Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 253 (2014) 86–138
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

The uniformization of certain algebraic hypergeometric functions

Robert S. Maier

Depts. of Mathematics and Physics, University of Arizona, Tucson, AZ 85721, USA

Received 25 June 2009; accepted 21 November 2013

Available online 24 December 2013

Communicated by Ezra Miller

Abstract

The hypergeometric functions ${}_nF_{n-1}$ are higher transcendental functions, but for certain parameter values they become algebraic, because the monodromy of the defining hypergeometric differential equation becomes finite. It is shown that many algebraic ${}_nF_{n-1}$'s, for which the finite monodromy is irreducible but imprimitive, can be represented as combinations of certain explicitly algebraic functions of a single variable; namely, the roots of trinomials. This generalizes a result of Birkeland, and is derived as a corollary of a family of binomial coefficient identities that is of independent interest. Any tuple of roots of a trinomial traces out a projective algebraic curve, and it is also determined when this so-called Schwarz curve is of genus zero and can be rationally parametrized. Any such parametrization yields a hypergeometric identity that explicitly uniformizes a family of algebraic ${}_nF_{n-1}$'s. Many examples of such uniformizations are worked out explicitly. Even when the governing Schwarz curve is of positive genus, it is shown how it is sometimes possible to construct explicit single-valued or multivalued parametrizations of individual algebraic ${}_nF_{n-1}$'s, by parametrizing a quotiented Schwarz curve. The parametrization requires computations in rings of symmetric polynomials.

MSC: 33C20; 33C80; 14H20; 14H45; 05E05

Keywords: Hypergeometric function; Algebraic function; Imprimitive monodromy; Trinomial equation; Binomial coefficient identity; Algebraic curve; Belyi cover; Uniformization; Symmetric polynomial

E-mail address: rsm@math.arizona.edu.
URL: http://math.arizona.edu/~rsm.

0001-8708/$ – see front matter
http://dx.doi.org/10.1016/j.aim.2013.11.013

1. Introduction

The hypergeometric functions ${}_nF_{n-1}(\zeta)$, $n \geqslant 1$, are parametrized special functions of fundamental importance. Each ${}_nF_{n-1}(\zeta)$ is a function of a single complex variable, and in general is a higher transcendental function. It is parametrized by complex numbers $a_1, \dots, a_n; b_1, \dots, b_{n-1}$, and is written as ${}_nF_{n-1}(a_1, \dots, a_n; b_1, \dots, b_{n-1}; \zeta)$. It is analytic on $|\zeta| < 1$, with definition

$$ {}_nF_{n-1}\left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_{n-1} \end{matrix} \,\middle|\, \zeta \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_n)_k}{(b_1)_k \cdots (b_{n-1})_k (1)_k} \zeta^k. \tag{1.1} $$

Here $(c)_k := c(c+1)\cdots(c+k-1)$, and the lower parameters $b_1, \dots, b_{n-1}$ may not be non-positive integers. The $n=1$ function ${}_1F_0(a_1; –; \zeta)$ equals $(1-\zeta)^{-a_1}$, and the $n=2$ function ${}_2F_1(a_1, a_2; b_1; \zeta)$ is the Gauss hypergeometric function.

If its parameters are suitably chosen, ${}_nF_{n-1}(\zeta)$ will become an *algebraic* function of ζ. Equivalently, if one regards ${}_nF_{n-1}$ as a single-valued function on a certain Riemann surface, defined by continuation from the disk $|\zeta| < 1$, then the surface will become compact. If $n=1$, this occurs when $a_1 \in \mathbb{Q}$. In the first nontrivial case $n=2$, the characterization of the triples $(a_1, a_2; b_1)$ for which ${}_2F_1(a_1, a_2; b_1; \zeta)$ is algebraic is a classical result of Schwarz. There is a finite list of possible *normalized* triples (the famous 'Schwarz list'), and ${}_2F_1$ is algebraic iff $(a_1, a_2; b_1)$ is a denormalized version of a triple on the list. Denormalization involves integer displacements of the parameters. For specifics, see [10, §2.7.2].

More recently, Beukers and Heckman [3] treated $n \geqslant 3$, and obtained a complete characterization of the parameters $(a_1, \dots, a_n; b_1, \dots, b_{n-1})$ that yield algebraicity. Like the $n=2$ result of Schwarz, their result was based on the fact that the function ${}_nF_{n-1}(a_1, \dots, a_n; b_1, \dots, b_{n-1}; \zeta)$ satisfies an order-n differential equation on the Riemann sphere $\mathbb{P}^1_\zeta$, called $E_n(a_1, \dots, a_n; b_1, \dots, b_{n-1}, 1)_\zeta$ below. In modern language, E_n specifies a flat connection on an n-dimensional vector bundle over $\mathbb{P}^1_\zeta$. It has singular points at $\zeta = 0, 1, \infty$, and its (projective) monodromy group is generated by loops about these three points. The monodromy, resp. projective monodromy group is a subgroup of $GL_n(\mathbb{C})$, resp. $PGL_n(\mathbb{C})$, and algebraicity occurs iff the monodromy is finite. Schwarz exploited the classification of the finite subgroups of $PGL_2(\mathbb{C})$. In a *tour de force*, Beukers and Heckman handled the $n \geqslant 3$ case by exploiting the Shephard–Todd classification of the finite subgroups of $GL_n(\mathbb{C})$ generated by complex reflections. Their characterization result, however, is non-constructive: it supplies an algorithm for determining whether a given function ${}_nF_{n-1}(a_1, \dots, a_n; b_1, \dots, b_{n-1}; \zeta)$ is algebraic in ζ, but it does not yield a polynomial equation (with coefficients polynomial in ζ) satisfied by the function.

In this paper, a new approach is taken to the problem of constructing equations satisfied by algebraic ${}_nF_{n-1}$'s. Several classes of hypergeometric function, known to be algebraic by Beukers–Heckman, are made explicit by being *uniformized*. Recall that an algebraic function F may be of genus zero, i.e., may have a 'uniformization,' or parametrization, by rational functions. That is, one may have $F(R_1(t)) = R_2(t)$ for certain rational functions R_1, R_2, so that formally, $F = R_2 \circ R_1^{-1}$. In this case the Riemann surface of $F = F(\zeta)$, on which ζ and F are single-valued meromorphic functions, is isomorphic to the Riemann sphere $\mathbb{P}^1_t$. A sample result of this nature, obtained below, is the following. Let $n = p+2$, where $p \geqslant 1$ is odd. Then

$$ {}_nF_{n-1}\left(\begin{matrix} \frac{a}{n}, \dots, \frac{a+(n-1)}{n} \\ \frac{a+1}{p}, \dots, \frac{a+p}{p}, \frac{1}{2} \end{matrix} \,\middle|\, \frac{4n^n}{p^p} \frac{t^2(1-t^2)^{2p}[(1+t)^p + (1-t)^p]^p}{[(1+t)^n + (1-t)^n]^n} \right) $$

$$= \frac{1}{2}\left[(1+t)^{-2a} + (1-t)^{-2a}\right]\left[\frac{(1+t)^n + (1-t)^n}{(1+t)^p + (1-t)^p}\right]^a. \tag{1.2}$$

Eq. (1.2) holds as an equality for all t in a neighborhood of $t = 0$, or equivalently if one expands each side about $t = 0$, as an equality between power series. It is a *hypergeometric identity*, which is more general than a uniformization of a single F. This is because $a \in \mathbb{C}$ is arbitrary. (If any lower hypergeometric parameter is a non-positive integer, the identity must be interpreted in a limiting sense.) If $a \in \mathbb{Q}$, it follows that ${}_nF_{n-1}(\frac{a}{n}, \dots, \frac{a+(n-1)}{n}; \frac{a+1}{p}, \dots, \frac{a+p}{p}, \frac{1}{2}; \zeta)$ is algebraic in ζ. If moreover $a \in \mathbb{Z}$, in which case (1.2) is a uniformization in the strict sense, this algebraic ${}_nF_{n-1}$ must be of genus zero.

Most of the algebraic ${}_nF_{n-1}$'s parametrized below, or more specifically uniformized, are of the following type. The ${}_nF_{n-1}$ in (1.2) is the $q = 2$ case of

$${}_nF_{n-1}\left(\begin{matrix} \frac{a}{n}, \dots, \frac{a+(n-1)}{n} \\ \frac{a+1}{p}, \dots, \frac{a+p}{p}, \frac{1}{q}, \dots, \frac{q-1}{q} \end{matrix} \,\middle|\, \zeta\right), \tag{1.3}$$

where $n = p + q$. The order-n differential equation E_n on $\mathbb{P}^1_\zeta$ associated to (1.3),

$$E_n\left(\frac{a}{n}, \dots, \frac{a+(n-1)}{n}; \frac{a+1}{p}, \dots, \frac{a+p}{p}, \frac{1}{q}, \dots, \frac{q-1}{q}, 1\right)_\zeta, \tag{1.4}$$

plays a role in the Beukers–Heckman analysis of algebraicity (see Section 2). The treatment of (1.3) and (1.4) relies on the following fact. If $\gcd(p, q) = 1$, the solution space of (1.4) is spanned by $x_1^\gamma(\zeta), \dots, x_n^\gamma(\zeta)$, where $\gamma := -qa$ and the n algebraic functions $x_1(\zeta), \dots, x_n(\zeta)$ are the roots of the *trinomial equation*

$$x^n - gx^p - \beta = 0. \tag{1.5}$$

Here $g \neq 0$ is arbitrary but fixed, and β is determined implicitly by

$$\zeta = (-)^q \frac{n^n}{p^p q^q} \frac{\beta^q}{g^n}. \tag{1.6}$$

(This statement assumes $\gamma \notin \mathbb{Z}$.) As defined by (1.6), ζ is a projectivized and normalized version of the *discriminant* of (1.5): if $\beta \neq 0$, the trinomial has coincident roots if and only if $\zeta = 1$.

Formulas relating such hypergeometric functions as (1.3) to trinomial roots can be traced to the 1758 work of Lambert, who expressed such roots as finite sums of hypergeometric series. (See [2, p. 72].) Once trinomial roots have been expressed in terms of hypergeometric series, typically by Lagrange inversion, formulas for hypergeometric functions in terms of trinomial roots can be derived. Two especially useful reversed formulas of this sort were obtained in the 1920s by Birkeland [4]. (See [1,27] for general reviews; the classical literature is cited in [18], and some explicit series for the roots of (1.5) are given in [9,13].) For example, Birkeland expressed the algebraic function (1.3) as a combination of only q of the n functions $x_1^\gamma(\zeta), \dots, x_n^\gamma(\zeta)$.

The first result of this paper is the extension of Birkeland's two formulas to the case when the ${}_nF_{n-1}$ being represented in terms of trinomial roots has a so-called parametric excess $S = \sum_{i=1}^{n-1} b_i - \sum_{i=1}^{n} a_i$ that does not equal $\frac{1}{2}$, as it does in (1.3), but rather is an arbitrary half-odd-integer. There are in fact two families of formulas indexed by $\ell \in \mathbb{Z}$, with $S = \frac{1}{2} - \ell$, that subsume Birkeland's '$\ell = 0$' formulas. The new families are shown to follow from a family of binomial coefficient identities, which is derived first and is of independent interest: it subsumes several combinatorial identities that are listed individually in [16]. The derivation of the family of binomial coefficient identities resembles that of Chu [6]. The technique used is not Lagrange inversion (explicitly) but rather the Vandermonde convolution transform of Gould [15].

The heart of this paper is a study of the trinomial equation (1.5), focused on basic algebraic geometry rather than on controlling or bounding its roots. (For the latter, see [11] and papers cited therein.) The key concept is that of a *Schwarz curve*. For each coprime $p, q \geqslant 1$ with $n := p+q$, this is defined following Kato and Noumi [21] to be a complex projective curve $\mathcal{C}_{p,q}^{(n)} \subset \mathbb{P}^{n-1}$ comprising all $[x_1 : \ldots : x_n] \in \mathbb{P}^{n-1}$ such that $x_1, \ldots, x_n$ are the roots of *some* equation of the form (1.5). That is, $\mathcal{C}_{p,q}^{(n)}$ is the common zero set of the $n-2$ elementary symmetric polynomials $\sigma_1, \ldots, \sigma_{q-1}$ and $\sigma_{q+1}, \ldots, \sigma_{n-1}$ in $x_1, \ldots, x_n$. Clearly $g = (-)^{q-1}\sigma_q$ and $\beta = (-)^{n-1}\sigma_n$, giving a formula for ζ; so there is a degree-$n!$ covering $\mathcal{C}_{p,q}^{(n)} \to \mathbb{P}_\zeta^1$. The curve $\mathcal{C}_{p,q}^{(n)}$ is irreducible [21, Cor. 4.7].

For each k, $n-1 \geqslant k \geqslant 2$, a subsidiary Schwarz curve $\mathcal{C}_{p,q}^{(k)} \subset \mathbb{P}^{k-1}$, also irreducible, is defined to include all $[x_1 : \ldots : x_k] \in \mathbb{P}^{k-1}$ such that $x_1, \ldots, x_k$ are k of the n roots of some equation of the form (1.5). The curve $\mathcal{C}_{p,q}^{(2)}$ is $\mathbb{P}^1$ itself, and one can introduce a final curve $\mathcal{C}_{p,q}^{(1)}$, also isomorphic to $\mathbb{P}^1$. These curves are joined by maps $\phi_{p,q}^{(k)}: \mathcal{C}_{p,q}^{(k)} \to \mathcal{C}_{p,q}^{(k-1)}$ with $\deg \phi_{p,q}^{(k)} = n-k+1$, i.e.,

$$\mathcal{C}_{p,q}^{(n)} \xrightarrow{\phi_{p,q}^{(n)}} \mathcal{C}_{p,q}^{(n-1)} \xrightarrow{\phi_{p,q}^{(n-1)}} \cdots \xrightarrow{\phi_{p,q}^{(3)}} \mathcal{C}_{p,q}^{(2)} \xrightarrow{\phi_{p,q}^{(2)}} \mathcal{C}_{p,q}^{(1)} \xrightarrow{\phi_{p,q}^{(1)}} \mathbb{P}_\zeta^1. \tag{1.7}$$

Each covering $\mathcal{C}_{p,q}^{(k)} \to \mathbb{P}_\zeta^1$ is a Belyĭ map, i.e., is ramified only over $\zeta = 0, 1, \infty$.

The Schwarz curves are not in general of genus zero, though $\mathcal{C}_{p,q}^{(2)}$ and $\mathcal{C}_{p,q}^{(1)}$ are. One can write $\mathcal{C}_{p,q}^{(2)} \cong \mathbb{P}_t^1$ and $\mathcal{C}_{p,q}^{(1)} \cong \mathbb{P}_s^1$, where t and s are rational parameters, and make the concrete choice $t = (x_1 + x_2)/(x_1 - x_2)$. It is a consequence of the rational parametrizations of $\mathcal{C}_{p,q}^{(2)}$ and $\mathcal{C}_{p,q}^{(1)}$, substituted into the appropriate Birkeland-style representation formulas, that when $q = 2$ or $q = 1$, the function ${}_nF_{n-1}$ of (1.3) can be parametrized by t or s, with its argument ζ being respectively a degree-$[n(n-1)]$ rational function of t, and a degree-n rational function of s. This turns out to be the origin of the sample identity (1.2). Its right side is the sum of two terms, proportional to $(1+t)^{-2a}$, $(1-t)^{-2a}$; and these come from the exponentiated roots x_1^γ, x_2^γ.

The fundamental fact is that if a hypergeometric function ${}_nF_{n-1}(\zeta)$ can be represented in terms of a k-tuple of roots of the trinomial equation (1.5) with $n = p+q$, any parametrization of the curve $\mathcal{C}_{p,q}^{(k)}$ will yield a parametrization of ${}_nF_{n-1}(\zeta)$; and if this curve is of genus zero, with a rational parameter, the parametrization of ${}_nF_{n-1}(\zeta)$ will be by rational functions of the parameter. Many interesting examples of this are worked out below, going well beyond the sample identity (1.2). The curves employed include 'top' Schwarz curves $\mathcal{C}_{p,q}^{(n)}$, as well as subsidiary curves $\mathcal{C}_{p,q}^{(k)}$ with $k = 1, 2, 3$.

To derive parametrizations of certain algebraic ${}_nF_{n-1}(\zeta)$'s, a more sophisticated technique is useful. If there is a representation of an ${}_nF_{n-1}(\zeta)$ in terms of k trinomial roots that is symmetric in the roots, the parametrization is really by a *quotiented* Schwarz curve, obtained from $\mathcal{C}_{p,q}^{(k)}$ by quotienting out the symmetric group $\mathfrak{S}_k$. Even if $\mathcal{C}_{p,q}^{(k)}$ is of positive genus, the quotiented curve may be of genus zero. Quotienting out the cyclic group $\mathfrak{C}_k$ or the dihedral group $\mathfrak{D}_k$ may also be useful. Several examples of explicit parametrizations of algebraic ${}_nF_{n-1}$'s that can be obtained by quotienting are worked out, by computation in rings of symmetric polynomials.

The paper is organized as follows. The first half focuses on hypergeometric equations and series, and series identities. In Section 2 the equations E_n are introduced, and their monodromy is discussed. In Section 3, the family of binomial coefficient identities is derived with the aid of the Vandermonde convolution transform. In Section 4, families of formulas of Birkeland type for various hypergeometric functions, including algebraic ones, are derived from these identities. The main results are Theorem 4.4, which extends Birkeland's two representations and is parametrized by $\ell \in \mathbb{Z}$, and Theorem 4.6. The former expresses ${}_nF_{n-1}$'s of a type that generalizes (1.3) in terms of trinomial roots. The latter expresses certain non-algebraic ${}_{n+1}F_n$'s in terms of ${}_2F_1$'s and ${}_3F_2$'s.

The second half is explicitly algebraic–geometric. In Sections 5 and 6, the Schwarz curves are introduced and studied. Their ramification structures and genera are computed, and the curves of genus zero are determined. (Theorem 6.3, the genus formula, generalizes a theorem of [21].) In Section 7, many hypergeometric identities with a free parameter $a \in \mathbb{C}$ are derived, including uniformizations of algebraic ${}_nF_{n-1}$'s of genus zero. A typical result is Theorem 7.3, of which Eq. (1.2) above is a special case; it comes from $\mathcal{C}^{(2)}_{p,2}$. In Section 8, the quotiented Schwarz curves are defined and in some cases parametrized, yielding parametrizations in which $a \in \mathbb{Q}$ is fixed. Some of these are multivalued parametrizations with radicals.

2. Monodromy and degeneracy

The parametrized functions ${}_nF_{n-1}(a_1,\dots,a_n;b_1,\dots,b_{n-1};\zeta)$ are locally defined by Eq. (1.1). Two sorts of degeneracy are of interest, for the functions themselves, and for the equations that they satisfy. The first is when an upper and a lower parameter equal each other. If so, they may be 'cancelled,' reducing the ${}_nF_{n-1}$ to an ${}_{n-1}F_{n-2}$. The second arises when one of the lower parameters is taken to an integer $-m$, and one of the upper ones to an integer $-m'$, with $-m \leqslant -m' \leqslant 0$. Let $[\zeta^{>m'}]F$ denote the sum of the terms proportional to ζ^k, $k > m'$, in the series for F; and let $(\mathfrak{a})$ and $(\mathfrak{b})$ denote $(a_1,\dots,a_{n-1})$ and $(b_1,\dots,b_{n-2})$. The following auxiliary lemma permits such hypergeometric identities as (1.2) to be interpreted in a limiting sense; it will be used in Section 8.

Lemma 2.1. *Let $a_n \to -m'$, $b_{n-1} \to -m$, with $(a_n+m')/(b_{n-1}+m) \to \alpha$. Then*

$$\begin{aligned}&[\zeta^{>m'}]\,{}_nF_{n-1}\left(\left.\begin{matrix}(\mathfrak{a}),a_n\\(\mathfrak{b}),b_{n-1}\end{matrix}\,\right|\,\zeta\right)\\ &\quad\longrightarrow \alpha(-)^{m-m'}\binom{m}{m'}^{-1}\frac{(\mathfrak{a})_{m+1}}{(\mathfrak{b})_{m+1}(1)_{m+1}}\zeta^{m+1}\,{}_nF_{n-1}\left(\left.\begin{matrix}(\mathfrak{a})+m+1,m-m'+1\\(\mathfrak{b})+m+1,m+2\end{matrix}\,\right|\,\zeta\right).\end{aligned}$$

If $m = m'$, this limit simplifies to

$$\alpha\cdot[\zeta^{>m'}]\,{}_{n-1}F_{n-2}\left(\left.\begin{matrix}(\mathfrak{a})\\(\mathfrak{b})\end{matrix}\,\right|\,\zeta\right).$$

Proof. Take the term-by-term limit of the series in (1.1). □

It is well known that the function ${}_nF_{n-1}(a_1,\dots,a_n;b_1,\dots b_{n-1};\zeta)$, defined on the disk $|\zeta|<1$, satisfies the $b_n = 1$ case of the order-n equation $\mathcal{D}_nF=0$, where (with $D_\zeta := \mathrm{d}/\mathrm{d}\zeta$)

$$\begin{aligned}\mathcal{D}_n &= \mathcal{D}_n(a_1,\dots,a_n;b_1,\dots,b_n)\\ &= (\zeta D_\zeta + b_1 - 1)\cdots(\zeta D_\zeta + b_n - 1) - \zeta(\zeta D_\zeta + a_1)\cdots(\zeta D_\zeta + a_n).\end{aligned}\tag{2.1}$$

The equation $\mathcal{D}_n F = 0$ is denoted by $E_n(a_1, \dots, a_n; b_1, \dots, b_n)$ here; a subscript ζ will be added as appropriate to indicate the independent variable. This equation has three regular singular points on $\mathbb{P}^1_\zeta$, namely $\zeta = 0, 1, \infty$, with respective characteristic exponents $1 - b_1, \dots, 1 - b_n$; and $0, 1, \dots, n-2, S$; and $a_1, \dots, a_n$. In this,

$$S = \left(\sum_{i=1}^{n} b_i\right) - \left(\sum_{i=1}^{n} a_i\right) - 1 \tag{2.2}$$

is the 'parametric excess,' which reduces to $\sum_{i=1}^{n-1} b_i - \sum_{i=1}^{n} a_n$ if $b_n = 1$. The canonical solution ${}_nF_{n-1}$, defined if none of $b_1, \dots, b_{n-1}$ is a non-positive integer, is analytic at $\zeta = 0$ and belongs to the exponent $1 - b_n = 0$. Allowing b_n to differ from unity in E_n is not a major matter, since adding δ to each parameter of E_n merely multiplies all solutions by $\zeta^{-\delta}$. The following is a standard fact.

Lemma 2.2. *If $b_j - b_{j'} \notin \mathbb{Z}$, $\forall j, j'$, so that the singular point $\zeta = 0$ is non-logarithmic, the solution space of E_n at $\zeta = 0$ is spanned by the functions*

$$\zeta^{1-b_j}\, {}_nF_{n-1}\left(\begin{matrix} a_1 - b_j + 1, \dots, a_n - b_j + 1 \\ b_1 - b_j + 1, \dots, \widehat{b_j - b_j} + 1, \dots, b_n - b_j + 1 \end{matrix} \,\middle|\, \zeta\right), \qquad j = 1, \dots, n.$$

The differential equation E_n on $\mathbb{P}^1_\zeta$ is the canonical order-n one with three regular singular points, the monodromy at one of them being special. This is seen as follows. Associated to any E_n is a monodromy representation of $\pi_1(X, \zeta_0)$, the fundamental group of the triply punctured sphere $X = \mathbb{P}^1_\zeta \setminus \{0, 1, \infty\}$. (The base point ζ_0 plays no role, up to a similarity transformation.) The image H of $\pi_1(X, \zeta_0)$ in $GL_n(\mathbb{C})$ is the monodromy group of the specified E_n. It is generated by h_0, h_1, h_∞, the monodromy matrices around $\zeta = 0, 1, \infty$, which satisfy $h_\infty h_1 h_0 = I$. Their eigenvalues are exponentiated characteristic exponents, so they have characteristic polynomials

$$\begin{aligned} &\det(\lambda I - h_\infty) = \prod_{j=1}^{n} (\lambda - \alpha_j), \qquad \det\bigl(\lambda I - h_0^{-1}\bigr) = \prod_{j=1}^{n} (\lambda - \beta_j), \\ &\det(\lambda I - h_1) = (\lambda - 1)^{n-1}\bigl(\lambda - e^{2\pi \mathrm{i} S}\bigr), \end{aligned} \tag{2.3}$$

where $\alpha_j = e^{2\pi \mathrm{i} a_j}$, $\beta_j = e^{2\pi \mathrm{i} b_j}$. Moreover, if $S \notin \mathbb{Z}$ then h_1 is diagonalizable [3], i.e., the singular point $\zeta = 1$ is not logarithmic. Hence if $S \notin \mathbb{Z}$, h_1 will act on $\mathbb{C}^n$ as multiplication by $e^{2\pi \mathrm{i} S}$ on a 1-dimensional subspace, and as the identity on a complementary $(n-1)$-dimensional subspace. It will be a *complex reflection*.

The monodromy representation is irreducible iff no upper parameter a_j and lower one $b_{j'}$ differ by an integer, i.e., $\alpha_j \neq \beta_{j'}$, $\forall j, j'$; and it is unaffected (up to similarity transformations) by integer displacements of parameters [3]. If exactly one upper and one lower parameter of ${}_nF_{n-1}(a_1, \dots, a_n; b_1, \dots, b_{n-1}; \zeta)$ are equal, permitting cancellation, then the corresponding E_n will have reducible monodromy. One can show that in this case, the differential operator $\mathcal{D}_n$ will have an order-$(n-1)$ right factor, which comes from the ${}_{n-1}F_{n-2}$ to which the ${}_nF_{n-1}$ reduces. In the theory of the Gauss function ${}_2F_1$, the reducible case, when a_1 or a_2 differs by an integer from b_1 or $b_2 = 1$, is often called the 'degenerate' case [10, §2.2]. Reducibility of E_2 facilitates the explicit construction of solutions [22].

A monodromy group $H < GL_n(\mathbb{C})$ of an equation E_n, if irreducible, is said to be *imprimitive* if there is a direct sum decomposition $V_1 \oplus \cdots \oplus V_d$ of $\mathbb{C}^n$ with $d \geqslant 2$ and $\dim V_j \geqslant 1$, such that each element of H permutes the spaces V_j. The following is a key theorem of Beukers and Heckman. It refers to the complex reflection subgroup H_r of H, which is generated by the elements $h_\infty^k \cdot h_1 \cdot h_\infty^{-k}$, $k \in \mathbb{Z}$. A 'scalar shift' by $\delta \in \mathbb{C}$ means the addition of δ to each parameter.

Theorem 2.3. *(Cf. [3], Thm. 5.8.) Suppose that H, H_r, computed from E_n, are irreducible in $GL_n(\mathbb{C})$. Then H will be imprimitive if and only if there exist relatively prime $p, q \geqslant 1$ with $p+q=n$, and $a \in \mathbb{C}$, such that E_n takes on one of the two forms*

$$E_n\left(\begin{matrix}\frac{a}{n}, \dots, \frac{a+(n-1)}{n} \\ \frac{a+1}{p}, \dots, \frac{a+p}{p}; \frac{1}{q}, \dots, \frac{q}{q}\end{matrix}\right), \qquad E_n\left(\begin{matrix}\frac{-a}{p}, \dots, \frac{-a+(p-1)}{p}; \frac{a}{q}, \dots, \frac{a+(q-1)}{q} \\ \frac{1}{n}, \dots, \frac{n}{n}\end{matrix}\right),$$

up to integer displacements of parameters, and up to a scalar shift by some $\delta \in \mathbb{C}$. (For irreducibility, one must have that $qa \notin \mathbb{Z}$, resp. $na \notin \mathbb{Z}$; this condition ensures that the upper and lower parameters satisfy $a_j - b_{j'} \notin \mathbb{Z}$, $\forall j, j'$.) Moreover, if $a \in \mathbb{Q}$ then H will be finite.

This theorem yields a class of algebraic ${}_nF_{n-1}$'s, including (1.3). One must choose the shift δ so that one of the lower parameters equals 1; e.g., $\delta = 0$. Beukers and Heckman do not compute the monodromy group H of the two E_n's in the theorem, but it follows readily from their proof that if, e.g., $a = -1/mq$, resp. $a = -1/mn$, with $m \geqslant 2$, then H is of order $m^{n-1}n!$, and is isomorphic to the 'symmetric' index-m subgroup of the wreath product $\mathfrak{C}_m \wr \mathfrak{S}_n$, where $\mathfrak{C}_m$ and $\mathfrak{S}_n$ are the usual cyclic and symmetric groups. So is the corresponding projective monodromy group $\overline{H} < PGL_n(\mathbb{C})$ (see [21, Cor. 4.6]).

If $a \in \mathbb{Z}$ then Theorem 2.3 does not apply, due to reducibility: in each E_n an upper and a lower parameter will differ by an integer. But, one has the following.

Theorem 2.4. *(Cf. [3], Prop. 5.9.) If there are* equal *upper and lower parameters in either E_n of Theorem 2.3, and $a \in \mathbb{Z}$ (e.g., if $a = \pm 1$), then the E_{n-1} obtained by cancelling them will have monodromy group $H < GL_{n-1}(\mathbb{C})$ isomorphic to $\mathfrak{S}_n$.*

This yields a class of algebraic ${}_{n-1}F_{n-2}$'s. The hypergeometric functions appearing in the following sections will include algebraic ${}_nF_{n-1}$'s of the type arising from Theorem 2.3, but for certain choices of parameter, they will reduce to ${}_{n-1}F_{n-2}$'s. On the power series level, Lemma 2.1 will perform some of these reductions.

3. Sequence transformations and coefficient identities

In this section and Section 4, it is shown that the solutions of hypergeometric equations E_n of the types introduced in Theorem 2.3, with monodromy that is irreducible but imprimitive, include algebraic functions that are solutions of trinomial equations. Explicit expressions for the associated ${}_nF_{n-1}$'s as combinations of algebraic functions will be derived, which extend those of Birkeland [4].

The chief result of this section is Theorem 3.4, which provides a family of binomial coefficient identities, the family being indexed by $\ell \in \mathbb{Z}$. They are derived from the standardized trinomial equation $y - 1 - zy^B = 0$, and will be used in Section 4. Theorem 3.7 provides two more identities, which are more sophisticated in the sense that they have an additional free parameter. Some of these identities were found by Lambert, Ramanujan, Pólya, and Gould, and the family indexed by $\ell \in \mathbb{Z}$ was explored by Chu [6] in a little-noticed paper. The tool employed in deriving the identities of Theorems 3.4 and 3.7 is the Vandermonde convolution transform of Gould [14,15], which is useful in deriving identities relating coefficients of related series, such as binomial coefficient identities.

