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CONGRUENCE DISTRIBUTIVITY IMPLIES BOUNDED WIDTH*

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Abstract. We show that a constraint language with compatible Jónsson terms (or, equivalently, associated with an algebra generating a congruence distributive variety) defines a constraint satisfaction problem solvable by the local consistency checking algorithm.

Key words. constraint satisfaction problem, bounded width, local consistency, congruence distributive variety, Jónsson terms

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1. Introduction. The constraint satisfaction problem (CSP) is of central importance in theoretical computer science. An instance of the CSP consists of variables and constraints and the aim is to determine whether the variables can be evaluated in such a way that all of the constraints are satisfied. The CSP provides a common framework for many problems in various areas of computer science; some of the most interesting algorithmic questions in database theory [31], machine vision recognition [26], temporal and spatial reasoning [30], technical design [28], scheduling [25], natural language comprehension [1], and programming language comprehension [27] are examples of CSPs.

The problem of solving CSPs with arbitrary constraints is NP-complete, and therefore, the research in this area is focused on solving CSPs with constraints taken from a fixed, finite set. More precisely, for any finite set of constraints (i.e., finitary relations) over a finite set, we seek to determine the complexity of solving CSPs restricted to instances with constraints coming exclusively from this set. The dichotomy conjecture of Feder and Vardi [17] postulates that every problem in such a family is NP-complete or solvable in polynomial time.

The dichotomy conjecture has proved to be a challenging question, and the advances using standard methods have been slow. A breakthrough in the development occurred when Jeavons, Cohen, and Gyssens [21] announced an algebraic approach to the problem. Their work, refined later by Bulatov, Jeavons, and Krokhin [12, 7], showed that the complexity of any particular CSP is fully determined by a set of functions—*polymorphisms* of the constraints. This allowed a rephrasing of the problem in algebraic terms and provided tools necessary to deal with conjectures open for years (see, for example, [3, 4]). More importantly, the algebraic approach allowed the authors to conjecture a precise boundary between polynomial-time solvable and

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NP-complete problems [12], and it pointed out classes of problems that have to be characterized before the dichotomy conjecture can be attacked.

A positive verification of the dichotomy conjecture requires a construction of an algorithm (or a class of algorithms) unifying all known algorithms. A characterization of applicability classes of existing algorithms is crucial for constructing such a unification. In particular, the class of *problems of bounded width* (i.e., problems solvable by the most widely known algorithm, the *local consistency checking* algorithm) has to be described. The only plausible conjecture on a structural characterization of this class was proposed by Larose and Zádori in [24]. They conjectured that a problem has bounded width if and only if the algebra associated with it (where operations of the algebra are polymorphisms of the constraints) generates a congruence meet semidistributive variety (the aforementioned terms will be defined later in the paper).

The part of this conjecture that received the most attention in recent years states that if the algebra associated with a set of constraints generates a congruence distributive variety, then the CSP has bounded width. Considering such a class of algebras is a natural first step toward verifying the conjecture of Larose and Zádori. Algebras generating congruence distributive varieties are equivalently described as those with a chain of Jónsson terms. The first attempts of an attack resulted in proving the conjecture for chains of three [23] and four [15] terms. We employ an approach different from the one from [23] and [15], and we prove bounded width for algebras with an arbitrary chain of Jónsson terms.

The methods developed in this paper turned out to be crucial for our recent result, which proves the conjecture of Larose and Zádori [2] in its full generality.

2. Algebraic preliminaries.

We briefly recall some universal algebraic notions and results, which will be needed in this article. For a more in-depth introduction to universal algebra, we recommend [14].

Throughout the article we use the following definitions. An n-ary *relation* on a set A is a subset of A^n, and an n-ary *operation* on A is a mapping from A^n to A.

2.1. Relational structures.

A *relational structure* is a tuple $\mathbb{A} = (A, R_0, R_1, \dots)$, where A is a set and $R_0, R_1, \dots,$ are relations on A. Relational structures $\mathbb{A} = (A, R_0, R_1, \dots)$, $\mathbb{B} = (B, S_0, S_1, \dots)$ have the same *type* if they have the same number of relations and the relation R_i has the same arity as S_i for every i (denoted by ar_i). In this situation, a mapping $f : A \to B$ is called *a homomorphism* from $\mathbb{A}$ to $\mathbb{B}$ if it preserves all the relations; i.e., for every i and every $(a_1, \dots, a_{\mathrm{ar}_i}) \in R_i$, we have $(f(a_1), \dots, f(a_{\mathrm{ar}_i})) \in S_i$. We say that $\mathbb{A}$ is *homomorphic* to $\mathbb{B}$ if there exists a homomorphism from $\mathbb{A}$ to $\mathbb{B}$. A relational structure $\mathbb{A}$ is said to be a *core* if every homomorphism $\mathbb{A} \to \mathbb{A}$ is bijective.

We say that a structure $\mathbb{A}$ is an *induced substructure* of $\mathbb{B}$ if $A \subseteq B$ and $R_i = S_i \cap A^{\mathrm{ar}_i}$ for every i. A *partial homomorphism* from $\mathbb{A}$ to $\mathbb{B}$ is a homomorphism from an induced substructure of $\mathbb{A}$ to $\mathbb{B}$. The following trivial observation will be used in the paper.

