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On the Recognition of Fan-Planar and Maximal Outer-Fan-Planar Graphs

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Abstract *Fan-planar* graphs were recently introduced as a generalization of 1-*planar* graphs. A graph is *fan-planar* if it can be embedded in the plane, such that each edge that is crossed more than once, is crossed by a bundle of two or more edges incident to a common vertex. A graph is *outer-fan-planar* if it has a fan-planar embedding in which every vertex is on the outer face. If, in addition, the insertion of an edge destroys its outer-fan-planarity, then it is *maximal outer-fan-planar*. In this paper, we present a linear-time algorithm to test whether a given graph is *maximal outer-fan-planar*. The algorithm can also be employed to produce an outer-fan-planar embedding, if one exists. On the negative side, we show that testing fan-planarity of a graph is NP-complete, for the case where the rotation system (i.e., the cyclic order of the edges around each vertex) is given.

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1 Introduction

A *simple drawing* of a graph is a representation of a graph in the plane, where each vertex is represented by a point and each edge is a Jordan curve connecting its end-points such that no edge contains a vertex in its interior, no two edges incident to a common end-vertex cross, no edge crosses itself, no two edges meet tangentially, and no two edges cross more than once. An important subclass of drawn graphs is the class of planar graphs, which have no edge crossings. Although planarity is among the most desirable properties when drawing a graph, many real-world graphs are in fact non-planar.

On the other hand, it is widely accepted that edge crossings have negative impact on the human understanding of a graph drawing [31] and simultaneously it is NP-complete in general to find drawings with minimum number of edge crossings [19]. This motivated the study of "almost planar" graphs which may contain crossings as long as they do not violate some prescribed forbidden crossing patterns. Typical examples of such graphs include k-planar graphs [32], k-quasi planar graphs [2], RAC graphs [12] and fan-crossing free graphs [9].

Fan-planar graphs were recently introduced in the same context [25]. Typically, a *fan-planar drawing* of graph $G = (V, E)$ is a simple drawing which allows for more than one crossing on an edge $e \in E$ if and only if the edges that cross e are incident to a common vertex on the same side of e. Such a crossing is called *fan-crossing* (see Fig. 1a). An equivalent definition can be stated by means of forbidden crossing patterns; see Fig. 1b–d. A graph is *fan-planar* if it admits a fan-planar drawing. Note that the class of fan-planar graphs is in a sense the complement of the class of fan-crossing free graphs [9], which simply forbid fan-crossings.

Kaufmann and Ueckerdt [25] showed that a fan-planar graph on n vertices can have at most $5n - 10$ edges and this bound is tight. Binucci et al. [6] proved that testing fan-planarity in the variable embedding setting is NP-complete. They gave tight bounds on the density of constrained versions of fan-planar drawings and examined the relationship between fan-planarity and k-planarity.

An *outer-fan-planar drawing* is a fan-planar drawing in which all vertices are on the outer face. A graph is *outer-fan-planar* if it admits an outer-fan-planar drawing. An outer-fan-planar graph is *maximal outer-fan-planar* if adding any edge to it yields a graph that is not outer-fan-planar. Note that the forbidden pattern II is irrelevant for outer-fan-planarity.

Our main contribution is a linear time algorithm for the recognition of maximal outer-fan-planar graphs and significant insights in their structural properties (see Sect. 2). As a byproduct we obtain that a 3-connected maximal outer-fan-planar graph with n vertices has exactly $2n$ or $3n - 6$ edges. Note that an outer-fan-planar graph with n vertices has at most $3n - 5$ edges [6]. We also prove that testing fan-planarity is NP-complete even if the *rotation system* (i.e., the circular order of the edges around each vertex) is given (see Sect. 3). We conclude in Sect. 4 with open problems and future work.

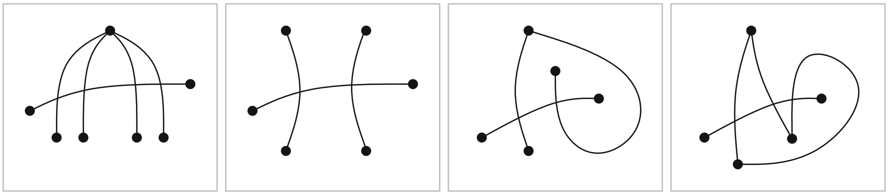


Fig. 1 (taken from [25]) **a** Illustration of a fan-crossing. **b** Forbidden crossing pattern I: an edge cannot be crossed by two independent edges. **c** Forbidden crossing pattern II: an edge cannot be crossed by two edges having their common end-point on different sides of it. **d** Forbidden crossing pattern II implies that an edge cannot be crossed by three edges forming a triangle

1.1 Related Work

As already stated, k-planar graphs [32], k-quasi planar graphs [2], RAC graphs [12] and fan-crossing free graphs [9] are closely related to the class of graphs we study. A graph is *k-planar*, if it can be embedded in the plane with at most k crossings per edge. Obviously, 1-planar graphs are also fan-planar. A 1-planar graph with n vertices has at most $4n - 8$ edges and this bound is tight [7,16,30]. Grigoriev and Bodlaender [21], and, independently Kohrzik and Mohar [27] proved that the problem of determining whether a graph is 1-planar is NP-hard and remains NP-hard, even if the deletion of an edge makes the input graph planar [8].

On the positive side, Eades et al. [13] presented a linear time algorithm for testing *maximal* 1-planarity of graphs with a given rotation system. Testing outer-1-planarity of a graph can be solved in linear time, as shown independently by Auer et al. [4] and Hong et al. [23]. Note that an outer-1-planar graph is always planar [4], while this is not true in general for outer-fan-planar graphs. Indeed, the complete graph K_5 is outer-fan-planar, but not planar.

The well-known Fary's theorem [17] proved that every plane graph admits a straight-line drawing. However, Thomassen [33] presented two forbidden subgraphs for straight-line drawings of 1-plane graphs. Hong et al. [24] gave a linear-time testing and drawing algorithm to construct a straight-line 1-planar drawing, if it exists. Recently, Nagamochi solved the more general problem of straight-line drawability for wider classes of embedded graphs [28]. On the other hand, Eggleton showed that every outer-1-planar graph admits an outer-1-planar straight-line drawing [15].

A drawn graph is called *k-quasi planar* if it does not contain k mutually crossing edges. Fan-planar graphs are 3-quasi planar, since they cannot contain three independent edges that mutually cross. It is conjectured that the number of edges of a k-quasi planar graph is linear in the number of its vertices. Pach et al. [29] and Ackerman [1] showed that this conjecture holds for 3- and 4-quasi planar graphs, respectively. Fox and Pach [18] showed that every k-quasi-planar graph with n vertices has at most $O(n \log^{1+o(1)} n)$ edges.

A different forbidden crossing pattern arises in RAC drawings where two edges are allowed to cross, as long as the crossings edges form right angles. Graphs that admit such drawings (with straight-line edges) are called *right-angle crossing graphs* or *RAC graphs*, for short. Didimo et al. [12] showed that a RAC graph with n vertices cannot have more than $4n - 10$ edges and that this bound is tight. It is also known that a RAC graph is quasi planar [12], while a maximally dense RAC graph (i.e., a RAC graph with n vertices and exactly $4n - 10$ edges) is 1-planar [14]. Testing whether a given graph is a RAC graph is NP-hard [3]. Dekhordi and Eades [10] proved that every outer-1-plane graph has a straight-line RAC drawing, at the cost of exponential area.

1.2 Preliminaries

We consider finite, undirected and simple graphs. A graph G is *connected* if for any pair of vertices there is a path connecting them. G is *simply connected* if it can be disconnected by removing only one vertex, which is called a *cut-vertex* of G. G is *biconnected* (or 2-*connected*) if removing any vertex leaves G connected, i.e. there is no cut-vertex in a biconnected graph. Two vertices v and w are a *separation pair* of a biconnected graph G if the graph that results from G by deleting v and w is not connected. A graph is *triconnected* (or 3-*connected*) if it contains more than three vertices but no separation pair.

The *rotation system* of a drawing is the counterclockwise order of the incident edges around each vertex. The *embedding* of a drawn graph consists of its rotation system and for each edge the sequence of edges crossing it. So an embedding not only describes the pairs of crossing edges, but also specifies the order in which the crossings occur along the edges. We denote by $V[G]$ the vertex set and by $E[G]$ the edge set of a graph G. For a vertex $v \in V[G]$, we denote by $G - \{v\}$ the graph that results from G by removing v.

We now briefly recall the SPQR-tree data structure [22]. An *SPQR-tree* $\mathcal{T}$ is a labeled tree that represents the decomposition of a biconnected graph G into its 3-connected components; in order to avoid confusion, we will refer to the elements of $\mathcal{T}$ as nodes and arcs, instead of vertices and edges, respectively. Each node ν of $\mathcal{T}$ is labeled with a multi-graph G_ν—called the *skeleton* of ν. There are four different types of labels with the following skeletons: (i) *S-nodes:* a simple cycle. (ii) *P-nodes:* three or more parallel edges. (iii) *R-nodes:* a simple 3-connected graph. (iv) *Q-nodes:* a single edge. No two S-nodes, nor two P-nodes are adjacent in an SPQR-tree. For each node ν of $\mathcal{T}$ there is a one-to-one correspondence of the edges of the skeleton of G_ν and the arcs of $\mathcal{T}$ incident to ν. Further, let $\{\nu, \mu\}$ be an arc of $\mathcal{T}$ and let e_ν and e_μ be the edges of G_ν and G_μ, respectively, that are assigned to arc $\{\nu, \mu\}$. Then, e_ν and e_μ have the same end-vertices.

Suppose we are given an SPQR-tree $\mathcal{T}$ of an unknown graph G, then G can be constructed by iteratively merging arcs of $\mathcal{T}$ as follows. For an arc $\{\nu, \mu\}$ of the current tree, let G_ν and G_μ be the skeletons currently associated with ν and μ, respectively. Remove the edge associated with $\{\nu, \mu\}$ from both G_ν and G_μ – except if they are Q-nodes. Let the graph associated with the node that results from merging ν and μ be the union of (the remaining parts of) G_ν and G_μ.

The edges of a skeleton of a node ν are called *virtual edges* if they correspond to an arc incident to ν that is not incident to a Q-node and *real edges* otherwise. Note that real edges correspond to the edges of the graph G represented by the SPQR-tree. Every biconnected graph has a unique SPQR-tree and the SPQR-tree of a biconnected graph can be constructed in linear time [22].

2 Recognizing and Drawing Maximal Outer-Fan-Planar Graphs

Throughout this section let $G = (V, E)$ be a graph on n vertices. We prove the following theorem.