In what follows,

$$\binom{a}{r} := (a-r+1)_r/r! \tag{3.1}$$

extends the binomial coefficient to the case of arbitrary upper parameter, and

$$(a)_r := \begin{cases} (a)\cdots(a+r-1), & r \geqslant 0; \\ [(a-s)\cdots(a-1)]^{-1}, & r=-s \leqslant 0, \end{cases} \tag{3.2}$$

extends the usual rising factorial, so that $(a)_r = [(a+r)_{-r}]^{-1}$ for all $r \in \mathbb{Z}$. The standard forward finite difference operator $\Delta_{A,B}$, which acts on functions of A, is defined by $\Delta_{A,B}[h](A) = h(A+B) - h(A)$. Its n'th power $\Delta_{A,B}^n$ satisfies

$$\Delta_{A,B}^n[h](A) = \sum_{k=0}^{n} (-)^{n-k} \binom{n}{k} h(A+kB). \tag{3.3}$$

Theorem 3.1. *(See [15].) Let $A, B \in \mathbb{C}$ be given, with $B \neq 0, 1$. Let $f(k)$, $k \geqslant 0$, be an infinite sequence of complex numbers, and let the sequence $\hat{f}(n)$, $n \geqslant 0$, be defined by*

$$\hat{f}(n) = \sum_{k=0}^{n} (-)^k \binom{n}{k} \binom{A+Bk}{n} f(k). \tag{3.4}$$

In a neighborhood of $z=0$, let y near 1 be defined implicitly by the standardized trinomial equation $y - 1 - zy^B = 0$. Then

$$\sum_{k=0}^{\infty} \binom{A+Bk}{k} f(k) z^k = y^A \sum_{n=0}^{\infty} \hat{f}(n) \left(\frac{1-y}{y}\right)^n \tag{3.5}$$

holds as an equality between power series in z; and hence, if either has a positive radius of convergence, as an equality in a neighborhood of $z=0$.

If the theorem applies, the sequence $\hat{f}(n)$, $n \geqslant 0$, will be called the (A,B)-Vandermonde convolution transform of the sequence $f(k)$, $k \geqslant 0$. If $f(0)=1$, then $\hat{f}(0)=1$ also. For the inverse transform, see [15].

Definition 3.2. For each $\ell \in \mathbb{Z}$, the sequence $f_\ell(A,B;k)$, $k \geqslant 0$, is defined by

$$f_\ell(A,B;k) := \frac{(A+Bk+1)_{\ell-1}}{(A+1)_{\ell-1}} = \frac{(A+\ell)_{1-\ell}}{(A+Bk+\ell)_{1-\ell}}. \tag{3.6}$$

It is normalized so that $f_\ell(A,B;0)=1$. Note that $f_1(A,B;k) \equiv 1$.

It is an immediate consequence of (3.3) that the (A,B)-Vandermonde convolution transform of $f_\ell(A,B;k)$, defined by Eq. (3.4), has the representation

$$\begin{aligned}\hat{f}_\ell(A,B;n) &= \frac{(-)^n}{(A+1)_{\ell-1}}\Delta^n_{A,B}\left[\binom{A}{n}(A+1)_{\ell-1}\right]\\ &= \frac{(-)^n}{n!(A+1)_{\ell-1}}\Delta^n_{A,B}\big[(A-n+1)_{n+\ell-1}\big]. \qquad (3.7)\end{aligned}$$

Here and in subsequent equations and identities, parameters such as A, B are constrained, sometimes tacitly, so that division by zero occurs on neither side.

Theorem 3.3. (i) *For each $\ell = -m \leqslant 0$, $\hat{f}_\ell(A,B;n)$ is nonzero only if $n = 0,\dots,m$, and is rational in A, B for each n.* (ii) *For each $\ell \geqslant 1$, $\hat{f}_\ell(A,B;n)$ equals a certain degree-$(\ell-1)$ polynomial in n with coefficients polynomial in A, B, multiplied by the function $(n+1)_{\ell-1}(-B)^n/(A+1)_{\ell-1}$ of n.*

Proof. Part (i) follows from the fact that if $n \geqslant m+1$, then $(A-n+1)_{n-m-1}$, appearing in (3.7), is a polynomial in A of degree $n-m-1$; hence its n'th finite difference must equal zero. Part (ii) comes from a finite difference computation (cf. [19, §76]). Explicitly, $\hat{f}_\ell(A,B;n)$ equals $(-B)^n/(A+1)_{\ell-1}$ times

$$\sum_{i=0}^{\ell-1}\sum_{j=0}^{i} s(n+\ell-1,n+i)S(n+i,n+j)(n+1)_j B^i\binom{(A+\ell-1)/B}{j}, \qquad (3.8)$$

where s, S are the Stirling numbers of the first and second kinds. But $s(\nu,\nu-\rho)$, $S(\nu,\nu-\rho)$ are degree-2ρ polynomials in ν that are divisible by $(\nu-\rho+1)_\rho$, so each summand is of degree at most $2(\ell-1)$ in n and is divisible by $(n+1)_{\ell-1}$. □

Remark 3.3.1. The formula given in the preceding proof, valid when $\ell \geqslant 1$, can be written as

$$\hat{f}_\ell(A,B;n) = \frac{(-)^n}{(A+1)_{\ell-1}}\sum_{j=0}^{\ell-1} C(n+\ell-1,n+j;B)(n+1)_j\binom{(A+\ell-1)/B}{j}, \qquad (3.9)$$

where the coefficients $C(n,k;B) := \sum_{i=k}^{n} s(n,i)B^i S(i,k)$ are 'C-numbers' [5]. Closed-form expressions for $C(n,k;B)$ when $B = -1, \frac{1}{2}, 2$ are known [7, p. 158]. However, these values of B play no special role in the present analysis.

Examples of transformed sequences $\hat{f}_\ell(A,B;n)$, $n \geqslant 0$, include

$$\hat{f}_{-1}(A,B;n) = \delta_{n,0} + \frac{AB}{A+B-1}\delta_{n,1}, \qquad (3.10a)$$

$$\hat{f}_0(A,B;n) = \delta_{n,0}, \qquad (3.10b)$$

$$\hat{f}_1(A,B;n) = (-B)^n, \qquad (3.10c)$$

$$\hat{f}_2(A,B;n) = \left[(A+1) + \frac{1}{2}(B-1)n\right]\frac{(n+1)(-B)^n}{A+1}. \qquad (3.10d)$$

These follow readily from the representation (3.7), or with more effort from (3.9). For later reference, note that

$$\hat{f}_2(A,B;n) + B\hat{f}_2(A,B;n-1) = \big[A+1+(B-1)n\big]\frac{(-B)^n}{A+1}. \qquad (3.11)$$

Table 1
The first few rational functions F_ℓ, with $S := \frac{1}{2} - \ell$.

ℓ	S	$F_\ell(A, B; y)$
-1	$\frac{3}{2}$	$\frac{AB-(A-1)(B-1)y}{(A+B-1)y}$
0	$\frac{1}{2}$	1
1	$-\frac{1}{2}$	$\frac{y}{(1-B)y+B}$
2	$-\frac{3}{2}$	$\frac{y^2[(B-1)(B-A-1)y-B(B-A-2)]}{(A+1)[(1-B)y+B]^3}$ $= (y^2 D_y)\{\frac{2(B-1)(B-A-1)y-B(2B-2A-3)}{2(A+1)(B-1)[(1-B)y+B]^2}\}$

Theorem 3.4. *For each $\ell \in \mathbb{Z}$, there is a certain rational function of y, $F_\ell(A, B; y)$, with coefficients polynomial in A, B, such that*

$$y^A F_\ell(A, B; y) = 1 + \sum_{k=1}^{\infty} \frac{(A + Bk + 1)_{\ell-1}}{(A + 1)_{\ell-1}} \binom{A + Bk}{k} z^k \tag{3.12}$$

holds in a neighborhood of $z = 0$, if y near 1 is the solution of the standardized trinomial equation $y - 1 - zy^B = 0$. (By assumption, $B \neq 0, 1$.) Specifically,

$$F_\ell(A, B; y) = \sum_{n=0}^{\infty} \hat{f}_\ell(A, B; n) \left(\frac{1 - y}{y}\right)^n, \tag{3.13}$$

where the sequence $\hat{f}_\ell$ was defined in (3.7) and characterized in Theorem 3.3.

The rational function F_ℓ has at most one pole on $\mathbb{P}^1_y$. If $\ell = -m < 0$, the pole is at $y = 0$ and is of order m; and if $\ell > 0$, it is at $y = B/(B - 1)$ and is of order $2\ell - 1$. If $\ell > 0$, F_ℓ can be written as $(y^2 D_y)^{\ell-1} \tilde{F}_\ell$, where $D_y = \mathrm{d}/\mathrm{d}y$ and $\tilde{F}_\ell$ is rational with a pole at $y = B/(B - 1)$ of order ℓ.

Proof. The identity (3.12) is the specialization of (3.5) to the case $f = f_\ell$, $\hat{f} = \hat{f}_\ell$. By Theorem 3.3, F_ℓ is a degree-m polynomial in $v := (1 - y)/y$ if $\ell = -m \leqslant 0$, and if $\ell > 0$, it is rational on $\mathbb{P}^1_v$ with a unique pole (of order $2\ell - 1$) at $v = (-B)^{-1}$, i.e., at $y = B/(B - 1)$. Since $(n + 1)_{\ell-1} \mid \hat{f}_\ell$, the function F_ℓ on $\mathbb{P}^1_v$ is the $(\ell - 1)$'th derivative of some rational function with a unique finite-v pole (of order ℓ) at $v = (-B)^{-1}$. The final claim thus follows from $D_v = -y^2 D_y$. $\square$

As examples of the use of formula (3.13), four of the rational functions F_ℓ are listed in Table 1. They come from the sequences $\hat{f}_\ell$ given in (3.10).

Theorem 3.4, with the aid of Table 1, yields the following binomial coefficient identities, which are indexed by $\ell = -1, 0, 1, 2$, respectively. They hold near $z = 0$, if y near 1 is defined by $y - 1 - zy^B = 0$.

$$y^A \left[\frac{AB - (A - 1)(B - 1)y}{(A + B - 1)y}\right] = 1 + \sum_{k=1}^{\infty} \frac{A - 1}{A + Bk - 1} \frac{A}{A + Bk} \binom{A + Bk}{k} z^k, \tag{3.14a}$$

$$y^A = 1 + \sum_{k=1}^{\infty} \frac{A}{A + Bk} \binom{A + Bk}{k} z^k, \tag{3.14b}$$

$$y^A\left[\frac{y}{(1-B)y+B}\right] = 1 + \sum_{k=1}^{\infty} \binom{A+Bk}{k} z^k, \tag{3.14c}$$

$$y^A\left\{\frac{y^2[(B-1)(B-A-1)y - B(B-A-2)]}{(A+1)[(1-B)y+B]^3}\right\}$$
$$= 1 + \sum_{k=1}^{\infty} \frac{A+Bk+1}{A+1} \binom{A+Bk}{k} z^k. \tag{3.14d}$$

The $\ell = 0$ identity (3.14b) and the $\ell = 1$ identity (3.14c) are well known. The former was derived by Lambert in 1758 (and by Ramanujan [2, p. 72]), and the latter by Pólya [28]. They can be proved by Lagrange inversion [25, Thm. 2.1], and imply each other [29, Lem. 1]. But the identities (3.14a), (3.14d) are less familiar. The $\ell = 3, 4, \dots$ and $\ell = -2, -3, \dots$ identities can also be worked out.

Each identity in this family indexed by $\ell \in \mathbb{Z}$ can be obtained by differentiating the preceding. This is formalized in the following recurrence, which is equivalent to one obtained by Chu [6]. By starting with $F_0 \equiv 1$ and iterating, one can generate all F_ℓ, $\ell > 0$. By doing the reverse, i.e., by integrating (and exploiting the fact that $F_\ell(1) = 1$, $\forall \ell \in \mathbb{Z}$, which determines each constant of integration), one can generate all F_ℓ, $\ell < 0$.

Theorem 3.5. *The rational functions $F_\ell(A, B; y)$, $\ell \in \mathbb{Z}$, satisfy*

$$F_{\ell+1}(A, B; y) = \frac{(A-B+2)_{\ell-1}}{(A+1)_\ell} \frac{y^{-A+B+1}}{(1-B)y+B} D_y\left[y^{A-B+1} F_\ell(A-B+1, B; y)\right].$$

Proof. Apply D_z to both sides of (3.12), using $D_z = \frac{y^{2B}}{y^B - B(y-1)y^{B-1}} D_y$, which follows from $y - 1 - zy^B = 0$. $\square$

In the binomial coefficient identities of the form (3.12) with $\ell = 3, 4, \dots$ and $\ell = -2, -3, \dots$, not given here, the left-hand functions $F_\ell(A, B; y)$ become increasingly complicated. But for all $\ell \in \mathbb{Z}$, the right-hand power series in z has radius of convergence $|(B-1)^{B-1}/B^B|$ if $B \neq 1$, and unity if $B = 1$. This is consistent with the presence (when $\ell > 0$, at least) of a pole at $y = B/(B-1)$ on the left-hand side, since $y = B/(B-1)$ corresponds to $z = (B-1)^{B-1}/B^B$. As $B \to 1$, the pole moves to $y = \infty$, i.e., to $z = 1$.

The following theorem reveals when the evaluation of the rational function $F_\ell(A, B; y)$, in any identity in this family, may lead to a division by zero.

Theorem 3.6. *The rational function $F_\ell(A, B; y)$ in (3.12) is of the form*

$$\frac{\tilde{\Pi}_m(A, B; y)}{(A+B-m)_m (A+2B-m)_{m-1} \cdots (A+mB-m)_1 y^m}, \quad \ell = -m \leqslant 0; \tag{3.15a}$$

$$\frac{y^\ell \Pi_{\ell-1}(A, B; y)}{(A+1)_{\ell-1}[(1-B)y+B]^{2\ell-1}}, \quad \ell \geqslant 1, \tag{3.15b}$$

where $\tilde{\Pi}_m$, $\Pi_{\ell-1}$ are polynomials in y, the subscripts indicating their degrees.

Proof. Iterate Theorem 3.5 toward negative and positive ℓ. $\square$

It should be mentioned that in the identities indexed by $\ell \geqslant 1$, the poles in A that are evident in (3.15b), located at $A = -1, \dots, -\ell + 1$, are removable. The denominator factor $(A+1)_{\ell-1}$ is present on the right as well as the left side of (3.12), and can simply be cancelled. But the poles in A present in F_ℓ, $\ell \leqslant -1$, which are evident in (3.15a), are less easily removed.

This family of identities can be generalized by modifying the initial sequence $f_\ell(A, B, k)$ of (3.6) to include one or more free parameters. A pair of such generalizations, involving the functions ${}_2F_1$ and ${}_3F_2$, will prove useful. Let

$$g_\ell(A, B, C; k) := \frac{A + C + \ell}{A + C + Bk + \ell} f_{\ell+1}(A, B; k), \quad k \geqslant 0, \tag{3.16}$$

so that g_ℓ is 'interpolating': it reduces to f_ℓ if $C = 0$ and to $f_{\ell+1}$ as $C \to \infty$. It follows from (3.3), much as in (3.7), that the (A, B)-transforms of g_0, g_1 are

$$\hat{g}_0(A, B, C; n) = (-)^n (A + C) \Delta^n_{A,B} \left[\binom{A}{n} \left(\frac{1}{A + C} \right) \right], \tag{3.17a}$$

$$\hat{g}_1(A, B, C; n) = (-)^n \left(\frac{A + C + 1}{A + 1} \right) \Delta^n_{A,B} \left[\binom{A}{n} \left(\frac{A + 1}{A + C + 1} \right) \right]. \tag{3.17b}$$

By a finite difference computation or an expansion in partial fractions (cf. [14, §6]), these transformed sequences satisfy

$$\hat{g}_0(A, B, C; n) = (-)^n \frac{(C)_n}{(\frac{A+C}{B} + 1)_n}, \tag{3.18a}$$

$$\hat{g}_1(A, B, C; n) + B \hat{g}_1(A, B, C; n-1) = (-)^n \frac{(C)_n (\frac{A+1}{B-1} + 1)_n}{(\frac{A+C+1}{B} + 1)_n (\frac{A+1}{B-1})_n}, \tag{3.18b}$$

for each $n \geqslant 0$, resp. $n \geqslant 1$, showing by comparison with (3.10) and (3.11) that $\hat{g}_0(A, B, C; n)$ reduces to $\hat{f}_0(A, B; n) = (-B)^n$ if $C = 0$ and to $\hat{f}_1(A, B; n) = \delta_{n,0}$ as $C \to \infty$; and that $\hat{g}_1(A, B, C; n)$ reduces to $\hat{f}_1(A, B; n) = 1$ if $C = 0$, and to $\hat{f}_2(A, B; n)$, given in (3.10d), as $C \to \infty$. To avoid division by zero in (3.18), one may assume that $B \neq 0, 1$ and that $(A+C)/B$ is not a negative integer.

The right sides of (3.18a), (3.18b) have the form of coefficients of ${}_2F_1$ and ${}_3F_2$ series. Applying Theorem 3.1 therefore yields a pair of hypergeometric identities.

Theorem 3.7. *Under the preceding assumptions, one has the* interpolating *identities*

$$y^A \, {}_2F_1\left(\left. \begin{matrix} C, 1 \\ \frac{A+C}{B} + 1 \end{matrix} \right| \frac{y-1}{y} \right) = 1 + \sum_{k=1}^{\infty} \frac{A + C}{A + C + Bk} \binom{A + Bk}{k} z^k,$$

$$y^A \left[\frac{y}{(1-B)y + B} \right] {}_3F_2\left(\left. \begin{matrix} C, \frac{A+1}{B-1} + 1, 1 \\ \frac{A+C+1}{B} + 1, \frac{A+1}{B-1} \end{matrix} \right| \frac{y-1}{y} \right)$$

$$= 1 + \sum_{k=1}^{\infty} \frac{A + C + 1}{A + C + Bk + 1} \frac{A + Bk + 1}{A + 1} \binom{A + Bk}{k} z^k,$$

which are valid near $z = 0$, if y near 1 is defined by $y - 1 - zy^B = 0$.

Of these two identities the first was found by Gould [14, §6], but the more complicated second one is new. The parametrized functions multiplied by y^A on their left sides will be denoted by $G_\ell(A,B,C;y)$, $\ell=0,1$, respectively.

The first identity reduces if $C=0$ to (3.14b), the $\ell=0$ identity of Lambert and Ramanujan, and as $C\to\infty$ to (3.14c), the $\ell=1$ identity of Pólya. The function $G_0(A,B,C;y)$ reduces to $F_0(A,B;y)\equiv 1$ and $F_1(A,B;y)=y/[(1-B)y+B]$, respectively. Similarly, the second identity reduces if $C=0$ to Pólya's identity, and as $C\to\infty$ to the $\ell=2$ identity (3.14d). The function $G_1(A,B,C;y)$ reduces respectively to $F_1(A,B;y)$ and $F_2(A,B;y)$.

4. Algebraic and hypergeometric functions

In this section certain algebraic functions, namely the solutions of trinomial equations, are expressed in terms of ${}_nF_{n-1}$'s that are solutions of E_n's with monodromy that is irreducible but imprimitive. In Theorem 4.4, which is the main result of this section, the ${}_nF_{n-1}$'s are expressed in terms of algebraic functions by 'inverting' the just-mentioned representations. If the parameter $\ell\in\mathbb{Z}$ in Theorem 4.4 is set to zero, the expressions reduce to those of Birkeland [4]. Theorem 4.6 is a partial extension of Theorem 4.4 that has an additional free parameter, and is less algebraic.

The chief result of Section 3 was Theorem 3.4, which provides a family of binomial coefficient identities indexed by ℓ. It was based on the *standardized* trinomial equation $y-1-zy^B=0$, and the solution y that is near 1 for z near 0. The theorem is now restated in terms of the *general* trinomial equation

$$x^n-gx^p-\beta=0. \tag{4.1}$$

Here $n=p+q$, for integers $p,q\geqslant 1$, and $g,\beta\in\mathbb{C}$ with at most one of g,β equal to zero. The condition $\gcd(p,q)=1$ will be added later.

Let the n solutions of (4.1), with multiplicity, be denoted $x_1,\dots,x_n$. In the limit $\beta\to 0$ with fixed $g>0$, one may choose

$$x_j=\begin{cases}\varepsilon_q^{-(j-1)}g^{1/q}, & j=1,\dots,q,\\ 0, & j=q+1,\dots,n.\end{cases} \tag{4.2}$$

In the limit $g\to 0$ with fixed $\beta>0$, one may choose

$$x_j=\varepsilon_n^{-(j-1)}\beta^{1/n},\qquad j=1,\dots,n. \tag{4.3}$$

Here and below, $\varepsilon_r:=\exp(2\pi\mathrm{i}/r)$ signifies a primitive r'th root of unity.

To reduce the first case of (4.1), i.e., that of β near zero, to $y-1-zy^B=0$, let

$$y=g^{-1}x^q,\qquad z=\varepsilon_q^{(j-1)n}g^{-n/q}\beta,\qquad B=-p/q. \tag{4.4}$$

To reduce the second case, i.e., that of g near zero, let

$$y=\beta^{-1}x^n,\qquad z=\varepsilon_n^{(j-1)q}\beta^{-q/n}g,\qquad B=p/n. \tag{4.5}$$

By undoing these two reductions, and letting $A=\gamma/q$, resp. $A=\gamma/n$, one obtains the following 'de-standardized' version of Theorem 3.4, in which the rational functions $F_\ell(A,B;y)$, $\ell\in\mathbb{Z}$, were defined.

Theorem 4.1. *The following hold for $\ell \in \mathbb{Z}$ and $\gamma \in \mathbb{C}$.*

(i) *In a neighborhood of $\beta = 0$ with fixed $g > 0$, and with the trinomial root x_j near $\varepsilon_q^{-(j-1)} g^{1/q}$ defined by (4.1), (4.2),*

$$\begin{aligned}&\left[\varepsilon_q^{(j-1)} g^{-1/q} x_j\right]^\gamma F_\ell\left(\gamma/q, -p/q; g^{-1} x_j^q\right)\\&\quad= 1 + \sum_{k=1}^{\infty} \frac{(\gamma/q - pk/q + 1)_{\ell-1}}{(\gamma/q + 1)_{\ell-1}} \binom{\gamma/q - pk/q}{k} \left[\varepsilon_q^{(j-1)n} g^{-n/q} \beta\right]^k,\end{aligned}$$

for $j = 1, \ldots, q$.

(ii) *In a neighborhood of $g = 0$ with fixed $\beta > 0$, and with the trinomial root x_j near $\varepsilon_n^{-(j-1)} \beta^{1/n}$ defined by (4.1), (4.3),*

$$\begin{aligned}&\left[\varepsilon_n^{(j-1)} \beta^{-1/n} x_j\right]^\gamma F_\ell\left(\gamma/n, p/n; \beta^{-1} x_j^n\right)\\&\quad= 1 + \sum_{k=1}^{\infty} \frac{(\gamma/n + pk/n + 1)_{\ell-1}}{(\gamma/n + 1)_{\ell-1}} \binom{\gamma/n + pk/n}{k} \left[\varepsilon_n^{(j-1)q} \beta^{-q/n} g\right]^k,\end{aligned}$$

for $j = 1, \ldots, n$.

In (i) *and* (ii), *$\gamma \in \mathbb{C}$ is constrained so that no division by zero occurs in the evaluation of either the rational function F_ℓ or the first factor of the summand.*

The connection to hypergeometric equations E_n with imprimitive monodromy, and their canonical solutions ${}_nF_{n-1}$, can now be made. Henceforth, let a Riemann sphere $\mathbb{P}^1_\zeta$ be parametrized by

$$\zeta := (-)^q \frac{n^n}{p^p q^q} \frac{\beta^q}{g^n}. \tag{4.6}$$

The following formulas extend those of Birkeland [4, §3], who considered only the case $\ell = 0$. Since $F_0 \equiv 1$, his formulas contain no left-hand F_ℓ factor.

Theorem 4.2. *The following hold for $\ell \in \mathbb{Z}$ and $a \in \mathbb{C}$, with ζ defined by (4.6).*

(i) *In a neighborhood of $\beta = 0$ with fixed $g > 0$, and with the trinomial root x_j near $\varepsilon_q^{-(j-1)} g^{1/q}$ defined by (4.1), (4.2),*

$$\begin{aligned}&\left[\varepsilon_q^{(j-1)} g^{-1/q} x_j\right]^{-qa} F_\ell\left(-a, -p/q; g^{-1} x_j^q\right)\\&\quad= \sum_{\kappa=0}^{q-1} \left(\varepsilon_q^n\right)^{(j-1)\kappa} \frac{(-)^\kappa (a)_{1-\ell}}{(a + n\kappa/q)_{1-\ell-\kappa} (1)_\kappa} \left[\frac{(-)^q p^p q^q}{n^n} \cdot \zeta\right]^{\kappa/q}\\&\qquad\times {}_nF_{n-1}\left(\left.\begin{matrix}\frac{a}{n} + \frac{\kappa}{q}, \ldots, \frac{a+(n-1)}{n} + \frac{\kappa}{q}\\ \frac{a-\ell+1}{p} + \frac{\kappa}{q}, \ldots, \frac{a-\ell+p}{p} + \frac{\kappa}{q}; \frac{1}{q} + \frac{\kappa}{q}, \ldots, \widehat{\frac{q-\kappa}{q}} + \frac{\kappa}{q}, \ldots, \frac{q}{q} + \frac{\kappa}{q}\end{matrix}\right| \zeta\right),\end{aligned}$$

for $j = 1, \ldots, q$.

(ii) *In a neighborhood of $g = 0$ with fixed $\beta > 0$, and with the trinomial root x_j near $\varepsilon_n^{-(j-1)}\beta^{1/n}$ defined by* (4.1), (4.3),

$$\begin{aligned}
&\left[\varepsilon_n^{(j-1)}\beta^{-1/n}x_j\right]^{-na} F_\ell\left(-a, p/n; \beta^{-1}x_j^n\right) \\
&\quad = \sum_{\kappa=0}^{n-1}\left(\varepsilon_n^q\right)^{(j-1)\kappa} \frac{(-)^\kappa (a)_{1-\ell}}{(a+q\kappa/n)_{1-\ell-\kappa}(1)_\kappa}\left[\frac{(-)^q p^p q^q}{n^n}\cdot\zeta\right]^{-\kappa/n} \\
&\qquad \times {}_nF_{n-1}\left(\left.\begin{matrix} \frac{-a+\ell}{p}+\frac{\kappa}{n}, \dots, \frac{-a+\ell+(p-1)}{p}+\frac{\kappa}{n}; \frac{a}{q}+\frac{\kappa}{n}, \dots, \frac{a+(q-1)}{q}+\frac{\kappa}{n} \\ \frac{1}{n}+\frac{\kappa}{n}, \dots, \widehat{\frac{n-\kappa}{n}+\frac{\kappa}{n}}, \dots, \frac{n}{n}+\frac{\kappa}{n} \end{matrix}\right| \zeta^{-1}\right),
\end{aligned}$$

for $j = 1, \dots, n$.

In (i) *and* (ii), *$a \in \mathbb{C}$ is constrained so that no division by zero occurs on either side, and* (*in* (i)) *so that no lower parameter of any ${}_nF_{n-1}$ is a non-positive integer.*

Remark 4.2.1. The quantity in square brackets on the right sides of (i), (ii) equals $g^{-n}\beta^q$. It is raised to the power κ/q, resp. power $-\kappa/n$. The branch should be chosen so that the result equals $g^{-\kappa n/q}\beta^\kappa$, resp. $\beta^{-\kappa q/n}g^\kappa$.

Proof. Partition the series of Theorem 4.1 into residue classes of k mod q, resp. k mod n. That is, let $k = \kappa + i \cdot q$, resp. $k = \kappa + i \cdot n$, and for each κ, sum over $i = 0, 1, \dots,$ using the identity

$$(t)_{i\cdot r} = r^r\left(\frac{t}{r}\right)_i\left(\frac{t+1}{r}\right)_i\cdots\left(\frac{t+r-1}{r}\right)_i. \tag{4.7}$$

Also, let $\gamma = -qa$, resp. $\gamma = -na$. (That is, $A = -a$.) $\square$

Corollary 4.3. *The following hold for $\ell \in \mathbb{Z}$ and $a \in \mathbb{C}$.*

(i) *Near $\zeta = 0$ on $\mathbb{P}^1_\zeta$, with the trinomial root x_j near $\varepsilon_q^{-(j-1)}g^{1/q}$ defined by* (4.1), (4.2), *the coefficient $g > 0$ being fixed and arbitrary and the coefficient β being defined in terms of ζ by* (4.6), *the quantities*

$$x_j^{-qa}F_\ell\left(-a, -p/q; g^{-1}x_j^q\right), \quad j = 1, \dots, q,$$

regarded as functions of ζ, lie in the solution space of the differential equation

$$E_n\left(\begin{matrix} \frac{a}{n}, \dots, \frac{a+(n-1)}{n} \\ \frac{a-\ell+1}{p}, \dots, \frac{a-\ell+p}{p}; \frac{1}{q}, \dots, \frac{q}{q} \end{matrix}\right)_\zeta.$$

(ii) *Near $\zeta = \infty$ on $\mathbb{P}^1_\zeta$, with the trinomial root x_j near $\varepsilon_n^{-(j-1)}\beta^{1/n}$ defined by* (4.1), (4.3), *the coefficient $\beta > 0$ being fixed and arbitrary and the coefficient g being defined in terms of ζ by* (4.6), *the quantities*

$$x_j^{-na}F_\ell\left(-a, p/n; \beta^{-1}x_j^n\right), \quad j = 1, \dots, n,$$

regarded as functions of the reciprocal variable $\tilde{\zeta} := \zeta^{-1}$, lie in the solution space near $\tilde{\zeta} = 0$ of the differential equation

$$E_n\left(\begin{matrix}\frac{-a+\ell}{p},\dots,\frac{-a+\ell+(p-1)}{p};\frac{a}{q},\dots,\frac{a+(q-1)}{q}\\ \frac{1}{n},\dots,\frac{n}{n}\end{matrix}\right)_{\tilde{\zeta}}.$$

In (i)*, it is assumed that $a\in\mathbb{C}$ is such that no pair of lower parameters differs by an integer*; *i.e., that the singular point $\zeta=0$ of the E_n is non-logarithmic.*

Proof. Immediate, by Lemma 2.2 applied to Theorem 4.2. □

If in either E_n of Corollary 4.3, a is chosen so that no upper parameter differs by an integer from a lower one, then the monodromy group of the E_n will be irreducible; and moreover, if $\gcd(p,q)=1$ then the E_n will be of the imprimitive irreducible type characterized in Theorem 2.3. The latter fact is obvious if $\ell=0$, though if $\ell\in\mathbb{Z}\setminus\{0\}$, integer displacements of parameters are needed.

