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On the existence of subalgebras of direct products with prescribed d-fold projections

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Abstract. Baker and Pixley have shown the equivalence of a number of conditions on a positive integer d and a variety V of algebras, one of which is that each subalgebra $S \subseteq A_1 \times \cdots \times A_r$ of any finite direct product algebra in V be uniquely determined by its projections in the d-fold subproducts $A_{i(1)} \times \cdots \times A_{i(d)}$. It is shown here that *under Baker and Pixley's conditions* this uniqueness result is complemented by an existence result: Suppose $A_1, \ldots, A_r \in V$, and that for every d-tuple $I = \{i(1), \ldots, i(d)\}$ a subalgebra $S_I \subseteq A_{i(1)} \times \cdots \times A_{i(d)}$ is given. Then these data are the projections of one subalgebra $S \subseteq A_1 \times \cdots \times A_r$ if and only if they are "consistent" on each $d+1$-tuple $\{i(1), \ldots, i(d+1)\}$.

In the case where each A_i is the lattice $\{0, 1\}$, these results lead to the well-known description of finite distributive lattices in terms of finite partially ordered sets.

Under appropriate hypotheses the above result generalizes to subalgebras of infinite direct products.

1. The main theorem

"Variety" will mean a variety (equational class) of algebras in the sense of universal algebra. The *operations* of a variety will mean the primitive *and derived* operations – called "polynomials" in [1]. But we will follow [1] in using the term *interpolating polynomial* for an *operation* which extends a given partial function. Our varieties may have infinitary operations, but only in §5 will we be explicitly concerned with these.

We begin by quoting (slightly reworded) the result we shall complement:

THEOREM B–P ([1] Theorem 2.1). *For a variety V and an integer $d \geq 2$, the following conditions are equivalent:*

(1) *V has a $(d+1)$-ary operation $m(x_1, \ldots, x_{d+1})$ which satisfies the identities*

$$m(x, \ldots, x, y, x, \ldots, x) = x \quad (\text{"near-unanimity identities"})$$

for all positions of the lone y.

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(2) *If* $S \subseteq A_1 \times \cdots \times A_r$ $(d \leq r < \infty)$ *is a subalgebra of a direct product of algebras* $A_i \in V$, *then* S *is uniquely determined by its images under the projections of* $A_1 \times \cdots \times A_r$ *onto the* $\binom{r}{d}$ *subproducts* $\prod_I A_i$ *with* $I \subseteq \{1, \ldots, r\}$, $|I| = d$.

(3) *In any* $A \in V$, *if a finite system of congruences,* $x \equiv a_i(\theta_i)$ $(i \leq r)$ *where* $d \leq r < \infty$ *are solvable (in* x*)* d *at a time, then they are solvable simultaneously.*

(4, 5) *Suppose* $A \in V$, n *is a cardinal, and* f *is a partial* n*-ary function on* A, $A^n \supseteq D_f \xrightarrow{f} A$, *with domain* D_f *of finite cardinality* r. *Then* f *has an interpolating polynomial* $p : A^n \to A$ *if and only if its restrictions to all subsets of* D_f *of cardinality* $\leq d$ *have interpolating polynomials. (Equivalently, if and only if all subalgebras of* A^d *are closed under the induced* n*-ary partial operation* f^d.)

Note that the conditions of the above Theorem for a given d imply the corresponding conditions for all $d' \geq d$. In particular, if m_{d+1} is a $(d+1)$-ary near unanimity operation as in (1), then a $(d'+1)$-ary near unanimity operation is given simply by $m_{d'+1}(x_1, \ldots, x_{d'+1}) = m_{d+1}(x_1, \ldots, x_{d+1})$, for any $d' \geq d$.

Now suppose V is a variety, $d \leq r$ are positive integers, and we are given algebras $A_1, \ldots, A_r$ in V, and for each family $I \subseteq \{1, \ldots, r\}$ with $|I| = d$, a subalgebra $S_I \subseteq \prod_I A_i$. For every J with $|J| \geq d$, let $S_J \subseteq \prod_J A_j$ denote the intersection, over all $I \subseteq J$ with $|I| = d$, of the inverse image of S_I under the natural map $\prod_J A_j \to \prod_I A_i$. We shall say that the given system of subalgebras $(S_I)_{|I|=d}$ is *consistent* on J if for every d-element subset $I \subseteq J$, the projection of S_J in $\prod_I A_i$ is all of S_I. This means that for every $I \subseteq J$, each d-tuple $a \in S_I \subseteq \prod_I A_i$ can be extended to a $|J|$-tuple in $\prod_J A_j$ each sub-d-tuple of which belongs to the appropriate subalgebra $S_{I'}$ $(I' \subseteq J)$.

Now let e be an integer, $d \leq e \leq r$. If we let J range over all subsets of $\{1, \ldots, r\}$ of cardinality e, the subalgebras S_J constructed as above form a system $(S_J)_{|J|=e}$, like the system $(S_I)_{|I|=d}$ that we started with, but with e in place of d. In particular, for $K \subseteq \{1, \ldots, r\}$ with $|K| \geq e$, this system induces a subalgebra of $\prod_K A_k$. But it is easy to see that this is just the subalgebra $S_K \subseteq \prod_K A_k$ that one obtains directly from the original subalgebras S_I. From this "transitivity" of the construction of the S_J we get the following "transitivity" result for the consistency property: if $d \leq e \leq f \leq r$, and the system $(S_I)_{|I|=d}$ is consistent on every e-tuple J, and the induced system $(S_J)_{|J|=e}$ is consistent on every f-tuple K, then the system $(S_I)_{|I|=d}$ is consistent on every f-tuple K.