Theorem 1 *There is a linear time algorithm to decide whether a graph is maximal outer-fan-planar and if so a corresponding straight-line drawing with all vertices on a circle can be computed in linear time.*

We first observe that biconnectivity is a necessary condition for maximal outer-fan-planarity, i.e., a simply connected graph (which is not biconnected) cannot be maximal outer-fan-planar. Indeed, if an outer-fan-planar drawing has a cut-vertex c, it is always possible to draw an outer edge, very close to the current drawing boundary, connecting two neighbors of c while preserving the outer-fan-planarity of the graph. The following lemma gives a useful property for devising a testing strategy: we only have to check whether G admits a straight-line fan-planar drawing on a circle $\mathcal{C}$; such a drawing is completely determined by the cyclic ordering of the vertices on $\mathcal{C}$.

Lemma 1 *A biconnected graph G is outer-fan-planar if and only if it admits a straight-line outer-fan-planar drawing in which the vertices of G are restricted on a circle $\mathcal{C}$.*

Proof Let G be an outer-fan-planar graph and let Γ be an outer-fan-planar drawing of G. We will only show that G has a straight-line outer-fan-planar drawing whose vertices lie on a circle $\mathcal{C}$ (the other direction is trivial). The order of the vertices along the outer face of Γ completely determines whether two edges cross, as in a simple drawing no two incident edges can cross and any two edges can cross at most once. Now, assume that two edges cross another edge in Γ. Then, both edges have to be incident to the same vertex; hence, cannot cross each other. So, the order of the crossings on an edge is also determined by the order of the vertices on the outer face. Hence, we can construct a drawing $\Gamma_{\mathcal{C}}$ by placing the vertices of G on a circle $\mathcal{C}$ preserving their order in the outer face of Γ and draw the edges as straight-line segments. □

We now show that a maximal outer-fan-planar graph is not only biconnected, but also Hamiltonian and the boundary of any outer-fan-planar drawing is a simple closed curve consisting of crossing-free edges.

Lemma 2 *Let Γ be a maximal outer-fan-planar drawing of a graph G with $n \geq 3$ vertices. Then, the following conditions hold:*

(i) G is Hamiltonian;
(ii) the boundary of Γ is a simple closed curve $\mathcal{C}$, which is a drawing of a Hamiltonian circuit C of G, and every edge of C is crossing-free in Γ.

Proof It is sufficient to prove condition (ii), because condition (i) is a direct consequence of it. Let $\mathcal{C}$ denote the boundary of Γ, and let u and v be two vertices that are consecutive on $\mathcal{C}$. We first observe that u and v must be adjacent in G. If not, it is easy to draw an open curve, within the outer face of Γ and very close to its boundary, connecting u and v and preserving the outer-fan-planarity. But this would imply that G is not maximal.

We now show that (u, v) must be a crossing-free outer edge of Γ, hence, it is entirely contained in $\mathcal{C}$. Clearly, (u, v) cannot be drawn as an inner edge, i.e. as an open curve $\mathcal{C}_{uv}$ completely contained in the open plane region enclosed by $\mathcal{C}$. More precisely, considering vertex u (an analogous argument applies also to v), there have to be two incident edges, say (u, x) and (u, x'), whose corresponding curves $\mathcal{C}_{ux}$ and $\mathcal{C}_{ux'}$ have non-empty intersection with $\mathcal{C}$; in particular, two specific portions of $\mathcal{C}_{ux}$ and $\mathcal{C}_{ux'}$, that are incident to u, must be contained in $\mathcal{C}$. Hence, $\mathcal{C}_{uv}$ must cross either $\mathcal{C}_{ux}$ or $\mathcal{C}_{ux'}$, but this is not allowed in a simple drawing.

Now, suppose that $\mathcal{C}_{uv}$ is not entirely contained within $\mathcal{C}$, thus it exits from $\mathcal{C}$ crossing some edge e of G. Then, $\mathcal{C}_{uv}$ cannot go back inside $\mathcal{C}$. Indeed, it cannot cross again e or some other edge e', because, in the first case, Γ would not be a simple drawing, while in the second case, one of the two end-vertices of e and of e' would be an inner vertex. On the other hand, if $\mathcal{C}_{uv}$ does not go back inside $\mathcal{C}$, then one of the two vertices x and x' would be an inner vertex as well, which is not possible in an outer-fan-planar drawing. Hence, all pairs of vertices that are consecutive on $\mathcal{C}$ are joined by a crossing-free curve entirely contained in $\mathcal{C}$. Thus, $\mathcal{C}$ is composed from the union of such curves, which proves condition (ii). □

Since fan-planar graphs with n vertices have at most $5n - 10$ edges [25], we may assume that the number of edges is linear in the number of vertices. We first consider the case that G is 3-connected (see Sect. 2.1) and then using SPQR-trees we show how the problem can be solved for biconnected graphs (see Sect. 2.2).

2.1 The 3-Connected Case

Assume that a straight-line drawing of a 3-connected graph G with n vertices on a circle $\mathcal{C}$ is given. Let $v_1, \ldots, v_n$ be the order of the vertices around $\mathcal{C}$. An edge $\{v_i, v_j\}$ is an *outer edge*, if $i - j \equiv \pm 1 \pmod{n}$, a *2-hop*, if $i - j \equiv \pm 2 \pmod{n}$, and a *long edge* otherwise. G is a *complete 2-hop graph*, if there are all outer edges and all 2-hops, but no long edges. Two crossing long edges are a *scissor* if their end-points form two consecutive pairs of vertices on $\mathcal{C}$. We say that a triangle is an *outer triangle* if two of its three edges are outer edges. We call an outer-fan-planar drawing *maximal*, if adding any edge to it yields a drawing that is not outer-fan-planar.

Our algorithm is based on the observation that if a graph is 3-connected maximal outer-fan-planar, then it is a complete 2-hop graph, or we can repeatedly remove any degree-3 vertex until only a triangle is left. In a second step, we reinsert the vertices maintaining outer-fan-planarity (if possible). It turns out that we have to check a constant number of possible embeddings. In the following, we prove some necessary properties. The next three lemmas are used in the proof of Lemma 7.

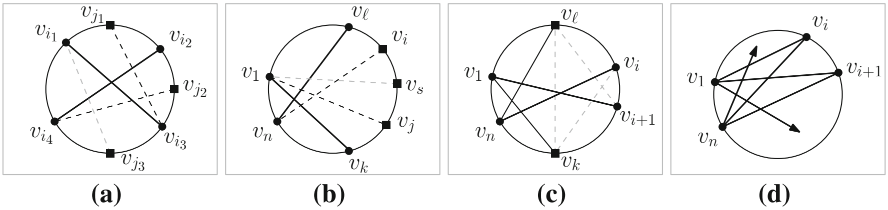


Fig. 2 Different configurations used in: **a** Lemma 3, **b** Lemma 4, **c**, **d** Lemma 5

Lemma 3 *Let G be a 3-connected outer-fan-planar graph embedded on a circle $\mathcal{C}$. If two long edges cross, then two of its end-points are consecutive on $\mathcal{C}$.*

Proof Assume to the contrary that there exist two long crossing edges $\{v_{i_1}, v_{i_3}\}$ and $\{v_{i_2}, v_{i_4}\}$, such that $2 \leq i_1 \leq i_2 - 2 \leq i_3 - 4 \leq i_4 - 6$ and $i_4 = n$; see Fig. 2a. Since G is 3-connected, there has to be a vertex v_{j_1} with $i_1 < j_1 < i_2$, such that v_{j_1} is adjacent to a vertex not in $\{v_{i_1}, \dots, v_{i_2}\}$. By outer-fan-planarity, this can only be v_{i_3} or v_{i_4}; say without loss of generality v_{i_3}. Likewise there is a vertex v_{j_2} with $i_2 < j_2 < i_3$, such that v_{j_2} is adjacent to a vertex not in $\{v_{i_2}, \dots, v_{i_3}\}$. By outer-fan-planarity this can now only be v_{i_4}. But now outer-fan-planarity does not permit to add an edge connecting the two parts separated by v_{i_3} and v_{i_4}. □

Lemma 4 *Let G be a 3-connected outer-fan-planar graph embedded on a circle $\mathcal{C}$. If there are two long crossing edges, then there is a scissor, as well.*

Proof Let e and e' be two long crossing edges. By Lemma 3, it follows that two of the end-points of e and e' are consecutive on $\mathcal{C}$. So, assume without loss of generality that the vertices on $\mathcal{C}$ are labeled such that $e = \{v_1, v_k\}$ and $e' = \{v_\ell, v_n\}$ for some $\ell < k$; see Fig. 2b. If $k = \ell + 1$, then the lemma holds. If this is not the case, then among all crossing long edges with end-vertices v_1 and v_n on one hand and end-vertices between v_ℓ and v_k on the other hand, let edges $\{v_1, v_j\}$ and $\{v_n, v_i\}$, with $\ell \leq i < j \leq k$ be the ones for which the difference $j - i$ is minimal. Obviously, if $j = i + 1$, then the edges $\{v_1, v_j\}$ and $\{v_n, v_i\}$ are a scissor. Assume now that $j > i + 1$. Since v_i and v_j cannot be a separation pair, there has to be an edge between a vertex v_s with $j < s < i$ and a vertex v_t with $t < i$ or $t > j$. By outer-fan-planarity $t = 1$ or $t = n$. This contradicts the initial choice of $\{v_1, v_j\}$ and $\{v_i, v_n\}$. □

Lemma 5 *Let G be a 3-connected graph embedded on a circle $\mathcal{C}$ with a maximal outer-fan-planar drawing. If G contains a scissor, then its end-vertices induce a K_4.*

Proof Assume without loss of generality that the vertices on $\mathcal{C}$ are labeled such that $\{v_1, v_{i+1}\}$ and $\{v_i, v_n\}$ is a scissor, for some $1 < i < n$. We have to show that $\{v_1, v_i\} \in E[G]$ and $\{v_n, v_{i+1}\} \in E[G]$. By outer-fan-planarity there cannot be an edge $\{v_\ell, v_k\}$, such that $1 < \ell < i$ and $i + 1 < k < n$; see Fig. 2c. Since v_1 and v_i cannot be a separation pair, there has to be an edge between v_n or v_{i+1} and a vertex v_ℓ with $1 < \ell < i$; say from v_n. Similarly, since v_n and v_{i+1} cannot be a separation pair, there has to be an edge between v_1 or v_i and a vertex v_k, with $i + 1 < k < n$. By outer-fan-planarity, this particular vertex can only be an edge from v_1, as otherwise edge

$\{v_1, v_{i+1}\}$ would be crossed by two independent edges; see Fig. 2d. As a consequence, there cannot be an edge between v_i and a vertex v_k, $i + 1 < k < n$ nor an edge between v_{i+1} and a vertex v_ℓ, $1 < \ell < i$. Hence, the edge $\{v_1, v_i\}$ is only crossed by edges incident to v_n. Moreover, any edge that is crossed by $\{v_i, v_1\}$ is already crossed by two edges incident to v_1. Since G is maximal outer-fan-planar, it must contain edge $\{v_i, v_1\}$. A similar argument holds for $\{v_n, v_{i+1}\}$. □

The proof of Lemma 5 actually shows that if there is a scissor then the vertices can be labeled such that $\{v_1, v_{i+1}\}$ and $\{v_i, v_n\}$ is the scissor and the only edge connecting $v_2, \ldots, v_i$ on one hand and $v_{i+1}, \ldots, v_{n-1}$ on the other hand is the edge $\{v_i, v_{i+1}\}$. We examine the case when there is no scissor.