The representations of Theorem 4.2 can now be inverted, to express the ${}_nF_{n-1}$'s satisfying these E_n's in terms of the trinomial roots $x_1,\dots,x_n$. In part (i) of the theorem below, only $x_1,\dots,x_q$ are needed; but in part (ii), all are needed.

Theorem 4.4. *If* $\gcd(p,q)=1$*, the following hold for each $\ell\in\mathbb{Z}$ and $a\in\mathbb{C}$.*

(i) *Near $\zeta=0$ on $\mathbb{P}^1_\zeta$, for arbitrary fixed $g>0$ and for $\kappa=0,\dots,q-1$,*

$$\begin{aligned}&\zeta^{\kappa/q}\,{}_nF_{n-1}\left(\left.\begin{matrix}\frac{a}{n}+\frac{\kappa}{q},\dots,\frac{a+(n-1)}{n}+\frac{\kappa}{q}\\ \frac{a-\ell+1}{p}+\frac{\kappa}{q},\dots,\frac{a-\ell+p}{p}+\frac{\kappa}{q};\frac{1}{q}+\frac{\kappa}{q},\dots,\widehat{\frac{q-\kappa}{q}+\frac{\kappa}{q}},\dots,\frac{q}{q}+\frac{\kappa}{q}\end{matrix}\,\right|\,\zeta\right)\\ &\quad=\frac{(-)^\kappa(1)_\kappa(a+n\kappa/q)_{1-\ell-\kappa}}{(a)_{1-\ell}}\left[\frac{(-)^q n^n}{p^p q^q}\right]^{\kappa/q}\\ &\qquad\times q^{-1}\sum_{j=1}^{q}\left(\varepsilon_q^{-n}\right)^{(j-1)\kappa}\left[\varepsilon_q^{(j-1)}g^{-1/q}x_j\right]^{-qa}F_\ell\left(-a,-p/q;g^{-1}x_j^q\right),\end{aligned}$$

in which the trinomial root x_j near $\varepsilon_q^{-(j-1)}g^{1/q}$, $j=1,\dots,q$, is defined by (4.1), (4.2)*, with the coefficient β determined by ζ according to* (4.6).

(ii) *Near $\zeta=\infty$ on $\mathbb{P}^1_\zeta$, for arbitrary fixed $\beta>0$ and for $\kappa=0,\dots,n-1$,*

$$\begin{aligned}&\zeta^{-\kappa/n}\,{}_nF_{n-1}\left(\left.\begin{matrix}\frac{-a+\ell}{p}+\frac{\kappa}{n},\dots,\frac{-a+\ell+(p-1)}{p}+\frac{\kappa}{n};\frac{a}{q}+\frac{\kappa}{n},\dots,\frac{a+(q-1)}{q}+\frac{\kappa}{n}\\ \frac{1}{n}+\frac{\kappa}{n},\dots,\widehat{\frac{n-\kappa}{n}+\frac{\kappa}{n}},\dots,\frac{n}{n}+\frac{\kappa}{n}\end{matrix}\,\right|\,\zeta^{-1}\right)\\ &\quad=\frac{(-)^\kappa(1)_\kappa(a+q\kappa/n)_{1-\ell-\kappa}}{(a)_{1-\ell}}\left[\frac{(-)^q n^n}{p^p q^q}\right]^{-\kappa/n}\\ &\qquad\times n^{-1}\sum_{j=1}^{n}\left(\varepsilon_n^{-q}\right)^{(j-1)\kappa}\left[\varepsilon_n^{(j-1)}\beta^{-1/n}x_j\right]^{-na}F_\ell\left(-a,p/n;\beta^{-1}x_j^n\right),\end{aligned}$$

in which the trinomial root x_j near $\varepsilon_n^{-(j-1)}\beta^{1/n}$, $j=1,\dots,n$, is defined by (4.1), (4.3)*, with the coefficient g determined by ζ according to* (4.6).

In (i) *and* (ii)*, $a\in\mathbb{C}$ is constrained so that no division by zero occurs, and* (*in* (i)) *so that no lower parameter of any ${}_nF_{n-1}$, for any κ, is a non-positive integer.*

Remark 4.4.1. A convention for choosing branches of fractional powers must be adhered to, in interpreting these formulas. In (i), there are q choices for each of

$$\zeta^{1/q}, \qquad \left[\frac{(-)^q n^n}{p^p q^q}\right]^{1/q}, \qquad \beta.$$

(The last is evident from (4.6).) Any choices will work, so long as the formal q'th root of Eq. (4.6), i.e.,

$$\zeta^{1/q} = \left[\frac{(-)^q n^n}{p^p q^q}\right]^{1/q} \cdot g^{-n/q} \cdot \beta, \tag{4.8}$$

is satisfied. This removes one degree of freedom. Part (ii) is similarly interpreted.

Proof. Each of the two formulas in Theorem 4.2 has the form of a linear transformation expressed as a matrix–vector product, i.e., $u_j = \sum_\kappa M_{j\kappa} v_\kappa$. Here, $\mathsf{M} = (M_{j\kappa}) = (\varepsilon^{(j-1)\kappa})$ is a $q \times q$, resp. $n \times n$ matrix, with $\varepsilon := \varepsilon_q^n$, resp. ε_n^q. If $\gcd(p,q) = 1$, or equivalently $\gcd(q,n) = 1$, then the root of unity ε will be a *primitive* q'th, resp. n'th, root, and M will be nonsingular. The v_κ are expressed in terms of the u_j by inverting M. But $\mathsf{M}^{-1} = q^{-1}\mathsf{M}^*$, resp. $\mathsf{M}^{-1} = n^{-1}\mathsf{M}^*$. Elementwise multiplication by M^* yields the claimed summation formulas. □

Corollary 4.5. *If* $\gcd(p,q) = 1$*, then the following holds for* $a \in \mathbb{C}$ *with* $na \notin \mathbb{Z}$*. Near* $\zeta = \infty$ *on* $\mathbb{P}^1_\zeta$*, if* x_j *near* $\varepsilon_n^{-(j-1)}\beta^{1/n}$*,* $j = 1,\dots,n$*, is defined by* (4.1)*,* (4.3)*, with the coefficient* g *determined by* ζ *according to* (4.6)*, then the exponentiated roots*

$$x_j^{-na}, \qquad j = 1,\dots,n,$$

regarded as functions of $\tilde{\zeta} := \zeta^{-1}$*, span the solution space near* $\tilde{\zeta} = 0$ *of the differential equation*

$$E_n\left(\begin{matrix} \frac{-a}{p}, \dots, \frac{-a+(p-1)}{p}; \frac{a}{q}, \dots, \frac{a+(q-1)}{q} \\ \frac{1}{n}, \dots, \frac{n}{n} \end{matrix}\right)_{\tilde{\zeta}}.$$

Proof. Immediate by Lemma 2.2, applied to the $\ell = 0$ case of Theorem 4.4(ii). (The condition $na \notin \mathbb{Z}$ ensures linear independence of the exponentiated roots.) □

The representations of certain ${}_nF_{n-1}$'s given in Theorem 4.4, in terms of the solutions of trinomial equations, are the point of departure for the rest of this paper. In each ${}_nF_{n-1}$, the parametric excess S (the sum of the lower parameters, minus the sum of the upper ones) equals $\frac{1}{2} - \ell$, by examination. As was mentioned, the $\ell = 0$ case of the theorem, when the F_ℓ factor on each right-hand side degenerates to unity, was previously obtained by Birkeland.

Many classical hypergeometric identities, such as the cubic transformations of ${}_3F_2$ found by Bailey, are restricted to ${}_nF_{n-1}$'s with $S = \frac{1}{2}$. Sometimes there are 'companion' identities that are satisfied by hypergeometric functions with $S = -\frac{1}{2}$. (For Bailey's identities, see [12, (5.3), (5.4) and (5.6), (5.7)].) It is remarkable that in the present context, the cases $S = \frac{1}{2}$ and $-\frac{1}{2}$, i.e., $\ell = 0$ and 1, are merely the most easily treated steps on an infinite ladder of possibilities.

The preceding theorems relating hypergeometric and algebraic functions, including Theorem 4.4, came ultimately from Theorem 3.4, which applied the Vandermonde convolution transform to the sequences f_ℓ given in Definition 3.2. One could start instead with Theorem 3.7, i.e., with the 'interpolating' sequences g_0, g_1 parametrized by C, which reduce to f_0, f_1 if $C = 0$ and to f_1, f_2 as $C \to \infty$. Deriving the following theorem, which interpolates between the $\ell = 0, 1$ and $1, 2$ cases of Theorem 4.4, is straightforward. (In it, c stands for $-C$.)

Theorem 4.6. *If* $\gcd(p, q) = 1$*, the following hold for* $\ell = 0, 1$ *and* $a \in \mathbb{C}$*, with*

$$G_0(A, B, C; y) := {}_2F_1\left(\begin{matrix} C, 1 \\ \frac{A+C}{B} + 1 \end{matrix} \middle| \frac{y-1}{y}\right),$$

$$G_1(A, B, C; y) := \left[\frac{y}{(1-B)y + B}\right] {}_3F_2\left(\begin{matrix} C, \frac{A+1}{B-1} + 1, 1 \\ \frac{A+C+1}{B} + 1, \frac{A+1}{B-1} \end{matrix} \middle| \frac{y-1}{y}\right).$$

(i) *Near* $\zeta = 0$ *on* $\mathbb{P}^1_\zeta$*, for arbitrary fixed* $g > 0$ *and for* $\kappa = 0, \dots, q-1$*,*

$$\begin{aligned} &\zeta^{\kappa/q}\, {}_{n+1}F_n\left(\begin{matrix} \frac{a}{n} + \frac{\kappa}{q}, \dots, \frac{a+(n-1)}{n} + \frac{\kappa}{q}; & \frac{a+c-\ell}{p} + \frac{\kappa}{q} \\ \frac{a-\ell}{p} + \frac{\kappa}{q}, \dots, \frac{a-\ell+(p-1)}{p} + \frac{\kappa}{q}; \frac{1}{q} + \frac{\kappa}{q}, \dots, \widehat{\frac{q-\kappa}{q} + \frac{\kappa}{q}}, \dots, \frac{q}{q} + \frac{\kappa}{q}; & \frac{a+c-\ell+p}{p} + \frac{\kappa}{q} \end{matrix} \middle| \zeta\right) \\ &= \frac{(-)^\kappa (1)_\kappa (a + n\kappa/q)_{-\ell-\kappa} (a + c - \ell + p\kappa/q)}{(a)_{-\ell} (a + c - \ell)} \left[\frac{(-)^q n^n}{p^p q^q}\right]^{\kappa/q} \\ &\quad \times q^{-1} \sum_{j=1}^{q} \left(\varepsilon_q^{-n}\right)^{(j-1)\kappa} \left[\varepsilon_q^{(j-1)} g^{-1/q} x_j\right]^{-qa} G_\ell\left(-a, -p/q, -c; g^{-1} x_j^q\right), \end{aligned}$$

in which the trinomial root x_j *near* $\varepsilon_q^{-(j-1)} g^{1/q}$*,* $j = 1, \dots, q$*, is defined by* (4.1), (4.2)*, with the coefficient* β *determined by* ζ *according to* (4.6).

(ii) *Near* $\zeta = \infty$ *on* $\mathbb{P}^1_\zeta$*, for arbitrary fixed* $\beta > 0$ *and for* $\kappa = 0, \dots, n-1$*,*

$$\begin{aligned} &\zeta^{-\kappa/n}\, {}_{n+1}F_n\left(\begin{matrix} \frac{-a+\ell+1}{p} + \frac{\kappa}{n}, \dots, \frac{-a+\ell+p}{p} + \frac{\kappa}{n}; \frac{a}{q} + \frac{\kappa}{n}, \dots, \frac{a+(q-1)}{q} + \frac{\kappa}{n}; & \frac{-a-c+\ell}{p} + \frac{\kappa}{n} \\ \frac{1}{n} + \frac{\kappa}{n}, \dots, \widehat{\frac{n-\kappa}{n} + \frac{\kappa}{n}}, \dots, \frac{n}{n} + \frac{\kappa}{n}; & \frac{-a-c+\ell+p}{p} + \frac{\kappa}{n} \end{matrix} \middle| \zeta^{-1}\right) \\ &= \frac{(-)^\kappa (1)_\kappa (a + q\kappa/n)_{-\ell-\kappa} (a + c - \ell - p\kappa/n)}{(a)_{-\ell} (a + c - \ell)} \left[\frac{(-)^q n^n}{p^p q^q}\right]^{-\kappa/n} \\ &\quad \times n^{-1} \sum_{j=1}^{n} \left(\varepsilon_n^{-q}\right)^{(j-1)\kappa} \left[\varepsilon_n^{(j-1)} \beta^{-1/n} x_j\right]^{-na} G_\ell\left(-a, p/n, -c; \beta^{-1} x_j^n\right), \end{aligned}$$

in which the trinomial root x_j *near* $\varepsilon_n^{-(j-1)} \beta^{1/n}$*,* $j = 1, \dots, n$*, is defined by* (4.1), (4.3)*, with the coefficient* g *determined by* ζ *according to* (4.6).

In (i) *and* (ii)*,* $a, c \in \mathbb{C}$ *are constrained so that no division by zero occurs, and so that no lower parameter of any* ${}_{n+1}F_n$*, for any* κ*, is a non-positive integer.*

The $\ell = 0, 1$ representations of ${}_{n+1}F_n$'s given in Theorem 4.6 reduce if $c = 0$ to the $\ell = 0, 1$ representations of ${}_nF_{n-1}$'s given in Theorem 4.4, by cancelling equal upper and lower parameters; and in the limit $c \to \infty$, to the $\ell = 1, 2$ ones.

It should be noted that unlike Theorem 4.4, which involves rational right-hand functions F_ℓ, $\ell \in \mathbb{Z}$, Theorem 4.6 is essentially non-algebraic: the right-hand functions G_ℓ, $\ell = 0, 1$, which depend on the additional parameter c, are hypergeometric and are generically transcendental.

5. Schwarz curves: Generalities

This section introduces what will be called Schwarz curves, which are projective algebraic curves that parametrize ordered k-tuples of roots (with multiplicity) of the general degree-n trinomial equation

$$x^n - gx^p - \beta = 0. \tag{5.1}$$

As in Section 4, it is assumed that $n = p + q$ for relatively prime integers $p, q \geqslant 1$, and that $g, \beta \in \mathbb{C}$ with at most one of g, β equaling zero. For each k, any ordered k-tuple of roots will trace out a Schwarz curve as

$$\zeta = (-)^q \frac{n^n}{p^p q^q} \frac{\beta^q}{g^n} \tag{5.2}$$

varies. Any Schwarz curve is a projective curve, since multiplying each root by any $c \neq 0$ will multiply g, β by c^q, c^n, but leave ζ invariant. The dependence on ζ, which is really a projectivized (and normalized) version of the discriminant of (5.1), will be interpreted as specifying a covering of $\mathbb{P}^1_\zeta$ by the Schwarz curve, the genus of which will be calculated. Determining the dependence of a point on the curve on the base point ζ will be of value when uniformizing the solutions of E_n's with imprimitive monodromy, since the formulas of Theorem 4.4(i), (ii) express many solutions in terms of q-tuples, resp. n-tuples of roots of (5.1).

Let $\sigma_l = \sigma_l(x_1, \dots, x_n)$ denote the l'th elementary symmetric polynomial in the roots $x_1, \dots, x_n$, so that $\sigma_0 = 1$, $\sigma_1 = \sum_{i=1}^n x_i$, etc. From (5.1),

$$\beta = (-)^{n-1}\sigma_n, \qquad g = (-)^{q-1}\sigma_q; \tag{5.3}$$

and $\sigma_l = 0$ for $l = 1, \dots, q-1$ and $q+1, \dots, n-1$. Parametrizing tuples of roots, given $\zeta \in \mathbb{P}^1$, means parametrizing them given the ratio $[{\sigma_n}^q : {\sigma_q}^n]$.

Definition 5.1. The top Schwarz curve $\mathcal{C}^{(n)}_{p,q} \subset \mathbb{P}^{n-1}$ is the algebraic curve comprising all points $[x_1 : \ldots : x_n] \in \mathbb{P}^{n-1}$ such that $x_1, \dots, x_n$ satisfy the $n-2$ homogeneous equations

$$\sigma_l(x_1, \dots, x_n) = 0, \quad l = 1, \dots, q-1 \text{ and } q+1, \dots, n-1. \tag{5.4}$$

That is, $\mathcal{C}^{(n)}_{p,q}$ comprises all $[x_1 : \ldots : x_n] \in \mathbb{P}^{n-1}$ such that $x_1, \dots, x_n$ are the n roots (with multiplicity) of some trinomial equation of the form (5.1). The curve $\mathcal{C}^{(n)}_{p,q}$ is stable under the action of the symmetric group $\mathfrak{S}_n$ on $[x_1 : \ldots : x_n]$. An associated degree-$n!$ covering map $\pi^{(n)}_{p,q} : \mathcal{C}^{(n)}_{p,q} \to \mathbb{P}^1_\zeta$ is defined by

$$\zeta = (-)^n \frac{n^n}{p^p q^q} \frac{{\sigma_n}^q}{{\sigma_q}^n}. \tag{5.5}$$

Proposition 5.2. *(See [21], Cor. 4.7.) The curve $\mathcal{C}^{(n)}_{p,q}$ is irreducible.*

Definition 5.3. For each k with $n > k \geqslant 2$, the subsidiary Schwarz curve $\mathcal{C}^{(k)}_{p,q} \subset \mathbb{P}^{k-1}$, also irreducible, is the image of $\mathcal{C}^{(n)}_{p,q}$ under the map $[x_1 : \ldots : x_n] \mapsto [x_1 : \ldots : x_k]$, which on $\mathcal{C}^{(n)}_{p,q}$ is $(n-k)!$-to-1. A more concrete definition of $\mathcal{C}^{(k)}_{p,q}$ is the following. It is the closure in $\mathbb{P}^{k-1}$ of the set of all points $[x_1 : \ldots : x_k] \in \mathbb{P}^{k-1}$ such that $x_1, \dots, x_k$ are k of the n roots (with multiplicity) of some trinomial equation of the form (5.1). The curve $\mathcal{C}^{(k)}_{p,q}$ is stable under the action of $\mathfrak{S}_k$ on $[x_1 : \ldots : x_k]$.

For each k with $n \geqslant k > 2$, let a projection $\phi^{(k)}_{p,q} : \mathcal{C}^{(k)}_{p,q} \to \mathcal{C}^{(k-1)}_{p,q}$ be defined by $\phi^{(k)}_{p,q}([x_1 : \ldots : x_k]) = [x_1 : \ldots : x_{k-1}]$, so that $\phi^{(k+1)}_{p,q} \circ \cdots \circ \phi^{(n)}_{p,q}$ is the projection $\mathcal{C}^{(n)}_{p,q} \to \mathcal{C}^{(k)}_{p,q}$. Since $\mathcal{C}^{(n)}_{p,q} \to \mathcal{C}^{(k)}_{p,q}$ is $(n-k)!$-to-1 for each k, it follows that for each k, $\phi^{(k)}_{p,q}$ is $(n-k+1)$-to-1, i.e., is a degree-$(n-k+1)$ map.

Remark 5.3.1. Each subsidiary curve $\mathcal{C}^{(k)}_{p,q} \subset \mathbb{P}^{k-1}$, $n > k \geqslant 2$, is the solution set of a system of $k-2$ homogeneous equations in $x_1, \ldots, x_k$, obtained by eliminating $x_{k+1}, \ldots, x_n$ from the system (5.4). The details of this will be given shortly.

Remark 5.3.2. As will be explained in Section 6.1, defining $\mathcal{C}^{(k)}_{p,q}$ as a closure in $\mathbb{P}^{k-1}$ appends at most a finite number of points to it. Specifically, if $k \leqslant p$, it appends each point $[x_1 : \ldots : x_k] \in \mathbb{P}^{k-1}$ in which $x_1, \ldots, x_k$ are distinct p'th roots of unity. There are $(p-k+1)_{k-1} = (p-k+1)_k/p$ such points in $\mathbb{P}^{k-1}$, none of which comes directly (via a k-tuple of roots) from a trinomial equation of the form (5.1). Rather, each is a limit in $\mathbb{P}^{k-1}$ of points that do.

Remark 5.3.3. The projections $\mathcal{C}^{(n)}_{p,q} \to \mathcal{C}^{(k)}_{p,q}$ and $\phi^{(k)}_{p,q}$ are really only *partial* maps, being undefined at a finite number of singular points, as is typical of maps between algebraic curves. (The symbol $\dashrightarrow$ could be used instead of $\to$.) Specifically, if $x_1 = \cdots = x_{k-1} = 0$ then $\phi^{(k)}_{p,q}([x_1 : \ldots : x_k])$ is undefined. The problems with zero tuples, associated to trinomials with $\beta = 0$, will be dealt with in Section 6, where each curve $\mathcal{C}^{(k)}_{p,q}$ will be lifted to a desingularized curve $\tilde{\mathcal{C}}^{(k)}_{p,q}$.

The reason for the term 'Schwarz curve' is this. The ratios of any n independent solutions $y_1, \ldots, y_n$ of an order-n differential equation on $\mathbb{P}^1_\zeta$ define a (multivalued) *Schwarz map* from $\mathbb{P}^1_\zeta$ to $\mathbb{P}^{n-1}$. Its image is a curve in $\mathbb{P}^{n-1}$, and in some cases the inverse map from the image is single-valued, i.e., supplies a covering of $\mathbb{P}^1_\zeta$ by the curve. Studying Schwarz maps is a standard way of computing the monodromy groups of differential equations [20,21], and goes back to Schwarz's classical work on E_2, the Gauss hypergeometric equation. In the present paper, solutions of E_n's with imprimitive monodromy have been expressed in terms of tuples of solutions of trinomial equations, which are algebraic in ζ; so a purely algebraic use of the Schwarz map and curve concepts seems warranted. The top Schwarz curve $\mathcal{C}^{(n)}_{p,q}$ was introduced and used by Kato and Noumi [21], though not under that name; the subsidiary Schwarz curves seem not to have been treated or exploited before.

It is evident that as defined, $\mathcal{C}^{(n)}_{p,q} \cong \mathcal{C}^{(n-1)}_{p,q}$ and $\mathcal{C}^{(2)}_{p,q} \cong \mathbb{P}^1_t$, where birational equivalence is meant. Here, t is any homogeneous degree-1 rational function of x_1, x_2. Henceforth the choice $t = (x_1 + x_2)/(x_1 - x_2)$ will be made, so $[x_1 : x_2] = [t+1 : t-1]$ will be a uniformization of $\mathcal{C}^{(2)}_{p,q}$ by the rational parameter t. It will also prove useful to define $\mathcal{C}^{(1)}_{p,q} := \mathbb{P}^1$. This will make it possible to define $\phi^{(2)}_{p,q} : \mathcal{C}^{(2)}_{p,q} \to \mathcal{C}^{(1)}_{p,q}$ and $\phi^{(1)}_{p,q} : \mathcal{C}^{(2)}_{p,q} \to \mathbb{P}^1_\zeta$ in such a way that

$$\pi^{(n)}_{p,q} = \phi^{(1)}_{p,q} \circ \cdots \circ \phi^{(n)}_{p,q}, \tag{5.6}$$

in the sense of being an equality on a cofinite domain. (See (5.18) and (5.21) below.) For each k with $n > k \geqslant 1$,

$$\pi_{p,q}^{(k)} = \phi_{p,q}^{(1)} \circ \cdots \circ \phi_{p,q}^{(k)} \tag{5.7}$$

will then define a subsidiary (partial) map $\pi_{p,q}^{(k)} : \mathcal{C}_{p,q}^{(k)} \to \mathbb{P}_\zeta^1$ of degree $(n-k+1)_k$.

One can derive a system of polynomial equations for each $\mathcal{C}_{p,q}^{(k)}$, $n > k > 2$, in terms of $x_1, \ldots, x_k$, by using resultants to eliminate $x_{k+1}, \ldots, x_n$. But one can also eliminate them by hand in the following way, which will incidentally indicate how best to define the curve $\mathcal{C}_{p,q}^{(1)}$. For any k, $0 < k < n$, let $\bar\sigma_m, \hat\sigma_m$ denote the m'th elementary symmetric polynomial in $x_1, \ldots, x_k$, resp. $x_{k+1}, \ldots, x_n$. Then

$$\sigma_l = \sum_{m=0}^{l} \bar\sigma_m \hat\sigma_{l-m}, \quad 0 \leqslant l \leqslant n, \tag{5.8}$$

it being understood that $\bar\sigma_m = 0$ if $m \notin \{0, \ldots, k\}$, and similarly, that $\hat\sigma_m = 0$ if $m \notin \{0, \ldots, n-k\}$, with $\bar\sigma_0 = \hat\sigma_0 = 1$. The defining equations of the top curve $\mathcal{C}_{p,q}^{(n)}$, by Definition 5.1, are

$$\begin{cases} 1 = \sigma_0 = \displaystyle\sum_{m=0}^{n} \bar\sigma_m \hat\sigma_{0-m}; \\ 0 = \sigma_1 = \displaystyle\sum_{m=0}^{n} \bar\sigma_m \hat\sigma_{1-m}, \qquad \ldots, \quad 0 = \sigma_{q-1} = \displaystyle\sum_{m=0}^{n} \bar\sigma_m \hat\sigma_{(q-1)-m}; \\ \sigma_q = \displaystyle\sum_{m=0}^{n} \bar\sigma_m \hat\sigma_{q-m}; \\ 0 = \sigma_{q+1} = \displaystyle\sum_{m=0}^{n} \bar\sigma_m \hat\sigma_{(q+1)-m}, \quad \ldots, \quad 0 = \sigma_{n-1} = \displaystyle\sum_{m=0}^{n} \bar\sigma_m \hat\sigma_{(n-1)-m}; \\ \sigma_n = \displaystyle\sum_{m=0}^{n} \bar\sigma_m \hat\sigma_{n-m}. \end{cases} \tag{5.9}$$

Lemma 5.4. *$\{\hat\sigma_m\}_{m=0}^{n-k}$ can be expressed in terms of $x_1, \ldots, x_k$ (and σ_n) by*

$$\hat\sigma_m = \begin{cases} (-)^m \sum_{\substack{m_1+\cdots+m_k=m \\ (\forall j)\ 0 \leqslant m_j \leqslant m}} x_1^{m_1} \cdots x_k^{m_k}, \qquad m = 0, \ldots, \min(q-1, n-k); \\ \frac{(-)^{n-k-m}\sigma_n}{(x_1\cdots x_k)^{n-k-m+1}} \sum_{\substack{m_1+\cdots+m_k=(k-1)(n-k-m) \\ (\forall j)\ 0 \leqslant m_j \leqslant n-k-m}} x_1^{m_1} \cdots x_k^{m_k}, \\ \qquad\qquad m = \max(q-k+1, 0), \ldots, n-k. \end{cases}$$

Note that $0 \leqslant m \leqslant \min(q-1, n-k)$ and $\max(q-k+1, 0) \leqslant m \leqslant n-k$, the m-ranges of validity of these two formulas, may overlap.

Proof. The first formula is proved by induction. The idea is that one solves the zeroth equation in (5.9) for $\hat\sigma_0$, then the first equation for $\hat\sigma_1$, etc.; stopping with the $\min(q-1, n-k)$'th equation, since the next one may involve σ_q, which is not known. But since the l'th equation, for $1 \leqslant l \leqslant \min(q-1, n-k)$, says that

$$\hat\sigma_l = -(\bar\sigma_1 \hat\sigma_{l-1} + \cdots + \bar\sigma_k \hat\sigma_{l-k}), \tag{5.10}$$

the inductive step amounts to verifying that

$$\begin{aligned} P(m) &= \bar\sigma_1 P(m-1) - \bar\sigma_2 P(m-2) + \cdots + (-)^{k-1}\bar\sigma_k P(m-k), \\ P(m) &:= \sum_{\substack{m_1+\cdots+m_k=m \\ (\forall j)\ 0\leqslant m_j\leqslant m}} x_1^{m_1}\cdots x_k^{m_k}. \end{aligned} \tag{5.11}$$

This is a well-known identity.

The second formula in the lemma is dual to the first and is proved by a similar induction, ‘downward.’ One solves the n’th equation in (5.9) for $\hat\sigma_{n-k}$, then the $(n-1)$’st equation for $\hat\sigma_{n-k-1}$, etc.; stopping with the $\max(q-k+1,0)$’th equation, since the next one may involve σ_q, which is not known. As with the first formula, the inductive step reduces to a verification of Eq. (5.11). □

The formulas of the lemma are accompanied by *constraints*. First, there are the implicit constraints coming from the equivalence between the two formulas for $\hat\sigma_m$, for each m in the doubly covered range

$$\max(q-k+1,0) \leqslant m \leqslant \min(q-1,n-k). \tag{5.12}$$

Second, there are the equations $\sigma_l = 0$ in (5.9), for each l in the ranges

$$\min(n-k+1,q) \leqslant l \leqslant q-1, \qquad q+1 \leqslant l \leqslant \max(k-1,q), \tag{5.13}$$

which were not exploited in the proof of the lemma. In all, there are $k-1$ constraint equations. They serve (if $k \geqslant 2$) to do two things: (i) they yield an expression for σ_n as a rational function of $\hat\sigma_1,\dots,\hat\sigma_{n-k}$, and hence as a rational function of $x_1,\dots,x_k$, homogeneous of degree n; and, (ii) they impose $k-2$ homogeneous conditions on $x_1,\dots,x_k$, which are the desired defining equations of the curve $\mathcal{C}^{(k)}_{p,q} \subset \mathbb{P}^{k-1}$.

The formula for σ_q in (5.9), as yet unused, yields an expression for σ_q as a rational function of $x_1,\dots,x_k$, homogeneous of degree q. Therefore, $\zeta \propto {\sigma_n}^q/{\sigma_q}^n$ is homogeneous of degree zero and is in the function field of $\mathcal{C}^{(k)}_{p,q}$, and

$$\mathbb{P}^{k-1} \supset \mathcal{C}^{(k)}_{p,q} \ni [x_1:\ldots:x_k] \mapsto \zeta \in \mathbb{P}^1 \tag{5.14}$$

yields a formula for the degree-$(n-k+1)_k$ (partial) map $\pi^{(k)}_{p,q}\colon \mathcal{C}^{(k)}_{p,q} \to \mathbb{P}^1_\zeta$, i.e., an explicit formula $\zeta = \zeta(x_1,\dots,x_k)$.

The just-sketched elimination procedure can be carried out by hand for the cases $k=2$ (see Lemma 5.5 below) and $k=3$ (see Theorem 6.5). When k increases further, carrying it out by hand becomes increasingly difficult.