We can now prove:

THEOREM 1. *Let* V *be a variety, and* d *a positive integer, satisfying the equivalent conditions of Theorem* B–P. *Let* $A_1, \ldots, A_r \in V$ $(r \geq d)$, *and for every subset* $I \subseteq \{1, \ldots, r\}$ *of cardinality* d, *let* S_I *be a subalgebra of* $\prod_I S_i$. *Then there* *exists a subalgebra* $S \subseteq A_1 \times \cdots \times A_r$ *whose projection in each* $\prod_I S_i$. ($|I| = d$) *is* S_I (*that is, the given system is consistent on* $\{1, \ldots, r\}$) *if and only if the given system* $(S_I)_{|I|=d}$ *is consistent on every* J *with* $|J| = d + 1$.

Proof. "Only if" is clear; we want "if."

For $d \leq e \leq f \leq r$, let us write $C(e, f)$ for the condition

$$(S_J)_{|J|=e} \quad \text{is consistent on all } K \text{ of cardinality } f. \tag{6}$$

We shall prove (using the conditions of Theorem B–P) that $C(e-1, e) \Rightarrow C(e, e+1)$ $(d < e < r)$. Hence from the hypothesis $C(d, d+1)$ we get $C(e, e+1)$ for all e, and linking these together by the above transitivity observation we get $C(d, r)$, which is the desired conclusion.

So assume $C(e-1, e)$, and let $J \subseteq K \subseteq \{1, \ldots, r\}$, with $|J| = e$, $|K| = e + 1$. We must prove that every $a \in S_J$ can be extended to an element of S_K. As a notational convenience we can clearly assume that $J = \{1, \ldots, e\}$ and $K = \{1, \ldots, e+1\}$.

Let $a = (a_1, \ldots, a_e) \in S_J$ $(a_i \in A_i)$. By $C(e-1, e)$, for each $i \in J$ we can extend the $(e-1)$-tuple $(a_1, \ldots, \hat{a}_i, \ldots, a_i) \in S_{J-\{i\}}$ (where denotes deletion) to an e-tuple

$$(a_1, \ldots, \hat{a}_i, \ldots, a_e, b_i) \in S_{K-\{i\}} \qquad (b_i \in A_{e+1};\ i \leq e). \tag{7}$$

Let $a_{e+1} = m_e(b_1, \ldots, b_e)$ (m_e as in Theorem B–P (1), with $e-1$ for d). We claim that $(a_1, \ldots, a_e, a_{e+1}) \in S_K$. To show this it will suffice to show that every e-element subsequence thereof, $(a_1, \ldots, \hat{a}_h, \ldots, a_{e+1})$ belongs to $S_{K-\{h\}}$. If $h = e+1$ this is our original sequence which lies in S_J by hypothesis. In the contrary case we may assume without loss of generality that $h = 1$. Let us then delete the A_1 term from each relation in (7), except the $i = 1$ case, which has no such term. We get:

$$(a_2, \ldots, a_e, b_1) \in S_{K-\{1\}} \quad (i = 1 \text{ case}) \tag{8}$$

$$(a_2, \ldots, \hat{a}_i, \ldots, a_e, b_i) \in S_{K-\{1,i\}} \quad (2 \leq i \leq e). \tag{9}$$

Now we again use $C(e-1, e)$, this time to "fill in" an i-term in each relation of (9). Calling the inserted term $c_i \in A_i$, we get

$$(a_2, \ldots, c_i, \ldots, a_e, b_i) \in S_{K-\{1\}} \quad (2 \leq i \leq e). \tag{9'}$$

We now apply m_e to the e elements of $S_{K-\{1\}}$ given by (8) and (9'). By the identities assumed for m_e, and the definition of a_{e+2}, we get

$$(a_2, \ldots, a_e, a_{e+1}) \in S_{K-\{1\}} = S_{K-\{h\}}$$

as claimed. This completes the proof that $(a_1, \ldots, a_{e+1}) \in S_K$, hence that $C(e-1, e) \Rightarrow C(e, e+1)$, hence of the Theorem. $\|$

2. Counterexamples

Let us give two examples which show how in the variety of abelian groups, which does not satisfy the conditions of Theorem B–P for any d, both the existence and uniqueness results for subalgebras of direct products fail. (The failure of uniqueness was also illustrated in [1].)

For nonuniqueness let $d=2$, $r=3$, $A_1=A_2=A_3=$ any nonzero abelian group A, and $S \subseteq A_1 \times A_2 \times A_3$ be $\{(a, b, c) \mid a+b+c=0\}$. It is easy to see that $S_{\{1,2\}} = S_{\{1,3\}} = S_{\{2,3\}} =$ all of $A \times A$. Thus the pairwise projections of S coincide with those of the full group $A_1 \times A_2 \times A_3$, though S is a proper subgroup thereof.