LEMMA 2.1. *Let $\mathbb{A}, \mathbb{B}$ be relational structures, let p be a number greater than or equal to the arity of every relation in $\mathbb{A}$, and let $f : A \to B$ be a mapping. If, for every $K \subseteq A$ with $|K| \leq p$, the mapping $f_{|K}$ is a partial homomorphism from $\mathbb{A}$ to $\mathbb{B}$, then f is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$.*

Note that all the relational structures in this paper are defined on a finite set and have a finite number of finitary relations.

2.2. Algebras and basic constructions. An *algebra* is a tuple $\mathbf{A} = (A, t_0, t_1, \dots)$, where A is a nonempty set (called a *universe*) and $t_0, t_1, \dots$ are operations on A. As with relational structures, algebras $\mathbf{A}, \mathbf{B}$ are of the same *type* if they have the same number of operations, and corresponding operations have equal arities. By abuse of notation, we denote operations of two algebras of the same type by the same symbols.

A mapping $f : A \to B$ is a homomorphism of algebras if it preserves all the operations, i.e., $f(t_i(a_1, \dots, a_{\mathrm{ar}_i})) = t_i(f(a_1), f(a_2), \dots, f(a_{\mathrm{ar}_i}))$ for any i and any $a_1, a_2, \dots, \in A$. A bijective homomorphism is called an *isomorphism*.

A set $B \subseteq A$ is a *subuniverse* of an algebra $\mathbf{A}$ if, for any i, the operation t_i restricted to B^{ar_i} has range contained in B. For a nonempty subuniverse B of an algebra $\mathbf{A}$, the algebra $\mathbf{B} = (B, t_0', \dots)$ (where t_i' is a restriction of t_i to B^{ar_i}) is called a *subalgebra* of $\mathbf{A}$. A *term function* of an algebra is any function that can be obtained as a composition using the operations of the algebra together with all the projections. A set $C \subseteq A$ *generates* a subuniverse B in an algebra $\mathbf{A}$ if B is the smallest subuniverse containing C—such a subuniverse always exists and can be obtained by applying all the term functions of the algebra $\mathbf{A}$ to all the choices of arguments coming from C.

Given algebras $\mathbf{A}, \mathbf{B}$ of the same type, a *product* $\mathbf{A} \times \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$ is the algebra with universe $A \times B$, and operations are computed coordinatewise. The product of algebras $\mathbf{A}_i$, $i \in I$, is defined in a similar manner for any set I. A *subdirect product* of $\mathbf{A}$ and $\mathbf{B}$ is a subalgebra $\mathbf{C}$ of $\mathbf{A} \times \mathbf{B}$ such that the projections of C to A and B are full (that is, the projection of C to the first coordinate is equal to A and the projection of C to the second coordinate is equal to B). For a set H, an *H-power* $\mathbf{A}^H$ of an algebra $\mathbf{A}$ has universe A^H (the set of mappings from H to A), and the operations are again computed coordinatewise.

An equivalence relation $\sim$ on A is called a *congruence* of an algebra $\mathbf{A}$ if $\sim$ is a subalgebra of $\mathbf{A} \times \mathbf{A}$. An equivalence relation is a congruence if and only if it is the kernel of some homomorphism from $\mathbf{A}$.

A *variety* is a class of algebras of the same type closed under forming subalgebras, products, and homomorphic images. The smallest variety containing an algebra $\mathbf{A}$ is called the variety *generated* by $\mathbf{A}$.

2.3. Congruence (semi)distributivity. The set of all congruences of an algebra $\mathbf{A}$ with the inclusion relation forms a lattice, i.e., a partially ordered set such that all two-element subsets $\{x, y\}$ have supremum $x \vee y$ and infimum $x \wedge y$. A lattice $\mathbb{L}$ is

- *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for every $a, b, c \in L$ (equivalently, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$);
- *join semidistributive* if $a \vee b = a \vee c$ implies $a \vee (b \wedge c) = a \vee b$;
- *meet semidistributive* if $a \wedge b = a \wedge c$ implies $a \wedge (b \vee c) = a \wedge b$.

A variety $\mathcal{V}$ is called congruence distributive (resp., join semidistributive, meet semidistributive) if all the algebras in $\mathcal{V}$ have distributive (resp., join semidistributive, meet semidistributive) congruence lattices. If a variety is congruence distributive, then it is also congruence join semidistributive; and if it is congruence join semidistributive, then it is also congruence meet semidistributive (but the latter implication is not true for a single lattice).

Congruence properties of a variety can often be characterized by the existence of certain term functions. In the case of congruence distributivity, such a characterization was given by Jónsson [22].

DEFINITION 2.2. *A sequence $t_0, t_1, \ldots, t_s$ of ternary operations on a set A is called a* Jónsson chain *if for every $a, b, c \in A$,*

$$\begin{array}{llll}
(J1) & t_0(a,b,c) = a, & \\
(J2) & t_s(a,b,c) = c, & \\
(J3) & t_r(a,b,a) = a & \text{for all } r \le s, \\
(J4) & t_r(a,a,b) = t_{r+1}(a,a,b) & \text{for all even } r < s, \\
(J5) & t_r(a,b,b) = t_{r+1}(a,b,b) & \text{for all odd } r < s.
\end{array}$$

An algebra $\mathbf{A} = (A, t_0, \ldots, t_s)$, where $t_0, \ldots, t_s$ is a Jónsson chain, will be called a $CD(s)$ algebra*.*

The following theorem connects the existence of Jónsson terms with the congruence distributivity of the algebras in a variety.

