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EMBEDDING FREE ALGEBRAS IN SKEW FIELDS

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ABSTRACT. This paper constructs a minimal element in the partial order on the set of skew fields generated by a free algebra, and shows that the partial order contains a certain sub partial order. Examples of embedding free algebras in skew fields of heights one and two are also given.

Let R be an integral domain (noncommutative) with identity. We will assume that the integral domain R is always embedded in some skew field (this need not be the case in general [**2**], [**3**], [**6**], [**7**]). There are many skew fields which contain R but in the commutative case there is only one field which is generated by R, namely the field of fractions. In the noncommutative case there may be several distinct skew fields generated by R [**4**, pp. 277]. For skew fields D_1 and D_2 generated by R, we say $D_1 \geqq D_2$ if there exists a place from D_1 to D_2 which extends the natural isomorphism between the embeddings of R. This paper shows that this is a partial order on the set $\mathcal{P}$ (we identify isomorphic embeddings) of skew fields generated by R and in the case where R is the free algebra on two generators we show that $\mathcal{P}$ contains the subposet $\mathcal{L}$ where $C(x)$ is the unique maximal element [**1**, Theorem 27] of $\mathcal{P}$ and K_i are distinct elements of $\mathcal{P}$ with K_2 minimal in $\mathcal{P}$. We also examine the height of an integral domain and give examples of embeddings of different heights of a free algebra in two skew fields.

I. The partial order and the chain of domains $Q_i(K, \phi(R))$.

DEFINITION. Let D be a skew field and ϕ an isomorphism of R into D. R is fully embedded in D if the smallest sub skew field of D containing $\phi(R)$ is D itself. We will denote a full embedding of $(D, \phi(R))$ by (D, ϕ).

DEFINITION. If D_1 and D_2 are two division rings we say ϕ is a place from D_1 to D_2 if ϕ is a homomorphism from a local subring S of D_1 onto D_2 [local means the set of nonunits is an ideal].

DEFINITION. For full embeddings (D, α) and (K, γ) we say $(D, \alpha) \geqq (K, \gamma)$ if there is a place ϕ from D to K such that $\phi^{-1}(K) \supset \alpha(R)$ and $\phi | \alpha(R) = \gamma\alpha^{-1}$.

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$(D, \alpha) \cong (K, \gamma)$ if there is an isomorphism ϕ between D and K such that $\phi | \alpha(R) = \gamma\alpha^{-1}$. If we identify isomorphic embeddings the class of embeddings is a set.

We also make the following notation. Suppose ϕ is an isomorphism of R into a skew field K.

$Q_0(K, \phi(R)) = \phi(R)$.

$Q_1(K, \phi(R)) =$ subring of K generated by $\{a, b^{-1}$ such that $a, b \in Q_0(K, \phi(R))$ with $b \neq 0\}$.

Inductively we define

$Q_i(K, \phi(R)) =$ subring of K generated by $\{a, b^{-1}$ such that $a, b \in Q_{i-1}(K, \phi(R))$ with $b \neq 0\}$.

When no confusion arises, we will write Q_i, $Q_i(K, \phi)$ or $Q_i(\phi(R))$ for $Q_i(K, \phi(R))$.

It is easy to see that $\bigcup_i Q_i(K, \phi(R))$ is the skew subfield of K generated by R and $(\bigcup Q_i, \phi)$ is a full embedding.

LEMMA 1. *The relation $\geqq$ is a partial order on the set of full embeddings $\mathcal{P}$ of R.*

PROOF. We will show that if $(D_1, \alpha) \geqq (D_2, \gamma)$ and $(D_2, \gamma) \geqq (D_1, \alpha)$ that $(D_1, \alpha) \cong (D_2, \gamma)$. The rest is clear. Let ϕ_1 be a place from S_1 onto D_2, $D_1 \supset S_1 \supset \alpha(R)$ and ϕ_2 a place from S_2 onto D_1, $D_2 \supset S_2 \supset \gamma(R)$. $\phi_1 | \alpha(R) = \gamma\alpha^{-1} = (\alpha\gamma^{-1})^{-1} = \phi_2^{-1} | \alpha(R)$. Hence $r \neq 0 \in Q_0(D_1, \alpha)$ implies $\phi_1(r) \neq 0 \Rightarrow r^{-1} \in S_1$ so that $S_1 \supset Q_1(D_1, \alpha)$. $\phi_1 | Q_1(D_1, \alpha)$ is an isomorphism since $\phi_1(\sum m_i) = 0$, where m_i is a product of elements of $\alpha(R)$ and inverses of elements of $\alpha(R)$, implies $\sum m_i = \phi_2(\sum \phi_2^{-1}(m_i)) = \phi_2(\sum \phi_1(m_i)) = \phi_2[\phi_1(\sum m_i)] = 0$. $\phi_1 = \phi_2^{-1}$ on $Q_1(D_1, \alpha)$. Similarly ϕ_1 is an isomorphism on all Q_i and hence

$$D_1 = \bigcup Q_i(D_1, \alpha) \overset{\phi_1}{\cong} \bigcup Q_i(D_2, \gamma) = D_2.$$