For a counterexample to the existence statement, consider a tetrahedron with faces labeled 1, 2, 3, 4, and define A_i $(i \leq 4)$ to be the group of all functions from the set of edges of the face i to the integers, having zero sum. (Thus each A_i is a free abelian group of rank 2, but there is not a natural identification of these four groups. To the algebraic topologist, each A_i is the group of integer 1-cocycles on the subcomplex i of our tetrahedron.) Given distinct faces i and j, let $S_{\{i,j\}}$ be the set of pairs $(a, b) \in A_i \times A_j$ which as functions on edges agree on the common edge of i and j. Then for any $I \subseteq \{1, 2, 3, 4\}$, the group S_I can clearly be identified with the group of functions on the set of edges of faces in I, which sum to zero over each face in I (1-cocycles on $\bigcup I$). One sees that for distinct i, j, k, any element of $S_{\{i,j\}}$ extends uniquely to an element of $S_{\{i,j,k\}}$: Its value on the edge of k not adjoining i or j is determined by the condition that the sum over edges of the face k be zero. Thus $C(2, 3)$ holds. But not every element of $S_{\{1,2,3\}}$ (and hence not every element of $S_{\{1,2\}}$) extends to $S_{\{1,2,3,4\}}$. For an element of $S_{\{1,2,3\}}$ is determined by specifying arbitrary values on the three nonbounding edges of $1 \cup 2 \cup 3$; the values on the bounding edges are negative pairwise sums of these numbers. Thus the sum of the values on the edges of face 4 is minus twice the sum of the original arbitrary values, which need not be zero. (If we replace "$\mathbb{Z}$-valued" by "A-valued" for any abelian group A in this example, we get counterexamples working for any nontrivial variety of groups – except groups of exponent 2, where the last five words fail. In this variety we can modify this example by letting A_4 be the group of *all* A-valued functions on the set of edges of face 4. Then not every member of A_4, hence not every member of $S_{\{1,4\}}$ extends to a member of $S_{\{1,2,3,4\}}$. These nonuniqueness and nonexistence examples are easily adapted to arbitrary d.)

3. A reformulation when $d = 2$

In this section we shall obtain a nice restatement of Theorem 1 when $d = 2$, which we will apply in the next section to the variety of lattices.

Let us begin with an observation valid for arbitrary d. We have been considering the determination of a subalgebra $S \subseteq \prod A_i$ from certain data, namely its (intended) images in the algebras $\prod_I A_i$ for all $I \subseteq \{1, \ldots, r\}$ of cardinality d. In some ways it is formally more convenient to look at the images S_I of S in the products $\prod_{m=1}^{d} A_{I(m)}$ for all *sequences* $I = (I(1), \ldots, I(d)) \in \{1, \ldots, r\}^d$. An obvious *disadvantage* is that this form of our data is more redundant than the earlier form, since each set of d elements of $\{1, \ldots, r\}$ is represented by $d!$ sequences, and there are also sequence with repeated terms, which correspond to subsets of cardinality $< d$. Thus there will be more consistency conditions needed in this formalism than in that of the preceding sections, stating how $S_{I'}$ is determined by S_I when the set of terms of I' is contained in that of I. But these conditions will be quite easy to work with.

A minor advantage of the new formulation is that the condition $d \leq r$, which was obviously needed in Theorem B–P (2) and Theorem 1, can here be dropped. (For when $d > r$, the set of subsets of $\{1, \ldots, r\}$ of cardinality d is empty, but this is not true of $\{1, \ldots, r\}^d$.) But the main advantage is the elegant form that the consistency condition $C(d, d+1)$ of the preceding section takes, at least for $d = 2$.

For any sequence $(i, j, \ldots, k)$, let us abbreviate $S_{(i,j,\ldots,k)}$ to $S_{ij\cdots k}$. Then for each 3-tuple of indices, (i, j, k), the condition that every element of S_{ik} extends to an element of S_{ijk} says that for every $(a, b) \in S_{ik} \subseteq A_i \times A_k$, there should exist $b \in A_j$ such that $(a, b, c) \in S_{ijk}$, i.e., such that $(a, b) \in S_{ij}$ and $(b, c) \in S_{jk}$. Now if we regard the sets S_{ij} as *relations* on the pairs of algebras A_i, A_j, this says that the relation S_{ik} is contained in the *composition* $S_{ij} \circ S_{jk}$.

For any algebra A, $Id_A \subseteq A \times A$ will denote the identity relation, that is, the diagonal subalgebra. The form that the $d = 2$ case of Theorem 1 now takes is:

COROLLARY 2. *Let V be a variety having a ternary operation m satisfying the identities*

$$m(x, x, y) = m(x, y, x) = m(y, x, x) = x.$$

Let $(A_i)_{i \in K}$ be a finite nonempty family of algebras in V, and for all $i, j \in K$, let S_{ij} be a subalgebra of $A_i \times A_j$. Then necessary and sufficient conditions for there to exist a subalgebra $S \subseteq \prod_K A_i$ such that each S_{ij} is the image of S under the map

$p_i \times p_j : \prod_K A_k \to A_i \times A_j$ are:

$$(\forall i, j, k \in K) \qquad S_{ik} \subseteq S_{ij} \circ S_{jk}, \tag{10}$$

$$(\forall i, j \in K) \qquad S_{ji} = S_{ij}^{-1}, \tag{11}$$

$$(\forall i \in K) \qquad S_{ii} \subseteq Id_{A_i}. \tag{12}$$

When these conditions are satisfied, the subalgebra S is uniquely determined by the S_{ij}. S will be a subdirect product of the A_i if and only if the above conditions hold with (12) *strengthened to*