Lemma 5.5. *Applying the elimination procedure to the case $k=2$ yields the formulas*

$$\sigma_n = (-)^n (x_1x_2)^p \frac{x_1^q - x_2^q}{x_1^p - x_2^p}, \qquad \sigma_q = (-)^{q-1} \frac{x_1^n - x_2^n}{x_1^p - x_2^p},$$
$$\zeta = (-)^n \frac{n^n}{p^p q^q}\frac{\sigma_n^q}{\sigma_q^n} = \frac{n^n}{p^p q^q}(x_1x_2)^{pq}\frac{(x_1^p - x_2^p)^p (x_1^q - x_2^q)^q}{(x_1^n - x_2^n)^n},$$

the last of which defines a degree-$[n(n-1)]$ covering map $\pi^{(2)}_{p,q}\colon \mathcal{C}^{(2)}_{p,q} \to \mathbb{P}^1_\zeta$.

Proof. By elementary algebra. □

Remark 5.5.1. If one simply defines $\pi^{(k)}_{p,q}\colon \mathcal{C}^{(k)}_{p,q} \to \mathbb{P}^1_\zeta$ as the composition of the two rational maps $[x_1 : \ldots : x_k] \mapsto [x_1 : x_2]$ and $\pi^{(2)}_{p,q}$, then it will follow immediately from Lemma 5.5 that ζ is in the function field of $\mathcal{C}^{(k)}_{p,q}$, without actually employing the just-sketched elimination procedure to derive an explicit rational formula $\zeta = \zeta(x_1, \ldots, x_k)$.

The case $k = 1$ obviously requires special treatment, since one cannot express σ_n, σ_q in terms of x_1 alone. If $k = 1$, the system (5.9) reduces to

$$\begin{cases} 0 = \sigma_1 = x_1 + \hat\sigma_1, & \ldots, \quad 0 = \sigma_{q-1} = x_1\hat\sigma_{q-2} + \hat\sigma_{q-1}; \\ \sigma_q = x_1\hat\sigma_{q-1} + \hat\sigma_q; & \\ 0 = \sigma_{q+1} = x_1\hat\sigma_q + \hat\sigma_{q+1}, & \ldots, \quad 0 = \sigma_{n-1} = x_1\hat\sigma_{n-2} + \hat\sigma_{n-1}; \\ \sigma_n = x_1\hat\sigma_{n-1}, & \end{cases} \tag{5.15}$$

the solution of which can be written as

$$\sigma_n = (-)^{n-q-1} x_1^{n-q}\hat\sigma_q, \qquad \sigma_q = \hat\sigma_q + (-)^{q-1} x_1^q. \tag{5.16}$$

This suggests focusing on s, the element of the function field of $\mathcal{C}^{(n)}_{p,q}$ defined by

$$s := (-)^{n-1}\sigma_n / x_1^n = \beta / x_1^n, \qquad 1 - s := (-)^{q-1}\sigma_q / x_1^q = g / x_1^q \tag{5.17}$$

(the two definitions being equivalent), and defining the special $k = 1$ Schwarz curve $\mathcal{C}^{(1)}_{p,q}$ to be $\mathbb{P}^1_s$.

There is then a degree-$(n-1)$ map $\phi^{(2)}_{p,q}\colon \mathcal{C}^{(2)}_{p,q} \to \mathcal{C}^{(1)}_{p,q}$ given by the map $t = (x_1 + x_2)/(x_1 - x_2) \mapsto s$, since both σ_n, σ_q are rational functions of x_1, x_2. By comparing the expressions for σ_n, σ_q in Lemma 5.5 with those for $\sigma_n/x_1^n, \sigma_q/x_1^q$ in (5.17), one finds that this degree-$(n-1)$ map $\phi^{(2)}_{p,q}$ is given by

$$\begin{aligned} s &= -\frac{x_2^p}{x_1^q}\frac{x_1^q - x_2^q}{x_1^p - x_2^p} = 1 - \frac{1}{x_1^q}\frac{x_1^n - x_2^n}{x_1^p - x_2^p} \\ &= -\frac{(t-1)^p}{(t+1)^q}\frac{(t+1)^q - (t-1)^q}{(t+1)^p - (t-1)^p} = 1 - \frac{1}{(t+1)^q}\frac{(t+1)^n - (t-1)^n}{(t+1)^p - (t-1)^p}. \end{aligned} \tag{5.18}$$

The degree-$(n-1)!$ composition $\phi^{(2)}_{p,q} \circ \cdots \circ \phi^{(n)}_{p,q}\colon \mathcal{C}^{(n)}_{p,q} \to \mathcal{C}^{(1)}_{p,q} \cong \mathbb{P}^1_s$ is made explicit by the ratios $z_j = x_j/x_1$, $j = 1, \ldots, n$, being the solutions z of

$$z^n - (1-s)z^p - s = 0, \tag{5.19}$$

so that $z_2, \ldots, z_n$ are the inverse images (with multiplicity) of s under the corresponding degree-$(n-1)$ rational function $s = s(z)$, which is

$$s = -\frac{z^p(1 - z^q)}{1 - z^p} = -\frac{z^p(1 + z + \cdots + z^{q-1})}{1 + z + \cdots + z^{p-1}}. \tag{5.20}$$

There is a final degree-n map $\phi^{(1)}_{p,q}\colon \mathcal{C}^{(1)}_{p,q} \cong \mathbb{P}^1_s \to \mathbb{P}^1_\zeta$, given by

$$\zeta = (-)^n \frac{n^n}{p^p q^q}\frac{{\sigma_n}^q}{{\sigma_q}^n} = (-)^q \frac{n^n}{p^p q^q}\frac{s^q}{(1-s)^n}. \tag{5.21}$$

It completes the sequence of maps leading from the top curve $\mathcal{C}^{(n)}_{p,q}$ down to $\mathbb{P}^1_\zeta$.

Pulling everything together yields the following theorem, which summarizes the results of this section.

Theorem 5.6. *For any pair of relatively prime integers $p, q \geqslant 1$ with $n = p + q$, there is a sequence of algebraic curves and (partial) maps*

$$\mathcal{C}^{(n)}_{p,q} \xrightarrow{\phi^{(n)}_{p,q}} \mathcal{C}^{(n-1)}_{p,q} \xrightarrow{\phi^{(n-1)}_{p,q}} \cdots \xrightarrow{\phi^{(3)}_{p,q}} \mathcal{C}^{(2)}_{p,q} \xrightarrow{\phi^{(2)}_{p,q}} \mathcal{C}^{(1)}_{p,q} \xrightarrow{\phi^{(1)}_{p,q}} \mathbb{P}^1_\zeta, \tag{5.22}$$

in which $\deg \phi^{(k)}_{p,q} = n - k + 1$. *For* $k = n, n-1, \ldots, 2$, *the curve* $\mathcal{C}^{(k)}_{p,q} \subset \mathbb{P}^{k-1}$, *prior to closure, comprises all* $[x_1 : \ldots : x_k]$ *in which* $x_1, \ldots, x_k$ *is a nonzero ordered* k*-tuple of roots (with multiplicity) of some trinomial equation of the form (5.1). Each partial map* $\phi^{(k)}_{p,q}$, $n \geqslant k > 2$, *takes* $[x_1 : \ldots : x_k]$, *where at least one of* $x_1, \ldots, x_{k-1}$ *is nonzero, to* $[x_1 : \ldots : x_{k-1}]$. *One writes* $\pi^{(k)}_{p,q}$ *for* $\phi^{(1)}_{p,q} \circ \cdots \circ \phi^{(k)}_{p,q}$, *which is of degree* $(n-k+1)_k$. *Any* $[x_1 : \ldots : x_k]$ *in the domain of* $\pi^{(k)}_{p,q}$ *is taken by it to* ζ, *computed from the associated trinomial equation by Eq. (5.2).*

The final two curves $\mathcal{C}^{(k)}_{p,q}$, $k = 2, 1$, *are of genus zero; i.e.,* $\mathcal{C}^{(2)}_{p,q} \cong \mathbb{P}^1_t$ *and* $\mathcal{C}^{(1)}_{p,q} \cong \mathbb{P}^1_s$, *where* $t := (x_1 + x_2)/(x_1 - x_2)$ *and* s *are rational parameters. The final two maps* $\phi^{(2)}_{p,q}, \phi^{(1)}_{p,q}$ *are given by*

$$\begin{aligned} s &= \phi^{(2)}_{p,q}(t) = -\frac{(t-1)^p}{(t+1)^q} \frac{(t+1)^q - (t-1)^q}{(t+1)^p - (t-1)^p}, \\ \zeta &= \phi^{(1)}_{p,q}(s) = (-)^q \frac{n^n}{p^p q^q} \frac{s^q}{(1-s)^n}, \end{aligned}$$

and their composition $\pi^{(2)}_{p,q} = \phi^{(1)}_{p,q} \circ \phi^{(2)}_{p,q}$ *by*

$$\begin{aligned} \zeta &= \pi^{(2)}_{p,q}(t) = \phi^{(1)}_{p,q}\bigl(\phi^{(2)}_{p,q}(t)\bigr) \\ &= \frac{n^n}{p^p q^q} \bigl(t^2 - 1\bigr)^{pq} \frac{[(t+1)^p - (t-1)^p]^p [(t+1)^q - (t-1)^q]^q}{[(t+1)^n - (t-1)^n]^n}. \end{aligned}$$

6. Schwarz curves: Ramifications and genera

The Schwarz curves $\mathcal{C}^{(k)}_{p,q}$ introduced in Section 5, in particular $\mathcal{C}^{(q)}_{p,q}, \mathcal{C}^{(n)}_{p,q}$, will be used in Section 7 to parametrize the solutions of hypergeometric differential equations (E_n's) with imprimitive monodromy. The explicit formulas of Theorem 5.6 will be especially useful. They exploit the parametrizations of $\mathcal{C}^{(2)}_{p,q}, \mathcal{C}^{(1)}_{p,q}$ by respective rational (i.e., $\mathbb{P}^1$-valued) parameters t, s, which exist because $\mathcal{C}^{(2)}_{p,q}, \mathcal{C}^{(1)}_{p,q}$ are of genus zero.

The question arises whether 'higher' $\mathcal{C}^{(k)}_{p,q}$, such as the family of projective plane curves $\mathcal{C}^{(3)}_{p,q} \subset \mathbb{P}^2$, are ever of genus zero. In principle, the genus of each $\mathcal{C}^{(k)}_{p,q} \subset \mathbb{P}^{k-1}$, $n \geqslant k > 2$, can be calculated from the Hurwitz formula applied to the map $\pi^{(k)}_{p,q} : \mathcal{C}^{(k)}_{p,q} \to \mathbb{P}^1_\zeta$. But some care is needed, since in general, $\mathcal{C}^{(k)}_{p,q}$ is a *singular* projective curve, and not being smooth, is not a Riemann surface. By resolving singularities, one must first construct a smooth desingularization $\tilde{\mathcal{C}}^{(k)}_{p,q} \to \mathcal{C}^{(k)}_{p,q}$. (Up to birational equivalence, $\tilde{\mathcal{C}}^{(k)}_{p,q}$ is unique.) The Hurwitz formula can then be applied to the lifted map $\tilde{\pi}^{(k)}_{p,q} : \tilde{\mathcal{C}}^{(k)}_{p,q} \to \mathbb{P}^1_\zeta$, which is a degree-$(n-k+1)_k$ holomorphic map of Riemann surfaces.

Table 2
Branching data for the ordinary multiple points on the algebraic curve $\mathcal{C}_{p,q}^{(k)} \subset \mathbb{P}^{k-1}$, $k \geqslant 2$. They are partitioned into types indexed by ν, with $\max(0, k-p) \leqslant \nu \leqslant \min(q,k)$. If nonzero, ν counts the j, $1 \leqslant j \leqslant k$, for which $x_j \neq 0$. There are $N_{p,q}^P(k,\nu)$ points of type ν.

	$N_{p,q}^P(k,\nu)$	$N_{p,q}^T(k,\nu)$	$M_{p,q}(k,\nu)$
$\nu = 0$	$(p-k+1)_{k-1}$	1	p
$0 < \nu < k$	$\binom{k}{\nu}(q-\nu+1)_{\nu-1}$	$(p-k+\nu+1)_{k-\nu-1}$	pq
$\nu = k$	$(q-k+1)_{k-1}$	1	q

Section 6.1 counts the singular points of $\mathcal{C}_{p,q}^{(k)}$ and determines their multiplicities. (See Table 2.) Section 6.2 determines the ramification structure of $\tilde{\pi}_{p,q}^{(k)}$. Section 6.3 explicitly parametrizes several plane curves $\mathcal{C}_{p,q}^{(3)} \subset \mathbb{P}^2$, which according to the resulting formula for the genus of $\mathcal{C}_{p,q}^{(k)}$ (Theorem 6.3) are of genus zero.

6.1. Desingularization

Recall that $\mathcal{C}_{p,q}^{(k)} \subset \mathbb{P}^{k-1}$, $n \geqslant k \geqslant 2$, is an algebraic curve including each of the points $[x_1 : \ldots : x_k] \in \mathbb{P}^{k-1}$ in which $x_1, \ldots, x_k$ are k of the n roots (with multiplicity) of some trinomial equation

$$x^n - gx^p - \beta = 0 \tag{6.1}$$

(with $n = p+q$, $\gcd(p,q) = 1$, and at most one of g, β equaling zero). Cremona inversion in $\mathbb{P}^{k-1}$, i.e., the substitution $x_j = 1/x'_j$, $j = 1, \ldots, k$, induces an equivalence $\mathcal{C}_{p,q}^{(k)} \cong \mathcal{C}_{q,p}^{(k)}$. It was noted (see Remark 5.3.2) that if $k \leqslant p$, $\mathcal{C}_{p,q}^{(k)}$ must really be defined as a closure in $\mathbb{P}^{k-1}$, and the taking of the closure appends a finite number of limit points, which do not come directly from any trinomial equation of the form (6.1). This will be elucidated below.

It is a standard fact (see Section 6.2) that if $\beta \neq 0$, at most two roots of (6.1) can coincide; and coincidence occurs if and only if

$$\zeta := (-)^q \frac{n^n}{p^p q^q} \frac{\beta^q}{g^n} \tag{6.2}$$

equals unity. Also, if a point $[x_1 : \ldots : x_k] \in \mathcal{C}_{p,q}^{(k)}$ comes directly from an equation of the form (6.1), the coefficients g, β are determined by the point, up to the scaling $(g,\beta) \mapsto (c^q g, c^n \beta)$ for some $c \neq 0$. (To see this, consider the system $x_1^n - gx_1^p - \beta = 0$, $x_2^n - gx_2^p - \beta = 0$, where by the preceding, $x_1 \neq x_2$ can be assumed. Since $\gcd(n,p) = 1$, if this system has a solution (g,β), it has a unique solution; but of course (x_1, x_2) can be multiplied by c.) The formula (6.2), which is unaffected by scaling, defines the map $\pi_{p,q}^{(k)} : \mathcal{C}_{p,q}^{(k)} \to \mathbb{P}_\zeta^1$.

The points on $\mathcal{C}_{p,q}^{(k)}$ of special interest here are those taken by $\pi_{p,q}^{(k)}$ to $\zeta = 0, 1, \infty$. Among points coming from trinomials, the "$\zeta = \infty$" points are ones with $g = 0$, and the "$\zeta = 0$" points are ones with $\beta = 0$. (It will be seen that each of the limit points present if $k \leqslant p$ is also a $\zeta = 0$ point, by continuity.) A 'generic' point on $\mathcal{C}_{p,q}^{(k)}$ is one that is taken by $\pi_{p,q}^{(k)}$ to some $\zeta \notin \{0, 1, \infty\}$.

Which points of $\mathcal{C}_{p,q}^{(k)} \subset \mathbb{P}^{k-1}$ are non-smooth will now be determined. First, consider any generic point $[a] = [a_1 : \ldots : a_k] \in \mathcal{C}_{p,q}^{(k)}$. That is, consider $a_1, \ldots, a_k$, distinct and nonzero, which are k of the roots (with multiplicity) of a unique trinomial $x^n - g_0 x^p - \beta_0$ with

$g_0, \beta_0 \neq 0$. Near $[a]$, treat β as a local parameter on $\mathcal{C}_{p,q}^{(k)}$. That is, for β near β_0, let $[a(\beta)] = [a_1(\beta) : \ldots : a_k(\beta)] \in \mathcal{C}_{p,q}^{(k)}$ come from the roots of $x^n - g_0 x^p - \beta$. For each j,

$$\frac{\mathrm{d}a_j}{\mathrm{d}\beta} = \left[\frac{\mathrm{d}}{\mathrm{d}x} \big(x^n - g_0 x^p\big)\bigg|_{x=a_j} \right]^{-1} = \big(n a_j^{n-1} - p g_0 a_j^{p-1}\big)^{-1}, \tag{6.3}$$

which (by distinctness of roots) is the reciprocal of a nonzero quantity, for β near β_0. Without loss of generality, assume $[a]$ is in the affine chart on $\mathbb{P}^{k-1}$ with coordinate $(z_1, \ldots, z_{k-1})$, where $z_j = x_j / x_k$. To show that $\mathcal{C}_{p,q}^{(k)}$ is smooth at $[a]$, it suffices to show that $(\mathrm{d}/\mathrm{d}\beta)(z_1, \ldots, z_{k-1})|_{\beta=\beta_0} \neq (0, \ldots, 0)$, with $z_j = a_j / a_k$. Suppose not; in fact, suppose merely that

$$\frac{\mathrm{d}}{\mathrm{d}\beta}(a_j / a_k) = a_k^{-2} \left(a_k \frac{\mathrm{d}a_j}{\mathrm{d}\beta} - a_j \frac{\mathrm{d}a_k}{\mathrm{d}\beta} \right) \tag{6.4}$$

is zero at $\beta = \beta_0$ for j equal to some j^*, $1 \leqslant j^* \leqslant k-1$. This is equivalent to $a_j^{-1} (\mathrm{d}/\mathrm{d}\beta) a_j |_{\beta=\beta_0}$ taking equal values when $j = j^*, k$, or (by (6.3)) $n a_j^n - p g_0 a_j^p$ taking equal values. But, equal values are also taken when $j = j^*, k$ by $a_j^n - g_0 a_j^p = \beta_0$. Hence $a_{j_*}^n = a_k^n$ and $a_{j_*}^p = a_k^p$; which since $\gcd(n, p) = 1$, contradicts $a_{j_*} \neq a_k$.

Points $[a] \in \mathcal{C}_{p,q}^{(k)}$ coming from trinomials with $g = g_0 = 0$ (i.e., "$\zeta = \infty$" points) can similarly be shown to be smooth, by using g rather than β as a local parameter. Any point $[a]$ coming from $x^n - \beta_0 = 0$ has $a_j = \beta_0^{1/n} \varepsilon_n^{\pi_j}$, where $\pi_1, \ldots, \pi_k$ are distinct elements of $\{0, 1, \ldots, n-1\}$. For g near $g_0 = 0$, let $[a(g)] = [a_1(g) : \ldots : a_k(g)] \in \mathcal{C}_{p,q}^{(k)}$ come from the roots of $x^n - g x^p - \beta_0$, and use the affine coordinate $(z_1, \ldots, z_{k-1})$ on $\mathbb{P}^{k-1}$, where $z_j = x_j / x_k$. Then

$$\frac{\mathrm{d}a_j}{\mathrm{d}g}\bigg|_{g=g_0=0} = \left[\frac{\mathrm{d}}{\mathrm{d}x} \big(x^q - \beta_0 x^{-p}\big)\bigg|_{x=a_j} \right]^{-1} = (n\beta_0)^{-1} a_j^{p+1}, \tag{6.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}g}(a_j / a_k)\bigg|_{g=g_0=0} = a_k^{-2} \left(a_k \frac{\mathrm{d}a_j}{\mathrm{d}g} - a_j \frac{\mathrm{d}a_k}{\mathrm{d}g} \right)\bigg|_{g=g_0=0} = (n\beta_0)^{-1} (a_j / a_k) \big(a_j^p - a_k^p\big). \tag{6.6}$$

The latter is zero for $j = 1, \ldots, k-1$ only if $a_1^p, \ldots, a_k^p$ are equal, but they are not. So, $(\mathrm{d}/\mathrm{d}g)(z_1, \ldots, z_{k-1})|_{g=g_0=0} \neq (0, \ldots, 0)$, and $\mathcal{C}_{p,q}^{(k)}$ is smooth at $[a]$.

The curve $\mathcal{C}_{p,q}^{(k)} \subset \mathbb{P}^{k-1}$ can also be shown to be smooth at any "$\zeta = 1$" point, i.e., any point coming from a trinomial with $\beta \neq 0$ that has a pair of coincident roots. Let $[a]$ be a point on $\mathcal{C}_{p,q}^{(k)}$ with $a_{j_1} = a_{j_2}$, coming from a trinomial $x^n - g_0 x^p - \beta_0$ with $\beta_0 \neq 0$. On a neighborhood of $[a]$, a parameter for the curve can be chosen to be $t = (\beta - \beta_0)^{1/2}$, so that $a_{j_1}(t), a_{j_2}(t)$ are $a_{j_1}(0) \pm Ct + O(t^2)$ for some $C \neq 0$, and $a_j(t) = a_j(0) + C_j t^2 + O(t^3)$ for each $j \neq j_1, j_2$. The derivative with respect to t of any affine coordinate $(z_j)_{j \neq j^*} = (x_j / x_{j^*})_{j \neq j^*}$ on $\mathbb{P}^{k-1}$, where j^* is chosen so that $j^* \neq j_1, j_2$, is defined and nonzero at $t = 0$. Hence, $\mathcal{C}_{p,q}^{(k)}$ is smooth at $[a]$.

In summary, a point in $\mathcal{C}_{p,q}^{(k)} \subset \mathbb{P}^{k-1}$ coming from a trinomial $x^n - g x^p - \beta$ can be non-smooth only if $g \neq 0$ and $\beta = 0$. That is, it must be one of the "$\zeta = 0$" points, which may display high-order coincidences of roots. Limit points on $\mathcal{C}_{p,q}^{(k)}$ coming not directly but indirectly from trinomials, via the taking of the closure in $\mathbb{P}^{k-1}$ (and in fact, from a $\beta \to 0$ limit), will be considered later.

For ease of understanding, the singular points of $\mathcal{C}_{p,q}^{(k)}$ will first be determined in the 'top' case $k = n$, in which the closure in $\mathbb{P}^{k-1}$ need not be taken. There are $n!/p!q$ points $\mathfrak{p} = [x_1 : \ldots : x_n]$

in $\mathcal{C}_{p,q}^{(n)} \subset \mathbb{P}^{n-1}$ that come from trinomials $x^n - gx^p - \beta$ with $g \neq 0$ and $\beta = 0$. One of them (cf. (4.2)) is $\mathfrak{p}_0$, defined by

$$x_j = \begin{cases} \varepsilon_q^{-(j-1)} g^{1/q}, & j = 1, \dots, q, \\ 0, & j = q+1, \dots, n. \end{cases} \tag{6.7}$$

Here $g^{1/q}$ signifies one of the q'th roots of g, chosen arbitrarily. Each point $\mathfrak{p}$ at which $\mathcal{C}_{p,q}^{(n)}$ can be non-smooth is obtained from $\mathfrak{p}_0$ by a permutation of $x_1, \dots, x_n$.

It will now be shown that $\mathfrak{p}_0$ is an *ordinary* $(p-1)!$*-fold multiple point* of the curve $\mathcal{C}_{p,q}^{(n)}$: there are exactly $(p-1)!$ branches (local irreducible components) of $\mathcal{C}_{p,q}^{(n)}$ at $\mathfrak{p}_0$, so that the curve is non-smooth at $\mathfrak{p}_0$ if and only $p > 2$. Moreover, in a neighborhood of $\mathfrak{p}_0$, each of the $(p-1)!$ branches at $\mathfrak{p}_0$ is holomorphically parametrized by $\xi := (-\beta)^{1/p}$. This is a consequence of the fact that near $\beta = \beta_0 = 0$ ($g \neq 0$ being held fixed), each of the n roots of $x^n - gx^p - \beta$ has a convergent Puiseux expansion in $(-\beta)^{1/p}$.

Assume the roots are ordered so that $[x_1 : \dots : x_n] \to \mathfrak{p}_0$ as $\beta \to 0$, i.e., so that for each j, the limit of x_j agrees with (6.7). (This assumption fixes the ordering of $x_1, \dots, x_q$, but leaves unspecified that of $x_{q+1}, \dots, x_n$.) Then the Puiseux expansions of $x_1, \dots, x_n$ in ξ will be of the form

$$x_j(\xi) = \begin{cases} \varepsilon_q^{-(j-1)} g^{1/q} [1 + \sum_{i=1}^{\infty} c_{ji} \xi^{pi}], & j = 1, \dots, q, \\ (\varepsilon_p^{\bar{\alpha}_j} \xi) g^{-1/p} [1 + \sum_{i=1}^{\infty} \bar{c}_i (\varepsilon_p^{\bar{\alpha}_j} \xi)^{qi}], & j = q+1, \dots, n, \end{cases} \tag{6.8}$$

where $(\bar{\alpha}_{q+1}, \dots, \bar{\alpha}_n)$ is an unspecified permutation of $(0, 1, \dots, p-1)$. Of these two types of expansion, the first is obtained from $x^n - gx^p - \beta = 0$ by applying the implicit function theorem to $x^n - gx^p + \xi^p = 0$, and the second by applying the theorem to $\xi^q y^n - gy^p + 1 = 0$ (where $y := x/\xi$), which is equivalent.

The branch of $\mathcal{C}_{p,q}^{(n)}$ at $\mathfrak{p}_0$ that results from (6.8), locally parametrized by ξ, is affected by the choice of the permutation $(\bar{\alpha}_{q+1}, \dots, \bar{\alpha}_n)$. But cyclically shifting $(\bar{\alpha}_{q+1}, \dots, \bar{\alpha}_n)$ by $(1, \dots, 1)$ is equivalent to multiplying ξ by ε_p, which does not alter the branch. Hence when counting branches, one must quotient out the group $\mathfrak{C}_p$ of cyclic permutations: there are not $p!$ but $(p-1)!$ distinct branches, with distinct tangent lines.

Like $\mathfrak{p}_0$, each of the $n!/p!q$ points $\mathfrak{p} \in \mathcal{C}_{p,q}^{(n)}$ coming from a trinomial with $\beta = 0$ is an ordinary $(p-1)!$-fold multiple point; therefore the curve $\mathcal{C}_{p,q}^{(n)} \subset \mathbb{P}^{n-1}$ is singular if and only if $p > 2$. Each such point $\mathfrak{p}$, including $\mathfrak{p}_0$, lifts to $(p-1)!$ points on the desingularization $\tilde{\mathcal{C}}_{p,q}^{(n)} \to \mathcal{C}_{p,q}^{(n)}$, and each of the other points on $\mathcal{C}_{p,q}^{(n)}$ lifts to a single point. Each point $\tilde{\mathfrak{p}} \in \tilde{\mathcal{C}}_{p,q}^{(n)}$ lifted from one of the $n!/p!q$ points $\mathfrak{p} \in \mathcal{C}_{p,q}^{(n)}$ has a neighborhood in $\tilde{\mathcal{C}}_{p,q}^{(n)}$ that is biholomorphic to a neighborhood of the point 0 on the complex ξ-line. That is, near each of these $(n!/p!q) \times (p-1)! = n!/pq$ lifted points $\tilde{\mathfrak{p}}$, ξ can be used as a local parameter on $\tilde{\mathcal{C}}_{p,q}^{(n)}$.

Composing $\tilde{\mathcal{C}}_{p,q}^{(n)} \to \mathcal{C}_{p,q}^{(n)}$ with the map $\pi_{p,q}^{(n)} : \mathcal{C}_{p,q}^{(n)} \to \mathbb{P}_\zeta^1$, yields the holomorphic lifted map $\tilde{\pi}_{p,q}^{(n)} : \tilde{\mathcal{C}}_{p,q}^{(n)} \to \mathbb{P}_\zeta^1$, which near any of these points $\tilde{\mathfrak{p}}$ is effectively a holomorphic function $\zeta = \zeta(\xi)$, defined on a neighborhood of $\xi = 0$. Since ζ as defined by (6.2) is proportional to β^q and hence to ξ^{pq}, the composition $\tilde{\pi}_{p,q}^{(n)}$ takes each of these points $\tilde{\mathfrak{p}}$ to $\zeta = 0$ with multiplicity pq.

The desingularization of any subsidiary Schwarz curve $\mathcal{C}_{p,q}^{(k)}$, $n > k \geqslant 2$, is similar to that of the top curve $\mathcal{C}_{p,q}^{(n)}$. With $g \neq 0$ fixed, consider any k-tuple of roots $x_1, \dots, x_k$ of the trinomial (6.1), and let ν denote the number of these k roots that tend to nonzero values as $\beta \to 0$.

Necessarily, $0 \leqslant \nu \leqslant k$ with $\nu \leqslant q$ and $k-\nu \leqslant p$. These k roots can be expanded as power series in $\xi := (-\beta)^{1/p}$, and up to a permutation of $x_1, \dots, x_k$, the expansions will be of the form

$$x_j(\xi) = \begin{cases} \varepsilon_q^{\alpha_j} g^{1/q}[1+\sum_{i=1}^{\infty} c_{ji}\xi^{pi}], & j = 1, \dots, \nu, \\ (\varepsilon_p^{\bar{\alpha}_j}\xi) g^{-1/p}[1+\sum_{i=1}^{\infty} \bar{c}_i(\varepsilon_p^{\bar{\alpha}_j}\xi)^{qi}], & j = \nu+1, \dots, k, \end{cases} \tag{6.9}$$

for certain distinct $\alpha_j \in \{0, \dots, q-1\}$ and certain distinct $\bar{\alpha}_j \in \{0, \dots, p-1\}$. In (6.9), it is the initial ν components of $(x_1, \dots, x_k)$ that do not tend to zero as $\beta \to 0$, and the final $k-\nu$ components that do.

The case $\nu = 0$ occurs only when $k \leqslant p$ and is clearly special, since the limit of $(x_1, \dots, x_k)$ as $\beta \to 0$ in this case is $(0, \dots, 0)$; but the limit of $[x_1 : \dots : x_k]$ as $\beta \to 0$ exists in $\mathbb{P}^{k-1}$ as the point $[\varepsilon_p^{\bar{\alpha}_1} : \dots : \varepsilon_p^{\bar{\alpha}_k}]$. This explains Remark 5.3.2, on the need to take the closure when defining $\mathcal{C}_{p,q}^{(k)} \subset \mathbb{P}^{k-1}$, if $k \leqslant p$. On the fibre of $\mathcal{C}_{p,q}^{(k)}$ over $\zeta = 0$, there are $(p-k+1)_{k-1} = (p-k+1)_k/p$ distinct limit points $[x_1 : \dots : x_k]$ of this $\nu = 0$ type, in which $x_1, \dots, x_k$ are distinct p'th roots of unity. The case $\nu = k$ occurs only when $k \leqslant q$, and is also rather special. On the fibre over $\zeta = 0$, there are $(q-k+1)_{k-1} = (q-k+1)_k/q$ distinct points of the $\nu = k$ type, in which $x_1, \dots, x_k$ are distinct q'th roots of unity.