LEMMA 2. *If (K, α) is a full embedding and $Q_i(K, \alpha)$ is simple for each $i \geqq 1$ then (K, α) is minimal in $\mathcal{P}$.*

PROOF. Suppose $(K, \alpha) \geqq (D, \gamma)$. Hence ϕ is a homomorphism from a local subring S of K onto D. $\phi | \alpha(R) = Q_0$ is an isomorphism. Suppose $\phi | Q_j$ is an isomorphism. Hence $Q_{j+1} \subset S$ and ϕ maps $Q_{j+1}(K, \alpha)$ onto $Q_{j+1}(D, \gamma)$. Since $Q_{j+1}(K, \alpha)$ is simple ϕ is an isomorphism on Q_{j+1}. Hence ϕ is an isomorphism from $\bigcup Q_i = K$ onto D.

Our last general observation is the following. Let I be an ideal of Q_{i+1}, and let i be the inclusion of Q_i into Q_{i+1}. This induces a monomorphism $\bar{\imath}$ from Q_i into Q_{i+1}/I. This is clear since every element of Q_i has an inverse in Q_{i+1} and hence in Q_{i+1}/I. Q_i and its inverses still generate Q_{i+1}/I.

II. **The K-height of a free algebra in two different skew fields.** Suppose ϕ is an isomorphism of R into a skew field K. The K-height of $\phi(R)$ is $h_K(\phi(R)) = \min\{i : Q_i(K, \phi(R)) \text{ is a skew field}\}$ or ∞ if Q_i is not a skew field for any i.

Denote the free algebra on a set X over the field F by $F[X]$. B. H. Neumann [**8**] gave an example of an embedding of $F[X]$ in a skew field D and conjectured that $h_D(F[X]) = \infty$. Jategaonkar [**5**] gives a different embedding. We show that this has height 2 and construct another embedding of height one.

Let Q be the rationals and let $F = Q(t_1, t_2, \cdots)$ be the commutative field with indeterminates $t_1, t_2, \cdots$. Denote the skew polynomial ring $F[x;\sigma]$ by S where σ is the monomorphism of F into F induced by mapping t_i into t_{i+1}, and where $\sigma(t_i)x = xt_i$. S is embedded in the division ring $K = \{x^{-n}\sum_{i=0}^{\infty} f_i x^i \text{ where } f_i \in F\}$. Since $xS + t_1xS$ is direct, the ring $R = Q[x, t_1x]$ is a free algebra over Q and embedded in K (Jategaonkar [**5**]).

THEOREM 1. $h_K(R) = 2$.

We will use the following series of lemmas to prove this.

LEMMA 3. *$Q_2(R)$ is a skew field.*

PROOF. The subring $Q_1(R)$ contains x and x^{-1} and hence contains $t_i = x^{i-1}(t_1x)x^{-i}$. Thus $Q_1(R) \supset Q[t_1, \cdots][x;\sigma]$ which is left Ore with a ring of quotients K_1. Thus taking quotients again we get $Q_2(R) \supset K_1 \supset Q_1(R)$. Since K_1 is a field, by taking quotients again we have $Q_3(R) \supset K_1 \supset Q_2(R)$ and the lemma is proven.

LEMMA 4. *If $r \in R$ then $r = \sum f_i x^i$ where $f_i = \sum_I \alpha_I t_1^{\epsilon_1} t_2^{\epsilon_2} \cdots t_i^{\epsilon_i}$, $\alpha_I \in Q$, where I ranges over all i-tuples $(\epsilon_1, \cdots, \epsilon_i)$ with $\epsilon_j = 1$ or 0.*

PROOF. Since R is a free algebra it is sufficient to examine a monomial m. We claim that if m is of degree i then $m = t_1^{\epsilon_1} t_2^{\epsilon_2} \cdots t_1^{\epsilon_i} x^i$ where $\epsilon_j = 1$ or 0. If $i = 1$ then this is true, since the only monomials we have are αx and $\beta t_1 x$ where $\alpha, \beta \in Q$. Suppose true for $i \leqq n$. If m is of degree $n+1$, then $m = m' t_1^{\epsilon_{n+1}} x$ where m' has degree n, and ϵ_{n+1} is either 1 or 0. By induction $m' = t_1^{\epsilon_1} t_2^{\epsilon_2} \cdots t_n^{\epsilon_n} x^n$ and $m = t_1^{\epsilon_1} t_2^{\epsilon_2} \cdots t_n^{\epsilon_n} t_{n+1}^{\epsilon_{n+1}} x^{n+1}$. Gathering monomials of same degree we have the lemma.

