D-Width: A More Natural Measure for Directed Tree Width

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Abstract. Due to extensive research on tree-width for undirected graphs and due to its many applications in various fields it has been a natural desire for many years to generalize the idea of tree decomposition to directed graphs, but since many parameters in tree-width behave very differently in the world of digraphs, the generalization is still in its preliminary steps.

In this paper, after surveying the main work that has been done on this topic, we propose a new simple definition for directed tree-width and prove a special case of the min-max theorem (duality theorem) relating haven order, bramble number, and tree-width on digraphs. We also compare our definition with previous definitions and study the behavior of some tree-width like parameters such as brambles and havens on digraphs.

1 Introduction

The notion of tree-width is considered as a generalization of trees (trees have tree-width 1) and many intractable problems are efficiently solvable on bounded tree-width graphs. Examples include *Hamiltonian cycle*, *graph isomorphism*, *vertex coloring*, *edge coloring*, and so on. Such problems arise in various fields including (but not restricted to) expert systems, telecommunication network design, VLSI design, Choleski factorization, natural language processing, etc. See [Bod93] for an overview of some such applications.

In 1996 Reed et al. [RRST96] proved Youngers's conjecture [You73] roughly saying that every directed graph has either a large set of disjoint directed circuits or a small set of vertices that cover all directed circuits. They considered an analogous definition of *well-linked* sets for directed graphs and since the size of the largest well-linked set in undirected graphs has close relationship with tree-width[Ree00] they suggested the idea that the analogous definition of tree-width for directed graphs might be very useful, as pointed out in [Ree97]. A proper definition should ideally measure the global connectivity of a digraph, for example the tree-width of a directed acyclic graph, DAG, is expectedly small because it is lowly connected.

Unfortunately finding an analogous definition for directed tree-width is not easy at all, and almost all concepts related to undirected tree-width behave

J. Jędrzejowicz and A. Szepietowski (Eds.): MFCS 2005, LNCS 3618, pp. 745–756, 2005.

differently in directed graphs. For example, the bramble number is equal to the haven order in undirected graphs, while they may differ by a factor of 2 in directed graphs. There is not even a single agreed-upon definition of tree-width for directed graphs.

For the first time Johnson, Robertson, Seymour, and Thomas[JRST01] gave a formal definition of directed tree-decomposition (called *arboreal-decomposition* in their paper) and directed tree-width, and proved some theorems relating directed tree-width and haven order.

Theorem 1. *[JRST01] For any digraph D, $H(D)-1 \leq$ tree-width(D) $\leq 3H(D) - 1$, where tree-width(D) and $H(D)$ are the tree-width and the haven order of D, respectively.*

They also show how their definition agrees width tree-width on undirected graphs in the sense that if we obtain a digraph D from an undirected graph G by replacing every edge (u, v) of G by two edges (u, v) and (v, u) in D, then the tree-width of G equals the directed tree-width of D. Their other results in that paper include relating the tree-widths of an Eulerian digraph and its underlying undirected graph and proposing a general algorithm for solving many hard problems like Hamiltonian cycle on digraphs of bounded tree-width. Finally they conjectured that digraphs with large tree-width have large grids as minors by defining directed variants of *grid* and *minor*. It is worth mentioning that the latter conjecture is a well-known fact for undirected graphs[RS86].

The other main work on the topic was by Reed[Ree00] who was also among those who proved Younger's conjecture in 1996. In his paper, Reed presents various global connectivity measures, such as bramble number($BN(D)$), $link(D)$, and $wlink(D)$, and proves that they are within a constant factor of each other in order to justify that all these terms are essentially measuring the same thing. Then, he presents another definition for directed tree-width which is very close to the definition of Johnson et al.[JRST01] (the two values differ by at most one for any digraph). The other parts of Reed's paper mainly discuss the hardness of obtaining results for directed graphs similar to those for undirected graphs.

In this paper we propose a new definition for directed tree-width which resembles the undirected version of tree-width in a natural way, and it seems to have the potential to form the basis for more efficient algorithms on some hard problems on digraphs. We have also studied, as an example to show how the close relationship of our definition and the undirected definition is useful, the situation in which our definition and the existing definition of Johnson et al.[JRST01] are equivalent by proving a min-max theorem relating haven order and d-width under certain conditions.