Lemma 6 *Let G be a 3-connected graph embedded on a circle $\mathcal{C}$ with a maximal outer-fan-planar drawing. Let e be a long edge that is not crossed by another long edge. Then e is crossed by exactly one 2-hop e' and the four end-vertices of e and e' induce a K_4.*

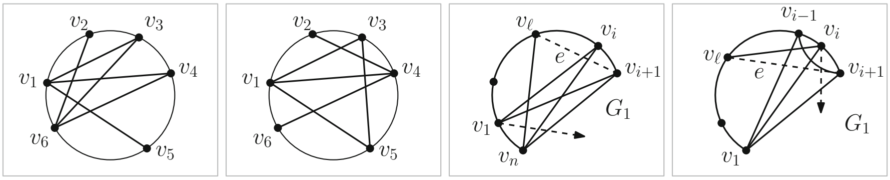
Proof If a long edge e was crossed by two 2-hops, then these two 2-hops would have to be independent. This is impossible in a fan-planar drawing. By 3-connectivity, e has to be crossed by at least one 2-hop e'. Let the vertices be labeled such that $e = \{v_1, v_i\}$ and $e' = \{v_{i-1}, v_{i+1}\}$. Then we could easily add the edges $\{v_1, v_{i-1}\}$ and $\{v_1, v_{i+1}\}$ along e and e'. □

Lemma 7 *Let G be a 3-connected graph with a maximal outer-fan-planar drawing that contains a long edge or let G be a K_5 minus a 2-hop. Then G contains exactly two vertices of degree three. Moreover, each of the two vertices of degree three is incident to two outer edges and a 2-hop and the neighborhood of a vertex of degree three induces a triangle.*

Proof We prove the lemma by induction on the number n of vertices. If $n = 5$ then G is a K_5 minus one edge and thus, the lemma is true. If G has at least six vertices, we distinguish whether G contains crossing long edges or not. Assume first that G contains two crossing long edges and thus, by Lemma 4 a scissor. Assume that the vertices are labeled such that $\{v_1, v_{i+1}\}$ and $\{v_i, v_n\}$ is a scissor and the only edge connecting $v_2, \ldots, v_i$ on one hand and $v_{i+1}, \ldots, v_{n-1}$ on the other hand is the edge $\{v_i, v_{i+1}\}$. Let G_1 be the subgraph of G induced by $\{v_1, \ldots, v_{i+1}, v_n\}$ and let G_2 be the subgraph of G induced by $\{v_1, v_i \ldots, v_n\}$.

We show that we can apply the induction hypothesis to G_1. If $i = 3$ then G_1 is a K_5 minus a 2-hop. Assume now that $i > 3$. G_1 is 3-connected. $\{v_1, v_i\}$ is a long edge. It remains to show that the outer-fan-planar drawing of G_1 is maximal. Assume that we could add an edge e to G_1 maintaining outer-fan-planarity. The only edges that would have prevented e to be present in G are edges connecting v_1 to a vertex in $\{v_{i+2}, \ldots, v_{n-1}\}$. So let $e' = \{v_1, v_j\}$ for some $i+1 < j < n$. We have to distinguish two cases: (a) e crosses e' or (b) e is not incident to v_1 but crosses an edge that is crossed by e'.

If e crosses e', then $e = \{v_n, v_\ell\}$ for some $1 < \ell < i$. Note that in this case e already crosses two edges incident to v_1 in G_1 and that e' already crosses two edges of G_1



(a) $n = 6$, scissor **(b)** $n = 6$, no scissor **(c)** Case: scissor **(d)** Case: no scissor

Fig. 3 Illustration of the proof of Lemma 7

that are incident to v_n. It follows that we could have added e already to G maintaining outer-fan-planarity.

If e is not incident to v_1 but crosses an edge e'' that is crosses by e' then $e = \{v_\ell, v_{i+1}\}$ for some $\ell = 2, \ldots, i-1$—otherwise e'' would cross the two independent edges $\{v_1, v_{i+1}\}$ and e in G_1. It follows that v_ℓ is the only neighbor of v_n among $v_2, \ldots, v_{i-1}$—otherwise e and an edge incident to v_n would be two independent edges crossing $\{v_1, v_i\}$ in G_1; see Fig. 3a, c. But then either v_1, v_ℓ or v_ℓ, v_i would be a separation pair.

Hence, by the inductive hypothesis, G_1 contains exactly two vertices of degree three and these two vertices have the desired properties. One of these two vertices is v_{i+1}. Hence, G_1 and, thus, G, contains exactly one vertex of degree three among $\{v_2, \ldots, v_{i-1}\}$. Applying the same argument to G_2, we obtain that G contains two vertices of degree three, one in $\{v_2, \ldots, v_{i-1}\}$ and one in $\{v_{i+2}, \ldots, v_{n-1}\}$ and they both have the desired properties.

Assume now that G contains no two crossing long edges. By Lemma 6 we may assume that the vertices are labeled such that $\{v_1, v_i\}$ is a long edge which is crossed by the 2-hop $\{v_{i-1}, v_{i+1}\}$. Further observe that by 3-connectivity v_i has to be adjacent to both, a vertex in $\{v_2, \ldots, v_{i-2}\}$ and a vertex in $\{v_{i+2}, \ldots, v_n\}$. Let G_1 be the subgraph of G induced by $\{v_1, \ldots, v_{i+1}\}$ and let G_2 be the subgraph of G induced by $\{v_1, v_{i-1} \ldots, v_n\}$. By similar arguments as in the first case, we obtain that the inductive hypothesis can be applied to G_1 and G_2; see Fig. 3b, d. Since v_{i+1} is a vertex of degree three in G_1 and v_{i-1} is a vertex of degree three in G_2, we conclude again that G fulfills the desired properties. □

Lemma 8 *Let G be a 3-connected graph with $n \geq 5$ vertices and let $v \in V[G]$ be a vertex of degree three that is contained in a K_4. Then, $G - \{v\}$ is 3-connected.*

Proof Let a, b, c and d be four arbitrary vertices of $G - \{v\}$. Since G was 3-connected, there was a path P from a to b in $G - \{c, d\}$. Assume that P contains v. Since v is only connected to vertices that are connected to each other, there is also another path from a to b in $G - \{c, d\}$ not containing v. Hence, a and b cannot be a separation pair in $G - \{v\}$. Since a and b were arbitrarily selected, $G - \{v\}$ is 3-connected. □

Lemma 9 *Let G be a 3-connected graph with $n > 6$ vertices, let v_1, v_2, v_3 and v_4 be four vertices that induce a K_4, such that the degree of v_3 is three. Then, $G - \{v_3\}$ has a maximal outer-fan-planar drawing if G has a maximal outer-fan-planar drawing.*

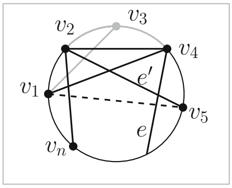


Fig. 4 Configuration used in Lemma 9

Proof Consider a maximal outer-fan-planar drawing of G on a circle $\mathcal{C}$ and let $v_1, v_2, v_3, v_4, \ldots, v_n$ be the order of the vertices on $\mathcal{C}$. Assume to the contrary that after removing v_3, we could add an edge e to the drawing; see Fig. 4. Observe that $\{v_3, v_1\}$ is the only edge incident to v_3 that crosses some edges of $G - \{v_3\}$. We have to consider the following two cases: (a) e crosses $\{v_3, v_1\}$ or (b) e is not incident to v_1 but there is an edge e' that crosses e and $\{v_3, v_1\}$.

In the first case, e is incident to v_2 and crosses $\{v_1, v_4\}$. Since $G - \{v_3\}$ plus e is outer-fan-planar it follows that all edges that cross e are incident to v_1 or v_4. Since G plus e is not outer-fan-planar, it follows that there is an edge incident v_4 that crosses e.

In the second case, it follows that e' has to be incident to v_2 and cross $\{v_1, v_4\}$. Hence, since $G - \{v_3\}$ plus e is outer-fan-planar it follows that e is incident v_4. Hence, in both cases we have two crossing edges e and e', one of which is incident to v_2 and one of which is incident to v_4.

Let now i be maximal so that there is an edge $\{v_2, v_i\}$ in $G - \{v_3\}$ plus e. If $i \neq n$, then v_1 and v_i is a separation pair: Any edge connecting $\{v_{i+1}, \ldots, v_{n-1}\}$ to $\{v_2, v_3, \ldots, v_{i-1}\}$ and not being incident to v_2 crosses $\{v_2, v_i\}$. But edges crossing $\{v_2, v_i\}$ can only be incident to v_1, a contradiction. Now, let $j > 4$ be minimum such that there is an edge $\{v_2, v_j\}$. We claim that $j = 5$. If this is not the case, then similarly to the previous case v_4 and v_j would be a separation pair in $G - \{v_3\}$ plus e, which is not possible due to Lemma 8.

Since G is outer-fan-planar, in G there cannot be an edge $\{v_4, v_k\}$ for some $k = 6, \ldots, n$, since it would cross $\{v_2, v_5\}$ which is crossed by $\{v_3, v_1\}$. This already excludes case (a). For case (b), we now conclude that G has to contain edge $\{v_1, v_5\}$: Observe that $\{v_1, v_5\}$ could only cross edges incident to v_2 that are already crossed by $\{v_3, v_1\}$ and $\{v_4, v_1\}$. Hence, $\{v_1, v_5\}$ could be added to G without violating outer-fan-planarity. Since e and $\{v_2, v_n\}$ both cross $\{v_1, v_5\}$ it follows that $e = \{v_4, v_n\}$. But now, v_5 and v_n has to be a separation pair. □

Lemma 10 *Let G be a graph with 6 vertices containing a vertex v of degree three. Then, G is maximal outer-fan-planar if and only if $G - \{v\}$ is a K_5 missing one 2-hop that connects a neighbor of v to one of the other two vertices.*

Proof Concerning the sufficient part, we preliminarily observe that no subgraph of K_6 with $n \geq 13$ vertices is outer-fan-planar. In other words, considering any drawing of K_6 with all vertices on a circle $\mathcal{C}$, the removal of any pair of inner edges cannot delete all forbidden patterns. Hence, an outer-fan-planar graph with six vertices has at most twelve edges, and thus, for proving the maximal outer-fan-planarity of G, that has twelve edges, it is sufficient to show the existence of an outer-fan-planar embedding. We now show that G is outer-fan-planar if $G - \{v\}$ is a K_5 minus a 2-hop. Consider an outer-fan-planar drawing of $G - \{v\}$ on a circle $\mathcal{C}$ and let v_i, $i = 1, \ldots, 5$, be the order of the vertices on $\mathcal{C}$. Without loss of generality assume that v_1 and v_3 denote the two vertices of degree three, i.e. the two end-vertices of the missing 2-hop, and that v is adjacent to v_1, v_2 and v_5 in G; observe that in the negative case, it is always possible to permute the positions of the vertices on $\mathcal{C}$ so that these conditions are fulfilled. Now, it is immediate to see that an outer-fan-planar drawing of G can be obtained from the one of $G - \{v\}$ by placing vertex v in any point of the circular arc of $\mathcal{C}$ delimited by v_1 and v_2.