The following notation will be used in the remaining lemmas. Define $\mathcal{S} = \{\sum_I \alpha_I t_1^{\epsilon_1} t_2^{\epsilon_2} \cdots t_i^{\epsilon_i} \neq 0$ where I ranges over all i-tuples $(\epsilon_1, \cdots, \epsilon_i)$ where $\epsilon_j = 1$ or 0, and i ranges over all integers$\}$. Let $\mathfrak{M}$ be all products of elements of $\mathcal{S}$. It is easy to see that $\sigma(\mathcal{S}) \subset \mathcal{S}$ and thus $\sigma(\mathfrak{M}) \subset \mathfrak{M}$. Since $\mathfrak{M}$ is multiplicatively closed, let $V = Q[t_1, t_2, \cdots]$ localized about $\mathfrak{M}$. $\sigma(V) \subset V$ and from this it is not difficult to see that $U = \{x^{-n}\sum_{i=0}^{\infty} v_i x^i \mid v_i \in V\}$ is a subring of K.

LEMMA 5. $Q_1(R) \subset U$.

PROOF. It is sufficient to show that $r \in R$ implies $r^{-1} \in U$. Let $r = \sum_{i=0}^{n} f_i x^{i+i_0}$ where $f_0 \neq 0$ and $f_i \in \mathcal{S}$. $r^{-1} = x^{-i_0}(\sum_{i=0}^{n} f_i x^i)^{-1} = x^{-i_0} \sum_{i=0}^{\infty} b_i x^i$ where $f_0 b_0 = 1$, and $b_n = -f_0^{-1} \sum_{i=1}^{n} f_i \sigma^i(b_{n-i})$. Since $b_0 = f_0^{-1} \in V$ and $\sigma(V) \subset V$ we have by induction that $b_n \in V$. Thus $r^{-1} \in U$.

LEMMA 6. *$(t_1 - t_2^2)^{-1}$ is not contained in U.*

PROOF. If $(t_1 - t_2^2)^{-1}$ were in U then it would be in V. Thus $(t_1 - t_2^2)^{-1} = f/g$ where $g \in \mathfrak{M}$ and $f \in Q[t_1, \cdots]$. Therefore $g = f(t_1 - t_2^2)$ where $t_1 - t_2^2$ is irreducible and hence divides $g = \pi p_i$ with $p_i \in \mathcal{S}$. This is impossible however since no p_i has terms with square factors.

The theorem now follows since $(t_1 - t_2^2)^{-1} \notin Q_1(R)$ implies $Q_1(R)$ is not a skew field.

We continue with the same notation, and construct an embedding of height one by factoring out a maximal ideal of $Q_1(K, R)$. Let P be the ideal of $Q[t_1, t_2, \cdots]$ generated by $t_1 - t_2^2, t_2 - t_3^2, \cdots$. It is clear that $\sigma(P) \subset P$. It is also not difficult to see that

$$Q[t_1, \cdots]/P \cong Q[\bar{t}_1, \bar{t}_1^{1/2}, \bar{t}_1^{1/4}, \cdots]$$

and hence P is a prime ideal. σ extends to an automorphism of $Q[\bar{t}_1, \bar{t}_1^{1/2}, \cdots]$ where $\sigma(\bar{t}_1^{1/2^n}) = \bar{t}_1^{1/2^{n+1}}$. In $Q[t_1, \cdots]/P$ every element can be written uniquely as $\sum_k \sum_I \alpha_{I,k} t^k \bar{t}_2^{\epsilon_1} \cdots \bar{t}_{n+1}^{\epsilon_n}$ so that $\mathfrak{M} \cap P = \varnothing$. Therefore factoring out the ideal of $Q_1(Q[x, t_1 x])$ generated by P we get the free algebra $Q[x, t_1 x]$ embedded in the Ore domain $Q[\bar{t}_1, \bar{t}_1^{1/2}, \bar{t}_1^{1/4}, \cdots][x; \sigma]$ and thus in a skew field, namely the domain's field of quotients K_2.

We will write t for $\bar{t}_1$.

THEOREM 2. *$Q_1(K_2, Q[x, tx])$ is a division ring.*

PROOF. Since x and x^{-1} are in Q_1, all integral powers of x are in Q_1. We first observe that if $p(x) \in Q[t, t^{1/2}, \cdots][x; \sigma]$ then there exists integers n and m such that $x^n p(x) x^m \in Q[x, tx]$. Suppose $p(x) = \sum_{i=0}^{k} a_i x^i$ where

$$a_i = \sum_{j=0}^{k_i} \left(\sum_I \alpha_{I,i,j} t^{\epsilon_1/2} \cdots t^{\epsilon_m/2^m} \right) t^j$$

where I runs over the m-tuples $(\epsilon_1, \cdots, \epsilon_m)$ with each $\epsilon_l = 0$ or 1. Pick n to be an integer such that $2^n \geqq \max\{k_i\}$.