$$(\forall i) \qquad S_{ii} = Id_{A_i}. \tag{12'}$$

Proof. For the S_{ij} to come from an S, the necessity of (10)–(12) is clear, as is their sufficiency in the trivial case $|K| = 1$. So suppose $|K| \geq 2$ and that we are given S_{ij}'s satisfying (10)–(12). It is clear from (11) that the data given by the subfamily $(S_{ij})_{i \neq j}$ is equivalent to giving a subalgebra of $\prod_I A_i$ for each two-element *set* $I = \{i, j\}$; and from the preceding discussion it is clear that (10) is equivalent to this system's satisfying the consistency condition $C(2, 3)$. So the first assertion will follow from Theorem 1 if we verify that the remaining algebras S_{ii} are "what they should be" for this data, namely that each for all i and j, S_{ii} is the diagonal image in $A_i \times A_i$ of the i^{th} projection of S_{ij}. That it is the diagonal image of some subalgebra of A_{ii} follows from (12). That this subalgebra contains the projection of S_{ij} follows from the inclusion $S_{ij} \subseteq S_{ii} \circ S_{ij}$; that it is contained in the projection follows from $S_{ii} \subseteq S_{ij} \circ S_{ji}$.

The next sentence follows from Theorem B–P, and the last sentence is clear. ||

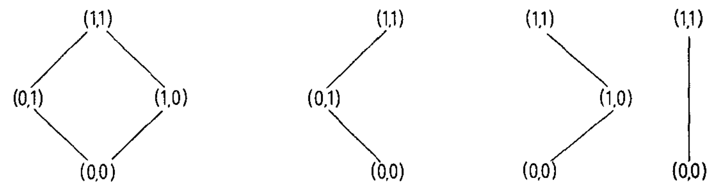
In the next section we will be considering a case where all the A_i are the same algebra. Let us call an expression of an algebra S as a subdirect product of copies of an algebra A, $S \subseteq A^K$, *irredundant* if no two projections $p_i \mid S$ $(i \in K)$ are equal. (This is weaker than meanings frequently given this term.) Then we see that in the situation of Corollary 2, if all $A_i = A$, S will be an irredundant subdirect product if and only if (12) can be strengthened to:

$$(\forall i, j) \qquad S_{ij} = Id_A \Leftrightarrow i = j. \tag{12''}$$

4. Subdirect products of 2

As was noted in [1], the variety of all lattices satisfies the conditions of Theorem B–P, with $d = 2$. Indeed, for $m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$, condition (1) is clear.

Let **2** denote the 2-element lattice $\{0, 1\}$. By the preceding results, any finite subdirect product $S \subseteq \mathbf{2}^K$ can be described by giving its pairwise projections in $\mathbf{2}\times\mathbf{2}$. The four pairwise subdirect products of **2** are shown below:



(13) (14) (15) (16)

Clearly, to describe each $S_{ij} \subseteq \mathbf{2}\times\mathbf{2}$, it will suffice to say whether or not it is contained in the sublattice (14) and whether or not it is contained in the sublattice (15). But by (11), S_{ij} lies in (15) if and only if S_{ji} lies in (14), so all we need to specify is which S_{ij}'s lie in (14). Let us denote the condition $S_{ij} \subseteq (14)$ by "$i \leq j$," since (14) looks like a "$\leqslant$" sign. (And less frivolously, the subalgebra (14) is the graph of the relation "$\leq$" on the lattice **2**.) One checks that in the above notation, condition (10) takes the form:

$$i \leq j,\, j \leq k \Rightarrow i \leq k. \tag{17}$$

Condition (11) was essentially absorbed in our convention to indicate inclusion in (14) but not inclusion in (15). (If we had written the latter as "$i \geq j$," then (11) would say $i \leq j \Leftrightarrow j \geq i$.) Condition (12′) says

$$(\forall i) \qquad i \leq i \tag{18}$$

while the stronger (12″) says

$$(\forall i, j) \qquad (i \leq j,\, j \leq i) \Leftrightarrow i = j. \tag{18′}$$

But (17) and (18) are the definition of a preorder "$\leqslant$" on K, while (17) and (18′) are the definition of a partial ordering!

The way in which $S \subseteq \mathbf{2}^K$ is (re)constructed as an intersection of cylinders on the S_{ij}'s takes the form

$$S = \{a \in \mathbf{2}^K \mid \forall i, j \in K,\ (i \leq j) \Rightarrow (a(i) \leq a(j) \text{ in } \mathbf{2})\}.$$

Hence:

COROLLARY 3. *A finite subdirect product of copies of the lattice* $\mathbf{2}$, $S \subseteq \mathbf{2}^K$, *is uniquely determined by a* preordering *"≤" on the index-set K. Here "≤" is described in terms of S by* $(i \le j) \Leftrightarrow (\forall a \in S) a(i) \le a(j)$, *and S can be reconstructed from "≤" as the lattice of all isotone (≤-respecting) maps* $K \to \mathbf{2}$. *"≤" is a* partial ordering *if and only if S is an irredundant subdirect product.* ||

It takes just a few more arguments to get from here to the duality between finite distributive lattices and finite partially ordered sets. But since the details depend on the previous development of distributive lattices, I leave these to the reader. Incidentally, the above result and Corollary 2 suggest that in working with an arbitrary finite family of lattices $(L(i))_{i \in K}$, it might be heuristically useful to think of a subdirect product of this family as an "L-valued preordering" on the index-set K.

While on the subject of subdirect powers of finite algebras, we note the following consequence of Theorem B–P:

COROLLARY 4. *Let A be a finite set, say* $|A| = n$, *and d a positive integer. Then the number of clones C of* finitary *operations on A, which contain at least one* $(d+1)$*-ary near-unanimity operation (as in (1)) is* $< 2^{2^{n^d}}$.