In the next section we state some preliminary definitions that are related to this paper. In section 3 we present our definition of directed tree-width and discuss its properties as well as its relationship with the definition of Johnson et al. [JRST01]. Section 4 is devoted to the min-max theorem on directed graphs. In that section we prove a weak version of the min-max theorem which is correct if the graph has a special separator property called *the augmenting condition.*

Although we believe that the augmenting condition holds for all digraphs, it still remains as an open part of our work.

2 Preliminaries

For general concepts of tree-decomposition and tree-width on undirected graphs including various related terms and algorithms the reader is referred to [Bod93]. Here we define some related concepts that we use in this paper.

A **tree-decomposition** of an undirected graph $G = (V, E)$ is a a pair (X, T), where $T = (I, F)$ is a tree and $X = \{X_i | i \in I\}$ is a family of subsets of V such that

- $\cup_{i \in I} X_i = V$
- For every edge $(u, v) \in E$ there exists some node i such that $u, v \in X_i$.
- For any vertex $u \in V$ the set of nodes $r \in I$ such that $u \in X_r$ induce a connected subtree in T.

The width of a T is $max_{i \in I}|X_i| - 1$, and the **tree-width** of G is the minimum width over all tree-decompositions of G.

A **haven** of order w in D (for integer w) is a function β which assigns to every subset X of less than w vertices of D a strongly connected component of $D \backslash X$ with the extra condition that if X and Y are two subsets of size less than w and X is a subset of Y, then $\beta(Y)$ is a subset of $\beta(X)$. The haven order of a digraph D, represented by $H(D)$, is the maximum w such that D has a haven of order w.

A **bramble** in a digraph D is a family of strongly connected subsets of D any two of which touch, that is, either have a vertex in common or there are edges from one to the other in both directions. The *order* of a bramble is the size of the minimum set cover of those strongly connected subsets. The *bramble number* of a graph D, represented by $BN(D)$, is the maximum bramble order over all brambles of D.

For example the digraph D which is depicted in Fig. 1 has 9 vertices. The three parts A, B, and C are undirected cliques (i.e. there is an edge between any two of their vertices in both directions) and all other edges are of the form (a_i, b_j), (b_i, c_j), or (c_i, a_j), for $i, j = 1, 2, 3$. The bramble $\varphi = \{\{a_1\}, \{a_2\}, \{a_2, b_2, c_2\}, \{a_3, b_3, c_3\}\}$ has order 3 because $\{a_1, a_2, a_3\}$ is its minimum cover. We can also compute a haven of order 3 by defining $\beta(Z)$ to be the strongly connected component in $D \backslash Z$ that has nonempty intersection with A. In Section 4 we show that $H(D) = 6$ whereas $BN(D) = 3$.

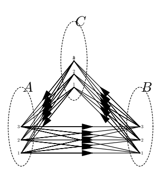

Fig. 1. A digraph with bramble number 3 and haven order 6

Notice that all definitions above have a corresponding version for undirected graphs.

2.1 Arboreal Decomposition

Since we use the definition of tree-width by Johnson et al.[JRST01] many times and compare our definition with theirs, it is worth mentioning their definition here.

An **arborescence** is formed from a rooted tree by directing all edges in the root-leaf direction, that is, for every vertex r in the arborescence there exists a unique path from the root to r.

If r and r' are two vertices of an arborescence R, we say $r > r'$ if $r \neq r'$ and there is a path in R from r' to r. For a vertex r and an edge $e = (t, r')$ we say $r > e$ if $r = r'$ or $r > r'$.

For two disjoint subsets Z and S of $V(D)$, S is called Z**-normal** if every walk in $D \backslash Z$ with first and last vertices in S contains no vertex of $D \backslash \{Z \cup S\}$.

