (2) vertices that are found in different branches of the tree and deleted from all branches except the rightmost one. This re-establishes (b). In Step (3) all invalidating vertices are removed from all nodes except the root. If this makes the label of a node z empty, then $\hat{I}_z$ is flashed in Step (4).

(1) for all $z \in Z$:
  $\ell(z) := \mathcal{R}(\delta d, \ell(z))$;
(2) for all $z, z' \in Z$ and all $v \in V$:
  if z is to the left of z' and $v \in \ell(z)$ and $v \in \ell(z')$, then remove v from $\ell(z)$;
(3) for all $z \in Z \backslash \{z^0\}$ and for all $v \in \ell(z)$:
  if $v \in I$, then remove v from $\ell(z)$;
(4) for all $z \in Z$ such that $\ell(z) = \emptyset$:
  remove z from Z and flash $\hat{I}_z$;
(5) for all $z \in Z \backslash \{z^0\}$:
  if $\ell(z) = \bigcup_{z' \text{ is a child of } z} \ell(z')$, then remove every descendant of z and flash $\hat{R}_z$;
(6) for all $z \in Z$ and for all $v \in \ell(z)$:
  if $v \in R$, $v \notin I$, and v is not in any $\ell(z')$, where z' is child of z, then add a new node z'' to Z, make z'' a left-most child of z, and let $\ell(z'') = \{v\}$.

Fig. 4. Extended Safra construction.

(Intuitively, since all paths in $\ell(z)$ have ended in an invalidating state or moved to another node, this part of the graph is good.) In Step (5) $\hat{R}_z$ is flashed and all descendants are removed when $\ell(z) = \bigcup_{z' \text{ is a child of } z} \ell(z')$ and z is not the root. This re-establishes requirement (c). As we shall see this means that all nodes in the current subset can be traced back along a backwards path in the $\ell(z)$ subsets passing through R at least once to a node when $\hat{R}_z$ was flashed last time. In Step (6) a node $v \in R \backslash I$ is moved up to a new child of z if not already in a child. Intuitively this is done because a promising part of $\mathcal{G}_\alpha(\mathcal{P}\mathfrak{U}, \tau)$, where $\mathrm{S}(R, I)$ is not fulfilled, may be narrowed down to contain only this vertex.

To finish the proof, we need to show that the following claim holds..

Claim 1. *$\mathcal{G}_\alpha(\mathcal{P}\mathfrak{U}, \tau) \models \mathrm{S}(R, I)$ if and only if α is a winning strategy for* A *in the game $(\mathcal{X}\mathcal{P}\mathfrak{U}SC, \tau)$.*

Proof. "$\Rightarrow$". Assume that for some z and for some path w through input tree τ, $\hat{R}_z$ is flashed infinitely often and $\hat{I}_z$ is flashed only finitely often. Then $z \neq z^0$ and it is sufficient to prove that for some path in the w-bundle of $\mathcal{G}_\alpha(\mathcal{P}\mathfrak{U}, \tau)$, R occurs infinitely often and I only finitely often.

Now since $\hat{I}_z$ can only be flashed in Step (4) and this happens a finite number of times, then from some point on, $\ell(z)$ is always nonempty and contains no I-invalidating state. Thus it suffices to find a path that is R-reconfirmed infinitely often in the part of the graph described by the $\ell(z)$ subsets along w. By assumption, the condition in Step (5) is satisfied infinitely many times. Consider two input positions δ and δ' where this happens and $\delta \prec \delta'$. Then for every state v' in $\ell(z)$ at position δ', it is possible to find a v in $\ell(z)$ at position δ and a finite path in $\mathcal{G}_\alpha(\mathcal{P}\mathfrak{U}, \tau)$ from (v, δ) to (v', δ') passing through the subsets $\ell(z)$ and at least one R-reconfirming state. This is because Step (6) ensures that a state is found in a child of $\ell(z)$ only if some ancestor was R-reconfirming, and therefore moved up one level, since $\hat{R}_z$ was flashed last time.

We can then use König's Lemma to show that the infinite subgraph made of these finite paths contains an infinite path that passes through R infinitely often. (Sketch of proof: make a new graph whose nodes correspond to the states in $\ell(z)$ when $\hat{R}_z$ is flashed and whose edges represent the finite paths above. This graph has an infinite path by König's Lemma. This path induces an infinite path in the $\ell(z)$ subsets passing through R infinitely often.)