To prove the necessary part, we consider a maximal outer-fan-planar drawing Γ of G and observe that Γ has to contain a long edge, otherwise G would be a complete 2-hop graph, which is not possible because there is a vertex v of degree three. We now distinguish two cases: (a) there exist two crossing long edges in Γ, or (b) every long edge is crossed only by a 2-hop in Γ. Suppose that Γ has two crossing long edges e and e', and w.l.o.g assume that $e = (v_1, v_4)$ and $e' = (v_3, v_6)$. Edges e and e' form a scissor in Γ, therefore, by Lemma 5, their end-vertices induce a K_4, which implies that v_1 and v_3 must be joined by a 2-hop as well as v_4 and v_6. Thus, there cannot be an edge connecting v_2 to v_5, because it would pass through the scissor, creating forbidden configurations. On the other hand, by 3-connectivity and outer-fan-planarity, v_2 and v_5 must be connected to either v_4 and v_3, respectively, or to v_6 and v_1, respectively. In both cases, G is a graph consisting of two vertices of degree five, two vertices of degree four and two vertices of degree three, from which immediately follows that $G - \{v\}$ is a K_5 minus a 2-hop.

We conclude the proof by examining case (b). Without loss of generality, assume that a long edge $e = (v_1, v_4)$ is crossed by a 2-hop $e' = (v_3, v_5)$ in Γ. By Lemma 6, the end-vertices of e and e' induce a K_4. Hence, v_1 and v_3 are adjacent as well as v_1 and v_5. Thus, there cannot be an edge connecting v_2 to v_6, otherwise e would be crossed by two independent edges. On the other hand, by 3-connectivity, v_2 (v_6, respectively) has a third incident edge, and this edge can only be $(v_2, v_4)((v_6, v_4)$, respectively), because (v_2, v_5) $((v_6, v_3)$, respectively) is a long edge that crosses e. Therefore, even in this case, G consists of two vertices of degree five, two vertices of degree four and two vertices of degree three, and thus $G - \{v\}$ is a K_5 minus a 2-hop. □

Lemma 11 *It can be tested in linear time whether a graph is a complete 2-hop graph. Moreover, if a graph is a complete 2-hop graph, then it has a constant number of outer-fan-planar embeddings and these can be constructed in linear time.*

Proof Let G be an n-vertex graph. We test whether G is a complete 2-hop graph as follows. If $n \in \{4, 5\}$, then G is either a K_4 or K_5. Otherwise, check first whether all vertices have degree four. If so, pick one vertex as v_1, choose a neighbor as v_2 and a common neighbor of v_1 and v_2 as v_3 (if no such common neighbor exists, then G is not a complete 2-hop graph.). Assume now that we have already fixed $v_1, \ldots, v_i$, $3 \leq i < n$. Test whether there is a unique vertex $v \in V \setminus \{v_1, \ldots, v_i\}$ that is adjacent to v_i and v_{i-1}. If so, set $v_{i+1} = v$. Otherwise reject. Do this for any possible choices of v_2 and v_3, i.e., for totally at most 12 choices. □

Remark 1 No degree-3 vertex can be added to a complete 2-hop at least 5 vertices.

We are now ready to describe our algorithm. If the graph is not a complete 2-hop graph, iteratively remove a vertex of degree 3. If G is maximal outer-fan-planar, Lemma 7 guarantees that such a vertex always exists in the beginning. Remark 1 guarantees that also in subsequent steps there is a long edge and, thus, Lemmas 8 and 9 guarantee that also in subsequent steps, we can apply Lemma 7 as long as we have at least six vertices. Lemma 10 guarantees that we can also remove two more vertices of degree 3 ending with a triangle.

At this stage, we already know that if the graph is outer-fan-planar, it is indeed maximal outer-fan-planar. Either, we started with a complete 2-hop graph or we iteratively removed vertices of degree three yielding a triangle. Note that in the latter case we must have started with $3n - 6$ edges. On the other hand, if we apply the above procedure to an n-vertex 3-connected maximal outer-fan-planar graph G, then it follows that G has exactly $2n$ or $3n - 6$ edges. We summarize this observation in the following lemma.

Lemma 12 *A 3-connected maximal outer-fan-planar graph with n vertices has exactly* $2n$ *or* $3n - 6$ *edges.*

Next, we try to reinsert the vertices in the reversed order in which we deleted them. By Lemma 7, the neighbors of a vertex of degree three have to be consecutive and we can insert the vertex of degree three only between two of its neighbors. Lemma 13 guarantees that in total, we have to check at most six possible drawings. A summary of our approach is also given in Algorithm 1.

Lemma 13 *When reinserting a sequence of degree 3 vertices starting from a triangle, we have at most three choices for the first vertex and at most two choices for the second vertex. All subsequent vertices can be inserted in at most one way maintaining outer-fan-planarity.*

Proof Let H be an outer-fan-planar graph and let three consecutive vertices v_1, v_2, v_3 induce a triangle. Assume, we want to insert a vertex v adjacent to v_1, v_2, v_3. By Lemma 7, we have to insert v between v_1 and v_2 or between v_2 and v_3 (If H contains only three vertices, we may also insert v between v_3 and v_1). Note that the edges that are incident to v_2 and cross $\{v_1, v_3\}$ are also crossed by an edge e incident to v. So, if there is an edge incident to v_2 that was already crossed twice before inserting v, this would uniquely determine whether e is incident to v_1 or v_3 and, thus, where to insert v.

We will now show that after the first insertion each relevant vertex is incident to an edge that is crossed at least twice. When we insert the first vertex we create a K_4. From the second vertex on, whenever we insert a new vertex, it is incident to an edge that is crossed at least twice. Also, after inserting the second degree 3 vertex, three among the four vertices of the initial K_4 are also incident to an edge that is crossed at least twice. The forth vertex of the initial K_4 is not the middle vertex of an outer triangle. It can only become such a vertex if its incident inner edges are crossed by a 2-hop. But then these inner edges are all crossed at least twice. □

Algorithm 1: 3-Connected Maximal Outer-Fan-Planarity

```
Input   : 3-connected graph G = (V, E), subset E' ⊆ E
Output  : TRUE if and only if G is maximal outer-fan-planar and
          has an outer-fan-planar drawing in which edges in E' are outer edges and
          if so all outer-fan-planar drawings of G in which edges in E' are outer edges

begin
    if G is a complete 2-hop graph then
        return all (at most 12) outer-fan-planar drawings of G with E' on outer face;
    if |V| = 5 then return FALSE while there is a vertex of degree 3 do
        let v be a vertex of degree 3;
        S.PUSH(v);
        remove v from G;
    if the remainder is not a triangle then return FALSE v ← S.POP;
    Let v_1, v_2, v_3 be the neighbors of v;
    for i = 1, 2, 3 do
        insert v between v_i and v_{(i+1) mod 3};
        v ← S.POP;
        Let v_1, v_2, v_3 be the neighbors of v in clockwise order;
        for i = 1, 2 do
            insert v between v_i and v_{i+1};
            S̃ ← S;
            while S̃ ≠ ∅ do
                v ← S̃.POP;
                Let v_1, v_2, v_3 be the neighbors of v in clockwise order (break if v_1, v_2, v_3 not outer
                triangle);
                if no edge incident to v_i, i = 1, 3 other than {v_1, v_3} intersects an edge incident to
                v_2 then
                    insert v between v_2 and v_{4−i} (break if neither possible);
            if S̃ = ∅ and all edges of E' on outer face then
                append constructed embedding to list L of possible embeddings;
    if L ≠ ∅ then return L return FALSE
end
```

Summarizing, we obtain the following theorem; in order to exploit this result in the biconnected case, it is also tested whether a prescribed subset (possibly empty) of edges can be drawn as outer edges.

Theorem 2 *Given a 3-connected graph G with a subset E' of its edge set, it can be tested in linear time whether G is maximal outer-fan-planar and has an outer-fan-planar drawing such that the edges in E' are outer edges. Moreover if such a drawing exists, it can be constructed in linear time.*

Proof Let n be the number of vertices. By Lemma 11, a complete 2-hop graph has only a constant number of outer-fan-planar embeddings which can be computed in linear time. In the other case, we can use buckets to sort the vertices by degree. While the minimum degree is three, we remove one of the degree three vertices and update the degrees of its neighbors.

To check whether the degree three vertices can be reinserted back in the graph, we only have to consider in total six different embeddings. Assume that we want to insert a vertex v into an outer triangle v_1, v_2, v_3. Then we just have to check whether v_1 or v_3

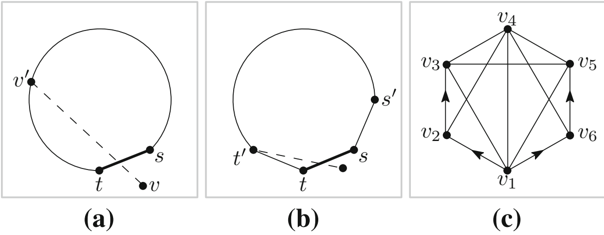


Fig. 5 **a** Illustration of the definition of porous edge. **b** Illustration of Lemma 14. **c** A maximal outer-fan-planar drawing with four porous edges, each porous edge is depicted as a straight-line segment with an *arrow* in the *middle pointing* the corresponding anchor; in particular, $\{v_1, v_2\}$ is porous around v_2 and its anchor is v_3, $\{v_2, v_3\}$ is porous around v_3 and its anchor is v_4, $\{v_1, v_6\}$ is porous around v_6 and its anchor is v_5, and $\{v_5, v_6\}$ is porous around v_5 and its anchor is v_4

are incident to edges other than the edge $\{v_1, v_3\}$ that are crossed by an edge incident to v_2. This can be done in constant time by checking only two pairs of edges. □

2.2 The Biconnected Case

We start by introducing some preliminary concepts that are fundamental to illustrate how the previous results for 3-connected graphs translate to the 3-connected components of G. Let Γ be an outer-fan-planar drawing of a 3-connected graph. We say that an outer edge $e = \{s, t\}$ in Γ is *porous* if we could connect an inner point of e to some vertex, distinct from s and t, preserving outer-fan-planarity; see Fig. 5a for an illustration. In other words, if e is porous, we can add a new vertex v in the outer face of Γ, and connect v to some vertex v', other than s and t, in such a way that no forbidden pattern is created and edge $\{v', v\}$ exits from Γ crossing its boundary only once, and, in particular, in some inner point of e. Vertex v' is called an *anchor* of porous the edge e; observe that we could add arbitrarily many vertices like v, and connect all these vertices to v' maintaining outer-fan-planarity. If Γ is a triangle, then all its edges are porous and have exactly one anchor, i.e. for each edge e of Γ, the corresponding anchor is the vertex that is not incident to e. Instead, if Γ is an outer-fan-planar drawing of a K_4, any outer edge is porous and has two anchors. The following lemma shows that v' is a vertex that either immediately precedes or immediately follows the pair s,t in Γ, see also Fig. 5b. Moreover, it also shows that a porous edge admits exactly one anchor if Γ is maximal and $n \neq 4$.