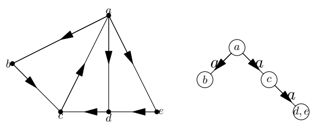

Fig. 2. A digraph with its arboreal decomposition

An **arboreal decomposition** of a digraph D is a triple (R, X, W), where R is an arborescence, and $X = \{X_e | e \in E(R)\}$ and $W = \{W_r | r \in V(R)\}$ are two families of subsets of $V(D)$, i.e. we assign to every edge and vertex of R a subset of vertices of D, with the following two conditions:

A1. W is a partition of $V(D)$ into nonempty sets.
A2. For any $e \in E(R)$, $\bigcup\{W_r | r > e\}$ is X_e-normal.

The width of (R, X, W) is the minimum w such that

$$\left| W_r \cup \bigcup_{e \sim r} X_e \right| \leq w + 1$$

for all vertices r, where $e \sim r$ means r is either the head or the tail of e. The *tree-width* of D is defined as the minimum width over all arboreal decompositions of D. Fig. 2 shows a digraph with an arboreal decomposition of it with width 2. To verify the condition A2, we need to verify that the sets $\{d, e\}$, $\{c, d, e\}$, and $\{b\}$ are $\{a\}$-normal.

3 Introducing D-Width

Let the triple $T = (R, X, W)$ be an arboreal decomposition of a digraph D. For any node $r \in R$ let $W'_r = W_r \cup \cup_{e \sim r} X_e$. Let S be a strongly connected set of D,

R_s be the set of vertices r of R such that $W_r \cap S \neq \emptyset$, and E_s be the minimum edges in R that are required to connect all vertices of R_s in R (ignore the edge directions for the moment). It is easy to verify that condition A2 implies all edges $e \in E_s$ have $X_e \cap S \neq \emptyset$. Let E'_s be the set of all edges $e = (r, r')$ in R for which $W'_r \cap W'_{r'} \cap S \neq \emptyset$. Obviously $E_s \subseteq E'_s$, so E'_s connects all the vertices of R_s. In general, E'_s induces one or more connected subtrees in R (ignore the direction of edges), but all vertices of R_s lie in exactly one of these components.

After considering many directed graphs we noticed that we might be able to restrict E'_s to form exactly one connected subtree without affecting the tree-width of the digraph. More formally, we define the d-decomposition and d-width of a graph as follows.

A **d-decomposition** of a digraph D is a pair (T, X), where T is a tree and X is a function that assigns to every node of T a subset of vertices of D such that:

B1 For any vertex v of D there exists some node i of T such that $v \in X_i$, i.e. $\bigcup_{i \in V(T)} X_i = V(D)$.

B2 For any strongly connected subset S of D, the nodes of T containing vertices of S form a connected subtree, i.e. if we select every node of T that contains a vertex of S and every edge $e = (i, j)$ of T such that $S \cap X_i \cap X_j \neq \emptyset$, then the result is a connected subtree of T.

Notice that the second condition yields the fact that the nodes of T containing a single vertex u of D form a subtree because every vertex of a digraph is a strongly connected set by itself.

The width of a d-decomposition is the minimum w such that $|X_i| \leq w+1$ for all $i \in T$, and the **d-width** of D is the minimum width over all d-decompositions of D.

A d-decomposition of width 1 of the directed graph in Fig. 2 is depicted in Fig. 3. The strongly connected sets of the digraph in Fig. 2 are $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{e\}$, $\{a, b, c\}$, $\{a, c, d\}$, $\{a, c, d, e\}$, $\{a, b, c, d\}$, and $\{a, b, c, d, e\}$, and it is easy to verify that condition B2 holds for all of them.

It's worth mentioning that if we replace 'strongly connected' in condition B2 above with 'connected', then the definition reduces to the tree-width definition for undirected graphs.

From now on we use the phrases tree-decomposition and tree-width to refer to the arboreal decomposition and tree-width defined in [JRST01] and the terms d-decomposition and d-width to refer to the concepts that we defined above.

For any digraph D, d-width(D) is at least as large as tree-width(D) because the digraph D must satisfy a more restrictive condition in order to have a given d-width than to have a given tree-width.