A straightforward extension of the preceding treatment of the top curve $\mathcal{C}_{p,q}^{(n)}$, which in retrospect was a treatment of the case $(k, \nu) = (n, q)$, yields the branching data in Table 2. $N_{p,q}^P(k, \nu)$ is the number of ordinary multiple points $\mathfrak{p} \in \mathcal{C}_{p,q}^{(k)} \subset \mathbb{P}^{k-1}$ of the type indexed by ν, coming from a trinomial equation with $\beta = 0$ (or in the $\nu = 0$ case, from a $\beta \to 0$ limit). For each such point, $N_{p,q}^T(k, \nu)$ is the multiplicity: the number of distinct branches at $\mathfrak{p}$, or equivalently the number of points $\tilde{\mathfrak{p}}$ to which $\mathfrak{p}$ lifts on the desingularization $\tilde{\mathcal{C}}_{p,q}^{(k)} \to \mathcal{C}_{p,q}^{(k)}$. For example, $N_{p,q}^P(n, q) = n!/p!q$ and $N_{p,q}^T(n, q) = (p-1)!$; these are the previously derived top-curve values.

The formula $\binom{k}{\nu}(q-\nu+1)_{\nu-1}$, given in the table for $N_{p,q}^P(k, \nu)$ if $0 < \nu < k$, comes from (i) choosing ν of the roots $x_1, \dots, x_k$ to be the ones with nonzero limits as $\beta \to 0$, and (ii) choosing the ν associated exponents α_j (i.e., powers of ε_q) to be distinct elements of $\{0, 1, \dots, q-1\}$, with the group $\mathfrak{C}_q$ of cyclic permutations quotiented out. At any $\mathfrak{p} \in \mathcal{C}_{p,q}^{(k)}$ specified in this way, there are $N_{p,q}^T(k, \nu) = (p-k+\nu+1)_{k-\nu-1}$ branches of $\mathcal{C}_{p,q}^{(k)}$. This is because a branch is specified by a choice of $k-\nu$ distinct exponents $\bar{\alpha}_j$ (i.e., powers of ε_p) from $\{0, 1, \dots, p-1\}$, with the group $\mathfrak{C}_p$ of cyclic permutations quotiented out.

For each ν with $0 < \nu < k$, each of the $N_{p,q}^P(k, \nu) \times N_{p,q}^T(k, \nu)$ lifted points $\tilde{\mathfrak{p}} \in \tilde{\mathcal{C}}_{p,q}^{(k)}$ has a neighborhood that is biholomorphic to a neighborhood of $\xi = 0$. The cases $\nu = 0$, resp. k are special, since the appropriate local parameter for $[x_1 : \dots : x_k] \in \mathcal{C}_{p,q}^{(k)}$ is not ξ but rather $\xi' := \xi^q$, resp. $\xi'' := \xi^p$, as is evident from (6.9). In both these cases there is only one branch, i.e., $N_{p,q}^T(k, 0) = 1$ (each appended limit point is a smooth point), resp. $N_{p,q}^T(k, k) = 1$.

The final quantity $M_{p,q}(k, \nu)$ in the table is the multiplicity with which each lifted point $\tilde{\mathfrak{p}} \in \tilde{\mathcal{C}}_{p,q}^{(k)}$ is taken to the point $\zeta = 0$ in $\mathbb{P}_\zeta^1$ by the lifted map $\tilde{\pi}_{p,q}^{(k)}$, which is locally a holomorphic function $\zeta = \zeta(\xi)$, or $\zeta = \zeta(\xi')$ resp. $\zeta = \zeta(\xi'')$. This multiplicity is usually pq, as previously seen in the top case, but in the special case $\nu = 0$, resp. k, it is p, resp. q, because $\zeta \propto \xi^{pq} = (\xi')^p = (\xi'')^q$.

Note that

$$\sum_{\nu=\max(0,k-p)}^{\min(q,k)} N_{p,q}^P(k, \nu) \cdot N_{p,q}^T(k, \nu) \cdot M_{p,q}(k, \nu) = (n-k+1)_k, \tag{6.10}$$

since the left side equals the number of points (with multiplicity) on the fibre of $\tilde{\pi}^{(k)}_{p,q}$ above $\zeta = 0$, i.e., $\deg \tilde{\pi}^{(k)}_{p,q} = (n-k+1)_k$. By substituting the data of Table 2 into (6.10), one obtains a familiar binomial coefficient identity.

6.2. *Genera*

It is now possible to determine, for each $k \geqslant 2$, the ramifications of the degree-$(n-k+1)_k$ holomorphic map of Riemann surfaces $\tilde{\pi}^{(k)}_{p,q}: \tilde{\mathcal{C}}^{(k)}_{p,q} \to \mathbb{P}^1_\zeta$, where $\tilde{\mathcal{C}}^{(k)}_{p,q} \to \mathcal{C}^{(k)}_{p,q}$ is the desingularization. Any $\tilde{\mathfrak{p}} \in \tilde{\mathcal{C}}^{(k)}_{p,q}$ can be mapped with nontrivial multiplicity to $\mathbb{P}^1_\zeta$ only if (i) its image, as in Section 6.1, is some point $\mathfrak{p} \in \mathcal{C}^{(k)}_{p,q}$ on the fibre of $\pi^{(k)}_{p,q}$ over $\zeta = 0$; *or*, (ii) its image $\mathfrak{p}$ is taken with nontrivial multiplicity by $\pi^{(k)}_{p,q}$ to some point $\zeta \neq 0$. Case (ii) can occur only if $\zeta = \pi^{(k)}_{p,q}(\mathfrak{p})$ is a critical value of $\pi^{(k)}_{p,q}$, i.e., only if $\pi^{(k)\,-1}_{p,q}\zeta$ comprises fewer than $(n-k+1)_k$ points on $\mathcal{C}^{(k)}_{p,q}$; i.e., only if $\pi^{(n)\,-1}_{p,q}\zeta$ comprises fewer than $n!$ points on $\mathcal{C}^{(n)}_{p,q}$.

Case (ii) can accordingly occur only if two of the n roots (with multiplicity) of the trinomial $x^n - gx^p - \beta$ giving rise to $\mathfrak{p}$ coincide; *or*, if the roots are distinct but are proportional to $\{\varepsilon_n^m\}_{m=0}^{n-1}$, the n'th roots of unity, so that there are only $(n-1)!$ distinct ordered n-tuples $[x_1 : \ldots : x_n] \in \mathbb{P}^{n-1}$. By evaluating the compact and elegant formula for the discriminant of the trinomial [17, 23], one finds that when $\beta \neq 0$, the first of these two subcases occurs only if

$$(pg/n)^n - (-p\beta/q)^q = 0, \tag{6.11}$$

which by (6.2) is equivalent to $\zeta = 1$. In this subcase, exactly two of the n roots coincide. (For a description of the monodromy of the roots around $\zeta = 1$, see [26, §2].) The second subcase occurs only if $g = 0$, i.e., $\zeta = \infty$. One concludes that $\tilde{\pi}^{(k)}_{p,q}: \tilde{\mathcal{C}}^{(k)}_{p,q} \to \mathbb{P}^1_\zeta$ can be ramified only over the three points $\zeta = 0, 1, \infty$. It is a so-called *Belyĭ cover*.

If $\zeta = 1$, i.e., Eq. (6.11) holds, without loss of generality one can take $g = n/p$, $\beta = -q/p$, so that the trinomial is proportional to $px^n - nx^p + q$, with a single doubled root at $x = 1$. There are $n!/2$ distinct points $[x_1 : \ldots : x_n] \in \mathcal{C}^{(n)}_{p,q}$ on the fibre over $\zeta = 1$: namely, points in which $x_1, \ldots, x_n$ are permutations of the roots of $px^n - nx^p + q$, including the single doubled root.

Definition 6.1. For each $p, q \geqslant 1$ with $\gcd(p,q) = 1$, a degree-$(p+q-2)$ polynomial with simple roots $T_{p,q}(x) = \sum_{j=0}^{p+q-2} t_j x^j$, satisfying

$$px^{p+q} - (p+q)x^p + q = (x-1)^2 T_{p,q},$$

is defined by

$$t_j = \begin{cases} (j+1)q, & 0 \leqslant j \leqslant p-1, \\ (p+q-1-j)p, & p-1 \leqslant j \leqslant p+q-2. \end{cases}$$

Its roots will be denoted $x^*_{p,q;\alpha}$, $1 \leqslant \alpha \leqslant p+q-2$.

Theorem 6.2. *For each k, $n \geqslant k \geqslant 2$, the degree-$(n-k+1)_k$ map of Riemann surfaces $\tilde{\pi}^{(k)}_{p,q}: \tilde{\mathcal{C}}^{(k)}_{p,q} \to \mathbb{P}^1_\zeta$ is a Belyĭ cover: it is ramified only over $\zeta = 0, 1, \infty$.*

1. *The fibre over $\zeta = 0$ contains $(p-k+1)_{k-1}$, resp. $(q-k+1)_{k-1}$ points of multiplicity p, resp. q, all other points being of multiplicity pq.*

2. *The fibre over* $\zeta = 1$ *contains* $(n-k-1)_k$ *points of unit multiplicity, all other points being of multiplicity* 2.
3. *The fibre over* $\zeta = \infty$ *contains* $(n-k+1)_{k-1}$ *points of multiplicity* n.

Remark 6.2.1. The case $k = n$ of this theorem, dealing with the top Schwarz curve $\mathcal{C}^{(n)}_{p,q}$, was proved by Kato and Noumi [21]. Note that the fibre over $\zeta = 0$ contains no points of multiplicity p if $k > p$, and none of multiplicity q if $k > q$; and that the fibre over $\zeta = 1$ contains no points of unit multiplicity if $k \geqslant n-1$.

Proof. The facts about the fibre over $\zeta = 0$ can be read off from Table 2.

Above $\zeta = 1$, the points of $\mathcal{C}^{(k)}_{p,q}$ that occur with unit multiplicity are the points $[x^*_{p,q;\chi(1)} : \ldots : x^*_{p,q;\chi(k)}]$, where the roots $x^*_{p,q;\alpha}$, $1 \leqslant \alpha \leqslant n-2$ were defined above, and $\chi(1), \ldots, \chi(k)$ are distinct integers selected from $1, \ldots, n-2$. The points of $\mathcal{C}^{(k)}_{p,q}$ that occur with multiplicity 2 are those in which, instead, one or two of the x_j's are equal to the double root 1.

As was already remarked, the points of $\mathcal{C}^{(k)}_{p,q}$ above $\zeta = \infty$ are the points $[x_1 : \ldots : x_k] \in \mathbb{P}^{k-1}$ in which $x_1, \ldots, x_k$ are distinct n'th roots of unity. There are exactly $(n-k+1)_{k-1} = (n-k+1)_k/n$ such points. $\square$

Theorem 6.3. *For each* $n \geqslant k \geqslant 1$, *the genus of* $\mathcal{C}^{(k)}_{p,q} \subset \mathbb{P}^{k-1}$ *as an algebraic curve, and the topological genus of the Riemann surface* $\tilde{\mathcal{C}}^{(k)}_{p,q}$, *are equal to*

$$1 + \left[\frac{(k-1)(2n-k-2)}{4(n-1)} - \frac{n}{2pq}\right](n-k+1)_{k-1} - \frac{q-1}{2q}(p-k+1)_{k-1} - \frac{p-1}{2p}(q-k+1)_{k-1}.$$

Proof. Apply the Hurwitz genus formula to the data given in Theorem 6.2 (which trivially extends to $k = 1$). The genus is stable under $p \leftrightarrow q$, since $\mathcal{C}^{(k)}_{p,q} \cong \mathcal{C}^{(k)}_{q,p}$, even though the singular points (i.e., their number and type) are not. $\square$

Corollary 6.4.

(i) *The following Schwarz curves, and only these, are rational, i.e., of genus zero.*
   - *The trivial curves* $\mathcal{C}^{(1)}_{p,q} \cong \mathbb{P}^1_s$ *and* $\mathcal{C}^{(2)}_{p,q} \cong \mathbb{P}^1_t$, *for all coprime* $p, q \geqslant 1$.
   - $\mathcal{C}^{(3)}_{p,q}$ *for* $\{p,q\} = \{1,2\}$ *and* $\{1,3\}$.
   - $\mathcal{C}^{(4)}_{p,q}$ *for* $\{p,q\} = \{1,3\}$.

   In consequence, $\mathcal{C}^{(q)}_{p,q}$ *can be rationally parametrized only if* $q = 1$, $q = 2$, *or* $(p,q) = (1,3)$; *and the top curve* $\mathcal{C}^{(n)}_{p,q}$ *only if* $\{p,q\} = \{1,1\}$, $\{1,2\}$, *or* $\{1,3\}$.

(ii) *The following Schwarz curves, and only these, are elliptic, i.e., of genus* 1.
   - $\mathcal{C}^{(3)}_{p,q}$ *for* $\{p,q\} = \{1,4\}$.

Proof. By Theorem 6.3, these curves are of genus 0 and genus 1 as claimed. The 'only these' statements remain to be proved.

It follows from Theorem 6.2 that for each k with $n > k > 2$, $\tilde{\mathcal{C}}^{(k)}_{p,q} \to \tilde{\mathcal{C}}^{(k-1)}_{p,q}$ is a covering with nontrivial branching. By the Hurwitz formula, if $g(\tilde{\mathcal{C}}^{(k-1)}_{p,q}) > 0$ then $g(\tilde{\mathcal{C}}^{(k)}_{p,q})$ must be strictly greater than $g(\tilde{\mathcal{C}}_{p,q}^{(k-1)})$. Therefore, to determine which Schwarz curves with $k \geqslant 3$ are of genus 0 or genus 1, one should focus on the case $k = 3$. Substituting $k = 3$ and $n = p + q$ into Theorem 6.3 yields

$$g\big(\tilde{\mathcal{C}}_{p,q}^{(3)}\big) = \big[\big(p^2 + 4pq + q^2\big) - 9(p + q) + 14\big]/2. \tag{6.12}$$

By examination, this equals 0 only if $\{p, q\} = \{1, 2\}$ or $\{1, 3\}$, and equals 1 only if $\{p, q\} = \{1, 4\}$. Moreover, by Theorem 6.3, $g(\tilde{\mathcal{C}}_{1,3}^{(4)}) = 0$ and $g(\tilde{\mathcal{C}}_{1,4}^{(4)}) > 1$; so the 'only these' statements are proved. □

6.3. Projective plane curves

The curves $\mathcal{C}_{p,q}^{(3)}$ are projective plane curves and include the first Schwarz curves of positive genus. The following theorem makes each $\mathcal{C}_{p,q}^{(3)} \subset \mathbb{P}^2$ and the associated covering map $\pi_{p,q}^{(3)} : \mathcal{C}_{p,q}^{(3)} \to \mathbb{P}_\zeta^1$ quite concrete.

Theorem 6.5. *For all coprime $p, q \geqslant 1$ with at least one of p, q greater than 1 and $n := p + q$, the curve $\mathcal{C}_{p,q}^{(3)} \subset \mathbb{P}^2$ has defining equation*

$$\frac{x_1^p(x_2^n - x_3^n) + x_2^p(x_3^n - x_1^n) + x_3^p(x_1^n - x_2^n)}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)} = 0, \tag{6.13}$$

where the left side is a symmetric homogeneous polynomial in x_1, x_2, x_3 that is of degree $n + p - 3$ and is of degree $n - 2$ in any single variable. The curve $\mathcal{C}_{p,q}^{(3)}$ goes through the fundamental points of $\mathbb{P}^2$ (i.e., $[1:0:0], [0:1:0], [0:0:1]$) if and only if $p \geqslant 2$, and is singular if and only if $p \geqslant 3$, in which case each fundamental point is an ordinary $(p-1)$-fold multiple point, and there are no other singular points. The (partial) covering map $\pi_{p,q}^{(3)} : \mathcal{C}_{p,q}^{(3)} \to \mathbb{P}_\zeta^1$ is given by

$$\sigma_n = \begin{cases} (-)^{n-1} x_1^p x_2^p x_3^p \left[\frac{x_1(x_2^{q+1} - x_3^{q+1}) + \text{cycl.}}{x_1(x_2^p x_3^{p+1} - x_2^{p+1} x_3^p) + \text{cycl.}}\right], & p > 1, \\ (-)^{n-1} x_1^p x_2^p x_3^p \left[\frac{x_1^q(x_2 - x_3) + \text{cycl.}}{x_1^{p+1}(x_2^p - x_3^p) + \text{cycl.}}\right], & q > 1; \end{cases}$$

$$\sigma_q = \begin{cases} (-)^{q-1} \left[\frac{x_1(x_2^p x_3^{n+1} - x_2^{n+1} x_3^p) + \text{cycl.}}{x_1(x_2^p x_3^{p+1} - x_2^{p+1} x_3^p) + \text{cycl.}}\right], & p > 1, \\ (-)^{q-1} \left[\frac{x_1^{p+1}(x_2^n - x_3^n) + \text{cycl.}}{x_1^{p+1}(x_2^p - x_3^p) + \text{cycl.}}\right], & q > 1; \end{cases}$$

$$\zeta = (-)^n \frac{n^n}{p^p q^q} \frac{{\sigma_n}^q}{{\sigma_q}^n};$$

and is of degree $n(n-1)(n-2)$.

Proof. To get (6.13), substitute the formulas for $\beta = (-)^{n-1}\sigma_n$, $g = (-)^{q-1}\sigma_q$ in Lemma 5.5 into the trinomial equation (6.1), and relabel x as x_3. To get the other formulas, apply the elimination procedure sketched in Section 5. (Cf. Lemma 5.5 itself, which results from applying the procedure when $k = 2$; the details when $k = 3$ are long and are omitted.) The singular points (0 or 3 in number) come from Table 2. If $p \geqslant 3$, there are singular points on the fibre over $\zeta = 0$: each fundamental point is a "$\nu = 1$" one, but there are none of types $\nu = 0, 2, 3$. □

Remark 6.5.1. The curves $\mathcal{C}^{(3)}_{p,q}, \mathcal{C}^{(3)}_{q,p} \subset \mathbb{P}^2$ are transforms of each other: the defining equation (6.13) of $\mathcal{C}^{(3)}_{p,q}$ is taken by a Cremona inversion in $\mathbb{P}^2$ (i.e., $x_i = 1/x'_i$) to itself, altered by $p \leftrightarrow q$; as is the formula for $\zeta = \zeta(x_1, x_2, x_3)$.

The genus of $\mathcal{C}^{(3)}_{p,q}$ was given in (6.12), but comes equally well by substituting the data of Theorem 6.5 into the standard genus–degree formula

$$g = \binom{N-1}{2} - \sum_i \binom{r_i}{2}. \tag{6.14}$$

Here $N = n + p - 3 = 2p + q - 3$ is the degree of $\mathcal{C}^{(3)}_{p,q}$, and the sum is over its three singular points, each (if $p \geqslant 3$) of multiplicity $r = p - 1$.

Parametrizing the plane curve $\mathcal{C}^{(3)}_{p,q}$ is a key step leading to a hypergeometric identity; especially, if either q or $n = p + q$ equals 3. The following examples of parametrizations are illustrative, and will be exploited in Section 7.

Example 6.6. $\{p, q\} = \{1, 2\}$, the simplest case. The top curves $\mathcal{C}^{(3)}_{1,2}, \mathcal{C}^{(3)}_{2,1} \subset \mathbb{P}^2$ are of genus zero (see Corollary 6.4). The defining equations are $x_1 + x_2 + x_3 = 0$ and $x_1x_2 + x_2x_3 + x_3x_1 = 0$. They are respectively a line, and a conic through the fundamental points of $\mathbb{P}^2$; and are related by a Cremona inversion.

In general, $\mathcal{C}^{(n)}_{p,q} \cong \mathcal{C}^{(n-1)}_{p,q}$, which is reflected in the fact that $\phi^{(n)}_{p,q} : \mathcal{C}^{(n)}_{p,q} \to \mathcal{C}^{(n-1)}_{p,q}$ is of degree 1. Hence $t := (x_1 + x_2)/(x_1 - x_2)$, used here to uniformize any $\mathcal{C}^{(2)}_{p,q}$, can be used as a rational parameter for $\mathcal{C}^{(3)}_{1,2}, \mathcal{C}^{(3)}_{2,1}$, yielding

$$[x_1 : x_2 : x_3] = \begin{cases} [t+1 : t-1 : -2t], & (p,q) = (1,2); \\ [-2t(t+1) : -2t(t-1) : t^2 - 1], & (p,q) = (2,1). \end{cases} \tag{6.15}$$

But to respect the $\mathfrak{S}_3$ symmetry it is better to use an alternative parameter $\tilde{t}$ (related to t by a Möbius transformation), thus:

$$\begin{aligned} &[x_1 : x_2 : x_3] \\ &\quad = \begin{cases} [\omega(1+\tilde{t}) : \bar{\omega}(1+\omega\tilde{t}) : (1+\bar{\omega}\tilde{t})], & (p,q) = (1,2); \\ [\omega(1+\omega\tilde{t})(1+\bar{\omega}\tilde{t}) : \bar{\omega}(1+\tilde{t})(1+\omega\tilde{t}) : (1+\bar{\omega}\tilde{t})(1+\tilde{t})], & (p,q) = (2,1), \end{cases} \end{aligned} \tag{6.16}$$

where $\omega := \varepsilon_3$ and $\bar{\omega} := {\varepsilon_3}^2$.

According to Theorem 6.5, the covering $\pi^{(3)}_{1,2} : \mathcal{C}^{(3)}_{1,2} \to \mathbb{P}^1_\zeta$ resp. $\pi^{(3)}_{2,1} : \mathcal{C}^{(3)}_{2,1} \to \mathbb{P}^1_\zeta$ is performed by a function $\zeta = \zeta(x_1, x_2, x_3)$, namely

$$\zeta = \begin{cases} -\frac{27}{4} \frac{(x_1x_2x_3)^2}{(x_1x_2+x_2x_3+x_3x_1)^3}, & (p,q) = (1,2); \\ -\frac{27}{4} \frac{x_1x_2x_3}{(x_1+x_2+x_3)^3}, & (p,q) = (2,1). \end{cases} \tag{6.17}$$

Substituting (6.16) yields a degree-6 rational map $\tilde{t} \mapsto \zeta$, i.e., the Belyĭ map

$$\zeta = \frac{(1+\tilde{t}^3)^2}{4\tilde{t}^3} = \begin{cases} \frac{27}{4} \frac{s^2}{(1-s)^3} \circ \frac{1-\tilde{t}+\tilde{t}^2}{(1+\tilde{t})^2}, & (p,q) = (1,2); \\ -\frac{27}{4} \frac{s}{(1-s)^3} \circ \frac{(1+\tilde{t})^2}{1-\tilde{t}+\tilde{t}^2}, & (p,q) = (2,1), \end{cases} \tag{6.18}$$

as the covering $\pi^{(3)}_{1,2}\colon \mathcal{C}^{(3)}_{1,2} \cong \mathcal{C}^{(2)}_{1,2} \to \mathbb{P}^1_\zeta$, resp. $\pi^{(3)}_{2,1}\colon \mathcal{C}^{(3)}_{2,1} \cong \mathcal{C}^{(2)}_{2,1} \to \mathbb{P}^1_\zeta$. Eq. (6.18) exhibits the compositions $\pi^{(2)}_{1,2} = \phi^{(1)}_{1,2} \circ \phi^{(2)}_{1,2}$ and $\pi^{(2)}_{2,1} = \phi^{(1)}_{2,1} \circ \phi^{(2)}_{2,1}$. The maps $\phi^{(2)}_{1,2}$ resp. $\phi^{(2)}_{2,1}$ and $\phi^{(1)}_{1,2}$ resp. $\phi^{(1)}_{2,1}$, i.e., $\tilde{t} \mapsto s$ and $s \mapsto \zeta$, are of degrees 2 and 3. They come from Theorem 5.6, if one takes account of the Möbius transformation relating t and $\tilde{t}$.

Example 6.7. $\{p,q\} = \{1,3\}$. The curves $\mathcal{C}^{(3)}_{1,3}$, $\mathcal{C}^{(3)}_{3,1}$ are of genus zero (see Corollary 6.4). First consider $(p,q) = (1,3)$. The defining equation of $\mathcal{C}^{(3)}_{1,3} \subset \mathbb{P}^2$ is

$$x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1 = 0, \tag{6.19}$$

which follows from Theorem 6.5, or more directly by eliminating x_4 from the equations $\sigma_1 = 0$, $\sigma_2 = 0$. This is a conic that does not go through the fundamental points of $\mathbb{P}^2$. It can be parametrized by inspection, with parameter $u \in \mathbb{P}^1$, as

$$[x_1 : x_2 : x_3] = \bigl[\omega(1-\bar{\omega}u)(1+2\bar{\omega}u) : \bar{\omega}(1-\omega u)(1+2\omega u) : (1-u)(1+2u)\bigr], \tag{6.20}$$

where $\omega := \varepsilon_3$. Substituting (6.20) into the function $\zeta = \zeta(x_1,x_2,x_3)$ of Theorem 6.5 yields a degree-24 rational map $u \mapsto \zeta$, i.e., the Belyĭ map

$$\zeta = -\frac{256}{27}\,\frac{(x_1x_2x_3)^3(x_1+x_2+x_3)^3}{(x_1+x_2)^4(x_2+x_3)^4(x_3+x_1)^4} \tag{6.21a}$$

$$= -256\,\frac{u^3(1-u^3)^3(1+8u^3)^3}{(1-20u^3-8u^6)^4} = 1 - \frac{(1+8u^6)^2(1+88u^3-8u^6)^2}{(1-20u^3-8u^6)^4} \tag{6.21b}$$

$$= -\frac{256}{27}\,\frac{s^3}{(1-s)^4} \circ \frac{(1-t)(1+3t^2)}{(1+t)^3} \circ \left[\left(\frac{\omega-\bar{\omega}}{3}\right)\left(\frac{1-2u-2u^2}{1+2u^2}\right)\right], \tag{6.21c}$$

as the covering $\pi^{(3)}_{1,3}\colon \mathcal{C}^{(3)}_{1,3} \cong \mathbb{P}^1_u \to \mathbb{P}^1_\zeta$. Eq. (6.21c) exhibits the composition $\pi^{(3)}_{1,3} = \phi^{(1)}_{1,3} \circ \phi^{(2)}_{1,3} \circ \phi^{(3)}_{1,3}$. The maps $\phi^{(3)}_{1,3}, \phi^{(2)}_{1,3}, \phi^{(1)}_{1,3}$, i.e., $u \mapsto t$, $t \mapsto s$, $s \mapsto \zeta$, are of degrees 2, 3, 4. They come from Theorem 5.6 and the fact that the parameter t on any curve $\mathcal{C}^{(2)}_{p,q}$ equals $(x_1+x_2)/(x_1-x_2)$.

The top Schwarz curve $\mathcal{C}^{(4)}_{1,3} \subset \mathbb{P}^3$ is birationally equivalent to $\mathcal{C}^{(3)}_{1,3}$, since the map $\phi^{(4)}\colon \mathcal{C}^{(4)}_{1,3} \to \mathcal{C}^{(3)}_{1,3}$ is of degree 1; so it too can be parametrized by u. Solving the equation $\sigma_1 = 0$ for $x_4 = x_4(u)$ yields $x_4 = -3u$, hence

$$\begin{aligned}&[x_1 : x_2 : x_3 : x_4]\\ &\quad = \bigl[\omega(1-\bar{\omega}u)(1+2\bar{\omega}u) : \bar{\omega}(1-\omega u)(1+2\omega u) : (1-u)(1+2u) : -3u\bigr]\end{aligned} \tag{6.22}$$

is an (asymmetric) rational parametrization of $\mathcal{C}^{(4)}_{1,3}$. Eq. (6.21b) can be viewed as defining the degree-24 covering $\pi^{(4)}_{1,3}\colon \mathcal{C}^{(4)}_{1,3} \cong \mathbb{P}^1_u \to \mathbb{P}^1_\zeta$.

The treatment of the case $(p,q) = (3,1)$ is similar. The genus-zero curve $\mathcal{C}^{(3)}_{3,1} \subset \mathbb{P}^2$ is obtained from $\mathcal{C}^{(3)}_{1,3} \subset \mathbb{P}^2$ by Cremona inversion ($x_i = 1/x_i'$), i.e., by the standard quadratic transformation ($x_i = x_j'x_k'$). It has defining equation

$$x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2 + x_1^2x_2x_3 + x_2^2x_3x_1 + x_3^2x_1x_2 = 0 \tag{6.23}$$

and is a trinodal quartic; it goes through the fundamental points $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, and each is an ordinary double point (‘node’). The smooth points $[1:\omega:\bar{\omega}]$, $[1:\bar{\omega}:\omega]$ are notable for being appended limit points in the sense of Section 6.1: they come indirectly rather than directly from trinomial roots.

By undoing the quadratic transformation, a rational parametrization by u of $\mathcal{C}^{(3)}_{3,1}$ can be obtained from (6.20), and one of $\mathcal{C}^{(4)}_{3,1}$ from (6.22). Thus the degree-24 rational map $u \mapsto \zeta$ given in (6.21b) can be used as $\pi^{(3)}_{3,1}: \mathcal{C}^{(3)}_{3,1} \cong \mathbb{P}^1_u \to \mathbb{P}^1_\zeta$ and $\pi^{(4)}_{3,1}: \mathcal{C}^{(4)}_{3,1} \cong \mathbb{P}^1_u \to \mathbb{P}^1_\zeta$.

Example 6.8. $\{p,q\} = \{1,4\}$. A discussion of the case $(p,q) = (1,4)$ will suffice. The defining equation of $\mathcal{C}^{(3)}_{1,4} \subset \mathbb{P}^2$ is

$$x_1^3 + x_2^3 + x_3^3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + x_2x_1^2 + x_3x_1^2 + x_3x_2^2 + x_1x_2x_3 = 0, \tag{6.24}$$

which follows from Theorem 6.5, or more directly by eliminating x_4, x_5 from the equations $\sigma_1 = 0$, $\sigma_2 = 0$, $\sigma_3 = 0$. This is a smooth cubic that does not go through the fundamental points of $\mathbb{P}^2$. It is elliptic, i.e., of genus 1, with Klein–Weber invariant $j = -5^2/2$. It can therefore be uniformized with the aid of elliptic functions, but it is easier to construct a multivalued parametrization with radicals. As usual, let $t = (x_1 + x_2)/(x_1 - x_2)$, so that $[x_1 : x_2] = [t+1 : t-1]$, and notice that as Theorem 6.5 predicts, Eq. (6.24) is of degree $n - 2 = 3$ in x_3. By symmetry, x_3, x_4, x_5 are the three roots, and they are computable in terms of radicals from t by Cardano's formula. It follows that each of $\mathcal{C}^{(3)}_{1,4}, \mathcal{C}^{(4)}_{1,4}, \mathcal{C}^{(5)}_{1,4}$ has a multivalued parametrization with radicals in terms of t. These parametrizations are respectively 3, 6, and 6-valued.