Proof. Let C be such a clone of operations on A. Suppose we know which subsets of A^d are subalgebras, under the induced operation of C on A^d. By Theorem B–P every subalgebra of A^r $(r \geq d)$ is an intersection of cylinders on such subalgebras, hence we know precisely which subsets of A^r are subalgebras. But this determines C, for the m-ary operations of C are the elements of the subalgebra of A^{A^m} generated by the m coordinate functions. Thus the number of such clones C is at most the number of sets of subsets of A^d, which is $2^{2^{n^d}}$. In fact, not all such subsets correspond to clones ($\varnothing$ does not) so one has strict inequality. ||

5. Subalgebras of infinite direct products

In this section we shall get versions of Theorems B–P and 1 for infinite direct products, assuming that the variety V possesses appropriate infinitary operations. These will apply, in particular, to varieties of α-complete lattices, for α a cardinal.

We shall understand a cardinal α to be the least ordinal of cardinality α.

DEFINITION 5. *Let X be a set and* α *a limit ordinal. An operation* $f : X^\alpha \to X$

will be called a <u>*weak α-limit*</u> *operation if for every $x \in X$, and α-tuple $y \in X^{\alpha}$ such that* $\sup\{\beta \in \alpha \mid y_\beta \neq x\} < \alpha$, *one has $f(y) = x$.*

Note that the condition that f be a weak α-limit operation is actually a family of identities. We shall speak of an operation in a *variety* as being a weak α-limit operation if those identities are identities of the variety.

EXAMPLE 6. If α is an infinite cardinal, then in the variety of α-complete lattices, the operation $f(y) = \bigvee_{\beta<\alpha} \bigwedge_{\beta<\gamma<\alpha} y_\gamma$ is a weak α-limit operation.

LEMMA 7. *Let A be an algebra and α a cardinal, such that for each infinite cardinal $\beta \leq \alpha$, A has a weak β-limit operation, f_β. Then any family of congruence classes of A which has the finite intersection property has the α-fold intersection property.*

Proof. For α finite, this is trivial, so assume α infinite. By induction we may suppose the result true for all $\beta < \alpha$. Let us be given a family of congruence relations $(\theta_\gamma)_{\gamma\in\alpha}$ on A, and congruence classes $x_\gamma\theta_\gamma \subseteq A$ having the finite intersection property.

By our inductive hypothesis we can find for each $\beta \in \alpha$ an element $y_\beta \in \bigcap_{\gamma<\beta} x_\gamma\theta_\gamma$. These form an α-tuple $y = (y_\beta)$; let $z = f_\alpha(y)$. Applying the weak α-limit property of f_α on each A/θ_γ, we see that $z\theta_\gamma = x_\gamma\theta_\gamma$, so $z \in \bigcap_{\gamma\in\alpha} x_\gamma\theta_\gamma$. ‖

Let us make the convention that if a is an element of a direct product of algebras $\prod_J A_j$, then for every subset $I \subseteq J$, the image of a in $\prod_I A_i$ will be denoted $a_{(I)}$. (When our indices are ordinals, it will be important to observe the distinction between $a_{(\beta)}$ and the β^{th} coordinate of a, a_β.)

LEMMA 8. *Let V be a variety of algebras and α an infinite cardinal. Let $(A_k)_{k\in K}$ be a family of algebras in V. Suppose that for every subset $J \subseteq K$ we have a subalgebra $S_J \subseteq \prod_J A_j$, and that*

(19) *$a \in S_J$ if and only if for all finite $I \subseteq J$, $a_{(I)} \in S_I$.*

(20) *Every element of S_I can be extended to an element of S_J whenever $I \subseteq J$ are* <u>*finite*</u> *subsets of K. (In the language of Theorem 1, our system satisfies $C(e,f)$ for all $e \leq f < \infty$.)*

(21) *For every cardinal $\beta < \alpha$, V has a weak β-limit operation f_β.*

Then for all subsets $I \subseteq J \subseteq K$ with $|I| < \alpha$, $|J| \leq \alpha$, every element of S_I can be extended to an element of S_J.

Proof. Assume inductively that the corresponding result is true for all cardinals $< \alpha$.

Let us be given $I \subseteq J \subseteq K$ with $|I| < \alpha$, $|J| \leq \alpha$. Let $|I| = \beta$, $|J - I| = \beta'$. We may assume for notational convenience that I and J are in fact the ordinals β and $\beta + \beta'$. ($\beta + \beta'$ means the ordinal with initial segment β and remainder of order-type β'. This is not necessarily a cardinal.) From the facts that $\beta < \alpha$, $\beta' \leq \alpha$, and α, β' are cardinals, α infinite, it follows that $\beta + \beta' \leq \alpha$ as ordinals. (This is stronger than the cardinality statement $|\beta + \beta'| \leq \alpha$.)

Now let $a \in S_\beta$ be an element which we wish to extend to an element of $S_{\beta+\beta'}$. The only case in which we need to compute is

CASE 1. $\beta' = 1$. If β is finite the desired result is our hypothesis (20), so assume β infinite. Since $\beta < \alpha$ our inductive hypothesis holds for the cardinal β, but it is not directly applicable to the present extension problem, because $|I|$ is not $< \beta$. However, for each $\gamma \in \beta$ we have $|\gamma| < \beta$, and we know $|J| = \beta$, so we *can* extend the γ-tuple $a_{(\gamma)}$ to a J-tuple $a^\gamma \in S_J$. Now applying the weak β-limit operation f_β to this β-tuple of elements of S_J, we get an element of S_J which is easily seen to extend a, as required.