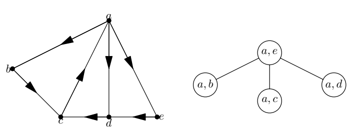

Fig. 3. A d-decomposition of width 1

Corollary 1. *For any digraph D, d-width(D) $\geq$ tree-width(D).*

d-width is a generalization of undirected tree-width in the sense that if we make a digraph D out of an undirected graph G in the most obvious way, i.e. replace every edge of G by two edges in D in both directions, then the d-width of D equals the tree-width of G. The reason is that every strongly connected set in D corresponds to a connected set of G, and vice versa. Johnson et al. [JRST01] also proved that tree-width(D) is equal to tree-width(G); thus, for the special class of undirected graphs, both definitions are equivalent. For this special case, the two decompositions can be efficiently transformed to each other in time $O(mn^2)$[Saf03], where m and n are the number of edges and vertices of G, respectively.

Corollary 2. *Let G be an undirected graph with (undirected) tree-width w and D be the digraph obtained by replacing every edge of G by two edges in both directions. Then, tree-width(D) = d-width(D) = w.*

If X, Y, and $X \cup Y$ are all strongly connected sets and $X \cap Y \neq \emptyset$, then the correctness of condition B2 for X and Y implies its correctness for $X \cup Y$. This means we only need to verify condition B2 for minimal strongly connected sets, that is, those sets S for which there is no strongly connected sets X and Y such that $S = X \cup Y$ and $X \cap Y \neq \emptyset$. That's why in the case of undirected tree-width it suffices to verify condition B2 for just edges and vertices because they are the only minimal connected sets in an undirected graph.

It seems to us that d-width and tree-width are equal on every digraph, though we are not able to prove this at the moment. In the following sections we relate tree-width and d-width and show some evidence that d-width seems to be a proper global measure of connectivity.

3.1 Properties

D-width is a nice measure because of its resemblance to the undirected tree-width, and is algorithmically useful at least because of its being a restricted version of directed tree-width defined by [JRST01].

Given a digraph D with a d-decomposition of width w, one can compute an undirected graph with tree-width at most w by connecting two vertices iff there exists some node in the d-decomposition that contains both the vertices.

Theorem 2. *Let $D = (V, E)$ be a digraph of d-width w. There exists an undirected graph $G = (V, E')$ whose tree-width is at most w such that every strongly connected set in D is a connected set in G.*

In regard to the algorithmic aspects of bounded d-width graphs, d-decomposition is at least as restrictive as tree-decomposition, so at least as many problems are efficiently solvable on digraphs of bounded d-width as on digraphs with bounded tree-width. This includes the linkage problem for fixed number of terminals, the Hamiltonian cycle and the Hamiltonian path problems, the Hamiltonian path problem with prespecified ends, the even cycle problem through a specified vertex, etc.

Since the suggested algorithm in [JRST01] for the above problems is not practically efficient[1], d-decomposition seems to be a good alternative in finding more efficient algorithms. However, the algorithmic properties of d-decomposition are not deeply studied yet.

One basic question related to our work is whether d-width equals tree-width or not. Since tree-width is believed to be equal to the haven order (minus one)[JRST01], we focus on the equality of haven order (minus one) and d-width in the next section. The weak min-max theorem which has been proved in the next section is a very nice example of how the close relationship between d-width and undirected tree-width is useful.

4 The Min-Max Theorem

In the case of undirected graphs there exists a very important theorem that relates haven order, bramble number, and tree-width together. This theorem is called the min-max theorem in [ST93] and the duality theorem in [Ree00].

Theorem 3. *[ST93, Ree00, BD02] For every undirected graph G, the tree-width of G equals the haven order of G minus one, and equals its bramble number minus one.*

Notice that tree-width is a minimized parameter, whereas haven order (or bramble number) is a maximized parameter. According to the above theorem the haven order (or the bramble number) of a digraph is at most w if and only if its tree-width is at least $w - 1$, which has a min-max flavor similar to the max-flow-min-cut theorem.