Lemma 14 *Let Γ be an outer-fan-planar drawing of a 3-connected graph G with $n \geq 3$ vertices, and let s', s, t, and t' be four consecutive vertices on the boundary of Γ, with s' and t' identical when G is a 3-cycle, such that $e = \{s, t\}$ is a porous edge. Then, only s' or t' can be an anchor of e. Moreover, if Γ is maximal and $n \neq 4$, then e admits exactly one anchor, which is either s' or t'.*

Proof If G is a 3-cycle, the statement is trivially true. So, suppose that G has at least four vertices, and without loss of generality assume that Γ is a straight-line drawing with all vertices placed on a circle $\mathcal{C}$. Also, denote by $v_1, \ldots, v_n$ the vertices of G in the order that appear around $\mathcal{C}$ such that $s' \equiv v_{n-1}$, $s \equiv v_n$, $t \equiv v_1$ and $t' \equiv v_2$. Now, according to the definition of porous edge, add a new vertex v in an inner point of the circular arc of $\mathcal{C}$ delimited by v_n and v_1, and let $\{v, v_i\}$ be a new edge ($i = 2, \ldots, n-1$) that can be added to Γ maintaining outer-fan-planarity. We first prove that $i = 2$ or $i = n - 1$. Suppose by contradiction that $3 \le i \le n - 2$. Since v_1,v_i cannot be a separation pair of G, there has to be an edge e from a v_k for some $k = 2, \ldots, i-1$ that crosses $\{v, v_i\}$. Since $\{v, v_i\}$ is already crossed by $\{v_1, v_n\}$ it follows that $e = \{v_k, v_n\}$. Symmetrically, since v_n,v_i cannot be a separation pair of G, there has to be an edge $\{v_1, v_l\}$ for some $l = i + 1, \ldots, n - 1$. But now there are three independent edges crossing. Hence, i is either 2 or or $n - 1$.

Assume now that Γ is maximal and $n \ge 5$. We show that i is equal either to 2 or to $n - 1$, i.e. there cannot be two anchors v_2 and v_{n-1}. Assume to the contrary that v_2 and v_{n-1} are two anchors of $\{v_n, v_1\}$. We will first prove that Γ has to contain $\{v_n, v_2\}$ and $\{v_{n-1}, v_1\}$. Let $3 \le k \le n - 1$ be the maximum integer for which edge $\{v_1, v_k\}$ exists. Since v_1,v_k cannot be a separation pair, there exists some vertex $v_{k'}$, with $k + 1 \le k' \le n$, that is adjacent to some vertex v_l with $l = 2, \ldots, k - 1$. Also, since v_2 is an anchor of $\{v_n, v_1\}$, by outer-fan-planarity it follows that $v_l \equiv v_2$. Further, $v_{k'}$ must coincide with v_n, otherwise, by maximality, there would exists edge $\{v_1, v_{k'}\}$ in Γ, contradicting the previous assumption on k. Hence, there exists edge $\{v_n, v_2\}$ in Γ. Symmetrically, it can be proved that also $\{v_{n-1}, v_1\}$ is an edge of Γ. But now, since $n \ge 5$ and v_2, v_{n-1} cannot be a separation pair, there has to be an edge from v_1 or v_n to a vertex v_j, with $3 \le j \le n - 2$, which violates the initial hypothesis that both v_2 and v_{n-1} are anchors of $\{v_n, v_1\}$. □

In what follows, we will say that e is *porous around* s if its anchor is closer to s than to t when traveling along the boundary of Γ, otherwise, if the anchor is closer to t, we will say that e is *porous around* t. Moreover, to indicate the porousness, we will often add an arrow in the middle of a porous edge, pointing s if e is porous around s or pointing t otherwise; see, e.g., Fig. 5c.

We are now ready to describe how maximal outer-fan-planarity affects the structure of the SPQR-tree and that of the skeletons of its nodes. We begin with a lemma listing some necessary conditions that are implied by the existence of a maximal outer-fan-planar drawing Γ of G; we recall this is not sufficient to state that G is maximal outer-fan-planar.

Lemma 15 *Let G be a biconnected graph that admits a maximal outer-fan-planar drawing. Then, the following conditions hold.*

(1) The skeleton of any R-node admits a maximal outer-fan-planar drawing in which all virtual edges are outer edges.
(2) No R-node is adjacent to an R-node or an S-node.
(3) All S-nodes have degree three.
(4) All P-nodes have degree three and are adjacent to a Q-node.

Proof Let Γ be a maximal outer-fan-planar drawing of G with all vertices on a circle $\mathcal{C}$, and let s,t be a separation pair of G. By Lemma 2, G is Hamiltonian and the outer boundary of Γ is a drawing of a Hamiltonian circuit of G. Therefore, the removal of s and t from G results in exactly two connected components G'_1 and G'_2. Moreover, let G_i $(i = 1, 2)$ be the subgraph of G induced by the vertices of G'_i along with s and t, and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the two circular arcs in which $\mathcal{C}$ is split by vertices s and t. Then, there cannot be an edge between a vertex v_1 in $\mathcal{A}_1$ and a vertex v_2 in $\mathcal{A}_2$ with $v_i \notin \{s, t\}$ for $i = 1, 2$, otherwise $\{s, t\}$ would not be a separation pair of G. This implies that all vertices of G_i $(i = 1, 2)$ are placed along circular arc $\mathcal{A}_i$ in Γ. Thus, by maximality, there must exist edge $\{s, t\}$, and this edge is drawn in Γ as a crossing free inner edge; i.e. every separation pair is indeed a separation edge. Therefore, G is composed from two non-trivial components G_1 and G_2 that are combined in parallel and whose intersection consists of the separation edge $\{s, t\}$, from which follows Condition 4.

Condition 3 is straightforward, because otherwise any two non-adjacent vertices of the skeleton of an S-node could be joined by an edge in Γ preserving outer-fan-planarity. Similarly, Condition 2 is also immediate. Indeed, there cannot be an R-node that is adjacent to an S-node, otherwise there would be a separation pair that is not a separation edge. Moreover, the absence of inner vertices in Γ implies that no two R-nodes can be adjacent. The last three conditions imply the following: for any R-node ν, there must exist a sub-drawing Γ_ν of Γ, which is a drawing of the skeleton G_ν where all virtual edges are outer edges. Of course, Γ_ν must be maximal outer-fan-planar, otherwise an edge that could be added to Γ_ν preserving its outer-fan-planarity would also preserve the outer-fan-planarity of Γ, which concludes the proof of Condition 1. □

Suppose now that G is maximal outer-fan-planar, this implies not only the existence of a maximal outer-fan-planar drawing, but also that all the other outer-fan-planar drawings of G, if any, are maximal. Therefore, in addition to condition 1) of Lemma 15, the skeleton of any R-node, and not just a drawing, must also be maximal outer-fan-planar, otherwise it would be possible to redraw a 3-connected component of Γ in such a way that the resulting drawing is still outer-fan-planar but not longer maximal. We now describe some other conditions that make it possible to redraw a maximal outer-fan-planar drawing of G into one that is not longer maximal, preserving outer-fan-planarity. All these conditions are expressed in terms of forbidden configurations involving the two skeletons G_1 and G_2 of the two neighbors of a P-node other than the Q-node.

To introduce the first forbidden configuration, we need to define the concept of *forbidden 2-hop*. Let $\{s, t\}$ be the common virtual edge of G_1 and G_2, and let Γ_2 be a maximal outer-fan-planar drawing of G_2. If $\{s, t\}$ is drawn as a 2-hop in Γ_2, we say that $\{s, t\}$ is a *forbidden 2-hop with respect to s* if (i) G_1 is the skeleton of an S-node (i.e. a 3-cycle) such that at most the edges incident to s are virtual, and (ii) $\{s, v\}$ is real and porous around v, where v is the vertex of Γ_2 between s and t; see Fig. 6 for an illustration. Of course, an analogous definition can be given for a *forbidden 2-hop with respect to t*. As depicted in Fig. 6a, G admits an outer-fan-planar embedding in which two non-adjacent vertices v and v' are consecutive, i.e. an outer edge is missing, which makes G non-maximal. In other words, if G is maximal outer-fan-planar, then there cannot exist a maximal outer-fan-planar drawing of G_i $(i = 1, 2)$ with one or more forbidden 2-hops and such that all the remaining virtual edges are outer edges.

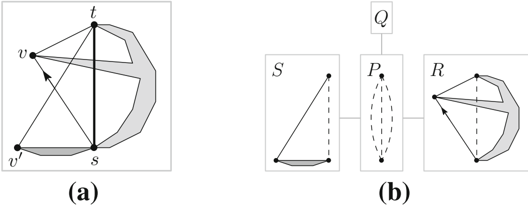


Fig. 6 Illustration of a forbidden 2-hop w.r.t. s: **a** an outer-fan-planar embedding with a missing outer edge, **b** the structure of the corresponding SPQR-tree

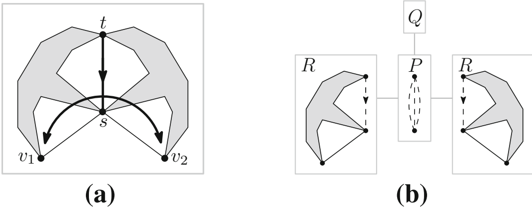


Fig. 7 Illustration of an edge $\{s, t\}$ that is simultaneously porous around s in both drawings of G_1 and G_2: **a** an embedding in which anchors v_1 and v_2 can be connected preserving outer-fan-planarity; (b) structure of the corresponding nodes of the SPQR-tree, where G_1 and G_2 are the skeletons of two R-nodes

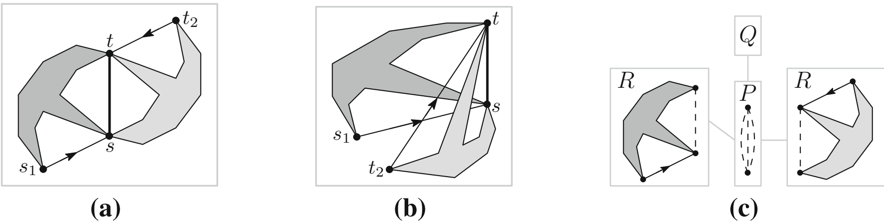


Fig. 8 Illustration of the third forbidden configuration: **a** a maximal outer-fan-planar embedding ε where edge $\{s_1, s\}$ is real and porous around s, and $\{t_2, t\}$ is real and porous around t, **b** an outer-fan-planar embedding ε', that is not longer maximal, obtained from ε by a "folding and stretching" transformation; (c) structure of the corresponding SPQR-nodes, where G_1 and G_2 are the skeletons of two R-nodes

Another forbidden configuration occurs when G_i $(i = 1, 2)$ admits a maximal outer-fan-planar drawing Γ_i such that $\{s, t\}$ is simultaneously porous around the same vertex, s or t, in both drawings Γ_1 and Γ_2. Indeed, in this case, it is possible to draw an edge connecting the anchor of $\{s, t\}$ in Γ_1 to the anchor in Γ_2 preserving outer-fan-planarity; see Fig. 7 for an illustration. Observe that G_i $(i = 1, 2)$ could be the skeleton of an R-node or of an S-node.