THEOREM 2.3 (see [22]). *An algebra $\mathbf{A}$ has a Jónsson chain of term functions if and only if the variety generated by $\mathbf{A}$ is congruence distributive.*

Similar conditions are available for join and meet semidistributivity [19, 32] as well. In both cases the characterization is obtained by weakening condition $(J3)$ from Definition 2.2.

3. CSP and polymorphisms. The CSP can be defined in several ways. In this paper we use a formulation using homomorphisms of relational structures. For different descriptions and more information about the algebraic approach to CSP, we recommend [7, 13].

Let $\mathbb{A}$ be a relational structure (with a finite universe and a finite number of relations, each of a finite arity). The CSP with template $\mathbb{A}$, CSP($\mathbb{A}$), is the following decision problem:

INPUT: A relational structure $\mathbb{X}$ of the same type as $\mathbb{A}$.
QUESTION: Does there exist a homomorphism $\mathbb{X} \to \mathbb{A}$?

Using this definition we can formulate the central problem in this area as follows.

CONJECTURE 3.1 (the dichotomy conjecture of Feder and Vardi [17]). *For every relational structure $\mathbb{A}$,* CSP($\mathbb{A}$) *is NP-complete or solvable in polynomial time.*

To state the algebraic dichotomy conjecture, we need to introduce the notion of compatible operation and polymorphism. An operation $f : A^m \to A$ is *compatible* with a relation $R \subseteq A^n$ if

$$(f(a_{11}, \ldots, a_{1m}), \ldots, f(a_{n1}, \ldots, a_{nm})) \in R$$

whenever $(a_{11}, \ldots, a_{n1}), \ldots, (a_{1m}, \ldots, a_{nm}) \in R$. An operation $f : A^m \to A$ is a *polymorphism* of a relational structure $\mathbb{A}$ if it is compatible with all the relations of $\mathbb{A}$.

To every relational structure $\mathbb{A}$ we associate an algebra $\mathbf{A}$ whose operations are all the polymorphisms of $\mathbb{A}$ (in an arbitrarily chosen order). It is easy to see that every projection is a polymorphism of any relational structure and that the set of all polymorphisms of a relational structure is closed under forming compositions. Therefore every term function of such a constructed $\mathbf{A}$ is an operation of $\mathbf{A}$.

One of the crucial observations in the development of the algebraic approach is that the complexity of CSP($\mathbb{A}$) depends only on $\mathbf{A}$ [21, 12]. "Nice" properties of $\mathbf{A}$ ensure the tractability of CSP($\mathbb{A}$), while "bad" properties of $\mathbf{A}$ cause the NP-completeness of CSP($\mathbb{A}$). Bulatov, Krokhin, and Jeavons [12] proved that if $\mathbb{A}$ is a core and the variety generated by $\mathbf{A}$ contains a G-set (i.e., an at least two-element algebra whose operations are of the form $f(x_1, \ldots, x_n) = g(x_i)$ for some permutation g), then CSP($\mathbb{A}$) is NP-complete. They conjectured that otherwise CSP($\mathbb{A}$) is tractable.

CONJECTURE 3.2 (the algebraic dichotomy conjecture). *Let $\mathbb{A}$ be a core relational structure and let* $\mathbf{A}$ *be the algebra associated with it. The* CSP($\mathbb{A}$) *is solvable in polynomial time if the variety generated by* $\mathbf{A}$ *does not contain a G-set. Otherwise,* CSP($\mathbb{A}$) *is NP-complete.*

Note that the assumption that $\mathbb{A}$ is a core is not restrictive at all, as it is easy to see that CSP($\mathbb{A}$) is equivalent to CSP($\mathbb{A}'$) for some suitably chosen core $\mathbb{A}'$.

All known results about the complexity of the CSP agree with Conjecture 3.2. It holds when A is a three-element set [10] (which generalizes the result of Schaefer for two-element relational structures [29]), $\mathbb{A}$ contains all unary relations [8], $\mathbf{A}$ has only exponentially many subalgebras of $\mathbf{A}^n$ with respect to n [5, 20] (which generalizes [6] and [16]), and $\mathbb{A}$ is a digraph with no sources or sinks [3, 4] (which generalizes [18]). In this paper we focus on yet another set of problems—CSPs of bounded width, i.e., problems that are solvable by local consistency checking.

4. Bounded width, main theorem.

Bounded width can be introduced in a number of equivalent ways (using duality, infinitary logic, pebble games, Datalog programs, or strategies); see [24, 11]. We define it using the notion of (k,l)-strategy.

DEFINITION 4.1. *Let $\mathbb{X}, \mathbb{A}$ be relational structures of the same type and let $k \le l$ be natural numbers. A set $\mathcal{F}$ of partial homomorphisms from $\mathbb{X}$ to $\mathbb{A}$ is called a* (k,l)-strategy *for $(\mathbb{X}, \mathbb{A})$ if it satisfies the following conditions:*

($S1$) $|\mathrm{dom}(f)| \le l$ *for any* $f \in \mathcal{F}$.

($S2$) *For any $f \in \mathcal{F}$ and any $K \subseteq \mathrm{dom}(f)$, the function $f_{|K}$ belongs to $\mathcal{F}$.*

($S3$) *For any $K \subseteq L \subseteq X$ with $|K| \le k$, $|L| \le l$, and $f \in \mathcal{F}$ with $\mathrm{dom}(f) = K$, there exists $g \in \mathcal{F}$ such that $\mathrm{dom}(g) = L$ and $g_{|K} = f$.*

For $K \subseteq X$ with $|K| \le l$, the set of all partial homomorphisms from $\mathcal{F}$ with domain K will be denoted by $\mathcal{F}_K$, i.e., $\mathcal{F}_K = \mathcal{F} \cap A^K$.