"$\Leftarrow$". Assume that $\mathscr{G}_\alpha \models \mathrm{S}(R, I)$ does not hold. Then some infinite path in $\mathscr{G}_\alpha \models \mathrm{S}(R, I)$ is R-reconfirmed infinitely often and I-invalidated only finitely often. Let w be the corresponding input path in τ and let $v_0 v_1 \cdots$ be the corresponding states. It suffices to prove that along w there is some z such that $\hat{R}_z$ is flashed infinitely often and $\hat{I}_z$ is flashed only finitely often.

Claim 2. *There is a node $z \neq z^0$ such that along w, z is almost always an ancestor of the top-node of v_k and is infinitely often identical to the top-node.*

Proof. From some point on $v_k \notin I$. From Steps (5) and (6) it can be seen that once the top-node then becomes different from the root, the top-node is always above the root. Thus from some point on we can observe the node z_k^1 that is defined as the child of the root on the path to the top-node of v_k. This node can move only to the right, and since nodes are inserted from the left, z_k^1 eventually settles down to some final value denoted z^1. If z^1 is infinitely often the top-node, the conclusion in Claim 2 follows.

Otherwise, a similar argument can be applied to obtain a child z^2 of z^1 that is almost always an ancestor of the top-node of v_k. If z^2 is not infinitely often a top node, we obtain a z^3 etc. Since the number of nodes is finite, we eventually obtain some node z satisfying Claim 2.

Let z be the node of Claim 2. Since the label of any ancestor of the top-node of v_k contains v_k and since z is almost always an ancestor of the top-node, $\hat{I}_z$ is flashed only finitely many times according to Step (4). Suppose now for a contradiction that $\hat{R}_z$ is flashed only finitely many times. Following the last such time and the last time v_k is I-invalidating, v_k is eventually moved to a child of z by Step (6), since v_k is R-confirming infinitely often. The state v_k may subsequently be passed towards the right from one child to another but will never be removed since v_k is not I-invalidating. Thus v_k will eventually remain in a fixed child of z, contradicting that z satisfies Claim 2. □

5.7. Putting it all together

Theorem 2 (*Complementation*). *There is an automaton $\bar{\mathfrak{U}}\mathfrak{S}\bar{C}$ accepting the complement of $L(\mathfrak{U}\mathfrak{S}C)$. Moreover, $|\bar{\mathfrak{U}}| = 2^{\mathrm{O}(n^2 \cdot m \log n)}$ and $|C| = n^2 \cdot m$, where $n = |\mathfrak{U}|$ and $m = |C|$. In particular, when $m = \mathrm{O}(n)$, $|\bar{\mathfrak{U}}| = 2^{\mathrm{O}(n^3 \log n)}$ and $|\bar{C}| = n^3$.*