The original proof appears in [ST93] but other authors like Ballenbaum and Diestel in [BD02] give shorter and cleaner proofs. The above equality relation between haven order, bramble number, and tree-width not only convinces us of the properness of tree-width as a measure of global connectivity, but also has some algorithmic and theoretical consequences. For the case of directed graphs, the story is very different. We know that the bramble number may not be equal to the haven order and may differ by a factor of 2 (see the next section) but we suspect that directed tree-width is equal to the haven order, though there is no proof or disproof at the moment. Johnson et al. [JRST01] were able to prove Theorem 1 which only bounds the two terms, directed tree-width and haven order, within a constant factor of each other.

Here we talk about the relations between bramble number, haven order, and d-width. In section 4.1, we present a tight inequality relating haven order and bramble number. Then, in section 4.2, we present a theorem, similar to the min-max theorem, for directed graphs.

[1] It is $\Omega(n^w)$ where w is the tree-width.

4.1 Brambles and Havens in Digraphs

In this section we prove that the haven order and the bramble number of a digraph are not necessarily equal, but are within a constant factor of each other.

Lemma 1. *For every digraph D, $BN(D) \leq H(D)$ and there exists some D for which the equality holds.*

Proof. Let $\varphi = \{\varphi_1, \varphi_2, \cdots, \varphi_m\}$ be a bramble of order w in D. We show that D has a haven of order at least w. For every subset Z of vertices of D such that $|Z| < w$, there exists some $\varphi_k \in \varphi$ such that Z fails to cover φ_k. Let $\beta(Z)$ be the strongly connected component of $D \backslash Z$ that contains φ_k. β has order at least w. By the way, if D is undirected, then the equality holds.

Lemma 2. *For any digraph D, $\left\lceil \frac{H(D)}{2} \right\rceil \leq BN(D)$.*

Proof. Let $\varphi = \{\beta(Z) | Z \subset V(D) \textit{ and } |Z| \leq \left\lceil \frac{w}{2} \right\rceil - 1\}$, where β is a haven of order w in D. φ is a bramble of order at least $\left\lceil \frac{w}{2} \right\rceil$.

Lemma 3. *There exist digraphs D for which $\left\lceil \frac{H(D)}{2} \right\rceil = BN(D)$.*

Proof. Let D be a digraph similar to the one depicted in Fig. 1 in which A, B, and C are undirected cliques, i.e. there is an edge between any two of their vertices in both directions, $|A| = |B| = |C| = k$, and all other edges are of the form (a_x, b_y), (b_x, c_y), or (c_x, a_y), for $x, y = 1, 2, \cdots, k$. It can be shown that its haven order is at least $2k$, whereas its bramble number is at most k. For the haven order, if we define $\beta(Z)$ to be the strongly connected component of $D \backslash Z$ containing $A - Z$ if $A - Z \neq \emptyset$, and the strongly connected component of $D \backslash Z$ that contains $B - Z$, otherwise, then β has order at least $2k$.

For the bramble number, it is easy to show that A or B or C are the cover for any bramble in D.

Our goal is now achieved as a direct consequence of the above lemmas.

Theorem 4. *For any digraph D, $\left\lceil \frac{H(D)}{2} \right\rceil \leq BN(D) \leq H(D)$ and both inequalities are tight.*

4.2 The Min-Max Theorem on Directed Graphs

For the proof of Theorem 3, Seymour and Thomas [ST93] use powerful properties of brambles and separators in undirected graphs. They prove the following lemma which reduces to Theorem 3 in the special case $\varphi = \emptyset$.

Lemma 4. *[ST93] Let G be an undirected graph with bramble number at most k and φ be a bramble in G. There exists a tree-decomposition T of G such that every node of size more than k fails to cover φ, that is, for every node X such that $|X| > k$ there exists some non-empty $\varphi_i \in \varphi$ such that $X \cap \varphi_i = \emptyset$.*

To prove lemma 4, Seymour and Thomas find a minimum covering of φ, say X, and examine the components of $G\backslash X$. For any component C_i of $G\backslash X$ they find a tree-decomposition T_i of $G[X \cup C_i]$ that satisfies the condition in lemma 4 with the additional property that there is some node in T_i that contains X. They finally join all of these tree-decompositions to form the final tree-decomposition for G. The join process is simply adding a new node containing X and connecting it to a node of T_i that contains X, for all i's.