A standard procedure [17] called (k,l)-consistency checking finds the greatest (with respect to inclusion) (k,l)-strategy $\mathcal{F}$ for $(\mathbb{X}, \mathbb{A})$. The algorithm starts by throwing initially in $\mathcal{F}$ all partial homomorphisms (from $\mathbb{X}$ to $\mathbb{A}$) with domain of size less than or equal to l. Then we repeatedly remove from $\mathcal{F}$ all the mappings falsifying conditions ($S2$) or ($S3$) until all the conditions are satisfied. It is not difficult to see that this algorithm runs in polynomial time with respect to $|X|$.

Observe that for any homomorphism $f : \mathbb{X} \to \mathbb{A}$ and any $K \subseteq X$ with $|K| \le l$, the partial homomorphism $f_{|K}$ belongs to the strategy returned by the (k,l)-consistency algorithm. Therefore, if the algorithm returns $\mathcal{F} = \emptyset$, then there is certainly no homomorphism from $\mathbb{X}$ to $\mathbb{A}$. The structure $\mathbb{A}$ is of *width* (k,l) if the converse is also true.

DEFINITION 4.2. *A relational structure $\mathbb{A}$ has* width (k,l) *if for every relational structure $\mathbb{X}$ of the same type, if there exists a nonempty (k,l)-strategy for $(\mathbb{X}, \mathbb{A})$, then $\mathbb{X}$ is homomorphic to $\mathbb{A}$. Moreover, $\mathbb{A}$ is said to be of* width k *if it has width (k,l) for some l, and to be of* bounded width *if it has width k for some k.*

In other words, a relational structure $\mathbb{A}$ has bounded width if there exist k and l such that we can use the (k,l)-consistency checking algorithm to solve CSP($\mathbb{A}$). As noted above, this algorithm works in polynomial time; thus, if $\mathbb{A}$ has bounded width, then CSP($\mathbb{A}$) is tractable.

Larose and Zádori [24] proved that if a core $\mathbb{A}$ has bounded width, then the variety generated by $\mathbf{A}$ is congruence meet semidistributive and conjectured the converse.

CONJECTURE 4.3 (the bounded width conjecture). *A core relational structure $\mathbb{A}$ has bounded width if and only if the variety generated by $\mathbf{A}$ is congruence meet semidistributive.*

The conjecture was verified in the case that $\mathbf{A}$ has a semilattice operation [17], a near-unanimity operation [17], a 2-semilattice operation [9], and a short Jónsson chain (see [23] for $CD(3)$ and [15] for $CD(4)$). Our main theorem confirms this conjecture for relational structures $\mathbb{A}$ such that $\mathbf{A}$ generates a congruence distributive variety, or, equivalently, $\mathbf{A}$ has a Jónsson chain of arbitrary length.

THEOREM 4.4. *Let $\mathbb{A}$ be a relational structure such that the variety generated by $\mathbf{A}$ is congruence distributive; then $\mathbb{A}$ has width $(2\lceil\frac{p}{2}\rceil, 3\lceil\frac{p}{2}\rceil)$, where p is the maximal arity of a relation in $\mathbb{A}$ ($\lceil x\rceil$ denotes the upper integer part of x).*

Actually, we prove a stronger, albeit more technical, version of the theorem above.

THEOREM 4.5. *For every relational structure $\mathbb{X}$ of the same type as $\mathbb{A}$, every $(2\lceil\frac{p}{2}\rceil, 3\lceil\frac{p}{2}\rceil)$-strategy $\mathcal{F}$ for $(\mathbb{X}, \mathbb{A})$, every $K \subseteq X$, $|K| \leq \lceil\frac{p}{2}\rceil$, and every $g \in \mathcal{F}_K$, there exists a homomorphism $f : \mathbb{X} \to \mathbb{A}$ such that $f_{|K} = g$.*

As mentioned in the introduction, Conjecture 4.3 was recently confirmed [2] via methods derived from the approach presented in this paper.

5. Reduction to $(2,3)$-systems. We prove Theorem 4.4 using a variation of the definition of a $(2,3)$-strategy for binary relational structures.

DEFINITION 5.1. *A* $(2,3)$-system *is a set of finite (nonempty) $CD(s)$ algebras (for some $s > 0$),*

$$\{\mathbf{B}_i, \mathbf{B}_{i,j} |\ i, j < n\},$$

such that, for any $i, j, k < n$,

($B1$) *$\mathbf{B}_{i,j}$ is a subdirect product of $\mathbf{B}_i$ and $\mathbf{B}_j$;*
($B2$) *$\mathbf{B}_{i,i}$ is a diagonal subalgebra, i.e., $B_{i,i} = \{(a, a)|\ a \in B_i\}$ for any i;*
($B3$) *$(a, b) \in B_{i,j}$ if and only if $(b, a) \in B_{j,i}$;*
($B4$) *if $(a, b) \in B_{i,j}$, then there exists $c \in B_k$ such that $(a, c) \in B_{i,k}$ and $(b, c) \in B_{j,k}$.*

A solution *of such a system is a tuple $(b_0, \dots, b_{n-1})$ such that $(b_i, b_j) \in B_{i,j}$ for any $i, j < n$.*

The following theorem is the core result of this paper.

