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UNDECIDABLE SEMIASSOCIATIVE RELATION ALGEBRAS

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Abstract. If K is a class of semiassociative relation algebras and K contains the relation algebra of all binary relations on a denumerable set, then the word problem for the free algebra over K on one generator is unsolvable. This result implies that the set of sentences which are provable in the formalism $\mathscr{L}w^\times$ is an undecidable theory. A stronger algebraic result shows that the set of logically valid sentences in $\mathscr{L}w^\times$ forms a hereditarily undecidable theory in $\mathscr{L}w^\times$. These results generalize similar theorems, due to Tarski, concerning relation algebras and the formalism $\mathscr{L}^\times$.

§**1. Introduction.** The formalism $\mathscr{L}^\times$ was introduced in [TG87] for the development of set theory without variables. The expressions of $\mathscr{L}^\times$ are built up from two basic symbols, namely, $\overset{\circ}{\mathbf{1}}$ and $\mathbf{E}$, by means of four operation symbols, $\odot$, $\breve{}$, $+$, and $^-$. $\mathbf{\Pi}$ is the set of expressions which can be obtained in this way. For example, $(\mathbf{E}^\smile \odot \mathbf{E}^- + \mathbf{E}^{\smile -} \odot \mathbf{E})^-$ is an expression in $\mathbf{\Pi}$. The basic symbols $\overset{\circ}{\mathbf{1}}$ and $\mathbf{E}$ are intended to denote the relations of identity and set membership, respectively, while the four operation symbols, $\odot$, $\breve{}$, $+$, and $^-$, are intended to denote the operations (on binary relations) of relative multiplication, conversion, union, and complementation, respectively. Every sentence of $\mathscr{L}^\times$ is an equation between two expressions in $\mathbf{\Pi}$. For example, the sentence $\overset{\circ}{\mathbf{1}} = (\mathbf{E}^\smile \odot \mathbf{E}^- + \mathbf{E}^{\smile -} \odot \mathbf{E})^-$ is the extensionality axiom for $\mathbf{E}$. The logical axiom schemata of $\mathscr{L}^\times$ are analogues of the equational postulates for relation algebras. The only rule of inference is the replacement of equals by equals. A theory in $\mathscr{L}^\times$ is a set of sentences (equations) of $\mathscr{L}^\times$ which contains all instances of the axiom schemata and is closed under the rule of replacing equals by equals. Every set of sentences Ψ generates a theory in $\mathscr{L}^\times$, called $\Theta\eta\Psi[\mathscr{L}^\times]$, or briefly, $\Theta\eta^\times\Psi$, which is the intersection of all theories containing Ψ. The smallest theory in $\mathscr{L}^\times$ is generated by the empty set. Theorem 4.7(vi) of [TG87] states that

(1) $\Theta\eta^\times\varnothing$ is an undecidable theory in $\mathscr{L}^\times$.

Because of the close connection between relation algebras and the construction of $\mathscr{L}^\times$, the minimal theory $\Theta\eta^\times\varnothing$ is essentially the same as the set of equations that are true in all relation algebras and that contain a single variable. Hence, there are two other ways of expressing theorem (1):

(1′) The set of one-variable equations true in all relation algebras is not recursive.

(1″) The word problem for the free relation algebra on one generator is recursively unsolvable.

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Theorem (1) is derived in [TG87] as a corollary of a much stronger result, namely, Theorem 4.7(v), which involves the formalism $\mathscr{L}^+$:

(2) $\Theta\eta^+\varnothing \cap \Sigma^\times$ is a hereditarily undecidable theory in $\mathscr{L}^\times$.

$\mathscr{L}^+$ is a formalism which contains both $\mathscr{L}^\times$ and the first-order language having $\mathbf{\Pi}$ as its set of binary predicates. The completeness theorem for first-order logic applies to $\mathscr{L}^+$ and implies that a sentence X of $\mathscr{L}^\times$ is in $\Theta\eta^+\varnothing$ if and only if X is logically valid. This fact, together with the semantical interpretations of the symbols $+, {}^-, \odot, {}^\smile$, and $\mathring{\mathbf{1}}$, as union, complementation, relative multiplication, conversion, and the identity relation, respectively, implies that $\Theta\eta^+\varnothing \cap \Sigma^\times$ is essentially the same as the set of equations that contain just one variable and that are true in every *representable* relation algebra. There are relation algebras that are generated by just one element and that are not representable. It follows that $\Theta\eta^+\varnothing$ is a proper subset of $\Theta\eta^+\varnothing \cap \Sigma^\times$, so $\mathscr{L}^\times$ is semantically incomplete. (For more details, see [TG87, p. 55].)

The fact that $\Theta\eta^+\varnothing \cap \Sigma^\times$ is undecidable, a consequence of (2), can be expressed in two other ways:

(3) The set of equations that contain just one variable and that are true in all representable relation algebras is not recursive.

(3′) The word problem for the free representable relation algebra with one generator is recursively unsolvable.

It is also possible, but not so easy, to formulate an algebraic statement of the fact that $\Theta\eta^+\varnothing \cap \Sigma^\times$ is a *hereditarily* undecidable theory in $\mathscr{L}^\times$, that is, all its subtheories are undecidable. Suppose Ψ is theory in $\mathscr{L}^\times$ and $\Psi \subseteq \Theta\eta^+\varnothing \cap \Sigma^\times$. Then Ψ may be viewed as a presentation (not necessarily finite or even recursive) of a one-generated relation algebra in which all the relations imposed on the generator are true for all binary relations. For example, $\Theta\eta^\times\{\mathbf{E} \odot \mathbf{E} + \mathbf{E} = \mathbf{E}\}$ is not such a theory because not every binary relation is transitive (which is what $\mathbf{E} \odot \mathbf{E} + \mathbf{E} = \mathbf{E}$ asserts about the interpretation of $\mathbf{E}$). To obtain a proper example of such a theory, one which is different from $\Theta\eta^\times\varnothing$, we need a sentence of $\mathscr{L}^\times$ that is provable in $\mathscr{L}^+$ but not provable in $\mathscr{L}^\times$. The smallest known example of such a sentence is

$$\begin{aligned}\mathring{\mathbf{1}} + \mathring{\mathbf{1}}^- = (\mathring{\mathbf{1}} + \mathring{\mathbf{1}}^-) \odot [(\mathbf{E}^- \odot \mathbf{E}^-)^- \\ + ([(\mathbf{E} \odot \mathbf{E}) + \mathring{\mathbf{1}} + ([(\mathbf{E} + \mathbf{E}^\smile)^- \odot (\mathbf{E} + \mathbf{E}^\smile)^-]^- \\ + \mathbf{E}^\smile]^- + (\mathbf{E} \odot \mathbf{E}^\smile)^-] \odot (\mathring{\mathbf{1}} + \mathring{\mathbf{1}}^-).\end{aligned}$$

This sentence is due to Givant, McNulty, and Tarski and arises from a nonrepresentable relation algebra found by McKenzie. (See [TG87, p. 55].) In view of these remarks we can reformulate theorem (2) as follows.

(2′) The word problem for a finitely presented relation algebra on one generator is recursively unsolvable if all the relations imposed on the generator are true for all binary relations.

In this paper we will extend theorems (1) and (2) to a weaker formalism called $\mathscr{L}w^\times$ which is introduced in [TG87, p. 89]. The only difference between $\mathscr{L}w^\times$ and $\mathscr{L}^\times$ is that some of the axioms of $\mathscr{L}^\times$ are not axioms of $\mathscr{L}w^\times$. One of the ten axiom schemata for $\mathscr{L}^\times$ expresses the associative law for $\odot$. The axiom schemata for $\mathscr{L}w^{\times}$ are obtained from those of $\mathscr{L}^{\times}$ by weakening the associative law of $\odot$. The reason for considering this weakening is that the associative law for $\odot$ is the only axiom schema of $\mathscr{L}^{\times}$ whose instances may require four variables to prove (in a sense made precise in Chapter 3 of [TG87]). All instances of the other axiom schemata can be proved using only three variables. The weakened associative law used in the construction of $\mathscr{L}w^{\times}$ can also be proved using only three variables. In fact, the formalism $\mathscr{L}w^{\times}$ corresponds to a certain natural restriction of first-order logic with binary predicates to only three variables. (A precise formulation and proof of this connection appears in [M78, Chapter 11].) The extensions of theorems (1) and (2) proved in this paper are

(4) $\Theta\eta\varnothing[\mathscr{L}w^{\times}]$ is an undecidable theory in $\mathscr{L}w^{\times}$.

(5) $\Theta\eta^{+}\varnothing \cap \Sigma^{\times}$ is a hereditarily undecidable theory in $\mathscr{L}w^{\times}$.

These results are mentioned in footnote 6 on page 138 of [TG87] and appear as Theorems 20 and 21 below. The formalism $\mathscr{L}w^{\times}$ is related to the class of semiassociative relation algebras in just the same way as $\mathscr{L}^{\times}$ is related to the class of relation algebras. Consequently, we have the following two other ways of expressing theorem (4):

(4′) The set of one-variable equations true in all semiassociative relation algebras is not recursive.

(4″) The word problem for the free one-generated semiassociative relation algebra is recursively unsolvable.

The main results of this paper are Theorems 11, 15, 20, and 21. Theorem 15 shows that the word problem for any given finite presentation of a semigroup can be reduced to the word problem for a suitable semiassociative relation algebra. The free semiassociative relation algebra on one generator happens to be suitable, which is how we obtain theorem (4″) (and hence theorems (4) and (4′)). By applying Theorem 11 to semiassociative relation algebras which are obtained from the formalism $\mathscr{L}w^{\times}$ and may not be free, we can also prove theorem (5).

§2. Semiassociative relation algebras.

Definition 1. Let $\mathfrak{A} = \langle A, +, \cdot, ^{-}, 0, 1, ;, \breve{}, 1'\rangle$ be an algebra with binary operations $+$, $\cdot$, and $;$, unary operations $^{-}$ and $\breve{}$, and distinguished elements 0, 1, and 1'. Then $\mathfrak{A}$ is *nonassociative relation algebra* if $\langle A, +, \cdot, ^{-}, 0, 1\rangle$ is a Boolean algebra (called the *Boolean part* of $\mathfrak{A}$), $\mathfrak{A}$ satisfies the *identity law*

(IL) $x = x\ ;\ 1' = 1'\ ;\ x$,

and $\mathfrak{A}$ satisfies the *cycle law*

(CL) $x \cdot y\ ;\ z = 0$ iff $y \cdot x\ ;\ \breve{z} = 0$ iff $z \cdot \breve{y}\ ;\ x = 0$.

NA is the class of all nonassociative relation algebras.