(1) $\tau \notin L(\mathfrak{A}\mathbf{S}C)$ iff	by definition of acceptance
(2) A does not have a w.s. in $(\mathfrak{A}\mathbf{S}C,\tau)$ iff	by (Borel Determinacy)
(3) PF has a w.s. in $(\mathfrak{A}\mathbf{S}C,\tau)$ iff	by (Immediate Determinacy)
(4) PF has a memoryless w.s. in $(\mathfrak{A}\mathbf{S}C,\tau)$ iff	by (Memoryless ρ-graph)
(5) $\exists\rho : \mathcal{G}^{\mathrm{ML}}_{\rho}(\mathfrak{A},\tau) \models \mathbf{R}C$ iff	by (Rabin Duality)
(6) $\exists\mu : \exists\rho : \mathcal{E}\mathcal{G}^{\mathrm{ML}}_{\rho}(\mathfrak{A},\tau) \models \mathbf{S}C^{\mu}$ iff	by (Edge Automaton)
(7) $\exists\mu : \exists\rho : \mathcal{G}^{\mathrm{ML}}_{\rho}(\mathcal{E}\mathfrak{A},\tau) \models \mathbf{S}C^{\mu}$ iff	by (Subset Construction)
(8) $\exists\mu : \exists\alpha : \mathcal{G}_{\alpha}(\mathcal{P}\mathcal{E}\mathfrak{A},\tau) \models \mathbf{S}C^{\mu}$ iff	see text
(9) $\exists\alpha : \mathcal{G}_{\alpha}(\mathcal{M}\mathcal{P}\mathcal{E}\mathfrak{A},\tau) \models \mathbf{S}C^{\mathcal{M}}$ iff	by definition of Streett acceptance
(10) $\exists\alpha : \forall(R,I)\in C^{\mathcal{M}} : \mathcal{G}_{\alpha}(\mathcal{M}\mathcal{P}\mathcal{E}\mathfrak{A},\tau) \models \mathbf{S}(R,I)$ iff	by (Extended Safra)
(11) $\exists\alpha : \forall(R,I)\in C^{\mathcal{M}} : \alpha$ is a w.s. in $(\mathcal{X}_{(R,I)}\mathcal{M}\mathcal{P}\mathcal{E}\mathfrak{A}\mathbf{S}\mathcal{C}_{(R,I)},\tau)$ iff	by (And'ing Extensions)
(12) $\exists\alpha : \alpha$ is a w.s. in $((\bigotimes_{(R,I)\in C^{\mathcal{M}}} \mathcal{X}_{(R,I)})\mathcal{M}\mathcal{P}\mathcal{E}\mathfrak{A}\mathbf{S}(\bigcup_{(R,I)\in C^{\mathcal{M}}} \mathcal{C}_{(R,I)}),\tau)$ iff	by definition of acceptance
(13) $\tau \in L((\bigotimes_{(R,I)\in C^{\mathcal{M}}} \mathcal{X}_{(R,I)})\mathcal{M}\mathcal{P}\mathcal{E}\mathfrak{A}\mathbf{S}(\bigcup_{(R,I)\in C^{\mathcal{M}}} \mathcal{C}_{(R,I)}))$	

Fig. 5. Putting it all together. (Here "w.s." abbreviates "winning strategy.")

Proof. The ingredients in the proof are put together in Fig. 5, according to the recipe in Section 5. Thus we define

$$\bar{\mathfrak{U}}\mathbf{S}\bar{C} = \Big(\bigotimes_{(R,I)\in C^{\mathscr{M}}} \mathscr{X}_{(R,I)}\Big)\mathscr{M}\mathscr{P}\mathscr{E}\mathfrak{U}\mathbf{S}\left(\bigcup_{(R,I)\in C^{\mathscr{M}}} \mathscr{C}_{(R,I)}\right),$$

where the notation $\mathscr{X}_{(R,I)}$ and $\mathscr{C}_{(R,I)}$ indicates that the Extended Safra construction is applied for each pair (R,I) in the pairs set $C^{\mathscr{M}}$ defined below. Let us explain more carefully how this automaton is obtained.

The equivalences among (1) to (5) are direct applications of the definition of acceptance and Lemmas 6 (Borel determinacy), 9 (Immediate determinacy), and 8 (Memoryless ρ-graph). Thus to show that τ is not accepted by $\mathfrak{U}\mathbf{S}C$, we use the necessary and sufficient condition "guess a memoryless strategy ρ such that the corresponding memoryless ρ-graph $\mathscr{G}^{\mathrm{ML}}_{\rho}(\mathfrak{U},\tau)$ satisfies $\mathbf{R}C$." The goal is to let the complement automaton implement this condition based on the subset construction, which allows it to guess and piecemeal represent a PF strategy ρ.

But if directly obtained in this way, the resulting automaton cannot use its Streett acceptance condition to check that all paths in each w-bundle of $\mathscr{G}^{\mathrm{ML}}_{\rho}(\mathfrak{U},\tau)$ satisfy the Rabin condition $\mathbf{R}C$. Thus we use Lemma 5 (Rabin duality) to transform the Rabin condition to a Streett condition on the edge graph. Thereby we obtain condition (7), which calls for guessing a reduced quasi Rabin measure $\mu : V \times \mathbb{B}^* \rightharpoonup \mathbb{B}$ and guessing a memoryless strategy ρ such that the edge graph of $\mathscr{G}^{\mathrm{ML}}_{\rho}(\mathfrak{U},\tau)$ satisfies the derived

Streett condition SC^{μ}. At this point we are ready to begin the construction of the complement automaton.