THEOREM 5.2. *Every $(2,3)$-system has a solution. Moreover, for any $i < n$ and any $b \in B_i$, there exists a solution $(b_0, \dots, b_{n-1})$ such that $b_i = b$.*

We present a proof of Theorem 4.4 (and Theorem 4.5) using Theorem 5.2, which we prove in the next section.

Proof of Theorem 4.4. Let $\mathbb{A}$ be a relational structure such that the variety generated by the associated algebra $\mathbf{A}$ is congruence distributive, let p be the maximal arity of a relation in $\mathbb{A}$, and let $q = \lceil\frac{p}{2}\rceil$. According to Theorem 2.3, there exist a number s and term functions $t_0, \dots, t_s$ of $\mathbf{A}$ (i.e., polymorphisms of $\mathbb{A}$) which form a Jónsson chain. Let $\mathbf{A}'$ denote the $CD(s)$ algebra $(A, t_0, \dots, t_s)$.

Let $\mathbb{X}$ be a relational structure of the same type as $\mathbb{A}$ and let $\mathcal{F}$ be a nonempty $(2q, 3q)$-strategy for $(\mathbb{X}, \mathbb{A})$. From the properties ($S2$), ($S3$) it easily follows that $\mathcal{F}_L$ is nonempty for all L with $|L| \leq 3q$. Let $\mathcal{G}_K$ be the subuniverse of $(\mathbf{A}')^K$ generated by $\mathcal{F}_K$ for every $K \subseteq X$, $|K| \leq 3q$. It is easy to see (and widely known) that the family $\mathcal{G} = \bigcup_{|K|\leq l} \mathcal{G}_K$ is also a $(2q, 3q)$-strategy. Thus we can, without loss of generality, assume that $\mathcal{F}_K$ is a nonempty subuniverse of $(\mathbf{A}')^K$ for every K such that $|K| \leq 3q$.

To prove that there exists a homomorphism $f : \mathbb{X} \to \mathbb{A}$, we assume that $|X| > 3q$ (since otherwise any $f \in \mathcal{F}_X$ is a homomorphism), and we define a $(2,3)$-system indexed by the q-element subsets of X (instead of natural numbers). Let $K, L \subseteq X$ be such that $|K| = |L| = q$. We define the universes of the algebras in the $(2,3)$-system to be the following:

- $B_K = \mathcal{F}_K$,
- $B_{K,L} = \{(f,g) \in B_K \times B_L : \exists h \in \mathcal{F}_{K\cup L}\ h_{|K} = f,\ h_{|L} = g\}$.

Since $\mathcal{F}_K$ is a subuniverse of $(\mathbf{A}')^K$, we can define $\mathbf{B}_K$ to be the subalgebra of $(\mathbf{A}')^K$ with universe B_K. As $\mathcal{F}_{K\cup L}$ is a subuniverse of $\mathbf{A}^{K\cup L}$, it follows that $B_{K,L}$ is a subuniverse of $\mathbf{B}_K \times \mathbf{B}_L$.

To prove that the projection of $B_{K,L}$ to the first coordinate is full, consider an arbitrary $f \in \mathcal{F}_K$. Property $(S3)$ of the strategy $\mathcal{F}$ tells us that there exists $h \in \mathcal{F}_{K\cup L}$ such that $h_{|K} = f$. From property $(S2)$ we get $h_{|L} \in \mathcal{F}$; hence $(f, h_{|L}) \in B_{K,L}$. By an analogous argument, the projection of $B_{K,L}$ to the second coordinate is also full and property $(B1)$ is proved. Property $(B4)$ can be proved similarly, and $(B2)$, $(B3)$ hold trivially.

By Theorem 5.2 there exists a solution $(f_K : K \subseteq X$ and $|K| = q)$ of this $(2,3)$-system. Note that for any K and L, if $i \in K \cap L$, then, since $(f_K, f_L) \in B_{K,L}$, we have $f_K(i) = f_L(i)$. Therefore there exists a well-defined function $f : X \to A$ such that $f_{|K} = f_K$ for any $K \subseteq X$ with $|K| \le q$. The construction immediately implies that, for any $K, L \subseteq X$ with $|K|, |L| \le q$, $f_{|K\cup L}$ is a partial homomorphism from $\mathbb{X}$ to $\mathbb{A}$ and, using Lemma 2.1, we conclude that f is the desired homomorphism.

Theorem 4.5 follows from the “moreover” part of Theorem 5.2. ∎

6. Proof of Theorem 5.2.

In this section we prove Theorem 5.2. Let $\{\mathbf{B}_i, \mathbf{B}_{i,j} | i,j < n\}$ be an arbitrary $(2,3)$-system. We recall that, in such a case, $\mathbf{B}_i = (B_i, t_0, \dots, t_s)$ and $\mathbf{B}_{i,j} = (B_{i,j}, t_0, \dots, t_s)$ are $CD(s)$ algebras satisfying $(B1) - (B4)$.

6.1. Patterns and realizations.

We require the following definitions.

Definition 6.1. *A* pattern *is a finite sequence of natural numbers smaller than n. A concatenation of patterns is performed in a natural way: for patterns w, v we write wv for a pattern equal to the concatenation of patterns w and v, and w^k for a pattern equal to a k-ary concatenation of w with itself. We write w^{-1} for a pattern w with reversed order and we set $w^{-k} = (w^{-1})^k$ for any $k > 0$.*

A pattern can be realized in a $(2,3)$-system as follows.