Let $\mathfrak{A} \in$ NA. Then $\mathfrak{A}$ is a *semiassociative relation algebra* if it satisfies the *semiassociative law*

(SL) $(x\ ;\ 1)\ ;\ 1 = x\ ;\ 1$,

and $\mathfrak{A}$ is a *relation algebra* if it satisfies the *associative law*

(AL) $(x\ ;\ y)\ ;\ z = x\ ;\ (y\ ;\ z)$.

SA and RA are the classes consisting of all semiassociative relation algebras and all relation algebras, respectively.

If $\mathfrak{A} \in$ NA, $x, y \in A$, and $n < \omega$, then

(1) $0' = \overline{1'}$,
(2) $x \diamond y = x \cdot \overline{y} + \overline{x} \cdot y$,
(3) $x^0 = 1'$, and
(4) $x^{n+1} = x^n ; x$.

The elements 0, 1, 1', and 0' are called the *zero element*, *unit element*, *identity element*, and *diversity element* of $\mathfrak{A}$, respectively, and $x \diamond y$ is the *symmetric difference* of x and y.

We omit parentheses from expressions denoting elements of an algebra $\mathfrak{A}$ according to the convention that the operations should be performed in the following order: $\breve{}\,, {}^{-}, ;, \cdot, \diamond, +$. Thus, for example, $v + w \diamond x ; \overline{y} \cdot \breve{z} = v + (w \diamond ((x ; (\overline{y})) \cdot (\breve{z})))$ and $x ; y \diamond z = (x ; y) \cdot \overline{z} + \overline{x ; y} \cdot z$. When the same binary operation occurs several times, the calculation proceeds from left to right. For example, $x + y + z = (x + y) + z$. The definition of x^n is consistent with this convention.

LEMMA 2. *Let* $\mathfrak{A} \in \mathsf{NA}$. *Then, for all* $x, y, z \in A$:

(i) $(x + y) ; z = x ; z + y ; z$,
(ii) $x ; (y + z) = x ; y + x ; z$,
(iii) *if* $x \leq y$, *then* $x ; z \leq y ; z$ *and* $z ; x \leq z ; y$,
(iv) $x \leq x ; 1$, $x \leq 1 ; x$, *and* $1 = 1 ; 1$,
(v) $\breve{\breve{x}} = x$,
(vi) $(x + y)\breve{} = \breve{x} + \breve{y}$,
(vii) $(x ; y)\breve{} = \breve{y} ; \breve{x}$,
(viii) $\breve{x} ; \overline{x ; y} \leq \overline{y}$,
(ix) $\overline{x ; y} ; \breve{y} \leq \overline{x}$,
(x) $x ; y \cdot z \leq (x \cdot z ; \breve{y}) ; y$,
(xi) $x ; y \cdot z \leq x ; (y \cdot \breve{x} ; z)$,
(xii) $x \diamond x = 0$,
(xiii) $x \diamond y = y \diamond x$,
(xiv) $x \diamond y + y \diamond z \geq x \diamond z$,
(xv) $x ; y \diamond x ; z \leq x ; (y \diamond z)$,
(xvi) $x ; z \diamond y ; z \leq (x \diamond y) ; z$.

PROOF. Parts (i)–(vii) are proved in Theorem 1.13 of [M82]. Note that (viii) is equivalent to

$$\breve{x} ; \overline{x ; y} \cdot y = 0,$$

which, by the cycle law, is equivalent to

$$x ; y \cdot \overline{x ; y} = 0.$$

Thus, (viii) follows from the cycle law by Boolean algebra and so does (ix). Parts (x) and (xi) also follow from the cycle law and Boolean algebra, by Theorem 1.4 of [M82]. Parts (xii)–(xiv) hold by Boolean algebra alone. The proofs of (xv) and (xvi) are similar. For (xv), first note that

$$\begin{aligned} x ; y \cdot \overline{x ; z} &\leq x ; (y \cdot \breve{x} ; \overline{x ; z}) && \text{part (xi)},\\ &\leq x ; (y \cdot \overline{z}) && \text{parts (iii), (viii)}. \end{aligned}$$

Similarly,

$$\overline{x ; y} \cdot x ; z \leq x ; (\overline{y} \cdot z),$$

so

$$\begin{aligned} x ; y \diamond x ; z &= x ; y \cdot \overline{x ; z} + \overline{x ; y} \cdot x ; z \\ &\leq x ; (y \cdot \overline{z}) + x ; (\overline{y} \cdot z) \\ &= x ; (y \cdot \overline{z} + \overline{y} \cdot z) && \text{part (ii)} \\ &= x ; (y \diamond z). \end{aligned}$$ □

The definition of RA used here (and in [J82], [JT52], [M78], [M82], and [M92]) differs from the definitions used in [CT51] and [TG87]. Relation algebras as defined in [CT51] are algebras of the form $\langle A, +, ^-, ;, ^\smile \rangle$. The postulates defining relation algebras in [CT51] are all equational, except for the postulate which asserts the existence of an identity element 1'. Relation algebras in [TG87] include the identity element as a distinguished element and, thus, are algebras of the form $\langle A, +, ^-, ;, ^\smile, 1' \rangle$. All the postulates defining relation algebras in [TG87] are equations, which makes it obvious that RA is a variety (closed under the formation of homomorphic images, subalgebras, and direct products). Relation algebras in [JT52] have the form $\langle A, +, 0, \cdot, 1, ;, 1', ^\smile \rangle$, while in [J82] they have the form $\langle A, +, 0, \cdot, 1, ^-, ;, 1', ^\smile \rangle$. These differences in similarity type derive essentially from the fact that Boolean algebras may be conceived as algebras with various different similarity types. More important is that fact that the definition of RA given here is not equational and looks rather different from the definitions in both [CT51] and [TG87]. However, Theorem 2.2 of [CT51] shows that the definition used here is equivalent to the definitions used in [TG87] and [CT51]. We now use that theorem to observe that NA and SA also have equational characterizations similar to the definition of RA in [TG87]. This fact will be useful later when we want to show that algebras obtained from certain formalisms happen to be semiassociative relation algebras.

THEOREM 3. *Let* $\mathfrak{A} = \langle A, +, \cdot, ^-, 0, 1, ;, ^\smile, 1' \rangle$. *Then* $\mathfrak{A} \in$ NA *iff* $\langle A, +, \cdot, ^-, 0, 1 \rangle$ *is a Boolean algebra and* $\mathfrak{A}$ *satisfies the following identities*:

(i) $(x + y) ; z = x ; z + y ; z$,
(ii) $x ; 1' = x$,
(iii) $\breve{\breve{x}} = x$,
(iv) $(x + y)^\smile = \breve{x} + \breve{y}$,
(v) $(x ; y)^\smile = \breve{y} ; \breve{x}$,
(vi) $\breve{x} ; \overline{x ; y} + \overline{y} = \overline{y}$.

PROOF. First, assume $\mathfrak{A} \in$ NA. Then by Definition 1 $\langle A, +, \cdot, ^-, 0, 1 \rangle$ is a Boolean algebra and satisfies identity (ii). All the other identities occur in Lemma 2.

Next assume $\langle A, +, \cdot, ^-, 0, 1 \rangle$ is a Boolean algebra and $\mathfrak{A}$ satisfies (i)–(vi). Then except for differences in similarity types, $\mathfrak{A}$ satisfies postulates 1.1(i) and 1.1(iii)–(vii) of Definition 1.1 in [CT51]. The proof of Theorem 2.1 of [CT51] does not use postulate 1.1(ii) (the associative law for ;). That theorem, therefore, holds for $\mathfrak{A}$, and it asserts that $\mathfrak{A}$ satisfies the cycle law. Similarly, the proof of Corollary 1.4 of [CT51] does not use postulate 1.1(ii), and Corollary 1.4 asserts that $\mathfrak{A}$ satisfies the identity law. Thus, $\mathfrak{A} \in$ NA. □

The proof just given did not refer to the proof of Theorem 2.2 of [CT51], because the proof of Theorem 2.2 has an apparent (but unnecessary) use of the associative law.

Not every semiassociative relation algebra is associative. Nevertheless, we can reduce word problems for semigroups to the word problem for the free one-generated semiassociative relation algebra, just because the equational theory of SA does contain a "sufficient amount of associativity". To express this precisely we use the next definition (taken from [M78, p. 35] and [M91, Definition 15]).

DEFINITION 4. For every nonassociative relation algebra $\mathfrak{A}$, let $P^{(\mathfrak{A})}$ be the unique function that maps finite sequences of elements of A to subsets of A and that satisfies the following conditions for all $x \in A$, $n \in \omega$, and $x_0, \ldots, x_n \in A$:

$$P^{(\mathfrak{A})}(x) = \{x\},$$
$$P^{(\mathfrak{A})}(x_0, \ldots, x_n) = \{y \mathbin{;} z : y \in P(x_0, \ldots, x_{m-1}), z \in P^{(\mathfrak{A})}(x_m, \ldots, x_n), \quad 0 < m \leq n\}.$$

The simplest cases of the definition of $P^{(\mathfrak{A})}$ are

$$P^{(\mathfrak{A})}(x) = \{x\},$$
$$P^{(\mathfrak{A})}(x, y) = \{x \mathbin{;} y\},$$
$$P^{(\mathfrak{A})}(x, y, z) = \{(x \mathbin{;} y) \mathbin{;} z, x \mathbin{;} (y \mathbin{;} z)\},$$
$$P^{(\mathfrak{A})}(w, x, y, z) = \{w \mathbin{;} x \mathbin{;} y \mathbin{;} z,\ w \mathbin{;} x \mathbin{;} (y \mathbin{;} z),\ w \mathbin{;} (x \mathbin{;} (y \mathbin{;} z)), \quad w \mathbin{;} (x \mathbin{;} y \mathbin{;} z),\ w \mathbin{;} (x \mathbin{;} y) \mathbin{;} z\}.$$

Obviously, if $\mathfrak{A} \in \mathsf{RA}$, then $P^{(\mathfrak{A})}(x_0, \ldots, x_n) = \{x_0 \mathbin{;} \cdots \mathbin{;} x_n\}$ for all $x_0, \ldots, x_n \in A$. The next theorem shows that this is still true in case $\mathfrak{A}$ is only a semiassociative relation algebra and $1 \in \{x_0, \ldots, x_n\}$. It is a special case of Theorem 25 of [M91] and was first stated in [M78, pp. 35–36].

THEOREM 5. *If* $\mathfrak{A} \in \mathsf{SA}$ *and* $1 \in \{x_0, \ldots, x_n\} \subseteq A$, *then* $|P^{(\mathfrak{A})}(x_0, \ldots, x_n)| = 1$.

Theorem 5 implies that any rearrangement of parentheses is possible in a term of the form $x_0 \mathbin{;} \cdots \mathbin{;} x_n$, assuming that one of the x_k's is the unit element. For example, $(((w \mathbin{;} x) \mathbin{;} y) \mathbin{;} z) \mathbin{;} 1 = w \mathbin{;} (x \mathbin{;} (y \mathbin{;} (z \mathbin{;} 1)))$ whenever w, x, y, z are elements of a semiassociative relation algebra. Theorem 5 is the main reason for the undecidability of the equational theory of semiassociative relation algebras, which was first proved in [M78, Theorem 2, p. 222].