Note that this shows that for *any* $I' \subseteq K$ of cardinality $< \alpha$, and $j \notin I'$, members of $S_{I'}$ can be extended to members of $S_{I' \cup \{j\}}$, even though, as will be the case in the next paragraph, I' may happen to be identified with some ordinal *other* than the cardinal $|I'|$. (In other words, in *applying* this case we can now discard the notational assumption that it was convenient to make for the proof.)

GENERAL CASE. Let P denote the set of pairs (I', a') such that I' is an ordinal with $I \leq I' \leq J$, and a' is an element of $S_{I'}$ extending $a \in S_I$. Partially order P by the relation "$I'' \geq I'$ and a'' extends a'." Now it is easy to see from (19) that P satisfies the hypothesis of Zorn's Lemma, hence must have a maximal element (I', a'). But if I' were strictly less than J, then we would have $|I'| < \alpha$ and by Case 1 we could extend a' to the successor ordinal of I', contradicting maximality. So $I' = J$, and a' is the desired extension of a. ‖

We now easily deduce:

THEOREM 9. *Let $d \geq 2$ be an integer, and V a variety, satisfying the equivalent conditions of Theorem* B–P; *and let α be a cardinal. Then*

(22) *If V has a weak α-limit operation for every infinite cardinal $\beta \leq \alpha$, then conditions* (2)–(5) *of Theorem* B–P *hold with the integer r replaced by the cardinal α.*

(23) *If (less stringently!) V has a weak β-limit operation for every infinite cardinal $\beta < \alpha$, the result of Theorem* 1 *holds with r replaced by α. In particular, Theorem* 1 *is true with $r = \aleph_0$ without assuming the existence of any infinitary operations.* ‖

Let us now apply Corollary 2 and this result to lattices. A β-complete sublattice of a β-complete lattice L will mean a sublattice closed under β-fold meets and joins. By $<\alpha$-complete we shall mean "β-complete for all cardinals $\beta<\alpha$." (Thus, all lattices are $<\aleph_0$-complete.)

THEOREM 10. *Let α be a cardinal. Then an α-complete sublattice of a direct product of α-complete lattices over an index set of cardinality $\leq\alpha$ is uniquely determined by its projections on the pairwise subproducts. Conversely, let us be given a family of (merely) $<\alpha$-complete lattices $(L_i)_{i\in K}$ on an index set K with $|K|\leq\alpha$, and for each $i, j\in K$, a $<\alpha$-complete sublattice $S_{ij}\subseteq L_i\times L_j$. Then* (10), (11), (12) *of Corollary* 2 *are the necessary and sufficient conditions for the S_{ij} to be the pairwise projections of a $<\alpha$-complete sublattice $S\subseteq\prod L_i$.* $\|$

6. Appendix: On some implications

Is the converse to Theorem 1 true? That is, given a variety V and an integer $d\geq 2$ for which the indicated result on the existence of subalgebras of direct products with specified d-fold projections is true, must V and d satisfy the equivalent conditions of Theorem B–P? We shall see below that if we allow ourselves to slightly strengthen the conclusion of Theorem 1, the answer is yes. But for the Theorem as stated, we shall find evidence suggesting a negative answer.

To state these results concisely, let us set up some notation. If V is a variety and d, e, e', f, f' are positive integers, we shall say that V satisfies

$$(d: C(e, e')\Rightarrow C(f, f'))$$

if for every family (A_i) of algebras of V, every system of subalgebras $S_I\subseteq\prod_I A_i$ ($|I|\subseteq d$) of the d-fold direct products, which satisfies the consistency condition $C(e, e')$ (see proof of Theorem 1) also satisfies $C(f, f')$. Thus, Theorem 1 says that if V satisfies the conditions of Theorem B–P, it satisfies

$$(d: C(d, d+1)\Rightarrow C(d, r)) \tag{24}$$

for all $r\geq d$. But we proved this by showing in fact that for all $e>d$, V satisfied

$$(d: C(e-1, e)\Rightarrow C(e, e+1)). \tag{25}$$

Now for Theorem 1 with (25) as conclusion, or even just the $e=d+1$ case thereof, the converse is true:

LEMMA 11. *For any variety V and integer $d \geq 2$, the condition $(d: C(d, d+1) \Rightarrow C(d+1, d+2))$ is equivalent to the conditions of Theorem* B–P.

Proof. Given a variety V and an integer $d \geq 2$ (about which we assume nothing to start with), let $K = \{1, \ldots, d+2\}$, $A_1 = \cdots = A_{d+1} = \langle x, y \rangle$, and $A_{d+2} = \langle z_1, \ldots, z_{d+1} \rangle$ (free algebras in V on 2 and $d+1$ indeterminates respectively). Define homomorphisms $f_i : A_{d+2} \rightarrow A_i$ $(i \leq d+1)$ by

$$f_i(z_j) = \begin{cases} x & \text{if} \quad i \neq j \\ y & \text{if} \quad i = j. \end{cases}$$

Let $T \subseteq \prod_K A_i$ be the "graph" $\{(f_1(a), \ldots, f_{d+1}(a), a) \mid a \in A_{d+2}\}$, and for all $I \subseteq K$ with $|I| = d$, let S_I be the image of T under the projection $\prod_K A_i \rightarrow \prod_I A_i$. Since the system (S_I) is induced by a subalgebra $T \subseteq \prod_K A_i$, it clearly satisfies the consistency conditions $C(d, d+1)$ and $C(d, d+2)$. But it need not a priori satisfy $C(d+1, d+2)$, since for sets J of cardinality $d+1$, S_J is not defined as the projection of T in $\prod_J A_i$, but rather (see section 1) as the subalgebra of $\prod_J A_i$ induced by the $(S_I)_{|I|=d}$.