Definition 6.2. *A sequence $a_0, \dots, a_l$ is called a* realization *of a pattern $w = (w_0, \dots, w_l)$ if $a_i \in B_{w_i}$ for all $i \le l$ and $(a_i, a_{i+1}) \in B_{w_i, w_{i+1}}$ for all $i < l$.*

We say that two elements $a \in B_i$, $b \in B_j$ are connected via a pattern $w = (i, \dots, j)$ if there exists a realization $a = a_0, a_1, \dots, a_l = b$ of the pattern w.

The following lemma is an easy consequence of property $(B4)$.

Lemma 6.3. *Let $(a, b) \in B_{i,j}$. Then a and b can be connected via any pattern beginning with i and ending with j.*

Proof. Let $(a, b) \in B_{i,j}$ and let $w = (i = w_0, \dots, w_m = j)$ be a pattern. Applying $(B4)$ from the definition of the $(2,3)$-system to (a, b) and the coordinates i, j, w_1, we obtain $c_0 \in B_{w_1}$ such that $(a, c_0) \in B_{i,w_1}$ and $(c_0, b) \in B_{w_1,j}$. The element c_0 is the second (after a) element of a realization of the pattern w. Continuing this reasoning, we apply $(B4)$ to $(c_0, b) \in B_{w_1,j}$ and the coordinates w_1, j, w_2 to obtain c_2—the third element of a realization of w. Repeated application of this reasoning produces a realization of the pattern w connecting a to b. ∎

The lemma implies the following corollary.

Corollary 6.4. *Suppose that a and b are elements of B_i which are connected via a pattern v. Let w be any pattern starting and ending with i such that all the numbers that occur in v also do in w. Then a and b are connected via a pattern w^M for a sufficiently large M.*

Proof. Let $v = (i = v_0, \dots, v_l = i)$ and let $a = a_0, \dots, a_l = b$ be a realization of the pattern v. Since v_1 appears in w, there exists an initial part of w, say w', starting with i and ending with v_1. Since $(a, a_1) \in B_{i,v_1}$, we use Lemma 6.3 to connect a to a_1 via w'. Since v_2 appears in w, there exists w'' such that $w'w''$ is an initial part of w^2 and such that w'' ends in v_2. Since $(a_1, a_2) \in B_{v_1,v_2}$, we use Lemma 6.3 again to connect a_1 to a_2 via a pattern v_1w''. Now a_0 and a_2 are connected via the pattern $w'w''$. By continuing this reasoning, we obtain the pattern w^M (for some M) connecting a to b (where at the last step we use Lemma 6.3 to connect b to itself via the remaining part of w, if necessary). $\square$

6.2. Absorbing systems. Our proof proceeds by reducing absorbing systems inside a fixed $(2, 3)$-system.

DEFINITION 6.5. *We say that $C \subseteq B_i$ absorbs B_i if $t_r(c, b, c') \in C$ for any $c, c' \in C$, $b \in B_i$, and $r \leq s$. (In particular, C is a subuniverse of $\mathbf{B}_i$.)*

For a subset C of B_i and $j < n$, we set

$$\gamma_{i,j}(C) = \{b \in B_j : \text{ there exists } a \in C \text{ such that } (a, b) \in B_{i,j}\}.$$

The following lemma lists some properties of absorbing sets.

LEMMA 6.6. *Let $i, j < n$ and $C, D \subseteq B_i$.*

(i) *The set $\{b\}$ absorbs B_i for any $b \in B_i$.*
(ii) *If C and D absorb B_i, then $C \cap D$ absorbs B_i as well.*
(iii) *If C absorbs B_i, then $\gamma_{i,j}(C)$ absorbs B_j.*

Proof.

(i) The proof follows from property $(J3)$ of the Jónsson chain.
(ii) The proof is obvious.
(iii) Let $e, e' \in \gamma_{i,j}(C)$, $d \in B_j$, and $r \leq s$. As $B_{i,j}$ is subdirect (see $(B1)$), there exists $b \in B_i$ such that $(b, d) \in B_{i,j}$. Since $e, e' \in \gamma_{i,j}(C)$, there exist $c, c' \in C$ such that $(c, e) \in B_{i,j}$ and $(c', e') \in B_{i,j}$. The set $B_{i,j}$ is a subuniverse of $\mathbf{B}_i \times \mathbf{B}_j$ (see $(B1)$ again), therefore $(t_r(c, b, c'), t_r(e, d, e')) \in B_{i,j}$. Now $t_r(c, b, c') \in C$ because C absorbs B_i; hence $t_r(e, d, e') \in \gamma_{i,j}(C)$. $\square$

We define a key concept of the proof—an absorbing system.

DEFINITION 6.7. *A sequence $C_0, C_1, \ldots, C_{n-1}$ of* nonempty *sets is an* absorbing system *if, for any numbers $i, j < n$,*

$(A1)$ *$C_i \subseteq B_i$,*
$(A2)$ *$\gamma_{i,j}(C_i) \supseteq C_j$, and*
$(A3)$ *C_i absorbs B_i.*

For such an absorbing system, we define

$$\delta_{i,j}(D) = \gamma_{i,j}(D) \cap C_j,$$

where $D \subseteq C_i$, and, similarly, for any pattern $w = (w_0, \ldots, w_l)$ and any $D \subseteq C_{w_0}$, we write

$$\delta_w(D) = \delta_{w_{l-1},w_l}(\cdots \delta_{w_1,w_2}(\delta_{w_0,w_1}(D)) \cdots).$$

Note that the sequence $B_0, B_1, \ldots, B_{n-1}$ is a trivial absorbing system. The following lemma states an important property of all absorbing systems.