Now we review the definitions leading to the representable relation algebras.

DEFINITION 6. For any equivalence relation E, let

$$\mathfrak{Sb}E = \langle SbE, \cup, \cap, {}^{\sim E}, \varnothing, E, |, {}^{-1}, \mathrm{Id}_{\mathrm{Fd}\,E} \rangle,$$

where

$$\mathrm{Sb}\,E = \{R \colon R \subseteq E\},$$
$$\mathrm{Fd}\,E = \{x \colon \text{for some } y,\ \langle x, y \rangle \in E \text{ or } \langle y, x \rangle \in E\},$$
$$\mathrm{Id}_{\mathrm{Fd}\,E} = \{\langle x, x \rangle \colon x \in \mathrm{Fd}\,E\},$$

and, for all $R, S \subseteq E$,

$$\begin{aligned} R^{\sim E} &= E \sim R, \\ R|S &= \{\langle x,z\rangle : \text{for some } y, \langle x,y\rangle \in R \text{ and } \langle y,z\rangle \in S\}, \\ R^{-1} &= \{\langle y,x\rangle : \langle x,y\rangle \in R\}. \end{aligned}$$

Note that $\mathrm{Sb}\, E$ is closed under $\cup, \cap, {}^{\sim E}, {}^{-1}$, and $|$. $\mathfrak{Sb}\, E$ is the (relation) *algebra of subrelations of* E. For any set U, let

$$\mathrm{Re}\, U = \mathrm{Sb}(U \times U) \quad \text{and} \quad \mathfrak{Re}\, U = \mathfrak{Sb}(U \times U).$$

$\mathrm{Re}\, U$ is the set of binary relations on U, and $\mathfrak{Re}\, U$ is the (relation) *algebra of relations on* U.

It is easy to check that $\mathfrak{Re}\, U$ and $\mathfrak{Sb}\, E$ are relation algebras. In addition to the identity relation Id_U over a set U, we will also use the diversity relation Di_U, which is defined by

$$\mathrm{Di}_U = \{\langle x,y\rangle : x, y \in U \text{ and } x \neq y\}.$$

Note that Id_U and Di_U are the identity and diversity elements of $\mathfrak{Re}\, U$, respectively. Also, in accordance with the definition of x^n in an arbitrary relation algebra, we let $R^0 = \mathrm{Id}_U$ and $R^{n+1} = R^n|R$ for all $n < \omega$, whenever $R \in \mathrm{Re}\, U$. The notation "R^0" is thus ambiguous. When using it we must have a specific algebra $\mathfrak{Re}\, U$ in mind. On the other hand, if $n \geq 1$, then R^n is actually independent of the algebra $\mathfrak{Re}\, U$, in the sense that if $R \in \mathrm{Re}\, U$ and $R \in \mathrm{Re}\, W$, then "R^n" denotes the same relation whether it is referred to $\mathfrak{Re}\, U$ or $\mathfrak{Re}\, W$. The ambiguity in "R^0" could be eliminated without affecting the meaning of "R^n" for $n \geq 1$ by setting $R^0 = \{\langle x,x\rangle : \text{for some } y, \langle x,y\rangle \in R\}$. The convention we have followed here imitates [TG87, p. 24].

DEFINITION 7. Let $\mathfrak{A} \in \mathsf{NA}$, and let E be an equivalence relation. A function f mapping A into $\mathrm{Sb}\, E$ is a *representation of* $\mathfrak{A}$ *over* E if f is an isomorphic embedding of $\mathfrak{A}$ into $\mathfrak{Sb}\, E$. In case $E = U \times U$ for some set U, we also say f is a representation of $\mathfrak{A}$ *based on* U. $\mathfrak{A}$ is *representable* if there is a representation of $\mathfrak{A}$ over some equivalence relation E. RRA is the class of all representable relation algebras.

§3. **Semigroups and word problems.** Let $0 < n \leq \omega$, and let $V_n = \{\mathbf{v}_k : k < n\}$ be an arbitrary set of n elements. Let $\mathfrak{T}_n = \langle T_n, \circ\rangle$ be the groupoid (that is, a set with a binary operation) which is absolutely freely generated by V_n. The elements of T_n are called terms. Every mapping of V_n into a groupoid has a unique extension to a homomorphism from $\mathfrak{T}_n$ into that groupoid. Every nonassociative relation algebra $\mathfrak{A} = \langle A, +, \cdot, ^-, 0, 1, ;, \breve{}, 1'\rangle$ has three natural groupoid reducts, namely, $\langle A, +\rangle$, $\langle A, \cdot\rangle$, and $\langle A, ;\rangle$. We will be especially interested in maps from $\mathfrak{T}_n$ to $\langle A, ;\rangle$ in case $\mathfrak{A}$ is semiassociative. In particular, every mapping from V_n into A has a unique extension to a homomorphism from $\mathfrak{T}_n$ into $\langle A, ;\rangle$.

DEFINITION 8. For every $P \subseteq T_n \times T_n$, let $\Xi(P)$ be the intersection of all binary relations E which satisfy the following conditions:

(i) $P \subseteq E \subseteq T_n \times T_n$,

(ii) E is an equivalence relation,
(iii) $\langle (t \circ u) \circ v, t \circ (u \circ v) \rangle \in E$ for all $t, u, v \in T_n$,
(iv) if $\langle u, v \rangle \in E$ and $t \in T_n$, then $\langle u \circ t, v \circ t \rangle \in E$ and $\langle t \circ u, t \circ v \rangle \in E$.

Let $P \subseteq T_n \times T_n$. It is easy to see that $\Xi(P)$ is a congruence relation on the algebra $\mathfrak{T}_n$. This is due to conditions 8(ii) and 8(iv). Because of 8(iii), the quotient algebra $\mathfrak{T}_n/\Xi(P)$ is a semigroup. Furthermore, $\mathfrak{T}_n/\Xi(P)$ is freely generated as a semigroup by $\{\mathbf{v}_k/\Xi(P) : k < n\}$ subject to the conditions imposed on these generators by P. In other words, if $\mathfrak{G} = \langle S, \circ \rangle$ is any semigroup and f is any mapping from V_n into S such that $f(r) = f(s)$ whenever $\langle r, s \rangle \in P$, then there is a unique homomorphism h from $\mathfrak{T}_n/\Xi(P)$ to $\mathfrak{G}$ such that $h(v/\Xi(P)) = f(v)$ for every $v \in V_n$. Thus, $\langle V_n, P \rangle$ is really just a presentation of the semigroup $\mathfrak{T}_n/\Xi(P)$. If n is finite, then the presentation $\langle V_n, P \rangle$ is finite just in case P is finite and is recursive just in case P is recursive. Not every countably generated semigroup has a finite (or even recursive) presentation. But if $\mathfrak{G}$ is a semigroup which is generated by n elements, then it does have some presentation, that is, there is some $P \subseteq T_n \times T_n$ such that $\mathfrak{G}$ is isomorphic to $\mathfrak{T}_n/\Xi(P)$. To get such a P, choose a map from $\{\mathbf{v}_k : k < n\}$ onto the generators, let h be the homomorphism extending this map, and let P be the kernel of h, namely, $P = h|h^{-1}$.

Given a presentation $\langle V_n, P \rangle$, its *word problem* is to determine, given terms $r, s \in T_n$, whether $\langle r, s \rangle \in \Xi(P)$. The word problem is *unsolvable* if there is no algorithm for doing this. The word problem is *recursively unsolvable* if $\Xi(P)$ is not recursive. By Church's Thesis, the two types of unsolvability coincide. The word problem for $\langle V_n, P \rangle$ may be unsolvable if P is itself not recursive, so the more interesting question is whether there are finite sets P for which $\langle V_n, P \rangle$ has an unsolvable word problem, i.e., are there finite presentations with unsolvable word problems. Of course, Post and Markov proved long ago that there are such presentations. In fact, there are such presentations for every finite $n \geq 2$ (see [D58, p. 98] and the references given there).

In the next section we will show how to translate questions of the form "is $\langle r, s \rangle \in \Xi(P)$" into questions of the form "is $\varepsilon(r, s, P)$ true in $\mathfrak{A}$", where $\mathfrak{A}$ is any suitable semiassociative relation algebra and $\varepsilon(r, s, P)$ is an equation which can be computed directly from r, s, and P, whenever P is finite.

Note that for every set U, $\langle \mathrm{Re}\, U, | \rangle$ is a semigroup. By a *representation* of a semigroup $\mathfrak{G}$ we mean an isomorphic embedding of $\mathfrak{G}$ into a semigroup of the form $\langle \mathrm{Re}\, U, | \rangle$. It is easy to show that every semigroup $\mathfrak{G}$ has a representation by imitating the proof of Cayley's Theorem for groups (see [MMT87, pp. 128–129]). The first step in the proof is to embed $\mathfrak{G}$ into a semigroup with unit if $\mathfrak{G}$ does not already have one (see [MMT87, Lemma 1, p. 129]). So assume that $\mathfrak{G}$ has a unit element $e \in S$. We just take $U = S$ and define the Cayley representation $g : \mathfrak{G} \to \langle \mathrm{Re}\, U, | \rangle$ defined for every $s \in S$ by

$$g(s) = \{\langle t, t \circ s \rangle : t \in S\}.$$

One can show that g is a homomorphism without reference to e, but the presence of e ensures that g will be one-to-one. It also gives something more that we will need later. We say that a relation $D \subseteq U \times U$ is a *test relation* for the representation g if, for all $s, t \in S$, $s = t$ iff $(U \times U)|D|g(s) = (U \times U)|D|g(t)$. It is easy to see that $U \times \{e\}$ is a test relation for the Cayley representation. In fact, any nonempty relation included in $U \times \{e\}$ is also a test relation, such as $\{\langle e, e\rangle\}$. Thus, every semigroup has a representation with a test relation.

§**4. Reducing word problems to algebras.** Let us assume that we are working with a fixed, but otherwise arbitrary, finite semigroup presentation $\langle V_n, P\rangle$ with $0 < n < \omega$, $|P| = m < \omega$, and $P = \{\langle p_0, q_0\rangle, \dots, \langle p_{m-1}, q_{m-1}\rangle\}$. (We allow $P = \varnothing$ and $m = 0$.) Let us also assume that we have a fixed but arbitrary semiassociative relation algebra $\mathfrak{A} = \langle A, +, ^-, ;, ^\smile, 1'\rangle$ and a way to translate the terms in T_n into elements of $\mathfrak{A}$, namely, a homomorphism $h\colon \mathfrak{T}_n \to \langle A, ;\rangle$. Note that $\langle A, ;\rangle$ is associative (and is a semigroup) just in case $\mathfrak{A} \in \mathsf{RA}$.