The above proof does not work for the case of directed graphs. First, unlike for undirected graphs the bramble number of a digraph is not equal to its d-width. Second, the behavior of separators on directed graphs is very different. There may be minimal strongly connected subsets of D that do not appear as strongly connected subsets of $D[X \cup C_i]$ for any strongly connected component C_i of $D\backslash X$. For example in Fig. 4, where $X = \{a\}$, the strongly connected component $\{a, b, c, d\}$ does not show up in $D[\{a, b\}]$ or $D[\{a, c\}]$ or $D[\{a, d\}]$, so the resulting d-decomposition may not be valid because the strongly connected set $\{a, b, c, d\}$ violates property B2.

In order to be able to prove a theorem similar to Theorem 3, we need to resolve the bad behavior of separators on digraphs and to use brambles properly. For separators, before we obtain the components of $D\backslash X$ we add (or remove) some edges to the graph, and obtain a new digraph D' so that all minimal (recall the concept of minimal from Section 3) strongly connected sets of D show up in some components of $D'\backslash X$. For example in Fig. 5 the original digraph of Fig. 4 has been changed to a newer digraph in which the set of minimal strongly connected components is $\{\{a, b\}, \{a, c\}, \{a, d\}\}$. Notice that every d-decomposition of this new digraph is a d-decomposition of the original digraph as well.

Our goal is to transform the original digraph D to a a digraph D' with the same haven order such that D' preserves all strongly connected sets of D', i.e., every strongly connected set in D is strongly connected in D', and D' has no bad separator.

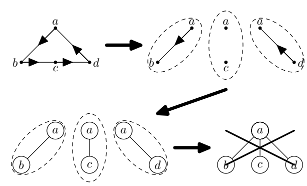

Fig. 4. The bad behavior of separators in digraphs

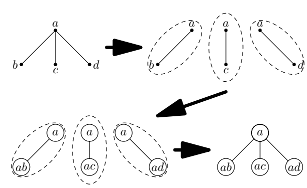

Fig. 5. Making a good separator

Notice that if d-width$(D) = H(D) - 1$, then such a digraph D' exists according to Theorem 2. In what follows we prove that the reverse is also true: The existence of such a digraph D' implies the min-max theorem.

A set X in a digraph D is called a **good-separator** if and only if for every minimal strongly connected set H of D, $V(H) \subseteq V(D[X \cup C])$, for some strongly connected component C of $D \backslash X$.

Augmenting Condition: A digraph D satisfies this condition if for any bramble ϕ and any minimum cover X of ϕ, there exists a digraph D' such that

- D and D' have the same haven order.
- X is a good separator in D'.
- Every strongly connected set of D is a strongly connected set of D'.

The latter condition guarantees that every d-decomposition of D' is a d-decomposition of D as well.

Notice that the augmenting condition requires only minimum bramble covers to be good separators rather than all subsets of vertices. This weaker condition may make it easier to establish the augmenting condition for some directed graphs.

We are now ready to prove the following version of min-max theorem.

Theorem 5. *Let D be a digraph with haven order w. If D satisfies the augmenting condition, then d-width$(D) \leq w - 1$.*

Proof. Our proof has a flavor very similar to the proof of Seymour and Thomas for min-max theorem on undirected graphs [ST93]. First we prove the following lemma.

Lemma 5. *Let D be a digraph of haven order at most w and φ be a bramble of D. Assume that D satisfies the augmenting condition. Then, there exists a d-decomposition T of D such that every node of size more than w fails to cover φ.*

Let X be a cover of minimum size for φ. Since D satisfies the augmentation condition, there exists some digraph D' with haven order at most w in which X is a good separator. Let $C_1, C_2, \cdots, C_k$ be the components of $D' \backslash X$. Assume there is no edge between a vertex of C_i and a vertex of C_j for any i, j such that $i \neq j$; otherwise, we can remove them without affecting any required condition. Now we make a d-decomposition for $D'[X \cup C_i]$ by using the following lemma.