LEMMA 6.8. *Let $C_0, \ldots, C_{n-1}$ be an absorbing system and let $D_0, \ldots, D_l$ and $m_0, \ldots, m_l$ be such that for any $0 \leq i < n$,*

- *$m_0 = m_l$ and $D_0 = D_l$,*
- *$D_i \subseteq C_{m_i}$,*
- *$\delta_{m_i,m_{i+1}}(D_i) = D_{i+1}$;*

then, for any $i, j < n$,

$$\delta_{m_i,m_j}(D_i) = D_j.$$

Proof. Let $w = (m_0, \dots, m_l)$ be the pattern derived from the sequence used in the statement of the lemma. Note that, under our assumptions, $\delta_w(D_0) = D_0$.

We will show first that for any $i, j < n$, we have $\delta_{m_i,m_j}(D_i) \subseteq D_j$. Suppose, for a contradiction (choosing without loss of generality the coordinate m_0), that there exists $a_0 \in C_{m_0} \setminus D_0$ such that for some k and some $a_1 \in D_k$ we have $(a_1, a_0) \in B_{m_k,m_0}$. Since $\gamma_{m_{l-1},m_0}(C_{m_{l-1}}) \supseteq C_{m_0}$, there exists $a_{-1} \in C_{m_{l-1}}$ such that $(a_{-1}, a_0) \in B_{m_{l-1},m_0}$ and, since $a_0 \notin D_0$ and $\delta_{m_{l-1},m_0}(D_{l-1}) = D_0$, we get $a_{-1} \in C_{m_{l-1}} \setminus D_{l-1}$. Repeating the same reasoning for a_{-1} instead of a_0, we obtain a_{-2}. Further on we obtain an infinite sequence of a_i's (for negative i's) and, as the set C_{m_0} is finite, we get $a' \in C_{m_0} \setminus D_0$ such that a' can be connected to itself via a pattern w^{-M} (for some number M) realized *fully* inside the absorbing system and can be connected to a_0 via a pattern containing only numbers from the set $\{m_0, \dots, m_l\}$. Reversing the direction of the realization of the pattern w^{-M}, we can connect a' to itself via the pattern w^M realized also fully in the absorbing system (and obviously M can be substituted by any multiple of M).

Moreover, since a_1 is in D_k and $\delta_{m_{k-1},m_k}(D_{k-1}) = D_k$, there exists an element $a_2 \in D_{k-1}$ such that $(a_2, a_1) \in B_{m_{k-1},m_k}$. Proceeding further as in the previous case, we obtain an infinite sequence of a_i's (with positive i's this time) and then an element $a'' \in D_0$ connected to itself via the pattern w^N (for some number N) realized *fully* inside the absorbing system and connected to a_1 via a pattern containing only numbers from the set $\{m_0, \dots, m_l\}$.

We know that a' is connected to a_0 via a pattern containing only numbers from the set $\{m_0, \dots, m_l\}$ and a'' is connected to a_1 via a pattern containing only numbers from $\{m_0, \dots, m_l\}$. Since $(a_1, a_0) \in B_{m_k,m_0}$, we obtain a connection from a' to a'' containing only numbers from $\{m_0, \dots, m_l\}$. Applying Corollary 6.4 to elements a', a'', and the pattern w^{MN} we immediately obtain two more numbers K and L such that a' can be connected to a'' via a pattern w^{MNK} and a'' can be connected to a' via a pattern w^{MNL} (none of the realizations of these patterns has to be inside the absorbing system).

Let $a'' = b_0, b_1, \dots, b_P = a''$, $a' = c_0, c_1, \dots, c_P = a'$, and $a'' = d_0, d_1, \dots, d_P = a'$ be realizations of the pattern w^{MNL}, where the elements $b_0, b_1, \dots, c_0, c_1, \dots$ are inside the absorbing system. From property $(B1)$ of the $(2,3)$-system, it follows that for any $r \le s$,

$$t_r(a'', a'', a') = t_r(b_0, d_0, c_0),\ t_r(b_1, d_1, c_1),\ \dots$$
$$\dots,\ t_r(b_P, d_P, c_P) = t_r(a'', a', a')$$

is a realization of the pattern w^{MNL}. The absorbing property $(A3)$ implies that this realization lies inside the absorbing system. Similarly, using a realization of the pattern connecting a' to a'', we infer that one can connect $t_r(a'', a', a')$ to $t_r(a'', a'', a')$ via a realization of the pattern w^{MNK} fully inside the absorbing system. By using the properties $(J1)$, $(J2)$, $(J4)$, $(J5)$ of the Jónsson chain, we obtain a realization of a big power of the pattern w connecting a'' to a' fully inside the absorbing system— $a'' = t_0(a'', a'', a')$ is connected to $t_0(a'', a', a')$ via w^{MNL}, $t_0(a'', a', a') = t_1(a'', a', a')$ is connected to $t_1(a'', a'', a')$ via w^{MNK}, and so on. Since $\delta_w(D_0) = D_0$ and $a'' \in D_0$, this contradicts $a' \notin D_0$.