Let $r, s \in T_n$. Consider the following two statements:

$$\textbf{(P)} \qquad \langle r, s\rangle \in \Xi(P),$$

$$\textbf{(R)} \qquad 1 ; [h(p_0) \diamond h(q_0) + \cdots + h(p_{m-1}) \diamond h(q_{m-1})] ; 1 \geq h(r) \diamond h(s).$$

The word problem for the presentation $\langle V_n, P\rangle$ is to determine whether **(P)** holds, that is, to determine whether r and s denote the same object in the semigroup $\mathfrak{T}_n/\Xi(P)$. The next theorem shows that, under certain assumptions on $\mathfrak{A}$ and h, **(P)** and **(R)** are equivalent and the word problem for $\langle V_n, P\rangle$ is thus reducible to a problem concerning the algebra $\mathfrak{A}$. Here is one of those assumptions.

(1) There is a semigroup representation $g\colon \mathfrak{T}_n/\Xi(P) \to \langle \mathrm{Re}\, U, |\rangle$ and a homomorphism $f : \mathfrak{A} \to \mathfrak{Re}\, U$, such that $f(h(\mathbf{v}_k)) = g(\mathbf{v}_k/\Xi(P))$ for every $k < n$.

THEOREM 9. *For any finite semigroup presentation $\langle V_n, P\rangle$, with $0 < n < \omega$, $|P| = m < \omega$, and $P = \{\langle p_0, q_0\rangle, \dots, \langle p_{m-1}, q_{m-1}\rangle\}$, any $\mathfrak{A} \in \mathsf{SA}$, and any homomorphism $h\colon \mathfrak{T}_n \to \langle A, ;\rangle$, the following statements hold*:

(i) *If $\mathfrak{A} \in \mathsf{RA}$, then* **(P)** *implies* **(R)**.

(ii) *If* (1), *then* **(R)** *implies* **(P)**.

PROOF. Let

$$z = 1 ; [h(p_0) \diamond h(q_0) + \cdots + h(p_{m-1}) \diamond h(q_{m-1})] ; 1$$

if $P \neq \varnothing$, and let $z = 0$ if $P = \varnothing$. Let

$$E = \{\langle r, s\rangle : r, s \in T_n \text{ and } z \geq h(r) \diamond h(s)\}.$$

The definitions of z and E are designed so that the inclusion $E \supseteq \Xi(P)$ holds if and only if **(P)** implies **(R)** for all terms $r, s \in T_n$. To prove that $E \supseteq \Xi(P)$, it suffices to verify that E satisfies conditions 8(i)–(iv). First, for every $i < m$ we have, by Lemma 2(iii)(iv),

$$h(p_i) \diamond h(q_i) \leq 1 ; [h(p_i) \diamond h(q_i)] \leq 1 ; [h(p_i) \diamond h(q_i)] ; 1 \leq z,$$

so $\langle p_i, q_i\rangle \in E$. Thus, $P \subseteq E$, so Definition 8(i) holds for E. It is similarly easy to show that E is reflexive by Lemma 2(xii), symmetric by Lemma 2(xiii), and transitive by Lemma 2(xiv). Thus, E is an equivalence relation and 8(ii) holds.

To prove 8(iii) we need the assumption that $\mathfrak{A} \in \mathsf{RA}$ so that we may use (AL), the associative law for ;. Suppose $t, u, v \in T_n$. Then

$$\begin{aligned} h(t \circ (u \circ v)) &= h(t) ; [h(u) ; h(v)] \\ &= [h(t) ; h(u)] ; h(v) \quad \text{(AL)} \\ &= h((t \circ u) \circ v), \end{aligned}$$

so $h(t \circ (u \circ v)) \diamond h((t \circ u) \circ v) = 0 \leq z$, and hence, $\langle t \circ (u \circ v), (t \circ u) \circ v \rangle \in E$. Finally, to prove 8(iv), we assume $\langle u, v \rangle \in E$ and $t \in T_n$. Then

$$\begin{aligned} h(t \circ u) \diamond h(t \circ v) &= h(t) ; h(u) \diamond h(t) ; h(v) \\ &\leq h(t) ; [h(u) \diamond h(v)] && \text{Lemma 2(xv)} \\ &\leq 1 ; z && \langle u, v \rangle \in E, \text{ Lemma 2(iii)} \\ &= z && \text{(AL), Lemma 2(iv)} \end{aligned}$$

and

$$\begin{aligned} h(u \circ t) \diamond h(v \circ t) &= h(u) ; h(t) \diamond h(v) ; h(t) \\ &\leq [h(u) \diamond h(v)] ; h(t) && \text{Lemma 2(xvi)} \\ &\leq z ; 1 && \langle u, v \rangle \in E, \text{ Lemma 2(iii)} \\ &= z && \text{(AL), Lemma 2(iv).} \end{aligned}$$

Hence, $\langle u \circ t, v \circ t \rangle \in E$ and $\langle t \circ u, t \circ v \rangle \in E$. This completes the proof of (i).

For (ii) assume (1) in (**R**). From (1) we have $f(h(\mathbf{v}_k)) = g(\mathbf{v}_k/\Xi(P))$ for every $k < n$. Since $\mathfrak{T}_n$ is generated by $\{\mathbf{v}_k : k < n\}$, it follows that $f(h(t)) = g(t/\Xi(P))$ for every $t \in T_n$. In particular, if $\langle u, v \rangle \in \Xi(P)$, then $f(h(u) \diamond h(v)) = \varnothing$. But $P \subseteq \Xi(P)$, so

$$\begin{aligned} f(z) &= f(1 ; [h(p_0) \diamond h(q_0) + \cdots + h(p_{m-1}) \diamond h(q_{m-1})] ; 1) \\ &= (U \times U)|\varnothing|(U \times U) = \varnothing. \end{aligned}$$

By (**R**), $z \geq h(r) \diamond h(s)$, so

$$\begin{aligned} \varnothing = f(z) \supseteq f(h(r) \diamond h(s)) &= f(h(r)) \diamond f(h(s)) \\ &= g(r/\Xi(P)) \diamond g(s/\Xi(P)). \end{aligned}$$

This is equivalent to $g(r/\Xi(P)) = g(s/\Xi(P))$, but g is one-to-one, so $r/\Xi(P) = s/\Xi(P)$ and (**P**) holds. □

The associative law was used three times in the proof of Theorem 9(i), but only the first application was essential. Theorem 5 can be used in place of the last two applications.

Let us say that $\mathfrak{A} \in \mathsf{RA}$ is an "undecidable relation algebra" if it has the property that every class of relation algebras containing $\mathfrak{A}$ has an undecidable equational theory. The next theorem describes a large class of finitely generated undecidable relation algebras that are all isomorphic to subalgebras of $\mathfrak{Re}\,\omega$.

THEOREM 10. *Suppose $\langle V_n, P\rangle$ is a semigroup presentation with an unsolvable word problem, g is a semigroup representation, $g\colon \mathfrak{T}_n/\Xi(P) \to \langle \mathrm{Re}\, U, |\rangle$, and $\mathfrak{B}$ is the subalgebra of $\mathfrak{Re}\, U$ generated by $\{g(\mathbf{v}_k/\Xi(P))\colon k < n\}$. If $\mathfrak{B} \in K \subseteq \mathsf{RA}$, then*

(i) *the free algebra over K on n generators has an unsolvable word problem,*

(ii) *the set of n-variable equations true in every algebra in K is not recursive,*

(iii) *the equational theory of K is undecidable.*

PROOF. We need only prove (i), since (ii) and (iii) follow from (i). Let $\mathfrak{A}$ be a relation algebra that is K-freely generated by $\{a_k\colon k < n\}$. Let h be the homomorphism on $\mathfrak{T}_n$ which maps the variables in V_n to the free generators of $\mathfrak{A}$, that is, $h(\mathbf{v}_k) = a_k$ for $k < n$. Since $\mathfrak{B} \in K$, there is a homomorphism $f : \mathfrak{A} \to \mathfrak{B}$, such that $f(a_k) = g(\mathbf{v}_k/\Xi(P))$ for $k < n$. Thus (1) holds, so (**R**) implies (**P**) by Theorem 9(ii). On the other hand, $\mathfrak{A} \in \mathsf{RA}$ since $K \subseteq \mathsf{RA}$, so (**P**) implies (**R**) by Theorem 9(i). This reduces the word problem for $\langle V_n, P\rangle$ to the word problem for $\mathfrak{A}$. Hence, $\mathfrak{A}$ has an unsolvable word problem. □

Semigroup presentations with only one variable have solvable word problems, so the smallest n for which Theorem 10 can be applied is $n = 2$. There do exist finite semigroup presentations on two variables that have undecidable word problems. If $\langle V_2, P\rangle$ is such a presentation, then the Cayley representation of $\mathfrak{T}_2/\Xi(P)$ yields an undecidable subalgebra of $\mathfrak{Re}(T_2/\Xi(P))$, the one generated by the representations of $\mathbf{v}_0/\Xi(P)$ and $\mathbf{v}_1/\Xi(P)$. By Theorem 10, any class K of relation algebras containing this algebra has an undecidable equational theory; moreover, the set of two-variable equations true in K is not recursive.

We wish to get similar results involving SA instead of RA and only one generator instead of two. We will deal with the reduction to one generator in the next section. In the remainder of this section we show that the reduction method can be applied to SA, in spite of the lack of full associativity, by replacing (**R**) with the more complicated statement (**S**). Unfortunately, this change will lead naturally to a result for SA which involves three generators, not one.

$$(\mathbf{S})\quad 1 \mathbin{;} [h(p_0) \diamond h(q_0) + \cdots + h(p_{m-1}) \diamond h(q_{m-1})] \mathbin{;} 1 \geq 1 \mathbin{;} x \mathbin{;} h(r) \diamond 1 \mathbin{;} x \mathbin{;} h(s).$$

Notice that (**R**) and (**S**) are the same except that the term $h(r) \diamond h(s)$ has been replaced by $1 \mathbin{;} x \mathbin{;} h(r) \diamond 1 \mathbin{;} x \mathbin{;} h(s)$. The presence of "1" in this latter term makes it possible to derive (**S**) from (**P**) without the hypothesis that $\mathfrak{A} \in \mathsf{RA}$, using Theorem 5 in place of (AL). To prove (**P**) from (**S**) we will need the two occurrences of "x", together with (1) and the following assumption which expresses the rôle of x in (**S**).

(2) $f(x)$ is a test relation for the representation g.

THEOREM 11. *For any finite semigroup presentation $\langle V_n, P\rangle$ with $0 < n < \omega$, $|P| = m < \omega$, and $P = \{\langle p_0, q_0\rangle, \ldots, \langle p_{m-1}, q_{m-1}\rangle\}$, any $\mathfrak{A} \in \mathsf{SA}$, and any homomorphism $h\colon \mathfrak{T}_n \to \langle A, ;\rangle$, the following statements hold*:

(i) *For every $x \in A$,* (**P**) *implies* (**S**).