The technique of the last example immediately yields the following theorem.

Theorem 6.9. *For all coprime $p \geqslant 1$, $q \geqslant 2$ with $n := p + q \leqslant 6$, one can construct multivalued parametrizations with radicals for the subsidiary curve $\mathcal{C}^{(q)}_{p,q} \subset \mathbb{P}^{q-1}$ and the top curve $\mathcal{C}^{(n)}_{p,q} \subset \mathbb{P}^{n-1}$, respectively $(p+1)_{q-2}$-valued and $(n-2)!$-valued.*

7. Identities with free parameters

The results of Sections 5 and 6 will now be put to use, by deriving an interesting collection of parametrized hypergeometric identities. The source of many is Theorem 4.4, which expressed certain ${}_nF_{n-1}$'s in terms of algebraic functions. The expressions in parts (i) and (ii) of that theorem involve the roots $x_1, \dots, x_q$, resp. $x_1, \dots, x_n$, of the trinomial equation

$$x^n - gx^p - \beta = 0, \tag{7.1}$$

with $n := p + q$ and $\gcd(p,q) = 1$. It follows that any parametrization of the Schwarz curve $\mathcal{C}^{(q)}_{p,q}$, resp. $\mathcal{C}^{(n)}_{p,q}$, will yield a hypergeometric identity. The curves $\mathcal{C}^{(q)}_{p,q}, \mathcal{C}^{(n)}_{p,q}$ of genus zero were classified in Corollary 6.4, and the resulting identities are given in Sections 7.1, 7.2, 7.3 below, the respective parameters used being s, t, u. These are respectively the rational parameters for any $\mathcal{C}^{(1)}_{p,q}$, any $\mathcal{C}^{(2)}_{p,q}$, and (by Example 6.7) the plane curves $\mathcal{C}^{(3)}_{1,3}, \mathcal{C}^{(3)}_{3,1}$.

Each identity derived from Theorem 4.4 in this section is really a family of identities: the represented ${}_nF_{n-1}$ depends on a discrete parameter $\ell \in \mathbb{Z}$, and the identity involves a rational function $F_\ell = F_\ell(A,B;y)$. The functions F_ℓ were defined in Section 3 (see Table 1 and Theorem 3.5). The reader will recall that in particular, $F_0 \equiv 1$ and $F_1(A,B;y) = y/[(1-B)y + B]$.

Each of the ${}_nF_{n-1}$'s also depends on a parameter $a \in \mathbb{C}$. If a is chosen so that no upper parameter of the corresponding differential equation E_n differs by an integer from a lower one, the monodromy group of the E_n will be of the imprimitive irreducible type characterized in Theorem 2.3; and if $a \in \mathbb{Q}$, the group will be finite. (The case of *equal* upper and lower parameters, permitting 'cancellation,' is possible only for a finite number of choices of a, such as $a = \pm 1$ when $\ell = 0$; it was mentioned in Theorem 2.4.) One must treat with care the possibility that one of the lower parameters may be a non-positive integer, in which case ${}_nF_{n-1}$ is undefined (though it may still be possible to interpret the identity in a limiting sense; cf. Lemma 2.1). *It is assumed for simplicity in this section that $a \in \mathbb{C}$ is chosen so that this does not occur, and so that no division by zero occurs.*

Several of the identities below are rationally parametrized formulas for ${}_{n+1}F_n$'s rather than ${}_nF_{n-1}$'s. They come from Theorem 4.6 rather than Theorem 4.4, and instead of the rational functions F_ℓ, $\ell \in \mathbb{Z}$, they involve interpolating functions G_0, G_1 that were defined in Theorem 4.6 in terms of ${}_2F_1$, ${}_3F_2$. Each of these identities has an additional free parameter $c \in \mathbb{C}$, and a similar caveat applies.

7.1. Parametrizations by s

The rational parametrization of the curve $\mathcal{C}^{(q)}_{p,q} = \mathcal{C}^{(1)}_{p,1}$ by $s \in \mathbb{P}^1$ given in Eqs. (5.17) and (5.21), when substituted into part (i) of each of Theorems 4.4 and 4.6, yields the following.

Theorem 7.1. *For each $p \geqslant 1$, with $n := p + 1$, define a degree-n Belyĭ map $\mathbb{P}^1_s \to \mathbb{P}^1_\zeta$ by*

$$\zeta = \zeta_{p,1}(s) := -\frac{n^n}{p^p}\frac{s}{(1-s)^n}.$$

Then for all $a \in \mathbb{C}$ and $\ell \in \mathbb{Z}$, one has that near $s = 0$,

$${}_nF_{n-1}\left(\left.\begin{matrix}\frac{a}{n}, \dots, \frac{a+(n-1)}{n}\\ \frac{a-\ell+1}{p}, \dots, \frac{a-\ell+p}{p}\end{matrix}\,\right|\, \zeta_{p,1}(s)\right) = (1-s)^a F_\ell\big(-a, -p; (1-s)^{-1}\big).$$

Moreover, for all $a \in \mathbb{C}$, $c \in \mathbb{C}$ and $\ell = 0, 1$, one has that near $s = 0$,

$$\begin{aligned}&{}_{n+1}F_n\left(\left.\begin{matrix}\frac{a}{n}, \dots, \frac{a+(n-1)}{n}; & \frac{a+c-\ell}{p}\\ \frac{a-\ell}{p}, \dots, \frac{a-\ell+(p-1)}{p}; & \frac{a+c-\ell+p}{p}\end{matrix}\,\right|\, \zeta_{p,1}(s)\right)\\ &\quad= (1-s)^a G_\ell\big(-a, -p, -c; (1-s)^{-1}\big).\end{aligned}$$

Proof. Straightforward, as $g^{-1}x_1 = (1-s)^{-1}$ by (5.17). $\square$

Remark 7.1.1. For each of $\ell = 0, 1$, the second identity reduces to the first when $c = 0$; and as $c \to \infty$, it reduces to a version of the first in which ℓ is incremented by 1. Interpolation of this sort is familiar from Theorem 4.6 and will be seen repeatedly.

Similarly, the parametrization of the top curve $\mathcal{C}^{(n)}_{p,q} = \mathcal{C}^{(2)}_{1,1} \cong \mathcal{C}^{(1)}_{1,1}$ by $s \in \mathbb{P}^1$, substituted into part (ii) of each of Theorems 4.4 and 4.6, yields the following.

Theorem 7.2. *As in Theorem 7.1, let*

$$\zeta_{1,1}(s) := -\frac{4s}{(1-s)^2}.$$

Then for all $a \in \mathbb{C}$ and $\ell \in \mathbb{Z}$, and $\kappa = 0, 1$, one has that near $s = 1$,

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix} -a+\ell+\frac{\kappa}{2}, a+\frac{\kappa}{2} \\ \frac{1}{2}+\kappa \end{matrix}\,\middle|\, \frac{1}{\zeta_{1,1}(s)}\right) \\
&\quad = \frac{(-)^\kappa (a+\kappa/2)_{1-\ell-\kappa}}{(a)_{1-\ell}}\left[-\zeta_{1,1}(s)/4\right]^{\kappa/2} \\
&\qquad \times \frac{1}{2}\left[s^a F_\ell\left(-a, \frac{1}{2}; s^{-1}\right) + (-)^\kappa s^{-a} F_\ell\left(-a, \frac{1}{2}; s\right)\right].
\end{aligned}$$

Moreover, for all $a \in \mathbb{C}$, $c \in \mathbb{C}$ and $\ell = 0, 1$, and $\kappa = 0, 1$, one has that near $s = 1$,

$$\begin{aligned}
&{}_3F_2\left(\begin{matrix} -a+\ell+1+\frac{\kappa}{2}, a+\frac{\kappa}{2}; & -a-c+\ell+\frac{\kappa}{2} \\ \frac{1}{2}+\kappa; & -a-c+\ell+1+\frac{\kappa}{2} \end{matrix}\,\middle|\, \frac{1}{\zeta_{1,1}(s)}\right) \\
&\quad = \frac{(-)^\kappa (a+\kappa/2)_{-\ell-\kappa}(a+c-\ell-\kappa/2)}{(a)_{-\ell}(a+c-\ell)}\left[-\zeta_{1,1}(s)/4\right]^{\kappa/2} \\
&\qquad \times \frac{1}{2}\left[s^a G_\ell\left(-a, \frac{1}{2}, -c; s^{-1}\right) + (-)^\kappa s^{-a} G_\ell\left(-a, \frac{1}{2}, -c; s\right)\right].
\end{aligned}$$

Proof. Straightforward, as $\beta^{-1/2}x_1 = s^{-1/2}$ and $\beta^{-1/2}x_2 = -s^{1/2}$. □

The cases $\ell = 0, 1$ of the identities of Theorem 7.1 were previously obtained by Gessel and Stanton using series manipulations [12]. (See their Eqs. (5.10)–(5.13), (5.15).) When $p = 2$, the $\ell = 0, 1$ cases of the first identity become one-parameter specializations of Bailey's first cubic transformation of ${}_3F_2$ and its companion.

By comparison, the two identities of Theorem 7.2 are elementary. It should be possible to derive them, or at least the first, by using contiguous function relations or other classical hypergeometric techniques.

7.2. Parametrizations by t

The rational parametrization of the curve $\mathcal{C}^{(q)}_{p,q} = \mathcal{C}^{(2)}_{p,2}$ by $t \in \mathbb{P}^1$ given in Eq. (5.3) and Lemma 5.5, and the fact that $[x_1 : x_2] = [t+1 : t-1]$, when substituted into part (i) of each of Theorems 4.4 and 4.6, yield the following.

Theorem 7.3. *For each odd $p \geqslant 1$, with $n := p + 2$, define a map $\mathbb{P}^1_t \to \mathbb{P}^1_\zeta$ by*

$$\zeta = \zeta_{p,2}(t) := \frac{4n^n}{p^p}\,\frac{t^2(1-t^2)^{2p}[(1+t)^p + (1-t)^p]^p}{[(1+t)^n + (1-t)^n]^n}.$$

Then for all $a \in \mathbb{C}$ and $\ell \in \mathbb{Z}$, and $\kappa = 0, 1$, one has that near $t = 0$,

$$\begin{aligned}
&{}_nF_{n-1}\left(\begin{matrix} \frac{a}{n}+\frac{\kappa}{2}, \dots, \frac{a+(n-1)}{n}+\frac{\kappa}{2} \\ \frac{a-\ell+1}{p}+\frac{\kappa}{2}, \dots, \frac{a-\ell+p}{p}+\frac{\kappa}{2}; \frac{1}{2}+\kappa \end{matrix}\,\middle|\, \zeta_{p,2}(t)\right) \\
&\quad = \frac{(-)^\kappa (a+n\kappa/2)_{1-\ell-\kappa}}{(a)_{1-\ell}}\left[\frac{4p^p}{n^n}\zeta_{p,2}(t)\right]^{-\kappa/2} \\
&\qquad \times \frac{1}{2}\left[(1+t)^{-2a} F_\ell\left(-a, -p/2; (1+t)^2\left[\frac{(1+t)^p + (1-t)^p}{(1+t)^n + (1-t)^n}\right]\right)\right.
\end{aligned}$$

$$+ (-)^{\kappa} (1-t)^{-2a} F_{\ell}\left(-a, -p/2; (1-t)^2 \left[\frac{(1+t)^p + (1-t)^p}{(1+t)^n + (1-t)^n}\right]\right)\Bigg]$$
$$\times \left[\frac{(1+t)^n + (1-t)^n}{(1+t)^p + (1-t)^p}\right]^a.$$

Moreover, for all $a \in \mathbb{C}$, $c \in \mathbb{C}$ and $\ell = 0, 1$, and $\kappa = 0, 1$, one has that near $t = 0$,

$$ {}_{n+1}F_n\left(\begin{matrix} \frac{a}{n}+\frac{\kappa}{2}, \dots, \frac{a+(n-1)}{n}+\frac{\kappa}{2}; & \frac{a+c-\ell}{p}+\frac{\kappa}{2} \\ \frac{a-\ell}{p}+\frac{\kappa}{2}, \dots, \frac{a-\ell+(p-1)}{p}+\frac{\kappa}{2}; \frac{1}{2}+\kappa; & \frac{a+c-\ell+p}{p}+\frac{\kappa}{2} \end{matrix} \,\middle|\, \zeta_{p,2}(t)\right)$$
$$= \frac{(-)^{\kappa}(a+n\kappa/2)_{-\ell-\kappa}(a+c-\ell+p\kappa/2)}{(a)_{-\ell}(a+c-\ell)} \left[\frac{4p^p}{n^n}\zeta_{p,2}(t)\right]^{-\kappa/2}$$
$$\times \frac{1}{2}\Bigg[(1+t)^{-2a} G_{\ell}\left(-a, -p/2, -c; (1+t)^2\left[\frac{(1+t)^p + (1-t)^p}{(1+t)^n + (1-t)^n}\right]\right)$$
$$+ (-)^{\kappa} (1-t)^{-2a} G_{\ell}\left(-a, -p/2, -c; (1-t)^2 \left[\frac{(1+t)^p + (1-t)^p}{(1+t)^n + (1-t)^n}\right]\right)\Bigg]$$
$$\times \left[\frac{(1+t)^n + (1-t)^n}{(1+t)^p + (1-t)^p}\right]^a.$$

Remark 7.3.1. The even function $\zeta = \zeta_{p,2}(t)$ is a degree-$[n(n-1)]$ Belyĭ map. The case $\ell = 0$, $\kappa = 0$ of the first identity appeared in Section 1 as the sample result (1.2). Note that if $\kappa = 0$, there is simplification: the right-hand prefactor becomes unity.

Similarly, the rational parametrization of the top curve $\mathcal{C}_{p,q}^{(n)} = \mathcal{C}_{1,2}^{(3)} \cong \mathcal{C}_{1,2}^{(2)}$ given in (6.16) by $\tilde{t} \in \mathbb{P}^1$ (related to t by a Möbius transformation), substituted into part (ii) of each of Theorems 4.4 and 4.6, yields the following.

Theorem 7.4. *Define a degree-6 Belyĭ map $\mathbb{P}^1_{\tilde{t}} \to \mathbb{P}^1_{\zeta}$ by*

$$\zeta = \zeta_{1,2}(\tilde{t}) := \frac{(1+\tilde{t}^3)^2}{4\tilde{t}^3} = 1 + \frac{(1-\tilde{t}^3)^2}{4\tilde{t}^3}.$$

Then for all $a \in \mathbb{C}$ and $\ell \in \mathbb{Z}$, and $\kappa = 0, 1, 2$, one has that near $\tilde{t} = 0$,

$$ {}_3F_2\left(\begin{matrix} -a+\ell+\frac{\kappa}{3}; \frac{a}{2}+\frac{\kappa}{3}, \frac{a+1}{2}+\frac{\kappa}{3} \\ b_1(\kappa), b_2(\kappa) \end{matrix} \,\middle|\, \frac{1}{\zeta_{1,2}(\tilde{t})}\right)$$
$$= \frac{(-)^{\kappa}(1)_{\kappa}(a+2\kappa/3)_{1-\ell-\kappa}}{(a)_{1-\ell}} \left[\frac{4}{27}\zeta_{1,2}(\tilde{t})\right]^{\kappa/3}$$
$$\times \frac{1}{3}\Bigg\{\left[\frac{(1+\tilde{t})^3}{1+\tilde{t}^3}\right]^{-a} F_{\ell}\left(-a, \frac{1}{3}; \frac{(1+\tilde{t})^3}{1+\tilde{t}^3}\right)$$
$$+ \bar{\omega}^{\kappa}\left[\frac{(1+\omega\tilde{t})^3}{1+\tilde{t}^3}\right]^{-a} F_{\ell}\left(-a, \frac{1}{3}; \frac{(1+\omega\tilde{t})^3}{1+\tilde{t}^3}\right)$$
$$+ \omega^{\kappa}\left[\frac{(1+\bar{\omega}\tilde{t})^3}{1+\tilde{t}^3}\right]^{-a} F_{\ell}\left(-a, \frac{1}{3}; \frac{(1+\bar{\omega}\tilde{t})^3}{1+\tilde{t}^3}\right)\Bigg\}.$$

Moreover, for all $a \in \mathbb{C}$, $c \in \mathbb{C}$ and $\ell = 0, 1, 2$, and $\kappa = 0, 1$, one has that near $\tilde{t} = 0$,

$$
\begin{aligned}
&{}_4F_3\left(\begin{matrix}-a+\ell+1+\frac{\kappa}{3};\ \frac{a}{2}+\frac{\kappa}{3},\ \frac{a+1}{2}+\frac{\kappa}{3}; & -a-c+\ell+\frac{\kappa}{3}\\ b_1(\kappa), b_2(\kappa); & -a-c+\ell+1+\frac{\kappa}{3}\end{matrix}\ \middle|\ \frac{1}{\zeta_{1,2}(\tilde t)}\right)\\
&\quad=\frac{(-)^\kappa(1)_\kappa(a+2\kappa/3)_{-\ell-\kappa}(a+c-\ell-\kappa/3)}{(a)_{-\ell}(a+c-\ell)}\left[\frac{4}{27}\zeta_{1,2}(\tilde t)\right]^{\kappa/3}\\
&\qquad\times\frac{1}{3}\left\{\left[\frac{(1+\tilde t)^3}{1+\tilde t^3}\right]^{-a}G_\ell\left(-a,\frac{1}{3},-c;\frac{(1+\tilde t)^3}{1+\tilde t^3}\right)\right.\\
&\qquad+\bar\omega^\kappa\left[\frac{(1+\omega\tilde t)^3}{1+\tilde t^3}\right]^{-a}G_\ell\left(-a,\frac{1}{3},-c;\frac{(1+\omega\tilde t)^3}{1+\tilde t^3}\right)\\
&\qquad\left.+\omega^\kappa\left[\frac{(1+\bar\omega\tilde t)^3}{1+\tilde t^3}\right]^{-a}G_\ell\left(-a,\frac{1}{3},-c;\frac{(1+\bar\omega\tilde t)^3}{1+\tilde t^3}\right)\right\}.
\end{aligned}
$$

In these two identities, the lower parameters b_1, b_2 *are* $\frac{1}{3}, \frac{2}{3}$; *or* $\frac{2}{3}, \frac{4}{3}$; *or* $\frac{4}{3}, \frac{5}{3}$; *depending on whether* $\kappa = 0$, 1, *or* 2. *Also,* $\omega := \varepsilon_3 = \exp(2\pi\mathrm{i}/3)$ *and* $\bar\omega := {\varepsilon_3}^2$.

Proof. Straightforward, as $\beta = \sigma_3 = x_1x_2x_3 = 1 + \tilde t^3$ by (6.16). □

One can also derive identities from the parametrization by $\tilde t \in \mathbb{P}^1$ of the birationally equivalent top curve $\mathcal{C}^{(3)}_{2,1} \cong \mathcal{C}^{(2)}_{2,1}$, given along with the parametrization of $\mathcal{C}^{(3)}_{1,2} \cong \mathcal{C}^{(2)}_{1,2}$ in (6.16). Details are left to the reader.

7.3. Parametrizations by u

The rational parametrization of the plane curve $\mathcal{C}^{(q)}_{p,q} = \mathcal{C}^{(3)}_{1,3}$ by $u \in \mathbb{P}^1$ given in (6.20), in Example 6.7, when substituted into part (i) of each of Theorems 4.4 and 4.6, yields the following.

Theorem 7.5. *Define a degree-*24 *Belyĭ map* $\mathbb{P}^1_u \to \mathbb{P}^1_\zeta$ *by*

$$
\zeta = \zeta_{1,3}(u) := -256\frac{u^3(1-u^3)^3(1+8u^3)^3}{(1-20u^3-8u^6)^4} = 1 - \frac{(1+8u^6)^2(1+88u^3-8u^6)^2}{(1-20u^3-8u^6)^4}.
$$

Then for all $a \in \mathbb{C}$ *and* $\ell \in \mathbb{Z}$, *and* $\kappa = 0, 1, 2$, *one has that near* $u = 0$,

$$
\begin{aligned}
&{}_4F_3\left(\begin{matrix}\frac{a}{4}+\frac{\kappa}{3},\ldots,\frac{a+3}{4}+\frac{\kappa}{3}\\ a-\ell+1+\frac{\kappa}{3}; b_1(\kappa), b_2(\kappa)\end{matrix}\ \middle|\ \zeta_{1,3}(u)\right)\\
&\quad=\frac{(-)^\kappa(1)_\kappa(a+4\kappa/3)_{1-\ell-\kappa}}{(a)_{1-\ell}}\left[-\frac{27}{256}\zeta_{1,3}(u)\right]^{-\kappa/3}\\
&\qquad\times\frac{1}{3}\left\{\left[\frac{(1-u)^3(1+2u)^3}{1-20u^3-8u^6}\right]^{-a}F_\ell\left(-a,-\frac{1}{3};\frac{(1-u)^3(1+2u)^3}{1-20u^3-8u^6}\right)\right.\\
&\qquad+\bar\omega^\kappa\left[\frac{(1-\omega u)^3(1+2\omega u)^3}{1-20u^3-8u^6}\right]^{-a}F_\ell\left(-a,-\frac{1}{3};\frac{(1-\omega u)^3(1+2\omega u)^3}{1-20u^3-8u^6}\right)\\
&\qquad\left.+\omega^\kappa\left[\frac{(1-\bar\omega u)^3(1+2\bar\omega u)^3}{1-20u^3-8u^6}\right]^{-a}F_\ell\left(-a,-\frac{1}{3};\frac{(1-\bar\omega u)^3(1+2\bar\omega u)^3}{1-20u^3-8u^6}\right)\right\}.
\end{aligned}
$$

Moreover, for all $a \in \mathbb{C}$, $c \in \mathbb{C}$ *and* $\ell = 0, 1$, *and* $\kappa = 0, 1, 2$, *one has that near* $u = 0$,

$$
\begin{aligned}
&{}_5F_4\left(\begin{matrix}\frac{a}{4}+\frac{\kappa}{3},\dots,\frac{a+3}{4}+\frac{\kappa}{3}; & a+c-\ell+\frac{\kappa}{3}\\ a-\ell+\frac{\kappa}{3};b_1(\kappa),b_2(\kappa); & a+c-\ell+1+\frac{\kappa}{3}\end{matrix}\,\middle|\,\zeta_{1,3}(u)\right)\\
&\quad=\frac{(-)^\kappa(1)_\kappa(a+4\kappa/3)_{-\ell-\kappa}(a+c-\ell+\kappa/3)}{(a)_{-\ell}(a+c-\ell)}\left[-\frac{27}{256}\zeta_{1,3}(u)\right]^{-\kappa/3}\\
&\qquad\times\frac{1}{3}\left\{\left[\frac{(1-u)^3(1+2u)^3}{1-20u^3-8u^6}\right]^{-a}G_\ell\left(-a,-\frac{1}{3},-c;\frac{(1-u)^3(1+2u)^3}{1-20u^3-8u^6}\right)\right.\\
&\qquad+\bar\omega^\kappa\left[\frac{(1-\omega u)^3(1+2\omega u)^3}{1-20u^3-8u^6}\right]^{-a}G_\ell\left(-a,-\frac{1}{3},-c;\frac{(1-\omega u)^3(1+2\omega u)^3}{1-20u^3-8u^6}\right)\\
&\qquad\left.+\omega^\kappa\left[\frac{(1-\bar\omega u)^3(1+2\bar\omega u)^3}{1-20u^3-8u^6}\right]^{-a}G_\ell\left(-a,-\frac{1}{3},-c;\frac{(1-\bar\omega u)^3(1+2\bar\omega u)^3}{1-20u^3-8u^6}\right)\right\}.
\end{aligned}
$$

In both identities, the lower parameters b_1, b_2 are defined as in the previous theorem.

Proof. Straightforward, as $g=\sigma_3=x_1x_2x_3=1-20u^3-8u^6$ by (6.20). □

One can similarly derive identities from the rational parametrizations of the pair of top curves $\mathcal{C}^{(4)}_{1,3}\cong\mathcal{C}^{(3)}_{1,3}$ and $\mathcal{C}^{(4)}_{3,1}\cong\mathcal{C}^{(3)}_{3,1}$, also given in Example 6.7, by substituting them into Theorems 4.4(ii) and 4.6(ii). The resulting identities involve ${}_4F_3$, ${}_5F_4$, like the preceding.

8. Identities without free parameters

In this final section an alternative approach to the parametrizing of certain ${}_nF_{n-1}$'s, based on computation in rings of symmetric polynomials, is sketched. The need for a strengthened approach is indicated by the nature of the preceding results. Each identity in Section 7 was derived from either Theorem 4.4 or Theorem 4.6: it had a free parameter $a\in\mathbb{C}$ and was based on a (rational) parametrization of a Schwarz curve, either $\mathcal{C}^{(q)}_{p,q}$ or $\mathcal{C}^{(n)}_{p,q}$. But not many such curves are of zero or low genus, making it difficult to derive large numbers of hypergeometric identities, even if multivalued parametrizations with radicals are allowed.

In Section 8.1, it is shown that if $a\in\mathbb{Z}$, the relevant curve is a *quotiented* Schwarz curve $\mathcal{C}^{(q;\text{symm.})}_{p,q}$ or $\mathcal{C}^{(n;\text{symm.})}_{p,q}$; and if $qa\in\mathbb{Z}$, resp. $na\in\mathbb{Z}$, it is another such curve $\mathcal{C}^{(q;\text{cycl.})}_{p,q}$, resp. $\mathcal{C}^{(n;\text{cycl.})}_{p,q}$. It is actions of the symmetric group $\mathfrak{S}_q$ resp. $\mathfrak{S}_n$ and the cyclic group $\mathfrak{C}_q$ resp. $\mathfrak{C}_n$ that are quotiented out, and quotienting may lower the genus; which facilitates explicit parametrization, rational or otherwise. The several interesting examples worked out in Section 8.2 and Section 8.3 illustrate this, though they provide representations of algebraic ${}_nF_{n-1}$'s arising from E_n's that are not of the imprimitive irreducible type characterized in Theorem 2.3, which was previously the focus. (The difference in type is due to reducible monodromy and/or a lower hypergeometric parameter being a non-positive integer.)

8.1. Theory

Theorem 8.2 below is a restatement of the 'Birkeland' $\ell=0$ case of Theorem 4.4, which is the simplest because when $\ell=0$, the right-hand function F_ℓ degenerates to unity. The restatement emphasizes the role played by symmetric functions of the trinomial roots $x_1,\dots,x_n$. As usual, $\sigma_l=\sigma_l(x_1,\dots,x_n)$ denotes the l'th elementary symmetric polynomial, and $\mathcal{C}^{(n)}_{p,q}\subset\mathbb{P}^{n-1}$ comprises all points $[x_1:\dots:x_n]\in\mathbb{P}^{n-1}$ such that $\sigma_1,\dots,\sigma_{q-1}$ and $\sigma_{q+1},\dots,\sigma_{n-1}$ equal zero.

Definition 8.1. For each $p, q \geqslant 1$ with $\gcd(p,q) = 1$ and $n := p + q$, let

$$\mathcal{F}^{(q;\kappa)}_{p,q}(a;\zeta) := {}_nF_{n-1}\left(\left.\begin{matrix} \frac{a}{n}+\frac{\kappa}{q},\dots,\frac{a+(n-1)}{n}+\frac{\kappa}{q} \\ \frac{a+1}{p}+\frac{\kappa}{q},\dots,\frac{a+p}{p}+\frac{\kappa}{q};\ \frac{1}{q}+\frac{\kappa}{q},\dots,\widehat{\frac{q-\kappa}{q}+\frac{\kappa}{q}},\dots,\frac{q}{q}+\frac{\kappa}{q}\end{matrix}\ \right|\ \zeta\right)$$

for $\kappa = 0, \dots, q-1$, and

$$\mathcal{F}^{(n;\kappa)}_{p,q}(a;\zeta) := {}_nF_{n-1}\left(\left.\begin{matrix} \frac{-a}{p}+\frac{\kappa}{n},\dots,\frac{-a+(p-1)}{p}+\frac{\kappa}{n};\ \frac{a}{q}+\frac{\kappa}{n},\dots,\frac{a+(q-1)}{q}+\frac{\kappa}{n} \\ \frac{1}{n}+\frac{\kappa}{n},\dots,\widehat{\frac{n-\kappa}{n}+\frac{\kappa}{n}},\dots,\frac{n}{n}+\frac{\kappa}{n}\end{matrix}\ \right|\ \zeta\right)$$

for $\kappa = 0, \dots, n-1$. For all $a \in \mathbb{C}$, each function is defined and analytic in a neighborhood of $\zeta = 0$, unless (in the first) a lower parameter is a non-positive integer. If exactly one lower parameter equals such an integer, $-m$, and an upper parameter equals an integer $-m'$ with $-m \leqslant -m' \leqslant 0$, then the function can be defined as a limit, with the aid of Lemma 2.1.

By a congruential argument, the differential equation E_n corresponding to each of these ${}_nF_{n-1}$'s will have reducible monodromy, i.e., have an upper and a lower parameter that differ by an integer, iff $qa \in \mathbb{Z}$, resp. $na \in \mathbb{Z}$. In fact, the E_n will have $\zeta = 0$, resp. $\zeta = \infty$ as a logarithmic point, i.e., have a pair of lower, resp. upper parameters that differ by an integer, iff $qa \in \mathbb{Z}$, resp. $na \in \mathbb{Z}$.

Theorem 8.2.

(i) *Near the "$\zeta = 0$" point $[x_1 : \ldots : x_n] \in \mathcal{C}^{(n)}_{p,q}$ with*

$$x_j = \begin{cases} \varepsilon_q^{-(j-1)}, & j = 1, \dots, q, \\ 0, & j = q+1, \dots, n, \end{cases}$$

for each $\kappa = 0, \dots, q-1$ one has

$$\mathcal{F}^{(q;\kappa)}_{p,q}(a;\zeta) = \frac{(-)^\kappa (1)_\kappa (a + n\kappa/q)_{1-\kappa}}{(a)_1}\left[(-)^q \frac{p^p q^q}{n^n}\zeta\right]^{-\kappa/q}\left[(-)^{q-1}\sigma_q\right]^a \times \left[\frac{1}{q}\sum_{j=1}^{q}\left(\varepsilon_q^{-n}\right)^{(j-1)\kappa}\left(\varepsilon_q^{(j-1)}x_j\right)^{-qa}\right].$$

(ii) *Near the "$\zeta = \infty$" point $[x_1 : \ldots : x_n] \in \mathcal{C}^{(n)}_{p,q}$ with*

$$x_j = \varepsilon_n^{-(j-1)}, \qquad j = 1, \dots, n,$$

for each $\kappa = 0, \dots, n-1$ one has

$$\mathcal{F}^{(n;\kappa)}_{p,q}\left(a;\zeta^{-1}\right) = \frac{(-)^\kappa (1)_\kappa (a + q\kappa/n)_{1-\kappa}}{(a)_1}\left[(-)^q \frac{p^p q^q}{n^n}\zeta\right]^{\kappa/n}\left[(-)^{n-1}\sigma_n\right]^a \times \left[\frac{1}{n}\sum_{j=1}^{n}\left(\varepsilon_n^{-q}\right)^{(j-1)\kappa}\left(\varepsilon_n^{(j-1)}x_j\right)^{-na}\right].$$

In both (i) *and* (ii)*,*

$$\zeta := (-)^n \frac{n^n}{p^p q^q} \frac{\sigma_n{}^q}{\sigma_q{}^n}, \qquad (-)^q \frac{p^p q^q}{n^n} \zeta = (-)^p \frac{\sigma_n{}^q}{\sigma_q{}^n},$$

as usual; and it is assumed that $a \in \mathbb{C}$ is such that the left-hand function is defined. When $\kappa > 0$, branches must be chosen appropriately.