By applying Lemma 10(3) (Edge automaton) and Lemma 11 (Subset construction), we obtain in (8) the automaton on $\mathscr{P}\mathscr{E}\mathfrak{U}$. Intuitively, each state of this automaton describes a part of the edge graph $\mathscr{E}\mathscr{G}_{\rho}^{\mathrm{ML}}(\mathfrak{U}, \tau)$ corresponding to some input position δ by defining a set of pairs of states of $\mathfrak{U}$ and a direction for PF to play for each pair. The dual pairs condition obtained from guessing $\mu: V \times \mathbb{B}^* \hookrightarrow \mathbb{B}$ is denoted C^{μ} for all three graphs and defined according to the discussion preceding Lemma 10.

Thus at this stage, τ is not accepted by $\mathfrak{U}SC$ if and only if there is a strategy α for $\mathscr{P}\mathscr{E}\mathfrak{U}$ such that for some reduced quasi measure μ, all paths in all w-bundles of $\mathscr{G}_{\alpha}(\mathscr{P}\mathscr{E}\mathfrak{U}, \tau)$ satisfy SC^{μ}.

In (9) we modify $\mathscr{P}\mathscr{E}\mathfrak{U}$ so that it guesses the reduced quasi measure μ piecemeal. By Lemma 10 (Edge automaton) it is sufficient to guess a progress value for each state occurring as a left component in a pair and to guess a progress value for each state occurring as a right component in a pair. The resulting automaton is denoted by $\mathscr{M}\mathscr{P}\mathscr{E}\mathfrak{U}$ and is called the *measure automaton*. For each strategy α this automaton defines an α-graph $\mathscr{G}_{\alpha}(\mathscr{P}\mathscr{E}\mathfrak{U}, \tau)$, which is the graph described by the underlying automaton $\mathscr{P}\mathscr{E}\mathfrak{U}$. Note that on a transition, the measure automaton must check that if (v, v') is in the old subset and (v', v'') is in the new subset, then $\mu(v')$ guessed for the old subset is the same as $\mu(v')$ guessed for the new subset. The dual pairs condition C^{μ}, consisting of $|C|$ new pairs, can be calculated from the state of the measure automaton according to Definition 5. Since the Streett condition SC^{μ} now is defined solely in terms of information in the measure automaton, it is now denoted $SC^{\mathscr{M}}$.

In (10) we have used that checking a Streett condition amounts to checking each pair in the pairs set.

Then in (11), for each pair (R, I) in $C^{\mathscr{M}}$, Lemma 13 (Extended Safra) is applied to the measure automaton $\mathscr{M}\mathscr{P}\mathscr{E}\mathfrak{U}$ by carrying out the subset construction of the Lemma on automaton $\mathscr{E}\mathfrak{U}$. Note that as formulated, Lemma 13 cannot be directly applied since (R, I) – even though essentially a basic pair on the state space of $\mathscr{E}\mathfrak{U}$ – is also dependent on the extra measure information guessed. It can be seen, however, that the construction of the Lemma can still be carried through.

Finally, in (12) the extensions are combined according to the Lemma 12 (And'ing extensions), and in (13) the definition of acceptance shows that the resulting complement automaton works as intended.

To calculate the size of the new automaton, we note that the size of $\mathscr{P}\mathscr{E}\mathfrak{U}$ is $2^{O(n^2)}$. Since $\mathscr{M}\mathscr{P}\mathscr{E}\mathfrak{U}$ guesses a progress value, which is a permutation of at most m colors, for at most $2 \cdot n$ nodes in $\mathscr{G}_{\rho}^{\mathrm{ML}}(\mathfrak{U}, \tau)$ at a time, the size of $\mathscr{M}\mathscr{P}\mathscr{E}\mathfrak{U}$ is $2^{O(n^2)} \cdot 2^{O(n^2 \cdot m \log n)}$. Since $|C^{\mathscr{M}}| = |C| = m$ and each application of the Extended Safra construction yields a size factor of $2^{O(n^2 \log n)}$, the total size factor for the cross product is $2^{O(n^2 \cdot m \log n)}$. The size of $\bar{\mathfrak{U}}$ is then $2^{O(n^2)} \cdot 2^{O(n \cdot m \log m)} \cdot 2^{O(n^2 \cdot m \log n)}$, which is $2^{O(n^2 \cdot m \log n)}$ (since there are at most 4^n different basic pairs, whence $\log m = O(n)$). Moreover, since each application of the Extended Safra construction yields a pairs set of size n^2, the size of $\bar{C}$ is $n^2 \cdot m$. □

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