(ii) *If* (1) *and* (2), *then* (**S**) *implies* (**P**).

PROOF. As in the proof of Theorem 9(i), we let $z = 1 \mathbin{;} [h(p_0) \diamond h(q_0) + \cdots + h(p_{m-1}) \diamond h(q_{m-1})] \mathbin{;} 1$ if $P \neq \varnothing$, and $z = 0$ if $P = \varnothing$. We alter the definition of E

to match (**S**) as follows:

$$E = \{\langle r, s\rangle : r, s \in T_n \text{ and, for every } x \in A,\ z \geq 1 \,;\, x \,;\, h(r) \diamond 1 \,;\, x \,;\, h(s)\}.$$

We show that E satisfies conditions 8(i)–(iv). For every $i < m$, we have

$$\begin{aligned} 1 \,;\, x \,;\, h(p_i) \diamond 1 \,;\, x \,;\, h(q_i) &\leq 1 \,;\, x \,;\, [h(p_i) \diamond h(q_i)] && \text{Lemma 2(xv)} \\ &\leq 1 \,;\, [h(p_i) \diamond h(q_i)] \,;\, 1 && \text{Lemma 2(iii)(iv)} \\ &\leq z && \text{Lemma 2(iii),} \end{aligned}$$

so $\langle p_i, q_i\rangle \in E$. Thus, 8(i) holds for E. It follows from Lemma 2(xii)–(xiv) that E is an equivalence relation. For 8(iii), suppose $t, u, v \in T_n$ and $x \in A$. Then

$$\begin{aligned} 1 \,;\, x \,;\, h(t \circ (u \circ v)) &= 1 \,;\, x \,;\, [h(t) \,;\, [h(u) \,;\, h(v)]] \\ &= 1 \,;\, x \,;\, [[h(t) \,;\, h(u)] \,;\, h(v)] && \text{Theorem 5} \\ &= 1 \,;\, x \,;\, h((t \circ u) \circ v), \end{aligned}$$

so $1 ; x ; h(t \circ (u \circ v)) \diamond 1 ; x ; h((t \circ u) \circ v) = 0 \leq z$. Hence, $\langle t \circ (u \circ v), (t \circ u) \circ v\rangle \in E$. To prove 8(iv), assume $\langle u, v\rangle \in E$ and $t \in T_n$. Then

$$\begin{aligned} &1 \,;\, x \,;\, h(u \circ t) \diamond 1 \,;\, x \,;\, h(v \circ t) \\ &\quad = 1 \,;\, x \,;\, [h(u) \,;\, h(t)] \diamond 1 \,;\, x \,;\, [h(v) \,;\, h(t)] \\ &\quad = 1 \,;\, x \,;\, h(u) \,;\, h(t) \diamond 1 \,;\, x \,;\, h(v) \,;\, h(t) && \text{Theorem 5} \\ &\quad \leq [1 \,;\, x \,;\, h(u) \diamond 1 \,;\, x \,;\, h(v)] \,;\, h(t) && \text{Lemma 2(xvi)} \\ &\quad \leq z \,;\, 1 && \langle u, v\rangle \in E,\ \text{Lemma 2(iii)} \\ &\quad = z && \text{Theorem 5, Lemma 2(iv)} \end{aligned}$$

and

$$\begin{aligned} &1 \,;\, x \,;\, h(t \circ u) \diamond 1 \,;\, x \,;\, h(t \circ v) \\ &\quad = 1 \,;\, x \,;\, [h(t) \,;\, h(u)] \diamond 1 \,;\, x \,;\, [h(t) \,;\, h(v)] \\ &\quad = 1 \,;\, [x \,;\, h(t)] \,;\, h(u) \diamond 1 \,;\, [x \,;\, h(t)] \,;\, h(v) && \text{Theorem 5} \\ &\quad \leq z && \langle u, v\rangle \in E. \end{aligned}$$

Hence, $\langle u \circ t, v \circ t\rangle \in E$ and $\langle t \circ u, t \circ v\rangle \in E$. This completes the proof of (i).

For part (ii), assume (1), (2), and (**S**). The first step is the same as in the proof of Theorem 9(ii): from (1) we get $f(z) = \varnothing$ and $f(h(t)) = g(t/\Xi(P))$ for every $t \in T_n$. Hence, by (**S**),

$$\begin{aligned}\varnothing &= f(z)\\ &\supseteq f(1\,;x\,;h(r) \diamond 1\,;x\,;h(s))\\ &= (U \times U)|f(x)|f(h(r)) \diamond (U \times U)|f(x)|f(h(s))\\ &= (U \times U)|f(x)|g(r/\Xi(P)) \diamond (U \times U)|f(x)|g(s/\Xi(P)),\end{aligned}$$

which implies

$$(U \times U)|f(x)|g(r/\Xi(P)) = (U \times U)|f(x)|g(s/\Xi(P)).$$

Hence, (**P**) holds by (2). □

Let us say that $\mathfrak{A} \in \mathsf{SA}$ is an "undecidable semiassociative relation algebra" if every class of semiassociative relation algebras containing $\mathfrak{A}$ has an undecidable equational theory. From Theorem 11 we obtain the following semiassociative version of Theorem 10. It describes a large class of finitely generated undecidable semiassociative relation algebras.

THEOREM 12. *Suppose $\langle V_n, P\rangle$ is a semigroup presentation with an unsolvable word problem, D is a test relation for the semigroup representation $g\colon \mathfrak{T}_n/\Xi(P) \to \langle \mathrm{Re}\, U, |\rangle$, and $\mathfrak{B}$ is the subalgebra of $\mathfrak{Re}\, U$ generated by $\{g(\mathbf{v}_k/\Xi(P))\colon k < n\} \cup \{D\}$. If $\mathfrak{B} \in K \subseteq \mathsf{SA}$, then*:
- (i) *the free algebra over K on $n+1$ generators has an unsolvable word problem,*
- (ii) *the set of $(n+1)$-variable equations true in every algebra in K is not recursive,*
- (iii) *the equational theory of K is undecidable.*

PROOF. Let $\mathfrak{A}$ be K-freely generated by $\{a_k\colon k < n+1\}$. Let h be the homomorphism from $\mathfrak{T}_n$ into $\langle \mathrm{Re}\, U, |\rangle$ such that $h(\mathbf{v}_k) = a_k$ for $k < n$. Since $\mathfrak{B} \in K$, there is a homomorphism $f\colon \mathfrak{A} \to \mathfrak{B}$, such that $f(a_k) = g(\mathbf{v}_k/\Xi(P))$ for $k < n$ and $f(a_n) = D$. Let $x = a_n$. Then (1) and (2) hold, so (**S**) implies (**P**) by Theorem 11(ii). But $\mathfrak{A} \in \mathsf{SA}$ since $K \subseteq \mathsf{SA}$, so (**P**) implies (**S**) by Theorem 11(i). The word problem for $\mathfrak{A}$ is, therefore, unsolvable. □

Applying Theorem 12 to the Cayley representation of a finitely presented semigroup $\mathfrak{T}_2/\Xi(P)$ with undecidable word problem yields a three-generated undecidable semiassociative relation algebra $\mathfrak{A}$ that is embeddable in $\mathfrak{Re}\,\omega$. Any class K of semiassociative relation algebras containing $\mathfrak{A}$ has an undecidable equational theory. This time, however, it is the set of three-variable equations true in K that is not recursive.

§**5. Reduction to one generator.** We have replaced RA with SA at the expense of increasing from two generators to three. We now reduce three to one.

DEFINITION 13. Let $n \leq \omega$. Suppose $\langle F_k\colon k < n\rangle$ is a countable sequence of binary relations on a set U. We say that a binary relation C *codes up* U *and* $\langle F_k\colon k < n\rangle$, if $\mathrm{Id}_U = C \cap \mathrm{Id}_{\mathrm{Fd}\, C}$ and $F_k = \mathrm{Id}_U|(C \cap \mathrm{Di}_{\mathrm{Fd}\, C})^{k+2}|\mathrm{Id}_U$ for every $k < n$.

Every binary relation codes up a set and a sequence of relations on the set. An arbitrary relation C can be split into its identity and diversity parts, namely, $C \cap \mathrm{Id}_{\mathrm{Fd}\,C}$ and $C \cap \mathrm{Di}_{\mathrm{Fd}\,C}$, respectively. If the field of the identity part of C is U, then, for every $n \leq \omega$, C codes up U and the sequence of relations $\langle \mathrm{Id}_U | (C \cap \mathrm{Di}_{\mathrm{Fd}\,C})^{k+2} | \mathrm{Id}_U \colon k < n\rangle$. The following lemma gives a converse to this fact. Clearly, if C codes up U and $\langle F_k \colon k < n\rangle$, then the subalgebra of $\mathfrak{Re}\,\mathrm{Fd}\,C$ generated by C includes all those relations in the subalgebra of $\mathfrak{Re}\,U$ that are generated by $\{F_k \colon k < n\}$. In fact, the latter algebra is a subalgebra of the relativization of the former algebra to $U \times U$.

LEMMA 14. *For every set U, every $n \leq \omega$, and every countable sequence $\langle F_k \colon k < n\rangle$ of relations on U, there is a relation C which codes up U and $\langle F_k \colon k < n\rangle$.*

PROOF. Let

$$Q = \{\langle j, k, a, b\rangle \colon j \leq k < n \text{ and } \langle a, b\rangle \in F_k\}.$$

Choose a set M so that $M \cap U = \varnothing$ and $|M| = |Q|$. The elements of M are called markers. Let m be a one-to-one correspondence mapping Q onto M. Denote the marker $m(j, k, a, b)$ by $m_j^k(a, b)$. We construct C by replacing each pair in each relation F_k by several new pairs. The idea is that for each pair $\langle a, b\rangle$ in each relation F_k, we have $k+1$ markers, say, $c_0, c_1, \ldots, c_k$, and we replace the pair $\langle a, b\rangle$ with $k+2$ pairs $\langle a, c_0\rangle$, $\langle c_0, c_1\rangle, \ldots, \langle c_k, b\rangle$. C is the set of pairs so obtained plus all pairs $\langle a, a\rangle$ with $a \in U$. More formally,

$$\begin{aligned} C = {} & \mathrm{Id}_U \cup \{\langle a, m_0^k(a, b)\rangle \colon k < n,\ \langle a, b\rangle \in F_k\} \\ & \cup \{\langle m_j^k(a, b), m_{j+1}^k(a, b)\rangle \colon j < k < n,\ \langle a, b\rangle \in F_k\} \\ & \cup \{\langle m_k^k(a, b), b\rangle \colon k < n,\ \langle a, b\rangle \in F_k\}. \end{aligned}$$

Incidentally, the relation C constructed in this proof satisfies the cardinality condition $|\mathrm{Fd}\,C| = |U| + \sum_{k<n}(k+2)|F_k|$, so C is finite whenever U and n are finite. □

Using Theorem 11 together with the coding device of Lemma 14, we now get a large class of one-generated undecidable semiassociative relation algebras embeddable in $\mathfrak{Re}\,\omega$.