Actually, from the definition of T we easily see that for any J containing the index $d+2$, S_J will indeed be the projection of T in $\prod_J A_i$. In particular $S_K = T$. But consider the case of $J = K - \{d+2\}$. Note that for every $i < d+2$, $S_{K-\{i,d+2\}}$ contains $(f_1(z_i), \ldots, \widehat{f_i(z_i)}, \ldots, f_{d+1}(z_i)) = (x, \ldots, x)$. Hence by construction $S_{K-\{d+2\}}$ contains $(x, \ldots, x)$.

Now suppose V satisfies $(d: C(d, d+1) \Rightarrow C(d+1, d+2))$. Then as $(S_I)_{|I|=d}$ satisfies $C(d, d+1)$, the $d+1$-tuple $(x, \ldots, x) \in S_{K-\{d+2\}}$ extends to a $d+2$-tuple $(x, \ldots, x, m) \in S_K = T$. It follows that the element $m \in S_{\{d+2\}} = \langle z_1, \ldots, z_{d+1} \rangle$ will be precisely the near-unanimity operation required for (1) of Theorem B–P.

The converse direction was proved in the course of proving Theorem 1.

(We could have gotten directly from $(d: C(d, d+1) \Rightarrow C(d+1, d+2))$ to any of the other conditions of Theorem B–P by very similar arguments. E.g., given A, $\theta_1, \ldots, \theta_{d+1}$ as in (3), set $A_{d+2} = A$, $A_i = A/\theta_i$ $(i \leq d+1)$, $T = \{(a\theta_1, \ldots, a\theta_{d+1}, a) \mid a \in A\}$ and proceed as above. Or given $A_1, \ldots, A_{d+1}$ as in (2), and S, $S' \subseteq \prod A_i$ with the same d-fold projections, take $A_{d+2} = S'$, form the obvious T, and proceed in the same way.) $\|$

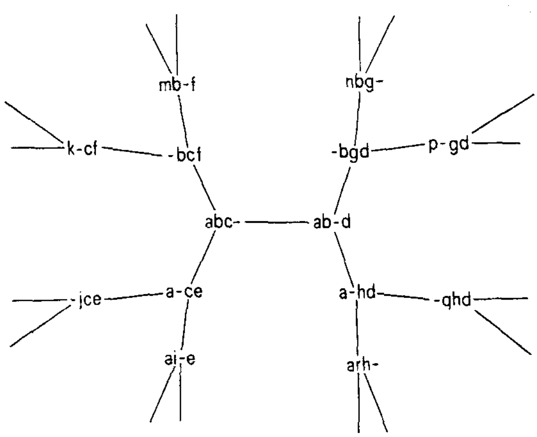
To approach the question of whether a V which satisfies the weaker conditions (24) for all $r \geq d$ must have the properties of Theorem B–P, let us write down the simplest of these conditions for the smallest possible d:

$$(2: C(2, 3) \Rightarrow C(2, 4)) \tag{26}$$

and see how a variety V can satisfy (26). Let A_1, A_2, A_3, A_4 be algebras in a variety V. It will be most convenient to write members of subproducts of $A_1 \times A_2 \times A_3 \times A_4$, such as $A_1 \times A_3 \times A_4$, not as 3-tuples (a, b, c) etc., but as 4-tuples with dashes in the missing positions, e.g. $(a, -, b, c)$, or for brevity when there is no risk of confusion, $a{-}bc$. Now given subalgebras $S_{ij} \subseteq A_i \times A_j$ $(1 \leq i < j \leq 4)$ satisfying $C(2, 3)$, and given an element $ab{-}{-} \in S_{12}$, we ask whether we can extend it to a 4-tuple

$$abuv \in S_{1234} \tag{27}$$

with the help of some properties of V. The construction of such a 4-tuple should go essentially like this: By our hypothesis $C(2, 3)$ we can extend $ab{-}{-}$ to 3-tuples $abc{-} \in S_{123}$ and $ab{-}d \in S_{124}$. The condition $abc{-} \in S_{123}$ (given that we already know $ab{-}{-} \in S_{12}$) means $a{-}c{-} \in S_{13}$ and ${-}bc{-} \in S_{23}$. We can now extend these pairs in new directions to get 3-tuples $a{-}ce \in S_{134}$, ${-}bcf \in S_{234}$. Going on like this, we get an infinite tree of 3-tuples, as shown in (28):



(28)

We should now obtain the elements $u \in A_3$, $v \in A_4$ needed for (27) by applying some operations of the variety V to the third and fourth coordinates of appropriately chosen elements of this tree. To establish (27) we must then prove that the u, v so obtained satisfy:

$$a{-}u{-} \in S_{13}, \qquad a{-}{-}v \in S_{14}, \qquad {-}bu{-} \in S_{23}, \qquad {-}b{-}v \in S_{24}, \qquad {-}{-}uv \in S_{34}.$$

We should get these relations by applying operations of V to families of pairs *known* to lie in these subalgebras; e.g. (let us return here to standard ordered-pair notation) (a, c), (a, h), (k, c), (n, g), $(p, g), \ldots \in S_{13}$, and referring to appropriate *identities* of V.