In parts (i) and (ii) of this theorem, the trinomial roots appear in the form x_j^{-qa}, resp. x_j^{-na}. In consequence, the case when $a \in \mathbb{Z}$, in particular when a is a negative integer, is especially nice. If $\kappa = 0$, which will be assumed henceforth, this is seen as follows. The identities of parts (i) and (ii) specialize if $\kappa = 0$ to

$$\mathcal{F}^{(q;0)}_{p,q}(a;\zeta) = \left[(-)^{q-1}\sigma_q\right]^a \cdot \frac{1}{q} \sum_{j=1}^{q} \left(\varepsilon_q^{(j-1)} x_j\right)^{-qa}, \tag{8.1a}$$

$$\mathcal{F}^{(n;0)}_{p,q}\left(a;\zeta^{-1}\right) = \left[(-)^{n-1}\sigma_n\right]^a \cdot \frac{1}{n} \sum_{j=1}^{n} \left(\varepsilon_n^{(j-1)} x_j\right)^{-na}. \tag{8.1b}$$

When $a \in \mathbb{Z}$, these reduce to

$$\mathcal{F}^{(q;0)}_{p,q}(a;\zeta) = \left[(-)^{q-1}\sigma_q\right]^a \cdot \frac{1}{q} \sum_{j=1}^{q} x_j^{-qa}, \tag{8.2a}$$

$$\mathcal{F}^{(n;0)}_{p,q}\left(a;\zeta^{-1}\right) = \left[(-)^{n-1}\sigma_n\right]^a \cdot \frac{1}{n} \sum_{j=1}^{n} x_j^{-na}. \tag{8.2b}$$

The right side of (8.2a), resp. (8.2b), is *single-valued* on $\mathcal{C}^{(q)}_{p,q} \subset \mathbb{P}^{q-1}$, resp. $\mathcal{C}^{(n)}_{p,q} \subset \mathbb{P}^{n-1}$. Therefore, when $a \in \mathbb{Z}$ the algebraic functions $\zeta \mapsto \mathcal{F}^{(q;0)}_{p,q}(a;\zeta)$ and $\zeta \mapsto \mathcal{F}^{(n;0)}_{p,q}(a;\zeta^{-1})$ are respectively *uniformized* by $\mathcal{C}^{(q)}_{p,q}, \mathcal{C}^{(n)}_{p,q}$.

In fact, a stronger result is true. There is an obvious permutation symmetry in (8.2a), (8.2b) under $\mathfrak{S}_q$, resp. $\mathfrak{S}_n$, which can be quotiented out, as will now be explained. Consider the action of $\mathfrak{S}_k$ on the curve $\mathcal{C}^{(k)}_{p,q} \subset \mathbb{P}^{k-1}$, for $k = 2, \dots, n$. It was shown in Section 5 that $\mathcal{C}^{(k)}_{p,q}$ is defined by a system of $k-2$ equations, each of which sets to zero a homogeneous polynomial in the invariants $\bar\sigma_1, \dots, \bar\sigma_k$, the elementary symmetric polynomials in $x_1, \dots, x_k$. The function field of $\mathcal{C}^{(k)}_{p,q}$ is the field of rational functions in $x_1, \dots, x_k$ that are homogeneous of degree zero, with these defining equations quotiented out.

The function field of $\mathcal{C}^{(k)}_{p,q}$ contains a subfield of $\mathfrak{S}_k$-stable functions, comprising all rational functions in $\bar\sigma_1, \dots, \bar\sigma_k$ that are homogeneous (in $x_1, \dots, x_k$) of degree zero; again, with the defining equations quotiented out. One element of this subfield is the function ζ, which gives the degree-$(n-k+1)_k$ covering $\pi^{(k)}_{p,q}\colon \mathcal{C}^{(k)}_{p,q} \to \mathbb{P}^1_\zeta$. Up to birational equivalence, define a quotiented curve $\mathcal{C}^{(k;\mathrm{symm.})}_{p,q} := \mathcal{C}^{(k)}_{p,q}/\mathfrak{S}_k$ that has this subfield of $\mathfrak{S}_k$-stable functions as its function field, so that

$$\mathcal{C}^{(k)}_{p,q} \longrightarrow \mathcal{C}^{(k;\mathrm{symm.})}_{p,q} \longrightarrow \mathbb{P}^1_\zeta \tag{8.3}$$

is a decomposition of $\pi^{(k)}_{p,q}$ into maps of respective degrees $k!$, $\binom{n}{k}$.

The genus-zero case $k = 2$ is illustrative. It is the case that $\mathcal{C}_{p,q}^{(2)} \cong \mathbb{P}_t^1$, where by convention $t = (x_1 + x_2)/(x_1 - x_2)$ is the rational parameter; and $\mathcal{C}_{p,q}^{(2;\text{symm.})} \cong \mathbb{P}_v^1$, where $v := t^2$. This is because the only nontrivial element of $\mathfrak{S}_2$ is the involution $x_1 \leftrightarrow x_2$, which on account of the parametrization $[x_1 : x_2] = [t + 1 : t - 1]$ is the involution $t \mapsto -t$; so the function field of $\mathcal{C}_{p,q}^{(2;\text{symm.})}$ comprises only functions even in t. When $k = 1$ or 0, $\mathcal{C}_{p,q}^{(k;\text{symm.})}$ will be defined as follows. By convention, $\mathcal{C}_{p,q}^{(1;\text{symm.})} \cong \mathcal{C}_{p,q}^{(1)} \cong \mathbb{P}_s^1$ and $\mathcal{C}_{p,q}^{(0;\text{symm.})} \cong \mathcal{C}_{p,q}^{(0)} \cong \mathbb{P}_\zeta^1$.

Lemma 8.3. *$\mathcal{C}_{p,q}^{(k;\text{symm.})} \cong \mathcal{C}_{p,q}^{(n-k;\text{symm.})}$ for $k = 0, \ldots, n$, due to the respective function fields being isomorphic. Here $n := p + q$, as usual.*

Proof. To prove the case $k = n$ (or $k = 0$), observe that the function field of $\mathcal{C}_{p,q}^{(n;\text{symm.})}$ comprises all rational functions of $\sigma_n{}^q/\sigma_q{}^n \propto \zeta$. (Also, $\mathfrak{S}_n$ is the covering group of $\pi_{p,q}^{(n)}: \mathcal{C}_{p,q}^{(n)} \to \mathbb{P}_\zeta^1$.) If $0 < k < n$, use the fact that any element of $\mathcal{C}_{p,q}^{(n-k;\text{symm.})}$ can be viewed as a quotient of two symmetric polynomials in $x_{k+1}, \ldots, x_n$ that are homogeneous and of the same degree. Any homogeneous symmetric polynomial in $x_{k+1}, \ldots, x_n$ can be expressed as a quotient of two such polynomials in $x_1, \ldots, x_k$ by the elimination procedure of Section 5; cf. Lemma 5.4. $\square$

Theorem 8.4. *For each $a \in \mathbb{Z}$:*

(i) *The algebraic function $\zeta \mapsto \mathcal{F}_{p,q}^{(q;0)}(a; \zeta)$ is uniformized by $\mathcal{C}_{p,q}^{(q;\text{symm.})}$.*
(ii) *The algebraic function $\zeta \mapsto \mathcal{F}_{p,q}^{(n;0)}(a; \zeta^{-1})$ is uniformized by $\mathcal{C}_{p,q}^{(n;\text{symm.})}$.*

Proof. Immediate, by the permutation symmetry of formulas (8.2a), (8.2b). $\square$

The following is a specialization of Theorem 8.4, with a constructive proof.

Theorem 8.5. *For each $a \in \mathbb{Z}$:*

(i) *If p or q equals 1, resp. 2, the algebraic function $\zeta \mapsto \mathcal{F}_{p,q}^{(q;0)}(a; \zeta)$ can be rationally parametrized by s, resp. $v := t^2$, where s, t are the rational parameters of $\mathcal{C}_{p,q}^{(1)} \cong \mathbb{P}_s^1$ and $\mathcal{C}_{p,q}^{(2)} \cong \mathbb{P}_t^2$. (This is because $\mathcal{C}_{p,q}^{(q;\text{symm.})} \cong \mathbb{P}_s^1$, resp. $\mathbb{P}_v^1$.)*
(ii) *The algebraic function $\zeta \mapsto \mathcal{F}_{p,q}^{(n;0)}(a; \zeta^{-1})$ is in fact rational, i.e., is uniformized by $\mathcal{C}_{p,q}^{(0)} := \mathbb{P}_\zeta^1$.*

In part (i), *the rational parametrization of the* argument *of the algebraic function, whether $\zeta = \pi_{p,q}^{(1)}(s)$ or $\zeta = \pi_{p,q}^{(2)}(t)$, was already given in Theorem* 5.6; *the latter is even in t, and hence, single-valued in v.*

Proof. Part (ii) is trivial, since $\mathcal{F}_{p,q}^{(n;0)}(a; \zeta)$ will be a polynomial in ζ if $a \in \mathbb{Z}$, because one of the upper hypergeometric parameters will then be a non-positive integer (see Definition 8.1). Also, the cases $q = 1, 2$ of part (i) are familiar from Section 5. If $q = 1$ then the right side of (8.2a) equals $(1 - s)^a$, and if $q = 2$ then one can use the parametrization $[x_1 : x_2] = [t + 1 : t - 1]$ and the formula for $\sigma_q = \sigma_q(x_1, x_2)$ given in Lemma 5.5 to express the right side of (8.2a) as a rational function of the parameter t; in fact, of $v = t^2$.

The cases $p = n - q = 1, 2$ of part (i) are more interesting. They can be viewed as consequences of $\mathcal{C}^{(q;\mathrm{symm.})}_{p,q} \cong \mathcal{C}^{(p;\mathrm{symm.})}_{p,q}$, which comes from Lemma 8.3. But a constructive rather than an abstract proof will be given, in the elimination-theoretic spirit of Lemma 5.4. Suppose the integer $\gamma = -qa$ is positive. Write

$$\sum_{j=1}^{q} x_j^{\gamma} = \begin{cases} p_\gamma - x_n^{\gamma}, & p = 1, \\ p_\gamma - x_n^{\gamma} - x_{n-1}^{\gamma}, & p = 2, \end{cases} \tag{8.4}$$

where $p_\gamma := \sum_{j=1}^{n} x_j^{\gamma}$ is a symmetric polynomial in $x_1, \dots, x_n$, the so-called γ'th power-sum symmetric polynomial. The elementary symmetric polynomials $\{\sigma_l\}_{l=1}^{n}$ in $x_1, \dots, x_n$ form an algebraic basis for the ring of symmetric ones, so the $\{p_\gamma\}_{\gamma=1}^{\infty}$ can be expressed as polynomials in the $\{\sigma_l\}_{l=1}^{n}$; e.g., by inverting the Newton–Girard formula [24, §I.2]

$$\sigma_l = l^{-1} \sum_{\gamma=1}^{l} (-)^{\gamma-1} p_\gamma \sigma_{l-\gamma}, \tag{8.5}$$

which holds for all $l \geqslant 1$, it being understood that $\sigma_l = 0$ if $l > n$. On the curve $\mathcal{C}^{(n)}_{p,q}$, each σ_l other than $\sigma_0, \sigma_q, \sigma_n$ equals zero, which introduces simplifications.

Now, introduce an alternative sequence of subsidiary Schwarz curves

$$\mathcal{C}^{(n)}_{p,q} \longrightarrow \mathcal{C}^{(n-1)\prime}_{p,q} \longrightarrow \cdots \longrightarrow \mathcal{C}^{(2)\,\prime}_{p,q} \longrightarrow \mathcal{C}^{(1)\,\prime}_{p,q} \longrightarrow \mathbb{P}^1_{\zeta}, \tag{8.6}$$

based on the successive elimination of $x_1, \dots, x_n$ rather than of $x_n, \dots, x_1$, so that $\mathcal{C}^{(1)\,\prime}_{p,q} \cong \mathbb{P}^1_{s'}$, where

$$\sigma_q = (-)^{q-1} x_n^q \cdot \left(1 - s'\right), \qquad \sigma_n = (-)^{n-1} x_n^n \cdot s' \tag{8.7}$$

(cf. (5.17)), and $\mathcal{C}^{(2)\,\prime}_{p,q} \cong \mathbb{P}^1_{t'}$ is parametrized by $t' := (x_n + x_{n-1})/(x_n - x_{n-1})$, so that $[x_n : x_{n-1}] = [t' + 1 : t' - 1]$. By using (8.4), (8.5) and (8.7), one can write the right side of (8.2a) as a rational function of s', resp. of t' (in fact of $v' := {t'}^2$, by invariance under $x_n \leftrightarrow x_{n-1}$). Moreover, the expressions for ζ in terms of s', t' are the same as those for ζ in terms of s, t.

The proof when $\gamma = -qa < 0$ is an easy modification of the preceding. □

Remark 8.5.1. Illustrations of Theorem 8.5(i), showing how the algorithm embedded in its proof is implemented, are given in Section 8.2 below.

Now consider what the two identities of Theorem 8.2 amount to, when a is *not* an integer, but nonetheless $qa \in \mathbb{Z}$, resp. $na \in \mathbb{Z}$. If $a = \frac{m}{q}$, resp. $a = \frac{m}{n}$, with $m \in \mathbb{Z}$, the $\kappa = 0$ specializations (8.1a), (8.1b) reduce to

$$\mathcal{F}^{(q;0)}_{p,q}\left(\frac{m}{q}; \zeta\right) = \left[(-)^{q-1}\sigma_q\right]^{m/q} \cdot \frac{1}{q} \sum_{j=1}^{q} \varepsilon_q^{-(j-1)m} x_j^{-m}, \tag{8.8a}$$

$$\mathcal{F}^{(n;0)}_{p,q}\left(\frac{m}{n}; \zeta^{-1}\right) = \left[(-)^{n-1}\sigma_n\right]^{m/n} \cdot \frac{1}{n} \sum_{j=1}^{n} \varepsilon_n^{-(j-1)m} x_j^{-m}. \tag{8.8b}$$

It follows that the power $[\mathcal{F}^{(q;0)}_{p,q}(\frac{m}{q}; \zeta)]^q$, resp. $[\mathcal{F}^{(n;0)}_{p,q}(\frac{m}{n}; \zeta^{-1})]^n$, is a rational function of $x_1, \dots, x_q$, resp. $x_1, \dots, x_n$, i.e., is in the function field of $\mathcal{C}^{(q)}_{p,q}$, resp. $\mathcal{C}^{(n)}_{p,q}$. Equivalently, these powers of algebraic functions are uniformized by $\mathfrak{C}^{(q)}_{p,q}$, $\mathfrak{C}^{(n)}_{p,q}$. But because of the presence of the coefficients $\varepsilon_q^{-(j-1)m}$, $\varepsilon_n^{-(j-1)m}$, they are not stable under the action of $\mathfrak{S}_q$ on $x_1,\dots,x_q$, resp. $\mathfrak{S}_n$ on $x_1,\dots,x_n$. Rather, they are stable under the associated subgroups of *cyclic* permutations. The group of cyclic permutations of $x_1,\dots,x_k$ will be denoted by $\mathfrak{C}_k$, and the group of dihedral ones (if $k>2$) by $\mathfrak{D}_k$. The relevant order and subgroup indices are $|\mathfrak{C}_k|=k$ and $(\mathfrak{D}_k:\mathfrak{C}_k)=2$, $(\mathfrak{S}_k:\mathfrak{D}_k)=(k-1)!/2$.

Definition 8.6. The quotient curves $\mathfrak{C}^{(k;\text{cycl.})}_{p,q}$, $\mathfrak{C}^{(k;\text{dihedr.})}_{p,q}$, $\mathfrak{C}^{(k;\text{symm.})}_{p,q}$ are defined to be $\mathfrak{C}^{(k)}_{p,q}/\Gamma$ with $\Gamma=\mathfrak{C}_k,\mathfrak{D}_k,\mathfrak{S}_k$, so that if $k>2$ one has the sequence of coverings

$$\mathfrak{C}^{(k)}_{p,q}\longrightarrow\mathfrak{C}^{(k;\text{cycl.})}_{p,q}\longrightarrow\mathfrak{C}^{(k;\text{dihedr.})}_{p,q}\longrightarrow\mathfrak{C}^{(k;\text{symm.})}_{p,q}\longrightarrow\mathbb{P}^1_\zeta,$$

which have respective degrees k, 2, $(k-1)!/2$, and $\binom{n}{k}$, as a decomposition of the degree-$(n-k+1)_k$ covering map $\pi^{(k)}_{p,q}:\mathfrak{C}^{(k)}_{p,q}\to\mathbb{P}^1_\zeta$. If $k=1,2$ then there is no dihedral curve, but $\mathfrak{C}^{(k;\text{cycl.})}_{p,q}\to\mathfrak{C}^{(k;\text{symm.})}_{p,q}$ has degree $(k-1)!$ in all cases.

Theorem 8.7. *For each $m\in\mathbb{Z}$:*

(i) *The algebraic function $\zeta\mapsto[\mathcal{F}^{(q;0)}_{p,q}(\frac{m}{q};\zeta)]^q$ is uniformized by $\mathfrak{C}^{(q;\text{cycl.})}_{p,q}$.*
(ii) *The algebraic function $\zeta\mapsto[\mathcal{F}^{(n;0)}_{p,q}(\frac{m}{n};\zeta^{-1})]^n$ is uniformized by $\mathfrak{C}^{(n;\text{cycl.})}_{p,q}$.*

Proof. Immediate, by the cyclic permutation symmetry of (8.8a), (8.8b). □

The following is a specialization of Theorem 8.7(i), with a constructive proof.

Theorem 8.8. *For each $m=-1,-2,\dots$:*

(i) *For each $q\geqslant 1$, the algebraic function $\zeta\mapsto[\mathcal{F}^{(q;0)}_{1,q}(\frac{m}{q};\zeta)]^q$ has a $(q-1)!$-valued parametrization by s, the rational parameter of $\mathfrak{C}^{(q;\text{symm.})}_{1,q}\cong\mathfrak{C}^{(1)}_{1,q}\cong\mathbb{P}^1_s$. That is, it satisfies a degree-$(q-1)!$ polynomial equation with coefficients in $\mathbb{Z}[s]$.*
(ii) *For each odd $q\geqslant 1$, the algebraic function $\zeta\mapsto[\mathcal{F}^{(q;0)}_{2,q}(\frac{m}{q};\zeta)]^q$ has a $(q-1)!$-valued parametrization by $v=t^2$, the rational parameter of $\mathfrak{C}^{(q;\text{symm.})}_{2,q}\cong\mathbb{P}^1_v$, i.e., by the square of the rational parameter of $\mathfrak{C}^{(2)}_{1,q}\cong\mathbb{P}^1_t$. That is, it satisfies a degree-$(q-1)!$ polynomial equation with coefficients in $\mathbb{Z}[v]$.*

The rational parametrizations $\zeta=\pi^{(1)}_{1,q}(s)$ and $\zeta=\pi^{(2)}_{2,q}(t)$ of the argument *of the algebraic function, for* (i), (ii)*, were already given in Theorem 5.6; the latter is even in t, and hence, single-valued in v.*

Proof. The rings of symmetric polynomials in $x_1,\dots,x_q$ and in $x_1,\dots,x_n$ have algebraic bases $\{\sigma_l^{(q)}\}_{l=1}^q$ or $\{p_\gamma^{(q)}\}_{\gamma=1}^q$, resp. $\{\sigma_l^{(n)}\}_{l=1}^n$ or $\{p_\gamma^{(n)}\}_{\gamma=1}^n$, of elementary or power-sum symmetric polynomials. Consider

$$G_{q,-m}(y):=\prod_\chi\left\{y-\left[\sum_{j=1}^q\varepsilon_q^{-(j-1)m}x_{\chi(j)}^{-m}\right]^q\right\},\tag{8.9}$$

the product being taken over one representative χ of each of the $(q-1)!$ cosets of $\mathfrak{C}_q$ in $\mathfrak{S}_q$, acting on $x_1,\dots,x_q$. By Eq. (8.8a),

$$\left[\mathcal{F}^{(q;0)}_{p,q}\left(\frac{m}{q};\zeta\right)\right]^q = q^{-q}\bigl[(-)^{q-1}\sigma_q\bigr]^m y_0, \tag{8.10}$$

where y_0 is a certain root of $G_{q,-m}(y)$. Each coefficient of the degree-$(q-1)!$ polynomial $G_{q,-m}(y)$ is stable under the action of $\mathfrak{S}_q$ on $x_1,\dots,x_q$, and can therefore be expressed as a polynomial in the $\{p^{(q)}_\gamma\}_{\gamma=1}^q$. But,

$$p^{(q)}_\gamma = \sum_{j=1}^q x_j^\gamma = \begin{cases} p^{(n)}_\gamma - x_n^\gamma, & p=1,\\ p^{(n)}_\gamma - x_n^\gamma - x_{n-1}^\gamma, & p=2. \end{cases} \tag{8.11}$$

Hence, each coefficient of $G_{q,-m}(y)$ is expressible in terms of the $\{\sigma^{(n)}_l\}_{l=1}^n$ and x_n, resp. x_n, x_{n-1}. But on $\mathcal{C}^{(n)}_{p,q}$, each of the $\{\sigma^{(n)}_l\}_{l=1}^n$ other than $\sigma^{(n)}_q, \sigma^{(n)}_n$ equals zero. By introducing the alternative sequence (8.6) of subsidiary Schwarz curves, one can express $\sigma^{(n)}_q, \sigma^{(n)}_n$ and x_n, resp. x_n, x_{n-1}, in terms of s' resp. t', the rational parameter of $\mathcal{C}^{(1)}_{1,p}{}'$ resp. $\mathcal{C}^{(2)}_{2,p}{}'$. And the expressions for ζ in terms of s', t' are the same as those for ζ in terms of s, t. Each rational function of t' encountered is a function of $v' := (t')^2$, by invariance under $x_n \leftrightarrow x_{n-1}$. $\square$

Remark 8.8.1. Illustrations of Theorem 8.8, showing how the algorithm embedded in its proof is implemented, are given in Section 8.3 below.

8.2. *Parametrizations when $a \in \mathbf{Z}$*

Theorems 8.9 and (especially) 8.11 below are sample applications of the algorithm embedded in the proof of Theorem 8.5(i). They show how it is possible rationally to parametrize many algebraic ${}_nF_{n-1}$'s with $a \in \mathbb{Z}$, due to their being uniformized by quotient curves $\mathcal{C}^{(q;\mathrm{symm.})}_{p,q}$ that are of genus zero. These examples also illustrate how lower parameters that are non-positive integers can be handled by taking a limit, and applying the auxiliary Lemma 2.1.

The first theorem is relatively simple, since the governing curve is $\mathcal{C}^{(q)}_{p,q} = \mathcal{C}^{(1)}_{n-1,1}$, which is of genus zero without quotienting; though Lemma 2.1 is used.

Theorem 8.9. *For each $n \geqslant 2$,*

$${}_{n-1}F_{n-2}\left(\begin{matrix} -\frac{1}{n};\ \frac{1}{n},\dots,\frac{n-2}{n} \\ \frac{1}{n-1},\dots,\frac{n-2}{n-1}\end{matrix}\ \middle|\ -\frac{n^n}{(n-1)^{n-1}}\frac{s}{(1-s)^n}\right) = \frac{(n-1)+s}{(n-1)(1-s)}$$

in a neighborhood of $s=0$.

Proof. As mentioned in the proof of Theorem 8.5(i), if $q=1$ then $\mathcal{F}^{(q;0)}_{p,q}(a;\zeta)$ equals $(1-s)^a$, where

$$\zeta = (-)^q \frac{n^n}{p^p q^q}\frac{s^q}{(1-s)^n} \tag{8.12}$$

is the usual map $\phi^{(1)}: \mathcal{C}^{(1)}_{p,q} \cong \mathbb{P}^1_s \to \mathbb{P}^1_\zeta$. Hence, $\mathcal{F}^{(1;0)}_{n-1,1}(-1;\zeta)$ equals $(1-s)^{-1}$. Formally,

$$\mathcal{F}^{(1;0)}_{n-1,1}(-1;\zeta) = {}_nF_{n-1}\left(\begin{array}{c} \frac{-1}{n}, \frac{0}{n}, \frac{1}{n}, \dots, \frac{n-2}{n} \\ \frac{0}{n-1}, \frac{1}{n-1}, \dots, \frac{n-2}{n-1} \end{array} \middle| \zeta\right), \tag{8.13}$$

but the right side is undefined, on account of the presence of a non-positive integer $-m = 0$ as a lower parameter (accompanied, fortunately, by a matching upper parameter, $-m' = 0$). One must interpret this in a limiting sense, i.e.,

$$\mathcal{F}^{(1;0)}_{n-1,1}(-1;\zeta) = \lim_{a\to -1} {}_nF_{n-1}\left(\begin{array}{c} \frac{a}{n}, \frac{a+1}{n}, \frac{a+2}{n}, \dots, \frac{a+(n-1)}{n} \\ \frac{a+1}{n-1}, \frac{a+2}{n-1}, \dots, \frac{a+(n-1)}{n-1} \end{array} \middle| \zeta\right). \tag{8.14}$$

By Lemma 2.1, this limit equals

$$\frac{1}{n} + \frac{n-1}{n} \cdot {}_{n-1}F_{n-2}\left(\begin{array}{c} -\frac{1}{n}; \frac{1}{n}, \dots, \frac{n-2}{n} \\ \frac{1}{n-1}, \dots, \frac{n-2}{n-1} \end{array} \middle| \zeta\right), \tag{8.15}$$

and the theorem now follows by a bit of algebra. □

Remark 8.9.1. Theorem 8.9 can also be viewed as a degenerate case of Theorem 7.1. Explicit representations for this algebraic ${}_{n-1}F_{n-2}$, for low n, were recently given by Dominici [8]. The differential equation E_{n-1} of which this ${}_{n-1}F_{n-2}$ is a solution has monodromy group $H < GL_{n-1}(\mathbb{C})$ isomorphic to $\mathfrak{S}_n$, by Theorem 2.4.

The following formula, of Girard type, facilitates computation in the function fields of any top Schwarz curve and its subsidiaries.

Lemma 8.10. *Consider the subset of $\mathbb{C}^n$, coordinatized by $x_1, \dots, x_n$, on which the elementary symmetric polynomials $\sigma_1, \dots, \sigma_{q-1}$ and $\sigma_{q+1}, \dots, \sigma_{n-1}$ equal zero. (The image of this subset under $\mathbb{C}^n \to \mathbb{P}^{n-1}$ is the top curve $\mathcal{C}^{(n)}_{p,q}$.) On this subset, one can express any power-sum symmetric polynomial $p_\gamma = \sum_{j=1}^n {x_j}^\gamma$, $\gamma \geqslant 1$, in terms of σ_q and σ_n by*

$$p_\gamma = \sum c_{m_q,m_n} {\sigma_q}^{m_q} {\sigma_n}^{m_n},$$

where the sum is over all $m_q, m_n \geqslant 0$ with $m_q q + m_n n = \gamma$, and

$$c_{m_q,m_n} = (-)^{\chi(q)m_q + \chi(n)m_n}\left[q\binom{m_q+m_n-1}{m_n} + n\binom{m_q+m_n-1}{m_q}\right],$$

with $\chi(l) := 1, 0$ if $l \equiv 0, 1 \pmod 2$.

Proof. This comes, e.g., from the determinantal formula

$$p_\gamma = \begin{vmatrix} \sigma_1 & 1 & 0 & \cdots & 0 \\ 2\sigma_2 & \sigma_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma\sigma_\gamma & \sigma_{\gamma-1} & \sigma_{\gamma-2} & \cdots & \sigma_1 \end{vmatrix}, \tag{8.16}$$

with $\sigma_l = 0$ if $l > n$. It is inverse to the Newton–Girard formula [24, §I.2]. □

Each hypergeometric function in the following theorem, with $a \in \mathbb{Z}$, is algebraic with governing curve $\mathcal{C}_{p,q}^{(q;\mathrm{symm.})} = \mathcal{C}_{2,3}^{(3;\mathrm{symm.})}$. This quotient curve is of genus zero, although the unquotiented curve $\mathcal{C}_{p,q}^{(q)} = \mathcal{C}_{2,3}^{(3)}$, which sextuply covers it, is of genus 3 by the formula of Theorem 6.3. The point of the theorem is that if $a \in \mathbb{Z}$, a uniformization by rational functions becomes possible.