THEOREM 15. *Suppose*
$\langle V_n, P\rangle$ is a finite presentation of the semigroup $\mathfrak{T}_n/\Xi(P)$,
D is a test relation for the semigroup representation $g \colon \mathfrak{T}_n/\Xi(P) \to \langle \mathrm{Re}\,U, |\rangle$,
C codes up U and $\langle g(\mathbf{v}_0/\Xi(P)), \ldots, g(\mathbf{v}_{n-1}/\Xi(P)), D\rangle$,
$\mathfrak{C}$ is the subalgebra of $\mathfrak{Re}\,\mathrm{Fd}\,C$ generated by C, and
K is any class of algebras such that $\mathfrak{C} \in K \subseteq \mathsf{SA}$.

Then the word problem for $\langle V_n, P\rangle$ is reducible to the word problem for the free algebra over K on one generator. If the word problem for $\langle V_n, P\rangle$ is unsolvable, then:
(i) *the word problem for the free algebra over K on one generator is unsolvable,*
(ii) *the set of one-variable equations true in every algebra in K is not recursive,*
(iii) *the equational theory of K is undecidable.*

PROOF. Let $\mathfrak{A}$ be K-freely generated by a. Since $\mathfrak{C} \in K$, there exists a homomorphism $f \colon \mathfrak{A} \to \mathfrak{Re}\,\mathrm{Fd}\,C$ such that $f(a) = C$. Note that f maps $\mathfrak{A}$ onto $\mathfrak{C}$. Consider the homomorphism $h \colon \mathfrak{T}_{n+1} \to \mathfrak{A}$ such that $h(\mathbf{v}_k) = (a \cdot 1') ; (a \cdot 0')^{k+2} ; (a \cdot 1')$

for every $k < n$ and $h(\mathbf{v}_n) = (a \cdot 1') ; (a \cdot 0')^{n+2} ; (a \cdot 1')$. Since C codes up U and $\langle g(\mathbf{v}_0/\Xi(P)), \ldots, g(\mathbf{v}_{n-1}/\Xi(P)), D\rangle$, we have

$$\begin{aligned} f(h(\mathbf{v}_k)) &= f((a \cdot 1') ; (a \cdot 0')^{k+2} ; (a \cdot 1')) \\ &= \mathrm{Id}_U | (C \cap \mathrm{Di}_{\mathrm{Fd}\, C})^{k+2} | \mathrm{Id}_U \\ &= g(\mathbf{v}_k/\Xi(P)) \end{aligned}$$

for every $k < n$, and setting $x = h(\mathbf{v}_n)$, we also have

$$f(x) = D.$$

By the assumption on D, (2) holds. To see that (1) holds with $\mathrm{Fd}\, C$ in place of U, just note that $g\colon \mathfrak{T}_n/\Xi(P) \to \langle \mathrm{Re}\, \mathrm{Fd}\, C, |\rangle$ because $U \subseteq \mathrm{Fd}\, C$. We have $\mathfrak{A} \in \mathsf{SA}$ because $K \subseteq \mathsf{SA}$. Therefore, by Theorem 11, (**P**) and (**S**) are equivalent and the word problem for $\langle V_n, P\rangle$ is reducible to the word problem for $\mathfrak{A}$. Note that by the choice of h and x, (**S**) is a statement involving just the generator a of $\mathfrak{A}$. So if the word problem for $\langle V_n, P\rangle$ is unsolvable then (i)–(iii) hold. □

In the following corollary we meet two varieties not previously mentioned. SQRA is the variety of all subalgebras of Q-relation algebras. A Q-relation algebra is a relation algebra which contains two elements a and b such that $\breve{a} ; a \leq 1'$, $\breve{b} ; b \leq 1'$, and $\breve{a} ; b = 1$. (See Definition 8.4(ii) of [TG87].) IRRA is the variety of all subdirect products of algebras in $\{\mathfrak{Re}\, U\colon U \text{ is infinite}\}$. (See Definition 8.4(viii) of [TG87].) It turns out that IRRA is a subvariety of SQRA. This depends on the deep and important result of Tarski that every Q-relation algebra is representable. (See Theorem 8.4(iii),(ix) of [TG87]). By Theorem 8.1 of [J82], IRRA is the variety generated by $\mathfrak{Re}\, \omega$, namely, $\mathbf{HSP}\{\mathfrak{Re}\, \omega\}$.

COROLLARY 16. *Suppose $\mathfrak{Re}\, \omega \in K \subseteq \mathsf{SA}$. Then the free algebra over K on one generator has an unsolvable word problem, and the one-variable equational theory of K is undecidable. In particular, this is true if K is any one of the varieties* SQRA, IRRA, RRA, RA, *or* SA.

PROOF. Any variety generated by any class K containing $\mathfrak{Re}\, \omega$ will contain all the subalgebras of $\mathfrak{Re}\, \omega$, so the first part of the corollary follows from Theorem 15. Each of the varieties SQRA, IRRA, RRA, RA, and SA happens to contain $\mathfrak{Re}\, \omega$. □

Corollary 16 was first proved by Tarski with RA in place of SA in the hypothesis and SA omitted from the conclusion. Indeed, this restricted version of Corollary 16 follows immediately from the (somewhat stronger) Theorem 8.5(xii)(β) in [TG87].

§6. The formalisms $\mathscr{L}^\times, \mathscr{L}w^\times, \mathscr{L}$, and $\mathscr{L}^+$.

In this section we give brief descriptions of four formalisms constructed in [TG87], namely, $\mathscr{L}^\times, \mathscr{L}w^\times, \mathscr{L}$, and $\mathscr{L}^+$, all of which are based on just one nonlogical binary predicate $\mathbf{E}$.

$\mathbf{\Pi}$ is the set of predicates of $\mathscr{L}^\times$ and of $\mathscr{L}w^\times$. It is the smallest set with the following properties: $\overset{\circ}{\mathbf{1}}, \mathbf{E}$ are in $\mathbf{\Pi}$, and if A, B are in $\mathbf{\Pi}$, then $A \odot B$, $A + B$, A^-, and $A^{\smile}$ are also in $\mathbf{\Pi}$. There is an equality symbol in $\mathscr{L}^\times$, namely, $=$, intended to denote equality between predicates. $\mathbf{\Sigma}[\mathscr{L}^{\times}]$ is the set of sentences of $\mathscr{L}^{\times}$ (and is also called $\mathbf{\Sigma}^{\times}$). $\mathbf{\Sigma}[\mathscr{L}^{\times}]$ is defined as follows:

$$\mathbf{\Sigma}^{\times} = \mathbf{\Sigma}[\mathscr{L}^{\times}] = \{A = B \colon A, B \in \mathbf{\Pi}\}.$$

The set $\mathbf{\Lambda}[\mathscr{L}^{\times}]$ ($= \mathbf{\Lambda}^{\times}$) of logical axioms of $\mathscr{L}^{\times}$ consists of all instances of the following sentences in which A and B may be any predicates in $\mathbf{\Pi}$.

(BI) $A + B = B + A$,
(BII) $A + (B + C) = (A + B) + C$,
(BIII) $(A^{-} + B)^{-} + (A^{-} + B^{-})^{-} = A$,
(BIV) $A \odot (B \odot C) = (A \odot B) \odot C$,
(BV) $(A + B) \odot C = (A \odot C) + (B \odot C)$,
(BVI) $A \odot \mathring{\mathbf{1}} = A$,
(BVII) $A^{\smile\smile} = A$,
(BVIII) $(A + B)^{\smile} = A^{\smile} + B^{\smile}$,
(BIX) $(A \odot B)^{\smile} = B^{\smile} \odot A^{\smile}$,
(BX) $A^{\smile} \odot (A \odot B)^{-} + B^{-} = B^{-}$.

$\mathscr{L}w^{\times}$ has the same sets of predicates and sentences as $\mathscr{L}^{\times}$, namely, $\mathbf{\Pi}$ and $\mathbf{\Sigma}^{\times}$, respectively. The logical axioms of $\mathscr{L}w^{\times}$ are all instances of schemata (BI)–(BIII), (BV)–(BX), together with all instances of the following schema (BIV′), which replaces (BIV).

(BBIV′) $A \odot (B \odot \mathbf{1}) = (A \odot B) \odot \mathbf{1}$.

Schema (BIV′) uses a special predicate $\mathbf{1}$ which is defined by $\mathbf{1} = \mathring{\mathbf{1}} + \mathring{\mathbf{1}}^{-}$. Some other special predicates and operations on predicates are given in the following definitions:

$\mathbf{0} = (\mathring{\mathbf{1}} + \mathring{\mathbf{1}}^{-})^{-}$,
$\mathring{\mathbf{0}} = \mathring{\mathbf{1}}^{-}$,
$A \cdot B = (A^{-} + B^{-})^{-}$ for all $A, B \in \mathbf{\Pi}$,
$A \diamond B = (A^{-} + B)^{-} + (A + B^{-})^{-}$ for all $A, B \in \mathbf{\Pi}$.

The only rule of inference in $\mathscr{L}^{\times}$ is the rule of replacement: infer $A' = B'$ from $A = B$ and $C = D$, whenever C occurs as a part of A or B, and $A' = B'$ is the equation which we obtain from $A = B$ when we replace some occurrence of C by D. For every $\Psi \subseteq \mathbf{\Sigma}^{\times}$ and every $X \in \mathbf{\Sigma}^{\times}$, we say X is derivable from Ψ in $\mathscr{L}^{\times}$, in symbols $\Psi \vdash X[\mathscr{L}^{\times}]$ or just $\Psi \vdash^{\times} X$, if X belongs to every set of sentences of $\mathscr{L}^{\times}$ which contains $\Psi \cup \mathbf{\Lambda}^{\times}$ and is closed under the rule of replacement. For any $\Psi \subseteq \mathbf{\Sigma}^{\times}$, the theory generated by Ψ in $\mathscr{L}^{\times}$ is

$$\Theta\eta^{\times}\Psi = \Theta\eta\ \Psi[\mathscr{L}^{\times}] = \{X \in \mathbf{\Sigma}^{\times} \colon \Psi \vdash^{\times} X\}.$$

We define $\Psi \vdash X[\mathscr{L}w^{\times}]$, $\Psi \vdash^{w} X$, and $\Theta\eta^{w}\ \Psi = \Theta\eta\ \Psi[\mathscr{L}w^{\times}]$ similarly, using the logical axioms of $\mathscr{L}w^{\times}$ in place of $\Lambda^{\times}$.