The last forbidden configuration prevents the possibility of folding one of the two drawings along $\{s, t\}$ and then suitably stretching it, in such a way that the overlap with the other drawing produces only fan-planar crossings and, at the same time, two non-adjacent vertices become consecutive on the outer boundary of the resulting overall drawing, which is therefore not longer maximal. An illustration of this configuration is depicted in Fig. 8, even in this case G_i $(i = 1, 2)$ could be the skeleton of an R-node or of an S-node.

The following theorem characterizes the biconnected graphs that are maximal outer-fan-planar, by showing that the previously described forbidden configurations, and the necessary conditions given in Lemma 15, provide an overall set of conditions that are also sufficient.

Theorem 3 *A biconnected graph is maximal outer-fan-planar if and only if the following conditions hold.*

(1) The skeleton of any R-node
  i. is maximal outer-fan-planar,
  ii. has an outer-fan-planar drawing in which all virtual edges are outer edges, and
  iii. it does not admit an outer-fan-planar drawing with one or more forbidden 2-hops and such that all the remaining virtual edges are outer edges.

(2) No R-node is adjacent to an R-node or an S-node.

(3) All S-nodes have degree three.

(4) All P-nodes have degree three and are adjacent to a Q-node.

(5) Let G_1 and G_2 be the skeletons of the two neighbors of a P-node other than the Q-node, and let $\{s, t\}$ be the common virtual edge of G_1 and G_2. Then, G_i ($i = 1, 2$) does not admit an outer-fan-planar drawing Γ_i in which all virtual edges are outer edges and such that
  i. edge $\{s, t\}$ is porous in both Γ_1 and Γ_2 around the same vertex, or
  ii. the edges $\{s_1, s\}$ and $\{t_2, t\}$, respectively, are both real and porous around s and t, respectively, or
  iii. the edges $\{t_1, t\}$ and $\{s_2, s\}$, respectively, are both real and porous around t and s, respectively.

  Where s_i, s, t, t_i are consecutive vertices around the outer face of Γ_i, with s_i and t_i coincident if Γ_i is a triangle.

Proof The necessary part is an immediate consequence of Lemma 15 and of the previous considerations on the forbidden configurations. We now prove the sufficient part, i.e. let G be a biconnected graph that satisfies all the conditions listed in the statement, we show that G is maximal outer-fan-planar. Clearly, if Conditions 1ii, 2, and 4 are fulfilled, then G is outer-fan-planar. Just merge skeletons at common virtual edges such that one skeleton is in the outer face of the other skeleton. We show maximality by induction on the number of inner nodes in the SPQR-tree.

If the SPQR-tree has only one inner node, then it is either an S-node or an R-node. Hence, by Condition 1i or 3, respectively, G is maximal outer-fan-planar. Assume now that the SPQR-tree of G has more than one inner node. Consider an inner node ν_1 of the SPQR-tree that would be a leaf if Q-nodes were omitted. Node ν_1 is an S-node or an R-node. By Condition 2 and 4, ν_1 is adjacent to a P-node which is adjacent to a Q-node and another node ν_2. Let $\{s, t\}$ be the edge associated with the Q-node and let G_i ($i = 1, 2$) be the skeleton of ν_i. Let G' be the graph that is obtained from G by deleting all vertices of G_1 except s and t. The SPQR-tree of G' has two inner nodes less than the one of G. Hence, G' is maximal outer-fan-planar.

Consider now an outer-fan-planar drawing Γ of G with all vertices on a circle $\mathcal{C}$. By the inductive hypothesis Γ induces a maximal outer-fan-planar drawing Γ' of G'.

Hence, all outer edges of G' are present. This implies that any separation pair of G' is connected by a crossing free inner edge in Γ', and this edge is an outer edge in any restriction of Γ' to a 3-connected component containing it. In the following, we consider the drawing $\Gamma_{1,2}$ that is the restriction of Γ to the union $G_{1,2}$ of G_1 and G_2 only, i.e. $\Gamma_{1,2} = \Gamma_1 \cup \Gamma_2$. We distinguish three cases.

ν_1 and ν_2 are R-nodes: Observe that G_1 and G_2 must be maximal outer-fan-planar and thus all outer edges of both Γ_1 and Γ_2 must be present. We now prove that *the vertices of* G_1 *and* G_2, *respectively, are consecutive on* $\mathcal{C}$. In this case, $\mathcal{C}$ contains circular arcs $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A}_i$ $(i = 1, 2)$ contains all vertices of G_i and no vertex of G_{3-i} except s and t.

Assume that the vertices of G_1 and G_2 are not both consecutive on $\mathcal{C}$. We first prove that then *any outer edge of* Γ_1 or Γ_2, *respectively, excluding* $\{s, t\}$, *is an outer edge or a 2-hop of* $\Gamma_{1,2}$. Let e be an outer edge of Γ_1 that is not an outer edge of $\Gamma_{1,2}$. If e is not incident to s or t then e can only be a 2-hop. Otherwise, e would cross two independent outer edges of Γ_2. Assume now that e is incident to one among s or t, i.e., w.l.o.g, let $e = \{v, s\}$ for some $v \neq t$.

Lemma 14 applied to Γ_2 implies that there can be at most one vertex, say w, of G_2 between v and s. Vertices s and v split circle $\mathcal{C}$ into two circular arcs $\mathcal{A}_1$ and $\mathcal{A}_2$. Since $\{s, v\}$ is an outer edge of Γ_1, it follows that the vertices of G_1 (excluding s and v) are completely contained in one of the two arcs, say in arc $\mathcal{A}_1$. Assume first that $\mathcal{A}_1$ contains w. Since t is the only vertex of G_2 in $\mathcal{A}_1$, it follows that $w = t$. If $G_1 - \{s\}$ would be completely contained between t and v, then the vertices of G_1 and G_2, respectively, would both be consecutive on circle $\mathcal{C}$; contradicting our assumption. So assume there is a vertex of G_1 between s and w. But then an inner edge of Γ_2 incident to t would be crossed by two independent edges, that is, by an inner edge of Γ_2 and by an edge of G_1 preventing $\{t, s\}$ from being a separation pair of G_1. So, Γ_1 is contained in the circular arc between s and t not containing w. So, e is a 2-hop.

Consider the two circular arcs into which circle $\mathcal{C}$ is split by s and t. If the vertices of G_1 (and G_2, respectively) are not consecutive, then there is at least one arc such that the outer edges of Γ_1 and Γ_2 are all 2-hops in $\Gamma_{1,2}$ except for some of the edges incident to s and t. More precisely, the vertices of G_1 and G_2 are alternating in this circular arc. We call the circular arc that contains alternating vertices the *upper arc* and the other the *lower arc*. If the vertices are alternating in both arcs, we call the arc containing more vertices the *upper arc*. Assume first that the upper arc contains at least four vertices, i.e., at least two vertices of G_1 and G_2, respectively. Let the vertices of the upper arc be enumerated $s = v_1, \ldots, v_\ell = t$ around the circle. Assume without loss of generality that v_3 is in G_1. Since t, v_3 cannot be a separation pair in G_1, it follows that there is an edge between s or a vertex of G_1 in the lower arc on one hand and another vertex w of G_1 in the upper arc on the other hand. But now any edge preventing s, w from being a separation pair of G_1 would cross two independent edges of $G_{1,2}$.

Assume now that there are exactly three vertices in the upper arc, say two vertices of G_2 and one vertex of G_1. Then there have to be some vertices of G_1 in the lower arc. Since s, t cannot be a separation pair of G_1, there has to be an edge between a vertex of the lower arc and the single vertex w of G_1 in the upper arc. But this edge would cross two independent edges of $G_{1,2}$.

This concludes the proof that the vertices of G_1 and G_2, respectively, are consecutive on circle $\mathcal{C}$. Thus, we know that $\{s, t\}$ must be an outer edge in both Γ_1 and Γ_2. Furthermore, the vertices of G_1 have to be inserted next to s or t into Γ_2. Consider now Conditions 5ii and 5iii if the outer edges incident to $\{s, t\}$ are not virtual and otherwise Condition 5i applied to the P-node associated with a virtual outer edge incident to $\{s, t\}$. These conditions together imply that G_1 must be inserted right between s and t. On the other hand, since G' is maximal outer-fan-planar, it follows that vertices not in G_1 cannot be placed between s and t. Hence, the only edge that could be inserted into the drawing Γ would be an edge crossing $\{s, t\}$. But this is prohibited by Condition 5i.

ν_1 is an S-node and ν_2 is an R-node: We first assume that $\{s, t\}$ is an outer edge in Γ_2. Let Γ_1 be the triangle on the vertices s, t, and w. Then it follows directly by Lemma 14 and—on one hand Conditions 5ii and 5iii if the outer edges incident to $\{s, t\}$ are not virtual or otherwise by Condition 5i applied to the P-node associated with the outer edge incident to $\{s, t\}$—that v has to be inserted into G_2 between s and t. Again, since G' is maximal outer-fan-planar, no other vertex of G can be between s and t. Hence, by Condition 5i, no edge can be added to the drawing.

Assume now that $\{s, t\}$ is not an outer edge of Γ_2. We show that in this case Condition 1 would be violated. Recall that by the maximality of G' all virtual edges of G_2 other than $\{s, t\}$ must be outer edges. Let Γ_1 be the triangle on the vertices s, t, w. Since $\{s, t\}$ is not an outer edge of Γ_2 it follows that at least one among $\{w, s\}$ or $\{w, t\}$—say $\{w, t\}$—must intersect an outer edge e of Γ_2. This implies that e is porous.

Lemma 14 implies that $\{s, t\}$ is a 2-hop and that $e = \{s, v\}$ where v is the only vertex of G_2 that is between s and t on $\mathcal{C}$. Furthermore, e cannot be virtual. If e would be contained in the skeleton of an S-node, then e could not be porous in Γ_2 due to Condition 5i. If e would be contained in the skeleton G_3 of another R-node then all vertices of G_3 are between s and v on $\mathcal{C}$. We distinguish three cases. If w is between s and all vertices of G_3 then $\{w, t\}$ would cross two independent edges, namely an inner edge of Γ_3 incident to s and an inner edge of Γ_2 incident to v. If w is between some vertices of G_3 then $\{w, t\}$ would cross two independent outer edges of Γ_3. Finally, if w is between v and all other vertices of G_3 then any inner edge of Γ_3 incident to v would be crossed by two independent edges, namely some inner edge of G_3 and edge $\{w, t\}$.