Theorem 8.11. *For each integer $a \leqslant -1$, the function $\mathcal{F}_{2,3}^{(3;0)}(a;\zeta)$ satisfies*

$$\mathcal{F}_{2,3}^{(3;0)}\big(a;\zeta(t)\big) = \frac{1}{3}\big[\mathfrak{s}_3(t)\big]^a\big[P_a\big(\mathfrak{s}_3(t),\mathfrak{s}_5(t)\big) - (t+1)^{-3a} - (t-1)^{-3a}\big] \tag{8.17}$$

in a neighborhood of $t=0$, where

$$\mathfrak{s}_3(t) = \frac{1+10t^2+5t^4}{2t}, \qquad \mathfrak{s}_5(t) = -\frac{(1-t^2)^2(1+3t^2)}{2t},$$
$$\zeta(t) = -\frac{5^5}{2^2 3^3}\frac{\mathfrak{s}_5{}^3(t)}{\mathfrak{s}_3{}^5(t)} = \frac{5^5}{3^3}\frac{t^2(1-t^2)^6(1+3t^2)^3}{(1+10t^2+5t^4)^5},$$

and $P_a \in \mathbb{Z}[\mathfrak{s}_3,\mathfrak{s}_5]$ is defined by, e.g.,

$$P_a(\mathfrak{s}_3,\mathfrak{s}_5) = \begin{cases} 3\mathfrak{s}_3^{-a}, & a=-1,-2,-3,-4;\\ 3\mathfrak{s}_3{}^5 + 5\mathfrak{s}_5{}^3, & a=-5.\end{cases}$$

For each a, the special function $\mathcal{F}_{2,3}^{(3;0)}(a;\zeta)$ can be expressed in terms of nondegenerate hypergeometric functions by Lemma 2.1; e.g.,

$$\mathcal{F}_{2,3}^{(3;0)}(-1;\zeta) = 1 - \frac{2^2 3^3}{5^5}\zeta\cdot{}_5F_4\left(\begin{matrix}\frac45;\ \frac65,\frac75,\frac85;\ 1\\ \frac32;\ \frac43,\frac53;\ 2\end{matrix}\ \middle|\ \zeta\right) = \frac35 + \frac25\cdot{}_4F_3\left(\begin{matrix}-\frac15;\ \frac15,\frac25,\frac35\\ \frac12;\ \frac13,\frac23\end{matrix}\ \middle|\ \zeta\right),$$
$$\mathcal{F}_{2,3}^{(3;0)}(-5;\zeta) = 1 - \frac{2^2 3^2}{5^4}\zeta - \frac{2^6 3^9}{5^{14}}\zeta^3\cdot{}_5F_4\left(\begin{matrix}\frac{11}{5},\frac{12}{5},\frac{13}{5},\frac{14}{5};\ 2\\ \frac32;\ \frac{10}{3},\frac{11}{3};\ 4\end{matrix}\ \middle|\ \zeta\right).$$

Proof. This is the $(p,q)=(2,3)$ case of Theorem 8.5(i), made explicit, and the identity (8.17) is a rational parametrization of the formula (8.2a) for $\mathcal{F}_{p,q}^{(q;0)}(a,\zeta)$ when $a \in \mathbb{Z}$. The functions $\mathfrak{s}_3(t),\mathfrak{s}_5(t)$ are $\sigma_3/x_5^3, \sigma_5/x_5^5$, i.e., $\sigma_q/x_n^q, \sigma_n/x_n^n$, expressed in terms of t, the rational parameter of the alternative Schwarz curve $\mathcal{C}_{p,q}^{(p)}{}' = \mathcal{C}_{2,3}^{(2)}{}'$. The given expressions and the formula for $\zeta = \pi_{p,q}^{(2)}(t)$ come from Lemma 5.5 and Theorem 5.6; though x_1,x_2 are to be replaced by x_5,x_4, and 't' is to be understood as $(x_5+x_4)/(x_5-x_4)$, not $(x_1+x_2)/(x_1-x_2)$, so that $[x_5 : x_4] = [t+1 : t-1]$.

For each integer $a \leqslant -1$, the quantity $P_a = P_a(\mathfrak{s}_3,\mathfrak{s}_5)$ is a normalized version of the power-sum symmetric polynomial p_γ, i.e.,

$$p_\gamma = \sum_{j=1}^{n} x_j^\gamma = \sum_{j=1}^{5} x_j^\gamma, \qquad \gamma := -qa = -3a, \tag{8.18}$$

expressed as a polynomial in σ_3,σ_5 by the Girard formula of Lemma 8.10. The subtracted terms $(t+1)^{-3a}$, $(t-1)^{-3a}$ in (8.17) come from subtracting x_4^γ, x_5^γ from p_γ to obtain $x_1^\gamma + x_2^\gamma + x_3^\gamma$, as in (8.4).

The removal from $\mathcal{F}_{2,3}^{(3,0)}(a;\zeta)$, which is a ${}_5F_4(\zeta)$, of lower hypergeometric parameters that are non-positive integers, is a straightforward application of Lemma 2.1. E.g., for $a = -1,-2,-3,-4,-5$, the relevant lower/upper parameters $(-m,-m')$ are respectively $(0,0)$, $(0,0)$, $(-1,0)$, $(-1,0)$, $(-2,-1)$. Only if $a=-1$ or -2 does $-m=-m'$, permitting the ${}_5F_4$ to be reduced to a ${}_4F_3$. $\square$

Remark 8.11.1. In the two cases $a=-1,-2$ in which an upper and a lower parameter of $\mathcal{F}_{2,3}^{(3,0)}(a;\zeta)$ can be cancelled, reducing it to a ${}_4F_3(\zeta)$, the E_4 of which this ${}_4F_3$ is a solution has monodromy group $H < GL_4(\mathbb{C})$ isomorphic to $\mathfrak{S}_5$, by Theorem 2.4.

If $a \leqslant -3$ then $\mathcal{F}_{2,3}^{(3,0)}(a;\zeta)$ is essentially a ${}_5F_4(\zeta)$, which is the solution of an E_5 that has reducible monodromy, due to an upper and a lower parameter differing by an integer. (See the typical case $a=-5$, below.)

Corollary 8.12. *Define a degree-*10 *Belyĭ map* $\mathbb{P}^1_v \to \mathbb{P}^1_\zeta$ *by*

$$\begin{aligned}\zeta = \zeta_{2,3}(v) &:= \frac{5^5}{3^3}\,\frac{v(1-v)^6(1+3v)^3}{(1+10v+5v^2)^5}\\ &= 1 - \frac{(1-35v-125v^2-225v^3)^2(27+115v+25v^2+25v^3)}{27(1+10v+5v^2)^5}.\end{aligned}$$

Then in a neighborhood of $v=0$,

$$\begin{aligned}&{}_4F_3\left(\begin{matrix}-\frac15;\,\frac15,\,\frac25,\,\frac35\\ \frac12;\,\frac13,\,\frac23\end{matrix}\;\middle|\;\zeta_{2,3}(v)\right) = \frac{3+5v^2}{3(1+10v+5v^2)},\\ &{}_5F_4\left(\begin{matrix}\frac{11}{5},\,\frac{12}{5},\,\frac{13}{5},\,\frac{14}{5};\,2\\ \frac32;\,\frac{10}{3},\,\frac{11}{3};\,4\end{matrix}\;\middle|\;\zeta_{2,3}(v)\right)\\ &\quad= \frac{(3+v)(5+10v+v^2)(1+28v+134v^2+92v^3+v^4)(1+10v+5v^2)^{10}}{15(1-v)^{18}(1+3v)^9}.\end{aligned}$$

Proof. These are the rational parametrizations of the nondegenerate hypergeometric functions obtained in the theorem (when $a=-1$, resp. -5). They are even in t and expressible in terms of $v := t^2$, as expected. $\square$

Remark 8.12.1. The uniformizing parameter v is a rational parameter for the genus-0 quotient curve $\mathcal{C}_{p,q}^{(q;\mathrm{symm.})} = \mathcal{C}_{2,3}^{(3;\mathrm{symm.})}$. The reason why the above ${}_4F_3(\zeta)$ and ${}_5F_4(\zeta)$ are 10-branched functions of ζ is that the maps

$$\mathcal{C}_{2,3}^{(3)} \longrightarrow \mathcal{C}_{2,3}^{(3;\mathrm{symm.})} \cong \mathbb{P}^1_v \longrightarrow \mathbb{P}^1_\zeta \tag{8.19}$$

have respective degrees 6, 10. The 10-sheetedness of the latter map comes from $\binom{n}{q} = \binom{p+q}{q} = \binom{5}{3} = 10$.

The subsidiary curve $\mathcal{C}_{2,3}^{(3)} \subset \mathbb{P}^2$ is of genus 3 and is a smooth quartic through the fundamental points of $\mathbb{P}^2$, with defining equation

$$\begin{aligned}&(x_1+x_2+x_3)(x_1x_2x_3) + (x_1x_2+x_2x_3+x_3x_1)^2\\ &\quad - (x_1+x_2+x_3)^2(x_1x_2+x_2x_3+x_3x_1) = 0,\end{aligned} \tag{8.20}$$

as follows from Theorem 6.5. A formula for the degree-6 quotient map $\mathcal{C}_{2,3}^{(3)} \to \mathcal{C}_{2,3}^{(3;\text{symm.})} \cong \mathbb{P}_v^1$ can be derived by eliminating x_4, x_5 from the defining equations $\sigma_1 = 0$, $\sigma_2 = 0$, $\sigma_4 = 0$ of the top curve $\mathcal{C}_{2,3}^{(5)} \subset \mathbb{P}^4$, using $v = t'^2$ with $t' = (x_5 + x_4)/(x_5 - x_4)$. It is

$$v = \frac{(x_1 + x_2 + x_3)^2}{4(x_1x_2 + x_2x_3 + x_3x_1) - 3(x_1 + x_2 + x_3)^2}. \tag{8.21}$$

8.3. Parametrizations when $qa \in \mathbf{Z}$

Theorems 8.13 and 8.14 below are sample applications of the algorithm embedded in the proof of Theorem 8.8, to the cases $(p,q) = (2,3)$ and $(1,4)$ with $a = -\frac{1}{q}$. They show how one can construct parametrizations of many algebraic ${}_nF_{n-1}$'s with $qa \in \mathbb{Z}$, due to powers of the ${}_nF_{n-1}$'s being uniformized by quotient curves $\mathcal{C}_{p,q}^{(q;\text{cycl.})}$ that if not rational, are at least low-degree covers of rational ones. In these two theorems the rational lower curves will respectively be $\mathcal{C}_{p,q}^{(q;\text{symm.})}$ and $\mathcal{C}_{p,q}^{(q;\text{dihedr.})}$, and the parametrizations of the ${}_nF_{n-1}$'s will for the first time involve radicals.

Theorem 8.13. *Define a degree-*10 *Belyĭ map $\mathbb{P}_v^1 \to \mathbb{P}_\zeta^1$ by the rational formula for $\zeta = \zeta_{2,3}(v)$ given in Corollary* 8.12. *Then in a neighborhood of $v = 0$,*

$$\begin{aligned}&\left(1 + 10v + 5v^2\right)^{1/3} {}_4F_3\left(\left.\begin{matrix} -\frac{1}{15}, \frac{2}{15}, \frac{8}{15}, \frac{11}{15} \\ \frac{1}{3}, \frac{2}{3}, \frac{5}{6} \end{matrix}\right| \zeta_{2,3}(v)\right) \\ &\quad = \left\{\frac{1}{2} + \frac{5}{3}v - \frac{5}{54}v^2 + \frac{1}{2}\left[(1+3v)\left(1 + \frac{115}{27}v + \frac{25}{27}v^2 + \frac{25}{27}v^3\right)\right]^{1/2}\right\}^{1/3}. \end{aligned} \tag{8.22}$$

Proof. This is the $(p,q) = (2,3)$ case of Theorem 8.8(ii), made explicit, and the identity (8.22) is a rational parametrization of $[\mathcal{F}_{p,q}^{(q;0)}(a,\zeta)]^q = [\mathcal{F}_{2,3}^{(3;0)}(a,\zeta)]^3$ when $m = -1$, i.e., $a = -1/3$ and $qa = -1$.

As in Theorem 8.11, the relevant top and subsidiary Schwarz curves are $\mathcal{C}_{p,q}^{(n)} = \mathcal{C}_{2,3}^{(5)}$ and $\mathcal{C}_{p,q}^{(q)} = \mathcal{C}_{2,3}^{(3)}$, and the complementary subsidiary curve is $\mathcal{C}_{p,q}^{(p)}{}' = \mathcal{C}_{2,3}^{(2)}{}'$. The homogeneous coordinates on $\mathcal{C}_{2,3}^{(3)} \subset \mathbb{P}^2$ are x_1, x_2, x_3 and those on $\mathcal{C}_{2,3}^{(2)}{}' \cong \mathbb{P}_t^1$ are x_4, x_5. The rational parameter on the latter curve is $t := (x_5 + x_4)/(x_5 - x_4)$, and if $\mathfrak{s}_3 := \sigma_3/x_5^3$, $\mathfrak{s}_5 := \sigma_5/x_5^5$, then

$$\mathfrak{s}_3 = \frac{1}{x_5^3}\frac{x_5^5 - x_4^5}{x_5^2 - x_4^2} = \frac{1 + 10t^2 + 5t^4}{2t}, \qquad \mathfrak{s}_5 = -\frac{x_4^2}{x_5^3}\frac{x_5^3 - x_4^3}{x_5^2 - x_4^2} = -\frac{(1-t^2)^2(1+3t^2)}{2t},$$
$$\zeta = -\frac{5^5}{2^2 3^3}\frac{\mathfrak{s}_5^3}{\mathfrak{s}_3^5} = \frac{5^5}{3^3}\frac{t^2(1-t^2)^6(1+3t^2)^3}{(1+10t^2+5t^4)^5}$$

follow from Lemma 5.5 (with x_1, x_2 relabeled as x_5, x_4). This was the origin of the formula $\zeta = \zeta_{2,3}(v)$ given in Corollary 8.12, with $v := t^2$.

By the formula in Eq. (8.8a),

$$\mathcal{F}_{2,3}^{(3;0)}\left(-\frac{1}{3};\zeta\right) := {}_5F_4\left(\left.\begin{matrix} -\frac{1}{15}, \frac{2}{15}, \frac{1}{3}, \frac{8}{15}, \frac{11}{15} \\ \frac{1}{3}, \frac{5}{6}; \frac{1}{3}, \frac{2}{3} \end{matrix}\right| \zeta\right) = {}_4F_3\left(\left.\begin{matrix} -\frac{1}{15}, \frac{2}{15}, \frac{8}{15}, \frac{11}{15} \\ \frac{1}{3}, \frac{2}{3}, \frac{5}{6} \end{matrix}\right| \zeta\right) \tag{8.23}$$

(an upper and a lower parameter being cancelled) has the representation

$$\mathcal{F}_{2,3}^{(3;0)}\left(-\frac{1}{3};\zeta\right)=\frac{1}{3}\sigma_3{}^{-1/3}\left[x_1+\varepsilon_3x_2+\varepsilon_3^2x_3\right], \tag{8.24a}$$

$$\left[\mathcal{F}_{2,3}^{(3;0)}\left(-\frac{1}{3};\zeta\right)\right]^3=\frac{1}{3^3}\sigma_3{}^{-1}\left[x_1+\varepsilon_3x_2+\varepsilon_3^2x_3\right]^3. \tag{8.24b}$$

Define a polynomial $G_{3,1}(y)$, symmetric in x_1,x_2,x_3, one of the roots of which is the final factor in (8.24b), as a product over the two cosets of $\mathfrak{C}_3$ in $\mathfrak{S}_3$, i.e.,

$$\begin{aligned}G_{3,1}(y)&=\left\{y-\left[x_1+\varepsilon_3x_2+\varepsilon_3^2x_3\right]^3\right\}\left\{y-\left[x_1+\varepsilon_3^2x_2+\varepsilon_3x_3\right]^3\right\}\\&=y^2+\left[-2\hat\sigma_1^3+9\hat\sigma_1\hat\sigma_2-27\hat\sigma_3\right]y+\left[\hat\sigma_1^2-3\hat\sigma_2\right]^3,\end{aligned} \tag{8.25}$$

as one finds by a bit of computation. Here, $\{\hat\sigma_l\}_{l=1}^3$ are the elementary symmetric polynomials in x_1,x_2,x_3 alone; the ones in x_4,x_5 will be denoted by $\{\bar\sigma_l\}_{l=1}^2$.

Each coefficient in $G_{3,1}$ can be expressed rationally and symmetrically in terms of x_4,x_5, as the function fields of $\mathcal{C}_{2,3}^{(3;\mathrm{symm.})}$, $\mathcal{C}_{2,3}^{(2;\mathrm{symm.})}$ are isomorphic. One way to do this is to exploit the formula given in Lemma 5.4. Another uses the structure of the ring of symmetric polynomials in $x_1,\dots,x_5$, and was sketched in the proof of Theorem 8.8. One can express $\{\hat\sigma_l\}_{l=1}^3$ in terms of the power-sum symmetric polynomials $\{\hat p_\gamma\}_{\gamma=1}^3$, which can be expressed in terms of the overall power-sum symmetric polynomials $\{p_\gamma\}_{\gamma=1}^5$ in x_1,x_2,x_3,x_4,x_5, along with x_4,x_5. But the $\{p_\gamma\}_{\gamma=1}^5$ can be expressed in terms of $\{\sigma_l\}_{l=1}^5$. Of these, the only nonzero ones are σ_3,σ_5, which were expressed above in terms of x_4,x_5.

Regardless of which technique one uses, one finds that

$$G_{3,1}(y)=y^2+\left[\frac{-7\bar\sigma_1^4+36\bar\sigma_1^2\bar\sigma_2-27\bar\sigma_2^2}{\bar\sigma_1}\right]y+\left[-2\bar\sigma_1^2+3\bar\sigma_2\right]^3. \tag{8.26}$$

Let F_3 denote the left side of Eq. (8.24b), so that $F_3=y/3^3\sigma_3$, where y is a root of $G_{3,1}$. It follows from (8.26) and the formula for $\mathfrak{s}_3=\sigma_3/x_5^3$ in terms of t, and $[x_5:x_4]=[t+1:t-1]$, that F_3 satisfies

$$F_3{}^2-\frac{27+90v-5v^2}{27(1+10v+5v^2)}F_3-\frac{4v(3+5v)^3}{729(1+10v+5v^2)^2}=0, \tag{8.27}$$

where $v=t^2$. The theorem now follows from the quadratic formula. □

Remark 8.13.1. The cube of $\mathcal{F}_{2,3}^{(3;0)}(-\frac{1}{3};\zeta)$, i.e., of the ${}_4F_3$ in Theorem 8.13, is uniformized by $\mathcal{C}_{2,3}^{(3;\mathrm{cycl.})}$, which doubly covers $\mathcal{C}_{2,3}^{(3;\mathrm{symm.})}\cong\mathbb{P}_v^1$. (This is clear from (8.24b).) In fact, the statement of the theorem implicitly contains a plane model of $\mathcal{C}_{2,3}^{(3;\mathrm{cycl.})}$ as a double cover of $\mathcal{C}_{2,3}^{(3;\mathrm{symm.})}\cong\mathbb{P}_v^1$, namely

$$w^2=(1+3v)\left(1+\frac{115}{27}v+\frac{25}{27}v^2+\frac{25}{27}v^3\right). \tag{8.28}$$

This affine quartic $\mathcal{C}_{2,3}^{(3;\mathrm{symm.})}\ni(v,w)$ is elliptic (of genus 1), with Klein–Weber invariant $j=-2^{12}5^2/3$. It is triply covered by the unquotiented Schwarz curve $\mathcal{C}_{2,3}^{(3)}\subset\mathbb{P}^2$, which is of genus 3 and has defining equation (8.20).

In summary, the maps

$$\mathcal{C}^{(3)}_{2,3} \longrightarrow \mathcal{C}^{(3;\text{cycl.})}_{2,3} \longrightarrow \mathcal{C}^{(3;\text{symm.})}_{2,3} \cong \mathbb{P}^1_v \longrightarrow \mathbb{P}^1_\zeta \tag{8.29}$$

have respective degrees 3, 2, 10. This diagram contrasts with

$$\mathcal{C}^{(3)}_{2,3} \longrightarrow \mathcal{C}^{(2)}_{2,3} \cong \mathbb{P}^1_t \longrightarrow \mathcal{C}^{(1)}_{2,3} \cong \mathbb{P}^1_s \longrightarrow \mathbb{P}^1_\zeta, \tag{8.30}$$

in which the maps have respective degrees 3, 4, 5.

An $\mathfrak{S}_3$-invariant formula $v = v(x_1, x_2, x_3)$ for the degree-6 quotient map $\mathcal{C}^{(3)}_{2,3} \to \mathcal{C}^{(3;\text{symm.})}_{2,3} \cong \mathbb{P}^1_v$, which is the composition of the first two maps in (8.29), was given in Eq. (8.21). The first of the two, the degree-3 map $\mathcal{C}^{(3)}_{2,3} \to \mathcal{C}^{(3;\text{cycl.})}_{2,3}$, is of the form $(x_1, x_2, x_3) \mapsto (v, w)$. A rational formula for $w = w(x_1, x_2, x_3)$, $\mathfrak{C}_3$-invariant rather than $\mathfrak{S}_3$-invariant, can also be worked out.

Theorem 8.14. *Define a degree-*15 *Belyĭ map* $\mathbb{P}^1_x \to \mathbb{P}^1_\zeta$ *by*

$$\begin{aligned}\zeta = \zeta_{1,4}(x) &:= \frac{1}{4}\frac{x(1-5x)^4(5+6x+5x^2)^4}{(1-x)^5(1+10x+5x^2)^5} = \frac{5^5}{4^4}\frac{s^4}{(1-s)^5} \circ \frac{-(1-5x)(5+6x+5x^2)}{64x} \\ &= 1 - \frac{(4-5x-10x^2-5x^3)(1-55x-5x^2-5x^3)^2(1+5x^2+10x^3)^2}{4(1-x)^5(1+10x+5x^2)^5}.\end{aligned}$$

Then in a neighborhood of $x = 0$*,*

$$\begin{aligned}&(1-x)^{1/4}\left(1+10x+5x^2\right)^{1/4} {}_4F_3\left(\left.\begin{matrix} -\frac{1}{20}, \frac{3}{20}, \frac{7}{20}, \frac{11}{20} \\ \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \end{matrix}\,\right|\, \zeta_{1,4}(x)\right) \\ &\quad = \left\{\frac{1}{2} - \frac{5}{16}x - \frac{5}{8}x^2 - \frac{5}{16}x^3 + \frac{1}{2}\left[1 - \frac{5}{4}x - \frac{5}{2}x^2 - \frac{5}{4}x^3\right]^{1/2}\right\}^{1/4}.\end{aligned} \tag{8.31}$$

Proof. This is the $(p, q) = (1, 4)$ case of Theorem 8.8(i), made explicit, and the identity (8.31) is a rational parametrization of $[\mathcal{F}^{(q;0)}_{p,q}(a, \zeta)]^q = [\mathcal{F}^{(4;0)}_{1,4}(a, \zeta)]^4$ when $m = -1$, i.e., $a = -1/4$ and $qa = -1$. The relevant top and subsidiary Schwarz curves are $\mathcal{C}^{(n)}_{p,q} = \mathcal{C}^{(5)}_{1,4}$ and $\mathcal{C}^{(q)}_{p,q} = \mathcal{C}^{(4)}_{1,4}$, and the complementary subsidiary curve is $\mathcal{C}^{(p)\prime}_{p,q} = \mathcal{C}^{(1)\prime}_{1,4} \cong \mathbb{P}^1_{s'}$.

The computations resemble those in the proof of Theorem 8.13 and will only be sketched. One defines a degree-6 polynomial $G_{4,1}(y)$, symmetric in x_1, x_2, x_3, x_4, as a product over the six cosets of $\mathfrak{C}_4$ in $\mathfrak{S}_4$. Each coefficient in $G_{4,1}$ can be expressed rationally in x_5 and the rational parameter $s' = \sigma_5/x_5^5$. This leads to a degree-6 polynomial equation for F_4, defined as the fourth power of $\mathcal{F}^{(4;0)}_{1,4}(-\frac{1}{4}; \zeta)$, i.e., of the ${}_4F_3$ in the theorem. The coefficients of the degree-6 polynomial are polynomial in s'. By examination, if one substitutes

$$s' = s'(x) = \frac{-(1-5x)(5+6x+5x^2)}{64x}, \tag{8.32}$$

this polynomial will factor. The only relevant factor is a quadratic one, namely

$${F_4}^2 - \frac{8-5x-10x^2-5x^3}{8(1-x)(1+10x+5x^2)}F_4 + \frac{25x^2(1+x)^4}{256(1-x)^2(1+10x+5x^2)^2} = 0. \tag{8.33}$$

The theorem follows from Eq. (8.33) by the quadratic formula. □

Remark 8.14.1. The fourth power of $\mathcal{F}_{1,4}^{(4;0)}(-\frac{1}{4};\zeta)$, i.e., of the ${}_4F_3$ in Theorem 8.14, is uniformized by $\mathcal{C}_{1,4}^{(4;\text{cycl.})}$, which doubly covers $\mathcal{C}_{1,4}^{(4;\text{dihedr.})}$, which in turn, triply covers $\mathcal{C}_{1,4}^{(4;\text{symm.})} \cong \mathcal{C}_{1,4}^{(1;\text{symm.})} \cong \mathbb{P}^1_{s'}$. Hence, x can be identified as a rational parameter for the genus-0 curve $\mathcal{C}_{1,4}^{(4;\text{dihedr.})}$. The statement of the theorem implicitly contains a plane model of $\mathcal{C}_{1,4}^{(4;\text{cycl.})}$ as a double cover of $\mathcal{C}_{1,4}^{(4;\text{dihedr.})} \cong \mathbb{P}^1_x$, namely

$$w^2 = 1 - \frac{5}{4}x - \frac{5}{2}x^2 - \frac{5}{4}x^3. \tag{8.34}$$

This affine cubic $\mathcal{C}_{1,4}^{(4;\text{cycl.})} \ni (x,w)$ is elliptic (of genus 1), with Klein–Weber invariant $j = -5^2/2$, like $\mathcal{C}_{1,4}^{(3)}$. (See Example 6.8; the equality of the j-invariants remains to be investigated.) It is quadruply covered by the top curve $\mathcal{C}_{1,4}^{(5)} \cong \mathcal{C}_{1,4}^{(4)} \subset \mathbb{P}^3$, which is of genus 4.

In summary, the maps

$$\mathcal{C}_{1,4}^{(5)} \cong \mathcal{C}_{1,4}^{(4)} \longrightarrow \mathcal{C}_{1,4}^{(4;\text{cycl.})} \longrightarrow \mathcal{C}_{1,4}^{(4;\text{dihedr.})} \cong \mathbb{P}^1_x \longrightarrow \mathcal{C}_{1,4}^{(4;\text{symm.})} \cong \mathbb{P}^1_{s'} \longrightarrow \mathbb{P}^1_\zeta \tag{8.35}$$

have respective degrees 4, 2, 3, 5. This diagram contrasts with

$$\mathcal{C}_{1,4}^{(5)} \cong \mathcal{C}_{1,4}^{(4)} \longrightarrow \mathcal{C}_{1,4}^{(3)} \longrightarrow \mathcal{C}_{1,4}^{(2)} \cong \mathbb{P}^1_t \longrightarrow \mathcal{C}_{1,4}^{(1)} \cong \mathbb{P}^1_s \longrightarrow \mathbb{P}^1_\zeta, \tag{8.36}$$

in which the maps have respective degrees 2, 3, 4, 5.

An $\mathfrak{S}_4$-invariant formula for the degree-24 quotient map $\mathcal{C}_{1,4}^{(4)} \to \mathcal{C}_{1,4}^{(4;\text{symm.})} \cong \mathbb{P}^1_{s'}$, which is the composition of the first three of the four maps in (8.35), can be derived by eliminating x_5 from the defining equations $\sigma_1 = 0$, $\sigma_2 = 0$, $\sigma_3 = 0$ of the top curve $\mathcal{C}_{1,4}^{(5)} \subset \mathbb{P}^4$, using $s' = \sigma_5/x_5^5$. It is

$$s' = \frac{x_1x_2x_3x_4}{(x_1+x_2+x_3+x_4)^4}. \tag{8.37}$$

Acknowledgment

We thank the anonymous referee for many suggestions that led to improvements in the exposition.

References

[1] G. Belardinelli, Risoluzione analitica delle equazioni algebriche generali, Rend. Sem. Mat. Fis. Milano 29 (1959) 13–45.
[2] B.C. Berndt, Ramanujan's Notebooks, Part I, Springer-Verlag, New York, Berlin, 1985.
[3] F. Beukers, G. Heckman, Monodromy for the hypergeometric function ${}_nF_{n-1}$, Invent. Math. 95 (1989) 325–354.
[4] R. Birkeland, Über die Auflösung algebraischer Gleichungen durch hypergeometrische Funktionen, Math. Z. 26 (1927) 566–578.
[5] C.A. Charalambides, A new kind of numbers appearing in the n-fold convolution of truncated binomial and negative binomial distributions, SIAM J. Appl. Math. 33 (1977) 279–288.
[6] W. Chu, Binomial convolutions and hypergeometric identities, Rend. Circ. Mat. Palermo (2) 43 (1995) 333–360.
[7] L. Comtet, Advanced Combinatorics, Reidel, Boston, Dordrecht, 1974.
[8] D. Dominici, Asymptotic analysis of generalized Hermite polynomials, Analysis (Munich) 28 (2008) 239–261.
[9] A. Eagle, Series for all the roots of a trinomial equation, Amer. Math. Monthly 46 (1939) 422–425.
[10] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi (Eds.), Higher Transcendental Functions, McGraw–Hill, New York, 1953–1955, also known as *The Bateman Manuscript Project*.

[11] H. Fell, The geometry of zeros of trinomial equations, Rend. Circ. Mat. Palermo (2) 29 (1980) 303–336.

[12] I. Gessel, D. Stanton, Strange evaluations of hypergeometric series, SIAM J. Math. Anal. 13 (1982) 295–308.

[13] M.L. Glasser, Hypergeometric functions and the trinomial equation, J. Comput. Appl. Math. 118 (2000) 169–173.

[14] H.W. Gould, Some generalizations of Vandermonde's convolution, Amer. Math. Monthly 63 (1956) 84–91.

[15] H.W. Gould, A series transformation for finding convolution identities, Duke Math. J. 28 (1961) 193–202.

[16] H.W. Gould, Combinatorial identities: A standardized set of tables listing 500 binomial coefficient summations, Morgantown, WV, 1972.

[17] G.R. Greenfield, D. Drucker, On the discriminant of a trinomial, Linear Algebra Appl. 62 (1984) 105–112.

[18] N.A. Hall, The solution of the trinomial equation in infinite series by the method of iteration, Natl. Math. Mag. 15 (1941) 219–229.

[19] C. Jordan, Calculus of Finite Differences, 3rd ed., Chelsea, New York, 1965.

[20] M. Kato, Minimal Schwarz maps of ${}_3F_2$ with finite irreducible monodromy groups, Kyushu J. Math. 60 (2006) 27–46.

[21] M. Kato, M. Noumi, Monodromy groups of hypergeometric functions satisfying algebraic equations, Tohoku Math. J. (2) 55 (2003) 189–205.

[22] T. Kimura, K. Shima, A note on the monodromy of the hypergeometric differential equation, Jpn. J. Math. (N. S.) 17 (1991) 137–163.

[23] P. Lefton, A trinomial discriminant formula, Fibonacci Quart. 20 (1982) 363–365.

[24] I.G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., Oxford Univ. Press, 1995, with contributions by A. Zelevinsky.

[25] D. Merlini, R. Sprugnoli, M.C. Verri, Lagrange inversion: When and how, Acta Appl. Math. 94 (2006) 233–249.

[26] E.N. Mikhalkin, On solutions of general algebraic equations by means of integrals of elementary functions, Sib. Math. J. 47 (2006) 301–306. Russian original in: Sibirsk. Mat. Zh. 47 (2006) 365–371.

[27] M. Passare, A. Tsikh, Algebraic equations and hypergeometric series, in: O.A. Laudal, R. Piene (Eds.), The Legacy of Niels Henrik Abel, Springer-Verlag, New York and Berlin, 2004, pp. 653–672.

[28] G. Pólya, Sur les séries entières, dont la somme est une fonction algébrique, Enseign. Math. 22 (1921–1922) 38–47.

[29] D. Zeitlin, A new class of generating functions for hypergeometric polynomials, Proc. Amer. Math. Soc. 25 (1970) 405–412.