Lemma 6. *For any i, there exists a d-decomposition T_i for $D'[X \cup C_i]$ such that any node of size more than w fails to cover φ and there exists a node r in T_i that contains X, i.e., $X \subseteq X_r$.*

Proof. There are two cases:

Case 1: C_i does not touch some $\varphi_j \in \varphi$. In this case, a d-decomposition T_i with two nodes u and v such that $X_u = X$ and $X_v = (C_i \cup X) - \varphi_j$ has the necessary conditions. First, $(C_i \cup X) - \varphi_j$ fails to cover φ_j and so fails to cover φ. Second, let S be a strongly connected subset in $(C_i \cup X)$. We have two cases:

1. $(X - \varphi_j) \cap S \neq \emptyset$. In this case, the vertices of S clearly form a connected subtree in T_i.
2. $(X - \varphi_j) \cap S = \emptyset$. In this case either $S \subset C_i$ or $S \subset \varphi_j$.

Case 2: C_i touches every member of φ. So $\varphi \cup C_i$ is a bramble and its order is at most w.

By the induction hypothesis (The induction is based on the number of connected sets in φ), there exists a d-decomposition T' for D' such that every node of size more than w fails to cover $\varphi \cup C_i$. If every node of size more than w fails to cover φ, then it directly implies Lemma 5. Otherwise, let y be a node of T' such that $Y = X_y$ has size more than w and fails to cover C_i, but covers every member of φ.

We now build an undirected graph G from T' by adding edge (u, v) if and only if $u, v \in X_r$ for some node r in T' and both are vertices of the same subgraph $D'[X \cup C_j]$, for some j. Since X is a good separator, it can be easily seen that every strongly connected subset of T, and in particular every member of φ, forms a connected subset in G. We now use the following useful lemma whose proof is similar to the undirected version proof in [ST93] and is stated in the appendix.

Lemma 7. *There are at least $|X|$ vertex-disjoint paths from X to Y in G.*

Let $X = \{x_1, x_2, \cdots, x_m\}$, and $\{P_1, P_2, \cdots, P_m\}$ be a set of m vertex-disjoint paths from X to Y such that P_j begins with x_j and ends with a vertex of Y. We assume, by truncating the path at the first vertex in Y, that P_j has exactly one vertex from Y.

Let $C = C_i \cup X$. Let T'' be obtained from T' by replacing every set X_t by $X'_t = (X_t \cap C) \cup \{x_i | P_i \cap X_t \neq \emptyset\}$, for every node t of T'. T'' has the following properties:

1. $X'_y = X$.
2. For any node t, $|X'_t| \leq |X_t|$.
3. Every node r in T'' of size more than w fails to cover φ.

The remaining steps of the proof are straightforward. We join all the T_j's and attach them to a new node that contains only X. The resulting d-decomposition satisfies all the requirements of Lemma 5.

□

Corollary 3. *For any digraph D that satisfies the augmentation condition tree-width(D) = d-width$(D) = H(D) - 1$.*

Proof. This is an easy consequence of Theorem 1, Corollary 1, and Theorem 5.

5 Conclusions and Future Work

Since the topic is very new there exists a huge number of open questions. Any research topic related to undirected tree-width may have a corresponding problem on directed graphs. Here we list just some of these:

– Corresponding definitions for separators, path-width, and branch-width.
– Identifying the class of digraphs whose tree-width is k for a constant k, $k = 2, 3, \cdots$. For $k = 0$, the answer is the class of directed acyclic graphs.
– Identifying the class of digraphs for which the augmenting condition holds. We believe all digraphs satisfy the augmenting condition, but proving it for some special classes of digraphs would also be a big step forward.
– Finding d-decomposition of small d-width for bounded d-width graphs.

Acknowledgement

The main part of this work was done when I was a graduate student at the school of Computer Science at the University of Waterloo. I should thank my supervisor, Prabhakar Ragde, for his many helps and comments during this research.

My sincere thanks to Will Evans for his contribution in preparing this paper. I would like to thank him for his invaluable hints on this paper.

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