ν_1 is an R-node and ν_2 is an S-node: The argumentation here is symmetric to the second case. The only difference is that when we consider the case that $\{s, t\}$ is not an outer edge of Γ_1, we have to make sure that $\{w, t\}$ is not a virtual edge of G_2. Otherwise, let G_3 be the skeleton of the R-node such that G_3 contains the edge $\{w, t\}$. Then Γ_3 would contain an inner edge e incident to t. This edge would be crossed by two independent edges, namely an inner edge of Γ_3 and the edge $\{v, s\}$ of G_1. □

It is easy to see that the conditions in Theorem 3, except those for checking whether a forbidden configuration occurs, i.e. Condition 1iii and Condition 5, can be tested in linear time, since the SPQR-tree of a biconnected graph can be computed in linear time. We now show how to test Condition 1iii and Condition 5 in overall linear time. The next lemma implies that a 3-connected maximal outer-fan-planar graph with a fixed outer-fan-planar drawing cannot have more than four porous edges.

Lemma 16 *Let Γ be an outer-fan-planar drawing of a 3-connected graph G that is maximal outer-fan-planar. Then, an outer edge e of Γ can only be porous around one of its end-vertices if e is incident to a vertex of degree three.*

Proof First observe that a complete 2-hop graph with more than four vertices has no porous edges. Moreover, the only maximal outer-fan-planar graph with $n \leq 5$ vertices is a K_n. Observe that K_3 is not 3-connected, K_4 has only vertices of degree three, and K_5 is a complete 2-hop graph.

Hence, suppose that G has $n \geq 6$ vertices and Γ contains at least a long edge. By Lemma 7, we know that G has exactly two vertices of degree 3. We may assume that Γ is a straight-line drawing and that the vertices $v_1, \ldots, v_n$ are placed in this order around a circle $\mathcal{C}$. Assume now that $e = \{v_1, v_n\}$ is porous around v_1.

Consider the graph G' resulting from G by adding a new vertex v_{n+1} and the edges $\{v_{n+1}, v_1\}$, $\{v_{n+1}, v_2\}$, and $\{v_{n+1}, v_n\}$. Since $e = \{v_1, v_n\}$ is porous around v_1, we can draw v_{n+1} between v_1 and v_n on $\mathcal{C}$ to obtain an outer-fan-planar drawing Γ' of G' with a long edge. Since Γ contains a long edge, G has $3n - 6$ edges. Hence, G' has $3(n+1) - 6$ edges and $n + 1$ vertices. Thus, G' is a 3-connected maximal outer-fan-planar graph and Γ' has a long edge, therefore G' must have exactly two vertices of degree three. Since v_{n+1} has degree three it follows that exactly one of the two vertices that had degree three in G must be v_1, v_2, or v_n.

Moreover, v_2 cannot have degree three in G: By 3-connectivity, v_1 has to have degree at least three. Let $2 < k < n$ be minimum such there is an edge $\{v_1, v_k\}$. Then $k = 3$: Otherwise, the edge $\{v_1, v_k\}$ would be crossed by two independent edges in Γ', namely by $\{v_{n+1}, v_2\}$ and by some edge e that prevents v_2, v_k from being a separation pair.

Assume now that v_2 has degree three. We first show that in that case v_1 would have to be adjacent to all other vertices: Otherwise, assume that there is an ℓ such that v_ℓ is not adjacent to v_1. Let $k_1 < \ell$ be maximum and $k_2 > \ell$ be minimum such that v_{k_1} and v_{k_2} are adjacent to v_1. Then, v_{k_1} and v_{k_2} would be a separation pair. But now v_3 can only be adjacent to v_1, v_2, v_4. Otherwise, $\{v_1, v_4\}$ would be crossed by two independent edges, by $\{v_{n+1}, v_2\}$ and an edge $\{v_3, v_k\}$, $k > 4$. But then we could add the edge $\{v_2, v_4\}$ contradicting the maximality of G. □

Lemma 17 *All porous edges of a 3-connected maximal outer-fan-planar graph can be determined in overall linear time.*

Proof Recall that a 3-connected maximal outer-fan-planar graph has a constant number of outer-fan-planar embeddings. In each of these embeddings, we have to check at most 8 edges, whether they are porous around one of their end-vertices. It remains to show that we can test in linear time whether one of these edges is porous around one of its end-vertices.

Assume that the vertices are labeled such that we want to determine whether $\{v_n, v_1\}$ is porous around v_1. Observe that $\{v_n, v_1\}$ is porous around v_1, if and only if the edges incident to v_1 are only crossed by edges incident to v_2. We can check this by looking at each edge at most twice as follows. Let the neighbors of v_1 be $v_2 = v_{k_1}, v_{k_2}, \ldots, v_{k_\ell}, v_n$ such that $k_1 < k_2 < \cdots < k_\ell < n$. Then $\{v_n, v_1\}$ is not porous around v_1 if and only if there is an edge $\{v_i, v_{i'}\}$ with $\max(3, k_j) \leq i < k_{j+1}$ and $i' > k_{j+1}$ for some $j = 1, \ldots, \ell - 1$. □

Combining the characterization in Theorem 3 with the previous lemma, we obtain the following corollary, which concludes the proof of Theorem 1.

Corollary 1 *Maximal outer-fan-planarity of a biconnected graph can be tested in linear time.*

3 The NP-completeness of the Fan-Planarity with Fixed Rotation System Problem

In this section, we study the Fan- Planarity with Fixed Rotation System problem (FP-FRS), that is, the problem of deciding whether a graph G with a fixed rotation system $\mathcal{R}$ admits a fan-planar drawing preserving $\mathcal{R}$.

Lemma 18 Fan- Planarity with Fixed Rotation System *is in NP.*

Proof Let $\langle G, \mathcal{R} \rangle$ be an instance of FP-FRS, where $G = (V, E)$ is a graph with n vertices and m edges. We show that a non-deterministic algorithm can test whether $\langle G, \mathcal{R} \rangle$ is a *Yes*-instance of FP-FRS in a time that is polynomial in n and m. Our proof, inspired by the one given in [20], relies on a non-deterministic generation of all *crossing structures* of E with k crossings, where $0 \leq k \leq \binom{m}{2}$. A *crossing structure* with k crossings is defined by (i) a set of k pairs of edges of E, each pair represents a crossing; (ii) the order in which crossings occur along edges involved in more than one crossing; (iii) the *rotation around each crossing*, i.e. for each pair of edges (u, v) and (x, y) forming a crossing c, it is specified one of the two possible circular orders of arc segments $\widetilde{uc}$, $\widetilde{cv}$, $\widetilde{xc}$ and $\widetilde{cy}$ around crossing c.

A crossing structure is *plausible* if it contains neither a pair of crossing edges sharing a common end-vertex nor a triple of edges forming a forbidden crossing pattern I or II. Crossing structures that are not plausible are discarded. Let $\mathcal{C}$ denote a plausible crossing structure. We conclude the proof by describing a polynomial-time transformation $T(\mathcal{C})$ that maps $\langle G, \mathcal{R} \rangle$ into an instance $\langle G'(\mathcal{C}), \mathcal{R}'(\mathcal{C}) \rangle$ of problem Planarity with Fixed Rotation System (P-FRS); we recall that P-FRS can be easily solved in linear time, for details see [11,26]. $T(\mathcal{C})$ essentially replaces crossings in $\mathcal{C}$ with dummy vertices.

Initially, $\langle G'(\mathcal{C}), \mathcal{R}'(\mathcal{C}) \rangle$ is set equal to $\langle G, \mathcal{R} \rangle$. Now, let (u, v) and (x, y) be two crossing edges whose crossing is the first one in the two orders specified in $\mathcal{C}$ for (u, v) and (x, y). Then, a dummy vertex c is added to $G'(\mathcal{C})$, edge (u, v) is replaced by edges (u, c) and (c, v), and edge (x, y) is replaced by edges (x, c) and (c, y). Of course, the circular orders of the edges that are incident to vertices u and v must be updated consistently after this edge replacement operation. Also, the circular order of edges incident to dummy vertex c is given by the rotation around the corresponding crossing, as specified in $\mathcal{C}$. The procedure for the insertion of dummy vertices is repeated until all crossings of $\mathcal{C}$ are removed from $G'(\mathcal{C})$, i.e. k times in total.

Suppose now that $\langle G, \mathcal{R}\rangle$ is a *Yes*-instance of FP-FRS, and let $\mathcal{C}$ be the crossing structure induced by any fan-planar drawing of G that preserves $\mathcal{R}$. By definition of $T(\mathcal{C})$, it is straightforward that $\langle G'(\mathcal{C}), \mathcal{R}'(\mathcal{C})\rangle$ is a *Yes*-instance of P-FRS. Conversely, if for some plausible crossing structure $\mathcal{C}$, graph $G'(\mathcal{C})$ admits a planar embedding that preserves rotation system $\mathcal{R}'(\mathcal{C})$, then, by construction, there is no forbidden crossing pattern I or II in $\langle G, \mathcal{R}\rangle$.

In conclusion, the statement follows because one only needs to guess a crossing structure $\mathcal{C}$ of a solution to FP-FRS, then its correctness can be verified in polynomial time, by preliminarily checking if $\mathcal{C}$ is a plausible crossing structure and, in case of a positive answer, by subsequently testing whether graph $G'(\mathcal{C})$ admits a planar embedding preserving $\mathcal{R}'(\mathcal{C})$. □

Theorem 4 FAN- PLANARITY WITH FIXED ROTATION SYSTEM *is NP-complete.*

Proof We already proved in Lemma 18 that the problem is in NP. We then prove the NP-hardness by using a reduction from 3- PARTITION (3P). An instance of 3P is a multi-set $A = \{a_1, a_2, \ldots, a_{3m}\}$ of $3m$ positive integers in the range $(B/4, B/2)$, where $B := 1/m \cdot \sum_{i=1}^{3m} a_i$ is an integer. 3P asks whether A can be partitioned into m subsets $A_1, A_2, \ldots, A_m$, each of cardinality 3, such that the sum of the numbers in each subset is B. As 3P is *strongly* NP-hard [19], it is not restrictive to assume that B is bounded by a polynomial in m.

Given an instance A of 3P, we show how to transform it into an instance $\langle G_A, \mathcal{R}_A\rangle$ of FP-FRS, by a polynomial-time transformation, such that the former is a *Yes*-instance of 3P if and only if the latter is a *Yes*-instance of FP-FRS.