Obviously, all instances of (BIV′) are instances of (BIV), so any sentence in $\mathbf{\Sigma}^{\times}$ which is provable in $\mathscr{L}w^{\times}$ is also provable in $\mathscr{L}^{\times}$, but it turns out that there are instances of (BIV) which are not provable in $\mathscr{L}^{\times}$. Thus, $\Theta\eta^{w}\varnothing \subset \Theta\eta^{\times}\varnothing$. The algebraic reason for this is simply that there are one-generated semiassociative relation algebras that are not relation algebras.

$\mathscr{L}$ is the formalism of first-order predicate logic with equality symbol $\mathring{\mathbf{1}}$ and one binary predicate $\mathbf{E}$. The set of variables of $\mathscr{L}$ is $\mathbf{\Phi} = \{\mathbf{v}_k : k < \omega\}$. For every $x \in \mathbf{\Phi}$, $\mathbf{in}\, x$ (the index of x) is the unique $k < \omega$ such that $x = \mathbf{v}_k$. The logical constants of $\mathscr{L}$ are the implication symbol $\rightarrow$, the negation symbol $\neg$, the universal quantifier $\forall$, and the equality symbol $\mathring{\mathbf{1}}$. ($\mathbf{E}$ is a nonlogical constant.) The existential quantifier and the other sentential connectives are introduced by abbreviations, e.g., $\exists_x X = \neg\forall_x\neg X$. Atomic formulas and arbitrary formulas are defined in the usual way. The set of all formulas of $\mathscr{L}$ is $\mathbf{\Psi}[\mathscr{L}]$ $(= \mathbf{\Psi})$. For any $X \in \mathbf{\Psi}$, $\mathbf{\Phi}\phi(X)$ is the set of variables that occur freely in X. So X is a sentence iff $\mathbf{\Phi}\phi(X) = \varnothing$. The set of all sentences of $\mathscr{L}$ is $\mathbf{\Sigma}[\mathscr{L}]$ $(= \mathbf{\Sigma})$. For every $X \in \mathbf{\Psi}$, $[X]$ is just X if $\mathbf{\Phi}\phi(X) = \varnothing$, and $[X]$ is $\forall_{x_0}\cdots\forall_{x_m}X$ if $\Upsilon\phi(X) = \{x_0, \dots, x_m\}$ and $\mathbf{in}(x_k) < \mathbf{in}(x_{k+1})$ for all $k < m$. Thus, $[X]$ is always a sentence, called the closure of X.

The set $\mathbf{\Lambda}[\mathscr{L}]$ $(= \mathbf{\Lambda})$ of logical axioms of $\mathscr{L}$ is the set of all instances of the following schemata, in which $X, Y, Z \in \Phi$ and $x, y \in \mathbf{\Phi}$:

(AI) $[(X \rightarrow Y) \rightarrow ((Y \rightarrow Z) \rightarrow (X \rightarrow Z))]$,

(AII) $[(\neg X \rightarrow X) \rightarrow X]$,

(AIII) $[X \rightarrow (\neg X \rightarrow Y)]$,

(AIV) $[\forall_x\forall_y X \rightarrow \forall_y\forall_x X]$,

(AV) $[\forall_x(X \rightarrow Y) \rightarrow (\forall_x X \rightarrow \forall_x Y)]$,

(AVI) $[\forall_x X \rightarrow X]$,

(AVII) $[X \rightarrow \forall_x X]$ where $x \notin \Upsilon\phi(X)$,

(AVIII) $[\exists_x(x\mathring{\mathbf{1}}y)]$,

(AIX) $[x\mathring{\mathbf{1}}y \rightarrow (X \rightarrow Y)]$, where X is any atomic formula in which x occurs, and Y is obtained from X by replacing a single occurrence of x with y.

The only rule of inference is *modus ponens*, i.e., infer Y from X and $X \rightarrow Y$. Thus, for every $\Psi \subseteq \mathbf{\Sigma}$ and every $X \in \mathbf{\Sigma}$, $\Psi \vdash X[\mathscr{L}]$ (briefly, $\Psi \vdash X$) if X belongs to every set of sentences of $\mathscr{L}$ which contains $\Psi \cup \Lambda$ and is closed under *modus ponens*. For every $\Psi \subseteq \mathbf{\Sigma}$, the theory generated by Ψ in $\mathscr{L}$ is

$$\Theta\eta\ \Psi = \Theta\eta\ \Psi[\mathscr{L}] = \{X \in \mathbf{\Psi} : \Psi \vdash X\}.$$

Now we turn to $\mathscr{L}^+$, which uses the infinite set $\mathbf{\Pi}$ of predicates in place of $\{\mathbf{E}, \mathring{\mathbf{1}}\}$ and also uses the additional equality symbol $=$ from $\mathscr{L}^\times$. In fact, $\mathscr{L}^+$ is a common extension of both $\mathscr{L}$ and $\mathscr{L}^\times$. The atomic formulas of $\mathscr{L}^+$ are not only all those of the form xAy, where $A \in \mathbf{\Pi}$ and $x, y \in \mathbf{\Phi}$, but also those of the form $A = B$, where $A, B \in \mathbf{\Pi}$. Thus, every sentence of $\mathscr{L}^\times$ (or $\mathscr{L}w^\times$) is an atomic formula of $\mathscr{L}^+$. The definitions of the set of formulas $\mathbf{\Psi}[\mathscr{L}^+]$ $(= \mathbf{\Psi}^+)$ and the set of sentences $\mathbf{\Sigma}[\mathscr{L}^+]$ $(= \mathbf{\Sigma}^+)$ of $\mathscr{L}^+$ are otherwise the same as $\mathscr{L}$.

The set of logical axioms $\mathbf{\Lambda}[\mathscr{L}^+]$ $(= \mathbf{\Lambda}^+)$ of $\mathscr{L}^+$ contains not only all instances of (AI)–(AIX) but also all instances of the following schemata, in which $A, B \in \mathbf{\Pi}$:

(DI) $\forall_{\mathbf{v}_0}\forall_{\mathbf{v}_1}(\mathbf{v}_0A + B\mathbf{v}_1 \leftrightarrow (\mathbf{v}_0A\mathbf{v}_1 \vee \mathbf{v}_0B\mathbf{v}_1))$,

(DII) $\forall_{\mathbf{v}_0}\forall_{\mathbf{v}_1}(\mathbf{v}_0A^-\mathbf{v}_1 \leftrightarrow \neg\mathbf{v}_0A\mathbf{v}_1)$,

(DIII) $\forall_{\mathbf{v}_0}\forall_{\mathbf{v}_1}(\mathbf{v}_0A \odot B\mathbf{v}_1 \leftrightarrow \exists_{\mathbf{v}_2}(\mathbf{v}_0A\mathbf{v}_2 \wedge \mathbf{v}_2B\mathbf{v}_1))$, (DIV) $\forall_{\mathbf{v}_0}\forall_{\mathbf{v}_1}(\mathbf{v}_0 A\breve{\ }\mathbf{v}_1 \leftrightarrow \mathbf{v}_1 A\mathbf{v}_0)$,

(DV) $A = B \leftrightarrow \forall_{\mathbf{v}_0}\forall_{\mathbf{v}_1}(\mathbf{v}_0 A\mathbf{v}_1 \leftrightarrow \mathbf{v}_0 B\mathbf{v}_1)$.

The only rule of inference in $\mathscr{L}^+$ is also *modus ponens*. For every $\Psi \subseteq \mathbf{\Sigma}^+$ and every $X \in \mathbf{\Sigma}^+$, $\Psi \vdash X[\mathscr{L}^+]$.if X belongs to every set of sentences in $\mathscr{L}^+$ that contains $\Psi \cup \mathbf{\Lambda}^+$ and is closed under *modus ponens*. The theory generated by Ψ in $\mathscr{L}^+$ is

$$\Theta\eta^+\Psi = \Theta\eta\ \Psi[\mathscr{L}^+] = \{X \in \mathbf{\Psi}^+ : \Psi \vdash^+ X\}.$$

In [TG87] it is shown that the formalisms $\mathscr{L}$ and $\mathscr{L}^+$ are equipollent in means of both expression and proof and that they are both more powerful than $\mathscr{L}^\times$. Thus,

$$\Theta\eta^\times\Psi \subseteq \Theta\eta\ \Psi \cap \mathbf{\Sigma}^\times = \Theta\eta^+\Psi \cap \mathbf{\Sigma}^\times$$

for every $\Psi \subseteq \mathbf{\Sigma}^\times$, but

$$\Theta\eta^\times\varnothing \subset \Theta\eta\varnothing \cap \mathbf{\Sigma}^\times = \Theta\eta^+\varnothing \cap \mathbf{\Sigma}^\times.$$

§**7. Algebras from formalisms.** In this section we follow a method, developed in Chapter 8 of [TG87], by which we obtain various algebras from the formalisms described in the previous section. We then obtain results about the formalisms by applying previous theorems to the algebras obtained from the formalisms.

DEFINITION 17. Let $\mathfrak{P} = \langle \mathbf{\Pi}, +, \cdot, ^-, \mathbf{0}, \mathbf{1}, \odot, \breve{\ }, \mathring{\mathbf{1}} \rangle$ so that $\mathfrak{P}$ is an absolutely free algebra of similarity type $\langle 2, 2, 1, 0, 0, 2, 1, 0\rangle$, and $\mathfrak{P}$ is generated by $\mathbf{E}$.

For any $\Psi \subseteq \mathbf{\Sigma}^\times$, let $\simeq^w_\Psi$ be the binary relation that holds between any $A, B \in \mathbf{\Pi}$ iff $\Psi \vdash A = B[\mathscr{L}w^\times]$. Let $\simeq^w$ be the relation obtained in case $\Psi = \varnothing$.

For any $\Psi \subseteq \mathbf{\Sigma}^+$, let $\simeq^+_\Psi$ be the binary relation that holds between any $A, B \in \mathbf{\Pi}$ iff $\Psi \vdash A = B[\mathscr{L}^+]$. Let $\simeq^+$ be the relation obtained in case $\Psi = \varnothing$.

The next two theorems are obtained from Theorem 8.2(ix)(x) of [TG87] by restricting to the case of a single binary predicate and by replacing $\mathscr{L}^\times$ with $\mathscr{L}w^\times$ and RA with SA. Their proofs are nearly the same as the proofs of Theorem 8.3(ix)(x).

THEOREM 18. *For every $\Psi \subseteq \mathbf{\Sigma}^\times$, $\simeq^w_\Psi$ is a congruence relation on the algebra $\mathfrak{P}$. The quotient algebra $\mathfrak{P}/\simeq^w_\Psi$ is a semiassociative relation algebra generated by $\mathbf{E}/\simeq^w_\Psi$.*

PROOF. It is easy to verify that $\simeq^w_\Psi$ is a congruence relation on $\mathfrak{P}$. Since (BI)–(BIII) are axiom schemata for $\mathscr{L}w^\times$, it follows that the Boolean part of $\mathfrak{P}/\simeq^w_\Psi$ actually is a Boolean algebra.