I claim in fact, that if a variety V satisfies (26), this *must* be a consequence of the existence of just such operations and identities. For let us construct *free* V-algebras A_1, A_2, A_3, A_4 on countable generating sets read out of the tree (28), e.g. $A_1 = \langle a, k, m, n, p, \ldots \rangle$, and let $S_{ij} \subseteq A_i \times A_j$ $(1 \leq i < j \leq 4)$ be the subalgebras generated by pairs of elements likewise read out of (28), e.g. S_{13} generated by $\{(a, c), (a, h), (k, c), (n, g), (p, g), \ldots\}$. As a consequence of the way (28) was constructed, this system (S_{ij}) will satisfy $C(2, 3)$. And for these algebras, (27) means precisely the existence of operations and identities in V with the required properties.

This approach gives Mal'cev-type necessary and sufficient conditions for (26) to hold – but stated in terms of operations of countable arity. If V is a variety with finitary operations, each such operation really depends on only finitely many of its arguments. If we take some finite set of the variables shown in (28) and assume that the operations to be used depend only on these, we get various finitary Mal'cev-type conditions, each *sufficient* for (26). The following result was obtained in this manner, using the 14 3-tuples actually shown in the above picture of (28). (In the statement of this Lemma, the variables are named so as to conform to the labeling of (28), though this has meant using many more variable-symbols in writing the identities (29) than are really necessary. The common values of the last two lines represent the "u" and the "v" of the above discussion.)

LEMMA 12. *A sufficient condition for a variety V to satisfy* (26) *is that it have* 4-*ary operations P, Q, R, S, T satisfying the identities*

$$\begin{aligned} P(a, a, k, n) &= \quad a \quad = Q(a, a, p, k) \\ R(b, b, j, q) &= \quad b \quad = S(b, b, q, i) \\ P(c, h, c, g) &= T(c, c, g, h) = R(c, g, c, h) \\ Q(d, e, d, f) &= T(e, f, d, d) = S(d, f, d, e). \quad \| \end{aligned} \tag{29}$$

Note that *if* a variety V has a ternary operation m satisfying $x = m(x, x, y) = m(x, y, x) = m(y, x, x)$ as in (1), then we can satisfy (29) by taking

$$\begin{aligned} P(u, v, w, x) &= Q(u, v, x, w) = R(u, v, w, x) \\ &= S(u, w, x, v) = T(v, w, u, x) = m(u, v, w) \end{aligned}$$

(The 4-tuple (27) that we get using these choices is $(a, b, c, m(d, e, f))$, as in the proof of Theorem 1.) But I claim that if we simply take for V the variety presented by five operations P, Q, R, S, T, and the identities (29), V will *not* contain any ternary operation m satisfying (1). To show this it suffices to exhibit a particular algebra in V in which no operation m satisfies (1). Let L be any nontrivial lattices (e.g. $L = \mathbf{2}$) and on L define the 4-ary operations

$$\begin{aligned} &P = Q = R = S = (x_1 \wedge x_2) \vee (x_1 \wedge x_3 \wedge x_4) \\ &T = (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_4) \vee (x_1 \wedge x_3 \wedge x_4) \vee (x_2 \wedge x_3 \wedge x_4). \end{aligned} \tag{30}$$

Then it is easy (really!) to verify (29). (The 4-tuple (27), incidentally, comes out $(a, b, (c \wedge g) \vee (c \wedge h), (d \wedge e) \vee (d \wedge f))$.) Now it is also easy to verify that the two 4-ary operations of L shown in (30) both have the property that every ternary operation obtainable by substituting some sequence of x's, y's and z's for (x_1, x_2, x_3, x_4) is, as a function $L^3 \to L$, less than or equal to one of the three coordinate functions (under the partial ordering on the lattice L). Indeed, without any substitutions we have $P(x_1, x_2, x_3, x_4) \leq x_1$, while for T we note that it is completely symmetric in its four variables, and $T(x, x, y, z) \leq x$. It is easily deduced that every ternary operation in the clone generated by these operations is $\leq$ one of the coordinate functions. But no ternary operation m on a nontrivial partially ordered set can both satisfy the near unanimity operation (1) and be identically $\leq$ one of the coordinate functions. By Lemma 12, V satisfies $(2 : C(2, 3) \Rightarrow C(2, 4))$, but by the above observations it does not satisfy the equivalent conditions of Theorem B–P for $d = 2$.

I do not know whether this V satisfies $(2 : C(2, 3) \Rightarrow C(2, r))$ for *all* $r \geq 2$, which is what we would need for it to be a true counterexample to the converse of Theorem 1. If not, then another good candidate for a counterexample would be the variety generated by the algebras with underlying set $\{0, 1\}$, and having as operations *all* finitary $f(x_1, \ldots, x_n)$ such that every ternary operation obtained by substituting a sequence of x's, y's and z's for $(x_1, \ldots, x_n)$ is $\leq$ one of the coordinate functions. (By Lemma 5 with $d = 3$ and $m = T$, this clone of operations on $\{0, 1\}$ is finitely generated!)

But we might instead take after Aesop's fox, and decide that condition (24) is not as nice anyway as (25), in terms of which we have a converse to Theorem 1 in Lemma 11!

The following supplement to Theorem B–P is worth noting:

LEMMA 13. *The equivalent conditions of Theorem* B–P *are also equivalent to the* (*formally weaker*) *conditions obtained from* (2)–(5) *by restricting* r *to the single value* $d + 1$.

Proof. The case $r = d + 1$ is all we need in proving condition (1). ||

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