Before describing our transformation in detail, we need to introduce the concept of *barrier gadget*. An n-vertex *barrier gadget* is a complete 2-hop graph of $n \geq 5$ vertices, i.e. it consists of an n-vertex cycle plus all its 2-hop edges; a barrier gadget is therefore a maximal outer-fan-planar graph. We make use of barrier gadgets in order to constraint the routes of some specific paths of G_A, as will be clarified soon. We exploit the following property of barrier gadgets. Let G be a biconnected fan-planar graph containing a barrier gadget G_b, and let Γ be a fan-planar drawing of G such that drawing Γ_b of G_b in Γ is maximal outer-fan-planar. Then, no path π of $G - G_b$ can enter inside the boundary cycle of Γ_b and cross a 2-hop edge. Indeed, every 2-hop edge e_b of Γ_b is crossed by two other 2-hop edges having an end-vertex in common, hence if e_b were crossed by π, then e_b would be crossed by two independent edges. On the other hand, if path π enters inside Γ_b without crossing any 2-hop edge, then it must cross twice a same boundary edge e'_b because of the biconnectivity of G; namely, if path π enters in Γ_b, then it must also exit from it passing through the same boundary edge. In this case, the only possibility that preserves the fan-planarity of Γ is that π crosses e'_b with two consecutive edges, thus forming a fan-crossing. Otherwise, e'_b would be crossed either by two independent edges of π or by a same edge of π twice, but both these cases are not allowed in a (simple) fan-planar drawing.

Now, we are ready to describe how to transform an instance A of 3P into an instance $\langle G_A, \mathcal{R}_A\rangle$ of FP-FRS. We start from the construction of graph G_A which will be always biconnected. First of all, we create a *global ring barrier* by attaching four barrier gadgets G_t, G_r, G_b and G_l as depicted in Fig. 9. G_t is called the *top beam* and contains exactly $3mK$ vertices, where $K = \lceil B/2 \rceil + 1$. G_r is the *right wall* and has only five vertices. G_b and G_r are called the *bottom beam* and the *left wall*, respectively, and they are defined in a specular way. Observe that G_t, G_r, G_b and G_l can be embedded so that all their vertices are linkable to points within the closed region delimited by the global ring barrier. Then, we connect the top and bottom beams by a set of $3m$ *columns*, see Fig. 9 for an illustration of the case $m = 3$. Each *column* consists of a stack of $2m - 1$ *cells*; a *cell* consists of a set of pairwise disjoint edges, called the *vertical edges* of that cell. In particular, there are $m-1$ *bottommost cells*, one *central cell* and $m - 1$ *topmost cells*. Cells of a same column are separated by $2m - 2$ barrier gadgets, called *floors*. Central cells (that are $3m$ in total) have a number of vertical edges depending on the elements of A. Precisely, the central cell C_i of the i-th column contains a_i vertical edges connecting its delimiting floors ($i \in \{1, 2, ..., 3m\}$). Instead, all the remaining cells have, each one, K vertical edges. Hence, a non-central cell contains more edges than any central cell. Further, the number of vertices of a floor is given by the number of its incident vertical edges minus two. Let u and v be the "central" vertices of the left and right walls, respectively (see also Fig. 9). We conclude the construction of graph G_A by connecting vertices u and v with m pairwise internally disjoint paths, called the *transversal paths* of G_A; each transversal path has exactly $(3m - 3)K + B$ edges.

Concerning the choice of a rotation system $\mathcal{R}_A$, we define a cyclic ordering of edges around each vertex that is compatible with the following constraints: (i) every barrier gadget can be embedded with all its 2-hop edges inside its boundary cycle; (ii) the global ring barrier can be embedded with only four vertices on the outer face; (iii) columns can be embedded inside the region delimited by the global ring barrier without crossing each other; (iv) vertical edges of cells can be embedded without creating crossings; (v) transversal paths are attached to the left and right walls such that the ordering of their edges around u is specular to the ordering around v; this choice makes it possible to avoid crossings between any two transversal paths. From what said, it is straightforward to see that an instance of 3P can be transformed into an instance of FP-FRS in polynomial time in m.

We now prove that a *Yes*-instance of 3P is transformed into a *Yes*-instance of FP-FRS, and vice-versa. Let A be a *Yes*-instance of 3P, we show that $\langle G_A, \mathcal{R}_A \rangle$ admits a fan-planar drawing Γ_A preserving $\mathcal{R}_{\mathcal{A}}$. We preliminarily observe that such a drawing is easy to compute if one omits all the transversal paths. It is essentially a drawing like that one depicted in Fig. 9, where columns are one next to the other within the closed region delimited by the global ring barrier. However, by exploiting a solution $\{A_1, A_2, \ldots, A_m\}$ of 3P for the instance A, also the transversal paths can be easily embedded without violating the fan-planarity. The idea is to route these paths in such a way that: (R.1) they do not cross each other; (R.2) they do not cross any barrier; (R.3) each path passes through exactly 3 central cells and $3m - 3$ non-central cells; and (R.4) each cell is traversed by at most one path. More precisely, each transversal path π_j is in bijection with a subset A_j ($j \in \{1, 2, \ldots, m\}$) and the three central cells it passes through have three sets of vertical edges whose cardinalities form a triple of integers matching the three integers of A_j. Paths are routed by sweeping columns from left to right and by proceeding as follows. Let C_1 be the 1-st central cell; C_1 has

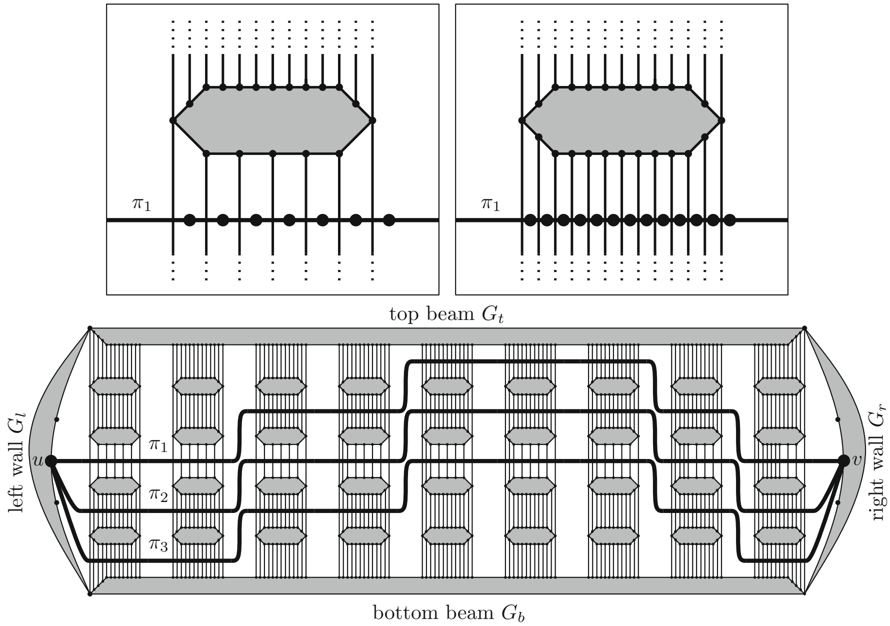


Fig. 9 Illustration of the reduction of FP-FRS from 3P, where $m = 3$, $A = \{7, 7, 7, 8, 8, 8, 8, 9, 10\}$ and $B = 24$. Transversal paths are routed according to the following solution of 3P: $A_1 = \{7, 7, 10\}$, $A_2 = \{7, 8, 9\}$ and $A_3 = \{8, 8, 8\}$. The *top-left* and *top-right boxes* show a zoom of the first central cell and of the non-central cell of the 3-rd column traversed by path π_1, respectively

a_1 vertical edges by construction. The transversal path passing through C_1 is a path $\pi_{j(1)}$ such that subset $A_{j(1)}$ contains an integer equal to a_1. The remaining transversal paths are routed until the 1-st column by preserving the cyclic edge-ordering around u and by respecting conditions R.1, R.2, R.3 and R.4; note that condition R.3 cannot be violated at this point. Suppose now that all paths have been already routed until the $(i-1)$-th column, for some $i \geq 2$, and suppose also that conditions R.k $(1 \leq k \leq 4)$ are satisfied. Then, there is at least a path $\pi_{j(i)}$ whose corresponding subset $A_{j(i)}$ contains an integer a_i that has not yet been considered. Path $\pi_{j(i)}$ is the next path that goes through the central cell C_i. The remaining paths are routed in such a way that their "vertical distances" to path $\pi_{j(i)}$, in terms of number of cells, are unchanged when passing from the $(i-1)$-th column to the i-th column. Eventually, each transversal path crosses exactly $(3m-3)K + B$ vertical edges, which is the same number of its edges. Therefore, it is possible to draw these paths by ensuring that each of their edges crosses exactly one vertical edge, which preserves the fan-planarity. Hence, eventually we get a fan-planar drawing Γ_A preserving the rotation system $\mathcal{R}_A$.

We conclude the proof by showing that if $\langle G_A, \mathcal{R}_A \rangle$ is a *Yes*-instance of FP-FRS, then A is a *Yes*-instance of 3P. Let Γ_A be a fan-planar drawing of G_A preserving the rotation system $\mathcal{R}_A$. We first observe that the top beam and the bottom beam are disjoint, otherwise there would be at least a 2-hope edge in one beam that is crossed by another edge of the other beam, thus violating the fan-planarity. We also note that columns can partially cross each other, but this does not actually affect the validity of the proof. Indeed, an edge e of a column L might cross an edge e' of another column L' only if e is incident to a vertex in the rightmost (leftmost) side of L, e' is a leftmost (rightmost) vertical edge of L', and L and L' are two consecutive columns. With a similar argument, it is immediate to see that vertices u and v must be separated by all the columns. Therefore, every transversal path satisfies conditions R.1, R.2 and it must pass through at least three central cells. Otherwise, it would cross a large number of pairwise disjoint edges. Hence, Γ_A would not be fan-planar.

On the other hand, because of condition R.4, which is obviously satisfied, there cannot be any transversal path passing through more than three central cells. Otherwise, there would be some other transversal path that traverses a number of central cells that is strictly less than three. Hence, also condition R.3 is satisfied. In conclusion, every transversal path π_j ($j \in \{1, 2, \ldots, m\}$) crosses $(3m - 3)K + B$ vertical edges and traverses exactly three central cells C_{1j}, C_{2j} and C_{3j}. If $m(C_{1j})$, $m(C_{2j})$ and $m(C_{3j})$ denote the number of edges of these cells, then $m(C_{1j}) + m(C_{2j}) + m(C_{3j}) = B$, because each non-central cell has K edges. Therefore, the partitioning of A defined by $A_1, A_2, \ldots, A_m$, where $A_j = \{m(C_{1j}), m(C_{2j}), m(C_{3j})\}$, is a solution of 3P for the instance A. □

4 Conclusions

In this paper, we showed that the problem of testing whether a graph is maximal outer-fan-planar is linear time solvable. On the negative side, we proved that testing whether a graph is fan-planar is NP-complete, for the case where the rotation system is fixed. There are two main problems that remain open in this context. The first one is whether it is possible to test (non-maximal) outer-fan-planarity efficiently. The second one is the problem of testing whether a graph is maximum fan-planar, that is, given a graph on n vertices and $5n - 10$ edges determine whether this graph is fan-planar (either in the fixed embedding setting or not).

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