The fact that (BV) is an axiom schema for $\mathscr{L}w^\times$ implies that $\mathfrak{P}/\simeq^w_\Psi$ satisfies identity (i) of Theorem 3. Similarly, the other axiom schemata (BVI)–(BX) imply that $\mathfrak{P}/\simeq^w_\Psi$ satisfies identities (ii)–(vi) of Theorem 3. It follows that $\mathfrak{P}/\simeq^w_\Psi$ is a nonassociative relation algebra.

Using Lemma 2(iv) together with axiom schema (BIV′), we conclude that the semiassociative law holds in $\mathfrak{P}/\simeq^w_\Psi$. Hence, $\mathfrak{P}/\simeq^w_\Psi$ is a semiassociative relation algebra.

It is easy to see that $\mathfrak{P}/\simeq^w_\Psi$ is generated by $\mathbf{E}/\simeq^w_\Psi$. □

THEOREM 19. *$\mathfrak{P}/\simeq^w$ is a semiassociative relation algebra which is* SA*-freely generated by $\mathbf{E}/\simeq^w$.*

PROOF. Imitate the proof of Theorem 8.2(x) of [TG87]. □

THEOREM 20. *$\Theta\eta^{w}\varnothing$ is an undecidable theory in $\mathscr{L}w^{\times}$.*

PROOF. This is an immediate consequence of Corollary 16 and Theorem 19. □

THEOREM 21. *$\Theta\eta^{+}\varnothing \cap \mathbf{\Sigma}^{\times}$ is a hereditarily undecidable theory in $\mathscr{L}w^{\times}$.*

PROOF. Let $\Psi \subseteq \Theta\eta^{+}\varnothing \cap \mathbf{\Sigma}^{\times}$. Then for all $A, B \in \mathbf{\Pi}$, $A \simeq^{w}_{\Psi} B$ implies $A \simeq^{+} B$. It follows that there is a homomorphism f_0 mapping $\mathfrak{P}/\simeq^{w}_{\Psi}$ onto $\mathfrak{P}/\simeq^{+}$ such that $f_0(A/\simeq^{w}_{\Psi}) = A/\simeq^{+}$ for every $A \in \mathbf{\Pi}$. Choose a finite semigroup presentation $\langle V_2, P\rangle$ with an unsolvable word problem, where $P = \{\langle p_0, q_0\rangle, \ldots, \langle p_{m-1}, q_{m-1}\rangle\}$. Choose a semigroup representation $g\colon \mathfrak{T}_2/\Xi(P) \to \langle \mathrm{Re}\, U, |\rangle$ with test relation D. By Lemma 14 there is a relation C which codes up U and $\langle g(\mathbf{v}_0/\Xi(P)), g(\mathbf{v}_1/\Xi(P)), D\rangle$. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{Re}\, \mathrm{Fd}\, C$ generated by C. By Theorem 8.3(viii) of [TG87], $\mathfrak{P}/\simeq^{+}$ is RRA-freely generated by $\mathbf{E}/\simeq^{+}$, so there is a homomorphism $f_1\colon \mathfrak{P}/\simeq^{+} \to \mathfrak{Re}\, \mathrm{Fd}\, C$ mapping $\mathfrak{P}/\simeq^{+}$ onto $\mathfrak{C}$ such that $f_1(\mathbf{E}/\simeq^{+}) = C$. Let $\mathfrak{A} = \mathfrak{P}/\simeq^{w}_{\Psi}$, $a = \mathbf{E}/\simeq^{w}_{\Psi}$, and $f = f_0|f_1$. Then $\mathfrak{A} \in \mathsf{SA}$ by Theorem 18, f is a homomorphism from $\mathfrak{A}$ onto $\mathfrak{C}$, and $f(a) = C$. Let h be the homomorphism from $\mathfrak{T}_2$ into $\langle A, ;\rangle$ such that $h(\mathbf{v}_0) = (a \cdot 1') ; (a \cdot 0')^2 ; (a \cdot 1')$ and $h(\mathbf{v}_1) = (a \cdot 1') ; (a, 0')^3 ; (a \cdot 1')$. Then

$$\begin{aligned} f(h(\mathbf{v}_0)) &= f((a \cdot 1') ; (a \cdot 0')^2 ; (a \cdot 1')) \\ &= (f(a) \cap \mathrm{Id}_{\mathrm{Fd}\, C})|(f(a) \cap \mathrm{Di}_{\mathrm{Fd}\, C})^2|(f(a) \cap \mathrm{Id}_{\mathrm{Fd}\, C}) \\ &= \mathrm{Id}_U|(C \cap \mathrm{Di}_{\mathrm{Fd}\, C})^2|\mathrm{Id}_U \\ &= g(\mathbf{v}_0/\Xi(P)) \end{aligned}$$

and similarly, $f(h(\mathbf{v}_1)) = g(\mathbf{v}_1/\Xi(P))$. Thus, (1) holds. Set $x = (a \cdot 1') ; (a \cdot 0')^4 ; (a \cdot 1')$. Then $f(x) = D$, so (2) also holds. By Theorem 11, $(\mathbf{P})$ and $(\mathbf{S})$ are equivalent. By the choice of $\langle V_2, P\rangle$, there is no algorithm for determining whether $(\mathbf{P})$ holds, so the same is true for $(\mathbf{S})$. We now convert $(\mathbf{S})$ into an equivalent condition involving provability in $\mathscr{L}w^{\times}$.

First, factor h through $\langle \mathbf{\Pi}, \odot\rangle$. Let h_0 be the homomorphism from $\mathfrak{T}_2$ to $\langle \mathbf{\Pi}, \odot\rangle$ such that $h_0(\mathbf{v}_0) = (\mathbf{E} \cdot \mathring{\mathbf{1}}) \odot (\mathbf{E} \cdot \mathring{\mathbf{0}})^2 \odot (\mathbf{E} \cdot \mathring{\mathbf{1}})$ and $h_0(\mathbf{v}_1) = (\mathbf{E} \cdot \mathring{\mathbf{1}}) \odot (\mathbf{E} \cdot \mathring{\mathbf{0}})^3 \odot (\mathbf{E} \cdot \mathring{\mathbf{1}})$. Let h_1 be the canonical quotient homomorphism from $\mathfrak{P}$ onto $\mathfrak{P}/\simeq^{w}_{\Psi}$ defined by $h_1(A) = A/\simeq^{w}_{\Psi}$ for every $A \in \mathbf{\Pi}$. Then $h(\mathbf{v}_0) = h_1(h_0(\mathbf{v}_0))$ and $h(\mathbf{v}_1) = h_1(h_0(\mathbf{v}_1))$, so $h = h_0|h_1$. Hence,

$$\begin{aligned} &1 ; [h(p_0) \diamond h(q_0) + \cdots + h(p_{m-1}) \diamond h(q_{m-1})] ; 1 \\ &\quad = (\mathbf{1} \odot [h_0(p_0) \diamond h_0(q_0) + \cdots + h_0(p_{m-1}) \diamond h_0(q_{m-1})] \odot \mathbf{1})/\simeq^{w}_{\Psi} \end{aligned}$$

and

$$\begin{aligned} &1 ; x ; h(r) \diamond 1 ; x ; h(s) \\ &\quad = ([\mathbf{1} \odot (\mathbf{E} \cdot \mathring{\mathbf{1}}) \odot (\mathbf{E} \cdot \mathring{\mathbf{0}})^4 \odot (\mathbf{E} \cdot \mathring{\mathbf{1}}) \odot h_0(r)] \\ &\qquad \diamond [\mathbf{1} \odot (\mathbf{E} \cdot \mathring{\mathbf{1}}) \odot (\mathbf{E} \cdot \mathring{\mathbf{0}})^4 \odot (\mathbf{E} \cdot \mathring{\mathbf{1}}) \odot h_0(s)])/\simeq^{w}_{\Psi}. \end{aligned}$$

Therefore, (**S**) is equivalent to

$$\begin{aligned}\Psi \vdash^{w}\ & \mathbf{1} \odot [h_0(p_0) \diamond h_0(q_0) + \cdots + h_0(p_{m-1}) \diamond h_0(q_{m-1})] \odot \mathbf{1} \\ &= \mathbf{1} \odot [h_0(p_0) \diamond h_0(q_0) + \cdots + h_0(p_{m-1}) \diamond h_0(q_{m-1})] \odot \mathbf{1} \\ &\quad + [\mathbf{1} \odot (\mathbf{E} \cdot \mathring{\mathbf{1}}) \odot (\mathbf{E} \cdot \mathring{\mathbf{0}})^4 \odot (\mathbf{E} \cdot \mathring{\mathbf{1}}) \odot h_0(r)] \\ &\quad \diamond [\mathbf{1} \odot (\mathbf{E} \cdot \mathring{\mathbf{1}}) \odot (\mathbf{E} \cdot \mathring{\mathbf{0}})^4 \odot (\mathbf{E} \cdot \mathring{\mathbf{1}}) \odot h_0(s)].\end{aligned}$$

There is no algorithm for determining whether this last condition holds for arbitrary $r, s \in T_2$, so Ψ is an undecidable theory in $\mathscr{L}w^{\times}$. □

§**8. Historical remarks.** Much of the material in this paper already appears in Chapter 12 of [M78]. Tarski's result, that every variety of relation algebras containing RRA has an undecidable equational theory, was proved in Theorem (1), p. 220, of [M78] by reducing word problems for semigroups on two generators to the word problem for the free relation algebra on two generators. The proof given there actually establishes Theorem 10 of this paper. The method was extended by the device of Theorem 11 to show, in Theorem (2), p. 222, [M78], that any variety of semiassociative relation algebras containing RRA has an undecidable equational theory, but the proof given there actually establishes Theorem 12 of this paper. (The stronger results established by the proofs of Theorems (1) and (2) are indicated on p. 234 of [M78].) The methods of [M78] were adapted to cylindric algebras in [M80] to prove that both the class of 3-dimensional cylindric algebras and the class of 3-dimensional diagonal-free cylindric algebras have undecidable equational theories, thus solving a fairly long-standing open problem. The proofs in [M80] show something more, namely, cylindric-algebraic versions of Theorems 10 and 12. In early 1986 the proofs in [M78] were slightly modified and used, together with Theorem 7.1(v) of [TG87], to prove theorems (4), (4′), and (4″) in the Introduction. I. Németi also proved a theorem from which theorems (4), (4′), and (4″) can be derived; see [N86, Remark 18(v)]. Another theory shows that theorems (4′) and (4″) cannot be generalized by replacing semiassociative relations algebras with nonassociative ones. In late 1987 the use of Theorem 7.1(v) of [TG87] was replaced by the simpler coding device of Lemma 14 and theorem (5) was proved.

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