We have proved that $\delta_{m_i,m_j}(D_i) \subseteq D_j$ for any $i, j < n$. Since $\gamma_{m_j,m_i}(C_{m_j}) \supseteq C_{m_i}$, then for any $a \in D_i$ there exists $b \in C_{m_j}$ such that $(a, b) \in B_{m_i,m_j}$. The element b lies necessarily in D_j as $\delta_{m_i,m_j}(D_i) \subseteq D_j$. This shows that $D_i \subseteq \delta_{m_j,m_i}(D_j)$; on the other hand, $\delta_{m_j,m_i}(D_j) \subseteq D_i$, and the lemma is proved. ∎

We compare absorbing systems by inclusion on all of the coordinates; i.e., an absorbing system $C_0, \ldots, C_{n-1}$ is *less than or equal* to an absorbing system $C'_0, \ldots, C'_{n-1}$ if, for all $i < n$, $C_i \subseteq C'_i$. The following lemma states that the minimal elements of this ordering are as small as possible.

LEMMA 6.9. *If $C_0, \ldots, C_{n-1}$ is an absorbing system minimal under inclusion, then every set C_i contains exactly one element.*

Proof. Suppose for a contradiction that one of the sets in the system, say C_0, has more than one element, and let $\{a\} \subsetneq C_0$. Let

$$Z = \{(E, i) : i < n,\ E \subsetneq C_i,\ E = \delta_w(\{a\}) \text{ for some pattern } w = (0, \ldots, i)\}.$$

Note that Z is nonempty (as $(\{a\}, 0) \in Z$) and that for any $(E, i) \in Z$ and any pattern w starting at i and ending at j, either $\delta_w(E) = C_j$ or $(\delta_w(E), j) \in Z$.

Let $(E, i) \in Z$ be arbitrary. The set E was obtained from $\{a\}$ by applying the operation γ and taking intersections with elements of the absorbing system $C_0, \ldots, C_{n-1}$. From Lemma 6.6 it follows that E absorbs B_i.

We define the relation $\preceq$ on the elements of Z in the following way:

$$(E, i) \preceq (E', i') \text{ iff } E' = \delta_w(E) \text{ for some pattern } w = (i, \ldots, i').$$

Since $\preceq$ is a quasi order (i.e., it is reflexive and transitive), we can choose $(E, k) \in Z$ to be one of its maximal elements. From the maximality, we get that for any $j < n$ and any pattern $w = (k, \ldots, j)$, either $\delta_w(E) = C_j$ or there exists a pattern $v = (j, \ldots, k)$ such that $\delta_v(\delta_w(E)) = E$.

We will show that the sequence $E_0 = \delta_{k,0}(E), \ldots, E_{n-1} = \delta_{k,n-1}(E)$ is an absorbing system smaller than $C_0, \ldots, C_{n-1}$. As $E_k = \delta_{k,k}(E) = E \subsetneq C_i$, the new system is smaller. We already know that E_i absorbs B_i for every $k < n$, so it remains for us to prove property $(A2)$ (i.e., that for any $i, j < n$ we have $\gamma_{i,j}(E_i) \supseteq E_j$). Let us fix an arbitrary i and j; if $\delta_{i,j}(E_i) = C_j$, then the inclusion holds trivially. In the other case, as observed above, there exists a pattern $v = (j = v_0, v_1, \ldots, v_l = k)$ such that

$$E = \delta_v(\delta_{i,j}(E_i)).$$

In such a case, Lemma 6.8, applied for the sequence

$$\begin{aligned} m_0 = k, D_0 = E, \quad m_1 = i, D_1 = E_i, \quad m_2 = j, D_2 = \delta_{i,j}(E_i),\\ m_3 = v_1, D_3 = \delta_{(i,j,v_1)}(E_i), \quad m_4 = v_2, D_4 = \delta_{(i,j,v_1,v_2)}(E_i), \ldots\\ \ldots, m_{l+2} = v_l = k, D_{l+2} = \delta_{(i,j,v_1,\ldots,v_l)}(E_i) = D_0, \end{aligned}$$

provides $\delta_{k,j}(D_0) = D_2$. However, $\delta_{k,j}(D_0) = E_j$ and $D_2 = \delta_{i,j}(E_i)$, and the proof is concluded. ∎

6.3. Q.E.D. We are ready to finish the proof of Theorem 5.2. Let $i < n$ and $b \in B_i$ be arbitrary and let $C_j = \gamma_{i,j}(\{b\})$ for every $j < n$. The sequence $C_0, \ldots, C_{n-1}$ is an absorbing system: from $(B1)$ it follows that the sets C_j are nonempty, $(A2)$ follows from $(B4)$, and $(A3)$ is a consequence of Lemma 6.6.

Let us choose any minimal (with respect to inclusion) absorbing system $D_0, \ldots, D_{n-1}$ such that $D_j \subseteq C_j$ for all $j < n$. By Lemma 6.9 it consists of one-element sets. Call b_j

the unique element of D_j. The $(A2)$ property of the absorbing system $D_0, \dots, D_{n-1}$ guarantees that $(b_j, b_k) \in B_{j,k}$ for all $j, k < n$, and therefore $b_0, \dots, b_{n-1}$ constitutes a solution of our $(2,3)$-system. Since $C_i = \{b\}$ $(= D_i)$, we have $b_i = b$, and the theorem is proved.

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