Therefore $x^n p(x) = \sum_{i=0}^{k} a_i' x^{i+n}$ where each

$$a'_i = \sum_{j=0}^{k_i}\left(\sum_I \alpha_{I,i,j} t^{\epsilon_1/2^{n+1}} \cdots t^{\epsilon_m/2^{n+m}}\right) t^{j/2^n}.$$

Each j can be written in the form $\gamma_n+\gamma_{n-1}2+\cdots+\gamma_0 2^n$ with $\gamma_i=0$, or 1. Hence

$$t^{j/2^n}t^{\epsilon_1/2^{n+1}} \cdots t^{\epsilon_m/2^{n+m}} = t^{\gamma_0}t^{\gamma_1/2} \cdots t^{\gamma_n/2^n}t^{\epsilon_1/2^{n+1}} \cdots t^{\epsilon_m/t^{n+m}}$$

where $(\gamma_0, \gamma_1, \cdots, \gamma_n, \epsilon_1, \cdots, \epsilon_m)$ is an $n+m$ tuple of zeros and ones. Hence $x^n p(x) x^m \in Q[x, tx]$.

Since $R=Q[t, t^{1/2}, \cdots][x; \sigma]$ is Ore and contained in Q_1 we may assume that an element of Q_1 is written $q(x)^{-1}p(x)$ with $q(x)$ and $p(x)\in R$.

By the above we can choose integers n and m such that both $x^n p(x) x^m$ and $x^n q(x) x^m$ are in $Q[x, tx]$.

Hence $(x^n p(x) x^m)^{-1} \in Q_1$ and

$$1 = (q(x)^{-1}p(x))\left[x^m(x^n p(x) x^m)^{-1}(x^n q(x) x^m)x^{-m}\right] \in Q_1$$

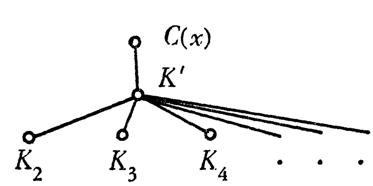
so that Q_1 is a division ring.

COROLLARY 1. $h_{K_2}(Q[x, tx])=1$.

III. **The poset of skew fields.** Let K' be the field of quotients of $Q(t_1, t_2, \cdots)[x;\sigma]$ where $\sigma(t_i)=t_{i+1}$ and $xf=\sigma(f)x$ for $f\in Q(t_1, t_2, \cdots)$. Let F_n be $Q[t_1, t_2, \cdots]$ localized about the prime ideal $P_n=\langle t_1-t_2^n, t_2-t_3^n, \cdots\rangle$ for a positive integer $n>1$. $\sigma(P_n)\subset P_n$ so we can localize $F_n[x;\sigma]$ about the ideal generated by P_n, to obtain S_n. S_n is a local ring which maps onto the field of quotients K_n of $Q(s, s^{1/n}, s^{1/n^2}, \cdots)[y;\sigma]$. If $Q[u, v]$ is the free algebra on two generators, the map $\phi(u)=x$ and $\phi(v)=t_1x$ maps $Q[u, v]$ onto a free algebra generating K'. Likewise $\phi_n(u)=y$ and $\phi_n(v)=sy$ maps $Q[u, v]$ onto a free algebra generating K_n.

THEOREM 3. *$(K_n, \phi_n)\not\geqq(K_m, \phi_m)$ for $n\neq m$.*

PROOF. The quotient field of $Q(s, s^{1/n}, s^{1/n^2}, \cdots)[y;\sigma]$ is K_n where $ys=s^{1/n}y$ and the quotient field of $Q(r, r^{1/m}, r^{1/m^2}, \cdots)[z;\sigma]$ is K_m where $zr=r^{1/m}z$. If $(K_n, \phi_n)\geqq(K_m, \phi_m)$ and γ is the place then $\gamma(s)=\gamma(syy^{-1})=rzz^{-1}=r$ but $\gamma(s^{1/n})=\gamma(ysyy^{-2})=zrzz^{-2}=r^{1/m}$. $\gamma[(s^{1/n})^n]=(r^{1/m})^n=r$ which implies $n=m$.



Hence we have $(K', \phi) > (K_n, \phi_n)$ for each $n > 1$. Furthermore, since $Q_1(K_2, \phi_2)$ is a skew field and hence simple, Lemma 2 says (K_2, ϕ_2) is minimal. By [1, Theorem 27], $\mathcal{P}$ has a maximal element $C(x)$ and we have the subposet $\mathcal{L}$ embedded in $\mathcal{P}$.

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